On the local multiset dimension of $m$-shadow graph

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Abstract. Let $G = (V, E)$ be a simple and connected graph with edge set $E$ and vertex set $V$. Suppose $W = \{s_1, s_2, \ldots, s_k\}$ is a subset of vertex set $V(G)$, the representation multiset of a vertex $v$ of $G$ with respect to $W$ is $r_m(v|W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$ where $d(v, s_i)$ is a distance between $v$ and the vertices in $W$ together with their multiplicities. The resolving set $W$ is a local resolving set of $G$ if $r_m(v|W) \neq r_m(u|W)$ for every pair $u, v$ of adjacent vertices of a graph $G$. The minimum local resolving set $W$ is a local multiset basis of $G$. If $G$ has a local multiset basis, then its cardinality is called local multiset dimension, denoted by $\mu_l(G)$. In this paper, we analyzed the local multiset dimension of $m$-shadow graph. The $m$-shadow of a connected graph $G$, denoted by $D_m(G)$, is constructed by taking $m$ copies of $G$, say $G_1, G_2, \ldots, G_m$, then join each vertex $u$ in $G_i$, to the neighbors of the corresponding vertex $v$ in $G_j$, where $1 \leq i$ and $j \leq m$. We will investigate the characterization and exact value of local multiset dimension of $m$-shadowing of cycle graph, $m$-shadowing of star graph, $m$-shadowing of path graph, and $m$-shadowing of complete graph.

1. Introduction

Graph theory has been studied for several years. There are so many applications of graph. Graph theory is used in designing printed circuit for use in electronic device, as a chart of the steps used in an algorithm for solving a certain problem using computer, etc. One of the topic in graph theory is metric dimension. This topic has been observed by many researchers nowadays. In this paper, we will discuss the development of metric dimension topic, namely ”local multiset dimension”. We focus on simple, connected and undirected graph. For further definition of graph, please see [11, 2, 11, 7, 9, 13].

The local multiset dimension of graph is the development of local metric dimension and multiset dimension study. In recent years, the local metric dimension has been studied by [6, 14, 17] and the related topic in resolving set has been studied by
[4][5][8][10][15][12]. Then, Simanjuntak et al [16] started the definition of multiset dimension of graph G. Let G be a connected graph with vertex set V(G). Suppose \(W = \{s_1, s_2, \ldots, s_k\}\) is a subset of vertex set \(V(G)\), the representation multiset of a vertex \(v\) of G with respect to W is \(r_m(v|W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}\) where \(d(v, s_i)\) is a distance between \(v\) and the vertices in \(W\) together with their multiplicities. The resolving set \(W\) is a resolving set of G if \(r_m(v|W) \neq r_m(u|W)\) for every pair of distances \(v, u\) of adjacent vertices of a graph G. The minimum resolving set \(W\) is a local multiset basis of G. If G has a multiset basis, then its cardinality is called multiset dimension, denoted by \(md(G)\). The concept of local multiset dimension was firstly introduced by Ridho et al [3]. The definition of local multiset dimension is as follows:

**Definition 1.1:** Let G be a connected graph with vertex set \(V(G)\). Suppose \(W = \{s_1, s_2, \ldots, s_k\}\) is a subset of vertex set \(V(G)\), the representation multiset of a vertex \(v\) of G with respect to W is \(r_m(v|W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}\) where \(d(v, s_i)\) is a distance between \(v\) and the vertices in \(W\) together with their multiplicities. The resolving set \(W\) is a local resolving set of G if \(r_m(v|W) \neq r_m(u|W)\) for every pair \(u, v\) of adjacent vertices of a graph G. The minimum local resolving set \(W\) is a local multiset basis of G. If G has a local multiset basis, then its cardinality is called local multiset dimension, denoted by \(\mu_l(G)\).

In this paper, the object of this research is \(m - \text{shadow of graph}\). The \(m - \text{shadow of a connected graph G, denoted by } D_m(G),\) is constructed by taking \(m\) copies of G, say \(G_1, G_2, \ldots, G_m\) then join each vertex \(u\) in \(G_i\) to the neighbors of the corresponding vertex \(v\) in \(G_j\), where \(1 \leq i \leq j \leq m\). We will investigate the characterization and exact value of local multiset dimension of \(m\)-shadowing of cycle graph, \(m\)-shadowing of star graph, \(m\)-shadowing of path graph, and \(m\)-shadowing of complete graph.

