ON PARTIAL SUMS OF NORMALIZED $q$-BESSEL FUNCTIONS

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Abstract. In the present investigation our main aim is to give lower bounds for the ratio of some normalized $q$-Bessel functions and their sequences of partial sums. Especially, we consider Jackson’s second and third $q$-Bessel functions and we apply one normalization for each of them.

1. Introduction

Let $A$ denote the class of functions of the following form:

\begin{equation}
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\end{equation}

which are analytic in the open unit disk

$$U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.$$

We denote by $S$ the class of all functions in $A$ which are univalent in $U$.

The Jackson’s second and third $q$-Bessel functions are defined by (see [4])

\begin{equation}
    J^{(2)}_\nu(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{q}\right)^{2n+\nu}}{(q; q)_n (q^{\nu+1}; q)_n} q^{n(n+\nu)}
\end{equation}

and

\begin{equation}
    J^{(3)}_\nu(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+\nu}}{(q; q)_n (q^{\nu+1}; q)_n} q^{\frac{1}{2}n(n+1)},
\end{equation}

where $z \in \mathbb{C}, \nu > -1, q \in (0, 1)$ and

$$\frac{a}{q} = 1, (a; q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}), (a, q)_\infty = \prod_{k=1}^{\infty} (1 - aq^{k-1}).$$

Here we would like to say that Jackson’s third $q$-Bessel function is also known as Hahn-Exton $q$-Bessel function.

Recently, the some geometric properties like univalence, starlikeness and convexity of the some special functions were investigated by many authors. Especially, in [1 5 6 8] authors have studied on the starlikeness and convexity of the some normalized $q$-Bessel functions. In addition, the some lower bounds for the ratio of some special functions and their sequences of partial sums were given in [3 7 10 11]. Moreover, results related with partial sums of analytic functions can be found in [2 9 12 13 14] etc.

Motivated by the previous works on analytic and some special functions, in this paper our aim is to present some lower bounds for the ratio of normalized $q$-Bessel functions to their sequences of partial sums.
Due to the functions defined by (1.2) and (1.3) do not belong to the class $A$, we consider following normalized forms of the $q$-Bessel functions:

\begin{equation}
 h^{(2)}_\nu (z; q) = 2^\nu c_\nu (q) z^{1-\frac{\nu}{2}} J^{(2)}_\nu (\sqrt{z}; q) = \sum_{n \geq 0} K_n z^{n+1}
\end{equation}

and

\begin{equation}
 h^{(3)}_\nu (z; q) = c_\nu (q) z^{1-\frac{\nu}{2}} J^{(3)}_\nu (\sqrt{z}; q) = \sum_{n \geq 0} T_n z^{n+1},
\end{equation}

where $K_n = \frac{(-1)^n q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n}$, $T_n = \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q; q)_n (q^{\nu+1}; q)_n}$ and $c_\nu (q) = (q; q)_\infty / (q^{\nu+1}; q)_\infty$. As a result of the above normalizations, all of the above functions belong to the class $A$.

### 2. Main Results

The following lemmas will be required in order to derive our main results.

**Lemma 1.** Let $q \in (0, 1)$, $\nu > -1$ and $4(1-q)(1-q^\nu) > q^\nu$. Then the function $h^{(2)}_\nu (z; q)$ satisfies the next two inequalities for $z \in U$:

\begin{equation}
 |h^{(2)}_\nu (z; q)| \leq \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu},
\end{equation}

\begin{equation}
 \left| (h^{(2)}_\nu (z; q))' \right| \leq \left( \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu} \right)^2.
\end{equation}

**Proof.** It can be easily shown that the inequalities

\begin{align*}
 q^{n(n+\nu)} &\leq q^{n\nu}, 
 (1-q)^n &\leq (q; q)_n \quad \text{and} \quad (1-q^\nu)^n \leq (q^{\nu+1}; q)_n
\end{align*}

are valid for $q \in (0, 1)$ and $\nu > -1$. Making use the above inequalities and well-known triangle inequality, for $z \in U$, we get

