NEW EXPLICIT CONSTRUCTION OF FOLD MAPS ON
GENERAL 7-DIMENSIONAL CLOSED AND
SIMPLY-CONNECTED SPIN MANIFOLDS

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Abstract. 7-dimensional closed and simply-connected manifolds have been
attractive as central and explicit objects in algebraic topology and differential
topology of higher dimensional closed and simply-connected manifolds, which
were studied actively especially in 1950s–60s.

Attractive studies of the class of these 7-dimensional manifolds were started
by the discovery of so-called exotic spheres by Milnor. It has influenced on the
understanding of higher dimensional closed and simply-connected manifolds
via algebraic and abstract objects. Recently this class is still being actively
studied via more concrete notions from algebraic topology such as concrete
bordism theory by Crowley, Kreck, and so on.

As a new kind of fundamental and important studies, the author has been
challenging understanding the class in constructive ways via construction of
fold maps, which are higher dimensional versions of Morse functions. The
present paper presents a new general method to construct ones on spin mani-

folds of the class.

1. Introduction and fold maps.

Closed (and simply-connected) manifolds whose dimensions are $m \geq 5$ have been
central objects in algebraic topology and differential topology around 1950s–60s.
They have been classified via algebraic and abstract objects. 7-dimensional closed
and simply-connected manifolds were explicit and central objects in this scene as
the Milnor’s discovery of 7-dimensional exotic spheres [15] and a related work [4]
show and the class has been attractive until now as [2], [3], [14], and so on, show:
an exotic (homotopy) sphere is a homotopy sphere which is not diffeomorphic to
any standard sphere.

1.1. Terminologies and notation on differentiable manifolds and maps,
bundles, and so on. Throughout the present paper, manifolds and maps be-
 tween manifolds are smooth (of class $C^\infty$) unless otherwise stated. We assume that
diffeomorphisms on manifolds are smooth. We define the diffeomorphism group
of a manifold as the group of all diffeomorphisms on the manifold. As bundles
whose fibers are manifolds, the structure groups are subgroups of the diffeomor-
phism groups or equivalently, the bundles are smooth, unless otherwise stated. A
linear bundle is a smooth bundle whose fiber is a unit sphere or disc and whose
structure group acts linearly in a natural way on the fiber.

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A singular point \( p \in X \) of a differentiable map \( c : X \to Y \) between two differentiable manifolds is a point at which the rank of the differential \( dc \) of the map is smaller than both the dimensions \( \dim X \) and \( \dim Y \) or equivalently, \( \operatorname{rank} dc_p < \min\{\dim X, \dim Y\} \) holds where \( dc_p \) denotes the differential at \( p \). \( S(c) \) denotes the set of all singular points of \( c \) and this is defined as the singular set of \( c \). \( c(S(c)) \) is the regular value set of \( c \). We define a singular (regular) value as a point in the singular (resp. regular) value set of \( c \).

For \( p \) in the Euclidean space \( \mathbb{R}^k \), \( ||p|| \) denotes the distance between the origin 0 and \( p \) where the underlying space is endowed with the standard Euclidean metric.

For positive integers \( k \) and \( r \), we denote the set \( \{ x \in \mathbb{R}^k | ||x|| \in \mathbb{N}, 1 \leq ||x|| \leq r \} \) by \( D_{n,r}^k \).

1.2. Fold maps.

Definition 1. Let \( m > n \geq 1 \) be integers. A smooth map \( f \) from an \( m \)-dimensional smooth manifold with no boundary into an \( n \)-dimensional smooth manifold with no boundary is said to be a fold map if at each singular point \( p \), the map has the form \( (x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, m-n \cdot x_k^2 - \sum_{k=m-i+1}^{m} x_k^2) \) for suitable coordinates and an integer \( 0 \leq i(p) \leq m-n+1 \).

Proposition 1. In Definition 1, the following properties hold.

1. For any singular point \( p \), \( i(p) \) is unique.
2. The set consisting of all singular points of a fixed index of the map is a closed submanifold of dimension \( n-1 \) with no boundary of the \( m \)-dimensional manifold.
3. The restriction map to the singular set is an immersion.

