Two-Mode Squeezed States and Their Superposition in the Motion of Two Trapped Ions

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Abstract

We propose a method to create two-mode squeezed states and their superposition in the center-of-mass mode and breathing mode of two-trapped ions. Each ion is illuminated simultaneously by two standing waves. One of the fields is tuned to excite resonantly and simultaneously both upper sidebands of the two normal modes, while the other field tuned to the corresponding lower sidebands.

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The superposition principle in quantum mechanics enables quantum states to have some interesting distinct characteristics than classical ones, such as coherence, squeezing and quantum entanglement. In recent years, much efforts have been devoted to the field of generating variety quantum states to test the validity of quantum mechanical fundamental predictions[1,2]. In the context of cavity QED, a number of schemes have been presented for the generation of nonclassical light fields[3-5]. Experimentally, squeezed light and sub-Poissonian light have been produced[6].

A trapped-ion system turns out to be an alternative candidate for realizing quantum-state preparation. The quantized vibrational motion of ions in the trap potential plays a role of boson mode. When the trapped ions interact with classical laser fields, its internal and external degrees of freedom are coupled via the exchange of momentum with the laser fields. This property provides the possibility of generating various nonclassical states in the vibrational motion of trapped ions by exciting ions with appropriate laser fields. So far, proposals to prepare various nonclassical motional states of a trapped ion, such as Fock states [7], squeezed state [8], even and odd coherent states [9,10] have been made. More important, schemes for the motional quantum-state engineering via Fock state superposition [11] and coherent state superposition on a line or on a circle [12] have been proposed, which allows one to approximate many quantum states [13] and provides a new way for quantum-state generation. In addition, the study on creating motional states of multiple trapped ions [14-16] has also been started. Experimentally, motional Fock states, squeezed states, coherent states [17] and Schrödinger Cat states[18] for the center-of-mass mode of a single trapped ion have been observed.

The two-mode squeezed states are very important in quantum optics, since several devices produce light which is correlated at two frequencies. Usually these frequencies are symmetrically placed at either side of a carrier frequency. The squeezing exists not in the single modes but in the correlated state formed by the two modes[19]. This kind of correlation violate certain classical inequalities and can be employed to explain EPR paradox[20]. More recently, this kind of correlation has also played a leading role in the quantum tele-
portation of continuous variables [21]. In addition, the superposition of two-mode squeezed states is also important, because it can approximate a variety of two-mode entangled states with different degrees of entanglement.

In reference [10], the author has assumed that an ion is trapped in a two-dimensional isotropic harmonic potential and driven by four lasers, two along $x$ direction and two along $y$ direction, tuned to excite resonantly the upper and lower vibrational sidebands in the two directions respectively. In that way, entangled two-mode coherent states in the two-dimensional normal modes of one ion have been prepared. While in reference [15], the authors have used four lasers to excite respectively the both upper and lower sidebands in the center-of-mass and breathing modes of N-trapped ions, the same vibrational states, but in two normal modes of N-trapped ions, have been produced. In this letter, we provide a method to create two-mode squeezed states and their superposition in the center-of-mass and breathing modes of two trapped ions, which needs only two lasers to excite simultaneously the both upper and lower sidebands of the two normal modes.

Let us consider two two-level ions of mass $m$ trapped in a linear trap which are strongly bounded in the $y$ and $z$ directions but weakly bounded in a harmonic potential in the $x$ direction. The two ions are placed symmetrically at either side of the origin of the $x$ axis and oscillate around their equilibrium positions, $x_{10} = -d/2$, $x_{20} = d/2$. We denote by $\hat{X} = (\hat{x}_2 + \hat{x}_1)/2$, $\hat{x} = (\hat{x}_2 - \hat{x}_1)/2$ the center-of-mass and breathing mode operators, respectively. Both ions are simultaneously illuminated by two classical homogeneous standing wave lasers $E^{(+)}_I = E_{0I} \cos(k_I x + \varphi_I) e^{-i\omega_I t}$ and $E^{(+)}_{II} = E_{0II} \cos(k_{II} x + \varphi_{II}) e^{-i\omega_{II} t}$, with same amplitudes $E_{0I} = E_{0II} = E_0$ and same effective wavevectors $k_I = k_{II} = k$, but with different frequencies $\omega_I$ and $\omega_{II}$. Experimentally, there are two ways to implement the two standing waves: One is to use two-photon stimulated-Raman transitions for each of the fields[22]. And the other, for single photon electric-dipole transitions, one can arrange appropriately two standing waves, which have wavevectors $\vec{k}'_I$, $\vec{k}'_{II}$ and frequencies $\omega_I$, $\omega_{II}$, so that the projective waves onto $\hat{x}$ direction have the required effective wavevectors $k_I = \vec{k}'_I \cdot \hat{x} = k_{II} = \vec{k}'_{II} \cdot \hat{x}$. We assume that each ion is located at the anti-nodes of both standing waves, so that $\varphi_I = \varphi_{II} = 0$. 
The Hamiltonian of this system can be written as,

