Black holes and particles with zero or negative energy

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We study properties of particles with zero or negative energy and a nonzero orbital angular momentum in the ergosphere of a rotating black hole. We show that the sign of the particle energy is uniquely determined by the angular velocity of its rotation in the ergosphere. We give a simple proof of the fact that extreme black holes cannot exist. We investigate the question of the possibility of an unlimited energy increase in the center-of-mass system of two colliding particles, one or both of which have negative or zero energy.

Key words: black hole, Kerr metric, negative-energy particle, particle collision, geodesic

1. Introduction

As is known, in the ergosphere of rotating black holes described by the Kerr metric, there are special geodesics along which particles with a negative or zero energy (with a nonzero projection of angular momentum) can move [1]–[3]. Such particles, of course, are not observed in the region outside the ergosphere, where there are only particles with positive energy.

Here, we systematically analyze properties of geodesics in the ergosphere of a black hole and obtain bounds for the energy and the projection of the orbital momentum of particles with any energy value. Based on these bounds, we prove the statement that there cannot be an extreme black hole with a critical value of the intrinsic angular momentum of the black hole rotation. We show that the geodesics of particles with zero energy in the ergosphere of a black hole escape from under the gravitational radius and then go back under the gravitational radius (just like the geodesics of particles with negative energy as we previously showed [4]). We obtain estimates for the angular velocity of particles with positive, negative, and zero energy.

In this paper, we use the system of units in which the gravitational constant and the speed of light are equal to unity: $G = c = 1$. 
2. Geodesics in the Kerr metric

The Kerr metric of a rotating black hole \(^5\) in the Boyer-Lindquist coordinates \(^6\) has the form

\[
ds^2 = \frac{\rho^2 \Delta}{\Sigma^2} \, dt^2 - \frac{\sin^2 \theta}{\rho^2 \Sigma^2} (d\varphi - \omega \, dt)^2 - \frac{\rho^2}{\Delta} \, dr^2 - \rho^2 d\theta^2. \tag{1}\]

where

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \\
\Sigma^2 = (r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta, \quad \omega = \frac{2Mra}{\Sigma^2},
\]

\(M\) is the mass of the black hole, and \(aM\) is its angular momentum. We assume that \(0 \leq a \leq M\). The event horizon of the Kerr black hole is the surface given by the equation

\[
r = r_H \equiv M + \sqrt{M^2 - a^2}. 
\]

The surface defined by the equation

\[
r = r_C \equiv M - \sqrt{M^2 - a^2}.
\]

is called the Cauchy horizon. The surface of the static limit is defined by the formula

\[
r = r_1 \equiv M + \sqrt{M^2 - a^2 \cos^2 \theta}.
\]

The region of spacetime between the static limit and the event horizon is called the ergosphere \(^1\) \(^3\). The quantity

\[
S(r, \theta) = r^2 - 2Mr + a^2 \cos^2 \theta
\]

vanishes on the ergosphere boundary, and \(S(r, \theta) < 0\) inside the ergosphere.

Using the relation

\[
S \Sigma^2 + 4M^2 r^2 a^2 \sin^2 \theta = \rho^4 \Delta, \tag{2}\]

one can write the equations of geodesics for the Kerr metric \(^1\) \(^2\) \(^3\) (see \(^2\), Sec. 62 or \(^3\), Sec. 3.4.1) in the form

\[
\rho^2 \frac{dt}{d\lambda} = \frac{1}{\Delta} \left( \Sigma^2 E - 2MraJ \right), \quad \rho^2 \frac{d\varphi}{d\lambda} = \frac{1}{\Delta} \left( 2MraE + \frac{SJ}{\sin^2 \theta} \right), \tag{3}
\]

\[
\rho^2 \frac{dr}{d\lambda} = \sigma_r \sqrt{R}, \quad \rho^2 \frac{d\theta}{d\lambda} = \sigma_\theta \sqrt{\Theta}, \tag{4}
\]

where

\[
R = \Sigma^2 E^2 - \frac{SJ^2}{\sin^2 \theta} - 4MraEJ - \Delta \left[ m^2 \rho^2 + \Theta \right], \tag{5}
\]

\[
\Theta = Q - \cos^2 \theta \left[ a^2 (m^2 - E^2) + \frac{J^2}{\sin^2 \theta} \right]. \tag{6}
\]

Here \(E = \text{const}\) is the energy of a moving particle (called the energy at infinity in \(^1\)), \(J\) is the conserved projection of the particle angular momentum on the black hole rotation axis, \(m\) is the rest mass of the moving particle, \(\lambda\) is an affine parameter along the geodesic
(λ = τ/m for a particle with m ≠ 0, where τ is its proper time), and Q is the Carter constant (Q = 0 when moving in the equatorial plane θ = π/2). The constants σ_r, σ_θ = ±1 determine the direction of motion with respect to the coordinates r and θ.

