Developments in perfect simulation of Gibbs measures through a new result for the extinction of Galton-Watson-like processes

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Abstract

This paper deals with the problem of perfect sampling from a Gibbs measure with infinite range interactions. We present some sufficient conditions for the extinction of processes which are like supermartingales when large values are taken. This result has deep consequences on perfect simulation, showing that local modifications on the interactions of a model do not affect simulability. We also pose the question to optimize over a class of sequences of sets that influence the sufficient condition for the perfect simulation of the Gibbs measure. We completely solve this question both for the long range Ising models and for the spin models with finite range interactions.

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1 Introduction

In this paper we deal with the problem of perfect simulation of Gibbs measures. The first algorithm of this kind was realized by [PW96]. This paper opened a new field of research which is evolving in different directions. In [MG98], the authors extended the results of [PW96] to continuous state space. In [HS00], the study of perfect sampling from a Gibbs measure started and in [DSP08] the authors showed the importance of percolation in perfect simulation algorithms for Gibbs measures with finite range interactions. In [CFF02], the authors dealt with long memory processes which means that the state of the process at a fixed time depends on all its past history. In [GLO10], the authors considered the problem of perfect sampling from a Gibbs measure with infinite range interactions.

We start from the paper [GLO10] and we pose new questions. The algorithm described in [GLO10] is based on a probability distribution that we improve. It, in our paper, depends on the choice of a sequence of growing sets having appropriate properties. In Section 3, we pose the question to optimize over this sequence. We completely solve the problem in the case of finite range interactions and in the case of infinite range Ising models (see Theorem 7 and Remark 2). In Theorem 6, we show that there always exists an optimal choice that in general one is not able to calculate. In Theorem 7 specialized for the Ising model, we make explicit the best sequence of these growing sets.

In Section 4, we present some sufficient conditions for the extinction of a discrete process with values in $\mathbb{N}$. Theorem 8 presents this result and it has applications in various areas. The assumptions of Theorem 8 are weaker than the ones for the extinction of Galton-Watson process which is solved as a particular case (see [Wil91] for Galton-Watson process). This result has implications for the perfect simulation algorithm, see Theorem 1, because it supplies a weaker sufficient condition for the applicability of the algorithm, than the condition given in [GLO10]. Finally, we establish an equivalence relation among interactions in the sense that two interactions are equivalent if they only differ on a finite region. By Theorem 4, we prove that, given two equivalent interactions, if one respects the sufficient condition for the perfect sampling, then the other one satisfies it too.

In Appendix A, we provide the pseudo code of the algorithm, for the Ising model, which calculates the optimal sequence of growing sets and, at the same time, builds a perfect sampling from the Gibbs measure observed on a finite window.
2 Synopsis

Let \( S = \{-1, 1\}^{\mathbb{Z}^d} \) be the set of spin configurations. We endow \( S \) with \( \mathcal{S} \), the \( \sigma \)-algebra generated by cylinders. A point \( v \in \mathbb{Z}^d \) is called vertex. Let \( \sigma(v) \in \{-1, 1\} \) be the value of the configuration \( \sigma \in S \) at vertex \( v \in \mathbb{Z}^d \), and let \( \sigma^v \in \{-1, 1\} \) be the value of the configuration modified in \( v \), i.e.

\[
\sigma^v(u) = \sigma(u) \text{ for all } u \neq v, \quad \sigma^v(v) = -\sigma(v).
\]

We write \( A \subset \mathbb{Z}^d \) to denote that \( A \) is a finite subset of \( \mathbb{Z}^d \). The cardinality of a set \( A \) is indicated with \( |A| \). An interaction is a collection of real numbers \( J = \{ J_B \in \mathbb{R} : B \subset \mathbb{Z}^d, |B| \geq 2 \} \) such that

\[
\sup_{v \in \mathbb{Z}^d} \sum_{B \ni v} |B||J_B| < \infty.
\]

(1)

We denote by \( J \) the collection of all the interactions. Note that in literature more general definitions of interactions are considered but in our paper we will only use this more restrictive definition, as done also in [GLO10].

For brevity of notation set \( \chi_B(\sigma) = \prod_{v \in B} \sigma(v) \) for any \( B \subset \mathbb{Z}^d \) and \( \sigma \in S \). A probability measure \( \pi \) on \((S, \mathcal{S})\) is said to be a Gibbs measure relative to the interaction \( J \in J \) if for all \( v \in \mathbb{Z}^d \) and for any \( \zeta \in S \)

\[
\pi(\sigma(v) = \zeta(v) | \sigma(u) = \zeta(u) \ \forall u \neq v) = \frac{1}{1 + \exp(-2\sum_{B : v \in B} (J_B \chi_B(\sigma)))} \ a.s.
\]

(2)

which are called local specifications.

Let us define the set \( A_v = \{ B \subset \mathbb{Z}^d : v \in B, J_B \neq 0 \} \), for \( v \in \mathbb{Z}^d \); the set \( A_v \) is finite or countable, therefore we can write \( A_v = \{ A_{i,v} : i < N_v + 1 \} \) where \( N_v = |A_v| \). We now introduce a sequence of sets with appropriate properties that will replace the balls with distance \( L^1 \) used in [GLO10].

Let \( B_v = (B_v(k) \subset \mathbb{Z}^d : k \in \mathbb{N}) \), for \( v \in \mathbb{Z}^d \), be a sequence of finite subsets in \( \mathbb{Z}^d \) such that

1) \( B_v(0) = \{v\} \);

2) \( B_v(k) \subset B_v(k + 1) \) and \( B_v(k + 1) \setminus B_v(k) \neq \emptyset \), for \( k \in \mathbb{N} \);
3) $\bigcup_{k \in \mathbb{N}} B_v(k) \supset \bigcup_{A \in A_v} A = \bigcup_{i < N_v + 1} A_{i,v}$.

We denote by $B_v$ the space of the sequences verifying 1), 2) and 3).

In [GLO10] a perfect simulation algorithm for a Gibbs measure $\pi$ with long range interaction is presented. It can be divided into two steps: the backward sketch procedure and the forward spin procedure. For the applicability of the algorithm they only have to assume a condition on the first part, i.e. on the backward sketch procedure. The algorithm is defined through a Glauber dynamics having $\pi$ as reversible measure. A process $(\sigma_t(v), v \in \mathbb{Z}^d, t \in \mathbb{R})$ taking values in $S$ and having such dynamics, will be constructed. For any $v \in \mathbb{Z}^d$, $\sigma \in S$ and $J \in J$ let $c_{v,J}(\sigma)$ be the rate at which the spin in $v$ flips when the system is in the configuration $\sigma$,

$$c_{v,J}(\sigma) = \exp \left( - \sum_{B: v \in B} J_B \chi_B(\sigma) \right).$$

The generator $G_J$ of the process is defined on cylinder functions $f : S \to \mathbb{R}$ as follows

$$G_J f(\sigma) = \sum_{v \in \mathbb{Z}^d} c_{v,J}(\sigma) [f(\sigma^v) - f(\sigma)].$$

Assumption (1) implies the uniform boundedness of the rates $c_{v,J}(\sigma)$ with respect to $v$ and $\sigma$, and

$$\sup_{v \in \mathbb{Z}^d} \sum_{u \in \mathbb{Z}^d} \sup_{\sigma \in S} |c_{v,J}(\sigma) - c_{v,J}(\sigma^u)| < \infty.$$

Hence, Theorem 3.9 of [Lig85] guarantees that $G_J$ is effectively the generator of a Markovian process $(\sigma_t(v), v \in \mathbb{Z}^d, t \in \mathbb{R})$ having $\pi$ as invariant measure.

The difficulty of dealing with a measure with long range interaction is overcome through a decomposition of the rates $c_v(\sigma)$ as a convex combination of local range rates.

To present the decomposition we define two probability distributions. The first one selects a random region of dependence and the second one updates the value of the spins. For $v \in \mathbb{Z}^d$, $J \in J$, let

$$\lambda_{v,J,B_v}(k) = \begin{cases} 
\exp(-2 \sum_{B: v \in B} |J_B|) & \text{if } k = 0, \\
\exp(-\sum_{B: v \in B, B \not\subset B_v(1)} |J_B|) - \exp(-2 \sum_{B: v \in B} |J_B|) & \text{if } k = 1, \\
\exp(-\sum_{B: v \in B, B \not\subset B_v(k)} |J_B|) - \exp(-\sum_{B: v \in B, B \not\subset B_v(k-1)} |J_B|) & \text{if } k \geq 2.
\end{cases}$$

(3)
Note that, for $v \in \mathbb{Z}^d$, $(\lambda_{v,J,B_v}(k) : k \in \mathbb{N})$ is a probability distribution on $\mathbb{N}$ because of properties 1), 2) and 3) of $B_v$.

Moreover, for each $v \in \mathbb{Z}^d$, $\sigma \in S$ and $J \in \mathcal{J}$ let $M_{v,J} = 2 \exp(\sum_{B,v \in B} |J_B|)$,

$$
p_{[0]}^{v,J,B_v}(1) = p_{[0]}^{v,J,B_v}(-1) = \frac{1}{2},
$$

$$
p_{v,J,B_v}^1(-\sigma(v)|\sigma) = \frac{1}{M_{v,J}} \exp(-\sum_{B,v \in B,B \subset B_v(1)} J_B \chi_B(\sigma)) \frac{1 - \exp(-2 \sum_{B,v \in B,B \subset B_v(1)} |J_B|) \exp(-\sum_{B,v \in B,B \subset B_v(1)} |J_B|)}{1 - \exp(-\sum_{B,v \in B,B \subset B_v(1)} |J_B|)},
$$

and for $k \geq 2$

$$
p_{v,J,B_v}^k(-\sigma(v)|\sigma) = \frac{\exp(-\sum_{B,v \in B,B \subset B_v(k-1)} J_B \chi_B(\sigma))}{M_{v,J}} \frac{1 - \exp(-\sum_{B,v \in B,B \subset B_v(k-1)} |J_B|)}{1 - \exp(-\sum_{B,v \in B,B \subset B_v(k-1)} |J_B|)}.
$$

Finally set for any $k \geq 1$

$$
p_{v,J,B_v}^k(\sigma(v)|\sigma) = 1 - p_{v,J,B_v}^k(-\sigma(v)|\sigma).
$$

It is possible with some calculations to prove that $p_{v,J,B_v}^k \in [0,1]$, thus $p_{v,J,B_v}^k(-\sigma(v)|\sigma)$ is a probability distribution on $\{-1,1\}$. The probabilities in (4)-(6) will be used in the forward spin procedure.

