ON UNCOUNTABLE STRONGLY CONCENTRATED SETS OF REALS

EILON BILINSKY
TEL AVIV UNIVERSITY

ABSTRACT. We construct new models of $ZF$ with an uncountable set of reals that has a unique condensation point. This addresses a question by Sierpiński from 1918.

1. Introduction

The real line is one of the most basic concepts in modern mathematics. In particular, questions about the topological properties of $R$ were studied extensively since the late 19th century and the beginning of the 20th century. Some of those basic questions were settled easily using Choice principles, but remain difficult in the absence of choice. In the 20th century, in view of the independence phenomena in $ZF$, people start asking about what knowledge the standard theories give us, in particular with respect to those questions about $R$.

In this paper we focus on questions related to the existence of condensation points of large subsets of the real line. Intuitively, since $R$ is separable and complete, one might expect that any uncountable subset of the real line will have more than one condensation points. Indeed, assuming the Axiom of Choice, this is provable.

In [12], Sierpiński asked whether some from of Choice is indeed required (see Problem 1 for exact formulation). This question can be reformulated as a question about concentrated sets. An uncountable set $X \subseteq R$ has a single condensation point if and only if it is concentrated on a singleton.

The goal of this paper is to give additional examples for models of $ZF$ which provide a positive answer for Sierpiński’s question. Other models in which there are large bounded sets of reals which are concentrated on a single point can be also obtained using the Feferman-Levy method [3], or a variation of Cohen’s method, [2].

In all these models there exists a bounded subset $X \subseteq R$ such that, in the model, $X$ has a unique condensation point. We will give two methods for obtaining a model in which such a set exists. In the first method $X$ is well orderable (and therefore by Lemma 1 $\aleph_1$ is singular) and in the second method $\aleph_1$ is regular. Moreover, the models which are obtained in the second method are closer (in some sense) to the model of choice we start with.

The structure of the paper is as follows. In Section 3 we will review some basic concepts and theorems which are relevant for the question. In Section 4 we will show that well orderable large strongly concentrated sets of reals exist in some of the Feferman-Levy models. In Section 5 we will construct a model of $ZF$ in which there is a large strongly concentrated set of reals and $\aleph_1$ is regular.
We work in $ZF$ and mention any use of the axiom of choice. Our notations are mostly standard. For basic facts about forcing and models with atoms (models of $ZFA$) we refer the reader to [7].

2. Acknowledgments

I would like to thank Lior Shalom, Michal Amir, Limor Friedman, Itamar Rosenfeld Rauch, Oren Yakir, Karina Samvelyan, Dor Elboim, Leonid Vishnevsky, Elad Zelingher, Peleg Michaeli, Ofir Gorodetsky, Asaf Cohen, Eyal Kaplan and Tom Benhamou for their help in the technical issues. I would like to thank Heike Mildenberger for her remarks and encouragement. I would like to thank William Chen, Assaf Rinot for reviewing a draft the paper. I would like to thank Asaf Kargila for pointing me to Sierpiński question. I would like to thank Yair Hayut for his help in the technical issues and for improving the style of the paper. I would like to thank Moti Gitik for his guidance and specific for his impotent suggestions.

Finally, I would like to thank the anonymous referee for their thorough reading of the paper and their critical suggestions that improved this paper significantly. In particular, their report broadened my historical and mathematical view and pointed me to some crucial issues regarding the topics of this paper, which I was not aware of.

3. Preliminaries

**Definition 1.** Let us define a class function $\alpha \mapsto \aleph_\alpha$ by: For all ordinal $\alpha$, let us define $\aleph_\alpha$ to be the cardinal of the set of all ordinals such that their cardinality is finite or equal to some $\aleph_\beta$ for some $\beta < \alpha$.

**Definition 2.** For all ordinal $\alpha$ let us define $\beth_\alpha = |V_{\omega+\alpha}|$.

**Definition 3.** For an ordinal $\alpha$ we define $\cf(\aleph_\alpha)$ to be the minimal $\aleph_\beta$ such that there exists a set $A$ of sets, such that the cardinality of each set in $A$ is less than $\aleph_\alpha$, $|A| = \aleph_\beta$ and $|\bigcup A| = \aleph_\alpha$.

For every $\aleph_\alpha$, $\cf(\aleph_\alpha)$ exists and $\cf(\aleph_\alpha) \leq \aleph_\alpha$.

**Definition 4.** A singular cardinal is $\aleph_\alpha$ in which $\cf(\aleph_\alpha) < \aleph_\alpha$.

**Definition 5.** A regular cardinal is $\aleph_\alpha$ in which $\cf(\aleph_\alpha) = \aleph_\alpha$.

The claim “$\aleph_1$ is a regular cardinal” is provable by the axiom of choice ([5, Form 34]).

**Definition 6.** For all set $X$ and an ordinal $\alpha$ let us define: $P_{\aleph_\alpha}(X) = \{Y \subseteq X \mid |Y| < \aleph_\alpha\}$.

**Definition 7.** D-infinite set is a set $A$ in which exist some $B \subset A$ ($B \neq A$) such that $|B| = |A|$.

**Definition 8.** D-finite set is a set $A$ such that $A$ is not D-infinite set.

A set $A$ is D-finite if and only if not exist an injection $f : \omega \rightarrow S$, namely $A$ has no infinite countable subset.

