On the Distribution of Cube–Free Numbers
with the Form $[n^c]$
all $c > 1$, the set $\mathcal{P}^c \cap \mathcal{P}$ is infinite in the sense of Lebesgue measure, he can not prove that $\mathcal{P}^c \cap \mathcal{P}$ is infinite for each fixed $c > 1$.

**Case 3.** $A = \mathbb{N}$, $B = \mathfrak{F}_2$.

Rieger [11] first investigated this case. He proved that the set $\mathbb{N}^c \cap \mathfrak{F}_2$ is infinite for $1 < c < 3/2$, which can be derived from the result of Deshouillers [6]. More precisely, Rieger proved that the asymptotic formula

$$\mathbb{N}^c \cap \mathfrak{F}_2(x) := \sum_{n \leq x, [n^c] \in \mathfrak{F}_2} 1 = 6 \pi^2 x + O(x^{2c+1} + \varepsilon)$$

holds for $1 < c < 3/2$. In 1998, Cao and Zhai proved that the asymptotic formula

$$\mathbb{N}^c \cap \mathfrak{F}_2(x) = 6 \pi^2 x + O(x^{\frac{2c+1}{2}} + \varepsilon)$$

holds for $1 < c < 61/36$. It is important to emphasise that Stux [13] proved that the set $\mathbb{N}^c \cap \mathfrak{F}_2$ is infinite for almost all $c \in (1, 2)$ in the sense of Lebesgue measure. However, his method can not determine the value of $c$ such that $\mathbb{N}^c \cap \mathfrak{F}_2$ is infinite. In 2008, Cao and Zhai [5] improved their earlier result in [3] and show that, for any fixed $1 < c < 149/87$, the set $\mathbb{N}^c \cap \mathfrak{F}_2$ is infinite.

In this paper, we consider the case $A = \mathbb{N}$, $B = \mathfrak{F}_3$. To be specific, we shall prove that, for a class of infinite sets $\mathcal{A} \subseteq \mathbb{N}$, $\mathcal{A}^c \cap \mathfrak{F}_3$ are infinite sets.

Let $c > 1$ be a real number and $\mathcal{A} \subseteq \mathbb{N}$ satisfying the following two conditions:

1. For any $\eta > 0$, there holds
   $$\mathcal{A}(x) := \sum_{\substack{n \leq x \\ n \in \mathcal{A}}} 1 \gg x^{1-\eta}; \quad (1.1)$$

2. Let $\eta > 0$ be an arbitrary small real number and $x > 1$ be a real number. If $\alpha = a/q$ is a rational number satisfying $(a, q) = 1$ and $2 \leq q \leq x^\eta$, then there exists positive constant $\delta$, which depends only on $c$, satisfying $\eta \leq \delta < 1/2$ such that there holds
   $$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} e(\alpha [n^c]) \ll x^{1-\delta}. \quad (1.2)$$

**Remark** There are many subsets of $\mathbb{N}$ satisfying (1.1) and (1.2). For instance, the sets $\mathbb{N}$, $\mathcal{D}$, $\mathfrak{F}_k$ $(k = 2, 3, \cdots)$, etc.

The main result is the following theorem.

**Theorem 1.1** Let $1 < c < 11/6$, $\gamma = c^{-1}$ and $0 < \varepsilon < 10^{-10}$ be a sufficiently small constant. Suppose that the set $\mathcal{A} \subseteq \mathbb{N}$ satisfies the condition (1.1) and (1.2). Then we have

$$\mathcal{A}^c \cap \mathfrak{F}_3(x) := \sum_{\substack{n \leq x \\ n \in \mathcal{A}, [n^c] \in \mathfrak{F}_3}} 1 = \frac{1}{\zeta(3)} \mathcal{A}(x) + O(x^{1-\varepsilon}).$$
Corollary 1.2 Let $1 < c < 11/6$, $\gamma = c^{-1}$ and $0 < \varepsilon < 10^{-10}$ be a sufficiently small constant. Then we have

\[
N^c \cap \mathcal{F}_3(x) = \frac{x}{\zeta(3)} + O(x^{1-\varepsilon}),
\]
\[
\mathcal{P}^c \cap \mathcal{F}_3(x) = \frac{1}{\zeta(3)} \int_2^x \frac{du}{\log u} + O(xe^{-c_1 \sqrt{\log x}}),
\]
\[
\mathcal{F}^c_3 \cap \mathcal{F}_3(x) = \frac{x}{\zeta^2(3)} + O(x^{1-\varepsilon}),
\]
where $c_1$ is an absolute constant.

