$\mathcal{F}$-equicontinuity and an Analogue of Auslander-Yorke Dichotomy Theorem

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Abstract

In this paper, we introduce an $\mathcal{F}$-equicontinuity and show an analogue of Auslander-Yorke dichotomy theorem for $\mathcal{F}$-sensitivity. Precisely, under the condition that $k \mathcal{F}$ is translation invariant, we prove that a transitive system is either $\mathcal{F}$-sensitive or almost $k \mathcal{F}$-equicontinuous, and so generalize the result of previous work. Also we show that $\mathcal{F}$-equicontinuity is preserved by an open factor map and consider the implication between $\mathcal{F}$-equicontinuity and mean equicontinuity.

Keywords: equicontinuity, sensitivity, mean equicontinuity, $\mathcal{F}$-sensitivity, mean sensitivity, Furstenberg family.

1 Introduction

A topological dynamical system $(X, T)$ means a compact metric space $(X, d)$ with a continuous self-surjection $T$ defined on it. Throughout this paper we are only interested in a nontrivial topological dynamical system, where the state space is a compact metric space without isolated points. Here a trivial dynamical system means that the state space is a singleton.

A dynamical system $(X, T)$ is deterministic in the sense that the evolution of the system is described by a map $T$, so that the present (the initial state) completely determines the future (the forward orbit of the state). Li and Yorke introduced the term “chaos” into mathematics in 1975 ([9]) and showed that a deterministic system has an unpredictable and complex behavior. And later many definitions of chaos have been introduced into mathematics by several scholars, and although there is no universal mathematical definition of chaos, it is generally agreed that a chaotic dynamical system should exhibit sensitive dependence on initial conditions, i.e., minor changes in the initial state lead
to completely different long-term behavior. A dynamical system \((X, T)\) is called sensitive [3] if there exists \(\varepsilon > 0\) such that for every \(x \in X\) and every neighborhood \(U_x\) of \(x\), there exist \(y \in U_x\) and \(n \in \mathbb{N}\) with \(d(T^n x, T^n y) > \varepsilon\).

The equicontinuity is opposite to the notion of sensitivity. A dynamical system \((X, T)\) is called equicontinuous if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that for every \(x \in X\) and every neighborhood \(U_x\) of \(x\), there exist \(y \in U_x\) and \(n \in \mathbb{N}\) with \(d(T^n x, T^n y) < \varepsilon\). 

Recall that a point \(x \in X\) is equicontinuous if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that for every \(y \in X\) with \(d(x, y) < \delta\), \(d(T^n x, T^n y) < \varepsilon\) for \(n = 0, 1, 2, \ldots\). A transitive system \((X, T)\) is called almost equicontinuous if there exists some equicontinuous points.

The well-known Auslander-Yorke dichotomy theorem [4] states that a minimal dynamical system is either sensitive or equicontinuous, which was supplemented in [2]: a transitive system is either sensitive or almost equicontinuous.

In [8], the authors introduced the notions of mean equicontinuity and mean sensitivity. A dynamical system \((X, T)\) is called mean equicontinuous if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that \(d(x, y) < \delta\) implies \(\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon\) and a point \(x \in X\) is called mean equicontinuous if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that for every \(y \in X\) with \(d(x, y) < \delta\), \(\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon\). A transitive system \((X, T)\) is called almost mean equicontinuous if there is at least one mean equicontinuous point.

A dynamical system \((X, T)\) is called mean sensitive if there exists \(\varepsilon > 0\) (sensitive constant) such that for every \(x \in X\) and every neighborhood \(U_x\) of \(x\), there is \(y \in U_x\) with \(d(T^n x, T^n y) > \varepsilon\).

And they showed an analogue of Auslander-Yorke theorem, which states that a transitive dynamical system is either almost mean equicontinuous (in the sense of containing some mean equicontinuous points) or mean sensitive and that a minimal system is either mean equicontinuous or mean sensitive.

Recently, some scholars considered various forms of sensitivity via Furstenberg family such as syndetic sensitivity, cofinite sensitivity and thickly sensitivity and so on.([6], [10], [11], [13-15])

In [7] the authors considered equicontinuity via a syndetic Furstenberg family and introduced a notion of syndetically equicontinuity and showed that an analogue of Auslander-Yorke dichotomy theorem could also be found for some stronger forms of sensitivity. Precisely, they proved that a minimal system is either thickly sensitive or syndetically equicontinuous.1. Concerning the study on analogue of Auslander-Yorke dichotomy theorem, recently we can find more result in [12] where is obtained an Auslander-Yorke type dichotomy theorem for r-sensitivity being stronger version of sensitivity.

