Extension of a key identity

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Abstract

In this article, we extend a certain key identity proved by J. Jorgenson and J. Kramer in [6] to noncompact hyperbolic Riemann orbisurfaces of finite volume. This identity relates the two natural metrics, namely the hyperbolic metric and the canonical metric defined on a Riemann orbisurface.

Introduction

Notation

Let $X$ be a noncompact hyperbolic Riemann orbisurface of finite volume $\text{vol}_{\text{hyp}}(X)$ with genus $g \geq 1$, and can be realized as the quotient space $\Gamma \backslash \mathbb{H}$, where $\Gamma \subset \text{PSL}_2(\mathbb{R})$ is a Fuchsian subgroup of the first kind acting on the hyperbolic upper half-plane $\mathbb{H}$, via fractional linear transformations.

Let $\mathcal{P}$ denote the set of cusps of $\Gamma$, and put $\overline{X} = X \cup \mathcal{P}$. Then, $\overline{X}$ admits the structure of a Riemann surface.

Let $\mu_{\text{hyp}}$ denote the $(1,1)$-form associated to hyperbolic metric, which is the natural metric on $X$, and of constant negative curvature minus one.

The Riemann surface $\overline{X}$ is embedded in its Jacobian variety $\text{Jac}(\overline{X})$ via the Abel-Jacobi map. Then, the pull back of the flat Euclidean metric by the Abel-Jacobi map is called the canonical metric, and the $(1,1)$-form associated to it is denoted by $\hat{\mu}_{\text{can}}$. We denote its restriction to $X$ by $\mu_{\text{can}}$.

Let $\Delta_{\text{hyp}}$ denote the hyperbolic Laplacian acting on smooth functions on $X$. Let $K_{\text{hyp}}(t; z, w)$ denote the hyperbolic heat kernel defined on $\mathbb{R}_{>0} \times X \times X$, which is the unique solution of the heat equation

$$
\left( \Delta_{\text{hyp},z} + \frac{\partial}{\partial t} \right) K_{\text{hyp}}(t; z, w) = 0,
$$

and the normalization condition

$$
\lim_{t \to 0} \int_X K_{\text{hyp}}(t; z, w)f(z)\mu_{\text{hyp}}(z) = f(w),
$$

for any fixed $w \in X$ and any smooth function $f$ on $X$. When $z = w$, for brevity of notation, we denote the hyperbolic heat kernel by $K_{\text{hyp}}(t; z)$.

Let $C_{\ell,\ell}(\overline{X})$ denote the space of singular functions, which are log-singular at finitely many points of $X$, and are log-log-singular at the cusps. With notation as above, we now state the main result.

Main result

With notation as above, for any $f \in C_{\ell,\ell}(\overline{X})$, we have the equality of integrals

$$
g \int_X f(z)\mu_{\text{can}}(z) = \\
\left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) \int_X f(z)\mu_{\text{hyp}}(z) + \frac{1}{2} \int_X f(z) \left( \int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt \right) \mu_{\text{hyp}}(z).
$$
The above relation, which relates the two natural metrics defined on a Riemann orbisurface has been proved for compact hyperbolic Riemann surfaces, as a relation of differential forms by J. Jorgenson and J. Kramer in [6]. The same authors have also extended the key identity to noncompact hyperbolic Riemann surfaces of finite volume in [5]. In this paper, the authors use different methods from [6], and study the behavior of the key identity over a family of degenerating compact hyperbolic Riemann surfaces.

Our main theorem can be seen as an extension of their result to elliptic fixed points and cusps at the level of currents acting on the space of singular function $C_{\mathcal{L},\mathcal{H}}(\overline{X})$. Our methods are different from the ones employed in [5], and are organized around the original line of proof in [6].

**Arithmetic significance** The key identity has been the most significant technical result of [6], which transforms a problem in Arakelov theory into that of hyperbolic geometry. The key identity has enabled J. Jorgenson and J. Kramer to derive optimal bounds for the canonical Green’s function defined on a compact hyperbolic Riemann surface $X$ in terms of invariants coming from the hyperbolic geometry of $X$. These bounds were essential for B. Edixhoven’s algorithm in [3] for computing certain Galois representations associated to a fixed modular form of arbitrary weight. Furthermore, using the key identity and the Polyakov formula, J. Jorgenson and J. Kramer have obtained optimal bounds for the Faltings delta function in [7]. The key identity is again the most important technical tool.

Using the key identity one can relate the holomorphic world of cusp forms with the $C^\infty$ world of Möss forms, via the spectral expansion of the hyperbolic heat kernel $K_{hyp}(t; z)$ in terms of Möss forms. In fact, J. Jorgenson and J. Kramer have derived a Rankin-Selberg $L$-function relation relating the Fourier coefficients of cusp forms with those of Möss forms in [5].

The extended version of the key identity enables us to extend the work of J. Jorgenson and J. Kramer to noncompact hyperbolic Riemann orbisurfaces of finite volume. In an upcoming article [2], using the key identity, we extend the bounds derived in [6].

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## 1 Background material

Let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H}$. Let $X$ be the quotient space $\Gamma \backslash \mathbb{H}$, and let $g$ denote the genus of $X$. The quotient space $X$ admits the structure of a Riemann orbisurface.

Let $\mathcal{E}, \mathcal{P}$ be the finite set of elliptic fixed points and cusps of $X$, respectively; put $S = \mathcal{E} \cup \mathcal{P}$. For $\epsilon \in \mathcal{E}$, let $m_\epsilon$ denote the order of $\epsilon$; for $p \in \mathcal{P}$, put $m_p = \infty$; for $z \in X \setminus \mathcal{E}$, put $m_z = 1$. Let $\overline{X}$ denote $X \cup S$.

Locally, away from the elliptic fixed points and cusps, we identity $\overline{X}$ with its universal cover $\mathbb{H}$, and hence, denote the points on $\overline{X} \setminus S$ by the same letter as the points on $\mathbb{H}$.

**Structure of $\overline{X}$ as a Riemann surface** The quotient space $\overline{X}$ admits the structure of a compact Riemann surface. We refer the reader to section 1.8 in [10], for the details regarding the structure of $\overline{X}$ as a compact Riemann surface. For the convenience of the reader, we recall the coordinate functions for the neighborhoods of elliptic fixed points and cusps.

Let $w \in U_\epsilon(\epsilon)$ denote a coordinate disk of radius $r$ around an elliptic fixed point $\epsilon \in \mathcal{E}$. Then, the coordinate function $\vartheta_\epsilon(w)$ for the coordinate disk $U_\epsilon(\epsilon)$ is given by

$$\vartheta_\epsilon(w) = \left(\frac{w - \epsilon}{w - \epsilon^*}\right)^{m_\epsilon}.$$
Similarly, let \( p \in \mathcal{P} \) be a cusp and let \( w \in U_r(p) \). Then \( \vartheta_p(w) \) is given by
\[
\vartheta_p(w) = e^{2\pi i \sigma_p^{-1}w},
\]
where \( \sigma_p \) is a scaling matrix of the cusp \( p \) satisfying the following relations
\[
\sigma_p \infty = p \quad \text{and} \quad \sigma_p^{-1} \Gamma_p \sigma_p = \langle \gamma \rangle, \quad \text{where} \quad \gamma \infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Gamma_p = \langle \gamma_p \rangle
\]
denotes the stabilizer of the cusp \( p \) with generator \( \gamma_p \).

