Kusuoka-Stroock gradient bounds for the solution of the filtering equation

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Abstract

We obtain sharp gradient bounds for perturbed diffusion semigroups. In contrast with existing results, the perturbation is here random and the bounds obtained are pathwise. Our approach builds on the classical work of Kusuoka and Stroock [7, 9, 10, 11], and extends their program developed for the heat semi-group to solutions of stochastic partial differential equations. The work is motivated by and applied to nonlinear filtering. The analysis allows us to derive pathwise gradient bounds for the un-normalised conditional distribution of a partially observed signal. It uses a pathwise representation of the perturbed semigroup in the spirit of classical work by Ocone [14]. The estimates we derive have sharp small time asymptotics.

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1 Introduction

In the eighties, Kusuoka and Stroock [7, 9, 10, 11] analysed the smoothness properties of the (perturbed) semigroup associated to a diffusion process. More precisely, let \((\Omega, \mathcal{F}, P)\) be a probability space on which we have defined a \(d_1\)-dimensional standard Brownian motion \(B\) and \(X^x = \{X^x_t, t \geq 0\}, x \in \mathbb{R}^N\) be the stochastic flow

\[
X^x_t = x + \int_0^t V_0(X^x_s)ds + \sum_{i=1}^{d_1} \int_0^t V_i(X^x_s) \circ dB^i_s, \quad t \geq 0,
\]

where the vector fields \(\{V_i, \ i = 0, \ldots, d_1\}\) are \(C^\infty\), by which mean they are smooth and bounded with bounded derivatives of all orders, and the stochastic integrals in \([11]\) are of Stratonovich type. The corresponding perturbed diffusion semigroup is then given by

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where $c \in C_b^\infty(\mathbb{R}^N)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ is an arbitrary bounded measurable function. The vector fields $\{V_i, \ i = 0, \ldots, d_1\}$ are assumed to satisfy Kusuoka’s so-called UFG condition. This condition states that the $C_b^\infty(\mathbb{R}^N)$–module $\mathcal{W}$ generated by the vector fields $\{V_i, \ i = 1, \ldots, d_1\}$ within the Lie algebra generated by $\{V_i, \ i = 0, \ldots, d_1\}$ is finite dimensional. In particular, the condition does not require that the vector space $\{W(x) \mid W \in \mathcal{W}\}$ is isomorphic to $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$. Hence, in this sense, the UFG condition is weaker than the uniform Hörmander condition.

Kusuoka, Stroock prove that, under the UFG condition, $P_t^c\varphi$ is differentiable in the direction of any vector field $W$ belonging to $\mathcal{W}$. Moreover, they deduce sharp gradient bounds of the following form: Given vector fields $W_i \in \mathcal{W}$, $i = 1, \ldots, m + n$ there exist constants $C > 0$, $l > 0$ such that

$$\|W_1 \ldots W_n P_t^c(W_{m+1} \ldots W_{m+n}\varphi)\|_p \leq C t^{-l}\|\varphi\|_p,$$

holds for any $\varphi \in C_b^\infty(\mathbb{R}^N)$, $t \in (0, 1]$ and $p \in [1, \infty]$. In fact, the constant $l$ depends explicitly on the vector fields $W_i$, $i = 1, \ldots, m + n$ and the small time asymptotics are sharp. In this paper we deduce a similar result for the randomly perturbed semigroup. More precisely, let

$$Y = \left\{(Y_t^i)_{i=1}^{d_2}, t \geq 0\right\}$$

be a $d_2$-dimensional standard Brownian motion independent of $X$, and define

$$\rho_t^Y(\varphi)(x) = \mathbb{E}\left[\varphi(X_t^x) Z_t^x \mid X_0 = x\right], \ t \geq 0, \ x \in \mathbb{R}^N,$$

where $Z^x = \{Z_t^x, t \geq 0\}$, $x \in \mathbb{R}^N$ is the stochastic process

$$Z_t^x = \exp\left(\sum_{i=1}^{d_2} \int_0^t h_i(X_s^x) \, dY_s^i - \frac{1}{2} \sum_{i=1}^{d_2} \int_0^t h_i(X_s^x)^2 \, ds\right), \ t \geq 0, \ x \in \mathbb{R}^N,$$

$h_i \in C_b^\infty(\mathbb{R}^N)$, $i = 1, \ldots, d_2$ and $\varphi$ is an arbitrary bounded measurable function on $\mathbb{R}^N$. Then we prove in the following that for the mapping $x \mapsto \rho_t^Y(\varphi)(x)$, there exists a $P$-almost surely finite random variable $\omega \rightarrow C(\omega)$ such that with $l$ the explicit constant in \[2\] we have

$$\|W_1 \ldots W_n \rho_t^Y(\varphi)(W_{m+1} \ldots W_{m+n}\varphi)\|_p \leq C(\omega) t^{-l}\|\varphi\|_p,$$

for any $\varphi \in C_b^\infty(\mathbb{R}^N)$, $t \in (0, 1], p \in [1, \infty]$.

We are interested in this particular perturbation as it provides the Feynman-Kac representation for solutions of linear parabolic stochastic partial differential equations (SPDEs)\(^1\).

\(^1\)We expect the methodology presented here can be extended to handle a wider class of random perturbations. We chose this particular perturbation because the corresponding randomly perturbed semigroup provides the Feynman-Kac representation for the solution of the filtering problem. See the Kallianpur-Striebel formula \([10]\) below.
More precisely, let $\rho^x = \{\rho^x_t, t \geq 0\}, x \in \mathbb{R}^N$ be the measure valued process defined on the probability space $(\Omega, \mathcal{F}, P)$ by the formula

$$(\rho^x_t(\omega))(\varphi) = \rho^Y_t(\varphi)(x),$$

where $\varphi$ is an arbitrary Borel measurable function. Then $\rho^x$ is the solution of the following linear parabolic SPDE (written here in its weak form):

$$d\rho^x_t(\varphi) = \rho^x_t(A\varphi)dt + \sum_{k=1}^{d_2} \rho^x_t(h_k\varphi)dY^k_t,$$

$$\rho^x_0 = \delta_x.$$  \hspace{1cm} (6)

Here, $\delta_x$ is Dirac delta distribution centered at $x \in \mathbb{R}^N$, $A = V_0 + \frac{1}{2} \sum_{i=1}^{d_1} V^2_i$ is the infinitesimal generator of $X$, and $\varphi$ is a suitably chosen test function. Equation (6) is called the Duncan-Mortensen-Zakai equation (cf. [5, 15, 16]). It plays a central rôle in nonlinear filtering: The normalised solution of (6) gives the conditional distribution of a partially observed stochastic process. We give details of this intrinsic connection in the second section.

Let us finally note that for a fixed $x \in \mathbb{R}^N$, and any suitably chosen test function $\varphi$, the application $Y(\omega) \mapsto \rho^Y_t(\varphi)(x)$ is a (locally) Lipschitz continuous function as defined on the space of continuous paths\footnote{Here we consider the space of continuous paths defined on $[0, \infty)$ with values $\mathbb{R}^{d_2}$ endowed with the topology of convergence in the supremum norm on compacts. The choice of the norm is important. See [3] for further details.}, see [2] for details. In this paper, we study the mapping $x \mapsto \rho^Y_t(\varphi)(x)$ for a fixed (Brownian) path $Y(\omega)$ and a suitably chosen test function $\varphi$.

The paper is structured as follows: In Section 2 we introduce the filtering problem and explain the connection with the randomly perturbed semigroup (RPS). In section 3 we state the main results of the paper, that is, we deduce sharp gradient bounds of the type (5) for the RPS. In addition, we also give direct corollaries on the smoothness properties of the solution of the filtering problem.

In Section 4, we derive an expansion of the RPS in terms of a classical perturbation series. The expansion is in terms of a series of (iterated) integrals with respect to the Brownian motion $Y$ and derived by exploiting the intrinsic connection between the RPS and the mild solution to the Zakai equation. We then proceed to prove the main theorem. The proof of the main theorem is contingent on two non-trivial regularity estimates for the terms appearing in the perturbation expansion of $\rho^Y_t(\varphi)$ (Propositions 7 and 8), which we prove in the remainder of the paper.

In a first step towards proving these two propositions we re-write in Section 5 the terms of the perturbation expansion iteratively using integration by parts to derive a pathwise representation of the RPS. In particular, this allows us to give an alternative proof of the robust formulation of the filtering problem. In Section 6 we then prove a priori regularity estimates for the terms in the perturbation series. For this, we derive Hölder type regularity

$$\text{Here we consider the space of continuous paths defined on } [0, \infty) \text{ with values } \mathbb{R}^{d_2} \text{ endowed with the topology of convergence in the supremum norm on compacts. The choice of the norm is important. See [3] for further details.}$$
estimates for each term in the pathwise representation of the perturbation expansion by carefully leveraging the gradient estimates for heat semi-groups due to Kusuoka and Stroock. The a priori estimates are asymptotically sharp estimates for the lower order terms in the expansion, but unfortunately not summable.

Finally, in Section 7 we rely on both the a prior estimates derived in Section 6 and arguments underlying the Extension Theorem - a fundamental result from rough path theory (see, e.g. [12, 13]) - to deduce factorially decaying Hölder type bounds for the terms in the perturbation expansion. To this end, we observe that the terms of the original series (as derived in Section 4), when regarded as bounded linear operators between suitable spaces that encode the derivatives, are multiplicative functionals. Such multiplicative functionals are more general than ordinary rough paths but arise similarly for example also in the context of the work of Deya, Gubinelli, Tindel et al (see e.g. [4]) where they analyse rough heat equations. The paper is completed with an appendix containing several useful lemmas and an explicit description of the first three terms in the pathwise representation or the perturbation expansion for one dimensional observations.

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2 The non-linear filtering problem

Let $C_{0}^{\infty}(\mathbb{R}^{N})$ denote the space of smooth bounded functions on $\mathbb{R}^{N}$ with bounded derivatives of all orders and $C_{0}^{\infty}(\mathbb{R}^{N})$ the space of compactly supported smooth functions on $\mathbb{R}^{N}$. The nonlinear filtering problem is stated on the probability space $(\Omega, \mathcal{F}, \tilde{P})$, where the new probability measure $\tilde{P}$ is related to the probability measure $P$ under which the triple $(X, Y, B)$ has been introduced in the previous section. More precisely, the probability measure $\tilde{P}$ is absolutely continuous with respect to $P$ and its Radon-Nikodym derivative is given by

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = Z_t, \quad t \geq 0,$$

where $Z = \{Z_t, t \geq 0\}$ is the exponential martingale defined in (4), that is,

$$Z_t = \exp \left( \sum_{i=1}^{d_2} \int_{0}^{t} h_i (X_s) \, dY_s^i - \frac{1}{2} \sum_{i=1}^{d_2} \int_{0}^{t} h_i (X_s)^2 \, ds \right), \quad t \geq 0.$$  

Under $\tilde{P}$ the law of the process $X$ remains unchanged. That is, $X$ satisfies the stochastic differential equation

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{d_1} V_i(X_t) \circ dB_t^i, X_0 = x \in \mathbb{R}^N \quad t \geq 0. \quad (7)$$

$^3$Throughout this section, we will omit the dependence on the initial condition $x \in \mathbb{R}^N$ for the processes $X^x$. The same applies to all other processes ($Z, W, \rho$ etc).
As in the previous section, we assume that the vector fields $\{V_i, i = 0, \ldots, d_1\}$ are smooth and bounded with bounded derivatives, i.e. $V_i \in C^\infty_b (\mathbb{R}^N, \mathbb{R}^N)$, and the stochastic integrals in $\mathbb{H}$ are of Stratonovich type. We denote by $\pi_0$ the initial distribution of $X$, $\pi_0 = \delta_x$.

Under $\tilde{P}$ the process $Y$ is no longer a Brownian motion, but becomes a semi-martingale. More precisely, $Y$ satisfies the following evolution equation

$$Y_t = \int_0^t h(X_s) \, ds + W_t, \tag{8}$$

where $W$ is a standard $\mathcal{F}_t$-adapted $d_2$-dimensional Brownian motion (under $\tilde{P}$) independent of $X$. Let $\{\mathcal{Y}_t, t \geq 0\}$ be the usual filtration associated with the process $Y$, that is $\mathcal{Y}_t = \sigma(Y_s, s \in [0, t])$.

Within the filtering framework, the process $X$ is called the signal process and the process $Y$ is called the observation process. The filtering problem consists in determining $\pi_t$, the conditional distribution of the signal $X$ at time $t$ given the information accumulated from observing $Y$ in the interval $[0, t]$, that is, for $\varphi$ Borel bounded function, computing

$$\pi_t (\varphi) = E[\varphi(X_t) | \mathcal{Y}_t]. \tag{9}$$

The connection between $\pi_t$, the conditional distribution of the signal $X_t$, and the randomly perturbed semigroup is given by the Kallianpur-Striebel formula. We have

$$\pi_t (\varphi) = \frac{\rho_t^Y(\varphi)}{\rho_t^Y(1)} \tilde{P}(\mathbb{P}) - a.s., \tag{10}$$

where $1$ is the constant function $1(x) = 1$ for any $x \in \mathbb{R}^N$. Equivalently, the Kallianpur-Striebel formula can be stated as

$$\pi_t = \frac{1}{c_t} \rho_t \tilde{P}(\mathbb{P}) - a.s.,$$

where $\rho_t$ is the measure valued process which solves the Duncan-Mortensen-Zakai equation $\mathbb{F}$ and $c_t = \rho_t(1)$. The Kallianpur-Striebel formula explains the usage of the term unnormalised for $\rho_t$ as the denominator $\rho_t(1)$ can be viewed as the normalizing factor for $\rho_t$. For further details of the filtering framework see, for example, [1] and the references therein.

### 3 The main theorem

In this section we will state the main results of our paper. Define the set of all multi-indices $\mathbb{A}$ by letting

$$\mathbb{A} = \bigcup_{k=0}^{\infty} \{0, \ldots, d_1\}^k.$$ 

Following Kusuoka [7] we define for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_l) \in \mathbb{A}$ a multiplication by setting

$$\alpha \ast \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l).$$
Furthermore we define a degree on a multi-index \( \alpha \) by \( \| \alpha \| = k + \text{card}(j : \alpha_j = 0) \). Let \( A_0 = \mathbb{A} \setminus \{0\} \), \( A_1 = \mathbb{A} \setminus \{(0,0)\} \) and \( A_1(j) = \{\alpha \in A_1 : \|\alpha\| \leq j\} \). We inductively define a family of vector fields indexed by \( A \) by taking

\[
V[0] = \text{Id}, \quad V[i] = V_i, \quad 0 \leq i \leq d_1
\]

\[
V[(\alpha \ast i)] = [V[\alpha], V[i]], \quad 0 \leq i \leq d_1, \alpha \in A.
\]

The following condition was introduced by Kusuoka and is weaker than the usual (uniform) Hörmander condition imposed on the vector fields defining the signal diffusion (see Kusuoka [7]).

**Definition 1** The family of vector fields \( V_i, i = 0, \ldots, d_1 \) is said to satisfy the condition (UFG) if the Lie algebra generated by it is finitely generated as a \( C^\infty_b \) left module, i.e. there exists a positive \( k \) such that for all \( \alpha \in A_1 \) there exist \( u_{\alpha, \beta} \in C^\infty_b (\mathbb{R}^N) \) satisfying

\[
V[\alpha] = \sum_{\beta \in A_1(k)} u_{\alpha, \beta} V[\beta]. \tag{11}
\]

From now on suppose that our system of vector fields \( V_i, i = 0, \ldots, d_1 \) satisfies the UFG condition and let \( \ell \) denote the minimal integer \( k \) for which the condition (11) holds. We are ready to formulate the main theorem.

