From Unified Field Theory to the Standard Model and Beyond

David J. Jackson

September 12, 2018

Abstract

One hundred years ago this year attempts began to generalise general relativity with the ambition of incorporating electromagnetism alongside gravitation in a unified field theory. These developments led to gauge theories and models with extra spatial dimensions that have greatly influenced the modern-day pursuit of a unification scheme incorporating the Standard Model of particle physics, again ideally together with gravity. In this paper we motivate a further natural generalisation from extra spatial dimensions at an elementary level which is found to much more directly accommodate distinctive features of the Standard Model. We also investigate the potential to uncover new physical phenomena, making a case in the neutrino sector for one left-handed neutrino state to be massless, and emphasise the opportunity for a close collaboration between theory and experiment. The new theory possesses a very simple interpretation regarding the underlying source of these empirical structures.

Contents

1 Introduction: Unified Field Theories 1
2 Elementary Extra Spatial Dimensions 3
3 Extra Dimensions Reconsidered 3
4 Utilising the Form of Time 5
5 $E_6$, $E_7$, $E_8$ and the Standard Model 6
6 Neutrinos and other New Physics 10
7 Interpretation and Opportunities 11

1 Introduction: Unified Field Theories

The now familiar pattern of elementary particle multiplets consisting of leptons, quarks, gauge bosons and the Higgs was termed the ‘Standard Model’ in the mid-1970s ([1] chapter 21(e)). The richness of this structure contrasts with the situation a hundred years ago, during the era of the ‘Bohr atom’, when the basic constituents of particle physics were simply the electron, the proton (identified with the hydrogen nucleus) and the photon; with electromagnetism and gravitation being the only two known fundamental forces ([2] chapter 17(a)). Early unified field theories were based on generalisations of Einstein’s theory of gravity, then only recently expounded [3], which itself possesses a simple geometric interpretation.

1email: david.jackson.th@gmail.com
For general relativity the assumption of an extended globally flat 4-dimensional Minkowski spacetime, the original arena of special relativity, was dropped. The theory was motivated by the apparent absence of the force of gravity in a local inertial reference frame, with special relativity still holding for all non-gravitational physics within such a local ‘free fall’ arena by the equivalence principle ([2] chapter 9). An infinitesimal ‘proper time’ interval $\delta s$ associated with any free fall trajectory at any location can by definition be expressed in a local inertial reference frame as:

$$ (\delta s)^2 = (\delta x^0)^2 - (\delta x^1)^2 - (\delta x^2)^2 - (\delta x^3)^2 = \eta_{ab} \delta x^a \delta x^b \quad (1) $$

in terms of local coordinates $\{x^a\}$, with the Lorentz metric $\eta = \text{diag}(+1, -1, -1, -1)$ and $a, b = 0, 1, 2, 3$, in a form invariant under local Lorentz transformations between inertial frames. With respect to extended general coordinates $\{x^\mu\}$, with $\mu, \nu = 0, 1, 2, 3$, a local proper time interval can be expressed as an invariant under general coordinate transformations with:

$$ (\delta s)^2 = g_{\mu\nu}(x) \delta x^\mu \delta x^\nu \quad (2) $$

The force of gravity is ascribed to the metric field function $g_{\mu\nu}(x)$ which generalises $\eta_{ab}$ on the global scale and describes the geometry of an extended curved spacetime. It was then natural to enquire whether a further generalisation from general relativity might also provide an explanation of electromagnetic phenomena on relaxing further assumptions regarding the 4-dimensional spacetime metric geometry. In 1918 the first attempt was made by Weyl on considering the path-independence of the length of a parallel transported 4-vector to be a last vestige of rigid Euclidean geometry in general relativity. On dropping that assumption and introducing a scale factor Weyl developed a theory of electromagnetism of geometric origin ([4], [5] chapters 1–3). While that theory was flawed, by 1929 the scale factor applied to the metric of general relativity was converted into a phase factor applied to a complex wavefunction in quantum mechanics, successfully describing a theory of electromagnetism ([6], [5] chapters 4–5). This ‘gauge theory’ for electromagnetism with gauge group U(1) was generalised for larger non-Abelian gauge symmetries in the 1950s ([5] chapters 8–10) and became central to the modern-day structure of the Standard Model and Grand Unified Theories. A degree of success has been achieved for unification groups such as $E_6$, $E_7$ and $E_8$ (see for example [7, 8]). However, while these exceptional Lie groups are also of particular interest owing to the high degree of symmetry they describe and the uniqueness of these mathematical structures, a clear underlying conceptual motivation, whether geometric or otherwise, for their application in particle physics is still lacking.

In the 1920s the unified field theory of Kaluza [9] and Klein [10] was also introduced ([2] chapter 17(c), [5] chapter 3), with the assumption that spacetime should be limited to the 4-dimensional arena of general relativity being dropped. While various equations of electrodynamics could be extracted from a 5-dimensional spacetime framework, providing an element of formal geometric unification with general relativity, no new phenomena were predicted. Nevertheless the elegance and unity of the Kaluza-Klein idea, together with the realisation that the geometry of extended spacetimes with further extra spatial dimensions could be adapted to incorporate non-Abelian gauge theory (see for example [11]), has motivated subsequent unification schemes. Since the 1970s the ambition has been to accommodate structures of the Standard Model of particle physics, or even a Grand Unified Theory, via the properties of the extra spatial dimensions over 4-dimensional spacetime.
2 Elementary Extra Spatial Dimensions