### 2. Main Results

In this section, we will discuss some results on local multiset dimension of \(m\)-shadowing of graph. There are several theorems related to local multiset dimension of \(m\)-shadowing of cycle graph, \(m\)-shadowing of star graph, \(m\)-shadowing of path graph, and \(m\)-shadowing of complete graph.

**Theorem 2.1.** Let \(D_m(P_n)\) be a \(m\)-shadowing path graph with \(n \geq 3\), the local multiset dimension of \(D_m(P_n)\) is \(\mu_l(D_m(P_n)) = 1\)

**Proof:** The \(m\)-shadowing of path graph \(D_m(P_n)\) is a graph with \(mn + n\) vertices. The vertex set \(V(D_m(P_n)) = \{v_i \cup v_j^i; 1 \leq i \leq n, 1 \leq j \leq m - 1\}\) and edge set \(E(D_m(P_n)) = \{v_i v_{i+1}; 1 \leq i \leq n - 1\} \cup \{v_i v_{i+j}; 1 \leq i \leq n - 1, 1 \leq j \leq m - 1\} \cup \{v_i v_{i+1}; 1 \leq i \leq n - 1, 1 \leq j \leq m - 1\}\). The cardinality of vertex set and edge set, respectively are \(|V(D_m(P_n))| = mn + n\) and \(|E(D_m(P_n))| = 3mn - 3m + n - 1\).

In order to proof the local multiset dimension of \(m\)-shadowing path graph is one or \(\mu_l(D_m(P_n)) = 1\), we proposed the proof of lower and upper bound local multiset dimension of \(m\)-shadowing path graph. First, we prove that the lower bound of local multiset dimension of \(m\)-shadowing path graph is one, which is \(\mu_l(D_m(P_n)) \geq 1\). By using contradiction, we suppose the local multiset dimension of \(m\)-shadowing path graph is \(\mu_l(D_m(P_n)) < 1\). It means that by assuming \(\mu_l(D_m(P_n)) < 1\), we will have the cardinality of resolving set is zero or in the other hand there are no local multiset
dimension of $m$-shadowing path graph. But, We will show the existence resolving set $W$. In this case, we know that there are a resolving set of $m$-shadowing path graph such that the representation of the vertices $v_i$ which is adjacent are distinct, thus $W$ is a local resolving set. For example, we have $W = \{x_1\}$ or $W = \{x_n\}$. It was contradiction. Thus we have $\mu(D_m(P_n)) \geq 1$.

Then, we will show that the upper bound of local multiset dimension of path $D_m(P_n)$ is one or we can write it as $\mu(D_m(P_n)) \leq 1$. Suppose $W = \{v_1\}$, the representation of vertices $v \in V(D_m(P_n))$ respect to $W$ is as follow:

$$r(v_1|W) = \{0\}$$
$$r(v_i|W) = \{i - 1\}; 1 \leq i \leq n$$
$$r(v_i'|W) = \{i - 1\}; 2 \leq i \leq n$$
$$r(v_j'|W) = \{2\}; 1 \leq j \leq m$$

From the representation above, It can be seen the representation of the vertices $v_i$ which is adjacent are distinct or $r_m(v_i|W) \neq r_m(v_{i+1}|W)$. Thus, we obtain the upper bound of local multiset dimension of $D_m(P_n)$ is one, $\mu(D_m(P_n)) \leq 1$. Because we have the lower and upper bound local multiset dimension of $m$-shadowing path graph is equal to one, We conclude that $\mu(D_m(P_n)) = 1$. 

\[\square\]

**Theorem 2.2.** Let $D_m(S_n)$ be a $m$-shadowing star graph with $n \geq 3$, the local multiset dimension of $D_m(S_n)$ is $\mu(D_m(S_n)) = 1$

**Proof:** The $m$-shadowing star graph $D_m(S_n)$ is a graph with $mn + m + n + 1$ vertices. The vertex set $V(D_m(S_n)) = \{x \cup x_i \cup x_i' \cup x_j; 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(D_m(S_n)) = \{xx_i \cup xx_i'; 1 \leq i \leq n, 1 \leq j \leq m\}$. The vertex $x$ is a central vertex and the vertices $x_i$ is pendant vertex with degree 1. The cardinality of vertex set and edge set, respectively are $|V(D_m(S_n))| = mn + m + n + 1$ and $|E(D_m(S_n))| = 2mn + n$.

