INTEGRAL $p$-ADIC HODGE THEORY IN THE IMPERFECT RESIDUE FIELD CASE

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Abstract. Let $K$ be a mixed characteristic complete discrete valuation field with residue field admitting a finite $p$-basis, and let $G_K$ be the Galois group. We first classify semi-stable representations of $G_K$ by weakly admissible filtered $(\varphi, N)$-modules with connections. We then construct a fully faithful functor from the category of integral semi-stable representations of $G_K$ to the category of Breuil-Kisin $G_K$-modules. Using the integral theory, we classify $p$-divisible groups over the ring of integers of $K$ by minuscule Breuil-Kisin modules with connections.

1. Introduction

1.1. Overview and main theorems. Let $p$ be a prime. In this paper, we study $p$-adic Hodge theory, where we use various (semi-)linear algebra data to classify $p$-adic Galois representations. The theory works particularly well if one considers the Galois group of a CDVF (complete discrete valuation field) of mixed characteristic $(0, p)$ with perfect residue field such as a finite extension of $\mathbb{Q}_p$.

To study $p$-adic local systems of a general rigid space – which arise naturally, e.g., in the study of Shimura varieties – one is lead to the theory of relative $p$-adic Hodge theory. In this context, mixed characteristic CDVFs with imperfect residue fields naturally arise. For example, let $R = \mathbb{Z}_p(T^{\pm 1})$ be the (integral) ring corresponding to a rigid torus, and let $\widehat{R}(p)$ be the $p$-adic completion of the localization $R(p)$, then $\widehat{R}(p)[1/p]$ is a CDVF with residue field $\mathbb{F}_p(T)$. Indeed, localizing $R$ at some other ideals also give rise to CDVFs with imperfect residue fields, cf. e.g. [Bri08, §3.3]. To understand $p$-adic representations of $\pi^1(R[1/p])$ (called “the relative case” in the following), it turns out to be crucial to understand representations for these CDVFs with imperfect residue fields (called “the imperfect residue field case” in the following), such as representations of the Galois group of $\widehat{R}(p)[1/p]$.

In fact, Brinon studies the Hodge-Tate, de Rham, and crystalline representations in the imperfect residue field case in [Bri06], which then pave the way for the studies in the relative case in [Bri08]. Throughout this paper, let $K$ be a CDVF of characteristic 0 with residue field $k_K$ of characteristic $p$ such that $[K_K : k_K] < \infty$, and let $G_K$ be the Galois group. Our first main result is the following.

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Theorem 1.1.1. (= Thm. 2.4.9) The category of semi-stable representations of $G_K$ is equivalent to the category of weakly admissible filtered $(\varphi, N)$-modules with connections (cf. Def. 2.4.5).

Remark 1.1.2. (1) When the residue field of $K$ is perfect, then this is the classical theorem of Colmez-Fontaine [CF00]; note that in the perfect residue field case, the connection is automatically a zero map.

(2) For general $K$, the crystalline case of Thm. 1.1.1 is obtained in [Bri06].

We now have quite good understanding of general $p$-adic Galois representations in the relative case; for example, the “relative $(\varphi, \Gamma)$-modules” (as well as their overconvergent versions) are constructed in [And06, AB08, KL15, KL]. Next thing in order is to study the Hodge-Tate, de Rham, crystalline and semi-stable representations in the relative case. Brinon initiated these studies in the Hodge-Tate, de Rham, and crystalline case in [Bri08]; indeed, with Thm. 1.1.1 established, we should be able to obtain some similar results in the semi-stable case. Recently, the rigidity theorem of Liu-Zhu [LZ17] provides a surprisingly simple description of de Rham local systems. Then, a rigidity theorem for almost Hodge-Tate local systems is established by Shimizu [Shi18]. Further, Shimizu obtains a variant of $p$-adic local monodromy theorem in the relative case in [Shi]. In summary, it seems that we now have some quite interesting understanding of local systems in the de Rham (and almost Hodge-Tate) case, but perhaps not so much in the crystalline or semi-stable case; for example, our understanding of the notion “weak admissibility” is not completely satisfactory so far, cf. the discussion on [Bri08, p. 136].

In the study of crystalline and semi-stable representations in the perfect residue field case, an important phenomenon is that we can give very precise classifications of integral crystalline and semi-stable representations, the study of which we call integral $p$-adic Hodge theory. The second main theorem in the paper is the following Thm. 1.1.3, which we regard as a very first step towards the study of relative integral $p$-adic Hodge theory.

Theorem 1.1.3. (= Thm. 4.4.4) There is a fully faithful functor from the category of integral semi-stable representations of $G_K$ to the category of Breuil-Kisin $G_K$-modules (cf. Def. 4.4.2).

Remark 1.1.4. (1) When the residue field of $K$ is perfect, then the main result of [Gao] says that the functor in the theorem is indeed an equivalence.

(2) In comparison with Thm. 1.1.1, it might seem tempting to construct some category of Breuil-Kisin $G_K$-modules with connections. However, it seems difficult to characterize the properties of the possible connection operator, cf. Rem. 4.4.5. Nonetheless, if we focus on integral crystalline representations with Hodge-Tate weights in $\{0, 1\}$, which correspond to $p$-divisible groups over $O_K$ (the ring of integers of $K$), then we can indeed construct some category with connection operators, cf. Thm. 1.1.5 in the following.

Using the integral theory developed in this paper, we can classify $p$-divisible groups over $O_K$.

Theorem 1.1.5. (= Thm. 6.3.3) The category of $p$-divisible groups over $O_K$ is equivalent to the category of minuscule Breuil-Kisin modules with connections.

Remark 1.1.6. (1) When $K$ has perfect residue field, the theorem is due to Kisin [Kis06] when $p > 2$, and independently to Kim [Kim12], Lau [Lau14] and Liu [Liu13] when $p = 2$.

(2) For general $K$, the theorem is known when $p > 2$ (cf. Rem. 6.3.4): one can deduce it by an easy combination of results in [BT08] and a theorem of Caruso-Liu [CL09]. (Alternatively, it can also be regarded as a special case of results of Kim [Kim15], where $p$-divisible groups for $p > 2$ in the relative case are classified).

(3) When $p = 2$ (and for general $K$), new ideas are needed. In fact, a key ingredient in the proof is that we can first show that the category of $p$-divisible groups
over \( \mathcal{O}_K \) is equivalent to the category of integral crystalline representations of \( G_K \) with Hodge-Tate weights in \( \{0, 1\} \). Then we use our integral theory to construct the equivalence with the category of minuscule Breuil-Kisin modules with connections.

(4) As a consequence of Thm. 1.1.5, we can classify finite flat group schemes over \( \mathcal{O}_K \), cf. Thm. 6.4.4.

We propose some speculations and questions for future investigations.

Remark 1.1.7. (1) What is the possible generalization of Thm. 1.1.3 in the relative case?

(2) What is the relation between Thm. 1.1.5 and the filtered prismatic Dieudonné modules in [ALB]?

(3) As mentioned in Rem. 1.1.6(2), \( p \)-divisible groups in the relative case are classified by Kim [Kim15] when \( p > 2 \); to study the \( p = 2 \) case, it seems likely the ideas (and possibly even the results) in this paper could be useful.

1.2. Structure of the paper. In §2, we use Fontaine modules (i.e., weakly admissible filtered \((\varphi, N)\)-modules with connections) to classify semi-stable representations. In §3, we build the link from Fontaine modules to Kisin’s \( \mathcal{A} \)-modules and \( \mathcal{S} \)-modules. In §4, we use Breuil-Kisin \( G_K \)-modules to study integral semi-stable representations. In §5, we review results about \( \varphi \)-modules over \( \mathcal{S} \) and \( S \). In §6, we classify \( p \)-divisible groups over \( \mathcal{O}_K \).

1.3. Some notations and conventions.

Convention 1.3.1. Categories and co-variant functors.

- In this paper we will define many categories of modules (with various structures); we will always omit the definition of morphisms for these categories, which are always obvious (i.e., module homomorphisms compatible with various structures).
- When we define functors relating various categories, we will always use co-variant functors. This makes the comparisons amongst them easier (i.e., using tensor products, rather than Hom’s).

Convention 1.3.2. Hodge-Tate weights, and Breuil-Kisin heights.

- In accordance with Convention 1.3.1, our \( D_{st}(V) \) is defined as \( (V \otimes \mathbb{Q}_p \mathcal{B}_{st})^{G_K} \), and hence the Hodge-Tate weight of the cyclotomic character \( \chi_p \) is \(-1\).
- Once we move on to study integral theory from §3, we will focus on representations with non-negative Hodge-Tate weights and Breuil-Kisin modules with non-negative \( E(u) \)-heights. For example, the Breuil-Kisin module associated to \( \chi^{−1} \) has \( E(u) \)-height 1.

Notation 1.3.3. Let \( H \) be a profinite group, then we use \( \text{Rep}_{\mathbb{F}_p}(H) \) (resp. \( \text{Rep}_{\mathbb{Z}_p}(H) \), resp. \( \text{Rep}_{\mathbb{Q}_p}(H) \)) denote the category of finite dimensional \( \mathbb{F}_p \)-vector spaces (resp. finite free \( \mathbb{Z}_p \)-modules, resp. finite dimensional \( \mathbb{Q}_p \)-vector spaces) \( V \) with continuous \( \mathbb{F}_p \)- (resp. \( \mathbb{Z}_p \)-, resp. \( \mathbb{Q}_p \)-) linear \( H \)-actions. Sometimes we put superscripts such as “cris, st, ≥ 0” etc. over \( \text{Rep}_{\mathbb{Z}_p}(H) \) or \( \text{Rep}_{\mathbb{Q}_p}(H) \) for the obvious meaning: i.e., crystalline representations, resp. semi-stable representations, resp. those with Hodge-Tate weights \( ≥ 0 \).

Notation 1.3.4. Throughout this paper, we reserve \( \varphi \) to denote Frobenius operator. We sometimes add subscripts to indicate on which object Frobenius is defined. For example, \( \varphi_M \) is the Frobenius defined on \( M \). We always drop these subscripts if no confusion arises. Given a homomorphism of rings \( \varphi : A \to A \) and given an \( A \)-module \( M \), denote \( \varphi^* M := A \otimes_{\varphi, A} M \). For a category \( \mathcal{C} \) whose morphism sets are \( \mathbb{Z}_p \)-modules, we use \( \mathcal{C} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) to denote its isogeny category.

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2. Fontaine modules and semi-stable representations

In this section, we first review the Fontaine rings and the Fontaine modules (in the imperfect residue field case). We then show that weakly admissible filtered \((\varphi, N)\)-modules with connections are admissible, and hence classify semi-stable representations.

2.1. Notations of base fields. Let \(K\) be a complete discrete valuation field of characteristic 0 with residue field \(k_K\) of characteristic \(p > 0\) such that \([k_K : k_p^d] = p^d\) where \(d \geq 0\). Fix an algebraic closure \(\overline{K}\) of \(K\) and let \(G_K = \text{Gal}(\overline{K}/K)\). We first set up notations for some other fields related with \(K\).

The notations below depend on various choices, particularly that of \(K_0\) and \(\varphi\). The way these choices are made are slightly different in the references [Bri06], [BT08], [Mor10, Mor14] and [Ohk13]. But they are all “equivalent” definitions, cf. the detailed discussions in [Ohk13, 1A, 1G]. Our exposition here largely follows that in [Bri06].

Fix a closed sub-field \(K_0 \subset K\) with residue field \(k_K \subset k_{K_0}\) and with \(p\) as a uniformizer. Note that such \(K_0\) is not unique (unless \(d = 0\) or \(p\) is a uniformizer of \(K\)); but the ramification index \(e_K = [K : K_0]\) is uniquely determined. Let \(\mathcal{O}_K, \mathcal{O}_{K_0}\) be the ring of integers. Let \(i_1, \cdots, i_d\) be a basis of \(k_K\) over \(k_p^d\), and fix some lifts \(t_1, \cdots, t_d \in \mathcal{O}_{K_0}\). Fix a Frobenius \(\varphi\) on \(\mathcal{O}_{K_0}\) which lifts the \(p\)-power map on \(k_K\); such \(\varphi\) is not unique.

Let \(k_k\) be the residue field of \(\overline{k}\), and let \(k\) be the radical closure of \(k_K \subset k_k\). The choices \(K_0\) and \(\varphi\) determines a \(\varphi\)-equivariant embedding
\[
i_{\varphi} : \mathcal{O}_{K_0} \hookrightarrow W(k),
\]
where \(W(k) = \text{the ring of Witt vectors with } \varphi_{W(k)}\) the usual Frobenius map. Let \(K_0 = W(k)[1/p]\), and fix \(\overline{k}\) an algebraic closure of \(K_0\). The map \(i_{\varphi}\) induces
\[
i_{\varphi} : \overline{K} \hookrightarrow \overline{k}
\]
with dense image and hence further induces
\[
i_{\varphi} : C \simeq \mathbb{C},
\]
where \(C\) (resp. \(\mathbb{C}\)) is the \(p\)-adic completion of \(\overline{K}\) (resp. \(\overline{k}\)). Let \(\mathcal{K} = \mathcal{K}_0 i_{\varphi}(K) \subset \overline{k}\), and let \(G_K = \text{Gal}(\overline{K}/K)\). Via \(i_{\varphi} : \overline{K} \hookrightarrow \overline{k}\), we can identify \(G_K\) as a subgroup of \(G_K\).

2.2. Fontaine rings. In this subsection, we give a quick review the Fontaine rings in the imperfect residue field case; the presentations here are not necessarily in the order they were first developed. We refer the readers to [Bri06] and [Mor14] for more details.

Notation 2.2.1. Starting from \(\mathcal{K}\) - a CDVF with perfect residue field– and the perfectoid field \(\mathbb{C}\), we use
\[
\text{(2.2.1)} \quad \mathcal{B}_{\text{dR}}^+, \mathcal{B}_{\text{dR}}^-, \mathcal{B}_{\text{HT}}, \mathcal{A}_{\text{cris}}^+, \mathcal{B}_{\text{cris}}^+, \mathcal{B}_{\text{cris}}^-, \mathcal{B}_{\text{st}}, \mathcal{A}_{\text{st}}, \mathcal{B}_{\text{st}}^+, \mathcal{B}_{\text{st}}^-
\]
to denote the “usual” Fontaine rings (in the perfect residue field case).

Notation 2.2.2. Indeed, all the rings in (2.2.1) depend only on the perfectoid field \(\mathbb{C}\) (and not \(\mathcal{K}\)). Hence starting from the perfectoid field \(\mathbb{C}\), we can also define the corresponding rings; denote them as
\[
\text{(2.2.2)} \quad \mathcal{B}_{\text{dR}}^{\nabla}, \mathcal{B}_{\text{dR}}^\nabla, \mathcal{B}_{\text{HT}}^{\nabla}, \mathcal{A}_{\text{cris}}^{\nabla}, \mathcal{B}_{\text{cris}}^{\nabla}, \mathcal{B}_{\text{cris}}^{\nabla}, \mathcal{A}_{\text{st}}^{\nabla}, \mathcal{B}_{\text{st}}^{\nabla}, \mathcal{B}_{\text{st}}^{\nabla}.
\]
(The superscripts \(\nabla\) will be explained in §2.4.2.) The isomorphism \(i_{\varphi} : C \simeq \mathbb{C}\) induces ring isomorphisms
\[
i_{\varphi} : \mathcal{B}_*^{\nabla} \simeq \mathcal{B}_*, \text{ for } * \in \{\text{dR, HT, cris, st}\}.
\]
These isomorphisms are compatible with various \(\varphi\)-, \(N\)- and filtration structures, and are \(G_K\)-equivariant (as we mentioned in §2.1, we regard \(G_K\) as a subgroup of \(G_K\)).
**Notation 2.2.3.** Since $O_C$ is a perfectoid ring, we can define its tilt $O_C^\psi$; let $W(O_C^\psi)$ be the ring of Witt vectors. Elements in $O_C^\psi$ are in bijection with sequences $(x^{(n)})_{n \geq 0}$ where $x^{(n)} \in O_C$ and $(x^{(n+1)})^p = x^{(n)}$. For each $1 \leq i \leq d$, fix some $t_i \in O_C^\psi$ with $\tilde{t}_i^{(0)} = t_i$, and let $[\tilde{t}_i]$ be its Teichmüller lift. Let

$$u_i = t_i \otimes 1 - 1 \otimes [\tilde{t}_i] \in O_K \otimes_{\mathbb{Z}} W(O_C^\psi).$$

Let

$$\theta: W(O_C^\psi) \to O_C$$

be the usual Fontaine’s map.

**Notation 2.2.4.** By scalar extension, the $\theta$-map induces a map

$$\theta_K: O_K \otimes_{\mathbb{Z}} W(O_C^\psi) \to O_C.$$

Let

$$A_{\text{inf}}(O_C/O_K) := \lim_{n \to \infty} \left( O_K \otimes_{\mathbb{Z}} W(O_C^\psi) \right) / \left( \theta_K^{-1}(pO_C^\psi) \right)^n.$$

Then the $\theta_K$-map extends to

$$\theta_K: A_{\text{inf}}(O_C/O_K) \to O_C \quad \text{and} \quad \theta_K: A_{\text{inf}}(O_C/O_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to C.$$

Let

$$B_{dR}^+: = \lim_{n \to 1} (A_{\text{inf}}(O_C/O_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) / (\text{Ker} \theta_K)^n.$$

There are natural maps $W(O_C^\psi) \to A_{\text{inf}}(O_C/O_K)$ and $\text{Ker} \theta \to \text{Ker} \theta_K$, and they induce a natural map

$$B_{dR}^+ \to B_{dR}^+;$$

By [Bri06, Prop. 2.9], it further induces an isomorphism of rings

$$(2.2.3) \quad B_{dR}^+[[u_1, \ldots, u_d]] \simeq B_{dR}^+. $$

This implies that $B_{dR}^+$ is a $\overline{K}$-algebra, and is a local ring with maximal ideal

$$m_{dR} = \text{Ker} \theta_K = (t, u_1, \ldots, u_d),$$

where $t \in B_{dR}^+$ is the usual element. Define a filtration on $B_{dR}^+$ where

$$\text{fil}^i B_{dR}^+ = m_{dR}^i, \forall i \geq 0.$$}

Using $\text{fil}^i$, define another filtration on $B_{dR} := B_{dR}^+[1/t]$ by:

$$\text{Fil}^0 B_{dR} = \sum_{n=0}^{\infty} t^{-n} \text{fil}^n B_{dR}^+ = B_{dR}^+[\frac{u_1}{t}, \ldots, \frac{u_d}{t}],$$

$$\text{Fil}^i B_{dR} = t^i \text{Fil}^0 B_{dR}, \forall i \in \mathbb{Z}.$$

$B_{dR}$ carries a natural $G_K$-action such that $\text{Fil}^i B_{dR}$ is stable under the action.

