Thermal resitivity due to electron viscosity in bulk antimony

Alexandre Jaoui,1,2 Benoît Fauqué,1 and Kamran Behnia2

1 JEIP, USR 3573 CNRS, Collège de France, PSL Research University, 11, Place Marcelin Berthelot, 75231 Paris Cedex 05, France
2 Laboratoire de Physique et Etude des Matériaux (CNRS/UPMC), Ecole Supérieure de Physique et de Chimie Industrielles, 10 Rue Vauquelin, 75005 Paris, France

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In a fermionic quantum liquid, both viscosity and thermal conductivity decrease upon warming due to the multiplication of fermion-fermion collisions. This effect, predicted and observed in 3He, has been elusive in electronic fluids because, in presence of an imperfect and finite lattice, most electron-electron collisions do not conserve momentum. Here, we report on the observation of this hydrodynamic feature through the scrutiny of the thermal and electrical resistivities of semi-metallic antimony when the carrier mean-free-path approaches the average diameter of the sample. We detect a size-dependent departure from the Wiedemann-Franz law, expected in the hydrodynamic picture (and not in the momentum-relaxing one) of transport in Fermi liquids. Our results permit a quantitative comparison of the rate of fermion-fermion collision in a metal and in liquid 3He.

Soon after the conception of Landau’s Fermi liquid theory, Abrikosov and Khalatnikov [1] calculated the transport coefficients of an isotropic Fermi liquid, focusing on liquid 3He below its degeneracy temperature. They showed that since the phase space for fermion-fermion scattering grows quadratically with temperature T, the viscosity η and the thermal conductivity κ both decrease upon warming following η ∝ T−2 and κ ∝ T−1. The theoretical picture [2] was confirmed by subsequent experiments. Specifically, Greywall [3] confirmed that in normal liquid 3He cooled down to 10mK, thermal conducitivity becomes inversely proportional to temperature: in zero-pressure liquid 3He above the superfluid transition, κT|0 ≃ 0.29mW.m−1.K−1 [3].

This viscosity-driven picture of transport has found little relevance in the case of metallic solids. While the phase space for collisions among electronic quasiparticles grows similarly as the square of temperature, the presence of a crystalline lattice in a metal alters the context. There, electron-electron collisions can degrade the flow of charge and heat by transferring momentum to the underlying lattice (through Umklapp processes for example). Nevertheless, the possibility of viscous electronic flow was suggested long ago by Gurzhi [4] and has attracted attention recently [5–7]. The quasi-particle (QP) flow profile is expected to change when momentum-conserving (MC) collisions among electrons outweigh scattering by boundaries as well as various momentum-relaxing (MR) collisions. In this case, momentum and energy of the QPs will be redistributed over a length much shorter than the resistive mean-free-path. As a consequence, the further away the electron is from the boundaries, the hardest the MC collisions will make it for the QP to make its way to the boundaries of the system. If boundary scattering becomes also more frequent than MR collisions, then the QPs the furthest away from the boundaries are less likely to undergo a dissipative collision. As a consequence, the QP flow becomes analogous to that of a viscous fluid in a channel. Experimentally, viscous corrections to electronic transport properties have been reported in mesoscopic ultra-pure metals [8–10]. Here, we turn our attention to thermal transport and show that, if the electronic mean-free-path is sufficiently long compared to the sample dimensions, a finite κT|0 caused by a viscous flow of electrons becomes accessible experimentally.

A fundamental correlation between the electronic thermal conductivity κe and the electrical conductivity σ is given by the Wiedemann-Franz (WF) law:

$$\frac{\kappa_e}{\sigma T} = \frac{\pi^2 k_B^2}{3 e^2}$$

(1)

The left hand of the equation is the (electronic) Lorenz number, L_e, which can be measured experimentally. The right hand side is a fundamental constant, called the Sommerfeld value $L_0 = 2.44 \times 10^{-8} \text{V}^2\text{K}^{-2}$. The WF law is expected to be valid when inelastic scattering is absent, i.e. at zero temperature.

Principi and Vignale (PV) [6] recently argued that in hydrodynamic electron liquids, the WF law is violated because MC electron-electron (e−e) scattering would degrade thermal current but not electrical current. As a consequence, by drastically reducing the $L_e/L_0$ ratio, electron hydrodynamics would lead to a finite-temperature departure from the WF law. However, the standard transport picture based on MR collisions expects a similar departure at finite temperature as a consequence of inelastic small-angle e−e scattering [14–18]. The two pictures differ in an important feature: the evolution of the $L_e/L_0$ ratio with the carrier lifetime. In the hydrodynamic picture, the deviation from the WF law becomes more pronounced with the relative abundance of MC e−e collisions, which can be amplified by reducing the weight of MR collisions (by enhancing purity or size).

