CLASSIFICATION OF BASE SEQUENCES $BS(n+1,n)$

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ABSTRACT. Base sequences $BS(n+1,n)$ are quadruples of $\{±1\}$-sequences $(A; B; C; D)$, with $A$ and $B$ of length $n+1$ and $C$ and $D$ of length $n$, such that the sum of their nonperiodic autocorrelation functions is a $δ$-function. The base sequence conjecture, asserting that $BS(n+1,n)$ exist for all $n$, is stronger than the famous Hadamard matrix conjecture. We introduce a new definition of equivalence for base sequences $BS(n+1,n)$, and construct a canonical form. By using this canonical form, we have enumerated the equivalence classes of $BS(n+1,n)$ for $n ≤ 30$. As the number of equivalence classes grows rapidly (but not monotonically) with $n$, the tables in the paper cover only the cases $n ≤ 13$.

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1. INTRODUCTION

Base sequences $BS(m,n)$ are quadruples $(A; B; C; D)$ of binary sequences, with $A$ and $B$ of length $m$ and $C$ and $D$ of length $n$, such that the sum of their nonperiodic autocorrelation functions is a $δ$-function. In this paper we take $m = n + 1$.

Sporadic examples of base sequences $BS(n+1,n)$ have been constructed by many authors during the last 30 years, see e.g. [5, 6, 7, 10] and the survey paper [9] and its references. A more systematic approach has been taken by the author in [2, 4]. The $BS(n+1,n)$ are presently known to exist for all $n ≤ 38$ (ibid) and for Golay numbers $n = 2^a10^b26^c$, where $a, b, c$ are arbitrary nonnegative integers. However the genuine classification of $BS(n+1,n)$ is still lacking. Due to the important role that these sequences play in various combinatorial constructions such as that for $T$-sequences, orthogonal designs, and Hadamard matrices [3, 5, 8], it is of interest to classify the base sequences of small length. Our main goal is to provide such classification for $n ≤ 30$.

Key words and phrases. Base sequences, nonperiodic autocorrelation functions, canonical form.
In section 2 we recall the basic properties of base sequences $BS(n + 1, n)$. We also recall the quad decomposition and our encoding scheme for this particular type of base sequences.

In section 3 we enlarge the collection of standard elementary transformations of $BS(n + 1, n)$ by introducing a new one. Thus we obtain new notion of equivalence and equivalence classes. Throughout the paper, the words “equivalence” and “equivalence class” are used in this new sense. We also introduce the canonical form for base sequences. By using it, we are able to compute the representatives of the equivalence classes.

In section 4 we introduce an abstract group, $G_{BS}$, of order $2^{12}$ which acts naturally on all sets $BS(n + 1, n)$. Its definition depends on the parity of $n$. The orbits of this group are just the equivalence classes of $BS(n + 1, n)$.

In section 5 we tabulate some of the results of our computations (those for $n \leq 13$) giving the list of representatives of the equivalence classes of $BS(n + 1, n)$. The representatives are written in the encoded form which is explained in the next section. For $n \leq 8$ we also include the values of the nonperiodic autocorrelation functions of the four constituent sequences. We also raise the question of characterizing the binary sequences having the same nonperiodic autocorrelation function. A class of examples is constructed, showing that the question is interesting.

The column “Equ” in Table 1 gives the number of equivalence classes in $BS(n + 1, n)$ for $n \leq 30$. The column “Nor” gives the number of normal equivalence classes (see Section 5 for their definition).
Table 1: Number of equivalence classes of $BS(n+1,n)$

| $n$ | Equ. | Nor. | $n$ | Equ. | Nor. |
|-----|------|------|-----|------|------|
| 0   | 1    | 1    | 16  | 1721 | 104  |
| 1   | 1    | 1    | 17  | 2241 | 0    |
| 2   | 1    | 1    | 18  | 1731 | 2    |
| 3   | 1    | 1    | 19  | 4552 | 2    |
| 4   | 3    | 2    | 20  | 3442 | 72   |
| 5   | 4    | 1    | 21  | 3677 | 0    |
| 6   | 5    | 0    | 22  | 15886| 0    |
| 7   | 17   | 6    | 23  | 6139 | 0    |
| 8   | 27   | 14   | 24  | 10878| 0    |
| 9   | 44   | 4    | 25  | 19516| 4    |
| 10  | 98   | 10   | 26  | 10626| 4    |
| 11  | 84   | 3    | 27  | 22895| 0    |
| 12  | 175  | 8    | 28  | 31070| 0    |
| 13  | 475  | 5    | 29  | 18831| 2    |
| 14  | 331  | 0    | 30  | 19640| 0    |
| 15  | 491  | 2    | 31  | ?    | 0    |

