NON-PERTURBATIVE RESULTS IN GLOBAL SUSY AND TOPOLOGICAL FIELD THEORIES

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In this lecture we briefly review the Seiberg–Witten model and explore some of its topological aspects.

1 Introduction

Many progresses have been made in recent years in the understanding of non–perturbative phenomena in globally supersymmetric (SUSY) gauge theories. In a seminal paper, Seiberg and Witten calculated all the non–perturbative contributions to the holomorphic effective action for an \( N = 2 \) Super Yang–Mills (SYM) theory. The low–energy action can in fact be expressed in terms of a unique holomorphic function \( \mathcal{F} \), called the effective prepotential. Arguing that the different phases of the theory are described by a polymorphic prepotential, Seiberg and Witten were able to translate a set of physical conditions in terms of the number of singularities and the monodromies of \( \mathcal{F} \). From this knowledge, it was thus possible to reconstruct the entire prepotential. The only physical information concerning non–perturbative effects fed to \( \mathcal{F} \) was the surviving quantum symmetry of the theory which, for gauge group \( SU(2) \), is \( \mathbb{Z}_2 \). To check the reliability of this framework, the non–perturbative contributions to \( \mathcal{F} \) were computed directly in the Coulomb phase of the theory (for gauge instantons of winding number one and two), by using a saddle point approximation for certain correlators of the relevant fields. The results were in agreement with those of Seiberg and Witten. These successful checks raise a number of questions which are the motivations of our investigation. The most compelling of them is: How come that a saddle point approximation, in which only quadratic terms are retained in the expansion of the action, is able to give the correct result? Why are higher–order corrections exactly zero? In the presence of SUSY a certain number of welcome simplifications allow the computation of some Green’s functions. The most remarkable simplification is that these correlators are given by a constant times the appropriate power of the renormalization group invariant (RGI) scale. This fact was exploited by Witten to argue that these computations are relevant for the determination of a class of invariants of four dimensional manifolds.

In the same paper it was shown that in the topological twisted version of \( N = 2 \) SYM the semiclassical expansion is exact. Here we suggest that these guidelines could help us give an answer to the problems previously raised.

2 A Review of the Seiberg–Witten Model

The Lagrangian density for the microscopic \( N = 2 \) SYM theory, in the \( N = 2 \) supersymmetric formalism is given by

\[
L = \frac{1}{16\pi} \text{Im} \int d^2\theta d^2\bar{\theta} \mathcal{F}(\Psi) .
\]

(1)

The chiral superfield \( \Psi \), which describes the vector multiplet of the \( N = 2 \) SUSY, transforms in the adjoint representation of the gauge group \( G \) (which will be \( SU(2) \) from now on). Re–expressing the Lagrangian density in the \( N = 1 \) formalism, we have

\[
L = \frac{1}{16\pi} \text{Im} \left[ \int d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}, V) + \int d^2\theta f_{ab}(\Phi) W^a W^b \right] ,
\]

(2)

where \( a, b \) are indices of the adjoint representation of \( G \). The Kähler potential \( K(\Phi, \bar{\Phi}, V) \) and the holomorphic function \( f_{ab}(\Phi) \) are given, in terms of \( \mathcal{F} \), by

\[
K(\Phi, \bar{\Phi}, V) = (\Phi e^{-2V})^a \partial \mathcal{F} / \partial \Phi^a \quad \text{and} \quad f_{ab}(\Phi) = \partial^2 \mathcal{F} / \partial \Phi^a \partial \Phi^b .
\]

The classical action is obtained by choosing for the holomorphic prepotential \( \mathcal{F} \) the functional form

\[
\mathcal{F}_{\text{cl}}(\Psi) = \frac{\tau_{\text{cl}}}{2} (\Psi^a \Psi^a) ,
\]

