Quantum Correction of the Wilson Line and Entanglement Entropy in the AdS$_3$ Chern-Simons Gravity Theory

Xing Huang$^{a,b}$, Chen-Te Ma$^{c,d}$, and Hongfei Shu$^{e,f}$

$^a$ Institute of Modern Physics, Northwest University, Xi’an 710069, China.
$^b$ Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xi’an 710069, China.
$^c$ Institute of Quantum Matter, School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou 510006, Guangdong, China.
$^d$ The Laboratory for Quantum Gravity and Strings, Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag, Rondebosch 7700, South Africa.
$^e$ Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden.
$^f$ Department of Physics, Tokyo Institute of Technology, Tokyo, 152-8551, Japan.

$^1$e-mail address: xingavatar@gmail.com
$^2$e-mail address: yefgst@gmail.com
$^3$e-mail address: hongfei.shu@su.se
Abstract

We compute the expectation value of the Wilson line in the AdS$_3$ Chern-Simons gravity theory and also the entanglement entropy in the boundary theory. Because the boundary theory with a quantum correction is not described by CFT, the entanglement entropy is supposed to deviate due to the new quantum correction. By comparing the results on both ends, we show that the Wilson line provides the equivalent description to the boundary entanglement entropy. This comparison leads to a concrete example of the AdS/non-CFT correspondence from the building of “minimum surface=entanglement entropy”.
1 Introduction

The holographic principle states that physical degrees of freedom in quantum gravity theory is fully encoded by the boundary [1]. Because Einstein gravity theory [2] loses renormalizability, a direct study in quantum gravity theory is not easy. The holographic principle provides the boundary perspective to study quantum gravity theory. In this direction, the most well-studied is the Anti-de Sitter/ Conformal Field Theory (AdS/CFT) correspondence [3].

The AdS/CFT correspondence was motivated and conjectured by the ultraviolet complete theory, string theory. Therefore, this conjecture is quite realizable. The AdS black hole solution [4] also provides an application to condensed matter systems [5]. A new application of the AdS/CFT correspondence is the equivalence between entanglement entropy of CFT [6] and the codimension-two minimum surface at a given time slice in the AdS background [7]. The replica trick [8] is the most general method for computing entanglement entropy, but it is still hard to obtain an exact solution. The holographic method [9] provides usefulness to studying entanglement entropy in strongly coupled CFT [10]. The proposal becomes more realizable by using the replica trick in the bulk gravity theory to reach the same conclusion [11]. Because the entanglement entropy needs to be defined by the decomposition, the gravity and gauge theories should suffer from the non-gauge invariant cutting. Nevertheless, borrowing the von Neumann algebra gives an interpretation for doing a partial trace operation without breaking the gauge symmetry in each sub-region [12]. Hence this holographic study gives a concretely useful application to the AdS/CFT correspondence.

What people are mostly interested in the holographic principle is pure Einstein gravity theory. Nevertheless, people cannot probe a quantum region due to the well-known issue, renormalizability. Nowadays, the closest route is the AdS$_3$ Einstein gravity theory [13], defined by a gauge theory [14], not by a metric formulation. The gauge formulation is the SL(2) Chern-Simons gravity theory [14]. This theory can be quantized and is renormalizable [15]. The gauge formulation is equivalent to the metric formulation only up to the classical level [16]. Two formulations are not equivalent exactly. This gauge formulation [17] is also interesting for the simple extension of higher spins [18] with a unified study from the SL($M$) groups [19].

Since the three-dimensional Einstein gravity theory does not have the local gravitation
fluctuation, a direct derivation of the boundary theory is possible. The original derivation points to CFT$_2$, the Liouville theory [20]. This theory does not have a normalizable vacuum. It contradicts a known truth. This contradiction goes away after one found two-dimensional Schwarzian theory [21]. This boundary theory breaks conformal symmetry in a quantum regime.

The Wilson line [22] provides the universal contribution [23] to the holographic entanglement entropy at the classical level [24]. The central question that we would like to address in this letter is the following: What is the quantum deformation of the minimum surface? Since the minimum surface is only defined at the classical level, we would like to study the quantum deformation using the computable gauge formulation.

We precisely compute [25] entanglement entropy for one-interval using the $n$-sheet partition function [26]. We then show that the bulk operator, Wilson line, is exactly dual to the boundary entanglement entropy. We provide the quantum deformation of the minimum surface [27] from the Wilson line with the dual of entanglement entropy [28]. This operator correspondence also realizes a concrete example of AdS/non-CFT correspondence.