Through the notion of syndetically equicontinuity in [7], we know that it can be generalized to \(\mathcal{F}\)-equicontinuity, where \(\mathcal{F}\) is a Furstenberg family. Also we know that the notion of mean equicontinuity in [8] is related to \(\mathcal{F}\)-equicontinuity.

In this paper we introduce an \(\mathcal{F}\)-equicontinuity and show an analogue of Auslander-Yorke dichotomy theorem for transitive system (Section 3) which generalize the result of Theorem 3.4 in [7].

We also show that the notion of mean equicontinuity introduced in [8] could be consid-
pered as an $\mathcal{F}$-equicontinuity (Section 4) and that $\mathcal{F}$-equicontinuity is preserved by open factor map.

This paper is organized as follows. In section 2, we provide some basic concepts and definitions in topological dynamical system. And we introduce the notion of $\mathcal{F}$-equicontinuity. In section 3, $\mathcal{F}$-equicontinuity and an analogue of Auslander-Yorke dichotomy theorem are discussed. In section 4, we show that $\mathcal{F}$-equicontinuity is preserved by open factor maps and discuss the implication between mean equicontinuity and $\mathcal{F}$-equicontinuity.

2 Preliminaries

2.1. Furstenberg family

In this section we recall some basic concepts related to Furstenberg family (more detail in [1]).

Denote by $\mathbb{Z}_+$ the set of all non-negative integers.

Let $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}_+$. A collection $\mathcal{F} \subset \mathcal{P}$ is Furstenberg family if $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$.

Given a Furstenberg family $\mathcal{F}$, define its dual family $k\mathcal{F}$ as follows:

$$k\mathcal{F} = \{ F \in \mathcal{P} : \mathbb{Z}_+ \setminus F \notin \mathcal{F} \} = \{ F \in \mathcal{P} : \text{for any } F' \in \mathcal{F}, F \cap F' \neq \emptyset \}.$$

Then it is easy to check that $\mathcal{F}$ is a Furstenberg family if and only if $k\mathcal{F}$ is so, and that $k(k\mathcal{F}) = \mathcal{F}$.

For $i \in \mathbb{Z}_+$ and $F \in \mathcal{P}$, let $F + i = \{ j + i : j \in F \}$ and $F - i = \{ j - i : j \in F \} \cap \mathbb{Z}_+$. A Furstenberg family $\mathcal{F}$ is called translation invariant if for any $F \in \mathcal{F}$ and any $i \in \mathbb{Z}_+$, $F + i \in \mathcal{F}$ and $F - i \in \mathcal{F}$.

Given two Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$, define

$$\mathcal{F}_1 \cdot \mathcal{F}_2 = \{ F_1 \cap F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2 \}.$$

A Furstenberg family $\mathcal{F}$ is said to be a filter if it satisfies $\mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}$ and it has the Ramsey property if $F_1 \cup F_2 \in \mathcal{F}$ implies $F_1 \in \mathcal{F}$ or $F_2 \in \mathcal{F}$.

It can be easily checked that Furstenberg family $\mathcal{F}$ has the Ramsey property if and only if $k\mathcal{F}$ is a filter.

Let $\mathcal{B}$ be the collection of all infinite subsets of $\mathbb{Z}_+$ and $\mathcal{F}_{cf}$ be the family of cofinite subsets, that is, the collection of subsets of $\mathbb{Z}_+$ with finite complements. It is easy to see that $\mathcal{F}_{cf} = k\mathcal{B}$.

Let $\mathcal{F}_t$ be the collection of the subsets of $\mathbb{Z}_+$ which contain arbitrary long runs of positive integers and denote its dual family by $\mathcal{F}_s$. The element of $\mathcal{F}_s$ is called syndetic set. And then the set $F \in \mathcal{P}$ is a syndetic set if and only if there is an $N \in \mathbb{N}$ such that $\{i, i+1, \cdots, i+N\} \cap F \neq \emptyset$ for every $i \in \mathbb{Z}_+$. Also the element of $\mathcal{F}_t$ is called thick set.

The set $F \in \mathcal{P}$ is thickly syndetic set if for every $N \in \mathbb{N}$ the positions where length $N$ runs begin form a syndetic set.
We recall the upper density of a set $F \subset \mathbb{Z}_+$ by

$$D(F) = \limsup_{n \to \infty} \frac{\#(F \cap [0, n - 1])}{n},$$

where $\#(\cdot)$ means the cardinality of a set([8]). For every $a \in [0, 1)$, set $D(a+) = \{ F \in \mathcal{B} : D(F) > a \}$.