Here, we present a study of heat and charge transport
in semi-metallic antimony (Sb) and find that κ and σ both increase with sample size. Sb is the most magnetoresistant semi-metal [19]. Its extremely mobile charge carriers present mean-free-paths ℓ₀ that depend on sample thickness at low temperature [20]. We begin by verifying the validity of the WF law in the zero-temperature limit and resolving a clear departure from it at finite temperature. This arises because of the inequality between the prefactors of the T-square electrical and thermal resistivities [17]. In contrast to its electrical counterpart, the T-square thermal resistivity (which is equivalent to κ ∝ T⁻¹), can be purely generated by MC scattering which sets the viscosity of the electronic liquid. We find that the departure from the WF law is amplified with the increase in the sample size and the carrier mean-free-path, in agreement with the hydrodynamic scenario [6]. We then quantify κT₀ and the quadratic lifetime of fermion-fermion collisions, τ₂, for electrons in Sb and compare it with that of ³He fermions.

All measurements were carried using a conventional 4-electrode (two thermometers, one heater and a heat sink) setup (further details are given in the supplementary material). The Sb crystals are presented in table I. Electrical and heat currents were applied along the bisectrix direction of all samples. The electrical resistivity, shown in figure 1a, displays a strong size dependence below T = 25K and saturates to larger values in the two thinner samples, as reported previously [20]. As seen in table I, the mean-free-path remains below the average thickness, but tends to increase with the sample average thickness. The thermal conductivity, κ, of the same samples is presented in figure 1b. κ presents a peak whose magnitude and position correlates with sample size and resistivity. In large samples the peak is larger in amplitude and occurs at lower temperatures. Semi-metallic antimony has one electron and one hole for ~600 atoms. The lattice and electronic contributions to the thermal conductivity are comparable in size. The inset of figure 1a shows the temperature dependence of the Seebeck co-

| Sample | Size (mm²) | RRR | \( \rho_0 \) (nΩ.cm) | \( \pi \) (µm) | \( \ell_0 \) (µm) | \( \rho_0 \) \( \pi \) (pΩ m²) | \( A_2 \) (nΩ.cm.K⁻²) | \( B_2 \) (nΩ.cm.K⁻²) |
|--------|------------|-----|----------------|-------------|----------------|----------------|----------------|----------------|
| 1      | (0.25±0.05 × 0.5 × 4.1) | 260 | 159 | 350 | 17 | 0.56 | 0.70 ± 0.03 | 0.81 ± 0.05 |
| 1b     | (0.2 × 0.5 × 4.6) | 250 | 164 | 320 | 16 | 0.49 | 0.73 ± 0.04 | - |
| 2      | (0.4 × 0.4 × 4.1) | 430 | 94.6 | 400 | 28 | 0.38 | 0.56 ± 0.03 | 0.74 ± 0.03 |
| 3      | (1.1 × 1.0 × 10.0) | 3000 | 13.4 | 1050 | 197 | 0.14 | 0.38 ± 0.03 | 0.68 ± 0.04 |
| 3’     | (1.1 × 1.0 × 7.0) | 3000 | 13.4 | 1050 | 197 | 0.14 | 0.38 ± 0.03 | - |
| 4      | (1.0 × 5.0 × 10.0) | 1700 | 24.1 | 2240 | 110 | 0.54 | 0.32 ± 0.04 | 0.63 ± 0.08 |
| 5      | (3.0 × 1.0 × 10.0) | 3700 | 11.1 | 1730 | 238 | 0.19 | 0.33 ± 0.03 | - |
| 6      | (1.7 × 1.8 × 10.0) | 4200 | 9.8 | 1800 | 270 | 0.18 | 0.33 ± 0.03 | - |

TABLE I. Sb crystals used in this study were oriented along the bisectrix axis. \( \pi = \sqrt{\text{width} \times \text{thickness}} \) represents the average diameter of the conducting cross-section. The residual resistivity ratio is defined as \( RRR = \frac{\rho_0}{\rho_{300K}} \). The carrier mean-free-path \( \ell_0 \) was calculated from the residual resistivity and the expression for Drude conductivity assuming three hole and electron pockets (See the supplement for more details). Also given is the product of \( \rho_0 \pi \), a measure of crystalline perfection (See the supplement). The last two columns give the electrical \( (A_2) \) and thermal \( (B_2) \) T²-resistivities prefactors.

