Some Supercongruences Occurring In Truncated Hypergeometric Series

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Abstract

For the purposes of this paper supercongruences are congruences between terminating hypergeometric series and quotients of \( p \)-adic Gamma functions that are stronger than those one can expect to prove using commutative formal group laws. We prove a number of such supercongruences by using classical hypergeometric transformation formulae. These formulae, most of which are decades or centuries old, allow us to write the terminating series as the ratio of products of \( \Gamma \)-values. At this point sums have become quotients. Writing these \( \Gamma \)-quotients as \( \Gamma_p \)-quotients, we are in a situation that is well-suited for proving \( p \)-adic congruences. These \( \Gamma_p \)-functions can be \( p \)-adically approximated by their Taylor series expansions. Sometimes there is cancelation of the lower order terms, leading to stronger congruences. Using this technique we prove, among other things, a conjecture of Kibelbek and a strengthened version of a conjecture of van Hamme.

Keywords: Supercongruences, Hypergeometric Series, \( p \)-adic Gamma functions.

1. Introduction

Set \( (a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \), the rising factorial or the Pochhammer symbol, where \( \Gamma(x) \) is the Gamma function. For \( r \) a nonnegative integer and \( \alpha_i, \beta_i \in \mathbb{C} \) with \( \beta_i \), the (generalized) hypergeometric series \( _{r+1}F_r \) defined by

\[
_{r+1}F_r \left[ \begin{array}{c} \alpha_1 \ldots \alpha_{r+1} \\ \beta_1 \ldots \beta_r \end{array} ; \lambda \right] := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_{r+1})_k}{(\beta_1)_k \cdots (\beta_r)_k} \cdot \lambda^k \frac{k!}{k!}
\]

converges for \( |\lambda| < 1 \) if it is well-defined. When any of the \( \alpha_i \) is a negative integer and none of the \( \beta_i \) is a negative integer larger than \( \alpha_i \), the above sum terminates. We set

\[
_{r+1}F_r \left[ \begin{array}{c} \alpha_1 \ldots \alpha_{r+1} \\ \beta_1 \ldots \beta_r \end{array} ; \lambda \right]_n := \sum_{k=0}^{n} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_{r+1})_k}{(\beta_1)_k \cdots (\beta_r)_k} \cdot \lambda^k \frac{k!}{k!},
\]

the truncation of the series after the \( \lambda^n \) term.

Hypergeometric series are of fundamental importance in many research areas including algebraic varieties, differential equations, Fuchsian groups and modular forms. For instance, periods of abelian varieties such as elliptic curves, certain K3 surfaces and other Calabi-Yau manifolds can be described by hypergeometric series ([6]). Indeed, the Euler integral representation of \( _2F_1 \) (Theorem 2.2.1 of [4])

\[
P_{(a,b;c)}(\lambda) := \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-\lambda t)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \cdot _2F_1 \left[ \begin{array}{c} a \ b \\ c \end{array} ; \lambda \right]
\]
holds when \( R_c > R_b > 0 \) and \( \lambda \in \mathbb{C} \setminus [1, \infty) \). When \( a = b = \frac{1}{2} \) and \( c = 1 \) the right side \( p_{(\frac{1}{2}, \frac{1}{2}; 1)}(\lambda) \) is a period of the Legendre family of elliptic curves \( E_\lambda : y^2 = x(x-1)(x-\lambda) \) parameterized by \( \lambda \). It is known that for general \( a, b, c \in \mathbb{Q} \) under some suitable assumptions \( p_{(a,b,c)}(\lambda) \) is a period of a generalized Legendre curve of the form \( y^N = x^i(1-x)^j(1-\lambda x)^k \) where \( i, j, k \) can be computed from \( a, b, c \), see [13]. It is worth mentioning that there are finite field analogues of hypergeometric series, in particular, the \( \mathbb{F}_q \) finite field analogues due to Greene can be used to compute the Galois representations of the generalized Legendre curves. For more information, see [13]. A generalization of the Euler integral representation to \( r+1 \) \( F \) is presented in [28, §4.1]. These periods are in general complicated transcendental numbers. They are much more predictable when the elliptic curve has complex multiplication (CM), e.g. \( \lambda = -1 \). The Selberg-Chowla formula predicts that any period of a CM elliptic curve is an algebraic multiple of a quotient of Gamma values [26]. For instance, using a formula of Kummer (see (3.9)) one can compute that \( p_{(\frac{1}{2}, \frac{1}{2}; 1)}(-1) = \frac{\sqrt{3}}{4} \frac{\Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2}} \). The hypergeometric series \( \pi^2 \cdot \; _3F_2 \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \lambda \right] \) is period of the \( K3 \) surface \( X_\lambda : s^2 = xy(x-1)(y-1)(x-\lambda y) \) (see [28, §4.1]) and [3]. Note that in equation (1) of [3] \( \lambda \) bears a sign opposite to our description of \( X_\lambda \).

This paper focuses on a \( p \)-adic analog of these complex periods computed from the hypergeometric series and Gamma function. We motivate our results using the following example. For any prime \( p \equiv 1 \mod 4 \), the elliptic curve \( E_{-1} \) has ordinary reduction at \( p \). From the theory of commutative formal group laws (CFGL) it is known that for any integer \( r \geq 1 \)

\[
\left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; -1 \right) \left( 1 \right) \equiv \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; -1 \right) \left( 1 \right) \mod p^r
\]

where \( \Gamma_p(.) \) stands for the \( p \)-adic Gamma function recalled in §2.1. This is closely related to Dwork’s unit root for \( E_{-1} \) at ordinary primes in terms of truncated hypergeometric series [15] and the \( a_p \)-values of the modular form corresponding to \( E_{-1} \). For more details see [17]. Here we are interested in the so-called supercongruences, congruences stronger than those predicted by CFGLs. For instance, using the arithmetic of elliptic curves and modular functions, Coster and van Hamme showed in [12] the congruence (1.2) holds mod \( p^{2r} \). This result can also be recovered by our method. For a more detailed discussion for how our method works in the case of \( r = 1 \), see Section 3 of [14]. In this paper, the authors explore the compatible and intertwining roles of hypergeometric series (as periods), truncated hypergeometric series (as unit roots for the ordinary case), and finite field analogues of hypergeometric series (as Galois representations). We refer interested readers to [14] for more related geometric and arithmetic backgrounds.

Various supercongruences have been conjectured by many mathematicians including Beukers [8], van Hamme [31], Rodriguez-Villegas [24], Zudilin [33], Chan et al. [9], and many more by Z.-W. Sun [29, 30, et al.]. Often the statements relate truncated hypergeometric series to Hecke eigenforms and hence to Galois representations. Some of these conjectures are proved using a variety of methods, including Gaussian hypergeometric series [1, 2, 18, 22, 21, et al.], the Wilf-Zeilberger method [33] and \( p \)-adic analysis [7, 23]. For more information on recent development on supercongruences, please see [23]. In this paper, we will show the following supercongruence result at \( \lambda = 2 \). The elliptic curve \( E_2 \) has CM and is isomorphic to \( E_{-1} \) over \( \mathbb{Q} \).

**Theorem 1.** For any prime \( p \equiv 1 \mod 4 \) and any integer \( r \geq 1 \),

\[
2F_1 \left[ \frac{1-p^r}{2}, 1; 2 \right] \equiv 2F_1 \left[ \frac{1-p^r-1}{2}, 1; 2 \right] \equiv (-1)^{p^r-1} \frac{p^r \left( \frac{1}{2} \right)^2}{\Gamma_p \left( \frac{1}{2} \right)} \mod p^{2r}. \tag{1.3}
\]

We outline the strategy of the proof. The strategies for the proof of Theorems 2 and 3 are outlined in §4 and 5. We use the Pfaff transformation and Kummer evaluation formula to obtain the equalities

\[
2F_1 \left[ \frac{1-p^r}{2}, 1; 2 \right] = \left( \frac{1}{2} \right)^{p^r-1} \frac{1}{(1)^{p^r-1}} 2F_1 \left[ \frac{1-p^r}{2}, 1-p^r; 2 \right] = \left( \frac{1}{2} \right)^{p^r-1} \frac{1}{(1)^{p^r-1}} \cdot \frac{1}{\Gamma \left( \frac{5-p^r}{4} \right)} \cdot \frac{\Gamma \left( \frac{5-p^r}{4} \right)}{\Gamma \left( \frac{5-p^r}{4} \right)}.
\]
The ratio on the left side of Theorem 1 then becomes a \( \Gamma \)-quotient times a power of 4. The \( \Gamma \)-quotient is converted to a \( \Gamma_p \)-quotient. We use the Kazandzidis supercongruence to \( p \)-adically approximate this power of \( 4 \text{ mod } p^{2r} \) and a Taylor expansion approximates the \( \Gamma_p \)-quotient. The coefficients of the first order \( p^r \) terms in the two approximations cancel, giving the desired supercongruence.

Compared to [19] we provide a systematic way to carry out the \( p \)-adic analysis using the local analytic behavior of the \( p \)-adic Gamma function that handles the ordinary and supersingular cases equally well. Our method also explains the similarity between \( p(\frac{1}{3}, \frac{1}{3}) \equiv 1 \) and the right side of (1.2). It is particularly effective when there are formulae for computing the complex periods as quotients of Gamma functions. For instance, we are able to prove the following result

**Theorem 2.** For \( p \geq 5 \) a prime the following congruence holds mod \( p^6 \):

\[
\begin{aligned}
7F6 \left[ \begin{array}{ccc}
\frac{7}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \\
\frac{1}{6} & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array} ; 1 \right] & = \sum_{k=0}^{p-1} (6k+1) \left( \frac{1}{3} \right)_k^6 \\
& \equiv \begin{cases}
-\frac{p\Gamma_p \left( \frac{1}{7} \right)^9}{27} & \text{if } p \equiv 1 \text{ mod } 6 \\
-\frac{10}{27} p^4 \Gamma_p \left( \frac{1}{3} \right)^9 & \text{if } p \equiv 5 \text{ mod } 6.
\end{cases}
\end{aligned}
\]

(1.4)

This result is stronger than a prediction of van Hamme in [31] which asserts a mod \( p^4 \) congruence for \( p \equiv 1 \text{ mod } 6 \). Computations yield that the only prime between 5 and 500 satisfying the above congruence mod \( p^7 \) is 19.

It is known that for \( \lambda \in \mathbb{Q} \) the zeta-function of the K3 surface \( X_\lambda \) can be computed from a weight-3 Hecke eigenform \( f_\lambda \) if and only if \( \lambda \) is in \( \{ \pm 1, 4, \frac{1}{4}, -8, -\frac{1}{8}, 64, \frac{1}{64} \} \), see [27] and [3]. For \( \lambda \) in this set and almost all primes \( p \) the following congruence holds

\[
3F2 \left[ \begin{array}{ccc}
\frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
1 & 1 & 1 \\
\end{array} ; \lambda \right]_{p-1} \equiv a_p(f_\lambda) \text{ mod } p^2
\]

where \( a_p(f_\lambda) \) is the \( p \)-th coefficient of an explicit weight-3 Hecke eigenform \( f_\lambda \) which can be derived from [17]. For instance, \( f_1 = \eta(4z)^6 \) where \( \eta(z) \) is the standard Dedekind eta function. We end the paper with a few supercongruences experimentally discovered by J. Kibelbek for ordinary primes and stated in [17]. For instance, we prove

**Theorem 3.** The following congruence holds modulo \( p^3 \)

\[
3F2 \left[ \begin{array}{ccc}
\frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\
1 & 1 & 1 \\
\end{array} ; 1 \right]_{p-1} \equiv \begin{cases}
-\frac{\Gamma_p \left( \frac{1}{7} \right)^4}{16} & \text{if } p \equiv 1 \text{ mod } 4 \\
-p^3 \Gamma_p \left( \frac{1}{3} \right)^4 & \text{if } p \equiv 3 \text{ mod } 4.
\end{cases}
\]

The corresponding mod \( p^2 \) congruence when \( p \equiv 1 \text{ mod } 4 \) was first established by van Hamme [31]. We obtain a similar result for \( -\frac{1}{4} \) at ordinary primes.

We briefly outline the paper. Section 2 includes the basic properties of \( p \)-adic Gamma functions we need, including Theorem 14 which we use repeatedly, and Lemma 17 which allows us to convert \( \Gamma \)-quotients to \( \Gamma_p \)-quotients, which can then be \( p \)-adically approximated by Theorem 14. We give a simple application of this in \( \S 2.2 \). In \( \S 3 \) we use the Kazandzidis supercongruence of binomial coefficients to prove Proposition 22, a relation between a \( p \)-adic logarithm and logarithmic derivatives of \( \Gamma_p \) that is used in the proof of Theorem 1. Section 4 is devoted to the proof of Theorem 2 and \( \S 5 \) to a corollary. In \( \S 6 \) we prove Theorem 3 and another result of similar type. The appendix, section 7, includes a few hypergeometric transformation and evaluation formulae that will be useful for our discussion. Many of the theorems here were motivated by computations carried out using the open source software Sage. We thank Heng Huat Chan for conversations which led to \( \S 2.2 \), Wadim Zudilin for inspiring discussions and the reference [28] and Jonas Kibelbek for his comments on an earlier version. Finally, thanks to Djordje Milicevic and Yujie Li for pointing out a gap in the proof of Theorem 2 in a previous version of the manuscript. In \( \S 4.1 \) we prove the needed result using Bailey’s formula. Milicevic and Li have since filled the gap using a different approach.
2. \( p \)-adic Gamma functions and immediate applications

2.1. Preliminaries

Throughout this paper \( p \geq 5 \) is a prime, \( v_p(\cdot) \) denotes the \( p \)-order and \( |x|_p = p^{-v_p(x)} \) the \( p \)-adic norm. The main results of this section that are used later in the paper are Theorem 14 and Lemma 17. If the reader wishes, she may assume these, noting that \( G_k(a) := \Gamma_p^{(k)}(a)/\Gamma_p(a) \), and proceed to §3.

For the sake of completeness, we recall some basic properties of the Morita \( p \)-adic Gamma function \( \Gamma_p(x) \) for \( x \in \mathbb{Z}_p \). None of the results of this subsection are new, but we gather them here for our convenience and hopefully that of the reader. For more details, see [20] and [11, §11.5]. As an immediate application we reprove and generalize a result from [10].

