Periodic orbits related to the equilibrium points in the potential of Irregular-shaped minor celestial bodies

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ABSTRACT

We presented an overview of detailed continuation results of periodic orbit families which emanate from the equilibrium points (EPs) of irregular-shaped minor celestial bodies (hereafter called minor bodies). The generation and annihilation of periodic orbits (POs) related to the EPs are discussed in detail. The branch points of families of POs are also investigated. We presented 3D bifurcation diagrams for periodic orbits families emanating from the EPs of minor bodies which have five EPs totally. Structures of the 3D bifurcation diagrams depend on the distribution of EPs with different topological classifications. We calculated orbit families emanating from the EPs of asteroids 433 Eros and 216 Kleopatra, including the Lyapunov orbit family, the Vertical orbit family, the orbit families bifurcating from the Vertical orbit family, as well as the nonplanar orbit family.

Introduction

The study of dynamics around irregular-shaped bodies is useful for understanding the motion of stars and clusters of galaxies, the origin, evolution, and stability of moonlets of multiple asteroid systems, as well as the design of deep space missions to minor celestial bodies. The dynamical behaviour around some simple model has been studied in several papers to investigate the dynamics around irregular bodies [1–5]. The model can be a triaxial ellipsoid, a straight segment, etc. Both the uniformly rotating triaxial ellipsoid and the straight segment have four equilibrium points outside the body [1,3]. The stability of the EPs depends on the shape and rotation parameters [5]. In addition, there are also works on periodic families around minor bodies based on the harmonic approximation of the gravity field truncated at finite orders [6–9]. The EPs around irregular bodies have several different topological cases [10], the calculation of topological cases of several minor bodies shows that three topological cases are most common, one is the linearly stable case and the other two are unstable cases [11–14]. Stability of POs around EPs is related to the stability of EPs [15]. Previous studies investigated the POs around EPs for several different models. Elipe and Lara [3] used the straight segment to model the gravitation and shape of asteroid 433 Eros. The zero velocity curves, equilibria and POs have been investigated. Vasikova [16] used a triaxial ellipsoid to model the gravitational field of an elongated asteroid and calculated the POs around the EPs. The parameter of the triaxial ellipsoid is chosen based on the asteroid 243 Ida. Romanov and Doedel [5] calculated periodic orbit families that originate from the EPs of a rotating triaxial ellipsoid and a rotating straight segment, and presented the bifurcation diagram of the orbit families.

Werner and Scheeres [17] developed the polyhedral model to calculate the gravitational field and irregular shapes of the minor celestial bodies. Using the polyhedral model with sufficient vertices and faces, the gravitational field and irregular shapes of the minor celestial body can be calculated more precise than simple models [18–31]. Scheeres [19] found six EPs outside the body of the asteroid 1580 Betulia and four EPs outside the body of the comet 67P/CG. Jiang et al. [10] presented a theoretical study on the linear stability and topological classification of EPs around asteroids. The submanifolds, periodic orbit families, and quasi-periodic orbit families near the equilibrium point are also discussed. Chanut et al. [12] plotted the zero velocity surfaces of asteroid 216 Kleopatra and analyzed the stability of the asteroid’s seven EPs.

This paper investigates the periodic orbit families originating from the EPs of an irregular-shaped celestial body, and analyzes the bifurcation diagram of the periodic orbit families. First, we discuss the stability and topological cases of EPs in Section 2. Normally there are two cases of the equilibrium points’ topological classification exist in a uniformly rotating celestial body. The linearized motion equations around the EPs are presented, which is suitable for both uniformly rotating and tumbling minor bodies. In Section 3, the generation and

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annihilation of POs related to the EPs are discussed. For minor bodies which have five EPs with four of them outside, the distribution of EPs with different topological classifications has two cases normally; the 3D bifurcation diagrams for POs related to the EPs have been presented for both of the two cases of distributions. The conclusions are suitable for minor celestial bodies which have seven EPs with four of them outside the body, such as asteroid 216 Kleopatra. POs related to the EPs around asteroids 433 Eros and 216 Kleopatra have been investigated.

