THE GOLDMAN-WOLPERT LIE ALGEBRA OF CURVES

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ABSTRACT. We first give a new direct geometric definition of a Lie bracket of two undirected curves on a surface which was found originally by Wolpert and Goldman in the eighties, and which is referred to here as the GW bracket. The geometric picture reveals readily certain nontrivial known results, unveils new ones and also motivates an unexpected but likely true conjecture. The conjecture reduces a hard to obtain geometric property, disjointness, to a structured algebraic calculation. The new result for the GW-bracket is that its center consists precisely of the obvious central elements, namely the Lie sub algebra generated by the class of the trivial loop and the classes of loops parallel to the boundary components or punctures.

We also prove in two short steps that there is no cancellation of geometric terms in the GW bracket of two curves if one curve is simple: firstly by showing that a cancelling pair of terms must have supplementary angles between geodesic representatives, by noting secondly, earthquaking along the simple geodesic changes these angles monotonically, this leading to the contradiction.

Concerning the conjecture, we have substantial computer evidence suggesting that if the GW-bracket of two unoriented curves is zero then these curves have disjoint representatives. This possibility was unanticipated by the history.

The GW-bracket and an initial non-cancellation result go back to more algebraic work of first Wolpert on Poisson Lie algebras and then Goldman on his Lie algebras of directed and of undirected curves, all in the 80’s.

1. INTRODUCTION

1.1. New definition of the GW-algebra. Let $\Sigma$ be an oriented Riemann surface, not necessarily of finite type, which carries a complete metric of constant curvature $-1$.

Given two curves $x$ and $y$ on $\Sigma$ and a transversal intersection point $P$ of $x$ and $y$, one can define a new curve $(x \ast_P y)_0$ as follows: orient $x$ and $y$ in such a way that the orientation of the surface agrees with the orientation determined by the two ordered oriented branches of $x$ and $y$ emanating from $P$, then cut the two branches at $P$ and reconnect the curves following the new orientations (that is, perform the usual loop product at $P$), and then discard the orientation of the composed curve.
A second new curve \((x \ast_P y)_\infty\) is defined similarly as \((x \ast_P y)_0\) but orientating \(x\) and \(y\) so that the orientation determined by the two oriented branches emanating from \(P\) disagrees with the orientation of the surface. Figure 1 shows the two possible re-connections around \(P\).

![Figure 1](image)

**Figure 1.** Local picture of the two smoothings associated with an intersection point \(P\) between two curves \(x\) and \(y\).

The free homotopy class of a curve \(x\) on \(\Sigma\) is denoted by \(\tilde{x}\). (Unless explicitly noted, all curves considered are unoriented). The set of free homotopy classes of unoriented closed curves is denoted by \(\tilde{\pi}\) and the free module generated by \(\tilde{\pi}\) over a ring \(K\) is denoted by \(K\tilde{\pi}\). We are now going to give a definition of a Lie algebra on \(\tilde{\pi}\), first by giving the bracket of a pair of element of the basis \(\tilde{\pi}\), and then, extending bilinearly.

Consider two curves \(x\) and \(y\) intersecting only in transversal double points. We now define geometrically \([\tilde{x}, \tilde{y}]\) as the sum over all the intersection points of \(x\) and \(y\) of the free homotopy classes of two signed smoothings associated with each intersection point of \(x\) and \(y\). In symbols,

\[
[\tilde{x}, \tilde{y}] = \sum_{P \in \Sigma} (\tilde{(x \ast_P y)_0} - (\tilde{x \ast_P y)_\infty}).
\]

Clearly, if \(\tilde{x}\) and \(\tilde{y}\) have disjoint representatives, \([\tilde{x}, \tilde{y}] = 0\). We call this structure, the *Goldman-Wolpert Lie algebra of undirected curves*, or briefly, the *GW-Lie algebra*. See history and background below. In Section 5, we prove that our definition coincides with that of Goldman [9]. In particular, this implies that the GW-algebra is well defined on homotopy classes and it is indeed a Lie algebra.

**Remark 1.1.** Note that the smoothings at an intersection point can be defined for each pair of branches if the intersection is transversal, but the point could have multiplicity larger than two. More about this in Subsection 2.1.

1.2. **Main results, road maps for the proofs and a conjecture.** Given two free homotopy classes of closed curves \(\tilde{x}\) and \(\tilde{y}\), the *geometric intersection number of \(\tilde{x}\) and \(\tilde{y}\),
denoted by \(i(\tilde{x}, \tilde{y})\), is defined to be the smallest number of crossings of a pair of representatives \(\tilde{x}\) and \(\tilde{y}\), that intersect in transversal double points. We prove that the GW-bracket “counts” intersections:

**Counting Intersection Theorem.** If \(x\) and \(y\) are closed, curves and \(x\) is simple then

\[
\text{the number of terms (counted with multiplicity) of } [\tilde{x}, \tilde{y}] \text{ is twice the intersection number, } 2i(\tilde{x}, \tilde{y}).
\]

One key point of the geometric definition of the GW-Lie bracket here and the resulting angle technique is a new generalized proof in two instant pictures proving no cancellation of terms in the GW-bracket: 1. Cancellation means that a certain pair of angles are supplementary. 2) earthquaking along the simple curve changes both angles strictly monotonically (both decreasing or both increasing). This is a contradiction.

Here is the road map of the proof. If there is cancellation in the bracket of a simple curve \(x\) and another curve \(y\), it must occur between two terms with opposite signs. Thus one of these cancelling terms is as in Figure I middle and the other as in Figure I right. Endow the surface with a hyperbolic metric. Since the cancelling free homotopy classes coincide, the Law of Cosines implies that the angles at each of the corresponding intersection points add up to \(\pi\). (The angle of an ordered pair of two geodesics \((x, y)\) at an intersection point \(P\) is the angle from \(y\) to \(x\) following the orientation of the surface). But this identity cannot occur for all metrics because by left-twisting the angles between two curves, one of them simple, decrease.

