Deep inelastic scattering and final state interaction in an exactly solvable relativistic model

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Abstract

In the theory of deep inelastic scattering (DIS) the final state interaction (FSI) between the struck quark and the remnants of the target is usually assumed to be negligible in the Bjorken limit. This assumption, still awaiting a full validation within nonperturbative QCD, is investigated in a model composed by two relativistic particles, interacting via a relativistic harmonic oscillator potential, within light-cone hamiltonian dynamics. An electromagnetic current operator whose matrix elements behave properly under Poincaré transformations is adopted. It is shown that: i) the parton model is recovered, once the standard parton model assumptions are adopted; and ii) when relativistic, interacting eigenfunctions are exactly taken into account for both the initial and final states, the values of the structure functions, averaged over small, but finite intervals of the Bjorken variable $x$, coincide with the results of the parton model in the Bjorken limit.

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1 Introduction

The parton model proposed by Bjorken and Feynman almost thirty years ago [1, 2] has turned out to be a convenient language for discussing different properties of deep inelastic scattering (DIS), although, according to the theory based on the operator product expansion [3], the parton model, even in the Bjorken limit, is accurate only up to anomalous dimensions and perturbative QCD corrections.

If $q$ is the momentum transfer in DIS, $Q = |q^2|^{1/2}$, $P'$ is the four-momentum of the target in the initial state and $m$ is its mass, then the Bjorken limit is defined by the condition that $Q \gg m$ and the Bjorken variable $x = Q^2/(2P'q)$ is not too close to 0 and 1. In the parton model, it is assumed that the final state interaction (FSI) of the struck quark with the remnants of the target is a higher twist effect, i.e., an effect which is suppressed at least as $m^2/Q^2$. The qualitative motivation of this assumption is that the time needed for the absorption of the virtual photon by the struck quark is much smaller than the time of its hadronization and therefore in the process of absorption the struck quark can be considered as approximately free. It has been also shown within the framework of the collinear expansion [4] that the Feynman diagrams describing the FSI are indeed suppressed at high $Q^2$, once proper assumptions are adopted.

It could be thought very surprising that FSI can be disregarded in presence of confinement (see, e.g., Ref. [5]). Indeed, at first glance the picture which follows from confinement fully differs from that given by the parton model, i.e., no interaction in the final state and a continuous spectrum. In particular, while the structure functions in the parton model are continuous, in models where confinement is taken into account the structure functions are linear combinations of delta functions. However the two models can be reconciled within the framework of the theory of distributions. An analogous situation takes place in the case of the reaction $e^+e^- \rightarrow \text{hadrons}$ (see, e.g., the discussion in Ref. [6] and references therein).

Of course the relevance of the FSI has to be studied in the framework of nonperturbative QCD, but in absence of a full solution it is desirable to consider models in which the structure functions can be calculated exactly and therefore it is possible to check whether the FSI is indeed a higher twist effect.

In the last years several nonrelativistic two-body models of inelastic scattering were considered (see, e.g., Refs. [7, 8]) and it was shown that the effects of the FSI in these models are in agreement with the standard parton model assumptions, even in the case of confining interactions. However, DIS in the Bjorken limit by no means can be considered nonrelativistically. Indeed, in this limit the energy transfer $q^0$ in the target rest frame has to satisfy the condition $q^0 \gg m$, while the nonrelativistic approach holds only if $q^0 \ll m$.

The FSI for a relativistic particle bound in an external field was considered in Ref. [7]. However, in this case one has a two-body problem, where one of the particles (the source of the field) has an infinitely large mass. Therefore this problem cannot be studied in the Bjorken limit, where $Q$ is much larger than all the masses involved in the problem. Another relativistic approach to the FSI has been considered in Ref. [8] in the framework of the Bethe-Salpeter formalism, but it was assumed, as usual, that only Feynman diagrams from
a certain class (so called handbag diagrams) are dominant. Then the result of Ref. [4] that
the FSI is a higher twist effect has been confirmed.

The aim of this paper is to investigate the role played by FSI in DIS for an exactly
solvable relativistic constituent quark model (CQM), within the light-cone (= front-form)
hamiltonian dynamics. As well known, in the framework of relativistic CQMs (see, e.g.,
Ref. [9]) confinement can be ensured by choosing a quark-quark potential such that the
mass operator of a system with a fixed number of relativistic constituent quarks has only
the discrete spectrum (while in QCD confinement is understood as the property of the quark
and gluon Green functions to have no poles for real values of the mass). Our purpose is to
verify whether the naive treatment of confinement in relativistic CQMs is compatible with
the parton model.

We consider a simple system composed by two relativistic particles interacting via
the relativistic harmonic oscillator potential. We adopt an electromagnetic current whose
matrix elements exhibit the correct properties under Poincaré transformations and fulfill the
current conservation, as shown in [21]. In the proper Breit frame, the relevant components
of the current are the same as in the parton model. Then, in the framework of the light-
cone hamiltonian dynamics, we can derive exact expressions for the DIS structure functions,
including the FSI effects, and show that in the Bjorken limit the exact results coincide with
those given by the parton model, after an average over small intervals of the scaling variable
x has been performed. This average features the finite detector resolution and allows us to
avoid some mathematical technicalities of the theory of distributions.

The paper is organized as follows. In Sec. 2 we explicitly define a model describing
the interaction of two relativistic spin 1/2 particles, then in Sec. 3 we calculate the DIS
structure functions of this system with the standard parton model assumptions and in Sec. 4
the structure functions are calculated using the exact two-body wave functions. Conclusions
are drawn in Sec. 5.

2 Relativistic harmonic oscillator potential

We consider a system of two different particles with the same mass, \( m_0 \), and spin 1/2.
To describe such a system it is necessary to choose first an explicit form of the unitary
irreducible representation (UIR) of the Poincaré group pertaining to each particle. There
are many equivalent ways to construct an explicit realization of such a representation [11].
We choose the realization in the front form of dynamics (see, e.g., Refs. [11] [12] [13] [14] [15]).

Let \( p \) be the particle 4-momentum, \( \hat{s} \) be the spin operator and \( \sigma \) be the spin projection
on the z axis (\( \sigma = \pm 1/2 \)). We define \( p^\pm = (p^0 \pm p^z)/\sqrt{2} \), and we use \( p_\perp \) to denote the projection of \( p \) onto the plane \( xy \). The one particle Hilbert space can be chosen as the space
of functions \( \phi(p_\perp, p^+, \sigma) \) such that

\[
(\phi, \phi) = \sum_\sigma \int |\phi(p_\perp, p^+, \sigma)|^2 dp(p_\perp, p^+) < \infty,
\]
\[ d\rho(\vec{p}_\perp, p^+) = \frac{d\vec{p}_i dp^+}{2(2\pi)^3 p^+} \]  

(1)

The Hilbert space \( \mathcal{H} \) for the representation of the Poincaré group describing a system of two free or interacting particles is realized in the space of functions \( \phi(\vec{p}_1, \vec{p}_1^+, \sigma_1, \vec{p}_2, \vec{p}_2^+, \sigma_2) \) such that

\[
\sum_{\sigma_1\sigma_2} \int |\phi(\vec{p}_1, \vec{p}_1^+, \sigma_1, \vec{p}_2, \vec{p}_2^+, \sigma_2)|^2 \prod_{i=1}^2 d\rho(\vec{p}_i, p_i^+) < \infty
\]  

(2)

Instead of the variables \( \vec{p}_1, p_1^+, \vec{p}_2, p_2^+ \), let us consider the variables \( \vec{P}_\perp, P^+, \vec{k}, \) where \( \vec{P}_\perp = \vec{p}_1 + \vec{p}_2, P^+ = p_1^+ + p_2^+ \) and the relative momentum \( \vec{k} \) is defined as follows [12, 14]. We first define the quantities

\[
\xi = \frac{p_1^+}{P^+}, \quad \vec{k}_\perp = \vec{p}_1 - \xi \vec{P}_\perp
\]  

(3)

and then define \( \vec{k} = (\vec{k}_\perp, k_z) \), where

\[
\xi - \frac{1}{2} = \frac{k_z}{2\omega(\vec{k})}, \quad \omega(\vec{k}) = (m_0^2 + k^2)^{1/2}.
\]  

(4)

A direct calculation shows that

\[
\prod_{i=1}^2 d\rho_i(\vec{p}_i, p_i^+) = d\rho(\vec{P}_\perp, P^+) d\rho(int),
\]

\[
d\rho(int) = \frac{d\vec{k}_\perp d\xi}{2(2\pi)^3 \xi(1 - \xi)} = \frac{d\vec{k}}{(2\pi)^3 \omega(\vec{k})}
\]  

(5)

If the particles do not interact with each other, the generators of the two-particle representation are equal to sums of the corresponding one-particle generators [12]. The result is that the free-mass operator of the system is \( M_0 = 2\omega(\vec{k}) \) and the two-body spin operator is equal to

\[
\vec{S} = U^{-1} [\hat{l}(\vec{k}) + \vec{s}_1 + \vec{s}_2] U, \quad U = v(\vec{k}, \vec{s}_1)v(-\vec{k}, \ vec{s}_2),
\]  

(6)

where \( v(\vec{k}, \vec{s}) \) is the Melosh matrix [16]. In the given context the Melosh matrix was first considered by Terent’ev [14]:

\[
v(\vec{k}, \vec{s}) = \exp\left(\frac{2\epsilon_1 \epsilon_2 \vec{s}_j \vec{s}_l}{k_\perp} \arctg \frac{k_\perp}{m_0 + \omega(\vec{k}) + k_z}\right)
\]  

(7)

with \( k_\perp = |\vec{k}_\perp|, j = 1, 2, l = 1, 2, \) and \( \epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0. \)

Let us define the “internal” Hilbert space \( \mathcal{H}_{int} \) as the space of functions \( \chi = \chi(\vec{k}, \sigma_1, \sigma_2) \) such that
\[
\sum_{\sigma_1, \sigma_2} \int |\chi(\tilde{k}, \sigma_1, \sigma_2)|^2 d\rho(\text{int}) < \infty
\] (8)

If particles 1 and 2 interact with each other, then the representation space is the same, but the generators differ from the free ones (see, e.g., Eqs. (A34-A36) of Ref. [13] or Eq. (2.34) of Ref. [15]). The interaction should be introduced in such a way that the new set of generators, as well as the set of free generators, satisfies the commutation relations of the Poincaré group Lie algebra. By analogy with Ref. [17], this can be done by replacing the free mass operator \( M_0 \) by an interaction dependent mass operator \( M \), which acts only through \( \tilde{k}, \sigma_1, \sigma_2 \) and commutes with the spin operator \( \tilde{S} \) given by Eq. (8). As well known, the system of generators obtained in such a way defines the front form of dynamics [18], in which only three generators are interaction dependent and the other seven generators are free.

