ON THE ESTIMATES OF WARPING FUNCTIONS ON ISOMETRIC IMMERSIONS

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Abstract. Using the results of [11], we get some estimates of warping functions on isometric immersions by replacing the target manifolds with some types of Riemannian manifolds: constant space forms and Hermitian symmetric spaces. And we obtain their applications.

1. Introduction

Let $(B,g_B)$ and $(F,g_F)$ be Riemannian manifolds. Given a warped product manifold $M = B \times_f F$ with a warping function $f$ (See [11]), we can consider an isometric immersion $\psi : M \mapsto (\overline{M}, \overline{g})$, where $(\overline{M}, \overline{g})$ is a Riemannian manifold.

In 2018, B. Y. Chen [5] proposed two Fundamental Questions on the isometric immersion $\psi : M \mapsto (\overline{M}, \overline{g})$ and gave some recent results on these problems where $(\overline{M}, \overline{g})$ is a Kähler manifold.

In 2014, as a generalization of Chen’s works ([3],[4]), the author [11] obtained two inequalities, which give the upper bound and the lower bound of the function $\triangle f$. Replacing the Riemannian manifold $(\overline{M}, \overline{g})$ with several types of Riemannian manifolds (i.e., real space forms, complex space forms, quaternionic space forms, Sasakian space forms, Kenmotsu space forms, Hermitian symmetric spaces: complex two-plane Grassmannians, complex hyperbolic two-plane Grassmannians, complex quadrics), we will obtain the upper bounds and the lower bounds of the functions $\triangle f$. And by using these results, we will get some applications.

We know that warped product manifolds take an important position in differential geometry and in physics, in particular in general relativity. And Nash’s result [9] implies that each warped product manifold can be isometrically embedded in a Euclidean space.

The paper is organized as follows. In section 2 we remind some notions, which will be used in the following section. In section 3 we estimate the upper bounds and the lower bounds of the functions $\triangle f$ by replacing the Riemannian manifold $(\overline{M}, \overline{g})$ with several types of Riemannian manifolds and give some applications.

2. Preliminaries

In this section we recall some notions, which will be used in the following section.

Let $(\overline{M}, \overline{g})$ be an $n$-dimensional Riemannian manifold and let $M$ be an $m$-dimensional submanifold of $(\overline{M}, \overline{g})$. We denote by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connections of $\overline{M}$ and $M$, respectively.

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Then we get the \textit{Gauss formula} and the \textit{Weingarten formula}
\begin{align}
\nabla_X Y &= \nabla_X Y + h(X, Y), \\
\nabla_X N &= -A_N X + D_X N,
\end{align}
respectively, for tangent vector fields \( X, Y \in \Gamma(TM) \) and a normal vector field \( N \in \Gamma(TM^\perp) \), where \( h, A, D \) denote the \textit{second fundamental form}, the \textit{shape operator}, the \textit{normal connection} of \( M \) in \( \overline{M} \), respectively.

Then we know
\begin{align}
G(A_N X, Y) &= \overline{g}(h(X, Y), N).
\end{align}
Fix a local orthonormal frame \( \{v_1, \ldots, v_n\} \) of \( TM \) with \( v_i \in \Gamma(TM) \), \( 1 \leq i \leq m \) and \( v_{\alpha} \in \Gamma(TM^\perp) \), \( m + 1 \leq \alpha \leq n \). We define the mean curvature vector field \( H \), the squared mean curvature \( H^2 \), the squared norm \( ||h||^2 \) of the second fundamental form \( h \) as follows:
\begin{align}
H &= \frac{1}{m} \text{trace} h = \frac{1}{m} \sum_{i=1}^{m} h(v_i, v_i), \\
H^2 &= \overline{g}(H, H), \\
||h||^2 &= \sum_{i,j=1}^{m} \overline{g}(h(v_i, v_j), h(v_i, v_j)).
\end{align}

We call the submanifold \( M \subset (\overline{M}, \overline{g}) \) \textit{totally geodesic} if the second fundamental form \( h \) vanishes identically. Denote by \( R, \overline{R} \) the Riemannian curvature tensors of \( M, \overline{M} \), respectively.

Let
\begin{align}
K(X \wedge Y) &= \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}, \\
\overline{K}(X \wedge Y) &= \frac{\overline{g}(\overline{R}(X, Y)Y, X)}{\overline{g}(X, X)\overline{g}(Y, Y) - \overline{g}(X, Y)^2}
\end{align}
for \( X, Y \in \Gamma(TM) \), where \( g \) denotes the induced metric on \( M \) of \((\overline{M}, \overline{g})\), i.e., given a plane \( V \subset T_p M \), \( p \in M \), spanned by vectors \( X, Y \in T_p M \), \( K(V) = K(X \wedge Y) \) and \( \overline{K}(V) = \overline{K}(X \wedge Y) \) denote the sectional curvatures of a plane \( V \) in \( M \) and in \( \overline{M} \), respectively.

Let
\begin{align}
(\inf \overline{K})(p) &= \inf\{\overline{K}(V) \mid V \subset T_p M, \dim V = 2\}, \\
(\sup \overline{K})(p) &= \sup\{\overline{K}(V) \mid V \subset T_p M, \dim V = 2\}.
\end{align}
Let \( \overline{R}(X, Y, Z, W) := \overline{g}(\overline{R}(X, Y)Z, W) \) for \( X, Y, Z, W \in \Gamma(TM) \).

