On the interpretation of quantum cosmology *

Franz Embacher
Institut für Theoretische Physik
Universität Wien
Boltzmannasse 5
A-1090 Wien

E-mail: fe@pap.univie.ac.at

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Abstract

We formulate a "minimal" interpretational scheme for fairly general (minisuperspace) quantum cosmological models. Admitting as few exact mathematical structure as is reasonably possible at the fundamental level, we apply approximate WKB-techniques locally in minisuperspace in order to make contact with the realm of predictions, and propose how to deal with the problems of mode decomposition and almost-classicality without introducing further principles. In order to emphasize the general nature of approximate local quantum mechanical structures, we modify the standard WKB-expansion method so as to rely on exact congruences of classical paths, rather than a division of variables into classical and quantum.

The only exact mathematical structures our interpretation needs are the space of solutions of the Wheeler-DeWitt equation and the Klein-Gordon type indefinite scalar product. The latter boils down to plus or minus the ordinary

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quantum mechanical scalar product in the local quantum structures. According to our approach all further structures, in particular the concepts encountered in conventional physics, such as observables, time and unitarity, are approximate. Our interpretation coincides to some extent with the standard WKB-oriented view, but the way in which the conventional concepts emerge, and the accuracy at which they are defined at all, are more transparent. Applying our scheme to the Hawking model, we find hints that the no-boundary wave function predicts a cosmic catastrophe with some non-zero probability.

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1 Introduction

The purpose of this paper is to formulate an interpretational scheme for quantum cosmology that relies on a minimal amount of pre-supposed mathematical structure. The fundamental level is provided by the space $IH$ of solutions to the Wheeler-DeWitt equation and the Klein-Gordon-type scalar product, denoted by $Q(\psi_1, \psi_2)$, whose power has probably been underestimated so far. In our formulation, we restrict ourselves to the minisuperspace approximation, but part of our program is of general conceptual nature, so that it might carry over to the full theory. (For a review of the field see e.g. Ref. [1]). As an example for illustration we use the "Hawking model", i.e. a closed Friedmann-Robertson-Walker universe filled with a homogeneous minimally coupled massive scalar field $\phi$. (Containing several basic features that have become paradigms in describing the evolution of the universe — "birth from nothing" by tunnelling, inflation, matter dominated expansion to a maximum size, and recontraction — this model is rich enough to provide a testing ground for conceptual questions and even for observational predictions). However, much of what will be said in this article applies to simpler and to more sophisticated minisuperspace models as well.

We begin, in Section 2, to sketch the basics necessary to talk about the interpretational issue, in particular the Wheeler-DeWitt equation in minisuperspace $M$. Sections 3 – 5 are devoted to two of the prominent interpretations, the WKB- and the Bohm-type approach, respectively. In Section 3, we review how the nature of rapid oscillations in wave functions of the WKB-type gives rise to the structure of usual physics, but only at an approximate level and locally in $M$. We put some emphasis on the question how conventional quantum physics (i.e. an approximate unitary Schrödinger-type evolution) may be reconstructed. In the standard WKB-expansion approach one divides the variables into classical and quantum ones by means of a limit of the type $\ell_P \to 0$. In order to obtain a general notation of approximate local quantum structures (Hilbert space and unitary evolution), it is convenient to have a modified WKB-expansion at hand, based on geometric quantities in $M$. The "(quasi)classical background", provided by the evolution of the classical ("macroscopic") variables as considered in the standard WKB-expansion is replaced by a congruence of exact classical trajectories. Section 3 contains an outline of this approach, whereas the details are described at the end of the article, in Section 12 (which serves as an appendix). Section 4 deals with the possibility that a given wave function, when written down within the local quantum framework
of some WKB-domain, may exhibit catastrophic exponential behavior that prevents interpretation. We introduce the distinction between true and false local quantum mechanics. Section 5 comments on the Bohm-interpretation.

Our conceptual point of view in this article will be that these WKB-techniques are the only means to make contact between solutions of the Wheeler-DeWitt equation and physical observations. In Section 6, we begin to present the idea for an alternative conceptual scheme which is not new from the point of view of mathematical structures involved, but which seems to have attained few attention so far, at least as far as minisuperspace quantum cosmology is concerned. The scalar product \( Q(\psi_1, \psi_2) \) between solutions of the Wheeler-DeWitt equation is proposed to be the only pre-supposed exact mathematical structure on \( \mathcal{H} \). All predictions must be expressed in terms of such numbers. Conventional physics is approximately identified by recognizing its conceptual structures as "hidden" in the fundamental framework defined by \( (\mathcal{H}, Q) \). In the WKB-approximation, \( Q \) boils down to plus or minus the local quantum mechanics scalar product. We try as radically as possible to reject any further fundamental structure or principle, and denote this program "minimal" interpretation. In using words like "approximate" and "local", we are well aware of the fact that there is not much mathematical rigor in these. Regrettably, to some extent this is unavoidable: We talk about the approximate identification of structures from some exact underlying framework. Determining the degree of accuracy to which this is possible will provide technical difficulties by its own — most of which we are not able to answer in detail — and will presumably depend on the particular model.

The most severe difficulty in our program is provided by the hyperbolic nature of the Wheeler-DeWitt equation or, likewise, by the indefiniteness of the scalar product \( Q \). In simple minisuperspace models, it is related to the fact that the universe may contract or expand. This is the decomposition problem. In Section 7 we sketch how it can be dealt with on account of the "minimal" interpretation. Provided certain assumptions on the model hold, the decomposition of wave functions into "incoming" and "outgoing" modes can be defined approximately and locally in certain regions of minisuperspace by means of the same WKB-techniques that serve for identifying observable quantities and local quantum structures. A deep conceptual problem arises by the fact that the universe at sufficiently large scales is experienced as a classical background, and that we need some structure of this type to state clear predictions. Thus, the possible breakdown of physics as the possibility to describe the universe in terms of quantitative predictions is related with the issue of clas-
sicality. Motivated by the "minimality" of our program, we formulate a condition which ensures that some almost-classical "macroscopic history" of the universe can be experienced. The appearance of a false local quantum mechanical situation is a hint for an "interruption" of such a history. The observational details of the split into classical and quantum is addressed to a decoherence mechanism acting locally in $\mathcal{M}$.

Having discussed the issues of the identification of observables, decomposition and almost-classicality, we are ready in Section 8 to assign approximate probabilities for alternative outcomes of observations, with respect to local WKB-domains. The space $\mathcal{IH}$ serves as collection of "reference" states with respect to which a given wave function of the universe can be interpreted. As compared to the standard WKB-interpretation, our scheme implies automatically the correct book-keeping in normalization issues as soon as several WKB-branches are involved, and includes a conceptually clearer treatment of quantum gravitational corrections and long-range unitarity-violating effects. The philosophy in treating the latter is essentially to WKB-analyze a given wave function in various adjacent WKB-domains (rather than to evolve them by means of a local Schrödinger equation). It is expected that this accounts for all quantum gravitational effects contained in some the model, to the degree of accuracy at which observational situations are defined at all.

Section 9 is devoted to three comments. Concerning the relation of our interpretation to the one referring to the conserved (Klein-Gordon) current of the Wheeler-DeWitt equation, we recall the conceptual differences and show that the latter gives in fact rise to Bohm’s interpretation of ordinary quantum mechanics in the local WKB-branches. As a second point we observe that, taking our proposal over to these branches, the slightly unusual feature arises that amplitudes should be expressed as scalar products between solutions of the time-dependent Schrödinger equation. However, due to the first order nature of the latter, this is equivalent to the usual formalism. Lastly, we outline how our framework could possibly be modified to yield a third-quantized theory.

In order to illustrate some of the features of our scheme, we look a bit closer to the Hawking model. The typical classical solutions in this model undergo an inflationary phase, followed by a matter dominated era in which the scalar field oscillates, until the scale factor reaches a maximum and the universe begins to recollapse. The matter energy $E$ is approximately conserved after inflation. In Section 10 we make explicit how approximate wave packet solutions of the Wheeler-DeWitt equation in the inflationary domain may be constructed (thereby working out an example of a
local quantum structure) and how predictions formulated in terms of these relate to the standard WKB-interpretation. Also, we discuss how the limited range of the WKB-approximation spoils the notation of a probability density on the set of classical trajectories, and just allows for a probability interpretation of sufficiently broad tubes of paths.

In Section 11, we present a set of wave functions in the Hawking model describing families of expanding and contracting universes to given values \( E \) of the matter energy in the post-inflationary domain. This amounts to an interpretation of the wave function of the universe in terms of observations of the energy. An example is provided by expanding the Hartle-Hawking (no boundary) wave function in terms of these states. Also, we display a further example of a local quantum structure. Moreover, the structure of the approximate energy eigenstates as a convenient basis for \( H \) gives some insight into the question what happens when the universe attains its maximum size. We encounter some hints that the no-boundary wave function predicts a cosmic catastrophe with some non-zero probability, but that no logical inconsistency is implied. A condition for a wave function to avoid such an event may be formulated in terms of the energy states.

Lastly, in Section 12 (which may be regarded as an appendix) we present our modified WKB-expansion to extract ordinary quantum mechanics from the Wheeler-DeWitt equation. It is based on the geometric structure of \( \mathcal{M} \) and the equal-time hypersurfaces of a congruence of classical trajectories. Although the notation of a quasiclassical background history is contained therein only implicitly and the relation to observation is less transparent, it provides a more general analysis of rapidly oscillating wave functions than the standard approach and is applicable in larger domains than the latter.

2 Minisuperspace quantum cosmology

We will consider the approach to quantum cosmology that is based on the Wheeler-DeWitt equation \( \mathcal{H}\psi = 0 \) for the wave function of the universe. Formally, this equation \( \mathcal{H}\psi = 0 \) is obtained as the quantum version of the Hamiltonian constraint \( \mathcal{H} = 0 \) of the underlying field theory, i.e. general relativity with some fundamental matter field sector. Despite the mathematical subtleties related to the transition from the classical constraint to a well-defined wave operator (the appearance of infinities and the operator ordering problem), one may state that the overall nature of the
Wheeler-DeWitt equation is that of a hyperbolic (Klein-Gordon-type) functional differential equation. The background on which it is defined is the space of all metrics $g_{ij}(x)$ and matter field configurations $\phi(x)$ on a given three-manifold $\Sigma$ which fixes the spatial topology. (Therefore, by the way, this approach does not naturally accomodate for the possibility of topology fluctuations. It relies on one of the assumptions that such fluctuations do not exist at all, are negligible for quantum cosmology, or must be taken into account by some generalization — see, e.g. Ref. [4] for a path integral approach). This space of configurations on $\Sigma$ is endowed with the DeWitt metric, which is defined by the quadratic form in which the momenta appear in the Hamiltonian constraint, and hence in which the second (functional) derivatives appear in the Wheeler-DeWitt equation. The DeWitt metric is ultralocal and at each point $x$ of $\Sigma$ has Lorentzian-type signature $(-,+,+,\ldots)$ [3].

The wave function is a functional $\psi[g_{ij}(x),\phi(x)]$. The classical momentum constraint $H_i = 0$ translates into the condition that $\psi$ is invariant under diffeomorphisms on $\Sigma$. This gives rise to factor out the diffeomorphism group, by which the space of configurations turns into the "superspace" (see Ref. [5] for a recent discussion of the geometry of this construction). Moreover, $\psi$ does not depend on an extra "time" (or evolution) parameter, as opposed to the Schrödinger wave function of ordinary quantum mechanics. Hence, there is no obvious Hilbert space structure, and no obvious notation of probabilities. The structure as well as the contents of all predictions of observations in the universe, including conventional quantum physics and the experience of an almost-classical evolution at sufficiently large scales, must be extracted from a functional depending only on three-geometries and matter field configurations. This raises the problem of how to interpret a given wave function in terms of observations.

This problem is retained at a conceptual level if one truncates the infinite number of degrees of freedom down to a finite "minisuperspace" (see e.g. Ref. [4]). Thereby one usually chooses a more or less symmetric ansatz for the four-metric and the matter fields in terms of a collection of parameters, typically $(N, y^0, y^1, \ldots y^{n-1}) \equiv (N, y^n)$, where $N$ represents the lapse function that accounts for the possibility of redefining the "time" parameter $t$ labelling the slices of the foliation of the four-manifold in a $3 + 1$-ADM-split [6] [7]. The usual type of the truncation for the four-metric is

$$ds^2 = -N(t)^2 dt^2 + g_{ij}(x, y(t))dx^i dx^j, \quad (2.1)$$

where $x \equiv (x^i)$, and the $g_{ij}$ are fixed functions. (To be even a bit more general, one may replace $g_{00}$ by $-b(x, y(t))^2 N(t)^2 dt^2$, with $b$ a fixed function. We will admit
this possibility, but refer to $N = 1$ as the ”proper time gauge” for simplicity). An analogous ansatz is imposed for the matter fields, symbolically $\phi \equiv \phi(x, y(t))$. We shall denote by $\mathcal{M}$ the parameter space labelled by the $y^\alpha$. (It is called minisuperspace, and is usually a manifold). If different $y \in \mathcal{M}$ describe non-diffeomorphic configurations $(g_{ij}, \phi)$, the omission of the momentum constraint (by putting $g_{0i} = 0$ from the outset) is heuristically justified. Upon inserting the ansatz into the classical Einstein-Hilbert action (with appropriate boundary term), and integrating over the surfaces $\Sigma$ of constant $t$ (which kills the appearance of $x$), one obtains a Lagrangian (and from it a Hamiltonian) formulation of a reduced model for a finite number of degrees of freedom. This serves as a starting point for (canonical) quantization à la Dirac [8]. The variable $N$ enters the Lagrangian only algebraically, hence defines the reduced analogue of the Hamiltonian constraint. The latter is in general of the form

$$H \equiv \frac{1}{2} \left( \gamma^{\alpha\beta} p_\alpha p_\beta + U_{cl}(y) \right) = 0,$$

(2.2)

where $U_{cl}$ is the (real valued) potential term originating from the curvature of $g_{ij}$ on $\Sigma$ and the matter couplings. Up to operator ordering ambiguities, one may translate this constraint into a minisuperspace Wheeler-DeWitt equation. The latter is now a second order partial differential equation for the wave function $\psi(y^0, y^1, \ldots, y^{n-1}) \equiv \psi(y)$, and thus provides a mathematically well-defined problem. The quadratic form $\gamma_{\alpha\beta}$ (the inverse of $\gamma^{\alpha\beta}$) defines the reduced version of the DeWitt metric, and has Lorentzian-type signature $(-, +, +, \ldots)$. This is true at least for the case of all interesting minisuperspace models. (With respect to this metric, we will talk about ”timelike” and ”spacelike” directions in $\mathcal{M}$ later on). In general, one may choose the operator ordering such that the minisuperspace Wheeler-DeWitt equation reads

$$(-\nabla_\alpha \nabla^\alpha + U(y)) \psi(y) = 0,$$

(2.3)

where $\nabla_\alpha$ is the covariant derivative with respect to the DeWitt metric. One may include the Ricci-scalar of the latter into the potential, thus slightly modifying $U_{cl}$ from (2.2), in order to make the formalism invariant under $y$-dependent rescalings of the lapse (conformal transformations of the DeWitt metric [1]).

Although we admit rather general models of this type, let us impose two requirements. First of all, $\mathcal{M}$ shall admit a foliation by ”spacelike” hypersurfaces. This implies that $\mathcal{M}$ is orientable. We suspect that this restriction still admits models that describe a reasonable range of three-geometries of topology $S^3$ or $\mathbb{R}^3$. It is not entirely clear to what extent the orientability of $\mathcal{M}$ is just a technical issue. We
leave it open whether it may be weakened or omitted in a more general scheme. As a second requirement we assume that the domain in \( \mathcal{M} \) defined by all \( y \) such that \( U(y) > 0 \) is the relevant one for drawing quantitative predictions. This is not a very precise formulation, but its motivation is essentially to exclude situations that are too far from the structure of general relativity and realistic matter fields. Usually, the domain \( U > 0 \) (which need not be connected) is the one in which the major oscillations in the wave functions take place. This seems to be true at least for models whose geometry is close to Friedmann-Robertson-Walker, and whose matter contents is of rather realistic type (in other words: models ”close” to the Hawking model). Regions with \( U < 0 \) are typically entered by a classical solution only for short periods of (proper) time. These \( U < 0 \) domains will be important for the problem of almost-classicality (the issue of returning universes), but not for the quantitative prediction of particular measurements. Also, one may encounter situations — as e.g. in the Hawking model; see Section 11 — in which the \( U < 0 \) domains may simply be ignored for most purposes.

Classically, such a truncation to minisuperspace may be consistent with the full dynamics. By this we mean that a solution \((N(t), y^\alpha(t))\) of the reduced classical system, when inserted into the truncation ansatz for the metric and the matter field, gives rise to a solution of the full classical Einstein plus matter field equations (possibly with \( U_{cl} \) replaced by \( U \)). This is the case for all reductions to homogeneous three-metrics and matter fields, and in particular for the most popular models of the Friedmann-Robertson-Walker type. In principle, we can think about more sophisticated models, the \( y^\alpha \) representing a good selection of the information contained in the (infinite) set of original degrees of freedom. Quantum mechanically, one looses information about the fluctuations off the minisuperspace configurations. Such fluctuations can be re-introduced to first order by means of a perturbative analysis around minisuperspace. Despite their approximate nature, various minisuperspace models have been proposed and discussed. On the one hand, they may serve for a preliminary orientation about the quantitative nature of predictions. On the other hand, they are particularly useful in attacking the conceptual problems of quantum cosmology. From now on we would like to restrict ourselves to the minisuperspace approaches.

In order to have an example at hand, we write down the Wheeler-DeWitt equation of the Hawking model. The minisuperspace variables are the scale factor \( a \) of a spatially isotropic closed universe and the spatially homogeneous value \( \phi \) of a minimally coupled scalar field with mass \( m \). The classical Hamiltonian constraint,
when expressed in terms of canonically conjugate variables \((a, p_a; \phi, p_\phi)\) is given by

\[
\mathcal{H} \equiv \frac{1}{2} \left( \frac{p_a^2}{a} + a \right) + \frac{1}{2} \left( \frac{p_\phi^2}{a^3} + m^2 a^3 \phi^2 \right) = 0.
\]  

(2.4)

The DeWitt metric \(ds^2_{DW} = \gamma_{\alpha\beta} dy^\alpha dy^\beta\) is thus

\[
ds^2_{DW} = a(-da^2 + a^2 d\phi^2),
\]

(2.5)

and the momenta are related to the derivatives with respect to the classical evolution parameter \(t\) by

\[
p_a = -\frac{a}{N} \frac{da}{dt} \quad \quad p_\phi = \frac{a^3}{N} \frac{d\phi}{dt}.
\]

(2.6)

Up to an overall numerical constant (see Ref. [13] for units), the classical space-time metric is

\[
ds^2 = -N(t)^2 dt^2 + a(t)^2 d\sigma_3^2
\]

with \(d\sigma_3^2\) the metric on the round unit three-sphere. In the operator ordering corresponding to (2.3) — the Ricci-scalar of \(ds^2_{DW}\) being zero and thus not producing any further ambiguity —, the Wheeler-DeWitt equation (after multiplication by \(a\)) reads [13] [14]

\[
\left( \partial_{aa} + \frac{1}{a} \partial_a - \frac{1}{a^2} \partial_{\phi\phi} + m^2 a^4 \phi^2 - a^2 \right) \psi(a, \phi) = 0.
\]

(2.8)

The potential term as appearing in (2.3) is \(U(a, \phi) = a(m^2 a^2 \phi^2 - 1)\). The minisuperspace manifold \(\mathcal{M}\) is given by all values \((a, \phi)\) such that \(a > 0\).

There have been made a number of proposals how physical information should be extracted from a given wave function \(\psi\), i.e. a solution of the Wheeler-DeWitt equation (in both full superspace and minisuperspace). There seem to be two main ideas along which the problem of interpretation may be attacked, both relying on mathematical structures related to the Wheeler-DeWitt equation. The first one tries to interpret \(|\psi|^2\) as a generalized probability density, whereas the second one uses the conserved (Klein-Gordon-type) current — to be introduced in equation (2.10) below — for extracting predictions. Let us first outline the former.

In terms of the minisuperspace version (2.3) of the Wheeler-DeWitt equation, one may start with the observation that there is a preferred measure \(d\mu = dp^3 y \sqrt{-\gamma}\)
on minisuperspace. (For the Hawking model, this is $a^2 d\alpha d\phi$). The proposal is that with any region $G$ of minisuperspace one associates the positive quantity

$$P_G = \int_G d\mu \psi^* \psi.$$  \hfill (2.9)

Predictions about observations should be stated in terms of these numbers. This interpretation has been advocated, among others \cite{13}, by Hawking and Page \cite{13} \cite{14}. In the words of these authors, $P_G$ is "the probability of finding a 3-metric and matter field configuration in a region" $G$.

The second of the above-mentioned ideas starts from the fact that equation (2.3) admits the conserved (Klein-Gordon type) current \cite{3}

$$j_\alpha = -\frac{i}{2} \left( \psi^* \partial_\alpha \psi - (\partial_\alpha \psi^*) \psi \right) \equiv -\frac{i}{2} \psi^* \leftrightarrow \partial_\alpha \psi.$$  \hfill (2.10)

Using the reality of the potential $U$ one easily checks

$$\nabla_\alpha j^\alpha = 0.$$  \hfill (2.11)

An analogous construction is sometimes possible if the Wheeler-DeWitt equation is defined by a more general operator ordering than (2.3). (In the Hawking model the contravariant components of the current are given by $j_\alpha = -j_\alpha/a$, $j_\phi = j_\phi/a^3$, and the conservation law (2.11) reads $a \partial_\alpha (a j_\alpha) - \partial_\phi j_\phi = 0$). Much like the magnetic field strength in classical electrodynamics (satisfying $\partial_j B^j = 0$), the current (2.10) defines a flow on minisuperspace and an associated measure on the flow lines. The flow lines are those curves $y_{\text{flow}}^\alpha(\tau)$ which are tangential to the current

$$\frac{d}{d\tau} y_{\text{flow}}^\alpha(\tau) = j^\alpha(y_{\text{flow}}(\tau)).$$  \hfill (2.12)

Suppose now that $j \neq 0$ in some domain of $\mathcal{M}$. If $\mathcal{B}$ is a "pencil" of flow lines, and $\Gamma$ a hypersurface intersecting any curve in $\mathcal{B}$ only once, the surface integral

$$P_\mathcal{B} = \int_{\Gamma \cap \mathcal{B}} ds^\alpha j_\alpha$$  \hfill (2.13)

is independent of the choice of $\Gamma$. This follows from integrating equation (2.11) over an appropriate $n$-dimensional domain. If the integrand in (2.13) is non-zero in some domain (which, however, is not always the case) one can manage $P_\mathcal{B}$ to be positive by choosing the orientation of the surface element on $\Gamma$. 

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Having made this observation, one may proceed in several ways in order to use this structure for an interpretational scheme. We will outline two prominent approaches relying on the current $j$. The first one is the realm of WKB-techniques (Section 3), the second one is a generalization of the Bohm interpretation of quantum mechanics (Section 5). The former provides a powerful tool in making contact with the structure of conventional quantum physics, and in order to account for a subtlety of interpretation that is often overlooked, some adaption to domains $U < 0$ of $\mathcal{M}$ (which sometimes play a ”Euclidean” role) will be necessary (Section 4).

## 3 WKB-interpretation and local quantum structure

Consider a wave function whose oscillations in some domain of $\mathcal{M}$ are mainly of the WKB-type, i.e. which is of the form

$$\psi(y) = A(y) e^{iS(y)},$$

(3.1)

where $S$ satisfies the Hamilton-Jacobi equation for the classical action

$$-(\nabla^\alpha S)\nabla_\alpha S = U$$

(3.2)

and $A$ is a slowly changing prefactor. (In general, this means $|\nabla A| \ll |A\nabla S|$ with $\nabla$ denoting a derivative off the hypersurfaces $S = \text{const}$). The quantity $S$ serves as the classical action of a congruence ($\equiv$ non-intersecting $(n - 1)$-parameter family) of solutions $(N(t), y^\alpha(t))$ to the classical equations of motion [1]. These trajectories (paths) are defined by

$$p_\alpha \equiv \gamma_{\alpha\beta} \frac{1}{N} \frac{dy^\beta(t)}{dt} = \partial_\alpha S,$$

(3.3)

where $N$ is the lapse whose choice fixes the ”time gauge”, i.e. the evolution coordinate $t$ in space-time. (If the operator ordering in (2.3) is such that the Ricci-scalar of $ds^2_{\text{DW}}$ is present in $U$, it is reasonable to consider it part of the classical equations of motion). In the Hawking model, the first equality in (3.3) is just given by (2.6).

