The Power of Amortized Recourse for Online Graph Problems

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Abstract

In this work, we study online graph problems with monotone-sum objectives. We propose a general two-fold greedy algorithm that references yardstick algorithms to achieve $t$-competitiveness while incurring at most $w_{\text{max}} \cdot (t+1) \cdot \min\{1,w_{\text{min}}\} \cdot (t-1)$ amortized recourse, where $w_{\text{max}}$ and $w_{\text{min}}$ are the largest value and the smallest positive value that can be assigned to an element in the sum. We further show that the general algorithm can be improved for three classical graph problems by carefully choosing the referenced algorithm and tuning its detailed behavior. For \textsc{Independent Set}, we refine the analysis of our general algorithm and show that $t$-competitiveness can be achieved with $t-1$ amortized recourse. For \textsc{Maximum Cardinality Matching}, we limit our algorithm’s greed to show that $t$-competitiveness can be achieved with $(2-t^*)\cdot(3-t^*) + \frac{t^*-1}{t^*}$ amortized recourse, where $t^*$ is the largest number such that $t^* = 1 + \frac{1}{j}$ for some integer $j$. For \textsc{Vertex Cover}, we show that our algorithm guarantees a competitive ratio strictly smaller than 2 for any finite instance in polynomial time while incurring at most 3.33 amortized recourse. We beat the almost unbreakable 2-approximation in polynomial time by using the optimal solution as the reference without computing it. We remark that this online result can be used as an offline approximation result (without violating the unique games conjecture [19]) to partially improve upon the constructive algorithm of Monien and Speckenmeyer [22].

1 Introduction

Graph optimization problems serve as stems for various practical problems. A solution for such a problem can be described as an assignment from the elements of the problem (e.g. vertices of a graph) to non-negative real numbers such that the constraints between the elements are satisfied. In the online setting, the most considered models are the vertex-arrival and edge-arrival models. That is, the graph is revealed vertex-by-vertex or edge-by-edge, and once an element arrives, the online algorithm has to immediately make an irrevocable decision on the new element. The performance of an online algorithm is measured by competitive ratio against the optimal offline solution. Many graph optimization problems are non-competitive: the larger the input size, the larger the competitive ratio of any deterministic online algorithm. In other words, a non-competitive problem has no constant-competitive online algorithm.

The pure online model is pessimistic, in that altering decisions may be possible (albeit expensive) or limited knowledge about the future may be available in the real world. In this work, we investigate online graph optimization problems in the recourse model. That is, decisions made by the online algorithm can be revoked. In particular, we aim at finding out the amount of amortized recourse that is sufficient and/or necessary for attaining a desirable competitive ratio for a given problem.

Uncertainty and amortized recourse. The competitive ratio can be seen as quantification of how far the quality of an online algorithm’s solution is from that of a conceptual optimal offline algorithm that has complete knowledge of the input and unlimited computational power. Therefore, the non-competitiveness of graph optimization problems suggests that uncertainty of the input is critical to these problems. However, the online algorithm may perform better when the irrevocability constraint is relaxed or knowledge about future inputs is available. It is intriguing to investigate to what extent these problems...
remain non-competitive under these conditions, in particular to determine how much revocability or knowledge the online algorithm needs in order to attain a desirable competitive ratio.

Beyond the practical motivation of relaxing irrevocability of online algorithms’ decisions, amortized recourse also provides insight on how a given online problem is affected by uncertainty. In particular, it captures how rapidly the structure of the offline optimal solution can change: the fewer elements required to do so, the larger the amortized recourse. Furthermore, the impact of uncertainty is directly correlated with this idea: the faster the optimal solution can change, the more impact uncertainty on future inputs will have. Different problems may attain constant competitive ratios using different amounts of (amortized) recourse, which implies variability in the impact of uncertainty. For example, to attain a constant competitive ratio, one needs exactly $O(\log n)$ recourse per edge for min-cost bipartite matching [20], while one only needs a constant amount of recourse per element for maximum independent set and minimum vertex cover [10].

Online monotone-sum problems. We study online graph problems in the vertex-arrival or the edge-arrival models. Along with the newly-revealed element, which can be a vertex or an edge according to the arrival model, there may be constraints imposed upon some subset of the currently-revealed elements that a feasible solution should satisfy. An algorithm aims at finding a feasible solution that maximizes (or minimizes) the objective. A problem is a sum problem if the objective is a sum of the values assigned to each element. If the value of the optimal solution of an instance is always greater than or equal to that of a subset of the instance, then the problem is a monotone problem.

An online algorithm makes decisions upon arrival of each element. In the recourse model, the online algorithm can also revoke an earlier decision that it made and pay for the revocation. We aim to reduce the competitive ratio with as little total recourse (i.e. as few revocations) as possible.

Our contribution. We propose a general online algorithm $TaS_t$ that provides a trade-off between amortized recourse and competitive ratio for arbitrary monotone-sum graph problems. In particular, we consider two measurements of recourse cost: number of reassigned elements, or the amount of change in the reassigned values. Our result works for both unweighted and weighted problems, and it even works for fractional optimization problems, where the smallest non-zero value assigned to a single element can be a real number between 0 and 1. The following is the main result of our work, where the bound of amortized recourse works for both measurements of recourse cost (Theorem 1 and Corollary 1).

Main Result (informal) 1. Using an optimal algorithm (resp. an incremental $\alpha$-approximation algorithm, defined formally in Section 2) as the yardstick, $TaS_t$ is $t$-competitive (resp. $(t \cdot \alpha)$-competitive) and incurs at most $\frac{w_{\text{max}}}{\min(1,w_{\text{min}})} \cdot \frac{(t+1)}{(t-1)}$ amortized recourse for any monotone-sum graph problem where $w_{\text{max}}$ and $w_{\text{min}}$ are the maximum and minimum non-zero values that can be assigned to an element.

$TaS_t$ is two-fold greedy. First, it assigns the value greedily once an element arrives. Second, the algorithm aligns its solution to the yardstick solution completely and incurs recourse when the current solution fails to be $t$-competitive against the yardstick solution.

In general, the $TaS_t$ algorithm works for any optimization problem. The challenge is to bound the amortized recourse that it incurs, as the complete alignment may require a vast amount of recourse. By looking closer at a specific problem, we can show a tighter bound on the amount of recourse needed. We use a sophisticated analysis for the INDEPENDENT SET problem and improve the recourse bound (Theorem 2).

The two-fold greedy algorithm may perform better when the greediness is relaxed. Moreover, by choosing different yardstick algorithms and tuning the alignment to the yardstick carefully, the amortized recourse can be further reduced. We show that for the MAXIMUM CARDINALITY MATCHING problem, partially aligning to the yardstick solution is more recourse-efficient (Theorem 5).

\(^1\)The DOMINATING SET and MATCHING WITH DELAYS problems are sum problems but not monotone. The COLORING PROBLEM is monotone but not a sum problem.

\(^2\)The bound of amortized recourse $\frac{w_{\text{max}}}{\min(1,w_{\text{min}})} \cdot \frac{(t+1)}{(t-1)}$ is larger when the elements can be assigned minimum non-zero values smaller than 1. For example, the fractional VERTEX COVER problem in [23].
For the Vertex Cover problem, we show that a special version of TaS\(t\) with \(t = 2 - \frac{2}{\OPT}\) incurs a very small amount of amortized recourse (Theorem 5) and is \((2 - \frac{2}{\OPT})\)-competitive, where \(\OPT\) is the size of the optimal vertex cover\(^4\) (Theorem 7). Our algorithm uses an optimal solution as a yardstick. The key to the polynomial time complexity is that instead of explicitly finding the yardstick assignment, we show that the yardstick cannot be too “far” from our solution at any moment if the target competitive ratio \(2 - \frac{2}{\OPT}\) is not already achieved. More specifically, by restricting the range of greedy choice, we can show that the yardstick solution can be aligned partially within a constant amount of amortized recourse. Thus, our result breaks the almost unbreakable 2-approximation for the Vertex Cover problem and improves upon that of Monien and Speckenmeyer \[22\] for a subset of the graphs containing odd cycles of length no less than \(2k + 3\) (for which \(2 - \frac{2}{\OPT} < 2 - \frac{1}{k+1}\)), using an algorithm that is also constructive.

Our results are summarized in Table 1 which illustrates the power of amortization.

| Problem | (Competitive ratio, amortized recourse) |
|---------|----------------------------------------|
| Maximum Matching | \((k, O(\frac{\log k}{t}) + 1)) \[10\] (Theorem 2, Corollary 1) |
| Maximum Independent Set | \((t, \frac{1}{t})\) \[2\] (Theorem 2) |
| Minimum Vertex Cover | \((2, 1) \[10\] \((2 - \frac{2}{\OPT} - \epsilon)\) (Theorem 8, P) |

Table 1: Summary of our results. Note that \(t\) can be any real number larger than 1. For Maximum Matching, \(t^*\) is the largest number such that \(t^* \leq t\) and \(t^* = 1 + \frac{1}{j}\) for some integer \(j\). The note P means that the algorithm is a polynomial-time online algorithm.

**Related work.** The closest previous result is the work by Boyar et al. \[10\]. The authors investigated the Independent Set, Maximum Cardinality Matching, Vertex Cover, and Minimum Spanning Forest problems, which are all non-competitive in the pure online model. The authors showed that the competitive ratio of these problems can be massively reduced to a constant by incurring at most 2 recourse for any single element. Note that the bounds of the worst case recourse are upper bounds of the amortized recourse. Moreover, the algorithms in \[10\] incur at least 1.5 amortized recourse for the Maximum Cardinality Matching problem and at least 0.5 amortized recourse for the Vertex Cover problem.

There is a line of research on online matching problems with recourse. Angelopoulos et al. \[2\] studied a more general setting for Maximum Cardinality Matching and showed that given that no element incurs more than \(k\) recourse, there exists an algorithm that attains a competitive ratio of \(1 + O(1/\sqrt{k})\). Megow and Nölke \[20\] showed that for the Min-Cost Bipartite Matching problem, constant competitiveness is achievable with amortized recourse \(O(\log n)\), where \(n\) is the number of requests. Bernstein et al. \[6\] showed that there exists an algorithm that achieves 1-competitiveness with \(O(\log^2 n)\) amortized recourse for the Bipartite Matching problem, where \(n\) is the number of vertices inserted. The result also shows that to achieve 1-competitiveness for Vertex Cover, any online algorithm needs at least \(O(n)\) amortized recourse per vertex.

In addition, there has been extensive work on online algorithms in the recourse model for a variety of different problems. For amortized recourse, studied problems include online bipartite matching \[6\], graph coloring \[9\], minimum spanning tree and traveling salesperson problem \[21\], Steiner tree \[12\], online facility location \[11\], bin packing \[13\], submodular covering \[14\], and constrained optimization \[3\].

Graph problems model various real-world issues whose performance guarantees are often abysmal, as they are notoriously non-competitive in the pure online model. Prior work has shown curiosity

\(^4\)Note that over all instances, \(\OPT\) can be arbitrarily large. Thus, there is no \(\varepsilon > 0\) for which \(2 - \frac{2}{\OPT} < 2 - \varepsilon\) over all instances. Therefore, our result does not violate the unique games conjecture \[19\].
about the conditions under which these problems become competitive, and these problems have been investigated under different models out of both practical and theoretical interests. Other than the recourse model, considered models include paying for a delay in the timing of decision making to achieve a better solution [5, 7]. Another model for delayed decision making is the reordering buffer model [1], where the online algorithm can delay up to \( k \) decisions by storing the elements in a size-\( k \) buffer.

The impact of extra knowledge about the input has also been studied. For example, once a vertex arrives, the neighborhood is known to the algorithm [16]. In the lookahead model, an online algorithm is capable of foreseeing the next events [1]. Predictions provided by machine learning are also considered for graph problems [1]. Finally, there are also works where the integral assignment restrictions are relaxed for vertex cover and matching problems [23].

Another major area of related work for practically any problem considered in the online model is polynomial-time approximation algorithms for the equivalent problem in the offline setting. The link between the two is particularly salient when considering a polynomial-time online algorithm, as this online algorithm can also be run in polynomial time in the offline setting by processing the graph as if it were revealed in an online manner.

In the case of minimum vertex cover, assuming the unique games conjecture, it is not possible to obtain an approximation factor of \( (2 - \varepsilon) \) for fixed \( \varepsilon > 0 \) [19]. However, results have been obtained for parameterized \( \varepsilon \). In particular, Halperin [15] showed an approximation factor of \( 2 - (1 - o(1)) \frac{2 \ln \ln \Delta}{\ln \Delta} \) on graphs with maximum degree \( \Delta \), and Karakostas [18] showed an approximation factor of \( 2 - \Theta \left( \frac{1}{\sqrt{\log n}} \right) \). Both of these results use semidefinite relaxations of the problem, whereas Monien and Speckenmeyer [22] had previously used a constructive approach to show an approximation factor of \( 2 - \frac{1}{k+1} \) for graphs without odd cycles of length at most \( 2k + 1 \).

