WILDER CONTINUA AND THEIR SUBFAMILIES AS COANALYTIC ABSORBERS

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Abstract. The family of Wilder continua in the cube $I^n$ and its two subfamilies—of continuum-wise Wilder continua and of hereditarily arcwise connected continua—are recognized as coanalytic absorbers in the hyperspace $C(I^n)$ of subcontinua of $I^n$ for $3 \leq n \leq \infty$. In particular, each of them is homeomorphic to the set of all nonempty countable closed subsets of the unit interval $I$.

1. Introduction

By a continuum we mean a nonempty Hausdorff compact connected space.

Definition 1.1. [16] A topological space $X$ has the Wilder property if it has at least three points and for any mutually distinct points $x, y, z \in X$ there exists a continuum $K \subset X$ containing $x$ and exactly one of the points $y, z$. A continuum that has the Wilder property is called a Wilder continuum. A continuum each of whose nondegenerate subcontinua has the Wilder property is called a hereditarily Wilder continuum.

The Wilder property was introduced by B. E. Wilder in [16] under the name of property $C$. However, in order to avoid confusion with a much more popular concept of a $C$-space existing in dimension theory (cf. [7]), we decided to change the name. It was shown by Wilder that both arcwise connected spaces (by an arc we mean a continuum with exactly two non-separating points) and aposyndetic continua have the Wilder property and plane examples were given of Wilder continua that are neither arcwise connected nor aposyndetic [16, 18]. Moreover, by [16, Corollary 2], one can observe that a nondegenerate continuum is

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a hereditarily Wilder continuum if and only if it is hereditarily arcwise
connected. Wilder continua were also studied in [17], [12] and [2].

Besides (hereditarily) Wilder continua, we introduce a narrower class
of \textit{continuum-wise Wilder continua} (in this vein, Wilder continua could
well be called point-wise Wilder).

\textbf{Definition 1.2.} A space \(X\) has the \textit{continuum-wise Wilder property},
if it contains at least three points and if for any mutually disjoint sub-
continua \(A, B, C\) of \(X\) there exists a subcontinuum \(L \supset A\) containing
exactly one of \(B, C\) and disjoint with the other one (i.e. either \(B \subset L\),
\(C \cap L = \emptyset\) or vice-versa).

\textbf{Example 1.3.} A plane Wilder continuum which is not continuum-wise
Wilder can be obtained from the set
\[
\{(−1, 1) \times [−1, 1]) \cup ([−1, 1] \times \{−1, 1\}) \cup \\
\{(x, \sin \frac{1}{x − 1}) : x \in (1, 2]\} \cup \{(x, \sin \frac{1}{x + 1}) : x \in (−2, −1]\}
\]
by identifying points \((2, \sin 1)\) and \((-2, \sin(-1))\).

Using Wilder’s result that each irreducible continuum with his prop-
erty is an arc [16], one can easily observe that in the realm of heredi-
tarily unicoherent continua all three types of Wilder continua coincide
with dendroids (\(≡\) nondegenerate hereditarily unicoherent arcwise con-
nected continua).

\textbf{Proposition 1.4.} If \(X\) is a nondegenerate hereditarily unicoherent
continuum, then the following statements are equivalent.

\begin{itemize}
  \item \(X\) is Wilder,
  \item \(X\) is a dendroid,
  \item \(X\) is hereditarily Wilder,
  \item \(X\) is continuum-wise Wilder.
\end{itemize}

Arguments similar to those for Wilder continua show that the class of
nondegenerate arcwise connected continua is properly contained in the
class of continuum-wise Wilder continua. This is not true for aposyn-
detic continua.

\textbf{Example 1.5.} It is known that the product of arbitrary nondegenerate
continua is aposyndetic. Let \(X\) be a continuum containing three points
\(a, b\) and \(c\) such that \(X\) is irreducible between \(a\) and \(b\) and between \(a\) and
\(c\) and let \(Y\) be a nondegenerate continuum. Then \(X \times Y\) is aposyndetic
but it is not continuum-wise Wilder, since, for subcontinua \(A = \{a\},
B = \{b\} \times Y\) and \(C = \{c\} \times Y\), if \(L \subset X \times Y\) is a continuum which
contains \(A \cup B\) then the projection \(L_X\) of \(L\) into \(X\) contains \(\{a, b\}\), so \(L_X = X\) and \(L \cap C \neq \emptyset\).

Henceforth, we will consider only separable metric spaces. The hyperspace \(2^X\) of all nonempty compact subsets of a space \(X\) is equipped with the Hausdorff metric and \(C(X)\) denotes its subspace consisting of all subcontinua of \(X\). Throughout the paper \(I = [0, 1]\) and cubes \(I^n\) are endowed with the Euclidean metrics.

