Trapped, Two-Armed, Nearly Vertical Oscillations in Disks with Toroidal Magnetic Fields

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Abstract

We have examined the trapping of two-armed ($m = 2$) nearly vertical oscillations (vertical p-mode) in vertically isothermal ($c_s = \text{const.}$) relativistic disks with toroidal magnetic fields. The magnetic fields are stratified so that the Alfvén speed, $c_A$, is constant in the vertical direction. The ratio $c_A^2/c_s^2$ in the vertical direction is taken to be a parameter that examines the effects of magnetic fields on wave trapping. We find that the two-armed nearly vertical oscillations are trapped in the inner region of the disks, and their frequencies decrease with an increase of $c_A^2/c_s^2$.

The trapped regions of the fundamental ($n = 1$) and the first-overtone ($n = 2$) are narrow (less than the length of the Schwarzschild radius, $r_g$), and their frequencies are relatively high (on the order of the angular frequency of disk rotation in the inner region). In contrast to this, the second overtone ($n = 3$) is trapped in a wide region (a few times $r_g$), and the frequencies are low and tend to zero in the limit of $c_A^2/c_s^2 = 2.0$.

Key words: accretion, accretion disks — neutron stars — quasi-periodic oscillations — two-armed disk oscillations — X-rays; stars

1. Introduction

Discoseismology is one of the important fields in studying astrophysical disks, since in some of them quasi-periodic oscillations (QPOs) have been observed, and most of them seem to be attributed to disk oscillations. In low-mass X-ray binaries where the central sources are neutron stars or black holes, for example, QPOs are often observed. They are classified into high-frequency QPOs ($\geq 100$ Hz), low-frequency QPOs and low-frequency complex (0.01–100 Hz), power-law components, and others (van der Klis 2004). The high-frequency and low-frequency QPOs will be, more or less, related to oscillation phenomena in the inner region of relativistic disks.

A first step to examine whether a disk oscillation mode can describe the observed QPOs is to compare the frequencies resulting from the mode with those of the observed QPOs. In disks that extend far outside, discrete frequencies of oscillations will be expected only when oscillatory perturbations are trapped in a particular finite region of the disks. In this sense, an examination of the trapping of disk oscillations and of their frequencies is an important subject in discoseismology.

In geometrically thin relativistic disks, there are many kinds of disk oscillation modes. Here, we classify them into four classes in terms of the node numbers in the vertical direction and the frequencies in the corotating frame, i.e., $p$-, $g$-, $c$-, and vertical $p$-mode oscillations (for details, see Kato et al. 2008; Kato 2001). (i) The $p$-mode is an inertial-acoustic mode with no node in the vertical direction$^1$ (nearly horizontal oscillations). The square of their frequencies in a corotating frame, $(\omega - m\Omega)^2$, is larger than the square of the horizontal epicyclic frequency, $k^2$, i.e., $(\omega - m\Omega)^2 > k^2$, where $\omega$ is the frequency of oscillation in the inertial frame, $m$ is the azimuthal wavenumber and $\Omega$ is the angular velocity of the disk rotation. (ii) The $g$-mode oscillation (inertial mode or r-mode) has at least one node in the vertical direction ($n \geq 1$) and $(\omega - m\Omega)^2$ is smaller than $k^2$, i.e., $(\omega - m\Omega)^2 < k^2$. (iii) The vertical $p$-mode oscillations are those that have at least one node in the vertical direction ($n \geq 1$) and have high frequencies in the corotating frame in the sense that $(\omega - m\Omega)^2 > \Omega^2_n$, where $\Omega_n$ is the vertical epicyclic frequency, which is always larger than $k$. This mode corresponds to the “breathing mode” of Blaes, Arras, and Fragile (2006) in oscillations of tori. (iv) Among oscillations formally belonging to $(\omega - m\Omega)^2 > \Omega^2_n$, the one-armed ($m = 1$) one with one node ($n = 1$) in the vertical direction has a particular position. The oscillation is called $c$-mode (corrugation mode). It is nearly incompressible motions changing the disk plane up and down with a corrugation pattern, corresponding to warp (or tilt).

The trapping of $p$-, $g$-, and $c$-mode oscillations has been extensively studied, e.g., by Kato and Fukue (1980) and Ortega-Rodríguez, Silbergleit, and Wagoner (2002) for $p$-mode oscillations; by Okazaki, Kato, and Fukue (1987), Nowak and Wagoner (1992) and Perez et al. (1997) for $g$-mode oscillations; and by Kato (1990) and Silbergleit, Wagoner, and Ortega-Rodriguez (2001) for $c$-mode oscillations. For reviews, see e.g., Wagoner (1999) and Kato (2001). Recent, much development has been made in examining the effects of the corotation resonance and magnetic fields on the trapping and excitation (or damping) of oscillations. That is, an amplification of non-axisymmetric $p$-mode oscillations by the corotation resonance

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$^1$ When we mention the node number in the vertical direction, it is the number of nodes of the density perturbation in the vertical direction. The node number associated with the vertical component of velocity is smaller than that of the density perturbation by one.
was found by Lai and Tsang (2009) and Tsang and Lai (2009b), which would be a refining of the Papaloizou–Pringle instability in different situations. Different from the p-mode, the non-axisymmetric g-mode and c-mode are heavily damped by the corotation resonance (Kato 2003; Li et al. 2003; Latter & Balbus 2009 for g-mode; Tsang & Lai 2009b for c-mode). Magnetic fields also have non-negligible effects on trapping. Fu and Lai (2009) showed that the magnetic fields act so as to destroy the general-relativistic self-trapping of axisymmetric g-mode oscillations. Considering these recent developments, Lai and Tsang (2009) suggested that among discoseismic modes a possible candidate of high-frequency QPOs would be axisymmetric p-mode oscillations.

Compared with the above-mentioned many studies on p-, g-, and c-mode oscillations, the trapping of vertical p-mode oscillations has been only little examined. An examination of the trapped vertical p-mode oscillations is, however, interesting in relation to QPOs, since their frequencies are in a wide range by differences of (i) the node numbers in the vertical and radial directions, and (ii) the disk parameters. Furthermore, general relativity is not essential for the trapping of vertical p-mode oscillations. Hence, the nearly vertical oscillations will be one of the good candidates to describe the QPOs in various disks (disks of low-mass X-ray binaries to those of dwarf-nanve) by an unified model.

It should be noted that the interesting oscillations of the vertical p-mode would be only those of two-armed ones, i.e., \( m = 2 \). This is because there would be no trapping in the oscillations of \( m = 1 \), and because for oscillations of \( m \geq 3 \) the frequencies of trapped oscillations are too high, except for cases of large \( n \). The oscillations with large \( n \), however, are not interesting from observational points of view. Considering these situations, we restrict our attention only to two-armed \(( m = 2 \) oscillations with \( n = 1 \) to \( n = 3 \).