Similarly, $D(F)$, the lower density of $F$, is defined by

$$D(F) = \liminf_{n \to \infty} \frac{\#(F \cap \{0, 1, \ldots, n - 1\})}{n}.$$ 

The upper Banach density $BD^*(F)$ is defined by

$$BD^*(F) = \limsup_{N - M \to \infty} \frac{\#(F \cap [M, N])}{N - M + 1}.$$ 

Similarly, we can define the lower Banach density $BD^*(F)$([8]). For every $a \in [0, 1)$, set $BD^*(a+) = \{ F \in \mathcal{B} : BD^*(F) > a \}$.

2.2. $\mathcal{F}$-equicontinuity

Firstly we recall some concepts of topological dynamical system.

A dynamical system $(X, T)$ is called transitive if for any nonempty open subsets $U, V \subset X$, $N_T(U, V) = \{ n \in \mathbb{Z}_+ : U \cap T^{-n}V \neq \emptyset \}$ is nonempty and a point $x \in X$ is transitive if its orbit $O^+(x) = \{ T^n(x) : n \in \mathbb{Z}_+ \}$ is dense in $X$. Denote by $\text{Tran}(X, T)$ the set of all transitive points of $(X, T)$. $(X, T)$ is transitive if and only if $\text{Tran}(X, T)$ is a dense $G_\delta$ subset of $X$.

A dynamical system $(X, T)$ is called minimal if $\text{Tran}(X, T) = X$.

Let $(X, d)$ be a compact metric space and $T : X \to X$ be a continuous map. And let $\mathcal{F}$ be a Furstenberg family. For given $x \in X$ and a subset $G \subset X$, set

$$N_T(x, G) = \{ n \in \mathbb{Z}_+ : T^n(x) \in G \}.$$ 

We write as follows:

$$\Delta_\varepsilon = \{ (x, y) \in X \times X : d(x, y) < \varepsilon \}, B(x, \delta) = \{ y \in X : d(x, y) < \delta \},$$

$$\overline{\Delta}_\varepsilon = \{ (x, y) \in X \times X : d(x, y) \leq \varepsilon \}.$$ 

Now we introduce the notion of $\mathcal{F}$-equicontinuity.

**Definition 2.1.** A dynamical system $(X, T)$ is said to be $\mathcal{F}$-equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$,

$$N_{T \times T}((x, y), \Delta_\varepsilon) \in \mathcal{F}. $$
A point $x \in X$ is called an $\mathcal{F}$-equicontinuous point (or $(X, T)$ is $\mathcal{F}$-equicontinuous at $x \in X$) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $y \in B(x, \delta)$, $N_{T \times T}((x, y), \Delta_\varepsilon) \in \mathcal{F}$.

A transitive system is called almost $\mathcal{F}$-equicontinuous if there is at least one $\mathcal{F}$-equicontinuous point. The set of all $\mathcal{F}$-equicontinuous points is denoted by $\text{Eq}_\mathcal{F}(T)$.

Set

$$\text{Eq}_\mathcal{F} := \{ x \in X | \text{there is a } \delta > 0 \text{ such that for any } y, z \in B(x, \delta), \quad N_{T \times T}((y, z), \Delta_\varepsilon) \in \mathcal{F} \}. $$

And we need more following definitions for our study.

**Definition 2.2.** ([6], [10-14]) A topological dynamical system $(X, T)$ is said to be $\mathcal{F}$-sensitive if there exists $\varepsilon > 0$ (F-sensitive constant) such that for any nonempty open subset $U$ of $X$

$$S_f(U, \varepsilon) = \{ n \in \mathbb{Z}_+ : \text{diam } T^n(U) > \varepsilon \} \in \mathcal{F}. $$

In addition, if $\mathcal{F}$ is $\mathcal{F}_s(\mathcal{F}_{cf}, \mathcal{F}_t$ respectively), then $(X, T)$ is called syndetically sensitive (cofinitely sensitive, thickly sensitive respectively).

**Definition 2.3.** ([8]) A dynamical system $(X, T)$ is said to be mean-L-stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$,

$$\overline{D}(\{ n \in \mathbb{Z}_+ : d(T^n x, T^n y) \geq \varepsilon \}) < \varepsilon. $$

**Definition 2.4.** ([8]) Let $X$ and $Y$ be topological spaces and $\pi : X \to Y$ be a map. The map $\pi$ is called open if the image of each nonempty open subset of $X$ is open in $Y$, and semi-open if the image of each nonempty open subset of $X$ has nonempty interior in $Y$. And $\pi$ is said to be open at a point $x \in X$ if for every neighborhood $U$ of $x$, $\pi(U)$ is a neighborhood of $\pi(x)$.

3 An analogue of Auslander-Yorke dichotomy theorem for $\mathcal{F}$-sensitivity

In this section we study on analogues of Auslander-Yorke theorem for $\mathcal{F}$-sensitivity using $\mathcal{F}$-equicontinuity.