**Notation 2.2.5.** Let $B_{HT}$ be the graded algebra associated to $\text{Fil}^i$ of $B_{dR}$. $B_{HT}$ is a $C$-algebra and it carries an induced $G_K$-action. Clearly

$$B_{HT} \simeq B_{HT}^+[\frac{u_1}{t}, \ldots, \frac{u_d}{t}] \simeq C[t, t^{-1}, \frac{u_1}{t}, \ldots, \frac{u_d}{t}].$$

**Notation 2.2.6.** Let

$$\theta_K: O_{K_0} \otimes_{\mathbb{Z}} W(O_C^\psi) \to O_C$$

denote the natural extension of $\theta: W(O_C^\psi) \to O_C$. Let $A_{\text{crys}}$ be the $p$-adic completion of the PD-envelope of $O_{K_0} \otimes_{\mathbb{Z}} W(O_C^\psi)$ with respect to $\text{Ker}(\theta_K)$. Define

$$B_{crys}^+ := A_{\text{crys}}[1/p], \quad B_{crys} := B_{crys}^+[1/t].$$

The ring $B_{crys}$ is the $K_0$-algebra endowed with natural commuting $G_K$-action and $\varphi$-action. Furthermore, by [Bri06, Prop. 2.47], there is a natural injection

$$K \otimes_{K_0} B_{crys} \to B_{dR},$$
and hence $K \otimes_{K_0} B_{\text{cris}}$ is endowed with a filtration induced by that of $B_{\text{dR}}$. By [Bri06, Prop. 2.39], there is an isomorphism of rings
\[(2.2.4) \quad f : \text{p-}
adic completion of $A_{\text{cris}}\langle u_1, \ldots, u_d \rangle_{\text{PD}} \simeq A_{\text{cris}}$
where $(\ast)_{\text{PD}}$ denotes PD-polynomials.

**Notation 2.2.7.** (cf. [Mor14, §2]) Fix an element $\tilde{p} = (s^{(n)} ) \in O_C$ such that $s^{(0)} = p$. Then, the series $\log(\tilde{p}/p)$ converges to an element $u$ in $B_{\text{dR}}^s$, and $u$ is transcendental over $B_{\text{cris}}$. Let
\[
A_{\text{st}} := A_{\text{cris}}[u], \quad B_{\text{st}}^+ := B_{\text{cris}}^+[u], \quad B_{\text{st}} := B_{\text{cris}}[u];
\]
all these are regarded as subrings of $B_{\text{dR}}$, and are independent of $\tilde{p}$. $B_{\text{st}}$ is $G_K$-stable inside $B_{\text{dR}}$, and $K \otimes_{K_0} B_{\text{st}}$ is endowed with a filtration induced by that of $B_{\text{dR}}$. We extend the Frobenius $\varphi$ on $B_{\text{cris}}$ to $B_{\text{st}}$ by setting $\varphi(u) = pu$. Furthermore, define the $B_{\text{cris}}$-derivation $N : B_{\text{st}} \to B_{\text{st}}$ by $N(u) = -1$. It is easy to verify $N\varphi = p\varphi N$. Induced by (2.2.4), we have an isomorphism
\[(2.2.5) \quad f : (\text{p-}
adic completion of $A_{\text{cris}}\langle u_1, \ldots, u_d \rangle_{\text{PD}})[1/p, u, 1/t] \simeq B_{\text{st}}$.

**Proposition 2.2.8.** There are canonical isomorphisms
\[(B_{\text{dR}})^{G_K} = K, \quad (B_{\text{HT}})^{G_K} = K, \quad (B_{\text{cris}})^{G_K} = K_0, \quad (B_{\text{st}})^{G_K} = K_0.
\]

*Proof.* See [Bri06, Prop. 2.16, Prop. 2.50] and [Mor14, Prop. 2.2].

2.3. **Fontaine functors.** Let $V$ be a $p$-adic representation of $G_K$. Define
\[D_*(V) = (B_\ast \otimes_{Q_p} V)^{G_K}, \quad \text{for } \ast \in \{\text{dR, HT, crs, st}\}.
\]
By [Bri06] for $\ast \in \{\text{dR, HT, crs}\}$ and by [Mor14] for $\ast = \text{st}$, $D_{\text{dR}}(V)$ (resp. $D_{\text{HT}}(V)$, resp. $D_{\text{cris}}(V)$, resp. $D_{\text{st}}(V)$) is a $K_0$-vector space of dimension $\leq \dim_{Q_p} V$.

**Definition 2.3.1.**
1. $V$ is a called a de Rham (resp. Hodge-Tate, resp. crystalline, resp. semi-stable) representation of $G_K$ if the aforementioned $K$- or $K_0$-vector space is of dimension $\dim_{Q_p} V$.
2. Say $V$ is potentially de Rham (resp. potentially Hodge-Tate, resp. potentially crystalline, resp. potentially semi-stable) if there exists a finite field extension $L/K$ such that $V|_{G_L}$ is a de Rham (resp. Hodge-Tate, resp. crystalline, resp. semi-stable) representation of $G_L$.

For the reader’s convenience, we summarize the relations of these notions in Thm. 2.3.2 and Thm. 2.3.3. These results will not be used in this paper.

**Theorem 2.3.2.** [Mor10, Mor14] Let $V$ be a $p$-adic representation of $G_K$.

1. $V$ is de Rham $\iff$ $V$ is potentially de Rham $\iff$ $V|_{G_K}$ is de Rham (as a representation of $G_K$, similar in the following).
2. $V$ is Hodge-Tate $\iff$ $V$ is potentially Hodge-Tate $\iff$ $V|_{G_K}$ is Hodge-Tate.
3. $V$ is potentially crystalline $\iff$ $V|_{G_K}$ is potentially crystalline.
4. $V$ is potentially semi-stable $\iff$ $V|_{G_K}$ is potentially semi-stable.

*Proof.* Items (1) and (2) are proved in [Mor10]; Items (3) and (4) are proved in [Mor14].

**Theorem 2.3.3.** Let $V$ be a $p$-adic representation of $G_K$. Then $V$ is de Rham $\iff$ $V$ is potentially semi-stable. (Hence conditions in Items (1) and (4) of Thm. 2.3.2 are all equivalent.)

*Proof.* The first proof is due to Morita: it is an easy consequence of Items (1) and (4) of Thm. 2.3.2, together with Berger’s $p$-adic local monodromy theorem [Ber02] (for de Rham representations of $G_K$). Alternatively, another proof which works even when $[k : k^p] = +\infty$ is supplied by [Ohk13].
2.4. Weakly admissible implies admissible. In this subsection, we prove the main result of this section, namely the Colmez-Fontaine theorem in the imperfect residue field case. In order to do so, we need to introduce connection operators on various rings and modules.

Notation 2.4.1. Let
\[
\hat{\Omega}_{K_0} := \varprojlim_{n>0} \Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}}/p^n\Omega^1_{\mathcal{O}_{K_0}/\mathbb{Z}}
\]
be the \(p\)-adically continuous Kähler differentials, and let
\[
\widehat{\Omega}_{K_0} = \hat{\Omega}_{K_0} \otimes_{\mathcal{O}_{K_0}} K_0.
\]
\(\widehat{\Omega}_{K_0}\) is a free \(\mathcal{O}_{K_0}\)-module with a set of basis \(\{d \log(t_i)\}_{1 \leq i \leq d}\), where \(t_1, \cdots, t_d \in \mathcal{O}_{K_0}\) are introduced in §2.1 as a lift of a \(p\)-basis of \(k_K\). Let
\[
d : \mathcal{O}_{K_0} \to \widehat{\Omega}_{K_0}
\]
be the canonical differential map. For \(D\) a \(\mathcal{O}_{K_0}\)-module, a connection on \(D\) is an additive map
\[
\nabla : D \to D \otimes_{\mathcal{O}_{K_0}} \hat{\Omega}_{K_0}
\]
satisfying Leibniz law with respect to \(d\). Let \(\hat{\Omega}_{K}\) and \(d : \mathcal{O}_K \to \hat{\Omega}_{K}\) be similar defined; then one can also define connections on \(\mathcal{O}_K\)-modules.

Notation 2.4.2. Recall that we have \(B_{\text{dR}}^+ = B_{\text{dR}}^+[[u_1, \cdots, u_d]]\). For \(1 \leq i \leq d\), let \(N_i\) be the unique \(B_{\text{dR}}^+\)-derivation of \(B_{\text{dR}}^+\) such that
\[
N_i(u_j) = \delta_{i,j} u_j
\]
where \(\delta_{i,j}\) is the Kronecker symbol; hence in particular we have \(N_i(t) = 0, \forall i\), and thus \(N_i\) extends to a \(B_{\text{dR}}^+\)-derivation of \(B_{\text{dR}}\). Define a map
\[
(2.4.1) \quad \nabla : B_{\text{dR}} \to B_{\text{dR}} \otimes_{K_0} \hat{\Omega}_{K_0}, \quad x \mapsto \sum_{i=1}^d N_i(x) \otimes d \log(t_i);
\]
the connection is integrable as \(N_i\)’s commute with each other. Note that
\[
(B_{\text{dR}})^{\nabla=0} = B_{\text{dR}}^\nabla,
\]
which explains the notation “\(B_{\text{dR}}^\nabla\)” in §2.2.2. We list some basic properties of \(\nabla\).

1. By [Bri06, Prop. 2.23], \(\nabla\) satisfies Griffith transversality, i.e.,
\[
\nabla(\text{Fil}^r B_{\text{dR}}) \subset \text{Fil}^{r-1} B_{\text{dR}} \otimes_{K_0} \hat{\Omega}_{K_0}.
\]
2. By [Bri06, Prop. 2.24], \(\nabla\) commutes with \(G_K\)-action.
3. By [Bri06, Prop. 2.25], \(\nabla|_K\) is precisely the canonical differential \(d : K \to K \otimes_{K_0} \hat{\Omega}_{K_0}\).

Notation 2.4.3. One can easily check (cf. [Bri06, p. 950]) that via \(B_{\text{cris}} \hookrightarrow B_{\text{dR}}\), \(\nabla\) on \(B_{\text{dR}}\) induces an integrable connection
\[
\nabla : B_{\text{cris}} \to B_{\text{cris}} \otimes_{K_0} \hat{\Omega}_{K_0}.
\]
Since \(u \in B_{\text{dR}}^\nabla\) and hence \(\nabla(u) = 0\), we further have an integrable connection
\[
\nabla : B_{\text{st}} \to B_{\text{st}} \otimes_{K_0} \hat{\Omega}_{K_0}.
\]
We have
\[
(B_{\text{cris}})^{\nabla=0} = B_{\text{cris}}^\nabla, \quad (B_{\text{st}})^{\nabla=0} = B_{\text{st}}^\nabla.
\]
By [Bri06, Prop. 2.58], \(\varphi \nabla = \nabla \varphi\) over \(B_{\text{cris}}\) (and hence also over \(B_{\text{st}}\)); here the \(\varphi\)-action on \(\hat{\Omega}_{K_0}\) extends the \(\varphi\)-action on \(K_0\) and \(\varphi(dt_i) = d(\varphi(t_i))\). In addition, we obviously have \(N \nabla = \nabla N\) over \(B_{\text{st}}\), where \(N = 0\) on \(\hat{\Omega}_{K_0}\).

Definition 2.4.4. [Bri06, Def. 4.3]
(1) Given a $p$-adic separated and complete $\mathcal{O}_{K_0}$-module $\mathcal{D}$, a connection $\nabla : \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{O}_{K_0}} \hat{\Omega}_{K_0}$ is called quasi-nilpotent if the induced connection

$$\nabla : \mathcal{D}/p\mathcal{D} \to \mathcal{D}/p\mathcal{D} \otimes_{\mathcal{O}_{K_0}} \hat{\Omega}_{K_0}$$

is nilpotent.

(2) Given a $K_0$-vector space $\mathcal{D}$, a connection $\nabla : \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{O}_{K_0}} \hat{\Omega}_{K_0}$ is called quasi-nilpotent if there exists an $\mathcal{O}_{K_0}$-submodule $\mathcal{D} \subset \mathcal{D}$ which is $p$-adic separated and complete such that $\mathcal{D}[1/p] = \mathcal{D}$ and

$$\nabla(\mathcal{D}) \subset \mathcal{D} \otimes_{\mathcal{O}_{K_0}} \hat{\Omega}_{K_0},$$

and such that the restricted connection $\nabla |_{\mathcal{D}}$ is quasi-nilpotent as in Item (1) above.

**Definition 2.4.5.** Let $\mathbf{MF}_{K/K_0}(\varphi, N, \nabla)$ be the category of filtered $(\varphi, N)$-modules with connections which consists of finite dimensional $K_0$-vector spaces $\mathcal{D}$ equipped with

1. a $\varphi_{K_0}$-semi-linear Frobenius $\varphi : \mathcal{D} \to \mathcal{D}$ such that $\varphi(D)$ generates $\mathcal{D}$ over $K_0$;
2. a monodromy $N : \mathcal{D} \to \mathcal{D}$, which is a $K_0$-linear map such that $N\varphi = p\varphi N$;
3. an integrable quasi-nilpotent connection $\nabla : \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{O}_{K_0}} \hat{\Omega}_{K_0}$, which commutes with $\varphi$ and $N$ (here on the right hand side, $\varphi = \varphi_{\mathcal{D}} \otimes \varphi_{\hat{\Omega}_{K_0}}$ and $N = N_{\mathcal{D}} \otimes 1$, cf. §2.4.3 for $\varphi$ and $N$ over $\hat{\Omega}_{K_0}$);
4. a decreasing, separated and exhaustive filtration $(\text{Fil}^i D_K)_{i \in \mathbb{Z}}$ on $D_K = \mathcal{D} \otimes_{\mathcal{O}_{K_0}} K$, by $K$-vector subspaces, such that

$$\nabla(\text{Fil}^i D_K) \subset \text{Fil}^{i-1} D_K, \forall i,$$

where $\nabla : D_K \to D_K \otimes_{\mathcal{O}_{K_0}} \hat{\Omega}_{K_0}$ is the induced connection (satisfying Leibniz law with respect to $d : K \to \hat{\Omega}_{K_0}$).

**Remark 2.4.6.** When $K$ has imperfect residue field, $\varphi$ on $K_0$ is not necessarily bijective; hence we need to require “$\varphi(D)$ generates $\mathcal{D}$ over $K_0$” in Def. 2.4.5. This implies that

$$K_0 \otimes_{\varphi, K_0} D \xrightarrow{1 \otimes \varphi} D$$

is a $K_0$-linear isomorphism.

**Proposition 2.4.7.** Let $\text{Rep}^{st}_{\mathbb{Q}_p}(G_K)$ be the category of semi-stable representations of $G_K$. Then $D_{st}$ induces a functor

$$\text{Rep}^{st}_{\mathbb{Q}_p}(G_K) \to \mathbf{MF}_{K/K_0}(\varphi, N, \nabla).$$

**Proof.** The $N = 0$ (i.e., crystalline) case is proved in [Bri06, Prop. 4.19]. The semi-stable case follows from similar argument; in particular, to show that the induced connection on $D_{st}(V)$ is quasi-nilpotent, one simply replaces the use of $\mathbf{A}_{\text{cris}}$ in loc. cit. by $\mathbf{A}_{st}$. 

**Definition 2.4.8.**

1. Let $\mathbf{MF}^{wa}_{K/K_0}(\varphi, N, \nabla)$ be the subcategory defined by the essential image of $D_{st}$; the objects in $\mathbf{MF}^{wa}_{K/K_0}(\varphi, N, \nabla)$ is called admissible objects.
2. For $D \in \mathbf{MF}_{K/K_0}(\varphi, N, \nabla)$, one can define the Newton number $t_N(D)$ and the Hodge number $t_H(D)$ with the usual recipe, cf. [Bri06, §4.1]. $D$ is called weakly admissible if $t_N(D) = t_H(D)$ and $t_N(D') \leq t_H(D')$ for any sub-object (in the category $\mathbf{MF}_{K/K_0}(\varphi, N, \nabla)$) $D' \subset D$. Denote the sub-category of weakly admissible objects as $\mathbf{MF}^{wa}_{K/K_0}(\varphi, N, \nabla)$.

**Theorem 2.4.9.** The functor $D_{st}$ induces an equivalence of categories

$$\text{Rep}^{st}_{\mathbb{Q}_p}(G_K) \simeq \mathbf{MF}^{wa}_{K/K_0}(\varphi, N, \nabla).$$

A quasi-inverse is given by

$$V_{st}(D) = (\mathbf{B}_{st} \otimes_{K_0} D)_{\varphi = 1, N = 0, \nabla = 0} \cap \text{Fil}^0(D_K \otimes_K \mathbf{B}_{dR}).$$
Proof. The proof follows the same strategy of [Bri06, Thm. 4.34] (the crystalline case). Here, we content ourselves by giving a sketch of the argument; in particular, we point out how the many ingredients in [Bri06, §4.2] still hold in the semi-stable case. In the following, let

\[ D \in \text{MF}_{K/K_0}^{wa}(\varphi, N, \nabla), \quad V = V_{st}(D). \]

**Step 1:** By the semi-stable version (see later) of [Bri06, Prop. 4.28], we know that \( \dim_{Q_p} V < \infty \), and \( V \) is semi-stable. Furthermore, if \( D' = D_{st}(V) \), then \( D' \subset D \) is a sub-object in \( \text{MF}_{K/K_0}^{wa}(\varphi, N, \nabla) \). Now, let us sketch why the semi-stable version of [Bri06, Prop. 4.28] holds:

1. The semi-stable version of [Bri06, Thm. 4.24] obviously holds; namely, \( D_{st} \) induces an equivalence of Tannakian categories

\[ \text{Rep}_{Q_p}^{st}(G_K) \xrightarrow{\simeq} \text{MF}_{K/K_0}^{st}(\varphi, N, \nabla), \]

and \( V_{st} \) is a quasi-inverse.

