Experimental evidence for the occurrence of \(E_8\) in nature and the radii of the Gosset circles

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Abstract. A recent experimental discovery involving the spin structure of electrons in a cold one-dimensional magnet points to a validation of a Zamolodchikov model involving the exceptional Lie group \(E_8\). The model predicts 8 particles and predicts the ratio of their masses. In more detail, the vertices of the 8-dimensional Gosset polytope identifies with the 240 roots of \(E_8\). Under the famous two-dimensional (Peter McMullen) projection of the polytope, the image of the vertices are arranged in 8 concentric circles, hereafter referred to as the Gosset circles. The Gosset circles are now understood to correspond to the 8 masses in the model, and in addition it is understood that the ratio of the their radii is the same as the ratio of the corresponding conjectural masses. A ratio of the two smallest circles (read 2 smallest masses) is the golden number. Marvelously, the conjectures have been now validated experimentally, at least for the first five masses.

The McMullen projection generalizes to any complex simple Lie algebra whose rank is greater than 1. The Gosset circles generalize as well, using orbits of the Coxeter element. Using results in [K-59], I found some time ago a very easily defined operator \(A\) whose spectrum is exactly the squares of the radii \(r_i\) of these generalized Gosset circles. As a confirmation, in the \(E_8\) case, using only the eigenvalues of a suitable multiple of \(A\), Vogan computed the ratio of the \(r_i\). Happily these agree with the corresponding ratio of the Zamolodchikov masses.

The operator \(A\) is written as a sum of \(\ell + 1\) rank 1 operators, parameterized by the points in the extended Dynkin diagram. Involved in this expansion are the coefficients \(n_i\) of the highest root. Suggestively, recalling the McKay correspondence, in the \(E_8\) case the \(n_i\), together with 1, are the dimensions of the irreducible representations of the binary icosahedral group.

Key Words: \(E_8\), Cartan subalgebras in apposition, Gosset circles, Ising chain in \(E_8\) symmetry, Zamolodchikov theory, 1-dimensional magnet, Coxeter element, Coxeter number, golden number, conformal field theory, particle physics.

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0. Introduction

0.1. Let \(g\) be a complex simple Lie algebra and let \((x, y)\) be the Killing form \(B\) on \(g\). Let \(\ell = \text{rank} \, g\) and let \(h\) be a Cartan subalgebra of \(g\). Let \(\Delta\) be the set of roots for \((h, g)\) and let \(\Delta^+ \subset \Delta\) be a choice of positive roots. For any \(\varphi \in \Delta\), let \(e_\varphi\) be a corresponding root vector. We assume choices are made so that \(\langle e_\varphi, e_{-\varphi} \rangle = 1\).

Let \(\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Delta^+\) be the set of simple positive roots. Let \(h\) be the Coxeter number of \(g\). Let \(w \in h\) be the unique element such that \(\langle \alpha_i, w \rangle = 1\), \(i = 1, \ldots, \ell\). Let \(G\) be a Lie group such that \(\text{Lie} \, G = g\) and let \(H \subset G\) be the subgroup corresponding to \(h\). Let \(c \in H\) be given by \(c = \exp 2\pi i w/h\). Then \(c\) is a regular semisimple element of \(G\) and its centralizer in \(g\) is given by

\[
g^c = h. \tag{0.1}\]
Let $\psi \in \Delta_+$ be the highest root and let $n_i$, $i = 1, \ldots, \ell$, be the coefficients (known to be positive) relative to the simple roots so that $\psi = \sum_{i=1}^{\ell} n_i \alpha_i$. Let

$$x(\beta) = e^{-\psi} + \sum_{i=1}^{\ell} \sqrt{n_i} e_{\alpha_i}. \quad (0.2)$$

Then results in [K-59] imply $x(\beta)$ is a regular semisimple element of $\mathfrak{g}$. Let $h(\beta)$ be the Cartan subalgebra of $\mathfrak{g}$ which contains $x(\beta)$. Furthermore let $\gamma = e^{2\pi i/h}$ so that $\gamma$ is a primitive $h$ root of unity. The following is proved in [K-59].

**Theorem 0.1** The Cartan subalgebra $h(\beta)$ is stable under $\text{Ad} c$. Furthermore if $\sigma_\beta = \text{Ad} c|_{h(\beta)}$, then $\sigma_\beta$ is a Coxeter element in the Weyl group $W(\beta)$ of $h(\beta)$. In addition

$$\sigma_\beta x(\beta) = \gamma x(\beta). \quad (0.3)$$

The two Cartan subalgebras $\mathfrak{h}$ and $h(\beta)$ are said to be in apposition, in the terminology of [K-59].

Let $\Delta(\beta)$ be the set of roots for the pair $(h(\beta), \mathfrak{g})$. Then $\Delta(\beta)$ decomposes into $\ell$ orbits, $O_i$, $i = 1, \ldots, \ell$, under the action of $\sigma(\beta)$ and each orbit $O_i$ has $h$ roots. We may choose root vectors $e_\nu$ for $\nu \in \Delta$ so that one has $c \cdot e_\nu = e_{\sigma_\beta \nu}$ for any $\nu \in \Delta(\beta)$.

**Theorem 0.2.** The elements $z_i$, $i = 1, \ldots, \ell$ in $\mathfrak{g}$, given by

$$z_i = \frac{1}{h} \sum_{\nu \in O_i} e_\nu$$

are a basis of $\mathfrak{h}$.

0.2. We assume from now on on $\ell > 1$ so that $h > 2$. Let $\text{Vec} h(\beta)$ be the real space of all hyperbolic elements in $h(\beta)$ so that $\text{Vec} h(\beta)$ is a $W(\beta)$-stable real form of $h(\beta)$. If conjugation in $h(\beta)$ is defined with respect to $\text{Vec} h(\beta)$, then $x(\beta)$ is a regular eigenvector of $\sigma_\beta$ with eigenvalue $\gamma$. Since $\gamma \notin \mathbb{R}$ one defines a real two-dimensional $\sigma_\beta$-stable subspace $Y$ of $\text{Vec} h(\beta)$ by putting

$$Y = \text{Vec} h(\beta) \cap (\mathbb{C}x(\beta) + \overline{\mathbb{C}x(\beta)}). \quad (0.4)$$

Let

$$Q : \text{Vec} h(\beta) \rightarrow Y \quad (0.5)$$

be the orthogonal projection (and $\sigma_\beta$ map) defined by (positive definite) $\mathcal{B}|\text{Vec} h(\beta)$. For any $\nu \in \Delta$ let $w_\nu \in \text{Vec} h(\beta)$ be the image of $\nu$ under the $W(\beta)$-isomorphism $h(\beta)^* \rightarrow h(\beta)$
defined by $B$. One defines circles $C_i$, $i = 1, \ldots, \ell$, in $Y$ of positive radius $r_i$, centered at the origin, by the condition that

$$Q(w_\nu) \in C_i, \ \forall \nu \in O_i.$$ 

In the special case where $G = E_8$ we will refer to the $C_i$ as Gosset circles.

