Majorana Qubit Readout Using Longitudinal Qubit-Resonator Interaction

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We introduce a quantum non-demolition Majorana qubit readout protocol based on parametric modulation of a longitudinal interaction between a pair of Majorana bound states and a resonator. The interaction can be modulated through radio frequency gate-voltage control of a quantum dot or through flux modulation, and constitutes a realization of longitudinal qubit readout. The qubit-resonator coupling is quantum non-demolition to exponential accuracy, a property inherited from the topological nature of the Majorana zero modes. The same mechanism as used for readout can also be used to enact long range entangling gates.

Introduction.— When quantum information is stored in Majorana zero modes in topological superconductors, two Majorana wavefunctions with appreciable overlap leads to a splitting of the ground space degeneracy. Quantum information can then in principle be read out by measuring the resulting localized charge. Promising progress has been made towards realizing Majorana zero modes in epitaxial semiconductor-superconductor nanowire hybrids [1,3] and in 2D hybrids with lithographically defined 1D channels [4,6]. However, many questions remain about how to best control and measure these systems. To fully harness the long lifetimes expected for quantum information stored in topological systems, it is crucial that the logical operations are fast and do not introduce new errors that break the inherent protection from noise. In particular, in a measurement-only approach where braiding is enacted through a series of measurements [7–9], measurement speed and readout fidelity are key metrics that determine the overall fidelity of the braids.

We propose an approach to readout of Majorana zero modes based on parametric modulation of a longitudinal interaction between a pair of Majorana modes and a single mode of a readout resonator. Majorana bound states are assumed to be realized in a network of quantum wires, and Majorana modes localized at the ends of two distinct quantum wires are made to overlap by tunnel coupling each wire to a common quantum dot, acting as a mediator. Coupling to the electromagnetic field is introduced by capacitively coupling the dot to the electric field of a resonator, leading to an interaction of the generic form $\hat{H}_{\text{int}} = -e\hat{V}_r \hat{n}_{\text{dot}}$, where $\hat{V}_r$ is the voltage bias of the dot due to the resonator and $\hat{n}_{\text{dot}}$ the total electron number on the dot. When the tunnel coupling is turned on between the dot and the two quantum wires, the interaction transforms to $\hat{H}_{\text{int}} = -e\eta \hat{V}_r \hat{\gamma}_i \hat{\gamma}_j + \ldots$, where $\hat{\gamma}_i = \hat{\gamma}_i^\dagger$ are Majorana operators satisfying $\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}$, and $\eta$ is proportional to the Majorana wavefunction overlap on the dot. The ellipses refer to other fermionic degrees of freedom which should ideally remain in the ground state during the measurement.

In itself, $\hat{H}_{\text{int}}$ is not directly useful for readout because there is no energy exchange between the resonator and the Majorana subsystem, and the interaction leads to a negligible response of the resonator. To enact a measurement we therefore propose to modulate the Majorana wavefunction overlap, $\eta \rightarrow \eta(t)$. In semiconducting systems this can be done through gate voltages controlling the dot energy and/or tunnel couplings. Alternatively, as we show below, in a Majorana box qubit setup [8] the coupling can also be controlled through an external magnetic flux through the qubit loop. If $\eta(t)$ is modulated at the resonator frequency, $\hat{H}_{\text{int}}$ takes the form of an on-resonance ac voltage drive of the resonator, with a Majorana state dependent phase. As a consequence, the resonator field is displaced in one of two diametrically opposite directions in phase space, depending on the eigenvalue $\pm 1$ of $i\hat{\gamma}_i \hat{\gamma}_j$, leading to a large response for the resonator.

The readout scheme is a realization of a recently proposed readout technique based on longitudinal (as opposed to transverse) qubit-resonator interaction [10]. While this approach was originally introduced in the context of conventional transmon qubits, we emphasize that it is challenging to realize the desired longitudinal interaction in a transmon architecture. In contrast, longitudinal coupling is the natural form for the light-matter interaction with Majorana qubits (see also [11]). Crucially, we expect spurious transverse coupling terms to be exponentially suppressed in the length of the quantum wires hosting the Majorana zero modes, leading to a readout which is quantum-non-demolition (QND) to exponential accuracy. The QND form of the interaction holds also with respect to coupling to environmental noise, and this is consequently a much stronger form of QND than what is possible for conventional qubits, where interaction with the environment invariably leads to state transitions in the measurement basis. We refer to this manifestation of the topological nature of Majorana bound modes in a measurement as topological QND (TQND) measurement.

We show below that with a modest modulation of the longitudinal coupling of a few MHz, extremely high fidelity can be reached on a time scale of hundreds of ns. These results do not include imperfections such as finite wire lengths, quasi-particle poisoning rates, and other
sources of decoherence. The readout fidelities we show below should therefore be understood as an indication of what is possible if topologically protected qubits with natural life-times much longer than the readout time can be realized.

Finally, we show how the same mechanism proposed for the readout protocol can be used to enact long-range entangling gates between Majorana qubits. In this situation, two Majorana qubits are independently coupled to the same resonator mode. The proposed two-qubit gate is non-topological, but is nevertheless compelling as an ingredient in a topological quantum computing platform. It can be used to generate entangled qubit pairs used in two-qubit gate teleportation [12], and the possibility of generating long-range entanglement can facilitate a more modular approach to topological quantum computing [13].

Dot mediated Majorana-resonator interaction.— We consider physical realizations of Majorana qubits where pairs of Majorana bound states are localized at the ends of a quasi one-dimensional electronic system, or quantum wire, and light-matter interaction arises due to capacitive coupling to the electric field of a resonator. To be specific, consider the schematic setup illustrated in Fig. 1. Two topological superconducting wires are tunnel coupled to a common semiconductor region, which we also refer to as a quantum dot in the following, effectively forming a topological superconductor-semiconductor-superconductor (TS-Sm-TS) junction. The semiconducting segment serves mainly as an effective tunable tunnel barrier between the two wires. One realization of this setup is a single nanowire with proximity induced superconductivity everywhere except in a middle segment which is not in contact with the superconductor. Another realization is two physically distinct nanowires tunnel coupled to a quantum dot, as in Refs. [8, 9]. Finally, both the wires and the semiconductor region can be capacitively coupled to the electromagnetic field of a resonator (in the figure, only capacitively coupled to the semiconducting region can be capacitively coupled to the resonator mode). The overlap of the Majorana wavefunctions on the dot can be controlled through gate-voltages ($V_g$), or through an external flux threading the superconducting loop ($\Phi_x$). A geometry with two parallel wires as in [8] can also be used.

We are here interested in the ideal situation where the wires are sufficiently long and we take $E_\alpha \to 0$ from here on. The term $\hat{H}_C = E_C(N - n_g)^2$ describes the total charging energy of the superconducting island, with $\hat{N}$ the electron occupancy of the island.

The two last terms in Eq. (1) are $\hat{H}_{dot} = \sum_j \varepsilon_j \hat{d}_j^\dagger \hat{d}_j$, where $\hat{d}_j^\dagger$ creates an excitation on the dot with energy $\varepsilon_j$, and describes tunneling between the dot and the two wires. Here $t_{ij} \geq 0$ are tunneling amplitudes proportional to the overlap of the corresponding Majorana mode functions and the dot orbitals [14]. The superconducting phase appearing in the tunneling Hamiltonian and the electron island number are canonical conjugate variables satisfying $[\hat{N}, e^{i\phi/2}] = e^{i\phi/2}$. Because the wire-dot-wire system forms a loop with the bulk superconductor, we also need to account for any external flux $\varphi_x = 2\pi \hat{\Phi}_x / \Phi_0$ threading the loop, where $\Phi_0 = h / 2e$ is the flux quantum.