Geodesic equations (3), (4) coincide with the Euler-Lagrange equations for the Lagrangian

\[ L = \frac{g_{ik} \, dx^i \, dx^k}{2 \, d\lambda / d\lambda}. \tag{7} \]

The corresponding generalized momenta are equal to

\[ p_i = \frac{\partial L}{\partial \left( \frac{dx^i}{d\lambda} \right)} = g_{ik} \frac{dx^k}{d\lambda}, \tag{8} \]

\[ p_t = E, \quad p_r = -\sigma_r \sqrt{\frac{R}{\Delta}}, \quad p_\theta = -\sigma_\theta \sqrt{\Theta}, \quad p_\varphi = -J. \tag{9} \]

It is easy to verify by direct calculation that

\[ p_i p_k g^{ik} = m^2. \]

We note that the signs of the momentum components in (9) correspond to the signs of the covariant components of the four-momentum in (7) (see Sec. 9): \( p_i = (E, -\mathbf{p}) \). The momenta \( p_t \) and \( p_\varphi \) are constant on geodesics, i.e. \( E \) and \( J \) are constant, which follows from the fact that the components of Kerr metric (1) are independent of the time \( t \) and angular coordinate \( \varphi \). The fact that the parameter \( E \) preserved on geodesics is interpreted as the particle energy can also be obtained based on the metric energy-momentum tensor of the classical action of a point particle [8].

As follows from (4), the parameters characterizing any geodesic should satisfy the conditions

\[ R \geq 0, \quad \Theta \geq 0. \tag{10} \]

For a geodesic corresponding to the trajectory of a moving test particle outside the event horizon, the condition of motion “forward in time” must be satisfied:

\[ dt / d\lambda > 0. \tag{11} \]

Conditions (11) and (11) lead to the following inequalities for the possible energy values \( E \) and the angular momentum projection \( J \) of a test particle at a point with the coordinates \((r, \theta)\) with a fixed value of \( \Theta \geq 0 \)[9].

- Outside the ergosphere \( S(r, \theta) > 0, \)

\[ E \geq \frac{1}{\rho^2} \sqrt{(m^2 \rho^2 + \Theta)S}, \quad J \in [J_-(r, \theta), \ J_+(r, \theta)], \tag{12} \]

\[ J_\pm(r, \theta) = \frac{\sin \theta}{S} \left[ -2rMaE \sin \theta \pm \sqrt{\Delta (\rho^4 E^2 - (m^2 \rho^2 + \Theta)S)} \right]. \tag{13} \]

- On the boundary of the ergosphere (for \( \theta \neq 0, \pi \))

\[ r = r_1(\theta) \implies E \geq 0, \tag{14} \]
\[ J \leq E \left[ \frac{Mr_1(\theta)}{a} + a \sin^2 \theta \left( 1 - \frac{m^2}{2E^2} - \frac{\Theta}{4Mr_1(\theta)E^2} \right) \right], \tag{15} \]

with the possible value \( E = 0 \) if \( m = 0 \) and \( \Theta = 0 \), in which case any value \( J < 0 \) is allowed.

- Inside the ergosphere for \( r_H < r < r_1(\theta) \) and \( S < 0 \),

\[ J \leq \frac{\sin \theta}{-S} \left[ 2r MaE \sin \theta - \sqrt{\Delta \left( \rho^4 E^2 - (m^2 \rho^2 + \Theta) \Sigma^2 \right)} \right], \tag{16} \]

and the particle energy, as is known, can have any value, both positive and negative.

As can be seen from inequalities (15) and (16), the angular momentum projection of a particle moving along a geodesic at the boundary and inside the ergosphere can be negative and arbitrarily large in absolute value for a fixed energy value. This property, found in [9, 10] for the Kerr metric, holds in the ergosphere of any black hole with an axially symmetric metric as was later shown in [11].