We define \( i(p) \) in Proposition 1 as the index of \( p \). A special generic map is a fold map such that the index of each singular point is 0. The class of special generic maps contains the class of Morse functions on closed manifolds with exactly two singular points, which are central objects in so-called Reeb's theorem, characterizing spheres topologically except the case where the manifold is 4-dimensional. A standard 4-dimensional sphere is characterized as this in the case. The class of special generic maps also contains canonical projections of unit spheres. Fold maps have been fundamental and strong tools in studying algebraic topological, differential topological and more general algebraic or geometric properties of manifolds in the branch of the singularity theory of differentiable maps as Morse functions have been in so-called Morse theory. Studies related to fold maps were essentially started in 1950s by Thom and Whitney ([24] and [25]). These studies are on smooth maps on manifolds whose dimensions are equal to or larger than 2 into the plane. After various studies, recently, Saeki, Sakuma and so on, have been found interesting facts on fold maps satisfying appropriate conditions, especially, special generic maps, and manifolds admitting them in [17], [18], [19], [20], [21], [22], and so on. [5], [6], [7], [8] and so on, of the author, are motivated by these studies.

1.3. Explicit fold maps on 7-dimensional closed and simply-connected manifolds of several classes and the main theorem.

Definition 2. For a fold map \( f : M \to \mathbb{R}^n \) on a closed and connected manifold \( M \), we also assume that \( f|_{S(f)} \) is an embedding and that for each connected component \( C \) of the singular value set and its small closed tubular neighborhood \( N(C) \), the
composition of \( f|_{f^{-1}(N(C))} : f^{-1}(N(C)) \rightarrow N(C) \) with a canonical projection to \( C \) gives a trivial bundle over \( C \). We say that \( f \) is \( S \)-trivial.

**Theorem 1** ([11] and [12]). Let \( A, B \) and \( C \) be free commutative groups of rank \( a, b \) and \( c \). Let \( \{a_{i,j}\}_{j=1}^{b} \) be a sequence of integers where \( 1 \leq i \leq b \) is an integer. Let \( p \in B \oplus C \). Let \( (h_{i,j}) \) be a symmetric \( b \times b \) matrix such that the \((i, j)\)-th component is an integer satisfying \( h_{i,i} = 0 \) for \( 1 \leq i \leq b \). In this situation, there exist a 7-dimensional closed and simply-connected spin manifold and a fold map \( f : M \rightarrow \mathbb{R}^4 \) such that the following properties hold.

1. \( H_*(M; \mathbb{Z}) \) is free. Isomorphisms \( H^2(M; \mathbb{Z}) \cong A \oplus B \) and \( H^4(M; \mathbb{Z}) \cong B \oplus C \) hold and by fixing suitable identifications we have the following properties.
   (a) Products of elements in \( A \oplus \{0\} \subset H^2(M; \mathbb{Z}) \) vanish.
   (b) Consider a suitable generator \( \{(a_j^*, 0)\}_{j=1}^{b} \in A \oplus \{0\} \subset H^2(M; \mathbb{Z}) \) and a suitable generator \( \{(0, b_j^*)\}_{j=1}^{b} \in \{0\} \oplus B \subset H^2(M; \mathbb{Z}) \). The product of \( (a_j^*, 0) \) and \( (0, b_j^*) \) is regarded as \( (a_{j_2,j_1}b_{j_1}^*, 0) \in B \oplus \{0\} \subset H^4(M; \mathbb{Z}) \). The product of \( (0, b_j^*) \) and \( (0, b_{j_2}^*) \) is regarded as \( (h_{j_1,j_2}, b_{j_1}^*, b_{j_2}^*, 0) \in H^4(M; \mathbb{Z}) \).
2. The 1st Pontryagin class of \( M \) is \( 4p \in H^4(M; \mathbb{Z}) \) where the identification before is considered.
3. The index of each singular point of \( f \) is 0 or 1 and preimages of regular values are disjoint unions of at most 3 copies of \( S^3 \). Furthermore, if \( (h_{i,j}) \) is the zero matrix, then we can construct this map \( f \) as an \( S \)-trivial map such that preimages of regular values are disjoint unions of at most 2 copies of \( S^3 \).

We explain several facts implying explicitly that for classes of 7-dimensional closed and simply-connected spin manifolds, fold maps into \( \mathbb{R}^4 \) are interesting.

