THE STRUCTURE JACOBI OPERATOR FOR HYPERSURFACES IN $\mathbb{CP}^2$ AND $\mathbb{CH}^2$

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Abstract. Using the methods of moving frames, we study real hypersurfaces in complex projective space $\mathbb{CP}^2$ and complex hyperbolic space $\mathbb{CH}^2$ whose structure Jacobi operator has various special properties. Our results complement work of several other authors who worked in $\mathbb{CP}^n$ and $\mathbb{CH}^n$ for $n \geq 3$.

1. Introduction

The complete simply connected Kähler manifolds of nonzero constant holomorphic curvature are the complex space forms $\mathbb{CP}^n$ and $\mathbb{CH}^n$. Takagi [14], for $\mathbb{CP}^n$ and Montiel [9], for $\mathbb{CH}^n$, catalogued a specific list of real hypersurfaces which may be characterized as the homogeneous Hopf hypersurfaces. Other characterizations of these hypersurfaces have been derived over the years, both in terms of extrinsic information (such as properties of the shape operator) and intrinsic information (such as properties of the curvature tensor). In both cases, the interaction of these geometric objects with the complex structure has played an important role.

Occurring as a real hypersurface in $\mathbb{CP}^n$ or $\mathbb{CH}^n$ places significant restrictions on the geometry of a Riemannian manifold $M$ and on the way it is immersed. For example, it is known that such an $M$ cannot be Einstein (an intrinsic condition) or umbilic (an extrinsic condition). In fact, neither the Ricci tensor nor the shape operator can be parallel. Nevertheless, elements of the lists of Takagi and Montiel enjoy many nice properties and geometers have been successful in characterizing them in terms of these properties.

Recently, the structure Jacobi operator has been an object of study and various non-existence and classification results are now known for $n \geq 3$. Unfortunately, the methods of proof used in establishing these results do not carry over to the case $n = 2$. In this paper, we obtain corresponding results for $\mathbb{CP}^2$ and $\mathbb{CH}^2$ using the method of moving frames, along with the theory of exterior differential systems.

In what follows, all manifolds are assumed connected and all manifolds and maps are assumed smooth ($C^\infty$) unless stated otherwise. Basic notation and historical information for hypersurfaces in complex space forms may be found in [10]. For more on moving frames and exterior differential systems, see the monograph [11] or the textbook [3].

1.1. Hypersurfaces in Complex Space Forms. Throughout this paper, we will take the holomorphic sectional curvature of the complex space form in question to be $4c$. The curvature operator $\tilde{R}$ of the space form satisfies

$$\tilde{R}(X, Y) = c(X \wedge Y + JX \wedge JY + 2\langle X, JY \rangle J)$$

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for tangent vectors $X$ and $Y$ (cf. Theorem 1.1 in [10]), where $X \wedge Y$ denotes the skew-adjoint operator defined by

$$(X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y.$$ 

We will denote by $r$ the positive number such that $c = \pm 1/r^2$. This is the same convention as used in ([10], p. 237).

A real hypersurface $M$ in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ inherits two structures from the ambient space. First, given a unit normal $\xi$, the structure vector field $W$ on $M$ is defined so that

$$JW = \xi, \quad W \in TM,$$

where $J$ is the complex structure. This gives an orthogonal splitting of the tangent space

$$TM = \text{span}\{W\} \oplus W^\perp.$$ 

Second, on the tangent space we define a linear operator $\varphi$ which is the complex structure $J$ followed by projection onto $TM$:

$$\varphi X = JX - \langle X, W \rangle \xi, \quad \varphi : TM \to TM.$$

Recall that, for a tangent vector field $V$ on a Riemannian manifold, the Jacobi operator $R_V$ is a tensor field of type $(1,1)$ satisfying

$$R_V(X) = R(X,V)V,$$

where $R$ denotes the Riemannian curvature tensor of type $(1,3)$. Note that, because of the symmetries of the curvature tensor, $R_V$ is self-adjoint and $R_VV = 0$. For a real hypersurface in a complex space form in particular, and $V = W$ (the structure vector), $R_W$ is called the structure Jacobi operator. In this paper, we will characterize certain hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ in terms of the structure Jacobi operator.

Some of the results we will state involve the notion of Hopf hypersurfaces. A hypersurface $M$ in a complex space form is said to be a Hopf hypersurface if the structure vector $W$ is a principal vector, (i.e. $AW = \alpha W$, where $A$ is the shape operator). It is a non-obvious fact (proved by Y. Maeda [7] for $\mathbb{C}P^n$ and by Ki and Suh [5] for $\mathbb{C}H^n$) that the principal curvature $\alpha$ is (locally) constant. We refer to $\alpha$ as the Hopf principal curvature following Martins [8]. For an arbitrary oriented hypersurface in a complex space form, we define the function

$$\alpha = \langle AW, W \rangle.$$ 

Of course, $\alpha$ need not be constant in general.

We also recall the notion of pseudo-Einstein hypersurfaces. A real hypersurface $M$ in a complex space form is said to be pseudo-Einstein if there are constants $\rho$ and $\sigma$ such that the Ricci $(1,1)$-tensor $S$ of $M$ satisfies

$$SX = \rho X + \sigma \langle X, W \rangle W$$

for all tangent vectors $X$.

