Log-dependent slope of scalar induced gravitational waves in the infrared regions

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We analytically calculate the scalar induced gravitational waves (SIGWs) and find a log-dependent slope of SIGW in the infrared regions \((f < f_c)\), namely \(n_{GW}(f) = 3 - 2/\ln(f_c/f)\), and \(n_{GW}(f) = 2 - 2/\ln(f_c/f)\) near the peak if the power spectrum of scalar curvature perturbation is quite narrow, where \(f_c\) is roughly the frequency at the peak of SIGW. Such a log-dependent slope can be taken as a new template for distinguishing SIGW from other sources.

Dark matter (DM) is one of the components which makes up around 26% of the total energy density in the Universe at present [1]. However, the nature of DM remains completely unknown. Even though there is a miracle for the weakly-interacting massive particles (WMIPs), the limits on them are tightening. Considering some alternative models to WMIPs becomes more and more important. Among the alternative models in literature, the primordial black holes (PBHs) have attracted much attentions in the past few years, in particular after the discovery of the gravitational waves (GWs) from the coalescence of two binary black holes by aLIGO [2] because PBHs are supposed to provide a possible explanation if the abundance of stellar-mass PBHs in DM is roughly \(\mathcal{O}(10^{-5})\) [3–6]. Up to now, there are a various observations which have put constraints on the abundance of PBH DM [5–25], but a substantial open window in the mass range of \([10^{-16}, 10^{-14}] \cup [10^{-13}, 10^{-12}]M_\odot\) is still allowed for PBHs composing of all of DM. See a recent summary in [5].

PBHs are supposed to form from the gravitational collapse of over-densed regions seeded by relatively large curvature perturbations [26, 27] on small scales which are less constrained by the CMB and large-scale structure observations. These over-densed regions are produced when curvature perturbations exceed a critical value and will collapse to form a PBH at about horizon size after the corresponding wavelength re-enters the horizon. However, how to test the postulation of PBH DM is still an open question. Actually, the curvature perturbations couple to the tensor perturbations at second-order, thus inevitably generating the scalar induced GWs (SIGWs) in the radiation dominated era [28–34]. The enhancement of scalar curvature perturbation for significantly forming PBHs will generate relatively large SIGWs which provide a new way to probe PBHs [35]. See some other related works in [36–57].

A normalized stochastic gravitational-wave background (SGWB) spectral energy density \(\Omega_{GW}(f)\) expresses the GW spectral energy density in terms of the energy density per logarithmic frequency interval divided by the cosmic closure density, namely [58, 59]

\[
\Omega_{GW}(f) \equiv \frac{1}{\rho_c} \frac{d \log \rho_{GW}}{d \log f} = \frac{2\pi^2}{3H_0^2} f^3 S_h(f),
\]

where \(\rho_{GW}\) and \(\rho_c\) are the energy density of GWs and critical density, \(f\) is the GW frequency, \(H_0\) is the Hubble constant, and \(S_h(f)\) is the spectral density. Conventionally, \(\Omega_{GW}(f)\) is modeled as a power law form, i.e.

\[
\Omega_{GW}(f) \propto f^{n_{GW}}
\]

with a slope \(n_{GW}\). The predicted SGWB from compact binary coalescences is well modeled by a power law of slope \(n_{GW} = 2/3\), and \(n_{GW} = 0\) corresponds to a scale-invariant energy. For the primordial GWs, \(\Omega_{GW}(f) \propto f^{n_t + \alpha_t \ln(f/f_{CMX})}/2\), where \(n_t\) is the spectral index of primordial GW power spectrum and \(\alpha_t\) is the running of spectral index [60–63]. For the white noise which corresponds to a random signal, the spectral density \(S_h\) is a constant, and then \(\Omega_{GW}(f) \propto f^3\). In this sense, the \(f^3\) behavior is a trivial result rather than a model-independent evidence for the detection of SIGWs. That is why there are many processes generating a SGWB scaling as \(f^3\) in the infrared limit, such as GWs from first-order phase transition [64] and GWs induced by an inflaton field [65]. In addition, \(\Omega_{GW}(k)\) drops down quickly in the infrared region from the peak, making it difficult to detect the \(\propto f^3\) slope.

