Low frequency dispersive estimates for the Schrödinger group in higher dimensions

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Abstract

For a large class of real-valued potentials, \( V(x), x \in \mathbb{R}^n, n \geq 4 \), we prove dispersive estimates for the low frequency part of \( e^{it(-\Delta + V)}P_{ac} \), provided the zero is neither an eigenvalue nor a resonance of \(-\Delta + V\), where \( P_{ac} \) is the spectral projection onto the absolutely continuous spectrum of \(-\Delta + V\). This class includes potentials \( V \in L^\infty(\mathbb{R}^n) \) satisfying \( V(x) = O \left( (x)^{-(n+2)/2-\epsilon} \right) \), \( \epsilon > 0 \). As a consequence, we extend the results in [4] to a larger class of potentials.

1 Introduction and statement of results

Let \( V \in L^\infty(\mathbb{R}^n), n \geq 4 \), be a real-valued function satisfying

\[
|V(x)| \leq C(x)^{-\delta}, \quad \forall x \in \mathbb{R}^n, \tag{1.1}
\]

with constants \( C > 0 \), \( \delta > (n + 2)/2 \). Denote by \( G_0 \) and \( G \) the self-adjoint realizations of the operators \(-\Delta\) and \(-\Delta + V\) on \( L^2(\mathbb{R}^n) \), respectively. It is well known that the absolutely continuous spectra of the operators \( G_0 \) and \( G \) coincide with the interval \([0, +\infty)\), and that \( G \) has no embedded strictly positive eigenvalues nor strictly positive resonances. However, \( G \) may have in general a finite number of non-positive eigenvalues and that the zero may be a resonance. We will say that the zero is a regular point for \( \frac{d}{dt} \left( e^{itG} \right) \left( \chi_{G} \right) \) if it is neither an eigenvalue nor a resonance, for potentials satisfying (1.1) with \( \delta > n + 4 \) as well as the condition

\[
\tilde{V} \in L^1. \tag{1.3}
\]

This was later improved by Yajima [9] for potentials satisfying (1.1) with \( \delta > n + 2 \). When \( n = 3 \), the estimate (1.2) in fact holds without (1.3). In this case, it was proved in [2] for potentials satisfying (1.1) with \( \delta > 3 \) and was later improved in [6] and [10] for potentials satisfying (1.1) with \( \delta > 5/2 \). Goldberg [1] has recently showed that (1.2) holds for potentials \( V \in L^{n/2-\epsilon} \cap L^{3/2+\epsilon}, 0 < \epsilon \ll 1 \), which includes potentials satisfying (1.1) with \( \delta > 2 \). When \( n = 2, (1.2) \) is proved by Schlag [3] for potentials satisfying (1.1) with \( \delta > 3 \).

Given any \( a > 0 \), set \( \chi_a(\sigma) = \chi_1(\sigma/a) \), where \( \chi_1 \in C^\infty(\mathbb{R}), \chi_1(\sigma) = 0 \) for \( \sigma \leq 1, \chi_1(\sigma) = 1 \) for \( \sigma \geq 2 \). Set \( \eta_a = \chi(1 - \chi_a) \), where \( \chi \) denotes the characteristic function of the interval \([0, +\infty)\). Clearly, \( \eta_a(G) + \chi_a(G) = P_{ac} \). When \( n \geq 4 \), dispersive estimates with loss of \((n - 3)/2\) derivatives for the operator \( e^{itG} \chi_a(G), \forall a > 0 \), have been recently proved in
under the assumption (1.1), only. The loss of derivatives in this case is a high frequency phenomenon and cannot be avoided unless one imposes some regularity condition on the potential (see [3]). The condition (1.3) in [3] plays this role but it seems too strong. The natural conjecture would be that we have dispersive estimates for $e^{itG} \chi_a (G)$ with loss of $\nu$ derivatives, $0 \leq \nu \leq (n - 3)/2$, provided $V \in C^{(n-3)/2 - \nu} (\mathbb{R}^n)$ (with a suitable decay at infinity). It turns out that no regularity on the potential is needed in order to get dispersive estimates for the low frequency part $e^{itG} \eta_a (G), a > 0$ small. One just needs some decay at infinity. In fact, the low frequency analysis turns out to be easier in dimensions $n \geq 4$ compared with the cases of $n = 2$ and $n = 3$, and can be carried out for a larger class of potentials satisfying (with some $0 < \epsilon \ll 1$)

$$\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \left( |x - y|^{-n+2} + |x - y|^{-(n-2)/2 + \epsilon} \right) |V(x)| dx \leq C < +\infty. \quad (1.4)$$

Clearly, (1.4) is fulfilled for potentials satisfying (1.1). Our main result is the following

**Theorem 1.1** Let $n \geq 4$, let $V$ satisfy (1.4) and assume that the zero is a regular point for $G$. Then, there exists a constant $a_0 > 0$ so that for $0 < a \leq a_0$ we have the estimate

$$\left\| e^{itG} \eta_a (G) \right\|_{L^1 \rightarrow L^\infty} \leq C |t|^{-n/2}, \ t \neq 0. \quad (1.5)$$

**Remark 1.** We expect that (1.5) holds true for the larger class of potentials satisfying

$$\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \left( |x - y|^{-n+2} + |x - y|^{-(n-1)/2} \right) |V(x)| dx \leq C < +\infty, \quad (1.6)$$

but the proof in this case would require a different approach.

Combining (1.5) with the estimates of [7], we obtain the following

**Theorem 1.2** Let $n \geq 4$, let $V$ satisfy (1.1) and assume that the zero is a regular point for $G$. Then, we have the estimates, $\forall t \neq 0, 0 < \epsilon \ll 1$,

$$\left\| e^{itG} P_{ac} f \right\|_{L^\infty} \leq C |t|^{-n/2} \left\| (G)^{(n-3)/4} f \right\|_{L^1}, \quad (1.7)$$

$$\left\| e^{itG} P_{ac} f \right\|_{L^\infty} \leq C \epsilon |t|^{-n/2} \left\| (G)^{n/4 + \epsilon} f \right\|_{L^2}. \quad (1.8)$$

**Remark 2.** The proof in [7] is based on uniform estimates for the operator $\psi (h^2 G), 0 < h \leq 1, \psi \in C^\infty_0 ((0, +\infty))$ (see Lemma 2.2 of [7] or Lemma 2.3 of [8]). In the proof of this lemma (which is given in [8]), however, there is a mistake. That is why, we will give a new proof in Appendix 1 of the present paper.

**Remark 3.** We conjecture that the estimates (1.7) and (1.8) hold true for potentials satisfying (1.1) with $\delta > (n+1)/2$.

Theorem 1.1 also allows to extend the results in [4] to a larger class of potentials. More precisely, we have the following

**Theorem 1.3** Let $n \geq 4$, let $V$ satisfy (1.1) with $\delta > n - 1$ as well as (1.3), and assume that the zero is a regular point for $G$. Then, the estimate (1.2) holds true.

Theorem 1.3 follows from (1.5) and the dispersive estimate for $e^{itG} \chi_a (G)$ proved in Appendix 2.

To prove (1.5) we adapt the semi-classical approach of [7] based on the semi-classical version of Duhamel's formula (which in our case is of the form (3.4) or (3.5)). While in [7] the estimates had to be uniform with respect to the semi-classical parameter $0 < h \leq 1$, in the case of low frequency we need to make them uniform for $h \gg 1$ (see (3.1)). This,
however, turns out to be easier (when \( n \geq 4 \)) as we can absorb the remaining terms taking \( h \) big enough (see Section 3). That is why, we do not need any more to work on weighted \( L^2 \) spaces (as in [4]), which in turn allows to cover a much larger class of potentials. As mentioned in Remark 1, the natural class of potentials for which the low frequency analysis works out (for \( n \geq 4 \)) is given by (1.6), and the fact that the crucial Proposition 2.1 below holds true under (1.6) is a strong indication for that. In fact, (1.4) is used in the proof of Proposition 2.3, only.

## 2 Preliminary estimates

Let \( \psi \in C_0^\infty((0, +\infty)) \). We will first prove the following

**Proposition 2.1** Let \( n \geq 4 \), let \( V \) satisfy (1.6) and assume that the zero is a regular point for \( G \). Then, there exist positive constants \( C, \beta \) and \( h_0 \) so that the following estimates hold

\[
\| \psi(h^2 G_0) \|_{L^1 \to L^1} \leq C, \quad h > 0, \tag{2.1}
\]
\[
\| \psi(h^2 G) \|_{L^1 \to L^1} \leq C, \quad h \geq h_0, \tag{2.2}
\]
\[
\| \psi(h^2 G) - \psi(h^2 G_0) T \|_{L^1 \to L^1} \leq C h^{-\beta}, \quad h \geq h_0, \tag{2.3}
\]
where the operator
\[
T = (1 - V \Delta^{-1})^{-1} : L^1 \to L^1 \tag{2.4}
\]
is bounded by assumption.