## 2 SL(2) Chern-Simons Gravity Theory

The action of the SL(2) Chern-Simons gravity theory is given by [14]

$$
S_G = \frac{k}{2\pi} \int d^3 x \, \epsilon^{tr\theta} \text{Tr} \left( A_t F_{r\theta} - \frac{1}{2} (A_r \partial_t A_\theta - A_\theta \partial_t A_r) \right) \\
- \frac{k}{2\pi} \int d^3 x \, \epsilon^{tr\theta} \text{Tr} \left( \bar{A}_t \bar{F}_{r\theta} - \frac{1}{2} (\bar{A}_r \partial_t \bar{A}_\theta - \bar{A}_\theta \partial_t \bar{A}_r) \right) \\
- \frac{k}{4\pi} \int dt d\theta \, \text{Tr}(A_\theta^2) \\
- \frac{k}{4\pi} \int dt d\theta \, \text{Tr}(\bar{A}_\theta^2),
$$

in which we assume that the boundary conditions of the gauge fields $A$ and $\bar{A}$ are: $A_- \equiv A_t - A_\theta = 0$ and $\bar{A}_+ = A_t + A_\theta = 0$. The variable $k$ is defined by $l/(4G_3)$, where $1/l^2 \equiv -\Lambda$. The cosmological constant is denoted by $\Lambda$, and the three-dimensional gravitational constant is denoted by $G_3$. The gauge fields are defined by the vielbein
We also choose the unit \( \Lambda = -\lambda \) we obtain the boundary conditions:

\[
\partial \equiv \frac{1}{g} \left( e^a_\mu + \omega^a_\mu \right), \quad \bar{A}_\nu \equiv \bar{A}^a_\nu J_a \equiv J_a \left( \frac{1}{g} e^a_\nu - \omega^a_\nu \right),
\]

in which the Lie algebra indices are labeled by \( a, \) and the indices are raised or lowered by \( \eta \equiv \text{diag}(-1, 1, 1). \) The spacetime indices are labeled by \( \mu \) and \( \nu. \) This bulk terms in this theory are equivalent to the Chern-Simons theory up to a boundary term. The measure in this gravitation theory is \( \int \mathcal{D}A \mathcal{D}\bar{A}. \)

The SL(2) \( \times \) SL(2) generators are given by the followings:

\[
J_0 \equiv \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad J_1 \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad J_2 \equiv \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},
\]

\[
\bar{J}_0 \equiv \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \bar{J}_1 \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad \bar{J}_2 \equiv \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.
\]

The generators satisfy the algebra: \([J^a, J^b] = \epsilon^{abc} J^c, \) \( \text{Tr}(J^a J^b) = \eta^{ab}/2, \) \([\bar{J}^a, \bar{J}^b] = -\epsilon^{abc} \bar{J}^c, \) and \( \text{Tr}(\bar{J}^a \bar{J}^b) = \eta^{ab}/2. \)

The AdS\(_3\) geometry is \( ds^2 = -(r^2 + 1) dt^2 + dr^2/(r^2 + 1) + r^2 d\theta^2, \) in which the ranges of coordinates are defined by that \(-\infty < t < \infty, 0 < r < \infty, \) and \( 0 < \theta \leq 2\pi. \) We also choose the unit \( \Lambda = -1. \) The metric is defined by the vielbein \( g_{\mu\nu} \equiv 2 \cdot \text{Tr}(e_\mu e_\nu). \)

We take the solution \((F_{r\theta} = 0)\) into the action, and use the asymptotic boundary condition to get: \( g^{-1}_{\text{SL}(2)} \partial_\theta g_{\text{SL}(2)} |_{r\to\infty} = A_\theta |_{r\to\infty} \) and \( g^{-1}_{\text{SL}(2)} \partial_\theta g_{\text{SL}(2)} |_{r\to\infty} = \bar{A}_\theta |_{r\to\infty}, \) which are fixed by the metric of AdS boundary. Using the SL(2) transformations:

\[
g_{\text{SL}(2)} = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix}, \quad \bar{g}_{\text{SL}(2)} = \begin{pmatrix} 1 & -F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{\Psi} & 1 \end{pmatrix},
\]

we obtain the boundary conditions: \( \lambda^2 \partial_\theta F = 2r, \) \( \partial_\theta \bar{F}/\partial_\theta F = -4r \Psi, \) \( \bar{\lambda}^2 \partial_\theta \bar{F} = 2r, \) and \( \partial_\theta^2 \bar{F}/\partial_\theta \bar{F} = -4r \bar{\Psi}, \) which eventually give the boundary theory, two-dimensional Schwarzian theory \[21\].

\[
e = \frac{S_G}{2\pi} \int dt d\theta \left( \frac{3}{2} (\partial_\theta \partial_\theta F)(\partial_\theta^2 \bar{F}) - (\partial_\theta^2 \bar{F}) \right) - \frac{k}{2\pi} \int dt d\theta \left( \frac{3}{2} (\partial_\theta \partial_\theta \bar{F})(\partial_\theta \bar{F}) - (\partial_\theta \partial_\theta \bar{F}) \right),
\]

(5)
where
\[ x^+ \equiv t + \theta, \quad x^- \equiv t - \theta, \] (6)
\[ \partial_+ = \frac{1}{2} \partial_t + \frac{1}{2} \partial_\theta, \quad \partial_- = \frac{1}{2} \partial_t - \frac{1}{2} \partial_\theta. \] (7)

The measure is \( \int dF d\bar{F} \left( \frac{1}{(\partial_\theta F \partial_\theta \bar{F})} \right) \).

