The minimum modulus of gap power series and h-measure of exceptional sets

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Abstract

For entire function of the form \( f(z) = \sum_{k=0}^{\infty} f_k z^{n_k} \), where \((n_k)\) is a strictly increasing sequence of non-negative integers, we establish conditions when the relations

\[
M_f(r) = (1 + o(1))m_f(r), \quad m_f(r) = (1 + o(1))\mu_f(r)
\]

is true as \( r \to +\infty \) outside some set \( E \) such that \( h\text{-meas}(E) = \int_E \frac{dh(r)}{r} < +\infty \) uniformly in \( y \in \mathbb{R} \), where \( h(r) \) is positive continuous function increasing to \(+\infty\) on \([0, +\infty)\) with non-decreasing derivative, and \( M_f(r) = \max\{|f(z)|: |z| = r\}, \quad m_f(r) = \min\{|f(z)|: |z| = r\}, \quad \mu_f(r) = \max\{|f_k|r^{n_k}: k \geq 0\} \) the maximum modulus, the minimum modulus and the maximum term of \( f \) respectively.

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1 Introduction

Let \( L \) be the class of positive continuous functions increasing to \(+\infty\) on \([0; +\infty)\). By \( L^+ \) we denote the subclass of \( L \) which consists of the differentiable functions with non-decreasing derivative, and \( L^- \) the subclass of functions with non-increasing derivative.
Let $f$ be an entire function of the form
\[ f(z) = \sum_{k=0}^{+\infty} f_k z^{n_k}, \tag{1} \]
where $(n_k)$ is a strictly increasing sequence of non-negative integers. For $r > 0$ we denote by $M_f(r) = \max\{|f(z)| : |z| = r\}$, $m_f(r) = \min\{|f(z)| : |z| = r\}$, $\mu_f(r) = \max\{|f_k r^{n_k} : k \geq 0\}$ the maximum modulus, the minimum modulus and the maximum term of $f$ respectively.

P.C. Fenton [1] (see also [2]) has proved the following statement.

**Theorem 1.1** ([1]). If
\[ \sum_{k=0}^{+\infty} \frac{1}{n_{k+1} - n_k} < +\infty, \tag{2} \]
then for every entire function $f$ of the form (1) there exists a set $E \subset [1, +\infty)$ of finite logarithmic measure, i.e. $\log\text{-}\text{meas} E := \int_E d\log r < +\infty$, such that relations
\[ M_f(r) = (1 + o(1))m_f(r), \quad M_f(r) = (1 + o(1))\mu_f(r) \tag{3} \]
hold as $r \to +\infty$ ($r \notin E$).

P. Erdős and A.J. Macintyre [2] proved that conditions (2) implies that (3) holds as $r = r_j \to +\infty$ for some sequence $(r_j)$.

Denote by $D(\Lambda)$ the class of entire (absolutely convergent in the complex plane) Dirichlet series of form
\[ F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}, \tag{4} \]
where $\Lambda = (\lambda_n)$ is a fixed sequence such that $0 = \lambda_0 < \lambda_n \uparrow +\infty$ ($1 \leq n \uparrow +\infty$).

Let us introduce some notations for $F \in D(\Lambda)$ and $x \in \mathbb{R}$: $\mu(x, F) = \max\{|a_n e^{x\lambda_n} : n \geq 0\}$ is the maximal term, $M(x, F) = \sup\{|F(x+iy)| : y \in \mathbb{R}\}$ is the maximum modulus, $m(x, F) = \inf\{|F(x+iy)| : y \in \mathbb{R}\}$ is the minimum modulus, $\nu(x, F) = \max\{n : |a_n e^{x\lambda_n} = \mu(x, F)\}$ is the central index of series (4).

In [3] (see also [4]) we find the following theorem.