We need following lemmas for it.

**Lemma 3.1.** Let $(X, T)$ be a dynamical system and $\mathcal{F}$ be a translation invariant Furstenberg family. Then $\text{Eq}_\mathcal{F}$ is an open subset of $X$ and $T^{-1}(\text{Eq}_\mathcal{F}) \subset \text{Eq}_\mathcal{F}$. Moreover if $\mathcal{F}$ is a filter then $\text{Eq}_\mathcal{F}(T) = \bigcap_{\varepsilon > 0} \text{Eq}_\mathcal{F}$. 

**Proof.** Assume that $\mathcal{F}$ is a translation invariant family. Fix any $x \in \text{Eq}_\mathcal{F}$ and then there exists a $\delta > 0$ such that for any $z, w \in B(x, \delta)$, $N_{T \times T}((z, w), \Delta_\varepsilon) \in \mathcal{F}$.

If $y \in B(x, \delta/2)$ and $z, w \in B(y, \delta/2)$ then $z, w \in B(x, \delta)$.

So $B(x, \delta/2) \subset \text{Eq}_\mathcal{F}$, thus $\text{Eq}_\mathcal{F}$ is an open set.
Next, if \( x \in T^{-1}(\text{Eq}_\mathcal{F}) \) then \( Tx \in \text{Eq}_\mathcal{F} \) and there is a \( \delta > 0 \) such that for any \( y', y'' \in B(Tx, \delta) \), \( N_{T \times T}((y', y''), \Delta_e) \in \mathcal{F} \).

Since \( T \) is continuous, there is a \( \eta > 0 \) such that \( y, z \in B(x, \eta) \) implies \( Ty, Tz \in B(Tx, \delta) \). So if \( y, z \in B(x, \eta) \) then \( N_{T \times T}((Ty, Tz), \Delta_e) \in \mathcal{F} \).

Since \( \mathcal{F} \) is a translation invariant and \( 1 + N_{T \times T}((Ty, Tz), \Delta_e) \subset N_{T \times T}((y, z), \Delta_e) \),

\[
N_{T \times T}((y, z), \Delta_e) \in \mathcal{F}.
\]

Therefore \( x \in \text{Eq}_\mathcal{F} \) and \( T^{-1}(\text{Eq}_\mathcal{F}) \subset \text{Eq}_\mathcal{F} \).

Finally, we are going to show that if \( \mathcal{F} \) is a filter then \( \text{Eq}_\mathcal{F}(T) = \bigcap_{\epsilon > 0} \text{Eq}_\epsilon^\mathcal{F} \).

If \( x \in \text{Eq}_\mathcal{F}(T) \) then for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( y \in B(x, \delta) \) implies \( N_{T \times T}((x, y), \Delta_{\epsilon/2}) \in \mathcal{F} \). So for every \( y, z \in B(x, \delta) \), since

\[
N_{T \times T}((y, z), \Delta_e) \supset N_{T \times T}((x, y), \Delta_{\epsilon/2}) \cap N_{T \times T}((x, z), \Delta_{\epsilon/2})
\]

and \( \mathcal{F} \) is a filter, \( N_{T \times T}((y, z), \Delta_e) \in \mathcal{F} \). Therefore \( x \in \text{Eq}_\epsilon^\mathcal{F} \) and \( \text{Eq}_\mathcal{F}(T) \subset \bigcap_{\epsilon > 0} \text{Eq}_\epsilon^\mathcal{F} \).

The proof of \( \text{Eq}_\mathcal{F}(T) \subset \bigcap_{\epsilon > 0} \text{Eq}_\epsilon^\mathcal{F} \) is clear. So \( \text{Eq}_\mathcal{F}(T) = \bigcap_{\epsilon > 0} \text{Eq}_\epsilon^\mathcal{F} \). \( \square \)

**Lemma 3.2.** Let \((X, T)\) be a dynamical system and \( \mathcal{F} \) be a filter. Then \((X, T)\) is \( \mathcal{F} \)-equicontinuous if and only if \( \text{Eq}_\mathcal{F}(T) = X \).

**Proof.** The proof of necessity is clear. We will prove the sufficiency.

Fix any \( \epsilon > 0 \). Since \( \text{Eq}_\mathcal{F}(T) = X \), for every \( x \in X \) there is a \( \delta_x > 0 \) such that \( y \in B(x, \delta_x) \) implies \( N_{T \times T}((x, y), \Delta_{\epsilon/2}) \in \mathcal{F} \).

