Spin squeezing and concurrence

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Received 7 October 2010
Published 10 December 2010
Online at stacks.iop.org/JPhysB/44/015501

Abstract
We study the relations between spin squeezing and concurrence, and find that they are qualitatively equivalent for an ensemble of spin-1/2 particles with exchange symmetry and parity. We exemplify the result by considering a superposition of two Dicke states.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

As an important resource of quantum information and computation, entanglement \cite{1, 2} has attracted much attention in recent years \cite{3–12}. How to measure and detect entanglement is crucial for both theoretical investigations and potential practical applications \cite{13, 14}. The entanglement of a two-qubit system can be well quantified by concurrence \cite{15, 16}. However, quantification of many-body entanglement is still an open question in quantum information.

It is well known that there are close relations between entanglement and spin squeezing \cite{17–25}. There are several definitions of spin squeezing parameters \cite{18–20}, which are studied in different papers. The squeezing parameter $\xi_S^2$ defined by Wineland et al is closely related to multipartite entanglement. It has been proven that \cite{18}, for an ensemble of spin-1/2 particles, if this squeezing parameter is less than 1, the state is entangled. The advantages of spin squeezing parameters in detecting entanglement have been shown in both theoretical and experimental aspects.

The squeezing parameter $\xi_S^2$ defined by Kitagawa and Ueda is relevant to pairwise entanglement \cite{21}. And for states with exchange symmetry and parity, a simple quantitative relation between $\xi_S^2$ and concurrence was given \cite{22}. Furthermore, it has been shown that the spin squeezing and pairwise entanglement are equivalent for states generated by the one-axis twisting Hamiltonian \cite{22}. However, even for states with a fixed parity, such as the states generated by the one-axis twisting Hamiltonian with a transverse field, $\xi_S^2$ is not always equivalent to concurrence \cite{26}. Inspired by recent works \cite{27, 28}, where a set of generalized spin squeezing inequalities are developed, one can define another spin squeezing parameter $\xi_T^2$ from one of the inequalities \cite{29}. Similar to the parameter $\xi_S^2$, one advantage of the parameter $\xi_T^2$ is that we can firmly say that the state is entangled if $\xi_T^2 < 1$. However, if the parameter $\xi_T^2 < 1$, we cannot say the state is entangled, although this parameter is closely related to entanglement.

In \cite{21} it was found that spin squeezing according to the parameter $\xi_T^2$ is equivalent to the minimal pairwise correlation $C_{\vec{n}, \vec{n}}$ along the direction $\vec{n} \perp$ (which is perpendicular to the mean spin direction) for symmetric states. It was further found in \cite{30} that for the symmetric states, the spin squeezing defined via $\xi_T^2$ is equivalent to the minimal pairwise correlation $C_{\vec{n}, \vec{n}}$ along an arbitrary direction $\vec{n}$. For states with a fixed parity, the relations between the two parameters $\xi_S^2$ and $\xi_T^2$ are more evident. It will be seen from section 3 that $\xi_T^2$ contains the term $\xi_S^2$, and the spin squeezing results from just the competition between pairwise correlation along the direction $\vec{n}$ and that along the mean spin direction. So, in this sense, the parameter $\xi_T^2$ is a natural generalization of $\xi_S^2$.

We find that for states with exchange symmetry and parity, the spin squeezing parameter $\xi_T^2$ is qualitatively equivalent to the concurrence in characterizing pairwise entanglement. In other words, the spin squeezing parameter and concurrence emerge and vanish at the same time. This finding is of significance for entanglement detection in experiments. As we all know, entanglement detectors such as entropy and...
concurrence are generally not easy to measure, especially for physical systems like BEC, for which we cannot address individual atoms. However, spin squeezing parameters are relatively easy to measure in experiments, since they only consist of expectations and variances of collective angular momentum operators. Nevertheless, the traditional spin squeezing parameter $\xi^2$ is not always equivalent to concurrence even for states with exchange symmetry and parity. As $\xi^2$ is qualitatively equivalent to concurrence for an ensemble of spin-1/2 particles with exchange symmetry and parity, it is better than $\xi^2$ in detecting pairwise entanglement.

