Research Article

The p-Adic Valuations of Sums of Binomial Coefficients

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In this paper, we prove three supercongruences on sums of binomial coefficients conjectured by Z.-W. Sun. Let $p$ be an odd prime and let $h \in \mathbb{Z}$ with $2h - 1 \equiv 0 \pmod{p}$. For $a \in \mathbb{Z}^+$ and $p^a > 3$, we show that

$$\sum_{k=0}^{p^a-1} \binom{h p^a - 1}{k} \left( \frac{2k}{p} \right) (-h/2)^k \equiv 0 \pmod{p^{a+1}}.$$

Also, for any $n \in \mathbb{Z}^+$, we have

$$\nu_p \left( \sum_{k=0}^{n-1} \binom{hn - 1}{k} \left( \frac{2k}{p} \right) (-h/2)^k \right) \geq \nu_p(n),$$

where $\nu_p(n)$ denotes the $p$-adic order of $n$. For any integer $m \equiv 0 \pmod{p}$ and positive integer $n$, we have

$$\left( \frac{1}{pn} \sum_{k=0}^{p^{r-1} - 1} \binom{pm - 1}{k} \left( \frac{2k}{p} \right) / (-m)^k \right) \equiv (m(m-4)/p) \sum_{k=0}^{p-1} \binom{n-1}{k} \left( \frac{2k}{p} \right) (-1)^k \mod Z_p,$$

where $(-)$ is the Legendre symbol and $Z_p$ is the ring of $p$-adic integers.

1. Introduction

Let $p$ be an odd prime. In 2006, Pan and Sun [1] proved the congruence

$$\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left( \frac{p-d}{3} \right) \pmod{p}, \quad \text{for } d = 0, \ldots, p-1 \tag{1}$$

via a curious combinatorial identity. For any positive integer $a$ and prime $p \geq 5$, later Sun and Tauraso [2] established the following general result:

$$\sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \left( \frac{p^a}{3} \right) \pmod{p^a}. \tag{2}$$

Guo [3] conjectured a $q$-analogue of (2), which was confirmed by Liu and Petrov [4] using a $q$-analogue of Sun–Zhao congruence on harmonic sums and a $q$-series identity. Guo and Zudilin [5] also gave the $q$-generalizations of (2).

Let $A, B \in \mathbb{Z}$. The Lucas sequence $u_n = u_n(A, B)$ $(n \geq 0)$ is given by

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = Au_n - Bu_{n-1} \quad (n \geq 1). \tag{3}$$

If $p$ is an odd prime not dividing $B$, then it is known that $p|u_{p-(\Delta/p)}$ (see, e.g., [6]). For a nonzero integer $n$ and a prime $p$, let $\nu_p(n)$ denote the $p$-adic valuation of $n$, i.e., $\nu_p(n)$ is the largest integer such that $p^{\nu_p(n)}|n$, especially $\nu_p(0) = +\infty$ and we define $\nu_p(m/n) = \nu_p(m) - \nu_p(n)$ for rational number $m/n$. For more developments on $p$-adic valuation, we refer the reader to the papers [7–10].

In 2011, Sun [11] proved that for any nonzero integer $m$ and odd prime $p$ with $pm$, there holds

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{1}{m^p} \equiv \left( \frac{\Delta}{p} \right) + u_{p-(\Delta/p)}(m-2, 1) \pmod{p^2}. \tag{4}$$

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where \( \Delta = m(m - 4) \). As a common extension of (4), Sun [12] showed that
\[
\frac{1}{p^m} \left( \sum_{k=0}^{m-1} \binom{2k}{k} \frac{1}{m^k} - \left( \frac{\Delta}{p} \right) \sum_{k=0}^{m-1} \binom{2k}{k} \frac{1}{m^k} \right) \in \mathbb{Z}_p, \tag{5}
\]
and furthermore
\[
\frac{1}{n} \left( \sum_{k=0}^{m-1} \binom{2k}{k} \frac{1}{m^k} - \left( \frac{\Delta}{p} \right) \sum_{k=0}^{m-1} \binom{2k}{k} \frac{1}{m^k} \right)
= \frac{2n}{n} \equiv \frac{n}{2m^{m-1}} u_{p-(\Delta/p)} (m-2,1) (\mod p^2).
\tag{6}
\]

