SPECTRAL GAPS OF THE HILL–SCHRÖDINGER OPERATORS WITH DISTRIBUTIONAL POTENTIALS

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Dedicated to Professor V. M. Adamyan on the occasion of his 75 birthday

Abstract. The paper studies the Hill–Schrödinger operators with potentials in the space \( H^\omega \subset H^{-1}(\mathbb{T},\mathbb{R}) \). The main results completely describe the sequences arising as the lengths of spectral gaps of these operators. The space \( H^\omega \) coincides with the Hörmander space \( H^T_\omega (\mathbb{T},\mathbb{R}) \) with the weight function \( \omega(\sqrt{1+\xi^2}) \) if \( \omega \) belongs to Avakumovich’s class \( \mathcal{O}R \). In particular, if the functions \( \omega \) are power, then these spaces coincide with the Sobolev spaces. The functions \( \omega \) may be nonmonotonic.

1. Introduction

Let us consider the Hill–Schrödinger operator

\[
S(q)u := -u'' + q(x)u, \quad x \in \mathbb{R},
\]

with 1-periodic real-valued potential

\[
q(x) = \sum_{k \in \mathbb{Z}} \hat{q}(k)e^{ik2\pi x} \in L^2(\mathbb{T},\mathbb{R}), \quad \mathbb{T} := \mathbb{R}/\mathbb{Z}.
\]

This condition means that

\[
\sum_{k \in \mathbb{Z}} |\hat{q}(k)|^2 < \infty \quad \text{and} \quad \hat{q}(k) = \overline{\hat{q}(-k)}, \quad k \in \mathbb{Z}.
\]

It is well known that the operator \( S(q) \) is lower semibounded and self-adjoint in the Hilbert space \( L^2(\mathbb{R}) \). Its spectrum is absolutely continuous and has a zone structure [22].

Spectrum of the operator \( S(q) \) is completely defined by the location of the endpoints of spectral gaps \( \{\lambda_0(q), \lambda_n(q)\}_{n=1}^\infty \), which satisfy the inequalities

\[
-\infty < \lambda_0(q) < \lambda_1(q) \leq \lambda_2(q) \leq \lambda_3(q) \leq \cdots.
\]

Some gaps may be degenerate, then the corresponding bands merge. For even/odd numbers \( n \in \mathbb{Z}_+ \), the endpoints of spectral gaps \( \{\lambda_0(q), \lambda_n^\pm(q)\}_{n=1}^\infty \) are eigenvalues of the periodic/semiperiodic problems on the interval \((0,1)\).

The interiors of spectral bands (the stability zones)

\[
\mathcal{B}_0(q) := (\lambda_0(q), \lambda_1^-(q)), \quad \mathcal{B}_n(q) := (\lambda_n^-\ast(q), \lambda_{n+1}^-\ast(q)), \quad n \in \mathbb{N},
\]

together with the collapsed gaps,

\[
\lambda = \lambda_{n}^--\lambda_{n}^+,
\]

are characterized as the set of those \( \lambda \in \mathbb{R} \), for which all solutions of the equation

\[
- u'' + q(x)u = \lambda u, \quad x \in \mathbb{R},
\]

are bounded on \( \mathbb{R} \). The open spectral gaps (instability zones)

\[
\mathcal{G}_0(q) := (-\infty, \lambda_0(q)), \quad \mathcal{G}_n(q) := (\lambda_n^-(q), \lambda_n^+\ast(q)) \neq \emptyset, \quad n \in \mathbb{N},
\]

form a set of those \( \lambda \in \mathbb{R} \) for which any nontrivial solution of the equation \( (3) \) is unbounded on \( \mathbb{R} \).

We study the behaviour of the lengths of spectral gaps,

\[
\gamma_q(n) := \lambda_1^-(q) - \lambda_n^-(q), \quad n \in \mathbb{N},
\]

of the operator \( S(q) \) in terms of behaviour of the Fourier coefficients \( \{\hat{q}(n)\}_{n\in\mathbb{N}} \) of the potential \( q \) with respect to test sequence spaces, that is in terms of potential regularity.

