Finding Euler Tours in One Pass in the W-Streaming Model with $O(n \log(n))$ RAM

Christian Glazik  Jan Schiemann  Anand Srivastav

Department of Computer Science
Kiel University
Christian-Albrechts-Platz 4
24118 Kiel, Germany
{cgl,jasc,asr}@informatik.uni-kiel.de

Abstract: We study the problem of finding an Euler tour in an undirected graph $G$ in the W-Streaming model with $O(n \text{ polylog}(n))$ RAM, where $n$ resp. $m$ is the number of nodes resp. edges of $G$. Our main result is the first one pass W-Streaming algorithm computing an Euler tour of $G$ in the form of an edge successor function with only $O(n \log(n))$ RAM which is optimal for this setting (e.g., Sun and Woodruff (2015)). The previously best-known result in this model is implicitly given by Demetrescu et al. (2010) with the parallel algorithm of Atallah and Vishkin (1984) using $O(m/n)$ passes under the same RAM limitation. For graphs with $\omega(n)$ edges this is non-constant.

Our overall approach is to partition the edges into edge-disjoint cycles and to merge the cycles until a single Euler tour is achieved. Note that in the W-Streaming model such a merging is far from being obvious as the limited RAM allows the processing of only a constant number of cycles at once. This enforces us to merge cycles that partially are no longer present in RAM. Furthermore, the successor of an edge cannot be changed after the edge has left RAM. So, we steadily have to output edges and their designated successors, not knowing the appearance of edges and cycles yet to come. We solve this problem with a special edge swapping technique, for which two certain edges per node are sufficient to merge tours without having all of their edges in RAM. Mathematically, this is controlled by structural results on the space of certain equivalence classes corresponding to cycles and the characterization of associated successor functions. For example, we give conditions under which the swapping of edge successors leads to a merging of equivalence classes. The mathematical methods of our analysis might be of independent interest for other routing problems in streaming models
1 Introduction

For the processing of large graphs, the graph streaming or semi streaming model introduced by Feigenbaum et al. [5] has been studied extensively over the last decade. In this model, a graph with $n$ nodes and $m$ edges is given as a stream of its edges. Random-access memory (RAM, also called internal memory) is restricted to $O(n \text{polylog}(n))$ edges at a time, see, e.g., the survey [6] for a detailed introduction. In consequence, the model cannot be applied to problems where the size of the solution exceeds this amount of memory. In the Euler tour problem, we are looking for a closed trail in an undirected graph such that each edge is visited exactly once. Since the size of an Euler tour is $m$, which might even be $\Theta(n^2)$, we need a relaxation of the model that allows us to store the output separated from the RAM.

1.1 Previous Work on W-Streaming

The W-Streaming model introduced by Demetrescu et al. [4] is a relaxation of the classical streaming model. At each pass, an output stream is written which then becomes the input stream of the next pass. In [4], a trade-off between internal memory and streaming passes is shown for undirected connectivity and the single-source shortest path problem in directed graphs. The One-Pass-Model plays a special role, since in this case the writing to the stream is only for output reasons, because the stream is only processed once. This is particularly interesting regarding problems with solutions which do not fit in RAM. The W-Streaming model originated as a more restrictive alternative to the StrSort model introduced by Aggarwal et al. [7,1].

Finding an Euler tour in trees has been studied in multiple papers (e.g., [3]), but to the best of our knowledge the general Euler tour problem has hardly been considered in a streaming model so far. However, there are some general results for transferring PRAM algorithms to the W-Streaming model. Atallah and Vishkin [2] presented a PRAM algorithm for finding Euler tours, using $O(\log(n))$ time and $n + m$ processors. Transferred to the W-Streaming model with the methods from [3], this algorithm computes an Euler tour in the form of a bijective successor function within $p = O(m \text{polylog}(n)/s)$ passes, where $s$ is the RAM-capacity. For a RAM size of $O(n \text{polylog}(n))$, this translates to $\Omega(m/n)$ passes which for any $m = \omega(n)$ is non-constant. Furthermore, Sun and Woodruff [8] showed that a one-pass streaming algorithm for verifying whether a graph is Eulerian needs $\Omega(n \log(n))$ RAM. This implies that a one pass W-streaming algorithm for finding an Euler tour with less RAM does not exist and therefore justifies our choice of the RAM size.
1.2 Our Contribution

We present the W-Stream algorithm Euler-Tour for finding an Euler tour in a graph in form of a bijective successor function or stating that the graph is not Eulerian, using only one pass and $O(n \log(n))$ bits of RAM. This is not only a significant improvement over previous results, but is in the view of the lower bound of Sun and Woodruff the first optimal algorithm in this setting. Usually, the W-Streaming model is restricted to sub-linear internal memory but in our case the output stream is used solely for storing the solution which needs $\Omega(m)$ memory. As in [2], our algorithm outputs the Euler tour as a successor function that for every edge gives the following edge of the tour. Atallah and Vishkin find edge disjoint tours (in our case cycles) and connect them by pairwise swapping the successor edges of suitable edges. This idea is easy to implement without memory restrictions but the implementation gets distinctly more complicated with limited memory space: We cannot store all cycles in RAM. Therefore, we have to output edges and their successors before finding resp. processing all cycles. Our idea is to keep specific edges of some cycles in RAM along with additional information so that we are able to merge following cycles regardless of their appearance with already processed tours which likely are no longer present in RAM.

We develop a mathematical foundation by partitioning the edges into equivalence classes induced by a given bijective successor function and prove structural properties that allow to iteratively change this function on a designated set of edges so that the modified function is still bijective. Translated to graphs this is a tour merging process. This mathematical approach is quite general and might be useful in other routing scenarios in streaming models.

1.3 Organization of the Article

In Section 2 we give some basic definitions. The main techniques of our algorithm are described in Section 3 in an intuitive manner. Section 4 contains the pseudo code of the algorithm. In the analysis in Section 5 we show the connection of the concepts of Euler tours and successor functions and then show that the required RAM of the algorithm does not exceed $O(n \log(n))$ and that the output actually depicts an Euler tour (Theorem theorem 10).

2 Preliminaries

Let $\mathbb{N} := \{1, 2, \ldots \}$ denote the set of natural numbers. For $n \in \mathbb{N}$ let $[n] := \{1, \ldots, n\}$. In the following, we consider a graph $G = (V, E)$ where $V$ denotes the set of nodes and $E$
the set of (undirected) edges. A trail in \( G \) is a finite sequence \( T = (v_1, \ldots, v_\ell) \) of nodes of \( G \) with \( \{v_i, v_{i+1}\} \in E \) and \( v_i = v_j \) implies \( v_{i+1} \notin \{v_{j-1}, v_{j+1}\} \) for all \( i \in \{1, \ldots, \ell - 1\} \) and \( j \in \{2, \ldots, \ell - 1\} \) with \( i \neq j \). The length of \( T \) is \( \ell - 1 \). The (directed) edge-set of \( T \) is \( E(T) := \{(v_i, v_{i+1}) | i \in [\ell - 1]\} \). We also write \( e \in T \) instead of \( e \in E(T) \). For a directed edge \( e \) we denote by \( e_{(1)} \) its first and by \( e_{(2)} \) its second component. A trail \( T = (v_1, \ldots, v_\ell) \) with \( v_1 = v_\ell \) is called a tour. In tours, we usually do not care about starting point and end point, so we slightly abuse the notation and write \( v_{i-k} \) resp. \( v_{i+k} \) for any \( k \in \mathbb{N} \), identifying \( v_0 := v_\ell \) and \( v_{\ell+1} := v_2 \) and so on. If additionally \( v_i \neq v_j \) holds for all \( i, j \in [\ell - 1], i \neq j \) (and \( \ell \geq 3 \)), we call \( T \) a cycle. An Euler tour of \( G \) is a tour \( T \) with \( E(T) = E \). Since in the streaming model the graph is represented as a set of edges, we often use the edges for the depiction of tours. With \( e_i := \{v_i, v_{i+1}\} \) for all \( i \in [\ell - 1] \), \( T \) can be written as \( T = (e_1, \ldots, e_\ell) \). Here, we also use the slightly abusive index notation. Note that for the tour \( T \) the edges are distinct. For \( i \in [\ell] \), we call \( e_{i+1} \) the successor edge of \( e_i \) in tour \( T \). Our algorithm outputs an Euler tour \( T = (v_1, \ldots, v_{|E|}, v_1) \) in form of a successor function, i.e., for every \( i \in [|E]| \), we output the triple \( (v_i, v_{i+1}, v_{i+2}) \), where \( \{v_{i+1}, v_{i+2}\} \) is the successor edge of \( \{v_i, v_{i+1}\} \) in \( T \).

