EISENSTEIN SERIES AND CONVOLUTION SUMS

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Abstract. We compute Fourier series expansions of weight 2 and weight 4 Eisenstein series at various cusps. Then we use results of these computations to give formulas for the convolution sums

\[\sum_{a+pb=n} \sigma(a)\sigma(b), \sum_{p_1a+p_2b=n} \sigma(a)\sigma(b)\]

and \(\sum_{a+p_1p_2b=n} \sigma(a)\sigma(b)\) where \(p, p_1, p_2\) are primes.

Keywords: sum of divisors function, convolution sums, Eisenstein series, Dedekind eta function, eta quotients, modular forms, cusp forms, Fourier series.

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1. Introduction

Let \(\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{C}\) and \(\mathbb{H}\) denote the sets of positive integers, non-negative integers, integers, rational numbers, complex numbers and the upper half plane, respectively. Throughout the paper we let \(z \in \mathbb{H}\) and \(q = e^{2\pi iz}\). Let \(N \in \mathbb{N}\). Let \(\Gamma_0(N)\) be the modular subgroup defined by

\[\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1, \ c \equiv 0 \pmod{N} \right\}.
\]

We note that the matrices

\[(1.1) \quad S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

are the generators of \(\Gamma_0(1) = SL_2(\mathbb{Z})\). An element \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(1)\) acts on \(\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}\) by

\[M(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty, \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}
\]

Let \(k \in \mathbb{N}\). We write \(M_k(\Gamma_0(N))\) to denote the space of modular forms of weight \(k\) for \(\Gamma_0(N)\), and \(E_k(\Gamma_0(N))\) and \(S_k(\Gamma_0(N))\) to denote the subspaces of Eisenstein forms and cusp forms of \(M_k(\Gamma_0(N))\), respectively. It is known (see for example [23 p. 83] and [22]) that

\[(1.2) \quad M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)).\]
Let \( A_c = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix} \), then the Fourier series expansion of \( f(z) \in M_k(\Gamma_0(N)) \) at the cusp \( \frac{1}{c} \in \mathbb{Q} \cup \{\infty\} \) is given by the Fourier series expansion of \( f(A_c^{-1}z) \) at the cusp \( \infty \), see [16, pg. 35]. If the Fourier series expansion of \( f(z) \) at the cusp \( \frac{1}{c} \) is given by the infinite sum

\[
(cz + 1)^k \sum_{n \geq 0} a_n e^{2\pi inz},
\]

then we use the notation \([n]_c f(z) = a_n\). It is to use \([n]\), instead of \([n]_0\), at the cusp \( \infty \) (= \(1/0\)).

For \( k \in \mathbb{N}_0 \) and \( n \in \mathbb{N} \) we define the sum of divisors function by

\[
\sigma_k(n) = \sum_{d \mid n} d^k
\]

where \( d \) runs through positive divisors of \( n \). If \( n \notin \mathbb{N} \) we set \( \sigma(n) = 0 \). When \( k = 1 \), we write \( \sigma(n) \) instead of \( \sigma_1(n) \). We define the weight 2 and the weight 4 Eisenstein series by

\[
\begin{align*}
E_2(z) &= 1 - 24 \sum_{n \geq 1} \sigma(n) q^n, \\
E_4(z) &= 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,
\end{align*}
\]

respectively. Let \( d \in \mathbb{N} \), we denote \( E_2(dz) \) by \( E_{(2,d)}(z) \) and \( E_4(dz) \) by \( E_{(4,d)}(z) \). For convenience let us define \( L_d(z) = E_2(z) - dE_{(2,d)}(z) \). It is known that

\[
\begin{align*}
L_d(z) \in E_2(\Gamma_0(d)), \\
E_4(dz) \in E_4(\Gamma_0(d)),
\end{align*}
\]

see [23, Theorem 5.8].

Let \( r, s \in \mathbb{N} \). We define

\[
W(r, s; n) = \sum_{\substack{a, b \in \mathbb{N}, \\
ar + bs = n}} \sigma(a) \sigma(b)
\]

to be the convolution of sum of divisors function. The formula

\[
W(1, 1; n) = \frac{5}{12} \sigma_3(n) + \frac{1 - 6n}{12} \sigma(n)
\]

appears in the works of Besge, Glaisher and Ramanujan, see [8, [12, 20], respectively. Since then \( W(r, s; n) \) has been calculated for various \((r, s)\). In the table below we grouped the known results according to nature of \( r \) and \( s \), together with the references.
In this paper we give formulas for $W(1, p; n)$, $W(p_1, p_2; n)$ and $W(1, p_1p_2; n)$. In Theorem 2.2 we give the precise expression for the Eisenstein part of the formulas for $W(1, p; n)$, $W(p_1, p_2; n)$ and $W(1, p_1p_2; n)$ for all primes $p, p_1, p_2$ such that $p_1 \neq p_2$. These formulas generalize the previously known formulas in rows 2–4 referenced in Table 1. Our method proves the existence of the cusp forms satisfying the formulas but fails to provide a general expression for the cusp part. Later on in the paper we will give the cusp part of the formulas for $W(r, s; n)$ with $(r, s) = (1, 2), (1, 3), (1, 5), (1, 7), (1, 11), (2, 3), (2, 5), (2, 7), (3, 5), (1, 6), (1, 10), (1, 14), (1, 15)$ in terms of linear combinations of eta quotients.

