New $\mathbb{Z}_3$ strings

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A B S T R A C T

We consider a Yang–Mills–Higgs theory with the gauge group SU(3) broken to its center $\mathbb{Z}_3$ by two scalar fields in the adjoint representation and obtain new $\mathbb{Z}_3$ strings asymptotic configurations with the gauge field and magnetic field in the direction of the step operators.

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1. Introduction

String or vortex solutions may appear naturally in non-Abelian theories if the vacuum manifold has a nontrivial fundamental group, i.e., $\pi_1(G/G_0) \neq 1$, where $G$ and $G_0$ are, respectively, the gauge and the unbroken gauge groups. These configurations are topological and may be relevant in many areas such as condensed matter physics [1], astrophysics and cosmology [2] and high energy physics, in special in Grand Unified Theories [3] and the quark confinement problem [4,5]. In order to have finite energy per unit length, or finite tension, the string solutions are asymptotically constructed by gauge transformations of the vacuum configuration and the associated group elements can be related to the fundamental group elements. This relation classify the solutions into topological classes, the strings belonging to the same equivalence class being homotopic to each other. These features can be seen in the Abelian Higgs Model [6] which couples a $U(1)$ gauge field to a complex scalar. This symmetry is spontaneously broken by a non vanishing vacuum configuration $\phi_{\text{vac}}$, which vanishes the potential. Hence the asymptotic scalar field can be written as $\phi(\varphi) = \exp(ia(\varphi))\phi_{\text{vac}}$ and therefore provides a mapping from the circle $S^1$ at spatial infinity to the circle $S^1$ in the internal space. These mappings can be classified into homotopy classes and this prevents non trivial solutions to be continuously deformed into the vacuum configuration. These string solutions may also appear in theories with non-Abelian groups and they were initially studied in [7–9].

It is believed that confinement in the strong coupling regime of QCD is due to chromoelectric strings called QCD strings. Many properties of QCD strings have been studied using lattice QCD. On the other hand, it is believed that QCD string in the strong coupling may be dual to chromomagnetic strings in weak coupling [4,5]. Since QCD strings appear in gauge theories with non-Abelian groups (without $U(1)$ factors) and are believed to be associated to center elements, in the last years we have analyzed some properties of chromomagnetic $\mathbb{Z}_N$ strings which appear in Yang–Mills–Higgs theories with arbitrary simple gauge group $G$ broken to its center $Z(G)$ by two scalar fields [10–12]. These $\mathbb{Z}_N$ strings are associated to coweights of $G$ and their topological sectors associated to center elements was analyzed in detail in [11]. Similarly to QCD strings, the tensions of the BPS $\mathbb{Z}_N$ strings can satisfy the Casimir law [10,11]. We showed also that the magnetic charges of the monopoles, which appear in the first symmetry breaking, are always integer linear combinations of the magnetic charges of the $\mathbb{Z}_N$ strings, which allows the monopole confinement by the $\mathbb{Z}_N$ strings.

In all our previous works we consider $\mathbb{Z}_N$ strings with gauge field and magnetic field in the direction of the Cartan subalgebra (CSA). However it is possible to have string or vortex solutions with gauge fields as combinations of step operators as has been done for strings in theories with gauge group $SO(10)$ broken to $SU(5) \times \mathbb{Z}_2$ [13], $SU(2)$ broken to $\mathbb{Z}_2$ [14] and $SU(2) \times U(1)$ [15]. In this paper we consider a Yang–Mills–Higgs theory with gauge group $SU(3)$ broken to its center $\mathbb{Z}_3$ and construct the asymptotic configurations with gauge fields and magnetic fields as a combination of the $su(3)$ step operators which we shall call E-strings.
The paper is organized as follows: in section 2 we define our conventions adopted through the paper and define a new basis for $su(3)$. In section 3 we choose a vacuum configuration that spontaneously breaks the symmetry and allows for the existence of $\mathbb{Z}_3$ strings. In section 4 we define group elements associated to generators which are not in the Cartan subalgebra, show that they belong to the center of $SU(3)$ and then obtain the asymptotic form of the vector and the scalar fields.

