Revisiting two classical results on graph spectra

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Abstract

Let \( \mu(G) \) and \( \mu_{\text{min}}(G) \) be the largest and smallest eigenvalues of the adjacency matrix of a graph \( G \). Our main results are:

(i) If \( H \) is a proper subgraph of a connected graph \( G \) of order \( n \) and diameter \( D \), then

\[
\mu(G) - \mu(H) > \frac{1}{\mu^{2D}(G)n}.
\]

(ii) If \( G \) is a connected nonbipartite graph of order \( n \) and diameter \( D \), then

\[
\mu(G) + \mu_{\text{min}}(G) > \frac{2}{\mu^{2D}(G)n}.
\]

These bounds have the correct order of magnitude for large \( \mu \) and \( D \).

Keywords: smallest eigenvalue, largest eigenvalue, diameter, connected graph, bipartite graph

1 Introduction

Our notation is standard (e.g., see [2], [3], and [5]). In particular, unless specified otherwise, all graphs are defined on the vertex set \([n] = \{1, ..., n\}\) and \( \mu(G) \) and \( \mu_{\text{min}}(G) \) stand for the largest and smallest eigenvalues of the adjacency matrix of a graph \( G \).

The aim of this note is to refine quantitatively two well-known results on graph spectra. The first one, following from Frobenius’s theorem on nonnegative matrices, asserts that if \( H \) is a proper subgraph of a connected graph \( G \), then \( \mu(G) > \mu(H) \). The second one, due to H. Sachs [7], asserts that if \( G \) is a connected nonbipartite graph, then \( \mu(G) > -\mu_{\text{min}}(G) \).

Our main result is the following theorem.

Theorem 1 If \( H \) is a proper subgraph of a connected graph \( G \) of order \( n \) and diameter \( D \), then

\[
\mu(G) - \mu(H) > \frac{1}{\mu^{2D}(G)n}.
\]  (1)
It can be shown that, for large $\mu$ and $D$, the right-hand of (1) gives the correct order of magnitude; examples can be constructed as in the proofs of Theorems 2 and 3 below.

**Theorem 2** If $G$ is a connected nonbipartite graph of order $n$ and diameter $D$, then

$$\mu(G) + \mu_{\text{min}}(G) > \frac{2}{\mu^{2D}(G)n}. \quad (2)$$

Moreover, for all $k \geq 3$, $D \geq 4$, and $n = D + 2k - 1$, there exists a connected nonbipartite graph $G$ of order $n$ and diameter $D$ with $\mu(G) > k$, and

$$\mu(G) + \mu_{\text{min}}(G) < \frac{4}{(k-1)^{2D-4}}.$$  

Theorem 2 shows that $\mu(G) + \mu_{\text{min}}(G)$ can be extremely small, although $G$ is nonbipartite and connected. Here is another viewpoint to this fact.

**Theorem 3** Let $0 < \varepsilon < 1/16$. For all sufficiently large $n$, there exists a connected graph $G$ of order $n$ with $\mu(G) + \mu_{\text{min}}(G) < n^{-\varepsilon}$ such that, to make $G$ bipartite, at least $(1/16 - \varepsilon)n^2$ edges must be removed.

The picture is completely different for regular graphs. In [4] it is proved that if $G$ is a connected nonregular graph of order $n$, size $m$, diameter $D$, and maximum degree $\Delta$, then

$$\Delta - \mu(G) > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}.$$

This result and Theorem 2 help deduce the following theorems; we omit their straightforward proofs.

**Theorem 4** If $H$ is a proper subgraph of a connected regular graph $G$ of order $n$ and diameter $D$, then

$$\mu(G) - \mu(H) > \frac{1}{n(D + 1)}.$$  

**Theorem 5** If $G$ is a connected regular nonbipartite graph of order $n$ and diameter $D$, then

$$\mu(G) + \mu_{\text{min}}(G) > \frac{2}{n(2D + 1)}.$$  

**Theorem 6** If $G$ is a connected, nonregular, nonbipartite graph of order $n$, diameter $D$, and maximum degree $\Delta$, then

$$\Delta + \mu_{\text{min}}(G) > \frac{1}{n(D + 1)} + \frac{1}{\mu^{2D}(G)n}.$$  

Note that the last two theorems give a fine tuning of a result of Alon and Sudakov [1].
2 Proofs

Our proof of Theorem\textsuperscript{1} stems from a result of Schneider \cite{8} on eigenvectors of irreducible nonnegative matrices; for graphs it reads as: if \( G \) is a connected graph of order \( n \) and \( x_{\min}, x_{\max} \) are minimal and maximal entries of an eigenvector to \( \mu(G) \), then

\[
\frac{x_{\min}}{x_{\max}} \geq \mu^{-n+1}(G).
\]

We reprove this inequality in a more flexible form that sheds some extra light on the original matrix result of Schneider as well. Hereafter we write \( \text{dist}(u,v) \) for the length of a shortest path joining the vertices \( u \) and \( v \).

