Breather-like pulses in a reduced Maxwell-Duffing model

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Abstract. Propagation of pulses with oscillating amplitude of the electromagnetic field (breathers) is considered in the framework of a model in which the medium is represented by anharmonic oscillators with cubic nonlinearities. Numerical simulations demonstrate that the high frequency breathers propagate as the steady-state solutions and are very robust against collisions with one-soliton solutions of this model. Evolution of low-frequency breather leads to dispersion spreading and conversion the pulse into quasiharmonic wave.

1. Introduction
Nowadays the methods of generation of the pulses consisting of one or just a few cycles of the electromagnetic field are well established [1-7]. These pulses are called the extremely short pulses (ESPs). In the papers [8-9] the analytic solutions for a system of Maxwell-Bloch equations were found, these solutions are obtained without an assumption about slowness of variation of the envelope of the electromagnetic pulse. In general, the wave equation possesses solutions, corresponding to oppositely propagating waves. However, if the polarization response is small, and backward scattering could be neglected, the unidirectional approximation might be used instead. This approximation allows to reduce the wave equation to the first-order one without setting the limitations on the duration of the pulse. The nonlinear dynamics of the medium under electromagnetic pulse action is often modeled by an ensemble of anharmonic oscillators. For example, the propagation of linearly polarized ESP in the cubic nonlinear medium (described by Duffing model) is considered in [2,10]. Evolution of ESP in dispersive nonlinear medium modeled with quadratic nonlinear oscillators is considered in [1], the case without dispersion is considered in [11]. In these works the oscillator characterizes high-frequency (electronic) response of the medium on action of the electromagnetic pulse. As the ESP spectrum lies in the low-frequency range, the low-frequency oscillations of the ion lattice could contribute significantly to medium polarization. Propagation of the femtosecond pulse in a medium with nonlinearity caused both by electronic and ionic vibrations is studied in [12]. In this paper the propagation of linearly polarized breathers in a nonlinear dispersive medium represented by anharmonic oscillators with cubic nonlinearities is considered. The oscillator represents a response of the high-frequency electron degree of freedom to the electromagnetic field. A model of the propagation of breathers...
in a medium with cubic nonlinearity is analyzed by numerical computation. A breather solution of modified Korteweg-de Vries equation has been taken as the initial condition. We found that the low-frequency breathers are quickly destroyed by dispersion spreading and results in periodic quasiharmonic wave. Variation of parameters increasing breather’s frequency leads to pulse stabilization. It is shown that the high-frequency breather evolution is very similar to behavior of the breather solutions in the completely integrable models. Moreover, an example of strong interaction illustrated by collisions of one-soliton solution of the Duffing model with the high-frequency breather, shows the stability of such a breather pulse.

2. System of Maxwell-Duffing equations
Evolution of the electric component $E$ of the electromagnetic pulse in cubic nonlinear medium with polarization $P$ of a unit volume of the medium is described by a wave equation

$$\frac{\partial^2 E}{\partial z^2} + \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 P}{\partial t^2}. \tag{1}$$

In the unidirectional approximation [13-14] it reduces to first-order equation

$$\frac{\partial E}{\partial z} + \frac{1}{c} \frac{\partial E}{\partial t} = -\frac{2\pi}{c} \frac{\partial P}{\partial t}. \tag{2}$$

To describe polarization response of nonlinear medium the oscillator model is used [15-17]:

$$\frac{\partial^2 X}{\partial t^2} + \omega_0^2 X + k_3 x^3 = \frac{e}{m_{\text{eff}}} E(z,t). \tag{2}$$

Then polarization of a unit volume of the medium is defined as $P = n_A e X$, $X$ - oscillator coordinate. Here $n_A$ - density of atoms, $e$ - electron charge, $m_{\text{eff}} = 3m/(\varepsilon + 2)$ - effective mass of an electron, $\omega_0$ - oscillator eigenfrequency, $k_3$ - nonlinearity coefficient.