**Paper organization.** Section 2 defines monotone-sum graph problems and the amortized recourse model. We propose a general algorithm \( \text{TAS}_t \) for finding the trade-off between the desired competitive ratio and the amortized recourse needed. Section 3 provides a refined analysis on the \( \text{TAS}_t \) algorithm on the \textsc{Independent Set} problem. Section 4 discusses an existing algorithm [2], which is a variant of \( \text{TAS}_t \) algorithm, for the \textsc{Maximum Cardinality Matching} problem that is less greedy in aligning its solution and obtains a better trade-off. Section 5 introduces a polynomial-time version of \( \text{TAS}_t \) algorithm for the \textsc{Vertex Cover} problem that limits both greedy aspects. This algorithm can also be used as a novel offline approximation algorithm for certain graph classes. Due to space constraints, proofs for all lemmas and theorems can be found in the appendix. We also provide proof ideas for some theorems in the main text.

## 2 Monotone-Sum Graph Problems and a General Algorithm

For an **online graph problem** \( Q \) on a graph \( G = (V, E) \), which is unknown a priori, we consider either the **vertex-arrival** model or the **edge-arrival** model. In the vertex-arrival model (resp. edge-arrival model), the elements in \( V \) (resp. elements in \( E \)) arrive one and a time, and an algorithm has to assign each element a non-negative value in \([0, w_{\text{max}}] \) such that the assignment satisfies certain properties associated with \( Q \).

Formally, the **assignment** is defined as \( \mathcal{A} : \mathcal{X} \rightarrow \mathbb{R}^+ \), where \( \mathcal{X} \) is \( V \) or \( E \), such that \( \mathcal{A}(\mathcal{X}) \) satisfies a set of properties \( P_Q \). The **value** of a feasible assignment \( \mathcal{A} \) is defined as a function \( \text{value} : \mathcal{X} \times \mathcal{A}(\mathcal{X}) \rightarrow \mathbb{R}^+ \), which should be minimized or maximized as appropriate. In this work, we focus on the problems with sum objectives, that is, \( \text{value}(\mathcal{X}, \mathcal{A}(\mathcal{X})) = \sum_{x \in \mathcal{X}} \mathcal{A}(x) \). Moreover, we concern ourselves about the impact of lacking information on the optimality of the solution. Therefore, we consider monotone sum graph problems where given a feasible assignment and a newly-arrived element, there is always a value in \([0, w_{\text{max}}] \) that can be assigned to the new element such that the new assignment is feasible.

We denote the assignment on input \( \mathcal{X} \) returned by the algorithm \( \text{ALG} \) by \( \text{ALG}(\mathcal{X}) \). We abuse the

\[4\] Classical graph problems such as \textsc{Independent Set}, \textsc{Maximum Cardinality Matching}, and \textsc{Vertex Cover} all satisfy this property.
notation \( \mathcal{X} \) to denote the graph revealed by the input \( \mathcal{X} \). We further abuse notation and denote the total value of the assignment by \( \text{ALG}(\mathcal{X}) \) as well. That is, \( \text{ALG}(\mathcal{X}) = \sum_{x_i \in \mathcal{X}} \text{ALG}(x_i) \). When the context is clear, the parameter \( \mathcal{X} \) is dropped.

We study the family of monotone-sum graph problems, which is defined as follows. Similarly, we define the family of incremental algorithms. Note that a monotone-sum problem can be a maximization or a minimization problem.

**Definition 1.** The projection of an assignment \( \mathcal{A}(G) \) on an induced subgraph \( H \) of \( G \) assigns to each element in \( H \) the same value that \( \mathcal{A}(G) \) does in \( G \).

**Definition 2.** Monotone-sum graph problems. A sum problem is monotone if for any graph \( G \) and any induced subgraph \( H \) of \( G \), 1) the projection of any feasible assignment \( \mathcal{A}(G) \) on \( H \) is also feasible, and 2) \( \text{OPT}(H) \leq \text{OPT}(G) \), where \( \text{OPT} \) is an optimal solution.

**Definition 3.** Incremental algorithms. An algorithm \( \text{ALG} \) is incremental if for any graph \( G \) corresponding to the instance \( \mathcal{X} \) and any induced subgraph \( H \) of \( G \), \( \text{ALG}(H) \leq \text{ALG}(G) \). Furthermore, the projection of \( \text{ALG}(\mathcal{X}) \) on a prefix \( \mathcal{X}' \) of instance \( \mathcal{X} \) does not have a better objective value than the assignment \( \text{ALG}(\mathcal{X}') \).\(^5\)

In this work, the performance of an online algorithm is measured by the competitive ratio. An online algorithm \( \text{ALG} \) attains a competitive ratio of \( t \) if \( \max \left( \frac{\text{OPT}(\mathcal{X})}{\text{ALG}(\mathcal{X})}, \frac{\text{OPT}(\mathcal{X})}{\text{ALG}(\mathcal{X})} \right) \leq t \) for any instance \( \mathcal{X} \), where \( \text{OPT} \) is the optimal offline algorithm that knows all information necessary for solving the problem. In the recourse model, the online algorithm can revoke its decisions and incurs recourse cost. There are two types of recourse cost considered in this paper:

- **Type-1:** The recourse cost is defined as the number of elements which assignment values are changed. Formally, \( \sum_{x_i \in \mathcal{X}} 1[A_1(x_i) \neq A_2(x_i)] \) when an assignment on instance \( \mathcal{X} \) is changed from \( A_1(\mathcal{X}) \) to \( A_2(\mathcal{X}) \).

- **Type-2:** The recourse cost is defined as the amount of change of the assignment value. Formally, \( \sum_{x_i \in \mathcal{X}} |A_1(x_i) - A_2(x_i)| \) when an assignment on instance \( \mathcal{X} \) is changed from \( A_1(\mathcal{X}) \) to \( A_2(\mathcal{X}) \).

We study the trade-off between the competitive ratio and the amortized recourse. That is, the total incurred recourse cost divided by the number of elements that should be assigned a value in the final instance. We define a family of algorithms for monotone-sum problems.

**Target-and-Switch (TaS\(_t\)) algorithm.** The TaS\(_t\) algorithm uses a yardstick algorithm \( \text{REF} \) as a reference, where the yardstick can be the optimal algorithm or an incremental \( \alpha \)-approximation algorithm. Throughout the process, TaS\(_t\) keeps track of the yardstick solution value. Once a new element arrives, TaS\(_t\) greedily assigns a feasible value\(^6\) to the newly-revealed element if this assignment remains \( t \)-competitive relative to the yardstick algorithm’s solution. Otherwise, TaS\(_t\) switches its assignment to the one by the yardstick algorithm and incurs recourse. (See Algorithm\(^\Box\) in Appendix.)

Now, we show that the TaS\(_t\) algorithm achieves the desired competitive ratio \( t \) with at most polynomial of \( t \) amortized recourse. In our analysis, we use the following observation heavily (including for Theorem\(^\Box\)).

**Observation 1.** For all \( x_i \geq 0 \) and \( y_i > 0 \), \( \sum_{i=1}^{n} x_i / y_i \leq \max_{i} x_i / y_i \).

**Theorem 1.** Using an optimal algorithm (resp. incremental \( \alpha \)-approximation algorithm) as the yardstick, TaS\(_t\) is \( t \)-competitive (resp. \(( t \cdot \alpha ) \)-competitive) and incurs at most \( \frac{w_{max}(t+1)}{t-1} \) Type-2 amortized recourse for any monotone-sum graph problem where \( w_{max} \) is the maximum value that can be assigned to an element. The bound also works for Type-1 amortized recourse.

\(^5\)For example, the Ramsey algorithm in \([8]\) is an incremental algorithm. Also note that any online algorithm is an incremental algorithm.

\(^6\)Note that there always exists a value such that the new assignment is feasible since the problem is monotone.
Proof. (Ideas.) We can show that any optimal solution satisfies the incremental property (see the full version) and thus can be seen as an incremental 1-approximation algorithm.

Since recourse is incurred only at the moments when a switch happens in the TaS algorithm, we partition the process of the algorithm into phases according to the switches. Phase \( i \) consists all the events after the \((i - 1)\)-th switch until the \( i \)-th switch. By Observation [1] the amortized recourse for the whole instance is bounded by the maximum amortized recourse incurred within a phase. Therefore, we consider the amortized recourse incurred by the \((i + 1)\)-th switch for arbitrary \( i \geq 0 \).

Let \( \text{REF}_i \) and \( \text{TaS}_i \) denote the value of the yardstick algorithm’s solution and the TaS algorithm’s solution right after the \( i \)-th switch, respectively. By the TaS algorithm, \( \text{TaS}_i = \text{REF}_i \). Let \( \text{ALG} \) denote the value of \( \text{TaS}_i \)’s solution right before the arrival of \( x \), which triggers the \((i + 1)\)-th switch. The total Type-2 recourse cost is at most \( \text{ALG} + \text{REF}_{i+1} \) (where the TaS algorithm changes the value on every element to zero and then changes it to the \( \text{REF} \) assignment).

The main ingredients for the proof are:

- **Property 1:** Monotonicity of the problem and the incremental nature of \( \text{REF} \) implies that \( \text{REF}_i \leq \text{REF}_{i+1} \).
- **Property 2:** The incremental nature of \( \text{TaS}_i \) during a phase implies that \( \text{ALG} \geq \text{REF}_i \).
- **Property 3:** By the switching condition of \( \text{TaS}_i \), \( \text{ALG} < \text{REF}_{i+1}/t \) for maximization problems, and \( \text{ALG} + w_{\text{max}} > t \cdot \text{REF}_{i+1} \) for minimization problems.

Maximization problems. By **Property 1** and the fact that the assigned values are at most \( w_{\text{max}} \), we can show that the adversary needs to release at least \( \frac{\text{REF}_{i+1} - \text{REF}_i}{w_{\text{max}}} \) elements such that the yardstick assignment value increases enough to trigger the switch. By **Property 2** and **Property 3**, \( \frac{\text{REF}_{i+1} - \text{REF}_i}{w_{\text{max}}} \geq (1 - 1/t) \cdot \text{REF}_{i+1} \). By **Property 3**, the total recourse incurred by the \((i + 1)\)-th switch is at most \( \text{ALG} + \text{REF}_{i+1} < (1 + 1/t) \cdot \text{REF}_{i+1} \). Hence, the Type-2 amortized recourse incurred in phase \( i + 1 \) is bounded by \( \frac{w_{\text{max}}(1+1/t) \cdot \text{REF}_{i+1}}{(1-1/t) \cdot \text{REF}_{i+1}} = \frac{w_{\text{max}}(t+1)}{t-1} \).

Minimization problems. In minimization problems, the \((i + 1)\)-th switch may be triggered by shifting the \( \text{REF} \) assignment completely but without changing its value. In this case, a massive amount of recourse is incurred by a single input. However, we can show by **Property 3** that in this case, the \( \text{ALG} \) value must be large enough to trigger the switch. Thus, we can bound the number of elements released during phase \( i + 1 \) by the change of \( \text{TaS}_i \) assignment’s total value. That is, it is at least \( \frac{\text{ALG} - \text{TaS}_i}{w_{\text{max}}} + 1 \), where the \( 1 \) is the element which triggers the switching. By **Property 1** and **Property 2**, the number is at least \( \frac{(1-1/t)(\text{ALG} + w_{\text{max}})}{w_{\text{max}}} \). By **Property 3**, the total recourse incurred by the \((i + 1)\)-th switch is at most \( \text{ALG} + \text{REF}_{i+1} < (1 + 1/t) \cdot \text{ALG} + w_{\text{max}}/t \). Therefore, the Type-2 amortized recourse incurred in phase \( i + 1 \) is bounded by \( \frac{w_{\text{max}}((1+1/t) \cdot \text{ALG} + w_{\text{max}}/t)}{(1-1/t)(\text{ALG} + w_{\text{max}})} = \frac{w_{\text{max}}(t+1)}{t-1} \).

The yardstick algorithm can be the optimal offline algorithm. Since the problem is monotone, our algorithm can be \( t \)-competitive for arbitrary \( t > 1 \). Furthermore, if we apply a polynomial-time incremental \( \alpha \)-approximation algorithm as the yardstick, then our algorithm also runs in polynomial time.

The results work for weighted versions of problems, and it also work for fractional assignment problems, where the value assigned to any element is in \([0, 1]\) (for example, the fractional Vertex Cover problem in [23]). In this case, the Type-2 amortized recourse is bounded above by the Type-1 amortized recourse:

**Corollary 1.** For a fractional monotone-sum problem, \( \text{TaS}_i \) is \((t \cdot \alpha)\)-competitive and incurs at most \( \frac{t+1}{w_{\text{max}}(t-1)} \) Type-1 amortized recourse using an incremental \( \alpha \)-approximation algorithm as the yardstick. The bound also works for Type-2 amortized recourse.

