DEVELOPABLE CUBICS IN $\mathbb{P}^4$ AND THE LEFSCHETZ
LOCUS IN $\text{GOR}(1, 5, 5, 1)$

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Abstract. We provide a classification of developable cubic hypersurfaces in $\mathbb{P}^4$. Using the correspondence between forms of degree 3 on $\mathbb{P}^4$ and Artinian Gorenstein $K$-algebras, given by Macaulay-Matlis duality, we describe the locus in $\text{GOR}(1, 5, 5, 1)$ corresponding to those algebras which satisfy the Strong Lefschetz property.

1. Introduction

We work over an algebraically closed field $K$ of characteristic zero. The projective space over $K$ of dimension $N$ will be denoted by $\mathbb{P}^N$. In this note we will focus our attention on the classification of developable cubic hypersurfaces on $\mathbb{P}^4$ as well as the Artinian Gorenstein algebras defined by them.

An irreducible projective variety $X \subset \mathbb{P}^N$ is called developable if it has degenerate Gauss map. Recent progress on the classification problem of developable varieties has been made via the focal locus of the ruling defined by fibers of the Gauss map. For instance, see [AG, MT] for a classification of developable threefolds. In Section 2 we proceed with a careful analysis of the focal locus to provide a finer classification of developable cubic hypersurfaces in $\mathbb{P}^4$. Our first goal is the following result.

Theorem 1.1. Let $X \subset \mathbb{P}^4$ be an irreducible cubic hypersurface. Assume that $X$ is not a cone. Then $X$ is developable if and only if it is projectively equivalent to a linear section of the secant variety of the Veronese surface.

A linear section of the secant variety of the Veronese surface is projectively equivalent to one of the following varieties (see Section 2.4):

1. the secant variety of the rational normal quartic curve;
2. the join of two irreducible conics sharing a single point; this point coincides with the intersection between the planes containing the conics.
3. the dual variety of the scroll surface $S(1, 2)$.

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Cases (1) and (2) correspond to cubic hypersurfaces which have nonvanishing Hessian while case (3) yields a cubic with vanishing Hessian.

To each cubic hypersurface above we can associate an Artinian Gorenstein \( \mathbb{K} \)-algebra. More generally, Macaulay-Matlis duality offers a correspondence between forms of degree \( d \) in \( N + 1 \) variables over \( \mathbb{K} \), not defining a cone, and standard graded Artinian Gorenstein \( \mathbb{K} \)-algebras of socle degree \( d \) and codimension \( N + 1 \). These algebras enjoy nice properties such as Poincaré duality in cohomology theory. We give a precise definition in Section 3.

Given a standard graded Artinian Gorenstein \( \mathbb{K} \)-algebra \( A = \bigoplus_{k=0}^{d} A_k \), AG algebra for short, its Hilbert vector is the vector \( \text{Hilb}(A) = (1, a_1, \ldots, a_d) \) where \( a_k = \dim_{\mathbb{K}} A_k \). We denote by \( \text{GOR}(T) \) the space which parametrizes AG algebras with Hilbert vector \( T \). It has been extensively studied in \([IK]\).

We are interested in algebras inside \( \text{GOR}(T) \) which satisfy the Strong Lefschetz property, SLP for short, which means that there exists a linear form \( l \in A_1 \) such that every multiplication map \( \mu_l : A_k \to A_{k+j} \) has maximal rank. This notion was introduced by R. Stanley and J. Watanabe, see \([St]\) and \([HW]\), and was inspired by the so called hard Lefschetz Theorem on the cohomology of smooth projective complex varieties, see for example \([GH]\).

The Lefschetz properties have attracted a lot of attention over the last years; we refer to \([HW]\) for a survey on the area.

Now we go back to cubic hypersurfaces in \( \mathbb{P}^4 \) which are not cones. They correspond to AG algebras with Hilbert vector \( (1, 5, 5, 1) \), via Macaulay-Matlis duality. We focus on the Strong Lefschetz property of algebras in \( \text{GOR}(1, 5, 5, 1) \) which come from a developable cubic hypersurface. Algebras associated to cases (1) and (2) above have the SLP, whereas it fails in case (3). Moreover, by the main result of \([MW]\), any AG algebra with Hilbert vector \( (1, 5, 5, 1) \) failing SLP comes from case (3). Sections 3 and 4 are devoted to the description of the locus of algebras in \( \text{GOR}(1, 5, 5, 1) \) failing the SLP, the main results of are summarized in the following theorem.

**Theorem 1.2.** The space \( \text{GOR}(1, 5, 5, 1) \) parametrizing AG algebras with Hilbert vector \( (1, 5, 5, 1) \) coincides with \( \mathbb{P}^3 \setminus C_4 \), where \( C_4 \) is the space of cubic cones in \( \mathbb{P}^4 \). Moreover, the following assertions hold:

1. The locus \( C_4 \) is the image of a projective bundle over \( \mathbb{P}^4 \) by a birational morphism, its dimension is 23 and its degree is 1365.
2. The locus of algebras failing SLP coincides with \( K \setminus C_4 \), where \( K \) is a rational projective variety of dimension 18 and degree 29960. More precisely, \( K \) is the image of a projective bundle over the Grassmannian \( \mathbb{G}(2, 4) \) by a birational morphism.
3. The intersection \( K \cap C_4 \) is a divisor in \( K \) of degree 116420.

We note that the locus of algebras in \( \text{GOR}(1, 5, 5, 1) \) failing SLP coincides with the locus of algebras with Jordan type \( 4^1 \oplus 2^3 \oplus 1^2 \), while any other algebra has Jordan type \( 4^1 \oplus 2^4 \). In particular \( 4^1 \oplus 2^3 \oplus 1^2 \) is the only possible
2. Developable Cubics in \( \mathbb{P}^4 \)

2.1. Basic definitions. Given a rational map \( \varphi : X \to Y \) between projective varieties, its image is the closure of \( \varphi(U) \) in \( Y \), where \( U \) is the maximal domain where \( \varphi \) is defined.

Let \( X \subset \mathbb{P}^N \) be a projective subvariety of dimension \( n \geq 1 \). Let \( (\mathbb{P}^N)^* \) denote the space of hyperplanes in \( \mathbb{P}^N \). We denote by \( \text{Con}_X \subset \mathbb{P}^N \times (\mathbb{P}^N)^* \) the conormal variety of \( X \): this is the closure of the set of pairs \((x,H)\) such that \( x \) is a regular point of \( X \) and \( H \) contains the tangent space \( T_xX \). Let \( X^* \) be the image of the projection in the second coordinate. It is the dual variety of \( X \). Given a point \( x \in \mathbb{P}^N \), we define \( x^* \subset (\mathbb{P}^N)^* \) as the set of hyperplanes passing through it.

Let \( \mathbb{G}(n,N) \) denote the Grassmannian of \( n \)-planes in \( \mathbb{P}^N \). The Gauss map \( \gamma : X \to \mathbb{G}(n,N) \) associates to each regular point \( x \in X \) the tangent space \( T_xX \in \mathbb{G}(n,N) \). We denote by \( X^\vee \) the image of the projection in the second coordinate. It is the dual variety of \( X \). Given a point \( x \in \mathbb{P}^N \), we define \( x^* \subset (\mathbb{P}^N)^* \) as the set of hyperplanes passing through it.

Let us denote by \( \text{Hess}_f \) the Hessian matrix of \( f \), namely the matrix of the second derivatives. Its determinant is the Hessian determinant. We shall say that \( X = V(f) \) is a hypersurface with vanishing Hessian if its Hessian determinant is null. Therefore \( \Phi_f \) is nondominant if and only if \( f \) has vanishing Hessian. This is equivalent to say that the derivatives \( f_0, \ldots, f_N \) of \( f \) are algebraically dependent. We summarize the above discussion in the following proposition.

**Proposition 2.1.** Let \( X = V(f) \subset \mathbb{P}^N \) be a hypersurface and \( Z \) its polar image. The following conditions are equivalent.

1. \( X \) has vanishing Hessian;
2. The partial derivatives of \( f \) are algebraically dependent;
3. \( Z \) is a proper subvariety of \( (\mathbb{P}^N)^* \).

The singular locus and the polar image of a hypersurface with vanishing Hessian have a relevant role. The following proposition gives a relation...
Proposition 2.2 ([P, Za2]). Let \( X \subset \mathbb{P}^N \) be a hypersurface with vanishing Hessian. Then \( Z^* \subset \text{Sing}(X) \).