**Theorem 2** Suppose the family of vector fields \( V_i, i = 0, \ldots, d_1 \) satisfies the UFG condition. Let \( m \geq j \geq 0, \alpha_1, \ldots, \alpha_j, \ldots, \alpha_m \in A_1(\ell), h \in C^\infty_b (\mathbb{R}^{d_2}) \). Then there exists a random variable \( C(\omega) \) almost surely finite such that the randomly perturbed semigroup \( p^Y(\omega, t) \) satisfies

\[
\left\| \left( V[\alpha_1] \cdots V[\alpha_j] p^Y(\omega) \left( V[\alpha_{j+1}] \cdots V[\alpha_m] \varphi \right) \right) (x) \right\|_\infty 
\leq C(\omega) t^{- (\| \alpha_1 \| + \cdots + \| \alpha_m \| ) / 2} \| \varphi \|_\infty
\]

for any \( \varphi \in C^\infty_b (\mathbb{R}^N) \), \( t \in (0, 1] \). If in addition \( h \in C^\infty_0 (\mathbb{R}^{d_2}) \) there exists \( C(\omega) \) a.s. finite such that

\[
\left\| \left( V[\alpha_1] \cdots V[\alpha_j] p^Y(\omega) \left( V[\alpha_{j+1}] \cdots V[\alpha_m] \varphi \right) \right) (x) \right\|_p 
\leq C(\omega) t^{- (\| \alpha_1 \| + \cdots + \| \alpha_m \| ) / 2} \| \varphi \|_p
\]

for all \( \varphi \in C^\infty_0 (\mathbb{R}^N) \), \( t \in (0, 1] \), \( p \in [1, \infty] \).

**Remark 3** The random variable \( C(\omega) \) only depends on \( Y(\omega) \) via it’s Hölder control as an Itō rough path (see Lemma 11 for details).

Before we begin the proof of our main theorem we explore some immediate consequences of the result. We first observe that we can obtain similar estimates for the normalised conditional density.
Corollary 4 Under the assumptions of Theorem 2 there exists a r.v. \( C(\omega) \) almost surely finite such that the normalised conditional density \( \pi_t \) satisfies
\[
\left\| \left( V_{[\alpha_1]} \cdots V_{[\alpha_j]} \pi_t \left( V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi \right) \right) \right\|_{\infty} \leq C(\omega) t^{-\left( \|\alpha_1\| + \cdots + \|\alpha_j\| \right)/2} \|\varphi\|_{\infty}
\]
for any \( \varphi \in C_c^\infty (\mathbb{R}^N) \), \( t \in (0, 1] \).

Proof. We have
\[
V_{[\alpha]} \pi_t \left( V_{[\beta]} \varphi \right) (x) = V_{[\alpha]} \left[ \frac{\rho_t^Y(\omega)(V_{[\beta]} \varphi)}{\rho_t^Y(1)} \right] (x)
\]

\[
= \frac{V_{[\alpha]} \rho_t^Y(\omega)(V_{[\beta]} \varphi)}{\rho_t^Y(1)} \frac{\rho_t^Y(1)}{\rho_t^Y(1)} - \rho_t^Y(\omega)(V_{[\beta]} \varphi) V_{[\alpha]} \rho_t^Y(1)
\]

\[
\leq C(\omega) t^{-\left( \|\alpha\| + \|\beta\| \right)/2} \left( 1/\rho_t^Y(1) \right)^2 \max \left( \|\varphi\|_{\infty}, 1 \right).
\]

which gives the estimates as, almost surely, (see the Appendix for a proof)
\[
\sup_{x \in \mathbb{R}^N} \left( 1/\rho_t^Y(1) \right) < \infty.
\]

Finally, the regularity estimates for the un-normalised conditional density allow us to deduce estimates for the smoothness of the density of the unnormalised conditional distribution of the signal with respect to the Lebesgue measure. Assume that the vector fields \( V_i, i = 0, \ldots, d_1 \) satisfy the uniform Hörmander condition and that \( \pi_0 = \delta_x \) is the Dirac measure at \( x \). Then
\[
\rho_x^t(\varphi) = \int_{\mathbb{R}^d} \varphi(y) \Psi_x^t(y) \rho_x^t(y) dy,
\]
where \( y \to \rho_x^t(y) \) is the density of the law of the signal \( X^t_x \) with respect to the Lebesgue measure and \( y \to \Psi_x^t(y) \) is the likelihood function
\[
\Psi_x^t(y) = \mathbb{E}[Z_t^x | X_t = y, \mathcal{Y}_t^x].
\]

We deduce from Theorem 2 that
\[
\| V_{[\alpha]}^* (\Psi_x^t \rho_x^t) \|_1 \leq C t^{-\frac{\|\alpha\|}{2}}, \quad t \in (0, 1]
\]
where \( V_{[\alpha]}^* \) is the adjoint operator of \( V_{[\alpha]} \) for any multi-index \( \alpha \in A_1(\ell) \).

4 Proof of the main theorem
As a first step in the proof of our main theorem we expand the unnormalised conditional distribution of the signal using its representation as the mild solution of the Zakai equation as seen for example in [14]. We have

\[
\rho_t^Y(\varphi)(x) = P_t(\varphi)(x) + \sum_{i=1}^{d_2} \int_0^t \rho_s^Y(h_i P_{t-s}(\varphi))(x) \, dY_s^i.
\]

To iterate this expansion we define the set of operators: \( R_{\bar{t},i} \) where \( \bar{t} = (t_1, t_2, \ldots, t_k) \) is a non-empty multi-index with entries \( t_0, t_1, \ldots, t_k \in [0, \infty) \) that have increasing values \( t_0 < t_1 < \ldots < t_k \) and \( i = (i_1, \ldots, i_{k-1}) \) is a multi-index with entries \( i_1, \ldots, i_{k-1} \in \{1, 2, \ldots, d_2\} \) defined

\[
R_{(t_0, t_1), \varphi}(\varphi) = P_{t_1-t_0}(\varphi)
\]

and, inductively, for \( k > 1, \)

\[
R_{(t_0, t_1, t_2, \ldots, t_k), (i_1, \ldots, i_{k-1})}(\varphi) = R_{(t_0, t_1, \ldots, t_{k-1}), (i_1, \ldots, i_{k-2})}(h_{i_{k-1}} P_{t_{k-1}}(\varphi))
\]

\[
= P_{t_1-t_0}(h_{i_1} P_{t_2-t_1} \cdots (h_{i_{k-1}} P_{t_{k-1}}(\varphi)))
\]

\[
= P_{t_1-t_0}(h_{i_1} R_{(t_1, t_2, \ldots, t_k), (i_2, \ldots, i_{k-1})}(\varphi))
\]

Note that the length of the multi-index \( \bar{t} \) is always two units more than \( \bar{i} \). In the following we will use the notation \( S(m) \) to denote the set of all multi-indices

\[
S(m) = \{(i_1, \ldots, i_m) \mid 1 \leq i_j \leq d_2, \quad 1 \leq j \leq m\}.
\]

and let \( S = \bigcup_{m=1}^{\infty} S(m) \).

**Lemma 5** We have almost surely that

\[
\rho_t^Y(\varphi)(x) = P_t(\varphi)(x) + \sum_{m=1}^{\infty} \sum_{i \in S(m)} R_{0, \bar{t}}^{i, \bar{i}}(\varphi) \tag{15}
\]

where, for \( \bar{i} = (i_1, \ldots, i_m), \)

\[
R_{0, \bar{t}}^{i, \bar{i}}(\varphi) = \int_0^t \int_0^{t_m} \cdots \int_0^{t_2} R_{(0, t_1, \ldots, t_m, t), \bar{i}}(\varphi)(x) \, dY_{t_1}^{i_1} \cdots dY_{t_m}^{i_m}.
\]

**Proof.** Arguing by induction it is easy to see that

\[
\rho_t^Y(\varphi)(x) = P_t(\varphi)(x) + \sum_{m=1}^{k} \sum_{i \in S(m)} R_{0, \bar{t}}^{m, \bar{i}}(\varphi) + \sum_{i \in S(k+1)} \text{Rem}^{k+1, \bar{i}}_{0, \bar{t}}(\varphi),
\]

where

\[
\text{Rem}^{k+1, \bar{i}}_{0, \bar{t}}(\varphi) = \int_0^t \int_0^{t_{k+1}} \cdots \int_0^{t_2} \rho_t^Y(h_{i_1} P_{t_2-t_1} h_{i_2} \cdots h_{i_{k+1}} P_{t_{k+1}}(\varphi))(x) \, dY_{t_1}^{i_1} \cdots dY_{t_{k+1}}^{i_{k+1}}.
\]
Using iteratively Jensen’s inequality and the Itô isometry we see that
\[
E \left[ \text{Rem}_{k+1}^t (\varphi) \right] \\
\leq \int_0^1 \int_0^{t_{k+1}} \cdots \int_0^{t_2} E \left[ \rho_t^Y (h_{i_1} P_{t_2-t_1} h_{i_2} \cdots h_{i_{k+1}} P_{t-t_{k+1}} (\varphi)) \right] dt_1 \cdots dt_{k+1} \\
\leq e^{t||h||_\infty} \frac{||h||^{2(k+1)}_\infty}{(k+1)!} \|\varphi\|_\infty^2,
\]

since, by Jensen’s inequality
\[
E \left[ \rho_t^Y (h_{i_1} P_{t_2-t_1} h_{i_2} \cdots h_{i_{k+1}} P_{t-t_{k+1}} (\varphi))^2 \right] \leq ||h||^{2(k+2)}_\infty E \left[ (Z_t^\varphi)^2 \right] \leq e^{t||h||_\infty} ||h||^{2(k+1)}_\infty.
\]

Before we can prove the main theorem we require three non-trivial estimates for the regularity of the terms appearing in the expansion (15) of \( \rho_t^Y (\varphi) \). The first is the aforementioned gradient estimate due Kusuoka and Strook for the heat semi-group. The following Theorem is due to Kusuoka-Stroock [11] under the uniform Hörmander condition and Kusuoka [7] under the UFG assumption.

**Theorem 6** Suppose the family of vector fields \( V_i, i = 0, \ldots, d_1 \) satisfies the UFG condition. Let \( m \geq j \geq 0, \alpha_1, \ldots, \alpha_j, \ldots, \alpha_m \in A_1 (\ell) \) then there exists a constant \( C \) such that
\[
\left\| V_{[\alpha_1]} \cdots V_{[\alpha_j]} P_t (V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi) \right\| \leq C t^{-((\|\alpha_1\| + \cdots + \|\alpha_m\|)/2) \|\varphi\|_p}
\]
for any \( \varphi \in C_0^\infty (R^N), t \in (0, 1) \) and \( p \in [1, \infty] \).

The second ingredient for the proof of the main theorem are the following regularity estimates for the terms \( R_{0,t}^{m,i} \).

**Proposition 7** Under the assumptions of Theorem 2 let \( \alpha, \beta \in A_1 (\ell) \), \( \gamma \in (1/3, 1/2) \) then there exist a r.v. \( C (\omega, m, \gamma) > 0 \) a.s. finite such that
\[
\left\| V_{[\alpha]} R_{0,t}^{m,i} V_{[\beta]} \varphi \right\|_{\infty} \leq C (\omega, m, \gamma) t^{-((\|\alpha\| + \|\beta\|)/2 + m\gamma) \|\varphi\|_\infty}
\]
for all \( \bar{t} \in S (m) \), \( \varphi \in C_b^\infty (R^N) \) and \( t \in (0, 1] \).

The preceding proposition implies that the short term asymptotics of the regularity of \( \rho_t \) are determined by the leading term of the expansion - the heat semi-group \( P_t f \) itself. The estimate is unfortunately not summable in \( m \) and will therefore only be used to control the regularity of \( R_{0,t}^{m,i} \) for small \( m \). Before we proceed we state a second set of a priori estimates that capture the regularity of the \( R_{0,t}^{m,i} \) in terms of operator norms on some carefully chosen
spaces. Note that these estimate do not lead to sharp short small time asymptotics and will therefore only be used to estimate the regularity of $R^{m,1}_{0,t}$ for sufficiently large values of $m$.

To derive the second set of factorially decaying estimates we regard the $R^{m, i}_{0,t}$ as linear operators acting on smooth functions endowed with suitable norms. Noting that the heat kernels and the multiplication operators defined by the sensor functions $h_i$ map $C^\infty_b (\mathbb{R}^N)$ functions we see that $R^{m, i}_{0,t}$ maps $C^\infty_b (\mathbb{R}^N)$ to $C^\infty_b (\mathbb{R}^N)$. We first define a distribution space appropriate for our problem. For $\varphi \in C^\infty_b (\mathbb{R}^N)$ let

$$
\| \varphi \|_{H^{-1}} := \inf \left\{ \sum_{\alpha \in A_0 (\ell)} \| \varphi_\alpha \|_\infty : \varphi = \sum_{\alpha \in A_0 (\ell)} V_\alpha \varphi_\alpha, \ varphi_\alpha \in C^\infty_b (\mathbb{R}^N) \right\}.
$$

Then $\| \cdot \|_{H^{-1}}$ defines a norm on $C^\infty_b (\mathbb{R}^N)$ that is bounded above by $\| \varphi \|_\infty$, but potentially smaller. Similarly we may define a Sobolev type norm on $C^\infty_b (\mathbb{R}^N)$ by letting

$$
\| \varphi \|_{H^1} := \sum_{\alpha \in A_0 (\ell)} \| V_\alpha \varphi \|_\infty.
$$

Recall in this context that the index set $A_0 (\ell)$ contains the empty set and we have set $V_\emptyset = Id$.

**Proposition 8** Under the assumptions of Theorem 2 there exist constants $\theta > 0$, $\gamma' \in (1/3, 1/2)$, $m_0 (\gamma') \in \mathbb{N}$ and a random variable $c(\gamma', \omega)$, almost surely finite, such that

$$
\left\| R^{m,i}_{0,t} \right\|_{H^{-1} \to H^1} \leq \frac{(c(\gamma', \omega) t)^{m\gamma'}}{\theta (m\gamma')!}
$$

(16)

for all $m \geq m_0, i \in S(m)$ and $t \in (0, 1]$.

Combining the previous estimates we are ready to prove our main theorem.

**Proof of Theorem 2** We are going to show that there exists a positive random variable $c(\omega)$ almost surely finite such that

$$
\sup_{x_0 \in \mathbb{R}^N} \left\| V_{[\alpha]} P_t (V_{[\beta]} \varphi) \right\|_\infty \leq c(\omega) t^\left(-\frac{\| \alpha \| + \| \beta \|}{2}\right) \| \varphi \|_\infty.
$$

for any $t \in (0, 1]$ and $\varphi \in C^\infty_b (\mathbb{R}^N)$.