The monotone-sum problem property captures many classical graph optimization problems such as Independent Set, Maximum Cardinality Matching, and Vertex Cover. The three problems can be interpreted as a special case of general monotone-sum problems as follows.
Independent Set problem in vertex-arrival model. Vertices arrive one at a time and should be assigned a value 0 or 1. Once a vertex is revealed, the edges between it and its previously-revealed neighbors are known. The goal is to find a maximum value assignment such that for any edge, the sum of values assigned to the two endpoints is at most 1.

Maximum Cardinality Matching problem in vertex/edge-arrival model. Edges or vertices arrive one at a time and each of the edges should be assigned a value 0 or 1. The goal is to find a maximum value assignment such that for any vertex, the sum of values assigned to its incident edges is at most 1.

Vertex Cover problem in vertex-arrival model. Vertices arrive one at a time and should be assigned a value 0 or 1. Once a vertex is revealed, the edges between it and its previously-revealed neighbors are known. The goal is to find a minimum value assignment such that for any edge, the sum of values assigned to its two endpoints is at least 1.

Since the available value for each element is either 0 or 1 in these three problems, we say that an element is \textit{accepted} if it is assigned a value 1. Similarly, an element is \textit{rejected} if it is assigned a value 0. An element is \textit{late-accepted} if its value is changed from 0 to 1 after its arrival, and \textit{late-rejected} if its value is changed from 1 to 0 after its arrival. Furthermore, since the value for any element only changes between 0 and 1, the \textbf{Type-1} recourse cost and \textbf{Type-2} recourse cost are equivalent in these three problems. Therefore, we have the following corollary.

\textbf{Corollary 2.} For Independent Set, Maximum Cardinality Matching, and Vertex Cover problems, the TaS\textsubscript{t} algorithm attains competitive ratio \(t > 1\) while incurring at most \(\frac{t+1}{t-1}\) (Type-1 or Type-2) amortized recourse.

3 Maximum Independent Set

For the maximum independent set problem in the vertex-arrival model, the algorithm proposed by Boyar et al. incurs at most 2 amortized recourse while maintaining a competitive ratio of 2.598 \cite{10}. By Theorem 1, the general TaS\textsubscript{t} algorithm incurs at most \(\frac{t+1}{t-1}\) amortized recourse and guarantees a competitive ratio of t. In this section, we show that the amortized recourse incurred by TaS\textsubscript{t} is even smaller by a more sophisticated analysis.

\textbf{(Instance reduction)} For any instance \((G, \sigma)\) of the maximum independent set problem, there exists an instance \((G', \sigma')\) for which any newly revealed vertex is either accepted by TaS\textsubscript{t} or is part of the optimal offline solution when TaS\textsubscript{t} incurs its next switch, but not both, such that the amortized recourse for \((G', \sigma')\) is at least that for \((G, \sigma)\).

Using Lemma 3, we can bound above the amortized recourse incurred by TaS\textsubscript{t} against any reduced instance, and thus the amortized recourse incurred against any instance.

\textbf{Theorem 2.} For the maximum independent set problem, given a target competitive ratio \(t > 1\), TaS\textsubscript{t} is \(t\)-competitive while incurring at most \(\frac{t}{t-1}\) amortized recourse.

\textbf{Proof. (Ideas.)} We show that, for any reduced instance from Lemma 3, TaS\textsubscript{t} will incur at most \(\frac{t}{t-1}\) amortized recourse, and thus that this upper bound holds for any instance. To do this, we use the same phase partition argument as in the proof of Theorem 1 combined with Observation 1. We consider a scheme in which each newly-revealed vertex carries budget \(B\), and the vertices revealed in phase \(i + 1\) must pay the full cost of the recourse incurred by switch \(i + 1\). If the total budget carried by these newly-revealed vertices is at least \(\text{ALG} + \text{OPT}_{i+1}\), the amortized recourse is \(B\).

We can show that the number of vertices revealed in phase \(i + 1\) that are part of \(\text{OPT}_{i+1}\) is bounded above by \(\text{OPT}_{i+1} - \text{OPT}_{i-1}\), which implies that it is sufficient for the budget to satisfy \(B \geq \frac{\text{ALG} + \text{OPT}_{i+1}}{\text{OPT}_{i+1} - \text{OPT}_{i-1}}\). Furthermore, we incorporate both the number of vertices in phase \(i + 1\) that are accepted by TaS\textsubscript{t} and the number of vertices revealed in phase \(i\) that are accepted by TaS\textsubscript{t} into our analysis and show that the lower bound on the required budget is largest when there are no such vertices.
We conclude that it is sufficient for each newly-revealed vertex to carry budget $B = \frac{t}{t-1}$. Thus, the $t$-competitive algorithm only incurs at most $\frac{1}{t-1}$ amortized recourse.

**Theorem 3.** For any $1 < t \leq 2$, $\varepsilon > 0$, and $t$-competitive deterministic online algorithm, there exists an instance for which the algorithm incurs at least $\frac{1}{t-1} - \varepsilon$ amortized recourse.

## 4 Maximum Cardinality Matching

The $t$-Greedy algorithm greedily aligns with the yardstick solution completely and incurs a lot of recourse. However, for some of the elements whose value is changed, the alignment may not contribute to the improvement of the competitive ratio as much as the alignment of other elements. This observation suggests that it may be possible to reduce the amount of amortized recourse while maintaining $t$-competitiveness by switching the solution only partially into the yardstick. In this section, we show that the $L$-Greedy algorithm by Angelopoulos et al. [2], which is in fact a $t$-Greedy algorithm that uses an optimal solution as the yardstick without aligning to it fully, incurs less amortized recourse for the Maximum Cardinality Matching problem.

**$L$-Greedy algorithm [2].** The algorithm is associated with a parameter $L$. Throughout the process, the $L$-Greedy algorithm partially switches its solution to the optimal once by eliminating all augmenting paths with length at most $2L + 1$. That is, it late rejects all the edges selected by itself and late accepts all the edges in the optimal solution on the path.

After applying late operations on all augmenting paths with at most $2L + 1$ edges, every remaining augmenting path has length at least $2L + 3$, and the ratio of the OPT solution value to the $L$-Greedy solution value is $\frac{OPT(P)}{L}$-Greedy($P$) $\leq \frac{L+2}{L+1}$ on the component $P$. Since the Maximum Cardinality Matching problem can be solved in $O(n^{2.5})$ time, the following theorem holds by selecting $L = \lceil \frac{1}{t-1} \rceil - 1$.

**Theorem 4.** The $L$-Greedy algorithm returns a valid matching with competitive ratio $\frac{L+2}{L+1}$ in $O(n^{2.5})$ time, where $n$ is the number of vertices in the final graph.

Since it was shown that to achieve 1.5-competitiveness, every vertex incurs at most 2 recourse, we consider a target competitive ratio $1 < t < 2$ and have the following theorem. Note that $1 < t^* < 2$, thus $0 < \frac{t^*-1}{3-t^*} < 1$.

**Theorem 5.** For the Maximum Cardinality Matching problem in the edge/vertex-arrival model, the $L$-Greedy algorithm is $t$-competitive for any $1 < t < 2$ and incurs at most $\frac{(2-t^*)}{(t-1)(3-t^*)} + \frac{t^*-1}{3-t^*}$ amortized recourse, where $t^*$ is the largest number such that $t^* \leq t$ and $t^* = 1 + \frac{1}{j}$ for some integer $j$.

**Proof. (Ideas.)** Consider the connected components generated by the union of edges chosen by $L$-Greedy or by OPT. Let $C_i$ be the components in the graph and $TR_i$ be the total recourse incurred by the elements in $C_i$ (from very beginning till the end), the amortized recourse given by the whole graph will be upper-bounded by $\sum_i \frac{TR_i}{|C_i|}$ (Observation 1).

By selecting $L = \lceil \frac{2}{t+1} \rceil$, the path eliminations only happen at odd-size components with length from 3 to $2l(\lceil \frac{1}{t-1} \rceil - 1) + 1$ (note that $\lceil \frac{1}{t-1} \rceil \geq 2$ since $1 < t < 2$). Moreover, for such a $(2k+1)$-edge augmenting path, the total recourse incurred by the $2k+1$ elements in the path is at most $1 + \sum_{i=1}^{k} \frac{1}{t^{i-1}} 2k$. Hence, the amount of amortized recourse incurred by this component is at most $\frac{2-t}{(t-1)(3-t)} + \frac{t-1}{3-t}$. It can be adapted to the case in which there is no integer $j$ such that $t = 1 + \frac{1}{j}$ by rounding down $t$ to the largest $t^* \leq t$ such that $t^* = 1 + \frac{1}{j}$ for some integer $j$. By eliminating all augmenting paths
Theorem 6. No deterministic $t$-competitive online algorithm can incur amortized recourse less than \( \frac{(2-t^*)}{(t^*-1)(3-t^*)} \) in the worst case.

5 Minimum Vertex Cover

In this section, we propose a special version of the TaS$_t$ algorithm, Duo-Halve, that attains a competitive ratio of \( 2 - \frac{2}{OPT} \) for the MINIMUM VERTEX COVER problem with optimal vertex cover size \( OPT \) in polynomial time. The Duo-Halve algorithm uses an optimal solution as the yardstick with \( t = 2 - \frac{2}{OPT} \). However, the computation of the optimal solution of VERTEX COVER is very expensive. Thus, we maintain a maximal matching greedily (as the well-known 2-approximation algorithm for VERTEX COVER) on the current input graph and only select vertices that are saturated by the matching. If the DH algorithm rejects two of these vertices, the competitive ratio is at most \( 2 - \frac{2}{OPT} \). We show that there always exists a feasible solution where the last two matched edges only contribute two vertices to the solution, or the optimal solution has size at least \( |M| + 1 \), where \( M \) is the current maximal matching, and the Duo-Halve algorithm is therefore \( (2 - \frac{2}{OPT}) \)-competitive.

In the following discussion, we use some terminology. Let ME1 and ME2 be the most and the second-most recently matched edges respectively. Also, let \( V_{M(X)} \) be the vertices saturated by the maximal matching \( M(X) \). The DH algorithm partitions the vertices into three groups: Group-1: the endpoints of ME1 or ME2, Group-2: the vertices in \( V_M \) but not in Group-1, and Group-3: the vertices in \( V \setminus V_M \).

Duo-Halve Algorithm (DH). When a new vertex \( v \) arrives, if an edge \((p,v)\) is added to \( M(X) \), then it introduces a new ME1 (namely \((p,v)\)). The algorithm first accepts all Group-2 vertices that are adjacent to \( v \). Then, the algorithm decides the assignment of ME1 and ME2 and minimizes the number of accepted endpoints of ME1 and ME2. If there is a tie, we apply the one that accepts fewer endpoints in ME1 and/or incurs less recourse.

The DH algorithm returns a feasible solution in \( O(n^3) \) time, where \( n \) is the number of vertices in the graph. Intuitively, the algorithm maintains a valid solution as it greedily covers edges using vertices in the maximal matching, with the exception of ME1 and ME2, where it carefully ensures that a feasible configuration is chosen. Furthermore, the most computationally-expensive component of the DH algorithm, which checks the validity of a constant number of configurations by looking at the neighborhoods of ME1 and ME2, runs in \( O(n^2) \) time for each new element.

We first show that if DH fails to produce a solution where it accepts only one vertex of ME1, then \( OPT \geq |M| + 1 \). The intuition is that if DH has to accept both endpoints of ME1, there must be at least one Group-3 vertex in each of the endpoints’ neighborhoods. Therefore, the optimal solution has to cover the corresponding edges with at least two vertices.

In the assignment of DH, if both endpoints of ME1 are selected, then the optimal solution must contain at least two vertices in \( ME1 \cup (V \setminus V_M) \), and \( DH \leq 2 - \frac{2}{OPT} \).

Theorem 7. The DH algorithm is \( (2 - \frac{2}{OPT}) \)-competitive.

Proof. (Ideas.) In any possible solution provided by DH, there are three states based on the configuration of ME1 and ME2: 1) both ME1 and ME2 are half, 2) ME1 is half and ME2 is full, and 3) ME1 is full. In state 1, we can directly show that the bound holds since \( DH \leq 2|M| - 2 \). The bound holds for state 3 by Lemma 5.

State 2 requires more involved analysis. If an endpoint of ME2 has a rejected Group-2 neighbor, then DH rejects at least two vertices in \( V_M \) (this Group-2 neighbor and 1 ME1 vertex) and \( DH \leq 2|M| - 2 \). Otherwise, if at least one endpoint of ME2 has no Group-3 neighbor, then we can show that there is no solution based on the maximal matching containing only 2 Group-1 vertices. This means that \( OPT \) must contain either a Group-3 vertex or 3 Group-1 vertices, and thus \( OPT \geq |M| + 1 \). Finally, if each endpoint
of ME2 has a Group-3 neighbor, then OPT must either select a Group-3 vertex or both endpoints of ME2, and \( OPT \geq |M| + 1 \). \(\square\)

For a single newly-revealed vertex, the amount of recourse incurred can be up to \( O(n) \). Even if we restrict our consideration to ME1 and ME2, a single new vertex can incur recourse at most 4. However, this cannot happen at every input. We use a potential function to show that the amortized recourse incurred by \( DH \) is at most 3.33.