\begin{align*}
 |h^{(2)}_\nu (z; q)| &= \left| z + \sum_{n \geq 1} \frac{(-1)^n q^{n(n+\nu)}}{4^n (q; q)_n (q^{\nu+1}; q)_n} z^{n+1} \right| \\
 &\leq 1 + \sum_{n \geq 1} \frac{q^n}{4^n (q; q)_n (q^{\nu+1}; q)_n} \\
 &\leq 1 + \sum_{n \geq 1} \left( \frac{q^\nu}{4(1-q)(1-q^\nu)} \right)^n \\
 &\leq 1 + \frac{q^\nu}{4(1-q)(1-q^\nu)} \sum_{n \geq 1} \left( \frac{q^\nu}{4(1-q)(1-q^\nu)} \right)^{n-1} \\
 &= \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu}.
\end{align*}
and

\[ \left| (h_\nu^{(3)}(z; q))' \right| = \left| 1 + \sum_{n \geq 1} \frac{(-1)^n(n + 1)q^{n(n+\nu)}}{4^n(q; q)_n(q^{\nu+1}; q)_n} z^n \right| \]
\[ \leq 1 + \sum_{n \geq 1} \frac{(n + 1)q^{n(n+\nu)}}{4^n(q; q)_n(q^{\nu+1}; q)_n} \]
\[ \leq 1 + \sum_{n \geq 1} (n + 1) \left( \frac{q^{\nu}}{4(1-q)(1-q^{\nu})} \right)^n \]
\[ = \left( \frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu}) - q^{\nu}} \right)^2. \]

Thus, the inequalities (2.1) and (2.2) are proved. \( \square \)

Lemma 2. Let \( q \in (0, 1) \), \( \nu > -1 \) and \( (1-q)(1-q^{\nu}) > \sqrt{q} \). Then the function \( h_\nu^{(3)}(z; q) \) satisfies the inequalities

(2.3) \[ \left| h_\nu^{(3)}(z; q) \right| \leq \frac{(1-q)(1-q^{\nu})}{(1-q)(1-q^{\nu}) - \sqrt{q}}, \]

and

(2.4) \[ \left| (h_\nu^{(3)}(z; q))' \right| \leq \left( \frac{(1-q)(1-q^{\nu})}{(1-q)(1-q^{\nu}) - \sqrt{q}} \right)^2 \]

for \( z \in \mathcal{U} \).

Proof. It is known that the inequalities

\[ q^{\frac{1}{2}n(n+1)} \leq q^{\frac{1}{2}n}, (1-q)^n \leq (q; q)_n \text{ and } (1-q^{\nu})^n \leq (q^{\nu+1}; q)_n \]

are valid for \( q \in (0, 1) \) and \( \nu > -1 \). Now, using the well-known triangle inequality for \( z \in \mathcal{U} \), we have

\[ \left| h_\nu^{(3)}(z; q) \right| = \left| z + \sum_{n \geq 1} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q; q)_n(q^{\nu+1}; q)_n} z^{n+1} \right| \]
\[ \leq 1 + \sum_{n \geq 1} \frac{q^{\frac{n}{2}}}{(1-q)^n(1-q^{\nu})^n} \]
\[ \leq 1 + \frac{\sqrt{q}}{(1-q)(1-q^{\nu})} \sum_{n \geq 1} \left( \frac{\sqrt{q}}{(1-q)(1-q^{\nu})} \right)^{n-1} \]
\[ = \frac{(1-q)(1-q^{\nu})}{(1-q)(1-q^{\nu}) - \sqrt{q}}. \]
Theorem 1. Let $\nu > -1, q \in (0, 1)$, the function $h^{(2)}_\nu : \mathcal{U} \to \mathbb{C}$ be defined by (1.4) and its sequences of partial sums by $h^{(2)}_\nu(z; q) = z + \sum_{n=1}^{\infty} K_n z^{n+1}$. If the inequality $2(1-q)(1-q^\nu) \geq q^\nu$, then the following inequalities hold true for $z \in \mathcal{U}$:

\[ \Re \left\{ \frac{h^{(2)}_\nu(z; q)}{(h^{(2)}_\nu)_m(z; q)} \right\} \geq \frac{4(1-q)(1-q^\nu) - 2q^\nu}{4(1-q)(1-q^\nu) - q^\nu}. \]  