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Hochstadt [6, 7], Marchenko and Ostrovskii [14], McKean and Trubowitz [12, 23] proved that the potential $q$ is an infinitely differentiable function if and only if the lengths of spectral gaps $\{\gamma_q(n)\}_{n=1}^{\infty}$ decrease faster than an arbitrary power of $1/n$:

$$q \in C^\infty(T) \Leftrightarrow \gamma_q(n) = O(n^{-k}), \quad n \to \infty, \quad k \in \mathbb{Z}_+.$$  

However, the scale of spaces $\{C^k(T)\}_{k \in \mathbb{N}}$ turned out unsuitable to obtain precise quantitative results. Marchenko and Ostrovskii [14] (see also [13, 11]) found that

$$q \in H^s(T) \Leftrightarrow \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^s(\mathbb{N}), \quad s \in \mathbb{Z}_+.$$  

The Sobolev spaces $H^s(T)$, $s \in \mathbb{R}$, of 1-periodic functions/generalized functions may also be defined by means of their Fourier coefficients

$$H^s(T) = \left\{ f = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ik2\pi x} \in \mathcal{D}(T) \mid \|f\|_{H^s(T)}^2 := \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} |\hat{f}(k)|^2 < \infty \right\}.$$  

Here by $\mathcal{D}(T)$ we denote the space of 1-periodic generalized functions on $T$.

We define the weighted sequence spaces $h^s(\mathbb{N})$, $s \in \mathbb{R}$, in the following way:

$$h^s(\mathbb{N}) := \left\{ a = \{a(k)\}_{k \in \mathbb{N}} \mid \|a\|_{h^s(\mathbb{N})}^2 := \sum_{k \in \mathbb{N}} (1 + |k|)^{2s} |a(k)|^2 < \infty \right\}.$$  

Marchenko–Ostrovskii theorem [41] can be extended to a more general scale of Hörmander spaces $\{H^\omega(T)\}_\omega$ [11, 12, 21], where $\omega = \{\omega(k)\}_{k \in \mathbb{Z}}$ is a weighted sequence. Recall that a sequence $a = \{a(k)\}_{k \in \mathbb{Z}}$ is called a weight or weighted sequence if it is positive and even, i.e., $a(k) \geq 0$ and $a(-k) = a(k)$ for $k \in \mathbb{Z}_+$.

However, complete description of the sequences that form lengths of the gaps with potentials from the given functional class, in particular the Hörmander space or the Sobolev space, remained an open question. This paper deals with this issue in more general situation of distributional potentials.

2. Main results

Let us start with necessary notations. The spaces $H^\omega(T)$ and $h^\omega(\mathbb{N})$ are defined similarly to the spaces $H^s(T)$ and $h^s(\mathbb{N})$:

$$H^\omega(T) := \left\{ f = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ik2\pi x} \in \mathcal{D}(T) \mid \|f\|_{H^\omega(T)}^2 := \sum_{k \in \mathbb{Z}} \omega^2(k)|\hat{f}(k)|^2 < \infty \right\},$$  

$$h^\omega(\mathbb{N}) := \left\{ a = \{a(k)\}_{k \in \mathbb{N}} \mid \|a\|_{h^\omega(\mathbb{N})}^2 := \sum_{k \in \mathbb{N}} \omega^2(k)|a(k)|^2 < \infty \right\}.$$  

We say that the weighted sequence $\omega = \{\omega(k)\}_{k \in \mathbb{Z}}$ belongs to the class $I_0$, if it satisfies the following condition:

$$|k|^s \ll \omega(k) \ll |k|^{1+s}, \quad s \in [0, \infty).$$  

The notation

$$b(k) \ll a(k) \ll c(k), \quad k \in \mathbb{N},$$  

means that there are positive constants $C_1$ and $C_2$ such that the following inequalities hold:

$$C_1 b(k) \leq a(k) \leq C_2 c(k), \quad k \in \mathbb{N}.$$  

We say that the weighted sequence $\omega = \{\omega(k)\}_{k \in \mathbb{Z}}$ belongs to the class $M_0$, if it satisfies the following conditions:

(i) $\omega(k) \uparrow \infty$, $k \in \mathbb{N}$; (monotonicity)

(ii) $\omega(k + m) \leq \omega(k)\omega(m), \quad k, m \in \mathbb{N}$; (submultiplicity)

(iii) $\frac{\log \omega(k)}{k} \downarrow 0, \quad k \to \infty$, (subexponentiality).