If the particle angular momentum projection \( J \) and a value \( \Theta \geq 0 \) are given, then from conditions (10) and (11), we obtain

\[ E \geq \frac{1}{\Sigma^2} \left[ 2Mr a J + \sqrt{\Delta \left( \frac{\rho^4 J^2}{\sin^2 \theta} + (m^2 \rho^2 + \Theta) \Sigma^2 \right)} \right]. \tag{17} \]

anywhere outside the horizon. The lower bound of the energy values here corresponds to \( R = 0 \). It can be seen from this inequality that the energy in the ergosphere can be negative only if the particle angular momentum projection is negative.

3. Effect of incident particles on the black hole rotation

As \( r \) tends to the horizon \( r_H \) (for \( \theta \neq 0, \pi \)), from (16) and (17), we obtain

\[ J \leq J_H = \frac{2Mr_H E}{a}, \quad E \geq \frac{aJ}{2Mr_H}. \tag{18} \]

Hence, \( J_H \) is the upper bound of values of the angular momentum projection of a particle with the energy \( E \) at the black hole event horizon.

We use the first inequality in (18) to estimate the effect of incident particles on the dimensionless angular momentum \( A = a/M \) of the black hole. From the energy and momentum conservation laws, we obtain the formula for the dimensionless angular momentum of the black hole after a particle falls through the event horizon:

\[ A' = \frac{aM + J}{(M + E)^2}. \tag{19} \]

Then

\[ A' - A = \frac{J - AE^2 - 2AME}{(M + E)^2} \leq \frac{J_H - AE^2 - 2AME}{(M + E)^2} = \frac{2E \left( \frac{Mr_H}{a} - a \right) - E^2 \frac{a}{M}}{(M + E)^2}. \tag{20} \]

Setting \( A = 1 \) in this relation and taking into account that \( r_H = a = M \) in this case, we obtain the proof of the following statement.
**Statement 1.** The fall of a particle into an extreme black hole with $A = 1$ leads to a decrease in its dimensionless angular momentum. The black hole becomes nonextreme.

This indicates that extreme black holes cannot exist in nature. We note that the estimate in [12] for the limit angular momentum of a black hole attainable with accretion of matter on it is $a = 0.998M$.

Taking into account that $J < 0$ at zero energy and using the left-hand side of inequality (20) for zero energy and the right-hand side of (20) for negative energies, we obtain the following statement.

**Statement 2.** The fall of a particle with zero or negative energy into a black hole with $A \leq 1$ leads to a decrease in its dimensionless angular momentum.

For test particles ($|E|/M \ll 1$) with $J \approx J_H$, we have

$$A' - A \approx \frac{2E}{(M + E)^2} \left( M \frac{a}{a} - a \right) \approx \frac{2E}{(M + E)^2} \left( M \frac{a}{a} - a \right).$$ \hspace{1cm} (21)

Therefore, we have the following statement.

**Statement 3.** The fall of a test particle with positive energy and an angular momentum projection close to the maximum value $J_H$ into a nonextreme black hole with $A < 1$ leads to an increase in its dimensionless angular momentum.

We here note that for nonextreme black holes, the admissible values of $J$ in the neighborhood outside the horizon are always strictly less than $J_H$ (see, e.g., [13]).

In the case of particles freely falling into a nonextreme black hole from infinity, we prove the following statement.

**Statement 4.** The dimensionless angular momentum of a nonextreme black hole can always be increased by the free fall from infinity of specially selected particles.

**Proof.** A simple analysis of the geodesic equations allows verifying that for nonrelativistic particles at infinity with $E = m$, the condition for a fall into a black hole from infinity in the equatorial plane is the inequality

$$-2(1 + \sqrt{1 + A}) \leq \frac{J}{mM} \leq 2(1 + \sqrt{1 - A}).$$ \hspace{1cm} (22)

In the fall of a test particle with $E = m \ll M$ and $J = 2(1 + \sqrt{1 - A})mM$, we have

$$A' - A = \frac{2(1 + \sqrt{1 - A})mM - Am^2 - 2AmM}{(M + m)^2} \approx \frac{2mM}{(M + m)^2} \left( 1 - A + \sqrt{1 - A} \right) > 0,$$ \hspace{1cm} (23)

which proves the statement.