Even though the behavior of SIGW \(\Omega_{GW}(f)\) is expected to be dependent on the power spectrum of the scalar curvature perturbations, we find \(n_{GW} = 3 - 2/\ln(f_c/f)\) for SIGW in the infrared region, and \(n_{GW} = 2 - 2/\ln(f_c/f)\) near the peak if the scalar power spectrum is very narrow. In the infrared limit \((f \to 0)\), \(n_{GW} \to 3\), indicating that the correlation of perturbations can be neglected and the signal behaves randomly. This log-dependent slope can be taken as a distinguishing feature for the SIGWs.
The perturbed Freedmann-Robert-Walker (FRW) metric for a perturbed universe in Newtonian gauge takes the form, [36],

\[ ds^2 = a^2 \left\{ -(1 + 2\phi) d\eta^2 + \left[ (1 - 2\phi) \delta_{ij} + \frac{h_{ij}}{2} \right] dx^i dx^j \right\}, \tag{3} \]

where \( \phi \) is the scalar perturbation and \( h_{ij} \) is the GW perturbation. In a radiation dominated universe without entropy perturbations, the equation of motion for \( \phi \) governed by Einstein equation reads

\[ \phi''(\eta) + \frac{4}{\eta} \phi'(\eta) + \frac{k^2}{3} \phi(\eta) = 0 \tag{4} \]

in Fourier space. This equation of motion has an attenuation solution given by, [37],

\[ \phi_k(\eta) \equiv \phi_k \frac{9}{(k\eta)^2} \left[ \frac{\sin(k\eta/\sqrt{3})}{k\eta/\sqrt{3}} - \cos(k\eta/\sqrt{3}) \right], \tag{5} \]

where \( \phi_k \) is the primordial perturbation whose value at \( \eta = 0 \) is given by inflation models. The equation of motion for the tensor components is given by Einstein equation at second-order, namely

\[ h_{ij}'' + 2\mathcal{H} h_{ij}' - \nabla^2 h_{ij} = -4\mathcal{T}_{ij}^{\text{em}} S_{\text{em}}, \tag{6} \]

where the prime denotes the derivative with respect to the conformal time \( \eta \), and \( \mathcal{H} = \dot{a}/a \) is the conformal Hubble parameter. The source term [36]

\[ S_{ij}^{(2)} = 4\phi \partial_i \phi_j + 2\partial_i \phi \partial_j \phi - \frac{1}{\mathcal{H}^2} \partial_i (\mathcal{H}\phi + \phi') \partial_j (\mathcal{H}\phi + \phi'), \tag{7} \]

is projected to transverse-traceless gauge by the projection operator \( \mathcal{T}_{ij}^{\text{em}} \), i.e.,

\[ \mathcal{T}_{ij}^{\text{em}} = \int \frac{d^3k}{(2\pi)^3/2} e^{ik \cdot x} \left[ e_{ij}(k) e^{\text{em}}(k) + \bar{e}_{ij}(k) e^{\text{em}}(k) \right]. \tag{8} \]

Here the two polarization tensors are defined by

\[ e_{ij}(k) \equiv \frac{1}{\sqrt{2}} \left[ e_i(k) e_j(k) - \bar{e}_i(k) \bar{e}_j(k) \right], \tag{9} \]

\[ \bar{e}_{ij}(k) \equiv \frac{1}{\sqrt{2}} \left[ \bar{e}_i(k) \bar{e}_j(k) + \bar{e}_i(k) e_j(k) \right], \]

where \( e(k) \) and \( \bar{e}(k) \) are two time-independent unit vectors orthogonal to \( k \). After solving Eq. (6) in Fourier space by Green function, one obtains, [37],

\[ h(k, \eta) = \frac{1}{k a(\eta)} \int d\eta' \sin(k\eta - k\eta') a(\eta') S_k(\eta), \tag{10} \]

where \( S_k(\eta) = -4 e^{ij}(k) \bar{S}_{ij}(k, \eta) \) with \( \bar{S}_{ij}(k, \eta) \) to be the Fourier transformed source term. Then the dimensionless power spectrum of the SIGWs, \( P_h(k) \), can be evaluated by the two point correlation

\[ \langle h(k, \eta) h(k', \eta) \rangle = \frac{2\pi^2}{k^4} P_h(k, \eta) \delta(k + k'). \tag{11} \]