It is known that there exist exactly 28 types of 7-dimensional oriented homotopy spheres (see [15] and see also [4]).

**Theorem 2** ([6] and so on.). Every 7-dimensional homotopy sphere admits an \( S \)-trivial fold map \( f \) into \( \mathbb{R}^4 \) satisfying the following properties.

1. \( f(S(f)) = D_{9,3}^4 \).
2. The index of each singular point is always 0 or 1.
3. For each connected component of the regular value set of \( f \), the preimage of a regular value is, empty, diffeomorphic to \( S^3 \), diffeomorphic to \( S^3 \sqcup S^3 \) and diffeomorphic to \( S^3 \sqcup S^3 \sqcup S^3 \), respectively.

Moreover, we can show the following two facts.

1. \( M \) admits an \( S \)-trivial fold map \( f \) into \( \mathbb{R}^4 \) satisfying the last two properties of the previous three properties and the first property replaced by \( f(S(f)) = D_{9,1}^4 \) if and only if \( M \) is a standard sphere.
2. \( M \) admits an \( S \)-trivial fold map \( f \) into \( \mathbb{R}^4 \) satisfying the last two properties of the previous three properties and the first property replaced by \( f(S(f)) = D_{9,2}^4 \) if and only if the homotopy sphere \( M \) is oriented and one of 16 types of the 28 types, where the standard sphere is one of the 16 types.

We review known results on special generic maps on homotopy spheres. As a specific case, if a homotopy sphere of dimension 7 admits a special generic map into \( \mathbb{R}^4 \), then it is diffeomorphic to a standard sphere.
Theorem 3. (1) ([1], [17], [18] and so on.) Every exotic homotopy sphere of dimension \( m > 3 \) admits no special generic map \( \mathbb{R}^k \) for \( k = m - 3, m - 2, m - 1 \).

(2) ([26].) 7-dimensional oriented homotopy spheres of 14 types of all the 28 types admit no special generic map into \( \mathbb{R}^3 \).

Theorem 4 ([11] and [12].). In the situation of Theorem 1, if at least one of the following two hold, then \( M \) admits no special generic map into \( \mathbb{R}^4 \).

(1) \( p \in B \oplus C \) is not zero.

(2) In the sequence \( \{a_{i,j}\}_{j=1}^n \) and the family \( \{h_{i,j}\} \) of all the entries of the \( b \times b \) matrix, at least one non-zero number exists.

In the present paper, the main theorem is the following.

Main Theorem. Let \( A \) and \( B \) be free commutative groups of rank \( a \) and \( b \). Let \( \{k_j\}_{j=1}^a \) be a sequence of integers such that integers in \( \{k_j + a\}_{j=1}^b \) are 0 or 1. Let \( Y_0 \) be a 4-dimensional closed and simply-connected spin manifold whose integral cohomology ring is isomorphic to that of a manifold represented as a connected sum of finitely many copies of \( S^2 \times S^2 \) and denote \( H^j(Y_0; \mathbb{Z}) \) by \( H^j \). In this situation, there exist a 7-dimensional closed and simply-connected spin manifold \( M \) and an \( S \)-trivial fold map \( f : M \to \mathbb{R}^4 \) such that the following properties hold.

(1) \( H_*(M; \mathbb{Z}) \) is free.

(2) Isomorphisms \( H^2(M; \mathbb{Z}) \cong A \oplus H^2, H^3(M; \mathbb{Z}) \cong B \oplus H^2, H^4(M; \mathbb{Z}) \cong B \oplus H^2 \) and \( H^5(M; \mathbb{Z}) \cong A \oplus H^2 \) hold and by fixing suitable identifications we have the following properties.

(a) Products of elements in \( A \oplus \{0\} \subset H^2(M; \mathbb{Z}) \) vanish.

(b) Products of elements in \( A \oplus \{0\} \subset H^2(M; \mathbb{Z}) \) and \( B \oplus \{0\} \subset H^3(M; \mathbb{Z}) \) vanish.