\[ H = H_0 + H_{\text{int}}, \] (1)

\[ H_0 = \mu a^+ a + \nu b^+ b + \omega_0 (\sigma_{z1} + \sigma_{z2})/2, \] (2)

\[ H_{\text{int}} = \sum_{i=1}^{2} \frac{\Omega}{2} \sigma_{+i} [(e^{-i\omega_1 t} + e^{-i\omega_2 t}) \cos(k\hat{x}_i) + H.C.] \] (3)

where \( \mu (\nu) \) and \( a (b) \) are the frequency and annihilation operator of the center-of-mass mode (breathing mode), \( \omega_0 \) is the energy difference between the ground state \(|g\rangle\) and the long-lived metastable excited state \(|e\rangle\) of each ion, \( \sigma_{zi} \) and \( \sigma_{+i} \) are Pauli operators describing the internal states of \( i \)th ion, \( \Omega \) is the Rabi frequency associated with both standing waves.

For simplicity, the Planck’s constant is set \( \hbar = 1 \). We start by taking the frequencies of the two standing wave lasers to excite resonantly the both upper sidebands and both lower sidebands of center-of-mass and breathing modes, e.g.

\[ \omega_I = \omega_0 - (\mu + \nu), \quad \omega_{\bar{I}} = \omega_0 + (\mu + \nu). \] (4)

If both the center-of-mass mode and breathing mode are cooled under Lamb-Dicke limit, then we can expand the Hamiltonian up to the order terms of \( \eta^2 \) or \( \eta_r^2 \) which represent the lowest couplings between internal and external degrees of freedom. Transforming the above Hamiltonian to the interaction picture with respect to \( H_0 \) and making use of rotating wave approximation, we have,

\[ H_{\text{int}}^I = \Omega \eta \eta_r (ab + a^+ b^+) (\sigma_{x1} - \sigma_{x2}) \] (5)

where \( \sigma_{xi} \) is the \( x \) component of Pauli operator describing the internal state of \( i \)th ion, and \( \eta = k\sqrt{1/4m\mu} \) and \( \eta_r = k\sqrt{1/4m\nu} \) are the Lamb-Dicke parameters corresponding to the center-of-mass and breathing modes.

It is worthwhile to point out that the Hamiltonian (5) can also be achieved by employing two classical running waves under similar conditions. The advantage of using standing waves
rather than running waves is that all the odd order terms in the expansion of Hamiltonian (3) are suppressed, so that, contrary to running waves, the incorrectness from those terms, particularly from the first and the third order terms, vanishes automatically. In this sense, the use of standing waves may make the scheme more reliable than the use of running waves. In addition, reference [23] advanced a method to produce similar coupling as (5) between the two-dimensional modes of a trapped ion and its internal states, but which needs more number of lasers. Observing the commutation relations of the spin and boson operators, the exact propagator for this Hamiltonian reduces to

$$U = \frac{1}{4} \left\{ [S^\dagger(G) + S(G)]^2 - \sigma_{x1}\sigma_{x2}[S^\dagger(G) - S(G)]^2 
+ (\sigma_{x1} - \sigma_{x2})[S^\dagger(G) + S(G)][S^\dagger(G) - S(G)] \right\} \tag{6}$$

where $G = -i\Omega_\eta_\eta_\nu t$ and $S(G) = \exp(G^\dagger a b - G a^\dagger b^\dagger)$ is the unitary two-mode squeezing operator.

