REAL DEGENERACY LOCI OF MATRICES AND PHASE RETRIEVAL

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Abstract. Let $A = \{A_1, \ldots, A_r\}$ be a collection of linear operators on $\mathbb{R}^m$. The degeneracy locus of $A$ is defined as the set of points $x \in \mathbb{P}^{m-1}$ for which $\text{rank}(\{A_1x, \ldots, A_rx\}) \leq m - 1$. Motivated by results in phase retrieval we study degeneracy loci of four linear operators on $\mathbb{R}^3$ and prove that the degeneracy locus consists of 6 real points obtained by intersecting four real lines if and only if the collection of matrices lies in the linear span of four fixed rank one operators. We also relate such quadrilateral configurations to the singularity locus of the corresponding Cayley cubic symmetroid. More generally, we show that if $A_i, i = 1, \ldots, m + 1$ are in the linear span of $m + 1$ fixed rank-one matrices, the degeneracy locus determines a generalized Desargues configuration which corresponds to a Sylvester spectrahedron.

1. Introduction

Given a collection of linear operators $A_1, \ldots, A_r$ on $\mathbb{R}^m$, the generalized phase retrieval problem [12] is the problem recovering a vector $x \in \mathbb{R}^m$ (up to sign) from the non-linear measurements $x^TA_1x, \ldots, x^T A_rx$. We say that a collection $A = \{A_1, \ldots, A_r\}$ has the phase retrieval property if every $x$ can be recovered up to sign from these non-linear measurements, or equivalently if the map $M_A : \mathbb{R}^m / \pm 1 \to \mathbb{R}^r, x \mapsto (x^TA_1x, \ldots, x^T A_rx)$ is injective.

In [12 Theorem 2.1] it is proved that when the $A_i$ are symmetric then $M_A$ is injective at $x$ if and only if the vectors $A_1x, \ldots, A_rx$ span $\mathbb{R}^m$ (When the $A_i$ are projections this was first proved in [7 Theorem 1.1]). Denote by $\mathbb{X}(A_1, \ldots, A_m)$ the subscheme of $\mathbb{P}^{m-1}$ defined by the condition that $x \in \mathbb{X}(A_1, \ldots, A_r)$ if and only if $\text{rank}(A_1x, \ldots, A_rx) < m$. The result of [7, 12] can then be restated as stating that the collection $A$ of symmetric matrices does not admit phase retrieval if and only if $\mathbb{X}(A_1, \ldots, A_r)$ contains a real point. In [7 Theorem 1.4], [12 Section 5] it is proved that if $r \geq 2m - 1$ then a generic collection has the phase retrieval property; i.e., $\mathbb{X}(A_1, \ldots, A_r)$ does not contain a real point. On the other hand the arguments of [7] imply that if $m = 2^k + 1$ and $r = 2m - 2$ then $\mathbb{X}(A_1, \ldots, A_r)$ always contains at least two real points [12 Theorem 5.1].

For this reason a natural question is to study when degeneracy loci of collections of matrices contain real points. In this paper our focus is on determining conditions on a collection of real matrices which ensures that the degeneracy locus is entirely real. When $m = 3$ and $r = 4$ we prove that the degeneracy locus consists of six points in $\mathbb{P}^2$ in quadrilateral configuration if and only if the matrices $A_i$ are in the linear span of four fixed rank-one matrices. We also prove that this is always the case if the four matrices are symmetric. This allows us to reinforce the connection between degeneracy loci of 4 symmetric matrices and the very real Cayley cubic symmetroid which we discuss in Section 2.4.

The second author was supported by Simons Collaboration Grants 315460, 708560.
When \( m = 4 \) the degeneracy locus of 5 general matrices is a one dimensional subscheme of degree 10 in \( \mathbb{P}^3 \). We prove that degeneracy locus consists of 10 real lines with 10 special rational points in \( \text{Desargues configuration} \) if the five matrices are in the linear span of five fixed rank-one matrices. More generally if matrices \( A_i, i = 1, \ldots, m+1 \) are in the linear span of \( m+1 \) fixed rank-one matrices then the degeneracy locus is a \textit{generalized Desargues configuration}, and corresponds to a (generalized) Sylvester symmetroid.

2. \textbf{Four Operators in \( \mathbb{R}^3 \): The Case \( m = 3 \)}

\textbf{Proposition 2.1.} For a generic four-tuple \((A_1, A_2, A_3, A_4)\) of linear operators on \( \mathbb{R}^3 \) the degeneracy locus \( \mathcal{X}(A_1, A_2, A_3, A_4) \) is a zero dimensional subscheme of degree 6 in \( \mathbb{P}^2 \).

\textbf{Proof:} This follows from the formula for the degree of a determinantal variety. We look at the locus of rank-two \( 3 \times 4 \) matrices. This is a 9-dimensional subscheme of \( \mathbb{P}^{11} \) defined by the vanishing of all \( 3 \times 3 \) minors. Using the formula (5.1) on page 95 of [1] we see that this has degree 6. For a given choice of \( A_1, \ldots, A_4 \) we obtain a 2-dimensional linear subspace in \( \mathbb{P}^{11} \) and so the intersection of this determinantal variety will, for general \( A_1, \ldots, A_4 \), be a zero-dimensional subscheme of \( \mathbb{P}^2 \).

For a generic four-tuple of matrices the degeneracy locus \( \mathcal{X}(A_1, \ldots, A_4) \subset \mathbb{P}^2 \) consists of six distinct points. Proposition [1,10] implies that if the \( A_i \) are all symmetric then the points of \( \mathcal{X} \) are rational functions of the entries of the matrices \( A_1, A_2, A_3, A_4 \).