1.2. Summary of results. We summarize results of Perez and collaborators on hypersurfaces satisfying conditions involving the structure Jacobi operator:

**Theorem 1** (Ortega-Perez-Santos [11]). Let $M^{2n-1}$, where $n \geq 3$, be a real hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$. Then the structure Jacobi operator $R_W$ cannot be parallel.
Theorem 2 (Perez-Santos [12]). Let $M^{2n-1}$, where $n \geq 3$, be a real hypersurface in $\mathbb{CP}^n$. Then the Lie derivative $\mathcal{L}_V R_W$ of the structure Jacobi operator cannot vanish for all tangent vectors $V$.

Weakening the hypothesis of Theorem 2, Perez et al. [13] were able to prove the following.

Theorem 3 (Perez-Santos-Suh). Let $M^{2n-1}$, where $n \geq 3$, be a real hypersurface in $\mathbb{CP}^n$. If $\mathcal{L}_W R_W = 0$, then $M$ is a Hopf hypersurface. If the Hopf principal curvature $\alpha$ is nonzero, then $M$ is locally congruent to a geodesic sphere or a tube over a totally geodesic $\mathbb{CP}^k$, where $0 < k < n - 1$.

In §3 we extend Theorem 1 to the case $n = 2$, while at the end of §4 we extend Theorem 2 to the case $n = 2$ for both $\mathbb{CP}^2$ and $\mathbb{CH}^2$. We find that the analogue of Theorem 3 for $n = 2$ is essentially the same, and is valid for $\mathbb{CH}^2$ as well as $\mathbb{CP}^2$. Specifically, in §4 we prove

Theorem 4. Let $M^3$ be a real hypersurface in $\mathbb{CP}^2$ or $\mathbb{CH}^2$. Then the identity $\mathcal{L}_W R_W = 0$ is satisfied if and only if $M$ is a pseudo-Einstein hypersurface.

It is not immediately obvious that Theorem 4 is, in fact, the extension of Theorem 3 to $\mathbb{CP}^2$ and to $\mathbb{CH}^2$. The analogue of Theorem 3 for $n = 2$ would say that a hypersurface $M^3$ in $\mathbb{CP}^2$ with $\mathcal{L}_W R_W = 0$ must be an open subset of a geodesic sphere or a Hopf hypersurface with $\alpha = 0$. However, the classification of pseudo-Einstein hypersurfaces in $\mathbb{CP}^2$ by Kim and Ryan [6] yields exactly the same list of hypersurfaces. The classification of pseudo-Einstein hypersurfaces in $\mathbb{CH}^2$ by Ivey and Ryan [4] yields an analogous list – open subsets of horospheres, geodesic spheres, tubes over $\mathbb{CH}^1$, and Hopf hypersurfaces with $\alpha = 0$.

It is not hard to check that every Hopf hypersurface with $\alpha = 0$ (in $\mathbb{CH}^n$ as well as in $\mathbb{CP}^n$) for $n \geq 2$, satisfies $\mathcal{L}_W R_W = 0$. The structure theory for Hopf hypersurfaces with $\alpha = 0$ is described in [2, 4, 6, 8]. Note that such hypersurfaces need not be pseudo-Einstein when $n \geq 3$. On the other hand, there are some pseudo-Einstein hypersurfaces in $\mathbb{CP}^n$, where $n \geq 3$, that do not satisfy $\mathcal{L}_W R_W = 0$. Thus one cannot restate Theorem 3 in terms of the pseudo-Einstein condition.

Finally, we observe that the condition considered in Theorem 2 is actually quite strong. In §5 we provide a new proof of this theorem that is also valid for $\mathbb{CH}^n$.

2. Basic Equations

In this and subsequent sections, we follow the notation and terminology of [10]: $M^{2n-1}$ will be a hypersurface in a complex space form $\tilde{M}$ (either $\mathbb{CP}^n$ or $\mathbb{CH}^n$) having constant holomorphic sectional curvature $4c \neq 0$. The structures $\xi$, $W$, and $\varphi$ are as defined in the Introduction. The $(2n-2)$-dimensional distribution $W^\perp$ is called the holomorphic distribution. The operator $\varphi$ annihilates $W$ and acts as complex structure on $W^\perp$. The shape operator $A$ is defined by

$$AX = -\nabla_X \xi$$

where $\nabla$ is the Levi-Civita connection of the ambient space. The Gauss equation expresses the curvature operator of $M$ in terms of $A$ and $\varphi$, as follows:

$$R(X, Y) = AX \wedge AY + c (X \wedge Y + \varphi X \wedge \varphi Y + 2 \langle X, \varphi Y \rangle \varphi).$$

In addition, it is easy to show (see [10], p. 239) that

$$\nabla_X W = \varphi AX,$$
where $\nabla$ is the Levi-Civita connection of the hypersurface $M$.