In the approximation of slowly varying $A$, the current is approximately proportional to the gradient of $S$,

$$j_\alpha \approx |A|^2 \partial_\alpha S.$$  

(3.4)
Hence, the classical trajectories $y(t)$ approximately coincide — up to a redefinition of the time parameter $t$ — with the flow lines $y_{\text{flow}}(\tau)$, and the construction (2.13) provides (locally in $\mathcal{M}$) a positive definite measure on the set of these curves. It can be used to interprete the wave function in terms of relative probabilities for pencils of classical paths. This interpretational approach has been advocated, among others, by Vilenkin [16] (see also Refs. [3][13][14]), and it plays an important role in the formulation of his outgoing mode proposal (tunnelling proposal) [17][19]. It is an approximate concept that relies on the WKB-form of $\psi$. When the gradient of $S$ becomes small or when neighbouring classical trajectories intersect each other, it will break down. (See e.g. Ref. [20] for a criticism of this philosophy altogether).

Hence, the applicability of techniques related to the WKB-form of $\psi$ is bound to particular domains of $\mathcal{M}$. Sometimes the description of the wave function in terms of classical trajectories and their action may be extended into a region where $\psi$ is still oscillating but (3.1) is no longer a good approximation and caustics appear (see e.g. Refs. [21][22]). In such regions the wave function usually becomes a superposition of several WKB-branches of the type (3.1), the physical reason behind such phenomena typically being that the number of variables in which $\psi$ rapidly oscillates (the "classical" variables, see below) depends on the location in $\mathcal{M}$. In contrast, domains in which $\psi$ behaves exponentially rather than oscillating are usually thought of as a hint for tunnelling phenomena which do not admit direct observation. Although a Euclidean version of the WKB-approach [1][3][4] may formally be applied, the direct relationship to predictions as in (2.13) is lost, and no classical paths representing observable universes can be associated with $\psi$ within these domains.

Pushing the analysis further, one finds by means of a WKB-expansion that the prefactor $A$ in (3.1) contains a Schrödinger-type wave function describing quantum fluctuations between the paths in a WKB-branch. We will first sketch the common approach to see this, and afterwards impose a modification in philosophy that fits our purposes better.

Usually, by treating the Planck length $\ell_P$ as the small parameter (or, equivalently, the Planck mass $m_P$ as the large parameter) in a WKB- (semiclassical) expansion, one uncovers an effective Schrödinger equation whose evolution parameter $t$ is associated with a family of classical background space-times [3][12][16], [23][27]. This is not an entirely unique procedure because some of the matter coupling constants may depend on the Planck mass, or certain field values may be very large. For example, in the Hawking model one has $m/m_P \approx 10^{-6} - 10^{-5}$ [2][28], and if one likes this ratio to remain intact, the matter sector must be treated on an equal footing with
gravitation in domains where $|\phi|$ is sufficiently large, or mimicked by an effective cosmological constant $\Lambda_{\text{eff}} \approx m|\phi|$. (In this case $\phi$ acts as an inflaton and drives the universe to expand exponentially). For pure gravity, the limits $\ell_P \to 0$ and $\hbar \to 0$ are equivalent \[26\]. The variables $y$ are divided (either by hand or by the way how $\ell_P$ appears in the Wheeler-DeWitt equation) into ”classical” and ”quantum” (”heavy” and ”light”) ones ($y_{cl}, y_q$). The former usually correspond to the large-scale geometric degrees of freedom, possibly in combination with some part of the matter sector, and provide a congruence of classical ”background” configurations in the space labelled by the $y_{cl}$ alone (cf. however Ref. \[16\], where it is emphasized that this division of variables into two groups may depend on the domain in $M$, e.g. when an inflaton $\phi$ is present. Also, one would not expect the graviton degrees of freedom to belong to the classical variables \[26\]). The background action $S_0 \equiv S_0(y_{cl})$ depends only on the classical variables and satisfies the Hamilton-Jacobi equation (3.2) to the first relevant order (which in practice is defined by a ”classical” part of the potential $U_0(y_{cl})$). After a suitable choice of the background lapse $N(y_{cl})$, these configurations are endowed with an evolution parameter $t$, and $\partial_t$ denotes the derivative along the background paths in the $y_{cl}$-manifold. This all is mathematically somewhat analogous to the non-relativistic ($c \to \infty$) limit of the flat Klein-Gordon equation (i.e. $\gamma_{\alpha\beta} = \eta_{\alpha\beta} = \text{diag}(-1,1,1,1)$ and $U = m^2$) by the ansatz (making $c$ and $\hbar$ explicit) $\psi(t, \vec{x}) = A(t, \vec{x}) \exp(-imc^2 t/\hbar)$, where formally $y_{cl} \equiv t$ and $y_q \equiv \vec{x}$. The technically most important feature in both cases is the idea that a contribution of the type $\partial_t A$ is neglected to lowest order in the small parameter, so that the remaining equation is of first order in $\partial_t$. We will not go into the details of these methods but just note that the prefactor $A$ and the phase $S$ from \[3.1\] are usually redefined by

$$\psi(y) = \tilde{\chi}(y_{cl}, y_q) D_0(y_{cl}) e^{i S_0(y_{cl})}$$

(3.5)

for some appropriate choice of the (real valued) factor $D_0$ which does not concern us here. Both functions $S_0$ and $D_0$ characterize the (”macroscopic” or quasiclassical) WKB-branch, whereas $\tilde{\chi}$ specifies a particular (quantum) state. To first order in the expansion (which is actually a Born-Oppenheimer type approximation), the Wheeler-DeWitt equation reduces to an effective Schrödinger equation \[3\] \[12\] \[16\], \[23\] \[27\] of the type

$$i \partial_t \tilde{\chi} = \tilde{H}_{\text{eff}} \tilde{\chi},$$

(3.6)

where $\tilde{\chi}$ may be associated with a Hilbert space structure (with respect to which it is symbolically denoted as $|\tilde{\chi}_t\rangle$), such that $\tilde{H}_{\text{eff}}$ is hermitan. (Its form depends
very strongly on the particular appearance of the small parameter — usually it is just the matter Hamiltonian —, and the hermiticity is achieved by a suitable choice of \( D_0 \). Hence, the semiclassical methods define an approximate unitary evolution and allow for a probability interpretation. The function \( \tilde{\chi} \) can be chosen freely, as long as it satisfies (3.6), is sufficiently concentrated inside the WKB-domain but does not undergo too rapid variations that would spoil the approximation. (3.6) is a minisuperspace version of the Tomonaga-Schwinger equation \([23][24][26]\), and thus it also reproduces the truncated form of relativistic quantum "field" theory. (We will not go into the details of recovering the full quantum field theory from the full superspace Wheeler-DeWitt equation). Thus, whenever we will talk about the extraction of "quantum physics" from quantum cosmology in this article, we automatically mean quantum field theory — in its minisuperspace reduction — as well.

Whereas the semiclassical expansion assumes that conventional quantum physics essentially relies on the smallness of \( \ell_P \) it is sometimes appropriate to have a modified and geometrically oriented notation of WKB-states. Let us write some wave function according to (3.1) as
\[
\psi(y) = \chi(y)D(y)e^{iS(y)},
\]
where \( S \) is the action (3.2) of a congruence of exact classical trajectories — as opposed to \( S_0 \) —, and define an approximation by assuming that \( S \) is rapidly varying as compared to \( D \) and \( \chi \). No \( \text{\`a priori} \) division into classical and quantum variables is performed. Maybe this is in spirit a bit closer to the early study of DeWitt [3] than to the modern WKB-approaches. The advantage of putting things like this is that all types of "paths" one may encounter (i.e. the exact classical trajectories and the flow lines of \( j \)) are trajectories in \( \mathcal{M} \) — as opposed to \( y_{cl}(t) \) in the standard approaches. In practice, the split into classical and quantum variables will emerge to some accuracy rather than being assumed. The classical trajectories described by \( S \) serve as the background which provides the effective quantum evolution parameter \( t \). However, the reconstruction of the experienced local Hilbert space structure (3.6) does not seem to be trivial. Hence, although the use of the form (3.7) seems to be more fundamental than (3.5), the relation to observation is less transparent. Logically, the modified formalism might turn out as an intermediate step between the full Wheeler-DeWitt equation and the standard semiclassical version of local quantum mechanics, although closer to the latter.

In Section 12, we will outline this modified approach in some detail, and so we are rather brief here. By appropriately defining the (real valued) factor \( D \) (which,
together with $S$, completely defines the WKB-branch, including all representation and normalization issues) and setting $\partial_t = N(\nabla^\alpha S)\nabla_\alpha$ — the lapse being specified as a function $N(y)$ on $\mathcal{M}$ —, one again encounters an effective Schrödinger equation

$$i \partial_t \chi = H_{\text{eff}} \chi$$

(3.8)

to the degree of accuracy defined by the nature of oscillations of $\exp(iS)$. In Section 12, the effective Hamiltonian is expressed in terms of geometric quantities on the equal-time hypersurfaces in $\mathcal{M}$, the result being equation (12.32). One may associate these states with an approximate Hilbert space structure (with respect to which the states are denoted as $|\chi_t\rangle$), just as in the semiclassical expansion (see equations (12.6) and (12.20) below). Also, we show in Section 12 that this approximation is but the first order of a modified WKB-expansion.

As before, the function $\chi$ can be chosen freely, as long as it satisfies (3.8), is sufficiently concentrated inside the WKB-domain but does not undergo too rapid variations in directions transverse to the classical trajectories. Large transverse variations give rise to large $\partial_{tt}\chi$ which would spoil the approximation. This also limits the width of wave packets from below (to be $\gg |U|^{-1/2}$ in terms of the distance defined by the DeWitt metric, as a rough estimate will yield in Section 12). In Section 10 we will discuss the consequences of this fact for the issue of interpretation. A heuristic condition for (3.8) to be a good approximation to the Wheeler-DeWitt equation (2.3) is that $H_{\text{eff}}$ (e.g. in the sense of the expectation value $\varepsilon = \langle \chi | H_{\text{eff}} | \chi \rangle$) is ”smaller” than $NU$, as will become clearer in Section 12.

In many practical situations, this approach should not be too different from the formulation (3.3). Heuristically, one would expect the relation between $\tilde{\chi}$ and $\chi$ to be essentially a ”picture changing” transformation that interpolates between $D_0 \exp(iS_0)$ and $D \exp(iS)$. However, since in the modified formalism no division of variables in two groups has been assumed, the classical character of some of them might emerge only after an average over (superposition of) neighbouring exact WKB-branches and the neglect of small terms. Also note that in general (3.8) may be expected to come closer to solutions of the full Wheeler-DeWitt equations than (3.0), i.e. $H_{\text{eff}}$ to be ”smaller” than $\tilde{H}_{\text{eff}}$. From the point of view of our formulation, a very small value of $\ell_P$ will automatically allow for rapid oscillations of $\psi$ in just those variables that have been assumed to be the classical ones in the standard WKB-expansion approach, although there may be wave functions which do not respect the pattern provided by the small parameters.
There is some ambiguity in both approaches. In the standard one, an $\ell_P$-dependent redefinition of the variables may lead to another, macroscopically equivalent (quasi)classical background. In the modified one, two slightly different WKB-branches may be macroscopically indistinguishable, and lead effectively to the same local quantum structure. Hence, to be precise, there is no unique notion of ”trajectories $y(t)$ or WKB-branches $S$ contributing” to a wave function. In any such statement the trajectory or WKB-branch must be considered as just representing a larger class of possible other ones. The effective equivalence of the according quantum structures restores the uniqueness of the approximation to a reasonable extent.

This problem of ambiguity of exact WKB-branches to characterize macroscopic situations has a counterpart in the standard semiclassical formalism. As mentioned above, it is conceivable that the ”picture changing” transformation between $\tilde{\chi}$ and $\chi$ may be understood as (or should be combined with) an average over macroscopically equivalent (and possibly also over close but non-equivalent) exact WKB-branches. Due to such an average (together with the omission of small terms), one ”looses” variables (note in particular that the semiclassical effective Hamiltonian $\tilde{H}^{\text{eff}}$ contains no derivatives with respect to the classical variables $y_{cl}$) and ends up with a wave function built around the rapidly oscillating part $\exp(iS_0(y_{cl}))$. In passing from the modified to the standard formalism, one thus looses the geometric significance of the local quantum structure with respect to $\mathcal{M}$. This implies a typical disadvantage of the conventional semiclassical approach: The ”background” (expressed as a family of quasiclassical evolutions $y_{cl}(t)$) does not define paths (or even local directions) in $\mathcal{M}$.

Both shortcomings (ambiguity of exact WKB-branches in the modified and semiclassical evolutions destroying geometry based on $\mathcal{M}$ in the standard approach) have the same origin, and both approaches should be regarded as closely related tools for recovering local quantum structures from the Wheeler-DeWitt equation. From the point of view of reconstructing the actual variables and representations of locally observed quantum mechanics, our framework might constitute an intermediate step. However, the remaining steps to recover (3.3), once the local quantum structure (3.8) has been revealed, should work inside (or near) the familiar realm of Hilbert space techniques. Hence, from the mere point of view of extracting a Hilbert space environment from the Wheeler-DeWitt equation, both approaches may even be regarded as equivalent.

One advantage for our present purposes of the modified version is that it is a
bit more global in spirit and can be applied to larger domains in $\mathcal{M}$: As already mentioned, it may happen that during the background evolution some classical variables become quantum and *vice versa* (as e.g. at the end of inflation, where the inflaton field becomes quantum). In the conventional scheme, this amounts to using different collections $y_{cl}$ in two adjacent domains (or, at the end of inflation, switching off an effective cosmological constant adiabatically) whereas in our approach there is a common structure for both, and just the numbers of variables in $\mathcal{H}^{\text{eff}}$ that contribute to rapid and slow oscillations in $\chi$ change. It thus applies to the *intermediate* domains as well, in which the status of some variables is just about to change (see also Ref. [29], p. 245, where the problematic nature of the split into classical and quantum variables was pointed out). Technically, our formalism allows us cover $\mathcal{M}$ by a pattern of overlapping local domains inside which congruences of paths exist and thus give rise to various local quantum structures of the type (3.8) that represent possible WKB-type oscillating behaviour there. The main advantage of this modified WKB-approach in the present context is however its geometric formulation, which enables us to make contact between exact and approximate issues without dividing variables into two groups (as e.g. the derivation of equation (6.5) below). From now on we will include all these and related methods to extract the structure of conventional classical and quantum physics from the Wheeler-DeWitt equation in the phrase ”WKB-techniques”.

Let us add the remark that although in minisuperspace there is only one lapse degree of freedom $N$, the redefinition of the evolution parameter along the trajectories is not a simple issue in the quantum context. In the full theory one has to deal with a lapse function $N(x)$ (and a shift vector $N_i(x)$) on the three-manifold $\Sigma$, and the question whether different coordinate choices yield the same effective four-dimensional space-time physics provides a highly non-trivial problem [29]–[31]. Probably, this is the most significant difference between the full and the minisuperspace framework, and it certainly limits the straightforward application to the former of all ideas we sketch in this article. However, it is also a non-trivial question in minisuperspace to what extent local quantum structures that are constructed from the same WKB-branch but with different time gauges $N(y)$ and different choices of equal-time hypersurfaces in $\mathcal{M}$ are equivalent. We suspect that they are in fact equivalent to a sufficient degree of accuracy if the equal-time hypersurfaces do not differ too drastically from the equal-action hypersurfaces.

When going beyond the accuracy assumed so far (i.e. by looking at even higher orders in a WKB-expansion) one typically encounters a ”corrected Schrödinger equa-
tion” with a non-hermitan part in the Hamiltonian (see e.g. Ref. [26] for the standard approach, and equations (12.43) and (12.44) below for our modified formalism). This implies violation of unitarity and the (conceptual) breakdown of the probabilistic interpretation (”loss of probability”). It is due to the fact that the probability flow defined by \( j \) (which is approximately tangential to the classical paths in the WKB-context) is exactly conserved only along the flow lines \( y_{\text{flow}}(\tau) \), and neither along the classical solutions \( y(t) \) nor along the quasiclassical trajectories \( y_{\text{cl}}(t) \), as appearing in the standard WKB-approach. We will explain in Section 7 how we handle the appearance of unitarity-violating terms in our approach to quantum cosmology. Let us just remark here that we do not consider approaches that redefine the wave function \( \tilde{\chi} \rightarrow \chi' \) by means of prefactors and phases that depend on \( t \) and on expectation values with respect to \( \tilde{\chi} \). In this way one may manage \( \langle \chi'_1 | \chi'_2 \rangle \) not to depend on \( t \) (see e.g. Ref. [32] for a recent discussion) but it destroys the linear structure (i.e. the possibility of superpositions) and the notation of a unitary evolution in a Hilbert space (for which also \( \langle \chi'_{1,1} | \chi'_{1,2} \rangle \) is a reasonable quantity and has to be independent of \( t \)). As a consequence, unitarity in the usual sense is still an approximate concept in such approaches.

Although the WKB-techniques prove useful in uncovering conventional physics locally in \( M \), they do not seem to be appropriate for giving quantum cosmology a fundamental, mathematically firm structure. We rather feel that all approximate methods referring to classical trajectories should work within a more basic framework. The philosophy behind the WKB-procedures as we use them is that the oscillations of the WKB-type wave functions make the identification of standard physical concepts possible as a kind of classical optics approximation, and that nothing at the fundamental level tells us about the observational significance of a wave function before this identification has been performed, and beyond its applicability. The wave functions as mathematical objects (and their oscillations in \( M \)) define ”the fundamental level” of the theory, and reconstructing observational physics is an approximate issue. Also, we do not regard the WKB-techniques as approximations to a path integral (see e.g. Ref. [33]) in this article.

Let us as a last remark in this Section comment on the size of the local WKB-domains necessary for our interpretation. Since the local physical concepts refer to particular WKB-branches contained in a wave function, the basic quantities limiting the size of such domains from below are the classical actions \( S \) associated with the branches. In order to identify a particular \( S \), the domain must at least allow for several oscillations of the phase \( \exp(iS) \). If \( t \) is the classical evolution parameter
along a trajectory, and since \( \frac{dS}{dt} = -NU \) (which is a consequence of the Hamilton-Jacobi equation (3.2)) the minimal time scale possible (below which no physical statement makes sense) is \( \delta t_{\text{min}} \approx (NU)^{-1} \). Since in the post-inflationary domain \( U \) is of the order of the total matter energy \( E \), this is a very small scale. In this way the extension of a WKB-domain in the direction tangential to the classical paths is limited from below. This condition may be translated into a coordinate independent statement. The "proper length squared" along a piece of the trajectory with respect to the DeWitt metric is given by \( ds_{\text{DW}}^2 = -\frac{dS^2}{U} \). Thus the minimal scale of "proper length" along the trajectory is \( s_{\text{min}} \approx |U|^{-1/2} \). The minimal size of a domain in the transverse directions is not that strictly governed, but it should allow one at least to identify the equal-action hypersurfaces. As already mentioned, the scale in "proper length" over which \( \chi \) changes considerably in a direction transverse to the classical paths must be much larger than \( |U|^{-1/2} \), thus limiting the widths of wave packets and the scale of transverse oscillations, and hence the transverse size of WKB-domains from below. For a given wave function, this should in general provide a fine pattern of domains covering the relevant part of \( \mathcal{M} \), inside each of which the WKB-analysis may be performed. On the other hand, for some purposes it may be necessary to choose domains as large as possible, even extending to "infinity" in some directions (e.g. if questions of normalizability or well-behavedness of \( \chi \) with respect to a probabilistic interpretation are concerned). In the region \( U > 0 \), such large domains can be chosen in the vicinity of spacelike hypersurfaces.

4 "True" and "false" local quantum mechanics

In the foregoing section we have sketched how local quantum structures are contained in an approximate and "hidden" way in the solutions of the Wheeler-DeWitt equation. When trying to give a clear meaning to this phrase, it turns out that there are actually two different possible motivations leading to the WKB-approximation. The first one is to rewrite the Wheeler-DeWitt equation under certain conditions, and to use the effective Schrödinger equations (3.6) or (3.8) in order to construct approximate solutions. In this case, one regards the local quantum structure (i.e. the Schrödinger-type form of the equation and the existence of the scalar product) as a tool for performing similar methods as in conventional quantum mechanics: to solve an initial value problem with prescribed \( \chi_{t=0} \). One will of course put emphasis on normalizable wave functions, i.e. one will begin with \( \bar{\chi} \)'s or \( \chi \)'s that behave reasonable under the scalar product. Imposing normalization on these functions (in-
side the WKB-domain) leads to solutions that are more or less of a wave packet-like type (see Refs. [34]–[37] for wave packets in minisuperspace). As in ordinary quantum mechanics one will also admit wave functions of a soft non-normalizable nature (analogous to ”distributions”, such as momentum eigenstates, associated with a Hilbert space).

However, it is a different issue if some wave function $\psi$ is given and shall be analyzed and interpreted by WKB-techniques. In such a case one would first determine the particular WKB-branches ($S, D$) the wave function $\psi$ can be associated with. If this is done, the particular Schrödinger-type wave function $\chi$ belonging to some branch may be computed straightforwardly from $\psi$. For any local quantum structure we define that true local quantum mechanics is recovered if $\chi$ admits a probabilistic interpretation without severe problems. Because of the local nature of all WKB-issues this is not a precise definition, but in practice it means that $\chi$ is normalizable with respect to the local $\langle | \rangle$ or at least of a harmless non-normalizable type, as mentioned above. One can in general expect this to be the case in WKB-domains in which $U(y) > 0$ (the classical trajectories being ”timelike”): Consider a ”spacelike” hypersurface $\Gamma$ that intersects the WKB-domain. Since the integral (2.13) — as well as the Klein-Gordon scalar product to be considered below — refers to hypersurfaces of this type, one may impose a reasonable behaviour on $\Gamma$ for all admissible wave functions from the outset (in some analogy with the requirement of normalizability in ordinary quantum mechanics). The hyperbolic structure of the Wheeler-DeWitt equation allows for integration, once the value of a wave function and its (normal) derivative on a ”spacelike” hypersurface are prescribed. Also, one may use local wave packet-like states (to be constructed inside the WKB-branch by means of solving the effective Schrödinger equation, as described above) in order to interpret $\psi$ in terms of reasonable relative probabilities of alternative observations.

The situation may change in a domain in which $U < 0$. It is a common feature of wave functions to exhibit exponential rather than oscillatory character in a ”time-like” coordinate in the $U < 0$ domains of a certain size [1]. If $\mathcal{M}$ is higher than just one-dimensional, some ”spacelike” variables will take over the role of providing the oscillations and thus the evolution parameter in the local Schrödinger equation (3.8) — cf. Refs. [34]–[37]. The equal-time hypersurfaces inhabiting the effective wave functions $\chi$ are no longer ”spacelike”. Still one may construct well-localized wave packets inside such a domain by imposing suitable initial conditions on the effective Schrödinger equation (3.8), but it is by no means guaranteed that the effective wave function $\chi$ of some given state $\psi$ behaves reasonably in a global sense.
often encounter functions that are exponentially decreasing or increasing in some of their variables. This provides a particularly clear division of wave functions if the domain extends to "infinity" in some direction of $\mathcal{M}$ (as e.g. in the Hawking model the domain defined by $|\phi| \lesssim 1/(ma)$ and $a \gtrsim a_{\text{min}}$, but $a$ arbitrarily large). Whereas the case of asymptotically vanishing $\chi$ will still admit a reasonable probabilistic interpretation (e.g. in terms of wave packets) — and hence fits into a true quantum mechanical scheme, the case of exponential increase in some unbounded domain will lead to a catastrophe, from the point of view of the local quantum structure. We will denote such a case by the terminus false local quantum mechanics. Again this is not a precise formulation, but in practice it should suffice to recognize the particular type of behaviour we just described. The prototypical situation is — in a one-dimensional analogue — a particle in a potential like $V(x) = V_0 \Theta(x)$ and a wave function with energy $0 < E < V_0$ of the type $\exp(ikx)$ for $x < 0$ that behaves exponentially increasing for $x > 0$. Does it describe a particle "tunnelling into nothing"? It is certainly beyond the scope of reasonable probabilistic prediction.

To summarize, the false case appears when — despite of the existence of the effective local quantum mechanical structure provided by $H^\text{eff}$ and $\langle | \rangle$ — no reasonable interpretation of a given state $\psi$ in terms of ordinary quantum mechanics is possible. We will encounter an explicit example of this type in Section 11, related to the question what happens when the universe reaches its maximum size in the Hawking model.

Let us add the speculation that the non-appearance of false local quantum mechanics related with a wave function $\psi$ might turn out to be related with simple mathematical properties, such as $|\psi|$ being bounded (at least in certain domains). However, since the interplay between the exact scalar product $Q$ — to be defined in Section 6 — and the local scalar product in domains with $U < 0$ is non-trivial, this is not entirely clear, and we will leave this point open.

5 Bohm-interpretation

So far we have dealt with the WKB-approximation. For completeness we note that the second possibility for the use of (2.13) in extracting predictions from a wave function is to assume that the flow lines $y_{\text{flow}}(\tau)$ themselves provide a congruence of "corrected" (and "real") trajectories, no longer "classical" in the sense of the original classical equations of motion, but "classical" in the sense of providing a
family of deterministic alternatives on which (2.13) defines a probability measure. This idea is essentially the application of Bohm’s (and de Broglie’s) interpretation of quantum mechanics [38]–[40] to quantum cosmology, and several attempts to formulate it in detail have been made [41]–[44]. We should add that the WKB- or Bohm-type approaches and the Hawking-Page approach are not equivalent even in the WKB-approximation, but there is a relation between them. In a WKB-domain, the probabilities (2.9) differ from (2.13) by the interval of proper time \(N = 1\) a trajectory spends within a certain region of minisuperspace [14].