2.1. \textbf{Configuration of Six Rational Points.} The goal of this section is to investigate when the points of the degeneracy locus are rational functions of the entries of the matrices \( A_1, \ldots, A_4 \). In this case we say that the degeneracy locus consists of six rational points.

\textbf{Proposition 2.2.} For generic matrices \( A_1, A_2, A_3, A_4 \) which are in the linear span of four fixed generic rank-one matrices, the degeneracy locus \( \mathcal{X}(A_1, \ldots, A_4) \) consists of 6 rational points obtained by intersecting 4 lines in \( \mathbb{P}^2 \).

\textbf{Proof.} By hypothesis we can write

\[
\begin{align*}
A_1 &= a_{1,1} E_1 + a_{1,2} E_2 + a_{1,3} E_3 + a_{1,4} E_4 \\
A_2 &= a_{2,1} E_1 + a_{2,2} E_2 + a_{2,3} E_3 + a_{2,4} E_4 \\
A_3 &= a_{3,1} E_1 + a_{3,2} E_2 + a_{3,3} E_3 + a_{3,4} E_4 \\
A_4 &= a_{4,1} E_1 + a_{4,2} E_2 + a_{4,3} E_3 + a_{4,4} E_4
\end{align*}
\]

(1)

where \( E_i, i = 1, \ldots, 4 \) are rank-one matrices, and the \( a_{ij} \) are chosen generically.

Since the \( E_i \) are rank one, \( \ker E_i \) is a 2-dimensional subspace of \( \mathbb{R}^3 \); or equivalently a line \( \mathbb{P}^2 \). Since the \( E_i \)'s are chosen generically, then for each pair of indices \( i, j \), \( \ker E_i \cap \ker E_j \) is a line in \( \mathbb{R}^3 \) or equivalently a point \( x_{ij} \in \mathbb{P}^2 \). If \( x \in \ker E_i \cap \ker E_j \) then the vectors \( A_1 x, A_2 x, A_3 x, A_4 x \) lie in the two dimensional subspace span the images of \( E_k \) and \( E_l \) where \( \{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\} \).

Since the degree of the degeneracy locus is 6, we see that \( \mathcal{X}(A_1, A_2, A_3, A_4) \) is the union of the \( x_{ij} \) which correspond to the pairwise intersections of the four projective lines \( l_i = \ker E_i \). Moreover each point in the degeneracy locus is a rational function of the entries of the \( E_i \).

\textbf{Remark 2.3.} Note that the degeneracy locus is independent of the of the choices of the parameters \( a_{ij} \) in (1).
Example 2.4. Suppose \( \text{Im } E_i = \langle e_i \rangle, i = 1, \ldots, 4 \) where 
\[
\begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]
\]
\[
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\] \( \in \mathbb{R}^3 \). Then the degeneracy locus is
\[
X = \{x_{12} = (1 : -1 : 0), x_{13} = (1 : 0 : -1), x_{14} = (0 : 1 : -1), x_{23} = (1 : 0 : 0), x_{24} = (0 : 1 : 0), x_{34} = (0 : 0 : 1)\}.
\]

Note that by construction the triples of points \( \{x_{12}, x_{13}, x_{14}\}, \{x_{12}, x_{23}, x_{24}\}, \{x_{13}, x_{23}, x_{34}\}, \{x_{14}, x_{24}, x_{34}\} \) are collinear. In projective algebraic geometry a configuration of six points \( (P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}) \in (\mathbb{P}^2)^6 \) is called a quadrilateral set (Fig. 1) if the triples \( P_{12}P_{13}P_{14}, P_{12}P_{23}P_{24}, P_{13}P_{23}P_{34}, \) and \( P_{14}P_{24}P_{34} \) are collinear. In the dual projective space this figure forms a complete quadrangle which is a system of geometric objects consisting of any four lines in a plane, no three of which are collinear, and of the six points connecting the six pairs of lines. See [8, Section 15, Chapter 3] for more details on the classically well-known topic of projective configurations.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{Complete Quadrilateral and Quadrangle Configurations: \(6_{243}3\) and \(4_{362}\)}
\end{figure}

The following lemma shows that any two quadrilateral configurations are projectively equivalent.

Lemma 2.5. The ordered six-tuples of rational points in \( \mathbb{P}^2 \) obtained as the intersections of \( A \) lines consist of a single orbit under the action of the group \( GL_3(\mathbb{P}^2) \).

Proof. A quadrilateral configuration is determined by 4 points \( l_1, l_2, l_3, l_4 \) in the dual projective space \( (\mathbb{P}^2)^* \) no three of which are collinear. Let \( A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \in GL_3 \) be the change of basis matrix such that \( Ae_i = l_i \) and for \( i = 1, 2, 3 \). Then \( A^{-1}l_4 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \), \( abc \neq 0 \)

(otherwise it will contradict that no three of the \( l_i \)'s are collinear). Set \( D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \) then \((AD)^{-1}\) moves the points \( l_1, l_2, l_3, l_4 \in (\mathbb{P}^2)^* \) to the points \( (1 : 0 : 0), (0 : 1 : 0), (0 : 1 : 0), (1 : 1 : 1) \). \( \square \)
The main result of this section is the converse to Proposition 2.2.

**Theorem 2.6.** The degeneracy locus $\mathbb{X} := \mathbb{X}(A_1, A_2, A_3, A_4)$ consists of 6 rational points with quadrilateral configuration in $\mathbb{P}^2$ if and only if $A_1, \ldots, A_4$ are in the linear span of four fixed rank-one matrices.

**Remark 2.7.** A natural question is whether it is possible to obtain as degeneracy loci other rational configurations of 6 points in $\mathbb{P}^2$ which are not quadrilateral configurations. Theorem 2.6 says that such a configuration does not arise from matrices which lie in the span of four rank one matrices.

