STATISTICAL MECHANICAL ENTROPY OF
two-dimensional black holes

J. Gegenberg
Dept. of Mathematics and Statistics, University of New Brunswick
Fredericton, New Brunswick E3B 5A3 Canada

G. Kunstatter
Dept. of Physics and Winnipeg Institute of Theoretical Physics, University of
Winnipeg, Manitoba R3B 2E9 Canada

T. Strobl
Institut für Theoretische Physik, RWTH-Aachen
Sommerfeldstr. 26-28, D52056 Aachen, Germany

We calculate the statistical mechanical entropy associated with boundary terms in
the two-dimensional Euclidean black holes in deSitter gravity.

PITHA - 96/22
UNB Technical Report 96-01

1 Introduction

Although the final word has certainly not been written, several calculations
have been performed which relate the Bekenstein-Hawking thermodynamic
entropy to the degeneracy of microscopic states. For Carlip and Balachandran and
his group, the microstates originate from treating the event horizon as the boundary of
the spacetime. In particular, degrees of freedom which are pure gauge in the interior
of spacetime become dynamical on the boundary.

In this talk, we apply Carlip’s program to an even lower dimensional black
hole — namely that occurring as a solution of the ‘constant curvature’ (CC)
theory of gravity in two-dimensional spacetime with a cosmological constant. The
field equations for this theory include $R = 2k$, for Ricci scalar $R$ and

1 cosmological constant $k \neq 0$. We will show that when the latter theory is constructed on a

1 manifold with a boundary that corresponds to the bifurcation two-sphere of

1 a black hole of fixed mass $M$, and provided the gauge group is analytically

1 continued to $SU(2)$ (or, equivalently, if we consider the Euclidean CC-model),

1 the surface term gives rise to a non-trivial mechanical system equivalent to two

1 harmonic oscillators with fixed energy. This system can be quantized only if
the mass of the black hole is quantized. The resulting degeneracy of quantum states on the boundary is proportional to the square root of the black hole mass, and hence the entropy is given by $\log \sqrt{M}$.

2 The 2-D Gravity Model

The CC theory is closely related to a gauge theory

$$S[\phi, A] = Tr \int_M \phi F(A) - Tr \oint_{\partial M} \phi A,$$  

where $F(A) = dA + A \wedge A$ is the curvature of $A$, a Lie-algebra valued connection on a principal bundle over $M$ and $\phi$ is a scalar field over $M$ which also takes its values in the same Lie algebra. In order to describe the Euclidean CC theory with cosmological constant $\kappa$, the appropriate Lie algebra is $so(2,1)$ for negative cosmological constant and $su(2)$ for positive $\kappa$, while for the Minkowskian theory it is $so(2,1)$ irrespective of the sign of $\kappa$. The gauge group thus coincides with the isometry group of the on-shell metric. The equivalence with the CC gravity model is achieved by the identification $A = A^i T^i = e^a T^a + \omega^3$, where $a, b, \ldots = 1, 2$ and $e^a, \omega$ are Zweibein and spin connection, respectively. The surface term is required in Eq.(1), if the spacetime is a manifold-with-boundary such that the field $\phi$ is fixed on the boundary.

We will henceforth suppose that the spacetime metric given by the dyad $e^a$ has positive-definite signature and that $\kappa$ is positive. Thus the appropriate gauge group is $SU(2)$. The black holes are to be identified with the manifold-with-boundary $M$ with interior the spherical cap and with boundary $\partial M$ the embedded circle. The metric is given locally by

$$ds^2 = \left( M - x^2 / \ell^2 \right) d\theta^2 + \left( M - x^2 / \ell^2 \right)^{-1} dx^2,$$  

where $M$ is an invariant dimensionless parameter defined by $M = Tr \phi^2 = \phi^a \phi_a + \phi^3 \phi^3$, which may be identified with the mass of the black hole. This is the dimensionally reduced Euclideanized BTZ black hole. The boundary in these coordinates is given by $x = x_0 > \sqrt{M} \ell$.

It turns out that on-shell the fields $\phi^a$ may be identified with the Killing vector $k$ of the black hole solution: $\phi^a = e^a_\mu k^\mu$. Thus to impose boundary conditions corresponding to the bifurcation surface of the event horizon we will require $\phi^a = 0$ and, as a consequence, $\phi_3 = \sqrt{M}$.

It is important to note that the surface term in Eq.(1) is not gauge invariant under $SU(2)$ gauge transformations. The gauge transformations generate spacetime diffeomorphisms, and since some diffeomorphisms shift the location of the boundary, one expects that some gauge degrees of freedom in the
interior of $\mathcal{M}$ are dynamical degrees of freedom on the boundary. Hence, as in Carlip’s discussion of the three-dimensional case, we parametrize the fields $\phi, A$ in terms of gauge-fixed fields $\bar{\phi}, \bar{A}$ and an element $g \in SU(2)$:

$$\phi = g^{-1}\bar{\phi} g; \quad A = g^{-1}\bar{A} g + g^{-1}dg.$$  \hspace{1cm} (3)

This yields the boundary action

$$S[g] = -Tr \oint_{\partial \mathcal{M}} \bar{\phi} dg \bar{g}^{-1}$$  \hspace{1cm} (4)

describing gauge modes that have become physical due to the imposed black hole boundary conditions. In the next section we will quantize the theory with action Eq.(4).