**Hyperbolic metric** We denote the \((1,1)\)-form corresponding to the hyperbolic metric of \( X \), which is compatible with the complex structure on \( X \) and has constant negative curvature equal to minus one, by \( \mu_{\text{hyp}}(z) \). Locally, for \( z \in X \setminus \mathcal{E} \), it is given by
\[
\mu_{\text{hyp}}(z) = \frac{i}{2} \cdot \frac{dz \wedge d\bar{z}}{\text{Im}(z)^2}.
\]
Let \( \text{vol}_{\text{hyp}}(X) \) be the volume of \( X \) with respect to the hyperbolic metric \( \mu_{\text{hyp}} \). It is given by the formula
\[
\text{vol}_{\text{hyp}}(X) = 2\pi \left( 2g - 2 + |\mathcal{P}| + \sum_{\epsilon \in \mathcal{E}} \left( 1 - \frac{1}{m_{\epsilon}} \right) \right).
\]
The hyperbolic metric \( \mu_{\text{hyp}}(z) \) is singular at the elliptic fixed points and at the cusps, and defines a singular and integrable \((1,1)\)-form on \( X \), which we denote by \( \tilde{\mu}_{\text{hyp}}(z) \). The rescaled hyperbolic metric
\[
\mu_{\text{shyp}}(z) = \frac{\mu_{\text{hyp}}(z)}{\text{vol}_{\text{hyp}}(X)}
\]
measures the volume of \( X \) to be one, and we denote the \((1,1)\)-form determined by \( \mu_{\text{shyp}}(z) \) on \( X \) by \( \tilde{\mu}_{\text{shyp}}(z) \). Furthermore, let us denote the \((1,1)\)-currents determined by \( \tilde{\mu}_{\text{hyp}}(z) \) and \( \tilde{\mu}_{\text{shyp}}(z) \) acting on smooth functions defined on \( X \) by \( [\tilde{\mu}_{\text{hyp}}(z)] \) and \( [\tilde{\mu}_{\text{shyp}}(z)] \), respectively.

Locally, for \( z \in X \), the hyperbolic Laplacian \( \Delta_{\text{hyp}} \) on \( X \) is given by
\[
\Delta_{\text{hyp}} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = -4y^2 \left( \frac{\partial^2}{\partial z \partial \bar{z}} \right).
\]
Recall that \( d = (\partial + \overline{\partial}) \), \( d^c = \frac{1}{4\pi i} (\partial - \overline{\partial}) \), and \( dd^c = -\frac{\partial \overline{\partial}}{2\pi i} \).

**Canonical metric** Let \( S_2(\Gamma) \) denote the \( \mathbb{C} \)-vector space of cusp forms of weight 2 with respect to \( \Gamma \) equipped with the Petersson inner product. Let \( \{ f_1, \ldots, f_g \} \) denote an orthonormal basis of \( S_2(\Gamma) \) with respect to the Petersson inner product. Then, the \((1,1)\)-form \( \mu_{\text{can}}(z) \) corresponding to the canonical metric of \( X \) is given by
\[
\mu_{\text{can}}(z) = \frac{i}{2g} \sum_{j=1}^{g} |f_j(z)|^2 dz \wedge d\bar{z}.
\]
The canonical metric \( \mu_{\text{can}}(z) \) remains smooth at the elliptic fixed points and at the cusps, and measures the volume of \( X \) to be one. We denote the smooth \((1,1)\)-form defined by \( \mu_{\text{can}}(z) \) on \( X \) by \( \tilde{\mu}_{\text{can}}(z) \), and the \((1,1)\)-current determined by \( \tilde{\mu}_{\text{can}}(z) \) acting on smooth functions defined on \( X \) by \( [\tilde{\mu}_{\text{can}}(z)] \).

**Canonical Green’s function** For \( z, w \in X \), the canonical Green’s function \( \tilde{\gamma}_{\text{can}}(z, w) \) is defined as the solution of the differential equation
\[
d_{\bar{z}}\bar{d}_{\bar{z}}\tilde{\gamma}_{\text{can}}(z, w) + \delta_w(z) = \tilde{\mu}_{\text{can}}(z), \quad (1)
\]
with the normalization condition
\[ \int_X \hat{g}_{\text{can}}(z, w) \hat{\mu}_{\text{can}}(z) = 0. \]

From equation (1), it follows that \( \hat{g}_{\text{can}}(z, w) \) admits a log-singularity at \( z = w \), i.e., for \( z, w \in \overline{X} \), it satisfies
\[ \lim_{w \to z} (\hat{g}_{\text{can}}(z, w) + \log |\theta_z(w)|^2) = O_z(1). \]

For a fixed \( w \in \overline{X} \), the canonical Green’s function \( \hat{g}_{\text{can}}(z, w) \) determines a current \( [\hat{g}_{\text{can}}(\cdot, w)] \) of type \((0,0)\) acting on smooth \((1,1)\)-forms defined on \( \overline{X} \). Furthermore, for a fixed \( w \in \overline{X} \), the current \( [\hat{g}_{\text{can}}(\cdot, w)] \) is a Green’s current satisfying the differential equation
\[ dz^p d\bar{z}^q \hat{g}_{\text{can}}(z, w)(f) + f(w) = [\hat{\mu}_{\text{can}}(z)](f), \]

where \( f \) is a smooth function defined on \( \overline{X} \). We refer the reader to [9] for the proof of the above equation. Let us denote the restriction of \( \hat{g}_{\text{can}}(z, w) \) to \( X \times X \) by \( g_{\text{can}}(z, w) \).

**Residual canonical metric on** \( \Omega^1_X \). Let \( \Omega^1_{\overline{X}} \) denote the cotangent bundle of holomorphic differential forms on \( \overline{X} \). For \( z \in \overline{X} \), we define
\[ \|d\theta_z\|^2_{\text{res,can}}(z) = \exp \left( \lim_{w \to z} (\hat{g}_{\text{can}}(z, w) + \log |\theta_z(w)|^2) \right). \]

From equation (2), it follows that the residual canonical metric is well defined and remains smooth on \( \overline{X} \). Furthermore, for \( z \in \overline{X} \), the first Chern form \( c_1(\Omega^1_{\overline{X}} \cdot \|_{\text{res,can}}) \) is given by the formula
\[ c_1(\Omega^1_{\overline{X}} \cdot \|_{\text{res,can}}) = -dz^p d\bar{z}^q \log \|d\theta_z\|^2_{\text{res,can}}(z) = (2g - 2) \hat{\mu}_{\text{can}}(z). \]

We refer the reader to [1] for the details of the proof of the above formula.

**Parabolic Eisenstein Series** For \( z \in X \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the parabolic Eisenstein series \( E_{\text{par}, p}(z, s) \) corresponding to a cusp \( p \in \mathcal{P} \) is defined by the series
\[ E_{\text{par}, p}(z, s) = \sum_{\eta \in \Gamma_p \backslash \Gamma} \text{Im}(\sigma_p^{-1} \eta z)^s. \]

The series converges absolutely and uniformly for \( \text{Re}(s) > 1 \). It admits a meromorphic continuation to all \( s \in \mathbb{C} \) with a simple pole at \( s = 1 \), and the Laurent expansion at \( s = 1 \) is of the form
\[ E_{\text{par}, p}(z, s) = \frac{1}{\text{vol}_{\text{hyp}}(X)} \frac{1}{s - 1} + \kappa_p(z) + O_s(s - 1), \]

where \( \kappa_p(z) \) the constant term of \( E_{\text{par}, p}(z, s) \) at \( s = 1 \) is called Kronecker’s limit function (see Chapter 6 of [4]).