**Remark.** If $\gamma'$ is another primitive $h$ root of unity, then one knows (A.J. Coleman) that $\gamma'$ occurs with multiplicity 1 as an eigenvalue of $\sigma_\beta$. If $0 \neq x(\beta)'$ is a corresponding (necessarily regular) eigenvector, then one may replace $(\gamma, x(\beta))$ by $(\gamma', x(\beta)')$ and replace $(Q, Y)$ by a corresponding $(Q', Y')$. However both cases are “geometrically” isomorphic. In particular the radii $r_i$ do not change. The reason for this is that one can show that if $Z(\sigma(\beta))$ is the cyclic group generated by $\sigma(\beta)$ and $N(\sigma(\beta))$ is the normalizer of $Z(\sigma(\beta))$ in $W(\beta)$, then

$$N(\sigma(\beta))/Z(\sigma(\beta)) \cong \Gamma_h,$$

where $\Gamma_h$ is the Galois group of the cyclotomic field spanned over $\mathbb{Q}$ by the $h$ roots of unity.

In the $E_8$ case the projection $Q$ of the Gosset polytope appears ubiquitously throughout the mathematical literature (see e.g., the frontispiece of [CX]). It has been described by Coxeter as the “most symmetric” two-dimensional projection of this polyhedron. But in fact it can be described precisely as the unique such projection, up to isomorphism, (there are 4 such projections) which commute with the action of the Coxeter element $\sigma_\beta$.

The following Theorem 0.3 below is our main result. It gives the radii $r_i$ of the circles $C_i$. Its significance in the $E_8$ case will be explained in §0.3. If $x, y \in \mathfrak{h}$, let $x \otimes y$ be the rank 1 operator on $\mathfrak{h}$, defined so that if $z \in \mathfrak{h}$, then $x \otimes y(z) = (x, z) y$. Also let $w_i \in \mathfrak{h}$ be the image of $\alpha_i$ in $\mathfrak{h}$ under the isomorphism $\mathfrak{h}^* \rightarrow \mathfrak{h}$ defined by the Killing form $B$. Now let $A$ be the operator on $\mathfrak{h}$, written as a sum of $\ell + 1$ rank 1 operators, given by putting

$$A = \sum_{i=0}^{\ell} n_i w_i \otimes w_i,$$

where $n_0 = 1$ and, we recall, $n_i, i > 0$, is the coefficient of $\alpha_i$ in the simple root expansion of the highest root $\psi$.

**Theorem 0.3.** The eigenvalues of $A$ are $r_i^2$, $i = 1, \ldots, \ell$, and $z_i$ (see Theorem 0.2) is an $A$-eigenvector for $r_i^2$.

0.3. I obtained the result Theorem 0.3 sometime (I believe) in the early 1990s. At that time publication seemed unwarranted since I believed there would be little interest in a knowledge of the radii $r_i$. However in the middle 1990s Peter McMullen’s image, by $Q$, in the $E_8$ case, of the Gosset polytope was very widely published and became well known, even to many in the general public, due no doubt, to the very extensive (and well
deserved) publicity, given to the determination by a large team of mathematicians, of the characters of the real forms of $E_8$. I showed Theorem 0.3 to David Vogan, one of the leading members of the aforementioned team. Applying a computer program to a Weyl group reformulation of a scalar multiple of the operator $A$, the following list, in increasing size, of the normalized 8 radii was obtained by Vogan. His normalization was to make the largest of the 8 radii equal to 1. To avoid decimals we took the liberty of multiplying his list by 1000. (What will be significant is the ratio of the radii and not the radii themselves).

\begin{align*}
209 \\
338 \\
416 \\
502 \\
618 \\
673 \\
813 \\
1000
\end{align*}

\( (0.8) \)

**Remark.** The first 7 numbers in (0.8) are the integral parts of the normalized radii and to that extent only approximate the normalized radii.

Having been directed by colleagues to the papers [Za] and [Co], we have only recently become aware of the fact that the ratio of the numbers in (0.8) have physical significance. In [Za] Zamolodchikov conjectures the existence of 8 particles in connection with a field theory associated with the Ising model. Happily his ratio of the masses of the conjectured particles “agrees” with the ratio of the normalized radii in (0.8). (See (1.8), p. 4237 in [Za].) The use of the quotation marks in “agrees”, and also below, is because of the statement in the remark above. Particular emphasis is made in [Za] of the fact that the ratio \( m_2/m_1 \) of the first two masses should be the golden number. Indeed \( 338/209 = \frac{1}{2}(1 + \sqrt{5}) \). Zamolodchikov makes a connection in his paper with $E_8$ at the bottom of p. 4247 but attributes the prediction of an $E_8$ connection to V. Fateev.

The recent nine person authored paper [Co] is an experimental discovery, using a very cold one-dimensional magnet, validating Zamolodchikov’s theory, at least for the first 5 particles. In particular the equality of \( m_2/m_1 \) (the ratio of the radii of the two inner Gosset circles) with the golden number is very clearly seen.

**0.4.** I wish to thank Nolan Wallach for many conversations regarding the subject matter in this paper. In particular for conversations regarding the two types of Gosset circles which appear in a number of publications. In addition I thank for him for the minicourse he gave me on the representations of the Virasoro algebra. Representations of this algebra make an appearance in [Za]. I also want to thank David Vogan for making the computations, using Theorem 0.3, which resulted in (0.8). Also David factorized a relevant characteristic polynomial into a product of two irreducible (over $\mathbb{Q}$) polynomials of degree 4. The irreducible polynomials relate directly to the two types of Gosset circles mentioned above.
1. Cartan subalgebras in apposition

1.1. Let \( g \) be a complex simple Lie algebra and let \((x, y)\) be the Killing form \( B \) on \( g \). Let \( \ell = \text{rank } g \) and let \( \mathfrak{h} \) be a Cartan subalgebra of \( g \). Let \( \Delta \) be the set of roots for \((\mathfrak{h}, g)\) and let \( \Delta_+ \subset \Delta \) be a choice of positive roots. For any \( \varphi \in \Delta \) let \( e_\varphi \) be a corresponding root vector. We assume choices are made so that

\[
(e_\varphi, e_{-\varphi}) = 1. \tag{1.1}
\]