Suppose that a weighted sequence $\omega = \{\omega(k)\}_{k \in \mathbb{Z}}$ belongs either to class $I_0$ or to the class $M_0$. Then

$$q \in H^\omega(T) \Leftrightarrow \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^\omega(\mathbb{N}).$$  

The statement (5) for the case $\omega \in I_0$ was proved by the authors [18], and the case $M_0$ was closely studied in [11, 21].
The statement \[13\] may be strengthened. It is well-known that the sequence of lengths of spectral gaps \(\{\gamma_q(n)\}_{n \in \mathbb{N}}\) of the Hill–Schrödinger operator \(S(q)\) with an \(L^2(\mathbb{T})\)-potential \(q\) belongs to the space 

\[h^0_+(\mathbb{N}) := \{a = \{a(k)\}_{k \in \mathbb{N}} \in L^2(\mathbb{N}) \mid a(k) \geq 0, \ k \in \mathbb{N}\}.\]

Let us consider the map 

\[\gamma : L^2(\mathbb{T}) \ni q \mapsto \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^0_+(\mathbb{N}).\]

Then due to Garnett and Trubowitz \[4, 5, 10\], 

\[\gamma (L^2(\mathbb{T})) = h^0_+(\mathbb{N}).\]

We introduce the following notations: 

\[h^\omega_+(\mathbb{N}) := \{a = \{a(k)\}_{k \in \mathbb{N}} \in h^\omega(\mathbb{N}) \mid a(k) \geq 0, \ k \in \mathbb{N}\}\]

**Theorem 1.** Suppose that \(q \in L^2(\mathbb{T})\) and that either \(\omega \in \mathbb{I}_0\) or \(\omega \in \mathbb{M}_0\). Then the map 

\[\gamma : L^2(\mathbb{T}) \ni q \mapsto \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^\omega_+(\mathbb{N})\]

satisfies the relations 

\[(i) \quad \gamma (H^\omega(\mathbb{T})) = h^\omega_+(\mathbb{N}),\]

\[(ii) \quad \gamma^{-1}(h^\omega_+(\mathbb{N})) = H^\omega(\mathbb{T}).\]

Now let us consider the Hill–Schrödinger operator \(S(q)\) with a 1-periodic real-valued distribution potential \(q\) that belongs to the negative Sobolev space:

\[q = \sum_{k \in \mathbb{Z}} \hat{q}(k)e^{ik2\pi x} \in H^{-1}(\mathbb{T}).\]

All real-valued pseudo-functions, measures, pseudo-measures and some more singular distributions on the circle satisfy this condition. For more detailed discussion of operators with strongly singular potentials see \[8\] and references therein.

Under the assumption \[6\] the operator \(14\) may be well defined in the complex Hilbert space \(L^2(\mathbb{R})\) in the following basic ways:

- as form-sum operator;
- as quasi-differential operators;
- as limit of operators with smooth 1-periodic potentials in the norm resolvent sense.

Equivalence of all these definitions was proved in the paper \[15\], more general case was treated in \[19\].

The Hill–Schrödinger operator \(S(q)\) with strongly singular potential \(q\) is lower semibounded and self-adjoint, its spectrum is absolutely continuous and has a band and gap structure as in the classical case \[9, 10, 15, 3, 20\].

The endpoints of spectral gaps satisfy the inequalities \[12\]. For even/odd numbers \(n \in \mathbb{Z}_+\) they are eigenvalues of the periodic/semiperiodic problems on the interval \([0, 1]\) \[15, Theorem C\].