(c) Consider a suitable generator \( \{(a_j, 0)\}_{j=1}^a \) of \( A \oplus \{0\} \subset H^2(M; \mathbb{Z}) \) and a suitable generator \( \{(b_j, 0)\}_{j=1}^b \) of \( B \oplus \{0\} \subset H^3(M; \mathbb{Z}) \). We also take a suitable generator \( \{(0, h_j^*)\}_{j=1}^{\text{rank}H^2} \) of \( \{0\} \oplus H^2 \subset H^4(M; \mathbb{Z}) \). The product of \( (a_j, 0) \) and \( (0, h_j^*) \) is regarded as \( (k_{j_1}, h_{j_2}, 0) \in \{0\} \oplus H^2 \subset H^4(M; \mathbb{Z}) \). The product of \( (b_j, 0) \) and \( (0, h_j^*) \) is regarded as \( (k_{a+j}, h_{j}, 0) \in \{0\} \oplus H^2 \subset H^5(M; \mathbb{Z}) \).

(d) For the suitable generator \( \{(0, h_j^*)\}_{j=1}^{\text{rank}H^2} \) of \( \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z}) \) just before, we have the following properties.

(i) \( \text{rank}H^2 \) is even.

(ii) We can take the generator so that the dual PD_\(X(h_j^*) \) of PD_\(X(h_j^*) \) is \( h_{a+j} \). For \( 1 \leq j \leq \frac{\text{rank}H^2}{2} \).

(iii) For the suitable generator before, the product of \( (0, h_{j_1}^*) \in \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z}) \) and \( (0, h_{j_2}^*) \in \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z}) \) vanishes unless \( |j_1 - j_2| = \frac{\text{rank}H^2}{2} \) and the product of \( (0, h_{j_1}^*) \in \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z}) \) and \( (0, h_{j_2}^*) \in \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z}) \) is \( \Sigma_{j=1}^{\text{rank}H^2} A_{a+j} \subset B \oplus \{0\} \cong H^4(M; \mathbb{Z}) \) where \( |j_1 - j_2| = \frac{\text{rank}H^2}{2} \).

For this class of manifolds, we can also show the following theorem.

Main Theorem. In Main Theorem 1.3, we can obtain manifolds which we cannot obtain in Theorem 1 under the constraint that the matrix \( (h_{i,j}) \) is the zero matrix.
1.4. The content of the present paper. The organization of the paper is as the following. In the next section, we demonstrate construction of new fold maps on 7-dimensional closed, simply-connected and spin manifolds and have Main Theorems (Theorems 5 and 6). Key methods resemble methods in the referred articles in Theorems 1 and 4 in a sense and are also mainly based on arguments in [9], [10] and so on. It is an important fact that these two articles of the author are mainly for studies of topological properties of Reeb spaces of fold maps. The Reeb space $W_c$ of a map $c : X \to Y$ is the quotient space $X/\sim_c$ obtained by considering the following equivalence relation $\sim_c$ on $X$: for two points $x_1, x_2 \in X$, $x_1 \sim_c x_2$ if and only if they are in a same connected component of a same preimage. Reeb spaces already appeared in [16] for example. For a fold map, the Reeb space has been shown to be a polyhedron whose dimension is equal to that of the target space in [13], [23], and so on, and Reeb spaces have been fundamental tools in studying algebraic or geometric properties, especially, (algebraic) topological properties of the manifolds. Note also that investigating the homology groups and the cohomology rings of the manifolds are different from investigating those of the Reeb spaces and more difficult. In considerable cases, Reeb spaces inherit topological invariants of the manifolds admitting the maps such as homology groups, cohomology rings, and so on, of the manifolds.

2. Construction of new family of fold maps on 7-dimensional closed and simply-connected manifolds of a new class.

Hereafter, $M$ denotes a closed and connected manifold of dimension $m$, $n < m$ is a positive integer and $f : M \to \mathbb{R}^n$ denotes a smooth map unless otherwise stated. For a topological space $X$ such as a manifold, which is regarded as a polyhedron in a canonical way, and a general polyhedron, let $c$ be a homology class. The class $c$ is represented by a closed and compact submanifold $Y$ with no boundary, if for a homology class $\nu_Y$ of degree $\dim Y$ canonically obtained from $Y$, $i_*(\nu_Y) = c$ where $i : Y \to X$ denotes the inclusion: in other words $\nu_Y$ is the fundamental class if $Y$ is connected, orientable and oriented and characterized as the generator of the homology group of degree $\dim Y$ respecting the orientation.