Equilibrium points of the Irregular-shaped celestial bodies

The body’s gravitational potential [17,32] can be calculated by

\[ U = G \int dV \rho \frac{1}{2} \text{G}\sigma \sum_{e\text{edges}} \vec{r}_e \cdot \vec{r}_e \cdot L_e - \frac{1}{2} \text{G}\sigma \sum_{f\text{faces}} \vec{F}_f \cdot \vec{r}_f \cdot \omega_f \]  

and the attraction as well as the gravitational gradient can be computed by

\[ \nabla U = -\text{G}\sigma \sum_{e\text{edges}} \vec{E}_e \cdot \vec{L}_e + \text{G}\sigma \sum_{f\text{faces}} \vec{F}_f \cdot \vec{r}_f \cdot \omega_f \]  

where \( G = 6.67 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2} \) is the Newtonian gravitational constant, \( \vec{r} \) is from the body’s mass center to the elemental volume, \( r \) is the norm of \( \vec{r} \), \( m \) represents the mass of the elemental volume, \( \sigma \) is the body's bulk density; \( \vec{r}_e \) and \( \vec{r}_f \) are body-fixed vectors while \( \vec{E}_e \) and \( \vec{F}_f \) are body-fixed tensors, \( \vec{E}_e \) and \( \vec{F}_f \) are from test points to edge \( e \) and face \( f \), respectively, \( \vec{L}_e \) represents the integration factor while \( \omega_f \) represents the signed angle.

For motion around the irregular bodies, the effective potential \( V \) is

\[ V(\vec{r}) = U(\vec{r}) - \frac{1}{2}(\omega \times \vec{r}) \cdot (\omega \times \vec{r}) \]  

The motion equation of the particle in the potential of a rotating body is

\[ \ddot{\vec{r}} + 2\omega \times \dot{\vec{r}} + \dot{\omega} \times \vec{r} + \frac{\partial V(\vec{r})}{\partial \vec{r}} = 0 \]  

For most of the irregular minor bodies, the rotation speed is a constant, then \( \dot{\omega} = 0 \) and the motion equation can be simplified to [20,33]

\[ \ddot{\vec{r}} + 2\omega \times \dot{\vec{r}} + \frac{\partial V(\vec{r})}{\partial \vec{r}} = 0 \]  

However, the rotation speeds for a few minor bodies are time-variant. For instance, the tumbling asteroid 4179 Toutatis [34] rotates about the minimum moment of inertia \( I_0 \). Its angular velocity vector varies between 2.1122 \times 10^{-5} \text{s}^{-1} and 2.1287 \times 10^{-5} \text{rad s}^{-1}, the angle of the rotational angular velocity vector and the x-axis varies between 21.904 and 20.162^\circ, and the angle of the angular momentum vector and the x-axis varies between 50.486 and 49.550^\circ.

The linearized motion equations around the equilibrium points of the asteroid follow

\[ \ddot{\vec{x}} + 2\omega_0 \cdot \dot{\vec{x}} - 2\omega_0 \cdot \vec{y} + \vec{V}_x + \vec{V}_\eta + \vec{V}_z + \vec{V}_\zeta = 0 \]  

\[ \ddot{\vec{y}} + 2\omega_0 \cdot \dot{\vec{y}} - 2\omega_0 \cdot \vec{x} + \vec{V}_y + \vec{V}_\eta + \vec{V}_z + \vec{V}_\zeta = 0 \]  

\[ \ddot{\vec{z}} + 2\omega_0 \cdot \dot{\vec{z}} - 2\omega_0 \cdot \vec{x} + \vec{V}_z + \vec{V}_\eta + \vec{V}_y + \vec{V}_\zeta = 0 \]  

Case O1: 3 pairs Case O2: 2 pairs Case O3: 1 pair Case O4: 0 pairs

Case O5: 1 pair Case O6: 3 pairs

\[ \xi = D^2 \vec{F} \]  

Thus the above equation can be written by

\[ \begin{bmatrix} D^2 + V_x & 2\omega_0 \cdot D + V_y & 2\omega_0 \cdot D + V_z \\ 2\omega_0 \cdot D + V_y & D^2 + V_y & -2\omega_0 \cdot D + V_z \\ -2\omega_0 \cdot D + V_z & 2\omega_0 \cdot D + V_z & D^2 + V_z \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = 0 \]  

