Gravitational waves: a foundational review

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Abstract. The standard linear approach to the gravitational waves theory is critically reviewed. Contrary to the prevalent understanding, it is pointed out that this theory contains many conceptual and technical obscure issues that require further analysis.

1 Introduction

For many years in the past, the existence of gravitational waves was a controversial issue. The discovery of a binary pulsar whose orbital period changes in accordance with the predicted gravitational wave emission [1] put an end to that controversy. In fact, that discovery provided a compelling evidence for the existence of gravitational waves (for a textbook reference, see Ref. [2]). That evidence, however, did not provide any clue on their form and effects. The only it has done was to confirm the quadrupole radiation formula. Despite this fact, together with the quadrupole radiation formula, the standard linear approach to the gravitational waves theory became widely considered a finished topic, a theory not to be questioned anymore (see, for example, Ref. [3], page 313). In other words, it became a dogma.

However, as a careful analysis of the current theory shows, it is actually plagued by many obscure points [4]. From one hand, owing to the nonlinear nature of gravitation, which makes it difficult to deal with, it is understandable the existence of some obscure, or even controversial points. On the other hand, these difficulties cannot be used as an excuse for our leniency with the established theory. In these notes, by using the potential (or Lagrangian) form of Einstein field equation, even at the risk of committing a heresy, I will critically review the foundations of the standard linear approach to the gravitational waves theory, pointing out precisely where it lacks consistency and why it requires further attention.

2 Gauge versus gravitational waves

It is well-known that, in order to transport its own charge (or source), a gauge field must satisfy a nonlinear field equation. For example, the gauge field equations of chromodynamics must be nonlinear to allow the field to transport color charge. In the language of differential forms, the Yang-Mills equation is written as

\[ dH - j = J, \] (1)

The Yang-Mills theory will be adopted as the paradigm of nonlinear gauge theories.
where \( H = -\partial L / \partial F \) is the excitation 2-form with \( L \) the gauge Lagrangian and \( F = DA \) the field strength of the gauge potential \( A \). In addition, \( j \) stands for the gauge pseudo-current, and \( J \) is the source current. Due to the property

\[
\text{dd} = 0, \tag{2}
\]

known as Poincaré lemma, the field equation implies the conservation of the total current:

\[
d(j + J) = 0. \tag{3}
\]

Electromagnetism is a particular case of Yang-Mills theories, with the Abelian unitary group \( U(1) \) as the gauge group. In this case, the Yang-Mills equation reduces to the linear Maxwell equation

\[
dH = J, \tag{4}
\]

where \( H = -\partial L / \partial F \) is the electromagnetic excitation 2-form, with \( L \) the Lagrangian and \( F = dA \) the electromagnetic field strength. The source \( J \) in this case is the electric current, which is conserved on account of the Poincaré lemma:

\[
dJ = 0. \tag{5}
\]

This conservation law says that a source cannot lose electric charge when emitting electromagnetic waves. In fact, remembering that currents are quadratic in the field variable, the linearity of Maxwell equation restricts the gauge self-current \( j \) to be also linear, and consequently to vanish

\[
j = 0. \tag{6}
\]

This is the reason why an electromagnetic wave is unable to transport its own source, that is, electric charge, a result consistent with the conservation law. Observe that the source current \( J \) is quadratic in the source field variables, but linear in the electromagnetic field. Differently from the self-current \( j \), therefore, the linearity of Maxwell equation does not restrict it to vanish. Then comes the crucial point: since neither energy nor momentum is source of the electromagnetic field, the energy-momentum current does not appear explicitly in the electromagnetic field equation, and for this reason the linearity of Maxwell equation does not restrict the energy-momentum tensor of the electromagnetic field to be linear. This means that, even though electromagnetic waves are unable to carry electric charge, they do carry energy and momentum, whose intensity is given by the (quadratic) Poynting vector.