3 Entanglement Entropy in the Two-Dimensional Schwarzian Theory

We first write down the boundary theory on the sphere manifold [21] and then compute the \( n \)-sheet partition function to obtain the entanglement entropy for one-interval, which shows the difference from the CFT\(_2\).

3.1 Boundary Effective Action on the Sphere Manifold

The bulk Euclidean AdS\(_3\) metric can be asymptotically written as \( ds^2_a = r^2 ds^2_s + dr^2/r^2 \), where \( ds^2_s = d\psi^2 + \sin^2 \psi d\theta^2 \), \( 0 \leq \psi < \pi \), and \( 0 \leq \theta < 2\pi \). The \( \psi \) is the Euclidean time defined by \( \psi \equiv \text{it} \). The line element \( ds^2_s \) is for the unit sphere. The asymptotic behaviors of the gauge fields for the Lorentzian AdS\(_3\) metric are:
\[ A_{r \to \infty} = \begin{pmatrix} \frac{dr}{2r} & 0 \\ -r E^- \frac{dr}{2r} & -\frac{dr}{2r} \end{pmatrix}, \quad \bar{A}_{r \to \infty} = \begin{pmatrix} -\frac{dr}{2r} & -r E^- \frac{dr}{2r} \\ 0 & \frac{dr}{2r} \end{pmatrix}, \] (8)

where \( E^+ \equiv E^\theta + E^t \) and \( E^- \equiv E^\theta - E^t \) are the boundary zweibein. Then we can find the below boundary condition by replacing \( r \to r E^\pm_\theta \): \( \lambda = \sqrt{2r E^+ \partial_\theta / \partial_\theta F}, \quad \bar{\lambda} = \sqrt{2r E^- \partial_\theta / \partial_\theta \bar{F}}, \) and \( \Psi = -(\partial_\theta \bar{F} / \partial_\theta F)/(4r E^+ \theta). \) For the sphere manifold, we have \( E^\psi = d\psi \) and \( E^\theta = \sin \psi d\theta \). Because we did the Wick rotation, we use the following coordinates: \( x^+ = -i\psi + \theta, \quad x^- = -i\psi - \theta, \quad \psi = i(x^+ + x^-)/2, \) and \( \theta = (x^+ - x^-)/2. \) The \( \theta \)-component of the boundary zweibein is defined by the \( E^\pm_\theta \). Therefore, we have \( E^\pm_\theta = E^\pm_\theta = \sin \psi. \) The boundary gauge-field in the Lorentzian manifold satisfies the conditions: \( E^\pm_\theta A^t - E^\pm_\theta A^\theta = 0 \) and \( E^\pm_\theta \bar{A}^t - E^\pm_\theta \bar{A}^\theta = 0. \) Therefore,
the AdS$_3$ gravitation action with the spherical asymptotic boundary is \[21\]

\[
S_{GS} = \frac{k}{2\pi} \int d^3x \, \epsilon^{tr\theta} \text{Tr} \left( A_t F_{tr\theta} - \frac{1}{2} (A_r \partial_t A_{\theta} - A_{\theta} \partial_t A_r) \right) \\
- \frac{k}{2\pi} \int d^3x \, \epsilon^{tr\theta} \text{Tr} \left( \tilde{A}_t \tilde{F}_{tr\theta} - \frac{1}{2} (\tilde{A}_r \partial_t \tilde{A}_{\theta} - \tilde{A}_{\theta} \partial_t \tilde{A}_r) \right) \\
+ \frac{k}{4\pi} \int dt d\theta \, \text{Tr} \left( \frac{E_t^+}{E_{\theta}^+} A_{\theta}^2 \right) \\
- \frac{k}{4\pi} \int dt d\theta \, \text{Tr} \left( \frac{E_t^-}{E_{\theta}^-} \tilde{A}_{\theta}^2 \right). \tag{9}
\]

Then we use the conditions $\lambda^2 \partial_\theta F = 2E_t^+ \gamma$ and $\tilde{\lambda}^2 \partial_\theta \tilde{F} = 2E_{\theta}^- \gamma$ to obtain the boundary effective action on the sphere manifold \[21\]

\[
S_{GS} = \frac{k}{\pi} \int dt d\theta \left( \frac{(\partial_\theta \lambda)(D_- \lambda)}{\lambda^2} - \frac{(\partial_\theta \tilde{\lambda})(D_+ \tilde{\lambda})}{\tilde{\lambda}^2} \right), \tag{10}
\]

where

\[
D_+ \equiv \frac{1}{2} \partial_t + \frac{1}{2} E_{\theta}^- \partial_\theta, \quad D_- \equiv \frac{1}{2} \partial_t + \frac{1}{2} E_{\theta}^+ \partial_\theta. \tag{11}
\]