Given a \( C^\infty \)-function \( f \in C^\infty(M) \), we define the \textit{Laplacian} \( \triangle f \) of \( f \) by
\begin{align}
\triangle f := \sum_{i=1}^{m} ((\nabla_v v_i) f - v_i^2 f).
\end{align}
Let \((B, g_B)\) and \((F, g_F)\) be Riemannian manifolds.

Throughout this paper, we will denote by \( (M, g) := (B \times_f F, g_B + f^2 g_F) \) the warped product manifold of Riemannian manifolds \((B, g_B)\) and \((F, g_F)\) with the warping function \( f : B \rightarrow \mathbb{R}^+ \) (See [11]).
3. Estimates of functions $\Delta f$

In this section, we will estimate the function $\Delta f$ of an isometric immersion $\psi$ from a warped product manifold $(M, g) = (B \times f, g_B + f^2 g_F)$ into a Riemannian manifold $(\mathcal{M}, \mathcal{g})$ where the manifold $(\mathcal{M}, \mathcal{g})$ is one of the following manifolds:

real space forms, complex space forms, quaternionic space forms, Sasakian space forms, Kenmotsu space forms, complex two-plane Grassmannians, complex hyperbolic two-plane Grassmannians, complex quadrics.

And by using the results, we will obtain some applications.

By Theorem 3.1, Theorem 3.4, and their proofs of [11], we have

**Lemma 3.1.** Let $(M, g) = (B \times f, g_B + f^2 g_F)$ be a warped product manifold and let $(\mathcal{M}, \mathcal{g})$ be a Riemannian manifold. Let $\psi : (M, g) \rightarrow (\mathcal{M}, \mathcal{g})$ be an isometric immersion. Then we get

$$
\frac{m_1 m_2}{2(m-1)} H^2 - m_1 \|h\|^2 + m_1 c \leq \frac{\Delta f}{f} \leq \frac{m^2}{4m_2} H^2 + m_1 c,
$$

where $m_1 = \dim B$ and $m_2 = \dim F$ with $m = m_1 + m_2$.

Using Lemma 3.1, we obtain

**Theorem 3.2.** Let $(M, g) = (B \times f, g_B + f^2 g_F)$ be a warped product manifold and $(\mathcal{M}, \mathcal{g}) = (\mathcal{M}(c), \mathcal{g})$ a real space form of constant sectional curvature $c$. Let $\psi : (M, g) \rightarrow (\mathcal{M}, \mathcal{g})$ be an isometric immersion. Then we have

$$
\frac{m_1 m_2}{2(m-1)} H^2 - m_1 \|h\|^2 + m_1 c \leq \frac{\Delta f}{f} \leq \frac{m^2}{4m_2} H^2 + m_1 c,
$$

where $m_1 = \dim B$ and $m_2 = \dim F$ with $m = m_1 + m_2$.

**Proof.** We know that the Riemannian curvature tensor $\mathcal{R}$ of $(\mathcal{M}, \mathcal{g})$ is given by

$$
\mathcal{R}(X, Y)Z = a(\mathcal{g}(Y, Z)X - \mathcal{g}(X, Z)Y)
$$

for $X, Y, Z \in \Gamma(\mathcal{T} \mathcal{M})$. Since $\inf \mathcal{K} = \sup \mathcal{K} = c$, by Lemma 3.1 we get the result. 

**Remark 3.3.** Furthermore, if we assume that the manifold $(M, g)$ is a totally geodesic submanifold of $(\mathcal{M}, \mathcal{g})$, then from (3.2), we obtain

$$
m_1 c \leq \frac{\Delta f}{f} \leq m_1 c
$$

so that

$$
\Delta f = m_1 c f,
$$

which implies that the warping function $f$ is an eigen-function with eigenvalue $m_1 c$.

In particular, if $c = 0$ (i.e., $(\mathcal{M}, \mathcal{g})$ is a Euclidean space $\mathbb{E}^n$), then the warping function $f$ is a harmonic function.

**Theorem 3.4.** Let $(M, g) = (B \times f, g_B + f^2 g_F)$ be a warped product manifold and $(\mathcal{M}, \mathcal{g}) = (\mathcal{M}(c), \mathcal{g}, J)$ a complex space form of constant holomorphic sectional curvature $c$. Let $\psi : (M, g) \rightarrow (\mathcal{M}, \mathcal{g})$ be an isometric immersion. Then we have

$$
\frac{m_1 m_2}{2(m-1)} H^2 - m_1 \|h\|^2 + m_1 c \leq \frac{\Delta f}{f} \leq \frac{m^2}{4m_2} H^2 + m_1 c,
$$

$c \geq 0$,

$$
\frac{m_1 m_2}{2(m-1)} H^2 - m_1 \|h\|^2 + m_1 c \leq \frac{\Delta f}{f} \leq \frac{m^2}{4m_2} H^2 + m_1 c,
$$

$c < 0$, 

where $m_1 = \dim B$ and $m_2 = \dim F$ with $m = m_1 + m_2$. 

where \( m_1 = \dim B \) and \( m_2 = \dim F \) with \( m = m_1 + m_2 \).

