The $\phi^4_3$ lattice field theory viewed from the high-temperature side

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Abstract

We analyze high-temperature series expansions of the two-point and four-point correlation-functions in the three-dimensional euclidean lattice scalar field theory with quartic self-coupling, which have been recently extended through twenty-fifth order for the simple-cubic and body-centered-cubic lattices. We conclude that the length of the present series is sufficient for a fairly accurate description of the critical behavior of the model and confirm the validity of universality, scaling and hyperscaling. In the case of the body-centered-cubic lattice, we determine the value of the quartic self-coupling for which the leading corrections to scaling approximately vanish and correspondingly the universal critical parameters can be determined with high accuracy. In particular, for the susceptibility and the correlation-length exponents we find $\gamma = 1.2373(2)$ and $\nu = 0.6301(2)$. For the four-point renormalized coupling we find $g = 23.56(3)$. In the case of the simple-cubic lattice our results are consistent with earlier estimates.

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I. INTRODUCTION

High-temperature (HT) expansions for scalar spin models with bilinear nearest-neighbor interaction and general single-spin measure, on 2- and 3-dimensional bipartite lattices, have been extended\textsuperscript{1,2,3,4,5,6} in the last few decades from order $K^{10}$ to $K^{25}$. Here $K = \frac{J}{kT}$ denotes the usual HT expansion variable, with $k$ the Boltzmann constant, $T$ the temperature and $J$ some energy scale characterizing the next-neighbor interaction. The main observables expanded, include the susceptibility $\chi_2$, the second moment of the correlation-function $\mu_2$, the zero-momentum four-point function $\chi_4$. The linked-cluster method used in these calculations expresses the series coefficients in a closed form as polynomials in the (normalized) moments of the single-spin measure thus making the analysis of a broad class of models possible. This fact entails two related benefits. First, it enables one to explore the extent of the Ising universality class to which these models should generally belong. Second, one can take advantage of the non-universality of the amplitudes of the corrections to scaling in the critical behavior to obtain very accurate estimates of universal critical parameters, such as exponents and scaling functions, by focusing the numerical analysis on those particular spin models, within the given universality class, which show vanishing (or quite small) leading corrections to scaling (LCS). This prescription, which can be effective only with rather long series, was suggested in Refs.\textsuperscript{7,8} and extensively tested by 21-term HT series on the body-centered-cubic lattice, for various single-spin measures.

In the initial studies, however, the available series were simply too short. In Ref.\textsuperscript{2} an analysis of the critical properties of the lattice $\phi^4$ euclidean field theory in two-, three- and four-dimensional space, using tenth-order HT series, suggested a complicated and puzzling critical behavior in the 3d case, which showed failures of universality and hyperscaling and cast doubt on the validity of key mathematical assumptions\textsuperscript{9} (see however Ref.\textsuperscript{10}) in the application of the renormalization group to the critical phenomena. This analysis was historically important in raising questions which could be answered only by numerical investigation of specific models and was a strong incentive to further extend the HT series and to repeat and improve simulation studies. The anomalous features observed in Ref.\textsuperscript{2} were not confirmed by the successive studies\textsuperscript{3,8,11} and were finally ascribed to the shortness of the series analyzed. Therefore over a decade ago the problem was considered as settled.

We have recently extended, through order $K^{25}$, the HT expansions of the moments of the correlation-function and, through order $K^{23}$, the expansion of the zero-momentum four-point correlation-function for a general scalar model defined on three-dimensional bipartite lattices (such as the simple-cubic (sc) and the body-centered-cubic (bcc) lattices). These expansions have already proved useful for accurately displaying the properties of critical scaling and universality of the spin-$S$ Ising model\textsuperscript{5,12} and have also produced high-precision estimates of critical exponents and universal combinations of critical amplitudes, based on the prescriptions of Refs.\textsuperscript{7,8}. Also following the lead of the same Authors\textsuperscript{7,8}, analogous expansions, independently obtained\textsuperscript{6} for the sc lattice case, through order 25 for $\chi_2$ and $\mu_2$ and through order 21 for $\chi_4$, were used to study the lattice $\phi^4_3$ model in a neighborhood of a particular value of the self-coupling for which the LCS are negligible. It remains to show that the present HT expansions are also adequate for an extensive and accurate study of the scalar field model for two different lattices and on the whole range of the self-coupling. From our study we conclude that the present HT series are sufficiently long that, by properly resumming them, our analysis can reach an accuracy comparable or higher than the most extensive MonteCarlo simulations. Moreover, we show that also in the case of the bcc
lattice, we can approximately determine a value of the quartic self-coupling such that the LCS vanish, and thereby we can obtain very accurate estimates of the universal critical parameters in complete agreement with the other cited recent computations.