**Theorem A** (O.B. Skaskiv, 1984). For every entire function $F \in D(\Lambda)$ relation
\[ F(x + iy) = (1 + o(1))a_{\nu(x, F)} e^{(x+iy)\lambda_{\nu(x, F)}} \tag{5} \]
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holds as \( x \to +\infty \) outside some set \( E \) of finite Lebesgue measure \((\int_E dx < +\infty)\) uniformly in \( y \in \mathbb{R} \), if and only if

\[
\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < +\infty.
\] (6)

Note, in the paper [5] was proved the analogues of other assertions from the article of Fenton [1] for the subclasses of functions \( F \in D(\Lambda) \) defined by various restrictions on the growth rate of the maximal term \( \mu(x, F) \).

The finiteness of Lebesgue measure of an exceptional set \( E \) in Theorem A is the best possible description. It follows from the such statement.

**Theorem B (T.M. Salo, O.B. Skaskiv, 2001 [6]).** For every sequence \( \lambda = (\lambda_k) \) (including those which satisfy (6)) and for every positive continuous differentiable function \( h: [0, +\infty) \to [0, +\infty) \) such that \( h'(x) \not\to +\infty \) \((x \to +\infty)\) there exist an entire Dirichlet series \( F \in D(\lambda) \), a constant \( \beta > 0 \) and a measurable set \( E_1 \subset [0, +\infty) \) of infinite \( h \)-measure (\( h\)-meas \((E_1) \) \( \triangleq \int_{E_1} dh(x) = +\infty \)) such that

\[
(\forall \: x \in E_1): \: M(x, F) > (1 + \beta)\mu(x, F), \: M(x, F) > (1 + \beta)m(x, F). \] (7)

Recently, Ya.V. Mykytyuk showed us, that in Theorem it is enough to require that the positive non-decreasing function \( h \) be such that

\[
h(x)/x \to +\infty \quad (x \to +\infty).
\]

From Theorem B follows that the finiteness of logarithmic measure of an exceptional set \( E \) in Fenton’s Theorem 1.1 also is the best possible description.

It is easy to see that the relation

\[
F(x + iy) = (1 + o(1))a_\nu(x, F)e^{(x+iy)\lambda_\nu(x, F)}
\]

holds as \( x \to +\infty \) \((x \notin E)\) uniformly in \( y \in \mathbb{R} \), if and only if

\[
M(x, F) \sim \mu(x, F) \quad \text{and} \quad M(x, F) \sim m(x, F) \quad (x \to +\infty, \: x \notin E). \] (8)

Due to Theorem B the natural question arises: what conditions must satisfy the entire Dirichlet series that relation (5) is true as \( x \to +\infty \) outside some set \( E_2 \) of finite \( h \)-measure, i.e.

\[
h\text{-meas} \( (E_2) < +\infty? \)
\]

In this paper we obtain the answer to this question when \( h \in L^+ \).
2 \( h \)-measure with non-decreasing density

According to Theorem B, in case \( h \in L^+ \) condition (6) must be fulfilled. Therefore, in subclass

\[
D(\Lambda, \Phi) = \{ F \in D(\Lambda) : \ln \mu(x, F) \geq x\Phi(x) \ (x > x_0) \}, \quad \Phi \in L,
\]

it should be strengthened. The following theorem indicates this.

**Theorem 2.1.** Let \( \Phi \in L, h \in L^+ \) and \( \varphi \) be the inverse function to the function \( \Phi \). If

\[
(\forall b > 0) : \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h'\left(\varphi(\lambda_k) + \frac{b}{\lambda_{k+1} - \lambda_k}\right) < +\infty, \quad (9)
\]

then for all \( F \in D(\Lambda, \Phi) \) holds (5) is true as \( x \to +\infty \) outside some set \( E \) of finite \( h \)-measure uniformly in \( y \in \mathbb{R} \).

**Proof of Theorem 2.1.** Note first that condition (9) implies the convergence of series (6). Denote \( \Delta_0 = 0 \) and for \( n \geq 1 \)

\[
\Delta_n = \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \sum_{m=j+1}^{+\infty} \left( \frac{1}{\lambda_m - \lambda_{m-1}} + \frac{1}{\lambda_{m+1} - \lambda_m} \right).
\]

Consider the function

\[
f_q(z) = \sum_{n=0}^{+\infty} \frac{a_n}{\alpha_n} e^{z\lambda_n},
\]

where \( \alpha_n = e^{q\Delta_n}, \ q > 0 \).

