The nondynamical r-matrix structure of the elliptic Ruijsenaars-Schneider model with N=2

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Abstract

We demonstrate that in a certain gauge the elliptic Ruijsenaars-Schneider model with N=2 admits a nondynamical r-matrix structure and the corresponding classical r-matrix is the same as that of its non-relativistic counterpart (Calogero-Moser model) in the same gauge. The relation between our (classical) Lax operator and the Lax operator given by Ruijsenaars is also obtained.

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1 Introduction

Ruijsenaars-Schneider (RS) model as a relativistic-invariant generalization of the well-known nonrelativistic Calogero-Moser (CM) model[1], has been constructed and shown to be integrable[2]. It describes a completely integrable system of N one-dimensional interacting relativistic particles. Its importance lies in the fact that it is related to the dynamics of solitons in some integrable relativistic field theories[2—5] and the corresponding quantum model is realized in terms of commuting difference operators—the Macdonald operator and its elliptic generalization[8,9]. Recent development was shown that it can be obtained by a Hamiltonian reduction of the cotangent bundle of some Lie group[6,7]. Among all type RS
model, the elliptic RS model is the most general one and the other type such as the rational, hyperbolic and trigonometric case is just the various degenerations of the elliptic one. So, the study of the elliptic RS model is of great importance in the completely integrable relativistic particles system.

The Lax operator of the elliptic RS model was found by Ruijsenaars[2]. On the classical level, the r-matrix structure was constructed in [10] for the degeneration case (the rational, hyperbolic and trigonometric case). The r-matrix structure for the most general case — elliptic RS model was found in [11]. There exists a specific feature that the fundamental Poisson bracket of the Lax operator is given in terms of a quadratic bracket (or Sklyanin bracket)[13] and the corresponding r-matrix is shown to be of dynamical type. Consequently, besides some difficulties presented by the dynamical aspect of the r-matrix[11,12], the dynamical Yang-Baxter type relations for RS model is still an open problem[11]. To overcome the above problems may be that whether a “good” Lax operator for the RS model which has a nondynamical r-matrix structure could be found. In our former work[12], we found such a “good” Lax operator for the elliptic $A_{N-1}^{(1)}$ CM model. Our main purpose in this paper will be to construct such a “good” Lax operator for the elliptic RS model in the case of $N=2$ (i.e. the two particle system). In this new Lax representation (cf. that of Ruijsenaars’), we find that its r-matrix is of numerical type (nondynamical type) and is the same as that of its nonrelativistic version — CM model[12]. The corresponding r-matrix satisfies the classical Yang-Baxter equation.

2 Review of the RS model

The Ruijsenaars-Schneider model is the system of $N$ one-dimensional relativistic particles interacting by the two-body potential. In terms of the canonical variables $p_i, q_i$ ($i = 1, \ldots, N$) enjoying in the canonical Poisson bracket

$$\{p_i, p_j\} = 0 \quad , \quad \{q_i, q_j\} = 0 \quad , \quad \{p_i, q_j\} = \delta_{ij}$$

the Hamiltonian of the system is expressed as [2]

$$H = mc^2 \sum_{j=1}^{N} \cosh p_j \prod_{k \neq j} \left\{ \frac{\sigma(q_{jk} + \gamma)\sigma(q_{jk} - \gamma)}{\sigma^2(q_{jk})} \right\}^{\frac{1}{2}} , \quad q_{jk} = q_j - q_k \quad (1)$$