3 Idea of the Algorithm

As the analysis of our algorithm is quite involved, in this section we try to explain the new algorithmic idea and where the mathematical analysis is required. First we explain how merging of subtours can be accomplished without RAM limitation clarifying why this does not work in W-streaming. Thereafter we explain our merging technique and its locality and RAM efficiency.

3.1 Subtour merging in unrestricted RAM

Recall that an Euler tour will be presented by giving for every edge the corresponding successor edge in the tour. Let \( G = (V, E) \) be an Eulerian graph and \( T, T' \) be edge-disjoint tours in \( G \). The tour induces an orientation of the edges in a canonical way. If \( T \) and \( T' \) have a common node \( v \), it is easy to merge them to a single tour: \( T \) has at least one in-going edge \((u, v)\) with a successor edge \((v, w)\), and \( T' \) has at least one in-going edge \((u', v)\) with a successor edge \((v, w')\). By changing the successor edge of \((u, v)\) from \((v, w)\) to \((v, w')\) and the successor edge of \((u', v)\) to \((v, w)\), we get a tour containing all edges of \( T \cup T' \) (see Figure 1). The same principle can be applied when merging more than two tours at once. When we have a tour \( T \) and tours \( T_1, \ldots, T_k \), \( k \in \mathbb{N} \), such that \( T, T_1, \ldots, T_k \) are pairwise edge-disjoint and for every \( j \in [k] \) there is a common node \( v_j \) of \( T \) and \( T_j \), switching the successor edges of two in-going edges per node \( v_j \) as described above creates a tour containing the edges of \( T \cup T_1 \cup \cdots \cup T_k \).
We can use this method as a simple algorithm for finding an Euler tour:

a) Find a partition of $E$ into edge disjoint cycles.

b) Iteratively pick a cycle $C$ and merge it with all tours encountered so far which have at least one common node with $C$.

Such a merging process certainly converges to a tour covering all nodes, if a subtour obtained by merging some subtours does not decompose later into some subtours again.

If we use a local swapping technique to merge tours, this can very well happen, if swapping is again applied to some other node of the merged tour (see Figure 2). In the RAM model we can keep all tours in RAM and avoid such fatal nodes.

In the W-stream model with $O(n \log n)$ RAM it is far from being obvious how to implement an efficient tour merging for the following reasons.

1. We cannot keep every intermediate tour in RAM, so we have to regularly remove some edges together with their successors from RAM, even if we do not know the edges yet to come. But on the other hand, we have to keep edges in RAM which are essential in later merging steps.
2. Sometimes we have to merge cycles with tours that had already left RAM. Therefore, we must keep track of common nodes and the related edges.

3.2 Subtour merging in limited RAM

Let us assume that we have found say four cycles $C_1, \ldots, C_4$ in that order, all sharing a common node $v$. (see Figure 3). Let $(u_1, v), \ldots, (u_4, v)$ be the respective in-going edges and $(v, w_1), \ldots, (v, w_4)$ be the respective out-going edges. By swapping the successor edges of $(u_1, v)$ and $(u_2, v)$ as explained before, we get a tour containing all edges from $C_1$ and $C_2$. We then merge this tour with $C_3$ swapping the successor edges of $(u_1, v)$ and $(u_3, v)$, and then with $C_4$ by swapping the successors of $(u_1, v)$ and $(u_4, v)$. The successor edges are now as follows:

$$(u_1, v) \rightarrow (v, w_4) \quad (u_2, v) \rightarrow (v, w_1)$$
$$(u_3, v) \rightarrow (v, w_2) \quad (u_4, v) \rightarrow (v, w_3)$$

For $i > 1$ and cycle $C_i$, the successor of the edge $(u_i, v)$ is edge $(v, w_{i-1})$, the out-going edge of $C_{i-1}$. The edge $(u_1, v)$ of the cycle $C_1$ has the out-going edge of the last cycle as its successor edge. The edge $(u_1, v)$ is the first in-going edge of $v$ called the first-in edge of $v$. Let us briefly show how this merging can be implemented in W-streaming. When $C_1$ is kept in RAM, we store the edge $(u_1, v)$, since we don’t know its final successor edge yet. We also keep the edge $(v, w_1)$ in RAM, because it will be the successor edge of $C_2$. We call such an edge the potential successor edge of $v$. We can remove every edge except $(u_1, v)$ together with their respective successor edges in $C_1$, since only the successor edge of $(u_1, v)$ will change over the course of the algorithm. Then iteratively, if we have a cycle $C_i$ for $i > 1$ in RAM, we assign the edge $(v, w_{i-1})$ as successor edge of $(u_i, v)$, replace $(v, w_{i-1})$ by $(v, w_i)$ in RAM as potential successor edge of $v$ for the next cycle and then remove $C_i$ with the respective successor edges from RAM. Finally, as no more cycles with node $v$ occur, we can remove $(u_1, v)$ together with the last successor edge left from RAM (in our case this is $(v, w_4)$).

Now, let us consider the more complicated case, where we wish to merge a cycle $C$ with multiple tours at several nodes. Consider a cycle $C$ and tours $T_1, \ldots, T_j$. Let $v_1, \ldots, v_j$, ...
C be nodes so that \( v_i \) belongs to \( T_i \) and \( C \) for all \( i \). We distinguish between merging at three types of nodes:

1. For the nodes \( v_1, \ldots, v_j \), we use the successor edge swapping.

2. Nodes in \( C \) and in \( T_1 \cup \cdots \cup T_j \setminus \{v_1, \ldots, v_j\} \): as only one successor edge swapping per tour is needed, these additional common nodes are not used, so for every \( v \in T_1 \cup \cdots \cup T_j \setminus \{v_1, \ldots, v_j\} \) the in-going edge \((u, v)\) of \( C \) keeps its successor edge, so nothing happens here.

3. Nodes in \( C \setminus (T_1 \cup \cdots \cup T_j) \). These nodes are visited by the algorithm for the first time. Since we might want to merge \( C \) with future cycles at these nodes, we store for every \( v \in C \setminus (T_1 \cup \cdots \cup T_j) \) the in-going edge \((u, v)\) of \( C \) as first-in edge and the out-going edge \((v, w)\) of \( C \) as potential successor edge.

Note that the very first cycle found by the algorithm consists only of type 3 nodes, so every edge will become a first-in edge.

The challenge in the analysis is on the one hand to choose sufficiently many nodes where merging is done in a *simultaneous* way in order to stay within the one-pass complexity, and on the other hand to ensure that simultaneous merging enlarges and never decomposes subtours. Here we need two lemmas. Lemma \[8\] is used to show that merging of equivalence classes of Euler subtours leads to equivalence classes of a new subtour, thus subtour merging is invariant w.r.t. the equivalence class relation. This lemma is needed to prove Lemma \[9\] which shows that the sequence of successor functions iteratively built by refining the equivalence relation are indeed Eulerian subtours. It also gives a criterion for belongingness of edges to such a subtour. This criterion is finally used to show that the successor function returned by our algorithm is equal to the successor function associated to the last and most refined equivalence relation, hence is an Eulerian tour for the graph.

