Supplementary Appendix to
New HEAVY Models for Fat-Tailed Realized
Covariances and Returns

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Appendix A: Proofs

We use the following matrix calculus results for a general matrix $X$,

$$\begin{align*}
    d X^{-1} &= -X^{-1} (d X) X^{-1}, \\
    \text{tr}(A'B) &= \text{vec}(A)' \text{vec}(B), \\
    b \otimes a &= \text{vec}(ab'),
\end{align*}$$

with $a, b \in \mathbb{R}^{k \times 1}$, and $\text{tr}()$ denoting the trace. See for instance Abadir and Magnus (2005) for these and other useful results.

**Proof of Proposition 1:** The general form of the score is given by (8) and (9). The relevant parts of the log-likelihood that depend on $V_t$ are

$$\begin{align*}
    \mathcal{L}_{y,t} &= -\frac{1}{2} \log |V_t| - \frac{\nu_0 + k}{2} \log \left(1 + \frac{y_t' V_t^{-1} y_t}{\nu_0 - 2}\right) = -\frac{1}{2} \log |V_t| - \frac{\nu_0 + k}{2} \log(\tilde{\nu}_t), \\
    \mathcal{L}_{RC,t} &= -\frac{\nu_1}{2} \log |V_t| - \frac{\nu_1 + \nu_2}{2} \log \left| I_k + \frac{\nu_1}{\nu_2 - k - 1} V_t^{-1} RC_t \right| = -\frac{\nu_1}{2} \log |V_t| - \frac{\nu_1 + \nu_2}{2} \log |\tilde{W}_t|,
\end{align*}$$

with $\tilde{\nu}_t = (1 + (\nu_0 - 2)^{-1} y_t' V_t^{-1} y_t)$ and $\tilde{W}_t = (I_k + \nu_1 (\nu_2 - k - 1)^{-1} V_t^{-1} RC_t)$. Using the matrix calculus results above, we obtain

$$\begin{align*}
    d \mathcal{L}_{y,t} &= -\frac{1}{2} \text{tr}(V_t^{-1} d V_t) - \frac{\nu_0 + k}{2} \tilde{\nu}_t^{-1} d \tilde{\nu}_t \\
    &= -\frac{1}{2} \text{tr}(V_t^{-1} d V_t) - \frac{\nu_0 + k}{2} \tilde{\nu}_t^{-1} \left( d \log V_t | y_t, V_t^{-1} RC_t \right) \\
    &= -\frac{1}{2} (\text{vec } V_t^{-1})' \frac{d \text{vec } V_t}{\nu_0 - 2} + \frac{\nu_0 + k}{2} \tilde{\nu}_t^{-1} \left( d \text{vec } V_t | y_t, V_t^{-1} RC_t \right) \\
    &= -\frac{1}{2} (\text{vec } V_t^{-1})' \frac{d \text{vec } V_t}{\nu_0 - 2} + \frac{\nu_0 + k}{2} \tilde{\nu}_t^{-1} \
    \quad \times \text{vec}(V_t^{-1} y_t V_t^{-1})',
\end{align*}$$

such that

$$\frac{\partial \mathcal{L}_{y,t}}{\partial \text{vec } V_t} = -\frac{1}{2} \text{vec } V_t^{-1} + \frac{\nu_0 + k}{\nu_0 - 2} \tilde{\nu}_t^{-1} \text{vec}(V_t^{-1} y_t V_t^{-1}).$$

Note that we have dealt with $V_t$ in the above derivations as a general rather than a symmetric matrix, for reasons explained in the main text. Omitting the vec operator and rewriting yields the desired result.

