THE HOROFUNCTION BOUNDARY OF THE LAMPLIGHTER GROUP $L_2$ WITH THE DIESTEL-LEADER METRIC

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Abstract. We fully describe the horofunction boundary $\partial_h L_2$ with the word metric associated with the generating set $\{t, at\}$ (i.e. the metric arising in the Diestel-Leader graph $DL_2(2)$). The visual boundary of $L_2$ with this metric is a subset of $\partial_h L_2$. The height function on $DL_2(2)$ provides a natural stratification of $\partial_h L_2$. Further, the height function and its negation are themselves non-Busemann functions in $\partial_h L_2$ and are global fixed points of the action of $L_2$.

1. Introduction

The horofunction boundary $\partial_h X$ of a proper complete metric space $(X,d)$ is in general defined as a subspace of the quotient of $C(X)$, the space of continuous $\mathbb{R}$-valued functions on $X$, by constant functions [1, Definition II.8.12]. For our purposes, it suffices to choose a base point $b$ in $X$ and use the embedding $i : X \hookrightarrow C(X)$ sending $z \in X \mapsto d(z, x) - d(z, b)$. Since $X$ is proper, the closure $\overline{i(X)}$ in $C(X)$ provides a compactification of $X$. We define $\partial_h X$ to be $X \setminus i(X)$. We call a point in $\overline{X}$ a horofunction, and given a sequence $(y_n)$ of points in $X$, one can define a horofunction associated to $(y_n)$ by

\[
 h_{y_n}(x) = \lim_{n \to \infty} d(y_n, x) - d(y_n, b)
\]

provided this limit exists.

Gromov defines the horofunction boundary, which he calls the ideal boundary, in the context of hyperbolic manifolds [5], but the definition applies to any complete metric space. In [1] Bridson and Haefliger use this construction in the context of CAT(0) spaces as a functorial construction of the visual boundary. The horofunction boundary also naturally arises in the study of group $C^*$-algebras, where Rieffel, referring to it as the metric boundary, demonstrates its usefulness particularly in determining the $C^*$-algebra he calls the cosphere algebra [11, §3].

In this paper, $X$ is a group with a word metric, which is $\mathbb{N}$-valued. In this setting, we may follow Walsh [13] and define a geodesic ray to be an isometric embedding $\mathbb{N} \to X$. We refer to point of $\partial_h X$ as a Busemann point if it corresponds to a sequence of points lying along a geodesic ray. We will refer to the space of

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1For us, $\mathbb{N}$ contains 0.
asymptotic classes of geodesic rays in \((X,d)\) as the visual boundary \(\partial_{\infty}X\). In CAT(0) spaces, all horofunctions correspond to Busemann points; in fact, we can extend \(i\) to \(\bar{i}: X \sqcup \partial_{\infty}X \to \overline{X}\), and this is a homeomorphism \([1, \S II.8.13]\). In general one cannot expect an injective, surjective, or even continuous map from \(\partial_{\infty}X\) to \(\partial_hX\). Rieffel brings up the question of determining for a given space \((X,d)\) which points of \(\partial_{\infty}X\) are Busemann points \([11, \text{after Definition 4.8}]\). As an interesting example of non-surjectivity, Reiffel demonstrates that there are no non-Busemann points in \(\partial_{h}\mathbb{Z}^n\) with the \(\ell_1\) norm, and there are countably many Busemann points \([11]\). However, Kitzmiller and Rathbun demonstrate that \(\partial_{\infty}\mathbb{Z}^n\) is uncountable \([7]\).

Others have studied the horofunction boundary of Cayley graphs of non-CAT(0) groups, often with variation in their terminology, though examples are still sparse. Develin extended Rieffel’s work to abelian groups (he refers to the horofunction boundary as a Cayley compactification of the group) \([2]\). Friedland and Freitas found explicit formulas for horofunctions for \(\text{GL}(n,\mathbb{C})/U_n\) with Finsler \(p\)-metrics (they use the term Busemann compactification) \([3]\). Webster and Winchester (using the term metric boundary as Rieffel) studied the action of a word hyperbolic group on its horofunction boundary and found it is amenable \([15]\). They also established necessary and sufficient conditions for an infinite graph to have non-Busemann points in its horofunction boundary \([16]\). Walsh has considered the horofunction boundaries of Artin groups of dihedral type \([13]\) and the action of a nilpotent group on its horofunction boundary \([14]\). Klein and Nikas have studied the horofunction boundary of the Heisenberg group equipped with different metrics \([8], [9]\); in particular, they determine the isometry group of the Heisenberg group with the Carnot-Carathéodory metric.

The lamplighter group is given by the presentation:

\[L_2 = \langle a, t \mid [a^i, a^j] \forall i, j \in \mathbb{Z} \rangle\]

Let \(S = \{t, at\}\). The generating set \(S\) naturally arises when viewing the lamplighter group as a group generated by a finite state automaton (FSA) \([4]\). This is a rare case where we are able to understand the Cayley graph of such a group with its FSA generating set. In this case, the Cayley graph is the Diestel-Leader graph \(DL_2(2)\) \([17]\). In \([6]\), the authors describe the visual boundary for Diestel-Leader graphs \(DL_d(q)\), which are certain graphs arising from products of \(d\) regular trees of valence \(q+1\). When \(d > 2\), the topology is indiscrete, but for \(d = 2\), \(DL_2(2)\) inherits enough structure from its component trees that \(\partial_{\infty}DL_2(2)\) is an interesting non-Hausdorff space. This then provides a boundary for \(L_2\) by using the corresponding word metric; again this is dependent on generating set.

The goal of this paper is to fully describe \(\partial_hL_2\) where the metric on \(L_2\) is the word metric from \(S\). In Section \([3]\) we provide some background on this metric, and in Section \([2]\) we discuss the relationship between \(\partial_{\infty}L_2\) and \(\partial_hL_2\), proving

**Theorem** (A - Corollary \([3.3]\) and Observation \([3.4]\)). There is an injection \(\partial_{\infty}L_2 \to \partial_hL_2\), but it is not continuous.

In Observation \([6.2]\) we point out that \(\partial_{\infty}L_2\) is naturally partitioned into two subspaces \(\partial_{\infty}L_2^\pm\), each of which does (continuously) embed in \(\partial_hL_2\). In Section \([4]\) we explicitly compute formulas for families of horofunctions, including Busemann points.
functions. It turns out the natural height map $H : L_2 \to \mathbb{Z}$ (see Def. 2.1) is a non-Busemann horofunction. Section 5 provides a classification of the points of $\partial_h L_2$, which is our main result.