Let \( \Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Delta_+ \) be the set of simple positive roots. Let \( h \) be the Coxeter number of \( g \). Let \( w \in \mathfrak{h} \) be the unique element such that for all \( i = 1, \ldots, \ell \),

\[
\langle \alpha_i, w \rangle = 1. \tag{1.2}
\]

For any \( \varphi \in \Delta \) put \( o(\varphi) = \langle \varphi, w \rangle \). If \( \psi \in \Delta_+ \) is the highest root, then one knows that

\[
o(\psi) = h - 1. \tag{1.3}
\]

Let \( G \) be a Lie group such that \( \text{Lie } G = g \) and let \( H \subset G \) be the subgroup corresponding to \( \mathfrak{h} \). If \( a \in H \) and \( \nu \in \mathfrak{h}^* \) is an \( H \)-weight, put \( a^\nu = e^{\langle x, \nu \rangle} \), where \( a = \exp x \). Let \( c \in H \) be given by

\[
c = \exp 2\pi i w/h. \tag{1.4}
\]

Also let

\[
\gamma = e^{2\pi i/h}.
\]

For \( g \in G \) and \( z \in g \) we will sometimes write \( g \cdot z \) for \( \text{Ad } g (z) \). Let

\[
g(\gamma) = \{x \in g \mid c \cdot x = \gamma x\}.
\]

As one readily notes, using e.g., (1.3),

**Proposition 1.1.** One has \( \dim g(\gamma) = \ell + 1 \) and in fact the elements \( e_{\alpha_i}, i = 1, \ldots, \ell \), and \( e_{-\psi} \) are a basis of \( g(\gamma) \).

For

\[
\beta = (\beta_1, \ldots, \beta_\ell, \beta_{-\psi}) \in \mathbb{C}^{\ell+1}
\]

let \( x(\beta) \in g(\gamma) \) be defined by putting

\[
x(\beta) = \beta_{-\psi} e_{-\psi} + \sum_{i=1}^\ell \beta_i e_{\alpha_i}. \tag{1.5}
\]

The following result was established in [K-59].
Theorem 1.2. $x(\beta)$ is regular semisimple if and only if $\beta \in (\mathbb{C}^\times)^{\ell+1}$.

If $\beta \in (\mathbb{C}^\times)^{\ell+1}$, let $\mathfrak{h}(\beta)$ be the Cartan subalgebra which contains $x(\beta)$. It is immediate that if $\beta \in (\mathbb{C}^\times)^{\ell+1}$, then $\mathfrak{h}(\beta)$ is stable under $\text{Ad}\, c$. Let $\sigma_\beta$ be the element of the Weyl group of $\mathfrak{h}(\beta)$ defined by $c$. One thus has

$$\sigma_\beta x(\beta) = \gamma x(\beta).$$

(1.6)

Part of the following is established in [K-59] and uses a result of A.J. Coleman.

Theorem 1.3. Let $\beta \in (\mathbb{C}^\times)^{\ell+1}$. Then $\sigma_\beta$ is a Coxeter element of the Weyl group of $\mathfrak{h}(\beta)$ and up to a scalar multiple $x(\beta)$ is the unique element of $\mathfrak{h}(\beta)$ satisfying (1.6).

1.2. Using the Killing form we may identify the algebra of polynomial functions on $\mathfrak{g}$ with the symmetric algebra $S(\mathfrak{g})$. This is done so that if $x, y \in \mathfrak{g}$, then $x^n(y) = (x,y)^n$. Also the Killing form extends naturally to a symmetric bilinear form $(p,q)$ on $S(\mathfrak{g})$. One has

$$\frac{1}{n!} (x^n, y^n) = x^n(y).$$

The algebra of $S(\mathfrak{g})^G$ of symmetric invariants is a polynomial ring $\mathbb{C}[J_1, \ldots, J_\ell]$ where the $J_k$ are homogeneous, say of degree $d_k$, and algebraically independent. Choose the ordering so that the $d_k$ are nonincreasing. In that case $d_1 = h$ and $d_k < h$ for $k > 1$. The definition of a cyclic element $x \in \mathfrak{g}$ was introduced in [K-59]. One has that $x$ is cyclic if and only if $J_1(x) \neq 0$ and $J_k(x) = 0$ for $k > 1$. This condition is independent of the choice of the $J_k$. It is established in [K-59] that cyclic elements are regular semisimple. We have also proved

Theorem 1.4. $x(\beta)$ is cyclic for any $\beta \in (\mathbb{C}^\times)^{\ell+1}$ and, up to conjugacy, any cyclic element in $\mathfrak{g}$ is of this form.

Let $n_i \in \mathbb{C}, i = 1, \ldots, \ell$, be defined so that

$$\psi = \sum_{i=1}^\ell n_i \alpha_i.$$

One knows that the $n_i$ are positive integers. It is immediate from the independence of the simple roots that if $k_i \in \mathbb{Z}_+, i = 1, \ldots, \ell$, and $k_- \psi \in \mathbb{Z}_+$ are such that $k_- \psi + \sum_{i=1}^\ell k_i = h$, then the monomial

$$e^{k_- \psi} e_{\alpha_1}^{k_1} \cdots e_{\alpha_\ell}^{k_\ell}$$

(1.7)

is in $S^h(\mathfrak{g})$ and

Proposition 1.5. The monomial (1.7) is a zero weight vector if and only if $k_- \psi = 1$ and $k_i = n_i$ for $i = 1, \ldots, \ell$. 

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It follows immediately from Theorem 1.4 and Proposition 1.5 that if \( \beta \in (C^\times)^{l+1} \), then

**Proposition 1.6.** With respect to the inner product in \( S(g) \) one has

\[
(J_1, e_{-\psi} e_{\alpha_1}^{m_1} \cdots e_{-\alpha_\ell}^{m_\ell}) \neq 0
\]

and

\[
J_1(x(\beta)) = \beta_{-\psi} \beta_1^{m_1} \cdots \beta_\ell^{m_\ell} \frac{1}{m_1! \cdots m_\ell!} (J_1, e_{-\psi} e_{\alpha_1}^{m_1} \cdots e_{-\alpha_\ell}^{m_\ell}).
\] (1.8)

It is clear that the set of cyclic elements of the form \( x(\beta) \) for \( \beta \in (C^\times)^{l+1} \) is stable under conjugation by \( H \). One readily defines an action of \( H \) on \( (C^\times)^{l+1} \) so that if \( \beta \in (C^\times)^{l+1} \) and \( a \in H \), then

\[
a \cdot x(\beta) = x(a \cdot \beta).
\]

