PARTICLE DYNAMICS IN A RELATIVISTIC INVARIANT STOCHASTIC MEDIUM

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Abstract

The dynamics of particles moving in a medium defined by its relativistically invariant stochastic properties is investigated. For this aim, the force exerted on the particles by the medium is defined by a stationary random variable as a function of the proper time of the particles. The equations of motion for a single one-dimensional particle are obtained and numerically solved. A conservation law for the drift momentum of the particle during its random motion is shown. Moreover, the conservation of the mean value of the total linear momentum for two particles repelling each other according to the Coulomb interaction also follows. Therefore, the results indicate the realization of a kind of stochastic Noether theorem in the system under study. Possible applications to the stochastic representation of Quantum Mechanics are advanced.

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I. INTRODUCTION

Stochastic processes made their appearance in research in Physics long time ago and their theory has played an important role in the description of systems which do not behave in a deterministic manner\cite{1}. In particular, the study of the dynamics of particles lying inside material media has been the object of high interest. A classical example is the study of the Brownian motion\cite{1}. A large amount of those investigations had a non-relativistic character and the random interactions with the background medium were considered as being dependent of the state of motion of the particle, that is, lacking invariance under the changes of the reference system\cite{1-3}. Another large class of studies in this field had been directed to show the equivalence with random processes of the solutions of quantum relativistic or non-relativistic equations, like the Klein-Gordon, Dirac and Schrodinger ones\cite{4-12}. Two basic additional subjects in connection with stochastic processes in Quantum Theory are: The attempts to derive the 'collapse' of the wave function during measurements from the existence of random perturbations in Quantum Mechanics (QM)\cite{14-16}, and the study of the decoherence processes and their role in spontaneous transitions from pure to mixed states\cite{13}.

The main objective of the present work is to investigate some consequences on the motion of a particle determined by the action exerted over it by a medium which random properties are defined in a relativistically invariant form. The basic motivation is simple: It is recognized that the Copenhagen interpretation of Quantum Mechanics (QM), is the most successful and dominant from all the existing ones. However, it is also accepted that its approach to measurements constitute one of its more puzzling aspects, which up to now is widely debated in the literature\cite{4-16}. Let us suppose for a moment, that in opposition to the Copenhagen interpretation and in accordance with Einstein expectations, the particles in Nature are in fact localized at definite points of the space at any moment. Then, the only way we can imagine for the quantum mechanical properties of the motion to emerge from a model, is that the action of the vacuum on the particles have a stochastic character. But, the relativistic invariance of the vacuum, leads to expect that the acceleration felt by the particle in its proper frame should be a stationary random variable as a function of the proper time. This circumstance motivates the study started here about the motion of particles inside a random media showing the above mentioned property. For the sake of simplicity the one dimensional motion is considered. It is not attempted to show the equivalence of the dynamics in the medium with the one predicted by the quantum mechanical equations. The purpose in this first step, being redundant, is to study general properties of the motion of one and two particles assuming two main conditions: a) They have a definite localization in the space at any moment and b) The forces acting on the particles have the above mentioned random properties which are independent of the observer’s inertial reference frame.

The work will proceed as follows. Firstly, the equations of motion of the particles under the action of the medium are formulated. For this purpose the properties which ensure the relativistic invariance of the motion under the action of the medium are stated by specifying the form of the random forces. Further, the equations of motion of a single particle are written and solved and a statistical analysis of the random properties is done. A main conclusion which follows is the existence of a conservation law for a mean drift momentum and kinetic energy of a 'free' particle propagating in the medium. It indicates the validity of a kind of stochastic Noether theorem which links the relativistic invariance of the stochastic motion with the conservation of the mean 4-momentum of the particle.

Further, the conservation law is studied for the mean of the addition of two four momenta associated to the scattering of two identical particles, which repel each other through an instantaneous Coulomb interaction. It is concluded that the random action of the medium does not
modify the usual conservation law, valid for the impact in the absence of external forces.

A review of the results and future extensions of the work are presented in a conclusion section. Some possibilities to extend the study are advanced. In general terms, our view is that the form of the analysis has the chance of being useful in the search for consistent hidden variables models. The study of these possibilities will be considered elsewhere.