Fix $\gamma \in (1/3, 1/2)$ and let $\gamma'$, $\theta$ and $m_0$ as in Proposition 8. We have by Lemma 5

$$
\left\| V_{[\alpha]} \rho_t^{(\omega)} (V_{[\beta]} \varphi) \right\|_\infty \leq \left\| V_{[\alpha]} P_t (V_{[\beta]} \varphi) (x) \right\|_\infty
$$

$$
\sum_{k=1}^{m_0} \sum_{i \in S(k)} \left\| V_{[\alpha]} R^{k,i}_{0,t} (V_{[\beta]} \varphi) \right\|_\infty + \sum_{k=m_0+1}^{\infty} \sum_{i \in S(k)} \left\| V_{[\alpha]} R^{k,i}_{0,t} (V_{[\beta]} \varphi) \right\|_\infty.
$$

(17)
Now
\[
\left\| V_\alpha R_{0,t}^k (V_\beta \varphi) \right\|_\infty \leq \left\| P_0^k (V_\beta \varphi) \right\|_{H^1} \\
\leq \left\| P_0^k \right\|_{H^{-1} \rightarrow H^1} \left\| V_\beta \varphi \right\|_{H^{-1}} \\
\leq \left\| P_0^k \right\|_{H^{-1} \rightarrow H^1} \| \varphi \|_\infty .
\]
Therefore using Theorem 6 for the first, Proposition 7 for the second and Proposition 8 for the third term in the sum on the right hand side of (17) we see that
\[
\left\| V_\alpha \rho_t (V_\beta \varphi) \right\|_\infty \leq t^{-((\|\alpha\|+\|\beta\|)/2)} \|\varphi\|_\infty + \sum_{k=1}^{m_0} c_k t^{-((\|\alpha\|+\|\beta\|)/2+k\gamma)} \|\varphi\|_\infty \\
+ \sum_{k=m_0+1}^{\infty} k^{k\gamma} c (\gamma', \omega, d_2) \frac{k\gamma'}{\theta (k\gamma')!} \|\varphi\|_\infty \\
\leq c (\omega) t^{-((\|\alpha\|+\|\beta\|)/2)} \|\varphi\|_\infty
\]
where
\[
c (\omega) = 1 + \sum_{k=1}^{m_0} c_k + \sum_{k=m_0+1}^{\infty} \frac{c (\gamma', \omega, d_2) k^{k\gamma'}}{\theta (k\gamma')!} .
\]
The proof may be generalised to higher derivatives by noting that all the estimates for the smoothness of the integral kernels in section 6 may be generalised using straightforward induction arguments. Similarly the Sobolev and distribution spaces $H^1$ and $H^{-1}$ may be generalised to accommodate higher derivatives. Clearly, the constants and the parameter $m_0$ in equation (17) will depend on the number of derivatives.

For the proof of the second part of the theorem, the general $L^p$ estimate, we follow Kusuoka [7]. First observe that
\[
\|\varphi\|_1 = \sup_{\|g\|_\infty \leq 1} \left| \int \varphi g \right| .
\]
Next we identify the (formal) adjoint of the heat semi group $P_t \varphi$. Let
\[
\tilde{c} = \text{div} (V_0) + \frac{1}{2} \sum_{j=1}^{d} V_j (\text{div} (V_j)) + \frac{1}{2} \sum_{j=1}^{d} (\text{div} (V_j))^2
\]
and
\[
\tilde{V}_0 = -V_0 + \frac{1}{2} \sum_{j=1}^{d} V_j (\text{div} (V_j)) .
\]
Let $\tilde{X}_t$ be the diffusion associated to the vector fields $\left( \tilde{V}_0, V_1, \ldots, V_d \right)$ and define for $x \in R^N$
\[
P_t^* \varphi (x) := E \left( \exp \left( \int_0^t \tilde{c} \left( \tilde{X}_s^x \right) ds \right) \varphi \left( \tilde{X}_t^x \right) \right) .
\]
The following Theorem is a particular case of a result that may be found in Kusuoka, Stroock [11].

**Theorem 9 (Kusuoka-Stroock)** Let \( \varphi \in C_0^\infty (R^N) \) and \( g \in C_0^\infty (R^N) \) then we have

\[
\int P_t \varphi (x) g (x) \, dx = \int \varphi (x) P_t^* g (x) \, dx,
\]

i.e. the semi group \( P_t^* \) is the (formal) adjoint to \( P_t \).

By Lemma [5] we may write

\[
\rho_t^{Y(\omega)} = P_t + \sum_{m=1}^\infty \sum_{i \in \mathcal{S}(m)} \int_{\Delta_{m,t}^N} P_t h_i P_{t_2-t_1} h_{i_2} \cdots H_{m_1} P_{t-m_1} dY_{i_1}^m \cdots dY_{i_m}^m,
\]

where \( H_i \) are the (self-adjoint) multiplication operators corresponding the (compactly supported) \( h_i \). Iteratively applying Theorem [9] to the expansion of \( \rho_t^{Y(\omega)} \) to identify the formal adjoint \( \rho_t^* \) as

\[
\rho_t^* = P_t^* + \sum_{m=1}^\infty \sum_{i \in \mathcal{S}(m)} \int_{\Delta_{m,t}^N} P_{t-m}^* H_{i_m} P_{t-m-1}^* H_{i_{m-1}} \cdots H_{i_1} P_{t_1}^* dY_{i_1}^m \cdots dY_{i_m}^m. \tag{19}
\]

Using (13) and (19) we see that

\[
\| V_{[\alpha]} \rho_t^* \left( V_{[\beta]} \varphi \right) (Y) \|_1 = \sup_{g \in C_0^\infty, \| g \|_\infty \leq 1} \left| \int \varphi (x) \rho_t^{Y(\omega)} \left( V_{[\beta]} \varphi (x) \right) dx \right|
\]

\[
= \sup_{g \in C_0^\infty, \| g \|_\infty \leq 1} \left| \int V_{[\beta]}^* \rho_t^* \left( V_{[\beta]}^* g (x) \right) \varphi (x) dx \right|
\]

\[
\leq \sup_{g \in C_0^\infty, \| g \|_\infty \leq 1} \left\| V_{[\beta]}^* \rho_t^* \left( V_{[\beta]}^* g (x) \right) \right\|_\infty \| \varphi \|_1,
\]

where the formal adjoint of a vector field \( V_{[\alpha]} \) is given by

\[
V_{[\alpha]}^* = -V_{[\alpha]} - \sum_{i=1}^N \frac{\partial}{\partial x^i} V_{[\alpha]}^i.
\]

The arguments in the proof of Proposition [7] generalise easily allowing us to deduce the relevant estimates for the terms in the expansion (19). Extending the proof of Proposition [8] requires some small modifications that are discussed in Remark [25]. Going through the steps in the proof of the first part of the theorem with \( \rho_t^* \) in place of \( \rho_t \) we deduce that

\[
\left\| V_{[\beta]}^* \rho_t^* \left( V_{[\alpha]}^* g \right) \right\|_\infty \leq c(\omega) t^{-((\| \alpha \| + \| \beta \|) / 2)} \| g \|_\infty,
\]

and the case of general \( p \in [1, \infty] \) is a consequence of classical Riesz-Thorin interpolation.
5 Pathwise representation of the perturbation expansion and some preliminary estimates

For the first step towards a proof of Proposition 7 we derive the pathwise representation for the multiple stochastic integrals $R_{m,i,j}^m(t)$ as a sum of Riemann integrals with integrands depending on the Brownian motion $Y$. We will require the following notation. For $k \in \mathbb{N}$ let $\Delta_{s,t}^k$ denote the simplex defined by the relation

$$s < t_1 < \cdots < t_k < t$$

and let

$$d\bar{t}_k := dt_1 \cdots dt_k.$$

For $\bar{i} = (i_1, \ldots, i_k) \in S(k)$ we set

$$dY_{\bar{i}} = dY_{t_1} \cdots dY_{t_k}.$$

and define iterated integrals $q_{s,t}^\bar{i}(Y)$ by setting

$$q_{s,t}^\bar{i}(Y) := \int_{\Delta_{s,t}^k} dY_{\bar{i}} = \int_s^t \int_s^{t_{i_1}} \cdots \int_s^{t_{i_k}} dY_{t_1} \cdots dY_{t_k}.$$

Let $q_{s,t}^{k_1,\ldots,k_r}(Y)$, $k_1,\ldots,k_r \in S$, $\bar{t} = (t_1,\ldots,t_r)$ be the products of iterated integrals

$$q_{s,t}^{k_1,\ldots,k_r}(Y) = \prod_{i=1}^r q_{s,t_i}^{k_i}(Y).$$

Next define the sets $\Theta(k)$

$$\Theta(k) = \text{sp} \left\{ q_{s,t}^{k_1,\ldots,k_r}(Y) : k_1,\ldots,k_r \in S, \bar{t} = (t_1,\ldots,t_r), \sum_{i=1}^r |k_i| = k \right\}$$

and let $\Theta := \bigcup_{k \in \mathbb{N}} \Theta(k)$. For $q \in \Theta$ we define its formal degree by setting $\text{deg}(q) := r$, where $r$ is the unique number such that $q \in \Theta(r)$. For $\bar{i} = (i_1,\ldots,i_k) \in S(k)$ define $\Phi_{\bar{i}}$, $\Psi_{\bar{i}}$, be the following operators

$$\Phi_{\bar{i}} \varphi = h_{i_1} \cdots h_{i_k} \varphi$$

$$\Psi_{\bar{i}} \varphi = [\Phi_{\bar{i}}, A](\varphi)$$

$$= A(h_{i_1} \cdots h_{i_k}) \varphi + \sum_{i=1}^d V_i(h_{i_1} \cdots h_{i_k}) V_i \varphi.$$
In the following proposition we obtain a pathwise representation of the terms in our expansion of the un-normalised conditional density. The proof will exploit integration by parts formulas of the form

\[
\int_0^t q_{0,s}^{(i_1,\ldots,i_k)}(Y) \left( \int_0^s Z_r \, dr \right) \, dY_{s,t}^{i_{k+1}} = q_{0,t}^{(i_1,\ldots,i_{k+1})}(Y) \int_0^t Z_s \, ds - \int_0^t q_{0,s}^{(i_1,\ldots,i_{k+1})}(Y) \, Z_s \, ds.
\]

**Proposition 10** Let \( \bar{i} = (i_1, \ldots, i_m) \in S(m) \). Then we have, almost surely, that

\[
R_{s,t}^{m,\bar{i}}(\varphi) = P_{t-s}(h_{i_1} \ldots h_{i_m} \varphi)(x) q_{s,t}^{\bar{i}}(Y)
\]

\[
+ \sum_{k=1}^{m-1} \sum_{j_1, \ldots, j_k, \ell_1, \ldots, \ell_k} e_{(s,t)}^{m,j_1,\ldots,j_k} \int_{\Delta_{s,t}^k} b_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(Y) \, R_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(\varphi)(x) \, df_k
\]

\[
+ \sum_{k=m}^{m} \sum_{j_1, \ldots, j_k, \ell_1, \ldots, \ell_k} \int_{\Delta_{s,t}^k} c_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(Y) \, R_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(\varphi)(x) \, df_k,
\]

(20)

and \( a_{(s,t)}^{m,j_1,\ldots,j_k}(Y), b_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(Y), c_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(Y) \in \Theta \) are linear combinations of (products of) iterated integrals of \( Y \) and \( R_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(\varphi) \), respectively

\[
P_{t-s}(\hat{\Phi}_1 P_{l_2-t_1} \ldots (\hat{\Phi}_k P_{l_k-t_k}(\varphi))),
\]

where \( \hat{\Phi}_p \in \Gamma_{j_p}, p = 1, \ldots, k \). Moreover we have

\[
\text{deg} \left( a_{(s,t)}^{m,j_1,\ldots,j_k}(Y) \right) + \text{deg} \left( b_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(Y) \right) = \text{deg} \left( c_{(s,t),x_1,\ldots,x_k}^{m,j_1,\ldots,j_k}(Y) \right) = m.
\]

(21)

**Proof.** The proof follows by induction. For \( m = 1 \) observe that

\[
R_{s,t}^{1,\bar{i}}(\varphi) = \int_s^t P_{t-s}(h_{i_1} P_{t-t_1}(\varphi))(x) dY_{s,t}^{i_1}
\]

\[
= P_{t-s}(h_{i_1} \varphi)(x) \int_s^t dY_{s,t}^{i_1} - \int_s^t \left( \int_s^t dY_{s,t}^{i_1} \right) \frac{d}{dt} P_{t-s}(h_{i_1} P_{t-t_1}(\varphi))(x) dt_1,
\]

where

\[
\frac{d}{dt} P_{t-s}(h_{i_1} P_{t-t_1}(\varphi))(x) = P_{t-s}(A(h_{i_1} P_{t-t_1}(\varphi)))(x) - P_{t-s}(h_{i_1} A P_{t-t_1}(\varphi))(x)
\]

\[
= P_{t-s}(\Psi(1) P_{t-t_1}(\varphi))(x).
\]

so (20) holds true with

\[
c_{(s,t)}^{1,(i_1)}(Y) = \int_s^t dY_{s,t}^{i_1}
\]

and, obviously (21) is satisfied. For the induction step, observe that for \( \bar{i} \ast \bar{i}_{m+1} \)

\[
R_{s,t}^{m+1,\bar{i}_{m+1}}(\varphi) = \int_s^t R_{s,t}^{m,\bar{i}}(h_{i_{m+1}} P_{t-t_{m+1}}(\varphi))(x) dY_{s,t}^{i_{m+1}}.
\]

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Hence, assuming that $R_{s,t}^{m,\bar{i} \imath s \bar{r} m+1}$ has an expansion of the form (20), it follows that

$$R_{s,t}^{m+1,\bar{i} \imath s \bar{r} m+1} (\varphi) = R_{s,t}^{1,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi) + R_{s,t}^{2,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi) + R_{s,t}^{3,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi), \quad (22)$$

where

$$R_{s,t}^{1,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi) = \int_s^t P_{m+1-s} (h_{i_1} \ldots h_{i_m+1} P_{t-m+1} (\varphi)) (x) \int_{\Delta_{s,t}^{m+1}} dY_t^s dY_{t_{m+1}}^{\bar{r}} \Delta_{s,t}^{m+1}$$

$$R_{s,t}^{2,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi) = \sum_{k=1}^{m-1} \sum_{\bar{i} = j_1 \ldots j_k} \int_s^t \int_{\Delta_{s,t}^{m \bar{i} \imath s \bar{r} m+1}} a_{k \bar{i} \imath s \bar{r} m+1} (Y) \int_{\Delta_{s,t}^{k \bar{i} \imath s \bar{r} m+1}} b_{m \bar{i} \imath s \bar{r} m+1} (Y) R_{s,t}^{k \bar{i} \imath s \bar{r} m+1} (h_{i_{m+1}+1} P_{t-m+1} (\varphi)) (x) dY_t^s dY_{t_{m+1}}^{\bar{r}}.$$

$$R_{s,t}^{3,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi) = \sum_{k=1}^{m} \int_s^t \int_{\Delta_{s,t}^{k \bar{i} \imath s \bar{r} m+1}} c_{m \bar{i} \imath s \bar{r} m+1} (Y) R_{s,t}^{k \bar{i} \imath s \bar{r} m+1} (h_{i_{m+1}+1} P_{t-m+1} (\varphi)) (x) dY_t^s dY_{t_{m+1}}^{\bar{r}}.$$