Of course \( (C^\times)^{l+1} \) is stable under multiplication by \( C^\times \). One defines another action of \( C^\times \) on \( (C^\times)^{l+1} \) where, for \( \lambda \in C^\times \) and \( \beta \in (C^\times)^{l+1} \), one lets \( \lambda \ast \beta \in (C^\times)^{l+1} \) be given so that

\[
(\lambda \ast \beta)_i = \beta_i, \ i = 1, \ldots, \ell,
\]

but

\[
(\lambda \ast \beta)_{-\psi} = \lambda \beta_{-\psi}.
\]

**Theorem 1.7.**

1. Two cyclic elements \( v, v' \) in \( g \) are \( G \)-conjugate if and only if \( J_1(v) = J_1(v') \).
2. Furthermore, if \( v = x(\beta), v' = x(\beta') \) where \( \beta, \beta' \in (C^\times)^{l+1} \), then \( v \) and \( v' \) are \( G \)-conjugate \( \iff \) they are \( H \)-conjugate.
3. Given a cyclic element \( v \in g \) and \( \beta \in (C^\times)^{l+1} \), there exists a unique \( \lambda \in C^\times \) such that \( v \) and \( x(\lambda \ast \beta) \) are \( G \)-conjugate.

**Proof.** (1) follows immediately from the fact that cyclic elements are regular semisimple. To prove (2), assume that \( x(\beta) \) and \( x(\beta') \) are \( G \)-conjugate. Clearly there exists \( a \in H \) so that \( (a \cdot \beta'_i) = \beta_i \) for \( i = 1, \ldots, \ell \). But then \( (a \cdot \beta')_{-\psi} = \beta_{-\psi} \) by (1.8) since \( J_1(x(\beta)) = J_1(a \cdot \beta') \). Thus \( a \cdot x(\beta') = x(\beta) \). (3) follows from (1) since \( J_1(x_{\lambda \ast \beta}) \) is linear in \( \lambda \) by (1.8). QED

1.2. Let \( G_{Ad} \) be the adjoint group. Then we recall from [K-59] there exists a unique conjugacy class \( C \) of regular elements of order \( h \) in \( G_{Ad} \). Furthermore if \( a \in G \) and \( Ad a \in C \), there exists a Cartan subalgebra \( a \) which is stable under \( Ad a \) and \( Ad a \mid a \) is a Coxeter element. In such a case we will say that \( a \) is Coxeter for \( a \). Conversely, if \( \sigma \) is a Coxeter element for a Cartan subalgebra \( a \) and \( Ad a \) normalizes \( a \) and induces \( \sigma \), then \( Ad a \in C \).
Recalling Theorem 1.3, one has that $c \in C$. Furthermore since any Cartan subalgebra which is Coxeter for $c$ necessarily has a regular eigenvector with eigenvalue $\gamma$ for $\text{Ad} \, c$, one has

**Proposition 1.8.** $\mathfrak{h}(\beta)$ is Coxeter for $c$ for any $\beta \in (\mathbb{C}^\times)^{l+1}$. Conversely, any Cartan which is Coxeter for $c$ is equal to $h(\beta)$ for some $\beta \in (\mathbb{C}^\times)^{l+1}$.

Now note that by Theorem 1.3, if $\beta, \beta' \in (\mathbb{C}^\times)^{l+1}$, then

$$h(\beta) = h(\beta') \iff \beta' = \lambda \beta \text{ for some } \lambda \in \mathbb{C}^\times. \tag{1.9}$$

On the other hand, if $a \in H$ and $\beta \in (\mathbb{C}^\times)^{l+1}$, then obviously

$$\text{Ad} \, a \, (h(\beta)) = h(a \cdot \beta). \tag{1.10}$$

**Theorem 1.9.** Let $c' \in C$. Then the set of all Cartan subalgebras which are Coxeter for $c'$ is an adjoint orbit for the (unique) Cartan subgroup which contains $c'$. In particular if $c = c'$, then $H$ is the Cartan subgroup which contains $c$ and the orbit is \( \{ h(\beta) \mid \beta \in (\mathbb{C}^\times)^{l+1} \} \).

**Proof.** Let $\beta, \beta' \in (\mathbb{C}^\times)^{l+1}$, We have only to show that there exists $a \in H$ such that

$$\text{Ad} \, a \, (h(\beta)) = h(\beta'). \tag{1.11}$$

But now for any $\lambda \in \mathbb{C}^\times$ one has

$$J_1(x(\lambda \beta')) = \lambda^h J_1(x(\beta')). \tag{1.12}$$

But then we can choose $\lambda$ so that

$$J_1(x(\lambda \beta')) = J_1(x(\beta')). \tag{1.13}$$

But then by Theorem 1.7 there exists $a \in H$ such that $a \cdot x(\beta) = x(\lambda \beta')$. But then clearly $\text{Ad} \, a \, (h(\beta)) = h(\lambda \beta')$. But $h(\lambda \beta') = h(\beta')$. QED

**1.3.** Let $\beta \in (\mathbb{C}^\times)^{l+1}$ and let $\Delta(\beta)$ be the set of roots for the pair $(\mathfrak{h}(\beta), \mathfrak{g})$. Then $\sigma_\beta = \text{Ad} \, c \, | \, h(\beta)$ is a Coxeter element. Let $O_i \subset \Delta(\beta), i = 1, \ldots, \ell$, be the orbits of $\sigma_\beta$. For any $\nu \in \Delta(\beta)$, let $e_\nu \in h(\beta)^\perp$ be a corresponding root vector. We assume the root vectors are chosen so that

$$c \cdot e_\nu = e_{\sigma_\beta \nu}.$$

We note, for $\nu \in O_i$, that we may write

$$e_\nu = z_i + \sum_{\varphi \in \Delta} d_{\nu, \varphi} e_\varphi, \tag{1.14}$$

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where \( z_i \in \mathfrak{h} \). One further notes that

\[
c \cdot e_\nu = z_i + \sum_{\varphi \in \Delta} \gamma^{\varphi} d_{\nu,\varphi} e_\varphi
\]

\[
= e_{\sigma \beta \nu}
\]

\[
= z_i + \sum_{\varphi \in \Delta} d_{\sigma \beta \nu,\varphi} e_\varphi.
\]

(1.14)

However since \( c \) is regular one has

\[
\mathfrak{g}^c = \mathfrak{h}.
\]

But \( \sum_{\nu \in C_i} e_\nu \) is an invariant of \( c \) and hence lies in \( \mathfrak{h} \). Thus from (1.14) one must have

\[
\sum_{\nu \in C_i} e_\nu = h z_i.
\]

(1.15)

That is, for any \( \varphi \in \Delta \),

\[
\sum_{\nu \in C_i} d_{\nu,\varphi} = 0,
\]

(1.16)

and in fact the orbits \( O_i \) consequently define a distinguished basis of \( \mathfrak{h} \).

