Self-Dual Chern-Simons Higgs Systems
with an N=3 Extended Supersymmetry

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Abstract

We construct and study an $N = 3$ supersymmetric Chern-Simons Higgs theory. This theory is the maximally supersymmetric one containing the self-dual models with a single gauge field and no gravity.

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1. Introduction

Recently there has been considerable interest in self-dual Chern-Simons-Higgs systems in 2+1 dimensional spacetime. With a specific sixth-order Higgs potential which has degenerate symmetric and asymmetric minima, it has been shown that there are topological vortices and nontopological solitons satisfying self-dual or Bogomolny-type equations. This Higgs potential is shown to be determined by requiring an \( N = 2 \) extended supersymmetry. The central charge of the extended supersymmetric algebra gives a quantum version of the Bogomolny bound. Renormalization group and finite temperature properties of this \( N = 2 \) supersymmetric theory has been studied.

It is well known that a theory with extended supersymmetry has less ultraviolet divergence. Some supersymmetric theories have been shown to be even finite. It would be interesting to know whether there is a supersymmetric Chern-Simons Higgs theory which is ultraviolet finite. It is not so hard to see that the \( N = 2 \) supersymmetric theory in Ref.[2] is not finite perturbatively. Motivated partially by this observation, we construct in this paper an \( N = 3 \) supersymmetric Chern-Simons Higgs theory and study some properties of it. This theory turns out to be the ‘maximally supersymmetric’ Chern-Simons Higgs theory with a single gauge field and no gravity.

In four dimensional spacetime, an \( N = 3 \) supersymmetric theory becomes \( N = 4 \) supersymmetric because of the \( CPT \) and rotational symmetry. (For a general reference for supersymmetry, see for example Ref.[6,7,8].) In three dimensional spacetime a particle and its antiparticle carry the same spin, say \( s \). Under rotation a state of angular momentum \( j \) transforms into itself with some phase. There is no need to have a state of spin \(-s\) in the theory. If we represent anyons as
charged particles in a Chern-Simons Higgs theory, both anyons and antianyons have spin \( s = 1/(4\pi\kappa) \) where \( \kappa \) is the coefficient of the Chern-Simons term. They can annihilate each other into neutral integer spin particles because the orbital angular momentum between them is \(-2s + \text{integer}\). Hence in three dimensional spacetime there can be an \( N = 3 \) supersymmetric theory.

In section 2 we construct an \( N = 2 \) supersymmetric Chern-Simons Higgs theory in three dimensional spacetime by applying the dimensional reduction method to the four dimensional \( N = 1 \) supersymmetric QED. This \( N = 2 \) theory does not have any central charge and is different from the theory in Ref.[2]. In section 3 we eliminate the auxiliary fields from the lagrangian and choose a special set of coupling constants, resulting in an additional global \( U(1) \) symmetry. We use this new global symmetry to complexify one of \( N = 2 \) supersymmetries, leading to an \( N = 3 \) extended supersymmetry. This is quite similar to the method used in Ref.[2] to get the \( N = 2 \) supersymmetry. In section 4 we study some properties of the theory. In section 5 we investigate the \( N = 3 \) supersymmetric operator algebra. There is a quantum version of the Bogomolny-type bound on the energy functional. We show that the maximally supersymmetric Chern-Simons Higgs theory with a single gauge field and no gravity is \( N = 3 \). In section 6, we conclude with some remarks.
2. N = 2 Supersymmetry

By the dimensional reduction method, one can obtain a theory of larger supersymmetry. For example, the $N = 1$ supersymmetric QED in four dimensional spacetime will lead to a $N = 2$ supersymmetric QED in three dimensional spacetime. The dimensional reduction method specifies both the lagrangian and the supersymmetric transformation law. We start by reviewing the $N = 1$ supersymmetric QED in four dimensional spacetime.\(^9\) The gauge multiplet is made of $A_\mu, \lambda, D$ and the chiral matter multiplet is made of $z_k, \psi_k, f_k$, where $k = 1, 2$ denote the components of a vector under the $U(1)$ gauge symmetry. The sign of the metric is chosen to be $(-, +, +, +)$. The gamma matrices in the Majorana representation are

\[
\begin{align*}
\gamma^0 &= -i \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix},
\gamma^1 &= \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix},
\gamma^2 &= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix},
\gamma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

We use the convention where $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ and $\bar{\psi} = \psi^\dagger \gamma^0$. The fermion fields are Majorana or $\lambda^* = \lambda, \psi_k^* = \psi_k$.

The covariant derivative of the matter multiplet will be given by $D_\mu z_k = \partial_\mu z_k - e A_\mu z^k$, where $z^1 = z_2$ and $z^2 = -z_1$. The standard $N = 1$ supersymmetric kinetic term for a charged matter multiplet coupled to the gauge multiplet is then

\[
\mathcal{L}_K = -|D_\mu z_k|^2 + |f_k|^2 - i\bar{\psi}_k \gamma^\mu D_\mu \psi_k + ieDz_k z^{k*} + 2ie\bar{\lambda}[\psi_k L z^{k*} + \psi_k R z^k] \quad (2.1)
\]

where $\psi_{L,R} = (1 \pm i\gamma_5)\psi/2$. Besides the $U(1)$ gauge symmetry, the kinetic term has an additional chiral $U(1)$ symmetry, which changes the phases of $z_k, f_k^*, \psi_k L, \psi_k R$ uniformly.
The self-interaction of the matter multiplet is given by the gauge invariant superpotential $W$, which as a function of $z_k$ is given by

$$W = \frac{m}{2}(z_k)^2 + \frac{g}{4}(z_k^2)^2$$

(2.2)

where the coupling constants $m, g$ are complex. From the above superpotential one can obtain the supersymmetric self-interaction lagrangian for the matter multiplet,

$$\mathcal{L}_P = \frac{\delta W}{\delta z_k} f_k - i \frac{\delta^2 W}{\delta z_k \delta z_l} \bar{\psi}_{kL} \psi_{lL} + h.c.$$  

(2.3)

This self-interaction becomes renormalizable only after dimensional reduction to three dimensional spacetime.