Notice that for each $a \in \{-1,1\}$, $p_{v,J,B_v}^{[0]}(a)$ does not depend on $v$ and that, by construction, for any $k \geq 1$, $p_{v,J,B_v}^k(-\sigma(v)|\sigma)$ depends only on the restriction of the configuration $\sigma$ to the set $B_v(k)$. This is an important property that links the backward sketch procedure to the forward spin procedure.

The announced decomposition of the rates $c_{v,J}(\sigma)$ is stated in [GLO10] in the following proposition.

**Proposition 1.** Under condition (1), the following decomposition holds for any $\sigma \in S$

$$
c_{v,J}(\sigma) = M_{v,J} \left[ \frac{\lambda_{v,J,B_v}(0)}{2} + \sum_{k=1}^{\infty} \lambda_{v,J,B_v}(k)p_{v,J,B_v}^k(-\sigma(v)|\sigma) \right].
$$

Now in [GLO10] there is a construction of an auxiliary process that links the Glauber dynamics with the perfect sampling algorithm through decomposition (7).
Later on, for brevity of notation, we will omit the indices $J, B_v$ when there is no ambiguity. The backward sketch procedure constructs a process that we are going to define. Let $M_v$ be the mass associated to each vertex $v$. Let $(C_n)_{n \in \mathbb{N}}$ be a process with homogeneous Markovian dynamics and which takes values on $\mathcal{C} = \{A \subseteq \mathbb{Z}^d\}$. Let $C_0 \subseteq \mathbb{Z}^d$ the set in which we want to observe the perfect sampling from the Gibbs measure with infinite range interaction. If $C_n = \emptyset$ then $C_{n+1} = \emptyset$. If $C_n \neq \emptyset$, then the set $C_{n+1}$ is constructed as follows.

A random vertex $W_n$ is selected, proportionally to its mass, with
\[
P(W_n = w|C_n) = \frac{M_w}{\sum_{z \in C_n} M_z}, \quad \text{for } w \in C_n.
\] (8)

Formula (8) will be used to define more general models in Section 5. Then a random value $K_{w,n}$ is drawn by using the probability distribution $\lambda_w$, thus
\[
P(K_{w,n} = k) = \lambda_w(k), \quad \text{for } k \in \mathbb{N}.
\]

If $K_{w,n} = 0$ then $C_{n+1} = C_n \setminus \{w\}$; if $K_{w,n} = k$, for $k \in \mathbb{N}_+$, then $C_{n+1} = C_n \cup B_w(K_{w,n}) = C_n \cup B_w(k)$. The procedure ends at the first time $m \in \mathbb{N}_+$ such that $C_m = \emptyset$. When this happens, the forward spin procedure begins. Now the value of the spin is assigned to all the vertices visited during the first stage, starting at the last vertex with $k = 0$. The assignment of spins is done by using the update probabilities $p^{[k]}_v$, coming back up to give the definitive value of the spin to the vertices belonging to $C_0$.

The following proposition characterizes the computability of the algorithm and shows that there is an unique condition on the backward sketch procedure and none on the second part of the algorithm.

Proposition 2. The perfect simulation algorithm in [GLO10] generates a random field with distribution $\pi$ if and only if for any $v \in \mathbb{Z}^d$
\[
\limsup_{n \to \infty} C_n = \emptyset \quad \text{a.s.}
\] (9)

Proof. Condition (9) is surely necessary by definition of algorithm. It is also sufficient because it means that the backward sketch procedure stops in a finite number of steps (almost surely), moreover conditions (1) and $p^{[k]}_v(\cdot|\sigma) \in [0, 1]$, which hold by hypothesis and by construction respectively, are sufficient for the forward spin procedure. \qed
A sufficient condition, given in [GLO10], for (9) is

\[(H1) \sup_{v \in \mathbb{Z}^d} \sum_{k=1}^{\infty} |B^*_v(k)|\lambda_v(k) < 1,\]

where $B^*_v(k)$ is the ball, in norm $L^1$, centered in $v$ with radius $k$.

We provide a weaker sufficient condition for (9) than (H1) that is presented in the following theorem.

**Theorem 1.** For a given $\mathbf{J} \in \mathcal{J}$,

(H2) if a collection $\{B_v \in B_v : v \in \mathbb{Z}^d\}$ such that

\[\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{v \notin \Lambda} \sum_{k=1}^{\infty} |B_v(k)|\lambda_v(k) < 1\]

can be constructed, then (9) holds. Hence (H2) is a sufficient condition for the perfect sampling from the Gibbs measure related to $\mathbf{J}$ (see Proposition 2).

In Section 5 we will give the proof of this theorem.

**Remark 1.** For a given $\mathbf{J} \in \mathcal{J}$, if

\[\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{v \notin \Lambda} \min_{B_v \in B_v} \sum_{k=1}^{\infty} |B_v(k)|\lambda_v(k) < 1,\]

then there exists a unique Gibbs measure verifying the local specifications (see (2)). Therefore (H2) can be seen also as a sufficient condition for the uniqueness of the Gibbs measure. In Theorem 6, we prove that the minimum in the previous expression exists. Hence the results on perfect simulation are important also for the study of the transition phase, a classical argument of the statistical mechanics.

### 3 Stochastic ordering for $\lambda_{v,\mathbf{J},B_v}$ and an optimization problem for the perfect simulation

In this section we deal with the optimal choice of $B_v \in B_v$, reaching concrete results. We start with some definitions.
Definition 1. For $v \in \mathbb{Z}^d$, the sequence $B_v \in \mathcal{B}_v$ is less refined than $B_v' \in \mathcal{B}_v$, in symbols $B_v \preceq B_v'$, if $B_v$ is a subsequence of $B_v'$.

This relation between two sequences of $\mathcal{B}_v$ is a partial order. The set $\mathcal{B}_v$ has no minimum, nor maximum, nor even minimal elements; nevertheless it has an uncountable infinite number of maximal elements, corresponding to the sequences of sets which increase by only one vertex at a time.

Let us define, for $v \in \mathbb{Z}^d$, a probability distribution obtained from $\lambda_{v,J,B_v}$ as follows

\[
\hat{\lambda}_{v,J,B_v}(|B_v(l)| - 1) = \lambda_{v,J,B_v}(l), \quad \text{for} \ l \in \mathbb{N},
\]

\[
\hat{\lambda}_{v,J,B_v}(i - 1) = 0, \quad \text{for} \ i \not\in \{|B_v(l)|, l \in \mathbb{N}\}.
\]

Theorem 2. Let $v \in \mathbb{Z}^d$, $J \in \mathcal{J}$, and $B_v, B_v' \in \mathcal{B}_v$ such that $B_v \preceq B_v'$. Then $\hat{\lambda}_{v,J,B_v'} \preceq_{st} \hat{\lambda}_{v,J,B_v}$.

Proof. For brevity of notation we write $\hat{\lambda}_v = \hat{\lambda}_{v,J,B_v}$ and $\hat{\lambda}'_v = \hat{\lambda}_{v,J,B_v'}$. To show the stochastic ordering $\hat{\lambda}'_v \preceq_{st} \hat{\lambda}_v$ we equivalently prove that for each $n \in \mathbb{N},$

\[
F'(n) = \sum_{l=0}^{n} \hat{\lambda}'_v(l) \geq \sum_{l=0}^{n} \hat{\lambda}_v(l) = F(n),
\]

(10)

The functions $F(n)$ and $F'(n)$ are the cumulative distribution functions relative to $\hat{\lambda}_v$ and $\hat{\lambda}'_v$ respectively. They are piecewise constant functions whose jumps occur only in the points of the set $\{|B_v(l)| - 1, l \in \mathbb{N}\}$ and $\{|B_v'(l)| - 1, l \in \mathbb{N}\}$ respectively, i.e.

\[
F(n) = \sum_{l=0}^{j} \hat{\lambda}_v(l) + \sum_{l=j}^{n} \lambda_v(l), \quad \text{where} \ j = \max\{l \in \mathbb{N} : |B_v(l)| - 1 \leq n\},
\]

\[
F'(n) = \sum_{l=0}^{j'} \hat{\lambda}'_v(l) + \sum_{l=j'}^{n} \lambda'_v(l), \quad \text{where} \ j' = \max\{l \in \mathbb{N} : |B_v'(l)| - 1 \leq n\}.
\]

Now we show that for each $m \in \{|B_v(l)| - 1, l \in \mathbb{N}\},$

\[
F(m) = F'(m).
\]

(11)

Let $m \in \{|B_v(l)| - 1, l \in \mathbb{N}\}$, then

\[
F(m) = \sum_{l=0}^{j} \lambda_v(l), \quad \text{where} \ j \text{ is the unique index such that } |B_v(j)| - 1 = m,
\]
\[ F'(m) = \sum_{l=0}^{j'} \lambda'_v(l), \] where \( j' \) is the unique index such that \( |B'_v(j')| - 1 = m, \) from which, by the hypothesis of the theorem,

\[ B_v(j) = B'_v(j'). \quad (12) \]

Note that the following sums are telescopic, hence

\[ \sum_{l=0}^{n} \lambda_v(l) = \exp \left( -\sum_{B,v\in B,B\not\subset B_v(n)} |J_B| \right), \]

\[ \sum_{l=0}^{n} \lambda'_v(l) = \exp \left( -\sum_{B,v\in B,B\not\subset B'_v(n)} |J_B| \right), \quad (13) \]

for \( n \in \mathbb{N}_+. \) Moreover \( F(0) = \hat{\lambda}_v(0) = \lambda_v(0) = \lambda_v(0) = \hat{\lambda}'_v(0) = F'(0). \)

From (12) and (13),

\[ \sum_{l=0}^{j} \lambda_v(l) = \sum_{l=0}^{j'} \lambda'_v(l) \]

immediately follows and it implies (11). Since \( F \) and \( F' \) are nondecreasing, from (11) and

\[ \{|B_v(l)|, l \in \mathbb{N}\} \subset \{|B'_v(l)|, l \in \mathbb{N}\} \]

we obtain (10).