FIG. 1. a) Electrical resistivity along the bisectrix direction, \( \rho \), plotted as a function of temperature for the various sizes of Sb samples presented in table I. Inset shows the zero-field thermopower \( S_{xx} \) as a function of the temperature of the same samples. b) Temperature dependence of the thermal conductivity, \( \kappa \), of the aforementioned Sb samples. Inset shows the Lorenz number \( L \) plotted as \( L/L_0 \), where \( L_0 \) is the Sommerfeld number, as a function of temperature. \( L/L_0 = 1 \) corresponds to the recovery of the Wiedemann-Franz law.
efficient in the same samples. The Seebeck coefficient remains below \(5 \mu V/K\), as reported previously \[21\], because of the cancellation between hole and electron contributions to the total Seebeck effect. The small size of the Seebeck response has two important consequences. First, it implies that the thermal conductivity measured in absence of charge current is virtually identical to the one measured in absence of electric field (which is the third Onsager coefficient \[22\]). The second is that the ambipolar contribution to the thermal transport is negligible and \(\kappa = \kappa_e + \kappa_{ph}\) (see the supplement for a discussion of both issues).

The temperature dependence of the overall Lorenz number \((L = (\kappa \rho / T))\) divided by \(L_0\), is plotted as a function of temperature in the inset of figure [4]b. For \(T < 4K\), \(L/L_0 \to 1\). The Wiedemann-Franz (WF) law is almost recovered below \(4K\) in all samples. At higher temperatures, \(L\) displays a non-monotonic and size-dependent temperature dependence resulting from two different effects: a downward departure from the WF law in \(\kappa_e\) and a larger share of \(\kappa_{ph}\) in the overall \(\kappa\).

The application of a magnetic field provides a straightforward way to separate \(\kappa_e\) and \(\kappa_{ph}\) in a semi-metal with very mobile carriers \[23\]. Indeed, under the effect of a magnetic field, the electronic conductivity drastically collapses (the low-temperature magnetoresistance in Sb reaches up to \(5.10^6\%\) at \(1T\), as shown in the supplement) while the lattice contribution is left virtually unchanged. This is visible in the field dependence of \(\kappa\), shown in figure 2a (for sample S4 at \(T = 0.56K\)). One can see a sharp drop in \(\kappa(B)\) below \(B^* \approx 0.5T\) and a saturation at higher fields. The initial drop represents the evaporation of \(\kappa_e\) due to the huge magnetoresistance of the system. The saturation represents the indifference of \(\kappa_{ph}\) towards magnetic field. This interpretation is confirmed by the logarithmic plot in the inset and is further proven by the study of the low-temperature thermal conductivity of Sb as a function of temperature under the effect of several fields presented in the supplement. Below \(B^* \approx 0.1T\), \(L_0T/\rho\) is close to \(\kappa\), indicating that in this field window, heat is carried mostly by electrons and the WF law is satisfied. However, by \(B^* \approx 1T\), \(L_0T/\rho\) is three orders of magnitude lower than \(\kappa\), implying that at this field, heat is now mostly carried by phonons with a vanishing contribution from electrons. The electronic component of thermal conductivity separated from the total thermal conductivity, \((\kappa_e(T) = \kappa(B = 0)(T) - \kappa(B = 1T)(T))\) is shown in figure 2b. One can see that, for all four samples and at sufficiently low temperature, \(\kappa_e/T\) becomes constant (and equal to \(L_0/\rho_0\)). It is the subsequent downward deviation at higher temperatures which will become the focus of our attention. We construct the electronic Lorenz ratio \(L_e = \kappa_e\rho/T\) and show its evolution with temperature in figure 3a. Below \(T < 4K\), \(L_e \approx L_0\) in all samples, save for S3, the cleanest. With increasing temperature, \(L_e/L_0\) diverges down and the deviation becomes larger as the samples become cleaner. Let us scrutinize separately the temperature dependence of the electrical and the thermal resistivities. The latter can be expressed in the familiar units of resistivity (i.e. \(\Omega.m\)), using \(WT = L_0T/\kappa_e\) as a shorthand. Figure 3b shows \(\rho\) and \(WT\) as a function of \(T^2\) for the four different samples. In the low-temperature limit, an asymptotic \(T^2\) behavior is visible in all samples and the two lines corresponding to \(\rho\) and \(WT\) have identical y-axis intercepts, thus confirming the recovery of the WF Law in the zero-temperature limit. In every case, the slope of \(WT(T^2)\) is larger than that of \(\rho(T^2)\), indicating that the prefactor of the thermal \(T\)-square resistivity (dubbed \(B_2\)) is larger than the prefactor of the electrical \(T\)-square resistivity (dubbed \(A_2\)). This behavior, observed for the first time.
in Sb, was previously reported in a handful of metals, namely W [15], WP2 [17], UPt3 [20] and CeRhIn5 [16].