Let \( p \) be an odd prime and let \( m \) be an integer with \( p \nmid m \). One can easily get the following formula:
\[
\sum_{k=0}^{m-1} \binom{p-1}{k} (-1)^k \binom{2k}{k} \frac{1}{m^k} \equiv \sum_{k=0}^{m-1} \binom{2k}{k} \frac{1}{m^k} \pmod{p}, \tag{7}
\]
since for any \( k \in \{1, 2, \ldots, p - 1\} \), we get
\[
\binom{p-1}{k} \equiv (-1)^k \pmod{p}.
\]
It looks like the left-hand side of (7) has some connection with the right-hand side. Motivated by (4) and (7), Sun [13] determined the sum
\[
\left( \sum_{k=0}^{p-1} \binom{hp^a - 1}{k} \binom{2k}{k} / m^k \right) \pmod{p^2},
\]
modulo \( p^2 \), where \( h \) is a \( p \)-adic integer and \( m \in \mathbb{Z} \) with \( p \nmid m \). For example, if \( h \equiv 0 \pmod{p} \) and \((2h \equiv 1 \pmod{p}) \) or \( p^a > 3 \), then
\[
\sum_{k=0}^{p-1} \binom{hp^a - 1}{k} \binom{2k}{k} \frac{1}{m^k} \equiv \left( 1 - 2h \frac{p^a}{p^a} \right) \left( 1 + h \left( 4 - \frac{2}{h} \right)^{p-1} - 1 \right) \pmod{p^2}.
\tag{8}
\]

It is interesting to consider whether there exists the supercongruence as (8) modulo the higher powers of \( p \) in the case \( 2h - 1 \equiv 0 \pmod{p^2} \) and \( p^a > 3 \). Sun [13] managed to investigate the above case and made the following conjecture. The first aim of this paper is to prove the conjectured results.

**Theorem 1.** Let \( p \) be an odd prime and let \( h \in \mathbb{Z} \) with \( 2h - 1 \equiv 0 \pmod{p} \). If \( a \in \mathbb{Z}^+ \) and \( p^a > 3 \), then
\[
\sum_{k=0}^{p^a-1} \binom{hp^a - 1}{k} \binom{2k}{k} \frac{1}{m^k} \equiv 0 \pmod{p^{a+1}}.
\tag{9}
\]

Also, for any \( n \in \mathbb{Z}^+ \), we have
\[
\gamma_p \left( \sum_{k=0}^{n-1} \binom{hn - 1}{k} \binom{2k}{k} \frac{1}{(-m^k)^2} \right) \geq \gamma_p(n). \tag{10}
\]

On the other hand, based on (5) and (7), Sun [12] conjectured the corresponding result with \( p^a \nmid m \). The second aim of this paper is to show the following result.

**Theorem 2.** Let \( p \) be an odd prime and let \( \Delta = m(m - 4) \). For any integer \( m \equiv 0 \pmod{p} \) and positive integer \( n \), we have
\[
\frac{1}{p^m} \left( \sum_{k=0}^{m-1} \binom{2k}{k} \frac{1}{(-m^k)^2} \right) \pmod{p^2}.
\tag{11}
\]

The remainder of the paper is organized as follows. In the next section, we give some lemmas. The proofs of Theorems 1 and 2 will be given in Section 3.

## 2. Some Lemmas

In the following section, for an assertion \( A \), we adopt the notation:
\[
[A] = \begin{cases} 
1, & \text{if } A \text{ holds,} \\
0, & \text{otherwise.} 
\end{cases} \tag{12}
\]

We know that \([m = n]\) coincides with the Kronecker symbol \( \delta_{mn} \).