**Proposition/Definition 4.** 1) \( \Gamma_p(0) = 1 \)

2) \( \frac{\Gamma_p(x + 1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } |x|_p = 1 \\ -1 & \text{if } |x|_p < 1. \end{cases} \)

3) \( \Gamma_p(x)\Gamma_p(1 - x) = (-1)^{a_0(x)} \) where \( a_0(x) \in \{1, 2, \ldots, p\} \) satisfies \( x - a_0(x) \equiv 0 \mod p. \)

4) \( \Gamma_p \left( \frac{1}{2} \right)^2 = (-1)^{\frac{p+1}{2}}. \)

We use Theorem 14 repeatedly in this paper. Its proof involves essentially reproving:

**Theorem 5 (Morita, Barsky).** For \( a \in \mathbb{Z}_p \) the function \( x \mapsto \Gamma_p(a + x) \) is locally analytic on \( \mathbb{Z}_p \) and converges for \( v_p(x) \geq \frac{1}{p} + \frac{1}{p - 1} \).

Recall the \( p \)-adic logarithm

\[
\log_p(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n},
\]

which converges for \( x \in \mathbb{C}_p \) with \( |x|_p < 1. \)

Set \( G_k(a) = \Gamma_p^{(k)}(a)/\Gamma_p(a) \). In particular, \( G_0(a) = 1. \)

**Corollary 6.** For \( a \in \mathbb{Z}_p \), \( G_1(a) = G_1(1 - a) \), and \( G_2(a) + G_2(1 - a) = 2G_1^2(a). \)

**Proof.** Take \( \log_p \) of 3) of Proposition/Definition 4 and, using that \( a_0(x) \) is constant in a small enough neighborhood of \( x \), differentiate this to obtain \( G_1(x) = G_1(1 - x) \). Rewriting this in terms of \( p \)-adic Gamma functions and differentiating again gives \( G_2(x) - G_1^2(x) = -G_2(1 - x) + G_1^2(1 - x) \). The second part follows from the first part.

**Theorem 7.** [Robert-Zuber, [25]] Assume \( p \geq 5 \) is a prime. The function \( \log_p \Gamma_p(x) \) is an odd analytic function on \( p\mathbb{Z}_p \) such that

\[
\log_p \Gamma_p(x) = \lambda_0 x - \sum_{m \geq 1} \frac{\lambda_m}{2m(2m + 1)} x^{2m+1}
\]

(2.1)

where \( \lambda_1 \in \mathbb{Z}_p \) and \( p\lambda_m \in \mathbb{Z}_p \) when \( m \geq 2 \).

**Corollary 8.** \( G_2(0) = G_1^2(0). \)

**Proof.** Differentiate (2.1) twice and plug in \( x = 0. \)

**Lemma 9.** The \( k \)-th derivative of \( \log_p \Gamma_p(x) \) is a polynomial with integer coefficients in the \( G_i(x) \) for \( i \leq k \) and the sum of the subscripts in each monomial is \( k \). The coefficient of \( G_k(x) \) is 1.
Proof. We induct on \( k \), the case \( k = 1 \) being trivial. Suppose the statement is true for \( k - 1 \), that is the \( k - 1 \)st derivative of \( \log_p \Gamma_p(x) \) is of the form \( \sum_{i} a_i \prod_{j=1}^{r_i} G_{i_j}(x) \) where \( a_i \in \mathbb{Z} \), the subscripts in each monomial add to \( k - 1 \) and the term \( G_{k-1}(x) \) appears with coefficient 1. The derivative of \( G_{k-1}(x) \) is

\[
\frac{\Gamma_p^{(k)}(x)\Gamma_p(x) - \Gamma_p^{(k-1)}(x)\Gamma_p^{(1)}(x)}{\Gamma_p(x)^2} = G_k(x) - G_{k-1}(x)G_1(x).
\]

Since no other term in the sum includes a \( k - 1 \)st derivative, this completes the induction for the coefficient of \( G_k(x) \). Completing the rest of the induction involves writing the monomial \( \prod_{j=1}^{r_i} G_{i_j}(x) \) as \( \frac{\Gamma_p^{(i_1)}(x)\Gamma_p^{(i_2)}(x)\cdots\Gamma_p^{(i_{r_i})}(x)}{(\Gamma_p(x))^{r_i}} \) and employing the quotient and product rules.

Lemma 10. \( v_p \left( \frac{1}{k!} \right) \geq -\frac{k}{p-1} \).

Proof. \( v_p(k!) = \sum_{r=1}^{\infty} \left[ \frac{k}{p^r} \right] \leq \sum_{r=1}^{\infty} \frac{k}{p^r} = \frac{k}{p-1} \).

Lemma 11. Let \( p \geq 5 \). Then \( v_p(G_1(0)) \geq 0 \). Also, \( v_p(G_i(0)) \geq -\left[ \frac{i}{p} \right] \) for \( i > 1 \). In particular, \( v_p(G_1(0)) \geq 0 \) for \( i < p \) and \( v_p(G_i(0)) = -1 \).

Proof. From Theorem 7 we know that

\[
\log \Gamma_p(x) = \lambda_0 x - \sum_{m \geq 1} \frac{\lambda_m}{2m(2m+1)} x^{2m+1}
\]

is analytic for \( x \in \mathbb{Z}_p \).

Note \( \lambda_0 = G_1(0) = \frac{\Gamma_p'(0)}{\Gamma_p(0)} = \Gamma_p'(0) = \lim_{r \to \infty} \frac{\Gamma_p(p^r) - 1}{p^r} \). Now \( \Gamma_p(p^r) = (-1)^{p^r} \prod_{k=1,(k,p)=1}^{p^r} k \) is by Wilson’s theorem \(-1)^{p^r} \equiv 1 \mod p^r\) so \( v_p(\Gamma_p'(0)) = v_p(G_1(0)) \geq 0 \). (D. Thakur has pointed out to us that the only known cases where this valuation is positive are \( p = 5, 13, 563 \) and in these cases \( v_p(\Gamma_1(0)) = 1 \).)

For \( m \geq 1 \) Lemma 1 of [25] gives \( \lambda_m \equiv B_{2m} \mod \mathbb{Z}_p \) where \( B_{2m} \) is the \( 2m \)th Bernoulli number. Take \( 2m \) derivatives of the expression for \( \log \Gamma_p(x) \), plug in \( x = 0 \) and use Lemma 9 to get that \( G_{2m}(0) \) is a polynomial with integer coefficients in the terms \( G_k(0) \) for \( k < 2m \). Similarly, \( G_{2m+1}(0) \) is the difference between such a polynomial with \( k < 2m + 1 \) and \( (2m+1)!\lambda_m = (2m-1)!\lambda_{m-1} \). The Clausen-von Staudt Theorem states that the denominator of \( B_{2m} \) is \( \prod_{\ell=1}^{2m} \ell \). So \( v_p(G_i(0)) = 0 \) for \( i < p \). The denominator of \( B_{p-1} \) is exactly divisible by \( p \) so \( v_p(G_p(0)) = -1 \). For \( p < i < 2p \), \( G_i(0) \) is the sum of a polynomial in the previous \( G_j(0) \) (with \( G_p(0) \) occurring to at most first degree in any monomial by Lemma 9) and \((i-2)!\lambda_{i-1} \) which is integral in \( p \). The formula for \( G_{2p}(0) \) contains the term \( G_1^2(0) \) which has \( p \)-adic valuation \(-2 \), so \( v_p(G_{2p}(0)) \geq -2 \). Continuing in this fashion gives \( v_p(G_i(0)) \geq -\left[ \frac{i}{p} \right] \).

We immediately see:

Corollary 12. For \( i < p \), \( v_p \left( \frac{\Gamma_p(i)(0)}{i!} \right) = v_p \left( \frac{G_i(0)}{i!} \right) = 0 \). For all \( i \), \( v_p \left( \frac{\Gamma_p(i)(0)}{i!} \right) = v_p \left( \frac{G_i(0)}{i!} \right) \geq -\left( \frac{1}{p} + \frac{1}{p-1} \right) \). For \( x \in \mathbb{C}_p \) satisfying \( v_p(x) > \left( \frac{1}{p} + \frac{1}{p-1} \right) \), the Taylor series \( \Gamma_p(x) = \sum_{k=0}^{\infty} \frac{\Gamma_p(k)(0)}{k!} x^k \) converges. Dividing by \( \Gamma_p(0) \) this becomes \( \sum_{k=0}^{\infty} \frac{G_k(0)}{k!} x^k \).
We ‘transfer’ this result to arbitrary \( a \in \mathbb{Q} \cap \mathbb{Z}_p \). As the proof of Proposition 13 is a routine exercise in expansions of nonarchimedean series about different points, we do not include it.

**Proposition 13.** Let \( p \geq 5 \) and \( a \in \mathbb{Q} \) with \( v_p(a) \geq 0 \). Then \( v_p \left( \frac{G_i(a)}{i!} \right) \geq -i \left( \frac{1}{p} + \frac{1}{p-1} \right) \). For \( i < p \), \( v_p \left( \frac{G_i(a)}{i!} \right) = 0 \). We may extend the domain of \( \Gamma_p(a+x) \) by setting \( \Gamma_p(a+x) = \Gamma_p(a) \cdot \sum_{k=0}^{\infty} \frac{G_k(a)}{k!} x^k \) for \( x \in \mathbb{C}_p \) with \( v_p(x) \geq \left( \frac{1}{p} + \frac{1}{p-1} \right) \). In particular,

\[
\frac{\Gamma_p(a+1+x)}{\Gamma_p(a+x)} = \begin{cases} -(a+x) & |a+x|_p = 1 \\ -1 & |a+x|_p < 1 \end{cases},
\]

and

\[
\Gamma_p(a+x)\Gamma_p(1-a-x) = (-1)^{a_0(a)}. \tag{2.3}
\]

Theorem 14 provides a \( p \)-adic approximation to \( \Gamma_p \)-quotients.

**Theorem 14.** For \( p \geq 5 \), \( r \in \mathbb{N} \), \( a \in \mathbb{Z}_p \), \( m \in \mathbb{C}_p \) satisfying \( v_p(m) \geq 0 \) and \( t \in \{0, 1, 2\} \) we have

\[
\frac{\Gamma_p(a + mp^t)}{\Gamma_p(a)} = \sum_{k=0}^{t} \frac{G_k(a)}{k!} (mp^t)^k \mod p^{(t+1)r}.
\]

The above result also holds for \( t = 4 \) if \( p \geq 11 \).

**Proof.** When \( t = 1 \) we need only show that all terms past the first degree term in the Taylor series have valuation at least \( 2r \). Since \( p \geq 5 \), Proposition 13 implies \( \frac{G_k(a)}{k!} \in \mathbb{Z}_p \) for \( k = 2, 3, 4 \) so \( v_p \left( \frac{G_k(a)}{k!} (mp^t)^k \right) = kr \geq 2r \) so there is no problem with the 2nd, 3rd and 4th terms. For \( k \geq 5 \) we have \( v_p \left( \frac{G_k(a)}{k!} (mp^t)^k \right) \geq kr - k \left( \frac{1}{p} + \frac{1}{p-1} \right) \). We need to check this is at least \( 2r \). A simple computation reduces this to the inequality

\[
r \geq \left( \frac{k}{k-2} \right) \left( \frac{1}{p} + \frac{1}{p-1} \right).
\]

For \( k, p \geq 5 \) the right side is less than \( 1 \) so the inequality holds for all \( r \in \mathbb{N} \).

When \( t = 2 \), the 3rd and 4th terms are handled as above. For \( k \geq 5 \) the relevant inequality is

\[
r \geq \left( \frac{k}{k-3} \right) \left( \frac{1}{p} + \frac{1}{p-1} \right).
\]

The right side is less than \( 1 \) when \( k \geq 5 \) and \( p \geq 7 \). It is also less than \( 1 \) when \( k \geq 6 \) and \( p = 5 \). It remains to check the \( k = p = 5 \) case, namely that \( v_p \left( \frac{G_5(a)}{5!} (m5^r)^5 \right) \geq 3r \). Proposition 13 gives this valuation is \( 5r - 2 \), which is at least \( 3r \) for all natural numbers \( r \).

When \( t = 4 \), \( p \geq 11 \), \( v_p \left( \frac{G_k(a)}{k!} (mp^t)^k \right) \geq kr \) for \( 5 \leq k \leq 10 \) by arguments similar to those above. Again, arguing as above for \( k \geq 11 \), we need

\[
r \geq \left( \frac{k}{k-5} \right) \left( \frac{1}{p} + \frac{1}{p-1} \right).
\]

As the right side is less than \( 1 \) when \( k, p \geq 11 \), we are done.

We omit the \( t = 0 \) case.

We will need some formulae for various \( \Gamma \) and \( \Gamma_p \)-quotients.
Lemma 16. Let \( a \in \left\{ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n} \right\} \) and \( p \equiv 1 \mod n \). Then \( a' = a \).

Proof. \( \frac{k}{n} + k \cdot \frac{p-1}{n} = kp \).

Lemma 17 below is the technical ingredient that allows us to replace \( \Gamma \)-quotients with \( \Gamma_p \)-quotients.

Lemma 17. Let \( a \in (0,1) \cap \mathbb{Q} \).