Since \( \Delta_n \geq 0 \), then \( f_q \in D(\Lambda) \) and \( \nu(x, f_q) \to +\infty \ (x \to +\infty) \).

Repeating the proof of Lemma 1 from [8], it is not difficult to obtain the following lemma.

**Lemma 2.1.** For all \( n \geq 0 \) and \( k \geq 1 \) inequality

\[
\frac{\alpha_n}{\alpha_k} e^{\tau_k(\lambda_n - \lambda_k)} \leq e^{-q|n-k|}, \quad (10)
\]

is true, where \( \tau_k = \tau_k(q) = qx_k + \frac{q}{\lambda_k - \lambda_{k-1}}, \ x_k = \frac{\Delta_{k-1} - \Delta_k}{\lambda_k - \lambda_{k-1}} \).

**Proof of Lemma 1.** Since

\[
\ln \alpha_n - \ln \alpha_{n-1} = q(\Delta_n - \Delta_{n-1}) = -qx_n(\lambda_n - \lambda_{n-1}),
\]
Then for \( n \geq k + 1 \) we have

\[
\ln \frac{\alpha_n}{\alpha_k} + \tau_k(\lambda_n - \lambda_k) = -q \sum_{j=k+1}^{n} x_j(\lambda_j - \lambda_{j-1}) + \tau_k \sum_{j=k+1}^{n} (\lambda_j - \lambda_{j-1}) =
\]

\[
= - \sum_{j=k+1}^{n} (qx_j - \tau_k)(\lambda_j - \lambda_{j-1}) \leq - \sum_{j=k+1}^{n} (qx_j - \tau_j)(\lambda_j - \lambda_{j-1}) =
\]

\[
= -q \sum_{j=k+1}^{n} 1 = -q(n - k).
\]

Similarly, for \( n \leq k - 1 \) we obtain

\[
\ln \frac{\alpha_n}{\alpha_k} + \tau_k(\lambda_n - \lambda_k) = - \ln \frac{\alpha_k}{\alpha_n} - \tau_k(\lambda_k - \lambda_n) =
\]

\[
= q \sum_{j=n+1}^{k} x_j(\lambda_j - \lambda_{j-1}) - \tau_k \sum_{j=n+1}^{k} (\lambda_j - \lambda_{j-1}) = - \sum_{j=n+1}^{k} (\tau_k - qx_j)(\lambda_j - \lambda_{j-1}) \leq
\]

\[
\leq - \sum_{j=n+1}^{k} (\tau_j - qx_j)(\lambda_j - \lambda_{j-1}) = -q \sum_{j=n+1}^{k} 1 = -q(k - n)
\]

and Lemma 1 is proved.

Let \( J \) be the range of central index \( \nu(x, f_q) \). Denote by \((R_k)\) the sequence of the jump points of central index, numbered in such a way that \( \nu(x, f_q) = k \) for all \( x \in [R_k, R_{k+1}] \) and \( R_k < R_{k+1} \). Then for all \( x \in [R_k, R_{k+1}] \) and \( n \geq 0 \) we have

\[
\frac{a_n e^{x\lambda_n}}{\alpha_n} \leq \frac{a_k e^{x\lambda_k}}{\alpha_k}.
\]

According to Lemma 1, for \( x \in [R_k + \tau_k, R_{k+1} + \tau_k] \) we obtain

\[
\frac{a_n e^{x\lambda_n}}{a_k e^{x\lambda_k}} \leq \frac{\alpha_n}{\alpha_k} e^{\tau_k(\lambda_n - \lambda_k)} \leq e^{-q|n-k|} \quad (n \geq 0).
\]

Therefore,

\[
\nu(x, F) = k, \quad \mu(x, F) = a_k e^{x\lambda_k} \quad (x \in [R_k + \tau_k, R_{k+1} + \tau_k]) \tag{11}
\]

and

\[
|F(x + iy) - a_{\nu(x,F)} e^{(x+iy)\lambda_{\nu(x,F)}}| \leq
\]

\[
\leq \sum_{n \neq \nu(x,F)} \mu(x, F) e^{-q|n-\nu(x,F)|} \leq 2 \frac{e^{-q}}{1 - e^{-q}\mu(x, F)} \tag{12}
\]
for all \( x \in [R_k + \tau_k, R_{k+1} + \tau_k] \) and \( k \in J \). Thus, inequality (12) holds for all \( x \notin E_1(q) \) defined as \( \bigcup_{k=0}^{+\infty} [R_{k+1} + \tau_k, R_{k+1} + \tau_{k+1}] \).