At the most elementary level of purely local structure adding extra spatial dimensions implies augmenting the quadratic expression for the 4-dimensional proper time interval $\delta s$ of equation 1 to the n-dimensional form:

$$(\delta s)^2 = (\delta x^0)^2 - (\delta x^1)^2 - (\delta x^2)^2 - (\delta x^3)^2 - (\delta x^4)^2 - \ldots - (\delta x^{n-1})^2 = \hat{\eta}_{ab}\delta x^a\delta x^b$$

where $\{x^4, \ldots, x^{n-1}\}$ are the $(n - 4)$ extra dimensions, $\hat{\eta} = \text{diag}(+1, -1, \ldots, -1)$ is the extended local Lorentz metric and $a, b = 0, \ldots, (n - 1)$. On dividing both sides by $(\delta s)^2$ and defining the components $v^a = \frac{\delta x^a}{\delta s}$ in the limit $\delta s \to 0$ this expression can be written as:

$$|v_n|^2 := (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 - (v^4)^2 - \ldots - (v^{n-1})^2 = \hat{\eta}_{ab}v^av^b = 1$$

in terms of the components of the 'n-velocity' vector $v_n = (v^0, \ldots, v^{n-1}) \in \mathbb{R}^n$. The simplest and most direct means of constructing a physical theory based on this structure is to assume the breaking of the $SO^+(1, n - 1)$ symmetry of equation 4 in projecting the first four components onto the local tangent space, $v_4 = (v^0, v^1, v^2, v^3) \in TM_4$, at any location on the 4-dimensional spacetime manifold $M_4$, upon which a preferred local external Lorentz $SO^+(1, 3) \subset SO^+(1, n - 1)$ symmetry acts. On taking the residual components of equation 4 to form the basis for 'matter fields' in the extended spacetime we directly deduce the following symmetry breaking pattern:

$$\text{SO}^+(1,n-1) \rightarrow \text{SO}^+(1,3) \times \text{SO}(n-4)$$

$$v_n \rightarrow \begin{cases} v_4 \in \mathbb{R}^4 & : 4\text{-vector invariant : tangent vector} \\ v_{n-4} \in \mathbb{R}^{n-4} & : \text{scalar } (n-4)\text{-vector : matter field} \end{cases}$$

In this simple picture the matter field $v_{n-4}(x)$ in spacetime $M_4$, as a Lorentz scalar that transforms under the $(n - 4)$-dimensional vector representation of the residual internal gauge symmetry $\text{SO}(n - 4)$, does not remotely resemble structures of the Standard Model of particle physics for any value of $n$. Rather than specifically adding more sophisticated structures we shall motivate an intrinsic generalisation that greatly improves this situation.

3 Extra Dimensions Reconsidered

Here we observe that since we do not perceive or navigate around the extra dimensions there is no compelling reason for the additional components to possess the local structure of $\{x^4, \ldots, x^{n-1}\}$ in equation 3 as a quadratic extension to the local 4-dimensional spacetime form of equation 1 (with the minus signs from the Lorentz metric signature convention). That is, the extra components in equation 3 have the 'spatial' property of adding quadratically to form local 'lengths' $\delta\Sigma$, with for example:

$$(\delta\Sigma)^2 = (\delta x^4)^2 + (\delta x^{n-1})^2$$

which via the Pythagorean theorem describes right-angled triangle structures as a basis for a local Euclidean geometry. This property is only required for the local geometric...
structure of \{\delta x^1, \delta x^2, \delta x^3\} in forming the basis of the extended external 3-dimensional space that we do perceive and move around in, while the assumption of this locally Euclidean form can be dropped for the extra components.

This unnecessary restriction seems even more artificial on considering large \(n\) since then almost all of the components on the right-hand side of equation 3 are not required to take a quadratic form as the \{\delta x^a\} for all \(a > 3\) do not represent a physical perceived space. However the left-hand side of equation 3 still describes a simple interval of proper time \((\delta s)\), invariant under \(\text{SO}^+(1, n-1)\) transformations, which is hence pivotal in threading together all of the basis-dependent components on the right-hand side and in defining this structure. In fact we can interpret equation 3 as simply representing a possible arithmetic expression for a real proper time interval \(\delta s \in \mathbb{R}\) and then ask what further possibilities there may be.

While intervals of time add linearly, as objectively recordable by a clock, on exploiting the basic arithmetic properties of the real numbers expressions for an infinitesimal interval can be written down for \((\delta s), (\delta s)^2, (\delta s)^3, \ldots\) or \((\delta s)^p\) in general, for \(p = 1, 2, 3, \ldots\) of which equation 3 represents only a particular case for \(p = 2\). This suggests that the functional form on the right-hand side of equation 3 can be generalised to a \(p\)th-order homogeneous polynomial expression in \(n\) components \{\delta x^a\}, with \(a, b, c = 0, \ldots, n-1\):

\[
(\delta s)^p = \alpha_{abc\ldots} \delta x^a \delta x^b \delta x^c \ldots \quad \text{with each } \alpha_{abc\ldots} \in \{-1, 0, 1\} \quad (8)
\]

provided we can extract an appropriate 4-dimensional quadratic substructure in four components \{\delta x^0, \delta x^1, \delta x^2, \delta x^3\}, in the form of the right-hand side of equation 1, as required to represent the local geometric structure of the external spacetime \(\mathcal{M}_4\). That is, we require that equation 8 can in general be written in the form:

\[
(\delta s)^p = \left[\eta_{ab} \delta x^a \delta x^b \right] (\delta x^4, \ldots, \delta x^{n-1})^{p-2} + (\delta x^0, \ldots, \delta x^{n-1})^p \quad (9)
\]

where here in the first term \(a, b = 0, 1, 2, 3\) in the first factor and the second factor represents a \((p-2)\)th-order polynomial in the remaining \((n-4)\) components, while the second term represents the further \(p\)th-order polynomial contributions to equation 8.