**Theorem (B - Corollary 5.15).** Every point in $\partial_h L_2$ is one of the horofunctions computed in Section 4, or is $\pm H$.

We describe the topology of $\partial_h L_2$ in Section 5 by determining the accumulation points, leading to the visualization in Figure 5 on page 14. Finally, Section 7 deals with some properties of the natural action of $L_2$ on $\partial_h L_2$, in particular noting that $\pm H$ are global fixed points.

**2. The Diestel-Leader metric on $L_2$**

Let $d$ denote the word metric on $L_2$ with generating set $S = \{t, at\}$. Since this is the metric on $L_2$ induced by the Cayley graph $DL_2(2)$, we refer to it as the Diestel-Leader metric on $L_2$. Whenever we refer to $\partial_\infty L_2$ or $\partial_h L_2$, we always mean with $d$. Stein and Taback have calculated the metric for general Diestel-Leader graphs [12], but in our case it is simple enough to review and provide a proof.

Each element of $L_2$ is associated with a “lamp stand”, which consists of an infinite row of lamps in bijective correspondence with $\mathbb{Z}$, finitely many of which are lit, and a marked lamp indicating the position of the lamplighter. Figure 1 illustrates a typical example. The lamps are boolean: either on or off. Right multiplying by $a$ toggles the lamp at the lamplighter’s position, while right multiplying by $t$ increments the position of the lamplighter. We think of this increment as a “step right” as in the figure. Using $S$, the actions are either “step (right or left)” for $t^{\pm 1}$, “toggle then step right” for $at$, or “step left then toggle” for $(at)^{-1} = t^{-1}a$.

**Definition 2.1.** For $g \in L_2$, we define $H(g)$ to be the position of the lamplighter in the lamp stand representing $g$, or equivalently the exponent sum of $t$ in a word representing $g$, or the height of $g$ in $DL_2(2)$.

We define $m(g)$ to be equal to the minimum position of a lit lamp in the lamp stand representation of $g$ if the set of lit lamps is non-empty, and equal to $-\infty$ otherwise. Similarly, we define $M(g)$ to be equal to the maximum position of a lit lamp in the lamp stand representation of $g$ if the set of lit lamps is non-empty, and equal to $-\infty$ otherwise.

For $g_1, g_2 \in L_2$, we define $m(g_1, g_2)$ to be the minimum position of a lamp whose status differs in the lamp stands of $g_1$ and $g_2$ if such a position exists, and equal to $+\infty$ otherwise. Similarly, $M(g_1, g_2)$ is the maximum position of a lamp whose status differs in the lamp stands of $g_1$ and $g_2$ if such a position exists, and is $-\infty$ otherwise.

**Lemma 2.2.** If $g_1, g_2 \in L_2$, then $d(g_1, g_2) = 2(B - A) - C$ where

- $A = \min\{m(g_1, g_2), H(g_1), H(g_2)\}$,
Figure 2. Distance between two elements of $L_2$ with Diestel-Leader metric.

- $B = \max\{M(g_1, g_2) + 1, H(g_1), H(g_2)\}$,
- $C = |H(g_2) - H(g_1)|$

See Figure 2 for an illustration of a typical path.

**Proof.** Since the Cayley graph is vertex transitive, without loss of generality we may assume that $g_1 = \text{id}$ and we denote $g_2$ simply by $g$. We will consider a geodesic from $\text{id}$ to $g$ on the lamp stand representations of the elements of $L_2$.

A geodesic will start at $\text{id}$ with no lit lamps and the lamplighter at position $H(\text{id}) = 0$. The lamplighter must move in one direction (either left or right) until it has gone as far as it needs to, it then travels to the other extremal position, and then finishes by moving to $H(g)$. The initial direction will be away from $H(g)$ in order to minimize the total distance. Notice that the minimum extremal position is given by $A$, which in this case is $A = \min\{m(g), H(g), 0\}$, and the maximal extremal position is given by $B$, which in this case is $B = \max\{M(g) + 1, H(g), 0\}$. Notice that we use $M(g) + 1$ and not $M(g)$ since to turn on the lamp at position $k$, the lamplighter must be at position $k + 1$ either immediately before turning on lamp $k$ (if using generator $(at)^{-1}$) or immediately after (if using generator $at$).

Thus, the second of the three segments of the geodesic will have length $B - A$. The lengths of the first and third segments will sum to less than $B - A$, and the amount less will be exactly equal to the distance between the starting and ending position, which in our case is $|H(g)|$.

3. Busemann points

In [6], the authors investigate the visual boundary of $DL_2(2)$ and find that as a set, it is a disjoint union of two punctured Cantor sets, but topologically it is not Hausdorff.

3.1. The visual boundary as a subset of the horoboundary. We now show that there is an injection from $\partial_\infty L_2 = \partial_\infty DL_2(2)$ into $\partial_h L_2$.

**Lemma 3.1** (Lemma 8.18(1) in Chapter II.8 of [1]). Let $\gamma$ be a geodesic ray in $DL_2(2)$ based at the identity. Then the sequence of points $(\gamma(n))$ defines a horofunction $b^\gamma$.

The horofunction $b^\gamma$ is called the Busemann function associated to $\gamma$. In a CAT(0) space, the Busemann functions of two rays are equal if and only if those two rays are asymptotic. Even though $DL_2(2)$ is not CAT(0), the same is true in our case.

**Lemma 3.2.** Let $\gamma, \gamma'$ be geodesic rays in the Cayley graph of $L_2$ based at the identity. The Busemann functions $b^\gamma$ and $b^{\gamma'}$ are equal if and only if $\gamma$ and $\gamma'$ are asymptotic to each other.
Proof. By [6, Lemma 3.5], asymptotic rays in DL₂(2) eventually merge. Thus, their Busemann functions are equal.

Now suppose that \( \gamma \) and \( \gamma' \) are not asymptotic to each other. Let \( \alpha \in [\gamma], \alpha' \in [\gamma'] \) (i.e. \( \alpha \) is in the asymptotic equivalence class of \( \gamma \)) so that \( \alpha \) and \( \alpha' \) have maximal shared initial segment. Say that this shared initial segment has length \( k \). Let \( x = \alpha(k + 1) \). Notice that by definition, \( b^\alpha(x) = -(k + 1) \). By our choice of \( \alpha \) and \( \alpha' \), \( b^\alpha'(x) = -(k - 1) \), so \( b^\alpha \neq b^{\alpha'} \). By the proof above of the other direction, \( b^{\gamma} = b^\alpha \) and \( b^{\gamma'} = b^{\alpha'} \) and we are done. \( \square \)

**Corollary 3.3.** The relation taking an asymptotic equivalence class of geodesic rays based at the identity to their Busemann functions is an injection of \( \partial_\infty L_2 \) into \( \partial_h L_2 \).