**Proof.** Set \( \varphi(\lambda) = \psi(\lambda^2) \). We are going to take advantage of the formula

\[
\psi(h^2 G) = \frac{2}{\pi} \int_{C} \frac{\partial \tilde{\varphi}}{\partial z}(z) (h^2 G - z^2)^{-1} z L(dz), \tag{2.5}
\]
where \( L(dz) \) denotes the Lebesgue measure on \( C \), \( \tilde{\varphi} \in C_0^\infty(C) \) is an almost analytic continuation of \( \varphi \) supported in a small complex neighbourhood of supp \( \varphi \) and satisfying

\[
\left| \frac{\partial \varphi}{\partial z}(z) \right| \leq C_N |\text{Im } z|^N, \quad \forall N \geq 1.
\]

For \( \pm \text{Im } z > 0 \), denote

\[
R_{0,h}^{\pm}(z) = (h^2 G_0 - z^2)^{-1}, \quad R_{h}^{\pm}(z) = (h^2 G - z^2)^{-1}.
\]

The kernel of the operator \( R_{0,h}^{\pm}(z) \) is of the form \( R_{h}^{\pm}(|x - y|, z) \), where

\[
R_{h}^{\pm}(\sigma, z) = \pm h^{-2} \frac{3e^{-2\nu}}{4(2\pi)^{\nu}} H_{\nu}^{\pm}(\sigma z/h) = h^{-n} R_{1}^{\pm}(\sigma h^{-1}, z),
\]
where \( \nu = (n - 2)/2 \), \( H_{\nu}^{\pm}(\lambda) = \lambda^{n} H_{\nu}^{\pm}(\lambda) \), \( H_{\nu}^{\pm} \) being the outgoing and incoming Henkel functions of order \( \nu \). It is well known that these functions satisfy the bound

\[
|H_{\nu}^{\pm}(\lambda)| \leq C(\lambda)^{(n-3)/2} e^{-|\text{Im } \lambda|}, \quad \forall \lambda, \pm \text{Im } \lambda \geq 0, \tag{2.6}
\]

while near \( \lambda = 0 \) they are of the form

\[
H_{\nu}^{\pm}(\lambda) = a_{\nu,1}^{\pm}(\lambda) + \lambda^{n-2} \log \lambda a_{\nu,2}^{\pm}(\lambda), \tag{2.7}
\]
where $a_{\nu,j}^{\pm}$ are analytic functions, $a_{\nu,2}^{\pm} = 0$ if $n$ is odd. By (2.6) and (2.7), we have
\[ |H^{\pm}_{\nu}(\lambda) - H^{\pm}_{\nu}(0)| \leq C|\lambda|^{1/2}(\lambda)^{(n-4)/2}, \quad \forall \lambda, \pm \text{Im} \lambda \geq 0. \] (2.8)
Hence, the functions $R_{h}^{\pm}$ satisfy the bounds (for $z \in C_{\varphi}^{\pm} := \{ z \in \text{supp} \varphi, \pm \text{Im} z \geq 0 \}, \sigma > 0, h \geq 1$)
\[ |R_{h}^{\pm}(\sigma, z)| \leq Ch^{-2} \left( \sigma^{-n+2} + \sigma^{-(n-1)/2} \right), \] (2.9)
\[ |R_{h}^{\pm}(\sigma, z) - R_{h}^{\pm}(\sigma, 0)| \leq Ch^{-5/2} \left( \sigma^{-n+5/2} + \sigma^{-(n-1)/2} \right). \] (2.10)
Using the above bounds we will prove the following

**Lemma 2.2** For $z \in C_{\varphi}^{\pm}$, we have
\[ \| VR_{0,h}(z) \|_{L^{1} \rightarrow L^{1}} \leq Ch^{-2}, \quad h \geq 1, \] (2.11)
\[ \| VR_{0,h}(z) - VR_{0,h}(0) \|_{L^{1} \rightarrow L^{1}} \leq Ch^{-5/2}, \quad h \geq 1, \] (2.12)
\[ \| VR_{h}(z) \|_{L^{1} \rightarrow L^{1}} \leq Ch^{-2}, \quad h \geq h_{0}, \] (2.13)
\[ \| R_{0,h}^{\pm}(z) \|_{L^{1} \rightarrow L^{1}} \leq C|\text{Im} z|^{-q}, \quad h > 0, \text{Im} z \neq 0, \] (2.14)
\[ \| R_{h}^{\pm}(z) \|_{L^{1} \rightarrow L^{1}} \leq C|\text{Im} z|^{-q}, \quad h \geq h_{0}, \text{Im} z \neq 0, \] (2.15)
with constants $C, q, h_{0} > 0$ independent of $z$ and $h$.

**Proof.** In view of (2.9), the norm in the LHS of (2.11) is upper bounded by
\[ \sup_{y \in \mathbb{R}^{n}} \int_{\mathbb{R}^{n}} |V(x)| |R_{h}^{\pm}([x - y], z)| \, dx \leq Ch^{-2} \sup_{y \in \mathbb{R}^{n}} \int_{\mathbb{R}^{n}} |V(x)| \left( |x - y|^{-n+2} + |x - y|^{-(n-1)/2} \right) \, dx \leq Ch^{-2}. \]
The estimate (2.12) follows in the same way using (2.10). To prove (2.14), we use (2.6) to get (for $z \in C_{\varphi}^{\pm}, \text{Im} z \neq 0$)
\[ |R_{1}^{\pm}(\sigma, z)| \leq C\sigma^{-2\nu}<\sigma>(n-3)/2 \exp{-\sigma|\text{Im} z|} \leq C\sigma^{-2\nu}<\sigma>^{-5/2}|\text{Im} z|^{-(n+2)/2}. \] (2.16)
By (2.16), the norm in the LHS of (2.14) is upper bounded by
\[ C \int_{0}^{\infty} \sigma^{n-1} |R_{1}^{\pm}(\sigma, z)| \, d\sigma = C \int_{0}^{\infty} \sigma^{n-1} |R_{1}^{\pm}(\sigma, z)| \, d\sigma \leq C|\text{Im} z|^{-(n+2)/2} \int_{0}^{\infty} \sigma^{-3/2} d\sigma \leq C|\text{Im} z|^{-(n+2)/2}. \]
To prove (2.13) and (2.15), we will use the identity
\[ R_{h}^{\pm}(z) \left( 1 + h^{2}VR_{0,h}^{\pm}(z) \right) = R_{0,h}^{\pm}(z), \quad \pm \text{Im} z > 0. \] (2.17)
Observe that $1 + h^{2}VR_{0,h}^{\pm}(0) = 1 - V\Delta^{-1}$, which is supposed to be invertible on $L^{1}$ with a bounded inverse denoted by $T$. Thus, it follows from (2.12) that there exists a constant $h_{0} > 0$ so that for $h \geq h_{0}$ the operator $1 + h^{2}VR_{0,h}^{\pm}(z)$ is invertible on $L^{1}$ with an inverse satisfying
\[ \left\| \left( 1 + h^{2}VR_{0,h}^{\pm}(z) \right)^{-1} \right\|_{L^{1} \rightarrow L^{1}} \leq C, \quad z \in C_{\varphi}^{\pm}, \] (2.18)
with a constant $C > 0$ independent of $z$ and $h$. Hence, we can write

$$R_h^\pm(z) = R_{0,h}^\pm(z) \left(1 + h^2V R_{0,h}^\pm(z)\right)^{-1}.$$  \hfill (2.19)

Now (2.13) follows from (2.11), (2.18) and (2.19), while (2.15) follows from (2.14), (2.18) and (2.19).

Clearly, (2.1) and (2.2) follow from (2.5) and (2.14), respectively. To prove (2.3) we rewrite the identity (2.19) in the form

$$R_h^\pm(z) - R_{0,h}^\pm(z)T$$

$$= R_{0,h}^\pm(z)T \left(h^2V R_{0,h}^\pm(z) - h^2V R_{0,h}^\pm(0)\right)T \left(1 + \left(h^2V R_{0,h}^\pm(z) - h^2V R_{0,h}^\pm(0)\right)\right)^{-1}.$$  \hfill (2.20)

By Lemma 2.2, (2.18) and (2.20) we conclude

$$\left\|R_h^\pm(z) - R_{0,h}^\pm(z)T\right\|_{L^1\to L^1} \leq Ch^{-\beta}\left|\text{Im } z \right|^{-q}, \quad h \geq h_0, \quad z \in C_\varphi^\pm, \quad \text{Im } z \neq 0,$$

with constants $C, q, \beta > 0$ independent of $z$ and $h$. Now (2.3) follows from (2.5) and (2.21).

Let $\psi_1 \in C_0^\infty((0, +\infty))$, $\psi_1 = 1$ on supp $\psi$.

**Proposition 2.3** Under the assumptions of Theorem 1.1, there exist positive constants $h_0$ and $\beta$ so that we have the estimates

$$\int_{-\infty}^{\infty} \left\|Ve^{itG_0}\psi(h^2G_0)\right\|_{L^1\to L^1} dt \leq Ch^{-\beta}, \quad h \geq 1,$$  \hfill (2.22)

$$\int_{-\infty}^{\infty} \left\|V\psi(h^2G_0)e^{itG_0}\psi_1(h^2G_0)\right\|_{L^1\to L^1} dt \leq Ch^{-\beta}, \quad h \geq h_0.$$  \hfill (2.23)

**Proof.** It is shown in [7] (Section 2) that the kernel of the operator $e^{itG_0}\psi(h^2G_0)$ is of the form $K_h(\|x - y\| t)$ with a function $K_h$ satisfying

$$K_h(\sigma, t) = h^{-n}K_1(\sigma h^{-1}, th^{-2}),$$

$$|K_1(\sigma, t)| \leq C|t|^{-s+1/2}\sigma^{s-(n-1)/2}, \quad 0 \leq s \leq (n-1)/2, \quad \sigma > 0, \quad t \neq 0.$$