From the field redefinition: $F \equiv F/E_{\theta}^+$ and $\tilde{F} \equiv \tilde{F}/E_{\theta}^-$, the gravitation action on the sphere manifold becomes \[21\]:

\[
S_{GS} = \frac{k}{4\pi} \int dt d\theta \left( \frac{(\partial_\theta^2 F)(D_- \partial_\theta F)}{(\partial_\theta F)^2} - \frac{(\partial_\theta^2 \tilde{F})(D_+ \partial_\theta \tilde{F})}{(\partial_\theta \tilde{F})^2} \right) \\
= \frac{k}{4\pi} \int dt d\theta \left[ \frac{(\partial_\theta^2 \phi)(D_- \partial_\theta \phi)}{(\partial_\theta \phi)^2} - (\partial_\theta \phi)(D_- \phi) \right] \\
- \frac{k}{4\pi} \int dt d\theta \left[ \frac{(\partial_\theta^2 \tilde{\phi})(D_+ \partial_\theta \tilde{\phi})}{(\partial_\theta \tilde{\phi})^2} - (\partial_\theta \tilde{\phi})(D_+ \tilde{\phi}) \right], \tag{12}
\]

in which we used $F \equiv \tan(\phi/2)$ and $\tilde{F} \equiv \tan(\tilde{\phi}/2)$. When we take the scale transformation on the the boundary zweibein, this theory is invariant. Therefore, we can use the scale transformation to compute the entanglement entropy as in CFT.

### 3.2 Entanglement Entropy for One-Interval

Now we want to compute the Rényi entropy $S_n \equiv (\ln Z_n - n \ln Z_1)/(1 - n)$ from the replica trick \[8\] on the $\theta$-direction, where $Z_n$ is the $n$-sheet partition function,
and $Z_1$ is same as the partition function. Because we only consider the computation up to the one-loop correction, we obtain that the partition function is a product of the classical partition-function ($Z_c$) and the one-loop partition-function ($Z_q$) $Z_n = Z_{n,c} \cdot Z_{n,q}$. When we take the logarithmic on the $n$-sheet partition function, we obtain $\ln Z_n = \ln Z_{n,c} + \ln Z_{n,q}$. Hence we can treat the classical and one-loop partition-functions separately in the computation of the Rényi entropy. Our motivation is to compare the entanglement entropy to the expectation value of the Wilson line. Therefore, we will take the limit $n \to 1$ in the Rényi entropy to obtain the entanglement entropy. Now we compute the $Z_n$ on the sphere manifold, and then the result corresponds to the entanglement entropy for one-interval.

We first perform the coordinate transformation to get $ds^2_s = \text{sech}^2(y)(dy^2 + d\theta^2)$, in which we used $\text{sech} y = \sin \psi$. In the $n$-sheet manifold, the range of the $\theta$ is $0 < \theta \leq 2\pi n$. The periodicity of this theory with respect to the $\theta$ is $2\pi n$. When we do the computation, we need to regularize the range of the $y$-direction. The range of the $y$-direction is $-\ln(L/\epsilon) < y \leq \ln(L/\epsilon)$. The periodicity of this theory with respect to the $y$ is $4\ln(L/\epsilon)$ because we assume the Dirichlet boundary condition in the $y$-direction. The $L$ is the length of an interval, and $\epsilon$ is the cut-off on the ending point of the interval.

Finally, we identify the sphere from the torus to determine the complex structure $\tau_n$ on the sphere. The coordinates of torus $z \equiv (\theta + iy)/n$ satisfy the identification: $z \sim z + 2\pi$ and $z \sim z + 2\pi \tau_n$. The boundary condition of the fields, $\phi$ and $\bar{\phi}$ is given by $\phi(y/n, \theta/n + 2\pi) = \phi(y/n, \theta/n) + 2\pi$, $\phi(y/n + 2\pi \cdot \text{Im}(\tau_n), \theta/n + 2\pi \cdot \text{Re}(\tau_n)) = \phi(y/n, \theta/n)$, $\bar{\phi}(y/n, \theta/n + 2\pi) = \bar{\phi}(y/n, \theta/n) + 2\pi$, and $\bar{\phi}(y/n + 2\pi \cdot \text{Im}(\tau_n), \theta/n + 2\pi \cdot \text{Re}(\tau_n)) = \bar{\phi}(y/n, \theta/n)$. Therefore, we can quickly find that the complex structure on the sphere is $\tau_n = (2i/(n\pi)) \ln(L/\epsilon)$. The fields on the sphere can be expanded from the way: $\phi = \theta/n + \epsilon(y, \theta)$ and $\bar{\phi} = -\theta/n + \bar{\epsilon}(y, \theta)$, where

$$
\epsilon(y, \theta) \equiv \sum_{j,k} \epsilon_{j,k} e^{i j \theta - k y}, \quad \epsilon^*_j \equiv \epsilon_{-j,-k},
$$

$$
\bar{\epsilon}(y, \theta) \equiv \sum_{j,k} \bar{\epsilon}_{j,k} e^{i j \theta - k y}, \quad \bar{\epsilon}^*_j \equiv \bar{\epsilon}_{-j,-k}, \quad (13)
$$