Recall that the center of a Lie algebra on \(V\) is the set of elements \(v \in V\) such that \([v, w] = 0\) for all \(w \in V\). In this paper, we prove:

**Center Theorem.** The center of the GW-Lie algebra of curves is linearly generated by the class of curves homotopic to a point, and the classes of curves winding multiple times around a single puncture or boundary component.

A difficulty that arises in the study of the center using topological or geometric tools is that formal linear combinations of classes of curves, (as opposed to single classes of curves) have to be considered. Thus, the characterization of the center requires more argument than that of the Counting Intersection Theorem.

Here is the idea of the proof of the Center Theorem: The intersection points of a union of geodesics \(y_1, y_2, ..., y_k\) and different powers of a simple geodesic \(x\) are “kind of” the same when the powers vary -they are the same “physical points” and the angles are the same.
If the bracket of $x^m$ with a linear combination of $y_1, y_2, ..., y_k$ is zero for enough values of $m$, then there are pairs of intersection points of $x$ with the union of $y_1, y_2, ..., y_k$ that yield terms of the corresponding brackets that cancel for different values of $m$. This implies pairs of angles at intersection points are supplementary or congruent for all metrics. By twisting along the simple curve $x$, and some hyperbolic geometry we show that both possibilities lead to a contradiction.

We state a new conjecture, verified computationally for as many classes of curves available computers can handle (see Section 4 for precise statements).

**Conjecture.** If the GW-Lie bracket of two classes of undirected curves is zero, then the classes have disjoint representatives.

This statement does not hold for the Goldman Lie bracket on directed curves. In other words, there are examples of pairs of classes of directed curves with Goldman bracket zero and without disjoint representatives, see for instance, or [4, Exercise 11.1].

There are also examples that show the GW-Lie bracket of two non-simple curves can have cancellation, for in Example 4.1.

1.3. History and background of the GW-bracket.

**Remark 1.2.** Wolpert [15, Theorem 4.8], discovered a sub-Lie algebra of vector fields, the twist lattice, on the linear span of the Fenchel-Nielsen vector fields associated to curves (which are the infinitesimal generators of the earthquakes along these curves) and gave a topological description of this Lie algebra. Goldman [9] showed that the twist lattice Lie algebra is the homomorphic image of a more basic Lie algebra, the GW-Lie algebra. Namely, he proved [9, Theorem 5.2 and §5.12] that an equivalent version of the GW-bracket (defined above) is skew-symmetric and satisfies the Jacobi identity. According to Goldman, the embedding of the GW-Lie algebra in the Goldman Lie algebra, which he used to define the GW-algebra, was first observed by Dennis Johnson.

The GW-Lie algebra, and its “cousin”, the Goldman Lie algebra (see Section 5 for a definition) are infinite-dimensional and still have many mathematical “secrets” to reveal.

1.4. History of the relation between the Goldman bracket, the GW-bracket and the geometric intersection and self-intersection number of closed curves.
Goldman [9, Theorem 5.7] proved that if either the Goldman or the GW-Lie bracket of two classes of curves is zero, and one of them is simple, then the two classes have disjoint representatives, that is, their geometric intersection number is zero. Goldman’s proof for the unoriented case (suggested by Wolpert) uses convexity of geodesic length functions along earthquake paths. Goldman asked for a topological proof of this (topological) result.

Chas gave a combinatorial group theory proof of this result and of a generalization in [3]: The number of terms (counted with multiplicity) of the Goldman bracket of two classes of directed curves, when one of them simple, counts the geometric intersection number of the two classes. She also proved an analogous result for the GW-Lie algebra. In both cases, the main tool was the use of free products with amalgamation and HNN structures on the fundamental group of the surface, determined by simple curves. In the case of the GW- bracket, Chas proof was complicated requiring several combinatorial technical lemmas.

Trying to understand the relation of the Goldman-Turaev Lie bialgebra with intersection and self-intersection of curves on surfaces Chas and Sullivan found String Topology [7], a structure that generalizes the Goldman Lie algebra and the Turaev Lie coalgebra to arbitrary orientable manifolds of all dimensions.

Chas and Krongold [6] proved that, on a surface with boundary, a non-power directed curve $x$ is simple if and only if the Goldman Lie bracket of $x$ with $x^m$ is non-zero, for any $m \geq 3$. Moreover, for any directed curve $x$ the number of terms, counted with multiplicity, of the Goldman Lie bracket of $x$ with $x^m$ is $2m$-times the geometric self-intersection number of $x$. (Computer evidence suggests these statements are also true when $m = 2$. We are working on a proof of this result.)

Chas and Gadgil [5] proved that if $x$ and $y$ are non-power directed curves, then for $p$ and $q$ large enough, the Goldman Lie bracket $[x^p, y^q]$ counts the geometric intersection number of the classes of $x$ and $y$.

Cahn and Tchernov [2] determined that the Andersen-Mattes-Reshetikhin Poisson bracket (a generalization of the Goldman Lie bracket) counts intersections of two classes of curves, when the classes are distinct.

Kawazumi and Kuno [12] proved that the center of the Goldman Lie algebra on directed curves on a surface with infinite genus and one boundary component is spanned by powers of loops parallel to the boundary component.
1.5. **Earlier proofs of the center of the Goldman Lie algebra.** Etingof [8] using known representation theory proved that the center of the Goldman Lie algebra of directed curves on a closed surface is generated by the class of the trivial loop. Kabiraj, in his Ph.D. thesis, analyzed the center of the Goldman Lie algebra of curves on surfaces with boundary. His proof treats all cases (closed or with boundary) and proved that the center is generated by the trivial loop together with curves parallel to the boundary components [10].

Let $\Sigma$ be a closed surface. Consider $\mathcal{M}_C^n$, the algebraic variety associated to the moduli space of representations of $\pi_1(\Sigma)$ into $GL(n,\mathbb{C})$ up to conjugation, which on its smooth part admits the Goldman symplectic structure [9]. This symplectic structure was also found independently by Atiyah and Bott for compact Lie groups using infinite dimensional symplectic reduction.