2. Conventions

Let us consider a Yang–Mills theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a G^{a\mu\nu} + \frac{1}{2} (D_\mu \phi) (D^\mu \phi) + V(\phi),$$

where $\phi$, $s = 1, 2$, are complex scalar fields in the adjoint representation of the gauge group $G$ whose Lie algebra is $\mathfrak{g}$. The covariant derivative is defined as

$$D_\mu \phi = \partial_\mu \phi + i [W_\mu, \phi],$$

and the field strength tensor is

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + i [W_\mu, W_\nu].$$

The so-called Cartan–Weyl basis decomposes the generators of a simple Lie algebra into the Cartan elements $H_i$ and the step operators $E_{\pm\alpha}$ satisfying

$$[H_i, H_j] = 0,$$

$$[H_i, E_{\pm\alpha}] = \alpha_i (H_i) E_{\pm\alpha},$$

where $\alpha$ is said to be a root of the algebra. Given an algebra of rank $r$, the roots belong to an $r$-dimensional vector space, $\Phi(\mathfrak{g})$, called root space. This space is dual to the space containing the weights $\lambda$ which satisfy $H_i(\lambda) = \lambda_i (H_i) \lambda$. The basis of these spaces are the simple roots $\alpha_i$ and the fundamental weights $\lambda_i$, respectively. The fundamental co-weights and simple co-roots are defined as $\lambda_i^c = 2\alpha_i / \alpha_i^2$ and $\alpha_i^c = 2\alpha_i / \alpha_i^2$, respectively, and they satisfy $\lambda_i^c \cdot \alpha = \lambda_i \cdot \alpha_i^c = \delta_{ij}$. The step operators have the following commutation relations among themselves,

$$[E_\alpha, E_{-\beta}] = \begin{cases} 
2\alpha / \alpha^c \cdot H, & \text{if } \alpha = -\beta, \\
N_{\alpha\beta} E_{\alpha+\beta}, & \text{if } \alpha + \beta \in \Phi(\mathfrak{g}), \\
0, & \text{if } \alpha + \beta \notin \Phi(\mathfrak{g}),
\end{cases}$$

where $N_{\alpha\beta}$ are antisymmetric coefficients.

We define the generators

$$T_3 = \sum_{i=1}^{r} \lambda_i^c \cdot H, \quad T_{\pm} = \sum_{i=1}^{r} \sqrt{2c_i} E_{\pm\alpha_i},$$

where $c_i = \sum_{j=1}^{r} (K^{-1})_{ij}$, with $K_{ij} = 2\alpha_i \cdot \alpha_j^c$ being the elements of the Cartan matrix associated to $\mathfrak{g}$. These generators form an $su(2)$ algebra

$$[T_3, T_{\pm}] = \pm T_{\pm}, \quad [T_+, T_-] = 2T_3,$$

which is called the principal $su(2)$ subalgebra of $\mathfrak{g}$.

From the so-called principal element,

$$S = \exp \left( \frac{2\pi i T_3}{\hbar} \right),$$

where $\hbar$ is the Coxeter number of $\mathfrak{g}$, one can show that the generators of the algebra satisfy

$$ST^{(n)}S^{-1} = \exp \left( \frac{2\pi i n}{\hbar} \right) T^{(n)}, \quad n = 0, 1, \ldots, h - 1,$$

which provides $\mathfrak{g}$ with a $\mathbb{Z}_h$ grading.

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{h-1}.$$
where $X_M^i$ has degree $M$ and satisfies $(X_M^i) = X_M^{i-M},$ with $[M]$ denoting $M$ modulo 3. This set of generators has the normalization in the adjoint representation

$$\text{Tr}(X_M^i X_M^j) = 2\delta_{ij} \delta_{MN}.$$ 

and the commutation relations

$$[E, X_M^i] = \epsilon_{ij} \sqrt{3} X_M^j, \quad [F^i, X_M^j] = -\epsilon_{ij} \sqrt{3} X_M^j.$$ 

From (7) and the fact that $X_M^0 = X_0^0$, the set $\{X_M^i\}$ has only seven non-vanishing independent commutation relations, which read

$$[X_M^0, X_N^0] = -X^0_{M+N}, \quad [X_M^1, X_N^1] = -X^1_{M+N}, \quad [X_M^2, X_N^2] = -X^2_{M+N}.$$ 

Note that by calculating $\text{Tr}(E X_M^i X_M^j)$ and $\text{Tr}(F^i X_M^j X_M^k)$ one can write the above commutation relations in a more compact form

$$[X_M^i, X_N^j] = C_{MN}^{ij} X^k_{M+N} + \frac{2}{\sqrt{3}} \epsilon_{ij} \delta_{[M+N+1]} 0 E^1 - \delta_{[M+N-1]} 0 E^2,$$

where $C_{MN}^{ij} = 0, \pm 1$, according to the above commutation relations. With this new basis it turned out to be easier to compute the commutators needed to obtain the asymptotic fields.

