Combinatorial Alphabet-Dependent Bounds for Locally Recoverable Codes

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Abstract

Locally recoverable codes (LRC) have recently been a subject of intense research due to the theoretical appeal and their applications in distributed storage systems. In an LRC, any erased symbol of a codeword can be recovered by accessing only few other symbols. For LRC codes over small alphabet (such as binary), the optimal rate-distance trade-off is unknown. We present several new combinatorial bounds on LRC codes including the locality-aware sphere packing and Plotkin bounds. We also develop an approach to linear programming (LP) bounds on LRC codes. The resulting LP bound gives better estimates in examples than the other upper bounds known in the literature. Further, we provide the tightest known upper bound on the rate of linear LRC codes with a given relative distance, an improvement over the previous best known bounds.

I. INTRODUCTION

We consider codes over a finite alphabet that have the usual property of error correction and the additional property of being able to recover one or more erased symbols of the codeword by accessing only a small number of other symbols. Codes of this kind are said to be locally recoverable (LRC), and they have applications in large-scale distributed storage systems. LRC codes were first defined in [13] and were studied in a number of subsequent papers in recent years.

A $q$-ary code $C$ of length $n$, cardinality $M$, and distance $d$ is a set of $M$ vectors over an alphabet $Q$, $|Q| = q$ with minimum pairwise Hamming distance $d$. The quantity $k = \log_q M$ is called the dimension of $C$. If $Q$ is a finite field and $C$ is a linear subspace of $Q^n$ then $k$ is the dimension of $C$ as a vector space. Below, $[n] \equiv \{1, \ldots, n\}$, and for any $x \in Q^n$, $x_i$ is the projection of $x$ in the $i$th coordinate. By extension, for any $I \subseteq [n]$, $x_I$ is the projection of $x$ onto the coordinates of $I$.

Definition 1. A code $C$ of length $n$ is said to have locality $r$ if every coordinate $i \in [n]$ is contained in a subset $\mathcal{R}_i \subset [n]$ of size at most $r + 1$ with the property that there exists a function $\phi_i$ such that for every codeword $c \in C$,

$$c_i = \phi_i(\{c_j, j \in \mathcal{R}_i \setminus \{i\}\}).$$

(1)

We use the notation $(n, k, r)$ to refer to a code of length $n$, dimension $k$ and locality $r$.

The definition of LRC codes was extended in several different ways. The following generalization is important for our purposes.

Definition 2. A code $C \subset Q^n$ of cardinality $q^k$ is said to have the $(\rho, r)$ locality property (to be an $(n, k, r, \rho)$ LRC code) where $\rho \geq 2$, if each coordinate $i \in [n]$ is contained in a subset $\mathcal{R}_i \subset [n]$ of size at most $r + \rho - 1$ such that the restriction $C|_{\mathcal{R}_i}$ of the code $C$ to the coordinates in $\mathcal{R}_i$ forms a code of distance at least $\rho$. Notice that the values of any $\rho - 1$ coordinates of $\mathcal{R}_i$ are determined by the values of the remaining $|\mathcal{R}_i| - (\rho - 1) \leq r$ coordinates, thus enabling local recovery.

This definition was first proposed in [24], [18] with the less demanding restriction of protecting only the information symbols of the codeword (see also [23] for a related but different notion). In the above definition we consider all-symbol locality, without differentiating between the information and parity symbols. The set $\mathcal{R}_i$ is called the repair group for the coordinate $i$.
Other extensions of the concept of LRC codes include codes with multiple disjoint repair groups for every coordinate, also called codes with the availability property \([33]\), codes with sequential repair of several erasures \([25]\), codes with cooperative repair \([20]\), local repair on graphs \([21]\), as well as other variations.

Problems of constructing LRC codes and bounding their parameters have been the subject of a considerable number of publications. Constructions of LRC codes obtained by combining some known code families without the locality property were suggested in \([28]\, [15]\, [32]\). A family of codes extending the construction of Reed Solomon codes with codes with locality was proposed in \([29]\) and further generalized to codes on algebraic curves in \([4]\). We refer to \([3]\) for a survey of some aspects of the algebraic theory of LRC codes.

Research on bounds for LRC codes was initiated in \([13]\) which showed that the distance \(d(C)\) of an \((n, k, r)\) LRC code \(C\) is bounded as follows:

\[
d(C) \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2.
\]

In \([17]\) this bound was extended to the case of arbitrary \(\rho \geq 2\). Namely, the distance of an \((n, k, r, \rho)\) LRC code satisfies the inequality

\[
d \leq n - k + 1 - \left( \frac{k}{r} \right) (\rho - 1).
\]

Bounds for codes with availability were established in \([33]\, [27]\, [30]\).

Note that the bounds \([2]\, [3]\) do not depend on the size of the code alphabet \(q\). A bound that accounts for the value of \(q\) was derived in \([17]\). It has the following form: For any \(q\)-ary LRC code with the parameters \((n, k, r)\) and distance \(d\),

\[
k \leq \min_{1 \leq s \leq n/(r+1)} \{ sr + \log_q M_q(n-s(r+1), d) \},
\]

where \(M_q(n, d)\) is the maximum cardinality of a \(q\)-ary length \(n\) code with distance \(d\). This bound can be used to derive asymptotic upper bounds on the rate of LRC codes with a given value of the distance (more on this below).

Asymptotic lower bounds (achievability results) on the rate of LRC codes, namely Gilbert-Varshamov (GV) type asymptotic bounds, was also derived independently in \([30]\, [7]\); in particular the later derives a bound for the case of availability 2 as well.

In this paper we focus on combinatorial upper bounds on the parameters of LRC codes, tightening prior results, and emphasizing the dependence between the parameters and the size of the code alphabet \(q\). We explore several general approaches to the derivation of the upper bounds, including recursive bounds, the linear programming approach, and the approach relying on the coset leader graph of the code.

Linear programming (LP) is a powerful technique that accounts for some of the best known upper bounds on the size of codes with a given distance. It was pioneered in \([11]\) and used in \([22]\) to derive the best currently known asymptotic upper bound on error correcting codes. These results rely on the approach to codes via association schemes and their eigenvalues combined with some analytic techniques. Incorporating the locality constraints into the LP problem in a way that yields closed-form bounds is a nontrivial problem. We suggest a way to address to it under the additional assumption that \(R_i \cap R_j = \emptyset\), \(i \neq j\), i.e., that different repair groups are disjoint, and the set of coordinates \([n]\) is a disjoint union of the repair groups. With this assumption, an association scheme that fits the locality constraints forms a Delsarte extension of the usual Hamming scheme. Relying on this observation, we derive an LP bound on \((n, k, \rho, r)\) LRC codes in a polynomial form and construct a polynomial that gives rise to a Singleton-like bound on such codes. We also compute numerical examples for \(\rho = 2\), which corresponds to the original definition of LRC codes, and show that the LP bound is sometimes better than the only other known alphabet-dependent bound \([4]\).

We note that linear programming bounds on linear LRC codes were earlier considered in \([9]\) which considered a standard LP problem \([11]\) with the additional constraint that every coordinate is contained in a codeword of the dual code of weight \(\leq r + 1\). At the same time \([9]\) gave no closed-form solutions of the LP problem or any numerical examples. LP bounds for cyclic LRC codes were considered in \([31]\).