We say that the weighted sequence \(\omega = \{\omega(k)\}_{k \in \mathbb{Z}}\) belongs to \(I_{-1}\), if it satisfies the following conditions:

\[(i) \quad \omega(k) = (1 + |k|)^{-1}, \quad s = 1,\]

\[(ii) \quad |k|^s \ll \omega(k) \ll |k|^{1+2s-\delta} \quad \forall \delta > 0, \quad s \in (-1, 0),\]

\[(iii) \quad |k|^s \ll \omega(k) \ll |k|^{1+s}, \quad s \in [0, \infty).\]

We say that the weighted sequence \(\omega = \{\omega(k)\}_{k \in \mathbb{Z}}\) belongs to \(M_{-1}\), if it can be represented as:

\[\omega(k) = \frac{\omega^*(k)}{1 + |k|}, \quad k \in \mathbb{Z}, \quad \omega^* = \{\omega^*(k)\}_{k \in \mathbb{Z}} \in \mathbb{M}_0.\]

Suppose that a weighted sequence \(\omega = \{\omega(k)\}_{k \in \mathbb{Z}}\) belongs either to the class \(I_{-1}\), or to the class \(M_{-1}\), then

\[q \in H^\omega(\mathbb{T}) \Rightarrow \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^\omega(\mathbb{N}).\]

The statement \[14\] for the case \(\omega \in I_{-1}\) is proved below (in a weaker form, this assertion was proved earlier by the authors \[17\]), also for the case \(\omega \in M_{-1}\) the statement \[17\] was proved in \[2\]. Note that \(I_0\) and \(M_0\), as well as \(I_{-1}\) and \(M_{-1}\), intersect, but do not cover each other.

Let us consider the map \(\gamma : q \mapsto \{\gamma_q(n)\}_{n \in \mathbb{N}}\). Then, according to Korotyaev \[10, Theorem 1.1\], map \(\gamma\) maps \(H^{-1}(\mathbb{T})\) onto \(h^\omega_+(\mathbb{N})\):

\[\gamma(H^{-1}(\mathbb{T})) = h^\omega_+(\mathbb{N}).\]
Theorem 2. Suppose that \( q \in H^{-1}(\mathbb{T}) \) and that either \( \omega \in I_{-1} \) or \( \omega \in M_{-1} \). Then the map
\[
\gamma : H^{-1}(\mathbb{T}) \ni q \mapsto \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^1_+(\mathbb{N})
\]
satisfies following equalities:
\[
\begin{align*}
(1) & \quad \gamma(H^s(\mathbb{T})) = h^s_+(\mathbb{N}), \\
(2) & \quad \gamma^{-1}(h^s_+(\mathbb{N})) = H^s(\mathbb{T}).
\end{align*}
\]

3. The Proofs

Proof of Theorem 1. Due to Garnett and Trubowitz it occurs that for any sequence \( \{\gamma(n)\}_{n \in \mathbb{N}} \in h^0_+(\mathbb{N}) \) we can place the open intervals \( I_n \) of the lengths \( \gamma(n) \) (to the length 0 corresponds point) on the positive semi-axis \((0, \infty)\) in a such single way that there exists a potential \( q \in L^2(\mathbb{T}) \) for which the sequence \( \{\gamma(n)\}_{n \in \mathbb{N}} \) is a sequence of the lengths of spectral gaps of the Hill–Schrödinger operator \( S(q) \), i.e., the map \( \gamma \) maps the space \( L^2(\mathbb{T}) \) onto the sequence space \( h^0_+(\mathbb{N}) \):
\[
\gamma(L^2(\mathbb{T})) = h^0_+(\mathbb{N}).
\]
And, as a consequence, we also have
\[
\gamma^{-1}(h^0_+(\mathbb{N})) = L^2(\mathbb{T}).
\]

The case \( \omega \in I_0 \) was investigated by the authors in [18].