2. The “semi-stable version” of [Bri06, Prop. 4.26] holds: in fact, an object \( E \in \text{MF}_{K/K_0}^{st}(\varphi, N, \nabla) \) of dimension 1 automatically satisfies \( N = 0 \)! Hence we automatically have \( V_{st}(E) = V_{\text{cris}}(E) \).

3. The semi-stable version of [Bri06, Prop. 4.27] holds: namely, if \( E \in \text{MF}_{K/K_0}^{st}(\varphi, N, \nabla) \) is admissible, then it is weakly admissible. The proof relies heavily on [Bri06, Prop. 4.26], which is still applicable as we just mentioned.

**Step 2:** Let \( D_{K_0} := K_0 \otimes_K D \), then we naturally get an object in \( \text{MF}_{K/K_0}^{st}(\varphi, N) \), where \( \text{MF}_{K/K_0}^{st}(\varphi, N) \) is the category defined “using \( K \) instead of \( K' \)” in Def. 2.4.5; note that \( \nabla \)-operators disappear because \( \Omega_{K_0} = 0 \). Note that a priori, it is not clear if \( D_K \) is weakly admissible in \( \text{MF}_{K/K_0}^{st}(\varphi, N) \) (although we will see later it is). We claim that there is an isomorphism of vector spaces:

\[ V \xrightarrow{\simeq} (\mathcal{B}_{st} \otimes_K D_{K_0})_{\varphi=1, N=0} = \text{Fil}^0(B_{\text{dR}} \otimes_K D_K) = \mathbb{V}_{st}(D_{K_0}), \]

where \( \mathbb{V}_{st} \) is the obvious functor \( \text{MF}_{K/K_0}^{st}(\varphi, N) \to \text{Rep}_{Q_p}(G_K) \). This is the semi-stable version of one of the displayed equations in [Bri06, Thm. 4.34], and we sketch the proof here:

1. By [Bri06, Cor. 4.31], we have an isomorphism

\[ \text{Fil}^0(B_{\text{dR}} \otimes_K D_K)^{\nabla=0} \simeq \text{Fil}^0(B_{\text{dR}} \otimes_K D_K) = \text{Fil}^0(B_{\text{dR}} \otimes_K D_K). \]

Note that [Bri06, Cor. 4.31] only concerns the de Rham period rings and \( \nabla \)-structure on modules; it has nothing to do with \( \varphi \)- or \( N \)-structures.

2. The semi-stable version of [Bri06, Prop. 4.32] and hence [Bri06, Cor. 4.33] still holds. To prove the semi-stable version of [Bri06, Prop. 4.32], it suffices to show that for any \( \nabla \)-modules over \( K_0 \), we have

\[ \text{Hom}_{K_0, \nabla}(D, \mathcal{B}_{st}) \simeq \text{Hom}_{K_0}(D, \mathcal{B}_{st}). \]

Note that we have decompositions

\[ \mathcal{B}_{st} = \bigoplus_{i \geq 0} (\mathcal{B}_{\text{cris}} \cdot u^i), \quad \mathcal{B}_{st} = \bigoplus_{i \geq 0} (\mathcal{B}_{\text{cris}} \cdot u^i), \]

which are both \( K_0 \)-linear; the first decomposition is \( \nabla \)-stable since \( \nabla(u) = 0 \). Hence it suffices to show that the decomposed (“crystalline”) pieces of (2.4.4) are isomorphic to each other, namely,

\[ \text{Hom}_{K_0, \nabla}(D, \mathcal{B}_{\text{cris}} \cdot u^i) \simeq \text{Hom}_{K_0}(D, \mathcal{B}_{\text{cris}} \cdot u^i), \quad \forall i; \]

these follow from [Bri06, Prop. 4.32]. Note that (2.4.4) implies the full semi-stable version of [Bri06, Prop. 4.32]. Thus the semi-stable version of [Bri06, Cor. 4.33] holds by some obvious duality argument; namely we have

\[ (\mathcal{B}_{st} \otimes_K D)^{\varphi=1, N=0, \nabla=0} \simeq (\mathcal{B}_{st} \otimes_K D_{K_0})^{\varphi=1, N=0}. \]

3. Finally, (2.4.2) holds by using (2.4.3) and (2.4.5).

**Step 3:** With Step 1 and Step 2 established, one can then use exactly the same argument as in the final two paragraphs of [Bri06, Thm. 4.34] (which uses [CF00, Thm. 4.3.(ii)]) to conclude that \( D \) is admissible. \( \square \)
3. Modification of Fontaine modules

In this section, we follow the idea of Kisin – modified by Brinon-Trihan in the imperfect residue field case – to study modification of Fontaine modules. The main theorem says that we can construct a fully faithful functor from the category of weakly admissible filtered \((\varphi, N)\)-modules with connections to a certain category of Breuil-Kisin modules equipped with \((\varphi, N, \nabla)\)-operators. The constructed Breuil-Kisin modules will be used in our integral theory in §4.

3.1. Fontaine modules and Kisin’s \(A\)-modules.

**Notation 3.1.1.** Let
\[
A = \{ f(u) = \sum_{i=0}^{+\infty} a_i u^i, a_i \in K_0 \mid f(u) \text{ converges, } \forall u \in m_{\mathcal{O}_K} \},
\]
where \(m_{\mathcal{O}_K}\) is the maximal ideal of \(\mathcal{O}_K\); i.e., it consists of series that converge on the entire open unit disk defined over \(K_0\). (The ring is denoted as \(\mathcal{O} = \mathcal{O}^{[0,1]}\) in [Kis06]; we follow the notation of [BT08] and use \(A\) here, because we use “\(\mathcal{O}_K, \mathcal{O}_{K_0}\)” etc to denote rings of integers in this paper.) Let \(\varphi\) be the Frobenius operator extending \(\varphi\) on \(K_0\) such that \(\varphi(u) = u^p\). Let \(\pi \in K\) be a fixed uniformizer, and let \(E(u)\) be its minimal polynomial over \(K_0\). Define an element
\[
\lambda = \prod_{n=0}^{+\infty} \varphi^n \left( \frac{E(u)}{E(0)} \right) \in A.
\]
Let \(N_{\nabla}\) be the \(K_0\)-linear differential operator \(u \frac{d}{du}\) on \(A\).

**Definition 3.1.2.** Let \(\text{Mod}_A(\varphi, N_{\nabla}, \nabla)\) be the category of the following data:

1. \(M\) is a finite free \(A\)-module;
2. \(\varphi : M \rightarrow M\) is a \(\varphi_A\)-semi-linear morphism such that the cokernel of \(1 \otimes \varphi : \varphi^* M \rightarrow M\) is killed by \(E(u)^h\) for some \(h \in \mathbb{Z}_{\geq 0}\);
3. \(N_{\nabla} : M \rightarrow M\) is a map such that \(N_{\nabla}(fm) = N_{\nabla}(f)m + fN_{\nabla}(m)\) for all \(f \in A\) and \(m \in M\), and \(N_{\nabla}(\varphi) = p E(u)/E(0) \varphi N_{\nabla}\);
4. \(\nabla : M/uM \rightarrow M / uM \otimes_{K_0} \hat{\Omega}_{K_0}\) is an integrable quasi-nilpotent connection which commutes with \(\varphi\) and \(N_{\nabla}\) (where \(N_{\nabla} = 0\) on \(\hat{\Omega}_{K_0}\)).

**Notation 3.1.3.** Fix a system of elements \(\pi_n \in \overline{K}\) such that \(\pi_0 = \pi\) and \(\pi_{n+1}^p = \pi_n, \forall n \geq 0\). For \(n \geq 0\), let \(K_{n+1} = K(\pi_n)\) (hence \(K_1 = K\)). Let
\[
\hat{\mathcal{O}} = \mathcal{O}_{K_0}[[u]] \subset A.
\]
Let \(\hat{\mathcal{Q}}_n\) be the completion of \(K_{n+1} \otimes_{W(k)} \hat{\mathcal{O}}\) at the maximal ideal \((u - \pi_n)\); \(\hat{\mathcal{Q}}_n\) is equipped with its \((u - \pi_n)\)-adic filtration, which extends to a filtration on the quotient field \(\hat{\mathcal{Q}}_n[1/(u - \pi_n)]\). There is a natural \(K_0\)-linear map
\[
A \rightarrow \hat{\mathcal{Q}}_n
\]
simply by sending \(u\) to \(u\). Let \(\ell_u\) be a formal variable, which behaves like \(\log u\). We can extend the map \(A \rightarrow \hat{\mathcal{Q}}_n\) to \(A[\ell_u] \rightarrow \hat{\mathcal{Q}}_n\) which sends \(\ell_u\) to
\[
\sum_{i=1}^{+\infty} (-1)^{i-1} i^{-1} \left( \frac{u - \pi_n}{\pi_n} \right)^i \in \hat{\mathcal{Q}}_n.
\]
We can naturally extend \(\varphi\) to \(A[\ell_u]\) by setting \(\varphi(\ell_u) = p \ell_u\), and extend \(N_{\nabla}\) to \(A[\ell_u]\) by setting \(N_{\nabla}(\ell_u) = -\lambda\). Finally, let \(N\) be the \(A\)-derivation on \(A[\ell_u]\) such that \(N(\ell_u) = 1\).

**Notation 3.1.4.** The above construction works for any CDVF \(K\), hence in particular for \(\mathbb{K}\). Hence for example, we can define
\[
A_\mathbb{K} = \{ f(u) = \sum_{i=0}^{+\infty} a_i u^i, a_i \in \mathbb{K}_0 \mid f(u) \text{ converges} , \forall u \in m_{\mathcal{O}_K} \}.
\]
Furthermore, since $\pi \in K$ is also a uniformizer of $K$, hence we can keep using the elements $\pi_n$ (regarded as elements in $\mathbb{K}$) and the polynomial $E(u)$ (regarded as the minimal polynomial of $\pi$ over $K_0$). Thus when we work with $K$, we get exactly the same $\lambda$, and the same $\varphi, N_{\nabla}$ operators. We then can define Mod$_A(\varphi, N_\nabla)$ (without $\nabla$-operators as $\widehat{\Omega}_K = 0$). Then we can define $\mathcal{G}_{K, K_n, \mathcal{G}_{K_n}}$ etc. together with the various operators. The formal variable $\ell_u$ behaves in the same way in this case.

**Construction 3.1.5.** Let

$$\text{MF}_{K/K_0}^\geq 0(\varphi, N) \subset \text{MF}_{K/K_0}(\varphi, N)$$

be the sub-category consisting of objects with Fil$_0^0 \mathcal{D}_K = \mathcal{D}_K$; here $\text{MF}_{K/K_0}(\varphi, N)$ is the category of usual Fontaine modules for $K$ (which has perfect residue field). For $D \in \text{MF}_{K/K_0}^\geq 0(\varphi, N)$, write $\iota_n$ for the following composite map:

$$(3.1.1) \quad \mathcal{A}_K[\ell_u] \otimes_{K_0} D \xrightarrow{\varphi^n \otimes \varphi^{-n}} \mathcal{A}_K[\ell_u] \otimes_{K_0} \mathcal{D} \rightarrow \mathcal{G}_{K,n} \otimes_{K_0} \mathcal{D} = \mathcal{G}_{K,n} \otimes_{K_0} \mathcal{D}_K$$

Here

- $\varphi_K : \mathcal{A}_K[\ell_u] \rightarrow \mathcal{A}_K[\ell_u]$ is the map which acts on $K_0$ by $\varphi$ and fixes $u$ and $\ell_u$;

- note that the map $\varphi^{-n} : D \rightarrow D$ is well-defined because $K$ has perfect residue field (compare with Rem. 2.4.6);

- the second map is induced from the map $\mathcal{A}_K[\ell_u] \rightarrow \mathcal{G}_{K,n}$.

The composite map extends to

$$(3.1.2) \quad \iota_n : \mathcal{A}_K[\ell_u, 1/\lambda] \otimes_{K_0} D \rightarrow \mathcal{G}_{K,n}[1/(u - \pi_n)] \otimes_{K_0} \mathcal{D}_K$$

Now, set

$$(3.1.3) \quad \mathcal{M}(D) := \{ x \in (\mathcal{A}_K[\ell_u, 1/\lambda] \otimes_{K_0} D)^{N = 0} : \iota_n(x) \in \text{Fil}_0^0 \left( \mathcal{G}_{K,n}[1/(u - \pi_n)] \otimes_{K_0} \mathcal{D}_K \right), \forall n \geq 1 \}$$

where the Fil$_0^0$ in (3.1.3) comes from tensor product of two filtrations.

**Theorem 3.1.6.** [Kis06, Thm. 1.2.15] $\mathcal{M}(D)$ is finite free over $\mathcal{A}_K$, is stable under $\varphi \otimes \varphi$-action and $N_\nabla \otimes 1$-action induced from $\mathcal{A}_K[\ell_u, 1/\lambda] \otimes_{K_0} D$. Indeed, $D \mapsto \mathcal{M}(D)$ induces an equivalence of categories

$$\text{MF}_{K/K_0}^\geq 0(\varphi, N) \rightarrow \text{Mod}_A(\varphi, N_{\nabla}).$$

**Construction 3.1.7.** The embedding $K_0 \hookrightarrow K_0$ extends to $\mathcal{A} \hookrightarrow \mathcal{A}_K$ where $u \mapsto u$; this embedding is equivariant with the operators $\varphi, N_\nabla$ defined above. The embedding also extends to $\mathcal{A}[\ell_u] \hookrightarrow \mathcal{A}_K[\ell_u]$ as well as to $\mathcal{G}_n \hookrightarrow \mathcal{G}_{K_n}$, which are compatible with all the various structures discussed above. Hence if $D \in \text{MF}_{K/K_0}^\geq 0(\varphi, N, \nabla)$, then we can regard $\mathcal{A}[\ell_u, 1/\lambda] \otimes_{K_0} D$ as a submodule of $\mathcal{A}_K[\ell_u, 1/\lambda] \otimes_{K_0} D$ compatible with various structures. Define

$$M(D) := (\mathcal{A}[\ell_u, 1/\lambda] \otimes_{K_0} D) \cap \mathcal{M}(D).$$

**Theorem 3.1.8.** $M(D)$ is finite free over $\mathcal{A}$, is stable under $\varphi$-action and $N_{\nabla}$-action induced from $\mathcal{M}(D)$, and there is a $(\varphi, N)$-equivariant isomorphism

$$M(D)/\text{uM}(D) \simeq D,$$

where $N$ acts on $M(D)/\text{uM}(D)$ by $N_\nabla$ (mod $u$). Equipping $\nabla$ on $M(D)/\text{uM}(D)$ from that on $D$, this makes $M(D)$ an object in $\text{Mod}_A(\varphi, N_{\nabla}, \nabla)$. Indeed, this construction induces an equivalence of categories

$$\text{MF}_{K/K_0}^\geq 0(\varphi, N, \nabla) \rightarrow \text{Mod}_A(\varphi, N_{\nabla}, \nabla).$$

**Proof.** This is analogue of [Kis06, Thm. 1.2.15], and the $N = 0$ case is proved [BT08, Prop. 4.11]. The general case here follows the same argument as in loc. cit. (by keeping track of the $N$- and $N_{\nabla}$-operators).
Let
\[ \mathcal{R} := \{ f(u) = \sum_{i=-\infty}^{+\infty} a_i u^i, a_i \in K_0, f(u) \text{ converges for all } u \in \mathcal{R} \} \]
(3.1.4)
be the Robba ring (with coefficients in \( K_0 \)).

**Definition 3.1.9.** Let \( \text{Mod}_A^0(\varphi, N_\nabla, \nabla) \) be the subcategory of \( \text{Mod}_A(\varphi, N_\nabla, \nabla) \) consisting of objects \( M \) such that \( \mathcal{R} \otimes_A M \) is *pure of slope* 0 in the sense of Kedlaya (cf. [Ked04, Ked05]).

**Theorem 3.1.10.** The functor in Thm. 3.1.8 induces a fully faithful functor
\[ \text{MP}^{\geq 0, \text{wa}}_{K/K_0}(\varphi, N, \nabla) \to \text{Mod}_A^0(\varphi, N_\nabla, \nabla). \]

**Proof.** This is the “analogue” of [Kis06, Thm. 1.3.8], and the \( N = 0 \) case is proved [BT08, Prop. 4.14]. The general case here follows the same argument as in loc. cit.; in particular, let us mention that the proof makes use of the “weakly admissible implies admissible” theorem in Thm. 2.4.9. Note that in the perfect residue field case [Kis06, Thm. 1.3.8], the functor is indeed an *equivalence of categories*, which we do not expect in the general case. \( \square \)

**Definition 3.1.11.** Let \( \text{Mod}_A(\varphi, N, \nabla) \) be the category of the following data:

1. \( M \) is a finite free \( A \)-module;
2. \( \varphi : M \to M \) is a \( \varphi_A \)-semi-linear morphism such that the cokernel of \( 1 \otimes \varphi : \varphi^* M \to M \) is killed by \( (u)^h \) for some \( h \in \mathbb{Z}_{\geq 0} \);
3. \( N : M/uM \to M/uM \) is a \( K_0 \)-linear map such that \( N \varphi = p \varphi N \);
4. \( \nabla : M/uM \to M/uM \otimes_{K_0} \hat{\Omega}_{K_0} \) is an integrable quasi-nilpotent connection which commutes with \( \varphi \) and \( N \).

Let \( \text{Mod}_A^0(\varphi, N, \nabla) \) be the subcategory consisting of modules which are pure of slope 0 (after tensoring with \( \mathcal{R} \)).

Given \( M \in \text{Mod}_A(\varphi, N_\nabla, \nabla) \), we can construct a module \( \tilde{M} \in \text{Mod}_A(\varphi, N, \nabla) \) by taking \( \tilde{M} = M \) equipped with the same \( \varphi \) and \( \nabla \), and take \( N \) to be the reduction of \( N_\nabla \) modulo \( u \).