**Lemma 1.** Let \( n, k, a \) be positive integers and \( p \) be a prime. Then,
\[
\binom{p^a n - 1}{k} = \binom{k}{p^a} \binom{k}{p} (-1)^{k-[kp]} \left( 1 - np^a \sum_{j=1}^{k} \frac{1}{j} \right) \pmod{p^{a+2}}.
\tag{13}
\]

**Proof.** Note that
\[
\binom{p^a n - 1}{k} = \prod_{j=1}^{k} \frac{p^a n - j}{j} = \binom{k}{p^a} \binom{k}{p} (-1)^{k-[kp]} \left( 1 - np^a \sum_{j=1}^{k} \frac{1}{j} \right) \pmod{p^{a+2}}.
\tag{14}
\]

\[
\binom{p^a n - 1}{k} = \binom{k}{p^a} \binom{k}{p} (-1)^{k-[kp]} \left( 1 - \sum_{j=1}^{k} \frac{p^a n}{j} \right) \pmod{p^{a+2}}.
\tag{14}
\]
This proves (13). The congruence (13) is a result of Beukers [14, Lemma 2].

Lemma 2. Let \( p \) be an odd prime. Then, for any integers \( a, b \) and positive integers \( r, s \), we have

\[
\left( \frac{p^r a}{p^s b} \right) \equiv 1 \pmod{p^{rs + \min(r,s) - \delta_{p,1}}}.
\]

(15)

This lemma is a well-known congruence due to Osburn et al., see, e.g., [15, (19)].

The following curious result is due to Sun [16].

Lemma 3 (see [16, Theorem 1]). Let \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z}^* \).
Suppose that \( p \) is an odd prime dividing \( m - 4 \). Then,

\[
\nu_p \left( \sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{m^k} \right) \geq \nu_p (n),
\]

(16)

Furthermore,

\[
\frac{1}{n} \sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{m^k} \equiv \frac{2n - 1}{4^{n-1}} + \delta_{p,3} [3|n] \frac{m - 4}{3} + \left( \frac{2n}{3^{\nu_3 (n)}} - 1 \right) (\mod p^{\nu_p (m-4)}),
\]

(17)

and also

\[
\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \frac{2k}{m^k} \equiv C_{n-1} \pmod{p^{\nu_p (m-4) - \delta_{p,1}}},
\]

(18)

where \( C_k \) denotes the Catalan number \( \frac{1}{(k+1) ! \binom{2k}{k}} = \binom{2k}{k} - \binom{2k}{k+1} \). Thus, for \( a \in \mathbb{Z}^* \), we have

\[
\frac{1}{p^a} \sum_{k=0}^{p^a-1} \binom{2k}{k} \frac{1}{m^k} \equiv 1 + \delta_{p,3} \frac{m - 4}{3} \equiv \frac{m - 1}{3} \pmod{p}.
\]

(19)

Lemma 4. Let \( p \) be an odd prime and let \( h \in \mathbb{Z} \) with \( 2h - 1 \equiv 0 \pmod{p} \). Let \( l, \alpha \) be nonnegative integers. If \( p \geq 5 \), then we have

\[
\sum_{|k| p^a = l} \left( \frac{2k}{k} \right) \frac{1}{2l} \equiv \frac{p^a}{4} \left( \frac{2l}{l} \right) \pmod{p^{a+1}}.
\]

(20)

If \( p = 3 \), then

\[
\sum_{|k| p^a = l} \left( \frac{2k}{k} \right) \frac{h^k}{2} \equiv 0 \pmod{p^a}.
\]

(21)

If \( p = 3 \) and \( l \equiv 1 \pmod{3} \), then

\[
\sum_{|k| p^a = l} \left( \frac{2k}{k} \right) \frac{h^k}{2} \equiv \frac{p^a}{4} \left( \frac{2l}{l} \right) + 2[l \geq 1] p^{a-1} \frac{(1 - 2h)}{h} \left( \frac{2l}{l} \right) \pmod{p^{a+1}}.
\]

(22)