Remark 2.3. Hypersurfaces with vanishing Hessian are developable. To see this, we assume that \( X \) has vanishing Hessian. First we note that \( X^* \) is a proper subvariety of \( Z \). In fact if \( X^* = Z \) then \( Z^* = X \), but this contradicts Proposition 2.2. The strict inclusions of irreducible varieties \( X^* \subset Z \subset (\mathbb{P}^N)^* \) imply that \( \dim X^* < N - 1 \), hence \( X \) is developable.

Given projective subvarieties \( V, W \subset \mathbb{P}^N \), we denote by \( S(V, W) \) the join between them. It is the closure of the union of lines in \( \mathbb{P}^N \) joining \( V \) to \( W \). In particular \( S(V) = S(V, V) \) is the secant variety of \( V \). A subvariety \( V \subset \mathbb{P}^N \) is a cone if there exists \( x \in V \) such that \( S(x, V) = V \). This motivates the definition of the vertex of \( V \)

\[
\text{Vert}(V) = \{ x \in V : S(x, V) = V \}.
\]

Cones are the simplest examples of hypersurfaces with vanishing Hessian. Now we state the following useful proposition the proof of which will be left to the reader.

Proposition 2.4. Let \( X = V(f) \subset \mathbb{P}^N \) be a hypersurface. Then the following conditions are equivalent:

(i) \( X \) is a cone;

(ii) The partial derivatives of \( f \) are linearly dependent;

(iii) \( Z \) is contained in a hyperplane of \((\mathbb{P}^N)^*\);

(iv) \( X^* \) is contained in a hyperplane of \((\mathbb{P}^N)^*\);

(v) Up to a projective transformation \( f \) depends on at most \( N \) variables.

There are many classical examples of varieties with vanishing Hessian which are not cones. The following example appears in the work of Gordan and Noether [GN] and Perazzo [P], called un esempio semplicissimo.

Example 2.5. Let \( X = V(f) \subset \mathbb{P}^4 \) be the irreducible hypersurface given by

\[
f = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2.
\]

We can check that \( X \) is not a cone, showing for example the linear independence between the partial derivatives. But since \( f_0f_2 = f_1^2 \) is an algebraic relation among them, \( X \) has vanishing Hessian.

2.2. Linearity of general fibers and focal locus. Let \( X \subset \mathbb{P}^N \) be an irreducible projective variety of dimension \( n \geq 1 \). Developable varieties are ruled by fibers of the Gauss map, which are union of finitely many linear spaces, see [Se, p. 95]. In fact, it has been proved by Zak that the closure of a general fiber is irreducible. For instance, see [Za1, Theorem 2.3] or [FP, p. 87].
**Theorem 2.6** ([Sc] Za1). Let \( X \subset \mathbb{P}^N \) be an irreducible projective variety. If \( X \) is developable then the closure of a general fiber of \( \gamma \) is a linear subspace. 

When \( X \) is a hypersurface with vanishing Hessian, the fibers of its polar map share this phenomenon of linearity. We state this result below, the proof can be found in [Za2, Proposition 4.9].

**Theorem 2.7** (Za2). Let \( X \subset \mathbb{P}^N \) be a reduced hypersurface with vanishing Hessian. The closure of the fiber of \( \Phi_f \) over a general point \( z \in Z \) is a union of finitely many linear subspaces passing through the subspace \( (T_z Z)^* \).

**Example 2.8.** We want to illustrate the linearity of fibers of \( \gamma \) and \( \Phi_f \) where \( f = x_0x_2^3 + x_1x_3x_4 + x_2x_4^2 \). In particular, we will see in this example that \( X^* \) is a scroll surface \( S(1, 2) \). See Figure I.

The singular locus of \( X = V(f) \), with reduced structure, is \( Y = V(x_3, x_4) \). Its polar image is the quadratic cone \( Z = V(y_0y_2 - y_1^2) \subset (\mathbb{P}^4)^* \) which has as vertex the line \( l = Y^* \). Therefore \( Z^* \) is a conic contained in \( Y \). Consider the plane \( P = V(y_3, y_4) \subset (\mathbb{P}^4)^* \), observe that \( C = Z \cap P \) is a conic and \( Z \) is the join between \( C \) and \( l \).

We denote by \( \mathbb{P}_t^3 \), \( t \in \mathbb{P}^1 \) the family of hyperplanes containing \( Y \) and for each \( t \in \mathbb{P}^1 \) let \( \eta_t \in (\mathbb{P}^4)^* \) be the corresponding point of \( l \). The reader can check that \( \mathbb{P}_t^3 \cap X \) is a union of a plane \( \mathbb{P}_t^2 \) and \( Y \), where \( Y \) appears with multiplicity two. A direct calculation shows that for a general point \( x \in \mathbb{P}_t^3 \) the closure of \( \Phi_f^{-1}(y), y = \Phi_f(x), \) is a line contained in \( \mathbb{P}_t^3 \) and passing through \( \xi_t = (T_y Z)^* \in Z^* \). In particular, for a general \( x \in \mathbb{P}_t^2 \) the closure of \( \gamma^{-1}(y) \) is a line contained in \( \mathbb{P}_t^2 \) passing through \( \xi_t \). Hence \( X \) is swept out by planes \( \mathbb{P}_t^2 \) and fibers of \( \gamma \) lying in \( \mathbb{P}_t^2 \) determine a star of lines passing through the point \( \xi_t \in Z^* \).

Now we prove that \( X^* \) is a scroll surface \( S(1, 2) \). For a general point \( x \in \mathbb{P}_t^2 \), the tangent space \( T_x X \) contains \( \mathbb{P}_t^2 \). Therefore the image of \( \mathbb{P}_t^2 \) by \( \gamma \) is \( (\mathbb{P}_t^2)^* \cong \mathbb{P}_t^1 \). Let \( \mu_t \in (\mathbb{P}^4)^* \) be the point corresponding to the unique hyperplane \( H_t \) containing \( \mathbb{P}_t^2 \) and \( P^* \) (as \( P^* \subset H_t, \mu_t \in Z \cap P = C \)). Observe that \( \mu_t \) is the line passing through \( \eta_t \in l \) and through \( \mu_t \in C \). This shows that \( X^* \) is a scroll \( S(1, 2) \) which has as rulings the lines passing through \( \eta_t \in Y^* = l \) and \( \mu_t \in C, t \in \mathbb{P}^1 \).

Let \( X \subset \mathbb{P}^N \) be a developable projective variety. By Theorem 2.6 \( X \) is ruled by linear subspaces (fibers of the Gauss map) of dimension \( k \), where

\[ k = \dim(X) - \dim(X^\vee). \]

Let \( U \subset X \) be the open subset where \( \gamma \) has maximal rank. For each \( x \in U \), let \( L_x \) be the \( k \)-dimensional subspace passing through \( x \) such that \( \gamma \) is constant along \( L_x \).

We will denote by \( B_x \) the closure in \( \mathbb{G}(k, N) \) of the set \( \{L_x : x \in U\} \). We shall say that \( B_{\gamma} \) is the family of \( k \)-dimensional subspaces determined
by fibers of $\gamma$. Let $B'_\gamma$ be a desingularization of $B_\gamma$ and $I \subset B'_\gamma \times \mathbb{P}^N$ the incidence variety of $B'_\gamma$ with natural projection $\psi : I \to X$. For a general $x \in X$ the fiber $\psi^{-1}(x)$ coincides with the point $(L_x, x) \in I$.

Let $R_\psi$ be the ramification divisor of $\psi$ and $\pi : I \to B'_\gamma$ the natural projection on the first coordinate. We can write $R_\psi = H_\psi + V_\psi$ where the restriction of $\pi$ to any irreducible component of the support of $H_\psi$ is dominant and of the support of $V_\psi$ is nondominant. We say that $H_\psi$ is the horizontal divisor and $V_\psi$ is the vertical divisor. The direct image by $\psi$ of the horizontal divisor, denoted by $\Delta = \psi_*(H_\psi)$, is called the focal locus of $X$.

We note that the restriction of $\psi$ to a general fiber of $\pi$

$$\psi|_{\pi^{-1}(L)} : \pi^{-1}(L) \to L$$

is an isomorphism. So the restriction of $H_\psi$ to $\pi^{-1}(L)$ defines a divisor in $L$ which coincides with the restriction of the focal locus of $X$ to $L$. This divisor will be denoted by $\Delta_L$.

One of the main results concerning developable varieties is the following. For the proof see [IL, Theorem 3.4.2].