**Theorem 8.** The amortized recourse incurred by \( DH \) is at most \( \frac{10}{3} \).

**Proof. (Ideas.)** We prove the theorem by using a potential function. To this end, we define an edge \((u, v)\) as being free if there exist feasible assignments both by either accepting \( u \) or by accepting \( v \). Also, we define a matched edge with only one endpoint selected as being expired if it is neither ME1 nor ME2. Finally, we define \( A \) as the set of vertices accepted by \( DH \). Using these terms, we define the potential function \( \Phi \) as

\[
\Phi := |\{(u, v)|(u, v) \text{ expired}\}| + \frac{1}{3}|A \cap (\text{ME1} \cup \text{ME2})| + \frac{2}{3} \cdot I[\text{ME2 is free}]
\]

Furthermore, at any given moment in the input sequence where the matching constructed by \( DH \) contains at least 2 edges, the status of ME1 and ME2 is characterized by one of 6 states according to their possible combinations of selection statuses of their endpoints. We also differentiate between the two half possibilities for ME1, since the newly-revealed vertex in ME1 can be accepted without incurring a late operation when there is a new ME1.

We show that, for any possible state transition triggered by a newly-revealed vertex, the number of incurred late operations \( LO \) added to the change in potential \( \Delta \Phi \) is bounded above by \( \frac{10}{3} \). Note that, for any newly-revealed vertex \( v \), \( v \) may be adjacent to \( k \geq 0 \) rejected vertices that are matched by some expired edge. This incurs \( k \) late operations, but also decreases \( \Phi \) by \( k \), so this may be ignored when computing \( LO + \Delta \Phi \). Since \( \Phi_0 = 0 \) and \( \Phi_t \geq 0 \), this allows us to conclude the statement of our theorem. \(\square\)

Moreover, we can show a lower bound by constructing a family of instances that alternates between incurring a late accept on a Group-2 vertex, and 4 late operations on ME1 and ME2. (See Figure 2 in Appendix.)

For any \( \varepsilon > 0 \), there exists an instance such that \( DH \) incurs amortized recourse strictly greater than \( \frac{5}{2} - \varepsilon \).

Finally, we show that the analysis in Theorem 4 is tight for a class of online algorithms where its solution only contains vertices saturated by the matching maintained throughout the process in an incremental manner. In other words, no online algorithm in this class achieves a lower competitive ratio, no matter how much amortized recourse it uses. This is done by using an adversary that constructs an arbitrary number of triangles that all share a common vertex \( v \). This common vertex is revealed last, so all edges not incident to \( v \) are added to the matching and \( v \) is rejected. (See Figure 3 in Appendix.)

**Theorem 9.** No deterministic incremental matching-based algorithm achieves a competitive ratio smaller than \( 2 - \frac{2}{\text{OPT}} \).

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A Algorithms and Figures

Algorithm 1 TaΣ\(_t\) algorithm for monotone-sum graph problems

\begin{verbatim}
ALG ← 0
while new element v arrives do
    g ← the best value from [0, w\(_{\text{max}}\)] such that no feasibility constraint is violated
    if the new assignment will fail to be \(t\)-competitive then
        SWITCH(OPT)
    else
        incorporate the greedy assignment
    end if
    ALG ← the value of TaΣ\(_t\)'s current assignment
end while

function SWITCH(assignment A)
    for every element x do
        if TaΣ\(_t\)(x) \(\neq\) A(x) then
            change the assignment of element x into A(x)
        end if
    end for
end function
\end{verbatim}

Figure 1: An illustration of the flow of \texttt{HalveBoth}(ME1, ME2).
Algorithm 2 Duo-Halve algorithm (DH) for Minimum Vertex Cover Problem

\[ \text{ME1} \leftarrow \emptyset, \text{ME2} \leftarrow \emptyset, V_M \leftarrow \emptyset \]

while new vertex \( v \) arrives do
  if there is a vertex \( p \in N(v) \cup (V \setminus V_M) \) then
    ME2 \( \leftarrow \) ME1
    ME1 \( \leftarrow (p, v) \) \( \triangleright (p, v) \) is a new matched edge. If there is more than one \( p \), choose one arbitrarily.
    add \( p \) and \( v \) into \( V_M \)
    LateAccept all rejected vertices in \( (V_M \setminus \{\text{vertices in ME1 or ME2}\}) \cap N(v) \)
    HalveBoth(ME1, ME2)
  else
    LateAccept all rejected vertices in \( V_M \cap N(v) \)
    HalveBoth(ME1, ME2)
  end if
end while

Function HalveBoth(matched edge ME1, matched edge ME2)

Among accept/reject configurations of ME1 and ME2 that yield a valid vertex cover, return one that maximizes the number of half edges among ME1 and ME2 with the minimum number of late operations (see Figure 1 for details).

end Function

\[ \begin{array}{c}
\begin{array}{c}
\text{a} \\
\rightarrow 0
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
b \\
\rightarrow a
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{a} \\
\rightarrow 2
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
b \\
\rightarrow a
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{c} \\
\rightarrow 0
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
b \\
\rightarrow a
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{c} \\
\rightarrow 0
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{d} \\
\rightarrow 0
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{b} \\
\rightarrow a
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{c} \\
\rightarrow d
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{d} \\
\rightarrow c
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{b} \\
\rightarrow a
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{c} \\
\rightarrow 1
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{d} \\
\rightarrow e
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{e} \\
\rightarrow f
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{b} \\
\rightarrow a
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{e} \\
\rightarrow c
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{d} \\
\rightarrow f
\end{array}
\end{array} \]

Figure 2: Adversarial instance for VERTEX COVER such that DH incurs asymptotic amortized recourse \( \frac{5}{2} \). Each arrow’s number denotes the number of late operations incurred by the next vertex’s reveal. The dotted ovals highlight the repeating structure.

\[ \begin{array}{c}
\begin{array}{c}
1 \\
\rightarrow 2
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
2 \\
\rightarrow 3
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
3 \\
\rightarrow 4
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
4 \\
\rightarrow n-2
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
n-2 \\
\rightarrow n-1
\end{array}
\end{array} \]

Figure 3: Adversarial instance for VERTEX COVER such that any incremental matching-based algorithm is exactly \( (2 - \frac{2}{OPT}) \)-competitive. Vertices are labeled by their release order. Any such algorithm must accept \( n - 1 \) vertices, whereas the optimal solution contains \( \frac{n-1}{2} + 1 \) vertices.
The bound also works for Type-1 amortized recourse for any monotone-sum graph problem where \( w_{\text{max}} \) is the maximum value that can be assigned to an element. The bound also works for Type-1 amortized recourse.

**Theorem 1.** Using an optimal algorithm (resp. incremental \((t \cdot \alpha)\)-competitive) and incurs at most \( \frac{w_{\text{max}}(t+1)}{t-1} \) Type-2 amortized recourse for any monotone-sum graph problem where \( w_{\text{max}} \) is the maximum value that can be assigned to an element. The bound also works for Type-1 amortized recourse.

**Proof.** First, note that in the TaS\(_t\) algorithm, recourse is incurred only at the moments when a switch happens. We partition the process of the algorithm into phases according to the switches. Phase \( i \) consists all the events after the \((i-1)\)-th switch until the \(i\)-th switch. Let \( \mathcal{X}_i \) denote the set of elements that are released during the \(i\)-th phase and can be assigned non-zero value. Let \( TR(\mathcal{X}_i) \) and \( AR(\mathcal{X}_i) \) denote the total recourse and the amortized recourse incurred at the \(i\)-th switch, respectively. The total amortized recourse is given by \( AR(\mathcal{X}) = \sum TR_i \frac{|\mathcal{X}_i|}{|\mathcal{X}|} \leq \max_i \frac{TR_i}{|\mathcal{X}_i|} \) (Observation 1). In the following, we prove that for any switch \( i \), \( \frac{TR_i}{|\mathcal{X}_i|} \leq \frac{w_{\text{max}}(t+1)}{t-1} \) and thus \( AR(\mathcal{X}) \leq \frac{w_{\text{max}}(t+1)}{t-1} \).

We first show that any optimal schedule is incremental. In this case, the optimal solution can be seen as an incremental 1-approximation algorithm. Given any graph \( G \) and its subgraph \( H \), since the problem is monotone, \( OPT(H) \leq OPT(G) \). Moreover, consider the projection \( OPT' \) of \( OPT(G) \) on \( H \), it is a feasible solution of \( H \) since the problem is monotone. Since \( OPT' \) is feasible, \( value(OPT(G)) \geq value(OPT') \) for maximization problems, and \( value(OPT(G)) \leq value(OPT') \) for minimization problems. That is, the projection \( OPT' \) of \( OPT(G) \) on \( H \) does not have a better objective value than the assignment \( OPT(H) \).

Let \( \text{REF} \) be the yardstick incremental \(\alpha\)-approximation algorithm referenced (where \(\alpha = 1\) if we reference the optimal exact solution). Let \( \text{REF}_i \) and TaS\(_i\) denote the value of the yardstick solution and the value of TaS\(_i\)’s solution right after the \(i\)-th switch, respectively. By the TaS\(_i\) algorithm, TaS\(_i\) = \( \text{REF}_i \). Consider the input \( x \) that triggers the \((i+1)\)-th switch. Let \( \text{ALG} \) denote the value of TaS\(_i\) algorithm right before the arrival of \( x \). Once \( x \) is revealed, the TaS\(_i\) algorithm attempts to greedily assign a value \( g \in [0, w_{\text{max}}] \) to \( x \), while the value of the REF assignment for the whole instance is \( \text{REF}_{i+1} \). By the definition of the TaS\(_i\) algorithm, the \((i+1)\)-th switch is triggered since \( max\{\frac{\text{ALG}+g}{\text{REF}_{i+1}}, \frac{\text{ALG}+g}{\text{REF}_{i+1}}\} > t \). In the worst case, the TaS\(_i\) algorithm changes the decision on every element. Since we check the \(t\)-competitiveness before assigning the value greedily, \( TR_{i+1} \leq \text{ALG} + \text{REF}_{i+1} \).

**Maximization problems.** According to the condition that triggers the switching, at the moment when the \((i+1)\)-th switch happens, \( \text{REF}_{i+1} > t \cdot (\text{ALG} + g) \). Hence, the total recourse \( TR_{i+1} \leq \text{ALG} + \text{REF}_{i+1} < (1 + 1/t) \cdot \text{REF}_{i+1} \).

Now, we bound the number of elements released between switch \( i \) and switch \( i+1 \) from below. Since \( \text{REF} \) is incremental, and the assigned values are non-negative, the adversary needs to release a sufficient number of elements such that the value of the referenced assignment increases enough to trigger the switch. Let \( \text{REF}' \) be the projection of \( \text{REF}_{i+1} \) on the graph at the beginning of the \(i+1\)-th phase. That is, \( \text{REF}' = \sum_j \sum x \text{ released in phase } j \cdot \text{REF}_{i+1}(x) \). Since the \( \text{REF} \) algorithm is incremental, \( \text{REF}' \leq \text{REF}_i \). Since every new element brings at most \( w_{\text{max}} \) additional cost in the \( \text{REF} \) assignment value, the minimum number of elements that the adversary has to release such that the \( \text{REF} \) value increases enough is at least \( \frac{\text{REF}_{i+1} - \text{REF}_i}{w_{\text{max}}} \).

Therefore, \( |\mathcal{X}_{i+1}| \geq \frac{\text{REF}_{i+1} - \text{REF}_i}{w_{\text{max}}} \). By the TaS\(_i\) algorithm and the fact that the assigned values are non-negative, \( |\mathcal{X}_{i+1}| \geq \frac{\text{REF}_{i+1} - \text{ALG}_{i+1}}{w_{\text{max}}} = \frac{\text{REF}_{i+1} - \text{ALG}_{i+1}}{\text{REF}_{i+1} - \text{REF}_{i+1}} \). By the switch triggering condition, \( \text{ALG} + g < \text{REF}_{i+1} / t \). Hence, \( |\mathcal{X}_{i+1}| \geq \frac{\text{REF}_{i+1} - (\text{ALG} + g)}{w_{\text{max}}} \geq \frac{\text{REF}_{i+1} - \text{REF}_{i+1}}{w_{\text{max}} - (1 - 1/t) \cdot \text{REF}_{i+1}} = \frac{\text{REF}_{i+1} - \text{REF}_{i+1}}{w_{\text{max}} - (1 - 1/t) \cdot \text{REF}_{i+1}} \).

