Capacity and Modulations with Peak Power Constraint

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Abstract

A practical communication channel often suffers from constraints on input other than the average power, such as the peak power constraint. In order to compare achievable rates with different constellations as well as the channel capacity under such constraints, it is crucial to take these constraints into consideration properly. In this paper, we propose a direct approach to compare the achievable rates of practical input constellations and the capacity under such constraints. As an example, we study the discrete-time complex-valued additive white Gaussian noise (AWGN) channel and compare the capacity under the peak power constraint with the achievable rates of phase shift keying (PSK) and quadrature amplitude modulation (QAM) input constellations.

Keywords: peak power constraint, capacity, AWGN channel, PSK, QAM, APSK.

1 Introduction

The channel capacity is defined as the supremum of the mutual information between input and output [10], where the supremum is generally taken under some constraint on the input. For a band-limited channel, a well-known result is the Shannon-Hartley theorem, which dictates the capacity $W \log(1+\text{SNR})$ (SNR: signal-to-noise ratio) for a complex-valued additive white Gaussian noise (AWGN) channel with a bandwidth of $W$ under the average power constraint on the input. We do not have to add anything about theoretical and conceptual importance of this celebrated formula, found in almost all textbooks on communication theory. One has to be a bit careful, however, about its practical significance because a real-world communication system suffers from limitations other than the average power constraint. For example, from an engineering viewpoint, the peak power constraint is important, because the power amplifier of a communication system physically has an absolute peak power (amplitude) limitation. It should also be noted that power efficiency of the amplifier largely depends on the peak value of the continuous-time input signal [8].

Although the importance of the peak power constraint has been realized, it has mostly been considered only indirectly via the peak-to-average power ratio (PAR) [12]. Indeed, typical conventional arguments define the capacity under the average power constraint, and discuss the peak power (or the power efficiency) only via PAR. The most crucial drawback with this indirect approach is that, under the peak

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power constraint, the quantity \( W \log(1 + \text{SNR}) \) is no longer the capacity and is therefore no more an appropriate performance measure.

In this paper, we propose a direct approach to study the achievable rates and the capacity under proper constraints on input. The practical significance with this direct approach is that it allows us to evaluate quantitatively how close the achievable rates to the theoretical limit posed by the capacity under a practical constraint on input. One can also expect that the proposed approach will provide a fresh look at the problem of comparing performance between different input modulation schemes. A major weakness with our approach would be that one can no longer expect a simple closed-form expression for the capacity, such as \( W \log(1 + \text{SNR}) \), so that evaluation of the capacity itself might be elaborate and computationally intensive. We nevertheless believe, despite this weakness, the significance of our approach in view of better understanding of the room for improvement in achievable rates toward the theoretical limit under practical constraints. In order to demonstrate the significance of our approach, we study, as an example, the capacity of a complex-valued AWGN channel under peak power constraint on discrete-time signal (as in [3, Sec. 12.5]) and compare it with the achievable rates with practical input constellations such as phase shift keying (PSK) and quadrature amplitude modulation (QAM). We then show how the achievable rates with PSKs and QAMs behave and reveal that they are surprisingly close to the capacity. Our results imply that a well-designed practical discrete adaptive modulation [3] can achieve a rate which is very close to the capacity.

2 Modulations and Constraints

2.1 AWGN Channel and Peak Power Constraints

We consider a discrete-time complex-valued AWGN channel, which is assumed memoryless and having an isotropic independent Gaussian noise,

\[
\begin{pmatrix}
Y_I \\
Y_Q
\end{pmatrix} = \begin{pmatrix}
X_I \\
X_Q
\end{pmatrix} + \frac{\sigma}{\sqrt{2}} \begin{pmatrix}
N_I \\
N_Q
\end{pmatrix}, \quad N_I, N_Q \sim \mathcal{N}(0, 1),
\]

where \( X_I \) and \( X_Q \) denote in-phase (I)- and quadrature (Q)-phase input components, respectively.