(2.5)

\[ \Re \left\{ \frac{(h^{(2)}_\nu)_m(z; q)}{h^{(2)}_\nu(z; q)} \right\} \geq \frac{4(1-q)(1-q^\nu) - q^\nu}{4(1-q)(1-q^\nu)}. \]  

(2.6)

Proof. From the inequality (2.1) we have that

\[ 1 + \sum_{n \geq 1} |K_n| \leq \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu}. \]  

(2.7)

The inequality (2.7) is equivalent to

\[ \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \sum_{n \geq 1} |K_n| \leq 1. \]  

(2.8)

In order to prove the inequality (2.5), we consider the function $w(z)$ defined by

\[ \frac{1+w(z)}{1-w(z)} = \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \left\{ \frac{h^{(2)}_\nu(z; q)}{(h^{(2)}_\nu)_m(z; q)} - \frac{4(1-q)(1-q^\nu) - 2q^\nu}{4(1-q)(1-q^\nu) - q^\nu} \right\} \]  

which is equivalent to

\[ \frac{1+w(z)}{1-w(z)} = 1 + \sum_{n=1}^{m} K_n z^n + \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} K_n z^n. \]  

(2.9)
By using the equality (2.9) we get

\[ w(z) = \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} K_n z^n \]

and

\[ |w(z)| \leq \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} |K_n|.
\]

The inequality

(2.10)

\[ \sum_{n=1}^{m} |K_n| + \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \sum_{n=m+1}^{\infty} |K_n| \leq 1 \]

implies that \(|w(z)| \leq 1\). It suffices to show that the left hand side of (2.10) is bounded above by

\[ \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu} \sum_{n \geq 1} |K_n|, \]

which is equivalent to

\[ \frac{4(1-q)(1-q^\nu) - 2q^\nu}{q^\nu} \sum_{n \geq 1} |K_n| \geq 0. \]

The last inequality holds true for \(2(1-q)(1-q^\nu) \geq q^\nu\).

In order to prove the result (2.6) we use the same method. Now, consider the function \(p(z)\) given by

\[ \frac{1 + p(z)}{1 - p(z)} = \left(1 + \frac{4(1-q)(1-q^\nu) - q^\nu}{q^\nu}\right) \left\{\frac{\left(h_{\nu}^{(2)}\right)(z, q)}{\left(h_{\nu}^{(2)}\right)(z, q)} - \frac{4(1-q)(1-q^\nu) - q^\nu}{4(1-q)(1-q^\nu)}\right\}.
\]

Then from the last equality we get

\[ p(z) = \frac{-4(1-q)(1-q^\nu) \sum_{n=m+1}^{\infty} K_n z^n}{2 + 2 \sum_{n=1}^{m} K_n z^n - \frac{4(1-q)(1-q^\nu)}{q^\nu} \sum_{n=m+1}^{\infty} K_n z^n} \]

and

\[ |p(z)| \leq \frac{4(1-q)(1-q^\nu) \sum_{n=m+1}^{\infty} |K_n|}{2 - 2 \sum_{n=1}^{m} |K_n| - \frac{4(1-q)(1-q^\nu)}{q^\nu} \sum_{n=m+1}^{\infty} |K_n|}.
\]

The inequality

(2.11)

\[ \sum_{n=1}^{m} |K_n| + \frac{4(1-q)(1-q^\nu)}{q^\nu} \sum_{n=m+1}^{\infty} |K_n| \leq 1 \]

implies that \(|p(z)| \leq 1\). Since the left hand side of (2.11) is bounded above by

\[ \frac{4(1-q)(1-q^\nu) - q^\nu \sum_{n=1}^{m} |K_n|}{q^\nu} \geq 0 \]

the proof is completed. \(\Box\)
Theorem 2. Let $\nu > -1, q \in (0, 1)$, the function $h^{(2)}_\nu : \mathcal{U} \to \mathbb{C}$ be defined by (1.4) and its sequences of partial sums by $(h^{(2)}_\nu)_m(z; q) = z + \sum_{n=1}^{m} K_n z^{n+1}$. If the inequality $(1-q)(1-q^\nu) \geq q^\nu$ is valid, then the following inequalities hold true for $z \in \mathcal{U}$:

$$
\Re \left\{ \left( \frac{h^{(2)}_\nu(z; q)}{(h^{(2)}_\nu)_m(z; q)} \right)' \right\} \geq \frac{16(1-q)(1-q^\nu) ((1-q)(1-q^\nu) - q^\nu) + 2q^{2\nu}}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}},
$$