2. Quadratic Decomposition and the Encoding Scheme

We denote finite sequences of integers by capital letters. If, say, $A$ is such a sequence of length $n$ then we denote its elements by the corresponding lower case letters. Thus

$$A = a_1, a_2, \ldots, a_n.$$  

To this sequence we associate the polynomial

$$A(x) = a_1 + a_2x + \cdots + a_nx^{n-1},$$

viewed as an element of the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$. (As usual, $\mathbb{Z}$ denotes the ring of integers.) The nonperiodic autocorrelation function $N_A$ of $A$ is defined by:

$$N_A(i) = \sum_{j \in \mathbb{Z}} a_j a_{i+j}, \quad i \in \mathbb{Z},$$

where $a_k = 0$ for $k < 1$ and for $k > n$. Note that $N_A(-i) = N_A(i)$ for all $i \in \mathbb{Z}$ and $N_A(i) = 0$ for $i \geq n$. The norm of $A$ is the Laurent polynomial $N(A) = A(x)A(x^{-1})$. We have

$$N(A) = \sum_{i \in \mathbb{Z}} N_A(i)x^i.$$
The negation, \(-A\), of \(A\) is the sequence
\[-A = -a_1, -a_2, \ldots, -a_n.\]

The reversed sequence \(A'\) and the alternated sequence \(A^*\) of the sequence \(A\) are defined by
\[A' = a_n, a_{n-1}, \ldots, a_1\]
\[A^* = a_1, -a_2, a_3, -a_4, \ldots, (-1)^{n-1}a_n.\]

Observe that \(N(-A) = N(A') = N(A)\) and \(N_{A^*}(i) = (-1)^i N_A(i)\) for all \(i \in \mathbb{Z}\). By \(A, B\) we denote the concatenation of the sequences \(A\) and \(B\).

A binary sequence is a sequence whose terms belong to \(\{-1, 1\}\). When displaying such sequences, we shall often write + for +1 and − for −1.

The base sequences consist of four binary sequences \((A; B; C; D)\), with \(A\) and \(B\) of length \(m\) and \(C\) and \(D\) of length \(n\), such that

\[N(A) + N(B) + N(C) + N(D) = 2(m + n).\]

Thus, for \(i \neq 0\), we have
\[N_A(i) + N_B(i) + N_C(i) + N_D(i) = 0.\]  

We denote by \(BS(m, n)\) the set of such base sequences with \(m\) and \(n\) fixed. From now on we shall consider only the case \(m = n + 1\).

Let \((A; B; C; D) \in BS(n+1, n)\). For convenience we fix the following notation. For \(n\) even (odd) we set \(n = 2m\) \((n = 2m+1)\). We decompose the pair \((A; B)\) into quads
\[
\begin{bmatrix}
  a_i & a_{n+2-i} \\
  b_i & b_{n+2-i}
\end{bmatrix}, \quad i = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor,
\]
and, if \(n\) is even, the central column \([ a_{m+1} \ b_{m+1} ]\). Similar decomposition is valid for the pair \((C; D)\).

Recall the following basic and well-known property [7, Theorem 1].

**Theorem 2.1.** For \((A; B; C; D) \in BS(n+1, n)\), the sum of the four quad entries is 2 (mod 4) for the first quad of the pair \((A; B)\) and is 0 (mod 4) for all other quads of \((A; B)\) and also for all quads of the pair \((C; D)\).

Thus there are 8 possibilities for the first quad of the pair \((A; B)\):
CLASSIFICATION OF BASE SEQUENCES $BS(n + 1, n)$

1' = \begin{bmatrix} - & + & - \\ + & - & - \end{bmatrix}, \quad 2' = \begin{bmatrix} + & - & + \\ + & + & - \end{bmatrix}, \quad 3' = \begin{bmatrix} + & - & - \\ + & + & + \end{bmatrix}, \quad 4' = \begin{bmatrix} + & - & + \\ - & - & + \end{bmatrix}, \quad 5' = \begin{bmatrix} + & - & + \\ - & - & - \end{bmatrix}, \quad 6' = \begin{bmatrix} + & - & + \\ - & - & - \end{bmatrix}, \quad 7' = \begin{bmatrix} + & - & + \\ - & - & - \end{bmatrix}, \quad 8' = \begin{bmatrix} + & - & + \\ - & - & - \end{bmatrix}.

These eight quads occur in the study of Golay sequences (see e.g. [1]) and we refer to them as the Golay quads.

There are also 8 possibilities for each of the remaining quads of $(A; B)$ and all quads of $(C; D)$:

1 = \begin{bmatrix} + & + & + \\ + & + & + \end{bmatrix}, \quad 2 = \begin{bmatrix} + & + & - \\ - & - & - \end{bmatrix}, \quad 3 = \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix}, \quad 4 = \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix}, \quad 5 = \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix}, \quad 6 = \begin{bmatrix} + & - & + \\ + & + & + \end{bmatrix}, \quad 7 = \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix}, \quad 8 = \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix}.