II. EQUATION OF MOTION

In this section we will obtain and solve the Newton equation of motion for a particle on which a random force \( F_p(\tau) \) is acting. A one dimensional system will be considered to make the discussion as simple as possible. The force will be defined as a vector in the proper reference frame of the particle and will depend on the proper time \( \tau \). That means, in each instant we will consider an inertial reference frame moving relative to the observer’s fixed frame (Lab frame) with the velocity of the particle \( \nu \) and having the origin of coordinates coinciding with it. In this system of reference, after a time increment \( d\tau \), it is possible to write

\[
F_p(\tau) \, d\tau = m_0 \, d\nu',
\]

where \( m_0 \) is the proper mass of the particle.

The particle reaches a small velocity \( d\nu' \) relative to this system and a new velocity respect to the Lab frame \( \nu + d\nu \), which are related by the equation

\[
\nu + d\nu = \frac{\nu + d\nu'}{1 + \frac{\nu}{c} \frac{d\nu'}{c^2}},
\]

\[
\approx (\nu + d\nu') \times \left(1 - \frac{\nu \, d\nu'}{c^2}\right),
\]

\[
\approx \nu + \left(1 - \frac{\nu^2}{c^2}\right) \, d\nu',
\]

where \( c \) is the velocity of light. Thus, the variation of speed in the Lab frame \( d\nu \) is

\[
d\nu = \left(1 - \frac{\nu^2}{c^2}\right) \, d\nu'.
\]

From expressions (1) and (4) the required differential equation for the motion is obtained:

\[
F_p(\tau) = \frac{m_0}{(1 - \frac{\nu^2}{c^2})} \frac{d\nu}{d\tau},
\]

\[
\nu = \frac{dx(t)}{dt} = \left(1 - \frac{\nu^2}{c^2}\right) \frac{dx}{d\tau}.
\]

It is useful to state the relation between the strength of the force in the Lab system and its proper frame counterpart, which is:

\[
F_p(\tau) = \sqrt{1 - \frac{\nu^2}{c^2}} F_L(\tau).
\]

However, since the relativistic invariance condition will be imposed on \( F_p(\tau) \) this will be the type of force mostly considered in what follows. Integrating equation (5) in the proper time it follows that

\[
\int F_p(\tau) \, d\tau + \dot{C} = m_0 \int \frac{1}{(1 - \frac{\nu^2}{c^2})} \, d\nu,
\]

\[
= \frac{m_0 \, c}{2} \ln \left(1 + \frac{\nu^2}{c^2}\right),
\]
which determines the velocity the Lab frame $\nu(\tau)$ as a function of the proper time $\tau$, only through the dependence of $\tau$, of the integral of the random force $F_p(\tau)$. The explicit form of $\nu$ becomes

$$\nu(\tau) = c \cdot \tanh\left( \frac{1}{m_0c} \cdot \left( \int F_p(\tau) \cdot d\tau + \dot{C} \right) \right),$$

where $\dot{C}$ is an arbitrary constant.

### III. THE RANDOM FORCE

As mentioned in the Introduction, the medium under study will be defined in the proper frame as randomly acting over the particle being at rest in it. That is, its action in this reference system will be given by a stochastic process showing no preferential spatial direction and assumed to be produced by an external relativistic system which dynamics is unaffected by the presence of the particle. It is also natural to impose the coincidence of the distribution function of the forces of the medium for a large sampling interval of proper time $T$ and the one obtained fixing the proper time $\tau$, produced by an ensemble of a large number of samples of the forces taken during long time intervals $T$.

![FIG. 1: The figure shows a realization of the force field corresponding to a spectrum of $N = 200$ frequencies. The horizontal coordinate is the time in seconds and the vertical one is the force in Newtons. The amplitude was fixed $f_0 = 1$ ($N_t = Newton$).](image)

These conditions, can be assured by a random force $F_p(\tau)$ being stationary, ergodic and symmetrical distributed about the zero value of the force. A numerical realization of a band limited white noise distribution obeying these properties is implemented in reference$^{17}$ and will be employed here. Concretely, the expression for the stochastic force given in the proper reference frame as a function of the proper time will be taken in the form

$$f_N(\tau, \varphi_i) = \frac{f_0}{N} \sum_{i=1}^{N} \cos(w_i(N) \cdot \varphi_i),$$

$$w_i(N) = \frac{8\pi i}{N}, \quad i = 1, \ldots N,$$

where the $N$ phases $\varphi_i, i = 1, \ldots N$, are randomly chosen with a uniform distribution between 0 and $2\pi$. The integer number $N$ is the number of frequency components of the numerical
band limited white noise distribution. The bandwidth will be chosen as a fixed one and equal to $\Delta \omega = 8\pi$. Clearly, the exact randomness for an arbitrarily large time interval only will be attained in the infinite limit of $N$. The parameter $f_0$ controls the amplitude of the force values. Note that the absence of a zero frequency component is implied by the condition of the process not showing a preferential direction in space.