**Observation 3.4.** The injection in Corollary 3.3 is not continuous.

**Proof.** The continuous injective image of a non-Hausdorff space like \( \partial_\infty L_2 \) must also be non-Hausdorff, while \( C(L_2) \) (and thus its subspace \( \partial_h L_2 \)) is Hausdorff. \( \square \)

### 3.2. Lampstand Interpretation of Busemann Points.

As in [6, Section 3.3], we can interpret Busemann points of the boundary in terms of the lamp stand model. Any Busemann point can be represented by a geodesic ray emanating from the identity which follows a sequence of steps wherein the lamplighter first moves one direction until reaching the extremal lit lamp in that direction (without lighting any lamps) then “turns around” and marches off towards \( \pm \infty \) toggling lamps as necessary. Thus, in the limit there is either a minimal lit lamp (if any are lit at all), and the lamplighter is at \( +\infty \); or there is a maximal lit lamp (if any are lit at all), and the lamplighter is at \( -\infty \). A “turning around” only occurs if the minimal lit lamp has negative index in the former case, or the maximal lit lamp has positive index in the latter. The choice of not lighting lamps prior to arriving at the minimal or maximal lit lamp specifies a unique geodesic ray to represent the Busemann function, but the infinite lamp stand is the same for asymptotic rays.

**Definition 3.5.** If for a geodesic ray \( \gamma \), the infinite lamp stand for its asymptotic class \( [\gamma] \) has the lamplighter at \( +\infty \), we say \( [\gamma] \in \partial_\infty L_2^+ \), set \( m(\gamma) \) to be the minimal position of a lit lamp (or \( +\infty \)) and \( H(\gamma) = +\infty \). If the lamplighter is at \( -\infty \), we say \( [\gamma] \in \partial_\infty L_2^- \), and set \( M(\gamma) \) to be the maximal position of a lit lamp (or \( -\infty \)) and \( H(\gamma) = -\infty \). Additionally, for \( g \in L_2 \), we define \( m(\gamma,g) \) (if \( H(\gamma) = +\infty \)) and \( M(\gamma,g) \) (if \( H(\gamma) = -\infty \)) as we defined \( m(g_1,g_2) \) and \( M(g_1,g_2) \) in Definition 2.1.

### 4. Model horofunctions

Below, we repeatedly apply Lemma 2.2 and Equation 1 to explicitly calculate certain horofunctions. We break \( \partial_h L_2 \) into four categories: the Busemann points, the spine, the ribs, and the two points \( \pm H \). The reader may skip ahead to Figure 5 on page [14] to preview a visualization of the boundary, illustrating our choice of terms.

#### 4.1. The Spine.
Figure 3. The element $s_n$ having only lamps at index $\pm n$ lit, and the lamplighter at position 0.

4.1.1. $s_0$. For $n \in \mathbb{N}$, let $s_n$ be the element of $L_2$ such that the lamps at $-n$ and $+n$ are lit, and $H(s_n) = 0$, as in Figure 3. Applying Lemma 2.2 with parameters $A = -n$, $B = n + 1$, and $C = 0$, we have

$$d(s_n, \text{id}) = 2(B - A) - C = 4n + 2.$$ 

Given $g \in L_2$, take $n > \max\{-m(g), M(g), |H(g)|\}$ and apply Lemma 2.2 to obtain

$$d(s_n, g) = 4n + 2 - |H(g)|.$$ 

The corresponding horofunction is

$$s_0^0(g) = h_{s_n}(g) = -|H(g)|,$$

where the 0 indicates that $H(s_n) = 0$ for all $n$.

4.1.2. $s_l$. For $l \in \mathbb{Z}$, consider the sequence $(s_n^l)$, defined as in $(s_n)$ except that $H(s_n^l) = l$ instead of 0. By Lemma 2.2, $d(s_n^l, \text{id}) = 4n + 2 - |l|$. Given $g \in L_2$, we take $n > \max\{-m(g), M(g), |H(g)|, |l|\}$ and apply Lemma 2.2 to obtain $d(s_n^l, g) = 4n + 2 - |l - H(g)|$. Applying Equation 1 gives the horofunction

$$s_l^l(g) = h_{s_n^l}(g) = |l| - |l - H(g)|.$$ 

Figure 4 shows the graphs of $s^{-1}$, $s^0$, $s^1$, and $s^2$, respectively, as functions of height. Since $\lim_{l \to \pm \infty} s_l^l(g) = \pm H(g)$, the height function and its negative are horofunctions as well.

4.2. The Ribs.
For $n \in \mathbb{N}$, let $r^+_n \in L_2$ have $H(r^+_n) = 0$ and only the lamp at position $n$ lit. From Lemma 2.2, we see $d(r^+_n, id) = 2n + 2$. Given $g \in L_2$, take $n > \max\{M(g), H(g)\}$. Using Lemma 2.2 with $A = \min\{0, H(g), m(g)\}$, $B = n + 1$, and $C = |H(g)|$, we arrive at:

$$d(g, r^+_n) = 2n + 2 - 2 \min\{0, H(g), m(g)\} - |H(g)|.$$  

The horofunction $h_{(r^+_n)}$ is therefore given by:

$$r^+(g) = h_{r^+_n}(g) = -2 \min\{m(g), H(g), 0\} - |H(g)|$$

Define $r^-_n$ as $r_n$, except lamp $-n$ is lit instead of lamp $n$. A similar calculation yields the horofunction:

$$r^-(g) = h_{r^-_n}(g) = 2 \max\{M(g) + 1, H(g), 0\} - |H(g)|$$

More generally, define $r^{+,-}_n$ and $r^{-,+}_n$ analogous to $r^+_n$ and $r^-_n$, except with $H(r^{+,-}_n) = l$, and one can verify the following calculations:

$$r^{+,-}_n(g) = h_{r^{+,-}_n}(g) = -2 \min\{m(g), H(g), l\} - |l - H(g)| + l$$

$$r^{-,+}_n(g) = h_{r^{-,+}_n}(g) = 2 \max\{M(g) + 1, H(g), l\} - |l - H(g)| - l$$

The functions given by Equations (0) and (4), $l \in \mathbb{Z}$, form copies of the spine. The spine horofunctions are bidirectional in the sense that the $n^{th}$ term of the sequence $(s^l)$ has lamps at $\pm n$ lit. So we think of $r^{+,-}$ as positive and $r^{-,+}$ as negative rib functions.