Let us consider now the gravitational case. Denoting by \( L_g \) the gravitational Lagrangian, and using the first half of the Latin alphabet \((a, b, c, \ldots)\) to denote algebraic (or tangent space) indices, the potential (or Lagrangian) form of Einstein equation reads \( \kappa = 8\pi G / c^4 \)

\[
dH_a - \kappa t_a = \kappa T_a, \tag{7}
\]

where

\[
H_a = -\frac{k}{e} \frac{\partial L_g}{\partial \partial e^a}. \tag{8}
\]

\(^2\)In Yang-Mills theory, as well as in electromagnetism, the field excitation 2-form coincides with the field strength. However, there are theories in which they do not coincide. This is the case, for example, of teleparallel gravity, a gauge theory for the translation group. \[8\]
is the gravitational field excitation 2-form (also called superpotential), with $e^a$ the tetrad (or coframe) field and $e = \det(e^a)$. In addition,

$$t_a = -\frac{1}{h} \frac{\partial L_g}{\partial e^a}$$

(9)

stands for the gravitational self-current, which in this case represents the gravitational energy-momentum pseudotensor, and

$$T_a = -\frac{1}{h} \frac{\partial L_s}{\partial e^a}$$

(10)

is the source energy-momentum current, with $L_s$ the source Lagrangian. Notice that in this form, Einstein equation (7) is similar, in structure, to the Yang-Mills equation (1). Its main property is to explicitly exhibit the complex defining the energy-momentum pseudo-current of the gravitational field. From the Poincaré lemma (2), the total energy-momentum density is found to be conserved as a consequence of the field equation:

$$d(t_a + T_a) = 0.$$  (11)

Now, owing to the weakness of the gravitational interaction, and considering that the sources of gravitational waves are at enormous distances from Earth, it is sensible to assume that the amplitude of a gravitational wave when reaching a detector on Earth will be very small. These facts allow the use of a perturbative analysis, where the gravitational field variable is expanded in powers of a small parameter. Namely,

$$e^a = \delta^a + e^a_{(1)} + e^a_{(2)} + \cdots,$$

(12)

where $\delta^a$ is a trivial tetrad related to Minkowski spacetime. In the linear, or first-order approximation, the gravitational field equation becomes mathematically similar to Maxwell equation. In fact, at this order the field equation (7) reduces to

$$dH_a^{(1)} = k T_a^{(1)},$$

(13)

which is mathematically similar to the Maxwell equation (4).

However, in spite of this similarity, there is a fundamental difference between the two cases. As we have already seen, the linearity of Maxwell equation restricts the electromagnetic self-current to vanish. As a consequence, an electromagnetic wave is unable to transport electric charge. Analogously, since the gravitational energy-momentum pseudotensor $t_a$ is at least quadratic in the field variables [9], it vanishes in the linear approximation:

$$t_a^{(1)} = 0.$$  (14)

This means that a linear gravitational wave is unable to transport its own source, that is, energy and momentum. This is consistent with the Poincaré lemma, which when applied to the first-order field equation (13) implies that the source energy-momentum tensor is conserved:

$$dT_a^{(1)} = 0.$$  (15)

Strictly speaking, this conservation law says that, at this order, a mechanical system cannot lose energy in the form of gravitational waves. Since any wave must have energy to exist, what this conservation law is saying is that linear (or dipole) gravitational radiation does not exist.
3 The standard approach to gravitational waves

3.1 Linear or nonlinear: that is the question

Even though there seems to be a certain agreement that the transport of energy and momentum by gravitational waves is a nonlinear phenomenon (see, for example, Ref. [10]), instead of going to the second order, the standard approach to gravitational waves follows a kind of “mixed procedure”, which consists basically in assuming that gravitational waves carry energy (are nonlinear), but at the same time, because the amount of energy transported is so small, it is also assumed that its dynamics can be approximately described by a linear equation [11]. More precisely, it can be described by the (sourceless version of the) linear wave equation [10].

Conceptually speaking, however, this is a questionable assumption. The reason is that either a gravitational wave does or does not carry energy. If it carries, it cannot satisfy a linear equation. If applied to a Yang-Mills propagating field, it would correspond to assume that, for a gauge field with small-enough amplitude, its evolution could be accurately described by a linear equation. Of course, this is plainly wrong: a Yang-Mills propagating field must be nonlinear to carry its own source, otherwise it is not a Yang-Mills field. Analogously, a gravitational wave must be nonlinear to transport its own source, otherwise it is not a gravitational wave. It is important to realize that this is not a matter of approximation, but a conceptual issue.

