SHARP ASYMPTOTICS FOR TOEPLITZ DETERMINANTS AND CONVERGENCE TOWARDS THE GAUSSIAN FREE FIELD ON RIEMANN SURFACES

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Abstract. We consider canonical determinantal random point processes with \( N \) particles on a compact Riemann surface \( X \) defined with respect to the constant curvature metric. We establish strong exponential concentration of measure type properties involving Dirichlet norms of linear statistics. This gives an optimal Central Limit Theorem (CLT), saying that the fluctuations of the corresponding empirical measures converge, in the large \( N \) limit, towards the Laplacian of the Gaussian free field on \( X \) in the strongest possible sense. The CLT is also shown to be equivalent to a new sharp strong Szegö type theorem for Toeplitz determinants in this context. One of the ingredients in the proofs are new Bergman kernel asymptotics providing exponentially small error terms in a constant curvature setting.

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1. Introduction

This paper is one in a series which deal with \( N \)-particle determinantal point processes on a polarized compact complex manifold \( X \), i.e. associated to high powers of an ample line bundle \( L \rightarrow X \). In the paper in [4] a general Large Deviation Principle (LDP) for such processes was established in the large \( N \)-limit showing that the empirical measures converge exponentially towards the deterministic pluripotential equilibrium measure. Moreover, in the paper [3] a Central Limit Theorem (CLT) was obtained, showing that the fluctuations in the “bulk” may be described by a Gaussian free field in the case of smooth test functions (linear statistics). In the present paper we specialize to the lowest dimensional case when \( X \) is a Riemann surface and the corresponding \( N \)-particle point processes are the “canonical” ones, i.e. the they are induced by the Kähler-Einstein metric on \( X \). In this setting we obtain sharp versions of the upper large deviation bound and show that the convergence towards the Gaussian free field holds in the strongest possible sense, i.e. for linear statistics with minimal regularity assumptions (finite Dirichlet norm). This CLT is equivalent to a new sharp strong Szegö type theorem for Toeplitz determinants in this context. The results are obtained from new “determinantal” Moser-Trudinger type inequalities, which imply strong concentration of measures properties. The proof of these latter inequalities is based on a convexity argument in the space of all Kähler metrics, combined with Bergman kernel asymptotics and potential theory.
1.1. The general setup. Let \( L \to X \) be an ample holomorphic line bundle over a compact complex manifold \( X \) of dimension \( n \). We will denote by \( H^0(X,L) \) the \( N \)--dimensional vector space of all global holomorphic sections of \( L \). Given the geometric data \((\nu,\|\cdot\|)\) consisting of a probability measure \( \nu \) on \( X \) and a continuous Hermitian metric \( \|\cdot\| \) on \( L \) one obtains an associated probability measure \( \mu^{(N)} \) on the \( N \)--fold product \( X^N \) defined as

\[
\mu^{(N)} := \frac{1}{Z_N} \|\det \Psi\|^2 (x_1, \ldots, x_N) \nu(x_1) \otimes \cdots \otimes \nu(x_N)
\]

where \( \det \Psi \) is a holomorphic section of the pulled-back line bundle \( L^{\otimes N} \) over \( X^N \) representing the \( N \)th (i.e. maximal) exterior power of \( H^0(X,L) \) and \( Z_N \) is the normalizing constant. Concretely, fixing a base \((\Psi_i)_{i=1}^N \) in \( H^0(X,L) \) we can take

\[
(\det \Psi)(x_1, \ldots, x_N) = \det(\Psi_i(x_j))
\]

We will denote \( \frac{i}{\pi} \) times the curvature two-form of the metric on \( L \) by \( \omega \) (compared with mathematical physics notation \( \omega = \frac{i}{2} F_A \) where \( A \) is the Chern connection induced by the metric on \( L \)). It will be convenient to take the pair \((\omega, \nu)\), which will refer to as a weighted measure, as the given geometric data.

The empirical measure of the ensemble above is the following random measure:

\[
(x_1, \ldots, x_N) \mapsto \delta_N := \sum_{i=1}^N \delta_{x_i}
\]

which associates to any \( N \)--particle configuration \((x_1, \ldots, x_N)\) the sum of the delta measures on the corresponding points in \( X \). In probabilistic terms this setting hence defines a determinantal random point process on \( X \) with \( N \) particles [23, 28].

If the corresponding \( L^2 \)--norm on \( H^0(X,L) \)

\[
\|\Psi\|^2_X = \langle \Psi, \Psi \rangle_X := \int_X \|\Psi(x)\|^2 d\nu(x)
\]

is non-degenerate (which will always be the case in this paper) then the probability measure \( \mu^{(N)} \) on \( X^N \) may be expressed as a determinant of the Bergman kernel of the Hilbert space \((H^0(X,L),\|\cdot\|_X)\), i.e. the integral kernel of the corresponding orthogonal projection \( \Pi \). A central role in this paper will be played by the logarithmic generating function (or free energy)

\[
\log \mathbb{E}(e^{-\sum_{i=1}^N \phi(x_i)})
\]

of the linear statistic

\[
\sum_{i=1}^N \phi(x_i),
\]

where \( \mathbb{E} \) denotes the expectation wrt the ensemble \((X^N, \mu^{(N)})\), i.e. \( \mathbb{E}(\cdot) = \int_{X^N} (\cdot) \mu^{(N)} \).

By a well-known formula going back to the work of Heine in the theory of orthogonal polynomials the expectation above can also be written as a Toeplitz determinant with symbol \( e^{-\phi} \):

\[
\mathbb{E}(e^{-\sum_{i=1}^N \phi(x_i)}) = \det(\left\langle e^{-\phi} \Psi_i, \Psi_j \right\rangle_X) = \det T[e^{-\phi}]
\]

where \((\Psi_i)_{i=1}^N\) is an orthonormal base in the Hilbert space \((H^0(X,L),\|\cdot\|_X)\) (and \( T[f] := \Pi(f) \)) is the corresponding Toeplitz operator on \( H^0(X,L) \) with symbol \( f \).

Replacing \( L \) with its \( k \)th tensor power, which we will write in additive notation as \( kL \), yields a sequence of point processes on \( X \) of an increasing number \( N_k \) of
particles. We will be concerned with the asymptotic situation when \( k \to \infty \). This corresponds to a large \( N \)–limit of many particles, since
\[
N_k := \dim H^0(X, kL) = V k^n + o(k^n)
\]
where the constant \( V \) is, by definition, the \textit{volume} of \( L \).

As shown in [4] the normalized empirical measure \( \delta_N/N_k \) converges towards a pluripotential equilibrium measure \( \mu_{eq} \), exponentially in probability. In particular, letting
\[
(1.7) \quad \epsilon_{N_k, \lambda}(\phi) := \text{Prob} \left\{ \left| \frac{1}{N_k} \left( \phi(x_1) + \ldots + \phi(x_{N_k}) \right) - \int_X \mu_{eq} \phi \right| > \lambda \right\}
\]
denote the \textit{tail of the linear statistic} determined by \( \phi \), at level \( k \), it was shown that \( \epsilon_{N_k, \lambda}(\phi) \to 0 \) as \( k \to \infty \) for any \( \lambda > 0 \) at a rate of the order \( e^{-k^{n+1}/C} \). In the case when \( X \) is a Riemann surface the curvature current \( \omega \) of the metric on \( L \) is semi-positive (so that \( \mu_{eq} = \omega \) the following more precise estimate was obtained:
\[
(1.8) \quad \epsilon_{N_k, \lambda}(\phi) \leq 2 \exp(-N_k^2 \left| \frac{2V \lambda^2}{\|d\phi\|^2_X} (1 + o(1)) \right|)
\]
where the error term \( o(1) \) denotes a sequence tending to zero as \( k \to \infty \) (but depending on \( \phi \)).

1.2. The canonical setting on a Riemann surface. Let now \( L \to X \) be a line bundle of positive volume (degree) \( V \) over a Riemann surface \( X \) of genus \( g \). It determines a particular sequence of determinantal point process that we will refer to as the \textit{canonical determinantal point process on} \( X \) \textit{associated to} \( kL \). These processes are obtained by letting \( (\nu, \omega) = (\omega/V, \omega) \) for \( \omega \) the the unique volume form on \( X \) of volume \( V \) such that Riemannian metric determined by \( \omega \) has constant scalar curvature. By the Riemann-Roch theorem we have (for \( k \) sufficiently large)
\[
(1.9) \quad N_k = kV - (g - 1)
\]
giving a simple relation between the level \( k \) and the corresponding number of particles \( N_k \). Accordingly, it will be convenient to talk about the \textit{canonical determinantal random point process on} \( X \) \textit{with} \( N \) \textit{particles} and use \( N \) as the asymptotic parameter. Strictly speaking \( N (= N_k) \) only determines \( L \) up to twisting by a flat line bundle, but the results will be independant of the flat line bundle. Physically, the canonical processes associated to \( kL \) represents the groundstate of a gas of spin-polarized free fermions in the “uniform” magnetic field \( kF_A \) where \( \omega = \frac{1}{2\pi} F_A \) and \( A \) is a unitary connection on \( L \) (see [4] and references therein). Equivalently, these processes are defined by the lowest Landau level of the corresponding magnetic Schrödinger operator.