T-square resistivity arises due to $e-e$ collisions. The common explanation for the experimentally observed $B_2 > A_2$ inequality is the under-representation of small-angle scattering in the electrical channel, which damps the electric prefactor $A_2$, but not its thermal counterpart $B_2$ [14-18]. However, as one can see in Figure 3b, the two slopes are further apart in the cleaner samples. The evolution of the two prefactors with sample dimensions is presented in Figure 3c. The figure includes, in addition to ours, previous data on the slope of electrical $T^2$-resistivity [19, 24, 25]. One can see the emergence of a consistent picture: the electrical ($A_2$) prefactor displays a significant size dependence and the $A_2/B_2$ ratio substantially decreases with the increase in sample size and electronic mean-free-path. This constitutes a critical signature of the hydrodynamic picture.

Because of momentum conservation, $e-e$ collisions cannot decay the momentum flow by themselves. Momentum can be relaxed by such collisions through two mechanisms, known as Umklapp and interband (or Baber) scattering. There is at least one established case of T-square resistivity in absence of either mechanisms [27]. Both these mechanisms can lead to a $A_2/B_2$ ratio lower than unity [18]. In the specific case of a com-

FIG. 3. a) Electronic fraction of the Lorenz number $L_e = \kappa_e \rho / T$ plotted as $L_e / L_0$, where $L_0$ is the Sommerfeld number, as a function of temperature. $L_e / L_0 = 1$ corresponds to the recovery of the Wiedemann-Franz law. b) Thermal (WT) and electrical ($\rho$) resistivities plotted as functions of $T^2$ for the four sizes of Sb samples. WT is featured as symbols while $\rho$ is shown as a solid line. All four graphs share a common x-axis and y-axis span. c) Evolution of the electrical and thermal $T^2$-resistivities prefactors with sample size in Sb. Data points from [19, 24, 25] are featured. d) The electronic Lorenz ratio at $T = 10K$, normalized by the Sommerfeld value, plotted as function of the electronic mean-free-path at various temperatures. The solid lines correspond to a fit given by equation $L_e / L_0 = 1 + \ell_0 \ell_{ee}$ as per the hydrodynamic scenario [6]. $\ell_0$ refers to the Drude mean-free-path while $\ell_{ee}(T)$ is the typical distance traveled by a charge carrier in-between two momentum-conserving collisions.
pensated metal, Li and Maslov [13] have argued that the ratio of the two prefactors (and therefore the deviation from the WF law) is tuned by two material-dependent parameters: i) the screening length and ii) the relative weight of interband and intraband scattering. In their picture, increasing the screening length would enhance \( B_2 \) and leave \( A_2 \) unchanged. Enhancing interband scattering would also reduce the Lorenz ratio. Given that neither of these two is expected to change with the crystal size or imperfection, the evolution seen in figure 3.c cannot be explained along either of these two lines.

In contrast, the Principi and Vignale scenario [6] predicts that the deviation from the WF law should become more pronounced with increasing carrier lifetime (or equivalently mean-free-path \( \ell_0 \)): \( L_0/L_0 = 1/(1 + \ell_0/\ell_{ee}) \). Such a picture provides a reasonable account of our observation, as seen in figure 3.d, which shows the variation of \( L_0/L_0 \) at different temperatures with carrier mean-free-path. In this picture, the evolution of the Lorenz ratio with \( \ell_0 \) would imply a mean-free-path for MC \( e-e \) scattering, \( \ell_{ee} \), which ranges from 0.15mm at 10K to 1.1mm at 3.5K. This is in agreement with \( \ell_{ee} \) extracted from the magnitude of \( B_2 \) and \( A_2 \). Assuming that MC \( e-e \) collisions generate the difference between them, one finds \( \ell_{ee}(10K) = 0.09mm \) and \( \ell_{ee}(3.5K) = 0.8mm \) and a reasonably consistent picture emerges. What is still missing in these pictures is an account of boundary scattering. The decrease in \( \rho_0 \) with sample size in elemental metals have been widely documented and analyzed by pondering the relative weight of specular and diffusive scattering [28]. This can also weigh on the magnitude of \( A_2 \) [29]. However, a quantitative account of the experimental data, by employing Soffer’s theory [30], remains unsuccessful [20] [29] [31]. The role of surface roughness acquires original features in the hydrodynamic regime [32], which are yet to be explored by experiments on samples with mirror/matt surface dichotomy.