Coupling to the electromagnetic field is introduced through a total Hamiltonian $\hat{H} = \hat{H}_{wdw} + \hat{H}_r + \hat{H}_{int}$ where $\hat{H}_r = \hbar \omega_r \hat{a}^\dagger \hat{a}$ and

$$\hat{H}_{int} = \sum_{\alpha} \hbar \omega_C \hat{N} (\hat{a}^\dagger - \hat{a}) + \sum_j \hbar \omega_j \hat{d}_j^\dagger (\hat{a}^\dagger - \hat{a}).$$

Here $\hat{a}^\dagger$ creates a resonator photon with energy $\hbar \omega_r$, and $\{\omega_C, \omega_j\}$ are coupling constants describing capacitive coupling of the resonator to the island charge and the dot, respectively.
In the proposed readout protocol, the tunnel coupling to the dot is gradually turned on such that the initial (near) zero energy logical qubit eigenstates adiabatically evolve into hybridized states partially localized on the dot. The logical states then become split in energy and couple to the resonator field. The key physics behind the effective Majorana-resonator coupling can be exposed by diagonalizing $\hat{H}_{\text{wrd}}$ and treating only a single dot-orbital $j = 0$. Since $\hat{H}_R$ only couples states $|N = n, n_d = 0\rangle$ and $|N = n - 1, n_d = 1\rangle$ within the charge-dot subsystem, where $N$ is an island charge eigenvalue and $n_d$ the dot occupancy, we can diagonalize $\hat{H}_{\text{wrd}}$ block by block. We here restrict our focus to $|n_d| \ll 1$ and large charging and dot energies, such that we can assume the initial state of the charge-dot subsystem to be $|N = 0, n_d = 0\rangle$, while the general case is given in [14]. Furthermore, treating the interaction $\hat{H}_{\text{int}}$ as a perturbation, we have a Hamiltonian for the low-energy subspace [14]

$$\hat{H} \simeq \hbar \omega_q \hat{a}^\dagger \hat{a} + \frac{\hbar \omega_q}{2} \hat{\sigma}_z + i \hbar g_z (\hat{\sigma}_z + 1)(\hat{a}^\dagger - \hat{a}),$$

(4)

where we have defined $\hat{\sigma}_z = i \gamma_{L2}\gamma_{R1}$, and to first order in the resonator couplings we have $\hbar \omega_q = J + \frac{1}{2} [f_+]\langle \phi_x \rangle - f_-\langle \phi_x \rangle]$ and $g_z = -\frac{\lambda c - \lambda_0}{4} \frac{\partial \omega_q}{\partial \delta}$, with $\delta = \varepsilon_0 + E_C$ and $f_\pm(\phi_x) = \sqrt{\delta^2 + \delta_0^2} \pm 2 \delta t_R t_L \cos \left( \frac{\phi_x}{2} \right)$. Furthermore, in the limit of small $t_L, t_R \ll \delta$ we have the simplified expressions $\hbar \omega_q \simeq t_L t_R \cos \left( \frac{\phi_x}{2} \right)/\delta$ and $h g_z \simeq \frac{\lambda c - \lambda_0}{4} t_L t_R \cos \left( \frac{\phi_x}{2} \right)/\delta^2$.

**Parametric qubit readout.**—The Majorana-resonator coupling in Eq. (4) is of a purely longitudinal nature. Longitudinal coupling here refers to a qubit-resonator Hamiltonian interaction of the form $\hat{H}_z = g_z \hat{\sigma}_z (\hat{a}^\dagger - \hat{a})$ that is proportional to the qubit Hamiltonian $\hat{H}_q \propto \hat{\sigma}_z$, while transversal coupling in contrast refers to an interaction $\hat{H}_x = g_x \hat{\sigma}_x (\hat{a}^\dagger - \hat{a})$ orthogonal to the qubit Hamiltonian. At first glance, the longitudinal nature of the qubit-resonator interaction might seem like a disadvantage, because there is no energy exchange between the two systems (note that the coupling terms are fast rotating in the interaction frame, and would thus average out on a short time-scale). In a proposal introduced in Ref. [17] this issue was overcome by working with relatively short wires, introducing terms of the form $E_\alpha i \gamma_{\alpha} \gamma_{\alpha 2}$ to the Hamiltonian. However, this leads to a readout that is not truly QND, and breaks the topological protection of the qubit. We instead propose to work in the long wire limit with purely longitudinal coupling (up to exponentially small corrections in the wire length), and introduce a simple cure to bridge the energy gap between the logical qubit states and the resonator [10]. By parametrically modulating the longitudinal coupling strength at the resonator frequency the longitudinal interaction takes the form of a resonant drive of the resonator, with a qubit-state dependent phase, leading to a large response for the resonator. Note that in contrast to readout based on transversal coupling, the standard approach for superconducting qubits [18], a longitudinal interaction leads to a fully quantum non-demolition readout. The readout protocol is illustrated conceptually in Fig. 2.

In semiconducting systems, the tunnel couplings $t_{\alpha j}$ and/or dot energies $\varepsilon_j$, can be controlled via gate voltages, giving us a mechanism for modulating the Majorana-resonator coupling, $\omega_q \rightarrow \omega_q(V_g)$. Effectively, this amounts to modulating the Majorana wavefunction overlap on the dot. Experimentally the simplest option might be to vary the dot energy while keeping the tunneling amplitudes fixed. This, however, has the disadvantage that the modulation amplitude is largest when the energy cost of moving an electron from the island to the dot is small [see Fig. 3 (a)]. It might therefore be more advantageous to modulate the tunnel couplings directly. Another attractive option is to modulate the external flux $\phi_x(t) = \phi_x + \phi_x(t)$. In Fig. 3 we show the parametric dependence of $g_z$ and $\omega_q$ on $\delta$, $t = t_L = t_R$ and external flux $\phi_x$, for a single dot level $j = 0$ as before. Notably, the coupling $g_z$ can be a large fraction of the bare dot-resonator coupling strength $\lambda_0$. The dot-resonator coupling strength itself can be boosted by using a high-impedance resonator to increase the vacuum fluctuations of the electric field. Dot-resonator coupling strengths in the 100 MHz range have been reached [19][21], suggesting that Majorana-resonator couplings of tens of MHz might be achievable. In Ref. [14] we moreover show that diagonalization of a microscopic nanowire model leads to a similar conclusion for the magnitude of the longitudinal coupling strength.
Following Ref. \[10\], we propose to modulate the effective coupling strength \( g_z(t) = \tilde{g}_z + \tilde{g}_z \cos(\omega_r t) \) at the resonator frequency. In the interaction picture and dropping fast rotating terms we then have \( \hat{H}_{\text{ideal}} = \frac{1}{2} \tilde{g}_z \hat{\sigma}_z (\hat{a}^\dagger - \hat{a}) \). The dynamics under this Hamiltonian is exactly solvable: In the long-time limit the resonator is displaced to one of two coherent states \( \{ \pm \alpha \} \) depending on the qubit state, with \( \alpha = \tilde{g}_z/\kappa \) where \( \kappa \) is the resonator decay rate \[10\].

As was demonstrated in Ref. \[10\] parametric modulation of longitudinal coupling can lead to extremely fast, QND readout. In Fig. 3(a) we show the readout infidelity 1 − \( F \) as a function of \( \tilde{g}_z/\kappa \) for two different measurement times \( \kappa \tau = 1, 2 \), and in Fig. 3(b) we show the measurement time and modulation amplitude needed to reach infidelities 1 − \( F = 10^{-3} \) and 1 − \( F = 10^{-6} \). One of the remarkable properties of longitudinal qubit readout is the fast rate at which qubit information is attained \[10\], as these results show. For example, for a readout rate of \( \kappa/(2\pi) = 1 \) MHz and a coupling modulation amplitude of \( \tilde{g}_z/(2\pi) = 5 \) MHz, an infidelity of \( 10^{-6} \) can be reached in about 300 ns. These are encouraging results for the prospects of performing high fidelity logical operations with Majorana qubits.

A distinguishing feature of the natural longitudinal coupling to the electromagnetic field that arises for Majorana bound states, is that any coupling to the electromagnetic environment beyond the readout resonator is necessarily longitudinal to exponential accuracy as well \[14\]. The electromagnetic environment thus only causes dephasing noise in the measurement basis, which is inconsequential for the measurement fidelity, leading to the notion of a TQND measurement. Purcell decay is exponentially suppressed for the same reason.