Both interpretations relying on (2.13) are limited by the fact that the current \(j\) associated with a particular solution to the Wheeler-DeWitt equation may vanish identically (as e.g. for any real \(\psi\)). Given that in some region a wave function \(\psi = A \exp(iS)\) (with slowly varying and approximately real \(A\)) represents (in the WKB-sense) a family of expanding universes. Then its complex conjugate \(\psi^* \approx A \exp(-iS)\) describes an analogous family of contracting universes, and one might expect that a superposition of the form \(\psi' = \psi + \psi^* \approx 2A \cos S\) still contains these two alternatives in a way reminiscent of ordinary quantum mechanics. But the Bohm-inspired interpretation would just predict a static universe [43][44]. It is reasonable to expect that the flow lines of each WKB-branch are still present in \(\psi'\) as a mathematical structure and as a physical information, but the expression (2.13) is useless for uncovering it in this case. The situation is even worse if a wave function is defined as a superposition between two WKB-branches that are only approximately complex conjugates of each other. Then \(j\) could be almost anything, and nothing in it would give us a hint towards the WKB-branches the wave function has been constructed from. We will keep in mind that the current (2.10) provides an exact mathematical structure related with the Wheeler-DeWitt equation, and that it coincides approximately with the tangent to the classical paths in a WKB-situation, but we will not adopt the point of view that it is the basic tool for predictions. The WKB-interpretation, on the other hand, when confronted with \(\psi'\), would have to reconstruct the original WKB-components \(\psi\) and \(\psi^*\), and apply (2.13) as approximate assignment for alternative classical paths inside each branch, as described above. We will try to imbed this latter method into a "minimal" but well-defined structure.
6 The fundamental structure \((\mathcal{H}, Q)\)

As we have seen in Sections 3 and 4, the interpretation of a wave function \(\psi\) in terms of physical observables — as identified by WKB-techniques — is possible only within local regions of \(\mathcal{M}\) at an approximate level. Assuming that these approximate identifications are the only ones at hand, it follows that concepts like observables, time, probability, the unitarity of quantum evolution (cf. Refs. [16][45]) and the Hilbert space structure (cf. Ref. [46]) are approximate and local in character. One of the goals of quantum cosmology is then to extract the conceptual structures of conventional physics (and also the accuracy to which they apply) from some underlying mathematical framework.

It is sometimes argued that the mathematical structure of a fundamental quantum theory of the universe is still to be found [47]. Although this may be so, it is not clear to what extent a general principle would tell one how conventional physics should be recovered. It is conceivable that the underlying theory is quite simple in structure, and that the extraction of information we need for all sorts of predictions operates at an approximate level. We would like to advocate a framework that has a small amount of pre-supposed mathematics, and which makes contact to the realm of observations by structurally identifying conventional classical and quantum physics, using WKB-techniques.

There is a scalar product related with the current (2.10) that seems to be still underexploited in this field. Due to the definition of \(j\), one has to insert the same wave function twice. However, if \(\psi_1\) and \(\psi_2\) are two (complex valued) solutions of the Wheeler-DeWitt equation (2.3), the current

\[
 j_\alpha(\psi_1, \psi_2) = -\frac{i}{2} (\psi_1^* \partial_\alpha \psi_2 - (\partial_\alpha \psi_1^*) \psi_2) \equiv -\frac{i}{2} \psi_1^* \partial_\alpha \psi_2
\]

is still conserved,

\[
 \nabla_\alpha j^\alpha(\psi_1, \psi_2) = 0,
\]

and admits a congruence of flow-lines analogous to (2.12). Hence, there is also a conserved quantity like (2.13) for any "pencil". In order to remove the reference to such a pencil, we recall our assumption that \(\mathcal{M}\) admits a foliation by "space-like" hypersurfaces. Supposing that at least one of the two wave functions \(\psi_{1,2}\) is sufficiently concentrated on such a hypersurface \(\Gamma\), the expression

\[
 Q(\psi_1, \psi_2) = \int_{\Gamma} ds^\alpha j_\alpha(\psi_1, \psi_2)
\]
will in fact be independent of $\Gamma$. (This is analogous to computing the integral of $-\frac{i}{2} \psi_1^* \partial_0 \psi_2 + \text{c.c.}$ over $\mathbb{R}^3$ at $t = \text{const} = \text{arbitrary}$ for the Klein-Gordon equation in Minkowski-space. For the full superspace version of the scalar product in the case of pure gravity, see Ref. [3]; for its use in analyzing the Klein-Gordon equation on a curved space-time background, see Ref. [48].) The scalar product $Q$ is indefinite (but non-degenerate). In the Hawking-Model, on can choose $\Gamma$ as any hypersurface of constant $a$, and

$$Q(\psi_1, \psi_2) = -\frac{i}{2} \int_{-\infty}^{\infty} d\phi \, a \left( \psi_1^* \stackrel{\leftrightarrow}{\partial_a} \psi_2 \right)$$

(6.4)

exists and is independent of $a$, as long as at least one of the two wave functions $\psi_{1,2}$ is sufficiently well-behaved as $|\phi| \to \infty$. If $\psi_2$ is understood as ”the” wave function of the universe and $\psi_1$ as a ”test” (or ”reference”) state (e.g. a wave packet or an approximate matter energy eigenstate in the post-inflationary era), this does not even imply a severe normalization condition for $\psi_2$.

Our proposal is that the space $\mathcal{H}$ of complex valued solutions to the Wheeler-DeWitt equation (2.3) and the scalar product $Q$ as defined by (6.3) provide the only exact mathematical structure at the fundamental level. We will not try to give a precise definition of $\mathcal{H}$ here but merely assume that it is reasonably large. Also, the mutual scalar product of two proper elements of $\mathcal{H}$ shall exist. In analogy to, say, the momentum eigenstates $\exp(i \vec{k} \cdot \vec{x})$ in ordinary quantum mechanics, it will also be necessary to consider wave functions $\psi$ of distributional character, such that $Q(\Xi, \psi)$ exists for all $\Xi \in \mathcal{H}$. Whereas the candidates for ”the” wave function of the universe will be of this latter type, we think of $\mathcal{H}$ as providing an environment for interpretation (hence the notation ”reference states” that we have used already above).

A key point in our argumentation is now that, in a typical WKB-situation, $Q$ reproduces the scalar product of the local quantum structure. This is valid at least for domains in which $U > 0$ (which we have assumed to be the relevant ones for almost all types of predictions). To see this, we consider two wave functions $\psi_1$ and $\psi_2$, both of the type (3.7) — inside the same WKB-branch — with effective Schrödinger type states $|\chi_{1,t}\rangle$ and $|\chi_{2,t}\rangle$. When computing $Q(\psi_1, \psi_2)$, one encounters an expression proportional to $\partial_\alpha S$ — analogous to (3.4) — as the dominant contribution. The corresponding scalar hypersurface element $ds^\alpha \partial_\alpha S$ on the (”spacelike”) hypersurface $\Gamma$ may be positive or negative definite, according to the orientation of the flow. Since $U > 0$, $\Gamma$ intersects each classical trajectory only once, inside a domain of $\mathcal{M}$ in which these form a congruence. By using the local Hilbert space
structure associated with this WKB-branch one finds

\[ Q(\psi_1, \psi_2) \approx \int ds^\alpha (\partial_\alpha S) D^2 \chi_1^* \chi_2 \equiv \pm \langle \chi_{1,t} | \chi_{2,t} \rangle . \tag{6.5} \]

This is in accordance with the fact that the right hand side of (6.5) is independent of \( t \) on account of the unitarity of the evolution (3.8) — see Section 12, and in particular equation (12.6), for details. In Ref. [3] an equation similar to (6.5) was derived in the context of wave packet-like states. The physical role of the two possible signs will concern us later. Just note here that complex conjugation of the whole WKB-branch (essentially \( S \to -S \)) interchanges them.

Thus, there is some WKB-evidence for using \( Q \) as basic structure. In analogy with ordinary quantum mechanics, we expect that — forgetting about the indefinite nature of the scalar product for the moment — predictions should be formulated in terms of expressions \( Q(\Xi, \psi) \), where \( \psi \) is "the" wave function of the universe and \( \Xi \) represents an observational device or a "question" or a "physical property" of the universe. However, in contrast to ordinary quantum mechanics, we have no external "time" parameter at hand, and thus we assume that \( \Xi \) satisfies the wave equation along with \( \psi \) in order to make \( Q(\Xi, \psi) \) a unique number. One may imagine that the restriction to scalar products between solutions of the Wheeler-DeWitt equation corresponds to the fact that in a closed system all observational devices are subject to the according evolution laws. Also, this implies that the notion of an "observable" as a well-defined concept (e.g. as an operator on a Hilbert space) is given up at the fundamental level, and is replaced by a wave function (or a collection of wave functions) that, in a sloppy way of putting it, encodes certain "physical properties" approximately. The concept of observables in the usual sense should appear only in a WKB-sense, when the structure of conventional quantum physics is found to be hidden within the bulk of data provided by \( (\mathcal{H}, Q) \). Thus, it is the overall structure of all (or many) \( Q(\Xi_1, \Xi_2) \) that is envisaged to provide an approximate interpretational environment.

Now we can rephrase the different attempts to interpret the wave function. The approach of Hawking and Page [14] assumes that all predictions can be formulated in terms of the numbers (2.9). The Bohm-type approach as described above [11]–[13] assumes that all predictions can be formulated in terms of the numbers (2.13). The WKB-interpretation [1] [12] [13] [14] uses numbers of the type (2.13) as well, possibly in addition to a reconstruction of proper WKB-branches. In contrast to these attempts, we will advocate the point of view that all resonable predictions should be
formulated in terms of the numbers \((6.3)\), i.e. within the mathematical framework provided by \((IH, Q)\). The concepts of conventional physics as well as any limitation of observations (e.g. the impossibility of any question that implicitly refers to an external observer) should emerge from this structure, together with appropriate WKB-techniques. Both objects \(IH\) and \(Q\) are well-known, and the underlying idea is not at all new. Here, we just try to pursue it as radical as possible. Thus, our approach closely communicates with the WKB-type (semiclassical) interpretation, but the theoretical status of the according approximations is shifted and those features which depend on particular domains of minisuperspace are logically separated from the basic structure \((IH, Q)\) (which, at the fundamental level, contains no reference to ”local” things).

In a sense our strategy is to envisage a ”minimal” interpretation (related in spirit to Vilenkin’s scheme \([16]\), but based on \((IH, Q)\), and to see how far one comes. From this point of view, the final solution of the problem to extract all possible predictions from a quantum theory of the universe will, to some extent, depend on the viability of the particular models at hand rather than on the search for more fundamental structures. In the following, we will not try to be too rigorous, but put emphasis on the main ideas, the realization of which will certainly encounter various technical problems when applied to realistic models. Any model can be viewed as providing its own way how (and to which accuracy) the structures of conventional physics may be uncovered. We will merely sketch what we believe are the general features of such a program.

7 Problems of observables, decomposition and almost-classicality

Before further considering the issue of interpretation, we will proceed rather formal for a moment and take into account the indefinite nature of the scalar product \(Q\) (which can be traced back to the fact that the Wheeler-DeWitt equation is of second order (Klein-Gordon) rather than of Schrödinger type). Since the complex conjugate \(\psi^*\) of any wave function satisfies the Wheeler-DeWitt equation along with \(\psi\), and since \(Q\) has the algebraic properties

\[
Q(\psi_1, \psi_2)^* = Q(\psi_2, \psi_1)
\]

\[
Q(\psi_1^*, \psi_2) = -Q(\psi_2^*, \psi_1),
\]

(7.1) \hspace{1cm} (7.2)
the space $\mathcal{H}$ may be decomposed into two subspaces $\mathcal{H}^\pm$ that are orthogonal to each other with respect to $Q$, and on which $Q$ is positive/negative definite, respectively. (This is in some analogy to the scalar product associated with the Klein-Gordon equation in curved space-time [48]). In other words, one can choose a basis $\{\Xi^+_r, \Xi^-_r\}$ of $\mathcal{H}$ that is normalized according to

$$Q(\Xi^+_r, \Xi^+_s) = \pm \delta_{rs}, \quad (7.3)$$

$$Q(\Xi^+_r, \Xi^-_s) = 0. \quad (7.4)$$

For convenience, one may require in addition

$$\left(\Xi^+_r\right)^* = \Xi^-_r. \quad (7.5)$$

Denoting the spaces spanned by the subsets $\{\Xi^+_r\}$ and $\{\Xi^-_r\}$ as $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively, we obtain the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Complex conjugation maps $\mathcal{H}^\pm$ into $\mathcal{H}^\mp$ in a bijective way. This construction is somehow analogous to the decomposition of solutions to the flat Klein-Gordon equation into negative and positive frequency states (as e.g. $\psi = \exp(\pm imc^2 t/\hbar)$ for zero three-momentum).

Expanding a wave function $\psi$ with respect to such a basis,

$$\psi = \sum_r (c^+_r \Xi^+_r + c^-_r \Xi^-_r) \equiv \psi^+ + \psi^- , \quad (7.6)$$

the coefficients may be found simply by

$$c^\pm_r = \pm Q(\Xi^\pm_r, \psi). \quad (7.7)$$

Thus, the expansion of a given wave function $\psi$ in terms of a basis of other solutions is traced back to the computation of numbers of the type $Q(\Xi, \psi)$, as long as the normalization condition $(7.3)-(7.4)$ is adopted. This supports the use of $(\mathcal{H}, Q)$ as the fundamental framework in our program towards a ”minimal” interpretational scheme.

We can formally simplify the notation of a decomposition by introducing the operator $K : \mathcal{H} \mapsto \mathcal{H}$ which acts as $\pm 1$ on $\mathcal{H}^\pm$. It has the properties

$$K^2 = 1 \quad (7.8)$$

$$K(\psi^*) = -(K\psi)^*. \quad (7.9)$$

By

$$Q_K(\psi_1, \psi_2) = Q(\psi_1, K\psi_2) \quad (7.10)$$
it defines a positive definite scalar product $Q_k$ on $\mathcal{H}$ (that coincides with $\pm Q$ on $\mathcal{H}^\pm$ and renders $\{\Xi^+_r, \Xi^-_r\}$ an orthonormal basis). Moreover, $K$ is a hermitean operator both with respect to $Q$ and $Q_k$, and the subspaces $\mathcal{H}^\pm$ are orthogonal to each other with respect to $Q_k$ as well. Conversely, given an operator $K : \mathcal{H} \mapsto \mathcal{H}$ which satisfies (7.8) and for which the scalar product $Q_k$ defined by (7.10) is positive definite, then a decomposition of the above type is induced by defining $\mathcal{H}^\pm$ as the eigenspaces of $K$ with respect to the eigenvalues $\pm 1$ (and (7.9) follows as a consequence). Hence, the notation of such a decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is identical to the notation of an operator $K$ with the properties listed above. We will call $K$ a decomposition (operator).

This construction is not unique. Any decomposition operator $K$ gives rise to a positive definite scalar product, and for any $K$ one may formally apply the standard quantum mechanical assignment of probabilities and give $\mathcal{H}$ (possibly redefined so as to contain all states with $Q_k(\Xi, \Xi) < \infty$) the role of a quantum mechanical Hilbert space. In this sense, any triple $(\mathcal{H}, Q, K)$ corresponds to a scheme upon which the interpretation of wave functions can be based. The identity (6.5) guarantees that $Q_k$ translates into the local quantum mechanical scalar product within a WKB-branch, as long as the whole branch (3.7) and its complex conjugate are associated with $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively.

The degree of non-uniqueness of $K$ is a Bogoljubov-type freedom [18]: Let $K$ and $\widetilde{K}$ be two decomposition operators and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- = \widetilde{\mathcal{H}}^+ \oplus \widetilde{\mathcal{H}}^-$ the corresponding decompositions. Then $\widetilde{K} = S K S^{-1}$, where the linear operator $S : \mathcal{H} \mapsto \mathcal{H}$ acts isometrically, $Q(S\psi_1, S\psi_2) = Q(\psi_1, \psi_2)$, and commutes with complex conjugation, $S(\psi^*) = (S\psi)^*$. If $S$ leaves $\mathcal{H}^\pm$ invariant, it is just a unitary transformation on these subspaces, leaving the decomposition invariant, $\widetilde{K} = K$. Otherwise we have $S : \mathcal{H}^\pm \mapsto \widetilde{\mathcal{H}}^\pm$ and a true change, $\widetilde{K} \neq K$. It may as well be expressed as a change of a basis of the type (7.3)–(7.5) to an other one by $S\Xi^\pm \equiv \Xi^\pm = \sum_s (a_{rs}\Xi^+_s + b_{rs}\Xi^-_s)$ for (at least some) non-zero $b_{rs}$. (In general, the collection of all $(a_{rs}, b_{rs})$ must obey a Bogoljubov-type normalization condition).

This non-uniqueness of $K$ provides a major obstacle in formulating a consistent quantum cosmological framework (see e.g. Ref. [30]). This is in contrast to the Klein-Gordon equation in Minkowski space, where the existence of a timelike Killing vector field ensures a unique Lorentz-invariant definition of negative/positive frequency wave functions, and thus a unique decomposition. One can imagine several possible ways out of the dilemma (which we call "decomposition problem").
To begin with, one may suspect that some fundamental principle should uniquely specify one particular decomposition $K_0$. It is hard to see how this can be done in general without introducing a lot of additional structure, but we cannot exclude it, and this matter is presumably model-dependent. There seems to be a naturally preferred decomposition in the Hawking model (see Section 11), relying on the analyticity structure of wave functions in the asymptotic region $a \to \infty$, but it is not clear to what extent this feature carries over to more sophisticated models. Another attempt to define a unique decomposition (based on a ”positive frequency” condition in the limit of zero spatial volume) was presented in Refs. [49] [50]. (As a further possible point of view, one may of course expect the fundamental problems of interpretation not to be solved in general but by the particular structure of the underlying classical theory: general relativity plus matter. Thus the (non)existence of a preferred $K_0$ might be envisaged as a criterion to discriminate between different models for matter). In case it is reasonably possible to single out a unique decomposition, the triple $(\mathcal{H}, Q, K_0)$ should provide a consistent basis for deriving all sorts of predictions. Note that even in this case the identification of wave functions with observational data is not pre-supposed, and local WKB-techniques will certainly still play some role. Also, the appearance of false local quantum mechanical situations in domains with $U < 0$ may occur.

Another possible line of thought is to recall that in the minisuperspace approximation we have truncated the major bulk of degrees of freedom contained in the full theory. Although classically these degrees of freedom may appear to be of ”small scale” type and are irrelevant for cosmology, this need not be the case in a quantum theory. The mere fact that such degrees of freedom are not included into observation may in principle — by means of a reduced density matrix formulation and a ”decoherence” mechanism, e.g. in a path integral or a consistent histories approach [51] [53] — give rise to a superselection rule [50] [51] at the minisuperspace level that simply forbids certain superpositions of wave functions to be observed, and hence to be used as a basis of reference states. (Recall that ”reference states” — at least in the strong sense of the word — should be related with possible observations). This in turn could show up as an additional effective structure, e.g. as a condition for preferred decompositions, possibly depending on the particular WKB-domain in $\mathcal{M}$ to which an observation refers.

We would prefer not to incorporate explicitly the bulk of original degrees of freedom, but merely work with solutions of the minisuperspace Wheeler-DeWitt
equation, and to implement the "trace over unobservables" locally. Let us try how far one comes if the "minimality" of our approach is retained, and the decomposition problem is treated at an equal footing with two other important questions: The interpretation of the reference states in $\mathcal{H}$, and the problem of the experience of an almost-classical (quasi-classical) world.

Although we reject any fundamental (mathematically exact) principle telling us what the "properties" represented by a particular state $\Xi \in \mathcal{H}$ are, we may structurally extract the physics of observations by WKB-techniques from certain types of states. This may appear to be a "logical circle" in that the wave function of the universe $\psi$ is interpreted in terms of reference states whose interpretation is not pre-supposed either! However, taking the closedness of the universe serious, such a "logical circle" is quite plausible (if not unavoidable). In the "minimal" interpretation the answer is that "properties" and concepts like observables, time and unitarity may be identified with limited accuracy just because they are no accurate concepts.

Even if for some wave function a WKB-interpretation has been performed successfully in some local domain, there is still a fundamental problem. To any WKB-type wave function there is a large collection of classical trajectories contributing, and the spread of $\chi$ over these may in principle be interpreted as "quantum fluctuations". Moreover, as we already know, given a WKB-type wave function $\psi$, there will be a range of WKB-branches $(D, S)$ that only differ slightly from each other and may all be associated with $\psi$ to any reasonable accuracy. However, trajectories that are distinguishable at large (macroscopic) scales are never experienced according to probabilistic laws, but rather as elements of a classical world, even if several repeated measurements are performed in order to find out "which trajectory is realized". It is only for very close trajectories that the spread of $\chi$ and the uncertainty in the WKB-branch is experienced as a genuine quantum feature. In terms of the standard WKB-split into classical and quantum variables, one would say that the universe primarily evolves according to one of the evolutions $y_{cl}(t)$ described by the action $S_0(y_{cl})$, and that the rest of the variables $y_q$ are described by a quantum theory on this background. In terms of our modified WKB-approach, one can express classicality by considering collections ("tubes") of various trajectories $y(t)$ that are close to each other so as to represent the same evolution with respect to macroscopic observations. Genuine quantum fluctuations are experienced only for alternatives within such tubes, i.e. with respect to small scale (microscopic) measurements. No matter how this fact is formulated in terms of variables, it just means
that macroscopic (quasiclassical) alternatives "decohere" (see Refs. [51]–[59]). This provides the possibility of "time" as an ordering structure, together with at least some almost-classical variables that altogether guarantee the experience of a world retaining its "identity" as an alternative with respect to "other worlds". Without such a structure we cannot formulate predictions in the sense of statements about observations that can be prescribed and performed. The reason behind the decoherence of macroscopic alternatives is commonly regarded as the fact that one does not observe all variables. In "tracing out" most of them, an according reduced density matrix becomes an almost-classical statistical mixture with respect to the quasiclassical variables. Partially, these questions should not provide conceptual problems for quantum cosmology. Inside each WKB-branch, associated with a local quantum structure, the non-observation of variables should be performed by tracing these out in the sense of a Hilbert space. However, there must one thing be guaranteed in order for these methods to work: The existence of a viable local quantum structure enabling one not only to perform traces formally but also to obtain results that are sufficiently well-behaved so as to allow for interpretation. This condition is far from being trivial (due to the approximate and local nature of the WKB-methods), and far from being of minor importance only.

Thus we have formulated three conceptual problems: (i) How is the bulk of states in $\mathcal{H}$ to be identified with physical quantities? (ii) How is the non-uniqueness of the decomposition $K$ to be dealt with? (iii) What can be inferred about almost-classicality on macroscopic scales which is necessary to formulate predictions in terms of local quantum physics?

We will try to treat these problems of an equal footing. Due to the "minimality" of the philosophy pursued here, the answer to all of them is given it terms of the identification of conventional physics by WKB-techniques. The way to attack (i) has been announced already: By identifying local quantum structures to the accuracy admitted by the WKB-techniques, the "properties" described by states $\Xi \in \mathcal{H}$ which display WKB-behaviour in some domain of $\mathcal{M}$, and the observables associated with them (within this domain) are uncovered. The principal limitation in the accuracy of this identification is provided by the nature of the oscillations in $\Xi$ (e.g. by the size of the "wave lengths", as compared to the size of the domain). Predictions are always stated with respect to a domain, hence locally in $\mathcal{M}$. There is no ultra-local (pointwise) formulation of physics, because any WKB-domain has a minimal size (below which the oscillations in $\Xi$ cannot be resolved). Thus, for a very deep reason, conventional physics is approximate in practically all its concepts.
To illustrate this point, two possible questions in the Hawking model are "What is the probability to observe the matter energy $E$ when the universe is in its post-inflationary $|\phi| \lesssim 1$ era?" and "What is the probability to find the universe expanding when it is in the region $|\phi| \lesssim 1$ and has $E \approx E_{\text{given}}$?". It might happen for some wave function that the answer to the second question is different from the answer to "What is the probability to find the universe expanding when it is in the region $|\phi| \gg 1$?" where the wave function may even make it necessary to specify the region in more detail (as e.g. prescribing $\phi \approx \phi_{\text{given}} \gg 1$, $a \approx a_{\text{given}} \gg 1/(m\phi)$). As a further example, a question like "What is the probability for the universe to be expanding?" is in general not well-posed since it must be assigned to a WKB-domain associated with the concrete observation whose outcome is to be predicted (although it might be admissible for certain wave functions). Questions of this type refer to the macroscopic (large scale) state and provide the typical motivations a theory of the universe is usually concerned with (see e.g. Ref. [62]). Another class of admissible questions may refer to particular things happening "in our universe", and this is just the realm of local quantum physics, where the word "our universe" refers to a particular domain as well.