Analogously to \cite{GLO10}, see (H1), we introduce the following quantity that will be used later; we call it birth-death expectation,

\[ \mu_{v,J}(B_v) = \sum_{l=1}^{\infty} |B_v(l)| \lambda_{v,J,B_v}(l) - 1, \]

for \( J \in \mathcal{J}, v \in \mathbb{Z}^d, B_v \in B_v. \)

We are now in the position to present our result concerning the birth-death expectation, it will be involved in conditions (H1) and (H2) for the perfect sampling.

**Corollary 1.** Let \( J \in \mathcal{J}, v \in \mathbb{Z}^d, B_v, B'_v \in B_v \) such that \( B_v \preceq B'_v. \) Then \( \mu_{v,J}(B'_v) \leq \mu_{v,J}(B_v). \)

**Proof.** Let \( J \in \mathcal{J}, v \in \mathbb{Z}^d, B_v, B'_v \in B_v \) such that \( B_v \preceq B'_v \) and let \( \hat{\lambda}_v = \hat{\lambda}_{v,J,B_v}, \)

\( \hat{\lambda}'_v = \hat{\lambda}_{v,J,B'_v} \) be the corresponding measures. Consider two random variables \( X_v \sim \mathcal{L} \hat{\lambda}_v \) and
$X'_v \sim \mathcal{L} \hat{\lambda}_v$. From Theorem 2, it follows that $\mathbb{E}(f(X_v)) \geq \mathbb{E}(f(X'_v))$ for each nondecreasing function $f : \mathbb{N} \to \mathbb{R}$. Note that

$$
\mu_{v,J}(B_v) = \sum_{l=1}^{\infty} |B_v(l)| \lambda_v(l) - 1 = \sum_{l=1}^{\infty} (|B_v(l)| - 1) \lambda_v(l) - \lambda_v(0) \tag{14}
$$

$$
= \sum_{l=1}^{\infty} (|B_v(l)| - 1) \hat{\lambda}_v(|B_v(l)| - 1) - \hat{\lambda}_v(0) = \sum_{i=1}^{\infty} (i - 1) \hat{\lambda}_v(i - 1) - \hat{\lambda}_v(0) = \sum_{i=1}^{\infty} i \hat{\lambda}_v(i) - \hat{\lambda}_v(0),
$$

therefore (14) is the expected value of the random variable $g(X_v)$ where,

$$
g(i) = \begin{cases} 
-1 & \text{if } i = 0, \\
i & \text{if } i \geq 1.
\end{cases} \tag{15}
$$

The function in (15) is nondecreasing. Thus, by the stochastic ordering, $\mu_{v,J}(B'_{v}) = \mathbb{E}(g(X'_v)) \leq \mathbb{E}(g(X_v)) = \mu_{v,J}(B_v)$. \hfill \Box

By the next two theorems, we will see that if an interaction $J$ verifies (H1), then all the interactions obtained from $J$ by changing them on a finite region and by lowering them in absolute value elsewhere, still verify (H2). By Theorem 1, all the Gibbs measures associated to these interactions are perfectly simulable.

**Theorem 3.** Let $v \in \mathbb{Z}^d$, $B_v \in B_v$, $J$, $\tilde{J} \in \mathcal{J}$ such that $|\tilde{J}_B| \leq |J_B|$ for each $B \in \mathbb{Z}^d$. Then $\lambda_{v,J,B_v} \preceq_{st} \lambda_{v,\tilde{J},B_v}$. Hence $\mu_{v,J}(B_v) \leq \mu_{v,\tilde{J}}(B_v)$.

**Proof.** For brevity of notation we write $\lambda_v = \lambda_{v,J,B_v}$ and $\tilde{\lambda}_v = \lambda_{v,\tilde{J},B_v}$. To show the stochastic ordering, we equivalently prove that for each $v \in \mathbb{Z}^d$, $n \in \mathbb{N}$

$$
\sum_{l=0}^{n} \tilde{\lambda}_v(l) \geq \sum_{l=0}^{n} \lambda_v(l).
$$

Since $|\tilde{J}_B| \leq |J_B|$ for each $B \in \mathbb{Z}^d$, then

$$
\tilde{\lambda}_v(0) = \exp\left(-2 \sum_{B:v \in B} |\tilde{J}_B|\right) \geq \exp\left(-2 \sum_{B:v \in B} |J_B|\right) = \lambda_v(0),
$$

and for $n \geq 1$

$$
\sum_{l=0}^{n} \tilde{\lambda}_v(l) = \exp\left(- \sum_{B:v \in B,B \notin B_v(n)} |\tilde{J}_B|\right) \geq \exp\left(- \sum_{B:v \in B,B \notin B_v(n)} |J_B|\right) = \sum_{l=0}^{n} \lambda_v(l).
$$

\hfill \Box
The following result is directly related to our sufficient condition (H2).

**Theorem 4.** Given the interactions $\mathbf{J}, \tilde{\mathbf{J}} \in \mathcal{J}$, if the cardinality of $\mathcal{C} = \{B \in \mathbb{Z}^d : |J_B| \neq |\tilde{J}_B|\}$ is finite, then for $v \in \mathbb{Z}^d$ and $B_v \in B_v$,

$$\limsup_{\Lambda \uparrow \mathbb{Z}^d} \mu_{v, \mathbf{J}}(B_v) = \limsup_{\Lambda \uparrow \mathbb{Z}^d} \mu_{v, \tilde{\mathbf{J}}}(B_v). \quad (16)$$

**Proof.** Note that the measures $\lambda_{v, \mathbf{J}, B_v}$, $\lambda_{v, \tilde{\mathbf{J}}, B_v}$ are equal for each $v$ such that all the finite subsets $B$ containing $v$ do not belong to $\mathcal{C}$. In fact if $\{B \in \mathbb{Z}^d : v \in B, B \in \mathcal{C}\} = \emptyset$, then for each $B$ including $v$ we have $|J_B| = |\tilde{J}_B|$, hence $\lambda_{v, \mathbf{J}, B_v} = \lambda_{v, \tilde{\mathbf{J}}, B_v}$ for each $k \geq 0$. Therefore for $\Lambda \supset \bigcup_{B \in \mathcal{C}} B$,

$$\sup_{v \notin \Lambda} \mu_{v, \mathbf{J}}(B_v) + 1 = \sup_{v \notin \Lambda} \sum_{k=1}^{\infty} |B_v(k)| \lambda_{v, \mathbf{J}, B_v}(k) = \sup_{v \notin \Lambda} \sum_{k=1}^{\infty} |B_v(k)| \lambda_{v, \tilde{\mathbf{J}}, B_v}(k) = \sup_{v \notin \Lambda} \mu_{v, \tilde{\mathbf{J}}}(B_v) + 1. \quad (17)$$

Since the cardinality of $\mathcal{C}$ is finite, then $\bigcup_{B \in \mathcal{C}} B$ is finite. Therefore, passing to the limit in (17) for $\Lambda \uparrow \mathbb{Z}^d$, we obtain (16). \hfill \Box

Condition (H2) says that $\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{v \notin \Lambda} \mu_{v, \mathbf{J}}(B_v) < 0$, therefore we are interested in finding the infimum value $\inf_{x \in B_v} \mu_{v, \mathbf{J}}(x)$.

We define $\mathcal{E}_v$ by distinguishing two cases $N_v = \infty$, $N_v < \infty$. In the first case let $\mathcal{E}_v$ be a subset of $\mathcal{B}_v$ such that each element $(B_v(l))_{l \in \mathbb{N}} \in \mathcal{E}_v$ has the property that there exists a sequence $(i_k)_{k \in \mathbb{N}}$ where $B_v(l) = \bigcup_{k=1}^{N_v} A_{i_k, v}$ for $l \in \mathbb{N}_+$. When $N_v < \infty$, let $\mathcal{E}_v$ be a subset of $\mathcal{B}_v$ such that each element $(B_v(l))_{l \in \mathbb{N}} \in \mathcal{E}_v$ has the property that

$$\exists \bar{l} : B_v(\bar{l}) = \bigcup_{k=1}^{N_v} A_{i_k, v}, \exists (i_1, \ldots, i_{\bar{l}}) : B_v(l) = \bigcup_{k=1}^{l} A_{i_k, v} \forall l \leq \bar{l}. \quad (18)$$

We notice that, for each $l > \bar{l}$, $\lambda_v(l) = 0$ for any choice of $B_v(l)$ verifying 2).