**Long-range interactions.**—The same mechanism as used for readout, i.e., parametric modulation of the longitudinal qubit-resonator interaction, can also be used to enact long-range two-qubit gates. Consider a two-qubit setup, where for each qubit a pair of Majorana operators are coupled a quantum dot, as before. Each dot is in turn coupled capacitively to a common resonator mode. Based on the results leading up to Eq. (4) we expect this setup to be well described by a Hamiltonian

\[
\hat{H} = \frac{\hbar}{2} \omega_{q1} \hat{i} \gamma_1 \hat{\gamma}_2 + \frac{\hbar}{2} \omega_{q2} \hat{i} \gamma_5 \hat{\gamma}_6 + \hbar \omega_r \hat{a} \hat{a}^\dagger + i \hbar g_1(t) \hat{i} \gamma_1 \hat{\gamma}_2 (\hat{a}^\dagger - \hat{a}) + i \hbar g_2(t) \hat{i} \gamma_5 \hat{\gamma}_6 (\hat{a}^\dagger - \hat{a}).
\]

(5)

Here, qubit one is defined in terms of Majorana operators \( \hat{\gamma}_1, \ldots, \hat{\gamma}_4 \) and qubit two in terms of \( \hat{\gamma}_5, \ldots, \hat{\gamma}_8 \). Modulating the coupling parameters at the resonator frequency leads to a readout of the two observables \( \hat{i} \gamma_1 \hat{\gamma}_2 \) and \( \hat{i} \gamma_5 \hat{\gamma}_6 \), as before. In contrast, by choosing a modulation frequency \( g_i(t) = \tilde{g}_i + \tilde{g}_i \cos(\omega_r t) \) which is far off-resonant \( |\omega_r - \omega_m| \gg \kappa \), Eq. (4) leads to an effective qubit-qubit interaction. As shown in \[22\], an exact unitary transformation maps the Hamiltonian Eq. (5) onto

\[
\hat{H}' = \hbar J \hat{i} \gamma_1 \gamma_2 \gamma_5 \gamma_6 + \hbar \omega_\text{m} \hat{a} \hat{a}^\dagger,
\]

where \( J \approx \tilde{g} \tilde{g}_2 (\omega_m - \omega_r) \). By modulating for a period of time \( t_g = \pi/|J| \) this gives a two-qubit entangling gate. In an encoding where \( \hat{i} \gamma_1 \hat{\gamma}_2 = \hat{\sigma}_{x1} \) and \( \hat{i} \gamma_5 \gamma_6 = \hat{\sigma}_{x2} \) the gate is, up to single qubit unitaries, equivalent to a controlled-Z (or CPHASE) gate \[22\]. If the dots are tunable such that one can selectively couple to different pairs of Majorana operators on each island, similar to the proposals in \[9\], it is furthermore possible to enact effective \( \hat{P}_i \otimes \hat{P}_j \) interactions where \( \hat{P}_i, \hat{P}_j \) are any of \( \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z \).

**Conclusions.**—We have introduced a readout protocol for Majorana qubits based on parametric modulation of a longitudinal Majorana-resonator interaction. Under modest assumptions about the magnitude of the coupling modulation, we have shown that very high-fidelity readout is possible in short measurement times. Moreover, the same mechanism can be used to generate long-range entanglement between Majorana qubits, thus making parametric modulation of longitudinal coupling an at-

**FIG. 3.** (a)–(c): Parametric dependence of longitudinal coupling on \( \delta/t \) (a) and \( t/\delta \) (b) for \( \varphi_x = 0 \), and on \( \varphi_x \) (c) for \( t/\delta = 1 \) (solid line) and \( t/\delta = 0.5 \) (dotted line). (d)–(e): Qubit splitting \( \hbar \omega_i/t \) (d), \( \hbar \omega_i/\delta \) (e) and (f) for the same parameters as in the top row. The dashed lines in (a), (b), (d) and (e) are approximations for small \( t/\delta \). In all panels we use \( \lambda_C = 0 \).

**FIG. 4.** Measurement infidelity for ideal longitudinal read-out with interaction \( \hat{H}_\text{ideal} \). (a) Infidelity as a function of modulation amplitude for two different measurement times \( \kappa \tau = 1, 2 \). (b) Measurement time needed to reach infidelities 1 − \( F = 10^{-3} \) and 1 − \( F = 10^{-6} \) as a function of modulation amplitude.
tractive ingredient in a topological quantum computing architecture.

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[1] V. Mourik, K. Zuo, S. M. Frolov, S. Plissard, E. P. Bakkers, and L. P. Kouwenhoven, Science 336, 1003 (2012).
[2] M. Deng, S. Vaitiekūnas, E. B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C. M. Marcus, Science 354, 1557 (2016).
[3] S. M. Albrecht, A. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. Marcus, Nature 531, 206 (2016).
[4] F. Nichele, A. C. C. Drachmann, A. M. Whiticar, E. C. T. O’Farrell, H. J. Suominen, A. Fornieri, T. Wang, G. C. Gardner, C. Thomas, A. T. Hatke, P. Krogstrup, M. J. Manfra, K. Flensberg, and C. M. Marcus, Phys. Rev. Lett. 119, 136803 (2017)
[5] H. Zhang, C.-X. Liu, S. Gazibegovic, D. Xu, J. A. Logan, G. Wang, N. Van Loo, J. D. Bonner, M. W. De Moor, D. Car, et al., Nature 556, 74 (2018).
[6] H. J. Suominen, M. Kjaergaard, A. R. Hamilton, J. Shahani, C. J. Palmstrøm, C. M. Marcus, and F. Nichele, Phys. Rev. Lett. 119, 176805 (2017).
[7] P. Bonderson, M. Freedman, and C. Nayak, Phys. Rev. Lett. 101, 010501 (2008).
[8] S. Plugge, A. Rasmussen, R. Egger, and K. Flensberg, New J. Phys. 19, 012001 (2017).
[9] T. Karzig, C. Knapp, R. M. Lutchyn, P. Bonderson, M. B. Hastings, C. Nayak, J. Alicea, K. Flensberg, S. Plugge, Y. Oreg, et al., Phys. Rev. B 95, 235305 (2017).
[10] N. Didier, J. Bourassa, and A. Blais, Phys. Rev. Lett. 115, 203601 (2015).
[11] M. C. Dartiailh, T. Kontos, B. Douçot, and A. Cottet, Phys. Rev. Lett. 118, 126803 (2017).
[12] S. Bravyi, Phys. Rev. A 73, 042313 (2006).
[13] N. H. Nickerson, Y. Li, and S. C. Benjamin, Nature Comm. 4, 1756 (2013).
[14] See Supplemental Material, which includes Refs. [23]–[29].
[15] C. Knapp, T. Karzig, R. M. Lutchyn, and C. Nayak, Phys. Rev. B 97, 125404 (2018).
[16] L. Fu, Phys. Rev. Lett. 104, 056402 (2010).
[17] C. Ohm and F. Hassler, Phys. Rev. B 91, 085406 (2015).
[18] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A 69, 062320 (2004).
[19] A. Stockklauser, P. Scarlino, J. V. Koski, S. Gasparinetti, C. K. Andersen, C. Reichl, W. Wegscheider, T. Ihn, K. Ensslin, and A. Wallraff, Phys. Rev. X 7, 011030 (2017).
[20] X. Mi, M. Benito, S. Putz, D. M. Zajac, J. M. Taylor, G. Burkard, and J. R. Petta, Nature 555, 599 (2018).
[21] N. Samkharadze, G. Zheng, N. Kalhor, D. Brousse, A. Sammak, U. Mendes, A. Blais, G. Scappucci, and L. Vandersypen, Science 359, 1123 (2018).
[22] B. Royer, A. L. Grimsmo, N. Didier, and A. Blais, Quantum 1, 11 (2017).
[23] M. Hell, J. Danon, K. Flensberg, and M. Leijnse, Phys. Rev. B 94, 035424 (2016).
[24] A. Cottet, T. Kontos, and B. Douçot, Phys. Rev. B 91, 205417 (2015).
[25] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
[26] R. M. Lutchyn, J. D. Sau, and S. D. Sarma, Phys. Rev. Lett. 105, 077001 (2010).
[27] G. Zhu, D. G. Ferguson, V. E. Manucharyan, and J. Koch, Phys. Rev. B 87, 024510 (2013).
[28] C. Gardiner, P. Zoller, and P. Zoller, Quantum noise: a handbook of Markovian and non-Markovian quantum stochastic methods with applications to quantum optics, Vol. 56 (Springer Science & Business Media, 2004).
[29] J. Cayao, E. Prada, P. San-Jose, and R. Aguado, Phys. Rev. B 91, 024514 (2015).
Supplemental Material for “Majorana Qubit Readout Using Longitudinal Qubit-Resonator Interaction”