In order to establish a convenient notation and avoid expressions with infinitesimal elements we can in turn generalise equation 4 by again defining an \(n\)-vector \(v_n \in \mathbb{R}^n\) with the generally finite components \(v^a = \frac{\delta v^a}{\delta s} \big|_{\delta s \to 0}\), and on dividing both sides of equation 8 by \((\delta s)^p\) we define:

\[
L_p(v_n)_{\hat{G}} := \alpha_{abc\ldots} \frac{\delta x^a \delta x^b \delta x^c \ldots}{\delta s \delta s \delta s \ldots \big|_{\delta s \to 0}} = \alpha_{abc\ldots} v^a v^b v^c \ldots = 1 \quad (10)
\]

again with \(a, b, c = 0, \ldots, n-1\) and each \(\alpha_{abc\ldots} \in \{-1, 0, 1\}\). Here \(L_p\) denotes a \(p\)th-order homogeneous polynomial function of the \(n\) components of \(v_n\) with the full symmetry group \(\hat{G}\). For \(p > 2\) particular values for \(p\) and \(n\) will be inherently preferred as unique mathematical structures which possess a high degree of symmetry, while subsuming equation 1, will be highlighted. In this sense the progression from ‘spacetime forms’ to ‘forms of time’ is both more general and yet more restrained, and in a manner that leads to well known unification groups as we describe below.
4 Utilising the Form of Time

As a means of explicitly embedding the 4-dimensional quadratic spacetime form inside a higher-order homogeneous polynomial form of time we rearrange equation 1 in the fashion of equation 10 and write the resulting expression as the determinant of a $2 \times 2$ Hermitian complex matrix:

$$L_2(v_4)_{SL(2, \mathbb{C})} = \eta_{ab} v^a v^b = \det(h) = \det \begin{pmatrix} v^0 + v^3 & v^1 - v^2i \\ v^1 + v^2i & v^0 - v^3 \end{pmatrix} = 1$$ (11)

with the Lorentz 4-vector $v_4 \equiv h \in \mathbb{h}_2 \mathbb{C}$. As indicated this determinant form is invariant under the action of the symmetry group $SL(2, \mathbb{C})$ as the double cover of the Lorentz group $SO^+(1, 3)$ ([12] equations 16 and 17). This 4-dimensional form can be embedded directly within the determinant of a $3 \times 3$ Hermitian complex matrix, which we interpret as a cubic form of time in nine components consistent with equation 10, now with an augmented $SL(3, \mathbb{C})$ symmetry:

$$L_3(v_9)_{SL(3, \mathbb{C})} = \det \begin{pmatrix} v^0 + v^3 & v^1 - v^2i & v^4 + v^5i \\ v^1 + v^2i & v^0 - v^3 & v^6 + v^7i \\ v^4 - v^5i & v^6 - v^7i & v^8 \end{pmatrix} = \det \begin{pmatrix} h & \psi \\ \psi^\dagger & n \end{pmatrix} = 1$$ (12)

$$= \det(v_9) = \left[ \eta_{ab} v^a v^b \right] n - 2h \cdot (\psi \psi^\dagger) = 1$$ (13)

with $v_9 \in \mathbb{h}_3 \mathbb{C}$, $h \in \mathbb{h}_2 \mathbb{C}$, $\psi \in \mathbb{C}^2$ and here $n = v^8 \in \mathbb{R}$ while $a, b = 0, 1, 2, 3$. In the final term $h \cdot (\psi \psi^\dagger)$ is the Lorentz inner product between the 4-vectors associated with $h, \psi \psi^\dagger \in \mathbb{h}_2 \mathbb{C}$ ([13] equations 23 and 70).

The full $SL(3, \mathbb{C})$ symmetry of $L_3(v_9)_{SL(3, \mathbb{C})} = 1$ is broken through a preferred external $SL(2, \mathbb{C}) \subset SL(3, \mathbb{C})$ symmetry acting upon a necessary choice of $v_4 = (v^0, v^1, v^2, v^3) \in TM_4$ subcomponents of $v_9 \in \mathbb{h}_3 \mathbb{C}$ projected onto a local inertial reference frame from equations 12–13. The extraction of this necessarily quadratic substructure to match the local geometry of the external spacetime, via the square brackets in equation 13, also leaves a residual internal $U(1) \subset SL(3, \mathbb{C})$ symmetry that can be interpreted as a gauge group underlying a theory of electromagnetism ([12] subsections 2.3 and 4.2). The broken symmetry reduces the full 9-dimensional vector $v_9 \in \mathbb{h}_3 \mathbb{C}$ to the three parts introduced in equation 12 with the Lorentz $SL(2, \mathbb{C})$ and internal $U(1)$ factors acting upon these subcomponents as:

$$SL(3, \mathbb{C}) \to SL(2, \mathbb{C}) \times U(1)$$ (14)

$$v_9 \to \begin{cases} h \in \mathbb{h}_2 \mathbb{C} : \text{vector} & 0 : \text{tangent vector} \\ \psi \in \mathbb{C}^2 : \text{L-spinor} & 1 : \text{matter field} \\ n \in \mathbb{R} : \text{scalar} & 0 : \text{matter field} \end{cases}$$ (15)

with the 2-component Weyl spinor $\psi$ taken to be left-handed by convention. Hence the internal $U(1)$ gauge symmetry of electromagnetism also acts non-trivially upon
a spin-$\frac{1}{2}$ field $\psi(x)$ in spacetime, as indicated by the normalised unit charge ‘1’ in equation 15, while $n(x)$ is a neutral scalar.

Being central to the symmetry breaking, and having a scalar magnitude $|h| = \sqrt{\det(h)}$ in the projection onto $TM_4$, the components of the vector field $h(x) \in h_2 \mathbb{C}$ of equation 15 are associated with a non-standard Higgs in this theory (also for the further reasons reviewed in [13] after figure 4). In general the symmetry breaking projection of $h \equiv v_4 \in TM_4$ out of the full set of components for the $n$-dimensional form of equation 10 breaks the original full symmetry $\hat{G}$ down to the subgroup:

$$\text{Lorentz} \times G \subset \hat{G}$$

where the Lorentz symmetry may be the double cover $\text{SL}(2, \mathbb{C})$ and $G$ is the internal symmetry, with $G = \text{U}(1)$ in equation 14. The components of $v_n \in \mathbb{R}^n$ are partitioned into subsets that transform under irreducible representations of this subgroup, as listed for example in equation 15 (and discussed in [12] for equation 23 there). At the same time the set of terms in the expansion of the corresponding form $L_p(v_n)_{\hat{G}} = 1$, which is invariant under $\hat{G}$, will be partitioned into subsets invariant under the Lorentz $\times G$ broken symmetry of equation 16, as for the two parts of equation 13. For the full theory such individually invariant parts of $L_p(v_n)_{\hat{G}} = 1$ which contain a factor of $h$, or a scalar composition such as $|h|$, are proposed to be associated with ‘mass terms’ in an effective Lagrangian deriving from the theory, in part motivating the kernel symbol ‘$\mathcal{L}$’ in equation 10.