2.2. **Proof of Theorem 2.6.** For a generic choice of four-tuple $\mathcal{A} = (A_1, A_2, A_3, A_4)$ of $3 \times 3$ matrices the degeneracy locus $\mathbb{X}_\mathcal{A} = \mathbb{X}(A_1, A_2, A_3, A_4)$ is a 0-dimensional subscheme of $\mathbb{P}^2$ of degree 6. Thus we can define a rational map $A^36 = (A^9)^4 \dashrightarrow (\mathbb{P}^2)^{[6]}, \mathcal{A} \mapsto \mathbb{X}_\mathcal{A}$. Here $(\mathbb{P}^2)^{[6]}$ denotes the Hilbert scheme of 0-dimensional subschemes of $\mathbb{P}^2$ of degree 6.

If $A \in \text{GL}_3$ then $\mathbb{X}(A_1A^{-1}, A_2A^{-1}, A_3A^{-1}, A_4A^{-1}) = A\mathbb{X}(A_1, A_2, A_3, A_4)$ so the rational map $\Phi$ commutes with the action of $\text{GL}_3$. Hence to prove the theorem it suffices to prove that if $\mathbb{X} \in (\mathbb{P}^2)^{[6]}$ is the point corresponding the configuration of six points in “standard” quadrilateral configuration

$$\mathbb{X} = \{(1 : -1 : 0), (1 : 0 : -1), (0 : 1 : -1), (1 : 1 : 0), (0 : 0 : 1), (0 : 1 : 0)\}$$

then the fiber $\Phi^{-1}(\mathbb{X})$ consists of four tuples of matrices which lie in the linear span of four fixed rank one matrices.

Our argument proceeds by dimension counting. Specifically we will show that $\Phi^{-1}(\mathbb{X})$ is a rational variety of dimension 24 and this equals the dimension of the variety of 4-tuples of matrices which lie in the span of four fixed rank one matrices and whose degeneracy locus is $\mathbb{X}$.

We begin with a proposition.

**Proposition 2.8.** The variety $X_1 := (A_1, A_2, A_3, A_4) \subset \mathbb{A}^{36}$ of 4-tuples of matrices which are in the linear span of four fixed rank one matrices is an irreducible variety of dimension 32. Moreover, the subvariety of $X_1$ parametrizing 4-tuples in the linear span of four rank one matrices with fixed null spaces is subvariety of dimension 24 obtained by intersecting $X_1$ with a linear subspace.

**Proof.** We will show $X_1$ has a rational parametrization. First observe that if $E$ is a matrix of rank one then $\Lambda E$ is of rank one and has the same null space and image. Hence we can assume that the coefficients, $a_{1,1}, \ldots, a_{1,4}$ in $[1]$ are equal to one; i.e., $A_1 = E_1 + E_2 + E_3 + E_4$.

The locus of rank one $3 \times 3$ matrices has dimension 5 since a rank one matrix is determined by a single column vector (3 parameters) and then the other columns are multiples of the first column (2 more parameters). Hence the first matrix depends on 20 parameters. The other 3 matrices are linear combinations of the first 4 rank ones giving 12 additional parameters. Therefore, the dimension of the variety $X_1$ is 32.

The dimension of the locus of rank one matrices with fixed null space is a codimension-two linear subspace of the locus of matrices of rank one. Hence the subvariety of $X_1$ parametrizing 4-tuples in the linear span of four rank one matrices with fixed null spaces is a subvariety of co-dimension $4 \times 2 = 8$; i.e. this locus has dimension 24. $\square$
We now complete the proof of the Theorem 2.6 by showing that the fiber of $\Phi$ at $P \in \mathbb{P}^2$ is rationaly parametrized variety of dimension 24. Suppose the 4-tuple of matrices

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A_2 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \quad A_3 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad A_4 = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

has degeneracy locus consisting of the six points $(1:0:0), (0:1:0), (0:0:1), (1:-1:0), (1:0:-1), (0:1:-1)$ corresponding to four lines $x = 0, y = 0, z = 0$ and $x + y + z = 0$.

The condition that the degeneracy locus contains the first three points $(1:0:0), (0:1:0), (0:0:1)$ is equivalent to the fact that the columns of $A_3, A_4$ can be expressed as the linear combination of the corresponding columns of $A_1$ and $A_2$, i.e., the $j$-th column of $A_3, A_4$ can be expressed as

$$A_3(j) = \alpha_{1j}A_1(j) + \alpha_{2j}A_2(j), \quad A_4(j) = \alpha_{3j}A_1(j) + \alpha_{4j}A_2(j), \quad j = 1, 2, 3$$

Thus, this locus is parametrized by $(18+6+6) = 30$ parameters. Now impose the additional conditions on the differences of the columns as the degeneracy locus also contains $(1:-1:0), (0:1:-1)$. We get for each of $A_3, A_4$ three determinantal conditions, in total 6 equations. Note that these six equations are cubic in $a_{ij}$ but linear equations in $\alpha_{ij}$.

For example, for $A_3$ the three determinantal equations are as follows.

$$\begin{align*}
(A_1(1) - A_1(2)) \wedge (A_2(1) - A_2(2)) \wedge (A_3(1) - A_3(2)) &= 0 \\
(A_1(1) - A_1(3)) \wedge (A_2(1) - A_2(3)) \wedge (A_3(1) - A_3(3)) &= 0 \\
(A_1(2) - A_1(3)) \wedge (A_2(2) - A_2(3)) \wedge (A_3(2) - A_3(3)) &= 0
\end{align*}$$