**Heat Kernels** For \( t \in \mathbb{R}_{>0} \) and \( z, w \in H \), the hyperbolic heat kernel \( K_H(t; z, w) \) on \( \mathbb{R}_{>0} \times H \times \mathbb{H} \) is given by the formula
\[ K_H(t; z, w) = \sqrt{2e^{-t/4}} \int_{d_H(z, w)}^\infty \frac{r e^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(d_H(z, w))}} dr, \]

where \( d_H(z, w) \) is the hyperbolic distance between \( z \) and \( w \). For \( t \in \mathbb{R}_{>0} \) and \( z, w \in X \), the hyperbolic heat kernel \( K_{\text{hyp}}(t; z, w) \) on \( \mathbb{R}_{>0} \times X \times X \) is defined as
\[ K_{\text{hyp}}(t; z, w) = \sum_{\gamma \in \Gamma} K_H(t; z, \gamma w). \]
For \( z, w \in X \), the hyperbolic heat kernel \( K_{\text{hyp}}(t; z, w) \) satisfies the differential equation
\[
\left( \Delta_{\text{hyp}, z} + \frac{\partial}{\partial t} \right) K_{\text{hyp}}(t; z, w) = 0,
\]
\( (6) \)

Furthermore for a fixed \( w \in X \) and any smooth function \( f \) on \( X \), the hyperbolic heat kernel \( K_{\text{hyp}}(t; z, w) \) satisfies the equation
\[
\lim_{t \to 0} \int_X K_{\text{hyp}}(t; z, w)f(z) \mu_{\text{hyp}}(z) = f(w).
\]
\( (7) \)

To simplify notation, we write \( K_{\text{hyp}}(t; z) \) instead of \( K_{\text{hyp}}(t; z, z) \), when \( z = w \).

**Automorphic Green’s function** For \( z, w \in \mathbb{H} \) with \( z \neq w \), and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \), the free-space Green’s function \( g_{\text{f}, s}(z, w) \) is defined as
\[
g_{\text{f}, s}(z, w) = g_{\text{f}, s}(u(z, w)) = \frac{\Gamma(s)^2}{\Gamma(2s)} u^{-s} F(s, s; 2s, -1/u),
\]
where \( u = u(z, w) = |z - w|^2/(4 \text{Im}(z) \text{Im}(w)) \) and \( F(s, s; 2s, -1/u) \) is the hypergeometric function.

There is a sign error in the formula defining the free-space Green’s function given by equation (1.46) in [4], i.e., the last argument \(-1/u\) in the hypergeometric function has been incorrectly stated as \(1/u\), which we have corrected in our definition. We have also normalized the free-space Green’s function defined in [4] by multiplying it by \(4\pi\).

For \( z, w \in X \) with \( z \neq w \), and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the automorphic Green’s function \( g_{\text{hyp}, s}(z, w) \) is defined as
\[
g_{\text{hyp}, s}(z, w) = \sum_{\gamma \in \Gamma} g_{\text{f}, s}(z, \gamma w).
\]

The series converges absolutely uniformly for \( z \neq w \) and \( \text{Re}(s) > 1 \) (see Chapter 5 in [4]).

For \( z, w \in X \) with \( z \neq w \), and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the automorphic Green’s function satisfies the following properties (see Chapters 5 and 6 in [4]):

1. For \( \text{Re}(s - 1) > 1 \), we have
\[
g_{\text{hyp}, s}(z, w) = 4\pi \int_0^\infty K_{\text{hyp}}(t; z, w) e^{-t(s-1)} dt.
\]
2. It admits a logarithmic singularity along the diagonal, i.e.,
\[
\lim_{w \to z} \left( g_{\text{hyp}, s}(z, w) + \log |z(w)|^2 \right) = O_{s,z}(1).
\]
3. The automorphic Green’s function \( g_{\text{hyp}, s}(z, w) \) admits a meromorphic continuation to all \( s \in \mathbb{C} \) with a simple pole at \( s = 1 \) with residue \( 4\pi/\text{vol}_{\text{hyp}}(X) \), and the Laurent expansion at \( s = 1 \) is of the form
\[
g_{\text{hyp}, s}(z, w) = \frac{4\pi}{s(s-1) \text{vol}_{\text{hyp}}(X)} + g_{\text{hyp}, s}^{(1)}(z, w) + O_{z,w}(s-1),
\]
where \( g_{\text{hyp}, s}^{(1)}(z, w) \) is the constant term of \( g_{\text{hyp}, s}(z, w) \) at \( s = 1 \).
4. Let \( p, q \in P \) be two cusps. Put
\[
C_{p,q} = \min \left\{ c > 0 \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_p^{-1} \Gamma \sigma_q \right. \right\},
\]
and \( C_{p,p} = C_p \). Then, for \( z, w \in X \) with \( \text{Im}(w) > \text{Im}(z) \) and \( \text{Im}(w) \text{Im}(z) > C_{p,q}^{-2} \), and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the automorphic Green’s function admits the Fourier expansion
\[
g_{\text{hyp}, s}(\sigma_p z, \sigma_q w) = \frac{4\pi \text{Im}(w)^{1-s}}{2s - 1} \mathcal{E}_{\text{par}, q}(\sigma_p z, s) - \delta_{p,q} \log \left| 1 - e^{2\pi i (w-z)} \right|^2 + O(e^{-2\pi \text{Im}(w) - \text{Im}(z)})).
\]
\( (8) \)

This equation has been proved as Lemma 5.4 in [4], and one of the terms was wrongly estimated in the proof of the lemma. We have corrected this error, and stated the corrected equation.
Hyperbolic Green’s function  For $z, w \in X$ and $z \neq w$, the hyperbolic Green’s function is defined as

$$g_{\text{hyp}}(z, w) = 4\pi \int_0^\infty \left( K_{\text{hyp}}(t; z, w) - \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) dt.$$  

For $z, w \in X$ with $z \neq w$, the hyperbolic Green’s function satisfies the following properties:

1. For $z, w \in X$, we have
   $$\lim_{w \to z} \left( g_{\text{hyp}}(z, w) + \log |\vartheta_z(w)|^2 \right) = O_{\text{Z}}(1).$$  

2. For $z, w \in X \setminus \mathcal{E}$, the hyperbolic Green’s function satisfies the differential equation
   $$d_z d_z^c g_{\text{hyp}}(z, w) + \delta_w(z) = \mu_{\text{shyp}}(z),$$
   with the normalization condition
   $$\int_X g_{\text{hyp}}(z, w) \mu_{\text{hyp}}(z) = 0.$$

3. For $z, w \in X$ and $z \neq w$, we have
   $$g_{\text{hyp}}(z, w) = g^{(1)}_{\text{hyp}}(z, w) = \lim_{s \to 1} \left( g_{\text{hyp},s}(z, w) - \frac{4\pi}{s(s-1)\text{vol}_{\text{hyp}}(X)} \right).$$

The above properties follow from the properties of the heat kernel $K_{\text{hyp}}(t; z, w)$ (equations (6) and (7)) or from that of the automorphic Green’s function $g_{\text{hyp},s}(z, w)$.