In order to proof the local multiset dimension of $m$-shadowing star graph is one or $\mu(D_m(S_n)) = 1$, we proposed the proof of lower and upper bound local multiset dimension of $m$-shadowing star graph. First, We prove that the lower bound of local multiset dimension of $m$-shadowing star graph is one, which is $\mu(D_m(S_n)) \geq 1$. By using contradiction, we suppose the local multiset dimension of $m$-shadowing star graph is $\mu(D_m(S_n)) < 1$. It means that by assuming $\mu(D_m(P_n)) < 1$, we will have the cardinality of resolving set is zero or in the other hand there are no local multiset dimension of $m$-shadowing star graph. We will show the existence resolving set $W$. In this case, we know that there are a resolving set of $m$-shadowing star graph such that the representation of the vertices $v_i$ which is adjacent are distinct, thus $W$ is a local resolving set. For example, we have $W = \{x\}$. It was contradiction. Thus we have $\mu(D_m(S_n)) \geq 1$.

Then, we will show that the upper bound of local multiset dimension of star $D_m(S_n)$ is one or we can write it as $\mu(D_m(S_n)) \leq 1$. Suppose $W = \{x\}$, the representation of vertices $v \in V(D_m(S_n))$ respect to $W$ is as follow:

$$r(x|W) = \{0\}$$
\[ r(x_i|W) = \{1\}; 1 \leq i \leq n \]

\[ r(x_i'|W) = \{1\}; 1 \leq i \leq n, 1 \leq j \leq m \]

\[ r(x_j|W) = \{2\}; 1 \leq j \leq m \]

From the representation above, it can be seen the representation of the vertices \(x_i\) which is adjacent are distinct or \(r_m(x_i|W) \neq r_m(x_{i+1}|W)\). Thus, we obtain the upper bound of local multiset dimension of \(D_m(S_n)\) is one, \(\mu(D_m(S_n)) \leq 1\). Because we have the lower and upper bound local multiset dimension of \(m\)-shadowing star graph is equal to one, it can be concluded that \(\mu_l(D_m(S_n)) = 1\). \(\square\)

**Theorem 2.3.** Let \(D_m(G_n)\) be a \(m\)-shadowing gear graph with \(n \geq 4\), the local multiset dimension of \(D_m(G_n)\) is \(\mu(D_m(G_n)) = 1\)

**Proof:** Proof. The \(m\)-shadowing gear graph \(D_m(G_n)\) is a connected graph with vertex set \(\{c\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\} \cup \{c'; 1 \leq j \leq m\} \cup \{x_i'; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_i'; 1 \leq i \leq n, 1 \leq j \leq m\}\) and edge set \(D_m(G_n) = \{cx_i; 1 \leq i \leq n\} \cup \{x_iy_i; 1 \leq i \leq n\} \cup \{y_ix_{i+1}; 1 \leq i \leq n-1\} \cup \{y_{n-1}x_1\} \cup \{cx_i'; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{x_iy_i'; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_{n-1}x_1'; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{ynx_i'; 1 \leq i \leq n, 1 \leq j \leq m\}\). The cardinality of vertex set and edge set, respectively, are \(|V(D_m(G_n))| = 2mn + 2n + m + 1\) and \(|E(D_m(G_n))| = 6mn + 3n\).

In order to proof the local multiset dimension of \(m\)-shadowing gear graph is one or \(\mu(D_m(G_n)) = 1\), we proposed the proof of lower and upper bound local multiset dimension of \(m\)-shadowing gear graph. First, we prove that the lower bound of local multiset dimension of \(m\)-shadowing gear graph is one, which is \(\mu_l(D_m(G_n)) \geq 1\). By using contradiction, we suppose the local multiset dimension of \(m\)-shadowing gear graph is \(\mu_l(D_m(G_n)) < 1\). It means that by assuming \(\mu_l(D_m(G_n)) < 1\), we will have the cardinality of resolving set is zero or in the other hand there are no local multiset dimension of \(m\)-shadowing gear graph. We will show the existence resolving set \(W\). In this case, we know that there are a resolving set of \(m\)-shadowing gear graph such that the representation of the vertices \(x_i\) and \(y_i\) which is adjacent are distinct, thus \(W\) is a local resolving set. For example, we have \(W = \{c\}\). It was contradiction. Thus we have \(\mu_l(D_m(G_n)) \geq 1\).