In a previous paper (Kato 2010: Paper I), we examined the trapping of two-armed vertical p-mode oscillations in disks with polytropic gas, and showed how the frequencies of trapped oscillations depend on a change of the polytropic index. In this paper we restrict our attention to vertically isothermal disks for simplicity, but assume that the disks are subject to toroidal magnetic fields, and examine how the frequency of trapped oscillations and the trapped region depend on the strength of the magnetic fields.

2. Unperturbed Disks and Equations Describing Disk Oscillations

We consider geometrically thin, relativistic disks. For mathematical simplicity, however, the effects of general relativity are taken into account only when we consider the radial distributions of \( \Omega(r) \), \( \kappa(r) \), and \( \Omega_{\perp}(r) \), which are, in turn, the angular velocity of disk rotation and the epicyclic frequencies in the radial and vertical directions. Except for them, the Newtonian formulations are adopted. Since geometrically thin disks are considered, \( \Omega_{\perp} \) is approximated to be the relativistic Keplerian angular velocity, \( \Omega_{\perp}(r) \), when its numerical values are necessary. Here, \( r \) is the radial coordinate of the cylindrical ones \(( r, \varphi, z \) ), where the \( z \)-axis is perpendicular to the disk plane and its origin is the disk center. The functional forms of \( \Omega_{\perp}(r) \), \( \kappa(r) \), and \( \Omega_{\perp}(r) \) are given in many publications (e.g., Kato et al. 2008).

2.1. Unperturbed Disks with Toroidal Magnetic Fields

The equilibrium disks are axisymmetric with toroidal magnetic fields. The fields are assumed to be purely toroidal with no poloidal component,

\[
B_0(r, z) = [0, B_0(r, z), 0].
\]  

We further assume that the gas is isothermal in the vertical direction and the magnetic fields, \( B_0 \), are distributed in such a way that the Alfvén speed, \( c_A \), is constant in the vertical direction, i.e., \((B_0^2/4\pi\rho_0)^{1/2} = \text{const.}\) in the vertical direction, where \( \rho_0(r, z) \) is the density in the unperturbed disks. Furthermore, the rotation is assumed to be cylindrical; e.g., the angular velocity of the disk rotation, \( \Omega \), is only a function of \( r \).

We now consider the vertical structure of the disks. The hydrostatic balance in the vertical direction is given by

\[
-\frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \rho_0 + \frac{B_0^2}{8\pi} \right) - \Omega_{\perp}^2 z = 0,
\]  

where \( \rho_0(r, z) \) is the pressure, and is related to \( \rho_0(r, z) \) by \( \rho_0 = \rho_0c_s^2 \), \( c_s \) being the isothermal acoustic speed.

Since both \( c_s \) and \( c_A \) are constant in the vertical direction, equation (2) can be integrated to give

\[
\rho_0(r, z) = \rho_{00}(r) \exp \left( -\frac{z^{2}}{2H^{2}} \right) \quad \text{and} \quad B_0(r, z) = B_{00}(r) \exp \left( -\frac{z^{2}}{4H^{2}} \right),
\]  

where the scale height, \( H \), is related to \( c_s, c_A \), and \( \Omega_{\perp} \) by

\[
H^2(r) = \frac{c_s^2 + c_A^2/2}{\Omega_{\perp}^2}.
\]  

2.2. Equations Describing Disk Oscillations

Now, small-amplitude perturbations are superposed on the equilibrium disk described above. The velocity perturbation over rotation is denoted by \( (u_r, u_\varphi, u_z) \), and the perturbed part of the magnetic field over the unperturbed one by \( (b_r, b_\varphi, b_z) \).
Then, the \( r-, \varphi-, \) and \( z-\) components of the linearized equation of motions are written, respectively, as

\[
\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) u_r - 2\Omega u_{\varphi} = -\frac{1}{\rho_0} \frac{\partial}{\partial r} \left( p_1 + B_0 b_r \right) + \frac{B_0}{4\pi \rho_0} \left( \frac{\partial b_r}{r \partial \varphi} - \frac{2b_r}{r} \right) + \frac{\rho_1}{\rho_0} \left[ \frac{\partial}{\partial r} \left( p_0 + B_0^2 \frac{2}{8\pi} \right) + \frac{B_0^2}{4\pi r} \right].
\]

(5)

\[
\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) u_{\varphi} + \frac{k^2}{2\Omega} u_r = -\frac{1}{\rho_0} \frac{\partial}{\partial r} \left( p_1 + B_0 b_{\varphi} \right) + \frac{B_0}{4\pi \rho_0} \left( \frac{\partial b_{\varphi}}{r \partial \varphi} + \frac{b_r}{r} \right) + \frac{1}{4\pi \rho_0} \left( b_r \frac{\partial}{\partial r} + b_z \frac{\partial}{\partial z} \right) B_0.
\]

(6)

\[
\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) u_z = -\frac{1}{\rho_0} \frac{\partial}{\partial z} \left( p_1 + B_0 b_z \right) + \frac{B_0}{4\pi \rho_0} \frac{\partial b_z}{r \partial \varphi} + \frac{\rho_1}{\rho_0} \frac{\partial}{\partial z} \left( p_0 + \frac{B_0^2}{8\pi} \right),
\]

(7)

where \( p_1 \) and \( \rho_1 \) denote the perturbed parts of the pressure and density, respectively. Similarly, the induction equation gives

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} \right) b_r = B_0 \frac{\partial u_r}{r \partial \varphi},
\]

(8)

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} \right) b_{\varphi} = r \frac{d\Omega}{dr} b_r - \frac{\partial}{\partial r} \left( B_0 u_r \right) - \frac{\partial}{\partial z} \left( B_0 u_z \right).
\]

(9)

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} \right) b_z = B_0 \frac{\partial u_z}{r \partial \varphi}.
\]

(10)

The equation of continuity is

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} \right) \rho_1 + \frac{\partial}{\partial r} (r \rho_0 u_r) + \frac{\partial}{\partial \varphi} (\rho_0 u_{\varphi}) + \frac{\partial}{\partial z} (\rho_0 u_z) = 0.
\]

(11)

Another relation that we need here is a relation between \( p_1 \) and \( \rho_1 \). Considering isothermal perturbations, we adopt

\[
p_1 = \rho_1 c_s^2.
\]

(12)