Finally, we study asymptotic upper bounds on linear LRC codes that corresponds to Definition \([1]\). The starting point of our study is an observation that a linear LRC code necessarily contains several low-weight parity checks. Another class of codes that has the same property is low-density parity check (LDPC) codes. A recent work \([16]\) derived new improved asymptotic bounds on the rate of LDPC codes by analyzing the coset graph of the code. While LDPC codes by definition contain only low-weight parity checks, LRC codes combine such checks with a large number of unrestricted parity check equations. Nevertheless, it is possible to combine the approach of \([16]\) with the recursive bound \([4]\) to obtain an asymptotic bound on linear LRC codes that is better than the asymptotic bound obtained from \([4]\). An even better bound can be obtained for linear LRC codes with disjoint repair groups.
The paper is organized as follows. In Section II we derive a general upper bound on the size of LRC codes that reduces the problem to bounds on codes with a given distance but without locality constraints. This result is conceptually similar to the bound (4) from (7) but relies on a different kind of recursion, and this reduction enables us to use known bounds on codes to derive new results for LRC codes. We also derive an asymptotic Gilbert-Varshamov (GV) bound on \((n, k, r, \rho)\) LRC codes and observe that it gives the exact value of the asymptotic code rate when \(d/n \to 0\). This result, proved earlier for \(\rho = 2\) in [7], is extended here to any \(\rho \geq 2\). In Section III we derive Delsarte’s linear programming (LP) bounds for LRC codes with disjoint repair groups, our results beat the shortening bound of (4). In the end, in Section IV, we consider linear LRC codes, and include a Singleton bound for LP codes. For the special case of usual LRC codes (Definition 1) with disjoint repair groups, our results beat the shortening bound of (4). In the end, in Section IV, we consider linear LRC codes, and by using a theory of coset-leader graphs combined with the approach of [7], are able to provide better asymptotic bounds on the rate-relative distance trade-off of LRC codes. The bound strengthens if we consider disjoint repair groups.

This paper is a result of merging and developing papers by S. Hu, I. Tamo, and A. Barg [14] and by A. Agarwal and A. Mazumdar [2], both devoted to the problem of deriving alphabet-dependent bounds on LRC codes.

II. NEW BOUNDS ON LRC CODES

In the next theorem we introduce a method of using upper bounds on codes with a given distance (without the locality property) to derive upper bound on LRC codes. Let \(B(l, \rho)\) be an upper bound on the cardinality of a code of length \(l\) and distance \(\rho\), which is a log-convex function of \(l\) and such that \(B(0, \rho) = 1\).

**Theorem 1.** Let \(C\) be an \((n, k, r, \rho)\) \(q\)-ary LRC code with distance \(d\). Let

\[
\mu = \mu(n, d, r, \rho) := \left\lceil \frac{n - (d - 1)}{N} \right\rceil + 1,
\]

where \(N = r + \rho - 1\). Then, for any \(\rho \geq 2\), we have

\[
k \leq \mu \log_q B(N, \rho).
\]

**Proof:** We begin with constructing a sequence of nonempty disjoint subsets \(X_i \subset [n]\) whose union is of size at least \(n - (d - 1)\). Starting with \(X_0 = \emptyset\), assume that the sets \(X_0, \ldots, X_i\), \(i \geq 0\) are already constructed. If \(|\bigcup_{i=0}^j X_i| \geq n - (d - 1)\), terminate the procedure. Otherwise let \(j\) be an arbitrary element in \([n]\) \(\bigcup_{i=0}^{j-1} X_i\), w.l.o.g. we can assume that \(j = i + 1\), and define

\[
X_{i+1} = R_{i+1} \setminus \bigcup_{i=0}^j X_i.
\]

Suppose that this procedure terminates after \(m\) steps, and let \(X_1, \ldots, X_m\) be the sequence of subsets constructed above. For \(i = 1, \ldots, m\) let \(X_{[i]} := \bigcup_{j=0}^i X_j\) and denote by \(C_i\) the restriction of \(C\) to the coordinates in \(X_{[i]}\). Note that by the construction, we have

\[
|X_{[m-1]}| < n - (d - 1),
\]

and

\[
|X_{[m]}| \leq |X_{[m-1]}| + N \leq \mu N.
\]

Let us prove by induction on \(i\) that

\[
|C_i| \leq \prod_{j=1}^i B(|X_j|, \rho) \text{ for all } i = 1, \ldots, m.
\]

For \(i = 1\), \(X_1 = R_1\) and by definition, the code \(C_1 = C_{R_1}\) has distance at least \(\rho\). Therefore, \(|C_1| \leq B(|X_1|, \rho)\). Now assume that (8) holds for \(C_{i-1}\). Let \(c\) be an arbitrary codeword of \(C_{i-1}\), and let \(S(c)\) be the set of codewords in \(C_i\) whose restriction to \(X_{[i-1]}\) equals \(c\). Since the restriction of \(S(c)\) to the coordinates in \(X_i\) \((\subseteq R_i)\) form a code of distance at least \(\rho\), we have \(|S(c)| \leq B(|X_i|, \rho)\). This completes the induction step.

Since the code \(C\) has distance \(d\) and \(|X_{[m]}| \geq n - (d - 1)\), it follows that

\[
|C| = |C_m| \leq \prod_{j=1}^m B(|X_j|, \rho).
\]
Suppose that \( i, j \) are such that \( 1 \leq |X_i| \leq |X_j| \leq N \), then using log-convexity, we obtain

\[
B(|X_i|, \rho)B(|X_j|, \rho) \leq B(|X_i| - 1, \rho)B(|X_j| + 1, \rho).
\]

This step can be repeated \( \min(|X_i|, N - |X_j|) \) times till either the larger subset is of the maximum possible size \( N \) or the smaller one becomes empty (in which case we put \( B(0, \rho) = 1 \)). Use this argument in (9) and successively reduce the number of factors on the right-hand side as many times as possible. On account of (7) we will obtain at most \( \mu \) factors, and in each of them the size of the coordinate subset \( X_{(j)} \) will be \( N \) or less. We conclude that

\[
\prod_{j=1}^{m} B(|X_j|, \rho) \leq B(N, \rho)^\mu. \tag{10}
\]

Now (6) follows by combining (9) and (10) and taking logarithms on both sides of the resulting equation.  

Theorem 1 provides a general upper bound on the size of LRC codes. Explicit results are obtained once we substitute a log-convex upper bound \( B(\cdot) \). Fortunately, many known bounds on codes are in fact log-convex. For instance, let us prove that this is the case for the Hamming (sphere-packing) bound.

**Lemma 2.** The function \( B_H(n, e) = q^n / \left( \sum_{i=0}^{e} \binom{n}{i} (q - 1)^i \right) \) is log-convex in \( n \).