The answer to (ii), the decomposition problem, may be given at a similar level. Suppose that in some domain $U > 0$. In this case, we have already encountered a relation between the fundamental structure $Q$ and local quantum physics: equation (6.5) and its double sign gives us a hint. Obviously, given a congruence of classical paths that make up a WKB-branch by their action $S$, there are two possible signs in the relation between the locally assigned Hilbert space structure and $Q$. Moreover, the $Q$-scalar product between two wave functions corresponding to two WKB-branches that are complex conjugate to each other is likely to vanish approximately if the oscillations are rapid enough. (By direct computation one may infer that the scalar product between complex conjugate branches lacks the "large" contribution from $\partial_a S$ in the integrand, as compared to products within one branch). Also, the proper (single-branch) WKB-states $\psi \sim A \exp(iS)$ are the only ones around which an effective local quantum mechanics (3.6) or (3.8) is defined (as opposed to their superpositions $\psi \sim \sum A_j \exp(iS_j)$). This is so because any local quantum structure needs an evolution parameter $t$, whereas several macroscopically different WKB-branches would provide several such parameters. Hence, requiring that all those wave functions around (or "near") which a local quantum structure is possible are in one of the two local subspaces $H^\pm$, we have almost solved the problem: The only reasonable candidates for elements of $H^\pm$ at the level of accu-
racy limited by the WBK-techniques are the (single-branch) states $\psi \sim A \exp(\pm iS)$.
Since we have assumed $U > 0$, the congruences of classical paths clearly fall into two classes: "past" (incoming) and "future" (outgoing) directed ones, with respect to the DeWitt metric. It is locally clear what the physical difference between these two possibilities is. In many models one may associate these with contracting and expanding universes. The details of such assignments depend on the structure of $\mathcal{M}$ and its geometry, but they have always a clear physical meaning. This provides a list of candidate elements of two locally and approximately defined subspaces $\mathcal{H}^\pm$ of $\mathcal{H}$. The superpositions thereof will be oscillating as well, but not of the proper (single-branch) WKB-type. There will also be elements of $\mathcal{H}$ containing only exponential rather than oscillating contributions within our domain. These are representing tunnelling phenomena and do not correspond to observable features (at least not in the domain under consideration). In terms of the $Q$-product, we expect all these states to be approximately orthogonal to the WKB-contributions, and for convenience we decompose the subspace of $\mathcal{H}$ corresponding to these tunnelling states as well (in fact it does not matter how, but there may be reasonable choices, e.g. dictated by smoothness with respect to adjacent domains). We end up with an approximate decomposition $\mathcal{H} \approx \mathcal{H}^+ \oplus \mathcal{H}^-$, associated with the local domain in which the analysis was performed. This provides the interpretational environment for the wave function of the universe with respect to the domain. In practice one even needs less: Since the $Q$-product between largely distinct WKB-branches is expected to be negligible, the choice of $K$ may be attached to the particular observation one is interested in, and restricted to the subspace of $\mathcal{H}$ spanned by the WKB-branches of interest and their complex conjugates. Note that, although a classically unique notion of outgoing/incoming paths exists, this does not enable us to assign outgoing/incoming properties for wave functions at the exact level, so the essential difference to the flat Klein-Gordon equation remains intact.

At the technical level, we may write down equations defining the approximate decomposition. Consider a domain $\mathcal{G}$ of $\mathcal{M}$ in which $U > 0$, and let $\Gamma$ be a "spacelike" hypersurface in $\mathcal{G}$. Then there is a uniquely defined family of (outgoing) classical trajectories starting from $\Gamma$ orthogonally. The corresponding action may be computed by solving the Hamilton-Jacobi equation (3.2) with initial condition $S|_{\Gamma} = 0$ in a vicinity of $\Gamma$. The sign of $S$ is fixed by choosing the time coordinate $t = -S$ along the paths, such that it increases in the outgoing direction. The orthogonal derivative along the unit normal of $\Gamma$ may expressed in this time gauge as $\partial_\perp = U^{1/2} \partial_t$ (but has of course a geometric significance independent of any gauge). Now
consider wave functions of the WKB-form (3.7). The prefactor $D$ may be computed in a vicinity of $\Gamma$ (it is unique only up to a function that is constant along each path, but this freedom will cancel out in what follows — it can always be chosen as $D|_\Gamma = 1$). Furthermore let us assume that $\chi$ approximately satisfies the effective Schrödinger equation (3.8) and dies off near the boundary of $\Gamma$, so that it lives only on a domain in which $U > 0$. All wave functions of this type make up a space $\mathcal{H}^{-}(\Gamma)$, which is thus defined approximately. An according space $\mathcal{H}^{+}(\Gamma)$ is provided by the complex conjugate wave functions, i.e. by $\psi = \chi D \exp(-iS)$, subject to the conjugate effective Schrödinger equation $i\partial_t \chi = -H^{\text{eff}} \chi$. Here we use the same time coordinate $t$ as before, hence produce an additional minus sign. It is a straightforward application of the formalism described in Section 12 to rewrite the equations $i\partial_t \chi = \mp H^{\text{eff}} \chi$ defining $\mathcal{H}^{\pm}(\Gamma)$ in terms of $\psi$. The result is

$$\partial_\perp \psi = \left( \pm i U^{1/2} - \frac{\partial_\perp U}{2U} + \frac{1}{2} K \pm i U^{-1/2} H^{\text{mod}} \right) \psi, \quad (7.11)$$

where $K$ is the trace of the extrinsic curvature on $\Gamma$, and $H^{\text{mod}}$ is the operator $N^{-1} D H^{\text{eff}} D^{-1}$ which — in the present gauge — reads $-\frac{1}{2} U^{1/2} D^{a} U^{-1/2} D_{a}$ (cf. equation (12.32) in Section 12). $D_{a}$ is the covariant derivative with respect to the induced metric on $\Gamma$. The identities necessary to establish this result are $\partial_\perp S = -U^{1/2}$ and

$$\frac{\partial_\perp D}{D} = -\frac{\partial_\perp U}{2U} + \frac{1}{2} K. \quad (7.12)$$

Each of the two equations (7.11) is just a restriction among the initial data $(\psi, \partial_\perp \psi)$ on $\Gamma$ for the Wheeler-DeWitt equation (2.3). The condition for $\chi$ to satisfy the effective Schrödinger equation to a good accuracy in a vicinity of $\Gamma$ is that $H^{\text{mod}} \psi$ is in a sense much "smaller" than $U \psi$. (Let us note in parentheses that — if the behaviour of $\chi$ at the boundary of $\Gamma$ is ignored — the simplest possibility to specify a wave function is to set $\psi|_\Gamma = \psi_0 = \text{const}$. At $\Gamma$, we thus have $H^{\text{mod}} \psi = 0$, and (7.11) defines initial conditions for an exact solution near $\Gamma$. In the case of the flat Klein-Gordon equation and $\Gamma$ a spacelike plane in Minkowski-space, $\psi$ turns out to be a negative/positive frequency plane wave). This procedure is performed for all "spacelike" hypersurfaces $\Gamma$ in $\mathcal{G}$. If $\mathcal{G}$ is not too large, we expect — to some reasonable accuracy — the scalar product $Q$ to be positive/negative definite on the linear span of the union $\cup \mathcal{H}^{\pm}(\Gamma)$ over all $\Gamma$. This defines an approximate decomposition for oscillating modes in a local domain in which $U > 0$.

If the potential is negative, $U < 0$, in some domain, any WKB-branch defines a local quantum structure as well. Although in this case the trajectories do not fall

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into two clearly separated categories (because they are "spacelike" now), the double structure of mutually complex conjugate WKB-branches still exists, but the relation between $Q$ and the local scalar product is lost due to the non-spacelike character of a hypersurface on which $\langle | \rangle$ is defined. Completing such a local hypersurface to a global one, $\Gamma$, which is admissible in the definition (6.3) of $Q$, a wave packet constructed by appropriate initial conditions for (3.8) might leave the $U < 0$ domain, but intersect $\Gamma$ somewhere else and produce a further contribution to $Q$ about which we have no control. However, we will use the $U < 0$ domains only for a particular condition that refers to the notation of true and false local quantum mechanics (see below), and there is no need for a decomposition of $\mathcal{H}$ there.

Summarizing, the decomposition problem is solved at the level of accuracy that enables one (a) to distinguish between time-reversed congruences of classical paths and (b) to implement these into local quantum structures. For the use of extracting predictions with reference to some domain, one defines the triple $(\mathcal{H}, Q, K)$, where $\mathcal{H}$ and $Q$ are the fundamental structure of the theory and $K$ is the local approximate decomposition. Although in practice a decomposition of the whole space $\mathcal{H}$ is not necessary, we will retain this notation for convenience. Due to our "minimal" point of view, there is no possibility to transcend the approximate character of this construction, just as there is no possibility to identify physical properties in an oscillating state better than allowed by the WKB-methods.

So far our scheme allows for local approximate probabilistic interpretation, but rather in the sense of conventional quantum mechanical expectation than quantum cosmology: With regards to large scales (semiclassical variables), a wave function contains macroscopic alternatives that never get mixed in observations. Let us consider now a state and several adjacent WKB-domains in $\mathcal{M}$. Inside each, one has a local quantum structure, and these structures are expected to "match" smoothly (e.g. by the existence of intermediate domains, covering part of two adjacent ones, such that there is a common local quantum structure). Mathematically, one could define adjacent domains by a local foliation such that any relevant classical trajectory intersects any foliation hypersurface only once. The particular domains are provided by appropriate neighbourhoods of the hypersurfaces (which may often, but not always, be chosen "spacelike"). One is tempted to thinking about short pieces of semiclassical evolution. Alternatively, one may consider families of macroscopically equivalent trajectories in $\mathcal{M}$, followed over a short period of time. These local structures should "match" so as to provide "the quasiclassical history" of the universe as it can be described in retrospect. Our point is now that the possibility
of predictions concerning the large-scale almost-classical variables implies the existence of a chain of adjacent local quantum structures without interruption. We admit that our formulation is not very precise, but the idea is that local quantum structures are considered "along" any piece of a local WKB-branch (e.g. by following the hypersurfaces of a foliation), or "along" any piece of "quasiclassical history" (expressed in terms of the classical variables $y_{cl}$). The crucial requirement may now be formulated as follows: All these adjacent local structures must be of the type of true local quantum mechanics, because otherwise we cannot talk about predictions "on the way" of the experienced macroscopic history of the universe. In principle, we should have formulated this statement already when giving the answer to (i). In order to emphasize the local character of the identification of quantum structures and "properties" of states, we have swept it under the carpet. In the present context, however, it seems unavoidable to impose a feature that is a bit more global in nature and ensures (or limits) the realm of predictions "in" a universe.

As already remarked at the end of Section 4, it is conceivable that this condition may be re-expressed in a mathematically more rigorous way, such as the requirement that the wave function $\psi$ be bounded ($|\psi| < k < \infty$), at least in certain domains. Such a condition is added to the tunneling proposal [18] as well, the motivation probably being a quite similar one than here. However, we do not consider it as an ultimate part of the definition of states. Any wave function that does not meet this criterion is admissible but — as soon as the false situation occurs "along" a quasiclassical history — provides an "interruption" that would not occur in a purely classical world.

The principle of relating predictability with true local quantum mechanics is important only in situations where the existence of $U < 0$ domains may not be ignored. Since the classical paths are "timelike" if $U > 0$, the only possibility for an outgoing trajectory to return and become an ingoing one (or vice versa) is to enter a region with $U < 0$ and leave it with reversed orientation. The typical situation for this is the case when the (classical) universe has reached its maximum size and starts to re-contract. Also, these returning points are just those where the scale factor becomes needless as a time parameter, and where the intrinsic nature of time in quantum cosmology must show up [34]. Our conclusion is thus that the fate of the universe near its maximum expansion depends on whether the wave function of the universe $\psi$ gives rise to true or false local quantum mechanics there. In Section 11, we will treat this problem as it appears in the Hawking model. In more sophisticated models, there may appear more general situations of this type. Let us just summarize that
a history of the universe that may be experienced macroscopically (and which is ex-
perienced "continuous" in evolution) requires the existence of a chain of reasonably
matching adjacent true local quantum mechanics structures. A given wave function
may describe several such "worlds", that must all be regarded as alternatives to each
other. In general, there will contribute many WKB-branches to a wave function in
a local domain, and many distinct large-scale histories are possible as alternatives.
However, it may happen that within one single WKB-branch, i.e. even along the
classical paths in this branch, a logical separation into alternatives occurs! In such a
case, the associated local quantum mechanics will typically be false. The appearance
of false local quantum mechanics in a wave function is by no means inconsistent or
forbidden, but is just an indication for a logical separation of macroscopic histo-
ries from each other that is not always recognized easily. (If false situations occur
well-separated from the domains in which the universe is quasi-classical, e.g. in a
domain describing the nucleation of the universe, they will be comparably harmless,
because there is nothing to be "interrupted" there). This possibility is missing in
many discussions about the interpretation of quantum cosmology, and our criterion
implies that the common interpretation of the no-boundary wave function is likely
to be wrong. By applying our criterion to the Hawking model in Section 11, we will
find hints that the no-boundary wave function allows for a cosmic catastrophe.

Let us mention that the true chain requirement is not a complete substitute for
a decoherence mechanism. We have already mentioned that macroscopically equiv-
alent trajectories ("tubes" of paths in $\mathcal{M}$) display observable quantum fluctuations,
whereas macroscopically distinct trajectories decohere. In order to work out the ac-
tual scales at which these phenomena take place, one should invoke the techniques of
tracing out unobserved quantities within the WKB-domains. This issue — although
of major importance — is not in particular our concern, once it is brought down to
familiar Hilbert space structures.

Thus we have presented our key idea how the three problems (i) – (iii) might
be solved in particular models, in an interpretational scheme that is as "minimal"
as possible.

Let us at the end of this section insert a comment on possible relations between
the selection of "the actual" wave function and the decomposition problem. In
the discussion about Vilenkin’s outgoing mode (tunnelling) proposal \[17\] – \[19\] it is
sometimes regretted that we do not know how to define precisely what an "outgoing
mode" is — in particular in domains of minisuperspace which may not be relevant
for observations. When interpreted with respect to our scheme this is clear, because
8 "Minimal" interpretation

We can now formulate how relative and approximate probabilities are assigned. Let $K$ be a decomposition associated with some local domain of $M$. We proceed according to the formal rules of ordinary quantum mechanics, using the scalar product $Q_K$. Therefore we assume some wave function $\psi$ to represent the actual quantum state of the universe. It need not be in $IH$ (neither $Q(\psi, \psi)$ nor $Q_K(\psi, \psi)$ are required to be finite), but it shall have well-defined scalar product with any $\Xi \in IH$. In terms of a basis $\{\Xi^r, \Xi^{-r}\}$, normalized according to (7.3)–(7.5), the assignment of approximate relative probabilities associated with each $\Xi^\pm$ is given by

$$P^\pm_r = |c^\pm_r|^2 \equiv |Q(\Xi^\pm_r, \psi)|^2. \quad \text{(8.1)}$$

In principle, the relative probabilities of finding the universe in the $\pm$ modes are

$$P^\pm = \sum_r |c^\pm_r|^2 \equiv \pm Q(\psi^\pm, \psi^\pm) \equiv Q_K(\psi^\pm, \psi^\pm), \quad \text{(8.2)}$$

although these two numbers may be infinite. All expressions above emerge when one formally expands

$$Q(\psi, \psi) = Q(\psi^+, \psi^+) + Q(\psi^-, \psi^-) \quad \text{(8.3)}$$
in terms of the components of $\psi$ and changes the sign of the $\psi^-$-term. Thus, one never has to deal with the original expression $Q(\psi, \psi)$, and therefore one will not encounter any problem if $j = 0$, as opposed to the Bohm-type interpretation. In

even if there is a classically well-defined definition of "outgoing" paths (which we have assumed) this provides only an approximate concept for wave functions. Hence the proposal can at best determine a wave function approximately. It is certainly preferable to have a proposal which singles out a unique wave function at the exact level, and logically independent of the approximate reconstruction of conventional physics. This would in any case require the introduction of an additional structure, and we leave it open how it can be done and what consequences it would have for the decomposition non-uniqueness problem. Also, including path integral arguments — as is necessary if the no-boundary wave function $[4]$ is considered as the "true" state — could enrich the mathematical possibilities for an alternative treatment of the problem (see also Refs. $[63]–[65]$).
general, one may assign a relative probability with respect to any normalized state $\Xi \in \mathcal{H}$, with arbitrary non-zero $+$ and $-$ components, just by $P(\Xi) = |Q_\kappa(\Xi, \psi)|^2$.

The type of predictions that may be stated in our formalism are to a large extent just those emerging from the standard WKB-treatment of a given wave function. For these, our scheme provides a clearer way of writing things down. In this sense our program can be regarded as an attempt to give the local WKB-techniques some common and global structure. However, as soon as different WKB-branches and different WKB-domains in $\mathcal{M}$ are involved, the standard WKB-formalism becomes less well-posed, and one is obliged to introduce more or less heuristic $ad$ hoc considerations which will in effect just uncover pieces of the structure defined by $\mathcal{H}$, $Q$ and $K$. As an example, Vilenkin, when defining his interpretational scheme \[16\] treats the (macroscopically) different WKB-branches separately, which implies that the orientation of the hypersurface element $ds^\alpha$ in (2.13) depends on the particular branch one is integrating. Hence, his argument that the cross-terms (involving different branches) in (2.13) are small due to rapid oscillations of the integrand is not really well-posed, since $ds^\alpha$ is not specified in this case. As a consequence, it is conceptually unclear how to estimate the accuracy of these cancellations. (In contrast, his cross-terms appear in our formalism in the form $Q(\psi_1, \psi_2)$, which is at least in principle well-posed and may serve as a check of orthogonality of the different branches). The technically relevant point in Vilenkin’s scheme is an adjustment of signs in order to set up a computational framework. In practice this adjustment does the same job as our decomposition $K$ of $(\mathcal{H}, Q)$, but it does not operate within a fundamental and (hopefully) appealing structure.

Once having identified elements of $\mathcal{H}$ with physical properties of the universe as observable in the respective domain, our interpretational scheme may reproduce the probability and Hilbert space structure as associated with the local WKB-branches and the effective Schrödinger equations (3.6) or (3.8). In this sense, the concept of probabilities in conventional quantum physics is recovered, and with it the standard WKB-interpretation if $\psi$ is locally of the WKB-type with a single branch, $\psi = A \exp(iS)$. In domains in which $\psi$ turns out to be a superposition

$$\psi \approx \sum_j A_j e^{is_j},$$

our scheme automatically implies the correct book-keeping of relative normalizations in the different branches. This is due to the positivity of the scalar product $Q_\kappa$. Moreover, our interpretation tells us in a simple way whether two observations

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associated with two states \( \Xi_1 \) and \( \Xi_2 \) are in fact *alternatives* to each other. This is the case if \( Q_K(\Xi_1, \Xi_2) = 0 \) (a non-trivial statement if the two states are elements of slightly different WKB-branches: it is only for actions \( S_1 \) and \( S_2 \) fairly different from each other that one can expect the scalar product between the whole branches to be zero on account of destructive interference in the integral \((6.3)\), and it shows us that the notation of alternatives in a measurement is a local one, depending on \( K \). Again, this provides the possibility of estimating the accuracy at which standard physics is valid. Clearly, our scheme admits \( \psi \) to be any superposition of elements of \( \mathcal{H} \), out of both (local) subspaces \( \mathcal{H}_\pm \). However, it is able to do even more, in particular when some wave function \( \psi \) is considered in different local WKB-domains. The treatment of \((\mathcal{H}, Q)\) as a fixed fundamental structure and \( K \) as a local one is crucial in identifying how local physics changes when the evolution times are long. The effects showing up here transcend to some extent the reach of the local WKB-interpretation. The identification of ”logical interruptions” along the large-scale history by means of the *true/false* quantum mechanics criterion is more than what is usually done, although it is achieved by no more than the application of WKB-techniques.

There is another type of effects related with long evolution times, and it is the final theoretical part in our scheme. So far, we have mainly talked about the reconstruction of local conventional quantum physics. The identification of the local scalar product and the properties of states works at the Schrödinger approximation \((1.6)\) or \((1.8)\). Only with respect to it we can talk about probabilities. However, at the next higher order in the (standard or modified) WKB-expansion we encounter a unitarity violating (as well as a hermitean) contribution to the effective Hamiltonian. How do we deal with this situation? Our answer is that we just have to apply the local WKB-techniques (in addition to a standard decoherence mechanism, i.e. taking traces in local Hilbert spaces) in order to account for unitarity-violating and quantum gravitational corrections! Suppose that a given wave function \( \Xi \) is analyzed in adjacent WKB-domains \( \mathcal{G}_j \), each one providing its local decomposition \( K_j \) and its (true) local quantum structure. In each domain one will find several relevant WKB-branches. The probabilities associated with these branches may (even if the branches exist in a domain larger than the individual \( \mathcal{G}_j \)'s) differ as one runs from one domain to another. This happens for example if \( \Xi \) represents a congruence of expanding universes in some domain, and a superposition of expanding and contracting universes in another one. Also, if only one congruence of classical paths suffices to describe \( \Xi \), the evolution of probabilities may be such that it does not
fit into a unitary wave equation when followed over many local WKB-domains. In other words, even if the unitarity-violating contribution to the Hamiltonian is small, it may give rise to observably large inconsistencies if (3.6) or (3.8) are retained over long times. Technically speaking, the evolution of the wave function according to the Schrödinger approximation may simply be in conflict with the fact that it solves the Wheeler-DeWitt equation. Note however that these effects — as well as the hermitean correction at the higher orders — are too small to show up at short (proper) time scales. The corrected Schrödinger equation — equations (12.43) or (12.44) in Section 12 — implies a quantum gravity correction to any energy scale encountered in local physics, but these corrections can only be verified experimentally during a long period of time. In Section 12, a rough estimate of the relevant time scales is given.

It is precisely features like this that are accounted for by our local decomposition construction. The crucial point with such situations is that one never has to propagate a wave function with respect to the local quantum structure. We suppose that $\Xi$ is prescribed, i.e. we know from the outset that it satisfies the Wheeler-DeWitt equation. The WKB-domains necessary for our interpretation must only be large enough to allow for the identification of WKB-branches (i.e. of $\chi$) and the definition of local physical concepts. In our universe, this corresponds to extremely small time scales, so that $\mathcal{M}$ can be thought of as being covered by a huge number of $\mathcal{G}_j$’s. The interpretation can be performed in any of these domains independently. As a consequence, in any domain a probability interpretation is at hand. The long-term change of local physics is incorporated automatically. The only problem that remains is how these structures match in order to describe a ”continuity” that can be experienced as the quasiclassical ”history” of the universe. However, applying a local decoherence mechanism, provides a local notation of quasiclassical history, for each domain $\mathcal{G}_j$ independently. In terms of the classical variables $y_{cl}$, as used in the standard WKB-expansion (supposing for simplicity that they remain intact over the total region considered), one would find a family of small pieces of background evolutions $y_{cl}(t)$, the ”length” of each piece never exceeding the size of the respective domain in these coordinates. Any piece is given a (relative) probability. These pieces match to a family of background histories $y_{cl}(t)$ that can be experienced. Unitarity-violation as well as quantum gravitational corrections should show up as a slight departure from the local unitary evolution (3.8) when one passes through a sufficiently large number of adjacent domains $\mathcal{G}_j$. Working out these features in detail is certainly a difficult issue in a realistic model. Concerning their conceptual
status, we expect that all uncertainties in realizing this program (with regards to the words "approximate", "local" and "smooth") arise just at the same level of accuracy which is limited by the WKB-philosophy altogether. There are exceptions such as the case of a flat DeWitt metric and positive constant potential $U$ (this is in fact the flat Klein-Gordon equation). There, the WKB-expansion can be performed to all orders without encountering any unitarity-violating term. As already remarked, this is due to the fact that a timelike Killing vector ensures the existence of a unique decomposition $K$ into negative/positive frequency states. Thus, the approximate nature of interpretation in quantum cosmology can be traced back to the very nature of any realistic Wheeler-DeWitt equation, as opposed to the flat Klein-Gordon case.

We thus do not allow for "jumps" between different (quasiclassical) backgrounds (cf. Ref. [66]). Unitarity-violation is usually considered as an indication of instability [26], and the relevant time scales are computed (see e.g. Ref. [37]). In our view only smooth transitions will occur, where the word "smooth" means that adjacent local quantum structures match, as described above. No abrupt jump between macroscopically different WKB-branches is possible. To some accuracy the slight change of local physics (the long-term departure of the quasiclassical history from classical solutions) may be described by the version of semiclassical gravity that includes the "back-reaction" according to the modified Einstein-equations $G_{\mu\nu} = \langle T_{\mu\nu} \rangle$ or some minisuperspace version thereof. This "principle of continuity" matches perfectly our interpretation of false quantum mechanical situations as logically separating pieces of trajectories from each other (as opposed to predicting a jump). This does not mean that the universe behaves peacefully: Drastic events are possible, if clearly predicted by a wave function (see Section 11 for an example). Also, sometimes very drastic tunnelling phenomena — such as the tunnelling of the universe into a de Sitter phase — are supposed to be possible in some models (see e.g. Ref. [4]), although with strongly suppressed probabilities. Most of them are admitted by our scheme as well, in particular if they arise from a (local) standard quantum mechanical interpretation (e.g. under-barrier tunnelling). Thus, our scheme does not at all truncate the variety of things that might happen.