Substituting the relations from equation (2) into the corresponding determinantal equations we get three equations which are linear in $(\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22}, \alpha_{13}, \alpha_{23})$ and $(\alpha_{12}, \alpha_{22}, \alpha_{13}, \alpha_{23})$ respectively. More precisely,

$$\begin{align*}
&\begin{bmatrix} a_{11} & (b_{11} - b_{12}) & (b_{21} - b_{22}) & (b_{31} - b_{32}) & (a_{11} - a_{12}) & (a_{12} - a_{13}) & (b_{11} - b_{12}) & (b_{12} - b_{13}) & (a_{11} - a_{12}) & (a_{12} - a_{13}) \\ a_{21} & (b_{21} - b_{22}) & (b_{11} - b_{12}) & (b_{31} - b_{32}) & (a_{21} - a_{22}) & (a_{11} - a_{12}) & (a_{12} - a_{13}) & (b_{21} - b_{22}) & (b_{11} - b_{12}) & (a_{21} - a_{22}) \\ a_{31} & (b_{31} - b_{32}) & (b_{21} - b_{22}) & (b_{11} - b_{12}) & (a_{31} - a_{32}) & (a_{11} - a_{12}) & (a_{12} - a_{13}) & (b_{31} - b_{32}) & (b_{11} - b_{12}) & (a_{31} - a_{32}) \\
\end{bmatrix} = 0
\end{align*}$$

This system of linear equations can be expressed as

$$Q \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{12} & \alpha_{22} & \alpha_{13} & \alpha_{23} \end{bmatrix}^T = 0$$

where $Q$ is a $3 \times 6$ matrix with full rank. For example, the random choice of $A_1 = \begin{bmatrix} 3 & 5 & 6 \\ 7 & 2 & 4 \\ 5 & 2 & 8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 4 & 3 \\ 5 & 3 & 2 \\ 5 & 1 & 7 \end{bmatrix}$, the matrix $Q = \begin{bmatrix} 90 & 64 & -90 & -64 & 0 & 0 \\ -6 & -12 & 0 & 0 & 6 & 12 \\ 0 & 0 & 68 & 37 & -68 & -37 \end{bmatrix}$

has rank 3. Similarly, there are 3 more independent determinantal equations associated with $A_4$ which provide three more free parameters. Thus, one needs to use $9 \times 2 + 3 + 3 = 24$ parameters to represent the fiber. Therefore, the dimension of the fiber is 24 for a specific choice of 6 points in the degeneracy locus. Moreover, $\varphi^{-1}(X)$ is an irreducible variety due to its rational parametrization. On the other hand by Proposition 2.8 the variety parametrizing matrices which are in the span of four fixed rank one matrices with fixed
null spaces is also 24. By Proposition 2.2, this variety is in contained in the fiber. Hence the proof.

Example 2.9. Here is a recipe to compute a 4-tuple of matrices from a given set of six points with quadrilateral configuration in \(\mathbb{P}^2\). Consider the six points

\[
(P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34})
\]

such that the triples \(P_{12}P_{13}P_{14}, P_{12}P_{23}P_{34}, P_{13}P_{23}P_{34},\) and \(P_{14}P_{24}P_{34}\) are collinear (see Fig 1). Let \(v_1 = P_{12} \times P_{13}\) (usual cross product in \(\mathbb{A}^3\) where the the point \(P_{i,j}\) is represented by a vector which is unique up to scaling), \(v_2 = P_{12} \times P_{23}, v_3 = P_{13} \times P_{23}, v_4 = P_{14} \times P_{34}\). Then the 4 lines of the quadrilateral configuration have equations \(v_i(x, y, z) = 0\), so the six points \(P_{i,j}\) form the degeneracy locus of the 4-tuple \((A_1, A_2, A_3, A_4)\) where \(A_i = \begin{bmatrix} v_i \\ 0 \\ 0 \\ 0 \end{bmatrix}\). If we write \(v_i = (a_i, b_i, c_i)\) then we can make the matrices symmetric setting \(A_i = u_i v_i^t\).

2.3. Degeneracy loci and cubic hypersurfaces. If \(S \subset \mathbb{P}^n\) is a hypersurface of deg \(d\) then a determinantal representation of \(S\) is a \(d \times d\) matrix of linear forms \(U = u_1A_0 + u_2A_2 + \ldots + u_{n+1}A_{n+1}\) such that \(S = \mathcal{V}(\det U)\). A classical fact in projective geometry states that any smooth cubic hypersurface \(S \subset \mathbb{P}^3\) has a determinantal representation meaning that if \(S = \mathcal{V}(f)\) then we can find a 4-tuple \(A = (A_1, A_2, A_3, A_4)\) of \(3 \times 3\) matrices such that \(f = \lambda \det((\sum_{i=1}^4 u_i A_i))\) for some scalar \(\lambda\). Moreover, a theorem of Clebsch states that a smooth cubic surface in \(\mathbb{P}^3\) has 72 non-equivalent determinantal representations. For an explicit construction of a representative of an equivalence class of determinantal representations see [5] and for more details on determinantal hypersurfaces see [3].

Given a determinantal representation \(U = u_1A_1 + u_2A_2 + u_3A_3 + u_4A_4\) the degeneracy locus \(X = X(A_1, A_2, A_3, A_4)\) relates to the cubic surface as follows: let \(L = [A_1x_2A_2x_3A_3x_4x]\) where \(x = (x_1, x_2, x_3)^T\). The maximal minors of \(L\) determine a linear system on \(\mathbb{P}^2\) whose base locus is \(X(A_1, A_2, A_3, A_4)\). The closure of the image of \(\mathbb{P}^2\) under the rational map \(\mathbb{P}^2 - \rightarrow \mathbb{P}^3\) defined by this linear system is a cubic hypersurface projectively equivalent to the hypersurface \(\mathcal{V}(\det U)\).