**Proposition 3.1.12.** The map \( M \mapsto \tilde{M} \) above induces a fully faithful functor:
\[ \text{Mod}_A(\varphi, N_\nabla, \nabla) \to \text{Mod}_A(\varphi, N, \nabla), \]
which then induces a fully faithful functor
\[ \text{Mod}_A^0(\varphi, N_\nabla, \nabla) \to \text{Mod}_A^0(\varphi, N, \nabla). \]

**Proof.** When \( K \) has perfect residue field, this is [Kis06, Lem. 1.3.10(2)]. The general case here follows exactly the same proof: the additional \( \nabla \)-operators here cause no trouble, as they remain unchanged under the functor. \( \square \)

### 3.2. Relation with modules over \( \mathcal{S} \) and \( \mathcal{S} \)
Recall \( \mathcal{S} = \mathcal{O}_{K_0}[[u]] \subset A \), it is stable under \( \varphi \).

**Definition 3.2.1.** Let \( \text{Mod}_\mathcal{S}(\varphi, N, \nabla) \) be the category consisting of \( (\mathfrak{M}, \varphi) \) where \( \mathfrak{M} \) is a finite free \( \mathcal{S} \)-module, and \( \varphi : \mathfrak{M} \to \mathfrak{M} \) is a \( \varphi_\mathcal{S} \)-semi-linear map such that the \( \mathcal{S} \)-linear span of \( \varphi(\mathfrak{M}) \) contains \( E(u)^h \mathfrak{M} \) for some \( h \geq 0 \). We say that \( \mathfrak{M} \) is of \( E(u) \)-height \( \leq h \).

**Definition 3.2.2.** Let \( \text{Mod}_\mathcal{S}(\varphi, N, \nabla) \) be the category of the following data:

1. \( (\mathfrak{M}, \varphi) \) is an object in \( \text{Mod}_\mathcal{S}^0 \);
2. a \( K_0 \)-linear map \( N : \mathfrak{M}/u\mathfrak{M}[1/p] \to \mathfrak{M}/u\mathfrak{M}[1/p] \) such that \( N \varphi = p \varphi N \) over \( \mathfrak{M}/u\mathfrak{M}[1/p] \);
3. an integral quasi-nilpotent connection \( \nabla : \mathfrak{M}/u\mathfrak{M}[1/p] \to \mathfrak{M}/u\mathfrak{M}[1/p] \otimes_{K_0} \hat{\Omega}_{K_0} \) which commutes with \( \varphi \) and \( N \).

Let \( \text{Mod}_\mathcal{S}(\varphi, N, \nabla) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) be its isogeny category.
Theorem 3.2.3. We have the following commutative diagram of categories. Here \( \simeq \) indicates an equivalence of categories, and a hooked arrow indicates a fully faithful functor.

\[
\begin{align*}
\text{MF}^{\geq 0}_{K/K_0}(\varphi, N, \nabla) & \quad \dashv \quad \text{MF}^{\geq 0}_{K/K_0}(\varphi, N, \nabla) \quad \dashv \quad \text{Mod}_A(\varphi, N, \nabla) \\
\simeq & \quad \dashv \quad \text{Mod}^0_A(\varphi, N, \nabla) \quad \dashv \quad \text{Mod}^0_A(\varphi, N, \nabla)
\end{align*}
\]

Proof. With Thm. 3.1.8, Thm. 3.1.10 and Prop. 3.1.12 established, it suffices to discuss the arrow labeled as \( \Theta \); the equivalence \( \Theta \) would induce the fully faithful functor labelled as \( (\ast) \).

The equivalence \( \Theta \) is simply defined via

\[ \mathcal{M} \mapsto \mathcal{M} \otimes_A \mathcal{A}. \]

The equivalence \( \Theta \) is the analogue of [Kis06, Lem. 1.3.13], and the \( N = 0 \) case is proved [BT08, Prop. 4.17]. The general case here follows the same argument as in loc. cit.; in particular, as mentioned in loc. cit., the proof makes use of slope filtration theorem of Kedlaya [Ked04, Ked05], which has no restriction on residue fields.

Notation 3.2.4. Let \( S \) be the \( p \)-adic completion of the PD envelope of \( \mathcal{O}_{K_0}[u] \) with respect to the ideal \( (E(u)) \). Explicitly,

\[ S = \{ x = \sum_{i \geq 0} a_i \frac{E(u)^i}{i!} \mid a_i \in \mathcal{O}_{K_0}[u], \lim_{i \to \infty} v_p(a_i) = +\infty \} \subset K_0[[u]]. \]

Proposition 3.2.5. Suppose \( D \in \text{MF}^{\geq 0, \text{wa}}_{K/K_0}(\varphi, N, \nabla) \), and it maps to \( \mathcal{M} \) (up to isogeny) via the functor labelled as \( (\ast) \) in Thm. 3.2.3. Then there is an \( \varphi \)-equivariant isomorphism

\[
D \otimes_{K_0} S[1/p] \simeq \mathcal{M} \otimes_{\varphi, S} S[1/p].
\]

Proof. This follows [Kis06, Lem. 1.2.6] (cf. also [Liu08, §3.2]) when \( K \) has perfect residue field, and follows the argument in [BT08, Prop. 5.7] in the general case. Note that in [BT08, Prop. 5.7], it was assumed that \( \mathcal{M} \otimes_{\varphi, S} S \) is an object in “\( \text{MF}^{\text{BT}}(\varphi, \nabla) \)” (cf. Def. 5.2.1) and in particular the \( E(u) \)-height of \( \mathcal{M} \) is \( \leq 1 \); but this assumption is irrelevant.

4. Integral \( p \)-adic Hodge theory

The main goal in this section is to study integral semi-stable representations. In §4.2, we study the the relation between étale \( \varphi \)-modules, Breuil-Kisin modules, and Galois representations; in §4.3 we strengthen some of these results in the semi-stable case. In §4.4, we construct a fully faithful functor from the category of integral semi-stable representations to the category of Breuil-Kisin \( G_K \)-modules. Some of the results in §4.2, as well as in §§5 and 6 (but not in §4.4) were in fact already established in the relative case in [Kim15]; hence in §4.1, we first review some notions from [Kim15] which will be mentioned later.

4.1. Relation with Kim’s work. We first review some notions from [Kim15].

Assumption 4.1.1. Let \( R \) be a \( p \)-adically complete and separated flat \( \mathbb{Z}_p \)-algebra. We list some assumptions on \( R \).

(i) \((p\text{-basis, cf. [Kim15, §2.2.1])}. \) Assume that \( R \cong R_0[u]/E(u) \), where \( R_0 \) is a \( p \)-adic flat \( \mathbb{Z}_p \)-algebra such that \( R_0/(p) \) locally admits a finite \( p \)-basis, and

\[ E(u) = p + \sum_{i=1}^e a_i u^i \]
for some integer $e > 0$, with $a_i \in R_0$ and $a_e \in R_0^\times$ (where $R_0^\times \subset R_0$ are the units). Let $\varpi \in R$ denote the image of $u \in R_0[u]$.

(i) $(p$-basis + Cohen subring, cf. [Kim15, §2.2.1]). In addition to (i), assume there is a Cohen subring $W \subset R_0$ such that $E(u) \in W[u]$.

(ii) $(p$-basis+formally finite-type, cf. [Kim15, §2.2.2]). In addition to (i), assume that $R$ is $J_R$-adically separated and complete for some finitely generated ideal $J_R$ containing $\varpi$, and $R/J_R$ is finitely generated over some field $k$.

(iii) $(p$-basis+formally finite-type + refined almost étaleness, cf. [Kim15, §2.2.3]). In addition to (ii), assume $R$ is a domain such that $R[1/p]$ is finite étale over $R_0[1/p]$ and we have $\widehat{\Omega}_{R_0} = \bigoplus_{i=1}^d R_0dt_i$ for some units $T_i \in R_0^\times$. Here, $\widehat{\Omega}_{R_0}$ is the module of $p$-adically continuous Kähler differentials.

Remark 4.1.2. In [Kim15, §2.2.4], a certain normality assumption is also introduced; it is weaker than Assumption (ii) above. Also, certain local complete intersection assumption is introduced in [Kim15, §2.2.5]; it is irrelevant in the current paper.

Lemma 4.1.3. $\mathcal{O}_K$ satisfies all the assumptions (including (i)') in §4.1.1.

Proof. For $R = \mathcal{O}_K$, we can use the subring $R_0 = \mathcal{O}_{K_0}$, the polynomial $\frac{1}{E(0)}E(u)$ (then $\varpi = \pi$), and $J_R = (\pi)$.

4.2. Étale $\varphi$-modules.

Notation 4.2.1. Recall $\mathcal{S} = \mathcal{O}_{K_0}[[u]]$. Let $\mathcal{O}_\mathcal{E}$ be the $p$-adic completion of $\mathcal{S}[1/u]$. We have defined $\pi_n$ in §3.1.3. Regard $\pi = \{\pi_n\}_{n \geq 0}$ as an element in $\mathcal{O}_\mathcal{E}$, and let $\widehat{\pi} \in W(\mathcal{O}_\mathcal{C})$ be its Teichmüller lift. The embedding $i_\varphi : \mathcal{O}_{K_0} \hookrightarrow W(k)$ induces an $\mathcal{O}_{K_0}$-linear and $\varphi$-equivariant embedding

$$\mathcal{S} \hookrightarrow W(\mathcal{O}_\mathcal{C})$$

by sending $u$ to $\widehat{\pi}$; it extends to an embedding

$$\mathcal{O}_\mathcal{E} \hookrightarrow W(\mathcal{O}_\mathcal{C}).$$

Let $\mathcal{O}_{\mathcal{E}ur} \subset W(\mathcal{O}_\mathcal{C}^\flat)$ be the maximal unramified extension of $\mathcal{O}_\mathcal{E}$, and let $\mathcal{O}_{\mathcal{E}ur}$ be its $p$-adic completion. Let $\widehat{\mathcal{S}}^{ur} = \mathcal{O}_{\mathcal{E}ur} \cap W(\mathcal{O}_\mathcal{C}^0)$.

Definition 4.2.2. An étale $\varphi$-module is a finite free $\mathcal{O}_\mathcal{E}$-modules $M$ equipped with a $\varphi \mathcal{O}_\mathcal{E}$-semi-linear endomorphism $\varphi_M : M \rightarrow M$ such that $1 \otimes \varphi : \varphi^*M \rightarrow M$ is an isomorphism. Let $\text{Mod}_{\mathcal{O}_\mathcal{E}}^\varphi$ denote the category of these objects.

Notation 4.2.3. Starting from $\mathbb{K}$, we can also define $\mathcal{S}_\mathbb{K} = \mathcal{O}_{\mathbb{K}_0}[[u]]$. Let $\mathcal{O}_{\mathcal{E}_\mathbb{K}}$ be the $p$-adic completion of $\mathcal{S}_\mathbb{K}[1/u]$. Similarly as in §4.2.1, we have $\mathcal{O}_{\mathbb{K}_0}$-linear embeddings

$$\mathcal{S}_\mathbb{K} \hookrightarrow W(\mathcal{O}_\mathcal{C}^0), \quad \mathcal{O}_{\mathcal{E}_\mathbb{K}} \hookrightarrow W(\mathcal{C}_\mathbb{K})$$

where $\mathcal{O}_\mathcal{C}^0$ and $\mathcal{C}_\mathbb{K}$ are the tiltings, and where we regard $\pi$ as an element in $\mathcal{O}_\mathcal{C}^0$. We can similarly define $\mathcal{O}_{\mathcal{E}ur}$ and $\widehat{\mathcal{S}}^{ur}_\mathbb{K}$. There are $\varphi$-equivariant embeddings

\[
\begin{array}{ccc}
\mathcal{S} & \hookrightarrow & \widehat{\mathcal{S}}^{ur} \hookrightarrow W(\mathcal{O}_\mathcal{C}) \\
\downarrow & & \downarrow_{\varphi, \approx} \\
\mathcal{S}_\mathbb{K} & \hookrightarrow & \widehat{\mathcal{S}}^{ur}_\mathbb{K} \hookrightarrow W(\mathcal{O}_\mathcal{C}^0) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathcal{E}_\mathbb{K}} & \hookrightarrow & \mathcal{O}_{\mathcal{E}ur} \hookrightarrow W(\mathcal{C}_\mathbb{K}) \\
\downarrow_{\varphi, \approx} & & \downarrow \\
\mathcal{O}_\mathcal{E} & \hookrightarrow & \mathcal{O}_{\mathcal{E}ur} \hookrightarrow W(\mathcal{C}_\mathbb{K}^0)
\end{array}
\]

(4.2.1)

Analogous to Def. 3.2.1 and Def. 4.2.2, we can define $\text{Mod}_{\mathcal{S}_\mathbb{K}}^\varphi$ and $\text{Mod}_{\mathcal{O}_{\mathcal{E}_\mathbb{K}}}^\varphi$. 

Construction 4.2.4. In the following, we will explain the content of the following commutative diagram of functors. Here \( \simeq \) signifies an equivalence of categories, and a hooked arrow signifies a fully faithful functor.

\[
\begin{array}{ccc}
\text{Mod}^\varphi_{O_\varphi} & \xrightarrow{(1)} & \text{Mod}^\varphi_{E_K} & \xrightarrow{(5)} & \text{Rep}_{Z_p}(G_{K_\infty}) \\
\downarrow & & \downarrow & & \downarrow (7) = \\
\text{Mod}^\varphi_{\tilde{O}_E} & \xrightarrow{(3) \simeq} & \text{Mod}^\varphi_{\tilde{O}_{E_K}} & \xrightarrow{(6) \simeq} & \text{Rep}_{Z_p}(G_{K_\infty})
\end{array}
\]

(4.2.2)

Here

- The functors (1)-(4) are defined via scalar extensions in (4.2.1); they are obviously well-defined and the left square is commutative.
- The functor (5) is defined via \( M \mapsto (M \otimes_{O_E \hat{K}} \tilde{O}_{E_K^{ur}})^{G_{K_\infty}} \).
- The functor (6) is defined via \( M \mapsto (M \otimes_{O_{E_K}} \tilde{O}_{E_K^{ur}})^{G_{K_\infty}} \).
- Note that \( \mathbb{K} \) has perfect residue field, hence the commutativity of the right square, as well as the equivalence of (6), the full faithfulness of (4) and (5), are proved in [Kis06, §2.1].

It remains to show the equivalence of (3) and the full faithfulness of (1), which is carried out in Prop. 4.2.5 and Prop. 4.2.7 respectively.

Proposition 4.2.5. We have tensor exact equivalences

\[
\text{Mod}^\varphi_{O_\varphi} \xrightarrow{(3)} \text{Mod}^\varphi_{\tilde{O}_{E_K}} \xrightarrow{(6)} \text{Rep}_{Z_p}(G_{K_\infty}).
\]

Given \( T \in \text{Rep}_{Z_p}(G_{K_\infty}) \), the corresponding objects \( M \in \text{Mod}^\varphi_{O_\varphi} \) and \( M_\mathbb{K} \in \text{Mod}^\varphi_{\tilde{O}_{E_K}} \) are

\[
M = (T \otimes_{Z_p} O_{\tilde{E}^{ur}})^{G_{K_\infty}}, \quad M_\mathbb{K} = (T \otimes_{Z_p} O_{E_K^{ur}})^{G_{K_\infty}}.
\]

Remark 4.2.6. Let \( R \) be as in Assumption 4.1.1. Suppose \( R \) is a domain, and satisfies Assumptions (i), (i)' and (ii) there. Then [Kim15, Prop. 7.7] proves the relative version of (and hence by Lem. 4.1.3, implies) Prop. 4.2.5. We nonetheless give a sketch of Kim’s proof, to illustrate the useful ideas.

Proof of Prop. 4.2.5. We first introduce some notations, let

\[
E^+_K := \mathfrak{G}/p\mathfrak{G} = k_K[[u]], \quad E^+_K = \mathfrak{G}_K/p\mathfrak{G}_K = k[[u]].
\]

Note that \( E^+_K \) and \( E^+_{K_\infty} \) have the same perfect closures as \( k \) is the perfect closure of \( k_K \); let \( \tilde{E}^+_{K_\infty} \) be the \( u \)-adic completion of their common perfect closure. Let

\[
E_{K_\infty} = E^+_K[1/u], \quad \tilde{E}_{K_\infty} = \tilde{E}^+_{K_\infty}[1/u], \quad \bar{E}_{K_\infty} = \bar{E}^+_{K_\infty}[1/u].
\]

By dévissage, to prove the proposition, it suffices to prove equivalences of the corresponding \( p \)-torsion categories. Namely, it suffices to show equivalences of categories

\[
\text{Mod}^\varphi_{E_{K_\infty}} \xrightarrow{(4.2.4)} \text{Mod}^\varphi_{\tilde{E}_{K_\infty}} \xrightarrow{(4.2.5)} \text{Rep}_p(G_{K_\infty}),
\]

where the categories are defined in the obvious fashion. Note that we have the following isomorphisms of Galois groups

\[
G_{K_\infty} \simeq G_{E_{K_\infty}} \simeq G_{\tilde{E}_{K_\infty}} \simeq G_{E_{K_\infty}}.
\]

Here, the first isomorphism follows from classical theory of field of norms (as \( \mathbb{K} \) has perfect residue field); the other isomorphisms follow from [GR03, Prop. 5.4.53] as both \( E^+_K \) and \( E^+_{K_\infty} \) are Henselian rings with respect to the ideal generated by \( u \). Thus we can apply [Kat73, Prop. 4.1.1] to conclude equivalences in (4.2.4). Finally, from (4.2.5), we have

\[
(O_{\tilde{E}^{ur}})^{G_{K_\infty}} = O_{\tilde{E}}, \quad (O_{E_K^{ur}})^{G_{K_\infty}} = O_{E_K}.
\]

This proves (4.2.3). \( \square \)
Proposition 4.2.7. We have a chain of fully faithful functors:

$$\text{Mod}_E \overset{(1)}{\rightarrow} \text{Mod}_{E_K} \overset{(5)}{\rightarrow} \text{Rep}_{\mathbb{Z}_p}(G_{K_{\infty}}).$$

Proof. Let $\mathcal{M}_1, \mathcal{M}_2 \in \text{Mod}_E^\varphi$. Let $\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2 \in \text{Mod}_{E_K}$, $T_1, T_2 \in \text{Rep}_{\mathbb{Z}_p}(G_{K_{\infty}})$, $M_1, M_2 \in \text{Mod}_{E_K}^\varphi$, $\widetilde{M}_1, \widetilde{M}_2 \in \text{Mod}_{O_{E_K}}^\varphi$ be the corresponding objects. Recall full faithfulness of (5) is known by [Kis06, Prop. 2.1.12]. The functor (1) is obviously faithful, hence it suffices to show fullness of the composite functor $(5) \circ (1)$. Given a morphism $T_1 \rightarrow T_2$, the equivalences in Prop. 4.2.5 supplies a unique corresponding morphism $M_1 \rightarrow M_2$ and a unique corresponding morphism $\widetilde{M}_1 \rightarrow \widetilde{M}_2$; the full faithfulness of (5) supplies a unique corresponding morphism $\widetilde{\mathcal{M}}_1 \rightarrow \widetilde{\mathcal{M}}_2$. Since

$$\mathcal{G} = \mathcal{O}_E \cap \mathcal{G}_K \subset \mathcal{O}_{E_K},$$

we have

$$\mathcal{M}_i = M_i \cap \widetilde{\mathcal{M}}_i \subset \widetilde{M}_i.$$

Hence the desired morphism $\mathcal{M}_1 \rightarrow \mathcal{M}_2$ is induced by the aforementioned morphisms. □

Notation 4.2.8. From now on, we denote the fully faithful functor (5) as $T_{E_K} : \text{Mod}_{E_K} \rightarrow \text{Rep}_{\mathbb{Z}_p}(G_{K_{\infty}})$, and denote the composite fully faithful functor $(5) \circ (1)$ above as $T_{E} : \text{Mod}_E \rightarrow \text{Rep}_{\mathbb{Z}_p}(G_{K_{\infty}})$.