For digital communications systems, choosing an appropriate discrete input-signal constellation is a key issue. Figure 1 shows the achievable rates with QPSK, 16PSK, and 16QAM input constellations for the AWGN channel and Shannon’s capacity \( \log(1 + \text{SNR}) \), which is the capacity for the AWGN channel under the average input power constraint. One observes from Fig. 1 that the achievable rate with 16QAM input constellation is almost always closer to the capacity than those with other PSK input constellations. The results shown here imply that, if we assume a system with adaptive modulation which can use QPSK, 16PSK, and 16QAM, then such a system should always choose 16QAM, if we do not think about complexity in implementation.

The above comparison is well-known, but not appropriate in practice from two reasons. Firstly, for the single-carrier transmitter, the average power consumption of the amplifier is dominated by back-off [8], and it is therefore more reasonable to compare different signal constellations by aligning their amplitudes in terms not of their average power but of their peak power. Indeed, for the same average power, the peak power of 16QAM constellation is 9/5 times larger than that of PSKs, which results in an unfair comparison under the peak power constraint. Secondly, the capacity \( \log(1 + \text{SNR}) \) is achieved only by
a Gaussian input distribution whose support is unbounded. This cannot be realized under peak power constraint.

Figure 2: Two types of peak power constraints.

There are two different forms of peak power constraint which are considered natural for wireless digital communications systems, according to different implementations of the transmitter front-end. Figure 2a shows the case in which each component has an amplifier separately. Assuming that the peak power of each amplifier is bounded equally, a natural form of peak power constraint is the component-wise constraint \( X_I^2, X_Q^2 \leq \frac{E_{\text{max}}}{2} \). We call this the \textit{box constraint}. Another implementation is shown in Fig. 2b, where the sum of the components is amplified at once. The peak power constraint in this case is formulated as \( X_I^2 + X_Q^2 \leq E_{\text{max}} \), which we call the \textit{circular constraint}.

Let \( p_{\text{SNR}} = \frac{E_{\text{max}}}{\sigma^2} \) denote the ratio of peak input power to the noise variance, which we call peak SNR. Under the peak power constraint, the capacity of the AWGN channel (1) must be smaller than \( \log(1 + p_{\text{SNR}}) \) for two reasons: 1) \( p_{\text{SNR}} \geq \text{SNR} \) holds and 2) the input distribution cannot be Gaussian. It is known that the capacity-achieving distribution (CAD) for the AWGN channel under peak power constraint often becomes discrete. This phenomenon was first shown for a scalar AWGN channel [11], and has been extended for many channels with different constraints [2, 6, 7, 9]. Using these results, we numerically evaluate the capacity under each of the box and circular constraints in the following subsections.
2.2 Box Constraint

Under the box constraint, the $I$- and $Q$-components of the channel (1) suffer from independent Gaussian channel noises as well as independent peak power constraints. Accordingly, the channel (1) is decomposed into two independent real-valued AWGN channels under the respective peak power constraints $X_I^2 \leq E_{\text{max}}/2$ and $X_Q^2 \leq E_{\text{max}}/2$. The capacity of the complex-valued AWGN channel is thus attained by the direct product of CADs of the two real-valued AWGN channels under peak power constraint.

The capacity of a real-valued scalar AWGN channel under the peak power constraint has been studied by Smith [11]. He has proved that the capacity is achieved by a discrete input distribution with a finite number of probability mass points. No analytical solution is known for the capacity itself, nor the CAD. We evaluated them numerically via the method described in [11] with Gauss-Hermite integration.

Figure 3a shows the positions of the probability mass points of the CAD versus $p\text{SNR}$. The points are symmetrically positioned around 0 and two points are always located at the boundaries $\pm \sqrt{E_{\text{max}}/2}$. The number of the probability mass points of the CAD is 2 for low enough peak SNRs, while it increases as $p\text{SNR}$ becomes larger. It is in contrast with the case under average power constraint, where the CAD is Gaussian and remains essentially the same irrespective of noise level.