**Proof:** Let \( Q = q - 1 \) and let \( f(n, e) = \sum_{i=0}^{e} \binom{n}{i} Q^i \). In all the expressions below \( e \) does not change, so to simplify the notation we write \( f(n) \) instead of \( f(n, e) \). For any \( n \geq 1 \) we have

\[
f(n) = \sum_{i=0}^{e} \left( \frac{n-1}{i} + \frac{n-1}{i-1} \right) Q^i = (1 + Q)f(n - 1) - \left( \frac{n-1}{e} \right) Q^{e+1}.
\]

We find

\[
\begin{align*}
f(n_1)f(n_2) - f(n_1 - 1)f(n_2 + 1) &= \left[ (1 + Q)f(n_1 - 1) - \left( \frac{n_1-1}{e} \right) Q^{e+1} \right] f(n_2) - f(n_1 - 1)\left[ (1 + Q)f(n_2) - \left( \frac{n_2}{e} \right) Q^{e+1} \right] \\
&= \left[ \left( \frac{n_2}{e} \right) f(n_1 - 1) - \left( \frac{n_1-1}{e} \right) f(n_2) \right] Q^{e+1} \\
&= Q^{e+1} \sum_{i=0}^{e} \left[ \left( \frac{n_1-1}{i} \right) \left( \frac{n_2}{e} \right) - \left( \frac{n_2}{i} \right) \left( \frac{n_1-1}{e} \right) \right] Q^i.
\end{align*}
\]

It is straightforward to check that with \( n_1 \leq n_2 \) each term inside the brackets on the last line is nonnegative, and so

\[
\frac{q^{n_1}}{f(n_1)} \cdot \frac{q^{n_2}}{f(n_2)} \leq \frac{q^{n_1-1}}{f(n_1 - 1)} \cdot \frac{q^{n_2+1}}{f(n_2 + 1)}.
\]

Similar (but simpler) checks can be performed to verify the log-convexity of the Plotkin and Singleton bounds, and we obtain the following corollary.

**Corollary 3.** Let \( C \) be an \( (n, k, r, \mu) \) LRC code over an alphabet of size \( q \) and minimum distance \( d \) and let \( \mu \) be defined in (5). The following bounds hold true:

1) **Locality-dependent Hamming bound:**

\[
k \leq \mu \left( N - \log_q \left( \sum_{e=0}^{\rho - \frac{1}{q} N} \binom{N}{e} (q - 1)^e \right) \right). \tag{11}
\]

2) **Locality-dependent Plotkin bound:** Let \( \rho > \frac{\sqrt{q} - 1}{q} N \), then

\[
k \leq \mu \log_q \frac{\rho}{\rho - \frac{\sqrt{q} - 1}{q} N}. \tag{12}
\]

3) **Locality-dependent Singleton bound:**

\[
k \leq \mu r. \tag{13}
\]
The bound (13) is slightly weaker than the Singleton-type bounds previously derived in (13) for $\rho = 2$ and in (24) for arbitrary $\rho \geq 2$.

Remark 1. Not all bounds on codes are log-convex in the code length. For example, let $A(l, \rho)$ be the maximum size of a binary code of length $l$ and distance $\rho$. We have $A(7, 4) = 8, A(8, 4) = 16, A(9, 4) = 20$, and so $A(8, 4)^2 > A(7, 4)A(9, 4)$, violating the log-convexity condition (which stipulates that the geometric average be greater than the “middle value”).

Assume that there exists a Gilbert-Varshamov type bound holds true: $R_{\rho}(n, k, r, \rho, \delta_n)$ of $M_n(n, k, r, \rho, \delta_n)$ is the maximum cardinality of the $(n, k, r, \rho, \delta_n)$ LRC code with minimum distance $\delta_n$. We finish this section by showing that the bound (13) can be combined with a known result to derive an exact value of $R_{\rho}$ for $\delta = 0$. The following lower asymptotic bound for LRC codes was obtained in (4).

Theorem 4. (2) Assume that there exists a $q$-ary MDS code of length $N$ and distance $\rho$. Then the following Gilbert-Varshamov type bound holds true:

$$R_{\rho}(r, \rho, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log_q M_q(n, r, \rho, \delta_n),$$

where $M_q(n, r, \rho, \delta_n)$ is the maximum cardinality of the $(n, k, r, \rho, \delta_n)$ LRC code with minimum distance $\delta_n$. We finish this section by showing that the bound (13) can be combined with a known result to derive an exact value of $R_{\rho}$ for $\delta = 0$. The following lower asymptotic bound for LRC codes was obtained in (4).

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where

$$M_q(n, r, \rho, \delta_n) = \max_{\delta \geq 0} \left\{ \log_q b_q(s) - \delta \log_q s \right\}$$

This result implies the following corollary, which for $q = 2$ was already established in (30).

Corollary 5. Assume that there exists a $q$-ary MDS code of length $N$ and distance $\rho$, then $R_{\rho}(r, \rho, 0) = \frac{r}{N}$.

Proof: The bound (13) implies the estimate $R_{\rho}(r, \rho, 0) \leq \frac{r}{N}$ while (14) gives the opposite inequality.

III. Algebraic Combinatorics of LRC Codes and LP Bounds

Delsarte’s linear programming bound is a powerful method of estimating the size of optimal codes in various metric spaces that satisfy a set of general assumptions (11). In this section we develop an adaptation of the approach in (11) to $(n, k, r, \rho)$ LRC codes.

A. Association Schemes and their Powers

1) Metric Association Schemes: We begin with a brief reminder about metric association schemes (11), (5). Let $X$ be a finite metric space with distance function $d$, and let $R = \{R_0, R_1, \ldots, R_n\}$ be a partition of $X \times X$ such that $R_i := \{(x, y) \in X^2 \mid d(x, y) = i\}$ for all $i$. The pair $(X, R)$ is called an association scheme if the intersection volume of two balls in $X$ depends only on the distance between their centers and the radii of the balls. For each $i$ denote by $A_i$ the $|X| \times |X|$ adjacency matrix of $R_i$, where $(A_i)_{x,y} = 1$ if $(x, y) \in R_i$ and 0 otherwise. The matrices $A_0, A_1, \ldots, A_n$ span a complex semisimple algebra of dimension $n + 1$, called the Bose-Mesner algebra of the scheme. Since each $A_i$ is symmetric, this algebra is commutative. It affords a dual basis of minimal idempotents $E_0, E_1, \ldots, E_n$. We can represent the matrix $A_i$ as a linear combination of the idempotents. The coefficients of this expansion form the first eigenmatrix the scheme $A$, denoted by $P$. A similar transition can be performed in the other direction, and the corresponding coefficients form the second eigenmatrix of $A$, denoted by $Q$. Namely, we have

$$A_i = \sum_{j=0}^{n} P_{ji} E_j, \quad 0 \leq i \leq n,$$

$$E_j = \frac{1}{|X|} \sum_{i=0}^{n} Q_{ij} A_i, \quad 0 \leq j \leq n.$$
2) **Products of Association Schemes:** Let $A = (Y, \mathcal{R})$ be a metric association scheme with eigenmatrices $P$ and $Q$, and let $X := Y^s$ be a Cartesian power of $Y$. We can define a product association scheme $A^{\otimes s} = (Y, \mathcal{R})^{\otimes s}$ by introducing the relations $R_{it} = \{(i_1, \ldots, i_s)\}$ on $X \times X$ in the following obvious way [11, p.17]:

$$R_{ij} = \{(x_{11}, \ldots, x_{1s}), (x_{21}, \ldots, x_{2s})\} \mid (x_{1j}, x_{2j}) \in R_{ij}, j = 1, \ldots, s\}.$$ 

The adjacency matrices of $A^{\otimes s}$ are formed of the Kronecker products $\otimes_{i=1}^s A_i$, where $A_i$ is an adjacency matrix of the $i$th copy of $Y$ in the product. The first (second) eigenmatrix of the scheme $A^{\otimes s}$ equals the $i$th Kronecker power of $P$ (resp., of $Q$).