For the readers convenience we give a high level description of our algorithm. A detailed description in pseudo code together with an outline of the analysis and the proof of the main theorem will follow in the next sections. We denote the set of first-in edges by \( F \).

1. Iteratively:
   
   1. Read edges from the input stream until the edges in RAM contain a cycle \( C \).
   2. If a node \( v \) of \( C \) is visited for the first time,
      
      a) store the in-going edge \((u, v)\) of \( C \) in \( F \) (we will process these \( \leq n \) edges in step 2),
b) remember the out-going edge \((v, w)\) as potential successor edge of \(v\).

3. Every node \(v\) that has already been visited, has thereby been assigned to a unique tour \(T\) with \(v \in C \cap T\). For each tour that shares a node with \(C\), choose exactly one common node.

4. For each node \(v\) chosen in step 3, “swap the successors”. That means, we write the in-going edge \(e\) to the stream and take the recent potential successor edge of \(v\) as successor edge for \(e\). Then, save the out-going edge as new potential successor edge of \(v\).

5. For each edge that has not been stored in \(F\) (step 2.) or written to the stream (step 4.) so far, write this edge to the stream and take as successor the following edge in \(C\).

6. All tours with common nodes together with all newly visited nodes are now assigned to a single tour.

2. After the end of the input stream is reached, all edges have either been written to the stream or stored in \(F\). For every edge \((u, v) \in F\), write it to the stream and take as its successor the potential successor edge of \(v\).

An example of how the algorithm works can be found in the appendix.

4 The Algorithm

To enable a clear and structured analysis, in this section we present the pseudo-code for our algorithm. For a better understanding it is split up into several procedures that correspond to the steps from our high level description in Section 3. Note that these procedures are not independent algorithms, since they access variables from the main algorithm. The output is an Euler tour on \(G\), given in the form of a successor function \(\delta^*\). To be more precise, the output is a stream of triples \((v_1, v_2, s)\) with \(v_1, v_2, s \in V\) and \(\{v_1, v_2\} \in E\). Each of these triples represents the information \(\delta^*((v_1, v_2)) = (v_2, s)\). If a triple \((v_1, v_2, s)\) is written to the stream, we say that the edge \((v_2, s)\) is marked as successor of the edge \((v_1, v_2)\). For every node we store two important values during the algorithm: The value \(t(v)\) that gives the tour \(v\) is assigned to at the moment and the value \(j(v)\) that indicates that \((v, j(v))\) is the potential successor edge of \(v\).
Algorithm 1: Euler-Tour

input: Undirected graph $G = (V, E)$, edge by edge on a stream $S$
output: Euler tour on $G$, i.e. a successor function $\delta^*$, if there is one

1 $c := 0; F := \emptyset; E_{\text{int}} := \emptyset; \text{ for every } v \in V: j(v) := 0, t(v) := 0;$
2 for every edge $e$ on $S$ do
3 \hspace{1em} $E_{\text{int}} := E_{\text{int}} \cup \{e\};$
4 \hspace{1em} if $G_{\text{int}} = (V, E_{\text{int}})$ contains a cycle $C$ then
5 \hspace{2em} node Merge-Cycle $(C);$ 
6 if $E_{\text{int}} = \emptyset$ then
7 \hspace{2em} ERROR: At least one node with odd degree;
8 if there exist $u, v$ with $t(u) \neq t(v) \neq 0$ then
9 \hspace{2em} ERROR: Graph not connected;
10 Write-$F;$

The algorithm searches the stream for cycles (Step 1 in our high level description) and whenever a cycle is found, we will run the procedure Merge-Cycle on this cycle. The procedure Merge-Cycle contains the steps 2 to 6, and Write-$F$ corresponds to step 2.

Procedure Merge-Cycle

input: Ordered cycle $C = (v_1, \ldots, v_k)$
1 NEW-NODES;
2 CONSTRUCT-J-M;
3 MERGE;
4 WRITE;
5 UPDATE;
6 for every edge $e \in C$ do
7 \hspace{1em} delete $e$ from $E_{\text{int}}$

The procedure NEW-NODES implements step 2. If a node $v$ is processed the very first time by the algorithm, this is indicated by $t(v) = 0$. If this is the case, we store the corresponding in-going edge in the set $F$ and store the next node on the cycle in $j(v)$ (this is, the edge $(v, v_{i+1})$ becomes the potential successor of $v$).

Procedure NEW-NODES

1 for $i = 1, \ldots, k$ do
2 \hspace{1em} if $t(v_i) = 0$ then
3 \hspace{2em} $j(v_i) = v_{i+1};$
4 \hspace{2em} $F = F \cup \{(v_{i-1}, v_i)\};$

The procedure CONSTRUCT-J-M is a realization of step 3. For every value $j \neq 0$, we pick exactly one node $v$ with $t(v) = j$ if there is one. These nodes are stored in $J$, their
values are stored in $M$. The nodes in $J$ are the “chosen” nodes we want to use for merging tours. If two nodes already have the same value in $t$, this means they are already part of the same tour (see Lemma 9) and we want to avoid using both of them for merging.

**Procedure** Construct-J-M

1. $M = \emptyset$; $J = \emptyset$;
2. for $j = 1, \ldots, |V|$ do
   3. if exists $i \in [k]$ with $t(v_i) = j$ then
   4. add exactly one $v_i$ with $t(v_i) = j$ to the set $J$;
   5. $M = M \cup \{j\}$;

In the following procedure **Merge**, we use the nodes from $J$ to merge all tours that share a node in the cycle $C$ by edge-swapping (step 4).

**Procedure** Merge

1. for each $v_i \in J$ do
2. write ($v_{i-1}, v_i, j(v_i)$) to the stream;
3. $j(v_i) = v_{i+1}$;

In the procedure **Write**, we take care of all the edges that have not been stored in $F$ and have not been written to the stream in the procedure **Merge** (Step 5).

**Procedure** Write

1. for each edge $(v_i, v_{i+1}) \in C$ that has not been written to the stream or added to $F$ do
2. write ($v_i, v_{i+1}, v_{i+2}$) to the stream;

In the procedure **Update** we update the $t$-values to implement step 6. After this step we can be sure that any two nodes $v, v' \in V$ with $t(v) = t(v') \neq 0$ belong to the same tour, whereas $t(v) = 0$ means that $v$ has not been processed so far.

**Procedure** Update

1. $a := 0$;
2. if $M = \emptyset$ then
3. $c := c + 1$;
4. $a := c$;
5. else
6. $a := \min(M)$;
7. for each $v \in V$ do
8. if $t(v) \in M$ then
9. $t(v) := a$;
10. for $i = 1, \ldots, k$ do
11. $t(v_i) = s$;
Finally, in the procedure Write-F (step 2.), the first-in edges that have been stored in $F$ during the algorithm are written to the stream with proper successors.

**Procedure Write-F**

1. for each edge $(u, v) \in F$ do
2. write $(u, v, j(v))$ to the stream;

## 5 Analysis

### 5.1 Subtour representation by equivalence classes

In this subsection we present some basic definitions and results that allow us to transfer the problem of tour merging in a graph to the language of equivalence relations. This will allow an elegant and clear analysis of our algorithm in Section 5.

**Definition 1.**

(i) Let $G = (V, E)$ be an undirected graph. An orientation of the edges of $G$ is a function $R : E \rightarrow V^2$ such that for every edge $\{u, v\} \in E$ either $R(\{u, v\}) = (u, v)$ or $R(\{u, v\}) = (v, u)$. So $R(G) := (V, R(E))$ is a directed graph.