Consider now the case $n = 2$, so that $M^3$ is a hypersurface in $\mathbb{CP}^2$ or $\mathbb{CH}^2$. Suppose that there is a point $p$ (and hence an open neighborhood of $p$) where $AW \neq \alpha W$. Then there is a positive function $\beta$ and a unit vector field $X \in W^\perp$ such that

$$AW = \alpha W + \beta X.$$ 

Let $Y = \varphi X$. Then there are smooth functions $\lambda$, $\mu$, and $\nu$ defined near $p$ such that with respect to the orthonormal frame $(W, X, Y)$,

\begin{equation}
A = \begin{pmatrix}
\alpha & \beta & 0 \\
\beta & \lambda & \mu \\
0 & \mu & \nu
\end{pmatrix}.
\end{equation}

A routine computation, using the Gauss equation (2), yields

\begin{equation}
R_W = \begin{pmatrix}
0 & 0 & 0 \\
0 & \alpha \lambda + \beta^2 & \alpha \mu \\
0 & \alpha \mu & \alpha \nu + c
\end{pmatrix}.
\end{equation}

Consider now a point where $AW = \alpha W$. Let $X$ be a unit principal vector in $W^\perp$ and let $Y = \varphi X$. Then there are numbers $\alpha$, $\lambda$, and $\nu$ such that equations (4) and (5) still hold at this point, but with $\beta = \mu = 0$.

In this connection, we recall the following useful fact ([10], p. 246.)

**Proposition 5.** Let $M^{2n-1}$, where $n \geq 2$, be a Hopf hypersurface in $\mathbb{CP}^n$ or $\mathbb{CH}^n$ with Hopf principal curvature $\alpha$. If $X$ is a unit vector in $W^\perp$ such that $AX = \lambda X$ and $A\varphi X = \nu \varphi X$, then

\begin{equation}
\lambda \nu = \frac{\lambda + \nu}{2} \alpha + c.
\end{equation}

3. **Parallelism of $R_W$**

3.1. **The condition** $\nabla R_W = 0$. We first show that this condition implies $R_W = 0$.

**Proposition 6.** Let $M^{2n-1}$ be a hypersurface in $\mathbb{CP}^n$ or $\mathbb{CH}^n$, where $n \geq 2$. If $\nabla R_W = 0$ on $M$, then $R_W = 0$.

*Proof.* Since $R_W$ is parallel, every curvature operator commutes with $R_W$. Then for any tangent vector $V$,

$$0 = R(V, W)R_W W = R_W R(V, W) W = R_W^2 V,$$

and thus $R_W^2 = 0$. So $R_W$, being self-adjoint, must also vanish. $\square$

3.2. **The condition** $R_W = 0$.

**Proposition 7.** There are no hypersurfaces in $\mathbb{CP}^2$ or $\mathbb{CH}^2$ such that the structure Jacobi operator $R_W$ vanishes identically.

*Proof.* We use the setup from [2] with $n = 2$. First look at possibility of a Hopf hypersurface with $R_W = 0$. We see from (5) with $\beta = 0$ that $\alpha \lambda + c = \alpha \nu + c = 0$, so that $\alpha \neq 0$ and $\nu = \lambda \neq 0$. However, in view of Proposition 5 we have $0 = \alpha \lambda + c = \lambda^2$, which is a contradiction.

The non-Hopf case is handled by the following proposition which follows directly from (4) and (5). Then Lemma 9 completes our proof. $\square$
Proposition 8. Suppose that $M^3$ is a non-Hopf hypersurface in $\mathbb{CP}^2$ or $\mathbb{CH}^2$ satisfying $R_W = 0$. Then, in a neighborhood of some point $p$, we have (using the basic setup of §2)
- $\beta$ and $\alpha$ are nonzero;
- $\mu = 0$;
- $\beta^2 = \alpha \lambda + c$;
- $\alpha \nu + c = 0$.
Conversely, every hypersurface satisfying these conditions will have $R_W = 0$.

Lemma 9. There does not exist a hypersurface in $\mathbb{CP}^2$ or $\mathbb{CH}^2$ satisfying the conditions of Proposition 8.

We prove this lemma in §6 using exterior differential systems.

4. Lie Parallelism of $R_W$

We begin by deriving a necessary condition for a hypersurface to satisfy $\mathcal{L}_W R_W = 0$.

Proposition 10. For any hypersurface in $\mathbb{CP}^n$ or $\mathbb{CH}^n$, where $n \geq 2$, satisfying $\mathcal{L}_W R_W = 0$, we must have

$$[R_W, [\varphi, A]] = 0. \tag{7}$$

Proof.

$$(\mathcal{L}_W R_W) V = \mathcal{L}_W (R_W V) - R_W (\mathcal{L}_W V) = \nabla_W (R_W V) - \nabla_{R_W V} W - R_W (\nabla_W V) + R_W (\nabla_V W) = (\nabla_W R_W) V - \varphi A (R_W V) + R_W (\varphi AV)$$

for all tangent vectors $V$. (Here we have used (3)). Thus $\mathcal{L}_W R_W = 0$ if and only if $\nabla_W R_W = -[R_W, \varphi A]$. Using the fact that $\nabla_W R_W$ is self-adjoint, we see that $\mathcal{L}_W R_W = 0$ implies that

$$(R_W \varphi A - \varphi A R_W)^t = (R_W \varphi A - \varphi A R_W),$$

which, once we use the fact that $A, R_W$ are self-adjoint while $\varphi$ is skew-adjoint, reduces to the desired identity. \hfill \Box

4.1. The non-Hopf case.

Proposition 11. Suppose that $M^3$ is a non-Hopf hypersurface in $\mathbb{CP}^2$ or $\mathbb{CH}^2$ satisfying $\mathcal{L}_W R_W = 0$. Then, in a neighborhood of some point $p$, we have (using the basic setup of §2)
- $\beta$ and $\alpha$ are nonzero;
- $\mu = 0$;
- $\lambda = \nu$;
- $\alpha \nu + c = 0$.

