Localized ASD moduli spaces based on the reduced cohomology group over Casson handles

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Introduction

One of the most important aspects in global analysis on non compact spaces is the choice of functional spaces over them. Of particular interest for us is analysis of elliptic differential operators over non compact manifolds, which deeply touches topological structure of their underlying spaces. When one uses standard Sobolev spaces or weighted ones as the analytic setting, Fredholm property often breaks, which happens when continuous spectrum appears near zero. Let us describe such situation in terms of the cohomology groups. Let $X$ be a space and:

$$
0 \rightarrow C^0(X) \xrightarrow{d_0} C^1(X) \xrightarrow{d_1} C^2(X) \rightarrow 0.
$$

be a bounded complex between topological cochains $C^*(X)$. One obtains two different types of cohomology groups from this complex, where one is the ordinary cohomology $H^i(X)$ and the another is the reduced one $\bar{H}^i(X)$:

$$
H^i(X) = \text{Ker } d_i / \text{im } d_{i-1}, \quad \bar{H}^i(X) = \text{Ker } d_i / \text{im } \bar{d}_{i-1}
$$

where $\text{im } \bar{d}_{i-1}$ is the closure of $\text{im } d_{i-1} \subset C^i(X)$. There is a canonical surjection $H^i(X) \rightarrow \bar{H}^i(X)$.

These coincide with each other when $X$ is compact, while let us consider the differential over $L^2$ functions on the universal covering space of a compact manifold. It was verified by Brooks that these coincide with each other, if and only if the fundamental group is non amenable. Such difference is reflected on the behavior of the spectrum of the Laplace operator near zero, whether continuous spectrum attains zero or it is isolated.

In this paper we introduce a new functional analytic framework which is based on the reduced cohomology groups of elliptic complexes over non compact
manifolds, and develop a deformation theory of the Fredholm operators on index theory. Our original motivation to use the reduced cohomology arose by the work by Ballmann, Brüning and Carron, which include index theory of elliptic differential operators over cylindrical manifolds with boundary ([BBC]).

We shall apply it to non linear PDE analysis and obtain a topological constrain of complexity of smooth structure over non compact open subsets which are embedded into compact smooth four manifolds.

Let $E_i \rightarrow X$ be Euclidean vector bundles over $X$ and fix a sufficiently large $k$. Let us consider a family of of elliptic complexes:

$$0 \rightarrow L^2_{k+1}(E_0) \xrightarrow{d^0} L^2_k(E_1) \xrightarrow{d^1} L^2_{k-1}(E_2) \rightarrow 0 \quad (*)$$

parametrized by $0 \leq \mu \leq \delta$, and introduce the Hilbert spaces $\mathcal{L}_{k+1}(E_i)_\mu$ as the functional spaces which we call the reduced Sobolev spaces with their norms:

$$\|u\|_{\mathcal{L}_{k+1}(E_0)_\mu}^2 = \|d^0_\mu(u)\|_{L^2_k}^2, \quad \|w\|_{\mathcal{L}_k(E_1)_\mu}^2 = \|w'\|_{L^2_k}^2 + \|d^1_\mu(w'')\|_{L^2_{k-1}}^2$$

with respect to the orthogonal decomposition:

$$L^2_k(E_1) \cong V \oplus V^\perp, \quad w = w' + w''$$

where $V$ is the closure of the image $d^0_\mu(L^2_{k+1}(E_0)) \subset L^2_k(E_1)$. We put $\mathcal{L}_{k-1}(E_2) = L^2_{k-1}(E_2)$ as the usual Sobolev space.

These functional spaces satisfy some non standard properties, for example both 0-th and 1-st cohomology groups are eliminated. On the other hand it turns out that both $\mathcal{L}_{k+1}(E_0)_\mu$ and $\mathcal{L}_k(E_1)_\mu$ admit the equivalent norms with $L^2_{k+1}$ and $L^2_k$ respectively at $\mu$, if it is of Fredholm such that their cohomology groups $H^*_\mu$ satisfy $H^0_\mu = H^1_\mu = 0$.

When the base space is non compact, the standard Sobolev spaces do not suffice to obtain Fredholm complexes by elliptic PDE systems. Often we use smaller functional spaces by replacing them by the weighted Sobolev spaces. A typical situation is the case of cylindrical manifolds. If we assign $\mu$ as the weight constants, then they constitute the Fredholm complexes over cylindrical manifolds for small and positive $\mu > 0$. In the case the functional spaces are the standard Sobolev spaces at $\mu = 0$, where they are no more of Fredholm.

Our first theorem is the following, which applies also to the case of cylindrical manifolds.
Theorem 0.1 Suppose the above family of elliptic complexes (*) are the filtered Fredholm complexes for $0 < \mu \leq \delta$ of non negative indices $m \geq 0$. Then the family of the induced complexes $\mathfrak{C}_\mu$:

$$
0 \longrightarrow \mathfrak{L}_{k+1}(E_0)_\mu \stackrel{d^0_\mu}{\longrightarrow} \mathfrak{L}_k(E_1)_\mu \stackrel{d^1_\mu}{\longrightarrow} L^2_{k-1}(E_2) \longrightarrow 0
$$

are also of Fredholm for all $0 \leq \mu \leq \delta$ whose indices are equal to:

$$\dim H^2_{\mu}.$$

In particular if $\dim H^0_\mu = 0$ hold for all $0 < \mu \leq \delta$, then the index of $\mathfrak{C}_0$ is larger than or equal to the Euler characteristics:

$$\text{ind } \mathfrak{C}_0 \geq - \dim H^1_\mu + \dim H^2_\mu.$$

In four manifold theory, Yang-Mills gauge theory is a fundamental tool to study topological structure of smooth four manifolds. It uses the moduli space which is given by the set of solutions to the ASD equation modulo gauge transformations. In general, the infinitesimal structure of the ASD moduli space is based on the Atiyah-Hitchin-Singer elliptic complex:

$$
0 \longrightarrow \mathcal{C}^\infty(X) \stackrel{d}{\longrightarrow} \mathcal{C}^\infty(X; \Lambda^1) \stackrel{d^+}{\longrightarrow} \mathcal{C}^\infty(X; \Lambda^2_+) \longrightarrow 0
$$

The first differential $\mathcal{C}^\infty(X) \stackrel{d}{\longrightarrow} \mathcal{C}^\infty(X; \Lambda^1)$ corresponds to the infinitesimal gauge group action, and closeness of the image corresponds to Hausdorff property in the construction of the global moduli space. In the theory, several functional spaces have been used so far, not only ordinary Sobolev spaces based on the ordinary cohomology theory, but also weighted Sobolev spaces. There is another development by use of families of Banach spaces in study of quasi-conformal mappings ([K4]).

In this paper we apply our functional analytic setting above, and introduce a new construction of the localized ASD moduli space which is based on the reduced cohomology theory. It turns out that this formalism makes the local construction of the moduli space quite canonical in a situation when the differentials of AHS complex do not have closed range, where the standard $L^2$ theory does not work directly. We think that our construction may have more chance to apply in another occasion, such as construction of instanton Floer homology groups ([F]).
Our main motivation is to study smooth complexity of Casson handles inside smooth four manifolds. Casson handle $CH(T)$ is an open four manifold with boundary, which is homeomorphic to the standard open 2 handle but far from diffeomorphic. It is parametrized by a signed and rooted infinite tree $T$, and if two such trees $T_1 \subset T_2$ admit embedding, then the corresponding Casson handles also admit smooth embedding $CH(T_2) \subset CH(T_1)$ in a reverse way, preserving their attaching regions. So growth structure of the tree directly reflects complexity of its smooth structure.

Casson handles arise when a simply connected, oriented and smooth four manifold is decomposed topologically with respect to its intersection form (see [K3]). Let $M$ be such a manifold with even type form. Then there exists an open subset $S \subset M$ homeomorphic to the connected sums of $S^2 \times S^2$ removed 4 cell, which is compatible with the decomposition of the form. $S$ admits the induced smooth structure and admits a smooth decomposition:

$$S \cong D^4 \cup_{i=1}^{2l} CH(T_i).$$

So these Casson handles are embedded inside $M$ smoothly, and both the end of $S$ and $S$ itself are simply connected.

We say that an open four manifold $S$ has tree-like end if there is a finite family of signed trees $T_1, \ldots, T_l$, such that $S$ is diffeomorphic to $D^4 \cup_{j=1}^l CH(T_j)$, where every $CH(T_j)$ is attached to the zero handle $D^4$ along the attaching $S^1$ of the first stage kinky handle which corresponds to the root in $T_j$.

In [K1], we have introduced a class of signed infinite trees which are called trees of bounded type. Any tree of bounded type grows polynomially.

Let $M$ be as above, and take an $SO(3)$ bundle $E \to M$. A standard form of the intersection form $< \ , \ > = k(-E_8) \oplus lH$, gives decomposition as $H_2(M : \mathbb{Z}) \cong \mathbb{Z}^{8k} \oplus \mathbb{Z}^{2l}$, where $H$ is the hyperbolic 2 by 2 matrix. We call such splitting a marking of the form. A marking gives an open four manifold $S = D^4 \cup_{j=1}^l CH(T_j) \subset M$. Notice that marking is not unique because of existence of lattice automorphisms. The trees, and hence the Casson handles change if we choose different markings. In [K3], we gave a proof of the following:

**Theorem 0.2** (1) For any marking on the K3 surface, the corresponding embedded Casson handles cannot be all of bounded type.

(2) Let $M$ be K3 surface or its logarithmic transforms $X_p$ for odd $p$. Then there is an $SO(3)$ bundle $E \to M$ with $w_2(E) \neq 0$ which admits non empty
generic markings, so that for any generic marking, the corresponding embedded Casson handles cannot be all of bounded type.

With respect to the decomposition above, the second Stiefel-Whitney class $w_2$ of $E$ splits as $w_2 = w_2^1 \oplus w_2^2$. We say that a marking is generic with respect to the $SO(3)$ bundle $E$, if both $w_2^1 \neq 0$ and $w_2^2 \neq 0$ do not vanish.

Our aim is to construct another approach to the proof of theorem 0.2 by use of the new functional analytic setting described above. (1) follows from (2) with [M] ([K3]). So we focus on the proof of (2) combining with [Kr] which concerns existence of ASD connections over $M$.

In [K1], we have explicitly introduced complete Riemannian metrics of bounded geometry on any Casson handles.

**Theorem 0.3** Let $S \cong D^4 \cup_{i=1}^{2l} CH(T_i)$ be the Riemannian-Casson handles of homogeneously bounded type. Then the induced AHS complex:

$$
0 \longrightarrow \mathfrak{L}_{k+1}(S) \xrightarrow{d} \mathfrak{L}_k(S; \Lambda^1) \xrightarrow{d^+} L^2_{k-1}(S; \Lambda^2_+) \longrightarrow 0.
$$

is of Fredholm whose index $I$ admits the bounds:

$$
l \leq I \leq 2l.
$$

This is a consequence of theorem 0.1. In [K1], we have constructed weighted Sobolev spaces with the weights $0 < \mu << 1$ such that the corresponding AHS complexes $\text{AHS}_\mu$ are filtered Fredholm. So the family of the Fredholm complexes satisfies the assumption in theorem 0.1.

Let us describe the construction of our variant of Yang-Mills moduli theory. Let $E \to S$ be an $SO(3)$ bundle whose trivialization near the end is fixed. A connection $A$ over $E$ is called the anti self dual (ASD), if its curvature $F_A$ satisfies the equation:

$$
F_A^+ \equiv F_A + *F_A = 0.
$$

The construction of our localized ASD moduli space at $A$ uses the functional spaces $\mathfrak{L}_k(A)$, however there causes a problem if one tries to use the ASD equation itself, since it will not be well defined to formulate the self dual curvature form for elements $A + a \in \mathfrak{L}_k(A)$. So we use a kind of regularization of the self dual curvature which replaces $F^+$ by $\tilde{F}^+$, by use of the spectral decomposition. Let us cut off the spectra near zero as $P_{f_\epsilon} = \int_0^\infty f_\epsilon(\lambda) dE(\lambda)$, where $f_\epsilon$ vanishes
on $[0, \epsilon]$. Then the regularization is given by:
\[
\tilde{F}^+(A + a) = d_A^+(a) + (Q_{f_\epsilon}(a) \wedge Q_{f_\epsilon}(a))^+
\]
where $Q_{f_\epsilon}(a) = (d_A^+)^* \Delta_A^{-1} P_{f_\epsilon}(d_A^+(a))$. Then we define the localized ASD moduli space by:
\[
\mathcal{M}_k(A) = \{A + a : \tilde{F}^+(A + a) = 0, \ a \in (\text{Ker } d_A^+) \perp \subset \mathfrak{L}_k(A)\}.
\]

Let us describe the idea of our approach to (2) in theorem 0.2 roughly. Let $M$ be a simply connected, oriented, closed and smooth four manifold of even type, equipped with a marking:
\[
\Phi : (H_2(M; \mathbb{Z}), <, >) \cong (\oplus^{8k} \mathbb{Z} \oplus^{2l} \mathbb{Z}, \ k(-E_8) \oplus lH)
\]
where $k \geq 2$ and $l \geq 3$. By Casson-Freedman theory, one finds an open four manifold $S$ with tree-like end, homeomorphic to the interior of $l(S^2 \times S^2) \setminus D^4$, and finds a smooth embedding $S \hookrightarrow M$ which induces an embedding of the form:
\[
(H^2(S : \mathbb{Z}), <, >) \cong (\oplus^{2l} \mathbb{Z}, lH) \hookrightarrow (H^2(M : \mathbb{Z}), <, >).
\]

Let us assume that the Donaldson’s invariant is non zero, and suppose the embedded Casson handles could be all of bounded type. Let us equip with the Riemannian metric $g$ on $S$ in theorem 0.3. We induce a contradiction as below. Let us choose an exhaustion on $S$ by compact subsets as $K_0 \subset K_1 \subset \cdots \subset S \subset M$. Choose a family of generic Riemannian metrics $\{g_i\}_i$ on $M$ such that $g_i|K_i \sim g|K_i$ are sufficiently near each other in $C^\infty$. Take a family of ASD connections $A_i$ with respect to $(M, g_i)$ which converges to an $L^2$ ASD connection $A$ with respect to $(S, g)$. Then we obtain the non empty ASD moduli space $A \in \mathcal{M}_k(A)$ over $S$, where we use the new functional spaces. $A$ may not be regular, and so we use perturbation of the regularized ASD equation, which is the simplest kind where we do not care about gauge group actions. One can find a solution to the perturbed equation near $A$ in the functional space $\mathfrak{L}_k(A)$. The index computation shows that the dimension should be negative, which gives a contradiction.

M.Tsukamoto pointed out to me that $L^2$ ASD connections can not be trivialized near infinity over general open four manifolds, and our argument in proposition 1.1 [K3] is not enough, to whom the author thanks. We also fill

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in the detail of this gap in section 3, and verify that the argument still works in the case of our Riemannian-Casson handles. We shall introduce a notion of $T^* \times \mathbb{N}$ structure on Riemannian manifolds, and verify that Casson handles admit such structure, and those Riemannian manifolds with a condition on the fundamental group admit such trivialization.

Theorem 0.2 implies that those Casson handles in $K3$ should grow much more than bounded type so that our framework of the Fredholm theory also breaks. On the other hand the $L^2$ ASD connection $A$ over the Riemannian-Casson handles inside $K3$ surface certainly exist by the above deformation process of the Riemannian metrics. It follows from our argument that the cokernel of:

$$d_A^+: \mathcal{L}_k(A) \rightarrow L^2_{k-1}(S : Ad(P) \otimes \Lambda^2_+)$$

should be 0 or infinite dimensional.

In the former case, the ASD moduli space near $A$ would consist of zero dimensional regular smooth manifold. However since the growth of the Casson handle inside $K3$ surface would be so wild, it seems reasonable to predict its behavior as:

**Conjecture 0.1:** Let $A$ be the $L^2$ ASD connection over the Riemannian-Casson handle inside $K3$ surface. Then the cokernel of $d_A^+$ will be of infinite dimension.

From geometric analysis viewpoint, it would be of interest for us to develop construction of the global ASD moduli space and the gauge group action over our functional spaces.

1 Reduced cohomology group

1.A Functional spaces: Let $(X, g)$ be a complete Riemannian manifold, and $E_i \rightarrow X$ be $SO(N)$ vector bundles which are trivialized near infinity for $i = 1, 2, 3$.

Suppose there is a complex between the Sobolev spaces:

$$0 \longrightarrow L^2_{k+1}(E_0) \xrightarrow{d^0} L^2_k(E_1) \xrightarrow{d^1} L^2_{k-1}(E_2) \longrightarrow 0.$$ 

by elliptic differential operators.
Example 1.1: Our main application is the Atiyah-Hitchin-Singer complex:

\[ 0 \rightarrow L_{k+1}^2(X) \xrightarrow{d} L_k^2(X; \Lambda^1) \xrightarrow{d^+} L_{k-1}^2(X; \Lambda^2_+) \rightarrow 0 \]

over smooth four manifolds, where \( d^+ \) is the composition of \( d \) with the projection to the self dual part on 2 form.