For $d \mathcal{L}_{RC,t}$ we have

$$\begin{align*}
    d \mathcal{L}_{RC,t} &= -\frac{\nu_1}{2} \text{tr}(V_t^{-1} d V_t) - \frac{\nu_1 + \nu_2}{2} \text{tr} \left( \tilde{W}_t^{-1} d \tilde{W}_t \right) \\
    &= -\frac{\nu_1}{2} (\text{vec } V_t^{-1})' \frac{d \text{vec } V_t}{\nu_2 - k - 1} + \frac{\nu_1 + \nu_2}{2} \text{tr} \left( \tilde{W}_t^{-1} d \tilde{W}_t | V_t^{-1} RC_t \right) \\
    &= -\frac{\nu_1}{2} (\text{vec } V_t^{-1})' \frac{d \text{vec } V_t}{\nu_2 - k - 1} + \frac{\nu_1 + \nu_2}{2} \text{tr} \left( \tilde{W}_t^{-1} d \tilde{W}_t | V_t^{-1} V_t^{-1} RC_t \right) \\
    &= -\frac{\nu_1}{2} (\text{vec } V_t^{-1})' \frac{d \text{vec } V_t}{\nu_2 - k - 1} + \frac{\nu_1 + \nu_2}{2} \text{vec} \left( \tilde{W}_t^{-1} V_t^{-1} RC_t \right) | V_t^{-1} \tilde{W}_t^{-1} V_t^{-1}).
\end{align*}$$
Consequently,
\[
\frac{\partial L_{RC,t}}{\partial \text{vec} V_t} = -\frac{\nu_1}{2} \text{vec} V_t^{-1} + \frac{\nu_1 + \nu_2}{2} \text{vec} \left( \frac{\nu_1}{\nu_2 - k - 1} V_t^{-1} R C_t \tilde{W}_t^{-1} V_t^{-1} \right).
\]

Again, removing the vec operator yields the desired result.

**Proof of Proposition 2:** We can rewrite (4) using (11) as
\[
V_{t+1} = \Omega + (\beta - \alpha) V_t + \alpha \frac{\nu_2 y_t}{\nu_1} + \alpha \frac{\nu_1 + \nu_2}{(\nu_1 + 1)(\nu_2 - k - 1)} R C_t \left( I_k + \frac{\nu_1}{\nu_2 - k - 1} \right)^{-1}.
\]
(A.1)

As \( \beta > \alpha > 0 \), the sum of the first two terms \( \Omega + (\beta - \alpha) V_t \) is positive definite if \( V_t \) is positive definite. As \( \alpha > 0 \), the third term is positive semi-definite. Finally, given that \( R C_t \) is positive semi-definite for every \( t \), also the last term is positive semi-definite. To see this, note that \( \nu_2 > k + 1 \) and let \( k_t \) denote the rank of \( R C_t \), with \( R C_t = U_t S_t U_t' \) the singular value decomposition of \( R C_t \), with \( U_t \) and \( S_t \) a \( k \times k_t \) and a \( k_t \times k_t \) matrix, respectively. For \( a \in \mathbb{R}^{k \times 1} \), \( c = \nu_1 (\nu_2 - k - 1)^{-1} > 0 \), and positive definite \( V_t \), we have
\[
a'R C_t (I_k + c V_t^{-1} R C_t)^{-1} a = a' U_t S_t U_t' (V_t + c U_t S_t U_t')^{-1} V_t a =
\]
\[
a'R C_t (I_k + c V_t^{-1} R C_t)^{-1} a = a'R C_t (I_k + c V_t^{-1} R C_t)^{-1} U_t' V_t a =
\]
\[
a'R C_t (I_k + c V_t^{-1} R C_t)^{-1} U_t' V_t a =
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a'R C_t (I_k + c V_t^{-1} R C_t)^{-1} U_t' V_t a =
\]
\[
a'R C_t (I_k + c V_t^{-1} R C_t)^{-1} U_t' V_t a = 0,
\]
because both \( S_t \) and \( U_t' V_t^{-1} U_t \) are positive definite for every \( t \). The proof then follows by induction from the assumption that \( V_t \) is positive definite.

