Quantum chaos and regularity in $\Phi^4$ theory
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We check the eigenvalue spectrum of the $\Phi^4_{1+1}$ Hamiltonian against Poisson or Wigner behavior predicted from random matrix theory. We discuss random matrix theory as a tool to discriminate the validity of a model Hamiltonian compared to an analytically solvable Hamiltonian or experimental data.

1. Motivation

The fluctuation properties of the eigenvalues of a Hamiltonian give much insight into the dynamics of the underlying system. In particular, the so-called nearest-neighbor spacing distribution $P(s)$, i.e., the distribution of spacings $s$ between adjacent eigenvalues plays an important role. According to the Bohigas-Giannoni-Schmit conjecture [1], quantum systems whose classical counterparts are chaotic have a $P(s)$ given by random matrix theory (RMT) whereas systems whose classical counterparts are integrable obey a Poisson distribution, $P(s) = e^{-s}$. In this sense, the form of $P(s)$ characterizes quantum chaos.

Today we know a number of physical systems where Wigner and Poisson behavior is observed [2]. Here we test a one-dimensional chain of $N_s$ coupled harmonic oscillators, with anharmonic perturbation [3]. Its Euclidean action is given by

$$S = \int dt \sum_{n=1}^{N_s} \left[ \frac{1}{2} \dot{\phi}_n^2 + \frac{\Omega_n^2}{2} (\phi_{n+1} - \phi_n)^2 + \frac{\Omega_0^2}{2} \phi_n^2 + \frac{\lambda}{2} \phi_n^4 \right]. \quad (1)$$

In the continuum formulation it corresponds to the scalar $\Phi^4_{1+1}$ model,

$$S = \int dt \int dx \left[ \frac{1}{2} (\frac{\partial \Phi}{\partial t})^2 + \frac{1}{2} (\nabla_x \Phi)^2 + \frac{m^2}{2} \Phi^2 + \frac{g}{4!} \Phi^4 \right]. \quad (2)$$

Introducing a space-time lattice with lattice spacing $a_s$ and $a_t$, this action becomes

$$S = \sum_{n=1}^{N_s} \sum_{k=0}^{N_t-1} a_t a_s \left[ \frac{1}{2} (\Phi(x, t_{k+1}) - \Phi(x, t_k))^2 \frac{\Omega_0}{a_s} + \frac{1}{2} (\Phi(x_{n+1}, t_k) - \Phi(x_n, t_k))^2 \frac{\Omega_n}{a_t} + \frac{m^2}{2} \Phi^2(x, t_k) + \frac{g}{4!} \Phi^4(x, t_k) \right]. \quad (3)$$

The actions given by Eq.(3) and Eq.(1) can be identified by posing $\phi = \sqrt{a_s} \Phi$, $\Omega = 1/a_s$, $\Omega_0 = m$, and $\lambda/2 = g/4!$.

For the underlying real and symmetric matrix one expects a correspondence to the Gaussian orthogonal ensemble (GOE). The RMT result for $P(s)$ is quite complicated; it can be expressed in terms of so-called prolate spheroidal functions, see Ref. [4] where $P(s)$ has also been tabulated.

A very good approximation to $P(s)$ is provided by

$$P(s) = \pi \frac{\text{se}^{-\pi s^2}}{2} \quad (4)$$

which is the Wigner surmise for the GOE.

We computed the eigenvalues of the Hamiltonian in Eq. (3) via the Monte Carlo Hamiltonian method using a stochastic basis [5]. This is a concept to calculate eigenvalues and eigenstates (in
Figure 1. Nearest-neighbor spacing distribution $P(s)$ for a chain with $N_s = 9$ harmonic oscillators from the analytical spectrum (left plot) and from the stochastic method (right plot). The Poisson and the Wigner distribution for the GOE are inserted.

Figure 2. $P(s)$ for a chain with $N_s = 3, 5, 7, 9$ anharmonic oscillators with $\lambda = 1$ using a stochastic basis.
some energy window) of field theories or many-body systems.

To construct $P(s)$, one first has to “unfold” the spectra [1]. This procedure is a local rescaling of the energy scale so that the mean level spacing is equal to unity on the unfolded scale. Ensemble and spectral averages (the latter is possible because of the spectral ergodicity property of RMT) are only meaningful after unfolding.

2. Results

Figure 1 shows the nearest-neighbor spacing distribution $P(s)$ for the chain of harmonic oscillators with $\lambda = 0$. This case reduces to the Klein-Gordon model which is classically integrable and thus a free theory. The quantized system of $N_s$ oscillators has a highly degenerate spectrum. We keep in mind that a single harmonic oscillator is an exceptional situation which would lead to a $\delta$-function at $s = 1$ after formal unfolding. The left plot in Fig. 1 depicts $P(s)$ of the analytically known spectrum for $N_s = 9$ oscillators. It might consist of a washed out peak from the harmonic oscillator spacing and of a peak around the origin from the Poisson distribution of the lifted degeneracies. The right plot in Fig. 1 presents $P(s)$ for the corresponding spectrum obtained from stochastic basis (breaking parity and translational invariance) with 1000 elements. It clearly exhibits Wigner behavior with the underlying symmetry of the GOE. It seems that the "holes" in the matrix produced by the method give effectively a random matrix. Although the individual exact eigenvalues are well reproduced [5] their fluctuations are different leading to a completely different dynamics.

Figure 2 shows the nearest-neighbor spacing distribution $P(s)$ for the oscillator chain when the anharmonic term is switched on, $\lambda = 1$. We vary the length of the chain from $N_s = 3$ to $N_s = 9$ and take $100 - 200$ states for the stochastic basis (with parity symmetry implemented) into account. The plots are inconclusive and might show a trend to uncorrelated eigenvalues compared to $\lambda = 0$ which could be an effect from the deeper quartic potential. It will be interesting to see if this is enhanced with increased $\lambda$.

In summary, we have analyzed the spectrum of a finite chain of oscillators, with and without a quartic coupling. The infinitely long chain of harmonic oscillators is equivalent to the Klein-Gordon model being a free theory. Its spectrum is known analytically also for finite $N_s$. We tested a stochastic method and obtained a spectrum corresponding to a Wigner distribution in contrast to the theoretical expectation. Only preliminary results could be collected for the anharmonic oscillators and conclusions have to be drawn in the future increasing the stochastic basis and the coupling space.

This study is part of a general concept where random matrix theory is used to distinguish between the quality of a model compared to a theory or experiment. An analysis in this respect is being performed with a quark potential model and the experimental hadron spectrum [6].

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