Effective field theory of a topological insulator and the Foldy-Wouthuysen transformation

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Abstract

Employing the Foldy-Wouthuysen transformation it is demonstrated straightforwardly that the first and second Chern numbers are equal to the coefficients of the 2+1 and 4+1 dimensional Chern-Simons actions which are generated by the massive Dirac fermions coupled to the Abelian gauge fields. A topological insulator model in 2+1 dimensions is discussed and by means of a dimensional reduction approach the 1+1 dimensional descendant of the 2+1 dimensional Chern-Simons theory is presented. Field strength of the Berry gauge field corresponding to the 4+1 dimensional Dirac theory is explicitly derived through the Foldy-Wouthuysen transformation. Acquainted with it the second Chern numbers are calculated for specific choices of the integration domain. A method is proposed to obtain 3+1 and 2+1 dimensional descendants of the effective field theory of the 4+1 dimensional time reversal invariant topological insulator theory. Inspired by the spin Hall effect in graphene, a hypothetical model of the time reversal invariant spin Hall insulator in 3+1 dimensions is proposed.

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1 Introduction

2 + 1 dimensional Dirac theory which was supposed to be physically unrealizable turned out to be indispensable to reveal the main features of graphene. At the Dirac points, in the low energy and long wavelength limit, the electrons of graphene effectively satisfy the 2 + 1 dimensional Dirac equation where the velocity of light \( c \) is substituted by the effective velocity \( v_F \) \([1, 2]\). This had enormous impact on condensed matter physics. Based on the 2 + 1 dimensional Dirac Hamiltonian Haldane\([3]\) discovered that it is possible to construct a model where an integer quantum Hall effect results without an external magnetic field. In spite of the fact that a magnetic field is not needed, it is a time reversal breaking (TRB) theory. Kane and Mele\([4]\) incorporated the spin degrees of freedom into the Haldane’s construction and formulated the time reversal invariant (TRI) spin Hall effect in graphene. This model\([4]\) paved the way to the theoretical prediction of the topological insulator phase in real materials\([5]\) which was observed for the first time in \([6]\). Topological insulators are usually defined to be ordinary insulators in the bulk which possess conducting states moving at the edge or boundary surface\([7, 8, 9]\). They can be classified by a new topological invariant called \( Z_2 \)\([10]\).

Topological invariants emerge already in the quantum Hall effect. The Hall conductivity is given by the first Chern number\([11, 12]\). Moreover, in 2 + 1 dimensions a topological gauge theory is generated by integrating out the massive Dirac fermion fields coupled to Abelian gauge fields in the related path integral\([13, 14, 15]\). It is described by the 2 + 1 dimensional Chern-Simons action whose coefficient is the winding number of the noninteracting massive fermion propagator which is equal to the first Chern number resulting from the Berry gauge field\([16, 17]\) of the quantum Hall states. Thus the Hall current can be acquired from a topological field theory which manifestly violates time reversal symmetry\([18, 19]\). One can also derive the TRI spin Hall current of the model introduced in \([4]\) by calculating the related first Chern numbers\([20]\).

The 4 + 1 dimensional Chern-Simons action which can be generated by the massive fermions coupled to Abelian gauge fields, is manifestly TRI and Qi-Hughes-Zhang\([21]\) designated it as the effective topological field theory of the fundamental TRI topological insulator in 4 + 1 dimensions. They demonstrated that for the band insulators which can be deformed adiabatically to a flat band model, the coefficient of the effective action is equal to the second Chern number given by the related non-Abelian Berry vector fields. We will prove the equivalence of the coefficients of the induced Chern-Simons actions with the Chern numbers in a straightforward manner by employing the Foldy-Wouthuysen transformation. Qi-Hughes-Zhang constructed the 3+1 and 2+1 dimensional descendant theories by dimensional reduction from the 4+1 dimensional action of the massive Dirac fields coupled to external gauge fields. We mainly follow the approach of \([21]\), though we only deal with the continuous Dirac theory and propose a slightly modified method to introduce the descendant theories which permits us to derive explicitly the related physical objects like polarizations. Moreover, we posit a hypothetical model of TRI spin Hall effect in 3 + 1 dimensions which may be useful to understand some aspects of physically realizable three dimensional topological insulators\([22, 23, 24]\).

In the next section, we introduce the Berry gauge fields corresponding to Dirac fermions through the Foldy-Wouthuysen transformation. Section 3 is devoted to demonstrate in a straightforward fashion that the coefficients of the the Chern-Simons actions generated by integrating out the massive Dirac spinor fields which are coupled to the Abelian gauge fields in the path integral are equal to the first and second Chern numbers given, respectively, by the Berry gauge fields in 2 + 1 and 4 + 1 dimensions. We consider the 2 + 1 dimensional topological field theory of the integer quantum Hall effect in Section 4 and recall how to construct the TRI spin quantum Hall effect in graphene which is a model of 2 + 1 dimensional topological insulator. Then, we present the dimensional reduction to
1 + 1 dimensions by obtaining the one dimensional charge polarization explicitly. In Section 5, we consider the 4+1 dimensional Chern-Simons field theory which was shown to describe the fundamental topological insulator. We derive the field strengths of the related Berry gauge fields to obtain the second Chern number and study dimensional reduction to 3+1 dimensions. By imitating the approach of [4] we theorized a hypothetical model in 4 + 1 dimensions which yields a TRI spin Hall current in 3 + 1 dimensions by means of the dimensional reduction. Slightly modifying the approach of [21] we proposed a dimensional reduction procedure to 2 + 1 dimensions which provides explicit forms of the gauge field components which take part in the descendant action. In the last section we discussed the results obtained.

2 The Foldy-Wouthuysen transformation and the Berry gauge field

We consider relativistic electrons of charge $e > 0$ with a characteristic velocity like the velocity of light $c$ or the effective velocity $v_F$ as in graphene. To retain the formulation general we work in units $\hbar = c = v_F = 1$, as well as $e = 1$ and recuperate them when needed. Thus, the free, massive electrons are described by the Dirac Hamiltonian

$$H = \alpha \cdot k + \beta m.$$  

In this section, vectors are $d$-dimensional, like the momentum $k$ whose components are denoted $k_I; I = 1, \cdots , d$. The Hamiltonian (2.1) can be diagonalized as

$$UHU^\dagger = E\beta,$$  

where $E$ is the total energy

$$E = \sqrt{k^2 + m^2},$$  

and $U$ is the unitary Foldy-Wouthuysen transformation

$$U = \frac{\beta H + E}{\sqrt{2E(E + m)}}.$$  

Through the transformation $U$ a pure gauge field can be introduced as

$$\mathcal{A}^U = iU(k) \frac{\partial U^\dagger(k)}{\partial k}.\tag{2.4}$$

The Berry gauge field $\mathcal{A}$ follows by projecting (2.4) onto the positive energy eigenstates of the Dirac Hamiltonian (2.1). One can be convinced that eliminating the negative energy states is equivalent to the adiabatic approximation by revoking its similarity to suppression of the interband interactions in molecular problems[25]. Thus, we define the Berry gauge field as

$$\mathcal{A} \equiv I_+ \mathcal{A}^U I_+,$$  

where $I_+$ is the projection operator onto the positive energy subspace. This definition of the Berry gauge field is valid irrespective of the dimensions of the Hamiltonian (2.1). In order to derive $\mathcal{A}$ explicitly let us adopt the following $2^N \times 2^N; N = [\frac{d}{2}]$, dimensional realizations of $\alpha$ and $\beta$

$$\alpha = \begin{pmatrix} 0 & \rho \\ \rho^\dagger & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

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Here \( \rho \) and the unit matrix 1 are \( 2^{N-1} \times 2^{N-1} \) dimensional. In representation (2.6) the gauge field (2.4) becomes
\[
\mathcal{A}_I^U = \frac{i}{2E^2(E + m)} \left[ E(E + m)\alpha_I\beta + \beta \alpha \cdot k k_I - iE\sigma_I k_J \right],
\]
(2.7)
where \( \sigma_{IJ} \equiv -\frac{i}{2}[\alpha_I, \alpha_J] \). Therefore, the Berry gauge field (2.5) results to be
\[
\mathcal{A}_I = -\frac{i}{4E(E + m)}(\rho_I \rho_J^\dagger - \rho_J \rho_I^\dagger) k_J.
\]
(2.8)
Although the field strength of (2.7) vanishes because of being a pure gauge field, the Berry curvature
\[
\mathcal{F}_{IJ} = \partial_I \mathcal{A}_J - \partial_J \mathcal{A}_I - i[\mathcal{A}_I, \mathcal{A}_J],
\]
(2.9)
is non-vanishing in general.