Residual hyperbolic metric on $\Omega^1_X$  For $z \in X$, we define

$$\|d\vartheta_z\|^2_{\text{res},\text{hyp}}(z) = \exp \left( \lim_{w \to z} \left( g_{\text{hyp}}(z, w) + \log |\vartheta_z(w)|^2 \right) \right).$$

From equation (9), it follows that the residual hyperbolic metric is well defined on $X$. Furthermore, from Proposition 3.3 in [6], for $z \in X \setminus \mathcal{E}$, we have

$$c_1(\Omega^1_X, \| \cdot \|_{\text{res},\text{hyp}}) = -d_z d_z^c \|d\vartheta_z\|^2_{\text{res},\text{hyp}}(z) = \frac{1}{2\pi} \mu_{\text{hyp}}(z) + \left( \int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt \right) \mu_{\text{hyp}}(z).$$

Convergence results  From Lemmas 5.2 and 6.3, and Proposition 7.3 in [8], the function

$$4\pi \int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt$$

is well defined on $X$ and remains bounded at the elliptic fixed points and at the cusps. Hence, it defines a smooth function on $X$, which we denote symbolically by

$$\int_0^\infty \Delta_{\text{hyp}} \hat{K}_{\text{hyp}}(t; z) dt.$$  

Key identity  For $z \in X \setminus \mathcal{E}$, we have the relation of differential forms

$$g \mu_{\text{can}}(z) = \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) \mu_{\text{hyp}}(z) + \frac{1}{2} \left( \int_0^\infty \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt \right) \mu_{\text{hyp}}(z).$$

This relation has been established as Theorem 3.4 in [6], when $X$ is compact. The proof given in [6] applies to our case where $X$ does admit elliptic fixed points and cusps, as long as $z \in X \setminus \mathcal{E}$.  

The space $C_{L, \ell}(X)$ Let $C_{L, \ell}(X)$ denote the set of complex-valued functions $f : X \to \mathbb{P}^1(\mathbb{C})$, which admit the following type of singularities at finitely many points $\text{Sing}(f) \subseteq X$, and are smooth away from $\text{Sing}(f)$:

1. If $s \in \text{Sing}(f) \setminus \mathcal{P}$, then as $z$ approaches $s$, the function $f$ satisfies
   \[ f(z) = c_{f,s} \log |\vartheta_s(z)| + O(z), \]
   for some $c_{f,s} \in \mathbb{C}$.

2. For $p \in \text{Sing}(f) \cap \mathcal{P}$, as $z$ approaches $p$, the function $f$ satisfies
   \[ f(z) = c_{f,p} \log (-\log |\vartheta_p(z)|) + O(z), \]
   for some $c_{f,p} \in \mathbb{C}$.

2 Extension of key identity

In this section, we extend the key identity, i.e., equation (14) to elliptic fixed points and cusps at the level of currents acting on the space of singular functions $C_{L, \ell}(X)$. In subsection 2.1, we investigate the behavior of the hyperbolic Green’s function at the cusps, and show that it defines a current acting on the space of singular functions $C_{L, \ell}(X)$. In subsection 2.2, we prove an auxiliary identity which is extending the key identity (14) to elliptic fixed points and cusps. In subsection 2.3 using the results from the previous two subsections, we extend the key identity.

2.1 Hyperbolic Green’s function as a Green’s current

Although it is obvious from the differential equation (10) that $g_{\text{hyp}}(z, w)$ is log log-singular at the cusps, the exact asymptotics derived in the following proposition come very useful in the upcoming articles (especially in [2]).

**Proposition 2.1.** With notation as in Section 1, for a fixed $w \in X$, and for $z \in X$ with $\text{Im}(\sigma_p^{-1}z) > \text{Im}(\sigma_p^{-1}w)$ and $\text{Im}(\sigma_p^{-1}z) \text{Im}(\sigma_p^{-1}w) > C_p^{-2}$, we have

\[
\begin{align*}
g_{\text{hyp}}(z, w) &= 4\pi \kappa_p(w) - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} \frac{\text{Im}(\sigma_p^{-1}z)}{\text{vol}_{\text{hyp}}(X)} - \frac{4\pi \log(\text{Im}(\sigma_p^{-1}z))}{\text{vol}_{\text{hyp}}(X)} \log[1 - e^{2\pi i(\Im(\sigma_p^{-1}z) - \Im(\sigma_p^{-1}w))}] + O(e^{-2\pi i(\Im(\sigma_p^{-1}z) - \Im(\sigma_p^{-1}w)))}. \\
\end{align*}
\]

**Proof.** As the limit in (12) converges uniformly, combining it with equation (8), for a fixed $w \in X$, for each $z \in X$ with $\text{Im}(\sigma_p^{-1}z) > \text{Im}(\sigma_p^{-1}w)$ and $\text{Im}(\sigma_p^{-1}z) \text{Im}(\sigma_p^{-1}w) > C_p^{-2}$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, we have

\[
\begin{align*}
g_{\text{hyp}}(z, w) &= 4\pi \lim_{s \to 1} \left( \frac{\text{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1} \mathcal{E}_{\text{hyp},p}(w, s) - \frac{1}{(s-1)\text{vol}_{\text{hyp}}(X)} \right) + \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} - \\
&\log[1 - e^{2\pi i(\Im(\sigma_p^{-1}z) - \Im(\sigma_p^{-1}w))}]^2 + O(e^{-2\pi i(\Im(\sigma_p^{-1}z) - \Im(\sigma_p^{-1}w)))}. \\
\end{align*}
\]

To evaluate the above limit, we compute the Laurent expansions of $\mathcal{E}_{\text{hyp},p}(w, s)$, $\text{Im}(\sigma_p^{-1}z)^{1-s}$, and $(2s-1)^{-1}$ at $s = 1$. The Laurent expansions of $\text{Im}(\sigma_p^{-1}z)^{1-s}$ and $(2s-1)^{-1}$ at $s = 1$ are easy to compute, and are of the form

\[
\text{Im}(\sigma_p^{-1}z)^{1-s} = 1 - (s-1)\log(\text{Im}(\sigma_p^{-1}z)) + O((s-1)^2), \quad \frac{1}{2s-1} = 1 - 2(s-1) + O((s-1)^2).
\]

Combining the above two equations with equation (5), we arrive at

\[
4\pi \lim_{s \to 1} \left( \frac{\text{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1} \mathcal{E}_{\text{hyp},p}(w, s) - \frac{1}{(s-1)\text{vol}_{\text{hyp}}(X)} \right) = \\
4\pi \kappa_p(w) - \frac{8\pi}{\text{vol}_{\text{hyp}}(X)} \frac{\text{Im}(\sigma_p^{-1}z)}{\text{vol}_{\text{hyp}}(X)},
\]

which together with equation (18) implies the proposition. \(\square\)
Corollary 2.2. For a fixed \( w \in X \), as \( z \in X \) approaches a cusp \( p \in \mathcal{P} \), we have

\[
g_{\text{hyp}}(z, w) = -\frac{4\pi \log (\text{vol}_{\text{hyp}}(X))}{\text{vol}_{\text{hyp}}(X)} + O_{z,w}(1) = -\frac{4\pi \log (\log |\theta_p(z)|)}{\text{vol}_{\text{hyp}}(X)} + O_{z,w}(1).
\]

\[\text{Proof.} \quad \text{The corollary follows from Proposition 2.1.}\]