And then, at the matter-radiation equality,

\[ \Omega_{GW, \text{eq}}(k) = 1 \frac{k^2}{12} P_h(k, \eta), \tag{12} \]

where we have summed over two polarization modes and take the oscillation average, and \( f = k/2\pi \). The present density parameter can be evaluated by, [47, 48],

\[ \Omega_{GW}(k) = \Omega_r \times \Omega_{GW, \text{eq}}(k) = \Omega_r \times \Omega_{GW}(\eta \rightarrow \infty, k) \]

\[ = \Omega_r \int_0^\infty dv \int_{|1-v|}^{1+v} du I(u, v) P_\phi(vk) P_\phi(uk), \tag{13} \]

where \( \Omega_r \) is the radiation density parameter at present, and \( P_\phi(k) \) is the power spectrum of \( \phi \). Here \( u \) and \( v \) are two dimensionless variables. The function \( I(u, v) \) comes from integrating the conformal time in the convolution of the source term, \( \langle S_k(\eta) S_{k'}(\eta) \rangle \). In radiation dominated era, \( I(u, v) \) is given by [47, 48]

\[ I(u, v) = \frac{1}{6} \left( \frac{3(u^2 + v^2 - 3)(-4uv + (1 - u^2 + v^2)^2)}{16a^4v^4} \right)^2 \left( -4uv + (u^2 + v^2)^2 \log \left| \frac{3 - (u + v)^2}{3 - (u - v)^2} \right|^2 + \pi^2(u^2 + v^2 - 3)^2\Theta(u + v - \sqrt{3}) \right), \tag{14} \]

where \( \Theta \) is the Heaviside function.

Let’s consider a scalar power spectrum \( P_\phi(k) \) which is peaked at \( k_s \) and is nonzero only for \( k_- < k < k_+ \), like that illustrated in Fig. 1. Since the amplitude of the scalar power spectrum for the formation of PBHs is supposed to be much larger than those at CMB scales, the power spectrum at CMB scales is neglected. For simplicity, we introduce a dimensionless parameter \( \Delta \) to quantify the width of the scalar power spectrum as follows

\[ \Delta \equiv \frac{k_+ - k_-}{k_s}. \tag{15} \]

The power spectrum is narrow if \( \Delta \ll 1 \). From Eq. (13), the density parameter of SIGWs for such a power spectrum reads

\[ \Omega_{GW, \text{eq}} = \int_{k_s}^{k_+} dv \int_{\max(k_s, \frac{1}{1-v})}^{\min(k_s, \frac{1}{1-v})} du I(u, v) P_\phi(uk) P_\phi(vk), \tag{16} \]

In this letter, we focus on the behavior of \( \Omega_{GW, \text{eq}} \) in the infrared region, namely \( k \ll k_- \), and then \( 1 \ll k_-/k < k_+/k \).
In the last step, we consider that both the absolute values \(\phi\) and \(\Delta\) are much smaller than one, and neglect the higher order corrections \(\sim O(\Delta^2)\) are neglected as well. Similarly, the slope of SIGW becomes

\[
n_{GW}(k) = 3 - \frac{4}{\ln \frac{4k^2}{3k^2}}.
\]  

In the infrared limit \((k \rightarrow 0)\), the slope approaches to 3, or equivalently \(\Omega_{GW}(k \rightarrow 0) \propto k^3\), due to the un-correlation of the perturbations at those scales. In addition, for the \(\delta\)-power spectrum corresponding to \(\Delta \rightarrow 0\), we only have \(\Delta \ll k/k_* \ll 1\), and then \(n_{GW} = 2 - \frac{4}{\ln \frac{4k^2}{3k^2}}\) which is consistent with what we have known.