It is opportune at this point to recall some properties of solitary waves, whose existence depends on a precise compensation between dispersion and nonlinearity [14]. In the specific case of surface waves in shallow water, solitary waves are obtained from the Navier-Stokes equation (for an inviscid fluid) through a perturbation scheme. At the first order, one obtains a linear wave-equation whose solution determines the dispersion relation of the system, not the physical wave. At the second order, the first-order solution appears multiplied by itself, giving rise to a nonlinear evolution equation—the so-called Korteweg-de Vries equation. The solitary wave, which is the physical wave observed in nature, is then obtained as a solution to this nonlinear equation. Of course, in order to obtain a more precise solution, one has to go to higher orders in the perturbation scheme. The important point is to observe that, even for very small wave amplitudes, a solitary wave can never be approximately described by a linear equation. This is a general property of nonlinear waves, of which gravitational waves are just an example.

3.2 Linear gravitational waves

The invariant formalism of differential forms is useful for theoretical discussions. When talking about experiments, however, which are always performed in a particular frame and using apparatuses that suppose a particular coordinate system, the use of a covariant formalism in terms of components is mandatory. As a consequence of the tetrad expansion [12], the metric tensor is expanded in the form

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{(1)\mu\nu} + \cdots, \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{(1)\mu\nu} + \cdots, \quad (16) \]

\[^3\text{It is interesting to remark that even the well-known exact plane gravitational wave solution of Einstein equations [12] transports neither energy nor momentum [13]. This is in accordance with the nonlinear nature of the transport of energy-momentum by gravitational waves.}\]
with $\eta_{\mu\nu}$ the Minkowski metric. In the class of harmonic coordinates, which at first order is represented by the condition
\begin{equation}
\partial_\mu h^{\mu}_{\nu}(1) = 0,
\end{equation}
the field equation (13) assumes the form
\begin{equation}
\Box h^{\mu}_{\nu}(1) = 0,
\end{equation}
with $\Box$ the flat spacetime d’Alambertian operator. A monochromatic plane-wave solution to this equation has the form
\begin{equation}
h^{\mu}_{\nu}(1) = A^{\mu}_{(1)} \nu \exp(ik^\rho x^\rho),
\end{equation}
where $A^{\mu}_{(1)} \nu$ is the (symmetric) polarization tensor, and the wave vector $k^\rho$ satisfies
\begin{equation}
k^\rho k^\rho = 0.
\end{equation}
The harmonic coordinate condition (17), on the other hand, implies
\begin{equation}
k_\mu h^{\mu}_{\nu}(1) = 0.
\end{equation}

Analogously to the Lorentz gauge of electromagnetism, it is possible to further specialize the harmonic class of coordinates to a particular coordinate system. Once this is done, the coordinate system becomes completely specified, and the components $A^{\mu}_{(1)} \nu$ turn out to represent only physical degrees of freedom. A quite convenient choice is the so called transverse-traceless coordinate system (sometimes called transverse-traceless gauge, in analogy to electromagnetism), in which
\begin{equation}
h^{\mu}_{(1)\mu} = 0 \quad \text{and} \quad h^{\mu}_{(1)\nu} U^\nu_{(0)} = 0,
\end{equation}
with $U^\nu_{(0)}$ an arbitrary, constant four-velocity. Now, although the coordinate system has already been completely specified, we still have the freedom to choose different local Lorentz frames $e^a = e^a_{\mu} dx^\mu$. In particular, it is always possible to choose a specific frame in which $U^\nu_{(0)} = \delta^\nu_0$ (see Section 4.3). In this frame, as can be seen from the second of the Eqs. (22),
\begin{equation}
h^{\mu}_{(1)0} = 0
\end{equation}
for all $\mu$. Linear waves satisfying these conditions are usually assumed to represent a plane gravitational wave in the transverse-traceless gauge.

### 3.3 Messing with the perturbation scheme

The next step of the standard approach is to compute the energy and momentum transported by these linear waves. The common procedure is to make use of the second-order energy-momentum pseudo-current [11]. The argument used to justify this procedure is that this is similar to the electromagnetic wave, which in spite of being linear, its energy-momentum tensor is quadratic in the field variables. However, this is a misleading argument. In fact, as discussed in Section 2, the linearity of Maxwell equation does not impose any restriction on the

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4The problem of the non-localizability of the energy and momentum of the gravitational field is not relevant for the present discussion, and will not be considered here.
(quadratic) energy-momentum tensor of the electromagnetic field, which can then be used to compute the energy and momentum transported by electromagnetic waves. In the gravitational case, on the other hand, the linearity of the first-order gravitational field equation (13) restricts the gravitational energy-momentum current to be linear, and consequently to vanish at this order. A quadratic energy-momentum pseudotensor can only appear in orders higher than one. This is the case, for example, of the second-order gravitational field equation (39) below, where \( \gamma^{(2)} \) represents the second-order gravitational energy-momentum pseudotensor. It must, for this reason, represent the energy and momentum transported by second-order gravitational waves. The simultaneous use of quantities belonging to different orders of the perturbation scheme constitutes a clear violation of the method itself, and is for this reason physically and mathematically unacceptable.