The simplest case of this setting occurs when \( g = 0 \), i.e. \( X \) is the Riemann sphere and then \( H^0(X, kL) \) may be identified with the space of all polynomials on the affine piece \( \mathbb{C} \) of degree at most \( k = N - 1 \) equipped with the usual \( SU(2) \)–invariant Hermitian product. Alternatively, embedding \( X \) as the unit-sphere in Euclidian \( \mathbb{R}^3 \) the \( N \)–point correlation function of the process, i.e. the density of the probability measure, may be explicitly expanded as
\[
(1.10) \quad \rho^{(N)}(x_1, \ldots, x_N) := \Pi_{1 \leq i < j \leq N} \|x_i - x_j\|^2 / Z_N
\]
where \( 1/Z_N = N^N (N-1)! \ldots (N\text{-}1)! / N! \). In the physics litterature this ensemble also appears as a \textit{Coulomb gas} of \( N \) unit-charge particles (i.e a one component plasma) confined to the sphere in a neutralizing uniform background \( \omega \) (see for example [11]). An interesting random matrix model for this process was recently given in [29]. In the higher genus case the role of polynomials are played by theta functions.
and modular (automorphic) forms on the universal covers \( \mathbb{C} \) and \( \mathbb{H} \) of \( X \) (when \( g = 1 \) and \( g > 1 \) respectively) equipped with their standard Hermitian products. See for example [17] for the case \( g = 1 \) in connection to fermions and bosonization. When \( g > 1 \) the Riemann surface \( X \) may be represented as the quotient \( \Gamma / \mathbb{H} \) of the upper half-plane with a suitable discrete subgroup \( \Gamma \). Taking \( L := \frac{1}{2} K_X \), where \( K_X \) denotes the canonical line bundle \( K_X = T^* X \) (using the induced spin structure to take the square root of \( K_X \)) realizes \( H^0(X, kL) \) as the Hilbert space of all modular forms of weight \( k \), i.e. all holomorphic functions on \( \mathbb{H} \) satisfying \( f((az + b)/(cz + d)) = (cz + d)^k f(z) \) equipped with the Petterson norm

\[
\|f\|_X^2 := \int_{\Gamma / \mathbb{H}} |f|^2 y^k \frac{dx \wedge dy}{y^2},
\]

integrating over a fundamental domain for \( \Gamma \). In special arithmetic situation the base \( (\Psi_i) \) in 1.2 may be represented by Hecke eigenfunctions (but note that we have assumed that \( X \) is smooth and compact and in particular there are no cusps)[30].

1.3. Statement of the main results. It will be convenient to use the following conformally invariant notation for the normalized Dirichlet norm of a function \( \phi \) on \( X \), i.e. the \( L^2 \)-norm of its gradient times \( 1/4\pi \):

\[
\|d\phi\|_X^2 := \int_X d\phi \wedge d\bar{\phi} := \left( \frac{i}{2\pi} \int_X \partial \phi \wedge \bar{\partial} \phi \right).
\]

We will obtain a very useful Moser-Trudinger type inequality for the canonical determinantal point processes, which generalizes Onofri’s sharp version of the Moser-Trudinger inequality [36] (obtained when \( X \) is the two-sphere and \( N = 1 \)).

**Theorem 1.1.** Let \( X \) be a genus \( g \) Riemann surface and consider the canonical determinantal point process on \( X \) with \( N \) particles. It satisfies the following Moser-Trudinger type inequality:

\[
\log \mathbb{E}(e^{-\sum_{i=1}^N (\phi(x_i) - f_X \phi(x_i))}) \leq \left( \frac{1}{1 + (1 - g)/N} + \epsilon_N \right) \frac{1}{2} \|d\phi\|_X^2 + \epsilon_N
\]

where the error term \( \epsilon_N \) is exponentially small, i.e., \( \epsilon_N \leq Ce^{-N\delta} \) for some positive number \( C \) and \( \delta \) independent of \( \phi \) and where \( \delta \) can be explicitly expressed in terms of the injectivity radius of \( (X, \omega) \) (see formula 2.4 in Prop 2.1). Similarly,

\[
\log \mathbb{E}(e^{-\sum_{i=1}^N (\phi(x_i) - \mathbb{E}(\phi(x_i)))}) \leq \left( \frac{1}{1 + (1 - g)/N} + \epsilon_N \right) \frac{1}{2} \|d\phi\|_X^2 + \epsilon_N \|\phi\|_{L^1(X)/\mathbb{R}} + \epsilon_N
\]

Moreover, when \( X \) is the Riemann sphere (i.e \( g = 0 \)) all the error terms above vanish identically.

An important ingredient in the previous proof is a convexity result of Berndtsson [8] which in this particular case essentially amounts to the positivity of a certain determinantal line bundle over the space of all Kähler metrics in the first Chern class of \( L \). The error terms \( \epsilon_N \) above come from the error terms in the Yau-Tian-Zelditch-Catlin expansion [44, 2, 33, 32] for the underlying Bergman kernel. As follows from Theorem 3.1 below these error terms are exponentially small, slightly refining previous recent results in [31, 32] (see section 3 for precise formulations).

As a simple consequence of the previous theorem we then obtain a sharp version of the tail estimate 1.8 for such canonical processes. The main point is that it shows that the error term \( o(1) \) appearing in the estimate 1.8 can be taken to be independent of the function \( \phi \). As a consequence the estimate holds with minimal regularity assumptions on \( \phi \):

\[
\frac{1}{2} \|d\phi\|_X^2 + \epsilon_N \|\phi\|_{L^1(X)/\mathbb{R}} + \epsilon_N
\]
Corollary 1.2. Let $X$ be a genus $g$ Riemann surface and consider the canonical determinantal point process on $X$ with $N$ particles. Let $\phi$ be a function on $X$ such that its differential $d\phi$ is in $L^2(X)$. Then the linear statistic defined by $\phi$ has an exponentially decaying tail:

$$
\epsilon_{N,\lambda}(\phi) \leq 2 \exp\left(-N^2 \left( \frac{2\lambda^2}{\|d\phi\|_X^2 \left(1 + \frac{(1-g)}{N} + \epsilon_N\right)} + \epsilon_N \right) \right)
$$

where the error terms $\epsilon_N$ are as in the previous theorem.

We will also show that the Moser-Trudinger inequality in Theorem 1.1 is in fact an asymptotic equality in the following sense:

**Theorem 1.3.** (strong Szegö type theorem). Let $X$ be a genus $g$ Riemann surface and consider the canonical determinantal point process on $X$ with $N$ particles. Let $\phi$ be a complex valued function on $X$ such that its differential is in $L^2(X,\mathbb{C})$, i.e. $\phi$ has finite Dirichlet norm. Then

$$
\log \mathbb{E}(e^{-\sum_{i=1}^N (\phi(x_i) - \int_X \phi \omega)}) \rightarrow \frac{1}{2} \int_X d\phi \wedge d^c \phi
$$

as $N \rightarrow \infty$ and the same convergence holds when the exponent above is replaced with the fluctuation of the linear statistic of $\phi$.

In [3] it was shown that, as long as $\omega > 0$ and $\phi$ is smooth an analogue of the convergence above holds in any dimension $n$ if the conformally invariant norm above is replaced by the Dirichlet norm wrt $\omega$. But it should be emphasized that when $n > 1$ the convergence does not hold if one relaxes the smoothness assumption on $\phi$ to allowing a gradient in $L^2$ (see section 2.4 for counter examples).

The previous theorem may be equivalently formulated as the following Central Limit Theorem (CLT), valid under minimal regularity assumptions:

**Corollary 1.4.** (CLT) The fluctuations $\delta_N - \mathbb{E}(\delta_N)$ of the empirical measure $\delta_N$ converge in distribution to the the Laplacian (or rather $dd^c$) of the Gaussian free field (GFF). In other words, for any $\phi \in L^1(X)$ with $d\phi \in L^2(X)$ the fluctuations

$$
\sum_{i=1}^N (\phi(x_i) - \mathbb{E}(\phi(x_i)))
$$

of the corresponding linear statistics converge in distribution to a centered normal random variable with variance $\|d\phi\|_X^2$.

The GFF is also called the massless bosonic free field in the physics litterature. Heuristically, this is a random function wrt the Gaussian measure on the Hilbert space of all $\phi \mod \mathbb{R}$ equipped with the Dirichlet norm $\|d\phi\|_X^2 / 2$. For the precise definition of the GFF and its Laplacian see [40] (Prop 2.13 and Remark 2.14) and for a comparison with the physics litterature on Coulomb gases see section 1.3 in [41].

### 1.4. Relations to previous results.

**Exponential concentration.** A determinantal Moser-Trudinger (M-T) inequality on $S^2$, but with non-optimal constants was first obtained by Fang [15] building on previous work by Gillet-Soulé concerning the $S^1$–invariant case [18], which in turn used the classical Moser-Truding (one-particle) inequality. The motivation came from arithmetic (Arakelov) geometry and spectral geometry. The optimal constants on $S^2$ were obtained by the author in [5] using methods further developed in the present paper. It would be interesting to know for which other (determinantal) random point processes similar inequalities hold, i.e. upper bounds on the
logarithmic moment generating function of the linear statistic defined by $\phi(x)$ in terms of the Dirichlet norm $\|d\phi\|^2_X$. The only previously known case seems to be the case when the measure measure $\nu$ is the invariant measure on $S^1$ (and $\omega = 0$), corresponding to the standard unitary random matrix ensemble. Then the corresponding inequalities follow from a simple monotonicity argument going back to the classical work of Szegö (see for example [25] and references therein). Recently, several works have been concerned with a weaker form of such moment inequalities where the role of the Dirichlet norm is played by the Lipschitz norm. These inequalities fit into a circle of ideas surrounding the “concentration of measure phenomena” in high dimensions. We refer to the survey [21] and the book [34] for precise references. Formulated in the present settings these latter inequalities hold for $\nu = 1 e^{-v(x)} dx$ with $v(x)$ strictly convex (satisfying $d^2 v/d^2 x > C$). As explained in [21], by the Bakry-Emery theorem and Klein’s lemma, the corresponding point processes satisfy a log Sobolev inequality, which by Herbst’s argument yields the desired inequality on the logarithmic moment generating function.