We expand each of the three terms in (22). For the first term we have

$$R_{s,t}^{1,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi) = P_{s-t} (h_{i_1} \ldots h_{i_m+1} \varphi) (x) \int_{\Delta_{s,t}^{m+1}} dY_t^s dY_{t_{m+1}}^{\bar{r}} \Delta_{s,t}^{m+1}$$

$$= P_{s-t} (h_{i_1} \ldots h_{i_m+1} \varphi) (x) \int_{\Delta_{s,t}^{m+1}} dY_t^s dY_{t_{m+1}}^{\bar{r}} \Delta_{s,t}^{m+1} \frac{d}{dt_{m+1}} P_{t_{m+1}-s} (h_{i_1} \ldots h_{i_m+1} P_{t_{m+1}-m+1} (\varphi)) (x) dt_{m+1}$$

$$= P_{s-t} (h_{i_1} \ldots h_{i_m+1} \varphi) (x) \int_{\Delta_{s,t}^{m+1}} dY_t^s dY_{t_{m+1}}^{\bar{r}} \Delta_{s,t}^{m+1}$$

$$- \int_s^t \left( \int_s^{t_{m+1}} dY_t^s dY_{t_{m+1}}^{\bar{r}} \right) \frac{d}{dt_{m+1}} P_{t_{m+1}-s} (h_{i_1} \ldots h_{i_m+1} P_{t_{m+1}-m+1} (\varphi)) (x) dt_{m+1} \ . \quad (23)$$

so the first term in the expansion of $R_{s,t}^{1,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi)$ gives us the first term in the expansion of (22) and the second term in the expansion of $R_{s,t}^{1,m+1,\bar{i} \imath s \bar{r} m+1} (\varphi)$ can be incorporated in the last term in the expansion of (22). Obviously,

$$\deg \left( \int_{\Delta_{s,t}^{m+1}} dY_t^s dY_{t_{m+1}}^{\bar{r}} \right) = m + 1$$
so (21) is satisfied. For the second term we have

\[
R_{s,t}^{2,m+1,\tilde{s}t^{m+1}}(\varphi) = \sum_{k=1}^{m-1} \sum_{j_k = j_1 \ldots j_k \in j_1 \ldots j_k} \int_s^t a_{(s,t,m+1)}^{m,j_1 \ldots j_k}(Y) \int_s^t b_{(s,t_1,\ldots,t_k)}^{m,j_1 \ldots j_k}(Y) \tilde{R}_{(s,t_1,\ldots,t_k,m+1)}^{m,j_1 \ldots j_k}(h_{i,m+1} P_{t-t_m+1}(\varphi))(x) dY_{t_m+1} dY_{t_m+1}
\]

\[
= \sum_{k=1}^{m-1} \sum_{j_k = j_1 \ldots j_k} \int_s^t a_{(s,t,m+1)}^{m,j_1 \ldots j_k}(Y) dY_{t_m+1} \int_s^t S_{s,t,m+1}^{2,m+1,j_1 \ldots j_k}(\varphi) dt_{m+1}
\]

\[
- \sum_{k=1}^{m-1} \sum_{j_k = j_1 \ldots j_k} \int_s^t \left( \int_s^t a_{(s,t,r)}^{m,j_1 \ldots j_k}(Y) dY_{t_m+1} \right) S_{s,t,m+1}^{2,m+1,j_1 \ldots j_k}(\varphi) dt_{m+1} (24)
\]

where

\[
S_{s,t,m+1}^{2,m+1,j_1 \ldots j_k}(\varphi) = \frac{d}{dt_{m+1}} \int_{\Delta_{s,t,m+1}} b_{(s,t_1,\ldots,t_k)}^{m,j_1 \ldots j_k}(Y) \tilde{R}_{(s,t_1,\ldots,t_k,m+1)}^{m,j_1 \ldots j_k}(h_{i,m+1} P_{t-t_m+1}(\varphi))(x) dY_{k-1}
\]

\[
= \int_{\Delta_{s,t,m+1}} b_{(s,t_1,\ldots,t_k)}^{m,j_1 \ldots j_k}(Y) \tilde{R}_{(s,t_1,\ldots,t_k,m+1)}^{m,j_1 \ldots j_k}(h_{i,m+1} P_{t-t_m+1}(\varphi))(x) dY_{k-1}
\]

\[
+ \int_{\Delta_{s,t,m+1}} b_{(s,t_1,\ldots,t_k)}^{m,j_1 \ldots j_k}(Y) \tilde{R}_{(s,t_1,\ldots,t_k,m+1)}^{m,j_1 \ldots j_k}(A(h_{i,m+1} P_{t-t_m+1}(\varphi)) - h_{i,m+1} A(P_{t-t_m+1}(\varphi)))(x) dY_{k-1}
\]

\[
= \int_{\Delta_{s,t,m+1}} b_{(s,t_1,\ldots,t_k)}^{m,j_1 \ldots j_k}(Y) \tilde{R}_{(s,t_1,\ldots,t_k,m+1)}^{m,j_1 \ldots j_k}(\Phi_{i,m+1} P_{t-t_m+1}(\varphi))(x) dY_{k-1}
\]

\[
+ \int_{\Delta_{s,t,m+1}} b_{(s,t_1,\ldots,t_k)}^{m,j_1 \ldots j_k}(Y) \tilde{R}_{(s,t_1,\ldots,t_k,m+1)}^{m,j_1 \ldots j_k}(\Psi_{i,m+1} P_{t-t_m+1}(\varphi))(x) dY_{k-1} (25)
\]

The first term in the expansion of \( R_{s,t}^{2,m+1,\tilde{s}t^{m+1}}(\varphi) \) contributes to the second term in the expansion of (22). The identity (21) is also satisfied as each of the terms \( a_{(s,t,m+1)}^{m,j_1 \ldots j_k}(Y) \) is replaced by

\[
\int_s^t a_{(s,t,m+1)}^{k,m,j_1 \ldots j_k}(Y) dY_{t_m+1}
\]

so the degree for each term increases by 1. Similarly, the second term in the expansion of \( R_{s,t}^{2,m+1,\tilde{s}t^{m+1}}(\varphi) \) contributes to the third term in the expansion of (22), whilst the identity
is also satisfied as each of the terms \( a_{(s,r)}^{m,j_1,...,j_k} (Y) \) is replaced by \[
\int_s^{t_{m+1}} a_{(s,r)}^{m,j_1,...,j_k} (Y) \, dY_r^{im+1}
\]
so, again, the degree for each term increases by 1. Similarly,

\[
R^{3,m+1,ir_{m+1}}_{s,t} (\varphi) = \sum_{k=1}^m \sum_{\vec{j}, i=1,...,j_k} \int_s^t \int_{\Delta^k_{s,t_{m+1}}} \sum_{m,j_1,...,j_k} c_{(s,t_1,...,t_k)} (Y) \hat{R}_{(s,t_1,...,t_k,t_{m+1})}^m (h_{im+1} P_{t-t_{m+1}} (\varphi)) (x) \, d\tilde{f}_k dY_r^{im+1}
\]

\[
= \sum_{k=1}^m \sum_{\vec{j}, i=1,...,j_k} \int_s^t \int_{\Delta^k_{s,t_{m+1}}} S_{s,t_{m+1}}^{3,m,j_1,...,j_k,im+1} (\varphi) \, dt_{m+1}
\]

(26)

where

\[
S_{s,t_{m+1}}^{3,m,j_1,...,j_k,im+1} (\varphi) = \frac{d}{dt_{m+1}} \int_{\Delta^k_{s,t_{m+1}}} c_{(s,t_1,...,t_k)} (Y) \hat{R}_{(s,t_1,...,t_k,t_{m+1})}^m (h_{im+1} P_{t-t_{m+1}} (\varphi)) (x) \, d\tilde{f}_k
\]

\[
= \int_{\Delta^{k-1}_{s,t_{m+1}}} c_{(s,t_1,...,t_k)} (Y) \hat{R}_{(s,t_1,...,t_k,t_{m+1})}^m (\Phi_{(im+1)} P_{t-t_{m+1}} (\varphi)) (x) \, d\tilde{f}_{k-1}
\]

\[
+ \int_{\Delta^k_{s,t_{m+1}}} c_{(s,t_1,...,t_k)} (Y) \hat{R}_{(s,t_1,...,t_k,t_{m+1})}^m (\Psi_{(im+1)} P_{t-t_{m+1}} (\varphi)) (x) \, d\tilde{f}_k
\]

(27)

The first term in the expansion of \( R^{3,m+1,ir_{m+1}}_{s,t} (\varphi) \) contributes to the second term in the expansion of \( (22) \). The identity \( (21) \) is also satisfied as we add \( \int_s^t dY_r^{im+1} \) to each of the terms so the total degree increases by 1. Similarly, the second term in the expansion of \( R^{3,m+1,ir_{m+1}}_{s,t} (\varphi) \) contributes to the third term in the expansion of \( (22) \), whilst the identity \( (21) \) is again satisfied as we add \( \int_s^t dY_r^{im+1} \) to each term.

The results now follow from \( (23) \), \( (24) \), \( (25) \), \( (26) \) and \( (27) \).

We will require a pathwise control of the iterated (Ito) integrals \( \dot{q}_{s,t} (Y) \) of the Brownian motion. It is well known that the Ito lift of Brownian motion is a Holder controlled rough path (see e.g. \[13\] or \[6\]), which immediately implies the following lemma.

---
Lemma 11 For any $1/3 < \gamma < 1/2$ there exists a positive random variable $c = c(\omega, \gamma)$ and some constant $\theta > 0$ such that, almost surely,

$$|q_{s,t}^\xi(Y)| \leq \frac{(c(\omega, \gamma) |s-t|)^{k\gamma}}{\theta (k\gamma)!}$$

for all $0 \leq s \leq t \leq 1$, $\xi \in S(k)$.

It is important to note that the operators $\Phi$ that arise when we recursively apply the integration by parts in the Proposition 10 only involve the vector fields $V_i$, $i = 1, \ldots, d_1$ (but not the vector field $V_0$) and these vector fields do not change if we consider the Ito or Stratonovich versions of the SDE defining the signal.

We have already seen that the $R_{s,t}^{\alpha,\beta}$ may be regarded as bounded linear operators. The next two lemmas show us how to deduce regularity estimates on $R_{s,t}^{\alpha,\beta}$ from regularity estimates on the integral kernels $R$ and $\tilde R$.

Lemma 12 With the notation of Lemma 11 Let $(W, \| \cdot \|)$ be a Banach space, $i \in S(m)$ and suppose $R_{s,t}^{\alpha,\beta} \in W$. For any $1/3 < \gamma < 1/2$ there exist random variables $c(\gamma, \omega)$ such that, almost surely

$$\|R_{s,t}^{\alpha,\beta}\| \leq (c(\gamma, \omega) |t-s|)^{m\gamma} \sum_{k=1}^m \sum_{j_1, \ldots, j_k = 1} \|R_{(s,t_1, \ldots, t_k)}^{m,j_1, \ldots, j_k}\| + \|R_{(s,t_1, \ldots, t_k)}^{m,j_1, \ldots, j_k}\| d\bar{\alpha}_k.$$  

Proof. It follows immediately from combining the Hölder estimates for the iterated integrals $q_{s,t}^\xi(Y)$ obtained in Lemma 11 and Proposition 10 that

$$\|R_{s,t}^{\alpha,\beta}\| \leq (c(\gamma, \omega) |t-s|)^{m\gamma} \sum_{k=1}^m \sum_{j_1, \ldots, j_k = 1} \|\int_{\Delta_{s,t}} R_{(s,t_1, \ldots, t_k)}^{m,j_1, \ldots, j_k} \circ d\alpha_k\| + \|\int_{\Delta_{s,t}} R_{(s,t_1, \ldots, t_k)}^{m,j_1, \ldots, j_k} \circ d\bar{\alpha}_k\|.$$  

In the following lemma we assume that the integral kernels $R$ and $\tilde R$ have bounds with integrable singularities. The control of the constants in the lemma is actually stronger than we will later require.

Lemma 13 Under the assumptions of Lemma 12 Let $i \in S(m)$, $m \geq 1$. Suppose there exists a constant $c$ such that for all $j_1, \ldots, j_k \in S$ satisfying $i = j_1 * \ldots * j_k$, $t_0 = 0 < t_1 < \cdots < t_k < t$ we have both

$$\|R_{(0,t_1, \ldots, t_k,t)}^{m,j_1, \ldots, j_k}\| \leq ct^{k_0} \frac{1}{\sqrt{t_1 - t_0}} \ldots \frac{1}{\sqrt{t_k - t_{k-1}}}$$

and

$$\|\tilde R_{(0,t_1, \ldots, t_k,t)}^{m,j_1, \ldots, j_k}\| \leq ct^{k_0} \frac{1}{\sqrt{t_1 - t_0}} \ldots \frac{1}{\sqrt{t_k - t_{k-1}}}$$

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for some \( k_0 \in \mathbb{R} \). Then

\[
\int_{\Delta_{s,t}^k} \left\| \tilde{R}_{(s,t_1, \ldots, t_k,t)}^{m,j_1, \ldots, j_k} \right\| \, d\tilde{t}_k \leq a_k \left| t - s \right|^{k/2+k_0}
\]

where

\[
a_k = \frac{4 (2\sqrt{\pi})^k c}{k \Gamma \left( \frac{k}{2} \right)}.
\]

**Proof.** First observe that

\[
\int_s^t \int_s^{t_1} \int_s^{t_2} \cdots \int_s^{t_k} \tilde{R}_{(s,t_1, \ldots, t_k,t)}^{m,j_1, \ldots, j_k}(\varphi) \, dt_1 \cdots dt_k = \int_0^{t-s} \int_0^{t-k} \int_0^{t-2} \cdots \int_0^{t-2} \tilde{R}_{(0,t_1, \ldots, t_k,t-s)}^{m,j_1, \ldots, j_k}(\varphi) \, dt_1 \cdots dt_k.
\]

Hence, it is sufficient to prove the result for \( s = 0 \). Writing \( t - s = u \) let \( \Lambda_u \) be the simplex

\[
\Lambda_u = \left\{ (a_1, \ldots, a_k) \in \mathbb{R}_+^k \left| \sum_{i=1}^k a_i \leq u \right. \right\}.
\]

We have

\[
\left\| P_{a_1} \left( \tilde{\Phi}_1 P_{a_2} \cdots P_{a_{k-1}} \tilde{\Phi}_{k-1} P_{a_k} \left( \tilde{\Phi}_k P_{u-(\sum_{j=1}^k a_j)} \right) \right) \right\| \leq c_k u^{k_0} \frac{1}{\sqrt{a_1}} \frac{1}{\sqrt{a_2}} \cdots \frac{1}{\sqrt{a_k}}
\]

and introduce the change of variable \( a_i = uz_i^2, \quad i = 1, \ldots, k \) with the determinant of its Jacobian being \( 2^k u^{k_1/2} z_1 z_2 \cdots z_k \). Then

\[
\int_{\Lambda_1^{k+1}} \left\| P_{a_1} \left( \tilde{\Phi}_1 P_{a_2} \cdots P_{a_{k-1}} \tilde{\Phi}_{k-1} P_{a_k} \left( \tilde{\Phi}_k P_{u-(\sum_{j=1}^k a_j)} \right) \right) \right\| \, da \leq c_k 2^k u^{k/2+k_0} \left( \Lambda_1^{k+1} \right),
\]

where

\[
\Lambda_1^{k+1} \subset \left\{ (z_1, \ldots, z_k) \in \mathbb{R}_+^k \left| \sum_{i=1}^k z_i^2 \leq 1 \right. \right\}
\]

In other words \( \Lambda_1^{k+1} \) is a subset of the unit hypersphere hence its volume less the volume of the sphere so

\[
l \left( \Lambda_1^{k+1} \right) \leq \frac{2\pi^{k/2}}{k \Gamma \left( \frac{k}{2} \right)}.
\]

A similar argument using \( \tilde{R}_{(s,t_1, \ldots, t_k,t)}^{m,j_1, \ldots, j_k} \) in place of \( \tilde{R}_{(s,t_1, \ldots, t_k,t)}^{m,j_1, \ldots, j_k} \) completes the proof.
6 Kusuoka-Stroock regularity estimates for the integral kernels

The aim of this section is to derive regularity estimates for the integral kernels $\hat{R}$ and $\bar{R}$ that arise in the pathwise representation of the expansion of the unnormalised conditional density. The use of these bounds is twofold. First, they will allow us to control directly the lower order terms in the expansion derived in section 5 and second provide us via Lemma 12 with bounds on the operator norms of the operators $R^m, \bar{R}$ acting on the spaces $H^1$ and $H^{-1}$ respectively.