1) If \( v_p(a) = 0 \) then \( \forall m, r \in \mathbb{N}, \)

\[
\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^{r-1})} = (-1)^mp^{mp^{r-1}} \frac{\Gamma_p(a + mp^r)(a')_{mp^{r-1}}}{\Gamma_p(a)(a')_{mp^{r-1}}}.
\]

2) Suppose \( a + mp^r \in \mathbb{N} \forall r \in \mathbb{N}. \) (Here, \( a, m \in \mathbb{Q} \) but need not be in \( \mathbb{Z} \).) Then

\[
\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^{r-1})} = (-1)^{a+mp^r}p^{a+mp^{r-1}-1} \frac{\Gamma_p(a + mp^r)}{\Gamma_p(d)}.
\]

3) Let \( d \in (0,1] \) satisfy \( a + mp^r - d, a + mp^{r-1} - d, d(p-1) \in \mathbb{Z} \) and \( v_p(d) = 0 \). Suppose also \( 0 < a < p/(p-1) \). Then

\[
\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^{r-1})} = (-1)^{a+mp^r-1-d}p^{a+mp^{r-1}-a} \frac{\Gamma_p(a + mp^r)}{\Gamma_p(d)}.
\]

4) Let \( a, b \in \mathbb{Q} \) and suppose \( a - b \in \mathbb{Z} \) and \( a, b \notin \mathbb{Z}_0 \). If none of the numbers between \( a \) and \( b \) that differ from both by an integer are divisible by \( p \) then

\[
\frac{\Gamma(a)}{\Gamma(b)} = (-1)^{a-b} \frac{\Gamma_p(a)}{\Gamma_p(b)}.
\]

Equivalently, if \( a \in \mathbb{Z}_p, n \in \mathbb{N} \) such that none of \( a, a+1, \ldots, a+n-1 \) in \( p\mathbb{Z}_p \), then

\[
(a)_n = (-1)^n \frac{\Gamma_p(a+n)}{\Gamma_p(a)}.
\]

5) If there are numbers \( x \) between \( a \) and \( b \) with \( x-a, x-b \in \mathbb{Z} \) and \( v_p(x) > 0 \), then the right side must include these multiples.

Proof. 1) As \( a' + mp^{r-1} > 0, \Gamma(a' + mp^{r-1}) \neq 0 \) so

\[
\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^{r-1})} = \frac{\Gamma(a + mp^r)}{\Gamma(a' + mp^{r-1})} \frac{\Gamma(a' + mp^{r-1})}{\Gamma(a + mp^{r-1})}
\]

which, by the fundamental property of the \( \Gamma \)-function equals

\[
\frac{\Gamma(a) \cdot a \cdot \ldots \cdot (a + mp^{r-1})}{\Gamma(a') \cdot a' \cdot \ldots \cdot (a' + mp^{r-1} - 1)} \cdot \frac{\Gamma(a' + mp^{r-1})}{\Gamma(a + mp^{r-1})}
\]

Now \( a + i \) is the first term in the numerator that is a multiple of \( p \) and this is \( pa' \). Similarly \( a + i + p = p(a' + 1), \ldots, a + i + mp^r - p = p(a' + mp^r - 1) \). The multiples of \( p \) in the numerator cancel exactly with the terms in the denominator leaving

\[
\frac{\Gamma(a)}{\Gamma(a')} \left( p^{mp^{r-1}} \prod_{j=0}^{mp^{r-1} - 1} (a + j) \right) \frac{\Gamma(a' + mp^{r-1})}{\Gamma(a + mp^{r-1})} = \frac{\Gamma(a)}{\Gamma(a')} \left( p^{mp^{r-1}} \frac{\Gamma_p(a)}{\Gamma_p(a')} \prod_{j=0}^{mp^{r-1} - 1} (a + j) \right) \frac{\Gamma(a' + mp^{r-1})}{\Gamma(a + mp^{r-1})}
\]

where the products are over terms prime to \( p \). By the definition of \( \Gamma_p \) this is

\[
\frac{\Gamma(a)}{\Gamma(a')} \left( p^{mp^{r-1}} \frac{\Gamma_p(a)}{\Gamma_p(a')} \right) \frac{\Gamma(a' + mp^{r-1})}{\Gamma(a + mp^{r-1})} = (-1)^mp^{mp^{r-1}} \frac{\Gamma_p(a + mp^{r-1})}{\Gamma_p(a + mp^{r-1})} \frac{\Gamma_p(a + mp^{r-1})}{\Gamma_p(a + mp^{r-1})}.
\]
where the last equality follows from using the definition of rising factorials twice.

2) This is like part 1), except here

\[
\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^r - 1)} = \frac{(a + mp^r - 1)!}{(a + mp^r - 1)!}.
\]

It is easy to see, using that \( a \in (0, 1) \cap \mathbb{Q} \), that each multiple of \( p \) in the numerator is \( p \) times an element of the denominator. Canceling these, introducing 1 in the form \( \frac{\Gamma_p(1)}{\Gamma_p(1)} \) and using the basic properties of \( \Gamma_p \), including that \( \Gamma_p(1) = -1 \), gives the result.

3) Write

\[
\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^r - 1)} = \frac{(a + mp^r - 1)(a + mp^r - 2) \ldots (d + 1)(d)\Gamma(d)}{(a + mp^r - 1)(a + mp^r - 2) \ldots (d + 1)(d)\Gamma(d)}.
\]

One cancels the \( \Gamma(d) \) factors and observes each term in the denominator cancels with \( \Gamma_p \) times itself in the numerator (this uses some of the hypotheses). The product that remains is just \( (-1)^{a + mp^r - 1 - d}\Gamma_p(a + mp^r) \)

4) and 5) follow immediately from the definitions of \( \Gamma \) and \( \Gamma_p \).

2.2. An immediate application

Lemma 2.1 of [10] asserts that for each prime \( p \equiv 1 \mod 4 \)

\[
\left(\frac{3}{4}\right)_p \equiv 3 \left(\frac{1}{4}\right)_p \mod p^3.
\] (2.4)

We generalize this as follows.

**Lemma 18.** Let \( a \in (0, 1) \cap \mathbb{Q} \) with \( v_p(a) = 0 \). For \( p \geq 5 \)

\[
\frac{(a)_{p^r}}{(a')_{p^r-1}} \equiv \frac{(1 - a)_{p^r}}{(1 - a')_{p^r-1}} \mod p^{r-1+2r}.
\] (2.5)

**Proof.** By definition \( (a)_{p^r} = a(a + 1) \ldots (a + p^r - 1) \). The set \( \{a + i\}_{i=0}^{p^r-1} \) contains \( p^r - 1 \) multiples of \( p \), which are \( a', (a' + 1)p, \ldots, (a' + p^r - 1)p \) respectively, combining with \( (a)_{p^r} = \frac{\Gamma(a + p^r)}{\Gamma(a)} \) we have \( (a)_{p^r} = (a')_{p^r-1} \frac{\Gamma_p(a + p^r)}{\Gamma_p(a)} \). By Theorem 14, we have

\[
\frac{(a)_{p^r}}{(a')_{p^r-1}} = p^{r-1}(-1)^{p^r} \frac{\Gamma_p(a + p^r)}{\Gamma_p(a)}.
\]

Similarly,

\[
\frac{(1 - a)_{p^r}}{(1 - a')_{p^r-1}} = p^{r-1}(-1)^{p^r} \frac{\Gamma_p(1 - a) + G_1(a)p^r + G_2(a)p^{2r}}{(1 - a')_{p^r-1}} \mod p^{r-1+2r}.
\]

The fact that \( G_1(a) = G_1(1 - a) \) gives the result.

**Corollary 19.** Fix \( n \in \mathbb{N} \). For any prime \( p \equiv 1 \mod n \),

\[
\left(\frac{1 - 1}{n}\right) \equiv (n - 1) \left(\frac{1}{n}\right) \mod p^3.
\] (2.6)

**Proof.** Set \( a = 1 - \frac{1}{n} \) in Lemma 18 and use Lemma 16. Setting \( n = 4 \) recovers the result of [10].
3. Kazandzidis supercongruences and the proof of Theorem 1

We prove Theorem 1. We first use the Kazandzidis supercongruence for binomial coefficients to $p$-adically approximate $4^{\frac{2r}{p^r}} \mod p^{2r}$ in Proposition 22. The Pfaff transformation formula allows us to rewrite our $2F_1$ as another such expression and then Kummer’s evaluation formula converts this $2F_1$ to a $\Gamma$-quotient. Thus the left side of Theorem 1 becomes a $\Gamma$-quotient multiplied by a power of $4^{\frac{2r}{p^r} - 1}$, which we already approximated. Lemma 17 expresses the $\Gamma$-quotient as a $\Gamma_p$-quotient and then we use Theorem 14 and the result for $4^{\frac{2r}{p^r} - 1}$ to approximate it.

3.1. Approximating $4^{\frac{2r}{p^r} - 1}$.

In 1968, Kazandzidis proved that for $0 \leq m \leq n$ and prime $p \geq 5$,

$$\binom{pm}{nm} \equiv \binom{n}{m} \mod p^3.$$ 

When $n = 2$ and $m = 1$, it is equivalent to Wolstenholme’s theorem. In [25], Robert and Zuber generalized this to

$$\binom{p^r n}{p^r m} \equiv \binom{p^{r-1} n}{p^{r-1} m} \mod p^{3r}. \quad (3.1)$$

In fact, they proved a stronger congruence, but the above is all we need. The expression $(\binom{p^r n}{p^r m})/(\binom{p^{r-1} n}{p^{r-1} m})$ can be written in terms of $p$-adic Gamma functions.

We need some easy lemmas whose proofs we omit.

Lemma 20. Let $p \geq 5$, $m \in \{1, 2\}$ and $r \geq 0$. Then $v_p \left( \binom{2mp^r}{mp^r} \right) = 0$.

Lemma 21. Let $p \geq 5$ and $x \in p^r \mathbb{Z}_p$. Then $\log_p(1 + x) \equiv x \mod p^{2r}$, $\log_p(1 + x) \equiv x - \frac{x^2}{2} \mod p^{3r}$, $e^x \equiv 1 + x \mod p^{2r}$ and $e^x \equiv 1 + x + \frac{x^2}{2!} \mod p^{3r}$.

We also need the Legendre duplication formula,

$$\Gamma(2z) = \frac{2^{2z-1} \Gamma(z) \Gamma(z + 1/2)}{\sqrt{\pi}}. \quad (3.2)$$

By the fundamental property of the Gamma function

$$\binom{2p^r}{p^r}/ \binom{2p^{r-1}}{p^{r-1}} = \frac{\Gamma(1 + 2p^r)}{\Gamma(1 + p^r)} \cdot \frac{\Gamma(1 + p^r)}{\Gamma(1 + p^{r-1})}^2.$$

Taking $2z = 1 + 2p^r$ and $2z = 1 + 2p^{r-1}$ in (3.2)

$$\binom{2p^r}{p^r}/ \binom{2p^{r-1}}{p^{r-1}} = \frac{2^{2p^r} \Gamma\left(\frac{1}{2} + p^r\right) \Gamma(1 + p^r)}{\sqrt{\pi} (\Gamma(1 + p^r))^2} \cdot \frac{\sqrt{\pi} (\Gamma(1 + p^{r-1}))^2}{2^{2p^{r-1}} \Gamma\left(\frac{1}{2} + p^{r-1}\right) \Gamma(1 + p^{r-1})} = 2^{2(p^r - p^{r-1})} \frac{\Gamma\left(\frac{1}{2} + p^r\right) \Gamma(1 + p^{r-1})}{\Gamma\left(\frac{1}{2} + p^{r-1}\right) \Gamma(1 + p^r)} = 4^{(p^r - p^{r-1})} \frac{\Gamma\left(\frac{1}{2} + p^r\right) \Gamma\left(\frac{1}{2} + p^{r-1}\right)}{\Gamma\left(\frac{1}{2} + p^r\right) \Gamma\left(\frac{1}{2} + p^{r-1}\right)} = 4^{(p^r - p^{r-1})} \frac{\Gamma\left(\frac{1}{2} + p^r\right) \Gamma\left(\frac{1}{2} + p^{r-1}\right)}{\Gamma\left(\frac{1}{2} + p^r\right) \Gamma\left(\frac{1}{2} + p^{r-1}\right)} \Gamma\left(\frac{1}{2} + p^r\right) \Gamma\left(\frac{1}{2} + p^{r-1}\right).$$

where the last equality follows from two applications of 1) of Lemma 17. The various powers of $(-1)$ cancel out.

Theorem 14 implies that taking Taylor expansions of these $\Gamma_p$’s and truncating mod $p^{2r}$ gives congruences mod $p^{2r}$. Then taking $\log_p$ and truncating mod $p^{2r}$ perpetuates these congruences by Lemma 21. Using (3.1) and invoking Lemma 20 to divide yields

$$p^{r-1} \log_p 4^{p-r-1} + \log_p \left[ 1 + G_1 \left( \frac{1}{2} \right) p^r + G_2 \left( \frac{1}{2} \right) \frac{(p^r)^2}{2} + \cdots \right] - \log_p \left[ 1 + G_1(0) p^r + G_2(0) \frac{(p^r)^2}{2} + \cdots \right] = 0 \mod p^{3r}. \quad (3.3)$$
We see
\[ p \cdot \frac{1}{p} \log_p 4^{p-1} + G_1 \left( \frac{1}{2} \right) p^r - G_1(0) p^r \equiv 0 \mod p^{2r} \]
holds \( \forall r \). Letting \( r \to \infty \) gives
\[ \frac{1}{p} \log_p 4^{p-1} + G_1 \left( \frac{1}{2} \right) - G_1(0) = 0. \]

(3.4)
The equation below is derived similarly from the Kazandzidis supercongruence. We go through two iterations of Legendre’s duplication formula (3.2) and four applications of 1) of Lemma 17 to get
\[ 1 \equiv \frac{(2p^r)^{p^r}}{(2p^{r-1})^{p^{r-1}}} = 2^{4(p^r - p^{r-1})} \frac{\Gamma_p \left( \frac{1}{2} + p^r \right) \Gamma_p \left( \frac{3}{4} + p^r \right) \Gamma_p \left( \frac{3}{4} - p^r \right)}{\Gamma_p(1 + p^r) \Gamma_p \left( \frac{1}{2} + p^r \right) \Gamma_p \left( \frac{1}{2} - p^r \right)} \mod p^{2r}. \]

Note that \( \Gamma_p \left( \frac{1}{4} \right) \Gamma_p \left( \frac{3}{4} \right) = (-1)^{a(1/4)} \), \( \Gamma_p(1) = -1 \) and \( \Gamma_p \left( \frac{1}{2} \right) \) is some 4th root of unity. Thus \( \log_p \) of the constant part of the expression above is 0. Taking \( \log_p \) and using Theorem 14 to reduce mod \( p^{2r} \) we see
\[ \frac{2}{p} \log_p (4^{p-1}) + G_1 \left( \frac{1}{4} \right) + G_1 \left( \frac{3}{4} \right) - G_1(1) - G_1 \left( \frac{1}{2} \right) = 0. \]