Since the experimentally-resolved \( T^2 \) resistivity is (at least partially) caused by thermal amplification of momentum exchange between fermionic quasi-particles, we are in a position to quantify \( \kappa T \). Its lower boundary is \( L_0/(A_2 - B_2) \) and the upper boundary \( L_0/B_2 \). This yields \( 3000 < \kappa T |_0 < 7900 \) in units of W.m\(^{-1}\). This is six orders of magnitude larger than in normal liquid \( ^3\)He [3] (see table [1]). Such a difference is not surprising since: i) \( \kappa T |_0 \) of a fermion liquid is expected to scale with the cube of the Fermi momentum \( (p_F) \) and the square of the Fermi velocity \( (v_F) \) [34]; and ii) \( ^3\)He is a strongly correlated Fermi liquid and Sb is not. More specifically \( \kappa T |_0 \) can be written in terms of the Fermi wave-vector \( (k_F) \) and the Fermi energy \( (E_F) \):

\[
\kappa T |_0 = \frac{1}{B_0} \frac{E_F^2 k_F}{\hbar} \tag{2}
\]

This equation is identical to equation 17 in ref. [34], with \( B_0 \) introduced as a dimensionless parameter (see the supplementary material for more details). In \(^3\)He, a strongly interacting and dense quantum fluid, which can be solidified upon a one-third enhancement in density, one finds \( B_0 \gg 1 \). The magnitude of \( \kappa T |_0 \) is consistent with the Landau parameters and the specific heat [34]. In contrast, the electronic fluid in antimony is a dilute gas of weakly interacting fermions and \( B_0 \) is two orders of magnitude lower, as one can see in table [1]. In the first approximation, the large difference in \( B_0 \) reflects the difference in collision cross section caused by the difference in density of the two fluids. Using the kinetic expression \( (\kappa = \frac{1}{3} C v^2 r) \), the \( T^2 \) fermion-fermion scattering rate can be extracted and \( \tau_e T^2 \) can be compared with the case of \(^3\)He [3] [35] [36] (see table [1]). As expected, electronic fermions collide much less frequently and the electronic fluid is many orders of magnitude more viscous than its much denser counterpart. A similar quantification is yet to be done in strongly-correlated electronic fluids.

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TABLE II. Comparison of two Fermi liquids - The density of atoms at ambient pressure in $^3$He is two orders of magnitude larger than the total density of electron-like and hole-like carriers (n+p) in Sb. Also listed in the table are the average Fermi temperature, the average Fermi momentum and the magnitude of the experimentally-resolved $\kappa T_0$ (W.m$^{-1}$). Its natural units are $\frac{E_F}{k_B}$. $B_0$ is defined in the main text. $\tau_0 T^2$ quantifies the rate of fermion-fermion collisions.

| System | Density (cm$^{-3}$) | $T_F$ (K) | $k_F$ (nm$^{-1}$) | $\kappa T_0$ (W.m$^{-1}$) | $\frac{E_F k_F}{\hbar}$ (W.m$^{-1}$) | $B_0$ | $\tau_0 T^2$ (s.K$^2$) |
|--------|---------------------|-----------|-----------------|------------------------|-------------------------------------|------|------------------------|
| $^3$He | $1.63 \times 10^{22}$ | 1.8       | 7.8             | 2.9 $\times 10^{-7}$      | 0.04                                | 137  | 3.9 $\times 10^{-14}$   |
| Sb     | $1.1 \times 10^{20}$ | 1100      | 0.8             | 3900-7900               | 1500                               | 0.4-0.8 | 1.5-3.0 $\times 10^{-8}$ |

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SUPPLEMENTARY MATERIAL FOR 'THERMAL RESISTIVITY DUE TO ELECTRON VISCOITY IN BULK ANTIMONY'

MATERIALS AND METHODS

Samples

Sb crystals were commercially obtained through MaTeck GmbH. Their dimensions are given in Table 1 of the main text. Samples S1, S1b and S2 were cut from a lingot of Sb using a wire saw. Samples S3, S4, S5 and S6 were prepared by MaTeck to the aforementioned dimensions: sample S4 was cut while samples S3, S5 and S6 were etched to these dimensions. Sample S3 was measured before and after a cut of a few mm perpendicular to the bisectrix direction. The long axis of all samples were oriented along the bisectrix direction.

The high mobility of charge carriers in Sb leads to a very large magnetoresistance, as reported in Ref.[19]. The samples presented in this study confirm this. As an example, the magnetoresistance of sample S6 at \( T = 2K \) and \( B = 9T \) is shown in figure S1.a. This large magnetoresistance translates into a suppression of the electronic thermal conductivity through the Wiedemann-Franz law. As a consequence, the separation of lattice and electronic contributions of \( \kappa \) becomes straightforward. The mobility and the magnetoresistance of the samples used in this study are shown in figure S1.b and compared to other semi-metals. One can see that carriers in Sb are extremely mobile compared to most other semi-metals.