**Proposition 7** If \( G \) is a connected graph of order \( n \) and \((x_1, \ldots, x_n)\) is an eigenvector to \( \mu(G) \), then

\[
\frac{x_i}{x_j} \geq (\mu(G))^{-\text{dist}(i,j)} \tag{3}
\]

for every two vertices \( i, j \in V(G) \).

**Proof** Clearly we can assume that \( i \neq j \). For convenience we also assume that \( i = 1 \) and the vertices \((1, \ldots, j)\) form a path joining 1 to \( j \). Then, for all \( u = 1, \ldots, j-1 \), we have

\[
\mu x_u = \sum_{uv \in E(G)} x_v \geq x_{u+1};
\]

hence, (3) follows by multiplying all these inequalities. \( \square \)

We shall need also the following simple bound.

**Proposition 8** If \( G \) is a connected graph of order \( n \geq 3 \) and diameter \( D \), then \( \mu^D(G) > n/\sqrt{3} \).

**Proof** Note that every two vertices can be joined by a walk of \( D \) or \( D+1 \) vertices. Hence, letting \( w_k(G) \) be the number of walks of \( k \) vertices, we find that \( w_D(G) + w_{D+1}(G) \geq n^2 \); therefore, by a result in \cite{3}, \( \mu^{D-1}(G) + \mu^D(G) \geq n \). Since \( \mu(G) > \sqrt{2} \), we see that

\[
\sqrt{3} \mu^D(G) > \frac{1}{\sqrt{2}} \mu^D(G) + \mu^D(G) \geq \mu^{D-1}(G) + \mu^D(G) \geq n,
\]

completing the proof. \( \square \)

**Proof of Theorem\textsuperscript{1}** Since \( \mu(H) \leq \mu(H') \) whenever \( H \subset H' \), we may assume that \( H \) is a maximal proper subgraph of \( G \), that is to say, \( V(H) = V(G) \) and \( H \) differs from \( G \) in a single edge \( uv \). Our proof is split into two cases: (a) \( H \) connected; (b) \( H \) disconnected.

**Case (a):** \( H \) is connected.
In this case we shall prove a stronger result than required, namely

$$\mu(G) - \mu(H) > \frac{2}{\mu^{2D}(G)n}. \quad (4)$$

Our first goal is to prove that, for every \( w \in V(H), \)

$$\text{dist}_H(w, u) + \text{dist}_H(w, v) \leq 2D. \quad (5)$$

Let \( w \in V(H) \) and select in \( H \) shortest paths \( P(u, w) \) and \( P(v, w) \) joining \( u \) and \( v \) to \( w \). Let \( Q(u, x) \) and \( Q(v, x) \) be the longest subpaths of \( P(u, w) \) and \( P(v, w) \) having no internal vertices in common. If \( s \in Q(u, x) \) or \( s \in Q(v, x) \), we obviously have

$$\text{dist}_H(w, s) = \text{dist}_H(w, x) + \text{dist}_H(s, x). \quad (6)$$

The paths \( Q(u, x), Q(v, x) \) and the edge \( uv \) form a cycle in \( G \); write \( k \) for its length. Assume that \( \text{dist}(v, x) \geq \text{dist}(u, x) \) and select \( y \in Q(v, x) \) with \( \text{dist}_H(x, y) = [k/2] \). Let \( R(w, y) \) be a shortest path in \( G \) joining \( w \) to \( y \); clearly the length of \( R(w, y) \) is at most \( D \). If \( R(w, y) \) does not contain the edge \( uv \), it is a path in \( H \) and, using (6), we find that

$$D \geq \text{dist}_G(w, y) = \text{dist}_H(w, y) = \text{dist}_H(w, x) + [k/2]$$

$$= \text{dist}_H(w, x) + \left\lfloor \frac{\text{dist}_H(x, u) + \text{dist}_H(x, v) + 1}{2} \right\rfloor$$