To sum up, our approach to long-term phenomena is simply to treat the higher orders in a WKB-expansion — a soon as unitarity-violation appears — as irrelevant inside a local WKB-domain.

It is conceivable that any further attempt to formulate predictions more accurate than allowed by WKB-techniques would mean to leave the realm of conventional
physics and all its concepts. We emphasize that this is a consequence of our conceptual ("minimal") point of view, and that other positions are of course possible. Also, we must say that there is certainly some form of "experience" possible in regions where all WKB-concepts begin to break down heavily (shortly "before" the big crunch, say). But it is hard to say whether this has anything to do with "observations" in the sense of physics. At least we feel that it would be a realm of nature in which questions formulated in terms of the language of conventional physics cannot be posed. This sort of possible breakdown of predictability has no counterpart in ordinary quantum mechanics (where, e.g., a momentum or energy eigenstate is an unambiguously well-defined concept). Formally, it arises because most relevant fundamental observables of interest (e.g. the momentum of the matter field \( p_\phi = -i\partial_\phi \)) are represented by operators that do not commute with the constraint operator \( \mathcal{H} \) defining the Wheeler-DeWitt equation (2.3). They are thus not operators acting on the space of solutions \( \mathcal{H} \), and their eigenfunctions are not allowed as wave functions. In terms of the general theory of quantization of parametrized systems, they do not even qualify as "observables" [29][30], although at the WKB-level there is certainly some information about their distribution in a wave function. Measuring "the value" of such an observable to a degree more accurate than allowed by its identification in a reasonable WKB-approximation is simply impossible, because at such an accuracy there is no unique notion of what it should mean in terms of observations. (If, on the other hand, an operator like \( p_\phi \) happens to commute with \( \mathcal{H} \) in a simple model, we have the exceptional case of a completely well-defined observable whose eigenstates satisfy the Wheeler-DeWitt equation and need no approximate WKB-arguments for their identification). Maybe a final answer to the question what kind of experience one could make in almost genuine quantum gravitational situations amounts to feed the theory with information about all the particular objects present there, in particular the human body, including the brain and the like.

On the conceptual level (leaving cosmology for the moment), one might object that at least the problem of the final state of black holes should provide an observational "window" towards a full quantum gravity [28]. This is certainly true, but the experimental device by which physical observations of quantum black holes are performed will, for example, be at some distance from these objects (or "separated" from them by the appearance of largely different energy scales) and thus in a WKB-type environment. Nevertheless, the wave function satisfies the Wheeler-DeWitt equation exactly. In this sense, the final state of black holes is "described" by full quantum gravity, and the non-WKB features of the wave function in certain
domains of (mini)superspace will of course be essential. Although the mathematical
details are far from being clear, this seems to be a beautiful example how an exact
underlying mathematical framework might interplay with the approximate nature
of extracting physical information. Posing questions that refer to a situation too
"close" to such a quantum gravity process would run into the problem that the
precise description of some "measurement" (that should be performed in order to
test the theory) becomes impossible on account of the nature of the process itself.

Summarizing, a "minimal" interpretation of quantum cosmology that relies on
\((\mathcal{H},Q)\) as the only fundamental mathematical structure makes a quite radical point
of view possible: Quantum cosmology (quantum gravity) virtually destroys the lan-
guage of physics and limits the realm of nature in which we can reasonably talk about
observations — independent of who performs them — and thus about physics. A
true "Planck scale physics" would exist primarily as a mathematical theory, and
its relation to "experience" is unclear and tends to transcend what is usually called
the "physical world". We have presented our arguments only in the minisuperspace
approximation, but in principle one can try to implement analogous ideas — in par-
ticular the "minimality" of the scheme — to a full quantum cosmological framework.
The main goal such a framework could possibly achieve is to show how the concepts
of usual physics are hidden therein, and to extract predictions for all observations
that can be formulated in the conventional physics' language and that actually can
be performed. We admit that our approach does not tell us why just these identifi-
cations between approximate mathematical structures and observations have to be
made. We leave it open whether a "completion" is possible and which philosophical
status it would have.

9 Comments

Despite the extensive use of WKB-techniques, the underlying structure of our pro-
gram consists of inserting solutions of the Wheeler-DeWitt equation into the scalar
product \(Q\). This was our starting point, and it raises the question how a formalism
based on numbers \(Q(\psi_1,\psi_2)\) relates to a Bohm-type interpretation using (2.13) as
a measure on the set of flow-lines \(y_{\text{flow}}(\tau)\) of \(j\). In terms of this interpretation, it is
possible to associate with each pencil of flow lines a relative probability, irrespec-
tive of how narrow the pencil is. However, the expression (2.13) cannot simply be
written down in terms of numbers of the type \(Q(\Xi,\psi)\), with \(\Xi\) being a solution of
the Wheeler-DeWitt equation. It is rather of the type $Q(\psi', \psi')$, where $\psi'(y)$ is a function that is defined to be identical to $\psi(y)$ for all points $y$ through which a trajectory of the pencil $B$ passes, and zero otherwise. Hence, $\psi'$ does not in general satisfy the Wheeler-DeWitt equation. Among all functions that arise from $\psi$ by cutting away its values outside a tube $T$, it is just the functions of the type $\psi'$ (i.e. with $T = B$) for which the integral (2.13) is independent of the hypersurface $\Gamma$. Hence $\psi'$ mimicks a solution without actually being one, but this is enough to define a conserved current. One may in fact use this situation as a definition of the flow lines of $\mathbf{j}$. One encounters precisely the same structure in ordinary quantum mechanics, where the trajectories considered in the Bohm-interpretation [38]–[40] just serve for the same purpose as $y_{\text{flow}}(\tau)$. Interpreting these flow lines as describing an actual motion (in particular when the wave function is not approximately of the WKB-type) is not necessary in quantum mechanics, nor is it in quantum cosmology. When this situation is examined within our scheme, it turns out that we cannot assign relative probabilities to pencils of flow-lines, because an arbitrarily cut-out pencil does not correspond to an exact solution of the Wheeler-DeWitt equation. In contrast, we can compute relative probabilities of the type $|Q(\Xi^\pm, \psi)|^2$, where $\Xi^\pm$ is a solution that represents an incoming or outgoing wave packet, concentrated near $T$. When this packet is considered to be very narrow on a hypersurface $\Gamma$ as used in (2.13), it will spread out very fast in a neighbourhood of $\Gamma$. It is this property which is associated with the reference state $\Xi$, and it has nothing to do with a pencil-like structure, nor with a conserved current. The current $\mathbf{j}$ may be viewed as a mathematical object which is of minor importance, just as this point of view is possible in ordinary quantum mechanics. In this sense, the interpretations based on (2.13) are not equivalent to ours. However, they have a reasonable place within our framework, and the relative probabilities (2.13) may be regarded as expressing a Bohm-type interpretation of the ordinary quantum mechanics structure (3.8) inside the local WKB-branches. This may be read as an argument in favour of the Bohm-interpretation, but at the level of quantum mechanics it is just a matter of talking about the formalism, and may (but need not) be adopted. However, all arguments we gave in this discussion rely on the WKB-approximation. There is a lower bound in the width of packets and initial conditions, below which the approximation gets spoiled. This provides a further conflict between the Bohm-interpretation and ours. In Section 10, we will discuss this point when having a simple example at hand.

We would like to add a remark concerning the formalism of ordinary quantum mechanics and the way it emerges. Adopt the phrase "everything shall be ex-
pressed in terms of $Q(\psi_1, \psi_2)$ and retain it even in the local WKB-branches and their associated quantum structure. To be specific, let $\psi_{1,2}$ be two approximate WKB-type wave functions of the type (3.7). As we have seen in equation (5.2), the scalar product $Q$ boils down locally to the scalar product of ordinary quantum mechanics, but with two solutions of the time-dependent Schrödinger equation inserted. Taking our proposal serious that probabilities should be expressed in this way, the question arises whether one can reconstruct the standard formulae for amplitudes which are rather of the type $\langle a | \chi_t \rangle$, where $|a\rangle$ is a time-independently defined eigenstate of some operator $A$ to an eigenvalue $a$. The answer is quite simple, and it makes use of the particular role time plays in quantum mechanics: Let $|\eta_t\rangle$ be a solution of the time-dependent Schrödinger equation that coincides with $|a\rangle$ at $t = t_0$. To the approximation at which quantum mechanics is valid, this state is unique. Then the scalar product $\langle \eta_t | \chi_t \rangle$ is just $\langle a | \chi_{t_0} \rangle$, and it is of the required form. Thus, the conventional formalism is in accordance with our proposal. The practical use of wave functions like $|\eta_t\rangle$ for predictions relies on our ability to ”prepare” states strictly at $t = t_0$. This is possible within the realm of conventional quantum mechanics, where time plays the role of an external parameter and, moreover, the underlying wave equation is of first order in $\partial_t$.

As a further remark — that shall prevent misunderstandings — we emphasize that the status of quantum cosmology in the approach using the Wheeler-DeWitt equation is that of a second-quantized field theory in the ”position representation”. Although in the minisuperspace approximation most of the degrees of freedom have been truncated, the wave function $\psi(y)$ is just a simplified version of $\psi[g_{ij}(x), \phi(x)]$. Thus, our particular interpretational scheme to use only scalar products of solutions to the field equation does not carry over to the Klein-Gordon equation, which may be regarded as the first-quantized model for a scalar field, or as a classical model whose quantization is second-quantized. In the wave functional $\psi$, the field $\phi(x)$ just serves as a variable and is never subjected to a Klein-Gordon equation. Whenever we talked about this equation so far, we had in mind the mathematical similarities and differences to the Wheeler-DeWitt equation, not an alternative interpretation of quantum field theory.

It is sometimes proposed that a proper theory of quantum cosmology should be third-quantized (see e.g. Refs. 68, 70). In such a case, our interpretation as it was formulated will clearly need some modification. Nevertheless, the idea of using reference wave functions whose physical contents is known as far as observations are concerned is in perfect accordance with the needs of third quantization. In an ex-
pansion of the type (7.6), the coefficients \( c_±^r \) would become operators associated with the "modes" \( \Xi^± \). A fundamental theory could be defined by a Fock space structure with respect to any decomposition \( K \) (the superscripts \( ± \) on the operators \( c_±^r \) denoting creation and annihilation, respectively). An appropriate set of commutation relations (for any given \( K \)) is provided by

\[
[c_±^r, c_±^s] = \delta_{rs}
\]

\[
[c_±^r, c_±^s] = 0 .
\]

(9.1)

A change of decomposition would have to be represented as Bogoljubov-transformation, just as what is usually done in the formalism of quantum field theory in curved space-time [48]. The notation of states, i.e. superpositions of objects of the type

\[
|\Phi\rangle = c_1^+ c_2^+ \cdots c_N^+ |0\rangle_K ,
\]

(9.2)

should thus exist at the exact fundamental level. Writing down a given state \( |\Phi\rangle \) in this way as element of a Fock space, a decomposition \( K \) must be selected. Here, the approximate nature of conventional physics comes in exactly as it does in the second-quantized version: Taking into account what kind of observations we have in mind, we choose the approximate local decomposition as constructed in Section 7. However, a third-quantized formalism as sketched here is usually regarded as a free theory, and the major new possibility one has is to introduce interactions ("between" different universes, such as joining and splitting). Such a construction could possibly accommodate for topology fluctuations and the creation and annihilation of baby-universes, thereby using as little pre-supposed mathematical structure as is reasonably possible, in accordance with the spirit of the rest of this work.

10 Wave packets in the Hawking model

So far, we have used the Hawking model just to illustrate some expressions in a simple context. Let us look at it now in a bit more detail. In this section we will analyze approximate wave functions in the inflationary domain — which is specified by the conditions \( |\phi| \gg 1 \) and \( m^2 a^2 \phi^2 \gg 1 \) — and work out the associated local quantum structure as an example. There is a family of classical paths that is particularly important, and we will confine our consideration to these. A typical trajectory of this family starts at some \( \phi = \phi_0 \), along one of the two curves (hypersurfaces) at which \( U = 0 \), i.e. \( m^2 a^2 \phi^2 = 1 \). In a time gauge \( N \neq 0 \), its initial condition is \( \dot{a} = \dot{\phi} = 0 \), and from its initial point \((a_0, \phi_0)\) it runs into the region \( U > 0 \). If \( |\phi_0| \gg 1 \), it
represents the inflationary era of a classically expanding universe, until eventually \(|\phi|\) becomes of order unity (and inflation comes to end, followed by oscillations of \(\phi\) with decreasing amplitude — see Section 11). These paths appear in the context of the no-boundary proposal as those dominating the Euclidean path integral in the inflationary domain \([4,13]\) (by which they may be called no-boundary trajectories), as well in the context of the outgoing mode proposal \([18]\) and chaotic inflation \([28]\).

Without loss of generality, we restrict ourselves to positive \(\phi\). In addition to the WKB-approximation (which is perfectly applicable within the inflationary region), we assume \(\phi \gg 1\) and \(U \approx m^2 a^3 \phi^2 \gg a\). The inflationary nature of the expansion tends to swallow information along the trajectories, and the mathematical aspect of this feature is that retaining a reasonable degree of accuracy is quite cumbersome. So we will write down only the very leading order in most expressions (and use the \(\approx\) sign only in case of very crude approximations). The trajectories defined above have been studied in very detail by Page \([21]\). The corresponding action is given by

\[
S = -\frac{1}{3m^2\phi^2} \left( m^2 a^2 \phi^2 - 1 \right)^{3/2} + \frac{\pi}{4} \approx -\frac{1}{3} m \phi a^3. 
\] (10.1)

The trajectories (which, according to our notation, are outgoing ones, describing expanding universes), are given by the curves

\[
a = \frac{1}{2m\phi_0^{2/3} \phi^{1/3}} \exp \left( \frac{3}{2} \phi_0^2 - \phi^2 \right) \approx \exp \left( \frac{3}{2} \phi_0^2 - \phi^2 \right). 
\] (10.2)

This expression is valid as long as \(1 \ll \phi \ll \phi_0 - 1/(3\phi_0)\). In the proper time gauge we have \(N = 1\), the evolution parameter along each trajectory being \(t_{\text{proper}} = 3(\phi_0 - \phi)/m\). We will now write down the local quantum structure associated with this congruence, and simply apply the general formalism as described in Section 12. The most convenient coordinates for the quantum case are \(\phi_0\) (corresponding to the \(\xi^a\) of Section 12) and \(t \equiv -S\). The transformation from \((a, \phi)\) to \((t, \phi_0)\) is provided by (10.1) and (10.2). To leading order, the DeWitt metric \((2.5)\) reduces to

\[
ds_{\text{DW}}^2 = -\frac{1}{a^3m^2\phi^2} dt^2 + \frac{a^3\phi_0^2}{\phi^2} d\phi_0^2.
\] (10.3)

The "lapse" of this metric is thus given by \(N = U^{-1/2}\), the "shift" (corresponding to \(N_a\) of Section 12) vanishes identically. The (physical) lapse defining the negative of \(S\) as time parameter is given by \(N = U^{-1}\), and the prefactor from \((3.7)\) —
see also equation (12.13) — is given by $D^2 = m^{-1} a^{-3} \dot{\phi}_0^{-1} f(\phi_0)$, where $f(\phi_0)$ is an arbitrary function defining the particular representation. Choosing $f(\phi_0) \equiv 1$, the wave function can be written as

$$\Xi(t, \phi_0) = \frac{m^{-1/2} a^{-3/2} \phi_0^{-1/2}}{\chi(t, \phi_0)} \exp(-it). \quad (10.4)$$

The prefactor coincides in this approximation with the one given in Ref. \[21\]. The local scalar product takes the form

$$Q(\Xi_1, \Xi_2) \approx -\int d\phi_0 \chi_1^*(t, \phi_0) \chi_2(t, \phi_0) \equiv -\langle \chi_1, t | \chi_2, t \rangle \quad (10.5)$$

where the minus sign comes from the fact that we are considering the $\mathcal{H}^-$ subspace of expanding universes. (The approximate local decomposition $K$ in terms of incoming/outgoing modes is quite clear here, and all wave functions constructed around this family of trajectories are in $\mathcal{H}^-$). Hence, the probability density associated with a state in the WKB-interpretation is just

$$dP(t, \phi_0) = |\chi(t, \phi_0)|^2. \quad (10.6)$$

This quantity is essentially the integrand of (2.13) and is usually computed by using curves of (small) constant $a$ or $U$ to play the role of $\Gamma$. It plays a major role in the discussion about the likeliness of sufficient inflation \[1\]–\[22\], \[71\]–\[73\] and the comparison of the no-boundary versus the tunnelling proposal. Inserting (10.4) into the WheelerDeWitt equation (2.8) — or likewise computing the effective Hamiltonian as given by (12.32) straightforwardly — we find as the dominant contribution

$$i \frac{\partial}{\partial t} \chi(t, \phi_0) = -\frac{1}{18} t^2 \left( \frac{\phi}{\phi_0} \right)^{1/2} \frac{\partial}{\partial \phi_0} \phi \frac{\partial}{\partial \phi_0} \left( \frac{\phi}{\phi_0} \right)^{1/2} \chi(t, \phi_0). \quad (10.7)$$

The Hamiltonian is hermitean with respect to (10.3). Also, recall that $\phi$ is understood as a function of $t$ and $\phi_0$ (one finds $\phi^2(t, \phi_0) \approx \phi_0^2 - 2 \ln(t)/9$). This is an example for a local quantum structure of the type (3.8), i.e. an approximate unitary evolution, whose Hamiltonian is not at all of the standard form. Since $t$ grows rapidly as the evolution goes on, the action of the effective Hamiltonian on wave functions will be suppressed, and $\chi$ becomes essentially a function of $\phi_0$ alone. The probability density (10.6) also becomes $dP(\phi_0)$, and this is the approximation that is usually invoked in the discussion of the inflationary domain. The fact that (10.4) contains a slight dependence on $t$ — and thus does not exactly provide a
probability density on the set of classical trajectories — just reflects the difference between classical paths and the flow lines of the current \((2.10)\).

Let us discuss whether the WKB- or Bohm-inspired interpretations which consider \(dP(\phi_0)\) as an approximate (relative) probability density on the set of classical paths is equivalent to ours. In Section 9, we have already considered an aspect of this question, and we would like to add another one here. We may use \((10.7)\) to construct wave packets. Note that this evolution equation implies a very tiny amount of spread if \(t\) is large. Given a wave function \(\psi\), the typical (relative) probabilities as used in our scheme are \(P = |Q(\Xi, \psi)|^2\), where \(\Xi\) is an outgoing (hence \(\mathbb{H}^-\)) wave packet-like state. One might expect that a notation of probability of this type, based on the use of various wave packets \(\Xi\), is equivalent to stating that \(dP(\phi_0)\) is a probability density. This would be perfectly true in conventional quantum mechanics.

However, in our case the WKB-approximation was invoked in order to obtain \((10.7)\). In this approximation, \(\chi(t, \phi_0)\) may be prescribed at \(t = t_0\), and the Schrödinger time evolution fixes \(\chi\) uniquely. This enables one to prescribe \(\chi\) to be non-zero only in an arbitrarily narrow interval of \(\phi_0\) at \(t = t_0\). (Compare the discussion of ordinary quantum mechanics in Section 9). The expression \(P = |Q(\Xi, \psi)|^2\) is just the standard quantum mechanical probability for finding \(\phi_0\) in the given interval, when measured at time \(t_0\). The drawback is that — as we have already mentioned — if the width of the packet becomes too narrow (i.e. the change of \(\chi\) being too large), the WKB-approximation based on \(S\) from \((10.1)\) will break down. Prescribing a wave function \(\Xi\) at the ”initial” curve \(t = t_0\) to be a narrow peak in some interval of \(\phi_0\), and using the full Wheeler-deWitt equation to describe the dynamics, some information about the time-derivative of \(\Xi\) is missing. Without adding this information to the initial conditions, the state \(\Xi\) is simply unknown! In a sense, this situation corresponds to the uncontrolled production of particles in quantum field theory if one likes to look at some process too accurate. In the WKB-approximation, our interpretation tells us that the time-derivative of \(\Xi\) at \(t_0\) must be chosen such that it is compatible with \((10.7)\). This makes \(dP(\phi_0)\) effectively a probability density for sufficiently large intervals of \(\phi_0\). Hence, the use of very narrow initial conditions at \(t_0\) is limited by the requirement of assigning a clear operational significance to the situation. As opposed, in the WKB-approximation such a significance is provided by our ability to identify the ”instant of time” \(t_0\). Thus, no predictions are possible for intervals of \(\phi_0\) whose size are below a certain bound. This is a conceptually important (though not very practical) difference to
the standard WKB-scheme based on the notation of a probability density. The limitation of observations — caused by the breakdown of local quantum mechanics — is reflected automatically in our interpretation: No prediction is possible that is not expressed in terms of the $Q$-product between solutions of the Wheeler-DeWitt equation. In this sense the breakdown of the WKB-approximation for very narrow wave packets is not just a shortcoming of a convenient formalism, but rather a crucial piece of our interpretation.

11 Approximate matter energy eigenstates in the Hawking model

The probabilistic interpretation of wave functions in the Hawking model is usually based on the variable $\phi_0$, labelling the no-boundary trajectories, as used in the foregoing Section. From the knowledge of a distribution in $\phi_0$, sometimes conclusions are drawn to other quantities, such as the density parameter \[14\]. It would however be convenient to have some basis of $\mathcal{H}$ at hand which is related more directly to features we can observe in the post-inflationary domain. Such a family is provided by the approximate eigenstates of the matter energy after inflation. Therefore, we consider trajectories that enter the domain $|\phi| \lesssim 1$ after an inflationary era (these trajectories may — but need not — be in the no-boundary family whose behaviour in the inflationary domain was analyzed in the foregoing Section). Any such trajectory begins to oscillate with decreasing amplitude. This forces the scale factor $a(t)$ to decelerate (matter dominated phase). Eventually the universe reaches a maximum size $a_{\text{max}}$ (this happens when the amplitude of $\phi$ falls into the $U < 0$ domain, i.e. inside the pair of curves $\phi = \pm 1/(am)$), and recontracts. The matter energy $E$ (defined to be measured in proper time, i.e. in the gauge $N = 1$) is approximately conserved during this era, and its value is related to the maximum size of the universe by $a_{\text{max}} = 2E$. Quantum mechanically, it is given by the operator

$$E = -\frac{1}{2a^3} \partial_{\phi\phi} + \frac{1}{2} a^3 m^2 \phi^2$$

which appears (multiplied by $2a$) in the Wheeler-DeWitt equation \[2.8\]. Since $E$ does not commute with the Hamiltonian constraint, there will be no exact energy eigenstates in $\mathcal{H}$. However, one may find approximate eigenstates, and this is what we sketch now. Since the details about this issue will be published elsewhere \[74\].
we will be brief, and mainly write down the results. The mass $m$ of the scalar field shall be of the order of magnitude $10^{-6}$ (in the dimensionless units of Ref. [13]), in order to account for the necessary amount of density fluctuations [12] [28].

Using the fact that (11.1) is of harmonic oscillator type in $\phi$, with frequency $m$ and mass $a^3$, we infer that the eigenstates of $E$ have eigenvalues $E_n = (n + \frac{1}{2})m$. The classical universes with energy $E_n$ enter the post-inflationary domain at a size $a_{\text{min}} \approx (n/m)^{1/3}$ and reach a maximum size $a_{\text{max}} \approx 2mn$. This immediately implies $n \gg m^{-2} \approx 10^{12}$ as condition for $E_n$ to describe the energy of a real universe. Any such eigenvalue is associated with two approximate solutions of the Wheeler-DeWitt equation, given by

$$\Xi_n^\pm(a, \phi) \approx \left(\frac{m}{2E_n}\right)^{1/4} \exp\left(\pm \frac{2i}{3} (2E_n)^{1/2} a^{3/2}\right) \Psi_n(m^{1/2}a^{3/2} \phi),$$

where $\Psi_n$ are the unit harmonic oscillator eigenfunctions (e.g. $\Psi_0(x) = \pi^{-1/4} \times \exp(-x^2/2)$). The above form is valid for $a_{\text{min}} \leq a \ll a_{\text{max}}$ and extends in $\phi$ over the range in which the $\Psi_n$ are oscillating. For even larger $|\phi|$, the $\Xi_n^\pm$ become strongly suppressed. These wave functions have been used by Kiefer [35] to construct wave packets, and a particular combination of $\Xi_0^\pm$ (called $\Omega_0^\pm$, see equation (11.3) below) is mentioned by Page [21] as the dominating part of the no-boundary wave function in the Euclidean domain $|\phi| < 1/(ma)$. Moreover, Hawking and Page have encountered wave functions related to these [75] in a wormhole context. The $\Xi_n^\pm$ are normalized according to (7.3)–(7.5). By appropriate continuation to $a > a_{\text{max}}$, one may infer the asymptotic as well as the analyticity structure of these wave functions. In Ref. [74] we provide evidence that $\Xi_n^\pm$ may be defined exactly. Also, small values of $n$ (integer and non-negative) may be included, but these wave functions are mere tunnelling states. One thus seems to end up with an exact basis of $IH$ into which any given wave function may be expanded.

