The $C^0$-convergence at the Neumann boundary for Liouville equations

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Abstract

In this paper, we study the blow-up analysis for a sequence of solutions to the Liouville type equation with exponential Neumann boundary condition. For interior case, i.e. the blow-up point is an interior point, Li (Commun Math Phys 200(2):421–444, 1999) gave a uniform asymptotic estimate. Later, Zhang (Commun Math Phys 268(1):105–133, 2006) and Gluck (Nonlinear Anal 75(15):5787–5796, 2012) improved Li’s estimate in the sense of $C^0$-convergence by using the method of moving planes or classification of solutions of the linearized version of Liouville equation. If the sequence blows up at a boundary point, Bao–Wang–Zhou (J Math Anal Appl 418:142–162, 2014) proved a similar asymptotic estimate of Li (1999). In this paper, we will prove a $C^0$-convergence result in this boundary blow-up process. Our method is different from Gluck (2013), Zhang (2006).

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1 Introduction

The compactness of a sequence of solutions to a nonlinear equation plays an important role in the study of the existence problem. For most of interesting geometric partial differential equations, one can not use variational methods directly to get the existence result, such as harmonic maps, minimal surface, Liouville equation and so on. The main reason lies in the lack of compactness of the solution space. To overcome this type of problem, an important method called blow-up analysis was employed to provide better information of solution space.

The blow-up analysis for a sequence of solutions to Liouville type equation has been widely studied since the work of Brezis–Merle [3] where a concentration-compactness phenomenon of solutions was revealed. Later, Li–Shafrir [10] initiated to study the blow-up value at the blow-up point, which is quantized, i.e. at each blow-up point, the blow-up value is \(8\pi m\) for some positive integer \(m\). We emphasize that there is no boundary condition in Li–Shafrir [10]. If certain boundary assumption is imposed, for example the oscillation on the boundary is uniformly bounded, then Li [9] proved that there is only one bubble at each blow-up point, which implies that the blow-up value is \(8\pi\). Roughly speaking, let \(u_k\) be a sequence solutions to Liouville equation

\[-\Delta u = e^u, \text{ in } B_1(0)\]  

with uniformly bounded energy

\[\int_{B_1(0)} e^{u_k(x)} \, dx \leq C\]  

and boundary condition

\[\text{osc}_{\partial B_1(0)} u_k \leq C.\]  

Suppose \(x = 0\) is the only blow-up point for this sequence. Then Li [9] proved that there exists a sequence of points \(x_k \to 0\) as \(k \to \infty\), such that passing to a subsequence, there holds

\[\left\| u_k(x) - u_k(x_k) - v \left( \frac{x - x_k}{e^{\frac{1}{2}u_k(x_k)}} \right) \right\|_{C^0(B_1(0))} \leq C, \quad \forall \, k,\]  

where \(v(x) = -2 \log(1 + \frac{1}{8} |x|^2)\). Furthermore, if we assume a stronger boundary condition that

\[\text{osc}_{\partial B_1(0)} u_k = o(1),\]

Chen–Lin [4], Zhang [13] and Gluck [6] established a type of \(C^0\)-convergence result of (1.4) as follows

\[\lim_{k \to \infty} \left\| u_k(x) - u_k(x_k) - v \left( \frac{x - x_k}{e^{\frac{1}{2}u_k(x_k)}} \right) \right\|_{C^0(B_1(0))} = 0.\]  

Motivated by the question of prescribed geodesic curvature, it is also interesting to study the Liouville equation with a Neumann boundary condition. For the blow-up analysis of Liouville type equation near the Neumann boundary, Guo–Liu [7] proved Brezis–Merle type concentration phenomenon and Li–Shafrir type quantization property. Later, Bao–Wang–Zhou [1] extended Li’s work [9] to the boundary case. More precisely, they proved that there
is only one bubble near a boundary blow-up point and a similar convergence result of (1.4) also holds near the boundary if the oscillation on the boundary is uniformly bounded.

In this paper, we want to establish a similar $C^0$-convergence of (1.5) near a boundary blow-up point. Before stating the main result, we make some notations.

Let $B_r(x_0)$ be the ball in $\mathbb{R}^2$ with radius $r$ centered at $x_0$. Let $\partial B_r(x_0)$ be the boundary of $B_r(x_0)$. Denote

$$B^+_r(x_0) := \{ y = (y^1, y^2) \in B_r(x_0) | y^2 > 0 \}$$
and

$$\partial^+ B^+_r(x_0) := \{ y = (y^1, y^2) \in \partial B_r(x_0) | y^2 > 0 \}, \quad \partial^0 B^+_r(x_0) := \{ y = (y^1, y^2) \in B_r(x_0) | y^2 = 0 \}.$$

Denote

$$\mathbb{R}^2_a := \{ y = (y^1, y^2) \in \mathbb{R}^2 | y^2 > -a \}$$
for some $a \geq 0$ and

$$\partial \mathbb{R}^2_a := \{ y = (y^1, y^2) \in \mathbb{R}^2 | y^2 = -a \}.$$

For simplicity of notations, we always denote $B_1(0), B^+_1(0), B_R(0), B^+_R(0)$ and $\mathbb{R}^2_0$ by $B, B^+, B_R, B^+_R$ and $\mathbb{R}^2_+$ respectively.

Now we consider the following equation

$$\begin{cases}
-\Delta u = e^u, & \text{in } B^+, \\
\frac{\partial u}{\partial n} = e^u, & \text{on } \partial^0 B^+,
\end{cases} \tag{1.6}$$

where $n$ is the unit outer normal vector on the boundary.

Let $u_k$ be a sequence of solutions of (1.6) with uniformly bounded energy

$$\int_B e^{u_k} dx + \int_{\partial^0 B^+} e^{u_k} ds_x \leq C < \infty \tag{1.7}$$

and 0 be its only blow-up point in $B^+_1(0)$, i.e.

$$\max_{K \subset \subset B^+} u_k \leq C(K), \quad \max_{B^+} u_k \to +\infty. \tag{1.8}$$

Assume the boundary condition as

$$\text{osc}_{\partial^+ B^+} u_k = o(1). \tag{1.9}$$

Our main result is as follows

**Theorem 1.1** Let $u_k$ be a sequence of solutions of (1.6) with conditions (1.7), (1.8) and (1.9). Then there exists a sequence of points $\{x_k\} \subset \overline{B^+}$ such that, passing to a subsequence, there hold:

1. $x_k \to 0$ and $u_k(x_k) = \sup_{B^+} u_k(x) \to +\infty$ as $k \to \infty$;
2. Denote $\lambda_k = e^{-\frac{1}{2}u_k(x_k)}$ and $\Omega_k := \{ x \in \mathbb{R}^2 | x_k + \lambda_k x \in B^+ \}$. Then
   $$\lim_{k \to \infty} \frac{\text{dist}(x_k, \partial^0 B^+)}{\lambda_k} = a < +\infty$$
   and $\Omega_k \to \mathbb{R}^2_a$ as $k \to \infty$.  

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(3) Denote $v_k(x) = u_k(x_k + \lambda_k x) + 2 \log \lambda_k$. Then
\[
\lim_{k \to \infty} \|v_k(x) - v(x)\|_{C^t(B_R(0) \cap \Omega_k)} = 0, \quad \forall R > 0,
\]
where
\[
v(x) = \log \frac{8 \lambda^2}{\lambda^2 + (x^1 + s_0)^2 + \left( x^2 + a + \frac{1}{\sqrt{2}} \right)^2},
\]
x = (x^1, x^2) ∈ ℝ^2
for some $\lambda > 0$ and $s_0 ∈ ℝ$, which satisfies $v(0) = 0$ and
\[
\begin{cases}
-\Delta v = e^v & \text{in } \mathbb{R}_+^2, \\
\frac{\partial v}{\partial n} = e^{\frac{v}{2}} & \text{on } \partial \mathbb{R}_+^2.
\end{cases}
\]

(4) The $C^0$-convergence:
\[
\lim_{k \to \infty} \left\| u_k(x) - u_k(x_k) - v\left( \frac{x - x_k}{\lambda_k} \right) \right\|_{C^0(B^+)} = 0.
\]

To prove Theorem 1.1, we focus on establishing the $C^0$-convergence in the blow-up process, i.e. the fourth conclusion of Theorem 1.1 since conclusions (1)–(3) are standard now by [1, 7, 14].

For conclusion (4), by a standard blow-up analysis, it is not hard to see that we only need to show
\[
\lim_{k \to \infty} \left\| u_k(x) - u_k(x_k) - v\left( \frac{x - x_k}{\lambda_k} \right) \right\|_{\text{osc}(B^+ \setminus B^+_{\frac{1}{2}R_k}(x_k))} = 0,
\]
for some $R_k \to +\infty$. We divide the oscillation estimate into two parts: $B^+_{\frac{1}{2}}(x_k) \setminus B^+_{\frac{1}{2}R_k}(x_k)$ and $B^+ \setminus B^+_{\frac{1}{2}}(x_k)$. For the first part, we will prove that both the tangential and radial oscillation are small. To control the tangential oscillation, we will use Green’s formula to derive a pointwise estimate of $\nabla v_k$, where there appears a bad term $|\log \text{dist}(x, \partial \mathbb{R}_+)|$. One may see that this term is not bounded if $x$ goes to the boundary even if $|x|$ is big. To overcome this obstacle, we will use the method of integration and the symmetry of Green’s function for the unit ball, see Corollary 3.3. To control the radial oscillation, the key point is to derive the following type energy decay
\[
\alpha(t) = 4 + \frac{O(1)}{t^p}, \quad t \in \left[ R_k, \frac{1}{2} \lambda_k^{-1} \right]
\]
for some $p > 0$, where $\alpha(t)$ is the energy defined by (2.13). This will be derived by combining Pohozaev’s type identity, the method of integration and above point estimate of $\nabla v_k$, see Lemma 3.4. With the help of energy decay estimate, we will use an ODE method to get the control of radial oscillation, see Proposition 3.5. To estimate the second part, we will use the PDE’s theory including $L^p$-theory, Green’s formula and maximal principle to prove the oscillation is small.

The rest of paper is organized as follows. In Sect. 2, we will prove some basic lemmas including a bounded oscillation lemma and a fast decay lemma near the boundary which will
be used in our later proof. In Sect. 3, we first derive a point estimate of $\nabla v_k$. Then additionally by using Pohozaev’s type identity and the method of integration, we will establish a energy decay lemma which is crucial in our proof. Finally, we will prove Theorem 1.1 at the end of this section.