Proof. Observe that

\[
\sum_{|k| p^a = l} \left( \frac{2k}{k} \right) \frac{h^k}{2} = \sum_{k=0}^{p^a-1} \left( \frac{2k}{k} \right) \frac{h^k}{2} - \sum_{k=0}^{p^a-1} \binom{2k}{k} \frac{h^k}{2}.
\]

(23)

Since \( (2/h) - 4 = 2(1 - 2h)/h \equiv 0 \pmod{p} \) and \( l \equiv 1 \pmod{3} \), by (17), we obtain that
\begin{equation}
\sum_{\lfloor k/p^\alpha \rfloor = l} \left( \begin{array}{c} 2k \\ k \end{array} \right) \left( \frac{h}{2} \right)^k \equiv (p^n l + p^n) \left( \frac{(2l + 2)p^n}{(l + 1)p^n + \frac{3}{4}l^2} + \delta_{p, l} \right) \frac{2(1 - 2h)}{3l + 1} \left( \frac{2(p^n l + p^n)}{3l + 1} - 1 \right) \left( \frac{2l}{3l + 1} - 1 \right) \left( \frac{2l}{3l + 1} - 1 \right) (\text{mod} p^{l+1}).
\end{equation}

If \( p \geq 5 \), by (15) and (24), we get

\begin{equation}
\sum_{\lfloor k/p^\alpha \rfloor = l} \left( \begin{array}{c} 2k \\ k \end{array} \right) \left( \frac{h}{2} \right)^k \equiv p^n (l + 1) \left( \frac{2l + 2}{4l^2} \right) - p^n l \left( \frac{l}{4l^2} - \frac{1}{4} \right) = p^n (2l + 1) \left( \frac{2l}{4l^2} - 2p^n l \left( \frac{l}{4l^2} - \frac{1}{4} \right) = p^n \left( \frac{2l}{4l^2} - \frac{l}{4l^2} \right) (\text{mod} p^{l+1}).
\end{equation}

Thus, (20) is proved. The congruence (21) is easily deduced from (24).

Finally, we will prove (22). The congruence (22) is trivial when \( \alpha = 0 \). Now we may assume \( \alpha \geq 1 \). With the help of (15), (17), and (24), for any nonnegative integer \( l \) with \( l \equiv 1 \text{(mod} 3) \), we have

\begin{equation}
\sum_{\lfloor k/3^\alpha \rfloor = l} \left( \begin{array}{c} 2k \\ k \end{array} \right) \left( \frac{h}{2} \right)^k \equiv 3^\alpha (l + 1) \left( \frac{2l + 2}{4l^2} + \frac{1}{3l + 1} \right) - 3^\alpha l \left( \frac{l}{4l^2} + \frac{1 - 2h}{3l + 1} \right) = 3^\alpha l \left( \frac{2l}{4l^2} - \frac{1}{4l^2} \right) (\text{mod} 3^{l+1}).
\end{equation}

This concludes the proof. \( \square \)
Lemma 5. Let \( p \) be an odd prime and \( l, s \) be nonnegative integers. Let \( \Delta = m(m - 4) \). For any integer \( m \equiv 0 \pmod{p} \), we have

\[
\sum_{k=p^l}^{p^{l+1}-1} \binom{2k}{k} \frac{1}{m^k} = \sum_{k=0}^{p^{l+1}-1} \binom{2k}{k} \frac{1}{m^k}
\]

Equation (27)

Substituting \( n = p^{l-1}(l + 1) \) and \( n = p^{l-1}l \) in (6), we obtain (27).

Proof. The proof is very similar to (20). Clearly,

\[
\sum_{k=p^l}^{p^{l+1}-1} \binom{2k}{k} \frac{1}{m^k} = \sum_{k=0}^{p^{l+1}-1} \binom{2k}{k} \frac{1}{m^k} - \sum_{k=0}^{p^{l-1}} \binom{2k}{k} \frac{1}{m^k}.
\]

(28)