Eqs. (7) and (9) are suitable for uniformly rotating minor bodies. If the parameters of the minor bodies vary [17], the position, stability and topological classification of equilibrium points vary. The parameters can be the rotation speed, the density and the shape. If the body-fixed frame is defined by \( \omega = \omega \vec{e}_z \), where \( \vec{e}_z \) is the unit vector of the z-axis, then the above characteristic equations can be simplified to

\[ P(\lambda) = \begin{bmatrix} \lambda^2 + V_x & -2\omega_0 \cdot \lambda + V_y & 2\omega_0 \cdot \lambda + V_z \\ -2\omega_0 \cdot \lambda + V_y & \lambda^2 + V_y & -2\omega_0 \cdot \lambda + V_z \\ 2\omega_0 \cdot \lambda + V_z & -2\omega_0 \cdot \lambda + V_z & \lambda^2 + V_z \end{bmatrix} = 0 \]  

In this paper, we consider the uniformly rotating minor bodies, and use the definition of \( \omega = \omega \vec{e}_z \). The EPs which are non-degenerate and non-resonant can be classified into several different cases [10,21]. Table 1 shows the distribution of eigenvalues of topological classifications of non-degenerate and non-resonant EPs. The eigenvalues of the equilibrium point have the form \( \pm (x + iy) \) for \( x, y \in \mathbb{R} \), \( x \neq 0 \), and \( x, \pm y \in \mathbb{R} \), \( x, \pm y \neq 0 \). Which implies that eigenvalues on the real axis or imaginary axis occur in pairs and eigenvalues on the complex plane occur in quadruple. If, for instance, an equilibrium point belongs to Case O5, then the eigenvalues of the equilibrium point are in the form of \( \pm (x \pm iy) \in \mathbb{R} \) and \( x, \pm y \neq 0 \).

If the equilibrium point belongs to Case O1, the equilibrium point is linearly stable and three periodic orbit families emanating from the equilibrium point. Suppose the gravity of the minor body is symmetric w.r.t. the x-y plane (or the equatorial plane), then the EPs are in the x-y plane. Two periodic orbit families lie in the x-y plane, which are similar to the planar families in the Circular Restricted 3-Body Problem. The other periodic orbit family has member orbits in three dimensional spaces, emanating from infinitesimal oscillations along the z-axis. However, the minor body’s gravity is usually not symmetric w.r.t. its equatorial plane, so the EPs are not in the x-y plane, neither are the “planar” families. Nevertheless, unless the asteroid’s shape is extremely peculiar, the EPs lies close to the x-y plane, i.e., the out-of-plane component of the position is smaller compared with the in-plane components, and the “planar” families, although they are in three dimensional space, are close to the x-y plane if they are not large enough. As a result, in the following studies, to simplify our notations of these families, we still classify them as “planar families” and “vertical family”. If the equilibrium point belongs to Case O2, the equilibrium point is unstable and two periodic orbit families emanating from the equilibrium point, including a planar family and a vertical family.

Table 1

| Topological classifications | imaginary axis | Real axis | Complex plane |
|----------------------------|---------------|----------|--------------|
| Case O1                    | 3 pairs       | 0        | 0            |
| Case O2                    | 2 pairs       | 1 pair   | 0            |
| Case O3                    | 1 pair        | 2 pairs  | 0            |
| Case O4                    | 0 pairs       | 1 group  | 0            |
| Case O5                    | 1 pair        | 0        | 1 group      |
| Case O6                    | 0 pairs       | 3 pairs  | 0            |
If the equilibrium point belongs to Case O4, the equilibrium point is unstable and only one periodic orbit family emanating from the equilibrium point. This periodic orbit family is the vertical family.

**Generation and annihilation of periodic orbits related to the equilibrium points**

In this section, we investigate the generation and annihilation of P0s related to the EPs during the continuation. Most of the minor celestial bodies have five EPs, one is inside and others are outside [10,11,35], such as asteroids 4 Vesta, 243 Ida, 4769 Castalia, and 6489 Golevka, comets 1P/Halley, 9P/Tempel1, and 103P/Hartley2, as well as satellites of planets J5 Amalthea, N8 Proteus, and S16 Prometheus. Asteroid 216 Kleopatra [10,11,12,36] have seven EPs, four of them are outside the body and three of them are inside. The names of these outside EPs are denoted by E1-E4 when rotate anticlockwise to see the EPs. The inner equilibrium point is denoted by E5. For asteroid 216 Kleopatra, the inner EPs are denoted by E5-E7.