Let us study analytic behavior of deformation of these elliptic complexes. Let \( \delta > 0 \) be a small and positive number, and consider a smooth deformation of the elliptic differential operators:

\[ 0 \rightarrow L_{k+1}^2(E_0) \xrightarrow{d^0} L_k^2(E_1) \xrightarrow{d^1} L_{k-1}^2(E_2) \rightarrow 0. \]

for \( 0 \leq \mu \leq \delta \). Let us denote by \( H^*_\mu \) as their (unreduced) cohomology groups.

Notice that the differential:

\[ d^0 : L_{k+1}^2(E_0) \rightarrow L_k^2(E_1). \]

does not have closed range in general. For example the differentials on functions do not have closed range over the cylindrical manifolds or \( \mathbb{R}^n \).

Let us introduce new functional spaces which we call the reduced Sobolev spaces.

**Definition 1.1**

1. The reduced Sobolev space \( \mathfrak{L}_{k+1}(E_0) \) (which depend on \( \mu \)) is given by the maximal extension of the domain \( C^\infty_c(E_0) \) of \( d^0_\mu \) with the norm:

\[ ||u||_{\mathfrak{L}_{k+1}}^2 = ||d^0_\mu(u)||_{L_k^2}^2. \]

2. \( \mathfrak{L}_k(E_1) \) is given by the closure of \( L_k^2(E_1) \) with the norm:

\[ ||w||_{\mathfrak{L}_k}^2 = ||w'||_{L_k^2}^2 + ||d^1_\mu(w'')||_{L_{k-1}^2}^2 \]

with respect to the orthogonal decomposition \( w = w' + w'' \) as:

\[ L_k^2(E_1) = \mathfrak{L}_{k+1}(E_0) \oplus \mathfrak{L}_{k+1}(E_0)^\perp. \]

Notice that \( \mathfrak{L}_{k+1}(E_0) \) is identified with the closed linear subspace \( d^0_\mu(\mathfrak{L}_{k+1}(E_0)) \subset L_k^2(E_1) \). Our choice of the norms induces the family of bounded complexes \( \mathfrak{C}_\mu \):

\[ 0 \rightarrow \mathfrak{L}_{k+1}(E_0) \xrightarrow{d^0_\mu} \mathfrak{L}_k(E_1) \xrightarrow{d^1_\mu} L_{k-1}^2(E_2) \rightarrow 0. \]
Lemma 1.1 Suppose $X$ is compact without boundary. Then there are canonical isomorphisms for $* = 0, 1$:

$$L^1_\mu(E_*) \cong L^2_\mu(E_*) / H^*_\mu$$

In particular there are embeddings:

$$L^1_\mu(E_*) \hookrightarrow L^2_\mu(E_*)$$

Proof: Let us consider the case $* = 0$. There is a continuous map from $L^2_k(E_0)$ to $L^2_{k+1}(E_0)$, since a priori estimates hold:

$$||v||_{L^2_{k+1}} \geq C' ||v||_{E_{k+1}}.$$ 

By the assumption, the spectrum of $\Delta^0 \equiv (d^0_\mu)^* d^0_\mu$ is discrete, and let us decompose $L^2_{k+1}(E_0) = \mathbb{R}^m \oplus V$, where $V = \text{Ker} (d^0_\mu)^\perp$. Then the estimates:

$$||v||_{L^2_{k+1}} \leq C' ||d^0_\mu(v)||_{L^2_k} = C' ||v||_{E_{k+1}}$$

hold for any $v \in V$ and for some constant $C'$. So the continuous map from $L^2_{k+1}(X)$ to $L^2_{k+1}$ is surjective and hence has closed range with kernel $= \mathbb{R}^m$.

Next consider $* = 1$ case. By definition of the norm, $L^1_\mu(E_1)$ splits as the direct sum of $d^1_\mu(L^1_\mu(E_0)) \subset L^2_\mu(E_1)$ with $d^1_\mu(L^2_\mu(E_1)) \subset L^2_{k-1}(E_2)$, both of which consist of the closure of the images of $d^0_\mu(L^2_{k+1}(E_0))$ and $d^1_\mu(L^2_k(E_1))$ respectively.

By the assumption, both $d^0_\mu$ and $d^1_\mu$ have closed range, and hence it splits as:

$$L^1_\mu(E_1) \cong d^0_\mu(L^2_{k+1}(E_0)) \oplus d^1_\mu(L^2_k(E_1)).$$

So it is enough to see the isomorphism:

$$L^2_k(E_1)/H^1_\mu \cong d^0_\mu(L^2_{k+1}(E_0)) \oplus d^1_\mu(L^2_k(E_1))$$

where we identify $H^1_\mu \cong \mathbb{R}^m \subset L^2_k(E_1)$.

Let $V = (d^0_\mu(L^2_{k+1}(E_0)) \oplus \mathbb{R}^m)^\perp \subset L^2_k(E_1)$. Then as in the case of $* = 0$, $d^1_\mu : V \cong d^1_\mu(L^2_k(E_1))$ gives the isomorphism. So we obtain the isomorphism:

$$L^2_k(E_1) \cong d^0_\mu(L^2_{k+1}(E_0)) \oplus d^1_\mu(L^2_k(E_1)) \oplus \mathbb{R}^m$$

which gives the desired isomorphism.

This completes the proof.

Below let us state an application of lemma 1.1. Let us say that an elliptic complex is reversible, if for any compact subset $K \subset X$, there is another compact submanifold $K \subset K' \subset X$ such that the restriction of the complex over $K'$ can be extended to another elliptic complex over the double $DK'$. 

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**Corollary 1.1** Suppose $X$ is possibly non compact, and the elliptic complex is reversible. Let $\varphi : X \to [0,1]$ be a cut off function with compact support.

Then there admit continuous functional:

$$\varphi : \mathcal{L}_l(E_*) \to \mathcal{L}_l(E_*)_c$$

which are induced from multiplication by $\varphi$.

**Proof:** Let $K \subset X$ be a compact subset which contains support of $\varphi$ in its interior. Let $\mathcal{L}_l(E_*)_K \subset \mathcal{L}_l(E_*)$ be the closure of the image of $L^2_l(E_*)_0$, and $DK$ be the double of $K$, which is a closed manifold. One can extend the differentials so that another elliptic complexes:

$$0 \to L^2_{k+1}(E_0|DK) \xrightarrow{(d^0_\mu)' } L^2_k(E_1|DK) \xrightarrow{(d^1_\mu)' } L^2_{k-1}(E_2|DK) \to 0.$$

are obtained, which extend the restriction of the original ones over $K$. Then we obtain the embeddings by lemma 1.1:

$$\Phi : \mathcal{L}_l(E_*)_K \hookrightarrow \mathcal{L}_l(E_*|DK) \hookrightarrow L^2_l(E_*|DK)$$

and the composition of the following gives the desired map:

$$\varphi(a) \equiv \varphi\Phi(a) \in L^2_l(E_*)_K \subset L^2_l(E_*)_c \to \mathcal{L}_k(E_*)_c.$$

This completes the proof.

Let $X$ be a complete Riemannian manifold.

**Lemma 1.2** (1) There are isometries of the Hilbert spaces:

$$d^0_\mu : \mathcal{L}_{k+1}(E_0) \cong d^0_\mu(\mathcal{L}_{k+1}(E_0)) \subset L^2_k(E_1).$$

$$\mathcal{L}(E_1) \cong d^0_\mu(\mathcal{L}_{k+1}(E_0)) \oplus d^1_\mu(\mathcal{L}(E_1)) \subset L^2_k(E_1) \oplus L^2_{k-1}(E_2).$$

(2) Suppose:

$$0 \to L^2_{k+1}(E_0) \xrightarrow{d^0_\mu} L^2_k(E_1) \xrightarrow{d^1_\mu} L^2_{k-1}(E_2) \to 0.$$

is a Fredholm complex with $H^0_\mu = H^1_\mu = 0$ at $\mu$.

Then there are constants $C_\mu > 0$ such that the estimates hold:

$$C^{-1}_\mu ||u||^2_{L^2_{k+1}} \leq ||u||^2_{\mathcal{L}_{k+1}(E_0)} \leq C_\mu ||u||^2_{L^2_{k+1}},$$

$$C^{-1}_\mu ||w||^2_{L^2_k} \leq ||w||^2_{\mathcal{L}_k(E_1)} \leq C_\mu ||w||^2_{L^2_k}.$$
Proof: (1) follows by definition (see the proof of lemma 1.1).

For (2), let us verify the first equivalence. The uniform bounds:

\[ C_\mu^{-1} ||u||_{L^2_{k+1}} \leq ||d^0_\mu(u)||_{L^2_k} \leq C_\mu ||u||_{L^2_{k+1}} \]

hold by the assumption, and so the conclusion holds.

Next let us consider the second one. If \( w = d^0_\mu(u) \) for some \( u \in \mathfrak{L}_{k+1}(E_0) \), then the estimates hold by definition of the norm.

Let \( w \in (d^0_\mu(\mathfrak{L}_{k+1}(E_0)))^\perp \). Then the estimates:

\[ C_\mu^{-1} ||d^1_\mu(w)||_{L^2_{k-1}} \leq ||w||_{L^2_k} \leq C_\mu ||d^1_\mu(w)||_{L^2_{k-1}} \]

holds since \( H^1_\mu(X) = 0 \) and Fredholmness of the complex. Any element \( w \) in \( \mathfrak{L}_k(E_1) \) can be given by the direct sum of these cases. This verifies the second inequalities. This completes the proof.

1.A.2 Filtration of functional spaces: Let us consider a smooth family of elliptic complexes for \( 0 \leq \mu \leq \delta \):

\[ 0 \rightarrow L^2_{k+1}(E_0) \xrightarrow{d^0_\mu} L^2_k(E_1) \xrightarrow{d^1_\mu} L^2_{k-1}(E_2) \rightarrow 0. \]

Definition 1.2 Let us say that the family of complexes are filtered, if there are infinite embeddings by Hilbert spaces:

\[ C^\infty_c(E_*) \subset W_\mu(E_*) \subset W_{\mu'}(E_*) \subset W_0 = L^2_l(E_*) \]

for any \( 0 \leq \mu' \leq \mu \leq \delta \) such that there are isomorphisms:

\[ I_\mu : L^2_l(E_*) \cong W_l(E_*)_\mu \]

with \( I_0 = id \), which are compatible with their complexes:

\[ 0 \rightarrow W_{k+1}(E_0)_\mu \xrightarrow{d^0_\mu} W_k(E_1)_\mu \xrightarrow{d^1_\mu} W_{k-1}(E_2)_\mu \rightarrow 0 \]

where both \( d^0 \) and \( d^1 \) are independent of \( \mu \).

We will describe in 1.B that AHS complexes with respect to the weighted Sobolev spaces are filtered over cylindrical manifolds.

Proposition 1.1 Let us consider the filtered complexes for \( 0 \leq \mu \leq \delta \), which are of Fredholm of the same indices with cohomology groups \( H^0_\mu = H^1_\mu = 0 \) and \( \dim H^2_\mu = m \) for all \( 0 < \mu \leq \delta \).
Let us consider the induced complexes \( \mathfrak{C}_\mu \):

\[
0 \rightarrow \mathfrak{L}_{k+1}(E_0)_\mu \xrightarrow{d^0_\mu} \mathfrak{L}_k(E_1)_\mu \xrightarrow{d^1_\mu} \mathfrak{L}_{k-1}(E_2) \rightarrow 0.
\]

Then it is of Fredholm whose cohomolgy groups \( \mathbf{H}^i_\mu \) satisfy:

\[
\mathbf{H}^0_\mu = \mathbf{H}^1_\mu = 0, \quad \dim \mathbf{H}^2_\mu = m
\]

for all \( 0 \leq \mu \leq \delta \). In particular it is also Fredholm at \( \mu = 0 \).

Moreover there is \( m \) dimensional linear subspace \( V \subset \mathfrak{L}^2_{k-1}(E_2) \) such that the projections:

\[
\pi : d^1_\mu(\mathfrak{L}_k(E_1)_\mu) \rightarrow \mathfrak{L}^2_{k-1}(E_2)/V
\]

are onto for all small \( 0 \leq \mu \).

Proof: Step 1: By definition, \( \mathbf{H}^0_\mu = 0 \) and \( \mathbf{H}^1_\mu = 0 \) hold for all \( 0 \leq \mu \leq \delta \).

Step 2: Suppose \( \mathbf{H}^2_\mu = 0 \) for all positive \( \mu > 0 \). For any \( v \in \mathfrak{L}^2_{k-1}(E_2) \), there is \( w \in \mathfrak{L}^2_k(E_1) \) with \( d^1_\mu(w) = v \). One can regard \( w \in \mathfrak{L}_k(E_1)_\mu \) by definition of the norm, and so \( \mathbf{H}^2_\mu = 0 \) hold for all \( 0 < \mu \leq \delta \).

Let us verify \( \mathbf{H}^2_\mu = 0 \). Notice that \( d^1_\mu : \mathfrak{L}_k(E_1)_\mu \rightarrow \mathfrak{L}^2_{k-1}(E_2) \) has closed range for all \( 0 \leq \mu \leq \delta \) by definition of the norm. So it is enough to see that the image is dense. Let:

\[
0 \rightarrow W_{k+1}(E_0)_\mu \xrightarrow{d^0_\mu} W_k(E_1)_\mu \xrightarrow{d^1_\mu} W_{k-1}(E_2)_\mu \rightarrow 0
\]

be the filtered complexes. Let us take any \( v \in \mathfrak{L}^2_{k-1}(E_2) \), and choose approximations \( v_\mu \in W_{k-1}(E_2)_\mu \) which converge to \( v \) in \( \mathfrak{L}^2_{k-1}(E_2) \). Since \( \mathbf{H}^2_\mu = 0 \) for all positive \( \mu > 0 \), there are \( w_\mu \in \mathfrak{L}_k(E_1)_\mu \) with \( d^1(w_\mu) = v_\mu \). In particular the image is dense as desired.

Step 3: Since the Fredholm indices are invariant under continuous deformations, \( \mathbf{H}^2_\mu \) have constant rank \( m \) for all positive \( \mu > 0 \).

By definition, \( d^1_\mu(\mathfrak{L}_k(E_1)_\mu) \subset \mathfrak{L}^2_{k-1}(E_2) \) are closed subspaces for all \( 0 \leq \mu \leq \delta \). We claim that there is a vector subspace \( V \subset \mathfrak{L}^2_{k-1}(E_2) \) of dimension \( m \) such that the projections:

\[
\pi : d^1_\mu(\mathfrak{L}_k(E_1)_\mu) \rightarrow \mathfrak{L}^2_{k-1}(E_2)/V
\]

are onto and hence isomorphic for all positive \( \mu > 0 \). Notice that the orthogonal complement \( V_\mu = d^1_\mu(\mathfrak{L}_k)^\perp \) in \( \mathfrak{L}^2_{k-1}(E_2) \) is a smooth family of \( m \) dimensional vector subspaces.
Suppose contrary. Then for any $m$ dimensional vector space $V$, there is a small $\mu_0 > 0$ such that $\pi$ are onto for $\mu > \mu_0$ but not the case at $\mu_0$. This implies that there is a line $l \subset V$ such that $l$ is contained in the image of $d^1_{\mu_0}$. So for any $\mu > 0$, there is smaller $\mu > \mu_0$ so that $V_\mu$ contain a line in $V_{\mu_0}^\perp$, which cannot happen.

Notice that the closure of $d^1(L^2_k(E_1))$ is equal to $d^1(\mathcal{L}_k(E_1))$ at $\mu = 0$. Let $W \subset L^2_{k-1}(E_2)$ be the co-kernel of $d^1(\mathcal{L}_k(E_1))$. We claim that dimension of $W$ does not exceed $m$. Let us consider the extension of the complexes:

$$0 \rightarrow L^2_{k+1}(E_0) \xrightarrow{(d^0,0)} L^2_k(E_1) \oplus V \xrightarrow{d^1_+ + \text{id}} L^2_{k-1}(E_2) \rightarrow 0.$$  

This is acyclic for $0 < \mu \leq \delta$, and one can obtain the filtered complexes:

$$0 \rightarrow W_{k+1}(E_0)_\mu \xrightarrow{(d^0,0)} W_k(E_1)_\mu \oplus I_\mu(V) \xrightarrow{d^1_+ + \text{id}} W_{k-1}(E_2)_\mu \rightarrow 0.$$  

Let us take any $v \in L^2_{k-1}(E_2)$. Then for any $\epsilon > 0$, there is some $\mu > 0$ and $v_\mu \in W_{k-1}(E_2)_\mu$ so that $\|v - v_\mu\|_{L^2_{k-1}} < \epsilon$ with $v_\mu \in d^1(W_k(E_1)_\mu) + I_\mu(V)$, where dimension of $I_\mu(V) = m$. This verifies the claim.

**Step 4:** The continuous family of the complexes $\mathcal{E}_\mu$:

$$0 \rightarrow \mathcal{L}^2_{k+1}(E_0)_\mu \xrightarrow{d^0_\mu} \mathcal{L}^2_k(E_1)_\mu \xrightarrow{d^1_\mu} L^2_{k-1}(E_2) \rightarrow 0.$$  

are Fredholm for all $0 \leq \mu \leq \delta$ by step 3.