In the next theorem we restrict the research of the infimum from $\mathcal{B}_v$ to $\mathcal{E}_v$. This produces a sensitive improvement when $N_v$ is finite for each vertex $v \in \mathbb{Z}^d$, in this case the infimum is a minimum because there is a finite number of choices in (18), and this fact allows us to calculate it. In any case, in Theorem 6 we will prove that the minimum of $\mu_{v, \mathbf{J}}(x)$ always exists.

We endow $\mathcal{B}_v$ with the discrete topology to consider the limit of a sequence in $\mathcal{B}_v$ in the next two theorems.
Theorem 5. Let $J \in \mathcal{J}$, $v \in \mathbb{Z}^d$, then

$$\inf_{x \in B_v} \mu_{v,J}(x) = \inf_{x \in E_v} \mu_{v,J}(x).$$

Proof. First we consider the case $N_v = \infty$. To prove the theorem we will show that for each $x \in B_v$ there exists $y \in E_v$ such that $\mu_{v,J}(y) \leq \mu_{v,J}(x)$. Starting from $x = (x(l))_{l \in \mathbb{N}} \in B_v$, we will construct a sequence of points $(x^{(n)})_{n \in \mathbb{N}}$ such that $x^{(0)} = x$ and $\lim_{n \to \infty} x^{(n)} = y \in E_v$. We will prove that, for $n \in \mathbb{N}$, $\mu_{v,J}(x^{(n+1)}) \leq \mu_{v,J}(x^{(n)})$ and then, by Fatou’s lemma, $\mu_{v,J}(y) \leq \liminf_{n \to \infty} \mu_{v,J}(x^{(n)})$, from which $\mu_{v,J}(y) \leq \mu_{v,J}(x)$.

Let $x^{(0)} = x = (x(l))_{l \in \mathbb{N}} \in B_v$, we now give the rules to construct $x^{(1)}$. Define

$$k_0 = 1 + \sup\{l \in \mathbb{N}_+ : \exists (i_1, \ldots, i_l) \text{ s.t. } x(j) = \bigcup_{k=1}^{j} A_{i_k,v} \text{ for any } j = 1, \ldots, l\},$$

if $k_0 = \infty$, then $x \in E_v$ and there is nothing to prove. If $k_0 < \infty$ then define the finite sets of indices

$$I = \{i \in \mathbb{N}_+ : A_{i,v} \subset x(k_0)\},$$

$$I^- = \{i \in \mathbb{N}_+ : A_{i,v} \subset x(k_0 - 1)\}.$$

If $I = I^-$ then eliminate $x(k_0)$ from the sequence obtaining $x^{(1)}(l) = x(l)$, for $l \leq k_0 - 1$, $x^{(1)}(l) = x(l + 1)$, for $l \geq k_0$. In this case $\mu_{v,J}(x^{(0)}) = \mu_{v,J}(x^{(1)})$.

If $I \neq I^-$, consider $j = \min\{i : i \in I \setminus I^-\}$, define $x^{(1)}(l) = x(l)$, for $l \leq k_0 - 1$, $x^{(1)}(k_0) = x(k_0 - 1) \cup A_{j,v}$, $x^{(1)}(l) = x(l - 1)$, for $l \geq k_0 + 1$. It is easy to check that the sequence $x^{(1)}$ verify the conditions 1), 2) and 3) defining $B_v$. In this case the sequence $x^{(0)}$ is less refined than $x^{(1)}$, therefore $\mu_{v,J}(x^{(0)}) \geq \mu_{v,J}(x^{(1)})$, by Corollary [□]

We repeat the procedure to construct $x^{(n+1)}$ from $x^{(n)}$, for any $n \in \mathbb{N}_+$. Obviously there exists $\lim_{n \to \infty} x^{(n)} = y \in E_v$. Since $\hat{\lambda}_{v,J,z}(0)$ does not depend on $z \in B_v$ we set $\hat{\lambda}_{v,J}(0) = \hat{\lambda}_{v,J,z}(0)$, therefore we can write

$$\mu_{v,J}(y) = -\hat{\lambda}_{v,J,y}(0) + \sum_{i=1}^{\infty} i\hat{\lambda}_{v,J,y}(i) = -\hat{\lambda}_{v,J}(0) + \sum_{i=1}^{\infty} \liminf_{n \to \infty} i\hat{\lambda}_{v,J,x^{(n)}}(i)$$

$$\leq -\hat{\lambda}_{v,J}(0) + \liminf_{n \to \infty} \sum_{i=1}^{\infty} i\hat{\lambda}_{v,J,x^{(n)}}(i) = \liminf_{n \to \infty} \mu_{v,J}(x^{(n)}) \leq \mu_{v,J}(x),$$

where the first inequality follows by Fatou’s lemma. The case $N_v < \infty$ is simpler and in a finite number $n_0$ of steps one obtains that $x^{(n_0)} \in E_v$. □
Now we state the theorem on the minimum that has a theoretical flavor but we will see that in some important cases the point realizing the minimum can be explicitly calculated.

**Theorem 6.** Let $J \in \mathcal{J}$, $v \in \mathbb{Z}^d$, then

$$\min_{x \in B_v} \mu_{v,J}(x) = \min_{x \in E_v} \mu_{v,J}(x).$$

**Proof.** The theorem is obviously true in the case of $N_v < \infty$, thus we consider $N_v = \infty$. First we prove the existence of $\min_{x \in B_v} \mu_{v,J}(x)$. If for each $x \in B_v$, $\mu_{v,J}(x) = \infty$, there is nothing to show. Suppose that for $x = (x(l))_{l \in \mathbb{N}}$, $\mu_{v,J}(x) \leq c < \infty$.

Define, for any $A_{k,v} \in A_v$,

$$\bar{l}(k) = \min \{ l \in \mathbb{N} : x(l) \supset A_{k,v} \},$$

therefore

$$A_{k,v} \not\subset x(\bar{l}(k) - 1), \ A_{k,v} \subset x(\bar{l}(k)). \quad (19)$$

We will prove that for each $k \in \mathbb{N}_+$

$$\bar{l}(k) \leq \frac{c + 1}{e^{-L+|J_{A_{k,v}}|} - e^{-L}} \lor 2, \quad (20)$$

where $L = \sum_{B:v \in B} |J_B|$. Since $\mu_{v,J}(x) \leq c$, then

$$(\bar{l}(k) + 1)\lambda_{v,J,x}(\bar{l}(k)) \leq \sum_{l=1}^{\infty} |x(l)|\lambda_{v,J,x}(l) = \mu_{v,J}(x) + 1 \leq c + 1, \quad (21)$$

where the first inequality is true because we have only taken a term of the sum and used that $|x(l)| \geq l + 1$.

Let $S_k = \sum_{B:v \in B, B \not\subset x(k-1)} |J_B|$. Since $S_k \in [0, L]$, then

$$e^{-L+|J_{A_{k,v}}|} - e^{-L} \leq e^{-S_k+|J_{A_{k,v}}|} - e^{-S_k} \leq \lambda_{v,J,x}(\bar{l}(k)), \quad (22)$$

where the last inequality follows from (19) and from the expression of $\lambda_{v,J,x}(l)$ (see (3)) for $l \geq 2$. By (21) and (22) we obtain

$$(\bar{l}(k) + 1)(e^{-L+|J_{A_{k,v}}|} - e^{-L}) \leq c + 1,$$
which implies (20).

Now define a sequence \((x^{(n)} \in B_v : n \in \mathbb{N})\), such that \(x^{(0)} = x\), the birth-death expectations \(\mu_v, J(x^{(n)})\) are nonincreasing in \(n\) and \(\lim_{n \to \infty} \mu_v, J(x^{(n)}) = \inf_{x \in B_v} \mu_v, J(x)\). Let \(\bar{l}^{(n)}(k) = \min \{ l \in \mathbb{N} : x^{(n)}(l) \supset A_{k,v}\}\) the analogous of \(\bar{l}(k)\). By (20) there exists a subsequence \((x_1^{(n)} \in B_v : n \in \mathbb{N})\) of \((x^{(n)} \in B_v : n \in \mathbb{N})\) such that \(\bar{l}^{(n)}(1)\) is constant in \(n\). Therefore, by using diagonal method, the sequence \((x_n^{(n)} \in B_v : n \in \mathbb{N})\) admits limit, i.e.

\[
\lim_{n \to \infty} x_n^{(n)} = y \in B_v.
\]

The sequence \((x_n^{(n)} \in B_v : n \in \mathbb{N})\) is a subsequence of the initial one \((x^{(n)} \in B_v : n \in \mathbb{N})\); hence \(\mu_v, J(x_n^{(n)})\) is nonincreasing in \(n\) and \(\lim_{n \to \infty} \mu_v, J(x_n^{(n)}) = \inf_{x \in B_v} \mu_v, J(x)\). Now, by Fatou’s Lemma,

\[
\mu_v, J(y) = -\dot{\lambda}_{v, J, y}(0) + \sum_{i=1}^{\infty} i \dot{\lambda}_{v, J, y}(i) = -\dot{\lambda}_{v, J}(0) + \sum_{i=1}^{\infty} \liminf_{n \to \infty} i \dot{\lambda}_{v, J, x_n^{(n)}}(i) \\
\leq -\dot{\lambda}_{v, J}(0) + \liminf_{n \to \infty} \sum_{i=1}^{\infty} i \dot{\lambda}_{v, J, x_n}(i) = \inf_{x \in B_v} \mu_v, J(x).
\]

Therefore

\[
\mu_v, J(y) = \inf_{x \in B_v} \mu_v, J(x),
\]

that implies the existence of the minimum on \(B_v\).