Here, the azimuthal and time dependencies of the perturbed quantities are taken to be proportional to \( \exp[i(\omega t - m \varphi)] \), where \( \omega \) and \( m \) are the frequency and the azimuthal wavenumber of the perturbations, respectively. The perturbations are assumed to be local in the sense that their characteristic radial wavelength, \( \lambda_c \), is shorter than the characteristic radial scale of the disks, \( \lambda_D \), i.e., \( \lambda_c < \lambda_D \). By using this approximation, we neglect such quantities as \( d \ln \rho_0 / d \ln r \), \( d \ln B_0 / d \ln r \), \( d \ln H / d \ln r \), and \( d \ln \Omega / d \ln r \), compared with terms on the order of \( r / \lambda_c \). Then, the \( r-, \varphi-, \) and \( z-\) components of the equation of motion, equations (5)–(7), are reduced to

\[
i(\omega - m \Omega) u_r - 2\Omega u_{\varphi} = -\frac{h_1}{2} - c_s^2 \frac{\partial}{\partial r} \left( \frac{b_{\varphi}}{B_0} \right).
\]

(13)

\[
i(\omega - m \Omega) u_{\varphi} + \frac{k^2}{2\Omega} u_r = 0.
\]

(14)

\[
i(\omega - m \Omega) u_z = -\left( \frac{\partial}{\partial z} + \frac{c_s^2}{2c_s^2 H^2} \right) h_1 - c_s^2 \left( \frac{\partial}{\partial z} - \frac{z}{H^2} \right) \left( \frac{b_{\varphi}}{B_0} \right) - i \frac{m}{r} c_s^2 \left( \frac{b_z}{B_0} \right).
\]

(15)

In the above equations, \( h_1 \), defined by \( h_1 = p_1 / \rho_0 = c_s^2 \rho_1 / \rho_0 \), has been introduced. Similarly, the \( r-, \varphi-, \) and \( z-\) components of the induction equation, equations (8)–(10), are reduced to

\[
i(\omega - m \Omega) \frac{b_r}{B_0} = -i \frac{m}{r} u_r,
\]

(16)

\[
i(\omega - m \Omega) \frac{b_{\varphi}}{B_0} = r \frac{d\Omega}{dr} \frac{b_r}{B_0} - \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} - \frac{z}{2H^2} \right) u_z.
\]

(17)

\[
i(\omega - m \Omega) \frac{b_z}{B_0} = -i \frac{m}{r} u_z.
\]

(18)

Finally, the equation of continuity, equation (11), is reduced to

\[
i(\omega - m \Omega) h_1 = -c_s^2 \left[ \frac{\partial u_r}{\partial r} + \left( \frac{\partial}{\partial z} - \frac{z}{H^2} \right) u_z \right].
\]

(19)
Now, we further simplify equations (15) and (17). The last term, \(-i (m/r)c_\lambda^2 (b_\lambda/b_0)\), of equation (15) can be expressed in terms of \(u_z\) by using equation (18). The result shows that the term \(-i (m/r)c_\lambda^2 (b_\lambda/b_0)\) is smaller than the left-hand term, \(i(\omega - m\Omega)u_z\), by a factor of \(c_\lambda^2 / r^2\Omega^2\). Considering this, we neglect the last term on the right-hand side of equation (15). Next, we consider equation (17). The first term on the right-hand side, \(r (d\Omega/dr)(b_\lambda/b_0)\), is smaller than the second term, \(-\partial u_z / \partial r\), by a factor of \(\lambda^2 / r\), which can be shown by expressing \(b_\lambda\) in terms of \(u_z\) by using equation (16). Hence, we neglect this term in the following analyses.

After introducing the above approximations into equations (15) and (17), we multiply \(i(\omega - m\Omega)\) to equation (15) in order to express \(h_1\) and \(b_\lambda/b_0\) in equation (15) in terms of \(u_z\) and \(u_r\) by using equations (19) and (17). Then, after changing the independent variables from \((r, z)\) to \((r, \eta)\), where \(\eta = z/H\), we have

\[
\left[ \frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial \eta} \right] u_z + H^2 \left[ \frac{\partial}{\partial \eta} - c_\lambda^2 / 2 \eta \right] u_r = 0.
\]

This is the basic wave equation to be solved in this paper.

3. Nearly Vertical Oscillations

Equation (20) is now solved by approximately decomposing into two equations describing the behaviors in the vertical and radial directions.

3.1. Nearly Vertical Oscillations

As mentioned before, we are interested in nearly vertical oscillations (i.e., vertical p-mode). The main terms in equation (20) are thus those of the first brackets, and the terms of the second brackets are small perturbed quantities. Although the terms of the second brackets are small quantities, they are of importance to determine the wave trapping, as is shown in subsequent subsections.

First, we should notice that the quantity \((\omega - m\Omega)^2 - \Omega_\perp^2\) depends weakly on the radius \(r\). Hence, in order to consider this weak \(r\)-dependence of \((\omega - m\Omega)^2 - \Omega_\perp^2\) as being a small perturbed quantity, the third term in the first brackets of equation (20) is now expressed as

\[
\frac{\omega - m\Omega)^2 - \Omega_\perp^2}{c_s^2 + c_\lambda^2} H^2 + \epsilon(r),
\]

where the subscript \(c\) represents the value at the capture radius, \(r_c\), which is the outer boundary of the propagation region of oscillations, and will be determined later, and \(\epsilon(r)\) is a small quantity depending on \(r\). The magnitude of \(\epsilon(r)\) is found from equation (21) when \(r_c\) and \(\omega\) are determined later by an eigenvalue problem in the radial direction (see the final subsection of this section). If the term \(\epsilon(r)\) is transported to terms of small perturbations, in the lowest order of approximations, equation (20), is written in the form

\[
\frac{\partial^2}{\partial \eta^2} u_z^{(0)} - \frac{\partial}{\partial \eta} u_z^{(0)} + \left[ \frac{(\omega - m\Omega)^2 - \Omega_\perp^2}{c_s^2 + c_\lambda^2} H^2 \right] u_z^{(0)} = 0,
\]

where the superscript \((0)\) is attached to \(u_z\) in order to emphasize that it is the quantity of the lowest order of approximations. By imposing the boundary condition that \(u_z^{(0)}\) does not grow exponentially at \(z = \pm\infty\), we find that the \(z\)-dependence of \(u_z^{(0)}\) can be expressed by a Hermite polynomial and the term in the large brackets in equation (22) is determined as the eigenvalue, and is found to be \(n - 1\), where \(n\) is a positive integer (Okazaki et al. 1987). That is, we have

\[
u_z^{(0)} = f(r) g^{(0)}(\eta),
\]

where

\[
g^{(n)}(\eta) = \mathcal{H}_{n-1}(\eta), \quad n = 1, 2, 3 \ldots
\]

and

\[
\left[ \frac{(\omega - m\Omega)^2 - \Omega_\perp^2}{c_s^2 + c_\lambda^2} H^2 \right] = n - 1.
\]