The kinetic and interacting terms of the matter multiplet in Eqs. (2.1) and (2.3) are invariant under the $N = 1$ supersymmetric transformation. The gauge multiplet transforms as

$$\delta A_\mu = i \bar{\alpha} \gamma_\mu \lambda$$

$$\delta \lambda = -\frac{1}{2} F_{\mu \nu} \gamma^{\mu \nu} \alpha + D_5 \gamma \alpha$$

$$\delta D = i \bar{\alpha} \gamma_5 \gamma^\mu \partial_\mu \lambda$$

(2.4)

where the parameter $\alpha$ is a Majorana spinor. The matter multiplet transforms as follow,

$$\delta z_k = 2i \bar{\alpha} \psi_{kL}$$

$$\delta \psi_{kL} = D_\mu z_k \gamma^\mu \alpha_R + f_k \alpha_L$$

$$\delta f_k = 2i \bar{\alpha} \gamma^\mu D_\mu \psi_{kL} - 2ie \bar{\alpha} \lambda R z^k$$

(2.5)

Dimensional reduction to three dimensional spacetime is done by assuming that the fields are independent of the third spatial coordinate. In three dimensional
spacetime, we keep the same notation for the gamma matrices. This will not cause any confusion because all the following derivations and calculation will be in three dimensional spacetime. The gamma matrices in the three dimensional spacetime are again in the Majorana representation and are given by

\[ \gamma^0 = -i\sigma^2, \gamma^1 = \sigma^3, \gamma^2 = \sigma^1, \]

which satisfy \( \gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu\rho} \gamma^\rho \) with \( \epsilon^{012} = 1 \). We then rename the third component of the gauge field and split four-component spinors into two two-component spinors in three dimension;

\[ A_3 = C, \quad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \]

Let us first apply the dimensional reduction method to the supersymmetric transformation of the gauge multiplet. The supersymmetric transformation (2.4) becomes

\[ \delta A_\mu = i\bar{\alpha}_a \gamma_\mu \lambda_a \]
\[ \delta \lambda_a = -B_\mu \gamma_\mu \alpha_a + \partial_\mu C \gamma^\mu \alpha^a + D \alpha^a \]
\[ \delta C = i\bar{\alpha}_a \lambda_a \]
\[ \delta D = i\bar{\alpha}_a \gamma^\mu \partial_\mu \lambda_a \] (2.6)

where \( B_\mu = \epsilon^{\mu\nu\rho} F_{\nu\rho}/2, a = 1, 2, \) and \( \alpha^1 = \alpha_2, \alpha^2 = -\alpha_1. \) With two independent parameters \( \alpha_1, \alpha_2, \) the supersymmetric transformation becomes \( N = 2 \) in three dimensional spacetime. \( A_\mu, \lambda_a, C, D \) form an \( N = 2 \) supersymmetric vector multiplet. We are interested in Chern-Simons rather than Maxwell kinetic term. The \( N = 2 \) supersymmetric version of the Chern-Simons kinetic term can be easily
guessed by using the supersymmetric transformation (2.6) and is

\[ \mathcal{L}_{CS} = \frac{\kappa}{2} (\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - i\bar{\lambda}_a \lambda_a + 2CD). \]  (2.7)

The dimensional reduction of the scalar multiplet is facilitated by redefining fields,

\[ \phi_1 = \text{Re} z_1 + i\text{Re} z_2 \]
\[ \phi_2 = \text{Im} z_1 + i\text{Im} z_2 \]
\[ F_1 = \text{Re} f_1 + i\text{Re} f_2 \]
\[ F_2 = -\text{Im} f_1 - i\text{Im} f_2 \]
\[ (\psi \chi) = \psi_1 + i\psi_2 \]  (2.8)

where \( \psi, \chi \) are two-component spinors. The complex fields \( \phi_a, \psi, \chi, F_a \), form an \( N = 2 \) supermultiplet in three dimensional spacetime and carry unit electric charge. The transformation (2.5) becomes the \( N = 2 \) supersymmetric transformation,

\[ \delta \phi_a = i(\bar{\alpha}^a \psi + \bar{\alpha}_a \chi) \]
\[ \delta \psi = D_\mu \phi_a \gamma^\mu \alpha_a + F_a \alpha_a + ieC \phi_a \alpha_a \]
\[ \delta \chi = D_\mu \phi_a \gamma^\mu \alpha_a + F_a \alpha^a - ieC \phi_a \alpha^a \]
\[ \delta F_a = i\bar{\alpha}_a \gamma^\mu D_\mu \psi + i\bar{\alpha}^a \gamma^\mu D_\mu \chi \]
\[ -eC(\bar{\alpha}^a \psi - \bar{\alpha}_a \chi) - e(\bar{\alpha}^a \lambda_b \phi_b + \bar{\alpha}_a \lambda^b \phi_b) \]  (2.9)