Here, $m$ denotes the particle mass, $c$ denotes the speed of light, $\gamma$ is the coupling constant and $\sigma(u)$ is some elliptic function defined in Eq. (9). The Hamiltonian Eq. (1) is known to be completely integrable[2]. The most effective way to show its integrability is to construct the Lax representation for the system (namely, to find the Lax operator, or the classical L-operator). One Lax representation for the elliptic RS
In this paper, we restrict ourselves to the case $N = 2$. Then, the Lax operator given by Ruijsenaars can be written as

$$L_R(u) = \left( \begin{array}{cc} \frac{\sigma(q_{ij} + \gamma)}{\sigma(q_{ij} + q_{ij})} & \frac{\sigma(q_{ij} - \gamma)}{\sigma(q_{ij} - q_{ij})} \\ \frac{\sigma(q_{ij} + q_{ij} + q_{ij})}{\sigma(q_{ij} + q_{ij})} & \frac{\sigma(q_{ij} - q_{ij} + q_{ij})}{\sigma(q_{ij} - q_{ij})} \end{array} \right)$$

(2)

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(3)

Here, we adopt another Lax operator $\tilde{L}_R(u)$, which is the same as that of Nijhoff et al in Ref. [11]

$$\tilde{L}_R(u) = \left( \begin{array}{cc} \frac{\sigma(q_{ij} + \gamma)}{\sigma(q_{ij} + q_{ij})} & \frac{\sigma(q_{ij} - \gamma)}{\sigma(q_{ij} - q_{ij})} \\ \frac{\sigma(q_{ij} + q_{ij} + q_{ij})}{\sigma(q_{ij} + q_{ij})} & \frac{\sigma(q_{ij} - q_{ij} + q_{ij})}{\sigma(q_{ij} - q_{ij})} \end{array} \right)$$

(4)

The relation of $\tilde{L}_R(u)$ with the standard Ruijsenaars $L_R(u)$ can be obtained from the following canonical transformation (or Poisson map)

$$q_i \rightarrow q_i , \quad p_i \rightarrow p_i + \frac{1}{2} \ln \prod_{k \neq i} \frac{\sigma(q_k + \gamma)}{\sigma(q_k - \gamma)}$$

(5)

The fundamental Poisson bracket of the Lax operator $\tilde{L}_R(u)$ can be given in the following quadratic r-matrix form [11, 13]

$$\{ \tilde{L}_R(u)_1, \tilde{L}_R(v)_2 \} = \tilde{L}_R(u)_1 \tilde{L}_R(v)_2 r_{12}^R(u, v) - r_{12}^L(u, v) \tilde{L}_R(u)_1 \tilde{L}_R(v)_2$$

$$+ \tilde{L}_R(u)_1 s_{12}^R(u, v) \tilde{L}_R(v)_2 - \tilde{L}_R(v)_2 s_{12}^L(u, v) \tilde{L}_R(u)_1$$

(6)

where

$$r_{12}^R(u, v) = a_{12}(u, v) - s_{12}(u) + s_{21}(v) , \quad r_{12}^L(u, v) = a_{12}(u, v) + u_{12}^+ + u_{12}^-$$

$$s_{12}^R(u, v) = s_{12}(u) + u_{12}^+ , \quad s_{12}^L(u, v) = s_{21}(v) - u_{12}^-$$

and

$$u_{12}^\pm = \sum_{i,j} \xi(q_{ij} \pm \gamma) e_{ij} \otimes e_{ij} , \quad a_{12}(u, v) = r_{12}^0(u, v) + \sum_{i=1}^2 \xi(u - v)e_{ii} \otimes e_{ii} + \sum_{i \neq j} \xi(q_{ij}) e_{ij} \otimes e_{ij}$$

$$r_{12}^0(u, v) = \sum_{i \neq j} \frac{\sigma(q_{ij} + u - v)}{\sigma(q_{ij})} e_{ij} \otimes e_{ij} , \quad s_{12}(u) = \sum_{i,j} (\tilde{L}_R(u) \partial_i \tilde{L}_R(u))^i_j e_{ij} \otimes e_{ij}$$

(7)

$\xi(x)$ is some elliptic function defined in Eq.(9). The matrix element of $e_{ij}$ is equal to $(e_{ij})^i_k = \delta_i^d \delta_{jk}$. It can be checked that the following symmetric condition hold for the r-matrices $r_{12}^\pm(u, v)$ and $s_{12}^\pm(u, v)$

$$r_{21}^\pm(u, v) = -r_{12}^\pm(u, v) , \quad s_{21}^\pm(u, v) = s_{12}^\pm(u, v)$$