Now we assume that, initially, both the center-of-mass mode and breathing mode are cooled to ground states, and the internal state are prepared in a superposition of ground and excited states (which can be realized by using carrier transition), so that the whole initial state is

$$|\psi_0\rangle = \frac{1}{2} (|e\rangle_1 - |g\rangle_1) (|e\rangle_2 + |g\rangle_2) |0\rangle_c |0\rangle_r, \tag{7}$$

where $|0\rangle_c$ and $|0\rangle_r$ refer to vacuum states of center-of-mass mode and breathing mode, respectively. By illuminating both the two ions simultaneously with the two standing waves discussed above, e.g., performing unitary transformation (6) on the initial state, after a interaction time $t$, the system evolves as,

$$|\psi_1\rangle = \frac{1}{2} (|e\rangle_1 - |g\rangle_1) (|e\rangle_2 + |g\rangle_2) S(2G) |00\rangle_{cr}. \tag{8}$$

At this moment, the internal and external degrees of freedom are unentangled and the motional state is then in a two-mode squeezed vacuum state,

$$|0, 0, G\rangle_{squ} = S(2G) |00\rangle_{cr}. \tag{9}$$
The degrees of squeezing is measured by the squeezing factor $2G$ which can be enhanced by simply increasing the interaction time $t$.

The unitary transformation (6) possesses an interesting configuration, which, for the initial state of (7), entangles the two kinds of external motion, but does not entangle the internal states of the two ions, furthermore, the internal and external degrees of freedom of the evolved state are always unentangled at any time. These properties enable us to produce the wanted two-mode squeezed state without any measurements on the internal states of the two ions.

The unitary transformation (6) can also be used to produce superposition states of two-mode squeezed states with different degrees of squeezing. For this purpose, we firstly prepare the initial state of the whole system as,

$$|\varphi_0\rangle = N_0 (|e\rangle_1 + p_1 |g\rangle_1) (|e\rangle_2 - p_2 |g\rangle_2) |00\rangle_{cr},$$  

where $N_0 = \left[\left(1 + |p_1|^2\right) \left(1 + |p_2|^2\right)\right]^{-1/2}$ with complex parameters $p_1$ and $p_2$ being the controlling weights of the internal levels of the two ions respectively. Then, we let the system experience a unitary evolution governed by (6) for a duration $T$, followed by a measurement on the internal states of the two ions. With no fluorescence being detected, the conditioned state of the system reads

$$|\varphi_1\rangle \sim 2 \prod_{i=1}^{2} [(1 - p_i)S(G) + (1 + p_i)S(-G)] |ee\rangle |00\rangle_{cr}. $$  

We now set the internal states of the two ions again being in a superposition form similar to (10), but with weight factors $p_3$ and $p_4$ for the two ions respectively. After an interaction time $T$ governed by the unitary evolution (6), we perform a measurement on each ion. Repeating this procedure $m$ cycles with each cycle having identical interaction time $T$, if no fluorescence being detected in all cycles, with probability $p = \prod_{i=1}^{2m} \frac{1}{4} \left(1 + |p_i|^2\right)^{-1}$, the final conditioned state for the vibratic motion of the two-ion system is

$$|\varphi_m\rangle \sim 2 \prod_{i=1}^{2m} [(1 - p_i)S(G) + (1 + p_i)S(-G)] |00\rangle_{cr}$$

$$= \sum_{k=0}^{2m} C^k_{2m} S [2(k - m)G] |00\rangle_{cr}, $$  

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with
\[ C_{2m}^k = \sum \prod_{i \in \{k\}} (1 - p_i) \prod_{j \in \{2m-k\}} (1 + p_j), \] (13)
where \( \{k\} \) denotes a set that picking \( k \) elements out of \( 2m \) natural numbers corresponding to \( 2m \) parameters \( p_i \), and \( \{2m - k\} \) denotes the complementary set to \( \{k\} \), e.g., the set constituted by the residual elements. The first \( \prod \) represents the multiplication of \( k \) factors of \((1 - p_i)\) with \( i \) being the element of set \( \{k\} \), and the second \( \prod \) represents the multiplication of \((2m - k)\) factors of \((1 + p_j)\) with \( j \) being the element of set \( \{2m - k\} \). The sum in (13) is for all possible different sets \( \{k\} \) that formed by picking \( k \) elements out of \( 2m \) natural numbers.

Eq.(12) is a superposition state which contains one vacuum state and \( 2m \) two-mode squeezed states with different squeezing factors. By adjusting the superposition coefficients, it can approximate many types of entangled two-mode states, which have different degrees of entanglement. This is also one of the methods to produce two-mode entanglement states. Of course, in order to produce required two-mode entanglement states, we are usually demanded to solve all of \( p_i \) from given superposition coefficients \( C_{2m}^k \). The detail description of treating this problem can be found in reference[16].