**Theorem 2.9.** Let $X \subset \mathbb{P}^N$ be a developable projective variety. If $X$ is not a linear subspace, then $X$ is singular and its focal locus is contained in $\text{Sing}(X)$. Moreover, for a general $L$ belonging to $B_\gamma$, the restriction of the focal locus to $L$ is a divisor $\Delta_L$ in $L$ of degree $\dim(B_\gamma)$.

### 2.3. Cubics with vanishing Hessian

Revisiting the work of Perazzo [P], in [GR] the authors provide a classification of cubic hypersurfaces with vanishing Hessian in $\mathbb{P}^N$, for $N \leq 6$. In this section we rebuild the classification for $N = 4$. This digression will be useful in the next section.

**Lemma 2.10.** Let $X \subset \mathbb{P}^N$, $N \geq 3$, be an irreducible cubic hypersurface. Assume there is a component $Y$ of $\text{Sing}(X)$, with $\dim Y = \dim X - 1$. Then $Y$ is a linear subspace.

**Proof.** Since $X$ has degree 3, the secant variety $S(Y)$ of $Y$ must be contained in $X$. Hence either $S(Y) = Y$, in this case $Y$ is a linear subspace, or
$S(Y) = X$. But the second case cannot occur, because the equality
\[ \dim S(Y) = \dim Y + 1 \]
implies that $X = S(Y)$ is a linear subspace, see [Ru, Proposition 1.2.2]. □

**Remark 2.11.** For cubic surfaces in $\mathbb{P}^3$ the picture turns out to be the following. It is well known that a developable surface must be either a cone or the tangent developable to a curve. In the last case the curve lies in the singular locus and by Lemma 2.10 this situation cannot occur.

**Lemma 2.12.** Let $X \subset \mathbb{P}^4$ be an irreducible cubic hypersurface. Assume that $X$ is not a cone. If $\text{Sing}(X)$ contains a linearly embedded $\mathbb{P}^2$ then $X$ is projectively equivalent to $V(f)$, where $f = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2$.

**Proof.** Let us suppose $Y = V(x_3, x_4) \subset \text{Sing}(X)$. If $f$ is an irreducible polynomial defining $X$ one can write $f = ax_3 + bx_4$, where $a$ and $b$ are polynomials of degree two. Since the derivatives of $f$ must vanish in $Y$ we can write
\[ f = l_0x_3^2 + l_1x_3x_4 + l_2x_4^2 \]
where $l_i$, $i = 0, 1, 2$, are linear forms. If $X$ is not a cone, then $l_0$, $l_1$ and $l_2$ are linearly independent, so there is a projective transformation such that
\[ f = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2. \]

The following proposition is a classical result of Perazzo [P].

**Proposition 2.13.** Let $X \subset \mathbb{P}^4$ be a cubic hypersurface. Assume that $X$ is not a cone. The following conditions are equivalent:

(i) $X$ has vanishing Hessian;

(ii) $X^*$ is projectively equivalent to the scroll surface $S(1, 2)$.

**Proof.** First we will show that we can assume $X$ irreducible. Suppose that $X = V(f)$ is reducible and is not a cone, then $f = hq$ where $h$ is a linear form and $q$ is homogeneous of degree 2. In this case its polar map $\Phi_f$ is dominant, as follows by a straightforward computation. This also can be proved (when $\mathbb{K} = \mathbb{C}$) by using the following identity
\[ d(X) = d(V(h)) + d(V(q)) + d(V(h) \cap V(q)) \]
where $d(V)$ denotes the degree of the polar map associated to $V$, see [FM, Corollary 4.3]. Since $V(h) \cap V(q)$ is a smooth conic, recall that we are assuming $X$ is not a cone, then the right side of the identity is positive, which implies that $\Phi_f$ is dominant.

Now we suppose that $X$ is irreducible and has vanishing Hessian. By Proposition 2.2 we get $Z^* \subset \text{Sing} X$. We can assume $\dim(Z^*) \geq 1$, otherwise $Z$ is contained in a hyperplane and Proposition 2.4 ensures that $X$ is a cone.
We will show that $Z^*$ cannot be a component of $\text{Sing}(X)$. Let us consider the Perazzo map

$$
P_X : \mathbb{P}^N \rightarrow \mathbb{G}((\text{codim}Z - 1, N)
$$

$$
x \mapsto (T_{\phi(x)}Z)^*.
$$

Since $X$ is an irreducible cubic hypersurface, the closure of a general fiber of $P_X$ is a linear space, see [GR, Theorem 2.5]. According with [GR, Proposition 2.16] this implies that $Z^*$ lies in the intersection of fibers of $P_X$. And from [GR, Proposition 2.13] this is equivalent to say that the linear span $< Z^* >$ lies in $\text{Sing}(X)$. Finally, this ensures that if $Z^*$ is a component of $\text{Sing}(X)$ then $Z^* = < Z^* >$, which implies that $Z^*$ is a linear subspace. But, in this case $X$ must be a cone.

So far we have proved that $\dim Z^* \geq 1$ and $Z^*$ cannot be component of $\text{Sing} X$. Hence, one may assume that $\text{Sing}(X)$ contains a two–dimensional component. It follows from Lemma 2.10 and Lemma 2.12 that $X$ is projectively equivalent to $V(f)$, where $f = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2$. This is enough to conclude that $X^* \simeq S(1, 2)$, see Example 2.8.

The converse is immediate. Since all scrolls $S(1, 2)$ are projectively equivalent, $X^* \simeq S(1, 2)$ implies that $X$ is projectively equivalent to $V(f)$, $f = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2$.

$\square$

2.4 Sections of the secant variety of the Veronese surface. Let us identify $\mathbb{P}^2$ with $\mathbb{P}h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and $\mathbb{P}^5$ with $\mathbb{P}h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. The image $\mathcal{V}$ of the Veronese map

$$
v : \mathbb{P}^2 \rightarrow \mathbb{P}^5
$$

$$
l \mapsto l^2
$$

is the Veronese surface. Its secant variety $S(\mathcal{V})$ is a cubic hypersurface. Using coordinates, the map above can be given by

$$(a : b : c) \mapsto (a^2 : 2ab : b^2 : 2bc : c^2)$$

so writing $(x_0 : \cdots : x_5)$ as coordinates for $\mathbb{P}^5$ we have $\mathcal{V}$, is given by the following ideal

$$(4x_0x_3 - x_1^2, 2x_0x_4 - x_1x_2, 4x_0x_5 - x_2^2, x_1x_4 - 2x_2x_3, 2x_1x_5 - x_2x_4, 4x_3x_5 - x_4^2)$$

and

$$S(\mathcal{V}) = V(4x_0x_3x_5 - x_0x_1^2 - x_1^2x_5 - x_2^2x_3 + x_1x_2x_4).$$

We want to describe all the sections of $S(\mathcal{V})$ up to projectivity. The preimage by $v$ of a hyperplane in $\mathbb{P}^5$ is a conic in $\mathbb{P}^2$. The hyperplane is tangent to $\mathcal{V}$ if and only if the conic is singular, thus the dual variety of $\mathcal{V}$ is isomorphic to the locus of singular conics. A pair of distinct lines corresponds to a section which is tangent to the surface at a single point and a double line corresponds to a section which is tangent along a conic. The first case yields a pair of conics in $\mathcal{V}$ sharing a single point. If we identify $(\mathbb{P}^5)^*$ with the projectivization of the space of symmetric matrices.
then $\mathcal{V}^* \subset (\mathbb{P}^5)^*$ is identified with the locus of singular matrices. Since any singular matrix can be written as a sum of two rank one matrices, $\mathcal{V}^*$ coincides with the secant variety of the locus $\tilde{\mathcal{V}}$ formed by matrices which have rank one. Note that $\tilde{\mathcal{V}}$ is isomorphic to $\mathcal{V}$. Also, it coincides with the dual variety of the secant variety of $\mathcal{V}$, $\tilde{\mathcal{V}} = S(\mathcal{V})^*$. Therefore one has $S(\mathcal{V})^* \simeq \mathcal{V}$ and $\mathcal{V}^* \simeq S(\mathcal{V})$.

The natural action of the algebraic group $\text{PGL}(3)$ on $(\mathbb{P}^5)^*$ gives three orbits:

- $U_1 = (\mathbb{P}^5)^* \setminus S(\tilde{\mathcal{V}})$, yielding sections which are transverse to $\mathcal{V}$;
- $U_2 = S(\tilde{\mathcal{V}}) \setminus \tilde{\mathcal{V}}$, corresponding to sections which are tangent to $\mathcal{V}$ at a single point; and
- the closed orbit $\mathcal{V}$, giving sections which are tangent to $\tilde{\mathcal{V}}$ along to a conic.