Goldman defined the homomorphism of Poisson algebras $S(\mathbb{C}\pi) \to \mathbb{C}[\mathcal{M}_C^n]$ defined by $\Phi_n(x) = \text{tr}_x$ (here, $\mathbb{C}\pi$ denotes the module spanned by the set $\pi$ of conjugacy classes of fundamental group of surface - that is, the set of free homotopy classes of directed curves- and $S(\mathbb{C}\pi)$ denotes the symmetric algebra of $\mathbb{C}\pi$). The map $\Phi_n$ is surjective [8] but not injective in general. However Etingof [8, Proposition 2.2] proved that given any finite dimensional subspace $V$ of $\mathbb{C}\pi$, there exists $N$ such that $\Phi_n|_V$ is injective for all $n \geq N$. Using this result together with the fact that Poisson center of $\mathbb{C}[\mathcal{M}_C^n]$ being $\mathbb{C}$, Etingof computed the center of $\mathbb{C}\pi$. In principle Etingof’s method could be used to compute the center of $\mathbb{K}\pi$ by replacing $GL_n(\mathbb{C})$ in the definition of $\mathcal{M}_C^n$ by one of the following groups: $O_n(\mathbb{C}), Sp_n(\mathbb{C})$ or $U_n(\mathbb{C})$ [9, Theorem 3.14 and Theorem 5.13]. However, as with the Etingof’s proof, this possible method will work only for closed surfaces and for coefficients $\mathbb{K} = \mathbb{C}$. Our proof, on the other hand, works for any complete hyperbolic surface with coefficients in any ring $\mathbb{K}$.

Chas and Gadgil [5] used geometric group theory to study the quasi-geodesic nature of the lifts of the terms of the Goldman bracket. Using this result and the facts that lifts of a simple closed curve are disjoint, in [10], the second author computed the center of $\mathbb{K}\pi$. Although, this method could be used to compute the center of $\mathbb{K}\pi$, there are two drawbacks. Firstly because the proof is based on case by case considerations, it is long, technical and geometrically less transparent than our proof here. Secondly because of the geometric group theory techniques, the various bounds obtained are qualitative not quantitative.
2. Intersection points, angles and earthquakes. Proof of the Counting Intersection Theorem

Denote by $\mathcal{T}$ the Teichmüller space associated with the surface $\Sigma$. A closed curve on $\Sigma$ is an $X$-geodesic if it is a geodesic for the metric $X \in \mathcal{T}$.

2.1. Intersection points and metrics. In order to be able to follow intersection points of two curves through homotopies of these curves, we need to refine the definition of intersection point: If $x$ and $y$ are two closed curves intersecting transversally, an $(x, y)$-intersection point is a point $P$ on the intersection of $x$ and $y$, together with a choice of a pair of small arcs, one of $x$ and the other of $y$, intersecting only at $P$.

Remark 2.1. For any two curves $x, y$ (possibly with intersection points of multiplicity larger than two) we have

$$i(x, y) = \min \# \{(x, y) - \text{intersection points} : x, y \in \mathcal{T}, x \text{ and } y \text{ intersect transversally}\}.$$ 

Here the number of intersection points counted with multiplicity, namely $k$ lines intersecting transversally at a point counts as $"k \choose 2$".

By Thurston’s Earthquake Theorem (see the appendix of [13] for a proof), there is a unique earthquake path between any pair $X, X'$ of elements of $\mathcal{T}$. The next lemma can be proved by following the $(x, y)$ – intersection points of two $X$-geodesics $x$ and $y$ along this unique geodesic path.

Lemma 2.2. Let $X, X' \in \mathcal{T}$ and let $x$ and $y$ be two $X$-geodesics. If $x', y'$ are two $X'$-geodesics such that both pairs $x, x'$ and $y, y'$ are homotopic then there is a canonical bijection between the $(x, y) – \text{intersection points}$ and the $(x', y') – \text{intersection points}$.

Remark 2.3. Lemma 2.2 can be also proved using that $\mathcal{T}$ is simply connected, instead of Thurston’s Earthquake Theorem.

2.2. Angles, metrics and earthquakes. Fix a metric $X \in \mathcal{T}$ and $P$, an $(x, y)$-intersection point of two $X$-geodesics $x$ and $y$. For each metric $Y \in \mathcal{T}$, the $Y$-angle of $x$ and $y$ at $P$, denoted by $\phi_P(Y)$, is defined as the angle at the intersection point corresponding to $P$ by Lemma 2.2 of the two $Y$-geodesics homotopic to $x$ and $y$, measured from the geodesic homotopic to $y$ to the geodesic homotopic to $x$, following the orientation of the surface (see Figure 2). Clearly, $\phi_P(X)$ is the angle at $P$, from $y$ to $x$ following the orientation of
the surface. Observe that $\phi_P(Y) \in (0, \pi)$. (The angle $\phi_P$ is defined from $y$ to $x$ as is done in [14]).

![Figure 2. The angle $\phi$](image)

Following [13], for each simple $X$-geodesic $x$ and each real number $t$, $E_x(t)$, is the element of $\mathcal{T}$ given by left twist deformation of $X$ along $x$ at the time $t$ starting at $X$. (Clearly, $E_x(t)$ also depends on $X$).

By [13, Proposition 3.5] and [11, Lemma 2.1] we have,

**Lemma 2.4.** If $X \in \mathcal{T}$, and $x$ and $y$ are two $X$-geodesics such that $x$ is simple, and $P$ is an $(x,y)$-intersection point then the function $\phi_P(E_x(t))$ is a strictly decreasing function of $t$.

**Remark 2.5.** The aforementioned Proposition 3.5 of [13] is proved assuming that both $x$ and $y$ are simple geodesics. In the same work, it is stated that it holds when $y$ is non-simple. An explicit proof of this fact can be found in [11, Lemma 2.1].

We include the following result from [1 Theorem 7.38.6] and its proof because both will be used later.