**Minimization problems.** The \((i+1)\)-th switch occurs when \( \text{ALG} + g \geq t \cdot \text{REF}_{i+1} \). Hence, \( TR_{i+1} \leq \text{ALG} + \text{REF}_{i+1} \leq \text{ALG} + (1/t) \cdot (\text{ALG} + g) \leq \frac{(1 + 1/t) \cdot \text{ALG} + w_{\text{max}} / t} {t-1} \).

Now, we bound the number of elements released between switch \( i \) and switch \( i+1 \) from below. Unlike in maximization problems, in minimization problems it is possible that \( \text{REF}_{i+1} = \text{REF}_i \), but the assignment at the \((i+1)\)-th switch is completely different from that at the \(i\)-th switch and causes massive amount of
recourse for $\text{TaS}_t$ to remain $t$-competitive. Therefore, we cannot bound the number of new elements by the change of $\text{REF}$ value but instead do so according to the change of $\text{TaS}_t$ assignment’s total value. Right before the input that triggers the $(i + 1)$-th switch, $\text{TaS}_t$ has value $\text{ALG} \geq \text{ALG}_i$ since the assigned values are non-negative (note that $\text{ALG} = \text{ALG}_i$ if it is the input that triggers the $i$-th switch, and the switches happen in a row). Together with the reveal of the input that triggers the $(i + 1)$-th switch, $|X_{i+1}|$ is at least $\frac{\text{ALG} - \text{TaS}_i}{w_{\text{max}} + 1} + 1$. Therefore, $|X_{i+1}| \geq \frac{\text{ALG} - \text{TaS}_i}{w_{\text{max}}} + 1 = \frac{\text{ALG} - \text{REF}_{i+1}}{w_{\text{max}}} + 1 \geq \frac{\text{ALG} - (1/t) \cdot (\text{ALG}_i + g)}{w_{\text{max}}} + 1 \geq \frac{\text{ALG} - (1/(t + 1)) \cdot (\text{ALG}_i + g)}{w_{\text{max}}} + 1$.

By definition, $\text{AR}_i = \frac{\text{TR}_{i+1}}{|X_{i+1}|} \leq \frac{w_{\text{max}}}{\text{ALG} + w_{\text{max}}} \cdot \frac{(1+1/t) \cdot \text{ALG} + w_{\text{max}}/t}{(1/(t + 1)) \cdot \text{ALG} + w_{\text{max}}/(t + 1)} = \frac{w_{\text{max}}}{\text{ALG} + w_{\text{max}}} \cdot \frac{(t + 1) \cdot \text{ALG} + w_{\text{max}}}{(t/(t - 1)) \cdot \text{ALG} + w_{\text{max}}/(t - 1)} = \frac{\text{ALG} + w_{\text{max}}}{t - 1} \cdot \frac{\text{ALG} + w_{\text{max}}}{t - 1}$.

**Corollary 1.** For a fractional monotone-sum problem, $\text{TaS}_t$ is $(t \cdot \alpha)$-competitive and incurs at most $\frac{t + 1}{w_{\text{min}}/(t - 1)}$ Type-1 amortized recourse using an incremental $\alpha$-approximation algorithm as the yardstick. The bound also works for Type-2 amortized recourse.

**Proof.** The proof is similar to the one of Theorem 1. The new ingredient of the proof is that, given the same amount of value changed, the number of elements whose assigned values are changed is upper-bounded by $\frac{\text{ALG} - \text{REF}_{i+1}}{w_{\text{min}}}$, the other arguments in the proof of Theorem 1 still follow.

**C Independent Set Full Proof**

*(Instance reduction)* For any instance $(G, \sigma)$ of the maximum independent set problem, there exists an instance $(G', \sigma')$ for which any newly revealed vertex is either accepted by $\text{TaS}_t$ or is part of the optimal offline solution when $\text{TaS}_t$ incurs its next switch, but not both, such that the amortized recourse for $(G', \sigma')$ is at least that for $(G, \sigma)$.

**Proof.** For any instance $(G, \sigma)$ of INDEPENDENT SET, any vertex $v \in G$ released after switch $i$ and before switch $i + 1$ belongs to one of the following four types:

- $v$ is selectable by $\text{TaS}_t$, and $v \in \text{OPT}_{i+1}$.
- $v$ is selectable by $\text{TaS}_t$, and $v \notin \text{OPT}_{i+1}$.
- $v$ is not selectable by $\text{TaS}_t$, and $v \in \text{OPT}_{i+1}$.
- $v$ is not selectable by $\text{TaS}_t$, and $v \notin \text{OPT}_{i+1}$.

Since $\text{TaS}_t$ accepts any newly-revealed vertex that it can, any vertex selectable by $\text{TaS}_t$ will be accepted. Then, consider any vertex $v$ of the fourth type. Such a vertex does not incur any late operation at switch $i + 1$, nor does it have any impact on either $\text{TaS}_t$’s solution or $\text{OPT}_{i+1}$. Thus, an instance $(G, \sigma')$ such that $\sigma'$ releases $v$ immediately after switch $i + 1$ and is otherwise identical to $\sigma$ will incur no less amortized recourse than $(G, \sigma)$. Furthermore, if $v$ would be released in $\sigma'$ after the last switch, it can instead be removed from $G$. This transformed instance $(G \setminus \{v\}, \sigma' \setminus \{v\})$ incurs the same recourse as $(G, \sigma)$ and contains one less vertex, and thus incurs more amortized recourse. In this way, we can transform any general instance $(G, \sigma)$ into an instance $(G', \sigma')$ containing no vertices of the fourth type such that the amortized recourse incurred by $(G', \sigma')$ is no less than that incurred by $(G, \sigma)$.

Furthermore, consider any vertex $v$ of the first type in an instance $(G', \sigma')$ containing no vertices of the fourth type. Such a vertex does not incur any late operation at switch $i + 1$, and decreases the ratio between the size of the optimal solution and $\text{TaS}_t$’s solution. Thus, $v$ cannot directly contribute towards an increase in amortized recourse incurred by switch $i + 1$. However, it may have an indirect impact on
TaS\textsubscript{t}'s solution. In particular, there may be some neighbor \( u \) of \( v \) that would be in TaS\textsubscript{t}'s solution if \( v \) were not present in the graph. Since TaS\textsubscript{t} accepts any selectable vertex, \( v \) must be released before \( u \) and thus \( u \) is not selectable by TaS\textsubscript{t} when it is released. Therefore, since \( u \) is not part of OPT\textsubscript{i+1}, \( u \) is a vertex of the fourth type. However, \((G', \sigma')\) contains no vertices of the fourth type, so there is no such vertex \( u \). Then, using the same approach as for vertices of the fourth type, we can transform any instance \((G', \sigma')\) containing no vertices of the fourth type into an instance \((G'', \sigma'')\) containing no vertices of the first or fourth types.

\[\square\]

**Theorem 2.** For the maximum independent set problem, given a target competitive ratio \( t > 1 \), TaS\textsubscript{t} is \( t \)-competitive while incurring at most \( \frac{t}{t-1} \) amortized recourse.

**Proof.** We show that, for any reduced instance from Lemma 3, TaS\textsubscript{t} will incur at most \( \frac{t}{t-1} \) amortized recourse, and thus that this upper bound holds for any instance. To do this, we show that for any switch \( i \),

\[\frac{\text{TR}}{|X_i|} \leq \frac{t}{t-1} \text{ and thus } \max \left( \frac{\text{TR}}{|X_i|} \right) \leq \frac{t}{t-1}.\]

Consider a scheme in which each newly-revealed vertex carries budget \( B \), and the vertices revealed between switch \( i \) and switch \( i+1 \) must pay the full cost of the recourse incurred by switch \( i+1 \). Then, the total budget carried by these newly-revealed vertices must be at least \( ALG + OPT \).

Furthermore, the total available budget at switch \( i+1 \) is at least \( B \) times the number of vertices revealed between switches \( i \) and \( i+1 \), which in turn is at least \( B \) times the number of these vertices of the third type. Note that OPT\textsubscript{i+1} can always be constructed by extending an existing independent set of size at least OPT\textsubscript{i-1}, so the number of vertices of the third type is bounded above by OPT\textsubscript{i-1} – OPT\textsubscript{i-1}. Therefore, to have sufficient budget, \( B \) must satisfy

\[B \geq \frac{ALG + OPT_{i+1}}{OPT_{i+1} - OPT_{i-1}}.\]

Denote \( k \) the number of vertices of the second type revealed between switches \( i \) and \( i+1 \), and \( k' \) the number of vertices of the second type revealed between switches \( i-1 \) and \( i \). Then \( ALG = OPT_i + k \) and \( ALG' = OPT_{i-1} + k' \), so OPT\textsubscript{i+1} \( \geq t \cdot OPT_i + k \), OPT\textsubscript{i+1} \( - 1 < t \cdot ALG \) which is equivalent to OPT\textsubscript{i+1} \( < t \cdot OPT_i + tk + 1 \), and OPT\textsubscript{i-1} \( 1 < t \cdot ALG' \) which is equivalent to OPT\textsubscript{i-1} \( > \frac{OPT_i}{t} - k' - \frac{1}{t} \).

Therefore, we have

\[B \geq \frac{ALG + OPT_{i+1}}{OPT_{i+1} - OPT_{i-1}} \geq \frac{OPT_i + tk + k}{t \cdot OPT_i + tk + \frac{k}{t} + \frac{1}{t}} = \frac{OPT_i + k}{t \cdot OPT_i + tk + k'} = \frac{tk + k}{tk + 1 + \frac{k}{t}}.\]

Note that this lower bound is maximized when \( k' = 0 \). Furthermore, note that \( \frac{tk + k}{tk + 1 + \frac{k}{t}} \geq \frac{tk + k}{tk + 1 + \frac{k}{t}} \) for any \( k \geq 0 \), so this lower bound is maximized when \( k = 0 \).

Therefore, for any instance, it is sufficient for each newly-revealed vertex to carry budget \( B = \frac{2i}{t} = \frac{t}{t-1} \).

Thus, TaS\textsubscript{t} is \( t \)-competitive while incurring at most \( \frac{t}{t-1} \) amortized recourse. \( \square \)

**Theorem 3.** For any \( 1 < t \leq 2, \varepsilon > 0 \), and \( t \)-competitive deterministic online algorithm, there exists an instance for which the algorithm incurs at least \( \frac{1}{t-1} - \varepsilon \) amortized recourse.

**Proof.** Consider any \( t \)-competitive online algorithm against an adversary that constructs a complete bipartite graph and only reveals new vertices in the partition which does not contain the algorithm’s current solution.

We assume that the algorithm is “sane”, in that it will not reduce the size of its solution. Thus, whenever the algorithm switches solution and changes partition, it incurs recourse of at least twice the size of its current solution. In addition, whenever such a switch occurs, the number of vertices in the graph is at most \( 2t \) times the size of the algorithm’s solution prior to switching. Thus, each partition-changing switch will incur at least \( \frac{1}{t} \) recourse amortized over the size of the revealed graph when the switch occurs.

Then, consider the largest possible increase in the size of the revealed graph between two consecutive switches. The least possible number of vertices at switch \( i \) relative to \( ALG \) is \((1 + t)ALG \), whereas the largest possible number of vertices at switch \( i + 1 \) is \((t^2 + t)ALG \). Thus, there are at most \( \frac{(t^2 + t)ALG}{(1 + t)ALG} = t \) times more vertices at switch \( i + 1 \) than at switch \( i \).
Combining the above two results, we derive a recurrence relation that bounds below the recourse incurred by all switches up to switch $i$ amortized over the size of the revealed graph when switch $i$ occurs: $f(i) = \frac{1}{t} + \frac{1}{t} f(i-1)$. Furthermore, consider the smallest possible amortized recourse incurred by the first switch. The online algorithm must select the first revealed vertex in order to remain competitive. Then, the algorithm can wait until at most $\lceil t \rceil + 1$ vertices are revealed in the other partition before its first switch, at which point it must construct a solution with at least two vertices for an amortized recourse of $\frac{3}{\lceil t \rceil + 1}$. Alternatively, the algorithm can switch earlier and construct a solution with only one vertex in the other partition, for an amortized recourse of at least $\frac{2}{\lceil t \rceil}$. If we assume that $t \leq 2$, then the first switch for any $t$-competitive online algorithm incurs at least 1 amortized recourse. Therefore, we can set the initial value of the aforementioned recurrence relation as $f(1) = 1$.

Solving this recurrence relation, we obtain that the recourse incurred by all switches up to switch $i$ amortized over the size of the revealed graph when switch $i$ occurs is bounded below by $\frac{(t-2)(\frac{1}{t})^{t-1}+1}{t-1}$. This lower bound applies for any online algorithm against the described adversary when said adversary terminates its input sequence after the algorithm's $i$-th switch.