The physical interpretation of the $\Xi_n^\pm$ is clear: they represent contracting and expanding universes, respectively, whose approximate value of the matter energy in the domain $|\phi| \lesssim 1$ is $E_n$. They also define a decomposition $K$ which corresponds to finding the universe contracting ($\Xi_n^+$) or expanding ($\Xi_n^-$) when the observation is done in the domain $|\phi| \lesssim 1$ (i.e. in the ”post”- or ”pre”-inflationary era, according to the orientation of the classical paths). Thus, we have a perfect set of reference states at hand (just note that for $n \lesssim m^{-2}$ a relation to an observation is not possible).

If it is actually possible to define a basis $\Xi_n^\pm$ exactly by means of analyticity arguments, one may think of considering the resulting decomposition $K$ as a pre-
ferred one upon which all interpretational issues are based. (We have pointed out this possibility in Section 7, the preferred decomposition was called $K_0$ there). It is however not clear whether $K$ coincides with the approximate local decompositions as constructed in Section 7 in other domains (i.e. for small $a$ and large $|\phi|$). Also, we do not know to what extent the appropriateness of the asymptotic analyticity structure for defining an exact decomposition carries over to other models. It will not be necessary here to specify the philosophy so as to consider $K$ as an exact or an approximate object, but this choice provides an interesting issue for future research.

As an example of an expansion of a state into $\Xi^\pm_n$, we consider the no-boundary wave function. In the inflationary domain it can be written as \[11.3\]

$$\psi_{NB}(a,\phi) \approx \frac{1}{\pi^{1/2}a\sqrt{a^2m^2\phi_0^2 - 1}} \left( \sqrt{6} + 2 \left( \exp\left(\frac{1}{3m^2\phi_0^2}\right) - 1\right) \right) \cos(S)$$

with $S$ from \[10.1\], and $\phi_0$ now interpreted as a function of $a$ and $\phi$. Expanding $\psi_{NB}$ in terms of the basis (see equations \[7.6\]–\[7.7\]) is a bit tricky because not much details about the wave function in the domain $|\phi| < \sim 1$ are known. Without any computation we just state that a rough estimate \[74\] yields for $n \gg m^{-2}$

$$c^+_n = (c^-_n)^* = \frac{1}{\sqrt{6\pi n}} \frac{1}{\sqrt{2\ln(nm^2)}} \left( \sqrt{6} + 2 \left( \exp\left(\frac{3}{2m^2\ln(nm^2)}\right) - 1\right) \right) k_n$$

for even $n$, where the $|k_n|$ are of order unity, and $c^+_n = 0$ for odd $n$. The first equality is exact, it stems from the fact that $\psi_{NB}$ is real. The result shows that very large $n$ are suppressed (note that the associated probabilities are $P^\pm_n = |c^\pm_n|^2$), at least as long as universes nucleating with densities above the Planck scale ($n \gtrsim m^{-2} \exp\left(\frac{9}{2m^2}\right)$) are excluded. This is a well-known problem of the no-boundary wave function to produce sufficient inflation and a universe of realistic size \[1\]–\[22\], \[71\]–\[73\]. (On the other hand, in the limit $n \to \infty$ the energy spectrum becomes $|c^\pm_n| \sim n^{-1}$, which makes arbitrarily large $n$ to dominate as long as no upper cutoff for $n$ is introduced. Also cf. Ref. \[14\]).

There are two particularly interesting real superpositions $\Omega^\pm_n$ of the $\Xi^\pm_n$, defined by

$$\Xi^\pm_n(a,\phi) = \exp \left( \pm i \left( \frac{\pi}{2} E^2_n + \frac{\pi}{4} \right) \right) \left( \Omega^+_n(a,\phi) \mp i\Omega^-_n(a,\phi) \right) .$$

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For $a \gg a_{\text{max}}$ they have the asymptotic behaviour $\Omega_n^\pm \sim \exp(\pm a^2/2 \mp aE_n)$. Expanding $\psi_{\text{NB}}$ in terms of these, one finds — up to some uncertainty in the approximations involved — that the expansion coefficients with respect to both types of states $\Omega_n^\pm$ are non-zero. This matches the well-known fact that $\psi_{\text{NB}}$ grows exponentially with $a$ when $|\phi|$ is sufficiently small. The dominating part for large $a$ and $\phi = 0$ is $\Omega_0^+$ (a pure tunnelling state), and it is just this function that has been displayed by Page [21] as the exponentially growing piece of $\psi_{\text{NB}}$.

Let us insert here a remark about the tunnelling wave function [18]. It is usually required to be bounded, $|\psi_T(a,\phi)| < \infty$. The outgoing mode proposal operates at the ”singular” boundary of $M$, i.e. it is concerned with the contribution of trajectories describing a singular behaviour $|\phi| \to \infty$ as $a \to 0$. This makes it difficult to compute $\psi_T(a,\phi)$ in the post-inflationary domain. However, if all necessary conditions may consistently be implemented (which does not seem evident to us; at least Hawking and Page [75] provide a hint for the existence of wave functions that are everywhere bounded), the tunnelling wave function is apparently a superposition of the $\Omega_m^-$ alone (although with complex coefficients).

We will now use the Hawking model to discuss the issue of true and false local quantum mechanics, as introduced in Section 4. The major part of the post-inflationary phase provides an example of a local quantum structure for which reasonable wave functions lead to true local quantum mechanics. Consider a family of universes with (post-inflationary) matter energy $E$, in the domain $a_{\text{min}} \approx (E/m^2)^{1/3} \ll a \ll a_{\text{max}} = 2E$. This automatically guarantees $|\phi| \ll 1$. The curvature part of the potential can completely be ignored, as well as the fact that the hypersurfaces of constant $a$ have a small part in which they are not ”spacelike”. The approximate evolution (in the proper time gauge $N = 1$) is given by

$$a(t) = \left( \frac{3}{2} \sqrt{2E} t \right)^{2/3} \tag{11.6}$$
$$\phi(t) = \pm F(a(t)) \sin(mt + m\xi) \tag{11.7}$$

where the (slowly changing) amplitude of the oscillations is

$$F(a) = \frac{\sqrt{2E}}{ma^{3/2}} \tag{11.8}$$

thus $F(a(t)) = 2/(3mt)$. The scale factor evolution is of the matter dominated type. (If $a$ becomes of the order of magnitude of $a_{\text{max}}$, the expansion settles down.
The equation describing this behaviour is $\dot{a} = \sqrt{(2E/a) - 1}$, but we will stick to the domain in which (11.6) is valid). The parameter $\xi$ labels the trajectories, and $E$ is regarded as fixed (the condition $E \gg m^{-1}$ ensures $a_{\text{min}} \ll a_{\text{max}}$). Adjacent trajectories intersect each other at the caustic $\phi = F(a)$, which allows us to follow a single path only during half a period in which the sign of $\dot{\phi}$ is non-zero. The action associated with these trajectories is given by

$$S_{\pm} = -\frac{2}{3} \sqrt{2E} a^{3/2} \pm \frac{E}{m} \left( \arcsin \left( \frac{\phi}{F(a)} \right) + \frac{\phi}{F(a)} \sqrt{1 - \left( \frac{\phi}{F(a)} \right)^2} \right),$$

(11.9)

where the double sign denotes the sign of $\dot{\phi}$. The sign of the first term is chosen such that $\dot{a} > 0$. This first part of the action — let us denote it by $S_0(a)$ — suggests itself as describing a (semiclassical) expansion around the matter dominated background evolution $a(t)$. It also appears in the exponential of (11.2). The second part $S_q(a, \phi)$ would then belong to the (quantum) fluctuations of $\phi$. Upon exponentiating $iS_q$ and taking the real part, one just recovers the $\Psi_n$-part in (11.2) with $n = E/m$ and small $\phi$. Hence, the $\Xi_n$ are associated with the two WKB-branches given by $S_{\pm}$. One may now straightforwardly apply the formalism of Section 12 to reveal the according local quantum structure. We use the coordinates $(t, \xi)$ from (11.6)–(11.7), and restrict ourselves to a small domain of $\xi$ such that $\cos(mt + m\xi) > 0$. The DeWitt metric becomes (approximately) decomposed as

$$ds^2_{\text{DW}} = -2E dt^2 + 2E \cos^2(mt + m\xi) (d\xi + dt)^2$$

(11.10)

(cf. equation (12.10)), and the effective Schrödinger equation reads (for both $S_{\pm}$ branches)

$$i \frac{\partial}{\partial t} \chi(t, \xi) = -\frac{1}{4E} (f(\xi) \cos(X))^{-1/2} \frac{\partial}{\partial \xi} \sin^2(X) \frac{\partial}{\partial \phi} \cos(X) \left( f(\xi) \right)^{1/2} \chi(t, \xi),$$

(11.11)

where $X \equiv m(t + \xi)$, and $f(\xi)$ is the arbitrary function related to the prefactor $D^2 = f(\xi)/(2E \cos(X))$ from (12.15) specifying the representation. This is a particular example for a local quantum structure (3.8). In order to display it in terms of more convenient coordinates, we rewrite this equation in terms of $(t, \phi)$. Setting $\chi(t, \xi) \equiv \tilde{\chi}(t, \phi)$ and $f \equiv 1$, the result is

$$i \frac{\partial}{\partial t} \tilde{\chi}(t, \phi) = \left( -\frac{m^2}{4E} (F(t)^2 - \phi^2) \right)^{1/2} \frac{\partial}{\partial \phi} \phi^2 \frac{\partial}{\partial \phi} (F(t)^2 - \phi^2)^{-1/2}$$
for the branch $S_\pm$, where $F(t)$ is sloppy for $F(a(t)) = 2/(3mt)$. This is a special case of equation (12.50). As compared to (11.11) it has the advantage that its range of applicability is not constrained to small tubes around a trajectory. It may be regarded as the Schrödinger equation for approximate eigenstates of $E$ in an expanding universe around trajectories with prescribed sign of $\dot{\phi}$. At $\phi = F(t)$ (the location of the caustic) the WKB-approximation breaks down. Classically, it is just here that the sign of $\dot{\phi}$ changes. For $\phi \ll F(t)$ and large $t$, the first term in the effective Hamiltonian becomes suppressed, and we get $(3t\partial_t \pm 2\partial_\phi)\tilde{\chi} = 0$, admitting a constant $\tilde{\chi}$. The physical wave function is a superposition of an $S_+$ and an $S_-$ solution that behaves reasonable for $\phi > F(t)$. For a given wave function (such as $\Xi_-$) this is clear, since it is well-behaved as $|\phi| \to \infty$. This setup may be interpreted as a true quantum mechanical structure.

If one wishes to further analyze each of the two equations (11.12) separately, some simplification is possible. The first part of the Hamiltonian is $H_{\text{eff}}$ from (11.11), rewritten in terms of the coordinates $(t, \phi)$. The total Hamiltonian may — by means of a simple integration — be rewritten as $e^{-iG_\pm}(H_{\text{eff}} + V)e^{iG_\pm}$ with $G_\pm \equiv G_\pm(t, \phi)$ and $V \equiv V(t, \phi)$ being pure functions. $V$ has a small imaginary part stemming from the particular operator ordering that may presumably be neglected. Redefining the wave function as $\chi = e^{iG_\pm}\tilde{\chi}$, and setting $W = V - \partial_t G_\pm$, the equations (11.12) attain the more familiar form $i\partial_t \chi(t, \phi) = (H_{\text{eff}} + W)\chi(t, \phi)$.

In order to pass over to the semiclassical picture, one would either identify $S_0 = -\frac{2}{3}\sqrt{2E_a^3/2} = -2Et$ as the classical "background" action or perform a picture changing transformation on the Hamiltonian from (11.12). The first method amounts to impose the semiclassical WKB-ansatz (11.3), i.e. (ignoring the prefactor $D_0$) $\psi = \tilde{\chi}\exp(iS_0)$ (although $S_0$ has not been constructed by an $\ell_P \to 0$ or some related limit). This results to leading order into (using (11.9) for the conversion between $t$ and $a$ as variables, hence $\tilde{\chi} \equiv \tilde{\chi}(t, \phi)$ or $\tilde{\chi} \equiv \tilde{\chi}(a, \phi)$, as one prefers)

$$i\partial_t \tilde{\chi} \equiv i\sqrt{\frac{2E}{a}}\partial_a \tilde{\chi} = (E^{\text{op}} - E)\tilde{\chi}, \quad (11.13)$$

where $E^{\text{op}}$ is the operator (11.1) for the matter energy, as opposed to $E$ as the classical parameter characterizing the background action $S_0$ and hence the quasiclassical
evolution (11.6). In a sense, this equation is a special case of a local quantum structure (3.6) as emerging in the conventional WKB-approach. Taking into account the harmonic oscillator type of $E^{op}$ and imposing an adiabatic approximation, we obtain solutions of the form

$$\tilde{\chi}(t, \phi) \approx \exp \left( -i(E_n - E)t \right) \Psi_n \left( \frac{3}{2} t\phi \sqrt{2Em} \right)$$

(11.14)

for any non-negative integer $n$ and $E_n = (n + \frac{1}{2})m \approx nm$. The WKB-approximation is good if the oscillations induced by the $t$-dependence of $\tilde{\chi}$ are milder than those of the prefactor $\exp(iS_0) \equiv \exp(-2iEt)$ by which $\tilde{\chi}$ is multiplied to give $\psi$, i.e. if $mn \approx E$. This in turn gives back wave functions of the type $\Xi_n^-$ (cf. equation (11.2)). Also, the physical significance of the proper time $t$ (defined by (11.6)) as appearing in the semiclassical context is clarified: it is the time variable associated with the difference $\tilde{H}_{eff} = E^{op} - E$.

The second method invokes a picture changing transformation on the Hamiltonian in order to put back a factor $\exp(i(S - S_0))$ into $\chi$. Naively one would expect to arrive at $\tilde{\chi}$ from (11.13). However, this is only possible if the two branches $S_\pm$ are macroscopically equivalent. A straightforward implementation of a picture changing transformation gives identical results for the two branches only to the very crude approximation of small $\phi/F(t)$ and large $t$. In order to recover the semiclassical picture at the same accuracy as above, one must perform in addition a superposition of (i.e. an average between) the two branches $S_\pm$, thereby unifying the two underlying modes (positive and negative $\dot{\phi}$) in terms of one single quasiclassical quantum structure. Technically, this should restore a peaceful behaviour for large $|\phi|$, as in (11.2) and (11.14). However, the details are far from being clear, and the issue deserves further study. (Also note that the "wave lengths" induced by the two pieces of the action (11.3) are of comparable order of magnitude when expressed in terms of the DeWitt metric, although $S_0$ varies in "timelike" and $S_q$ varies mainly in "spacelike" direction. This might indicate that the above interpretation of the sum $S_0 + S_q$ leads us close to the breakdown of the semiclassical WKB-approximation, and trying another decomposition of the action into two parts might be reasonable. Moreover, it is not totally clear to what extent the use of the proper time gauge is appropriate).

Let us now consider another question and ask for the significance of the exponentially increasing part of $\psi_{NB}$ for small $|\phi|$ and large $a$. This part shows up as the contribution with respect to the basis elements $\Omega^+_n$. The simplest philosophy is
that an exponential contribution in a wave function does not refer to observations, and so may essentially be ignored. This would imply (and has often been stated for the no-boundary wave function) that any classical trajectory provides a possible quasiclassical (macroscopic) "history" of the universe, including the feature that it "returns" after attaining maximal size, and recontracts. The according probabilities may be computed by WKB-methods in the post-inflationary or even in the inflationary domain. On the other hand, it has been argued by Zeh and Kiefer [34][35][76] that a viable wave function $\psi$ should asymptotically vanish for $a \gg a_{\text{max}}$, and they talk about a "final condition". Only in this case, $\psi$ has a chance to be decomposed into wave packet-like states that are peaked around classical trajectories describing returning universes (i.e. paths in which $a(t)$ attains its maximum value and thereafter decreases again). These wave packets should provide a true intrinsic notation of "time" in a universe. However, the issue of expanding $\psi$ near the turning region into proper wave packets provides problems in some models (in particular in the Hawking model, as has been shown by Kiefer [35]). How does our interpretational scheme judge on this situation?

Restricting oneself to one particular energy value $n \gg m^{-2}$, the asymptotics of the wave functions $\Omega_n^\pm \sim \exp(\pm a^2/2)$ for $a$ exceeding the maximum radius associated with $E_n$ clearly shows two different sorts of behaviour. Without going into the details, it is likely that $\Omega_n^\pm$, when WKB-analyzed with respect to the local quantum structure for the domain $a \gtrsim a_{\text{max}}$ and $|\phi| \lesssim 1/(ma)$, provides effective wave functions with the asymptotics $\chi^\pm \sim \exp(\pm a^2/2)$. Note that $a$ is now a coordinate labelling the classical paths, whereas $\phi$ can be chosen as "time" coordinate $t$. Thus, in case the argument holds, $\Omega_n^-$ gives rise to a true, and $\Omega_n^+$ belongs to a false local quantum mechanical situation. In other words, $\Omega_n^-$ admits expansion, maximum size and recontraction to be experienced as the macroscopic history of the universe. The fact that wave packets cannot be formed by superpositions of $\Omega_n^-$ for various $n$ does not necessarily spoil this argument, but just adresses the details to a decoherence mechanism, as also remarked by Kiefer in his study [35] (although this issue seems to be still somewhat open: loosely speaking, it is not yet clear whether decoherence effects near the turning point are "strong enough" to ensure almost-classicality there [54]; cf. also Ref. [76]; also cf. Ref. [77] for a recent discussion of the validity of the semiclassical approximation near turning points). On the other hand, $\Omega_n^+$ interrupts the chain of reasonably true local quantum structures just in the region of return.
Hence, we conclude that any wave function

$$\psi(a, \phi) = \sum_{n=1}^{\infty} (C_n^+ \Omega_n^+(a, \phi) + C_n^- \Omega_n^-(a, \phi))$$

(11.15)

contains several possibilities for the experience of a macroscopic history of the universe. If all $C_n^+ = 0$, the universe will be experienced as expanding and recontracting. If $C_n^- = 0$, all (or most) possible macroscopic histories are likely to consist of "broken" almost-classical evolutions (the details depending on a proper decoherence analysis). The broken histories have the following properties: Either, the universe is created (i.e. becomes real) in the inflationary domain, near the curve $U = 0$, expands and reaches a maximum size, thereby undergoing a transition into a pure tunnelling state (i.e. "tunnelling into nothing"). As already mentioned in Section 8 we cannot tell much about what actually will be experienced when conventional physics is just about to break down. Or — as a logical alternative — the universe is created at its maximum size, contracts, and disappears into "nothing" at the edge of the inflationary domain. According to our interpretation, no relation between these alternatives can be experienced. Even the question whether the second possibility occurs "after" the first one has no meaning. The radicality in this statement relies on the type of scheme we apply, and we admit that a different interpretation of quantum cosmology may match just those macroscopic histories that we have separated.

As an example, the no-boundary wave function $\psi_{\text{NB}}$ has both non-zero $C_n^+$ and $C_n^-$, and thus contains all three variants (one type of "unbroken" and two types of "broken" histories) with according probabilities (which we suspect to be roughly of equal order). In contrast, any bounded wave function (such as the tunnelling wave function, if it exists) has $C_n^+ = 0$ and should thus contain only "unbroken" histories. Due to our interpretation, the false contributions in a wave function imply a "crack of doom" — the breakdown of physical laws in the universe in a way unexpected by its inhabitants (cf. Ref. [78] for the use of this notion in Kaluza-Klein theory, where a singularity in the compactified dimension leads to a totally unexpected cosmic catastrophe). However, this does by no means provide a logical inconsistency. With regards to the question whether "our universe" will survive its era of maximum expansion, one just can hope that the wave function is only a superposition of the $\Omega_n^-$. 
12 Modified WKB-expansion

This final Secton serves as an appendix. We outline the treatment of rapidly oscillating solutions of the Wheeler-DeWitt equation (2.3), and the local reconstruction of an approximate unitary evolution of the standard quantum mechanical type. Our procedure does not assume an à priori separation of the coordinates $y^\alpha$ into classical (heavy) and quantum (light) ones, nor does it assume a particular structure of the Hamiltonian with respect to small parameters such as $\ell_P$ or $\hbar$. Moreover, it is formulated in terms of geometric quantities on $\mathcal{M}$ and on the equal-time hypersurfaces of the classical trajectories, and may in this respect be considered more appealing than the standard (Born-Oppenheimer type) approach.

Let $\psi$ be written as in (3.7) and suppose that $S$ is a classical action function, i.e. it satisfies the exact Hamilton-Jacobi equation (3.2) in some domain $\mathcal{G}$ of $\mathcal{M}$ and therefore defines a congruence of classical paths. Inside $\mathcal{G}$ the trajectories do not intersect each other, and no caustics are present. The sign of the potential $U$ may be of either type, or even changing, as long as the trajectories still form a congruence, i.e. as long as they altogether cross the zero-potential surface without mutual intersection or turning points. However, for simplicity, we will only deal with the cases where $U > 0$ or $U < 0$ in the whole of $\mathcal{G}$, and lay emphasis on the former.

By choosing the lapse function $N(y)$, an evolution parameter $t$ is defined for each path via equation (3.3). The derivative of a scalar function $F(y)$ on $\mathcal{G}$ along the trajectories with respect to $t$ is then provided by $\dot{F} = N(\nabla^{\alpha}S)\nabla_\alpha F$. Moreover, suppose that $D$ is real and satisfies the conservation law

$$\nabla_\alpha(D^2\nabla^\alpha S) = 0.$$  \hspace{1cm} (12.1)

This fixes $D(y)$ only up to $D(y) \rightarrow k(y)D(y)$ where $k$ is constant along the classical paths (which corresponds to a pure "picture changing" transformation on the effective wave function $\chi$). The local WKB-branch is defined by $S$, and the representation of the hidden quantum mechanical structure will be fixed by a choice of $D$. The prefactor $\chi$ represents the particular state.

Inserting (3.7) into the Wheeler-DeWitt equation (2.3), we easily obtain an exact equivalent of the latter

$$i\dot{\chi} = K\chi.$$  \hspace{1cm} (12.2)
where

$$K = -\frac{1}{2} ND^{-1} \nabla^\alpha D \equiv -\frac{1}{2} ND^{-1} \left( D \nabla^\alpha + 2(\nabla^\alpha D) \nabla^\alpha + (\nabla^\alpha \nabla^\alpha D) \right). \quad (12.3)$$

Note that we distinguish operators like $\nabla^\alpha D : F \to \nabla^\alpha (DF)$ from $(\nabla^\alpha D) : F \to (\nabla^\alpha D)F$ by appropriate brackets. The origin of the operator $K$ may be illustrated by condensing the manipulations performed above in the form

$$D^{-1} e^{-iS} (\nabla^\alpha \nabla^\alpha - U) De^{iS} = \frac{2}{N} (iN (\nabla^\alpha S) \nabla^\alpha - K) \quad (12.4)$$

which is valid when acting on a scalar function on $\mathcal{M}$. It represents a canonical transformation.

In order to make contact with ordinary quantum mechanics we define $\Gamma_t$ to be the hypersurfaces of constant $t$, thus defining a foliation of $\mathcal{G}$. The scalar product $Q$ leads us the way to the hidden Hilbert space structure. Let $\psi_1$ and $\psi_2$ be two solutions of the Wheeler-DeWitt equation of the type (3.7) for the same $S$ and $D$, but with two functions $\chi_1$ and $\chi_2$, respectively. Then (12.3) is a sum

$$Q(\psi_1, \psi_2) = \langle \chi_1 | \chi_2 \rangle_{\Gamma} + \langle \chi_1 | \hat{\chi}_2 \rangle_{\Gamma} \quad (12.5)$$

where

$$\langle \chi_1 | \chi_2 \rangle_{\Gamma} = \int_{\Gamma} ds^\alpha (\partial_\alpha S) D^2 \chi_1^* \chi_2, \quad (12.6)$$

$$\langle \chi_1 | \hat{\chi}_2 \rangle_{\Gamma} = -\frac{i}{2} \int_{\Gamma} ds^\alpha D^2 (\chi_1^* \partial_\alpha \chi_2). \quad (12.7)$$

For the moment, we leave the orientation of the hypersurface element $ds^\alpha$ unspecified. Obviously, the sum (12.5) is independent of $\Gamma$, whereas the individual terms are in general not. By setting $\Gamma = \Gamma_t$ inside $\mathcal{G}$, and by assuming that the contribution to the integral from the part of $\Gamma$ that lies outside $\mathcal{G}$ is negligible, the scalar product (12.3) becomes a function of $t$. As we will see, this already defines the Hilbert space structure associated with the WKB-branch $(S,D)$: The identification is based on the fact that for WKB-type wave functions $\exp(iS)$ oscillating much faster than $D$ and the $\chi$’s) the scalar product $\langle \chi_1 | \chi_2 \rangle_{\Gamma_t}$ is almost independent of $t$, and $\langle \chi_1 | \hat{\chi}_2 \rangle_{\Gamma_t}$ from (12.7) is negligible.