where \( D_\mu \phi_a = \partial_\mu \phi_a + ieA_\mu \phi_a \), etc. The kinetic term (2.1) becomes

\[ \mathcal{L}_K = -|D_\mu \phi_a|^2 - |F_a|^2 - i\bar{\psi} \gamma^\mu D_\mu \psi - i\bar{\chi} \gamma^\mu D_\mu \chi \]
\[ -e^2 C^2 |\phi_a|^2 + eC(\bar{\chi} \psi - \bar{\psi} \chi) + ieD_\mu \phi_a \phi^{a*} \]
\[ + e\bar{\lambda}_a [\psi \phi^{a*} - \psi^* \phi^a - \chi \phi_a^* + \chi^* \phi_a] \]  (2.10)
The dimensional reduction of the potential energy (2.3) leads to

$$\mathcal{L}_P = \frac{1}{2}[m + g(\phi_1 + i\phi_2)(\phi_1^* + i\phi_2^*)][(\phi_a + i\phi^a)F_a^* + (\phi_a^* + i\phi^a*)F_a]$$

$$+ \frac{i}{2}[m + 2g(\phi_1 + i\phi_2)(\phi_1^* + i\phi_2^*)][i\bar{\psi}\psi - i\bar{\chi}\chi - \bar{\psi}\psi - \bar{\chi}\chi]$$

$$+ \frac{ig}{4}(\phi_1 + i\phi_2)^2(i\bar{\psi}\psi^* - i\bar{\chi}\chi^* - 2\bar{\psi}\psi)$$

$$+ \frac{ig}{4}(\phi_1^* + i\phi_2^*)^2(i\bar{\psi}^*\psi - i\bar{\chi}^*\chi - 2\bar{\psi}^*\psi) + h.c. \tag{2.11}$$

The $N = 2$ supersymmetric lagrangian in three dimensional spacetime is then the sum of the kinetic terms in Eqs. (2.7) and (2.10) and the potential term (2.11), which is given symbolically by

$$\mathcal{L}_{N=2} = \mathcal{L}_{CS} + \mathcal{L}_K + \mathcal{L}_P \tag{2.12}$$

Note that $\mathcal{L}_{CS} + \mathcal{L}_K$ are invariant under the global $O(2)$ rotation in $a$-indices. In contrast to theory in Ref. [2], this $R$-symmetry is broken by $\mathcal{L}_P$. 
3. N=3 Supersymmetry

We now have an \( N = 2 \) supersymmetric Chern-Simons-Higgs theory (2.12), which has a \( U(1) \) gauge symmetry. Let us eliminate the auxiliary fields \( \lambda, C, D, F \), from the \( N = 2 \) supersymmetric lagrangian (2.11) by using their field equations,

\[
\lambda_a = -\frac{ie}{\kappa} (\phi^a \psi - \phi^a \psi^* - \phi_a^* \chi + \phi_a \chi^*) \\
C = -\frac{ie}{\kappa} \phi_a \phi^a \\
D = -2\frac{ie^3}{\kappa^2} |\phi_a|^2 \phi_a \phi^a - \frac{e}{\kappa} (\bar{\chi} \psi - \bar{\psi} \chi) \\
F_a = -\phi_a [m_R + g_R (|\phi_1|^2 - |\phi_2|^2) - g_I (\phi_1 \phi_2^* + \phi_2 \phi_1^*)] \\
+ \phi_a^* [m_I + g_I (|\phi_1|^2 - |\phi_2|^2) + g_R (\phi_1 \phi_2^* + \phi_2 \phi_1^*)]
\]

where the subscripts \( R, I \) to coupling constants denotes the real and imaginary parts.

The terms depending on the gaugino field \( \lambda_a \) become

\[
\mathcal{L}_\lambda = \frac{ie^2}{2\kappa} \left[ 2|\phi_a|^2 (\bar{\psi} \psi + \bar{\chi} \chi) - (\phi_a)^2 (\bar{\psi} \psi^* + \bar{\chi} \chi^*) - (\phi_a^*)^2 (\bar{\psi} \psi + \bar{\chi} \chi^*) - 2\phi_a \phi^a (\bar{\chi} \psi - \bar{\psi} \chi) \right]
\]

The terms depending on \( C, D \), become

\[
\mathcal{L}_{CD} = \frac{e^4}{\kappa^2} |\phi_a|^2 (\phi_a \phi^a)^2 - \frac{ie^2}{\kappa} \phi_a \phi^a (\bar{\chi} \psi - \bar{\psi} \chi)
\]

The terms depending on \( F_a \) become

\[
\mathcal{L}_F = -|\phi_a|^2 \left\{ |m|^2 + |g|^2 (|\phi_1|^2 - |\phi_2|^2)^2 + |g|^2 (\phi_1 \phi_2^* + \phi_2 \phi_1^*)^2 \\
+ (mg^* + m^* g)(|\phi_1|^2 - |\phi_2|^2) - i(mg^* - m^* g)(\phi_1 \phi_2^* + \phi_2 \phi_1^*) \right\}
\]
From Eqs. (2.7) and (2.10), we obtain the kinetic part of the lagrangian,

$$\mathcal{K} = \frac{\kappa}{2} \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho + |D_\mu \phi_a|^2 - i \bar{\psi} \gamma^\mu D_\mu \psi - i \bar{\chi} \gamma^\mu D_\mu \chi$$  \hspace{1cm} (3.5)