We note that because of the simplicity of the proofs of our statements, there is no need to use the laws of black hole mechanics formulated in the famous paper [14] by analogy with thermodynamics.
4. Particles with zero energy in the Kerr metric

The study of the properties of a zero-energy particle is practically absent from the literature on black holes. We consider the features of geodesics for such particles.

The value $E = 0$ is possible at the boundary and inside the ergosphere. From the condition $\Theta \geq 0$ (see (10)) and expression (6), we obtain

$$E = 0 \Rightarrow Q \geq 0,$$

i.e., the Carter constant is nonnegative for test particles with zero energy.

From condition (11) and expression (3) for $\frac{dt}{d\lambda}$, we obtain

$$E = 0, \quad \rho^2 \frac{dt}{d\lambda} = -\frac{2rMaJ}{\Delta} > 0 \Rightarrow J < 0,$$

i.e., the angular momentum projection of zero-energy particles is negative. From inequality (16) for zero-energy particles inside the ergosphere, we have

$$E = 0 \Rightarrow J \leq -\sin \theta \sqrt{\frac{\Delta(m^2\rho^2 + \Theta)}{-S}}. \quad (26)$$

In the case of zero energy of a particle,

$$E = 0 \Rightarrow R = -\Delta(m^2\rho^2 + \Theta) - \frac{J^2}{\sin^2 \theta} S(r, \theta). \quad (27)$$

Therefore, the upper point of the trajectory of a massive particle with zero energy is inside the ergosphere, $r < r_1(\theta)$.

Motion along the coordinate $\theta$ can continue as long as $\Theta(\theta)$ does not vanish. Therefore, in the case $m = 0$, we find from (27) that the upper point of the trajectory of a massless zero-energy particles is on the ergosphere boundary, $r = r_1(\theta)$.

It follows from the geodesic equation at zero energy that

$$\frac{d\varphi}{dt} = \frac{-S(r)}{2rMa \sin^2 \theta} = \frac{r^2 - 2rM + a^2 \cos^2 \theta}{-2rMa \sin^2 \theta}. \quad (28)$$

Therefore, the angular velocity of any zero-energy particle is independent of the mass and momentum of the particle and is given by (28). If a zero-energy particle reaches the ergosphere boundary (hence, $m = 0$ necessarily), then $S(r) = 0$ and

$$r \to r_1(\theta) \Rightarrow \frac{d\varphi}{dt} \to 0. \quad (29)$$

For the radial velocity of a particle with zero energy, we have

$$E = 0 \Rightarrow \frac{dr}{dt} = \frac{\sigma_r \Delta}{2Mra} \sqrt{\frac{-S}{\sin^2 \theta} - \frac{\Delta m^2\rho^2 + \Theta}{J^2}}. \quad (30)$$

Therefore, the radial velocity, generally speaking, depends on the angular momentum. Namely, in the case $m^2\rho^2 + \Theta > 0$ at a point with a given $r$ in the ergosphere, the larger $|J|$ is, the larger the radial velocity in absolute value, and it can range from

$$\frac{dr}{dt} = 0 \text{ at } J = -\sin \theta \sqrt{\frac{\Delta(m^2\rho^2 + \Theta)}{-S}}.$$
to
\[
\frac{dr}{dt} = \frac{\sigma r \Delta \sqrt{-S}}{2M r a \sin \theta} \quad \text{at} \quad J \to -\infty.
\]

The trajectory of photons in the equatorial plane \((m = 0, \Theta = 0)\) is independent of the angular momentum, as can be seen from Eqs. (3) and (4). In polar coordinates, the equation of this trajectory can be written in elementary functions and has the simplest form at \(a = M\):

\[
\varphi(r) - \varphi(2M) = \pm \left( \arcsin \left( \frac{r}{M} - 1 \right) + \frac{\sqrt{(2M - r)r}}{r - M} - \frac{\pi}{2} \right).
\]  

(31)

We show part of the trajectory in Fig. 1a, where we use the radial coordinate \(r/M\). In Fig. 1b, we show the trajectory of a photon moving in the equatorial plane along a geodesic with positive energy and the orbital angular momentum projection \(J = ME\). The equation of this geodesic at \(a = M\) has the form

\[
\varphi(r) - \varphi(2M) = \pm \frac{2M - r}{r - M}.
\]  

(32)

The presented trajectories, like trajectories of any particles crossing the event horizon of the black hole, wrap around the horizon infinitely many times. Indeed, it follows from geodesics equations (3) and (4) that

\[
\frac{d\varphi}{dr} \sim \frac{1}{\Delta} = \frac{1}{(r - r_H)(r - r_C)}
\]  

(33)

near the horizon. We note that except in the case where \(a = M\) and \(J = J_H > 0\), a massive particle makes an infinite number of turns in a finite proper time!