The structure of the paper is as follows. In the next section we give the statements of main results. In the third section we derive Fourier series expansions of weight 2 and weight 4 Eisenstein series at various cusps. Then, in Section 4 we use the first terms of Fourier series expansions of Eisenstein series at certain cusps to prove the main results. In Section 5 we give a description for the cusp forms $C_{(r,s)}(z)$ with $(r, s) = (1, 2), (1, 3), (1, 5), (1, 7), (1, 11), (2, 3), (2, 5), (2, 7), (3, 5), (1, 6), (1, 10), (1, 14), (1, 15)$ in terms of eta quotients. Finally in Section 6 for the interested reader, we describe how to extend the results of this paper to give formulas for $W(r, s; n)$ with $\text{lcm}(r, s)$ a square-free, two times a square-free number or four times a square-free number.

2. MAIN RESULTS

In this section we state the main results.

**Theorem 2.1.** Let $p$ be a prime. Then there exists a cusp form $C_{(1,p)}(z) \in S_4(\Gamma_0(p))$ such that

$$
(2.1) \quad (L_p(z))^2 = \frac{(p-1)^2}{p^2+1}E_4(z) + \frac{p^2(p-1)^2}{p^2+1}E_4(pz) + C_{(1,p)}(z).
$$

\[3\]
Let \( p_1, p_2 \) be primes such that \( p_1 \neq p_2 \). Then there exists cusp forms \( C_{(p_1,p_2)}(z) \), \( C_{(1,pip2)}(z) \in S_4(\Gamma_0(p_1p_2)) \) such that

\[
L_{p_1}(z)L_{p_2}(z) = \frac{(p_1^2 - p_1 + 1)(p_2^2 - p_2 + 1)}{(p_1^2 + 1)(p_2^2 + 1)} E_4(z) - \frac{p_1^3(p_2 - p_2 + 1)}{(p_1^2 + 1)(p_2^2 + 1)} E_4(p_1z)
\]
\[
- \frac{p_2^3(p_1 - p_1 + 1)}{(p_1^2 + 1)(p_2^2 + 1)} E_4(p_2z) + \frac{p_1p_2^3}{(p_1^2 + 1)(p_2^2 + 1)} E_4(p_1p_2z) + C_{(p_1,p_2)}(z).
\]

(2.3)

\[
(L_{p_1p_2}(z))^2 = \left(1 - \frac{2p_1p_2}{(p_1^2 + 1)(p_2^2 + 1)}\right) E_4(z) - \frac{2p_1^3p_2}{(p_1^2 + 1)(p_2^2 + 1)} E_4(p_1z)
\]
\[
- \frac{2p_1p_2^3}{(p_1^2 + 1)(p_2^2 + 1)} E_4(p_2z) + \left(p_1^2p_2^2 - \frac{2p_1^3p_2^3}{(p_1^2 + 1)(p_2^2 + 1)}\right) E_4(p_1p_2z)
+ C_{(1,pip2)}(z).
\]

We compare the coefficients of \( q^n \) on both sides of equations (2.2)–(2.3) to get the following theorem.

**Theorem 2.2.** Let \( p, p_1, p_2 \) be primes such that \( p_1 \neq p_2 \), then for all \( n \in \mathbb{N} \) we have

\[
W(1, p; n) = \frac{5}{12(p^2 + 1)}\sigma_3(n) + \frac{5p^2}{12(p^2 + 1)}\sigma_3(n/p) + \frac{p - 6n}{24p}\sigma(n)
+ \frac{1}{24}\sigma(n/p) - \frac{1}{1152p}[n]C_{(1,p)}(z),
\]

\[
W(p_1, p_2; n) = \frac{5}{5p_1^2} \sigma_3(n) + \frac{5p_2^2}{5p_2^2} \sigma_3(n/p_1)
+ \frac{1}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n/p_2) + \frac{5p_1^2p_2^2}{12(p_2^2 + 1)}\sigma_3(n/p_1p_2)
+ \frac{p_2 - 6n}{24p_2}\sigma(n/p_1) + \frac{p_1 - 6n}{24p_1}\sigma(n/p_2) - \frac{1}{1152p_1p_2}[n]C_{(p_1,p_2)}(z).
\]