**Theorem 1.10.** The \( z_i, i = 1, \ldots, \ell \), given by (1.15), are a basis of \( \mathfrak{h} \).

**Proof.** Since \( \sigma_\beta \) has no nontrivial invariant in \( \mathfrak{h} \) the only contribution to \( \mathfrak{g}^c \) must come from (1.15). But this proves the theorem since there are \( \ell \) orbits and \( \dim \mathfrak{h} = \ell \).

QED

1.4. Now note that (1.1) implies

\[
[e_\varphi, e_{-\varphi}] = w_\varphi
\]

(1.17)

for any \( \varphi \in \Delta \), where \( w_\varphi \in \mathfrak{h} \) is such that for any \( x \in \mathfrak{h} \),

\[
(w_\varphi, x) = \langle \varphi, x \rangle.
\]

(1.18)

Now recalling the notation of (1.8) one has by (1.18),

\[
\sum_{i=1}^{\ell} n_i w_{\alpha_i} = w_\psi.
\]

(1.19)

Now any Cartan subalgebra \( \mathfrak{h}_1 \) is the sum of its vector part \( \text{Vec}\mathfrak{h}_1 \) (split real Cartan subalgebra) and its toroidal part \( \text{Tor}\mathfrak{h}_1 = i \text{Vec}\mathfrak{h}_1 \) (Cartan subalgebra of a compact real form). In particular, for any \( \beta \in (\mathbb{C}^\times)^{\ell+1} \) one has

\[
\mathfrak{h}(\beta) = \text{Vec}\mathfrak{h}(\beta) + i \text{Vec}\mathfrak{h}(\beta).
\]

(1.20)
In particular
\[
x(\beta) = \Re x(\beta) + i \Im x(\beta),
\]  \hspace{1cm} (1.21)
where \( \Re x(\beta), \Im x(\beta) \in \text{Vec} \, h(\beta) \). But now recalling (1.6) one has
\[
\gamma x(\beta) = (\Re \gamma + i \Im \gamma)(\Re x(\beta) + i \Im x(\beta))
\]  \hspace{1cm} (1.22)
\[
= (\Re \gamma \Re x(\beta) - \Im \gamma \Im x(\beta)) + i (\Re \gamma \Im x(\beta) + \Im \gamma \Re x(\beta)).
\]
Since \( c_\beta \) stabilizes both vector and toroidal parts of \( h(\beta) \) one has
\[
c_\beta \Re x(\beta) = \Re \gamma \Re x(\beta) - \Im \Im x(\beta)
\]  \hspace{1cm} (1.23)
\[
c_\beta \Im x(\beta) = \Re \gamma \Im x(\beta) + \Im \Re x(\beta).
\]
Now put
\[
\overline{x(\beta)} = \Re x(\beta) - i \Im x(\beta)
\]  \hspace{1cm} (1.24)
so that \( \overline{x(\beta)} \in h(\beta) \). Also, for \( \nu \in \Delta(\beta) \), let
\[
\nu_\beta = \langle \nu, x(\beta) \rangle.
\]  \hspace{1cm} (1.25)
One notes that
\[
\sigma_\beta \overline{x(\beta)} = \overline{\gamma x(\beta)}
\]  \hspace{1cm} (1.26)
\[
\langle \nu, \overline{x(\beta)} \rangle = \overline{\nu_\beta}.
\]
Indeed the first equation in (1.26) is obvious from (1.23). The second follows from the fact that \( \nu \) takes real values on \( \text{Vec} \, h(\beta) \).
Now let
\[
x_-(\beta) = \frac{1}{\beta_\psi} e_\psi + \sum_{i=1}^l \frac{n_i}{\beta_i} e_{-\alpha_i}.
\]  \hspace{1cm} (1.27)
It is immediate from (1.3) that
\[
c \cdot x_-(\beta) = \overline{\gamma} x_-(\beta).
\]  \hspace{1cm} (1.28)

**Theorem 1.11.** One has \( x_-(\beta) \in h_\beta \) and
\[
\sigma_\beta x_-(\beta) = \overline{\gamma} x_-(\beta).
\]  \hspace{1cm} (1.29)
In fact by Coleman’s uniqueness theorem there exists \( t_\beta \in \mathbb{C}^\times \) such that
\[
t_\beta \overline{x(\beta)} = x_-(\beta).
\]  \hspace{1cm} (1.30)
Proof. Since \( x(\beta) \in \mathfrak{h}(\beta) \) and is regular it clearly suffices to prove that \( x_-(\beta) \) commutes with \( x(\beta) \). Here one recalls (1.26) and (1.28). But

\[
[x(\beta), x_-(\beta)] = [\beta_\psi e_{-\psi} + \sum_{i=1}^{\ell} \beta_i e_{\alpha_i}, 1_{\beta_\psi} e_\psi + \sum_{i=1}^{\ell} n_i e_{-\alpha_i}]
\]

\[
= [e_{-\psi}, e_\psi] + \sum_{i=1}^{\ell} n_i [e_{\alpha_i}, e_{-\alpha_i}]
\]

\[
= w_{-\psi} + \sum_{i=1}^{\ell} n_i w_{\alpha_i}
\]

\[
= 0 \text{ by (1.19).} \]

QED

Normalize (Weyl’s normal form) the choice of the \( e_\varphi, \varphi \in \Delta \) so that

\[
\theta(e_\varphi) = -e_{-\varphi},
\]

where \( \theta \) is an involution of \( \mathfrak{g} \) such that \( \theta = -1 \) on \( \mathfrak{h} \). In particular there exists a compact form \( \mathfrak{g}_u \) of \( \mathfrak{g} \)

\[
e_{\varphi} - e_{-\varphi} \text{ is contained in } \mathfrak{g}_u \text{ for all } \varphi \in \Delta.
\]

Let \( \beta^{(1)} \in (\mathbb{C}^\times)^{\ell+1} \) be defined so that \( \beta_{-\psi}^{(1)} = 1 \) and \( \beta_i^{(1)} = \sqrt{n_i}, i = 1, \ldots, \ell \). Then one has

**Theorem 1.12.** One has \( x_-(\beta^{(1)}) = \overline{x(\beta^{(1)})} \) so that \( t_{\beta^{(1)}} = 1 \).

**Proof.** One has

\[
x(\beta^{(1)}) = e_{-\psi} + \sum_{i=1}^{\ell} \sqrt{n_i} e_{\alpha_i}
\]

and

\[
x_-(\beta^{(1)}) = e_\psi + \sum_{i=1}^{\ell} \sqrt{n_i} e_{-\alpha_i}.
\]