(2.12) \hspace{1cm}

$$
\Re \left\{ \left( \frac{(h^{(2)}_\nu)_m(z; q)}{(h^{(2)}_\nu(z; q))' \right) \right\} \geq \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}},
$$

(2.13)

Proof. From the inequality (2.2) we have that

$$
1 + \sum_{n \geq 1} (n+1) |K_n| \leq \left( \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu} \right)^2.
$$

(2.14)

The inequality (2.14) is equivalent to

$$
\frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n \geq 1} (n+1) |K_n| \leq 1.
$$

(2.15)

In order to prove the inequality (2.12), we consider the function $h(z)$ defined by

$$
\frac{1 + h(z)}{1 - h(z)} = \frac{(1-h(z))(1-q^\nu)}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n \geq 1} (n+1) |K_n|,
$$

where $\delta = \frac{16(1-q)(1-q^\nu)((1-q)(1-q^\nu) - q^\nu) + 2q^{2\nu}}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}}$. The last equality is equivalent to

$$
\frac{1 + h(z)}{1 - h(z)} = 1 + \sum_{n=1}^{m} (n+1) K_n z^n + \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) K_n z^n.
$$

(2.16)

By using the equality (2.13) we get

$$
h(z) = \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) K_n z^n
$$

and

$$
|h(z)| \leq \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n|.
$$

The inequality

$$
\sum_{n=1}^{m} (n+1) |K_n| + \frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n| \leq 1
$$

(2.17)

implies that $|h(z)| \leq 1$. It suffices to show that the left hand side of (2.17) is bounded above by

$$
\frac{(4(1-q)(1-q^\nu) - q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n \geq 1} (n+1) |K_n|,
$$

(2.17)
which is equivalent to
\[
\delta \sum_{n \geq 1} (n+1) |K_n| \geq 0.
\]

Thus, the result (2.12) is proved.

To prove the result (2.13), consider the function \( k(z) \) defined by
\[
1 + k(z) = \left\{ 1 + \frac{(4(1-q)(1-q^\nu) - q^{2\nu})^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \right\} \left\{ \frac{\left( h_{\nu}^{(2)} (z; q) \right)' - (4(1-q)(1-q^\nu) - q^{2\nu})^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \right\}.
\]

The last equality is equivalent to
\[
(2.18) \quad \frac{1 + k(z)}{1 - k(z)} = 1 + \sum_{n=1}^{m} (n+1) K_n z^n - \frac{(4(1-q)(1-q^\nu) - q^{2\nu})^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) K_n z^n.
\]

From the equality (2.17) we have
\[
(2.19) \quad \sum_{n=1}^{m} (n+1) |K_n| + \frac{(4(1-q)(1-q^\nu) - q^{2\nu})^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n| \leq 1
\]

implies that \( |k(z)| \leq 1 \). Since the left hand side of (2.19) is bounded above by \( \frac{(4(1-q)(1-q^\nu) - q^{2\nu})^2}{8(1-q)(1-q^\nu)q^\nu - q^{2\nu}} \sum_{n=m+1}^{\infty} (n+1) |K_n| \),

which is equivalent to
\[
\delta \sum_{n=m+1}^{\infty} (n+1) |K_n| \geq 0,
\]

the proof of result (2.13) is completed. □

**Theorem 3.** Let \( \nu > -1, q \in (0,1) \), the function \( h_{\nu}^{(3)} : U \to \mathbb{C} \) be defined by (1.5) and its sequences of partial sums by \( (h_{\nu}^{(3)})_m(z; q) = z + \sum_{n=1}^{m} T_n z^{n+1} \). If the inequality \((1-q)(1-q^\nu) \geq 2\sqrt{q}\) is valid, then the next two inequalities are valid for \( z \in U \):
\[
\Re \left\{ \frac{h_{\nu}^{(3)} (z; q)}{(h_{\nu}^{(3)})_m (z; q)} \right\} \geq \frac{(1-q)(1-q^\nu) - 2\sqrt{q}}{\sqrt{q}},
\]
\[
\Re \left\{ \frac{(h_{\nu}^{(3)})_m (z; q)}{h_{\nu}^{(3)} (z; q)} \right\} \geq \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}}.
\]
Proof. From the inequality (2.23) we have that

\[ 1 + \sum_{n \geq 1} |T_n| \leq \frac{(1 - q)(1 - q^\nu)}{(1 - q)(1 - q^\nu) - \sqrt{q}}. \]