(ii) Let $\vec{G} = (V, \vec{E})$ be a directed graph. A successor function on $\vec{G}$ is a function $\delta : \vec{E} \rightarrow \vec{E}$ with $\delta(e)(1) = e(2)$ for all $e \in \vec{E}$.

(iii) Let $\vec{G} = (V, \vec{E})$ be a directed graph with successor function $\delta$. We define the relation $\equiv_\delta$ on $\vec{E}$ by $e \equiv_\delta e' :\Leftrightarrow \exists k \in \mathbb{N} : \delta^k(e) = e'$, where $\delta^k$ denotes the $k$-wise composition of $\delta$.

So $e \equiv_\delta e'$ means that $e'$ can be reached from $e$ by iteratively applying $\delta$.

**Lemma 1.** Let $\delta$ be a bijective successor function on a directed graph $\vec{G} = (V, \vec{E})$. Then $\equiv_\delta$ is an equivalence relation on $\vec{E}$.

**Proof.** Reflexivity: Let $e \in \vec{E}$. Since $\vec{E}$ is finite, there exists $k \in \mathbb{N}$ with the following property: There exists $k' \in \mathbb{N}$ with $k' < k$ and $\delta^k(e) = \delta^{k'}(e)$. Let $k$ be minimal with this property. Since $\delta$ is injective, it follows that $\delta^{k-1}(e) = \delta^{k'-1}(e)$ and the minimality of $k$ enforces that $k' - 1 \notin \mathbb{N}$. So $k' = 1$, therefore $\delta^k(e) = \delta(e)$ and by injectivity of $\delta$ we have $\delta^{k-1}(e) = e$.

Symmetry: Let $e, e' \in \vec{E}$ with $e \equiv_\delta e'$. Then there exists a minimal $k \in \mathbb{N}$ with $\delta^k(e) = e'$. As shown above, there also exists a $k' \in \mathbb{N}$ with $\delta^{k'}(e) = e$. Because $k$ is minimal, we have $k < k'$. It follows that $\delta^{k'-k}(e') = \delta^{k'}(e) = e$. 

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Transitivity: Let \(e, e', e'' \in \tilde{E}\) with \(e \equiv_\delta e'\) and \(e' \equiv_\delta e''\). Then there exist \(k_1, k_2 \in \mathbb{N}\) with \(\delta^{k_1}(e) = e'\) and \(\delta^{k_2}(e') = e''.\) So we have \(\delta^{k_1 + k_2}(e) = e''\). \(\square\)

We denote the equivalence class of an edge \(e \in \tilde{E}\) w.r.t. \(\equiv_\delta\) by \([e]_\delta\).

The following lemma is necessary to show that the equivalence classes of \(\delta\) always form tours on \(\tilde{G}\).

**Lemma 2.** Let \(\tilde{G} = (V, \tilde{E})\) be a directed graph with bijective successor function \(\delta\) and the related equivalence relation \(\equiv_\delta\). Then we have:

(i) Let \(e \in \tilde{E}\) and \(k_1, k_2 \in \mathbb{N}_0\) with \(k_1 \neq k_2\) and \(\delta^{k_1}(e) = \delta^{k_2}(e)\). Then \(|k_1 - k_2| \geq |[e]_\delta|\).

(ii) For any \(e \in \tilde{E}\) we have \(\delta^{|[e]_\delta|}(e) = e\).

Proof. (i): Assume for a moment that there exist \(e \in \tilde{E}\) and \(k_1, k_2 \in \mathbb{N}\) with \(\delta^{k_1}(e) = \delta^{k_2}(e)\) and \(0 < |k_1 - k_2| < |[e]_\delta|\). Without loss of generality let \(k_1 > k_2\). We have \(\delta^{k_1 - k_2}(\delta^{k_2}(e)) = \delta^{k_1}(e) = \delta^{k_2}(e)\) and via induction for every \(s \in \mathbb{N}\), we get \(\delta^{s(k_1 - k_2)}(\delta^{k_2}(e)) = \delta^{k_2}(e)\). For the set \(M := \{\delta^k(e) | k_2 \leq k < k_1\}\), we have \(|M| \leq k_1 - k_2 < |[e]_\delta|\).

But on the other hand, we also have \([e]_\delta \subseteq M\): Let \(e' \in [e]_\delta = [\delta^{k_2}(e)]_\delta\). Let \(n \in \mathbb{N}\) with \(e' = \delta^n(\delta^{k_2}(e))\). Then there exist unique \(s, r \in \mathbb{N}_0\) with \(0 \leq r < k_1 - k_2\) and \(n = s(k_1 - k_2) + r\). So

\[e' = \delta^n(\delta^{k_2}(e)) = \delta^r(\delta^{s(k_1 - k_2)}(\delta^{k_2}(e))) = \delta^r(\delta^{k_2}(e)) = \delta^{k_2 + r}(e) \in M.\]

Now we have \(|M| \leq k_1 - k_2 < |[e]_\delta| \leq |M|\), a contradiction.

(ii): Assume that there exists \(e \in \tilde{E}\) with \(\delta^{|[e]_\delta|}(e) = e' \neq e\). Define \(M := \{\delta^k(e) | 1 \leq k \leq |[e]_\delta|\}\). Clearly, \(M \subseteq [e]_\delta\).

Case 1: \(e \in M\). Then \(\delta^0(e) = e = \delta^k(e)\) for some \(k\) with \(1 \leq k < |[e]_\delta|\), in contradiction to (i).

Case 2: \(e \notin M\). Then \(|M| < |[e]_\delta|\). By pigeon hole principle, there exist \(k_1, k_2 \leq |[e]_\delta|\) with \(\delta^{k_1}(e) = \delta^{k_2}(e)\) in contradiction to (i). \(\square\)

**Theorem 3.** Let \(\tilde{G} = (V, \tilde{E})\) be a directed graph with bijective successor function \(\delta\) such that \(e \equiv_\delta e'\) for all \(e, e' \in \tilde{E}\). Then \(\delta\) determines an Euler tour on \(\tilde{G}\) in the following sense: For every \(e \in \tilde{E}\) the sequence \((e(1), \delta(e)(1), \ldots, \delta^{|E|}(e)(1))\) is an Euler tour on \(\tilde{G}\).
Proof. Let $e \in \tilde{E}$. Note that $[e]_{k} = \tilde{E}$. The sequence $(e_{(1)}, \delta(e)_{(1)}, \ldots, \delta|\tilde{E}|(e)_{(1)})$ consists of $|\tilde{E}|$ edges, namely $e, \delta(e), \ldots, \delta|\tilde{E}|-1(e)$. These edges are pairwise distinct: Otherwise, we would have $\delta^{k1}(e) = \delta^{k2}(e)$ for some $k_{1}, k_{2} \in \{0, \ldots, |\tilde{E}|-1\}$. Hence, $|k_{1} - k_{2}| < |\tilde{E}|$ in contradiction to Lemma 2 (i). So the sequence is a trail. By applying Lemma 2 (ii), we get $e = \delta[e]_{k}(e) = \delta|\tilde{E}|(e)$, thus the trail is a tour on $\tilde{G}$ and since it has length $|\tilde{E}|$, it is an Euler tour on $\tilde{G}$. \hfill \Box

Before we start with a detailed memory- and correctness analysis, we show that at the end of the algorithm, every edge $\{u, v\} \in E$ has been written to the output stream exactly once, either in the form $(u, v)$ or in the form $(v, u)$. We also show that $|E_{\text{int}}| \leq n$ all the time.

Lemma 4. (i) After each processing of an edge (lines 2 to 5 in Euler-Tour) in the algorithm, the graph $G_{\text{int}} = (V, E_{\text{int}})$ is cycle-free so $|E_{\text{int}}| \leq n$. If all nodes from $V$ have even degree in $G$, after completion of Euler-Tour, $E_{\text{int}} = \emptyset$.