The paper is organized as follows. In section 2, we give definitions of the two spin squeezing parameters and concurrence. In section 3, we give the concrete forms of the spin squeezing parameters and the concurrence for states with exchange symmetry and parity. The relations between these two parameters and concurrence were given in section 4. We exemplify the result by considering superpositions of Dicke states in section 5. Finally, section 6 is devoted to conclusions.

2. Spin squeezing parameters and concurrence

To study spin squeezing, we consider an ensemble of $N$ spin-1/2 particles. For the sake of describing many-particle systems, we use the total angular momentum operators

$$ J_n = \frac{1}{2} \sum_{k=1}^{N} \sigma_{n\alpha}(\alpha = x, y, z) $$

where $\sigma_{n\alpha}$ are the Pauli matrices for the $n$th spin. Now, we give definitions of the two spin squeezing parameters. The first one is defined as [19]

$$ \xi^2 = \frac{4 \min(\Delta J_{n\alpha})^2}{N}, $$(2)

where $\Delta J_{n\alpha}$ is perpendicular to the mean spin direction $\overrightarrow{J} = \frac{J}{|J|}$. Since the system has the exchange symmetry, the total angular momentum is $j = \frac{N}{2}$. For spin coherent states [19], $\Delta J_{n\alpha} = \frac{1}{2}$ and $\xi^2 = 1$. In the following, we consider states with exchange symmetry.

The next spin squeezing parameter is based on the generalized spin squeezing inequalities given by Tóth et al [28], and is defined as [29]

$$ \xi^2 = \frac{\lambda_{\min}}{(J^2) - \frac{N}{2}}, $$

where $\lambda_{\min}$ is the minimum eigenvalue of

$$ \Gamma = \frac{N(J^2)}{N\langle J_x,J_x \rangle} \begin{pmatrix} N\langle J_y,J_y \rangle & 2 & \langle J_z \rangle \\ 2 & 0 & 0 \\ \langle J_z \rangle & 0 & N(\Delta J_x)^2 + \langle J_z \rangle^2 \end{pmatrix} $$

where $\langle A, B \rangle_n = AB + BA$, and equation (3) reduces to [29]

$$ \xi^2 = \min \{\xi_1^2, \xi_2^2\}, $$

where

$$ \xi^2 = \frac{4}{N^2} \left[ N(\Delta J_x)^2 + \langle J_x \rangle^2 \right] $$

$$ = 1 + (N - 1) \left( \langle \sigma_1 \sigma_2 \rangle - \langle \sigma_1 \rangle^2 \right) $$

$$ = 1 + (N - 1)C_{zz}, $$

(9)

with $C_{zz}$ being the two-spin correlation function along the $z$ direction. The explicit form of $\xi^2$ could be obtained as [22]

$$ \xi^2 = \frac{2}{N} \left( \langle J^2 \rangle - \langle J_x \rangle^2 \right) $$

$$ = 1 - 2(N - 1) \left[ |\langle \sigma_1 \sigma_2 \rangle| - \frac{1}{4} (1 - \langle \sigma_1 \sigma_2 \rangle) \right], $$

(10)

where we have used the following relations:

$$ \langle J_x \rangle = \frac{N}{2} \langle \sigma_{1\alpha} \rangle, $$

$$ \langle J^2 \rangle = \frac{N}{4} + \frac{N(N - 1)}{4} \langle \sigma_{1\alpha} \sigma_{2\alpha} \rangle, $$

$$ \langle J_z \rangle = N(N - 1) \langle \sigma_1 \sigma_2 \rangle, $$

(11)

which connect the local expectations with collective ones.