(8)
It can be seen that the classical r-matrices \( r_{12}^\pm(u,v) \), \( s_{12}^\pm(u,v) \) are of dynamical ones (i.e. the matrix element of theirs do depend upon the dynamical variables \( q_i \)). The quadratic Poisson bracket Eq. (6) and the the symmetric conditions Eq. (7)—Eq. (8) lead to the evolution integrals \( tr(\tilde{L}_R(u))^n \) of the motion.

Due to the r-matrices depending on the dynamical variables, the Poisson bracket of \( \tilde{L}_R(u) \) is no longer closed. The complexity of the r-matrices Eq. (6) results in that it is still an open problem to check the generalized Yang-Baxter relations for the RS model. The same situation also occur for the standard Lax operator \( L_R(u) \) [11].

3 The new Lax representation for RS model and its nondynamical r-matrix structure

The dynamical r-matrix structure for the Lax operator of \( L_R(u) \) and its Poisson-equivalence \( \tilde{L}_R(u) \) for the elliptic RS model lead to some difficulties [11] in the investigation of the RS model: the Poisson algebra of the Lax operator is no longer closed and the generalized Yang-Baxter relations is still an open problem. However, the choice of a gauge for the Lax operator is quite important in that it influences to a great extent the complexity of the associated r-matrix structure [12]. This motivate us to find a new Lax representation of the RS model. Following the success in finding a “good” Lax operator of the elliptic CM model, we construct a new Lax operator (we call it as a “good” Lax operator in the sense that it has a nondynamical r-matrix structure). When taking the nonrelativistic limit of this Lax operator, we can obtain the “good” Lax operator for the corresponding CM model [12].

First, let us define some elliptic functions

\[
\begin{align*}
\theta^{(1)}(u) &= \theta \left[ \frac{1}{2} - \frac{u}{2} \right] (u, 2\tau) \\
\sigma(u) &= \theta \left[ \frac{1}{2} \frac{1}{2} \right] (u, \tau) \\
\theta \left[ \begin{array}{c}
a \\
b 
\end{array} \right] (u, \tau) &= \sum_{m=-\infty}^{\infty} \exp \left\{ \sqrt{-1} \pi \left[ (m + a)^2 \tau + 2(m + a)(z + b) \right] \right\} \\
\theta^{(2)}(u) &= \partial_u \{ \theta^{(1)}(u) \} \quad \xi(u) = \partial_u \{ \ln \sigma(u) \}
\end{align*}
\]  

where \( \tau \) is a complex number with \( \text{Im}(\tau) > 0 \). We find that there exist another Lax representation for the RS model and denote it by \( L(u) \)

\[
L(u)_j^i = \sum_{k=1}^{2} \frac{1}{\sigma(\gamma)} A(u + 2\gamma; q)_{ik} A^{-1}(u; q)_{jk} e^{\rho_k}, \quad i, j = 1, 2
\]

where the matrix \( A(u; q) \) is

\[
A(u; q) = \begin{pmatrix}
\theta^{(1)}(u + q_{12} + \frac{1}{2}) & \theta^{(1)}(u + q_{21} + \frac{1}{2}) \\
\theta^{(2)}(u + q_{12} + \frac{1}{2}) & \theta^{(2)}(u + q_{21} + \frac{1}{2})
\end{pmatrix}
\]
The new Lax operator given by us in Eq.(10) can be obtained from the $L_R(u)$ through a dynamical similarity transformation as follows

$$ L(u) = g(u) \tilde{L}_R(u) g^{-1}(u) $$

(11)

where

$$ g(u) = \begin{pmatrix} \theta^{(1)}(u + q_{12} + \frac{1}{2}) & -\theta^{(1)}(u + q_{21} + \frac{1}{2}) \\ \theta^{(2)}(u + q_{12} + \frac{1}{2}) & -\theta^{(2)}(u + q_{21} + \frac{1}{2}) \end{pmatrix} $$