3) **The Linear Programming Bound:** Let $(X, A)$ be an association scheme with $n$ classes and let $C$ be a code (any subset $C \subset X$). The distance distribution of $C$ is given by $a = (a_0, a_1, \ldots, a_n)$, where $a_i = |(C \times C) \cap R_i|/|C|$ is the average number of codewords at distance $i$ from a given codeword of $C$. Clearly, $a_0 = 1$ and $\sum a_i = |C|$. The vector $aQ$, called the MacWilliams transform of the distance distribution of $C$, satisfies the Delsarte inequalities $(aQ)_i \geq 0, i = 1, \ldots, n$. This gives rise to the linear programming bound on codes: let $C \subset X$ be a code, then

$$|C| \leq \max \left\{ \sum_{i=0}^n a_i \text{ s.t. } aQ \geq 0, a_0 = 1, a_i \geq 0, 1 \leq i \leq n \right\}.$$ 

This bound also applies to product schemes. Indeed, let $A^{\otimes s} = (X, A)^{\otimes s}$ and let $C \subset X^s$ be a code. Let $a = (a_{ij})$, where $\mathbf{i} = (i_1, \ldots, i_s)$ and $i_j = 0, 1, \ldots, n$ for all $j$, be the distance distribution of $C$. We have

$$|C| \leq \max \left\{ \sum_{\mathbf{i}} a_{ij} \text{ s.t. } \sum_{\mathbf{i}} a_{ij}Q_{\mathbf{i}j} \geq 0 \text{ for all } j; \quad a_0 = 1, a_i \geq 0 \right\}$$  

(15)

where $Q$ is the second eigenmatrix of $A^{\otimes s}$. More details about product schemes are given in [11, Sec. 2.5] as well as in more recent works [6, 20].

4) **The Hamming Scheme:** The following classic example will be useful below in the context of LRC codes. Let $F$ be a set of cardinality $q$ ($q \geq 2$) and let $X = F^n$ be a Cartesian power of $Y$. We specialize the definition of the metric scheme by assuming that $d$ is a Hamming metric on $X$. Namely, let $R_i := \{(x, y) \in X^2 \mid d(x, y) = i\}$ where $d$ is the Hamming distance. We obtain a symmetric association scheme with $n$ classes, denoted by $H(n, q)$. The eigenvalues of $H(n, q)$ are given by $Q_{ij} = K_j^{(n)}(i)$ [11], where

$$K_j^{(n)}(x) = \sum_{l=0}^{j}(-1)^l(q-1)^{j-l} \binom{n-x}{j-l} \binom{x}{l}$$

is the Krawtchouk polynomial. Also we have $P = Q$.

The Hamming scheme $H(n, q)$ also carries the structure of a product scheme for $n > 2$. Consider the Hamming scheme $H(m + n, q)$ as being obtained from the product of $H(m, q)$ and $H(n, q)$ by merging all relations $R_{i' + i''}$ with $i' + i'' = i$ into one relation $R_{ij}$. We have $A_i = \sum_{i' + i'' = i} A_{i'}A_{i''}, E_i = \sum_{i' + i'' = i} E_{i'}E_{i''}$ and $Q_{ij} = \sum_{i' + i'' = i} Q_{i'}Q_{i''}$ for any pair $(i', i'')$ with $i' + i'' = i$. Also we can view all three association schemes involved as merged versions of powers of $H(1, q)$.

Clearly, $H(n, q) = (H(1, q))^{\otimes n}$, and similarly $H(st, q) = (H(t, q))^{\otimes s}$ for any $s \geq 2$. We conclude that the eigenvalues of the scheme $H(st, q)$ have the form

$$K_{\mathbf{i}}^{(t)}(\mathbf{j}) = \prod_{p=1}^{s} K_{j_p}^{(t)}(i_p),$$

(16)

where the multi-indices $\mathbf{i}, \mathbf{j}$ are the indices of the relations of the scheme. As is the case with the original Hamming scheme, the obtained scheme $H(st, q)$ is also self-dual. It is this setting that we apply to the analysis of LRC codes in the next section.

B. **The Linear Programming Bound for LRC Codes with Disjoint Repair Groups**

1) **General Bound:** We begin with stating a general LP bound for codes with locality. Let $C$ be an $(n, k, r, \rho)$-LRC code with minimum distance $d$. Suppose that $n = sN$. For $0 \leq t \leq s - 1$, define the interval

$$\mathcal{R}_{t+1} = [tN + 1, (t + 1)N],$$

The adjacency matrices of $A^{\otimes s}$ are formed of the Kronecker products $\otimes_{i=1}^s A_i$, where $A_i$ is an adjacency matrix of the $i$th copy of $Y$ in the product. The first (second) eigenmatrix of the scheme $A^{\otimes s}$ equals the $i$th Kronecker power of $P$ (resp., of $Q$).
so the coordinate set is a disjoint union of these intervals:
\[ [n] = \bigcup_{i=1}^{s} \mathcal{R}_i. \]

For \( I \subset [n] \) denote by \( \mathcal{C}|_I \) the projection of \( \mathcal{C} \) on the coordinates in \( I \). Throughout this section we will assume that the code \( \mathcal{C} \) has the property that
\[ d(\mathcal{C}|_{\mathcal{R}_i}) \geq \rho \text{ for all } i = 1, \ldots, s. \]

In accordance with (16), define the following polynomials of \( s \) discrete variables \( x = (x_1, \ldots, x_s) \)
\[ K^{(N)}_\mathcal{I}(x) = \prod_{p=1}^{s} K^{(N)}_{\mathcal{I}_p}(x_p), \]
where \( \mathcal{I} = (j_1, \ldots, j_s) \). The polynomials \( K^{(N)}_\mathcal{I} \) are orthogonal on the set \([N]^s\):
\[ \sum_{\mathcal{I} \in [N]^s} f(\mathcal{I}) K^{(N)}_\mathcal{I}(\mathcal{I}') K^{(N)}_{\mathcal{I}}(\mathcal{I}) = q^n f(\mathcal{I}) \delta_{\mathcal{I},\mathcal{I}'} \]
where
\[ f(\mathcal{I}) = K^{(N)}_\mathcal{I}(0) = \prod_{p=1}^{s} \left( \binom{N}{i_p} (q-1)^{i_p} \right). \] (17)

Let \( a = (a_\mathcal{I}, \mathcal{I} \in [N]^s) \) be the distance distribution of \( \mathcal{C} \), where each \( a_\mathcal{I} = (a_{i_1}, a_{i_2}, \ldots, a_{i_s}) \) is an \( s \)-tuple. In words, \( a_\mathcal{I} \) is the number of pairs of codewords \( c, c' \in \mathcal{C} \) such that the Hamming distance \( d(c_{i_j}, c'_{i_j}) = a_{i_j}, j = 1, \ldots, s \), normalized by the cardinality of the code \( q^k \). Note that the codewords \( c_{i_j}, c'_{i_j} \) are contained in the code \( \mathcal{C}|_{\mathcal{R}_j} \). By definition, we have \( a_0 = 1 \).