(3.5)
Subtracting (3.4) from (3.5) and using Corollary 6 gives

**Proposition 22.** \( \frac{1}{p} \log_p 4^{p-1} = 2G_1 \left( \frac{1}{2} \right) - 2G_1 \left( \frac{1}{4} \right) \). Multiplying by \( \frac{p^r}{4} \) and exponentiating, this becomes
\[ 4^{p^r - p^{r-1}} \equiv (-1)^{\frac{p^r - p^{r-1}}{8}} \left[ 1 + \left( G_1 \left( \frac{1}{2} \right) - G_1 \left( \frac{1}{4} \right) \right) \frac{p^r}{2} \right] \mod p^{2r}. \]

We need (3.6) below for use in (6.15) and (6.16). We only sketch its derivation. It requires two applications of the duplication formula, Euler’s reflection formula (3.10) and 2) and 5) of Lemma 17. Multiplying the congruences \( \frac{2p}{p} \equiv 1 \mod p^3 \) and \( \frac{4p}{2p} \equiv 1 \mod p^3 \), we have
\[ 64^{p-1} \frac{\Gamma_p \left( \frac{1}{4} + p \right) \Gamma_p \left( \frac{3}{4} + p \right)}{\Gamma_p(1 + p^2) \Gamma_p \left( \frac{1}{4} + p \right) \Gamma_p \left( \frac{3}{4} + p \right)} \equiv 1 \mod p^3. \]

Thus
\[ 64^{p-1} \equiv 1 + \left( 2G_1(0) - 2G_1 \left( \frac{1}{4} \right) \right) p + 2 \left( G_1(0) - G_1 \left( \frac{1}{4} \right) \right)^2 p^2 \mod p^3 \]

It follows again from \( \left( \frac{2}{p} \right) = (-1)^{\frac{p^r - p^{r-1}}{8}} \) and Lemma 21 that
\[ 2^{\frac{p^r - p^{r-1}}{2}} \equiv (-1)^{\frac{p^r - p^{r-1}}{8}} \left[ 1 + \frac{1}{6} \left( G_1(0) - G_1 \left( \frac{1}{4} \right) \right) p + \frac{1}{72} \left( G_1(0) - G_1 \left( \frac{1}{4} \right) \right)^2 p^2 \right] \mod p^3 \]

(3.6)
and cubing this we have
\[ 2^{\frac{3(p^r - p^{r-1})}{2}} \equiv (-1)^{\frac{p^r - p^{r-1}}{8}} \left[ 1 + \frac{1}{2} \left( G_1(0) - G_1 \left( \frac{1}{4} \right) \right) p + \frac{1}{8} \left( G_1(0) - G_1 \left( \frac{1}{4} \right) \right)^2 p^2 \right] \mod p^3 \]

(3.7)

3.2. Approximating the \( _2F_1 \) ratio and the Proof of Theorem 1

We are ready to prove Theorem 1. We will need the following Pfaff transform (see [4, eq. (2.3.14), pp. 79])
\[ _2F_1 \begin{bmatrix} -n & b \\ c & c \end{bmatrix} : x = \frac{(c-b)n}{(c)n} \ _2F_1 \begin{bmatrix} -n & b \\ b+1-n-c & 1-x \end{bmatrix}. \]

(3.8)
as well as Kummer’s evaluation formula, see (2.11) of [32]
\[ _2F_1 \begin{bmatrix} a & b \\ 1+a-b & -1 \end{bmatrix} = \frac{\Gamma(1+a-b)\Gamma \left( 1+\frac{a}{2} \right)}{\Gamma \left( 1+\frac{a}{2}-b \right) \Gamma(1+a)}. \]

(3.9)
Proof (Proof of Theorem 1). Recall $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and Euler’s reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad (3.10)$$

We let $x = 2$ and $n = \frac{p^r - 1}{2}$, $b = \frac{1}{2}$, $c = 1$ in (3.8).

\[
2 F_1 \left[ \frac{1-p^r}{2} \frac{1}{2} ; 2 \right] = \left( \frac{1}{2} \right) \frac{p^r - 1}{2} 2 F_1 \left[ \frac{1-p^r}{2} \frac{1}{2} ; -1 \right]
\]

(by (3.8))

\[
\left( \frac{1}{2} \right) \frac{p^r - 1}{2} \Gamma\left( \frac{1}{2} - \frac{p^r}{2} \right) \Gamma\left( \frac{5-p^r}{4} \right)
\]

(by (3.9))

\[
= \Gamma\left( \frac{p^r}{2} \right) / \Gamma\left( \frac{1}{2} \right) \Gamma\left( 1 - \frac{p^r}{2} \right) \Gamma\left( \frac{5-p^r}{4} \right) \Gamma\left( \frac{3-p^r}{2} \right)
\]

(definition of rising factorials)

\[
= \Gamma\left( \frac{p^r}{2} \right) \Gamma\left( 1 - \frac{p^r}{2} \right) 2 \frac{p^r - 1}{2} \sqrt{\pi} \Gamma\left( 1 - \frac{p^r}{2} \right)
\]

(set $z = (3-p^r)/4$ in (3.2))

\[
= \frac{\Gamma\left( \frac{p^r}{2} \right) \Gamma\left( 1 - \frac{p^r}{2} \right)}{\Gamma\left( \frac{1+p^r}{2} \right) \Gamma\left( \frac{3-p^r}{4} \right)}
\]

(by (3.10))

Recall $p \equiv 1 \mod 4$ so $\Gamma_p \left( \frac{1}{2} \right)^2 = -1$. Then

\[
\frac{2 F_1 \left[ \frac{1-p^r}{2} \frac{1}{2} ; 2 \right]}{2 F_1 \left[ \frac{1-p^r}{2} \frac{1}{2} ; 2 \right]} = \frac{4^{p^r-p^r-1}}{4^{p^r-p^r-1}} \frac{\Gamma\left( \frac{1+p^r}{2} \right) \Gamma\left( \frac{3-p^r}{4} \right)^2}{\Gamma\left( \frac{1+p^r}{2} \right) \Gamma\left( \frac{3-p^r}{4} \right)^2}
\]

(by (3.10))

\[
= \frac{4^{p^r-p^r-1}}{4^{p^r-p^r-1}} \frac{\left[ \Gamma_p \left( \frac{1+p^r}{2} \right) \right]^2}{\left[ \Gamma_p \left( \frac{1+p^r}{2} \right) \right]^2}
\]

(2 of Lemma 17)

\[
= \frac{4^{p^r-p^r-1}}{4^{p^r-p^r-1}} \frac{\left[ (-1)^{1+p^r} p^{1-p^r-1} \Gamma_p \left( \frac{1+p^r}{2} \right) \right]^2}{\left[ (-1)^{1+p^r} p^{1-p^r-1} \Gamma_p \left( \frac{1+p^r}{2} \right) \right]^2}
\]

(3 of Lemma 17)

\[
= \frac{4^{p^r-p^r-1}}{4^{p^r-p^r-1}} \frac{\Gamma_p \left( \frac{1+p^r}{4} \right)^2}{\Gamma_p \left( \frac{1+p^r}{4} \right) \Gamma_p \left( \frac{1+p^r}{2} \right)}
\]

\[
= \frac{4^{p^r-p^r-1}}{4^{p^r-p^r-1}} \frac{\left[ \Gamma_p \left( \frac{1+p^r}{4} \right) \right]^2}{\Gamma_p \left( \frac{1+p^r}{4} \right) \Gamma_p \left( \frac{1+p^r}{2} \right)}
\]
To continue, we apply Proposition 22 to the last term to conclude

\[
\begin{align*}
2F_1 & \left[ \frac{1-p^r}{1}, \frac{1}{2}; 2 \right] \\
2F_1 & \left[ \frac{1-p^r}{1}, \frac{1}{2}; 2 \right] \\
\equiv & (-1)^{\frac{p^r-1}{2}} \left[ 1 + \left( G_1 \left( \frac{1}{2} \right) - G_1 \left( \frac{1}{4} \right) \right) \frac{p^r}{2} \right] \\
& \quad \cdot \frac{\Gamma_p \left( \frac{1}{4} \right)^2}{\Gamma_p \left( \frac{1}{2} \right)} \frac{1 + G_1 \left( \frac{1}{4} \right) \frac{p^r}{2}}{1 + G_1 \left( \frac{1}{2} \right) \frac{p^r}{2}} \mod p^{2r} \\
\equiv & (-1)^{\frac{p^r-1}{2}} \frac{\Gamma_p \left( \frac{1}{4} \right)^2}{\Gamma_p \left( \frac{1}{2} \right)} \mod p^{2r}
\end{align*}
\]

the last congruence following as the coefficients of the $p^r$ terms cancel.

4. **Proof of a strengthened conjecture of van Hamme and a Proposition**

We prove Theorem 2 in this section. We first prove Lemma 23, a mod $p^6$ congruence between the $7F_6$ of Theorem 2 and a version truncated even earlier at $\frac{tp-1}{3}$ in §4.1. We then prove a mod $p^6$ congruence between this highly truncated $7F_6$ and a perturbed $7F_6$ that has a Galois symmetry with respect to the 5th roots of unity. This perturbed $7F_6$ involves the variable $x$ and is easily seen to belong to $\mathbb{Z}_p[[x^5]]$. Next, we show it is congruent mod $p$ to a $9F_8$ which lies in $p\mathbb{Z}_p[[x]]$ by a hypergeometric transformation formula. Thus the perturbed $7F_6$ belongs to $p\mathbb{Z}_p[[x^5]]$ and evaluating it at $x = 0$ and $x = \frac{tp}{3}$ gives the mod $p^6$ congruence between the $7F_6$ of Theorem 2 and the perturbed $7F_6$ evaluated at $x = \frac{tp}{3}$. In §4.2 we evaluate this perturbed $7F_6$ in terms of $p$-adic Gamma functions to obtain Theorem 2.

4.1. A congruence with a perturbed $7F_6$.

For $p \equiv 1 \mod 6$, we have $\frac{1}{3} + \frac{p-1}{3} = p \cdot \frac{1}{3} \cdot \left( \frac{1}{3} \right)^t = \frac{1}{3}$. For $p \equiv 5 \mod 6$ we have $\frac{1}{3} + \frac{2p-1}{3} = p \cdot \frac{2}{3}$ so $\left( \frac{1}{3} \right)^t = \frac{2}{3}$ in this case. For convenience we write $\left( \frac{1}{3} \right)^t = \frac{t}{3}$ where $t = 1$ or $2$ as appropriate. In either case $\frac{1}{3} - \frac{tp}{3} \in -\mathbb{N}$ so when this quantity appears in the top row of an $aF_b$ expression the series terminates.

**Lemma 23.**

\[
7F_6 \left[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} ; \frac{1}{6}, 1, 1, 1, 1, 1 \right] \mod p^6.
\]

**Proof.** This congruence holds since for $\frac{tp-1}{3} < k < p$ the expression $\left( \frac{1}{3} \right)_k$ contains the factor $\frac{1}{3} + \frac{tp-1}{3} = \frac{tp}{3} \equiv 0 \mod p$ and one easily sees $(1)_k \not\equiv 0 \mod p$.

**Lemma 24.**

\[
\frac{(1-x)_k}{(1+x)_k} = \left( \frac{1}{3} \right)_k \left[ 1 + a_{1,1}x + a_{2,2}x^2 + \cdots \right] \in \mathbb{Z}_p[[x]] \quad \text{for} \quad k \leq \frac{tp-1}{3}.
\]

**Proof.** For $k = 0$ note $(a)_k = 1$. Henceforth $k \geq 1$. From

\[
\left( \frac{1}{3} - x \right)_k = \prod_{j=0}^{k-1} \left( \frac{1}{3} + j - x \right) = \prod_{j=0}^{k-1} \left( \frac{1}{3} + j \right) \left( 1 - \frac{x}{\left( \frac{1}{3} + j \right)} \right) = \left( \frac{1}{3} \right)_k \left[ 1 - \sum_{j=0}^{k-1} \frac{3x}{1+3j} + \sum_{0 \leq \ell < j \leq k-1} \frac{9x^2}{(1+3\ell)(1+3j)} + \cdots \right] \quad (4.1)
\]
we see that for \( k \) in the specified range that none of the denominators in (4.1) are multiples of \( p \). It is trivial to see that for these \( k \) the constant term of the polynomial \((1 + x)_k\) is not divisible by \( p \). Thus its reciprocal, when viewed as a power series, has \( p \)-integral coefficients.

The last claim follows from Definition 15.