\[
W(1, p_1p_2; n) = \frac{5}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n) + \frac{5p_1^2}{12(p_1^2 + 1)(p_2^2 + 1)}\sigma_3(n/p_1)
+ \frac{1}{12(p_2^2 + 1)}\sigma_3(n/p_2) + \frac{5p_1^2p_2^2}{12(p_2^2 + 1)}\sigma_3(n/p_1p_2)
+ \frac{p_1p_2 - 6n}{24p_1p_2}\sigma(n) + \frac{1 - 6n}{24}\sigma(n/p_1p_2) - \frac{1}{1152p_1p_2}[n]C_{(1,pip2)}(z),
\]

where \( C_{(1,p)}(z) \), \( C_{(1,pip2)}(z) \) and \( C_{(p_1p_2)}(z) \) are the cusp forms from Theorem 2.1.

Note that Chan and Cooper in [9] gave the equation for \( W(1, p; n) \), valid for \( p = 3, 7, 11, 23 \). The closed form they gave for Eisenstein part of the formula is
the same as Eisenstein part of the formula for $W(1,p;n)$ given in the previous theorem.

3. Fourier series expansions of weight 2 and weight 4 Eisenstein series at various cusps

In this section we find the Fourier series expansion of weight 2 and weight 4 Eisenstein series at various cusps. The results of this section will be used to prove Theorem 2.1. The transformation formula for $E_2(z)$ under the matrices $T$ and $S$ are given by

(3.1) \[ E_2(Tz) = E_2(z+1) = E_2(z), \]

(3.2) \[ E_2(Sz) = E_2(-1/z) = z^2 \left( E_2(z) - \frac{1}{2\pi iz} \right), \]

see [15, Prop. 2.9].

Theorem 3.1. Let $1 < t \in \mathbb{N}$. The Fourier series expansion of $L_t(z)$ at cusp $1 \in \mathbb{Q}$ is given by

(3.3) \[ L_t(A_1^{-1}z) = (z+1)^2 \left( E_2(z) - \frac{1}{t} E_2 \left( \frac{z+1}{t} \right) \right). \]

Let $p_1, p_2$ be prime. The Fourier series expansions of $L_{p_1p_2}(z)$ at cusps $1/p_1, 1/p_2 \in \mathbb{Q}$ are given by

(3.4) \[ L_{p_1p_2}(A_{p_1}^{-1}z) = (p_1z+1)^2 \left( E_2(z) - \frac{p_1}{p_2} E_2 \left( \frac{p_1z+1}{p_2} \right) \right) \]

(3.5) \[ L_{p_1p_2}(A_{p_2}^{-1}z) = (p_2z+1)^2 \left( E_2(z) - \frac{p_2}{p_1} E_2 \left( \frac{p_2z+1}{p_1} \right) \right), \]

respectively.

Proof. Let $t \in \mathbb{N}$ and the matrices $S$ and $T$ are as in (1.1). We have

\[ E_{(2,t)}(A_1^{-1}z) = E_2 \left( S^2 T^t S \left( \frac{z+1}{t} \right) \right) \]

\[ = E_2 \left( T^t S \left( \frac{z+1}{t} \right) \right) \]

\[ = E_2 \left( S \left( \frac{z+1}{t} \right) \right) \]

\[ = \left( \frac{z+1}{t} \right)^2 \left( E_2 \left( \frac{z+1}{t} \right) - \frac{t}{2\pi i (z+1)} \right). \]

where in the first, second and third steps we use $E_2(S^2(z)) = E_2(z)$, $E_2(T^t(z)) = E_2(z)$ and (3.2), respectively. Let $1 < t \in \mathbb{N}$. Then we use (3.6) to get