The identification of the local geometric structure of the spacetime manifold itself with a quadratic substructure extracted from equation 10 implies the complete distinction between the external and internal components. Hence the full symmetry $\hat{G}$ of equation 10, with which we begin in the mathematics of the theory, is broken absolutely to the product of the external Lorentz and internal $G$ symmetry in equation 16 to describe all physics that can be defined in 4-dimensional spacetime, consistent with the demands of the Coleman-Mandula theorem ultimately for the relativistic quantum theory limit ([12] subsection 5.3).

5 E$_6$, E$_7$, E$_8$ and the Standard Model

While the Lie algebras, including the five exceptional cases of G$_2$, F$_4$, E$_6$, E$_7$ and E$_8$, were classified by Killing and Cartan in the late 19$^{th}$ century ([14] section 4 opening) an understanding of explicit expressions for certain representations of these algebras developed from the mid-20$^{th}$ and continues into the 21$^{st}$ century. For example the smallest non-trivial representation of E$_6$ can be expressed by the space of $3 \times 3$ Hermitian octonion matrices $h_3 \mathbb{O}$, as employed in 1950 [15], with the corresponding determinant preserving $E_6 \equiv \text{SL}(3, \mathbb{O})$ group action described in explicit detail more recently [16].

In the context of the present theory, while we obtained equation 12 from equation 11 by a natural minimal extension from the $2 \times 2$ to the $3 \times 3$ matrix case, we can also augment equation 12 by a natural generalisation from the complex numbers $\mathbb{C}$ to the octonions $\mathbb{O}$, which with eight real components uniquely form the largest division
algebra ([14] sections 1 and 1.1), to obtain the cubic 27-dimensional form:

\[ L_3(v_{27})_{E_6} = \det(v_{27}) = \det \begin{pmatrix} X & \theta \\ \theta^\dagger & n \end{pmatrix} = 1 \]  

(17)

with \( v_{27} \in h_3 \mathbb{O} \), \( X \in h_2 \mathbb{O} \), \( \theta \in \mathbb{O}^2 \) and again \( n \in \mathbb{R} \). The SL(3, \mathbb{C}) symmetry of equation 12 is augmented correspondingly to SL(3, \mathbb{O}) \equiv E_6. On identifying an external symmetry SL(2, \mathbb{C}) \subset E_6 acting upon a projected Lorentz 4-vector \( v_4 \in TM_4 \), now identified with \( h \in h_2 \mathbb{C} \) extracted from subcomponents of \( X \in h_2 \mathbb{O} \) in equation 17, a symmetry breaking pattern is determined with ([13] subsection 4.2):

\[ E_6 \to \text{SL}(2, \mathbb{C}) \times \text{SU}(3)_c \times \text{U}(1)_Q \]  

(18)

\[
\begin{aligned}
    &\{ X \in h_3 \mathbb{O} : \} \\
    &\{ \theta \in \mathbb{O}^2 : \} \\
    &n \in \mathbb{R} : \\
    &\text{vector} \quad 1 \quad 0 \quad : \nu\text{-lepton}/h \\
    &\text{scalar} \quad 3 \quad \frac{2}{3} \quad : u\text{-quark} \\
    &\text{L-spinor} \quad 1 \quad 1 \quad : e\text{-lepton} \\
    &\text{L-spinor} \quad 3 \quad \frac{1}{3} \quad : d\text{-quark} \\
    &\text{scalar} \quad 1 \quad 0
\end{aligned}
\]

(19)

Through this natural augmentation from equations 14–15 we hence find an internal non-Abelian symmetry, which is identified with the colour gauge group SU(3)$_c$, alongside the original Abelian gauge group of electromagnetism, now denoted U(1)$_Q$. The pattern of U(1)$_Q$ relative charge magnitudes determined and listed in equation 19 as aligned with the SU(3)$_c$ singlets 1 and triplets 3 leads to the provisional ‘matter field’ interpretation of the subcomponent decomposition of \( v_{27} \in h_3 \mathbb{O} \) under the broken symmetry as representing a generation of Standard Model leptons and quarks, as listed in the final column of equation 19.

However, with respect to the external SL(2, \mathbb{C}) symmetry, while the \( d\)-quark and \( e\)-lepton states transform uniformly as a set of four 2-component left-handed Weyl spinors, the ‘\( u\)-quark’ states transform as Lorentz scalars and the neutral components most naturally associated with the neutrino, with respect to the internal symmetry, are incorporated into a Lorentz 4-vector. Compounding these discrepant features, this natural slot for the ‘\( \nu\)-lepton’ in equation 19 is already occupied specifically by the 4-vector \( h \in h_2 \mathbb{C} \) subcomponent of \( X \in h_2 \mathbb{O} \), which is projected onto the tangent space of the external spacetime and associated with the Higgs as noted for equation 15.

The smallest non-trivial representation of the next largest exceptional Lie group E$_7$ can be described by an action on a related 56-dimensional space [17], which we denote \( F(h_3 \mathbb{O}) \), as again has been studied in detail more recently (see for example [18]). The E$_7$ action preserves a homogeneous quartic form \( q \) on the space \( F(h_3 \mathbb{O}) \) which, while not expressed as a determinant function itself, contains the E$_6$ action on the 27-dimensional cubic form of equation 17 and hence can be considered as a further natural augmentation consistent with equation 10 with ([13] equations 30 and 63):
\[ L_4(v_{56})_{E_7} = q(\mathcal{X}, \mathcal{Y}, \alpha, \beta) = 1 \]