The general bound of (15) in our case takes the following form.

**Theorem 6** (Primal LP bound). Let \( \mathcal{C} \) be a \( q \)-ary \( (n, k, r, \rho) \) LRC code with distance \( d \). Define
\[ T := \{ \mathcal{I} = (i_1, \ldots, i_s) \mid i_1 + \cdots + i_s \geq d, \]
\[ i_p \in \{0, \rho, \rho+1, \ldots, N\} \text{ for all } p = 1, \ldots, s. \} \]
Then the cardinality of \( \mathcal{C} \) satisfies
\[ |\mathcal{C}| \leq 1 + \sum_{\mathcal{I} \in T} a_\mathcal{I}, \text{ where the vector } (a_\mathcal{I}, \mathcal{I} \in T) \text{ is a solution of the following LP problem} \]
\[
\begin{align*}
\text{maximize} & \sum_{\mathcal{I} \in T} a_\mathcal{I} \\
\text{subject to} & a_\mathcal{I} \geq 0, \quad \mathcal{I} \in T, \\
& \sum_{\mathcal{I} \in T} a_\mathcal{I} Q_{\mathcal{I}} \geq -K^{(N)}_{\mathcal{I}}(0), \quad \mathcal{I} \in [N]^s \setminus \{0\}. 
\end{align*}
\]

The dual problem of the LP problem in Theorem 6 has the following form.

**Theorem 7** (Dual LP bound). Let \( \mathcal{C} \) and \( T \) be as defined in Theorem 6. The cardinality of \( \mathcal{C} \) satisfies
\[ |\mathcal{C}| \leq 1 + \min_{\mathcal{I} \in [N]^s \setminus \{0\}} f(\mathcal{I}) K^{(N)}_{\mathcal{I}}(0) \] (18)
subject to
\[ f(\mathcal{I}) \geq 0, \quad \mathcal{I} \in [N]^s \setminus \{0\}, \] (19)
\[ f(\mathcal{I}) \leq 0, \quad \mathcal{I} \in T. \] (20)

As in the classical case (cf. [11, p.53], [19]), instead of solving this LP problem, we construct feasible solutions which provide upper bounds for the minimum. We state the result in polynomial form, which is obvious from (18)-(20).

**Corollary 8.** Let \( \mathcal{C} \) be a \( q \)-ary \( (n, k, r, \rho) \) LRC code with distance \( d \). Let \( f(\mathcal{I}) = f(x_1, \ldots, x_s) \) be a polynomial whose Krawtchouk expansion has the form
\[ f(\mathcal{I}) = 1 + \sum_{\mathcal{I} \in [N]^s \setminus \{0\}} f(\mathcal{I}) K^{(N)}_{\mathcal{I}}(0), \]
where the coefficients \( f_\lambda \) satisfy the conditions in (19)-(20). Then \(|C| \leq f(\emptyset)\).

2) **The Singleton Bound:** The bounds in Corollary[3] can be proved using the polynomial approach of this theorem. To exemplify this claim, we give another proof of the Singleton bound. The original form of this bound in [23] is as follows:

\[
d \leq n - k + 1 - \left( \frac{k}{r} \right)(\rho - 1).
\]

(21)

Assume that \( d = tN + \partial \) for some \( \partial, 1 \leq \partial \leq N \). Relaxing (21) by omitting the ceiling function, we obtain

\[
k \leq r(s - t) + \frac{r(\rho - \partial)}{N}.
\]

(22)

Recall that in the classical case the Singleton bound is proved using the polynomial [11, p. 54], [19, p. 544] (the "annihilator" of the weight distribution). Following this approach, define the polynomial \( f(x) \) in the form

\[
f(x) = q^{n-d} + \prod_{j=d}^{n} \left( 1 - \frac{x}{j} \right)
\]

(23)

(24)

We will prove that the polynomial \( f \) is a feasible solution of the dual LP problem, i.e., that it satisfies the conditions in (19)-(20). Consider the expansion

\[
f(x) = 1 + \sum_{j \in [N] \setminus \{\emptyset\}} f_j K^{(N)}(x).
\]

On account of (23) we conclude that \( f_j \geq 0 \) for all \( j \), so (19) is indeed true.

To prove (20), we will show that \( f(\hat{i}) \leq 0 \) for \( \hat{i} \in \hat{T} \). First suppose that \( \rho \leq \partial \). Choose any \( \hat{i} = (i_1, \ldots, i_s) \in \hat{T} \), then \( i_1 + \cdots + i_s \geq d \). If \( i_{s-t} + \cdots + i_s \geq d \), then we must have \( i_{s-t} \geq \partial \), implying that \( f(\hat{i}) = 0 \). If \( i_{s-t} + \cdots + i_s < d \), then there must exist some nonzero \( i_l, 1 \leq l \leq s - t - 1 \), which implies that \( i_l \geq \rho \) and again \( f(\hat{i}) = 0 \). The case \( \rho > \partial \) can be analyzed using similar arguments.

Therefore, by Theorem [7] we obtain the bound

\[
|C| \leq f(\emptyset) = \begin{cases} q^{r(s-t)+\rho-\partial} & \text{if } \rho \leq \partial \\ q^{r(s-t)} & \text{if } \rho > \partial. \end{cases}
\]

This estimate is an LP version of the Singleton bound, and it is slightly better than (22).

3) **Bounds for \((n, k, r, 2)\) LRC Codes:** It is interesting to apply the LP approach to bounds on LRC codes for \( \rho = 2 \), i.e., the case of single-symbol locality (Definition [1]). The known bounds that apply in this case include the Singleton bound [13], which does not depend on \( q \), and a shortening bound of [7]. For the ease of reading we reproduce the bound [4]: For any \((n, k, r, 2)\) LRC code with distance \( d \), we have

\[
k \leq \min_{1 \leq s \leq n/(r+1)} \left\{ sr + \log_q M_q(n - s(r + 1), d) \right\}.
\]

where \( M_q(n, d) \) is the maximum cardinality of a \( q \)-ary code of length \( n \) and distance \( d \).

We computed this bound and the bound of Theorem [7] using \( \rho = 2 \) in the definition of the index set \( T \). The results are summarized in Tables I–IV. We perform the computations using the GAP package GUAVA and the package GLPK in the symbolic computations system SageMath [11]. Each result was verified using the package COIN-OR, also available in SageMath.

In all the above examples the LP bound either matches the shortening bound or is tighter than it.
### IV. Asymptotic Bounds for Binary Linear LRC Codes

In this section we study asymptotic bounds for binary linear LRC codes that can recover locally from one erasure. However, the results extends to the case of general \( q \)-ary alphabet quite straightforwardly. Throughout the section we assume that the code satisfies Definition 1. For \( \delta \in [0, 1/2) \) define the functions

\[
R(r, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log_2 M(n, r, \delta n)
\]

\[
R^{\text{lin}}(r, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log_2 M^{\text{lin}}(n, r, \delta n),
\]

where \( M(n, r, d) \) is the maximum cardinality of the code (resp., of the linear code) of length \( n \), distance \( d \) and locality \( r \). Clearly, \( R(r, \delta) \geq R^{\text{lin}}(r, \delta) \).