One easily sees \( \left( \frac{\zeta}{b} \right)_k = (1 + 6k) \). Let \( \zeta_5 \) be any primitive 5th root of unity. For \( k < \frac{tp - 1}{3} \) we have \((1 + \zeta_5^p x)_k \in \mathbb{Z}_p[[5]][x]\) has a unit constant term so \( \frac{1}{(1 + \zeta_5^p x)_k} \in \mathbb{Z}_p[[5]][x] \). These two facts along with Galois symmetry imply

\[
\begin{pmatrix}
\frac{1}{3} & \frac{7}{6} & \frac{1}{6} - \zeta_5 x & \frac{1}{3} - \zeta_5^2 x & \frac{1}{3} - \zeta_5^3 x & \frac{1}{3} - \zeta_5^4 x & \frac{1}{3} - x \\
\frac{1}{6} & 1 + \zeta_5 x & 1 + \zeta_5^2 x & 1 + \zeta_5^3 x & 1 + \zeta_5^4 x & 1 + x \\
\end{pmatrix} ; 1 \in \mathbb{Z}_p[[5]] \] (4.2)

Next we show the above series is in \( p\mathbb{Z}_p[[x]] \). We’ll use the following identity of Bailey that relates two terminating \( gF_8 \) series. See (7.21) of [5]. If \( n \) is a positive integer, \( r = 2a - b - c - d + 1 \) and \( b + c + d + e + f + g - n = 3a + 2 \) then

\[
\begin{pmatrix}
a & \frac{n}{2} + 1 & b & c & d & e & f & g & -n \\
a - b - 1 & a - c - 1 & a - d + 1 & a - e + 1 & a - f + 1 & a - g + 1 & a + n + 1 & 1 \\
\end{pmatrix} = \frac{(a + 1)n(r - e + 1)n(r - f + 1)n(r - g + 1)n}{(r + 1)n(a - e + 1)n(a - f + 1)n(a - g + 1)n}
\times \begin{pmatrix}
r & \frac{n}{2} + 1 & r & b & a & r & c & a & r & d & a & e & f & g & -n \\
\frac{r}{2} & a - b + 1 & a - c + 1 & a - d + 1 & r & e & 1 & r & f & 1 & r & g & 1 & r & n + 1 & 1 \\
\end{pmatrix} \] (4.3)

Let

\[
a = \frac{1}{3}, b = 1 - \frac{2tp}{3}, c = \frac{5}{3}, d = \frac{1}{3} - \zeta_5 x, e = \frac{1}{3} - \zeta_5^2 x, f = \frac{1}{3} - \zeta_5^3 x, g = \frac{1}{3} - x + tp \text{ and } n = \frac{tp - 1}{3}
\]

so \( r = x(\zeta_5 + \zeta_5^2) + \frac{2tp}{3} \). Then (4.3) becomes

\[
\begin{pmatrix}
\frac{1}{3} & \frac{7}{6} & \frac{1}{6} - \zeta_5 x & \frac{1}{3} - \zeta_5^2 x & \frac{1}{3} - \zeta_5^3 x & \frac{1}{3} - \zeta_5^4 x & \frac{1}{3} - x + tp & \frac{1 - tp}{3} ; 1 \\
\frac{1}{6} & 1 + \frac{2tp}{3}, 1 + \zeta_5 x, 1 + \zeta_5^2 x, 1 + \zeta_5^3 x, 1 + \zeta_5^4 x, 1 + x - tp, 1 + \frac{tp}{3} & \end{pmatrix}
\times \begin{pmatrix}
0 + \cdots + 1 + \cdots + \frac{2}{3} + \cdots + 0 + \cdots + 0 + \cdots + \frac{1}{3} + \cdots + \frac{1}{3} + \cdots + \frac{1}{3} + \cdots + \frac{1 - tp}{3} ; 1 \\
0 + \cdots + \frac{1}{3} + \cdots + 1 + \cdots + \frac{2}{3} + \cdots + \frac{2}{3} + \cdots + \frac{2}{3} + \cdots + \frac{2}{3} + \cdots + \frac{2}{3} + \cdots + \frac{2}{3} + \cdots + \frac{2}{3} + \cdots + \frac{2}{3} + \cdots \end{pmatrix}
\] (4.4)

where in expressions of the form \( c + \cdots \) the dots indicate an element of the maximal ideal of \( \mathbb{Z}_p[[5]][x] \) and \( c \) is the \( \mod \ p \) constant term. Reducing the the left side of (4.4) \( \mod \ p \) and cancelling the identical \( \mod \ p \) terms in the top and bottom rows we see it is congruent to (4.2) \( \mod \ p \). We now show the right side lies in \( p\mathbb{Z}_p[[5]][x] \). On the right side \((a + 1)n = \frac{4}{3}\) \( \mod \ p \) is a multiple of \( p \) as \( \frac{4}{3} + \frac{tp - 1}{3} - 1 = \frac{tp}{3} \).

We study the first \( gF_8 \) in (4.4). As mentioned above, the six \( 1 + \cdots \) terms in the bottom row are never a problem for invertibility. The \( \frac{1}{6} \) mostly cancels with the \( \frac{7}{6} \) as mentioned above. This leaves the \( \frac{1 + 2tp}{3} \) term. When \( p \equiv 1 \mod 6 \) (and \( t = 1 \)) adding \( \frac{p - 1}{3} \) gets us to next multiple of \( p \) but the most we ever add in a rising factorial is \( \frac{p - 4}{3} \) so there is no problem. When \( p \equiv 5 \mod 6 \) (and \( t = 2 \)) the next multiple of \( p \)
past \( \frac{1 + 4p}{3} \) is \( \frac{5p}{3} \) but these differ by \( \frac{p - 1}{3} \) which is not an integer, so we go to the next multiple of \( p, \frac{6p}{3} \), which differs from \( \frac{1 + 4p}{3} \) by \( \frac{2p - 1}{3} \) which is too big. The \( g \mathcal{F}_8 \) on the left side of (4.4) is a power series in \( \mathbb{Z}_p[\zeta_5][[x]] \).

We now consider the right side of (4.4). All the rising factorials in the denominator are of the form \((1 + \cdots ) r_{p-1} \) and thus have unit constant terms and are invertible. It remains to consider the bottom row of the second \( g \mathcal{F}_8 \) term.

For \( k \geq 1 \) that \( (r)_k \frac{\left( \frac{p}{3} + 1 \right)_k}{\left( \frac{2}{3} \right)_k} = (r)_k \frac{r + 2k}{r} = (r + 1)_{k-1} (r + 2k) \). Thus the first two terms in the top row of the second \( g \mathcal{F}_8 \) term cancel the \( \frac{r}{2} \) term in the bottom row so the \( 0 + \cdots \) term in the bottom row of (4.4) presents no invertibility problems. Nor do the \( 1 + \cdots \) terms for reasons mentioned earlier.

Let \( p \equiv 1 \mod 6 \). For the constant term of \( \left( \frac{1}{3} + \cdots \right)_k \) to be a multiple of \( p \) we must have \( k > \frac{p - 1}{3} \) which does not happen. Similarly, for the constant term of \( \left( \frac{2}{3} + \cdots \right)_k \) to be a multiple of \( p \) we must have \( k > \frac{2p - 1}{3} \) which also does not happen. Thus the product of all the terms on the right side of (4.4) excluding the term \( \left( \frac{4}{3} \right) r_{p-1} \) is integral. Since the excluded factor is a multiple of \( p \) we have proved, for \( p \equiv 1 \mod 6 \), the left side of (4.4) is a multiple of \( p \) and thus so is our perturbed \( \tau \mathcal{F}_6 \).

Finally, we consider the \( p \equiv 5 \mod 6 \) case. We again need only study the terms in the bottom row in the right \( g \mathcal{F}_8 \) of (4.4). The \( 0 + \cdots \) and \( 1 + \cdots \) terms are not a problem for reasons already presented. For the constant term of \( \left( \frac{1}{3} + \cdots \right)_k \) to be a multiple of \( p \) we must have \( k > \frac{2p - 1}{3} \) which does not happen. On the other hand \( \left( \frac{2}{3} + \cdots \right)_k \) has constant term a multiple of \( p \) when \( k > \frac{p - 2}{3} \). Going back to (4.3) the first three \( \frac{2}{3} + \cdots \) terms correspond to \( r - e + 1, r - f + 1, r - g + 1 \) in the bottom row. In each of these \( \frac{2}{3} + \cdots \) terms, the unique factor that makes the constant term a multiple of \( p \) cancels with the corresponding factor of the rising factorial, e.g. of \((r - e + 1)n\) on the right side of (4.3). Thus we need only consider the factor coming from \( r + n + 1 = \frac{2}{3} + x(\zeta_5 + \zeta_5^2) + tp \).

When we add \( \frac{p - 2}{3} \) to \( r + n + 1 \) we get \( \frac{p}{3} + x(\zeta_5 + \zeta_5^2) + tp = \frac{7p}{3} + x(\zeta_5 + \zeta_5^2) \). Call this linear polynomial \( P(x) \). We know the left side of (4.4) lies in \( \mathbb{Z}_p[[x]] \). In the sum for the \( g \mathcal{F}_8 \) on the right side the terms up to \( k = \frac{p - 2}{3} \) have invertible denominator and the terms past \( \frac{p - 2}{3} \) may have, after inverting the other factors, denominator \( P(x) \). Thus the right side of (4.4) becomes, after factoring out the factor of \( p \) from \( \left( \frac{4}{3} \right) r_{p-1} \),

\[
p \left( A_1(x) + \frac{A_2(x)}{P(x)} \right)
\]

where \( A_i(x) \in \mathbb{Z}_p[\zeta_5][[x]] \). By the Weierstrass preparation theorem \( A_i(x) = p^{i} B_i(x) U_i(x) \) where \( B_i(x) \) is a distinguished polynomial and \( U_i(x) \) is a unit power series. The right side of (4.4) becomes

\[
p \left( p^{i} B_1(x) U_1(x) + \frac{p^{j} B_2(x) U_2(x)}{P(x)} \right).
\]

If \( B_2(x) \) is not a multiple of \( P(x) \) then (4.4) has a pole at \( x = \frac{-7p}{3(\zeta_5 + \zeta_5^2)} \in p \mathbb{Z}_p[\zeta_5] \). As the left side of (4.4) is defined on all of \( p \mathbb{Z}_p[\zeta_5] \) we have a contradiction so \( P(x) \mid B_2(x) \) and both sides of (4.4) are in \( p \mathbb{Z}_p[\zeta_5][[x]] \) so (4.2) is a multiple of \( p \) when \( p \equiv 5 \mod 6 \) as well.
4.2. Dougall’s formula, a $\Gamma_p$-quotient, Galois symmetry, the proof of Theorem 2 and a proposition.

To prove Theorem 2 it remains to show the left side of (4.2) is congruent to the right side in Theorem 2 mod $p^6$. We use Dougall’s formula (c.f. Theorem 3.5.1 of [4] or (8.2) of [32]) which asserts that for $f$ a negative integer and $1 + 2a = b + c + d + e + f$ that

$$7F_6\left[\begin{array}{cccccc}
a & 1 + \frac{a}{2} & b & c & d & e & f \\
\frac{a}{2} & 1 + a - b & 1 + a - c & 1 + a - d & 1 + a - e & 1 + a - f; 1 \\
\end{array}\right]$$

$$= (a + 1)_f(a - b - c + 1)_f(a - b - d + 1)_f(a - c - d + 1)_f(a - a - 1)$$

(4.5)

Set $a = \frac{1}{3}, b = 1 - \zeta_5 tp, c = 1 - \zeta_5^2 tp, d = \frac{1 - \zeta_5^3 tp}{3}, e = \frac{1 - \zeta_5^4 tp}{3}$, and $f = \frac{1 - tp}{3}$. As $\zeta_5$ is a primitive 5th root of unity, it satisfies $1 + \zeta_5 + \cdots + \zeta_5^4 = 0$ and hence $1 + 2a = b + c + d + e + f$ so Dougall’s formula applies. We get

$$7F_6\left[\begin{array}{cccccc}
\frac{1}{3} & \frac{7}{6} & 1 - \zeta_5 tp & 1 - \zeta_5^2 tp & 1 - \zeta_5^3 tp & 1 - \zeta_5^4 tp \\
\frac{1}{6} & 1 + \zeta_5 tp & 1 + \zeta_5^2 tp & 1 + \zeta_5^3 tp & 1 + \zeta_5^4 tp; 1 \\
\end{array}\right]$$

$$= \left(\frac{1}{3}\right)^{\frac{1}{t+1}} \left(\frac{2 + \zeta_5 tp + \zeta_5^2 tp}{3}\right)^{\frac{1}{t-1}} \left(\frac{2 + \zeta_5 tp + \zeta_5^2 tp}{3}\right)^{\frac{1}{t-1}}$$

(4.6)

Regardless of whether $t = 1$ or 2, none of the rising factorials in the denominator contain a multiple of $p$. We substitute $\Gamma$-quotients for the rising factorials above and then replace these by $\Gamma_p$-quotients, necessarily introducing the factor $(-1)^{\frac{tp-1}{2}} = 1$ along the way. Observe

$$\left(\frac{1}{3}\right)^{\frac{1}{t+1}} = \frac{\Gamma\left(\frac{tp+3}{3}\right)}{\Gamma\left(\frac{3}{3}\right)} = \frac{tp}{3} \frac{\Gamma\left(\frac{tp}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} = (-1)^{\frac{tp-1}{2}} t^{\frac{tp-1}{2}} p^{\frac{tp}{3}} \frac{\Gamma_p\left(\frac{tp}{3}\right)}{\Gamma_p\left(\frac{1}{3}\right)}.$$

When $t = 1$, that is $p \equiv 1 \mod 6$, the remaining rising factorials in the numerator contain no multiples of $p$ and, when viewed as $\Gamma$-quotients can be replaced directly by $\Gamma_p$-quotients.