When we deal with \( 2n + 1 \) dimensional space-time coordinates where \( n = 1, 2 \cdots \), the Berry curvature (2.9) can be employed to define the Chern number which is the integrated Chern character, as[26]
\[
N_n = \frac{1}{(4\pi)^n n!} \int_{\mathcal{M}_{2n}} d^{2n} k \epsilon_{I_1 I_2 \cdots I_{2n}} \text{tr} \left\{ \mathcal{F}_{I_1 I_2} \cdots \mathcal{F}_{I_{2n-1} I_{2n}} \right\}.
\]
(2.10)
For \( 2 + 1 \) dimensional systems the Berry gauge field is Abelian, so that \( \mathcal{F}_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a \), where \( a, b = 1, 2 \), and the first Chern number is
\[
N_1 = \frac{1}{4\pi} \int d^2 k \epsilon_{ab} \text{tr} \mathcal{F}_{ab}.
\]
(2.11)
In 4 + 1 dimensions one introduces the second Chern number as
\[
N_2 = \frac{1}{32\pi^2} \int d^4 k \epsilon_{ijkl} \text{tr} \{ \mathcal{F}_{ij} \mathcal{F}_{kl} \},
\]
(2.12)
where \( i, j, k, l = 1, 2, 3, 4 \).

3 Topological field theories and the Chern numbers

Field theory of electrons interacting with the external Abelian gauge field \( A_\alpha \) is given by the Dirac Lagrangian density
\[
\mathcal{L}(\psi, \bar{\psi}, A) = \bar{\psi} \left[ i \gamma^\alpha (p_\alpha + A_\alpha) - m \right] \psi,
\]
(3.1)
where \( \alpha = 0, 1 \cdots d \). By integrating out the fermionic degrees of freedom in the related path integral one formally gets the action of the external fields as
\[
S[A] = -i \ln \det[i\gamma^\alpha (\partial_\alpha - iA_\alpha) - m].
\]
(3.2)
For \( d = 2n \) one of the terms which it gives rise to is
\[
T[A^{n+1}] = \int [dq_1] \cdots [dq_{n+1}] A^{\alpha_1}(q_1) \cdots A^{\alpha_{n+1}}(q_{n+1}) \pi_{\alpha_1 \cdots \alpha_{n+1}}(q_1 \cdots q_{n+1}).
\]
\[ dq \] denotes the integral over the related phase space. At the order of first loop

\[ \pi_{\alpha_1 \cdots \alpha_{n+1}}(q_1 \cdots q_{n+1}) = \int \frac{d^{2n+1}k}{(2\pi)^{2n+1}} \text{tr}\{G(k)\lambda_{\alpha_1}(k, k - q_1)G(k - q_1) \cdots \lambda_{\alpha_{n+1}}(k + q_{n+1}, k)\}, \]

where \( G(k) \) is the one particle Green function of the free Dirac field and \( \lambda_{\alpha} \) is the photon vertex.

\[ T[A^{n+1}] \] generates the \( 2n + 1 \) dimensional Chern-Simons term

\[ S_{\text{eff}}^{2n+1}[A] = C_n \int d^{2n+1}x \epsilon_{\alpha_1 \cdots \alpha_{2n+1}} A_{\alpha_1} \partial_{\alpha_2} A_{\alpha_3} \cdots \partial_{\alpha_{2n}} A_{\alpha_{2n+1}}, \]

which can be taken as the effective topological action in the low energy limit. In the weak field approximation the coefficient \( C_n \) can be written as [27]

\[ C_n = \frac{(-i)^n \epsilon_{\alpha_1 \cdots \alpha_{n+1}}}{(n+1)(2n+1)!} \partial_{(1)}^\alpha \cdots \partial_{(n)}^\alpha \pi_{\alpha_1 \cdots \alpha_{n+1}}(q_1 \cdots q_{n+1})|_{q_i=0}, \]

where \( \partial_{(n)}^\alpha \equiv \partial/\partial q_n^\alpha \). Substituting the photon vertex with

\[ \lambda_{\alpha}(k, k) = -i\partial_{\alpha} G^{-1}(k), \]

where \( G^{-1}(k) \) is the inverse of \( G(k) \), one can express \( C_n \) as the winding number of the fermion propagator [27]:

\[ C_n = \frac{(-i)^n \epsilon_{\alpha_1 \cdots \alpha_{2n+1}}}{(n+1)(2n+1)!} \int \frac{d^{2n+1}k}{(2\pi)^{2n+1}} \text{tr}\{[G(k)\partial_{\alpha_1} G(k)^{-1}] \cdots [G(k)\partial_{\alpha_{2n+1}} G(k)^{-1}]\}. \]

In the rest of this section we will demonstrate that for the \( 2 + 1 \) and \( 4 + 1 \) dimensional Dirac theories the topological invariant winding numbers of the fermion propagator (3.4) are equal to the Chern numbers of the Berry gauge fields (2.10), up to constant factors. Obviously, these results can be obtained by other means; however we think that the following derivations are quite straightforward and clear.

By virtue of the Foldy-Wouthuysen transformation \( U \), one can invert (2.2) to write the Dirac Hamiltonian as \( H = EU^\dagger \beta U \), which is suitable to express it in a projector form. Indeed, for \( k^\alpha = (w, k) \) we can write the inverse of the propagator as

\[ G^{-1}(k) = w + (E + i\varepsilon)(P_- - P_+), \]

where the projection operators \( P_+ \) and \( P_- \), satisfying

\[ P_+^2 = P_+, \quad P_-^2 = P_-, \quad P_+ + P_- = 1, \]

are given explicitly as

\[ P_+ = U^\dagger I_+ U, \quad P_- = U^\dagger I_- U. \]

The operators \( I_+ \) and \( I_- \) project, respectively, onto the positive and the negative energy states which are obtained in representation (2.6) as

\[ I_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad I_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]
Now, (3.5) can easily be inverted to obtain the Green function as
\[ G(k) = \frac{P_+}{w - (E + i\varepsilon)} + \frac{P_-}{w + (E + i\varepsilon)}. \]

In the sequel we will not explicitly write the positive, infinitesimal parameter \( \varepsilon \) unless necessary. Derivative of \( G^{-1} \) with respect to \( k_0 = w \) is
\[ \frac{\partial G^{-1}(k)}{\partial w} = 1. \quad (3.8) \]

Moreover, owing to the projector form (3.5) and the energy-momentum relation (2.3), it satisfies
\[ \frac{\partial G^{-1}(k)}{\partial k_I} = \frac{k_I}{E}(P_- - P_+) - 2E \frac{\partial P_+}{\partial k_I}. \quad (3.9) \]

The following relations between the projection operators \( P_+ \) and \( P_- \)
\[ P_+ \frac{\partial P_-}{\partial k_I} = -\frac{\partial P_+}{\partial k_I} P_-; \quad P_- \frac{\partial P_-}{\partial k_I} = -P_- \frac{\partial P_+}{\partial k_I} P_+, \]
can easily be derived by inspecting their basic properties (3.6).

