A rigorous definition of mass in special relativity

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Summary. — The axiomatic definition of mass in classical mechanics, outlined by Mach in the second half of 19th century and improved by several authors, is simplified and extended to the theory of special relativity. According to the extended definition presented here, the mass of a relativistic particle is independent of its velocity and coincides with the rest mass, i.e., with the mass defined in classical mechanics. Then, force is defined as the product of mass and acceleration, both in the classical and in the relativistic framework.

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1. – Introduction

The problem of stating a rigorous definition of mass in classical mechanics and special relativity has not yet received a widely shared solution. Several university textbooks still employ the traditional definition of mass based on the concept of force. Some treatments of this kind, chosen among the most accurate, will be briefly commented.

Halliday, Resnick and Walker [1] define force as an interaction that can cause acceleration and a unit of force as the force which causes a unit acceleration when applied to a reference body \( R \), to which the value \( m_R = 1 \) of a property \( m \) called mass is assigned. By definition, the magnitude of a force applied to \( R \) equals the magnitude of the acceleration that it produces. In order to measure the mass \( m_A \) of a body \( A \), one should apply the same force to \( A \) and to \( R \) and measure the ratio of the magnitudes of the accelerations of \( R \) and \( A \). A similar argument is presented by Beiser [2]. Indeed, by this method forces are defined quantitatively only when applied to \( R \); a general rule to establish when the same force is applied to different bodies, such as \( R \) and \( A \), is not stated; moreover, one should prove that the ratio of two masses, \( m_B/m_A \), is independent of the choice of the reference body \( R \).

The treatment presented by Wellner [3] is as follows. One chooses a reference object with unit mass, called standard mass; one also states that a fraction \( 1/q \) (in volume) of
the standard mass has mass \( m = 1/q \). Then one defines force as the product of mass and acceleration, namely \( \mathbf{f} = m \mathbf{a} \). At this stage mass, and thus force, is defined only for the standard mass and parts thereof. By means of the standard mass and of the definition of force, one calibrates a spring scale. Thus, force is defined (for any body) by the position of an index in the calibrated spring scale. By employing the calibrated spring scale, one measures the magnitude \( f_A \) of the force which acts on a body \( A \) and hence the mass \( m_A \) of the body as \( m_A = f_A/a_A \), where \( a_A \) is the magnitude of the acceleration of \( A \). Indeed, one should prove: that the procedure is reproducible; that the mass \( m_A \) of a body \( A \) is independent of both the position and the velocity of \( A \); that the ratio of two masses, \( m_B/m_A \), is independent of the choices of both the spring scale and the standard mass.

As clearly pointed out by Jammer [4] and by Lindsay and Margenau [5], any definition of mass which employs the concept of force is vitiated by a logical circularity, because any rigorous and general definition of force is based on the concept of mass. As a consequence, the definition of mass must be founded only on kinematic quantities. Similarly, the statement of the inertia principle and the definition of an inertial reference frame cannot involve the concept of force.

A definition of mass based only on kinematic quantities was conceived by Mach [6], in the last decades of 19th century, and has been improved by several authors [5], [7]–[9]. Thus, a rigorous definition of mass in classical mechanics, which will be called axiomatic definition of mass, is now available; in some textbooks, it is outlined after a preliminary traditional treatment [10], [11].

The axiomatic definition of mass is stated along the following lines. An inertial reference frame is defined, without employing the concept of force, and the existence of inertial reference frames is stated (first law of dynamics). Then, the second law of dynamics is stated as follows [5], [8].

Let \( A \) and \( B \) be two material particles placed far away from all the others, so that they can be considered as isolated; let \( \mathbf{a}_{AB} \) be the acceleration of \( A \) due to \( B \), with magnitude \( a_{AB} \), and let \( \mathbf{a}_{BA} \) be the acceleration of \( B \) due to \( A \), with magnitude \( a_{BA} \). Then, \( \mathbf{a}_{AB} \) and \( \mathbf{a}_{BA} \) are opposite vectors and the ratio \( m_{BA} = a_{AB}/a_{BA} \) is a constant, which is called mass ratio of \( B \) with respect to \( A \). Moreover, if three pairs of isolated material particles, \((A, B), (A, C)\) and \((B, C)\) are considered, the mass ratios \( m_{BA}, m_{CA} \) and \( m_{AB} \) fulfil the equation

\[
m_{BC} = m_{BA} m_{AC} = \frac{m_{BA}}{m_{CA}}.
\]

On account of the second law, the mass \( m_B \) of \( B \) can be defined as follows. Let us consider \( A \) as a reference material particle, to which an arbitrarily chosen value of mass, \( m_A \), is assigned. Then, the mass \( m_B \) of \( B \) is defined as

\[
m_B = m_A m_{BA}.
\]