Let \( \omega \in M_0 \). From statement (5) we get
\[
\gamma(H^s(\mathbb{T})) \subset h^s_+(\mathbb{N}).
\]
To establish the equality (i) of Theorem 1 it is necessary to prove the inverse inclusion in formula (11). So, let \( \{\gamma(n)\}_{n \in \mathbb{N}} \) be an arbitrary sequence in the space \( h^s_+(\mathbb{N}) \). Then \( \{\gamma(n)\}_{n \in \mathbb{N}} \in h^s_+(\mathbb{N}) \). Due to (9) a potential \( q \in L^2(\mathbb{T}) \) exists, such that the sequence \( \{\gamma(n)\}_{n \in \mathbb{N}} \in h^s_+(\mathbb{N}) \) is sequence of the lengths of spectral gaps. Since by assumption \( \{\gamma(n)\}_{n \in \mathbb{N}} \in h^s_+(\mathbb{N}) \) due to (5) we conclude that \( q \in H^s(\mathbb{T}) \) and as a consequence \( \{\gamma(n)\}_{n \in \mathbb{N}} \in \gamma(H^s(\mathbb{T})) \). Therefore the inclusion
\[
\gamma(H^s(\mathbb{T})) \supset h^s_+(\mathbb{N})
\]
holds.

Inclusions (11) and (12) give the equality (i).

Now, let us prove the equality (ii) of Theorem 1. Let \( q \) be an arbitrary function in the space \( H^s(\mathbb{T}) \). Then, due to statement (5), we have \( \gamma_q = \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^s_+(\mathbb{N}) \) and as a consequence \( q \in \gamma^{-1}(h^s_+(\mathbb{N})) \). Therefore
\[
\gamma^{-1}(h^s_+(\mathbb{N})) \supset H^s(\mathbb{T}).
\]

Conversely, let \( \{\gamma(n)\}_{n \in \mathbb{N}} \) be an arbitrary sequence from the space \( h^s_+(\mathbb{N}) \). Then due to (10) we have \( \gamma^{-1}(\{\gamma(n)\}_{n \in \mathbb{N}}) \subset L^2(\mathbb{T}) \). Taking into account (5) we conclude that \( \gamma^{-1}(\{\gamma(n)\}_{n \in \mathbb{N}}) \subset H^s(\mathbb{T}) \), that is
\[
\gamma^{-1}(h^s_+(\mathbb{N})) \subset H^s(\mathbb{T}).
\]

Inclusions (13) and (14) give the equality (ii) of Theorem 1.

The proof of Theorem 1 is complete.

Proof of Formula (7). Notice that for the case \( \omega(k) = (1 + |k|)^s \), \( s \in [-1, \infty) \), the relation (7) has the form
\[
q \in H^s(\mathbb{T}) \iff \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^s(\mathbb{N}), \quad s \in [-1, \infty).
\]

The limiting case \( s = -1 \) was treated by Korotyaev [10]. The proof of statement (15) was completed in [2]. Earlier [18] was established by the authors [16] under a stronger assumption \( q \in H^{-1+}(-\mathbb{T}) \) and \( s > -1 \).

Furthermore, if \( q \in H^s(\mathbb{T}), \ s \in [-1, \infty) \), then for the lengths of spectral gaps the following asymptotic formula hold [10, 16]:
\[
\begin{align*}
(16) & \quad \gamma_q(n) = 2|\hat{q}(n)| + h^{-1}(n) \quad \text{if} \quad s = -1, \\
(17) & \quad \gamma_q(n) = 2|\hat{q}(n)| + h^{1+2s-\delta}(n) \quad \forall \delta > 0 \quad \text{if} \quad s \in (-1, 0), \\
(18) & \quad \gamma_q(n) = 2|\hat{q}(n)| + h^{1+s}(n) \quad \text{if} \quad s \in [0, \infty).
\end{align*}
\]

Let us also recall that if \( \omega_1 \gg \omega_2 \), i.e., \( \omega_1(k) \gg \omega_2(k) \), \( k \in \mathbb{Z} \), then
\[
H^{\omega_1}(\mathbb{T}) \hookrightarrow H^{\omega_2}(\mathbb{T}), \quad h^{\omega_1}(\mathbb{N}) \hookrightarrow h^{\omega_2}(\mathbb{N}).
\]
Let \( q \in H^s(\mathbb{T}) \) and \( \omega \in \mathbb{I}_1 \), then taking into account \([19]\) we have \( q \in H^s(\mathbb{T}) \), \( s \in [-1, \infty) \), as \( \omega(k) \gg |k|^s \). Taking into account that \( \omega \in \mathbb{I}_1 \), together with \([13]\), from \([16] - [18]\) we get:

\[
\gamma_q(n) = 2|\bar{q}(n)| + h_\omega(n),
\]
i. e., \( \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h_\omega(\mathbb{N}) \).