For \(y \in B_v\) as in (23), there exists \(z \in E_v\) such that \(\mu_v, J(y) \geq \mu_v, J(z)\) (see the proof of Theorem [5]). It is immediately seen that \(\mu_v, J(z) = \inf_{x \in E_v} \mu_v, J(x)\). \(\square\)

In some cases it is possible to identify the sequence \(z_v \in B_v\) such that \(\mu_v, J(z_v) = \min_{x \in B_v} \mu_v, J(x)\). We present a result on the Ising models in which it occurs. Let \(J_2 \subset J\) be the set of the interactions such that

\[
J_B \neq 0 \Rightarrow |B| = 2.
\]

Note that under condition (11) one gets

\[
\limsup_{\Lambda \uparrow \mathbb{Z}^d, u \notin \Lambda} J_{\{v,u\}} = 0 \text{ for all } v \in \mathbb{Z}^d,
\]

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therefore it can be written, for a fixed vertex $v$,

$$|J_{\{v,w_1\}}| \geq |J_{\{v,w_2\}}| \geq \ldots \geq |J_{\{v,w_i\}}| \geq \ldots$$

(24)

where $\bigcup_{i=1}^{\infty} w_i = \mathbb{Z}^d$ and $w_i \neq w_j$ if $i \neq j$. Let $(w_n \in \mathbb{Z}^d : n \in \mathbb{N}_+)$ be the sequence written in (24) and define $z_v \in B_v$ such that, for $i \in \mathbb{N}_+$,

$$z_v(i) = \{v, w_1, \ldots, w_i\}.$$

Let also $J^{(n)}_v = |J_{\{v,w_n\}}|$ for $n \in \mathbb{N}_+$. We remark that, given an interaction $J \in J_2$, the sequence $(w_n \in \mathbb{Z}^d : n \in \mathbb{N}_+)$ is not in general unique.

**Theorem 7.** Let $J \in J_2$, for each $v \in \mathbb{Z}^d$,

$$\mu_{v,J}(z_v) = \min_{x \in B_v} \mu_{v,J}(x) = -2e^{-2\sum_{i=1}^{\infty} J^{(i)}_{\{v,w_i\}}} + e^{-\sum_{i=2}^{\infty} J^{(i)}_{\{v,w_i\}}} + \sum_{l=2}^{\infty} l \left( e^{-\sum_{i=l+1}^{\infty} J^{(i)}_{\{v,w_i\}}} - e^{-\sum_{i=l}^{\infty} J^{(i)}_{\{v,w_i\}}} \right).$$

(25)

**Proof.** First notice that $z_v$ is a maximal element of $B_v$ and $z_v \in E_v$, moreover $\lambda_{v,J,z_v} = \hat{\lambda}_{v,J,z_v}$. To prove the theorem we will show that, for each maximal element $x \in B_v$, we obtain $\lambda_{v,J,z_v} \preceq_{st} \lambda_{v,J,x}$. Hence by Theorem 2 we will get the first equality of (25).

Let $x$ be a maximal element of $B_v$, as in the proof of Theorem 2, we show that for any $n \in \mathbb{N}$

$$\sum_{l=0}^{n} \lambda_{v,J,z_v}(l) \geq \sum_{l=0}^{n} \lambda_{v,J,x}(l),$$

(26)

which guarantees the stochastic ordering. The l.h.s. of (26) is

$$\exp \left( -\sum_{B:v \in B,B \not\subset z_v(n)} |J_B| \right) = \exp \left( -\sum_{l=n+1}^{\infty} |J_{\{v,w_l\}}| \right).$$

Consider the sequence of distinct vertices $\{u_n \in \mathbb{Z}^d : n \in \mathbb{N}_+\}$ such that $u_0 = \{v\}$ and $u_n = x(n) \setminus x(n-1)$. Since the r.h.s. of (26) can be written

$$\exp \left( -\sum_{B:v \in B,B \not\subset x(n)} |J_B| \right) = \exp \left( -\sum_{l=n+1}^{\infty} |J_{\{v,u_l\}}| \right).$$

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then inequality (26) is equivalent to

\[
\exp \left( - \sum_{l=n+1}^{\infty} |J_{\{v,w\}}| \right) \geq \exp \left( - \sum_{l=n+1}^{\infty} |J_{\{v,u\}}| \right)
\]
or

\[
\exp \left( \sum_{l=1}^{n} |J_{\{v,w\}}| \right) \geq \exp \left( \sum_{l=1}^{n} |J_{\{v,u\}}| \right)
\]
that is obviously true by using the definition of sequence \{w_n\} (see (24)).

The second equality in (25) follows by elementary calculations.

**Remark 2.** If for any \(v \in \mathbb{Z}^d\) the number \(N_v\) is small and if it can be proved that for some \((x_v \in B_v)_{v \in \mathbb{Z}^d}\)

\[
\limsup_{\Lambda \uparrow \mathbb{Z}^d, v \notin \Lambda} \mu_v, J(x_v) < 0,
\]
then the perfect simulation algorithm can be run. Proving (24) is a little easier than proving condition (H1), and in both cases it should be done a priori. In the backward sketch procedure a random vertex \(w\) is selected with probability (8), now the algorithm calculates all the \(\hat{x}_w\)'s belonging to \(\arg\min_{x \in E_w} \mu_w, J(x)\) with a finite number of elementary operations because, for any \(x \in E_w\), \(\lambda_w, J, x(l)\) must be calculated for \(l = 1, \ldots, N_w\) and also all the sums involved in the definition of \(\lambda_w, J, x\) and of \(\mu_w, J(x)\) are finite. Moreover \(|E_w| \leq N_w!\). By comparing the finite list (having at most \(N_w!\) elements) of \(\mu_w, J(x)\) with \(x \in E_w\), the algorithm finds all the \(\hat{x}_w\)'s belonging to \(E_w\) such that \(\mu_w, J(\hat{x}_w) = \min_{x \in E_w} \mu_w, J(x)\). This procedure is repeated for all the selected vertices, which are almost surely finite. Hence the problem is computable and the previous procedure is really an algorithm. The computability is guaranteed by the fact that \(N_v\) is finite, further the algorithm runs in reasonable time if \(N_v\) is small.

If \(N_v\) is large or equal to infinity, if one succeeds in calculating a \((x_v \in B_v)_{v \in \mathbb{Z}^d}\) such that condition (27) is satisfied, then the algorithm can use this particular choice.

The case \(N_v = \infty\) is in some sense theoretical but there are models in which a change of the first terms of a given sequence \(B_v\) may produce a sensitive improvement for \(\mu_v, J(B_v)\), i.e. it goes from positive values to negative values. For simplicity of the exposition we only consider translation invariant models. Let us assume that, for a fixed \(\hat{B}_v \in B_v\),

\[
\sum_{k=1}^{\infty} |\hat{B}_v(k)| \lambda_v, J, \hat{B}_v(k) < \infty.
\]
Hence, for \( N \in \mathbb{N} \), we consider the finite subset \( \Upsilon_N(\hat{\mathcal{B}}_v) \) of \( \mathcal{B}_v \) made by all the sequences verifying these rules:

- \( \hat{\mathcal{B}}_v(0) \subset \hat{\mathcal{B}}_v(1) \subset \ldots \subset \hat{\mathcal{B}}_v(L) = \hat{\mathcal{B}}_v(N) \), with \( L = |\hat{\mathcal{B}}_v(N)| - 1 \);
- \( \tilde{\mathcal{B}}_v(L + i) = \hat{\mathcal{B}}_v(N + i) \), for \( i \geq 1 \).

It is easy to calculate

\[
\min_{x \in \Upsilon_N(\hat{\mathcal{B}}_v)} \mu_v, J(x),
\]

and for each \( x \in \Upsilon_N(\hat{\mathcal{B}}_v) \) there exists \( y \in \Upsilon_{N+1}(\hat{\mathcal{B}}_v) \) that is more refined than \( x \). Therefore increasing \( N \) the minimum in the previous formula can only decrease, via Theorem 2.

We conclude with a more explicit example. Let \( d = 2 \), consider \( \mathcal{B}_v^* \) (the sequence of balls chosen in \([GLO10]\)), suppose that \((28)\) holds and take \( N = 1 \). Note that \( \mathcal{B}_v^* \) is less refined than each sequence in \( \Upsilon_1(\mathcal{B}_v^*) \). With simple calculations we obtain

\[
\mu_v, J(\hat{\mathcal{B}}_v) = \mu_v, J(\mathcal{B}_v^*) - \sum_{i=1}^{3} (|\mathcal{B}_v^*(1)| - |\tilde{\mathcal{B}}_v(i)|) \lambda_v, J, \tilde{\mathcal{B}}_v(i),
\]

where \( \tilde{\mathcal{B}}_v \in \Upsilon_1(\mathcal{B}_v^*) \).

Notice that \( |\mathcal{B}_v^*(1)| = 5 \) and \( (|\mathcal{B}_v^*(1)| - |\tilde{\mathcal{B}}_v(i)|) \geq 1 \) for \( i = 1, 2, 3 \). Now one can take \( \tilde{\mathcal{B}}_v(1), \tilde{\mathcal{B}}_v(2), \tilde{\mathcal{B}}_v(3), \tilde{\mathcal{B}}_v(4) \) among the \( 4! \) possible choices selecting the one which maximizes the sum in \((29)\). For some interactions \( J \in \mathcal{J} \) the next inequalities \( \mu_v, J(\mathcal{B}_v^*) > 0 \) and \( \mu_v, J(\hat{\mathcal{B}}_v) < 0 \) hold.

4 A general result on the extinction of a population

The following theorem gives a generalization of the extinction result on Galton-Watson’s process and it applies to processes that behave like a supermartingale when they assume large values.

In the following theorem we will write for brevity of notation \( i_k^h \) in place of the vector \((i_h, \ldots, i_k)\), for \( h \leq k \). Furthermore, the equalities or inequalities between conditioned probabilities have to be considered valid only if the conditioning events have positive measure.