When $t = 2$, that is $p \equiv 5 \mod 6$, the remaining rising factorials in the numerator each contain exactly one multiple of $p$, namely $\frac{2 + tp}{3}$. In this case, when replacing the rising factorials by $\Gamma_p$-quotients we have to include the factor

$$\frac{p}{3}(1 + 2\zeta_5 + 2\zeta_5^2)\frac{P}{3}(1 + 2\zeta_5 + 2\zeta_5^3)\frac{P}{3}(1 + 2\zeta_5^2 + 2\zeta_5^3) = \frac{5p^3}{27}.$$

Set $A_1 = 1$ and $A_2 = \frac{5p^3}{27}$. Then (4.6) becomes

$$tpA_1 \Gamma_p\left(\frac{tp}{3}\right) \Gamma_p\left(\frac{1 + \zeta_5 tp + \zeta_5^2 tp + tp}{3}\right) \Gamma_p\left(\frac{1 + \zeta_5 tp + \zeta_5^2 tp + tp}{3}\right) \Gamma_p\left(\frac{1 + \zeta_5 tp + \zeta_5^2 tp + tp}{3}\right) \times$$

$$\Gamma_p\left(\frac{1 + \zeta_5 tp}{3}\right) \Gamma_p\left(1 + \zeta_5^2 tp\right) \Gamma_p\left(1 + \zeta_5^3 tp\right) \Gamma_p\left(1 + \zeta_5^4 tp\right) \Gamma_p\left(1 + \zeta_5 tp + \zeta_5^2 tp + \zeta_5^3 tp + \zeta_5^4 tp\right)$$

(4.7)

As $1 + \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 = 0$, we have

$$\Gamma_p\left(\frac{\zeta_5 tp + \zeta_5^2 tp + \zeta_5^3 tp + \zeta_5^4 tp}{3}\right) = \Gamma_p\left(\frac{\zeta_5^4 tp}{3}\right) = (-1)^{\alpha_0(0)} \Gamma_p\left(1 + \frac{\zeta_5^4 tp}{3}\right)^{-1}$$
by the functional equation (see Proposition 13). We apply the functional equation to the rest of the denominator terms to get (grouping the $\Gamma_p(1 + \ast)$ terms at the end)

$$\begin{align*}
(-1)^{p+6a_0(\frac{2}{3})} tpA_t \frac{\Gamma_p(\frac{tp}{3})}{\Gamma_p(\frac{1}{3})} \frac{1 + \zeta_5tp + \zeta_5^2tp + tp}{3} \frac{1 - \zeta_5tp - \zeta_5^2tp}{3} \frac{1 + \zeta_5tp + \zeta_5^3tp + tp}{3} \\
\times \frac{\Gamma_p(1 - \zeta_5tp - \zeta_5^2tp)}{3} \frac{1}{3} \frac{1 + \zeta_5tp + \zeta_5^3tp + tp}{3} \frac{1 - \zeta_5tp - \zeta_5^3tp}{3} \frac{1 - \zeta_5tp}{3} \\
\times \frac{\Gamma_p(1 + \zeta_5tp + \zeta_5^3tp + \zeta_5^5tp)}{3} \frac{4}{3} \frac{1}{3} \frac{1 + \zeta_5tp}{3} \frac{1 - \zeta_5tp}{3} \frac{1}{3} \\
= (-1)^{p+6a_0(\frac{2}{3})} tpA_t \prod_{0 \leq i < j \leq 4} \frac{1}{3} \frac{1 - \zeta_5tp - \zeta_5^3tp}{3} \frac{1}{3} \frac{1 + \zeta_5tp}{3} (4.8)
\end{align*}$$

Note that $p + 6a_0(\frac{2}{3})$ is odd and $\Gamma_p(\frac{tp}{3}) = -\Gamma_p(1 + \frac{tp}{3})$. Place this with the four terms at the end. We have symmetry with respect to the 5th roots of unity so

$$\Gamma_p\left(1 + \frac{\zeta_5tp}{3}\right) \Gamma_p\left(1 + \frac{\zeta_5^2tp}{3}\right) \Gamma_p\left(1 + \frac{\zeta_5^3tp}{3}\right) \Gamma_p\left(1 + \frac{\zeta_5^4tp}{3}\right) \Gamma_p\left(1 + \frac{tp}{3}\right)$$

has, by the $t = 4, p \geq 11$ case of Theorem 14, Taylor series expansion $\Gamma_p(1^5[1 + O(p^5)] = -1 + O(p^5))$. The overall expression (4.6) has symmetry with respect to 5th roots of unity as does the remaining part,

$$(-1)^{1+1} \frac{tpA_t}{\Gamma_p(\frac{1}{3})} \prod_{0 \leq i < j \leq 4} \Gamma_p\left(1 + \frac{\zeta_5tp}{3} - \frac{\zeta_5^3tp}{3}\right)$$

Thus above product has Taylor series expansion $tpA_t \Gamma_p\left(\frac{1}{3}\right)^9 [1 + O(p^5)]$ as well. Multiplying by $-1 + O(p^5)$ we get

$$\begin{align*}
\sum_{k=0}^{p-1} (6k + 1) \left(\frac{1}{3}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{1}{3}\right)_k \\
= 7F_6 \left[ \frac{1}{3} \frac{1}{6} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} : 1 \right] \\
= 7F_6 \left[ \frac{1}{3} \frac{1}{6} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} : 1 \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \mod p^6 \right] \\
= -tpA_t \Gamma_p\left(\frac{1}{3}\right)^9 \mod p^6.
\end{align*}$$

We have proved Theorem 2 for $p \geq 11$. We have verified the cases $p = 5, 7$ by hand. \qed

5. Another result

The next result is obtained by the same method as the previous proof. It uses a formula due to Pfaff-Saalschütz.

**Proposition 25.** For any prime $p > 3$,

$$\begin{align*}
3F_2 \left[ \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} : 1 \right] \equiv \left\{ \begin{array}{ll}
\Gamma_p\left(\frac{1}{3}\right)^6 \mod p^3 & \text{if } p \equiv 1 \mod 6 \\
-\frac{p^3}{3} \Gamma_p\left(\frac{1}{3}\right)^6 \mod p^3 & \text{if } p \equiv 5 \mod 6
\end{array} \right. \end{align*}$$
PROOF. The Pfaff-Saalschütz Theorem (Theorem 2.2.6 of [4]) says for $n \in \mathbb{N}$
\[
3F_2 \left[ \begin{array}{c} -n \ a \ b \\ c \ 1 + a + b - c - n \end{array} ; 1 \right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \tag{5.1}
\]
As in the proof of Theorem 2, set $t = 1$ or 2 as $p \equiv 1$ or 5 mod 6. Letting $n = \frac{tp - 1}{3}, a = \frac{1 - \zeta_3 tp}{3}$ and $b = \frac{1 - \zeta_3^2 tp}{3}$, where $\zeta_3$ is a primitive cube root of unity, we get
\[
3F_2 \left[ \begin{array}{c} 1 - \frac{tn}{3} \frac{1 - \zeta_3 tp}{3} \frac{1 - \zeta_3^2 tp}{3} \\ 1 \ 1 \ 1 \end{array} ; 1 \right] = \frac{ \left( 2 + \zeta_3 tp \right) \Gamma_p}{\left( 1 - \frac{tp - 1}{3} \right) \Gamma_p \Gamma_p(1 - \frac{tp}{3})}.
\tag{5.2}
\]
By the symmetry with respect to cube roots of unity the left side, a finite sum, agrees with $3F_2 \left[ \begin{array}{c} t_p - 1 \frac{1}{3} \frac{1}{3} \frac{1}{3} \\ 1 \ 1 \ 1 \end{array} ; 1 \right]_{tp-1}$ mod $p^3$ and hence with $3F_2 \left[ \begin{array}{c} t_p - 1 \frac{1}{3} \frac{1}{3} \frac{1}{3} \\ 1 \ 1 \ 1 \end{array} ; 1 \right]_{tp-1}$ as the terms $k = \frac{tp - 1}{3}$ in this series are all divisible by $p^3$.

When $p \equiv 1$ mod 6 ($t = 1$) the rising factorials on the right side of (5.2) are all units in $\mathbb{Z}_p$. When $p \equiv 5$ mod $p$ ($t = 2$), $\left( 2 + \zeta_3 tp \right) \Gamma_p$ and $\left( 2 + \zeta_3^2 tp \right) \Gamma_p$ contain, respectively, the multiples of $p \left( 2 + \zeta_3 tp + \frac{p - 2}{3} = \frac{2\zeta_3 + 1}{3} + \frac{2 + \zeta_3^2 tp}{3} + \frac{p - 2}{3} = \frac{2\zeta_3^2 + 1}{3} p \right.$. Set $B_1 = 1$ and $B_2 = \frac{2\zeta_3 + 1}{3} - \frac{2\zeta_3^2 + 1}{3} p = \frac{p^2}{3}$.

We rewrite the right side of (5.2) using $\Gamma_p$-quotients to get (there are four factors of $(-1)^{t_p/3}$ which cancel)
\[
B_1 = \frac{\Gamma_p(1 + t_p + \zeta_3 tp) \Gamma_p(1 + t_p + \zeta_3^2 tp) \Gamma_p(1 - t_p)}{\Gamma_p(2 + \zeta_3 tp) \Gamma_p(2 + \zeta_3^2 tp) \Gamma_p(2 + t_p)} = \frac{\Gamma_p(1 - t_p) \Gamma_p(1 - \zeta_3 tp) \Gamma_p(1 - \zeta_3^2 tp) \Gamma_p(1 - t_p)}{\Gamma_p(2 + t_p) \Gamma_p(2 + \zeta_3 tp) \Gamma_p(2 + \zeta_3^2 tp) \Gamma_p(2 + t_p)}
\tag{5.3}
\]
where the first equality uses that $1 + \zeta_3 + \zeta_3^2 = 0$, the second that $\Gamma_p(1 + pb) \Gamma_p(2 - pb) = (-1)^{t_p(2/3)}$ and the third uses the symmetry with respect to the cube roots of unity and that $\frac{2}{3} = \frac{p + 2}{3} \equiv 2 \cdot \frac{p + 1}{3} \equiv 1 \mod p$ where $\frac{p + 2}{3} \in \mathbb{N}$ when $p \equiv 1 \mod 6$ and $2 \cdot \frac{p + 1}{3} \in \mathbb{N}$ when $p \equiv 5 \mod 6$. Thus $(-1)^{t_p(2/3)} = (-1)^{t}$.

The series $\frac{2x}{3} \cdot 3F_2 \left[ \begin{array}{c} \frac{1}{3} \frac{1}{3} \frac{1}{3} \\ 1 \ 1 \ 1 \end{array} ; 1 \right]$ and its ‘companion’ series $\frac{2x}{3} \cdot 3F_2 \left[ \begin{array}{c} \frac{2}{3} \frac{2}{3} \frac{2}{3} \\ 1 \ 1 \ 1 \end{array} ; 1 \right]$ are both periods of the algebraic variety $\gamma^3 = (x_1 x_2)^2 (1 - x_1)(1 - x_2)(x_1 - x_2)$. In particular for any prime $p \equiv 1 \mod 3$
\[
3F_2 \left[ \begin{array}{c} \frac{2}{3} \frac{2}{3} \frac{2}{3} \\ 1 \ 1 \ 1 \end{array} ; 1 \right]_{tp-1} \equiv -\Gamma_p(\frac{1}{3})^3 \mod p^2.
\]
See [14] for a proof and the corresponding discussion on the Galois representations attached to this variety.

6. The proof of Theorem 3 and other applications

We prove Theorem 3. We separate the proof into the cases where $p \equiv 3 \mod 4$ and $p \equiv 1 \mod 4$. For $p \equiv 3 \mod 4$ we write our $3F_2$ as the constant term of a Taylor expansion of a perturbed $3F_2$ - actually we
write the constant term in terms of the rest of the series and the perturbed \( _3F_2 \). We truncate the series to get a congruence and using Propositions 27 and 28 convert this congruence to one with \( \Gamma_p \)-values, proving the result. The strategy for \( p \equiv 1 \mod 4 \) is similar, but the computations are somewhat different.

In this section, we will consider supercongruences occurring for \( _3F_2 \left[ \frac{1}{2} \frac{1}{2} \frac{1}{2} ; \lambda \right] \) at values of \( \lambda \) related to CM elliptic curves over \( \mathbb{Q} \). A brief description of the geometric background was given in the introduction.

**Lemma 26.** Let \( k \in \mathbb{N} \) with \( k \leq \frac{p - 1}{2} \) and \( M \in \mathbb{Z} \). Set

\[
A_k = \sum_{j=0}^{k-1} \frac{1}{2j+1}, \quad B_k = \sum_{0 \leq i \leq j \leq k-1} \frac{1}{(2i+1)(2j+1)}.
\]

(6.1)

Note \( A_k \) is defined for \( k \geq 1 \) and \( B_k \) is defined for \( k \geq 2 \). Then

\[
\left( \frac{1 - Mp}{2} \right)_k = \left( \frac{1}{2} \right)_k (1 - MpA_k + (Mp)^2B_k) \mod p^3
\]

and

\[
(1 - Mp)_k \equiv (1)_k(1 - MpE_k + (Mp)^2F_k) \mod p^3.
\]

**Proof.** Observe

\[
\left( \frac{1 - Mp}{2} \right)_k = \left( \frac{1}{2} - \frac{Mp}{2} \right) \left( \frac{3}{2} - \frac{Mp}{2} \right) \cdots \left( \frac{2k-1}{2} - \frac{Mp}{2} \right).
\]

Multiplying this out gives an expansion of the form

\[
\left( \frac{1}{2} \right)_k \left( 1 - Mp \right) \sum_{j=0}^{k-1} \frac{1}{2j+1} + (Mp)^2 \sum_{0 \leq i < j \leq k-1} \frac{1}{(2i+1)(2j+1)} + O(p^3).
\]

The big ‘\( O \)’ term is justified as no multiples of \( p \) occur in any of the denominators of any of the sums as all indices are at most \( \frac{p - 3}{2} \). The proof of the \( (1 - Mp)_k \) result is identical.

**Proposition 27.** Set \( A_k \) and \( B_k \) as in (6.1). Set \( _3F_2(C) = _3F_2 \left[ \frac{1-Cp}{2} \frac{1+(C-2)p}{2} \frac{1-p}{2} ; 1 \right] \). Then

\[
_3F_2(C) - _3F_2(D) \equiv p^2 \sum_{k=0}^{p-1} \frac{1}{(k!)^3} [(2C^2 - 2D^2 - 4C + 4D)B_k + (-C^2 + D^2 + 2C - 2D)A_k^2] \mod p^3.
\]

**Proof.** We apply Lemma 26 for \( \frac{1}{2} \) (resp. 1) with \( M = C, 2 - C \) and 1 (resp. \( M = 1 \) and \( \frac{1}{2} \)). So

\[
_3F_2(C) = _3F_2 \left[ \frac{1-Cp}{2} \frac{1+(C-2)p}{2} \frac{1-p}{2} ; 1 \right] = \sum_{k=0}^{p-1} \frac{1}{(k!)^3} \frac{1-Cp}{2k} \frac{1+(C-2)p}{2k} \frac{1-p}{2k} \left[ 1 - \frac{p}{2} E_k + \frac{p^2}{4} F_k \right]^{-1}
\]

\[
\equiv \sum_{k=0}^{p-1} \frac{1}{(k!)^3} \left[ 1 - CpA_k + C^2pB_k \right] \left[ 1 + (C - 2)pA_k + (C - 2)^2p^2B_k \right] \left[ 1 - pA_k + p^2B_k \right] \mod p^3
\]

\[
\equiv \sum_{k=0}^{p-1} \frac{1}{(k!)^3} \left[ 1 - 3pA_k + p^2 \left[ (2C^2 - 4C + 5)B_k + (-C^2 + 2C + 2)A_k^2 \right] \times \left[ 1 - \frac{3}{2} pE_k + \frac{5}{4} p^2 F_k + \frac{p^2}{2} E_k \right]^{-1} \right]
\]

\[
\equiv \sum_{k=0}^{p-1} \frac{1}{(k!)^3} \left[ 1 + p \left[ -3A_k + \frac{3}{2} E_k \right] + p^2 X \right] \mod p^3.
\]
where \( X = \left[ (2C^2 - 4C + 5)B_k + (-C^2 + 2C + 2)A_k^2 - \frac{3}{2} F_k + \frac{7}{4} E_k^2 - \frac{9}{2} A_k E_k \right]. \) The result follows by simply performing the subtraction \( 3F_2(C) - 3F_2(D) \) as most terms are independent of \( C \) and \( D \) and therefore cancel.