4 Problems and obscure points

On account of unjustifiable assumptions, as well as of a misuse of the perturbation scheme, the standard approach to gravitational waves becomes plagued by many inconsistencies and obscure points. In this section, some of these points are discussed.

4.1 The question of the gravitational wave frequency

Gravitational waves are generated, and act on free particles through tidal effects (see, for example, Ref. [3], page 310). These effects, described by the geodesic deviation equation, are well-known to be produced by inhomogeneities in the gravitational field, and like the ocean tides on Earth, occur twice for each complete cycle of the system. In fact, according to the quadrupole radiation formula, gravitational radiation comes out from the source with a frequency that is twice the source frequency \([15]\). However, the plane gravitational wave that emerges from the standard linear approach propagates with the same frequency of the source. To circumvent this problem, one has to artificially adjust by hands the wave frequency, as explained in side-note 8, page 105 of Ref. [2]. This is an inconsistency of the linear approach, which seems to say that the first-order wave does not represent the physical, quadrupole gravitational wave.

4.2 The question of linear curvature

The existence of a non-vanishing first-order Riemann tensor is perhaps one of the main puzzles of the gravitational wave theory. The usual lore is that, if a linear gravitational wave produces a non-vanishing linear curvature tensor, it must exist physically. However, there are a number of points that should be considered. As discussed in Section 2 due to the fact that the gravitational energy-momentum current is at least quadratic in the field variables, the energy-momentum density associated to any linear gravitational field configuration must vanish. A non-vanishing energy density can only appear in orders higher than one. This does not mean that the first-order gravitational field is physically meaningless. As a matter of fact, at the second-order the first-order solutions will appear multiplied by itself, giving rise to nonlinear field configurations with non-vanishing energy-momentum density.
Furthermore, recall that the components of the Riemann tensor $R^{\alpha\beta\mu\nu}$ are not physically meaningful in the sense that they are different in different coordinate systems. For example, starting with the “electric components” $R^{i0j0}$ of the Riemann tensor, through a general coordinate transformation one can get non-vanishing “magnetic components” $R^{i0jk}$. By inspecting the components of the Riemann tensor, therefore, it is not possible to know whether they represent a true gravitomagnetic field produced by a rotating source, or just effects of coordinates. This can only be done by inspecting the invariants constructed out of the Riemann tensor (see Ref. [16], page 355). Now, as a simple computation shows, all invariants constructed out of the first-order Riemann tensor of the linear gravitational waves vanish identically [17–20]. This includes the Kretschmann and the pseudo-scalar invariants, given respectively by

$$R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$$

and

$$\star R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu},$$

with $\star$ denoting the Hodge dual. Considering that these invariants are proportional to the mass/energy and angular momentum of the field configuration (see Ref. [16], page 356-357), it can be immediately concluded that first-order gravitational waves are empty of physical meaning as neither mass/energy nor angular momentum can be attributed to them.

4.3 The question of the effects on free particles

Similarly to the electromagnetic wave, the field components of the first-order gravitational wave (in transverse-traceless coordinates) are orthogonal to the propagation direction. As a consequence, by using the geodesic deviation equation it is concluded that, when passing through two vertically separated particles, the first-order gravitational wave would make them to oscillate orthogonally around the original point. A circumference of free particles would be distorted in such a way that it would become an ellipse, first (let us say) vertically, then horizontally, and so on. The question then arises: how a strictly attractive field like gravitation could give rise to orthogonal oscillations around the original position? This orthogonal oscillation can be easily understood in the electromagnetic case, where the Lorentz force is either attractive or repulsive depending on the sign of the field component. However, in the strictly attractive case of gravitation, it is not clear at all how such orthogonal particles oscillation could be possible.

