Twisted $p$-adic $(h, q)$-$L$-functions

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Abstract

Abstract. By using $q$-Volkenborn integral on $\mathbb{Z}_p$, we ([29], [30]) constructed new generating functions of the $(h, q)$-Bernoulli polynomials and numbers. By applying the Mellin transformation to the generating functions, we constructed integral representation of the twisted $(h, q)$-Hurwitz function and twisted $(h, q)$-two-variable $L$-function. By using these functions, we construct twisted new $(h, q)$-partial zeta function which interpolates the twisted $(h, q)$-Bernoulli polynomials and generalized twisted $(h, q)$-Bernoulli numbers at negative integers. We give relation between twisted $(h, q)$-partial zeta functions and twisted $(h, q)$-two-variable $L$-function. We construct twisted new $(h, q)$-partial zeta function which interpolates the twisted $(h, q)$-Bernoulli polynomials:

$$L^{(h)}_{\xi, p, q}(1 - n, t, \chi) = -\frac{B_{n, \chi, \xi}(p^s t, q) - \chi(n)p^{n-1}B_{n, \chi, \xi}(p^{-1}p^s t, q^p)}{n}.$$ 

2000 Mathematics Subject Classification. 11B68, 11S40, 11S80, 11M99, 30B50, 44A05.

Key Words and Phrases. $q$-Bernoulli numbers and polynomials, twisted $q$-Bernoulli numbers and polynomials, $q$-zeta function, $p$-adic $L$-function, twisted $q$-zeta function, twisted $q$-$L$-functions, $q$-Volkenborn integral.

1. Introduction, Definitions and Notations

In [8], Kim constructed $p$-adic $q$-$L$-functions. He gave fundamental properties of these functions. By $p$-adic $q$-integral he also constructed generating function of Carlitz’s $q$-Bernoulli number. Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denoted by the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively, (see [9], [11]). Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}_p$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $1 - q |_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $| x |_p \leq 1$. If $q \in \mathbb{C}$, then we normally assume $| q | < 1$.

Kubota and Leopoldt proved the existence of meromorphic functions, $L_p(s, \chi)$, which is defined over the $p$-adic number field. $L_p(s, \chi)$ is defined by

$$L_p(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \frac{(n, p) = 1}{(n, p) = 1} = (1 - \chi(p)p^{-s})L(s, \chi),$$
where \( L(s, \chi) \) is the Dirichlet \( L \)-function which is defined by

\[
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.
\]

\( L_p(s, \chi) \) interpolates the values

\[
L_p(1-n, \chi) = \frac{(1-\chi(p)p^{n-1})}{n} B_{n, \chi}, \text{ for } n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\},
\]

where \( B_{n, \chi} \) denotes the \( n \)-th generalized Bernoulli numbers associated with the primitive Dirichlet character \( \chi \), and \( \chi_n = \chi w^{-n} \), with the Teichmüller character cf. (3, 2, 5, 3, 7, 18, 17, 21, 22, 24, 23).

Kim, Jang, Rim and Pak [20] defined twisted \( q \)-Bernoulli numbers by using \( p \)-adic invariant integrals on \( \mathbb{Z}_p \). They gave twisted \( q \)-zeta function and \( q \)-\( L \)-series which interpolate twisted \( q \)-Bernoulli numbers. In [27], the author gave relations between these functions and numbers. In [15], by using \( p \)-adic integral representation for the two-variable \( p \)-adic \( L \)-functions, he used the integral representation to extend the \( L \)-function to the large domain, in which it is a meromorphic function in the first variable and an analytic element in the second. These integral representations imply systems of congruences for the generalized Bernoulli polynomials. In [15], by using \( q \)-Volkenborn integration, Kim constructed the new \((h, q)\)-extension of the Bernoulli numbers and polynomials. He defined \((h, q)\)-extension of the \( z \)-eta functions which are interpolated new \((h, q)\)-extension of the Bernoulli numbers and polynomials. In [29], the author defined twisted \((h, q)\)-Bernoulli numbers, \( z \)-eta functions and \( L \)-function. The author also gave relations between these functions and numbers. In [19], Kim and Rim constructed two-variable \( L \)-function, \( L(s, x | \chi) \). They showed that this function interpolates the generalized Bernoulli polynomials associated with \( \chi \). By the Mellin transforms, they gave the complex integral representation for the two-variable Dirichlet \( L \)-function. They also found some properties of the two-variable Dirichlet \( L \)-function. In [10], Kim constructed the two-variable \( p \)-adic \( q \)-\( L \)-function which interpolates the generalized \( q \)-Bernoulli polynomials associated with Dirichlet character. He also gave some \( p \)-adic integrals representation for this two-variable \( p \)-adic \( q \)-\( L \)-function and derived \( q \)-extension of the generalized formula of Diamond and Ferro and Greenberg for the two variable \( p \)-adic \( L \)-function in terms of the \( p \)-adic gamma and log gamma function. In [32], Simsek, D. Kim and Rim defined \( q \)-analogue two-variable \( L \)-function. They generalized these functions.

In [17], Kim constructed the new \( q \)-extension of generalized Bernoulli polynomials attached to \( \chi \) due to his work [15] and derived the existence of a specific \( p \)-adic interpolation function which interpolate the \( q \)-extension of generalized Bernoulli polynomials at negative integers. He gave the values of partial derivative for this function. In this study, we construct twisted version of Kim’s \( p \)-adic \( q \)-\( L \)-function.