We previously showed [4] that in the ergosphere, the geodesics of particles with negative energies begin and end on the surface \(r_H\). We prove here that in the ergosphere, geodesics of zero-energy particles also begin and end on the surface \(r_H\). For this, we define the effective potential by the formula

\[
V_{\text{eff}} = -\frac{R}{2\rho^2}.
\]  

(34)
In accordance with Eq. (4), we have

\[ \frac{1}{2} \left( \frac{dr}{d\lambda} \right)^2 + V_{\text{eff}} = 0, \quad \frac{d^2 r}{d\lambda^2} = -\frac{dV_{\text{eff}}}{dr}. \] (35)

The necessary condition of existence of orbits with a constant \( r \) (spherical orbits) can be written as

\[ V_{\text{eff}} = 0, \quad \frac{dV_{\text{eff}}}{dr} = 0. \] (36)

To prove the above statement, it suffices to prove (see [4]) that

\[ E = 0, \quad r > r_H, \quad V_{\text{eff}}(r) = 0 \Rightarrow V'_{\text{eff}}(r) > 0. \] (37)

Differentiating (27), we obtain

\[ E = 0 \Rightarrow R'(r) = -2 \left[ (r - M) \left( \frac{J^2}{\sin^2 \theta} + m^2 \rho^2 + \Theta \right) + m^2 r \Delta \right], \] (38)

whence it clearly follows that condition (37) holds. This proves our statement and, in particular, shows the absence of spherical orbits for zero-energy particles.

We note that the finiteness of the proper time of motion of zero-energy particles in the ergosphere (or the affine parameter \( \lambda \) for photons) follows from (37), as shown in [4].

As can be seen from Fig. 1, there are trajectories of photons with positive energy (and also, as can be seen from similar figures, with negative energy) that visually differ little from the trajectory of a zero-energy photon. We further consider the question of which simple properties of the particle motion in the ergosphere allow distinguishing particles with negative, zero, and positive energy.

5. The angular velocity of particles in the ergosphere

We find the constraints on the angular velocity of particles in the ergosphere from the condition \( ds^2 \geq 0 \). We have

\[ g_{00} \, dt^2 + 2g_{0\varphi} \, dt \, d\varphi + g_{\varphi\varphi} \, d\varphi^2 \geq 0, \] (39)

and the angular velocity \( \Omega = d\varphi/dt \) of any particle satisfies the constraints [15]

\[ \Omega_1 \leq \Omega \leq \Omega_2, \quad \Omega_{1,2} = \frac{g_{0\varphi} \pm \sqrt{g_{0\varphi}^2 - g_{00}g_{\varphi\varphi}}}{-g_{\varphi\varphi}}. \] (40)

On the ergosphere boundary, \( g_{00} = 0 \) and \( \Omega_1 = 0 \). Inside the ergosphere, \( g_{00} < 0, \Omega_{1,2} > 0, \) and all the particles move in the direction of the black hole rotation [11–3]. In approaching the event horizon,

\[ \lim_{r \to r_H} \Omega_1(r) = \lim_{r \to r_H} \Omega_2(r) = \omega_{\text{Bh}} = \frac{a}{2Mr_H}. \] (41)

The value \( \omega_{\text{Bh}} \) is called the angular velocity of the black hole rotation.
Substituting the components of metric (1) in (40), we obtain the limit values for the angular velocity of a Kerr rotating black hole:

\[ \Omega_{1,2} = \frac{2Mra\sin\theta \mp \rho^2\sqrt{\Delta}}{\sin\theta \Sigma^2} = \omega \mp \frac{\rho^2\sqrt{\Delta}}{\sin\theta \Sigma^2}. \] 

(42)

From the geodesic equations for the angular velocity of freely moving particles, we obtain

\[ \frac{d\varphi}{dt} = \frac{2MraE + \frac{SJ}{\sin^2\theta}}{\Sigma^2 E - 2MraJ} = \omega + \frac{\Delta \rho^4 J}{\sin^2\theta\Sigma^2(\Sigma^2 E - 2MraJ)}. \] 