**Proof.** We see that the Riemannian curvature tensor \( \overline{R} \) of \((\overline{M}, \overline{g})\) is given by

\[
\overline{R}(X, Y)Z = \frac{c}{4}(\overline{g}(Y, Z)X - \overline{g}(X, Z)Y + \overline{g}(JY, Z)JX - \overline{g}(JX, Z)JY - 2\overline{g}(JX, Y)JZ)
\]

for \( X, Y, Z \in \Gamma(TM) \). Given orthonormal vectors \( X, Y \in \mathcal{T}_pM, p \in \overline{M} \), we get

\[
\overline{R}(X \wedge Y) = \overline{R}(X, Y, Y, X) = \frac{c}{4}(1 + 3\overline{g}(JX, Y)^2)
\]

so that since \( 0 \leq |\overline{g}(JX, Y)| \leq 1 \), we easily obtain

\[
\frac{c}{4} \leq \overline{R}(X \wedge Y) \leq c, \quad c \geq 0,
\]

\[
c \leq \overline{R}(X \wedge Y) \leq \frac{c}{4}, \quad c < 0.
\]

From Lemma 3.1, the result follows. \( \square \)

**Remark 3.5.** 1. If the manifold \((M, g)\) is a totally geodesic totally real submanifold of \((\overline{M}, \overline{g})\) (i.e., \(J(TM) \subset TM^\perp\)), then from Lemma 3.1 and (3.7), we have

\[
m_1 \frac{c}{4} \leq \frac{\triangle f}{f} \leq m_1 \frac{c}{4}
\]

so that

\[
\triangle f = m_1 \frac{c}{4} f.
\]

So, the warping function \( f \) is an eigen-function with eigenvalue \( m_1 c \).

2. If the manifold \((M, g)\) is a 2-dimensional totally geodesic complex submanifold of \((\overline{M}, \overline{g})\) (i.e., \(J(TM) = TM\)), then from Lemma 3.1 and (3.7), we get

\[
m_1 c \leq \frac{\triangle f}{f} \leq m_1 c
\]

so that

\[
\triangle f = m_1 c f.
\]

Hence, the warping function \( f \) is an eigen-function with eigenvalue \( m_1 c \).

**Theorem 3.6.** Let \((M, g) = (B \times_f F, g_B + f^2 g_F)\) be a warped product manifold and \((\overline{M}, \overline{g}) = (\overline{M}(c), E, \overline{g})\) a quaternionic space form of constant quaternionic sectional curvature \( c \). Let \( \psi : (M, g) \mapsto (\overline{M}, \overline{g}) \) be an isometric immersion. Then we obtain

\[
\frac{m_1 m_2}{2(m - 1)} H^2 - \frac{m_1}{2} ||h||^2 + m_1 \frac{c}{4} \leq \frac{\triangle f}{f} \leq \frac{m_2}{4m_2} H^2 + m_1 c, \quad c \geq 0,
\]

\[
\frac{m_1 m_2}{2(m - 1)} H^2 - \frac{m_1}{2} ||h||^2 + m_1 c \leq \frac{\triangle f}{f} \leq \frac{m_2}{4m_2} H^2 + m_1 \frac{c}{4}, \quad c < 0,
\]

where \( m_1 = \dim B \) and \( m_2 = \dim F \) with \( m = m_1 + m_2 \).

**Proof.** We know that the Riemannian curvature tensor \( \overline{R} \) of \((\overline{M}, \overline{g})\) is given by

\[
\overline{R}(X, Y)Z = \frac{c}{4}(\overline{g}(Y, Z)X - \overline{g}(X, Z)Y
\]

\[
+ \sum_{\alpha=1}^{3} (\overline{g}(J_\alpha Y, Z)J_\alpha X - \overline{g}(J_\alpha X, Z)J_\alpha Y - 2\overline{g}(J_\alpha X, Y)J_\alpha Z))
\]
for $X,Y,Z \in \Gamma(TM')$. Given orthonormal vectors $X,Y \in T_pM'$, $p \in M'$, we have

$$\kappa(X \wedge Y) = \kappa(X,Y,Y,X) = \frac{c}{4}(1 + 3 \sum_{\alpha=1}^{3} \gamma(J_{\alpha}X,Y)^2).$$

(3.11)

Since $\{J_1X,J_2X,J_3X\}$ is orthonormal, we get $0 \leq \sum_{\alpha=1}^{3} \gamma(J_{\alpha}X,Y)^2 \leq |Y|^2 = 1$ so that

$$\frac{c}{4} \leq \kappa(X \wedge Y) \leq c, \quad c \geq 0,$$

$$c \leq \kappa(X \wedge Y) \leq \frac{c}{4}, \quad c < 0.$$

From Lemma 3.1 we obtain the result. □

**Remark 3.7.** 1. If the manifold $(M,g)$ is a totally geodesic totally real submanifold of $(M',\gamma)$ (i.e., $J_\alpha(TM) \subset TM'$, $\forall \alpha \in \{1,2,3\}$), then from Lemma 3.1 and (3.11), we have

$$m_1c \leq \frac{\Delta f}{f} \leq m_1c$$

so that

$$\Delta f = m_1c f,$$

which implies that the warping function $f$ is an eigen-function with eigenvalue $\frac{m_1c}{4}$.