Recall that $\Delta_{s,t}^k$ denotes the simplex defined by the relation $s < t_1 < \cdots < t_k < t$. In a first step we would like to obtain estimates for the kernels of the form (30) that are (essentially) uniform across the simplex over which we are integrating. The basic idea is that for any $(t_1, \ldots, t_k) \in \Delta_{0,t}^k$ there exists always at least one time interval $[t_j, t_{j-1}]$ that is of length at least $t/(k+1)$. We then use the Kusuoka-Stroock regularity estimates (Theorem 6) to deduce smoothness of the heat semigroup over this particular interval. The proof of Theorem 6 employs the methods of Malliavin calculus. As we will in the following draw on elements of their method we recall some basic concepts of the Malliavin calculus.

Let $(\Theta, H, \mu)$ be the abstract Wiener space and let $L$ denote the Ornstein Uhlenbeck operator defined as in Kusuoka [8]. Denote by $G(L)$ the set of arbitrarily often Malliavin differentiable real valued random variables on $\Theta$ and denote by $D^p_s$, the usual Kusuoka-Stroock Sobolev spaces based on the Ornstein-Uhlenbeck operator (see e.g. Kusuoka [8] or [7] for details). The following definition is take from Kusuoka [7], p.267.

Definition 14 Let $r \in \mathbb{R}$ and $K_r$ denote the set of functions $f : (0, 1] \times \mathbb{R}^N \rightarrow G(L)$ satisfying the following conditions

1. $f(t, x)$ is smooth in $x$ and $\frac{\partial^{\nu} f}{\partial x^\nu}$ is continuous in $(t, x) \in (0, 1] \times \mathbb{R}^N$ with probability one for any multi-index $\nu$

2. 

$$\sup_{t \in (0,1], x \in \mathbb{R}^N} t^{-r/2} \left\| \frac{\partial^{\nu} f}{\partial x^\nu} (t, x) \right\|_{D^p_s} < \infty$$

for any $s \in \mathbb{R}$, $p \in (1, \infty)$.

For $\Phi \in K_r$, $\varphi \in C_b^\infty$ define $P^\Phi t \varphi = E(\Phi(t, x) \varphi(X_t(x)))$. An important ingredient in the proof of Theorem 6 which we will use repeatedly is the following Lemma (Kusuoka [7] Corollary 9).

Lemma 15 (Kusuoka) Let $r \in \mathbb{R}$, $\Phi \in K_r$ and $\alpha \in A_1(\ell)$. Then there are $\Phi_{\alpha,1}, \Phi_{\alpha,2} \in K_{r-\|\alpha\|}$ such that

$$P^\Phi_t V_{[\alpha]} = P^\Phi_{t \alpha,1} \quad \text{and} \quad V_{[\alpha]} P^\Phi_t = P^\Phi_{t \alpha,2}. \quad (32)$$
Moreover there exists $C$ such that
\[ \| P_t^\phi \varphi \|_\infty \leq t^{r/2} \| \varphi \|_\infty \]
for any $\varphi \in C^\infty_b (\mathbb{R}^N)$ and $t \in (0, 1]$.

Before we proceed we gather some simple properties of the spaces $\mathcal{K}_r$. The following Lemma may be found in Kusuoka \cite{Kusuoka} (Lemma 7).

\textbf{Lemma 16} Let $r_1, r_2 \in \mathbb{R}$. Then

1. If $f_1 \in \mathcal{K}_{r_1}$ and $f_2 \in \mathcal{K}_{r_2}$ then $f_1 f_2 \in \mathcal{K}_{r_1 + r_2}$

2. If $\varphi \in C^\infty_b (\mathbb{R}^N)$ then $\varphi (X_t (x)) \in \mathcal{K}_0$

3. For any $\alpha, \beta \in A_1 (\ell)$ there exist $a^\beta_\alpha, b^\beta_\alpha \in \mathcal{K}_{(1, \| \beta \| - \| \alpha \|)}$ such that
\[
((X^{-1} t)_* V_\alpha) (x) = \sum_{\beta \in A_1 (t)} a^\beta_\alpha (t, x) V_\beta (x)
\]
and
\[
V_\alpha (x) = \sum_{\beta \in A_1 (t)} b^\beta_\alpha (t, x) ((X^{-1} t)_* V_\beta) (x).
\]

\textbf{Proof.} The claims (2) and (3) are shown in \cite{Kusuoka} (Lemma 7). For (1) note that the space $\bigcap_{1 < p < \infty} D^p_\alpha (\mathbb{R})$ is an algebra (Kusuoka \cite{Kusuoka} Lemma 2.13) and $\| fg \|_{D_p^q} \leq \| f \|_{D^r_p} \| g \|_{D^s_q}$ for $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$, and any $f, g \in \bigcap_{1 < p < \infty} D^p_\alpha$ and
\[
\sup_{t \in [0,1], x \in \mathbb{R}^N} t^{-(r_1 + r_2)/2} \left\| \frac{\partial (f_1 f_2)}{\partial x} (t, x) \right\|_{D^p_q} \leq \sum_{1 \leq i, j \leq 2, i \neq j} \| f_i \|_{D^r_p} \sup_{t \in [0,1], x \in \mathbb{R}^N} t^{-r_i/2} \left\| \frac{\partial f_j}{\partial x} (t, x) \right\|_{D^s_q} < \infty
\]
The generalisation to higher derivatives is clear and the claim follows. $\blacksquare$

In particular the Lemma implies that for any multi-index $\gamma$, $p \in [1, \infty)$ and $T > 0$
\[
\sup_{x \in \mathbb{R}^N} E \left[ \sup_{t \in [0, T]} \left| \frac{\partial^{\gamma}}{\partial x^\gamma} a^\beta_\alpha (t, x) \right|^p \right] < \infty
\]
and
\[
\sup_{x \in \mathbb{R}^N} E \left[ \sup_{t \in [0, T]} \left| \frac{\partial^{\gamma}}{\partial x^\gamma} b^\beta_\alpha (t, x) \right|^p \right] < \infty.
\]

Let $J^{ij}_t (x) = \frac{\partial}{\partial x_i} X^j_t (t, x)$ and note that for any $C^\infty_b$ vector field $W$ we have
\[
((X_t)_* W)^i (X_t (x)) = \sum_{j=1}^N J^{ij}_t (x) W^j,
\]

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Suppose \( \Phi \in \mathcal{K}_r \). Then

\[
V_{[\alpha]} P_t^\Phi \varphi (x) = E \left[ V_{[\alpha]} \Phi \varphi (X_t (x)) + \sum_{i,j=1}^{N} \Phi V_{[\alpha]}^j (x) \left( \frac{\partial}{\partial x^i} \varphi \right) (X_t (x)) J_t^{ij} (x) \right]
\]

It is straightforward to see that \( V_{[\alpha]} \Phi \in \mathcal{K}_r \) and for the second term in the sum we have

\[
E \left[ \sum_{i,j=1}^{N} \Phi V_{[\alpha]}^j (x) \left( \frac{\partial}{\partial x^i} \varphi \right) (X_t (x)) J_t^{ij} (x) \right]
\]

\[
= E \left[ \Phi \sum_{\beta \in A_1 (t)} b_\beta^0 (t, x) \sum_{i=1}^{N} \left( (X_t)_\beta, V_{[\beta]} \right) (x) \left( \frac{\partial}{\partial x^i} \varphi \right) (X_t (x)) \right]
\]

\[
= E \left[ \Phi \sum_{\beta \in A_1 (t)} b_\beta^0 (t, x) \sum_{i=1}^{N} V_{[\beta]} (X_t (x)) \left( \frac{\partial}{\partial x^i} \varphi \right) (X_t (x)) \right]
\]

\[
= \sum_{\beta \in A_1 (t)} P^{\Phi^\beta_0} (V_{[\beta]} \varphi) (x)
\]

Note that by Lemma 16 \( \Phi^\beta_0 (t, x) \in \mathcal{K}_{(||\beta||_0 + r)} \). We have just proved the following Lemma (see e.g. Kusuoka [7] Corollary 9).

**Lemma 17** Let \( \Phi \in \mathcal{K}_r \) and \( \alpha \in A_1 (t) \) then \( V_{[\alpha]} \Phi \in \mathcal{K}_r \) and there exist \( \Phi^\beta_0 \in \mathcal{K}_{(||\beta||_0 + r)} \) such that we have

\[
V_{[\alpha]} P_t^\Phi \varphi (x) = P^{V_{[\alpha]} \Phi} \varphi (x) + \sum_{\beta \in A_1 (t)} P^{\Phi^\beta_0} (V_{[\beta]} \varphi) (x),
\]

for all \( \varphi \in C^\infty_b (\mathbb{R}^N) \).

The following Lemma is an immediate consequence of Lemma 15.

**Lemma 18** Let \( \Phi \in \mathcal{K}_r \) and \( \alpha \in A_1 (t) \) then there exists \( C > 0 \) such that

\[
\| V_{[\alpha]} P_t^\Phi \varphi \|_\infty \leq C \sum_{\beta \in A_0 (t)} \min \left( t^{r/2}, t^{(||\beta||_0 + r)/2} \right) \| V_{[\beta]} \varphi \|_\infty
\]

for all \( t \in (0, 1] \), \( \varphi \in C^\infty_b (\mathbb{R}^N) \). In particular, if \( H \) is of the form \( H = u V_i + v \) for some \( u, v \in C^\infty_b \), \( i \in \{ 1, \ldots, d_1 \} \) and \( \Phi \in \mathcal{K}_0 \) we have

\[
\| V_{[\alpha]} P_t^\Phi H \varphi (x) \|_\infty \leq C \sum_{\beta \in A_0 (t)} \min \left( t^{-1/2}, t^{(||\beta||_0 + r)/2 - 1/2} \right) \| V_{[\beta]} \varphi \|_\infty.
\]
Proof. By Lemma 17 there exist \( \Phi_\beta \in \mathcal{K}_{(\|\beta\| - \|\alpha\|)0 + r} \), such that
\[
\left\| V_\alpha P_t^\Phi \varphi (x) \right\|_\infty
\leq \sum_{\beta \in A_0 (t)} \left\| P_t^\Phi V_\beta \varphi \right\|_\infty
\leq C \sum_{\beta \in A_0 (t)} \min \left( t^{r/2}, t^{(\|\beta\| - \|\alpha\|)/2 + r/2} \right) \left\| V_\beta \varphi \right\|_\infty .
\]

The last inequality is a consequence of Lemma 15 (2). To deduce the second claim from the first of the proposition we note that by 7 Corollary 9 (2) if \( \Phi \in \Phi_a \in \mathcal{K}_r \) there exists \( \Phi_a \in \mathcal{K}_{r - \|\alpha\|} \) such that \( P_t^\Phi V_0 = P_t^\Phi a \). ■

Intuitively the preceding lemma provides us with a uniform (for small times) bound when we move derivatives through the heat kernel from the outside to the inside.

We now consider the reverse situation in which we move the vector fields from the inside to the outside. We have the following Lemma.

Lemma 19 Let \( \Phi \in \mathcal{K}_r \) and \( \alpha \in A_1 (t) \) then there exist \( \Phi_\beta \in \mathcal{K}_r \) and \( \Phi a^\beta_\alpha \in \mathcal{K}_{(\|\beta\| - \|\alpha\|)0 + r} \) such that
\[
(P_t^\Phi V_\alpha \varphi)(x) = \sum_{\beta \in A_1 (t)} \left\{ V_\beta P_t^\Phi \varphi (x) - P_t^\Phi \varphi \right\},
\]
for all \( \varphi \in C_b^\infty (\mathbb{R}^N) \).

Proof. We have using Lemma 16 (3)
\[
(P_t^\Phi V_\alpha \varphi)(x) = E \left[ \Phi \sum_{i=1}^N V_i^\alpha (x) \left( \frac{\partial}{\partial x_i} \varphi \right) (X_t (x)) \right]
= E \left[ \Phi \sum_{i=1}^N ((X_t)_* (X_t^{-1})_* V_\alpha)_i \varphi (X_t (x)) \left( \frac{\partial}{\partial x_i} \varphi \right) (X_t (x)) \right]
= E \left[ \Phi \sum_{i,j=1}^N ((X_t)_* (X_t^{-1})_* V_\alpha)_j \varphi (X_t (x)) J_{ij} (x) \left( \frac{\partial}{\partial x_i} \varphi \right) (X_t (x)) \right]
= E \left[ \Phi \sum_{\beta \in A_1 (t)} a^\beta_\alpha (t, x) \sum_{i=1}^N V_i^\beta (x) \sum_{i=1}^N J_{ij} (x) \left( \frac{\partial}{\partial x_i} \varphi \right) (X_t (x)) \right]
= \sum_{\beta \in A_1 (t)} E \left[ \Phi a^\beta_\alpha (t, x) \sum_{j=1}^N V_j^\beta (x) \frac{\partial}{\partial x_j} \varphi (X_t (x)) \right],
\]
where \( \Phi a^\beta_\alpha \in \mathcal{K}_{(\|\beta\| - \|\alpha\|)0 + r} \). On the other hand we have
\[
V_\beta P_t^{\alpha^\beta_\alpha} \varphi (x)
= E \left[ \Phi a^\beta_\alpha (t, x) \sum_{j=1}^N V_j^\beta (x) \frac{\partial}{\partial x_j} \varphi (X_t (x)) \right] + E \left[ V_\beta \left( \Phi a^\beta_\alpha \right) (t, x) \varphi (X_t (x)) \right]
\]
and deduce that

\[ (P_t^\Phi V_{[a]}\varphi)(x) = \sum_{\beta \in A_1(\ell)} \left\{ V_{[\beta]} P_t^{\alpha_\beta}\varphi(x) - E \left[ V_{[\beta]} \left( \Phi a_\beta^0 \right) (t,x) \varphi(X(t,x)) \right] \right\}, \]

where \( V_{[\beta]} \left( a_\beta^0 \Phi \right) (t,x) \in \mathcal{K}_r \) and \( a_\beta^0 \in \mathcal{K}_{(\|\beta\| - \|\alpha\|v_0)} \).

The representation obtained in the previous lemma generalises to multiple heat kernels as we observe in the following proposition.