**Notation** In this paper, we use $[x], x$ and $\|x\|$ to denote the integral part of $x$, the fractional part of $x$ and the distance from $x$ to the nearest integer, respectively; $\mu(n)$ denotes Möbius function; $e(t) = e^{2\pi i t}$; $\psi(x) = x - [x] - 1/2$; $n \sim N$ denotes $N < n \leq 2N$.

2 Preliminary Lemmas

In order to prove Theorem we need the following two lemmas.

**Lemma 2.1** For any $J \geq 2$, we have

\[
\psi(t) = \sum_{1 \leq |h| \leq J} a(h)e(ht) + O\left(\sum_{|h| \leq J} b(h)e(ht)\right), \quad a(h) \ll \frac{1}{|h|}, \quad b(h) \ll \frac{1}{J}.
\]

**Proof.** See pp. 116 of Graham and Kolesnik [7] or Vaaler [14].

**Lemma 2.2** For any $H \geq 1$, we have

\[
\psi(\theta) = -\sum_{0 < |h| \leq H} \frac{e(\theta h)}{2\pi i h} + (g(\theta, H)),
\]

where

\[
g(\theta, H) = \min\left(1, \frac{1}{H\|\theta\|}\right) = \sum_{h=-\infty}^{+\infty} a_h e(\theta h)
\]

and

\[
a_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{|h|^2}\right).
\]

**Proof.** See pp. 245 of Heath–Brown [8].

**Lemma 2.3** Let $y$ be not an integer, $\alpha \in (0, 1)$, $H \geq 3$. Then we have

\[
e(-\alpha \{y\}) = \sum_{|h| \leq H} c_h(\alpha)e(hy) + O\left(\min\left(1, \frac{1}{H\|\theta\|}\right)\right),
\]

where

\[
c_h(\alpha) := \frac{1 - e(-\alpha)}{2\pi i(h + \alpha)}.
\]

**Proof.** See the thesis of Buriev [2].
3 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. Let $c$ and $\varepsilon$ satisfy the conditions in Theorem 1.1. It is well known that the characteristic function of cube–free numbers is

$$
\sum_{d^3\mid n} \mu(d) = \begin{cases} 
0, & \exists m \text{ s.t. } m^3\mid n,
1, & \text{others.}
\end{cases}
$$

Then, we can write

$$
\mathcal{A}^c \cap \mathfrak{S}_3(x) := \sum_{n \leq x} \sum_{[n^3] \in \mathfrak{S}_3} \mu(d)
= \sum_{d \leq x^\varepsilon} \sum_{n \in \mathcal{A}} \mu(d) + \sum_{d > x^\varepsilon} \sum_{n \in \mathcal{A}} \mu(d)
=: \Sigma_1 + \Sigma_2.
$$

From the formula

$$
\sum_{k=1}^q e\left(\frac{hn}{q}\right) = \begin{cases} 
q, & \text{if } q \mid n,
0, & \text{if } q \nmid n,
\end{cases}
$$

we get

$$
\Sigma_1 = \sum_{m \leq x} \sum_{d \leq x^\varepsilon} \mu(d) \sum_{\ell=1}^{d^3} e\left(\frac{\ell|m^c}{d^3}\right) = \sum_{d \leq x^\varepsilon} \mu(d) \sum_{m \leq x} \sum_{\ell=1}^{d^3} e\left(\frac{\ell|m^c}{d^3}\right)
= \sum_{d \leq x^\varepsilon} \frac{\mu(d)}{d^3} \mathfrak{A}(x) + \sum_{2 \leq d \leq x^\varepsilon} \frac{\mu(d)}{d^3} \sum_{\ell=1}^{d^3-1} \sum_{m \leq x} e\left(\frac{\ell|m^c}{d^3}\right).
$$

Taking $\eta = 2\varepsilon$ in (1.2), then there exists $\delta$ satisfying $2\varepsilon \leq \delta \leq 1/2$. From (1.2) we obtain

$$
\sum_{2 \leq d \leq x^\varepsilon} \frac{\mu(d)}{d^3} \sum_{\ell=1}^{d^3-1} \sum_{m \leq x} e\left(\frac{\ell|m^c}{d^3}\right) \ll x^{1-\delta} \sum_{2 \leq d \leq x^\varepsilon} \frac{d^3 - 1}{d^3} \ll x^{1-\varepsilon}.
$$