We rewrite the fermion-boson interacting part of Eq. (2.11),

$$\mathcal{L}_{fi} = \left\{- \left\{ m_I + 2 g_I (|\phi_1|^2 - |\phi_2|^2) + 2 g_R (\phi_1 \phi_2^* + \phi_2 \phi_1^*) \right\} (i \bar{\psi} \psi - i \bar{\chi} \chi) 
- \left\{ m_R + 2 g_R (|\phi_1|^2 - |\phi_2|^2) - 2 g_I (\phi_1 \phi_2^* + \phi_2 \phi_1^*) \right\} (i \bar{\chi} \psi + i \bar{\psi} \chi) 
- \frac{1}{2} \left\{ g_I \left[ (\phi_1)^2 - (\phi_2)^2 \right] + 2 g_R \phi_1 \phi_2 \right\} (i \bar{\psi} \psi^* - i \bar{\chi} \chi^*) 
- \frac{1}{2} \left\{ g_I \left[ (\phi_1^*)^2 - (\phi_2^*)^2 \right] + 2 g_R \phi_1^* \phi_2^* \right\} (i \bar{\psi} \psi^* - i \bar{\chi} \chi^*) 
- \{ g_R \left[ (\phi_1^*)^2 - (\phi_2^*)^2 \right] - 2 g_I \phi_1 \phi_2 \} i \bar{\chi} \psi^* 
- \{ g_R \left[ (\phi_1^*)^2 - (\phi_2^*)^2 \right] - 2 g_I \phi_1 \phi_2 \} i \bar{\chi} \psi \right\}$$  \hspace{1cm} (3.6)

After we eliminate the auxiliary fields, the $N = 2$ supersymmetric lagrangian (2.12) becomes a sum,

$$\mathcal{L}_{N=2} = \mathcal{K} + \mathcal{L}_{CD} + \mathcal{L}_\lambda + \mathcal{L}_F + \mathcal{L}_{fi}$$  \hspace{1cm} (3.7)

For a moment, let us consider the part of the lagrangian (3.7) which depends on bosonic fields only. Besides the kinetic terms, there are self-interacting terms from Eqs. (3.3) and (3.4). If $m = v^2 g$ with a real number $v^2$, the last term in Eq. (3.4) vanishes. If $|g| = e^2 / \kappa$, there is a partial cancellation between the first term of Eq. (3.3) and the third term of Eq. (3.4), which implies that the phases of $\phi_1$ and $\phi_2$ can be rotated independently in the bosonic part of the lagrangian (3.7). Let us restrict ourselves from now on to the case where coupling constants satisfy

$$|g| = \frac{e^2}{\kappa}$$

$$m = v^2 g$$  \hspace{1cm} (3.8)

where $v^2$ is a real number. The Higgs potential is given by the bosonic part in the
sum of $-\mathcal{L}_{CD}$ and $-\mathcal{L}_F$, which is

$$U(\phi_a) = \frac{e^4}{\kappa^2} |\phi_a|^2 [(|\phi_1|^2)^2 + 2v^2(|\phi_1|^2 - |\phi_2|^2) + v^4]$$  (3.9)

Note that $U(\phi_a) \geq 0$ regardless of the sign of $v^2$. Without losing any generality, we will assume that $v^2 > 0$. The Higgs potential has two degenerate minima, the symmetric one where $< \phi_a > = 0$ and the asymmetric one where $< \phi_1 > = 0, < \phi_2 > = v$.

Let us now examine the fermionic parts of the lagrangian (3.7). The fermion mass term in the symmetric phase is the $m$ dependent terms in Eq. (3.6). One can see easily that the phases of $\psi, \chi$ can be rotated independently if $m$ is pure imaginary, which with Eq.(3.8) implies that $g = \pm ie^2/\kappa$. Let us choose the positive sign as one can see that choosing the negative sign is equivalent to our choice if $\psi$ and $\chi$ are exchanged. With our choice of the coupling constant,

$$g = \frac{m}{v^2} = \frac{ie^2}{\kappa}$$  (3.10)

the fermionic part of $\mathcal{L}_\lambda + \mathcal{L}_{CD} + \mathcal{L}_{fi}$ becomes the Yukawa interaction,

$$\mathcal{Y} = \frac{-e^2}{\kappa}v^2(i\bar{\psi}\psi - i\bar{\chi}\chi) + \frac{3e^2}{\kappa}(|\phi_1|^2 + |\phi_2|^2)(i\bar{\psi}\psi + i\bar{\chi}\chi)$$

$$- \frac{4ie^2}{\kappa}(\phi_1\bar{\psi} - \phi_2\bar{\chi})(\phi_1^*\psi - \phi_2^*\chi)$$

$$- \frac{ie^2}{\kappa}(\phi_1\bar{\psi} - \phi_2\bar{\chi})(\phi_1^*\psi - \phi_2^*\chi^*) - \frac{ie^2}{\kappa}(\phi_1^*\bar{\psi}^* - \phi_2^*\bar{\chi}^*)(\phi_1^*\psi - \phi_2^*\chi)$$  (3.11)