The following special generic maps play important roles.

Example 1. Let $l \geq 0$ be an integer. Let $m > n \geq 2$ be integers. A closed and connected manifold $M$ of dimension $m$ represented as a connected sum of manifolds of a family $\{S^{l_j} \times S^{m-l_j}\}_{j=1}^l$ satisfying $1 \leq l_j \leq n-1$ admits a special generic map $f : M \to \mathbb{R}^n$ satisfying the following properties.

(1) $f|_{S(f)}$ is an embedding.

(2) $f(M)$ is a compact submanifold and represented as a boundary connected sum of manifolds of a family $\{S^{l_j} \times D^{n-l_j}\}$.

(3) The following two submersions, or smooth maps with no singular point, give a trivial liner bundle whose fiber is $D^{m-n+1}$ and a smooth bundle whose fiber is $S^{m-n}$, respectively.

(a) The composition of the restriction of $f$ to the preimage of a small collar neighborhood of $\partial f(M)$ with the canonical projection to $\partial f(M)$.

(b) The restriction of $f$ to the preimage of the complementary set of the interior of the collar neighborhood before in $f(M)$.

(4) We denote the surjection obtained by restricting the target space of $f$ to $f(M)$ by $f_M$. The homomorphism $f_{M*}$ between the homology groups maps
a class represented by \( S^i_j \times \{ *_{j,1} \} \subset S^i_j \times S^{m-l_j} \) in the connected sum to a class represented by \( S^i_j \times \{ *_{j,2} \} \subset S^i_j \times D^{n-l_j} \subset S^i_j \times D^{n-l_j} \) in the boundary connected sum \( f(M) \).

The following proposition is a fundamental fact and rigorous proofs are left to readers.

**Proposition 2.** In the situation of Example 1, let \((m, n) = (7, 4)\), \(l > 0\) and \(a, b, c \geq 0\) be integers satisfying \( l_a + l_b = l \) and let \( l_j = 2\) for \( 1 \leq j \leq l_a \) and \( l_1 = 3 \) for \( l_a + 1 \leq j \leq l \). We choose a suitable class represented by \( S^i_j \times \{ *_{j,1} \} \subset S^i_j \times S^{m-l_j} \) in the connected sum by \( \nu_j \). We have the following two.

1. For a copy \( X \) of \( S^3 \), put a generator \( \nu_{X,3} \) of its 3rd integral homology group, isomorphic to \( \mathbb{Z} \), or a fundamental class. Let \( \{ a_{j+1} \}_{j=1}^{l} \) be a sequence of integers of length \( l \) such that the integers are 0 or 1. In this situation, there exists an embedding \( i_{X,f(M)} \) of \( X \) into the interior of \( f(M) \) such that
\[
i_{X,f(M)}(\nu_{X,3}) = \sum_{j=1}^{l} a_{j+1} f_M(\nu_j).
\]
2. For a copy \( X \) of \( S^2 \times S^1 \), put a generator \( \nu_{X,2} \) of its 2nd integral homology group, isomorphic to \( \mathbb{Z} \), and a generator \( \nu_{X,3} \) of its 3rd integral homology group, isomorphic to \( \mathbb{Z} \), or a fundamental class. Let \( \{ a_j \}_{j=1}^{l} \) be a sequence of integers of length \( l \) such that the integers in \( \{ a_{j+1} \}_{j=1}^{l} \) are 0 or 1. In this situation, there exists an embedding \( i_{X,f(M)} \) of \( X \) into the interior of \( f(M) \) such that
\[
i_{X,f(M)}(\nu_{X,2}) = \sum_{j=1}^{l} a_j f_M(\nu_j)
\] and that
\[
i_{X,f(M)}(\nu_{X,3}) = \sum_{j=1}^{l} a_j f_M(\nu_j+1).
\]

For a compact manifold \( X \), let there exist a closed and connected manifold \( X_0 \) such that \( X \) is obtained by removing the interior of the union of two smoothly and disjointly embedded unit discs of \( \dim X_0 \) and a Morse function \( c : X_0 \to \mathbb{R} \) such that at distinct singular points the values are distinct, that there exist exactly two local extrema \( a < b \), that their preimages are in the two embedded unit discs and that on the distinctly embedded unit discs there exists no other singular point. We denote the restriction \( c|_X \) by \( c_{X,(X_0,x)} \).