Since \( \tau_{k+1} - \tau_k = 2q/(\lambda_{k+1} - \lambda_k) \), and by the Lagrange theorem
\[
h(R_{k+1} + \tau_{k+1}) - h(R_{k+1} + \tau_k) = (\tau_{k+1} - \tau_k)h'(R_{k+1} + \tau_k + \theta_k(\tau_{k+1} - \tau_k)),
\]
where \( \theta_k \in (0; 1) \), then for every \( q > 0 \) we have

\[
\text{h-meas}\left(E_1(q)\right) = \sum_{k=0}^{+\infty} \int_{R_{k+1} + \tau_k}^{R_{k+1} + \tau_{k+1}} dh(x) =
\]
\[
= \sum_{k=0}^{+\infty} (h(R_{k+1} + \tau_{k+1}) - h(R_{k+1} + \tau_k)) \leq
\]
\[
\leq 2q \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h'(R_{k+1} + \tau_k + 2q \frac{1}{\lambda_{k+1} - \lambda_k}).
\]

Here we applied the condition \( h \in L^+ \).

For \( F \in D(\Lambda, \Phi) \) as \( x > \max\{x_0, 1\} \) we have

\[
x \Phi(x) \leq \ln \mu(x, F) = \ln \mu(1, F) + \int_1^x \lambda_{\nu(x,F)} dx \leq \ln \mu(1, F) + (x-1)\lambda_{\nu(x-0,F)},
\]
and for all \( x \geq x_1 \geq x_0 \) it implies

\[
x \Phi(x) \leq x\lambda_{\nu(x-0,F)},
\]

i.e.

\[
x \leq \varphi\left(\lambda_{\nu(x-0,F)}\right) \quad (x \geq x_1).
\]

Thus, according to (11) for \( k \geq k_0 \) we obtain

\[
R_{k+1} + \tau_k \leq \varphi(\lambda_{\nu(R_{k+1} + \tau_k - 0,F)} = \varphi(\lambda_k).
\]

Applying the previous inequality to inequality (13), by the condition \( h \in L^+ \)
we have

\[
\text{h-meas}\left(E_1(q)\right) \leq 2q \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h'(\varphi(\lambda_k) + 2q \frac{1}{\lambda_{k+1} - \lambda_k}).
\]

Therefore, using (9) we conclude that \( \text{h-meas}\left(E_1(q)\right) < +\infty \).
Let \( q_k = k \). Since \( \text{h-meas}(E_1(q_k)) < +\infty \), then

\[
\text{h-meas}(E_1(q_k) \cap [x, +\infty)) = o(1) \quad (x \to +\infty),
\]
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thus it is possible to choose an increasing to $+\infty$ sequence $(x_k)$ such that

$$h\text{-meas} \left( E_1(q_k) \cap [x_k; +\infty) \right) \leq \frac{1}{k^2}$$

for all $k \geq 1$. Denote $E_1 = \bigcup_{k=1}^{+\infty} (E_1(q_k) \cap [x_k; x_{k+1}])$. Then

$$h\text{-meas} (E_1) = \sum_{k=1}^{+\infty} h\text{-meas} (E_1(q_k) \cap [x_k; x_{k+1})) \leq \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty,$$

On the other hand from inequality (12) we deduce for $x \in [x_k; x_{k+1}) \setminus E_1$

$$|F(x + iy) - a_{\nu(x,F)} e^{(x+iy)\lambda_{\nu(x,F)}}| \leq 2 \frac{e^{-q_k}}{1 - e^{-q_k}} \mu(x, F),$$

whence, as $x \to +\infty$ ($x \notin E_1$) we obtain (5). Theorem 2.1 is proved. $\Box$

Note, if $h(x) \equiv x$ then condition (9) turn into condition (6), and $h$ - measure of the set $E$ is it’s Lebesgue measure.