Hence, for all $0 \leq s \leq (n-1)/2, \sigma > 0, \quad t \neq 0, \quad h > 0$, we have

$$|K_h(\sigma, t)| \leq Ch^{s-(n-1)/2}|t|^{-s+1/2}\sigma^{s-(n-1)/2},$$

which together with (1.4) imply

$$\left\|Ve^{itG_0}\psi(h^2G_0)\right\|_{L^1\to L^1} \leq Ch^{s-(n-1)/2}|t|^{-s+1/2}, \quad 1/2 - \epsilon \leq s \leq 1/2 + \epsilon,$$  \hfill (2.24)

where $0 < \epsilon \ll 1$. Clearly, (2.22) follows from (2.24). Furthermore, using (2.5), (2.13), (2.14) and (2.24), we get

$$\left\|V(\psi(h^2G) - \psi(h^2G_0))e^{itG_0}\psi_1(h^2G_0)\right\|_{L^1\to L^1} \leq Ch^2 \sum_{\pm} \int_{C_\varphi^\pm} \left|\frac{\partial^2 \tilde{F}}{\partial z}(z)\right| \left\|VR_h^\pm(z)Ve^{itG_0}\psi_1(h^2G_0)R_{0,h}^\pm(z)\right\|_{L^1\to L^1} L(dz).$$
\[
\leq Ch^2 \sum_{\pm} \int_{C^\pm} \left| \frac{\partial \tilde{\phi}}{\partial z}(z) \right| \left| VR_f(z) \right|_{L^1 \rightarrow L^1} \left| V e^{i t G_0} \psi_1 (h^2 G_0) \right|_{L^1 \rightarrow L^1} \left| R_0^\pm (z) \right|_{L^1 \rightarrow L^1} L(dz)
\]

\[
\leq Ch^{s-(n-1)/2} |t|^{-s-1/2} \sum_{\pm} \int_{C^\pm} \left| \frac{\partial \tilde{\phi}}{\partial z}(z) \right| |\text{Im } z|^{-\delta} L(dz)
\]

\[
\leq Ch^{s-(n-1)/2} |t|^{-s-1/2}, \quad 1/2 - \epsilon \leq s \leq 1/2 + \epsilon,
\]

which clearly implies (2.23).

\[\square\]

3 Proof of Theorem 1.1

Denote

\[
\Psi(t, h) = e^{i t G} \psi(h^2 G) - T^* e^{i t G_0} \psi(h^2 G_0) T,
\]

\[T\] being given by (2.4). We will first show that (1.5) follows from the following

Proposition 3.1 Under the assumptions of Theorem 1.1, there exist positive constants \(C, h_0\) and \(\beta\) so that for \(h \geq h_0\) we have

\[
\| \Psi(t, h) \|_{L^1 \rightarrow L^\infty} \leq Ch^{-\beta} |t|^{-n/2}, \quad t \neq 0.
\]

(3.1)

Recall that \(\chi_a(\sigma) = \chi_1(\sigma/a), a > 0\) small. Then we can write the function \(\eta_\sigma\) as follows

\[
\eta_\sigma(\sigma) = \int_a^\infty \psi(\sigma \theta) \frac{d\theta}{\theta}, \quad \sigma > 0,
\]

where \(\psi(\sigma) = \sigma \chi_1(\sigma) \in C_0^\infty((0, +\infty))\). Thus, we obtain from (3.1),

\[
\| e^{i t G} \eta_\sigma(G) - T^* e^{i t G_0} \eta_\sigma(G_0) T \|_{L^1 \rightarrow L^\infty} \leq \int_{a-1}^\infty \| \Psi(t, \sqrt{\theta}) \|_{L^1 \rightarrow L^\infty} \frac{d\theta}{\theta}
\]

\[
\leq C|t|^{-n/2} \int_{a-1}^\infty \theta^{-1/2 - \beta/2} d\theta \leq C|t|^{-n/2},
\]

(3.2)

provided \(a\) is taken small enough. Clearly, (1.5) follows from (3.2).

Proof of Proposition 3.1. We will first prove the following

Proposition 3.2 Under the assumptions of Theorem 1.1, there exist positive constants \(C, h_0\) and \(\beta\) so that for \(h \geq h_0\) we have

\[
\int_{-\infty}^\infty \| V e^{i t G} \psi(h^2 G) \|_{L^1 \rightarrow L^1} dt \leq Ch^{-\beta}.
\]

(3.3)

Proof. Using Duhamel’s formula

\[
e^{i t G} = e^{i t G_0} + i \int_0^t e^{i(t-\tau)G} V e^{i \tau G_0} d\tau,
\]

we get the identity

\[
e^{i t G} \psi(h^2 G) = \psi(h^2 G) e^{i t G_0} \psi_1(h^2 G_0) T + e^{i t G} \psi(h^2 G) (\psi_1(h^2 G) - \psi_1(h^2 G_0) T)
\]

\[
+ i \int_0^t \psi(h^2 G) e^{i(t-\tau)G} V e^{i \tau G_0} \psi_1(h^2 G_0) T d\tau.
\]

(3.4)
Using Propositions 2.1 and 2.3, (3.4) together with Young’s inequality we obtain
\[
\int_{-\infty}^{\infty} \|Ve^{itG}\psi(h^2G)\|_{L^1 \to L^1} dt \leq Ch^{-\beta} + Ch^{-\beta} \int_{-\infty}^{\infty} \|Ve^{itG}\psi(h^2G)\|_{L^1 \to L^1} dt
\]
\[
+ \int_{-\infty}^{\infty} \int_0^t \|Ve^{i(t-\tau)G}\psi_1(h^2G_0)\|_{L^1 \to L^1} \|Ve^{irG0}\psi_1(h^2G_0)\|_{L^1 \to L^1} d\tau dt
\]
\[
\leq Ch^{-\beta} + Ch^{-\beta} \int_{-\infty}^{\infty} \|Ve^{itG}\psi(h^2G)\|_{L^1 \to L^1} dt,
\]
which clearly implies (3.3) if we take \( h \) large enough.

Using Duhamel’s formula
\[
e^{itG} = e^{itG_0} + i \int_0^t e^{i(t-\tau)G_0} Ve^{i\tau G} d\tau,
\]
we get the identity
\[
\Psi(t; h) = \sum_{j=1}^{2} \Psi_j(t; h), \tag{3.5}
\]
where
\[
\Psi_1(t; h) = T^*\psi_1(h^2G_0)e^{itG_0} (\psi(h^2G) - \psi(h^2G_0)T) + (\psi_1(h^2G) - T^*\psi_1(h^2G_0)) e^{itG}\psi(h^2G),
\]
\[
\Psi_2(t; h) = i \int_0^t T^*\psi_1(h^2G_0)e^{i(t-\tau)G_0} Ve^{i\tau G}\psi(h^2G) d\tau.
\]
By (2.1) and (2.3) together with the well known estimate
\[
\|e^{itG_0}\|_{L^1 \to L^\infty} \leq C|t|^{-n/2},
\]
we get
\[
\|\Psi_1(t; h)f\|_{L^\infty} \leq Ch^{-\beta}|t|^{-n/2}\|f\|_{L^1} + Ch^{-\beta} \|\Psi(t; h)f\|_{L^\infty}, \quad t \neq 0. \tag{3.6}
\]
By Propositions 2.3 and 3.2, \( \forall f \in L^1, \) \( t > 0, \) we have
\[
t^{n/2} \|\Psi_2(t; h)f\|_{L^\infty}
\]
\[
\leq C \int_0^{t/2} (t-\tau)^{n/2} \|\psi_1(h^2G_0)e^{i(t-\tau)G_0}\|_{L^1 \to L^\infty} \|Ve^{i\tau G}\psi(h^2G)f\|_{L^1} d\tau
\]
\[
+ C \int_{t/2}^t \|\psi_1(h^2G_0)e^{i(t-\tau)G_0} V\|_{L^\infty \to L^\infty} \tau^{n/2} \|e^{i\tau G}\psi(h^2G)f\|_{L^\infty} d\tau
\]
\[
\leq C \int_{-\infty}^{\infty} \|Ve^{i\tau G}\psi(h^2G)f\|_{L^1} d\tau
\]
\[
+ C \sup_{t/2 \leq \tau \leq t} \tau^{n/2} \|Ve^{i\tau G}\psi(h^2G)f\|_{L^\infty} \int_{-\infty}^{\infty} \|Ve^{i\tau G}\psi_1(h^2G_0)\|_{L^1 \to L^1} d\tau
\]
\[
\leq Ch^{-\beta} \|f\|_{L^1} + Ch^{-\beta} \sup_{t/2 \leq \tau \leq t} \tau^{n/2} \|e^{i\tau G}\psi(h^2G)f\|_{L^\infty}. \tag{3.7}
\]
Combining (3.5), (3.6) and (3.7), we conclude, \( \forall f \in L^1, \) \( t > 0, \)
\[
t^{n/2} \|\Psi(t; h)f\|_{L^\infty} \leq Ch^{-\beta} \|f\|_{L^1} + Ch^{-\beta} t^{n/2} \|\Psi(t; h)f\|_{L^\infty}
\]
\[
+ Ch^{-\beta} \sup_{t/2 \leq \tau \leq t} \tau^{n/2} \|\Psi(t; h)f\|_{L^\infty}. \tag{3.8}
\]
Taking \( h \) big enough we can absorb the second and the third terms in the RHS of (3.8), thus obtaining (3.1). Clearly, the case of \( t < 0 \) can be treated in the same way.
A Appendix 1

We will prove the following

**Lemma A.1** Let \( \psi \in C_0^\infty((0, +\infty)) \). Then, for all \( h > 0, s \geq 0 \), we have the estimates

\[
\|\psi(h^2G_0)\|_{L^1} \leq C, \tag{A.1}
\]

\[
\| \langle x \rangle^s \psi(h^2G_0) \langle x \rangle^{-s} \|_{L^1} \leq Ch^{-n/2} \langle h \rangle^s, \tag{A.2}
\]

where the constant \( C \) is of the form

\[
C = C' \sup_{0 \leq k \leq k_0} \sup_{\lambda \in \mathbb{R}} |\partial_\lambda^k \psi(\lambda)|, \tag{A.3}
\]

with some integer \( k_0 \) independent of \( \psi \) and a constant \( C' > 0 \) depending only on the support of \( \psi \). Furthermore, if \( V \) satisfies (1.1) with \( \delta > n/2 \), we have the estimates (for \( 0 < h \leq 1 \))

\[
\|\psi(h^2G) - \psi(h^2G_0)\|_{L^1} \leq Ch^2, \tag{A.4}
\]

\[
\| \langle x \rangle^\delta (\psi(h^2G) - \psi(h^2G_0)) \|_{L^1} \leq Ch^{2-n/2}. \tag{A.5}
\]