While the theory of absorbing sets is well established ([1] and [13] are good references), only a few examples of coanalytic absorbers in hyperspaces of cubes have been recognized. They include:

- the Hurewicz set of all nonempty closed countable subsets of \(I\) in \(2^I\) [4],
- the set of all hereditarily decomposable subcontinua of \(I^n\) in \(C(I^n), 3 \leq n \leq \infty\) [15],
- the set of all strongly countable-dimensional subcontinua of \(I^\infty\) of dimension \(\geq 2\) in \(C(I^\infty)\) [11],
- the sets of all weakly infinite-dimensional subcontinua of \(I^\infty\) of dimension \(\geq 2\) and of \(C\)-subcontinua of \(I^\infty\) of dimension \(\geq 2\) in \(C(I^\infty)\) [10].

In this paper we consider the sets
\[
\mathcal{W} \supseteq \mathcal{CW} \supseteq \mathcal{HA}
\]
of Wilder continua, continuum-wise Wilder continua and nondegenerate hereditarily arcwise connected continua, resp., in \(I^n\) for \(2 \leq n \leq \infty\) as the subspaces of \(C(I^n)\) as \(I^\infty\). All of them are characterized in the next section as coanalytic absorbers in \(C(I^n)\) if \(n \geq 3\). It means, in particular, that they are homeomorphic to the Hurewicz set.

The characterization of \(\mathcal{HA}\) seems to be of particular interest because closely related families such as arcwise connected subcontinua, hereditarily locally connected subcontinua or dendroids so far escape known methods of recognizing absorbers. The sets of hereditarily locally connected subcontinua and of dendroids are coanalytic complete in \(C(I^\infty)\) (see [6], [3]), so they are natural candidates. The class \(\mathcal{AC}\) of nondegenerate arcwise connected continua in \(I^n\) is \(\Pi_1^2\)-complete for \(n \geq 3\) (see [9] Theorem 37.11]). It was claimed in [5] that \(\mathcal{AC}\) is a \(\Pi_2^1\)-absorber in \(C(I^n)\) but, seemingly, the proof was never published. We observe that \(\mathcal{AC}\) is strongly coanalytic-universal for \(n \geq 2\) and is covered by a \(\sigma\mathbb{Z}\)-set in \(C(I^n)\) if \(n \geq 3\).
For notions undefined in this paper the reader is referred to the standard books [9], [13] and [14].

2. Main results

Let \( X \) be a topological Hilbert cube with a metric \( d \). Recall that a closed subset \( A \subset X \) is called a Z-set in \( X \) if for any \( \epsilon > 0 \) there exists a continuous mapping \( f : X \to X \) such that \( f(X) \cap A = \emptyset \) and \( \tilde{d}(f, \text{id}_X) := \sup\{d(f(x), x) : x \in X\} < \epsilon \). A countable union of Z-sets in \( X \) is called a \( \sigma Z \)-set in \( X \).

A subset \( A \) of \( X \) is strongly coanalytic-universal if for each coanalytic subset \( M \) of the Hilbert cube \( I^\infty \) and each compact set \( K \subset I^\infty \), any embedding \( f : I^\infty \to X \) such that \( f(K) \) is a Z-set in \( X \) can be approximated arbitrarily closely (in the sense of the uniform convergence) by an embedding \( g : I^\infty \to X \) such that \( g(I^\infty) \) is a Z-set in \( X \), \( g|K = f|K \) and \( g^{-1}(A) \setminus K = M \setminus K \).

Finally, \( A \) is a coanalytic absorber in \( X \) provided that

1. \( A \) is a coanalytic set,
2. \( A \) is contained in a \( \sigma Z \)-set in \( X \),
3. \( A \) is strongly coanalytic-universal.

All coanalytic absorbers are mutually homeomorphic. Moreover, if \( A \) and \( B \) are coanalytic absorbers in a Hilbert cube \( X \), then there is a homeomorphism \( h : X \to X \) such that \( h(A) = B \) which is arbitrarily close (in the sense of metric \( \tilde{d} \)) to the identity [13, Theorem 5.5.2].

**Proposition 2.1.** If \( Y \) is a compact space, then the families \( \mathcal{W}, \mathcal{CW} \) and \( \mathcal{HA} \) in \( Y \) are coanalytic subsets of the hyperspace \( C(Y) \).

**Proof.** A direct evaluation of the descriptive complexity of the formula:

\[
W \in \mathcal{W} \text{ iff } \begin{align*}
|W| > 1 & \land \quad \forall x, y, z \in W \ \exists K \in C(Y) \\
K \subset W \land x \in K \land (y \in K \land z \notin K) \lor (y \notin K \land z \in K)
\end{align*}
\]

immediately gives coanalyticity of \( \mathcal{W} \). Similar evaluations work for two other families. \( \square \)

**Proposition 2.2.** The families \( \mathcal{W}, \mathcal{CW}, \mathcal{AC} \) and \( \mathcal{HA} \) in \( I^n \) are contained in the family \( \mathcal{D}(I^n) \) of all decomposable subcontinua of \( I^n \) which is a \( \sigma Z \)-set in \( C(I^n) \) if \( n \geq 3 \).