Let $\Phi \in L$. Consider the classes

$$D_0(\Lambda, \Phi) = \{ F \in D(\Lambda) : (\exists K > 0) [\ln \mu(x, \Phi) \geq K x \Phi(x) \ (x > x_0)]\},$$

$$D_1(\Lambda, \Phi) = \{ F \in D(\Lambda) : (\exists K_1, K_2 > 0) [\ln \mu(x, \Phi) \geq K_1 x \Phi(K_2 x) \ (x > x_0)]\}.$$

**Theorem 2.2.** Let $\Phi_0 \in L$, $h \in L^+$ and $\varphi_0$ be the inverse function to the function $\Phi_0$. If

$$(\forall b > 0) : \sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} h' \left( \varphi_0(b\lambda_n) + \frac{b}{\lambda_{n+1} - \lambda_n} \right) < +\infty, \quad (16)$$

then for each function $F \in D_0(\Lambda, \Phi_0)$ relation (5) holds as $x \to +\infty$ outside some set $E$ of finite $h$ - measure uniformly in $y \in \mathbb{R}$.

**Theorem 2.3.** Let $\Phi_1 \in L$, $h \in L^+$, and $\varphi_1$ be the inverse function to the function $\Phi_1$. If

$$(\forall b > 0) : \sum_{n=0}^{+\infty} \frac{h'(b\varphi_1(b\lambda_n))}{\lambda_{n+1} - \lambda_n} < +\infty, \quad (17)$$

then for every function $F \in D_1(\Lambda, \Phi_1)$ relation (5) holds as $x \to +\infty$ outside some set $E$ of finite $h$-measure uniformly in $y \in \mathbb{R}$. 

Proof of Theorems 2 and 3. Theorems 2 and 3 immediately follow from Theorem 2.1.

Indeed, if $F \in D_0(\Lambda, \Phi_0)$ then $F \in D(\Lambda, \Phi)$ as $\Phi(x) = K\Phi_0(x)$. But in this case $\varphi(x) = \varphi_0(x/K)$ and thus condition (9) follows from condition (16). It remains to apply Theorem 2.1.

Similarly, if $F \in D_1(\Lambda, \Phi_1)$ then $F \in D(\Lambda, \Phi)$ as $\Phi(x) = K_1\Phi_1(K_2x)$. But in this case $\varphi(x) = \varphi_1(x/K_1)/K_2$ and thus condition (9) follows from condition (17). It remains to apply Theorem 2.1 again.

Remark 2.1. It is easy to see, that for every fixed functions $h \in L^+$ and $\Phi \in L$ there exists a sequence $\Lambda$ such that conditions (9), (16) and (17) hold.

The following theorem indicates that condition (17) is necessary for relations (5), (8) to hold for every $F \in D_1(\Lambda, \Phi_1)$ as $x \to +\infty$ outside a set of finite $h$ - measure. Here we assume that condition (6) is satisfied.

Theorem 2.4. Let $\Phi_1 \in L$, $h \in L^+$, and $\varphi_1$ is the inverse function to the function $\Phi_1$. For every sequence $\Lambda$ such that

$$\tag{18} (\exists b > 0) : \sum_{n=0}^{+\infty} \frac{h'(b\varphi_1(b\lambda_n))}{\lambda_{n+1} - \lambda_n} = +\infty,$$

there exist a function $F \in D_1(\Lambda, \Phi_1)$, a set $E \subset [0, +\infty)$ and a constant $\beta > 0$ such that inequalities (7) hold for all $x \in E$ and $h$-meas $(E) = +\infty$.