For a graded commutative algebra \( A \) over \( \mathbb{Z} \), we define the \( i \)-th module as the module consisting of all elements of degree \( i \) of \( A \). We also assume that the 0-th module is isomorphic to \( \mathbb{Z} \).

For a non-negative integer \( i \geq 0 \), we define the \( \leq i \)-part \( A_{\leq i} \) of \( A \) as a graded commutative algebra over \( \mathbb{Z} \) as the following and as a graded module, this is regarded as a submodule of the module \( A \).

1. The \( j \)-th module is same as that of \( A \) for any \( j \leq i \).
2. The product of two elements such that the sum of the degrees is smaller than or equal to \( i \) is same as that in the case of \( A \).
3. The \( j \)-th module is the zero module for any \( j > i \).

**Proposition 3.** In the situation just before, let \( X_0 \) be orientable satisfying \( \dim X_0 > 1 \) and let \( i_X : X \to X_0 \) denote the inclusion.

1. The restriction of \( i_X^* : H^*(X_0; \mathbb{Z}) \to H^*(X; \mathbb{Z}) \) to \( H^*(X_0; \mathbb{Z})_{\leq \dim X - 2} \) is an isomorphism between the graded commutative algebras \( H^*(X_0; \mathbb{Z})_{\leq \dim X - 2} \) and \( H^*(X; \mathbb{Z})_{\leq \dim X - 2} \).
2. The restriction of \( i_X^* : H^*(X_0; \mathbb{Z}) \to H^*(X; \mathbb{Z}) \) to \( H^*(X_0; \mathbb{Z})_{\leq \dim X - 1} \) gives a monomorphism between the graded commutative algebras \( H^*(X_0; \mathbb{Z})_{\leq \dim X - 1} \).
and $H^*(X; \mathbb{Z})_{\leq \dim X-1}$ and $H^*(X; \mathbb{Z})_{\leq \dim X-1}$ is represented as the internal direct sum of the image of the monomorphism and a commutative subgroup $G$ of $H^*(X; \mathbb{Z})_{\leq \dim X-1}$ isomorphic to $\mathbb{Z}$.

We need several notions and explain them. For a closed, connected and oriented manifold $X$, we denote the so-called Poincaré dual to an integral (co)homology class $c$ by $\text{PD}_X(c)$. If for a (compact) topological space $X$ whose integral homology group is free, then the dual $c^* \in H^i(X; \mathbb{Z})$ of a homology class $c \in H_j(X; \mathbb{Z})$ cannot represent as a form $kc'$ such that $k \neq 0, 1, -1$ is an integer and that $c'$ is not zero is defined as a unique element satisfying the following two.

(1) $c^* (c) = 1$.
(2) For any subgroup $G$ of $H_j(X; \mathbb{Z})$ such that $H_j(X; \mathbb{Z})$ is the internal direct sum of the group generated by $c$ and $G$, $c^*(G) = 0$.

Moreover, $\cong$ between groups denotes an isomorphism.

**Theorem 5.** Let $A$ and $B$ be free commutative groups of rank $a$ and $b$. Let $\{k_j\}_{j=1}^{a+b}$ be a sequence of integers such that integers in $\{k_{j+a}\}_{j=1}^b$ are 0 or 1. Let $Y_0$ be a 4-dimensional closed and simply-connected spin manifold whose integral cohomology ring is isomorphic to that of a manifold represented as a connected sum of finitely many copies of $S^2 \times S^2$ and denote $H^i(Y_0; \mathbb{Z})$ by $H^i$. In this situation, there exist a 7-dimensional closed and simply-connected spin manifold $M$ and an S-trivial fold map $f : M \to \mathbb{R}^4$ such that the following properties hold.

(1) $H_*(M; \mathbb{Z})$ is free.
(2) Isomorphisms $H^2(M; \mathbb{Z}) \cong A \oplus B^2$, $H^3(M; \mathbb{Z}) \cong B \oplus H^2$, $H^4(M; \mathbb{Z}) \cong B \oplus H^2$ and $H^5(M; \mathbb{Z}) \cong A \oplus H^2$ hold and by fixing suitable identifications we have the following properties.