To see how such effects are obtained in the usual approach to gravitational waves, let us consider two nearby particles separated by the four-vector $\xi^\alpha$. This vector obeys the geodesic deviation equation

$$\nabla_U \nabla_U \xi^\alpha = R^\alpha_{\mu\nu\beta} U^\mu U^\nu \xi^\beta,$$

where $U^\mu = dx^\mu / ds$ is the four-velocity of the particles, with $ds = (g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$. Now, each order of the Riemann tensor expansion

$$R^\alpha_{\mu\nu\beta} = \varepsilon R^\alpha_{(1)\mu\nu\beta} + \varepsilon^2 R^\alpha_{(2)\mu\nu\beta} + \ldots,$$

which follows naturally from [12], will give rise to a different contribution to $\xi^\alpha$. For consistence reasons, therefore, this vector must also be expanded:

$$\xi^\alpha = \xi^\alpha_{(0)} + \varepsilon \xi^\alpha_{(1)} + \varepsilon^2 \xi^\alpha_{(2)} + \ldots.$$
In this expansion, \( \xi^\alpha_{(0)} \) represents the initial, that is, undisturbed separation between the particles. Of course, as the four-velocity \( U^\mu \) depends on the gravitational field, it must also be expanded,

\[
U^\mu = U^\mu_{(0)} + \varepsilon U^\mu_{(1)} + \varepsilon^2 U^\mu_{(2)} + \ldots ,
\]

where \( U^\mu_{(0)} \) is a constant arbitrary four-velocity, which depends on the frame from which the phenomenon will be observed and measured. Choosing a frame fixed at one of the particles, usually called proper frame, the four-velocity \( U^\mu_{(0)} \) can be expressed in terms of the observer proper time \( s_{(0)} \), that is, \( U^\mu_{(0)} = dx^\mu/ds_{(0)} \), where

\[
ds^2_{(0)} = \eta_{\mu\nu}dx^\mu dx^\nu
\]
is the flat spacetime quadratic interval. Since in the proper frame the particles are initially at rest, we have that

\[
U^\mu_{(0)} \equiv \delta^\mu_0 = (1, 0, 0, 0),
\]

which means that, in this frame, the proper time \( s_{(0)} \) coincides with the coordinate \( x^0 \).

At first order, the geodesic deviation equation (24) assumes the form

\[
\frac{d^2 \xi^\alpha_{(1)}}{ds^2_{(0)}} + U^\rho_{(0)} \partial_\rho \left( \Gamma^\alpha_{(1)\beta\gamma} U^{\gamma}_{(0)} \right) \xi^\beta_{(0)} = R^\alpha_{(1)\mu\nu\beta} U^\mu_{(0)} U^\nu_{(0)} \xi^\beta_{(0)}
\]
with

\[
\Gamma^\alpha_{(1)\beta\gamma} = \frac{1}{2} \eta^{\alpha\rho} \left( \partial_\beta h_{(1)\gamma\rho} + \partial_\gamma h_{(1)\rho\beta} - \partial_\rho h_{(1)\beta\gamma} \right)
\]

the first-order Christoffel connection. Substituting \( U^\mu_{(0)} \) given by Eq. (29), it reduces to

\[
\frac{d^2 \xi^\alpha_{(1)}}{ds^2_{(0)}} + \partial_0 \Gamma^\alpha_{(1)\beta0} \xi^\beta_{(0)} = R^\alpha_{(1)00\beta} \xi^\beta_{(0)}.
\]

Using then the first-order Riemann tensor

\[
\begin{align*}
R^\alpha_{(1)\mu\nu\beta} &= \partial_\nu \Gamma^\alpha_{(1)\mu\beta} - \partial_\beta \Gamma^\alpha_{(1)\mu\nu},
\end{align*}
\]
it becomes

\[
\frac{d^2 \xi^\alpha_{(1)}}{ds^2_{(0)}} + \partial_0 \Gamma^\alpha_{(1)\beta0} \xi^\beta_{(0)} = \left( \partial_0 \Gamma^\alpha_{(1)\beta0} - \partial_\beta \Gamma^\alpha_{(1)00} \right) \xi^\beta_{(0)}.
\]

This is the geodesic deviation equation as seen from the proper frame. Although the frame has already been chosen, the coordinate system remains unspecified.