**Proposition 28.** Let \( C, D > 0. \)

For \( C \equiv D \equiv p \equiv 1 \mod 4, \)

\[
3F_2(C) - 3F_2(D) \equiv \left[ -\frac{p^2}{16} \Gamma_p \left( \frac{1}{4} \right) \right] \cdot \left( (2C^2 - 2D^2 - 4C + 4D)G_2 \left( \frac{1}{4} \right) + (-2C^2 + 2D^2 + 4C - 4D)G_1 \left( \frac{1}{4} \right) \right)^2 \mod p^3. \tag{6.2}
\]

Similarly, when \( C \equiv D \equiv p \equiv 3 \mod 4, \)

\[
3F_2(C) - 3F_2(D) \equiv \frac{(C^2 - D^2 - 2C + 2D)}{16} p^2 \Gamma_p \left( \frac{1}{4} \right)^4 \mod p^4.
\]

**Proof.** By Clausen’s formula (7.1), which holds as long as both sides converge and the Gauss summation formula (7.2) which applies as \( \Re \left( 1 - \frac{p}{2} - \frac{1}{4} \right) \Gamma_p \left( \frac{1}{2} \right) = \Re \left( \frac{1}{2} \right) = \frac{1}{2} > 0, \)

\[
3F_2 \left[ \frac{1-Cp}{2} \frac{1+(C-2)p}{2} \frac{1-p}{2} ; 1 \right] = 3F_2 \left[ \frac{1-Cp}{2} \frac{1+(C-2)p}{2} \frac{1-p}{2} ; 1 \right]^2 = \frac{\Gamma \left( \frac{1}{2} - \frac{p}{2} \right)^2 \Gamma \left( \frac{1}{2} \right)^2}{\Gamma \left( \frac{3+C-2p}{4} \right)^2 \Gamma \left( \frac{3-Cp}{4} \right)^2}.
\tag{6.3}
\]

In both pairs \( \left( \frac{1}{2}, \frac{3 + (C-2)p}{4} \right) \) and \( \left( \frac{3-Cp}{4}, \frac{1-p}{2} \right) \) the elements differ by an integer. When \( p \equiv 1 \mod 4, \) write \( C = 4k + 1. \) The numbers between both elements of the first pair that differ from each by an integer and are multiples of \( p \) are \( \frac{2p}{2}, \frac{6p}{4}, \ldots, \frac{4k - 2}{4} - p. \) For the second pair the corresponding list is simply the negatives of the numbers above. As in 5) of Lemma 17, when we want to replace a quotient of \( \Gamma \)-functions by the same quotient of \( \Gamma_p \)-functions, care must be taken with the signs and multiples of \( p. \) Here everything will be squared so we need not worry about signs, and the multiples of \( p \) of concern end up canceling in the numerator and denominator so we may simply replace all \( \Gamma \)-functions by \( \Gamma_p. \) Thus

\[
3F_2 \left[ \frac{1-Cp}{2} \frac{1+Cp}{2} \frac{1+p}{2} ; 1 \right] = \frac{\Gamma \left( 1 - \frac{p}{2} \right)^2 \Gamma \left( \frac{1}{2} \right)^2}{\Gamma \left( \frac{3+C-2p}{4} \right)^2 \Gamma \left( \frac{3-Cp}{4} \right)^2} = \frac{\Gamma_p \left( 1 - \frac{p}{2} \right)^2 \Gamma_p \left( \frac{1}{2} \right)^2}{\Gamma_p \left( \frac{3+C-2p}{4} \right)^2 \Gamma_p \left( \frac{3-Cp}{4} \right)^2}
\]

which by arguments which are now standard in this paper becomes

\[
\Gamma_p \left( 1 - \frac{p}{2} \right)^2 \Gamma_p \left( \frac{1}{2} \right)^2 \Gamma_p \left( 1 + (2 - C)p \right)^2 \Gamma_p \left( 1 + Cp \right)^2 = \left[ -\Gamma_p \left( 1 - \frac{p}{2} \right)^2 \Gamma_p \left( \frac{1}{4} \right)^4 \right]
\cdot \left[ 1 + G_1 \left( \frac{1}{4} \right) \left( 2 - C \right) \right] + \frac{G_2 \left( \frac{1}{4} \right)}{2} \left( 2 - C \right) \Gamma_p \left( \frac{1}{4} \right)^2 \right]^2 \left[ 1 + G_1 \left( \frac{1}{4} \right) \left( C \right) \right] \frac{G_2 \left( \frac{1}{4} \right)}{2} \left( C \right) \left( \frac{1}{4} \right)^2 \mod p^3.
\tag{6.4}
\]

This simplifies to

\[
-\Gamma_p \left( 1 - \frac{p}{2} \right)^2 \Gamma_p \left( \frac{1}{4} \right)^4 \left[ 1 + pG_1 \left( \frac{1}{4} \right) + \frac{p^2}{16} Y \right]
\]

where \( Y = \left[ (2C^2 - 4C + 4)G_2 \left( \frac{1}{4} \right) + (-2C^2 + 2C + 4)G_1 \left( \frac{1}{4} \right)^2 \right]. \) So

\[
3F_2(C) - 3F_2(D) \equiv \left( -\Gamma_p \left( 1 - \frac{p}{2} \right)^2 \Gamma_p \left( \frac{1}{4} \right)^4 \frac{p^2}{16} \right) \cdot Y \mod p^3.
\tag{6.5}
\]
By a simple application of Theorem 14 we see $\Gamma_p\left(1 - \frac{p}{2}\right) \equiv \Gamma_p(1) \equiv -1 \mod p$. The result follows for $p \equiv 1 \mod 4$.

When $p \equiv 3 \mod 4$, write $C = 4k+3$, then within $\left(\frac{1}{2}, \frac{3 + (C - 2)p}{4}\right)$ the multiples of $p$ are $\frac{2p}{4}, \cdots, \frac{4k - 2}{4}p$. The multiples of $p$ within $\left(\frac{3 - Cp}{4}, 1 - \frac{p}{2}\right)$ include (up to sign) those just listed and one more, $-\left(\frac{(C + 2)p}{4}\right) = -\left(\frac{(C - 1)p}{4}\right)$. Thus

$$3F_2(C) = \frac{(C-1)^2p^2}{4^2} \frac{\Gamma_p\left(1 - \frac{p}{2}\right)^2 \Gamma_p\left(\frac{1}{2}\right)^2}{\Gamma_p\left(\frac{3+(C-2)p}{4}\right)^2 \Gamma_p\left(\frac{3-Cp}{4}\right)^2} = \frac{(C-1)^2p^2\Gamma_p\left(\frac{1}{2}\right)^4}{4^2} \mod p^3.$$  

It follows that $3F_2(C) - 3F_2(D) = \frac{(C - D)(C + D - 2)p^2}{16} \Gamma_p\left(\frac{1}{4}\right)^4 \mod p^3$.

We are now ready to prove Theorem 3. Proposition 27 gives a formula for $3F_2(C) - 3F_2(D)$ in terms of sums involving $A_k$ and $B_k$. Proposition 28 uses Clausen’s formula to change this to a $\Gamma_p$-quotient. The expansion (6.8) comes from Lemma 26 and expresses our desired quantity in terms of a perturbed $3F_2$ and an expression involving $A_k$ and $B_k$ that we have computed as a $\Gamma_p$-quotient. When $p \equiv 3 \mod 4$, the perturbed term is 0 and we are done. For $p \equiv 1 \mod 4$ our standard approach applies: the perturbed term is reduced to a $\Gamma_p$-quotient by use of Clausen’s formula and Gauss’ summation formula. It is then changed to a $\Gamma_p$-quotient by Lemma 17.

**Proof (Proof of Theorem 3).** We first handle the $p \equiv 3 \mod 4$ case. We need a vanishing result in this case. Using the first equation on [4, pp 142] which holds when $n$ is a positive integer so both sides converge

$$3F_2\left[-n \begin{array}{c} a \\ d \\ c \end{array} ; 1\right] = \frac{(e-a)n}{(e)n} 3F_2\left[-n \begin{array}{c} a - b \\ d \\ a + 1 - n - e \end{array} ; 1\right] \quad (6.6)$$

we know when $p \equiv 3 \mod 4$,

$$3F_2\left[\begin{array}{c} \frac{1+p}{2} \\ 1 \\ \frac{1}{2} \end{array} ; 1\right] = 0 \quad (6.7)$$

Thus $3F_2\left[\begin{array}{c} \frac{1+p}{2} \\ 1 \\ \frac{1}{2} \end{array} ; 1\right] = 0$ in this case.

By Lemma 26

$$3F_2\left[\begin{array}{c} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{array} ; 1\right] = 3F_2\left[\begin{array}{c} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{array} ; 1\right] + p^2 \sum_{k=0}^{p-1} \frac{1}{k!^3} [A_k^2 - 2B_k] \mod p^3. \quad (6.8)$$

By Propositions 27 and 28

$$3F_2(3) - 3F_2(7) \equiv 32p^2 \sum_{k=0}^{p-1} \frac{1}{k!^3} [A_k^2 - 2B_k] \mod p^3$$

and

$$3F_2(3) - 3F_2(7) \equiv -2p^2 \Gamma_p\left(\frac{1}{4}\right)^4 \mod p^3.$$  

Thus

$$3F_2\left[\begin{array}{c} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{array} ; 1\right] = 0 + \sum_{k=0}^{p-1} \frac{1}{k!^3} [A_k^2 - 2B_k] \equiv \frac{p^2}{16} \Gamma_p\left(\frac{1}{4}\right)^4 \mod p^3.$$  

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This settles Theorem 3 for \( p \equiv 3 \mod 4 \).

Now we assume \( p \equiv 1 \mod 4 \). Observe that by definition \( {}_3 F_2 \left[ \frac{1}{2}, \frac{1}{2}, 1 ; 1 \right] = \frac{\Gamma(1)^{3}}{\Gamma \left( \frac{3-p}{4} \right) \Gamma \left( \frac{3+p}{4} \right)} \).

Let \( C = 5 \) and \( D = 9 \) in Propositions 27 and 28. Taking care with signs, the differences \( {}_3 F_2(C) - {}_3 F_2(D) \) are

\[
48p^2 \sum_{k=0}^{\frac{p-1}{2}} \left( \frac{1}{2} \right)_k (2B_k - A_k^2) \equiv -p^2 \Gamma_p \left( \frac{1}{4} \right)^4 \left[ 6G_1 \left( \frac{1}{4} \right)^2 - 6G_2 \left( \frac{1}{4} \right) \right] \mod p^3. \tag{6.9}
\]

We will compute \( {}_3 F_2 \left[ \frac{1-p}{2}, \frac{1+p}{2}, \frac{1}{2} ; 1 \right] \) in two different ways. Since

\[
\left( \frac{1-p}{2} \right)_k \left( \frac{1+p}{2} \right)_k \left( \frac{1}{2} \right)_k \equiv \left( \frac{1}{2} \right)_k \left( 1 - p^2 \sum_{j=0}^{k-1} \frac{1}{(2j+1)^2} \right) \equiv \left( \frac{1}{2} \right)_k \left( \frac{1}{2} \right)_k \left( A_k^2 - 2B_k \right) \mod p^3.
\]

The first congruence below is just (6.8) while the second follows from (6.9).

\[
{}_3 F_2 \left[ \frac{1-p}{2}, \frac{1+p}{2}, \frac{1}{2} ; 1 \right] = {}_3 F_2 \left[ \frac{1}{2}, \frac{1}{2}, 1 ; 1 \right] - p^2 \sum_{k=0}^{\frac{p-1}{2}} \left( \frac{1}{2} \right)_k \left( A_k^2 - 2B_k \right) \mod p^3. \tag{6.10}
\]

To estimate the left side (6.10) we need \( p \)-adic Gamma functions. We use Clausen’s formula, Gauss’ summation formula, that \( p \equiv 1 \mod 4 \) and note the elements of the pair \( \left( \frac{3-p}{4}, \frac{1}{2} \right) \) differ by an integer, as do the elements of \( \left( 1, \frac{3+p}{4} \right) \). In neither pair are there any multiples of \( p \) between the numbers and an integral distance from both endpoints, so we can use 4) of Lemma 17 directly to see

\[
{}_3 F_2 \left[ \frac{1-p}{2}, \frac{1+p}{2}, \frac{1}{2} ; 1 \right] = {}_2 F_1 \left[ \frac{1-p}{4}, \frac{1+p}{4} ; 1 \right] = \frac{\Gamma(1) \Gamma \left( \frac{1}{4} \right)^2}{\Gamma \left( \frac{3-p}{4} \right) \Gamma \left( \frac{3+p}{4} \right)} \left[ \frac{\Gamma_p(1) \Gamma_p \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3-p}{4} \right) \Gamma_p \left( \frac{3+p}{4} \right)} \right]^2
\]

\[
\equiv -\Gamma_p \left( \frac{1}{4} \right)^4 \left[ 1 + G_2 \left( \frac{1}{4} \right)^2 - G_1 \left( \frac{1}{4} \right)^2 \right] \mod p^3.
\]

The fourth equality and first congruence involve what are for us standard arguments with \( \Gamma_p \)-functions. Comparing with (6.10), the \( p \equiv 1 \mod 4 \) case follows and Theorem 3 is proved.

We prove one more supercongruence. We will restrict ourselves to the ordinary primes.
Theorem 29. When \( p \equiv 1 \mod 4 \) is a prime, then
\[
3F2 \left[ \begin{array}{ccc}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 1 & 1 \\
-\frac{1}{8}
\end{array} : -1 \right] \equiv (-1)^{\frac{p-1}{8}} \Gamma_p \left( \frac{1}{4} \right)^4 \mod p^3.
\]

PROOF. Here, we will outline the proof and will skip some details if the analysis is similar to what we have used above.