2 Some basic Lemmas

In this section, we will recall some classical blow-up analysis for Liouville type equation, and prove some lemmas which will be used in our later proof, such as a bounded oscillation estimate near the boundary, a fast decay lemma near the boundary and so on.

We start by recalling some standard theory in the blow-up analysis of Liouville type equation. See [1, 3, 7, 9, 10]. Let $x_k \in \overline{B^+}$ be the point such that

$$u_k(x_k) = \sup_{B^+} u_k(x).$$

By (1.8), it is easy to check that

$$x_k \to 0 \ and \ u_k(x_k) \to +\infty \ as \ k \to \infty.$$

Set $\lambda_k := e^{-\frac{1}{2} u_k(x_k)}$ and

$$v_k(x) := u_k(x_k + \lambda_k x) + 2 \log \lambda_k.$$

Denote

$$\Omega_k := \{ x \in \mathbb{R}^2 | x_k + \lambda_k x \in B^+ \}$$

and

$$d_k := \text{dist}(x_k, \partial B^+).$$

Then we have the following two cases.

**Case 1:** $\lim_{k \to \infty} \frac{d_k}{\lambda_k} = \infty$.

In this case, it is easy to check that $v_k(x)$ is well defined in the domain $\Omega_k$ which converges to the whole plane $\mathbb{R}^2$. By the standard theory of Liouville equation, it is well known that $v_k(x) \to v(x)$ in $C^2_{loc}(\mathbb{R}^2)$, where $v(x)$ satisfies Liouville equation

$$-\Delta v(x) = e^v, \ in \ \mathbb{R}^2 \ with \ \int_{\mathbb{R}^2} e^v \, dx \leq C < +\infty. \quad (2.1)$$

It follows from the classification result in [5] that

$$v(x) = -2 \log \left( 1 + \frac{|x|^2}{8} \right), \ \int_{\mathbb{R}^2} e^v \, dx = 8\pi.$$

**Case 2:** $\lim_{k \to \infty} \frac{d_k}{\lambda_k} = a < \infty$.

In this case, the domain $\Omega_k$ converges to $\mathbb{R}^2_a := \{ x = (x^1, x^2) \in \mathbb{R}^2 | x^2 \geq -a \}$. Since

$$v_k(0) = 0, \ v_k(x) \leq 0, \ \forall x \in \Omega_k$$

and

$$|\Delta v_k(x)| \leq C, \ \left| \frac{\partial v_k}{\partial n} (x) \right| \leq C,$$
by a standard elliptic theory involving a Harnack inequality near a Neumann-type boundary (see Lemma A.2 in [8]), we claim that
\[ \| v_k \|_{L^\infty(B^+_R)} \leq C(R) \text{ uniformly in } B^+_R. \tag{2.2} \]

In fact, denoting \( \tilde{x}_k = (0, -\frac{d_k}{\lambda_k}) \) and setting 
\[ \tilde{v}_k(x) := v_k(x + \tilde{x}_k), \]
we can check that \( \tilde{v}_k(x) \leq 0, \tilde{v}_k(-\tilde{x}_k) = 0 \) and
\[ \begin{cases} -\Delta \tilde{v}_k = e^{\tilde{v}_k}, & \text{in } B^+_1, \\ \frac{\partial \tilde{v}_k}{\partial n} = e^{\tilde{v}_k}, & \text{on } \partial B^+_1. \end{cases} \]

Let \( w_k \) be the solution of
\[ \begin{cases} -\Delta w_k = -\Delta \tilde{v}_k, & \text{in } B^+_2, \\ \frac{\partial w_k}{\partial n} = 0, & \text{on } \partial B^+_2, \\ w_k = 0, & \text{on } \partial B^+_2. \end{cases} \]

Extend \( w_k \) evenly, then \( \| w_k \|_{L^\infty(B^+_2)} \leq C(R). \) Set \( \eta_k = \tilde{v}_k - w_k \), then
\[ \begin{cases} -\Delta \eta_k = 0, & \text{in } B^+_2, \\ \frac{\partial \eta_k}{\partial n} = \frac{\partial \tilde{v}_k}{\partial n}, & \text{on } \partial B^+_2, \\ \eta_k = \tilde{v}_k, & \text{on } \partial B^+_2. \end{cases} \]

By the Harnack inequality of Lemma A.2 in [8]) and the fact that \( |\eta_k(-\tilde{x}_k)| \leq C < +\infty \), we have \( \| \eta_k \|_{L^\infty(B^+_2)} \leq C(R), \) which implies the claim (2.2) immediately.

By (2.2) and the standard elliptic theory, we know that \( v_k \) converges in \( C^2(B_R(0) \cap \Omega_k \cap \mathbb{R}^2) \) to a function \( v \) which satisfies
\[ \begin{cases} -\Delta v = e^v, & \text{in } \mathbb{R}^2, \\ \frac{\partial v}{\partial n} = e^\frac{v}{2}, & \text{on } \partial \mathbb{R}^2, \end{cases} \]

and
\[ \int_{\mathbb{R}^2} e^v dx + \int_{\partial \mathbb{R}^2} e^\frac{v}{2} ds \leq C < \infty. \]

Using the classification result in [11], we know
\[ v(x) = \log \frac{8\lambda^2}{\left( \lambda^2 + (x^1 + s_0)^2 + \left( x^2 + a + \frac{s_0}{\sqrt{2}} \right)^2 \right)^2}, \quad x = (x^1, x^2) \in \mathbb{R}^2 \]
for some \( \lambda > 0 \) and \( s_0 \in \mathbb{R} \) and
\[ \int_{\mathbb{R}^2} e^v dx + \int_{\partial \mathbb{R}^2} e^\frac{v}{2} ds = 4\pi. \]
Moreover, we can choose a sequence $R_k \to \infty$ such that $\lambda_k R_k \to 0$ and passing to a subsequence, there holds
\[
\|v_k(x) - v(x)\|_{C^1(B_{2R_k} \cap \Omega_k \cap \mathbb{R}^2)} = o(1) \quad \text{and} \quad \int_{B_{R_k} \cap \Omega_k} e^{v_k(x)} \, dx \\
+ \int_{B_{R_k} \cap \partial^0 \Omega_k} e^{\frac{1}{2} v_k(x)} \, ds_x = 4\pi + o(1),
\]
where $\partial^0 \Omega_k := \{ x = (x^1, x^2) \in \overline{\Omega_k} \mid x^2 = -\frac{d_k}{\lambda_k} \}$.

**Lemma 2.1** Under the assumptions of Theorem 1.1, case (1) will not happen, i.e. only case (2) holds. Moreover, we have
\[
\int_{\Omega_k \setminus B_{R_k}} e^{v_k(x)} \, dx + \int_{\partial^0 \Omega_k \setminus B_{R_k}} e^{\frac{1}{2} v_k(x)} \, ds_x = o(1).
\]

**Proof** It follows from the results in [1] or [14] that
\[
\int_{B^+} e^{u_k(x)} \, dx + \int_{\partial B^+} e^{\frac{u_k(x)}{2}} \, ds_x = 4\pi + o(1).
\]
If case (1) holds, then it is easy to get that
\[
\int_{B^+} e^{u_k(x)} \, dx \geq 8\pi,
\]
which is a contradiction. Thus case (1) will not happen. Since case (2) implies (2.3), the second conclusion of the lemma follows immediately from (2.4).

**Remark 2.2** So far, one can see that we have proved conclusions (1)–(3) of Theorem 1.1. In the sequel, we focus on the proof of conclusion (4).

Since the solution has only one bubble, by the selection of bubbling areas in [10], we have the following lemma.

**Lemma 2.3** Under the assumptions of Theorem 1.1, we have
\[
u_k(x) + 2 \log |x - x_k| \leq C.
\]

**Proof** From (2.3), we see
\[
v_k(x) + 2 \log |x| \leq C, \quad |x| \leq 2R_k, \ x \in \Omega_k,
\]
which implies
\[
u_k(x) + 2 \log |x - x_k| \leq C, \quad |x - x_k| \leq 2\lambda_k R_k, \ x \in B^+.
\]
Now we claim
\[
u_k(x) + 2 \log |x - x_k| \leq C, \quad x \in B^+.
\]
Otherwise, there exists $y_k \in \overline{B^+}$ such that $|y_k - x_k| \geq 2\lambda_k R_k$ and
\[
u_k(y_k) + 2 \log |y_k - x_k| \rightarrow +\infty.
\]
Denote $t_k := \frac{1}{2} |x_k - y_k|$ and
\[
\phi_k(x) := u_k(x) + 2 \log(t_k - |x - y_k|).
\]

\[Springer\]
Let $p_k$ be the maximal point such that
\[ \phi_k(p_k) = \max_{x \in B^+_{y_k}} \phi_k(x). \]

It is easy to check that
\[ \phi_k(p_k) \geq \phi_k(y_k) \to +\infty, \]
which also implies $u_k(p_k) \to +\infty$.

Denote
\[ s_k := \frac{1}{2}(t_k - |p_k - y_k|) \quad \text{and} \quad \epsilon_k := e^{-\frac{1}{2}u_k(p_k)}. \]

Since $\phi_k(p_k) \to +\infty$, we have
\[ \frac{s_k}{\epsilon_k} \to +\infty. \]

Noting that for any $x \in B^+_{s_k}(p_k)$, there hold
\[ u_k(x) + 2 \log |t_k - |x - y_k|| \leq \phi_k(p_k) = u_k(p_k) + 2 \log (2s_k) \]
and
\[ t_k - |x - y_k| = t_k - |x - p_k| - |p_k - y_k| \geq s_k, \]
we get
\[ u_k(x) \leq u_k(p_k) + 2 \log 2, \quad \forall x \in B^+_{s_k}(p_k). \]

Set
\[ w_k(x) = u_k(p_k + \epsilon_k x) + 2 \log \epsilon_k, \quad p_k + \epsilon_k x \in B^+. \]

It is easy to check that
\[ p_k + \epsilon_k x \in B^+, \quad w_k(0) = 0, \quad w_k(x) \leq 2 \log 2, \quad \forall |x| \leq \frac{s_k}{\epsilon_k}. \]

Now, we distinguish the following two cases.