Figure 1 shows the force distribution $f_N$ for a realisation of the ensemble of forces for $N = 200$. A realisation here is called the time evolution of the force, for the set of randomly fixed phases $\varphi_i$ at the beginning. The picture qualitatively shows the stationary character of the random force. Figure 2 depicts an interpolation curve of the data for the distribution function $w(f)$ corresponding to an ensemble of random realisations of the force. The white noise frequency spectrum was defined by $N = 250$ frequency components $w_i = \frac{8\pi}{250} i (s^{-1})$, $i = 1, \ldots, 250$.

The force amplitude fixed was as $f_0 = 1$ ($N_t$). The picture corresponds to a large sampling time $T$ (it is sufficient to be at most equal to the period of the smallest frequency of the spectrum). Notice the even character of the distribution and its rapid decay to zero. Of course, this occurs inside the interval defined by $f_{1,2} = \pm 1 N_t$. This result is natural due to the fact that the force is normalized and its absolute value can not exceed $f_0$.

IV. A PARTICLE IN THE MEDIUM

In this Section the existence of a conservation law for the mean momentum (to be also called, from now on, the drift momentum) will be shown for a particle moving in the previously defined random medium. For this purpose let us employ the solution (8). We will combine this expression with the results obtained from the definition of the random force ensemble in equation (9), to explicitly determine $v = v(\tau)$. The resulting relation links the velocity of the particle in the Lab frame with the proper time measured by a clock fixed to it. Once having $v(\tau)$ we will comment about its stationary random behavior. The existence of a non vanishing conserved mean value of the velocity $<v>$ will be shown. Starting from the method defined above, the velocity distribution function will be determined for the frequency spectrum defined before and a few representative values of the arbitrary constant $\hat{C}$. After that, the position of the particle $x(t)$ as a function of the time $t$ measured at the Lab frame will be evaluated.

Taking $F_p(\tau)$ to be given by the white noise force $f_N(\tau)$ we have:

$$F_p(\tau) = f_N(\tau) = \frac{f_0}{N} \sum_{i=1}^{N} \cos\left(\frac{8\pi i}{N} \tau + \varphi_i\right).$$

(11)
FIG. 3: The velocity of the particle in the laboratory frame against the proper time and the arbitrary constant $\hat{C}$. The parameters for the random force were $\frac{f_0}{8\pi\eta m c} = 0.1 \, (s^{-1}), \, w_i = \frac{8\pi i}{N} \, (s^{-1}), \, i = 1, \ldots, 250$.

Then, integrating with respect to $\tau$ produces

$$ I_{F_p}(\tau) = \int_0^\tau F_p(\tau) \, d\tau = \frac{f_0}{N} \int_0^\tau \left[ \sum_{i=1}^N \cos\left(\frac{8\pi i}{N} \tau + \varphi_i \right) \right] \, d\tau, $$

$$ = \frac{f_0}{8\pi} \sum_{i=1}^N \frac{1}{i} \sin\left(\frac{8\pi i}{N} \tau + \varphi_i \right). \tag{12} $$

Substituting (12) into (8), gives

$$ \nu(\tau) = c \, \tanh\left[ \frac{1}{m \eta c} (I_{F_p}(\tau) + \hat{C}) \right], \tag{13} $$

$$ = c \, \tanh\left[ \frac{1}{m \eta c} \left( \frac{f_0}{8\pi} \sum_{i=1}^N \frac{1}{i} \sin\left(\frac{8\pi i}{N} \tau + \varphi_i \right) + \hat{C} \right) \right]. \tag{14} $$

FIG. 4: The velocities of the particles (divided by $c$) in the laboratory system vs. the proper time for four specific values of $\hat{C}$: a) $\hat{C} = 0$ b) $\hat{C} = 0.3$ c) $\hat{C} = -0.2$ and d) $\hat{C} = -0.8$. Note the non vanishing value of the mean velocity for $\hat{C}$ different from zero.