where \( v_{56} \equiv (\mathcal{X}, \mathcal{Y}, \alpha, \beta) \in F(h_3O) \), with \( \mathcal{X}, \mathcal{Y} \in h_3O \) and \( \alpha, \beta \in \mathbb{R} \). Given the straightforward embedding of the subgroup \( E_6 \) with the 27-dimensional representation \( 27 \) and its complex conjugate \( 27 \) corresponding respectively to the \( \mathcal{X} \in h_3O \) and \( \mathcal{Y} \in h_3O \) subcomponents of \( F(h_3O) \) in equation 20, the main consequences of this \( E_7 \) augmentation follow directly from equation 19. On now projecting the four components of \( v_4 \equiv h \in h_2C \) onto the local tangent space of the external 4-dimensional spacetime from the \( \mathcal{Y} \in h_3O \) subcomponents the breaking pattern for the \( E_7 \) symmetry of equation 20 is determined ([13] subsection 4.3):

\[
E_7 \rightarrow \text{SL}(2,\mathbb{C}) \times \text{SU}(3)_c \times \text{U}(1)_Q \quad (21)
\]

\[
v_{56} \rightarrow \begin{cases}
\mathcal{X} \in h_3O : \\
\mathcal{Y} \in h_3O : \\
\alpha, \beta \in \mathbb{R} : \\
\end{cases}
\begin{align*}
\text{vector} & : 1 & 0 & : '\nu_L' \\
\text{scalar} & : 3 & \frac{2}{3} & : 'u_L' \\
L\text{-spinor} & : 1 & 1 & : e_L \\
L\text{-spinor} & : 3 & \frac{1}{3} & : d_L \\
\text{scalar} & : 1 & 0 & : n \\
\text{vector} & : 1 & 0 & : h \\
\text{scalar} & : 3 & \frac{2}{3} & : 'u_R' \\
R\text{-spinor} & : 1 & 1 & : e_R \\
R\text{-spinor} & : 3 & \frac{1}{3} & : d_R \\
\text{scalar} & : 1 & 0 & : N \\
\text{scalar} & : 1 & 0 & : \alpha, \beta \\
\end{align*}
\quad (22)
\]

In addition to the four left-handed spinors of equation 19, reproduced in the \( \mathcal{X} \in h_3O \) subcomponents above, a corresponding set of four right-handed spinors is identified in the \( \mathcal{Y} \in h_3O \) subcomponents. Hence the \( \mathcal{X} \) and \( \mathcal{Y} \) components of equation 22 are referred to as the ‘left-handed’ and ‘right-handed’ sectors of the theory. With the internal symmetry transformations being the same for both sectors, the 2-component Weyl spinors for the \( e \) and \( d \) states in equation 19 have been augmented to 4-component Dirac spinors in equation 22. Corresponding \( L \) and \( R \) subscripts are also added to the ‘\( u \)-quark’ and ‘\( \nu \)-lepton’ states in equation 22, albeit within quotation marks since the need to identify a Lorentz spinor structure for these states will require yet further augmentation.

However we can observe at this stage that the embedding of the external 4-vector \( h \in h_2C \), closely linked with the Higgs, within the \( \mathcal{Y} \in h_3O \) components prohibits the accommodation of a neutrino state ‘\( \nu_R \)’ in the right-handed sector while implying that the slot is now open for a left-handed neutrino ‘\( \nu_L \)’ in the corresponding components of \( \mathcal{X} \in h_3O \), without the conflict described for equation 19.

More generally we note that the branching patterns obtained for this elementary symmetry breaking for natural augmentations of the form of time in equation 10
over the local structure of 4-dimensional spacetime, leading to equations 15, 19 and 22, provide a far better template for the direct emergence of the Standard Model elementary particle multiplet structure than the equivalent analysis applied for the restricted case of extra spatial dimensions via the quadratic terms of equation 4, as described for equations 5–6, and with very little redundancy. This strongly suggests that in place of equation 3 the generalisation to equation 8 affords a more appropriate core basis for a unified theory, particularly since in the latter case we begin by discarding an assumption and the extraction of a necessarily quadratic substructure for external spacetime might underlie the mechanism of symmetry breaking itself.

Nevertheless further structure is still needed to describe the full Standard Model. In addition to the required spinor structure for the \( \nu \)-lepton and \( u \)-quark states in equations 21–22 the principle features that remain to be accounted for are an electroweak \( SU(2)_L \times U(1)_Y \) symmetry (that breaks to \( U(1)_Q \)) and a full three generations of leptons and quarks. Ideally a further natural mathematical generalisation might incorporate these features. This leads to the proposal of a possible symmetry action of \( E_8 \), uniquely the largest exceptional Lie group, on a homogeneous polynomial form:

\[
L_8(v_{248})_{E_8} = 1
\]

as the ultimate instantiation for equation 10. This provisional form is potentially of octic order with \( p = 8 \) (see for example [19]), and a close connection with the smallest non-trivial \( E_8 \) representation with \( n = 248 \) dimensions is here presumed; although other values for \( p \) and \( n \) might be conceivable. The nature of this structure and the plausibility of encompassing the principle remaining Standard Model features in a correlated manner is the main topic of [13].

As a unification symmetry the Lie group \( E_8 \) itself is comfortably able to incorporate a broken symmetry corresponding to a product of the external Lorentz group and internal Standard Model gauge groups in the form of equation 16 with:

\[
\text{Lorentz} \times SU(3)_c \times SU(2)_L \times U(1)_Y \subset E_8
\]

On the other hand as an extension from the \( 27 \) representation of \( E_6 \) underlying equation 19, as combined with the complex conjugate \( \overline{27} \) for equation 22, a possible factor of three for three generations of leptons and quarks is suggested by the subgroup embedding of \( E_6 \) in \( E_8 \) with the representation branching pattern:

\[
E_8 \supset E_6 \times SU(3): \quad 248 \rightarrow (27, 3) + (\overline{27}, \overline{3}) + (78, 1) + (1, 8)
\]

However, as explained in [13], unlike the case for the embedding of the \( E_6 \subset E_7 \) action the embedding of \( E_6 \) and \( E_7 \) in the \( E_8 \) action on \( L_8(v_{248})_{E_8} = 1 \) is expected to be less direct than that suggested by equation 25 if the needed spinor structures for \( \nu \)-leptons and \( u \)-quarks and a complete electroweak theory are also to be identified compatible with the symmetry breaking pattern of equation 24. As a generalisation from the \( E_6 \) action on equation 17 and the \( E_7 \) action on equation 20 the proposed \( E_8 \) action is also presumed to incorporate octonion composition in an essential way, with the properties of octonion ‘triality’ ([13] section 5) expected to be at the heart of unravelling the full Standard Model spinor structure for a full three generations of leptons and quarks.
Neutrinos and other New Physics