Proof. Note that

\[
\frac{p^{l-1}}{j!} \sum_{j=1, p \nmid j}^{p^{l+1}-1} \frac{1}{j} \equiv \frac{p^{l+1}}{j!} \sum_{j=1, p \nmid j}^{p^{l-1}} \frac{1}{j} \equiv \frac{2}{p^{l-1}} \sum_{j=1}^{p^{l+1}-1} \frac{1}{j} \pmod{p},
\]

Equation (29)

With the help of Lucas’ theorem (cf. [17], p. 44), it follows that

\[
\sum_{k=0}^{p^{l-1}} \binom{2k}{k} \frac{1}{m^k} \equiv \sum_{x=0}^{p^{l-1}/2} \left( \frac{2s}{x} \right) \left( \frac{2t}{x} \right) \left( \frac{h}{x} \right)^{x^{st}} \sum_{x=1}^{1} \frac{1}{j}
\]

Equation (30)

\[+ \sum_{x=0}^{p^{l-1}/2} \left( \frac{2s+1}{x} \right) \left( \frac{2t-p}{x} \right) \left( \frac{h}{x} \right)^{x^{st}} \sum_{x=1}^{1} \frac{1}{j} \equiv \sum_{x=0}^{p^{l-1}/2} \left( \frac{2s}{x} \right) \left( \frac{h}{x} \right)^{x^{st} \pmod{p}}.
\]

Equation (31)

Since \( (2/h) - 4 \equiv 0 \pmod{p} \), in view of (16), then

\[
\sum_{x=0}^{p^{l-1}/2} \left( \frac{2s}{x} \right) \left( \frac{h}{x} \right)^{x^{st}} \equiv 0 \pmod{p^{l+1}}.
\]

Equation (32)
The congruence (29) with \( a \geq 2 \) holds by (31) and (32). Now suppose that \( a = 1 \). In fact, for \( k = 0, \ldots, (p - 1)/2 \), we clearly have

\[
\binom{2k}{k} = \left( \frac{-1}{2} \right)^k \equiv \left( \frac{p - 1}{2} \right)^k \pmod{p}.
\]  

(33)

Observing that \( 2h \equiv 1 \pmod{p} \), we obtain

\[
\sum_{t=1}^{(p-1)/2} \binom{2t}{t} \ell \sum_{j=1}^{t} \frac{(-1)^j}{j} \equiv \sum_{t=1}^{(p-1)/2} \frac{(-1)^j}{j} \sum_{j=1}^{t} \frac{(-1)^j}{j}(t - j) \pmod{p}.
\]

(34)

Thus,

\[
\sum_{j=1}^{(p-1)/2} \frac{(-1)^j}{j} \sum_{t=j}^{(p-1)/2} \left( \frac{p - 1}{2} \right)_{t} \left(1 \pmod{p} \right).
\]

(36)

Therefore, (29) with \( a = 1 \) is proved by (31), (34), and (36).

By the above, we have completed the proof of Lemma 6. \( \square \)

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. We first prove (10). Let \( v_p(n) = a \). (10) is evidently trivial when \( a = 0 \). Next, we suppose that \( a \geq 1 \). With the help of (13), we have
\[
\sum_{k=0}^{n-1} \binom{hn - 1}{k} \left( \frac{2k}{k} \right) \left( \frac{-h}{2} \right)^k = \sum_{k=0}^{n-1} \sum_{\frac{k}{p}} \binom{\frac{h-1}{p}}{\frac{k}{p}} (-1)^{k-\lfloor k/p \rfloor} \left( \frac{2k}{k} \right) \left( \frac{-h}{2} \right)^k
\]
\[
= \sum_{l=0}^{(n/p)-1} \left( \frac{n}{p} h - 1 \right) \left( \frac{2k}{k} \right) \left( \frac{h}{2} \right)^k \equiv 0 \pmod{p^a}.
\]

By (20) and (21), for any odd prime \( p \), we get
\[
\sum_{\lfloor k/p \rfloor = 1} \left( \frac{2k}{k} \right) \left( \frac{h}{2} \right)^k \equiv 0 \pmod{p}.
\] (38)