Here, it is noted that the eigenfunction is taken to be \(\mathcal{H}_{n-1}\), not \(\mathcal{H}_n\). The reason is that in many previous studies \(h_1\) was adopted as the independent variable (not \(u_z\)), and the \(z\)-dependence of \(h_1\) is taken to be proportional to \(\mathcal{H}_n\) (e.g., Okazaki et al. 1987). The node number of \(u_z\) in the vertical direction is usually smaller than that of \(h_1\) by unity (e.g., see the equation of the \(z\)-component of equation of motion). Considering this, we have adopted \(\mathcal{H}_{n-1}(\eta)\) instead of \(\mathcal{H}_n(\eta)\) for \(u_z\). As shown in equation (23),

2 In nearly vertical oscillations, we have \(u_z \sim h_1/c_s\) [equation (15)]. Since the radial component of equation of motion [equation (13)] shows that \(u_r \sim (1/\lambda^2) h_1\), we have \(u_r \sim (H/\lambda) u_z\), i.e., \(u_r\) is smaller than \(u_z\) by a factor of \(H/\lambda\). Hence, the terms with the second brackets of equation (20) is smaller than the terms with the first brackets by a factor of \((H/\lambda)^2\).
in the lowest order of approximations, \( u_z^{(0)}(r, \eta) \), is expressed in a separable form with respect to \( r \) and \( \eta \). The \( r \)-dependence of \( u_z^{(0)} \) is free at this stage, which is denoted by \( f(r) \) in equation (23). It will be determined later by solving an eigenvalue problem in the radial direction.

Equation (25) can be rewritten in the form
\[
\left( \frac{\omega - m\Omega}{\Omega_L} \right)^2 = \left[ \frac{c_0^2 + \frac{c_A^2}{2}}{c_0^2 + c_A^2 / 2} \right] (n - 1) + 1. \tag{26}
\]

In the limit of \( c_A^2 = 0 \), this equation is reduced to \( (\omega - m\Omega)^2 = n\Omega_L^2 \), which is the expected result from the local dispersion relation in isothermal disks, i.e., \( [(\omega - m\Omega)^2 - \kappa^2][(\omega - m\Omega)^2 - n\Omega_L^2] = c_0^2 k_0^2 (\omega - m\Omega)^2 \) (Okazaki et al. 1987), where \( k \) is the radial wavenumber of oscillations. This dispersion relation shows that in non-magnetized isothermal disks, the propagation region of the nearly vertical oscillations is described by \( (\omega - m\Omega)^2 > n\Omega_L^2 \). This means that for oscillations with \( \omega \), one of their propagation regions is \( \omega < m\Omega - n^{1/2}\Omega_L \). That is, the outer boundary of the propagation region on the \( \omega-r \) plane is given by \( \omega = m\Omega - n^{1/2}\Omega_L \). As shown later, equation (26) suggests that in the present magnetized disks, the outer boundary of the propagation region is given by
\[
\omega = m\Omega - \left[ \frac{c_0^2 + \frac{c_A^2}{2}}{c_0^2 + c_A^2 / 2} (n - 1) + 1 \right]^{1/2} \Omega_L. \tag{27}
\]

This is really the case, as is shown later.

3.2. Derivation of Equation Describing Radial Behavior

We now proceed to take into account the deviation of the oscillations from purely vertical ones as perturbations. We soon see that separation of \( u_z \) into two functions of \( r \) and \( \eta \) is no longer valid. Hence, we consider the effects of small perturbed quantities by introducing a weak \( r \)-dependence in \( g \). That is, \( u_z \) is now written as
\[
u_z(r, \eta) = f(r) g^{(0)}(\eta) + g^{(1)}(r, \eta) + \cdots. \tag{28}
\]

Then, from equation (20) we obtain, as an equation describing \( f g^{(1)} \),
\[
f(r) \left( \frac{\partial^2}{\partial r^2} - \frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} + n - 1 \right) g^{(1)}(r, \eta) = -\epsilon(r) f(r) g^{(0)}(\eta) - H \left[ \frac{\partial}{\partial \eta} - \frac{c_A^2 / 2}{c_0^2 + c_A^2 / 2} \eta \right] \frac{\partial u_r^{(0)}}{\partial r}, \tag{29}
\]

where \( u_r^{(0)} \) is the lowest-order expression for \( u_r(r, \eta) \).

The next subject is to solve equation (29). To do so, \( u_r^{(0)} \) is expressed in terms of \( u_z^{(0)}[= f(r) g^{(0)}(\eta)] \). First, eliminating \( u_\psi \) from equations (13) and (14), we have
\[
[-(\omega - m\Omega)^2 + \kappa^2] u_r^{(0)} = -i (\omega - m\Omega) \left[ \frac{\partial h_1^{(0)}}{\partial r} + c_0^2 \frac{\partial}{\partial r} \left( \frac{b_0^{(0)}}{B_0} \right) \right]. \tag{30}
\]

In the lowest order of approximations of nearly vertical oscillations, the term \( \partial u_r / \partial r \) on the right-hand side of equation (19) can be neglected when evaluating \( h_1 \), compared with the term \( (\partial / \partial z - z / H^2) u_z \). Hence, by using equation (19) we can express \( \partial h_1^{(0)} / \partial r \) on the right-hand side of equation (30) directly by \( u_z \). Furthermore, the main term on the right-hand side of equation (17) is the term with \( u_r^{(0)} \). Hence, by using equation (17) the term of \( \partial (b_0^{(0)}/B_0) / \partial r \) on the right-hand side of equation (30) can also be expressed in terms of \( u_r^{(0)} \). Consequently, in the lowest-order approximations of nearly vertical oscillations, \( u_r^{(0)} \) can be expressed in terms of \( u_z^{(0)} \) alone from equation (30). After some manipulations we finally have
\[
u_r^{(0)} = L_s \left( \frac{\partial}{\partial \eta} - \eta \right) u_z^{(0)} + L_A \left( \frac{\partial}{\partial \eta} - \frac{1}{2} \eta \right) u_z^{(0)}, \tag{31}
\]

where \( L_s \) and \( L_A \) are operators defined by
\[
L_s = \frac{c_0^2 k_0^2}{-(\omega - m\Omega)^2 + \kappa^2} \left[ \frac{\partial}{\partial \eta} - \frac{\partial \ln (\omega - m\Omega)}{\partial r} \right] \tag{32}
\]
and
\[
L_A = -\frac{c_A^2 k_0^2}{-(\omega - m\Omega)^2 + \kappa^2} \left[ \frac{\partial}{\partial \eta} - \frac{\partial \ln (\omega - m\Omega)}{\partial r} \right]. \tag{33}
\]