For \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{ f \mid f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \} \), the \( p \)-adic \( q \)-integral (or \( q \)-Volkenborn integration) was defined by

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} q^x f(x), \quad (1.1)
\]

where

\[
\mu_q(a + dp^N \mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}, \quad N \in \mathbb{Z}^+ \]

and

\[
[x]_q = \begin{cases} 
\frac{1-q^x}{1-q}, & q \neq 1 \\
{x}, & q = 1
\end{cases} \quad \text{cf. (9, 10, 12, 13, 32).}
\]
\[ I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \] (1.2)

cf. ([9], [12]).

If we take \( f_1(x) = f(x + 1) \) in (1.2), then we have
\[ I_1(f_1) = I_1(f) + f'(0), \] (1.3)
where \( f'(0) = \frac{d}{dx} f(x) \bigg|_{x=0} \), cf. ([14], [11]).

Let \( p \) be a fixed prime. For a fixed positive integer \( f \) with \( (p, f) = 1 \), we set (see [9])
\[
\begin{align*}
\mathbb{X} &= \mathbb{X}_f = \lim_{N \to \infty} \mathbb{Z}/fp^N\mathbb{Z}, \\
\mathbb{X}_1 &= \mathbb{Z}_p, \\
\mathbb{X}^* &= \mathbb{X}_1 \cup \{ a + fp\mathbb{Z}_p : 0 < a < fp, (a, p) = 1 \}
\end{align*}
\]
and
\[ a + fp^N\mathbb{Z}_p = \{ x \in \mathbb{X} | x \equiv a (\text{mod } fp^N) \}, \]
where \( a \in \mathbb{Z} \) satisfies the condition \( 0 \leq a < fp \). For \( f \in UD(\mathbb{Z}_p, \mathbb{C}_p) \),
\[ \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \int_{\mathbb{X}} f(x) d\mu_1(x), \] (1.4)
(see [10], [13], for details). By (1.3), we easily see that
\[ I_1(f_b) = I_1(f) + \sum_{j=0}^{b-1} f'(j), \] (1.5)
where \( f_b(x) = f(x + b), b \in \mathbb{Z}^+ \).

Let
\[ T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n}, \]
where \( C_{p^n} = \{ \xi \in \mathbb{C}_p | \xi p^n = 1 \} \) is the cyclic group of order \( p^n \). For \( \xi \in T_p \), we denote by \( \phi_\xi : \mathbb{Z}_p \to \mathbb{C}_p \) the locally constant function \( x \to \xi^x \) ([6], [20]).

By using \( q \)-Volkenborn integration ([9], [10], [11], [12], [13], [15]), the author [29] constructed generating function of the twisted \((h, q)\)-extension of Bernoulli numbers \( B^{(h)}_{n, \xi}(q) \) by means of the following generating function
\[ F^{(h)}_{\xi, q}(t) = \frac{\log q^h + t}{\xi q^h e^t - 1} = \sum_{n=0}^{\infty} B^{(h)}_{n, \xi}(q) \frac{t^n}{n!}. \]

By using the above equation, and following the usual convention of symbolically replacing \((B^{(h)}_{\xi}(q))^n \) by \( B^{(h)}_{n, \xi}(q) \), we have
\[ B^{(h)}_{0, \xi}(q) = \frac{\log q^h}{\xi q^h - 1} \] (1.6)
\[ \xi q^h (B^{(h)}_{\xi}(q) + 1)^n - B^{(h)}_{n, \xi}(q) = \delta_{1, n}, \ n \geq 1, \]
where \( \delta_{1, n} \) is denoted Kronecker symbol. We note that if \( \xi \to 1 \), then \( B^{(h)}_{n, \xi}(q) \to B^{(h)}_{n}(q) \) and \( F^{(h)}_{\xi, q}(t) \to F^{(h)}_{q}(t) = \frac{\log q t}{q^t e^t - 1} \) (see [15]). If \( \xi \to 1 \) and \( q \to 1 \), then \( F^{(h)}_{\xi, q}(t) \to F(t) = \frac{1}{e^t - 1} \) and \( B_{n, \xi}(q) \to B_n \) are the usual Bernoulli numbers (see [33]).
Remark 1. Shiratani and Yamamoto\[24\] constructed a $p$-adic interpolation $G_{\mu}(s, u)$ of the Frobenius-Euler numbers $H_{n}(u)$ and as its application, they obtained an explicit formula for $L_{p}(u, \chi)$ with any Dirichlet character $\chi$. Let $u$ be an algebraic number. For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler numbers $H_{n}(u)$ belonging to $u$ are defined by means of of the generating function
\[
\frac{1 - u}{e^t - u} = e^{H(u)t}
\]
with usual convention of symbolically replacing $H^{n}(u)$ by $H_{n}(u)$. Thus for the Frobenius-Euler numbers $H_{n}(u)$ belonging to $u$, we have (see\[23\])
\[
\frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_{n}(u) \frac{t^{n}}{n!}.
\]
By using the above equation, and following the usual convention of symbolically replacing $H^{n}(u)$ by $H_{n}(u)$, we have
\[
H_{0} = 1 \text{ and } (H(u) + 1)^{n} = uH_{n}(u) \text{ for } (n \geq 1).
\]