(ii) If all nodes from $V$ have even degree in $G$, after completion of Euler-Tour every edge $\{u, v\} \in E$ has been written to the stream either in the form $(u, v, s)$ or in the form $(v, u, s)$ for some $s \in V$.

Proof. We start by proving the first part of (i) via induction over the number of already processed edges. If there are no edges processed so far, then $E_{\text{int}} = \emptyset$, so $G_{\text{int}}$ is cycle-free. Now let $k \in |E| \cup \{0\}$, let $G_{k}, G_{k+1}$ denote $G_{\text{int}}$ after $k$ resp. $k + 1$ edges have been processed and let $G_{k}$ be cycle-free. Let $e$ denote the $(k + 1)$-th processed edge. When $e$ is added to $G_{\text{int}}$, it may produce a cycle $C$. If $e$ does not produce a cycle, then $G_{k+1} = G_{k} \cup \{e\}$ is cycle-free and we are done. If $e$ produces a cycle $C$, then (at lines 6, 7 in Merge-Cycle) $C$ is deleted from $E_{\text{int}}$ and because $e \in C$, we get $G_{k+1} = (G_{k} \cup \{e\}) \setminus C \subseteq G_{k}$ and we are done by the induction hypothesis.

Now assume for a moment that $E_{\text{int}} \neq \emptyset$ at the end of Euler-Tour. We know that $G_{\text{int}}$ is cycle-free at this time, so $G_{\text{int}}$ contains a node with odd degree in $G_{\text{int}}$. Because we always delete whole cycles, the degree of this node in $G$ has to be odd as well, but then $G$ is not an Eulerian graph. In this case we might output a message that $G$ does not contain an Euler tour.

About (ii). During Euler-Tour, every edge from $E$ is added to $E_{\text{int}}$ at some point of time and there is only one way for an edge to be deleted from $E_{\text{int}}$ again, namely in line 7 of Merge-Cycle. At that point of time, the edge has either been written to the stream in Merge or Write (in which case we are done) or it has been added to $F$ in New-Nodes. In that case it is written to the stream in Write-F. Because, according to (i), $E_{\text{int}} = \emptyset$ at the end of Euler-Tour, at this point of time, every edge must have been written to the stream in exactly one of the two ways. \hfill \Box
The idea of (i) is that every time a cycle occurs in $E_{\text{int}}$, we delete this cycle so we assure that $E_{\text{int}}$ becomes cycle-free again (since we only add one edge at a time).

5.2 Memory Requirement

For the memory estimation, we have to consider the variables $j(v), t(v)$ for all $v \in V$, the sets $F, E_{\text{int}}, J, M$, and the counter $c$. By Lemma 4(i), $|E_{\text{int}}| \leq n$ and, with some straightforward considerations, we can estimate the memory requirement for the other parameters leading to the following lemma.

Lemma 5. Algorithm Euler-Tour needs at most $\mathcal{O}(n \log n)$ bits of RAM.

Proof. We consider the different parameters.

About $c$: We show that $c \leq n/3$ at every time, so $\log n$ bits suffice to store $c$. $c$ is initiated with $0$ and changed in the procedure UPDATE if and only if $M = \emptyset$ at that point of time. This only happens if for every node $v$ of the considered cycle, we have $t(v) = 0$, which means that none of the cycle nodes was considered before. This case can occur at most $n/3$ times during the algorithm, because there can be no more than $n/3$ node disjoint cycles in $G$, so $c \leq n/3$.

About $j(v)$: In this variable we store the label of a node, so for fixed $v$, $\log n$ bits suffice and altogether $n \log n$ bits suffice.

About $t(v)$: We prove that for any $v \in V$ $t(v) \leq n$ at any time: Assume for a moment that this is not the case. Consider the first point of time $T$ in which $t(v)$ is set to a value $> n$ for some $v \in V$. $t(v)$ is only changed in the procedure UPDATE, line 9 or 11. In both cases the value is set to $r$ which is either $c$ (line 4) or $\min(M)$ (line 6). We already showed $c < n$. Hence, by our assumption, $\min(M) > n$ at that point of time. But this implies that at the time of the construction of $M$, there already existed a node $u \in V$ with $t(u) > n$ in contradiction to the choice of $T$.

About $E_{\text{int}}, F, J, M$: Because a single element of each of these sets can be stored in $\log n$ bits, it suffices to show that the cardinalities of these sets do not exceed $n$. For $E_{\text{int}}$, this is shown in Lemma 4. For $J$ and $M$, it follows directly from the construction (see Procedure CONSTRUCT-J-M). In the set $F$, for every node we collect the first edge that leads into this node (see Procedure NEW-NODES, lines 2 and 4), so clearly $|F| \leq n$.  

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5.3 Correctness

In this subsection, we prove that \( \delta^* \) determines an Euler tour on \( G \), provided that \( G \) is Eulerian (Theorem 10). This is done with the help of Theorem 3, where bijectivity of \( \delta^* \) and the condition that \( \delta^* \) induces only one equivalence class is required. In the following, we show that these assumptions are true for \( \delta^* \) by generating a sequence of bijective successor functions \( \delta_0^*, \ldots, \delta_N^* \) such that \( \delta_0^* \) is bijective, \( \delta_N^* = \delta^* \) and \( \delta_{i+1}^* \) emerges from \( \delta_i^* \) by swapping of edge successors.

Lemma 4 (ii) induces an orientation on \( E \) which we call \( R^* \): For all \( \{u, v\} \in E \), we define

\[
R^*([u, v]) := (u, v) \text{ if } (u, v) \text{ has been written to the output stream}.
\]

From now on, let \( C_1, \ldots, C_N \) denote the cycles found in \( E_{\text{int}} \) by the algorithm in chronological order. For \( k \in \{0, \ldots, N\} \) and a variable \( x \), we denote by \( x_k \) the value of \( x \) after the \( k \)-th call of Merge-Cycle. For \( k = 0 \), this means the initial value of \( x \).

**Definition 2.** For each \( i \in [N] \), let \( C_i = (v_1^{(i)}, \ldots, v_{\ell_i}^{(i)}) \) be the form of the cycle given to Merge-Cycle. Define \( \delta^c_i : E(C_i) \rightarrow E(C_i) \) by \( \delta^c_i(v_j^{(i)}, v_{j+1}^{(i)}) := (v_{j+1}^{(i)}, v_{j+2}^{(i)}) \) for every \( j \in [\ell_i] \) and let \( \delta^c : R^*(E) \rightarrow R^*(E) \) denote the unique successor function with \( \delta^c|_{E(C_i)} = \delta^c_i \) for all \( i \in [N] \).

So \( \delta^c \) is the natural successor function induced by the cycles \( C_1, \ldots, C_N \).

**Lemma 6.** The successor function \( \delta^c \) is bijective and for any two edges \( e, e' \) we have \( e \equiv_{\delta^c} e' \Leftrightarrow \exists i \in [N] : e, e' \in C_i \).

**Proof.** We first show that \( \delta^c \) is surjective: Let \( e \in R^*(E) \). Then there exist \( k \in [N] \) and \( i \in \mathbb{N} \) such that \( C_k = (v_1, \ldots, v_{\ell_k}) \) and \( e = (v_i, v_{i+1}) \). Then \( \delta(v_{i-1}, v_i) = (v_i, v_{i+1}) = e \). Because \( R^*(E) \) is finite, \( \delta^c \) is bijective.

Now let \( e, e' \in R^*(E) \) with \( e \equiv_{\delta^c} e' \). Let \( i \in [N] \) such that \( e \in C_i \). Since \( \delta^c(C_i) = C_i \), it follows that \( e' \in C_i \).