**Proof.** Clearly, all the families are contained in \( \mathcal{W} \) and each Wilder continuum is decomposable. Moreover, \( \mathcal{D}(I^n) \) is a \( \sigma Z \)-set in \( C(I^n) \) for \( n \geq 3 \) by [15, Corollary 4.4]. \( \square \)
Proposition 2.3. The families $\mathcal{W}, \mathcal{CW}, \mathcal{AC}$ and $\mathcal{HA}$ in $I^n$ are strongly coanalytic-universal in $C(I^n)$ if $n \geq 2$.

Proof. For simplicity, we assume that $n < \infty$ but one can easily adapt the proof to the case of the Hilbert cube $I^\infty$. Fix an arbitrary coanalytic set $M \subset I^\infty$, a closed $K \subset I^\infty$, an embedding $f : I^\infty \to C(I^n)$ such that $f(K)$ is a Z-set and $\epsilon > 0$. We are going to define a Z-embedding which agrees with $f$ on $K$, is $\epsilon$-close to $f$ and satisfies

$$g^{-1}(W) \setminus K = g^{-1}(CW) \setminus K = g^{-1}(AC) \setminus K = g^{-1}(HA) \setminus K = M \setminus K.$$ 

We sketch main steps of a construction of $g$ following ideas from [8] and [15]. First we associate with $M$ a continuous map $\xi : I^\infty \to C(I^n)$ as in [15, Lemma 3.4]. It has the following property:

Property 2.4. For each $q \in I^\infty$, $\xi(q)$ is a hereditarily unicoherent 1-dimensional continuum such that

$$\left( (I \times \{0\}) \cup (\{0\} \times I) \right) \times \{0, \ldots, 0\} \subset \xi(q) \subset I^2 \times \{0, \ldots, 0\}$$

and if $q \in M$, then $\xi(q)$ is a dendroid which is a union of countably many arcs emanating from the segment $\{0\} \times I \times \{0, \ldots, 0\}$ while if $q \notin M$, then $\xi(q)$ contains a pseudoarc meeting $\{0\} \times I \times \{0, \ldots, 0\}$ at a single point.

Next, we modify map $\xi(q)$ by aggregating countably many copies of $\xi(q)$ in $I^2 \times \{0, \ldots, 0\}$:

$$\Xi(q) = \bigcup_i \alpha_i(\xi(q)) \cup (I \times \{1\} \times \{0, \ldots, 0\})$$

where

$$\alpha_i(x_1, x_2, 0, \ldots, 0) = (x_1, \frac{1}{i(i+1)} x_2 + \frac{i-1}{i} x_1 + 0, \ldots, 0), \quad i = 1, 2, \ldots$$

(Figure 1).

Hence, for $q \in M$, $\Xi(q) \in \mathcal{HA} \subset \mathcal{AC} \subset \mathcal{CW} \subset \mathcal{W}$. For $q \notin M$, $\Xi(q) \notin \mathcal{W}$ because of the subsequent property.

Property 2.5. If $P \subset \Xi(q)$ is a pseudoarc and $a, b, c \in P$ are points such that $P$ is irreducible between any two of them, then for any continuum $L \subset \Xi(q)$ containing $a, b$, the continuum $L \cap P$ must contain $c$. 


This shows that

\[
(2.3) \quad \Xi^{-1}(W) = \Xi^{-1}(CW) = \Xi^{-1}(AC) = \Xi^{-1}(HA) = M.
\]

Apart from \(\Xi\), we need another map \(\theta : I^\infty \to C([-1,1]^n)\) sending \(q = (q_i)\) to

\[
\theta(q) = \left(\left([-1,0] \times \{0\}\right) \cup S((\frac{1}{2},0); \frac{1}{2}) \cup \bigcup_{i=1}^{\infty} S(a_i; r_i(q))) \times \{(0,\ldots,0)\},
\]

where \(S(x;r)\) denotes the circle in the plane centered at \(x\) with radius \(r\), \(a_i = (-1 + 2^{-i}, 0) \in \mathbb{R}^2\) and \(r_i(q) = 4^{-(i+1)(1 + q_i)}\).

The figure \(\theta(q)\) is the union of countably many disjoint circles and of the diameter segment of the largest circle lying in \([-1,0] \times [-1,1] \times \{(0,\ldots,0)\}\). The map \(\theta\) is a continuous embedding.