We conclude that $H \in U_1$ yields a section $X = H \cap S(\mathcal{V})$ projectively equivalent to the secant variety of the rational normal curve in $\mathbb{P}^4$. If $H \in U_2$, $X = H \cap S(\mathcal{V})$ is projectively equivalent to a join $S(C_1, C_2)$ between two irreducible conics $C_1$ and $C_2$ sharing a single point. If $H \in \tilde{\mathcal{V}}$, $X = H \cap S(\mathcal{V})$ is projectively equivalent to $V(f)$, where $f = x_0x_3^2 + x_1x_3x_4 + x_2x_1^2$.

The rest of this section is devoted to the proof of Theorem 1.1. For this purpose we will need some preliminary lemmas.

2.5. Preliminary Lemmas. Let $B_\gamma$ be the family of linear subspaces determined by fibers of the Gauss map $\gamma$. If the dual variety $X^*$ of $X$ has dimension two, then $B_\gamma$ is a two-dimensional family of lines. Theorem 2.9 ensures that the restriction of the focal locus $\Delta$ of $X$ to a general line $L$ belonging to $B_\gamma$ is a divisor $\Delta_L$ of degree two in $L$. If $X^*$ has dimension one then $B_\gamma$ is a 1-dimensional family of 2-linear subspaces. Applying Theorem 2.9 again, we see that $\Delta_L$ is a divisor of degree one in $L$. The next results will be useful in the proof of Theorem 1.1.

Lemma 2.14. Let $X \subset \mathbb{P}^4$ be an irreducible developable cubic hypersurface with nonvanishing Hessian, then $X^*$ has dimension 2.

Proof. Since $X$ is not a linear subspace, then the dimension of $X^*$ is at least one. We will show that if $X^*$ has dimension 1 then $X$ has vanishing Hessian. Assume that $\dim X^* = 1$. For a general element $L \in B_\gamma$, $\Delta_L$ is a divisor of degree one in $L$ and $\Delta$ is contained is Sing($X$). If $\Delta_L \simeq \mathbb{P}^1$ varies with $L$, then the dimension of the singular set of $X$ is at least two. Lemma 2.10 and Lemma 2.12 will imply that $X$ has vanishing Hessian. If $\Delta_L$ is a fixed line, say $l \simeq \mathbb{P}^1$, when $L$ varies in $B_\gamma$ then we will show that $X$ must be a cone which vertex contains $l$. This contradicts our hypothesis. Given $y \in l$, let $z \in S(y, X)$ general: $z \in \langle y, x \rangle$ for general $x \in X$. We are assuming that the linear subspace $L_x \in B_\gamma$ passing through $x$ contains $l$. In particular $\langle y, x \rangle \subset L_x \subset X$. This implies that $z \in X$. Since $z \in S(y, X)$ is general, we get $S(y, X) \subset X$. This shows that $S(y, X) = X$. 

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and consequently \( y \in \text{Vert}(X) \), which is a contradiction. Therefore \( X^* \) has dimension 2 and this concludes the proof. \( \square \)

**Lemma 2.15.** Let \( X \subseteq \mathbb{P}^4 \) be an irreducible developable cubic hypersurface such that the support of the focal locus \( \Delta \) is an irreducible curve. Besides, assume that \( X^* \) has dimension two. If the restriction of \( \Delta \) to a general line belonging to \( B_\gamma \) is one point of multiplicity two, then \( X \) has vanishing Hessian.

**Proof.** Let us denote by \( C \) the support of \( \Delta \). Given a general point \( x \in C \), let \( V_x \) be the cone determined by lines of \( B_\gamma \) passing through it. Notice that \( X \) is a union of these \( V_x \) when \( x \) varies in \( C \). The developable hypothesis on \( X \) implies that these cones are tangent planes to \( C \), see [MT, p. 454]. Therefore \( C \) cannot be a line, otherwise \( X \) would be a cone.

Since \( X \) has degree three, the secant variety of \( C \), \( S(C) \), is contained in \( X \). Now we analyze the dimension of \( S(C) \). If it has dimension three, then \( S(C) = X \) and by Terracini Lemma \( \Delta_L \) coincides with two distinct points, this contradicts our hypothesis on \( \Delta_L \). If the dimension of \( S(C) \) equals two, we will show that \( S(C) \) is contained in the singular set of \( X \), which implies that \( X \) has vanishing Hessian. So, assume that

\[
\dim(S(C)) = 2 = \dim(C) + 1.
\]

In this situation, \( S(C) \) is a linearly embedded \( \mathbb{P}^2 \), see [Ru, Proposition 1.2.2]. Suppose that \( S(C) \simeq \mathbb{P}^2 \) is not contained in \( \text{Sing}(X) \). Let \( q \in S(C) \) be a smooth point of \( X \) and take a tangent line \( l_x \) of \( C \) at \( x \in C \) passing through \( q \). Since \( q \) belongs to the plane \( V_x \), the tangent space \( T_q X \) must contain \( V_x \). Thus, \( T_q X \) is the join between \( S(C) \) and \( V_x \), that is

\[
T_q X = S(S(C), V_x).
\]

Hence, the tangent space of \( X \) is constant along \( l_x \). But, two lines \( l_x \) and \( l_{x'} \), for distinct points \( x \) and \( x' \), must intersect at one point. Therefore the tangent space of \( X \) is constant along \( S(C) \). If \( H \) denotes the tangent space at one general point \( q \in S(C) \), then we have \( V_x \subset H \) for a general point \( x \in C \). In this case, we must have \( H = X \) and this contradicts our hypothesis \( \deg X = 3 \).

Therefore \( S(C) \simeq \mathbb{P}^2 \) is contained in \( \text{Sing}(X) \). Lemma 2.10 and Lemma 2.12 imply that \( X \) has vanishing Hessian. \( \square \)

For proof of Theorem 1.1, we also need the following result.

**Lemma 2.16.** Let \( C \subset \mathbb{P}^4 \) be a non-degenerate irreducible curve whose secant variety, \( S(C) \), is a cubic hypersurface. Then \( C \) is a rational normal quartic curve.

**Proof.** It is enough to show that \( \deg C = 4 \). Let \( x \in C \) be a smooth point, \( L = T_x C \) the tangent line at \( x \) and \( P = \mathbb{P}^2 \subset \mathbb{P}^4 \) a linear space skew to \( L \), that is, \( P \cap L = \emptyset \). We consider the projection \( \pi : C \to P \) from \( L \) which
sends $y \in C \setminus (C \cap L)$ to $\pi(y) = < L, y > \cap P$.

Note that $\tilde{C} = S(L, C) \cap P$ is the closure of the image of $C$ by $\pi$. We will show that $\tilde{C}$ has degree 2 and this implies that $C$ has degree 4.

Let $\mathcal{C}_x$ be the tangent cone of $S(C)$ at $x$. It has $S(L, C)$ as an irreducible component, see [CR, Theorem 3.1]. Since $\deg S(C) = 3$, we get $\deg \mathcal{C}_x = 2$. But $\mathcal{C}_x$ cannot be decomposed as product of hyperplanes, otherwise $C$ would be degenerate. This shows that $S(L, C) = \mathcal{C}_x$ and therefore $\tilde{C}$ has degree 2. This concludes the proof. $\square$

2.6. Proof of Theorem 1.1. Assume that $X \subset \mathbb{P}^4$ is a developable irreducible cubic hypersurface which is not a cone. If $X$ has vanishing Hessian, then Proposition 2.13 yields $X^* \simeq S(1, 2)$. Therefore $X$ corresponds to a section of $S(V)$ which is tangent to $V$ along a conic (see Section 2.4).

Suppose that $X$ has nonvanishing Hessian. By Lemma 2.14 we get that $X^*$ has dimension two. Thus, $B_\gamma$ is a two-dimensional family of lines. As consequence of Theorem 2.9 the restriction of the focal locus to a general line $L \in B_\gamma$ is a divisor of degree two in $L$.

We first remark that any irreducible component of the support $|\Delta|$ of $\Delta$ has dimension one. In fact, if there exists a zero dimensional component, say $x_0 \in X$, then $X$ must be a cone because every line $L \in B_\gamma$ must pass through $x_0$. Besides that, since $\Delta \subset \text{Sing}(X)$ then from Lemma 2.10 and Lemma 2.12 the existence of a two-dimensional component of $|\Delta|$ will imply that $X$ has vanishing Hessian.