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I. DOT-MEDIATED INTERACTION BETWEEN A PAIR OF MAJORANA MODES AND A RESONATOR

A. A single superconducting island hosting two quantum wires

We start by considering two quantum wires, hosting a pair of bound Majorana modes per wire, on a single superconducting island with charging energy $E_C$. Two possible setups are illustrated in Fig. 1, but we for now ignore the tunnel coupling to the dot (labelled “Sm” in the figure). We return to the tunnel coupling in the next section. The superconducting island hosting the two quantum wires can be described by a Hamiltonian

$$\hat{H}_{\text{island}} = E_C(\hat{N} + \hat{n}_e - n_g) + \sum_{\alpha=L,R} \hat{H}_{e,\alpha},$$

(1)

where the first term is the charging energy of the superconducting island, with $\hat{N}$ equal to two times the number of Cooper pairs and

$$\hat{n}_e = \sum_{\alpha=L,R} \sum_{\sigma=\uparrow,\downarrow} \int d^3r \hat{\psi}^\dagger_{\alpha\sigma}(r) \hat{\psi}_{\alpha\sigma}(r),$$

(2)

counts any unpaired electrons [2], which in a low-energy approximation only includes near-zero energy Majorana fermions. Here $\hat{\psi}_{\alpha\sigma}(r)$ is an electron field for the quantum wire labeled $\alpha \in \{L, R\}$ (with spin $\sigma = \uparrow, \downarrow$) satisfying $\{\hat{\psi}_{\alpha\sigma}(r), \hat{\psi}_{\beta'\sigma'}^\dagger(r')\} = \delta_{\alpha,\beta'}\delta_{\sigma,\sigma'}\delta(r - r')$. Finally, $n_g$ is an offset charge which is in principle gate controllable, but also unavoidably undergoes random fluctuations in any realistic setting.

The last two terms in Eq. (1) are BCS Hamiltonians for each individual quantum wire

$$\hat{H}_{e,\alpha} = \sum_{\sigma=\uparrow,\downarrow} \int d^3r \hat{\psi}^\dagger_{\alpha\sigma}(r) \hat{h}_{\alpha}(r) \hat{\psi}_{\alpha\sigma}(r) + \text{H.c.}$$

(3)

$$+ \int d^3r \left[ \Delta e^{-i\bar{\varphi}} \hat{\psi}^\dagger_{\alpha\uparrow}(r) \hat{\psi}^\dagger_{\alpha\downarrow}(r) + \text{H.c.} \right].$$

Here, the first line describes single-electron physics and the second line are superconducting pairing terms with a superconducting gap $\Delta$ and phase $\bar{\varphi}$, whose physical origin is proximity induced superconductivity due to contact with a nearby bulk superconductor. A physical realization of a quantum wire can be, e.g., a semiconducting nanowire with an epitaxially grown superconducting shell coating the wire. The superconducting phase and Cooper

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pair number are canonical conjugate variables satisfying \([N, e^{i\phi/2}] = e^{i\phi/2}\). Crucially, we assume that both wires are in contact with a common conventional s-wave superconductor (illustrated with blue color in Fig. 1) shunting the two wires such that the entire system behaves as a single superconducting island with a uniform superconducting phase. This configuration, first proposed in [1], has been dubbed a Majorana box qubit.

The single-electron Hamiltonian \(h_\alpha(r)\) is of the generic form

\[
h_\alpha(r) = -\frac{\hbar^2}{2m} \nabla^2 - eV(r) + \ldots,
\]

where \(m\) is the effective electron mass, \(V(r)\) refers to the total potential experienced by the electrons due to the confining potential of the nano-circuit as well as any (classical) gate voltages [3]. The trailing ellipses in Eq. (4) refer to spin-orbit coupling, Zeeman fields and any other single-electron physics necessary for the existence bound near-zero energy Majorana modes in the wire [4, 5]. See Sec. II for an explicit example of a topological nanowire model. We here simply assume that each wire is in the topological regime with a pair of bound Majorana modes localized at the respective wire ends.

Coupling to the quantized electromagnetic field is introduced through an interaction Hamiltonian

\[
\hat{H}_{\text{tot}} = \hat{H}_{\text{island}} + \hat{H}_r + \hat{H}_{\text{island, int}},
\]

where \(\hat{H}_r\) is the electromagnetic Hamiltonian, which in a single-mode approximation becomes \(\hat{H}_r = \hbar \omega_r \hat{a} \hat{a}^\dagger\), with \(\omega_r\) the resonator frequency and \(\hat{a}\) the annihilation operator for the resonator mode. The light-matter interaction describes capacitive coupling between the electrons on the island and the electric field of the resonator [3]

\[
\hat{H}_{\text{island, int}} = i\hbar \lambda_C \hat{N} (\hat{a}^\dagger - \hat{a}) - e \sum_{\alpha=L,R} \sum_{\sigma=\uparrow,\downarrow} \int d^2 r \hat{V}_\sigma(r) \hat{\psi}_\alpha^\dagger(r) \hat{\psi}_\alpha(r),
\]

where the coupling strength can be expressed as

\[
\lambda_C = -\omega_r \sqrt{\frac{\pi Z_r}{R_K C_{\text{island}}}},
\]

with \(C_c\) the coupling capacitance between the island and the resonator, \(C_{\text{island}} = \epsilon^2/(2EC)\) the island’s capacitance, \(Z_r\) the resonator’s characteristic impedance and \(R_K = \hbar/e^2\) the quantum of resistance. Equation (6) can be read as a quantized contribution to the gate voltage biasing the island, where in a single mode approximation the resonator voltage is

\[
\hat{V}_r(r) = i \omega_r \sqrt{\frac{\hbar Z_r}{2}} \frac{C_c}{C_{\text{island}}} u(r)(\hat{a}^\dagger - \hat{a}),
\]

with \(u(r)\) a dimensionless resonator mode function describing the spatial dependence of the voltage biasing the island. Note that if the island is small compared to any spatial variation of the resonator mode function, we can take \(u(r) = 1\) and the Hamiltonian takes the simpler form

\[
\hat{H}_{\text{island, int}} = i\hbar \lambda_C (\hat{N} + \hat{n}_e)(\hat{a}^\dagger - \hat{a}).
\]

We are interested in the low-energy physics of \(\hat{H}_{\text{island}}\). It is convenient to perform a unitary transformation [2] such that

\[
\hat{U} \hat{\psi}_{\alpha\sigma}(r) \hat{U}^\dagger = e^{-i\hat{\phi}/2} \hat{\psi}_{\alpha\sigma}(r),
\]

\[
\hat{U}^\dagger \hat{\psi}_{\alpha\sigma}(r) \hat{U} = e^{-i\hat{\phi}/2} \hat{\psi}_{\alpha\sigma}(r),
\]

where we take \(u(r) = 1\) across the entire island, the resonator voltage decouples from the unpaired electrons in any other single-electron physics necessary for the existence bound near-zero energy Majorana modes [4, 5]. See Sec. II for an explicit example of a topological nanowire model. We here simply assume that each wire is in the topological regime with a pair of bound Majorana modes localized at the respective wire ends.