Notwithstanding the above caveat regarding the need to incorporate a neutrino spinor structure, given the embedding of the E$_6$ level of equation 19 within the E$_7$ level of equation 22 we might anticipate some of the implications for neutrino physics of a further embedding within a three generation pattern at the E$_8$ level, if we assume in broad terms the simplest further progression for the neutrino sector:

\[
\begin{align*}
E_6 & : [\nu_L/h] \quad \text{(equation 19)} \\
E_7 & : [\nu_L] \quad \text{and} \quad [h] \quad \text{(equation 22)} \\
E_8 & : [\nu_L \nu_L \nu_L \nu_R \nu_R] \quad \text{(26)}
\end{align*}
\]

This schematic augmentation incorporates three generations of left-handed neutrinos, as for the left and right-handed charged leptons and quarks, and suggests the accommodation of two right-handed neutrinos alongside the original external $h \in h_2C$ projection, which now prohibits the identification of a third $\nu_R$ state. With the components of $h \equiv v_4 \in TM_4$ being associated with the Higgs and the origin of mass (as discussed before and following equation 16) a clear origin for a mass asymmetry in the neutrino sector is also implied for this provisional structure at the E$_8$ level in equation 26, which suggests some form of ‘seesaw’ imbalance between the left and right-handed states. In a standard neutrino seesaw mechanism model (see for example [20] section 2 and references therein) each $\nu_R$ state generates one $\nu_L$ mass. Hence with only two $\nu_R$ states available in the provisional scheme of equation 26 there is an indication that the lightest $\nu_L$ mass state may in fact be massless, that is $m_{\min} = 0$.

While ongoing and future experiments on tritium beta decay [21] and neutrinoless double-beta decay [22] will improve the corresponding constraints on the mass of the lightest $\nu_L$ state the most stringent test of a predicted $m_{\min} = 0$ may be provided by the cosmological observations limiting the total mass of the three $\nu_L$ states, currently with an upper bound of around $m_{\text{tot}} < 0.20\text{eV}$ ([23] section 64). Given the two empirically established $\nu_L$ mass differences from solar and atmospheric neutrino oscillations the lowest possible value is $m_{\text{tot}} \simeq 0.06\text{eV}$ (see also [20] section 2). Although model-dependent, future prospects for standard neutrinos within the ΛCDM cosmological model (with Λ the cosmological constant and CDM cold dark matter) are for $m_{\text{tot}} = 0.06\text{eV}$ to be detectable at the 3–4$\sigma$ level in the coming years ([23] sections 25.4 and 64). This implies that the case for $m_{\min} = 0$ is testable in that it could be disfavoured with statistical significance in the foreseeable future.

Without the guide of equation 26 a more symmetric proposal would be for the introduction of three $\nu_R$ states as for the ‘Neutrino Minimal Standard Model’, or $\nu$MSM ([20] section 7, [24, 25]), proposed as a simple economical extension from the Standard Model (for which all three $\nu_L$ states are massless and there are no $\nu_R$ states). Two of the $\nu_R$ states in the $\nu$MSM have nearly degenerate masses in the range of around 1–100 GeV which generate $\nu_L$ masses via the seesaw mechanism consistent with the solar and atmospheric neutrino oscillation data and also in principle accounting for the baryon asymmetry of the universe through $CP$-violating oscillations of these two $\nu_R$ states in its early history. Hence the $\nu$MSM implies that the two $\nu_R$ states in equation 26 may also be sufficient to account for these phenomena.
The third $\nu_R$ of the $\nu$MSM, with a mass of a few keV, acts as a warm dark matter candidate but with a Yukawa coupling too small to make a significant contribution to the $\nu_L$ masses and hence leaving the lightest $\nu_L$ state practically massless. However here from equation 26 there is no room to accommodate a third $\nu_R$ state, suggesting that $m_{\text{min}} = 0$, and a third $\nu_R$ is not needed for dark matter since such candidates are provided by the scalar invariant components $n, N, \alpha$ and $\beta$ listed in equation 22. These augment the original single scalar invariant $n \in \mathbb{R}$ of equations 15 and 19. The four scalar invariants at the $E_7$ level in equation 22 may generalise into a broader 'dark sector' involving further components at the full $E_8$ symmetry level and offer the possibility to observationally test this new physics by exploring the corresponding cosmological implications.

From equation 26 the Standard Model Higgs, deriving from the components underlying $h \equiv v_4 \in TM_4$, is clearly intimately connected with the neutrino sector, and hence some of their properties may be closely correlated. We note for example that at the upper end of the 1~100 GeV range suggested by the $\nu$MSM the mass scale for two $\nu_R$ states would be close to the observed value of $M_H \simeq 125$ GeV for the Higgs [23]. The need for a spinor structure for both the $\nu_L$ and $\nu_R$ states to be incorporated under the Lorentz $\subset E_8$ action of equation 24 on the components of $L_8(v_{248})_{E_8} = 1$ for equation 26 suggests that $h$ may itself also have an underlying spinor composition. Indeed spinor components can be combined to form both vector objects, as for $\psi \psi^\dagger$ in equation 13, and scalar objects as for composite models for the Higgs (see for example [26] and references therein). For composite Higgs models the coupling of the Higgs to fermion pairs can deviate from the Standard Model expectation by of order 10%, sufficient for this new physics to be observable at a 250 GeV $e^+ e^-$ linear collider ([27] section 5). At such a machine invisible Higgs decays to a dark or hidden sector, such as the $n, N, \alpha, \beta$ or further scalar invariant components discussed above, can also be inferred via a visible recoiling $Z^0$ boson decay ([27] section 6).