Then, let us define the internal space \( \mathcal{H}_{\text{int}}' \), whose elements \( \Psi(\tilde{k}, \sigma_1, \sigma_2) = \langle \tilde{k}, \sigma_1, \sigma_2 | \Psi \rangle \) are normalized as in the nonrelativistic quantum mechanics:

\[
\sum_{\sigma_1, \sigma_2} \int |\Psi(\tilde{k}, \sigma_1, \sigma_2)|^2 \frac{d^3 \tilde{k}}{(2\pi)^3} < \infty
\] (9)

As follows from Eqs. (8), (8) and (8), if the relation between the spaces \( \mathcal{H}_{\text{int}} \) and \( \mathcal{H}_{\text{int}}' \) is defined as

\[
\chi(\tilde{k}, \sigma_1, \sigma_2) = \langle \tilde{k}, \sigma_1, \sigma_2 | \chi \rangle = \langle \tilde{k}, \sigma_1, \sigma_2 | U^{-1} | \Psi \rangle \omega(\tilde{k})^{1/2},
\] (10)

then the two-body spin operator in \( \mathcal{H}_{\text{int}}' \) has the standard ”non-relativistic” form \( \tilde{S}_{nr} = \tilde{l}(\tilde{k}) + \tilde{s}_1 + \tilde{s}_2 \) (that also coincides with the instant-form one, see, e.g., [14, 15]).

Now we use the observation which is the essence of the ”minimal relativity principle” [19]: if \( \tilde{M} \) is the mass operator in \( \mathcal{H}_{\text{int}}' \) and the interaction operator \( V \) in \( \mathcal{H}_{\text{int}}' \) is defined as \( \tilde{M}^2 = M_0^2 + V \), then the equation \( \tilde{M}^2 \Psi_n = M_n^2 \Psi_n \) for the eigenvalues \( M_n \) and eigenfunctions \( \Psi_n \) of the operator \( \tilde{M} \) has the same form as the nonrelativistic Schroedinger equation in momentum representation:

\[
\frac{\tilde{k}^2}{m_0} + V \Psi_n(\tilde{k}, \sigma_1, \sigma_2) = E_n \Psi_n(\tilde{k}, \sigma_1, \sigma_2)
\] (11)

where

\[
V = V/4m_0, \quad E_n = (M_n^2 - 4m_0^2)/4m_0
\] (12)

The operator \( V \) should satisfy the same conditions as in the nonrelativistic quantum mechanics: the operator \( (\tilde{k}^2/m_0) + V \) should be selfadjoint and \( V \) should commute with \( \tilde{S}_{nr} \).

We can formally introduce the operator \( \tilde{r} = i\partial/\partial \tilde{k} \) which is canonically conjugated with \( \tilde{k} \). It is well known that in the relativistic case there is no operator which has all the properties of the position operator. Therefore \( \tilde{r} \) has not all the properties of the operator of the relative radius-vector between particles 1 and 2; it has such properties only in the
nonrelativistic or classical limit. Nevertheless we can choose the operator $V$ as the operator of multiplication by a function $V(r)$, where $r = |\vec{r}|$. In particular the function $V(r)$ can be chosen in such a way that the operator $\hat{M}$ (and hence $M$) has only a discrete spectrum, i.e., the property which in CQMs is associated with confinement.

We choose the function $V(r)$ in the form $V(r) = a^4 r^2 / m_0$, where $a$ is some constant with the dimension $\text{GeV}$. Then Eq. (11) is the well-known equation for the harmonic oscillator. We stress, once more, that in Eq. (11) the mass operator is a fully relativistic operator $\hat{\chi}$, chosen in such a way that the operator $\tilde{\chi}$ has the property which in CQMs is associated with confinement.

In order to calculate the DIS structure functions we have to know the wave function of the two-body system in the space $\mathcal{H}$ when the internal wave function $\chi_{\vec{n},\sigma'_1,\sigma'_2}$ in the space of spin eigenstates is described by the products $\Psi_{\vec{n},\sigma'_1,\sigma'_2} = \psi_{\vec{n}}(\vec{k}) \varphi_{\sigma'_1,\sigma'_2}(\sigma_1, \sigma_2)$, where $\varphi_{\sigma'_1,\sigma'_2}(\sigma_1, \sigma_2)$ is the spin eigenfunction of particles 1 and 2 with eigenvalues $\sigma'_1, \sigma'_2$, respectively, and

\[
\psi_{\vec{n}'}(\vec{k}) = \psi_{n_1}(k_x) \psi_{n_2}(k_y) \psi_{n_3}(k_z), \quad \vec{n} = (n_x, n_y, n_z),
\]

\[
\psi_{n_1}(k_i) = \left( \frac{2 \sqrt{\pi}}{a} \right)^{1/2} \Phi_{n_1}(\frac{k_i}{a}),
\]

\[
\Phi_{n_1}(t) = \frac{1}{(2^{n_1} n_1 !)^{1/2}} \exp\left( - \frac{t^2}{2} \right) H_{n_1}(t),
\]

\[
i = x, y, z. \quad \text{Eq. (13)}
\]

In Eq. (13) $n_i = 0, 1, 2, \ldots$, and $H_{n_i}$ is the Hermite polynomial of the $n_i$-th order. The normalization of $\psi_{n_i}$ is $\int_{-\infty}^{\infty} \left| \psi_{n_i}(k_i) \right|^2 \frac{dk_i}{2\pi} = 1$. As follows from Eqs. (11) and (12), the eigenvalues of the mass operator are equal to

\[
M_n = 2[m_0^2 + a^2(2n + 3)]^{1/2}
\]

with $n = n_x + n_y + n_z$.

In order to calculate the DIS structure functions we have to know the wave function of the twobody system in the space $\mathcal{H}$ when the internal wave function $\chi_{\vec{n},\sigma'_1,\sigma'_2}$ in the space $\mathcal{H}_{\text{int}}$ is related to $\Psi_{\vec{n},\sigma'_1,\sigma'_2}$ according to Eq. (11) and the system as a whole is in the eigenstate of the operators $\vec{P}_\perp$ and $P^+$ with the eigenvalues $\vec{P}_\perp$ and $P^+$, respectively. Let us first normalize as follows the one-particle eigenstates with the four-momentum $p'$ and spin projection $\sigma'$:

\[
\langle p'', \sigma'' | p', \sigma' \rangle = 2(2\pi)^3 \delta^{(2)}(\vec{p}'_\perp - \vec{p}'_\perp) \delta(p''^+ - p'^+) \delta_{\sigma'' \sigma'}(\delta_{\sigma'' \sigma'}) \quad \text{(the usual Kronecker symbol). Then the state} \quad |\vec{P}'_\perp, P^+, \chi_{\vec{n},\sigma'_1,\sigma'_2} \rangle \quad \text{is described by the wave function}
\]

\[
\langle P^+_\perp, P^+, \vec{k}, \sigma_1, \sigma_2 | P^+_\perp, P^+, \chi_{\vec{n},\sigma'_1,\sigma'_2} \rangle = 2(2\pi)^3 \delta^{(2)}(\vec{P}'_\perp - \vec{P}'_\perp) \delta(p''^+ - P'^+) \chi_{\vec{n},\sigma'_1,\sigma'_2}(\vec{k}, \sigma_1, \sigma_2)
\]

From Eqs. (11) and (13) it is clear that $||\chi_{\vec{n},\sigma'_1,\sigma'_2}|| = 1$, if the spin eigenfunctions are properly normalized.
3 DIS structure functions in the parton model

Let us consider in some detail how the structure functions can be obtained within the parton model in our light-cone framework. The results are well known (see, e.g., Ref. [20]), but the purpose of the following derivation is to make a comparison with the one given in Sec. 4, where the final states are described by the exact harmonic oscillator wave functions. For simplicity we will consider the case where the total spin of the initial system is $S = 0$ (this is by no way a restriction: indeed in the case where $S = 1$ one can obtain the same results as well). Then the internal wave function of the initial state will be

$$\chi_0(\vec{k}, \sigma_1, \sigma_2) = \langle \vec{k}, \sigma_1, \sigma_2 | \chi_0 \rangle = \omega(\vec{k})^{1/2} \psi_0(\vec{k}) \langle \sigma_1, \sigma_2 | U^{-1} | \varphi_{S=0} \rangle \quad (17)$$

Let $J^\mu(x)$ be the electromagnetic current operator for the system under consideration ($\mu = 0, 1, 2, 3$), where $x$ is a point in Minkowski space. It is well known that if the initial state is $|\vec{P}_\perp', P^+, \chi_0\rangle$ then the DIS hadronic tensor is given by

$$W^{\mu\nu} = \frac{1}{4\pi} \int \exp(iqx) \cdot \langle \vec{P}_\perp', P^+, \chi_0 | J^\mu(x) J^\nu(0) | \vec{P}_\perp', P^+, \chi_0 \rangle d^4x \quad (18)$$

The coordinate dependence of the current operator is fully defined by translational invariance, according to which

$$J^\mu(x) = \exp(iPx) J^\mu(0) \exp(-iPx) \quad (19)$$

where $P$ is the total four-momentum operator. Therefore, as follows from Eqs. (18) and (19),

$$W^{\mu\nu} = \frac{1}{4\pi} \sum_X (2\pi)^4 \delta^4(P' + q - P_X) \cdot \langle \vec{P}_\perp', P^{'+}, \chi_0 | J^\mu(0) | X \rangle \langle X | J^\nu(0) | \vec{P}_\perp', P^{'+}, \chi_0 \rangle \quad (20)$$

where a sum is taken over all possible final states $|X\rangle$ and $P_X$ is the four-momentum of the state $|X\rangle$. It is also well known that, as a consequence of Poincaré invariance and current conservation, the unpolarized hadronic tensor has the form

$$W^{\mu\nu}(P', q) = \left( \frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} \right) F_1(x, Q) + \frac{1}{(P'q)} \left( \frac{q^\mu (P'q)}{q^2} - \frac{q^\nu (P'q)}{q^2} \right) F_2(x, Q), \quad (21)$$

where $g^{\mu\nu}$ is the Minkowski tensor.
In the parton model one assumes that in Bjorken limit the operator $J^\mu(0)$ can be taken in the impulse approximation, i.e., $J^\mu(0) = \sum_{i=1}^n J^\mu_i(0)$ where the current operator $J^\mu_i(0)$ acts as follows

$$
\langle p_i, \sigma_i | J^\mu_i(0) | p_i', \sigma_i' \rangle = \bar{w}(p_i, \sigma_i) \gamma^\mu w(p_i', \sigma_i')
$$

(22)

where $\gamma^\mu$ are the Dirac $\gamma$ matrices and $w(p_i, \sigma_i)$ is the light-cone Dirac spinor.