$$\geq \text{dist}_H(w, x) + \frac{\text{dist}_H(x, u) + \text{dist}_H(x, v)}{2} = \frac{\text{dist}_H(w, u) + \text{dist}_H(w, v)}{2},$$

implying (5). Let now \( R(w, y) \) contain the edge \( uv \). Assume first that \( v \) occurs before \( u \) when traversing \( R(w, y) \) from \( w \) to \( y \). Then

$$\text{dist}_H(w, u) + \text{dist}_H(w, v) \leq 2\text{dist}_H(w, x) + \text{dist}_H(x, u) + \text{dist}_H(x, v)$$

$$\leq 2(\text{dist}_H(w, x) + \text{dist}_H(x, v)) < \text{dist}_G(w, y) \leq 2D,$$

implying (5). Finally, if \( u \) occurs before \( v \) when traversing \( R(w, y) \) from \( w \) to \( y \), then

$$D \geq \text{dist}_G(w, y) \geq \text{dist}_H(w, u) + 1 + \text{dist}_H(v, y)$$

$$= \text{dist}_H(w, x) + \text{dist}_H(x, u) + 1 + \text{dist}_H(v, y) = \text{dist}_H(w, x) + [k/2]$$

$$\geq \text{dist}_H(w, x) + \frac{\text{dist}_H(x, u) + \text{dist}_H(x, v)}{2} = \frac{\text{dist}_H(w, u) + \text{dist}_H(w, v)}{2},$$

implying (5). Thus, inequality (5) is proved in full.

Let now \( x = (x_1, ..., x_n) \) be a unit eigenvector to \( \mu(H) \) and let \( x_w \) be a maximal entry of \( x \). In view of (4) and (5), we have

$$\frac{x_u x_v}{x_w^2} \geq \frac{1}{\mu^{\text{dist}(u, w) + \text{dist}(v, w)}(H)} \geq \frac{1}{\mu^{2D}(H)}.$$
Hence, in view of $x_w^2 \geq 1/n$, we see that

$$
\mu(G) \geq 2 \sum_{ij \in E(G)} x_ix_j = 2x_u x_v + \mu(H) \geq \frac{2x_w^2}{\mu^{2D}(H)} + \mu(H) > \frac{2}{\mu^{2D}(G)n} + \mu(H),
$$

completing the proof of (4) and thus of (1).

**Case (b): $H$ is disconnected.**

Since $G$ is connected, $H$ is union of two connected graphs $H_1$ and $H_2$ such that $v \in H_1$, $u \in H_2$. Assume $\mu(H) = \mu(H_1)$, set $|H_1| = k$, and let $x = (x_1, ..., x_k)$ be a unit eigenvector to $\mu(H_1)$. Since any maximal entry of $x$ is at least $k^{-1/2}$ and diam $H_1 \leq$ diam $G \leq D$, Proposition 7 implies that $x_v \geq \mu^{-D}(H)k^{-1/2}$. Set $t = \mu^{-D}(H)k^{-1/2}$ and consider the unit vector

$$
(y_1, ..., y_k, y_u) = \left(x_1, \sqrt{1-t^2}, ..., x_k, \sqrt{1-t^2}, t\right).
$$

Then

$$
\mu(G) \geq \mu(H_1 + u) \geq 2 \sum_{ij \in E(H_1 + u)} y_i y_j \geq 2t \sum_{uj \in E(H_1 + u)} y_j + 2t \sum_{ij \in E(H_1)} x_i x_j \\
\geq 2t \sqrt{1-t^2} x_v + (1-t^2) \mu(H) = \frac{1}{\mu^{2D}(H)k} \left(2 \sqrt{1 - \frac{1}{\mu^{2D}(H)k}} - 1\right) + \mu(H).
$$

For $k \geq 3$, Proposition 8 implies that

$$
\frac{1}{\mu^{2D}(H)k} \left(2 \sqrt{1 - \frac{1}{\mu^{2D}(H)k}} - 1\right) > \frac{1}{\mu^{2D}(H)k} \left(2 \sqrt{1 - \frac{3}{k^3}} - 1\right) \\
\geq \frac{1}{\mu^{2D}(H)(k+1)} > \frac{1}{\mu^{2D}(G)n}.
$$

Finally, if $k = 3$, then $\mu(H_1) = 1$, $\mu(G) \geq \sqrt{2}$, $D \geq 2$, and $n \geq 3$; hence,

$$
\mu(G) - \mu(H) \geq \sqrt{2} - 1 > \frac{1}{3 (\sqrt{2})^4} \geq \frac{1}{\mu^{2D}(G)n},
$$

completing the proof.