**Definition 9.** The axiom $CUT(\mathbb{R})$ is the axiom that for every set $A$ if $|A| = \aleph_0$ and every element in $A$ is a countable subset of $\mathbb{R}$ then $\bigcup A$ is countable.

During this paper we will use the following convention:
Definition 10. A set $A$ is large if and only if $A$ is not finite and not countable. A set $A$ is uncountable if and only if $A$ is large and D-infinite.

Definition 11. Let $A \subseteq \mathbb{R}, r \in \mathbb{R}$. Then $r$ is a condensation point of $A$ if and only if for every neighborhood $U$ of $r$, $A \cap U$ is large.

The following classic definition is due to Besicovitch:

Definition 12 (Besicovitch, [1]). A set $A \subseteq \mathbb{R}$ is concentrated on a set $D \subseteq \mathbb{R}$ if and only if for every neighborhood $G$ of $D$, $|A \setminus G| \leq \aleph_0$.

The following theorem is classical:

Theorem 1 (Existence of a condensation point). For every bounded $A \subseteq \mathbb{R}$, if $A$ is large then $A$ has at least one condensation point.

Proof. Since $A$ is bounded, there exist $a, b \in \mathbb{R}$ such that $A \subseteq [a, b]$. Let us define the following two sequences $a_n, b_n$:

- $a_0 = a, b_0 = b$
- $c_n = \frac{a_n + b_n}{2}$
- If $A \cap [a_n, c_n]$ is large, $a_{n+1} = a_n, b_{n+1} = c_n$. Otherwise, $a_{n+1} = c_n, b_{n+1} = b_n$.

Observe that for every $n \in \mathbb{N}, a_n < b_n$, $A \cap [a_n, b_n]$ is large. Also note that

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$$

and in particular,

$$\lim_{n \to \infty} (b_n - a_n) = 0.$$ 

Thus, from Cantor’s lemma, there exists a unique point $c \in \bigcap_{n \in \mathbb{N}} [a_n, b_n]$.

Let us claim that $c$ is a condensation point of $A$. Indeed, for every neighborhood $U$ of $c$ there exists $n \in \mathbb{N}$ such that $[a_n, b_n] \subseteq U$, and since $A \cap [a_n, b_n]$ is large, the claim follows. \[\square\]

Theorem 2 (Sierpiński, [12], [10], [5, Form 6]). The following are equivalent:

1. $CUT(\mathbb{R})$.
2. Every large and bounded subset of $\mathbb{R}$ has at least two condensation points (equivalently, every strongly concentrated set is countable).
3. Every large subset of $\mathbb{R}$ has a condensation point.
4. For all $A \subseteq \mathbb{R}^n$ if $A \cap B$ is countable for every bounded $B \subseteq \mathbb{R}^n$, then $A$ is countable.

Proof. The equivalence 1 $\iff$ 4 holds by Theorem 5 in [4].

1 $\implies$ 2:

Suppose that any union of countably many countable sets of real numbers is countable. Let $A \subseteq \mathbb{R}$, be a large and bounded set. From Theorem 1 it follows that there is $c \in \mathbb{R}$ which is a condensation point of $A$. Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be
sequences of real numbers, such that \(a_n\) is strictly increasing and \(\lim_{n \to \infty} a_n = c\), and \(b_n\) is strictly decreasing and \(\lim_{n \to \infty} b_n = c\).

If for every \(n \in \mathbb{N}\) we have that \(A \cap [a_n, a_{n+1}]\) and \(A \cap [b_{n+1}, b_n]\) are both not large then \(A\) is the union of at most countably many sets of reals, each one of them is at most countable, and therefore, \(A\) is at most countable, a contradiction to the assumption. Thus, there exists a natural number \(n\) such that \(A \cap ([a_n, a_{n+1}] \cup [b_{n+1}, b_n])\) is large. Thus, by Theorem \(\ref{thm:large_set}\) there exists \(c' \in \mathbb{R}\) which is a condensation point of \(A \cap ([a_n, a_{n+1}] \cup [b_{n+1}, b_n])\) and in particular of \(A\). \(c' \neq c\) because \(c' \in [a_n, a_{n+1}] \cup [b_n + 1, b_n]\) and \(c \notin [a_n, a_{n+1}] \cup [b_n + 1, b_n]\).

2 \implies 3:

We prove that the negation of 3 implies the negation of 2.

Let \(A^* \subset \mathbb{R}\) a large set with no condensation point. By Theorem \(\ref{thm:large_set}\) for every \(a < b \in \mathbb{R}\) the set \(\{x \in A^* \mid a < x < y\}\) is not large. Let us define \(A = \{x \mid x \in A^*\}\) \(A\) is large with no condensation point, and every element in \(A\) is bigger then \(-1\). There is a function \(f: \mathbb{R} \to \{x \in \mathbb{R} \mid 0 < x < 1\}\) which is an order isomorphism.