The following representation for the $\gamma$ matrices has been adopted

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\tau_i \\ \tau_i & 0 \end{pmatrix}
$$

(23)

where $i = 1, 2, 3$ and $\tau_i$ are the Pauli matrices. Then the light-cone Dirac spinor can be written as

$$
\bar{w}(p, \sigma) = \sqrt{m_0} \begin{pmatrix} \beta(\vec{g}_\perp, g^+) \chi(\sigma) \\ [\beta(\vec{g}_\perp, g^+)]^T \chi(\sigma) \end{pmatrix}
$$

(24)

where $\chi(\sigma)$ is the ordinary spinor describing the state with the spin projection on the z axis equal to $\sigma$ and the matrix $\beta(\vec{g}_\perp, g^+)$ (where $g = p/m_0$) has the components

$$
\beta_{11} = \beta_{22} = 2^{1/4}(g^+)^{1/2}, \quad \beta_{12} = 0, \quad \beta_{21} = (g_x + ig_y)\beta_{22}
$$

(25)

For simplicity we assume that particle 1 has unit electric charge and particle 2 is chargeless. With this assumption, from Eqs. (19), (14) and (22), we have

$$
\langle p_1'', \sigma_1'' | J^\mu(0) | P^\perp, P^+; \chi_0 \rangle = \frac{1}{\xi'} \sum_{\sigma_1} \bar{w}(p_1'', \sigma_1'') \gamma^\mu w(p_1', \sigma_1') \chi_0(\vec{p}', \sigma_1', \sigma_2'')
$$

(26)

where $\xi'$ and $\vec{p}'$ are constructed by means of Eqs. (14) from the vectors $p_1'$ and $P'$, where $p_1'' = \vec{p}_1'' - \vec{p}_2''$, $p_1'^+ = P^+ - p_2''^+$.

As noted above, one of the major parton-model assumptions is that the FSI of the struck quark with the target remnants can be neglected. This implies that in our case the final states $X$ in Eq. (26) are the states of two free particles:

$$
|X\rangle = |p_1'', \sigma_1''\rangle |p_2'', \sigma_2''\rangle
$$

(27)

and therefore Eq. (26) for the hadronic tensor can be written as

$$
W^{\mu\nu} = \frac{1}{4\pi} \sum_{\sigma_1''\sigma_2''} \int (2\pi)^4 \delta^{(4)}(P' + q - P'') \cdot
$$

$$
\langle \vec{p}_1''^+, P''^+, \chi_0 | J^\mu(0) | p_1'', \sigma_1'', p_2'', \sigma_2'' \rangle \cdot
$$

$$
\langle p_1'', \sigma_1'', p_2'', \sigma_2'' | J^\nu(0) | \vec{p}_1'', P''^+, \chi_0 \rangle \cdot
$$

$$
\prod_{i=1}^2 dp_i(p_{i\perp}, p_i^+)
$$

(28)
where \( P'' = p''_1 + p''_2 \). In turn, as follows from Eqs. (13), (26) and (27), Eq. (28) can be written as

\[
W^{\mu\nu} = \frac{1}{4(P'^{\mu} + q'^{\mu})} \sum_{\sigma_1, \sigma_2} \int \delta(P'^{\nu} + q'^{\nu} - P''^{\nu}) \cdot \\
\chi_0(k'_\perp, \xi', \sigma_1'', \sigma_2'', [\bar{w}(p'_1, \sigma_1'') \gamma^{\mu}(p''_1 + m_0) \gamma^{\nu} w(p'_1, \sigma'_1)] \cdot \\
\chi_0(k''_1, \xi', \sigma'_1, \sigma''_1) \frac{d\xi''}{2(2\pi)^3 (\xi')^2 \xi''(1 - \xi'')} \tag{29}
\]

where \( \not{p}' = p'^{\mu} \gamma_{\mu} \) and, as follows from Eqs. (3) and (4),

\[
\begin{align*}
\vec{p}'_\perp &= \vec{k}'_\perp + \xi' \vec{P}'_\perp, & \vec{p}_\perp &= \vec{p}'_\perp + \vec{q}_\perp = \vec{k}'_\perp + \xi'' (\vec{P}'_\perp + \vec{q}_\perp), \\
\vec{p}_1' &= \xi' P'^{\perp}, & \vec{p}_1' + q'^{+} &= \xi'' (P'^{+} + q'^{+}), \\
\vec{p}_2'' &= -\vec{k}'_\perp + (1 - \xi'')(\vec{P}'_\perp + \vec{q}_\perp), & \vec{p}_2'' &= (1 - \xi'') P'^{+} + (1 - \xi'')(P'^{+} + q'^{+}) \tag{30}
\end{align*}
\]

Therefore the relation between \( (\vec{k}'_\perp, \xi') \) and \( (\vec{k}''_\perp, \xi'') \) is

\[
\vec{k}''_\perp = \vec{k}'_\perp + (\xi' - \xi'') \vec{P}'_\perp + (1 - \xi'') \vec{q}_\perp, \quad \xi'' (P'^{+} + q'^{+}) = \xi' P'^{+} + q'^{+} \tag{31}
\]

In the parton model the initial hadron is considered in the infinite momentum frame (IMF), i.e., in a reference frame where \( P'_z \) is positive and very large. It is also assumed that the transverse momenta of the constituents in the initial and final states are restricted by some value (say 300 MeV/c). In view of this, a suitable choice of the reference frame is such that \( \vec{P}'_\perp = \vec{q}_\perp = 0 \). Then, from Eqs. (30) and (31), one has

\[
\begin{align*}
\vec{p}'_\perp &= \vec{p}'_\perp = \vec{k}'_\perp, & \vec{p}_2' &= -\vec{k}'_\perp \tag{32}
\end{align*}
\]

The above conditions do not define the reference frame uniquely, since one can still boost this frame along the \( z \) axis and choose the Breit frame, where \( \vec{P}' + \vec{P}' = 0 \), in order to study DIS (let us recall that such boosts along the \( z \) axis are kinematical in the front form). It is important to point out that, as shown in Ref. [21], in the Breit frame the one-body current operator, Eq. (26), is fully compatible with the Poincaré trasformation properties.

In the chosen frame, as follows from the definition of the Bjorken variable \( x \), one has in the Bjorken limit

\[
\begin{align*}
q^0 &= 2P'_z(1 - x), & P'^{+} &= \sqrt{2} P'_z, & q'^{+} &= -\sqrt{2} P'_z x, \\
P''^{+} &= \sqrt{2} P'_z (1 - x) \tag{33}
\end{align*}
\]

and the relation between \( \xi' \) and \( \xi'' \) (see Eq. (31)) becomes

\[
\xi' = x + (1 - x) \xi'' \tag{34}
\]
Furthermore, by a direct calculation using Eq. (33), one obtains that the relevant elements of the hadronic tensor \([21]\) are given in the Bjorken limit by the following expressions

\[
W^{jl} = \delta^{jl} F_1(x, Q), \quad (j, l = 1, 2)
\]

\[
W^{++} = \frac{1}{4(2-x)}[F_2(x, Q) - 2xF_1(x, Q)],
\]

\[
W^{+-} = W^{-+} = \frac{1}{4x} [F_2(x, Q) - 2xF_1(x, Q)],
\]

\[
W^{--} = \frac{2-x}{4x^2} [F_2(x, Q) - 2xF_1(x, Q)]
\]

Finally, in the Bjorken limit the argument of the delta-function in Eq. (29) becomes proportional to

\[
\frac{Q^2(1-x)}{x} - \frac{(m_0^2 + \bar{k}_{\perp}^2)}{\xi''(1-\xi'')}
\]

Therefore we have two solutions for \(\xi''\), the first of which is close to 0 and the second is close to 1. A direct calculation based on Eqs. (24), (25), (29), (32) and (34) shows that in the Bjorken limit the contribution of the second solution is negligible. Indeed, when \(\xi''\) is close to 1, then \(\xi'\) is close to 1 too (see Eq. (34)) and \(k_z'\) goes to infinity (see Eq. (4)), making vanishing \(\chi_0\) in Eq. (29). In what follows we describe only the contribution of the first solution (\(\xi'' \) close to 0).