We also use two deformations. The first one \(H_0 : 2^I \times I \to 2^I\) is such that, for any \((A, t) \in 2^I \times (0, \frac{1}{2}]\), \(H_0(A, t)\) is finite, \(\text{dist}(A, H_0(A, t)) \leq 2t\) (\(\text{dist}\) is the Hausdorff distance) and \(H_0(A, t) \subset [t, 1-t]^n\). The second \(H : C(I^n) \times [0, \frac{1}{2}] \to C(I^n)\) is defined by

\[
H(A, t) = H_0(A, t) \cup \bigcup_{a, b \in H_0(A, t)} (a\overline{b} \cap (\overline{B}(a; 2t) \cup \overline{B}(b; 2t)))
\]

where \(\overline{B}(a; \alpha)\) is the closed \(\alpha\)-ball in \(I^n\) around \(a\) and \(a\overline{b}\) is the line segment in \(I^n\) from \(a\) to \(b\).

It is known that for any \((A, t) \in C(I^n) \times (0, \frac{1}{2})\), \(H(A, t)\) is a connected graph in \([t, 1-t]^n\) and \(\text{dist}(A, H(A, t)) \leq 4t\) (see [8] and [15] for details and other properties of \(H\)).
For each \( q \in I^\infty \), put
\[
\mu(q) = \frac{1}{12} \min \{ \epsilon, \min \{ \text{dist}(f(q), f(q')) : q' \in f(K) \} \}
\]
and define our approximation
\[
(2.4) \quad g(q) = H(f(q), \mu(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\theta(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\Xi(q))
\]
(we use linear operations of addition and scalar multiplication in (2.4)). It is shown in the proof of [15, Lemma 3.2] that \( g \) (denoted there by \( G \); the definition of \( G \) in [15] should be corrected to the effect as in (2.4)) is a \( Z \)-embedding which is \( \epsilon \)-close to \( f \) and coincides with \( f \) on \( K \).

In order to see that (2.1) is satisfied, suppose that \( q \notin K \). Then \( \mu(q) > 0 \). If \( q \in M \), then one can easily see that \( g(q) \in \mathcal{HA} \).

Assume now that \( q \notin M \). It is convenient to consider the vertex
\[
v = (v_1, v_2, v_3, \ldots, v_n) \in H_0(f(q), \mu(q))
\]
which is maximal in \( H_0(f(q), \mu(q)) \) with respect to the lexicographic order \( \prec \) on \( I^n \). Observe that the copy \( v + \mu(q)\Xi(q) \) of \( \Xi(q) \) can be intersected only by finitely many other copies \( v' + \mu(q)\Xi(q) \) for \( v' = (v'_1, v'_2, v'_3, \ldots, v'_n) \prec v \). Also, \( v + \mu(q)\Xi(q) \setminus \{v\} \) is disjoint from
\[
H(f(q), \mu(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\theta(q)).
\]
It follows that there exists an open in \( g(q) \) neighborhood \( U \) of the point \( v + \mu(q)(1, 1, 0, \ldots, 0) \) such that
\[
\overline{U} \subset v + \mu(q)(\Xi(q) \setminus \{0\} \times I \times \{0, \ldots, 0\})_{n-2}.
\]

By definition (2.2) of \( \Xi \) and since \( q \notin M \), some component of \( \overline{U} \) is a pseudoarc \( P \). Take an open subset \( V \) of \( U \) such that \( V \cap P \neq \emptyset \) and \( \nabla \subset U \). Let \( P' \) be a component of \( \nabla \) contained in \( P \), \( a, b, c \in P' \) be points from distinct composants of \( P' \) and let \( L \subset g(q) \) be a continuum containing \( a, b \). If \( L \subset \overline{U} \), then \( L \subset P \), hence \( P' \subset L \) by the hereditary indecomposability of \( P \). In case \( L \setminus \overline{U} \neq \emptyset \), the component \( L_a \) of \( L \cap \overline{U} \) containing point \( a \) meets the boundary of \( U \) by the Janiszewski Boundary Bumping Theorem [14, 5.4], so \( L_a \cap P' \neq \emptyset \neq L_a \setminus P' \) and, of course, \( L_a \subset P \). Thus, \( P' \subset L_a \) as \( P \) is hereditarily indecomposable. In any case, we get \( c \in L \).

Consequently, \( g(q) \notin \mathcal{W} \).

Our main theorem follows from Propositions 2.1, 2.2, 2.3.
Theorem 2.6. The families $\mathcal{H}A$ and $\mathcal{CW}$, $\mathcal{W}$ in $I^n$ are coanalytic absorbers in the hyperspace $C(I^n)$ if $n \geq 3$.

We do not know if Theorem 2.6 remains true for $n = 2$.

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