and $\tau = n\tau_n$. Because this theory has the $\text{SL}(2)$ redundancy, the variables have the constraints: $\epsilon_{j,k} = 0$ and $\bar{\epsilon}_{j,k} = 0$ when $j = -1, 0, 1$. To compute the partition function on the sphere, we need to do the Wick rotation $t = -i\psi$, and then the derivative
becomes:

\[
\begin{align*}
D_+ &= \frac{1}{2} \partial_t + \frac{1}{2} \frac{E^\psi}{E_\theta} \partial_\theta = -\frac{i}{2} \cosh(y) \partial_y - \frac{1}{2} \cosh(y) \partial_\theta, \\
D_- &= \frac{1}{2} \partial_t + \frac{1}{2} \frac{E^{\bar{\psi}}}{E_\theta} \partial_\theta = -\frac{i}{2} \cosh(y) \partial_y + \frac{1}{2} \cosh(y) \partial_\theta, \\
\end{align*}
\]

and the gravitation action becomes

\[
S_{GS} = k \frac{4}{\pi} \int_{\frac{\pi}{2} \text{Im}(\tau)}^{\frac{\pi}{2} \text{Im}(\tau)} dy \int_0^{2\pi} d\theta \sech(y) \left[ \frac{(\partial^2_{\theta} \phi)(D_- \partial_\theta \phi)}{(\partial_\theta \phi)^2} - (\partial_\theta \phi)(D_- \phi) \right] \\
- k \frac{4}{\pi} \int_{\frac{\pi}{2} \text{Im}(\tau)}^{\frac{\pi}{2} \text{Im}(\tau)} dy \int_0^{2\pi} d\theta \sech(y) \left[ \frac{(\partial^2_{\bar{\theta}} \bar{\phi})(D_+ \partial_{\bar{\theta}} \bar{\phi})}{(\partial_{\bar{\theta}} \bar{\phi})^2} - (\partial_{\bar{\theta}} \bar{\phi})(D_+ \bar{\phi}) \right].
\]

(14)

Substituting the saddle-points into the action, we obtain

\[
S_{GS} = -\left( \frac{c}{6n} \right) \ln(L/\epsilon),
\]

where \(c = 6k\) is the central charge of the CFT \(_2\) [23]. Therefore, we obtain \(\ln Z_{n,c} = \left( \frac{c}{6n} \right) \ln(L/\epsilon)\). The R\'enyi entropy from the saddle-points is given by:

\[
S_{n,c} = \frac{c}{1 - n} \left( \frac{1}{6n} - \frac{n}{6} \right) \ln \frac{L}{\epsilon} = \frac{c(1 + n)}{6n} \ln \frac{L}{\epsilon}.
\]

(16)

When we take \(n \to 1\), we obtain \(S_{1,c} = (c/3) \ln(L/\epsilon)\).

Now we consider the perturbation \(\epsilon(y, \theta)\) to obtain the one-loop effect. Because the \(\phi\)-part and \(\bar{\phi}\)-part are the same, we can only consider the field \(\phi\) to obtain the one-loop correction in the R\'enyi entropy. The expansion from the \(\epsilon\) in the gravitation action for the \(\phi\)-part is

\[
\begin{align*}
&\frac{k}{4\pi} \int_{\frac{\pi}{2} \text{Im}(\tau)}^{\frac{\pi}{2} \text{Im}(\tau)} dy \int_0^{2\pi} d\theta \left( n^2 (\partial^2_{\theta} \epsilon(y, \theta) (\bar{\partial} \partial_{\theta} \epsilon(y, \theta)) - (\partial_\theta \epsilon(y, \theta)) (\bar{\partial} \epsilon(y, \theta)) \right) \\
&= -i k \frac{8}{\pi} \sum_{j,k} j(j^2 - 1) \left( k + \frac{j}{n} \right) |\epsilon_{j,k}|^2,
\end{align*}
\]

(17)

where \(\bar{\partial} \equiv (-i \partial_y + \partial_\theta)/2\). Therefore, we obtain

\[
\partial_\tau \ln Z_{n,q} = - \sum_{j \neq 0, \pm 1} \sum_{k = -\infty}^{\infty} \frac{j}{k + \frac{j}{n}}.
\]

(18)

Now we use the following useful equation \(\tilde{\psi}(1 - x) - \tilde{\psi}(x) = \pi \cot(\pi x)\), in which the digamma function is defined by

\[
\tilde{\psi}(a) \equiv - \sum_{n=0}^{\infty} \frac{1}{n + a}.
\]
Therefore, we obtain:

\[
\sum_{m=-\infty}^{\infty} \frac{1}{m-x} = -\sum_{m=0}^{\infty} \frac{1}{m+x} + \sum_{m=0}^{\infty} \frac{1}{m+1-x} = \tilde{\psi}(x) - \tilde{\psi}(1-x) = -\pi \cdot \cot(\pi x). \tag{20}
\]

Hence we get

\[
\partial_\tau \ln Z_{n,q} = -2\pi \sum_{j=2}^{\infty} \left( \frac{j}{n} \right) \cdot \cot \left( \frac{j\pi \tau}{n} \right) \tag{21}
\]