The only large component of the momentum of particle 1 in the initial state is \(p_1^+\) and the only large component of the momentum of this particle in the final state is \(p_1^-\). Then, as easily seen from Eq. (29), the spin structure of the tensor \(W^{\mu\nu}\) in the Bjorken limit is

\[
W^{\mu\nu} \sim Tr(\gamma^- \gamma^\mu \gamma^+ \gamma^\nu)
\]

where \(Tr\) stands for trace. This implies that all the longitudinal components of \(W^{\mu\nu}\) are equal to zero, i.e.,

\[
W^{+\nu} = W^{-\nu} = W^{\mu+} = W^{\mu-} = 0,
\]

since \((\gamma^+)^2 = (\gamma^-)^2 = 0\). The only non-vanishing components are the transverse ones, viz.

\[
W^{jl} = \sum_{\sigma_1, \sigma_2} \int \chi_0(\bar{k}_{\perp}, x, \sigma_1, \sigma_2) \delta^{jl} \delta_{\sigma_1 \sigma_1'} + 2i \epsilon_{jl}(s_1^z)_{\sigma_1 \sigma_1'} \cdot \chi_0(\bar{k}_{\perp}, x, \sigma_1', \sigma_2') \frac{d\bar{k}_{\perp}}{4(2\pi)^3 x(1-x)}
\]

In obtaining Eq. (38) we have used the fact that, if \(\xi''\) is close to 0, then in the Bjorken limit \(\xi' = x\) (see Eq. (34)). As follows from Eq. (3), the quantity \(\xi'\) is the momentum fraction of particle 1 in the initial state. Therefore we obtain the well-known result that the Bjorken variable \(x\) has the meaning of the momentum fraction of the struck quark in the IMF.
The structure function \( F_1(x, Q) \) is given by the symmetrical part of \( W^{ji} \) and the structure function \( F_2(x, Q) \) by the longitudinal components. Let us introduce the notation

\[
\rho(x) = \sum_{\sigma_1, \sigma_2} \int |\chi_0(\vec{k}_\perp, x, \sigma_1, \sigma_2)|^2 \frac{d\vec{k}_\perp}{2(2\pi)^3 x(1-x)} = \int |\chi_0(\vec{k}_\perp, x)|^2 \frac{d\vec{k}_\perp}{2(2\pi)^3 x(1-x)}
\]

(39)

where (see Eqs. (14) and (13)) \( \chi_0(\vec{k}_\perp, \zeta) = \omega(\vec{k})^{1/2} \psi_0(\vec{k}) \). Then, as follows from Eqs. (3), (5) and (8), \( \rho(x) dx \) is the probability to have a momentum fraction of particle 1, in the initial state, falling in the interval \((x, x+dx)\). By using Eqs. (35), (37), (38) and (39), one obtains

\[
F_1(x) = \frac{1}{2} \rho(x) \quad \text{(40)}
\]

and the Callan-Gross relation \([22]\]

\[
F_2(x) = 2xF_1(x) \quad \text{(41)}
\]

Therefore in the Bjorken limit the structure functions \( F_1 \) and \( F_2 \) do not depend on \( Q \), namely one has Bjorken Scaling.

4 DIS structure functions with the relativistic harmonic oscillator wave functions

Let us consider the exact hadronic tensor for two particles, interacting via the relativistic harmonic oscillator potential introduced in Sec. 2, and let the initial state be the \( S = 0 \) state as in Sec. 3. Therefore in Eq. (20) we use, as final states \(|X\rangle\), the exact eigenfunctions defined by Eqs. (10), (13), (16). Then the exact expression for the hadronic tensor of the system under consideration is

\[
W^{\mu\nu} = \frac{1}{4\pi} \sum_{\sigma_1, \sigma_2} \sum_{n_x n_y n_z} \int (2\pi)^4 \delta^{(4)}(P' + q - P''_n) \cdot \\
\langle \vec{P}'_\perp, P'^+, \chi_0|J^{\mu}(0)|\vec{P}''_\perp, P''^+, \chi_{\bar{n}, \sigma_1^*, \sigma_2^*} \rangle \cdot \\
\langle \vec{P}'_\perp, P'^+, \chi_{\bar{n}, \sigma_1^*, \sigma_2^*}|J^{\nu}(0)|\vec{P}''_\perp, P''^+, \chi_0 \rangle \cdot \\
d\rho(\vec{P}'_\perp, P'^+)
\]

(42)

where the four-vectors \( P''_n = P' + q \) have the components \( P''^+_n = P''^+, \vec{P}'_\perp = \vec{P}''_\perp, \)

\[
P''^-_n = (M^2_n + \vec{P}'_\perp^2)/2P'^+,
\]

\[
|\chi_{\bar{n}, \sigma_1^*, \sigma_2^*}(\vec{k})\rangle = \omega(\vec{k})^{1/2} \phi_{\bar{n}}(\vec{k})U^{-1}|\phi_{\sigma_1^*, \sigma_2^*}\rangle,
\]

(43)
and \(|\varphi_{\sigma_1,\sigma_2}\rangle\) is the normalized spin eigenstate of particles 1 and 2 with the eigenvalues \(\sigma_1,\sigma_2\), respectively:

\[ ||\varphi_{\sigma_1,\sigma_2}|| = 1. \tag{44} \]

We will calculate the tensor (42) in the same reference frame as in the preceding section, i.e., in the Breit frame \((\vec{P}^\prime) + \vec{P}^\prime = 0\) with \(\vec{P}_\perp = \vec{q}_\perp = 0\). We assume \(J^\mu(0) = J^\mu_1(0)\) for \(\mu = +,1,2\) and use current conservation to define \(J^-_1(0)\). As it is shown in \([21]\), this current operator is an allowed choice, which is fully compatible in our reference frame with Poincaré transformation properties, and trivially fulfills current conservation. Then, from Eqs. (4), (6), (16), (21) and (26),

\[
F_1(x, Q) = \frac{1}{2} \sum_{\sigma_1,\sigma_2} \sum_{n_2n_\sigma} \delta(m^2 + \frac{Q^2(1-x)}{x} - M^2_n) \cdot \\
| \sum_{\sigma_1,\sigma_2} \int \chi_{\bar{n}_1,\sigma_1} \chi_{\sigma_2} (\vec{k}_\perp, \xi') \sigma_1, \sigma_2 | \omega(p_1, \sigma_1) \gamma^1 w(p', \sigma_1) \rangle \\
\chi_0 (\vec{k}_\perp, \xi, \sigma_1, \sigma_2) \frac{d\vec{k}_\perp d\xi'}{2(2\pi)^3 \delta^2 \xi\xi' (1 - \xi')} \tag{45} \]

where \(m = 2[m_0^2 + 3\alpha^2]^{1/2}\) is the ground state eigenvalue of the mass operator and the relations between the quantities \(p_1, p_1'\), \(\vec{k}_\perp, \xi, \vec{k}_\perp, \xi'\) are given by Eqs. (34) and (37). In the reference frame under consideration, \(\vec{k}_\perp = \vec{k}_\perp\) as in the parton model, and the relation between \(\xi\) and \(\xi'\) is still given by Eq. (34).

The spin sums in Eq. (13) can be easily done taking into account the following expressions for the Clebsh-Gordan coefficient in the initial state, the Melosh rotation and the matrix element of the parton current, viz.

\[
\langle \sigma_1, \sigma_2 | 00 \rangle = i \langle \sigma_1 | \frac{\sigma_y}{\sqrt{2}} | \sigma_2 \rangle \\
v(\vec{k}, \vec{s}) = \frac{m_0 + \omega(\vec{k}) + k_x - \omega(\vec{k}) \times \vec{k}_\perp}{\sqrt{2(\omega(\vec{k}) + m_0)(\omega(\vec{k}) + k_x)}} \\
\bar{w}(p_1, \sigma_1) \gamma_x w(p', \sigma_1') = \frac{1}{\sqrt{p_1 + p_1'}} \langle \sigma_1 | \left[ q^+ (i\sigma y m + k_x) + ik_y \sigma_x \right] + 2p_1' k_x' \rangle | \sigma_1' \rangle \tag{46} \]

Then Eq. (13) becomes

\[
F_1(x, Q) = \frac{1}{2} \sum_{n_2n_\sigma} \delta(m^2 + \frac{Q^2(1-x)}{x} - M^2_n) \cdot \\
\left[ \int \chi_{\bar{n}}(\vec{k}_\perp, \xi) \frac{1}{\sqrt{p_1 + p_1'}} q^+ \left[ m_0 + \frac{k^2 y}{\omega(\vec{k}) + m_0} \right] \chi_0(\vec{k}_\perp, \xi') \frac{d\vec{k}_\perp d\xi}{2(2\pi)^3 \delta^2 \xi\xi' (1 - \xi')} \right] \tag{47} \]

12
\[ | \int \chi_{\bar{R}}(\bar{k}_\perp, \xi) \frac{1}{\sqrt{p_1^{n'} + p_1^+}} \frac{q^+ k_y k_z}{[\omega(\bar{k}) + m_0]} \chi_0(\bar{k}_\perp, \xi') \frac{d\bar{k}_\perp d\xi}{2(2\pi)^3 \xi' \xi (1 - \xi)^2} | \]

\[ | \int \chi_{\bar{R}}(\bar{k}_\perp, \xi) \frac{1}{\sqrt{p_1^{n'} + p_1^+}} (q^+ + 2p_1^+) k_x \chi_0(\bar{k}_\perp, \xi') \frac{d\bar{k}_\perp d\xi}{2(2\pi)^3 \xi' \xi (1 - \xi)^2} | \]

\[ | \int \chi_{\bar{R}}(\bar{k}_\perp, \xi) \frac{1}{\sqrt{p_1^{n'} + p_1^+}} q^+ k_y k_z \chi_0(\bar{k}_\perp, \xi') \frac{d\bar{k}_\perp d\xi}{2(2\pi)^3 \xi' \xi (1 - \xi)^2} | \]

(47)

where in order to simplify the notation, the variables \( \bar{k}'' \) and \( \xi'' \) of Eq. (43) have been replaced by \( \bar{k}_\perp \) and \( \xi \); moreover (see Eqs. (34) and (43))

\[ \chi_{\bar{R}}(\bar{k}) = \omega(\bar{k})^{1/2} \psi_{\bar{R}}(\bar{k}), \quad \chi_0(\bar{k}') = \omega(\bar{k}')^{1/2} \psi_0(\bar{k}'), \quad \xi' = x + (1 - x) \xi, \]

and the relation between \( \bar{k} \) (\( \bar{k}' \)) and \( (\bar{k}_\perp, \xi) \) ((\( \bar{k}_\perp, \xi' \))) is given by Eq. (4).