2. If the manifold $(M,g)$ is a 4-dimensional totally geodesic quaternionic submanifold of $(M',\gamma)$ (i.e., $J_\alpha(TM) = TM$, $\forall \alpha \in \{1,2,3\}$), then from Lemma 3.1 and (3.11), we get

$$m_1c \leq \frac{\Delta f}{f} \leq m_1c$$

so that

$$\Delta f = m_1cf.$$ 

Hence, the warping function $f$ is an eigen-function with eigenvalue $m_1c$.

**Theorem 3.8.** Let $(M,g) = (B \times F, g_B + f^2g_F)$ be a warped product manifold and $(M',\gamma) = (M(c),\phi,\xi,\eta,\gamma)$ a Sasakian space form of constant $\phi$-sectional curvature $c$. Let $\psi : (M,g) \rightarrow (M',\gamma)$ be an isometric immerssion. Then we obtain

$$\frac{m_1m_2^2}{2(m-1)}H^2 - \frac{m_1}{2}||h||^2 + m_1 \leq \frac{\Delta f}{f} \leq \frac{m_2^2}{4m_2}H^2 + m_1c, \quad c \geq 1,$$

(3.12)

$$\frac{m_1m_2^2}{2(m-1)}H^2 - \frac{m_1}{2}||h||^2 + m_1c \leq \frac{\Delta f}{f} \leq \frac{m_2^2}{4m_2}H^2 + m_1, \quad c < 1,$$

(3.13)

where $m_1 = \text{dim} B$ and $m_2 = \text{dim} F$ with $m = m_1 + m_2$.

**Proof.** We see that the Riemannian curvature tensor $\kappa$ of $(M',\gamma)$ is given by

$$\kappa(X,Y)Z = \frac{c + 3}{4}(\gamma(Y,Z)X - \gamma(X,Z)Y)$$

$$+ \frac{c - 1}{4}(\eta(Y)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)\gamma(X,Z)Y - \eta(Y)\gamma(Y,Z)X)$$

$$+ \gamma(\phi Y,Z)\phi X - \gamma(\phi X,Z)\phi Y - 2\gamma(\phi X,Y)\phi Z$$

for $X,Y,Z \in \Gamma(TM')$. Given orthonormal vectors $X,Y \in T_pM'$, $p \in M'$, we have

$$\kappa(X \wedge Y) = \kappa(X,Y,Y,X) = \frac{c + 3}{4} + \frac{c - 1}{4}(-\eta(Y)^2 - \eta(X)^2 + 3\gamma(\phi X,Y)^2).$$

(3.15)
Since \( \xi \in \text{Span}(X, Y) \) \( \Rightarrow -\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2 = -1 \) and \( Y = \phi X, \eta(X) = 0 \Rightarrow -\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2 = 3 \), we get \(-1 \leq -\eta(Y)^2 - \eta(X)^2 + 3\overline{g}(\phi X, Y)^2 \leq 3\) so that
\[
1 \leq \overline{R}(X \wedge Y) \leq c, \quad c \geq 1,
\]
\[
c \leq \overline{R}(X \wedge Y) \leq 1, \quad c < 1.
\]

From Lemma 3.1, the result follows. \( \square \)

**Remark 3.9.** 1. If the manifold \((M, g)\) is a totally geodesic \(\phi\)-totally real submanifold of \((\overline{M}, \overline{g})\) with \(\xi \in \Gamma(TM)\) (i.e., \(\phi(TM) \subset TM\)), then from Lemma 3.1 and (3.16), we get
\[
m_1 \frac{c + 3}{4} \leq \frac{\Delta f}{f} \leq m_1 \frac{c + 3}{4}
\]
so that
\[
\Delta f = \frac{m_1(c + 3)}{4} f.
\]
The warping function \(f\) is an eigen-function with eigenvalue \(m_1(c + 3)\).

2. If the manifold \((M, g)\) is a 2-dimensional totally geodesic submanifold of \((\overline{M}, \overline{g})\) with \(\xi \in \Gamma(TM)\), then from Lemma 3.1 and (3.16), we obtain
\[
m_1 \cdot 1 \leq \frac{\Delta f}{f} \leq m_1 \cdot 1
\]
so that
\[
\Delta f = m_1 f.
\]
The warping function \(f\) is an eigen-function with eigenvalue \(m_1\).

3. If the manifold \((M, g)\) is a 2-dimensional totally geodesic \(\phi\)-invariant submanifold of \((\overline{M}, \overline{g})\) with \(\xi \in \Gamma(TM)\) (i.e., \(\phi(TM) = TM\)), then from Lemma 3.1 and (3.16), we have
\[
m_1 c \leq \frac{\Delta f}{f} \leq m_1 c
\]
so that
\[
\Delta f = m_1 cf.
\]
The warping function \(f\) is an eigen-function with eigenvalue \(m_1 c\).