Table S1 compares the electronic properties of Sb with a few other semi-metals. Figure S2 shows the magnitude of the electrical \( T^2 \)-resistivity prefactor \( A_2 \) in four different semi-metals. One can see that \( A_2 \) decreases with increasing Fermi temperature, as previously noted in the case of numerous dilute metals [37]. The correlation between \( A_2 \) and \( E_F \) is an extension of the Kadowaki-Woods correlation [38] to low-density systems.

### Table S1

| Semi-metal | \( n = p \) (cm\(^{-3}\)) | \( \rho_0 \) (\( \mu \Omega \cdot cm \)) | \( \mu_0 \) (m\(^2\)V\(^{-1}\)s\(^{-1}\)) | \( m^* / (m_0) \) | \( T_{F,e} \) (K) | \( T_{F,h} \) (K) | References |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|----------|
| Sb         | \( 5.5 \times 10^{19} \) | \( \sim 0.05 \) | \( \sim 100 \) | 0.07 - 1 | 1080 | 980 | [21, 33] |
| Bi         | \( 3.0 \times 10^{17} \) | \( \sim 1 \) | \( \sim 1000 \) | 0.001 - 0.612 | 320 | 120 | [33, 39] |
| WP\(_2\)   | \( 2.5 \times 10^{21} \) | \( \sim 0.005 \) | \( \sim 400 \) | 0.7 - 1.9 | 3000 | 2000 | [40, 41] |
| WTe\(_2\)  | \( 6.8 \times 10^{19} \) | \( \sim 1 \) | \( \sim 5 \) | 0.1 - 1.2 | 480 | 230 | [42] |

Table S1. Comparison of prominent semi-metals. We compare the electronic concentration \( n \), typical residual resistivity \( \rho_0 \), typical electronic mobility \( \mu_0 \), carriers effective mass \( m^* \) and typical Fermi temperature of electrons \( T_{F,e} \) and holes \( T_{F,h} \) in these materials. References used to construct this table are featured in the last column.

FIG. S1. a) Magnetoresistance of sample S6 at \( T = 2K \). b) Magnetoresistance of various semi-metals at \( B = 9T \) and \( T = 2K \) as a function of the mobility \( \mu_0 = 1/(\rho_0(n+p)e) \) where \( e \) is the elementary charge, \( n \) and \( p \) are the electron and hole densities and \( \rho_0 \) the zero field resistivity at \( T = 2K \). \( \mu_0 \) is expressed in Tesla\(^{-1}\) = \( 10^4 \)cm\(^2\)V\(^{-1}\)s\(^{-1}\).
FIG. S2. Electrical $T^2$-resistivity prefactor ($A_2$) plotted as a function of the Fermi temperature for the semi-metals discussed in table S1. $T_F$ was taken for both electrons and holes for Sb [21, 33], Bi [33, 39], WTe$_2$ [42] and WP$_2$ [40, 41] and the prefactor $A_2$ for Sb was taken from this work, Bi [21], WTe$_2$ [42] and WP$_2$ [17].

Measurement method

The thermal conductivity measurements were performed with a home-built one-heater-two-thermometers set-up. The thermometers were either Cernox 1030 or RuO$_2$ chips. Our setup was designed to allow the measurement of both the thermal conductivity $\kappa$ and the electrical resistivity $\rho$ with the same electrodes. The thermometers were either directly glued to the samples with Dupont 4922N silver paste or contacts were made using 25$\mu$m-diameter silver wires connected to the samples via silver paste (Dupont 4922N). Contact resistance was inferior to 1Ω. The thermometers were thermally isolated from the sample holder by manganin wires with a thermal conductance several orders of magnitude lower than that of the Sb samples and silver wires. The samples were connected to a heat sink (made of copper) with Dupont 4922N silver paste on one side and to a RuO$_2$ chip resistor serving as a heater on the other side. Both heat and electrical currents were applied along the bisectrix direction. The heat current resulted of an applied electrical current $I$ from a DC current source (Keithley 6220) to the RuO$_2$ heater. The heating power was determined by $I \times V$ where $V$ is the electric voltage measured across the heater by a digital multimeter (Keithley 2000). The thermal conductivity was checked to be independent of the applied thermal gradient by changing $\Delta T/T$ in the range of 10%. Special attention was given not to exceed $\Delta T/T_{\text{max}} = 10\%$. The thermometers were calibrated in-situ during each experiment and showed no evolution with thermal cycling. Special attention was given to suppress any remanent field applied to the sample and self-heating effects. The accuracy of our home-built setup was checked by the recovery of the Wiedemann-Franz law in an Ag wire at $B = 0$T and $B = 10$T through measurements of the thermal conductivity and electrical resistivity. At both magnetic fields, the WF was recovered at low temperatures with an accuracy of $1\%$ [17].