But then

\[
x(\beta^{(1)}) - x_-(\beta^{(1)}) = e_{-\psi} - e_{-\psi} + \sum_{i=1}^{\ell} \sqrt{n_i}(e_{\alpha_i} - e_{-\alpha_i}).
\]

But then \( x(\beta^{(1)}) - x_-(\beta^{(1)}) \in \mathfrak{g}_u \) by (1.32) and hence in particular, the corresponding operator, for the adjoint representation, has pure imaginary spectrum. Thus \( x(\beta^{(1)}) - x_-(\beta^{(1)}) \in \mathfrak{g}(\beta^{(1)}) \). But then

\[
\overline{x(\beta^{(1)})} - x_-(\beta^{(1)}) \in \mathfrak{g}(\beta^{(1)})
\]
Hence
\[(t_\beta(1) - 1)(x_-(\beta^{(1)})) \in \mathfrak{g}(\beta^{(1)})\]
by (1.30). That is, if \(s = i(t_\beta(1) - 1)\), then
\[s(x_-(\beta^{(1)})) \in \mathfrak{g}(\beta^{(1)})\]
On the other hand \(\mathfrak{g}(\beta^{(1)})\) is stable under \(\sigma_\beta(1)\). But this implies that \(s = 0\) since otherwise one has the contradiction that \(s(x_-(\beta^{(1)}))\) is an eigenvector for \(\sigma_\beta(1)\) with eigenvalue \(\gamma\) by (1.29). Hence \(t_\beta(1) = 1\). The theorem then follows from (1.30). QED

1.5. We recall the notation of the first paragraph of §1.3. Let \(\mathfrak{b}\) be the Borel subalgebra defined by \(\Delta_+\) and let \(\mathfrak{n}\) be the nilradical of \(\mathfrak{b}\). Let \(\beta \in (\mathbb{C}^\times)^{l+1}\) and let \(\beta' \in (\mathbb{C}^\times)^{\ell}\) be defined by deleting the last entry \(\beta_\psi\) from \(\beta\). Let \(x(\beta') = x(\beta) - \beta_\psi e_\psi\) so that \(x(\beta') \in \mathfrak{n}\) is principal nilpotent. For the opposed direction let \(\overline{\mathfrak{b}} = \theta \mathfrak{b}\) and let \(\overline{\mathfrak{n}} = \theta \mathfrak{n}\). Then let \(x_-(\beta') = x_-(\beta) - \frac{1}{\beta_\psi} e_\psi\) so that \(x_-(\beta')\) is principal nilpotent in \(\overline{\mathfrak{n}}\). Let \(i \in \{1, \ldots, \ell\}\) and let \(\nu \in O_i\). We will now see that the root vector \(e_\nu\) for \(h(\beta)\) is completely determined by its component \(z_i \in \mathfrak{h}\) and the number \(\nu_\beta = \langle \nu, x(\beta) \rangle\) (see (1.25). Recall that the regularity of \(x(\beta)\) guarantees that
\[\nu_\beta \neq 0.\] (1.36)
For \(k \in \mathbb{Z}\) let
\[\mathfrak{g}(k) = \{x \in \mathfrak{g} \mid [w, x] = k x\}\]
so that one has the direct sum
\[\mathfrak{g} = \bigoplus_{k=-\infty}^{\infty} \mathfrak{g}(k),\] (1.37)
and let
\[P_k : \mathfrak{g} \to \mathfrak{g}(k)\]
be the projection defined by (1.36). We will let \(e_\nu(k) = P_k e_\nu\) so that
\[e_\nu = \sum_{k=-\infty}^{\infty} e_\nu(k),\] (1.38)
noting that
\[e_\nu(0) = z_i.\] (1.39)
Of course \(\mathfrak{g}(k) = 0\) for \(|k| \geq h\) so that \(e_\nu(k) = 0\) for \(|k| \geq h\). Also \(\mathfrak{g}(0) = \mathfrak{h}\) and \(e_\nu(0) = z_i\).

**Theorem 1.13.** Let \(\beta \in (\mathbb{C}^\times)^{l+1}\) and \(\nu \in O_i\). Then for any positive integer \(k\) one has
\[e_\nu(k) = \frac{1}{\nu_\beta} [x(\beta'), e_\nu(k-1)] = \frac{1}{\nu_\beta} (\text{ad } x(\beta'))^k z_i.\] (1.40)
Proof. By induction we have only to prove the first line of (1.40). But now from the root vector property
\[ e_\nu = \frac{1}{\nu_\beta} [x(\beta), e_\nu]. \] (1.41)
But since \( e_\nu(j) = 0 \) for \( j \geq h \) the only contribution to \( e_\nu(k) \) on the left side of (1.41) is where \( x(\beta') \) replaces \( x(\beta) \) on the right side of (1.41) and \( e_\nu(k - 1) \) replaces \( e_\nu \). QED

Remark. If one considers the principle TDS defined by \( \frac{1}{\nu_\beta} x(\beta') = e' \) and \( w \), then note that Theorem 1.12 asserts that \( e_\nu(k) \) for \( k > 0 \) are just the elements in the cyclic \( e' \)-module generated by \( z_i \).

But we can also reverse direction. By (1.26) and (1.30)
\[ \langle \nu, x_-(\beta) \rangle = t_{\beta} \langle \nu, x(\beta) \rangle = t_{\beta} \nu_\beta. \]
But then \( e_\nu \) is an eigenvector for \( \text{ad} (t_{\beta \nu_\beta})^{-1} x_-(\beta) \) with eigenvalue 1. An argument similar to that in the proof of Theorem 1.12 yields

Theorem 1.14. Let the notation be as in Theorem 1.12 and (1.30). Then for any positive integer \( k \) one has
\[ e_\nu(-k) = [(t_{\beta \nu_\beta})^{-1} x_-(\beta'), e_\nu(-k + 1)] = ((\text{ad} (t_{\beta \nu_\beta})^{-1} x_-(\beta')))^k z_i. \] (1.42)

Lemma 1.15. Let \( z \in \mathfrak{h} \) and \( \beta \in (\mathbb{C}^\times)^{l+1} \). Then the maximum positive integer \( M_z \) such that \( (\text{ad} x(\beta'))^M z \neq 0 \) is independent of \( \beta \). Moreover \( M_z \) is the maximum integer such that \( (\text{ad} x_-(\beta'))^M z \neq 0 \). In fact if \( a \) is any principal TDS containing \( w \), then \( 2M_z + 1 \) is the dimension of the maximal dimensional irreducible component of the \( \text{ad} a \) submodule generated by \( z \).