Finally the $N = 2$ supersymmetric lagrangian (3.7) becomes the sum $\mathcal{L} =$
\( \mathcal{L} = \frac{\kappa}{2} e^{\mu \nu \rho} A_\mu \partial_\nu A_\rho - |D_\mu \phi_a|^2 - \frac{i}{2} \bar{\psi} \gamma^\mu \tilde{D}_\mu \psi - \frac{i}{2} \bar{\chi} \gamma^\mu \tilde{D}_\mu \chi \\
abla \psi \phi_1 - |\phi_2|^2 + 2v^2(|\phi_1|^2 - |\phi_2|^2) + v^4 - \frac{e^2}{\kappa} v^2 (i\bar{\psi} \psi - i\bar{\chi} \chi) \\
abla \phi_1 - |\phi_2|^2 (i\bar{\psi} \psi + i\bar{\chi} \chi) - \frac{4ie^2}{\kappa} (\phi_1 \bar{\psi} - \phi_2 \bar{\chi})(\phi_1^* \psi - \phi_2^* \chi) \\
abla (\phi_1 \bar{\psi} - \phi_2 \bar{\chi})(\phi_1 \psi^* - \phi_2 \chi^*) - \frac{ie^2}{\kappa} (\phi_1^* \bar{\psi} - \phi_2^* \chi^*)(\phi_1^* \psi - \phi_2^* \chi) \quad (3.12)\)

where \( \tilde{D}_\mu \) is defined so that \( \bar{\psi} \tilde{D}_\mu \chi = \bar{\psi} D_\mu \chi - (D_\mu \bar{\psi}) \chi \) and \( \mathcal{L} \) is hermitian. Let us consider the internal symmetry of this lagrangian. There is a \( U(1) \) local gauge symmetry inherited from the four dimensional super QED, which rotates simultaneously the phases of \( \phi_1, \phi_2, \psi, \chi \). There is a new global \( U(1) \) symmetry due to our special choice of coupling constants in Eq.(3.10) which rotates the phases of \( \phi_1, \phi_2^*, \psi, \chi^* \) uniformly. The N=2 extended supersymmetric theory in Ref.[2] can be obtained easily from the lagrangian (3.12) by discarding the terms depending on \( \phi_1 \) and \( \chi \).

Let us ask what happens to the supersymmetric transformation of the lagrangian (3.12). To get the on-shell \( N = 2 \) supersymmetric transformation, we remove the auxiliary fields from the supersymmetric transformation by using their equations of motion with the special choice of the coupling constants. Eqs.(2.6), (3.1) and (3.10) lead to the supersymmetric transformation of the gauge field,

\[ \delta A_\mu = \frac{e}{\kappa} \bar{\alpha}_1 \gamma_\mu (\psi \phi_2^* - \psi^* \phi_2 - \chi \phi_1^* + \chi^* \phi_1) \]

\[ + \frac{e}{\kappa} \bar{\alpha}_2 \gamma_\mu (-\psi \phi_1^* + \psi^* \phi_1 - \chi \phi_2^* + \chi^* \phi_2) \quad (3.13) \]

This supersymmetric transformation is compatible with the local gauge symmetry. However the new global transformation which rotates the phases of \( \phi_1, \psi, \phi_2^*, \chi^* \)
uniformly is not compatible with the $\alpha_1$ dependent part of the supersymmetric transformation because $A_\mu$ is neutral. To remedy this conflict, let us assume that the parameter $\alpha_1$ is a complex rather than Majorana spinor and transforms like $(\phi_1)^2$ under the new global transformation. Then the supersymmetric transformation for the gauge field can be modified to be consistent with the new global symmetry and is given by

$$\delta A_\mu = \frac{e}{\kappa} \bar{\alpha}_1 \gamma_\mu (\psi \phi_2^* + \chi^* \phi_1) - \frac{e}{\kappa} \bar{\alpha}_1^* \gamma_\mu (\psi^* \phi_2 + \chi \phi_1^*)$$

$$+ \frac{e}{\kappa} \bar{\alpha}_2 \gamma_\mu (-\psi \phi_1^* + \psi^* \phi_1 - \chi \phi_2^* + \chi^* \phi_2)$$

Similarly one can modify the supersymmetric transformations for the scalar and fermion fields so that they are compatible with the new symmetry. They become

$$\delta \phi_1 = i(\bar{\alpha}_1^* \chi + \bar{\alpha}_2 \psi)$$

$$\delta \phi_2 = i(-\bar{\alpha}_1 \psi + \bar{\alpha}_2 \chi)$$

and

$$\delta \psi = -\gamma^\mu \alpha_1 D_\mu \phi_2 + \gamma^\mu \alpha_2 D_\mu \phi_1$$

$$+ \frac{e^2}{\kappa} (\alpha_1 \phi_2 - \alpha_2 \phi_1)(v^2 + |\phi_1|^2 - |\phi_2|^2)$$

$$+ 2 \frac{e^2}{\kappa} (\alpha_1^* (\phi_1)^2 \phi_2^* + \alpha_2 \phi_1 |\phi_2|^2)$$

$$\delta \chi = \gamma^\mu \alpha_1^* D_\mu \phi_1 + \gamma^\mu \alpha_2 D_\mu \phi_2$$

$$+ \frac{e^2}{\kappa} (\alpha_1^* \phi_1 + \alpha_2 \phi_2)(v^2 + |\phi_1|^2 - |\phi_2|^2)$$

$$+ 2 \frac{e^2}{\kappa} (-\alpha_1 \phi_1^* (\phi_2)^2 + \alpha_2 \phi_1^2 |\phi_2|^2)$$