Conversely, given a set \(X\) of 6 points in \(\mathbb{P}^2\) (or more generally a 0-dimensional subscheme of length 6) the Hilbert-Burch theorem implies that there is a \(3 \times 4\) matrix \(L\) of linear forms in the variables \(x_1, x_2, x_3\) such that \(X\) is defined by the vanishing of the maximal minors \(L\). Given such an \(L\) we can define a \(3 \times 3\) linear matrix \(U\) in the variables \(u_1, u_2, u_3, u_4\) by the condition that

\[
L \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = U \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

Again \(\mathcal{V}(\det U)\) will be a cubic hypersurface which is projectively equivalent to the closure of the image of the rational map defined the linear system generated by the minors of \(L\).

If \(S\) is a cubic surface with determinantal representation \(S = \mathcal{V}(\det U)\) then the singular locus of \(S\) is the set of points \((u_1: u_2: u_3: u_4)\) where the matrix \(U\) has rank 1. The cubic surface \(S\) is non-singular if and only if the degeneracy locus \(X(A_1, A_2, A_3, A_4)\) consists of six distinct points no three of which are collinear. In this case the rational map \(\mathbb{P}^2 - \rightarrow S\) given by the linear system of cubics with base locus \(X(A_1, A_2, A_3, A_4)\)
extends to an isomorphism of \( \widetilde{\mathbb{P}^2} \to S \) where \( \widetilde{\mathbb{P}^2} \) is the blowup of \( \mathbb{P}^2 \) at the six points of \( \mathbb{X}(A_1, A_2, A_3, A_4) \).

2.4. Symmetric matrices and the Cayley symmetroid. A hypersurface \( X \subset \mathbb{P}^n \) of degree \( d \) is called a symmetric if it has a determinantal representation \( X = V(\det(\sum_{i=1}^{n+1} u_i A_i)) \) where the \( A_i \) are (complex) symmetric \( d \times d \) matrices. If the coefficients of the form \( \det(\sum_{i=1}^{n+1} u_i A_i) \) are real then \( X \) is called a real symmetric and if the \( A_i \) are real symmetric matrices then \( X \) is called a very real symmetric. A cubic symmetroid in \( \mathbb{P}^3 \) is called a Cayley symmetroid.

In this section we discuss the connection between the degeneracy loci of 4-tuples of \( 3 \times 3 \) symmetric matrices and very real Cayley symmetroids. We begin with the following result which we have not found previously stated in the literature.

**Proposition 2.10.** Any four-tuple \( \mathcal{A} = (A_1, A_2, A_3, A_4) \) of symmetric matrices is in the linear span of four rank one symmetric matrices. In particular the degeneracy locus \( \mathbb{X}(A_1, A_2, A_3, A_4) \) consists of 6 points in quadrilateral configuration.

**Proof.** The locus of symmetric \( 3 \times 3 \) matrices is isomorphic to \( \mathbb{A}^6 \), so the locus of 4-tuples of symmetric matrices is identified with \( \mathbb{A}^{24} \). On the other hand a symmetric rank one matrix is uniquely determined by a single non-zero row or column. Hence the locus of \( 3 \times 3 \) rank one symmetric matrices has dimension 3. The same argument used in the proof of Proposition 2.8 shows that the locus of matrices in the span of 4 rank one symmetric matrices is \( 3 \times 4 + 4 \times 3 = 24 \) which equals the dimension of locus of 4-tuples of \( 3 \times 3 \) symmetric matrices. Hence every 4-tuple of symmetric matrices is in the linear span of four fixed rank one symmetric matrices. \( \square \)

Let \( \mathcal{V} \) be locus of cubic Cayley symmetroids viewed as a subset of the \( \mathbb{P}^{19} \) of cubic hypersurfaces. The following fact is well known but we include a proof because of a lack of a reference.

**Lemma 2.11.** \( \mathcal{V} \) is irreducible and has codimension 4 in \( \mathbb{P}^{19} \).

**Proof.** Each of the four symmetric matrices \( A_i \) needs 6 free parameters. Since the polynomial \( f \) can be expressed as the determinant of some linear matrix pencil, so the 20 coefficients of \( f \) can be expressed as cubic polynomials in the 24 parameters, and hence defines a rational map \( \mathbb{P}^{23} \to \mathbb{P}^{19} \). Our variety \( \mathcal{V} \) is the Zariski closure of the image of this map, so it is irreducible. To compute its dimension, we form the \( 20 \times 24 \) Jacobian matrix of the parametrization. By evaluating at a generic point \( (A_1, \ldots, A_4) \) using NumericalImplicitization package [6], we find that the Jacobian matrix has rank 16. Hence the dimension of the symmetroid variety \( \mathcal{V} \subset \mathbb{P}^{19} \) is 15. \( \square \)

A Cayley symmetroid is singular since it has 4 nodes. These nodes can be understood in terms of the degeneracy locus \( \mathbb{X}(A_1, A_2, A_3, A_4) \) as follows. After blowing up \( \mathbb{P}^2 \) at six points in quadrilateral configuration the strict transforms of the lines defining the quadrilateral have self-intersection \( -2 \). The Cayley symmetroid is the surface in \( \mathbb{P}^3 \) obtained by blowing down these four lines to nodes.
Example 2.12. If we take \( A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), then the corresponding Cayley symmetroid in equation (3) is

\[
\begin{array}{c}
\begin{vmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{vmatrix}
\end{array}
\]

which has nodes at points \((1:0:0),(0:1:0),(0:0:1)\in\mathbb{P}^2\) and the degeneracy locus \(X(A_1, A_2, A_3, A_4)\) consists of the six intersection points of the four lines \(x = 0, y = 0, z = 0, x + y + z = 0\) in \(\mathbb{P}^2\).