**Proof.** The estimate (A.1) is proved in Section 2 using the formula (2.5) and (2.14). It can be also seen by using the fact that the kernel of the operator \( \psi(h^2G_0) \) is of the form \( k_h(|x - y|) \) with a function \( k_h \) satisfying

\[
k_h(\sigma) = h^{-n}k_1(\sigma/h), \tag{A.6}
\]

\[
|k_1(\sigma)| \leq C_m(\sigma)^{-m}, \quad \forall \sigma > 0, \tag{A.7}
\]

for all integers \( m \geq 0 \), with a constant \( C_m \) of the form

\[
C_m = C'_m \sup_{0 \leq j \leq j_m} \sup_{\lambda \in \mathbb{R}} |\partial_\lambda^j \psi(\lambda)|, \tag{A.8}
\]

where \( j_m \) is some integer independent of \( \psi \), while \( C'_m > 0 \) depends on the support of \( \psi \). By Young’s inequality, the norm in the LHS of (A.1) is upper bounded by

\[
\int_{\mathbb{R}^n} |k_h(|\xi|)|d\xi = \int_{\mathbb{R}^n} |k_1(|\xi|)|d\xi \leq C_{n+1}.
\]

The norm in the LHS of (A.2) is upper bounded by

\[
\sup_{y \in \mathbb{R}^n} \left( \int_{\mathbb{R}^n} \langle x \rangle^{2s} \langle y \rangle^{-2s} |k_h(|x - y|)|^2 dx \right)^{1/2} \leq \left( \int_{\mathbb{R}^n} \langle x - y \rangle^{2s} |k_h(|x - y|)|^2 dx \right)^{1/2}
\]

\[
\leq C \langle h \rangle^s \left( \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |k_h(|\xi|)|^2 d\xi \right)^{1/2}
\]

\[
= C \langle h \rangle^s h^{-n/2} \left( \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |k_1(|\xi|)|^2 d\xi \right)^{1/2} \leq C_{s_n} \langle h \rangle^s h^{-n/2},
\]

where \( s_n \) is some integer depending on \( n \) and \( s \). To prove (A.4) observe that by (2.5) we have

\[
\psi(h^2G) - \psi(h^2G_0) = \frac{2h^2}{\pi} \int_{C} \frac{\partial^2 \varphi}{\partial z^2}(z) (h^2G_0 - z^2)^{-1} V(h^2G - z^2)^{-1} z L(dz). \tag{A.9}
\]
Clearly, (A.4) would follow from (A.9), (2.14) and the estimate (for \( z \in \text{supp} \tilde{\varphi} \))

\[
\| (h^2 G - z^2)^{-1} \|_{L^1 \to L^1} \leq C |\text{Im} \, z|^{-q}, \quad 0 < h \leq 1, \, z \neq 0. \tag{A.10}
\]

Let \( \phi \in C_0^{\infty}([1,2]) \) be such that \( \int \phi(\theta^2) \theta^{-1} d\theta = 1 \). Given a parameter \( 0 < \varepsilon \ll 1 \), we decompose the free resolvent as follows

\[
(h^2 G_0 - z^2)^{-1} = A_\varepsilon(z; h) + B_\varepsilon(z; h), \tag{A.11}
\]

where

\[
A_\varepsilon(z; h) = \int_0^1 f ((\varepsilon \theta h)^2 G_0; (\varepsilon \theta)^2; z) \frac{d\theta}{\theta},
\]

\[
B_\varepsilon(z; h) = \int_1^\infty f ((\varepsilon \theta h)^2 G_0; (\varepsilon \theta)^2; z) \frac{d\theta}{\theta},
\]

where

\[
f(\lambda; \mu; z) = \frac{\phi(\lambda)}{\lambda \mu^{-1} - z^2}.
\]

It is easy to see that there exist constants \( 0 < \mu_1 < \mu_2 \) so that the function \( f \) satisfies the following bounds

\[
\left| \partial^j_\lambda f(\lambda; \mu; z) \right| \leq C_j \mu, \quad 0 < \mu \leq \mu_1, \tag{A.12}
\]

\[
\left| \partial^j_\lambda f(\lambda; \mu; z) \right| \leq C_j' |\text{Im} \, z|^{-j-1}, \quad \mu_1 \leq \mu \leq \mu_2, \tag{A.13}
\]

\[
\left| \partial^j_\lambda f(\lambda; \mu; z) \right| \leq C_j'', \quad \mu \geq \mu_2, \tag{A.14}
\]

for every integer \( j \geq 0 \). By (A.1), (A.3) and (A.12), we have

\[
\| f ((\varepsilon \theta h)^2 G_0; (\varepsilon \theta)^2; z) \|_{L^1 \to L^1} \leq C(\varepsilon \theta)^2, \quad 0 < \theta \leq 1, \tag{A.15}
\]

provided \( \varepsilon > 0 \) is taken small enough. We deduce from (A.15),

\[
\| A_\varepsilon(z; h) \|_{L^1 \to L^1} \leq C \varepsilon^2, \quad z \in \text{supp} \tilde{\varphi}, \tag{A.16}
\]

with a constant \( C > 0 \) independent of \( z, h \) and \( \varepsilon \). By (A.2), (A.3), (A.12)-(A.14), we have

\[
\| (x)^q f ((\varepsilon \theta h)^2 G_0; (\varepsilon \theta)^2; z) \langle x \rangle^{-q} \|_{L^1 \to L^2} \leq C(\varepsilon \theta h)^{-n/2}(\varepsilon \theta h)^q |\text{Im} \, z|^{-q}, \tag{A.17}
\]

with constants \( C, q > 0 \) independent of \( z, \theta, h \) and \( \varepsilon \). We deduce from (A.17),

\[
\| B_\varepsilon(z; h) \|_{L^1 \to L^2} \leq C \varepsilon h^{-n/2} \int_1^\infty \theta^{-1-n/2} d\theta \leq C \varepsilon h^{-n/2} |\text{Im} \, z|^{-q}, \quad z \in \text{supp} \tilde{\varphi}, \tag{A.18}
\]

with a constant \( C_\varepsilon > 0 \) independent of \( z \) and \( h \). It follows from (A.16) that the operator \( 1 + h^2 VA_\varepsilon(z; h) \) is invertible on \( L^1 \), provided \( \varepsilon > 0 \) is taken small enough, independent of \( h \). Therefore, we can write the identity

\[
(h^2 G - z^2)^{-1} = (h^2 G_0 - z^2)^{-1} + h^2 (h^2 G - z^2)^{-1} V (h^2 G_0 - z^2)^{-1}, \tag{A.19}
\]

in the form

\[
(h^2 G - z^2)^{-1} = M(z; h) + h^2 (h^2 G - z^2)^{-1} N(z; h), \tag{A.20}
\]

where the operators

\[
M(z; h) = (h^2 G_0 - z^2)^{-1} (1 + h^2 VA_\varepsilon(z; h))^{-1},
\]
\[ \mathcal{N}(z; h) = VB_e(z; h) \left(1 + h^2VA_e(z; h)\right)^{-1}, \]

satisfy the estimates

\[ \|M(z; h)\|_{L^1 \rightarrow L^1} + \|\mathcal{N}(z; h)\|_{L^1 \rightarrow L^1} \leq C|\text{Im } z|^{-q}, \quad (A.21) \]
\[ \|\mathcal{N}(z; h)\|_{L^1 \rightarrow L^2} \leq C h^{-n/2} |\text{Im } z|^{-q}. \quad (A.22) \]

By (A.20) we have

\[ (h^2G - z^2)^{-1} = \sum_{j=0}^{J-1} M(z; h)\mathcal{N}(z; h)^j + h^2J (h^2G - z^2)^{-1}\mathcal{N}(z; h)^J \]
\[ \quad + h^{2J+2} (h^2G_0 - z^2)^{-1} V (h^2G - z^2)^{-1}\mathcal{N}(z; h)^J, \quad (A.23) \]

for every integer \( J \geq 1 \). By (A.22) and (2.14), we obtain

\[ \left\| (h^2G_0 - z^2)^{-1} V (h^2G - z^2)^{-1}\mathcal{N}(z; h) \right\|_{L^1 \rightarrow L^1} \]
\[ \leq \|V\|_{L^2} \left\| (h^2G_0 - z^2)^{-1}\right\|_{L^1 \rightarrow L^1} \left\| (h^2G - z^2)^{-1}\right\|_{L^2 \rightarrow L^2} \|\mathcal{N}(z; h)\|_{L^1 \rightarrow L^2} \]
\[ \leq C h^{-n/2} |\text{Im } z|^{-q_2}. \quad (A.24) \]

Now, (A.10) follows from (A.21), (A.23) and (A.24).