**4.2.2. General rib horofunctions.** The rest of the model rib horofunctions will be defined by sequences parameterized by certain elements of $L_2$. There are two cases.

First, consider $f \in L_2$ satisfying $M(f) < H(f)$ and having at least one lit lamp. $(r^{+,-}_n) \subset L_2$ for $n > H(f)$ where $r^{+,-}_n$ agrees with $f$, except it also has lamp $n$ lit. This sequence defines a horofunction $h_{r^{+,-}_n}$. The condition $M(f) < H(f)$ ensures uniqueness — any $f' \in L_2$ with $H(f) = H(f')$ and lamps in positions less than $H(f)$ agreeing with $f$ will define the same horofunction. This will be proved in Lemma 5.7.

We first calculate $d(r^{+,-}_n, id)$ for large $n$. Applying Lemma 2.2 with $A = \min\{m(f), 0\}$, $B = n + 1$, and $C = |H(f)|$,

$$d(r^{+,-}_n, id) = 2(n + 1) - 2 \min\{m(f), 0\} - |H(f)|.$$  

Now let $g \in L_2$ be given. For $n$ large enough, we apply Lemma 2.2 with $A = \min\{m(g, f), H(g)\}$, $B = n + 1$, and $C = |H(f) - H(g)|$ to obtain

$$d(r^{+,-}_n, g) = 2(n + 1) - 2 \min\{m(g, f), H(g)\} - |H(f) - H(g)|.$$  

Setting $l = H(f)$, we have the horofunction:

$$r^{+,-}(g) = 2(\min\{m(f), 0\} - \min\{m(f, g), H(g)\}) + s^{+,-}(g)$$

Now consider $f \in L_2$ satisfying $M(f) \geq H(f)$ and at least one lamp is lit. Define $(r^{-,+}_n) \subset L_2$ where $r^{-,+}_n$ agrees with $f$, except it also has lamp $-n$ lit. This sequence defines a horofunction $h_{r^{-,+}_n}$. The calculation is similar to the previous case, and (again using $l = H(f)$) we arrive at:

$$r^{-,+}(g) = 2(\max\{M(f, g) + 1, H(g)\} - \max\{M(f) + 1, 0\}) + s^{-,+}(g)$$
4.3. Busemann Functions. Given a geodesic ray $\gamma$ with $\gamma(0) = id$, let $b^\gamma$ denote its horofunction. Let $g \in L_2$. As discussed in Definition 3.3 Section 3.2, we either have $\gamma \in \partial L_2^+$ and $m(\gamma)$ and $m(\gamma, g)$ are defined, or $\gamma \in \partial L_2^-$ and $M(\gamma)$ and $M(\gamma, g)$ are defined. The formula for $b^\gamma$ depends on the direction of $\gamma$, so we use $b^{+\gamma} = b^\gamma$ when $\gamma \in \partial L_2^+$ and $b^{-\gamma} = b^\gamma$ when $\gamma \in \partial L_2^-$, to be clear.

In the $\gamma \in \partial L_2^+$ case, for $n$ large enough, we apply Lemma 2.2 to obtain:

\[d(\gamma(n), id) = 2(H(\gamma(n)) - \min\{m(\gamma), 0\}) - H(\gamma(n))\]
\[d(\gamma(n), g) = 2(H(\gamma(n)) - \min\{m(\gamma, g), H(g)\}) + H(g) - H(\gamma(n))\]

The $\gamma \in \partial L_2^-$ case can be similarly calculated. Thus the Busemann function corresponding to $\gamma$ is given by

\[b^{+\gamma}(g) = 2(\min\{m(\gamma), 0\} - \min\{m(\gamma, g), H(g)\}) + H(g)\]
if $\gamma \in \partial L_2^+$, and
\[b^{-\gamma}(g) = 2(\max\{M(\gamma, g) + 1, H(g)\} - \max\{M(\gamma) + 1, 0\}) - H(g)\]
if $\gamma \in \partial L_2^-$. 

**Observation 4.1.** Equations 3, 6, 7, 8, 9, 10, and 11 yield distinct horofunctions. Thus the spine and rib horofunctions are non-Busemann, as are $\pm H$. Moreover, within each equation, different parameters yield distinct horofunctions.

5. Classification of Horofunctions

We will now prove that the functions referred to in Observation 4.1 constitute all of $\partial_h L_2$.

**Definition 5.1.** Given a sequence $(g_n) \subset L_2$, we say that the lamp at position $k$ in the lamp stands of these elements stabilizes if there exists $N \in \mathbb{N}$ such that the lamp in position $k$ for the lamp stand representing $g_n$ has the same status (i.e. on or off) for all $n > N$.

We say that the lamp at position $k$ is flickering if it does not stabilize.

**Definition 5.2.** We say that sequence $(g_n)$ of elements of $L_2$ has an upper uniform bound on stabilization of lamps (upper u.b.s.l.) if there exists $N \in \mathbb{N}$ and $M \in \mathbb{Z}$ such that for all $k > M$, the lamp at position $k$ for the lamp stand representing $g_n$ has the same status (i.e. on or off) for all $n > N$.

We define lower uniform bound on stabilization of lamps (lower u.b.s.l.) similarly.

**Observation 5.3.** If a sequence $(g_n) \subset L_2$ does not have an upper u.b.s.l., then there exists a subsequence $(g_{n_k})$ such that the sequence $(M(g_{n_k}))$ is increasing without bound.

Similarly, if a sequence $(g_n) \subset L_2$ does not have a lower u.b.s.l., then there exists a subsequence $(g_{n_k})$ such that the sequence $(m(g_{n_k}))$ is decreasing without bound.

**Proof.** If the sequence fails to have an upper u.b.s.l., then $\sup\{M(g_n) \mid n \in \mathbb{N}\} = +\infty$ since if this supremum were a finite value $M_0 \in \mathbb{Z}$, then by setting $N = 0$ and $M = M_0$, the sequence would satisfy the definition for having an upper u.b.s.l. The existence of the desired subsequence is then guaranteed.

