On final-state effects in $t\bar{t}$ production at threshold

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Abstract

We apply the relativistic Bethe-Salpeter formalism to the calculation of final-
state effects in the production of a $t\bar{t}$ pair at threshold. We find that final-state
rescattering does not affect the momentum distribution of the $t\bar{t}$ pair to lowest
order in the strong coupling constant. This result correctly extends earlier results
based on the non-relativistic Lippmann-Schwinger equation.

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As was first pointed out by Fadin and Khoze [1], the structure of the $t\bar{t}$ threshold region is very interesting. The cross-section is enhanced by $t\bar{t}$ resonances and depends strongly on the mass of the top quark $m_t$, its decay width $\Gamma_t$, and the strong coupling constant $\alpha_s(m_t)$. Unlike the other heavy quarks (b and c), $t\bar{t}$ pairs cannot form narrow resonances. This is due to the large value of $m_t$, which implies that the dominant decay mode is the weak decay $t \to W^+b$ with a width $\Gamma_t \sim 1$ GeV. Thus $\Gamma_t$ exceeds $\Lambda_{QCD}$ and as a result the $t\bar{t}$ pair decays before it has time to hadronize, which makes perturbation theory applicable. Since theoretical predictions are possible, it would be extremely interesting to measure the cross-section for threshold $t\bar{t}$ production at the next linear $e^+e^-$ collider (NLC).

In the threshold region, the $t\bar{t}$ pair is produced with little kinetic energy. Therefore, one can employ non-relativistic techniques to calculate the cross-section. In analogy with the Hydrogen atom, the $t\bar{t}$ pair is (weakly) bound together by a QCD Coulomb-like effective potential. The cross-section for $t\bar{t}$ production is related to the Green function, which obeys an appropriate Lippmann-Schwinger equation, by the optical theorem [2]. Although it is possible to use this method to calculate higher-order corrections to the cross-section [2], there are subtleties which ought to be taken into account. As was shown by Kummer and Mödritsch [3], these problems can be avoided if one employs the relativistic Bethe-Salpeter formalism, in analogy with the abelian positronium case [4].

In ref. [5], we extended the results of ref. [3] by showing that electroweak corrections to the decay width of toponium are suppressed by at least four powers of the strong coupling constant, $\alpha_s$. To this end, we employed the covariant Lorentz gauge and perturbed around the solution to the Bethe-Salpeter equation in the instantaneous approximation. The calculation was manifestly gauge-invariant and correctly took into account all contributions to the decay rate (Coulomb enhancement and phase-space reduction effects).

Here, we wish to apply the argument of ref. [5] to the calculation of radiative corrections to the momentum distribution of the $t\bar{t}$ pair produced at threshold at a future $e^+e^-$ collider.
One-loop corrections to the total cross-section of $t\bar{t}$ threshold production have already been shown to vanish to first-order in $\alpha_s$ by Melnikov and Yakovlev [6], who employed the non-relativistic potential formalism. By using the relativistic Bethe-Salpeter formalism, we shall show that there are no $o(\alpha_s)$ corrections to the differential cross-section either.

We are interested in one-loop corrections to the process (Fig. 1)

$$e^+e^- \rightarrow \gamma, Z^0 \rightarrow t\bar{t} \rightarrow W^+bW^-\bar{b},$$

where at threshold the dominant contribution is due to the $t\bar{t}$ bound states (toponium). To simplify the discussion, we shall ignore $t\bar{t}$ production via the $Z^0$ boson. We shall also neglect the Higgs boson exchange between $t$ and $\bar{t}$, which contributes to the formation of bound states. Our calculation can easily be extended to accommodate these additional effects.

Let us first review the non-relativistic potential formalism in order to demonstrate its power as well as its limitations. The $t\bar{t}$ system at threshold forms short-lived bound states which are eigenstates of the Hamiltonian

$$H = \frac{p^2}{M_t} - \frac{C_F\alpha_s}{r} + 2M_t,$$

where we have included the effects of the decay in the mass parameter $M_t = m_t + i\Gamma_t$. The Green function for this Hamiltonian,

$$G(x, x'; E) = -\sum_n \frac{\psi_n(x)\psi_n^*(x')}{E - E_n - i\Gamma_n},$$

is related to the cross-section of $t\bar{t}$ production by the optical theorem,

$$\sigma(\gamma \rightarrow t\bar{t}) = \frac{6\pi^2Q_t^2\alpha^2}{m_t^4} Im\ G(0,0; E),$$

where $E = \sqrt{s} - 2m_t$. This expression for the cross-section includes the contributions of all ladder diagrams containing an arbitrary number of Coulomb-like gluon exchanges between the two quarks ($t$ and $\bar{t}$). This class of diagrams dominates in the threshold region. By splitting the Hamiltonian (2) in its real and imaginary parts, $H = H_0 + i\Gamma$, and expanding,
\( \frac{1}{H - E} = \frac{1}{H_0 - E} \left( 1 + \frac{1}{H_0 - E} i\Gamma \right) + \ldots, \) \( (5) \)

we may write to first-order in the Fermi constant [1]

\[ \sigma(\gamma \rightarrow t\bar{t}) = \frac{3\pi^2 Q_t^2 \alpha^2}{m_t^4} \int \frac{d^3 p}{(2\pi)^3} |\tilde{G}(p; E)|^2 \Gamma(p; E). \] \( (6) \)