**Theorem 2.6.** Let $a$ and $b$ be hyperbolic isometries of the hyperbolic plane, whose axes intersect at a point $P$. Denote by $\beta$ the angle at $P$ of these axes in the forward direction of $a$ and $b$. Then the product $a.b$ is hyperbolic and

$$
\cosh \left( \frac{t_{a,b}}{2} \right) = \cosh \left( \frac{t_a}{2} \right) \cosh \left( \frac{t_b}{2} \right) + \sinh \left( \frac{t_b}{2} \right) \sinh \left( \frac{t_a}{2} \right) \cos(\beta),
$$

where $t_\alpha$ denotes the translation length of $\alpha$.

**Proof.** Denote by $Q$ the point on the axis of $a$ at distance $t_a/2$ of $P$ in the positive direction of the axis of $a$ and by $R$ the point on the axis of $b$ at distance $t_b/2$ of $P$ in the negative
direction of the axis of $b$. The axis of $a.b$ is the geodesic containing the oriented line from $R$ to $Q$ and the translation length of $a.b$ equals twice the distance between $R$ and $Q$. (see Figure 3).

**Figure 3. Theorem 2.6**

For each closed curve $x$ on $\Sigma$, the length of the unique $X$-geodesic in $\tilde{x}$ is denoted by $\ell_x(X)$.

**Figure 4. Lifts of $(x \ast_p y)_0$ and $(x \ast_p y)_\infty$.**

Given two $X$-geodesics $x$ and $y$ and an $(x, y)$-intersection point $P$, a lift of $(x \ast_p y)_0$ (respectively of $(x \ast_p y)_\infty$) to the upper half plane $\mathbb{H}$ is a bi-infinite piecewise geodesic (see Figure 4) consist of alternative geodesic segments of lift of the geodesics $x$ and $y$ (denoted
by \( x_* \) and \( y_* \) in Figure 4, respectively. The geodesic segments of lift of the geodesics \( x \) and \( y \) intersect each other in the lifts of the point \( P \) (denoted by \( P_* \) in Figure 4).

The next lemma follows directly from Theorem 2.6 and from Figure 4 by adding an appropriate orientation to the geodesics \( x \) and \( y \).

**Lemma 2.7.** If \( x \) and \( y \) are two closed \( X \)-geodesics and let \( P \) be an \((x,y)\)-intersection point. Then we have

\[
\cosh \left( \frac{\ell(x_*P_0y)}{2} \right) = \cosh \left( \frac{\ell_x}{2} \right) \cosh \left( \frac{\ell_y}{2} \right) - \sinh \left( \frac{\ell_x}{2} \right) \sinh \left( \frac{\ell_y}{2} \right) \cos(\phi_P) \\
\cosh \left( \frac{\ell(x_*P_0y)}{2} \right) = \cosh \left( \frac{\ell_x}{2} \right) \cosh \left( \frac{\ell_y}{2} \right) + \sinh \left( \frac{\ell_x}{2} \right) \sinh \left( \frac{\ell_y}{2} \right) \cos(\phi_P),
\]

where all lengths and angles are computed with respect to any metric \( Y \) in \( T \).

The next lemma states that, a 0-term of the bracket of two curves, and an \( \infty \)-term of the bracket of the same curves are always distinct. It is easier to state it in terms of geodesics (instead of free homotopy classes of curves).

**Lemma 2.8.** Let \( X \) be a hyperbolic metric on \( \Sigma \). If \( x \) and \( y \) are closed, \( X \)-geodesics, such that \( x \) is simple, and \( P \) and \( Q \) are two (not necessarily distinct) \((x,y)\)-intersection points then \( (x_*P_0y)_0 \neq (x_*Q_0y)_\infty \).

**Proof.** We argue by contradiction. If \( (x_*P_0y)_0 = (x_*Q_0y)_\infty \) then \( \ell(x_*P_0y)_0(Y) = \ell(x_*Q_0y)_\infty(Y) \) for any \( Y \in T \). By Lemma 2.7 we have

\[
\cos(\phi_P(Y)) = -\cos(\phi_Q(Y)),
\]

which implies,

\[
\phi_P(Y) + \phi_Q(Y) = \pi. \quad (1)
\]

for all \( Y \in T \). On the other hand, as \( x \) is simple, by Lemma 2.4 by twisting the metric \( X \) about the geodesic \( x \), both terms on the right side of Equation (1) strictly decrease. Since they add up to a constant, this is not possible. Hence, the proof is complete. \( \square \)

### 2.3. Proof of the counting intersection theorem.
If \( \tilde{x} \) and \( \tilde{y} \) have disjoint representatives, the result follows directly. Assume that \( i(\tilde{x}, \tilde{y}) > 0 \). From the definition of the bracket, it follows that

\[
[\tilde{x}, \tilde{y}] = \sum_{P \in x \cap y} (\tilde{x_*P_0y})_0 - (\tilde{x_*P_0y})_\infty.
\]
Fix a metric $X \in \mathcal{T}$. In order to simplify the notation, assume that $x$ and $y$ are $X$-geodesics. This implies that $x$ and $y$ intersect in $i(\overline{x}, \overline{y})$ points, the geometric intersection number of the class.

Suppose that the number of terms of the bracket is strictly smaller than $2i(\overline{x}, \overline{y})$. Hence, there exist two (not necessarily distinct) $(x,y)$-intersection points $P$ and $Q$ such that a pair of terms corresponding $P$ and $Q$ cancel.

The terms corresponding to $P$ are $(\overline{x \ast_P y})_0 - (\overline{x \ast_P y})_\infty$ and the terms corresponding to $Q$ are $(\overline{x \ast_Q y})_0 - (\overline{x \ast_Q y})_\infty$.

The assumption of cancellation implies that either $(\overline{x \ast_P y})_0 = (\overline{x \ast_Q y})_\infty$ or $(\overline{x \ast_P y})_\infty = (\overline{x \ast_Q y})_0$ which is not possible by Lemma 2.8. Thus, the proof is complete.

**Remark 2.9.** In the case of the Goldman bracket of two oriented curves, cancellation of two terms (regardless whether they are simple or not) implies that the corresponding oriented angles are congruent, [11, Theorem 5.1]. (The oriented angle between two oriented geodesics intersecting at a point $P$ is the angle between the positive direction of both curves).