In the following paragraphs we shall first recall the definitions of the quantities whose HT expansions are studied. We shall then very briefly comment on the results of the series analysis. We shall map out the phase diagram of the model and exhibit its universality properties along the critical line, such as the independence of the critical exponents $\gamma$ and $\nu$ on the self-coupling of the field and on the lattice structure. Then, we shall give evidence that the critical renormalized coupling constant is universal and non-zero, thus verifying the validity of hyperscaling. Finally we shall report our best estimates of the critical exponents $\gamma$ and $\nu$ and of the renormalized four-point coupling from the analysis of the model with minimal LCS on the bcc lattice. The figures summarizing our results are among the main motivations of this study, since some of them qualitatively differ from the analogous ones appearing in Ref. 2, which, to the best of our knowledge, have never been revised in the literature.

II. THE MODEL

The lattice $\phi^4_3$ model is defined by the Hamiltonian

$$KH\{\phi\} = -K \sum_{<i,j>} \phi_i \phi_j + \sum_i (\phi_i^2 + g_0 (\phi_i^2 - 1)^2)$$

(1)

Here $i$ and $j$ are integer-component (multi)indices denoting the sites of a three-dimensional lattice and the first sum extends to all nearest-neighbor sites.

This Hamiltonian is obtained from the lattice discretization of the euclidean action of the continuum real field $\psi(x)$ in 2 + 1 dimensions.

$$S = \int \left[ \frac{1}{2} (\partial_\mu \psi)^2 + \frac{1}{2} m^2 \psi^2 + \frac{\lambda}{4!} \psi^4 \right] dx$$

(2)

after setting $\psi = \sqrt{2K} \phi$, $m^2 = \frac{1 - 2g_0}{K} - 6$ and $\lambda = \frac{6g_0}{K^2}$.

We will analyze the HT expansion of the moments of the two-point connected correlation-function $<\phi_i \phi_j>_c$ and of the zero-momentum connected four-point correlation-function $\chi_4(K, g_0) = \sum_{i,j,k} <\phi_0 \phi_i \phi_j \phi_k>_c$.

The susceptibility is defined by the zero-order moment of $<\phi_0 \phi_i>_c$

$$\chi_2(K, g_0) = \sum_i <\phi_0 \phi_i>_c$$

(3)

Its behavior, as $K$ tends from below to the critical value $K_c(g_0)$, is expected to be

$$\chi_2(K, g_0) = A(g_0) \tau(g_0)^{-\gamma(g_0)} [1 + a(g_0)\tau(g_0) + ...]$$

(4)

where $A(g_0)$ is the critical amplitude of the susceptibility, $\tau(g_0) = 1 - K/K_c(g_0)$ is called the reduced (inverse) temperature, $\gamma(g_0)$ is the critical exponent of the susceptibility, $a(g_0)$ is the amplitude of the LCS and $\theta(g_0)$ is the exponent of the LCS.

The square of the correlation-length is expressed in terms of the ratio of the second moment of the correlation-function $\mu_2(K, g_0) = \sum_i i^2 <\phi_0 \phi_i>_c$ and the susceptibility as

$$\xi^2(K, g_0) = \frac{\mu_2(K, g_0)}{6\chi_2(K, g_0)}$$

(5)
and is expected to show the critical behavior
\[
\xi^2(K, g_0) = B(g_0) \tau(g_0)^{-2\nu(g_0)}[1 + b(g_0)\tau^\theta(g_0) + ...] \tag{6}
\]
where \(\nu(g_0)\) is the critical exponent of the correlation-length.

The critical renormalized coupling constant \(g(g_0)\) is expressed in terms of \(\chi_4(K, g_0)\), \(\chi_2(K, g_0)\) and \(\xi^2(K, g_0)\) as the value of the quantity
\[
g(K, g_0) = \frac{-\nu\chi_4(K, g_0)}{\xi^{3/2}(K, g_0)\chi_2^2(K, g_0)} \tag{7}
\]
when \(K\) tends from below to \(K_c(g_0)\). Here \(v\) denotes the volume per lattice site. We have \(v = 1\) for the sc lattice and \(v = 4/3\sqrt{3}\) for the bcc lattice. The expected critical behavior of \(g(K, g_0)\) is
\[
g(K, g_0) = g(g_0)\tau(g_0)^{\gamma(g_0)+3\nu(g_0)-2\Delta_4(g_0)}[1 + c(g_0)\tau^\theta(g_0) + ...] \tag{8}
\]
where \(\Delta_4(g_0)\) is the gap exponent. If \(\gamma(g_0) + 3\nu(g_0) - 2\Delta_4(g_0) = 0\), we say that hyperscaling holds and the critical renormalized coupling \(g(g_0)\) is the finite positive5 critical limit of \(g(K, g_0)\). Of course, the exponents \(\gamma(g_0), \nu(g_0), \Delta_4(g_0), \theta(g_0)\) and the critical renormalized coupling \(g(g_0)\), as determined from the analysis of the HT series, will appear to be independent of \(g_0\), if universality is valid along the critical line.