To prove (A.5) we rewrite (A.20) in the form

\[ (h^2G - z^2)^{-1} - (h^2G_0 - z^2)^{-1} = \sum_{j=1}^{3} \mathcal{F}_j(z; h), \quad (A.25) \]

where

\[ \mathcal{F}_1(z; h) = h^2A_e(z; h)VA_e(z; h) \left(1 + h^2VA_e(z; h)\right)^{-1}, \]
\[ \mathcal{F}_2(z; h) = h^2B_e(z; h)VA_e(z; h) \left(1 + h^2VA_e(z; h)\right)^{-1}, \]
\[ \mathcal{F}_3(z; h) = h^2 (h^2G - z^2)^{-1} VB_e(z; h) \left(1 + h^2VA_e(z; h)\right)^{-1}. \]

It is easy to see that we have the estimate

\[ \left\| \langle x \rangle^s \left( h^2G - z^2 \right)^{-1} \langle x \rangle^{-s} \right\|_{L^2 \rightarrow L^2} \leq C|\text{Im } z|^{-q}, \quad z \in \text{supp } \varphi, \quad 0 < h \leq 1, \quad (A.26) \]

for every \( s \geq 0 \) with constants \( C, q > 0 \) depending on \( s \) but independent of \( z \) and \( h \). By (A.18) and (A.26),

\[ \left\| \langle x \rangle^s \mathcal{F}_3(z; h) \right\|_{L^1 \rightarrow L^2} \leq C h^{2-n/2} |\text{Im } z|^{-q}, \quad z \in \text{supp } \varphi, \quad 0 < h \leq 1. \quad (A.27) \]

Observe now that we can write the operator \( A_e(z; h) \) in the form

\[ A_e(z; h) = \chi_e^3(h^2G_0) \left( h^2G_0 - z^2 \right)^{-1}, \]

where

\[ \chi_e^3(\sigma) = \int_0^{\sigma^{1/2}} \phi(\theta^2) \frac{d\theta}{\theta}. \]
Similarly, we can decompose the operator $\mathcal{B}_\varepsilon(z; h)$ as $\mathcal{B}_\varepsilon^{(1)} + \mathcal{B}_\varepsilon^{(2)}$, where

$$
\mathcal{B}_\varepsilon^{(j)}(z; h) = \chi_\varepsilon^{(j)}(h^2G_0) (h^2G_0 - z)^{-1}, \quad j = 1, 2,$$

$$
\chi_\varepsilon^{(1)}(\sigma) = \int_{\varepsilon^{-1}\sigma^{1/2}}^{\varepsilon} \phi(\theta^2) \frac{d\theta}{\theta}, \quad \chi_\varepsilon^{(2)}(\sigma) = \int_{\varepsilon^{-1}\sigma^{1/2}}^{\infty} \phi(\theta^2) \frac{d\theta}{\theta}.
$$

Taking $\varepsilon > 0$ small enough we can arrange that $\text{supp} \chi_\varepsilon^{(j)} \cap \text{supp} \varphi = \emptyset$, $j = 2, 3$, so the operator-valued functions $A_\varepsilon(z; h)$ and $\mathcal{B}_\varepsilon^{(2)}(z; h)$ are analytic on $\text{supp} \varphi$. Therefore, we can write (A.9) in the form

$$
\psi(h^2G) - \psi(h^2G_0) = \frac{2}{\pi} \sum_{j=3}^{4} \int_{C} \frac{\partial \varphi}{\partial z}(z) \mathcal{F}_j(z; h) zL(dz), \quad (A.28)
$$

where

$$
\mathcal{F}_j(z; h) = h^2\mathcal{B}_\varepsilon^{(1)}(z; h) V A_\varepsilon(z; h) (1 + h^2 V A_\varepsilon(z; h))^{-1}.
$$

By (A.17) (with $s = \delta$), we have

$$
\| (x)^{\delta} \mathcal{F}_4(z; h) \|_{L^1 \to L^2} \leq C h^2 \| (x)^{\delta} \mathcal{B}_\varepsilon^{(1)}(z; h) (x)^{-\delta} \|_{L^1 \to L^2}
$$

$$
\leq C h^{2-n/2} |\text{Im} z|^{-q}, \quad z \in \text{supp} \varphi, \quad 0 < h \leq 1. \quad (A.29)
$$

Now (A.5) follows from (A.27)-(A.29).

**B Appendix 2**

Combining some ideas from [6], [7] and [4] we will prove the following

**Theorem B.1** Let $n \geq 4$, let $V$ satisfy (1.1) with $\delta > n - 1$ as well as (1.3). Then, for every $a > 0$ we have the estimate

$$
\| e^{itG} \psi(G) \|_{L^1 \to L^\infty} \leq C |t|^{-n/2}, \quad t \neq 0. \quad (B.1)
$$

**Remark.** Note that (B.1) is proved in [4] for potentials satisfying (1.1) with $\delta > n$, the condition (1.3) as well as an extra technical assumption. Here we eliminate this extra assumption.

**Proof.** The key point in the proof in [4] is the bound

$$
\| e^{-itG_0} V e^{itG_0} \|_{L^1 \to L^1} \leq \| \mathcal{V} \|_{L^1}, \quad \forall t. \quad (B.2)
$$

Combining (B.2) with Duhamel’s formula one easily gets

$$
\| e^{-itG_0} V e^{itG} \|_{L^1 \to L^1} \leq C, \quad |t| \leq 1, \quad (B.3)
$$

with a constant $C > 0$ independent of $t$. In what follows we will derive (B.1) from (B.2) and (B.3). To this end, given a function $\psi \in C_0^\infty((0, +\infty))$ and a parameter $0 < h \leq 1$, as in [6], [7], denote

$$
\Psi(t, h) = e^{itG} \psi(h^2G) - e^{itG_0} \psi(h^2G_0),
$$

$$
F(t) = i \int_0^t e^{i(t-\tau)G_0} V e^{i\tau G_0} d\tau.
$$

As in these papers, it is easy to see that (B.1) follows from the following
Theorem B.2: Under the assumptions of Theorem B.1, there exist constants $C, \beta > 0$ so that we have the estimates (for $0 < h \leq 1, t \neq 0$)
\[
\|F(t)\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-n/2}, \tag{B.4}
\]
\[
\|\Psi(t,h) - F(t)\phi(h^2 G_0)\|_{L^1 \rightarrow L^\infty} \leq C h^\beta |t|^{-n/2}. \tag{B.5}
\]

Proof. Clearly, (B.4) follows from (B.2) for $|t| \leq 2$. Let $|t| \geq 2$. Without loss of generality we may suppose $t \geq 2$. Write $F = F_1 + F_2$, where
\[
F_1(t) = i \int_1^{t-1} e^{i(t-\tau)G_0} V e^{irG_0} d\tau,
\]
\[
F_2(t) = i \left( \int_1^t + \int_{t-1}^1 \right) e^{i(t-\tau)G_0} V e^{irG_0} d\tau.
\]

It follows from (B.2) that $F_2(t)$ satisfies (B.4). To deal with the operator $F_1(t)$, observe that its kernel is of the form
\[
c_n \int \mathbb{R}^n U(|x - \xi|^2/4, |y - \xi|^2/4, t) V(\xi) d\xi,
\]
where $c_n$ is a constant and
\[
U(\sigma_1, \sigma_2, t) = \int_1^{t-1} e^{i\sigma_1/(t-\tau) + i\sigma_2/\tau} (t - \tau)^{-n/2} \tau^{-n/2} d\tau.
\]

To prove that $F_1(t)$ satisfies (B.4), it suffices to show that
\[
|U(\sigma_1, \sigma_2, t)| \leq C t^{-n/2} (\sigma_1^{-1/2} + \sigma_2^{-1/2}), \quad \forall \sigma_1, \sigma_2 > 0, t \geq 2. \tag{B.6}
\]

To do so, observe that
\[
U(\sigma_1, \sigma_2, t) = t^{-n+1} (u(\sigma_1 t^{-1}, \sigma_2 t^{-1}, t^{-1}) + u(\sigma_2 t^{-1}, \sigma_1 t^{-1}, t^{-1})), \tag{B.7}
\]

where
\[
u(\sigma', \sigma', \kappa) = \int_{\kappa}^{1/2} e^{i\sigma'/((1-\tau') + i\sigma'/\tau'} (1 - \tau')^{-n/2} (\tau')^{-n/2} d\tau'.
\]

It is easy to see that (B.6) follows from (B.7) and the bound
\[
|u(\sigma'_1, \sigma'_2, \kappa)| \leq C \kappa^{-(n-3)/2} (\sigma'_2)^{-1/2}, \quad \forall \sigma'_1, \sigma'_2 > 0, 0 < \kappa \leq 1/2. \tag{B.8}
\]

To prove (B.8), we make a change of variables $\mu = 1/\tau'$ and write the function $u$ in the form
\[
u(\sigma'_1, \sigma'_2, \kappa) = \int_2^{\kappa^{-1}} e^{i\varphi(\mu, \sigma'_1, \sigma'_2)} \left( \frac{\mu}{\mu - 1} \right)^{n/2} \mu^{n/2-2} d\mu,
\]

where
\[
\varphi(\mu, \sigma'_1, \sigma'_2) = \mu \sigma'_2 + \frac{\mu}{\mu - 1} \sigma'_1.
\]

We have
\[
|u(\sigma'_1, \sigma'_2, \kappa)| \leq C \int_2^{\kappa^{-1}} \mu^{n/2-2} d\mu \leq C \kappa^{-(n-2)/2}. \tag{B.9}
\]

Furthermore, observe that
\[
\varphi'(\mu) = \frac{d\varphi}{d\mu} = \sigma'_2 - \frac{\sigma'_1}{(\mu - 1)^2}.
\]
so $\phi'$ vanishes at $\mu_0 = 1 + (\sigma'_1/\sigma'_2)^{1/2}$. We will consider now two cases.

Case 1. $\mu_0 \not\in [3/2, 3\kappa^{-1}/2]$. Then, we have

$$|\phi'(\mu)| \geq \sigma'_2 \frac{|\mu - \mu_0|}{\mu - 1} \geq \frac{\sigma'_2}{10}, \quad \mu \in [2, \kappa^{-1}].$$

Therefore, integrating by parts, we obtain

$$u(\sigma'_1, \sigma'_2, \kappa) = \int_2^{\kappa^{-1}} (i\phi')^{-1} \left( \frac{\mu}{\mu - 1} \right)^{n/2} \mu^{n/2-2} d\mu = e^{i\phi} (i\phi')^{-1} \left( \frac{\mu}{\mu - 1} \right)^{n/2} \mu^{n/2-2} \left| - \int_2^{\kappa^{-1}} e^{i\phi} f(\mu) d\mu, \right.$$

where

$$f(\mu) = \frac{d}{d\mu} \left( (i\phi')^{-1} \left( \frac{\mu}{\mu - 1} \right)^{n/2} \mu^{n/2-2} \right) = (i\phi')^{-1} \frac{d}{d\mu} \left( \left( \frac{\mu}{\mu - 1} \right)^{n/2} \mu^{n/2-2} \right) + \frac{i\phi''}{\phi'} \frac{d}{d\mu} \left( \left( \frac{\mu}{\mu - 1} \right)^{n/2} \mu^{n/2-2} \right).$$

Since

$$\left| \phi'' \right| \leq \frac{2\sigma'_1(\mu - 1)^{-2}}{(\mu - 1)|\sigma'_2 - \sigma'_1(\mu - 1)|^{-2}} \leq \frac{2}{\mu - 1} \left( 1 + \frac{\sigma'_2}{\phi'} \right) \leq \frac{22}{10},$$

we have (for $\mu \geq 2$)

$$|f(\mu)| \leq C(\sigma'_2)^{-1} \mu^{n/2-3}. \quad (B.10)$$

By (B.10) and (B.11),

$$|u(\sigma'_1, \sigma'_2, \kappa)| \leq C(\sigma'_2)^{-1} \kappa^{-(n-4)/2}. \quad (B.12)$$

Clearly, in this case (B.8) follows from (B.9) and (B.12).