### 4.1 A pedagogical example

In order to begin with a simple, pedagogical model that allows us to introduce the mathematical tools for the general case (to be considered below), we assume for the moment \( a \ll m_0 \). This assumption implies that the relevant momenta \( \bar{k}' \) in the wave function of the ground state satisfy the condition \( |\bar{k}'| \ll m_0 \) (see Eq. (13)), but does not destroy the relativistic nature of the highly excited final states. Then Eq. (47) greatly simplifies, putting in evidence the relevant integral over \( \xi \). With the help of Eq. (33), \( F_1(x, Q) \) becomes

\[ F_1(x, Q) = \frac{m_0^2 x^2}{2(1 - x)} \sum_{n_x \geq n_y \geq n_z} \delta(m^2 + \frac{Q^2(1 - x)}{x} - M_n^2) \cdot \]

\[ | \int \chi_{\bar{R}}(\bar{k}_\perp, \xi) \chi_0(\bar{k}_\perp, \xi') \frac{d\bar{k}_\perp d\xi}{2(2\pi)^3 \xi' \xi (1 - \xi)^2} | \]

(49)

So far we have not used the explicit expressions for the harmonic oscillator wave functions and therefore, if the FSI is neglected in Eq. (49), the result for \( F_1(x, Q) \) should be the same as in Eq. (44). Disregarding the FSI in Eq. (49) implies the following replacements

\[ \sum_{n_x \geq n_y \geq n_z} \rightarrow \int \frac{d\bar{k}_\perp d\xi''}{2(2\pi)^3 \xi'' (1 - \xi'')} , \quad M_n^2 \rightarrow \frac{m_0^2 + \bar{k}_\perp^2}{\xi'' (1 - \xi'')} , \]

\[ \chi_{\bar{R}}(\bar{k}_\perp, \xi) \rightarrow 2(2\pi)^3 \xi'' (1 - \xi'') \delta(1 - \xi'')(\bar{k}_\perp - \bar{k}_\perp') \delta(\xi - \xi'') \]

(50)

Then it is easy to see that we again arrive at Eq. (44) for \( F_1(x, Q) \).

Let us go back to Eq. (49), where the exact final state wave functions are used. Since only the values \( |\bar{k}_\perp| \ll m_0 \) are important in Eq. (49), we can neglect \( \bar{k}_\perp \) in \( \omega(\bar{k}) \). Then, as
follows from Eqs. (4), (13) and (48), the integration over \( \vec{k}_\perp \) in Eq. (49) is trivial and as a result

\[
F_1(x, Q) = \frac{m_0^2 x^2}{2(1 - x)} \sum_n \delta(m^2 + \frac{Q^2(1 - x)}{x} - M_n^2) \cdot 
\]

\[
| \int_0^1 (m_0^2 + k_z^2)^{1/4} \psi_n(k_z) \chi_0(\xi') \frac{d\xi}{4\pi(\xi'\xi)^{2/3}(1 - \xi)} |^2 \) (51)
\]

where \( n \) replaces \( n_z, \chi_0(\xi') = m_0^{1/2} \psi_0(k_z^2) \), the relation between \( \xi \) and \( \xi' \) is given by Eq. (48) and one-to-one relations \( \xi \leftrightarrow k_z \) and \( \xi' \leftrightarrow k_z' \) can be obtained from Eq. (4). In particular if the dependence of \( \omega(\vec{k}) \) on \( \vec{k}_\perp \) is neglected, then the relation between \( \xi \) and \( k_z \) is given by

\[
\xi - \frac{1}{2} = \frac{k_z}{2(m_0^2 + k_z^2)^{1/2}}, \quad k_z = \frac{m_0(\xi - 1/2)}{[\xi(1 - \xi)]^{1/2}}, 
\]

\[
d\xi = \frac{dk_z}{2\xi(1 - \xi)} = \frac{m_0^2 + k_z^2)^{1/2}}{(m_0^2 + k_z^2)^{1/2}} \) (52)
\]

and the relation between \( \xi' \) and \( k_z' \) is the same.

As a consequence of Eq. (52), Eq. (51) can be written in the form

\[
F_1(x, Q) = \frac{m_0^2 x^2}{8\pi^2(1 - x)} \sum_n \delta(m^2 + \frac{Q^2(1 - x)}{x} - M_n^2) \cdot 
\]

\[
| \int_{-\infty}^{\infty} \psi_n(k_z) \chi_0(\xi') dk_z |^2 \) (53)
\]

This is the last stage where we still can return back to the parton model. Indeed, as follows from Eqs. (50) and (51), neglecting FSI in Eq. (53) implies the following replacements

\[
\sum_n \rightarrow \int \frac{dk_z}{2\pi(m_0^2 + k_z^2)^{1/2}}, \quad M_n^2 \rightarrow 4(m_0^2 + k_z^2)^2, 
\]

\[
\psi_n(k_z) \rightarrow 2\pi(m_0^2 + k_z^2)^{1/4} \delta(k_z - k_z'), \) (54)
\]

With these replacements it is easy to see that, in the Bjorken limit, Eq. (53) leads to Eq. (11) for \( F_1(x, Q) \), if \( \chi_0(\vec{k}) \) is given by Eq. (13) and \( a \ll m_0 \).

In the last part of this section we will consider Eq. (53), with the function \( \psi_n(k_z) \) given by the \emph{exact eigenfunction}, i.e. by Eq. (13).

As follows from Eqs. (14) and (53), \( n = \frac{Q^2(1 - x)}{8a^2x} \) and therefore, if \( Q \) is large, only large values of \( n \) are important in Eq. (53). Taking into account Eq. (14) we can write Eq. (53) in the form

\[
F_1(x, Q) = \frac{m_0^2 x^4}{8\pi^2(1 - x)Q^2} \sum_n \delta(x - \frac{Q^2}{Q^2 + 8a^2n}) f(n, x)^2 \) (55)
\[ f(n, x) = (-1)^n \int_{-\infty}^{\infty} \frac{\psi_n(kz)\chi_0(\xi')dkz}{\xi'(\xi')^{1/2}(m_0^2 + k_z^2)^{1/4}} \]  

(56)

We will show in Appendix A that \( f(n, x) \) is a smooth function of \( x \) and has a finite limit for \( n \to \infty \).

Now the following question arises. While in the parton model \( F_1(x) \) and \( F_2(x) \) are continuous functions of \( x \), it is clear from Eq. (55) that at fixed \( Q \) the function \( F_1(x, Q) \) is a linear combination of delta-functions, which are not equal to zero only for discrete values of \( x \). As noted in Sec. 1, the first impression is that a correspondence between discrete and continuous cases cannot exist. However, Eq. (55) is meaningful only in the realm of the distributions. Within such a framework, the correspondence between the discrete and continuous cases could be shown in the Bjorken limit, since the discrete values of \( x \), where \( F_1(x, Q) \) is not zero, become closer and closer as \( Q \) increases.

In order to simplify the mathematical discussion, let us note that experiments allow one to determine not the very function \( F_1(x, Q) \), but its average values over some bins in \( x \) and \( Q \). Therefore, let us consider the integral

\[ \bar{F}_1(\bar{x}, Q) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} F_1(x, Q)dx, \]  

(57)

where \( \bar{x} \) belongs to the small interval \([x_1, x_2]\), such that \( x_2 - x_1 \ll \bar{x} \). It is clear that for large \( Q \) there exist many values of \( n \) such that \( Q^2/(Q^2 + 8a^2n) \in [x_1, x_2] \), even for a small, but finite value of \( x_2 - x_1 \). Therefore the integral (57) is a smooth function of \( Q \). Let us define \( n_1 \) and \( n_2 \) as follows

\[ n_1 = \frac{Q^2(1 - x_1)}{8a^2x_1}, \quad n_2 = \frac{Q^2(1 - x_2)}{8a^2x_2}. \]  

(58)

As a consequence \( n_1 - n_2 = (x_2 - x_1)n_2/[x_1(1 - x_2)] \ll n_2 \). From Eqs. (55) and (57) one has

\[ \bar{F}_1(\bar{x}, Q) = \frac{m_0^2\bar{x}^4}{8\pi^2(1 - \bar{x})(x_2 - x_1)Q^2} \sum_{n=n_2}^{n_1} f(n, \bar{x})^2 \]  

(59)

The scaling property of the function \( \bar{F}_1 \) holds if \( f(n, x) \to g(x) \) when \( n \to \infty \). As a matter of fact, in this case the quantity \( \sum_{n=n_2}^{n_1} f(n, \bar{x})^2 \) for large values of \( n_1 \) and \( n_2 \) becomes \((n_1 - n_2)g(\bar{x})^2\) and therefore from Eq. (58) one has

\[ \bar{F}_1(\bar{x}, Q) = \frac{m_0^2\bar{x}^2g(\bar{x})^2}{64\pi^2a^2(1 - \bar{x})}. \]  

(60)

Moreover, as follows from Eqs. (39), (40) and (52), the result (60) for the structure function will be equal to the parton model one if

\[ \lim_{n \to \infty} f(n, x) = g(x) = \frac{(8\pi)^{1/2}a\chi_0(x)}{m_0x^{3/2}} \]  

(61)
where \( \chi_0(x) = m_0^{1/2} \psi_0(k_\perp(x)) \), with \( k_\perp(x) = m_0(x - 1/2)[x(1 - x)]^{-1/2} \). Indeed in our case, where the initial state is a harmonic oscillator ground state with \( a \ll m_0 \), the structure function \( F_1(x) \) in Eq. (44) becomes

\[
F_1(x) = \frac{\chi_0(x)^2}{8\pi x(1 - x)}. \tag{62}
\]

In Appendix A it is shown how Eq. (61) can be proved. In conclusion our result for \( F_1(x, Q) \) (Eq. (55)) is indeed compatible with the parton model, once an average over bins of \( x \) is performed.