**Case 1:** $\lim_{k \to \infty} \frac{\text{dist}(p_k, \partial^0 B^+)}{\epsilon_k} = +\infty$.

In this case, we see that $w_k(x)$ is well defined in a domain which converges to the whole plane $\mathbb{R}^2$ as $k \to \infty$. Then by the standard theory of Liouville type equation, we know that $w_k(x) \to w(x)$ in $C^2_{\text{loc}}(\mathbb{R}^2)$, where $w(x)$ satisfies Liouville equation (2.1). Moreover, we can select a sequence $R_k^1 \to \infty$ such that $R_k^1 \epsilon_k = o(1)s_k$ and passing to a subsequence, there holds
\[ \|w_k(x) - w(x)\|_{C^0(B^+_{R_k^1})} \to 0. \]

It is easy to check that
\[ B^+_{s_k R_k}(x_k) \cap B^+_{\epsilon_k R_k^1}(p_k) = \emptyset. \]

Hence
\[ \int_{B^+} e^{u_k(x)} \, dx \geq \int_{B^+_{\epsilon_k R_k^1}} e^{u_k(x)} \, dx = 8\pi + o(1), \]
which is a contradiction.

Case 2: \( \lim_{k \to \infty} \text{dist}(p_k, \partial^0 B^+) = b < +\infty. \)

In this case, the domain \( \Omega^1_k = \{ x \in \mathbb{R}^2 \mid p_k + \epsilon_k x \in B^+ \} \) converges to \( \mathbb{R}^2_b \) as \( k \to \infty \).

By a similar argument as before, we know that \( w_k(x) \to w(x) \) in \( C^1(B_R(0) \cap \Omega^1_k) \), where \( w(x) \) is a solution of

\[
\begin{cases}
-\Delta w = e^w & \text{in } \mathbb{R}^2_b, \\
\frac{\partial w}{\partial n} = e^\frac{w}{2} & \text{on } \partial \mathbb{R}^2_b.
\end{cases}
\]

Moreover, we can select a sequence \( R_k^+ \to \infty \) such that \( R_k^+ \epsilon_k \to o(1) \) and passing to a subsequence, there holds

\[ \| w_k(x) - w(x) \|_{C^1(B_{R_k^+} \cap \Omega^1_k)} \to 0. \]

It is easy to check that

\[ B^+_{\delta_k R_k}(x_k) \cap B^+_{\epsilon_k R_k}(p_k) = \emptyset. \]

As a result,

\[
\begin{aligned}
\int_{B^+} e^{u_k(x)} \, dx + \int_{\partial^0 B^+} e^{\frac{1}{2} u_k(x)} \, ds_x \\
&\geq \int_{B^+_{\delta_k R_k}(x_k)} e^{u_k(x)} \, dx + \int_{\partial^0 B^+_{\delta_k R_k}(x_k)} e^{\frac{1}{2} u_k(x)} \, ds_x + \int_{B^+_{\epsilon_k R_k}(p_k)} e^{u_k(x)} \, dx \\
&\quad + \int_{\partial^0 B^+_{\epsilon_k R_k}(p_k)} e^{\frac{1}{2} u_k(x)} \, ds_x = 8\pi + o(1),
\end{aligned}
\]

which is also a contradiction.

We proved the lemma. \( \square \)

With the help of Lemma 2.3, we have the following bounded oscillation estimate near the boundary.

**Lemma 2.4** Let \( u_k \) be a sequence of solutions of (1.6) satisfying (1.7)–(1.8). Then

\[ \text{osc}_{B^+_{\frac{1}{2} d_k(x)}} (x) u_k \leq C, \quad \forall x \in B^+ \setminus \{x_k\}, \]

where \( d_k(x) = |x - x_k| \) and \( C \) is a universal constant independent of \( k \) and \( x \).

**Proof** Let \( w_k \) be the solution of

\[
\begin{cases}
-\Delta w_k = -\Delta u_k, & \text{in } B^+, \\
\frac{\partial w_k}{\partial n} = 0, & \text{on } \partial^0 B^+, \\
w_k = 0, & \text{on } \partial^+ B^+.
\end{cases}
\]

Define

\[ \phi_k(x) := -\frac{1}{\pi} \int_{\partial^0 B^+} \log|x - y| e^{\frac{u_k(y)}{2}} \, ds_y, \quad x \in B^+. \quad (2.5) \]
Then we can check
\[
\begin{cases}
-\Delta \phi_k = 0, & \text{in } B^+, \\
\frac{\partial \phi_k}{\partial n} = e^{\frac{u_k(x)}{2}}, & \text{on } \partial B^+.
\end{cases}
\tag{2.6}
\]

Letting \( \eta_k = u_k - w_k - \phi_k \), there holds
\[
\begin{cases}
-\Delta \eta_k = 0, & \text{in } B^+, \\
\frac{\partial \eta_k}{\partial n} = 0, & \text{on } \partial B^+, \\
\eta_k = u_k, & \text{on } \partial B^+.
\end{cases}
\tag{2.7}
\]

Extending \( \eta_k \) evenly, by the maximal principle, we have
\[
osc_{B^+} \eta_k(x) \leq osc_{\partial B^+} u_k(x) = o(1).
\]

Extending \( w_k \) and \( u_k \) evenly (we still use the same notations), by Green’s formula, we have
\[
w_k(x) = \int_{B_1(0)} G(x, y)e^{u_k(y)}dy,
\]
where
\[
G(x, y) = -\frac{1}{2\pi} \log |x - y| + H(x, y)
\]
is the Green’s function on \( B_1 \) with respect to the Dirichlet boundary and \( H(x, y) \) is a smooth harmonic function.

For any \( x \in B^+ \setminus \{x_k\} \) and \( p_1, p_2 \in B^+_3d_k(x), \) we have
\[
w_k(p_1) - w_k(p_2) + \phi_k(p_1) - \phi_k(p_2)
\]
\[
= \int_{B_1(0)} (G(p_1, y) - G(p_2, y)) e^{u_k(y)}dy - \frac{1}{\pi} \int_{\partial B^+} \log |p_1 - y| e^{\frac{u_k(y)}{2}}ds_y
\]
\[
= -\frac{1}{2\pi} \int_{B} \log |p_1 - y| e^{u_k(y)}dy - \frac{1}{\pi} \int_{\partial B^+} \log |p_1 - y| e^{\frac{u_k(y)}{2}}ds_y + O(1)
\]
\[
= -\frac{1}{2\pi} \int_{B_3d_k(x)} \log |p_1 - y| e^{u_k(y)}dy - \frac{1}{\pi} \int_{\partial B^+_3d_k(x)} \log |p_1 - y| e^{\frac{u_k(y)}{2}}ds_y + O(1)
\]
\[
= I + II + O(1).
\]

We first estimate \( II \). Noting that \( |y - x| \geq \frac{3}{4}d_k(x) \), we find
\[
\left| \log \frac{|p_1 - y|}{|p_2 - y|} \right| \leq C,
\]
which implies \( II = O(1) \).

For \( I \), a direct computation yields
\[
|I| \leq C \int_{B_3d_k(x)} \log \frac{|p_1 - y|}{|p_2 - y|} e^{u_k(y)}dy + C \int_{\partial B^+_3d_k(x)} \log \frac{|p_1 - y|}{|p_2 - y|} e^{\frac{u_k(y)}{2}}ds_y
\]
By (2.3), we know one can easily find that
\[ |x + d_k(x)y - x_k| \geq |x - x_k| - |d_k(x)y| \geq \frac{1}{4}d_k(x), \; \forall y \in B_r^+(0), \]
by Lemma 2.3, we have
\[ u_k(x + d_k(x)y) \leq C - 2 \log |x + d_k(x)y - x_k| \leq C - 2 \log d_k(x), \; \forall y \in B_r^+(0). \]
From
\[ \left| \frac{p_1 - x}{d_k(x)} \right| \leq \frac{1}{2}, \; \left| \frac{p_2 - x}{d_k(x)} \right| \leq \frac{1}{2}, \]
one can easily find that \( I = O(1) \), which implies the conclusion of the lemma. \( \square \)

At the end of this section, we prove the following fast decay lemma, which will be used in the next section to estimate a upper bound of solutions and to control the radial oscillation of solutions. This type of fast decay definition was firstly introduced in [12].

**Lemma 2.5** There exists a number sequence \( \{N_k\} \) which tends to \( +\infty \) as \( k \to \infty \), such that
\[ v_k(x) + 2 \log |x| \leq -N_k, \; |x| \geq R_k, \; x \in \Omega_k. \]

**Proof** By (2.3), we know
\[ v_k(x) + 2 \log |x| \leq -2 \log R_k + C, \; \forall R_k \leq |x| \leq 2R_k, \; x \in \Omega_k. \]
Denote \( \tilde{x}_k := (0, -\frac{d_k}{\lambda_k}) \) and
\[ \tilde{v}_k(x) = v_k(x + \tilde{x}_k), \; x \in B_r^+ \setminus B_{\frac{1}{2}\lambda^{-1}}. \] (2.8)
Noting that \( |\tilde{x}_k| \leq C \), it is easy to check that
\[ \tilde{v}_k(x) + 2 \log |x| \leq -2 \log R_k + C, \; \forall x \in B_{2R_k} \setminus B_{R_k}^+. \] (2.9)
Set
\[ \tilde{v}_k^+(r) := \frac{1}{\pi r} \int_{\partial^+ B_r^+} \tilde{v}_k(x) d\theta, \; 0 < r \leq \frac{3}{4}\lambda^{-1}_k. \] (2.10)
On one hand, by Lemma 2.4, we have
\[ osc_{\partial B_r^+} \tilde{v}_k(x) \leq osc_{\frac{3}{4}r \leq |x| \leq \frac{3}{2}r, x \in \Omega_k} v_k(x) \leq C, \; \forall r \in \left[R_k, \frac{1}{4}\lambda^{-1}_k\right]. \] (2.11)
Combining (2.9) and (2.11), we obtain
\[ \tilde{v}_k^*(2R_k) + 2 \log(2R_k) \leq -2 \log R_k + C. \] (2.12)