**Theorem 3.10.** Let \((M, g) = (B \times_f F, g_B + f^2 g_F)\) be a warped product manifold and \((\overline{M}, \overline{g}) = (\overline{M}(c, \phi, \xi, \eta, \overline{g}))\) a Kenmotsu space form of constant \(\phi\)-sectional curvature \(c\). Let \(\psi : (M, g) \mapsto (\overline{M}, \overline{g})\) be an isometric immersion. Then we obtain
\[
\frac{m_1 m^2}{2(m - 1)} H^2 - \frac{m_1}{2} ||h||^2 - m_1 \leq \frac{\Delta f}{f} \leq \frac{m^2}{4m_2} H^2 + m_1 c, \quad c \geq -1,
\]
\[
\frac{m_1 m^2}{2(m - 1)} H^2 - \frac{m_1}{2} ||h||^2 + m_1 c \leq \frac{\Delta f}{f} \leq \frac{m^2}{4m_2} H^2 - m_1, \quad c < -1,
\]
where \(m_1 = \dim B\) and \(m_2 = \dim F\) with \(m = m_1 + m_2\).

**Proof.** We know that the Riemannian curvature tensor \(\overline{R} \overline{7}\) of \((\overline{M}, \overline{g})\) is given by
\[
\overline{R}(X, Y) Z = \frac{c - 3}{4} (\overline{g}(Y, Z) X - \overline{g}(X, Z) Y)
\]
\[
+ \frac{c + 1}{4} (\eta(X) \eta(Z) Y - \eta(Y) \eta(Z) X + \eta(Y) \overline{g}(X, Z) \xi - \eta(X) \overline{g}(Y, Z) \xi
\]
\[
+ \overline{g}(\phi Y, Z) \phi X - \overline{g}(\phi X, Z) \phi Y - 2 \overline{g}(\phi X, Y) \phi Z).
\]
for \( X, Y, Z \in \Gamma(TM) \). Given orthonormal vectors \( X, Y \in T_pM \), \( p \in M \), we have

\begin{equation}
(3.19) \quad K(X \wedge Y) = R(X, Y, Y, X) = \frac{e - \frac{3}{4} + \frac{1}{4}}{(-\eta(Y))^2 - \eta(X)^2 + 3g(\phi X, Y)^2). \end{equation}

so that since \(-1 \leq -\eta(Y)^2 - \eta(X)^2 + 3g(\phi X, Y)^2 \leq 3\), we get

\begin{align*}
-1 \leq K(X \wedge Y) &\leq c, \quad c \geq -1, \\
c \leq K(X \wedge Y) &\leq -1, \quad c < -1.
\end{align*}

From Lemma 3.1, we obtain the result. □

**Remark 3.11.**

1. If the manifold \((M, g)\) is a 2-dimensional totally geodesic submanifold of \((\overline{M}, \overline{g})\) with \(\xi \in \Gamma(TM)\), then from Lemma 3.1 and (3.19), we have

\[ m_1 \cdot -1 \leq \triangle f \leq m_1 \cdot -1 \]

so that

\[ \triangle f = -m_1 f. \]

The warping function \(f\) is an eigen-function with eigenvalue \(-m_1\).

2. If the manifold \((M, g)\) is a 2-dimensional totally geodesic \(\phi\)-invariant submanifold of \((\overline{M}, \overline{g})\) with \(\xi \in \Gamma(TM^+)\) (i.e., \(\phi(TM) = TM\)), then from Lemma 3.1 and (3.19), we get

\[ m_1 c \leq \triangle f \leq m_1 c \]

so that

\[ \triangle f = m_1 cf. \]

The warping function \(f\) is an eigen-function with eigenvalue \(m_1 c\).

3. If the manifold \((M, g)\) is a totally geodesic \(\phi\)-totally real submanifold of \((\overline{M}, \overline{g})\) with \(\xi \in \Gamma(TM^+)\) (i.e., \(\phi(TM) \subset TM^+\)), then from Lemma 3.1 and (3.19), we obtain

\[ m_1 \frac{c - 3}{4} \leq \frac{\triangle f}{f} \leq m_1 \frac{c - 3}{4} \]

so that

\[ \triangle f = \frac{m_1 (c - 3)}{4} f. \]

The warping function \(f\) is an eigen-function with eigenvalue \(\frac{m_1 (c - 3)}{4}\).

**Theorem 3.12.** Let \((M, g) = (B \times F, g_B + f^2 g_F)\) be a warped product manifold and \((\overline{M}, \overline{g}) = G_2(C^{m+2}) = SU_{m+2}/SU_m U_2\) the complex two-plane Grassmannian. Let \(\phi : (M, g) \rightarrow (\overline{M}, \overline{g})\) be an isometric immersion. Then we have

\begin{equation}
(3.20) \quad \frac{m_1 m_2}{2(m - 1)} H^2 - \frac{m_1}{2} \|h\|^2 - m_1 \leq \frac{\triangle f}{f} \leq \frac{m_2^2}{4m_2} H^2 + 8m_1,
\end{equation}

where \(m_1 = \dim B\) and \(m_2 = \dim F\) with \(m = m_1 + m_2\).
Proof. We see that the Riemannian curvature tensor $\overline{R}$ of $(\bar{M}, \overline{g})$ is given by

$$\overline{R}(X,Y)Z = \overline{g}(Y, Z)X - \overline{g}(X, Z)Y + \overline{g}(JY, Z)JX - \overline{g}(JX, Z)JY - 2\overline{g}(JX, Y)JZ$$