Lemma 4.2.9. Let $\mathcal{M} \in \text{Mod}_E^\varphi$, let $V = T_{E}(\mathcal{M}) \otimes_{E_p} \mathbb{Q}_p \in \text{Rep}_{\mathbb{Q}_p}(G_{K_{\infty}})$, and let $M[1/p] = \mathcal{M} \otimes_{\mathcal{E}} \mathcal{E}$ where $\mathcal{E} = \mathcal{O}_{E_p}[1/p]$. Then the map $\mathfrak{m} \mapsto T_{E}(\mathfrak{m})$ induces a bijection between $\varphi$-stable $\mathcal{G}$-submodules $\mathfrak{m} \in M[1/p]$ such that $\mathcal{E} \otimes_{E_p} \mathfrak{m} = M[1/p]$ and $\mathfrak{m}/\varphi^*(\mathfrak{m})$ is killed by a power of $E(u)$, and $G_{K_{\infty}}$-stable $\mathbb{Z}_p$-lattices $L \subset V$.

Proof. This is the analogue of [Kis06, Lem. 2.1.15], and the proof follows the same argument, by using our Prop. 4.2.5 and Prop. 4.2.7. □

The following lemma will be used later.

Lemma 4.2.10. \((1)\) For $\widetilde{\mathcal{M}} \in \text{Mod}_{E_K}^\varphi$, there is an $G_{K_{\infty}}$-equivariant isomorphism

$$T_{E_K}(\widetilde{\mathcal{M}}) \simeq (\widetilde{\mathcal{M}} \otimes_{E_K} W(C^\varphi))^{\varphi=1}.$$

\((2)\) For $\mathcal{M} \in \text{Mod}_E^\varphi$, there is an $G_{K_{\infty}}$-equivariant isomorphism

$$T_{E}(\mathcal{M}) \simeq (\mathcal{M} \otimes_{E} W(C^\varphi))^{\varphi=1}.$$

Proof. Item (1) follows from [GL, Lem. 2.1.4, Lem. 2.3.1]. Item (2) then follows. Note that the $\varphi$-equivariant isomorphism $W(C^\varphi) \simeq W(C^\varphi)$ induced by $i_\varphi$ is $G_K$-equivariant. □

4.3. Semi-stable representations and Breuil-Kisin modules. Given $\mathcal{M} \in \text{Mod}_G^\varphi$ or $M \in \text{Mod}_{O_{E_K}}^\varphi$, then the functor in (4.2.3) implies that $\mathcal{M}$ or $M$ are fixed by $G_{K_{\infty}}$ inside $T \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_p}$. In this subsection, we show that when $\mathcal{M}$ and $M$ come from semi-stable representations, then they are furthermore fixed by $G_{K_{\infty}}$ (inside some suitable space). This is important for development in the next subsection.

We first very briefly recall some notions of overconvergent elements, they are well-known and we only use some very elementary properties of them, cf. e.g. [Ber02] for more details.

Notation 4.3.1. Denote

$$\tilde{A} = W(C^\varphi), \quad \tilde{B} = W(C^\varphi)[1/p],$$
and let \( \widetilde{A}^\ddagger, \widetilde{B}^\ddagger \) be the sub-ring of overconvergent elements, cf. [Ber02, §1.3]; indeed, 
\( \widetilde{B}^\ddagger = \widetilde{A}^\ddagger [1/p] \) is even a sub-field of the field \( \widetilde{B} \). Recall that (cf. [Ber02, §2.1])
\[
\widetilde{B}^\ddagger = \bigcup_{n \geq 0} \tilde{B}_{[r_n, +\infty]},
\]
where \( r_n := (p - 1) p^{n - 1} \) and \( \tilde{B}_{[r_n, +\infty]} \) is defined to be the subring of overconvergent elements with suitable range of overconvergence. By [Ber02, §2.2], for each \( n \geq 0 \), there is a \( G_K \)-equivariant embedding
\[
t_n : \tilde{B}_{[r_n, +\infty]} \hookrightarrow \tilde{B}^\ddagger.
\]

**Definition 4.3.2.**

1. Let \( \text{Mod}_{\tilde{A}}^{\varphi, G_K} \) be the category of the following data:
   - (a) a finite free \( \tilde{A} \)-module \( M \);
   - (b) a \( \varphi \)-semi-linear and bijective map \( \varphi : M \to M \);
   - (c) a \( \varphi \)-commuting \( G_K \)-action on \( M \), which is semi-linear with respect to the \( G_K \)-action on \( \tilde{A} \).

2. Let \( \text{Mod}_{\tilde{A}}^{\varphi, G_K} \) be similarly defined.

**Lemma 4.3.3.** There are equivalences of categories
\[
\text{Rep}_{\varphi}(G_K) \to \text{Mod}_{\tilde{A}}^{\varphi, G_K} \to \text{Mod}_{\tilde{A}}^{\varphi, G_K},
\]
where the first functor sends \( T \) to \( T \otimes_{\tilde{Z}_p} \tilde{A}^\ddagger \), and the second functor is defined by base-change.

**Proof.** This is well-known; essentially it is because \( C^\varphi \) is algebraically closed and is a perfectoid field, cf. e.g., [KL15, Thm. 8.5.3]. □

**Proposition 4.3.4.** Recall that in Thm. 3.2.3, we have constructed a fully faithful functor
\[
(*) : \text{MF}_{K_0}^{0, w_{\varphi}}(\varphi, N, \nabla) \to \text{Mod}_{\varphi}(\varphi, N, \nabla) \otimes_{\tilde{Z}_p} \tilde{Q}_p.
\]
Suppose \( D \in \text{MF}_{K_0}^{0, w_{\varphi}}(\varphi, N, \nabla) \) maps to \( (\mathfrak{M}, \varphi, N, \nabla) \) (up to isogeny), then there is a canonical \( G_{K_\infty} \)-equivariant isomorphism
\[
T_{\varphi}(\mathfrak{M}) \otimes_{\tilde{Z}_p} \tilde{Q}_p \simeq V_{\text{st}}(D)|_{G_{K_\infty}}
\]
**Proof.** \( T_{\varphi}(\mathfrak{M}) = T_{\varphi}(\mathfrak{M} \otimes_{\varphi} \mathfrak{G}_K) \), hence the proposition follows from [Kis06, Prop. 2.1.5]. □

**Proposition 4.3.5.** Let \( D \) and \( \mathfrak{M} \) be as in Prop. 4.3.4, namely, \( \mathfrak{M} \) (which is defined up to isogeny) comes from a semi-stable representation \( V := V_{\text{st}}(D) \). Prop. 4.3.4 and Lem. 4.2.10 induces a \( \varphi \)-equivariant isomorphism
\[
(3.1) \quad \mathfrak{M} \otimes_{\varphi} \widetilde{B} \simeq V \otimes_{\tilde{Q}_p} \widetilde{B};
\]
and the \( G_K \)-action on the right hand side induces a \( G_K \)-action on the left hand side. Then \( \mathfrak{M} \) is fixed by \( G_{K_\infty} \) under this action.

**Proof.** We have the following \( B_{\text{st}} \)-linear isomorphisms
\[
(3.2) \quad V \otimes_{\tilde{Q}_p} B_{\text{st}} \simeq D \otimes_{K_0} B_{\text{st}},
(3.3) \quad \simeq (D \otimes_{K_0} S[1/p]) \otimes_{S[1/p]} B_{\text{st}},
(3.4) \quad \simeq (\mathfrak{M} \otimes_{\varphi \otimes S} S[1/p]) \otimes_{S[1/p]} B_{\text{st}}.
\]
Here (3.2) is \( G_K \)-equivariant with trivial \( G_K \)-action on \( D \). In (3.3), \( D \otimes_{K_0} S[1/p] \) is fixed by \( G_{K_\infty} \) since \( S \subset (B_{\text{st}})^{G_{K_\infty}} \). Thus in (3.4) – which holds by Prop. 3.2.5 – the subset \( (\mathfrak{M} \otimes_{\varphi \otimes S} S[1/p]) \) is also fixed by \( G_{K_\infty} \). Thus \( \varphi^* \mathfrak{M} \) is fixed by \( G_{K_\infty} \) in the following base change:
\[
(3.5) \quad V \otimes_{\tilde{Q}_p} B_{\text{dR}} \simeq \varphi^* \mathfrak{M} \otimes_{\varphi} B_{\text{dR}}.
\]
Lem. 4.3.3 and Eqn. (3.1) implies that we have a \( \varphi \)-equivariant isomorphism (we can use \( \varphi^* \mathfrak{M} \) here because \( \varphi \) is bijective on \( \tilde{B} \) and \( \tilde{B}^\ddagger \))
\[
V \otimes_{\tilde{Q}_p} \tilde{B}^\ddagger \simeq \varphi^* \mathfrak{M} \otimes_{\varphi} \tilde{B}^\ddagger.
\]
Since \( \widetilde{\mathcal{B}}^t \) is a field, there exists some \( n \gg 0 \) such that
\[
V \otimes_{\mathbb{Q}_p} \widetilde{\mathcal{B}}_{[n, +\infty]} \simeq \varphi^* \mathcal{M} \otimes_{\mathcal{E}} \widetilde{\mathcal{B}}_{[n, +\infty]}.
\]
Apply the \( G_K \)-equivariant base changes \( \iota_n : \mathcal{B}_{[n, +\infty]} \to \mathcal{B}^{\nabla+}_{\mathbb{Q}_p} \) and \( \mathcal{B}^{\nabla+}_{\mathbb{Q}_p} \to \mathcal{B}_{\mathbb{Q}_p} \), we recover \( (4.3.5) \); thus \( \varphi^* \mathcal{M} \) and hence \( \mathcal{M} \) is fixed by \( G_{K_\infty} \).

4.4. Breuil-Kisin \( G_K \)-modules. In this subsection, we study integral semi-stable representations of \( G_K \). The following lemma will be repeatedly used.

**Lemma 4.4.1.** The two subgroups \( G_\mathbb{K} \) and \( G_{K_\infty} \) generate \( G_K \).

**Proof.** This is noted in the proof of [BT08, Prop. 3.8]; indeed \( \mathbb{K} \cap K_\infty = K \) as \( K_\infty \) is totally ramified.

Now we introduce the Breuil-Kisin \( G_K \)-modules. Let \( \mathfrak{m}_{\mathcal{O}_C} \) be the maximal ideal of \( \mathcal{O}_C \), and let \( W(\mathfrak{m}_{\mathcal{O}_C}) \) be the ideal of Witt vectors.

**Definition 4.4.2.** Let \( \text{Mod}^{\varphi, G_K}_{\mathfrak{m}_{\mathcal{O}_C}, W(\mathcal{O}_C)} \) be the category consisting of data which we call the (effective) Breuil-Kisin \( G_K \)-modules:

1. \( (\mathcal{M}, \varphi_\mathcal{M}) \in \text{Mod}^{\varphi, G_K}_{\mathfrak{m}_{\mathcal{O}_C}} 
2. G_K \) is a continuous \( W(\mathcal{O}_C) \)-semi-linear \( G_K \)-action on \( \widehat{\mathcal{M}} := W(\mathcal{O}_C) \otimes_{\mathfrak{m}} \mathcal{M} \);
3. \( G_K \) commutes with \( \varphi_\mathcal{M} \) on \( \widehat{\mathcal{M}} \);
4. \( \mathcal{M} \subset \widehat{\mathcal{M}}^{G_{K_\infty}} \) via the embedding \( \mathcal{M} \hookrightarrow \widehat{\mathcal{M}} \);
5. \( \mathcal{M}/u\mathcal{M} \subset (\widehat{\mathcal{M}}/W(\mathfrak{m}_{\mathcal{O}_C})\widehat{\mathcal{M}})^{G_K} \) via the embedding \( \mathcal{M}/u\mathcal{M} \hookrightarrow \widehat{\mathcal{M}}/W(\mathfrak{m}_{\mathcal{O}_C})\widehat{\mathcal{M}} \).

**Theorem 4.4.3.** Starting from \( \mathbb{K} \), let \( \text{Mod}^{\varphi, G_K}_{\mathfrak{m}_{\mathcal{O}_C}, W(\mathcal{O}_C)} \) be the category defined similarly as Def. 4.4.2: namely, one changes the notations

\[
\left\{ \mathfrak{S}, \mathcal{O}_C, G_K, G_{K_\infty} \right\}
\]

there to

\[
\left\{ \mathfrak{S}_\mathbb{K}, \mathcal{O}_C^\flat, G_K, G_{K_\infty} \right\}.
\]

Then the functor
\[
(\mathcal{M}, \widehat{\mathcal{M}}) \mapsto (\mathcal{M} \otimes W(\mathcal{O}_C^\flat))^{\varphi=1}
\]
induces an equivalence of categories
\[
\text{Mod}^{\varphi, G_K}_{\mathfrak{m}_{\mathcal{O}_C}, W(\mathcal{O}_C)} \simeq \text{Rep}_{Z_p}^{\text{st} \geq 0}(G_K).
\]

**Proof.** This follows from the main theorem of [Gao].

The following is our main theorem on integral \( p \)-adic Hodge theory in the imperfect residue field case.

**Theorem 4.4.4.** There is a fully faithful functor
\[
\text{Rep}_{Z_p}^{\text{st} \geq 0}(G_K) \to \text{Mod}^{\varphi, G_K}_{\mathfrak{m}_{\mathcal{O}_C}, W(\mathcal{O}_C)},
\]
such that if \( T \) maps to \( (\mathcal{M}, \widehat{\mathcal{M}}) \), then there is a \( G_K \)-equivariant isomorphism
\[
T \simeq (\mathcal{M} \otimes_{W(\mathcal{O}_C^\flat)} W(\mathcal{O}_C^\flat))^{\varphi=1}.
\]

**Proof.** Given \( T \in \text{Rep}_{Z_p}^{\text{st} \geq 0}(G_K) \), Prop. 4.3.4 and Lem. 4.2.9 imply that there is a unique \( \mathcal{M} \in \text{Mod}^{\varphi, G_K}_\mathcal{E} \) corresponding to \( T|_{G_{K_\infty}} \). The assignment \( T \mapsto \mathcal{M} \) is functorial because \( T_\mathcal{E} \) is fully faithful.

Now we want to construct a natural \( G_K \)-action on \( \mathcal{M} \otimes_{\mathcal{E}} W(\mathcal{O}_C^\flat) \). First, consider \( T \) as an object in \( \text{Rep}_{Z_p}^{\text{st} \geq 0}(G_K) \); by the construction in §4.2.4 and Thm. 4.4.3, \( \mathcal{M}_\mathbb{K} := \mathcal{M} \otimes_{\mathcal{E}} \mathfrak{S}_\mathbb{K} \) is the corresponding object in \( \text{Mod}^{\varphi, G_K}_{\mathfrak{m}_{\mathcal{O}_C}, W(\mathcal{O}_C)} \) and
\[
(4.4.1) \quad \mathcal{M}_\mathbb{K} \otimes_{\mathfrak{S}_\mathbb{K}} W(\mathcal{O}_C^\flat) \hookrightarrow T \otimes_{Z_p} W(\mathcal{O}_C^\flat) \text{ is } G_\mathbb{K} \text{-stable.}
Consider the image of the following \( \varphi \)-equivariant inclusions (where the second isomorphism follows from Lem. 4.2.10 and Lem. 4.3.3):
\[
\mathcal{M} \otimes \mathcal{S} W(\mathcal{O}_C^\ast) \subset \mathcal{M} \otimes \mathcal{S} W(C^\ast) \simeq T \otimes_{Z_p} W(C^\ast).
\]
Using (4.4.1) and the \( G_K \)-equivariant isomorphism \( W(\mathcal{O}_C^\ast) \simeq W(\mathcal{O}_C^\ast) \) induced by \( i_\varphi \) in \$2.1\$, we see that the image of (4.4.2) is \( G_K \)-stable; it is also \( G_{K^\infty} \)-stable because Prop. 4.3.5 implies that \( G_{K^\infty} \) acts trivially on \( \mathcal{M} \). Thus the image is \( G_K \)-stable by Lem. 4.4.1, and this induces the desired \( G_K \)-action on \( \mathcal{M} \otimes W(\mathcal{O}_C^\ast) \). The construction of \( G_K \)-action is functorial with respect to \( T \) by Lem. 4.3.3. To show that \( \mathcal{M}/u\mathcal{M} \) is fixed by \( G_K \), simply consider \( G_K \)- and \( G_{K^\infty} \)-actions again, and use Lem. 4.4.1.