In the case of the GW-bracket, cancellation of two terms implies that the unoriented angles are supplementary.

### 3. GW-Lie Bracket of powers of curves and proof of the Center Theorem

Let $X \in \mathcal{T}$. If $P$ is an $(x, y)$-intersection point of two $X$-geodesics $x$ and $y$ then for each positive integer $m$, the geodesic $x^m$ (that goes $m$ times around $x$) and $y$ also intersect at $P$.

**Remark 3.1.** The angles at $P$ of $x$ and $y$ and of $x^n$ and $y$ are congruent (they are the same angle). Thus, we can (and will) consider $P$ as $(x^m, y)$-intersection point and the angle $\phi_P(Y)$ will also denote the angle at $P$ of the $Y$-geodesic homotopic to $x^m$ and the $Y$-geodesic homotopic to $y$.

**Proposition 3.2.** Let $X \in \mathcal{T}$ be hyperbolic metric on $\Sigma$. Let $x, y$ and $z$ be three $X$-geodesics. Let $P$ and $Q$ be two $(x, y)$ and $(x, z)$-intersection points respectively.
Lemma 3.3. Let (\(x^m \ast_P y\))\( \subset \) = (\(x^m \ast_Q z\))\( \subset \) then \(\ell_y(Y) = \ell_z(Y)\) and \(\phi_P(Y) = \phi_Q(Y)\), for all \(Y \in T\).

Proof. We prove (1); the proof of (2) is analogous. From now on, we will fix a metric \(Y \in T\). To simplify the notation, we will not write the dependence on \(Y\) (for instance, we will write \(\cos(\phi_P)\) instead of \(\cos(\phi_P)(Y)\)). We follow the notation indicated in Remark 3.1.

Since (\(x^m \ast_P y\))\( \subset \) = (\(x^m \ast_Q z\))\( \subset \), \(\ell_{(x^m \ast_P y)_0} = \ell_{(x^m \ast_Q z)_0}\). By Lemma 2.7 we have

\[
\cosh\left(\frac{1}{2} \ell_{x^m}\right) \cosh\left(\frac{1}{2} \ell_y\right) - \sinh\left(\frac{1}{2} \ell_{x^m}\right) \sinh\left(\frac{1}{2} \ell_y\right) \cos(\phi_P) =
\]

\[
\cosh\left(\frac{1}{2} \ell_{x^m}\right) \cosh\left(\frac{1}{2} \ell_z\right) - \sinh\left(\frac{1}{2} \ell_{x^m}\right) \sinh\left(\frac{1}{2} \ell_z\right) \cos(\phi_Q).
\]

This implies

\[
\coth\left(\frac{1}{2} \ell_{x^m}\right)\{\cosh\left(\frac{1}{2} \ell_y\right) - \cosh\left(\frac{1}{2} \ell_z\right)\} = \sinh\left(\frac{1}{2} \ell_y\right) \cos(\phi_P) - \sinh\left(\frac{1}{2} \ell_z\right) \cos(\phi_Q).
\]

Note that that the right-hand side of the above equation does not depend on \(x\). Also, if \(m_1\) and \(m_2\) are distinct positive integers then \(\coth\left(\frac{1}{2} \ell_{x^{m_1}}\right) \neq \coth\left(\frac{1}{2} \ell_{x^{m_2}}\right)\). This implies \(\cosh\left(\frac{1}{2} \ell_y\right) - \cosh\left(\frac{1}{2} \ell_z\right) = 0\), and so, \(\ell_y = \ell_z\). Hence, \(\cos(\phi_P) = \cos(\phi_Q)\), which implies the equality of the corresponding angles, as desired.

Lemma 3.3. Let \(X\) in \(T\) and let \(x, y\) and \(z\) be three \(X\)-geodesics in \(\Sigma\) such that \(x\) is simple, \(\ell_y(Y) = \ell_z(Y)\) for all \(Y \in T\) and there exist an \((x, y)\)-intersection point \(P\) and a \((x, z)\)-intersection point \(Q\) such that for some positive integer \(m\), (\(x^m \ast_P y\))\( \subset \) = (\(x^m \ast_Q z\))\( \subset \), then either \(y = z\) or there exist an \((x, y)\)-intersection point \(R\) such that \(\phi_R > \phi_P\).

Proof. We prove the result for \(m = 1\). Combining Remark 3.1 with the equality \(\ell_{x^m} = m\ell_x\) the proof for \(m > 1\) follows by a similar argument.

Since (\(x \ast_P y\))\( \subset \) = (\(x \ast_Q z\))\( \subset \), there exists two lifts to the universal cover of the surface, the hyperbolic plane \(\mathbb{H}\), one of the piecewise geodesics (\(x^m \ast_P y\))\( \subset \) and the other of (\(x^m \ast_Q z\))\( \subset \) with the same endpoints. Denote these two lifts by \(C\) and \(D\) respectively, and by \(L\) the geodesic line joining their common endpoints. The two piecewise geodesics \(C\) and \(D\) zigzag about their line \(L\). The zigzag curve \(C\) (resp. \(D\)) is composed of alternating segments of lifts of \(x\) of length \(\ell_x\), and lifts of \(y\) (resp. of \(z\)) of length \(\ell_y\). In Figure 5
“laps” of lifts of $x$ are represented in blue, “laps” of lifts of $y$ in green and “laps” of lifts of $z$ in red. The line $L$ intersects each of these lap segments in their midpoints. (See Theorem 2.6 and its proof).

Consider a segment $S$ of the zigzag curve $C$, which is a lift of $x$. Denote the intersection point of $S$ and $L$ by $U$. Choose an endpoint of $S$ and denote it by $P_1$. Denote by $V$ the intersection of $L$ with the other segment of $C$ with endpoint $P_1$ (this last segment is a lift of $y$). These three points determine a triangle $UVP_1$

Consider the triangle $U'V'Q_1$, analogous to $UVP_1$, but with sides included in the zigzag curve $D$.