$$+ \sum_{\alpha=1}^{3} (\overline{g}(J_{\alpha}Y, Z)J_{\alpha}X - \overline{g}(J_{\alpha}X, Z)J_{\alpha}Y - \overline{g}(J_{\alpha}X, Y)J_{\alpha}Z)$$

$$+ \sum_{\alpha=1}^{3} (\overline{g}(J_{\alpha}JY, Z)J_{\alpha}JX - \overline{g}(J_{\alpha}JX, Z)J_{\alpha}JY)$$

for $X, Y, Z \in \Gamma(T\bar{M})$. Given orthonormal vectors $X, Y \in T_{p}\bar{M}, \ p \in \bar{M}$, we get

$$\overline{K}(X \wedge Y) = \overline{R}(X, Y, Y, X) = 1 + 3\overline{g}(JX, Y)^{2}$$

$$+ \sum_{\alpha=1}^{3} (3\overline{g}(J_{\alpha}X, Y)^{2} + \overline{g}(J_{\alpha}JY, Y)\overline{g}(J_{\alpha}JX, X) - \overline{g}(J_{\alpha}JX, Y)^{2}).$$

With simple computations, we obtain

$$\overline{g}(JX, Y)^{2} \leq |JX|^{2}|Y|^{2} = 1,$$

$$\sum_{\alpha=1}^{3} \overline{g}(J_{\alpha}X, Y)^{2} \leq |Y|^{2} = 1 \quad \text{(since \{J_{1}X, J_{2}X, J_{3}X\} is orthonormal)},$$

$$\left| \sum_{\alpha=1}^{3} \overline{g}(J_{\alpha}JY, Y)\overline{g}(J_{\alpha}JX, X) \right| \leq \sqrt{\sum_{\alpha=1}^{3} \overline{g}(J_{\alpha}JY, Y)^{2}} \cdot \sqrt{\sum_{\alpha=1}^{3} \overline{g}(J_{\alpha}JX, X)^{2}}$$

$$\leq \sqrt{|Y|^{2}} \sqrt{|X|^{2}} = 1 \quad \text{(by Cauchy-Schwarz inequality and since \{J_{1}JY, J_{2}JY, J_{3}JY\} and \{J_{1}JX, J_{2}JX, J_{3}JX\} are orthonormal)}$$

$$\Rightarrow -1 \leq \sum_{\alpha=1}^{3} \overline{g}(J_{\alpha}JY, Y)\overline{g}(J_{\alpha}JX, X) \leq 1,$$

$$\sum_{\alpha=1}^{3} \overline{g}(J_{\alpha}JX, Y)^{2} \leq |Y|^{2} = 1 \quad \text{(since \{J_{1}JX, J_{2}JX, J_{3}JX\} is orthonormal)}.$$

By using the above relations, we obtain

$$\overline{K}(X \wedge Y) \leq 1 + 3 \cdot 1 + 3 \cdot 1 + 1 = 8.$$

On the other hand, by the above relations, we have

$$\overline{K}(X \wedge Y) \geq 1 + \sum_{\alpha=1}^{3} (\overline{g}(J_{\alpha}JY, Y)\overline{g}(J_{\alpha}JX, X) - \overline{g}(J_{\alpha}JX, Y)^{2})$$

$$\geq 1 - 1 - 1 = -1.$$

From Lemma 3.1, by using (3.23) and (3.24), the result follows.\]

Remark 3.13. 1. Choose orthonormal vectors $X, Y \in T_{p}\bar{M}, \ p \in \bar{M}$ such that $Y = JX$ and $X$ is a singular vector. i.e., conveniently, $JX = J_{1}X$ (See [1]). From (3.22), we get

$$\overline{K}(X \wedge Y) = 1 + 3 + 3 + 1 + 0 = 8.$$

So, the upper bound of the function $\overline{K}(X \wedge Y)$ is rigid.
2. If the manifold \((M, g)\) is a 2-dimensional totally geodesic \(J\)-invariant submanifold of \((\overline{M}, \overline{g})\) with a singular vector field \(X \in \Gamma(TM)\) (i.e., \(J(TM) = TM\)), then from Lemma 3.1 and (3.22), we get

\[
m_1 \cdot 8 \leq \frac{\nabla f}{f} \leq m_1 \cdot 8
\]

so that

\[
\nabla f = 8m_1 f.
\]

The warping function \(f\) is an eigen-function with eigenvalue \(8m_1\).

**Theorem 3.14.** Let \((M, g) = (B \times F, g_B + f^2 g_F)\) be a warped product manifold and \((\overline{M}, \overline{g}) = SU_{2, m}/S(U_2 \cdot U_m)\) the complex hyperbolic two-plane Grassmannian. Let \(\psi : (M, g) \mapsto (\overline{M}, \overline{g})\) be an isometric immersion. Then we obtain

\[
m_1 m_2^2 \frac{H^2}{2(m-1)} - \frac{m_1}{2} ||h||^2 - 4m_1 \leq \frac{\nabla f}{f} \leq \frac{m_2^2}{4m_2} H^2 + \frac{1}{2} m_1,
\]

where \(m_1 = \dim B\) and \(m_2 = \dim F\) with \(m = m_1 + m_2\).