Then, we will show that the upper bound of local multiset dimension of \(D_m(G_n)\) is one or we can write it as \(\mu_u(D_m(G_n)) \leq 1\). Suppose \(W = \{c\}\), the representation of vertices \(v \in V(D_m(G_n))\) respect to \(W\) is as follow:

\[ r(c|W) = \{0\} \]

\[ r(c'|W) = \{2\}; 1 \leq j \leq m \]

\[ r(x_i|W) = \{1\}; 1 \leq i \leq n \]

\[ r(x_i'|W) = \{1\}; 1 \leq i \leq n, 1 \leq j \leq m \]

\[ r(y_i|W) = \{2\}; 1 \leq i \leq n \]

\[ r(y_i'|W) = \{2\}; 1 \leq i \leq n, 1 \leq j \leq m \]
From the representation above, it can be seen the representation of the vertices $x_i$ which is adjacent are distinct or $r_m(x_i|W) \neq r_m(x_{i+1}|W)$. Thus, we obtain the upper bound of local multiset dimension of $D_m(G_n)$ is one, $\mu(D_m(G_n)) \leq 1$. Because we have the lower and upper bound local multiset dimension of $m$-shadowing complete graph is equal to one, we can conclude that $\mu(D_m(S_n)) = 1$. \hfill $\square$

**Theorem 2.4.** Let $D_m(K_n)$ be a $m$-shadowing complete graph with $n \geq 3$, the local multiset dimension of $D_m(K_n)$ is $\mu(D_m(K_n)) = \infty$.

**Proof:** The $m$-shadowing complete $D_m(K_n)$ is a regular graph with $mn + n$ vertices. The vertex set $V(D_m(K_n)) = \{v_i \cup v_i^j; 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(D_m(K_n)) = \{v_i v_{i+k} \cup v_i v_{i+k}^j; 1 \leq i \leq n, 1 \leq k \leq n - i, 1 \leq j \leq m\}$.

We prove this theorem by contradiction. Assume that all vertices in $W$ is distance 1 and $W$ is a local resolving set of $m$-shadowing complete graph $D_m(K_n)$. There is some condition as follows.

- If we take $W = \{v_1\}$, then $r(v_1|W) = \{0\}$, $r(v_2|W) = r_m(v_2|W) = \cdots = r(v_{n-1}|W) = r(v_n|W) = \{1\}$, $r(v_2^1|W) = \{2\}$, $r(v_3^1|W) = r(v_3^2|W) = \cdots = r(v_{n-1}^1|W) = r(v_n^1|W) = \{1\}$, we know that $v_2, v_3, \ldots, v_n$ are adjacent, $v_3, v_3^1, \ldots, v_n^1$ are also adjacent such that it is a contradiction.

- If we take $W = \{v_1, v_2\}$, then $r(v_1|W) = r(v_2|W) = \{0, 1\}$, $r(v_3|W) = \cdots = r(v_{n-1}^1|W) = r(v_n^1|W) = \{1, 2\}$, $r(v_3^1|W) = r(v_3^2|W) = \cdots = r(v_{n-1}^2|W) = r(v_n^2|W) = \{0, 1\}$ and $r(v_3^1|W) = \cdots = r(v_n^1|W) = \{1, 2\}$ we know that $v_3, \ldots, v_n$ are adjacent $v_3, v_3^1, \ldots, v_n^1$ are also adjacent such that it is a contradiction.

- If we take $W = \{v_1, v_2, \ldots, v_k\}$ for $2 \leq k \leq n - 1$, then $r(v_1|W) = \cdots = r(v_k|W) = \{0, 1^{k-1}\}$ and $r(v_{k+1}|W) = \cdots = r(v_n|W) = \{1^{k}\}$, $r(v_1^1|W) = \cdots = r(v_k^1|W) = \{0, 1^{k-1}\}$ and $r(v_{k+1}^1|W) = \cdots = r(v_n^1|W) = \{1^{k}\}$ we know that $v_{k+1}, \ldots, v_n$ are adjacent, $v_{k+1}, \ldots, v_n^1$ are also adjacent such that it is a contradiction.

- If we take $W = \{v_1^1, v_2^1, \ldots, v_k^1\}$ for $2 \leq k \leq n - 1$, then $r(v_1|W) = \cdots = r(v_k|W) = \{0, 1^{k-1}\}$ and $r(v_{k+1}|W) = \cdots = r(v_n|W) = \{1^{k}\}$ we know that $v_{k+1}, \ldots, v_n$ are adjacent such that it is a contradiction.