Let us define \(B = \{y \in \mathbb{R} \mid 0 < y < 1, \exists x \in A, f(x) = y\}\). \(B\) is bounded. \(B\) is a large set because \(f\) is bijection and thus \(|B| = |A|\). For every \(r \in \mathbb{R}\) if \(r \neq 1\) then \(r\) is not a condensation point of \(B\) because for every \(D \subset \mathbb{R}\) if \(D\) is closed and \(1 \in \mathbb{R}\setminus D\) then \(\{x \in A \mid f(x) \in D\}\) is not large set.

3 \implies 1:

We prove that the negation of 1 implies the negation of 3.

We assume there is an uncountable subset of \(\mathbb{R}\) which this set is a result of a countable union of countable sets. \(|\mathbb{R}| = |\{x \in \mathbb{R} \mid 0 < x < 1\}|\) therefore there exists an uncountable set \(A \subseteq \{x \in \mathbb{R} \mid 0 < x < 1\}\) and a sequence of pairwise-disjoint and countable sets \((A_n)_{n \in \omega}\) such that \(A = \bigcup_{n \in \omega} A_n\). For all \(n \in \omega\) we define \(B_n = \{x \in \mathbb{R} \mid x - n \in A_n\}\). \(|B_n| = |A_n| = \aleph_0\). Let us define \(B = \bigcup_{n \in \omega} B_n\). \(|B| = |A|\) and therefore \(B\) is uncountable. \(B\) has no condensation points because every bounded subset of \(B\) is either finite or countable.

\(\square\)

By Theorem \(\ref{thm:strongly_concentrated}\) ZFC proves that any strongly concentrated set of reals is at most countable.

**Problem 1.** (Sierpiński) \(\Box\) Is it true that one cannot prove, without using choice, that every bounded and large set \(A \subseteq \mathbb{R}\), has at least two condensation points?

In this paper we interpret this question as follows:

Does ZF prove that every large and bounded set \(A \subseteq \mathbb{R}\), has at least two condensation points? Equivalently, does ZF prove that any strongly concentrated set of reals is at most countable?

In the standard examples of failure of CUT(\(\mathbb{R}\)) such as the Feferman-Levy model \((\mathbb{R})\), the obtained strongly concentrated set of reals is not well orderable. Yair Hayut asked the following:

**Problem 2.** Is it true that one cannot prove in ZF that every bounded, well orderable and large set \(A \subseteq \mathbb{R}\), has at least two condensation points?

We will isolate two models of ZF. In both models there is a large bounded subset of \(\mathbb{R}\) with a unique condensation point. In the first one, this set is well orderable, and in the second one \(\aleph_1\) is regular.
4. Well ordered large strongly concentrated sets

In this section we will show that there is a large well orderable strongly concentrated set of reals if and only if \( \aleph_1 \) is singular and there is an injection of \( \aleph_1 \) into the reals.

**Theorem 3.** The following are equivalent:

- There is a well orderable strongly concentrated set of real numbers.
- \( \aleph_1 < 2^{\aleph_0} \) (exist a one to one function from \( \omega_1 \) to \( \mathbb{R} \)) and \( \text{cf} (\aleph_1) = \aleph_0 \).

The conjunction of the following two lemmas implies the theorem.

**Lemma 1.** Assume that there is a bounded, well orderable set \( A \subseteq \mathbb{R} \), with a unique condensation point. Then \( \text{cf} (\aleph_1) = \aleph_0 \) and \( |A| = \aleph_1 \). In particular, there is an injection \( f : \omega_1 \to \mathbb{R} \).

**Proof.** Clearly, \( A \) is uncountable, because \( A \) has a condensation point. Therefore, since \( A \) can be well ordered, \( |A| \geq \aleph_1 \).

Let us show that there is \( B \subseteq \mathcal{P} (\aleph_1) (\mathbb{R}) \) such that \( |B| = \aleph_0 \) and \( A = \bigcup B \). This is done by imitating the proof of Theorem 2.

Namely, let \( c \) be the unique condensation point of \( A \). Let \( B_n = A \setminus \left( c - \frac{1}{n}, c + \frac{1}{n} \right) \) and define \( B = \{ B_n \mid n \in \mathbb{N} \setminus \{0\} \} \). If there is a natural number \( n \) such that \( B_n \) is large, then \( B_n \) has a condensation point. This condensation point cannot be \( c \), since \( c \) is not in the closure of \( B_n \).

Let us use the following lemma:

**Claim 1.** The cardinality of a countable union of countable sets of ordinals is at most \( \aleph_1 \).

**Proof.** Let \( B \) be a set which is a countable union of countable sets of ordinals. We claim that \( |B| \leq \aleph_1 \). Suppose otherwise. Let \( B \) be a counterexample. Passing to the cardinality of \( B \), we can replace it by an \( \aleph_\alpha \) with \( \alpha \geq 2 \).

Let us fix a countable sequence of countable subsets of \( \aleph_\alpha \), \( \{ B_n^* \mid n \in \omega \} \), such that \( \aleph_\alpha = \bigcup_{n<\omega} B_n^* \). We define a sequence of sets

\[
B_n = B_n^* \setminus \left( \bigcup_{k<n} B_k^* \right)
\]

for each \( n \in \omega \).