(3)

where we define \( \tau_{\text{cl}} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \). The classical action of the theory contains the scalar potential \( S_{\text{pot}} = \int d^4x \text{Tr}[\phi, \phi] \). The most general (supersymmetric) classical vacuum configuration is then

\[
\phi_0 = a \left( \Omega \frac{\sigma_3}{2} \Omega \right) , \quad a \in \mathbb{C} , \quad \Omega \in SU(2) .
\]

(4)

When \( a \neq 0 \) the \( SU(2) \) gauge symmetry is spontaneously broken to \( U(1) \). The classical vacuum “degeneracy” is
lifted neither by perturbative nor by non–perturbative quantum corrections, so that one has a fully quantum moduli space. The effective Lagrangian for the massless $U(1)$ fields $\Phi$, $W_\alpha$, will be the $U(1)$ version of Eq. (3) and reads, in $N = 1$ notation,

$$L_{\text{eff}} = \frac{1}{16\pi} \text{Im} \left[ \int d^2 \theta F''(\Phi) W^\alpha W_\alpha + \int d^2 \theta d^2 \bar{\theta} F'(\Phi) \right].$$

(5)

The low–energy dynamics are then governed by the effective prepotential $F(\Phi)$, whose crucial property is holomorphicity. The effective coupling constant is given by $\tau(\Phi) = F''(\Phi)$. The description of the low–energy dynamics in terms of $\Phi$ and $W_\alpha$ is not appropriate for all vacuum configurations. The quantum moduli space $\mathcal{M}_{SU(2)}$ is better described in terms of the variable $a$ and its dual $a_D = \partial_\alpha F$. When the gauge group is $SU(2)$, we can describe $\mathcal{M}_{SU(2)}$ in terms of the gauge–invariant coordinate $a = <\text{Tr}\phi^2>$. Then $\mathcal{M}_{SU(2)}$ is the Riemann sphere with punctures at $u = \infty$ and $u = \pm \Lambda^2$, where $\Lambda$ is the RGI scale in the normalization of Seiberg and Witten. At the classical level

$$F_{\text{cl}}(a) = \frac{\tau_{\text{cl}}}{2} a^2;$$

(6)

however perturbative as well as non–perturbative effects modify the expression of the prepotential, so that

$$F(a) = F_{\text{pert}}(a) + F_{\text{np}}(a).$$

(7)

The perturbative term has been calculated by Seiberg and is

$$F_{\text{pert}}(a) = \frac{i}{2\pi} \Psi^2 \ln \frac{\Psi^2}{M^2},$$

(8)

where $M$ can be fixed by assigning the value of the coupling constant at some subtraction point. The $R$–symmetry of the theory constrains the non–perturbative prepotential to be

$$F_{\text{np}}(a) = \sum_{k=1}^{\infty} F_k \left( \frac{\Lambda}{a} \right)^{4k} a^2;$$

(9)

and similarly

$$u(a) = \frac{1}{2} a^2 + \sum_{k=1}^{\infty} G_k \left( \frac{\Lambda}{a} \right)^{4k} a^2.$$

(10)

The 1–instanton contribution to $u(a)$ was found to be

$$<\text{Tr}\phi^2>_{k=1} = \frac{\Lambda_{PV}^2}{g^2 a^2},$$

(11)

where $\Lambda_{PV} = \Lambda/\sqrt{2}$ is the Pauli–Villars RGI scale, which naturally arises when performing instanton calculations.

Matone’s relation expresses then the $F_k$’s as functions of the $G_k$’s,

$$2i\pi k F_k = G_k.$$  

(12)

By making some hypotheses on the structure of the moduli space and on the monodromies of $\tau$ around its singularities, Seiberg and Witten obtained the expressions of $a(u)$ and $a_D(u),$

$$a(u) = \frac{\sqrt{2}}{\pi} \int_{-\Lambda^2}^{\Lambda^2} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - \Lambda^4}},$$

(13)

$$a_D(u) = \frac{\sqrt{2}}{\pi} \int_{-\Lambda^2}^{\Lambda^2} dx \frac{\sqrt{x - u}}{\sqrt{x^2 - \Lambda^4}}.$$  

(14)

From Eqs. (13), (14) one can derive the $F_k$’s and check whether they agree with those computed via instanton calculations, which we study in the next Section.