Note that the length of both segments, $UP_1$ and $UQ_1$ is $\ell_x/2$. Also the length of $VP_1$ and $V'Q_1$ is $\ell_y/2$. By Lemma 2.7 the length of $UV$ and $U'V'$ is half the length of the geodesic in $(x \ast_p y)_0$. Therefore, these two triangles $UVP_1$ and $U'V'Q_1$ are congruent. Hence, there is an isometry mapping one triangle to the other. If this isometry is orientation reversing, then it maps $U$ to $U$, $V$ to $V'$ and $P_1$ to $Q_1$. This implies that $\phi_P(X) + \phi_Q(X) = \pi$, see Figure 5 a.

If we perturb the metric $X$ slightly, we can repeat the above argument, and obtain that $\phi_P(Y) + \phi_Q(Y) = \pi$, for all $Y$ in a neighborhood of $X$. (The orientation reversing isometry for the corresponding $Y$-geodesics must exist by continuity). Since $x$ is simple, this is not possible by Lemma 2.4 Thus, there is an orientation preserving isometry mapping $U$ to $U$, $V$ to $V'$ and $P_1$ to $Q_1$. Now, there are two possibilities: either the midpoint of a lap of a lift of $x$ is also a midpoint of lap of a lift or $y$ (Figure 5 right) or not (Figure 5 middle.)

If $C = D$, then $y = z$ and the proof is complete. Hence, we can assume $C \neq D$. There are then two cases left, depicted in Figure 5, b. and c. In the case illustrated in b., a segment lifting of $z$ intersects the interior of the triangle $UVP_1$, and determines a triangle $WVR$ as in the figure. Since the area of $WVR$ is smaller than that of $UVP_1$, and two of the angles of of $WVR$ are congruent to two of the angles of $UVP_1$, the angle at $P_1$, is smaller than the angle at $R$, so the proof of this case is complete.

In the case illustrated in Figure 5 c., $\phi_P + \alpha + \beta < \pi = \phi_R + \alpha + \beta$, and $\phi_P < \phi_R$, as desired.

\begin{proposition}
Let $X \in T$ be a metric on $\Sigma$ and $x, y, z$ be three pairwise distinct $X$-geodesics such that $x$ is simple and let $P$ and $Q$ be $(x, y)$ and $(x, z)$-intersection points respectively. The following holds.
\end{proposition}
(1) The equality \( \widetilde{x}^m \ast P y_0 = (\widetilde{x}^m \ast Q z)_\infty \) holds for at most one positive value of \( m \).

(2) Either the equality \( \widetilde{x}^m \ast P y_0 = (\widetilde{x}^m \ast Q z)_0 \) (resp. \( \widetilde{x}^m \ast P y_\infty = (\widetilde{x}^m \ast Q z)_\infty \)) holds for at most one positive value of \( m \) or there exist an \((x,y)\)-intersection point \( R \) such that \( \phi_P(X) < \phi_R(X) \).

**Proof.** Suppose \( \widetilde{x}^m \ast P y_0 = (\widetilde{x}^m \ast Q z)_\infty \) for two distinct values of \( m \). By Proposition \ref{prop:3.2}(2), \( \phi_P(Y) + \phi_Q(Y) = \pi \), for all \( Y \in \mathcal{T} \). By Lemma \ref{lem:2.4} this is not possible. Thus, (1) is proved.

If \( (\widetilde{x}^m \ast P y)_0 = (\widetilde{x}^m \ast Q z)_0 \), for more than two values of \( m \) by Proposition \ref{prop:3.2}(1), we have that \( \ell_y(Y) = \ell_z(Y) \) and \( \phi_P(Y) = \phi_Q(Y) \) for all \( Y \in \mathcal{T} \).

Fix \( X \in \mathcal{T} \), any \( m \) and consider a geodesic lift \( A \) of the geodesic in the free homotopy class of \( \widetilde{x}^m \ast P y_0 = (\widetilde{x}^m \ast Q z)_0 \).

The result follows them by Lemma \ref{lem:3.3}.

The proof for the case \( (\widetilde{x}^m \ast P y)_\infty = (\widetilde{x}^m \ast Q z)_\infty \) is similar. Hence, result is proved. \( \square \)

**Lemma 3.5.** Let \( \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_k \) be pairwise distinct free homotopy classes of closed curves such that \( \widetilde{x} \) contains a simple representative. Let \( \widetilde{y} = \sum_{i=1}^{k} c_i \widetilde{y}_i \), where the coefficients \( c_1, c_2, \ldots, c_k \) are in the ring \( \mathbb{K} \). Then either \( i(x, y_i) = 0 \) for all \( i \in \{1, \ldots, k\} \) or there exists a positive integer \( m_0 \) such that \( [\widetilde{x}^m, \widetilde{y}] \neq 0 \) for all \( m \geq m_0 \).
Proof. From the definition of the bracket, for any \( m \in \mathbb{N} \),
\[
[x^m, \tilde{y}] = \sum_{i=1}^{k} c_i [x^m, \tilde{y}_i]
\]
\[
= m \sum_{i=1}^{k} c_i \sum_{P \in x \cap y_i} \left( x^m *_P y_i \right)_0 - \left( x^m *_P y_i \right)_\infty.
\]

Fix a metric \( X \) and assume that \( x, y_1, \ldots, y_k \) are \( X \)-geodesics. For each \( m \), the sum \( \sum_{i=1}^{k} c_i \sum_{P \in x \cap y_i} \left( x^m *_P y_i \right)_0 - \left( x^m *_P y_i \right)_\infty \) has \( I = 2(\sum_{i=1}^{k} (\tilde{x}, \tilde{y}_1) + \sum_{i=2}^{k} (\tilde{x}, \tilde{y}_2) + \cdots + \sum_{i=k} (\tilde{x}, \tilde{y}_k)) \) terms, before performing possible cancellations.

Choose a metric \( X \) in \( \mathcal{T} \). Consider \( P \), one of the intersection points of the \( X \)-geodesic in \( x \) and \( \bigcup_{1 \leq i \leq k} y_i \) and choose one of the terms of the bracket associated with \( P \), 0 or \( \infty \). For simplicity, assume that the chosen term is \( (x *_P y_1)_0 \).