\[ L_t(A_1^{-1}z) = E_2(A_1^{-1}z) - tE_{(2,t)}(A_1^{-1}z) \]
\[
(z + 1)^2 \left( E_2(z + 1) - \frac{1}{2\pi i (z + 1)} \right) \\
- t \left( \frac{z + 1}{t} \right)^2 \left( E_2 \left( \frac{z + 1}{t} \right) - \frac{t}{2\pi i (z + 1)} \right) \\
= (z + 1)^2 \left( E_2(z + 1) - \frac{1}{t} E_2 \left( \frac{z + 1}{t} \right) \right),
\]
which proves (3.3). Similarly, by using \( E_2(T^t(z)) = E_2(z) \) and (3.2) we find
\[
E_2(A^{-1}_{p_1} z) = E_2(ST^{-p_1} S z) \\
= \left( \frac{-p_1 z - 1}{z} \right)^2 \left( E_2(S z) - \frac{z}{2\pi i (-p_1 z - 1)} \right) \\
= (p_1 z + 1)^2 \left( E_2(z) - \frac{p_1}{2\pi i (p_1 z + 1)} \right),
\]
and by \( E_2(S^2(z)) = E_2(z) \), \( E_2(T^t(z)) = E_2(z) \) and (3.2) we find
\[
E_{(2,p_1p_2)}(A^{-1}_{p_1} z) = E_2 \left( S^2 T^{-p_2} S \left( \frac{p_1 z + 1}{p_2} \right) \right) \\
= E_2 \left( S \left( \frac{p_1 z + 1}{p_2} \right) \right) \\
= \left( \frac{p_1 z + 1}{p_2} \right)^2 \left( E_2 \left( \frac{p_1 z + 1}{p_2} \right) - \frac{p_2}{2\pi i (p_1 z + 1)} \right).
\]
Combining (3.7) and (3.8) we have
\[
L_{p_1p_2}(A^{-1}_{p_1} z) = E_2(A^{-1}_{p_1} z) - p_1p_2 E_{(2,p_1p_2)}(A^{-1}_{p_1} z) \\
= (p_1 z + 1)^2 \left( E_2(z) - \frac{p_1}{2\pi i (p_1 z + 1)} \right) \\
- p_1p_2 \left( \left( \frac{p_1 z + 1}{p_2} \right)^2 \left( E_2 \left( \frac{p_1 z + 1}{p_2} \right) - \frac{p_2}{2\pi i (p_1 z + 1)} \right) \right) \\
= (p_1 z + 1)^2 \left( E_2(z) - \frac{p_1}{p_2} E_2 \left( \frac{p_1 z + 1}{p_2} \right) \right),
\]
which proves (3.4). The proof of (3.5) is similar. \(\square\)

The following theorem is a special case of Theorem 4.1 from [1].

**Theorem 3.2.** Let \( t, c \in \mathbb{N} \) be such that \( c \mid t \). Then Fourier series expansion of \( E_{(4,t)}(z) \) at the cusp \( 1/c \) is given by
\[
E_{(4,t)}(A^{-1}_c z) = \left( \frac{c}{t} \right)^4 (cz + 1)^4 E_4 \left( \frac{cz + c}{t} \right).
\]
Proof. Let $t, c \in \mathbb{N}$ be such that $c \mid t$. Let $L = \begin{pmatrix} -t/c & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma_0(1)$ and $\gamma = \begin{pmatrix} -t & 0 \\ -c & -1 \end{pmatrix}$ be matrices. Then since $E_4(z) \in M_4(\Gamma_0(1))$, we have

$$E_{(4,t)}(A_c^{-1}z) = E_4(\gamma z) = E_4\left(L\left(\frac{c^2 z + c}{t}\right)\right) = \left(\frac{c^2 z + c}{t}\right)^4 E_4\left(\frac{c^2 z + c}{t}\right),$$

which proves the assertion. □

The following cusps in Table 2 are equivalent in the given modular subgroups.

Table 2: Equivalence of certain cusps

| Modular Subgroups | Equivalent cusps |
|-------------------|------------------|
| $\Gamma_0(1)$     | $1 \sim 1/p_1 \sim 1/p_2 \sim \infty$ |
| $\Gamma_0(p_1)$   | $1 \sim 1/p_2$ and $1/p_1 \sim \infty$ |
| $\Gamma_0(p_2)$   | $1 \sim 1/p_1$ and $1/p_2 \sim \infty$ |

We construct Tables 3 – 5 below by using (1.3), Theorems 3.1, 3.2, the definition of cusp forms and equivalence of the cusps given in Table 2. Tables 3, 4 and 5 will be used to prove (2.1), (2.2) and (2.3), respectively.

Table 3: First terms of modular forms at certain cusps

| Cusp $1/c$ | $[0]_c(L_p(z))^2$ | $[0]_cE_4(z)$ | $[0]_cE_4(pz)$ | $[0]_cC_p(z)$ |
|------------|-------------------|----------------|----------------|----------------|
| $c = 0$    | $(1 - p)^2$       | 1              | 1              | 0              |
| $c = 1$    | $\left(\frac{1-p}{p}\right)^2$ | 1              | $\frac{1}{p}$ | 0              |

Table 4: First terms of modular forms at certain cusps

| Cusp $1/c$ | $[0]_cL_{p_1}(z)L_{p_2}(z)$ | $[0]_cE_4(z)$ | $[0]_cE_4(p_1z)$ | $[0]_cE_4(p_2z)$ | $[0]_cE_4(p_1p_2z)$ | $[0]_cC_{(p_1,p_2)}(z)$ |
|------------|-----------------------------|----------------|-----------------|-----------------|-------------------|------------------------|
| $c = 0$    | $(1 - p_1)(1 - p_2)$        | 1              | 1              | 1              | 1                 | 0                      |
| $c = 1$    | $(1 - p_1)(1 - p_2)$        | 1              | 1              | 1              | 1                 | 0                      |
| $c = p_1$  | $p_1p_2$                    | $p_1$          | $p_2$          | $p_1p_2$       | 0                 | 0                      |
| $c = p_2$  | $(1 - p_1)(p_2 - 1)$        | $p_1$          | $p_2$          | $p_1p_2$       | 0                 | 0                      |
| $c = p_2$  | $(p_1 - 1)(1 - p_2)$        | $p_1$          | $p_2$          | $p_1p_2$       | 0                 | 0                      |
Solving equations (4.4) and (4.5) for $c$ obtain the sets of Eisenstein series (4.2) constitute bases for $E_M$ (4.1).