On the other hand, defining
\[ \alpha(r) := \frac{1}{\pi} \left( \int_{B^+_r} e^{\tilde{v}_k(x)} dx + \int_{\partial^0 B^+_r} \frac{\partial}{\partial n} \tilde{v}_k(x) ds_x \right) \] (2.13)
and making a direct computation, we have
\[ \frac{d}{dr} \tilde{v}_k^*(r) = \frac{1}{\pi r} \int_{\partial^+ B^+_r} \frac{\partial}{\partial n} \tilde{v}_k(x) d\theta \]
\[ = \frac{1}{\pi r} \int_{B^+_r} \Delta \tilde{v}_k(x) dx - \frac{1}{\pi r} \int_{\partial^0 B^+_r} \frac{\partial}{\partial n} \tilde{v}_k(x) ds_x \]
\[ = -\frac{1}{\pi r} \int_{B^+_r} e^{\tilde{v}_k(x)} dx - \frac{1}{\pi r} \int_{\partial^0 B^+_r} \frac{\partial}{\partial n} \tilde{v}_k(x) ds_x := -\frac{1}{r} \alpha(r). \] (2.14)
Thus, for any \( t \in [2R_k, \frac{1}{4} \lambda_k^{-1}] \), there holds
\[ \tilde{v}_k^*(t) + 2 \log t = \tilde{v}_k^*(2R_k) + 2 \log(2R_k) + \int_{2R_k}^t \frac{d}{dr} \left( \tilde{v}_k^*(r) + 2 \log r \right) dr \]
\[ = \tilde{v}_k^*(2R_k) + 2 \log(2R_k) + \int_{2R_k}^t \frac{2 - \alpha(r)}{r} dr \]
\[ \leq \tilde{v}_k^*(2R_k) + 2 \log(2R_k) \leq -2 \log R_k + C, \]
where the last second inequality follows from
\[ \alpha(r) \geq \frac{1}{\pi} \left( \int_{B^+_r} e^{\tilde{v}_k(x)} dx + \int_{B^+_r \cap \partial^0 \Omega_k} e^{\tilde{v}_k(x)} ds_x \right) = \frac{4\pi + o(1)}{\pi} = 4 + o(1) \]
and the last inequality follows from (2.12).

Combining this with (2.11) and (2.9), we get
\[ \tilde{v}_k(x) + 2 \log |x| \leq -2 \log R_k + C, \quad \forall x \in B^{+, -1}_{\frac{1}{\lambda_k}} \setminus B^+_{R_k}, \]
and hence
\[ v_k(x) + 2 \log |x| \leq -2 \log R_k + C, \quad \forall x \in B^{+, -1}_{\frac{1}{\lambda_k}} \setminus B_{R_k}, \quad x \in \Omega_k. \]
Combining this with the fact that
\[ \text{osc}_{x \in B_{\frac{1}{\lambda_k}} \setminus B_{\frac{1}{\lambda_k}}^{+, -1}} v_k \leq C, \]
and using Lemma 2.4, we conclude that
\[ v_k(x) + 2 \log |x| \leq -2 \log R_k + C, \quad \forall x \in B^{+, -1}_{\frac{1}{\lambda_k}} \setminus B_{R_k}, \quad x \in \Omega_k. \]
Therefore the conclusion of the lemma follows immediately by taking \( N_k = 2 \log R_k + C \).
3 Proof of Theorem 1.1

In this section, we first derive a uniformly upper bound of \( v_k \) in its whole definition domain and then prove a pointwise estimate of \( \nabla v_k \) when \(|x|\) is large, which yields the control of tangential oscillation. We will also derive a energy decay lemma which is crucial in our proof. The proof of Theorem 1.1 will be given at the end of this section.

Lemma 3.1  For any \( \delta \in (0, \frac{1}{2}) \), we have
\[
v_k(x) \leq -(4 - 2\delta) \log |x| + C, \quad x \in \Omega_k, \quad |x| \geq 2,
\]
for \( k \) large enough.

Proof The proof of the lemma is more or less standard now. See for example [1, 2]. For reader’s convenience and our later proof, here we give a detailed proof.

By Lemma 2.4, we know
\[
osc_{\Omega_k \setminus B_{\frac{1}{2}\lambda_k - 1}} v_k(x) \leq C.
\]
So it suffices to prove that
\[
v_k(x) \leq -(4 - 2\delta) \log |x| + C, \quad 2 \leq |x| \leq \frac{1}{2}\lambda_k - 1, \quad x \in \Omega_k.
\]

By (2.3), we know
\[
v_k(x) \leq -4 \log |x| + C, \quad 2 \leq |x| \leq R_k, \quad x \in \Omega_k.
\]
Hence, we only need to show that
\[
v_k(x) \leq -(4 - 2\delta) \log |x|, \quad R_k \leq |x| \leq \frac{1}{2}\lambda_k - 1, \quad x \in \Omega_k.
\]
Considering \( \lim_{k \to \infty} \frac{d_k}{\lambda_k} = a < \infty \), we find that the above estimate is equivalent to
\[
\tilde{v}_k(x) \leq -(4 - 2\delta) \log |x|, \quad x \in B^+_{\frac{1}{2}\lambda_k - 1} \setminus B^+_{R_k}, (3.1)
\]
where \( \tilde{v}_k(x) \) is defined by (2.8) which satisfies
\[
\begin{align*}
-\Delta \tilde{v}_k(x) &= e^{\tilde{v}_k(x)}, \quad \text{in } B^+_{\frac{1}{2}\lambda_k - 1}, \\
\frac{\partial \tilde{v}_k(x)}{\partial \vec{n}} &= e^{\frac{\tilde{v}_k(x)}{2}}, \quad \text{on } \partial^0 B^+_{\frac{1}{2}\lambda_k - 1}, \\
\tilde{v}_k(x) &\leq 0, \quad \text{on } \partial^+ B^+_{\frac{1}{2}\lambda_k - 1}.
\end{align*}
\]

Let \( w_k \) be the solution of
\[
\begin{align*}
-\Delta w_k &= -\Delta \tilde{v}_k, \quad \text{in } B^+_{\frac{1}{2}\lambda_k - 1}, \\
\frac{\partial w_k}{\partial \vec{n}} &= 0, \quad \text{on } \partial^0 B^+_{\frac{1}{2}\lambda_k - 1}, \\
w_k &= 0, \quad \text{on } \partial^+ B^+_{\frac{1}{2}\lambda_k - 1}.
\end{align*}
\]

Define
\[
\phi_k(x) := -\frac{1}{\pi} \int_{\partial^0 B^+_{\frac{1}{2}\lambda_k - 1}} \log |x - y| e^{\tilde{v}_k(y)} ds_y, \quad x \in B^+_{\frac{1}{2}\lambda_k - 1}, (3.3)
\]
It is easy to see that
\[
\begin{aligned}
-\Delta \phi_k &= 0, \quad \text{in } B^+_{\frac{1}{2}\lambda_k}, \\
\partial \phi_k / \partial n &= e^{\frac{\bar{\nu}_k(x)}{x}}, \quad \text{on } \partial B^+_{\frac{1}{2}\lambda_k}.
\end{aligned}
\] (3.4)

Letting \(\eta_k = \bar{v}_k - w_k - \phi_k\), there holds
\[
\begin{aligned}
-\Delta \eta_k &= 0, \quad \text{in } B^+_{\frac{1}{2}\lambda_k}, \\
\partial \eta_k / \partial n &= 0, \quad \text{on } \partial B^+_{\frac{1}{2}\lambda_k}, \\
\eta_k &= \bar{v}_k, \quad \text{on } \partial^+ B^+_{\frac{1}{2}\lambda_k}.
\end{aligned}
\] (3.5)

Extending \(\eta_k\) evenly, by maximal principle, we have
\[
\eta(x) - \eta_k(0) \leq \text{osc}_{B^+_{\frac{1}{2}\lambda_k}} \eta_k(x) \leq \text{osc}_{\partial B^+_{\frac{1}{2}\lambda_k}} \bar{v}_k(x) \leq \text{osc}_{B^+ \setminus B^+_{\frac{1}{2}\lambda_k}} u_k(x) \leq C
\] (3.6)

where the last inequality follows from Lemma 2.4.

Now we show that
\[
w_k(x) + \phi_k(x) - w_k(0) - \phi_k(0) \leq -(4 - 2\delta) \log |x| + O(1), \quad R_k \leq |x| \leq \frac{1}{2}\lambda_k^{-1},
\] (3.7)

which implies (3.1) since \(\bar{v}_k(0) \leq C\).

Extending \(w_k\) evenly, by Green’s formula, we have
\[
w_k(x) = \int_{B^+_{\frac{1}{2}\lambda_k}} G(2\lambda_k x, 2\lambda_k y)e^{\bar{v}_k(y)}dy, \quad \forall \, x \in B^+_{\frac{1}{2}\lambda_k}.
\]

A direct computation yields
\[
w_k(x) + \phi_k(x) - w_k(0) - \phi_k(0)
= -\frac{1}{2\pi} \int_{B^+_{\frac{1}{2}\lambda_k}} \log \frac{|x-y|}{|y|} e^{\bar{v}_k(y)}dy - \frac{1}{\pi} \int_{\partial B^+_{\frac{1}{2}\lambda_k}} \log \frac{|x-y|}{|y|} e^{\frac{1}{2}\bar{v}_k(y)}ds_y + O(1)

= -\log |x| \left[ \frac{1}{2\pi} \int_{B^+_{\frac{1}{2}\lambda_k}} e^{\bar{v}_k(y)}dy + \frac{1}{\pi} \int_{\partial B^+_{\frac{1}{2}\lambda_k}} e^{\frac{1}{2}\bar{v}_k(y)}ds_y \right]

- \frac{1}{2\pi} \int_{B^+_{\frac{1}{2}\lambda_k}} \log \frac{|x-y|}{|xy|} e^{\bar{v}_k(y)}dy

- \frac{1}{\pi} \int_{\partial B^+_{\frac{1}{2}\lambda_k}} \log \frac{|x-y|}{|xy|} e^{\frac{1}{2}\bar{v}_k(y)}ds_y + O(1)

= a \left( \frac{1}{2}\lambda_k^{-1} \right) - \frac{1}{2\pi} \int_{B^+_{\frac{1}{2}\lambda_k}} \log \frac{|x-y|}{|xy|} e^{\bar{v}_k(y)}dy

- \frac{1}{\pi} \int_{\partial B^+_{\frac{1}{2}\lambda_k}} \log \frac{|x-y|}{|xy|} e^{\frac{1}{2}\bar{v}_k(y)}ds_y + O(1),
\] (3.8)

where \(a(r)\) is defined by (2.13).
We claim that
\[- \frac{1}{2 \pi} \int_{B_1^{-1}} \frac{1}{|x-y|} \log |x-y| e^{\frac{1}{2} \tilde{v}_k(y)} dy = o(1) \log |x|, \quad \forall R_k \leq |x| \leq \frac{1}{2} \lambda_k^{-1} \quad (3.9)\]
and
\[- \frac{1}{\pi} \int_{\partial B_1^{-1}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} ds_y = o(1) \log |x|, \quad \forall R_k \leq |x| \leq \frac{1}{2} \lambda_k^{-1}. \quad (3.10)\]