The sets \( \{ B_n \mid n \in \omega \} \) are pairwise disjoint. Set \( \beta_n = \text{otp} (B_n) \), for every \( n < \omega \). \( \beta_n < \omega_1 \), since \( B_n \) is countable, and therefore so is \( \beta_n \). Define by induction a sequence of countable ordinals \( \langle \gamma_n \mid n < \omega \rangle \) as follows:

\( \gamma_0 = 0_\omega \), and for all \( n < \omega \), \( \gamma_{n+1} \) is the least ordinal \( \gamma \) such that \( \text{otp} (\gamma \setminus \gamma_n) \) has order type \( \beta_n+1 \). Clearly, for every \( n < \omega \), \( \gamma_n \) is countable and uniquely determined. Set \( \gamma^* = \bigcup_{n<\omega} \gamma_n \). Then \( \gamma^* \leq \aleph_1 \).

Let us denote by \( \pi_{X,Y} \) the unique order isomorphism between sets of ordinals \( X,Y \).

Finally, let us define a bijection \( f : \aleph_\alpha \to \gamma^* \) as follows: for every \( \nu < \aleph_\alpha \) there exists a unique \( n^* \) such that \( \nu \in B_n^{*} \). If \( n^* = 0 \), set \( f (\nu) = \pi_{B_0,\gamma_0} (\nu) \). Otherwise, \( n^* = n + 1 \) for some \( n < \omega \), set \( f (\nu) = \pi_{B_{n+1},\gamma_{n+1}\setminus\gamma_n} (\nu) \).

\[ \square \]
This concludes the proof of Lemma. \[\square\]

**Lemma 2.** If $\aleph_1$ is singular and there is an injection $g: \omega_1 \to \mathbb{R}$ then there is a bounded, well orderable, set $A \subseteq \mathbb{R}$ with a unique condensation point.

**Proof.** Identify $\mathbb{R}$ with $\omega^2$. By the assumption of the theorem, there is a function

$$\nu: \omega \to \omega_1$$

such that for all $n < m$, $\nu(n) < \nu(m)$ and $\bigcup_{n \in \mathbb{N}} \nu(n) = \omega_1$. We define a function $\rho: \omega_1 \to \omega$ by

$$\rho(\alpha) = \min\{n \in \omega \mid \nu(n) > \alpha\}.$$

Let $g: \omega_1 \to \mathbb{R}$ be an injection. Let us define a function $f: \omega_1 \to \mathbb{R}$ by:

$$f(\alpha)(n) = \begin{cases} 1 & \rho(\alpha) > n \\ 0 & \rho(\alpha) = n \\ g(\alpha)(n - \rho(\alpha) - 1) & \rho(\alpha) < n \end{cases}$$

Thus the real number $f(\alpha)$ is obtained by adding $\rho(\alpha)$ 1-s and a single zero at the beginning of $g(\alpha)$. $f$ is an injection since for all $\alpha < \beta < \omega_1$, $\rho(\alpha) \leq \rho(\beta)$.

If $\rho(\alpha) < \rho(\beta)$ then

$$f(\alpha)(\rho(\alpha)) = 0 \neq 1 = f(\beta)(\rho(\alpha))$$

and if $\rho(\alpha) = \rho(\beta)$ then since $g$ is one to one there is some $n \in \omega$ such that $g(\alpha)(n) \neq g(\beta)(n)$.

Let $A$ be im$f$.

$A \subseteq \mathbb{R}$ is a large set (since $|A| = \aleph_1$). By Theorem, $\aleph_1$ $A$ has a condensation point.

For every $y \in \mathbb{R}$ if there is $n \in \omega$ such that $y(n) = 0$ then there is some $\alpha \in \omega_1$ such that for every $\beta < \omega_1$, $\beta > \alpha$ and every $k < n + 2$, $f(\beta)(k) = 1$. Thus, $y$ is not a condensation point of $A$.

The assumptions of lemma hold in a Feferman-Lévy model. Namely, let $V$ be a well founded model of ZFC such that $\aleph_\omega < 2^{\aleph_0}$ (this can be arranged, for example, by adding $\aleph_\omega$ Cohen reals). Use the Feferman-Lévy construction over $V$ (See Chapter 10) to get a model $M$ of ZF. $M \models \aleph_1^M = \aleph_\omega$. In $M$, there is an injection $f: \omega_1 \to \mathbb{R}$ and $\aleph_1$ is a singular cardinal.

5. **LARGE STRONGLY CONCENTRATED SETS WITH REGULAR $\aleph_1$**

By the previous section, if $\aleph_1$ is singular and injects into the reals, then there is a large, well-orderable and strongly concentrated set. The existence of a large strongly concentrated set is consistent with the regularity of $\aleph_1$. This statement for example holds in Sageev’s Model. In this section we represent other way to get a model with this feature. One notable difference between the method which is introduced in the previous section and the method that we introduce in this section that while the method of the previous section collapse all uncountable cardinals below $\beth_\omega$ to $\aleph_0$, the current method preserves all cardinals above $\beth_1$ as cardinals.

Let us start with a well founded model of ZFC, $W$. In particular, $(2^{\aleph_0})^+$ is a regular cardinal in $W$.

Let $V$ be a model of ZFA + AC and let $A$ be the set of all atoms in $V$. Let us assume that $|A| > \aleph_0$. 


Definition 14. Let $S$ to be the group of all bijection $\pi : A \to A$.