When \( a \) is not a negative integer, \((7.3)\) says
\[
3F2 \left[ \begin{array}{ccc}
3a-1 & a & 1-a \\
2a & a+\frac{1}{2} & -\frac{1}{8}
\end{array} : -1 \right] = 2^{3a-3} \left( \frac{\Gamma(a+\frac{1}{2})\Gamma(\frac{a}{2})}{\Gamma(\frac{3a}{2})\Gamma(\frac{1}{2})} \right)^2. \tag{6.11}
\]

Setting \( a = \frac{1+p}{2} \) in \((6.11)\)
\[
3F2 \left[ \begin{array}{ccc}
\frac{1+3p}{2} & \frac{1+p}{2} & \frac{1-p}{2} \\
n+1 & 1+\frac{p}{2} & -\frac{1}{8}
\end{array} : -1 \right] = 2^{3p-3} \left( \frac{\Gamma(1+\frac{p}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{3+3p}{4})\Gamma(\frac{1}{2})} \right)^2. \tag{6.12}
\]

The number \( \frac{1}{2} \) and \( 1 + \frac{p}{2} \) differ by an integer and \( \frac{p}{2} \) is the only multiple of \( p \) an integral distance from both of them. The same holds for \( \frac{1+p}{4} \) and \( \frac{3+3p}{4} \) so
\[
2^{3p-3} \left( \frac{\Gamma(1+\frac{p}{2})\Gamma(\frac{1+p}{2})}{\Gamma(\frac{3+3p}{4})\Gamma(\frac{1}{2})} \right)^2 = 2^{3p-3} \left( \frac{\Gamma_p(1+\frac{p}{2})\Gamma_p(\frac{1+p}{2})}{\Gamma_p(\frac{3+3p}{4})\Gamma_p(\frac{1}{2})} \right)^2.
\]

Expanding the above using local analytic property of \( \Gamma_p(\cdot) \) and (3.6), we have
\[
3F2 \left[ \begin{array}{ccc}
\frac{1+3p}{2} & \frac{1+p}{2} & \frac{1-p}{2} \\
n+1 & 1+\frac{p}{2} & -\frac{1}{8}
\end{array} : -1 \right] \equiv (-1)^{\frac{p-1}{8}} \Gamma_p(\frac{1}{4})^4 \Gamma_p(\frac{1}{4})^2 \left[ 1 + \frac{3}{2} X p + \frac{1}{8} Y p^2 \right] \mod p^3. \tag{6.13}
\]

where
\[
X = G_1(0) - G_1 \left( \frac{1}{4} \right) \quad \text{and} \quad Y = -18G_1(0)G_1 \left( \frac{1}{4} \right) + 4G_1^2 \left( \frac{1}{4} \right) + 5G_2 \left( \frac{1}{4} \right) + 9G_4^2(0).
\]

Letting \( a = \frac{1-p/3}{2} \) in \((6.11)\)
\[
3F2 \left[ \begin{array}{ccc}
\frac{1-p}{2} & \frac{1+p/3}{2} & \frac{1-p/3}{2} \\
1-p & 1+\frac{p}{2} & -\frac{1}{8}
\end{array} : -1 \right] = 2^{3p-3} \left( \frac{\Gamma(1-\frac{p}{6})\Gamma(\frac{1-p/3}{4})}{\Gamma(\frac{1-p/3}{2})\Gamma(\frac{1}{2})} \right)^2. \tag{6.14}
\]

When \( p \equiv 1 \mod 4 \), one of \( \frac{1-p}{6} \) and \( \frac{1-p/3}{4} \) is in \( \frac{1}{6} + Z \) and the other one is in \( \frac{5}{6} + Z \). Thus \( \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1-p/3}{4})}{\Gamma(1-\frac{p}{6})\Gamma(\frac{1-p/3}{4})} \) is a product of 2 rising factorials and \( \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})^2} \) is itself a rising factorial. None of them contains multiples of \( p \).

Consequently,
\[
\left( \frac{\Gamma(1-\frac{p}{6})\Gamma(\frac{1-p/3}{4})}{\Gamma(\frac{1-p/3}{2})\Gamma(\frac{1}{2})} \right)^2 = \left( \frac{\Gamma(1-\frac{p}{6})\Gamma(\frac{1-p/3}{4})}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{6})} \right)^2 \left( \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})^2} \right)^2 = \left( \frac{\Gamma_p(1-\frac{p}{6})\Gamma_p(\frac{1-p/3}{4})\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{5}{6})\Gamma_p(\frac{1}{2})} \right)^2 \tag{6.15}
\]

as \( \left( \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})^2} \right)^2 = 4 \). So,
\[
3F2 \left[ \begin{array}{ccc}
\frac{1-p}{2} & \frac{1+p/3}{2} & \frac{1-p/3}{2} \\
1-p & 1+\frac{p}{2} & -\frac{1}{8}
\end{array} : -1 \right] = 2^{3p-3} \left( \frac{\Gamma_p(1-\frac{p}{6})\Gamma_p(\frac{1-p/3}{4})\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{5}{6})\Gamma_p(\frac{1}{2})} \right)^2.
\]
Applying (3.6) to (2) together using (3.6), we have

\[ 3F_2 \left[ \begin{array}{c} 1-p/2 \\ 1-p/2 \\ 1-p/2 \\ \frac{-1}{2} \end{array} ; \frac{-1}{8} \right] = (-1)^{\frac{x^2-1}{2}} \frac{\Gamma_p(\frac{1}{4})^4}{\Gamma_p(\frac{1}{2})^2} \left[ 1 - \frac{1}{2} Xp + \frac{1}{18} Y^2 \right] \mod p^3 \]  

(6.16)

where X and Y as before.

When \( a = \frac{1}{2} - p \), by (7.4)

\[ 3F_2 \left[ \begin{array}{c} 1-3p/2 \\ 1-p \\ 1-p \\ \frac{-1}{2} \end{array} ; \frac{-1}{8} \right] = 2^{\frac{3(p-1)}{2}} \frac{\Gamma_p(\frac{1}{2}) \Gamma_p(1-p) \Gamma_p(\frac{1+3p}{3}) \Gamma_p(\frac{1+p}{4})^2}{\Gamma_p(\frac{1}{2})^2} \]  

(6.17)

Applying (3.6) to \( 2^{\frac{3(p-1)}{2}} \) and the analytic expansion to \( \Gamma_p(\cdot) \), we have

\[ 3F_2 \left[ \begin{array}{c} 1-3p/2 \\ 1-p \\ 1-p \\ \frac{-1}{2} \end{array} ; \frac{-1}{8} \right] = (-1)^{\frac{x^2-1}{2}} \frac{\Gamma_p(\frac{1}{4})^4}{\Gamma_p(\frac{1}{2})^2} \left[ 1 - \frac{3}{2} Xp + \frac{1}{18} Y^2 \right] \mod p^3 \]  

(6.18)

Now we rewrite the right sides of (6.13), (6.16), and (6.18) in terms harmonic sums. So mod \( p^3 \) we get a linear system of 3 equations of the form

\[ A_0 + iA_1p + i^2 A_2 p^2 \equiv (-1)^{\frac{x^2-1}{2}} \frac{\Gamma_p(\frac{1}{4})^4}{\Gamma_p(\frac{1}{2})^2} \left[ 1 + iXp + \frac{i^2}{18} Y^2 \right] \mod p^3 \]

with \( i = 3/2, -1/2, -3/2 \) and \( A_0 = \sum_{k=0}^{\frac{x^2-1}{2}} \left( \frac{1}{7} \right)^k \left( -\frac{1}{8} \right)^k \) and \( A_1, A_2 \) involving harmonic sums like (6.1).

When \( p \geq 5 \) the matrix \( M = \begin{pmatrix} 1 & 3/2 & 1/8 \\ 1 & -1/2 & 1/72 \\ 1 & -3/2 & 1/8 \end{pmatrix} \) is invertible in \( \mathbb{Z}/p^3 \mathbb{Z} \), thus

\[ A_0 \equiv (-1)^{\frac{x^2-1}{2}} \frac{\Gamma_p(\frac{1}{4})^4}{\Gamma_p(\frac{1}{2})^2} = -(-1)^{\frac{x^2-1}{2}} \frac{\Gamma_p(\frac{1}{4})^4}{\Gamma_p(\frac{1}{2})^2} \mod p^3. \]

7. Appendix: Hypergeometric formulae

We list a couple extra formulae that we need for the last claim.

- **Clausen formula.** The following formula holds as formal power series and hence holds when both sides converge. See [28, §2.5] and Page 116 of [4].

\[ \left( \begin{array}{c} a \\ a+b+\frac{1}{2} \end{array} : x \right) = 3F_2 \left[ \begin{array}{c} 2a \ 2b \\ 2a + 2b \ a + b + \frac{1}{2} \end{array} ; x \right]. \]  

(7.1)

- **Gauss summation formula.** Thereof 2.2.2 of [4] For \( \Re(c-a-b) > 0 \),

\[ \left( \begin{array}{c} a \\ c \end{array} : 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \]  

(7.2)

- **If \( a \notin \mathbb{Z} < 0 \), then**

\[ 3F_2 \left[ \begin{array}{c} 3a-1 \\ 2a \ a+\frac{1}{2} \end{array} ; -\frac{1}{8} \right] = 2^{3a} \left( \frac{\Gamma(a+\frac{1}{2})\Gamma(\frac{a}{2})}{\Gamma(a)\Gamma(\frac{1}{2})} \right)^2 \]  

(7.3)
This is entry 1 in Table 1 of [16] by Karlsson. When \( a \in \mathbb{Z}_{<0} \), the value is different. In particular, when \( p \equiv 1 \mod 4 \) and \( a = \frac{1-p}{2} \),

\[
3 F_2 \left[ \frac{3a-1}{2a} \frac{a}{a+\frac{1}{2}}, \frac{1-a}{2a+\frac{1}{2}} ; -\frac{1}{8} \right] = -2^{3(p-1)} \left( \Gamma_p(1-p) \Gamma_p \left( \frac{1+3p}{4} \right) \Gamma_p \left( \frac{1+p}{4} \right) \right)^2 . \tag{7.4}
\]

To see why there are two cases, we first recall another formula of Pfaff

\[
2 F_1 \left[ \frac{a}{2}, \frac{b-a}{2} ; \frac{z^2}{4(z-1)} \right] = \begin{cases} 
(1-z)^{-b} 2 F_1 \left[ \frac{a}{2}, \frac{b-a}{2} ; \frac{z}{z-1} \right] & \text{if } b \in \mathbb{Z}_{<0} \text{ and } b < \mathcal{R}(a) \\
(1-z)^{-a} 2 F_1 \left[ \frac{a}{2}, \frac{b-a}{2} ; \frac{z}{z-1} \right] & \text{otherwise.}
\end{cases}
\tag{7.5}
\]

For the first case, see [4, pp 79], which can be verified using our argument below. The second case can be deduced from the argument in [4, pp 76]. In particular, we note that in this case, both sides are polynomials in \( z \). We also need a quadratic formula (which can be derived from the quadratic formula on pp 130, right above (3.1.16) [4] and the above Pfaff formula (7.5))

\[
2 F_1 \left[ \frac{a}{2}, \frac{\beta-a}{2} ; \frac{z^2}{4(z-1)} \right] = \begin{cases} 
(1-z)^\frac{b}{2} 2 F_1 \left[ \frac{a}{2}, \frac{b-a}{2} ; \frac{z}{z-1} \right] & \text{if } b \in \mathbb{Z}_{<0} \text{ and } b < \mathcal{R}(a) \\
(1-z)^\frac{a}{2} 2 F_1 \left[ \frac{a}{2}, \frac{b-a}{2} ; \frac{z}{z-1} \right] & \text{otherwise.}
\end{cases}
\tag{7.6}
\]

Another ingredient we need is how to deduce Kummer’s formula (3.9) from Gauss summation (7.2). This formula appears on page 126 of [4] which says

\[
2 F_1 \left[ \frac{a}{2}, \frac{b}{2} ; \frac{1}{1+a-b} ; -1 \right] = 2^{-a} 2 F_1 \left[ \frac{a}{2}, \frac{a+1-b}{2} ; \frac{1}{1+a-b} ; 1 \right]. \tag{7.7}
\]

Now we will show case 2 of (7.3). When \( p \equiv 1 \mod 4 \) and \( a = \frac{1-p}{2} \),

\[
3 F_2 \left[ \frac{1-3p}{2} \frac{1-p}{2} \frac{1+p}{2} ; -\frac{1}{8} \right] \tag{7.1} = 2 F_1 \left[ \frac{1-3p}{4} \frac{1+p}{4} ; -\frac{1}{8} \right]^2 
\]

\[
\text{(3.9)} = \left( 1 - \frac{1}{2} \right)^{\frac{1+p}{2}} 2 F_1 \left[ \frac{1+p}{2} \frac{1-p}{2} ; \frac{1}{2} \right]^2 
\]

\[
\text{(7.5)} = \left( 1 - \frac{1}{2} \right)^{\frac{p-1}{2}} (1+1)^{\frac{p-1}{2}} 2 F_1 \left[ \frac{1+p}{2} \frac{1-p}{2} ; -1 \right]^2 
\]

\[
\text{(7.7)} = \left( 2^{\frac{(p-1)}{2}} \right)^2 
\]

\[
\text{(7.2)} = 2^{\frac{3(p-1)}{2}} \left( \frac{3-p}{4} \right)^{\frac{p-1}{2}} 
\]

\[
4) \text{ of Lemma 17} = 2^{\frac{3(p-1)}{2}} \left( \Gamma_p \left( \frac{1}{2} \right) \right)^2 
\]

\[
\text{\Gamma_p} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{\frac{3-3p}{4}} 
\]

\[
= -2^{\frac{(p-1)}{2}} \left( \Gamma_p \left( 1-p \right) \Gamma_p \left( \frac{1+3p}{4} \right) \Gamma_p \left( \frac{1+p}{4} \right) \right)^2 
\]

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