3 Instanton Calculus in SUSY Gauge Theories

We now briefly review the strategy to perform semiclassical computations in supersymmetric gauge theories, in the context of the constrained instanton method.

We expand the action functional around a properly chosen field configuration, which is the solution of the equations

$$D_\mu F_{\mu\nu} = 0,$$

(15)

$$D^2(A)\phi_{\text{cl}} = 0, \text{lim}_{|x|\to\infty} \phi_{\text{cl}} = a \frac{\sigma_3}{2i}.$$  

(16)

The boundary condition in Eq. (16) is dictated by Eq. (14). Equation (15) admits instanton solutions. When $a = 0$, Eq. (16) has only the trivial solution $\phi = 0$. On the other hand, when $a \neq 0$, we shall decompose the fields $\phi$, $\phi^\dagger$ as

$$\phi = \phi_{\text{cl}} + \phi_Q , \phi^\dagger = (\phi_{\text{cl}})^\dagger + \phi_Q^\dagger ,$$

(17)

and integrate over the quantum fluctuations $\phi_Q, \phi_Q^\dagger$. We want to remark that Eqs. (15), (16) are just approximate saddle point equations of the $N = 2$ action functional. This is a tricky point of the constrained instanton method.

The integration over the bosonic zero–modes can be traded for an integration over the collective coordinates, at the cost of introducing the corresponding Jacobian. The existence of fermion zero–modes is the way by which Ward identities related to the group of chiral symmetries of the theory come into play. When $a = 0$, the anomalous $U(1)_R$ symmetry

$$\lambda \to e^{i\alpha}\lambda , \phi \to e^{2i\alpha}\phi$$

(18)

and gauge invariance allow a nonzero result for the Green’s functions with $n$ insertions of the gauge–invariant quantity $(\phi^\sigma\phi^\sigma)(x)$ only when $n = 2k$. These
correlators possess the right operator insertions needed to saturate the integration over the Grassmann parameters and, by supersymmetry, they are also position-independent. On the other hand, when \( a \neq 0 \), the correlator \( \langle \phi^a \phi^n \rangle \) admits a complete expansion in terms of instanton contributions.

Fermion zero-modes are found by solving the equation \( D^a_{\mu} \bar{\sigma}_{\mu \beta} \lambda_{\beta A} = 0 \), where \( A = 1, 2 \) is the supersymmetry index. For instantons of winding number \( k = 1 \) the whole set of solutions of this equation is obtained via SUSY and superconformal transformations, which yield

\[
\lambda^a_{\alpha A} = \frac{1}{2} F^a_{\mu \nu} (\sigma_{\mu \nu})^\beta \bar{\lambda}^\beta_{\beta A},
\]

where \( \zeta = \xi + (x - x_0)_\mu \sigma_\mu \eta / \sqrt{2} \), \( \xi, \eta \) being two arbitrary quaternions of Grassmann numbers.

The correct fermionic integration measure is given by the inverse of the determinant of the matrix whose entries are the scalar products of the fermionic zero-modes and it reads \( d^4 \xi d^4 \eta \left( \frac{g^2}{32 \pi^2} \right)^4 \), where \( d^4 \xi d^4 \eta = d^2 \xi d^2 \eta d^2 \xi d^2 \eta \).