The proof for the lower uniform bound case is similar. \qed
Lemma 5.4. Suppose that a sequence \((g_n) \subset L_2\) with \(H(g_n) \to l \in \mathbb{Z} \cup \{+\infty\}\) has a lower u.b.s.l. If \((g_n)\) is associated with some horofunction \(h_{g_n}\), then for every \(k < l\), the lamp at position \(k\) stabilizes.

Proof. If \((g_n)\) has no flickering lamps, then we are done. So assume the sequence has some flickering lamps, and let \(k \in \mathbb{Z}\) be the minimum position of a flickering lamp. Assume \(k < l\).

Let \(y \in L_2\) such that \(H(y) = k\), \(y\) agrees with the stabilization of lamps of \((g_n)\) on the positions \(k - 1\) and below, and the lamp at position \(k\) is off. Let \(x \in L_2\) be exactly as \(y\), except that \(H(x) = k + 1\). Let \(n\) be sufficiently large so that the lamps at positions \(k - 1\) and below of \(g_n\) have achieved their eventual status and \(H(g_n) > k\).

Suppose the lamp at position \(k\) is lit in the lamp stand for \(g_n\). In Lemma 2.2 when computing \(d(g_n, x)\), \(C = H(g_n) - (k + 1)\), but when computing \(d(g_n, y)\), \(C = H(g_n) - k\), while the values for \(A\) and \(B\) remain the same (in this case, \(A = k\) for both). Thus \(d(g_n, x) = d(g_n, y) + 1\).

Now suppose the lamp at position \(k\) is not lit in the lamp stand for \(g_n\). Using Lemma 2.2 again, when computing \(d(g_n, x)\), \(A = k + 1\), \(C = H(g_n) - (k + 1)\), while when computing \(d(g_n, y)\), \(A = k\), \(C = H(g_n) - k\), and \(B\) remains the same. In this case, we have \(d(g_n, x) = d(g_n, y) - 1\).

By Equation 1
\[
h_{g_n}(x) - h_{g_n}(y) = \lim_{k \to \infty} d(g_n, x) - d(g_n, y)
\]
which by the above, does not exist. But we assumed \(h_{g_n}\) exists. Hence, our assumption that \(k < l\) is incorrect, and we have the desired result. \(\square\)

Lemma 5.5. Suppose that a sequence \((g_n) \subset L_2\) with \(H(g_n) \to l \in \{\infty\} \cup \mathbb{Z}\) has an upper u.b.s.l. If \((g_n)\) is associated with some horofunction \(h = h_{g_n}\), then for every \(k > l\), the lamp at position \(k\) stabilizes.

Proof. The proof for this lemma is the same as for Lemma 5.4. The asymmetry in the inequalities (one is strict, while the other is not) comes from the asymmetry of our generating set (including \(a\) but not \(ta\)). \(\square\)

Lemma 5.6. Suppose that a sequence \((g_n) \subset L_2\) has both an upper and a lower u.b.s.l. and that \(H(g_n) \to l \in \mathbb{Z}\). If \((g_n)\) is associated with some horofunction \(h_{g_n}\), then \(h_{g_n}\) is an element of the image of \(L_2\) in \(\overline{L_2}\).

Proof. By Lemmas 5.4 and 5.5 all the lamps in \((g_n)\) stabilize. Since there is u.b.s.l. to both sides, we in fact have the existence of some \(N \in \mathbb{N}\) such that the set of lit lamps in \(g_n\) is constant for all \(n > N\). Since the lamplighter limits to \(l\) by hypothesis and since \(\mathbb{Z}\) is a discrete set, we have that the sequence \((g_n)\) is eventually constant. \(\square\)

Lemma 5.7. Suppose that a sequence \((g_n) \subset L_2\) has either an upper or a lower u.b.s.l., but not both, and that \(H(g_n) \to l \in \mathbb{Z}\). If \((g_n)\) is associated with some horofunction \(h_{g_n}\), then \(h_{g_n}\) is a rib, i.e. one of \(r^{\pm\cdot l}, l \in \mathbb{Z}\) or \(r^{\pm\cdot f}\), \(f \in L_2\).

Proof. We consider the case where the sequence \((g_n)\) has a lower, but not upper, u.b.s.l. The other case is similar.
By Lemma 5.4, there exists \( N \in \mathbb{N} \) such that the the lamps below position \( l \) are stable and \( H(g_n) = l \) for all \( n > N \). Let \( r \) be the rib horofunction that matches this stabilization. Set \( (r_n) \) to be the example sequence the generates this horofunction.

By Observation 5.3, we may take a subsequence \( (g_{n_k}) \) such that \( (M(g_{n_k})) \) is increasing with \( M(g_{n_k}) > k \) for all \( k \). Choose a subsequence \( (r_{n_k}) \) of our model sequence such that \( M(r_{n_k}) = M(g_{n_k}) \).

Let \( x \in L_2 \). Choose \( K \in \mathbb{N} \) such that \( K > \max\{|l|, |M(x)|, |H(x)|\} \), and let \( k > K \).

Let \( A, B, C \) be as in Lemma 2.2 for the computation of \( d(g_{n_k}, x) \) and let \( A', B', C' \) be as in Lemma 2.2 for the computation of \( d(r_{n_k}, x) \). Notice that \( A = A' \) since the lamp stands for \( g_{n_k} \) and \( r_{n_k} \) are the same below the position \( H(g_{n_k}) = H(r_{n_k}) \), \( B = M(g_{n_k}) + 1 = M(r_{n_k}) + 1 = B' \) by our choice of \( K \), and \( C = C' \) since \( H(g_{n_k}) = H(r_{n_k}). \) Thus, \( d(g_{n_k}, x) = d(r_{n_k}, x) \).

For \( x = id \), we have that \( d(g_{n_k}, id) = d(r_{n_k}, id) \). Hence, \( h_{g_{n_k}} = h_{r_{n_k}} \) and so therefore \( h_{g_n} = r \). □

**Lemma 5.8.** Suppose that a sequence \( (g_n) \subset L_2 \) has neither an upper nor a lower u.b.s.l. and that \( H(g_n) \rightarrow l \in \mathbb{Z} \). If \( (g_n) \) is associated with some horofunction \( h_{g_n} \), then \( h_{g_n} = g' \).

**Proof.** Suppose that there exists a subsequence \( (g_{n_k}) \) such that for all \( N \in \mathbb{N} \) there exists \( K_N \in \mathbb{N} \) such that for all \( k > K_N \) we have that \( M(g_{n_k}) > N \) and \( m(g_{n_k}) < -N \). Then let \( x \in L_2 \), and let \( N \in \mathbb{N} \) such that \( N > \max\{|M(x)|, |m(x)|, |l|\} \). Let \( K = \max\{K_N, K_1\} \) where \( K_N \) is as given above and \( K_1 \) is an integer such that for all \( k > K_1 \), \( H(g_{n_k}) = l \) (recall that \( H(g_n) \rightarrow l \) and the integers are a discrete set).