It can be seen that the summation within the argument of the hyperbolic tangent is symmetrically distributed around its zero value. Since the $\tanh(x)$ is an anti-symmetric function,
it follows that when \( \dot{C} \neq 0 \) there will be a nonvanishing mean velocity of the particle in the medium. As the mean value of the kinetic energy is also conserved, it can be said that the mean 4-momentum of the particle is conserved. Moreover, the relativistic invariance of the system also implies that the mean 4-momentum of the solutions for any two values of \( \dot{C} \) should be linked by certain Lorentz transformation. Thus, the particle trajectories for the various values of \( \dot{C} \) are simply a fixed trajectory (in the stochastic sense) after being Lorentz transformed into a certain moving frame.

The picture in figure 3 shows the behavior of \( \nu = \nu(\tau, \dot{C}) \) for the following values of the parameters \( \frac{\hbar}{8\pi \sqrt{m} \gamma} = 0.1 \text{s}^{-1}, \; w_i = \frac{8\pi \gamma}{N} \text{s}^{-1}, \; i = 1, ..., 250 \). Note that for fixed values of \( \dot{C} \) (that means, for certain initial conditions) the velocity rapidly oscillates around a value being close to the quantity \( \tanh[\dot{C}] \). It illustrates the mentioned conservation of the mean drift velocity for the particle in spite of the random oscillations of the medium. Figure 4 shows the same dependence for specific values of \( \dot{C} \).

A. Distributions

Let us now consider the numerical evaluation of the distribution function \( w(v) \) (for measuring a given value of the velocity \( v \)) for a few representative values of \( \dot{C} \). This function is found on the basis of the ergodic property of the system, that is, by performing a large number of evaluations of the velocity’s expression with time running from zero to a "sampling" value \( T \), for to afterwards compute the number of times, for which \( v \) takes values in a small neighborhood of a given value. Further, after interpolating the results, the distribution functions are obtained after dividing by the fixed size of the mentioned neighborhood. In figure (5) the distribution \( w(v) \) is plotted for a few values of \( \dot{C} \).

These pictures correspond to the frequency spectrum \( w_i = \frac{8\pi i}{250}, \; i = 1...250 \), but with the amplitude fixed by \( \frac{\hbar}{4\gamma} = 1 \). The almost independence of the form of these curves on the size of a large "sampling" time \( T \) employed (whenever \( N \) is sufficiently large) indicates the approximate validity of the ergodic property of the white noise implementation employed.

Note the rapid decay away from the mean value and the deformation of the symmetry around it, of the distribution \( w(v) \) for different values of \( \dot{C} \). These distributions allow to calculate the mean velocities in each of the cases. This method will be employed in the next sections to find the mean values as integrals over the domain of the quantity, of its value times the distribution
B. Particle velocities and trajectories

It can be noted that the explicit solution for the velocity obtained in (8) corresponds to this velocity in terms of the proper time $\tau$. However, in order to integrate the velocity to find the random particle trajectories as functions of the time in the laboratory frame, it becomes necessary to know the functional relationship $\tau(t, \dot{C})$ for the considered trajectory. Then, let us consider now the numerical determination of this relation. The parameters of the random force will be $\frac{f_0}{\sqrt{\pi m_0 c}} = 0.1 \, (s^{-1})$, $w_i = \frac{8\pi i}{N} \, (s^{-1})$, $i = 1, 2, \ldots, 250$. Using the solution $\nu = \nu(\tau, \dot{C})$ it follows

$$t(\tau, \dot{C}) = \int_0^\tau \frac{d\tau'}{(1 - \frac{\nu(\tau', \dot{C})^2}{c^2})^{1/2}},$$

$$t(0) = 0,$$

from which, the values of $t(\tau, \dot{C})$ were numerically found. Afterwards, finding the inverse mapping $\tau = \tau(t, \dot{C})$, the obtained function is depicted in figure 6 with $\dot{C}$ as a parameter. Finding the composed function for various values of $\dot{C}$, the observer’s time dependence of the particle velocity $\nu(t, \dot{C})$ follows. The results are illustrated in figure 7, for the chosen values of $\dot{C}$.

Further, after integrating the calculated velocities with respect to $t$ (starting at $t = 0$ and imposing $x(0) = 0$) the values of the positions $x(t)$ with respect to the laboratory reference frame are obtained. The trajectories of the particles are shown in figure 8, for the same set of values of $\dot{C}$.