We also note that
\[
H_{n}(-1) = \mathcal{E}_{n},
\]
where $\mathcal{E}_{n}$ denotes the aforementioned Tsumura version (see\[23\]) of the classical Euler numbers $E_{n}$ which we recalled above. Let $\xi^{\ast} = 1, \xi \neq 1$.
\[
\frac{t}{\xi e^{t} - 1} = \sum_{n=0}^{\infty} B_{n, \xi} \frac{t^{n}}{n!} \text{ cf. }[6]
\]
\[
\frac{t}{\xi e^{t} - 1} = \frac{1 - \xi^{-1}}{e^{t} - \xi^{-1}} = \frac{t}{\xi^{-1}} \frac{1 - \xi^{-1}}{e^{t} - \xi^{-1}} = \frac{1}{\xi - 1} \sum_{n=0}^{\infty} (n + 1)H_{n}(\xi^{-1}) \frac{t^{n+1}}{(n + 1)!}
\]
By comparing the coefficients on both sides of the above equations, we easily see that
\[
B_{n+1, \xi} = \frac{1}{\xi - 1} (n + 1)H_{n}(\xi^{-1}).
\]

Therefore, if $\xi \neq 1$, then we obtain relations between Frobenius-Euler numbers, $H_{n}(\xi^{-1})$ and twisted Bernoulli numbers, $B_{n, \xi}$. If $\xi = 1$, then twisted Bernoulli numbers, $B_{n, \xi}$ are reduced to classical Bernoulli numbers, $B_{n}$, for detail about this numbers and polynomials see cf.\[9, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 33, 1\].

Twisted $(h, q)$-extension of Bernoulli polynomials $B_{n, \xi}^{(h)}(z, q)$ are defined by means of the generating function\[29\]
\[
F_{\xi, q}^{(h)}(t, z) = \frac{(t + \log q^{h}) e^{tz}}{q^{h} e^{t} - 1} = \sum_{n=0}^{\infty} B_{n, \xi}^{(h)}(z, q) \frac{t^{n}}{n!},
\]
where $B_{n, \xi}^{(h)}(0, q) = B_{n, \xi}^{(h)}(q)$. By using Cauchy product in \[17\], we have
\[
B_{n, \xi}^{(h)}(z, q) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} B_{k, \xi}^{(h)}(q).
\]
We summarize our paper as follows:

In section 2, by applying the Mellin transformation to the generating functions of the Bernoulli polynomials and generalized Bernoulli polynomials, we give integral representation of the twisted $(h, q)$-Hurwitz function and twisted $(h, q)$-two-variable $L$-function. By using these functions, we construct twisted new $(h, q)$-partial
zeta function which interpolates the twisted \((h, q)\)-Bernoulli polynomials at negative integers. We give relation between twisted \((h, q)\)-partial zeta functions and twisted \((h, q)\)-two-variable \(L\)-function.

In section 3, we construct \(p\)-adic twisted \((h, q)\)-functions \((L_{\xi,\rho,q}^{(h)}(s, t, \chi))\), which are interpolate the twisted generalized \((h, q)\)-Bernoulli polynomials at negative integers. We calculate residue of \(L_{\xi,\rho,q}^{(h)}(s, t, \chi)\) at \(s = 1\). We also give fundamental properties of this functions.

2. \((h, q)\)-PARTIAL ZETA FUNCTIONS

Our primary aim in this section is to define twisted \((h, q)\)-partial zeta functions. We give the relation between generating function in \([17]\) and twisted \((h, q)\)-Hurwitz zeta function\([30]\). In this section, we assume that \(q \in \mathbb{C}\) with \(|q| < 1\). For \(s \in \mathbb{C}\), by applying the Mellin transformation to \([17]\), we have

\[
\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} \xi_q(t, x)dt = \zeta^{(h)}(s, x).
\]

By using the above equation, we\([29]\), \([30]\) defined twisted \((h, q)\)-Hurwitz zeta function as follows:

**Definition 1.** Let \(s \in \mathbb{C}\), \(x \in \mathbb{R}^+\). We define

\[
\zeta_{\xi,q}^{(h)}(s, x) = \sum_{n=0}^{\infty} \frac{q^n \zeta^{(h)}(n, x)}{(n + x)^s} - \frac{h \log q}{s - 1} \sum_{n=0}^{\infty} \frac{q^n}{(n + x)^{s-1}}.
\] (2.1)

**Remark 2.** Observe that when \(w \to 1\), \(\zeta_{\xi,q}^{(h)}(s, x)\) reduces to

\[
\zeta_{\xi,q}(s, x) = \sum_{n=0}^{\infty} \frac{q^n}{(n + x)^s} - \frac{h \log q}{s - 1} \sum_{n=0}^{\infty} \frac{q^n}{(n + x)^{s-1}}
\]

(see \([15]\)). When \(q \to 1\), \(\xi \to 1\), \(\zeta_{\xi,q}^{(h)}(s)\) reduces to \(\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\), Riemann zeta function and \(\zeta_{\xi,q}^{(h)}(s, x)\) reduces to \(\zeta(s, x) = \sum_{n=1}^{\infty} \frac{1}{(n + x)^s}\), Hurwitz zeta function. Observe that when \(x = 1\) in \([27]\), we easily see that \(\zeta_{\xi,q}^{(h)}(s, 1) = \zeta_{\xi,q}^{(h)}(s)\), which denotes twisted zeta function (see \([29]\)). We also note that \(\zeta_{\xi,q}^{(h)}(s)\) are analytically continued for \(\text{Re}(s) > 1\). \(\lim_{\xi \to 1} \zeta_{\xi,q}^{(h)}(s) = \zeta_{\xi,q}(s)\), which is given in \([15]\).