Now, we return to equation (29). The equation is an inhomogeneous equation with respect to \( g^{(1)}(r, \eta) \). The right-hand side of the equation is now expressed in terms of \( f g^{(0)} \) by using equation (31). As is done in a standard perturbation method, \( g^{(1)}(r, \eta) \) is now expressed in a series of orthogonal functions of the zeroth order equations as
\[ g^{(1)}(r, \eta) = \sum_m C_m(r) \mathcal{H}_m(\eta). \]  

(34)

The quantity \( \epsilon(r) \) is then obtained from the solvability condition of equation (29), using the orthogonality of the Hermite polynomials, which is

\[
\begin{align*}
&f(\mathcal{H}_{n-1}^2(\eta)) \epsilon(r) \\
&+ H \frac{d}{dr} L_n(f) \left( \frac{d}{d\eta} \mathcal{H}_{n-1} \left( \frac{d}{d\eta} - \frac{1}{2} \frac{c_A^2}{2 + c_s^2 + c_A^2} \right) \right) \left( \frac{d}{d\eta} \mathcal{H}_{n-1} \right) \\
&+ H \frac{d}{dr} L_n(f) \left( \frac{d}{d\eta} \mathcal{H}_{n-1} \left( \frac{d}{d\eta} - \frac{1}{2} \frac{c_A^2}{2 + c_s^2 + c_A^2} \right) \right) \left( \frac{d}{d\eta} \mathcal{H}_{n-1} \right) = 0,
\end{align*}
\]

(35)

where \( \langle A(\eta) B(\eta) \rangle \) is the integration of \( A(\eta) B(\eta) \) with respect to \( \eta \) in the range of \(( -\infty, \infty) \) with the weight \( \exp(-\eta^2/2) \).

This solvability condition leads to an ordinary differential equation of \( f(r) \), when the integration with respect to \( \eta \) is performed. After some manipulation we can write the results in the following form:

\[
\begin{align*}
n A \left( c_s^2 + \frac{1}{2} c_A^2 \right) & \frac{d}{dr} \left( \frac{\omega - m \Omega}{(\omega - m \Omega)^2 - \kappa^2} \frac{d}{dr} \left[ f(r) \right] \right) + \epsilon(r) f(r) = 0, \\
\end{align*}
\]

(36)

where \( A(r) \) is given by

\[
A(r) = 1 - \frac{1}{2} \frac{c_A^2}{c_s^2 + c_A^2} n c_s^2 + c_A^2 / 2.
\]

(37)

In previous studies (Paper I) on nearly vertical oscillations in non-magnetized disks, we have adopted \( h_1 \) (not \( u_z \)) as a dependent variable. To compare our present results with those in the previous ones, we introduce here a new variable, \( \bar{f} \), defined by \( \bar{f} = f / (\omega - m \Omega) \). Then, equation (36) is reduced to

\[
\begin{align*}
\frac{1}{\omega - m \Omega} & \frac{d}{dr} \left[ \frac{\omega - m \Omega}{(\omega - m \Omega)^2 - \kappa^2} \frac{d}{dr} \bar{f} \right] + \frac{\epsilon}{n A \Omega_+^2 H^2} \bar{f} = 0.
\end{align*}
\]

(38)

In the limit \( c_A^2 = 0 \), this equation becomes formally the same as that used in Paper I.

3.3. Radial Eigenvalue Problems

Next, we solve equation (38) as an eigenvalue problem to know where the oscillations are trapped and how much the eigen-frequency of the trapped oscillations is. The same WKB procedures as Silbergleit, Wagoner, and Ortega-Rodriguez (2001) used are adopted here (see also Paper I). That is, we introduce a new independent variable, \( \tau(r) \), defined by

\[
\tau(r) = \int_{r_i}^r \frac{\omega^2(r') - \kappa^2(r')}{\omega(r')} dr', \quad \tau_c \equiv \tau(r_c).
\]

(39)

where \( \bar{\omega} \) is defined by \( \bar{\omega} = \omega - m \Omega \), and \( r_i \) is the inner edge of disks where a boundary condition is imposed. Then, equation (38) is written in the form

\[
\frac{d^2 \bar{f}}{d \tau^2} + Q \bar{f} = 0,
\]

(40)

where

\[
Q(\tau) = \frac{\omega^2}{\omega^2 - \kappa^2} \frac{\epsilon}{n A \Omega_+^2 H^2}.
\]

(41)

Equations (40) and (41) show that the propagation region of oscillations is the region where \( Q > 0 \), which is the region of \( \epsilon > 0 \). The region of \( \epsilon(r) > 0 \) is found to be inside of \( r_c \) from the following considerations. Let us tentatively assume that \( r_c \) is at a certain radius, although it should be determined after solving equation (40). Since we take it such that equation (25) holds at \( r_c \), equation (21) defining \( \epsilon \) gives

\[
\omega = m \Omega - \left[ \frac{c_s^2 + c_A^2}{c_s^2 + c_A^2} (n - 1 + \epsilon) + 1 \right]^{1/2} \Omega_+.
\]

(42)

\footnote{In the lowest order of approximations, the equation of continuity gives \( i(\omega - m \Omega) h_1 + (c_s^2 / H) \partial / \partial \eta \) \( u_z = 0 \). If \( u_z \) is taken to be proportional to \( \mathcal{H}_{n-1} (\eta) \), i.e., \( u_z = f(r) \mathcal{H}_{n-1} (\eta) \), the above continuity relation shows that \( h_1 \) has a component proportional to \( \mathcal{H}_{n-1} (\eta) \), i.e., \( h_1 = f_0(\tau) \mathcal{H}_{n-1} (\eta) \) and \( f(r) \) and \( f_0(\tau) \) is related by \( i(\omega - m \Omega) f_0 = (c_s^2 / H) f \).}

\footnote{In Paper I, polytropic disks are considered. Hence, even in the limit of \( c_A^2 \), equation (38) does not become identical with equation (33) in this paper.}
Fig. 1. Frequency–radius plane (i.e., propagation diagram) showing the propagation region of two-armed \((m = 2)\) nearly vertical oscillations (i.e., vertical p-mode oscillations) in vertically isothermal disks with toroidal magnetic fields. The value of \(c_s^2 + c_A^2 = 1.0\) has been adopted. The propagation region of the oscillation modes with \(n\) is below the boundary curve labelled by \(m\Omega - [(n - 1)(c_s^2 + c_A^2)/(c_s^2 + 0.5c_A^2) + 1]^{1/2}\Omega_L\). The boundary curve is shown for two cases of \(n = 1\) and \(n = 2\). In the case of \(n = 1\) (and \(n_t = 0\)), the capture (trapped) zone and the frequency of the trapped oscillations are shown by the upper thick horizontal line (the frequency \(\omega = 775\) Hz and capture radius \(r_c = 3.80r_g\)). In oscillations with \(n = 2\), the trapped oscillation with \(n_r = 0\) is shown by the lower thick horizontal line. The trapped frequency, \(\omega = 349\) Hz and \(r_c = 3.92r_g\). The inner-boundary condition adopted at \(r_i\) is \(Q_f = 0\). This inner boundary condition is adopted in all cases in this paper, except for in figure 5. The central star is assumed to have no spin. The mass of the central star is taken to be \(2M_\odot\) in all cases shown in the figures in this paper.