Precise empirical predictions will require a full understanding of the structure of the theoretically predicted $E_8$ symmetry action on the full form of time $L_8(v_{248})_{E_8} = 1$ and the resulting symmetry breaking pattern. Since the further required features of the Standard Model beyond equations 21–22 are closely correlated it is plausible that they may all be uncovered together in one further augmentation from the $E_7$ form of equation 20 to the proposed $E_8$ form described for equation 23 (as considered in detail in [13]). If these required features emerge at the $E_8$ level this will provide a very firm basis for investigating a wealth of new physics beyond the Standard Model.

7 Interpretation and Opportunities

While connecting with current and future programs in both experimental particle physics and observational cosmology the theory presented here can be motivated in a similar spirit as for the early unified field theories dating from a hundred years ago briefly discussed in the opening of this paper. Following those initial proposals, in the early 1930s and still early in his personal quest for such a theory, Einstein summed up the nature of the problem of seeking a unifying extension to general relativity with the question [28]:

11
Is there a theory of the continuum in which a new structural element appears side by side with the metric such that it forms a single whole together with the metric?

Here equation 8, equivalent to equation 10, is proposed as such a ‘single whole’, deriving from the continuum of proper time, which can incorporate the local 4-dimensional spacetime metric, as described for equation 9 and exemplified by the cubic form of equations 12–13, side by side with additional structures that are interpreted as a basis for matter fields in spacetime.

Historically in progressing from a Newtonian absolute space and absolute time to the spacetime of special and then general relativity in the early 20th century a central role was played by the conception of time, with a different proper time carried at rest in each inertial reference frame. In general relativity the inertial frames are strictly local with each proper time interval $\delta s$ invariant under local Lorentz transformations and taking the form of equation 1. On the global scale the Lorentz metric $\eta_{ab}$ is supplanted by the general metric $g_{\mu\nu}(x)$, identified with the gravitational field, with the local proper time interval in equation 2 invariant under general coordinate transformations.

It is this distinctive invariant role for time that we have focussed upon and generalised in leading from equation 1, via equation 3, to equations 8 and 10. Matter fields originate at the most elementary level through a simple symmetry breaking analysis for equation 10, deriving directly from the extraction of a Lorentz $\subset\hat{G}$ external symmetry acting on the subcomponents $v_4 \in TM_4$ projected from $v_n \in \mathbb{R}^n$ onto the local external 4-dimensional spacetime. Through natural augmentations the properties of the residual components are found to resemble structures of the Standard Model at the elementary particle level of matter as described for equations 15, 19 and 22, without needing to contrive or postulate these features. The contrast with general relativity, which considers the global geometry of spacetime through equation 2 rather than the purely local generalisation for proper time $\delta s$ from equation 1 to equation 8, is depicted in figure 1.

![Figure 1: The relation between gravitation, described by the metric field $g_{\mu\nu}(x)$ on $M_4$ in generalising from the global Lorentz metric $\eta_{ab}$ of a flat 4-dimensional spacetime, and matter fields, originating from the broken symmetry of a complementary direct generalisation of the local Lorentz metric expression for a proper time interval $\delta s$.](image)
As implied in figure 1 we can zoom into an infinitesimal local inertial reference frame anywhere on the spacetime manifold \( M_4 \) and generalise equation 1 to equations 8 and 10 to explore the microscopic structure of matter deriving from the residual components. The theory is essentially founded upon the continuous flow of time that can be associated with any local inertial frame at any location in spacetime. Analysing this simple ‘one-dimensional’ starting point a finite duration of proper time \( s \in \mathbb{R} \) can be decomposed down to infinitesimal intervals:

\[
s = \delta s + \delta s + \delta s + \ldots \quad \text{with substructure} \quad \delta s = \sqrt[p]{\alpha_{abc\ldots} \delta x^a \delta x^b \delta x^c \ldots}
\] (27)

for \( \delta s \to 0 \) as the \( p^{th} \)-root of a homogeneous polynomial of order \( p \) in \( n \) components \( \{\delta x^a\} \in \mathbb{R}^n \); with \( a, b, c = 1, \ldots, n \) (or \( 0, \ldots, n - 1 \)) and each \( \alpha_{abc\ldots} \in \{-1, 0, 1\} \).

Here we are simply exploiting the basic arithmetic properties of the real numbers, representing the continuum of time with \( \delta s \in \mathbb{R} \), which together with addition include the operations of multiplication and extracting roots. The right-hand expression in equation 27 is precisely equivalent to equation 8, which as a natural generalisation from the expression for extra spatial dimensions in equation 3 forms the basis of the whole theory. In equations 27 and 8 we are not adding anything to time, nor replacing time with anything, but simply expressing an intrinsic arithmetic substructure that is carried simultaneously with time. The simplicity of this interpretation then provides a further motivation for the theory (the historical and philosophical aspects of which are discussed in [29]).

In summary there are three main supporting arguments for this theory:

- The assumption that further components augmenting the local 4-dimensional spacetime form for a proper time interval should have a spatial structure can be dropped, provided that we can identify a 4-dimensional quadratic spacetime substructure from this generalisation as described for equations 9 and 13. The generalisation to cubic and higher order expressions is permitted since we do not perceive the additional components as extra spatial dimensions.

- An underlying simplicity is achieved with the theory interpreted as deriving from the continuum of time alone, as described for equation 27. With potentially no further substructure either possible or needed, and with the flow of time infusing all experiments and observations, there is a suggestion of reaching the ultimate ‘bedrock’ underlying the structure of matter as observed in spacetime, rather than the next of an indefinite sequence of substrata.

- Significant connections with the empirical properties of the Standard Model are obtained from the natural mathematical development of the theory, including the identification of Lorentz spinors, colour SU(3)_c singlets and triplets with the appropriate electromagnetic U(1)_Q fractional charges and an intrinsic left-right asymmetry, which is particularly marked for the neutrino sector, and with very little redundancy of structure as described for equation 22.