The function \( \Gamma(p; E) \) represents the decay rate of toponium. To lowest order, \( \Gamma = 2\Gamma_t \). Higher-order corrections are due to time dilatation and are negligible [3,5]. Thus, the momentum distribution of the top quark to lowest order is given by

\[ \frac{d\sigma}{d|p|} = \frac{3Q_t^2 \alpha^2}{2m_t^4} p^2 |\tilde{G}(p; E)|^2. \] \( (7) \)

Higher-order corrections to the cross-section (4) can be calculated in this formalism using perturbation theory. However, one needs to be careful in applying ordinary perturbation theory, because the decay products (e.g., b and \( \bar{b} \)) feel the binding QCD potential and therefore, they cannot be described by plane waves. To avoid such subtleties, one may use the relativistic Bethe-Salpeter formalism which is manifestly gauge-invariant.

To calculate the differential cross-section, we note that it factorizes into two parts, describing \( t\bar{t} \) production (\( e^+ e^- \rightarrow \gamma , \ Z^0 \rightarrow t\bar{t} \)), and scattering and decay of the \( t\bar{t} \) pair (\( t\bar{t} \rightarrow bW^+ \bar{b}W^- \)), respectively. We shall concentrate on the latter part. Let \( k^\mu \) be the total incoming momentum, and \( p^\mu \) be the relative momentum of the \( t\bar{t} \) pair. After we integrate over the final-state phase space, we can use the optical theorem to express the cross-section as

\[ \frac{d\sigma}{d|p|} \sim Im A_{t\bar{t}}(k, p, p'), \] \( (8) \)

where \( A_{t\bar{t}} \) is the amplitude for \( t\bar{t} \) scattering, and \( p'^\mu \) is the relative momentum of the final \( t\bar{t} \) pair. This four-point amplitude satisfies the inhomogeneous Bethe-Salpeter equation,

\[ \text{The absorptive part of the Hamiltonian, } \Gamma, \text{ is proportional to } G_F. \]
\[ \Pi^{(1)}(p_+)\Pi^{(2)}(p_-)A_{\bar{t}t}(k, p, p') = 1 + \int \frac{d^4p''}{(2\pi)^4} V(p, p''; k)A_{\bar{t}t}(k, p'', p') , \]  

where \( \Pi(p) \) is the complete inverse fermion propagator,

\[ \Pi(p) = \not{p} - M_t - \Sigma(p) , \]

and we have defined momenta

\[ p_\pm = \frac{k}{2} \pm p . \]

\( V(p, p'; k) \) is a potential function which consists of the two-fermion irreducible graphs. For our purposes, the mass is complex,

\[ M_t = m_t + i\Gamma_t . \]

The solution to Eq. (9) has poles which are due to bound states. Near a pole,

\[ A_{\bar{t}t}(k, p, p') \sim \frac{i\chi_k(p)\chi_k(p')}{k^2 - M^2} . \]

The bound-state wavefunctions \( \chi_k(p) \) satisfy the homogeneous Bethe-Salpeter equation,

\[ \Pi^{(1)}(p_+)\Pi^{(2)}(p_-)\chi_k(p) + \int \frac{d^4p'}{(2\pi)^4} V(p, p'; k)\chi_k(p') = 0 . \]

To lowest order in \( \alpha_s \) and neglecting electroweak interactions, the potential is

\[ V_0(p, p'; k) = C_F 4\pi\alpha_s \gamma^{(1)}_\mu G^{\mu\nu}(p - p')\gamma^{(2)}_\nu , \]

where \( C_F = 4/3 \) is the Casimir operator, and \( G^{\mu\nu}(q) \) is the lowest-order gluon propagator. In the Feynman gauge (omitting group theory factors),

\[ G^{\mu\nu}(q) = \frac{\eta^{\mu\nu}}{q^2 + i\epsilon} , \]
and the potential $V_0(p, p'; k)$ is independent of $k^\mu$. At threshold, the quarks move with non-relativistic velocities and the Bethe-Salpeter equation can be approximated by the non-relativistic Schrödinger equation in momentum space, and then solved. To this end, we shall work in the total rest frame in which the overall momentum is $k^\mu = (E, \vec{0})$. In the instantaneous approximation, the potential becomes

$$V_0^{\text{inst}}(p, p'; k) = C_F 4\pi \alpha_s \frac{1}{(\vec{p} - \vec{p}')^2} \frac{1}{\gamma_0^{(1)} \gamma_0^{(2)}}. \quad (17)$$