Definition 15. For $\pi \in S$ and $x \in V \setminus A$ we define $\pi(x)$ recursively as $\pi(x) = \{\pi(t) \mid t \in x\}$.

Definition 16. For all $x \in V$ we define $\text{sym}_S(x) = \{\pi \in S \mid \pi(x) = x\}$.

Definition 17. For all $C \in P_{\aleph_1}(A)$ we define: $S_C = \{\pi \in S \mid \forall a \in C, \pi(a) = a\}$.

Definition 18. We define $F = \{H \leq S \mid \exists C \in P_{\aleph_1}(A), S_C \leq H\}$, $F$ is a filter of subgroups over $S$.

Definition 19. We define $\text{mys} = \{x \in V \mid \text{sym}_S(x) \in F\}$. $\text{mys}$ is the class of all symmetric elements. We define $B = \{x \in V \mid TC(x) \subseteq \text{mys}\}$. $B$ is the class of all hereditary symmetric elements.

By a well known theorem of Fraenkel (see [7]) $B$ is a model of $ZFA$.

Definition 20. For all $x \in V$ we define $St(x) = \{C \in P_{\aleph_1}(A) \mid S_C \subseteq \text{sym}_S(x)\}$.

Work in $B$.

Definition 21. We define a forcing $Q = \{h : D \to \{0, 1\} \mid D \in P_{\aleph_0}(\omega)\}$. We say that $h_0$ is stronger than $h_1$ or equal to $h_1$ if and only if $\text{dom}h_1 \subseteq \text{dom}h_0$ and $\forall d \in \text{dom}h_1, h_0(d) = h_1(d)$.

$Q$ is essentially the Cohen forcing.

Definition 22. We define $I = \{h \in Q \mid \exists n \in \text{dom}h, h(n) = 0\}$.

Definition 23. Let $n \in \omega$.

We define $P_n$ to be the set of all functions $f : A \to Q$ such that:

1. For all $a \in A$, $\text{dom}f(a) = n$.
2. $f^{-1}(I) \in P_{\aleph_1}(A)$.
3. $\forall t : n \to \{0, 1\}$, there are infinitely many $a \in A$ such that $f(a) = t$.

Definition 24. We define a forcing $P = \bigcup_{n \in \omega} P_n$.

We order $P$ by:

$\forall f_0, f_1 \in P, f_0 \leq f_1 \iff \forall a \in A, f_0(a) \leq_Q f_1(a)$.
Definition 25. For all $n \in \omega$ we define
\[ D_n = \bigcup_{k \in \omega \setminus n} P_k. \]

Definition 26. For all $C \in P_{\aleph_1}(A)$ we define
\[ Z_C = \{ f \in P \mid \forall a \in C, f(a) \in I \}. \]

Let $G$ be a generic filter for $P$.

Definition 27. For all $a \in A$ we define
\[ G_a = \{ h \in Q \mid \exists f \in G, f(a) = h \}. \]

Let us define $g_a = \bigcup G_a$.

Definition 28. For all $C \in P_{\aleph_1}(A)$, let $g^*_C : C \to \{0, 1\}$, be the function $g^*_C(c) = g_c$, for every $c \in C$. Let $g_C$ be a name which is forced by the weakest condition to be $g^*_C$.

Definition 29. For all $C \in P_{\aleph_1}(A)$ we define $Res_C^c : I \to \omega \cup \{\aleph_0\}$ which for all $i \in I$, $Res_C^c(i) = |\{a \in A \setminus C \mid i \in g^*(a)\}|$. Let $Res_C$ a name which is forced by every condition to be $Res_C^c$.

Definition 30. We define a function $num : P \to \omega$ by
\[ num(f) = \min \{ n \in \omega \mid f \in D_n \}. \]

Lemma 3. $2^{\aleph_0}$ of $B$ is countable in $B[G]$.

Proof. Work in $B$. Fix a sequence $\langle \ell_k \mid k < \omega \rangle$ of injective functions from $\omega$ to $A$ with disjoint images.

Claim 2. For every $h : \omega \to \{0, 1\}$ the following set is dense:
\[ D_h = \{ f \in P \mid \exists n < num(f), m < \omega, \forall k < \omega, f(\ell_m(k)) (n) = h(k) \}. \]

Proof. Let $f^* \in P$ and $n = num(f^*)$. Let $T_n = \{0, 1\}$, the set of all functions $\ell : n \to \{0, 1\}$. Define a function $F : \omega \to \mathcal{P}(T_n)$ as follows:
\[ F(k) = \{ t \in T_n \mid \exists a \in im \ell_k, f^*(a) = t \}. \]

Note that $F(k) = \text{im} f^* \circ \ell_k$, and hence it is never empty. $F$ defines a partition of $\omega$ into finitely many pieces. Hence at least one of them must be infinite. So there is $x \subseteq T_n$ and an infinite $Y \subseteq \omega$ such that for every $k \in Y$,
\[ F(k) = x. \]

Which means, for every $t \in x$ and $k \in Y$ there is $a \in im \ell_k$ such that
\[ f^*(a) = t. \]

Let $k^* = \min Y$. Note that for any $t \in T_n$ there are infinitely many $a \in A \setminus im \ell_{k^*}$ such that $f^*(a) = t$. Extend $f^*$ to a condition $f \in P_{n+1}$ as follows: for all $a \in im \ell_{k^*}$, $f(a)(n) = h(\ell_{k^*}^{-1}(a))$. For elements in $A \setminus im \ell_{k^*}$ define $f$ such that requirement 3 in Definition 23 will be satisfied. This is possible, since for every $t \in T_n$ there are infinitely many members of $A$ which are not in im $\ell_{k^*}$ such that $f^*(a) = t$. 

\[ \Box \]
Lemma 4. For all $n \in \omega$ and $f^* \in P_n$ there is $f \in P_{n+1}$ such that $f$ is stronger than $f^*$.