**Proposition 20** Let \( k \in \mathbb{N} \), \( \Phi_k \in \mathcal{K}_r \), \( \Phi_j \in \mathcal{K}_0 \) for \( 1 \leq j < k \), \( \alpha \in A_1(\ell) \), and \( H_j = u_j V_j + \nu_j \), where \( 1 \leq i_j \leq d \), \( u_j, \nu_j \in C^\infty_b(\mathbb{R}^N) \), \( j = 1, \ldots, k - 1 \). Then there exist \( \Phi_{\beta_1} \in \mathcal{K}_{r_1}, \ldots, \Phi_{\beta_k} \in \mathcal{K}_{r_k} \) such that \( r_k \geq r, r_1, \ldots, r_k \geq -1/2 \) and

\[ r_1 + r_2 + \cdots + r_k \geq \left( \| \beta_1 \| - \| \alpha \| \right) + 0 - (k - 1)/2 + r \tag{33} \]

and

\[
\begin{align*}
P_{t_1}^{\Phi_{\beta_1}} H_1 P_{t_2}^{\Phi_{\beta_2}} \cdots H_{k-1} P_{t_k}^{\Phi_{\beta_k}} V_{[a]} \varphi(x) \\
= \sum_{\beta_1 \in A_0(\ell)} \cdots \sum_{\beta_k \in A_0(\ell)} V_{[\beta]} P_{t_1}^{\Phi_{\beta_1}} P_{t_2}^{\Phi_{\beta_2}} \cdots P_{t_k}^{\Phi_{\beta_k}} \varphi(x)
\end{align*}
\]

holds for all \( \varphi \in C^\infty_b(\mathbb{R}^N) \).

Before we begin the proof of this proposition we examine the meaning of the assumptions on the \( r_j \). The assumptions \( r_1, \ldots, r_k \geq -1/2 \) imply that singularities in the bounds

\[ \left\| P_{t}^{\Phi_{\beta_j}} \varphi \right\|_{\infty} \leq t^{r_j/2} \| \varphi \|_{\infty} \]

in Lemma 15 are integrable. The inequality (33) can be interpreted as follows: The left hand side is the total regularity of the resulting expression in the proposition. For every application of an operator \( H \) we loose \( 1/2 \) regularity reflected in the term \( -(k - 1)/2 \). The degree of a singularity introduced by differentiating by \( V_{[\alpha]} \) depends on \( \| \alpha \| \). Thus if \( \| \beta \| > \| \alpha \| \) and we replace a \( V_{[\alpha]} \) by \( V_{[\beta]} \) we expect a compensating term, which is captured in \( \left( \| \beta_1 \| - \| \alpha \| \right) \) \( \lor \) 0.

**Proof.** As before it is by linearity sufficient to consider the case \( H_j = u_j V_j \), for some \( u_j \in C^\infty_b(\mathbb{R}^N) \) the case of the multiplication operator \( v_j \) following by a similar but easier calculation. We argue by induction, the base case being covered by Lemma 19. For the inductive step we note that if \( \Phi_k \in \mathcal{K}_0 \) then by Lemma 15 there exists \( \Phi_k \in \mathcal{K}_{-1/2} \) such that \( P_{t}^{\Phi_k} u V_1 = P_{t}^{\Phi_k} \). Combining this fact with Lemma 19 we see

\[ P_{t_1}^{\Phi_{\beta_1}} H_1 P_{t_2}^{\Phi_{\beta_2}} \cdots H_{k-1} P_{t_k}^{\Phi_{\beta_k}} H P_{t}^{\Phi_{\beta_k}} V_{[a]} \varphi(x) \]

\[ = P_{t_1}^{\Phi_{\beta_1}} H_1 P_{t_2}^{\Phi_{\beta_2}} \cdots H_{k-1} P_{t_k}^{\Phi_{\beta_k}} u V_1 P_{t}^{\Phi_{\beta_k}} V_{[a]} \varphi(x) \]

\[ = P_{t_1}^{\Phi_{\beta_1}} H_1 P_{t_2}^{\Phi_{\beta_2}} \cdots H_{k-1} P_{t_k}^{\Phi_{\beta_k}} \bar{P}_{t}^{\Phi_{\beta_k}} V_{[a]} \varphi(x) \]

\[ = \sum_{\beta \in A_0(\ell)} P_{t_1}^{\Phi_{\beta_1}} H_1 P_{t_2}^{\Phi_{\beta_2}} \cdots H_{k-1} P_{t_k}^{\Phi_{\beta_k}} V_{[\beta]} P_{t}^{\Phi_{\beta_k}} \varphi(x), \]

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where $\Phi_{\beta} \in \mathcal{K}_r(\|\beta\| - \|\alpha\|) \cup 0 + r$ and $\Phi_k \in \mathcal{K}_{-1/2}$. Using the inductive hypothesis we get
\[
\sum_{\beta \in A_0(t)} P_{t_1}^{\Phi_1} H_1 P_{t_2}^{\Phi_2} \cdots H_{k-1} P_{t_k}^{\Phi_k} V[\beta] P_{t_1}^{\phi_1} \varphi(x)
\]
\[
= \sum_{\beta^0 \in A_0(t)} \cdots \sum_{\beta^k \in A_0(t)} V[\beta^0] P_{t_1}^{\phi_1} P_{t_2}^{\phi_2} \cdots P_{t_k}^{\phi_k} P_{t_1}^{\phi_k} \varphi(x).
\]
From the inductive hypothesis we know that $\Phi_{\beta^1} \in \mathcal{K}_{r_1}$, ..., $\Phi_{\beta^k} \in \mathcal{K}_{r_k}$ such that $r_1, \ldots, r_k \geq -1/2$ (using that $\Phi_k \in \mathcal{K}_{-1/2}$) and
\[
r_1 + r_2 + \cdots + r_k \geq (\|\beta^1\| - \|\beta\|) \vee 0 - k/2.
\]
Hence, as required
\[
[\|\beta\| - \|\alpha\|] \vee 0 + r + r_1 + r_2 + \cdots + r_k
\]
\[
\geq [(\|\beta\| - \|\alpha\|) \vee 0] + r + (\|\beta^1\| - \|\beta\|) \vee 0 - k/2
\]
\[
\geq (\|\beta^1\| - \|\alpha\|) \vee 0 - k/2 + r.
\]

We are ready to prove the first main regularity estimate Proposition 7.

**Proof of Proposition 7.** Note that arguing as in the proof of Lemma 13 it is sufficient to show
\[
\|V[\alpha] \hat{P}^{m,j_1,\ldots,j_k}_{(0,t_1,\ldots,t_k,t)} V[\beta] \varphi\|_{\infty} \leq c_m t^{-\frac{(\|\alpha\| + \|\beta\|)/2}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}}} \|\varphi\|_{\infty}
\]
for some constant $c_m$ (the bounds on $\hat{P}^{m,j_1,\ldots,j_k}_{(0,t_1,\ldots,t_k,t)}$ follow by using the same arguments). The functions $V[\alpha] \hat{P}^{m,j_1,\ldots,j_k}_{(0,t_1,\ldots,t_k,t)} V[\beta] \varphi$ are linear combination of terms of the form
\[
V[\alpha] P_{t_1}^{\phi} H_1 P_{t_2}^{\phi} \cdots H_{k-1} P_{t_k}^{\phi} V[\beta] \varphi
\]
for some $\Phi \in \mathcal{K}_0$ and $H_j = u_j V_{t_j} + v_j$ with $u_j, v_j \in C^\infty_b$. Recall the convention $t = t_{k+1}$.

Suppose $[t_{j-1}, t_j]$ is the maximal subinterval, i.e. satisfies
\[
t_j - t_{j-1} = \max_{i=1,\ldots,k+1} (t_i - t_{i+1}) \geq \frac{t}{k} \quad (34)
\]
For notational reasons we have to treat the case $j = k + 1$ separately, however it will be clear from the proof that the same arguments apply in this case.

Suppose now that $j \in \{1, \ldots, k\}$, then by Proposition 20 we observe that
\[
P_{t_{j+1} - t_j}^{\phi} H_{j+1} P_{t_{j+2} - t_{j+1}}^{\phi} \cdots H_k P_{t_k - t_{k-1}}^{\phi} V[\beta] \varphi(x)
\]
\[
= \sum_{\beta^{j+1} \in A_0(t)} \cdots \sum_{\beta^k \in A_0(t)} G_{\beta^{j+1},\ldots,\beta^k} (x),
\]
where
\[
G_{\beta^{j+1},\ldots,\beta^k} := V[\beta^{j+1}] P_{t_{j+1} - t_j}^{\phi^{j+1}} P_{t_{j+2} - t_{j+1}}^{\phi^{j+2}} \cdots P_{t_k - t_{k-1}}^{\phi^{k+1}} \varphi,
\]

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for some functionals \( \Phi_{\beta^{j+1}} \in \mathcal{K}_{r_1}, \ldots, \Phi_{\beta^{k+1}} \in \mathcal{K}_{r_k} \) with \( r_{k+1} \geq 0, r_{j+1}, \ldots, r_k \geq -1/2 \) and \( r_{j+1} + r_2 + \cdots + r_{k+1} \geq \left( \| \beta^{j+1} \| - \| \beta^{j} \| \right) \vee 0 - (k - j) / 2 \).

It follows from the maximality of \([t_j, t_{j-1}]\) that

\[
(t_{j+1} - t_j)^{r_{j+1}} \cdots (t_k - t_{k-1})^{r_k} \\
\leq (t_{j+1} - t_j)^{-1/2} \cdots (t_k - t_{k-1})^{-1/2} (t_j - t_{j-1})^{(\| \beta^{j+1} \| - \| \beta^{j} \|)\vee 0}/2.
\]

On the other hand, to pass the derivative \( V\varphi_{\alpha} \) to \( P_{t_1-t_{j-1}}^\Phi \) we will iteratively use Lemma 18. Once again by maximality of \([t_j, t_{j-1}]\) it follows that

\[
(t_1 - t_0)^{-1/2\nu(\| \beta^{1} \| - \| \alpha \|)\vee 0}/2 \cdots (t_{j-1} - t_{j-2})^{-1/2\nu(\| \beta^{j-1} \| - \| \beta^{j-2} \|)\vee 0}/2 \\
\leq (t_1 - t_0)^{-1/2} \cdots (t_{j-1} - t_{j-2})^{-1/2} (t_j - t_{j-1})^{(\| \beta^{j-1} \| - \| \alpha \|)\vee 0}/2.
\]

Using Lemma 18 iteratively we see from our preceding observations that

\[
\left\| V\varphi_{\alpha} P_{t_1}^\Phi H_1 P_{t_2-t_1}^\Phi \cdots P_{t_k-t_{k-1}}^\Phi H_k P_{t_{k-1}}^\Phi V\varphi_{\beta} \right\|_{\infty} \\
= \left\| \sum_{\beta^{j+1} \in A_0(\ell)} \cdots \sum_{\beta^{k} \in A_0(\ell)} V\varphi_{\alpha} P_{t_1}^\Phi H_1 \cdots H_{j-1} P_{t_j-t_{j-1}}^\Phi H_j G_{\beta^{j+1}} \cdots, \beta^k \right\|_{\infty} \\
\leq \tilde{C}^k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} \| \varphi \|_{\infty} \\
\sum_{\beta^{j-1}, \beta^{j+1} \in A_0(\ell)} (t_j - t_{j-1})^{(\| \beta^{j-1} \| - \| \alpha \|)\vee 0}/2 \left\| V\varphi_{\alpha} P_{t_j-t_{j-1}}^\Phi H_j G_{\beta^{j+1}} \cdots, \beta^k \right\|_{\infty} \\
\leq C^k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} t^{-\| \alpha \|}/2 \| \varphi \|_{\infty},
\]

where the penultimate inequality used Lemma 15.

7 Proof of Proposition 8

Factorial decay of the integral summands via rough path techniques

7.1 Some preliminary estimates

Before we can proceed with the proof of Proposition 8 we explore some of the consequences of the estimates derived in the proof of Proposition 7.
Lemma 21 With the notation of Lemma\ref{lemma20} for any $0 < \gamma < 1/2$, $m > 0$ there exist random variables $c(\gamma, m, \omega)$ such that, almost surely
\[
\| R_{s,t}^{m,\bar i} \|_{H^{-1} \to H^{-1}} \leq c(\gamma, m, \omega) |t - s|^{m\gamma}.  
\]
(35)
\[
\| R_{s,t}^{m,\bar i} \|_{H^1 \to H^1} \leq c(\gamma, m, \omega) |t - s|^{m\gamma}.  
\]
(36)
and finally
\[
\| R_{s,t}^{m,\bar i} \|_{H^{-1} \to H^{-1}} \leq c(\gamma, m, \omega) |t - s|^{m\gamma - 2\ell}.  
\]
(37)
for all $\bar i \in S(m)$, $0 < s < t < 1$.

Proof. For all $j_1, \ldots, j_k \in S$ such that $\bar i = j_1 \ast \cdots \ast j_k$ we note that for any $0 < t \leq 1$ we have by iteratively applying Lemma\ref{lemma18}
\[
\| R_{s,t}^{m,\bar i} (\varphi) \|_{H^1} = \sum_{a \in A_{0(\ell)}} \| V[a] (R_{s,t}^{m,\bar i}) \|_\infty 
\leq C_k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} \sum_{\beta \in A_{0(\ell)}} \| V[\beta] \varphi \|_\infty.
\]
The bound on $\| R_{s,t}^{m,\bar i} (\varphi) \|_{H^1 \to H^1}$ now follows by applying Lemmas\ref{lemma12} and\ref{lemma13}. Finally to show inequalities \ref{eq:36} and \ref{eq:37} we let $\varphi \in H^{-1}$. Then there exist for every $\varepsilon > 0$ functions $\varphi^\beta$ such that
\[
\varphi = \sum_{\beta \in A_0(\ell)} V[\beta] \varphi^\beta
\]
and $\sum_{\beta \in A_0(\ell)} \| \varphi^\beta \|_\infty \leq \| \varphi \|_{H^{-1} + \varepsilon}$. First we have
\[
\| R_{s,t}^{m,j_1,\ldots,j_k} \|_{H^{-1}} = \sum_{\beta \in A_0(\ell)} \| R_{s,t}^{m,j_1,\ldots,j_k} V[\beta] \varphi^\beta \|_{H^{-1}}
\]
and by Proposition\ref{prop20} for each $\beta \in A_0(\ell)$ there exist functionals $\Phi_{\beta 1} \in K_{r_1}, \ldots, \Phi_{\beta k} \in K_{r_k}$ such that $r_k \geq 0, r_1, \ldots, r_{k-1} \geq -1/2$ and
\[
\| R_{s,t}^{m,j_1,\ldots,j_k} V[\beta] \varphi^\beta \|_{H^{-1}} \leq \sum_{\beta \in A_0(\ell)} \sum_{\beta \in A_0(\ell)} \cdots \sum_{\beta \in A_0(\ell)} \| P_{t_{l-0}}^{\Phi_{\beta 1}} P_{t_{l-1}}^{\Phi_{\beta 2}} \cdots P_{t_{l-k}}^{\Phi_{\beta k}} \varphi \|_{H^{-1}}
\]
We deduce from Lemma\ref{lemma15} that
\[
\| R_{s,t}^{m,j_1,\ldots,j_k} V[\beta] \varphi^\beta \|_{H^{-1}} \leq \sum_{\beta \in A_0(\ell)} \sum_{\beta \in A_0(\ell)} \cdots \sum_{\beta \in A_0(\ell)} \| P_{t_{l-0}}^{\Phi_{\beta 1}} P_{t_{l-1}}^{\Phi_{\beta 2}} \cdots P_{t_{l-k}}^{\Phi_{\beta k}} \varphi \|_{H^{-1}}
\]
\[
\leq C_k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} \| \varphi^\beta \|_\infty
\]

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and consequently
\[
\left\| \hat{R}_{(t_0, t_1, \ldots, t_k, t)}^{m, j_1, \ldots, j_k} \varphi \right\|_{H^{-1}} \leq c_k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} \sum_{\beta \in A_0(t)} \left\| \varphi^{\beta} \right\|_{\infty}
\]
\[
\leq c_k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} \left\| \varphi \right\|_{H^{-1} + \varepsilon}.
\]

To demonstrate the last inequality observe that arguing exactly as in the proof of Proposition 7 we have,
\[
\left\| \hat{R}_{(t_0, t_1, \ldots, t_k, t)}^{m, j_1, \ldots, j_k} \varphi \right\|_{H^1} = \sum_{\alpha \in A_0(t)} \left\| V_{[\alpha]} \left( \hat{R}_{(t_0, t_1, \ldots, t_k, t)}^{m, j_1, \ldots, j_k} \varphi \right) \right\|_{\infty}
\]
\[
\leq \sum_{\alpha \in A_0(t)} \sum_{\beta \in A_0(t)} \left\| V_{[\alpha]} \left( \hat{R}_{(t_0, t_1, \ldots, t_k, t)}^{m, j_1, \ldots, j_k} \varphi^{\beta} \right) \right\|_{\infty}
\]
\[
\leq c_k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} \sum_{\alpha \in A_0(t)} \sum_{\beta \in A_0(t)} t^{-\frac{\|a\| + \|\beta\|}{2}} \left\| \varphi^{\beta} \right\|_{\infty}
\]
\[
\leq c_k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} t^{-2\ell} \sum_{\beta \in A_0(t)} \left\| \varphi^{\beta} \right\|_{\infty}
\]
\[
\leq c_k \frac{1}{\sqrt{t_1 - t_0}} \cdots \frac{1}{\sqrt{t_k - t_{k-1}}} t^{-2\ell} \left\| \varphi \right\|_{H^{-1} + \varepsilon},
\]

where \(c_k\) are constants changing from line to line. The claim in both cases now follows once again from Lemmas 12 and 13. As before we note that the same estimates apply to \(\hat{R}_{(t_0, t_1, \ldots, t_k, t)}^{m, j_1, \ldots, j_k}\) in place of \(\hat{R}_{(t_0, t_1, \ldots, t_k, t)}^{m, j_1, \ldots, j_k}\).