As a consequence of Eqs. (1) and (2), the ratio \( m_B/m_C \) between the mass of \( B \) and that of \( C \) is independent of the choice of \( A \) and can be measured directly, i.e.

\[
m_B/m_C = m_{BC}.
\]

The definition outlined above is rigorous. However, it is neither universally considered as the true definition of mass nor widely adopted in textbooks. In our opinion, the
axiomatic definition of mass has obtained only a partial success for the following reasons.

a) The axiomatic statement of the second law of dynamics is rather involved and looks more mathematical than physical; moreover, the whole formalism appears as difficult for didactic presentations.

b) The axiomatic statement of the second law of dynamics does not hold in the framework of special relativity. In fact, the Lorentz electromagnetic force does not fulfil the third law of dynamics [12], [13]. Although some authors still support an alternative theory of electrodynamics which fulfils the third law [14], it is widely accepted that the standard theory of electrodynamics is correct, as is confirmed also by a recent experiment [15]. Thus, for a pair \((A, B)\) of isolated and electrically charged particles, if the magnitude of the relative velocity of the particles is comparable with light speed, the accelerations \(a_A\) and \(a_B\) of the particles have not, in general, opposite directions. As a consequence, the axiomatic statement of the second law of dynamics does not hold for a pair of relativistic material points.

In the present paper, the axiomatic definition of mass is simplified and extended to the special theory of relativity. In the framework of classical mechanics, the definitions of inertial reference frame and of isolated pair of material particles are improved and the definition of mass is simplified. Then, the mass of a material point \(A\) endowed with a relativistic speed \(v\) with respect to an inertial reference frame \(O\) is defined by considering a pair of isolated material points \((A, B)\) such that, at an instant \(t\), \(B\) has the same classical velocity \(v\) as \(A\). It is proved that, at time \(t\), the four-acceleration of \(A\) is proportional to that of \(B\), namely \(\alpha_A = -k\alpha_B\), and that the positive constant \(k\) is independent of \(v\), \(i.e.,\) has the same value same as in the classical limit. The mass of \(A\) is defined as \(m_A = m_B/k\), where \(m_B\) is the classical mass of \(B\). According to this definition, the mass of \(A\) in special relativity coincides with the classical mass of \(A\), in agreement with recent investigations on the concept of mass in relativistic physics [16]–[18].

Both in the classical and in the relativistic framework, force is defined as the product of mass and acceleration.

2. – Mass and force in non-relativistic classical mechanics

2’1. Definition: inertial reference frame. – Let \(A\) be an arbitrarily chosen material point, whose motion is observed with respect to a reference frame \(O\). If the velocity of \(A\) is constant whenever \(A\) is placed far away from any other physical object, then \(O\) will be called an inertial reference frame.

2’2. First law of dynamics. – Inertial reference frames exist.

2’3. Conditions and symbols. – In the whole paper, only inertial reference frames will be considered. In this section, we will assume that the speed \(v\) of every material point is much smaller than light speed \(c\). The Galilean transformation of coordinates ensures that, if \(O\) is an inertial reference frame and \(O'\) is any reference frame which moves with a constant velocity with respect to \(O\), then \(O'\) is an inertial reference frame as well. We will denote by \(a_A\) the acceleration of a material point \(A\) and by \(a_A\) the magnitude of \(a_A\).

2’4. Definition: isolated system of material points. – Let \((A, B, C, ...\)) be a set of material points. If each material point of the set has a constant velocity whenever all the others are removed and placed far away from the region of space considered, then \((A, B, C, ...\)) will be called an isolated system of material points.
2.5. Second law of dynamics. – Let \((A, B)\) be an isolated pair of material points. At any time instant \(t\), if \(\mathbf{a}_A\) is non-vanishing also \(\mathbf{a}_B\) is non-vanishing; moreover, \(\mathbf{a}_A\) and \(\mathbf{a}_B\) are parallel to the straight line from \(A\) to \(B\), with opposite directions, and the ratio \(\frac{\mathbf{a}_A}{\mathbf{a}_B}\) is a constant determined uniquely by the choice of \(A\) and \(B\) (i.e., independent of the positions and the velocities of \(A\) and \(B\)).