Proof. By Proposition 10, we get $R_W (\varphi A - A \varphi) W = 0$ which implies that $R_W \varphi AW = 0$. In the setup of §2 with $\beta > 0$, this gives $R_W Y = 0$. From equation (5), we get $\alpha \mu = 0$ and $\alpha \nu + c = 0$; the latter guarantees that $\alpha \neq 0$, and hence $\mu = 0$. Following the same procedure with $X$, we get $R_W (\varphi A - A \varphi) X = R_W (\lambda - \nu) Y = 0$. Therefore, $(\varphi A - A \varphi) R_W X = 0$, which reduces to $(\alpha \lambda + c - \beta^2)(\lambda - \nu) = 0$. If $\lambda \neq \nu$ at some point, then $\alpha \lambda + c - \beta^2$ vanishes in a neighborhood of this point and $R_W = 0$ there. This contradicts Proposition 7 so we must conclude that $\lambda = \nu$ and that in a neighborhood of $p$, we have...
$$A = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & -\frac{c}{\alpha} & 0 \\ 0 & 0 & -\frac{c}{\alpha} \end{pmatrix}$$

and

$$R_W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

However, the situation described in Proposition 11 cannot, in fact, occur.

**Lemma 12.** There does not exist a hypersurface in $\mathbb{CP}^2$ or $\mathbb{CH}^2$ satisfying the conditions listed in Proposition 11.

We prove this in §6 using exterior differential systems. Thus, we have,

**Proposition 13.** Let $M^3$ be a real hypersurface in $\mathbb{CP}^2$ or $\mathbb{CH}^2$ such that $\mathcal{L}_W R_W = 0$. Then $M$ must be a Hopf hypersurface.

We classify such hypersurfaces in the next section.

4.2. **The Hopf case.** Now consider a Hopf hypersurface $M^3$ in $\mathbb{CP}^2$ or $\mathbb{CH}^2$. At any point of $M$, let $X$ be a unit principal vector in $W^\perp$ and let $Y = \varphi X$. Then, with respect to the frame $(W, X, Y)$, we have

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

and

$$R_W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha \lambda + c & 0 \\ 0 & 0 & \alpha \nu + c \end{pmatrix}.$$ 

By a straightforward calculation, we obtain

$$[R_W, [\varphi, A]]W = 0,$$

$$[R_W, [\varphi, A]]X = -\alpha(\lambda - \nu)^2 Y,$$

$$[R_W, [\varphi, A]]Y = \alpha(\lambda - \nu)^2 X.$$

We are now ready to prove the following proposition.

**Proposition 14.** Let $M^3$ be a Hopf hypersurface in $\mathbb{CP}^2$ or $\mathbb{CH}^2$. Then the identity $\mathcal{L}_W R_W = 0$ is satisfied if and only if at each point of $M$, one of the following holds:

- $\alpha = 0$, $\lambda \neq \nu$ and $\lambda \nu = c$;
- $\alpha = 0$, $\lambda = \nu$, and $\lambda^2 = c$;
- $\alpha^2 + 4c = 0$ and $\lambda = \nu = \frac{c}{2}$; or
- $\alpha \neq 0$, $\alpha^2 + 4c > 0$, $\lambda = \nu$, and $\lambda^2 = \alpha \lambda + c$. 


Proof. The necessity of these conditions follows immediately from (12), Proposition 10 and Proposition 5. Now suppose that these conditions are satisfied.

If \( \alpha = 0 \), we see that \( R_W V = c V \) for all \( V \in W^\perp \). If \( \alpha^2 + 4c = 0 \), then \( \lambda = \nu = \alpha/2 \) and \( \alpha \lambda + c = -c \), so that \( R_W V = -c V \) for all \( V \in W^\perp \). In the remaining case, \( R_W V = \lambda^2 V \) for all \( V \in W^\perp \). In each case, there is a nonzero constant \( k \) such that the identity \( R_W V = k V \) holds globally for all \( V \in W^\perp \). Then for any vector field \( V \in W^\perp \), we have, using (3)

\[
(L_W R_W)V = L_W(kV) - k(L_W V - \langle L_W V, W \rangle W)
\]

(13)

Since \( (L_W R_W)W = 0 \) automatically, we have \( L_W R_W = 0 \) as required. \( \square \)

Note that according to Propositions 2.13 and 2.21 of [6], the conditions in Proposition 14 are precisely the conditions for \( M \) to be a pseudo-Einstein hypersurface. Thus we have completed the proof of Theorem 4.

With a little additional work, we can now prove the analogue of Theorem 2 for \( n = 2 \).

Theorem 15. Let \( M^3 \) be a real hypersurface in \( \mathbb{CP}^2 \) or \( \mathbb{CH}^2 \). Then the Lie derivative \( L_V R_W \) of the structure Jacobi operator cannot vanish for all tangent vectors \( V \).