Proof. Immediate from observation that any two such principal TDS are conjugate under the action of \( s\text{Ad exp} \mathfrak{h}. \) QED

Recalling the notation and statements of Theorems 1.12 and 1.14, put \( M_i = M_{z_i} \) so that
\[ e_\nu(k) = 0 \text{ for } |k| > M_i. \] (1.43)
Now we may write \( e_\nu(h - 1) = r_\nu e_\psi \) for some scalar \( r_\nu \). Also by (1.42)
\[
[x(\beta'), e_\nu(-1)] = [x(\beta'), [(t_\beta \overline{\nu_\beta})^{-1} x_-(\beta'), z_i]] = (t_\beta \overline{\nu_\beta})^{-1} [x(\beta'), [x_-(\beta'), z_i]] = (t_\beta \overline{\nu_\beta})^{-1} \sum_{j=1}^\ell \langle \alpha_j, z_i \rangle n_j w_{\alpha_j}
\]
so that
\[
z_i = \frac{1}{\nu_\beta} ([x(\beta'), e_\nu(-1)] + \beta_\psi r_\nu [e_-\psi, e_\psi]) = \frac{1}{\nu_\beta} \left((t_\beta \overline{\nu_\beta})^{-1} \sum_{j=1}^\ell \langle \alpha_j, z_i \rangle n_j w_{\alpha_j} - \beta_\psi r_\nu \psi\right)
\]
\[
= \frac{1}{\nu_\beta} \left(\sum_{j=1}^\ell \left((t_\beta \overline{\nu_\beta})^{-1} \langle \alpha_j, z_i \rangle - \beta_\psi r_\nu \right) n_j w_{\alpha_j}\right).
\]

Next one recalls that since
\[
[x_-(\beta), e_\nu] = t_\beta [x(\beta), e_\nu],
\]
computing the component in \( g_{h-1} \), one has
\[
\frac{1}{\beta_\psi} [e_\psi, z_i] = t_\beta \overline{\nu_\beta} r_\nu e_\psi
\]
so that
\[
\frac{\langle -\psi, z_i \rangle}{\beta_\psi} = t_\beta \overline{\nu_\beta} r_\nu.
\]
That is,
\[
-\beta_\psi r_\nu = (t_\beta \overline{\nu_\beta})^{-1} \langle \psi, z_i \rangle
\]
so that
\[
\frac{|\nu_\beta|^2}{t_\beta} z_i = \sum_{j=1}^\ell \langle \alpha_j + \psi, z_i \rangle n_j w_{\alpha_j}.
\]

**1.6.** We will identify \( \text{End} \ h \) with \( h \otimes h \), where if \( x, y \in h \), then \( x \otimes y \in \text{End} \ h \) is that operator such that if \( z \in h \), then
\[
x \otimes y(z) = (x, z) y
\]
Then if \( A \in \text{End} \ h \) is given by
\[
A = \sum_{j=1}^\ell (w_{\alpha_j} + w_\psi) \otimes n_j w_{\alpha_j}
\]
, then for \( z \in \mathfrak{h} \) one has
\[
Az = \sum_{j=1}^{\ell} \langle \alpha_j + \psi, z \rangle n_j w_{\alpha_j}.
\]
Then (1.49) is the statement

**Proposition 1.16.** For \( i = 1, \ldots, \ell \), one has that \( z_i \) is an eigenvector of \( A \) with eigenvalue
\[
|\nu_\beta|^2/t_\beta. \tag{1.50}
\]

We now want to simplify the expression for \( A \). Indeed
\[
\sum_{j=1}^{\ell} (w_{\alpha_j} + w_\psi) \otimes n_j w_{\alpha_j} = \sum_{j=1}^{\ell} w_{\alpha_j} \otimes n_j w_{\alpha_j} + \sum_{j=1}^{\ell} w_\psi \otimes n_j w_{\alpha_j}
\]
\[
= \sum_{j=1}^{\ell} (w_{\alpha_j} \otimes n_j w_{\alpha_j}) + w_\psi \otimes (\sum_{j=1}^{\ell} n_j w_{\alpha_j}) \tag{1.51}
\]
\[
= \sum_{j=1}^{\ell} n_j (w_{\alpha_j} \otimes w_{\alpha_j}) + w_\psi \otimes w_\psi.
\]
Thus if we consider the extended Dynkin diagram adding another node \( \alpha_0 = -\psi \) and define \( m_0 = 1 \) as in the McKay correspondence, we have proved

**Theorem 1.17.** One has
\[
A = \sum_{j=0}^{\ell} n_j w_{\alpha_j} \otimes w_{\alpha_j}. \tag{1.52}
\]

1.7. Henceforth we fix \( \beta \) so that \( \beta = \beta^{(1)} \) (see Theorem 1.11) so that \( t_\beta = 1 \). Also assume \( \mathfrak{g} \) is not of type \( A_1 \) so that \( \psi \) is not simple. One then has (see (1.5))
\[
x(\beta) = e_{-\psi} + \sum_{i=1}^{\ell} \sqrt{n_i} e_{\alpha_i} \tag{1.53}
\]
and (see (1.30) and (1.27))
\[
\overline{x(\beta)} = e_\psi + \sum_{i=1}^{\ell} \sqrt{n_i} e_{-\alpha_i} \tag{1.54}
\]
Recalling (1.21) and (1.24) one notes that then
\[
\Re x(\beta) = (x(\beta) + \overline{x(\beta)})/2
\]
\[
= (e_\psi + e_{-\psi})/2 + \sum_{i=1}^{\ell} \sqrt{n_i} (e_{\alpha_i} + e_{-\alpha_i})/2, \tag{1.55}
\]
\[
15
\]
and hence
\[(\Re x(\beta), \Re x(\beta)) = h/2. \tag{1.56}\]

But by (1.21) and (1.24) one has
\[
\Im x(\beta) = -i/2((x(\beta) - \overline{x(\beta)})
= -i/2((e_\psi - e_{-\psi}) + \sum_{i=1}^\ell \sqrt{\mu_i} (e_{\alpha_i} - e_{-\alpha_i})),
\tag{1.57}
\]
and hence
\[(\Im x(\beta), \Im x(\beta)) = h/2. \tag{1.58}\]

But clearly (1.55) and (1.57) imply
\[(\Re x(\beta), \Im x(\beta)) = 0. \tag{1.59}\]