The inequality (2.22) is equivalent to

\[ \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n \geq 1} |T_n| \leq 1. \]

In order to prove the inequality (2.20), we consider the function \( \phi(z) \) defined by

\[ \frac{1 + \phi(z)}{1 - \phi(z)} = \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \left\{ \frac{h^{(3)}(z; q)}{(h^{(3)}_m(z; q) - (1 - q)(1 - q^\nu) - 2\sqrt{q})} \right\}, \]

which is equivalent to

\[ \frac{1 + \phi(z)}{1 - \phi(z)} = \frac{1 + \sum_{n=1}^{m} T_nz^n + \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_nz^n}{1 + \sum_{n=1}^{m} T_nz^n}. \]

From the equality (2.24) we obtain

\[ \phi(z) = \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{2 + 2\sum_{n=1}^{m} T_nz^n + \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_nz^n}, \]

and

\[ |\phi(z)| \leq \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{2 - 2\sum_{n=1}^{m} T_nz^n - \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_nz^n} \sum_{n=m+1}^{\infty} |T_n|. \]

The inequality

\[ \sum_{n=1}^{m} |T_n| + \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n| \leq 1 \]

implies that \( |\phi(z)| \leq 1 \). It suffices to show that the left hand side of (2.25) is bounded above by

\[ \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n \geq 1} |T_n|, \]

which is equivalent to

\[ \frac{(1 - q)(1 - q^\nu) - 2\sqrt{q}}{\sqrt{q}} \sum_{n=1}^{m} |T_n| \geq 0. \]

The last inequality holds true for \( (1 - q)(1 - q^\nu) \geq 2\sqrt{q} \).

In order to prove the result (2.21), we consider the function \( \varphi(z) \) given by

\[ \frac{1 + \varphi(z)}{1 - \varphi(z)} = \left( 1 + \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \right) \left\{ \frac{(h^{(3)}_m(z; q) - (1 - q)(1 - q^\nu) - \sqrt{q})}{h^{(3)}(z; q)} \right\}. \]

Then from the last equality we get

\[ \varphi(z) = \frac{-\frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_nz^n}{2 + 2\sum_{n=1}^{m} T_nz^n - \frac{(1 - q)(1 - q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} T_nz^n}. \]
and

\[|\varphi(z)| \leq \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n|.\]

The inequality

\[
\sum_{n=1}^{m} |T_n| + \frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=m+1}^{\infty} |T_n| \leq 1
\]

implies that \(|\varphi(z)| \leq 1\). Since the left hand side of (2.26) is bounded above by

\[
\frac{(1-q)(1-q^\nu) - \sqrt{q}}{\sqrt{q}} \sum_{n=1}^{m} |T_n|,
\]

which is equivalent to

\[
\frac{(1-q)(1-q^\nu) - 2\sqrt{q}}{\sqrt{q}} \sum_{n=1}^{m} |T_n| \geq 0.
\]

This completes the proof of the theorem.

\[\square\]

**Theorem 4.** Let \(\nu > -1, q \in (0,1)\), the function \(h_{\nu}^{(3)} : U \to \mathbb{C}\) be defined by (1.3) and its sequences of partial sums by \((h_{\nu}^{(3)})_{m}(z; q) = z + \sum_{n=1}^{m} T_n z^{n+1}\). If the inequality \((1-q)(1-q^\nu) \geq 4\sqrt{q}\), then the next two inequalities are valid for \(z \in U\):

\[
\Re\left\{ \left(\frac{h_{\nu}^{(3)}(z; q)}{(h_{\nu}^{(3)}(z; q))'}\right) \right\} \geq \frac{(1-q)^2(1-q^\nu)^2 - 4(1-q)(1-q^\nu)\sqrt{q} + 2q}{2(1-q)(1-q^\nu)\sqrt{q} - q},
\]

\[
\Re\left\{ \left(\frac{(h_{\nu}^{(3)})_{m}(z; q)}{(h_{\nu}^{(3)}(z; q))'}\right) \right\} \geq \frac{(1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q}.
\]