Due to the transformation Eq.(11) being dependent up the dynamical variables $q_i$, the corresponding classical r-matrix structure would be changed drastically. Through the direct calculation, we find that the fundamental Poisson of $L(u)$ can be written in the standard quadratic Poisson-Lie bracket with a purely numerical r-matrix

$$ \{L_1(u), L_2(v)\} = [r_{12}(u - v), L_1(u)L_2(v)] $$

(12)

and the numerical r-matrix $r(u)$ is the same as that of Lax operator of the nonrelativistic counterpart—the elliptic CM model given by us[12]

$$ r(u) = \begin{pmatrix} a(u) & b(u) & c(u) & d(u) \\ b(u) & c(u) & a(u) & d(u) \\ c(u) & a(u) & b(u) & d(u) \\ d(u) & c(u) & b(u) & a(u) \end{pmatrix} $$

(13)

and

$$ a(u) = \frac{\theta^{(0)}(u)}{\theta^{(0)}(u)} - \frac{\sigma'(u)}{\sigma(u)} , \quad b(u) = \frac{\theta^{(1)}(u)}{\theta^{(1)}(u)} - \frac{\sigma'(u)}{\sigma(u)} \\
\quad c(u) = \frac{\theta^{(0)}(0)\theta^{(1)}(u)}{\theta^{(0)}(0)\theta^{(1)}(0)} , \quad d(u) = \frac{\theta^{(0)}(0)\theta^{(0)}(u)}{\theta^{(1)}(0)\theta^{(1)}(0)} $$

where $a(u), b(u), c(u), d(u)$ are all independent upon dynamical variable. The numerical r-matrix $r(u)$ defined in Eq.(13) satisfies the classical Yang-Baxter equation

$$ [r_{12}(u - v), r_{13}(u - \eta)] + [r_{12}(u - v), r_{23}(v - \eta)] + [r_{13}(u - \eta), r_{23}(v - \eta)] = 0 $$

(14)

and enjoys in the antisymmetric properties

$$ r_{12}(u) = -r_{21}(-u) $$

(15)

The nondynamical r-matrix $r(u)$ is equivalent to that of Sklyanin in [14] up to some scalar factor independent on the dynamical variable.
The standard quadratic Poisson-Lie bracket Eq.(12) of the Lax operator \( L(u) \) and the numerical r-matrix \( r(u) \) enjoying in the classical Yang-Baxter equation Eq.(14) and antisymmetry Eq.(15), make it possible to construct the quantum version of Eq.(12)

\[
R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v)
\]  (16)

where \( R_{12}(u - v) \) is the eight-vertex Baxter’s R-matrix and satisfies the quantum Yang-Baxter equation

\[
R_{12}(u - v)R_{13}(u - \eta)R_{23}(v - \eta) = R_{23}(v - \eta)R_{13}(u - \eta)R_{12}(u - v)
\]  (17)

The quantum version of Lax operator \( L(u) \) given by us is the quantum L-operator \( T(u) \) which satisfy the quantum Sklyanin algebra[14]. Moreover, the classical numerical r-matrix \( r(u) \) is the semi-classical limit of the quantum Baxter’s R-matrix \( R(u) \)

\[
R(u) = 1 + ur(u) + o(w^2)
\] , when the crossing parameter \( w \to 0 \)  (18)

the r-matrix \( r(u) \) given by us could also be obtained from that of Nijhoff et al through some kind classical twisting procedure of r-matrix[12].

Discussions

In this paper, we only consider the special case of \( N = 2 \) for the RS model. However, the results can be generalized to the generic case of \( 2 \leq N \). We will present the further results in the further paper. Moreover the same nondynamical r-matrix structure could be constructed for the rational, hyperbolic and trigonometric RS model.

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