Let \(Y \subset \text{Vec} \mathfrak{h}(\beta)\) be the two real-dimensional plane spanned by the orthogonal vectors \(\Re x(\beta)\) and \(\Im x(\beta)\), and let
\[Q : \text{Vec} \mathfrak{h}(\beta) \to Y\]
be the orthogonal projection. Thus if \(x \in \text{Vec} \mathfrak{h}(\beta)\), then
\[Qx = 2/h ((x, \Re x(\beta)) \Re x(\beta) + (x, \Im x(\beta)) \Im x(\beta)). \tag{1.60}\]

But this implies
\[(Qx, Qx) = 2/h ((x, \Re x(\beta))^2 + (x, \Im x(\beta))^2). \]

But now if \(z = (x, x(\beta))\), then
\[(x, \Re x(\beta)) = \Re z \]
\[(x, \Im x(\beta)) = \Im z \]
by the top lines in (1.55) and (1.57). Hence we have proved

**Proposition 1.18.** For any \(x \in \text{Vec} \mathfrak{h}(\beta)\) one has
\[|Qx|^2 = 2/h |(x, x(\beta))|^2. \tag{1.61}\]

Now for any \(\nu \in \Delta(\beta)\) (see §1.3) let \(w_\nu \in \mathfrak{h}(\beta)\) be defined, so that for any \(x \in \mathfrak{h}(\beta)\), one has \(\langle \nu, x \rangle = (w_\nu, x)\). Then as a consequence of (1.25) and Proposition 1.16 (where now \(t_\beta = 1\) by Theorem 1.12) and Proposition 1.18, one has

**Proposition 1.19.** Let \(\nu \in \Delta(\beta)\). Then
\[|Qw_\nu|^2 = \frac{2}{h} |\nu_\beta|^2 \tag{1.62}\]
and our main result on the radius of the two-dimensional orbit projections.

**Theorem 1.20.** Let $\beta \in \mathbb{C}^\times$ be fixed so that $\beta = \beta^{(1)}$ is given as in Theorem 1.12. Let $O_i$, $i = 1, \ldots, \ell$, be an orbit of the Coxeter element $\sigma_\beta$ on the set $\Delta(\beta)$ of roots of $(h(\beta), g)$. Let $z_i$ be the corresponding basal element of $h$ defined as in (1.15). Then, where $h$ is the Coxeter number, $z_i$ is an eigenvector of the operator (on $h$)

$$
2/h \sum_{i=0}^{\ell} n_i w_{\alpha_i} \otimes w_{\alpha_i}
$$

and the corresponding eigenvalue is $|Q w_\nu|^2$ where $\nu$ is any root in the orbit $O_i$.

**2. The special case of E_8**

**2.1.** Assume now that $g$ is of type $E_8$. Then $\ell = 8$ and the cardinality of the set $\Delta$ of roots is 240. The Coxeter number $h$ is 30. The group is unique up to isomorphism. In particular $G \cong G_{ad}$. Let $\beta \in (\mathbb{C}^\times)^9$ be as in Theorem 1. The Gosset polytope (see e.g., [Go]) published in 1900 may be taken to be the boundary of the convex hull of the vectors $w_\gamma, \gamma \in \Delta(\beta)$, in the 8-dimensional real space $\text{Vec} h(\beta)$. The Coxeter element $\sigma_\beta$ decomposes $\Delta(\beta)$ into 8 orbits $O_i, i = 1, \ldots, 8$, where each orbit contains 30 roots. Peter McMullen made a drawing of a two real-dimensional projection of the Gosset polytope. It appears as the frontispiece of Coxeter’s book [CX]. The projection is now quite famous and appears in many places in the literature. The image of the orbits in $\text{Vec} h(\beta)$ corresponding to the $O_i$ appears as 8 concentric circles, which, by abuse of notation, we will refer to as the Gosset circles. Our main objective here is to determine the ratio of the radii of the Gosset circles. That Theorem 1.20 accomplishes this is a consequence of John Conway’s identification of McMullen’s projection with the map $Q$.

**Remark.** One is forced into Conway’s identification if one demands that the projection commutes with the action of the Coxeter element. Indeed since in the $E_8$ case all the 8 eigenvalues of $\sigma_\beta$ are primitive 30th roots of unity, the corresponding eigenvectors are cyclic elements and hence are Weyl group conjugate by elements which normalize the cyclic group generated by $\sigma_\beta$. It follows that there are only 4 two-dimensional real projections which commute with the action of the Coxeter element $\sigma_\beta$ and all four are isomorphic to $Q$.

**2.2. Remark.** As one knows the $E_8$ root lattice can be constructed from the golden number and the embedding of the 120 element binary icosahedral group in the group of unit quaternions. It therefore may be more than a coincidence to note that the $n_i$ appearing in the construction of $A$ are, by the McKay correspondence, the dimensions of the irreducible representations of the binary icosahedral group.
David Vogan reexpressed the operator $A$ as an element $A'$ in the group algebra of the Weyl group. Letting $F$ be the characteristic polynomial of a convenient multiple of $A'$, he found that $F$ factors into a product of 2 irreducible (over $\mathbb{Q}$) degree 4 polynomials $F_1$ and $F_2$, where
\begin{align}
F_1(x) &= x^4 - 15x^3 + 75x^2 - 135x + 45 \\
F_2(x) &= x^4 - 15x^3 + 60x^2 - 90x + 45.
\end{align}

Vogan then computed the integral part of the radii of the Gosset circles normalized so that the maximal integral part is 1000. They are, in increasing size,
\begin{align}
209 \\
338 \\
416 \\
502 \\
618 \\
673 \\
813 \\
1000
\end{align}

(2.2)

The use of quotation marks in the following statements is a consequence of the statement in the Remark of §0.3.

“**Theorem** 2.1. The ratio of the normalized radii in (2.2) “agrees” with the ratio of the conjectured 8 masses in [Za]. See (1.8) in [Za].

We later found out that the ratio of the smallest Gosset circles (the larger over the smaller) should be the Golden number $R = \frac{1}{2}(1 + \sqrt{5})$. Finding this to be the case experimentally was the key discovery in [Co]. The decomposition $F = F_1F_2$ implies that the set of Gosset circles decomposes into two sets of 4 Gosset circles. The radii of one set can be expressed in terms of the radii of the other set using $R$ and $1/R$ as follows:
\begin{align}
209 \times R &\quad = \quad 338 \\
673 \times 1/R &\quad = \quad 416 \\
813 \times 1/R &\quad = \quad 502 \\
618 \times R &\quad = \quad 1000
\end{align}

(2.30)

Here the first column is filled with the normalized radii of the Gosset circles defined by $F_1$ and the last column is filled with the normalized radii of the Gosset circles defined by $F_2$.

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