For such states, the significance of $\xi^2$ and $\xi^2$ and the relations between them are clear. According to the parameter $\xi^2$, a state is squeezed when the minimum variance of the angular momentum in the $\overrightarrow{J}$-plane is smaller than $\frac{1}{2}$, while according to $\xi^2$, the variance in the mean spin direction $\overrightarrow{J}$ is also considered. Equation (9) represents the pairwise correlation along the mean spin direction, and this can be viewed as a complement of $\xi^2$, which only considers squeezing in the $\overrightarrow{J}$-plane. Thus, $\xi^2$ can be regarded as a generalization of $\xi^2$, and when $\xi^2 < \xi^2$, the parameter $\xi^2$ reduces to $\xi^2$.

To calculate concurrence, we first need to calculate the two-body reduced density matrix, which can be written as [22]

$$ \rho = \begin{pmatrix} v^* & 0 & 0 & u^* \\ 0 & y & 0 & 0 \\ 0 & y & 0 & 0 \\ u & 0 & 0 & v \end{pmatrix}, $$

(12)
we consider the squeezing parameter ξ

Firstly, we prove that for a state with exchange symmetry

concurrence

where

J. Phys. B: At. Mol. Opt. Phys. 44 (2011) 015501

in the basis \([|00\rangle, |01\rangle, |10\rangle, |11\rangle]\). Then, the concurrence is given by

\[ C = 2 \max(|u| - y, y - \sqrt{v_u v_\omega}). \] (14)

One key observation is that

\[ y^2 - v_u v_\omega = -\frac{1}{2} C_\omega. \] (15)

Thus,

\[ \xi^2 = 1 - 4(N - 1)(y + \sqrt{v_u v_\omega})(y - \sqrt{v_u v_\omega}). \] (16)

From equations (9), (10) and (13), we obtain

\[ \xi^2 = 1 - 2(N - 1)(|u| - y), \] (17)

\[ 1 - 4(N - 1)(y + \sqrt{v_u v_\omega})(y - \sqrt{v_u v_\omega}). \] (18)

Now, one can see that the squeezing parameters are related to the concurrence shown in equation (14). The relations between \(\xi^2\) and \(C\) have been studied [22]. In the following, we consider the squeezing parameter \(\xi^2\), and prove that it is qualitatively equivalent to the concurrence in detecting pairwise entanglement.

### 4. Relations between spin squeezing parameters and concurrence

Firstly, we prove that for a state with exchange symmetry and parity, if concurrence \(C > 0\), it must be spin squeezed according to the criterion \(\xi^2 < 1\). From equation (14) we note that when \(C > 0\), there are two cases, \(C = |u| - y > 0\) and \(C = y - \sqrt{v_u v_\omega} > 0\). However, since the density matrix \(\rho\) is positive, we find \(\sqrt{v_u v_\omega} \geq |u|\), then immediately

\[ (|u| - y)(y - \sqrt{v_u v_\omega}) \leq 0, \] (19)

which means \(|u| - y\) and \(y - \sqrt{v_u v_\omega}\) cannot be positive simultaneously. Therefore, if \(C > 0\), we have [31]

\[ C = \begin{cases} 2(|u| - y), & |u| > y, \\ 2(y - \sqrt{v_u v_\omega}), & y > \sqrt{v_u v_\omega}. \end{cases} \] (20)

According to equations (8) and (17), we get the following relations:

\[ \xi^2 = \begin{cases} 1 - (N - 1)C, & |u| > y, \\ 1 - 2(N - 1)(y + \sqrt{v_u v_\omega})C, & y > \sqrt{v_u v_\omega}, \end{cases} \] (21)

Since \(C > 0\), there always be \(\xi^2 < 1\).

Now, we prove that if the state is spin squeezed (\(\xi^2 < 1\)), concurrence \(C > 0\). If \(\xi^2 < 1\), there are two cases, \(\xi^2 = \xi^2 < 1\) and \(\xi^2 = \xi^2 < 1\). As discussed above, according to equations (17) and (19), \(\xi^2 < 1\) and \(\xi^2 < 1\) could not occur simultaneously. Therefore, if \(\xi^2 = \xi^2 < 1\), we have [32]

\[ C = \frac{1 - \xi^2}{N - 1}, \] (22)

while if \(\xi^2 = \xi^2 < 1\), we have

\[ C = \frac{1 - \xi^2}{2(N - 1)(y + \sqrt{v_u v_\omega})}. \] (23)

Therefore, if the state is squeezed, concurrence \(C > 0\).