If for all \( 1 \leq m \leq I+1 \), we have that \([x^m, \tilde{y}] = 0\) then there exist \( m_1, m_2 \in \{1, 2, \ldots, I+1\} \), and an \( X \)-geodesic \( y_i \), such that one of the following holds:

(a). \( (x^m *_P y_1)_0 = (x^m *_Q y_i)_0 \) for \( m = m_1 \) and \( m = m_2 \).

(b). \( (x^m *_P y_1)_0 = (x^m *_Q y_i)_\infty \) for \( m = m_1 \) and \( m = m_2 \).

By Proposition 3.4(1), (a) is not possible. Hence (b) holds and by Proposition 3.4(2), there exists a point \( R \) in \( x \cap y_1 \), such that the angle \( \phi \) at \( P \) is strictly smaller than the angle \( \phi \) at \( R \) (in symbols, \( \phi_P < \phi_R \)). This is not possible because \( P \) was chosen arbitrarily, and so the proof is complete. \( \square \)

We will make use of the following well known result. (This result is usually stated for finite type surfaces but it can be generalized to all Riemann surfaces using the fact that every closed curve is included in a compact subsurface).

Lemma 3.6. If \( \Sigma \) is an orientable surface and \( y \) is a closed curve on \( \Sigma \) such that \( i(x, y) = 0 \) for every simple closed curve \( x \). Then \( y \) is either homotopically trivial or homotopic to a boundary curve or homotopic to a puncture.

3.1. Proof of the Center Theorem. Suppose that \( \tilde{y} = \sum_{i=1}^{k} c_i \tilde{y}_i \) belongs to the center, where the free homotopy classes of \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k \) are pairwise distinct and \( c_i \in \mathbb{K} \) for \( i \in \{1, 2, \ldots, k\} \). Let \( x \) be any simple closed curve. By definition of center, \([x^n, \tilde{y}] = 0 \) for all positive integers \( n \). Therefore Lemma 3.5 implies that \( i(x, y_i) = 0 \) for all \( i \in \{1, 2, \ldots, k\} \).
Hence Lemma 3.6 implies that each $\tilde{y}_i$ is either homotopically trivial or homotopic to a boundary curve or homotopic to a puncture, which completes the proof.

**Remark 3.7.** Let $K\pi$ be the Goldman Lie algebra (see Section 5). We can use similar methods to compute the center of $K\pi$ and to obtain another proof of [10, Main Theorem]. We have to replace the unoriented angle by oriented angle which is the angle at an intersection point between the positive direction of both geodesic. These methods can also be extended to compute the centers of the universal enveloping algebra and symmetric algebras of both $K\pi$ and $K\tilde{\pi}$. The details will appear in our future work.

4. **Concrete examples of the GW-bracket**

Recall that, on a surface with boundary, the set of free homotopy classes of oriented closed curves is in one to one correspondence with conjugacy classes of the fundamental group. Thus, the set of conjugacy classes of the fundamental group minus the conjugacy class of the trivial loop is in two-to-one correspondence with the set of free homotopy classes of oriented closed curves minus the class of the trivial loop. We will use this correspondence in this section, first by choosing a set of minimal generators of the fundamental group, and second, denoting each class of curves by a word in these generators and their inverses. To simplify the notation, the inverse of a generator $x$ will be denoted by $X$. If $w$ is word in this alphabet, (that is, an element of the fundamental group) we will denote by $\tilde{w}$ the free homotopy class of a representative of $w$ after removing the orientation and the basepoint. (We are abusing notation in order to have less symbols).

![Figure 6. A choice of generators of the pair of pants (left) and the punctured torus (right)](image)

**Example 4.1.** Consider the fundamental group of the triply punctured sphere or pair of pants with generators given by the classes $a$ and $b$ of two curves parallel to two of the boundary components, oriented so that $ab$ goes around the third boundary component (see Figure 6, left).

$$[\tilde{aab}, \tilde{aB}] = \tilde{baaBa} - \tilde{Baaba},$$
Example 4.1 shows the need of the hypothesis of one of the curves being simple in the Intersection Counting Theorem since $i(\widetilde{aab}, \widetilde{aB}) = 2$ but the number of terms of the bracket is less than 4.

**Example 4.2.** If we consider the punctured torus with fundamental group with standard generators labeled $a$ and $b$ then $[\widetilde{ab}A\widetilde{b}, \widetilde{aB}] = \widetilde{aB}B\widetilde{B} - \widetilde{A}B\widetilde{aB}A\widetilde{b} + \widetilde{A}B\widetilde{aB} - \widetilde{aB}A\widetilde{B}a\widetilde{B}a\widetilde{B}$.

Example 4.2 illustrates Intersection Counting Theorem. In this case, $i(\widetilde{aB}, \widetilde{aab}) = 2$ and the number of terms of the bracket $[\widetilde{aab}, \widetilde{ab}]$ is 4.
Computational Theorem. Consider two classes of curves $\tilde{x}$ and $\tilde{y}$. If one of the following holds

- The word length of $\tilde{x}$ and $\tilde{y}$ is less than or equal to 12 and $x$ and $y$ are in the punctured torus.
- The word length of $\tilde{x}$ and $\tilde{y}$ is less than or equal to 11 and $x$ and $y$ are in the pair of pants.
- The word length of $\tilde{x}$ and $\tilde{y}$ is less than or equal to 8 and $x$ and $y$ are in the four holed sphere.
- The word length of $\tilde{x}$ and $\tilde{y}$ is less than or equal to 7 and $x$ and $y$ are in the punctured genus two surface with surface word $abABcdCD$.

and the bracket of $\tilde{x}$ and $\tilde{y}$ is zero then $\tilde{x}$ and $\tilde{y}$ have disjoint representatives. In symbols, if $[\tilde{x}, \tilde{y}] = 0$ then $i(\tilde{x}, \tilde{y}) = 0$.