Case 2. $\mu_0 \in [3/2, 3\kappa^{-1}/2]$. Denote $I(\mu_0) = [9\mu_0/10, 11\mu_0/10] \cap [2, \kappa^{-1}]$. We write the function $u$ as $u_1 + u_2$, where

$$u_1 = \int_{I(\mu_0)} e^{i\phi} \left( \frac{\mu}{\mu - 1} \right)^{n/2} \mu^{n/2-2} d\mu = \mu_0^{n/2} \int_{I(\mu_0)} e^{i\lambda\phi(z)} g(z) dz, \quad (B.13)$$

where we have made a change of variables $\mu = \mu_0(1 + z)$, $I(\mu_0) \subset [-1/10, 1/10]$, $\lambda = \mu_0\sigma'_2$,

$$g(z) = \left( \frac{1 + z}{1 + z - \mu_0^{-1}} \right)^{n/2} (1 + z)^{n/2-2},$$

$$\phi(z) = (1 + z) \left( 1 + \frac{(\mu_0 - 1)^2}{\mu_0(1 + z) - 1} \right) = \mu_0 + \frac{\mu_0}{\mu_0 - 1} z^2 + O(z^3), \quad |z| \ll 1,$$

uniformly in $\mu_0$. It is easy to see that we have the estimate

$$\left| \int_0^a e^{i\lambda\phi(z)} g(z) dz \right| \leq C\lambda^{-1/2}, \quad |a| \leq 1/10. \quad (B.14)$$

Indeed, the functions $g(z)$ and $\phi(z)$ are analytic in $|z| \leq 1/10$ with $|g(z)|$ bounded there uniformly in $\mu_0$. Therefore, we can change the contour of integration to obtain (with some $0 < \gamma \ll 1$)

$$\left| \int_0^a e^{i\lambda\phi(z)} g(z) dz \right| \leq \left| \int_0^a e^{i\lambda\phi(e^{i\gamma}y)} g(e^{i\gamma}y) dy \right| + |a| \int_0^\gamma e^{i\lambda\phi(e^{i\eta}a)} g(e^{i\theta}a) d\theta.$$
\[\leq C_1 \int_0^\gamma e^{-C_1^2 \lambda \theta} d\theta + C_1^2 \int_0^{\gamma} e^{-C_1^2 \lambda \theta} d\theta = O(\lambda^{-1/2}),\]

with some constants $C, C', C_1, C_1'$ > 0. By (B.13) and (B.14) we conclude

\[|u_1| \leq C(\sigma'_{2})^{-1/2} \mu_0^{(n-3)/2} \leq C(\sigma'_{2})^{-1/2} \kappa^{-(n-3)/2}. \tag{B.15}\]

On the other hand, if $\mu \in [2, \kappa^{-1}] \setminus I(\mu_0)$, then

\[\frac{|\mu - \mu_0|}{\mu - 1} \geq C > 0,\]

so we can bound from below $|\varphi'(\mu)|$. Therefore, the function $u_2$ can be treated in the same way as does $u$ in Case 1. Thus, $u_2$ satisfies (B.8) and hence, in view of (B.15), so does $u$. This completes the proof of (B.4).

It suffices to prove (B.5) for $0 < h < h_0$ with some constant $0 < h_0 \leq 1$, since for $h_0 \leq h \leq 1$ it follows from (B.4) and the estimate of the $L^1 \to L^\infty$ norm of $\Psi(t, h)$ proved in [7] for the larger class of potentials satisfying (1.1) with $\delta > (n + 2)/2$ (without using (1.3)). Without loss of generality we may suppose $t > 0$. Now, using Duhamel’s formula as in [6], [7] we get the identity

\[\Psi(t, h) - F(t)\psi(h^2 G_0) = \sum_{j=1}^5 \Psi_j(t, h), \tag{B.16}\]

where

\[\begin{align*}
\Psi_1(t, h) &= \psi_1(h^2 G_0) e^{i t G_0} (\psi(h^2 G) - \psi(h^2 G_0)) \\
&+ (\psi_1(h^2 G) - \psi_1(h_1 G_0)) e^{i t G_0} \psi(h^2 G_0) + (\psi_1(h^2 G) - \psi_1(h_1 G_0)) \Psi(t, h), \\
\Psi_2(t, h) &= i \left( \int_0^\gamma + \int_{\gamma-t}^\gamma \right) \psi_1(h^2 G_0) e^{i(t-\tau) G_0} e^{i \tau G} \psi(h^2 G) d\tau, \\
\Psi_3(t, h) &= -i \left( \int_0^\gamma + \int_{\gamma-t}^\gamma \right) e^{i(t-\tau) G_0} e^{i \tau G} \psi(h^2 G_0) d\tau, \\
\Psi_4(t, h) &= i \int_{\gamma-t}^\gamma \psi_1(h^2 G_0) e^{i(t-\tau) G_0} e^{i \tau G} \psi(h^2 G_0) d\tau, \\
\Psi_5(t, h) &= -i \int_{\gamma-t}^\gamma (1 - \psi_1(h^2 G_0)) e^{i(t-\tau) G_0} e^{i \tau G} \psi(h^2 G_0) d\tau,
\end{align*}\]

where $\psi_1 \in C_0^\infty((0, +\infty))$, $\psi_1 = 1$ on supp $\psi$, and $0 < \gamma \ll 1$ is a parameter to be fixed later on, depending on $h$. In view of (A.4), we have

\[\|\Psi_1(t, h)f\|_{L^\infty} \leq C h^2 t^{-n/2} \|f\|_{L^1} + C h^2 \|\Psi(t, h)f\|_{L^\infty}, \quad \forall f \in L^1. \tag{B.17}\]

By (B.2) and (B.3),

\[\|\Psi_j(t, h)f\|_{L^\infty} \leq C \gamma t^{-n/2} \|f\|_{L^1} + C \gamma \|\Psi(t, h)f\|_{L^\infty}, \quad \forall f \in L^1, \quad j = 2, 3, \tag{B.18}\]

with a constant $C > 0$ independent of $t$, $h$ and $\gamma$.

**Proposition B.3** Let $V$ satisfy (1.1) with $\delta > n - 1$. Then, there exist constants $C, \beta_1 > 0$ so that for $0 < h \leq 1$, $t \geq 2\gamma$, we have the estimate

\[\|\Psi_4(t, h)\|_{L^1 \to L^\infty} \leq C h^{\beta_1 \gamma} t^{-(n-3)/2} t^{-n/2}. \tag{B.19}\]

**Proof.** We will make use of the following estimates proved in [7].
Proposition B.4 Let $V$ satisfy (1.1) with $\delta > (n + 2)/2$. Then, for every $0 < \epsilon \ll 1$, $1/2 - \epsilon/4 \le s \le (n - 1)/2$, $0 < h \le 1$, $t \neq 0$, we have the estimates

$$\left\| \psi(h^2 G_0) e^{itG_0} \langle x \rangle^{-s-1/2-\epsilon} \right\|_{L^2 \to L^\infty} \le Ch^{s-(n-1)/2} |t|^{-s-1/2}, \quad (B.20)$$

$$\left\| \Psi(t, h) \langle x \rangle^{-s-1/2-\epsilon} \right\|_{L^2 \to L^\infty} \le Ch^{s-(n-3)/2-\epsilon/4} |t|^{-s-1/2}. \quad (B.21)$$

By (B.20) and (B.21), we get (with some $0 < \epsilon_0 \ll 1$)

$$\| \Psi_4(t, h) \|_{L^1 \to L^\infty} \le C \int_0^{t/2} \left\| \psi_1(h^2 G_0) e^{i(t-t')G_0} \langle x \rangle^{-n/2-\epsilon_0} \right\|_{L^2 \to L^\infty} \left\| \langle x \rangle^{-(n-2)/2-\epsilon_0} \Psi(t, h) \right\|_{L^1 \to L^2} dt$$

$$+ C \int_{t/2}^{t-\gamma} \left\| \psi_1(h^2 G_0) e^{i(t-t')G_0} \langle x \rangle^{-(n-2)/2-\epsilon_0} \right\|_{L^2 \to L^\infty} \left\| \langle x \rangle^{-(n-2)/2-\epsilon_0} \Psi(t, h) \right\|_{L^1 \to L^2} dt$$

$$\le Ch^{\epsilon_0/4} t^{-n/2} \int_0^{t/2} \tau^{-(n-2)/2} d\tau + Ch^{\epsilon_0/4} t^{-n/2} \int_{t/2}^{t-\gamma} (t - \tau)^{-(n-2)/2} d\tau$$

$$\le Ch^{\epsilon_0/4} \gamma^{-(n-3)/2} t^{-n/2}.$$

□

Proposition B.5 Let $V$ satisfy (1.1) with $\delta > n - 1$. Then, for every $0 < \epsilon \ll 1$, $0 < h \le 1$, $t \ge 2\gamma$, we have the estimate

$$\| \Psi_5(t, h) \|_{L^1 \to L^\infty} \le C h^\epsilon \gamma^{-(n-3)/2-\epsilon} t^{-n/2}. \quad (B.22)$$