Let $p, p_1, p_2$ be primes such that $p_1 \neq p_2$. By (1.5), we have $(L_p(z))^2 \in M_4(\Gamma_0(p))$, and $L_{p_1}(z)L_{p_2}(z), \ (L_{p_1p_2}(z))^2 \in M_4(\Gamma_0(p_1p_2))$. By [23, Theorem 5.9], the sets of Eisenstein series

$$\{E_4(z), E_4(pz)\},$$

$$\{E_4(z), E_4(p_1z), E_4(p_2z), E_4(p_1p_2z)\}$$

constitute bases for $E_4(\Gamma_0(p))$ and $E_4(\Gamma_0(p_1p_2))$, respectively. Then by (1.2), we obtain

(4.1) $(L_p(z))^2 = a_1 E_4(z) + a_2 E_4(pz) + C_{(1,p)}(z)$,

(4.2) $L_{p_1}(z)L_{p_2}(z) = b_1 E_4(z) + b_2 E_4(p_1z) + b_3 E_4(p_2z) + b_4 E_4(p_1p_2z) + C_{(1,p_1p_2)}(z)$,

(4.3) $(L_{p_1p_2}(z))^2 = b_1 E_4(z) + b_2 E_4(p_1z) + b_3 E_4(p_2z) + b_4 E_4(p_1p_2z) + C_{(1,p_1p_2)}(z)$,

for some $a_i, b_j \in \mathbb{C}, C_{(1,p)}(z) \in S_4(\Gamma_0(p)), C_{(p_1,p_2)}(z), \text{ and } C_{(1,p_1p_2)}(z) \in S_4(\Gamma_0(p_1p_2))$.

We use the values in Table 3 to compare first terms of Fourier series expansions of the functions on both sides of equation (4.1) to obtain the following linear equations.

(4.4) \( (1 - p)^2 = a_1 + a_2 \),

(4.5) \( \left( \frac{p - 1}{p} \right)^2 = a_1 + a_2 \frac{1}{p^4} \).

Solving equations (4.4) and (4.5) for $a_1$ and $a_2$ we get the desired result in (2.4).

We similarly prove (2.2) and (2.3) using the values in Table 4 in (1.2) and Table 5 in (4.3), respectively.

5. The Cusp Forms $C_{(r,s)}(z)$ for $(r,s) = (1,2), (1,3), (1,5), (1,7), (1,11), (2,3), (2,5), (2,7), (3,5), (1,6), (1,10), (1,14), (1,15)$

In this section we express $C_{(r,s)}(z) (r,s) = (1,2), (1,3), (1,5), (1,7), (1,11), (2,3), (2,5), (2,7), (3,5), (1,6), (1,10), (1,14), (1,15)$ as linear combinations of...
eta quotients. The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H}$ by the product formula

\begin{equation}
\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i nz}).
\end{equation}

Let $N \in \mathbb{N}$, an eta quotient (of level $N$) is defined to be a finite product of the form

\begin{equation}
f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z),
\end{equation}

where $\delta$ runs through positive divisors of $N$ and $r_\delta \in \mathbb{Z}$, not all zeroes. For convenience we use the notation

\[ \prod_{\delta | N} \eta^{r_\delta}(\delta z) = \eta[1^r_1, \ldots, \delta^r_\delta, \ldots, N^r_N](z), \]

for an eta quotient.