The previous Computational Theorem lead us to the following conjecture.

Conjecture. If the GW-Lie bracket of two classes of undirected curves is zero, then the classes have disjoint representatives.

5. Goldman’s definition of the GW-bracket

In this section we review the definition of the Goldman Lie algebra on directed curves, Goldman’s definition of the GW-Lie algebra of undirected curves and prove that there is an isomorphism between Goldman’s and the definition of the GW-Lie algebra we gave in the introduction.

Denote by $\pi$ the set of free homotopy classes of directed closed curves on $\Sigma$, by $\langle \alpha \rangle$ the free homotopy class of a directed closed curve $\alpha$, and by $\mathbb{K}\pi$ the free module spanned by $\pi$.

The Goldman Lie bracket on $\mathbb{K}\pi$ is the linear extension to $\mathbb{K}\pi$ of the of the bracket of two free homotopy classes $\langle \alpha \rangle$ and $\langle \beta \rangle$ defined by

$$[\langle \alpha \rangle, \langle \beta \rangle] = \sum_{P \in \alpha \cap \beta} \varepsilon_P \langle \alpha *_P \beta \rangle.$$ 

Here, the representatives $\alpha$ and $\beta$ are chosen so that they intersect transversely in a set of double points $\alpha \cap \beta$, $\varepsilon_P$ denotes the sign of the intersection between $\alpha$ and $\beta$ at an intersection point $P$, and $\alpha *_P \beta$ denotes the loop product of $\alpha$ and $\beta$ at $P$. 
Goldman [9] proved that this bracket is well defined, skew-symmetric and satisfies the Jacobi identity on \( K\pi \). In other words, \( K\pi \) is a Lie algebra.

There is a natural involution \( \iota : \pi \longrightarrow \pi \) defined by \( \iota(\langle \alpha \rangle) = \langle \bar{\alpha} \rangle \) where \( \bar{\alpha} \) denotes the curve \( \alpha \) with opposite orientation. By extending \( \iota \) linearly to \( K\pi \) we obtain a \( K \)-linear involution \( \iota : K\pi \longrightarrow K\pi \). The invariant subspace of \( \iota \), denoted by \( S \) is a Lie subalgebra of \( K\pi \) [9, Subsection 5.12].

Now we prove that the subalgebra \( S \) is isomorphic to the GW-Lie algebra \( K\tilde{\pi} \).

First observe that \( S \) is generated by the elements of the form \( \langle \alpha \rangle + \langle \bar{\alpha} \rangle \). A straightforward computation in \( K\tilde{\pi} \) shows that

\[
\langle \alpha \rangle + \langle \bar{\alpha} \rangle + \langle \beta \rangle + \langle \bar{\beta} \rangle = \langle \alpha \rangle + \langle \beta \rangle + \langle \bar{\alpha} \rangle + \langle \bar{\beta} \rangle
\]

(2)

where the “change direction” operator \( \bar{\cdot} \) is extended to \( K\tilde{\pi} \) by linearity.

Next define a map from \( S \) to \( K\tilde{\pi} \), by sending each element of \( S \) of the form \( \langle \alpha \rangle + \langle \bar{\alpha} \rangle \) (that is, each element of a the geometric basis of \( S \)) to the undirected free homotopy class \( u_{\langle \alpha \rangle} \) defined by “forgetting” the direction of \( \alpha \), and considering the free homotopy class. Extend \( u \) to \( S \) by linearity.

\[
\begin{array}{c}
\vphantom{1}\\
\vphantom{1}\\
\vphantom{1}\\
\vphantom{1}\\
\vphantom{1}\\
\vphantom{1}\\
\vphantom{1}\\
\vphantom{1}\\
\vphantom{1}\\
\vphantom{1}\\
\end{array}
\]

\textbf{Figure 9.} Sign of the elements.

Observe that (Figure 9)

\[
\varepsilon_P(\alpha, \beta) = \varepsilon_P(\bar{\alpha}, \bar{\beta}) = -\varepsilon_P(\alpha, \bar{\beta}) = -\varepsilon_P(\bar{\alpha}, \beta).
\]

Also, if \( \varepsilon_P(\alpha, \beta) = 1 \) then

\[
u_{(\alpha* P \beta)} = (\alpha* P \beta)_0 \text{ and } u_{(\alpha* P \beta)} = (\alpha* P \beta)_\infty.
\]

and if \( \varepsilon_P(\alpha, \beta) = -1 \) then

\[
u_{(\alpha* P \beta)} = (\alpha* P \beta)_\infty \text{ and } u_{(\alpha* P \beta)} = (\alpha* P \beta)_0.
\]
Therefore computing the bracket in $K\tilde{\pi}$, we observe

$$\left[u_{\langle\alpha\rangle}, u_{\langle\beta\rangle}\right] = \sum_{P \in \alpha \cap \beta, \varepsilon_P = 1} \left( u_{\langle\alpha \ast P \beta\rangle} - u_{\langle\alpha \ast P \bar{\beta}\rangle} \right) - \sum_{P \in \alpha \cap \beta, \varepsilon_P = -1} \left( u_{\langle\alpha \ast P \bar{\beta}\rangle} - u_{\langle\alpha \ast P \beta\rangle} \right)$$

$$= \sum_{P \in \alpha \cap \beta} \varepsilon_P \left( u_{\langle\alpha \ast P \beta\rangle} + u_{\langle\alpha \ast P \bar{\beta}\rangle} \right) = u_{\langle\alpha\rangle, \langle\beta\rangle} + u_{\langle\alpha\rangle, \langle\bar{\beta}\rangle}$$

On the other hand, by Equation (2) by applying $u$ to the bracket $\left[\langle\alpha\rangle + \langle\bar{\alpha}\rangle, \langle\beta\rangle + \langle\bar{\beta}\rangle\right]$ we obtain $u_{\langle\alpha\rangle, \langle\beta\rangle} + u_{\langle\alpha\rangle, \langle\bar{\beta}\rangle}$. This shows that $u$ is a Lie algebra isomorphism, as desired.

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