The following abstract lemma finishes the proof of proposition 1.1:

**Lemma 1.3** Let $V_t \subset H$ be a smooth family of cofinite dimensional vector subspaces. Suppose

$$D_t : V_t \rightarrow H^2$$  

is a family of uniformly bounded maps with closed range for $t \in [0,1]$ such that they are isomorphic for all $t \in (0,1]$. If $D_0$ is surjective, then $D_0$ is injective.

**Proof:** Suppose contrary, and consider the isomorphism $D_0 : V_0 / \ker D_0 \cong H_2$, and extend it as:

$$D_t : V_t / P_t(\ker D_0) \cong H_2$$  

for small $t \in [0,\epsilon)$, where $P_t$ are the projections to $V_t$. It should be a family of isomorphisms, since it is an open condition, which cannot happen.

This completes the proof.

Now we verify the following:
Theorem 1.1 Let us consider the filtered complexes for \(0 \leq \mu \leq \delta\) which are of Fredholm of non negative indices \(m \geq 0\) for \(0 < \mu \leq \delta\).

Then the family of the induced complexes \(C_{\mu}:\)

\[
0 \rightarrow \mathfrak{L}^2_{k+1}(E_0)_{\mu} \xrightarrow{d_{\mu}^0} \mathfrak{L}^2_k(E_1)_{\mu} \xrightarrow{d_{\mu}^1} L^2_{k-1}(E_2) \rightarrow 0
\]

are also of Fredholm for all \(0 \leq \mu \leq \delta\) whose indices are equal to \(\dim H^2_{\mu}\).

Proof: Step 1: We have seen the conclusion in proposition 1.1 for the special case when \(H^0_{\mu} = H^1_{\mu} = 0\) hold for all \(0 < \mu \leq \delta\).

Let us consider the Fredholm complexes for \(\mu > 0:\)

\[
0 \rightarrow L^2_{k+1}(E_0) \xrightarrow{d_{\mu}^0} L^2_k(E_1) \xrightarrow{d_{\mu}^1} L^2_{k-1}(E_2) \rightarrow 0.
\]

Let us choose finite dimensional vector spaces:

\[
V^0_{\mu} = \text{Ker } d_{\mu} \subset L^2_{k+1}(E_0), \quad V^1_{\mu} \subset \text{Ker } d_{\mu}^1 \subset L^2_k(E_1), \quad V^2_{\mu} \subset L^2_{k-1}(E_2)
\]

which represent \(H^0_{\mu}, H^1_{\mu}\) and \(H^2_{\mu}\) respectively.

Step 2: For \(\mu > 0\), suppose \(\dim V^2_{\mu}\) is less than or equal to the Fredholm index. Then \(\dim V^0_{\mu} \geq \dim V^1_{\mu}\) holds. We verify that the index of \(C_{\mu}\) is equal to \(\dim H^2_{\mu}\).

Let us prepare another vector space \(W\) with the acyclic complexes between finite dimensional spaces:

\[
0 \rightarrow V^0_{\mu} \xrightarrow{f_{\mu}} V^1_{\mu} \oplus W \rightarrow 0 \rightarrow 0.
\]

Then one can add the extra vector spaces in the complex:

\[
0 \rightarrow L^2_{k+1}(E_0) = (V^0_{\mu})^\perp \oplus V^0_{\mu} \xrightarrow{d_{\mu}^0 \oplus f_{\mu}} L^2_k(E_1) \oplus W = (V^1_{\mu})^\perp \oplus V^1_{\mu} \oplus W \xrightarrow{d_{\mu}^1} L^2_{k-1}(E_2) \rightarrow 0.
\]

Let \(I_{\mu} : L^2_\ell(E_*) \cong W_\ell(E_*)_{\mu}\) be the isomorphisms, and put \(\tilde{f}_{\mu} = I_{\mu}^{-1} f_{\mu} I_{\mu}\). Then consider:

\[
0 \rightarrow W_{k+1}(E_0)_{\mu} \xrightarrow{d_{\mu}^0 \oplus \tilde{f}_{\mu}} W_k(E_1)_{\mu} \oplus W \xrightarrow{d_{\mu}^1} W_{k-1}(E_2)_{\mu} \rightarrow 0
\]

which is Fredholm complex with \(H^0_{\mu} = 0\) and \(H^1_{\mu} = 0\), and the index is equal to \(\dim H^2_{\mu}\).
Now the induced complex:

\[ 0 \longrightarrow \mathcal{L}_{k+1}(E_0)_\mu \oplus V^0_\mu \xrightarrow{d_\mu \oplus f_\mu} \mathcal{L}_k(E_1)_\mu \oplus V^1_\mu \oplus W \xrightarrow{d_\mu^1} L^2_{k-1}(E_2) \longrightarrow 0. \]

is of Fredholm, which is chain homotopy equivalent to the original induced complex. Its index is equal to \( \dim H^2_\mu \) by lemma 1.2.

**Step 3:** Suppose \( \dim V^2_\mu \) is larger than the Fredholm index. Then the inequality \( \dim V^0_\mu \leq \dim V^1_\mu \) holds. Let us take another vector space \( W \) with the acyclic complex between finite dimensional spaces:

\[ 0 \longrightarrow V^0_\mu \xrightarrow{f_\mu} V^1_\mu \xrightarrow{g_\mu} W \longrightarrow 0. \]

Then one can add the extra vector spaces in the complex:

\[ 0 \longrightarrow L^2_{k+1}(E_0)_\mu = (V^0_\mu)^\perp \oplus V^0_\mu \xrightarrow{d_\mu \oplus f_\mu} L^2_k(E_1)_\mu = (V^1_\mu)^\perp \oplus V^1_\mu \xrightarrow{d_\mu^1 \oplus g_\mu} L^2_{k-1}(E_2)_\mu \oplus W \longrightarrow 0. \]

The index of this complex is \( \dim H^2_\mu \). By the same way as step 2, the induced complex:

\[ 0 \longrightarrow \mathcal{L}^2_{k+1}(E_0)_\mu \oplus V^0_\mu \xrightarrow{d_\mu \oplus f_\mu} \mathcal{L}^2_k(E_1)_\mu \oplus V^1_\mu \xrightarrow{d_\mu^1 \oplus g_\mu} L^2_{k-1}(E_2)_\mu \oplus W \longrightarrow 0. \]

is of Fredholm of the index \( \dim H^2_\mu \), which is chain homotopy equivalent to the original induced complex.

**Step 4:** Let us verify that \( \dim H^2_\mu \) are constant for all small \( 0 < \mu \). In fact there is a finite dimensional vector subspace \( V \subset W_{k-1}(E_2)_\mu \subset W_{k-1}(E_2)_{\mu'} \) such that for any \( v \in W_{k-1}(E_2)_\mu \), there is some \( a \in V \) such that \( v - a = d^1(w) \) holds for some \( w \in W_k(E_1)_\mu \).

So \( \dim H^2_\mu \geq \dim H^2_{\mu'} \) hold for any \( \mu \geq \mu' \), since any elements in \( W_{k-1}(E_2)_{\mu'} \) can be approximated by elements in \( W_{k-1}(E_2)_\mu \).

**Step 5:** Combining with step 2, 3, 4, it follows that the indices of the induced complexes are equal to \( \dim H^2_\mu \) for all \( 0 < \mu \leq \delta \). It follows from step 3 and step 4 in proposition 1.1 that it is a family of Fredholm complexes for all \( 0 \leq \mu \leq \delta \), whose indices coincide with \( \dim H^2_\mu \).

This completes the proof.

**Corollary 1.2** Consider the situation in theorem 1.1.
If \( \dim H^0_\mu = 0 \) hold for all \( 0 < \mu \leq \delta \), then \( \mathcal{C}_\mu \) are of Fredholm for all \( 0 \leq \mu \leq \delta \) whose indices are larger than or equal to the Euler characteristics:

\[- \dim H^1_\mu + \dim H^2_\mu \leq \text{ind} \mathcal{C}_\mu = \dim H^2_\mu.\]

1.B AHS complexes over cylindrical manifolds: The Atiyah-Hitchin-Singer complex is the elliptic differential complex over a Riemannian four manifold \( X \):

\[
0 \longrightarrow L^2_{k+1}(X) \xrightarrow{d} L^2_k(X; \Lambda^1) \xrightarrow{d^+} L^2_{k-1}(X; \Lambda^2_+) \longrightarrow 0
\]

where \( d^+ \) is the composition of the differential with the projection to the self dual 2 forms.

Let \( X \) be a complete Riemannian manifold such that it is isometric to the product \( M \times [0, \infty) \) except a compact subset \( K \subset X \). Such space is called as a cylindrical manifold.

In 1.B we verify the following:

**Proposition 1.2** Let \( X \) be a cylindrical four manifold.

(1) There is a filtered AHS complexes over \( X \) and positive \( \delta > 0 \), which are of Fredholm for all \( 0 < \mu \leq \delta \).

(2) Suppose the indices are non negative. Then:

\[
0 \longrightarrow \mathfrak{L}^2_{k+1}(X) \xrightarrow{d_\mu} \mathfrak{L}^2_k(X; \Lambda^1) \xrightarrow{d^+_\mu} \mathfrak{L}^2_{k-1}(X; \Lambda^2_+) \longrightarrow 0
\]

is a family of Fredholm complexes of the same indices for all \( 0 \leq \mu \leq \delta \).

**Proof:** (2) follows from (1) with corollary 1.2. Note that \( H^0_\mu = 0 \) always hold over non compact manifolds.

For (1), we review the construction of the weighted Sobolev spaces for convenience. For the details of the analysis, we refer to [K1].

**Step 1:** Let \( (M, g) \) be a closed Riemannian 3 manifold, and denote the product metric by \( g + dt \) on \( M \times \mathbb{R} \). By use of formally \( L^2 \) adjoint operator, we obtain the elliptic operator \( P = d^* \oplus d^+ \) from \( \Lambda^1(M \times \mathbb{R}) \) to \( \Lambda^0(M \times \mathbb{R}) \oplus \Lambda^2_+(M \times \mathbb{R}) \). One can canonically identify:

\[
\Lambda^1(M \times \mathbb{R}) = p^*(\Lambda^1(M)) \oplus p^*(\Lambda^0(M)), \quad \Lambda^2_+(M \times \mathbb{R}) = p^*(\Lambda^1(M))
\]
where \( p : M \times \mathbb{R} \mapsto M \) is the projection, and the isomorphisms are given by:

\[
(u + vdt) \leftrightarrow (u, v), \quad *_M u + u \wedge dt \leftrightarrow u.
\]

Then \( P : p^*(\Lambda^1(M) \oplus \Lambda^0(M)) \mapsto p^*(\Lambda^1(M) \oplus \Lambda^0(M)) \) is represented as:

\[
P = -\frac{d}{dt} + \left( *_M \frac{d}{dt} \frac{d}{d\tau} 0 \right) \equiv -\frac{d}{dt} + Q
\]

where \( Q \) is an elliptic self adjoint differential operator on \( L^2(M; \Lambda^1 \oplus \Lambda^0) \).

**Step 2:** Let us fix a small and positive \( \delta > 0 \). Then for \( 0 < \mu \leq \delta \), define:

\[
\tau : M \times [0, \infty) \mapsto [0, \infty), \quad \tau(m, t) = \mu t.
\]

Let \( X \) be a cylindrical manifold whose end is isometric to \( M \times [0, \infty) \). Then we fix the weight function \( \tau : X \mapsto [0, \infty) \) so that it coincides with \( \tau(m, t) \) on the end of \( X \). Then we define the weighted Sobolev \( k \) norms on \( X \) by:

\[
||u||_{(L^2_k)_\mu} = (\Sigma_{l \leq k} \int_X \exp(\tau) |\nabla^l u|^2)^{\frac{1}{2}}.
\]

We write by \( (L^2_k)_\mu \) as the space of the completion of \( C^\infty_c(X) \) with respect to the norm, since the isomorphism class of the function spaces is determined by \( \mu > 0 \), rather than \( \tau \) itself.

By this way, we obtain a filtration of the Sobolev spaces \( \{(L^2_k)_\mu\}_{0 \leq \mu \leq \delta} \), which satisfy the inclusions whenever \( \mu' \leq \mu \):

\[
C^\infty_c \subset (L^2_k)_\mu \subset (L^2_k)_{\mu'} \subset L^2_k
\]

such that the inclusions \( (L^2_k)_\mu \subset L^2_k \) are dense for all \( 0 \leq \mu \leq \delta \).

**Step 3:** Let us introduce the isometries:

\[
I_\mu : L^2(X, \Lambda^*) \mapsto (L^2)_\mu(X, \Lambda^*)
\]

by \( I_\mu(u) = \exp(-\frac{\tau}{2})u \), which induce the isomorphisms:

\[
I_\mu : L^2(X) \cong (L^2)_\mu(X).
\]

Let \( d^*_\tau \) be the \((L^2)_\mu\) adjoint operator so that \( <u, d(v)>_{(L^2)_\mu} = <d^*_\tau(u), v>_{(L^2)_\mu} \) holds, and put:

\[
P_\tau = d^*_\tau \oplus d^+ : L^2_{k+1}(X : \Lambda^1) \to L^2_k(X : (\Lambda^0 \oplus \Lambda^2_+))
\]
Then we have the following expression on the end $M \times [0, \infty)$:

$$I^{-1}_\tau P_\tau I_\tau = -\frac{d}{dt} + \left( *Md \frac{d}{d^*} - \frac{d}{d\tau} \frac{d\tau}{dt} \right) + \frac{1}{2} \frac{d\tau}{dt} \equiv -\frac{d}{dt} + Q_\mu$$

The following lemma is well-known. Theorem 1.1 with lemma 1.4 below finishes the proof of (1):

**Lemma 1.4** The $AHS_\mu$ complexes:

$$0 \rightarrow (L^2_{k+1})_\mu(X) \xrightarrow{d} (L^2_k)_\mu(X; \Lambda^1) \xrightarrow{d^+} (L^2_{k-1})_\mu(X; \Lambda^2_+) \rightarrow 0.$$

are of Fredholm with $H^0_\mu = 0$ for all $0 < \mu \leq \delta$.

**Proof:** [K1] computed spectral behavior of $Q_\mu$ near 0. For convenience let us outline how to verify this. It follows from the straightforward calculations that:

$$Q_\mu = \left( *Md \frac{d}{d^*} - \frac{d}{\frac{d\mu}{2}} \right) : \frac{L^2_{k+1}(M, \Lambda^1 \oplus \Lambda^0)}{\Delta} \rightarrow \frac{L^2_k(M, \Lambda^1 \oplus \Lambda^0)}{\Delta}$$

give isomorphisms for all small $\mu > 0$. In particular $I^{-1}_\tau P_\tau I_\tau$ and hence $P_\tau$ both give the invertible operators on the end. This completes the proof.

## 2 Analysis over Casson handles

### 2.A Casson handles: Casson handles are open smooth four manifolds with the attaching regions. They are inductively constructed by taking end connected sums and obtained as their direct limits. Each building block is called a kinky handle which is diffeomorphic to a finite number of the end connected sums $\natural(S^1 \times D^3)$ with two attaching regions, where one is a tubular neighborhood of bunch sums of Whitehead links (this is connected with the previous block), and the other is a disjoint union of the standard open subsets $S^1 \times D^2$ in $\natural S^1 \times S^2 = \partial(\natural S^1 \times D^3)$ (this is connected with the next block). The number of end-connected sums is exactly the one of self-intersections of the immersed two handle.

By construction, each Casson handle corresponds to the infinite rooted trees with sign on each edge. We attach a Casson handle to the zero handle along the first stage attaching circle and denote it by $S = D^4 \cup CH$. This is a smooth open four manifold. We refer to [K1] for detailed description on Casson handles.
2.A.2 Kinky handles as Riemannian manifolds: There are two simplest Casson handles with respect to signs. Both have $S^1 \times D^3$ as their building blocks, and are given by taking end connected sums of infinitely many kinky handles periodically.

We have introduced the complete Riemannian metrics on kinky handles in [K1] (section 2). Let us briefly describe their properties.

Let $W_1 \cong S^1 \times D^3$ be the simplest kinky handle. The Riemannian metric on $W_1$ has the properties:

1. $W_1$ contains two disjoint Riemannian subspaces:
   
   $$N \times (-\infty, 0] \sqcup M \times [0, \infty)$$

   where $N$ and $M$ are cylindrical three manifolds with their ends $\Sigma$, which are mutually isometric.

2. $W_1$ contains another Riemannian subspace:
   
   $$V \times [0, \infty)$$

   where $V$ is a non compact Riemannian three manifold with two ends which are isometric to $\Sigma \times (-\infty, 0] \sqcup \Sigma \times [0, \infty)$.