### 3.1 Relation between \( C_1 \) and \( N_1 \)

In \( 2 + 1 \) dimensions integration of the massive Dirac fermions in the related path integral with the Lagrangian density (3.1) leads to the effective topological action
\[ S_{\xi}^{2+1} = C_1 \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \quad (3.10) \]

where \( \mu, \nu, \rho = 0, 1, 2 \). The coefficient \( C_1 \) is given by
\[ C_1 = -\frac{1}{12} \epsilon^{\mu\nu\rho} \int \frac{d^2kdw}{(2\pi)^3} \text{tr}\{[G(k)\partial_\mu G(k)^{-1}][G(k)\partial_\nu G(k)^{-1}][G(k)\partial_\rho G(k)^{-1}]. \quad (3.11) \]

Making use of (3.8) we can express (3.11) as
\[ C_1 = -\frac{1}{4} \epsilon_{ab} \int \frac{d^2kdw}{(2\pi)^3} \text{tr}\{G(k)G(k)\partial_a G(k)^{-1}G(k)\partial_b G(k)^{-1}\}. \]

When we write the integrand explicitly by employing (3.9), obviously the quadratic terms in \( k_i \) vanish, so that we get
\[ C_1 = -\frac{1}{4} \epsilon_{ab} \int \frac{d^2kdw}{(2\pi)^3} \text{tr}\{G(k)G(k)\partial_a G(k)^{-1}G(k)\partial_b G(k)^{-1}\}. \]

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One can observe that \( k_a \) and \( k_b \) terms combine to vanish
\[ 2 \epsilon_{ab} \text{tr}\{[\frac{P_+}{(w - E)^3} - \frac{P_-}{(w + E)^3}][k_a \partial_b P_+ + k_b \partial_a P_+]\} = 0, \]
even before performing the $w$ integration. The remaining terms, after revoking the infinitesimal parameter $\varepsilon$, become

$$C_1 = -\epsilon_{ab} \int \frac{d^2k dw}{(2\pi)^3} E^2 \text{tr}\{(w - E - i\varepsilon)^2(w + E + i\varepsilon) + (w - E - i\varepsilon)(w + E + i\varepsilon)^2\partial_a P_+ \partial_b P_+\}.$$ 

Integration over $w$ yields

$$C_1 = \frac{i}{8\pi^2} \epsilon_{ab} \int d^2k e^{i\epsilon_{ab}} P_+ \partial_a P_+ \partial_b P_+.$$ 

Making use of definitions (3.7), one can easily observe that

$$\epsilon_{ab}\text{tr}\{P_+ \partial_a P_+ \partial_b P_+\} = \epsilon_{ab}\text{tr}\{(I_+ U \partial_a U^\dagger)(I_+ U \partial_b U^\dagger) + I_+ \partial_a U \partial_b U^\dagger I_+\} = \epsilon_{ab}\text{tr}\{I_+ \partial_a U \partial_b U^\dagger I_+\}.$$ 

On the other hand, plugging the field strength (2.9) of the Abelian Berry gauge field (2.5) into (2.11) leads to

$$N_1 = \frac{i}{2\pi} \int d^2k \epsilon_{ab} \text{tr}\{I_+ \partial_a U \partial_b U^\dagger I_+\}.$$ 

Therefore, we conclude that

$$C_1 = \frac{N_1}{4\pi} \quad (3.12)$$

### 3.2 Relation between $C_2$ and $N_2$

When we deal with $4 + 1$ dimensions, the effective topological action (3.3) becomes

$$S_{eff}^\epsilon[A] = C_2 \int d^5x \epsilon^{ABCDEF} A_A \partial_B A_C \partial_D A_E,$$

where $A, B, \cdots = 0, \cdots, 4$. $C_2$ is given by (3.4) as

$$C_2 = \frac{i}{3 \times 5!} \int \frac{d^4k dw}{(2\pi)^5} \epsilon^{ABCD} \text{tr}\{G \partial_A G^{-1} \partial_B G^{-1} \partial_C G^{-1} \partial_D G^{-1} \partial_E G^{-1}\}. \quad (3.13)$$

Plugging (3.8) and (3.9) into (3.13) leads to

$$C_2 = \frac{i}{3 \times 4!} \epsilon_{ijkl} \int \frac{d^4k dw}{(2\pi)^5} \text{tr}\{G[G(k_j E (P_- - P_+) - 2E \partial_l P_+)]G(k_l E (P_- - P_+) - 2E \partial_k P_+)]G(k_i E (P_- - P_+) - 2E \partial_E P_+)]\}.$$ 

Because of the symmetry properties the terms depending on $k$ at the second or higher order vanish. Thus, we need to consider only the following terms

$$C_2 = \frac{i}{9(2\pi)^5} \epsilon_{ijkl} \int d^4k dw tr\{2E^4 G \partial_l P_+ G \partial_j P_+ G \partial_k P_+ G \partial_l P_+$$

$$-E^2[k_j G G (P_- - P_+) \partial_j P_+ G \partial_k P_+ G \partial_l P_+ + k_j G G \partial_j P_+ G (P_- - P_+) \partial_k P_+ G \partial_l P_+$$

$$+ k_j G G \partial_j P_+ G \partial_k P_+ G (P_- - P_+) \partial_l P_+ + k_j G G \partial_j P_+ G \partial_k P_+ G \partial_l P_+ G (P_- - P_+)]\}.$$
Inspecting the symmetry properties one can easily observe that the second and the fifth terms are summed to give a vanishing contribution. Similarly, one can show that gathered together contributions of the third and the fourth terms vanish by making use of the equalities

\[ \partial_i P_+ G(P_- - P_+) = \left( \frac{P_+}{w+E} - \frac{P_-}{w-E} \right) \partial_i P_+, \quad (P_- - P_+) G \partial_i P_+ = \partial_i P_+ \left( \frac{P_+}{w+E} - \frac{P_-}{w-E} \right). \]

Hence, we get

\[ C_2 = \frac{2i}{9(2\pi)^3} \epsilon_{ijkl} \int d^4 k dw E^4 \text{tr}[GG \partial_i P_+ \partial_j P_+ \partial_k P_+ \partial_l P_+]. \]

After restoring \( \varepsilon \) this can be expressed as

\[ C_2 = \frac{2i}{9(2\pi)^3} \epsilon_{ijkl} \int d^4 k dw E^4 \text{tr}\left[ \frac{P_+}{(w-E-i\varepsilon)^2} + \frac{P_-}{(w-E+i\varepsilon)^2} \right] \partial_i P_+ \partial_j P_+ \partial_k P_+ \partial_l P_+. \]

By performing the \( w \) integration we obtain

\[ C_2 = -\frac{1}{12} \epsilon_{ijkl} \int \frac{d^4 k}{(2\pi)^4} \text{tr}\{ \partial_i P_+ \partial_j P_+ \partial_k P_+ \partial_l P_+ \}. \quad (3.14) \]

In terms of the explicit forms of \( P_+ \) and \( P_- \) given by (3.7) we can express the integrand of (3.14) as

\[ \epsilon_{ijkl} \text{tr}\{ \partial_i P_+ \partial_j P_+ \partial_k P_+ \partial_l P_+ \} = \epsilon_{ijkl} \text{tr}\{ \left( I_+ I_\rho \partial_i U + \partial_i U^\dagger I_+ U \right) P_- \left( I_+ \partial_j U + \partial_j U^\dagger I_+ U \right) \}
\]

\[ \left( I_+ \partial_k U + \partial_k U^\dagger I_+ \right) P_- \left( I_+ \partial_l U + \partial_l U^\dagger I_+ \right) \}
\]

\[ = \epsilon_{ijkl} \text{tr}\{ I_+ \partial_i \partial_j U^\dagger I_+ \partial_k U \partial_l U^\dagger I_+ \}. \quad (3.15) \]

Now, the Berry gauge field (2.5) is non-Abelian and its curvature \( F_{ij} \) can be written as

\[ F_{ij} = iI_+ \partial_i U \partial_j U^\dagger I_+ + iI_+ \partial_i U \partial_j U^\dagger I_+ \partial_j U \partial_j U^\dagger I_+ - i \leftrightarrow j \]

\[ = iI_+ \partial_i \partial_j U^\dagger I_+ - iI_+ \partial_i U^\dagger \partial_j U^\dagger I_+. \]

Inserting it into the definition of the second Chern number (2.12) and inspecting (3.14) and (3.15) one concludes that

\[ C_2 = \frac{N_2}{24\pi^2}. \quad (3.16) \]

Generalizing this method to higher dimensions is straightforward.