As an example of instanton calculation we now consider the correlator \( \langle \phi^a \phi^a \rangle \) in the semiclassical approximation around an instanton background of winding number \( k = 1 \),

\[
\langle \phi^a \phi^a \rangle = \int d^4 x_0 d\rho \left( \frac{2 \pi^6 \rho^3}{g^8} \right) e^{-\frac{4}{\rho^2} - 4\pi^2 |a|^2 \rho^2}
\]

\[
\int [\delta Q \delta \lambda] \delta \lambda \bar{\phi}^a \delta \bar{\phi}^a \delta \bar{\phi} \delta \bar{\phi}^a
\]

\[
\exp \left[ -S_H[\phi_Q, \phi^Q_1, A^{cl}] - S_F[\lambda, \bar{\lambda}, A^{cl}] + \frac{1}{2} \int d^4 x Q_{\mu} M_{\mu \nu} Q_{\nu} - \int d^4 x e D^2 (A^{cl})_c \right]
\]

\[
\int d^4 \xi d^4 \eta \left( \frac{g^2}{32 \pi^2} \right)^4 \left[ \xi \bar{\xi} + \eta \bar{\eta} \right]
\]

\[
\exp \left[ -S_V[\phi_{cl} + \phi_Q, (\phi_{cl})^\dagger, \phi_Q^\dagger, \lambda^{(0)}, \bar{\lambda} = 0] \right]
\]

\[
\left( \phi_{cl} + \phi_Q \right)^a (\phi_{cl} + \phi_Q)^a (x).
\]

Let us now explain where the different terms in Eq. \( (20) \) come from:

1. \( d^4 x_0 d\rho \left( \frac{2 \pi^6 \rho^3}{g^8} \right) \) is the bosonic measure after the integration over \( SU(2)/\mathbb{Z}_2 \) global rotations in color space has been performed. \( x_0 \) and \( \rho \) are the center and the size of the instanton.

2. \( S_H[\phi_{cl}, (\phi_{cl})^\dagger, A^{cl}] = 4\pi^2 |a|^2 \rho^2 \), is the contribution of the classical Higgs action.

3. The third and fourth lines include the quadratic approximation of the different kinetic operators for the quantum fluctuations of the fields, and the symbol \( [2 \bar{\lambda} \delta \lambda Q] \) denotes integration over nonzero-modes. \( \bar{c} \) and \( c \) are the usual ghost fields, \( \int d^4 x e D^2 (A^{cl})_c \) being the corresponding term in the action.

4. \( S_V[\phi, \phi^\dagger, \lambda^{(0)}, \bar{\lambda} = 0] \) is the Yukawa action calculated with the complete expansion of the fermionic fields replaced by their projection over the zero-mode subspace. It reduces to \( \sqrt{2} g e^{abc} \int \phi^{\dagger a} (\lambda^{(0)}_1 \bar{\lambda}^{(0)}_2) c \).

After the integration over \( \phi, \phi^\dagger \) and the nonzero-modes, the \( \phi_Q \) insertions get replaced by \( \phi_{inh} \), where

\[
\phi_{inh}^a = \sqrt{2} g e^{abc} [(D^2)^{-1}]^{ab} (\lambda^{(0)}_1 \bar{\lambda}^{(0)}_2 c),
\]

and the determinants of the various kinetic operators cancel against each other. The r.h.s. of Eq. \( (20) \) now reads

\[
\Lambda_{PV}^4 \int d^4 x_0 d\rho \left( \frac{2 \pi^6 \rho^3}{g^8} \right) e^{-4\pi^2 |a|^2 \rho^2}
\]

\[
\int d^4 \xi d^4 \eta \left( \frac{g^2}{32 \pi^2} \right)^4 \exp \left[ -\sqrt{2} g e^{abc} \int \phi^{\dagger a} (\lambda^{(0)}_1 \bar{\lambda}^{(0)}_2 c) \right]
\]

\[
(\phi_{cl} + \phi_{inh})^a (\phi_{cl} + \phi_{inh})^a (x),
\]

where \( \Lambda_{PV}^4 = \mu^{8-4} e^2 \frac{g^2}{32 \pi^2} \cdot \mu \) comes from the Pauli-Villars regularization of the determinants and the exponent is \( b_1 k = (n_B - n_F)/2 \) where \( n_B, n_F, b_1 \) are the number of bosonic, fermionic zero-modes and the first coefficient of the \( \beta \)-function of the theory.