From now on, we generalize our former discussion to the wide scalar power spectrum. In this case there is no available region of \(\Delta \ll k/k_* \ll 1\) any more. Similar to the previous case of \(k/k_* \ll \Delta\), one can easily find

\[
\Omega_{GW}(k) = \int_{k_-}^{k_+} dv \int_{v_-}^{v_+} dv + \int_{v_-}^{v_+} dv \int_{k_-}^{k_+} dv + \int_{k_-}^{k_+} dv \int_{v_-}^{v_+} dv \left( I(u,v) P_{\phi}(uk) P_{\phi}(vk) \right) (21)
\]

Considering \(1 \ll k_-/k < k_+/k\), both the first and the third integrations in the bracket of above equation are much small compared to the second integration which is approximately given by

\[
\Omega_{GW}(k) = \int_{k_-}^{k_+} dv I(u,v) P_{\phi}(vk). \quad (22)
\]

Since \(v \in [k_-/k, k_+/k] \gg 1\) and

\[
I(u,v) \simeq \frac{3}{2} v^{-4} \ln^2 \frac{4v^2}{3}, \quad (23)
\]

defining a new variable \(y\) related to \(v\) by \(v = \frac{k}{k}(1 + y)\), we have

\[
\Omega_{GW}(k) \approx 3 \left( \frac{k}{k_*} \right)^3 \int_{k_-/k_*}^{k_+/k_*} dy (1 + y)^{-4} \times \ln^2 \left( \frac{4k^2}{3k^2} (1 + y + \frac{x^2}{2}) \right) P_{\phi}(k_*(1 + y)),
\]

\[
\simeq 3 \left( \frac{k}{k_*} \right)^3 \ln^2 \left( \frac{4k^2}{3k^2} \right) P_{\phi}^2(k_*) \Delta^2. \quad (24)
\]

where the higher order corrections \(\sim O(\Delta^2)\) are neglected as well. Similarly, the slope of SIGW becomes

\[
n_{GW}(k) = 3 - \frac{4}{\ln \frac{4k^2}{3k^2}}. \quad (25)
\]

In the infrared limit \((k \rightarrow 0)\), the slope approaches to 3, or equivalently \(\Omega_{GW}(k \rightarrow 0) \propto k^3\), due to the un-correlation of the perturbations at those scales. In addition, for the \(\delta\)-power spectrum corresponding to \(\Delta \rightarrow 0\), we only have \(\Delta \ll k/k_* \ll 1\), and then \(n_{GW} = 2 - \frac{4}{\ln \frac{4k^2}{3k^2}}\) which is consistent with what we have known.

From now on, we generalize our former discussion to the wide scalar power spectrum. In this case there is no available region of \(\Delta \ll k/k_* \ll 1\) any more. Similar to the previous case of \(k/k_* \ll \Delta\), one can easily find

\[
\Omega_{GW}(k) = 3 \int_{k_-}^{k_+} dv v^{-4} \ln^2 \left( \frac{4v^2}{3} \right) P_{\phi}^2(v). \quad (26)
\]

\[
= 3k^3 \int_{k_-}^{k_+} dq q^{-4} \ln^2 \left( \frac{4q^2}{3k^2} \right) P_{\phi}^2(q). \quad (27)
\]
and then taking the derivative of \( \log \Omega_{GW,eq} \) with respect to \( \log k \), we obtain

\[
n_{GW}(k) = 3 - 4 \int_{k_-}^{k_+} dp \, p^{-1} \ln \left( \frac{4p^2/3k^2}{P_\phi^2(p)} \right) \int_{k_-}^{k_+} dq \, q^{-1} \ln^2 \left( \frac{4q^2/3k^2}{P_\phi^2(q)} \right), \tag{28}
\]

\[
= 3 - 4 \int_{k_-/k}^{k_+/k} du \, u^{-1} \ln \left( \frac{4u^2/3}{P_\phi(au)} \right) \int_{k_-/k}^{k_+/k} dv \, v^{-1} \ln^2 \left( \frac{4v^2/3}{P_\phi(vk)} \right). \tag{29}
\]