### III. NUMERICAL RESULTS

#### A. The phase diagrams

In Figs. 1 and 2 we have mapped out the phase diagrams of the \(\phi^4_3\) model on the sc and the bcc lattices, respectively. There are two phases separated by a line \(K = K_c(g_0)\) of second-order critical points. In the disordered phase, below the critical line, the reflection symmetry \(\phi \rightarrow -\phi\) is unbroken. For \(g_0 = 0\) the model reduces to the Gaussian model and we have \(K_c(0) = 2/q = 1/3\) on the sc lattice \((K_c(0) = 2/q = 1/4)\) on the bcc lattice, where \(q\) is the lattice coordination number. For \(g_0 \rightarrow \infty\) the model reduces to the usual spin-1/2 Ising model and we have \(K_c(\infty) = 0.221655(2)\) on the sc lattice \((K_c(\infty) = 0.1573725(10)\) on the bcc lattice) as indicated in our general study5 of the spin-S Ising model. In Table 1 we have reported numerical estimates of \(K_c(g_0)\) for a few values of \(0 < g_0 < \infty\) in the sc and the bcc lattice cases. These estimates are obtained by the following simple procedure. Using the susceptibility expansion, for each value of \(g_0\), we form the sequence \(\{K_c^{(n)}(g_0)\}\) of Zinn-Justin modified-ratio approximants (MRA)14 of the critical point. We observe that these sequences approach smoothly their expected asymptotic behavior5
\[
K_c^{(n)}(g_0) = K_c(g_0)[1 - C(\gamma, \theta)a(g_0)/n^{1+\theta(g_0)} + o(1/n^{1+\theta(g_0)})] \tag{9}
\]
where \(a(g_0)\) and \(\theta(g_0)\) are, respectively, the amplitude and the exponent of the LCS appearing in eq.(1) and \(C(\gamma, \theta)\) is a positive constant14 depending on the values of \(\gamma(g_0)\) and \(\theta(g_0)\). We expect that the exponent \(\theta(g_0)\) be universal, namely independent of \(g_0\) and indeed, to a good approximation, the asymptotic behavior of the MRA sequences is consistent with the value \(\theta = 0.517(4)\) suggested by a recent simultaneous study15 of a set of models in the Ising universality class. Since, for all values of \(g_0\), the contributions of the higher-order terms in eq.(1) appear to be small, particularly so in the bcc lattice case, we shall assume
that the MRA sequences \( \{K_c^{(n)}(g_0)\} \) can be extrapolated to infinite length \( n \) of the series simply by fitting the highest-order (alternate) approximants to the first two terms of the asymptotic expansion \([9]\). The values of \( K_c(g_0) \) determined by this prescription are quite stable, both under small variations of \( \theta \) and of the fitting procedure, such as including in the fit higher-order terms of the expansion \([9]\). Our conclusion is that at least five decimal figures of the result are reliable. These estimates are then refined by comparing the results of the extrapolations with the determination of \( K_c(g_0) \) from first-, second- and third-order differential approximants \([DA]^{14,16}\) and finally error bars are inferred which are likely to be only upper bounds. As expected, these uncertainties, which reflect the differences between the estimates by extrapolated MRA sequences and by DAs, depend strongly on the size of the corrections to scaling and therefore on the value of \( g_0 \), but of course they are generally invisible on the scale of figures \([1] \) and \([2] \).