If we integrate over $p^0$, we can write the Bethe-Salpeter equation (14) in terms of the wave-function $\Phi(\vec{p}) = \int \frac{dp^0}{2\pi} \chi(p)$ as

$$(H^{(1)} + H^{(2)} - E)\Phi(\vec{p}) = \left(\Lambda_+^{(1)} \Lambda_+^{(2)} - \Lambda_-^{(1)} \Lambda_-^{(2)}\right) C_F 4\pi \alpha_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\vec{p} - \vec{p}')^2} \Phi(\vec{p'}) \quad , \quad (18)$$

where $H$ is the Dirac Hamiltonian and $\Lambda_+ (\Lambda_-)$ is the projection operator onto positive (negative) energy states. In the non-relativistic limit, this reduces to the Schrödinger equation in momentum space

$$\left(\frac{\vec{p}^2}{M_t} + 2M_t - E\right)\Phi(\vec{p}) = C_F 4\pi \alpha_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\vec{p} - \vec{p}')^2} \Phi(\vec{p'}) \quad . \quad (19)$$

Thus, we obtain the energy levels

$$E_n = 2M_t - \frac{M_t C_F^2 \alpha_s^2}{4n^2} + o(\alpha_s^4) \quad , \quad (20)$$

which are the Bohr levels of the Coulomb-like QCD potential (17). Therefore, the first-order QCD correction to the decay rate of toponium is

$$\Gamma_{t\bar{t}} = 2\Gamma_t \left(1 - \frac{C_F^2 \alpha_s^2}{8n^2}\right) \quad , \quad (21)$$

which may be attributed to time dilatation [3]. The spherically symmetric $S = 0$ states are given by

$$\Phi_n(\vec{p}) = (M_t C_F \alpha_s)^{-3/2} \frac{L_n(n^2y)}{(1 + n^2y)^{n+1}} \quad , \quad y = \frac{4\vec{p}^2}{M_t^2 C_F \alpha_s^2} \quad . \quad (22)$$
where $L_n$ is a polynomial of order $n - 1$ related to the Laguerre polynomials. For $n = 1$, we have $L_1 = 16\sqrt{2}\pi$.

To discuss final-state corrections, we need to consider the diagrams in Fig. 3. They give first-order corrections to the differential cross-section for $t\bar{t}$ decay, according to the unitarity theorem. This may be seen by cutting these diagrams to produce the graphs of Fig. 4. In the positronium case, these loop graphs represent magnetic moment effects and contribute to a $o(\alpha^5)$ shift in the energy levels (poles). We have shown that there is no $W$ boson contribution to the color magnetic moment of the top quark [3]. This is due to the fact that $W$ only couples to a left-handed current. We shall sketch the proof of this statement for completeness.

The energy level shifts can be calculated by perturbing around the solution to the Schrödinger equation (19). The potential to be treated perturbatively is $V - V_0^{inst}$. There is also a contribution from the disconnected diagrams which are due to the self-energy terms in the fermion propagators (Eq. (10)), but they can be absorbed in the potential if we make use of the Schrödinger equation. Thus, according to the Bethe-Salpeter formalism [4], the first-order energy level shift is

$$\Delta E_n = \langle \Phi_n | D_k(p) \left( H^{(1)} + H^{(2)} - E_n \right) \left( V - V_0^{inst} \right) \left( H^{(1)} + H^{(2)} - E_n \right) D_k(p) | \Phi_n \rangle,$$  

where the inner product involves an integral over the four-momentum. $H$ is the Dirac Hamiltonian, and $D_k$ is the product of two free fermion propagators (cf. Eq. (14)), which can be expressed in terms of the projection operators $\Lambda_\pm$ as

$$D_E(p) = \sum_{\pm} \frac{\Lambda_{\pm}^{(1)} \Lambda_{\pm}^{(2)}}{[E/2 + p^0 \pm (E_\rho - i\epsilon)] [E/2 - p^0 \pm (E_\rho - i\epsilon)],}$$

where $E_\rho = \sqrt{\vec{p}^2 + m_t^2}$ is the energy of the quark on the mass shell.