These elements of the Standard Model have been accounted for through a rigorous analysis of the \( E_6 \) and \( E_7 \) levels underlying equations 17–22. The technical mathematical details are described extensively in ([13] and references therein) and naturally lead to the prediction of an \( E_8 \) symmetry of the form of time described for
equation 23 to uncover the full particle multiplet structure of the Standard Model and beyond. In particular the mathematical pursuit of a full symmetry action for \( E_8 \) on the form \( L_8(v_{248})E_8 = 1 \) and the resulting breaking pattern may elucidate the origin of the esoteric properties of neutrinos, building upon the schematic generation structure of equation 26. Mutually, existing empirical knowledge of the neutrino sector as established in recent decades, as expressed for example in the \( \nu \) MSM, might also be utilised as a key component in unlocking the detailed structure of the specific application for \( E_8 \) that is proposed.

More generally the mathematical possibilities for equation 10 are open to exploration, as are the wider implications of the theory in relation to Kaluza-Klein models and quantisation schemes on the technical side and cosmology and particle physics on the empirical side. The manner in which this theory has been able to reproduce a series of basic features of the Standard Model, and has already yielded provisional connections with neutrino and other new physics beyond, demonstrates this open opportunity to further develop this fundamental unified theory in parallel with advances in our empirical understanding of the elementary composition of the physical world.

References

[1] Abraham Pais, ‘Inward Bound: Of Matter and Forces in the Physical World’, Oxford University Press (1986, 1988, 1994).
[2] Abraham Pais, ‘“Subtle is the Lord…”: The Science and the Life of Albert Einstein’, Oxford University Press (1982, 2005).
[3] Albert Einstein, ‘The Foundation of the General Theory of Relativity’, Annalen Phys. 49 (7), 769–822 (1916) [Annalen Phys. 14 (Supplement), 517–571 (2005)], translated by Alfred Engel in ‘The Collected Papers of Albert Einstein’ 6, 146–200, Princeton University Press (1997).
[4] Hermann Weyl, ‘Gravitation and Electricity’, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1918, 465 (1918).
[5] L. O’Raifeartaigh, ‘The Dawning of Gauge Theory’, Princeton University Press (1997).
[6] Hermann Weyl, ‘Electron and Gravitation’, Z. Phys. 56, 330 (1929) [Surveys High Energ. Phys. 5, 261 (1986)].
[7] F. Gürsey, P. Ramond and P. Sikivie, ‘A Universal Gauge Theory Model based on \( E_6 \)’, Phys. Lett. B 60 (2), 177–180 (1976).
[8] F. Gürsey and P. Sikivie, ‘\( E_7 \) as a Universal Gauge Group’, Phys. Rev. Lett. 36 (14), 775–778 (1976).
[9] T. Kaluza, ‘On the Problem of Unity in Physics’, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1921, 966 (1921).
[10] O. Klein, ‘Quantum Theory and Five-Dimensional Relativity’, Z. Phys. 37, 895 (1926).
Y. M. Cho, ‘Higher-Dimensional Unifications of Gravitation and Gauge Theories’, J. Math. Phys. 16 (10), 2029 (1975).

D. J. Jackson, ‘Construction of a Kaluza-Klein type Theory from One Dimension’, arXiv:1610.04456 [physics.gen-ph] (2016).

D. J. Jackson, ‘Time, E8, and the Standard Model’, arXiv:1709.03877 [physics.gen-ph] (2017).

J. C. Baez, ‘The Octonions’, Bull. Am. Math. Soc. 39, 145–205 (2002) [arXiv:math/0105155 [math.RA]].

Claude Chevalley and Richard D. Schafer, ‘The Exceptional Simple Lie algebras F4 and E6’, Proc. Natl. Acad. Sci. USA 36, 137–141 (1950).

A. Wangberg and T. Dray, ‘E6, the Group: The Structure of SL(3, O)’, J. Algebra Appl. 14 (6), 1550091 (2015) [arXiv:1212.3182 [math.RA]].

Hans Freudenthal, ‘Lie Groups in the Foundations of Geometry’, Adv. Math. 1, 145–190 (1964).

M. Rios, ‘Jordan C*-Algebras and Supergravity’, arXiv:1005.3514 [hep-th] (2010).

M. Cederwall and J. Palmkvist, ‘The Octic E8 Invariant’, J. Math. Phys. 48, 073505 (2007) [arXiv:hep-th/0702024].

M. Drewes, ‘The Phenomenology of Right Handed Neutrinos’, Int. J. Mod. Phys. E 22, 1330019 (2013) [arXiv:1303.6912 [hep-ph]].

J. Wolf [KATRIN Collaboration], ‘The KATRIN Neutrino Mass Experiment’, Nucl. Instrum. Meth. A 623, 442–444 (2010) [arXiv:0810.3281 [physics.ins-det]].

M. Agostini, G. Benato and J. Detwiler, ‘Discovery Probability of Next-Generation Neutrinoless Double-β Decay Experiments’, Phys. Rev. D 96 (5), 053001 (2017) [arXiv:1705.02996 [hep-ex]].

M. Tanabashi et al. [Particle Data Group], ‘Review of Particle Physics’, Phys. Rev. D 98 (3), 030001 (2018), [available at http://pdg.lbl.gov].

T. Asaka, S. Blanchet and M. Shaposhnikov, ‘The νMSM, Dark Matter and Neutrino Masses’, Phys. Lett. B 631, 151 (2005) [arXiv:hep-ph/0503065].

T. Asaka and M. Shaposhnikov, ‘The νMSM, Dark Matter and Baryon Asymmetry of the Universe’, Phys. Lett. B 620, 17 (2005) [arXiv:hep-ph/0505013].

A. Azatov, ‘Status of Composite Higgs’, PoS EPS-HEP2017, Venice (5–12 July 2017), 255 (2017) [doi:10.22323/1.314.0255].

K. Fujii et al., ‘Physics Case for the 250 GeV Stage of the International Linear Collider’, arXiv:1710.07621 [hep-ex] (2017).

Albert Einstein, ‘The Problem of Space, Ether, and the Field in Physics’, essay in the volume ‘Mein Weltbild’, translated by Alan Harris in ‘The World as I see it’, John Lane The Bodley Head, London (first published in England in 1935). [The passage quoted appears in the first of the final seven paragraphs which were typically cut for later volumes incorporating this essay.]

D. J. Jackson, ‘The Structure of Matter in Spacetime from the Substructure of Time’, arXiv:1804.00487 [physics.gen-ph] (2018).