Therefore, for any $1 < t \leq 2$, $\epsilon > 0$, and $t$-competitive deterministic online algorithm, there exists an instance for which the algorithm incurs at least $\frac{1}{t-1} - \epsilon$ amortized recourse.

\[\square\]

### D Maximum Matching Full Proof

**Theorem 4.** The $L$-Greedy algorithm returns a valid matching with competitive ratio $\frac{L+2}{L+1}$ in $O(n^{3.5})$ time, where $n$ is the number of vertices in the final graph.

**Proof.** After applying the late operations on all the augmenting path with at most $2L + 1$ edges, every remaining augmenting path has length at least $2L + 3 = (L + 2) + (L + 1)$, and the ratio of the OPT size to the $L$-Greedy size $\frac{\text{OPT}(P)}{L\text{-Greedy}(P)} \leq \frac{L+2}{L+1}$ on the component $P$.

By selecting $L = \left\lceil \frac{1}{t-1} \right\rceil - 1$, the $L$-Greedy algorithm eliminates all augmenting paths that has length at most $2L + 1 = 2\left\lceil \frac{1}{t-1} \right\rceil - 1$.

Afterwards, every the remaining augmenting path has length at least $2\left\lceil \frac{1}{t-1} \right\rceil + 1$, and the algorithm attains a competitive ratio at most $\frac{\left\lceil \frac{1}{t-1} \right\rceil + 1}{\left\lceil \frac{1}{t-1} \right\rceil}$.

**$t$-competitiveness.** We first show that after switching every augmenting path with length at most $2L + 1$, the $L$-Greedy algorithm attains a competitive ratio of $t$. Consider the disjunctive union of the algorithm’s matching $M$ and the optimal matching $M^*$, $M \Delta M^*$. The disjunctive union consists of connected components. Let $M_i$ and $M_i^*$ be the edges in the $i$-th component from $M$ and from $M^*$, respectively. According to our algorithm, there are two properties of the connected components:

**(P1)** In any component with odd size, $\frac{|M_i^*|}{|M_i|} \leq \frac{\left\lceil \frac{1}{t-1} \right\rceil + 1}{\left\lceil \frac{1}{t-1} \right\rceil}$.

**(P2)** In any component with even size, $|M_i^*| = |M_i|$.

Therefore, the competitive ratio by the $L$-Greedy algorithm is at most $\frac{\sum_i |M_i^*|}{\sum_i |M_i|} \leq \max_i \frac{|M_i^*|}{|M_i|} \leq \frac{\left\lceil \frac{1}{t-1} \right\rceil + 1}{\left\lceil \frac{1}{t-1} \right\rceil} = 1 + \frac{1}{\left\lceil \frac{1}{t-1} \right\rceil} \leq 1 + \frac{1}{1/(t-1)} = 1 + (t-1) = t$. \[\square\]

**Theorem 5.** For the Maximum Cardinality Matching problem in the edge/vertex-arrival model, the $L$-Greedy algorithm is $t$-competitive for any $1 < t < 2$ and incurs at most $\frac{(2-t^*)}{(t^*-1)(3-t^*)} + \frac{t^*-1}{3-t^*}$ amortized recourse, where $t^*$ is the largest number such that $t^* \leq t$ and $t = 1 + \frac{1}{j}$ for some integer $j$. 

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Proof. Consider the connected component generated by the union of edges chosen by $L$-Greedy or by OPT. Let $C_i$ be the components in the graph and $\text{TR}_i$ be the total recourse incurred by the elements in $C_i$ (from very beginning till the end), the amortized recourse given by the whole graph will be upper-bounded by $\max_{C_i} \text{TR}_i$ (Observation 1).

Now, we analyze the upper bound of the amortized recourse for any connected component. First we observe that the path eliminations only happen at odd-size components with length 3, 5, 7, \cdots, 2\lceil \frac{1}{t-1} \rceil - 1) + 1 (note that we only consider the case when $1 < t < 2$, thus $\lceil \frac{1}{t-1} \rceil \geq 2$). Moreover, in our algorithm, we first check the competitiveness before any actual movement. Hence, for an augmenting path with $2n+1$ edges that triggers a path elimination, the recourse incurred by this path elimination is $2n$. Therefore, for such a $2n+1$-edge augmenting path, the total recourse incurred by the $2n+1$ elements in the path is at most $1 + \sum_{i=1}^{n} 2i = 1 + n \cdot (n+1)$ (the 1 is from the first edge in this path, which may be late accepted). Hence, the amount of amortized recourse incurred by this component is at most $1 + n \cdot (n+1)$. Since the amortized recourse incurred by an augmenting path increases as the augmenting path gets longer and the $L$-Greedy algorithm eliminates augmenting paths with length at most $2(\lceil \frac{1}{t-1} \rceil - 1) + 1$, the amortized recourse for a component is at most $\frac{\lceil \frac{1}{t-1} \rceil - 1)(\lceil \frac{1}{t-1} \rceil - 1+1)+1}{2(\lceil \frac{1}{t-1} \rceil - 1)} = \frac{\lceil \frac{1}{t-1} \rceil - 1+1}{2(\lceil \frac{1}{t-1} \rceil - 1)}$.

There are two cases of the target competitive ratio $t$, $t = 1 + \frac{1}{j}$ for some integer $j$ or otherwise. First, we consider the case when $t = 1 + \frac{1}{j}$. In this case, the amortized recourse for a component is at most $\frac{\lceil \frac{1}{t-1} \rceil - 1)(\lceil \frac{1}{t-1} \rceil - 1+1)}{2(\lceil \frac{1}{t-1} \rceil - 1)} = \frac{\lceil \frac{1}{t-1} \rceil - 1+1}{2(\lceil \frac{1}{t-1} \rceil - 1)} = \frac{1}{3-t} \cdot \frac{2-t+t-1}{(t-1)(3-t)} = \frac{2-t+t-1}{2(t-1)} + t \cdot \frac{t-1}{3-t}.

For the case when there is no integer $j$ such that $t = 1 + \frac{1}{j}$, we round down $t$ to the largest $t^* \leq t$ such that $1 = 1 + \frac{1}{j}$ for some integer $j$. By eliminating all augmenting paths that have length at most $\frac{2}{t^*-1} - 1 = \frac{2}{1+\frac{1}{j}} - 1 = 2j - 1$, the amount of incurred amortized recourse is at most $\frac{2-t^*}{(t^*-1)(3-t^*)} + t^* \cdot \frac{t-1}{3-t}$, and the algorithm attains a competitive ratio of $t^* \leq t$.

\[ \frac{(2-t^*)}{(t^*-1)(3-t^*)} \]

Theorem 6. No deterministic $t$-competitive online algorithm can incur amortized recourse less than $\frac{(2-t^*)}{(t^*-1)(3-t^*)}$ in the worst case.

Proof. If $\frac{n+2}{n+1} \leq t < \frac{n+1}{n}$ for some integer $n \geq 1$, we release a sequence of $2n+1$ edges that form a path.

Invariants. Consider any $1 \leq k \leq n$, the following invariants hold for any $t$-competitive algorithm:

1. For a path with length $2k+1$, a $t$-competitive algorithm has to accept $k+1$ edges.
2. For a path with length $2k$, a $t$-competitive algorithm has to accept $k$ edges.
3. When an instance is increased from a $2(k-1)+1$ path to a $2k+1$ path, a $t$-competitive algorithm incurs at least $2k$ amount of recourse.

Given the invariants 1, 2, and 3, the $2n+1$-path increases total recourse with total amount at least $\sum_{k=1}^{n} (2k) = n \cdot (n+1)$. Therefore, any $t$-competitive algorithm incurs at least $\frac{n(n+1)}{2n+1}$ amortized recourse for this $2n+1$-path instance. Let $t^* = \frac{n+2}{n+1} \leq t$. It follows that $n = \frac{2-t^*}{t^*-1}$. The amortized recourse is at least $\frac{n(n+1)}{2n+1} = \frac{2-t^*}{2(2-t^*)+1} = \frac{1}{t^*-1} \cdot \frac{(2-t^*)(2-t^*)(t^*-1)}{2(2-t^*)+1} = \frac{1}{t^*-1} \cdot \frac{2-t^*}{3-t^*} = \frac{(2-t^*)}{(t^*-1)(3-t^*)}$. Therefore, the total amortized recourse is at least $\frac{n(n+1)}{2n+1} = \frac{(2-t^*)}{(t^*-1)(3-t^*)}$.

Proof of the invariants. Now we prove that the invariants are true throughout the instance. Recall that $\frac{n+2}{n+1} \leq t < \frac{n+1}{n}$ and $1 \leq k \leq n$.

1. Proof of (I1): For any path with $2k+1$ edges, an $t$-competitive algorithm should accept at least $\frac{\text{OPT}}{t}$ edges, where $\text{OPT}$ is the cost of the offline optimal solution and has $k+1$ edges. Hence, the online algorithm should accept $\geq \text{OPT} + \frac{1}{t} > \frac{k+1}{(n+1)/n} = \frac{n(k+1)}{n+1} \geq \frac{k(k+1)}{k+1} = k$ edges (the last inequality is
because $k \leq n$). There is a strictly greater and hence the online algorithm should accept $k + 1$ edges for this $2k + 1$-path. Since the optimal solution has $k + 1$ edges, this means that the algorithm must accept exactly $k + 1$ edges.

- Proof of (I2): For any path with $2k$ edges, the optimal solution has $k$ edges. Therefore, a $t$-competitive algorithm should accept at least $\frac{\OPT}{t} = \frac{k}{t}$ edges. Since $t < \frac{n+1}{n}$, then $\frac{k}{t} > \frac{k}{(n+1)/n} = \frac{n\cdot k}{n+1} \geq \frac{k}{k+1} = \frac{(k+1)(k-1)+1}{k+1} = (k-1)+\frac{1}{k+1}$. Since $k > 0$, then $0 < \frac{1}{k+1} < 1$. Therefore, the algorithm should accept at least $(k-1)+\left\lceil \frac{1}{k+1} \right\rceil = k$ edges. Since the optimal solution has $k$ edges, this means that the algorithm must accept exactly $k$ edges.

- The invariant (I3) can be proven by a two-step growing for a $2k-1$ path to a $2k+1$ one. First, for the $2k-1$ path, from (I1), the algorithm accepts $k$ edges. When one more edge arrives and the instance is a $2k$-path, the algorithm has to accept exactly $k$ edges. At this step, the algorithm might incur recourse or not. Finally, when another edge arrives and the instance is a $2k+1$-path, the algorithm has to accept $k+1$ edges, which form a disjoint set with the previously accepts ones. Hence, the algorithm incurs $2k$ recourse at this step (if the algorithm is smart enough to accept the last revealed edge instead of late accepting it).

\[ \square \]

## E Vertex Cover (Full Version)

In this section, we propose a special version of the TaS$_s$ algorithm, Duo-Halve, that attains a competitive ratio of $2 - \frac{2}{\OPT}$ for the MINIMUM VERTEX COVER problem with optimal vertex cover size OPT. During the process, the algorithm maintains a maximal matching $M(X)$ on the current input graph $X$ to construct a solution $\DH(X)$ (we omit the parameter $X$ when the context is clear). The algorithm only selects vertices that are saturated by the matching and rejects as many vertices as possible from the two latest matched edges. It is clear that if we reject 2 such vertices, the competitive ratio of the algorithm is at most $2 - \frac{2}{\OPT}$ since $\OPT \geq |M|$. We show that if we cannot reject 2 such vertices, the optimal solution size must be big so the competitive ratio is still $2 - \frac{2}{\OPT}$ (Theorem 7).

In the following discussion, we use some terminology. Let ME1 and ME2 be the most and the second-most recently matched edges respectively. Also, let $V_{M(X)}$ be the vertices saturated by the maximal matching $M(X)$. The $\DH$ algorithm partitions the vertices into three groups: Group-1: the endpoints of ME1 or ME2, Group-2: the vertices in $V_M$ but not in Group-1, and Group-3: the vertices in $V \setminus V_M$.

A matched edge is full if both of its endpoints are selected by the algorithm $\DH$. Otherwise, the edge is half. The algorithm halves a matched edge $(u, v)$ by producing a valid vertex cover while only accepting either $u$ or $v$. A matched edge $(u, v)$ is halvable if there exists a vertex cover that contains exactly one of $u$ and $v$. A half edge flips if the accept/reject status of its endpoints is swapped. A configuration of a set of edges is a set of accept/reject statuses associated with each endpoint of those edges.

**Duo-Halve Algorithm ($\DH$).** When a new vertex $v$ arrives, if an edge $(p, v)$ is added to $M(X)$, then it introduces a new ME1 (namely $(p, v)$). The algorithm first accepts all Group-2 vertices that are adjacent to $v$. Then, the algorithm decides the assignment of ME1 and ME2 by testing if they can be both halved by the HalveBoth procedure as follows.

If ME1 is half or one of its endpoints is $v$, the $\DH$ algorithm halves ME2 if it is valid giving the current configuration of ME1 or $v$ being accepted. Otherwise, $\DH$ halves ME2 if it is valid by flipping ME1 or late-accepting $p$. Otherwise, in the vertex cover returned by $\DH$, ME1 is full and ME2 is unchanged. (See Figure 1 in Appendix for a detailed flow diagram about this HalveBoth procedure.)