It is straightforward to show that the lagrangian (3.12) is invariant under the supersymmetric transformation given by Eqs. (3.14), (3.15), (3.16) and (3.17). Since $\alpha_1$ is now a complex spinor, the supersymmetry has been extended to $N = 3$. The lagrangian (3.12) is the $N = 3$ supersymmetric Chern-Simons-Higgs lagrangian.
4. Elementary Properties

The energy functional of the bosonic part is given by

$$E = \int d^2 x \left[ |D_0 \phi_a|^2 + |D_i \phi_a|^2 + U \right]$$  \hspace{1cm} (4.1)

With the Gauss law for the bosonic part,

$$\kappa F_{xy} = -i (D_0 \phi_a^\dagger \phi_a - \phi_a^\dagger D_0 \phi_a)$$  \hspace{1cm} (4.2)

one can write the energy functional as

$$E = \int d^2 x \left[ |D_x \phi_a \pm i D_y \phi_a|^2 + |D_0 \phi_a \pm \frac{i e^2}{\kappa} \phi_a (|\phi_a|^2 - v^2)|^2 \right. $$

$$ \left. + \frac{4 e^4}{\kappa^2} v^2 |\phi_1|^2 |\phi_a|^2 \right] \mp e v^2 \Psi \hspace{1cm} (4.3)$$

where $\Psi = \int d^2 x F_{xy}$ is the total magnetic flux. Eq.(4.3) implies a bound on the energy functional,

$$< E > \geq e v^2 |\Psi|$$  \hspace{1cm} (4.4)

The energy bound is saturated by configurations satisfying the Gauss law and the following equations,

$$\phi_1 = 0$$

$$D_1 \phi_2 \pm i D_2 \phi_2 = 0$$  \hspace{1cm} (4.5)

$$D_0 \phi_2 \pm \frac{i e^2}{\kappa} \phi_2 (|\phi_2|^2 - v^2) = 0$$

The equations for $\phi_2$ are identical to those of the self-dual Chern-Simons system. In this theory, there are nontopological solitons in symmetric phase and topological vortices in asymmetric phase saturating the energy bound as shown in Ref. [1].
What are the elementary excitations of the theory (3.12)? In the symmetric phase, the quadratic terms in (3.12) with gauge interaction become

\[
\mathcal{L}_{\text{symm}} = \frac{\kappa}{2} \epsilon^\mu\nu\rho A_\mu \partial_\nu A_\rho - |D_\mu \phi_a|^2 - i \bar{\psi} \gamma^\mu D_\mu \psi - i \bar{\chi} \gamma^\mu D_\mu \chi
\]

\[
- \frac{m^2}{2} |\phi_a|^2 - \frac{m}{2} (i \bar{\psi} \psi - i \bar{\chi} \chi)
\]

where \( m \equiv 2e^2v^2/\kappa \). There is no particle excitation related to the gauge field. There are two complex scalar and two spinor fields of mass \( m/\sqrt{2} \). It is well known that charged particles carry fractional spin and satisfy fractional statistics due to the Chern-Simons term. The excitation related to the scalar field has spin \( s = 1/(4\pi k) \). The sign difference between the two mass terms of \( \psi \) and \( \chi \) implies that the spins of \( \psi \) and \( \chi \) are \( s + 1/2 \) and \( s - 1/2 \), respectively.

In the asymmetric phase, we use the unitary gauge where \( \phi_2 = f/\sqrt{2} + v \) with a real \( f \). By neglecting matter self-interaction, we get from Eq.(3.12) a lagrangian

\[
\mathcal{L}_{\text{asym}} = \frac{\kappa}{2} \epsilon^\mu\nu\rho A_\mu \partial_\nu A_\rho - |\partial_\mu \phi_1|^2 - \frac{1}{2} (\partial_\mu f)^2 - i \bar{\psi} \gamma^\mu \partial_\mu \psi
\]

\[
- \frac{i}{2} [\bar{\chi}_R \gamma^\mu \partial_\mu \chi_R + \bar{\chi}_I \gamma^\mu \partial_\mu \chi_I] - \frac{\kappa m}{2} (A_\mu)^2 - m^2 |\phi_1|^2 - \frac{m^2}{2} f^2
\]

\[
+ m i \bar{\psi} \psi - \frac{m}{2} (i \bar{\chi}_R \chi_R - i \bar{\chi}_I \chi_I)
\]

\[
+ e A_\mu \{-i (\partial_\mu \phi_1 \phi_1 - \phi_1^* \partial_\mu \phi_1) + \bar{\psi} \gamma^\mu \psi + i \bar{\chi}_R \gamma^\mu \chi_I\}
\]

\[
- e^2 (A_\mu)^2 [|\phi_1|^2 + \frac{1}{2} (2\sqrt{2}e^2vf + e^2f^2)]
\]

where \( \chi = \chi_R + i \chi_I \) with \( \chi_{R,I} \) being Majorana spinors. Due to the Higgs mechanism, there is a physical degree of freedom for \( A_\mu \), with mass \( m \) and spin 1. There are also three spin 1/2 particles for \( \psi, \chi_I \), three spin zero particles for \( f, \phi_2 \), and one spin \(-1/2\) particles for \( \chi_R \), all with the same mass \( m \).
5. Symmetry Algebra