Then comes the crucial step. According to the standard approach, one can always choose a local coordinate system in which the Christoffel connection can be made to vanish not only at a point, but along the whole world-line of the particle. This result is then used to eliminate the connection-term from the left-hand side of Eq. (34), which reduces to

\[
\frac{d^2 \xi^\alpha_{(1)}}{ds^2_{(0)}} = \left( \partial_0 \Gamma^\alpha_{(1)\beta0} - \partial_\beta \Gamma^\alpha_{(1)00} \right) \xi^\beta_{(0)}.
\]
Specializing to transverse-traceless coordinates, and denoting the time derivative with a dot, one gets finally
\[ \frac{d^2 \xi^\alpha}{ds^2_{(0)}} = \frac{1}{2} \partial_0^2 h^{\alpha}_0 \xi^\beta_{(0)}, \]  
from where the usual, orthogonal effects of gravitational waves on free particles are obtained.

However, the above procedure is clearly self-inconsistent. The standard argument used to eliminate the connection-term from the left-hand side of Eq. (34), but not the very same connection-term appearing in the right-hand side of the equation is that, since the two terms on the right hand side make up the Riemann tensor, which owing to its tensorial character can be described in any coordinate system, covariance arguments do not allow the elimination of just a piece of this tensor. Of course, this argument is misleading because the left-hand side as a whole is also a tensor, and the same argument would equally forbid the elimination of just a piece of this tensorial quantity.

Furthermore, any experimental apparatus presupposes fully specified frame and coordinate system. In this particular case, the frame has already been chosen to be the proper frame. On the other hand, since the wave equation for \( h^{(1)\rho}_0 \) was solved in transverse-traceless coordinates, and considering that ultimately one is going to use that solution in the geodesic deviation equation to obtain the effects of gravitational waves on free particles, mathematical consistency does require that this equation be considered in the same coordinate system. One cannot use a coordinate system to eliminate some terms, and then use another coordinates to obtain the physical effects on free particles. Note, in particular, that the Christoffel connection does not vanish in transverse-traceless coordinates.

If instead of the tortuous procedure of the standard approach we accept the first-order geodesic deviation equation in the form it emerges from the perturbation scheme, without choosing any coordinate system we can simply cancel out the connection-term \( \partial_0 \Gamma^\alpha_{(1)\beta} \xi^\beta_{(0)} \) appearing on both sides of that equation. Taking into account that in transverse-traceless coordinates the components \( h^{\gamma}_0 \) vanish, the last term on the right-hand side of Eq. (34) vanishes, and we get
\[ \ddot{\xi}^\alpha_{(1)} = 0, \]  
where we have denoted time derivative with a dot. This equation says that linear gravitational waves are unable to produce any effect on free particles, a result consistent with the fact that linear gravitational waves transport neither energy nor momentum. The natural way to proceed is then to go to the next order.

5 Second-order gravitational waves theory

5.1 Second-order gravitational waves

At the second order, the source energy-momentum tensor is found to be conserved in the covariant sense,
\[ DT^a_{(2)} = dT^a_{(2)} + \Gamma^a_{(1)b} \wedge T^b_{(1)} = 0, \]  
with \( \Gamma^a_{(1)b} \) the first-order spin connection. As is well-known, it is not a true conservation law, but just an identity, called Noether identity, which rules the exchange of energy and...
momentum between the source and gravitation. This means that, differently from what happens at first order, at the second order a mechanical system can lose energy in the form of gravitational waves. The amount of energy and momentum released is that correctly predicted by the quadrupole radiation formula.

At this order, instead of the Maxwell-type field equation, the gravitational field equation acquires the Yang-Mills form

\[ dH_a^{(2)} - \kappa t_a^{(2)} = \kappa T_a^{(2)}, \]

with \( t_a^{(2)} \) the second-order gravitational energy-momentum pseudotensor, which (we repeat) is quadratic in the first-order field variable. Similarly to a gluon field, which is able to transport color charge, the radiative solution of the sourceless version of the field equation is able to transport energy and momentum. It must, for this reason, represent the physical gravitational wave.

In general, the amplitude of a second-order gravitational wave is assumed to fall off with \( 1/r^2 \), where \( r \) is the distance from the source. Due to the large distances from the sources, second-order effects are usually considered to be negligible. However, owing to the intricacies of nonlinear equations, the monochromatic traveling-wave solution to the sourceless version of the field equation has two parts, one that falls off as \( 1/r \) and another that falls off as \( 1/r^2 \). For large enough distances, the dominant part of the solution is of the form

\[ h_{\mu\nu}^{(2)} = B_{\mu\nu}^{(2)} \sin(2k_{\rho}x^\rho), \]

where

\[ B_{\mu\nu}^{(2)} = \frac{A_{\alpha\beta}^{(1)} A_{\alpha\beta}^{(1)}}{8} \frac{K_\alpha x^\alpha}{K_\sigma k^\sigma} k^\mu k^\nu, \]