The resulting low-energy approximation to \(\hat{H}_{\text{island}} = \hat{U}^\dagger \hat{H}_{\text{island}} \hat{U}\) is

\[
\hat{H}'_{\text{island}} \approx E_C (\hat{N} - n) + \sum_{\alpha=L,R} E_\alpha \hat{\gamma}_\alpha \hat{\gamma}_\alpha^\dagger - \frac{1}{2} \sum_{\alpha=L,R} E_\alpha \hat{\gamma}_\alpha \hat{\gamma}_\alpha^\dagger - \frac{1}{2} \sum_{\alpha=L,R} E_\alpha \hat{\gamma}_\alpha \hat{\gamma}_\alpha^\dagger.
\]

Here \(E_\alpha\) is the energy splitting of the Majorana fermion, proportional to the Majorana mode function overlap which we assume to be exponentially small in the wire length \(L_\alpha\), that is \(E_\alpha \propto e^{-L_\alpha/k}\) with \(x_\alpha\) a characteristic coherence length for the wire [6]. We assume that \(L_\alpha\) is large enough that we can set \(E_\alpha = 0\) from here on.

Similarly, the interaction Hamiltonian \(\hat{H}_{\text{island, int}} = \hat{U}^\dagger \hat{H}_{\text{island, int}} \hat{U}\) is in new frame given by

\[
\hat{H}_{\text{island, int}} = i\hbar \lambda_C (\hat{a}^\dagger - \hat{a})
\]

\[
- \sum_{\alpha=L,R} \sum_{\sigma=\uparrow,\downarrow} \int d^2 r \delta \mu_\alpha(r) \hat{\psi}_{\alpha\sigma}^\dagger(r) \hat{\psi}_{\alpha\sigma}(r),
\]

where we have defined

\[
\delta \mu_\alpha(r) = \omega_r \sqrt{\frac{\pi Z_r}{R_K C_{\text{island}}}} [u_\alpha(r) - 1].
\]

We see that in the usual lumped element approximation where we take \(u(r) = 1\) across the entire island, the resonator voltage decouples from the unpaired electrons in...
this frame. However, it is worth noting that we can relax this assumption when the only relevant fermionic modes are the low-energy bound Majorana modes, since in the same low-energy approximation as before we have

\[ \hat{H}_{\text{island,int}} \simeq i\hbar \lambda \hat{N}(\hat{a}^\dagger - \hat{a}) + \sum_{\alpha=L,R} i\lambda_{\alpha}^{2} \gamma_{\alpha 1} \gamma_{\alpha 2}(\hat{a}^\dagger - \hat{a}) + i\hbar A(\hat{a}^\dagger - \hat{a}) \]

(16)

where

\[ \lambda_{\alpha} = -2\sqrt{\frac{\pi Z_{\alpha}}{R K}} \frac{C_{\alpha}}{C_{\text{island}}} \sum_{\sigma} \int d^{3}r \left[ u_{\sigma}(r) - 1 \right] \]

(17)

\[ A = -\omega_{r} \sqrt{\frac{\pi Z_{r}}{R K}} \frac{C_{r}}{C_{\text{island}}} \sum_{\sigma} \int d^{3}r \left[ u_{\sigma}(r) - 1 \right] \]

(18)

\[ \times \left[ f_{\sigma}(r) f_{\sigma}(r) + g_{\sigma}(r) g_{\sigma}(r) \right]. \]

Crucially, \( \lambda_{\alpha} \) vanishes in the topological regime where the Majorana mode function overlap is exponentially small. In the long wire limit we therefore take \( \lambda_{\alpha} \to 0 \). Moreover, this is holds independently of the detailed form of \( u(r) \), and we therefore expect the electromagnetic field to decouple from the Majorana modes even in a more general situation where the resonator mode function varies significantly over the qubit island. The term proportional to \( A \) gives a small displacement of the resonator \( \sim A/\omega_{r} \), which can be absorbed into a re-definition of \( \hat{a} \). We simply ignore this term in the following.

The disappearance of \( \hat{n}_{e} \) from the charging energy and the capacitive coupling to the resonator means that in the frame defined by Eq. (9), \( \hat{N} \) effectively counts the total charge on the island. This will become more clear in the next section when we introduce tunneling of electrons on and off the island.

B. Wire-dot-wire setup

To be able to couple to products of Majorana operators such as \( \gamma_{L2} \gamma_{R1} \), where the two corresponding Majorana mode functions are localized on different wires, we introduce a quantum dot as a mediator. An effective topological superconductor-semiconductor-superconductor (TS-Sm-TS) junction is formed by tunnel-coupling two wires to a common (in general multi-electron) quantum dot, as illustrated in Fig. 1. The dot segment is described by a set of electronic orbitals

\[ \hat{H}_{\text{dot}} = \sum_{j} \epsilon_{j} \hat{d}_{j}^{\dagger} \hat{d}_{j}, \]

(19)

The dot also couples capacitively to the resonator field, described by a Hamiltonian

\[ \hat{H}_{\text{dot,int}} = \sum_{j} i\hbar \lambda_{j} \hat{d}_{j}^{\dagger} \hat{d}_{j}(\hat{a}^\dagger - \hat{a}), \]

(21)

with \( \lambda_{j} \) a coupling strength for dot orbital \( j \).

Upon performing the unitary transformation Eq. (9) and the low-energy approximation as before, one can write a total Hamiltonian for the system in the transformed frame

\[ \hat{H}_{\text{tot}} = \hat{H}_{\text{island}} + \hat{H}_{\text{island,int}} + \hat{H}_{\text{dot}} + \hat{H}_{r} \]

\[ + \hat{H}_{\text{dot,int}} + \hat{H}_{T} \]

(22)

where the low-energy approximation to the tunneling Hamiltonian is

\[ \hat{H}_{T} \simeq \sum_{j} \left[ \frac{i\hbar L_{j}}{2} e^{i\phi/2} \hat{\gamma}_{L2} \hat{d}_{j} \right. \]

\[ - \left. \frac{t_{Rj}}{2} e^{i\phi/2} \hat{\gamma}_{R1} \hat{d}_{j} + \text{H.c.} \right], \]

(23)

with

\[ t_{Lj} = 2 \sum_{\sigma=\uparrow,\downarrow} \int d^{3}r \left| L_{j}\sigma(r) \right|^{2} \]

(24)

\[ t_{Rj} = 2 \sum_{\sigma=\uparrow,\downarrow} \int d^{3}r \left| R_{j}\sigma(r) \right|^{2}. \]

(25)

We have here assumed that the overlap between the tunnel couplings \( t_{\sigma j}(r) \) and the Majorana mode functions at the far ends of the wires, i.e., \( f_{L\sigma}(r) \) and \( g_{R\sigma}(r) \), is negligible.

The form of \( \hat{H}_{T} \) clarifies the role of \( \hat{N} \) in the frame defined by Eq. (9). The operator \( e^{i\phi/2} \) increases \( \hat{N} \) by one, i.e., \( e^{i\phi/2} |N\rangle = |N+1\rangle \) for \( |N\rangle \) an eigenstate of \( \hat{N} \) with eigenvalue \( N \). By introducing Majorana fermion operators for each wire \( \hat{f}_{L} = \frac{1}{2}(\hat{\gamma}_{L1} + i\hat{\gamma}_{L2}) \), \( \hat{f}_{R} = \frac{1}{2}(\hat{\gamma}_{R1} + i\hat{\gamma}_{R2}) \), such that

\[ \hat{\gamma}_{L2} = i\hat{f}_{L}^\dagger - i\hat{f}_{L}, \]

\[ \hat{\gamma}_{R1} = \hat{f}_{R}^\dagger + \hat{f}_{R}, \]

(26)

(27)

we see that the action of an operator like \( e^{i\phi/2} \hat{\gamma}_{L2} \hat{d}_{j} \) is to remove one electron from the dot, increase \( \hat{N} \) by one, and flip the state of the Majorana on the left wire. Thus, in this frame, \( \hat{N} \) counts the total charge on the island and the Majorana fermion operators are effectively charge-less.