2.6. Definition: mass of a material point. – Let us consider an isolated pair of material points \((A, R)\), where \(R\) is a reference material point, and a time instant \(t\) such that \(\mathbf{a}_R\) is non-vanishing. We will call mass of \(A\) the quantity \(m_A\) defined as follows:

\[
m_A = m_R \frac{\mathbf{a}_R}{\mathbf{a}_A},
\]

where \(m_R\) is a positive real number that will be called mass of \(R\). Since \(R\) and \(m_R\) are fixed once and for all, the second law of dynamics ensures that \(m_A\) has a unique value, which is strictly positive.

2.7. Comment. – Clearly, at this stage the definition of mass is incomplete. The mass of a material point could be measured only by employing the reference material point \(R\).

2.8. Third law of dynamics. – Let \((A, B)\) be an isolated pair of material points and let \(m_A\) and \(m_B\) be the masses of \(A\) and of \(B\), measured with respect to a reference material point \(R\). Then, at any time instant,

\[
m_A \mathbf{a}_A + m_B \mathbf{a}_B = 0.
\]

2.9. Direct measurement of the ratio of two masses. – Let \((A, B)\) be an isolated pair of material points. Let \(m_A\) and \(m_B\) be the masses of \(A\) and of \(B\). Then, at any time instant \(t\) chosen so that \(\mathbf{a}_B\) is non-vanishing, Eq. (5) implies that

\[
\frac{m_A}{m_B} = \frac{\mathbf{a}_B}{\mathbf{a}_A}.
\]

2.10. Comment. – We have proved that the ratio \(m_A/m_B\) between the mass of \(A\) and that of \(B\) is independent of the choice of the reference material point and can be measured directly. Thus, the definition of mass of a material point has been completed.

2.11. Definition: force which acts on a material point. – Let \(A\) be a material point with mass \(m_A\). We will call force which acts on \(A\), at a time instant \(t\), the vector

\[
\mathbf{f}_A = m_A \mathbf{a}_A,
\]

where \(\mathbf{a}_A\) is the acceleration of \(A\) with respect to an inertial reference frame \(O\) at the instant \(t\).

3. – Mass and force in special relativity

Let us denote the time coordinate by \(x_0 = ct\), where \(t\) is time and \(c\) is light speed in free space, and the space coordinates by \(x_i\) \((i = 1, 2, 3)\). Let \(A\) be a material point in
motion with respect to a reference frame $O$. At any time instant $t$, one can measure the speed $v$ of $A$ with respect to $O$,
\begin{equation}
    v = \left[ \left( \frac{dx_1}{dt} \right)^2 + \left( \frac{dx_2}{dt} \right)^2 + \left( \frac{dx_3}{dt} \right)^2 \right]^{1/2},
\end{equation}
as well as the dimensionless parameters
\begin{equation}
    \beta = \frac{v}{c}, \quad \gamma = \left( 1 - \beta^2 \right)^{-1/2}.
\end{equation}

A time interval $dt$, measured by an observer at rest with respect to $O$, corresponds to a proper-time interval $d\tau$, measured by an observer at rest with respect to $A$, given by
\begin{equation}
    d\tau = \frac{dt}{\gamma}.
\end{equation}

We will call velocity of $A$, at an instant $t$, the four-vector
\begin{equation}
    u_\mu = \frac{dx_\mu}{d\tau},
\end{equation}
whose components are given by
\begin{equation}
    u_0 = \gamma c, \quad u_i = \gamma \frac{dx_i}{dt}.
\end{equation}

We will call acceleration of $A$, at an instant $t$, the four-vector
\begin{equation}
    \alpha_\mu = \frac{du_\mu}{d\tau},
\end{equation}
whose components are given by
\begin{equation}
    \alpha_0 = \frac{v \cdot a}{c^2} \gamma^4, \quad \alpha_i = \gamma^2 \left( a_i + \frac{v \cdot a}{c^2} \gamma^2 v_i \right),
\end{equation}
as is easily obtained from standard expressions \cite{19} through the equality
\begin{equation}
    \frac{d\gamma}{dt} = \frac{v \cdot a}{c^2} \gamma^3.
\end{equation}

In Eq. (14), $v$ is the three–dimensional classical velocity, with components $v_i = dx_i/dt$, and $a$ is the three–dimensional classical acceleration, with components $a_i = d^2x_i/dt^2$.

If $O'$ is a reference frame which moves with a constant classical velocity $v$ with respect to a reference frame $O$, the space–time coordinates $x'_\mu$ of events observed by $O'$ can be obtained from the coordinates $x_\mu$ of the same events observed by $O$ by means of the Lorentz transformation
\begin{equation}
    x'_\mu = L_{\mu\nu} x_\nu,
\end{equation}
where the transformation coefficients $L_{\mu \nu}$ are constants. By differentiating twice Eq. (16) with respect to proper time, one obtains

$$\alpha'_{\mu} = L_{\mu \nu} \alpha_{\nu}.$$  \hfill (17)

The definition of inertial reference frame, the first law of dynamics and the definition of an isolated pair of material particles can be taken from Section 2 without changes.