For each null event \( A \) we pose \( \mathbb{P}(\cdot \mid A) = 1 \), in this way we can write the infimum in place of the essential infimum.
Theorem 8. Let $X = (X_n : n \in \mathbb{N})$ be a stochastic process over \( \mathbb{N} \). Suppose that there exists \( N \in \mathbb{N} \) such that the following relations hold:

1) \( \mathbb{P}(X_{n+1} = 0|X_n = 0) = 1 \), for \( n \in \mathbb{N} \);

2) for \( i \leq N \) there exists \( n_i \in \mathbb{N}_+ \) such that

\[
q_i = \inf_{m \in \mathbb{N}, i_0, \ldots, i_{m-1} \in \mathbb{N}_+} \mathbb{P}(X_{m+n_i} = 0|X_0 = i_0, \ldots, X_m = i_m) > 0, \ i_m = i;
\]

3) \( \mathbb{E}(X_{n+1}|X_0 = i_0, \ldots, X_n = i_n) \leq i_n \) a.s. for \( n \in \mathbb{N}, i_0, \ldots, i_{n-1} \in \mathbb{N}, i_n > N \);

4) \( p_i = \inf_{m \in \mathbb{N}, i_0, \ldots, i_{m-1} \in \mathbb{N}_+} \mathbb{P}(X_{m+1} \neq i|X_0 = i_0, \ldots, X_m = i_m) > 0, \ i_m = i > N \).

Then

\[
\lim_{n \to \infty} X_n = 0 \text{ a.s.}
\]

Proof. Let \( A = \{0, 1, \ldots, N\} \), \( B = \{N+1, N+2, \ldots\} \) where \( N \) is given in the theorem. Let us define

\[
T_{A \to B}^{(l)} = \inf\{n \geq 0 : X_n \in B\}, \quad T_{B \to A}^{(l)} = \inf\{n > T_{A \to B}^{(l)} : X_n \in A\},
\]

\[
T_{A \to B}^{(h)} = \inf\{n > T_{B \to A}^{(h-1)} : X_n \in B\}, \quad T_{B \to A}^{(h)} = \inf\{n > T_{A \to B}^{(h)} : X_n \in A\},
\]

for \( h \geq 2 \).

The random variables \( T_{A \to B}^{(h)}, T_{B \to A}^{(h)} \), for each \( h \geq 1 \), are stopping time. We put \( T_{A \to B}^{(h)} = \infty \) if the set, on which the infimum is defined, is empty or if \( T_{B \to A}^{(h-1)} = \infty \). Similarly we write \( T_{B \to A}^{(h)} = \infty \) if the set, on which the infimum is defined, is empty or if \( T_{A \to B}^{(h)} = \infty \). The following inequalities are obtained directly by definitions in \( (30) \)

\[
T_{A \to B}^{(1)} \leq T_{B \to A}^{(1)} \leq T_{A \to B}^{(2)} \leq \cdots \leq T_{A \to B}^{(h)} \leq T_{B \to A}^{(h)} \leq \cdots
\]

The previous inequalities are strict until one of these stopping times becomes infinite.

Let us define the stopped process \( (Y_n^{(m)} = X_{n\wedge T_{A \to B}^{(m)}} : n \in \mathbb{N}) \) on \( \{T_{A \to B}^{(m)} < \infty\} \), for \( m \in \mathbb{N}_+ \). We do a partition of \( \{T_{A \to B}^{(m)} < \infty\} \) in the sets \( \{T_{A \to B}^{(m)} = k : k \in \mathbb{N}_+\} \). On every set \( \{T_{A \to B}^{(m)} = k\} \), the elements of \( A \) are absorbing states for \( Y_n^{(m)} \) when \( n \geq k \), therefore \( \{Y_n^{(m)}\}_{n \geq k} \) is a non-negative supermartingale on \( \{T_{A \to B}^{(m)} = k\} \), by hypothesis 3). Thus, see [Wi91], there exists

\[
\lim_{n \to +\infty} Y_n^{(m)} < \infty \text{ on } \{T_{A \to B}^{(m)} < \infty\} \text{ a.s.} \quad (31)
\]
We will prove that the limit in (31) belongs to \( A \) almost surely.

Given \( k \in \mathbb{N}_+ \), we prove (31) on the set \( \{ T_{A \rightarrow B}^{(m)} = k \} \). In fact if \( i \in B \)

\[
\mathbb{P} \left( \lim_{n \to +\infty} Y_n^{(m)} = i \mid T_{A \rightarrow B}^{(m)} = k \right) = \mathbb{P} \left( \bigcup_{h=k+1}^{\infty} \bigcap_{n=h}^{\infty} \{ Y_n^{(m)} = i \} \mid T_{A \rightarrow B}^{(m)} = k \right) \leq \sum_{h=k+1}^{\infty} \prod_{r=h}^{\infty} \mathbb{P}(X_r = i \mid X_h = \ldots = X_{r-1} = i, T_{A \rightarrow B}^{(m)} = k) ,
\]

where the last equality is a consequence of the fact that, if the limit belonged to \( B \), then the process \( (X_n)_{n \geq k} \) would never visit \( A \) and so, in this case, the processes \( (Y_n^{(m)})_{n \geq k} \) and \( (X_n)_{n \geq k} \) would coincide. Now, by using hypothesis 4) and a standard argument on the partition of the trajectories, we obtain the following upper bound for (32)

\[
\sum_{h=k+1}^{\infty} \prod_{r=h}^{\infty} (1 - p_i) = 0. 
\]

Hence we get that

\[
\lim_{n \to +\infty} Y_n^{(m)} \in A \text{ a.s.}
\]
or equivalently that

\[
\mathbb{P} \left( \{ T_{A \rightarrow B}^{(m)} < \infty \} \setminus \{ T_{B \rightarrow A}^{(m)} < \infty \} \right) = 0,
\]
from which

\[
\mathbb{P}(\cdot \mid T_{A \rightarrow B}^{(m-1)} < \infty) = \mathbb{P}(\cdot \mid T_{B \rightarrow A}^{(m-1)} < \infty).
\]

Notice that, if the numbers \( n_i \), for \( i = 0, \ldots, N \), verify hypothesis 2) of the theorem, then, by taking \( n \geq \max\{n_i : i \leq N\} \), condition 2) is still verified. In fact, if the process visits the state zero, then it indefinitely remains in zero, which directly follows by hypothesis 1). Therefore let us define \( \tilde{n} = \max\{n_i : i \leq N\} \in \mathbb{N}_+ \), then hypothesis 2) is satisfied by using \( \tilde{n} \) instead of \( n_i \) where the values of the \( q_i \)'s can only increase by replacing all the \( n_i \)'s with \( \tilde{n} \). Hence all the \( q_i \)'s calculated setting \( n_i = \tilde{n} \) are greater than some positive constant \( q \) which can be chosen equal to \( \inf\{q_i : i = 1, \ldots, N\} \).
Then we get, by \((34)\), that for \(k \in \mathbb{N}_+\) 
\[
\mathbb{P}(T_{A \rightarrow B}^{(k+1)n} = \infty | T_{A \rightarrow B}^{(k)n} < \infty) = \mathbb{P}(T_{A \rightarrow B}^{(k+1)n} = \infty | T_{B \rightarrow A}^{(k)n} < \infty).
\]
By denoting the set of trajectories \(M_{n,k} = \{i_0^n \in \mathbb{N}^n : \{X^n_0 = i_0^n\} \subset \{T_{B \rightarrow A}^{(k)n} = n\}\}\), from the previous relation we obtain 
\[
\mathbb{P}(T_{A \rightarrow B}^{(k+1)n} = \infty | T_{B \rightarrow A}^{(k)n} < \infty) = \sum_{n=1}^{\infty} \sum_{i_0^n \in M_{n,k}} \mathbb{P}(T_{A \rightarrow B}^{(k+1)n} = \infty | T_{B \rightarrow A}^{(k)n} = n, X^n_0 = i_0^n, T_{B \rightarrow A}^{(k)n} < \infty) \mathbb{P}(T_{B \rightarrow A}^{(k)n} = n, X^n_0 = i_0^n | T_{B \rightarrow A}^{(k)n} < \infty) \geq q > 0.
\]

Thus indicating \(m = \lfloor n/\tilde{n} \rfloor\) for a generic \(n \in \mathbb{N}_+\), we obtain the following relation 
\[
\mathbb{P}(T_{A \rightarrow B}^{(n)} < \infty) \leq \prod_{k=2}^{m} \mathbb{P}(T_{A \rightarrow B}^{(k)n} = \infty | T_{A \rightarrow B}^{((k-1)n)} < \infty) \leq (1 - q)^{m-1}.
\]
Since, for each \(n \in \mathbb{N}_+, T_{A \rightarrow B}^{(n)} < \infty \supset T_{A \rightarrow B}^{(n+1)} < \infty\), by the monotone convergence theorem 
\[
\mathbb{P}\left(\bigcap_{n=1}^{\infty} \{T_{A \rightarrow B}^{(n)} < \infty\}\right) = \lim_{n \to +\infty} \mathbb{P}(T_{A \rightarrow B}^{(n)} < \infty) \leq \lim_{n \to +\infty} (1 - q)^{\lfloor n/\tilde{n} \rfloor - 1} = 0.
\]
Hence almost surely there exists a finite random index \(S = 2, 3, \ldots\) such that \(T_{A \rightarrow B}^{(S-1)} < \infty, T_{B \rightarrow A}^{(S-1)} < \infty\) and \(T_{A \rightarrow B}^{(S)} = \infty\), then \(X_n \in A\) for any \(n \geq T_{B \rightarrow A}^{(S-1)}\). It remains to show that the process can not stay indefinitely in \(\{1, 2, \ldots, N\}\).