Then we do the re-summation for the above series:

\[
\partial_\tau \ln Z_{n,q} = -2\pi \sum_{j=2}^{\infty} \left( \frac{j}{n} \right) \cdot \cot \left( \frac{j\pi \tau}{n} \right) \\
= -2\pi \sum_{j=2}^{\infty} \frac{j}{n} \cdot \left[ \cot \left( \frac{j\pi \tau}{n} \right) + i \right] + 2\pi i \sum_{j=2}^{\infty} \frac{j}{n}. \tag{22}
\]

By using the regularization

\[
\sum_{j=1}^{\infty} j \rightarrow -\frac{1}{12}, \tag{23}
\]

we obtain:

\[
\partial_\tau \ln Z_{n,q} = -2\pi \sum_{j=2}^{\infty} \frac{j}{n} \cdot \left[ \cot \left( \frac{j\pi \tau}{n} \right) + i \right] + 2\pi i \sum_{j=2}^{\infty} \frac{j}{n} \\
\rightarrow -2\pi \sum_{j=2}^{\infty} \frac{j}{n} \cdot \left[ \cot \left( \frac{j\pi \tau}{n} \right) + i \right] - i \frac{13\pi}{6n}. \tag{24}
\]

After integrating out the \( \tau \), we obtain:

\[
\ln Z_{n,q} = -2 \sum_{j=2}^{\infty} \left[ \ln \sin \left( \frac{\pi j \tau}{n} \right) + i \frac{j\pi \tau}{n} \right] - i \frac{13\pi \tau}{6n} + \cdots, \tag{25}
\]

where \( \cdots \) is independent of the \( \tau \). The above series is convergent for the \( \text{Im}(\tau) > 0 \). When we take the limit \( L/\epsilon \rightarrow \infty \), we obtain \( \ln Z_{n,q} = (13/(3n)) \ln(L/\epsilon) \). The Rényi entropy for the one-loop correction is: \( S_{n,q} = (\ln Z_{n,q} - n \ln Z_{1,q})/(1-n) = (13(n+1)/(3n)) \ln(L/\epsilon) \). Therefore, we obtain the Rényi entropy \( S_n = ((c+26)(n+1)/(6n)) \ln(L/\epsilon) \) and the entanglement entropy \( S_{EE} = ((c+26)/3) \ln(L/\epsilon) \).
4 Wilson Line

The entanglement entropy in the two-dimensional Schwarzian theory gives the conformal deviation from the quantum correction. Here we want to obtain a bulk description of the entanglement entropy. Since the Wilson lines \[22\]

\[W(P, Q) \equiv \text{Tr} \left[ \mathcal{P} \exp \left( \int_Q^P \bar{A} \right) \mathcal{P} \exp \left( \int_Q^P A \right) \right], \quad (26)\]

can provide the entanglement entropy in the CFT\(_2\), we begin from this operator to study. The \(\mathcal{P}\) denotes the path-ordering, \(P\) and \(Q\) are the two-ending points of the Wilson lines at a time slice. Here the trace operation acts on the representation.

We extend the Wilson line to the following form \[22\]

\[W_\mathcal{R}(C) = \int DUDPD\lambda \exp \left[ \int_C ds \left( \text{Tr}(PU^{-1}D_jU) + \lambda(s)(\text{Tr}(P^2) - c_2) \right) \right], \quad (27)\]

where \(U\) is an SL(2) element, \(P\) is its conjugate momentum, \(\sqrt{2c_2} \equiv c(1 - n)/6\), and the covariant derivative is defined as that: \(D_\mu U \equiv dU/ds + A_\mu U + U\bar{A}_\mu\) and \(A_\mu \equiv A_\mu \cdot (dx^\mu /ds).\) The equations of motion are \(i(k/(2\pi))F_{\mu_1\mu_2} = -\int ds \left( dx^{\nu_3}/ds \right)\epsilon_{\mu_1\mu_2\nu_3}\delta^3(x - x(s))UPU^{-1}\) and \(i(k/(2\pi))\tilde{F}_{\mu_1\mu_2} = \int ds \left( dx^{\nu_3}/ds \right)\epsilon_{\mu_1\mu_2\nu_3}\delta^3(x - x(s))P.\) A solution of the equations of motion is that \[22\]: \(A = g^{-1}ag + g^{-1}dg, \ g = \exp(L_1z)\exp(\rho L_0); \ \bar{A} = -\bar{g}^{-1}a\bar{g} - \bar{g}^{-1}d\bar{g}, \ \bar{g} = \exp(L_{-1}\bar{z})\exp(-\rho L_0),\) where the gauge field is given as \(a = \sqrt{c_2/2} \cdot (1/k) \cdot (dz/z - d\bar{z}/\bar{z})L_0.\) The SL(2) algebra is defined by that: \([L_j, L_k] = (j - k)L_{j+k}, \ j, k = 0, \pm 1; \ \text{Tr}(L_j) = 1/2, \ \text{Tr}(L_{-1}L_1) = -1,\) and the traces of other bilinears vanish. Here we choose \(z \equiv r \exp(i\Phi)\) and \(\bar{z} \equiv r \exp(-i\Phi).\) Then the space-time interval is \(ds^2 = d\rho^2 + \exp(2\rho)(dt^2 + n^2r^2d\Phi^2)\) \[22\]. This solution corresponds to \(U(s) = 1, \ P(s) = \sqrt{2c_2}L_0\) with the curve \(z(s) = 0\) and \(\rho(s) = s.\) Hence we find that including the Wilson line directly gives the \(n\)-sheet geometry \[22\]. When this geometry approaches the boundary, it is the \(n\)-sheet cylinder \((dt^2 + n^2d\Phi^2)\) up to a scale transformation by using \(r \equiv \exp(t).\)