Local periodic orbit families originate from equilibrium points; each local periodic orbit is located around only one equilibrium point. Other periodic orbit families are global periodic orbit families, which are located at least two equilibrium points. The structure of the periodic orbit families includes the path of the continuation of periodic orbit families as well as bifurcation diagrams of periodic orbit families. Resonant periodic orbit families are a kind of special family of orbits. A resonant periodic orbit here means the ratio of the period of the orbit and the rotational period of the asteroid is a rational number. For a resonant periodic orbit family, each periodic orbit in the family is a resonant periodic orbit. Kang et al. [22] investigated the continuation, the variety of resonant ratio, and bifurcations of resonant periodic orbit families. In this section, we only investigate bifurcation diagrams of periodic orbit families with non-resonance.

**3D bifurcation diagrams of periodic orbit families**

We first consider the distribution of topological classifications of external EPs. The external EPs [10,11,37] of asteroids 243 Ida, 433 Eros, 1620 Geographos, 1996 HW1, comet 103P/Hartley 2, and the satellites of planets J5 Amalthea and S16 Prometheus belong to Case O2 and Case O5; and the external EPs belong to Case O2 and O5 have a staggered distribution, i.e. the topological cases of EPs are Case O2, Case O5, Case O2, and Case O5 when we count EPs anticlockwise. For these minor bodies, each of the two EPs which belong to Case O2 have two periodic orbit families emanating from the equilibrium point, one planar family and one vertical family; and each of the two EPs which belong to Case O5 have one periodic orbit family emanating from the equilibrium point, the vertical family. The 3D bifurcation diagram for P0s related to the EPs of these minor bodies are shown in Fig. 1. Using 3D bifurcation diagrams, one can see the global structure of the periodic orbit families. In addition, 3D bifurcation diagrams make the relationship between different periodic orbit families clear. In Fig. 1, Ei represents EPs \((i = 1, 2, 3, \ldots)\), Li represents the Lyapunov family originating from the equilibrium point Ei \((i = 1, \ldots, 4)\), Vi represents the Vertical orbit family which originates from the equilibrium point Ei \((i = 1, \ldots, 4)\), Lij represents the j-th branch point along the orbit family Li, Vij represents the j-th branch point along the orbit family Vi, Aij represents the Axial orbit family connecting Li and Vi, Dij represents the out-of-plane orbit family bifurcating from Vi, F0 represents the out-of-plane orbit family connecting L1 and L3, and R0 represents the planar orbit family connecting four families Li \((i = 1, \ldots, 4)\). T2 and T4 represent planar orbit families which bifurcate from R0 and end in a homoclinic connection with EPs E2 and E4, respectively. S2 and S4 represent short-period planar orbit families which emanate from EPs E2 and E4, respectively.

Besides, the outside EPs [10,11] of asteroids 4 Vesta, 6489 Golevka, comets 1P/Halley and 9P/Tempel 1, and satellites of planets N8 Proteus and S9 Phoebe belonging to Case O1 and Case O2; and the outside EPs belong to different topological cases have a staggered distribution. For these minor bodies, each of the two EPs which belong to Case O1 have three periodic orbit families emanating from the equilibrium point, two planar families and one vertical family; and each of the two EPs which belong to Case O2 have two periodic orbit families emanating from the equilibrium point, one planar family and one vertical family. The 3D bifurcation diagram for P0s related to the EPs of these minor bodies is shown in Fig. 2.

Although asteroid 216 Kleopatra [11,12,36] have seven EPs, it only has four EPs outside. The 3D bifurcation diagram for P0s related to the outside EPs and the central equilibrium point which is near the mass centre of asteroid 216 Kleopatra is the same as Fig. 1.