So far, we have established a priori Hölder type estimates for \(R_{s,t}^m (\varphi)\), but the estimates in their current form are not yet summable. The following proof of Proposition 8 relies on a fundamental rough path technique to improve on these bounds and demonstrate that the operator norms of \(R_{s,t}^m\) decay in fact factorially in \(m\).

### 7.2 Proof of Proposition 8

To make the presentation more transparent we introduce some additional notations for the following arguments. Recall that \(\Delta_{s,t}^k\) denotes the simplex defined by the relation \(s < t_1 < \cdots < t_k < t\) and the \(H_i\) are the operators corresponding to multiplication by the sensor function \(h_i\). For any \(0 \leq s < t \leq T\) define \(R_{s,t}^0 := 1\) and recall the linear operators \(R_{s,t}^{k,i}\) may be written as
\[
R_{s,t}^{k,i} = \int_{\Delta_{s,t}^k} P_{t-s} H_{i_{1}} P_{t_{1}-t_{1}} H_{i_{2}} \cdots H_{i_{n}} P_{t_{n}-t_{n}} dY_{t_{1}}^{i_{1}} \cdots dY_{t_{n}}^{i_{n}}.
\]

for all \(\vec{i} = (i_1, \ldots, i_n) \in S, n \geq 1\).

Let \(W := \mathbb{R}^{d_2}\) and \(\varepsilon_1, \ldots, \varepsilon_{d_2}\) a Basis for \(W\). For \(\vec{i} = (i_1, \ldots, i_j) \in S (j)\) let \(\varepsilon_{\vec{i}} = \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_j}\) and note that the \(\varepsilon_{\vec{i}}\) are a basis for the space \(W^{\otimes j}\). Finally, let \(V\) be a Banach algebra (i.e. a
Banach space with a multiplication and a submultiplicative norm). We define $\mathcal{P}_{d_1,k} (V)$ the space of non-commutative polynomials in $d_2$ variables of degree at most $k$ over $V$ by letting

$$\mathcal{P}_{d_1,k} (V) := \left\{ \sum_{j=0}^{k} \sum_{\bar{i} \in S(j)} c_{\bar{i}} \varepsilon_{\bar{i}} : c_{\bar{i}} \in V \right\}.$$ 

Define a multiplication for $a = \sum_{j=0}^{k} a_{\bar{i}} \varepsilon_{\bar{i}}$ and $b = \sum_{j=0}^{k} b_{\bar{i}} \varepsilon_{\bar{i}}$ by setting

$$ab := \sum_{v=0}^{k} \sum_{j=0}^{v} a_{\bar{i}} b_{\bar{i}-j} := \sum_{v=0}^{k} \sum_{j=0}^{v} \sum_{l=0}^{v-j} \sum_{\bar{l} \in S(v-j)} a_{\bar{i}} b_{\bar{l}} \varepsilon_{\bar{i}+\bar{l}}. \quad (38)$$

Further note that

$$\sum_{j=0}^{v} \sum_{l=0}^{v-j} \sum_{\bar{i} \in S(j)} \sum_{\bar{m} \in S(v)} a_{\bar{i}} b_{\bar{m}} \varepsilon_{\bar{i}+\bar{m}} = \sum_{v=0}^{k} \sum_{j=0}^{v} \sum_{\bar{l} \in S(v-j)} a_{\bar{i}} b_{\bar{l}} \varepsilon_{\bar{i}+\bar{l}} \quad (39)$$

and define for $k \geq i \geq 1$ the projection $\pi_{i}$ by setting $\pi_{i}(a) = a_{i}$. We impose a norm on $\mathcal{P}_{d_1,k} (V)$ by setting

$$\left\| \sum_{j=0}^{k} \sum_{\bar{i} \in S(j)} c_{\bar{i}} \varepsilon_{\bar{i}} \right\| = \sup \{ \|c_{\bar{i}}\| : j \in \{0, \ldots, k\}, \bar{i} \in S(j) \}$$

Let $Q_{s,t}^{0} = 1$ and $Q_{s,t}^{j}$ for $j \in \mathbb{N}$ be given by

$$Q_{s,t}^{j} = \sum_{\bar{i} \in S(j)} R_{s,t}^{j,\bar{i}} \varepsilon_{\bar{i}}.$$ 

Finally, we may set

$$Q_{s,t}^{[n]} = \sum_{i=0}^{n} Q_{s,t}^{i}.$$ 

Observe that for any $s < u < t$ and $k \in \mathbb{N}$ and $\bar{i} = (i_1, \ldots, i_k) \in S(k)$, we have partitioning
the simplex $\Delta_{s,t}^k$

$$R_{s,t}^{k,\bar{i}} = \int_{\Delta_{s,u}^k} P_{t_1-s} H_{i_1} P_{t_2-t_1} H_{i_2} \cdots H_{i_k} P_{t-t_k} \, dY_{t_1}^{i_1} \cdots dY_{t_k}^{i_k}$$

$$+ \int_{\Delta_{u,t}^k} P_{t_1-s} H_{i_1} P_{t_2-t_1} H_{i_2} \cdots H_{i_k} P_{t-t_k} \, dY_{t_1}^{i_1} \cdots dY_{t_k}^{i_k}$$

$$+ \sum_{j=1}^{k-1} \int_{\Delta_{s,u}^k} P_{t_1-s} H_{i_1} \cdots P_{t_j-t_{j-1}} H_{i_j} P_{u-t_j} \, dY_{t_1}^{i_1} \cdots dY_{t_j}^{i_j}$$

$$\int_{\Delta_{u,t}^{k-j}} P_{t_{j+1}-u} H_{i_{j+1}} P_{t_{j+2}-t_{j+1}} \cdots H_{i_k} P_{t-t_k} \, dY_{t_{j+1}}^{i_{j+1}} \cdots dY_{t_k}^{i_k}$$

$$= \sum_{j=0}^{k} R_{s,u}^{j,(i_1,\ldots,i_j)} R_{u,t}^{k-j,(i_{j+1},\ldots,i_k)}$$

$$= \sum_{\bar{m} \preceq \bar{i}} R_{s,u}^{[\bar{m}],\bar{i}} R_{u,t}^{[\bar{i}],\bar{i}}$$

(40)

and therefore using (39)

$$Q_{s,t}^{[k]} = \sum_{v=0}^{k} \sum_{\bar{i} \in S(v)} R_{s,\bar{i}}^{v,\bar{i}}$$

$$= \sum_{v=0}^{k} \sum_{\bar{i} \in S(v)} \sum_{\bar{m} \preceq \bar{i}} R_{s,\bar{m}}^{v,\bar{m}} R_{u,\bar{i}}^{[\bar{m}],\bar{i}}$$

$$= \sum_{v=0}^{k} \sum_{j=0}^{v} \sum_{\bar{i} \in S(v)} \sum_{\bar{l} \in S(v-j)} R_{s,u}^{j,\bar{i}} R_{u,\bar{l}}^{v-j,\bar{l}}$$

$$= \sum_{v=0}^{k} \sum_{j=0}^{v} Q_{s,u}^{v-j}$$

or equivalently

$$Q_{s,t}^{[k]} = \sum_{v=0}^{k} \sum_{\bar{i} \in S(v)} R_{s,\bar{i}}^{v,\bar{i}}$$

(41)

Analogous to the corresponding rough path concept we will refer to (41) as the multiplicative property. We recall that by Lemma 5

$$\rho_t = P_t + \sum_{n=1}^{\infty} \sum_{i \in S(n)} R_{0,t}^{n,i}$$

(42)

The following proposition demonstrates that it suffices to obtain Holder type controls on finitely many of the $Q_{s,t}^{[n]}$ to control the infinite series in (42). The proof utilises techniques of the classical extension theorem for rough paths due to Lyons (see e.g. [13] p.45f) and exploits the multiplicative structure of the operator valued integrands.
Lemma 22. Let $q \geq 1$ and let $[q]$ denote the integer part of $q$ and $V$ be a Banach algebra with norm $|| \cdot ||$. Suppose $Q^{[q]} = \sum_{j=0}^{[q]} Q^{j} \in \mathcal{P}_{d,2}([q]) (V)$ satisfies the multiplicative property (41). Suppose there exists a constant $C > 0$ such that for all $(s, t) \in \Delta_{[0,1]}$, $j = 1, \ldots, [q]$,

$$
\|Q^{j}_{s,t}\| \leq (C |t-s|)^{j/q} \frac{1}{\theta (j/q)!},
$$

where $\theta = \left( q^{2} + \sum_{r=3}^{\infty} \frac{|q|+1}{r-2} \right)$. Then for all $m > [q]$ there exists a multiplicative extension $1 + Q^{1}_{s,t} + \cdots + Q^{[q]}_{s,t} + \tilde{Q}^{[q] + 1}_{s,t} + \cdots + \tilde{Q}^{m}_{s,t}$ on $\mathcal{P}_{d,2,m} (V)$ such that (43) holds for all $j \in \{1, \ldots, m\}$, $(s, t) \in \Delta_{[0,1]}^{2}$. Moreover if $\overline{Q}^{j}_{s,t}$ is another multiplicative extension such that $\|\overline{Q}^{j}_{s,t}\| \leq C (j) |t-s|^{j/q}$ for all $(s, t) \in \Delta_{[0,1]}^{2}$, then $\overline{Q}^{j}_{s,t} = \tilde{Q}^{j}_{s,t}$ for all $j \in \{1, \ldots, m\}$.

Before we begin the proof of the lemma we recall the neo-classical inequality from [12] (Lemma 2.2.2).

Theorem 23 (Neo-classical inequality, Lyons 98) For any $q \in [1, \infty)$, $n \in \mathbb{N}$ and $s, t \geq 0$

$$
\frac{1}{q^{2}} \sum_{i=0}^{n} \frac{s+t}{\frac{n}{q}} \frac{n-i}{\frac{n}{q}} \leq (s+t)^{n/q}.
$$

Proof of Lemma 22. We will inductively construct $Q^{[n]}_{s,t}$ for $n > [q]$, the base case of the induction following from the assumption on the $Q^{j}_{s,t}$, $j = 1, \ldots, [q]$. The proof closely follows the proof of the classical extension theorem for rough paths (see [13] p.45f). To extend from $n - 1 \geq [q]$ to $n$ first let on $\mathcal{P}_{d,2,n} (V)$

$$
\hat{Q}_{s,t} := \sum_{j=1}^{n-1} Q^{j}_{s,t}.
$$

Given any finite partition $\mathcal{D}$ of the interval $[s, t]$ define $Q^{[n]}_{s,t} \mathcal{D}$ by setting

$$
Q^{[n]}_{s,t} \mathcal{D} := \prod_{(s, t]} \hat{Q}_{t_{j},t_{j+1}}.
$$

By the pigeon hole principle there is $t_{j}$ such that

$$
(t_{j+1} - t_{j-1}) \leq 2 \frac{2}{|\mathcal{D}| - 1} (t - s)
$$

and we may coarsen the partition by dropping $t_{j}$ and write $\mathcal{D} : = \mathcal{D} \setminus \{t_{j}\}$. Then

$$
Q^{[n]}_{s,t} \mathcal{D} - Q^{[n]}_{s,t} \mathcal{D}' = \hat{Q}_{s,t_{1}} \cdots \left( \hat{Q}_{t_{j-1}, t_{j}} \hat{Q}_{t_{j}, t_{j+1}} - \hat{Q}_{t_{j-1}, t_{j+1}} \right) \cdots \hat{Q}_{t_{|\mathcal{D}|-1},t}
$$
and noting that \( \hat{Q}_{t,j-1} Q_{t,j,t+1} - \hat{Q}_{t,j-1,t,j+1} \) is a homogeneous polynomial of degree \( n \) we see that

\[
Q^{[n],D}_{s,t} - Q^{[n],D'}_{s,t} = \sum_{i=1}^{n-1} Q^i_{t,j-1,t,j} Q^{n-i}_{t,j,t+1}.
\]

Therefore using the submultiplicative property for the norm, the inductive hypothesis and finally the neo-classical inequality we see that

\[
\| \pi_n \left( Q^{[n],D}_{s,t} - Q^{[n],D'}_{s,t} \right) \| \leq \sum_{i=1}^{n-1} \| Q^i_{t,j-1,t,j} \| \| Q^{n-i}_{t,j,t+1} \|
\]

\[
\leq \sum_{i=1}^{n-1} \left( \frac{C |t_j - t_{j-1}|^{i/q}}{\theta (i/q)!} \right) \left( \frac{C |t_{j+1} - t_j|^{(n-i)/q}}{\theta ((n-i)/q)!} \right)
\]

\[
\leq \frac{q^2}{\theta} \left( \frac{2}{|D| - 1} \right)^{n} \left( C |t - s| \right)^{n} \frac{\theta}{\theta (n/q)!}.
\]