Currently the best known asymptotic bounds on binary LRC codes are as follows.

**Theorem 9.** We have

\[
R^{\text{lin}}(r, \delta) \geq 1 - \min_{0 < s \leq 1} \left\{ \frac{1}{r + 1} \log_2 ((1 + s)^{r+1} + (1 - s)^{r+1}) - \delta \log_2 s \right\}
\]

\[
R(r, \delta) \leq \min_{0 \leq \sigma < 1/(r + 1)} \left\{ \sigma + (1 - \sigma(r + 1)) R_{\text{opt}} \left( \frac{\delta}{1 - \sigma(r + 1)} \right) \right\},
\]

where \( R_{\text{opt}}(\delta) \) is any asymptotic upper bound on the rate of codes with relative distance \( \delta \). These bounds imply that

\[
R^{\text{lin}}(r, \delta) > 0 \iff 0 < \delta < 1/2;
\]

\[
R(r, 0) = R^{\text{lin}}(r, 0) = \frac{r}{r + 1}.
\]

The lower bound (25) is of the Gilbert-Varshamov type and was derived in [7] and [30], while the bound (26) is obtained from (14) by passing to the limit of large block length \( n \) (see [7]). To obtain the tightest possible bound in (26) we substitute the best known bound on \( R_{\text{opt}}(\delta) \), i.e., the McEliece et al. bound [22]:

\[
R_{\text{opt}}(\delta) \leq 1 + \min_{0 < \alpha \leq 1 - 2\delta} \left( g(\alpha^2) - g(\alpha^2 + 2\delta \alpha + 2\delta) \right),
\]

where \( g(x) := h(\frac{1}{2} - \frac{1}{2}\sqrt{1 - x}) \) and \( h(x) := -x \log_2 x - (1 - x) \log_2 (1 - x) \) is the binary entropy function.

**Remark 2.** Even though in Sec. III-B3 we showed by example that the LP bound is better than the bound [4] for finite length, it is difficult to derive a closed-form asymptotic version of the LP bound. The problem occurs because to derive the asymptotic version of the LP bound it would be easier to have a small number of local codes whose distance \( \rho \) grows in proportion to \( n \). In reality we have to deal with a growing number of local codes with distance \( \rho = 2 \).
Here we will prove the following theorem which improves upon the bound (26) in the case of linear codes with linear recovering functions.

**Definition 3.** A code $C$ is said to have linear locality $r$ if every coordinate $i$ is contained in a subset $R_i \subset [n]$ of size at most $|R_i| \leq r+1$ with the property that there exists a linear function $\phi_i$ such that for every codeword $c \in C$

$$c_i = \phi_i(\{c_j, j \in R_i \setminus \{i\}\}).$$

(28)

With a slight abuse of notation, we denote by $R^{(\text{lin})}(r, \delta)$ the rate of linear LRC codes with linear recovery functions throughout the rest of the paper. Our first asymptotic result is the following.

**Theorem 10** (Linear LRC codes).

$$R^{(\text{lin})}(r, \delta) \leq \min_{0 \leq s \leq \log_2 \frac{1 - \delta}{\delta}} \left\{ s + (1 - s(r + 1))R_0 \left( r, \frac{1 - s(r + 1)}{1 - \delta} \right) \right\};$$

(29)

where $R_0(r, \delta) := h(\tau) - c(r + 1, \tau), c(w, \tau) := \frac{\log_2 e}{w} \left( \frac{w}{\tau} \right)^{w+1}$, and $\tau := \frac{1}{3} - \sqrt{\delta(1 - \delta)}$.

Bound (29) improves upon the bound in (26) for all values of relative distance, however the improvement is really mild, and can barely be seen in a plot.

The proof of this theorem consists of two steps. First we observe that the approach of [16] applies to LRC codes, but this is an artifact of the LDPC setting rather than an essential element of the proof. In [16] shows that the last condition is not needed for it to be valid. It is true that this condition holds for LDPC codes, but this is an artifact of the LDPC setting rather than an essential element of the proof.

Now note that by definition of the LRC code (Definition 3) the dual code $C^\perp$, the dual code of $C$, contain a set of vectors of weight at most $r+1$ the union of whose supports equals $[n]$. Namely, for some natural number $m$

$$\exists \{v_1, \ldots, v_m\} \subset C^\perp \text{ such that } \text{wt}(v_i) \leq r+1, i = 1, \ldots, m; \bigcup_{i=1}^{m} \text{supp}(v_i) = [n].$$

(30)

Note that for linear LRC with linear recovery functions, this condition is satisfied. Also, note that [16] in addition states that these vectors form a basis for the code $C^\perp$; however, close examination of the proof of Theorem 1.2 in [16] shows that the last condition is not needed for it to be valid. It is true that this condition holds for LDPC codes, but this is an artifact of the LDPC setting rather than an essential element of the proof.

Now note that by definition of the LRC code (Definition 3) the dual code $C^\perp$ satisfies the assumption (30). As a consequence of this discussion, we obtain the following bound:

$$R^{(\text{lin})}(r, \delta) \leq R_0(r, \delta).$$

(31)

This bound does not improve on (26), but it is possible to establish a recursion that will lead to an improvement. Namely, we combine (31) with the shortening argument of [7] to obtain (29).

**Proposition 12.** Let $C$ be a binary LRC code of length $n$, locality $r$ and distance $d$, then

$$|C| \leq \min_{s \geq 1} (sr + \log_2 M(n - s(r + 1), d, r))$$

(32)

where $M(m, d, r)$ is the maximum cardinality of a linear LRC code of length $m$, distance $d$, and linear locality $r$. Therefore,

$$R^{(\text{lin})}(r, \delta) \leq \min_{0 \leq s \leq \log_2 \frac{1}{1 - \delta}} \left\{ \sigma + (1 - \sigma(r + 1))R^{(\text{lin})} \left( r, \frac{\delta}{1 - \sigma(r + 1)} \right) \right\};$$

(33)

Proof: Eq. (33) is an obvious consequence of (32), so let us prove (32). The proof is a minor modification of the argument in [7]. The proof in [7] relies on the fact that for any $s, 1 \leq s \leq k/r$ there exists a subset of coordinates $I \subset [n], |I| = s(r + 1)$ such that $\log |C_I| \leq sr$ ([7] Lemma 1). That such a subset exists can be shown relying on the locality property of the code $C$. Having found such an $I$, we shorten the code on these coordinates, obtaining a code $C_{I^c}$ of length $n - s(r + 1)$, dimension at least $k - sr$, and distance $d$ ([7] Lemma 2)). This code is obtained by taking all the codewords that contain zeros in the coordinates in $I$ and discarding these coordinates.
The only added element in our claim is that the shortened code $C_{I'}$ itself is LRC. Indeed, let $i \in I'$. Referring to Def. [1], we need to prove that for any coordinate $i \in [n] \setminus I$ there exists a function $\phi_i$ that depends on at most $r$ other coordinates and computes the value of the $i$th coordinate of the codeword $c \in C_{I'}$. There are two cases:

(i) The repair group $R_i$ of $i$ does not intersect the subset $I$. In this case there is nothing to prove.