The Yukawa action does not contain the Grassmann parameters of the zero-modes coming from SUSY transformations. As a consequence the only nonzero contributions are provided by picking out the terms in the \( \phi_{inh} \) insertions which contain the SUSY solutions of the Dirac equation. This amounts to say

\[
(\phi_{cl} + \phi_{inh})^a (\phi_{cl} + \phi_{inh})^a \rightarrow -\xi_1^2 \xi_2^2 (F_{\mu \nu} F^{\mu \nu})^a.
\]

Equation \( (20) \) now becomes

\[
\langle \phi^a \phi^a \rangle = \Lambda_{PV}^4 \int d\rho \left( \frac{2 \pi^6 \rho^3}{g^8} \right) e^{-4\pi^2 |a|^2 \rho^2}
\]

\[
\int d^4 x_0 \left( F_{\mu \nu}^a F^{\mu \nu}_a \right)
\]

\[
\frac{g^2}{2} (a^* a)^2 \int d^4 \xi \left( \frac{g^2}{32 \pi^2} \right)^2 \xi_1^2 \xi_2^2.
\]

\(^{a}\)This property generally holds for all the multi-instanton contributions to \( u(a) \).
Moreover, the twist transforms the scalar functions. When formulated on a generic manifold $M$ by redefining the generators of the Lorentz group in a suitably twisted fashion, the $N = 2$ SYM theory gives rise to the so-called Topological Yang–Mills theory (TYM)⁴. All the observables, included the partition function itself, are in this case topological invariants, in the sense that they are independent of the metric on $M$. With respect to the twisted Lorentz group, SUSY charges decompose as a scalar $Q$, an antisymmetric tensor $Q_{\mu \nu}$ and a vector $Q_\mu$:

\[ Q_{\alpha}^4 \rightarrow Q \oplus Q_{\mu \nu} \ , \]

\[ Q_{\alpha}^3 \rightarrow Q_{\mu} \ . \]  

Moreover, the twist transforms the $N = 2$ SYM fields as

\[ A_\mu \rightarrow A_\mu \ , \]

\[ \xi_\alpha^4 \rightarrow \eta \oplus \chi_{\mu \nu} \ , \]

\[ \xi_\alpha^3 \rightarrow \psi_{\mu} \ , \]

\[ \phi \rightarrow \phi \ , \]  

where the anticommuting fields $\eta, \chi_{\mu \nu}, \psi_{\mu}$ are respectively a scalar, a self–dual two–form and a vector. The scalar supersymmetry charge of TYM plays a major rôle, in that it is preserved on any (differentiable) four–manifold $M$, and has the crucial property of being nilpotent modulo gauge transformations. This allows one to interpret it as a BRST–like charge. Actually, in order to have a strictly nilpotent BRST charge, one needs to include gauge transformations with the appropriate ghost $c$. The BRST transformations are then

\[ sA = \psi - Dc \ , \]

\[ s\psi = -[c, \psi] - D\phi \ , \]

\[ sc = -[c, c] + \phi \ , \]

\[ s\phi = -[c, \phi] \ . \]  

This algebra can be read as the definition and the Bianchi identities for the curvature $\hat{F} = F + \psi + \phi$ of the connection $\hat{A} = A + c$ of the universal bundle $P \times A / G$ (\( P, A, G \) are respectively the principal bundle over $M$, the space of connections and the group of gauge transformations). The exterior derivative on the base manifold $M \times A / G$ is given by $d = d + s$.