Let us then consider the quantum correction of the Wilson line. Since we should choose the smooth fluctuation, we still obtain the two-dimensional Schwarzian theory by integrating out the time-component gauge fields. The \(n\)-sheet geometry can be used in the smooth region. Hence computing the Wilson line \(W_\mathcal{R}\) in the Chern-Simons gravity theory is equivalent to computing the \(Z_n/Z_1^n\) in the two-dimensional Schwarzian
theory. In other words, the entanglement entropy is

\[ S_{EE} = \lim_{n \to 1} \frac{1}{1 - n} \ln \langle W_\mathcal{R} \rangle, \tag{28} \]

where \( \langle W_\mathcal{R} \rangle \) is the expectation value of the Wilson line. Substituting the classical solution of the two-dimensional Schwarzian theory into the Wilson line, it provides the entanglement entropy of CFT\(_2\) [24], which implies that the Wilson line can be seen as the geodesic line at the on-shell level. Moreover, the equivalence between the Wilson line and the entanglement entropy is exact, which is not only restricted to the one-loop order. Hence the Wilson line can be seen as the appropriate operator to provide one equivalent description of the minimum surface even at the quantum level.

5 Outlook

The Wilson line was used in the AdS\(_3\) Einstein gravity theory for obtaining the entanglement entropy of CFT\(_2\) [8] at the classical level [22, 24]. It will be promoted to an operator at a quantum level. We first computed the entanglement entropy in the boundary theory, two-dimensional Schwarzian theory [21], and this leads to the contribution of the non-CFT. Then we used the Wilson line to obtain the bulk description for the boundary entanglement entropy. This shows that the Wilson line is a suitable operator providing an equivalent description of the minimum surface in the usual correspondence of “minimum surface=entanglement entropy”. We realized a concrete example of AdS/non-CFT correspondence, and the holographic method [22] could be applied to non-CFT without a modification. The CFT is very restricted in the real world. Hence the validity of non-CFT extends the applicability of the holographic principle like quantum chromodynamics and condensed matter systems.

Acknowledgments

We would like to thank Jan de Boer, Kristan Jensen, and Ryo Suzuki for their useful discussion. Xing Huang acknowledges the support of NWU Starting Grant No.0115/338050048 and the Double First-class University Construction Project of Northwest University. Chen-Té Ma was supported by the Post-Doctoral International Exchange Program and China Postdoctoral Science Foundation, Postdoctoral General Funding: Second Class (Grant No. 2019M652926), and would like to thank Nan-Peng Ma for his encouragement. Hongfei Shu was supported by the JSPS Research Fellowship 17J07135 for
Young Scientists, from Japan Society for the Promotion of Science (JSPS) and the grant “Exact Results in Gauge and String Theories” from the Knut and Alice Wallenberg foundation. We would like to thank the Jinan University, Institute of Physics at the University of Amsterdam, Shing-Tung Yau Center at the Southeast University, Institute of Theoretical Physics at the Chinese Academy of Sciences, Shanghai University, Shanghai Jiao Tong University, and National Center for Theoretical Sciences at the National Tsing Hua University. Discussions during the workshops, “Jinan University Gravitational Frontier Seminar”, “Quantum Information and String Theory”, “Amsterdam Summer Workshop on String Theory”, “Youth Symposium on Theoretical High Energy Physics in Southeast University”, “Workshop on Holography and Quantum Matter”, and “East Asia Joint Workshop on Fields and Strings 2019 and 12th Taiwan String Theory Workshop”, were useful to complete this work.

References

[1] G. ’t Hooft, “Dimensional reduction in quantum gravity,” Conf. Proc. C 930308, 284 (1993) [gr-qc/9310026].

[2] P. Kraus, “Lectures on black holes and the AdS(3) / CFT(2) correspondence,” Lect. Notes Phys. 755, 193 (2008) [hep-th/0609074].

[3] B. Oblak, “BMS Particles in Three Dimensions,” doi:10.1007/978-3-319-61878-4 arXiv:1610.08526 [hep-th].

[4] M. Banados, “Three-dimensional quantum geometry and black holes,” AIP Conf. Proc. 484, no. 1, 147 (1999) doi:10.1063/1.59661 [hep-th/9901148].