B. The critical exponents

In Figs. \([3] \) and \([4] \) referring to the sc and the bcc lattices respectively, we have plotted our estimates of the exponent of the susceptibility \( \gamma(g_0) \) vs. \( G1 = \frac{g_0}{g_0+1} \). Also in this case, using the susceptibility HT expansion, for each value of \( g_0 \), we can form the sequences \( \{\gamma^{(n)}(g_0)\} \) of MRAs of the exponent \( \gamma(g_0) \) and compare them to their expected asymptotic expansion

\[
\gamma^{(n)}(g_0) = \gamma(g_0) - D(\gamma, \theta) a(g_0)/n^{\theta(g_0)} + o(1/n^{\theta(g_0)}). \tag{10}
\]

Here \( D(\gamma, \theta) \) is a positive constant depending on \( \gamma(g_0) \) and \( \theta(g_0) \) and \( a(g_0) \) is the amplitude of the LCS in eq.\([4] \). We can observe that also these MRA sequences are smooth, but a simple minded extrapolation based on a fit of the sequences \( \{\gamma^{(n)}(g_0)\} \) to the first two terms of the asymptotic expansion \([11] \) cannot lead to very accurate estimates of \( \gamma(g_0) \), because, even at the present order of expansion, the contribution of higher-order corrections is not sufficiently small. In the figures \([3] \) and \([4] \) we have represented by solid lines the estimates \( \gamma(g_0) \) obtained from the MRAs \( \gamma^{(17)}(g_0) \), \( \gamma^{(21)}(g_0) \) and \( \gamma^{(25)}(g_0) \), which use the susceptibility series only up to the orders 17, 21 and 25 respectively. These estimates show a rapid crossover from the Gaussian value \( \gamma(0) = 1 \) to a behavior which, for \( g_0 > 0 \), tends to become independent of \( g_0 \) and to approach the dashed line in the figure indicating the central value \( \gamma = 1.2371 \) of our estimate in Ref.\([5] \) for the Ising universality class. If we include higher correction terms in the asymptotic expansion \([10] \) somewhat more accurate estimates of \( \gamma(g_0) \) can be obtained. The simplest possibility is to introduce a single effective higher correction \( O(1/n^s) \) with \( g_0 \)-independent exponent \( s \). Choosing the value \( s = 4 \), we obtain the estimates represented by the dotted curve. The accuracy of the approximation can be further improved at the expense of the simplicity of the fitting procedure, but here it is sufficient to indicate only the qualitative trend. In the same figures, we have also reported estimates of \( \gamma(g_0) \) obtained by simple first-order unbiased DAs using all available series coefficients. In order to keep the figures readable, these data are restricted to a smaller range of values of \( g_0 \) of particular interest. Completely analogous results are obtained starting with the HT series for the correlation-length squared \( \xi^2 \) to determine the critical exponent \( \nu \) but, for brevity, they will not be reported here.
C. Leading corrections to scaling and accurate estimates

In Fig. 5 we have plotted vs. \( g_0 \) the amplitude \( a(g_0) \) of the LCS in (11), as obtained by fitting the MRA sequence \( \{ K_c^{(n)}(g_0) \} \) to the asymptotic expansion (9) in the bcc lattice case. Of course, the determination of \( a(g_0) \) is much more sensitive than \( K_c(g_0) \) to the inclusion of higher-order corrections in the fit to the expansion (9). The most convincing results are obtained when a single additional correction \( O(1/n^{6+\epsilon}) \) is included. This three-parameter fit changes the results of the two-parameter fit for \( K_c(g_0) \) only within the estimated uncertainties and does not alter the qualitative structure of \( a(g_0) \), but only brings its evaluation into closer agreement with the corresponding estimate of \( a(g_0) \) from a similarly improved fit to eq. (10) and with the values of \( a(\infty) \) obtained in earlier works. The uncertainty of the final results should not exceed 5 \(-\) 10\%. In the case of the sc lattice, the behavior of \( a(g_0) \) is similar, but \( a(g_0) \) cannot be determined with comparable accuracy by this straightforward method because it is even more sensitive to the presence of higher-order corrections in (9). The important fact is however that, both in the sc and the bcc lattice cases, when \( g_0 \) increases from 0 to \( \infty \), the amplitude \( a(g_0) \) of the LCS in eq. (11) varies from positive to negative values. In the sc lattice case \( a(g_0) \) vanishes at \( \hat{g}_0^{sc} \approx 1.10(2) \). We recall that the above cited numerical analysis of the \( \phi^4 \) HT series on the sc lattice was indeed performed precisely at \( g_0 = 1.10 \). In the bcc lattice case we find that the zero of \( a(g_0) \) occurs at \( \hat{g}_0^{bcc} \approx 1.85(5) \). In Ref. 5 \( \hat{g}_0^{sc} \) had been determined by an appropriate MonteCarlo simulation. In our approach no preliminary MonteCarlo simulation is needed to determine \( \hat{g}_0^{bcc} \), since, knowing the structure of the function \( a(g_0) \), our study of the model on the whole range of values of \( g_0 \) directly gives a sufficiently accurate indication. For \( g_0 \) in a neighborhood of the zero of the LCS amplitude, we can observe that the various approximations of \( \gamma(g_0) \), shown in Figs. 3 and 4 tend to coincide and come nearest to our earlier estimate for the spin Ising model: \( \gamma = 1.2371(1) \) shown in the figure by a dashed line. Eq. (11) and the measured behavior of \( a(g_0) \) simply explain why, for \( g_0 < \hat{g}_0^{sc} \), (or \( g_0 < \hat{g}_0^{bcc} \) in the bcc lattice case) the MRAs of Figs. 3 and 4 approach their constant limiting value from below and otherwise from above. Of course, the same mode of approach to the limit can be observed for the estimates obtained from DAs, when varying the number of HT series coefficients used in the approximants, however, for clarity we have reported in the figures only the highest order results.