Given \( y, z \in B(x, \delta_x) \), we have

\[
N_{T \times T}((y, z), \Delta_e) \supset N_{T \times T}((x, y), \Delta_{\epsilon/2}) \cap N_{T \times T}((x, z), \Delta_{\epsilon/2})
\]

by the triangular inequality. And since \( \mathcal{F} \) is a filter, we have \( N_{T \times T}((y, z), \Delta_e) \in \mathcal{F} \).

By the compactness of \( X \), there are finite points \( x_1, x_2, \cdots, x_N \in X \) such that

\[
\bigcup_{i=1}^N B(x_i, \delta_{x_i}/2) = X.
\]

Set \( \delta = \min\{\delta_{x_1}/2, \cdots, \delta_{x_N}/2\} \).

Now we are going to prove the \( \mathcal{F} \)-equicontinuity of \((X, T)\).

Let \( u, v \in X \) be two points with \( d(u, v) < \delta \). Then there exists an \( i \in \{1, 2, \cdots, N\} \) such that \( u \in B(x_i, \delta_{x_i}/2) \subset B(x_i, \delta_{x_i}) \). Also \( d(u, v) < \delta \leq \delta_{x_i}/2 \) implies \( v \in B(x_i, \delta_{x_i}) \).

So \( N_{T \times T}((u, v), \Delta_e) \in \mathcal{F} \) and therefore \((X, T)\) is \( \mathcal{F} \)-equicontinuous. \( \square \)

Next proposition shows a property of the set of \( \mathcal{F} \)-equicontinuous points for a transitive dynamical system.

**Proposition 3.1.** Let \((X, T)\) be a transitive dynamical system and \( \mathcal{F} \) be a translation invariant Furstenberg family. Then the set of \( \mathcal{F} \)-equicontinuous points is either empty or residual. If in addition \((X, T)\) is almost \( \mathcal{F} \)-equicontinuous then \( \text{Tran}(X, T) \subset \text{Eq}_\mathcal{F}(T) \).

Moreover if in addition \( \mathcal{F} \) is a filter, and \((X, T)\) is minimal and almost \( \mathcal{F} \)-equicontinuous then it is \( \mathcal{F} \)-equicontinuous.
Proof. By Lemma 2.1, the set $\text{Eq}_k^F$ is open and $T^{-1}(\text{Eq}_k^F) \subset \text{Eq}_k^F$. If $\text{Eq}_k^F$ is not empty then for any nonempty open subset $U$ of $X$, by the transitivity of $(X, T)$, there exists an $n \in \mathbb{Z}_+$ such that $\emptyset \neq T^{-n}(\text{Eq}_k^F) \cap U \subset \text{Eq}_k^F \cap U$. So since $\text{Eq}_k^F$ intersects with any nonempty open subset, $\text{Eq}_k^F$ is dense in $X$.

Thus by the Baire Category theorem, $\text{Eq}_k^F(T)$ is empty or residual because $\text{Eq}_k^F(T) = \bigcap_{\varepsilon > 0} \text{Eq}_k^F$. 

If $\text{Eq}_k^F(T)$ is residual then for any $\varepsilon > 0 \text{Eq}_k^F$ is open and dense. If $x \in \text{Tran}(X, T)$ then there exists $n \in N$ such that $T^n(x) \in \text{Eq}_k^F$. So $x \in T^{-n}(\text{Eq}_k^F) \subset \text{Eq}_k^F$. Thus $x \in \bigcap_{\varepsilon > 0} \text{Eq}_k^F = \text{Eq}_k^F(T)$.

If $(X, T)$ is minimal then $\text{Tran}(X, T) = X$ and so $\text{Eq}_k^F(T) = X$. By Lemma 2.2 $(X, T)$ is $\mathcal{F}$-equicontinuous.

Next dichotomy theorem and corollary are analogues of the Auslander-Yorke’s dichotomy theorem for $\mathcal{F}$-sensitivity.

Theorem 3.1. Let $(X, T)$ be a transitive dynamical system and $\mathcal{F}$ be a Furstenberg family such that its dual family $k\mathcal{F}$ is translation invariant. Then $(X, T)$ is either $\mathcal{F}$-sensitive and $\text{Eq}_{k,F}(T) = \emptyset$, or almost $k\mathcal{F}$-equicontinuous and $\text{Tran}(X, T) \subset \text{Eq}_{k,F}(T)$.

Proof. It suffices to show that if $(X, T)$ is not $\mathcal{F}$-sensitive then $\text{Tran}(X, T) \subset \text{Eq}_{k,F}(T)$. Assume that $(X, T)$ is not $\mathcal{F}$-sensitive. Then for any $\varepsilon > 0$ there exists an nonempty open subset $U$ of $X$ such that $S_T(U, \varepsilon/2) \notin \mathcal{F}$. So

$$F = \{n \in \mathbb{Z}_+ : \text{diam}\, T^n(U) \leq \varepsilon/2 \} = \mathbb{Z}_+ \setminus S_T(U, \varepsilon/2) \in k\mathcal{F}.$$ 

Take any $x \in U$ and then there is a $\delta > 0$ with $B(x, \delta) \subset U$.