## 4 2 + 1 dimensional theory and dimensional reduction to 1 + 1 dimensions

One can observe that by employing (3.12) in (3.10) the effective topological action of external gauge fields coupled to massive Dirac electrons living in 2 + 1 dimensions becomes

\[ S_{\text{eff}}^{2+1} = \frac{N_1}{4\pi} \int d^3 x e^\mu_{\rho\sigma} A_\mu \partial_\nu A_\rho. \quad (4.1) \]

To calculate the related first Chern number (2.11), let us choose the representation \( \alpha = (\sigma_x, \sigma_y) \), where \( \sigma_a \) are the Pauli spin matrices. This corresponds to set \( \rho_a = (1, -i) \) in (2.8). Thus, the Abelian Berry gauge field can be written as

\[ A_a = \frac{\epsilon_{ab} k_b}{2E(E + m)}. \quad (4.2) \]
It yields the Berry curvature
\[
F_{12} = \left( \frac{\partial A_2}{\partial k_1} - \frac{\partial A_1}{\partial k_2} \right) = -\frac{m}{2E^3}.
\] (4.3)

We plug (4.3) into (2.11) and perform the change of variable by (2.3) to express the related first Chern number as
\[
N_1 = -\frac{m}{2} \int_D \frac{dE}{E^2},
\] (4.4)

where the domain of integration \(D\) will be specified according to the model considered. If it is required to treat the \(E > 0\) and \(E < 0\) domains on the same footing, we can deal with
\[
N_1 = -\frac{m}{2} \int_{-\infty}^{m} \frac{dE}{E^2} - \frac{m}{2} \int_{-m}^{\infty} \frac{dE}{E^2} = 1.
\] (4.5)

### 4.1 A model for 2 + 1-dimensional topological insulator

Before presenting the graphene model of [4], let us briefly recall the interconnection between the quantum Hall effect and the Chern-Simons action in 2 + 1 dimensions. For electrons moving on a surface in the presence of the external in-plane electric field \(\mathbf{E} = (E_x, E_y, 0)\) and the perpendicular magnetic field \(\mathbf{B} = (0, 0, B_z)\) the Hall current is given by
\[
j_a = \sigma_H \epsilon_{ab} E_b.
\] (4.6)

Ignoring the spin of electrons the Hall conductivity is a topological invariant[11, 12]:
\[
\sigma_H = \frac{e^2}{h} N_1.
\] (4.7)

Here \(N_1\) is the first Chern number resulting from the field strength \(F_B\) of the Berry gauge field obtained from the single particle Bloch wave functions which are solutions of the Schrödinger equation in the presence of the external magnetic field \(B_z\), integrated over the states up to the Fermi level \(E_F\) as
\[
N_1 = \int_{E_F} d^2k \frac{d^2k}{(2\pi)^2} F_B.
\] (4.8)

A field theoretic description is possible in terms of the Chern-Simons action (4.1) with definition (4.8). In fact, the current obtained from the topological field theory (4.1),
\[
j_\mu = \frac{N_1}{2\pi} \epsilon_{\mu\nu\rho} \partial' A^\rho,
\]
gives for \(E_a = \partial_a A_0 - \partial_0 A_a\) the Hall current (4.6). It also leads to the charge density \(j_0 = \sigma_H B\), where the induced magnetic field is \(B = \partial_x A_y - \partial_y A_x\). Note that \(B\) would also be generated by the Hall current (4.6) through the current conservation condition \(\partial_a j_a = -\partial_t j_0\). We would like to emphasize the fact that the field theory (4.1) is not aware of the external magnetic field \(B_z\). External magnetic field is responsible of creating the energy spectrum whose consequences are encoded in the calculation of the first Chern number (4.8).

By employing the Berry gauge field derived from the Dirac equation (4.2), we can still get the Hall conductivity as in (4.7) by an appropriate choice of the domain of integration \(D\) in (4.4). This construction does not necessitate an external magnetic field. For the first time in [3], Haldane described
how to obtain the quantum Hall effect without a magnetic field (vanishing in the average) through a Dirac like theory. To calculate the Hall conductivity following from the Dirac equation we let all the negative energy levels be occupied up to the Fermi level $E_F = m$ in (4.4), so that

$$\sigma_H = \frac{e^2}{\hbar} \left( -\frac{m}{2} \int_{-\infty}^{m} \frac{dE}{E^2} \right) = \frac{e^2}{2\hbar}. \quad (4.9)$$

In [4], Kane and Mele incorporated the spin of electrons into the Haldane model[3] and proposed the following Hamiltonian for graphene

$$H_G = \sigma_k k + \sigma_y k y + m\sigma_z \tau_z s_z, \quad (4.10)$$

which leads to a TRI spin current. The mass term is generated by a spin-orbit coupling. The Pauli spin matrices $\sigma_{x,y,z}$ act on the states of sublattices. The matrix $\tau_z = \text{diag}(1, -1)$ denotes the Dirac points $K, K'$ which should be interchanged under the time reversal transformation. The other Pauli matrix $s_z = \text{diag}(1, -1)$ describes the third component of the spin of electrons which should also be inverted under time reversal transformation. Thus the time reversal operator is given by $T = UK$ where we can take $U = \tau_y s_y$ and $K$ takes the complex conjugation as well as maps $k \rightarrow -k$. Therefore (4.10) is TRI:

$$TH_G T^{-1} = H_G.$$ 

The Abelian Berry gauge field obtained from the Hamiltonian (4.10) can be written as[20]

$$A_a = \frac{1}{2E(E + m)} \epsilon_{ab} k_b 1_\tau s_z, \quad (4.11)$$

where $1_\tau$ is the unit matrix in the $\tau_z$ space. The corresponding field strength is

$$F_{12} \equiv \text{diag}(F^\uparrow_{+}, F^\uparrow_{-}, F^\downarrow_{+}, F^\downarrow_{-}) = -\frac{m}{2E^3} 1_\tau s_z. \quad (4.12)$$

The indices $\uparrow\downarrow$ and $\pm$ label, respectively, the third component of the spin and $\tau_z$. The spin current defined as

$$j^s = j^\uparrow_{+} + j^\downarrow_{-} - j^\downarrow_{+} - j^\uparrow_{-},$$

leads to the spin Hall current

$$j^s_a = \sigma_{SH} \epsilon_{ab} E_b.$$

The difference of the related first Chern numbers

$$\Delta N_1 = \frac{1}{2\pi} \int_{E=-\infty}^{E=m} d^2 k \left[ (F^\uparrow_{+} + F^\uparrow_{-}) - (F^\downarrow_{+} + F^\downarrow_{-}) \right]$$

$$= \left( \frac{1}{2} + \frac{1}{2} \right) - \left( -\frac{1}{2} - \frac{1}{2} \right) = 2, \quad (4.13)$$

gives the spin Hall conductivity $\sigma_{SH}$ as

$$\sigma_{SH} = \frac{e}{4\pi} \Delta N_1 = \frac{e}{2\pi}. \quad (4.14)$$

10
4.2 Dimensional reduction to $1 + 1$ dimensions

We would like to discuss dimensional reduction from $2 + 1$ to $1 + 1$ dimensions by slightly modifying the procedure described in [21]. The dimensionally reduced theory can be defined through the $1 + 1$ dimensional Lagrangian density

$$L_{1+1}(\psi, \bar{\psi}, A) = \bar{\psi} \left[ \gamma_r (p_r + A_r) + \gamma_2 \zeta_y - m \right] \psi,$$

where $r = t, x$ and the external field $\zeta_y(t, x)$ is the reminiscent of the gauge field $A_y$. We define $\zeta_y = k_y + \zeta$, where $k_y$ is a parameter which permits us to deal with one particle Green function of the $2 + 1$ dimensional theory to derive the effective action of the external fields as in Section 3. In fact, integrating out the spinor fields $\psi, \bar{\psi}$ in the related path integral yields the effective action

$$S_{1+1}^{\text{eff}} = G_{1D}(k_y) \int dx dt \zeta(x, t) \epsilon_{rs} \partial_r A_s.$$