We can now take advantage of our estimate of \( \hat{g}_0^{bcc} \) to compute some universal critical parameters of the \( \phi^4 \) model on the bcc lattice with approximately vanishing LCS. The HT series coefficients of \( \chi_2, \mu_2 \) and \( \chi_4 \) on the bcc lattice, for \( g_0 = \hat{g}_0^{bcc} \), are reported in Table II. (The corresponding expansions in the sc lattice case, for \( g_0 = \hat{g}_0^{sc} \), can be found in Ref. 5.) Very accurate estimates are thus obtained for the critical exponents and the critical renormalized coupling. In particular, using first- and second-order DAs (either unbiased or biased with the critical value \( K_c(\hat{g}_0^{bcc}) = 0.2441357(5) \)), we find \( \gamma = 1.2373(2), \nu = 0.6301(2), \) \( g = 23.56(3) \) in good agreement with our previous estimates and with the results of Ref. 6.

Our estimated errors account also for the \( \approx 3\% \) uncertainty in the estimate of \( \hat{g}_0^{bcc} \). Since our HT series for \( g(K, g_0) \) is two terms longer, we have also repeated the analysis of Ref. 5 with our sc lattice series computed at \( \hat{g}_0^{sc} = 1.10 \). The final estimates: \( \gamma = 1.2372(2), \nu = 0.6301(2) \) and \( g = 23.56(4) \), evaluated at \( K_c(\hat{g}_0^{sc}) = 0.375097(1) \), are consistent both with our results for the bcc lattice and with those of Ref. 6.
D. The renormalized coupling

In Figs. 6 and 7, which refer to the sc and the bcc lattices respectively, we have plotted the renormalized four-point coupling constant \( g(\xi^2, g_0) \) vs. \( X4 = \frac{\xi^2}{\xi^2 + 1} \) for several fixed values of \( g_0 \). For fixed \( g_0 \) and \( \xi^2 \to \infty \), all curves appear to tend to a \( g_0 \)- and lattice-independent limiting value \( g(\infty, g_0) \) which is consistent with our estimate in Ref. 12 of the critical renormalized coupling in the Ising universality class \( g = 23.52(5) \) indicated in the figure by the dashed line. This fact suggests that the critical renormalized coupling is universal with respect to the self-coupling \( g_0 \) and to the structure of the lattice and moreover that it is non-zero (so that hyperscaling is valid). The curves in the figures 6 and 7 are obtained in a most straightforward way from the highest-order non-defective “simplified differential approximants” (SDA)\(^1\) biased with the values of \( \theta \) and \( K_c(g_0) \). Completely equivalent results are obtained also, slightly more laboriously, using ordinary biased DAs instead of SDAs. The estimated width of the error bars is comparable to the thickness of the lines. Also for \( g(\xi^2, g_0) \), we can observe that, as \( g_0 \) varies from 0 to \( \infty \), the amplitude \( c(g_0) \) of the LCS in \( \xi \), which is negative for small \( g_0 \), changes sign at \( \hat{g}_0 \), as it should, because the ratio \( c(g_0)/a(g_0) \) is expected to be universal. Correspondingly, as shown clearly by these figures, the curves tend to their common limiting value from below if \( g_0 < \hat{g}_0 \) and otherwise from above.