As follows from the analysis of Appendix A (see in particular the discussion about \( f_2(n, x) \), the main contribution to the hadronic tensor is given by the region where \( k_\perp \) is negative and \( |k_\perp| \) is very large (recall that \( t = k_\perp/a \)). In this region \( \xi \) is small and \( \to 0 \) in the Bjorken limit. In turn, the quantity \( \xi' \) is close to \( x \) (see Eq. (34)). We see that the usual interpretation of the Bjorken variable \( x \) as the momentum fraction of the struck quark is still valid if the FSI is not neglected.

We recall that, with our choice of the reference frame, \( |\vec{p}_{1\perp}| \) and \( |\vec{p}_{1\perp}'| \) can be neglected with respect to \( m_0 \) (see Eq. (30)) in the evaluation of the matrix elements of \( \gamma^\mu \) between light-cone Dirac spinors. Therefore a direct calculation using Eqs. (24) and (25) shows that the matrix elements \( \bar{w}(p_{1\perp}, \sigma_1)\gamma^+ w(p_{1\perp}', \sigma_1') \), up to constant quantities, are proportional to \( \delta_{\sigma_1 \sigma_1'}(p^++p'^+)1/2 \). Then, as shown in Appendix C, \( W^{++} \) is vanishing in the Bjorken limit, and from Eq. (33) one obtains again that the Callan-Gross relation (11) holds.

Finally, using once more Eq. (33), one obtains that all the longitudinal components of the hadronic tensor become negligible in the Bjorken limit, with the exact wave functions for the final states as well as in the parton model.

### 4.2 The general case

In the general case, i.e. for any value of the harmonic oscillator strenght \( a \), the parton model can be recovered once again. The starting point is Eq. (47) that is exact. After averaging over the \( x \)-bins, as in Eq. (57), one has

\[
\bar{F}_1(\bar{x}, Q) = \frac{\bar{x}^4}{2(1 - \bar{x})(x_2 - x_1)Q^2} \sum_{n=2}^{n_1} \sum_{n_x+n_y+n_z=n} \mathcal{F}(\vec{n}, \bar{x}) \tag{63}
\]

where in the sum all the values of \( n_i \) constrained by \( n = n_x + n_y + n_z \) are allowed, and \( n_{1(2)} \) is defined in Eq. (58). By using in Eq. (47) the expressions of \( p_{1\perp}^+ \) and \( p_{1\perp}'' \) obtained from Eqs. (30) and (33), \( \mathcal{F}(\vec{n}, \bar{x}) \) is given by

\[
\mathcal{F}(\vec{n}, \bar{x}) = \left[ |\int \psi_{\bar{n}}(\vec{k})\frac{1}{\sqrt{\xi^2}} \left[ m_0 + \frac{k^2}{\omega(\vec{k}) + m_0} \right] \chi_0(\vec{k}) \frac{d\vec{k}}{(2\pi)^3 \xi' \omega(\vec{k})^{1/2}} \right]^2 + \left[ |\int \psi_{\bar{n}}(\vec{k})\frac{1}{\sqrt{\xi^2}} \frac{k_y k_z}{[\omega(\vec{k}) + m_0]} \chi_0(\vec{k}) \frac{d\vec{k}}{(2\pi)^3 \xi' \omega(\vec{k})^{1/2}} \right]^2 + \ldots
\]
\[
\left| \int \psi_{n}(\tilde{k}) \frac{1}{\sqrt{\xi \xi'}} k_{x}(1 - 2 \frac{\xi'}{x}) \chi_{0}(\tilde{k}) \frac{dk}{(2\pi)^{3} \xi' \omega(\tilde{k})^{1/2}} \right|^{2} +
\left| \int \psi_{n}(\tilde{k}) \frac{1}{\sqrt{\xi \xi'}} k_{y} k_{x} \chi_{0}(\tilde{k}) \frac{dk_{z}}{(2\pi)^{3} \xi' \omega(\tilde{k})^{1/2}} \right|^{2}
\right]
\]  

(64)

It is important to note that using the completeness over \(n\) and \(n_{y}\) one has

\[
\sum_{n_{x}=0}^{\infty} \sum_{n_{y}=0}^{\infty} \mathcal{F}(\bar{n}, \bar{x}) = \int \int \frac{d\bar{k}_{\perp}}{(2\pi)^{2}} |\psi_{0}(k_{x}) \psi_{0}(k_{y})|^{2}.
\]

(65)

with \(\chi_{0}(k_{z}') = \omega(k_{z}', \bar{k}_{\perp})^{1/2} \psi(0, k_{z}')\). Then applying similar arguments as in Appendix A, where \(m_{0}\) has to be replaced by \(\sqrt{m_{0}^{2} + k_{z}^{2}}\), one obtains, as in the pedagogical example, that only the negative and large values of \(k_{z}\) contribute to the integrals in Eq. (63) in the Bjorken limit, and therefore \(\xi' \to x\). Finally one can find

\[
\lim_{n_{z} \to \infty} \sum_{n_{x}=0}^{\infty} \sum_{n_{y}=0}^{\infty} \mathcal{F}(\bar{n}, \bar{x}) = \frac{8a^{2}}{\bar{x}^{3}} \int \int \frac{d\bar{k}_{\perp}}{(2\pi)^{3}} |\chi_{0}(\bar{k}_{\perp}, \bar{x})|^{2} = \frac{8a^{2} \rho(\bar{x})(1 - \bar{x})}{\bar{x}^{2}}
\]

(66)

We explicitly note that in Eq. (64) the order of \(\lim_{n_{z} \to \infty}\) and of the integration over \(k_{\perp}\) has been exchanged, thanks to the presence in Eq. (63) of the gaussians \(\psi_{0}(k_{x}) \psi_{0}(k_{y})\) and to the power-law behaviour in \(\bar{k}_{\perp}\) of the remaining terms, for \(k_{x(y)} \to \pm \infty\). Equation (64) implies that the positive function \(\mathcal{F}(\bar{n}, \bar{x})\) is bounded by a quantity which does not depend upon \(\bar{n}\).

Let us come back to Eq. (63), that can be rewritten as

\[
\tilde{F}_{1}(\bar{x}, Q) = \frac{\bar{x}^{2}}{16a^{2}(1 - \bar{x})(n_{1} - n_{2})} \sum_{n_{x}=0}^{n_{1}-n_{x}} \sum_{n_{y}=0}^{n_{1}-n_{x}} \sum_{n_{z}=\text{MAX}(0, n_{2} - n_{x} - n_{y})}^{n_{1}-n_{x}-n_{y}} \mathcal{F}(\bar{n}, \bar{x})
\]

(67)

First of all, let us demonstrate that

\[
\lim_{n_{x} \to \infty} \mathcal{F}(\bar{n}, \bar{x}) = 0 \quad \lim_{n_{y} \to \infty} \mathcal{F}(\bar{n}, \bar{x}) = 0
\]

(68)
Indeed in Eq. (64), there are integrals of the form

\[ \int dk_i \psi_n(k_i) \psi_0(k_i) g_i(k_i) \quad i = x, y \]  

(69)

where it is easy to show that \( \psi_0(k_i) g_i(k_i) \) is i) continuous, ii) bounded and iii) \( \in \mathcal{L}_1 \) (due to the gaussian form of \( \psi_0(k_i) \) and the power-law behaviour of \( g_i(k_i) \) for \( k_i \to \pm \infty \)). Furthermore also the derivative of \( \psi_0(k_i) g_i(k_i) \) belongs to \( \mathcal{L}_1 \). Therefore, we can exploit the same arguments presented in Appendix C and the boundedness of \( \mathcal{F}(\vec{n}, \vec{x}) \) (see Eq. (60)) to show that for large \( n_x \) and \( n_y \) one has

\[ \mathcal{F}(\vec{n}, \vec{x}) \leq \frac{C}{n_x^{3/2} n_y^{3/2}} \]  

(70)

where \( C > 0 \) does not depend upon \( n_z \). Actually, using the property of the Hermite polynomials \( H_n'(-1) = 2(n + 1)H_n(-1) \), and the gaussian behaviour of \( \psi_0(k_i) g_i(k_i) \), it is possible to show that the fall-off of \( \mathcal{F}(\vec{n}, \vec{x}) \) is faster than any power of \( n_x n_y \).

By using the upper bound of Eq. (70) one can show that even for \( Q \to \infty \) (i.e. \( n_{1(2)} \to \infty \)) the sum over \( n_x \) in Eq. (67) contains only a finite number of relevant terms. As a matter of fact, one has

\[
\frac{1}{(n_1 - n_2)} \sum_{n_x = N_x + 1}^{n_1} \sum_{n_y = 0}^{n_1 - n_x} \sum_{n_z = \text{MAX}(0, n_2 - n_x - n_y)}^{n_1 - n_x - n_y} \mathcal{F}(\vec{n}, \vec{x}) \leq \\
\frac{1}{(n_1 - n_2)} \sum_{n_x = N_x + 1}^{n_1} \sum_{n_y = 0}^{n_1 - n_x} \sum_{n_z = \text{MAX}(0, n_2 - n_x - n_y)}^{n_1 - n_x - n_y} \frac{C}{n_x^{3/2} n_y^{3/2}} \leq \\
\frac{1}{(n_1 - n_2)} \sum_{n_x = N_x + 1}^{n_1} \sum_{n_y = 0}^{n_1 - n_x} \sum_{n_z = \text{MAX}(0, n_2 - n_x - n_y)}^{n_1 - n_x - n_y} \frac{1}{n_x^{3/2} n_y^{3/2}} \leq \epsilon
\]  

(71)

provided that \( N_x \) is large enough. By analogous arguments one can show the same property for the sum over \( n_y \). Furthermore, the indices \( n_1 - n_x - n_y \) and \( n_2 - n_x - n_y \) in the sum over \( n_x \) can be replaced by \( n_1 \) and \( n_2 \), respectively, by using once more the upper limit in Eq. (70). As a matter of fact, for large values of \( n_1 \) and \( n_2 \) and fixed values of \( N_x \) and \( N_y \), one has

\[
\frac{1}{(n_1 - n_2)} \sum_{n_x = 0}^{N_x} \sum_{n_y = 0}^{N_y} \sum_{n_z = n_{2(1)} - n_x - n_y}^{n_{2(1)}} \mathcal{F}(\vec{n}, \vec{x}) \leq \\
\frac{1}{(n_1 - n_2)} \sum_{n_x = 0}^{N_x} \sum_{n_y = 0}^{N_y} \sum_{n_z = n_{2(1)} - n_x - n_y}^{n_{2(1)}} \frac{C}{n_x^{3/2} n_y^{3/2}} \leq \\
\frac{C}{(n_1 - n_2)} \sum_{n_x = 0}^{N_x} \sum_{n_y = 0}^{N_y} \frac{(n_x + n_y)}{n_x^{3/2} n_y^{3/2}} \leq \epsilon
\]  