Proof. We will make use of the fact that the kernel of the operator $e^{itG_0} \psi(h^2 G_0)$ is of the form $K_h(|x - y|, t)$, where

$$K_h(\sigma, t) = \frac{\sigma^{-2\nu}}{(2\pi)^{\nu+1}} \int_0^\infty e^{it\sigma^2} J_\nu(\sigma^2) \psi(h^2 \sigma^2) \lambda d\lambda = h^{-n} K_1(\sigma h^{-1}, th^{-2}), \quad (B.23)$$

where $J_\nu(z) = z^\nu J_\nu(z)$, $J_\nu(z) = (H_\nu^+(z) + H_\nu^-(z))/2$ is the Bessel function of order $\nu = (n - 2)/2$. So, the kernel of the operator $\Psi_5$ is of the form

$$\int_{\mathbb{R}^n} W_h(|x - \xi|, |y - \xi|, t, \gamma) V(\xi) d\xi,$$

where

$$W_h(\sigma_1, \sigma_2, t, \gamma) = -i \int_{\gamma}^{t-\gamma} \bar{K}_h(\sigma_1, t - \tau) K_h(\sigma_2, \tau) d\tau$$

$$= h^{-2n+2} W_1(\sigma_1 h^{-1}, \sigma_2 h^{-1}, th^{-2}, \gamma h^{-2}), \quad (B.24)$$

where $\bar{K}_h$ is defined by replacing in the definition of $K_h$ the function $\psi$ by $1 - \psi_1$. It is easy to see that (B.22) follows from the bound (for all $\sigma_1, \sigma_2, \gamma > 0$, $0 < \epsilon \ll 1$, $t \ge 2\gamma$)

$$|W_h(\sigma_1, \sigma_2, t, \gamma)| \le C h^\epsilon \gamma^{-(n-3)/2-\epsilon} t^{-n/2} \left( \sigma_1^{-n+2} + \sigma_1^{-1+\epsilon} + \sigma_2^{n+2} + \sigma_2^{1+\epsilon} \right). \quad (B.25)$$

In view of (B.24), it suffices to prove (B.25) with $h = 1$. Now, observe that $W_1 = W_1^{(1)} - W_1^{(2)}$, where

$$W_1^{(1)}(\sigma_1, \sigma_2, t, \gamma) = \frac{(\sigma_1 \sigma_2)^{-2\nu}}{4(2\pi)^{2\nu+2}} \int_0^\infty \int_0^\infty e^{i(t-\gamma)\lambda_1^2 + i\gamma\lambda_2^2} \rho(\lambda_1^2, \lambda_2^2) J_\nu(\sigma_1 \lambda_1) J_\nu(\sigma_2 \lambda_2) d\lambda_1^2 d\lambda_2^2,$$
where

\[ b \]

Moreover, the functions

\[ \rho(\lambda_1^2, \lambda_2^2) = \frac{(1 - \psi_1)(\lambda_2^2)}{\lambda_2^2 - \lambda_1^2} = \frac{(1 - \psi_1)(\lambda_2^2)}{\lambda_2^2 - \lambda_1^2} \psi(\lambda_1^2) - \psi(\lambda_2^2) \]

satisfies the bound

\[ I_{\lambda_1^2, \lambda_2^2} \rho(\lambda_1^2, \lambda_2^2) \leq C_{\alpha, \sigma_2}(\lambda_1^2)^{-1 - \alpha_1}, \quad \forall (\lambda_1, \lambda_2). \]  

(B.26)

Given any integers \(0 \leq k, m < n/2\), since \(J_\nu(z) = O(z^{n-2})\) as \(z \to 0\), we can integrate by parts to get

\[
W_1^{(1)}(\sigma_1, \sigma_2, t, \gamma) = e^{-m-k(t - \gamma)} \int_0^\infty \int_0^\infty e^{i(x - \gamma)(\lambda_1^2 + i\epsilon)}/\lambda_2 \lambda_2^2 d\lambda_2 d\lambda_2^2,
\]

\[
W_1^{(2)}(\sigma_1, \sigma_2, t, \gamma) = e^{-m-k(t - \gamma)} \int_0^\infty \int_0^\infty e^{i(x - \gamma)(\lambda_1^2 + i\epsilon)}/\lambda_2 \lambda_2^2 d\lambda_2 d\lambda_2^2,
\]

Using the inequality

\[ \int_\infty^{\infty} e^{-i\epsilon^2} \varphi(\lambda) d\lambda \leq C|\epsilon|^{-1/2}\|\varphi\|_{L^1}, \]  

we obtain (for \(t \geq 2\gamma\))

\[
W_1^{(1)}(\sigma_1, \sigma_2, t, \gamma) \leq C t^{-k/2} e^{-m(\sigma_1 \sigma_2)^{2\nu}} \int_0^\infty \int_0^\infty e^{i(x - \gamma)(\lambda_1^2 + i\epsilon)}/\lambda_2 \lambda_2^2 d\lambda_2 d\lambda_2^2,
\]

\[
W_1^{(2)}(\sigma_1, \sigma_2, t, \gamma) \leq C t^{-k/2} e^{-m(\sigma_1 \sigma_2)^{2\nu}} \int_0^\infty \int_0^\infty e^{i(x - \gamma)(\lambda_1^2 + i\epsilon)}/\lambda_2 \lambda_2^2 d\lambda_2 d\lambda_2^2,
\]

(B.28)

(B.29)

Recall now that the function \(J_\nu\) is of the form \(J_\nu = e^{izb_\nu^+(z)} + e^{-izb_\nu^-(z)}\), where \(b_\nu^+(z)\) are symbols of order \((n - 3)/2\) for \(z \geq 1\), while near \(z = 0\) the function \(J_\nu(z)\) is equal to \(z^{2\nu}\) times an analytic function. Therefore, it satisfies the bounds

\[ \left| \partial_\nu^2 J_\nu(z) \right| \leq C z^{n-2-j} \left| z \right|^{-j} \left( n-2-j \right), \quad \forall z > 0, 0 \leq j \leq n - 2, \]  

\[ \left| \partial_\nu^2 J_\nu(z) \right| \leq C z^{(n-3)/2}, \quad \forall z > 0, j \geq 0. \]  

Moreover, the functions \(b_\nu^\pm(z)\) are of the form (near \(z = 0\))

\[ b_\nu^\pm(z) = b_\nu^{\pm, 1}(z) + z^{n-2} \log z b_\nu^{\pm, 2}(z), \]

where \(b_\nu^{\pm, 2}(z)\) are analytic functions, \(b_\nu^{\pm, 2}(z) \equiv 0\) if \(n\) is odd. Therefore, we have

\[ \left| \partial_\nu^2 b_\nu^\pm(z) \right| \leq C, \quad 0 < z \leq 1, 0 \leq j \leq n - 3, \]

\[ \left| \partial_\nu^2 b_\nu^\pm(z) \right| \leq C z^{-\epsilon}, \quad 0 < z \leq 1, j = n - 2, \]
\[ \left| \partial_j^b(x) \right| \leq C_j z^{n-2-j}, \quad 0 < z \leq 1, \quad j \geq n - 1, \]

which imply
\[ \left| \partial_j^b(x) \right| \leq C(z) \frac{(n-3)/2-j}{(n-2-j)}, \quad \forall z > 0, \quad 0 \leq j \leq n - 3, \quad (B.32) \]
\[ \left| \partial_j^b(x) \right| \leq C z^{-\epsilon} \frac{\tau^{(n-1)/2+j}}{\tau^{(n-2-j)}}, \quad \forall z > 0, \quad j = n - 2, \quad (B.33) \]
\[ \left| \partial_j^b(x) \right| \leq C_n^{-n-2-j} \frac{\tau^{(n-1)/2-j}}{\tau^{n-2-j}}, \quad \forall z > 0, \quad j \geq n - 1. \quad (B.34) \]

Set
\[ A_{\pm}^1(\lambda_1, \lambda_2, \sigma_1, \sigma_2) = \lambda_1 e^{\mp i \sigma_1 \lambda_1} \partial_{\lambda_1}^k \partial_{\lambda_2}^m \left( \rho(\lambda_1^2, \lambda_2^2) e^{\pm i \sigma_1 \lambda_1} b_{\nu}(\sigma_1 \lambda_1) \mathcal{J}_\nu(\sigma_2 \lambda_2) \right), \]
\[ A_{\pm}^2(\lambda_1, \lambda_2, \sigma_1, \sigma_2) = \lambda_2 e^{\mp i \sigma_2 \lambda_2} \partial_{\lambda_2}^k \partial_{\lambda_1}^m \left( \rho(\lambda_1^2, \lambda_2^2) e^{\pm i \sigma_2 \lambda_2} b_{\nu}(\sigma_2 \lambda_2) \right). \]

By (B.26), (B.30)-(B.34), we have (with \( \ell = 0, 1 \))
\[ \left| \partial_{\lambda_1}^k A_{\pm}^1(\lambda_1, \lambda_2, \sigma_1, \sigma_2) \right| \leq C(\sigma_1)^{k+(n-3)/2} \sigma_2^{n-2} \langle \sigma_2 \rangle^{m-(n-1)/2} \langle \lambda_1 \rangle^{(n-3)/2-k-1}, \quad \forall (\lambda_1, \lambda_2), \quad (B.35) \]
\[ \left| \partial_{\lambda_2}^k A_{\pm}^2(\lambda_1, \lambda_2, \sigma_1, \sigma_2) \right| \leq C \sigma_1^{n-2} \langle \sigma_1 \rangle^{m-(n-1)/2} \langle \sigma_2 \rangle^{k+(n-3)/2} \langle \lambda_1 \rangle^{(n-3)/2-m-2}, \quad \forall (\lambda_1, \lambda_2). \quad (B.36) \]