In fact,
\[
\int_{B_1^{-1}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} dy = \int_{\{y \in B_1^{-1} \mid |y| \geq |x|\}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} dy
+ \int_{\{y \in B_2^{-1} \mid |y-x| \geq \frac{1}{2}|x|, |y| \geq R_k\}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} dy
+ \int_{\{|y-x| \leq \frac{1}{2}|x|\}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} dy + \int_{\{|y| \leq R_k\}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} dy
= I_1 + I_2 + I_3 + I_4.
\]

Similarly,
\[
\int_{\partial B_1^{+}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} ds_y = \int_{\{y \in \partial B_1^{+} \mid |y| \geq |x|\}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} ds_y
+ \int_{\{y \in \partial B_2^{+} \mid |y-x| \geq \frac{1}{2}|x|, |y| \geq R_k\}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} ds_y
+ \int_{\{y \in \partial B_1^{+} \mid |y-x| \leq \frac{1}{2}|x|\}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} ds_y
+ \int_{\{|y| \leq R_k\}} \frac{1}{|x-y|} e^{\frac{1}{2} \tilde{v}_k(y)} ds_y
= II_1 + II_2 + II_3 + II_4.
\]

For $I_1$ and $II_1$, since $|y| \geq 2|x|$, it holds
\[
\frac{1}{2|x|} \leq \frac{1}{|x|} - \frac{1}{|y|} \leq \frac{|x-y|}{|x||y|} \leq \frac{1}{|x|} + \frac{1}{|y|} \leq \frac{3}{2|x|},
\]
which implies
\[
|I_1| \leq C \log |x| \int_{B_1^{-1}} e^{\tilde{v}_k(y)} dy = o(1) \log |x|
\]
and

$$|\mathbf{I}_1| \leq C \log |x| \int_{y \in \partial B_1^{+}, |y| \geq R_k} e^{\frac{1}{2} v_k(y)} ds_y = o(1) \log |x|.$$  

For $\mathbf{I}_2$ and $\mathbf{II}_2$, since $|y| \leq 2|x|$ and $|y - x| \geq \frac{1}{2}|x|$, it holds

$$\frac{1}{2} |y| \leq \frac{|x - y|}{|x||y|} \leq \frac{1}{2} + \frac{1}{|y|} \leq \frac{3}{|y|},$$

which implies

$$\left| \log \frac{|x - y|}{|y|} \right| \leq C \log |y| \leq C \log |x|.$$  

Hence,

$$|\mathbf{I}_2| \leq C \log |x| \int_{B_{\frac{1}{2} R_k}^{+} \setminus \overline{B}_{R_k}} e^{\int v_k(y)} dy = o(1) \log |x|$$

and

$$|\mathbf{II}_2| \leq C \log |x| \int_{y \in \partial B_1^{+}, |y| \geq R_k} e^{\frac{1}{2} v_k(y)} ds_y = o(1) \log |x|.$$  

For $\mathbf{I}_3$ and $\mathbf{II}_3$, since $|y - x| \leq \frac{1}{2}|x|$, it holds that $\frac{1}{2}|x| \leq |y| \leq \frac{3}{2}|x|$. Hence we have

$$\mathbf{I}_3 = \int_{|y - x| \leq \frac{1}{2}|x|} \log |x - y| e^{v_k(y)} dy - \int_{|y - x| \leq \frac{1}{2}|x|} (\log |x| + \log |y|) e^{v_k(y)} dy$$

$$= \int_{|y - x| \leq \frac{1}{2}|x|} \log |x - y| e^{v_k(y)} dy + o(1) \log |x|$$

$$= \int_{|y - x| \leq \frac{1}{2}|x|} \log |x - y| \frac{o(1)}{|y|^2} dy + o(1) \log |x|$$

$$= \frac{o(1)}{|x|^2} \int_{|y - x| \leq \frac{1}{2}|x|} \log |x - y| dy + o(1) \log |x| = o(1) \log |x|$$

where we have used the fact that $v_k(y) + 2 \log |y| \leq -N_k$ which follows from Lemma 2.5. Similarly,

$$\mathbf{II}_3 = \int_{y \in \partial B_1^{+}, |y - x| \leq \frac{1}{2}|x|} \log |x - y| e^{\frac{1}{2} v_k(y)} ds_y$$

$$= \int_{y \in \partial B_1^{+}, |y - x| \leq \frac{1}{2}|x|} (\log |x| + \log |y|) e^{\frac{1}{2} v_k(y)} ds_y$$

$$= \int_{y \in \partial B_1^{+}, |y - x| \leq \frac{1}{2}|x|} \log |x - y| e^{\frac{1}{2} v_k(y)} ds_y + o(1) \log |x|$$

$$= \int_{y \in \partial B_1^{+}, |y - x| \leq \frac{1}{2}|x|} \log |x - y| \frac{o(1)}{|y|} ds_y + o(1) \log |x|$$
\[\frac{o(1)}{|x|} \int_{y \in \partial^0 B^+_{2 \lambda^{-1}_k}, |y - x| \leq \frac{1}{2}|x|} \log |x - y| ds_y + o(1) \log |x| = o(1) \log |x|.
\]

For \(I_4\) and \(\Pi_4\), since \(|y| \leq R_k\) and \(|x| \geq 2R_k\), we have
\[\frac{1}{2|y|} \leq \frac{1}{|y|} - \frac{1}{|x|} \leq \frac{|x - y|}{|x||y|} \leq \frac{1}{|y|} + \frac{1}{|y|} \leq 3 \frac{1}{2|y|}.\]

Since \(\tilde{u}_k(x) \leq 0\), we get
\[|I_4| \leq C \int_{|y| \leq R_k} \log |y| |e^{\tilde{u}_k(y)}| dy \]
\[\leq C \int_{|y| \leq 2} \log |y| |e^{\tilde{u}_k(y)}| dy + C \int_{2 \leq |y| \leq R_k} \log |y| |e^{\tilde{u}_k(y)}| dy \]
\[\leq C \int_{|y| \leq 2} \log |y| dy + C \int_{2 \leq |y| \leq R_k} \log |y| \frac{1}{|y|^4} dy \leq C\]

and
\[|\Pi_4| \leq C \int_{y \in \partial^0 B^+_{2 \lambda^{-1}_k}, |y| \leq R_k} \log |y| |e^{\tilde{u}_k(y)}| ds_y \]
\[\leq C \int_{y \in \partial^0 B^+_{2 \lambda^{-1}_k}, |y| \leq 2} \log |y| |e^{\tilde{u}_k(y)}| ds_y + C \int_{y \in \partial^0 B^+_{2 \lambda^{-1}_k}, 2 \leq |y| \leq R_k} \log |y| |e^{\tilde{u}_k(y)}| ds_y \]
\[\leq C \int_{y \in \partial^0 B^+_{2 \lambda^{-1}_k}, |y| \leq 2} \log |y| ds_y + C \int_{y \in \partial^0 B^+_{2 \lambda^{-1}_k}, 2 \leq |y| \leq R_k} \log |y| \frac{1}{|y|^4} ds_y \leq C.\]

As a result, (3.9) and (3.10) hold true. Therefore, by (3.8), we get
\[w_k(x) + \phi_k(x) - w_k(0) - \phi_k(0) = -\alpha \left(\frac{1}{2} \lambda^{-1}_k\right) \log |x| + o(1) \log |x| + O(1), \quad R_k \]
\[\leq |x| \leq \frac{1}{2} \lambda^{-1}_k,\]

which yields (3.7) since
\[\alpha \left(\frac{1}{2} \lambda^{-1}_k\right) = 4 + o(1)\]
by Lemma 2.1. We proved the lemma. \(\square\)

With the help of Lemma 3.1, we get a pointwise estimate of first derivative as follows.

**Lemma 3.2** We have
\[\nabla \tilde{v}_k(x) - \nabla \int_{B^+_{2 \lambda^{-1}_k}} H(2\lambda_k x, 2\lambda_k y)e^{\tilde{u}(y)} dy \]
\[= -\frac{x}{|x|^2} \alpha(|x|) + \frac{O(1)}{|x|^{5/4}} + \frac{O(1) \log \text{dist}(x, \partial \mathbb{R}^2_+)}{|x|^{3/2}} + o(1) \lambda_k, \quad x \in B^+_{\frac{3}{2} \lambda^{-1}_k} \setminus B^+_{R_k}\]
and
\[\nabla \tilde{v}_k(x) = -\frac{x}{|x|^2} \alpha(|x|) + \frac{O(1)}{|x|^{5/4}} + \frac{O(1) \log \text{dist}(x, \partial \mathbb{R}^2_+)}{|x|^{3/2}} + O(1) \lambda_k, \quad x \in B^+_{\frac{3}{2} \lambda^{-1}_k} \setminus B^+_{R_k}.\]
**Proof** We just need to show the first conclusion, since the second one follows from the fact
\[
\nabla \int_{B_{\frac{1}{2}^{+}}_{k}} H(2\lambda_k x, 2\lambda_k y) e^{\tilde{\eta}(y)} dy = O(1)\lambda_k, \quad x \in B_{\frac{3}{8}^{+}}_{k} \setminus B_{R_k}^{+}.
\]

We use the notations as in Lemma 3.1. Let \( \eta \) be defined as before. Extending \( \eta(k) \) evenly and by using Green’s formula, we have
\[
\eta(k) - \eta(0) = \int_{\partial B_{\frac{1}{2}^{+}}_{k}} \frac{\partial}{\partial r} (G(2\lambda_k x, 2\lambda_k y)) (\eta(y) - \eta(0)) ds_y,
\]
which implies
\[
\nabla \eta(k) = o(1)\lambda_k, \quad \forall |x| \leq \frac{3}{8} \lambda_k^{-1}.
\]  

A direct computation yields
\[
\nabla w_k(x) + \nabla \phi_k(x) - \nabla \int_{B_{\frac{1}{2}^{+}}_{k}} H(\lambda_k x, \lambda_k y) e^{\tilde{\eta}(y)} dy
\]
\[
= -\frac{1}{2\pi} \int_{B_{\frac{1}{2}^{+}}_{k}} \frac{x - y}{|x - y|^2} e^{\tilde{\eta}_k(y)} dy - \frac{1}{\pi} \int_{\partial B_{\frac{1}{2}^{+}}_{k}} \frac{x - y}{|x - y|^2} e^{\frac{1}{2} \tilde{\eta}_k(y)} ds_y
\]
\[
= -\frac{1}{2\pi} \int_{|y| \leq 2|x|, |y - x| \geq \frac{1}{2}|x|, |y| \geq |x|^t} \frac{x - y}{|x - y|^2} e^{\tilde{\eta}_k(y)} dy
\]
\[
- \frac{1}{\pi} \int_{|y| \leq 2|x|, |y - x| \geq \frac{1}{2}|x|, |y| \geq |x|^t} \frac{x - y}{|x - y|^2} e^{\frac{1}{2} \tilde{\eta}_k(y)} ds_y
\]
\[
- \frac{1}{2\pi} \int_{|y - x| \leq |x|^t} \frac{x - y}{|x - y|^2} e^{\tilde{\eta}_k(y)} dy - \frac{1}{\pi} \int_{|y - x| \leq |x|^t} \frac{x - y}{|x - y|^2} e^{\frac{1}{2} \tilde{\eta}_k(y)} ds_y
\]
\[
= \text{III}_1 + \text{III}_2 + \text{III}_3 + \text{III}_4,
\]
where \( s \in (0, 1) \) is a constant which will be chosen later.