The relations between spin squeezing and concurrence are displayed in table 1, and we can see that, for a symmetric state, \(\xi^2 < 1\) is qualitatively equivalent to \(C > 0\), which means that spin squeezing according to \(\xi^2\) is equivalent to pairwise entanglement. Although \(\xi^2 < 1\) indicates \(C > 0\), when \(C = 2(y - \sqrt{v_u v_\omega}) > 0\), we find \(\xi^2 > 1\). Therefore, a spin-squeezed state (\(\xi^2 < 1\)) is pairwise entangled, while a pairwise entangled state may not be spin squeezed according to the squeezing parameter \(\xi^2\). Then, we come to the conclusion that for states with exchange symmetry and parity, the spin squeezing parameter \(\xi^2\) is qualitatively equivalent to the concurrence in characterizing pairwise entanglement. In the following, we will give some examples and applications of our result.

### 5. Examples and applications

We first consider a superposition of Dicke states with parity, and then consider states without a fixed parity. The states under consideration are all based on Dicke states [33] and defined as

\[ |n\rangle_N = \left| \frac{N}{2} - \frac{N}{2} + n \right\rangle, \quad n = 0, \ldots, N, \] (24)

where \(|0\rangle\) denotes a state for which all spins are in the ground states, and \(n\) is the excitation number of spins. Such states are elementary in atomic physics, and may be conditionally prepared in experiments with a quantum non-demolition technique [34–36].

As we consider the state with even parity, we choose a simple superposition of Dicke states as

\[ |\psi_D\rangle = \cos \theta |n\rangle_N + e^{i\varphi} \sin \theta |n + 2\rangle_N, \quad n = 0, \ldots, N - 2, \] (25)

with the angle \(\theta \in [0, \pi]\) and the relative phase \(\varphi \in [0, 2\pi]\). We can easily check that, for the superposition state in equation (25), the mean spin direction is along the z-axis. The
expressions for the relevant spin expectation values can be obtained as
\[
J_z = m + 2 \sin^2 \theta, \\
J_z^2 = m^2 + (4m + 1) \sin^2 \theta, \\
J_z^2 = J_z^2 = \frac{1}{2} e^{i\phi} \sin 2\theta \sqrt{\mu_n},
\]
where \(m = n - \frac{N}{2}\) and \(\mu_n = (n+1)(n+2)(N-n)(N-n-1)\).

By substituting equations (26) into equations (9) and (10), it is easy to get
\[
\xi_2^3 = 1 - \frac{2}{N} \left\{ \sin \theta \cos \theta |\sqrt{\mu_n} \\
- 4[m^2 + 4(m + 1) \sin^2 \theta] - N^2 \right\}
\]
and
\[
\xi_2 = \frac{4}{N} [m^2 + 4(m + 1) \sin^2 \theta] - \frac{4(N-1)}{N^2} [m + 2 \sin^2 \theta]^2.
\]

From the results in [37] we can easily get [31]
\[
u = \frac{e^{i\phi} \sin 2\theta}{2N(N-1)} \sqrt{\mu_n},
\]
\[
y = \frac{N}{4(N-1)} - \frac{1}{N(N-1)} [m^2 + 4(m + 1) \sin^2 \theta],
\]
\[
\sqrt{\nu \psi D} = \frac{\sqrt{(N^2 - 2N + 4|J_z|^2)}^2 - 16(N-1)^2 (J_z^2)^2}{4N(N-1)}.
\]

Inserting equation (29) into equation (14), one can get the expression of concurrence.