To lowest order, the potential is $V_0 - V_0^{inst}$ (Eqs. (15) and (17)). This is analogous to the positronium case, and produces an $o(\alpha_s^4)$ shift in the energy levels. The first-order electroweak correction is
\[ V_1(p, p'; k) = 4\pi C_F \alpha_s \alpha_W \left( \Lambda^{(1)}_{\mu}(p_+, p'_+) G^{\mu\nu}(p - p') \gamma^{(2)}_{\nu} + \gamma^{(1)}_{\mu} G^{\mu\nu}(p - p') \Lambda^{(2)}_{\nu}(p_-, p'_-) \right), \]  

(25)

where \( p_\pm = k/2 \pm p, \) \( p'_\pm = k/2 \pm p', \) and we have made explicit the electroweak coupling constant \( \alpha_W \sim G_F M_W^2, \) where \( G_F \) is the Fermi constant and \( M_W \) is the mass of the \( W \) boson. The vertex function \( \Lambda_\mu(p, p') \) consists of the diagrams shown in fig. 3. It is guaranteed to give a gauge invariant contribution by the Ward identity satisfied by the one-particle irreducible function,

\[ (p - p')^\mu \Gamma_\mu(p, p') = \Pi(p) - \Pi(p'). \]

(26)

Since we are only interested in first-order corrections, we may replace \( M_t \) by its real part \( m_t. \) The contribution of \( V_1 \) to the energy level shift (Eq. (23)) can then be written as

\[ \Delta E^\text{W}_n = \frac{C_F^2 \alpha_s^2 \alpha_W}{16 m_t} \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \frac{\eta^{\mu\nu}}{(p - p')^2 + i\epsilon} \frac{\mathcal{L}_n(n^2 y)}{(1 + n^2 y)^n} \frac{\mathcal{L}_n(n^2 y')}{(1 + n^2 y')^n} \times \left\langle D_k(p') \left( \Lambda^{(1)}_{\mu}(p_+, p'_+) \gamma^{(2)}_{\nu} + \gamma^{(1)}_{\mu} \Lambda^{(2)}_{\nu}(p_-, p'_-) \right) D_k(p) \right\rangle, \]

(27)

where \( y = 4p^2/m_t^2 C_F^2 \alpha_s^2 \) and \( y' = 4p'^2/m_t^2 C_F^2 \alpha_s^2. \) A simple scaling argument shows that the lowest-order contribution to the integral comes from the small three-momentum region. Momentum insertions contribute additional powers of \( \alpha_s. \) At low momentum transfer, the three-point vertex \( \Lambda_\mu \) may be written in general as

\[ \Lambda_\mu(p, p') = k^2 \mathcal{F}_1(k^2) + \sigma_{\mu\nu} k^\nu \mathcal{F}_2(k^2), \]

(28)

where \( k = p - p', \) and \( \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]. \) The form factors \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are regular as \( k^2 \to 0. \) In the positronium case, \( \mathcal{F}_2 \) gives an \( o(\alpha^5) \) contribution to the energy level shift, and is due to the magnetic moment interaction. In our case, we need to multiply the gamma matrices by the projection operator \( \frac{1}{2}(1 - \gamma_5), \) due to parity violation of weak interactions. A straightforward explicit calculation shows that the form factor \( \mathcal{F}_2(k^2) \) vanishes to lowest order in \( k^2. \) It follows that the three-point vertex is proportional to \( (p - p')^2 \) (recall that \( p_+ - p'_+ = p_- - p'_- = p - p' \)).
Having established the leading-order behavior of $\Lambda_\mu$, we can now estimate the integral in Eq. (27). As we just showed, $\Lambda_\mu$ contributes a factor $(p - p')^2$. This factor cancels the gluon propagator. Then the integral over $p_0$ and $p'_0$ can be easily done, because of the respective poles in the operators $D(p)$ and $D(p')$. The resulting expression contains six three-momentum factors implying that the integral is $o(\alpha_8^8)$. Therefore, the electroweak correction to the decay width is negligible. Of course, no conclusion can be drawn regarding the exact value of the electroweak correction, because such a high order is beyond the scope of first-order perturbation theory.

Having shown that the poles do not get shifted due to first-order final-state corrections, we deduce that there are no first-order corrections to the amplitude either. Therefore, the total cross-section as well as the differential cross-section of $t\bar{t}$ production and decay do not get corrected to first-order by final-state rescattering.

In conclusion, we have presented a simple physical argument showing that final-state threshold effects vanish to first-order in perturbation theory. This is true for the differential cross-section of $t\bar{t}$ production, which strengthens a previous result on the total cross-section by Melnikov and Yakovlev [6]. Our argument was based on the relativistic Bethe-Salpeter formalism. No special gauge-fixing procedure was required and the calculation was manifestly gauge-invariant.
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FIG. 1. Threshold $t\bar{t}$ production at an $e^+e^-$ collider (NLC).

FIG. 2. The homogeneous Bethe-Salpeter equation.
FIG. 3. First-order electroweak corrections to $t\bar{t}$ production.

FIG. 4. Final-state $t\bar{t}$ rescattering.