**Theorem 1.** \([30]\) Let \(n \in \mathbb{Z}^+\). We obtain

\[
\zeta_{\xi,q}^{(h)}(1 - n, x) = -\frac{B_{n,q}(x, q)}{n}.
\] (2.2)

Twisted \((h, q)\)-\(L\)-function is defined as follows:

**Definition 2.** \([29]\) Let \(s \in \mathbb{C}\). Let \(\chi\) be a Dirichlet character of conductor \(f \in \mathbb{Z}^+\). We define

\[
L_{\xi,q}^{(h)}(s, \chi) = \sum_{n=1}^{\infty} \frac{q^n \chi(n \xi)}{n^s} - \frac{\log q}{(s - 1)} \sum_{n=1}^{\infty} \frac{q^n \chi(n) \zeta^{(h)}(n, x)}{n^{s-1}}.
\] (2.3)

Observe that if \(\xi \to 1\), \(2.3\) reduces to \(L_q^{(h)}(s, \chi)\) function (see \([15]\)).

**Theorem 2.** \([29]\) Let \(\chi\) be a Dirichlet character of conductor \(f \in \mathbb{Z}^+\). Let \(n \in \mathbb{Z}^+\). We have

\[
L_{\xi,q}^{(h)}(1 - n, \chi) = -\frac{B_{n+1,\chi,q}(q)}{n + 1}.
\]

Relation between \(\zeta_{w,q}^{(h)}(s, z)\) and \(L_{w,q}^{(h)}(s, \chi)\) are given by the following theorem\([29]\):

**Theorem 3.** Let \(s \in \mathbb{C}\). Let \(\chi\) be a Dirichlet character of conductor \(f \in \mathbb{Z}^+\). We have

\[
L_{\xi,q}^{(h)}(s, \chi) = \frac{1}{f} \sum_{a=0}^{f-1} q^{ha} \chi(a) \zeta_{\xi,q}^{(h)}(s, a f).
\] (2.4)
The generalized twisted \((h, q)\)-extension of Bernoulli polynomials \(B_{n,\chi,\xi}^{(h)}(z, q)\) are defined by means of the generating function\,[30]::

\[
F_{\chi,\xi,q}^{(h)}(t, z) = \sum_{n=0}^{\infty} B_{n,\chi,\xi}^{(h)}(z, q) \frac{t^n}{n!}, \quad \text{cf. (29), (30)}.
\]

We are now ready to define the new twisted two-variable \((h, q)\)-functions. For \(s \in \mathbb{C}\), we consider the below integral which is known the Mellin transformation of \(F_{\chi,\xi,q}^{(h)}(t, z)\)[30].

\[
\frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-2} F_{\chi,\xi,q}^{(h)}(-t, z) dt = F_{\chi,\xi,q}^{(h)}(s, z, \chi).
\] (2.8)

We are now ready to define the new twisted two-variable \((h, q)\)-function. By using the above integral representation we arrive at the following definition:
Definition 3. [30] Let \( s \in \mathbb{C} \). Let \( \chi \) be a Dirichlet character of conductor \( f \in \mathbb{Z}^+ \). We define

\[
L_{\xi,q}^{(h)}(s,z,\chi) = \sum_{m=0}^{\infty} \frac{\chi(m)\phi_{\xi}(m)q^{hm}}{(z+m)^s} - \frac{\log q^h}{s-1} \sum_{m=0}^{\infty} \frac{\chi(m)\phi_{\xi}(m)q^{hm}}{(z+m)^{s-1}}. \tag{2.9}
\]

Relation between \( c_{\xi,q}^{(h)}(s,z) \) and \( L_{\xi,q}^{(h)}(s,z,\chi) \) is given by the following theorem:

Theorem 4. [30] We have

\[
L_{\xi,q}^{(h)}(s,z,\chi) = \frac{1}{f} \sum_{a=1}^{f} q^{ha} \xi^a \chi(a) c_{\xi^a,q}(s, a + \frac{z}{f}). \tag{2.10}
\]

Theorem 5. Let \( \chi \) be a Dirichlet character of conductor \( f \in \mathbb{Z}^+ \). Let \( n \in \mathbb{Z}^+ \). We have

\[
L_{\xi,q}^{(h)}(1-n,z,\chi) = \frac{\pi^{(h)}}{n}. \tag{2.11}
\]

Proof. Substituting \( s = 1 - n, n \in \mathbb{Z}^+ \) into (2.10), we obtain

\[
L_{\xi,q}^{(h)}(1-n,z,\chi) = \sum_{m=0}^{\infty} \frac{\chi(m)\phi_{\xi}(m)q^{hm}}{(z+m)^{1-n}} - \frac{\log q^h}{s-1} \sum_{m=0}^{\infty} \frac{\chi(m)\phi_{\xi}(m)q^{hm}}{(z+m)^{n-1}}.
\]

Substituting (2.9) into the above equation, we arrive at the desired result. \( \square \)

Remark 3. Note that Proof of (2.11) runs parallel to that of Theorem 8 in [33], for \( s = 1 - n, n \in \mathbb{Z}^+ \) and by using Cauchy Residue Theorem in (2.9), we arrive at the another proof the above theorem[30]. Observe that \( \lim_{p \to 0} L_{\xi,q}^{(h)}(s,1,\chi) = L_{q}^{(h)}(s,\chi) \). For \( q \to 1 \) and \( z = 1 \), then relations (2.9) reduces to the following well-known definition:

Let \( r \in \mathbb{Z}^+ \), set of positive integers, let \( \chi \) be a Dirichlet character of conductor \( f \in \mathbb{Z}^+ \), and let \( \xi^r = 1, \xi \neq 1 \). Twisted \( L \)-functions are defined by [22]

\[
L_{\xi}(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)\xi^n}{n^s}.
\]

Since the function \( n \to \chi(n)\xi^n \) has period \( fr \), this is a special case of the Dirichlet \( L \)-functions. Koblitz[22] and the author gave relation between \( L(s,\chi,\xi) \) and twisted Bernoulli numbers, \( B_{n,\chi,\xi} \) at non-positive integers(see [21], [22], [26], [25]).