Let us denote:

$$\tilde{W}_1 = W_1 \setminus \{ N \times (-\infty, -2) \sqcup M \times (2, \infty) \}.$$

By taking the end connected sum of $N \times \{-2\}$ with $M \times \{2\}$, we obtain the cylindrical four manifold:

$$Y_1 \equiv \tilde{W}_1 / \{ N \times \{-2\} \sim M \times \{2\} \}.$$ 

Let us describe the periodic cover of $Y$. Let us prepare infinitely many copies of $\tilde{W}_1$ and assign indices as $\{\tilde{W}_i\}_{i \in \mathbb{N}}$. Then we take the end connected sums of $M_i \equiv M \times \{2\}$ in $\tilde{W}_i$ with $N^{i+1} \equiv N \times \{-2\}$ in $\tilde{W}_{i+1}$. By this way, we obtain the half periodic Riemannian manifold:

$$\tilde{Y}_1^0 \equiv CH(\mathbb{N}) \equiv \tilde{W}_1^0 \cup \ldots \cup \tilde{W}_i \cup_{M_i \equiv N^{i+1}} \tilde{W}_{i+1} \cup \ldots$$

which is the simplest periodic Casson handle, where the attaching region lies in the boundary of $\tilde{W}_1^0$. 

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Notice that the periodic Riemannian manifold:
\[ \tilde{Y}_1 \equiv CH(\mathbb{Z}) \equiv \cdots \cup \tilde{W}_1^i \cup_{M^i \cong N^{i+1}} \tilde{W}_1^{i+1} \cup \ldots \]
is obtained by use of indices as \( \{ \tilde{W}_1^i \}_{i \in \mathbb{Z}} \), which is \( \mathbb{Z} \) covering of \( Y_1 \). This space is less interesting from the viewpoint of smooth structure, since it is in fact diffeomorphic to the standard four disc. However its Riemannian structure plays an important role in the analysis of Fourier-Laplace transform over the period cover. In fact we verify the following:

**Proposition 2.1** The AHS complex:
\[
0 \longrightarrow \mathcal{L}^2_{k+1}(\tilde{Y}_1; \Lambda^0) \xrightarrow{d^-} \mathcal{L}^2_k(\tilde{Y}_1; \Lambda^1) \xrightarrow{d^+} L^2_{k-1}(\tilde{Y}_1; \Lambda^2_+) \longrightarrow 0
\]
is acyclic Fredholm.

We verify this at the end of 2.B below.

**2.B Fourier-Laplace transform:** Let \( X \) be a complete Riemannian manifold, and \( D : C_c^\infty(E) \rightarrow C_c^\infty(F) \) be a differential operator between vector bundles over \( X \).

Let \( \tilde{X} \) be a periodic cover of \( X \) with the group \( \mathbb{Z} \). Then both \( E, F \) and \( D \) lift canonically as the invariant operator:
\[
\tilde{D} : C_c^\infty(\tilde{E}) \rightarrow C_c^\infty(\tilde{F})
\]
where \( \tilde{E} \rightarrow \tilde{X} \) is the natural lift, equipped with the shift isomorphism \( T : \tilde{E} \cong \tilde{E} \) which corresponds to \( 1 \in \mathbb{Z} \).

Let \( \tilde{W} \subset \tilde{X} \) be a fundamental domain with respect to \( \mathbb{Z} \) action, with the boundary \( \partial \tilde{W} = N \cup M \).

**Definition 2.1** Let us take any \( \psi \in C_c^\infty(\tilde{E}) \) and \( z \in \mathbb{C}^* \). The Fourier Laplace transform of \( \psi \) is given by:
\[
\hat{\psi}_z(\ ) = \sum_{n=-\infty}^{\infty} z^n (T^n \psi)(\ )
\]
over the restriction \( \hat{\psi}|\tilde{W} \), which determines a section of the vector bundle:
\[
E' \equiv [\tilde{E} \otimes_{\mathbb{R}} \mathbb{C}] / \mathbb{Z} \rightarrow X \times \mathbb{C}^*
\]
where \( 1 \in \mathbb{Z} \) sends \( (\rho, \lambda) \in \tilde{E} \otimes_{\mathbb{R}} \mathbb{C} \) to \( (T\rho, z\lambda) \).
Remark 2.1: $E'(z)$ is a family of bundles over $X$ with $E'(1) = E$. Every $E'(z)$ is isomorphic to $E$ ([K1] section 4).

The Fourier Laplace inversion formula is given as follows; for any smooth section $\hat{\eta} \in C^\infty_c(E')$ with $\hat{\eta}_z \in C^\infty_c(E'(z))$, let us take the lift and restrict it on $\tilde{W}$. Then for any $s \in (0, \infty)$ and $x \in \tilde{W}$,

$$T^n\eta(x) \equiv \frac{1}{2\pi i} \int_{|z|=s} z^{-n}\hat{\eta}_z(\pi(x)) \frac{dz}{z}$$

defines a smooth section over $\tilde{E} \to \tilde{X}$, where $\pi : \tilde{W} \to X$ is the projection. These are converses each other.

Let us define the family of differential operators over $E'(z)$ by:

$$\tilde{D}_z\hat{\psi}_z \equiv (\hat{D}\psi)_z$$

passing through the Fourier Laplace transform.

Suppose $X = Y$ is a cylindrical manifold and $\tau$ be a weight function on $Y$ with weight $\mu > 0$. Then the weight function canonically extends on the periodic cover $\tilde{Y}$, and hence Sobolev spaces $(L^2_k)_\mu(\tilde{Y})$ are obtained. Let us extend $\tilde{D}$ and $D_z$ over $(L^2_k)_\mu(\tilde{E})$ and $(L^2_k)_\mu(E'(z))$ respectively.

**Lemma 2.1 (K1) $\tilde{D}$ is invertible over $\tilde{Y}$, if $D_z$ are invertible for all $z \in C(1) = \{z \in \mathbb{C} : |z| = 1\}$.**

The assumption is satisfied, which we will explain in 2.B.2 below.

**2.B.2 Excision analysis:** Let us introduce some new analytic method to bridge various functional spaces over different spaces, which makes it convenient to perform excision process.

Let us prepare two complete Riemannian manifolds $X$ and $Y$ which satisfy the following conditions:

1. There are open sub manifolds $A, X_0 \subset X$ and $B, Y_0 \subset Y$ such that $X'_0 \equiv X \setminus A$ and $Y'_0 \equiv Y \setminus B$ are both manifolds with the product ends near boundary:

$$X_0 \setminus X'_0 \cong [-3, -2] \times N, \quad Y_0 \setminus Y'_0 \cong [2, 3] \times M$$

which are isometric $M \cong N$ mutually.

2. The Dirac operators $D_X$ and $D_Y$ are equipped over $X$ and $Y$, such that the isomorphism holds:

$$D_X|[-3, -2] \times N = D_Y|[2, 3] \times M.$$
Let us denote the end connected sum:

\[ Z = X_0 \cup Y_0 / \{ [-3, -2] \times N \sim [2, 3] \times M \} \]

and denote by \( D_Z \) as the induced Dirac operator over \( Z \).

Let us consider \( L^2_k(Y_0)_0 \) and regard it as a closed subspace in \( L^2_k(Z) \).

**Definition 2.2** The orthogonal complement of \( L^2_k(Y_0)_0 \) in \( L^2_k(X) \) is given by:

\[ L^2_k(Y_0)^\perp_0 = \{ w \in L^2_k(X) : < w, w' >= 0 \text{ for any } w' \in L^2_k(Y_0)_0 \cap L^2_k(X) \} \]

**Lemma 2.2** \( L^2_k(Y_0)^\perp_0 \) splits into two closed linear subspaces:

\[ L^2_k(Y_0)^\perp_0 = H_1 \oplus H_2 \]

with \( \text{supp } H_1 \subset X_0 \) and \( \text{supp } H_2 \subset A \), such that the orthogonal decomposition:

\[ L^2_k(Z) = L^2_k(Y_0)_0 \oplus H_1 \]

holds.

**Proof:** Any \( u \in L^2_k(Z) \) can be expressed as a union \( u_1 + u_2 \in L^2_k(X_0)_0 + L^2_k(Y_0)_0 \). Let us take \( v \in L^2_k(Y_0)^\perp_0 \), and verify that it defines a continuous linear functional:

\[ F_v : L^2_k(Z) \to \mathbb{R}, \quad F_v(u) = < v, u >_{L^2_k(X)} \]

Let us check its well-definedness. Choose two different decompositions \( u = u_1 + u_2 = u'_1 + u'_2 \). Then \( u_1 - u'_1 = u'_2 - u_2 \in L(X_0)_0 \cap L^2_k(Y_0)_0 \), and so:

\[ < v, u_1 - u'_1 >_{L^2_k(X)_0} = < v, u'_2 - u_2 >_{L^2_k(X)_0} = 0. \]

Thus we may regard \( F_v = v_1 \in L^2_k(X_0)^* = L^2_k(X)_0 \subset L^2_k(Z) \). This assignment gives the closed linear subspace \( H_1 \subset L^2_k(X)_0 \).

Notice that \( X \setminus \{ [2, 3] \times N \} \) splits into the disjoint union \( X'_0 \cup A \) respectively. Now any \( u \in L^2_k(X) \) can be expressed as a union:

\[ u_1 + u_2 + u_3 \in L^2_k(X_0)_0 + L^2_k(Y_0)_0 \cap L^2_k(X_0)_0 + L^2_k(A)_0. \]

\( < v, u_2 > = 0 \) holds for \( v \in L^2_k(Y_0)^\perp_0 \). Let us define another continuous linear functional:

\[ G_v : L^2_k(X) \to \mathbb{R}, \quad G_v(u) = < v - v_1, u >_{L^2_k(X)} = < v - v_1, u_3 >_{L^2_k(X)} \]

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So \( G_v = v_2 \in L_k^2(A_0)_0 = L_k^2(A_0)_0 \subset L_k^2(X) \), which gives another closed linear subspace \( H_2 \subset L_k^2(A_0)_0 \).

In total we have the decomposition:

\[
v = v_1 + v_2 \in L_k^2(X)_0 \oplus L_k^2(A)_0.
\]

which gives the closed linear subspaces \( H_1 \oplus H_2 \subset L_k^2(X) \) with the desired properties. This completes the proof.

**Proposition 2.2** Suppose both \( D_Y \) and \( D_X \) are of Fredholm. Then \( D_Z \) has closed range.

**Proof:** 

**Step 1:** Suppose contrary. Then the spectrum of \( D_Z \) accumulates near 0. Let us choose an orthonormal sequence \( \{u_i\}_i \subset L_{k+1}^2(Z) \) with \( \|u_i\|_{L_{k+1}^2} = 1 \) and \( \|D(u_i)\|_{L_k^2} \to 0 \), where the spectra of \( u_i \) lie within \((-\lambda_i, \lambda_i)\) with \( \lambda_i \to 0 \).

Let us decompose \( u_i = u_i^1 + u_i^2 \in L_{k+1}^2(Y_0)_0 \oplus H_1 \) in lemma 2.1.

We claim that there is positive \( \epsilon > 0 \) with the uniform lower bounds:

\[
\inf_i \{ \|u_i^1\|_{L_{k+1}^2}, \|u_i^2\|_{L_{k+1}^2} \} > \epsilon.
\]

Suppose contrary, and assume \( \|u_i^1\|_{L_{k+1}^2} \to 0 \).

Then both \( \|u_i^2\|_{L_{k+1}^2} \to 1 \) and \( \|D(u_i^2)\|_{L_k^2} \to 0 \) hold, while support of \( u_i^2 \) lie inside \( X_0 \subset Z \). Moreover the inner products satisfy asymptotic orthogonality:

\[
<u_i^2, u_j^2>_{L_{k+1}^2} \to 0
\]

holds as \( i, j \to \infty \).

If we regard \( u_i^2 \in L_{k+1}^2(X) \), then there is \( m \geq 0 \) and \( a_1^2 + \cdots + a_m^2 = 1 \) such that the uniform lower bound should hold:

\[
\|\sum_{i=1}^m a_i D(u_i^2)\|_{L_k^2} \geq \delta > 0
\]

since \( D \) is of Fredholm over \( X \). However it follows from the assumption that \( \|\sum_{i=1}^m a_i D(u_i^2)\|_{L_k^2} \to 0 \) should hold as \( i, j \to \infty \), which is a contradiction.

We can argue another case \( \|u_i^2\|_{L_{k+1}^2} \to 0 \) by the same way.

This verifies the claim.

**Step 2:** Each \( u_i \) are smooth since their spectra are small. Let us decompose:

\[
D(u_i) = w_i^1 \oplus w_i^2 \in L_{k+1}^2(Y_0)_0 \oplus H_1
\]

in lemma 2.2. Both \( w_i^1 \) and \( w_i^2 \) converge to 0 in \( L_{k+1}^2 \) as \( i \to \infty \).
We claim that $D(u^1_l)$ converge to 0 in $L^2_{k+1}$ for $l = 1, 2$. Notice the equality $D(u^1_l) + D(u^2_l) = w^1_l + w^2_l$, and so:

$$\delta_i \equiv D(u^1_l) - w^2_l = w^1_l - D(u^2_l) \in L^2_{k+1}(Z).$$

For any $v \in C^\infty_c(Y_0)$,

$$| < \delta_i, v >_{L^2_{k+1}} | = | < w^1_l, v > - < D(u^2_l), v > |$$

$$= | < w^1_l, v > - < u^2_l, D^*(v) > |$$

$$= | < w^1_l, v > | \leq ||w^1_l||_{L^2_{k+1}} ||v||_{L^2_{k+1}} \to 0.$$

Thus if we regard $\delta_i \in (L^2_{k+1}(Y_0))^* \cong L^2_{k+1}(Y_0)_0$, then it converges to zero. So $D(u^1_l) = \delta_i + w^2_l$ converge to zero in $L^2_{k+1}(Z)$. However this leads to a contradiction by arguing as in step 1. This completes the proof.

Let us apply proposition 2.2 to analysis of the parametrized elliptic operators over $Y$. Let us consider the periodic cover:

$$\tilde{Y} = \ldots \tilde{W}^0 \cup \tilde{W}^1 \cup \ldots$$

where $\tilde{W}^i$ are the copies of the same $\tilde{W}$. Then we obtain the parametrized differential operators for $z \in C(1) \subset \mathbb{C}$:

$$D_z : L^2_{k+1}(E(z)) \to L^2_k(F(z)).$$

**Corollary 2.1** Suppose $D : L^2_{k+1}(E) \cong L^2_k(F)$ gives an isomorphism. Then $D_z$ have closed range for all $z \in C(1)$.

**Proof:** It is true for $z = 1$ by the assumption. Let us denote $Y = \tilde{W}/N(-2) \sim M(2)$, and $H, H_1 \subset L^2_{k+1}(Y; E(z))$ be closed subspaces as in lemma 2.2 satisfying:

1. $H \oplus H_1 = L^2_{k+1}(Y; E(z))$
2. $H = L^2_{k+1}(\tilde{W}; E(z))_0$ and $\text{Supp} H_1 \subset N \times [-2, 0] \cup M \times [0, 2]$.

Then one can identify $H \subset L^2_{k+1}(Y; E)$.

For $u \in H_1$, one can associate $u' \in L^2_{k+1}(Y; E)$ by:

$$u'|N \times [-2, 0] = u, \quad u'|M \times [0, 2] = z^{-1}u.$$

Let us denote by $\tilde{H}_1 \subset L^2_{k+1}(Y; E)$ the corresponding subspace to $H_1$. Since this assignment is isometric, $\tilde{H}_1$ is also a closed subspace of $L^2_{k+1}(Y; E)$. Since
$D_i(\tilde{H}_1)$ is closed by the assumption, it follows that $D_i(H_1)$ is also a closed subspace. Then we can follow the proof of proposition 2.2 and obtain the result. This completes the proof.

Let $Y$ be the cylindrical manifold given by the end connected sum of the kinky handles in 2.A.2, and let $\Lambda^*(z)$ for $z \in C(1) = \{z : |z| = 1\}$ be the twisted differential forms in 2.B. The differentials canonically extend over the twisted AHS complexes. Moreover the weighted $L^2$ inner products are also induced from the one at $z = 1$. In [K1], we have verified the following:

**Lemma 2.3 (K1)** There is small $\delta > 0$ so that the twisted AHS$_\mu$ complexes:

$$0 \longrightarrow L^2_{k+1}(Y;\Lambda^0(z)) \xrightarrow{d} L^2_{k}(Y;\Lambda^1(z)) \xrightarrow{d^*} L^2_{k-1}(Y;\Lambda^2_+(z)) \longrightarrow 0$$

are acyclic for all $0 < \mu \leq \delta$.

**Proof of proposition 2.1:** Combining with lemma 2.1 and 2.3, it follows that AHS$_\mu$ complexes are acyclic Fredholm over the periodic cover $\tilde{Y}$ for all $0 < \mu \leq \delta$. Then proposition 2.1 follows from proposition 1.1 (see also proof of proposition 1.2). This completes the proof.

The above method is the basis for the analysis over the higher stage Casson handles.

**2.C Riemannian manifolds of the second stage:** Let $(W_1, N, M)$ be the simplest Riemannian kinky handle as described in 2.A.2. Let us recall $Y_1 = \tilde{W}_1/\{N \sim M\}$ and its half periodic cover:

$$\tilde{Y}_1 = \tilde{W}_1^0 \cup_{M_i \sim N^i} \cdots \cup \tilde{W}_1^{i-1} \cup_{M_i \sim N^i} \tilde{W}_1^i \cup \cdots$$

equipped with the attaching region $N^0 \subset \tilde{W}_1^0 \subset \tilde{Y}_1$. The half periodic Casson handle can be expressed by the half real line $T^0_1 = \mathbb{R}_{\geq 0} \equiv \mathbb{R}_0$ assigned with the same sign on each edge. Let $T^0_2$ be the tree:

$$T^0_2 = \mathbb{R}_0 \cup_{n \in \mathbb{N}} \mathbb{R}_0$$

which is obtained from $T^0_1$ attached with the infinite number of the same $(T^0_1)'$ at the root with each integers $\mathbb{N} \subset \mathbb{R}_0$.