The coefficient $G_{1D}(k_y)$ is required to satisfy

$$\int G_{1D}(k_y) dk_y = N_1,$$  \hspace{1cm} (4.15)

where the first Chern number $N_1$ is given by (4.4). Instead of the Cartesian coordinates we prefer to work with the polar coordinates $k, \theta$, where $k_x = k \cos \theta, k_y = k \sin \theta$. Similar to (4.15) we would like to introduce $G(\theta)$ satisfying

$$\int_0^{2\pi} G(\theta) d\theta = N_1,$$  \hspace{1cm} (4.16)

and define the $(1 + 1)$-dimensional effective action as

$$S_{1+1} = G(\theta) \int dx dt \zeta(x, t) \epsilon_{rs} \partial_r A_s.$$  \hspace{1cm} (4.17)

Although it can be deduced directly from definition (4.16), we can also obtain $G(\theta)$ by writing the components of the Abelian Berry gauge field (4.11) in polar coordinates:

$$A_\theta = \frac{-k}{2E(E + m)}, \quad A_k = 0.$$

The Berry curvature remains the same

$$F_{k\theta} = \frac{1}{k} \left[ \frac{\partial (k A_\theta)}{\partial k} - \frac{\partial A_k}{\partial \theta} \right] = \frac{m}{2E^3},$$

and allows us to calculate explicitly $G(\theta)$ as

$$G(\theta) = \frac{1}{2\pi} \int kdk F_{k\theta} = -\frac{m}{4\pi} \int_0^\infty \frac{dE}{E^2} = -\frac{N_1}{2\pi}.$$

Now, one can define the one dimensional charge polarization[28, 29] $P(\theta)$ by

$$\frac{\partial P(\theta)}{\partial \theta} \equiv G(\theta),$$  \hspace{1cm} (4.18)
Adopting the first Chern number calculated in (4.5), \( N_1 = 1 \), we solve (4.18) by

\[
P(\theta) = \frac{\theta}{2\pi}.
\]  

(4.19)

The physical observable is not directly the charge polarization given by \( P(\theta) \) but the adiabatic change in \( P(\theta) \) along a loop, which is equal to

\[
\Delta P = P(2\pi) - P(0) = 1.
\]

The (1 + 1)-dimensional action (4.17) becomes

\[
S_{1+1} = \frac{1}{2\pi} \int dx dt \epsilon_{rs} \partial_s \zeta(x,t),
\]  

(4.20)

for \( N_1 = 1 \). Action (4.20) leads to the current

\[
j_r = \frac{1}{2\pi} \epsilon_{rs} \partial_s \zeta(x,t),
\]

known as the Goldstone-Wilczek formula[30] and gives the charge

\[
Q = \frac{1}{2\pi} \int \frac{\partial \zeta(x,t)}{\partial x} dx = \frac{1}{2\pi} \Delta \zeta.
\]

In fact, it corresponds to solitons on polyacetylene with charge \( Q = 1/2 \) for \( \zeta \) changing from 0 to \( \pi \) and \( Q = 1/3 \) for \( \Delta \zeta = 2\pi/3 \) as it was obtained in [30].

5 4+1 dimensional topological insulator and dimensional reduction to 3+1 and 2+1 dimensions

The topological field theory

\[
S_{eff}^{4+1}[A] = \frac{N_2}{24\pi^2} \int d^5 x \epsilon ABCDE A_A \partial_B A_C \partial_D A_E,
\]  

(5.1)

is designated as the effective action of the 4 + 1 dimensional TRI topological insulators in [21]. It follows from (3.3) by making use of relation (3.16). To derive the related second Chern number \( N_2 \) we deal with the 4 + 1 dimensional realization of the Dirac Hamiltonian (2.1) which is provided by

\[
\alpha_{1,2,3} = \left( \begin{array}{ccc} 0 & i\sigma_{1,2,3} & 0 \\ -i\sigma_{1,2,3} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \alpha_4 = \left( \begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right), \quad \beta = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
\]  

(5.2)

Observing that \( \rho_i = (i\sigma_1, i\sigma_2, i\sigma_3, -1) \), the non-Abelian Berry gauge fields can be obtained from (2.8) as

\[
A_1 = \frac{\sigma_3 k_2 - \sigma_2 k_3 - \sigma_1 k_4}{2E(E + m)}, \quad A_2 = \frac{-\sigma_3 k_1 + \sigma_1 k_3 - \sigma_2 k_4}{2E(E + m)},
\]

\[
A_3 = \frac{\sigma_2 k_1 - \sigma_1 k_2 - \sigma_3 k_4}{2E(E + m)}, \quad A_4 = \frac{\sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 k_3}{2E(E + m)}.
\]  

(5.3) (5.4)
Plugging them into (2.12) and taking the trace yield
\[ d \]

One can obtain the 3 + 1 dimensional effective action as

\[ \text{Dimensional reduction to 3 + 1 dimensions} \]

By definition the Berry gauge field corresponding to the 4 + 1 dimensional Dirac Hamiltonian can also be derived by considering the explicit solutions of the Dirac equation as was done in [31]. They work in the chiral representation, so that the Berry gauge field components which they obtain differ from (5.3),(5.4).

\[ \text{From (5.3),(5.4).} \]

To calculate it explicitly, we would like to deal with the four dimensional polar coordinates given by

\[ k_1 = k \cos \phi_1, \ k_2 = k \sin \phi_1 \cos \phi_2, \ k_3 = k \sin \phi_1 \sin \phi_2 \cos \phi_3 \text{ and } k_4 = k \sin \phi_1 \sin \phi_2 \sin \phi_3, \]

where the angles \( \phi_1, \phi_2, \phi_3 \), respectively, take values in the intervals \([0, \pi], [0, \pi], [0, 2\pi]\). The volume element is \( d^4k = k^3 \sin^2 \phi_1 \sin \phi_2 dk d\phi_1 d\phi_2 d\phi_3 \). Hence, after the change of variable by (2.3), one can show that (5.5) can be written as

\[ \text{When } D \text{ is taken to be an overlap of the } E > 0 \text{ and } E < 0 \text{ domains, we may deal with} \]

\[ \text{Dimensional reduction to 3 + 1 dimensions} \]

Dimensional reduction of the 4 + 1 dimensional effective action given by (3.1) to 3 + 1 dimensions can be described by the Lagrangian density

\[ \mathcal{L}_{3+1}[\psi, \bar{\psi}, A] = \bar{\psi} \left[ \gamma^\alpha (\gamma_0 p_\alpha + A_\alpha) + \gamma_4 \theta - m \right] \psi, \]

where \( \alpha = 0, \cdots, 3 \). The external field \( \theta(x_\alpha) \) is the reminiscent of the gauge field \( A_4 \). \( \psi, \bar{\psi} \) fields can be integrated out as in Section 3 through the one particle Green function of 4 + 1 dimensional theory introducing the parameter \( k_4 \) by setting \( \theta = k_4 + \theta(x_\alpha) \). By keeping track of the phase space volume one can obtain the 3 + 1 dimensional effective action as

\[ S_{\text{eff}}^{3+1} = \frac{G_{3D}(k_4)}{4\pi} \int d^4x \theta e^{\alpha \beta \eta \gamma} \partial_\alpha A_\beta \partial_\gamma A_\eta, \]
where the coefficient is given through the condition
\[ \int G_{3D}(k_4) dk_4 = N_2. \] (5.10)