Restrictly, \( n_{GW} \) given in the above equation certainly depends on the scalar power spectrum even for \( k < k_- < k_+ \). Notice that both \( u \) and \( v \) are much larger than one because \( k_-/k > k_-/k \gg 1 \) in the infrared region for wide scalar power spectrum, and both \( \ln(4u^2/3) \) and \( \ln(4v^2/3) \) can be roughly taken as a constant if \( k_+ \) is not larger than \( k_- \) too much. In this sense, the above equation roughly gives

\[
n_{GW}(k) \approx 3 - \frac{4}{\ln(4k^2/3k^2)}, \tag{30}
\]

where \( k_* \) denotes a pivot scale for counting the integrations in Eq. (29).

Before closing this letter, for example, we consider a simple broken power spectrum parameterized by

\[
P_\phi(k) = A \times \begin{cases} 
\frac{k - k_-}{k_+ - k_-}, & \text{for } k_- < k < k_*, \\
\frac{k_+ - k}{k_+ - k_-}, & \text{for } k_* < k < k_+,
\end{cases} \tag{31}
\]

and \( P_\phi(k) = 0 \) for \( k < k_- \) and \( k > k_+ \). In order to check our former analytic results, we consider two cases, one narrow power spectrum with \( k_- = 0.995k_*, \ k_+ = 1.005k_* \) and then \( \Delta = 0.01 \); the other one is a wide power spectrum with \( k_- = 0.5k_*, \ k_+ = 10.5k_* \) and then \( \Delta = 10 \). Both the analytic (orange and red solid lines) and numerical results (blue dotted and dashed lines) are shown in Fig. 2. For the narrow power spectrum (\( \Delta = 0.01 \)), the blue dotted line shows that \( n_{GW} \) is roughly equal to \( 2 - 4/\ln(4k^2/3k^2) \) for \( \Delta \ll k_*/k_* \ll 1 \) and approaches to \( 3 - 4/\ln(4k^2/3k^2) \) for \( k_*/k_* \ll \Delta \). For the wide power spectrum (\( \Delta = 10 \)), the blue dashed line roughly recovers our analytic result (the red line). Fig. 2 indicates that our analytic results are nicely consistent with the numerical results.

To summarize, there are various GW sources, including the scalar curvature perturbation, which generate a SGWB in the Universe. It is important to figure out some features of SIGW for distinguishing it from other sources. In this letter, we calculate the SIGW in the infrared region and find a log-dependent slope of SIGW, namely

\[
n_{GW}(f) = 3 - \frac{2}{\ln(f_c/f)}, \tag{32}
\]

and \( n_{GW}(f) = 2 - 2/\ln(f_c/f) \) near the peak if the scalar power spectrum is quite narrow. Here \( f < f_c \) and \( f_c \) is roughly the frequency at the peak. Even though the slope of SIGW approaches to 3 in the infrared limit, the correction of \( 2/\ln(f_c/f) \) approaches to zero slowly, and the amplitude of \( \Omega_{GW}(f) \) in the region corresponding to \( \Omega_{GW}(f) \propto f^3 \) should be very small and could not be detected.

Actually, such a log-dependent slope is quite generic for SIGW. It comes from the oscillating behavior of the evolution of the scalar perturbations (the sine and cosine terms in Eq. (5)). Integrating over these sine and cosine terms results in a cosine integral, \( Ci(x) \), and

\[
\lim_{x \to 0^+} \frac{\text{Ci}(|A|/x) - \text{Ci}(|B|/x) - \ln(|A/B|)}{x}, \tag{33}
\]

which finally leads to the logarithmic terms in \( I(u, v) \). In this sense, the log-dependent slope of the SIGW spectra is a result of the evolution of the scalar perturbations in a radiation dominated universe. It implies that such a log-dependent slope is a unique feature for the SIGW which can be used to distinguish SIGW from other sources.

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