Now let \( e, e' \in C_i \) for some \( i \in [N] \), for instance \( C_i = (v_1, \ldots, v_{\ell_i}) \) and \( j, k \in [\ell_i] \) with \( e = (v_j, v_{j+1}) \) and \( e' = (v_k, v_{k+1}) \). W.l.o.g. let \( j < k \) and set \( r := k - j \). Then \( (\delta^c)^r(e) = e' \), so \( e \equiv_{\delta^c} e' \). \( \square \)

Let now \( k \in \{0, \ldots, N\} \). We consider the point of time right after the \( k \)-th iteration of Merge-Cycle (for \( k = 0 \) this means the very beginning of the algorithm). We call edges
from $\bigcup_{i=1}^{k} E(C_i)$ processed edges, since those edges have already been loaded into and then deleted from $E_{\text{int}}$. All processed edges can be divided into two types:

- **Type A**: The edge has been written on the stream with a dedicated successor.
- **Type B**: The edge has been added to $F$.

These are the only possible cases for processed edges because an edge is deleted from $E_{\text{int}}$ is either written to the stream or added to $F$. This leads to the following definition.

**Definition 3.** For every $k \in \{0, \ldots, N\}$ define the function $\delta_k : \bigcup_{i=1}^{k} E(C_i) \to \bigcup_{i=1}^{k} E(C_i)$ by

$$(u, v) \mapsto \begin{cases} e' & \text{if } (u, v) \text{ is of type A with successor } e' \\ (v, j_k(v)) & \text{if } (u, v) \text{ is of type B} \end{cases}$$

and define $\delta_k^c := \begin{cases} \delta_k & \text{on } \bigcup_{i=1}^{k} E(C_i) \\ \delta^c & \text{on } \bigcup_{i=k+1}^{N} E(C_i). \end{cases}$

Note that $\delta_0^c = \delta^c$ and $\delta_N^c = \delta^*.$

**Lemma 7.** Let $k, \ell \in \{0, \ldots, N\}$ with $k < \ell$. Then for any $v, v' \in V$, $e \in R^*(E)$, we have

(i) If $t_k(v) = t_k(v') \neq 0$, then $t_\ell(v) = t_\ell(v').$

(ii) If $e \in C_\ell$, then $[e]_{\delta_k^c} = [e]_{\delta^c}.$

**Proof.** About (i). Let $v, v' \in V$ with $t_k(v) = t_k(v') \neq 0$. Assume for a moment that $t_\ell(v) \neq t_\ell(v')$. Then there exists $k' \leq k' < \ell$ such that $t_{k'}(v) = t_{k'}(v')$ and $t_{k'+1}(v') \neq t_{k'+1}(v)$. Furthermore, $t_\ell(v) \neq 0$, because $t_k(v) \neq 0$ and the value $t(v)$ is never set to 0 after its initiation. We take a closer look at the $(k' + 1)$-th call of Merge-Cycle. If for a node its $t$-value is changed in this call, it is set to $a_{k'+1}$ (line 9 or 11 in Update), so we may assume that $t_{k'+1}(v) = a_{k'+1} \neq t_{k'+1}(v')$. But this implies that $t_{k'}(v) \in M$ or $v \in C_{k' + 1}$, in which case also $t_{k'}(v) \in M$ (since $t_k(v) \neq 0$). But then, $t_{k'}(v') = t_{k'}(v) \in M$ and therefore $t_{k'+1}(v') = a_{k'+1} = t_{k'+1}(v)$, in contradiction to our assumption.

About (ii). Let $e \in C_\ell$. With Lemma 6, we get $[e]_{\delta_k^c} = E(C_\ell)$. Since $\ell > k$, $\delta_k^c(e') = \delta^c(e')$ for any $e' \in C_\ell$. Hence, $\delta_k^c(e) = \delta^c(e)$ and using $\delta^c(e) \in C_\ell$, by induction $(\delta_k^c)^j(e) = (\delta^c)^j(e)$ for any $j > 1$, which proves the claim. \qed
The following two lemmata form the technical foundation of our analysis. In Lemma 8 we repeat in a formal way the basic idea of tour-merging given in Section 3. It is needed for the proof of Lemma 9.

**Lemma 8.** Let $\tilde{G} = (V, \tilde{E})$ be a directed graph with bijective successor function $\delta$ and the related equivalence relation $\equiv_\delta$. Let $r \in \mathbb{N}$ and $e_1, \ldots, e_r \in \tilde{E}$ with $e_i \equiv_\delta e_j$ for every $i, j \in [r]$. Let $e'_1, \ldots, e'_r \in \tilde{E}$ with $e'_i \not\equiv_\delta e'_j$ and $e_i \not\equiv_\delta e'_i$ for every $i, j \in [r]$. Let $\delta'$ be a successor function on $\tilde{G}$ with $\delta'(e) = \delta(e)$ for every $e \in \tilde{E} \setminus \{e_1, \ldots, e_r, e'_1, \ldots, e'_r\}$ and $\delta'(e_i) = \delta(e'_i)$ and $\delta'(e'_i) = \delta(e_i)$ for any $i \in [r]$. Then, $\delta'$ is bijective and

\[
[e_1]_{\delta'} = \bigcup_{i=1}^{r} [e'_i]_{\delta} \cup [e_1]_{\delta} \quad \text{(P1)}
\]

\[
[e]_{\delta'} = [e]_{\delta} \text{ for any } e \in \tilde{E} \setminus [e_1]_{\delta'} \quad \text{(P2)}
\]

**Proof.** Via induction over $r$. First of all notice that because of the definition of $\delta'$ and because $\delta$ is bijective, $\delta'$ is bijective as well. For $r = 1$ to shorten notation, we write $e$ and $e'$ instead of $e_1$ and $e'_1$. We first show

\[
[e]_{\delta'} \subseteq [e]_{\delta} \cup [e']_{\delta} \quad \text{(1)}
\]

First we show that for any $e'' \in [e]_{\delta} \cup [e']_{\delta}$, we have $\delta'(e'') \in [e]_{\delta} \cup [e']_{\delta}$. Let $e'' \in [e]_{\delta} \cup [e']_{\delta}$. Then there exists $k \in \mathbb{N}$ such that $e'' = \delta^k(e)$ or $e'' = \delta^k(e')$. If $e'' \in \{e, e'\}$, then $\delta'(e'') = \delta(e)$ and $\delta'(e'') = \delta(e')$ and otherwise $\delta'(e'') = \delta(e''') = \delta(k+1)(e)$ or $\delta'(e'') = \delta(k+1)(e')$, respectively. So in each case we have $\delta'(e'') \in [e]_{\delta} \cup [e']_{\delta}$. Since $e \in [e]_{\delta} \cup [e']_{\delta}$, it follows by induction on $n$ that $(\delta')^n(e) \in [e]_{\delta} \cup [e']_{\delta}$ for any $n \in \mathbb{N}$, so $[e]_{\delta'} \subseteq [e]_{\delta} \cup [e']_{\delta}$.