Thus whenever \( \theta \geq q^2 \left( 1 + 2^{n/q} \left( \zeta \left( \frac{[q]+1}{q} \right) - 1 \right) \right) \) the maximal inequality implies that

\[
\| \pi_n \left( Q^{[n],D}_{s,t} \right) \|^n \leq \frac{|t - s|^{n}}{\theta (n/q)!}
\]

holds for any partition of \([s, t]\). It remains to verify the existence of the limit \( \lim_{|D| \to 0} Q^{n,D}_{s,t} \). We proceed as in [13] and exhibit the Cauchy property for the sequence. Suppose \( D = \{t_j\} \) and \( \tilde{D} \) are two partitions of mesh size less than \( \delta \). Let \( \tilde{D} \) denote the common refinement of the two partitions and let \( \tilde{D}_j = [t_j, t_{j+1}] \cap \tilde{D} \). Then

\[
Q^{n,\tilde{D}}_{s,t} - Q^{n,D}_{s,t} = \sum_{j=1}^{n} Q^{n,\tilde{D}_j}_{t_{j-1},t_{j}} \left( Q^{n,\tilde{D}_j}_{t_j,t_{j+1}} - \hat{Q}_{t_j,t_{j+1}} \right) \ldots Q^{n,\tilde{D}_1}_{t_{1},t_{1}}
\]

As seen before this is a sum of homogeneous polynomials of degree \( n \) and by the maximal inequality

\[
\| \pi_n \left( Q^{n,\tilde{D}}_{s,t} - Q^{n,D}_{s,t} \right) \| \leq \sum_{\tilde{D}} \frac{|t_{j+1} - t_j|^{n}}{\theta (n/q)!} \leq \frac{|t - s|^{n}}{\theta (n/q)!} \delta^{n-1}
\]

as \( \frac{n}{q} - 1 > 0 \) we have a uniform estimate in \( \delta \) independent of the choice of partition. Going through the same argument for the partition \( \tilde{D} \) and using the triangle inequality the Cauchy property is established and the existence of the limit follows. The uniqueness of the limit follows as in [13]. The difference of two multiplicative functionals that agree up to level \([q]\) is
additive (see Lyons [12] Lemma 2.2.3) As the difference of the extensions is also a continuous path and by assumption
\[
\left\| \tilde{Q}_{s,t}^{\lfloor q \rfloor + 1} - \hat{Q}_{s,t}^{\lfloor q \rfloor + 1} \right\| \leq C (\lfloor q \rfloor + 1) |t - s|^\frac{\lfloor q \rfloor + 1}{q}
\]
it follows that $\tilde{Q}_{s,t}^{\lfloor q \rfloor + 1} - \hat{Q}_{s,t}^{\lfloor q \rfloor + 1}$ is identically zero. A simple induction now completes the proof. ■

**Lemma 24** For any $1/3 < \gamma < 1/2$ there exist a constant $\theta > 0$ and random variables $c(\gamma, \omega)$, almost surely finite, such that
\[
\left\| R_{s,t}^{n,\bar{i}} \right\|_{H^{-1} \to H^1} \leq \frac{(c (\gamma, \omega) |t - s|)^n \gamma}{\theta (n \gamma)!}.
\]
and
\[
\left\| R_{s,t}^{n,\bar{i}} \right\|_{H^{-1} \to H^{-1}} \leq \frac{(c (\gamma, \omega) |t - s|)^n \gamma}{\theta (n \gamma)!}
\]
for all $\bar{i} \in S(n), n \in \mathbb{N}, 0 < s < t \leq 1$.

**Proof.** We now take for $V$ the space of bounded linear operators on (the completion of) $H^1$ and $H^{-1}$ respectively. From the a priori estimates we know that $Q_{s,t}^{[n]} \in \mathcal{P}_{d_2,n}(V)$ for all $n \geq 1$. First note that by Lemma [21] $(Q_{s,t}^1, Q_{s,t}^2)$ satisfies the assumptions of Proposition [22] with $3 > q = 1/\gamma$ and therefore has a multiplicative extension $\tilde{Q}_{s,t}^j$ controlled in the sense of (43). Once again by Lemma [21] the uniqueness part of Proposition [22] applies and we deduce that $Q_{s,t}^j = \tilde{Q}_{s,t}^j$ for $j \in \mathbb{N}$. ■

Armed with these two factorially decaying a priori estimates we are finally ready proof a regularity estimate for $Q^n$ that decays factorially in $n$. When considering $R_{s,t}^{n,\bar{i}}$, $\bar{i} \in S(n)$ as an operator from $H^{-1}$ to $H^1$ we cannot directly apply Lemma [22] as the a priori bounds in Lemma [21] have singularities for small $n$. Instead we exploit that there is more than one way to estimate the operator norm of the composition of such operators. Together with the estimates already obtained in Lemma [21] this will be sufficient to proof factorially decaying bounds for $n$ sufficiently large. We recall Proposition [8] and restate it in the notation of the current section.

**Proposition 8** Let $1/3 < \gamma < 1/2$ be fixed. There exists $\theta > 0$, $\gamma' \in (1/3, \gamma)$, $m_0 \in \mathbb{N}$ and random variables $c(\gamma', \omega)$, almost surely finite, such that
\[
\left\| R_{s,t}^{n,\bar{i}} \right\|_{H^{-1} \to H^1} \leq \frac{(c (\gamma', \omega) |t - s|)^n \gamma'}{\theta (n \gamma')!}
\]
for all $n \geq m_0$.

Before we begin the proof note that by choosing $\gamma' < \gamma$ we have for $n$ sufficiently large by Lemma [21]
\[
\left\| R_{s,t}^{n,\bar{i}} \right\|_{H^{-1} \to H^1} \leq c (\gamma, n, \omega) |t - s|^{n \gamma - 2\ell} \leq c (\gamma, n, \omega) |t - s|^{n \gamma'}
\]
for all $0 < s < t < 1$.

**Proof of Proposition 8.** Choose $m_0$ and $0 < \gamma' \leq \gamma$ such that $\gamma n - \ell \geq \gamma'n$ for all $n \geq m_0$. Using Corollary 24 and Lemma 21 (with $\gamma = \gamma'$) we can find $c(\gamma', \omega)$ such that simultaneously (47) holds for all $n \in [m_0, 2m_0]$ and the two inequalities (45) and (46) hold for all $n \in \mathbb{N}$. Note that this also serves as the base case for our induction argument. For this lemma we set $V$ to be the space of bounded linear operators from $H^{-1}$ to $H^1$.

We argue now exactly as in the proof Lemma 22 to extend the functional from level $n \geq 2m_0$ to $n+1$, with the only difference being that we have no direct control over $W_{\ell}^i$, respectively. The bounds for $\sup_{i \in S(k), k < m_0}$ and $\sup_{i \in S(k), k = n+1}$ follow (for the appropriate values of $i$) from the inductive hypothesis. With this modification in place arguing exactly as in the proof of Lemma 22 yields the result. Note that the extension is only carried out for $n \geq 2m_0$. For $m_0 \leq n < 2m_0$ the estimates use the a priori bounds.

**Remark 25** To extend the proof of Proposition 8 to cover the terms in the expansion of $\rho^*_n$ we make the following modifications. In place of $R_{s,t}^{n,i}$ we have

$$X_{s,t}^{n,i} = \int_{\Delta_{s,t}}^{\Delta_{s,t}} Pt-t_nH_{t_n}P_{t-n_1}H_{l_1} \ldots H_{l_n} P_{l_1-s}dY_{l_1} \ldots dY_{l_n},$$
i.e. the order of non-commutative product in the integrand is reversed. We therefore define \( \tilde{\mathcal{P}}_{d_2,k}(V) \) as \( \mathcal{P}_{d_2,k}(V) \) but with the multiplication in (48) replaced by

\[
ab := \sum_{v=0}^{k} \sum_{j=0}^{v} \sum_{i \in S(j)} b_{i} a_{v-i} z_{v-i}.
\]

(49)

With this modification (49) becomes

\[
X_{s,t}^{k_i} = \int_{\Delta_{s,t}^{k_i}} P_{t-k} H_{t-k} P_{t-k-1} H_{t-k-1} \cdots H_{t_1} P_{t_1} dY_{t_1} \cdots dY_{t_k}
\]

\[
+ \int_{\Delta_{s,t}^{k_0}} P_{t-k} H_{t-k} P_{t-k-1} \cdots H_{t_1} P_{t_1} dY_{t_1} \cdots dY_{t_k}
\]

\[
+ \sum_{j=1}^{k-1} \int_{\Delta_{s,t}^{k-j}} P_{t-k} H_{t-k} P_{t-k-1-1} \cdots P_{t_j+1} dY_{t_j+1} \cdots dY_{t_k}
\]

\[
+ \int_{\Delta_{s,t}^{k_0}} P_{u-t} H_{j} P_{j-1} \cdots H_{j} P_{t_1} dY_{t_1} \cdots dY_{t_k}
\]

\[
= \sum_{j=0}^{k} X_{u,t}^{k-j} X_{s,u}^{j}.
\]

Combining this identity with the modified multiplication (49) we see that (11) holds on \( \tilde{\mathcal{P}}_{d_2,k}(V) \), i.e. our functional \( Q_{s,t}^{[n]} = \sum_{j=0}^{n} \sum_{i \in S(j)} X_{s,t}^{j,i}z_{i} \) has the multiplicative property. Going through the same steps as before with these modifications in place the proof of Proposition 8 may now be completed.

8 Appendix:

8.1 Proof of the inequality (13)

As \( Z_t^x \geq 0 \), we have, by Jensen’s inequality, that \( \mathbb{E} [Z_t^x | Y^x]^{-1} \leq \mathbb{E} [(Z_t^x)^{-1} | Y^x] \). Then observe that, by integration by parts

\[
- \sum_{i=1}^{d_2} \int_{0}^{t} h^i(X^x_s) dY^x_s i = \sum_{i=1}^{d_2} \left( -h^i(X^x_t) Y^x_t i + \int_{0}^{t} Y^x_s i A h^i(X^x_s) ds + \sum_{j=1}^{d} \int_{0}^{t} Y^x_s i V_j h^i(X^x_s) dB^j_s \right)
\]

\[
\leq \sum_{i=1}^{d_2} \sup_{s \in [0,t]} |Y^x_s i| \left( \|h^i\|_{\infty} + t \|A h^i\|_{\infty} \right)
\]

\[
+ \frac{d_2 t}{2} \sum_{i=1}^{d_2} \sup_{s \in [0,t]} |Y^x_s i|^2 \left( \sum_{j=1}^{d} \|V_j h^i\|^2 \right) + \eta^x_t,
\]

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where

\[ \eta^x_t = \sum_{j=1}^{d} \int_0^t \left( \sum_{i=1}^{d_2} Y_s^x i V^j h^i (X_s^x) \right) dB^j_s - \sum_{j=1}^{d} \frac{1}{2} \int_0^t \left( \sum_{i=1}^{d_2} Y_s^x i V^j h^i (X_s^x) \right)^2 ds. \]

Since, \( \tilde{E} [\exp \eta^x_t | Y^x_t] = 1 \), we get that

\[ (1/\rho^x_t (1)) < \exp C \left( \sup_{s \in [0,t]} |Y_s^x|^i + \sup_{s \in [0,t]} |Y_s^x|^2 \right), \]

where \( C \) is a constant independent of \( x \), \( C = \max_{i=1,...,d} \left( |h^i| \right)_\infty + t \left( |Ah^i| \right)_\infty + \frac{dt}{2} \sum_{j=1}^{d} \left( |V^j h^i| \right)^2 \).

Inequality (13) follows as \( \sup_{x \in R^N} \sup_{s \in [0,t]} |Y_s^x|^i \) is finite almost surely.

### 8.2 The expansion of the first three iterated integrals

For the first integral we can express it using the following two terms:

\[
\int_0^{t_2} R_{(t_1, t_2)}(\phi)dY_{t_1} = \int_0^{t_2} P_{t_1} (hP_{t_2-t_1}(\phi)) dY_{t_1} = q^1_{t_2}(Y) P_{t_2}(h\phi) - \int_0^{t_2} q^1_{t_1}(Y) P_{t_1} (\Psi_1 P_{t_2-t_1}(\phi)) dt_1.
\]

For the second iterated integral we end up with the following five terms (2+3)

\[
\int_0^{t_3} \int_0^{t_2} R_{(t_i,t_2,t_3)}(\phi)dY_{t_1}dY_{t_2} = \int_0^{t_3} \int_0^{t_2} R_{(t_1,t_2)} (hP_{t_3-t_2}(\phi)) dY_{t_1}dY_{t_2}
\]

\[
= \int_0^{t_3} q^1_{t_2}(Y) P_{t_1} (h^2 P_{t_3-t_2}(\phi)) dY_{t_1}dY_{t_2}
\]

\[
- \int_0^{t_3} q^1_{t_1}(Y) P_{t_1} (\Psi_1 P_{t_2-t_1}(h P_{t_3-t_2}(\phi))) dt_1dY_{t_2}
\]

\[
= q^2_{t_3}(Y) P_{t_3}(h^2 \phi) - \int_0^{t_3} q^1_{t_2}(Y) P_{t_2} (\Psi_2 P_{t_3-t_2}(\phi)) dt_2
\]

\[
- q^1_{t_3}(Y) \int_0^{t_3} q^1_{t_2}(Y) P_{t_2} (\Psi_1 P_{t_3-t_2}(h \phi)) dt_2
\]

\[
+ \int_0^{t_3} q^1_{t_2}(Y)^2 P_{t_2} (\Psi_1 (h P_{t_3-t_2}(\phi))) dt_2
\]

\[
+ \int_0^{t_3} q^1_{t_2}(Y) \int_0^{t_2} q^1_{t_1}(Y) P_{t_1} (\Psi_1 P_{t_2-t_1}(\Psi_1 P_{t_3-t_2}(\phi))) dt_1 dt_2
\]

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For the third iterated integral we end up with the following 14 terms (2+3x4)

\[
\int_0^{t_4} \int_0^{t_3} \int_0^{t_2} R_{(t_1, t_2, t_3)}(\varphi) dY_{t_1} dY_{t_2} dY_{t_3} = \int_0^{t_4} q_{t_3}^2(Y) P_{t_3} \left( h^3 P_{t_4-t_3}(\varphi) \right) dY_{t_3} \\
- \int_0^{t_4} \int_0^{t_3} q_{t_2}^2(Y) P_{t_2} \left( \Psi_2 P_{t_3-t_2}(h P_{t_4-t_3}(\varphi)) \right) dt_2 dY_{t_3} \\
- \int_0^{t_4} \int_0^{t_3} q_{t_2}^1(Y) \int_0^{t_3} q_{t_2}^1(Y) P_{t_2} \left( \Psi_1 P_{t_3-t_2}(h^2 P_{t_4-t_3}(\varphi)) \right) dt_2 dY_{t_3} \\
+ \int_0^{t_4} \int_0^{t_3} q_{t_2}^1(Y)^2 P_{t_2} \left( \Psi_1(h P_{t_3-t_2}(h P_{t_4-t_3}(\varphi))) \right) dt_2 dY_{t_3} \\
+ \int_0^{t_4} \int_0^{t_3} q_{t_2}^1(Y) \int_0^{t_2} q_{t_1}^1(Y) P_{t_1} \left( \Psi_1 P_{t_2-t_1}(h P_{t_3-t_2}(h P_{t_4-t_3}(\varphi))) \right) dt_1 dt_2 \\
= q_{t_4}^3(Y) P_{t_4} \left( h^3 \varphi \right) - \int_0^{t_4} q_{t_3}^2(Y) P_{t_3} \left( \Psi_3 P_{t_4-t_3}(\varphi) \right) dt_3 \\
- q_{t_4}^1(Y) \int_0^{t_4} q_{t_3}^2(Y) P_{t_3} \left( \Psi_2 P_{t_4-t_3}(h \varphi) \right) dt_3 \\
+ \ldots \]

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