Example 2.13. Here is a construction of the very real Cayley symmetroid starting with six points in quadrilateral configuration. By Lemma 2.5, we can assume that six points of the degeneracy locus \(X(A_1, A_2, A_3, A_4)\) consists of the six intersection points of the four lines \(x = 0, y = 0, z = 0, x + y + z = 0\) in \(\mathbb{P}^2\). A cubic curve passing through \((1:0:0),(0:1:0),(0:0:1)\) can be written as

\[
ax^2y + bxz^2 + cxy^2 + dxz^2 + eyz^2 + fxyz.
\]

Since the curve also passes through three more points \((1:-1:0),(1:0:-1),(0,1:-1)\), we have

\[
-a + c = 0, -b + d = 0, -c + f = 0.
\]

Equation (5) becomes

\[
axy(x + y) + bxz(x + z) + cxyz(y + z) + gxyz = 0.
\]

Thus, \(\{f_0 := xy(x + y), f_1 := xz(x + z), f_2 := yz(y + z), f_3 := xzy\} \) can be chosen as a basis for the four-dimensional linear system of cubics passing through these six points. The closure of the image of the map

\[
\mathbb{P}^2 \setminus \{P_1, \ldots, P_6\} \quad \longrightarrow \quad \mathbb{P}^3
\]

is the cubic surface defined by

\[
2u_4^2 + u_2^2(u_1 + u_2 + u_3) - u_1u_2u_3 = \begin{vmatrix}
u_1 & u_4 & u_4 \\
u_4 & -u_2 & u_4 \\
u_4 & u_4 & -u_3
\end{vmatrix} = 0.
\]

3. Five Operators in \(\mathbb{R}^4\): Case \(m = 4 = 2^1 + 2\)

We now study the degeneracy locus of five operators in \(\mathbb{R}^4\).

Proposition 3.1. For general 5-tuple \(\{A_1, \ldots, A_5\}\) of linear operators on \(\mathbb{R}^5\) the degeneracy locus \(X(A_1, A_2, A_3, A_4, A_5)\) is a one-dimensional sub-scheme of degree 10 in \(\mathbb{P}^3\).

Proof. We look at the locus of rank 3, \(4 \times 5\) matrices. This is a 17-dimensional subscheme of \(\mathbb{P}^{19}\) defined by the vanishing of all \(4 \times 4\) minors. Using the formula (5.1) on page 95 of [1] we see that this has degree 10. For a given choice of \(A_1, \ldots, A_5\), we obtain a 3-dimensional linear subspace in \(\mathbb{P}^{19}\) and so the intersection of this determinantal variety will, for general \(A_1, \ldots, A_5\), be a one-dimensional subscheme of \(\mathbb{P}^3\). \(\square\)
Remark 3.2. Since the degeneracy locus $\mathcal{X}(A_1, A_2, A_3, A_4, A_5)$ is defined by the vanishing of the maximal minors of the $4 \times 5$ matrix $L = [A_1 x \ A_2 x \ A_3 x \ A_4 x \ A_5 x]$ it is a Cohen-Macaulay subscheme of $\mathbb{P}^3$ of codimension-two and degree 10. The Hilbert-Burch theorem implies that any Cohen-Macaulay subscheme of $\mathbb{P}^3$ of codimension-two and degree 10 will be defined by the vanishing of the maximal minors of a $4 \times 5$ matrix of linear forms.

3.1. Configuration of Ten lines and ten points. In this subsection, we characterize the collections of matrices for which the degeneracy locus $\mathcal{X}$ consists of ten real lines in Desargues configuration. One can see the proof of Desargues theorem in $\mathbb{P}^3$ in many books, for example [11] Section 2.4, pg-30].

Proposition 3.3. If the matrices $A_1, \ldots, A_5$ are in the linear span of five fixed rank-one matrices, $E_1, E_2, E_3, E_4, E_5$ then the degeneracy locus $\mathcal{X}(A_1, \ldots, A_5)$ consists of ten real lines intersecting in 10 points. Such a configuration is called a Desargues configuration.

Proof. By assumption we know

\begin{equation}
A_i = \sum_{j=1}^{5} a_{i,j} E_j, \ i = 1, \ldots, 5
\end{equation}

for a collection of generic rank one matrices $E_1, E_2, E_3, E_4, E_5$ and scalars $a_{i,j}$. Let $l_{i,j}$ be the line in $\mathbb{P}^3$ corresponding to the two-dimensional linear subspace $\ker E_i \cap \ker E_j$ in $\mathbb{R}^4$. The union of the lines $l_{i,j}$ is a degree 10 one-dimensional subscheme of $\mathcal{X}(A_1, A_2, A_3, A_4, A_5)$ so by Proposition 3.1 this equals the degeneracy locus. The ten lines intersect in 10 points in $\mathbb{P}^3$ corresponding to the lines in $\mathbb{A}^4$ determined by the triple intersections $\ker E_i \cap \ker E_j \cap \ker E_k$. This gives the 10 special points of the Desargues configuration as shown in Fig. 3.4 below. \hfill \Box

Remark 3.4. Each of the five planes contains four of the lines (one corresponding to each of the other four planes) and each such line contains three of the points. Note that each of the five planes contains four lines and six points in quadrilateral configuration. The Desargues configuration is also known as the complete pentahedron in three dimensions [2] Chapter 4, pg-95.

Remark 3.5. Note that the degeneracy locus is same for any choice $a_{i,j}$ in (7) for a fixed tuple of rank-one matrices $E_i, i = 1, \ldots, 5$.

The same argument used in the proof of Lemma 2.5 yields the following result.

Lemma 3.6. The set of ten-tuples of lines in $\mathbb{P}^3$ in Desargues configuration is a single orbit under the group action of $\text{GL}_4$. \hfill \Box

Proposition 3.7. The variety $X_1 := (A_1, A_2, A_3, A_4, A_5) \subset \mathbb{A}^{80}$ of 5-tuples of matrices which are in the linear span of five fixed rank one matrices is an irreducible variety of dimension 55. Moreover, the subvariety of $X_1$ parametrizing 5-tuples in the linear span of five rank one matrices with fixed null spaces is a linear subspace of dimension 40.