(a) Products of elements in $A \oplus \{0\} \subset H^2(M; \mathbb{Z})$ vanish.
(b) Products of elements in $A \oplus \{0\} \subset H^2(M; \mathbb{Z})$ and $B \oplus \{0\} \subset H^3(M; \mathbb{Z})$ vanish.
(c) Consider a suitable generator $\{(a_j, 0)\}_{j=1}^a$ of $A \oplus \{0\} \subset H^2(M; \mathbb{Z})$ and a suitable generator $\{(b_j, 0)\}_{j=1}^b$ of $B \oplus \{0\} \subset H^3(M; \mathbb{Z})$. We also take a suitable generator $\{(0, h_j^*)\}_{j=1}^{\text{rank} H^2}$ of $\{0\} \oplus H^2 \subset H^2(M; \mathbb{Z})$. The product of $(a_j, 0)$ and $(0, h_j^*)$ is regarded as $(0, k_j, h_j^*) \in \{0\} \oplus H^2 \subset H^4(M; \mathbb{Z})$. The product of $(b_j, 0)$ and $(0, h_j^*)$ is regarded as $(0, k_{a+j}, h_j^*) \in \{0\} \oplus H^2 \subset H^5(M; \mathbb{Z})$.
(d) For the suitable generator $\{(0, h_j^*)\}_{j=1}^{\text{rank} H^2}$ of $\{0\} \oplus H^2 \subset H^2(M; \mathbb{Z})$ just before, we have the following properties.

(i) $\text{rank} H^2$ is even.
(ii) We can take the generator so that the dual PD$_X(h_j^*)^*$ of PD$_X(h_j^*)$ is $h_{a+j, 0}^{\text{rank} H^2}$ for $1 \leq j \leq \frac{\text{rank} H^2}{2}$.
(iii) For the suitable generator before, the product of $(0, h_j^*) \in \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z})$ and $(0, h_j^*) \in \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z})$ vanishes unless $|j_1 - j_2| = \frac{\text{rank} H^2}{2}$ and the product of $(0, h_j^*) \in \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z})$ and $(0, h_j^*) \in \{0\} \oplus H^2 \subset H^2(M; \mathbb{Z})$ is $\Sigma_{j=1}^{\text{rank} H^2}(k_{a+j}, h_j^*, 0) \in B \oplus \{0\} \cong H^4(M; \mathbb{Z})$ where $|j_1 - j_2| = \frac{\text{rank} H^2}{2}$.

**Proof.** First we construct a special generic map $f_0$ on an 7-dimensional manifold $M_0$ into $\mathbb{R}^4$ in Example 1 by setting $(l_a, l_b) = (a, b)$ in Proposition 2. We replace
a_j by k_j there. Take a copy X in Proposition 2 and remove the interior of a small
closed tubular neighborhood N(X) and its preimage.

Replace this removed map by a product map of c_{(Y_0, y)} before where X and
X_0 are replaced by a suitable compact manifold Y and Y_0 with a suitable Morse
function c respectively and the identity map on a manifold diffeomorphic to X.
Note that this is regarded as a finite iteration of normal bubbling operations in [9],
[10], and so on. Thus we have a desired fold map f.

We observe the integral homology group and the integral cohomology ring of
the manifold M to complete the proof. First we show the first statement (1) and
the second statement (2) on the cohomology group. H_2(M_0; \mathbb{Z}) and H_3(M_0; \mathbb{Z})
are generated by classes represented by standard spheres in M_0 and apart from
f_0^{-1}(N(X)) and from these standard spheres and homology classes represented
by them we have subgroups isomorphic to H_2(M_0; \mathbb{Z}) and H_3(M_0; \mathbb{Z}) respectively.
We put A := H^2(M_0; \mathbb{Z}) and B = H^3(M_0; \mathbb{Z}); we take the duals of classes in the
generators. Consider a generator of H_2(Y_0; \mathbb{Z}) \cong H_2(Y; \mathbb{Z}) such that we can define
the dual of each of the elements in the generator. We go to (2d). Since Y_0 is closed
and simply-connected, the homology groups and the cohomology groups of Y and
Y_0 are free and the first two statements hold (for H^2). Furthermore, we can obtain
the generator(s) by first obtaining a generator of H_2(Y_0; \mathbb{Z}) \cong H_2(Y; \mathbb{Z}) for each
of which we can define the dual and taking the duals. By considering elements of
the generators of A and H^2 as integral cohomology classes of M in a canonical
way, we can observe the structure of H^2(M; \mathbb{Z}) and obtain a desired identification.
H^4(M; \mathbb{Z}) is regarded as the internal direct sum of the following two subgroups
A_{4,1} and A_{4,2}.