**Proof.** From the inequality (2.4) we have that

\[
1 + \sum_{n=1}^{m} (n+1)|T_n| \leq \left(\frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu) - \sqrt{q}}\right)^2.
\]

The inequality (2.29) is equivalent to

\[
\frac{(1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q} \sum_{n=1}^{m} (n+1)|T_n| \leq 1.
\]

In order to prove the inequality (2.27), we consider the function \(\psi(z)\) defined by

\[
\frac{1 + \psi(z)}{1 - \psi(z)} = \frac{(1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q} \left\{ \left(\frac{h_{\nu}^{(3)}(z; q)}{(h_{\nu}^{(3)}(z; q))'}\right) - \lambda \right\},
\]

where \(\lambda = \frac{(1-q)^2(1-q^\nu)^2 - 4(1-q)(1-q^\nu)\sqrt{q} + 2q}{2(1-q)(1-q^\nu)\sqrt{q} - q}\). The last equality is equivalent to

\[
\frac{1 + \psi(z)}{1 - \psi(z)} = 1 + \sum_{n=1}^{m} (n+1) T_n z^n + \frac{(1-q)(1-q^\nu) - \sqrt{q})^2}{2(1-q)(1-q^\nu)\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n+1) T_n z^n.
\]
By using the equality (2.31) we get

$$
\psi(z) = \frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n + 1) T_n z^n
$$

and

$$
|\psi(z)| \leq \frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n + 1) |T_n|
$$

The inequality

$$
\sum_{n=1}^{m} (n + 1) |T_n| + \frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n + 1) |T_n| \leq 1
$$

implies that $|\psi(z)| \leq 1$. It suffices to show that the left hand side of (2.32) is bounded above by

$$
\frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q} \sum_{n=1}^{\infty} (n + 1) |T_n|,
$$

which is equivalent to

$$
\lambda \sum_{n=1}^{m} (n + 1) |T_n| \geq 0.
$$

Thus, the result (2.27) is proved.

To prove the result (2.28), consider the function $\rho(z)$ defined by

$$
\frac{1 + \rho(z)}{1 - \rho(z)} = \left\{1 + \frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q}\right\} \left\{\frac{\left(h^{(3)}_{\nu}(z; q)\right)'}{\left(h^{(3)}_{\nu}(z; q)\right)} - \frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q}\right\}.
$$

The last equality is equivalent to

$$
\frac{1 + \rho(z)}{1 - \rho(z)} = \frac{1 + \sum_{n=1}^{m} (n + 1) T_n z^n - \frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n + 1) T_n z^n}{1 + \sum_{n=1}^{\infty} (n + 1) T_n z^n}.
$$

From the equality (2.33) we get

$$
\rho(z) = -\frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n + 1) T_n z^n
$$

and

$$
|\rho(z)| \leq \frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n + 1) |T_n|.
$$

The inequality

$$
\sum_{n=1}^{m} (n + 1) |T_n| + \frac{\{(1-q)(1-q^\nu) - \sqrt{q}\}^2}{2(1-q)(1-q^\nu)/\sqrt{q} - q} \sum_{n=m+1}^{\infty} (n + 1) |T_n| \leq 1
$$
implies that $|\rho(z)| \leq 1$. Since the left hand side of (2.31) is bounded above by
\[
\frac{(1-q)(1-q^{\nu})-\sqrt{q}}{2(1-q)(1-q^{\nu})\sqrt{q}-q} \sum_{n \geq 1} (n+1)|T_n|,
\]
which is equivalent to
\[
\frac{(1-q)(1-q^{\nu}) ((1-q)(1-q^{\nu})-4\sqrt{q}) + 2q}{2(1-q)(1-q^{\nu})\sqrt{q}-q} \sum_{n=1}^{m} (n+1)|T_n| \geq 0,
\]
the proof of result (2.28) is completed.
\[\square\]

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