(ii) Some number of the coordinates of $R_i$ are inside the subset $I$. In this case the value of the discarded coordinates for every codeword of $C_{I'}$ is equal to 0. Suppose that $\psi_i(\{c_j : j \in R_i \setminus I\})$ is the recovery function of the original code $C$. We claim that the recovery group of the coordinate $i \in I'$ in the code $C_{I'}$ is the subset $R_i \setminus J_i$, and the recovery function is obtained from $\phi_i$ by substituting zeros for all the arguments in $J_i$. Note that the function $\phi_i$ essentially depends on $|R_i| - |J_i| - 1 \leq r$ coordinates of the codeword $c$, conforming with the locality requirement.

**Remark 3.** While we need this result only for linear codes, the claims of Proposition 12 are still valid if we omit the linearity assumption (with obvious modifications to the statement).

Now Theorem 10 follows immediately by using (31) in the estimate (33).

In the next two subsections, we improve on Theorem 10 for the case of disjoint repair groups.

A. The Approach of Iceland and Samorodnitsky [16] to Bounds on LDPC Codes

Coset graphs of linear codes have been often used as a tool to study combinatorial properties of codes and to obtain bounds on their parameters [8], [10], [12]. Given a linear code $C$, define a graph $\Gamma(V, E)$, where $V = F^n_q/\perp C$, i.e., the vertices of $\Gamma$ correspond to the cosets of the code, and two cosets are connected by an edge if the Hamming distance between them is one.

Let $C$ be a linear code. Throughout this section we use the coset graph of the dual code $C^\perp$, so all the references to the coset graph below are with respect to the dual code. The length of the shortest path between a pair of vertices equals the Hamming distance between the corresponding cosets. Given a vertex $v \in V$, denote by $B_\Gamma(v, t)$ the ball of radius $t$ around it in the graph. Since the graph is vertex-transitive, the volume of the ball does not depend on $v$, and we will use the notation $B(t) := |B_\Gamma(v, t)|$ where $v$ is an arbitrary vertex. Clearly, $B(t)$ equals the number of cosets whose leaders are of weight at most $t$.

The starting point of the argument in [16] is the following result from [12].

**Theorem 13 ([12]).** Consider a sequence of linear codes $C_i, i = 1, 2, \ldots$ of length $n_i \to \infty$ and let $B^{(i)}(t)$ be the number of cosets of weight at most $t$ in $C_i$. Then

$$ |C_i| \leq 2^{o(n_i)} B^{(i)}(\tau n_i), \quad \text{where } \tau = \frac{1}{2} - \sqrt{\delta(1 - \delta)}. \quad (34) $$

To shorten the notation, below we write $n$ instead of $n_i$ and assume that $n \to \infty$. Using the obvious estimate $B(t) \leq \sum_{i=0}^{t} \binom{n}{i}$, $t = \tau n$, one obtains a bound valid for any code $C$. The main idea in [16] is that it is possible to obtain a tighter estimator for $B(t)$ in the case when $C$ is an LDPC code, leading to an improved bound on the rate of such codes compared to the universal bounds of [22].

**Lemma 14 ([16]).** Let $x \in \{0, 1\}^n$ be a random vector with independent Bernoulli coordinates $x_i$ such that $P(x_i = 1) = p, P(x_i = 0) = 1 - p, p < 1/2$, and let $\pi_p$ be the probability that $x$ is a coset leader of $C^\perp$. Then

$$ B(pn) \leq \pi_p \left( \sqrt{2n} \sum_{i=0}^{t} \binom{n}{i} \right). \quad (35) $$

**Proof:** We include a very short proof. Limiting ourselves to the vectors $x$ with at most $pn$ ones, we have

$$ \pi_p \geq B(pn)p^{pn}(1 - p)^{(1-p)n} = B(pn)2^{-h(p)n} \geq B(pn)\frac{1}{\sqrt{2n} \sum_{i=0}^{t} \binom{n}{i}}. $$

The next step, which is the main technical ingredient of the result in [16], is to show that $\pi_p$ is an exponentially declining function of $n$. Let us assume that the dual code $C^\perp$ contains a set vectors $v_1, \ldots, v_m$ such that $\wt(v_i) = w$ for all $i$ and that $\bigcup_i \supp(v_i) = [n]$, see (30).

Construct a partition $I_w$ of the coordinate set $[n]$ into $w$ disjoint sets $I_w, I_{w-1}, \ldots, I_1$ as follows. Suppose that $I_w, I_{w-1}, \ldots, I_{k+1}$ are already defined. Set $I_k = \emptyset$. For each of the vectors $v_i, i = 1, \ldots, m$ consider the set of coordinates $L_{k,i} := \supp(v_i) \setminus \bigcup_{i=k}^{w} I_i$, and if the size of $L_{k,i}$ is exactly $k$, put $I_k \leftarrow I_k \cup L_{k,i}$. 

It is easy to observe that each block $I_i$ in the partition $I_w$ is a disjoint union of $t = |I_i|/i$ subsets $U_{i,j}$ such that $|U_{i,j}| = j$ for all $j$, and each $U_{i,j}$ is contained in the support of a different vector $v_j$. Moreover, the set $\operatorname{supp}(v_j)\setminus U_{i,j}$ is a subset of $\bigcup_{k=j+1}^w I_i$.

An important property of the partition $I_w$ is as follows.

**Lemma 15** ([16] Lemma 2). Let $A = 2/p^w$. There exists an index $k \in \{1, 2, \ldots, w\}$ such that

$$|I_k| \geq \max \left\{ A \sum_{j=k+1}^w |I_j|, \frac{n}{2^{w+1}} \right\}.$$

Now let $k$ be the index whose existence is guaranteed by this lemma. We have that $|I_k| \geq \max \left\{ A \sum_{j=k+1}^w |I_j|, \frac{n}{2^{w+1}} \right\}$ and the set $I_k$ is a disjoint union of $t = |I_k|/k$ sets $U_{k,j}$. Now consider a coset leader $x$. An easy argument shows that $\operatorname{supp}(x)$ contains no more than $tp^k/2$ subsets $U_{k,j}$. Indeed, let $S = \{j| U_{k,j} \subseteq \operatorname{supp}(x)\}$ and consider a vector $y$ from the same coset given by

$$y = x + \sum_{j \in S} v_{k,j}.$$

If $|S| \geq tp^k/2$ then we would obtain $w(y) < w(x)$, a contradiction.

The final step is to take a random vector $y$ as in Lemma 15 and to estimate the probability that it is a coset leader of the code $C$. First observe that for $k$ given by Lemma 15 we have

$$p^k t = \frac{p^k |I_k|}{n} n \geq \frac{p^w}{2^{w+1}} n = \frac{1}{2} \left( \frac{p^w}{2} \right)^{w+1} n.$$

Now let $Y \sim \operatorname{Binom}(t, p^k)$, then the Chernoff bound gives

$$\pi_p \leq P \left( Y \leq \frac{p^k t}{2} \right) \leq \exp \left( - \frac{p^k t}{8} \right) \leq \pi_p \leq 2^{-c(w,p)n}, \quad (36)$$

where $c(\cdot, \cdot)$ is defined in Theorem 10.