(1) The classes represented by standard spheres in M_0 and apart from f_0^{-1}(N(X))
and forming a generator of H_3(M_0; \mathbb{Z}). Consider the classes represented by
these spheres in M and their Poincaré duals. We define A_{4,1} as the sub-
group generated by all the Poincaré duals. This is isomorphic to B.
(2) We take the image of a section of the trivial bundle define by the restriction
of f_0 to f_0^{-1}(X). Each element of the suitable generator of H_2(Y_0; \mathbb{Z}) \cong
H_2(Y; \mathbb{Z}) used to obtain the generator of H^2 before is represented by a
2-dimensional closed submanifold with no boundary and we can obtain a
product of the image of the section over S^2 \times \{\ast\} \subset X = S^2 \times S^1 and this
submanifold in M in a canonical way. We can take the dual of the 4-th
integral homology class represented by the submanifold as a 4-th integral
cohomology class of M. A_{4,2} is defined as a subgroup generated by the set
of all such classes. This is isomorphic to H^2.

By considering elements of the generator of B and H^2, isomorphic to A_{4,2}, as the
image of the map corresponding each element of A_{4,2} to the Poincaré dual, we can
observe the structure of H^3(M; \mathbb{Z}) and obtain a desired identification. H^5(M; \mathbb{Z}) is
regarded as the internal direct sum of the following two subgroups A_{5,1} and A_{5,2}.

(1) The classes represented by standard spheres in M_0 and apart from f_0^{-1}(N(X))
and forming a generator of H_3(M_0; \mathbb{Z}). Consider the classes represented by
these spheres in M and their Poincaré duals. We define A_{5,1} as the sub-
group generated by all the Poincaré duals. This is isomorphic to A.
(2) We take the image of a section of the trivial bundle define by the restriction
of f_0 to f_0^{-1}(X). Each element of the suitable generator of H_2(Y_0; \mathbb{Z}) \cong
H_2(Y; \mathbb{Z}) used to obtain the generator of H^2 before is represented by a
2-dimensional closed submanifold with no boundary and we can obtain a product of the image of the section over $X = S^2 \times S^1$ and this submanifold in $M$ in a canonical way. We can take the dual as a 5-th integral cohomology class of $M$. $A_{5,2}$ is defined as a subgroup generated by the set of all such classes. This is isomorphic to $H^2$.

Note also that we can demonstrate this construction so that $M$ is simply-connected and spin.

We can show (2a) and (2b) easily by the structures of the cohomology groups and (2c) follows by the structures of the cohomology groups and Proposition 2. The third statement of (2d) follows by the structures of the cohomology groups and Propositions 2 and 3. This completes the proof.

□

Theorem 6. In Theorem 5, we can obtain manifolds which we cannot obtain in Theorem 1 under the constraint that the matrix $(h_{i,j})$ is the zero matrix.

Proof. In Theorem 5, set $a = 1$, $b = 1$, $k_1 \neq 0$, $k_2 \neq 0$ and $Y_0 := S^2 \times S^2$. We can easily see that $H^j(M; \mathbb{Z})$ is free and of rank 3 for $j = 2, 3, 4, 5$. We cannot take a submodule of rank 2 of $H^2(M; \mathbb{Z})$ consisting of elements such that the squares are zero. In Theorem 1 under the constraint that the matrix $(h_{i,j})$ is the zero matrix, we consider a 7-dimensional closed and simply-connected manifold such that $H^j(M; \mathbb{Z})$ is free and of rank 3. Products of elements in $A \oplus \{0\} \subset H^2(M; \mathbb{Z})$ and products of elements in $\{0\} \oplus B \subset H^2(M; \mathbb{Z})$ vanish. This implies that we can take a submodule of rank 2 of $H^2(M; \mathbb{Z})$ consisting of elements such that the squares are zero.

This completes the proof. □

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