For any $y \in B(x, \delta) \subset U$, since $N_{T \times T}((x, y), \Delta_x) \supset N_{T \times T}((x, y), \overline{\Delta}_x/2) \supset F$, $N_{T \times T}((x, y), \Delta_x) \in k\mathcal{F}$. Thus $x \in \text{Eq}_{k,F}(T)$ and by Proposition 3.1, $(X, T)$ is almost $k\mathcal{F}$-equicontinuous and $\text{Tran}(X, T) \subset \text{Eq}_{k,F}(T)$.

Remark 1. Theorem 3.1 coincides with Theorem 3.4 in [7] if the family $\mathcal{F}$ is replaced with thick family $\mathcal{T}$. So Theorem 3.1 is a generalization of Theorem 3.4 in [7].

Corollary 3.1. Assume that $\mathcal{F}$ has Ramsey property and its dual family $k\mathcal{F}$ is translation invariant. If $(X, T)$ is minimal then it is either $\mathcal{F}$-sensitive or $k\mathcal{F}$-equicontinuous.

Remark 2. Corollary 3.1 is a generalization of Auslander-Yorke’s theorem which is obtained by replacing the family $\mathcal{F}$ with the family $\mathcal{B}$.

4 The implication between mean equicontinuity and $\mathcal{F}$-equicontinuity

In this section we discuss some relations between mean equicontinuity and $\mathcal{F}$-equicontinuity.

In [8] is proved that mean equicontinuity is preserved by factor maps.

Here we are going to show that $\mathcal{F}$-equicontinuity is preserved by open factor maps.
Lemma 4.1. Let \((X, T)\) and \((Y, S)\) be topological dynamical systems and \(\pi : X \to Y\) be a factor map. If \(x \in X\) is an \(\mathcal{F}\)-equicontinuous point of \(T\) and \(\pi\) is open at \(x \in X\) then \(y = \pi(x)\) is an \(\mathcal{F}\)-equicontinuous point of \(S\).

Proof. Since \(\pi\) is continuous, for any \(\varepsilon > 0\) there is a \(\delta > 0\) such that \(d_X(x, x') < \delta\) implies \(d_Y(\pi(x), \pi(x')) < \varepsilon\). Here \(d_X\) and \(d_Y\) respectively denote the metric of \(X\) and \(Y\). And we write

\[
\Delta^X_\delta = \{(x, x') \in X \times X : d_X(x, x') < \delta\}, \Delta^Y_\delta = \{(y, y') \in Y \times Y : d_Y(y, y') < \delta\}.
\]

And since \(x\) is an \(\mathcal{F}\)-equicontinuous point of \(T\), for the above \(\delta > 0\) there is a \(\delta_1 > 0\) such that for any \(x' \in B(x, \delta_1)\), \(F = N_{T \times T}((x, x'), \Delta^X_\delta) \in \mathcal{F}\).

So if \(n \in F\) then \(\pi(T^n x, T^n x') < \delta\) and this implies

\[
d_Y(\pi(T^n x), \pi(T^n x')) = d_Y(S^n(\pi(x)), S^n(\pi(x'))) < \varepsilon.
\]

Thus \(N_{S \times S}((y, \pi(x')), \Delta^Y_\delta) \supset F\) and this implies \(N_{S \times S}((y, \pi(x')), \Delta^Y_\delta) \in \mathcal{F}\).

Since \(\pi\) is open at \(x \in X\), \(\pi(B(x, \delta_1))\) is a neighborhood of \(y = \pi(x)\) and so there is a \(\delta_2 > 0\) with \(B(y, \delta_2) \subset \pi(B(x, \delta_1))\).

Therefore for every \(y' \in B(y, \delta_2)\), \(N_{S \times S}((y, y'), \Delta^Y_\delta) \in \mathcal{F}\), that is, \(y\) is an \(\mathcal{F}\)-equicontinuous point of \(S\).

\(\Box\)

Theorem 4.1. Let \((X, T)\) and \((Y, S)\) be transitive dynamical systems and \(\pi : X \to Y\) be a semi-open factor map. And let \(\mathcal{F}\) be a translation invariant family. If \((X, T)\) is almost \(\mathcal{F}\)-equicontinuous then so is \((Y, S)\).