### 3. Vacuum configuration

A vacuum solution

$$\Phi_1^{\text{vac}} = v \cdot H,$$ 

with $v = v_i \lambda^i$ having all $v_i$ different from zero, commute only with the generators in the Cartan subalgebra and therefore spontaneously breaks the gauge symmetry to the maximal torus $U(1)^I$. The elements belonging to this unbroken group are then written as $\exp \left( \text{i} c_i \lambda^i \cdot H \right)$ with $c_i$ being real parameters. We can further consider another scalar field vacuum

$$\Phi_2^{\text{vac}} = \sum_{j=1}^r b_j E_{a_j},$$

with all $b_j \neq 0$, and by Baker–Campbell–Hausdorff (BCH) formula one can show that the only elements $\exp \left( \text{i} c_i \lambda^i \cdot H \right)$ leaving this vacuum invariant are

$$\exp (2\pi i \omega \cdot H),$$

with $\omega$ belonging to the co-weight lattice of $G$. Since these are just the center elements of $G$, we see that this vacuum configuration produces a spontaneous symmetry breaking pattern

$$G \rightarrow U(1)^I \rightarrow \mathbb{Z}(G).$$

Let us consider the same potential discussed in [10,11],

$$V = \frac{1}{2} d_0^2, \quad d_0 = \frac{e}{2} \sum_{a=1}^2 \left[ \phi_1^a + \phi_2^a \right],$$

where $m$ is a real mass parameter, which accept vacuum solutions of the form (8) and (9). In the case of $G = SU(3)$ these vacuum solutions can be written as

$$\phi_1^{\text{vac}} = a_1 \psi \cdot H = a_1 X_0^0.$$ 

where $\psi = a_1 + \lambda_1 + \lambda_2$ is the highest root of $SU(3)$, and

$$\phi_2^{\text{vac}} = a_2 \left( E_{a_1} + E_{a_2} \right) = \frac{a_2}{3} \left( \sqrt{3} X_1^1 + 2 E \right).$$

where $a_1$ and $a_2$ are real constants. This vacuum configuration spontaneously breaks the gauge symmetry in the pattern

$$SU(3) \rightarrow U(1) \times U(1) \rightarrow Z_3,$$

giving rise to a multiply connected vacuum manifold which allows the existence of $Z_3$ strings. 

### 4. Asymptotic $Z_3$ string solutions

For a theory with gauge group $G$ broken to its center $Z(G)$, the energy per unit length or string tension of a static topological non-Abelian string, considering $W_0 = 0 = W_3$, is given by

$$T = \int d^2x \left[ \frac{1}{4} G^2_{ij} G^i_j + \frac{1}{2} |D \phi_s|^2 + V(\phi_s) \right],$$

where $i = 1, 2$ denotes directions perpendicular to the string. In order to the string tension be finite, the asymptotic form of the fields must be related to the vacuum configuration by a gauge transformation,

$$W_i(\phi) = \frac{i}{\epsilon} (\delta_i g(\phi)) g(\phi)^{-1},$$

$$\phi_s(\phi) = g(\phi) \phi_s^{\text{vac}} g(\phi)^{-1}.$$ (11)

In order to be single valued configurations the group element $g(\phi)$ satisfies

$$g(\phi + 2\pi) g(\phi)^{-1} \in Z(G).$$ (12)

Notice that by assuming $g(0)$ as the identity, then the above condition leads to $g(2\pi) \in Z(G)$. We can consider that

$$g(\phi) = \exp (i \varphi M).$$ (13)

Then $\exp(2\pi i M) \in Z(G)$ and $M$ must be diagonalizable which implies that $M$ must be a normal generator, that is, $[M, M^T] = 0$. In order to fulfill this condition, one can consider that $M = \omega H$ is a linear combination of Cartan generators. Then, the vector $\omega$ is in the co-weight lattice of the gauge group to guarantee that $\exp(2\pi i \omega \cdot H) \in Z(G)$, as was considered in [10,11]. As it can be seen from equations (11) and (13), the asymptotic gauge field is in the Cartan subalgebra of $g$. However, this is not the only possible choice for $M$. From Eq. (2) we can see that we can consider

$$M = p_1 \lambda_1^* + p_2 \lambda_2^*, \quad p_1, p_2 \in \mathbb{Z}.$$ (14)

recalling that the algebra $su(3)$ is simply laced so that $\lambda_i^* = \lambda_i$. Solving (4) for $\lambda_1 \cdot H$ we find

$$M_1 = \lambda_1 \cdot H = \frac{E + E_1^1}{3}, \quad M_2 = \lambda_2 \cdot H = \frac{e^{-ix/3} E + e^{-ix/3} E_1^1}{3}.$$ (15)