Let us describe the corresponding Riemannian-Casson handle $CH(T_2) = Y_2$. Let $(W_2, N, M_1, M_2)$ be a kinky handle with 2 kinks. $W_2$ is obtained by the end
connected sum of two copies of $W_1$ along the disks on the boundary:

$$W_2 = W_1 \natural W_1.$$ 

The Riemannian structure on $W_2$ satisfies the following properties:

1. $W_2$ contains three disjoint Riemannian subspaces:

$$N \times (-\infty, 0] \cup M_1 \times [0, \infty) \cup M_2 \times [0, \infty)$$

where $N$ and $M_1, M_2$ are cylindrical manifolds with their ends $\Sigma$, which are mutually isometric.

2. $W_2$ contains another Riemannian subspace:

$$V_2 \times [0, \infty)$$

where $V_2$ is a non compact Riemannian three manifold with three ends which are isometric to three disjoint union of $\Sigma \times [0, -\infty)$.

Let us denote:

$$\tilde{W}_2 = W_2 \setminus \{N \times (-\infty, -2) \cup M_1 \times (2, \infty) \cup M_2 \times (2, \infty)\}$$

and attach $CH(\mathbb{N})$ in 2.A.2 by identifying $M_2 \times \{2\}$ with $N^0 \times \{-2\}$ in $\tilde{W}_1^0 \subset CH(\mathbb{N})$:

$$CW_2 = \tilde{W}_2 \cup_{M_2 \sim N^0} CH(\mathbb{N}).$$

If we take the end connected sum along $N \times \{-2\}$ with $M_1 \times \{2\}$ as:

$$Y_2 \equiv CW_2 / \{N \times \{-2\} \sim M_1 \times \{2\}\}$$

then $Y_2$ is a complete Riemannian manifold without boundary.

Let us consider its half periodic cover:

$$CH(T_2^0) = \tilde{Y}_2^0 = CW_2^0 \cup_{M_1^0 \sim N^1} \cdots \cup CW_i^j \cup_{M_i^j \sim N_i+1} CW_{i+1}^j \cup \cdots$$

where $\{CW_i^j\}_{i \in \mathbb{N}}$ are the infinite number of the copies of $CW_2$. This is the Casson handle corresponding to $T_2^0$ with the attaching region $N^0 \subset CW_2^0$.

As above, we denote the periodic cover of $Y_2$ by:

$$\tilde{Y}_2 = \cdots \cup CW_i^j \cup_{M_i^j \sim N_i+1} CW_{i+1}^j \cup \cdots$$

Combining proposition 2.1 and 2.2, AHS$_\mu$ complex has closed range for $0 < \mu \leq \delta$ over $Y_2$. It has been verified to be acyclic by use of an asymptotic method.
in [K1].

2.D Casson handles of higher stages: By use of Fourier-Laplace transform with the parallel argument to use Fourier-Laplace transform with acyclicity of AHS\(_\mu\) complexes over \(Y_2\) above, it follows that the AHS\(_\mu\) complexes over \(\tilde{Y}_2\) are also acyclic Fredholm.

Let us consider the third stages. Let:

\[
CH(T^0_0) = Y^0_2 = CW^0_2 \cup \cdots \cup CW^i_{M_i \cong N_{i+1}} \cup CW^{i+1}_2 \cup \ldots
\]

be the second stage of the Casson handle, and put:

\[
CW_3 = \tilde{W}_2 \cup_{M_2 \sim N^0} CH(T^0_2), \quad Y_3 \equiv CW_3 / \{N \times \{-2\} \sim M_1 \times \{2\} \}.
\]

Let us consider: \(T^0_2 = \mathbb{R}_+ \cup_{n \in \mathbb{N}} \mathbb{R}_+\), and put:

\[
T_3 = \mathbb{R} \cup_{n \in \mathbb{Z}} T^0_2.
\]

The induced periodic cover is the Casson handle which correspond to \(T_3\):

\[
\tilde{Y}_3 = CH(T_3).
\]

The same argument as above verifies that the AHS\(_\mu\) complexes over both \(Y_3\) and \(\tilde{Y}_3\) are acyclic Fredholm.

By this way, we obtain the complete Riemannian Casson handles \(Y_N\) so that the AHS\(_\mu\) complexes over both \(Y_N\) and \(\tilde{Y}_N\) are acyclic Fredholm.

So far we have described the rooted trees with at most trivalent branch. The above construction works for the rooted trees with more branches. In [K1], we have introduced a class of homogeneous trees of bounded type. In our notation, the half periodic trees \(T^0_k\) are expressed as \(T^0_{k+1} = T^0_{2,2,\ldots,2,1}\), where 2 appear \(k\) times. Let \(n_1, \ldots, n_k \in \{1, 2, \ldots\}\) be positive integers. Then using kinky handles with \(n_j\) kinks, one has a natural extension, and gets the homogeneous tree of bounded type \(T^0_{n_1,\ldots,n_k,1}\) which is a rooted infinite tree and admits the corresponding Riemannian-Casson handle \(CH(T^0_{n_1,\ldots,n_k,1})\).

By iterating the previous process, one can verify that the AHS\(_\mu\) complexes over:

\[
CH(T_{(n_1,\ldots,n_k,1)}) \equiv \tilde{Y}_{(n_1,\ldots,n_k,1)}
\]

are all acyclic Fredholm.
For practical application, one considers open four manifolds given by the 0 handle attached with Casson handles. Recall that \( l(S^2 \times S^2) \setminus \text{pt} \) is homotopy equivalent to the wedges of \( S^2 \), and also that it is described pictorially by \( l \) disjoint unions of Hopf links with 0 framings, where each \( S^1 \) component corresponds to the attaching region of the Casson handle. So there is a diffeomorphism:

\[
l(S^2 \times S^2) \setminus \text{pt} \cong D^4 \sqcup \bigcup_{i=1}^{2l} (D^2 \times D^2).
\]

Let \( T_1, \ldots, T_{2l} \) be signed homogeneous trees of bounded type. Let us equip with a complete Riemannian metric on \( D^4 \) with boundary \( M \times \{0\} \), which contains \( M \times [0, 1] \) isometrically. By the end connected sum, one obtains the Riemannian-Casson handle:

\[
S \equiv D^4 \sqcup \bigcup_{i=1}^{2l} CH(T_i).
\]

**Theorem 2.1 (K1)** Let \( S = D^4 \sqcup \bigcup_{j=1}^{2l} CH(T_j) \) be the Riemannian-Casson handle whose trees are homogeneous of bounded type. Then there is a complete Riemannian metric \( g \) of bounded geometry on \( S \) and positive \( \delta > 0 \) so that the bounded complexes:

\[
0 \rightarrow (L^2_{k+1})_{\mu}(S, g) \xrightarrow{d} (L^2_k)_{\mu}((S, g); \Lambda^1) \xrightarrow{d^+} (L^2_{k-1})_{\mu}((S, g); \Lambda^2_+) \rightarrow 0
\]

are Fredholm for all the weight \( 0 < \mu \leq \delta \) with their cohomology groups:

\[
H^0_{\mu} = 0, \quad H^1_{\mu} = 0,
\]

\[
l = \dim H^2_+(S : \mathbb{R}) \leq \dim H^0_{\mu} \leq 2l = \dim H^2(S : \mathbb{R}).
\]

Now combining theorem 2.1 with corollary 1.2, we obtain:

**Corollary 2.2** The AHS complex over \( S \):

\[
0 \rightarrow \mathcal{L}^2_{k+1}(S) \xrightarrow{d} \mathcal{L}^2_k(S; \Lambda^1) \xrightarrow{d^+} L^2_{k-1}(S; \Lambda^2_+) \rightarrow 0.
\]

is of Fredholm with the same index above.

## 3 Trivializing at infinity

### 3.A Tree like structure on Riemannian manifolds

Let \( X \) be a complete Riemannian four manifold of bounded geometry, and \( E \rightarrow X \) be an \( SO(3) \) bundle over \( X \). Each \( x \in X \) admits local chart onto \( \delta_0 > 0 \) ball in \( \mathbb{R}^4 \).
A covering \( \{U_i\}_i \) on a Riemannian manifold \( X \) is bounded, if (1) \( \sup_i \text{diam } U_i < \infty \) and (2) \( \sup_i \# \{ j : U_i \cap U_j \neq \phi \} < \infty \) hold. Later on we assume that \( U_i \) are diffeomorphic to the disks, which actually gives no extra conditions on the existence of bounded coverings, when \( X \) is of bounded geometry.

Let \( T^* \) be a connected and rooted tree with the root \( * \), and introduce the canonical tree metric on it. For any vertex \( v \in T^* \), let \( g(v) \subset T^* \) be the set of vertices which lie on the geodesic from the root to \( v \).

**Definition 3.1**

(1) A Riemannian manifold \( X \) admits \( T^* \times \mathbb{N} \) covering, if there exists a bounded covering \( \{U_i\}_i \) and a one to one correspondence \( I : T^* \times \mathbb{N} \cong \mathbb{N} \) such that \( I \) satisfies:

\[
U_{I(k,i)} \cap U_{I(l,j)} = \phi \text{ for } d(l, k) \geq 2,
\]

\[
\sup_{i,j} \{|i - j| : U_{I(i)} \cap U_{I(j)} \neq \phi\} < \infty.
\]

(2) \( T^* \times \mathbb{N} \) covering over \( X \) is trivial at infinity, if for any \( C \), there are \( C_0, k_0, j_0 \) such that for any \( v \in T^* \) with \( |v| \geq k_0 \) and \( j \geq j_0 \), \( C \) neighborhood of:

\[
\bigcup_{v' \in g(v)} U_{I(v',j)} \cup_{0 \leq i \leq j} U_{I(v,i)}
\]

is \( \pi_1 \)-null in its \( C_0 \) neighborhood.

Let us see particular cases:

(1) A Riemannian manifold \( X \) admits \( \mathbb{N} \) covering, if there exists a bounded covering \( \{U_i\}_i \) and a one to one correspondence \( I : \mathbb{N} \cong \mathbb{N} \) with:

\[
\sup_{i,j} \{|i - j| : U_{I(i)} \cap U_{I(j)} \neq \phi\} < \infty.
\]

**Example 3.1:** Any cylindrical manifolds admit \( \mathbb{N} \) covering.

(2) A Riemannian manifold \( X \) admits \( \mathbb{N}^2 \) covering, if there exists a bounded covering \( \{U_i\}_i \) and a one to one correspondence \( I : \mathbb{N}^2 \cong \mathbb{N} \) with:

\[
U_{I(k,i)} \cap U_{I(l,j)} = \phi \text{ for } |l - k| \geq 2,
\]

\[
\sup_{k,m,l,i} \{|i - m| : U_{I(i)} \cap U_{I(k,m)} \neq \phi\} < \infty.
\]

**Lemma 3.1** Let \( S \equiv D^4 \sharp \bigcup_{i=1}^m \text{CH}(T_i) \) be the Riemannian-Casson handle in 2.D. Then \( S \) admit \( T^* \times \mathbb{N} \) covering which is trivial at infinity.

In particular the half periodic Casson handles admit \( \mathbb{N}^2 \) covering which is trivial at infinity.
**Proof:** Step 1: Let us describe how the half periodic Casson handles admit $\mathbb{N}^2$ covering. Let:

$$S = D^4 \wr CH(\mathbb{N}) = D^4 \wr \tilde{W}_1^0 \wr \tilde{W}_1^1 \wr \tilde{W}_1^2 \wr \ldots$$

where $\tilde{W}_1^i$ contain two disjoint product ends $N \times [-1, 0] \sqcup M \times [0, 1]$, and $D^4$ has one product end $N \times [-1, 0]$. On the other hand $D^4$ and $\tilde{W}_1^i$ are manifolds with boundary which contain cylindrical ends $B \times [0, \infty)$ and $V^i \times [0, \infty)$ respectively, where the Riemannian three manifolds $V^i$ are all mutually isometric with two boundary components.

Let us decompose both $B$ and $V$ by open discs $B = \bigcup_{j=1}^{m_0} U_j$ and $V = \bigcup_{j=1}^{m_0} U'_j$. Let $I : \mathbb{N}^2 \rightarrow S$ be a map which satisfies the following properties:

1. The image $I(0, N) \subset D^4$ is a net such that $I(0, i m_0) \in B \times \{i + i_0\}$ for some $i_0$.
2. the image $I(k, N) \subset \tilde{W}_1^k$ is a net such that $I(k, i m_0) \in V^k \times \{i + i_0\}$.

It is immediate to see that $I$ satisfies the required conditions.

Let us consider the general case, and let $m$ be the number of the trees in the Casson handles. For $m = 1$, $S$ clearly admits $T_1^* \times \mathbb{N}$ covering in a parallel way.

For $m \geq 2$, let $T_1^*$ be the rooted tree which attach all $T_i$ at their roots. Then the construction of $T^* \times \mathbb{N}$ covering can be reduced to the case $m = 1$.

**Step 2:** Let us consider triviality at infinity. For simplicity of the notation, we verify the half periodic case $S = D^4 \wr CH(\mathbb{N})$ only. Let:

$$CH(\mathbb{N}) \equiv \tilde{W}_1^0 \cup \ldots \cup \tilde{W}_1^k \cup \tilde{W}_1^{k+1} \cup \ldots$$

be the half periodic Riemannian-Casson handle equipped with $\mathbb{N}^2$ covering. The end of $\tilde{W}_1^k$ is isometric to $V^k \times [0, \infty)$.

It is enough to see that for any $i, k$,

$$(\cup_{0 \leq t \leq k} V^t) \times \{i\} \cup M_k$$

are $\pi_1$-null in:

$$(\cup_{0 \leq t \leq k+1} V^t) \times \{i\} \cup (\tilde{W}_1^{k+1} \setminus V^{k+1} \times [i + 1, \infty)).$$

This follows from the construction of the kinky handles, where the embedding of the solid torus $T^1 \equiv S^1 \times D^2 \hookrightarrow S^1 \times D^2 \equiv T^2$ is given by the Whitehead double, and hence $T^1$ is contractible in $T^2$. This completes the proof.
Let $Y$ and $Z$ be two metric spaces. They are mutually quasi-isometric, if there is a map $f : Y \to Z$ (not necessarily continuous) and some $C, D$ so that (1) $C$ neighborhood of the image of $f$ covers $Z$, and (2) the inequalities hold:

$$C^{-1}d_Y(m, m') - D \leq d_Z(f(m), f(m')) \leq Cd_Y(m, m') + D$$

for any $m, m' \in Y$.

**Corollary 3.1** Any map $f : T^* \times \mathbb{N} \to S$ which satisfies $f(v, m) \in U_{I(v, m)}$ gives a quasi-isometry:

$$T^* \times \mathbb{N} \cong S = D^4_\mathbb{H} \cup_{l=1}^{m} C H(T_l)$$

with respect to $T^* \times \mathbb{N}$ structure in lemma 3.1.

**Proof:** This follows from the construction of the Riemannian metrics and $T^* \times \mathbb{N}$ structure equipped above. This completes the proof.

**3.A.2: Example:** Let us start from the following:

**Lemma 3.2** Hyperbolic space $\mathbb{H}^4$ does not admit quasi-isometry with $T^* \times \mathbb{N}$.

**Proof:** The asymptotic dimension is a numerical invariant for metric spaces, which is preserved under quasi-isometry ([Gr]). So their dimensions should coincide with each other, if they could admit quasi-isometry. However as-dim $\mathbb{H}^4$ is in fact equal to 4, while as-dim $(T^* \times \mathbb{N})$ is 2. This completes the proof.

M.Tsukamoto observed the following:

**Lemma 3.3** Let $\mathbb{H}^4$ be as above. Then for any small $r \in [0, 1)$, there exists an $L^2$ ASD connection $A_r$ with

$$||F_{A_r}||L^2(\mathbb{H}^4) = r.$$  

In particular it should not necessarily integer.

**Proof:** Let $A$ be a non trivial ASD connection with $e = ||F_A||L^2(D) > 0$ over the unit disc $D \subset \mathbb{R}^4$. Then for any $0 < a \leq e$, there exists some $0 < r \leq 1$ with $a = ||F_A||L^2(D_r)$, where $D_r \subset D$ is $r$-disc.

Notice that the ASD condition and $L^2$ norm are both preserved under conformal change of the metrics. Let us equip with Poincaré metric on $D_r$ which is isometric to $\mathbb{H}^4$. Then the restriction $A|D_r$ with the metric is the desired one.

**3.B Trivializing near infinity:** Let us recall the topology of the end of $S$:
**Theorem 3.1 (Fr)** The end of Casson handle $S = D^4 \cup \bigcup_{j=1}^{2l} CH(T_j)$ admits a topological color $\cong S^3 \times [0, \infty)$.

In particular $S$ is simply connected and simply connected at infinity.