Next, we show

\[
[e]_{\delta} \cup [e']_{\delta} \subseteq [e']_{\delta'} \quad \text{(2)}
\]

Let $e'' \in [e']_{\delta}$. Then there exists $k \in \{1, \ldots, |[e']_{\delta}|\}$ with $e'' = \delta^k(e')$. Since $e \not\in [e']_{\delta}$ and $\delta^k(e') \not\equiv_\delta e'$ for all $\ell \in \{1, \ldots, k-1\}$ (follows from Lemma 2(i)), we have

\[
\delta^k(e') = \delta(\delta^{k-1}(e')) = \delta'(\delta^{k-1}(e')) = \delta'(\delta^k(\delta^{k-2}(e'))) = \cdots = (\delta')^{k-1}(\delta(e')).
\]

Hence $e'' = \delta^k(e') = (\delta')^{k-1}(\delta(e')) = (\delta')^{k-1}(\delta'(e)) = (\delta')^k(e) \in [e]_{\delta'}$. So we have

\[
[e']_{\delta} \subseteq [e]_{\delta'} \quad \text{(3)}
\]

and analogously we get

\[
[e]_{\delta} \subseteq [e']_{\delta'} \quad \text{(4)}
\]
Because $\delta(e') \in [e]_{\delta} \subseteq [e]_{\delta'}$, we have $[\delta(e')]_{\delta'} = [e]_{\delta'}$ and it follows that

$$[e]_{\delta'} = [\delta(e')]_{\delta'} = [\delta'(e)]_{\delta} = [e']_{\delta}. \quad (5)$$

Combining (3), (4), and (5), we proved (2). With (1), (2), and (5), we have

$$[e]_{\delta'} \subseteq [e]_{\delta} \cup [e']_{\delta} \subseteq [e']_{\delta'} = [e]_{\delta'},$$

so property (P1) is proven. For (P2), note that $\delta^k(e''_{i}) = (\delta')^k(e'')$ for any $e'' \neq \delta e$, $e'' \neq \delta e'$ and any $k \in \mathbb{N}$.

**Induction step:** Now let $r \in \mathbb{N}$ and let the claim be true for all $k \leq r \in \mathbb{N}$. Let $e_1, \ldots, e_{r+1} \in \vec{E}$ with $e_i \equiv \delta e_j$ for every $i, j \in [r+1]$. Let $e'_1, \ldots, e'_{r+1} \in \vec{E}$ with $e'_i \neq \delta e'_j$ and $e'_i \neq \delta e_i$ for every $i \neq j \in [r+1]$. Let $\delta'$ be a successor function on $\vec{G}$ with $\delta'(e) = \delta(e)$ for every $e \in \vec{E} \setminus \{e_1, e_{r+1}, e'_1, \ldots, e'_{r+1}\}$ and $\delta'(e_i) = \delta(e'_i)$ and $\delta'(e'_i) = \delta(e_i)$ for every $i \in [r+1]$. We define a successor function $\delta_r$ for $\vec{G}$ by

$$\delta_r := \begin{cases} 
\delta' \text{ on } \vec{E} \setminus \{e_{r+1}, e'_{r+1}\} \\
\delta \text{ on } \{e_{r+1}, e'_{r+1}\}.
\end{cases}$$

With the induction hypothesis applied to $\delta$ and $\delta_r$, we get by (P1)

$$[e_1]_{\delta_r} = \bigcup_{i=1}^{r} [e'_i]_{\delta} \cup [e_1]_{\delta} \quad (6)$$

and by (P2)

$$[e'_r]_{\delta_r} = [e'_{r+1}]_{\delta}. \quad (7)$$

Now we apply the induction hypothesis to $\delta_r$ and $\delta'$ as follows: We take $\delta_r$ instead of $\delta$, $\delta'$ remains, $r = 1$, $e_1$ resp. $e'_1$ are replaced by $e_{r+1}$ resp. $e'_{r+1}$. This gives

$$[e_{r+1}]_{\delta_r} = [e'_{r+1}]_{\delta_r} \cup [e_{r+1}]_{\delta_r}. \quad (8)$$

Since $e_1 \equiv \delta e_{r+1}$, we get with (3)

$$e_{r+1} \in [e_{r+1}]_{\delta} = [e_1]_{\delta} \subseteq [e_1]_{\delta_r}$$

which implies

$$[e_{r+1}]_{\delta_r} = [e_1]_{\delta_r} \quad (9)$$
Summarizing, we have
\[ [e_{r+1}]_{\delta'} = [e_{r+1}]_{\delta_r} \cup [e_1]_{\delta_r} \]
\[ [e_{r+1}]_{\delta_r} \cup [e_1]_{\delta_m} \]
\[ [e_{r+1}]_{\delta_r} \cup \left( \bigcup_{i=1}^{r} [e'_i]_{\delta} \cup [e_1]_{\delta} \right) \]
\[ [e_{r+1}]_{\delta_r} \cup \left( \bigcup_{i=1}^{r+1} [e'_i]_{\delta} \cup [e_1]_{\delta} \right) \]
\[ = \bigcup_{i=1}^{r+1} [e'_i]_{\delta} \cup [e_1]_{\delta}. \]  
(11)

So (P1) is proved, if \([e_{r+1}]_{\delta'} = [e_1]_{\delta'}\). By (11) \([e_1]_{\delta} \subseteq [e_{r+1}]_{\delta'}\), so \([e_1]_{\delta} \in [e_{r+1}]_{\delta'}\) and hence
\[ [e_{r+1}]_{\delta'} = [e_1]_{\delta'}. \]  
(12)

For the proof of (P2), let \(e \in \bar{E} \setminus [e_1]_{\delta'}\). Since \(e \notin [e_1]_{\delta'}\), by (10) and (12) \(e \notin [e_1]_{\delta_r}\). Applying (P2) of the induction hypothesis to \(\delta\) and \(\delta'\), gives us \([e]_{\delta_r} = [e]_{\delta}\). We know \([e_{r+1}]_{\delta'} = [e_1]_{\delta'}\), so \(e \notin [e_{r+1}]_{\delta'}\). As above, we apply the induction hypothesis to \(\delta_r\) and \(\delta'\) and get \([e]_{\delta'} = [e'_1]_{\delta_r}\). Altogether \([e]_{\delta'} = [e]_{\delta_r} = [e]_{\delta}\). \(\square\)

Note that \(\delta'\) emerges from \(\delta\) by swapping of successors as explained in the beginning of Section 3. The restriction \(e'_i \neq e'_j\) reflects the fact that we have to choose exactly one common node per tour for merging, as already explained in Section 3, see Figure 2.

Lemma 9. Let \(k \in \{0, \ldots, N\}\). Then, \(\delta^*_k\) is bijective and for any \((u, v), (u', v')\) \(\in R^*(E)\), we have

(i) If \((u, v), (u', v')\) are processed edges, then \((u, v) \equiv_{\delta^*_k} (u', v') \iff t_k(u) = t_k(u')\).

(ii) If \((u, v)\) is a processed edge, then \(t_k(u) = t_k(v)\).

(iii) If \(t_k(u) = 0\), then \((u, v) \equiv_{\delta^*_k} (u', v') \iff (u, v) \equiv_{\delta^r} (u', v')\).

Claim (i) says that the procedure \(\text{Update}\) works correctly, i.e., that the \(t\)-value of a node (if it isn’t 0) always represents the tour it currently is associated to. Claim (ii) says that after an edge has been processed, both of their nodes are associated to the same tour. So after the algorithm has finished, every node of \(G\) is in the same tour as its neighbor.
Proof. We prove all claims via one induction over \( k \). For \( k = 0 \) we have \( \delta_0^* = \delta^e \) which is bijective (Lemma 6). Moreover, no edge has been processed so far, so (i) and (ii) are trivially fulfilled and (iii) follows directly from \( \delta_0^* = \delta^e \).

Now let all of the claims be true for \( k \in \{0, \ldots, N-1\} \). We start with proving the bijectivity and (i) for \( k+1 \).

For this we take a closer look at the \((k+1)\)-th call of \textsc{Merge-Cycle}.

If \( \delta_k^* \neq \delta_{k+1}^* \), this change has to be happening in one of the procedures \textsc{New-Nodes}, \textsc{Merge} or \textsc{Write}, since these are the only procedures in which edges are written to the stream or added to \( F \). First, note that for every edge \( e \) written to the stream during \textsc{Write} or added to \( F \) in \textsc{New-Nodes} it holds \( \delta^*_{k+1}(e) = \delta^*_{k+1}(e) \):

If \( e = (v_i, v_{i+1}) \) is written to the stream during \textsc{Write}, it is written in the form \((v_i, v_{i+1}, v_{i+2})\), so we have \( \delta^*_{k+1}(e) = (v_{i+1}, v_{i+2}) = \delta^e(e) = \delta^*_k(e) \).