is the second-order polarization tensor, with \( K_\alpha \) an arbitrary wave number four–vector. Observe that the amplitude \( B_{\mu\nu}^{(2)} \) depends linearly on the distance \( x^\alpha \) from the source, which makes its overall decrease with distance proportional to \( 1/r \). It depends also on the wave number, or equivalently, on the frequency of the wave, a typical property of nonlinear waves. As a simple inspection shows, it satisfies the conditions

\[ B_{(2)\mu}^{\mu} = 0 \quad \text{and} \quad k_\mu B_{(2)\nu}^{\mu} = 0, \]

where we have used the dispersion relation. Notice that, owing to the quadratic nonlinearity of the wave equation, the second-order gravitational wave naturally emerges propagating with a frequency that is twice the frequency of the source. This result constitutes a clear evidence that it is \( h_{\mu\nu}^{(2)} \) (and not \( h_{\mu\nu}^{(1)} \)) that represents the physical, quadrupole gravitational wave (see the discussion presented in Section 4.1).

We consider now a laboratory frame endowed with a Cartesian coordinate system, from which the wave will be observed. If we consider, for example, a wave traveling in the \( z \) direction of the Cartesian system, for which \( k^\mu = (\omega/c, 0, 0, \omega/c) \), the coefficient \( B_{(2)\nu}^{\mu} \) will be of the form

\[ (B_{(2)\nu}^{\mu}) = \frac{A_{\alpha\beta}^{(1)} A_{\alpha\beta}^{(1)}}{8} \frac{K_\alpha x^\alpha \omega^2}{K_\sigma k^\sigma c^2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \]
We see from this expression that the physical components of the wave are
\[ h^0_{(2),0} = -h^\perp_{(2),z} \quad \text{and} \quad h^0_{(2),z} = h^\perp_{(2),0}. \] (44)

### 5.2 Effects on free particles

At second order, the geodesic deviation equation (24) assumes the form
\[ \frac{d^2 \xi^\alpha_{(2)}}{ds^2_{(0)}} = \Gamma^\alpha_{(1),00} \Gamma^\gamma_{(1),00} \xi^\beta_{(0)} + \partial_0 \Gamma^\alpha_{(2),00} \xi^\beta_{(0)} = R^\alpha_{(2),00} U^\mu U^\nu \xi^\beta_{(0)} \] (45)

with
\[ R^\alpha_{(2),00} = \partial_0 \Gamma^\alpha_{(2),00} - \partial_\beta \Gamma^\alpha_{(2),00} + \Gamma^\alpha_{(1),00} \Gamma^\gamma_{(1),00} - \Gamma^\alpha_{(1),00} \Gamma^\gamma_{(1),00} \] (46)

the second-order curvature tensor. Considering that in transverse-traceless coordinates \( \Gamma^\gamma_{(1),00} = 0 \), the geodesic deviation equation reduces to
\[ \frac{d^2 \xi^\alpha_{(2)}}{ds^2_{(0)}} = (\frac{1}{2} \partial_\beta \partial^\alpha_{(2),00} - \partial_\beta \partial^\alpha_{(2),00}) \xi^\beta_{(0)}. \] (47)

Let us suppose now two particles separated initially in the \( x \) direction by a distance \( \xi^x_{(0)} \), that is,
\[ \xi^\beta_{(0)} = (0, \xi^x_{(0)}, 0, 0). \] (48)

Considering a gravitational wave traveling in the \( z \) direction of the laboratory Cartesian coordinates, whose physical components are given by Eq. (44), it is then easy to verify that in this case the resulting equations of motion are
\[ \ddot{\xi}^x_{(2)} = \ddot{\xi}^y_{(2)} = \ddot{\xi}^z_{(2)} = 0, \] (49)

where we have used that in the proper frame \( s_{(0)} = ct \). Of course the same result is obtained for two particles separated initially in the \( y \) direction. We consider now two particles separated initially in the \( z \) direction by a distance \( \xi^z_{(0)} \), that is,
\[ \xi^\beta_{(0)} = (0, 0, 0, \xi^z_{(0)}). \] (50)