In figure 1, we plot these two spin squeezing parameters and concurrence versus \(\theta\) in one period. We observe that for \(\theta \in (0, \pi/3) \cup (2\pi/3, \pi)\), \(\xi_2^3 < 7/3 < 1\). Therefore, the state is spin squeezed in the x-y plane; moreover, as \(C > 0\), the state is pairwise entangled. For \(\theta \in (\pi/3, 2\pi/3)\), it is obvious that the state is also pairwise entangled, since \(C > 0\), while spin squeezing occurs in the z-axis, \(\xi_2^3 < 1\) and \(\xi_2^3 > 1\). The results clearly show that \(\xi_2^3 < 1\) is equivalent to \(C > 0\). But if we adopt \(\xi_2^3 < 1\) as a squeezing parameter, the spin squeezing is not qualitatively equivalent to concurrence.

The equivalence of \(\xi_2^3 < 1\) and \(C > 0\) for states with parity has been demonstrated above. Here, we discuss states without parity to see the relations between spin squeezing and entanglement. For simplicity, we choose
\[
|\psi_D\rangle = \cos \theta |n\rangle_N + e^{i\phi} \sin \theta |n+1\rangle_N, \quad n = 0, \ldots, N - 1.
\]

Specifically, if \(\theta = \frac{x}{2}\), \(n = 0\) or \(n = N - 2\), the above state degenerates to the W state. Moreover, when \(N = 3\), equation (30) reduces to
\[
|\psi_D\rangle = \frac{1}{\sqrt{3}} (|110\rangle + |101\rangle + |011\rangle).
\]

The two-qubit reduced density matrix becomes
\[
\rho = \frac{1}{3} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
and using equation (14), we find \(C = \frac{2}{3}\). We can also get the expectations of spin components, \(\langle J_z \rangle = -\frac{1}{2}, \langle J_z^2 \rangle = \frac{5}{2}\) and \(\langle J_z^3 \rangle = \frac{1}{2}\), and then we can easily get the spin squeezing parameters, \(\xi_2^3 = \frac{7}{3}\) and \(\xi_2^3 = \frac{3}{2}\). The numerical results for \(\xi_2^3\) are displayed in figure 2, which coincide with the special result.

It is interesting to see that, although \(|\psi_D\rangle\) has no parity, the state is entangled (\(C > 0\)) and spin squeezed according to \(\xi_2^3\) in the entire interval. However, according to the parameter \(\xi_2^3\), the state is not squeezed in the middle region. Therefore, we find that \(\xi_2^3\) is more effective than \(\xi_2^3\) in detecting pairwise entanglement.

6. Conclusion
In conclusion, we have studied the relations between spin squeezing and pairwise entanglement. We have considered two types of spin squeezing parameters \(\xi_2^3\) and \(\xi_2^3\), and the pairwise entanglement is characterized by concurrence \(C\). We find that, for states with exchange symmetry and parity, spin squeezing according to \(\xi_2^3\) is qualitatively equivalent to pairwise entanglement. In detecting pairwise entanglement, the parameter \(\xi_2^3\) is more effective than the parameter \(\xi_2^3\).

It is important to emphasize that the above conclusion can be extended to the states without (even or odd) parity.
For states with properties $\langle J_\alpha \rangle = 0$, $\langle J_\alpha J_\beta \rangle = \langle J_\beta J_\alpha \rangle = 0$ and $\alpha = x, y$, we can have the same conclusion that spin squeezing and pairwise entanglement are qualitatively equivalent. The following superposition of Dicke states are examples:

$$|\psi_D\rangle = \cos \theta |n\rangle^N + e^{i\phi} \sin \theta |n + n'\rangle^N, n = 0, \ldots, N - n', \text{ for all } n' \geq 3.$$  

As we have seen, the parameter $\xi^2_S$ is a key factor in $\xi^2_T$ for our states. The present results imply that spin squeezing has more intimate relations with pairwise entanglement.

**Acknowledgments**

This work is supported by the National Natural Science Foundation of China (NSFC) with grant nos 11025527, 10874151 and 10935010.

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