The paragraph above gives the desired functor. It is faithful because \( T_\mathcal{S} \) is fully faithful. To show fullness, let \( T_1, T_2 \in \text{Rep}_{Z_p}^{st, \geq 0}(G_K) \), and let \( \widehat{\mathcal{M}}_1, \widehat{\mathcal{M}}_2 \in \text{Mod}_{\mathcal{S}, W(\mathcal{O}_C^\ast)}^{\varphi \in \mathcal{M}} \) be the corresponding objects. A morphism \( \widehat{\mathcal{M}}_1 \to \widehat{\mathcal{M}}_2 \) induces a morphism
\[
\widehat{\mathcal{M}}_1 \otimes_{W(\mathcal{O}_C^\ast)} W(C^\ast) \to \widehat{\mathcal{M}}_2 \otimes_{W(\mathcal{O}_C^\ast)} W(C^\ast)
\]
in \( \text{Mod}_{\mathcal{A}}^{\varphi, G_K} \), and hence induces the desired morphism \( T_1 \to T_2 \) by Lem. 4.3.3. \( \square \)

**Remark 4.4.5.** (1) We do not know if the functor in Thm. 4.4.4 is essentially surjective. Indeed, take an object \( (\mathcal{M}, \mathcal{M}) \in \text{Mod}_{\mathcal{S}, W(\mathcal{O}_C^\ast)}^{\varphi, G_K} \), and let \( T = (\mathcal{M} \otimes_{W(\mathcal{O}_C^\ast)} W(C^\ast))_{\varphi = 1} \). Then base change along \( \mathcal{S} \to \mathcal{S}_K \) (cf. (4.4.1)) gives an object in \( \text{Mod}_{\mathcal{S}, W(\mathcal{O}_C^\ast)}^{\varphi, G_K} \); hence Thm. 4.4.3 implies that \( T|_{G_K^\infty} \) is semi-stable. Thus by Thm. 2.3.2(4), \( T \) is *potentially* semi-stable. However, we do not know if \( T \) is semi-stable.

(2) Let \( T \in \text{Rep}_{Z_p}^{st, \geq 0}(G_K) \), and let \( (\mathcal{M}, \varphi_{\mathcal{M}}, G_K) \) be the corresponding Breuil-Kisin \( G_K \)-module. It might be tempting to add some connection operator \( \nabla \), say on \( \mathcal{M}/u\mathcal{M} \) or just on \( \mathcal{M}/u\mathcal{M} \otimes_{Z_p} \mathbb{Q}_p \). The later map
\[
\nabla: \mathcal{M}/u\mathcal{M} \otimes_{Z_p} \mathbb{Q}_p \to (\mathcal{M}/u\mathcal{M} \otimes_{Z_p} \mathbb{Q}_p) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_0}
\]
certainly exists as it corresponds to the connection on the Fontaine module of \( T[1/p] \); but then it loses integrality information. Furthermore, it is not clear if the above map induces
\[
\nabla: \mathcal{M}/u\mathcal{M} \to \mathcal{M}/u\mathcal{M} \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_0}.
\]

5. Modules over \( \mathcal{S} \) and \( S \)

In this section, we review some results about \( \varphi \)-modules over \( \mathcal{S} \) and \( S \). Our exposition here is more general than what is needed in \$6\$; we use this opportunity to discuss various related results in the literature.

5.1. On \( \varphi \)-modules over \( \mathcal{S} \) and \( S \). Recall in Notation 3.2.4, we defined \( S \). Explicitly,
\[
S = \{ x = \sum_{i \geq 0} a_i \frac{E(u)^i}{i!} \mid a_i \in \mathcal{O}_{K_0}[u], \lim_{i \to \infty} v_p(a_i) = +\infty \} \subset K_0[[u]].
\]
Let \( \text{Fil}^r S \) be the ideal topologically generated by \( (\frac{E(u)^i}{i!}, i \geq r) \). One can check \( S/\text{Fil}^1 S \simeq \mathcal{O}_K \) via \( u \mapsto \pi \). Let \( I_u \subset S \) be the kernel of the \( \mathcal{O}_{K_0} \)-linear map \( f_0 : S \to \mathcal{O}_{K_0} \) where \( u \mapsto 0 \), then \( S/I_u \simeq \mathcal{O}_{K_0} \). Let \( \varphi \) be the Frobenius operator on \( S \) extending \( \varphi \) on \( \mathcal{O}_{K_0} \) such that \( \varphi(u) = u^p \). For \( 1 \leq r \leq p-1 \), one can check \( \varphi(\text{Fil}^r S) \subset p^r S \), and hence one can define
\[
\varphi_r = \frac{\varphi}{p^r} : \text{Fil}^r S \to S.
\]

**Definition 5.1.1.** Suppose \( 1 \leq r \leq p-1 \). Let \( \text{Mod}_{\varphi}^{\mathcal{S}} \) be the category of the following data:

1. \( M \) is a finite free \( S \)-module;
2. \( \text{Fil}^r M \subset M \) is a sub-\( S \)-module which contains \( \text{Fil}^r S \cdot M \).
(3) $\varphi_r : \Fil^r M \to M$ is a $\varphi_{\mathcal{S}}$-semi-linear map such that the linearization map $\varphi^* \Fil^r M \to M$ is surjective.

**Construction 5.1.2.** Let $\Mod_{\mathcal{S}}^{\varphi, r}$ be the sub-category of $\Mod_{\mathcal{S}}$ consisting of objects of $E(u)$-height $\leq r$. There is a natural functor

$$\mathcal{M}_S^r : \Mod_{\mathcal{S}}^{\varphi, r} \to \Mod_{\mathcal{S}}^{\varphi, r}$$

defined as follows. Given $(\mathfrak{M}, \varphi) \in \Mod_{\mathcal{S}}^{\varphi, r}$, let

$$M = \mathcal{M}_S^r(\mathfrak{M}) := S \otimes_{\varphi, \mathcal{S}} \mathfrak{M},$$

and let

$$\Fil^r M = \{m \in M, (1 \otimes \varphi)(m) \in \Fil^r S \otimes_{\varphi, \mathcal{S}} \mathfrak{M}\},$$

and let $\varphi_r : \Fil^r M \to M$ be the composite:

$$\Fil^r M \xrightarrow{1 \otimes \varphi} \Fil^r S \otimes_{\varphi, \mathcal{S}} \mathfrak{M} \xrightarrow{\varphi_r \otimes 1} S \otimes_{\varphi, \mathcal{S}} \mathfrak{M} = M.$$ 

The following theorem and remark collect various results and arguments in [CL09, Kim15, Gao17].

**Theorem 5.1.3.** Suppose $1 \leq r \leq p - 1$.

1. When $p > 2$ and $1 \leq r \leq p - 2$, the functor $\mathcal{M}_S^r$ is an equivalence of categories.
2. For any $p$ and $1 \leq r \leq p - 1$, $\mathcal{M}_S^r$ is essentially surjective.
3. For any $p$ and $1 \leq r \leq p - 1$, $\mathcal{M}_S^r$ induces an equivalence of isogeny categories

$$\Mod_{\mathcal{S}}^{\varphi, r} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \Mod_{\mathcal{S}}^{\varphi, r} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

**Proof.** Item (1) is proved in [CL09, Thm. 2.2.1], under the assumption that $K$ has perfect residue field; but the proof is still valid for general $K$. Items (2) and (3) are proved in [Kim15, Prop. 6.6], which even works in the relative case (cf. Rem. 5.2.5). The proof in [Kim15, Prop. 6.6] uses similar ideas as in [CL09, Thm. 2.2.1], but carries out more refined analysis. Actually, the statement in [Kim15, Prop. 6.6] only deals with the case when $r = 1$, but the proof works for general $r$ as stated here: indeed, the only subtle case (in [Kim15, Prop. 6.6] and in here) is when $p = 2$ (and hence $r = 1$ then). 

**Remark 5.1.4.** In this remark, we first point out a gap in [BT08] related with Thm. 5.1.3 above. We then discuss some related results in [Gao17]: these discussions are not used in the current paper, but we hope it serves to clarify the relation between various results.

1. In [BT08, Prop. 6.8], it is claimed that $\mathcal{M}_S^1$ is fully faithful (including $p = 2$); unfortunately there is a gap in the argument. Using notations in loc. cit., the matrix $\Phi_1$ (similarly $\Phi_2$) is defined for the linearization of the Frobenius operator on $\mathfrak{M}_1$, hence instead of the claim that

$$\Phi_1 \in \text{GL}_{r_1}([\mathcal{S}[E(u)^{-1}]) \text{ and } E(u) \cdot \Phi_1^{-1} \in M_{r_1}(\mathcal{S}),$$

we only have

$$\Phi_1 \in \text{GL}_{r_1}([\varphi(E(u))^{-1}]) \text{ and } \varphi(E(u)) \cdot \Phi_1^{-1} \in M_{r_1}(\mathcal{S}).$$

Thus [BT08, Lem. 6.9] becomes irrelevant. Furthermore, the “modified” statement of loc. cit.:

- “if $f \in S$ and $\varphi(E(u))^N \cdot f \in \mathcal{S} + p^N S$ for all $N \geq 0$, then $f \in \mathcal{S}$”

is false; e.g., $f = \frac{E(u)^p}{p}$ is a counter-example. Indeed, we do not know if $\mathcal{M}_S^1$ is fully faithful when $p = 2$; see also next item.

2. As part of [Gao17, Thm. 2.5.6], it is shown that

- (Statement A): $\mathcal{M}_S^{p-1}$ (including $p = 2$) is fully faithful when restricted to the unipotent sub-categories.

For the notion of unipotency, we refer the reader to [Gao17, Def. 2.1.1, Def. 2.2.2], and also Part 0 in the proof of [Gao17, Thm. 2.5.6]. In fact, a stronger result is proved there:
• (Statement B): the “mod p functor” of $\mathcal{M}^p_{S^{-1}}$, denoted as

\[ \mathcal{M}_{S^{-1}} : \text{Mod}_{S^{-1}}^p \rightarrow \text{Mod}_{S^{-1}}^p \]

in loc. cit. is an equivalence of categories when restricted to the unipotent sub-categories.

In fact, one can directly prove the weaker (Statement A) using similar strategy as in (Statement B). However, if one examines the proof, it seems very unlikely that one can remove unipotency condition there.

(3) The other part of [Gao17, Thm. 2.5.6] shows that $\mathcal{M}^p_{S^{-1}}$ is essentially surjective (hence an equivalence) when restricted to the unipotent sub-categories; this can be regarded as supplement to results listed in Thm. 5.1.3.

The proof of essentially surjectivity in [Gao17, Thm. 2.5.6] uses similar strategy as [CL09, Thm. 2.2.1] and [Kim15, Prop. 6.6], but the result is weaker than Thm. 5.1.3(2) stated here. Indeed, in the proof of [Gao17, Thm. 2.5.6], we make use of a ring “$\Sigma$” $\subset S$, hence the iteration procedure actually becomes harder to converge: this explains why in loc. cit. we did not obtain Thm. 5.1.3(2).

5.2. Modules with connections.

**Definition 5.2.1.** Let $\text{MF}^\text{BT}_S(\varphi, \nabla)$ be the category of the following data:

1. $(M, \text{Fil}^1 M, \varphi_1) \in \text{Mod}_{S^{-1}}^{p, 1}$;
2. $\nabla : M/I_u M \rightarrow M/I_u M \otimes_{O_{K_0}} \widehat{\Omega}_{O_{K_0}}$ is a quasi-nilpotent integrable connection which commutes with $\varphi$-action on $M/I_u M$.

**Definition 5.2.2.** Let $\text{MF}^\text{BT}_S(\varphi, \nabla)$ be the category of the following data (which are called minuscule Breuil-Kisin modules with connections):

1. $(\mathfrak{M}, \varphi) \in \text{Mod}_{S^{-1}}^{p, 1}$;
2. $\nabla : \varphi^*(\mathfrak{M}/u\mathfrak{M}) \rightarrow \varphi^*(\mathfrak{M}/u\mathfrak{M}) \otimes_{O_{K_0}} \widehat{\Omega}_{O_{K_0}}$ is a quasi-nilpotent integrable connection which commutes with $\varphi$-action on $\varphi^*(\mathfrak{M}/u\mathfrak{M})$, where

\[ \varphi^*(\mathfrak{M}/u\mathfrak{M}) = \mathfrak{M}/u\mathfrak{M} \otimes_{\varphi, O_{K_0}} O_{K_0}. \]

**Remark 5.2.3.** (1) In Def. 5.2.2, we need to define $\nabla$ over the “Frobenius twist” $\varphi^*(\mathfrak{M}/u\mathfrak{M})$ so that the it is compatible with the $\nabla$ in Def. 5.2.1 under the functor (5.2.1) below. Our definition is compatible with that in [Kim15, Def. 6.1]. Note however that in [BT08, Def. 4.16], the $\nabla$-operator is defined over $\mathfrak{M}/u\mathfrak{M}$; however, it seems likely to be a wrong definition, see next item.

(2) As noted in Rem. 2.4.6, for a Fontaine module $D$, there is a $K_0$-linear isomorphism $\varphi^*(D) \simeq D$; hence to define $\nabla$ over $D$ is equivalent to do so over $\varphi^*(D)$. Indeed, this is why we do not need “Frobenius twist” in the $\nabla$-operators in Def. 2.4.5 and all the various categories in §3, as all of them are defined on the $K_0$-level. However, on the integral level, we do no expect $\varphi^*(\mathfrak{M}/u\mathfrak{M}) \simeq \mathfrak{M}/u\mathfrak{M}$ in general.

The functor in Construction 5.1.2 obviously induces a functor

(5.2.1) $\text{BT}_S^S : \text{MF}^\text{BT}_S(\varphi, \nabla) \rightarrow \text{MF}^\text{BT}_S(\varphi, \nabla)$

The following corollary follows obviously from Thm. 5.1.3.

**Corollary 5.2.4.** Consider the functor $\text{BT}_S^S$. It is an equivalence of categories when $p > 2$. It is essentially surjective when $p = 2$. For any $p$, it induces an equivalence of the isogeny categories.

**Remark 5.2.5.** Let $R$ be as in Assumption 4.1.1, and suppose it satisfies Assumption (i) there. Then [Kim15, Prop. 6.6] proves the relative version of Thm. 5.1.3 and hence Cor. 5.2.4.
6. Breuil-Kisin modules and \(p\)-divisible groups

In this section, we study \(p\)-divisible groups over \(\mathcal{O}_K\). The first two subsections §6.1, §6.2 contain results established by Brinon-Trihan (except Thm. 6.2.4), on the relationship between \(p\)-divisible groups, \(S\)-modules and crystalline representations. In §6.3, we classify \(p\)-divisible groups by Breuil-Kisin modules with connections; the case when \(p > 2\) is already known, the case \(p = 2\) requires new ideas and our approach is informed by the integral theory developed in §4. Finally in §6.4, we classify finite flat group schemes over \(\mathcal{O}_K\).

6.1. \(p\)-divisible groups and \(S\)-modules. Let \(\text{BT}(\mathcal{O}_K)\) be the category of \(p\)-divisible groups over \(\mathcal{O}_K\). Let \(G \in \text{BT}(\mathcal{O}_K)\); it corresponds to a compatible system \((G_n)_{n>0}\) where \(G_n\) is a Barsotti-Tate group over \(\mathcal{O}_K/p^n\mathcal{O}_K\) for each \(n\). Let \(D^*(G_n)\) be the co-variant (cf. Convention 1.3.1) Dieudonné crystals, and consider its evaluation on the thickening \(S \to \mathcal{O}_K/p^n\mathcal{O}_K\). Define
\[
\mathbf{M}(G) = D^*(G)(S \to \mathcal{O}_K) := \lim_{n \to 0} D^*(G_n)(S \to \mathcal{O}_K/p^n\mathcal{O}_K).
\]

**Theorem 6.1.1.** The above construction gives rise to a functor
\[
\mathbf{M} : \text{BT}(\mathcal{O}_K) \to \text{MF}^{\text{BT}}_S(\varphi, \nabla).
\]

The functor is an equivalence if \(p > 2\); it induces an equivalence of isogeny categories if \(p = 2\).

**Proof.** This is [BT08, Prop. 5.9, Thm. 5.10]. \(\square\)

**Proposition 6.1.2.** Let \(G \in \text{BT}(\mathcal{O}_K)\), the functor in Thm. 6.1.1 induces a \((\varphi, \nabla)\)-equivariant isomorphism of \(\mathcal{O}_{K_0}\)-modules:
\[
D^*(G \otimes_{\mathcal{O}_K} k)(\mathcal{O}_{K_0}) = \mathbf{M}(G) \otimes_S \mathcal{O}_{K_0} = \mathbf{M}(G)/I_u \mathbf{M}(G).
\]

Furthermore, we have a \(\varphi\)-equivariant isomorphism of \(\mathcal{O}_{K_0}\)-modules:
\[
\mathbf{M}(G)[1/p] \simeq D^*(G \otimes_{\mathcal{O}_K} k)(\mathcal{O}_{K_0}) \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_0}[1/p].
\]

**Proof.** (6.1.1) is extracted from the proof of [BT08, Prop. 5.9], and (6.1.2) is [BT08, Lem. 6.1]. \(\square\)

**Remark 6.1.3.** Let \(R\) be as in Assumption 4.1.1, and suppose it satisfies Assumption (ii) there. Then [Kim15, Thm. 3.17] proves the relative version of Thm. 6.1.1; it implies the relative version of (6.1.1). The relative version of (6.1.2) also follows from similar argument as in [BT08, Lem. 6.1].

6.2. \(p\)-divisible groups and crystalline representations. Let \(\text{Rep}^{\text{cris},1}_{\mathcal{Q}_p}(G_K)\) (resp. \(\text{Rep}^{\text{cris},1}_{\mathcal{Z}_p}(G_K)\)) be the category of crystalline (resp. integral crystalline) representations of \(G_K\) with Hodge-Tate weights in \(\{0, 1\}\). In this subsection, we show that \(\text{Rep}^{\text{cris},1}_{\mathcal{Q}_p}(G_K)\) is equivalent to \(\text{BT}(\mathcal{O}_K)\).

**Proposition 6.2.1.** [BT08, Cor. 6.4] Let \(G \in \text{BT}(\mathcal{O}_K)\), and let \(T_p(G)\) be its co-variant Tate module. Then \(T_p(G) \in \text{Rep}^{\text{cris},1}_{\mathcal{Z}_p}(G_K)\). Furthermore, if we let \(V_p(G) = T_p(G)[1/p]\), then there is a \((\varphi, \nabla)\)-equivariant isomorphism of \(K_0\)-vector spaces:
\[
D_{\text{cris}}(V_p(G)) = D^*(G_K)[1/p].
\]

**Remark 6.2.2.** Let \(R\) be a normal domain (the general assumption in [Kim15, §4, §5]), and suppose it satisfies Assumption 4.1.1(iii), then [Kim15, Thm. 5.2] and particularly [Kim15, Cor. 5.3] proves the relative version of Prop. 6.2.1.