The focal locus $\Delta$ is the direct image of the horizontal divisor $H_\psi$. Recall that the restriction of $\psi : \mathcal{I} \rightarrow X$ to $\pi^{-1}(L)$ gives an isomorphism $\pi^{-1}(L) \simeq L$, for general $L$. The restriction of $H_\psi$ to $\pi^{-1}(L)$ is a divisor of degree two which corresponds to $\Delta_L$, via this isomorphism. Therefore, the support of $H_\psi$ has at most two irreducible components. A fortiori, the number of irreducible components of $|\Delta|$ is at most two.

We will see that if $|\Delta|$ has two irreducible components then it is a linear section of $S(V)$, $X = H \cap S(V)$ with $H \in V^* \setminus S(V)^*$. Suppose $|\Delta|$ is a union of two distinct irreducible curves, say $C_1$ and $C_2$. Hence $X$ must be the join between them, $X = S(C_1, C_2)$. We first remark that $C_1$ and $C_2$ are plane curves. Indeed, if for example $S(C_1)$ has dimension 3 then $X = S(C_1)$ and $|\Delta| = C_1$, contradicting our hypothesis on $\Delta$. If $C_1$ and $C_2$ are disjoint, one has (see [H, p. 235])

$$3 = \deg(X) = \deg(C_1)\deg(C_2)$$

which means that at least one of these curves is a line and then $X$ is a cone. Let us suppose that $C_1$ and $C_2$ are not disjoint and have degree at least two. The two planes containing $C_1$ and $C_2$ must share exactly one point. Otherwise, $X$ coincides with the $\mathbb{P}^3$ spanned by them. We denote by $p$ the intersection point of $C_1$ and $C_2$. Now we proceed with the same argument of [H, p. 236 Calculation III]. If $\Gamma \subset \mathbb{P}^4$ is a general line, we may describe
the intersection $\Gamma \cap X$ by considering a general projection $\pi_\Gamma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ from $\Gamma$. Let $\tilde{C}_1, \tilde{C}_2 \subset \mathbb{P}^2$ be the image of $C_i$ by $\pi_\Gamma$, $i = 1, 2$, and $q = \pi_\Gamma(p)$. The points of $\Gamma \cap X$ correspond to the points of $\tilde{C}_1 \cap \tilde{C}_2$ distinct of $q$. We note that the intersection outside $q$ is transverse, thus

\begin{equation}
3 = \deg(X) = \deg(C_1)\deg(C_2) - I_q
\end{equation}

where $I_q = I(\tilde{C}_1, \tilde{C}_2; q)$ denotes the intersection multiplicity at $q$. Since $C_1$ and $C_2$ do not share a tangent line, for a general choice of $\Gamma$ we may assume the same for $\tilde{C}_1$ and $\tilde{C}_2$. Then $I_q$ is the product of the algebraic multiplicity of $\tilde{C}_1$ and $\tilde{C}_2$ at $q$, say $I_q = a_1a_2$. By the inequality $a_i \leq \deg(C_i) - 1$ and from (1) one obtains $\deg(C_1) + \deg(C_2) \leq 4$.

Hence $X$ is the join between the conics $C_1$ and $C_2$. The reader can check that $X$ is uniquely determined up to a projective transformation. We conclude that $X \cong H \cap S(V)$ with $H \in V^* \setminus S(V)^*$ and this concludes the case where $|\Delta|$ has two irreducible components.

Let us assume that $|\Delta| = C$ is an irreducible curve. If $L$ is a general line belonging to $B_\gamma$, we have two possibilities:

1. $\Delta_L = 2p$;
2. $\Delta_L = p + q$, with $p \neq q$.

From Lemma 2.15, the first case cannot happen because we are assuming that $X$ has nonvanishing Hessian. If we are in case (2), then a general line of $B_\gamma$ is secant to the non-degenerate curve $C$ and then $X = S(C)$. By Lemma 2.16 $X = S(C)$ where $C$ is a rational normal curve. This corresponds to the case where $X \cong H \cap S(V)$ where $H \in (\mathbb{P}^5)^* \setminus V^*$. This finishes the proof of Theorem 1.1.

3. The Lefschetz locus in GOR(1, 5, 5, 1)

3.1. Artinian Gorenstein algebras and the Lefschetz property. Let $A = \bigoplus_{i=0}^d A_i$, be a graded Artinian $\mathbb{K}$-algebra with $A_d \neq 0$, we say that $A$ is standard graded if $A_0 = \mathbb{K}$ and $A$ is generated by $A_1$ as algebra. The integer $d$ is called the socle degree of $A$. The codimension of $A$ coincides with its embedding dimension, that is $\dim A_1$. If $A = \mathbb{K}[X_0, \ldots, X_N]/I$ is a standard graded Artinian $\mathbb{K}$-algebra, where $I$ is an ideal with $I_1 = 0$, then $\text{codim} A = N + 1$. The Hilbert vector of $A$ is $\text{Hilb}(A) = (1, a_1, \ldots, a_d)$, where $a_k = \dim_\mathbb{K} A_k$.

It is a well known fact that $A$ is a Gorenstein algebra if and only if $\dim_\mathbb{K} A_d = 1$ and the restrictions of the multiplication in $A$ to complementary degrees $A_k \times A_{d-k} \to A_d$ is a perfect pairing, see [HW] Theorem...
For standard graded Artinian Gorenstein algebras, the Hilbert vector is symmetric: $a_i = a_{d-i}$.

We say that a standard graded Artinian Gorenstein algebra $A$ has the *Strong Lefschetz Property*, or simply $A$ has the SLP, if there exists a linear form $L \in A_1$ such that

\begin{itemize}
  \item $L^{d-2i} : A_i \to A_{d-i}$
\end{itemize}

is an isomorphism for every $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$. In this case $L$ is called a *strong Lefschetz element*.

The following is the model for standard graded Artinian Gorenstein algebras. Let $R = \mathbb{K}[x_0, \ldots, x_N]$ be the polynomial ring in $N+1$ indeterminates and $Q = \mathbb{K}[X_0, \ldots, X_N]$ the associated ring of differential operators. Using the classical identification $X_i := \frac{\partial}{\partial x_i}$, the ring $R$ has a natural structure of $Q$-module. In fact, differentiation induces a natural action $Q \times R \to R$, given by $(\alpha, f) \mapsto \alpha(f)$. Let $f \in R_d = \mathbb{K}[x_0, \ldots, x_N]_d$ be a homogeneous polynomial of degree $\deg(f) = d \geq 1$. We define the *annihilator ideal* by

$$\text{Ann}_f = \{ \alpha \in Q : \alpha(f) = 0 \} \subset Q.$$ 

The homogeneous ideal $\text{Ann}_f$ of $Q$ is also called the Macaulay dual of $f$. We define

$$A_f = \frac{Q}{\text{Ann}_f}.$$ 

One can verify that $A_f$ is a standard graded Artinian Gorenstein $\mathbb{K}$-algebra of socle degree $d$, see [MW, Section 1.2]. We assume, without loss of generality, that $(\text{Ann}_f)_1 = 0$. This is equivalent to say that the partial derivatives of $f$ are linearly independent, which means that $X = V(f)$ is not a cone.

By the theory of inverse systems, we get the following characterization of standard graded Artinian Gorenstein $\mathbb{K}$-algebras. It is also called Macaulay-Matlis duality. For a more general discussion of Macaulay-Matlis duality see [HW, Section 2.2], [IK, Section 1.1], [BS, Chapter 10] and [E, Chapter 21].

**Theorem 3.1. (Double annihilator theorem of Macaulay)**

Let $I$ be an ideal of $Q$ such that $Q/I$ is a standard graded Artinian $\mathbb{K}$-algebra of socle degree $d$. Then $Q/I$ is Gorenstein if and only if there exists $f \in R_d$ such that $I = \text{Ann}_f$.

Let $f \in R_d$ be a homogeneous polynomial and $A_f = \frac{Q}{\text{Ann}_f} = \bigoplus_{i=0}^d A_i$ the standard graded Artinian Gorenstein algebra associated to $f$.

Let $\{\alpha_1, \ldots, \alpha_s\}$ be an ordered $\mathbb{K}$-basis of $A_k$, with $k \leq \frac{d}{2}$. The $k$-th *Hessian* of $f$ is the matrix

$$\text{Hess}_f^k = [\alpha_i(\alpha_j(f))]_{1 \leq i, j \leq s}.$$ 

Its determinant will be denoted $\text{hess}_f^k$.