Fig. 2. Same as figure 1, except that the oscillations with \(n = 3\) are considered here. The propagation region of the oscillations is below the curve labelled by \(m\Omega - [(n - 1)(c_s^2 + c_A^2)/(c_s^2 + 0.5c_A^2) + 1]^{1/2}\Omega_L\), where \(n = 3\) is taken. Trapping of three modes of oscillations with \(n_t = 0, 1,\) and \(2\) are shown by three horizontal thick lines. The sets of frequency and capture radius for these three oscillation modes are, in turn, \((47.0\) Hz, \(4.76r_g\)), \((25.6\) Hz, \(7.13r_g\)), and \((14.7\) Hz, \(10.3r_g\)). It is noted that \(c_s^2 + c_A^2 = 1.0\) is adopted here, but in the case of \(c_s^2 + c_A^2 = 2.0\), there is no trapped oscillations.

This equation gives the \(\omega - r\) relation for a given \(\epsilon\). The \(\omega - r\) relation for \(\epsilon = 0\) is equation (27), and is shown in figures 1 and 2. The curve monotonically increases inwards as \(r\) decreases (see figures 1 and 2). This means that if \(r\) decreases from \(r_c\), while keeping \(\epsilon = 0\), the frequency given by equation (27) becomes larger than the frequency determined by \(r_c\). Hence, to satisfy equation (42) while keeping \(\omega\) at the value determined by \(r_c\), we must take a positive \(\epsilon\). If \(r\) increases from \(r_c\) while keeping \(\omega\); on the other hand, equation (42) can be satisfied by taking a negative \(\epsilon\). In summary, we have \(\epsilon > 0\) inside of \(r_c\), while \(\epsilon < 0\) outside of \(r_c\).

We solved equation (40) by a standard WKB method with the relevant boundary conditions (for details, see Silbergleit et al. 2001). The WKB approximation shows that the solution of equation (40) can be represented as

\begin{equation}
\frac{\partial^2\chi}{\partial \xi^2} + \frac{\partial\chi}{\partial \xi} \left( \frac{\partial\chi}{\partial \xi} - \frac{\omega}{\epsilon} \right) + \frac{\partial^2\chi}{\partial x^2} = 0
\end{equation}

\textit{It is noted, however, that in the case where the curve of } \omega = m\Omega - [(n - 1)(c_s^2 + c_A^2)/(c_s^2 + 0.5c_A^2) + 1]^{1/2}\Omega_L \textit{monotonically decreases inwards, the situations are changed (this is realized, for example, in the case where } n = 3 \text{ and } c_s^2/c_A^2 > 2\text{. That is, we have } \epsilon > 0 \text{ outside of } r_c \text{ and } \epsilon < 0 \text{ inside of } r_c. \text{ This means that a propagation region of waves is outside of } r_c \text{ and the waves are not trapped.}
The frequency \( \nu \) curve given by

\[
\frac{c_s^2}{c_A^2} = \frac{1}{\Omega^2} \left[ \frac{1}{\nu^2} \right] \quad \text{or} \quad \frac{c_s^2}{c_A^2} = \frac{1}{\nu^2} \left[ \frac{1}{\Omega^2} \right]
\]

is the conventional viscosity parameter, \( \alpha = \nu \Omega \) and \( \alpha = \nu \Omega \). The mass of neutron stars, \( M \), any solution of equation (45) specifies \( r_c \), which gives \( \omega \) of the trapped oscillation through equation (26). In other words, \( \omega \) and \( r_c \) are related by equation (26), i.e., \( \omega = \omega (r_c) \) or \( r_c = r_c (\omega) \), and the trapping condition determines \( r_c \) or \( \omega \) as functions of such parameters as \( c_s^2, c_A^2, a_s, \) and \( M \).

4. Numerical Results

To obtain numerical values of the frequency, \( \omega \), and the capture radius, \( r_c \), of trapped oscillations, we must specify the radial distribution of the acoustic speed, i.e., \( c_s (r) \). The final results of numerical calculations show that the trapped region is in the inner region of the disks. Hence, we consider the temperature distribution in the standard disk where the gas pressure dominates over the radiation pressure, and the opacity mainly comes from the free–free processes, and adopt (e.g., Kato et al. 2008)

\[
e^2 = 1.83 \times 10^{16} (\text{cm})^{-1/5} m^{-3/5} r^{-9/10} \text{cm}^2 \text{ s}^{-2},
\]

where \( \alpha \) is the conventional viscosity parameter, \( m (\equiv M / M_\odot)^6 \) and \( \dot{m} = \dot{M} / \dot{M}_{\text{crit}} \), \( \dot{M}_{\text{crit}} \) being the critical mass-flow rate, defined by

\[
\dot{M}_{\text{crit}} = \frac{L_E}{c_s^2} = 1.40 \times 10^{17} \text{ m g s}^{-1},
\]

where \( L_E \) is the Eddington luminosity. The parameters \( \alpha \) and \( \dot{m} \) affect the frequencies of trapped oscillations only through the magnitude of \( c_{s0} \). We adopt, throughout this paper, \( \alpha = 0.1 \) and \( \dot{m} = 0.3 \). A parameter specifying the strength of the magnetic field is \( c_A^2 / c_s^2 \). In this paper, we consider disks where the parameter \( c_A^2 / c_s^2 \) is in the range of \( c_A^2 / c_s^2 = 0 - 2 \). Other parameters specifying the disk-star system are \( m (\equiv M / M_\odot) \) and \( a_s \). We consider the cases \( M / M_\odot = 2.0 \) and \( a_s = 0 - 0.3 \).

We only consider two-armed oscillations with one, two, or three node(s) in the vertical direction, i.e., \( n = 1, 2, \) or \( 3 \). Oscillations with more nodes in the vertical direction are less interesting from the viewpoint of observability. The inner boundary of the oscillations is taken at the radius \( \kappa = 0 \), i.e., at the radius of the marginally stable circular orbit. In this paper, \( u_z = 0 \) (i.e., \( \tilde{f} = 0 \)) is adopted at the radius as a boundary condition, except in figure 5. In figure 5, boundary condition of \( d u_z / d r \) (i.e., \( d \tilde{f} / d r \sim 0 \)) is considered as well as \( u_z = 0 \) in order to see the effects of the boundary condition on the results. We find that the differences in the boundary condition bring about quantitative differences in results, but there is no essential differences in the parameter dependences of the results. Hence, except in figure 5, we adopt \( u_z = 0 \) as the inner-boundary condition throughout this paper. The horizontal node number, \( n_z \), of the oscillations that we consider is mainly \( n_z = 0 \), and supplementally \( n_z = 1 \) and 2.