From the above corollary, it follows that for a fixed \( w \in X \), as a function in the variable \( z \), the hyperbolic Green’s function \( g_{\text{hyp}}(z, w) \) has log-log-growth at the cusps. Hence, for a fixed \( w \in \overline{X} \setminus \mathcal{P} \), as a function in the variable \( z \), it defines a singular function \( g_{\text{hyp}}(z, w) \) on \( \overline{X} \) with log-log-singularity cusps and log-singularity at \( z = w \). So for a fixed \( w \in \overline{X} \), the hyperbolic Green’s function \( g_{\text{hyp}}(z, w) \) determines a current \( \lfloor g_{\text{hyp}}(\cdot, w) \rfloor \) of type (0,0) acting on smooth (1,1)-forms defined on \( \overline{X} \).

Remark 2.3. For any \( f \in C_{L,\ell}(\overline{X}) \), from standard arguments from analysis, it follows that \( d_z d^c_z f(z) \) defines an integrable (1,1)-form on \( \overline{X} \). Furthermore, for a fixed \( w \in \overline{X} \setminus (\text{Sing}(f) \cup \mathcal{P}) \), the following integral exists

\[
\int_{\overline{X}} g_{\text{hyp}}(z, w) d_z d^c_z f(z).
\]

In the following lemma, we show that the hyperbolic Green’s function defines a Green’s current acting on the space of singular functions \( C_{L,\ell}(\overline{X}) \).

Lemma 2.4. Let \( f \in C_{L,\ell}(\overline{X}) \), then for a \( w \in \overline{X} \setminus (\text{Sing}(f) \cup \mathcal{P}) \) fixed, we have the equality of integrals

\[
\int_{\overline{X}} g_{\text{hyp}}(z, w) d_z d^c_z f(z) + f(w) + \sum_{s \in \text{Sing}(f)} \sum_{g \in \mathcal{P}} \frac{c_{f,s}}{2} g_{\text{hyp}}(s, w) = \int_{\overline{X}} f(z) \hat{\mu}_{\text{shyp}}(z).
\]

\[\text{Proof.} \quad \text{Let } w \in \overline{X} \setminus (\text{Sing}(f) \cup \mathcal{P}) \text{ and let } U_r(w), U_r(s), \text{ and } U_r(p) \text{ denote open coordinate disks of radius } r \text{ around } w, s \in \text{Sing}(f), \text{ and a cusp } p \in \mathcal{P}, \text{ respectively. Put}
\]

\[
Y_r = \overline{X} \setminus \left( \bigcup_{s \in \text{Sing}(f)} \bigcup_{g \in \mathcal{P}} U_r(s) \cup U_r(p) \right).
\]

From equation (10) and Stokes’s theorem, it follows that it suffices to prove that

\[
\int_{Y_r} g_{\text{hyp}}(z, w) d_z d^c_z f(z) - \int_{Y_r} f(z) \hat{\mu}_{\text{shyp}}(z) = \]

\[
\int_{\partial U_r(w)} g_{\text{hyp}}(z, w)(-d^c_z f(z)) - \int_{\partial U_r(w)} f(z)(-d^c_z g_{\text{hyp}}(z, w)) +
\]

\[
\sum_{s \in \text{Sing}(f)} \left( \int_{\partial U_r(s)} g_{\text{hyp}}(z, w)(-d^c_z f(z)) - \int_{\partial U_r(s)} f(z)(-d^c_z g_{\text{hyp}}(z, w)) \right) +
\]

\[
\sum_{p \in \mathcal{P}} \left( \int_{\partial U_r(p)} g_{\text{hyp}}(z, w)(-d^c_z f(z)) - \int_{\partial U_r(p)} f(z)(-d^c_z g_{\text{hyp}}(z, w)) \right) \to 0 \text{ as } r \to 0.
\]

Recall that \( d^c_z \) in polar coordinates is given by

\[
d^c_z = \frac{r}{2} \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} - \frac{1}{4\pi} \frac{\partial}{\partial \theta} \frac{\partial}{\partial r}.
\]

Hence, as \( w \notin \text{Sing}(f) \), using equation (9) we derive

\[
\int_{\partial U_r(w)} g_{\text{hyp}}(z, w)(-d^c_z f(z)) - \int_{\partial U_r(w)} f(z)(-d^c_z g_{\text{hyp}}(z, w)) =
\]

\[
\int_0^{2\pi} r \log r \frac{\partial f}{\partial r} \frac{d\theta}{2\pi} - \int_0^{2\pi} f(z)r \frac{\partial \log r}{\partial r} \frac{d\theta}{2\pi} + O(r) \to -f(w).
\]
As \( w \not\in \text{Sing}(f) \), the hyperbolic Green's function \( g_{\text{hyp}}(z, w) \) remains smooth at \( s \in \text{Sing}(f) \setminus \mathcal{P} \). So for any \( s \in \text{Sing}(f) \) and \( s \not\in \mathcal{P} \), using equation (15) and from similar computations as in (19), we get

\[
\int_{\partial U, (s)} g_{\text{hyp}}(z, w)(-d_z^c f(z)) - \int_{0}^{2\pi} f(z)(-d_z^c g_{\text{hyp}}(z, w)) = \\
- c_{f,s} \left( \int_{0}^{2\pi} g_{\text{hyp}}(z, w) \frac{r \log r}{2 \pi} \right. + \int_{0}^{2\pi} \frac{r \log r}{2 \pi} \frac{\partial g_{\text{hyp}}(z, w)}{\partial r} \left. \right) + O(r) \xrightarrow{r \to 0} - \frac{c_{f,s}}{2} g_{\text{hyp}}(s, w).
\]

Finally for any cusp \( p \in \mathcal{P} \), using Corollary 2.2 and equation (10), we compute

\[
\int_{\partial U, (p)} g_{\text{hyp}}(z, w)(-d_z^c f(z)) - \int_{\partial U, (p)} f(z)(-d_z^c g_{\text{hyp}}(z, w)) = \\
\frac{4\pi c_{f,p}}{\text{vol}_{\text{hyp}}(X)} \left( \int_{0}^{2\pi} \log \left( r \log r \right) \frac{\partial \log \left( r \log r \right)}{\partial r} \frac{d\theta}{2\pi} - \int_{0}^{2\pi} \frac{\partial \log \left( r \log r \right)}{\partial r} \frac{d\theta}{2\pi} \right) + \\
O(1/\log r) = O(1/\log r) \xrightarrow{r \to 0} 0.
\]

Combining equations (19), (20), and (21) completes the proof of the lemma.