Proof. We suppose that \( L_V R_W \) vanishes for all \( V \) and derive a contradiction. First note that we must have \( L_W R_W = 0 \). By Proposition 13, \( M \) must be Hopf. Thus, the classification of Proposition 14 can be applied. For any unit vector field \( V \in W^\perp \), and \( U = \varphi V \), consider

\[
\langle (L_V R_W)U, W \rangle = \langle (L_V (R_W U) - R_W (L_V U)), W \rangle
\]

(14)

\[
= \langle (k L_V U - k (L_V U - \langle L_V U, W \rangle W)), W \rangle
\]

\[
= k \langle L_V U, W \rangle
\]

\[
= k \langle \nabla V U, W \rangle - k \langle \nabla U V, W \rangle
\]

\[
= -k \langle U, \varphi AV \rangle + k \langle V, \varphi AU \rangle.
\]

Now fix a particular point \( p \), and suppose that \( V \) is principal at \( p \), with \( AV = \lambda V \). Then \( U \) must also be principal at \( p \). Writing \( AU = \nu U \), we get

\[
\langle (L_V R_W)U, W \rangle = -k(\lambda + \nu).
\]

at \( p \). The right side of this equation is nonzero unless \( \lambda = -\nu \). Except possibly for the first case \( (\alpha = 0, \lambda \neq \nu, \lambda \nu = c) \) in Proposition 14 we have an immediate contradiction. In the remaining case, our argument shows that the principal curvatures sum to zero everywhere (since \( p \) was arbitrary). However, \( \lambda = -\nu \) locally would give \( \lambda^2 = -c \) and force \( \lambda \) and \( \nu \) to be locally constant. Since the well-known list of Hopf hypersurfaces with constant principal curvatures does not admit this possibility (see Theorem 4.13 of [10]), our proof is complete. \( \square \)

5. Lie Parallelism for \( n \geq 3 \)

The condition of “Lie parallelism” (see [12], p. 270) is very strong. In fact, a tensor field of type \((1, 1)\) will be Lie parallel if and only if it is a constant multiple of the identity.

Lemma 16. Let \( T \) be a tensor field of type \((1, 1)\) on a manifold \( M^n \), where \( n \geq 2 \). Then the Lie derivative \( L_X T \) vanishes for all vector fields \( X \) if and only if \( T \) is a constant multiple of the identity.
Proof. Let $X$ and $Y$ be vector fields and $f$ a real-valued function defined on an open set $U \subset M$. Then, it is easy to check that the identity

\[(\mathcal{L}_fX)Y = f(\mathcal{L}_X)Y - df(TY)X + df(Y)TX\]

holds.

Suppose now that $\mathcal{L}_V T = 0$ for all vector fields $V$. Then

\[df(TY)X = df(Y)TX\]

for all $X, Y, f$. For a suitable choice of $Y$ and $f$, we can assume that $df(Y)$ is nonvanishing on $U$, so that we can write $TX = \tau X$ for a function $\tau = df(TY)/df(Y)$. Since $\tau$ can depend only on $X$ and $Y$, there is a real-valued function $\tau$ such that $T = \tau I$. Finally, for any vector field $V$, we have

\[0 = (\mathcal{L}_V T)Y = \mathcal{L}_V(\tau Y) - \tau \mathcal{L}_V Y = d\tau(V)Y\]

so that $\tau$ must be locally constant. Conversely, the same equation shows that if $T$ is a constant multiple of the identity, then $\mathcal{L}_V T = 0$. \qed

We are now in a position to prove our theorem.

**Theorem 17.** Let $M^{2n-1}$, where $n \geq 3$, be a real hypersurface in $\mathbb{CP}^n$ or $\mathbb{CH}^n$. Then the Lie derivative $\mathcal{L}_V R_W$ of the structure Jacobi operator cannot vanish for all tangent vectors $V$.

**Proof.** Suppose that $\mathcal{L}_V R_W = 0$ for all $V$. Applying the preceding lemma to the $(1, 1)$ tensor field $R_W$, we get that $R_W$ is a constant multiple of the identity. Since $R_W W = 0$, we have, in fact, that $R_W = 0$. Our result is now immediate from Theorem 1. \qed

We could proceed similarly in the $n = 2$ case, invoking Proposition 6. This would provide an alternative proof of Theorem 15.

### 6. Differential Forms Calculations

In this section, we prove Lemmas 9 and 12 by analyzing the conditions that a moving frame along the hypersurface would have to satisfy, as a section of the orthonormal frame bundle of the relevant complex space form $\tilde{M} = \mathbb{CP}^2$ or $\mathbb{CH}^2$. The conditions proposed in the lemmas will imply that the sections are integral submanifolds of certain exterior differential systems on the frame bundle. The generators of these systems are defined in terms of the natural coframing on the frame bundle, which we will briefly review.