To apply the above argument to LRC codes, we note that it depends only on the existence of low-weight parity checks of the code $C$ whose supports jointly contain every coordinate in $[n]$, see (30). By definition, an LRC code has parity checks (repair groups) of weight at most $r+1$ that satisfy this condition. Therefore, we can use inequalities (34), (35), and (36) (where we put $p = \tau$, noting that $\tau < 1/2$ for all $\delta \in (0,1/2)$), obtaining the estimate (29) for $R_{\text{lin}}^\tau(r, \delta)$.

**B. Upper Bound on LRC Codes with Disjoint Repair Groups**

Upper bounds for LRC codes with disjoint repair groups were already considered in Sec. III. Here we consider the asymptotic version of this problem, noting that the general result of Theorem 10 in this case admits a significant improvement.

Assume that $n = (r+1)m$ and consider a binary linear LRC $C(n, \delta n, r)$ with disjoint repair groups $R_j, j = 1, \ldots, m$. For every $j$ the repair group $R_j$ corresponds to a vector $v_j$ of weight $r+1$ in $C^\perp$, and these vectors have pairwise disjoint supports and trivially satisfy the conditions in (30) (even if the repair groups are of smaller size, we can add redundant coordinates in them to bring the size to $r+1$).

Recall that $B(\tau n) = B_T(0, \tau n)$ is the set of coset leaders of $C$ of weight at most $\tau n$ and let $x \in B(\tau n)$. The vector $x$ satisfies the following constraints:

$$\min_{T \subseteq [m]} \operatorname{wt} \left( x + \sum_{i \in T} v_i \right) \leq \tau n \quad (37a)$$

$$\operatorname{wt}(x) \leq \tau n \quad (37b)$$

$$|B(\tau n) \cap (x + \operatorname{span}\{v_i, i = 1, \ldots, m\})| \leq 1 \quad \text{for all } x \in \{0,1\}^n \quad (37c)$$

Eqs. (37a)-37c can be used to derive an upper bound on $R(C)$ which is given in the following theorem.

**Theorem 16** (Linear LRC codes with disjoint repair groups). Let $R_{\text{lin},\text{dis}}^\tau(r, \delta)$ be the largest possible asymptotic rate of linear LRC codes with disjoint repair groups. Let $t := \left\lceil \frac{1}{\tau^2} \right\rceil$. We have

$$R_{\text{lin},\text{dis}}^\tau(r, \delta) \leq \tau \log(x) + \frac{\log(\sum_{i=0}^t \beta_i(x))}{r+1} \bigg|_{x=\mu} \quad (38)$$
where
\[
\beta_i(x) := \begin{cases} 
\frac{(r+1)}{i} x^i & \text{if } i < t \\
\frac{(r+1)}{i} \cdot 2^r \mod 2 x^i & \text{if } i = t,
\end{cases}
\]
(39)

where \( \mu = 1 \) if \( \sum_{i=0}^{t} i \cdot \beta_i(x) \bigg|_{x=1} \leq (r + 1) \tau \) and
\[
\mu = \text{Root}^+ \left( (r + 1) \tau \sum_{i=0}^{t} \binom{r + 1}{i} x^{r-i} - \sum_{i=0}^{t} \binom{r + 1}{i} i x^{r-i} \right)
\]
(40)
otherwise. Here \( \text{Root}^+(f(x)) \) denotes the unique positive zero of the polynomial \( f(x) \).

Proof: We will again use inequality (34). To bound above the \( B(\tau n) \) we use the constraints in (37). In particular, (37b) and (37a), we obtain
\[
B(\tau n) \subseteq \left\{ x \in \{0,1\}^n \mid \text{wt}(x) \leq \tau n, \text{ and for all } i, |\text{supp}(x) \cap R_i| \leq \left\lfloor \frac{r + 1}{2} \right\rfloor \right\}.
\]
(41)
The number of vectors \( x \) that satisfy (37b) and (37a) is therefore given by the coefficient of \( x^k \) in the following expression:
\[
\sum_{j \leq \tau n} \text{Coeff}_{x^j}(1 + \sum_{i=0}^{t} \binom{r + 1}{i} x^i)^m.
\]
(42)
Now let us in addition use (37c). Since \( \text{wt}(x) = \text{wt}(x + v_i) \) for all \( i \) and they are in the same coset, it suffices to count only one of them on the right-hand side of (41). Therefore if \( r + 1 \) is even, we obtain
\[
B(\tau n) \leq \sum_{j=0}^{\tau} n \text{Coeff}_{x^j}(g(x))
\]
where
\[
g(x) = \left( 1 + \sum_{i=0}^{t} \binom{r + 1}{i} x^i + \frac{(r+1)}{2^r \mod 2} x^i \right)^m.
\]
One can see that asymptotically for \( n \to \infty \) the dominating term in this expression is given by the largest coefficient, i.e.,
\[
\frac{1}{n} \log \sum_{0 \leq j \leq \tau n} \text{Coeff}_{x^j}(g(x)) \sim \frac{1}{n} \max_{0 \leq j \leq \tau n} \log \text{Coeff}_{x^j}(g(x)).
\]
Thus we have
\[
R^{(\text{lim.dis})}(r, \delta) \leq \max_{0 \leq j \leq \tau n} \log \text{Coeff}_{x^j}(g(x))
\]
which translates into the following convex maximization problem:
\[
\max \frac{1}{r + 1} H(P) + \sum_{i=0}^{t-1} \alpha_i \log \binom{r + 1}{i} + \alpha_i \frac{(r+1)}{2^r \mod 2}
\]
where the distribution \( P = (r + 1)(\alpha_0, \ldots, \alpha_t) \) satisfies the constraints
\[
\sum_{i=0}^{t} \alpha_i = \frac{1}{r + 1}; \sum_{i=0}^{t} i \alpha_i \leq \tau
\]
The maximum is found by differentiation (setting up a Lagrange function) and we obtain
\[
\alpha_i = \frac{\beta_i(x)}{(r + 1) \sum_{i=1}^{t} \beta_i(x)} \bigg|_{x=\mu}, \quad i = 0, 1, \ldots, t
\]
for \( \beta_i(x) \) and \( \mu \) as defined in (39), (40). Substituting these values of \( \alpha_i \), we find the bound in the form given in (38).

A numerical evaluation of the improvements in the asymptotic rate of this results over the previous existing results is shown in Fig. 1. The improvement can be seen for larger values of the relative distance. For instance, for \( r = 2 \) the improvement is obtained for \( \delta \geq 0.38 \), and this range increases for larger values of \( r \).
| Improvement in Rate | Difference from earlier best result |
|---------------------|-------------------------------------|
| 0.0000              | 0.0001                              |
| 0.0002              | 0.0004                              |
| 0.0005              | 0.0006                              |

Fig. 1. Difference between the bound on rate in eq. (38) and earlier best result (26) (cf. [7]).

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