Proof. The proof is very similar to the proof of Proposition 2.8. We again show that $X_1$ is rationally parametrized. The locus of rank one $4 \times 4$ matrices has dimension 7, so the first matrix gives $5 \times 7 = 35$ parameters, and the four other matrices depend on $4 \times 5 = 20$ additional parameters, so $\dim X_1 = 55$. If we fix the null spaces of the rank one matrices, then the first matrix gives $5 \times 4 = 20$ parameters. Hence, the locus of 5-tuples of matrices which are in the linear span of 5 rank one matrices with fixed null spaces has dimension $20 + 20 = 40$. \hfill \Box
Theorem 3.8. If $A_i, 1 \leq i \leq 5$ are five generic symmetric matrices in the linear span of five generic rank one matrices $E_i, 1 \leq i \leq 5$ then $E_i, 1 \leq i \leq 5$ are also symmetric.

Proof. Suppose that $A_i, 1 \leq i \leq 5$ are symmetric matrices in the linear span of rank one matrices $E_i, 1 \leq i \leq 5$ and let $X(A_1, A_2, A_3, A_4, A_5)$ be the corresponding degeneracy locus of the ten lines $l_{i,j}$ in $\mathbb{P}^3$ determined by two-dimensional linear subspace $\ker E_i \cap \ker E_j$ in $\mathbb{R}^4$. Since $A_i = A_i^T$, so $A_i, 1 \leq i \leq 5$ are also in the linear span of $E_i^T, 1 \leq i \leq 5$. It follows that the $X(A_1, A_2, A_3, A_4, A_5)$ is also the union of the lines $l'_{i,j} = \ker E_i^T \cap \ker E_j^T$.

Thus the sets of hyperplanes $\{\ker E_1, \ldots, \ker E_5\}$ and $\{\ker E_1^T, \ldots, \ker E_5^T\}$ are equal. By assuming that the $E_i$‘s are generic we can assume that $\ker E_i \neq \ker E_j^T$ for any $i \neq j$.

It follows that $\ker E_i = \ker E_i^T$. Assume for simplicity that $(E_i)_{1,1}$ is non-zero. Then $E_i$

$$
\begin{bmatrix}
v_i \\
\lambda v_i \\
\mu v_i \\
\delta v_i 
\end{bmatrix}
$$

is the matrix for some scalars $\lambda, \mu, \delta$ where $v_i = \langle a_i, b_i, c_i, d_i \rangle$ spans $\ker E_i$. Then $\ker E_i^T$ is spanned by $\langle a_i, \lambda a_i, \mu a_i, \delta a_i \rangle$ since we assume $a_i \neq 0$. Hence $\lambda a_i = b_i$, $\mu a_i = c_i$ and $\delta a_i = d_i$ and the matrix $E_i$ is necessarily symmetric. If $(E_i)_{1,1} = 0$ the above argument is readily modified to reach the same conclusion.

Corollary 3.9. If $m = 4$, then a 5-tuple of symmetric matrices is not in general in the span of 5 rank one matrices.

Proof. This can be seen by a dimension counting argument. By Theorem 3.8 we know that if 5 symmetric matrices are in the linear span of 5 rank one 4 x 4 matrices, then those rank one matrices can be chosen to be symmetric. Note that the dimension of the locus of matrices which are in the linear span of 5 rank one symmetric matrices is $5 \times 4 + 4 \times 5 = 40$ and it is smaller than the dimension of the locus of 5-tuples of symmetric 4 x 4 matrices which is $5 \times 10 = 50$. □
Here we conjecture that the necessary condition for the degeneracy locus to satisfy Desargues configuration is the tuple of 5 matrices in the linear span of five fixed rank-one matrices.

**Conjecture 3.10.** If the degeneracy locus $X := X(A_1, \ldots, A_5)$ consists of ten real lines in $\mathbb{P}^3$ which intersect in 10 points corresponding to a Desargues configuration, then 5-tuple of matrices are in the linear span of the five fixed rank-one matrices.

**Remark 3.11.** There is a natural strategy to prove Conjecture 3.10. By Lemma 3.6 and Proposition 3.7 it suffices to prove that the locus of 5-tuples of matrices whose degeneracy locus corresponds to a “standard” Desargues configuration is an irreducible variety of dimension 40.

### 3.2. Quartic Symmetroids in $\mathbb{P}^4$

A 5-tuple of $4 \times 4$ matrices $(A_1, A_2, A_3, A_4, A_5)$ determines a quartic hypersurface $S = \mathbb{V}(\det(A_1u_1 + A_2u_2 + A_3u_3 + A_4u_4 + A_5u_5))$. If we let $L = [A_1x_2A_2x_3A_3x_4A_4x_5]$ where $x = (x_1, x_2, x_3, x_4)^T$ then the maximal minors of $L$ determine a linear system on $\mathbb{P}^3$ whose base locus is $X(A_1, A_2, A_3, A_4)$ and $S$ is the closure of the image of the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^4$.

When the matrices $(A_1, A_2, A_3, A_4, A_5)$ are symmetric then the image is a quartic symmetroid in $\mathbb{P}^4$. Note that, unlike quartic symmetroids in $\mathbb{P}^3$, a quartic symmetroid in $\mathbb{P}^4$ is rational since it is the image of a rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^4$.

Let $\mathcal{V}$ be the variety of quartic symmetroids in $\mathbb{P}^4$. We can use the method of [4, Lemma 9] to find the dimension of the variety of three dimensional quartic symmetroids.