**Proof.** We know that the Riemannian curvature tensor \(\overline{R}(X, Y)Z = -\frac{1}{2}(\nabla (Y, Z)X - \nabla (X, Z)Y + \nabla (JX, Z)JY - \nabla (JY, Z)JX - 2\nabla (JX, Y)JZ + \sum_{\alpha=1}^{3} (\nabla (J_{\alpha}X, Z)J_{\alpha}X - \nabla (J_{\alpha}X, Z)J_{\alpha}Y - \nabla (J_{\alpha}Y, Z)J_{\alpha}X - \nabla (J_{\alpha}Y, Z)J_{\alpha}Y))\)

for \(X, Y, Z \in \Gamma(TM)\).

Hence, in a similar way with Theorem 3.12, we easily get the result. \(\square\)

**Remark 3.15.** 1. We choose orthonormal vectors \(X, Y \in T_p \overline{M}, p \in \overline{M}\), such that \(Y = JX\) and \(X\) is a singular vector. i.e., conveniently, \(JX = J_1 X\) (See [2]). In a similar way with Remark 3.13, we obtain

\[
\overline{K}(X \wedge Y) = -4.
\]

So, the lower bound of the function \(\overline{K}(X \wedge Y)\) is rigid.

2. If the manifold \((M, g)\) is a 2-dimensional totally geodesic \(J\)-invariant submanifold of \((\overline{M}, \overline{g})\) with a singular vector field \(X \in \Gamma(TM)\), then similarly, we have

\[
m_1 \cdot -4 \leq \frac{\nabla f}{f} \leq m_1 \cdot -4
\]

so that

\[
\nabla f = -4m_1 f.
\]

The warping function \(f\) is an eigen-function with eigenvalue \(-4m_1\).

**Theorem 3.16.** Let \((M, g) = (B \times F, g_B + f^2 g_F)\) be a warped product manifold and \((\overline{M}, \overline{g}) = Q^m = SO_{m+2}/SO_m SO_2\) the complex quadric. Let \(\psi : (M, g) \mapsto (\overline{M}, \overline{g})\) be an isometric immersion. Then we get

\[
m_1 m_2^2 \frac{H^2}{2(m-1)} - \frac{m_1}{2} ||h||^2 - 2.3m_1 \leq \frac{\nabla f}{f} \leq \frac{m_2^2}{4m_2} H^2 + 5m_1,
\]
where \( m_1 = \dim B \) and \( m_2 = \dim F \) with \( m = m_1 + m_2 \).

**Proof.** We see that the Riemannian curvature tensor \( \overline{R} \) of \((\overline{M}, \overline{\eta})\) is given by
\[
\overline{R}(X, Y)Z = \overline{\eta}(Y, Z)X - \overline{\eta}(X, Z)Y + \overline{\eta}(JY, Z)JX - \overline{\eta}(JX, Z)JY + \overline{\eta}(JA_Y, Z)JAX - \overline{\eta}(JAX, Z)JAY
\]
for \( X, Y, Z \in \Gamma(T\overline{M}) \). Given orthonormal vectors \( X, Y, \pi \in T_p\overline{M}, p \in \overline{M} \), we obtain
\[
\overline{K}(X \wedge Y) = \overline{R}(X, Y, X) = 1 + 3\overline{\eta}(JX, Y)^2 + \overline{\eta}(AY, Y)\overline{\eta}(AX, X) - \overline{\eta}(AX, Y)^2 + \overline{\eta}(JAY, Y)\overline{\eta}(JAX, X) - \overline{\eta}(JAX, Y)^2.
\]

Since \( A \) is an involution (i.e., \( A^2 = \text{id} \)), we get the following decompositions
\[
X = a\overline{x}_1 + b\overline{x}_2, \\
Y = c\overline{y}_1 + d\overline{y}_2,
\]
where \( |\overline{x}_1| = |\overline{x}_2| = |\overline{y}_1| = |\overline{y}_2| = 1 \), \( \overline{x}_1, \overline{y}_1 \in V(A) = \{ Z \in T_p\overline{M} | AZ = Z \}, \overline{x}_2, \overline{y}_2 \in JV(A) \) (See [13]) so that
\[
1 = |X|^2 = a^2 + b^2, \\
1 = |Y|^2 = c^2 + d^2, \\
0 = \overline{\eta}(X, Y) = a\overline{\eta}(\overline{x}_1, \overline{y}_1) + b\overline{\eta}(\overline{x}_2, \overline{y}_2).
\]

Conveniently, let \( (a, b) = (\cos \alpha, \sin \alpha) \) and \( (c, d) = (\cos \beta, \sin \beta) \).