For \( \text{III}_1 \), since \(|y| \geq 2|x|\), by Lemma 3.1, we have
\[
\left| \int_{|y| \leq 2|x|, |y - x| \leq |x|^t} \frac{x - y}{|x - y|^2} e^{\tilde{\eta}_k(y)} dy \right| \leq \frac{C}{|x|} \int_{|y| \leq 2|x|, |y - x| \leq |x|^t} \frac{1}{|y|^{4-2\delta}} dy \leq \frac{C}{|x|^{3-2\delta}}
\]
and
\[
\int_{\{y \in \partial^0 B^{+}_{\frac{1}{2}k}, |y| \geq 2|x|\}} \frac{x - y}{|x - y|^2} e^{\frac{1}{2} \hat{v}_k(y)} dy \leq C \left| x \right| \int_{\{y \in \partial^0 B^{+}_{\frac{1}{2}k}, |y| \geq 2|x|\}} \frac{1}{|y|^{2-\delta}} ds_y \leq C \left| x \right|^{2-\delta},
\]
which implies
\[
III_1 = \frac{O(1)}{|x|^{2-\delta}}.
\]

For \(III_2\), since \(|y - x| \geq \frac{1}{2}|x|\), by Lemma 3.1, we see
\[
\int_{\{|y| \leq 2|x|, |y - x| \geq \frac{1}{2}|x|, |y| \geq |x|^\epsilon\}} \frac{x - y}{|x - y|^2} e^{\frac{1}{2} \hat{v}_k(y)} dy \leq C \left| x \right| \int_{\{|y| \leq 2|x|, |y - x| \geq \frac{1}{2}|x|, |y| \geq |x|^\epsilon\}} \frac{1}{|y|^{4-2\delta}} dy \leq C \left| x \right|^{1+(2-2\delta)x}
\]
and
\[
\int_{\{y \in \partial^0 B^{+}_{\frac{1}{2}k}, |y| \leq 2|x|, |y - x| \geq \frac{1}{2}|x|, |y| \geq |x|^\epsilon\}} \frac{x - y}{|x - y|^2} e^{\frac{1}{2} \hat{v}_k(y)} ds_y \leq C \left| x \right| \int_{\{y \in \partial^0 B^{+}_{\frac{1}{2}k}, |y| \leq 2|x|, |y - x| \geq \frac{1}{2}|x|, |y| \geq |x|^\epsilon\}} \frac{1}{|y|^{2-\delta}} ds_y \leq C \left| x \right|^{1+(1-\delta)x},
\]
which implies
\[
III_2 = \frac{O(1)}{|x|^{1+(1-\delta)x}}.
\]

For \(III_3\), since \(|y - x| \leq \frac{1}{2}|x|\), it holds \(\frac{1}{2}|x| \leq |y| \leq \frac{3}{2}|x|\). Then by Lemma 3.1, we get
\[
\int_{\{|y - x| \leq \frac{1}{2}|x|\}} \frac{x - y}{|x - y|^2} e^{\frac{1}{2} \hat{v}_k(y)} dy \leq C \left| x \right| \int_{\{|y - x| \leq \frac{1}{2}|x|\}} \frac{1}{|y - x|} dy \leq C \left| x \right|^{3-2\delta}
\]
and
\[
\int_{\{y \in \partial^0 B^{+}_{\frac{1}{2}k}, |y - x| \leq \frac{1}{2}|x|\}} \frac{x - y}{|x - y|^2} e^{\frac{1}{2} \hat{v}_k(y)} ds_y \leq C \left| x \right|^{1-\delta} \int_{\{y \in \partial^0 B^{+}_{\frac{1}{2}k}, |y - x| \leq \frac{1}{2}|x|\}} \frac{1}{|y - x|} ds_y \leq C \left| x \right|^{2-\delta} \left( \log dist(x, \partial \mathbb{R}^2_+) + \log |x| \right),
\]
which implies
\[
III_3 = \frac{O(1) \log |x|}{|x|^{2-\delta}} + O(1) \frac{|\log dist(x, \partial \mathbb{R}^2_+)|}{|x|^{2-\delta}}.
\]
For $\textbf{III}_4$, since

$$\frac{x - y}{|x - y|^2} - \frac{x}{|x|^2} = O(1)$$

we find

$$-\frac{1}{2\pi} \int_{|y| \leq |x|^\beta} \frac{x - y}{|x - y|^2} e^\tilde{v}_k(y) \, dy = -\frac{1}{2\pi} \frac{x}{|x|^2} \int_{|y| \leq |x|^\beta} e^\tilde{v}_k(y) \, dy + O(1)$$

$$= -\frac{1}{2\pi} \frac{x}{|x|^2} \left( \int_{|y| \leq |x|} e^\tilde{v}_k(y) \, dy - \int_{|x'| \leq |y| \leq |x|} e^\tilde{v}_k(y) \, dy \right)$$

$$+ O(1) + \frac{O(1)}{|x|^{2-s}} + \frac{O(1)}{|x|^{1+(1-\delta)s}} + \frac{O(1)}{|x|^{2-s}},$$

and

$$-\frac{1}{\pi} \int_{\{y \in \partial B_{\frac{1}{2}\epsilon_k}^{+} \mid |y| \leq |x|^\beta \}} \frac{x - y}{|x - y|^2} e^\tilde{v}_k(y) \, ds_y$$

$$= -\frac{1}{\pi} \frac{x}{|x|^2} \int_{\{y \in \partial B_{\frac{1}{2}\epsilon_k}^{+} \mid |y| \leq |x|^\beta \}} e^{\frac{1}{2} \tilde{v}_k(y)} \, ds_y + \frac{O(1)}{|x|^{2-s}}$$

$$= -\frac{1}{\pi} \frac{x}{|x|^2} \left( \int_{\{y \in \partial B_{\frac{1}{2}\epsilon_k}^{+} \mid |y| \leq |x| \}} e^{\frac{1}{2} \tilde{v}_k(y)} \, ds_y - \int_{\{y \in \partial B_{\frac{1}{2}\epsilon_k}^{+} \mid |x'| \leq |y| \leq |x| \}} e^{\frac{1}{2} \tilde{v}_k(y)} \, ds_y \right)$$

$$+ \frac{O(1)}{|x|^{2-s}} = -\frac{1}{\pi} \frac{x}{|x|^2} \int_{\{y \in \partial B_{\frac{1}{2}\epsilon_k}^{+} \mid |y| \leq |x| \}} e^{\frac{1}{2} \tilde{v}_k(y)} \, ds_y + \frac{O(1)}{|x|^{1+(1-\delta)s}} + \frac{O(1)}{|x|^{2-s}},$$

where we have used the fact that

$$\int_{|x'| \leq |y| \leq |x|} e^{\tilde{v}_k(y)} \, dy \leq C \int_{|x'| \leq |y| \leq |x|} \frac{1}{|y|^{4-2\delta}} \, dy = O(1)$$

and

$$\int_{\{y \in \partial B_{\frac{1}{2}\epsilon_k}^{+} \mid |x'| \leq |y| \leq |x| \}} e^{\frac{1}{2} \tilde{v}_k(y)} \, ds_y \leq C \int_{\{y \in \partial B_{\frac{1}{2}\epsilon_k}^{+} \mid |x'| \leq |y| \leq |x| \}} \frac{1}{|y|^{2-\delta}} \, ds_y = O(1).$$

Thus,

$$\textbf{III}_4 = -\frac{1}{2\pi} \frac{x}{|x|^2} \int_{|y| \leq |x|} e^\tilde{v}_k(y) \, dy$$

$$-\frac{1}{\pi} \frac{x}{|x|^2} \int_{\{y \in \partial B_{\frac{1}{2}\epsilon_k}^{+} \mid |y| \leq |x| \}} e^{\frac{1}{2} \tilde{v}_k(y)} \, ds_y + \frac{O(1)}{|x|^{1+(1-\delta)s}} + \frac{O(1)}{|x|^{2-s}}.$$
Taking $\delta = s = \frac{1}{2}$, we deduce

\[
\nabla w_k(x) + \nabla \phi_k(x) - \nabla \int_{B_{\frac{1}{2} \lambda_k}^+} H(\lambda_k x, \lambda_k y) e^{\tilde{v}(y)} dy = -\frac{x}{|x|^2} \alpha(|x|) 
+ O(1) \frac{|\log \text{dist}(x, \partial \mathbb{R}^2_+) - \log \text{dist}(0, \partial \mathbb{R}^2_+)|}{|x|^{3/2}}.
\]

which implies

\[
\nabla \tilde{v}_k(x) - \nabla \int_{B_{\frac{1}{2} \lambda_k}^+} H(\lambda_k x, \lambda_k y) e^{\tilde{v}(y)} dy = -\frac{x}{|x|^2} \alpha(|x|) + O(1) \frac{|\log \text{dist}(x, \partial \mathbb{R}^2_+) - \log \text{dist}(0, \partial \mathbb{R}^2_+)|}{|x|^{3/2}} + o(1) \lambda_k.
\]

As a corollary of the above lemma, we can control the tangential oscillation as follows.