C. Diagonalizing the wire-dot-wire Hamiltonian

In the proposed readout protocol, the coupling to the dot is gradually turned on such that the initial near-zero energy logical qubit eigenstates evolve into hybridized
states partially localized on the dot. The logical states then become split in energy and couple to the resonator field. The key physics can be exposed by diagonalizing the Hamiltonian for the wire-dot-wire system, excluding the coupling to the resonator, i.e.,

$$\hat{H}_{\text{wdw}} = \hat{H}_{\text{island}}' + \hat{H}_{\text{dot}}' + \hat{H}_T'. \tag{28}$$

We only consider a single dot orbital $j = 0$ in this section, for simplicity.

It is convenient to first combine $\hat{\gamma}_{L2}$ and $\hat{\gamma}_{R1}$ in a single fermion $\hat{f} = \frac{1}{2}(\hat{\gamma}_{L2} + i\hat{\gamma}_{R1})$ such that the tunneling Hamiltonian can be written

$$\hat{H}_T' = \frac{it_-}{2} e^{i\varphi/2} \hat{f} \hat{d} \hat{f} + \frac{it_+}{2} e^{i\varphi/2} \hat{f} \hat{d} \hat{f} + \text{H.c.}, \tag{29}$$

where we have defined $t \equiv t_L e^{i\varphi/2} \pm t_R$. We here take $t_L$ and $t_R$ to be real and positive without loss of generality. Flux quantization around the loop formed by the superconducting island and the dot (see Fig. 1) is accounted for by including the external flux contribution $\varphi = \Phi_0 / \Phi_0$, with $\Phi_0 = \hbar/2e$ the flux quantum, in the tunneling amplitudes $t_L$.

To diagonalize Eq. (28), we first note that the Hamiltonian only contains transitions within a two-level subspace $\{|0_n\} \equiv \{|N = n, n_d = 0\}, \{|1_n\} \equiv \{|N = n - 1, n_d = 1\}$ of the charge-dot subsystem, where $n_d$ denotes the occupancy of the dot. We can thus treat each subspace labeled by $n$ independently. First define a new lowering operator

$$\hat{c}_n = |0_n\rangle \langle 1_n|, \tag{30}$$

such that in the $\{|0_n\}, \{|1_n\}\}$ subspace the Hamiltonian becomes

$$\hat{H}_{\text{wdw},n} = \delta(n) \hat{c}_n \hat{c}_n + E_C(n - n_d)^2 + \left(\frac{it_-}{2} \hat{f} \hat{c}_n + \frac{it_+}{2} \hat{f} \hat{c}_n + \text{H.c.}\right), \tag{31}$$

where $\delta(n) = \varepsilon_0 - 2E_C(n - n_d) + E_C$. This quadratic Hamiltonian can be diagonalized exactly by a unitary transformation $U_n = e^{-S_n}$ with

$$\hat{S}_n = \alpha^{(n)}_L \hat{c} \hat{f} - \alpha^{(n)}_R \hat{c} \hat{f}. \tag{32}$$

With the choice

$$\tan(2|\alpha^{(n)}_L|) = \frac{|t_+|}{\delta(n)}, \quad \frac{|\alpha^{(n)}_L|}{|\alpha^{(n)}_R|} = -\frac{it_+}{|t_+|}, \tag{33}$$

we find

$$\hat{H}_{\text{wdw},n} = \hat{U}_n^\dagger \hat{H}_{\text{wdw}} \hat{U}_n = \varepsilon_c(n) \hat{c} \hat{c}_n + \varepsilon_f(n) \hat{f} \hat{f} + E(n) \tag{34}$$

with

$$\varepsilon_c(n) = \frac{\text{sgn}(\delta(n))}{2} [f_+(n) + f_-(n)], \tag{35}$$

$$\varepsilon_f(n) = \frac{\text{sgn}(\delta(n))}{2} [f_+(n) - f_+(n)], \tag{36}$$

$$E(n) = E_C(n - n_d)^2 + \frac{1}{2} [\delta(n) - \varepsilon_c(n) - \varepsilon_f(n)], \tag{37}$$

For notational convenience we have defined $f_\pm(n) = \sqrt{\delta(n)^2 + t_\pm^2} \pm 2t_L t_R \cos \left(\frac{\delta(n)}{2}\right)$.

To consider the coupling to the resonator we need to also transform the interaction Hamiltonian. The total interaction Hamiltonian in the $n$th subspace is

$$\hat{H}_{\text{int},n} = i\hbar(\lambda - \lambda_C) \hat{c} \hat{d}_n (\hat{a}^\dagger - \hat{a}) + i\hbar n \lambda_C (\hat{a}^\dagger - \hat{a}), \tag{38}$$

which transforms to

$$\hat{H}_{\text{int},n} = i\hbar \left[ g_c(n) \delta(n) + g_f(n) [f_+(n) \hat{f} + f_+(n) \hat{f} + \hat{f} \hat{f} + H.c.] \right] \hat{a} \hat{a} \hat{a} \hat{a}, \tag{39}$$

with

$$g_c(n) = \frac{\lambda - \lambda_C}{2} \left( \frac{\delta(n)}{f_-(n)} + \frac{\delta(n)}{f_+(n)} \right), \tag{40}$$

$$g_f(n) = \frac{\lambda - \lambda_C}{2} \left( \frac{\delta(n)}{f_-(n)} - \frac{\delta(n)}{f_+(n)} \right), \tag{41}$$

$$w_\pm(n) = \left( \lambda - \lambda_C \right) \frac{it_\pm}{2\delta(n)}. \tag{42}$$

The advantage of this change of frame is that all terms that are off-diagonal in the electron operators are now of order $|w_\pm(n)|$. We can treat $w_\pm(n)$ as small parameters, assuming that the relevant electronic transition are all far detuned from the resonator energy, and treat the second line of $H_{\text{int},n}$ systematically in a Schrieffer-Wolff expansion [7]. We here, however, simply drop these terms on the basis that they are “fast rotating” in the interaction picture, which is equivalent to truncating the Schrieffer-Wolff expansion at first order. Some care has to be taken in practice, however, since we intend to modulate some of the parameters. In particular, we wish to modulate $g_f(n) \sim \cos(\omega_t t)$, but this will also modulate $w_\pm(n)$, and one has to take care not to accidentally bring any unwanted transitions into resonance. Second order corrections that we neglect are roughly speaking of order $\sim |w_\pm(n)|^2/|\varepsilon_c(n)| \pm |\varepsilon_f(n)| - \delta_m$, where $\delta_m = \omega_r \pm \omega_m$ with $\omega_m$ a modulation frequency for $w_\pm(n)$.

Resuming, the Hamiltonians $H_{\text{wdw}} = \sum_n \hat{H}_{\text{wdw},n} \hat{P}_n$ and $H_{\text{int}} = \sum_n \hat{H}_{\text{int},n} \hat{P}_n$, where $\hat{P}_n$ is a projector onto the $\{|0_n\}, \{|1_n\}\}$ subspace, can be written

$$\hat{H}_{\text{wdw}} = \varepsilon_\varepsilon(\hat{N} + 1) \hat{d} \hat{d} + \varepsilon_f(\hat{N}) \hat{f} \hat{f} + E(\hat{N}) + [E(\hat{N} + 1) - E(\hat{N})] \hat{d} \hat{d}, \tag{43}$$

and

$$\hat{H}_{\text{int}} \approx \frac{\text{ih} \gamma(n) \hat{N} + 1 + \lambda_C \hat{d} \hat{d} \hat{a} \hat{a} \hat{a} \hat{a}}{\lambda_C \hat{N} (\hat{a} \hat{a} \hat{a} \hat{a})}, \tag{44}$$
where the various functions of $\hat{N}$ are diagonal operators in the charge basis defined through $f(\hat{N} + a) = \sum_{n} f(n + a) |n\rangle \langle n|$. We emphasize that the diagonalization of $\hat{H}_{\text{dw}}''$ is exact, and the only approximation is made in $\hat{H}_{\text{int}}''$.