Let \( s \) be a complex variable, \( a \) and \( f \) be integers with \( 0 < a < f \). Then we define new twisted \((h,q)\)-partial zeta function as follows:

Definition 4.

\[
H_{\xi,q}^{(h)}(s,a:f) = \sum_{n \equiv a \pmod{f} \atop n > 0} q^{nh} \xi^n \frac{n^s}{s-1} - \sum_{n \equiv a \pmod{f} \atop n > 0} q^{nh} \xi^n \frac{n^s}{s^2 - 1}.
\]

By using the above definition, relation between \( H_{\xi,q}^{(h)}(s,a:f) \) and \( c_{\xi,q}^{(h)}(s,x) \) are given by

Theorem 6.

\[
H_{\xi,q}^{(h)}(s,a:f) = q^{ha} \xi^a f^{-s} c_{\xi^a,q}(s, a/f). \tag{2.12}
\]
Remark 4. The function $H_{\xi,q}^{(h)}(s,a : f)$ is meromorphic function for $s \in \mathbb{C}$ with simple pole at $s = 1$, having residue, $\text{Re} \ z_{s=1}H_{\xi,q}^{(h)}(s,a : f)$:

$$\text{Re} \ z_{s=1}H_{\xi,q}^{(h)}(s,a : f) = \lim_{s \to 1} (s - 1)H_{\xi,q}^{(h)}(s,a : f) = \frac{q^{ha} \zeta^a \log q^h}{q^hf \xi^f - 1}.$$ 

By (2.12) and (2.13), we have

**Corollary 1.** Let $n \in \mathbb{Z}^+$. We have

$$H_{\xi,q}^{(h)}(1 - n,a : f) = - \frac{q^{ha} \zeta^a f^{n-1}B_{n,\xi^f}(\frac{a}{f}, q^f)}{n}. \quad (2.13)$$

We modify the twisted $(h,q)$-extension of the partial zeta function as follows:

**Corollary 2.** Let $s \in \mathbb{C}$. We have

$$H_{\xi,q}^{(h)}(s,a : f) = \frac{a^s - q^{ha} \zeta^a}{(s-1)f} \sum_{k=0}^{\infty} \left( \frac{1}{k} \right) B_{k,\xi^f}(q^f). \quad (2.14)$$

**Proof.**

By using (13) and (2.13), we have

$$H_{\xi,q}^{(h)}(1 - n,a : f) = - \frac{q^{ha} \zeta^a f^{n-1}}{n} \sum_{k=0}^{n} \left( \frac{n}{k} \right) B_{k,\xi^f}(q^f).$$

Substituting $s = 1 - n$, and after some elementary calculations, we arrive at the desired result. \qed

Observe that if $\xi = 1$, then $H_{q}^{(h)}(s,a : f)$ is reduced to the following equation cf. [17]:

$$H_{q}^{(h)}(s,a : f) = \frac{a^s - q^{ha} \zeta^a}{(s-1)f} \sum_{k=0}^{\infty} \left( \frac{1}{k} \right) B_{k,\xi^f}(q^f).$$

By using (2.1), (2.12) and (2.14), we arrive at the following theorem:

**Theorem 7.** Let $s \in \mathbb{C}$. Let $\chi \ (\chi \neq 1)$ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$.

$$L_{\xi,q}^{(h)}(s,\chi) = \sum_{a=1}^{f} \chi(a)H_{\xi,q}^{(h)}(s,a : f)$$

$$= \frac{1}{(s-1)f} \sum_{a=1}^{f} \chi(a) a^s - q^{ha} \zeta^a \sum_{k=0}^{\infty} \left( \frac{1}{k} \right) B_{k,\xi^f}(q^f).$$

We now define new twisted $(h,q)$-partial Hurwitz zeta function as follows:

**Definition 5.**

$$H_{\xi,q}^{(h)}(s,x + a : f) = \sum_{n \equiv a \ (\text{mod} \ f)}^{\infty} \frac{q^{ha} \zeta^n}{(x + n)^s} - \log q^{h} \sum_{n \equiv a \ (\text{mod} \ f)}^{\infty} \frac{q^{ha} \zeta^n}{(x + n)^{s-1}}.$$  