Proof. Since \(\pi\) is semi-open, by Lemma 2.1 in [5], the set \(\{x \in X : \pi\) is open at \(x\}\) is residual in \(X\). So we can take a transitive point \(x \in X\) such that \(\pi\) is open at \(x \in X\). Since \((X, T)\) is almost \(\mathcal{F}\)-equicontinuous, by Proposition 3.1 \(x \in X\) is an \(\mathcal{F}\)-equicontinuous point of \(T\) and by Lemma 3.1 \(y = \pi(x)\) is also an \(\mathcal{F}\)-equicontinuous point of \(S\). Thus \((Y, S)\) is also almost \(\mathcal{F}\)-equicontinuous.

\(\Box\)

Let \((X, d_X)\) and \((Y, d_Y)\) be the metric spaces. The metric on the product space \(X \times Y\) is defined by \(d((x, y), (x', y')) = \sqrt{(d_X(x, x'))^2 + (d_Y(y, y'))^2}\). Then the following lemma holds.

Lemma 4.2. Let \((X, d)\) be a compact metric space and \(U\) be a nonempty open subset of \(X \times X\). Let \(\Delta^X\) be a diagonal of \(X \times X\), that is, \(\Delta^X = \{(x, x) : x \in X\}\). If \(\Delta^X \subset U\) then there exists a \(\delta > 0\) such that

\[
\Delta^X_\delta = \{(x, y) \in X \times X : d_X(x, y) < \delta\} \subset U.
\]

Proof. For every \((x, y) \in X \times X\), clearly \(d((x, y), \Delta^X) = d((y, x), \Delta^X)\). Since \(\Delta^X\) is closed in \(X \times X\), there is a \(\delta > 0\) such that

\[
B(\Delta^X, \delta) = \{(x, y) \in X \times X : d((x, y), \Delta^X) < \delta\} \subset U.
\]

If \((x, y) \in \Delta^X_\delta\) then \(\delta > d_X(x, y) = d((x, y), (y, y)) \geq d((x, y), \Delta^X)\) and this implies \((x, y) \in B(\Delta^X, \delta)\). Therefore \(\Delta^X_\delta \subset B(\Delta^X, \delta) \subset U\).

\(\Box\)
Theorem 4.2. Let \((X, T)\) and \((Y, S)\) be topological dynamical systems and \(\pi : X \to Y\) be an open factor map. And let \(\mathcal{F}\) be a Furstenberg family. If \((X, T)\) is \(\mathcal{F}\)-equicontinuous then so is \((Y, S)\).

Proof. Since \(\pi\) is continuous, for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that \(d_X(x, x') < \delta\) implies \(d_Y(\pi(x), \pi(x')) < \varepsilon\). For the above \(\delta > 0\), since \((X, T)\) is \(\mathcal{F}\)-equicontinuous, there exists a \(\delta_1 > 0\) such that \(d_X(x, x') < \delta_1\) implies \(F = N_{T \times T}((x, x'), \Delta^X_{\delta_1}) \in \mathcal{F}\). So if \(n \in F\) then \(d_X(T^n x, T^n x') < \delta\) and this implies

\[
d_Y(\pi(T^n x), \pi(T^n x')) = d_Y(S^n(\pi(x)), S^n(\pi(x'))) < \varepsilon.
\]

Thus \(N_{S \times S}((\pi(x), \pi(x')), \Delta^Y_{\varepsilon}) \supseteq F\).

Since \(\pi\) is open, \(\pi \times \pi : X \times X \to Y \times Y\) is also open. So \(\pi \times \pi(\Delta^X_{\delta_1})\) is open in \(Y \times Y\) and contains a diagonal of \(Y \times Y\), that is, \(\pi \times \pi(\Delta^X_{\delta_1}) \supseteq \Delta^Y_{\varepsilon}\). Then by Lemma 3.2 there exists \(\delta_2 > 0\) such that \(\Delta^Y_{\delta_2} \subseteq \pi \times \pi(\Delta^X_{\delta_1})\).

Thus for every \((y, y') \in \Delta^Y_{\delta_2}\) there exists a pair \((x', x') \in \Delta^X_{\delta_1}\) such that \(y = \pi(x), y' = \pi(x')\). Since \(d_X(x', x') < \delta_1, N_{S \times S}((y, y'), \Delta^Y_{\varepsilon}) \supseteq F\) and this implies

\[
N_{S \times S}((y, y'), \Delta^Y_{\varepsilon}) \in \mathcal{F}.
\]

Therefore \((Y, S)\) is also \(\mathcal{F}\)-equicontinuous. \(\square\)

Following lemma 4.3 and 4.4 show some implications between mean equicontinuity and \(\mathcal{F}\)-equicontinuity.