Proof of Theorem 2.4. Denote $\varkappa_1 = \varkappa_2 = 1$, $\varkappa_n = \sum_{k=1}^{n-2} r_k$ $(n \geq 3)$, where

$$r_1 = \max \left\{ b\varphi_1(b\lambda_2), \frac{1}{\lambda_2 - \lambda_1} \right\},$$

$$r_k = \max \left\{ b\varphi_1(b\lambda_{k+1}) - b\varphi_1(b\lambda_k), \frac{1}{\lambda_{k+1} - \lambda_k} \right\} \quad (k \geq 2),$$

and also choose $a_0 = 1$, $a_n = \exp \left\{ -\sum_{k=1}^{n} \varkappa_k (\lambda_k - \lambda_{k-1}) \right\}$ $(n \geq 1)$. We prove that the function $F$ defined by series (4) of the so-defined coefficients $(a_n)$ and indices $(\lambda_n)$ belongs to class $D_1(\Lambda, \Phi_1)$.

Since $\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < +\infty$ implies $n = o(\lambda_n)$ $(n \to +\infty)$, thus $\frac{\ln n}{\lambda_n} \to 0 \quad (n \to +\infty)$. By the construction $\varkappa_n = \frac{\ln a_{n-1} - \ln a_n}{\lambda_n - \lambda_{n-1}}$ $(n \geq 1)$ and $\varkappa_n \uparrow +\infty \quad (n \to +\infty)$, therefore Stolz’s theorem yields $\frac{\ln a_n}{\lambda_n} \to +\infty \quad (n \to +\infty)$ and by Valiron’s theorem [9, p.85]) the abscissa of absolute convergence of series (4) is equal to $+\infty$, i.e. $F \in D(\Lambda)$. 

Moreover, it is known that in case \( \kappa_n \uparrow +\infty \) \((n \to +\infty)\)
\[
\forall x \in [\kappa_n, \kappa_{n+1}) : \quad \mu(x, F) = a_n e^{x\lambda_n}, \quad \nu(x, F) = n. \tag{19}
\]
Since by the construction
\[
\kappa_n \leq b\varphi_1(b\lambda_{n-1}) + \sum_{k=1}^{n-2} \frac{1}{\lambda_{k+1} - \lambda_k} \leq 2b\varphi_1(b\lambda_{n-1}) \quad (n > n_0),
\]
for sufficiently large \( n \) for all \( x \in [\kappa_n, \kappa_{n+1}) \)
\[
\ln \mu(2x, F) = \ln \mu(x, F) + \int_2^x \lambda_\nu(t) dt \geq x\lambda_\nu(x) =
\]
\[
= x\lambda_n \geq \frac{x}{b}\Phi_1 \left( \frac{\kappa_{n+1}}{2b} \right) \geq \frac{x}{b}\Phi_1 \left( \frac{x}{2b} \right).
\]
Hence, for \( x \geq x_0 \) we have
\[
\ln \mu(x, F) \geq \frac{1}{2b}x\Phi_1 \left( \frac{x}{4b} \right)
\]
and thus \( F \in D_1(\Lambda, \Phi_1) \).

Note that
\[
\kappa_{n+1} - \kappa_n = \kappa_{n-1} \geq \frac{1}{\lambda_n - \lambda_{n-1}} \quad (n \geq 1).
\]
For \( x \in \left[ \kappa_n, \kappa_n + \frac{1}{\lambda_n - \lambda_{n-1}} \right] \) we have
\[
a_{n-1} e^{x\lambda_{n-1}} \mu(x, F) = a_{n-1} e^{x\lambda_{n-1}} = \exp \left\{ (\lambda_n - \lambda_{n-1})(\kappa_n - x) \right\} \geq e^{-1} := \beta, \tag{20}
\]
and, therefore, for \( x \in E = \bigcup_{n=1}^{\infty} \left[ \kappa_n, \kappa_n + \frac{1}{\lambda_n - \lambda_{n-1}} \right] \), choosing \( n = \nu(x, F) \), we obtain
\[
F(x) \geq a_{n-1} e^{x\lambda_{n-1}} + a_n e^{x\lambda_n} = \mu(x, F) \left( 1 + \frac{a_{n-1} e^{x\lambda_{n-1}}}{a_n e^{x\lambda_n}} \right) \geq (1 + \beta)\mu(x, F),
\]
hence inequalities (7) are true.