Besides producing above two-mode squeezed vacuum states and their superposition, we can also produce more general two-mode squeezed states and their superposition. In reference [16], we have let two trapped ions experience simultaneously two running waves. By adjusting the frequencies of the two running waves to be resonant with the first red and blue side-bands of center-of-mass mode, in Lamb-Dicke Limits and under rotating wave approximation conditions, we have obtained the unitary propagator,
\[ U_{cc} = \frac{1}{4} \prod_{i=1}^{2} \left\{ [D^+(\beta_c) + D(\beta_c)] - \sigma_y[D^+(\beta_c) - D(\beta_c)] \right\}, \] (14)
with \( \beta_c = i\eta \Omega t \) and \( D(\beta_c) = \exp(\beta_c a^+ - \beta_c^* a) \) being displacement operator for center-of-mass mode. Similarly, if we adjust the frequencies of the two running waves to be resonant with the first red and blue side-bands of breathing mode rather than center-of-mass mode, in the same way, we can obtain a propagator,
$$U_{re} = \frac{1}{4} \left[ (D^+(\beta_r) + D(\beta_r))^2 - \sigma_{y1}\sigma_{y2}[D^+(\beta_r) - D(\beta_r)]^2 \right.$$  
\left. + (\sigma_{y1} - \sigma_{y2})[D^+(\beta_r) + D(\beta_r)][D^+(\beta_r) - D(\beta_r)] \right]$$

(15)

with $\beta_r = i\eta_r\Omega t'$ and $D(\beta_r) = \exp(\beta_r b^+ - \beta_r^* b)$ being displacement operator for breathing mode.

Now let us describe how to create a general two-mode squeezed state. After the state (8) is created, we let the two ions experience the following unitary carrier transition

$$U_e = \frac{1}{4} \left( 1 - i\sigma_{x1} + i\sigma_{y1} + i\sigma_{z1} \right) \left( 1 - i\sigma_{x2} - i\sigma_{y2} - i\sigma_{z2} \right)$$

(16)

to prepare the state of the whole system as,

$$|\psi_2\rangle = \frac{1}{2} \left[ \langle e \rangle_1 - i \langle g \rangle_1 \right] \left[ \langle e \rangle_2 - i \langle g \rangle_2 \right] S(2G) |00\rangle_{cr}.$$  

(17)

By performing unitary transformation (14) on this state for a time $t$, it evolves,

$$|\psi_3\rangle = \frac{1}{2} \left[ \langle e \rangle_1 - i \langle g \rangle_1 \right] \left[ \langle e \rangle_2 - i \langle g \rangle_2 \right] D(2\beta_c) S(2G) |00\rangle_{cr}.$$  

(18)

Then we only let the ion 1 experience a carrier transition

$$U_e' = -\sigma_{z1}$$

(19)

to reprepare the state as,

$$|\psi_4\rangle = \frac{1}{2} \left[ \langle e \rangle_1 + i \langle g \rangle_1 \right] \left[ \langle e \rangle_2 - i \langle g \rangle_2 \right] D(2\beta_c) S(2G) |00\rangle_{cr}.$$  

(20)

By performing unitary transformation (15) on it for a time $t'$, the final state of the whole system becomes

$$|\psi_5\rangle = \frac{1}{2} \left[ \langle e \rangle_1 + i \langle g \rangle_1 \right] \left[ \langle e \rangle_2 - i \langle g \rangle_2 \right] \otimes$$

$$D(2\beta_c) D(2\beta_r) S(2G) |00\rangle_{cr}.$$  

(21)

and the motional state is just a general two-mode squeezed state

$$|\beta_c, \beta_r, G\rangle_{squ} = D(2\beta_c) D(2\beta_r) S(2G) |00\rangle_{cr}.$$  

(22)
Following this way produced the superposition of two-mode squeezed vacuum states, we can also prepare the superposition of (22) type of states i.e. a superposition of general two-mode squeezed states.

In conclusion, we have studied the properties of two trapped ions interacting simultaneously with two standing waves, where each ion is located at the anti-nodes of both lasers. In Lamb-Dicke limits and under rotating wave approximation, by adjusting the frequencies of the two standing waves to drive resonantly the both upper and lower sidebands of center-of-mass and breathing modes, we have successfully realized the preparation of two-mode squeezed states. We have also presented a procedure to generate the superposition of several two-mode squeezed states with different degrees of squeezing, by using a combination of the methods developed here and previous papers.

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