Relation between $\zeta_{\xi,q}^{(h)}(s,x)$ and $H_{\xi,q}^{(h)}(s,x + a : f)$ are given by

$$H_{\xi,q}^{(h)}(s,x + a : f) = \frac{q^{ha} a^s}{f} \zeta_{\xi,q}^{(h)}(s, \frac{a + x}{f}). \quad (2.15)$$
Let $n \in \mathbb{Z}^+$. Substituting (2.2) in the above and using (1.8), we obtain
\[
H^{(h)}_{\xi,q}(1-n,x+a : f) = \frac{H^{(h)}_{\xi,q}(a+x,q^f)}{n - \frac{q^h a x^n}{f}} - n \sum_{k=0}^{n} \binom{n}{k} \left( \frac{x+a}{f} \right)^{n-k} B^{(h)}_{n,k}(\frac{x+a}{f},q^f).
\]
Thus, by the above equation, we obtain
\[
H^{(h)}_{\xi,q}(s,x+a : f) = \frac{1-s^q a^s}{(s-1)f} \sum_{k=0}^{\infty} \binom{1-s}{k} \left( \frac{f}{x+a} \right)^k B^{(h)}_{k,\xi}(q^f).
\]
By (2.10) and (2.13), we obtain the following relations:
\[
L^{(h)}_{\xi,q}(s,x,\chi) = \sum_{a=1}^{f} \chi(a) H^{(h)}_{\xi,q}(a+x,q^f) = \frac{1}{(s-1)f} \sum_{a=1}^{f} \chi(a)(x+a)^{1-s} q^h a^s \sum_{k=0}^{\infty} \binom{1-s}{k} \left( \frac{f}{x+a} \right)^k B^{(h)}_{k,\xi}(q^f).
\]
By the above equation, $L^{(h)}_{\xi,q}(s,x,\chi)$ is an analytic for $x \in \mathbb{R}$ with $0 < x < 1$ and $s \in \mathbb{C}$ except $s = 1$.

**Remark 5.** Observe that if $\xi = 1$, then $L^{(h)}_{\xi,q}(s,x,\chi)$ is reduced to the following equation cf. [17]:
\[
L^{(h)}_{q}(s,x,\chi) = \frac{1}{(s-1)f} \sum_{a=1}^{f} \chi(a)(x+a)^{1-s} q^h a^s \sum_{k=0}^{\infty} \binom{1-s}{k} \left( \frac{f}{x+a} \right)^k B^{(h)}_{k}(q^f).
\]
By (2.11), the values of $L^{(h)}_{\xi,q}(s,x,\chi)$ at negative integers are algebraic, hence may be regarded as lying in an extension of $\mathbb{Q}_p$. Consequently, we investigate a $p$-adic function which agrees with at negative integers in the next section.

Substituting $s = 0$ into (2.10), we obtain
\[
L^{(h)}_{\xi,q}(0,x,\chi) = \frac{1}{f} \sum_{a=1}^{f} \chi(a)(x+a)^{q^h a^s} \left( \frac{f q^h \log q^h - (x+a)(\xi f q^h - 1) \log q^h}{(x+a)(\xi f q^h - 1)^2} \right).
\]

### 3. Twisted $p$-adic interpolation function for the $q$-extension of the generalized Bernoulli polynomials

In this section, we can use some notations which are due to Kim [17] and Washington [34]. The integer $p^*$ is defined by $p^* = p$ if $p > 2$ and $p^* = 4$ if $p = 2$ cf. (7, 17, 35). Let $w$ denote the Teichmüller character, having conductor $f_w = p^*$. For an arbitrary character $\chi$, we define $\chi_n = \chi \circ w^{-n}$, where $n \in \mathbb{Z}$, in the sense of the product of characters. In this section, if $q \in \mathbb{C}_p$, then we assume $|1-q|_p < p^{-\frac{1}{p^*}}$. Let $<a> = w^{-1}(a) a = \frac{a}{w(a)}$. We note that $<a> \equiv 1(\text{mod } p^* \mathbb{Z}_p)$. Thus, we see that
\[
< a + p^* t > = w^{-1}(a + p^* t)(a + p^* t) = w^{-1}(a) a + w^{-1}(a)(p^* t) \equiv 1(\text{mod } p^* \mathbb{Z}_p[t]),
\]
where $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, $(a,p) = 1$. The $p$-adic logarithm function, $\log_p$, is the unique function $\mathbb{C}_p^\times \to \mathbb{C}_p$ that satisfies the following conditions:

1) $\log_p(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$, $|x|_p < 1$,
ii) \( \log_p(xy) = \log_p x + \log_p y, \forall x, y \in \mathbb{C}_p^* \),

iii) \( \log_p p = 0. \)

Let

\[
A_j(x) = \sum_{n=0}^{\infty} a_{n,j} x^n, a_{n,j} \in \mathbb{C}_p, j = 0, 1, 2, \ldots
\]

be a sequence of power series, each of which converges in a fixed subset

\[
D = \{ s \in \mathbb{C}_p : | s |_p \leq | p^* |^{-1} p^{-\frac{1}{p-1}} \}
\]

of \( \mathbb{C}_p \) such that

1) \( a_{n,j} \to a_{n,0} \) as \( j \to \infty \), for \( \forall n, \)

2) for each \( s \in D \) and \( \epsilon > 0 \), there exists \( n_0 = n_0(s, \epsilon) \) such that \( | \sum_{n \geq n_0} a_{n,j} s^n | < \epsilon \) for \( \forall j \). Then \( \lim_{j \to \infty} A_j(s) = A_0(s) \) for all \( s \in D \). This is used by Washington [34] to show that each of the function \( w^{-s}(a)a^s \) and

\[
\sum_{k=0}^{\infty} \left( \begin{array}{c} s \\ k \end{array} \right) \left( \frac{F}{a} \right)^k B_k
\]

where \( F \) is the multiple of \( p^* \) and \( f = f_\chi \), is analytic in \( D \). We consider the twisted \( p \)-adic analogs of the twisted two variable \( q \)-L-functions, \( L^{(h)}_{\xi,q}(s,t,\chi) \). These functions are the \( q \)-analogs of the \( p \)-adic interpolation functions for the generalized twisted Bernoulli polynomials attached to \( \chi \). Let \( F \) be a positive integral multiple of \( p^* \) and \( f = f_\chi \).