On the orthonormal frame bundle $\mathcal{F}_o$ of a $n$-dimensional Riemannian manifold $\tilde{M}$, we define the *canonical 1-forms* $\omega^i$ and the *connection 1-forms* $\omega_j^i$ (where $1 \leq i, j, k \leq n$) by the following properties: if $(e_1, \ldots, e_n)$ is any orthonormal frame defined on an open set $U \subset \tilde{M}$, and $f : U \to \mathcal{F}_o$ is the corresponding local section, then

\[v = (v \cdot f^* \omega^k) e_k,\]

\[\tilde{\nabla}_v e_j = (v \cdot f^* \omega_j^k) e_k,\]

for any tangent vector $v$ at a point in $U$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$ and we use the summation convention. The connection forms satisfy $\omega^i_j = -\omega^i_j$. The forms
\(\omega^i\) and \(\omega^j\) (for \(i > j\)) together form a basis for the cotangent space of \(F_o\) at each point. They satisfy the structure equations

\begin{align}
d\omega^i &= -\omega_j^j \wedge \omega^i, \tag{19} \\
d\omega_j &= -\omega_k^k \wedge \omega_j^j + \Phi_j^j, \tag{20}
\end{align}

where the 2-forms \(\Phi_j^j\) pull back along any section to give the components of the curvature tensor with respect to the corresponding frame, i.e., \(f^*\Phi_j^j(e_k, e_\ell) = \langle e_\ell, \tilde{\mathbf{R}}(e_k, e_\ell) e_j \rangle\).

In our case, \(n = 4\) and \(\widetilde{M}\) is a complex space form. We will use moving frames that are adapted to the complex structure on \(\widetilde{M}\) in the following way:

\[e_4 = Je_1, \quad e_3 = Je_2.\]

We will refer to these as unitary frames, and let \(F_u \subset F_o\) be the sub-bundle of such frames. We restrict the canonical and connection forms to \(F\) without change of notation. The structure group of this sub-bundle is the 4-dimensional group \(U(2) \subset SO(4)\). Because \(J\) is parallel, only the connection forms \(\omega_2^3, \omega_1^4, \omega_2^4, \omega_3^4\) are linearly independent, the remaining forms satisfying the relations

\[
\omega_1^2 = -\omega_3^3, \quad \omega_1^3 = \omega_2^4.
\]

Using (1) and the structure equations, we find that the curvature forms on \(F_u\) satisfy

\[
\begin{align*}
\Phi_2^3 &= c(4\omega^3 \wedge \omega^2 + 2\omega^4 \wedge \omega^1), \\
\Phi_1^4 &= c(4\omega^4 \wedge \omega^1 + 2\omega^3 \wedge \omega^2), \\
\Phi_2^4 &= \Phi_3^3 = c(\omega^3 \wedge \omega^1 + \omega^4 \wedge \omega^2), \\
\Phi_3^4 &= \Phi_2^1 = c(\omega^1 \wedge \omega^2 + \omega^4 \wedge \omega^3).
\end{align*}
\]

Along a real hypersurface \(M \subset \widetilde{M}\), we will use an adapted moving frame, meaning a unitary frame such that \(e_4\) is normal to the hypersurface (and thus \(e_1\) is the structure vector). It follows from (17) that \(f^*\omega^4 = 0\) and \(f^*(\omega^1 \wedge \omega^2 \wedge \omega^3)\) is a nonzero 3-form at each point. It also follows from (18) that

\[
f^*\omega_i^4 = h_{ij} f^*\omega_j, \quad 1 \leq i, j \leq 3,
\]

where \(h_{ij}\) are functions that give the components of the shape operator of \(M\). In particular, working in a neighborhood of a point where \(AW \neq \alpha W\), let \(W, X, Y\) be the unit vector fields defined in §2. Then \(e_1 = W, e_2 = X, e_3 = Y\) and \(e_4 = \xi\) give the components of an adapted framing, and the \(h_{ij}\) are the entries of the matrix given by (4).

We now have all the tools necessary to prove the two lemmas.

**Proof of Lemma** 3. Again, let \(W, X, Y\) be unit vector fields on an open set \(U \subset M\), as in §2 and let \(f\) be the adapted moving frame such that \(e_1 = W, e_2 = X, e_3 = Y\). Then \(f\) immerses \(U\) as a three-dimensional submanifold of \(F_u\) on which \(\omega^4 = 0\) and the \(\omega_i^4\) satisfy

\[
\begin{align*}
\omega_1^4 &= \alpha \omega^1 + \beta \omega^2, \\
\omega_2^4 &= \beta \omega^1 + \lambda \omega^2 + \mu \omega^3, \\
\omega_3^4 &= \mu \omega^2 + \nu \omega^3,
\end{align*}
\]

(21)
for some functions $\alpha, \beta, \lambda, \mu, \nu$ satisfying the conditions in the lemma. Because we assume that $\alpha$ is nowhere vanishing, these conditions can be expressed as

$$\alpha, \beta \neq 0, \quad \lambda = \frac{\beta^2 - c}{\alpha}, \quad \mu = 0, \quad \nu = -\frac{c}{\alpha}.$$  

Under these conditions, the functions $\alpha$ and $\beta$ completely determine the second fundamental form (and hence, determine the hypersurface up to rigid motion). The proof will proceed by deriving an overdetermined system of differential equations that these functions must satisfy, and showing that no solutions exist satisfying the nonvanishing conditions.