**Theorem 5.1.** We express $C_{(r,s)}(z) \; (r, s) = (,)$ in terms of eta quotients as follows.

\begin{align*}
C_{(1,2)}(z) &= 0, \\
C_{(1,3)}(z) &= 0, \\
C_{(1,5)}(z) &= \frac{576}{13} \eta[1^4, 5^4](z), \\
C_{(1,7)}(z) &= \frac{576}{5} \eta[1^5, 2^{-1}, 7^5, 14^{-1}](z) + \frac{2304}{5} \eta[1^2, 2^2, 7^2, 14^2](z), \\
C_{(1,11)}(z) &= \frac{17280}{61} \eta[1^6, 2^{-2}, 11^6, 22^{-2}](z) + \frac{118656}{61} \eta[1^4, 11^4](z) \\
&\quad + \frac{276480}{61} \eta[1^2, 2^2, 11^2, 22^2](z) + \frac{276480}{61} \eta[2^4, 22^4](z), \\
C_{(2,3)}(z) &= -\frac{144}{5} \eta[1^2, 2^2, 3^2, 6^2](z), \\
C_{(2,5)}(z) &= \frac{48}{13} \eta[1^{-2}, 2^8, 5^2](z) - \frac{6000}{13} \eta[1^2, 5^{-2}, 10^8](z), \\
C_{(2,7)}(z) &= \frac{288}{5} \eta[1^5, 2^{-1}, 7^5, 14^{-1}](z) - \frac{336}{25} \eta[1^{-2}, 2^6, 7^6, 14^{-2}](z), \\
&\quad + \frac{8064}{25} \eta[1^2, 2^2, 7^2, 14^2](z) - \frac{5136}{25} \eta[1^6, 2^{-2}, 7^{-2}, 14^6](z), \\
C_{(3,5)}(z) &= \frac{108}{13} \eta[1^{11}, 2^{-5}, 3^{-5}, 5^{-5}, 6^3, 10^3, 15^{11}, 30^{-5}](z) \\
&\quad - \frac{1584}{13} \eta[1^2, 2^{-3}, 3^{-4}, 5^{-3}, 6^6, 10^4, 15^7, 30^{-4}](z).
\end{align*}
\[
\begin{align*}
& -\frac{2376}{13} \eta[1^4, 2^{-3}, 3^{-2}, 5^2, 6^5, 10^4, 15^0, 30^4](z) \\
& -\frac{5832}{13} \eta[1^1, 2^{-2}, 3^{-3}, 5^2, 6^8, 10^{-1}, 15^2, 30^1](z) \\
& + \frac{11448}{13} \eta[1^2, 2^{-2}, 3^0, 5^{-2}, 6^2, 10^6, 15^0, 30^2](z) \\
& + \frac{33264}{13} \eta[1^1, 2^2, 3^4, 5^3, 6^{-3}, 10^{-2}, 15^{-4}, 30^7](z) \\
& + \frac{10080}{13} \eta[1^1, 2^0, 3^{-1}, 5^{-3}, 6^0, 10^2, 15^7, 30^2](z) \\
& + \frac{83520}{13} \eta[1^1, 2^{-1}, 3^{-2}, 5^1, 6^4, 10^{-1}, 15^0, 30^6](z) \\
& + \frac{145584}{13} \eta[1^0, 2^1, 3^0, 5^2, 6^{-1}, 10^{-3}, 15^2, 30^7](z) \\
& - \frac{117720}{13} \eta[1^{-2}, 2^3, 3^5, 5^3, 6^{-4}, 10^{-4}, 15^{-4}, 30^{11}](z) \\
& - 1944\eta[1^1, 2^0, 3^1, 5^3, 6^0, 10^{-4}, 15^{-5}, 30^{12}](z) \\
& + \frac{158688}{13} \eta[1^0, 2^1, 3^1, 5^2, 6^{-2}, 10^{-3}, 15^{-3}, 30^{12}](z) \\
& + \frac{216000}{13} \eta[1^{-1}, 2^3, 3^1, 5^3, 6^{-1}, 10^{-5}, 15^{-7}, 30^{15}](z) \\
& + \frac{316224}{13} \eta[1^0, 2^1, 3^2, 5^2, 6^{-3}, 10^{-3}, 15^{-8}, 30^{17}](z), \\
C_{(1,6)}(z) &= \frac{288}{5} \eta[1^2, 2^2, 3^2, 6^2](z), \\
C_{(1,10)}(z) &= \frac{2976}{13} \eta[1^{-1}, 2^5, 5^5, 10^{-1}](z) - \frac{480}{13} \eta[1^5, 2^{-1}, 5^{-1}, 10^5](z), \\
C_{(1,14)}(z) &= \frac{10272}{25} \eta[1^{-2}, 2^6, 7^6, 14^{-2}](z) - \frac{4608}{25} \eta[1^2, 2^2, 7^2, 14^2](z) \\
& + \frac{672}{25} \eta[1^6, 2^{-2}, 7^{-2}, 14^6](z), \\
C_{(1,15)}(z) &= \frac{5976}{13} \eta[1^{11}, 2^{-5}, 3^{-5}, 5^{-5}, 6^3, 10^3, 15^{11}, 30^{-5}](z) \\
& + \frac{74592}{13} \eta[1^2, 2^{-3}, 3^{-4}, 5^{-3}, 6^9, 10^4, 15^7, 30^{-4}](z) \\
& - \frac{137808}{13} \eta[1^4, 2^{-3}, 3^{-2}, 5^2, 6^5, 10^4, 15^0, 30^4](z) \\
& - \frac{85968}{13} \eta[1^1, 2^{-2}, 3^{-3}, 5^2, 6^8, 10^{-1}, 15^2, 30^4](z) \\
& + \frac{153072}{13} \eta[1^2, 2^{-2}, 3^0, 5^{-2}, 6^2, 10^6, 15^0, 30^2](z) \\
\end{align*}
\]
Sturm Theorem (or Sturm Bound). To illustrate the proof we show these two modular forms are equal, see ([15, Theorem 3.13]) for