The following theorem yields a connection between Lefschetz properties and higher Hessians.
Theorem 3.2. [MW] Consider \( A_f \), where \( f \in R \) is a homogeneous polynomial. An element \( L = a_0X_0 + \ldots + a_NX_N \in A_1 \) is a strong Lefschetz element of \( A_f \) if and only if \( \text{hess}_f^k(a_0, \ldots, a_N) \neq 0 \) for all \( 0 \leq k \leq \lfloor d/2 \rfloor \).

Next we discuss the SLP for standard graded Artinian Gorenstein algebras of socle degree \( d = 3 \). For such an algebra the Hilbert vector is \( \text{Hilb}(A) = (1, N + 1, N + 1, 1) \).

We denote by \( \text{GOR}(1, N + 1, N + 1, 1) \) the space of standard graded Artinian Gorenstein algebras of socle degree 3. By Theorem 3.2, \( A_f \in \text{GOR}(1, N + 1, N + 1, 1) \) has the SLP if and only if \( \text{hess}_f \neq 0 \).

By Gordan-Noether Theorem, if \( N \leq 3 \), then \( \text{hess}_f = 0 \) if and only if \( X = V(f) \) is a cone (see [GN]). Therefore, every standard graded Artinian Gorenstein algebra of socle degree 3 and codimension \( \leq 4 \) has the SLP.

When \( A \) has codimension 5, Proposition 2.13 yields the following result.

Proposition 3.3. Let \( A \) be a standard graded Artinian Gorenstein \( \mathbb{K} \)-algebra of Hilbert vector \( \text{Hilb}(A) = (1, 5, 5, 1) \). Assume that \( A \) does not satisfy the SLP. Then \( A \) is isomorphic to the following algebra

\[
\mathbb{K}[X_0, X_1, X_2, X_3, X_4] / ((X_0, X_1, X_2)^2, X_0X_4, X_2X_3, X_1X_3 - X_2X_4, X_0X_3 - X_1X_4, (X_3, X_4)^3).
\]

Proof. The algebra is of the form \( A_f = \mathbb{K}[X_0, X_1, X_2, X_3, X_4] / \text{Ann}_f \), for some \( f \in \mathbb{K}[x_0, x_1, x_2, x_3, x_4] \), not a cone. Theorem 3.2 implies that SLP fails if and only if \( \text{hess}_f = 0 \). By Proposition 2.13 we can assume that \( f \) has equation \( f = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2 \). The desired isomorphism can be obtained using this explicit equation. \( \square \)

3.2. Jordan types. Let \( A = \bigoplus_{i=0}^{d} A_i \) be a standard graded Artinian \( \mathbb{K} \)-algebra. For \( l \in A_1 \) consider the map \( \mu_l : A \to A \) given by \( \mu_l(x) = lx \). Since \( l^{d+1} = 0 \), \( \mu_l \) is a nilpotent \( \mathbb{K} \)-linear map. The Jordan decomposition of such a map is given by Jordan blocks with 0 in the diagonal, therefore it induces a partition of \( \dim \mathbb{K} A \) which we denote \( J_{A,l} \). Indeed, the nilpotent linear map \( \mu_l : A \to A \) induces a direct sum decomposition of \( A \) into cyclic \( \mu_l \)-invariant subspaces \( A = \bigoplus_{i=0}^{m} C_i \). The partition \( J_{A,l} \) is given by the length \( k_i = \dim \mathbb{K} C_i \). Without loss of generality we consider the partition in a non-increasing order.

Given a partition \( P = p_1 \oplus \ldots \oplus p_s \) of \( \dim \mathbb{K} A \) with \( p_1 \geq \ldots \geq p_s \), we denote \( P' \) the dual partition obtained from \( P \) exchanging rows and columns in the Ferrer diagram (diagram of dots). If \( P' = p'_1 \oplus \ldots \oplus p'_t \) is another
partition of $\dim K A$ with $p'_1 \geq \ldots \geq p'_{t}$, we will write $P \preceq P'$ in the dominance order if for all $k$ we get

$$p_1 + \ldots + p_k \leq p'_1 + \ldots + p'_k.$$ 

If the partition $P$ has repeated terms, say $f_1, f_2, \ldots, f_r$ with multiplicity $e_1, e_2, \ldots, e_r$ respectively, we write

$$P = f_1^{e_1} \oplus \cdots \oplus f_r^{e_r}.$$ 

Since $K$ is a field of characteristic zero, there is a non empty Zariski open subset of $U \subset A_1$ where $J_{A,l}$ is constant for $l \in U$, we call it the $Jordan type$ of $A$ and we denote it $J_A$.

The following proposition is a special case of [HW, Proposition 3.64]. It shows that SLP can be described by the Jordan type of $A$.

**Proposition 3.4.** Suppose that $A = \bigoplus_{i=0}^{d} A_i$ is a standard graded Artinian $K$-algebra with $A_d \neq 0$. Then $A$ has the SLP, if and only if, $J_A = \Hilb(A) \vee$.

Since the generic AG algebra in $\text{GOR}(1, 5, 5, 1)$ satisfies the SLP, then the generic Jordan type is $(1, 5, 5, 1) \vee = 4^1 \oplus 2^4$.

**Example 3.5.** The algebra $A = Q/\Ann f$, $f = x_0x_3^2 + x_1x_3x_4 + x_2x_3^2$, has Hilbert vector $\Hilb(A) = (1, 5, 5, 1)$ and Jordan type $J_A = 4^1 \oplus 2^3 \oplus 1^2 \prec 4^1 \oplus 2^4$.

The following proposition is a consequence of Proposition 3.3 and Example 3.5.

**Proposition 3.6.** Let $A$ be a standard graded Artinian Gorenstein $K$-algebra of Hilbert vector $\Hilb(A) = (1, N + 1, N + 1, 1)$. If $N \leq 3$, then $A$ has the SLP. If $N = 4$, then the possible Jordan types of $A$ are: either $J_A = 4^1 \oplus 2^4$ if $A$ has the SLP or $J_A = 4^1 \oplus 2^3 \oplus 1^2$ if the SLP fails.

3.3. **The Lefschetz locus in $\text{GOR}(1, 5, 5, 1)$.** The affine scheme $\text{Gor}(T)$ parametrizing AG algebras with Hilbert vector $T$ was described by A. Iarrobino and V. Kanev and a great account of their work can be found in [IK]. In their context, $\text{Gor}(T)$ stands for the affine cone of the projective variety denoted $\text{GOR}(T)$ here. As we have seen, by Macaulay-Matlis duality the scheme $\text{GOR}(1, N + 1, N + 1, 1)$ can be identified with the parameter space of degree 3 homogeneous polynomials $f \in K[x_0, \ldots, x_N]$, up to scalars, such that $A_f$ has Hilbert vector $\Hilb(A_f) = (1, N + 1, N + 1, 1)$. Therefore, we have an identification

$$\text{GOR}(1, N + 1, N + 1, 1) \simeq \mathbb{P}^{\nu(N)} \setminus C_N$$

where $\nu(N) = \binom{N+3}{3} - 1$ and $C_N$ is the parameter space of cubic cones in $\mathbb{P}^N$. In particular

$$\text{GOR}(1, 5, 5, 1) \simeq \mathbb{P}^{34} \setminus C_4.$$
By Theorem 3.2, an AG algebra $A$ of socle degree 3 has the SLP if and only if its dual generator $f$ satisfies $\text{hess}_f \neq 0$. Proposition 3.6 gives a description of their Jordan types. We summarize this discussion in the following proposition.

**Proposition 3.7.** The space $\text{GOR}(1, N+1, N+1, 1)$ can be identified with $\mathbb{P}^\nu(N) \setminus \mathcal{C}_N$, where $\nu = \binom{N+3}{3}$ and $\mathcal{C}_N$ is the space of cubic cones in $\mathbb{P}^N$. For $N \leq 3$ all the algebras in $\text{GOR}(1, N+1, N+1, 1)$ have the SLP. For $N = 4$, the locus in $\text{GOR}(1, 5, 5, 1)$ of algebras satisfying SLP is $\text{GOR}(1, 5, 5, 1) \setminus Y$ where $Y$ can be identified with the locus formed by $f \in \mathbb{P}^\nu(N) \setminus \mathcal{C}_N$ with vanishing Hessian.

We proceed to give a more precise description of $\mathcal{C}_4$ and $Y$, and compute their dimension and degree.