Proof. We prove the lemma by induction on $f$.

For $n = 0$ the claim is true by the definition of $D_0 = P$.

We assume the validity of the claim for $n$. Let $f^* \in P_n$ by the induction hypothesis there exists $f^+ \in D_n$ such that $f^+$ is stronger than $f^*$ or equal to $f^*$.

If $\text{num}(f^+) > n$ we define $f = f^+$, and get that $f \in D_{n+1}$.

If $\text{num}(f^+) = n$ then by lemma 3 there is a condition $f \in P_{n+1}$ stronger than $f^+$.

Thus $f \in D_{n+1}$ and $f$ is stronger than $f^*$.

We conclude that $\forall a \in A, n \in \omega, \exists f \in G$ such that $n \in \text{dom } f(a)$.

Lemma 5. In $B$, for all function $\psi: P \to \{2^{\mathbb{R}_0}\}^+$, $|\text{im } \psi| \leq 2^{\mathbb{R}_0}$.

Proof. Let $C \subseteq \text{St}(\psi)$.

We define a function

$$\vartheta_0: P \to \{h_0: C \to Q\}$$

by

$$\vartheta_0(f)(c)(n) = f(c)(n).$$

We define

$$\vartheta_1: P \to \{h_1: I \to \omega \cup \{\aleph_0\}\}$$

by

$$\vartheta_1(f)(i) = |\{a \in A \setminus C \mid f(a) = i\}|.$$ 

We define

$$\vartheta: P \to \{h_0: C \to Q\} \times \{h_1: I \to \omega \cup \{\aleph_0\}\}$$

by

$$\vartheta(f) = (\vartheta_0(f), \vartheta_1(f)).$$

Lemma 7. For all $f_0, f_1 \in P$ if $\vartheta(f_0) = \vartheta(f_1)$ then $\psi(f_0) = \psi(f_1)$.

Proof. We define

$$\Delta_0 = \{a_0 \in A \setminus C \mid f_0(a_0) \in I\}$$

and

$$\Delta_1 = \{a_1 \in A \setminus C \mid f_1(a_1) \in I\}.$$ 

$\Delta_0, \Delta_1$ are finite or countable and for all $i \in I$,

$$|\{a_0 \in \Delta_0 \mid f_0(a_0) = i\}| = |\{a_1 \in \Delta_1 \mid f_1(a_1) = i\}|$$

since $\vartheta_1(f_0) = \vartheta_1(f_1)$. Let $D \subseteq A \setminus (C \cup \Delta_0 \cup \Delta_1)$ be countable. Let us define a permutation of $D, \Theta$, such that for all $a \in D$,

$$f_0(a) = f_1(\Theta(a)).$$

Let us extend $\Theta$ to a bijection $\Lambda: A \to A$ by defining $\Lambda(a) = a$ for all $a \notin D$. 

In particular, for all $a \in C$, $A (a) = a$. Therefore:

$$A (\psi) = \psi.$$ 

and

$$A (f_0) = f_1.$$ 

and

$$\psi (f_1) = A (\psi) (A (f_0)) = A (\psi (f_0)) = \psi (f_0).$$

We conclude that $| \text{im} \psi | \leq | \{ h_0 : C \to Q \} | \cdot | \{ h_1 : I \to \omega + 1 \} | = 2^{\aleph_0}$. 

**Lemma 8.** In $\mathcal{B}$, for all $P$-name $\Upsilon$ and $f^* \in P$ if

$$f^* \Vdash \Upsilon = \langle \alpha_n \mid n \in \omega \rangle, \ \alpha_n \in \big( (2^{\aleph_0})^+ \big)^{\mathcal{B}},$$

then $f^* \Vdash \bigcup_{n \in \omega} \alpha_n \in \big( (2^{\aleph_0})^+ \big)^{\mathcal{B}}$.

**Proof.** For all $n \in \omega$ we define

$$P^\Upsilon_n = \left\{ f \in P \mid \exists \alpha \in \big( (2^{\aleph_0})^+ \big)^{\mathcal{B}}, f \Vdash \alpha_n = \bar{\alpha} \right\}$$

and

$$\Psi_n : P^\Upsilon_n \to \big( (2^{\aleph_0})^+ \big)^{\mathcal{B}}$$

such that for all $f \in P^\Upsilon_n$,

$$f \Vdash \alpha_n = \Psi_n (f).$$

By lemma $| \text{im} \Psi_n | \leq 2^{\aleph_0}$ (in $\mathcal{B}$), since by the regularity of $(2^{\aleph_0})^+$,

$$\alpha_n \leq \beta_n = \sup \text{im} \Psi_n < (2^{\aleph_0})^+.$$ and

$$\bigcup_{n \in \omega} \beta_n < (2^{\aleph_0})^+.$$ 

**Lemma 9.** $\mathcal{N}_{1}^{B[G]} = \big( (2^{\aleph_0})^+ \big)^{\mathcal{B}}$ and it is a regular cardinal in the generic extension.