We would like to modify this construction by working with the four dimensional polar coordinates and proposing that the action describing the descendant theory is given by
\[ S_{3+1} = \frac{G_3(\phi_3)}{4\pi} \int d^4x \theta \epsilon^{\alpha\beta\gamma\eta} \partial_\alpha A_\beta \partial_\gamma A_\eta, \]
whose coefficient, like (5.10), is required to satisfy the condition
\[ \int_0^{2\pi} G_3(\phi_3) d\phi_3 = N_2. \]

Thus, the coefficient \( G_3(\phi_3) \) can be obtained as
\[ G_3(\phi_3) = \frac{1}{32\pi^2} \int \epsilon_{ijkl} \text{tr}(F_{ij}F_{kl}) k^3 \sin^2 \phi_1 \sin \phi_2 dk d\phi_1 d\phi_2 = \frac{N_2}{2\pi}, \] (5.11)
with definition (5.6) of the second Chern number \( N_2 \).

Similar to the one-dimensional charge polarization (4.18) one can associate the coefficient \( G_3(\phi_3) \) to \( P_3(\phi_3) \) through the relation \[ \int_0^{2\pi} \frac{\partial P_3(\phi_3)}{\partial \phi_3} d\phi_3 \equiv \int_0^{2\pi} G_3(\phi_3) d\phi_3 = N_2. \]

Hence the “magnetoelectric polarization” can be obtained as
\[ P_3(\phi_3) = \frac{N_2}{2\pi} \phi_3. \] (5.12)

Observe that like the one-dimensional case, for \( \Delta \phi_3 = 2\pi \) it changes by \( \Delta P_3 = 1 \) if we choose \( N_2 = 1 \) as it is calculated in (5.7). \( P_3(\phi_3) \) depends linearly on \( \phi_3 \) due to the fact that the second Chern character corresponding to free Dirac particle depends only on \( k \). Interacting Dirac particles may give rise to polarizations which would not be linearly dependent on \( \phi_3 \).

By inserting definition (5.11) into (5.9), the effective action becomes
\[ S_{3+1} = \frac{N_2}{8\pi^2} \int d^4x \theta \epsilon^{\alpha\beta\gamma\eta} \partial_\alpha A_\beta \partial_\gamma A_\eta. \] (5.13)

It can be written equivalently as
\[ S_{3+1} = \frac{1}{4\pi} \int d^4x P_3(\theta) \epsilon^{\alpha\beta\gamma\eta} \partial_\alpha A_\beta \partial_\gamma A_\eta, \]
where \( P_3(\theta) = N_2 \theta/2\pi \). This describes the axion electrodynamics which is invariant under the shift \( \theta \to \theta + 2\pi \) [32, 33].
5.2 A hypothetical model for $3+1$ dimensional topological insulators

In spite of the fact that the underlying topological gauge theory (5.1) is manifestly TRI, the theory given by the descendant action (5.13) is TRB except for the values $\theta = 0, \pi$. Nevertheless, we may deal with the TRB action (5.13) but introduce a TRI hypothetical model generalizing the spin Hall effect for graphene[4]. The current following from action (5.13) is

$$j^a = \frac{N_2}{(2\pi)^2} \epsilon^{\alpha\beta\gamma\eta} \partial_\beta \theta \partial_\gamma A_\eta.$$  

Assuming $\theta = \theta(z)$ and considering the in-plane electric field $E_a(x,y); a = 1, 2$, we obtain the current[34]

$$j_a = \frac{N_2}{(2\pi)^2} \partial_z \theta(z) \epsilon_{ab} E_b(x,y).$$  

(5.14)

The Hall current can be introduced by integrating (5.14) along the coordinate $z$ as

$$J_a(x,y) \equiv \int j_a dz = \sigma_H \epsilon_{ab} E_b(x,y).$$  

(5.15)

It leads to the surface Hall conductivity $\sigma_H [9, 21]$

$$\sigma_H = \frac{e^2}{h} \frac{N_2}{(2\pi)^2} \int \partial_z \theta dz = \frac{e^2}{h} \frac{N_2}{(2\pi)^2} \Delta \theta.$$  

(5.16)

Obviously we defined $\Delta \theta = \theta(\infty) - \theta(-\infty)$, which is non-vanishing for an adequate domain wall or at an interface plane between two samples.

Now we should define the second Chern number (5.6) appropriately. We suppose that all negative energy states are occupied till the Fermi level taken as the first positive energy value $E_F = m$, so that we get

$$N_2 = \frac{3m}{4} \int_{-\infty}^{m} \frac{m^2 - E^2}{E^4} dE = 1/2.$$

(5.17)

Considering a plane of interface which yields $\Delta \theta = 2\pi$ the Hall conductivity becomes

$$\sigma_H = \frac{e^2}{2h}.$$

In representation (5.2), the $4 + 1$ dimensional Dirac Hamiltonian (2.1) is TRI where the time reversal operator can be taken as $T_{4+1} = \alpha_2 \alpha_4 \tilde{K}$. However, the Hamiltonian corresponding to action (5.8) for $A_\alpha = 0$,

$$H_{3+1} = \alpha \cdot k + \alpha_4 \tilde{\theta} + m \beta,$$

(5.18)

violates time reversal symmetry. We will present a hypothetical model which is time reversal invariant emulating the spin Hall effect for graphene. Let $\alpha_i$ act on sublattices with two Dirac points. Assume that around these points which are interchanged under time reversal transformation, electrons are described by the Hamiltonians as in (5.18). Moreover, the third component of the spin given by the Pauli matrix $s_z = \text{diag}(1, -1)$ is included and conserved. Thus, we propose to consider the Hamiltonian

$$\tilde{H}_{3+1} = \tilde{\alpha} \cdot k + \tilde{\alpha}_4 \tilde{\theta} + \tau_z s_z \beta m,$$

(5.19)
where \( \tau_z = \text{diag}(1, -1) \) and in terms of \( \alpha_i \) and \( \beta \) given by (5.2) we defined

\[
\tilde{\alpha}_i = (\alpha_1, \tau_z \alpha_2, \alpha_3, \alpha_4).
\]

Now, as in Section 4.1, the time reversal operator interchanging the Dirac points and the third components of the spin can be defined by \( T = \tau_y s_y K \), so that (5.19) is TRI. Obviously, we can obtain (5.19) through the dimensional reduction from the \( 4 + 1 \) dimensional action corresponding to the following free Hamiltonian

\[
\tilde{H} = \tilde{\alpha}_i \cdot k_i + \tau_z s_z \beta m \equiv \text{diag}(\tilde{H}_1^{\uparrow+}, \tilde{H}_1^{\downarrow-}, \tilde{H}_2^{\uparrow+}, \tilde{H}_2^{\downarrow-}).
\]  

(5.20)

As we show in Appendix, the four dimensional Hamiltonians defined by (5.20) correspond to the second Chern numbers

\[
N_2^{\uparrow+} = N_2^{\downarrow-} = -N_2^{\downarrow+} = -N_2^{\downarrow-} = N_2,
\]

where \( N_2 \) is given by (5.6). Repeating the procedure yielding (5.15)-(5.17) in the presence of a domain wall we can obtain the dissipationless spin current as

\[
J_a^s = J_a^{\uparrow+} + J_a^{\downarrow-} - J_a^{\downarrow+} - J_a^{\downarrow-} = \sigma_{SH} \epsilon_{ab} E_b(x, y),
\]

with the spin Hall conductivity

\[
\sigma_{SH} = \frac{e}{4\pi} \left( N_2^{\uparrow+} + N_2^{\uparrow-} - N_2^{\downarrow+} - N_2^{\downarrow-} \right) = \frac{e}{2\pi},
\]

for \( \Delta\theta = 2\pi \). It is equal to the spin Hall conductivity for graphene (4.14).