**Szegö type limits and CLT’s.** The convergence in Theorem 1.3 (and its Corollary) in the case when $X = S^2$ was first obtained by Ryder-Virag [38], using combinatorial (and diagrammatic) arguments to estimate the cumulants (i.e. the coefficients in the Taylor expansion of the logarithmic moment generating function), combined with estimates on the 2-point functions. They also obtained analogous results for the homogenous determinantal point processes on the other two simply connected Riemann surfaces, i.e on $\mathbb{C}$ and $\mathbb{H}$. However, in the latter cases the processes have an infinite number of particles and are hence different from the sequence of non-homogenous ones considered in the present paper on a compact Riemann surfaces of genus $g > 0$. In the circle case (referred to above), assuming $\phi$ smooth, the analogue of the convergence in Thm 1.3 is the celebrated Szegö strong limit theorem from 1952. In this case the Dirichlet norm of $\phi$ has to be replaced by the Dirichlet norm of the harmonic extension of $\phi$ to the unit-disc. The result of Szegö was motivated by Onsager’s work on phase transitions for the 2D Ising model. The case of a general $\phi$ was eventually shown by Ibragimov [24]. A new proof was then given by Kurt Johansson [25], who also pointed out the relation to a CLT for the unitary random matrix ensemble. See also [14] for generalizations of the latter CLT using explicit moment calculations and harmonic analysis. We refer to the survey [42] for an interesting account of the history of Szegö’s theorem. It is also interesting to compare the appearance of exponentially small error terms in the inequalities 1.1 with the exponentially small error terms obtained in [42] in the context of the classical strong Szegö theorem. The proof in the Riemann surface cases in the present paper is partly inspired by the argument in [25], where the determinantal Moser-Trudinger inequalities on $S^1$ (referred to above) were used to reduce the upper bound in the convergence to the smooth case, also using analytic continuation. There are also similar convergence results for other weighted measures in the plane appearing in Random Matrix Theory, but regularity assumptions on $\phi$ are then imposed [25, 26]. It should be emphasizes that the classical strong Szegö theorem has previously been extended and extensively studied in various other directions, notably in the context of pseudo-differential operators and in particular Schrödinger operators (see [42] and references therein). Compared to the present paper the role of Schrödinger operators is here played by magnetic Schrödinger operators. Finally, it may also be interesting to compare the CLT above with the central limit theorem and variance asymptotics obtained very recently in [35] for non-smooth linear statistics in the different context of random point processes defined by zeroes of Gaussian entire functions.
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1.5. Notation. Let $L \to X$ be a holomorphic line bundle over a compact complex manifold $X$.

1.5.1. Metrics on $L$. We will fix, once and for all, a Hermitian metric $\|\cdot\|$ on $L$. Its curvature form times the normalization factor $\frac{1}{2\pi}$ will be denoted by $\omega$. The normalization is made so that $[\omega]$ defines an integer cohomology class, i.e. $[\omega] \in H^2(X, \mathbb{Z})$. The local description of $\|\cdot\|$ is as follows: let $s$ be a trivializing local holomorphic section of $L$, i.e. $s$ is non-vanishing on a given open set $U$ in $X$. Then we define the local weight $\Phi$ of the metric $\|\cdot\|$ by the relation

$$\|s\|^2 = e^{-\Phi}$$

The (normalized) curvature current $\omega$ may now be defined by the following expression:

$$\omega = \frac{i}{2\pi} \partial \bar{\partial} \Phi := dd^c \Phi,$$

(where we, as usual, have introduced the real operator $d^c := i \left( -\partial + \bar{\partial} \right) / 4\pi$ to absorb the factor $\frac{1}{2\pi}$). The point is that, even though the function $\Phi$ is merely locally well-defined, the form $\omega$ is globally well-defined (as any two local weights differ by $\log |g|^2$ for $g$ a non-vanishing holomorphic function). The current $\omega$ is said to be positive if the weight $\Phi$ is plurisubharmonic (psh). If $\Phi$ is smooth this simply means that the Hermitian matrix $\omega_{ij} = \left( \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right)$ is positive definite (i.e. $\omega$ is a Kähler form) and in general it means that, locally, $\Phi$ can be written as a decreasing limit of such smooth functions.

1.5.2. Holomorphic sections of $L$. We will denote by $H^0(X, L)$ the space of all global holomorphic sections of $L$. In a local trivialization as above any element $\Psi$ in $H^0(X, L)$ may be represented by a local holomorphic function $f$, i.e.

$$\Psi = fs$$

The squared point-wise norm $\|\Psi\|^2(x)$ of $\Psi$, which is a globally well-defined function on $X$, may hence be locally written as

$$\|\Psi\|^2(x) = (|f|^2 e^{-\Phi})(x)$$

It will be convenient to take the curvature current $\omega$ as our geometric data associated to the line bundle $L$. Strictly speaking, it only determines the metric $\|\cdot\|$ up to a multiplicative constant but all the geometric and probabilistic constructions that we will make are independent of the constant.

1.5.3. Metrics and weights vs $\omega$– psh functions. Having fixed a continuous Hermitian metric $\|\cdot\|$ on $L$ with (local) weight $\Phi_0$ any other metric may be written as

$$\|\cdot\|^2_{\phi} := e^{-\phi} \|\cdot\|^2$$

for a continuous function $\phi$ on $X$, i.e. $\phi \in C^d(X)$. In other words, the local weight of the metric $\|\cdot\|_{\phi}$ may be written as $\Phi = \phi + \Phi_0$ and hence its curvature current may be written as

$$dd^c \Phi = \omega + dd^c \phi := \omega_{\phi}$$

\(^1\)general references for this section are the books [19, 13]. See also [1] for the Riemann surface case.
This means that we have a correspondence between the space of all (singular) metrics on \(L\) with positive curvature current and the space \(PSH(X,\omega)\) of all upper-semi continuous functions on \(X\) such that \(\omega_\phi \geq 0\) in the sense of currents. Note for example, that if \(\Psi \in H^0(X,L)\) then \(\log \|\Psi\|^2 \in PSH(X,\omega)\). In particular, in the Riemann surface case \(PSH(X,\omega)(= SH(X,\omega))\) is the space of all usc functions \(\phi\) such that \(\Delta_\omega \phi \geq -1\), where \(\Delta_\omega\) denotes the Laplacian wrt the Riemannian metric corresponding to \(\omega\), i.e.
\[
\Delta_\omega \phi = (dd^c \phi)/\omega
\]
(where by our normalizations \(\Delta_\omega = \frac{1}{4\pi}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\) in the case when \(\omega\) is locally Euclidean).

2. Canonical point processes (Proofs of the main results)

For a general Kähler manifold \((X,\omega)\) there is well-known energy type functional which may be written as
\[
(2.1) \quad \mathcal{E}_\omega(\phi) := \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \omega_j^\phi \wedge (\omega)^{n-j}
\]
Up to normalization it can be defined as the primitive of the Monge-Ampère operator seen as a one-form on the space of all Kähler potentials \(\phi\) (and it was in this form it was first introduced by Mabuchi in Kähler geometry; see [4] and references therein). This means that \(d\mathcal{E}_\omega(\phi) = \omega_\phi^\phi/V\) (in the sense of formula 2.6 below)

We now turn to the case when \(X\) is a Riemann surface, i.e. \(n = 1\). In particular, after an integration by parts \(\mathcal{E}_\omega\) can then be expressed in terms of the usual Dirichlet energy on a Riemann surface:
\[
(2.2) \quad V \mathcal{E}_\omega(\phi) = -\frac{1}{2} \int d\phi \wedge d^c \phi + \int \phi \omega
\]
Following [5] it will also be convenient to consider a variant of the setting given in the introduction of the paper where the Hilbert space is the space \(H^0(X,kL + K_X)\) of holomorphic one-form with values in \(L\) equipped with the canonical Hermitian product induced by the weight \(\Phi\) on \(L:\)
\[
(2.3) \quad \langle \Psi, \Psi \rangle_X := i \int_X \Psi \wedge \Psi e^{-k\Phi}
\]
(equivalently, one picks a volume form \(\mu\) on \(X\) and takes \(1/\mu\) as the metric on \(K_X\)). We will call this the adjoint setting and the corresponding process on \(X\) the adjoint determinantal point process at level \(k\). Anyway, as explained below, the adjoint and the canonical point processes coincide when the curvature form \(\omega\) of \(\Phi\) has constant curvature. We note that if \(\delta_N\) denotes the empirical measure for the adjoint process with \(N\) particles then
\[
\mathbb{E}(\delta_N) = i \sum_{i=1}^{N} \Psi_i \wedge \Psi_i e^{-\Phi},
\]
for an orthonormal base \((\Psi_i)_i\), i.e. \(\mathbb{E}(\delta_N)\) is equal to the restriction to the diagonal of the Bergman kernel \(K_\Phi(x,y)\) of \(H^0(X,kL + K_X)\); see section 3.

**Proposition 2.1.** Let \(L \to X\) be a line bundle of degree \(V\) over a Riemann surface of genus \(g\). Assume that \(L\) is equipped with a metric \(e^{-\Phi}\) with strictly positive curvature form \(\omega(= dd^c \Phi)\) such that the Riemannian metric on \(X\) defined by \(\omega\) has constant scalar curvature \(R(= 2 - 2g/V)\). Then the canonical determinantal point processes associated to \(kL\) (with \(N(= N_k\) particles) satisfy
\[
\sup_X \frac{\mathbb{E}_N(\delta_N/N)}{\omega/V} - 1 \leq \epsilon_N,
\]
where $\epsilon_N$ is exponentially small, i.e. $\epsilon_N \leq C e^{-\delta N}$. In the case $g = 0$ we have $\epsilon_N = 0$ and when $g > 0$ the constant $\delta$ can be taken to be arbitrarily close to

$$
\frac{2}{RV} \log \left( \cosh \left( \sqrt{\frac{\pi R}{2}} I(X) \right) \right)
$$

where $I(X))$ is the injectivity radius of $X$ (which coincides with half the length of the shortest geodesic on $X$).