**Theorem 6.2.3.** [BT08, Thm. 6.10] The functor \(V_p\) induces an equivalence of categories
\[
\text{BT}(\mathcal{O}_K) \otimes_{\mathcal{Z}_p} \mathcal{Q}_p \to \text{Rep}^{\text{cris},1}_{\mathcal{Q}_p}(G_K)
\]
Proof. In the beginning of the proof of [BT08, Thm. 6.10], it says that “we already know that the functor is fully faithful”; it is not clear to us what the authors mean: it is certainly not obvious that \( V_p \) is fully faithful (at that point; also see our exposition in the following). (As a side note, it seems to us the authors are not pointing to [BT08, Prop. 6.8], whose proof is wrong as we mentioned in Rem. 5.1.4(1).)

In any case, we choose to give an exposition of the proof. It particularly serves as a good place to see how to piece together the various module comparisons we have obtained so far.

Let \( \text{MF}^\omega_{K/K_0}(\varphi, \nabla) \subset \text{MF}^\omega_{K/K_0}(\varphi, N, \nabla) \) be the sub-category consisting of objects with \( N = 0 \) and with Hodge-Tate weights in \( \{0,1\} \). Consider the composite of functors in the following diagram:

\[
\begin{align*}
\text{BT}(O_K) \otimes_{Z_p} Q_p & \xleftarrow{\simeq, \text{Thm. 6.1.1}} \text{MF}^\text{BT}_{S}(\varphi, \nabla) \otimes_{Z_p} Q_p \\
\text{Rep}^\text{cris}_p(G_K) & \xrightarrow{\simeq, \text{Thm. 5.1.3}} \text{MF}^\omega_{K/K_0}(\varphi, \nabla) \\
& \xrightarrow{\simeq, \text{Thm. 2.4.9}} \text{MF}^\omega_{K/K_0}(\varphi, \nabla) \\
& \xrightarrow{\text{Thm. 3.2.3}} \text{MF}^\text{BT}_{S}(\varphi, \nabla) \otimes_{Z_p} Q_p
\end{align*}
\]

(6.2.1)

The only caveat in the diagram is the fully faithful functor

\[ \text{MF}^\omega_{K/K_0}(\varphi, \nabla) \rightarrow \text{MF}^\text{BT}_{S}(\varphi, \nabla) \otimes_{Z_p} Q_p. \]

Indeed, the functor labelled as (*) in Thm. 3.2.3 implies there is a fully faithful functor

\[ \text{MF}^\omega_{K/K_0}(\varphi, \nabla) \rightarrow \text{Mod}^1_{\text{cris}}(\varphi, \nabla) \otimes_{Z_p} Q_p, \]

where \( \text{Mod}^1_{\text{cris}}(\varphi, \nabla) \) is the sub-category of \( \text{Mod}_{\text{cris}}(\varphi, N, \nabla) \) consisting of modules with \( E(u) \)-height \( \leq 1 \) and \( N = 0 \). Note that the \( \nabla \) for \( \text{Mod}^1_{\text{cris}}(\varphi, \nabla) \) in Def. 3.2.2 is defined on the \( K_0 \)-level, whereas \( \nabla \) for \( \text{MF}^\text{BT}_{S}(\varphi, \nabla) \) in Def. 5.2.2 is defined on the “Frobenius-twisted” \( O_{K_0} \)-level. Nonetheless, we have an equivalence of isogeny categories:

\[ \text{MF}^\text{BT}_{S}(\varphi, \nabla) \otimes_{Z_p} Q_p \simeq \text{Mod}^1_{\text{cris}}(\varphi, \nabla) \otimes_{Z_p} Q_p, \]

by the discussions in Rem. 5.2.3.

To prove the theorem, it suffices to show that \( V_p \) is a quasi-inverse of the composite in diagram (6.2.1), and hence completes the diagram. The argument in the following is the same as in [BT08, Thm. 6.10].

Let \( V \in \text{Rep}^\text{cris}_p(G_K) \), and suppose it maps to \( D, \mathfrak{M}, M, G \) in the corresponding categories using the functors above (where \( \mathfrak{M}, M, G \) are constructed up to isogeny).

\[
\begin{align*}
G & \leftarrow M \\
V & \rightarrow \mathfrak{M} \\
D & \rightarrow \mathfrak{M}
\end{align*}
\]

(6.2.3)

We have the following comparisons, which are always \( (\varphi, \nabla) \)-equivariant and compatible with filtrations

1. \( V \otimes_{Q_p} B_{\text{cris}} \simeq D \otimes_{K_0} B_{\text{cris}} \), since \( V \) is crystalline;
2. \( \mathfrak{M} \otimes_{\varphi, S} S \simeq M \), by construction;
3. \( D \otimes_{K_0} S[1/p] \simeq M[1/p] \), by Prop. 3.2.5, where the compatibility with filtrations is discussed in the proof of [BT08, Thm. 6.10];
4. \( M[1/p] \simeq D^\ast(G_k)(O_{K_0}) \otimes_{O_{K_0}} S[1/p] \), by (6.1.2);
5. \( D^\ast(G_k)(O_{K_0})[1/p] \otimes_{K_0} B_{\text{cris}} \simeq V_p(G) \otimes_{Q_p} B_{\text{cris}} \) by Prop. 6.2.1.
Combining all these comparisons, we get a \((\varphi, \nabla)\)-equivariant and filtration-compatible isomorphism
\[(6.2.4)\]
\[V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{cris}} \simeq V_p(G) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{cris}}.\]
Note that \textit{a priori}, we do not know if it is \(G_K\)-equivariant, essentially because there is no \(G_K\)-action on \(\mathfrak{M}\) or \(M\). However, if we equip trivial \(G_{K_\infty}\)-actions on \(\mathfrak{M}\) and \(M\) (and on \(\mathfrak{S}\) and \(S\)) along all the listed comparisons (1)-(5), then it is easy to see that (6.2.4) is \(G_{K_\infty}\)-equivariant. Taking \(\varphi = 1, \nabla = 0\) in the \(\text{Fil}^0\) pieces of (6.2.4), we see that \(V \simeq V_p(G)\) as \(G_{K_\infty}\)-representations; they are furthermore isomorphic as \(G_K\)-representations by \([BT08, \text{Prop. 3.8}]\), which says that the restriction functor
\[
\text{Rep}_{\mathbf{Q}_p}^\text{cris}(G_K) \to \text{Rep}_{\mathbf{Q}_p}(G_{K_\infty})
\]
is fully faithful.

**Theorem 6.2.4.** The functor \(T_p\) from \(\text{BT}(\mathcal{O}_K)\) to \(\text{Rep}_{\mathbf{Z}_p}^\text{cris,1}(G_K)\) is an equivalence.

**Proof.** With Thm. 6.2.3 (the analogue of \([Kis06, \text{Cor. 2.2.6}]\)) established, the proof follows the same argument in \([Liu13, \text{Thm. 2.2.1}]\). In particular, Tate’s isogeny theorem \([Tat67, \text{p. 181, Cor. 1}]\) is still applicable because \(\mathcal{O}_K\) is an integrally closed, Noetherian, integral domain with \(\text{char}K = 0\) as required in \([Tat67, \text{p. 180, Thm. 4}]\). Also, \([Ray74, \text{Prop. 2.3.1}]\) is still applicable because \(\mathcal{O}_K\) is a mixed characteristic discrete valuation ring as required in the beginning of \([Ray74, \text{p. 259, \S 2}]\) and \([Ray74, \text{p. 261, \S 2.3}]\). \(\square\)

**Remark 6.2.5.** It is far from clear if the \textit{relative} version of Thm. 6.2.3 (let alone Thm. 6.2.4) should still hold. Nonetheless, recently, Liu and Moon \([LM]\) have shown that these results hold when the “ramification index” \(e\) in Assumption 4.1.1(1) satisfies \(e < p - 1\).

6.3. \(p\)-divisible groups and Breuil-Kisin modules. The content of Thm. 6.2.3 completes the diagram (6.2.1); hence we obtain the following diagram (with top arrow direction reversed) of \textit{equivalences of categories}:

\[(6.3.1)\]
\[
\begin{array}{ccc}
\text{BT}(\mathcal{O}_K) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p & \longrightarrow & \text{MF}_{S}^{\text{BT}}(\varphi, \nabla) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \\
\downarrow & & \uparrow \\
\text{Rep}_{\mathbf{Z}_p}^{\text{cris,1}}(G_K) & \text{MF}_{S}^{\text{BT}}(\varphi, \nabla) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p & \text{MF}_{K/K_0}^{\text{wa,1}}(\varphi, \nabla) \\
\downarrow & & \downarrow \\
& \text{MF}_{\mathfrak{S}}^{\text{BT}}(\varphi, \nabla) & \end{array}
\]

All the categories in (6.3.1) except the bottom one have \textit{integral} avatars, and the content of this subsection is to complete the following diagram of \textit{integral categories}:

\[(6.3.2)\]
\[
\begin{array}{ccc}
\text{BT}(\mathcal{O}_K) & \longrightarrow & \text{MF}_{S}^{\text{BT}}(\varphi, \nabla) \\
\downarrow & & \uparrow \\
\text{Rep}_{\mathbf{Z}_p}^{\text{cris,1}}(G_K) & \longrightarrow & \text{MF}_{\mathfrak{S}}^{\text{BT}}(\varphi, \nabla) \\
\end{array}
\]

In particular, we will show that the (to-be-constructed) dotted arrow is an equivalence of categories: this implies that \(\text{MF}_{\mathfrak{S}}^{\text{BT}}(\varphi, \nabla)\) is equivalent to \(\text{BT}(\mathcal{O}_K)\).

**Proposition 6.3.1.** Let \(T \in \text{Rep}_{\mathbf{Z}_p}^{\text{cris,1}}(G_K)\), and let \(G \in \text{BT}(\mathcal{O}_K)\) be the corresponding \(p\)-divisible group via Thm. 6.2.4. Let \(\mathfrak{M}(T)\) be the Breuil-Kisin module attached to \(T\) via Thm. 4.4.4. Then we have a \(\varphi\)-equivariant isomorphism
\[
\mathfrak{M}(T) \otimes_{\varphi, \mathfrak{S}} S \simeq M(G).
\]
Theorem 6.3.3. The functor $\mathcal{M}$ is an equivalence, and hence we have equivalences of categories:

$$\text{BT}(\mathcal{O}_K) \xrightarrow{T_p} \text{Rep}^{\text{cris},1}(G_K) \xrightarrow{\mathcal{M}} \text{MF}^\text{BT}_K(\varphi, \nabla).$$

Remark 6.3.4. (1) When $p > 2$, the theorem follows by combining Thm. 6.1.1 (due to Brinon-Trihan) and Cor. 5.2.4 (which is essentially Thm. 5.1.3, and is due to Caruso-Liu). Our argument here works for any $p$, using only the equivalence up to isogeny in Thm. 6.1.1 as input; namely, we avoid using Thm. 5.1.3.

(2) Suppose $R$ satisfies Assumption 4.1.1(ii) and $p > 2$. As mentioned in Rem. 6.2.5 and Rem. 6.1.3 respectively, [Kim15, Prop. 6.6] and [Kim15, Thm. 3.17] prove the relative version of Cor. 5.2.4 and Thm. 6.1.1 respectively; hence there is an equivalence of categories

$$\text{BT}(R) \xrightarrow{\sim} \text{MF}^\text{BT}_{K^p}(\varphi, \nabla)$$

in this context. Note however the category $\text{Rep}^{\text{cris},1}(G_R)$ is not involved in this equivalence, cf. Rem. 6.2.5.

First, recall that Thm. 6.3.3 holds in the perfect residue field case.

Theorem 6.3.5. We have equivalences of categories:

$$\text{BT}(\mathcal{O}_K) \rightarrow \text{Rep}^{\text{cris},1}(G_K) \rightarrow \text{MF}^\text{BT}_K(\varphi).$$

Proof. This is due to Kisin [Kis06] when $p > 2$, and independently to Kim [Kim12], Lau [Lau14] and Liu [Liu13] when $p = 2$. 

We start with an easy lemma on connections.

Lemma 6.3.6. Let

$$F : (\mathcal{M}, \nabla_1) \rightarrow (\mathcal{M}_2, \nabla_1)$$

be a morphism in $\text{MF}^\text{BT}_K(\varphi, \nabla)$. Suppose the restricted morphism $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ in $\text{Mod}^\varphi_K$ is injective, and induces an isomorphism in $\text{Mod}^\varphi_K \otimes \mathbb{Z}_p \mathbb{Q}_p$. Then $f$ induces an injection

$$\mathcal{M}_1/u\mathcal{M}_1 \hookrightarrow \mathcal{M}_2/u\mathcal{M}_2,$$

and hence $\varphi^*(\mathcal{M}_1/u\mathcal{M}_1) \hookrightarrow \varphi^*(\mathcal{M}_2/u\mathcal{M}_2)$.

Furthermore, $\nabla_2$ on $\varphi^*(\mathcal{M}_2/u\mathcal{M}_2)$ admits $\nabla_1$ on $\varphi^*(\mathcal{M}_1/u\mathcal{M}_1)$. 

Proof. When $K$ has perfect residue field, this is [Kim12, Prop. 2.4]. In the general case, restricting the representation to $G_K$ implies

$$(\mathcal{M}(T) \otimes_{\mathbb{E}_K} \mathcal{S}_K) \otimes_{\varphi, \mathcal{S}_K} \mathcal{S}_K \simeq \mathcal{M}(G) \otimes_S \mathcal{S}_K,$$

where $\mathcal{S}_K$ is the “$\mathbb{K}$-version” of $\mathcal{S}$ defined in §5. Note that the objects in (6.2.3) correspond to each other, up to isogeny, via the equivalences of categories in (6.3.1), hence we must have

$$\mathcal{M}(T) \otimes_{\varphi, \mathcal{S}} \mathcal{S}[1/p] \simeq \mathcal{M}(G) \otimes_S \mathcal{S}[1/p].$$

We can conclude using the fact that $\mathcal{S}_K \cap \mathcal{S}[1/p] = \mathcal{S}$. 

Proof. Let $T \in \text{Rep}^{\text{cris},1}_p(G_K)$, and let $G \in \text{BT}(\mathcal{O}_K)$ be the corresponding $p$-divisible group. By Prop. 6.3.1, $\mathcal{M}(T) \otimes_{\varphi, \mathcal{S}} \mathcal{S} \simeq \mathcal{M}(G)$, hence the $\nabla$-operator on $\mathcal{M}(G)/\mathcal{I}_p \mathcal{M}(G)$ induces a $\nabla$-operator on $\varphi^*(\mathcal{M}(T)/u\mathcal{M}(T))$. This gives rise to an object in $\text{MF}^\text{BT}_K(\varphi, \nabla)$, and obviously defines the desired functor. 

Proposition 6.3.2. There exists a natural functor

$$\mathcal{M} : \text{Rep}^{\text{cris},1}_p(G_K) \rightarrow \text{MF}^\text{BT}_K(\varphi, \nabla),$$

which makes (3.2) commutative, and which after $\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ fits into (3.1). 

Proof. This is due to Kisin [Kim12] respectively, and obviously defines the desired functor.
Proof. The quotient $\mathcal{M}_2/\mathcal{M}_1$ is killed by some $p$-power and hence is $u$-torsion free by [Liu07, Prop. 2.3.2] (which is still applicable in our case). This implies that $\mathcal{M}_1/u\mathcal{M}_1 \hookrightarrow \mathcal{M}_2/u\mathcal{M}_2$. Then it is clear that $\nabla_2$ induces $\nabla_1$ by looking at the commutative diagram (recall that $\Omega_{\mathcal{O}_K}$ is finite free over $\mathcal{O}_K$)

$$
\phi^*(\mathcal{M}_1/u\mathcal{M}_1) \xrightarrow{\nabla_1} \phi^*(\mathcal{M}_1/u\mathcal{M}_1) \otimes_{\mathcal{O}_K} \hat{\Omega}_{\mathcal{O}_K} \\
\phi^*(\mathcal{M}_2/u\mathcal{M}_2) \xrightarrow{\nabla_2} \phi^*(\mathcal{M}_2/u\mathcal{M}_2) \otimes_{\mathcal{O}_K} \hat{\Omega}_{\mathcal{O}_K}.
$$

\[\square\]

Proof of Thm. 6.3.3. Part 1: full faithfulness. $\mathfrak{M}$ is faithful because the functor $T_\phi : \text{Mod}_G \rightarrow \text{Rep}_{z_p}(G_{K_\infty})$ is fully faithful. To show fullness, suppose given $T_1, T_2 \in \text{Rep}_{z_p}^{cris,1}(G_K)$, and the corresponding $(\mathcal{M}_1, \nabla_1), (\mathcal{M}_2, \nabla_2) \in \text{MF}^{BT}(\phi, \nabla)$.

Let $\mathfrak{f} : (\mathcal{M}_1, \nabla_1) \rightarrow (\mathcal{M}_2, \nabla_2)$ be a morphism. Since $\mathfrak{M}$ induces an equivalence between the isogeny categories, $\mathfrak{f}$ induces a unique $G_K$-equivariant morphism $f : T_1[1/p] \rightarrow T_2[1/p]$.

It suffices to show that $f(T_1) \subset T_2$ (as subsets). One can simply consider the induced morphism $\mathfrak{f}_\mathfrak{M} : \mathcal{M}_1 \otimes_{\mathfrak{M}} \mathfrak{G}_K \rightarrow \mathcal{M}_2 \otimes_{\mathfrak{M}} \mathfrak{G}_K$, then the $G_{K_\infty}$-equivariant $f$ must satisfies $f(T_1) \subset T_2$ by Thm. 6.3.5.

Part 2: essential surjectivity. Let $(\mathcal{M}, \nabla) \in \text{MF}^{BT}(\phi, \nabla)$. Since $\mathfrak{M}$ induces an equivalence between the isogeny categories, we can choose some $L \in \text{Rep}_{z_p}^{cris,1}(G_K)$ such that if we let $\mathfrak{M}(L) = (\mathfrak{M}, \nabla_{\mathfrak{M}}) \in \text{MF}^{BT}(\phi, \nabla)$ be the associated object, then we have a morphism $(\mathfrak{M}, \nabla) \rightarrow (\mathfrak{M}, \nabla_{\mathfrak{M}})$ such that $\mathfrak{M} \rightarrow \mathfrak{M}$ is an injective morphism in $\text{Mod}_G$ which becomes isomorphism after inverting $p$. By Lem. 6.3.6, $\nabla_{\mathfrak{M}}$ induces $\nabla$.