**Proof of Proposition 3:** Let \( e_i \) and \( v_i^* \) denote the \( i \)th column of the \( k \times k \) unit matrix and of \( V^{-1} \), respectively. Define \( J = V^{-1} \otimes V^{-1} \). Note that the columns of \( S \) are of the form \( (1 + \delta_{i,j})(e_i \otimes e_j + e_j \otimes e_i) \), with \( \delta_{i,j} \) denoting the Kronecker delta. If \( S_\perp \) denotes the null space of \( S \), i.e., \( S_\perp S = 0 \), then we can take the columns of \( S_\perp \) equal to \((e_i \otimes e_j - e_j \otimes e_i) \) for \( i \neq j \). Define \( S = (S, S_\perp) \), where \( S \) is invertible. Using these definitions, (12) holds if
\[
0 = J^{-1} \text{vec}(\nabla) - S(S' J S)^{-1} S' \text{vec}(\nabla) = (J^{-1} - S(S' J S)^{-1} S') \text{vec}(\nabla) = J^{-1}(1 - J S(S' J S)^{-1} S) S \text{vec}(\nabla) = J^{-1}(S')^{-1} (S' S - S' J S(S' J S)^{-1} S' S) \text{vec}(\nabla) = J^{-1}(S')^{-1} \left( \begin{pmatrix} S' S & S' S \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} S' S \\ 0 \end{pmatrix} \begin{pmatrix} S' S \\ 0 \end{pmatrix} \right) \text{vec}(\nabla) = J^{-1}(S')^{-1} \left( \begin{pmatrix} S' S \\ 0 \end{pmatrix} \begin{pmatrix} S' S \\ 0 \end{pmatrix} \right) \text{vec}(\nabla),
\]
which is true for arbitrary \( \text{vec}(\nabla) \) if and only if \( S_\perp J S = 0 \). Given the form of \( J, S \), and \( S_\perp \), we have for
\( i \neq j \) that
\[
S_i' J S = (e_i \otimes e_j - e_j \otimes e_i)^{\prime} \frac{(e_k \otimes e_\ell + e_\ell \otimes e_k)}{1 + \delta_{h=\ell}} = (v_i^{\ast} \otimes v_j^{\ast} - v_j^{\ast} \otimes v_i^{\ast})^{\prime} \frac{(e_k \otimes e_\ell + e_\ell \otimes e_k)}{1 + \delta_{h=\ell}} \\
= \frac{v_{ik}^{\ast} v_{j\ell}^{\ast} - v_{j\ell}^{\ast} v_{ik}^{\ast} + v_{i\ell}^{\ast} v_{jk}^{\ast} - v_{jk}^{\ast} v_{i\ell}^{\ast}}{1 + \delta_{h=\ell}} = 0.
\]

**Proof of Proposition 4:** Note that the recursion in (4) can be written as
\[
V_{t+1} = \Omega + \beta V_t + \alpha(V_t)^{1/2} V_t^{-1/2} S_t (V_t')^{-1/2} (V_t')^{1/2} + \eta_t = \Omega + (\beta - \alpha) V_t + \alpha(V_t)^{1/2} \eta_t (V_t')^{1/2},
\]
with \( \eta_t = V_t^{-1/2} S_t (V_t')^{-1/2} + I \). From (13), we note that \( \eta_t \) is i.i.d. with expectation I. The recursion in (A.2) can now be recognized as a semi-polynomial Markov chain as defined in Boussama (2006). The result then follows directly by an application of his Theorem 2, noting that his requirement \(|(\beta - \alpha) + \alpha| < 1\) together with the positivity constraint \( \beta > 0 \) implies \( 0 < \beta < 1 \). We remark that the fact that Boussama uses \( \tilde{\eta}_t \tilde{\eta}_t' \) for i.i.d. vector-valued \( \tilde{\eta}_t \) with mean zero and covariance matrix I rather than a matrix valued \( \eta_t \) with expectation I, is immaterial for his result to hold. The only important feature in the proof of Boussama (2006) for our current purposes is that (A.2) is quadratic in \( V_t^{1/2} \), and that \( \eta_t \) is i.i.d. with mean I.

**References**
Abadir, K.M. and J.R. Magnus (2005), *Matrix Algebra*, Cambridge University Press.

Boussama, F. (2006), Ergodicité des chaînes de markov à valeurs dans une variété algébrique: application aux modèles GARCH multivariés, *Comptes Rendus Mathematique, Académie Science Paris, Serie I* 343(4), 275–278.