Let us investigate the symmetries of the lagrangian (3.12). The local gauge symmetry implies the Gauss law constraint,

\[ \mathcal{G} \equiv \kappa F_{xy} + e \rho_c = 0, \tag{5.1} \]

where the electric charge density is given by

\[ \rho_c = i(D_0 \phi_a^* \phi_a - \phi_a^* D_0 \phi_a) - \psi^\dagger \psi - \chi^\dagger \chi. \tag{5.2} \]

The local gauge transformation is generated by the operator \( \mathcal{G} \). By integrating Eq.(5.1) over space, we see that the total electric charge \( Q = \int d^2 x \rho_c \) and total magnetic flux \( \Psi = \int d^2 x F_{xy} \) are related by

\[ \kappa \Psi = -eQ. \tag{5.3} \]

There is an additional global symmetry mentioned in section 3. The total charge for this symmetry, which transforms the phases of \( \phi_1, \psi, \phi_2^*, \chi^* \) uniformly, is given by

\[ T = \int d^2 x \left[ i(D_0 \phi_1^* \phi_1 - \phi_1^* D_0 \phi_1) - i(D_0 \phi_2^* \phi_2 - \phi_2^* D_0 \phi_2) - \psi^\dagger \psi + \chi^\dagger \chi \right]. \tag{5.4} \]

There are also spacetime symmetries. The symmetric energy momentum tensor is given by

\[ T_{\mu \nu} = D_\mu \phi_\nu^* D_\nu \phi_a + D_\nu \phi_a^* D_\mu \phi_a + \frac{i}{4} \bar{\psi} (\gamma_\mu \tilde{D}_\mu + \gamma_\nu \tilde{D}_\nu) \psi \\
+ \frac{i}{4} \bar{\chi} (\gamma_\mu \tilde{D}_\mu + \gamma_\nu \tilde{D}_\nu) \chi + \eta_{\mu \nu} \mathcal{L} \]  

\[ \tag{5.5} \]
The generators for the spacetime translation are the conserved three momentum,

\[ P^\mu = \int d^2x T^{\mu 0} \quad (5.6) \]

The generators of the Lorentz transformation form a three vector,

\[ M_\mu = \int d^2x \epsilon_{\nu\rho} x^{\nu T^\rho 0} \quad (5.7) \]

In particular, the total angular momentum is given by

\[ J = M^0 = \int d^2x (x^1 T^{20} - x^2 T^{10}) \quad (5.8) \]

There are three independent generators for supersymmetric transformations, given by \( \delta \phi_a = i[\bar{\alpha}_1 \mathcal{R} + i \mathcal{R} \alpha_1 + \bar{\alpha}_2 \mathcal{S}, \phi_a] \), etc. One can obtain these generators by using the supersymmetric transformations (3.14), (3.15), (3.15) and (3.16). Corresponding to the complex spinor \( \alpha_1 \), there is a complex spinor operator,

\[ \mathcal{R} = \int d^2x \left\{ \left[ D_\mu \phi_2^* \gamma^\mu + \frac{e^2}{\kappa} \phi_2^* (v^2 + |\phi_1|^2 - |\phi_2|^2) \right] \gamma^0 \psi + \frac{2e^2}{\kappa} (\phi_1^2 \phi_2^* \gamma^0 \psi^*) 
- \left[ D_\mu \phi_1^* \gamma^\mu - \frac{e^2}{\kappa} \phi_1 (v^2 + |\phi_1|^2 - |\phi_2|^2) \right] \gamma^0 \chi^* - \frac{2e^2}{\kappa} \phi_1 (\phi_2^2 \gamma^0 \chi^*) \right\} \quad (5.9) \]

For the Majorana spinor parameter \( \alpha_2 \), there is a Majorana generator

\[ \mathcal{S} = \int d^2x \left\{ -\left[ D_\mu \phi_1^* \gamma^\mu + \frac{e^2}{\kappa} \phi_1^* (v^2 + |\phi_1|^2 - 3|\phi_2|^2) \right] \gamma^0 \psi 
- \left[ D_\mu \phi_2^* \gamma^\mu - \frac{e^2}{\kappa} \phi_2^* (v^2 + 3|\phi_1|^2 - |\phi_2|^2) \right] \gamma^0 \chi + h.c. \right\} \quad (5.10) \]

Let us consider the algebra of symmetry operators. The equal time commutation relation between fields are standard, for example, \( [A_1(\bar{x}), A_2(\bar{y})] = (1/\kappa)i \delta(\bar{x} - \bar{y}) \).
The interesting and nontrivial ones are those between supersymmetric operators. The complex spinor generator $\mathcal{R}$ satisfies,

$$[i\bar{\alpha}_1 \mathcal{R}, i\bar{\alpha}_1 \beta_1] = \bar{\alpha}_1 \gamma^\mu \beta_1 \mathcal{P}_\mu + i\bar{\alpha}_1 \beta_1 Z \quad (5.11)$$

where the central charge $Z$ is given by

$$Z = -\int d^2 x \left[ eF_{xy}(|\phi_1|^2 - |\phi_2|^2) + \frac{e^2}{\kappa} \rho_c (v^2 + |\phi_1|^2 - |\phi_2|^2) \right] \quad (5.12)$$

The Gauss law constraint (5.1) implies that the central charge can be expressed as

$$Z = ev^2 \Psi. \quad (5.13)$$

The Majorana supersymmetry operator $\mathcal{S}$ satisfies

$$[i\bar{\alpha}_2 \mathcal{S}, i\bar{\alpha}_2 \beta_2] = 2\bar{\alpha}_2 \gamma^\mu \beta_2 \mathcal{P}_\mu. \quad (5.14)$$