If necessary, by replacing \( \overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2 \) with \(-\overline{x}_1, -\overline{x}_2, -\overline{y}_1, -\overline{y}_2\), respectively, we may assume
\[
0 \leq \alpha, \beta \leq \frac{\pi}{2}.
\]

Thus, with a simple calculation, we have
\[
\overline{K}(X \wedge Y) = 1 + 2\overline{\alpha}^2 \cos^2 \alpha \sin^2 \beta + 2\overline{\beta}^2 \sin^2 \alpha \cos^2 \beta + \cos 2\alpha \cos 2\beta + 2\overline{\alpha} \sin 2\alpha \sin 2\beta + \overline{\beta} \sin 2\beta - \overline{\beta}^2 \cos^2 \alpha \cos^2 \beta,
\]
where
\[
\overline{\alpha} = \overline{\eta}(\overline{x}_1, J\overline{y}_2) \\
\overline{\beta} = \overline{\eta}(\overline{x}_2, J\overline{y}_1) \\
\overline{\beta} = \overline{\eta}(J\overline{y}_1, \overline{y}_2) \\
\overline{\beta} = \overline{\eta}(J\overline{x}_1, \overline{x}_2) \\
\overline{\beta} = \overline{\eta}(\overline{x}_1, \overline{y}_1).
\]

We see
\[
-1 \leq \overline{\alpha}, \overline{\beta}, \overline{\beta}, \overline{\beta}, \overline{\beta} \leq 1.
\]
Consider the function
\[
S(x, y) = 2\overline{\alpha}^2 \cos^2 x \sin^2 y + 2\overline{\beta}^2 \sin^2 x \cos^2 y + \cos 2x \cos 2y + 2\overline{\alpha} \sin 2x \sin 2\beta + \overline{\beta} \sin 2\beta - \overline{\beta}^2 \cos^2 x \cos^2 y
\]
for \((x, y) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]\).
ON THE ESTIMATES OF WARping FUNCTIONS

\[ z = \cos(2x + 2y) \pm 2\sin(2x)\sin(2y) - \cos(x)^2\cos(y)^2, \quad z = \pm 3.2 \]

\[ \begin{array}{cccccccc}
0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 & 1.2 & 1.4
\end{array} \]

\[ \begin{array}{cccccccc}
0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 & 1.2 & 1.4
\end{array} \]

(a) \( z = h(x, y) \) and \( z = -3.2 \)

(b) \( z = h(x, y) \) and \( z = -3.3 \)

**Figure 1.** The lower bound of \( h(x, y) \)

Since \( \sin 2x \sin 2y \geq 0 \), by (3.32), we obtain

\[
S(x, y) \leq 2\cos^2 x \sin^2 y + 2\sin^2 x \cos^2 y \\
+ \cos 2x \cos 2y + 2 \sin 2x \sin 2y + \sin 2x \sin 2y \\
= 2(\cos x \sin y + \sin x \cos y)^2 + \cos(2x - 2y) + \sin 2x \sin 2y \\
= 2\sin^2(x + y) + \cos(2x - 2y) + \sin 2x \sin 2y \\
\leq 4
\]
and
\begin{equation}
S(x, y) \geq \cos 2x \cos 2y - 2 \sin 2x \sin 2y \\
- \sin 2x \sin 2y - \cos^2 x \cos^2 y \\
= \cos(2x + 2y) - 2 \sin 2x \sin 2y - \cos^2 x \cos^2 y.
\end{equation}

Consider the function \( h(x, y) = \cos(2x + 2y) - 2 \sin 2x \sin 2y - \cos^2 x \cos^2 y \) for \((x, y) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]\).

We see
\begin{equation}
h(x, y) \geq -3.3 \quad (\text{See Figure 1}).
\end{equation}

From Lemma 3.1, by using (3.31), (3.33), (3.34), (3.35), and (3.36), the result follows. □

Remark 3.17. 1. We get \( h(\pi/4, \pi/4) = -3.25 \). But \( h_x(\pi/4, \pi/4) = -1/2 \neq 0 \) and \( h_y(\pi/4, \pi/4) = 1/2 \neq 0 \), which implies that \((\pi/4, \pi/4)\) is not a critical point of \( h(x, y) \).

2. If the manifold \((M, g)\) is a 2-dimensional totally geodesic \(J\)-invariant submanifold of \((\overline{M}, \overline{g})\) with a non-vanishing vector field \( X \in \Gamma(TM) \cap V(A) \), then by using (3.29), we have
\[
m_1 \cdot 2 \leq \frac{\Delta f}{f} \leq m_1 \cdot 2
\]
so that
\[
\Delta f = 2m_1 f.
\]
The warping function \( f \) is an eigen-function with eigenvalue \( 2m_1 \).

3. If the manifold \((M, g)\) is a 2-dimensional totally geodesic submanifold of \((\overline{M}, \overline{g})\) with \( TM \subset V(A) \), then by using (3.29), we get
\[
m_1 \cdot 2 \leq \frac{\Delta f}{f} \leq m_1 \cdot 2
\]
so that
\[
\Delta f = 2m_1 f.
\]
The warping function \( f \) is an eigen-function with eigenvalue \( 2m_1 \).

4. If the manifold \((M, g)\) is a 2-dimensional totally geodesic submanifold of \((\overline{M}, \overline{g})\) with \( TM \perp J(TM) \) and \( \dim(TM \cap V(A)) = \dim(TM \cap JV(A)) = 1 \), then by using (3.29), we obtain
\[
m_1 \cdot 0 \leq \frac{\Delta f}{f} \leq m_1 \cdot 0
\]
so that
\[
\Delta f = 0.
\]
The warping function \( f \) is a harmonic function.

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