Figures 3b and 3c show CADs for the complex-valued AWGN channel under the box constraint with different $p\text{SNR}$ values. They show that QPSK is the CAD for small enough $p\text{SNR}$ and that 16QAM is similar to the CAD around the $p\text{SNR}$ level used in Fig. 3c. The number of the probability mass points is $m^2$, where $m \geq 2$, $m \in \mathbb{N}$, and probability masses on these points are generally not equal for $m > 2$.

Figure 4 shows the achievable rates with $n$QAMs and the capacity under the box constraint. One can observe that each of the achievable rates comes very close to the capacity around intermediate $p\text{SNR}$ values, and that the $p\text{SNR}$ range in which the achievable rate with $n$QAM comes close to the capacity shifts rightwards as $n$ increases. This observation is ascribed to the fact that the $n$QAMs are similar in their shapes to the CADs under the box constraint in the respective $p\text{SNR}$ ranges.

The above results also indicate that appropriate switching between $n$QAMs with different $n$ will
achieve rates that are close to the capacity under the box constraint. For example, among QPSK, 16QAM, and 64QAM, QPSK is the best for $\text{pSNR} < 5.4$, 16QAM is the best for $\text{pSNR}$ values between 5.4 and 33, and 64QAM is the best for larger $\text{pSNR}$. From Fig. 4, the degradation of the achievable rate with the above discrete adaptive modulation from the capacity in terms of $\text{pSNR}$ for the rates 1, 2, and 3 are 0.0, 1.0, and 0.012 dB, respectively.

### 2.3 Circular Constraint

The capacity and the joint distribution of $X_I$ and $X_Q$ which achieves the capacity under the circular constraint have been studied in [9]. The result is best described with the polar coordinate. Reparameterizing $X_I$ and $X_Q$ with the radius $r$ and the phase $\phi$, the CAD is uniform for $\phi$ and discrete with a finite number of probability mass points for $r$. Consequently, the CAD consists of concentric circles centered at the origin. The number of the circles and their radii, as well as their probability weights vary with $\text{pSNR}$. The analytical solution is not available, and we computed the capacity and the CAD numerically via the method described in [9] with Gauss-Laguerre integration.

Figure 5 shows the numerically computed CADs under the circular constraint $X_I^2 + X_Q^2 \leq E_{\text{max}}$. Figure 5a shows radial positions of probability masses of the CADs. The number of the radial points is 1 for small enough peak SNRs, while it increases as $\text{pSNR}$ becomes larger. One of the points is always located at the boundary $r = \sqrt{E_{\text{max}}}$. Accordingly, the CAD is a single circle for a small $\text{pSNR}$ (Fig. 5b) and multiple concentric circles for a larger $\text{pSNR}$ (Fig. 5c).

The achievable rates with different constellations are shown along with the capacity under the circular constraint in Fig. 6. The achievable rate with QPSK is very close to the capacity for small $\text{pSNR}$s. One also observes that 16PSK, although not popular in current communications systems, has the achievable rate closer to the capacity up to a moderate value of $\text{pSNR}$. Increasing the number $n$ of signal points in $n$PSK makes the achievable rate closer to the capacity up to a yet larger $\text{pSNR}$ value, but the rate becomes falling off from the capacity beyond that $\text{pSNR}$ value (Fig. 6). Figure 6a explains the reason. As the number $n$ of $n$PSK increases, the input distribution approaches a single circle, while the number of the circles increases for the CAD.
As is the case under the box constraint, one can expect that a higher rate should be achievable by designing the input distribution so as to make it similar to the CAD under the circular constraint. The shapes of CADs under the circular constraint imply that amplitude and phase shift keying (APSK)-type modulations work better than PSKs for a larger pSNR. As an example, we consider 16APSK whose constellation is shown in Fig. 6a which is intended to mimic the CAD with pSNR ~ 10 consisting of two circular components. The achievable rate with 16APSK is compared with those of PSKs and the capacity under the circular constraint in Fig. 6b. As we have expected, the achievable rate with 16APSK is worse than those of PSKs for a pSNR less than around 5 but is very close to the capacity under the circular constraint for a larger pSNR up to around 16. Note that pSNR of 5 corresponds to the point where the number of the circles of the CAD increases from 1 to 2 in Fig. 5a. Thus we expect that the APSK with more amplitude shifts would have very good achievable rates for a larger pSNR and that curves like those shown in Fig. 6a would indicate the corresponding pSNR to switch between them.