### 4. Parameter spaces

#### 4.1. Parameter space for cubic cones in $\mathbb{P}^4$

In this section we find a parameter space for cubic cones in $\mathbb{P}^4$ and compute its dimension and degree.

Throughout this chapter we denote by $V$ the vector space $\mathbb{K}^5$. A cubic cone in $\mathbb{P}^4$ is determined by a point $x \in \mathbb{P}^4$ and a cubic hypersurface in the $\mathbb{P}^3$ projectivization of the quotient $V/x$. Consider the tautological sequence on $\mathbb{P}^4 = \mathbb{P}(V)$:

$$
0 \to \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathcal{O}_{\mathbb{P}^4} \otimes V \to \mathcal{P} \to 0.
$$

The fiber of $\mathbb{P}(\mathcal{P})$ over $x \in \mathbb{P}^4$ can be identified with the $\mathbb{P}^3 = \mathbb{P}(V/x)$. Therefore $\mathbb{P}(\text{Sym}_3(\mathcal{P}^*))$ parametrizes cubic hypersurfaces lying in each $\mathbb{P}^3$, where $\mathcal{P}^*$ denotes the dual of the vector bundle $\mathcal{P}$. Note that $\mathcal{F} = \text{Sym}_3(\mathcal{P}^*)$ is a subbundle of $\mathcal{O}_{\mathbb{P}^4} \otimes \text{Sym}_3(V^*)$. Thus we have two projections

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{F}) & \xrightarrow{p_1} & \mathbb{P}^4 \\
\downarrow & & \downarrow \text{id} \\
\mathbb{P}(\text{Sym}_3(V^*)) & \xrightarrow{p_2} & \mathbb{P}(\text{Sym}_3(\mathcal{P}^*))
\end{array}
$$

where $p_2$ is generically injective and $\mathcal{C}_4$ is the image of $p_2$. We have the following result.

**Proposition 4.1.** Let $\mathcal{C}_4 \subset \mathbb{P}^{34}$ be the space of cubic cones in $\mathbb{P}^4$. Then the dimension of $\mathcal{C}_4$ is 23; its degree is given by the Segre class $s_4(\mathcal{F})$ and is equal to 1365.

**Proof.** The dimension can be computed by

$$
dim(\mathcal{C}_4) = dim(\mathbb{P}(\mathcal{F})) = 4 + rk(\mathcal{F}) - 1
$$

and $rk(\mathcal{F}) = 20$. 


To compute the degree, write \( H \) for the hyperplane class of \( \mathbb{P}^3 \). We have 
\[
p_2^*H = c_1 \mathcal{O}_F(1) =: h.
\]
We may compute
\[
\deg C_4 = \int_{\mathbb{P}^3} H^{23} \cap C_4 = \int_{\mathbb{P}^3} h^{23} = \int_{\mathbb{P}^4} p_1^*(h^{23}) = \int_{\mathbb{P}^4} s_4(F).
\]
Using Macaulay2 [GS] we find 
\[
s_4(F) \cap \mathbb{P}^4 = 1365 \tag{4.4}
\]
□

4.2. Parameter space for cubics with vanishing Hessian. This section is devoted to the description of the parameter space for cubic hypersurfaces with vanishing Hessian in \( \mathbb{P}^4 = \mathbb{P}(V) \).

Let \( H \subset \mathbb{P}^3 \) be the locus of hypersurfaces with vanishing Hessian. As in Proposition 3.7 we denote 
\[
Y = H \setminus C_4.
\]

Remark 4.2. Let \( K \subset \mathbb{P}^3 \) be the locus formed by \( f \in \mathbb{P}^3 \) such that 
\[
f \in I^2_W,
\]
where \( I^2_W \) is the ideal of some 2-plane \( W \subset \mathbb{P}^4 \). It is an irreducible subvariety of \( \mathbb{P}^3 \), actually we will show below that it is the image by a morphism of a projective bundle over \( G(2, 4) \). We claim that \( \overline{Y} = K \). Note that \( K \subset Y \) by Lemma 2.12. Reciprocally Proposition 2.13 yields \( Y \subset K \).

This concludes the claim.

We shall find a parameter space for \( \overline{Y} \) using the above characterization.

Let \( G = G(2, 4) \) denote the Grassmannian of 2-planes in \( \mathbb{P}^4 \). We have the following tautological sequence on \( G \):
\[
0 \to \mathcal{T} \to \mathcal{O}_G \otimes V \to Q \to 0
\]
where \( \mathcal{T} \) is a subbundle of rank 3 and \( Q \) is a bundle of rank 2.

Consider the multiplication map 
\[
\varphi : \text{Sym}_2 Q^* \otimes V^* \to \mathcal{O}_G \otimes \text{Sym}_3 (V^*).
\]
It defines a map of vector bundles whose image parametrizes the set of pairs \((W, f) \in G \times \text{Sym}_3 (V^*)\) such that \( f \in I^2_W \).

We claim that the kernel of \( \varphi \) is exactly \( \wedge^2 Q^* \otimes Q^* \). Therefore we have an exact sequence
\[
0 \to \wedge^2 Q^* \otimes Q^* \to \text{Sym}_2 Q^* \otimes V^* \to \mathcal{E} \to 0
\]
where \( \mathcal{E} = \text{Im} \varphi \).

Let us prove the claim. Consider the following exact diagram

\[
\begin{array}{cccccc}
\wedge^2 Q^* \otimes Q^* & \longrightarrow & \text{Ker} \varphi & \longrightarrow & \text{Ker} \bar{\varphi} \\
\downarrow m & & \downarrow & & \downarrow \\
\text{Sym}_2 Q^* \otimes Q^* & \longrightarrow & \text{Sym}_2 Q^* \otimes V^* & \longrightarrow & \text{Sym}_2 Q^* \otimes \mathcal{T}^* \\
\downarrow & \downarrow \varphi & \downarrow & \downarrow \bar{\varphi} \\
\text{Sym}_3 Q^* & \longrightarrow & \text{Im} \varphi & \longrightarrow & \text{Im} \bar{\varphi},
\end{array}
\]
To prove the claim it suffices to prove that $\bar{\varphi}$ is injective. As $G$ is a homogeneous variety, we can assume $W = V(x_3, x_4)$. Assume $\bar{f}$ lies in the fiber of $\text{Sym}_2 Q^* \otimes \mathcal{T}^*$ over $W$, then it can be written as

$$\bar{f} = x_3^2 \otimes f_0 + x_3x_4 \otimes f_1 + x_4^2 \otimes f_2$$

where $f_i$ are homogeneous polynomials of degree one, $f_i \in \langle x_0, x_1, x_2 \rangle$. Hence $\bar{\varphi}(\bar{f}) = 0$ means that $\varphi(f)$ lies in the fiber of $\text{Sym}_3 Q^*$, i.e. $x_3^2f_0 + x_3x_4f_1 + x_4^2f_2 \in \text{Sym}_3(x_3, x_4)$, therefore $f_0 = f_1 = f_2 = 0$.

We note that $E$ corresponds to a vector bundle over $G$ whose projectivization coincides with the incidence variety

$$\mathbb{P}(E) = \{(W, f) \in G \times \mathbb{P}^4 : f \in I_W\}.$$ Let us denote by $p_1 : \mathbb{P}(E) \to G$ and $p_2 : \mathbb{P}(E) \to \mathbb{P}^4$ the natural projections. We see that $K$ is the image of $p_2$ and $p_2$ is generically injective. Hence one obtains the following result.

**Proposition 4.3.** The locus $K$ is the birational image of a projective bundle over the Grassmannian $G$. The dimension of $K$ is 18 and its degree is 29960, the degree of the Segre class $s_6(E)$.

**Proof.** The dimension can be computed as

$$\dim(K) = \dim(\mathbb{P}(E)) = \dim G + \text{rk}(E) - 1.$$ Since $\text{rk}(E) = 13$ and $\dim G = 6$ the result follows.

We shall write $H$ for the hyperplane class of $\mathbb{P}^4$. We have $p_2^*H = c_1\mathcal{O}_E(1) =: h$. The degree of $K$ is given by

$$\int_{\mathbb{P}(E)} h^{18} = \int_G p_1^*(h^{18}) = \int_G s_6(E).$$

By sequence (4) we have

$$s(E) = s(\text{Sym}_2 Q^* \otimes V^*)c(\wedge^2 Q^* \otimes Q^*) = s(\text{Sym}_2 Q^*)^5c(\wedge^2 Q^* \otimes Q^*).$$

We can compute these characteristic classes using Macaulay2: $s_6(E) = 29960$ (see the Scripts in [1.4]).