It is easy to see that

$$
\sum_{d \leq x^\varepsilon} \frac{\mu(d)}{d^3} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^3} + O(x^{-2\varepsilon}) = \frac{1}{\zeta(3)} + O(x^{-2\varepsilon}).
$$

From (3.3) and the fact that $\mathcal{A} \subseteq \mathbb{N}$, we get

$$
\Sigma_1 = \frac{1}{\zeta(3)} \mathfrak{A}(x) + O(x^{1-2\varepsilon}).
$$
Now we estimate $\Sigma_2$. We have

$$\Sigma_2 = \sum_{m \leq x} \sum_{d^3 \leq \ell \leq d^3 + 1} \mu(d) \ll \sum_{m \leq x} \sum_{d^3 \leq 2 \leq m} \frac{1}{d \geq x^\varepsilon}$$

$$\ll \sum_{m \leq x} \sum_{(d^3 \ell - 2) \gamma < m \leq (d^3 \ell + 2) \gamma} 1 \ll \sum_{d^3 \ell \leq x^\varepsilon} \left( [(d^3 \ell + 2) \gamma] - [(d^3 \ell - 2) \gamma] \right)$$

$$\ll (\log x)^2 \sum_{d \sim D, \ell \sim L} \left( [(d^3 \ell + 2) \gamma] - [(d^3 \ell - 2) \gamma] \right)$$

(3.5)

for some pair $(D, L)$, where $x^\varepsilon \ll D \ll x^{\varepsilon/3}$, $1 \ll L \ll x^{\varepsilon-3\varepsilon}$, $D^3L \ll x^\varepsilon$.

If $d \sim D$, $\ell \sim L$, then, by Lagrange's mean value theorem, we get

$$(d^3 \ell + 2) \gamma - (d^3 \ell) \gamma < 2\gamma (d^3 \ell) \gamma - 1 < 2\gamma (D^3L) \gamma - 1 < 4\gamma (D^3L) \gamma - 1$$

and

$$(d^3 \ell) \gamma - (d^3 \ell - 2) \gamma < 2\gamma (d^3 \ell - 2) \gamma - 1 < 2\gamma (d^3 \ell/2) \gamma - 1 < 4\gamma (D^3L) \gamma - 1.$$ 

Therefore, we obtain

$$\sum_{d \sim D, \ell \sim L} \left( [(d^3 \ell + 2) \gamma] - [(d^3 \ell - 2) \gamma] \right)$$

$$= \sum_{d \sim D, \ell \sim L} 1 \ll \sum_{d \sim D, \ell \sim L} [(d^3 \ell) \gamma + 4\gamma (D^3L) \gamma - 1] - [(d^3 \ell) \gamma - 4\gamma (D^3L) \gamma - 1]$$

$$\ll (D^3L) \gamma - 1 DL + \left| \sum_{d \sim D, \ell \sim L} \psi \left( (d^3 \ell) \gamma - 4\gamma (D^3L) \gamma - 1 \right) \right|$$

$$+ \left| \sum_{d \sim D, \ell \sim L} \psi \left( (d^3 \ell) \gamma + 4\gamma (D^3L) \gamma - 1 \right) \right|$$

$$=: (D^3L) \gamma - 1 DL + |T_+(D, L)| + |T_-(D, L)|$$

$$\ll x^{1-2\varepsilon} + |T_+(D, L)| + |T_-(D, L)|,$$

(3.6)

where the last step uses the following estimate

$$(D^3L) \gamma - 1 DL = (D^3L) \gamma - 1 D^3L \cdot D^{-2} = (D^3L) \gamma \cdot D^{-2} \ll x^{1-2\varepsilon}.$$ 

From (3.1), (3.4), (3.5) and (3.6), we can see that it is sufficient to show

$$T_+(D, L) \ll x^{1-2\varepsilon}, \quad T_-(D, L) \ll x^{1-2\varepsilon}.$$ 

(3.7)

Set $N := D^3L$. We shall prove

$$T_+(D, L) \ll N^{\gamma-2\varepsilon}, \quad T_-(D, L) \ll N^{\gamma-2\varepsilon},$$

(3.8)
Thus, from what follows, we always assume that \( D \ll N^{(1-\gamma+2\epsilon)/2} \), then we have

\[
T_\pm(D, L) \ll DL = D^3 L \cdot D^{-2} = ND^{-2} \ll N^{1-(1-\gamma+2\epsilon)} = N^{\gamma-2\epsilon}.
\]

Thus, from what follows, we always assume that \( D \ll N^{(1-\gamma+2\epsilon)/2} \) and \( DL \geq 100N^{\gamma-2\epsilon} \).