Let $V = L[1/p]$. By Lem. 4.2.10 and Lem. 4.3.3, there exist $\phi$-equivariant isomorphisms

\begin{equation}
\mathfrak{M} \otimes_{\mathfrak{M}} W(C^p)[1/p] \simeq \mathfrak{M} \otimes_{\mathfrak{M}} W(C^p)[1/p] \simeq V \otimes_{q_p} W(C^p)[1/p].
\end{equation}

Note that $(\mathfrak{M} \otimes_{\mathfrak{M}} \mathfrak{G}_K, \phi \otimes \phi)$ corresponds to some $p$-divisible group over $\mathfrak{O}_K$ and hence to some integral crystalline representation of $G_K$ via Thm. 6.3.5; thus by Thm. 4.4.3,

\begin{equation}
\mathfrak{M} \otimes_{\mathfrak{M}} W(O_{C^p}) = (\mathfrak{M} \otimes_{\mathfrak{M}} \mathfrak{G}_K) \otimes_{\mathfrak{G}_K} W(O_{C^p}) \text{ is } G_{K_\infty}\text{-stable.}
\end{equation}

Using the identification $C^p \simeq \mathbb{C}$ via $i_\phi$ in §2.1, we have a $\phi$-equivariant isomorphism

\begin{equation}
\mathfrak{M} \otimes_{\mathfrak{M}} W(C^p)[1/p] \simeq V \otimes_{q_p} W(C^p)[1/p],
\end{equation}

and hence the $G_K$-action on $V \otimes_{q_p} W(C^p)[1/p]$ induces a $G_K$-action on $\mathfrak{M} \otimes W(C^p)[1/p]$. Under this action, $\mathfrak{M} \otimes W(O_{C^p})$ is $G_{K_\infty}$-stable by (6.3.4). Prop. 4.3.5 implies $G_{K_\infty}$ acts trivially on $\Omega[1/p] = \mathfrak{M}[1/p]$ as $\mathfrak{M}$ comes from a crystalline representation; hence $\mathfrak{M} \otimes W(O_{C^p})$ is also $G_{K_\infty}$-stable. Thus Lem. 4.4.1 implies that $\mathfrak{M} \otimes W(O_{C^p})$ is $G_K$-stable. Thus, in particular, $\mathfrak{M} \otimes W(C^p)$ is $G_K$-stable, and hence gives rise to a $G_K$-stable $Z_p$-lattice $T \subset L \subset V$. Let $\mathfrak{M}'(T) = (\mathfrak{M}' \otimes_{\mathfrak{M}} \mathfrak{G}_K)$, $\nabla'$. Obviously $\mathfrak{M}' = \mathfrak{M}$ (e.g., by Lem. 4.2.9). The injective morphism $T \hookrightarrow L$ induces a morphism $(\mathfrak{M}, \nabla') \rightarrow (\mathfrak{M}, \nabla_{\mathfrak{M}})$, such that $\mathfrak{M} \rightarrow \mathfrak{M}$ is injective. Hence Lem. 6.3.6 implies $\nabla'$ is also induced by $\nabla_{\mathfrak{M}}$, and hence $\nabla' = \nabla$. Namely, $\mathfrak{M}(T) = (\mathfrak{M}, \nabla)$. \[\square\]
6.4. Finite flat group schemes. Let $\text{FF}(\mathcal{O}_K)$ be the category of finite flat group schemes over $\mathcal{O}_K$. In this subsection, we classify $\text{FF}(\mathcal{O}_K)$ by torsion minuscule Breuil-Kisin modules with connections.

**Definition 6.4.1.** (1) Let $\text{Mod}^{\text{tor},\varphi,1}_S$ be the category consisting of $(\mathfrak{M}, \varphi)$ where $\mathfrak{M}$ is a finite generated $\mathfrak{S}$-module killed by $p^n$ for some $n \geq 1$, and $\varphi : \mathfrak{M} \to \mathfrak{M}$ is a $\varphi_{\mathfrak{S}}$-semi-linear map such that the $\mathfrak{S}$-span of $\varphi(\mathfrak{M})$ contains $E(u) \cdot \mathfrak{M}$.

(2) Let $\text{MF}_{\varphi}^{\text{tor},\text{BT}}(\varphi, \nabla)$ be the category consisting of the following data:

(a) $(\mathfrak{M}, \varphi) \in \text{Mod}^{\text{tor},\varphi,1}_S$.

(b) $\nabla : \varphi^*(\mathfrak{M}/u\mathfrak{M}) \to \varphi^*(\mathfrak{M}/u\mathfrak{M}) \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{K_0}$ is a quasi-nilpotent integrable connection which commutes with $\varphi$.

**Definition 6.4.2.** For an exact category $\mathcal{C}$, let $D^b(\mathcal{C})$ be the bounded derived category.

1. Let $(\text{Mod}^{\varphi,1}_S)^\bullet$ be the full subcategory of $D^b(\text{Mod}^{\varphi,1}_S)$ consisting of two-term complexes $\mathfrak{M}^\bullet = \mathfrak{M}_1 \to \mathfrak{M}_2$ concentrated in degree 0 and -1, such that the morphism $\mathfrak{M}_1 \to \mathfrak{M}_2$ is injective and $H^0(\mathfrak{M}^\bullet)$ is killed by some $p$-power.

2. Let $(\text{BT}(\mathcal{O}_K))^\bullet$ be the full subcategory of $D^b(\text{BT}(\mathcal{O}_K))$ consisting of isogenies of $p$-divisible groups $G^\bullet = G_1 \to G_2$.

**Proposition 6.4.3.** (1) The functor $\mathfrak{M}^\bullet \mapsto H^0(\mathfrak{M}^\bullet)$ induces an equivalence between $(\text{Mod}^{\varphi,1}_S)^\bullet$ and $\text{Mod}^{\text{tor},\varphi,1}_S$.

(2) The functor $G^\bullet \mapsto \text{Ker}(G^\bullet)$ induces an equivalence between $(\text{BT}(\mathcal{O}_K))^\bullet$ and $\text{FF}^{\text{tor, BT}}(\varphi, \nabla)$.

**Proof.** Item (1) is [Kis06, Lem. 2.3.4]. Item (2) is proved in the proof of [Kis06, Thm. 2.3.5]; the argument uses [BBM82, Thm. 3.1.1], which is applicable for our $\mathcal{O}_K$.

**Theorem 6.4.4.** There is an equivalence of categories

$$\text{FF}(\mathcal{O}_K) \xrightarrow{\simeq} \text{MF}_{\varphi}^{\text{tor, BT}}(\varphi, \nabla).$$

**Remark 6.4.5.** Let $R$ be as in Assumption 4.1.1 and suppose it satisfies Assumption (ii) there, then [Kim15, Thm. 9.8] proves the relative version of our Thm. 6.4.4 when $p > 2$. The argument there also proves our Thm. 6.4.4 even when $p = 2$, as our Thm. 6.3.3 is available when $p = 2$. Note that the proof in [Kim15] makes crucial use of Vasiu’s theory of “moduli of connections” [Vas13], which is applicable when $p = 2$.

**Proof of Thm. 6.4.4.** For the reader’s convenience, we give a sketch of Kim’s argument mentioned in Rem. 6.4.5, specialized in the case $R = \mathcal{O}_K$. This should only serve as a very brief road-map to Kim’s argument; in particular, we do not review Vasiu’s theory.

Our Thm. 6.3.3 is stronger than the assumption in [Kim15, Prop. 9.5], and our Prop. 6.4.3(2) is stronger than [Kim15, Lem. 9.7]; hence the argument in [Kim15, Prop. 9.5] easily shows that we can construct a fully faithful functor $\mathfrak{M}^\bullet : \text{FF}(\mathcal{O}_K) \to \text{MF}_{\varphi}^{\text{tor, BT}}(\varphi, \nabla)$.

The difficulty is to show the above functor is essentially surjective; the proof here follows similar strategy as in [Kim15, §10.4]. Let $(\mathfrak{M}, \varphi, \nabla) \in \text{MF}_{\varphi}^{\text{tor, BT}}(\varphi, \nabla)$. Unlike Prop. 6.4.3(1), it seems very difficult to directly lift $(\mathfrak{M}, \varphi, \nabla)$ to a finite free object with a connection. Nonetheless, Prop. 6.4.3(1) (playing the role of [Kim15, Lem. 10.10] here) implies that there exists a finite free $\mathfrak{N} \in \text{Mod}^\varphi_S$ of height 1 and a $\varphi$-equivariant surjective morphism $\mathfrak{N} \to \mathfrak{M}$. But

$$\mathcal{O}_{K_0} : = \mathfrak{N} \otimes_{\varphi, \mathfrak{S}} \mathcal{O}_{K_0} = \varphi^*(\mathfrak{M}/u\mathfrak{M}).$$

One can use the same argument as in [Kim15, Prop. 10.11] – which critically uses Vasiu’s theory of moduli of connections – to show that there exists a faithfully flat ind-étale map

$$\mathcal{O}_{K_0} \to A_0$$

such that the following are satisfied:
(1) $A_0$ is a CDVR with $p$ as a uniformizer and with residue field admitting a finite $p$-basis; here is the argument:

- in our context, the “$R_{0,k}$” in [Kim15, p. 8219] is simply $k_K$ (which is a field admitting a finite $p$-basis).
- Hence the étale algebra “$Q_{1,k}$” over “$R_{0,k}$” in [Kim15, p. 8219] is nothing but a finite product of finite separable extensions of $k_K$ (hence all admitting finite $p$-bases); hence so are all the inductively defined étale algebras “$Q_{n+1,k}$” over “$Q_{n,k}$” for all $n \geq 1$, as well as all the “$Q'_{n,k}$” in [Kim15, p. 8227] which are quotients of “$Q_{n,k}$”.
- The “$A_0$” in [Kim15, p. 8228] chosen in our context is nothing but a finite product of CDVRs each with $p$ as a uniformizer and with residue field admitting a finite $p$-basis; we could simply take one factor as that already satisfies the requirement Spec $A_0/(p)$ surjects to Spec $k_K$ as asked in [Kim15, p. 8228].

(2) And there is a connection operator

\[
\nabla : \mathcal{N}_0 \otimes_{\mathcal{O}_K} A_0 \to (\mathcal{N}_0 \otimes_{\mathcal{O}_K} A_0) \otimes_{A_0} \mathcal{\hat{N}}_A,
\]

such that the induced map

\[
\mathcal{N}_0 \otimes_{\mathcal{O}_K} A_0 \to \varphi^*(\mathcal{M}/u\mathcal{M}) \otimes_{\mathcal{O}_K} A_0
\]

is compatible with connections on both sides.

Let $A = A_0 \otimes_{\mathcal{O}_K} \mathcal{O}_K$. Then $A[1/p]$ is a CDVF whose residue field admits a finite $p$-basis. Then one can define $\mathcal{S}_A$ and define the relevant Breuil-Kisin modules and related categories. Let

\[
\mathcal{M}_A := \mathcal{S}_A \otimes \mathcal{M}, \quad \mathcal{\mathcal{M}}_A := \mathcal{S}_A \otimes \mathcal{M}
\]

and denote

\[
\mathcal{\mathcal{M}}'_A = \text{Ker}(\mathcal{M}_A \to \mathcal{\mathcal{M}}_A).
\]

By [Kim15, Prop. 10.3], $\mathcal{M}_A$ together with the connection (6.4.1) is an object in $\text{MF}_{\mathcal{S}_A}^\text{BT}(\varphi, \nabla)$. Because of (6.4.2), $\mathcal{\mathcal{M}}'_A$ can be equipped with an induced connection, and hence becomes an object in $\text{MF}_{\mathcal{S}_A}^\text{BT}(\varphi, \nabla)$ as well. Since $A[1/p]$ is a CDVF whose residue field admits a finite $p$-basis, Thm. 6.3.3 implies that $\mathcal{\mathcal{M}}'_A \hookrightarrow \mathcal{M}_A$ corresponds to an isogeny $G'_A \to G_A$ of $p$-divisible groups over $A$. Let

\[
H_A := \text{Ker}(G'_A \to G_A).
\]

Clearly, $\mathcal{M}^*(H_A) = \mathcal{M} \otimes \mathcal{S}_A$. Finally, a fpqc descent argument as in [Kim15, p. 8228] shows that $H_A$ descends to some finite flat group scheme over $\mathcal{O}_K$, giving the desired object mapping to $\mathcal{M}$. \hfill \Box

References

[AB08] Fabrizio Andreatta and Olivier Brinon. Surconvergence des représentations $p$-adiques: le cas relatif. Number 319, pages 39–116. 2008. Représentations $p$-adiques de groupes $p$-adiques. I. Représentations galoisiennes et $(\varphi, \Gamma)$-modules.

[ALB] Johannes Anschütz and Arthur-César Le Bras. Prismatic Dieudonné theory. preprint.

[And06] Fabrizio Andreatta. Generalized ring of norms and generalized $(\varphi, \Gamma)$-modules. Ann. Sci. École Norm. Sup. (4), 39(4):599–647, 2006.

[BBM82] Pierre Berthelot, Lawrence Breen, and William Messing. Théorie de Dieudonné cristalline. II, volume 930 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1982.

[Ber02] Laurent Berger. Représentations $p$-adiques et équations différentielles. Invent. Math., 148(2):219–284, 2002.

[Bri06] Olivier Brinon. Représentations cristallines dans le cas d’un corps résiduel imparfait. Ann. Inst. Fourier (Grenoble), 56(4):919–999, 2006.

[Bri08] Olivier Brinon. Représentations $p$-adiques cristallines et de de Rham dans le cas relatif. Mém. Soc. Math. Fr. (N.S.), (112):vi+159, 2008.

[BT08] Olivier Brinon and Fabien Trihan. Représentations cristallines et $F$-cristaux: le cas d’un corps résiduel imparfait. Rend. Semin. Mat. Univ. Padova, 119:141–171, 2008.

[CF00] Pierre Colmez and Jean-Marc Fontaine. Construction des représentations $p$-adiques semi-stables. Invent. Math., 140(1):1–43, 2000.
[CL09] Xavier Caruso and Tong Liu. Quasi-semi-stable representations. *Bull. Soc. Math. France*, 137(2):185–223, 2009.

[Gao] Hui Gao. Breuil-Kisin modules and integral $p$-adic Hodge theory. *preprint*.

[Gao17] Hui Gao. Galois lattices and strongly divisible lattices in the unipotent case. *J. Reine Angew. Math.*, 728:263–299, 2017.

[GL] Hui Gao and Tong Liu. Loose crystalline lifts and overconvergence of étale $(\varphi, \tau)$-modules. *to appear, Amer. J. Math*.

[GR03] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[Kat73] Nicholas M. Katz. $p$-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 69–190. Lecture Notes in Mathematics, Vol. 350, 1973.

[Ked04] Kiran S. Kedlaya. A $p$-adic local monodromy theorem. *Ann. of Math. (2)*, 160(1):93–184, 2004.

[Ked05] Kiran S. Kedlaya. *Slope filtrations revisited*. *Doc. Math.*, 10:447–525, 2005.

[Kim12] Wansu Kim. The classification of $p$-divisible groups over 2-adic discrete valuation rings. *Math. Res. Lett.*, 19(1):121–141, 2012.

[Kim15] Wansu Kim. The relative Breuil-Kisin classification of $p$-divisible groups and finite flat group schemes. *Int. Math. Res. Not. IMRN*, (17):8152–8232, 2015.

[Kis06] Mark Kisin. Crystalline representations and $F$-crystals. In *Algebraic geometry and number theory*, volume 253 of *Lecture Notes in Mathematics*, pages 459–496. Birkhäuser Boston, Boston, MA, 2006.

[KL] Kiran Kedlaya and Ruochuan Liu. Relative $p$-adic Hodge theory, II: imperfect period rings. *preprint*.

[KL15] Kiran S. Kedlaya and Ruochuan Liu. Relative $p$-adic Hodge theory: foundations. *Astérisque*, (371):239, 2015.

[Lau14] Eike Lau. Relations between Dieudonné displays and crystalline Dieudonné theory. *Algebra Number Theory*, 8(9):2201–2262, 2014.

[Liu07] Tong Liu. Torsion $p$-adic Galois representations and a conjecture of Fontaine. *Ann. Sci. École Norm. Sup.* (4), 40(4):633–674, 2007.

[Liu08] Tong Liu. On lattices in semi-stable representations: a proof of a conjecture of Breuil. *Compos. Math.*, 144(1):61–88, 2008.

[Liu13] Tong Liu. The correspondence between Barsotti-Tate groups and Kisin modules when $p = 2$. *J. Théor. Nombres Bordeaux*, 25(3):661–676, 2013.

[LM] Tong Liu and Yong Suk Moon. Relative crystalline representations and $p$-divisible groups in the small ramification case. *to appear, Algebra & Number Theory*.

[LZ17] Ruochuan Liu and Xinwen Zhu. Rigidity and a Riemann-Hilbert correspondence for $p$-adic local systems. *Invent. Math.*, 207(1):291–343, 2017.

[Mor10] Kazuma Morita. Hodge-Tate and de Rham representations in the imperfect residue field case. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(2):341–356, 2010.

[Mor14] Kazuma Morita. Crystalline and semi-stable representations in the imperfect residue field case. *Asian J. Math.*, 18(1):143–157, 2014.

[Ohk13] Shun Ohkubo. The $p$-adic monodromy theorem in the imperfect residue field case. *Algebra Number Theory*, 7(8):1977–2037, 2013.

[Ray74] Michel Raynaud. Schémas en groupes de type $(p, \ldots, p)$. *Bull. Soc. Math. France*, 102:241–280, 1974.

[Shi] Koji Shimizu. A $p$-adic monodromy theorem for de Rham local systems. *preprint*.

[Shi18] Koji Shimizu. Constancy of generalized Hodge-Tate weights of a local system. *Compos. Math.*, 154(12):2606–2642, 2018.

[Tat67] J. T. Tate. $p$-divisible groups. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.

[Vas13] Adrian Vasiu. A motivic conjecture of Milne. *J. Reine Angew. Math.*, 685:181–247, 2013.

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