Figs. 8 and 9 for the sc and the bcc lattices respectively, offer a different view of the same data. Here the renormalized coupling \( g(\xi^2, g_0) \) is plotted vs. \( G1 = \frac{g_0}{g_0 + 1} \) for several fixed values of \( \xi^2 \). All curves are monotonically increasing in \( g_0 \) and show a rapid crossover from the Gaussian value \( g(\xi^2, 0) = 0 \) to a shape increasingly flatter as \( \xi^2 \) becomes large. For \( g_0 < \hat{g}_0 \), as \( \xi^2 \to \infty \), the sequence of curves tends from below to the constant \( g(\infty, g_0) \), for \( g_0 > \hat{g}_0 \) the same value is approached from above. We have already stressed, that this pattern of behavior simply reflects the fact that the amplitude \( c(g_0) \) of the LCS in \( g(K, g_0) \) is negative for \( g_0 < \hat{g}_0 \) and positive for \( g_0 > \hat{g}_0 \). Here we should also recall that in a similar plot reported in Ref. 12 (only the case of the bcc lattice case is discussed in that study) the curves show quite a different structure for \( \xi^2 > 64 \). In particular, the curves \( g(\xi^2, g_0) \) in Ref. 12 are not monotonic in \( g_0 \) and their limiting behavior as \( \xi^2 \to \infty \) does not appear to be universal and non-zero. We can see no reason for these anomalous features other than the insufficient length of the ten-term HT series used in the analysis of Ref. 12.

One more view of the same data is presented in Figs. 10 and 11 for the sc and the bcc lattices respectively, showing contour plots of \( g(\xi^2, g_0) \) in the \( \xi^2 = \frac{\xi^2}{\xi^2 + 1} \), \( \hat{g}_0 = \frac{g_0}{g_0 + 1} \) plane. Also these figures are qualitatively different from the corresponding ones of Ref. 12 which show a spurious saddle-point structure and suggest a failure of universality. Nothing like that can be inferred from our updated figures.

E. Conclusions

In conclusion, we have analyzed HT expansions for a one-parameter family of continuous spin models which interpolate between the Gaussian and the spin-1/2 Ising models. In the case of the bcc lattice, we have taken advantage of the parameter dependence of the amplitudes of the LCS to improve the accuracy in the determination of some universal critical quantities. Moreover, we have shown that, in the light of our extended series, all puzzling and unexpected features which emerged from the old HT analysis of Ref. 12 can only
be ascribed to numerical inaccuracies deriving from the use of too short series.

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1 J. P. Van Dyke and W. J. Camp, Phys. Rev. Lett. 35, 323 (1975).
2 G. A. Baker and J. M. Kincaid, J. Stat. Phys. 24, 469 (1981).
3 B.G. Nickel, in Phase Transitions: Cargese 1980, edited by M. Levy, J. C. Le Guillou and J. Zinn-Justin, (Plenum, New York, 1982); B. G. Nickel and J. J. Rehr, J. Stat. Phys. 61, 1 (1990).
4 R. Roskies and P. Sackett, J. Stat. Phys. 49, 447 (1987); M. Lüscher and P. Weisz, Nucl. Phys. B 300, 325 (1988); M. Klomfass, Nucl. Phys. B 421, 621 (1994); H. Meyer-Ortmanns and T. Reisz, [hep-lat/9604006](http://arxiv.org/abs/hep-lat/9604006).
5 P. Butera and M. Comi, Phys. Rev. B 65, 144431 (2002).
6 M. Campostrini, A. Pelissetto, P. Rossi, E. Vicari, Phys. Rev. E 65, 066127 (2002).
7 J. Zinn-Justin, J. Physique (France), 42, 783 (1981).
8 J. H. Chen, M. E. Fisher and B. G. Nickel, Phys. Rev. Lett. 48, 630 (1982); M. E. Fisher and J. H. Chen, J. Physique (France) 46, 1645 (1985).
9 R. Schrader and E. Tränkle, J. Stat. Phys. 25, 269 (1981).
10 C. Bagnuls and C. Bervillier, Phys. Rev. B 41, 402 (1990).
11 M. Barma and M. E. Fisher, Phys. Rev. Lett. 53, 1935 (1984) and Phys. Rev. B 31, 5954 (1985); M. J. George and J. J. Rehr, Phys. Rev. Lett. 53, 2063 (1984); J. K. Kim and A. Patrascioiu, Phys. Rev. D 47, 2588 (1993); G.A. Baker and N. Kawashima, Phys. Rev. Lett. 75, 994 (1995).
12 P. Butera and M. Comi, J. Stat. Phys. 109, 311 (2002); P. Butera, M. Comi and A. J. Guttmann, Phys. Rev. B 67, 054402 (2003); P. Butera and M. Comi, Phys. Rev. B 69, 174416 (2004).
13 J. Lebowitz, Comm. Math. Phys. 35, 87 (1974); J. Glimm and A. Jaffe, Ann. Inst. H. Poincaré A 22, 97 (1975); R. Schrader, Phys. Rev. B 14, 172 (1976); M. Aizenman, Comm. Math. Phys. 86, 1 (1982).
14 A. J. Guttmann, in Phase Transitions and critical Phenomena, edited by C. Domb and J. Lebowitz, (Academic, New York, 1989) Vol. 13.
15 Y. Deng and H.W.J. Blöte, Phys. Rev. E 68, 036125 (2003).
16 D. L. Hunter and G. A. Baker, Phys. Rev. B 7, 3346, 3377 (1973), and *ibid.* B 19, 3808 (1979); M. E. Fisher and H. Au-Yang, J. Phys. A 12, 1677 (1979) and *ibid.* A 13, 1517 (1980).
17 P. Butera and M. Comi, Phys. Rev. B 60, 6749 (1999).
FIG. 1: Phase diagram of $\phi^4_3$ on the sc lattice. The critical line $K = K(g_0)$ is plotted vs. $G1 = \frac{g_0}{g_0 + 1}$.