Taking \( J = [D^2LN^{2\epsilon-\gamma}] \) in Lemma 2.1, then we have

\[
T_\pm(D, L) = \sum_{d \sim D} \sum_{\ell \sim L} \left( \sum_{1 \leq |h| \leq J} a(h) e\left(h((d^3\ell)^\gamma \pm 4\gamma(D^3L)^{\gamma-1})\right) + O\left(\sum_{|h| \leq J} b(h) e\left(h((d^3\ell)^\gamma \pm 4\gamma(D^3L)^{\gamma-1})\right)\right) \right) = I + II,
\]

where

\[
I = \sum_{1 \leq |h| \leq J} a(h) \sum_{d \sim D} \sum_{\ell \sim L} e\left(h((d^3\ell)^\gamma \pm 4\gamma(D^3L)^{\gamma-1})\right) \ll \sum_{1 \leq h \leq J} \frac{1}{h} \sum_{d \sim D} \sum_{\ell \sim L} \left| e(h(d^3\ell)^\gamma) \right|, \tag{3.10}
\]

\[
II \ll \sum_{d \sim D} \sum_{\ell \sim L} \sum_{|h| \leq J} b(h) e\left(h((d^3\ell)^\gamma \pm 4\gamma(D^3L)^{\gamma-1})\right) \ll \frac{DL}{J} + \sum_{1 \leq h \leq J} \left| b(h) \right| \sum_{d \sim D} \sum_{\ell \sim L} \left| e(h(d^3\ell)^\gamma) \right| \ll N^{\gamma-2\epsilon} + \sum_{1 \leq h \leq J} \frac{1}{h} \sum_{d \sim D} \sum_{\ell \sim L} \left| e(h(d^3\ell)^\gamma) \right| \ll N^{\gamma-2\epsilon} + \sum_{1 \leq h \leq J} \frac{1}{h} \sum_{d \sim D} \sum_{\ell \sim L} \left| e(h(d^3\ell)^\gamma) \right|. \tag{3.11}
\]

Combining (3.9), (3.10) and (3.11), we derive

\[
T_\pm(D, L) \ll N^{\gamma-2\epsilon} + \sum_{1 \leq h \leq J} \frac{1}{h} \sum_{d \sim D} \sum_{\ell \sim L} \left| e(h(d^3\ell)^\gamma) \right| \ll N^{\gamma-2\epsilon} + \frac{1}{H} \left| S(H, D, L) \right| \cdot \log J, \tag{3.12}
\]

for some \( 1 \ll H \ll J \), where

\[
S(H, D, L) := \sum_{h \sim H} \sum_{d \sim D} \sum_{\ell \sim L} \left| e(h(d^3\ell)^\gamma) \right|.
\]
Therefore, we only need to show
\[ S(H, D, L) \ll HN^{\gamma-2\varepsilon}. \] (3.13)

Let \( F = HN^\gamma \). Thus, we have \( N^\gamma \ll F \ll D^2LN^{2\varepsilon} \). Next, in order to prove (3.13), we shall consider the following three cases.

**Case 1** If \( D \ll N^{2\gamma-1-6\varepsilon} \), we use exponential pair \((1/2, 1/2)\) to estimate the inner sum over \( \ell \) and apply trivial estimate to the sum over \( h \) and \( d \). Thus, we get
\[
S(H, D, L) \ll \frac{HD}{HD^{3/2}L^{3/2}} + HD((HD^{3\gamma}L^{\gamma-1})^{1/2}L^{1/2}) = \frac{DL}{N^{\gamma}} + HD((HN^\gamma L^{-1})^{1/2}L^{1/2}) \ll N^{1-\gamma} + HD(JN^\gamma)^{1/2} \ll N^{1-\gamma} + HD(D^2L)^{1/2}N^{\varepsilon} = N^{1-\gamma} + HD^{1/2}(D^3L)^{1/2}N^{\varepsilon} \ll N^{1-\gamma} + HD^{1/2}N^{1/2+\varepsilon} \ll HN^{\gamma-2\varepsilon}.
\]