**Corollary 2.5.** Let \( f \in C_{L,\ell}(\overline{X}) \), then for a fixed \( w \in X \setminus (\text{Sing}(f) \cap X) \), we have the equality of integrals

\[
\int_X g_{\text{hyp}}(z, w) dz \wedge d_z^c f(z) + f(w) + \sum_{s \in \text{Sing}(f), s \not\in \mathcal{P}} \frac{c_{f,s}}{2} g_{\text{hyp}}(s, w) = \int_X f(z) \mu_{\text{hyp}}(z).
\]

**Proof.** The proof follows from Lemma 2.4 and the fact that there are only finitely many cusps of \( X \).

\[
\square
\]

### 2.2 An auxiliary identity

In this subsection, we drive an auxiliary identity, which is useful in proving the key identity in next subsection.

**Lemma 2.6.** There exists a unique integrable function \( \hat{\Phi}(z) \) defined on \( \overline{X} \), which satisfies the differential equation

\[
d_z d_z^c [\hat{\Phi}(z)] = [\hat{\mu}_{\text{hyp}}(z)] - [\hat{\mu}_{\text{can}}(z)],
\]

with the normalization condition

\[
\int_X \hat{\Phi}(z) \mu_{\text{can}}(z) = 0,
\]

where \([\hat{\Phi}(z)]\) is the current determined by \( \hat{\Phi}(z) \) acting on smooth \((1,1)\)-forms defined on \( \overline{X} \).

**Proof.** Since the cohomology classes of \([\hat{\mu}_{\text{hyp}}(z)]\) and \([\hat{\mu}_{\text{can}}(z)]\) are equal in \( H^2(\overline{X}, \mathbb{Z}) \cong \mathbb{Z} \), the difference \([\hat{\mu}_{\text{hyp}}(z)] - [\hat{\mu}_{\text{can}}(z)]\) is a \( d \)-exact current on \( \overline{X} \). Hence, from the \( \partial \bar{\partial} \)-lemma for currents, we can conclude that there exists an integrable function \( \hat{\Phi}(z) \) defined on \( \overline{X} \) such that

\[
d_z d_z^c [\hat{\Phi}(z)] = [\hat{\mu}_{\text{hyp}}(z)] - [\hat{\mu}_{\text{can}}(z)],
\]

which proves the existence of \( \hat{\Phi}(z) \). The normalization condition (23) ensures the uniqueness of \( \hat{\Phi}(z) \).

\[
\square
\]
Lemma 2.7. Let us denote the restriction of \( \hat{\Phi}(z) \) to \( X \) by \( \Phi(z) \). Then, for \( z, w \in X \), we have
\[
g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) = \frac{1}{2} \left( \Phi(z) + \Phi(w) + \int_X g_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) + \int_X g_{\text{hyp}}(w, \zeta) \mu_{\text{can}}(\zeta) \right). \tag{24}
\]
Proof. For a fixed \( w \in X \), consider the function
\[
F_w(z) = g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) - \int_X g_{\text{hyp}}(w, \zeta) \mu_{\text{can}}(\zeta)
\]
defined on \( X \). As \( g_{\text{can}}(z, w) \) and \( g_{\text{hyp}}(z, w) \) define currents \([g_{\text{can}}(\cdot, w)]\) and \([g_{\text{hyp}}(\cdot, w)]\) of type (0,0) on \( \mathbb{X} \), respectively, the function \( F_w(z) \) determines a current
\[
F_w = [\hat{g}_{\text{hyp}}(\cdot, w)] - [\hat{g}_{\text{can}}(\cdot, w)] - \int_X g_{\text{hyp}}(w, \zeta) \mu_{\text{can}}(\zeta)
\]
of type (0,0) acting on smooth smooth (1,1)-forms defined on \( \mathbb{X} \). For a fixed \( w \in X \), using equation (3) and Lemma 2.4, it is easy to see that \( F_w \) satisfies equations (22) and (23). Hence, from the uniqueness of \( \Phi(z) \), we get
\[
\Phi(z) = F_w(z) = g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) - \int_X g_{\text{hyp}}(w, \zeta) \mu_{\text{can}}(\zeta),
\]
which implies that \( F_w(z) \) is independent of \( w \in X \). Hence, from the above equation and from the symmetry of the Green’s functions \( g_{\text{hyp}}(z, w) \) and \( g_{\text{can}}(z, w) \), we deduce that
\[
g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) = \frac{1}{2} \left( \Phi(z) + \Phi(w) + \int_X g_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) + \int_X g_{\text{hyp}}(w, \zeta) \mu_{\text{can}}(\zeta) \right),
\]
which proves the lemma. \( \square \)

Proposition 2.8. For \( z, w \in X \), we have
\[
g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) = \phi(z) + \phi(w),
\]
where
\[
\phi(z) = \int_X g_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) - \frac{1}{2} \int_X \int_X g_{\text{hyp}}(\xi, \zeta) \mu_{\text{can}}(\zeta) \mu_{\text{can}}(\xi).
\]
Proof. For all \( z, w \in X \), combining Lemma 2.7 and equations (1) and (23), we obtain
\[
2 \int_X \left( g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) \right) \mu_{\text{can}}(w) = 2 \int_X g_{\text{hyp}}(z, w) \mu_{\text{can}}(w) = \int_X \left( \Phi(z) + \Phi(w) + \int_X g_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) + \int_X g_{\text{hyp}}(w, \zeta) \mu_{\text{can}}(\zeta) \right) \mu_{\text{can}}(w) = \Phi(z) + \int_X g_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) + \int_X \int_X g_{\text{hyp}}(\xi, \zeta) \mu_{\text{can}}(\zeta) \mu_{\text{can}}(\xi).
\]
Hence, we arrive at
\[
\Phi(z) = \int_X g_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) - \int_X \int_X g_{\text{hyp}}(\xi, \zeta) \mu_{\text{can}}(\zeta) \mu_{\text{can}}(\xi).
\]
Substituting the above formula for \( \Phi(z) \) in equation (24), we get
\[
g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) = \int_X g_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) + \int_X g_{\text{hyp}}(w, \zeta) \mu_{\text{can}}(\zeta) - \int_X \int_X g_{\text{hyp}}(\xi, \zeta) \mu_{\text{can}}(\zeta) \mu_{\text{can}}(\xi).
\]
The proof of the proposition follows by setting
\[
\phi(z) = \int_X g_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) - \frac{1}{2} \int_X \int_X g_{\text{hyp}}(\xi, \zeta) \mu_{\text{can}}(\zeta) \mu_{\text{can}}(\xi).
\]
\( \square \)
Corollary 2.9. As \( z \in X \) approaches a cusp \( p \in \mathcal{P} \), we have

\[
\phi(z) = -\frac{4\pi \log \left( -\log |\vartheta_p(z)| \right)}{\text{vol}_{\text{hyp}}(X)} + O_z(1).
\]

Proof. For a fixed \( w \in X \), as a function in the variable \( z \), the canonical Green’s function \( g_{\text{can}}(z, w) \) remains smooth at the cusps. So the proof of the corollary follows directly from combining Proposition 2.8 and Corollary 2.2.

In the following proposition, we show that the residual hyperbolic metric is log-log-singular at the cusps.