In this case, the geodesic deviation equation (47) yields
\[ \ddot{\xi}^x_{(2)} = \ddot{\xi}^y_{(2)} = 0 \] (51)

and
\[ \ddot{\xi}^z_{(2)} = c^2 \left( \partial_\beta \partial^\beta \partial_{(2),z} h_{(2),00} - \frac{1}{2} \partial_\beta \partial_{(2),z} h_{(2),00} \right) \xi^z_{(0)}. \] (52)

In order to obtain the form of the physical effects produced by the second-order gravitational waves on free particles, it is necessary to solve the geodesic deviation equation (52) for the solution (40-41). To begin with, considering the arbitrariness of the wave vector \( K_\rho \), without loss of generality we can choose it in such a way that \( K_0 = K_1 = K_2 = 0 \). In this case, the geodesic deviation equation (52) assumes the form
\[ \ddot{\xi}^z_{(2)} = \xi^z_{(0)} A^{(1)}_{(1)} A^{(1)}_{\alpha\beta} z \omega^3 4c \sin[2\omega(t - z/c)]. \] (53)
Although the wave amplitude decreases with distance, it can be assumed to be constant in the region of the experience. We then write

\[ \ddot{\xi}^{(2)} = \frac{1}{4} \Gamma^{(2)} \omega^2 \sin[2(\omega t - z/\lambda)], \]

(54)

where

\[ \Gamma^{(2)} = A_{(1)}^{(1)} A_{(0)}^{(1)} \frac{z}{\lambda} \]

(55)

with \( \lambda = c/\omega \) the reduced wavelength. Assuming that the particles are initially \( (t = 0) \) at rest, and considering gravitational waves with wavelength much larger than the particle separation \( \xi^{(2)} \), the solution is found to be

\[ \xi^{(2)} \sim -\xi^{(0)} \Gamma^{(2)} \sin(2\omega t). \]

(56)

We see from this solution that, when a gravitational wave passes through two particles separated by a distance \( \xi^{(0)} \) along the direction of propagation of the wave, both particles begin moving towards the source due to the attraction of gravitation. In addition, owing to the inhomogeneity of the gravitational field of the wave, the distance between them will oscillate with frequency \( 2\omega \) as they move. This means that second-order gravitational waves are longitudinal waves. Such properties are consistent with the quadrupole radiation formula, the tidal origin of gravitational waves, as well as with the strictly attractive character of gravitation. According to the second-order approach, therefore, this longitudinal oscillation is the signal to be looked for when searching for gravitational waves.

### 6 Final remarks

Owing to its mathematical similarity to the Yang-Mills equation, the potential (or Lagrangian) form of Einstein equation unveils important aspects of general relativity. In particular, because the energy-momentum pseudotensor of the gravitational field appears explicitly in this field equation, the similarities and differences in relation to Maxwell equation become much more visible. The reason why such form of Einstein equation has seldom been used in gravitation is probably related to the non-existence of an invariant Lagrangian for general relativity that depends on the tetrad and on the first derivative of the tetrad only. As is well known, the Einstein-Hilbert Lagrangian depends, besides on the tetrad and on the first derivative of the tetrad, also on the second derivative of the tetrad.

Taking advantage of the potential form of Einstein equation, a critical review of the foundations of the standard approach to the gravitational waves theory has been made. The existence of obscure points and inconsistencies in the standard approach based on the usual form of Einstein equation has been pointed. A natural way to circumvent these problems is to resignly accept as correct all results emerging from the first-order expansion of Einstein equation, which in turn amounts to accept that gravitational waves cannot be described by a linear equation, even approximately. One should then go to the second order, where a sound and consistent gravitational wave theory shows up. The ensuing second-order gravitational wave is found to be longitudinal, in agreement with both the tidal origin of gravitational waves and the strictly attractive character of gravitation. The second-order wave is in addition found to propagate...
with a frequency that is twice the source frequency, in agreement with the quadrupole radiation formula. Similarly to what happens in the solitary-waves theory, the first-order solution is just a mathematical structure on which the physical solution is constructed. It defines, furthermore, the dispersion relation (20) of the system.

It is clear by now that none of the existing antennas has succeeded in detecting any sign of gravitational waves. Of course, it is possible that the detectors did not meet the necessary sensibility to detect them, or that the magnitude of the gravitational waves when reaching a detector on Earth is smaller than originally expected. However, it is also possible that a faulty approach has led all detectors to look for the wrong sign. The analysis presented in these notes, whose purpose is to call the attention for potential problems in the currently accepted theory, suggests that this possibility should not be neglected. After all, one call always ask: where are the waves?

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