FIG. 2: Phase diagram of $\phi^4_3$ on the bcc lattice. The critical line $K = K(g_0)$ is plotted vs. $G1 = \frac{g_0}{g_0 + 1}$.
FIG. 3: The susceptibility exponent $\gamma(g_0)$ as a function of $G_1 = \frac{g_0}{g_0+1}$ for the sc lattice. The continuous curves represent the estimates obtained by the MRAs of orders 17, 21 and 25 respectively. The dashed curve indicates the central value of our estimate of $\gamma$ for the Ising universality class. The dotted curve shows the results of an extrapolation of the MRA sequences by a fit to the asymptotic expansion $O(1/n^4)$ supplemented by a single higher-order correction $O(1/n^4)$. The triangles show estimates of $\gamma(g_0)$ by simple unbiased first-order DAs.

FIG. 4: The same as in Fig.3 but for the bcc lattice.
FIG. 5: The amplitude $a(g_0)$ of the leading correction to scaling in $\phi_1$ vs. $G_1 = \frac{g_0}{g_0 + 1}$, as obtained from a fit to the asymptotic expansion $\phi_0$ supplemented by a single higher-order correction $O(1/n^{6+\theta})$ in the bcc lattice case.

FIG. 6: The renormalized coupling $g(\xi^2, g_0)$ as function of $X4 = \frac{\xi^4}{\xi^2 + 4}$ for several fixed values of the $\phi^4$ self-coupling $g_0$ on the sc lattice. The values of $g_0$ increase from the lowest curve up. The dashed line represents the central value of our previous estimate $g = 23.52(5)$, obtained in Ref. 5, of the renormalized coupling in the Ising universality class.
FIG. 7: The same as in Fig. 6 but for the bcc lattice.

FIG. 8: The renormalized coupling $g(\xi^2, g_0)$ as function of $G_1 = \frac{g_0}{g_0 + 1}$ for several values of the correlation-length square $\xi^2$ on the sc lattice. The values of $\xi^2$ are indicated by the symbols attached to the corresponding curves. The asterisk labels the curve for $\xi^2 = 1$, the empty rhomb for $\xi^2 = 2$, the full rhomb for $\xi^2 = 4$, the empty circle for $\xi^2 = 9$, the full circle for $\xi^2 = 16$, the empty square for $\xi^2 = 36$, the full square for $\xi^2 = 100$, the empty triangle for $\xi^2 = 256$, the full triangle for $\xi^2 = 900$. The dashed line represents the central value of our estimate $g = 23.52(5)$, obtained in Ref. 5 of the renormalized coupling in the Ising universality class.
FIG. 9: The same as in Fig. 8 but for the bcc lattice.

FIG. 10: Contour plot of the renormalized coupling $g = g(\xi^2, g_0)$ in the $G_4$, $X_4$ plane for the sc lattice. Here $G_4 = \frac{g_0}{g_0 + 4}$ and $X_4 = \frac{\xi^2}{\xi^2 + 4}$. The values of $g(\xi^2, g_0)$ are shown beside the corresponding level curves.
FIG. 11: The same as in Fig.10 but for the bcc lattice.
TABLE I: $K_c(g_0)$ for a few values of of $g_0$ in the case of the sc and bcc lattices.