Let \( k > K \). Then by choice of \( N \) and definition of \( K \) and using Lemma 2.2\( d(g_{n_k}, x) = 2(M(g_{n_k}) + 1 - m(g_{n_k})) - |l - H(x)| \) and specifically \( d(g_{n_k}, id) = 2(M(g_{n_k}) + 1 - m(g_{n_k})) - |l| \). Thus, by Equation 1, \( h_{g_{n_k}} = g'(x) \), and we are done.

Now suppose that such a subsequence does not exist. By Observation 5.3, since \( (g_n) \) has no lower u.b.s.l., there exists a subsequence \( (g_{n_i}) \) such that \( m(g_{n_i}) < -i \) for all \( i \) and \( (m(g_{n_i})) \) is decreasing. Also by Observation 5.3, since \( (g_n) \) has no upper u.b.s.l., there exists a subsequence \( (g_{n_j}) \) such that \( M(g_{n_j}) > j \) for all \( j \) and \( (M(g_{n_j})) \) is increasing. Since these are both subsequences of \( (g_n) \), both give rise to horofunctions, and \( h_{g_{n_i}} = h_{g_{n_j}} = h_{g_n} \).

Notice that the subsequence \( (g_{n_i}) \) must have an upper u.b.s.l., otherwise we would be able to find a subsequence as in the first part of the proof. Similarly, the subsequence \( (g_{n_j}) \) must have a lower u.b.s.l.

By Lemma 5.7, \( h_{g_{n_i}} \) is equal to one of the rib examples with stable component above the lamplighter. But also by Lemma 5.7, \( h_{g_{n_j}} \) is equal to one of the rib examples with stable component below the lamplighter. By inspecting Equations 8 and 9 we see that these two horofunctions cannot be equal, so \( h_{g_n} \) does not exist. □

**Lemma 5.9.** Suppose that a sequence \( (g_n) \subset L_2 \) has a lower u.b.s.l. and \( H(g_n) \rightarrow +\infty \). If \( (g_n) \) is associated with some horofunction \( h_{g_n} \), then \( h_{g_n} \) is equal to a Busemann function \( \beta^\gamma \) with \( \gamma \in \partial_\infty L_2^+ \).

**Proof.** By Lemma 5.4, there are no flickering lamps in \( (g_n) \), so consider the infinite lamp stand of the stabilization of lamps in \( (g_n) \). Since there is a lower u.b.s.l., if
there are any lamps lit in this infinite lamp stand, there is a minimum such lamp. Thus, there exists \([\gamma] \in \partial_\infty L^+_2\) with infinite lamp stand equal to this stabilization.

Take a subsequence \((g_{n_k})\) such that for every positive integer \(K\), for all \(k > K\) the lamps at positions at most \(K\) in the lamp stand for \(g_{n_k}\) have achieved their eventual status and \(H(g_{n_k}) > K\).

Let \(x \in L_2\). Set \(K\) to be sufficiently large so that \(K \geq \max \{m(\gamma), M(x), H(x)\}\), and for the finite values of \(m(\gamma)\) and \(m(\gamma, x)\), \(K \geq \max \{m(\gamma), m(\gamma, x)\}\) as well.

Let \(k > K\). Assume that \(h_{g_n}\) exists (and is therefore equal to \(h_{g_{n_k}}\)) and use Lemma 2.2 and Equation 1:

\[
\begin{align*}
 h_{g_{n_k}}(x) &= \lim_{k \to \infty} 2(\max \{M(g_{n_k}, x) + 1, H(g_{n_k}), H(x)\} \\
 &\quad - \min \{m(g_{n_k}, x), H(x)\} - (H(g_{n_k}) - H(x)) \\
 &\quad - [2(\max \{M(g_{n_k}) + 1, H(g_{n_k}), 0\} - \min \{m(\gamma), 0\} - H(g_{n_k})]
\end{align*}
\]

Notice that if \(\max \{M(g_{n_k}), M(g_{n_k}, x)\} > H(g_{n_k})\), then since \(H(g_{n_k}) > M(x)\), we have that \(M(g_{n_k}, x) = M(g_{n_k})\). Since \(H(g_{n_k}) \geq \max \{H(x), 0\}\), we have that \(\max \{M(g_{n_k}, x) + 1, H(g_{n_k}), H(x)\} = \max \{M(g_{n_k}) + 1, H(g_{n_k}), 0\}\). Therefore,

\[
\begin{align*}
 h_{g_{n_k}}(x) &= \lim_{k \to \infty} 2(\min \{m(\gamma), 0\} - \min \{m(g_{n_k}, x), H(x)\}) + H(x)
\end{align*}
\]

Now notice that if \(m(g_{n_k}) < 0\) or \(m(\gamma) < 0\), then \(m(\gamma) = m(g_{n_k})\). Similarly, if \(m(g_{n_k}, x) < H(x)\) or \(m(\gamma, x) < H(x)\), since \(H(x) < K\), then \(m(g_{n_k}, x) = m(\gamma, x)\). So by Equation 10 and the above, \(h_{g_n} = b^\gamma\).

**Lemma 5.10.** Suppose that a sequence \((g_n) \subset L_2\) has no lower u.b.s.l. and \(H(g_n) \to +\infty\). If \((g_n)\) is associated with some horofunction \(h_{g_n}\), then \(h_{g_n} = H\), the height function.

**Proof.** By Observation 5.3, \((g_n)\) has a subsequence \((g_{n_i})\) such that \((m(g_{n_i}))\) is decreasing with \(m(g_{n_i}) < -i\) for all \(i\). We still have \(H(g_{n_i}) \to +\infty\), so we can further take a subsequence \((g_{n_k})\) such that for all \(k\), \(m(g_{n_k}) < -k\) and \(H(g_{n_k}) > k\).