On the other hand, let

Proofs of all these equalities are similar. We basically find combinations of eta quotients whose first couple terms in Fourier series expansion at \( \infty \) agrees with the first couple terms in Fourier series expansion of \( \mathcal{C}_{(r,s)}(z) \) at \( \infty \). Then by Sturm Theorem these two modular forms are equal, see ([15, Theorem 3.13]) for Sturm Theorem (or Sturm Bound). To illustrate the proof we show

\[
\mathcal{C}_{(1,11)}(z) = \frac{17280}{61} \eta[6^1, 2^2, 11^6, 22^{-2}](z) + \frac{118656}{61} \eta[4^1, 11^4](z)
\]

(5.3)

By Theorem 2.1 we have

\[
\mathcal{C}_{(1,11)}(z) = (L_{11}(z))^2 - \frac{50}{61} E_4(z) - \frac{6050}{61} E_4(11z) \in S_4(\Gamma_0(11)) \subset S_4(\Gamma_0(22)).
\]

We use MAPLE to expand \( (L_{11}(z))^2, E_4(z) \) and \( E_4(11z) \). Then we find

\[
\mathcal{C}_{(1,11)}(z) = \frac{17280}{61} q + \frac{14976}{61} q^2 - \frac{8064}{61} q^3 - \frac{73728}{61} q^4 - \frac{1152}{61} q^5 - \frac{213120}{61} q^6
\]

(5.4)

+ \frac{182016}{61} q^7 - \frac{80640}{61} q^8 + \frac{361728}{61} q^9 + \frac{411264}{61} q^{10} - \frac{190080}{61} q^{11}

- \frac{377856}{61} q^{12} + O(q^{13}).

On the other hand, let

\[
f(z) = \frac{17280}{61} \eta[6^1, 2^2, 11^6, 22^{-2}](z) + \frac{118656}{61} \eta[4^1, 11^4](z) + \frac{276480}{61} \eta[2^1, 2^2, 11^2, 22^2](z)
\]
We have $f(z) \in S_4(\Gamma_0(22))$, see [15, Theorem 5.7, p. 99], [16, Corollary 2.3, p. 37], [13, p. 174] and [18]. Again using MAPLE we also compute

$$f(z) = 17280q^2 + 14976q^3 - 8064q^4 - 1152q^5 - 213120q^6 + 182016q^7 - 80640q^8 + 361728q^9 + 411264q^{10} - 377856q^{12} + O(q^{13}).$$

That is, by (5.4) and (5.5), we have

$$f(z) - C_{(1,1,1)}(z) = O(q^{13}).$$

Thus by Sturm Theorem (see [15, Theorem 3.13]), we have

$$f(z) - C_{(1,1,1)}(z) = 0,$$

which proves (5.3). □

6. FURTHER EXTENSIONS

Let $N_1 \in \mathbb{N}$ be a square-free number and $N = N_1, 2N_1$, or $4N_1$. Let $r, s \in \mathbb{N}$ be such that $\text{lcm}(r, s) \mid N$. We can use similar arguments to give formulas for $W(r, s; n)$. We consider $L_r(z)L_s(z) \in M_4(\Gamma_0(N))$. Then by [23, Theorem 5.9] and (1.2), we have that there exists $c_d \in \mathbb{C}$ and $C_{(r,s)}(z) \in S_4(\Gamma_0(N))$, such that

$$L_r(z)L_s(z) = \sum_{d \mid N} c_d E_4(dz) + C_{(r,s)}(z).$$

Now in order to compute $c_d$ we need to find the first terms of the modular forms on both sides of equation (6.1) at the cusps of $\Gamma_0(N)$. Note that, because of the choice of $N$, the set

$$\left\{ \frac{1}{c} : c \mid N \right\}$$

is a set of cusps of $\Gamma_0(N)$. Then comparing first coefficients of Fourier series expansions of modular forms in (6.1), we will find a set of linear equations. We can determine $c_d$ by solving these equations. Also note that in many cases $C_{(r,s)}(z)$ can be given in terms of eta quotients. Finally a formula for $W(r, s; n)$ can be given by comparing coefficients of $q^n$ on both sides of the equation (6.1).