Using the inequality
\[ \| \hat{\varphi}(\tau) \|_{L^1} \leq C\| \langle \tau \rangle \hat{\varphi}(\tau) \|_{L^2} \leq C \sum_{\ell=0}^{1} \| \partial_{\lambda_1}^\ell \varphi(\lambda) \|_{L^2} \leq C \sum_{\ell=0}^{1} \sup_{\lambda} \langle \lambda \rangle \left| \partial_{\lambda_1}^\ell \varphi(\lambda) \right|, \]

we obtain from (B.28) and (B.35) (if \( k > (n-3)/2 \))
\[ \left| W_{\pm}^1(\sigma_1, \sigma_2, t, \gamma) \right| \leq \sum_{\lambda} C t^{-k-1/2} \gamma^{-m}(\sigma_1 \sigma_2)^{-2\nu} \]
\[ \times \int_0^\infty \int_0^\infty e^{i \gamma \lambda_1 + i \gamma \lambda_2} A_{\pm}^1(\lambda_1, \lambda_2, \sigma_1, \sigma_2) d\lambda_1 d\lambda_2 d\tau \]
\[ \leq \sum_{\ell=0}^{1} C t^{-k-1/2} \gamma^{-m}(\sigma_1 \sigma_2)^{-2\nu} \sup_{\lambda_1, \lambda_2} \langle \lambda_1 \rangle \left| \partial_{\lambda_1}^\ell A_{\pm}^1(\lambda_1, \lambda_2, \sigma_1, \sigma_2) \right| \]
\[ \leq C t^{-k-1/2} \gamma^{-m} \langle \sigma_1 \rangle^{(n-2)/2} \langle \sigma_2 \rangle^{k+(n-3)/2} \langle \lambda_1 \rangle^{(n-3)/2-m-2}, \quad (B.37) \]

where we have made a change of variables \( \tau \to \tau \pm \sigma_1 \). Similarly, by (B.29) and (B.36), we get (if \( m > (n-3)/2 \))
\[ \left| W_{\pm}^2(\sigma_1, \sigma_2, t, \gamma) \right| \leq \sum_{\lambda} C t^{-k-1/2} \gamma^{-m}(\sigma_1 \sigma_2)^{-2\nu} \]
\[ \times \int_0^\infty \int_0^\infty e^{i \gamma \lambda_2 + i \gamma \lambda_1} A_{\pm}^2(\lambda_1, \lambda_2, \sigma_1, \sigma_2) d\lambda_2 d\lambda_1 d\tau \]
\[ \leq \sum_{\ell=0}^{1} C t^{-k-1/2} \gamma^{-m}(\sigma_1 \sigma_2)^{-2\nu} \sup_{\lambda_2} \int_0^\infty \left| \partial_{\lambda_2}^\ell A_{\pm}^2(\lambda_1, \lambda_2, \sigma_1, \sigma_2) \right| d\lambda_1^2 \]
\[ \leq C t^{-k-1/2} \gamma^{-m} \langle \sigma_2 \rangle^{(n-2)/2} \langle \sigma_1 \rangle^{k+(n-3)/2} \langle \lambda_1 \rangle^{(n-3)/2-m-2}. \quad (B.38) \]
We would like to apply (B.37) and (B.38) with $k = (n-1)/2$, $m = (n-3)/2 + \epsilon$, $0 < \epsilon \ll 1$. To this end, we need to show that these estimates are valid for all real $(n-3)/2 < m \leq (n-2)/2$, $(n-2)/2 \leq k < n/2$ if $n$ is even, and for $k = (n-1)/2$ and all real $(n-3)/2 < m \leq (n-1)/2$ if $n$ is odd. This can be done by interpolation as follows. Let $\phi \in C_0^\infty(\mathbb{R})$, $\phi(\lambda) = 1$ for $|\lambda| \leq 1$, $\phi(\lambda) = 0$ for $|\lambda| \geq 2$. Decompose $W_1^{(j)}$ as $X^{(j)} + Y^{(j)}$, $j = 1, 2$, where $X^{(j)}$ and $Y^{(j)}$ are defined by replacing in the definition of $W_1^{(j)}$ the function $\rho$ by $\phi(\lambda_1)\rho$ and $(1 - \phi)(\lambda_1)\rho$, respectively. Clearly, the functions $X^{(j)}$ satisfy (B.37) and (B.38), respectively, for all integers $0 \leq k, m < n/2$, while the functions $Y^{(j)}$ satisfy (B.37) and (B.38) for all integers $k > (n-3)/2$, $(n-3)/2 < m < n/2$, respectively. When $n$ is odd, this is fulfilled with $k = (n-1)/2$. To show this in the case of even $n$, we write the function $\phi$ as

$$\phi(\lambda) = \sum_{p=0}^\infty \phi_1(2^p\lambda),$$

with some function $\phi_1 \in C_0^\infty(\mathbb{R})$, $\phi_1(\lambda) = 0$ in a neighbourhood of $\lambda = 0$. Thus,

$$X^{(j)} = \sum_{p=0}^\infty X_p^{(j)},$$

where $X_p^{(j)}$ is defined by replacing in the definition of $X^{(j)}$ the function $\phi(\lambda_1)$ by $\phi_1(2^p\lambda_1)$. As above, one can see that the functions $X_p^{(j)}$, $j = 1, 2$, satisfy (B.37) and (B.38), respectively, with an extra factor in the RHS of the form $2^{p(k-n/2)}$ for all integers $k \geq (n-2)/2$, and hence, by interpolation, for all real $k \geq (n-2)/2$. Therefore, summing up these estimates we conclude that $X^{(j)}$, $j = 1, 2$, satisfy (B.37) and (B.38), respectively, for all real $(n-2)/2 \leq k < n/2$, and in particular for $k = (n-1)/2$. Hence, so do the functions $W_1^{(j)}$. Furthermore, $W_1^{(1)}$ satisfies (B.37) for all integers $0 \leq m < n/2$, and hence, by interpolation, for all real $0 \leq m \leq (n-1)/2$ if $n$ is odd, and for all real $0 \leq m \leq (n-2)/2$ if $n$ is even. In particular, this is valid with $m = (n-3)/2 + \epsilon$. To show that the function $W_1^{(2)}$ satisfies (B.38) with $m = (n-3)/2 + \epsilon$, we decompose it as $Z + N$, where $Z$ and $N$ are defined by replacing in the definition of $W_1^{(2)}$ the function $\rho$ by $\phi(\lambda_1)\rho$ and $(1 - \phi)(\lambda_1)\rho$, respectively. Clearly, the function $Z$ satisfies (B.37) for all integers $0 \leq m < n/2$, and hence, by interpolation, for all real $0 \leq m \leq (n-1)/2$ if $n$ is odd, and for all real $0 \leq m \leq (n-2)/2$ if $n$ is even. To deal with the function $N$, we write the function $1 - \phi$ as

$$(1 - \phi)(\lambda) = \sum_{p=0}^\infty \phi_2(2^{-p}\lambda),$$

with some function $\phi_2 \in C_0^\infty(\mathbb{R})$, $\phi_2(\lambda) = 0$ in a neighbourhood of $\lambda = 0$. Thus,

$$N = \sum_{p=0}^\infty N_p,$$

where $N_p$ is defined by replacing in the definition of $N$ the function $(1 - \phi)(\lambda_1)$ by $\phi_2(2^{-p}\lambda_1)$. Now, the functions $N_p$ satisfy (B.38) with an extra factor in the RHS of the form $2^{-p(m-(n-3)/2)}$ for all integers $0 \leq m < n/2$, and hence, by interpolation, for all real $0 \leq m \leq (n-1)/2$ if $n$ is odd, and for all real $0 \leq m \leq (n-2)/2$ if $n$ is even. Therefore, summing up these estimates we conclude that $N$ satisfies (B.38) for all real $(n-3)/2 < m \leq (n-1)/2$ if $n$ is odd, and for all real $(n-3)/2 < m \leq (n-2)/2$ if $n$ is even. In particular, this is valid with $m = (n-3)/2 + \epsilon$.

By (B.37) and (B.38) with $k = (n-1)/2$, $m = (n-3)/2 + \epsilon$, we obtain

$$|W_1(\sigma_1, \sigma_2, t, \gamma)|$$
\[ C t^{-n/2} \gamma^{-(n-3)/2-\epsilon} \left( \sigma_1^{-n+2} \langle \sigma_1 \rangle^{n-2} (\sigma_2)^{-1+\epsilon} + \sigma_2^{-n+2} (\sigma_2)^{n-2} (\sigma_1)^{-1+\epsilon} \right) \]
\[ \leq C t^{-n/2} \gamma^{-(n-3)/2-\epsilon} \left( \sigma_1^{-n+2} + \sigma_2^{-n+2} + \langle \sigma_1 \rangle^{-1+\epsilon} + \langle \sigma_2 \rangle^{-1+\epsilon} \right) \]
\[ \leq C t^{-n/2} \gamma^{-(n-3)/2-\epsilon} \left( \sigma_1^{-n+2} + \sigma_2^{-n+2} + \sigma_1^{-1+\epsilon} + \sigma_2^{-1+\epsilon} \right), \]

which is the desired bound. \(\square\)

Taking \(\gamma = h^{\beta'}\) with a suitably chosen constant \(\beta' > 0\), we deduce from (B.4), (B.16)-(B.19) and (B.22),
\[ \| \Psi(t; h)f - F(t)\psi(h^2 G_0)f \|_{L^\infty} \leq C h^{\beta} t^{-n/2} \|f\|_{L^1} + Ch^{\beta} \| \Psi(t; h)f - F(t)\psi(h^2 G_0)f \|_{L^\infty}, \quad \forall f \in L^1, \] (B.39)
with some constant \(\beta > 0\). Taking \(h\) small enough, we can absorb the second term in the RHS of (B.39), thus obtaining (B.5). \(\square\)

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