Take $(\alpha, \beta)$ as coordinates on $\mathbb{R}^2$, and let $\Sigma \subset \mathbb{R}^2$ be the subset where $\alpha \neq 0$ and $\beta \neq 0$. On $F_u \times \Sigma$ define the 1-forms

$$\theta_0 = \omega^4,$$
$$\theta_1 = \omega_1^4 - \alpha \omega_1 - \beta \omega^2,$$
$$\theta_2 = \omega_2^4 - \beta \omega_1 - \frac{(\beta^2 - c)}{\alpha} \omega^2,$$
$$\theta_3 = \omega_3^4 + \frac{c}{\alpha} \omega^3.$$

Then for any adapted frame $f$ along $M$, the image of the map $p \mapsto (f(p), \alpha(p), \beta(p))$ is a 3-dimensional submanifold in $F_u \times \Sigma$ which is an integral of the Pfaffian exterior differential system generated by $\theta_0, \theta_1, \theta_2, \theta_3$. In other words, all 1-forms in this span pull back to be zero on this submanifold. We will now investigate the set of such submanifolds, satisfying the independence condition $\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0$, which is implied by (17).

Along any such submanifold, the exterior derivatives of the $\theta_i$ must also vanish (i.e., they pull back to the submanifold to be zero). Therefore, we will obtain additional differential forms that must vanish along integral manifolds if we compute the derivatives of the 1-form generators modulo the algebraic ideal (under wedge product) generated by those 1-forms. In this case, we compute $d\theta_0 \equiv 0$ and

$$-d\theta_1 \equiv \pi_1 \wedge \omega^1 + \pi_2 \wedge \omega^2 + \pi_3 \wedge \omega^3,$$
$$-d\theta_2 \equiv \pi_2 \wedge \omega^1 + \left(\frac{2\beta}{\alpha} \pi_2 - \frac{(\beta^2 - c)}{\alpha^2} \pi_1\right) \wedge \omega^2 + \frac{\beta}{\alpha} \pi_3 \wedge \omega^3,$$
$$-d\theta_3 \equiv \pi_3 \wedge \left(\omega^1 + \frac{\beta}{\alpha} \omega^2\right) + \frac{c}{\alpha^2} \pi_1 \wedge \omega^3$$

where

$$\pi_1 := d\alpha + \frac{3\beta(\alpha^2 - c)}{\alpha} \omega^3,$$
$$\pi_2 := d\beta + \frac{(3\alpha^2 \beta^2 + c^2 - c\beta^2)}{\alpha^2} \omega^3,$$
$$\pi_3 := \beta \omega_2^3 + 4\alpha \beta \omega^1 + \frac{(4\alpha^2 \beta^2 - c^2 + c\beta^2)}{\alpha^2} \omega^2.$$

On any integral submanifold satisfying the independence condition, $\pi_1, \pi_2, \pi_3$ must restrict to be linear combinations of $\omega_1, \omega_2, \omega_3$ at each point. The possibilities for these linear combinations are determined by the requirement that the right-hand sides in (22) must be zero.
In fact, there is only one parameter’s worth of possible values for the \( \pi \)'s, given by
\[
\begin{align*}
\pi_1 &= \rho(\alpha \omega^1 + \beta \omega^2), \\
\pi_2 &= \rho \left( \beta \omega^1 + \frac{\beta^2 + c}{\alpha} \omega^2 \right), \\
\pi_3 &= \frac{\rho c}{\alpha} \omega^3
\end{align*}
\] (23)
in terms of the single parameter \( \rho \). In other words, along each submanifold there will be a function \( \rho \) such that the above equations hold. (To see why, note that the vanishing of the third line of (22) implies that \( \pi_1, \pi_3 \) must be linear combinations of \( \omega^3 \) and \( \alpha \omega^1 + \beta \omega^2 \). On the other hand, linearly combining the first two lines to eliminate the \( \pi_3 \wedge \omega^3 \) term reveals that \( \pi_1, \pi_2 \) must be linear combinations of \( \omega^1, \omega^2 \). Thus, \( \pi_1 \) must be a multiple of \( \alpha \omega^1 + \beta \omega^2 \).

By substituting this into the right-hand sides of (22), we see that this multiple determines the values of \( \pi_2 \) and \( \pi_3 \) at any point.)

Just as we did with \( \alpha \) and \( \beta \), we introduce \( \rho \) as a new coordinate, and define the following 1-forms on \( F_u \times \Sigma \times \mathbb{R} \):
\[
\begin{align*}
\theta_4 &= \pi_1 - \rho(\alpha \omega^1 + \beta \omega^2), \\
\theta_5 &= \pi_2 - \rho \left( \beta \omega^1 + \frac{\beta^2 + c}{\alpha} \omega^2 \right), \\
\theta_6 &= \pi_3 - \frac{\rho c}{\alpha} \omega^3.
\end{align*}
\]
Then for any adapted framing \( f \) along \( M \) satisfying our assumptions, the image of the map \( p \mapsto (f(p), \alpha(p), \beta(p), \rho(p)) \) is an integral submanifold of the Pfaffian system defined by the 1-forms \( \theta_0, \ldots, \theta_6 \). (In technical terms, this system is the prolongation of the previous one.)