**Case 2** If \( N^{2\gamma-1-6\varepsilon} \ll D \ll N^{6\gamma-3-22\varepsilon} \), by Theorem 7 of Cao and Zhai [4] with parameters \((M, M_1, M_2) = (L, H, D)\), we obtain
\[
N^{-\varepsilon} \cdot S(H, D, L) \ll (F^2L^3H^{10}D^7)^{1/8} + (F^4H^{18}D^{10}L^{54}D^{54})^{1/58} + (F^{10}L^{20}H^{100}D^{100})^{1/108} + (F^{11}L^{24}H^{92}D^{92})^{1/98} + (F^{16}L^{27}H^{27}D^{27})^{1/29} + (F^{111}L^{86}H^{294}D^{294})^{1/336} + (F^{103}L^{74}H^{266}D^{266})^{1/304} + (F^{119}L^{74}H^{294}D^{294})^{1/336} + (F^{80}L^{19}H^{188}D^{188})^{1/200} + (F^{149}L^{34}H^{344}D^{344})^{1/368} + (F^{43}L^{5}H^{94}D^{94})^{1/100} + (F^{2}L^{H}H^{6}D^{6})^{1/6} + (F^{4}L^{-1}H^{8}D^{8})^{1/8} + F^{-1/2}LHD \ll HN^{\gamma-3\varepsilon}.
\]

**Case 3** If \( D \gg N^{6\gamma-3-22\varepsilon} \), by Theorem 3 of Robert and Sargos [12] with parameters \((H, N, M) = (H, D, L)\), we deduce that
\[
N^{-2\varepsilon} \cdot S(H, D, L) \ll (HDL)\left(\left(\frac{F}{HD^{1/2}}\right)^{1/2} + \frac{1}{L^{1/2}} + \frac{1}{F}\right) \ll HN^{\gamma/4}D^{3/4}L^{1/2} + HDL^{1/2} + DLN^{-\gamma} \ll HN^{\gamma-4\varepsilon}
\]
under the condition \( D \gg N^{(2-3\gamma+16\varepsilon)/3} \). Moreover, by noting the fact that \( \gamma > 6/11 \), there must hold \( N^{(2-3\gamma+16\varepsilon)/3} \ll N^{6\gamma-3-22\varepsilon} \).

Combining the above three cases, we have finished the proof of Theorem 1.1.
4 Proof of Corollary 1.2

In this section, we shall prove Corollary 1.2. Take \( \eta = 2\varepsilon \) in (1.2). Suppose that \( 2 \leq d \leq x^\varepsilon \), \( 1 \leq \ell \leq d^2 - 1 \), then there exist a pair of integers \( a \) and \( q \) satisfying \( 2 \leq q \leq x^\eta \), \( 1 \leq a \leq q - 1 \), \( (a, q) = 1 \). Denote \( \alpha = a/q \) and

\[
S_c(x; A, \alpha) := \sum_{n \leq x \atop n \in A} e(\alpha [n^c]).
\]

In order to prove Corollary 1.2, we need to prove the following lemma.

**Lemma 4.1** Let \( 1 < c < 2 \) and \( 0 < \eta(c - 1)/100 \) be a sufficiently small constant, then we have

\[
S_c(x; N, \alpha) \ll x^{1-(3-c)/7} \log x, \quad (4.1)
\]

\[
S_c(x; \mathcal{A}, \alpha) \ll x^{1-(5-2c)/90} \log^{19} x \quad (4.2)
\]

\[
S_c(x; \mathcal{F}_3, \alpha) \ll x^{1-(11-4c)/22} \log^2 x. \quad (4.3)
\]

**Proof.** We only need to prove (4.3), since (4.1) and (4.2) are from Lemma 2 of [5]. Obviously, it is sufficient to prove that the following estimate

\[
S^*_c(N; \mathcal{F}_3, \alpha) := \sum_{n \sim N \atop n \in \mathcal{F}_3} e(\alpha [n^c]) \ll N^{1-\delta} \log^\omega N \quad (4.4)
\]

holds with \( x^{3/4} \ll N \ll x \).