**Proof.** By lemma $\mathcal{N}_{1}^{B[G]} \geq \big( (2^{\aleph_0})^+ \big)^{\mathcal{B}}$.

By lemma $\mathcal{N}_{1}^{B[G]} \leq \big( (2^{\aleph_0})^+ \big)^{\mathcal{B}}$ is a regular cardinal in $\mathcal{B}[G]$. 

**Lemma 10.** For all $C \in P_{\aleph_{1}} (A)$ the set $Z_C$ is dense.

**Proof.** Let $f^*$ be a condition in $P$. Let us denote $n = \text{num} (f^*)$ and

$$E = \{ a \in C \mid \forall k \in n, f^* (a) (k) = 1 \}.$$

For all $t : n \to \{ 0, 1 \}$ we choose $\ell_t \subseteq \{ a \in A \mid f^* (a) = t \}$ such that $| \ell_t | = \aleph_0$,

$\{ a \in A \mid f^* (a) = t \} \setminus \ell_t$ is infinite, and $\ell = \bigcup_{t : n \to \{ 0, 1 \}} \ell_t \cup E.$
We define $f \in P$ by:

$$f(a)(k) = \begin{cases} f^*(a)(k) & k \neq n \\ 0 & k = n, a \in \ell \\ 1 & k = n, a \notin \ell \end{cases}$$

For all $a \in A, \text{dom} f^*(a) \cap \{n\} = \emptyset$ and by first line of the definition of $f$ we get that if $k \in \text{dom} f^*(a)$ then $(k \neq n) f(a)(k) = f^*(a)(k)$ therefore $f$ is stronger than $f^*$.

For all $a \in C$ if $a \notin E$ then $\exists k \in n (k \neq n)$ such that $f(a)(k) = f^*(a)(k) = 0$ and if $a \in E$ then $f(a)(n) = 0$. \hfill\square

**Theorem 4.** $\forall a, b \in A, a \neq b \Rightarrow \exists f \in G, n \in \omega, f(a)(n) = 0, f(b)(n) = 1$.

**Proof.** Let $f^* \in P$ by definition exist $n \in \omega$ such that $f^* \in P_n$, we chose $f^+ \in P_{n+1}$ such that $f^+$ stronger than $f^*(\text{exist by lemma 4})$.

We define $f : A \rightarrow Q$ by

$$f(c)(k) = \begin{cases} f^+(c)(k) & c \in A - \{a, b\} \lor k \neq n \\ 0 & c = a \land k = n \\ 1 & c = b \land k = n \end{cases}$$

\hfill\square

**Theorem 5.** For all $n \in \omega$ the set $\{a \in A : \exists f \in G, k \in n, f(a)(k) = 0\}$ is a countable set in the ground model.

**Proof.** By lemma $5$ the set $D_n$ is dense, then exist $f^* \in D_n \cup G$, by definition of the forcing $|\{a \in A \mid \exists k \in n, f^*(a) = 0\}| = \aleph_0$.

Let $a \in A$ and $f \in G$ and if exist $k \in n$ such that $f(a)(k) = 0$ then $(f \parallel f^* \text{and } k \in \text{dom} f(a)) f^*(a)(k) = 0$. \hfill\square

**Theorem 6.** $A$ is not countable in the generic extension.

**Proof.** Let $f^*$ be a condition in $P$ and let $h$ be a $P$-name such that $f^* \Vdash h : \omega \rightarrow A$, and let $C \in \text{St}(h)$.

Let $G \subseteq P$ generic such that $f^* \in G$. By lemma $10$ there exists a condition $f \in ZC \cap G$ such that $f$ is stronger than $f^*$.

Claim 3. $f \parallel \text{Im} h \subseteq X_f = \{a \in A \mid \exists n \in \omega, f(a)(n) = 0\}$.

**Proof.** Suppose otherwise.

By the definition of $P, X_f$ is at most countable and in particular, $X_f \neq A$. By the assumption, there is $a \in A \setminus X_f, k, n \in \omega$ and $f^+ \in D_n$ stronger than $f$ such that $f^+ \Vdash h(k) = a$.

By the definition of $P$ the set

$$S_{f^+, a} = \{b \in A \mid f^+(b) = f^+(a)\}$$

is an infinite set.

We claim that $S_{f^+, a} \cap X_f = \emptyset$. For all $b \in S_{f^+, a}$ and for all $\ell \in \text{num}(f)$ then $f(b)(\ell) = f(a)(\ell) = 1$ since $a \notin X_f$. Therefore $f^+(b)(\ell) = 1$ and for all $c \in X_f$ exists $\ell \in \text{num}(f)$ such that $f(c)(\ell) = 0$. Thus $f^+(c)(\ell) = 0$.