Let \(x \in L_2\), let \(K = \max \{M(x), |m(x)|, |H(x)|\}\), and consider \(k > K\). By Lemma 2.2 there exists \(B \in \mathbb{Z}\) such that

\[
d(g_{n_k}, x) = 2(B - m(g_{n_k})) - |H(g_{n_k}) - H(x)|
\]

and

\[
d(g_{n_k}, id) = 2(B - m(g_{n_k})) - |H(g_{n_k})|.
\]

Thus,

\[
h_{g_{n_k}}(x) = \lim_{k \to \infty} |H(g_{n_k})| - |H(g_{n_k}) - H(x)| = H(x)
\]

**Lemma 5.11.** Suppose that a sequence \((g_n) \subset L_2\) has an upper u.b.s.l. and \(H(g_n) \to -\infty\). If \((g_n)\) is associated with some horofunction \(h_{g_n}\), then \(h_{g_n}\) is equal to a Busemann function \(b^\gamma\) with \([\gamma] \in \partial_\infty L^+_2\).

**Proof.** As in the proof of Lemma 5.9 but the Busemann function will have the lamplighter at \(-\infty\) instead of \(+\infty\).

**Lemma 5.12.** Suppose that a sequence \((g_n) \subset L_2\) has no upper u.b.s.l. and \(H(g_n) \to -\infty\). If \((g_n)\) is associated with some horofunction \(h_{g_n}\), then \(h_{g_n} = -H\), the negation of the height function.
Proof. Similar to the proof of Lemma 5.10 \( \square \)

**Theorem 5.13.** Suppose that a sequence \( (g_n) \subset L_2 \) has \( (H(g_n)) \) converging to some value \( l \in \mathbb{Z} \cup \{ \pm \infty \} \). If \( (g_n) \) is associated with some horofunction \( h_{g_n} \), then:

1. If \( l \in \mathbb{Z} \) and \( (g_n) \) has both a lower and an upper u.b.s.l., then the sequence is eventually constant and \( h_{g_n} \) is in the image of \( L_2 \) in \( \overline{L_2} \).
2. If \( l \in \mathbb{Z} \) and \( (g_n) \) has either a lower u.b.s.l. or an upper u.b.s.l., but not both, then \( h_{g_n} \) is equal to one of the rib examples.
3. If \( l \in \mathbb{Z} \) and \( (g_n) \) has neither a lower nor an upper u.b.s.l., then \( h_{g_n} = s^l \).
4. If \( l = +\infty \) and \( (g_n) \) has a lower u.b.s.l., then \( h_{g_n} = b^\gamma \) for some \( [\gamma] \in \partial_L^{-\infty} L_2^{-\infty} \).
5. If \( l = +\infty \) and \( (g_n) \) has no lower u.b.s.l., then \( h_{g_n} = H \), the height function.
6. If \( l = -\infty \) and \( (g_n) \) has an upper u.b.s.l., then \( h_{g_n} = b^\gamma \) for some \( [\gamma] \in \partial_L^{-\infty} L_2^{-\infty} \).
7. If \( l = -\infty \) and \( (g_n) \) has no upper u.b.s.l., then \( h_{g_n} = -H \), the negation of the height function.

Proof. If \( l \in \mathbb{Z} \), then apply one of Lemmas 5.6, 5.7, or 5.8 as appropriate for the existence of lower or upper u.b.s.l. If \( l = +\infty \), then apply either Lemma 5.9 or 5.10 depending on the existence of a lower u.b.s.l. If \( l = -\infty \), then apply either Lemma 5.11 or 5.12 depending on the existence of an upper u.b.s.l. \( \square \)

**Lemma 5.14.** Suppose for a sequence \( (g_n) \subset L_2 \), \( (H(g_n)) \) does not converge in \( \mathbb{Z} \cup \{ \pm \infty \} \). Then \( (g_n) \) is not associated with a horofunction.

Proof. By our hypotheses, \( (g_n) \) has subsequences \( (g_{n_i}) \) and \( (g_{n_j}) \) such that \( (H(g_{n_i})) \) and \( (H(g_{n_j})) \) converge in \( \mathbb{Z} \cup \{ \pm \infty \} \), but to distinct values. By Theorem 5.13 and Observation 4.1, since these limits are distinct, \( h_{g_{n_i}} \neq h_{g_{n_j}} \). Thus \( h_{g_n} \) cannot exist. \( \square \)

**Corollary 5.15.** Let \( h \in \overline{L_2} \), and choose a sequence \( (g_n) \subset L_2 \) such that \( h = h_{g_n} \). Then \( (H(g_n)) \) converges to some value \( l \in \mathbb{Z} \cup \{ \pm \infty \} \), and \( h_{g_n} \) can be categorized as in Theorem 5.13.

6. **Topology of the horofunction boundary**

The topology of \( \partial_h L_2 \) is the topology of uniform convergence on compact sets. The standard basis is the collection of sets of the form

\[
B_K(h, \epsilon) = \{ h' \in \partial_h L_2 \mid |h(x) - h'(x)| < \epsilon \text{ for all } x \in K \}
\]

where \( K \subset L_2 \) is compact and \( \epsilon > 0 \) \([10]\). By restricting to \( 0 < \epsilon < 1 \), we obtain an equivalent basis. Since the minimum distance between distinct points in \( L_2 \) is 1, we may use the following sets as a basis:

\[
B_K(h) = \{ h' \in \partial_h L_2 \mid h(x) = h'(x) \text{ for all } x \in K \}
\]

where \( K \subset L_2 \) is finite. Notice that pointwise convergence implies convergence in our topology since compact sets of \( L_2 \) are finite.

With the explicit descriptions of the horofunctions found in Section 4, we can establish the accumulation points of \( \partial_h L_2 \). We begin by recalling that since \( s^l(g) = |l| - |l - H(g)| \), we have

**Observation 6.1.** \( s^l \to \pm H \) as \( l \to \pm \infty \).
Observation 6.2. The injective map that takes elements of $\partial_\infty L_2^+$ to their Busemann functions in $\partial_\infty L_2$ is continuous, and the same is true of $\partial_\infty L_2^-$. 

Contrast this result with Observation 3.4 which states that the injection of the union of these two sets into the horofunction boundary is not continuous.

Proof. Let $[\gamma] \in \partial_\infty L_2^+$, and consider $B_K(b^\gamma)$ for some finite $K \subset L_2$. Let $M = \max\{M(g), H(g) \mid g \in K\}$, and let $k \in \mathbb{Z}$ such that $k > M + 2|m(\gamma)|$ if $m(\gamma) < 0$ or $k > M$ otherwise. Consider the set

$$B_{[0,k]}([\gamma], \epsilon) = \{\gamma' \in \partial_\infty L_2^+ \mid \sup\{d(\gamma(x), \gamma'(x)) \mid x \in [0, k]\} < \epsilon\}$$

for $0 < \epsilon < 1$. In [6] Observation 4.1, the authors observe that $B_{[0,k]}([\gamma], \epsilon)$ is an open set in $\partial_\infty L_2^+$. Notice that if $\gamma' \in B_{[0,k]}([\gamma], \epsilon)$, then the lamp stands of $\gamma$ and $\gamma'$ agree on all lamps at positions $M$ or below. Thus, by Equation 10 $b^\gamma(g) = b^{\gamma'}(g)$ for all $g \in K$. Therefore, $b^\gamma \in B_K(b^{\gamma'})$ for all $\gamma' \in B_{[0,k]}([\gamma], \epsilon)$, and so our injection is continuous [10 Theorem 18.1].