We verify the following:

**Proposition 3.1** Let $X$ be a complete Riemannian manifold of bounded geometry equipped with $T^* \times \mathbb{N}$ covering. Let $E \to X$ be a bundle on $X$ and $A$ be an ASD connection over $E$ with $||F_A|| L^2(X) < \infty$.

If the covering is trivial at infinity, then $A$ is approximated by a compactly supported smooth connection, after gauge transformation.

**Proof:** Notice that if $A$ is an ASD connection whose curvature $F_A$ is in $L^2$, then local $L^2$ norms are sufficiently small $||F_A||_{B_{\delta_0}(x)} < \epsilon$ near infinity $x \in X$.

In order to verify this, one uses the following.

**Sublemma 3.1 (U1)** Let $A$ be an ASD connection over $E$, and choose a local trivialization $E|_{B_{2\delta_0}(x)} \cong \mathbb{R}^3$. Then there exist $\epsilon_0 > 0$ and $C_k$ for $k \geq 3$ such that if $||F_A|| L^2(B_{2\delta_0}(x)) < \epsilon_0$ holds, then there exists a gauge transform $g \in \text{Aut } E|_{B_{2\delta_0}(x)}$ with:

$$g^*(A) = d + A', \quad ||A'|| L^2_k(B_{\delta_0}(x)) \leq C_k ||F_A|| L^2(B_{2\delta_0}(x)).$$

This is Uhlenbeck’s theorem based on the construction of the Coulomb gauges. In particular $A$ is smooth by the Sobolev embedding.

**Proof of proposition**: We split the proof into 5 steps, where we will not use triviality at infinity on $X$ until step 4.

**Step 1:** Let $\{U_i\}_i$ be a bounded covering, where each $U_i$ is diffeomorphic to the disc. Let us choose any frame at $m \in U_i$, and construct the local frame:

$$\psi_i : E|_{U_i} \cong U_i \times \mathbb{R}^3.$$ 

by use of the parallel transport with respect to $A$.

We claim that for any $\epsilon > 0$, there exists $i_0$ such that for any $i \geq i_0$, the restriction can be expressed as:

$$A|_{U_i} = d + a_i, \quad ||a_i|| L^2_k(U_i) < \epsilon$$

with respect to the trivialization over $U_i$. In fact $||F_A|| L^2(U_i) < \epsilon$ hold for all large $i >> 1$ by the assumption. Then it follows from sublemma 3.1 that there
is a local trivialization $\varphi_i : E|U_i \cong U_i \times \mathbb{R}^3$ such that the above estimates hold with respect to $\varphi_i$. One may assume that $\psi_i$ and $\varphi_i$ coincide at $m \in U_i$. Recall that the parallel transport is given uniquely by the ODE:

$$\nabla \dot{x}(t)\xi = \Sigma \left\{ \frac{d\xi^\lambda(x(t))}{dt} + \Sigma \Gamma_{\mu,i}^\lambda(x(t)) \frac{dx^i}{dt} \xi^\mu(x(t)) \right\} e_\lambda$$

where $\nabla e_\lambda = \Sigma \omega_\lambda^\mu e_\mu$, $A = (\omega_\lambda^\mu)$ with $\omega_\lambda^\mu = \Sigma \Gamma_{\lambda,i}^\mu dx^i$. In particular we may assume that $\Gamma_{\lambda,i}^\mu$ have sufficiently small norms in $L^2_k$ with respect to $\varphi_i$. This verifies the claim, since the parallel transport is independent of choice of local coordinates.

**Step 2:** Recall that if $x_1(t)$ and $x_2(t)$ are two homotopic paths with the same end points, then the difference of their parallel transports $\xi^1$ and $\xi^2$ with the same initial vector $\xi$ can be estimated by $l^1$ norms of the curvature of $A$ over any surface which span $x_1 \cup x_2$.

As a test case, let us consider a simple situation with $X = U_1 \cup U_2$, and trivializations $\varphi_i : E|U_i \cong U_i \times \mathbb{R}^3$ are given over $U_i$. Let us choose any $m \in U_1 \cap U_2$. If $g_{12} = \varphi_1^{-1}\varphi_2(m) \in SO(3)$ is sufficiently near the identity, then small modification of $\varphi_2|U_1 \cap U_2$ gives the extension of the trivialization $\varphi_1$ over $U_1 \cup U_2$. Notice that this is the case under the situation in step 1, by change of the trivialization over $U_2$ by constant if necessarily.

Next let us consider $X = U_1 \cup U_2 \cup U_3$ with the trivializations $\varphi_i : E|U_i \cong U_i \times \mathbb{R}^3$ over $U_i$ as above. If $X = U_1 \cap U_2 \cap U_3 \neq \phi$, then we can replace their trivializations over $U_i$ by constants so that small modification of their new trivializations over $U_i$ gives the global one over $X$. Suppose $X = U_1 \cap U_2 \cap U_3 = \phi$, and extensions of the trivializations over $U_1 \cup U_2$ and $U_2 \cup U_3$ are given as above. Let us choose any $m \in U_1 \cap U_3$ and consider $g_{13} = \varphi_1^{-1}\varphi_3(m) \in SO(3)$. If $\varphi_1^{-1}\varphi_3$ takes value sufficiently near the constant $g_{13}$, one can modify $\varphi_3$ on $U_1 \cap U_3$ so that the result of the local trivializations give flat structure over $X$. Notice that if $g_{13}$ is away from constant, then the flat structure may not be able to extend over $U_1 \cup U_2 \cup U_3$.

**Step 3:** Let us consider a case with many number of the coverings $X = U_1 \cup \cdots \cup U_i \cup \ldots$, and suppose:

1. there is some $s_0$ such that $|i - j| \leq s_0$ hold whenever $U_i \cap U_j \neq \phi$,
2. the curvature of $A$ has small $L^2$ norms over each $U_i$ as in step 1.

Then one can construct flat structure over $X$ inductively by use of the method
in step 2 as below.

Let us draw a bi-Lipschitz line $l : [1, \infty) \to X$ so that $l(i) \in U_i$ hold. Choose any frame at $l(1) \in U_1$ and fix the trivializations on all $l(i) \in U_i$ by use of the parallel transport. By use of the trivializations at $l(i)$, one obtains the local trivializations over $U_i$ as in step 1. Let us extend flat structure inductively. Suppose it is given over $U^{m-1} \equiv U_1 \cup \cdots \cup U_{m-1}$, and let us extend it over $U^m$. If $U_i \cap U_m \neq \phi$, then $i \geq m - s_0$ must hold.

As a simple case, let us assume moreover:

(3) there is some constant $C$ such that $\cup_{i \leq j \leq i+s_0} U_j$ are contractible in $C$ neighborhood of them for all $i$.

In this case we obtain the global trivialization over $X$, since the transition functions $\varphi^{-1}_i \varphi_m$ over $U_i \cap U_m$ must take values sufficiently near the identity, as we noticed at the first paragraph of step 2, and the trivialization can be extended as in step 2.

Now let us remove the condition (3), and construct flat structure over $X$. Let us fix a constant $C >> \sup_i \text{diam} \{U_i \cup \cdots \cup U_{i+s_0}\}$. Suppose $x \in U_i \cap U_m \neq \phi$ for some $i \leq m$. Let us consider the loop $l_{i,m}$ given by the union of $l[i, m]$ with another line between $l(i)$ and $l(m)$ in $U_i \cup U_m$.

If $l_{i,m}$ is contractible in $C$ neighborhood of $l(m)$, then $\varphi^{-1}_i \varphi_m$ take values near the identity, and one can modify $\varphi|U_i \cap U_m$ slightly so that trivialization is extended over $U_i \cap U_m$ to $U_i \cup U_m$.

If $l_{i,m}$ is not contractible in $C$ neighborhood of $l(m)$, then one can modify $\varphi|U_i \cap U_m$ slightly so that flat structure is extended over $U_i \cap U_m$ to $U_i \cup U_m$ as in step 2.

Next choose another $i' \leq m$ with $U_{i'} \cap U_m \neq \phi$. If $U_i \cap U_{i'} \cap U_m = \phi$, then we can extend the flat structure over $U_{i'} \cup U_m$ as above. Suppose $U_i \cap U_{i'} \cap U_m \neq \phi$. If $l_{i',m}$ is homotopic to $l_{i,m}$ in $C$ neighborhood of $l(m)$, then $\varphi^{-1}_{i'} \varphi_i$ take values near the identity, and so one can modify $\varphi|U_{i'} \cap U_m$ slightly, preserving it over $U_i \cap U_{i'} \cap U_m$, so that one can extend the trivialization over $U_{i'} \cap U_m$ to $U_m$. If $l_{i',m}$ is not homotopic to $l_{i,m}$ in $C$ neighborhood of $l(m)$, then one can modify $\varphi|U_{i'} \cap U_m$ slightly so that one can extend flat structure over $U_{i'} \cup U_m$ to $U_m$ as in step 2.

One can repeat this process at most $s_0$ times, and obtain the extension of the flat structure over $U^m$. This finishes the induction step.

Notice that if $U_i \cap U_j \neq \phi$ could happen with large $|i - j|$, then the corre-
sponding transition functions will vary very large. This is the key aspect where we have introduced uniformity of the multiplicity of coverings.

It is a basic fact that for a compact $X$, two bundles over $X$ are mutually homotopic, if their transition functions are sufficiently near. In fact there is some $n$ such that there are bundle surjections $\varphi : X \times \mathbb{R}^n \to E$ and $\varphi' : X \times \mathbb{R}^n \to E'$ by use of the trivializations over $U_i$ respectively. Since their transition functions are mutually near, the homotopy:

$$x \in X \to t \text{ Ker } \varphi(x) + (1 - t) \text{ Ker } \varphi'(x)$$

gives a family of bundles over $X$. In particular Ker $\varphi$ and Ker $\varphi'$ are mutually isomorphic. Then $E \cong (X \times \mathbb{R}^n)/ \text{ Ker } \varphi$ is isomorphic to $E'$ (see [A] p 29).

Suppose $X$ admits $\mathbb{N}$ covering. Let us check that $A$ can be approximated by a smooth connection which is flat at infinity.

Let us choose $j_0$ with $I(j) \geq i_0$ for all $j \geq j_0$, where $i_0$ is in step 1. Let us take a trivialization over $U_{I(j_0)}$ as in step 1.

By step 3, one obtains the flat structure on the end of $E$ with the estimates:

$$A| U_{j_0 \leq j} I(j) = d + a, \quad ||a||L^2_k(\cup_{j \geq j_0} U_{I(j)}) < \epsilon.$$  

**Step 4:** Suppose $X$ admits $\mathbb{N} \times \mathbb{N}$ covering which is trivial near infinity. Let us choose $j_0$ and $k_0$ such that $L^2$ norm of $F_A$ is less than small $\epsilon > 0$ over:

$$\cup_{j \geq j_0 \leq j_0} U_{I(j,k)} \cup_{j \geq 0 \leq k \geq k_0} U_{I(j,k)}.$$  

Let us put the set:

$$L(j_0, k_0) = \{(j, k_0) : 0 \leq j \leq j_0\} \cup \{(j_0, k) : 0 \leq k \leq k_0\}$$

and $L(j_0, k_0, s_0) = \cup_{j_0-s_0 \leq j \leq j_0+s_0} \cup_{k_0-s_0 \leq k \leq k_0+s_0} L(j, k)$. Correspondingly, we put:

$$U(j_0, k_0, s_0) = \cup_{j_0-s_0 \leq j \leq j_0+s_0} \cup_{k_0-s_0 \leq k \leq k_0+s_0} U_{I(j,k)}.$$  

Let us fix $s_0'' >> s_0 >> 0$ such that $U(j_0, k_0, s_0)$ is $\pi_1$-null in $U(j_0, k_0, s_0)'.

Let $\bar{L}(j_0, k_0)$ be the connected line in $[0, \infty) \times [0, \infty)$ which connects neighbor points by each straight line in $L(j_0, k_0)$. Let us choose a bi-Lipschitz line:

$$l : \bar{L}(j_0, k_0) \to X, \quad l(j, k) \in U_{I(j,k)}$$

so that it connects the points in $U_{I(j_0,k)}$ and $U_{I(j_0,k+1)}$ for all $0 \leq k \leq k_0 - 1$, and the points in $U_{I(j,k_0)}$ and $U_{I(j+1,k_0)}$ for all $0 \leq j \leq j_0 - 1$.  

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Let us choose a frame at a point in $U_{I(j_0,0)}$ as in step 1, and extend the trivialization along the line by parallel transport.

By step 1, extend the trivialization over $U_{I(j,k)}$ for all $(i,j) \in L(j_0, k_0, s'_0)$.

By step 3, extend the flat structure inductively over $U(j_0, k_0, s'_0)$ along the line. By restriction, we obtain the trivialization over $U(j_0, k_0, s_0)$:

$$A|U(j_0, k_0, s_0) = d + a, \quad ||a||L_k^2(U(j_0, k_0, s_0)) < \epsilon.$$ 

It follows from sub lemma 3.1 that $A$ can be approximated by the trivial connection over $\cup_{(j,k) \in L(j_0, k_0, s_0)} U_{I(j,k)}$ by use of cut off function on the region.

**Step 5:** Let us consider the case of $T^* \times \mathbb{N}$ covering which is trivial at infinity. The idea is similar to step 4. Let us put $T_k = \{ v \in T^* : d(v, \ast) \leq k \}$ with $\partial T_k = \{ v \in T^* : d(v, \ast) = k \}$. Then we choose $j_0$ and $k_0$ such that $L^2$ norm of $F_A$ is less than $\epsilon$ over:

$$T(j_0, k_0) = \{ (j,v) : 0 \leq j \leq j_0, v \in \partial T_{k_0} \} \cup \{ (j_0, v) : v \in T_{k_0} \}$$

Let us put $T(j_0, k_0, s_0) = \cup_{j_0-s_0 \leq j \leq j_0+s_0} \cup_{k_0-s_0 \leq k \leq k_0+s_0} U(j,k)$, and:

$$U(j_0, k_0, s_0) = \cup_{j_0-s_0 \leq j \leq j_0+s_0} \cup_{k_0-s_0 \leq k \leq k_0+s_0} U_{I(j,k)}.$$ 

Let $\tilde{T}(j_0, k_0)$ be the connected tree in $T^* \times [0, \infty)$ which connects neighbor points by each straight line in $T(j_0, k_0)$. Let us choose a bi-Lipschitz map $l : \tilde{T}(j_0, k_0) \to X$ with $l(j,k) \in U_{I(j,k)}$ for all $(j,k) \in T(j_0, k_0)$. Then as in step 4, we choose trivialization by parallel transport along $\tilde{T}(j_0, k_0)$. The rest process is the same as step 4.

This completes the proof.

**Corollary 3.2** Suppose $X$ admits $T^* \times \mathbb{N}$ structure which is trivial at infinity.

Then $p_1(A) = \frac{1}{4\pi^2} \int_Y tr(F_A \wedge F_A)$ is an integer.

**Proof:** Let us choose the trivialization as above. Then there is a compact subset $K \subset X$, so that the bundle over $X \setminus K$ is trivial and $A|X \setminus K = d + a$ with $||a||L_k^2(X \setminus K) < \epsilon$ is sufficiently small.

By cut off, $A$ is approximated by a compactly supported smooth connection. One may assume that the boundary of $K$ is a smooth submanifold of codimension 1. Let us consider the double of $K$, and extend the approximated connection over it, where it is trivial over the extra $K$. Then $p_1$ must take integer value over the double, which is equal to the $p_1$ value for the approximated connection over $X$. In particular $p_1(A)$ itself also takes integer value.
This completes the proof.

4 Local analysis of the moduli space of ASD connections

Let $X$ be a non compact smooth four manifold which is simply connected and simply connected at infinity. Let $g$ be a complete Riemannian metric of bounded geometry so that injectivity radius is uniformly bounded from below by a positive constant $\epsilon > 0$, and the curvature operator satisfies uniform bound from above as $\sup_{x \in X} |\nabla^l R| < \infty$ for any $l \geq 0$.

Let $E \to X$ be an $SO(3)$ vector bundle which is trivial over $X \setminus K$ for some compact subset $K$, equipped with a fixed connection $\nabla_0$ which is trivial near infinity with respect to the trivialization. $E$ is determined by $w_2(E) \in H^2(X : \mathbb{Z}_2)$ and $p_1(E) \in \mathbb{Z}$. Let $P$ be the corresponding principal $G$ bundle with $P \times_G \mathbb{R}^3 = E$, and put the adjoint bundle by:

$$\text{Ad}(P) = P \times_G \mathfrak{g}$$

where $\mathfrak{g}$ is the Lie algebra of $G$.

4.A Connection spaces: Let $A$ be an ASD connection over $E$ such that $a_0 \equiv A - \nabla_0 \in L^2_k(X; \text{Ad}(P) \otimes \Lambda^1)$ for any $k \geq 0$. We call such a connection as an $L^2$ ASD connection.

The Atiyah-Hitchin-Singer complex (AHS complex) is given by:

$$0 \to C^\infty_c(X; \text{Ad}(P)) \xrightarrow{d_A} C^\infty_c(X; \text{Ad}(P) \otimes \Lambda^1) \xrightarrow{d^+_A} C^\infty_c(X; \text{Ad}(P) \otimes \Lambda^2_+) \to 0$$

where $d^+_A = (1 + *) \circ d_A$.