**Lemma 3.12.** The variety $\mathcal{V} \subset \mathbb{P}^{69}$ is irreducible of dimension 34.

**Proof:** Each of the five symmetric matrices $A_i$ needs 10 free parameters. Since the polynomial $f$ is the determinant of a linear matrix pencil the 70 coefficients of $f$ are quartic polynomials in the 50 parameters determined by the five symmetric matrices. Hence our variety is the Zariski closure of the rational map $\mathbb{P}^{49} \dashrightarrow \mathbb{P}^{69}$ and is therefore irreducible. To compute its dimension, we form the $70 \times 50$ Jacobian matrix of the parametrization. By evaluating at a generic point $(A_1, \ldots, A_4)$ using NumericalImplicitization package [6], we find that the Jacobian matrix has rank 35. Hence the dimension of the symmetroid variety $\mathcal{V} \subset \mathbb{P}^{69}$ is 34. $\square$

**Example 3.13.** Consider the case where the $A_i$’s are in the span of five rank one matrices $v_i u_i^T$ where $v_1 = [1, 0, 0, 0], v_2 = [0, 1, 0, 0], v_3 = [0, 0, 1, 0], v_4 = [0, 0, 0, 1], v_5 = [1, 1, 1, 1]$. The matrix

$$L(x) = \begin{bmatrix} x_1 & 0 & 0 & 0 & x_1 + x_2 + x_3 + x_4 \\ 0 & x_2 & 0 & 0 & x_1 + x_2 + x_3 + x_4 \\ 0 & 0 & x_3 & 0 & x_1 + x_2 + x_3 + x_4 \\ 0 & 0 & 0 & x_4 & x_1 + x_2 + x_3 + x_4 \end{bmatrix}$$

and

$$\sum_{i=1}^{5} u_i A_i = \begin{bmatrix} u_1 + u_5 & u_5 & u_5 & u_5 \\ u_5 & u_2 + u_5 & u_5 & u_5 \\ u_5 & u_5 & u_3 + u_5 & u_5 \\ u_5 & u_5 & u_5 & u_4 + u_5 \end{bmatrix}.$$ 

In this case, the defining equation of the quartic symmetroid is a quartic determinantal polynomial which is the first derivative of the quintic hyperbolic polynomial $u_1 u_2 u_3 u_4 u_5$ in the sense of [10, Section 1]. We refer it as Sylvester’s quartic symmetroid in $\mathbb{P}^4$ based on its connection with *Sylvester Pentahedral Theorem* [9][10].
4. Generalization: $m + 1$ matrices in $\mathbb{R}^m$

**Proposition 4.1.** The degeneracy locus of an $(m + 1)$-tuple of linear operators on $\mathbb{R}^m$ is a subscheme of $\mathbb{P}^{m-1}$ of dimension $(m - 3)$ and degree $\binom{m+1}{2}$.

**Proof.** We look at the locus of rank $m - 1$, $(m + 1) \times m$ matrices. This is a $(m - 1)(2m + 1 - m + 1) - 1 = m^2 + m - 3$-dimensional subscheme of $\mathbb{P}^{m^2 + m - 1}$ defined by the vanishing of all $m \times m$ minors. Using the formula (5.1) on page 95 of [1] we see that this has degree $\frac{m^2 + m}{2} = \binom{m+1}{2}$. For a given choice of $A_1, \ldots, A_{m+1}$ we obtain an $(m - 1)$-dimensional linear subspace in $\mathbb{P}^{m^2 + m - 1}$ and so the intersection of this determinantal variety will, for general $A_1, \ldots, A_{m+1}$, be an $(m - 3)$-dimensional subscheme of $\mathbb{P}^{m-1}$. □

Using arguments similar to those made in the proof of Proposition 3.3 and Lemma 3.6 we derive the following results.

**Proposition 4.2.** If $A_1, \ldots, A_{m+1}$ are in the linear span of $m + 1$ fixed rank-one matrices, (i.e., $A_i = \sum_{j=1}^{m+1} a_{i,j} E_j$ where the $E_j$ are rank one matrices), then the degeneracy locus $X$ consists of $\binom{m+1}{2}$ $(m - 3)$-dimensional linear subspaces in generalized Desargues configuration in $\mathbb{P}^{m-1}(\mathbb{R})$. □

**Lemma 4.3.** The set of $\binom{m+1}{2}$-tuples of $m - 3$ dimensional linear subspaces of $\mathbb{P}^{m-1}$ which are in generalized Desargues configuration consists of a single orbit under the group action of $\text{GL}_m$. □

**Proposition 4.4.** The locus of $(m + 1)$-tuples of matrices which are in the linear span of $m + 1$ rank one matrices is an irreducible variety of dimension $3m^2 + 2m - 1$. If matrices $A_i, i = 1, \ldots, m + 1$ have fixed null spaces, the dimension of the parametrized variety is $2(m^2 + m)$. □

**Remark 4.5.** The proof of Theorem 3.8 is easily adapted to show that if $A_1, \ldots, A_{m+1}$ are symmetric matrices in the linear span of $m + 1$ rank one matrices $E_1, \ldots, E_{m+1}$ then the $E_i$ are necessarily symmetric. Adapting the argument of Corollary 3.1 we can conclude that the general $m + 1$-tuple $(A_1, \ldots, A_{m+1})$ of symmetric matrices is not in the linear span of $m + 1$ rank one matrices.

**Conjecture 4.6.** The degeneracy locus $X$ consists of $\binom{m+1}{2}$ $(m - 3)$-dimensional linear subspaces with generalized Desargues configuration in $\mathbb{P}^{m-1}(\mathbb{R})$ if and only if $(A_1, \ldots, A_{m+1})$ are in the linear span of the $m + 1$ fixed rank-one matrices

5. Acknowledgements

The first author would like to thank Justin Chen for his helpful suggestions while using NumericalImplicitization package in M2.

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