(43)

From restriction (16) previously found for particles with negative energy, we obtain the estimate

\[ \frac{J}{EM} \geq \frac{\sin\theta}{-S(r)} \left[ 2ra \sin\theta + \frac{1}{M} \sqrt{\Delta \left( \rho^4 - \left( \frac{m^2}{E^2}\rho^2 + \frac{\Theta}{M^2E^2} \right) S(r) \right) } \right], \] 

(44)

and for particles with positive energy in the ergosphere, we obtain the estimate

\[ \frac{J}{EM} \leq \frac{\sin\theta}{-S(r)} \left[ 2ra \sin\theta - \frac{1}{M} \sqrt{\Delta \left( \rho^4 - \left( \frac{m^2}{E^2}\rho^2 + \frac{\Theta}{M^2E^2} \right) S(r) \right) } \right]. \] 

(45)

Substituting the boundary values of expressions (44) and (45) for \( m = 0 \) and \( \Theta = 0 \) in (43), we obtain expressions for \( \Omega_1 \) and \( \Omega_2 \) (see formulas (12)). Taking into account that the angular velocity \( d\varphi/dt \) according to (43) increases as \( J/(EM) \) on each of the continuity intervals \((-\infty, \Sigma^2/(2M^2ra)) \) and \( (\Sigma^2/(2M^2ra), +\infty) \), we obtain a partition of the possible angular velocities in the ergosphere into two subdomains: the lower for particles with negative energy and the upper for particles with positive energy (see Fig. 2).

Passing to the limit \( |J/(EM)| \to \infty \) in equality (43), we obtain expression (28). Therefore, the boundary of the areas for particles with positive and negative energy is a line corresponding to the angular velocity of zero-energy particles. In the case of rotation in the equatorial plane, this is a straight line. Therefore, the value of the angular velocity of a particle in the ergosphere at a given value of the radial coordinate clearly indicates the sign of the particle energy and the value of the ratio \( J/(EM) \). Zero-energy particles are particles that move in the ergosphere with the angular velocity determined by (28). They separate particles in the ergosphere with negative energy from the particles with positive energy by the value of the angular velocity at a given \( r \) Particle with negative energy are particles that rotate in the ergosphere with an angular velocity less than velocity (28)!

From formulas (12), it is easy to obtain that

\[ \theta = \frac{\pi}{2} \Rightarrow \frac{\partial \Omega_2}{\partial r} \bigg|_{r=2M} = \frac{1 - 2A^2}{(2 + A^2)^2}, \]

and the tangent to the graph of the function \( \Omega_2(r) \) at the point with \( r = 2M \) (on the ergosphere boundary) is therefore horizontal at \( A = 1/\sqrt{2} \), as seen in Fig. 2b. The light green line line in Figs. 2c and 2d corresponds to circular equatorial orbits of massive
Figure 2: Domains of possible values of the angular velocities of particles in the ergosphere for $\theta = \pi/2$ with (a) $a/M = 0.5$, (b) $a/M = 1/\sqrt{2}$, (c) $a/M = 0.95$, (d) $a/M = 1$: the red area corresponds to the angular velocity of positive-energy particles and the blue area corresponds to the angular velocity of negative energy particles.

particles, and its limit point on the boundary of possible angular velocities corresponds to a circular photon orbit. Circular equatorial orbits in the ergosphere are only possible for particles with positive energy and only if $A \geq 1/\sqrt{2}$. As is known [2], [3], [16], the smallest possible value of the radius of a circular equatorial orbit at a given $A$ corresponds to the circular orbit of photons with $J > 0$ and is equal to

$$r_+ = 2M \left[ 1 + \cos \left( \frac{2}{3} \arccos \frac{a}{M} \right) \right].$$