In order to reveal the relation between the definition (12.4) and the conservation law (12.1), we note that

$$\frac{d}{dt} \langle \chi_1 | \chi_2 \rangle_{\Gamma_t} = \langle \dot{\chi}_1 | \chi_2 \rangle_{\Gamma_t} + \langle \chi_1 | \dot{\chi}_2 \rangle_{\Gamma_t}. \quad (12.8)$$
There are many ways to see this easily. Maybe the most explicit one is to introduce coordinates \( \xi^a (a = 1, 2, \ldots n-1) \) that label the trajectories \( (\dot{\xi}^a = 0) \). Choosing \( (y^\alpha) \equiv (t, \xi^a) \) as coordinate system in \( \mathcal{G} \), makes \( \dot{F} \equiv \partial_t F \equiv \partial_\alpha F \) just the partial derivative and \( \xi^a \) a coordinate system on each of the hypersurfaces \( \Gamma_t \). (Analogous coordinates have, in the standard semiclassical formulation, been used by Cosandey for the sector of ”classical variables” [27]). The identity (12.4) then becomes

\[
D^{-1} e^{-iS} (\nabla_\alpha \nabla^\alpha - U) D e^{iS} = \frac{2}{N} (i \partial_t - \mathcal{K}) .
\]

(12.9)

From now on we have to distinguish two cases.

If \( U(y) > 0 \) inside \( \mathcal{G} \), the trajectories are ”timelike” and the \( \Gamma_t \) should reasonably be chosen as ”spacelike” with respect to the DeWitt metric \( ds^2_{\text{DW}} = \gamma_{\alpha\beta} dy^\alpha dy^\beta \). (Note that, even if \( N \) has been fixed, there is still the freedom to reset the evolution parameter as \( t \to t-t_0 \) in each trajectory individually). As a consequence, the metric induced on \( \Gamma_t \) will be positive definite. If \( U(y) < 0 \) inside \( \mathcal{G} \), the trajectories are ”spacelike”, and one will reasonably choose the \( \Gamma_t \) so as to contain one ”timelike” and \( n-2 \) ”spacelike” directions. In this case the induced metric will be of Lorentzian-type signature \( (-,+,+,\ldots) \). Although these two cases may appear quite similar from the local point of view (refering to \( \mathcal{G} \) only), they are related differently to the global structure: In the \( U > 0 \) case, any \( \Gamma_t \) may be considered as just the part of a globally ”spacelike” hypersurface \( \Gamma \) — as used in (12.3) — that happens to lie in \( \mathcal{G} \). Also, one may construct wave functions with negligible values on the part of \( \Gamma \) outside \( \mathcal{G} \) (just by prescribing appropriate initial conditions for the Wheeler-DeWitt equation on \( \Gamma \)). This establishes the approximate relation between \( Q \) and \( \langle | \rangle_{\Gamma_t} \).

Also, when analyzing a given WKB-state with respect to this structure, one does not expect to encounter catastrophic (e.g. exponential) non-normalizibility effects. In the case \( U < 0 \) however, a similar construction is not possible because \( \Gamma_t \) is no longer ”spacelike”. One may of course extend \( \Gamma_t \) to a hypersurface \( \Gamma \) that qualifies for being used in the computation of (6.3). But it will in general pass through other domains (outside \( \mathcal{G} \)), giving rise to non-zero contributions, and the relation between \( Q \) and \( \langle | \rangle_{\Gamma_t} \) becomes less transparent. The latter may still be defined by (12.6), and the reconstruction of local quantum structures is to some extent possible as well, although a given WKB-state, when analyzed with respect to this structure, may display unacceptable (false) — see Section 4 — quantum behaviour with respect to normalization issues, and the interpretation in terms of observations may be impossible. In Section 7, we have related this possibility to the issue of interpretation.
For the (local) purpose of this Section, the two cases are not very different. From now on, no reference to global issues like \( \Gamma \) and \( Q \) will be important. We first discuss the \( U > 0 \) case, and afterwards just comment on the changes necessary to carry the results over to \( U < 0 \). The case of \( \mathcal{G} \) containing a hypersurface on which \( U = 0 \) is a bit trickier (and so we will not explicitly go into its details), but in principle there is no objection against it either.

As announced, we specify \( \Gamma_t \) to be "spacelike". With respect to the coordinates \( (y^a) \equiv (t, \xi^a) \), we can write down the DeWitt metric in an \((n-1) + 1\)-ADM-type decomposition

\[
ds_{\text{DW}}^2 = -N^2 dt^2 + h_{ab}(d\xi^a + N^a dt)(d\xi^b + N^b dt)
\]

where \( N \) and \( N_a \equiv h_{ab} N^b \) play the role of lapse and shift (all these quantities depending on \((t, \xi)\), and the former not to be confused with the lapse \( N \) of the space-time metric). The (positive definite) induced metric (first fundamental form) on \( \Gamma_t \) is \( h_{ab}(t, \xi) \), its determinant being denoted by \( h \), the inverse by \( h^{ab} \). Hence \( \gamma = -N^2 h \).

The volume element on \( \Gamma_t \) is \( d^{n-1} \xi \sqrt{h} \). The hypersurface element \( ds_\alpha \) is given by \( d^{n-1} \xi \sqrt{h \gamma} \), where \( n^\alpha \) is the unit \( (n^\alpha n_\alpha = -1) \) vector orthogonal to \( \Gamma_t \). We choose its orientation to be \( n_0 > 0 \), hence \( n^0 < 0 \). This makes \((12.6)\) a positive definite scalar product. Note that with this choice the hypersurface element is correlated with the orientation of the classical trajectories, as defined by increasing \( t \). Under complex conjugation \((S \rightarrow -S, \text{hence } t \rightarrow -t)\), it becomes reversed (although \( n_0 > 0 \) remains intact when expressed with respect to the new coordinates \((-t, \xi)\)).

If one likes to work with a prescribed orientation of \( \Gamma \) in \((6.3)\), this has to be taken into account by changing the orientation for all outgoing WKB-branches, say. The result is just the \( \pm \) sign in \((6.3)\), which does not show up here. Moreover, we denote the covariant derivative with respect to \( h_{ab} \) by \( \mathcal{D}_a \), and set of course \( h^{ab} \mathcal{D}_b = \mathcal{D}^a \).

The trace of the extrinsic curvature (second fundamental form) on \( \Gamma_t \) is given by

\[
K = \nabla_\alpha n^\alpha = \frac{1}{N} \mathcal{D}_a N^a - \frac{\dot{h}}{2N h},
\]

and

\[
\partial_\perp = -n^a \nabla_a = \frac{1}{N} \partial_t - \frac{N^a}{N} \partial_a \equiv -N \nabla^0
\]

is the derivative orthogonal to \( \Gamma_t \).

The definition of the coordinate system immediately implies

\[
\begin{align*}
N_a &= N \partial_a S, \\
N^2 &= N^2 U + N_a N^a = N^2 (U + (\mathcal{D}_a S) \mathcal{D}^a S) = N^2 (\partial_\perp S)^2,
\end{align*}
\]

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the second chain of equalities just expressing the Hamilton-Jacobi equation (3.2) in several forms. Also we find $\dot{S} = -NU$ and $\partial_{\perp} S = -N'/N$. The components of the DeWitt metric are $\gamma_{00} = -N^2U$, $\gamma_{0a} = N_a$ and $\gamma_{ab} = h_{ab}$. The inverse metric is $\gamma^{00} = -N^{-2}$, $\gamma^{0a} = N^{-2}N^a$, and $\gamma^{ab}$ is given by equation (12.33) below. Note that if $N_a \neq 0$ the vector field $\partial_t$ is not orthogonal to $\Gamma_t$. Hence, the surfaces of constant $S$ need not coincide with the surfaces of constant $t$.

The equation (12.1) for $D$ may now easily be written down in these coordinates, the general solution being

$$D(y)^2 = \frac{N(y)f(\xi)}{N'(y)\sqrt{h(y)}}$$

(12.15)

with $f$ an arbitrary (positive) function of constant value along each of the classical paths. We are now in position to display some relations that clarify the structure emerging here. Remember that the total volume element on $M$ with respect to the DeWitt metric is given by $d\mu = d^ny\sqrt{-\gamma}$. Using the previous results, we find

$$ds^\alpha \partial_\alpha S = d^{n-1}\xi \sqrt{h} n^\alpha \partial_\alpha S = d^{n-1}\xi \sqrt{h} \frac{N}{N}$$

(12.16)

and

$$d\mu = dt N d^{n-1}\xi \sqrt{h} = dt N ds^\alpha (\partial_\alpha S).$$

(12.17)

Hence, any surface integral with measure as in (12.6) is given by $(F \equiv F(y) \equiv F(t, \xi))$

$$\int_{\Gamma_t} ds^\alpha (\partial_\alpha S) D^2 F = \int d^{n-1}\xi f(\xi) F(t, \xi)$$

(12.18)

where the second integral is over the whole parameter space labelled by the $\xi^a$ (in fact, we require all integrands to be well-concentrated on $\Gamma_t$, so that neither the local nature of the construction nor the possible existence of boundaries of $\Gamma_t$ give rise to any problem). Differentiating with respect to $t$, it becomes clear that $\partial_t$ acts on $F$ alone

$$\frac{d}{dt} \int_{\Gamma_t} ds^\alpha (\partial_\alpha S) D^2 F = \int d^{n-1}\xi f(\xi) \dot{F}(t, \xi) = \int_{\Gamma_t} ds^\alpha (\partial_\alpha S) D^2 \dot{F},$$

(12.19)

from which (12.8) follows immediately. We thus obtain an alternative form of the scalar product (12.6)

$$\langle \chi_1 | \chi_2 \rangle_{\Gamma_t} = \int d^{n-1}\xi f(\xi) \chi_1^*(t, \xi) \chi_2(t, \xi) \equiv \int d^{n-1}\xi \sqrt{h} \frac{N'D^2}{N} \chi_1^* \chi_2,$$

(12.20)
where the second expression reflects the covariant character of the integral. One may achieve a result analogous to (12.19) by computing the difference between two integrals of the type (12.18) for two hypersurfaces \( \Gamma_{t_1} \) and \( \Gamma_{t_2} \). Denoting by \( G \) the region between these, and using Gauss’ theorem, one encounters the volume integral

\[
\int_G d\mu \nabla^\alpha \left( D^2 (\partial_\alpha S) F \right).
\]

(12.21)

Using (12.1), one again sees that \( N^{-1} \partial_t \equiv (\nabla^\alpha S) \partial_\alpha \) just acts on \( F \). Using (12.16) and (12.17), one recovers (12.19) in various forms. It also becomes clear how the function \( f(\xi) \) specifies the representation of operators and the local scalar product. From (12.20) we see that the choice \( f(\xi) \equiv 1 \) makes \( \langle \mid \rangle \) the standard scalar product on (a domain of) \( \mathbb{R}^{n-1} \). Also, once a choice of \( f \) has been fixed, one may redefine the \( \xi^a \) among themselves so as to achieve \( f \equiv 1 \) in the new variables.

The Wheeler-DeWitt equation (12.2) looks like a Schrödinger equation, but the operator \( \mathcal{K} \) contains derivatives off \( \Gamma_t \). The general structure we encounter here is that any hermitean operator acting on the space of square integrable functions \( \chi(\xi) \) (with respect to the scalar product (12.20)) defines a unitary evolution. Let us emphasize that such a unitary evolution is associated with only one Schrödinger equation for \( \chi(t, \xi) \). This provides a difference to the standard semiclassical formalism: There, one is faced with several Schrödinger equations, belonging to several classical backgrounds and thus several definitions of the classical time flow. Thus, the effective Schrödinger equation (3.4) of the standard approach is essentially unique only if the classical backgrounds contributing are strongly peaked. This may be implemented by a peak in the prefactor \( D_0 \) from (3.3); cf. Ref. [27]. No such peak in \( D \) is needed here, when a local quantum structure is formally extracted from the Wheeler-DeWitt equation. This confirms the nature of our formalism as an intermediate structure between the full Wheeler-DeWitt equation and observational quantum mechanics (which operates in a quasiclassical background), as has been mentioned in Section 3.

We thus raise the question whether — to some approximation — \( \mathcal{K} \) may be replaced by an operator \( \mathcal{H}^{\text{eff}} \) such that (i) \( \mathcal{H}^{\text{eff}} \) acts purely tangential to \( \Gamma_t \) (i.e. for any given \( t \) it acts in the space of functions on \( \Gamma_t \) — in practice, it is a combination of multiplication operators and ”spatial” derivatives \( \partial_a \)), and (ii) for any \( t \) it is hermitean with respect to the scalar product (12.6) or, equivalently, (12.20). The second condition just means that \( f \mathcal{H}^{\text{eff}} \) is hermitean with respect to the standard scalar product in \( \mathbb{R}^{n-1} \). It may be replaced by a ”covariant” version, namely by
requiring that for any \( t \) the combination \( N^{-1} \mathcal{N} D^2 H^{\text{eff}} \equiv f h^{-1/2} H^{\text{eff}} \) is hermitean with respect to the scalar product

\[
\int_{\Gamma_t} d^{n-1} \xi \sqrt{h} \chi_1^* \chi_2
\]

defined by the proper volume on \( \Gamma_t \). If this can be achieved, the Wheeler-DeWitt equation (12.2) approximately becomes (3.8), and the unitarity of the evolution defined by \( H^{\text{eff}} \) is evident.

In order to find \( H^{\text{eff}} \), we begin writing down the \((n-1)+1\)-decomposition of the full Laplacian (when acting on scalars functions on \( \mathcal{M} \))

\[
\nabla_\alpha \nabla^\alpha = -\partial_\perp \partial_\perp + K \partial_\perp + \frac{1}{N} D_a \mathcal{N} D^a.
\]

(12.23)

Here, \( D_a \) acts on everything to the right of it. This may be inserted into \( \mathcal{K} \). By explicitly shuffling the derivatives \( \partial_t \) to the right, one achieves the decomposition

\[
\mathcal{K} = \mathcal{U} \partial_{tt} + \mathcal{V} \partial_t + \mathcal{W}
\]

(12.24)

with \( \mathcal{U}, \mathcal{V} \) and \( \mathcal{W} \) operators that act purely tangential to \( \Gamma_t \). Before evaluating them, it is appropriate to discuss the kind of expansion we will perform.

Rapid oscillations in a WKB-type wave function \( \psi \) are induced by a rapidly varying action \( S \). In the conventional WKB-expansion approach this is accommodated by assuming that the action \( S_0 \) of the classical variables is large. One usually sets \( S_0(y_{cl}) = \sigma_0(y_{cl})/\epsilon \) with \( \epsilon \) a small expansion parameter (e.g. \( \ell_p \) or \( \hbar \)). We will not introduce such a quasiclassical "background" action, but work with a solution \( S(y) \) of the exact Hamilton-Jacobi equation (3.2). The latter is expressed in the \((n-1)+1\)-split by equation (12.14). We may assume that \( S \) is large (even if it contains contributions in addition to the "true" large part \( S_0 \)). Setting \( S(y) = \sigma(y)/\epsilon \) amounts to formally assume \( U(y) = u(y)/\epsilon^2 \) in a small \( \epsilon \) expansion in order to retain the form of (3.2). By virtue of (12.13) and (12.14) this makes \( N^{-1} \mathcal{N} \sim \epsilon^{-1} \sim N^{-1} \mathcal{N}_a \) large quantities. Dividing the Wheeler-DeWitt equation (12.2) by \( N \), the time derivative at the left hand side appears in the combination \( iN^{-1} \partial_t \). At the right hand side, we find all time derivatives to occur in the combination \( \mathcal{N}^{-1} \partial_t \) which is \( (\mathcal{N}^{-1} \mathcal{N}) N^{-1} \partial_t \) and thus scales as \( \epsilon N^{-1} \partial_t \) in a small \( \epsilon \) expansion. This is the mechanism that suppresses the time derivatives at the right hand side (as opposed to the one at the left hand side). We will use \( \epsilon \) as a formal bookkeeping parameter indicating small contributions. \( D \) appears in the Wheeler-DeWitt
equation homogeneous of degree zero, and so will not contribute any $\epsilon$. The ratio $N_a/N$ will be kept in all expressions, although it may be small. The net effect is thus to give any time-derivative at the right hand side a factor $\epsilon$. This is achieved by expressing the Wheeler-DeWitt equation in terms of $h_{ab}, N, D, N', N_a$ and the derivatives $\partial_t$ and $\partial_a$, and by formally substituting

$$N \to \frac{1}{\epsilon} N, \quad N_a \to \frac{1}{\epsilon} N_a, \quad N' \to \frac{1}{\epsilon} N',$$

leaving all other quantities unchanged. This does not necessarily mean that $N$ and $N_a$ are large, but simply accounts for the insertion of appropriate $\epsilon$’s in the operator $K$. Thus, we have to expand $K$ with respect to $\epsilon$. The only relevant pieces getting $\epsilon$’s are the contributions $\partial_\perp$ and $K$ in (12.23). Both terms decompose into an $O(\epsilon)$ and an $O(\epsilon^0)$ contribution. Note that the $\epsilon^{-1}$ from $N$ cancel in the last expression of (12.23). In terms of (12.24) one finds

$$U = \frac{\epsilon^2 N}{2N^2},$$

$$V = \frac{\epsilon^2 N}{2N^2} \left( \frac{2\dot{D}}{D} + \frac{\dot{h}}{2h} - \frac{\dot{N}}{N} \right) - \frac{\epsilon N}{2N} D^{-1} A D,$$

$$W = \frac{\epsilon^2 N}{2N^2} \left( \frac{\ddot{D}}{D} + \left( \frac{\dot{h}}{2h} - \frac{\dot{N}}{N} \right) \frac{\ddot{D}}{D} \right) - \frac{\epsilon N}{2N} \left( D^{-1} A \dot{D} + D^{-1} A^a D_a D \right) + H_{\text{eff}}$$

(12.28)

where

$$A = D_a \frac{N^a}{N} + \frac{N^a}{N} D_a$$

(12.29)

is an anticommutator (both $D_a$’s acting on every function to the right of them), and

$$A^a = \frac{1}{\sqrt{h}} \partial_t \left( \sqrt{h} \frac{N^a}{N} \right)$$

(12.30)

is a vector field, with $\partial_t$ acting only on the expression in the bracket. Note that the rule of assigning a factor $\epsilon$ to any time-derivative in $K$ includes the dot-terms in the above expressions. When time-derivatives of $D$ are involved, one may use the identity

$$\frac{\dot{D}}{D} = \frac{\dot{N}}{2N} - \frac{\dot{h}}{4h} - \frac{\dot{N}}{2N}$$

(12.31)

67
which follows from (12.15). The last operator in (12.28) is given by

\[ H^{\text{eff}} = -\frac{N}{2ND}\partial_a N\gamma^{ab}\partial_b D \equiv -\frac{N}{2ND\sqrt{\hbar}}\partial_a \sqrt{\hbar}N\gamma^{ab}\partial_b D , \]  

(12.32)

where the "spatial" contravariant components of the DeWitt metric, when expressed as a tensor object on \(\Gamma_t\), read

\[ \gamma^{ab} = h^{ab} - N^a N^b \frac{N^2}{N^2} . \]  

(12.33)

We have thus expanded \( K \) with respect to \( \epsilon \) and explicitly displayed the coefficient operators. Setting \( U = \epsilon^2 U_2, V = \epsilon^2 V_2 + \epsilon V_1 \) and \( W = \epsilon^2 W_2 + \epsilon W_1 + H^{\text{eff}} \), the Wheeler-DeWitt equation — with the replacement (12.25) performed — reads

\[ i\dot{\chi} = \epsilon^2 U_2 \dot{\chi} + (\epsilon^2 V_2 + \epsilon V_1)\dot{\chi} + (\epsilon^2 W_2 + \epsilon W_1 + H^{\text{eff}})\chi . \]  

(12.34)

Setting \( \epsilon = 1 \) gives back its exact form (12.2). Also, after performing a small \( \epsilon \) expansion to arbitrary order, in the end one should set \( \epsilon = 1 \) as well, because we have introduced \( \epsilon \) in a formal way as a book-keeping parameter.

There is still the question open how to choose the coordinate \( t \), i.e. the lapse \( N \) and the hypersurfaces \( \Gamma_t \). In general, one should choose \( t \) as a coordinate that conveniently serves as ”time” in the local description of the evolution of the universe and the laws of nature. In some situations a ”good” choice of time is quite evident, in others one may have a lot of freedom. In fact, there is a wide range of admissible choices, and only too drastic transformations thereof would spoil the structure of the small \( \epsilon \) expansion. The \( \Gamma_t \) should approximately coincide with the surfaces of constant \( S \), so as to shift the dominant oscillations from \( \chi \) to the WKB-phase \( \exp(iS) \). The according local quantum structures are essentially equivalent (just as the scalar product \( \langle | \rangle \) is largely independent of the hypersurface at which it is computed). The time-derivative \( \partial_t \) is along the trajectories, and it transforms under a change of time gauge and a change of the equal-time hypersurfaces simply as \( \partial_t = (N/N)\partial_\tau \). The operators acting purely tangential to \( \Gamma_t \) may be regarded as time-dependent operators acting on a space of functions which are defined on the set of trajectories. Sufficiently small transformations just interpolate between different representations (or "pictures") of this situation. If the equal-action hypersurfaces differ too much from the equal-time hypersurfaces, the pattern defined by the small \( \epsilon \) expansion may break down. Once the foliation provided by the \( \Gamma_t \) has been fixed,
a further change of time coordinate leaving this foliation invariant (i.e. \( t \leftrightarrow \tilde{t} \)), hence \( N(y)/\overline{N}(y) \) being a function of \( t \) only) yields an identical structure with respect to \( \epsilon \) (apart from a mixture between \( \mathcal{U} \) and \( \mathcal{V} \) which corresponds to the transformation of \( \partial_{tt} \) into a \( \partial_{\tilde{t}\tilde{t}} \) and a \( \partial_{\tilde{t}} \) term). In this case one finds \( N^{-1}H^{\text{eff}} = \overline{N^{-1}H^{\text{eff}}} \). In order to understand the emergence of quantum mechanics better, it is certainly necessary to study the issue of the appropriate time gauge and the accuracy of the approximations involved in more detail.

In the time gauge provided by the condition \( \mathcal{N}_a = 0 \) our formalism becomes somewhat simplified. On account of (12.13), this is equivalent to \( S = S(t) \), i.e. the evolution coordinate \( t \) along each trajectory is only a function of \( S \). As a consequence we find \( \mathcal{A} = A^a = 0 \), and (12.14) reduces to \( \mathcal{N}^2 = N^2U \). The relation between \( t \) and \( S \) is provided by \( \dot{S}(t) = -NU \), which implies that the combinations \( NU \) and \( N^{-1}N^2 \) depend only on \( t \) as well. The operators (12.26)–(12.28) simplify, and we obtain \( \mathcal{V}_1 = \mathcal{W}_1 = 0 \), hence all contributions to \( O(\epsilon) \) vanishing.

One may specify such a time gauge by setting \( S(t) = -t \), i.e. by using the (negative of the) classical action as time parameter along the paths. In this case, the \((n-1) + 1\)-split as well as the space-time lapse \( N \) become uniquely determined, and one finds (in addition to \( \mathcal{N}_a = \mathcal{A} = A^a = 0 \)) the further simplifications \( N = N^2 = U^{-1} \) and \( D^2 = U^{-1/2}h^{-1/2}f(\xi) \). The operators (12.26)–(12.28) become

\[
\mathcal{U} = \frac{\epsilon^2}{2}, \quad \mathcal{V} = 0, \quad \mathcal{W} = H^{\text{eff}} + \frac{\epsilon^2}{2} \left( \frac{\dot{D}}{D} - 2 \frac{D^2}{D^2} \right). \tag{12.35}
\]

The \( O(\epsilon^2) \)-term in \( \mathcal{W} \) may also be written as \( -\epsilon^2D(\partial_{tt}D^{-1})/2 \). This is just a real function of the coordinates \((t, \xi)\), and thus represents a hermitean operator.

Ordinary quantum mechanics is now recovered easily. To lowest order \((\epsilon \to 0)\), the operator \( \mathcal{K} \) is just \( H^{\text{eff}} \) as given in (12.32), and the Wheeler-DeWitt equation (12.34) reduces to the effective Schrödinger equation (12.8). \( H^{\text{eff}} \) acts purely tangential to \( \Gamma_t \) which is just condition (i), as defined above. Moreover, the combination \( N^{-1}\mathcal{N}D^2H^{\text{eff}} \) is clearly hermitean with respect to the scalar product (12.22) and thus obeys condition (ii). As a consequence, (12.8) defines a unitary evolution of the conventional quantum mechanical type. As already mentioned, no severe normalization problems are expected with the effective Schrödinger wave functions \( \chi \) in the case \( U > 0 \) that we have dealt with so far. Constructing a sufficiently large collection of wave packets that nicely fit into the approximate Hilbert space structure, a given
wave function can be expanded (and hence interpreted) in terms of these. This is what we have denoted true local quantum mechanics.