Let us introduce the corresponding functional spaces, which heavily depend on choice of $A$:

**Definition 4.1** The functional spaces $\mathfrak{L}_l(A)$ are given by the maximal extension of the domain $C^\infty_c(X; \text{Ad}(P) \otimes \Lambda^*)$ with their norms:

$$\|u\|_{\mathfrak{L}^2_{k+1}(A)} = \|d_A(u)\|_{L^2_k}, \quad \|w\|_{\mathfrak{L}^2_{k}(A)} = \|d_A(u)\|_{L^2_k} + \|d^+_A w'\|_{L^2_{k-1}}$$

with respect to the orthogonal decomposition $w = d_A(u) + w'$ as:

$$L^2_k(X; \text{Ad}(P) \otimes \Lambda^1) = d_A(\mathfrak{L}^2_{k+1}(X; \text{Ad}(P))) \oplus d_A(\mathfrak{L}^2_{k+1}(X; \text{Ad}(P)))^\perp.$$

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Lemma 4.2: Suppose \( d_A : \mathfrak{L}_k(A) \cong \mathfrak{L}_k(A') \subset L^2_k(X : Ad(P) \otimes \Lambda^1), \) \( \mathfrak{L}_k(A) \cong d_A(\mathfrak{L}_k(A)) \oplus d_A^+(\mathfrak{L}_k(A)) \) \( \subset L^2_k(X : Ad(P) \otimes \Lambda^1) \oplus L^2_k(X : Ad(P) \otimes \Lambda^2_+). \)

Remark 4.1: In general one cannot integrate \( d_A \) in order to obtain gauge group actions in a straightforward way, since \( g^{-1}ag \) will not be in \( \mathfrak{L}_k(A) \) for any gauge group \( g \) and \( a \in \mathfrak{L}_k(A) \). If we try to approximate \( a \) by compactly supported smooth forms, their \( C^0 \) norms may grow unboundedly, even though \( d_A^+(a) \) keep bounded \( L^2_{k-1} \) norms. Such situation will happen when the spectrum of the Laplacian on 1 form contain the continuous part near zero.

4.B ASD connections: Let \( A \) be an \( L^2 \) ASD connection. Then the isomorphisms of the Hilbert spaces hold by lemma 1.2:

\[
d_A : \mathfrak{L}_{k+1}(A) \cong d_A(\mathfrak{L}_{k+1}(A)) \subset L^2_k(X : Ad(P) \otimes \Lambda^1),
\]

\[
\mathfrak{L}_k(A) \cong d_A(\mathfrak{L}_{k+1}(A)) \oplus d_A^+(\mathfrak{L}_k(A)) \subset L^2_k(X : Ad(P) \otimes \Lambda^1) \oplus L^2_k(X : Ad(P) \otimes \Lambda^2_+).
\]

Lemma 4.1 ( Ker \( d_A^+ \)) \( \subset L^2_k(X : Ad(P) \otimes \Lambda^1) \) is dense in ( Ker \( d_A^+ \)) \( \subset \mathfrak{L}_k(A) \).

Proof: ( Ker \( d_A^+ \)) \( \subset \mathfrak{L}_k(A) \) is isomorphic to \( d_A^+(\mathfrak{L}_k(A)) \), and the closure of \( d_A^+(\text{ Ker } d_A^+) \subset L^2_k(X : Ad(P) \otimes \Lambda^2_+) \) is \( d_A^+(\mathfrak{L}_k(A)) \).

This completes the proof.

Let \( A, A' \) be two \( L^2 \) ASD connections with:

\[ ||a||L^2_k(X) \equiv ||A - A'||L^2_k(X) < \infty \]

Lemma 4.2 Suppose \( [A] = [A'] \in \mathfrak{L}_k(A) \). Then

1. \( A_t = A + t(A - A') \) is a parametrized ASD connections.
2. The curvatures are all the same \( F_{A_t} = F_A \).

Proof: The assumption implies \( d_A^+(a) = (a \wedge a)^+ = 0 \) hold. In particular for any \( t \in \mathbb{R} \), \( A_t = A + ta \) gives a family of ASD connections, since the equalities hold:

\[ F_{A_t}^+ = F_A^+ + td_A^+(a) + t^2(a \wedge a)^+ = 0. \]

On the other hand \( ||F_{A_t}||L^2(X) \) must be constant, which implies \( d_A(a) = a \wedge a = 0 \) must hold. So \( F_A = F_{A_t} \) hold. This completes the proof.

4.C Regularization: Let \( A \) be an \( L^2 \) ASD connection and consider:

\[ F^+ : A + L^2_k(X : Ad(P) \otimes \Lambda^1) \rightarrow L^2_{k-1}(X : Ad(P) \otimes \Lambda^2_+). \]
In general it would be impossible to extend it as a continuous functional from $\mathcal{L}_k(A)$, since kernel of $d^+_A$ will affect to determine output. Even when we restrict it on the orthogonal complement of the kernel, still spectra near zero will affect when the image of $d^+_A$ is not closed.

Let us fix an $L^2$ ASD connection $A$ and $\epsilon > 0$, and consider the spectral decomposition:

$$d^+_A \circ (d^+_A)^* = \Delta_A = \int_0^\infty \lambda^2 dE(\lambda)$$

on $L^2(X : Ad(P) \otimes \Lambda^2_+^+)$. Let $f : [0, \infty) \to \mathbb{R}$ be a smooth function, and put:

$$P_f = \int_0^\infty f(\lambda)dE(\lambda),$$

$$Q_f(a) = (d^+_A)^* \Delta_A^{-1} P_f(d^+_A(a)).$$

Then we introduce a deformation of smooth functionals:

$$F^+_f : L^2_k(X : Ad(P) \otimes \Lambda^1) \to L^2_{k-1}(X : Ad(P) \otimes \Lambda^2_+^+)$$

$$d^+_A(a) + (Q_f(a) \wedge Q_f(a))^+ \equiv d^+_A(a) + B_f(a, a)$$

As a particular case, let $f_\epsilon : [0, \infty) \to [0, 1]$ be a smooth function with:

$$f_\epsilon(\lambda) = \begin{cases} 0 & \lambda \in [0, \epsilon] \\ 1 & \lambda \geq 2\epsilon \end{cases}$$

**Lemma 4.3** (1) Suppose $d^+_A : L^2_k \to L^2_{k-1}$ has closed range. If we restrict $F^+_f$ over $(\text{ker } d^+_A)^\perp$, then it coincides with the standard self-dual curvature functional for all sufficiently small $\epsilon > 0$.

(2) $Q_{f_\epsilon} : \mathcal{L}_k(A) \to L^2_k$ gives a bounded linear functional:

$$||Q_{f_\epsilon}(\cdot)||L^2_k \leq C_k \epsilon^{-1}||\mathcal{L}_k(A)$$

where $C_k$ depends only on $k$.

(3)

$$F^+_f : \mathcal{L}_k(A) \to L^2_{k-1}(X : Ad(P) \otimes \Lambda^2_+^+)$$

defines a uniformly Lipschitz functional. In particular the differential at $A$ is isometric onto its image over $(\text{Ker } d^+_A)^\perp$.

So the operator norm $||(d^+_A)^{-1}|| = 1$ holds, when $d^+_A$ is an isomorphism on $(\text{Ker } d^+_A)^\perp$. 

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Lemma 4.4

(1) If \( \epsilon < \lambda_1 \) smaller than the first eigenvalue of \( \Delta_A \) on \( L^2(X : Ad(P) \otimes \Lambda^2_+) \), then \( Q_{f_{\epsilon}}(a) = a \) hold for all \( a \in (\ker d_A^+)^\perp \).

For (2), notice the estimate:

\[
< (d_A^+) \Delta_A^{-1}(P_{f_{\epsilon}}(b)), (d_A^+) \Delta_A^{-1}(P_{f_{\epsilon}}(b)) >_{L^2} = < P_{f_{\epsilon}}(b), \Delta_A^{-1}(P_{f_{\epsilon}}(b)) >_{L^2} \leq \epsilon^{-1} < b, b >_{L^2}.
\]

Then it follows from the estimates below:

\[
\|(d_A^+) \Delta_A^{-1}P_{f_{\epsilon}}(d_A^+(a))\|_{L^2_k} \leq C_k \epsilon^{-1}\|P_{f_{\epsilon}}(d_A^+(a))\|_{L^2_{k-1}} \leq C_k \epsilon^{-1}\|d_A^+(a)\|_{L^2_{k-1}} = C_k \epsilon^{-1}\|a\|_{\mathfrak{L}_k(A)}.
\]

(3) Let \( a_t = a + tb \in \mathfrak{L}_k(A) \), and consider the difference:

\[
F^+_{f_{\epsilon}}(a + tb) - F^+_{f_{\epsilon}}(a) = td_A^+(b) + B_{f_{\epsilon}}(a, a) + t(B(a, b) + B_{f_{\epsilon}}(b, a)) + t^2 B_{f_{\epsilon}}(b, b).
\]

The following estimates hold by the Hölder estimate and (2):

\[
\|B_{f_{\epsilon}}(a, b)\|_{L^2_{k-1}} \leq C_k\|Q_{f_{\epsilon}}(a)\|_{L^2_k}\|Q_{f_{\epsilon}}(b)\|_{L^2_k} \leq C_k \epsilon^{-1}\|a\|_{\mathfrak{L}_k(A)}\|b\|_{\mathfrak{L}_k(A)}
\]

where the constant \( C_k \) is independent of \( a, b \in \mathfrak{L}_k(A) \).

In particular the Lipschitz constant of the differential \( d_A^+ \) at \( A \) is 1, by definition of the norm on \( \mathfrak{L}_1(A) \). This completes the proof.

Definition 4.2 Let us fix \( \epsilon > 0 \). The regularization of \( F^+ \) at \( A \) is given by:

\[
\tilde{F}^+ : \mathfrak{L}_k(A) \rightarrow L^2_{k-1}(X : Ad(P) \otimes \Lambda^2_+),
\]

\[
\tilde{F}^+(A + a) = d_A^+(a) + (Q_{f_{\epsilon}}(a) \wedge Q_{f_{\epsilon}}(a))^+
\]

The following holds by the implicit function theorem:

Lemma 4.4

(1) If 0 is regular value of both:

\[
F^+_{f_{\epsilon}}, \ F^+ : L^2_k(X : Ad(P) \otimes \Lambda^1) \rightarrow L^2_{k-1}(X : Ad(P) \otimes \Lambda^2_+)
\]

then there is a cobordism between regular smooth manifolds:

\[
(F^+_{f_{\epsilon}})^{-1}(0), \ (F^+)^{-1}(0) \subset A + L^2_k(X : Ad(P) \otimes \Lambda^1).
\]

(2) If 0 is a regular value of \( \tilde{F}^+ \) over \( \mathfrak{L}_k(A) \), then \( (\tilde{F}^+)^{-1}(0) \subset \mathfrak{L}_k(A) \) is a regular smooth manifold.
Proof: We only have to verify (1). Let \( f^t_{\epsilon} : [0, \infty) \rightarrow [0, 1] \) be a smooth path of functions such that \( f^1_{\epsilon} = f_{\epsilon} \) and \( f^0_{\epsilon} \equiv 1 \). Then one obtains a family of the smooth functional:

\[
F^+_t : L^2_k(X : \text{Ad}(P) \otimes \Lambda^1) \rightarrow L^2_{k-1}(X : \text{Ad}(P) \otimes \Lambda^2_+)
\]

by \( d^+_A(a) + (Q_{f^t_{\epsilon}(a)} \wedge Q_{f^t_{\epsilon}(a)})^+ \). Then the conclusion follows by the inverse function theorem. This completes the proof.

Corollary 4.1 Let \( A \) be an \( L^2 \) ASD connection over \( X \), and assume it is regular so that \( d^+_A \) is surjective. Then for any \( \epsilon > 0 \), there is \( \delta > 0 \) which is independent of \( X \) such that for any small perturbation \( s \) with \( s(b) \in L^2_{k-1}(X : \text{Ad}(P) \otimes \Lambda^2_+) \) with \( \| s(b) \|_{L^2_{k-1}} \leq \delta \), there is a solution to the perturbed equation in \( \mathfrak{L}_k(A) \):

\[
(\tilde{F}^+ + s)(a) = 0.
\]

Proof: \( a = 0 \) is the solution for \( b = 0 \). Then this follows from lemma 4.3(3) and the inverse function theorem. This completes the proof.

Remark 4.1: This uniformity plays a key role for our proof of theorem 0.2. The above property holds even if we replace \( \tilde{F}^+ \) by the linear functional \( d^+_A \).

4.D Slices and index formula: Let \( U_\epsilon \subset \mathfrak{L}_k(A) \) be \( \epsilon \) neighborhood of \( A \), and put:

\[
U^\perp_\epsilon \equiv \{ A + v \in \mathfrak{L}_k(A) : v \in d_A(\mathfrak{L}_{k+1}(A))^\perp \cap U_\epsilon \}.
\]

Let \( A \) be an \( L^2 \) ASD connection, and consider \( ( \text{Ker } d^+_A)^\perp \subset L^2_k \). Since closure of \( \text{im } d_A \) is contained in \( \text{Ker } d^+_A \), there is an embedding:

\[
( \text{Ker } d^+_A)^\perp \subset \mathfrak{L}_k(A)
\]

Let \( V_\epsilon \subset L^2_k(X : \text{Ad}(P) \otimes \Lambda^1) \) be \( \epsilon \) neighborhood, and put:

\[
V^\perp_\epsilon = V_\epsilon \cap ( \text{Ker } d^+_A)^\perp.
\]

Then the embedding \( V^\perp_\epsilon \hookrightarrow U^\perp_\epsilon \) has dense image by lemma 4.1.

Proposition 4.1 Let \( S = D^4_{2l} \cup^{2l}_{i=1} CH(T^i) \) is the Riemannian-Casson handle homogeneously of bounded type, and choose an \( L^2 \) ASD connection \( A \).

Then for any \( L^2 \) ASD connection \( A \),

\[
0 \rightarrow \mathfrak{L}^2_{k+1}(A) \xrightarrow{d_A} \mathfrak{L}^2_k(A) \xrightarrow{d^+_A} L^2_{k-1}(X ; \text{Ad}(P) \otimes \Lambda^2_+) \rightarrow 0
\]
is of Fredholm whose index is larger than or equal to:
\[
2p_1(P) + 3 \dim H^2_\mu \geq 2p_1(P) + 3l
\]
if the number is non negative.

Proof: This follows from corollary 1.2 with the next lemma. The index computation below uses the excision principle, or relative index theorem ([GL]).

**Lemma 4.5 (K1)** Let \( S \) and \( A \) be as above. Then for each small \( \mu > 0 \), the \( AHS_\mu \) complex:
\[
0 \rightarrow (L^2_{k+1})_\mu(S; \text{Ad}(P)) \xrightarrow{d_A^-} (L^2_k)_\mu(S; \text{Ad}(P) \otimes \Lambda^1)
\]
\[
\xrightarrow{d_A^+} (L^2_{k-1})_\mu(S; \text{Ad}(P) \otimes \Lambda^2) \rightarrow 0
\]
is a Fredholm complex whose index satisfies the bounds:
\[
2p_1(P) + 3l \leq 2p_1(P) + 3 \dim H^2_\mu \leq 2p_1(P) + 6l.
\]

See also theorem 2.1 on the AHS complex without coefficient.

**Corollary 4.2** If \( d_A^+ : \mathfrak{L}_k(A) \rightarrow L^2_{k-1} \) is surjective, then the local moduli spaces:
\[
\mathfrak{M}(A)_{loc} = \{ A' = A + a \in U^\perp_c \subset \mathfrak{L}_k(A) : \tilde{F}^+(A') = 0 \}
\]
is a regular manifold whose dimension is smaller than or equal to:
\[
-2p_1(P) - 3 \dim H^2_\mu.
\]

In particular if \( 2p_1(P) + 3l > 0 \) is positive, then the regular manifold should have negative dimension, and hence \( d_A^+ \) can not be surjective in the case.

## 5 Local perturbation and transversality

**5.A Convergence process:** Let us fix \( \epsilon > 0 \) and \( \delta > 0 \) in corollary 4.1.

Let \( M \) be a closed smooth four manifold, and choose a family of Riemannian metrics \( g_i \) of bounded geometry on \( M \) such that they converge on each compact subset to a complete Riemannian metric \( h \) on an open subset \( S \subset M \).

**Process 1:** Let \( E \) be an \( SO(3) \) bundle, and take a family of regular ASD connections \( A_i \) with respect to \((M,g_i)\) so that \( d_{A_i}^+ \) are surjective and hence give isomorphisms on \(( \ker d_{A_i}^+ )^\perp \).
By taking a subsequence of \( \{ A_i \} \), they converge to an \( L^2 \) ASD connection \( A \) over \( S \) on each compact subset.

Let \( b \in C_c^\infty(S : Ad(P) \otimes \Lambda^2_+) \) with \( ||b|| \leq \delta \) be a smooth self dual 2 form on \( S \) with sufficiently small support. It follows from corollary 4.1 that there are family of solutions \( A_i + a_i \) to the equations:

\[
\tilde{F}^+_{A_i}(a_i) = b
\]
such that \( a_i \) have uniformly bounded \( L^k(A) \) norms over \( (M, g_i) \).