| $g_0$ | $K_{c}^{sc}(g_0)$ | $K_{c}^{bcc}(g_0)$ |
|-------|-------------------|-------------------|
| 0.000 | 0.3333333        | 0.250000          |
| 0.010 | 0.34055(1)        | 0.25471(1)        |
| 0.050 | 0.35940(1)        | 0.26724(1)        |
| 0.100 | 0.37340(1)        | 0.27655(1)        |
| 0.150 | 0.38238(1)        | 0.28242(1)        |
| 0.200 | 0.38842(1)        | 0.28627(1)        |
| 0.250 | 0.39250(1)        | 0.28876(1)        |
| 0.300 | 0.39517(1)        | 0.29029(1)        |
| 0.350 | 0.396782(4)       | 0.29108(1)        |
| 0.400 | 0.397585(3)       | 0.29131(1)        |
| 0.500 | 0.397397(3)       | 0.29055(1)        |
| 0.650 | 0.394135(3)       | 0.28739(1)        |
| 0.760 | 0.390343(3)       | 0.284131(4)       |
| 0.900 | 0.384503(3)       | 0.279335(3)       |
| 1.060 | 0.377035(2)       | 0.273375(3)       |
| 1.100 | 0.375096(2)       | 0.271844(2)       |
| 1.140 | 0.373137(3)       | 0.270306(2)       |
| 1.300 | 0.365220(3)       | 0.264137(2)       |
| 1.600 | 0.350584(3)       | 0.252883(1)       |
| 1.800 | 0.341319(3)       | 0.245830(1)       |
| 1.850 | 0.339085(3)       | 0.2441357(5)      |
| 2.000 | 0.332596(3)       | 0.239230(1)       |
| 3.000 | 0.297990(3)       | 0.213328(1)       |
| 4.100 | 0.274459(3)       | 0.195919(1)       |
| 5.000 | 0.262538(3)       | 0.187155(2)       |
| 6.400 | 0.251303(3)       | 0.178926(2)       |
| 10.000| 0.238853(4)       | 0.169848(2)       |
| 15.000| 0.232587(4)       | 0.165296(2)       |
| 45.000| 0.225112(4)       | 0.159875(2)       |
TABLE II: The HT expansions of the susceptibility $\chi_2$, the second moment of the correlation-function $\mu_2$ and the zero-momentum four-point function $\chi_4$ evaluated at $g_0 = 1.85$ on the bcc lattice.

| order | $\chi_2$   | $\mu_2$     | $\chi_4$       |
|-------|------------|-------------|----------------|
| 0     | 0.607989217187788034965 | 0.000000000000000000000 | -0.53014972377688222012 |
| 1     | 2.957207105732954317729 | 2.957207105732954317729 | -10.3141007042348207915 |
| 2     | 13.09429900741101048493 | 28.76720523422789148965 | -116.861963540111359343 |
| 3     | 57.79323569148811046867 | 197.714175506477337902 | -1037.09279795438675000 |
| 4     | 248.2179229506115615651 | 1185.408945561549742349 | -7930.113134914392838073 |
| 5     | 1064.858590911418926754 | 6551.599918767001108148 | -54882.22830020753646424 |
| 6     | 4512.534104619384245591 | 34421.021354593939522 | -353349.351394346529616 |
| 7     | 19111.2595155995320831 | 174321.7172035645509269 | -2155038.3928645744076 |
| 8     | 80383.21948173478800467 | 859681.096551254253637 | -12593879.17023276266910 |
| 9     | 337958.2776471983168127 | 415152.3524934172704111 | -7112251.1674907214398 |
| 10    | 1414677.216907545784892 | 19721977.30824040566487 | -390463929.572427479526 |
| 11    | 5919934.51172996630729 | 92423661.55059636566134 | -2093673545.07037911244 |
| 12    | 24698142.58226079299701 | 428322758.4354179679972 | -11002848880.38386524575 |
| 13    | 103016201.64360732234346 | 1966128942.161860887810 | -56833474652.325228348381 |
| 14    | 428735652.9640138248084 | 8952466557.867528601072 | -28918105755.1432464575 |
| 15    | 1783970290.009394230477 | 40476319807.17866188775 | -145214474869.861166239 |
| 16    | 7410672453.259276817146 | 18188533500.9131802296 | -7207354502435.721114810 |
| 17    | 30779127150.29510373162 | 812879825813.3598744239 | -35401801340419.05097854 |
| 18    | 12766834328.7040885522 | 3615487616835.276408009 | -17227386971328.5865152 |
| 19    | 529480497199.5111942304 | 16011252152185.13150869 | -831302587175201.5552329 |
| 20    | 2193593476094.224975766 | 70632026759664.65740219 | -3980914502340770.0707282 |
| 21    | 9086796325630.050694943 | 310488939709344.3260940 | -18931663273718966.90898 |
| 22    | 37608562966496.06439966 | 1360524482339925.373685 | -89460744784166538.46043 |
| 23    | 15563884991947.9034983 | 5944225695917982.872040 | -420283001120399961.6139 |
| 24    | 643624490532272.2532159 | 2590150603362878.72895 | -2661386657660504.869714 |
| 25    | 2661386657660504.869714 | 112585550276346019.8717 |