Lemma 4.3. If \((X, T)\) is mean equicontinuous then for any \(a < 0, 1\), it is \(k\overline{D}(a+)-\)equicontinuous. Also if \((X, T)\) is almost mean equicontinuous, then for any \(a < 0, 1\), it is almost \(k\overline{D}(a+)-\)equicontinuous.

Proof. Assume that there exists an \(a < 0, 1\) such that \((X, T)\) is not \(k\overline{D}(a+)-\)equicontinuous. Then there is \(\varepsilon_0 > 0\) such that for any \(\delta > 0\), there exist \(x, y \in X\) with \(d(x, y) < \delta\) such that \(N_{T \times T}((x, y), \Delta_{\varepsilon_0/a}) \notin k\overline{D}(a+)\). So

\[
Z_+ \setminus N_{T \times T}((x, y), \Delta_{\varepsilon_0/a}) = \{n \in Z_+ : d(T^n x, T^n y) \geq \varepsilon_0/a\} \in \overline{D}(a+).
\]

Set \(F = \{n \in Z_+ : d(T^n x, T^n y) \geq \varepsilon_0/a\}\), and then

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) = \limsup_{n \to \infty} \frac{1}{n} \left( \sum_{i \in [0, n-1] \cap F} d(T^i x, T^i y) + \sum_{i \in [0, n-1] \cap F^c} d(T^i x, T^i y) \right) \\
\geq \limsup_{n \to \infty} \frac{\#([0, n-1] \cap F)}{n \cdot \varepsilon_0/a} \geq \varepsilon_0.
\]

This contradicts to the mean equicontinuity of \((X, T)\). The proof of second part is similar to this. \(\square\)
Lemma 4.4. If \((X, T)\) is \(k\overline{D}(0+)\)-equicontinuous then it is mean equicontinuous. Also if \((X, T)\) is almost \(k\overline{D}(0+)\)-equicontinuous then it is almost mean equicontinuous.

Proof. Since \((X, T)\) is \(k\overline{D}(0+)\)-equicontinuous, for every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that whenever \(x, y \in X\) with \(d(x, y) < \delta\), \(N_{T \times T}((x, y), \Delta_\varepsilon) \in k\overline{D}(0+)\), that is,

\[
\{n \in \mathbb{Z}_+: d(T^n x, T^n y) \geq \varepsilon\} = \mathbb{Z}_+ \setminus N_{T \times T}((x, y), \Delta_\varepsilon) \notin \overline{D}(0+).
\]

So \(\overline{D}(\{n \in \mathbb{Z}_+: d(T^n x, T^n y) \geq \varepsilon\}) = 0 < \varepsilon\), that is, \((X, T)\) is mean-L-stable.

Thus by Lemma 3.1 in [8], \((X, T)\) is mean equicontinuous. The proof of second part is similar to this.

Following lemma shows an implication between mean sensivity and \(\mathcal{F}\)-sensitivity.

**Lemma 4.5.** If \((X, T)\) is mean sensitive with sensitive constant \(\delta > 0\) then for any \(a \in \left[0, \frac{\delta}{\text{diam}(X)}\right]\), it is \(\overline{D}(a+)\)-sensitive.

Proof. For any \(a \in \left[0, \frac{\delta}{\text{diam}(X)}\right]\), set \(\delta' = \delta - a \cdot \text{diam}(X) > 0\). Now we are going to show that \((X, T)\) is \(\overline{D}(a+)\)-sensitive with sensitive constant \(\delta' > 0\).

Fix any nonempty open subset of \(X\) and set \(F = S_T(U, \delta')\). We choose \(x \in U\) and then by the mean continuity of \((X, T)\), there exists \(y \in U\) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.
\]

So

\[
\delta < \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) = \limsup_{n \to \infty} \frac{1}{n} \left( \sum_{i \in [0, n-1] \cap F} d(T^i x, T^i y) + \sum_{i \in [0, n-1] \cap F^c} d(T^i x, T^i y) \right) \leq \limsup_{n \to \infty} \frac{\#([0, n-1] \cap F)}{n} \cdot \text{diam}(X) + \delta' = \overline{D}(F) \cdot \text{diam}(X) + \delta'.
\]

Therefore \(\overline{D}(F) > \frac{\delta - \delta'}{\text{diam}(X)} = a\), that is \(F = S_T(U, \delta') \in \overline{D}(a+)\).

As a consequence of the above consideration, the following proposition holds. It is immediately followed by Theorem 5.4 in [8].

**Proposition 4.1.** Let \((X, T)\) be a topological dynamical system. If \((X, T)\) is transitive, then there is an \(a > 0\) such that \((X, T)\) is either \(\overline{D}(a+)\)-sensitive or \(k\overline{D}(a+)\)-equicontinuous.

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