**Proof.** To simplify the notation we set $V = 1$ (the case $V \neq 1$ follows from trivial scalings). First we recall that in the general setting where 1.4 defines a Hilbert norm on $H^0(X, kL)$ we have the basic relation $E_N(\delta_N) = B_k(x)d\nu$ where $B_k(x)$ is the point-wise norm of the corresponding Bergman kernel and hence Theorem 3.1 below (or its corollary) gives $E_N(\delta_N) = k + R/2 + O(e^{-\delta k})$. Since, $N = \int B_k(x)d\nu$ it follows that $N = k + R/2$ for $k \gg 1$ (a special case of the Riemann-Roch theorem) concluding the proof of the proposition. \hfill \Box

2.1. **Proof of Theorem 1.1 (determinantal Moser-Trudinger inequality).** We will start by proving the following non-asymptotic inequality.

**Proposition 2.2.** Let $L \to X$ be a line bundle over a Riemann surface equipped with a smooth metric with strictly positive curvature form $\omega$. Consider the corresponding adjoint determinantal point process. Then the following estimate holds

$$
\frac{1}{N} \log E(e^{\phi}) - E_\omega(\phi) \leq \sup_X \left| \frac{E(\delta/N)}{\omega/V} - 1 \right| (-E_\omega(\phi - \sup \phi))
$$

for any smooth function $\phi$ satisfying $\omega_\phi := dd^c\phi + \omega \geq 0$, where $\delta(=\delta_N)$ denotes the empirical measure of the process with $N$ particles.

The proof is a simple modification of the proof Theorem 33 in [5]. As a courtesy to the reader we will recall the argument in [5]. An important ingredient in the proof is the notion of a $C^0$-geodesic (wrt the Mabuchi metric) connecting $\phi_0$ and $\phi_1$ in $C^0(X) \cap PSH(X, \omega)$. This may be defined as the continuous path $\phi_t(=\phi(., t))$ connecting $\phi_0$ and $\phi_1$ in $C^0(X) \cap PSH(X, \omega)$ obtained as the upper envelope of all $S^1$-invariant $\pi^*\omega$-psh extensions to the $n+1$-dimensional complex manifold with boundary $M := X \times ([0, 1] \times S^1)$ (where $\pi$ denotes the projection from $X \times [0, 1] \times S^1$ to $X$). In particular, $\phi_t$ is convex in the real parameter $t \in [0, 1]$ and satisfies the homogenous Monge-Ampère equation in the interior of $M$:

$$
\partial_t \partial_t \phi_t - |(\partial_X(\partial_t \phi_t))|^2_{\omega_\phi_t} = 0
$$

in the weak sense of pluripotential theory (see [5] for the precise construction). The following variational formulae are well-known (and straight-forward):

$$
(i) \frac{1}{N} d(\log E(e^{\phi_t})/dt = \left\langle E_{\omega_\phi_t}(\delta/N), d\phi_t/dt \right\rangle, \quad (ii) dE_{\omega_\phi_t}(\phi_t)/dt = \frac{1}{V} \left\langle \omega_\phi_t, d\phi_t/dt \right\rangle
$$

Moreover, if $\phi_t$ is a $C^0$-geodesic in $Psh(X, \omega)$ then

(i') $\log E(e^{-\phi_t})$ is concave, (ii') $E_{\omega_\phi_t}(\phi_t)$ is affine

in the real parameter $t$ (note however that $\log E(e^{-\phi_t})$ is convex along affine curves; compare Remark 2.5 below). The item (i') above follows from the Toeplitz determinant representation 1.6 combined with the positivity results for direct image bundles in [8]. See also the appendix in [5] for another proof of (i') using the structure of determinantal point processes. The key point is the following formula

$$
\partial_t^2 \log E(e^{-\phi_t}) = Tr \left( T[\partial_t \partial_t \phi_t] + \left( (T[\partial_t \phi_t])^2 - T[(\partial_t \phi_t)^2] \right) \right),
$$
where Tr denotes the trace and $T[f]$ is the Toeplitz operator with symbol $f$ wrt the perturbed weight $\Phi + \phi_t$:

$$
T[f] = \int_X f(y)K_{\Phi + \phi_t}(\cdot, y)(= \Pi_{\Phi + \phi_t}(f))
$$

One then uses the geodesic equation 2.5 to replace $\partial_t \partial_t \phi_t$ with $|\partial_X(\partial_t \phi_t)|^2_{\omega_{\phi_t}}$ in the first term in 2.7 and finally apply the Hörmander-Kodaira $L^2$–estimate for the inhomogenous $\partial_X$–equation (see 3.16 below) to deduce that $\partial^2_t \log E(e^{-\phi_t}) \leq 0$.

2.1.1. The proof of Proposition 2.2. Now consider the following functional on $C^0(X)$, which is invariant under addition of constants:

$$
\mathcal{F}_\omega(\phi) := \mathcal{E}_\omega(\phi) + \frac{1}{N} \log E(e^{-\phi})
$$

For any given $\phi \in C^0(X) \cap \text{Psh}(X, \omega)$ we let $\phi_t$ be the $C^0$–geodesic such that $\phi_0 = 0$ and $\phi_1 = \phi$. By the concavity of $\mathcal{F}_\omega(\phi_t)$ (resulting from $(i')$ combined with $(ii')$ above) and since $\mathcal{F}_\omega(\phi_0) = 0$ we have

$$
\mathcal{F}_\omega(\phi) \leq d(\mathcal{F}_\omega(\phi_t))/dt_{t=0} = \int (V\mathcal{E}(\delta/N)/\omega - 1)\frac{1}{V} \omega(-d\phi_t/dt)_{t=0}
$$

Next, note that, since the inequality in the theorem that we are about to prove is invariant under $\phi \to \phi + C$ we may as well assume that $\sup_X \phi = 0$. Since $\phi_t$ is convex in $t$ we have $-d\phi_t/dt \leq \phi_1 - \phi_0 = \phi$ (we are using right derivatives, which always exist by convexity) and hence

$$
\mathcal{F}_\omega(\phi) \leq \sup_X (V\mathcal{E}(\delta/N)/\omega - 1)\frac{1}{V} \left(\int \omega(-d\phi_t/dt)_{t=0}\right)
$$

Next, note that, combining $(ii)$ and $(ii')$ above gives

$$
\left(\int \omega(-d\phi_t/dt)_{t=0}\right) = d\mathcal{E}_\omega(\phi_t)/dt_{t=0} = -\mathcal{E}_\omega(\phi)
$$

and hence

$$
\mathcal{F}_\omega(\phi) \leq \sup_X (V\mathcal{E}(\delta/N)/\omega - 1)(-\mathcal{E}_\omega(\phi))
$$

Finally, replacing $\phi$ with $\phi - \sup_X \phi$ finishes the proof of the proposition.

2.1.2. The psh projection $P_\omega$. To reduce the case of a general smooth function $\phi$ to an $\omega$–psh one we will make use of the psh-projection $P_\omega$ mapping smooth functions to $\omega$–psh ones:

$$
P_\omega(\phi)(x) := \sup\{\psi(x) : \psi \in PSH(X, \omega), \psi \leq \phi \text{ on } X\}
$$

It is not hard to see that $P_\omega \phi$ is continuous when $\phi$ is and moreover that the following “orthogonality relation” holds [6]

$$
\int_X (\phi - P_\omega \phi) d\mathcal{E}(P_\omega \phi) = 0
$$

(as a consequence of the maximum principle for the Laplacian).

**Proposition 2.3.** Let $(X, \omega)$ be a Riemann surface with a Kähler. Then

$(i)$ $\mathcal{E}_\omega(\phi) \leq \mathcal{E}_\omega(P_\omega \phi)$, $(ii)$ $\|d(P_\omega \phi)\|^2_X \leq \|d\phi\|^2_X$

for any $\phi \in C^\infty(X)$.
Proof. (i) was proved in [5] and (ii) is proved in a similar way, as we will next see. Integrating by parts (which is allowed, for example using that \( P_\omega \phi \) is \( C^{1,1} - \) smooth [5]) gives

\[
\|d(P_\omega \phi)\|^2_X = \int (-P_\omega \phi)dd^c(P_\omega \phi) = \int (-P_\omega \phi)(dd^c P_\omega \phi + \omega) + \int (P_\omega \phi)\omega
\]

Next, since \( P_\omega \phi = \phi \) a.e. with respect to \( (dd^c P_\omega \phi + \omega) \) (by formula 2.9) this means that

\[
\|d(P_\omega \phi)\|^2_X = \int (-\phi)(dd^c P_\omega \phi + \omega) + \int (P_\omega \phi)\omega = \int (-\phi)(dd^c P_\omega \phi) + \int (P_\omega \phi - \phi)\omega
\]

But since \( (P_\omega \phi - \phi) \leq 0 \) and \( \omega \geq 0 \) the last term above is non-positive and hence

\[
\|d(P_\omega \phi)\|^2_X \leq \|d(P_\omega \phi)\|_X \|d\phi\|_X ,
\]

also using the Cauchy-Schwartz inequality for the first term above. Dividing out \( \|d(P_\omega \phi)\|_X \) (which is always non-zero if \( \phi \) is) proves Step 2. \( \square \)

2.1.3. End of proof of Theorem 1.1. We start with the proof of the inequality 1.10. Consider the line bundle \( kL \) with \( \Phi \) the weight of a metric on \( L \) with curvature \( \omega := dd^c \Phi > 0 \) and decompose

\[
kL =: L_k + K_X, \quad k\Phi =: \Phi_k + \Phi_{\omega}
\]

where \( \Phi_{\omega} := \log\left( \frac{1}{dd^c \lambda \omega \omega} \right) \) defines the weight of a metric on \( K_X \). Then the Hilbert space \( H^0(kL) \) associated to the weighted measure \( (\hat{\omega}, \omega) \) is naturally isomorphic to the Hilbert space \( H^0(L_k + K_X) \) associated to the weight \( \Phi_k \) in the adjoint setting, just using that, by definition,

\[
e^{-k\Phi \omega} = e^{-\Phi_k idz \wedge d\bar{z}}
\]

We will write \( \omega_k := dd^c \Phi_k \) (and we let \( N_k \) be the dimension of \( H^0(kL) \) and \( V_k \) the volume (degree) of \( L_k \). Then

\[
(2.10) \quad \omega_k/V_k = \omega/V
\]

and in particular \( \omega_k > 0 \). This follows immediately from the fact that the forms in rhs and the lhs above both integrate to one over \( X \) and moreover, by assumption, \( \omega \) satisfies the Kähler-Einstein equation:

\[
dd^c \omega := -\text{Ric} \omega = \lambda \omega
\]

for some constant \( \lambda \), so that \( \omega_k \) is proportional to \( \omega \).