Each point on the lines in the bifurcation diagram in Fig. 1 and Fig. 2 represents a periodic orbit. Hexagons represent EPs while the circle dots represent branch points. We use branch points to express the transcritical and pitchfork bifurcations. The bifurcations of P0s in this paper are different from the four kinds of bifurcations of P0s during the

![Fig. 1. 3D bifurcation diagrams for P0s related to the EPs around minor bodies, the minor bodies have five EPs and the topological classifications of the external EPs belong to Case O2 and Case O5.](image)

![Fig. 2. 3D bifurcation diagrams for P0s related to the EPs around minor bodies, the minor bodies have five EPs and the topological classifications of the external EPs belong to Case O1 and Case O2.](image)
The four kinds of bifurcations of POs, the period-doubling, tangent, Neimark-Sacker, and real saddle bifurcations are only the bifurcations in periodic orbit families; during the continuation, the topological cases of the POs vary and lead to these four kinds of bifurcations. There is no generation or annihilation of periodic orbit families in these four kinds of bifurcations. The bifurcations of POs in this paper are between different periodic orbit families. The difference between the bifurcations in periodic orbit families and the bifurcations between different periodic orbit families is whether the periodic orbit families have generation or annihilation.

The Folquet theory can be used to analyze the stability and topological classifications of the periodic orbits (POs). The motion equation around the irregular body can be expressed by

$$\ddot{X} = h(X)$$ \hspace{1cm} (11)

Denote the matrix $Vh := \frac{\partial h}{\partial X}$, then the state transition matrix [20,38] for periodic orbit $p$ can be calculated using the matrix $Vh$

$$\Phi(t) = \int_0^t \frac{\partial h}{\partial X}(p(r))dr$$ \hspace{1cm} (12)

the monodromy matrix of the periodic orbit $p$ is

$$M = \Phi(T)$$ \hspace{1cm} (13)

where $T$ is the period of the periodic orbit $p$. The Floquet multipliers are the eigenvalues of the monodromy matrix. If the periodic orbit $p$ has at least a Floquet multiplier $|\lambda| > 1$, the periodic orbit is unstable; otherwise, all the Floquet multipliers are in the unit circle and the periodic orbit is linearly stable.

**The Lyapunov orbit family L1**

Here we extend the result of POs related to the EPs in the potential of a rotating massive straight segment [9,39–41] to an arbitrary irregular celestial body. Asteroids 433 Eros and 216 Kleopatra are chosen for instance. The irregular shape and gravitational potential of asteroid 433 Eros and 216 Kleopatra are calculated by the polyhedral model [17,32] using radar observations from Gaskell [42] and Neese [43], respectively. For asteroid 433 Eros, the estimated bulk density is $2.67 \text{ g cm}^{-3}$ [44], the rotational period is $5.27025547 \text{ h}$ [45], and its overall dimensions are $(36 \times 15 \times 13) \text{ km}$ [45]. For asteroid 216 Kleopatra, the estimated bulk density is $3.6 \text{ g cm}^{-3}$ [46], the rotational period is $5.385 \text{ h}$ [46], and is a dumbbell-shaped body with overall dimensions of $217 \times 94 \times 81 \text{ km}$ [47].

Asteroid 433 Eros has 5 EPs, with 1 internal equilibrium point, and 4 external EPs [11]. Asteroid 216 Kleopatra has 7 EPs, with 3 internal EPs, and 4 external EPs [11]. Topological cases of the external EPs E1, E2, E3, and E4 are Case O2, Case O 5, Case O 2, and Case O 5. From Fig. 1, one can see that the structure of periodic orbit families emanating from the equilibrium point E1–E4 of asteroid 433 Eros is similar with the structure of them of asteroid 216 Kleopatra. The Lyapunov orbit family L1 emanates from the equilibrium point E1 of asteroids 433 Eros and 216 Kleopatra is presented in Fig. 3. The shapes of orbits of the Lyapunov orbit family L1 of these two asteroids look similar. The Lyapunov orbit family L1 is unstable, and the characteristic multipliers of POs in L1 are in the form of 1, 1, $\exp(i\phi), \exp(-i\phi)$, $\varepsilon, \frac{1}{\varepsilon}$, where $0 < \phi < \pi$ and $\varepsilon > 1$. During the continuation, the amplitude of orbits in L1 becomes larger, and the period of the POs varies. The equilibrium point E1 has two pairs of purely imaginary eigenvalues, i.e. $\pm i\beta_1, \pm i\beta_2$. When the amplitude of orbits becomes zero, the limit of periods of the POs in the Lyapunov orbit family L1 converges to one of $\frac{2\pi}{\beta_1}$ and $\frac{2\pi}{\beta_2}$, and the limit of periods of the POs in the Vertical orbit family converges to the other one of $\frac{2\pi}{\beta_1}$ and $\frac{2\pi}{\beta_2}$.