Now we prove that \( h\text{-meas}(E) = +\infty \). By the construction \( (\kappa_n) \) for all \( n \geq 1 \) we have
\[
\kappa_n \geq b\varphi_1(b\lambda_{n-1}). \tag{21}
\]
Taking into account the Lagrange theorem, condition \( h \in L^+ \) and inequality (21) we obtain
\[
\text{h-meas}(E) = \sum_{n=1}^{+\infty} \int_{\kappa_n}^{\kappa_n + \frac{1}{\lambda_n - \lambda_{n-1}}} dh(x) = \sum_{n=1}^{+\infty} \left( h(\kappa_n + \frac{1}{\lambda_n - \lambda_{n-1}}) - h(\kappa_n) \right) \geq
\]
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\[ \sum_{n=1}^{+\infty} \frac{h'(z_n)}{\lambda_n - \lambda_{n-1}} \geq \sum_{n=1}^{+\infty} \frac{h'(b\varphi_1(b\lambda_n))}{\lambda_n - \lambda_{n-1}} = +\infty. \]

Theorem 2.4 is proved. \(\square\)

The following criterion immediately follows from Theorems 2.3 and 2.4.

**Theorem 2.5.** Let \( \Phi_1 \in L, h \in L^+ \) and \( \varphi_1 \) be the inverse function to the function \( \Phi_1 \). For every entire function \( F \in D_1(\Lambda, \Phi_1) \) relation (5) holds as \( x \to +\infty \) outside some set \( E \) of finite \( h \)-measure uniformly in \( y \in \mathbb{R} \) if and only if (17) be true.

It is worth noting that if condition (16) of Theorem 2.2 is not fulfilled, that is
\[ (\exists b_1 > 0) : \sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} h'(\varphi_0(b_1\lambda_n) + \frac{b_1}{\lambda_{n+1} - \lambda_n}) = +\infty, \]
then for \( b = \max\{b_1; 2\} \) we have
\[ \sum_{n=0}^{+\infty} \frac{h'(b\varphi_0(b\lambda_n))}{\lambda_{n+1} - \lambda_n} = +\infty. \]

Therefore, condition (18) holds and according to Theorem 2.4 there exists a function \( F \in D_1(\Lambda, \Phi_0) \), a set \( E \subset [0, +\infty) \) and a constant \( \beta > 0 \) such that inequalities (7) hold for all \( x \in E \) and \( h\)-\meas \((E) = +\infty \).

Since for \( \Phi_0(x) = x^\alpha \) (\( \alpha > 0 \)) we have \( D_0(\Lambda, \Phi_0) = D_1(\Lambda, \Phi_0) \), then from Theorem 2.2 and 2.4 we obtain the following theorem.

**Theorem 2.6.** Let \( \Phi_0(x) = x^\alpha \) (\( \alpha > 0 \)), \( h \in L^+ \). For every entire function \( F \in D_0(\Lambda, \Phi_0) \) relation (5) holds as \( x \to +\infty \) outside some set \( E \) of finite \( h \)-measure uniformly in \( y \in \mathbb{R} \) if and only if
\[ (\forall b > 0) : \sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} h'(b(\lambda_n)^{1/\alpha} + \frac{b}{\lambda_{n+1} - \lambda_n}) < +\infty, \]
is true.

### 3 \( h \)-measure with non-increasing density

Note that for every differentiable function \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) with the bounded derivative \( h'(x) \leq c < +\infty \) (\( x > 0 \))
\[ \int_E dh(x) = \int_E h'(x)dx \leq c \int_E dx, \]
thus, the finiteness of Lebesgue measure of the set $E \subset \mathbb{R}_+$ implies $h$-meas $(E) < +\infty$. Therefore, according to Theorem A, condition (6) is sufficient to have the exceptional set $E$ of finite $h$-measure. However, we express an assumption that for $h \in L^-$ in the subclass

$$D_\varphi(\Lambda) = \{ F \in D(\Lambda) : (\exists n_0)(\forall n \geq n_0)[|a_n| \leq \exp\{-\lambda_n \varphi(\lambda_n)\}] \}, \quad \varphi \in L,$$

condition (6) can be weakened significantly. The following conjecture seems to be true.