Corollary 2.10. As \( z \in X \) approaches a cusp \( p \in \mathcal{P} \), we have

\[
\log \|d\vartheta_z\|_{\text{res,hyp}}^2(z) = -\frac{8\pi \log \left( -\log |\vartheta_p(z)| \right)}{\text{vol}_{\text{hyp}}(X)} + O_z(1).
\]

Proof. From Proposition 2.8 we have

\[
\lim_{w \to z} \left( g_{\text{hyp}}(z, w) + \log |\vartheta_z(w)|^2 \right) = \lim_{w \to z} \left( g_{\text{can}}(z, w) + \log |\vartheta_z(w)|^2 \right) + 2\phi(z).
\]

The proof of the corollary follows directly from combining equation (2) and Corollary 2.9.

From Corollary 2.10, it follows that \( \log \|d\vartheta_z\|_{\text{res,hyp}}^2(z) \) is smooth on \( \overline{X \setminus \mathcal{P}} \), and admits a log-log-singularity at the cusps. So for any \( f \in C_{c,\text{cyl}}(\overline{X}) \), the following integral exists

\[
\int_{\overline{X}} \log \|d\vartheta_z\|_{\text{res,hyp}}^2(z) d_z d_{\overline{z}} f(z).
\]

2.3 Key identity

In this subsection, we extend relation (14) to elliptic fixed points and cusps at the level of currents acting on the space of singular functions \( C_{c,\text{cyl}}(\overline{X}) \).

Let \( U_{r_0}(s) \) denote an open coordinate disk of fixed radius \( r_0 \) around \( s \in \text{Sing}(f) \cup \mathcal{S} \), and \( r_0 \) is small enough such that any two coordinate disks are disjoint. Put

\[
U_{r_0} = \bigcup_{s \in \text{Sing}(f) \cup \mathcal{S}} U_{r_0}(s) \quad \text{and} \quad \overline{X \setminus U_{r_0}}.
\]

Furthermore, for \( 0 < r < r_0 \), let \( U_r(s) \) denote an open coordinate disk of radius \( r \) around \( s \in \text{Sing}(f) \cup \mathcal{S} \), and let \( U_{r_0} \setminus U_r(s) \) denote the annulus \( U_{r_0}(s) \setminus U_r(s) \). Put

\[
U_r = \bigcup_{s \in \text{Sing}(f) \cup \mathcal{S}} U_r(s) \quad \text{and} \quad U_{r_0} \setminus U_r.
\]

Proposition 2.11. Let \( f \in C_{c,\text{cyl}}(\overline{X}) \), then we have the equality of integrals

\[
-\int_{U_{r_0}} \log \|d\vartheta_z\|_{\text{res,can}}^2(z) d_z d_{\overline{z}} f(z) = (2g - 2) \int_{U_{r_0}} f(z) \mu_{\text{can}}(z) + \sum_{s \in \text{Sing}(f)} \frac{\epsilon_f s}{2} \log \|d\vartheta_z\|_{\text{res,can}}^2(s) - \\
\int_{\partial U_{r_0}} \log \|d\vartheta_z\|_{\text{res,can}}^2(z) d_{\overline{z}} f(z) + \int_{\partial U_{r_0}} f(z) d_{\overline{z}} \log \|d\vartheta_z\|_{\text{res,can}}^2(z) + \int_{\partial U_{r_0}} f(z) d_{\overline{z}} \log \|d\vartheta_z\|_{\text{res,can}}^2(z)
\]

(25)

Proof. From equation (14), it follows that for any \( r > 0 \), it suffices to prove that

\[
-\int_{U_{r_0} \setminus U_r(s)} \log \|d\vartheta_z\|_{\text{res,can}}^2(z) d_z d_{\overline{z}} f(z) + \int_{U_{r_0} \setminus U_r(s)} f(z) d_z d_{\overline{z}} \log \|d\vartheta_z\|_{\text{res,can}}^2(s) - \\
\sum_{s \in \text{Sing}(f)} \frac{\epsilon_f s}{2} \log \|d\vartheta_z\|_{\text{res,can}}^2(s) - \\
\int_{\partial U_{r_0} \setminus U_r(s)} \log \|d\vartheta_z\|_{\text{res,can}}^2(z) d_{\overline{z}} f(z) + \int_{\partial U_{r_0} \setminus U_r(s)} f(z) d_{\overline{z}} \log \|d\vartheta_z\|_{\text{res,can}}^2(s) + \int_{\partial U_{r_0} \setminus U_r(s)} f(z) d_{\overline{z}} \log \|d\vartheta_z\|_{\text{res,can}}^2(s).
\]
Using Stokes’s theorem, we find that the left-hand side of the above limit simplifies to the following expression:

\[
\int_{\partial U_r} \log \|d\theta_z\|_{res,can}^2 f(z) - \int_{\partial U_r} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,can}^2 - \\
\int_{\partial U_{r_0}} \log \|d\theta_z\|_{res,can}^2 f(z) + \int_{\partial U_{r_0}} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,can}^2
\]

Furthermore, from the construction of the open sets \(U_r\), we have

\[
\int_{\partial U_r} \log \|d\theta_z\|_{res,can}^2 f(z) - \int_{\partial U_r} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,can}^2 = \\
\sum_{s \in Sing(f)} \left( \int_{\partial U_r(s)} \log \|d\theta_z\|_{res,can}^2 f(z) - \int_{\partial U_r(s)} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,can}^2 \right) + \\
\sum_{p \in P} \left( \int_{\partial U_r(p)} \log \|d\theta_z\|_{res,can}^2 f(z) - \int_{\partial U_r(p)} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,can}^2 \right).
\]

As \(\log \|d\theta_z\|_{res,can}^2\) remains smooth on \(\overline{X}\), from arguments as in Lemma 2.3, we derive

\[
\sum_{s \in Sing(f)} \left( \int_{\partial U_r(s)} \log \|d\theta_z\|_{res,can}^2 f(z) - \int_{\partial U_r(s)} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,can}^2 \right) \xrightarrow{r \to 0} 0
\]

\[
\sum_{p \in P} \left( \int_{\partial U_r(p)} \log \|d\theta_z\|_{res,can}^2 f(z) - \int_{\partial U_r(p)} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,can}^2 \right) \xrightarrow{r \to 0} 0,
\]

which completes the proof of the proposition. \(\square\)

**Proposition 2.12.** Let \(f \in C_{\ell,\ell}(\overline{X})\), then we have the equality of integrals

\[
- \int_{U_{r_0}} \log \|d\theta_z\|_{res,hyp}^2 f(z) + \frac{1}{2\pi} \int_{U_{r_0}} f(z) \hat{\mu}_{hyp}(z) + \\
\int_{U_{r_0}} f(z) \left( \int_0^\infty \Delta_{hyp} \hat{K}_{hyp}(t; z) dt \right) \hat{\mu}_{hyp}(z) + \sum_{s \in Sing(f)} \frac{c_f}{2} \log \|d\theta_z\|_{res,hyp}(s) - \\
\int_{\partial U_{r_0}} \log \|d\theta_z\|_{res,hyp}^2 f(z) + \int_{\partial U_{r_0}} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,hyp}^2.
\]

**Proof.** From Corollary 2.10, we know that \(\log \|d\theta_z\|_{res,hyp}^2\) is smooth on \(X\) and is log log-singular at the cusps. So the proof of the proposition follows from equation (13) and employing similar arguments as in Proposition 2.11. \(\square\)