If \( e = (v_{i-1}, v_i) \) is added to \( F \) during \textsc{New-Nodes}, it becomes a type-B-edge at this point, so \( \delta^*_{k+1}(e) = (v_i, j(v_i)) \). Moreover, \( j(v_i) \) is set to \( v_{i+1} \) in line 3, so \( \delta^*_{k+1}(e) = (v_i, v_{i+1}) = \delta^e(e) = \delta^*_k(e) \).

So we may concentrate on the procedure \textsc{Merge}: Here we process every node from the set \( J_{k+1} \). Let \( r := |J_{k+1}| \), for instance \( J = \{w_1, \ldots, w_r\} \). Each of these nodes \( w_i \) has been processed before, hence, there is a unique edge in \( F_k \) that ends in \( w_i \) and which we denote by \( e_i \). Moreover, there is a unique edge in \( C_{k+1} \) that ends in \( w_i \) and which we denote by \( e'_i \). Now let \( i \in [r] \). We process \( w_i \) in two steps:

Step 1: \((w_i, j(w_i))\) is marked as successor of \( e'_i \). So directly after this step, \( e'_i \) and \( e_i \) share the same successor, while the out-going edge of \( w_i \) in \( C_{k+1} \) has lost its predecessor.

Step 2: \( j(w_i) \) is set to the next node in the cycle, so that the out-going edge of \( w_i \) becomes the successor of \( e_i \).

In these two steps we swapped the successors of \( e_i \) and \( e'_i \) and did not change anything else, so what we get is

\[
\delta^*_{k+1}(e) = \delta^*_k(e) \text{ for any } e \in \bar{E} \setminus \{e_1, \ldots, e_r, e'_1, \ldots, e'_r\}
\]

and for any \( i \in [r] \)

\[
\delta^*_{k+1}(e_i) = \delta^*_k(e'_i) \text{ and } \delta^*_{k+1}(e'_i) = \delta^*_k(e_i).
\]

Let \( i, j \in [r] \) with \( i \neq j \). We have \( e'_i \equiv \delta^*_k e'_j \), because \( e'_j \in C_{k+1} \equiv [e'_j]_{\delta^e} = [e'_j]_{\delta^*_k} \). We also have \( e_i \neq \delta^*_k e_j \), which follows from \( t_k(w_i) \neq t_k(w_j) \) (\textsc{Construct-J-M}, line 4) together
with the induction hypothesis. Finally we have \( e_i \not\in \delta^*_k \) and \( e_i' \not\in E(C_{k+1}) = [e_i]_{\delta^*} = [e_i]_{\delta^*_k} \).

So we can apply Lemma 8 with \( \delta = \delta^*_k \) and \( \delta' = \delta^*_{k+1} \) and get the bijectivity of \( \delta^*_{k+1} \) and for every processed edge \( e \)

\[
e \in [e_1]_{\delta^*_{k+1}} \iff e \in \bigcup_{i=1}^r [e'_i]_{\delta^*_k} \cup [e_1]_{\delta^*_k} \iff t_k(e(1)) \in M_k \land e \in C_{k+1} \iff t_{k+1}(e(1)) = a_{k+1}
\]

and

\[
e \not\in [e_1]_{\delta^*_{k+1}} \iff e \not\in \bigcup_{i=1}^r [e'_i]_{\delta^*_k} \cup [e_1]_{\delta^*_k} \iff t_k(e(1)) \not\in M_k \land e \not\in C_{k+1}
\]

\[
\iff t_{k+1}(e(1)) = t_k(e(1)) \neq a_{k+1}.
\]

Now we are able to complete the proof of (i): Let \((u, v), (u', v')\) be processed edges.

Case 1: \((u, v), (u', v') \in [e_1]_{\delta^*_{k+1}}\). Then \((u, v) \equiv_{\delta^*_{k+1}} (u', v')\) and \(t(u) = a_{k+1} = t(u')\).

Case 2: \((u, v) \in [e_1]_{\delta^*_{k+1}}, (u', v') \not\in [e_1]_{\delta^*_{k+1}}\). Then \((u, v) \not\equiv_{\delta^*_{k+1}} (u', v')\) and \(t(u) = a_{k+1} \neq t(u')\).

Case 3: \((u, v) \not\in [e_1]_{\delta^*_{k+1}}, (u', v') \in [e_1]_{\delta^*_{k+1}}\). Analog to case 2.

Case 4: \((u, v), (u', v') \not\in [e_1]_{\delta^*_{k+1}}\). Then \(t_{k+1}(u) = t_k(u), t_{k+1}(u') = t_k(u')\) and (P2) of Lemma 8 \([(u, v)]_{\delta^*_{k+1}} = [(u, v)]_{\delta^*_k}\) and \([(u', v')]_{\delta^*_{k+1}} = [(u', v')]_{\delta^*_k}\). So we have

\[
(u, v) \equiv_{\delta^*_{k+1}} (u', v') \iff (u, v) \equiv_{\delta^*_k} (u', v') \iff t_k(u) = t_k(u') \iff t_{k+1}(u) = t_{k+1}(u').
\]

About (ii). Let \((u, v)\) be a processed edge. If \((u, v) \in C_{k+1}\), then at the end of MERGE-CYCLE both \(t(u)\) and \(t(v)\) are set to the same value \(a\). If \((u, v) \not\in C_{k+1}\), then \((u, v)\) already was a processed edge before so by induction hypothesis and Lemma 7 we are finished.

About (iii). Let \(u \in V\) with \(t_{k+1}(u) = 0\). That means that \(u\) is not processed in the first \(k+1\) calls of MERGE-CYCLE. Especially we have \((u, v) \equiv_{\delta^*_k} (u', v') \iff (u, v) \equiv_{\delta^*_k} (u', v') \iff (u, v) \equiv_{\delta^*} (u', v')\) by induction hypothesis.

These results suffice to proof our main result, given in the following.

**Theorem 10.** If \(G\) is Eulerian, \(\delta^*\) determines an Euler tour on \(G\).
Proof. According to Theorem 3, it suffices to show that \( \delta^* \) is bijective and that \( e \equiv_{\delta^*} e' \) for any \( e, e' \in R^*(E) \). Remember that \( \delta^* = \delta^*_N \), so by Lemma 9 \( \delta^* \) is bijective. For the second property, let \( e, e' \in R^*(E) \) with \( e = (u, v) \) and \( e' = (u', v') \). If \( G \) is Eulerian, it is connected, so there exists a \( u-u' \)-path \( P \) in \( G \). For every edge on \( P \), either the edge itself or the corresponding reversed edge has been processed during the algorithm Euler-Tour. By Lemma 9 (ii), \( t_N(x) = t_N(y) \) for all nodes \( x, y \) of \( P \), hence, \( t_N(u) = t_N(u') \) and by Lemma 9 (i), we get \( e \equiv_{\delta^*_N} e' \). Since \( \delta^*_N = \delta^* \), we are done. □

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Appendix

On the following two pages we present a working example for the method MERGE-CYCLE that corresponds to the steps 2 to 6 in our high level description. Note that every node has at most one in-going first-in edge and one out-going potential successor edge at a time.

A cycle $C$ has been found.

Step 2. For every new node the in-going edge becomes a first-in edge and the out-going edge becomes a potential successor.

Step 3. For each intersecting Tour $T_1$, $T_2$, $T_3$ we choose one common node.

Step 4. The successors of the chosen nodes are swapped with potential successor edges.
Step 5. For the rest of the edges the successor stays the same.

Step 6. The tours and the cycle have been merged to one tour.