Step one: scaling by \( k \) and assuming \( (\omega_k)_\phi := (\omega_k + dd^c \phi) \geq 0 \).

Applying Prop 2.2 and Prop 2.1 to \( (L_k, \omega_k) \) and \( \phi \) and using formula 2.2 gives, using 2.10,

\[
\frac{1}{N_k} \log \mathbb{E}(e^{-\phi}) + \mathcal{E}_{\omega_k}(\phi) \leq \epsilon_k \left( \frac{1}{2V_k} \|d\phi\|^2_X + \int_X (\sup X \phi - \phi) \frac{\omega}{V} \right)
\]

Next, we recall the following basic inequality: there is a constant \( C \) (only depending on \( \omega \)) such that

\[
\sup X \psi \leq \int X \psi \omega + C
\]

for any \( \psi \) such that \( \omega_\psi \geq 0 \) (as follows immediately from Green’s formula; see [20] for more general inequalities). Setting \( \psi = \phi/k \) and applying the previous inequality to the rhs in the preceeding inequality gives, since \( \omega_k/k \sim \omega \), that

\[
\frac{1}{N_k} \log \mathbb{E}(e^{-\phi}) + \mathcal{E}_{\omega_k}(\phi) \leq \epsilon_k (\|d\phi\|^2_X + kC)
\]

Step two: using \( P_{\omega_k} \)
Let now $\phi$ be a general smooth function. Since $P(\omega_k)\phi \leq \phi$ we have $\frac{1}{N} \log E(e^{-\phi}) \leq \frac{1}{N} \log E(e^{-P(\omega_k)\phi})$ and hence the previous step applied to $P_{\omega_k}\phi$ combined with (i) in the previous proposition and step one gives

$$
\frac{1}{N_k} \log E(e^{-\phi}) + E_{\omega_k}(\phi) - \epsilon_k \leq \epsilon_k \|d(P_{\omega_k}\phi)\|_X^2 \leq \epsilon_k \|d\phi\|_X^2
$$

also using (ii) in the previous proposition in the last inequality. Finally, using the scaling property

$$(2.11) \quad \log E(e^{-(\psi+c)})/N = -c + \log E(e^{-\psi})/N$$

together with formula 2.2 and the identity 2.10 we can rewrite

$$
\frac{1}{N_k} \log E(e^{-\phi}) + E_{\omega_k}(\phi) = \frac{1}{N_k} \log E(e^{-(\phi-\int_x \phi d\phi)}) - \frac{1}{V_k} \frac{1}{2} \|d\phi\|_X^2
$$

All in all this means that

$$
\log E(e^{-\phi}) \leq \left( \frac{N_k}{V_k} + \epsilon_k \right) \|d\phi\|_X^2 + \epsilon_k
$$

Finally, by the Riemann-Roch theorem

$$
\frac{N_k}{V_k} = \frac{k \deg(L) - \deg(K_X)/2}{k \deg(L) - \deg(K_X)} = \frac{N_k}{N_k - \deg(K_X)/2} = \frac{N_k}{N_k - (1 - g)}
$$

finishing the proof of the inequality 1.10.

To prove the second inequality 1.11 in the theorem we first note that

$$
\int \phi(\omega/V - E(\delta/N)) \leq \epsilon_N \|\phi\|_{L^1(X)/\mathbb{R}} := \epsilon_N \inf_{c \in \mathbb{R}} \|\phi + c\|_{L^1(X)}
$$

Indeed, the lhs above is invariant under the action of $\mathbb{R}$, $\phi \rightarrow \phi + c$, and hence the inequality follows immediately from Prop 2.1. The inequality 1.10 then follows immediately from the fact that $\phi \rightarrow d\phi$ is invariant under the action of $\mathbb{R}$ combined with the scaling property 2.11 (just take $\psi = \phi - \int \phi \omega$ and $c = \int \phi(\omega/V - E(\delta/N))$).

2.2. Proof of Cor 1.2. The proof is a standard application of Markov’s inequality: for any given $t > 0$ we have

$$
\text{Prob}\{Y > 1\} = \text{Prob}\{e^{tY} > e^t\} \leq e^{-tE(e^{tY})},
$$

where in our case $Y = \frac{1}{N_k} \sum (\phi(x_1) + ... + \phi(x_N))$. By the previous theorem the rhs above is bounded by $e^{-t+c^2/2e^{cN}}$ for $c = (a_N + \epsilon_N) \|d(\frac{1}{N_k}\phi)\|_X^2$. Taking $t = 1/c$ shows that the first factor may be estimated by $e^{-\frac{1}{2}}$ which finishes the proof of the corollary.

2.3. Proof of Theorem 1.3 (Sharp Szegö type limit theorem). We will use the following notation for the fluctuation of the linear statistic determined by a function $\phi$ on $X$:

$$
\tilde{\phi} := \sum_{i=1}^N (\phi(x_i) - E(\phi(x_i)))
$$

We start by proving the following universal bound on the variance for the canonical processes, which is of independent interest.

**Proposition 2.4.** For any given function $\phi$ on $X$ the following upper bound on the variance of the corresponding linear statistic holds:

$$
E(|\tilde{\phi}|^2)/4 \leq (1 + \epsilon_N) \|d\phi\|_X^2 + \epsilon_N \|\phi\|^2_{L^1(X)/\mathbb{R}}
$$
where \(\epsilon_N\) denotes a sequence, independent of \(\phi\), tending to zero. In particular, if \(\phi \in L^1(X)\) and \(d\phi \in L^2(X)\) then the variance is uniformly bounded from above by a constant independent of \(N\).

**Proof.** We will denote by \(\epsilon_N\) a sequence tending to zero, which may change from line to line. By the second inequality in Theorem 1.1 we have

\[
\mathbb{E}(e^{-t\hat{\phi}}) \leq e^{(1+\epsilon_N)t^2 \|d\phi\|_X^2 + \epsilon_N t\|\phi\|_{L^1(X)/R}^2} e^{\epsilon_N t}
\]

Using \(2ab \leq a^2 + b^2\) hence gives

\[
\mathbb{E}(e^{-t\hat{\phi}}) \leq e^{\frac{1}{2}t^2 f_N e^{\epsilon_N}}, \quad f_N = \left( (1 + \epsilon_N) \|d\phi\|_X^2 + \epsilon_N \|\phi\|_{L^1(X)/R}^2 + \epsilon_N \right)
\]

Repeating the argument in the proof of 1.2 (involving Markov’s inequality) hence gives

\[
\text{Prob}\{ (\hat{\phi} > \lambda) \} \leq e^{-\lambda^2 \frac{1}{2} t N} e^{\epsilon_N}
\]

Now using the push-forward formula for the integral in \(\mathbb{E}(|\hat{\phi}|^2)\) we can write

\[
\mathbb{E}(|\hat{\phi}|^2) = \int_0^\infty (\text{Prob}\{ \hat{\phi}^2 > \lambda \}) d\lambda^2 + \int_0^\infty (\text{Prob}\{ (-\hat{\phi})^2 > \lambda \}) d\lambda^2 
\]

\[
\leq 2 \cdot 2 f_N e^{\epsilon_N}
\]

where we used that \(\int_0^\infty e^{-\frac{1}{2} s} ds = 2a\) in the last step, finishing the proof. \(\square\)

As shown in [3] (see also the Remark below) we have for any fixed smooth function \(\phi\) and \(t \in \mathbb{R}\)

\[
(2.12) \quad \mathbb{E}(e^{it\hat{\phi}}) \to e^{-\frac{1}{2} t^2 \int_X d\phi \wedge d^c \phi}
\]

as \(N \to \infty\). Using the variance estimate above we can extend the previous convergence to the case when we merely assume that \(\|d\phi\|_X < \infty\) (and hence \(\|\phi\|_{L^1(X)} < \infty\)). To this end take a sequence \(\phi_j \in C^\infty(X)\) such that \(d(\phi_j - \phi)_X \to 0\) and \(\|\phi_j - \phi\|_{L^1(X)} \to 0\). Since

\[
\|\mathbb{E}(e^{it\hat{\phi}_j}) - \mathbb{E}(e^{it\hat{\phi}})\|^2 \leq \mathbb{E}(|\hat{u}|^2)
\]

for \(u = \phi_j - \phi\) (just using \(1 - e^{is} \leq |s|\)) we deduce that

\[
\|\mathbb{E}(e^{it\hat{\phi}_j}) - \mathbb{E}(e^{it\hat{\phi}})\|^2 \leq C(\|d(\phi - \phi_j)\|_X^2 + \|\phi - \phi_j\|_{L^1(X)}^2)
\]

for \(N >> 1\) and hence letting first \(N\) and then \(j\) tend to infinity proves the convergence 2.12 in the non-smooth case as well.