3 Conclusion

When designing constellation in digital communication, it is important to properly take into consideration practical constraints such as peak power constraint, but capacities of channels have conventionally been argued in terms of average SNR, and peak power constraints have mostly been considered indirectly via PAR. In this paper, we have proposed a direct approach to compare achievable rates and capacity of a channel under practical constraints on input. To demonstrate significance of the proposed approach, we have studied, as an example, a discrete-time complex-valued AWGN channel under peak power constraint, and have shown how close achievable rates with commonly used PSKs and QAMs are to the capacity under peak power constraint. In order to accomplish it, the achievable rates with PSKs and QAMs for a complex-valued AWGN channel have been evaluated and aligned according to the peak power, and
(a) Constellation of 16APSK.

(b) Achievable rates with QPSK, 16PSK, and 16APSK compared with capacity under circular constraint.

Figure 6: Constellation and achievable rate with APSK.
compared with the capacities numerically evaluated under the two different forms (box and circular) of peak power constraint.

We have observed that the achievable rates with $n$QAM are very close to the capacity under the box constraint for some range of $pSNR$, and that the range shifts to larger $pSNR$ as the modulation level of $n$QAM increases. We have also observed that the achievable rates with $n$PSK are very close to the capacity under the circular constraint for small $pSNR$. Our results have also suggested that APSK-type modulation is expected to have an achievable rate close to the capacity under the circular constraint for larger $pSNR$. The achievable rate with 16APSK has been computed to support this expectation. These results show that appropriate design of practical discrete adaptive modulation will bring us a very efficient modulation.

In this paper, we have considered only the case with a single (peak power) constraint on input. As an extension, we can consider more complicated forms of constraints, such as the cases in which both average power and peak power are simultaneously constrained. It is straightforward to study such complicated cases by rewriting the conditions, and similar results will be observed. This is because the CAD becomes discrete in many cases for many types of channels and constraints. Similar comparisons between achievable rates and capacities under those conditions will provide useful guidelines for adaptive modulations. This could not be possible with the indirect approach in which the capacity is derived in terms of average power of input and constraints are discussed separately.

We have also restricted our discussion to the case with discrete-time input. To the best of our knowledge, there has been no direct comparison in the literature between the achievable rates with different constellations and the capacity under peak power constraint even in the case with a discrete-time AWGN channel, and as we have demonstrated, the direct comparison for the discrete-time channel has yielded several novel quantitative observations regarding gaps between achievable rates with practical constellations and the capacity under peak power constraint. On the other hand, it has now been a common practice in the indirect approach to study the PAR in continuous time domain, typically in terms of the complementary cumulative distribution function (CCDF) of PAR values. Another important direction of extending our analysis is therefore to consider peak power constraint on continuous-time input in the direct approach as well.

Finally, we would like to mention that it has been shown from results in Bayesian statistics that the CAD for an AWGN channel under the box constraint should approach the Jeffreys prior as $pSNR$ becomes infinity, which in this case is a uniform distribution. This observation directly implies that $n$QAM and well designed $n$APSK with a large $n$ would be a good approximation of the CAD under the peak power constraint for large $pSNRs$.

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