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References

[1] A. Alaca, S. Alaca and Z. S. Aygin, A family of eta quotients and an extension of the Ramanujan-Mordell theorem, arXiv:1603.09412v2 [math.NT], preprint (2016).
[2] A. Alaca, S. Alaca and K. S. Williams, Evaluation of the convolution sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{l+4m=n} \sigma(l)\sigma(m)$, Adv. Theor. Appl. Math. 1 (2006) 27–48.
[3] A. Alaca, S. Alaca and K. S. Williams, Evaluation of the convolution sums $\sum_{l+18m=n} \sigma(l)\sigma(m)$ and $\sum_{l+9m=n} \sigma(l)\sigma(m)$, Int. Math. Forum 2 (2007) 45–68.
[4] A. Alaca, S. Alaca and K. S. Williams, Evaluation of the convolution sums $\sum_{l+24m=n} \sigma(l)\sigma(m)$ and $\sum_{l+8m=n} \sigma(l)\sigma(m)$, Math. J. Okayama Univ. 49 (2007) 93–111.
[5] A. Alaca, S. Alaca and K. S. Williams, The convolution sum $\sum_{m<n/16} \sigma(m)\sigma(n-16m)$, Canad. Math. Bull. 51 (2008) 3–14.
[6] S. Alaca and Y. Kesicioğlu, Evaluation of the convolution sums $\sum_{l+27m=n} \sigma(l)\sigma(m)$ and $\sum_{l+32m=n} \sigma(l)\sigma(m)$, J. Number Theory 1 (2016) 1–13.
[7] S. Alaca and K. S. Williams, Evaluation of the convolution sums $\sum_{l+6m=n} \sigma(l)\sigma(m)$ and $\sum_{l+3m=n} \sigma(l)\sigma(m)$, J. Number Theory 124 (2007) 491–510.
[8] M. Besge, Extrait d’une lettre de M. Besge à M. Liouville, J. Math. Pures Appl. 7 (1862) 256.
[9] H. H. Chan and S. Cooper, Powers of theta functions, Pacific J. Math. 235 (2008) 1714.
[10] S. Cooper and P. C. Toh, Quintic and septic Eisenstein series, Ramanujan J. 19 (2009) 163–181.
[11] S. Cooper and D. Ye, Evaluation of the convolution sums $\sum_{l+20m=n} \sigma(l)\sigma(m)$, $\sum_{l+5m=n} \sigma(l)\sigma(m)$ and $\sum_{l+5m=n} \sigma(l)\sigma(m)$, Int. J. Number Theory 6 (2014) 1385–1394.
[12] J. W. L. Glaisher, On the square of the series in which the coefficients are the sums of the divisors of the exponents, Mess. Math. 14 (1885) 156–163.
[13] B. Gordon and D. Sinor, Multiplicative properties of $q$-products, Lecture Notes in Math, vol.1395 Springer-Verlag, New York (1989), 173-200.
[14] J. G. Huard, Z. M. Ou, B. K. Spearman and K. S. Williams, Elementary evaluation of certain convolution sums involving divisor functions, Number Theory for the Millennium, II (A. K. Peters, Natick, MA, 2002), pp. 229–274.
[15] L. J. P. Kilford, Modular Forms, A classical and computational introduction, Imperial College Press, London, 2008.
[16] G. Köhler, Eta Products and Theta Series Identities, Springer Monographs in Mathematics, Springer, 2011.
[17] M. Lemire and K. S. Williams, Evaluation of two convolution sums involving the sum of divisor functions, Bull. Aust. Math. Soc. 73 (2005) 107–115.
[18] G. Ligozat, Courbes modulaires de genre 1, Bull. Soc. Math. France 43 (1975), 5-80.
[19] B. Ramakrishnan and B. Sahu, Evaluation of the convolution sums $\sum_{l+15m=n} \sigma(l)\sigma(m)$ and $\sum_{l+5m=n} \sigma(l)\sigma(m)$ and an application, Int. J. Number Theory 9 (2013) 799–809.
[20] S. Ramanujan, On certain arithmetical functions, Trans. Cambridge Philos. Soc. 22 (1916) 159–184.
[21] E. Royer, Evaluating convolution sums of the divisor function by quasimodular forms, Int. J. Number Theory 3 (2007) 231–261.
[22] J.-P. Serre, Modular forms of weight one and Galois representations, Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London, 1977, 193-268.
[23] W. A. Stein, Modular Forms, A Computational Approach, Amer. Math. Soc., Graduate Studies in Mathematics 79 (2007).
[24] K. S. Williams, *The convolution sum* $\sum_{m<n/9} \sigma(m)\sigma(n - 9m)$, Int. J. Number Theory 1 (2005) 193–205.

[25] K. S. Williams, *The convolution sum* $\sum_{m<n/8} \sigma(m)\sigma(n - 8m)$, Pacific J. Math. 228 (2006) 387–396.

[26] E. X. W. Xia, X. L. Tian and O. X. M. Yao, *Evaluation of the convolution sum* $\sum_{l+25m=n} \sigma(l)\sigma(m)$, Int. J. Number Theory 10, 1421 (2014).

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