Figures 1 and 2 are propagation diagrams for oscillations of \( n = 1 \) and 2 (figure 1) and \( n = 3 \) (figure 2), respectively, in the disks with \( c_A^2 / c_s^2 = 1 \) and \( a_s = 0 \). Only the oscillations of \( n_z = 0 \) are shown in figure 1, but three modes of oscillations, i.e., \( n_z = 0, 1, \) and 2, are shown in figure 2. The propagation regions of oscillations on the frequency-radius diagram is below the curve given by

\[
\omega = m \Omega - (n - 1)(c_s^2 + c_A^2) / (c_A^2 + c_s^2 / 2) + 1 \frac{1}{2} \Omega \Omega_{\text{crit}} \geq \Omega_{\text{crit}} \Omega_{\text{crit}} \text{[see equation (27)]}. \]

The results of numerical calculations show that the oscillations of \( n = 1 \) with \( n_z = 0 \) are trapped in the radial range shown by the upper thick horizontal line in figure 1. The frequency \( \omega \) and the capture radius \( r_c \) are, respectively, \( \omega = 775 \text{ Hz} \) and \( r_c = 3.80 r_g \). Outside \( r_c \), the oscillation is spatially damped. The radial range of trapped oscillations of \( n = 2 \) with \( n_z = 0 \) is shown by the lower thick horizontal line in figure 1. The frequency and the capture radius in this case are \( \omega = 349 \text{ Hz} \) and \( r_c = 3.92 r_g \).

In this section and hereafter, \( m \) is often used to denote \( M / M_\odot \) without confusion with the azimuthal wavenumber \( m \) of oscillations.

In oscillations with \( n \), we have \( h_{1}(r, z) \propto \hat{h}_{1}(z / H) \) and \( u_z (r, z) \propto \hat{h}_{1-1}(z / H) \). That is, in oscillations with \( n = 3 \), \( u_z \) is plane-symmetric with respect to the equatorial plane, and has one node (where \( u_z = 0 \)) above and below the equator.
Trapped oscillations of \( n = 3 \) have frequencies lower than those of \( n = 1 \) and 2, since on the propagation diagram the curve of \( \omega = m\Omega - [(n - 1)(c_s^2 + c_A^2)/(c_s^2 + c_A^2/2) + 1]^{1/2}\Omega_\perp \) is below those in the cases of \( n = 1 \) or \( n = 2 \). In figure 2, the frequency and the radial extent of trapped oscillations of \( n = 3 \) are shown for three modes in the radial direction; the fundamental mode (i.e., \( n_r = 0 \)) and the first two overtones (i.e., \( n_r = 1 \) and 2). The sets of the frequency and capture radius for these three modes of \( n_r = 0, 1, \) and 2 are, respectively, \( (47.0 \text{ Hz}, 4.76 r_g), (25.6 \text{ Hz}, 7.13 r_g), \) and \( (14.7 \text{ Hz}, 10.3 r_g) \) in the disks with \( c_s^2/c_A^2 = 1.0 \) and \( a_s = 0.2 \).

Figure 3 shows the \( c_s^2/c_A^2 \)-dependence of the capture radius, \( r_c \). As a typical case, the dependence is shown for oscillations of \( n_r = 0 \). No spin of the central source is adopted. It is noted that when \( c_s^2/c_A^2 > 2 \), \( \omega \) given by the above relation is negative and is a monotonically increasing function outwards on the \( \omega-r \) plane. Then, the region of \( \epsilon > 0 \) (i.e., propagation region) is in the outer region, and there is no trapping (see the previous section). The frequency–\( c_s^2/c_A^2 \) relations are summarized in figure 4 for two disks with \( a_s = 0 \) and \( a_s = 0.2 \). The modes of oscillations adopted are \( n = 1, 2, \) and 3. In all cases, \( n_r \) is taken to be \( n_r = 0 \). As mentioned before, the oscillations with \( n = 3 \) have low frequencies. In order to examine the characteristics of these low-frequency oscillations in more detail, the frequency–\( c_s^2/c_A^2 \) relation of trapped oscillations for some values of the vertical node number, \( n \), and spin parameter, \( a_s \). The oscillations with no node in the radial direction (\( n_r = 0 \)) are considered.
relation in the case of \( n = 3 \) is again shown in figure 5, including cases where the other parameter values are adopted. That is, in addition to oscillations with \( n_r = 0 \), oscillations with \( n_r = 1 \) and 2 are considered in figure 5. In addition, the cases where \( (d f / dr)_h = 0 \) is adopted at \( r_i \) as the inner boundary condition are shown by thin curves. In figure 6, the frequency–spin relation is shown for three modes of oscillations with \( n = 1, 2, \) and 3, where \( n_r = 0 \) and \( c_A^2 / c_s^2 = 1.0 \) are adopted.

5. Discussion

In this paper we have examined the characteristics of the trapping of two-armed (\( m = 2 \)), nearly vertical oscillations (vertical \( p \)-mode oscillations), assuming that the disk is isothermal in the vertical direction and is subject to purely toroidal magnetic fields. For mathematical simplicity, the ratio of the Alfvén speed to the acoustic speed is constant in the vertical direction, i.e., \( c_A^2 / c_s^2 \) is constant in the vertical direction. The effects of magnetic fields on the characteristics of the oscillation are measured by taking \( c_A^2 / c_s^2 \) as a parameter.

A difference from Paper I is that in Paper I we have considered a polytropic gas (not an isothermal gas), and examined how the characteristics of trapping of the nearly vertical oscillations depend on the polytropic index characterizing the gas. In this paper, the disk gas is taken to be isothermal, for simplicity, different from Paper I, but is subject to toroidal magnetic fields. Under these situations, we examine how the characteristics of nearly vertical oscillations change by a change of strength of the magnetic fields.
In this paper a partial differential equation [equation (20)] has been solved by a perturbation method. In this procedure, we have assumed that $\epsilon(r)$ defined by equation (21) is a small positive quantity (i.e., $\epsilon < 1$) in the wave-propagation region. The final results show that this is really acceptable as a first step to examine the qualitative behavior of trapping. To do quantitative argument, however, the approximation should be improved, especially in the cases of $n = 1$ and $n = 2$. That is, $\epsilon(r)$ is zero at $r = r_c$, by definition, and increases inwards monotonically and becomes a maximum at $r = r_s$. The results of calculations show that the maximum value of $\epsilon$ is 0.50 for $n = 1$, 0.37 for $n = 2$, and 0.12 for $n = 3$ when $c_\lambda^2/c_s^2 = 1.0$, $a_s = 0$ and $n_r = 0$. The value slightly increases with a decrease of $c_\lambda^2/c_s^2$, and increases of $a_s$ and $n_r$.