ESTIMATION OF THE ELECTRONIC MEAN-FREE-PATH

We evaluated the mean-free-path of charge carriers by assuming the Fermi surface of Sb to be composed of three hole and three electron pockets in the spherical approximation. The average Fermi wavevector was taken to be $k_F = (3\pi^2(n/3))^{1/3} = 0.82$nm$^{-1}$ from a carrier concentration $n = p = 5.5 \times 10^{19}$ cm$^{-3}$. $n$ and $p$ are respectively the electron and hole carrier concentrations, which are equal down to $10^{-4}$ in Sb [19]. The mean-free-path was then
FIG. S3. Thermal conductivity plotted as $\kappa/T$ as a function of $T^2$ in sample S4 for three successive magnetic fields. The symbols show the experimental values of $\kappa/T$ and the dotted lines are the linear fit while the value of $L_0/\rho_B$ are featured as arrows. The recovery of the Wiedemann-Franz law at each magnetic field is shown by the intercept of the dotted lines and arrows.

evaluated through equation S1 where $\rho_0$ is the residual resistivity of the sample.

$$\ell_0 = \frac{\pi}{4} \left( \frac{1}{\rho_0} \right) \left( \frac{\hbar}{e^2} \right) \left( \frac{1}{k_F^2} \right)$$  \hspace{1cm} (S1)

Note that in this estimation, the anisotropy of the Fermi surface pockets is neglected. If the crystal is ‘perfect’, that is, if the electrons do not encounter any scattering center along their trajectories, then the mean-free-path is set by the finite size of the sample. In this case, the size of the Fermi surface (in other words the carrier concentration) sets a lower boundary:

$$(\rho_0^s)_{\text{min}} = \frac{\pi}{4} \left( \frac{\hbar}{e^2} \right) \left( \frac{1}{k_F^2} \right) = 0.03 \text{p}\Omega\text{m}^2$$  \hspace{1cm} (S2)

As seen in Table 1 of the main text, our best Sb crystal (S3) is not ‘perfect’, because $\rho_0^s = 0.14 \text{p}\Omega\text{m}^2$. The lowest reported value for a Sb crystal, which we found in scientific literature was $(\rho_0^s \approx 0.1 \text{p}\Omega\text{m}^2)$ slightly lower than ours.

LOW FIELD & LOW TEMPERATURE RECOVERY OF THE WIEDEMANN-FRANZ LAW IN SB

Figure S3 shows the thermal conductivity plotted as $\kappa/T$ as a function of $T^2$ in sample S4 in the low temperature region (where we showed the WF law to be satisfied in the main text) for three different magnetic fields. The arrows point to the value of $L_0/\rho_B$. We observe that the arrow and y-axis intercept of the linear fit match for the three magnetic fields : the WFL is recovered under the effect of these three fields. Furthermore, the slope of the linear fit to $\kappa/T(T^2)$ remains similar for the different fields. This implies that the magnetic field does not affect the lattice thermal conductivity.

THERMAL CONDUCTIVITY AND THE THIRD ONSAGER COEFFICIENT

What we have measured is the thermal conductivity measured in absence of charge current. It is to be distinguished from the thermal conductivity measured in absence of electric field, which is a pure diagonal Onsager coefficient. 

However, in our case, the distinction is totally negligible. The heat current density, $J^Q$ and the particle flow density, $J^N$ are Onsager fluxes responding to Onsager forces: $\nabla \mu / T$ and $1 / T \nabla \mu$.

$$- J^N = L_{11} \frac{1}{T} \nabla \mu + L_{12} \nabla \frac{1}{T}$$  \hspace{1cm} (S3)

$$J^Q = L_{12} \frac{1}{T} \nabla \mu + L_{22} \nabla \frac{1}{T}$$  \hspace{1cm} (S4)

The thermal conductivity, $\kappa$, in absence of charge current ($J^e = 0$) and the one, $\kappa'$ in absence of potential gradient ($\nabla \mu = 0$) are to be distinguished. The latter is inversely proportional to the Onsager coefficient $L_{22}$:

$$\kappa' = \frac{1}{T^2} L_{22}$$  \hspace{1cm} (S5)

The former is a combination of all three Onsager coefficients and its magnitude is equal to:

$$\kappa = \kappa'(1 - \frac{S^2 \sigma T}{\kappa}) = \kappa'(1 - \frac{S^2}{L})$$  \hspace{1cm} (S6)

In our case, since $S < 5 \times 10^{-6} V/K$ and $L \sim L_0 = 2.45 \times 10^{-8} V^2/K^2$, one has $\frac{S^2}{T} < 0.001$, implying a negligible difference.