Note the randomness of the motion in the case of small absolute values of $\dot{C}$, which is not the case for the larger ones. This property can be understood analyzing the pictures in figure (7). In the cases in which the velocity constantly changes from positive to negative values and vice versa, the randomness is more evident. Thus, the aleatory effect is more visible in one case than in the other, because the curve changes from a monotonous to a non monotonous behavior. But, after taking into account the relativistic invariance of the model, as noticed before, it can be seen that the curves for different values of $\dot{C}$ should transform any one into another by a Lorentz
V. TWO PARTICLES IN THE MEDIUM

Let us consider in this final section the conservation of the mean total momentum of a ‘closed’ system of two particles which travel in the medium by also interacting between them. A repulsive interaction between the two identical particles 1 and 2 will introduced. The forces between the particles will be defined in the laboratory frame for studying the impact between them there. They will be chosen as satisfying the Law of Action and Reaction and having the Coulomb form. If, for example, the forces have an electromagnetic origin, the retardation effects will be disregarded in order to assure that the field will not deliver momentum to the set of two particles. This approximation is appropriate for low particle velocities in its random motion in the Lab frame.

The system of differential equations will be written and numerically solved. The results for the position and velocity of the two particles \((x_1(t), v_1(t))\) and \((x_2(t), v_2(t))\), will show how after
the impact, any of them deliver to the other its mean drift velocity and its type of randomness. As it was mentioned above, the expression for the repulsive force in the Lab frame will have the Coulomb form

\[ F_{\text{rep}}(x_1, x_2) = -F_{\text{rep}}(x_1, x_2) = \frac{\alpha}{|x_1 - x_2|^2}(x_1 - x_2). \]  

(17)

Consider now the relativistic Newton equations for both particles in their respective proper frames and also the two transformations between the common laboratory time \( t \) and the two different proper times \( \tau_1 \) and \( \tau_2 \). These relations may be written as

\[ \frac{m_0}{(1 - \frac{x_1'(t)^2}{c^2})^{\frac{3}{2}}} \frac{d^2 x_1}{dt^2} = F_{p1}(\tau_1) + (1 - \frac{x_1'(t)^2}{c^2})^{\frac{1}{2}} F_{\text{rep}}(x_1, x_2), \]  

(18)

\[ \frac{m_0}{(1 - \frac{x_2'(t)^2}{c^2})^{\frac{3}{2}}} \frac{d^2 x_2}{dt^2} = F_{p2}(\tau_2) + (1 - \frac{x_2'(t)^2}{c^2})^{\frac{1}{2}} F_{\text{rep}}(x_1, x_2), \]  

(19)

\[ \frac{d\tau_1}{dt} = (1 - \frac{x_1'(t)^2}{c^2})^{\frac{1}{2}}, \quad \tau_1(t_0) = \tau_{10}, \quad x_1(t_0) = x_{10}, \quad x_1'(t_0) = v_{10}, \]  

(20)

\[ \frac{d\tau_2}{dt} = (1 - \frac{x_2'(t)^2}{c^2})^{\frac{1}{2}}, \quad \tau_2(t_0) = \tau_{20}, \quad x_2(t_0) = x_{20}, \quad x_2'(t_0) = v_{20}. \]  

(21)

In order to be consistent with the non retarded approximation for the Coulomb repulsion, as mentioned above, initial velocities for the particles being relatively small with respect to \( c \) were taken. This assumption makes the considered approximation to be satisfied if the intensity of the repulsive force is weak so that the velocities after the impact will also be small. Employing

FIG. 9: The trajectories of the particles \( x_1(t) \) and \( x_2(t) \) during the impact for the repulsion parameter \( \alpha = 0.01 \ (N \times m^2) \) and initial conditions \( x_1(500) = x_2(500) = 0, \ v_1(500) = 0.015, \ v_2(500) = 0.15, \ \tau_1(500) = 0, \ \tau_2(500) = 0. \) The curve at the bottom is the same as the one at the top, but at a lower scale in the vertical coordinate.