Hence, $W$ is not a local resolving set of $m$-shadowing complete graph $D_m(K_n)$. It is conclude that $\mu(D_m(K_n)) = \infty$. \hfill $\square$

**Theorem 2.5.** Let $D_m(C_n)$ be a $m$-shadowing cycle graph with $n \geq 3$, the local multiset dimension of $D_m(C_n)$ is

$$
\mu(D_m(C_n)) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}
$$

**Proof:** The cycle $D_m(C_n)$ is a cyclic graph with $mn + n$ vertices. The vertex set $V(D_m(C_n)) = \{v_1, v_2, \ldots, v_n \cup v_1^1, v_2^1, \ldots, v_n^1; 1 \leq j \leq m\}$ and edge set $E(D_m(C_n)) = \{v_{1i}, v_{1i+1}, v_{2i}, v_{2i+1}; 1 \leq i \leq n - 1, 1 \leq j \leq m\}$. The cardinality of vertex set and edge set, respectively are $|V(D_m(C_n))| = mn + n$ and $|E(D_m(C_n))| = 3mn + n$. The
proof divided into two cases as follows.

**Case 1:** For $n$ is even, in order to proof the local multiset dimension of $m$-shadowsing cycle graph is one or $\mu_l(D_m(C_n)) = 1$, we proposed the proof of lower and upper bound local multiset dimension of $m$-shadowsing cycle graph. First, We prove that the lower bound of local multiset dimension of $m$-shadowsing cycle graph is one, which is $\mu_l(D_m(C_n)) \geq 1$. By using contradiction, we suppose the local multiset dimension of $m$-shadowsing cycle graph is $\mu_l(D_m(C_n)) < 1$. It means that by assuming $\mu_l(D_m(C_n)) < 1$, we will have the cardinality of resolving set is zero or in the other hand there are no local multiset dimension of $m$-shadowsing cycle graph. We will show the existence resolving set $W$. In this case, we know that there are a resolving set of $m$-shadowsing cycle graph such that the representation of the vertices $x_i$ and $y_i$ which is adjacent are distinct, thus $W$ is a local resolving set. For example, we have $W = \{v_1\}$ or $W = \{v_2\}$ or $W = \{v_3\}$, etc. It was contradiction. Thus we have $\mu_l(D_m(C_n)) \geq 1$ for $n$ is even. Furthermore, we will show that the upper bound of local multiset dimension of $m$-shadowsing cycle graph is $\mu_l(D_m(C_n)) \leq 1$. Suppose $W = \{v_1\}$, the representation of vertices $v \in V(D_m(C_n))$ respect to $W$ as follows.

$$r(v_1|W) = \{0\}$$
$$r(v_i|W) = \{i - 1\}; 2 \leq i \leq \frac{n}{2} + 1$$
$$r(v_i|W) = \{n - i + 1\}; \frac{n}{2} + 2 \leq i \leq n$$
$$r(v_j^1|W) = \{2\}; 1 \leq j \leq m$$
$$r(v_j^2|W) = \{i - 1\}; 2 \leq i \leq \frac{n}{2} + 1, 1 \leq j \leq m$$
$$r(v_j^3|W) = \{n - i + 1\}; \frac{n}{2} + 2 \leq i \leq n, 1 \leq j \leq m$$

It can be seen that $r_m(v_i|W) \neq r_m(v_{i+1}|W)$ with $v_i$ and $v_{i+1}$ are adjacent for $1 \leq i \leq n - 1$. Thus, we obtain the upper bound of local multiset dimension of $m$-shadowsing cycle graph $D_m(C_n)$ is one, $\mu_l(D_m(C_n)) \leq 1$. It can be concluded that $\mu_l(D_m(C_n)) = 1$ for $n$ is even.