We start our analysis by showing that the $\DH$ algorithm is valid, and that it runs in polynomial time. Intuitively, the algorithm maintains a valid solution as it greedily covers edges using vertices in the maximal matching, with the exception of ME1 and ME2, where it carefully ensures that a feasible configuration is chosen. Furthermore, the most computationally-expensive component of the $\DH$ algorithm, which checks
the validity of a constant number of configurations by looking at the neighborhoods of ME1 and ME2, runs in polynomial time.

The DH algorithm always returns a valid vertex cover in $O(n^3)$ time, where $n$ is the number of vertices in the graph. The DH algorithm always returns a valid vertex cover in $O(n^3)$ time, where $n$ is the number of vertices in the graph.

**Proof.** We prove the validity of the DH algorithm by induction on the size of the input graph. Initially, when the graph is empty, the solution returned by DH is empty and feasible. Suppose that the vertex cover maintained by DH is valid, we show that after the arrival of a new vertex $v$, the new assignment by DH is also valid.

Let $V_M$ and $V$ denote the vertices saturated by the maximum matching maintained by DH and the set of vertices that have been released before the arrival of $v$. There are three classes of the new edges $(u,v)$ incident to the new vertex $v$: 1) $u \in V_M$ but not in ME1 or ME2, 2) $u \in$ ME1 or in ME2, or 3) $u \in V \setminus V_M$.

By the definition of the DH algorithm, all class-1 edges $(u,v)$ will be covered by $u \in V_M$. By the definition of the HalveBoth procedure in the DH algorithm, all new class-2 edges $(u,v)$ will be covered by $v$ (if there is a new matched edge introduced by the arrival of $v$) or by $u$. Class-3 edges only occur when there is a new matched edge $(p,v)$, where $p$ may or may not be $u$. In this case, the HalveBoth procedure makes sure that the edge is covered by $v$ (or by $u$ if the edge $(u,v)$ is added into the matching). Therefore, the DH algorithm returns a valid vertex cover.

Whenever a new vertex $v$ arrives, the DH algorithm checks all its adjacent vertices $u$, accepts them if $(u,v)$ is in class 1, and then runs procedure HalveBoth. The HalveBoth procedure checks the coverage of edges where at least one of their endpoints is in ME1 or ME2 for a constant number of possible configurations of ME1 and ME2 being full or half. Hence, the total time complexity incurred by the arrival of a single vertex is $O(|E|)$ where $|E|$ is the number of edges in the final graph. Therefore, the total time complexity is $O(n^3)$.

After establishing the correctness of the DH algorithm, we describe a simple yet crucial observation about the relationship between $|M|$, OPT, and $\text{DH OPT}$.

**Observation 2.** If $\text{OPT} \geq |M| + 1$, then $\text{DH OPT} \leq 2 - \frac{2}{\text{OPT}}$.

The above observation allows us to show another condition under which $\text{OPT} \geq |M| + 1$ and thus our desired bound of $2 - \frac{2}{\text{OPT}}$ on the competitive ratio is achieved. The contrapositive of this lemma also provides us with a necessary characteristic of any “problematic” instance.

If $\text{OPT} \setminus V_M \neq \emptyset$ or there exists an edge $(u,v)$ in $M$ such that $\{u,v\} \subseteq \text{OPT}$, then $\text{DH OPT} \leq 2 - \frac{2}{\text{OPT}}$.

**Proof.** Assume there exists at least one vertex in OPT but not in ALG. Since OPT must select one vertex from each matched edges in $M$, OPT $\geq |M| + 1$.

If there is a matched edge whose endpoints are both selected by OPT, since OPT must select one vertex from each of the other matched edges, OPT $\geq 2 + (|M| - 1) = |M| + 1$.

By Observation 2, $\text{DH OPT} \leq 2 - \frac{2}{\text{OPT}}$ in both of the cases.

**Corollary 3.** If $2 - \frac{2}{\text{OPT}} < \text{DH OPT}$, then OPT $\subseteq V_M$.

We then present another observation, which helps with our analysis of DH by providing a feasibility guarantee for any newly-revealed ME1.

**Observation 3.** Immediately after an edge is added to the matching, ME1 is halvable by accepting the newly-revealed vertex.
Theorem 7. The DH algorithm is \((2 - \frac{2}{\OPT})\)-competitive.

Proof. We prove the theorem by considering three possible states for DH’s solution: 1) both ME1 and ME2 are half, 2) ME1 is half and ME2 is full, and 3) ME1 is full. In each case, we show that either \(\OPT\) fails to produce a solution in which it accepts only one vertex of ME1, or there is no possible solution contained in \(V_M\) that only accepts one vertex of ME1. In particular, this applies to \(\OPT\) as well, which by Lemma \[\text{Lemma}\] means that our desired bound on the competitive ratio is achieved.

We achieve this by considering the rejected vertices in the neighborhoods of the endpoints of ME1. Specifically, we show that if ME1 is full, then each of these neighborhoods must contain a vertex outside of \(V_M\), and thus \(\OPT\) must either contain a vertex outside of \(V_M\) or both endpoints of ME1.

In the assignment of DH, if both endpoints of ME1 are selected, then the optimal solution must contain at least two vertices in \(ME1 \cup (V \setminus V_M)\), and \(\frac{\DH}{\OPT} \leq 2 - \frac{2}{\OPT}\).

Proof. Assume that ME1 = \((p, v)\) (where \(v\) was revealed later) is full and consider the “blocking set” of each of its endpoints, i.e., the set of rejected vertices in its neighborhood. We partition the possible blocking set vertices into three classes: 1) vertices in \(V_M\) but not in ME2, 2) vertices in ME2, and 3) vertices in \(V \setminus V_M\). We prove the lemma by showing the following claim.

Claim: If ME1 is full, then both the blocking set of \(p\) and of \(v\) contain at least one vertex in \(V \setminus V_M\).

Let \(x\) and \(y\) be a class-3 vertex in the blocking set of \(p\) and of \(v\), respectively. If the claim is true, then the optimal solution has to choose at least two vertices in \(\{p, v, x, y\}\), since the three edges \((p, v)\), \((p, x)\), and \((y, v)\) must be covered. This proves the lemma.

Now, we prove the correctness of the claim by showing that the blocking sets of \(p\) and of \(v\) cannot contain class-1 vertices, and cannot contain only a class-2 vertex. Note that a blocking set cannot contain more than one class-2 vertex, as that would imply that both endpoints of ME2 are rejected,

First, consider any class-1 blocking set vertex \(x\), which is in \(V_M\) but not in ME2. Immediately prior to \((p, v)\) being added to the matching, all edges \((p, x)\) are covered by \(x\) since \(x\) must be accepted. When \(v\) is revealed, all of the considered vertices \(x\) are (late-)accepted, and all edges \((v, x)\) are covered by \(x\). Furthermore, none of the considered vertices will be late-rejected after \((p, v)\) is added to the matching (by definition of DH). Thus, all edges between \(\{p, v\}\) and the considered vertices are covered. Hence, none of the considered vertices can be in the blocking set of either \(p\) or \(v\).

Next, assume that either blocking set contains only a vertex in ME2. By the definition of the HalveBoth function in the DH algorithm, if a valid configuration exists such that ME1 is half, then it would have been chosen by DH. However, if either \(p\) or \(v\) is blocked only by a vertex in ME2, ME1 is halvable by making ME2 full. This is a contradiction, so each blocking set must contain at least one vertex outside of \(V_M\), and the claim is proven.

Corollary 4. In the assignment of DH, if ME1 is full, then \(\frac{\DH}{\OPT} \leq 2 - \frac{2}{\OPT}\).

After taking care of the above preliminaries, we show that our desired bound of \(2 - \frac{2}{\OPT}\) on the competitiveness of the DH algorithm is indeed attainable.

Theorem 7. The DH algorithm is \((2 - \frac{2}{\OPT})\)-competitive.
First, consider the vertices in $V_M$ that are not in ME1. If any of these vertices is in the blocking set of either endpoint of ME2, then there must be at least one half edge in $M$ other than ME1, which implies that $\text{DH} \leq 2|M| - 2 \leq 2\text{OPT} - 2$.

Then, assume that either blocking set contains only a vertex in ME1. By the definition of the HalveBoth function in the DH algorithm, if a valid configuration exists such that both ME1 and ME2 are half (with all other edges in $M$ full as per the previous paragraph), then it would have been chosen by DH. Therefore, any vertex cover must contain either both vertices in ME1 or both vertices in ME2. Thus, $\text{OPT} \geq |M| + 1$, which implies $\text{DH} \leq 2 - \frac{2}{\text{OPT}}$ by Observation 2.

Finally, if the blocking set for each vertex in ME2 contains a vertex outside of $V_M$, $\text{OPT}$ must contain either a vertex outside of $V_M$ or both vertices in ME2. This implies that $\text{OPT} \geq |M| + 1$ and thus $\text{DH} \leq 2 - \frac{2}{\text{OPT}}$ by Observation 2.

Therefore there is no instance such that $\text{DH} \text{OPT} > 2 - \frac{2}{\text{OPT}}$, and thus the DH algorithm is $(2 - \frac{2}{\text{OPT}})$-competitive.

With the upper bound on the competitive ratio of the DH algorithm proven, we then show an upper bound on the amortized recourse incurred by DH. Prior to doing so, we show a restriction on the possible changes to the DH algorithm’s solution when a new edge is added to the maximal matching $M$. By extension, this limits the possible number of late operations incurred in this scenario.

When an edge shifts from ME1 to ME2, it either goes from full to half or remains unchanged.

**Proof.** First, note that when a shift occurs, it is because a new edge ME1 is added to the matching, and this edge is incident to the newly-revealed vertex $v$.

Let $C$ denote the vertices accepted by DH prior to the arrival of $v$. The DH algorithm maintains a valid vertex cover throughout the process (by Lemma F), so by Observation 3, $C \cup \{v\}$ is a valid vertex cover. Therefore, the DH algorithm accepts $v$ and does not change the assignment of ME2 if ME2 was half (by the procedure HalveBoth). That is, the accept/reject status of endpoints of ME2 is not swapped, and ME2 cannot go from half to full.

Therefore, when an edge shifts from ME1 to ME2, it either goes from full to half or remains unchanged.

In particular, Lemma F tells us that a half edge shifting from ME1 to ME2 will not incur any late operations (i.e. it won’t become full and it won’t flip).

In the worst case, there can be multiple late accepts on Group-2 vertices, and 4 late operations on vertices in ME1 and ME2. In the following theorem, we use a potential function to prove that the amortized late operations per vertex is at most $\frac{10}{3}$.

**Theorem 8.** The amortized recourse incurred by DH is at most $\frac{10}{3}$.

**Proof.** We prove the theorem by using a potential function. For each vertex $v_i$, let $\text{LO}_i$ denote the number of incurred late operations when it arrives. Assume there exists a potential function $\Phi$ such that the potential after the arrival of $v_i$ and all the incurred late operations is $\Phi_i = \Phi_0$, and $\text{LO}_i + \Phi_i - \Phi_{i-1} \leq c$. Then, the total number of late operations is upper-bounded by $c \cdot n + \Phi_0 - \Phi_n \leq c \cdot n$, where $n$ is the number of vertices. Thus, the amortized recourse is bounded above by $c$.

At any given point in the input sequence where the matching constructed by DH contains at least 2 edges, the status of ME1 = $(p, v)$ and ME2 = $(u, w)$ is characterized by one of the following 6 states, where $v$ is the vertex of ME1 revealed last:

1. $u, w$, and $p$ are accepted
2. $u, w$, and $v$ are accepted
3. $u, w, p$, and $v$ are accepted
4. $p$ and one of $u$ and $w$ are accepted
5. $v$ and one of $u$ and $w$ are accepted
6. $p$, $v$, and one of $u$ and $w$ are accepted

Furthermore, define a half edge $(u, v)$ as being free if it is halvable both by accepting $u$ and by accepting $v$. Also, define a half edge as being expired if it is neither ME1 nor ME2. Finally, define $A$ as the set of vertices accepted by $\mathcal{D}H$. Then, define the potential function $\Phi$ as

$$
\Phi := \left| \{(u, v) \mid (u, v) \text{ expired half} \} \right| + \frac{1}{3} |A \cap (\text{ME1} \cup \text{ME2})| + \frac{2}{3} \cdot 1[\text{free half ME2}]
$$

Now, we show that, for any possible state transition triggered by a newly-revealed vertex, the number of incurred late operations $LO$ added to the change in potential $\Delta \Phi$ is bounded above by $\frac{10}{3}$. Note that, for any newly-revealed vertex $v$, $v$ may be adjacent to $k \geq 0$ rejected vertices that are matched by some expired edge. This incurs $k$ late operations, but also decreases $\Phi$ by $k$, so this may be ignored when computing LO + $\Delta \Phi$.