Since $C \subseteq X_f$, we conclude that $S_{f^+, a} \cap C = \emptyset$. 

We define \( t \in S \) by

\[
t(c) = \begin{cases} 
    b & c = a \\
    a & c = b \\
    c & c \notin \{a, b\}
\end{cases}
\]

\( t \) is an automorphism of \( V \) and since

\( f^+ \models h(k) = a \)

we get that

\( t(f^+) \models t(h)(t(k)) = t(a) \).

Moreover, since

\begin{itemize}
    \item \( t(f^+) = f^+ \).
    \item \( t(h) = h \).
    \item \( t(k) = k \)
\end{itemize}

we conclude that

\( f^+ \models h(k) = b \)

contradicting the fact that \( h \) is a function. \( \square \)

Working in \( V \), we conclude that \( \text{Im} \ h \) is forced by \( f^+ \) to be a subset of the countable set \( X_f \). In particular, \( \text{Im} \ h \neq A \). \( \square \)

By the general theory of \( ZFA \), there is a model of \( ZF \) with similar properties (see [9, 8]). For completeness, let us describe a concrete way to obtain such a model of \( ZF \) in our case:

**Definition 31.** In \( B[G] \) we define

\begin{itemize}
    \item \( C_0 = \emptyset \).
    \item For a successor ordinal \( \alpha = \beta + 1 \), \( C_\alpha = \mathcal{P}(C_\beta) \).
    \item For a limit ordinal \( \alpha \), \( C_\alpha = \bigcup_{\beta < \alpha} C_\beta \).
\end{itemize}

Let

\[
C = \{ x \in B[G] \mid \exists \alpha \in \text{Ord}, x \in C_\alpha \}.
\]

**Theorem 7.** It is consistent with \( ZF \) that \( \aleph_1 \) is regular and there is a large set \( A^* \subseteq R \) which is bounded and has a single condensation point.

**Proof.** Let \( A^* = \{ g_a \mid a \in A \} \). Note that \( A^* \in C \).

\( C \models ZF \). Working in \( C \), there exists a set \( A^* \subseteq R \) such that \( A^* \subseteq [0, 1] \), \( |A^*| > \aleph_0 \), and for all \( r \in R \) if \( r \in (0, 1) \) then \( |A^* \cap [0, r]| = \aleph_0 \).

Thus, \( A^* \subseteq R \) is bounded and large and the point 1 is the unique condensation point of \( A^* \).

By lemma [8] \((\aleph_1)^C = (\aleph_1)^{B[G]} = \left( (2^{\aleph_0})^+ \right)^B \) is a regular cardinal in \( B[G] \) and therefore also in \( C \). \( \square \)

**Corollary.** It is consistent that exist some set \( A \subseteq R \) with unique condensation point and any set \( A \subseteq R \) with unique condensation point not have a well order.

**Proof.** By theorem [7] it is consistent that exist some set \( A \subseteq R \) with unique condensation point and \( \aleph_1 \) is regular and by theorem [8] if \( \aleph_1 \) is regular then any set \( A \subseteq R \) with unique condensation point not have a well order. \( \square \)


References

[1] A. S. Besicovitch, Concentrated and Rarified Sets of Points, Acta Math. 62 (1933), no. 1, 289–300. MR 1555386
[2] Paul Cohen, The Independence of the Continuum Hypothesis, Proc. Nat. Acad. Sci. U.S.A. 50 (1963), 1143–1148. MR 0157890
[3] Solomon Feferman and Azriel Levy, Independence Results in Set Theory by Cohen’s Method II, Notices of the American Mathematical Society 10 (1963), 593.
[4] Paul Howard, Kyriakos Keremedis, Jean E. Rubin, Adrienne Stanley, and Eleftherios Tachtisis, Non-constructive properties of the real numbers, MLQ Math. Log. Q. 47 (2001), no. 3, 423–431. MR 1847458
[5] Paul Howard and Jean E. Rubin, Consequences of the Axiom of Choice, Mathematical Surveys and Monographs, vol. 59, American Mathematical Society, Providence, RI, 1998, With 1 IBM-PC floppy disk (3.5 inch; WD). MR 1637107
[6] T. Jech, The Axiom of Choice, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, Studies in Logic and the Foundations of Mathematics, Vol. 75. MR 0396271
[7] T. Jech, Set Theory, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The Third Millennium Edition, Revised and Expanded. MR 1940513
[8] T. Jech and A. Sochor, Applications of the Θ-model, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 14 (1966), 351–355. MR 0228337
[9] T. Jech, On Θ-Model of the Set Theory, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 14 (1966), 297–303. MR 0202579
[10] Gregory H. Moore, Zermelo’s Axiom of Choice, Studies in the History of Mathematics and Physical Sciences, vol. 8, Springer-Verlag, New York, 1982, Its Origins, Development, and Influence. MR 679315
[11] Gershon Sageev, An independence result concerning the axiom of choice, Ann. Math. Logic 8 (1975), 1–184. MR 0366668
[12] W. Sierpiński, L’axiome de M. Zermelo et son rôle dans la théorie des ensembles et de l’analyse, Bulletin international de l’Académie des sciences de Cracovie, Classe des Sciences Mathématiques et naturelles, Série A. (1918), 97–152.