### 5.3 Dimensional reduction to \( 2 + 1 \)-dimensions

The \( 2 + 1 \) dimensional Lagrangian density

\[
L_{2+1}[\psi, \bar{\psi}, A] = \bar{\psi} [\gamma^\mu (p_\mu + A_\mu) + \gamma_3 \zeta_3 + \gamma_4 \zeta_4 - m] \psi,
\]

describes the dimensionally reduced theory. The fields \( \zeta_3(x_\mu), \zeta_4(x_\mu) \) are the reminiscent of the gauge fields \( A_3, A_4 \), of the \( 4 + 1 \) dimensionally theory whose action is given by (3.1) for \( d = 4 \). By setting \( \zeta_3(x_\mu) = k_3 + \tilde{\phi}(x_\mu) \) and \( \zeta_4(x_\mu) = k_4 + \tilde{\theta}(x_\mu) \), where \( k_3, k_4 \) are parameters playing the role of the momentum components in one particle Green functions, one can follow the approach of Section 3 to derive the \( 2 + 1 \) dimensional effective action as

\[
S_{eff}^{2+1} = G_{2D}(k_3, k_4) \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu \tilde{\phi} \partial_\rho \tilde{\theta}.
\]

Its coefficient should fulfill the condition

\[
\int G_{2D}(k_3, k_4) dk_3 dk_4 = N_2.
\]  

(5.21)

As in Section 5.1, we consider the four dimensional polar coordinates and propose that the action

\[
S_{2+1} = G_2(\phi_2, \phi_3) \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu \tilde{\phi} \partial_\rho \tilde{\theta},
\]

(5.22)

describes the \( 2 + 1 \) dimensional descendant theory. Obviously, like (5.21) we pose the condition

\[
\int_0^\pi \int_0^{2\pi} d\phi_2 d\phi_3 G_2(\phi_2, \phi_3) = N_2.
\]
This can be solved by
\[ G_2(\phi_2, \phi_3) = \frac{N_2}{4\pi} \sin \phi_2. \]

Proceeding as in [21] we introduce the vector field
\[ \Omega_\mu \equiv \Omega_\theta \partial_\mu \theta + \Omega_\phi \partial_\mu \phi, \]
however by adopting the definitions
\[ \Omega_\theta = -\frac{N_2}{4} \cos \phi, \quad \Omega_\phi = -\frac{N_2}{4} \sin \phi. \]

The field strength of the field \( \Omega_\mu \) is
\[ \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu = \frac{N_2}{2} \sin \phi \left( \partial_\nu \theta \partial_\mu \phi - \partial_\nu \phi \partial_\mu \theta \right). \]

Let us define \( \phi = \phi_2 + \tilde{\phi} \) and \( \theta = \phi_3 + \tilde{\theta} \) as slowly varying fields, so that at the first order in derivatives we can write \( G_2(\phi, \theta) \partial_\mu \theta \partial_\nu \phi \approx G_2(\phi_2, \phi_3) \partial_\mu \tilde{\theta} \partial_\nu \tilde{\phi} \). Therefore, action (5.22) can be written in the form
\[ S_{2+1} = \frac{1}{2\pi} \int d^3x \epsilon_{\mu\nu\rho} A_\mu \partial_\nu \Omega_\rho. \] (5.23)

The current generated by the field \( \Omega_a \), \( a = 1, 2 \), is
\[ j^\Omega_a = \frac{e}{2\pi} \epsilon_{ab} E_b, \]
where the electric field is given by \( E_a = \partial_a A_0 - \partial_0 A_a \). It can be interpreted as the spin current yielding the spin Hall conductivity \( \sigma_{SH} = e/2\pi \). Hence, by attributing the adequate time reversal transformation properties to the gauge field \( \Omega_\mu \), action (5.23) corresponds to the TRI 2+1 dimensional model of [4] which we discussed in Section 4.1.

Action (5.23) generates the electric current
\[ j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu \Omega_\rho. \]

For fields satisfying \( \phi = \phi(x), \theta = \theta(y) \) it gives the total charge
\[ Q = e \frac{N_2}{4\pi} \int dx dy \sin \phi \partial_x \phi \partial_y \theta = e \frac{N_2}{4\pi} \int_0^\pi \sin \phi d\phi \int_0^{2\pi} d\theta = eN_2. \]

On the other hand the three dimensional Skyrmion field \( n \) coupled to Dirac fermion in 2+1 dimensions yields the current[35]
\[ j^\mu_T = \frac{1}{8\pi} \epsilon^{\mu\nu\rho} n \cdot \partial^\nu n \times \partial^\rho n. \]

The Skyrmion field configuration discussed in [36] satisfying \( n^2 = 1 \) possesses the charge \( Q^T = 2e \). Hence, if we deal with \( N_2 = 1 \), the Skyrmion theory can be described for the field configurations satisfying
\[ \sin \phi (\partial_\nu \theta \partial_{\mu} \phi - \partial_{\mu} \theta \partial_\nu \phi) = \frac{1}{4} n \cdot \partial_\nu n \times \partial_\mu n, \]
which leads to \( j_T^\mu = 2j_\mu \). In principle, this condition can be solved to obtain \( n \) in terms of the fields \( \phi \) and \( \theta \).

Observe also that for the field configurations \( \phi = \phi(t), \ \theta = \theta(y) \) the net charge flow in \( x \) direction is

\[
\int dt\, dy \, j_x = -N_2. \tag{5.24}
\]

Moreover, we can introduce the magnetoelectric polarization in the form given (5.12) by defining it as

\[
P_3(\theta) = -\int_0^\pi d\phi \Omega_\phi / \pi = \frac{N_2}{2\pi} \theta.
\]

Then the pumped charge (5.24) can also be written as

\[
\Delta Q = \int dP_3 = N_2
\]

which gives \( \Delta Q = 1/2 \) for \( N_2 = 1/2 \), as it is given for the Hall effect (5.17).

6 Discussions

The Foldy-Wouthuysen transformation which diagonalizes the Dirac Hamiltonian is proven to be a powerful tool to perform calculations in the effective field theory of the 4 + 1 dimensional TRI topological insulator. The Foldy-Wouthuysen transformation is employed to obtain the Berry gauge fields of Dirac Hamiltonians and through them we derived the first and second Chern characters explicitly. Then we demonstrated in a transparent manner that the winding numbers of Dirac propagators are equal to the coefficients of the effective Chern-Simons actions in 2 + 1 as well as in 4 + 1 dimensions. This construction can be generalized to higher odd dimensions straightforwardly. In the line of the graphene model [4] we introduced a hypothetical model leading to a dissipationless spin current in 3 + 1-dimensions. It can be helpful to understand some aspects of three dimensional TRI topological insulators if we can show that it is somehow related to some realistic models. Moreover, it seems that in terms of our explicit constructions one can discuss \( \mathbb{Z}_2 \) topological classification of TRI insulators in a tractable fashion. In principle our approach can be generalized to interacting Dirac particles where the related Foldy-Wouthuysen transformation at least perturbatively exists.