Since $\hat{H}_{\text{tot}}'' = \hat{H}_{\text{dw}}'' + \hat{H}_{\text{int}}'' + \hat{H}_{e}$ conserves the charge number and dot occupation at this level of approximation, we can assume that the charge and dot degrees of freedom remain in a definite state, and replace $\hat{N} \to N$, $d^{\dagger}d \to n_{d}$. This amounts to an “adiabatic elimination” of the dot and charge degrees of freedom. In particular, for $n_{d} \ll 1$ and large $E_{C}, \varepsilon_{0}$, we can assume the charge-dot subsystem to be in the state $|N = 0, n_{d} = 0\rangle$ and drop terms proportional to $\hat{N}$ and $d^{\dagger}d$ in $\hat{H}''$. The Hamiltonian in Eq. (4) of the main letter is then finally found by defining $\hat{\sigma}_{z} = 2f^{\dagger}f - 1$

$$\hat{H}_{\text{tot}}'' \simeq \hbar \omega\hat{a}^{\dagger}\hat{a} + \frac{\hbar \omega_{q}}{2} \hat{\sigma}_{z} + i\hbar g_{z}(\hat{\sigma}_{z} + 1)(\hat{a}^{\dagger} - \hat{a}). \tag{45}$$

with $\hbar \omega_{q} = \varepsilon_{f}(0)$, $g_{z} = g_{f}(0)/2$, and we have dropped a constant term.

It is also insightful to consider approximation expressions for the parameters in Eq. (45) in the limit $t_{L}, t_{R} \ll \delta(n)$. The expressions greatly simplify in this limit, which can be useful to gain physical insight, but we emphasize that this is not the main regime of interest for the readout protocol, where we rather want a strong hybridization of the dot orbitals and Majorana modes. The relevant parameters are in the small tunneling regime approximated by

$$\hbar \omega_{q} \simeq \frac{t_{L}t_{R} \cos(\varphi_{x}/2)}{\delta(n)}, \tag{46}$$

$$\hbar g_{z} \simeq \frac{\lambda_{C} - \lambda_{0}}{4} \frac{t_{L}t_{R} \cos(\varphi_{x}/2)}{\delta(n)^{2}}. \tag{47}$$

D. Noise during measurement

Noise in system parameters such as $n_{d}$, $t_{L}$, $\varepsilon_{f}$, or $\varphi_{x}$, leads to fluctuations in $\omega_{q}$ and $g_{z}$, but preserves the general form of Eq. (45). Understanding noise due to coupling to the surrounding electromagnetic environment is also straightforward in the sense that there is nothing special about the resonator mode that was singled out in the above treatment. In other words, other modes of the electromagnetic field couples in exactly the same way, such that in the same low-energy approximation as before, we expect the coupling to the environment to be of the form

$$\hat{H}_{\text{env, int}} = \hat{\sigma}_{z} \hat{B}(t), \tag{48}$$

where $\hat{B}(t)$ could include both classical stochastic processes, describing noise in system parameters, and quantum noise through a bath operator of the generic form [8]

$$\hat{B}(t) = i \int_{0}^{\infty} d\omega \lambda_{\omega}(\hat{b}_{\omega}^{\dagger} e^{\omega t} - \text{H.c.}), \tag{49}$$

where $\lambda_{\omega}$ are coupling constants and $[\hat{b}_{\omega}, \hat{b}_{\omega'}^{\dagger}] = \delta(\omega - \omega')$.

Based on the diagonal form of any noise process in the logical basis, we expect that although the measurement process necessarily opens the system to new noise channels, the environment can only cause dephasing in the measurement basis, which is inconsequential for the measurement fidelity. This holds to exponential accuracy in the length of the quantum wires, following the same line or arguments that leads to the conclusion of a topologically protected degeneracy. We refer to this strong form of quantum non-demolition coupling as topological quantum non-demolition (TQND) coupling.

II. NUMERICAL DIAGONALIZATION OF A MICROSCOPIC MODEL

The above treatment of light-matter interaction in a topological superconductor-semiconductor-superconductor setup used only very general arguments, without specifying any details of the underlying physics giving rise to Majorana bound modes in the first place. To gain further insight, we here consider a specific example of a simple one-dimensional model of a semiconducting nanowire with proximity induced superconductivity, spin orbit coupling, and a magnetic field [4, 5].

The nanowire model we consider is given by the following Hamiltonian

$$\hat{H}_{e} = \int_{0}^{L} dx \hat{\psi}^{\dagger}(x) h(x) \hat{\psi}(x)$$

$$+ \int_{0}^{L} dx \left[ \Delta(x) \hat{\psi}^{\dagger}(x) i\sigma_{y} \hat{\psi}(x)^{T} + \text{H.c.} \right], \tag{50}$$

where $\hat{\psi}(x) = [\hat{\psi}_{\uparrow}(x), \hat{\psi}_{\downarrow}(x)]^{T}$, $\Delta(x)$ is the induced superconducting pairing potential, and

$$h(x) = \left( -\frac{\hbar}{2m} \right) \frac{d^{2}}{dx^{2}} - \mu(x) + i\alpha R \sigma_{y} \frac{d}{dx} + B \sigma_{x}. \tag{51}$$

The $\sigma$-matrices here act on the spin degree of freedom, $m$ is the effective electron mass, $\alpha R$ the spin-orbit coupling, $B$ the magnetic field, and $\mu(x) = \mu_{0}(x) + e \sum V_{g}(x)$ is here to be understood as the chemical potential, including all the various gate voltages along the wire except the resonator. The resonator appears as a quantized contribution to the voltage bias at position $x$

$$\hat{H}_{\text{int}} = -e \int_{0}^{L} dx \hat{V}_{r}(x) \hat{\psi}^{\dagger}(x) \hat{\psi}(x), \tag{52}$$

with $\hat{V}_{r}(x) = i\omega_{r} \sqrt{\hbar Z_{r}/2u}(\hat{a}^{\dagger} - \hat{a})$.

To model a TS-Sm-TS setup as above, we take the following spatial dependence for the chemical and pairing
potentials
\[ \mu(x) = \begin{cases} 
0 & \text{for } 0 < x < L_1 \\
\mu_1 & \text{for } L_1 < x < L_2 \\
0 & \text{for } L_2 < x < L,
\end{cases} \] (53)
\[ \Delta(x) = \begin{cases} 
\Delta e^{i\varphi_x} & \text{for } 0 < x < L_1 \\
0 & \text{for } L_1 < x < L_2 \\
\Delta & \text{for } L_2 < x < L,
\end{cases} \] (54)

where \( \Delta > 0 \) is real and \( \varphi_x \) is the superconducting phase difference across the TS-Sm-TS junction.

We could at this stage diagonalize the middle segment independently, leading to a Hamiltonian of the form Eq. (19) for this segment and a tunneling Hamiltonian of the form Eq. (29). However, since Eq. (50) is quadratic, it is much more convenient in the present context to diagonalize the full Hamiltonian directly. The approach we take here is thus rather different than what we did in the previous section, but of course leads to the same conclusions.