Repeating the above process \( a - 1 \) times, we obtain that
\[
\sum_{\lfloor k/p \rfloor = 1} \left( \frac{2k}{k} \right) \left( \frac{h}{2} \right)^k \equiv 0 \pmod{p^a}.
\] (39)

Let us turn to (9). We assume that \( p \geq 5 \). In view of (13), we obtain
\[
\sum_{k=0}^{p^a-1} \left( \frac{hp^a - 1}{k} \right) \left( \frac{2k}{k} \right) \left( \frac{-h}{2} \right)^k
\]
\[
= \sum_{k=0}^{p^a-1} \left( \frac{hp^a - 1}{k} \right) \left( \frac{2k}{k} \right) \left( \frac{-h}{2} \right)^k
\]
\[
= \sum_{l=0}^{p^a-1} \left( \frac{h}{l} \right) \left( \frac{-h}{2} \right)^k \left( 1 - hp^a \sum_{j=1,p\mid j} \frac{1}{j} \right)
\]
\[
+ \sum_{l=0}^{p^a-1} \sum_{\lfloor k/p \rfloor = 1} \left( \frac{2k}{k} \right) \left( \frac{h}{2} \right)^k \left( 1 - hp^a \sum_{j=1,p\mid j} \frac{1}{j} \right) \pmod{p^{a+1}}.
\] (40)
For any positive integer \( a \), we have
\[
\left( h^{p^{a-1} - 1} l \right)(-1)^l \equiv 0 \pmod{p}.
\] (41)

Therefore,
\[
\sum_{k=0}^{p^{a-1}} \binom{h^{p^{a-1} - 1}}{k} \binom{2k}{k} \left( \frac{h}{2} \right)^k = \sum_{l=0}^{p^{a-1}-1} \binom{h^{p^{a-1} - 1}}{l} (-1)^l \sum_{\{l/p\}=l} \binom{2k}{k} \left( \frac{h}{2} \right)^k
\]

\[
= \sum_{l=0}^{p^{a-1}-1} \binom{h^{p^{a-1} - 1}}{l} (-1)^l \sum_{\{l/p\}=l} \binom{2k}{k} \left( \frac{h}{2} \right)^k
\]

\[
= \sum_{l=0}^{p^{a-1}-1} \binom{h^{p^{a-1} - 1}}{l} (-1)^l \sum_{\{l/p\}=l} \binom{2k}{k} \left( \frac{h}{2} \right)^k
\]

If \( a = 1 \), by (20), (29), and (42), then
\[
\sum_{k=0}^{p^{a-1}} \binom{h^{p^{a-1} - 1}}{k} \binom{2k}{k} \left( \frac{h}{2} \right)^k
\]

\[
= \sum_{k=0}^{p^{a-1}-1} \binom{2k}{k} \left( \frac{h}{2} \right)^k - 2hp \equiv p(1-2h) \equiv 0 \pmod{p^2}.
\] (44)

The congruence (9) holds with \( a = 1 \) and \( p \geq 5 \). If \( a \geq 2 \), combining (42) and (43), we have modulo \( p^{a+1} \),
\[
\sum_{l=0}^{p^{a+1}-1} \binom{h^{p^{a+1} - 1}}{l} (-1)^l \sum_{\{l/p\}=l} \binom{2k}{k} \left( \frac{h}{2} \right)^k
\]

Repeating this process \( s - 1 \) times, we have
\[
\sum_{l=0}^{p^{a-1}} \binom{h^{p^{a-1} - 1}}{l} (-1)^l \sum_{\{l/p\}=l} \binom{2k}{k} \left( \frac{h}{2} \right)^k
\]

\[
= \sum_{m=0}^{p^{s+1}-1} \binom{h^{p^{s+1} - 1}}{m} (-1)^m \sum_{\{l/p\}=m} \sum_{\{l/p\}=l} \binom{2k}{k} \left( \frac{h}{2} \right)^k
\]

\[
- h^{p^{s+1} - 1} \sum_{m=0}^{p^{s+1}-1} \sum_{\{l/p\}=m} \sum_{\{l/p\}=l} \frac{1}{j} \sum_{\{l/p\}=l} \binom{2k}{k} \left( \frac{h}{2} \right)^k.
\] (46)