(72)
Therefore, in the Bjorken limit, namely for large $n_1$ and $n_2$, Eq. (67) becomes

$$\bar{F}_1(\bar{x}, Q) = \frac{\bar{x}^2}{16a^2(1 - \bar{x})(n_1 - n_2)} \sum_{n_x=0}^{N_x} \sum_{n_y=0}^{N_y} \sum_{n_z=n_2}^{n_1} F(\vec{n}, \bar{x})$$

(73)

with finite values $N_x$ and $N_y$ which do not depend upon $Q$. Then, adding to Eq. (73) the following sum (that is vanishing for large values of $N_x$ and $N_y$ due to Eq. (70))

$$\bar{S}_1(\bar{x}, Q) = \frac{\bar{x}^2}{16a^2(1 - \bar{x})(n_1 - n_2)} \left[ \sum_{n_x=N_x+1}^{\infty} \sum_{n_y=0}^{N_y} + \sum_{n_x=0}^{\infty} \sum_{n_y=N_y+1}^{\infty} \right] \cdot \sum_{n_z=n_2}^{n_1} F(\vec{n}, \bar{x})$$

(74)

one has

$$\bar{F}_1(\bar{x}, Q) = \frac{\bar{x}^2}{16a^2(1 - \bar{x})(n_1 - n_2)} \sum_{n_x=0}^{\infty} \sum_{n_y=0}^{\infty} \sum_{n_z=n_2}^{n_1} F(\vec{n}, \bar{x}) = \rho(\bar{x}) \frac{1}{2}$$

(75)

where Eq. (66) has been used in the last step.

Arguments analogous to those presented in this subsection can be applied in order to show that $W^{++}$ is vanishing in the Bjorken limit also in the general case.

5 Conclusion

Within front-front dynamics we have studied DIS in the Bjorken limit for a relativistic model system composed by two spin 1/2 particles interacting via the relativistic analog of the harmonic oscillator potential (see Sec. 2).

First of all, we have explicitly calculated the DIS structure functions at $Q^2 \to \infty$ for our model system with the standard parton model assumptions in a Breit frame where the total transverse momentum $\vec{P}'_\perp$, as well as the transverse momentum transfer $\vec{q}_\perp$ are zero (i.e., in an infinite momentum frame). We have adopted as usual a one-body electromagnetic current, and have shown that the structure functions are indeed given by the standard formulas of the parton model. It should be pointed out that a one-body choice for the current operator in the Breit reference frame is compatible with Poincaré invariance (see Ref. [21]).

Then we have introduced the exact final state wave functions which properly take into account the FSI, and we have calculated the structure functions in the same Breit frame. In this frame, the components $J^+, J^1, J^2$ of the current operator have been taken in the one-body form, while $J^-$ has been defined through the current conservation. As shown in Ref. [21], this choice of the current is compatible with Poincaré invariance and trivially fulfills the current conservation. We have shown that if one takes average values of the exact structure functions over small, finite intervals $[x_1, x_2]$ of the Bjorken variable $x$, so that $(x_2 - x_1)/x_1 \ll 1$, then at
high $Q^2$ there exist many values of $n$ such that $Q^2/(Q^2 + 8a^2 n) \in [x_1, x_2]$ and in the Bjorken limit such average values coincide with those given by the parton model. This averaging procedure corresponds to the finite resolution of the experimental measurements and allows to avoid some mathematical technicalities of distributions.

It is the first time that the effects of the final state interactions in DIS have been exactly calculated in a relativistic model. It is worth noting that the relativistic calculation differs from the nonrelativistic ones considered in Ref. [8] in several aspects. In particular, the Bjorken limit implies that one gets a finite contribution to the structure functions only from excited states with $n \to \infty$, while the nonrelativistic approach is valid only if $n \ll (m_0/a)^2$ (see Eq. (14)). Another crucial difference between the two cases is that the nonrelativistic relation $k'_z = k'_z + q_z/2$ considerably differs from the corresponding relativistic expression that can be obtained by Eq. (34). Furthermore in the infinite momentum frame only the transverse components of the hadronic tensor survive in the Bjorken limit (see Sec. 3), while in the nonrelativistic case the component $W_{00}$ is the dominant one.

One might think that the main result of this paper is a consequence of the fact that the states with given quantum numbers $(n_x, n_y, n_z)$, as well as the states with given values of $\vec{k}$, form complete sets in $H_{int}$ if the spin variables are dropped. Since the hadronic tensor is determined by a sum over all possible final states, the sum over $(n_x, n_y, n_z)$ in Eq. (42) should give the same result as the integration over $\vec{k}$ in Eq. (20), as a consequence of the above mentioned completeness. This is not the case due to the presence of the delta function in Eq. (42). Indeed, at fixed values of $x$ and $Q$, the sum over $(n_x, n_y, n_z)$ is carried out only at some fixed values of the three indices (see Eq. (45)). Then the calculation of the average values of the structure functions (see Eqs. (57) and (59)) involves only a small part of all values of $n \equiv n_z$. Therefore formally the completeness property cannot be used. Heuristically, one can say that the eigenstates of the relativistic analog of the harmonic oscillator potential are equivalent to the free states in the relevant part of the Hilbert space. It represents an interesting topic to be investigated whether the same property holds for different confining interactions. An analogous result on the equivalence between interacting and free eigenstates has been obtained for the reaction $e^+ e^- \to hadrons$ in the model considered in [6].

Our results could be considered as an argument in favor of the ”common wisdom”, according to which the FSI in the Bjorken limit is a higher twist effect (in this connection it could be interesting to calculate the terms of order $1/Q^2$ in the structure functions and to see how they depend on the confinement radius). As noted above, our choice of the current is a possible choice, compatible with Poincaré invariance and current conservation. However, these requirements do not determine the current operator uniquely and many body components could be present in $J^\mu(0)$. Therefore one should study whether the results of the parton model can still be recovered in the Bjorken limit if the operator $J^\mu(0)$ contains many-body interaction terms. This problem will be considered elsewhere within a more refined model, including possibly a three particle system.
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Appendix A

In this Appendix it will be proved that

\[ \lim_{n \to \infty} f(n, x) = \frac{(8\pi)^{1/2}a\chi_0(x)}{m_0x^{3/2}} \]  

(A.1)

From Eqs. (13) and (56) one has

\[ f(n, x) = f_1(n, x) + f_2(n, x) + f_3(n, x) + f_4(n, x) + f_5(n, x) \]  

(A.2)

where

\[ f_1(n, x) = \frac{a\chi_0(x)}{m_0x^{3/2}(8\sqrt{\pi})^{1/2}}f(n), \]  

(A.3)

\[ f_2(n, x) = (-1)^n(2\sqrt{\pi})^{1/2} \int_{-b_1}^{b_2} \Phi_n(t) \cdot \frac{\chi_0(\xi')}{\xi'(\xi')^{1/2}(t^2 + m_0^2/a^2)^{1/4}} - \frac{2a\chi_0(x)|t|^{1/2}}{m_0x^{3/2}}dt, \]  

(A.4)

\[ f_3(n, x) = (-1)^n(2\sqrt{\pi})^{1/2} \int_{-b_1}^{b_2} \Phi_n(t) \cdot \frac{\chi_0(\xi')}{\xi'(\xi')^{1/2}(t^2 + m_0^2/a^2)^{1/4}} - \frac{2a\chi_0(x)|t|^{1/2}}{m_0x^{3/2}}dt, \]  

(A.5)

\[ f_4(n, x) = (-1)^n(2\sqrt{\pi})^{1/2} \int_0^{b_2} \Phi_n(t) \frac{\chi_0(\xi')}{\xi'(\xi')^{1/2}(t^2 + m_0^2/a^2)^{1/4}}dt, \]  

(A.6)

\[ f_5(n, x) = (-1)^n(2\sqrt{\pi})^{1/2} \int_{b_2}^{\infty} \Phi_n(t) \frac{\chi_0(\xi')}{\xi'(\xi')^{1/2}(t^2 + m_0^2/a^2)^{1/4}}dt, \]  

(A.7)

with \( f(n) \) given by

\[ f(n) = \int_0^{\infty} \Phi_n(t)t^{1/2}dt, \]  

(A.8)

and \( \Phi_n(t) \) defined in Eq. (13).

So far no approximation is made and \( b_1 \) and \( b_2 \) are arbitrary positive numbers.
In order to prove Eq. (A.11), we have to show that for any $\epsilon > 0$ there exists $N$ such that

$$|f(n, x) - \frac{(8\pi)^{1/2}a\chi_0(x)}{m_0 x^{3/2}}| < \epsilon \quad \text{for all} \quad n > N$$ \quad (A.9)

Let us first consider $f_5(n, x)$. When $t = k_z/a$ is positive and large then, as follows from Eq. (52), $\xi$ is close to 1. In turn, as follows from Eq. (48), $\xi' - \xi$ is small. Therefore for large $t$ we can replace $\xi'$ by $\xi$ and then $\chi_0(\xi')$ by $\chi_0(\xi)$, which falls off exponentially when $t \to \infty$. Let us use the property (see Sec. 10.18 of Ref. [23])

$$|\Phi_n(t)| < K$$ \quad (A.10)

where $K \approx 1.086435$. Therefore, for a given value of $\epsilon$ it is possible to find $b_2$ such that $|f_5(n, x)| < \epsilon/5$ for all $n$. Due to Eq. (A.10), the quantity $b_2$ depends only on $\epsilon$, but not on $n$.