**Corollary 3.3** We have

\[
\sup_{R_k \leq t \leq \frac{3}{8} \lambda_k^{-1}} \text{osc}_{t^+ B_t^+} \tilde{v}_k = o(1).
\]

**Proof** For any $R_k \leq t \leq \frac{1}{2} \lambda_k$ and any two points $(t, \theta_1), (t, \theta_2) \in \partial^+ B_t^+$, by Lemma 3.2, it holds

\[
|\tilde{v}_k(t, \theta_1) - \tilde{v}_k(t, \theta_2)| = \left| \int_{\theta_2}^{\theta_1} \frac{\partial \tilde{v}_k(t, \theta)}{\partial \theta} d\theta \right|
\leq \int_{\partial^+ B_t^+} \frac{1}{t} \left| \frac{\partial \tilde{v}_k(y)}{\partial \theta} \right| ds_y
\leq C \int_{\partial^+ B_t^+} \left( \frac{1}{t^{5/4}} + \frac{|\log \text{dist}(y, \partial \mathbb{R}^2_+) - \log \text{dist}(0, \partial \mathbb{R}^2_+)|}{t^{3/2}} + \lambda_k \right) ds_y
= C \int_{\partial^+ B_t^+} \frac{|\log \text{dist}(y, \partial \mathbb{R}^2_+) - \log \text{dist}(0, \partial \mathbb{R}^2_+)|}{t^{3/2}} ds_y + o(1).
\]

Taking the coordinate notation $y = (y^1, y^2)$ and noting that

\[
\int_{\partial^+ B_t^+} |\log \text{dist}(y, \partial \mathbb{R}^2_+)| ds_y = \int_{\partial^+ B_t^+} |\log |y^2|| ds_y
= \int_{y \in \partial^+ B_t^+, |y^2| \leq 1} |\log |y^2|| ds_y + \int_{y \in \partial^+ B_t^+, |y^2| \geq 1} |\log |y^2|| ds_y
\leq C \int_0^1 |\log y^2| dy^2 + C \int_{y \in \partial^+ B_t^+, |y^2| \geq 1} \log t ds_y \leq Ct \log t, \quad (3.12)
\]

we get

\[
\tilde{v}_k(t, \theta_1) - \tilde{v}_k(t, \theta_2) = o(1), \quad R_k \leq t \leq \frac{1}{2} \lambda_k^{-1}.
(3.13)
\]
For any $\lambda_k^{-\frac{1}{2}} \leq t \leq \frac{3}{8} \lambda_k^{-1}$ and any two points $z_1 = (t, \theta_1), z_2 = (t, \theta_2) \in \partial^+ B_i^+$, set
\[
F(x) = \int_{B_{\frac{1}{2}\lambda_k^{-1}}} H(2\lambda_k x, 2\lambda_k y)e^{\tilde{\phi}(y)} dy.
\]

On one hand, by Lemma 3.2, we have
\[
|\tilde{v}_k(t, \theta_1) - F(z_1) - \tilde{v}_k(t, \theta_2) + F(z_2)| = \left| \int_{\partial B_1^+} \frac{\partial}{\partial \theta} (\tilde{v}_k(t, \theta) - F(x)) d\theta \right| \\
\leq \int_{\partial^+ B_i^+} \left| \frac{1}{t} \frac{\partial}{\partial \theta} (\tilde{v}_k(x) - F(x)) \right| dx \\
\leq C \int_{\partial^+ B_i^+} \left( \frac{1}{t^{5/4}} + \frac{|\log \text{dist}(x, \partial R_k^2)|}{t^{3/2}} \right) + o(1)dx \\
= C \int_{\partial^+ B_i^+} \left( \frac{|\log \text{dist}(x, \partial R_k^2)|}{t^{3/2}} \right) + o(1) = o(1).
\]

On the other hand, by the formula of Green’s function for the unit ball, we know
\[
H(x, y) = \frac{1}{2\pi} \log \left| |x| \left( y - \frac{x}{|x|^2} \right) \right|.
\]

Hence, we see
\[
|F(z_1) - F(z_2)| \leq \int_{B_{R_k}} |H(2\lambda_k z_1, 2\lambda_k y) - H(2\lambda_k z_2, 2\lambda_k y)) e^{\tilde{\phi}(y)} dy| \\
+ C \int_{B_{\frac{1}{2}\lambda_k^{-1}} \setminus B_{R_k}} e^{\tilde{\phi}(y)} dy \\
= \frac{1}{2\pi} \int_{B_{R_k}} \log \left| \frac{4\lambda_k^2 y^2 - z_1}{4\lambda_k^2 y^2 - z_2} \right| e^{\tilde{\phi}(y)} dy \\
+ C \int_{B_{\frac{1}{2}\lambda_k^{-1}} \setminus B_{R_k}} e^{\tilde{\phi}(y)} dy \\
\leq \frac{1}{2\pi} \int_{B_{R_k}} \log \left| \frac{4\lambda_k^2 y^2 - z_1}{4\lambda_k^2 y^2 - z_2} \right| e^{\tilde{\phi}(y)} dy \\
+ C \int_{B_{\frac{1}{2}\lambda_k^{-1}} \setminus B_{R_k}} \frac{1}{|y|^{4-2\tilde{\phi}}} dy = o(1),
\]

where we have used Lemma 3.1 and the following fact that
\[
\log \left| \frac{4\lambda_k^2 y^2 - z_1}{4\lambda_k^2 y^2 - z_2} \right| = o(1)
\]
as $k \to \infty$ since $\lambda_k^{-\frac{1}{2}} \leq t \leq \frac{3}{8} \lambda_k^{-1}$, $|4\lambda_k^2 y^2| \leq CR_k$ and $\lambda_k R_k = o(1)$.

By (3.14) and (3.15), we get
\[
\tilde{v}_k(t, \theta_1) - \tilde{v}_k(t, \theta_2) = o(1), \quad \lambda_k^{-\frac{1}{2}} \leq t \leq \frac{3}{8} \lambda_k^{-1}.
\]

Combining this with (3.13), we proved the corollary. \qed

With the help of Lemma 3.2, using Pohozaev’s type identity, we get the following energy decay estimate.

**Lemma 3.4** There holds
\[
\alpha(|x|) = 4 + \frac{O(1)}{|x|^4}, \quad \forall R_k \leq |x| \leq \frac{3}{8} \lambda_k^{-1}.
\]
Proof By the Pohozaev identity, we have
\[
2 \left( \int_{B^+_t} e^{\tilde{v}_k} \, dx + \int_{\partial^0 B^+_t} e^{\frac{1}{2} \tilde{v}_k} \, ds_x \right) = t \int_{B^+_t} \left( \frac{\partial \tilde{v}_k}{\partial r} \right)^2 - \frac{1}{2} |\nabla \tilde{v}_k|^2 + e^{\tilde{v}_k} \right) \, ds_x + 2te^{\frac{1}{2} \tilde{v}_k(t,0)} + 2te^{\frac{1}{2} \tilde{v}_k(-t,0)}, \quad 0 < t \leq \frac{3}{8} \lambda_k^{-1}.
\]

Taking \( t = |x| \) in above equality, by Lemmas 3.2, 3.1 and (3.12), we find
\[
2\pi \alpha(|x|) = \frac{1}{2} \pi \alpha^2(|x|) + \alpha(|x|) \frac{O(1)}{|x|^4} + \frac{O(1)}{|x|^2} + \frac{O(1)}{|x|^{2-\delta}} + \frac{O(1)}{|x|^{1-\delta}}.
\]

Then taking \( \delta = \frac{1}{2} \), it holds
\[
\alpha(|x|) = 4 + \frac{O(1)}{|x|^4}, \quad \forall R_k \leq |x| \leq \frac{3}{8} \lambda_k^{-1}. \tag{3.16}
\]

Using the above energy decay estimate, we can control the oscillation of radial part as follows.

Proposition 3.5 We have
\[
\lim_{k \to \infty} \| v_k(x) - v(x) \|_{C^0(B_{\frac{3}{8} \lambda_k^{-1}} \cap \Omega_k)} = 0.
\]

Proof By (2.3), one can see that the proposition follows from the following estimate
\[
\lim_{k \to 0} osc_{R \leq |x| \leq \frac{3}{8} \lambda_k^{-1}, x \in \Omega_k} (v_k(x) - v(x)) = 0.
\]

Noting that
\[
v(x) + 4 \log |x| = o(1), \quad \forall |x| \geq R_k,
\]
we just need to show that
\[
\lim_{k \to 0} osc_{R \leq |x| \leq \frac{3}{8} \lambda_k^{-1}, x \in \Omega_k} (v_k(x) + 4 \log |x|) = 0,
\]
which is equivalent to
\[
\lim_{k \to 0} osc_{x \in B_{\frac{3}{8} \lambda_k^{-1}} \setminus B_{\lambda_k^{-1}}^+} (\tilde{v}_k(x) + 4 \log |x|) = 0, \tag{3.17}
\]
where \( \tilde{v}_k \) was defined by (2.8).

In fact, for any two points \( p_1 = (t_1, \theta_1), p_2 = (t_2, \theta_2) \in B_{\frac{3}{8} \lambda_k^{-1}} \setminus B_{\lambda_k}^+ \) (without loss of generality, we assume \( t_1 \leq t_2 \)), there holds
\[
|\tilde{v}_k(p_1) + 4 \log |p_1| - \tilde{v}_k(p_2) - 4 \log |p_2| | \\
\leq |\tilde{v}_k(t_1) + 4 \log |p_1| - \tilde{v}_k(t_2) - 4 \log |p_2| | + osc_{B_{t_1}^+ \setminus B_{\lambda_k}} \tilde{v}_k + osc_{B_{t_2}^+ \setminus B_{\lambda_k}} \tilde{v}_k. \tag{3.18}
\]

where \( \tilde{v}_k^* \) was defined by (2.10).