The TYM action can be interpreted as a pure gauge–fixing action which localizes the universal connection to

\[ \hat{A} = A + c = U^0 d + s U \ . \]  

$A$ is then an anti–self–dual (ASD) connection which we have written in the ADHM formalism. Once $\hat{A}$ is given, the components $F, \psi, \phi$ of $\hat{F}$ are in turn determined. $F$ is anti–self–dual ($F = F^-$), and $\psi$ is an element of the tangent bundle $T_A M^-$, where $M^-$ is the instanton moduli space. Moreover, the scalar field $\phi$ is the solution to the equation $D^2 \phi = [\psi, \psi]$. The explicit expression for $\phi$ is then given by the twisted version of Eq. (21). For the following discussion it is important to point out that the field $\phi$ has trivial boundary conditions ($\phi = 0$) at spatial infinity.

In this geometrical framework, the BRST operator $s$ has a very nice explicit realization as the exterior derivative on $M^-$. This leads us to compute correlators of $s$–exact operators as integrals of forms on $\partial M^-$.⁵ For example, we can write

\[ \text{Tr} \phi^2 = sK_c \ , \ K_c = \text{Tr} \left( \csc + \frac{2}{3} scc \right) \ , \]  

an expression which parallels the well–known relation

\[ \text{Tr} F^2 = sK_A \ , \ K_A = \text{Tr} \left( A dA + \frac{2}{3} AAA \right) \ . \]  

For winding number $k = 1$, the top form on the (eight–dimensional) instanton moduli space is $\text{Tr} \phi^2(x_1)\text{Tr} \phi^2(x_2)$, and one can compute

\[ \int_{M^-} \text{Tr} \phi^2 \text{Tr} \phi^2 = \int_{\partial M^-} \text{Tr} \phi^2 K_c = \frac{1}{2} \ . \]  

5 Topological Aspects of the Seiberg–Witten Model

From the comparison between the Seiberg–Witten ansatz for the $N = 2$ low–energy effective action and explicit instanton calculations, we learn two important lessons. On one hand, this comparison provides us with a consistency check of the scenario proposed by Seiberg and Witten. On the other hand, it strongly suggests that the semiclassical approximation around the instanton background saturates the non–perturbative sector. This calls for an explanation. A key property of the TYM theory is the exactness of the semiclassical limit; we are

⁴We use the definitions and conventions of Sec. II of [1].
then naturally led to explore the Coulomb phase of the $N = 2$ SYM theory starting from its topological twisted counterpart.

The first problem one must face is that nontrivial boundary conditions for the scalar field are not compatible with the BRST algebra in Eq. (28). This is because a nonzero v.e.v. for $\phi$ implies the existence of a (nonzero) central charge $Z$ in the SUSY algebra which acts on the fields as a $U(1)$ transformation with gauge parameter $\phi$. This new symmetry has to be included in an appropriate extension of the BRST operator, and it is implemented through the introduction of a new global ghost field $\Lambda$.

The resulting algebra is then

$$
\begin{align*}
  sA &= \psi - D(c + \Lambda) , \\
  s\psi &= -[c + \Lambda, \psi] - D\phi , \\
  s(c + \Lambda) &= -[c + \Lambda, c + \Lambda] + \phi , \\
  s\phi &= -[c + \Lambda, \phi] .
\end{align*}
$$

(33)

Once the universal connection is projected onto Eq. (29) by the gauge-fixing TYM action, the above extended algebra includes scalar fields with nonvanishing boundary conditions. Therefore, we can see that in this picture the field configurations dictated by the constrained instanton method (see Eqs. (15), (16)) naturally come into play, without resorting to any approximation procedure.

The TYM action gets now a nonzero boundary contribution from the term

$$
S_{\text{inst}} = \int_{R^4} d^4x \, \partial^\mu s \text{Tr}(\phi^\dagger \psi_\mu) .
$$

(34)

The explicit ADHM expressions for the fields in Eq. (34) can be derived from the extended algebra in Eq. (33) starting from the universal connection $U^\dagger (d + s) U$. It is easy to see that, when inserted into Eq. (34), they yield the multi-instanton action for the Seiberg–Witten model. This provides a natural and simplifying framework for further studies of non-perturbative effects in $N = 2$ theories.

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