In this paper we do not quantitatively consider the effects of $c_{\delta 0}$ on the frequency. An increase of $c_{\delta 0}$ without any change of the other parameters leads to a decrease of the frequency of trapped oscillations. The reason is that an increase of $c_{\delta 0}$ decreases $Q$. Hence, to satisfy the trapping condition (45), an increase of $r_c$ is necessary, which leads to a decrease of the frequency (see figures 1 and 2). The effects of changes of other various parameter values on the frequencies of trapped oscillations are qualitatively the same as those in Paper I (see table 1 in Paper I).

The purpose of this paper is to demonstrate the importance of two-armed ($m = 2$), nearly vertical oscillations as one of the possible candidates of disk oscillations describing quasi-periodic oscillations observed in low-mass X-ray binaries (LMXBs). One of the reasons why we pay attention to these oscillations is that they can be trapped in the inner region of disks, and their frequencies can cover a wide range of frequency by (i) the difference of modes ($n$ and $n_r$), and (ii) a change in the disk structure. As a change of disk structure, we considered a change of polytropic index in Paper I, while we consider here a change of the magnetic fields.

The parameters describing the difference in the oscillation modes are $n(= 1, 2, 3, \cdots)$ and $n_r(= 0, 1, 2, \cdots)$, where $n$ and $n_r$ are, respectively, the node number of $h_1$ in the vertical and horizontal directions. As in Paper I, the trapped oscillations of $n = 1$ and $n = 2$ have frequencies on the order of kHz QPOs, while those of $n = 3$ have lower frequencies, and on the order of the horizontal branch and the normal branch oscillations. An interesting result obtained in this paper is that the frequency of trapped oscillations decreases as the magnetic fields become stronger (figures 4 and 5). This is particularly so for oscillations of $n = 3$ (figure 5). That is, the trapped region of oscillations of $n = 3$ becomes wide, and their frequencies decrease as $c_\lambda^2/c_s^2$ increases, and finally there is no trapping for $c_\lambda^2/c_s^2 > 2$ (figure 5). This situation is similar to the case where the polytropic index is increased in polytropic gases (see Paper I).

The decrease in the frequency of trapped oscillations with an increase of $c_\lambda^2/c_s^2$ can be understood by considering the following situations. Since toroidal magnetic fields are considered here, the nearly vertical oscillations are the fast mode among three MHD oscillation modes. Their frequencies seen from the corotating frame are thus higher than those of pure acoustic oscillations in non-magnetized disks. If we consider purely vertical oscillations, neglecting $u_r$ and $u_{\phi r}$, in vertically isothermal disks with $c_\lambda^2/c_s^2 = \text{const.}$, we easily find that the eigen-frequency of the oscillations in the corotating frame, $\omega - m\Omega$, is given by

$$\Omega_\perp^2 = \left[ \frac{c_\lambda^2 + c_\lambda^2}{c_s^2 + c_\lambda^2/2}(n - 1) + 1 \right] \Omega_\perp^2$$

(49)

[see equation (27)], the right-hand side being larger than $n\Omega_\perp^2$ due to the presence of toroidal magnetic fields, as expected. If the terms neglected in deriving equation (49) are taken into account, $(\omega - m\Omega)^2$ becomes larger than the right-hand side of equation (49). Hence, we see that one of the propagation region of the nearly vertical oscillation is given by

$$\omega < m\Omega - \left[ \frac{c_\lambda^2 + c_\lambda^2}{c_s^2 + c_\lambda^2/2}(n - 1) + 1 \right]^{1/2} \Omega_\perp.$$  

(50)

This shows that the upper-boundary curve specifying the propagation region of oscillations, i.e., $\omega = m\Omega - [(n - 1)(c_\lambda^2 + c_\lambda^2)/(c_s^2 + 0.5c_s^2) + 1]^{1/2}\Omega_\perp$, moves downwards on the propagation diagram $(\omega - r$ plane) as $c_\lambda^2/c_s^2$ increases (see figures 1 and 2).

In the case of the oscillations of $n = 3$, the boundary curve tends to be close to $\omega \approx 0$ (as $c_\lambda^2/c_s^2$ approaches 2.0), since the term inside the brackets of equation (50) becomes 4. This is the reason why we have no trapped oscillations for $c_\lambda^2/c_s^2 \approx 2$ when $n = 3$ and $a_s = 0$.

The frequencies of QPOs observed in LMXBs have a time change, distinct from those in galactic black-hole candidates. As far as trapped oscillations are concerned, a large change in their frequencies cannot be expected, unless magnetic fields are considered. Global toroidal magnetic fields in accretion disks are time-dependent by amplification due to winding, and by damping due to reconnection. In the present model of QPOs, the time change of QPOs is attributed to the time change of global magnetic fields, but careful discussions are necessary concerning whether this is consistent with observations.

There are some important problems that remain to be clarified. First, it is not clear whether the innermost part of the disks can be regarded as being a boundary where oscillations are reflected back onwards. In disks of standard or ADAF disks, the innermost part of disks will reflect incoming waves, since the density decreases there sharply inwards. [See Kato et al. (1988) and Manmoto et al. (1996a, 1996b) for the reflection of waves in the inner edge of the disks.] In slim disks, however, there will be no sharp density decrease inwards near the transonic radius, and thus the reflection of waves in the innermost region will not be so efficient. In the case where the central source is a neutron star, the stellar surface, or a transition region near to the surface, will at least partially reflect incoming waves.
Whether nearly vertical oscillations are really excited on disks is also a problem to be examined, since they will not be excited by thermal and viscous overstable processes (e.g., Kato 1978), because of the presence of node(s) in the vertical direction. Most conceivable processes will be stochastic processes of turbulence (Goldreich & Keely 1977a, 1977b). In many stars with various characteristics (e.g., different evolutionary stages, effective temperatures ...), solar-like (non-radial) oscillations have been observed. The so-called k-mechanism cannot excite all of these oscillations. Their origin is now known to be stochastic processes of turbulence [see Samadi (2009) for a recent review of stochastic excitation of the oscillations]. Compared with in stars, much stronger MHD turbulence by the magneto-rotational instability (MRI) are expected in accretion disks, especially in the inner region of the disks. Hence, it is natural to suppose that in accretion disks many trapped oscillations are simultaneously excited by turbulence. This may be one of the causes of a variety of QPOs in LMXBs. This is a problem to be examined in the future.

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