Under the global transformation $T$, only $\mathcal{R}$ transforms nontrivially,

$$[T, \mathcal{R}] = 2\mathcal{R}. \quad (5.15)$$

As said in Sec.3, the operator $\mathcal{R}$ transforms like $(\phi_1)^2$ under the new global transformation. Thus, the above symmetry operators form an $N = 3$ extended superalgebra.
The physical implication of the central charge $Z$ can be reached by considering
\[ R_{\pm} = \frac{1 \pm i\gamma^0}{2} \mathcal{R} \]

\[ = \int d^2 x \left\{ \frac{1 \pm i\gamma^0}{2} \left[ -\psi (D_0 \phi_2^* \pm i\phi_2^* X) + \chi^* (D_0 \phi_1 \mp i\phi_1 X) \right] \right. 
\left. \pm i \frac{\gamma^1 \pm i\gamma^2}{2} \left[ \psi (D_1 \phi_2^* \mp iD_2 \phi_2^*) - \chi^* (D_1 \phi_1 \mp iD_2 \phi_1) \right] \right. 
\left. \pm i \frac{1 \pm i\gamma^0}{2} \left[ -\psi^* 2e^2 \kappa (\phi_1)^2 \phi_2^* + i\chi^* 2e^2 \kappa \phi_1 (\phi_2^*)^2 \right] \right\} \]

(5.16)

where $X = v^2 + |\phi_1|^2 - |\phi_2|^2$. After multiplying $1 \pm i\gamma^0/2$ to Eq. (5.11) and taking trace, we obtain

\[ \mathcal{P}^0 = \mp Z + \sum_A \{ R_{\pm A}, R_{\pm A}^\dagger \} \]

(5.17)

Hence there is a bound on the expectation value of the energy,

\[ < \mathcal{P}^0 > \geq | < Z > | \]

(5.18)

The inequality is saturated if the coefficient of each fermion field in Eq.(5.16) vanishes. These conditions are independent of the sign of $v^2$ and can be easily shown to be equivalent to the self-dual equations (4.4) for positive $v^2$. (For negative $v^2$, one has to rederive the self-dual equations.)

What are the possible representations under this $N = 3$ supersymmetric algebra? Let us consider only massive states. In the rest frame where $P^\mu = (M, 0, 0)$, Eqs.( 5.11) and (5.14) become

\[ \{ R_{\pm 1}, R_{\pm 1}^\dagger \} = M \pm Z \]

\[ \{ S_1, S_2 \} = 2M \]

(5.19)

Note that $R_{\pm 2} = \pm iR_{\pm 1}$ from Eq. (5.16). After a suitable normalization, we can build fermionic creation and annihilation operators. By applying them on a
Clifford vacuum, one can construct a representation. Let us first consider charged states which saturate the bound (5.18). From a Clifford vacuum $|\Omega>$ of angular momentum $j$, one can build two $j + 1/2$ states and one $j + 1$ state. Elementary particles in the symmetric phase saturate the energy bound and form such a representation. For nontopological solitons in symmetric phase and topological vortices in asymmetric phase, the classical solution saturates the energy bound. Hence, one expects these solitons to form the same representation as do elementary particles in the symmetric phase. This remains to be seen since the quantization of solitons are somewhat involved. Let us now consider states which do not saturate such bound. An irreducible representation for those states is made of one angular momentum $j$ state, three $j + 1/2$ states, three $j + 1$ states, and one $j + 3/2$ states. Elementary excitations in asymmetric phase form such a representation.

With the Chern-Simons term as the only gauge kinetic term, there is no physical degree for the gauge field in the symmetric phase. In the broken phase there is only one massive vector boson of spin 1 and it is neutral. A neutral nontrivial representation of the $N = 4$ supersymmetry is made of one spin 1, four spin 1/2, six spin 0, four spin -1/2, and one spin -1 states. As there is no candidate for spin -1 state, there is no $N = 4$ supersymmetric Chern-Simons Higgs theory with a single gauge field. Hence our $N = 3$ theory is the ‘maximally supersymmetric’ theory with a single gauge field and no gravity.
6. Conclusion

We have constructed an $N = 3$ extended supersymmetric Chern-Simons-Higgs theory and studied its properties. In addition to the original abelian gauge symmetry there exists a new global $U(1)$ symmetry. The self-dual Chern-Simons Higgs systems studied before $^{1,2}$ are part of this $N = 3$ supersymmetric theory. This theory turns out to be the maximally supersymmetric with a single gauge field and no gravity.

Let us conclude by mentioning several open questions. The $N = 4$ supersymmetric nonabelian theory in four dimensional spacetime is the maximal supersymmetric renormalizable theory and is known to be free of ultraviolet divergence. As the $N = 3$ theory discussed in this paper is maximally supersymmetric in three dimensional spacetime, it would be interesting to find out whether this theory is also ultraviolet finite.

One very interesting question is about the finite shift of the coefficient of Chern-Simons term, which is supposed to be due to fermion one loop correction $^{10}$. In the symmetric phase, the two fermions have opposite mass and there is no correction. In the asymmetric phase one of the fermion mass terms changes as shown in Sec.4, leading to non zero shift of the coefficient. It would be interesting to find out whether there is nontrivial contribution from Higgs field to keep the total contribution to be identical to that in the symmetric phase.

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