**Conjecture 3.1.** Let $\varphi \in L$, $h \in L^-$. If

$$\sum_{n=0}^{+\infty} \frac{h'(\varphi(\lambda_n))}{\lambda_{n+1} - \lambda_n} < +\infty,$$

then for all $F \in D_\varphi(\Lambda)$ relation (5) is true as $x \to +\infty$ outside some set $E$ of finite $h$-measure uniformly in $y \in \mathbb{R}$.

## 4 $h$-measure and lacunary power series

The important corollaries for entire functions represented by lacunary power series of the form (1) ensue from the well-proven theorems.

For entire function $f$ of the form (1) we put $F(z) = f(e^z)$, $z \in \mathbb{C}$.

Note that for $x = \ln r$, $y = \varphi$

$$F(x + iy) = F(\ln r + i\varphi) = f(re^{i\varphi})$$

and $M(x, F) = M_f(r)$, $m(x, F) = m_f(r)$, $\mu(x, F) = \mu_f(r)$, $\nu(x, F) = \nu_f(r)$. In addition, for $E_2 \overset{\text{def}}{=} \{ r \in \mathbb{R} : \ln r \in E_1 \}$ and $h_1$ such that $h'_1(x) = h'(e^x)$ it is true

$$h\text{-log-meas}(E_2) \overset{\text{def}}{=} \int_{E_2} \frac{dh(r)}{r} = \int_{E_1} \frac{dh(e^x)}{e^x} = \int_{E_1} dh_1(x) = h_1\text{-meas}(E_1).$$

Hence, the next corollary follows from Theorem B.

**Corollary 4.1.** For every sequence $(n_k)$ such that condition (6) holds and for every function $h \in L^+$ there exist an entire function $f$ of the form (1), a constant $\beta > 0$ and a set $E_2$ of infinite $h$-log-measure, i.e. ($\int_{E_2} \frac{dh(r)}{r} = +\infty$) such that

$$(\forall r \in E_2) : M_f(r) \geq (1 + \beta)\mu_f(r), \quad M_f(r) \geq (1 + \beta)m_f(r). \quad (22)$$

In turn, from Theorem 2.1 we obtain the following consequence.
Corollary 4.2. Let $\Phi \in L$, $h \in L^+$ and $\varphi$ be the inverse function to the function $\Phi$. If for an entire function $f$ of the form (1)
\[ \ln \mu_f(r) \geq \ln r \Phi(\ln r) \quad (r \geq r_0) \] (23)
and
\[ (\forall b > 0) : \sum_{k=0}^{+\infty} \frac{1}{n_{k+1} - n_k} h'(\exp \{ \varphi(n_k) + \frac{b}{n_{k+1} - n_k} \}) < +\infty, \] (24)
then relation
\[ f(re^{i\varphi}) = (1 + o(1))\alpha_{\nu_f(r)} r^{\nu_f(r)} e^{i\varphi \nu_f(r)} \] (25)
holds as $r \to +\infty$ outside some set $E_2$ of finite $h$-log-measure uniformly in $\varphi \in [0, 2\pi]$.

In fact, from condition (23) it follows that $F \in D(\Lambda, \Phi)$ with $\Lambda = (n_k)$ and it remains to apply Theorem 2.1 with the function $h_1$.

Denote by $\mathcal{E}$ the class of entire functions of positive lower order, i.e.
\[ \lambda_f := \lim_{r \to +\infty} \frac{\ln \ln M_f(r)}{\ln r} > 0. \]

Immediately from Theorem 2.5 we obtain following assertion.

Corollary 4.3. Let $h \in L^+$. In order that relations (3) hold for every function $f \in \mathcal{E}$ of the form (1) as $r \to +\infty$ outside a set of finite $h$-log-measure, necessary and sufficient
\[ (\forall b > 0) : \sum_{k=0}^{+\infty} \frac{1}{n_{k+1} - n_k} h'((n_k)^b) < +\infty. \]

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