**AMBIPOLAR THERMAL CONDUCTIVITY**

The electronic thermal conductivity of a semi-metal includes monopolar contributions from both electrons ($\kappa_e$) and holes ($\kappa_h$) as well as an ambipolar one associated with electron-hole pairs ($\kappa_{eh}$). This last contribution is negligible in Sb at $T \ll T_F$.

Heremans *et al.* showed that the ambipolar contribution to thermal conductivity $\kappa_{eh}$ can be written as equation $S7$.

\[ \kappa_{eh} = \frac{\pi^2 k_B}{3e} \left( \frac{T}{\sigma_e \sigma_h} \right) \left( \frac{k_B T}{E_{F,e}} + \frac{k_B T}{E_{F,h}} \right)^2 \]  \hspace{1cm} (S7)

In the temperature range of interest of the present study, $T < 10K$, the Fermi energy of holes and electrons in Sb (featured in table SI) leads to $\left( \frac{k_B T}{E_{F,i}} \right)^2 \approx 10^{-4}$. This implies, at best, an ambipolar correction to the Lorenz number $L_{eh} = 5.10^{-4} L_0$ at $T = 10K$. Such a correction falls within the experimental error bars of this study and is consequently neglected in our discussion. The small magnitude of the Seebeck coefficient confirms this conclusion.

**VISCOSITY AND THERMAL CONDUCTIVITY IN FERMI LIQUIDS**

Abrikosov and Khatalnikov [1] in their 1959 seminal paper calculated the viscosity of a Fermi liquid and found that:

\[ \eta T^2 = \frac{64 \hbar^3 p_F^3}{45 m^*} < W_\eta > \]  \hspace{1cm} (S8)

Here $< W_\eta >$ is a temperature-independent parameter representing the angular average of scattering amplitude for viscosity, $\eta$, expected to decrease with warming as $T^{-2}$. The same collisions lead to a thermal conductivity expressed as:

\[ \kappa T = \frac{8 \pi^2 \hbar^3 p_F^3}{3 m^*} < W_\kappa > \]  \hspace{1cm} (S9)

$< W_\kappa >$, like $< W_\eta >$, is neither dimensionless nor universal. The amplitude of both depends on the strength and the anisotropy of interaction and, in the case of $^3$He, strongly depends on the spin components of the overlapping wave-functions. Numerous experiments confirmed that $\eta \propto T^{-2}$ [55, 45, 46] and $\kappa \propto T^{-1}$ [3, 47]. In the case of
thermal conductivity, the most elaborate set of measurements performed by Greywall [3] found that at zero pressure, the asymptotic value for $\kappa T$ is $\kappa T|_0 = 2.9 \times 10^{-4}$ W m$^{-1}$. Calkoen and van Weert [34] showed that in the zero temperature limit, one can write:

$$\kappa T|_0 = \frac{5}{18\pi^3} \frac{p_F^2 v_F^2}{A^2} \quad (S10)$$

In this equation, one has taken $\hbar = 1$. Our equation 2, in the main text, is based on this one. In order to enhance clarity, we have introduced the dimensionless parameter $B_0$, which is simply proportional to $A^2$:

$$B_0 = \frac{9\pi^3 A^2}{10\hbar^2} \quad (S11)$$

Calkoen and van Weert [34] found that in $^3$He, a nearly ferromagnetic liquid, the magnitude of $A$ and its variation with pressure is compatible with the Landau parameters extracted from specific heat data [48]. The very low magnitude of $B_0$ in Sb (three orders of magnitude lower than in $^3$He) indicates that fermion-fermion collisions are rare in this dilute electron gas.

**QUASI-PARTICLE RELAXATION TIME IN FERMI LIQUIDS**

The fundamental reason behind the temperature dependence of $\eta$ and $\kappa$ is the quadratic temperature dependence of the relaxation time, which can be written as [36]:

$$\frac{\hbar}{\tau_{qp}} = \frac{(\pi k_B T)^2}{32 E_F} < A >_{\theta, \phi} \quad (S12)$$

Here $< A >_{\theta, \phi}$ represents the angular averages of quasi-particle scattering amplitudes for transition between spin singlet and spin triplet states [35]. In the case of $^3$He, measurements of viscosity [35] and thermal conductivity [3] have found values for $\tau_\kappa T^2$ and $\tau_\eta T^2$ close to each other.

In the case of the thermal conductivity, $\tau_\kappa T^2$ can be extracted from the heat capacity per volume $C_v$, using:

$$\tau_\kappa = \frac{3 \kappa}{C_v v_F^2} \quad (S13)$$

As in the case of $^3$He, we have used the electronic specific heat of Sb ($\gamma = 0.105$ mJ mol$^{-1}$ K$^{-2}$ [49]), together with the average Fermi velocity to calculate $\tau_\kappa T^2$ in Sb.