Appendix

The Hamiltonian (5.20) which comprises \( \tau_z \) and spin degrees of freedom denoted \( \uparrow \downarrow \), respectively, yields the 4 + 1 dimensional Dirac Hamiltonians

\[
\begin{align*}
\tilde{H}_{\uparrow}^{\uparrow} &= \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 + m\beta, \\
\tilde{H}_{\downarrow}^{\uparrow} &= \alpha_1 k_1 - \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 - m\beta, \\
\tilde{H}_{\uparrow}^{\downarrow} &= \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 - m\beta, \\
\tilde{H}_{\downarrow}^{\downarrow} &= \alpha_1 k_1 - \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 + m\beta. 
\end{align*}
\] \tag{A.1}

(A.2)

Let us first consider the two spin up Hamiltonians (A.1). They yield slightly different non-Abelian Berry gauge fields

\[
\begin{align*}
A_{\uparrow}^{1\pm} &= \frac{1}{2E(E+m)} (\pm \sigma_3 k_2 - \sigma_2 k_3 \mp \sigma_1 k_4), \\
A_{\uparrow}^{2\pm} &= \frac{1}{2E(E+m)} (\mp \sigma_3 k_1 \pm \sigma_1 k_3 - \sigma_2 k_4), \\
A_{3\pm}^{3\pm} &= \frac{1}{2E(E+m)} (\sigma_2 k_1 \mp \sigma_1 k_2 \mp \sigma_3 k_4), \\
A_{4\pm}^{4\pm} &= \frac{1}{2E(E+m)} (\pm \sigma_1 k_1 + \sigma_2 k_2 \pm \sigma_3 k_3).
\end{align*}
\]
The corresponding field strengths can be calculated as

\[
\mathcal{F}_{12}^{\pm} = \frac{1}{2E^3(E + m)} \left[ \mp \sigma_3 (E(E + m) - k_1^2 - k_2^2) + \sigma_2 (k_1 k_4 - k_2 k_3) \mp \sigma_1 (k_2 k_4 + k_1 k_3) \right], \\
\mathcal{F}_{13}^{\pm} = \frac{1}{2E^3(E + m)} \left[ \sigma_2 (E(E + m) - k_1^2 - k_2^2) \mp \sigma_1 (k_1 k_2 - k_3 k_4) \mp \sigma_3 (k_1 k_4 + k_2 k_3) \right], \\
\mathcal{F}_{14}^{\pm} = \frac{1}{2E^3(E + m)} \left[ \pm \sigma_1 (E(E + m) - k_1^2 - k_2^2) - \sigma_2 (k_1 k_2 + k_3 k_4) \mp \sigma_3 (k_1 k_3 - k_2 k_4) \right], \\
\mathcal{F}_{23}^{\pm} = \frac{1}{2E^3(E + m)} \left[ \pm \sigma_1 (E(E + m) - k_2^2 - k_3^2) - \sigma_2 (k_1 k_2 + k_3 k_4) \mp \sigma_3 (k_1 k_3 - k_2 k_4) \right], \\
\mathcal{F}_{24}^{\pm} = \frac{1}{2E^3(E + m)} \left[ \sigma_2 (E(E + m) - k_2^2 - k_3^2) \mp \sigma_1 (k_1 k_2 - k_3 k_4) \mp \sigma_3 (k_1 k_4 + k_2 k_3) \right], \\
\mathcal{F}_{34}^{\pm} = \frac{1}{2E^3(E + m)} \left[ \pm \sigma_3 (E(E + m) - k_3^2 - k_4^2) + \sigma_2 (k_1 k_4 - k_2 k_3) \mp \sigma_1 (k_2 k_4 + k_1 k_3) \right].
\]

Although they are different, they generate the same second Chern number equal to (5.5):

\[
N_2^{\pm} = \frac{1}{32\pi^2} \int d^4k \epsilon_{ijkl} \text{tr} \left[ \mathcal{F}_{ij}^{\pm} \mathcal{F}_{kl}^{\pm} \right] = \frac{3}{4\pi^2} \int (-\frac{m}{2E^3}) d^4k. \tag{A.3}
\]

The non-Abelian Berry gauge fields corresponding to the two spin down Hamiltonians (A.2) can be shown to satisfy

\[
\mathcal{A}_i^{\pm}(k_1, k_2, k_3, k_4) = (-1)^{\delta_{i4}} \mathcal{A}_i^{\pm}(k_1, k_2, k_3, -k_4),
\]

without summation over the repeated indices. Thus, the components of the related Berry curvature are

\[
\mathcal{F}_{ij}^{\pm}(k_1, k_2, k_3, k_4) = (-1)^{\delta_{i4} + \delta_{j4}} \mathcal{F}_{ij}^{\pm}(k_1, k_2, k_3, -k_4).
\]

They yield the same second Chern number which is given by (A.3) up to a minus sign: \( N_2^{\pm} = -N_2^{\pm} \).

References

[1] G. W. Semenoff, Phys. Rev. Lett. 53 (1984) 2449.
[2] D. P. DiVincenzo and E. J. Mele, Phys. Rev. B 29 (1984) 1685.
[3] F. D. M. Haldane, Phys. Rev. Lett. 61 (1988) 2015.
[4] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95 (2005) 226801.
[5] B.A. Bernevig, T.L. Hughes and S-C. Zhang, Science 314 (2006) 1757.
[6] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science 318 (2007) 766.
[7] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82 (2010) 3045.
[8] X-L. Qi and S-C. Zhang, Topological insulators and superconductors, arXiv:1008.2026.
[9] M.Z. Hasan and J.E. Moore, Ann. Rev. Condens. Matter Phys. 2 (2011) 55.
[10] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95 (2005) 146802.
[11] D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs, Phys. Rev. Lett. 49 (1982) 405.
[12] J. E. Avron, R. Seiler and B. Simon, Phys. Rev. Lett. 51 (1983) 51.
[13] A. J. Niemi, G. W. Semenoff, Phys. Rev. Lett. 51 (1983) 2077.
[14] A. N. Redlich, Phys. Rev. Lett. 52 (1984) 18.
[15] A. N. Redlich, Phys. Rev. D 29 (1984) 2366.
[16] M. V. Berry, Proc. R. Soc. A 392 (1984) 45.
[17] D. Xiao, M.-C. Chang and Q. Niu, Rev. Mod. Phys. 82 (2010) 1959.
[18] S. C. Zhang, T. H. Hansson and S. Kivelson, Phys. Rev. Lett. 62 (1989) 1988.
[19] S. C. Zhang, Int. J. Mod. Phys. B 6 (1992) 25.
[20] Ö. F. Dayi and E. Yunt, Phys. Lett. A 375 (2011) 2484.
[21] X.-L. Qi, T. L. Hughes and S.-C. Zhang, Phys. Rev. B 78 (2008) 195424.
[22] L. Fu, C. L. Kane and E. J. Mele, Phys. Rev. Lett. 98 (2007) 106803.
[23] J. E. Moore and L. Balents, Phys. Rev. B 75 (2007) 121306(R).
[24] R. Roy, Phys. Rev. B 79 (2007) 195322.
[25] A. Berard and H. Mohrbach, Phys. Lett. A 352 (2006) 190.
[26] M. Nakahara, Geometry, Topology and Physics (Adam Hilger, Bristol, 1990).
[27] M. F. L. Golterman, K. Jansen and D. B. Kaplan, Phys. Lett. B 301 (1993) 219.
[28] R. D. King-Smith and D. Vanderbilt, Phys. Rev. B 47 (1993) 1651.
[29] G. Ortiz and R. M. Martin, Phys. Rev. B 49 (1994) 14202.
[30] J. Goldstone and F. Wilczek, Phys. Rev. Lett. 47 (1981) 986.
[31] S. Ryu, A. P. Schnyder, A. Furusaki and A. W. W. Ludwig, New Journal of Physics 12 (2010) 065010.
[32] F. Wilczek, Phys. Rev. Lett. 58 (1987) 1799.
[33] M. M. Vazifeh and M. Franz, Phys. Rev. B 82 (2010) 233103.
[34] D. Boyanovsky, E. Dagotto and E. Fradkin, Nucl. Phys. B 285(FS19) (1987) 340.
[35] T. Jaroszewicz, Phys. Lett. B 146 (1984) 337.
[36] T. Grover and T. Senthil, Phys. Rev. Lett. 100 (2008) 156804.