the white noise parameters \( w_i = \frac{8\pi^2 i}{250}, \ i = 1, \ldots, 250 \) and \( \frac{f_0}{125} = 1 \), we obtained the numerical solutions shown in figure 9 for various initial conditions. The conservation law of the total mean momentum of the two particles can be noticed. Figure 10 shows the time variation of the
FIG. 10: The time dependence for the velocities (divided by \(c\)) of both particles: a) \(v_1(t)/c\) and b) \(v_2(t)/c\) in the considered shock. Note, the exchange of their mean velocity values.

velocities of the particles. The process of exchange of their drift momenta is clearly illustrated and therefore the corresponding conservation of the total momentum. This outcome follows in spite of the strong oscillations produced by the action of the medium, in opposition to what happens within a standard material media. In these systems the stochastic action normally tends to stop the motion of the particle making the drift velocity to vanish, if accelerating external fields are absent. The calculated values for the mean velocities before and after the shock are

\[
\begin{align*}
< v_{1,0} > & = 0, \quad < v_{2,0} > = 0.152, \\
< v_{1,f} > & = 0.152, \quad < v_{2,f} > = 0,
\end{align*}
\]

which numerically confirms the conservation of the total momentum in the shock of two particles forming a ‘closed’ system immersed and moving in a relativistic random medium.

VI. CONCLUSIONS

The one-dimensional equation of motion for a particle moving in a medium having relativistic invariant stochastic properties is formulated and numerically solved. The velocity of the particles in the medium is a function of the proper time only through the integral of the force in the proper reference frame. This relation directly shows the existence of a stochastic conservation law: A free particle in the defined random medium conserves its mean momentum and kinetic energy along its motion.

The problem of two particles moving in the medium is also investigated after considering a momentum conserving Coulomb repulsion between them. The evaluated solutions generalize the conservation law: The sum of the two drift momenta of two particles moving in the medium without any other external action conserves after a shock.

An interesting outcome is that for two different shocks with identical initial conditions for both particles when they are far apart, the drift velocities before the impact are not uniquely determined by the initial conditions. These velocities also depend on the array of phases utilized for the realization of the force. Therefore, it follows that the results of the impact will show a dependence on the specific realization of the random force. This circumstance implies that the
probability distribution associated to an ensemble of particles all situated at the beginning at
the same point and with a fixed value of the velocity, will not define only one direction of the
trajectory for large times. The probability distributions of such an ensemble, on the other hand,
should evolve in space and time in a relativistically invariant manner, as the Lorentz invariance
of the system indicates. The same conclusion can be traced for any other sort of fixed boundary
conditions. Therefore, the set of possible initial conditions for the particles (considering also
that they can be placed in the medium at different initial spacial and temporal points for the
construction of the ensemble) should be expected to be equivalent to the set of all possible
space time evolving ensembles that can be observed. The above remarks suggest some possible
extensions of the present work, which are described below:

• The indicated dependence of the ending results of the shocks, not only of the initial
conditions, but also of the concrete realization of the random force, led us to think in
extending the results to the 2D and 3D cases. The idea is to study the spatial and temporal
behavior of the ensemble of outcomes of a series of shocks ”prepared ” with fixed initial
mean velocities for both particles, and to compare the results with the corresponding
predictions of the quantum scattering.

• Another task of interest which is suggested by this study is to investigate the existence
of preferential bounded states in the case that the Coulomb interaction is considered as
attractive. A particular simpler situation could be to assume one of the particles as very
massive, that is, merely acting as an attracting center. In both cases the study could
consist in determining the statistical properties of the stationary trajectories given the
initial condition.

In conclusion, we would like to underline that it seems not without sense that the realization
of the above proposed studies could be of use in the justification or search for hidden variable
approaches to Quantum Mechanics (QM). As it can be seen from the discussion, the resulting
models could not show the limitations of the Bohm approach (like the absence of predictions
for the ”guided” motions of the particles for real wave functions, for example). Moreover,
the analysis seems of interest in seeking for a theory not requiring to fix a particular random
process to each stationary state, but one in which all the statistical properties can emerge from a
general framework. In the imagined outcome, the particle could propagate having a probability
for transit or stay into each one of a set of stationary random motions which could be associated
(but now within the general context) to the particular eigenfunctions of the system. As for the
allowance of the necessary generalizations needed to make contact with reality, it can imagined
that the possible generalization of the statistical Noether properties (detected in the simple
model considered here) could lead to the conservation of the mean values for the angular
momentum and other internal quantities, in analogy with the case of the linear momentum.
We think these possibilities deserve further examinations that are expected to be considered
elsewhere.
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