[5] C. T. Ma, “Parity Anomaly and Duality Web,” Fortsch. Phys. 66, no. 8-9, 1800045 (2018) doi:10.1002/prop.201800045 [arXiv:1802.08959 [hep-th]].

[6] P. Calabrese and J. Cardy, “Entanglement entropy and conformal field theory,” J. Phys. A 42, 504005 (2009) doi:10.1088/1751-8113/42/50/504005 [arXiv:0905.4013 [cond-mat.stat-mech]].
[7] S. Ryu and T. Takayanagi, “Holographic derivation of entangle-
ment entropy from AdS/CFT,” Phys. Rev. Lett. 96, 181602 (2006) 
doi:10.1103/PhysRevLett.96.181602 [hep-th/0603001].

[8] C. Holzhey, F. Larsen and F. Wilczek, “Geometric and renormalized entropy in conformal field theory,” Nucl. Phys. B 424, 443 (1994) doi:10.1016/0550-3213(94)90402-2 [hep-th/9403108].

[9] T. Takayanagi, “Entanglement Entropy from a Holographic Viewp oint,” Class. Quant. Grav. 29, 153001 (2012) doi:10.1088/0264-9381/29/15/153001 
[arXiv:1204.2450 [gr-qc]].

[10] T. Nishioka, S. Ryu and T. Takayanagi, “Holographic Entangleme nt Entropy: An Overview,” J. Phys. A 42, 504008 (2009) doi:10.1088/1751-8113/42/50/504008 
[arXiv:0905.0932 [hep-th]].

[11] A. Lewkowycz and J. Maldacena, “Generalized gravitational entropy,” JHEP 1308, 090 (2013) doi:10.1007/JHEP08(2013)090 [arXiv:1304.4926 [hep-th]].

[12] C. T. Ma, “Entanglement with Centers,” JHEP 1601, 070 (2016) 
doi:10.1007/JHEP01(2016)070 [arXiv:1511.02671 [hep-th]].

[13] S. Carlip, “Conformal field theory, (2+1)-dimensional gravity, and the BTZ black hole,” Class. Quant. Grav. 22, R85 (2005) doi:10.1088/0264-9381/22/12/R01 
gr-qc/0503022.

[14] E. Witten, “(2+1)-Dimensional Gravity as an Exactly Soluble System,” Nucl. 
Phys. B 311, 46 (1988). doi:10.1016/0550-3213(88)90143-5

[15] S. Elitzur, G. W. Moore, A. Schwimmer and N. Seiberg, “Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory,” Nucl. Phys. B 326, 108 (1989). doi:10.1016/0550-3213(89)90436-7

[16] E. Witten, “Three-Dimensional Gravity Revisited,” [arXiv:0706.3359 [hep-th]].
[17] M. A. Vasiliev, “Progress in higher spin gauge theories,” [hep-th/0104246].

[18] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, “Nonlinear higher spin theories in various dimensions,” [hep-th/0503128].

[19] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, “Black holes in three dimensional higher spin gravity: A review,” J. Phys. A 46, 214001 (2013) doi:10.1088/1751-8113/46/21/214001 [arXiv:1208.5182 [hep-th]].

[20] O. Coussaert, M. Henneaux and P. van Driel, “The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant,” Class. Quant. Grav. 12, 2961 (1995) doi:10.1088/0264-9381/12/12/012 [gr-qc/9506019].

[21] J. Cotler and K. Jensen, “A theory of reparameterizations for $\text{AdS}_3$ gravity,” JHEP 1902, 079 (2019) doi:10.1007/JHEP02(2019)079 [arXiv:1808.03263 [hep-th]].

[22] M. Ammon, A. Castro and N. Iqbal, “Wilson Lines and Entanglement Entropy in Higher Spin Gravity,” JHEP 1310, 110 (2013) doi:10.1007/JHEP10(2013)110 [arXiv:1306.4338 [hep-th]].

[23] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” Commun. Math. Phys. 104, 207 (1986). doi:10.1007/BF01211590

[24] J. de Boer and J. I. Jottar, “Entanglement Entropy and Higher Spin Holography in $\text{AdS}_3$,” JHEP 1404, 089 (2014) doi:10.1007/JHEP04(2014)089 [arXiv:1306.4347 [hep-th]].

[25] R. J. Szabo, “Equivariant localization of path integrals,” [hep-th/9608068].

[26] X. Huang and C. T. Ma, “Analysis of the Entanglement with Centers,” [arXiv:1607.06750 [hep-th]].
[27] C. T. Ma, “Discussion of Entanglement Entropy in Quantum Gravity,” Fortsch. Phys. 66, no. 2, 1700095 (2018) doi:10.1002/prop.201700095 [arXiv:1609.03651 [hep-th]].

[28] C. T. Ma, “Theoretical Properties of Entropy in a Strong Coupling Region,” Class. Quant. Grav. 35, no. 23, 235011 (2018) doi:10.1088/1361-6382/aaec3b [arXiv:1609.04550 [hep-th]].