**Process 2:** Let us verify that a subsequence of \( \{ a_i \} \) converge weakly to \( a \in \mathcal{L}_k(A) \) over \( S \). Let \( K \subset S \) be a compact subset.

Let us consider a family of bounded linear maps:

\[
\varphi_i \in L^2_{k-1}(K)^* \quad u \rightarrow <u, d^+_{A_i}(a_i)>
\]

where \( ||\varphi_i|| < \infty \) are uniformly bounded from above independently of \( K \). Then there is \( \varphi \in L^2_{k-1}(K)^* \) which is a weak limit of \( \varphi_i \). Let:

\[
\Psi : \mathcal{L}_k(A) \rightarrow L^2_{k-1}(K)^* \quad a \rightarrow u \rightarrow <d^+_{A}(a), u>
\]

be the bounded linear functional.

We claim that \( \varphi_i \) are approximated by the image of \( \Psi \). In fact there are \( a'_i \in L^2_{k-1}(M, g_i) \) which approximate \( a_i \) by lemma 4.1. Then by use of the cut off function which are equal to one on \( K \), one may regard them as elements in \( L^2_{k-1}(S, h) \) and hence in \( \mathcal{L}_k(A) \), which verifies the claim.

**Process 3:** Next we claim that \( \Psi \) has closed range, if:

\[
d^+_{A} : \mathcal{L}_k(A) \rightarrow L^2_{k-1}(S : Ad(P) \otimes \Lambda^2_+)
\]

has finite codimension. This follows from the following abstract argument. Let \( H \) be a Hilbert space and \( V \) be a finite dimensional vector space. Let \( L \subset H \oplus V \) be a closed linear subspace, and consider \( d : H \rightarrow L^* \) by the same way as above. Let us see the image of \( d \) is closed. Then it follows from this abstract property that \( \Psi \) has closed range.

Let \( P : H \oplus V \rightarrow H \) be the projection. Since \( d(h)(l) = <h, l> = <h, P(l)> \) hold, it is enough to see that \( P(L) \subset H \) is closed. One may assume \( L \cap V = 0 \), since \( P(L \cap V) = 0 \) holds.

So suppose \( P(l_i) \) converge to \( h \in H \). Let us decompose \( l_i = h_i \oplus v_i \). Then \( h_i \) converge to \( h \). Suppose \( v_i \) are bounded sequence in \( V \). Then by finite
dimensionality, a subsequence converge to $v \in V$, and hence $h + v \in L$. Assume $v_i$ could be unbounded. Then by rescaling by constants so that $||l_i|| = 1$ with $h_i \to 0$ and so $||v_i|| \to 1$. Then a subsequence converge to $v \in L$ which contradicts to the assumption, and we are done.

**Process 4:** Then $\varphi_i$ lie in the image of $\Psi$, and there is some $a \in \mathcal{L}_k(A)$ with:

$$\lim_{i} \varphi_i(u) = \langle u, d^+_A(a) \rangle_{L^2_{k-1}(S)}$$

hold for all $u \in L^2_{k-1}(K)$. Notice that for a closed linear subspace $L \subset H$ in a Hilbert space, if a sequence $a_i \in L$ weakly converge to some $a \in H$, then $a \in L$ holds, since $\langle a, v \rangle = \lim_i \langle a_i, v \rangle = 0$ hold for all $v \in L^\perp$.

$||a||\mathcal{L}_k(A)$ is uniformly bounded independent of choice of $K$, and denote it by $a_K$.

Let us choose an exhaustion of $S$ by compact subsets:

$$K_0 \subset K_1 \subset \cdots \subset S$$

and choose the corresponding $a_i \equiv a_{K_i} \in \mathcal{L}_k(A)$ which consists of uniformly bounded sequence. Again there is a weak limit $a = w - \lim_i a_i \in \mathcal{L}_k(A)$.

We claim that $a$ solves the equation:

$$\tilde{F}^+(a) = b.$$  

Let $k^i(x, y)$ and $k(x, y)$ be the smooth kernels of $\Delta^{-1}_{A_i} P_{f_i}(\Delta_{A_i})$ and $\Delta^{-1}_A P_{f}(\Delta_A)$ over $(M, g_i)$ and $(S, g)$ respectively. Then $k^i$ converge to $k$ smoothly on each compact subset in $S \times S$. It follows from the Sobolev estimate that $\tilde{F}^+(a_i)$ weakly converge to $\tilde{F}^+(a)$ so that the equality holds:

$$\langle u, \tilde{F}^+(a) \rangle = \langle u, b \rangle$$

for any $u \in L^2_{k-1}(S : Ad(P) \otimes \Lambda^2_{+})$. Since $\tilde{F}^+(a)$ itself is in $L^2_{k-1}$, it is equal to $b$. This verifies the claim.

**5.B Transversality:** In order to obtain regular moduli spaces, we have to perform transversality argument. Uhlenbeck's metric perturbation method does not seem to work for our case, since the functional spaces we have introduced, may change their structure heavily under small perturbation of metrics.

Another approach by holonomy perturbation surely does not change the structure of our functional spaces, since perturbation is local. However they
are given by the equivalent classes of connections, which can not determine holonomy.

For our purpose of proof of theorem 0.2, we do not require gauge invariant perturbations. This makes the situation quite simple, and we take a simplest way just to perturb the self dual 2 forms directly.

Let \( D \subset S \) be a small disk, and put the perturbation space \( B \)
\[
B = L_{k-1}^2(D : \text{Ad}(P) \otimes \Lambda^2_+)_0
\]
with the inclusion \( i : B \hookrightarrow L_{k-1}^2(S : \text{Ad}(P) \otimes \Lambda^2_+) \).

**Lemma 5.1** The functional:
\[
\tilde{F}^+ + i : \mathfrak{L}_k(A) \times B \to L_{k-1}^2(X : \text{Ad}(P) \otimes \Lambda^2_+)
\]
gives the surjective differential at \((A, 0)\).

**Proof:** Elements of the cokenel of the image \( d^+_A(\mathfrak{L}_k(A)) \subset L_{k-1}^2 \) can be assumed to satisfy the equation \((d^+_A)^*(u) = 0\), since the image is the closure of \( d^+_A(L_k^2(X : \text{Ad}(P) \otimes \Lambda^1)) \).

So elements in the cokenel of \( d(\tilde{F}^+ + i)|_{(A,0)} : \mathfrak{L}_k(A) \times B \to L_{k-1}^2(X : \text{Ad}(P) \otimes \Lambda^2_+) \) also satisfy the equation \((d^+_A)^*(u) = 0\). By unique continuation property, the restriction \( u|D \) does not vanish. On the other hand one can choose \( b \in B \) with \( <b, u> \neq 0\), which gives a contradiction. This completes the proof.

Let \( U_\perp^\perp \subset \mathfrak{L}_k(A) \) be in 4.D. It follows from lemma 5.1 with the infinite dimensional inverse function theorem, that the map:
\[
\tilde{F}^+ + i : U_\perp^\perp \oplus B \to L_{k-1}^2(S : \text{Ad}(P) \otimes \Lambda^2_+)
\]
has \( 0 \in U_\perp^\perp \oplus B \) as a regular point. So the inverse:
\[
\mathfrak{M}B \equiv (\tilde{F}^+ + i)^{-1}(0) \subset \mathfrak{L}_k \oplus B
\]
is the infinite dimensional Hilbert manifold near zero.

**Corollary 5.1** There is an open neighborhood \( U \subset U_\perp^\perp \oplus B \) and a Baire set \( \tilde{B} \subset B \) such that \( \tilde{B} \) is the regular set over \( U \):
\[
\mathfrak{M}(A, b) = \{ a \in \mathfrak{L}_k(A) : (a, b) \in U, \tilde{F}^+(a) + b = 0 \}
\]
is a regular and finite dimensional smooth manifold for any $b \in \tilde{B}$. Its dimension is equal to the codimension of $d_A^+ : \mathfrak{L}_k(A) \to L^2_{k-1}(S : \text{Ad}(P) \otimes \Lambda^2_+)$, which is smaller than or equal to:

$$-2p_1(P) - 3 \dim H^2_{\mu}.$$

Proof: Let $\pi : \mathfrak{M}B \to B$ be the projection, which is of Fredholm by proposition 4.1. Then the conclusion follows by the Sard-Smale theorem and corollary 4.2.

This completes the proof.

5.C Proof of theorem 0.2 and dimension counting: Let us recall the ideas of the argument in [K3]. Let $M$ be $K3$ surface, and choose an $SO(3)$ bundle $E$ over $M$ such that the Donaldson’s invariant does not vanish over $E$ ([Kr]). So there are always ASD connections over $E$ and generic ASD moduli spaces have 0 dimensional.

Let us proceed by contradiction argument to see that Casson handles of bounded type cannot be embedded into $K3$ surface. Suppose it could be, then by definition, Casson handles of homogeneously bounded type can also be embedded. Let us denote the Riemannian-Casson handle by $(S, h)$.

Let us choose a family of generic metrics $g_i$ over $M$ such that they converge to $h$ on each compact subset on $S \subset M$.

Let us choose any ASD connections $A_i$ with respect to $(M, g_i)$, which converges to an $L^2$ ASD connection $A$ over $S$, and fix a trivialization near infinity so that $A = d + m$ with $||m||_{L^2_k} < \infty$ holds. So $A_i = d + m_i$ with $||m_i||_{(L^2_k)_{\text{loc}}} < \infty$ on each compact subset of $S \subset M$ with respect to the trivialization.

Let us apply process 1 in 5.A. Let $B$ be the Banach perturbation space which consists of local sections on $D \subset S$ in 5.B, and choose solutions $\tilde{F}^+_A(a_i) = b$ for $a_i \in \mathfrak{L}_k(A_i)$ for generic $b \in \tilde{B}$ over $\mathfrak{L}_k(A)$ in 5.B. $d_A^+ : \mathfrak{L}_k(A) \to L^2_{k-1}(S : \text{Ad}(P) \otimes \Lambda^2_+)$ has finite codimension, and so $\{a_i\}_i$ converge weakly to $a \in \mathfrak{L}_k(A)$ by processes 2 − 4. $a$ is a solution to the equation $\tilde{F}^+_A(a) + b = 0$, and so the space:

$$\mathfrak{M}(A, b) = \{a \in \mathfrak{L}_k(A) \cap U : \tilde{F}^+_A(a) + b = 0\}$$

would be a non regular non empty smooth manifold, whose dimension is smaller than or equal to $-2p_1(P) - 3 \dim H^2_{\mu} \leq -2p_1(P) - 9$ by proposition 4.1.

Let us estimate its formal dimension. Over $K3$ surface or its logarithmic transforms, we consider the case with $p_1 = -6$ with $l = 3$. After this deforma-
tion process of metrics, one obtains another bundle whose absolute value of the first Pontryagin number strictly decreases (see the argument also in [K1]). So $|p_1(A)| \leq |p_1(A_i)| - 2 = 4$ and hence we obtain negativity:

$$-2p_1(P) - 3 \dim H^2_{\mu} \leq 8 - 9 = -1$$

which gives a contradiction. This completes the proof of theorem 0.2.

6 Some aspects on global analysis of moduli spaces

6.A Deformation on self dual curvature functionals: We assume that $X$ is compact in 6.

Let us recall the smooth function $f_\epsilon$ and the regularization in 4.C. Let $A$ be an $L^2$ ASD connection. For $a \in L^2_k(X : Ad(P) \otimes \Lambda^1) \cap (\text{Ker} \ d_A^+)^{\perp}$, the equality:

$$\tilde{F}_A^+(a) = d_A^+(a) + (Q_{f_\epsilon}(a) \wedge Q_{f_\epsilon}(a))^+ = F^+(A + a)$$

holds for sufficiently small $\epsilon > 0$.

Let us introduce a family of Hilbert spaces which are obtained by completion of $(\text{Ker} \ d_A^+)^{\perp} \subset L^2_k$ by:

$$||a||_{\hat{L}_k(A)_\epsilon} = ||P_{f_\epsilon}(d_A^+(a))||_{L^2_{k-1}}$$

Notice that the spectra of $\Delta_A$ on $L^2(X : Ad(P) \otimes \Lambda^2_+)$ is discrete, since $X$ is assumed to be compact. If spectrum $\delta$ of $\Delta_A$ lie between $0 < 2\epsilon_0 < \delta < \epsilon_1$, then there is a finite dimensional vector space $H_{\epsilon_0, \epsilon_1}$ such that the isomorphism holds:

$$\hat{L}_k(A)_{\epsilon_1} \oplus H_{\epsilon_0, \epsilon_1} \cong \hat{L}_k(A)_{\epsilon_0}.$$ 

Let $\delta_1 > 0$ be the first eigenvalue, and choose $\epsilon_0$ and $\epsilon_1$ as above. Notice the equalities:

$$\hat{L}_k(A)_{\epsilon_0} = \hat{L}_k(A) \equiv (\text{Ker} \ d_A^+)^{\perp} \subset L^2_k(X : Ad(P) \otimes \Lambda^1).$$

For $0 < t \leq \epsilon_1$, let us consider the smooth path $f_t$, and define:

$$\tilde{F}_t^+ : \hat{L}_k(A) \to L^2_{k-1}(X : Ad(P) \otimes \Lambda^2_+)$$

$$\tilde{F}_t^+(a) = \begin{cases} 
  d_A^+(a) + (Q_{f_t}(a) \wedge Q_{f_t}(a))^+ & t \leq \epsilon_1 \\
  d_A^+(a) + (Q_{f_t}(a') \wedge Q_{f_t}(a'))^+ & t \geq \epsilon_1 
\end{cases}$$
where \( a = a' + a'' \in \hat{\mathcal{L}}_k(A)_{\epsilon_1} \oplus H_{\epsilon_0, \epsilon_1} \). This is a uniformly bounded family of functionals.

**Lemma 6.1** Suppose \( A \) is regular. Then there is a smooth path \( a_t \in \mathfrak{L}_k(A) \) so that they satisfy the solutions \( \tilde{F}^+_t(a_t) = 0 \) with \( a_0 = 0 \).

In particular there is a canonical path between \( A \) and a solution to the regularized equation \( \tilde{F}^+(a) = 0 \).

**Proof:** This follows by lemma 4.3 and the above deformation of the ASD equations. This completes the proof.

**Remark 6.1:** Let \((S, h)\) be the Riemannian-Casson handle homogeneously of bounded type. Let \( A \) be a smooth \( L^2 \) ASD connection over \( S \), and consider a small neighborhood \( U \subset A + L^2_k(S; \text{Ad}(P) \otimes \Lambda^1) \). Then the codimension of:

\[
d^+_A : \mathfrak{L}_k(A') \to L^2_{k-1}(S; \text{Ad}(P) \otimes \Lambda^2_+)\]

is finite dimensional for any \( A' \in U \).

Let us denote \( \text{codim}(A') = \text{codim}\{d^+_A : \mathfrak{L}_k(A') \to L^2_{k-1}(S; \text{Ad}(P) \otimes \Lambda^2_+)\} \), and put:

\[
\text{codim}(U) = \sup_{A' \in U} \text{codim}(A') < \infty.
\]

If \( \text{codim}(U) = \text{codim}(A) \) holds, then \( \text{codim}(U) \) is finite dimensional for any \( A' \in U \), there is \( N \) such that the isomorphisms \( I_{A'} : \mathfrak{L}_k(A) \oplus \mathbb{R}^N \cong \mathfrak{L}_k(A') \) hold.

**6.B Yang-Mills functional:** Let us say that \( A' \in \mathfrak{L}_k(A) \) is ASD, if there is a convergent sequence \( A'_i \to A' \in \mathfrak{L}_k(A) \) with:

\[
A'_i \in A + L^2_k(X : \text{Ad}(P) \otimes \Lambda^1), \quad \tilde{F}^+_{A'_i} \to 0 \in L^2_{k-1}.
\]

It would be of interest for us to ask existence of an ASD element \( A' \in \mathfrak{L}_k(A) \) such that \( A' \) is not contained in the image of \( L^2_k(X : \text{Ad}(P) \otimes \Lambda^1) \to \mathfrak{L}_k(A) \).

For \( A' \in A + L^2_k(X : \text{Ad}(P) \otimes \Lambda^1) \), the Yang-Mills functional is given by:

\[
\int_S \|F^+(A')\|^2 \text{vol}.
\]

Notice that this cannot be directly defined on \( \mathfrak{L}_k(A) \).

For \( A' \in \mathfrak{L}_k(A) \), let us denote by \( [a_i] \) where \( a_i \in A + L^2_k(X : \text{Ad}(P) \otimes \Lambda^1) \) converge to \( A' \) in \( \mathfrak{L}_k(A) \). Then the Yang-Mills functional on \( \mathfrak{L}_k(A) \) is defined
by:

$$YM_A(A') = \inf_{||a_i||} \liminf_i \int_S ||F^+(a_i)||^2 \text{vol.}$$

When the bundle $E \rightarrow S$ admits a minimal ASD connection and if $YM_A(A') = 0$ holds for some $A' \in \mathcal{L}_k(A)$, then it would be of interest for us to ask whether $A'$ lies in the image of $L^2_k$ by use of sub lemma 3.1.

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