We diagonalize Eq. (50) numerically by approximating the integral over \( x \) as a Riemann sum and using
\[
\frac{d\psi(x)}{dx} \simeq \frac{\psi(x + a) - \psi(x - a)}{2a},
\]
\[
\frac{d^2\psi(x)}{dx^2} \simeq \frac{\psi(x + a) + \psi(x - a) - 2\psi(x)}{a^2},
\]
where \( a \) is the spatial unit cell. One readily finds
\[
\hat{H}_c = \sum_{n=0}^{N-1} \hat{c}_n^\dagger \hat{h}_n \hat{c}_n + \sum_{n=0}^{N-2} \left( \hat{c}_n^\dagger \hat{c}_{n+1} + \hat{c}_{n+1}^\dagger \hat{c}_n \right) + \sum_{n=0}^{N-1} \left( \hat{c}_n^\dagger \Delta_n \hat{c}_n + \hat{c}_n \Delta_n^\dagger \hat{c}_n \right),
\]
where
\[
\hat{c}_n = [\hat{c}_{1,n}, \hat{c}_{2,n}]^T = \sqrt{a}[\hat{\psi}_1(x_n), \hat{\psi}_2(x_n)]^T
\]
and
\[
\hat{h}_n = \begin{pmatrix} 2t - \mu(x_n) & B(x_n) \\ B^\dagger(x_n) & 2t - \mu(x_n) \end{pmatrix},
\]
\[
v = \begin{pmatrix} -t & \alpha \\ -\alpha & -t \end{pmatrix},
\]
\[
\Delta_n = \begin{pmatrix} 0 & \Delta(x_n) \\ -\Delta(x_n) & 0 \end{pmatrix},
\]
with \( t = \hbar^2/(2ma^2) \) and \( \alpha = \alpha_R/(2a) \).

The Hamiltonian can now be diagonalized by first writing it in matrix form, \( \hat{H}_c = \frac{1}{2} \vec{c} \hat{H}_M \vec{c}^\dagger \), where \( \vec{c} = [c_{1,0}, c_{1,1}, \ldots, c_{1,0}, c_{1,1}, \ldots, c_{1,0}, c_{1,1}, \ldots]^T \) and \( \hat{H}_M \) is a \( 4N \times 4N \) Hermitian matrix of the block form
\[
\hat{H}_M = \begin{pmatrix} A & B \\
-B^\dagger & -A^\dagger \end{pmatrix},
\]
where \( A^\dagger = A \) and \( B^T = -B \) are \( 2N \times 2N \) matrices [9].

The matrix \( \hat{H}_M \) can be diagonalized with a unitary transformation, \( \hat{U} \hat{H}_M \hat{U}^\dagger \) with \( \hat{U} \hat{U}^\dagger = I \) and
\[
D = \text{diag}[E_0, \ldots, E_{4N}].
\]
We can consequently write \( \hat{H} = \frac{1}{\sqrt{2}} \vec{b} \hat{D} \vec{b}^\dagger \), where \( \vec{b} = \hat{U} \vec{c} \). We have here used the standard approach of formally doubling the degrees of freedom, and the eigenvalues consequently come in pairs \( E_k = -E_{2N+k} \) [9], such that we can write after normal ordering
\[
\hat{H}_c = \sum_{k=0}^{2N-1} E_k \hat{b}_k^\dagger \hat{b}_k,
\]
dropping a constant term. We order the eigenvalues such that \( 0 \leq E_0 \leq E_1 \leq E_2 \leq \cdots \leq E_{2N-1} \).

In the top panel of Fig. 2, we show the first six eigenvalues \( E_k \) of \( H_M \) with smallest magnitude, for a set of parameters in the topological regime. Note that we plot both positive and negative eigenvalues of the matrix \( H_M \), while only positive values enter the physical spectrum, Eq. (62). We also plot the spatial Majorana mode functions, which are found through the inverse Bogoliubov transformation \( \vec{c} = \hat{U} \vec{b} \). The \( l \)th column of \( U \) is the eigenvector of \( H_M \) with eigenvalues \( E_l \), and keeping only the eigenvectors corresponding to the two lowest eigenvalues we can write
\[
\hat{c}_k = U_{k,0} \hat{b}_0 + U_{k,2N} \hat{b}_1 + U_{k,1} \hat{b}_1 + U_{k,2N+1} \hat{b}_1 + \cdots
\]
\[
= f_{1k} \hat{\gamma}_1 + i f_{4k} \hat{\gamma}_4 + f_{3k} \hat{\gamma}_3 + i f_{2k} \hat{\gamma}_2 + \cdots,
\]
where the dots refer to higher energy Bogoliubov modes that we drop, and in the second line we have expressed \( \hat{b}_0 = e^{i\theta_1} \hat{c}_0 + i \hat{\gamma}_4 \), \( \hat{b}_1 = e^{i\theta_2} \hat{c}_0 + i \hat{\gamma}_2 \), and defined corresponding Majorana mode functions \( f_{ik} \) with \( i = 1, 2, 3, 4 \) (the angles \( \theta_{1,2} \) are arbitrary in a numerical diagonalization). Here the index \( k \) labeling the fermionic operators \( \hat{c}_k \) runs over both spatial position label \( n \) and spin \( \sigma = \uparrow, \downarrow \), such that \( f_{ik} = f_{i\sigma}(x_n) \). The index \( i \) is chosen such that \( f_{ik} \) is localized to the left of \( f_{jk} \) of \( i < j \). In the lower panel of Fig. 2, we show the spatial dependence of the wave functions \( f_{ik} \) for an example parameter set.

The interaction with the resonator becomes after discretizing space
\[
\hat{H}_{\text{int}} = -\frac{e}{N} \sum_{n=0}^{N-1} \hat{V}_r(x_n) \hat{c}_n^\dagger \hat{c}_n.
\]

Keeping only the two lowest energy Bogoliubov modes we have
\[
\hat{H}_{\text{int}} \simeq i \hbar g_2 \hat{\gamma}_2 \hat{\gamma}_3 (\hat{a}^\dagger - \hat{a}) + i \hbar A(\hat{a}^\dagger - \hat{a}),
\]
with
\[
g_z = -2 \omega_r \sqrt{\frac{\pi Z_r}{R_K}} u(x_0) \text{Re} [f_2^* \cdot f_3],
\]
\[
A = -\omega_r \sqrt{\frac{\pi Z_r}{R_K}} u(x_i) \sum_i (f_i^* \cdot f_i),
\]
where we define an inner product \( f_i^* \cdot f_j = \sum_{k=0}^{2N} f_{ik}^* f_{jk} \), and neglect the overlap of mode functions \( f_{ik}, f_{jk} \) at the
far ends of the nanowire with any other mode functions, as well as the spatial variation of the resonator mode $u(x)$ across any Majorana mode function overlap.

In Fig. 3 we show the splitting of the two lowest lying energy eigenstates and the Majorana mode function overlap $\Re[f_2^* f_3]$ as a function of the potential barrier in the middle segment, $\mu_1$, for the same parameters as in Fig. 2. On the right hand side of the lower panel we also show the corresponding coupling strength $g_z$ for a $Z_r = 50 \, \Omega$ resonator with $\omega_r/(2\pi) = 7.5 \, \text{GHz}$ and for a maximum of the resonator voltage $u(x_0) = 1$.

[1] S. Plugge, A. Rasmussen, R. Egger, and K. Flensberg, New J. Phys. 19, 012001 (2017).
[2] M. Hell, J. Danon, K. Flensberg, and M. Leijnse, Phys. Rev. B 94, 035424 (2016).
[3] A. Cottet, T. Kontos, and B. Douçot, Phys. Rev. B 91, 205417 (2015).
[4] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
[5] R. M. Lutchyn, J. D. Sau, and S. D. Sarma, Phys. Rev. Lett. 105, 077001 (2010).
[6] C. Knapp, T. Karzig, R. M. Lutchyn, and C. Nayak, Phys. Rev. B 97, 125404 (2018).
[7] G. Zhu, D. G. Ferguson, V. E. Manucharyan, and J. Koch, Phys. Rev. B 87, 024510 (2013).
[8] C. Gardiner, P. Zoller, and P. Zoller, Quantum noise: a handbook of Markovian and non-Markovian quantum stochastic methods with applications to quantum optics, vol. 56 (Springer Science & Business Media, 2004).
[9] Y. Nagai, Y. Shinohara, Y. Futamura, Y. Ota, and T. Sakurai, Journal of the Physical Society of Japan 82, 094701 (2013).
[10] J. Cayao, E. Prada, P. San-Jose, and R. Aguado, Phys. Rev. B 91, 024514 (2015).
FIG. 3: The two lowest eigenvalues and the Majorana mode function overlap closest to the TS-Sm-TS junction, as a function of chemical potential in the junction region. The parameters are otherwise as in Fig. 2.