We define

\[
L^{(h)}_{\xi,p,q}(s,t,\chi) = \frac{1}{(s-1)F} \sum_{a=1 \atop (a,p) = 1}^{F} \chi(a) < a + p^* t > 1-s q^{ha} \xi^a \sum_{k=0}^{\infty} \left( \begin{array}{c} 1-s \\ k \end{array} \right) \left( \frac{F}{a + p^* t} \right)^k B^{(h)}_{k,\xi^a}(q^F).
\]

Then \( L^{(h)}_{\xi,p,q}(s,t,\chi) \) is analytic for \( t \in \mathbb{C}_p \) with \( | t |_p \leq 1 \), provided \( s \in D \), except \( s = 1 \) when \( \chi \neq 1 \). For \( t \in \mathbb{C}_p \) with \( | t |_p \leq 1 \), we see that

\[
\sum_{k=0}^{\infty} \left( \begin{array}{c} 1-s \\ k \end{array} \right) \left( \frac{F}{a + p^* t} \right)^k B^{(h)}_{k,\xi^a}(q^F)
\]

is analytic for \( s \in D \). By definition of \( < a + p^* t > \), it is readily follows that

\[
< a + p^* t >^s = < a >^s \sum_{k=0}^{\infty} \left( \begin{array}{c} s \\ k \end{array} \right) (a^{-1} p^* t)^k
\]

is analytic for \( t \in \mathbb{C}_p \) with \( | t |_p \leq 1 \) when \( s \in D \). Since \( (s-1)L^{(h)}_{\xi,p,q}(s,t,\chi) \) is a finite sum of products of these two functions, it must also be analytic for \( t \in \mathbb{C}_p \) with \( | t |_p \leq 1 \), whenever \( s \in D \).

Observe that

\[
\lim_{s \to 1} (s-1)L^{(h)}_{\xi,p,q}(s,t,\chi) = \frac{1}{F} \sum_{a=1 \atop (a,p) = 1}^{F} \chi(a)q^{ha} \xi^a B^{(h)}_{0,\xi^a}(q^F).
\]
Substituting $\chi = 1$ in the above, then we have
\[
\lim_{s \to 1} (s - 1) L_{\xi; p, q}^{(h)}(s, t, \chi) = \frac{B_{0, F}^{(h)}(q^F)}{F} \sum_{a = 1}^{F} q^{h a \xi^a} \\
= \frac{B_{0, F}^{(h)}(q^F)}{F} \left( \frac{1 - q^{h F \xi^F}}{1 - q^{h F}} - \frac{1 - q^{h p F}}{1 - q^{p F}} \right).
\]

By definition of $B_{0, F}^{(h)}(q^F)$ in (1.6), we obtain
\[
\Re z = 1 L_{\xi; p, q}^{(h)}(s, t, \chi) = \lim_{s \to 1} (s - 1) L_{\xi; p, q}^{(h)}(s, t, \chi) = \frac{\log q^h}{q^h \xi - 1} \left( \frac{1 - q^{h F \xi^F}}{1 - q^{h F}} - \frac{1 - q^{h p F}}{1 - q^{p F}} \right),
\]
when $\chi = 1$. Let $n \in \mathbb{Z}^+$ and $t \in \mathbb{C}_p$ with $|t|_p \leq 1$. Since $F$ must be a multiple of $f = f_{x_n}$, by (2.7), we obtain
\[
B_{n, x_n, \xi}^{(h)}(p^s t, q) = F^{n-1} \sum_{a = 0}^{F} \chi_n(a) \xi^a q^{h a} B_{n, F}^{(h)} \left( \frac{a + p^s t}{F}, q^F \right).
\]
If $\chi_n(p) = 0$, then $(p, f_{x_n}) = 1$, so that $\frac{F}{p}$ is a multiple of $f_{x_n}$. Consequently, we get
\[
\chi_n(p) p^{n-1} B_{n, x_n, \xi}^{(h)}(p^{-1} p^s t, q^F) = F^{n-1} \sum_{\substack{a = 0 \\ p \mid a}}^{F} \chi_n(a) \xi^a q^{h a} B_{n, F}^{(h)} \left( \frac{a + p^s t}{F}, q^F \right).
\]

The difference of (3.2) and (3.3), we have
\[
B_{n, x_n, \xi}^{(h)}(p^s t, q) - \chi_n(p) p^{n-1} B_{n, x_n, \xi}^{(h)}(p^{-1} p^s t, q^F) = F^{n-1} \sum_{\substack{a = 0 \\ p \mid a}}^{F} \chi_n(a) \xi^a q^{h a} B_{n, F}^{(h)} \left( \frac{a + p^s t}{F}, q^F \right).
\]

By using (1.8), we obtain
\[
B_{n, F}^{(h)} \left( \frac{a + p^s t}{F}, q^F \right) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{a + p^s t}{F} \right)^{n-k} B_{k, F}^{(h)}(q^F) = (a + p^s t)^n F^{-n} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{F}{a + p^s t} \right)^k B_{k, F}^{(h)}(q^F).
\]