As far as $f_4(n, x)$ is regarded, we note that, as follows from Eqs. (13), (48) and (52), there exist positive quantities $c_1(x)$ and $c_2(x)$ such that

$$c_1(x) \leq \frac{\chi_0(\xi')}{\xi'(\xi')^{1/2}(m_0^2 + k_z^2)^{1/4}} \leq c_2(x)$$ \quad (A.11)

when $t \in [0, b_2]$. Now we use the fact [23] that, uniformly in any bounded interval,

$$\Phi_n(t) = \frac{1}{(2\pi n!)^{1/2}} \frac{\Gamma(n + 1)}{(n^2/2 + 1)^{1/2}} \left[ \cos(N^{1/2}t - n \pi/2) + O(n^{-1/2}) \right]$$ \quad (A.12)

where $N = 2n + 1$. Therefore, using the Stirling formula [23], for large $n$ one has

$$\Phi_n(t) = \left( \frac{2}{\pi n} \right)^{1/4} \left[ \cos(N^{1/2}t - n \pi/2) + O(n^{-1/2}) \right].$$ \quad (A.13)

Then by the Riemann-Lebesgue theorem (see, e.g., Ref. [24]) it is possible to find $N_1$ such that $|f_4(n, x)| < \epsilon/5$ for all $n > N_1$.

Let us now consider $f_2(n, x)$. When $t$ is negative and $|t|$ is large, the quantity $\xi$ is small and $\xi^{-1/2}$ becomes $2a|t|/m_0$ (see Eq. (52)). Then, as follows from Eq. (48), $\xi' - x$ is small and for $t$ negative and $|t|$ large, one has

$$\frac{\chi_0(\xi')}{\xi'(\xi')^{1/2}(t^2 + m_0^2/a^2)^{1/4}} \to \frac{2a\chi_0(x)|t|^{1/2}}{m_0 x^{3/2}}.$$

An expansion of the difference between the above quantities as a Taylor series of $1/t^2$, shows that this difference behaves as $1/|t|^{3/2}$ for large negative values of $t$. Therefore, using again Eq. (A.10), we conclude that it is possible to find $b_1$ such that $|f_2(n, x)| < \epsilon/5$ for all $n$. The quantity $b_1$ depends only on $\epsilon$, but not on $n$.

By analogy with the previous considerations we did for $f_4(n, x)$, it is possible to find $N_2$, such that $|f_3(n, x)| < \epsilon/5$ for all $n > N_2$. 

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Finally, consider \( f_1(n, x) \). Using Eq. (7.376) in Ref. [25] and the Stirling formula, for large, odd values of \( n \) one has

\[
f(n) = 2 \frac{n}{\pi}^{1/4} (-1)^{(n-1)/2} \Gamma(5/4) F(-(n-1)/2, 5/4, 3/2, 2)
\]

(A.14)

where \( F(a, b, c, z) \) is the hypergeometric function. In the literature (see, e.g., [23]) the asymptotic behaviour of \( F(a, b, c, z) \) for \( |a| \to \infty \) is usually given in the case \( |z| < 1 \). In Appendix B we obtain the asymptotic behaviour of \( F(-m, b, c, z) \) for \( c > b > 0 \) and for \( z = 2 \). Then, using this asymptotic behaviour, one finds \( f(n) \to \pi^{1/4} \) when \( n \to \infty \).

For even values of \( n \) it is not convenient to use directly Eq. (7.376) of Ref. [25], since it involves \( F(a, b, c, z) \) with \( b > c \). However, from the relation [23]

\[
H_n(t) = \frac{1}{2t} \left[ H_{n+1}(t) + 2nH_{n-1}(t) \right]
\]

(A.15)

one has

\[
f(n) = \frac{1}{(2n!)^{1/2}} \int_0^\infty \exp\left(-\frac{t^2}{2}\right) t^{-1/2} \left[ \frac{1}{2} H_{n+1}(t) + nH_{n-1}(t) \right] dt
\]

(A.16)

Using again Eq. (7.376) of Ref. [25], for odd values of the Hermite polynomial index, and the Stirling formula Eq. (A.16) can be cast in the following form for large, even values of \( n \)

\[
f(n) = \frac{\Gamma(3/4)(-1)^{n/2}}{(\pi n)^{1/4}} \left[ (n+1)F(-n/2, 3/4; 3/2; 2) - nF(-n/2 + 1, 3/4; 3/2; 2) \right]
\]

(A.17)

Then by using the asymptotic expression for \( F(-m, b, c, z) \) given in Appendix B, also for even values of \( n \) one finds \( f(n) \to \pi^{1/4} \) when \( n \to \infty \).

Therefore it is possible to find \( N_3 \) such that

\[
|f_1(n, x) - (8\pi)^{1/2} a\chi_0(x) m_0 x^{3/2}| < \epsilon/5 \quad \text{for all} \quad n > N_3
\]

(A.18)

In conclusion, if \( N = \max\{N_1, N_2, N_3\} \), the condition (A.9) is satisfied and then Eq. (61) is proved.

**Appendix B**

In this Appendix we will investigate the asymptotic behavior of \( F(-m, b, c, 2) \) for \( m \to \infty \). In order to obtain this behavior let us use the well-known fact (see, e.g., Eqs. (9.111) and (8.834) of [25]) that, if \( c > b > 0 \), then

\[
F(-m, b, c) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{c-1}(1-t)^{c-b-1}(1-tz)^m dt
\]

(B.1)
For \( z = 2 \), due to the factor \((1 - 2t)^m\) only neighborhoods of \( t = 0 \) and \( t = 1 \) contribute to the integral when \( m \to \infty \). We can replace \((1 - t)^{c-b-1}\) by 1 and \((1 - 2t)^m\) by \((1 - t)^{2m}\) in the first neighborhood, and \( t^{b-1}\) by 1 and \((1 - 2t)^m\) by \((-1)^m t^{2m}\) in the second one. Then Eq. (B.1) becomes

\[
F(-m, b, c, 2) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \left[ \int_0^1 t^{b-1}(1 - t)^{2m} dt + \int_0^1 (1 - t)^{c-b-1}(-1)^m t^{2m} dt \right],
\]

since in the first integral there is a non-zero contribution only in the neighborhood of \( t = 0 \) and in the second integral only in the neighborhood of \( t = 1 \). Then, using the definition of the function \( B(x, y) \) (see, e.g., Eq. (8.380) of [25]) and its expression in terms of the function \( \Gamma(z) \) (see, e.g., Eq. (8.384) of [25]) one obtains

\[
F(-m, b, c, 2) = \frac{\Gamma(c)\Gamma(2m+1)}{\Gamma(c-b)(2m+1)} + \frac{\Gamma(c)\Gamma(2m+1)}{\Gamma(b)\Gamma(c-b+2m+1)} (-1)^m
\]

Finally, using the Stirling formula, one finds the following asymptotic behaviour for \( m \to \infty \)

\[
F(-m, b, c, 2) = \frac{\Gamma(c)}{\Gamma(c-b)(2m+1)} + (-1)^m \frac{\Gamma(c)}{\Gamma(b)(2m+1)(c-b)}.
\]

**Appendix C**

In this Appendix it will be proved that \( W^{++} \) is vanishing in the Bjorken limit.

Since from Eqs. (30) and (33) one has \( p^+ + p'^+ = \xi'Q(1-x)/x \) and from the delta function on the masses (see, e.g., Eq. (55)) one obtains \( Q = 2a\sqrt{2n} \sqrt{x/(1-x)} \), then apart from constant factors the relevant integral in the calculation of \( W^{++} \) is

\[
I_n = \sqrt{n} \int_{-\infty}^{\infty} \Phi_n(t) \sqrt{\xi'/(t^2 + m_0^2/a^2)^{1/4}} dt = \sqrt{n} \int_{-\infty}^{\infty} \Phi_n(t) \Psi(t) dt = \sqrt{n} \lim_{b \to \infty} \lim_{c \to \infty} \int_c^b \Phi_n(t) \Psi(t) dt
\]

with \( \Psi(t) = \sqrt{(\xi/\xi')} \chi_0(\xi')/(t^2 + m_0^2/a^2)^{1/4} \). Let us investigate the limit

\[
\lim_{n \to \infty} \sqrt{n} \int_{-c}^b \Phi_n(t) \Psi(t) dt
\]
We will show that this limit is uniform with respect to the extrema. First of all, let us observe that the function $\Psi(t)$ is i) continuous, ii) bounded and iii) $\in L_1$. Indeed, it falls exponentially for $t \to \infty$ and as $|t|^{-3/2}$ for $t \to -\infty$ (as it can be argued from the discussion in Appendix A). Furthermore also the derivative of $\Psi(t) \in L_1$, due to the behavior of $\Psi(t)$ for $t \to \pm\infty$. Since we have a bounded interval in (C.2), we can use the asymptotic expression (A.13) for $\Phi_n(t)$

$$\Phi_n(t) = \left(\frac{2}{\pi n}\right)^{1/4}[\cos(N^{1/2}t - \frac{n\pi}{2}) + O(n^{-1/2})]$$

(C.3)

with $N = 2n + 1$. The second term in Eq. (C.3) gives an integral in Eq. (C.2) that uniformly vanishes as $1/n^{1/4}$ with respect to the integration extrema, due to the property iii). Also the first term in Eq. (C.3) produces an integral in Eq. (C.2) that uniformly vanishes as $1/n^{1/4}$ with respect to the integration extrema. This can be shown with an integration by parts and exploiting the property ii) and the integrability of $|d\Psi(t)/dt|$.

Therefore the limit in Eq. (C.2) is zero and it is uniform with respect to the integration extrema. Then one has

$$\lim_{b \to \infty} \lim_{c \to \infty} \lim_{n \to \infty} \sqrt{n} \int_{-c}^{b} \Phi_n(t)\Psi(t)dt =$$

$$\lim_{n \to \infty} \lim_{b \to \infty} \lim_{c \to \infty} \sqrt{n} \int_{-c}^{b} \Phi_n(t)\Psi(t)dt$$

(C.4)

and finally

$$\lim_{n \to \infty} I_n = 0$$

(C.5)

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