Now, let \( \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h_\omega(\mathbb{N}) \), then due to \([19]\) we have \( \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h_\omega^s(\mathbb{N}) \), \( s \in [-1, \infty) \), and, as consequence, from \([15]\) we get \( q \in H^s(\mathbb{T}) \), \( s \in [-1, \infty) \), and the asymptotics \([16] - [18]\) hold. Taking into account that \( \omega \in \mathbb{I}_1 \) and \([19]\) we have:

\[
\gamma_q(n) = 2|\bar{q}(n)| + h_\omega(n),
\]

from where we get necessary result \( q \in H^\omega(\mathbb{T}) \).

The statement \([7]\) for the case \( \omega \in \mathbb{I}_1 \) is completely proved. \(\square\)

**Proof of Theorem 2.** From statement \([7]\) we get

\[
\gamma(H^\omega(\mathbb{T})) \subset h_\omega^s(\mathbb{N}).
\]

To establish the equality (i) of Theorem 2 it is necessary to prove the inverse inclusion in formula \([20]\). So, let \( \{\gamma(n)\}_{n \in \mathbb{N}} \) be an arbitrary sequence in the space \( h_\omega^s(\mathbb{N}) \). Then \( \{\gamma(n)\}_{n \in \mathbb{N}} \in h^s_\omega(\mathbb{N}) \). Due to \([8]\) a potential \( q \in H^{-1}(\mathbb{T}) \) exists, such that the sequence \( \{\gamma(n)\}_{n \in \mathbb{N}} \in h^{-1}_\omega(\mathbb{N}) \) is its sequence of the lengths of spectral gaps. Since by assumption \( \{\gamma(n)\}_{n \in \mathbb{N}} \in h_\omega^s(\mathbb{N}) \) due to \([7]\) we conclude that \( q \in H^\omega(\mathbb{T}) \) and as consequence \( \{\gamma(n)\}_{n \in \mathbb{N}} \in \gamma(H^\omega(\mathbb{T})) \). Therefore the inclusion

\[
\gamma(H^\omega(\mathbb{T})) \supset h_\omega^s(\mathbb{N})
\]

holds.

Inclusions \([20]\) and \([21]\) give the equality (i).

Now, let us prove the equality (ii) of Theorem 2. Let \( q \) be an arbitrary function in the space \( H^\omega(\mathbb{T}) \). Then, due to statement \([7]\), we have \( \gamma_q = \{\gamma_q(n)\}_{n \in \mathbb{N}} \in h^s_\omega(\mathbb{N}) \), i. e., \( q \in \gamma^{-1}(h_\omega^s(\mathbb{N})) \). Therefore

\[
\gamma^{-1}(h^s_\omega(\mathbb{N})) \supset H^\omega(\mathbb{T}).
\]

Conversely, let \( \{\gamma(n)\}_{n \in \mathbb{N}} \) be an arbitrary sequence from the space \( h^s_\omega(\mathbb{N}) \). Then due to \([8]\) we have \( \gamma^{-1}(h^s_\omega(\mathbb{N})) = H^{-1}(\mathbb{T}) \), and therefore \( \gamma^{-1}(\{\gamma(n)\}_{n \in \mathbb{N}}) \subset H^{-1}(\mathbb{T}) \). Taking into account \([7]\) we conclude that \( \gamma^{-1}(\{\gamma(n)\}_{n \in \mathbb{N}}) \subset H^\omega(\mathbb{T}) \), that is

\[
\gamma^{-1}(h^s_\omega(\mathbb{N})) \subset H^\omega(\mathbb{T}).
\]

Inclusions \([22]\) and \([23]\) give the equality (ii) of Theorem 2.

The proof of Theorem 2 is complete. \(\square\)

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