Taking \( H = N^\delta \), \( \delta = (11 - 4c)/22 \) in Lemma 2.3, we get

\[
S^*_c(N; \mathcal{F}_3, \alpha) = \sum_{n \sim N \atop n \in \mathcal{F}_3} e(\alpha n^c - \alpha \{ n^c \})
\]

\[
= \sum_{n \sim N \atop n \in \mathcal{F}_3} e(\alpha n^c) \left( \sum_{|h| \leq H} c_h(\alpha) e(h n^c) + O\left( \min\left(1, \frac{1}{H n^c}\right) \right) \right)
\]

\[
= \sum_{|h| \leq H} c_h(\alpha) \sum_{n \sim N \atop n \in \mathcal{F}_3} e((h + \alpha) n^c) + O\left( \sum_{n \sim N} \min\left(1, \frac{1}{H n^c}\right) \right).
\]

From Lemma 2.2, we obtain

\[
\sum_{n \sim N} \min\left(1, \frac{1}{H n^c}\right) = \sum_{n \sim N} \sum_{h = -\infty}^{+\infty} a(h) e(h n^c) = \sum_{h = -\infty}^{+\infty} a(h) \sum_{n \sim N} e(h n^c)
\]

\[
= N \cdot a(0) + \sum_{h = -\infty \atop h \neq 0}^{+\infty} a(h) \sum_{n \sim N} e(h n^c)
\]

\[
\ll N |a(0)| + \sum_{h = 1}^{\infty} |a(h)| \left| \sum_{n \sim N} e(h n^c) \right|
\]

8
\[
\begin{aligned}
&\ll N^{1-\delta} \log N + \sum_{h=1}^{\infty} \min\left( \frac{1}{h^2}, H \right) (hn^{-1})^{4/18} N^{11/18} \\
&\ll N^{1-\delta} \log N + \sum_{h \leq H} h^{-7/9} N^{(4c+7)/18} + \sum_{h > H} \frac{H}{h^{16/9}} N^{(4c+7)/18} \\
&\ll N^{1-\delta} \log N + N^{(4c+7)/18} H^{2/9} \\
&\ll N^{1-\delta} \log N + N^{(4c+7+4\delta)/18} \ll N^{1-\delta} \log N,
\end{aligned}
\]
where we use the exponential pair \((4/18, 11/18)\) to estimate the sum over \(n\).

Since \(2 \leq q \leq x^\eta, 1 \leq a \leq q-1, (a,q) = 1\), then \(x^{-\eta} \leq \alpha = a/q \leq (q-1)/q = 1 - 1/q \leq 1 - x^{-\eta}\). Thus, for \(|h| \leq H\), there holds \(x^{-\eta} \leq |h + \alpha| \ll N^\delta\). Noting that \(0 < \eta < (c-1)/100\), we get \(|h + \alpha| N^{c-1} \gg x^{-\eta} x^{3(c-1)/4} \gg 1\).

Therefore, we deduce that
\[
\begin{aligned}
\sum_{n \sim N \atop n \in \delta_n} e((h + \alpha)n^c) &= \sum_{n \sim N} \left( \sum_{m^3 \mid n} \mu(m) \right) e((h + \alpha)n^c) \\
&= \sum_{N < m^3 \leq 2N} \mu(m) e((h + \alpha)m^{3c}n^c) \\
&= \sum_{N < m^3 \leq 2N} \mu(m) e((h + \alpha)m^{3c}n^c) + O(N^{1-2\delta}) \\
&\ll \sum_{m \leq N^\delta} \left( |h + \alpha|m^{3c} \left( \frac{N^{c-1}}{m^3} \right) \right)^{4/18} \left( \frac{N^{c-1}}{m^3} \right)^{11/18} + N^{1-2\delta} \\
&\ll N^{(4c+7+4\delta)/18} \left( \sum_{m \leq N^\delta} m^{-7/6} \right) + N^{1-2\delta} \\
&\ll N^{(4c+7+\delta)/18} + N^{1-2\delta} \ll N^{1-\delta},
\end{aligned}
\]
where we use the exponential pair \((4/18, 11/18)\) to estimate the sum over \(n\). From (26) of Cao and Zhai [5], we have
\[
\sum_{|h| \leq H} c_h(\alpha) \sum_{n \sim N \atop n \in \delta_n} e((h + \alpha)n^c) \ll N^{1-\delta} \log N.
\]
This completes the proof of Lemma 4.1. ■

From the three following formulas
\[
\begin{aligned}
N(x) &= x + O(1), \\
\mathfrak{s}_3(x) &= \frac{x}{\zeta(3)} + O(x^{1/2+\epsilon}), \\
\mathcal{P}(x) &= \int_2^x \frac{du}{\log u} + O(xe^{-c_0 \sqrt{\log x}}),
\end{aligned}
\]
9
Theorem 1.1 and Lemma 4.1, we know that Corollary 1.2 holds.

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