Since $\chi_n(a) = \chi(a) w^{-n}(a)$, $(a, p) = 1$, and $t \in \mathbb{C}_p$ with $|t|_p \leq 1$, we have
\[
B_{n, x_n, \xi}^{(h)}(p^s t, q) - \chi_n(p) p^{n-1} B_{n, x_n, \xi}^{(h)}(p^{-1} p^s t, q^F) = \frac{1}{F} \sum_{a = 1}^{F} \chi(a) < a + p^s t > q^{h a \xi^a} \sum_{k=0}^{\infty} \binom{n}{k} \left( \frac{F}{a + p^s t} \right)^k B_{k, F}^{(h)}(q^F).
Substituting $s = 1 - n$, $n \in \mathbb{Z}^+$, into \([3.1]\), we obtain
\[
L^{(h)}_{\xi,p,q}(1-n,t,\chi) = -\frac{B^{(h)}_{n,\chi,\xi}(p^*t,q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi,\xi}(p^{-1}p^*t,q^p)}{n}.
\]
Consequently, we arrive at the following main theorem:

**Theorem 8.** Let $F$ be a positive integral multiple of $p^*$ and $f = f_{\xi,\chi}$, and let
\[
L^{(h)}_{\xi,p,q}(s,t,\chi) = \frac{1}{(s-1)^F} \sum_{a=1}^{F} \chi(a) < a + p^*t >^{1-s} q^{ha} \sum_{k=0}^{\infty} \left( \frac{1}{k} \right) (F)_{a+p^*t}^k B^{(h)}_{\xi}(q^F).
\]
Then $L^{(h)}_{\xi,p,q}(s,t,\chi)$ is analytic for $h \in \mathbb{Z}^+$ and $t \in \mathbb{C}_p$ with $t \mid p \leq 1$, provided $s \in D$, except $s = 1$. Also, if $t \in \mathbb{C}_p$ with $t \mid p \leq 1$, this function is analytic for $s \in D$ when $\chi \neq 1$, and meromorphic for $s \in D$, with simple pole at $s = 1$ having residue
\[
\frac{\log q^h}{q^h - 1} \left( \frac{1 - q^{hF} }{1 - q^h} - \frac{1 - q^{hpF}}{1 - q^p} \right)
\]
when $\chi = 1$. In addition, for each $n \in \mathbb{Z}^+$, we have
\[
L^{(h)}_{\xi,p,q}(1-n,t,\chi) = \frac{B^{(h)}_{n,\chi,\xi}(p^*t,q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi,\xi}(p^{-1}p^*t,q^p)}{n}.
\]

**Remark 6.** Observe that $\lim_{q^{-1}} L^{(h)}_{\xi,p,q}(s,t,\chi) = L^{(h)}_{\xi,p,q}(s,t,\chi)$ cf. \([17]\). $\lim_{q^{-1}} L^{(h)}_{\xi,p,q}(s,0,\chi) = L^{(h)}_{\xi,p,q}(s,\chi)$ cf. \([7,8]\). $\lim_{q^{-1}} L^{(h)}_{\xi,p,q}(s,\chi) = L^{(h)}_{\xi,p,q}(s,\chi)$, cf. \([2,3,5,21,22,34,31]\).

We defined Witt’s formula for $B^{(h)}_{n,\chi,\xi}(z,q)$ polynomials as follows:
\[
B^{(h)}_{n,\chi,\xi}(q) = \int_X \chi(x)q^{hz} d\mu_q(x), \text{ cf. } \([29,30]\)
\]
where $|1 - q|_p \leq p^{-\frac{1}{m-1}}$.

By using this formula, we define
\[
L^{(h)}_{\xi,p,q}(s,\chi) = \frac{1}{s-1} \int_X \chi(x)q^{hz} < x >^{s-1} q^{hz} d\mu_q(x),
\]
where $s \in \mathbb{C}_p$.

Substituting $s = 1 - n$, $n \in \mathbb{Z}^+$, into the above, we have
\[
L^{(h)}_{\xi,p,q}(1-n,\chi) = -\frac{1}{n} \int_X \chi(x)q^{hz} < x >^{n-1} q^{hz} d\mu_q(x) = -\frac{1}{n} \left( \int X \chi_n(x)q^{hz} x^m d\mu_q(x) - \int_{p^X} \chi_n(px)q^{p^h} x^m d\mu_q(px) \right) = -\frac{B^{(h)}_{n,\chi,\xi}(q) - \chi_n(p)p^{n-1}B^{(h)}_{n,\chi,\xi}(p^p)}{n}.
\]
Consequently, we arrive at the following theorem:

**Theorem 9.** Let $s \in \mathbb{C}_p$ and let
\[
L^{(h)}_{\xi,p,q}(s,\chi) = \frac{1}{s-1} \int_X \chi(x)q^{hz} < x >^{s-1} q^{hz} d\mu_q(x).
\]
For $n \in \mathbb{Z}^+$, we have
\[
L^{(h)}_{\xi,p,q}(1-n,\chi) = -\frac{1}{n} \int_{X, < x >} \xi^x \chi(x) < x >^{n-1} q^{hx} d\mu_q(x) = -B^{(h)}_{n,\xi,p,q}(q) - \chi_n(p)p^{n-1} B^{(h)}_{n,\xi,p,q}(q^p)\
\]

Acknowledgement 1. This paper was supported by the Scientific Research Project Administration Akdeniz University.

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