Let us introduce the variables $t = z/l$, $x = \omega_0(t - z/l)$ and fields $e = E/A_0$, $q = X/X_0$. The normalizing coefficients are $A_0 = m_{\text{eff}} \omega_0^2 X_0/e = m_{\text{eff}} \omega_0^3 e^{-1}(2\mu/|k_3|)^{1/2}$, $X_0 = (2\mu \omega_0^2/|k_3|)^{1/2}$, $l^{-1} = 2\pi n_A e^2/(m_{\text{eff}} \omega_0)$, plasma frequency $\omega_p = (4\pi n_A e^2/m_{\text{eff}})^{-1/2}$, $\mu = k_3 X_0^2/2\omega_0^2$. The normalized system of Maxwell-Duffing equations in the unidirectional approximation, i.e. reduced Maxwell-Duffing (RMD) model becomes following:

$$\frac{\partial e}{\partial t} = -\frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2} + q + 2\mu q^3 = e. \tag{3}$$

3. Numerical simulation
In the numerical simulation the following system of equation was studied:

$$\frac{\partial e}{\partial t} = -p, \frac{\partial q}{\partial x} = p, \frac{\partial p}{\partial x} = e - q - 2\mu q^3. \tag{3}$$

The subject of the present study was numerical simulation of the breather-like pulses formation in the RMD model. To find the breather-like solitary wave in a RMD model the breather solution of the modified Korteweg-de Vries (mKdV) equation was employed as the initial condition:

$$e(x,t = 0) = -\frac{4\beta}{\alpha_1} \cos \theta_2 c \theta_1 - \beta \sin \theta_1 \sin \theta_2 \alpha_1 \frac{c h^2 \theta_1}{ch^2 \theta_1 + (\frac{\beta}{\alpha_1})^2 \sin^2 \theta_2}, \tag{4}$$

and

$$\theta_2 = 2\beta(x - x_0) + 8\beta(\beta^2 - 3\alpha_1^2 - 0.25)t. \tag{4}$$
Figure 1. Evolution of a mKdV breather in a RMD model. Figure (a) shows evolution of a low-frequency breather characterized by the parameter $\alpha_1 = 0.5$, figure (b) corresponds to $\alpha_1 = 1.5$.

Figure 2. Evolution of a mKdV breather in a RMD model. Figure (a) shows evolution of a low-frequency breather characterized by the parameter $\alpha_1 = 2$, figure (b) corresponds to $\alpha_1 = 5$.

\[ \theta_2 = 2\alpha(x - x_{20}) + 8\alpha_1(\alpha^2 - 3\beta^2 + 0.25)t. \]  

(5)

The parameter $\beta = 0.5$, $\alpha_1$ which defines breather’s frequency is varied.

The first figure illustrates spreading of the low-frequency mKdV breathers in a RMD model. Dispersion leads to broadening of the initial pulse and transformation into quasiperiodic waves. The localized breather-like pulses do not form here. Increase for the parameter $\alpha_1$, which increases the frequency of the initial pulse, leads to stabilization of the wave packet, it is illustrated by the figure 2. Collision of such high-frequency pulses with steady-state solutions of the RMD model demonstrates their stability (figure 3).

An initial high-frequency pulse obtained by modulation of the steady-state pulse of the RMD model with harmonic wave shows considerable robustness during its evolution (figure 4), which is similar to that for high-frequency mKdV breathers evolving in RMD model. The frequency spectra of such pulses is located at the frequency of the carrier wave, it remains almost unchanged under pulse propagation. These modulated pulses also show stability against collisions with steady state pulses of RMD model.

4. Conclusion

The propagation of the breather-like pulses in a cubic nonlinear medium was studied numerically. The initial conditions were chosen corresponding to breather solutions of mKdV equation. It is shown that low-frequency breather-like pulses transform into periodic quasiharmonic waves during propagation due to dispersion effect. Increase for the frequency of the initial pulse leads
Figure 3. Collision of a RMD solitary wave (characterized by $\alpha = 3$, $\mu = 0.1$) with a high-frequency pulse evolved from mKdV breather. Figure (a) shows evolution of a low-frequency breather characterized by the parameter $\alpha_1 = 3$, figure (b) corresponds to $\alpha_1 = 5$.

Figure 4. Evolution of a pulse set as $e(x, 0) = \alpha \sqrt{\frac{\alpha - 1}{\mu}} \text{sech}(\sqrt{\alpha - 1}(x - \frac{t}{\alpha} - x_0)) \cos(5(x - x_0))$ at $\alpha = 2$, $\mu = 0.3$. The Fourier spectra of the pulse doesn’t change during propagation (figure (b)).

to its stabilization, these high-frequency waves are stable against collisions with steady state solitary waves of RMD model.

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