Next, we observe that the convergence 2.12 moreover holds for any \(t \in \mathbb{C}\). Indeed,

\[
f_k(t) := \mathbb{E}(e^{it\hat{\phi}})
\]

is a sequence of holomorphic functions on \(\mathbb{C}\) such that for \(t\) in a fixed compact subset \(K\) of \(\mathbb{C}\)

\[
|f_k(t)| \leq \mathbb{E}(e^{-(1m(t))\hat{\phi}}) \leq C_K
\]

using the second inequality in Theorem 1.1. Since \(f_k\) converges point-wise to the holomorphic function \(f(t) = e^{-t^2 \int_X d\phi \wedge d^c \phi}\) for \(t \in \mathbb{R}\) it hence follows (e.g. by Vitali’s theorem) that \(f_k\) converges to \(f\) everywhere on \(\mathbb{C}\). In other words we have now proved Theorem 1.3 for the case of real and imaginary \(\phi\). Finally, if \(\phi\) is complex valued we consider \(\phi_s = u + su\) where \(\phi = \phi_s\) for \(s = i\). The previous convergence shows that \(\mathbb{E}(e^{-\phi_s})\) converges to an explicit holomorphic function (as above) for \(s \in \mathbb{R}\). Moreover, since the upper bound on \(|f_k(s)|\) still holds (by the same argument) the previous argument also shows that the convergence holds for any \(s \in \mathbb{C}\) and in particular for \(s = i\).
Remark 2.5. For completeness we briefly indicate a “self-contained” proof of 2.12 in the case when \( \phi \) is smooth. Since we already have established the upper bound on \( \log \mathbb{E}(e^{-\hat{t}\phi}) \) it will be enough to establish the lower bound (and as above we may assume that \( t \) is real). To this end we use that, at level \( k \),

\[
\partial_t^2 \log \mathbb{E}(e^{-\hat{t}\phi}) = \frac{1}{2} \int_{X \times X} |K_{k\Phi+t\phi}(x,y)|^2 e^{-(k\Phi+t\phi)(x)+(k\Phi+t\phi)(y)} (\phi(x) - \phi(y))^2
\]

(which follows from 2.7 using that the first term vanishes and by re-writing the second term). Next we restrict the integration to \( A_k := \{d(x,y) \leq \log k/k^{1/2}\} \subset X \times X \). Let \( z \) denote local holomorphic coordinates centered at \( x \in X \) and a trivialization of \( L \) such that \( \Phi(z) = |z|^2 + O(|z|^3) \). Then it is well-known that

\[
K_{k\Phi+t\phi}(x+z/k^{1/2}, x+w/k^{1/2}) = ke^{z\bar{w}} + o(1)
\]

uniformly in \( k \) and \( t \) (for \( t = 0 \) this follows immediately from Theorem 3.1 below and the general case is obtained from the same proof since the perturbation \( \phi \) does not effect the leading term).

Finally integrating first over \( y \) (or rather \( w \)) and then over \( y \) gives the lower bound \( \|d\phi\|^2 \). Since, the first derivative of \( \log \mathbb{E}(e^{-\hat{t}\phi}) \) at \( t = 0 \) vanishes, using (i) in 2.6, this finishes the proof of the lower bound (by general integration theory).

2.4. A brief account of the higher dimensional case. Let us now come back to the case when \( X \) is \( n \)-dimensional and fix a Kähler form \( \omega \) on \( X \). In [3] the analogue of the convergence in Theorem 1.3 was shown to hold as long as \( \phi \) is smooth. More precisely, in the convergence statement \( \phi \) has to be replaced by \( k^{-(n-1)/2}\phi \) and the norm \( \|d\phi\|^2 \) by

\[
\|d\phi\|^2_{(X,\omega)} = \int_X d\phi \wedge d^*\phi \wedge \frac{\omega^{n-1}}{(n-1)!} \left( \int |\nabla\phi|^2 dV \right).
\]

However, when \( n > 1 \) there are integrable functions \( \phi \) with \( \int_X |
abla\phi|^2 \omega^n < \infty \), but \( \int e^{-\phi} dV = \infty \) (as is well-known in the context of Sobolev inequalities). As a consequence, it is not hard to check that for such a function \( \phi \) we have \( \mathbb{E}(e^{-(\phi(z_1)+\cdots)}) = \infty \) and in particular the analogue of the convergence in Theorem 1.3 cannot hold (after perhaps scaling \( \phi \)). Moreover, the corresponding analogue of the Moser-Trudinger inequality in Theorem 1.1 fails when \( n > 1 \) (as is seen by approximating \( \phi \) as above with a monotone smooth sequence \( \phi_j \)). Explicit counter-examples are obtained, already when \( N = 1 \), by letting \( X = \mathbb{P}^n(\supset C^\infty \) and \( \omega \) be the standard \( SU(n+1) \)-invariant metric on \( \mathbb{P}^n \) and taking \( \phi_j(z) := m \log(1/j^{1/2}+|z|^{2/2}) \) (for a fixed \( m \geq n \)) decreasing to \( \phi(z) \). Note that \( \phi_j \) is even \( \omega \)-psh.

On the other hand, another variant of the determinantal Moser-Trudinger inequality in Theorem 1.1 does hold in higher dimensions. More precisely, \( \frac{1}{2} \|d\phi\|^2 \) has to be replaced by Aubin’s \( J \)-functional (which is comparable to \( \int d\phi \wedge d^*\phi \wedge (\omega_0)^n \)). Moreover \( \phi \) has to be assumed \( \omega \)-psh (i.e. \( \omega_\delta \geq 0 \)) (otherwise there are counter-examples, as explained in [5]) When \( X = \mathbb{P}^n \) (or more generally \( X \) is a rational homogenous manifold) the corresponding inequality is the content of Cor 2 in [5], with vanishing error terms \( \epsilon_N \). More generally, the arguments in Step one in the proof of Theorem 1.1 extend in a straight-forward manner to the higher-dimensional case when the Kähler metric \( \omega \) has a constant scalar curvature (but then the error terms \( \epsilon_N \) are then of the order \( O(1/k) \)).

3. Bergman kernel asymptotics with exponentially small error terms

In this section we will prove the following theorem used in the proof of Proposition 2.1 above (for an explicit description of \( \delta \) below, see Remark 3.5).
Theorem 3.1. Let $L \to X$ be a line bundle over a Riemann surface equipped with a metric $e^{-\Phi}$ with positive curvature form $\omega(=d\bar{d}\Phi)$ such that the Riemannian metric on $X$ defined by $\omega$ has constant scalar curvature $R$ close to $x$. Then there is a neighbourhood of $\{x\} \times \{x\}$ in $X \times X$ such that the corresponding Bergman kernel $K_k$ satisfies

$$K_k(z, w) = (k + \frac{1}{2}R)e^{k\psi(z, w)} + \epsilon_k$$

where $\psi$ is the local holomorphic function such that $\psi(z, w) = \Phi(z)$ and $\epsilon_k$ denotes a smooth section of $kL \boxtimes kL$ whose point-wise norm is of the order $O(e^{-\delta k})$. In particular,

$$B_k(x) := \|K_k(x, x)\| = k + \frac{1}{2}R + O(e^{-\delta k})$$

(in the case when $X$ is the two-sphere and $R$ is constant on all of $X$ our arguments will give the well-known fact that the error terms vanish identically). Here the Berman kernel $K_k \in H^0(X \times \bar{X}, kL \boxtimes kL)$ denotes the integral kernel of the orthogonal projection from $C^\infty(X, L)$ onto the Hilbert space $H^0(X, kL)$ using the $L^2$-norm defined by the metric on $L$ and volume form $dv = \omega$ (formula 1.4). The normalization of $R$ has been chosen so that $R = \deg(TX)(=2g-2)$ when it is globally constant (and hence $O(e^{-\delta k})$ against $\omega$ over $X$ gives the Riemann-Roch relation 1.9 for $k$ large). We recall that a sequence $a_k$ is said to be exponentially small, written as $a_k = O(e^{-k\delta})$ if $|a_k| \leq Ce^{-k\delta}$ for some numbers $C, \delta > 0$ (if $a_k$ functions then, by definition, the estimate holds uniformly).

The case of larger error terms of the form $O(e^{-(\log k)^2\delta})$ in 3.2 was previously obtained in [31, 32] using Tian’s method of peak sections. It was also pointed out there that the case of even larger error terms of the form $O(k^{-\infty})$ can be deduced from the results in [33] concerning the Yau-Tian-Zelditch-Catlin expansion of $B_k$, but that one may expect exponentially small error terms (as confirmed in the theorem above). In the case when $L = K_X$ and the scalar curvature is constant on all of $X$ (in particular $X$ then has genus at least two and $R < 0$) the error term $O(e^{-\delta k})$ in 3.2 could also be obtained by writing $X = \Gamma/\mathbb{H}$ for a Fuchsian group $\Gamma$ and using that that $B_k$ is constant in the non-compact case setting of $X = \mathbb{H}$ and then estimate the effect of the “$\Gamma$-periodization” coming from a Poincaré theta series (as pointed out to the author by Steve Zelditch) A similar periodization argument was used in [16] in the case when $X$ is the torus.

One motivation to consider the situation when $R$ is not globally constant is to allow applications to the setting of constant curvature metrics with conical singularities and cusps. For example, in the hyperbolic setting this means that the Kähler form $\omega$ is the unique solution to

$$\text{Ric } \omega = -\omega + \sum_i c_i \delta_{P_i}$$

for given coefficients $c_i \in [0,1] \cap \mathbb{Q}$ and a finite number of points $P_i$ in $X$ ([22], Thm 21.1). Equivalently, $\omega$ has constant scalar curvature $-1$ on $X - \{P_i\}$ with conical singularities at an angle $2\pi(1-c_i)$ at any $P_i$ such that $c_i < 1$ and a cusp at any $P_i$ such that $c_i = 1$. \footnote{The classical case when $X - \{P_i\} = \Gamma/\mathbb{H}$ for a Fuchsian group $\Gamma$ corresponds to the the case when $c_i = 1 - 1/m$ for $m$ a positive integer or infinity and then $\omega$ is induced from the hyperbolic metric on $\mathbb{H}$}

Letting $D = \sum_i c_i O_{P_i}$ be the corresponding $\mathbb{Q}$– line bundle we then have the following
Corollary 3.2. Let $L = K_X + D$ and let $\omega$ be the unique (singular) metric on $X$ above. Then the Bergman kernel expansions 3.1 and 3.2 hold for any $x \in X - \{P_i\}$. Moreover, the positive number $\delta$ appearing in 3.2 may be taken to be arbitrarily close to

$$2 \log(\cosh(\pi I(x)/\sqrt{2}))$$

where $I(x)$ is the injectivity radius in $X - \{P_i\}$ at $x$ (which coincides with half the length of the shortest closed and simple geodesic on $X$, passing through $x$, when $D = 0$).