3 Quantum Mechanics on the Boundary of Spacetime

As discussed above, we choose $\phi = \bar{\phi}$ to be: $\bar{\phi} = \sqrt{M} T_3$, where $T_3$ is the constant diagonal $2 \times 2$ $su(2)$ matrix. This is a solution of the equations of motion on the boundary $\partial \mathcal{M}$, in the limit as $x_0 \to \sqrt{M} \ell$.

The theory Eq.(4) is a classical mechanical one: it describes a particle moving on the homogenous space $SU(2)/U(1)$ (equivalently, on the coadjoint orbit of $su(2)$) with periodic time $t$. This is a result of the fact that a $g$ in the diagonal subgroup $U(1)$ of $SU(2)$ does not contribute to the action – such terms are total divergences. In the following we quantize this point particle system in an elementary way, without resort to more elaborate procedures such as, e.g., geometric quantization. We parametrize $g$ by

$$g = \left[ \begin{array}{cc} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{array} \right],$$  \hspace{1cm} (5)

where

$$z_1 = \frac{1}{2} \left( \frac{q_1}{\sqrt{M}} - i p_1 \right), \quad z_2 = \frac{1}{2} \left( p_2 + i \frac{q_2}{\sqrt{M}} \right),$$  \hspace{1cm} (6)

with the $p$'s and $q$'s real. Then the action Eq.(4) becomes simply $S[q,p] = \oint_{\partial \mathcal{M}} dt (p_1 \dot{q}_1 + p_2 \dot{q}_2)$. However, since $g \in SU(2)$, there is a first-class constraint

$$|z_1|^2 + |z_2|^2 \equiv (p_1)^2 + (p_2)^2 + \frac{1}{M} ((q_1)^2 + (q_2)^2) \approx 1.$$  \hspace{1cm} (7)

Implementation of this constraint on the quantum level (plus normalizability of the physical wave functions) maps the system to the quantum system of two identical harmonic oscillators with fundamental frequency $\omega \propto 1/\sqrt{M}$.
and total energy constrained to be 1. This obviously is consistent only if $M$ is discrete (quantized). Counting of the degeneracy of the energy eigenvalue $1 = \hbar \omega (n + 1/2)$, $n \in \mathbb{N}$, is elementary now. It is easy to see that it yields the announced result of an entropy $\sim \log \sqrt{M}$. We observe in parenthesis that the period associated to the above oscillator frequency, $2\pi/\omega = 2\pi \sqrt{M}$, produces the correct periodicity in Euclidean time for a black hole with Hawking temperature $T_H = \sqrt{M}/2\pi$.

It is fairly well-known that the Bekenstein-Hawking entropy of the CC black holes goes as $\sqrt{M}$. Hence there is an insufficient degeneracy of states on the boundary to account for the Bekenstein-Hawking entropy in the case of the 2d Euclidean CC black holes.

Seemingly this changes into the opposite when looking at the Minkowskian CC theory: There the non-compactness of the gauge group gives rise to an infinite number of quantum states living at the boundary. In the framework of Poisson $\sigma$-models, furthermore, much of the above investigation may be generalized to generic 2d dilaton gravity theories. Remarkably, in many cases this yields infinitely many boundary states even for Euclidean signature black holes. Work on this is in progress and shall be reported on elsewhere.

**Acknowledgment** The authors would like to thank Steve Carlip for useful discussions. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada as well as by the Austrian FWF Project 10.221-PHY.

1. J.D. Bekenstein, Phys. Rev. D7, 2333 (1973); S.W. Hawking, Nature 248, 30 (1974).
2. S. Carlip, Phys. Rev. D51, 632 (1995).
3. A.P. Balachandran, L. Chandar and A. Momen, “Edge States in Gravity and Black Hole Physics”, gr-qc/9412019 (1994); “Edge States in Canonical Gravity”, gr-qc/9506006 (1995).
4. B.M. Barbashov, V.V. Nesterenko and A.M. Chervjakov, Theor. Math. Phys. 40, 15 (1979); R. Jackiw in *Quantum Theory of Gravity*, ed. by S. Christensen, Adam Hilger, Bristol (1984); C. Teitelboim in *Quantum Theory of Gravity*, ed. by S. Christensen, Adam Hilger, Bristol (1984).
5. T. Fukuyama and K. Kamimura, Phys. Lett. B160, 259 (1985); K. Isler and C. Trugenberger, Phys. Rev. Lett. 63, 834 (1989); A. Chamseddine and D. Wyler, Phys. Lett. B228, 75 (1989); cf. also P. Schaller and T. Strobl, Phys. Lett. B337, 266.
6. M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992); M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48, 1506 (1993).
7. J. Gegenberg, G. Kunstatter and D.Louis-Martinez, Phys. Rev. D51, 1781 (1995).
8. P. Schaller and T. Strobl, Mod. Phys. Letts. A9, 3129 (1994); “A Brief Introduction to Poisson σ-Models”, hep-th/9507020 (1995).