By (20) and (29), we get
At last, we only need to think about the case \( s = a - 1 \).
From (45)–(47), we have
\[
\sum_{k=0}^{p^a-1} \left( h p^{a-1} - 1 \right) \binom{2k}{k} \left( -\frac{h}{2} \right)^k \equiv h p^a \sum_{l=1}^{p^a-1} \left( \frac{2l}{4} \right) \sum_{j=1, p|l} \frac{1}{j} \equiv h p^a \sum_{l=1}^{p^a-1} \left( \frac{2l}{4} \right) \sum_{j=1, p|l} \frac{1}{j}.
\]

\[\equiv (1 - 2h) p^a \equiv 0 \mod p^{a+1}.\]  
(48)

\[
h_3^{a-s} \sum_{m=0}^{3^{a-s}-1} \sum_{j=1,3|j} \frac{1}{j} \sum_{l=0}^{3m-1} \left( \sum_{j=1,3|j} \frac{2k}{k} \left( \frac{h}{2} \right)^k \right) \]

\[= h_3^{a-s} \sum_{m=0}^{3^{a-s}-1} \sum_{j=1,3|j} \frac{1}{j} \sum_{l=3m}^{3m+2} \left( \sum_{j=1,3|j} \frac{2k}{k} \left( \frac{h}{2} \right)^k \mod 3^{a+1} \right),\]  
(49)

From (22), we obtain
\[
\sum_{j=1,3|j} \frac{1}{j} = \sum_{j=1,3|j} \frac{1}{j} + \frac{1}{3m + 1} \equiv 1 \mod 3,
\]

\[
\sum_{j=1,3|j} \frac{1}{j} \equiv 1 + \frac{1}{2} \equiv 0 \mod 3.
\]

With the help of (22), we obtain
have any integer proof of (11) is very similar to (10). With the help of (13), for integer

\[
\text{Proof of Theorem 2. Let } n = p^{s-1}d \text{ with } p \nmid d \text{ and } a \geq 1. \text{ The proof of (11) is very similar to (10). With the help of (13), for any integer } s \in [0, a - 1] \text{ and nonnegative integer } k, \text{ now we have}
\]

\[
\left(-1\right)^k \left(\frac{p^{s-1}d - 1}{k}\right) - \left(-1\right)^{k/p} \left(\frac{k}{p}\right) \in p^{s-i}d\mathbb{Z}_p
\]

\[
\text{(55)}
\]

Therefore,
\[
\frac{1}{d} \left( \sum_{k=0}^{p^d-1} \binom{\frac{k}{l} d - 1}{k} \binom{2k}{k} \frac{1}{(-m)^k} \right) \left( \frac{\Delta}{p} \right) \sum_{k=0}^{p^d-1} \left( \frac{p^d - 1}{k} \binom{2k}{k} \frac{1}{(-m)^k} \right)
\]

\[
\equiv \frac{1}{d} \left( \sum_{k=0}^{p^d-1} \binom{p^d - 1}{l} \binom{2k}{k} \frac{1}{(-m)^k} \right) \left( \frac{\Delta}{p} \right) \left( \frac{2l}{l} \right) \quad \text{(mod } p^d) \]  

(56)

In light of Lemma 5, we have

\[
\frac{1}{d} \left( \sum_{l=0}^{d-1} \binom{p^d - 1}{l} \binom{2k}{k} \frac{1}{(-m)^k} \right) \left( \frac{\Delta}{p} \right) \left( \frac{2l}{l} \right)
\]

\[
\equiv \frac{1}{d} \left( \sum_{l=0}^{d-1} \binom{p^d - 1}{l} \binom{2k}{k} \frac{1}{(-m)^k} \right) \left( \frac{\Delta}{p} \right) \left( \frac{2l}{l} \right) \quad \text{(mod } p^d) \]  