Equation (17) ensures that, if $O$ is an inertial reference frame and $O'$ is any reference frame which moves with a constant velocity with respect to $O$, then $O'$ is an inertial reference frame as well. Moreover, it yields the following corollary.

3.1. Corollary 1. – If the four-accelerations $\alpha_A\mu$ and $\alpha_B\mu$ of two material points $A$ and $B$ are proportional with respect to an inertial reference frame $O$, i.e. $\alpha_A\mu = k \alpha_B\mu$, they are proportional also with respect to a reference frame $O'$ which moves with respect to $O$ with a constant classical velocity $v$, and the proportionality constant $k$ is the same, i.e. $\alpha'_A\mu = k \alpha'_B\mu$.

Be means of corollary 1, we will prove the validity of the following extended statement of the second law of dynamics.

3.2. Second law of dynamics. – Let $(A, B)$ be an isolated pair of material points which, at a time instant $t$, have the same classical velocity $v$ with respect to an inertial reference frame $O$. Then, at time $t$, the accelerations $\alpha_A\nu$ and $\alpha_B\nu$ are related by the equation

$$\alpha_A\nu = -k \alpha_B\nu ,$$  \hfill (18)

where $k$ is a positive constant determined uniquely by the choice of $A$ and $B$ (i.e., independent of the positions of $A$ and $B$ and of their velocity $v$).

3.3. Proof. – Let $A$ and $B$ be material points which, at time $t$, have the same classical velocity $v$ with respect to an inertial reference frame $O$. Let $O'$ be an inertial reference frame which moves with respect to $O$ with the same classical velocity $v$ as $A$ and $B$, i.e., such that $A$ and $B$ are at rest with respect to $O'$ at the time instant $t'$ which corresponds to $t$.

On account of Eq. (14), the time components of the four-accelerations of $A$ and $B$ with respect to $O'$, at the instant $t'$, are vanishing, while the space components coincide with the components of the classical accelerations, i.e.

$$\alpha'_{A0} = 0 \ , \ \alpha'_{Ai} = a'_{Ai}.$$  \hfill (19)

$$\alpha'_{B0} = 0 \ , \ \alpha'_{Bi} = a'_{Bi}.$$  \hfill (20)

Moreover, the classical statement of the second law of dynamics holds with respect to $O'$ and yields

$$a'_{Ai} = -k a'_{Bi}.$$  \hfill (21)

Equations (19), (20) and (21) yield

$$\alpha'_{A\mu} = -k \alpha'_{B\mu}.$$  \hfill (22)
Equation (22) and corollary 1 yield Eq. (18).

3.4. **Definition: mass of a material point.** – Let \( A \) be a material point with classical velocity \( \mathbf{v} \) with respect to an inertial reference frame \( O \). Let us couple \( A \) with a material point \( B \) which, at time \( t \), has the same classical velocity \( \mathbf{v} \) as \( A \) and such that the pair \((A, B)\) is isolated. On account of the second law of dynamics, at time \( t \), \( \alpha_{A\nu} = -k \alpha_{B\nu} \). We will call mass of \( A \) the quantity \( m_A \) defined as follows:

\[
m_A = \frac{1}{k} m_B ,
\]

where \( m_B \) is the mass of \( B \) defined in non-relativistic classical mechanics.

3.5. **Comment.** – Since \( k \) is independent of velocity, \( m_A \) coincides with the mass of \( A \) defined in non-relativistic classical mechanics.

3.6. **Definition: force which acts on a material point.** – Let \( A \) be a material point with mass \( m_A \), whose motion is observed with respect to an inertial reference frame \( O \). We will call force which acts on \( A \), at a time instant \( t \), the four-vector

\[
K_{A\nu} = m_A \alpha_{A\nu} ,
\]

where \( \alpha_{A\nu} \) is the acceleration of \( A \) with respect to \( O \) at the instant \( t \).

4. – Conclusions

An axiomatic definition of mass in classical mechanics has been available for several decades; however, it is neither widely employed in textbooks nor stated in a form which applies to the broader framework of relativistic dynamics.

In this paper, the axiomatic definition of mass has been simplified and extended to the special theory of relativity. According to the definition proposed here, the mass of a material point \( A \) in special relativity coincides with the classical mass of \( A \). Both in the classical and in the relativistic framework, force has been defined as the product of mass and acceleration.

Thus, a natural extension to special relativity of the axiomatic definitions of mass and force has been obtained.

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