---

4.3. The locus $K \cap C_4$. In this section we describe the intersection $K \cap C_4$. Let us consider again the tautological sequence on $G(2, 4)$:

$$0 \to \mathcal{T} \to \mathcal{O}_G \otimes V \to Q \to 0.$$ Note that $\mathcal{T}$ corresponds to a vector bundle whose projectivization coincides with the incidence variety

$$\mathbb{P}(\mathcal{T}) = \{(W, p) \in G \times \mathbb{P}^4 : p \in W\}.$$ Let $q_1 : G \times \mathbb{P}^4 \to G$ and $q_2 : G \times \mathbb{P}^4 \to \mathbb{P}^4$ be the natural projections and denote by $\pi_1 = q_1|_{\mathbb{P}(\mathcal{T})}$ and $\pi_2 = q_2|_{\mathbb{P}(\mathcal{T})}$ their restrictions to $\mathbb{P}(\mathcal{T})$. 


We will construct a vector bundle \( E_1 \) over \( \mathbb{P}(T) \) with a birational morphism \( \mathbb{P}(E_1) \rightarrow K \cap C_4 \). Consider the tautological sequence on \( \mathbb{P}^4 = \mathbb{P}(V) \):
\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^4} \otimes V \rightarrow \mathcal{P} \rightarrow 0.
\]
Over \( \mathbb{P}(T) \) we have the following multiplication map:
\[
\xi : \pi_1^* \text{Sym}_2 Q^* \otimes \pi_2^* P^* \rightarrow \mathcal{O}_{\mathbb{P}(T)} \otimes \text{Sym}_3 (V^*).
\]
Its kernel is exactly \( \pi_1^*(\wedge^2 Q^* \otimes Q^*) \). This can be proved using the same argument we applied to the multiplication map \( \varphi \) that appears in Section 4.2. Therefore we have an exact sequence
\[
0 \rightarrow \pi_1^*(\wedge^2 Q^* \otimes Q^*) \rightarrow \text{Sym}_2 \pi_1^* Q^* \otimes \pi_2^* P^* \rightarrow E_1 \rightarrow 0
\]
where \( E_1 = \text{Im} \xi \). Note that \( E_1 \) defines a vector bundle whose fiber over a point \((W,p) \in \mathbb{P}(T)\) is equal to the vector space
\[
\{ f \in \text{Sym}_3 (V^*) : f \text{ is a cone with vertex } p \text{ and } f \in I_W \}.
\]
Let us consider the natural projections \( p_1 : \mathbb{P}(E_1) \rightarrow \mathbb{P}(T) \) and \( p_2 : \mathbb{P}(E_1) \rightarrow \mathbb{P}^{34} \). We see that \( p_2 \) is generically injective and its image is \( K \cap C_4 \). Hence one obtains the following result.

**Proposition 4.4.**

1. \( K \cap C_4 \) is a divisor in \( K \).
2. The degree of \( K \cap C_4 \) is 116420. It is determined by
\[
3s_6(E) + (c_1(\text{Sym}_2 Q^*) + c_1(Q))s_5(E) \cap [G]
\]
where \( E \) is the vector bundle of Section 4.2.

**Proof.** The dimension of \( K \cap C_4 \) coincides with
\[
\dim(\mathbb{P}(E_1)) = \dim \mathbb{P}(T) + \text{rk}(E_1) - 1.
\]
Since \( \text{rk}(E_1) = 10 \) and \( \dim \mathbb{P}(T) = 8 \) the result follows. In order to compute the degree, write \( H \) for the hyperplane class of \( \mathbb{P}^{34} \). We have \( p_2^* H = c_1 \mathcal{O}_{E_1}(1) =: h \). The degree of \( K \cap C_4 \) is given by
\[
\int_{\mathbb{P}(E_1)} h^{17} = \int_{\mathbb{P}(T)} p_1^*(h^{17}) = \int_{\mathbb{P}(T)} s_8(E_1).
\]
We claim that the following identity occurs in the Chow ring of \( G \times \mathbb{P}^4 \):
\[
[\mathbb{P}(T)] = c_2(q_2^* \mathcal{O}_{\mathbb{P}^4}(1) \otimes q_1^* Q) \cap [G \times \mathbb{P}^4].
\]
Indeed, exact sequences \( (2) \) and \( (3) \) yield a map \( \theta : q_2^* \mathcal{O}_{\mathbb{P}^4}(-1) \)
\[
q_2^* \mathcal{O}_{\mathbb{P}^4}(-1) \rightarrow \mathcal{O}_{G \times \mathbb{P}^4} \otimes V \rightarrow q_1^* Q
\]
which has \( \mathbb{P}(T) \) as zeros. Then it induces a regular section \( \sigma : G \times \mathbb{P}^4 \rightarrow q_2^* \mathcal{O}_{\mathbb{P}^4}(1) \otimes q_1^* Q \) which has \( \mathbb{P}(T) \) as zeros. This proves identity \( (7) \).
Putting together sequences (4) and (6) we get
\[ s(E_1) = s(\pi_1^*E)c(G). \]
Where \( G = \text{Sym}_2 \pi_1^*Q^* \otimes \pi_5^*O_{\mathbb{P}^4}(1) \). Using this, we deduce:
\[ s_8(E_1) = s_8(\pi_1^*E) + s_7(\pi_1^*E)c_1(G) + s_6(\pi_1^*E)c_2(G) + s_5(\pi_1^*E)c_3(G). \]
We note that \( s_i(\pi_1^*E) = 0 \) for \( i > 6 \) because \( E \) is a vector bundle over \( G \) which has dimension 6. From this and (7) one obtains that
\[ s_8(E_1) \cap [\mathbb{P}^4(\mathcal{T})] \]
coincides with
\[ (s_6(\pi_1^*E)c_2(G) + s_5(\pi_1^*E)c_3(G)) \cap (k^2 + kc_1(Q) + c_2(\pi_1^*Q)) \cap [G \times \mathbb{P}^4] \]
where \( k = c_1(g_2^*O_{\mathbb{P}^4}(1)) \).
In what follows we will omit the pull-back. Computing the Chern classes of a tensor product, we obtain
\[
\begin{align*}
  c_2(G) &= c_2(\text{Sym}_2 Q^*) + 2c_1(\text{Sym}_2 Q^*)k + 3k^2 \\
  c_3(G) &= c_3(\text{Sym}_2 Q^*) + c_2(\text{Sym}_2 Q^*)k + c_1(\text{Sym}_2 Q^*)k^2 + k^3.
\end{align*}
\]
Observe that
\[ c_i(Q)s_j(E) = c_i(\text{Sym}_2 Q^*)s_j(E) = 0, \quad i + j > 6. \]
Using this and (5), we can reduce the computation of \( s_8(E_1) \cap [\mathbb{P}(\mathcal{T})] \) to:
\[ (s_6(E)3k^2 + s_5(E)(c_1(\text{Sym}_2 Q^*)k^2 + k^3)) \cap (k + c_1(Q)) \cap [G \times \mathbb{P}^4]. \]
Finally, since \( k^5 = 0 \) and \( G \) has dimension 6 we have
\[ s_8(E_1) \cap [\mathbb{P}(\mathcal{T})] = 3s_6(E) + (c_1(\text{Sym}_2 Q^*) + c_1(Q))s_5(E) \cap [G]. \]
This number can be computed using Macaulay2/Schubert2, we find 116420. This concludes the proof of proposition. □

Now the Theorem 1.2 of the introduction is consequence of Propositions 3.7, 4.1, 4.3 and 4.4.

4.4. Scripts.

```
loadPackage "Schubert2"
G=flagBundle({1,4})
-- Grassmannian of lines in 5-space
(S,Q)=G.Bundles
-- names the sub and quotient bundles on G
R=dual (Q)
F=symmetricPower(3,R)
--Computes the classes in Proposition 4.1:
integral(segre(4,F))
```

```
loadPackage "Schubert2"
G=flagBundle({3,2})
-- Grassmannian of 3-planes in 5-space
(S,Q)=G.Bundles
R=dual (Q)
```
A=symmetricPower(2,R)
B=A^5
C=exteriorPower(2,R)*R
E=B-C
--Computes the classes in Proposition 4.3:
integral(segre(6,E))

--Computes the classes in Proposition 4.4:
integral(3*segre(6,E)+(chern(1,A)+chern(1,Q))*segre(5,E))

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