Let us define 
\[
\tilde{X}_k = X_{k\tilde{n}}, \text{ for } k \in \mathbb{N}.
\]
Note that for the process \(\tilde{X} = (\tilde{X}_n : n \in \mathbb{N})\) there exists a random time almost surely finite 
\[
\tilde{T}_A = \inf\{n : \tilde{X}_k \in A, \text{ for } k \geq n\},
\]
such that the process remains indefinitely in \(A\) after \(\tilde{T}_A\). Moreover observe that \(\tilde{T}_A\) is not a stopping time and it shall be taken into account the information provided by the value of \(\tilde{T}_A\). Directly from hypothesis 2) it follows that 
\[
\tilde{q} = \inf_{m \in \mathbb{N}, i_0, i_1, \ldots, i_{m-1} \in \mathbb{N}, i_m \in A} \mathbb{P}(\tilde{X}_{m+1} = 0 | \tilde{X}_0^m = i_0^m)
\]
is positive.

Now we will show that for each \( n \in \mathbb{N}_+ \),
\[
\inf_{m \geq n, i_0^n \in \mathbb{N}^{n-1}, i_{n-1} \in B, i_n, \ldots, i_m \in A} \mathbb{P}(\tilde{X}_{m+1} = 0 | \tilde{T}_A = n, \tilde{X}_0^m = i_0^m) \geq \tilde{q} > 0.
\]

We notice that for \( i_0^n \in \mathbb{N}^{n-1}, i_{n-1} \in B, i_n, \ldots, i_m \in A \),
\[
\{\tilde{X}_0^m = i_0^m, \tilde{X}_{m+1} = 0\} \subset \{\tilde{T}_A = n\},
\]
from which
\[
\mathbb{P}(\tilde{X}_0^m = i_0^m, \tilde{X}_{m+1} = 0) \leq \mathbb{P}(\tilde{T}_A = n).
\]

Hence
\[
\mathbb{P}(\tilde{X}_{m+1} = 0 | \tilde{T}_A = n, \tilde{X}_0^m = i_0^m) = \frac{\mathbb{P}(\tilde{T}_A = n, \tilde{X}_0^m = i_0^m, \tilde{X}_{m+1} = 0)}{\mathbb{P}(\tilde{T}_A = n, \tilde{X}_0^m = i_0^m)} \geq \frac{\mathbb{P}(\tilde{X}_0^m = i_0^m, \tilde{X}_{m+1} = 0)}{\mathbb{P}(\tilde{X}_0^m = i_0^m)} = \mathbb{P}(\tilde{X}_{m+1} = 0 | \tilde{X}_0^m = i_0^m).
\]

From which by taking the infimum,
\[
\inf_{m \geq n, i_0^n \in \mathbb{N}^{n-1}, i_{n-1} \in B, i_n, \ldots, i_m \in A} \mathbb{P}(\tilde{X}_{m+1} = 0 | \tilde{T}_A = n, \tilde{X}_0^m = i_0^m) \\
\geq \inf_{m \geq n, i_0^n \in \mathbb{N}^{n-1}, i_{n-1} \in B, i_n, \ldots, i_m \in A} \mathbb{P}(\tilde{X}_{m+1} = 0 | \tilde{X}_0^m = i_0^m) \\
\geq \inf_{m \in \mathbb{N}, i_0, i_1, \ldots, i_m \in \mathbb{N}, i_n, \ldots, i_m \in A} \mathbb{P}(\tilde{X}_{m+1} = 0 | \tilde{X}_0^m = i_0^m) = \tilde{q} > 0.
\]

Analogously to (33), by the latter inequalities and standard arguments on the partition of trajectories, one obtains that the process \( \tilde{X} \) is eventually equal to zero. Obviously the same property is obtained for the original process \( X \), i.e. \( \lim_{n \to +\infty} X_n = 0 \) a.s. \( \square \)

**Remark 3.** We note that, in the previous theorem, the process \( (X_n)_{n \in \mathbb{N}} \) could be a non-homogeneous Markov chain. In particular, one can consider a culture of bacteria in which the number of its population affects the ability of reproduction of the bacteria by changing the probability that the cell dies before its mitosis. In some way we can think that a process \( (X_n)_{n \in \mathbb{N}} \), verifying the assumptions of Theorem 8, can be chosen as a model for these biological cultures. Therefore the bacteria cultures will die in a finite time.
5 Applications of Theorem 8 to perfect simulation

Let us consider a probability distribution \( \psi_v \) indexed by \( v \in \mathbb{Z}^d \) and let \( \sum_{l=0}^{\infty} \psi_v(l) = 1 \).

Moreover, for each \( v \in \mathbb{Z}^d \), let \( \psi_v(0) > 0 \).

Let us associate to each vertex \( v \in \mathbb{Z}^d \) a sequence \( S_v = (S_v(l) \in \mathbb{Z}^d : l \in \mathbb{N}_+) \) and a mass \( M_v \) such that \( \inf_{v \in \mathbb{Z}^d} M_v \geq 1 \).

Let \( v \in \mathbb{Z}^d \) and \((D_n)_{n \in \mathbb{N}}\) be a homogeneous Markov chain with countable state space \( \mathcal{C} = \{ A \in \mathbb{Z}^d \} \).

At time zero the Markov chain has a initial measure \( \nu(0) \). The rules of the dynamics are given in Section 2, it only needs to replace \( C_n, B_v, \lambda_v \) with \( D_n, S_v, \psi_v \) respectively.

Let us define, for each \( v \in \mathbb{Z}^d \),
\[
\eta_v = -\psi_v(0) + \sum_{l=1}^{\infty} |S_v(l)| \psi_v(l),
\]
which is similar to the birth-death expectation and plays the same role.

We are now in the position to present our result on the extinction of the processes above defined.

**Corollary 2.** Let \( \eta_v \) as in (35), if \( \lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{v \in \Lambda} \eta_v < 0 \), then \( \limsup_{n \to \infty} D_n = \emptyset \) almost surely.

**Proof.** Let \( X_n = |D_n| \), we want to show that the process \((X_n)_{n \in \mathbb{N}}\) verifies all the hypotheses of Theorem 8. Hypothesis 1) is trivially verified because if \( D_n = \emptyset \), then \( D_{n+1} = \emptyset \). We now verify hypothesis 3). First of all note that from the assumption of the corollary it follows the existence of a \( \delta > 0 \) such that the set
\[
R_\delta = \{ v \in \mathbb{Z}^d : \eta_v > -\delta \}
\]
has finite cardinality.

Fix \( \delta > 0 \) such that \( |R_\delta| < \infty \), and define \( \alpha = \max\{0, M_v \eta_v : v \in R_\delta\} \). Consider \( D_n \neq \emptyset \), we easily see that
\[
\mathbb{E}(X_{n+1}|D_n) = \mathbb{E}(|D_{n+1}| |D_n) \leq |D_n| + \sum_{u \in D_n} \sum_{v \in D_n} \frac{M_u}{M_v} \eta_v.
\]
Under the assumption of the corollary and since $M_v \geq 1$ for each $v \in \mathbb{Z}^d$, we obtain
\[
\mathbb{E}(X_{n+1}|D_n) \leq |D_n| + \frac{1}{\sum_{u \in D_n} M_u} [a|R_\delta| - \delta(|D_n| - |R_\delta|)].
\]

We get that if
\[
X_n = |D_n| \geq \left\lceil \frac{a|R_\delta|}{\delta} + |R_\delta| \right\rceil \equiv N,
\]
then $\mathbb{E}(X_{n+1}|D_n) \leq X_n$. Since
\[
\mathbb{E}(X_{n+1}|X_0^n = i_0^n) = \sum_{A \in \mathbb{Z}^d:|A|=i_n} \mathbb{E}(X_{n+1}|D_n = A)\mathbb{P}(D_n = A|X_0^n = i_0^n),
\]
we have that (37) is lesser or equal to $X_n = i_n$ when $i_n \geq N$. Hence hypothesis 3) is obtained by choosing $N$ as in (36), because all the summands in (37) are non-positive.

Now we show that
\[
\xi = \inf_{v \in \mathbb{Z}^d} \psi_v(0) > 0.
\]
Note that
\[
\rho = \inf\{\psi_v(0): v \in R_\delta\} > 0
\]
because it is an infimum on a finite set of positive numbers. Moreover, from (35), it follows
\[
\rho' = \inf\{\psi_v(0): v \in R'_\delta\} \geq \delta > 0.
\]
Hence
\[
\xi = \min\{\rho, \rho'\} > 0.
\]
Therefore hypothesis 2) is verified for $n_i = N$ and the $q_i$’s are larger or equal than $\xi N > 0$, for $i \leq N$.

We also obtain 4) observing that $p_i \geq \xi > 0$ for each $i \in \mathbb{N}_+$.

Thus, from Theorem 8,
\[
\lim_{n \to +\infty} X_n = 0 \text{ a.s.}
\]
There exists an almost surely finite random time $Y$ such that $C_Y = \emptyset$. \hfill \Box

Given $J \in \mathcal{J}$, $v \in \mathbb{Z}^d$, $B_v \in \mathcal{B}_v$, set
\[
S_v(l) = B_v(l) \setminus \{v\} \text{ for } l \in \mathbb{N}_+,
\]
and $\psi_v = \lambda_{v,J,B_v}$, then, by a simple calculation, $\eta_v = \mu_{v,J}(B_v)$. Putting $M_u = 2 \exp(\sum_{B,u \in B} |J_B|)$, for each $u \in \mathbb{Z}^d$, and $\nu^{(0)} = \delta_{C_0}$ the process $(D_n)_n$ coincides with $(C_n)_n$ defined in Section 2.
Proof of Theorem 1. The first part of Theorem 1 is a direct consequence of Corollary 2.