- $1 \to 1$, without shift: $LO + \Delta \Phi \leq 0 + (0 + 0 + 0) = 0$
- $1 \to 1$, with shift: not possible, because an edge shifting from ME1 to ME2 won’t go from half to full (Lemma $E$)
- $1 \to 2$, without shift: $LO + \Delta \Phi \leq 2 + (0 + 0 + 0) = 2$
- $1 \to 2$, with shift: not possible, because an edge shifting from ME1 to ME2 won’t go from half to full (Lemma $E$)
- $1 \to 3$, without shift: $LO + \Delta \Phi \leq 1 + (0 + \frac{1}{3} + 0) = \frac{4}{3}$
- $1 \to 3$, with shift: not possible, because ME1 is always halvable after a shift (Observation $3$)
- $1 \to 4$, without shift: $LO + \Delta \Phi \leq 1 + (0 - \frac{1}{3} + \frac{2}{3}) = \frac{4}{3}$
- $1 \to 4$, with shift: $LO + \Delta \Phi \leq 1 + (0 - \frac{1}{3} + \frac{2}{3}) = \frac{4}{3}$
- $1 \to 5$, without shift: $LO + \Delta \Phi \leq 3 + (0 - \frac{1}{3} + \frac{2}{3}) = \frac{10}{3}$
- $1 \to 5$, with shift: $LO + \Delta \Phi \leq 0 + (0 - \frac{1}{3} + \frac{2}{3}) = \frac{1}{3}$
- $1 \to 6$, without shift: not possible, because $\mathcal{D}H$ won’t halve ME2 if ME1 is full
- $1 \to 6$, with shift: not possible, because $\mathcal{D}H$ prioritizes halving ME1 and this is always possible after a shift (Observation $3$)
- $2 \to 1$, without shift: $LO + \Delta \Phi \leq 2 + (0 + 0 + 0) = 2$
- $2 \to 1$, with shift: not possible, because an edge shifting from ME1 to ME2 won’t go from half to full (Lemma $E$)
- $2 \to 2$, without shift: $LO + \Delta \Phi \leq 0 + (0 + 0 + 0) = 0$
- $2 \to 2$, with shift: not possible, because an edge shifting from ME1 to ME2 won’t go from half to full (Lemma $E$)
- $2 \to 3$, without shift: $LO + \Delta \Phi \leq 1 + (0 + \frac{1}{3} + 0) = \frac{4}{3}$
2 \rightarrow 3, with shift: not possible, because ME1 is always halvable after a shift (Observation 3)

2 \rightarrow 4, without shift: \( LO + \Delta \Phi \leq 3 + (0 - \frac{1}{3} + \frac{2}{3}) = \frac{10}{3} \)

2 \rightarrow 4, with shift: \( LO + \Delta \Phi \leq 1 + (0 - \frac{1}{3} + \frac{2}{3}) = \frac{4}{3} \)

2 \rightarrow 5, without shift: \( LO + \Delta \Phi \leq 1 + (0 - \frac{1}{3} + \frac{2}{3}) = \frac{4}{3} \)

2 \rightarrow 5, with shift: \( LO + \Delta \Phi \leq 0 + (0 - \frac{1}{3} + \frac{2}{3}) = \frac{1}{3} \)

2 \rightarrow 6, without shift: not possible, because DH won’t halve ME2 if ME1 is full

2 \rightarrow 6, with shift: not possible, because DH prioritizes halving ME1 and this is always possible after a shift (Observation 3)

3 \rightarrow 1, without shift: \( LO + \Delta \Phi \leq 1 + (0 - \frac{1}{3} + 0) = \frac{2}{3} \)

3 \rightarrow 1, with shift: \( LO + \Delta \Phi \leq 1 + (0 - \frac{1}{3} + 0) = \frac{2}{3} \)

3 \rightarrow 2, without shift: \( LO + \Delta \Phi \leq 1 + (0 - \frac{1}{3} + 0) = \frac{2}{3} \)

3 \rightarrow 2, with shift: \( LO + \Delta \Phi \leq 0 + (0 - \frac{1}{3} + 0) = -\frac{1}{3} \)

3 \rightarrow 3, without shift: \( LO + \Delta \Phi \leq 0 + (0 + 0 + 0) = 0 \)

3 \rightarrow 3, with shift: not possible, because ME1 is always halvable after a shift (Observation 3)

3 \rightarrow 4, without shift: \( LO + \Delta \Phi \leq 2 + (0 - \frac{2}{3} + \frac{2}{3}) = 2 \)

3 \rightarrow 4, with shift: \( LO + \Delta \Phi \leq 2 + (0 - \frac{2}{3} + \frac{2}{3}) = 2 \)

3 \rightarrow 5, without shift: \( LO + \Delta \Phi \leq 2 + (0 - \frac{2}{3} + \frac{2}{3}) = 2 \)

3 \rightarrow 5, with shift: \( LO + \Delta \Phi \leq 1 + (0 - \frac{2}{3} + \frac{2}{3}) = 1 \)

3 \rightarrow 6, without shift: not possible, because DH won’t halve ME2 if ME1 is full

3 \rightarrow 6, with shift: not possible, because DH prioritizes halving ME1 and this is always possible after a shift (Observation 3)

4 \rightarrow 1, without shift: \( LO + \Delta \Phi \leq 1 + (0 + \frac{1}{3} + 0) = \frac{4}{3} \)

4 \rightarrow 1, with shift: not possible, because an edge shifting from ME1 to ME2 won’t go from half to full (Lemma E)

4 \rightarrow 2, without shift: \( LO + \Delta \Phi \leq 3 + (0 + \frac{1}{3} + 0) = \frac{10}{3} \)

4 \rightarrow 2, with shift: not possible, because an edge shifting from ME1 to ME2 won’t go from half to full (Lemma E)

4 \rightarrow 3, without shift: \( LO + \Delta \Phi \leq 2 + (0 + \frac{2}{3} + 0) = \frac{8}{3} \)

4 \rightarrow 3, with shift: not possible, because ME1 is always halvable after a shift (Observation 3)

4 \rightarrow 4, without shift: \( LO + \Delta \Phi \leq 2 + (0 + 0 - \frac{2}{3}) = \frac{4}{3} \)

4 \rightarrow 4, with shift: \( LO + \Delta \Phi \leq 1 + (1 + 0 + \frac{2}{3}) = \frac{8}{3} \)

4 \rightarrow 5, without shift: \( LO + \Delta \Phi \leq 4 + (0 + 0 - \frac{2}{3}) = \frac{10}{3} \)
• 4 → 5, with shift: \( \text{LO} + \Delta \Phi \leq 0 + \left(1 + 0 + \frac{2}{3}\right) = \frac{5}{3} \)

• 4 → 6, without shift: \( \text{LO} + \Delta \Phi \leq 3 + \left(0 + \frac{1}{3} - \frac{2}{3}\right) = \frac{8}{3} \)

• 4 → 6, with shift: not possible, because \( DH \) prioritizes halving ME1 and this is always possible after a shift (Observation 3)

• 5 → 1, without shift: \( \text{LO} + \Delta \Phi \leq 3 + \left(0 + \frac{1}{3} + 0\right) = \frac{10}{3} \)

• 5 → 1, with shift: not possible, because an edge shifting from ME1 to ME2 won't go from half to full (Lemma E)

• 5 → 2, without shift: \( \text{LO} + \Delta \Phi \leq 1 + \left(0 + \frac{1}{3} + 0\right) = \frac{4}{3} \)

• 5 → 2, with shift: not possible, because an edge shifting from ME1 to ME2 won't go from half to full (Lemma E)

• 5 → 3, without shift: \( \text{LO} + \Delta \Phi \leq 2 + \left(0 + \frac{2}{3} + 0\right) = \frac{8}{3} \)

• 5 → 3, with shift: not possible, because ME1 is always halvable after a shift (Observation 3)

• 5 → 4, without shift: \( \text{LO} + \Delta \Phi \leq 4 + \left(0 + 0 - \frac{2}{3}\right) = \frac{10}{3} \)

• 5 → 4, with shift: \( \text{LO} + \Delta \Phi \leq 1 + \left(1 + 0 + \frac{2}{3}\right) = \frac{8}{3} \)

• 5 → 5, without shift: \( \text{LO} + \Delta \Phi \leq 2 + \left(0 + 0 - \frac{2}{3}\right) = \frac{4}{3} \)

• 5 → 5, with shift: \( \text{LO} + \Delta \Phi \leq 0 + \left(1 + 0 + \frac{2}{3}\right) = \frac{5}{3} \)

• 5 → 6, without shift: \( \text{LO} + \Delta \Phi \leq 3 + \left(0 + \frac{1}{3} - \frac{2}{3}\right) = \frac{8}{3} \)

• 5 → 6, with shift: not possible, because \( DH \) prioritizes halving ME1 and this is always possible after a shift (Observation 3)

• 6 → 1, without shift: \( \text{LO} + \Delta \Phi \leq 2 + \left(0 + 0 + 0\right) = 2 \)

• 6 → 1, with shift: \( \text{LO} + \Delta \Phi \leq 1 + \left(1 + 0 + 0\right) = 2 \)

• 6 → 2, without shift: \( \text{LO} + \Delta \Phi \leq 2 + \left(0 + 0 + 0\right) = 2 \)

• 6 → 2, with shift: \( \text{LO} + \Delta \Phi \leq 0 + \left(1 + 0 + 0\right) = 1 \)

• 6 → 3, without shift: \( \text{LO} + \Delta \Phi \leq 1 + \left(0 + \frac{1}{3} + 0\right) = \frac{4}{3} \)

• 6 → 3, with shift: not possible, because ME1 is always halvable after a shift (Observation 3)

• 6 → 4, without shift: \( \text{LO} + \Delta \Phi \leq 3 + \left(0 - \frac{1}{3} - \frac{2}{3}\right) = 2 \)

• 6 → 4, with shift: \( \text{LO} + \Delta \Phi \leq 2 + \left(1 - \frac{1}{3} + \frac{2}{3}\right) = \frac{10}{3} \)

• 6 → 5, without shift: \( \text{LO} + \Delta \Phi \leq 3 + \left(0 - \frac{1}{3} - \frac{2}{3}\right) = 2 \)

• 6 → 5, with shift: \( \text{LO} + \Delta \Phi \leq 1 + \left(1 - \frac{1}{3} + \frac{2}{3}\right) = \frac{7}{3} \)

• 6 → 6, without shift: \( \text{LO} + \Delta \Phi \leq 2 + \left(0 + 0 - \frac{2}{3}\right) = \frac{4}{3} \)

• 6 → 6, with shift: not possible, because \( DH \) prioritizes halving ME1 and this is always possible after a shift (Observation 3)
Thus, for any possible state transition, $LO + \Delta \Phi \leq \frac{10}{3}$. Furthermore, $\Phi_0 = 0$ and $\Phi_i \geq 0$. Therefore, the amortized recourse incurred by $DH$ is bounded above by $\frac{10}{3}$.

After establishing an upper bound on the amortized recourse incurred by the $DH$ algorithm, we show a lower bound by constructing a family of instances that alternates between incurring a late accept on a Group-2 vertex, and 4 late operations on ME1 and ME2. This is illustrated in Figure 2 (see Appendix).

For any $\varepsilon > 0$, there exists an instance such that $DH$ incurs amortized recourse strictly greater than $\frac{5}{2} - \varepsilon$.

Finally, we show that the upper bound on the competitive ratio provided by Theorem 7 is tight not only for the $DH$ algorithm, but for a class of online algorithms defined below. This is done by using an adversary that constructs an arbitrary number of triangles that all share a common vertex $v$. This common vertex is revealed last, so all edges not incident to $v$ are added to the matching and $v$ is rejected. This is illustrated in Figure 3.

**Definition 4.** An algorithm for vertex cover is incremental matching-based if it maintains a maximal matching throughout the process in an incremental manner, and its solution only contains vertices saturated by the matching.

**Theorem 9.** No deterministic incremental matching-based algorithm achieves a competitive ratio smaller than $2 - \frac{2}{OPT}$.

**Proof.** Consider any incremental matching-based algorithm against an instance where $k$ disconnected edges are revealed via their endpoints. By nature of the incremental matching-based algorithm, each of these $k$ edges will be added to the matching, and at least one vertex from each pair will be accepted. Then, a final vertex is revealed, adjacent to all previously-revealed vertices. The incremental matching-based algorithm will not accept this vertex, as it is not matched, but must accept all other vertices for a vertex cover of size $2k$. However, the optimal solution consists of the last revealed vertex, and one vertex from each pair. Thus, against this instance, any incremental matching-based algorithm will be $\frac{2k}{k+1} = 2 - \frac{2}{OPT}$ times worse than the optimal solution. Therefore, no incremental matching-based algorithm can achieve a competitive ratio smaller than $2 - \frac{2}{OPT}$. □