The Bergman kernel in the previous corollary is, as usual, defined wrt the subspace of $H^0(X, kL)$ consisting of all “cusp forms”, i.e. sections vanishing at the cusps and it is well-defined for all $k$ such that $k\sigma_i \in \mathbb{Z}$ for all $i$.

The rest of the section is devoted to the proof of the theorem above; following the scheme in [2] we first prove a local variant of the expansion and then globalize. The main point here is the observation that the local expansion may be obtained using the “local symmetry” of $L$ (as opposed to the general case treated in [2]) which leads to a precise controle of the error terms.

As is well-known the local constant curvature condition implies that there exists a local holomorphic coordinate $w$ centered at $x$ on some simply connected neighbourhood $U$ such that

$$\omega := \frac{i}{2\pi} 2(1 + R|w|^2)^{-2} dw \wedge d\bar{w} = (dd^c \Phi_0)$$

where $\Phi_0(w) = R^{-1} \log(1 + R|w|^2)$ for $R \neq 0$ and $\Phi_0 = |w|^2$ for $R = 0$ (obtained in the limit $R \to 0$). Now fix a local holomorphic section $s$ of $L$ close to $x$ and write $\|s\|^2 = e^{-\Phi}$ for a local function $\Phi$, recalling that $\omega = dd^c \Phi$ (see section 1.5). By the previous relation this means that $\Phi - \Phi_0$ is a harmonic function on $U$ and hence we may write $e^{-\Phi} = |h|^2 e^{-\Phi_0}$ for some non-vanishing holomorphic function $h$ on $U$. Accordingly, after replacing $s$ with $h^{-1}s$ we may as well assume that we are in the model case $\Phi = \Phi_0$.

3.1. Local Bergman kernels for the model cases. Given a smooth function $\Phi$ on a domain $U$ in $\mathbb{C}$ containing 0 we let

$$\langle f, g \rangle_{U, k\Phi} := \int_U f \bar{g} e^{-k\Phi} dd^c \Phi$$

and denote by $H_{k\Phi}(U)$ the space of all holomorphic functions on $U$ such that $\|f\|^2_{U, k\Phi} := \langle f, f \rangle_{U, k\Phi} < \infty$. Following [2] we will say that $K_{(k)}(z, \zeta)$ is a (local) Bergman kernel mod $O(e^{-k\delta})$ (with respect to $\Phi$) if it is holomorphic in $\zeta$ and there exists number $\delta > 0$ such that for any $f \in H_{k\Phi}(U)$ we have, for all $z$ in some neighbourhood $V \subset U$ of 0 that

$$(3.4) \quad f_k(z) = \langle f_k, \chi K_{(k)}(z, \cdot) \rangle_{U, k\Phi} + \|f\|_{U, k\Phi} O(e^{-k\delta}) e^{k\Phi/2}$$

where $\chi$ denotes a smooth function $\chi$ compactly supported on $U$ which is equal to one on $\frac{1}{2} U$.

Proposition 3.3. Let $\Phi(w) = -2 \log(1 + R|w|^2)/R$. Then the function $K_{(k)}(z, \zeta) = (k + \frac{R}{2})(1 + R\zeta)^{2k/R}$ is a local Bergman kernel mod $O(e^{-k\delta})$ (wrt $\Phi$) when $R \neq 0$ and $K_{(k)}(z, \zeta) = k e^{\zeta}$ when $R = 0$ (coinciding with the limit when $R \to 0$).

Proof. Let $\epsilon$ be a fixed (small) positive number. It will be convenient to let $\delta$ be a small positive number (depending on $\epsilon$) whose value may change from line to line. First we not that for any $f \in H_{k\Phi}(U)$ we have

$$(3.5) \quad f_k(0) = (k + \frac{R}{2}) \int_{|w| < \epsilon} f_k e^{-k\Phi} dd^c \Phi + \|f_k\|_{U, k\Phi} O(e^{-k\delta}) e^{k\Phi/2},$$
Indeed, applying the mean-value property of holomorphic functions to $f_k(re^{i\theta})$ for $r$ fixed and then integrating over $r$ (using that $\Phi$ only depends on $r$) gives

$$ f_k(0) = c_k \int_{|w|<\epsilon} f e^{-k\Phi} dd^c \Phi $$

where

$$ (3.6) \quad 1/c_k = \int_{|w|<\epsilon} e^{-k\Phi} dd^c \Phi = \int_{|w|<C} e^{-k\Phi} dd^c \Phi - \int_{\epsilon<|w|<C} e^{-k\Phi} dd^c \Phi $$

and where we take $C^2 = -1/R$ when $R < 0$ and $C = \infty$ otherwise. Since, with $s = r^2$:

$$ e^{-k\Phi} dd^c \Phi = \frac{2}{\pi} (1 + R^2)^{-2k/R-2} \frac{1}{2} d(r^2)d\theta = \frac{d}{ds} \left( \frac{1}{k + R/2} (1 + Rs)^{-2k/R-1} \right) ds \wedge \frac{d\theta}{2\pi} $$

the first integral in $3.6$ equals $1/(k + R/2)$ and

$$ (3.7) \quad \int_{|w|<\epsilon} e^{-k\Phi} dd^c \Phi = O(e^{-k\delta}), \quad \delta = \Phi(\epsilon^2) $$

The formula $3.5$ then follows from the trivial relation $(1 + O(e^{-k\delta}))^{-1} = 1 + O(e^{-k\delta})$ combined with the Cauchy-Schwartz inequality.

Next, we fix $z$ in a given (small) neighbourhood $V$ of $U$ and define

$$ (3.8) \quad F_z(w) := \zeta := (z - w)/(1 + R\bar{z}w) $$

(which is invertible with $w = F_w(\zeta)$ mapping $0$ to $z$ and $g_z(w) := f_k(\zeta)e^{\psi(\bar{z},w)}$, $\psi(z, w) := R^{-1}\log(1 + R\bar{z}w)$ for a given $f_k \in H_k\Phi(U)$. Then

$$ (3.9) \quad |g_z(w)|^2 e^{-k\Phi(w)} e^{-k\Phi(z)} = |f_k(\zeta)|^2 e^{-k\Phi(\zeta)} $$

as follows immediately from the relation

$$ (3.10) \quad \psi(\bar{z}, w) + \psi(z, \bar{w}) - \Phi(w) - \Phi(z) = -\Phi(\zeta) $$

(see section 3.1.1 below). This shows in particular that $g_z \in H_k\Phi(U)$. Applying the formula $3.5$ to $f_k := g_z$ hence gives

$$ f_k(z) = (k + \frac{R}{2}) \int_{|w|<\epsilon} f_k(\zeta)e^{\psi(\bar{z},w)-k\Phi(w)} dw d\bar{w} \Phi + \|g_z\|_{\frac{1}{2}U,k\Phi} O(e^{-k\delta}), $$

To rewrite this we first note that $dd^c \Phi$ is invariant under the map $F_z$ (as follows immediately from differentiating $3.10$) and hence $3.9$ gives that

$$ e^{-k\Phi(z)} \int_{w \in \frac{1}{2}U} |g_z(w)|^2 e^{-k\Phi(w)} dw d\bar{w} \Phi = \int_{w \in \frac{1}{2}U} |f_k(\zeta)|^2 e^{-k\Phi(\zeta)} d\zeta d\bar{\zeta} \Phi, $$

i.e. that

$$ \|g_z\|_{\frac{1}{2}U,k\Phi} e^{-k\Phi(z)/2} = \|f\|_{F_z(\frac{1}{2}U'),k\Phi} (\leq \|f\|_{U,k\Phi}) $$

Moreover, the relation

$$ (3.11) \quad \psi(\bar{z}, w) - \Phi(w) = \psi(z, \zeta) - \Phi(\zeta) $$

(see section 3.1.1 below) then gives that

$$ (3.12) \quad f_k(z) = (k + \frac{R}{2}) \int_{F_z(\epsilon D)} f_k(\zeta)e^{\psi(\bar{z},\zeta)-k\Phi(\zeta)} d\zeta d\bar{\zeta} \Phi + \|f\|_{U,k\Phi} e^{k\Phi(z)/2} O(e^{-k\delta}). $$

Next, we note that by the Cauchy-Schwartz inequality (applied to $f_k$ and $e^{i\psi}$) and the relation $3.10$ (applied to $w = \zeta$) we have

$$ \int |f_k(\zeta)e^{i\psi(\bar{z},\zeta)-k\Phi(\zeta)} d\zeta d\bar{\zeta} \Phi|^2 e^{-\Phi(z)} \leq \int |f_k(\zeta)|^2 e^{-k\Phi(\zeta)} d\zeta d\bar{\zeta} \Phi \|f\|^2_{U,k\Phi} \int e^{-k\Phi(w)} dw d\bar{w} \Phi |^2 $$
By 3.7 the second factor in the rhs above is exponentially small when integrating over the complement of a small disc centered at \( w = 0 \) (i.e. a small neighbourhood of \( z \) in the \( \zeta \) coordinates). Hence, we may as well replace the integration region \( F_\varepsilon(\varepsilon D) \) in 3.12 with all of \( U \) at the expense of introducing the cut-off function \( \chi \), concluding the proof of the proposition. 