Let us now comment on the case $U < 0$. If the $\Gamma_t$ are chosen so as to make the induced metric Lorentzian, the $(n - 1) + 1$-split (12.10) has to be replaced by

$$ds_{DM}^2 = N^2 dt^2 + h_{ab}(d\xi^a + N^a dt)(d\xi^b + N^b dt).$$

(12.36)

The following analysis applies almost identically, and most of the formulae are correctly transformed by replacing $N^2 \rightarrow -N^2$ and $\sqrt{N} \rightarrow \sqrt{-N}$. In particular, (12.13) remains valid, (12.14) becomes

$$N^2 = -N^2 U - N_a N^a = -N^2 (U + (D_a S)D^a S)$$

(12.37)

and the contravariant components of the DeWitt metric tangential to $\Gamma_t$ (equation (12.33)) carry over to

$$\gamma^{ab} = h^{ab} + \frac{N^a N^b}{N^2}.$$  

(12.38)

With this definition, the expression for the effective Hamiltonian is still given by (12.32), and the various forms of the Wheeler-DeWitt equation as well as the general discussion still apply. The unit normal is of course changed into $n^\alpha n_\alpha = 1$, and the transformation of all formulae containing it (and $\partial_\perp$) explicitly is a simple exercise. Its two possible orientations imply a double sign like in (5.5) if a fixed hypersurface $\Gamma$ is used, and the $Q$-products are modified so as to integrate over $\Gamma$ only locally. As in the $U > 0$ case, one may construct a collection of local wave packets that fit into the approximate Hilbert space structure. However, a given state, when expanded into such reference wave packets, may spoil any reasonable local normalizability. Such a feature could possibly prevent any probabilistic interpretation. Hence, as outlined in Section 4, the $U < 0$ case may give rise to true as well as to false local quantum mechanics, depending on the wave function under consideration. For the rest of this Section we will again restrict ourselves to the $U > 0$ case.

When the issue of operator ordering is ignored, the structure of the effective Hamiltonian (12.32) may be understood heuristically in terms of the classical system. Let us for simplicity restrict ourselves to the case $S = -t$. Choose a solution $S$ of the Hamilton-Jacobi equation (3.2), parametrize the according paths by $t = -S$ and select variables $\xi^a$ to label them. Then $(y^a) \equiv (t, \xi^a)$ is a coordinate system on $M$. If the classical system is rewritten in terms of these coordinates, an evolution (the parameter being denoted by $\tau$) is given by $(t(\tau), \xi^a(\tau))$. The special cases $t(\tau) = \tau$, 70
\(\xi^a(\tau) = \text{const}\) give back the original family of paths, but these are of course not the only possible ones. Rescaling the lapse as \(N = \nu U^{-1}\), the Lagrangian of the classical system reads (the dot denoting \(d/d\tau\))

\[
\mathcal{L} = \frac{1}{2} \left(-\nu^{-1} \dot{t}^2 - \nu + \nu^{-1} U h_{ab} \dot{\xi}^a \dot{\xi}^b\right).
\] (12.39)

Variation with respect to \(\nu\) yields the constraint equation, which, in the Hamiltonian form, reads

\[
-p_t^2 + 1 + U^{-1} h^{ab} p_a p_b = 0.
\] (12.40)

In the gauge \(\nu = 1\) (which we fix once the constraint is established), we find \(p_t = -\dot{t}\).

Upon identifying \(p_a\) with \(-i\partial_a\) and ignoring operator ordering, the last term in (12.40) just has the structure of \(2H_{\text{eff}}\) (recall \(U^{-1} = N\) for \(\nu = 1\)) and thus shall be denoted by the same name. Redefining the variables as \(t(\tau) = \tau + T(\tau)\) and the momenta as \(p_t = -1 + p_T\), hence \(p_T = -\dot{T}\), we obtain the constraint in the form

\[
-p_T = -\frac{1}{2} p_T^2 + H_{\text{eff}}.
\] (12.41)

Quantizing by means of the substitution \(p_T \rightarrow -i\partial_T, p_a \rightarrow -i\partial_a\) gives

\[
i \dot{x} = \frac{1}{2} \ddot{x} + H_{\text{eff}} x
\] (12.42)

which has the overall structure of (12.34) with \(\epsilon = 1\) in the gauge \(S = -t\) (the additional operator \(W_2\) appearing there may be considered as arising by operator ordering and related issues). The above redefinition of the momenta corresponds to the transition from \(\psi\) to \(x\) (or likewise to the ansatz \(\psi = x \exp(-it)\)). The WKB-approximation declares \(\dot{x}\) to be negligible, and we have reproduced the essentials of (3.8). The classical counterpart of the effective Hamiltonian thus governs the evolution of those degrees of freedom which describe the deviation of generic trajectories from the paths generated by \(S\). The classical counterpart of the WKB-approximation is the condition \(p_T^2 \ll |p_T|\), thus \(|p_T| \ll 1\) (or equivalently \(|\dot{T}| \ll 1\)). As a consequence, we must have \(|H_{\text{eff}}| \ll 1\) in order to be close to the original \(S\)-trajectories. A similar analysis may be performed using a general gauge for the \(S\)-trajectories, the classical effective Hamiltonian becoming \(H_{\text{eff}} = \frac{1}{2} N \gamma^{ab} p_a p_b\) (cf. equation (12.33)). It arises as part of \(\gamma^{ab} p_a p_b\), which explains the appearance of \(\gamma^{ab}\) instead of \(h^{ab}\).

If one tries to go beyond the Schrödinger approximation derived so far, one may iteratively get rid of the time-derivatives of \(x\) in (12.34) to any order in \(\epsilon\). In order to
find the \(O(\epsilon)\) approximation, one may just insert (12.34) itself into the term \(\epsilon V_1 \dot{\chi}\), and the result is

\[ i \dot{\chi} = \left( H_{\text{eff}} + \epsilon (W_1 - i V_1 H_{\text{eff}}) \right) \chi \]  

(12.43)

with \(\epsilon = 1\) inserted. This provides the first post-quantum mechanics approximation. As is clear if \(N_a \neq 0\), the Hamiltonian appearing here does not meet the hermiticity condition \((ii)\). This follows from the fact that part of \(W_1\) is a first order "spatial" derivative operator with real coefficients, and thus provides an anti-hermitean contribution that cannot be cancelled by any other term.

One may proceed even further. Iteratively differentiating (12.34) with respect to \(t\) and performing appropriate substitutions yields a Schrödinger type equation without time-derivative on the right hand side (but with non-hermitean Hamiltonian) to arbitrary order in \(\epsilon\). If one uses a time gauge in which \(N_a = 0\) (see above), the \(O(\epsilon)\) corrections in (12.43) are in fact absent, because \(V_1 = W_1 = 0\). In this case we obtain at \(O(\epsilon^2)\)

\[ i \dot{\chi} = \left( H_{\text{eff}} + \epsilon^2 (B + iC) \right) \chi \]  

(12.44)

with

\[
\begin{align*}
B &= -U_2 (H_{\text{eff}})^2 + W_2 \\
C &= -U_2 H_{\text{eff}} - V_2 H_{\text{eff}} \\
&\equiv -\partial_t \left( \frac{D^2 \sqrt{h}}{2 N^2} H_{\text{eff}} \right) \equiv -\partial_t \left( \frac{N}{2 N^2} f(\xi) H_{\text{eff}} \right)
\end{align*}
\]  

(12.45)

(12.46)

and \(\epsilon = 1\) inserted. The operators \(B\) and \(C\) are hermitean with respect to the local scalar product. (In order to see this, one has to use the fact that \(N^{-1} N^2\) depends only on \(t\) in this gauge). As a consequence, the term \(iC\) violates unitarity. Using in addition the gauge \(S = -t\) (see above), further simplifications occur, and one finds \(U_2 = 0\), \(V_2 = 0\) and \(W_2 = -\frac{1}{2} D(\partial_t D^{-1})\) (see equation (12.35)). In this gauge, the corrections are negligible if — in a sloppy formulation — \((H_{\text{eff}})^2\), \(\dot{H}_{\text{eff}}\) and \(W_2\) are "smaller" than \(H_{\text{eff}}\). Note that the first of these statements (which seems to be the essential one) directly corresponds to the classical analogue of the WKB-approximation in the gauge \(\nu = 1\), as was discussed above (below equation (12.42)).

Thus we see that unitarity violation appears at \(O(\epsilon)\) or, by relating the time parameter to the action, at \(O(\epsilon^2)\). Although we did not make use of higher orders in our interpretational scheme, they may be of some help if one likes to demonstrate
explicitly how accurate the effective Schrödinger equation (3.8) describes the local time evolution, and of what order of magnitude the deviations are (as compared to solutions of the full Wheeler-DeWitt equation).

Let us at this point perform a rough estimate of the order of magnitude of the correction terms for the case that the physical time parameter $t$ depends mainly on $S$. The dominant contribution may be inferred from (12.44). Upon performing the expectation value (acting with $\langle \chi |$ from the left), setting $\varepsilon = \langle H^{\text{eff}} \rangle$, assuming widths to be small (omitting the brackets) and neglecting $V_2$ and $W_2$, this implies a quantum gravitational shift of the effective Hamiltonian

$$\delta \varepsilon \approx -\frac{\varepsilon^2}{2NU}$$

(12.47)

and an imaginary contribution $i\varepsilon_I$ with

$$\varepsilon_I \approx -\frac{\dot{\varepsilon}}{2NU}$$

(12.48)

indicating — in the usual language — an instability. The approximation $S \approx S(t)$ is in particular well-applicable during inflation if $t$ is the proper time (the exponential expansion suppresses the dependence of the action on variables other than the scale factor). This is the situation of a quantum field in de Sitter space. Formally, we can use as background the framework of the Hawking model as described in Section 10, in the domain $|\phi| \gg 1$ with $\Lambda^{\text{eff}} \approx m^2 \phi^2 \approx H_0^2$, $H_0$ the Hubble constant, $U \approx a^3 H_0^2$ and $N = 1$. From the inflationary (outgoing) trajectories (equation (10.2)) we find $\delta \varepsilon \approx -a^{-3} H_0^{-2} \varepsilon^2$. This expression agrees in structure with results achieved in the conventional WKB-expansion and by using more accurate methods [67]. However, one should not forget that our $\varepsilon$ does not coincide with $\langle H_{\text{matter}} \rangle$ from the semiclassical approach.

We can now estimate under which condition the hermitean correction (12.47) is small. The WKB-approximation itself is characterized mainly by the condition $|\dot{\chi}| \ll |\chi \dot{S}|$, which means in the language used above that $\varepsilon \ll NU$. The order of magnitude of the correction term (12.47) relative to the uncorrected Hamiltonian in the Schrödinger approximation is $\delta \varepsilon / \varepsilon \approx \varepsilon / (NU)$ Hence, the hermitean corrections are small if and only if the WKB-approximation holds, i.e. if the change of $S$ is large enough. Denoting by $\delta t_\varepsilon \approx \varepsilon^{-1}$ the typical time scale set by the effective Schrödinger equation, and by $\delta t_{\text{min}} \approx (NU)^{-1}$ the minimal resolution time in the WKB-approximation (cf. the estimate at the end of Section 3), we find $\delta \varepsilon / \varepsilon \approx \sqrt{73}$.
Thus $\delta t \gg \delta t_{\text{min}}$, which states that the time scale relevant for the local quantum structure is much above the minimum. Moreover, the correction itself may be associated with a time scale $\delta t_{\text{corr}} \approx \varepsilon^{-2}NU$. At least from a heuristic point of view, this is the time that must be available in order to measure an effect stemming from the hermitean correction to the Hamiltonian. Thus $\delta t_{\text{corr}}/\delta t_{\text{min}} \approx (\varepsilon/\delta \varepsilon)^2 \gg 1$, and we expect never to run into the problem that the correction plays a role in sufficiently small local WKB-domains. If this is actually true (the details may be model-dependent), the quantum gravity corrections can be neglected in the local effective Schrödinger equation, as long as it is followed over moderate time intervals.

The anti-hermitean correction (12.48) can be expected to be small when the prefactor $D$ varies slowly as compared to the WKB-phase, $|\dot{D}| \ll |D\dot{S}|$, because, roughly (cf. (12.31)), $\dot{\varepsilon}/\varepsilon \sim \dot{D}/D$. This is reasonable, because $D$ was chosen so as to guarantee the local unitarity. Hence, if the WKB-approximation is good, both corrections to the effective Schrödinger equation are locally suppressed. This does not necessarily mean that quantum gravity effects are unobservably small, but the time scales relevant for observations may be very large. What we suggest is that these effects are automatically incorporated by following the time evolution through a large number of adjacent domains, as described in Section 8. The particular modification of the (hermitean) Hamiltonian and the local quantum structure for long-range time evolution (as long as possible) and the time scale above which unitary quantum mechanics would not account for viable predictions may in principle be determined by this procedure.

There is an alternative way to look at the condition under which the WKB-approximation is good, and the corrections to the effective Schrödinger equation are small. The essential inequality is that $(NU)^{-1}(H^{\text{eff}})^2$ is negligible as compared to $H^{\text{eff}}$. This in turn implies that $H^{\text{eff}}$ is ”small” as compared to $NU$, and we obtain a scale in the coordinates $\xi^a$ below which $\chi$ must not change significantly. Using only the essential structure of $H^{\text{eff}}$, and applying it to a wave function of the form $\chi \sim \exp(-A_{ab}\delta \xi^a\delta \xi^b)$ with $\delta \xi^a = \xi^a - \bar{\xi}^a$, we find that the maximum eigenvalue of $h^{ac}A_{ch}$ must be much smaller than $U$. As a consequence, the width of wave packets is bounded from below. In terms of the ”proper length” defined by the DeWitt metric we have $s_{\text{min}} \gg U^{-1/2}$, a result that was also mentioned in Section 3. Once the variations of the wave function $\chi$ in $\xi^a$ are well below the bound, we expect the correction terms to be negligible and the WKB-approximation to hold. This implicitly assumes that the equal-time hypersurfaces do not differ too drastically from
the equal-action hypersurfaces, and that the contributions to $H^{\text{eff}}$ stemming from operator ordering issues (i.e. $D$, $\mathcal{N}$ and $\sqrt{\hbar}$ in equation (12.32)) are suppressed as well. A more detailed analysis will depend on the particular situation and the model one is considering.

Ultimately, the result of iterating the elimination of time-derivatives at the right hand side of (12.34) beyond any finite order in $\epsilon$ is an equation of the form

$$i\dot{\chi} = H^{\text{tot}}\chi,$$

where $H^{\text{tot}}$ is an infinite sum of generally non-hermitean operators acting purely tangential on $\Gamma_t$ and being of arbitrarily high order in the "spatial" derivatives. The validity of such an expansion is guaranteed if the sum converges in some sense, even after setting $\epsilon = 1$. (A similar — although much simpler — procedure arises when the flat Klein-Gordon equation is written down in the Schrödinger form by expanding the square root $\sqrt{m^2 - \Delta} = m - \Delta/(2m) - \Delta\Delta/(8m^3) + \ldots$. In this example, unitarity is not violated, but this is due to the fact that negative/positive frequencies may unambiguously be identified). We do not know for which wave functions $\chi$ the expansion converges (we don’t know either much about convergence in the standard WKB-approach), but at least formally we have obtained a unique expansion scheme to all orders in $\epsilon$. In terms of physics, this is probably not of much value because unitarity breaks down already at $O(\epsilon)$ or $O(\epsilon^2)$.

Nevertheless, one could think of integrating $i\dot{\chi} = H^{\text{tot}}\chi$ and its complex conjugate equation for some initial value $\chi(0,\xi)$ in order to obtain solutions of the Wheeler-DeWitt equation that may be associated with $\mathcal{H}^-$ and $\mathcal{H}^+$, respectively. This would in fact be the attempt to exactify equation (7.11) of Section 7. In the case of the flat Klein-Gordon equation this leads to the formal appearance of the operator $\pm\sqrt{m^2 - \Delta}$, which can be given a well-defined meaning in the Fourier representation (i.e. with respect to a particular regularization, or with respect to a basis in which $\Delta$ is diagonal) and reduces to the function $\pm(m^2 + \mathbf{k}^2)^{1/2}$. The result is the standard decomposition into negative and positive frequencies. In the general case of the Wheeler-DeWitt equation, one would not expect to find a distinguished regularisation for $H^{\text{tot}}$, and there seems to be very little chance to obtain a tool that would distinguish in some exact sense between outgoing and incoming modes at the quantum level. However, we cannot exclude that a more intense study of the expansion leading to $H^{\text{tot}}$ sheds new light on the problem.

The local quantum structure, based on the effective Schrödinger equation (3.8), has so far been formulated in terms of coordinates $(t, \xi^a)$ on $\mathcal{M}$. The $\xi^a$ are conserved quantities along any classical trajectory belonging to the congruence described by $S$ (although they are not conserved along other classical trajectories). This defines...
a very special type of formulation of quantum mechanics, which is only in few cases in the familiar representation. (It may be illustrated by describing the harmonic oscillator \( L = (\dot{x}^2 - x^2)/2 \) in terms of the variable \( \xi \), defined by \( x = \sin(t + \xi) \), instead of \( x \). Then \( \xi(t) = \text{const} \) provides a family of classical solutions, but all other solutions in the domain \( |x| \leq 1 \) emerge as well). One may however reformulate the effective Schrödinger equation (3.8) in terms of other coordinates \((t, z^a)\). This gives us the possibility to define a local quantum structure in a quite general coordinate system on \( \mathcal{M} \), provided the small \( \epsilon \) expansion holds (i.e., provided the \( \Gamma_t \) do not differ too drastically from the hypersurfaces of constant action). The systematic way to achieve this (after \( S \) has been selected) is to choose a coordinate \( t \) (i.e., a scalar function \( t(y) \) on \( \mathcal{M} \)) such that the equal-time hypersurfaces \( \Gamma_t \) are spacelike. (Even this condition may be somewhat relaxed). After having made this choice, the lapse \( N(y) \) may be determined. The coordinates \( \xi^a \) labelling the classical trajectories may be introduced as auxiliary quantities, and the \((n - 1) + 1\)-split of the DeWitt metric as well as the determination of \( D \) (choice of \( f(\xi) \)) and the definition of the scalar product and the effective Hamiltonian \( H^{\text{eff}} \) proceed as above. One may now select coordinates \( z^a \) such that \((t, z^a)\) make up a coordinate system on \( \mathcal{M} \). In other words, the \( z^a \) serve as coordinates on each individual \( \Gamma_t \), and we may write down their definition as \( z^a \equiv z^a(t, \xi) \). From now on, the effective Schrödinger equation may simply be re-written in terms of these new coordinates. Introducing

\[
\hat{\chi}(t, z) \equiv \chi(t, \xi), \tag{12.49}
\]

the result is

\[
i \partial_t \hat{\chi} = (H^{\text{eff}} + i C^a D_a) \hat{\chi}, \tag{12.50}
\]

where \( C^a D_a \equiv C^a \partial_a \) is the vector field \( -(\partial z^a/\partial t) \partial/\partial z^a \) on \( \Gamma_t \). Note that in this formulation \( \partial_t \) operates at constant \( z^a \), and is thus no longer the derivative along the trajectories. It differs from the previously defined dot-operator \( \partial_t \) at constant \( \xi^a \) just by \( C^a \partial_a \). Since \( H^{\text{eff}} \) acts purely tangential to \( \Gamma_t \) and is a three-covariant object, it may be rewritten in the new coordinates. It is thus still given by (12.32) and (12.33), with \( h_{ab} \) denoting the induced metric on \( \Gamma_t \), \( N_a \) the vector field on \( \Gamma_t \) defined by (12.13), and \( N \) the scalar function (12.14). If an \((n - 1) + 1\)-split of the DeWitt metric is performed in terms of \((t, z^a)\), the "lapse" is still given by \( N \), whereas the "shift" vector field \( \tilde{N}^a \) is related to its analogue \( N^a \) from the \((t, \xi^a)\) split (12.10) (with appropriately transformed components) by \( \tilde{N}^a = N^a + C^a \). This is a consequence of the identity \( dz^a = (\partial z^a/\partial \xi^b) d\xi^b - C^a dt \) and may serve — together with (12.13) — as an alternative definition of \( C^a \). The physical significance of such a
coordinate transformation may be illustrated by performing a Galilei transformation
\( \vec{x}^{\text{new}} = \vec{x}^{\text{old}} - \vec{v} t \) on the variables of the non-relativistic Schrödinger equation of a free particle \( (H^{\text{old}} = -\triangle/(2m)) \). The new Hamiltonian becomes \( H^{\text{new}} = -\triangle/(2m) + i\vec{v} \nabla \), which agrees with the overall structure of (12.50). Redefining the wave function by extracting the phase factor \( \exp(i m \vec{v} \vec{x}) \) — which just amounts to a unitary transformation — yields the new Hamiltonian in the representation \( -\triangle/(2m) - m \vec{v}^2/2 \). This shows that the net effect of the coordinate transformation is just a shift of the energy, according to the velocity of the new "observer". In the case of (12.50), it might be necessary to perform similar redefinitions in order to cast the result into a familiar form.

The standard WKB-expansion approach should be recovered by recalling that the (positive) contribution \( (D_0 S)D_0 S \) in (12.14) — although it may be large — can have small parts that correspond to the slow oscillations in the quantum variables. Trying to neglecting these, one is tempted to introduce the notation of classical and quantum variables, and to decompose \( S \) into the rapidly changing part \( S_0(y_{cl}) \) and some (slowly changing) rest \( S_q(y_{cl}, y_q) \). It is not always obvious how to do this, but whenever a reasonable identification of the two parts is possible, one should be able to recover the effective Schrödinger equation (3.6), with \( \tilde{H}^{\text{eff}} \) being the "quantum" part of the original Hamiltonian. In simple cases we expect \( \tilde{H}^{\text{eff}} \) to be \( V^{-1}H^{\text{eff}}V \) with \( V = D^{-1}D_0 \exp(-iS_q) \) and possibly some small contributions omitted. In general, some average over neighbouring WKB-branches may be involved. In Section 11 we have encountered a situation where the average over two branches is necessary. Let us remark that an average over infinitely many neighbouring branches — although this may not be necessary in our context — might mathematically relate to path integral ideas. The details of this would be an interesting subject to pursue.

Let us comment once more on the advantages and disadvantages of the modified formalism, as presented here. In contrast to the standard approach, we have retained in \( S \) and \( H^{\text{eff}} \) also small and slow contributions. Also, there may be situations in which the split into \( S = S_0 + S_q \) is not an obvious task (cf. also the discussion of the semiclassical approximation in Ref. [29], p. 245). Moreover, in the standard approach, the separation into classical and quantum variables is usually prior to considering states. Only those wave functions that fit the pattern (3.3) may emerge. In our approach, we can say that the action \( S \) is "the most classical" coordinate on \( \mathcal{M} \), and this is determined only by the particular WKB-branch. We are mainly motivated by adjusting this WKB-branch according to the oscillations in a given state, and this can be regarded as a slightly more general and fundamental point of view.
(although — or rather therefore — in a larger distance from actual observations). Our version of local quantum structures thus demonstrates that ”a clean division of variables into classical and quantum” [10] is not a necessary starting point for recovering an approximate unitary evolution from the Wheeler-DeWitt equation. Maybe one could state that approximate unitarity shows up at a slightly more fundamental level than the split between classical and quantum. Another advantage of our formalism becomes clear by comparing (3.5) and (3.7). Since $S$ solves the exact Hamilton-Jacobi equation, in many situations the effective Schrödinger equation of our formalism will come closer to full solutions of the Wheeler-DeWitt equation than the semiclassical approach. In this sense it is ”more accurate” than the latter.

The disadvantage of our formalism for many purposes might be that some enlarged degree of ambiguity appears. Whereas in the standard approach the classical ”background” action $S_0(y_{cl})$ has some definite ”macroscopic” significance, our exact action $S(y)$ is just one among many that are macroscopically (classically) equivalent. In the (rather trivial) example of the flat Klein-Gordon equation, one may use several actions of the type (with $\hbar = 1$ but $c$ displayed explicitly) $S = \vec{k}\vec{x} - c t (\vec{k}^2 + m^2 c^2)^{1/2}$ with $\vec{k}$ fixed and $\vec{k}^2 \ll m^2 c^2$ in order to find identical results: non-relativistic quantum mechanics of a free particle. We leave it open to what extent it is possible to ”factor out” this ambiguity in a generic situation by means of geometric concepts based on $\mathcal{M}$ and its foliation(s) by equal-time hypersurfaces. The notation of classical and quantum variables should emerge from such an attempt rather than being assumed. (Vilenkin’s condition [16] that the size of the coefficients of the DeWitt metric $\gamma_{\alpha\beta}$ between classical and quantum variables are small — symbolically $\gamma(y_{cl}, y_q) \ll \gamma(y_{cl}, y_{cl})$ — is likely to emerge as a special case, but there should be some dependence on the wave function as well). On the other hand, if a division into classical and quantum variables is assumed, it should be possible to make contact with the standard semiclassical formalism by using coordinates analogous to our $\xi^a$ for the classical sector alone (cf. Ref. [27]).

To sum up, if an exact wave function of the form $\psi = \chi D \exp(iS)$ (with $S$ and $D$ being defined as above) has the property that in some domain of $\mathcal{M}$ the phase factor $\exp(iS)$ oscillates much faster than $D$ and $\chi$, we can reasonably assume that the coefficients of the $\epsilon^2$- and $\epsilon$-terms in (12.34) are negligible, and $\chi$ is subject to the Schrödinger equation (3.8) with Hamiltonian (12.32). This generates a unitary evolution with respect to the scalar product (12.6), or equivalently (12.20).
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