(57)

\[
\frac{1}{d} \left( \sum_{l=0}^{d-1} \binom{p^d - 1}{l} \binom{2k}{k} \frac{1}{(-m)^k} \right) \left( \frac{\Delta}{p} \right) \left( \frac{2l}{l} \right)
\]

By Lemma 5 and (55), we get

\[
\frac{1}{d} \left( \sum_{l=0}^{d-1} \binom{p^d - 1}{l} \binom{2k}{k} \frac{1}{(-m)^k} \right) \left( \frac{\Delta}{p} \right) \left( \frac{2l}{l} \right)
\]

\[
\equiv \frac{1}{d} \left( \sum_{l=0}^{d-1} \binom{d - 1}{l} \binom{2k}{k} \frac{1}{(-m)^k} \right) \left( \frac{\Delta}{p} \right) \left( \frac{2l}{l} \right) \equiv 0 \quad \text{(mod } p^d) \]  

(58)

where the last result comes from Lemma 5.

In view of the above, we have completed the Proof of Theorem 2.

\[\blacksquare\]

**Data Availability**

No data were used to support this study.
Disclosure

An earlier version of this article has been presented as preprint on arXiv which can be accessed from the following link: https://arxiv.org/abs/1911.00005.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] H. Pan and Z.-W. Sun, “A combinatorial identity with application to Catalan numbers,” Discrete Mathematics, vol. 306, no. 16, pp. 1921–1940, 2006.
[2] Z.-W. Sun and R. Tauraso, “On some new congruences for binomial coefficients,” International Journal of Number Theory, vol. 7, no. 3, pp. 645–662, 2011.
[3] V. J. W. Guo, “Proof of a q-congruence conjectured by Tauraso,” International Journal of Number Theory, vol. 15, no. 1, pp. 37–41, 2019.
[4] J.-C. Liu and F. Petrov, “Congruences on sums of q-binomial coefficients,” Advances in Applied Mathematics, vol. 116, p. 102003, 2020.
[5] V. J. W. Guo and W. Zudilin, “Dwork-type supercongruences through a creative q-microscope,” Journal of Combinatorial Theory-Series A, vol. 178, p. 105362, 2021.
[6] Z.-W. Sun, “Binomial coefficients and quadratic fields,” Proceedings of the American Mathematical Society, vol. 134, no. 8, pp. 2213–2222, 2006.
[7] S. F. Hong and M. Qiu, “On the p-adic properties of Stirling numbers of the first kind,” https://arxiv.org/pdf/1908.05594.
[8] P. Leonetti and C. Sanna, “On the p-adic valuation of stirling numbers of the first kind,” Acta Mathematica Hungarica, vol. 151, no. 1, pp. 217–231, 2017.
[9] H. Pan and Z.-W. Sun, “On 2-adic orders of some binomial sums,” Journal of Number Theory, vol. 130, no. 12, pp. 2701–2706, 2010.
[10] M. Qiu and S. Hong, “2-adic valuations of Stirling numbers of the first kind,” International Journal of Number Theory, vol. 15, no. 9, pp. 1827–1855, 2019.
[11] Z. Sun, “Binomial coefficients, Catalan numbers and Lucas quotients,” Science China Mathematics, vol. 53, no. 9, pp. 2473–2488, 2010.
[12] Z.-W. Sun, “Supercongruences involving Lucas sequences,” Monatshefte für Mathematik, pp. 1–30, 2021.
[13] Z.-W. Sun, “On sums of binomial coefficients modulo $p^s$,” Colloquium Mathematicum, vol. 127, no. 1, pp. 39–54, 2012.
[14] F. Beukers, “Some congruences for the Apery numbers,” Journal of Number Theory, vol. 21, no. 2, pp. 141–155, 1985.
[15] R. Osburn, B. Sahu, and A. Straub, “Supercongruences for sporadic sequences,” Proceedings of the Edinburgh Mathematical Society, vol. 59, no. 2, pp. 503–518, 2016.
[16] Z.-W. Sun, “p-adic valuations of some sums of multinomial coefficients,” Acta Arithmetica, vol. 148, no. 1, pp. 63–76, 2011.