3.1.1. Proofs of the relations 3.10 and 3.11 by lifting. The relations 3.10 and 3.11 are without doubt well-known (and trivial for \( R = 0 \)), but for completeness we give a brief proof here. To this end we use a standard lifting argument. Geometrically, this amounts to lifting \( L \) above to an isometry of line bundles: \( L \otimes L_0 \to L \otimes L_z \).

Consider the vector space \( C^2 \) equipped with the diagonal Hermitian bi-linear form with eigenvalues \( (R, 1) \) so that the corresponding squared pseudo-norm \( \| \cdot \|^2_R \) is given by \( \|(w_1, w_2)\|^2_R = R|w_1|^2 + |w_2|^2 \). Let

\[
M_z := \begin{pmatrix} -1 & z \\ R \bar{z} & 1 \end{pmatrix}, \quad \pi(w_1, w_2) = w_1/w_2
\]

(assuming \( w_2 \neq 0 \)), where clearly \( M_z \) preserves \( \| \cdot \|^2_R \) modulo the scaling factor \( \det(M_z) = \|(z, 1)\|^2_R \). In particular, \( \|M_z(w, 1)\|^2_R = \|(z, 1)\|^2_R \|w, 1\|^2_R \) and since \( \zeta := F_\varepsilon(w) = \pi(M(z, 1)) \) this proves (upon taking logarithms) the relation 3.10. The relation 3.11 now follows by substituting the relation \( (1 + R|z|^2) = (1 + R\bar{z}w)(1 + R\bar{z}\zeta) \) into 3.10. In turn, this latter relation can be obtained by first calculating \( d\zeta/dw = -(1 + R|z|^2)/(1 + R\bar{z}w)^2 \) and similarly for \( \zeta \) replaced with \( w \) (using that \( w = F_\varepsilon(\zeta) \)). Since \( d\zeta/dw = (dw/d\zeta)^{-1} \) this forces the previous relation, finishing the proof of 3.11.

3.2. Globalization.

**Proposition 3.4.** Let \( L \to X \) be a positive Hermitian holomorphic line bundle over a compact complex manifold \( X \) and let \( x \in X \) be a fixed point such that the local weight \( \Phi \) of the metric wrt some trivialization of \( L \) around \( x \) is real-analytic ad admits a local Bergman kernel \( K(k) \) mod \( O(e^{-k\delta}) \) such that

\[
K(k)(z, \zeta) = a_k e^{k\phi(z, \bar{z})}
\]

(3.13)

for some sequence \( a_k \) with sub-exponential growth (i.e. \( |a_k| \leq C\delta e^{k\delta} \) for any \( \delta > 0 \)). Then the (global) Bergman kernel \( K_k \) associated to \( kL \) satisfies the uniform estimate

\[
\|K - K(k)\|_{k, \Phi} \leq Ce^{-\delta k}
\]

on some neighbourhood \( U \times U \) of \( \{x\} \times \{x\} \) for some numbers \( C, \delta > 0 \).

**Proof.** Take local holomorphic coordinates \( w \) centered at \( x \). The proof of the proposition is essentially contained in the globalization argument used in [2]. For completeness we recall the argument. Fixing \( z \) and applying the defining formula 3.4 \( u_k := K_{k,z} := K_k(z, \cdot) \) gives

\[
K_{k,z} = (\chi K_{k,z}, K_{(k)})_{U, k, \Phi} + O(e^{-k\delta})e^{\Phi(z)/2}
\]

where we have used that \( \|K_{k,z}\|^2_{U, k, \Phi} \leq \|K_{k,z}\|_{X, k, \Phi} = K_k(z, z) \leq Ck^\delta e^{k\Phi(z)} \) by a standard estimate for Bergman functions (as can be see from a simple argument using the mean value property of holomorphic functions, just as below). Next, we note that the difference \( u_{k,z} := K_{(k),z} - (\chi K_{k,z}, K_{(k)})_{U, k, \Phi} \) is the \( L^2 \)-minimal solution to the \( \bar{\partial} \)-equation

\[
\bar{\partial}u = g,
\]

(3.14)
with \( g = \bar{\partial}(\chi K_{(k), z}) \), which by the Hörmander-Kodaira \( L^2 \)-estimate satisfies
\[
(3.15) \quad \|u_{k,z}\|_{k}^2 \leq C \|g\|_{k}^2 = \| (\bar{\partial}\chi)K_{(k), z} \|_{k}^2,
\]
(recall that \( \bar{\partial}\chi \) is supported in a neighbourhood of \( w = 0 \), vanishing close to \( w = 0 \)). Hence the assumption 3.13 combined with the general basic fact that \( e^{\kappa t}\bar{\partial}e^{\omega}(z) - \phi(z) \) is exponentially concentrated around \( w = z \) (when \( d\bar{\partial}\phi > 0 \)) show that
\[
\|u_{k,z}\|_{k, X} e^{-k\Phi(z)} \leq C e^{-\delta k}.
\]
(in the case of Theorem 3.1 we get the same \( \delta \) as in 3.7 using 3.10 as above).

It is now a standard matter to convert this \( L^2 \)-estimate to an \( L^\infty \)-estimate for \( |u_{k,z}(\zeta)|^2 \) when \( \zeta \) is close to \( z \). Indeed restricting the integration in the previous inequality to a small disc \( D_k(\zeta) \) of radius \( \epsilon k^{-1/2} \) centered at \( \zeta \) gives
\[
\int_{D_k(\zeta)} |u_{k,z}(w)|^2 dw \wedge d\bar{w} \leq C' e^{-\delta k}.
\]

Finally, since the integral in the lhs above may, by the mean value property of holomorphic functions, be estimated from below by \( c|u_{k,z}(\zeta)|^2/k \) this finishes the proof of the proposition.

\[\square\]

Remark 3.5. Tracing through the arguments above in fact gives an explicit expression for the exponent \( \delta \) appearing in the error terms in Theorem 3.1. Indeed, if we take \( U \) as a disc \( D_r \) of radius \( r \) then \( \delta \) can be chosen to be arbitrary close to \( \Phi(r^2) \). To see this just let the cut-off function \( \chi \) instead be supported on \( (1 - \epsilon')U \) for a given \( \epsilon' \). Then \( \epsilon \) appearing in 3.7 can be taken arbitrarily close to \( r(1 - \epsilon') \). Note that if \( l \) is the radius of \( D_r \) in the metric \( \omega \) then, if \( R \) is globally constant, the optimal choice of \( r \) above corresponds to \( l \) being the injectivity radius of \( X \) at \( x \). A direct computation gives \( r = \frac{1}{\sqrt{-R}} \text{tanh} \left( \frac{\pi R l}{2} \right) \) and hence \( \Phi(r^2) = \frac{2}{\sqrt{-R}} \log \left( \cosh \left( \frac{\pi R l}{2} \right) \right) \).

Moreover, the proof of Theorem 3.1 also goes through, word for word, in any dimension \( n \) (so that \( w = (w_1, ..., w_n) \) etc) if one assumes that the Kähler metric \( \omega \) has constant holomorphic sectional curvature (and in particular constant scalar curvature := \( R \)). Indeed, using the normal coordinates in [10] one reduces to \( \Phi(z) = -(n + 1) \log(1 + R|w|^2) \) as before. Computing the integrals in 3.6 then shows that the \( k \)-dependent leading constant \( k + R/2 \) in the theorem has to be replaced by a constant which is an explicit polynomial in \( k \) which may be expanded as \( k^n + \frac{R}{2} + O(k^{n-1}) \). Also note that the global positivity (i.e. on \( X - U \)) of the line bundle in the previous proposition was only used in the Hörmander-Kodaira \( L^2 \)-estimate (and that \( C = C_k \) has sub-exponential growth in \( k \)). For example, it holds as long as \( L \) is ample and globally semi-positively curved (and positively curved on \( U \)).

3.3. Proof of the Corollary. The corollary follows from Theorem 3.1 and the remark above. Indeed, writing \( \omega = d\bar{\partial}\Phi \) we have by assumption that \( d\bar{\partial}\Phi = ce^{-\Phi}d\bar{\partial}z \wedge dz \) where \( \Phi_D = \log |s_D|^2 \) for \( s_D \) a holomorphic (multi-)section of \( D \). Hence, \( e^{-k\Phi}d\bar{\partial}e^{-\Phi} = e^{-\Phi_k} \), where \( \Phi_k = (k - 1)\Phi + \Phi_D \) is the weight of a singular metric on \( L_k := (k - 1)L + D \) (i.e. \( kL = L_k + K_X \)) with positive curvature current. It then follows from Demailly’s singular version of the Hörmander-Kodaira \( L^2 \)-estimates [13] that the \( L^2 \)-minimal solution \( u \) to 3.14 satisfies
\[
(3.16) \quad \left( \int_X |u|^2 e^{-k\Phi} d\bar{\partial}\Phi \right) = \int_X |u|^2 e^{-\Phi_k} \leq \int_X \frac{g\bar{g}}{d\bar{\partial}\Phi_k} e^{-\Phi_k}
\]
where \( g \) is seen as a \((0, 1)\)-form with values in \( L_k + K_X \). In our case we take \( g = \bar{\partial}(\chi K_{(k), z}) \) which is supported where \( \Phi \) is smooth and hence the previous estimates go through word for word (with a constant \( C \) only depending
on the fixed point \( x \)). Finally, since the space of all “cusp sections” of \( kL \) coincides with the subspace of \( H^0(X, kL) \) of all sections which are in \( L^2 \) wrt the \( L^2 \)-norm defined by \( \Phi_k \) this finishes the proof. This last fact follows from the well-known fact \[22\] that \( \Phi \) has only a mild singularity at any cusp (corresponding to \( z = 0 \)):

\[
\Phi \sim -\log(-\log|z|).
\]

Hence, since \( dd^c \Phi = ce^{\Phi} \frac{1}{|z|^2} dz \wedge d\bar{z} \), a local holomorphic function \( u_k \) is locally integrable square wrt \( e^{-k\Phi} dd^c \Phi \) iff \( u_k(0) = 0 \).

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