**Proof of Theorem** Let $x = (x_1, ..., x_n)$ be an eigenvector to $\mu_{\min}(G)$ and let $V_1 = \{u: x_u < 0\}$. Let $H$ be the maximal bipartite subgraph of $G$, containing all edges with exactly one vertex in $V_1$. It is not hard to see that $H$ is connected proper subgraph of $G$, $V(H) = V(G)$, and $\mu_{\min}(H) < \mu_{\min}(G)$. Finally, let $H'$ be a maximal proper subgraph of $G$ containing $H$. We have

$$
\mu(G) + \mu_{\min}(G) \geq \mu(G) + \mu_{\min}(H) = \mu(G) - \mu(H) \geq \mu(G) - \mu(H').
$$
and (2) follows from case (a) of the proof of Theorem 1.

To construct the required example, set $G_1 = K_3$, $G_2 = K_{k,k}$, join $G_1$ to $G_2$ by a path $P$ of length $n - 2k - 2$, and write $G$ for the resulting graph; obviously $G$ is of order $n$ and diameter $n - 2k + 1$. Set $\mu = \mu(G)$ and note that $\mu(G) > k$. Let $V(G_1) = \{u_1, u_2, v_1\}$ and $P = (v_1, \ldots, v_{n-2k-1})$, where $v_{n-2k-1} \in V(G_2)$. Let $\mathbf{x}$ be a unit eigenvector to $\mu(G)$ and assume that the entries $x_1, x_2, x_3, \ldots, x_{n-2k+1}$ correspond to $u_1, u_2, v_1, \ldots, v_{n-2k-1}$. Clearly $x_1 = x_2$, and so, from $\mu x_2 = x_2 + x_3$, we find that $x_1 = x_2 = x_3/(\mu - 1)$. Furthermore,

$$\mu x_3 = 2x_2 + x_4 = \frac{2x_3}{\mu - 1} + x_4 < x_3 + x_4,$$

and by induction we obtain $x_i < (\mu - 1)x_{i+1}$ for all $3 \leq i \leq n - 2k$. Therefore,

$$x_1 = x_2 \leq (\mu - 1)^{-n+2k+1}x_{n-2k+1} < (k - 1)^{D+2},$$

and by Rayleigh’s principle we deduce that

$$\mu(G) + \mu_{\min}(G) \leq 4x_1x_2 < \frac{4}{(k - 1)^{2D-4}},$$

completing the proof. \hfill \Box

Proof of Theorem 3 Set $r = \lceil n/4 \rceil + 1$, $s = \lceil (1/2 - \varepsilon)n \rceil$, select $G_1 = K_{r,r}$, $G_2 = K_s$, join $G_1$ to $G_2$ by a path $P$ of length $n - 2r - s + 1$ and write $G$ for the resulting graph. Note first that, to make $G$ bipartite, we must remove at least

$$\left(\frac{s}{2}\right) - \left\lfloor \frac{s^2}{4} \right\rfloor > \frac{s^2}{4} - \frac{s}{2} > \frac{(1/2 - \varepsilon)^2 n^2}{4} - \frac{s}{2} \geq \left(\frac{1}{16} - \varepsilon\right) n^2$$

edges, for $n$ large enough. Note also that

$$n - 2\left\lceil \frac{n}{4} \right\rceil - 2 - \left\lceil \left(\frac{1}{2} - \varepsilon\right)n \right\rceil + 1 > n - \frac{n}{2} - \left(\frac{1}{2} - \varepsilon\right)n - 4 = \varepsilon n - 4,$$

so the length of $P$ is greater than $\varepsilon n - 4$.

Let $\mathbf{x}$ be a unit eigenvector to $\mu(G)$. Clearly the entries of $\mathbf{x}$ corresponding to vertices from $V(G_1) \setminus V(P)$ have the same value $\alpha$. Like in the proof of Theorem 2 we see that $\alpha < (n/4)^{-\varepsilon n + 5}$. Hence, by Rayleigh’s principle, for $n$ large enough, we deduce that

$$\mu(G) + \mu_{\min}(G) \leq 4\alpha^2 \left(\frac{s}{2}\right) < (n/4)^{-2\varepsilon n + 10} \frac{n^2}{2} < (n/4)^{-2\varepsilon n + 12} < n^{-\varepsilon n},$$

completing the proof. \hfill \Box

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