The proof for the injection of $\partial_\infty L_2^-$ is similar. □

The topology of each of these sets is a punctured Cantor set, but in $\partial_\infty L_2$ these punctures are “filled” by the height function and its negative, as we now show.

Observation 6.3. If $([\gamma_n]) \subset \partial_\infty L_2^+$ with $m(\gamma_n) \to -\infty$, then $b^{\gamma_n} \to H$. Similarly, if $([\gamma_n]) \subset \partial_\infty L_2^+$ with $M(\gamma_n) \to +\infty$, then $b^{\gamma_n} \to -H$.

Proof. Let $([\gamma_n]) \subset \partial_\infty L_2^+$ with $\lim m(\gamma_n) = -\infty$. By Equation 10

$$b^{\gamma_n}(g) = 2 \min\{m(\gamma_n), 0\} - \min\{m(\gamma_n, g), H(g)\} + H(g)$$

Fix $g$ and take $n$ large enough so that $m(\gamma_n) < \min\{0, m(g), H(g)\}$, then

$$b^{\gamma_n}(g) = 2 \{m(\gamma_n) - m(\gamma_n)\} + H(g)$$

Thus, $b^{\gamma_n} \to H$. The other proof is similar. □

For a given $l \in \mathbb{Z}$, the following observation remarks that the spine is an accumulation point of the positive and negative (general) rib functions. The proofs are calculations similar to those in Observation 6.3.

Observation 6.4. Let $(f_n^l) \subset L_2$ be a sequence satisfying $M(f_n^l) < H(f_n^l) = l$ and $m(f_n^l) \to -\infty$ as $n \to \infty$. Then $r^{+,f_n^l} \to s^l$.

Similarly, if $(f_n^l) \subset L_2$ is a sequence satisfying $m(f_n^l) \geq H(f_n^l) = l$ and $M(f_n^l) \to \infty$ as $n \to \infty$, then $r^{-,f_n^l} \to s^l$.

Finally, the ribs accumulate to Busemann functions:

Observation 6.5. For a geodesic ray $r$, with $r(0) = \text{id}$, set $f_n = r(n)$. If $r \in \partial_\infty L_2^+$, then for large enough $n$, each $f_n$ defines $r^{+,f_n}$ and $r^{+,f_n} \to b^+-r$. If $r \in \partial_\infty L_2^-$, then for large enough $n$ each $f_n$ defines $r^{-,f_n}$ and $r^{+,f_n} \to b^+-r$.

Proof. We consider the $r \in \partial_\infty L_2^+$ case. For large enough $n$, each $f_n$ satisfies the requirements for defining $r^{+,f_n}$. Let $g \in L_2$ be given, and consider Equations 8 and 10. Again for large enough $n$, $m(f_n) = m(r)$ and $m(f_n, g) = m(r, g)$. Thus

$$r^{+,f_n} - b^+r = s^{H(f_n)}(g) - H(g) \to 0 \text{ as } n \to \infty$$

With Observations 6.1, 6.3, 6.4 and 6.5, we have the picture of the horofunction boundary illustrated in Figure 5.
7. Action of $L_2$ on the horofunction boundary

An isometric action of a group $G$ on a metric space $(X,d)$ with base point $b$ can be extended to the horofunction boundary $\partial_h X$ in the following way: For $g \in G$ and $(y_n) \subset X$ giving rise to a horofunction, we have that

$$h_g \cdot y_n(x) = h_{g \cdot y_n}(x) = \lim_{n \to \infty} d(g \cdot y_n, x) - d(g \cdot y_n, b)$$

In our setting, the action of $L_2$ on itself is by left multiplication.

**Observation 7.1.** Let $g \in L_2$, $h \in \mathcal{T}_{L_2}$, and choose $(g_n) \subset L_2$ such that $h = h_{g_n}$.

- $(H(g_n))$ converges to an integer if and only if $(H(g \cdot g_n))$ converges to an integer.
- $H(g_n) \to +\infty$ if and only if $H(g \cdot g_n) \to +\infty$.
- $H(g_n) \to -\infty$ if and only if $H(g_n) \to -\infty$.
- $(g_n)$ has a lower u.b.s.l. iff $(g \cdot g_n)$ has a lower u.b.s.l.
- $(g_n)$ has an upper u.b.s.l. iff $(g \cdot g_n)$ has an upper u.b.s.l.

**Proof.** These statements all follow from the fact that the lamp stand for $g$ has only finitely many lit lamps and the lamplighter at a finite position.

Combining Observation 7.1 and Corollary 5.15, we see that each of the seven categories of horofunctions in $L_2$ described in Theorem 5.13 is left invariant by the action of $L_2$. We now consider the action of $L_2$ on each of these sets.

Let $g \in L_2$. The action of $g$ on $\partial_\infty L_2$ is described in [6, §3.4 and §4.6]. If $H(g) \neq 0$, then the action of $g$ on $\partial_\infty L_2$ has two fixed points, which are given the notation $g^\infty$ and $g^{-\infty}$ in [6]. If $H(g) > 0$, then $g^\infty \in \partial_\infty L_2^+$ and $g^{-\infty} \in \partial_\infty L_2^-$. Otherwise, the reverse is true.

For a spinal horofunction $s^l \in \partial_h L_2$, $l \in \mathbb{Z}$, the action of $g$ on $s^l$ is given by $g \cdot s^l = s^{H(g) + l}$. We see similar behavior on the ribs of $\partial_h L_2$, but there is additional structure in this case.
Most interestingly, though, we see the following:

**Corollary 7.2.** The action of $L_2$ on $\partial_h L_2$ has two global fixed points: the height function $H$ and its negation.

**Observation 7.3.** For $g \in L_2$ with $H(g) \neq 0$, the action of $g$ on $\partial_h L_2$ has four fixed points: $H, -H, b^g\infty, b^g-\infty$.

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