It can be checked explicitly that $g(2\pi) \in Z(G)$ and $\phi_s(\phi)$ belongs indeed to the center of $SU(3)$. In effect, in the 3 dimension representation

$$E = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},$$

and a direct calculation shows that the powers of $M_1$ are given by...
\[ M^k_1 = \frac{J_k(3M_1 + 1) + (-1)^k}{3^k}, \quad k \geq 0, \]

where

\[ J_k = \frac{2^k - (-1)^k}{3}, \quad k \geq 0, \]

are the Jacobsthal Numbers. Then it is easy to compute

\[ \exp(2\pi i M_1) = \exp \left( \frac{4\pi i n}{3} \right) \mathbb{1} \in \mathcal{Z}(SU(3)). \]

In a similar way,

\[ M^k_2 = \frac{(-1)^k J_k(1 - 3M_2) + 1}{3^k}, \quad k \geq 0, \]

and

\[ \exp(2\pi i M_2) = \exp \left( \frac{2\pi i n}{3} \right) \mathbb{1} \in \mathcal{Z}(SU(3)). \]

Therefore the group elements given by (14) are in the center of SU(3).

For simplicity we shall adopt the notation

\[ \lambda E + \lambda^{-1} E^\dagger, \]

where \( \lambda = 1 \) or \( \lambda = \exp(-i\pi/3) \). Let us consider the group element \( g(\varphi) = \exp(\varphi n M_\lambda), \) \( n \in \mathbb{Z} \). Then the asymptotic gauge field reads

\[ W_i(\varphi) = -\frac{e_i X_i^J}{\rho} \pi n M_\lambda, \]

or in polar coordinates

\[ W_\rho(\varphi) = 0, \quad W_\varphi(\varphi) = -\frac{1}{\rho} n M_\lambda. \]

Let us define the generators

\[ X_\lambda^1 = \frac{\sqrt{3}}{3} \left( -\lambda X_1^1 + \lambda^{-1} X_2^1 \right), \]

\[ X_\lambda^2 = \frac{1}{3} \left( 2 X_2^0 - \lambda^{-2} X_1^2 - \lambda^2 X_2^2 \right), \]

which satisfy

\[ [M_\lambda, S_\lambda^1] = S_\lambda^2, \quad [M_\lambda, S_\lambda^2] = S_\lambda^1. \]

Using these relations and the BCH formula we obtain the asymptotic form of \( \phi_1 \) as

\[ \phi_1(\varphi) = \phi_1^{vac} + a_1 \left\{ i \sin(n\varphi) S_\lambda^1 + [\cos(n\varphi) - 1] S_\lambda^2 \right\}. \]  

(16)

Similarly we define

\[ T_1^1 = \frac{\sqrt{3}}{9} \left( 2 X_1^1 - \lambda^{-2} X_0^0 - \lambda^2 X_2^1 \right), \]

\[ T_1^2 = \frac{1}{3} \left( \lambda X_2^0 - \lambda^{-1} X_2^0 \right), \]

such that these generators obey

\[ [M_\lambda, T_\lambda^1] = T_\lambda^2, \quad [M_\lambda, T_\lambda^2] = T_\lambda^1, \]

and then we obtain

\[ \phi_2(\varphi) = \phi_2^{vac} + a_2 \left\{ i \sin(n\varphi) T_\lambda^1 + [\cos(n\varphi) - 1] T_\lambda^2 \right\}. \]  

(17)

Considering that

\[ W_i(\varphi, \rho) = -g(\varphi) \frac{e_i X_i^J}{\rho} n M_\lambda, \]

where \( g(\varphi) \) is a radial function with \( g(\infty) = 1 \) and \( g(0) = 0 \), then the non-vanishing component of the magnetic field has the form

\[ B_3(\varphi, \rho) = -G_{12}(\varphi, \rho) = g(\rho) \frac{n M_\lambda}{\rho}. \]

Therefore, for these strings, the gauge field and the magnetic field take value in the direction of the step operators and we will call them E-strings. Since we are considering the same potential as in our previous works, this theory also have \( Z_3 \) string in the direction of the Cartan subalgebra, which we shall call H-strings. Due to relation (2), E-strings belong to the same three topological sectors as the H-string. Which one is stable will depend on the form of the ansatz and energetic considerations which we will analyse in another work. It is interesting to note that in the symmetry breaking pattern (10), there appear monopoles in the first breaking, which get confined in the second breaking [10]. However, since the monopoles have magnetic flux in the Cartan direction, they will get confined only by H-strings. Then, in order to have finite energy the E-strings should not end in monopoles but close into itself.

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