As before, we compute the exterior derivatives of these 1-forms modulo themselves. We find that
\[
d \theta_4 \wedge (\alpha \omega^1 + \beta \omega^2) \equiv \frac{8c(\alpha^2 - c)\rho}{\alpha} \omega^1 \wedge \omega^2 \wedge \omega^3
\]
modulo \( \theta_0, \ldots, \theta_6 \), indicating that any integral submanifold satisfying the independence condition must have \( \rho(\alpha^2 - c) = 0 \) at each point. (Recall that the ambient curvature \( c \) is nonzero.) If \( \rho \neq 0 \) at a point on the submanifold, then \( \alpha^2 = c \) on an open set about that point. However, we compute
\[
d (\alpha \theta_5 - \beta \theta_4) \wedge \omega^2 \equiv \frac{2c(\beta^2 - 2(\alpha^2 - c))\rho}{\alpha} \omega^1 \wedge \omega^2 \wedge \omega^3,
\]
which shows that \( \beta \) must vanish on that open set, a contradiction. Therefore, we conclude that \( \rho \) must be identically zero on any integral satisfying the independence condition. We restrict the system to the submanifold where \( \rho = 0 \). Then we compute
\[
\begin{align*}
d (\alpha \theta_5 - \beta \theta_4) &\equiv \frac{c(2\beta^2 + c)(4\alpha^2 + \beta^2 - c)}{\alpha^2} \omega^1 \wedge \omega^2, \\
\theta_6 \wedge \omega^2 &\equiv \frac{c(10\alpha^2 \beta^2 - c(4\alpha^2 + \beta^2 - c))}{\alpha^3} \omega^1 \wedge \omega^2 \wedge \omega^3.
\end{align*}
\]
The first line can vanish only if \( 4\alpha^2 + \beta^2 = c \), whereupon the vanishing of the last line implies that one of \( \alpha \) or \( \beta \) must be zero, a contradiction. Thus, no hypersurfaces exist satisfying the hypotheses of the lemma. \( \square \)
Proof of Lemma 12. Again, let $W, X, Y$ be unit vector fields on an open set $U \subset M$, satisfying the conditions given in §2 and let $f$ be the adapted moving frame such that $e_1 = W, e_2 = X, e_3 = Y$. Then $f$ immerses $U$ as a 3-dimensional submanifold of $F_u$. We note that $f^*\omega^4 = 0$ and (21) hold for functions $\alpha, \beta, \lambda, \mu, \nu$ satisfying the conditions in the lemma, which can be expressed as

$$\beta, \lambda \neq 0, \quad \alpha = -c/\lambda, \quad \nu = \lambda, \quad \mu = 0.$$  

Thus, we set up an exterior differential system $\mathcal{I}$ on $F_u \times \Sigma$ (where now $\beta, \lambda$ are the nonzero coordinates on the second factor) generated by 1-forms

$$\theta_0 = \omega^4,$$

$$\theta_1 = \omega_1^4 + (c/\lambda)\omega^1 - \beta\omega^2,$$

$$\theta_2 = \omega_2^4 - \beta\omega^1 - \lambda\omega^2,$$

$$\theta_3 = \omega_3^4 - \lambda\omega^3.$$  

Then for any adapted frame $f$ along $M$, the image of the map $p \mapsto (f(p), \beta(p), \lambda(p))$ will be a 3-dimensional integral submanifold of $\mathcal{I}$ satisfying the usual independence condition.

We compute $d\theta_0 \equiv 0$ and

$$-d\theta_1 \equiv \frac{c}{\lambda^2} \pi_1 \wedge \omega^1 + \pi_2 \wedge \omega^2 + \pi_3 \wedge \omega^3,$$

$$-d\theta_2 \equiv \pi_2 \wedge \omega^1 + \pi_1 \wedge \omega^2,$$

$$-d\theta_3 \equiv \pi_3 \wedge \omega^1 + \pi_1 \wedge \omega^3$$

mod $\theta_0, \theta_1, \theta_2, \theta_3$,

(24)

where

$$\pi_1 := d\lambda - 3\beta\lambda\omega^3,$$

$$\pi_2 := d\beta + (\lambda^2 - \beta^2)\omega^3,$$

$$\pi_3 := \beta\omega_2^3 - \frac{\beta(3\lambda^2 + 4c)}{\lambda} \omega^1 - (\beta^2 + \lambda^2)\omega^2.$$  

In order for the pullbacks of the right-hand sides in (24) to vanish, $\pi_1, \pi_2$ and $\pi_3$ must be multiples of $\omega^1, \omega^2, \omega^3$ respectively—and moreover the multiples must all be the same at each point. In other words, there must be a single function $\rho$ such that $\pi_i = \rho \omega^i, i = 1\ldots3$, at each point.

Therefore, we define the prolongation of $\mathcal{I}$ on $F_u \times \Sigma \times \mathbb{R}$, with $\rho$ as new coordinate on the last factor, as the Pfaffian system generated by $\theta_0, \ldots, \theta_3$ and the new 1-forms

$$\theta_4 = \pi_1 - \rho \omega^1, \quad \theta_5 = \pi_2 - \rho \omega^2, \quad \theta_6 = \pi_3 - \rho \omega^3.$$  

Now we compute

$$d\theta_5 \wedge \omega^3 + d\theta_6 \wedge \omega^2 \equiv 24 \frac{c\beta^2}{\lambda} \omega^1 \wedge \omega^2 \wedge \omega^3$$

modulo $\theta_0, \ldots, \theta_6$. Since $\beta \neq 0$, this shows that no integral submanifold of the prolongation can satisfy the independence condition. Hence no hypersurfaces exist satisfying the hypothesis of the lemma.  

□
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