**The Vertical orbit family V1 and V2**

The Vertical orbit family V1 emanating from the equilibrium point

![Fig. 3. The Lyapunov orbit family L1 emanates from the equilibrium point E1 of asteroids. (a) 433 Eros; (b) 216 Kleopatra.](image-url)
During the continuation, the period of the POs varies when the amplitude of orbits in V1/V2 becomes larger. The equilibrium point E2 has one pair of purely imaginary eigenvalues, i.e. $\pm i\beta$. The limit of periods of the POs in the Vertical orbit family V2 converges to $\frac{2\pi}{\beta}$ when the amplitude of orbits becomes zero.

The orbit family D41 and D42 and the nonplanar orbit family E0

The orbit family D41 which bifurcates from the Vertical orbit family V4 of the equilibrium point E4 of asteroid 216 Kleopatra has been presented in Fig. 6. Here, the number of the out-of-plane orbit family which bifurcates from the Vertical orbit family V4 is larger than 1. The orbit family D42 which bifurcates from the Vertical orbit family V4 of the equilibrium point E4 of asteroid 216 Kleopatra has been presented in Fig. 7. Both of the orbit families D41 and D42 terminate in collision orbits.

During the continuation, topological cases of the POs in the orbit families D41 and D42 vary. For the orbit family D41, during the continuation, the distribution of Floquet multipliers changes from $\{1, 1, e, \frac{1}{e}, \cdots, \frac{1}{e^n}\}$ to $\{1, 1, 1, 1, e, \frac{1}{e}\}$ to $\{1, 1, \exp(i\phi), \exp(-i\phi), e^{i\phi}, e^{-i\phi}\}$ to $\{1, 1, \exp(i\phi), \exp(-i\phi), \exp(i\theta), \exp(-i\theta)\}$, where $0 < \phi, \theta < \pi$ and
Thus period-doubling bifurcation and tangent bifurcation for the POs exist in the orbit families D41. After that, two quasi-period-doubling bifurcations also exist. For the orbit family D42, during the continuation, the same vary of topological cases of POs occurs. Both of these two orbit families D41 and D42 are non-resonant.

The nonplanar orbit family E0 which bifurcates from the Lyapunov orbit family L3 and connects to the Lyapunov orbit family L1 around asteroid 216 Kleopatra. The periods of the POs in this nonplanar orbit family E0 is in the interval of $0.447 T_{216}$ to $0.457 T_{216}$, where $T_{216}$ is the rotational period of asteroid 216 Kleopatra. During the continuation, the distribution of Floquet multipliers changes from $\{1, 1, \exp(i\phi), \exp(-i\phi), \frac{1}{2}\exp(i\phi), \frac{1}{2}\exp(-i\phi)\}$ to $\{1, 1, \xi, \frac{1}{\xi}, -\xi, -\frac{1}{\xi}\}$ or $\{1, 1, -\xi, -\frac{1}{\xi}, -\xi, -\frac{1}{\xi}\}$, where $\xi > 1, \xi > 1$. Orbits in the nonplanar orbit family E0 are unstable.

Conclusions

In this paper we investigate the periodic orbit families emanating from the equilibrium points (EPs) of minor bodies. We presented the 3D bifurcation diagram for periodic orbits families in the potential of minor bodies which have five or seven EPs. The asteroids 433 Eros and 216 Kleopatra have been investigated for instance. The generation and annihilation of POs related to the EPs have been investigated. Structures of the 3D bifurcation diagram of the periodic orbit families depend on the number and topological cases of the external EPs of the minor bodies. Both of the Lyapunov orbit family and the Vertical orbit family emanating from the EPs of 433 Eros and 216 Kleopatra are unstable. The period-doubling bifurcation, tangent bifurcation, and quasi-period-doubling bifurcation occurs during the continuation of POs in the orbit families D41 and D42 which bifurcates from the Vertical orbit family.

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Appendix A. Supplementary material

Supplementary data to this article can be found online at https://doi.org/10.1016/j.rinp.2018.11.049.

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