**Proposition 2.13.** Let \(f \in C_{\ell,\ell}(\overline{X})\), then we have the equality of integrals

\[
- \int_{U_{r_0}} \log \|d\theta_z\|_{res,hyp}^2 f(z) = \\
2g \int_{U_{r_0}} f(z) \hat{\mu}_{can}(z) - 2 \int_{U_{r_0}} f(z) \hat{\mu}_{hyp}(z) + \sum_{s \in Sing(f)} \frac{c_f}{2} \log \|d\theta_z\|_{res,hyp}(s) - \\
\int_{\partial U_{r_0}} \log \|d\theta_z\|_{res,hyp}^2 f(z) + \int_{\partial U_{r_0}} f(z) d\theta_z^2 \log \|d\theta_z\|_{res,hyp}^2. \tag{26}
\]
Proof. Subtracting equation (25) from the desired equality in (20), it follows that for any \( r > 0 \), it suffices to prove that

\[
- \int_{U_{0},r} \log \| d\theta \|^2_\text{res, hyp}(z) d\theta_z f(z) + \int_{U_{0},r} \log ||d\theta_z||^2_\text{res, can}(z) d\theta_z f(z) - \\
2 \int_{U_{0},r} f(z) \mu_{\text{can}}(z) + 2 \int_{U_{0},r} f(z) \mu_{\text{hyp}}(z) \rightarrow_{r \to 0} \\
\sum_{s \in \text{Sing}(f)} \frac{c_{f,s}}{2} \left( \log \| d\theta \|^2_\text{res, hyp}(s) - \log \| d\theta \|^2_\text{res, can}(s) \right) - \\
\int_{\partial U_{0}} \log \| d\theta \|^2_\text{res, hyp}(z) d\theta_z f(z) + \int_{\partial U_{0}} f(z) d\theta_z \log \| d\theta \|^2_\text{res, hyp}(z) + \\
\int_{\partial U_{0}} \log \| d\theta \|^2_\text{res, can}(z) d\theta_z f(z) - \int_{\partial U_{0}} f(z) d\theta_z \log \| d\theta \|^2_\text{res, can}(z). \tag{27}
\]

From Proposition 2.8 for any \( r > 0 \) and \( z \in U_{0,r}, \) we have

\[
\mu_{\text{can}}(z) - \mu_{\text{hyp}}(z) = -d_z d^c_s \phi(z), \tag{28}
\]

\[
\log \| d\theta \|^2_\text{res, hyp}(z) - \log \| d\theta \|^2_\text{res, can}(z) = \lim_{w \to z} \left( g_{\text{hyp}}(z, w) - g_{\text{can}}(z, w) \right) = 2\phi(z). \tag{29}
\]

Therefore, using the above two equations and Stokes’s theorem, the left-hand side of limit (24) simplifies to give

\[
2 \int_{\partial U_r} \phi(z) d^c_s f(z) - 2 \int_{\partial U_r} f(z) d^c_s \phi(z) - \\
\int_{\partial U_r} \log \| d\theta \|^2_\text{res, hyp}(z) d\theta_z f(z) + \int_{\partial U_r} f(z) d\theta_z \log \| d\theta \|^2_\text{res, hyp}(z) + \\
\int_{\partial U_r} \log \| d\theta \|^2_\text{res, can}(z) d\theta_z f(z) - \int_{\partial U_r} f(z) d\theta_z \log \| d\theta \|^2_\text{res, can}(z). \]

From Corollary 2.9 we know that \( \phi(z) \) is smooth on \( X \) and is \( \log \log \)-singular at the cusps. So employing similar arguments as in Proposition 2.11 and using equation (20), we compute

\[
2 \int_{\partial U_r} \phi(z) d^c_s f(z) - 2 \int_{\partial U_r} f(z) d^c_s \phi(z) \rightarrow_{r \to 0} \sum_{s \in \text{Sing}(f)} \frac{c_{f,s}}{2} \phi(s) = \\
\sum_{s \in \text{Sing}(f)} \frac{c_{f,s}}{2} \left( \log \| dz \|^2_\text{res, hyp}(s) - \log \| dz \|^2_\text{res, can}(s) \right),
\]

which completes the proof of the proposition. \( \square \)

Theorem 2.14. Let \( f \in C_{L, \ell}(X) \), then we have the equality of integrals

\[
g \int_X f(z) \hat{\mu}_{\text{can}}(z) = \\
\left( \frac{1}{4\pi} + \frac{1}{\text{vol}_\text{hyp}(X)} \right) \int_X f(z) \hat{\mu}_{\text{hyp}}(z) + \frac{1}{2} \int_X f(z) \left( \int_0^\infty \Delta_{\text{hyp}} \hat{K}_{\text{hyp}}(t; z) dt \right) \hat{\mu}_{\text{hyp}}(z). \tag{30}
\]

Proof. From the equality of differential forms described in equation (14), for any \( f \in C_{L, \ell}(X) \), we have the desired equality of integrals (30) on the compact subset \( Y_{0,r} \).

For any \( f \in C_{L, \ell}(X) \), combining Propositions 2.12 and 2.13 proves the desired equality of integrals (30) on \( U_{0,r} \), and completes the proof of the theorem. \( \square \)
Corollary 2.15. Let $f \in C_{\ell,\ell}(\mathcal{X})$, then we have the equality of integrals

$$g \int_{\mathcal{X}} f(z) \mu_{\text{can}}(z) = \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X)} \right) \int_{\mathcal{X}} f(z) \mu_{\text{hyp}}(z) + \frac{1}{2} \int_{\mathcal{X}} f(z) \left( \int_{0}^{\infty} \Delta_{\text{hyp}} K_{\text{hyp}}(t; z) dt \right) \mu_{\text{hyp}}(z).$$

Proof. The proof follows from Theorem 2.14 and the fact that there are only finitely many cusps of $X$. \qed

Remark 2.16. Observe that for a fixed $w \in X$, as a function in the variable $z$, the hyperbolic Green’s function $g_{\text{hyp}}(z, w) \in C_{\ell,\ell}(\mathcal{X})$. Hence, combining Corollary 2.15 and Proposition 2.8, we find

$$\phi(z) = \frac{1}{2g} \int_{\mathcal{X}} g_{\text{hyp}}(z, \zeta) \left( \int_{0}^{\infty} \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta) - \frac{1}{8g^2} \int_{\mathcal{X}} g_{\text{hyp}}(\zeta, \xi) \left( \int_{0}^{\infty} \Delta_{\text{hyp}} K_{\text{hyp}}(t; \zeta) dt \right) \left( \int_{0}^{\infty} \Delta_{\text{hyp}} K_{\text{hyp}}(t; \xi) dt \right) \mu_{\text{hyp}}(\zeta) \mu_{\text{hyp}}(\xi).$$

The above equation allows us to express the canonical Green’s function $g_{\text{can}}(z; w)$ in terms of expressions involving only the hyperbolic heat kernel $K_{\text{hyp}}(t; z; w)$ and the hyperbolic metric $\mu_{\text{hyp}}(z)$. Hence, in the upcoming article [2], equation (31) serves as a starting point for the extension of the bounds for the canonical Green’s function $g_{\text{can}}(z; w)$ from [6] to noncompact hyperbolic Riemann orbisurfaces of finite volume.

Furthermore, as stated before the key identity has been the most crucial tool in the work of J. Jorgenson and J. Kramer. We hope that the extended version of the key identity leads to the extension of their work.

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