**Case 2:** For $n$ is odd, we will show that lower bound of the local multiset dimension of $D_m(C_n)$ is three, $\mu_l(D_m(C_n)) \geq 3$. Assume that $\mu_l(D_m(C_n)) < 3$. Suppose we have the cardinality of resolving set is equal to two. We assume the resolving set $W = \{u, v\}$ so that there will be some condition as follows:

a) If $u, v \in W$ are adjacent, then $r_m(u|W) = r_m(v|W) = \{0, 1\}$, it is a contradiction.

b) If $u, v \in W$ aren’t adjacent then there is at most two path $P_1$ and $P_2$ between two vertices $u$ and $v$. If $|V(P_1)| = k_1$ with $k_1$ is odd, then $|V(P_2)| = k_2$ with $k_2$ is even.

c) We take the cardinality $|V(P_2)| = k_2$ with $k_2$ is even and the vertices in $P_2$ includes path graph.
d) Let the vertices in $P_2$ be $v_1, \ldots, v_{2l} \in V(P_2)$ for $l \in \mathbb{Z}^+$ such that $d(v_l, v_1) = d(v_{l+1}, v_2)$.

e) We obtain that $d(v_1, u) = 1$ and $d(v_2l, v) = 1$, based on point d) that $r_m(v_l|W) = \{d(v_l, v_1) + d(v_1, u), d(v_{l+1}, v_1) + d(v_1, u) + 1\} = \{d(v_l, v_1) + 1, d(v_l, v_1) + 2\}$ and $r_m(v_{l+1}|W) = \{d(v_{l+1}, v_2l) + d(v_2l, v), d(v_{l+1}, v_2l) + d(v_2l, v) + 1\} = \{d(v_{l+1}, v_2l) + 1, d(v_{l+1}, v_2l) + 2\}$.

f) Based on point d), e) that $r_m(v_l|W) = \{d(v_l, v_1) + 1, d(v_l, v_1) + 2\} = \{d(v_{l+1}, v_2l) + 1, d(v_{l+1}, v_2l) + 2\} = r_m(v_{l+1}|W)$ and we know that $v_l$ is adjacent to $v_{l+1}$, it is a contradiction.

Based on point a),b),c),d),e), f), it can be concluded that the lower bound of local multiset dimension of $D_m(C_n)$ is three, $\mu_l(D_m(C_n)) \geq 3$. Furthermore, we will show the upper bound of the local multiset dimension of $D_m(C_n)$ is $\mu_l(D_m(C_n)) \leq 3$. Suppose the resolving set $W = \{v_1, v_3, v_4\}$, we can obtain the representation $r$ respect to $W$ as follows

- $r_m(v_1|W) = \{0, 2, 3\}$
- $r_m(v_1^j|W) = \{2, 2, 3\}$
- $r_m(v_3|W) = \{0, 1, 2\}$
- $r_m(v_3^j|W) = \{2, 1, 2\}$
- $r_m(v_4|W) = \{0, 1, 3\}$
- $r_m(v_4^j|W) = \{2, 1, 3\}$
- $r_m(v_2|W) = r_m(v_2^j|W) = \{1, 1, 2\}$
\[ r_m(v_i | W) = r_m(v_i^j | W) = \{i - 4, i - 3, i - 1\}; \quad 5 \leq i \leq \frac{n + 1}{2} \]

\[ r_m(v_i | W) = r_m(v_i^j | W) = \{i - 4, i - 3, i - 2\}; \quad i = \frac{n + 3}{2} \]

\[ r_m(v_i | W) = r_m(v_i^j | W) = \{i - 4, i - 3, i - 4\}; \quad i = \frac{n + 5}{2} \]

\[ r_m(v_i | W) = r_m(v_i^j | W) = \{n - i + 1, n - i + 3, n - i + 3\}; \quad i = \frac{n + 7}{2} \]

\[ r_m(v_i | W) = r_m(v_i^j | W) = \{n - i + 1, n - i + 3, n - i + 4\}; \quad \frac{n + 9}{2} \leq i \leq n \]

The representation of the vertices \( v_i \) which is adjacent are distinct such that \( W \) is a local resolving set of \( D_m(C_n) \). Thus, we obtain the upper bound of the local multiset dimension of \( D_m(C_n) \) is three, \( \mu_l(D_m(C_n)) \leq 3 \). Thus, it can be concluded that \( \mu_l(D_m(C_n)) = 3 \) for \( n \) is odd.

\[ \square \]

3. Concluding Remarks
In this paper, we have investigated the characterization and exact value of local multiset dimension of \( m \)-shadowing of cycle graph, \( m \)-shadowing of star graph, \( m \)-shadowing of path graph, and \( m \)-shadowing of complete graph. Hence, the following open problem arises naturally.

**Open Problem 1.** Determine the local multiset dimension of some families of graph namely related wheel related, generalized petersen graph, related cycle and others.

**Open Problem 2.** Determine the local multiset dimension of some graph operation results.

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