We conclude the paper discussing an example in which an interaction verifies hypothesis (H2) but does not verify (H1). The example is constructed by using the property of universality described in Theorem 4. Let $B_v$’s be fixed, let us consider an interaction $J \in \mathcal{J}$ such that the inequality in (H1) is verified. For a given $B_0 \in \mathbb{Z}^d$ such that $O \in B_0$, define $J^{(L)}_B$ as

$$J^{(L)}_B = \begin{cases} J_B & \text{if } B \neq B_0; \\ LJ_B & \text{if } B = B_0; \end{cases}$$

where $L \in \mathbb{R}$. By elementary calculations, for a sufficiently large $L > 0$, it occurs that $\mu_{O, J^{(L)}_B}(B_O) > 0$, hence $\sup_{v \in \mathbb{Z}^d} \mu_{v, J^{(L)}_B}(B_v) > 0$. Instead $\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{v \notin \Lambda} \mu_{v, J^{(L)}_B}(B_v)$ does not depend on $L$, therefore it is less than zero.

Other examples, verifying (H2) but not (H1), can be naturally constructed for each result in Section 3 following the scheme of the proofs and choosing suitable values for the $J$’s and the $B_v$’s.

We notice that Theorem 8, by eliminating anyone of its assumptions, becomes false; examples can be easily constructed.

To finish we stress that condition (H2) differs from (H1) for two reasons. First, the replacement of the supremum by the limit superior improves the sufficient condition for the applicability of the algorithm, but does not change the algorithm; second the different choice of the sets $B_v$’s improves the algorithm and its applicability.

A Algorithm for the infinite range Ising model

We present the algorithm for the infinite range Ising model showing how to implement the result presented in Theorem 7 in a pseudo code. First one has to prove that, given the interaction $J \in \mathcal{J}$,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{v \notin \Lambda} -2e^{-2\sum_{i=1}^{\infty} j_v^{(i)}} + e^{-\sum_{i=2}^{\infty} j_v^{(i)}} + \sum_{l=2}^{\infty} l \left( e^{-\sum_{i=l+1}^{\infty} j_v^{(i)}} - e^{-\sum_{i=1}^{\infty} j_v^{(i)}} \right) < 0.$$

If one does not use the finite range approximation presented in [GL010], it is important that the sums $G_J(v) = \sum_{v' \in \mathbb{Z}^d \setminus v} |J_{\{v, v'\}}|$ are calculable for each $v \in \mathbb{Z}^d$ and that these
values are given as input in the algorithm. The easiest case is the translational one where \( J_{u,v} = J_{u+w,v+w} \), for any \( u, v, w \in \mathbb{Z}^d \).

\[ D, H, I, K, L, N, N_{STOP}, R \] are variables taking values in \( \mathbb{N} \);  
\( U \) is a variable taking values in \( [0, 1] \);  
\( V \) is a variable taking values in \( \mathbb{Z}^d \);  
\( Y \) is a variable taking values in \( \{-1, 1\} \);  
\( M, M_0 \) are variables taking values in \( \mathbb{R} \);  
\( C, S, W, Z_-, Z_+ \) are arrays of elements of \( \mathbb{Z}^d \);  
\( Q \) is an array of elements of \( \mathbb{Z}^d \times \mathbb{N} \times \{ E \in \mathbb{Z}^d \}^2 \);  
\( G \) is a function from \( \mathbb{Z}^d \) to \( \mathbb{R} \);  
\( F \) is a function from \( \mathbb{Z}^d \times \mathbb{N} \) to \( [0, 1] \);  
\( P \) is a function from \( \mathbb{N} \times \mathbb{Z}^d \times \{ E \in \mathbb{Z}^d \}^2 \times \{ -1, 1 \} \mathbb{Z}^d \) to \( [0, 1] \);  
\( T \) is a bijective function from \( \mathbb{N} \) to \( \mathbb{Z}^d \);  
\( X \) is a function from \( \mathbb{Z}^d \) to \( \{-1, 1\} \cup \Delta \) where \( \Delta \) is an extra symbol that does not belong to \( \{-1, 1\} \) and it is called cemetery state;  
\( \text{RANDOM} \) is a uniform random variable in \( [0, 1] \).

Algorithm 1: backward sketch procedure plus construction of optimal \( \mathbf{B}_v \)’s

Input: \( J \in J_2; C = (V_1, \ldots, V_{|C|}); G(V) = \sum_{V' \in \mathbb{Z}^d - V} |J_{V,V'}| \)  
Output: \( N_{STOP}; Q; \)

1. \( N \leftarrow 0; \ N_{STOP} \leftarrow 0; \ Q \leftarrow \emptyset; \ D \leftarrow |C|; \)
2. \( \text{WHILE } C \neq \emptyset \)
3. \( N \leftarrow N + 1; \ R \leftarrow 1; \ M \leftarrow 0; \ M_0 \leftarrow 0; \ S \leftarrow \emptyset; \)
4. \( U \leftarrow \text{RANDOM}(); \)
5. \( \text{WHILE } \sum_{H=1}^{R} 2 \exp(G(V_H))/\sum_{I=1}^{|C|} 2 \exp(G(V_I)) < U \)
6. \( R \leftarrow R + 1; \)
7. \( \text{END WHILE} \)
8. \( K \leftarrow 0; \)
9. \( F(V_R, 0) \leftarrow \exp(-2G(V_R)); \)
10. \( \text{WHILE } F(V_R, K) < U \)
11. \( K \leftarrow K + 1; \ L \leftarrow 1; \)
12. \( \text{WHILE } G(V_R) - M - \sum_{I=1}^{L} |J_{V_R, T_I + V_R}| \mathbf{1}(T_I \notin S) > \)
max\{|J_{V_R,T_I+V_R}| : I = 1, \ldots, L, T_I \notin S\}
13. \( L \leftarrow L + 1; \)
14. \( M_0 \leftarrow \max\{|J_{V_R,T_I+V_R}| : I = 1, \ldots, L, T_I \notin S\}; \)
15. END WHILE
16. \( A \leftarrow \min\{I = 1, \ldots, L, T_I \notin S : |J_{V_R,T_I+V_R}| = M_0\}; \)
17. \( M \leftarrow M + |J_{V_R,T_A+V_R}|; \)
18. \( S \leftarrow S \cup (T_A + V_R); \)
19. \( W_K \leftarrow T_A + V_R; \)
20. \( F(V_R,K) \leftarrow \exp(-G(V_R) + \sum_{I=1}^{K} |J_{V_R,W_I}|); \)
21. END WHILE
22. IF \( K = 0 \)
23. \( C \leftarrow C \setminus V_R; \)
24. ELSE
25. FOR \( I = 1, \ldots, L; \)
26. \( C \leftarrow C \cup W_I; \)
27. END FOR
28. END IF
29. \( Q(N) \leftarrow (V_R, K, \bigcup_{I=1}^{L-1} W_I, \bigcup_{I=1}^{L} W_I); \)
30. END WHILE
31. \( N_{STOP} \leftarrow N; \)
32. RETURN \( N_{STOP}; Q. \)

Algorithm 2: forward spin assignment procedure

Input: \( N_{STOP}; Q; \)
Output: \( \{X(V_1), \ldots, X(V_D)\}; \)
33. \( N \leftarrow N_{STOP}; \)
34. \( X(j) \leftarrow \Delta \text{ for all } j \in \mathbb{Z}^d; \)
35. WHILE \( N \geq 1 \)
36. \( (V,K,Z_-, Z_+) \leftarrow Q(N); \)
37. \( U \leftarrow \text{RANDOM}(); \)
38. IF \( 0 \leq U \leq P_{V,Z_-,Z_+}^{[K]}(-X(V)|X) \)
39. $Y = -1$;
40. ELSE $Y = 1$;
41. END IF
42. $X(V) \leftarrow Y \cdot 1(K = 0) + X(V) \cdot Y \cdot 1(K > 0)$;
43. $N \leftarrow N - 1$;
44. END WHILE
45. RETURN \{X(V_1), \ldots, X(V_D)\}.

We write some comments to facilitate the understanding of the pseudo code.

Line 2. the b.s.p. ends when the set $C$ becomes empty.
Lines 5.-7. a random vertex $V_R$ in $C$ is chosen with probability given in (8).
Lines 10.-21. a random value $K$, related to the vertex $V_R$, is selected by Skorohod representation that uses $F_{V_R}(K)$ the cumulative distribution of $\lambda$ (see (3)). Notice that, for each $k$, $F_{V_R}(k)$ can be calculated with a finite number of elementary operations, when $G(v)$ is known.
Lines 12.-15. it is a small algorithm that finds for a positive sequence $\{a_n\}_{n \in \mathbb{N}}$ with $L = \sum_{n \in \mathbb{N}} a_n < \infty$ the biggest element $a_{\bar{n}} = \max\{a_n : n \in \mathbb{N}\}$ and the index $\bar{n}$. We stress that it is done in a finite number of steps. Iteratively the second biggest element is calculated and so on.
Line 38. The probabilities $p_{v,J,B_{\sigma},\sigma}^{[k]}(-\sigma(v)|\sigma)$ defined in (5)-(6) depend on the finite sets $B_v(k - 1)$ and $B_v(k)$ that in the pseudo code are $Z_-$ and $Z_+$ respectively. In the pseudo code these probabilities are $P_{v,Z_-,Z_+}^{[K]}(-X(V)|X)$. In (5) and (6) all the sums have a finite number of elements, except one in (5) that can be rewritten as $-G(v) + \sum_{u \in B_v(1)} |J_{\{v,u\}}|$, which has a finite number of addenda.

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