It follows from (2.14) and Lemma 3.4 that
\[
\frac{d \tilde{v}_k^*}{dr} = -4 + O(1) \frac{r^5}{r}. 
\]
which implies
\[
\left| \tilde{v}_k^p(t_1) + 4 \log |p_1| - \tilde{v}_k^p(t_2) - 4 \log |p_2| \right| = \left| \int_{t_1}^{t_2} \frac{d}{dr}(\tilde{v}_k^p(r) + 4 \log r) dr \right|
\leq C \int_{t_1}^{t_2} \frac{1}{r^{3/4}} dr \leq \frac{C}{R_k^{1/4}} = o(1).
\]
Combining this with (3.18) and Corollary 3.3, we get (3.17). \hfill \Box

**Proof of Theorem 1.1** By Proposition 3.5, we need to show that
\[
\lim_{k \to \infty} \| v_k(x) - v(x) \|_{C^0(\Omega_k \setminus B_\lambda^{-1}(\frac{\lambda_k}{\lambda_k}))} = 0,
\]
which is equivalent to
\[
\lim_{k \to \infty} \text{osc}_{\Omega_k \setminus B_\lambda^{-1}(\frac{\lambda_k}{\lambda_k})} (v_k(x) + 4 \log |x|) = 0.
\] (3.19)
Noting that \( \Omega_k \setminus B_\lambda^{-1}(\frac{\lambda_k}{\lambda_k})(0) \subset \Omega_k \setminus B_\lambda^{-1}(\frac{\lambda_k}{\lambda_k}) \), one can see that (3.19) is a consequence of the following fact
\[
\lim_{k \to \infty} \text{osc}_{B_\lambda^{-1}(\frac{\lambda_k}{\lambda_k}) \setminus B_\lambda^{-1}(\frac{\lambda_k}{\lambda_k})} (\tilde{v}_k(x) + 4 \log |x|) = 0,
\] (3.20)
where \( \tilde{x}_k := -\frac{(x_k, 0)}{\lambda_k} \) and \( \tilde{v}_k \) was defined by (2.8).

Let \( w_k^1 \) be the solution of
\[
\begin{align*}
-\Delta w_k^1 &= -\Delta \tilde{v}_k, \quad \text{in } B_\lambda^{-1}(\tilde{x}_k) \setminus B_\lambda^{-1}(\tilde{x}_k), \\
\frac{\partial w_k^1}{\partial n} &= 0, \quad \text{on } \partial^0 B_\lambda^{-1}(\tilde{x}_k) \setminus \partial^0 B_\lambda^{-1}(\tilde{x}_k), \\
w_k^1 &= 0, \quad \text{on } \partial^+ B_\lambda^{-1}(\tilde{x}_k) \cup \partial^+ B_\lambda^{-1}(\tilde{x}_k).
\end{align*}
\]
Extending \( w_k^1 \) evenly, by standard elliptic theory and Lemma 3.1, we can check
\[
\| w_k^1 \|_{C^0(\Omega_k \setminus B_\lambda^{-1}(\tilde{x}_k) \setminus B_\lambda^{-1}(\tilde{x}_k))} \leq C \| \lambda_k^{-2} e^{\tilde{v}_k(\lambda_k^{-1}x + \lambda_k \tilde{x}_k)} \|_{L^2(B_\lambda^1(0) \setminus B_\lambda^1(0))}
\leq C \| \lambda_k^{-2} \|_{L^2(B_\lambda^1(0) \setminus B_\lambda^1(0))}
\leq C \| \lambda_k^{-2} \|_{L^2(B_\lambda^1(0) \setminus B_\lambda^1(0))} \leq C \lambda_k^{-2} = o(1),
\]
since \( x_k = (x_k^1, x_k^2) \to 0 \) as \( k \to \infty \).

Denote
\[
w_k^2(x) := -\frac{1}{\pi} \int_{\partial^0 B_\lambda^{-1}(\tilde{x}_k)} \log |x - y| g(y) ds_y, \quad x \in B_\lambda^{-1}(\tilde{x}_k),
\]
where \( g(y) = 0 \) if \( y \in \partial^0 B_\lambda^{-1}(\tilde{x}_k) \) and \( g(y) = e^{\frac{1}{2} \tilde{v}_k(y)} \) if \( y \in \partial^0 B_\lambda^{-1}(\tilde{x}_k) \setminus \partial^0 B_\lambda^{-1}(\tilde{x}_k) \).

It is easy to check that
\[
\begin{align*}
-\Delta w_k^2 &= 0, \quad \text{in } B_\lambda^{-1}(\tilde{x}_k), \\
\frac{\partial w_k^2}{\partial n} &= g, \quad \text{on } \partial^0 B_\lambda^{-1}(\tilde{x}_k).
\end{align*}
\]
Moreover, we claim
\[
\|w_k^2\|_{C^0(B^{+}_{\lambda_k^{-1}}(\bar{x}_k))} = o(1).
\]
In fact, for any \(x \in B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k)\), if \(\text{dist}(x, \partial \mathbb{R}^2_{+}) > 1\), then by Lemma 3.1, we have
\[
|w_k^2(x)| \leq C \log \lambda_k^{-1} \int_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k)} |g(y)| ds_y
= C \log \lambda_k^{-1} \int_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus \partial B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k)} e^{\frac{1}{2}\tilde{g}_k(y)} ds_y
\leq C \log \lambda_k^{-1} \int_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus \partial B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k)} \frac{1}{|y|^{1-\delta}} ds_y
= C \lambda_k^{1-\delta} \log \lambda_k^{-1} = o(1).
\]
If \(\text{dist}(x, \partial \mathbb{R}^2_{+}) \leq 1\), from Lemma 3.1, it holds
\[
|w_k^2(x)| = \left| \frac{1}{\pi} \int_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k)} \log |x - y| g(y) ds_y \right|
\leq C \lambda_k^{2-\delta} \int_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus \partial B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k)} \log |x - y| ds_y
\leq C \lambda_k^{2-\delta} \int_{\{y \in \partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus \partial B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k) \mid |y - x| \leq 1\} \setminus \{y \in \partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus \partial B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k) \mid |y - x| \geq 1\}} \log |x - y| ds_y
+ C \lambda_k^{2-\delta} \int_{\{y \in \partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus \partial B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k) \mid |y - x| \leq 1\}} \log |x - y| ds_y
\leq C \lambda_k^{2-\delta} (1 + \lambda_k^{-1} \log \lambda_k^{-1}) = o(1),
\]
which implies the claim.

Now let \(w_k^2 = \tilde{v}_k + 4 \log |x| - w_k^1(x) - w_k^2(x)\). We have
\[
\begin{cases}
-\Delta w_k^3 = 0, & \text{in } B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k),
\frac{\partial w_k^3}{\partial n} = 0, & \text{on } \partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \setminus \partial B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k),
\end{cases}
\]
\(w_k^3 = \tilde{v}_k + 4 \log |x| - w_k^2(x)\), on \(\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k) \cup \partial B^{+}_{\frac{1}{2}\lambda_k^{-1}}(\bar{x}_k)\).

Extend \(w_k^3\) evenly and note that
\[
\text{osc}_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k)}(\tilde{v}_k + 4 \log |x| - w_k^2(x)) \leq \text{osc}_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k)} w_k(x)
+ \text{osc}_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k)} \log |x| + \text{osc}_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k)} w_k^2(x) = o(1)
\]
and
\[
\text{osc}_{\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k)}(\tilde{v}_k + 4 \log |x| - w_k^2(x)) \leq \|\tilde{v}_k(x) + 4 \log |x|\|_{C^0(\partial B^{+}_{\lambda_k^{-1}}(\bar{x}_k))}.\]
\[ + \| w_k^2(\bar{x}_k) \|_{C^0(\partial^+ B^+_{42} - 1(\bar{x}_k))} = o(1), \]

where we have used the fact that \( \| \tilde{v}_k(x) + 4 \log |x| \|_{C^0(\partial^+ B^+_{42} - 1(\bar{x}_k))} = o(1) \), which follows from Proposition 3.5.

By maximal principle, we deduce

\[ o_{C^0(\partial^+ B^+_{42} - 1(\bar{x}_k) \setminus B^+_{42} - 1(\bar{x}_k))} w_k^3 = o(1). \]

Combining these together, we get (3.20).

Consequently, we complete the proof.

\[ \square \]

**Data Availability** No data, models, or code were generated or used during the study.

**References**

1. Bao, J., Wang, L., Zhou, C.: Blow up analysis for solutions to Neumann boundary value problem. J. Math. Anal. Appl. 418, 142–162 (2014)
2. Bartolucci, D., Chen, C., Lin, C., Tarantello, G.: Profile of blow-up solutions to mean field equations with singular data. Commun. Part. Differ. Equ. 29(7–8), 1241–1265 (2004)
3. Brezis, H., Merle, F.: Uniform estimate and blow up behaviour for solutions of \(-\Delta u = V(x) e^{u}\) in two dimensions. Commun. Part. Differ. Equ. 16, 1223–1253 (1991)
4. Chen, C., Lin, C.: Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. Commun. Pure Appl. Math. 55(6), 728–771 (2002)
5. Chen, W., Li, C.: Classification of solutions of some nonlinear elliptic equations. Duke Math. J. 63(3), 615–622 (1991)
6. Gluck, M.: Asymptotic behavior of blow up solutions to a class of prescribing Gauss curvature equations. Nonlinear Anal. 75(15), 5787–5796 (2012)
7. Guo, Y., Liu, J.: Blow-up analysis for solutions of the Laplacian equation with exponential Neumann boundary condition in dimension two. Commun. Contemp. Math. 8, 737–761 (2006)
8. Jost, J., Wang, G., Zhou, C., Zhu, M.: The boundary value problem for the super-Liouville equation. Ann. Inst. H. Poincare Anal. Non Lineaire 31(4), 685–706 (2014)
9. Li, Y.: Harnack type inequality: the method of moving planes. Commun. Math. Phys. 200(2), 421–444 (1999)
10. Li, Y., Shafrir, I.: Blow-up analysis for solutions of \(-\delta u = V(x) e^{u}\) in dimension two. Indiana Univ. Math. J. 43(4), 1255–1270 (1994)
11. Li, Y., Zhu, M.: Uniqueness theorems through the method of moving spheres. Duke Math. J. 80, 383–417 (1995)
12. Lin, C., Wei, J., Zhang, L.: Classification of blowup limits for SU(3) singular Toda systems. Anal. PDE 8, 807–837 (2015)
13. Zhang, L.: Blowup solutions of some nonlinear elliptic equations involving exponential nonlinearities. Commun. Math. Phys. 268(1), 105–133 (2006)
14. Zhang, T., Zhou, C.: Liouville type equation with exponential Neumann boundary condition and with singular data. Calc. Var. Part. Differ. Equ. 57(6), 32 (2018)

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