6. The collision energy of particles in the center-of-mass system

We find the energy $E_{c.m.}$ in the center-of-mass system of two colliding particles with the rest masses $m_1$ and $m_2$ by squaring the expression

$$(E_{c.m.},0,0,0) = p^{i}_{(1)} + p^{i}_{(2)},$$

where $p^{i}_{(n)}$ is the four-momentum of the particles ($n = 1, 2$). Because $p^{i}_{(n)}p^{i}_{(n)i} = m^2_{n}$, we have

$$E^2_{c.m.} = m^2_{1} + m^2_{2} + 2p^{i}_{(1)}p^{i}_{(2)i}.$$  

For free-falling particles with the energies $E_1$ and $E_2$ (at infinity) and the angular momenta
\( J_1 \) and \( J_2 \), we obtain

\[
E_{c.m.}^2 = m_1^2 + m_2^2 - \frac{2}{\rho^2} \sigma_{1\theta} \sigma_{2\theta} \sqrt{\Theta_1 \Theta_2} + \frac{2}{\Delta \rho^2} \left[ E_1 E_2 \Sigma^2 - 2 M r (E_1 J_2 + E_2 J_1) - J_1 J_2 \frac{S}{\sin^2 \theta} - \sigma_{1r} \sigma_{2r} \sqrt{R_1 R_2} \right]
\]

(48)

from the geodesic equations. If the motion of the particles is codirected along the radial coordinate \((\sigma_{1r}, \sigma_{2r} = 1)\), then in the limit \( r \to r_H \), resolving the uncertainty of the type 0/0 in the last term in (48), we obtain

\[
E_{c.m.}^2 = m_1^2 + m_2^2 - \frac{2}{\rho^2} \sigma_{1\theta} \sigma_{2\theta} \sqrt{\Theta_1 \Theta_2} + \frac{2}{\Delta \rho^2} \left[ m_1^2 + \frac{\Theta_{1H}}{\rho_{1H}^2} \right] J_{2H} - J_2 + \frac{\rho_{2H}^2}{4 M^2 r_{2H}^2 \sin^2 \theta} (J_{1H} J_2 - J_{2H} J_1)^2
\]

(49)

for collisions on the horizon. If one of the particles has the angular momentum projection \( J = J_H \) (critical particle) and the second has \( J \neq J_H \), then the energy of collision on the horizon diverges. This was found in the paper of Bañados-Silk-West [17] for extreme rotating black holes. For nonextreme black holes in the neighborhood of the horizon, the value \( J = J_H \) is inadmissible, but with multiple collisions, generally speaking, it is possible [18, 19] that \( J \) reaches values arbitrarily close to \( J_H \) as \( r \to r_H \). We can obtain an arbitrarily high collision energy if the particle acquires an angular momentum projection that is large in absolute value but negative as a result of multiple collisions or the influence of external fields [9, 10], and this is possible with a fixed value of the particle energy in accordance with inequality (16).

The energy of head-on collisions \((\sigma_{1r}, \sigma_{2r} = -1)\) on the horizon diverges,

\[
E_{c.m.}^2 \sim \frac{4 a^2}{\Delta \rho^2} (J_{1H} - J_1)(J_{2H} - J_2) \to \infty, \quad r \to r_H,
\]

(50)

if \( J_i \neq J_{iH} \). This option to achieve ultrahigh collision energy is available for particles moving along the geodesic of a white hole [20, 21].

Formulas (46)–(50) hold for any of colliding particles with both positive and negative energy. For all values of the particle energy, the energy in the center-of-mass system always satisfies

\[
E_{c.m.} \geq m_1 + m_2,
\]

because the colliding particles in the center-of-mass system are moving toward each other at a certain speed. Three of the above ways to achieve infinitely large collision energy can also be realized for particles with a negative (zero) energy.

Calculations of the collision energy of two particles of equal mass \( m \), of which one falls from infinity into a black hole with \( a = 0.95 M \) and the other has positive, zero, or negative energy are shown in Fig. 3. At the selected value \( J \approx -16.487 m M \) of the angular momentum projection, the radial component of the velocity of the negative-energy particle is equal to zero at the point with \( r = 1.8 M \) for a black hole with \( a = 0.95 M \). Therefore, the collision energy is maximum here. The other particles fall with a nonzero radial velocity in the range \( r \in (M, 1.8M) \), and the collision energy is less.
Figure 3: Collision energy of a particle with $E = m$ and $J = 0$ and a second particle with $J = -16.487mM$ and the energy $E_2 = m$ (curve 1), $E_2 = 0$ (curve 2), or $E_2 = -m$ (curve 3).

Acknowledgements. This research was supported by the Russian Foundation for Basic Research (Grant No. 15-02-06818-a) and by funds from a subsidy provided in the framework of government support of Kazan Federal University for the purpose of increasing its competitiveness among leading world science education centers.

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