We shall refer to these eight quads as the BS-quads. We say that a BS-quad is symmetric if its two columns are the same, and otherwise we say that it is skew. The quads 1, 2, 7, 8 are symmetric and 3, 4, 5, 6 are skew. We say that two quads have the same symmetry type if they are both symmetric or both skew.

There are 4 possibilities for the central column:

0 = \begin{bmatrix} + & + & + \\ + & + & + \end{bmatrix}, \quad 1 = \begin{bmatrix} + & - & + \\ + & + & + \end{bmatrix}, \quad 2 = \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix}, \quad 3 = \begin{bmatrix} - & + & - \\ - & + & - \end{bmatrix}.

We encode the pair $(A; B)$ by the symbol sequence

(2.2) \quad p_1p_2 \cdots p_mp_{m+1},

where $p_i$ is the label of the $i$th quad except in the case where $n$ is even and $i = m + 1$ in which case $p_{m+1}$ is the label of the central column.

Similarly, we encode the pair $(C; D)$ by the symbol sequence

(2.3) \quad q_1q_2 \cdots q_m \text{ respectively } q_1q_2 \cdots q_m q_{m+1}

when $n$ is even respectively odd. Here $q_i$ is the label of the $i$th quad for $i \leq m$ and $q_{m+1}$ is the label of the central column (when $n$ is odd).

3. THE EQUIVALENCE RELATION

We start by defining five types of elementary transformations of base sequences $BS(n + 1, n)$. These elementary transformations include the standard ones, as described in [7] and [2]. But we also introduce one additional elementary transformation, see item (T4) below, which made its first appearance in [3] in the context of near-normal sequences. The
quad notation was instrumental in the discovery of this new elementary operation.

The elementary transformations of \((A; B; C; D) \in BS(n + 1, n)\) are the following:

1. Negate one of the sequences \(A; B; C; D\).
2. Reverse one of the sequences \(A; B; C; D\).
3. Interchange the sequences \(A; B\) or \(C; D\).
4. Replace the pair \((C; D)\) with the pair \((\tilde{C}; \tilde{D})\) which is defined as follows: If \((2.3)\) is the encoding of \((C; D)\), then the encoding of \((\tilde{C}; \tilde{D})\) is \(\tau(q_1)\tau(q_2)\cdots\tau(q_m)q_{m+1}\) or \(\tau(q_1)\tau(q_2)\cdots\tau(q_m)\) depending on whether \(n\) is even or odd, where \(\tau\) is the transposition \((45)\). In other words, the encoding of \((\tilde{C}; \tilde{D})\) is obtained from that of \((C; D)\) by replacing each quad symbol 4 with the symbol 5, and vice versa. (We verify below that \(N_{\tilde{C}} + N_{\tilde{D}} = N_C + N_D\).)
5. Alternate all four sequences \(A; B; C; D\).

In order to justify (T4) one has to verify that \(N_{\tilde{C}} + N_{\tilde{D}} = N_C + N_D\). For that purpose let us fix two quads \(q_k\) and \(q_{k+i}\) and consider their contribution \(\delta_i\) to \(N_{C}(i) + N_{D}(i)\). We claim that \(\delta_i\) is equal to the contribution \(\tilde{\delta}_i\) of \(\tau(q_k)\) and \(\tau(q_{k+i})\) to \(N_{\tilde{C}}(i) + N_{\tilde{D}}(i)\). If neither \(q_k\) nor \(q_{k+i}\) belongs to \(\{4, 5\}\), then \(\tau(q_k) = q_k\) and \(\tau(q_{k+i}) = q_{k+i}\) and so \(\delta_i = \tilde{\delta}_i\). If \(q \in \{4, 5\}\) then \(\tau(q)\) is the negation of \(q\). Hence if \(\{q_k, q_{k+i}\} \subseteq \{4, 5\}\) then again \(\delta_i = \tilde{\delta}_i\). Otherwise, say \(q_k \in \{4, 5\}\) while \(q_{k+i} \notin \{4, 5\}\), and it is easy to verify that \(\delta_i = 0 = \tilde{\delta}_i\). The pairs \(q_k\) and \(q_{n+1-k+i}\) also make a contribution to \(N_{C}(i) + N_{D}(i)\), but they can be treated in the same manner. Finally, if \(n\) is odd then the pair \((C; D)\) also has a central column with label \(q_{m+1}\). In that case, if \(k = m + 1 - i\) and \(q_k \in \{4, 5\}\), the contribution of \(q_k\) and \(q_{m+1}\) to \(N_{C}(i) + N_{D}(i)\) is 0. This completes the verification.

We say that two members of \(BS(n + 1, n)\) are equivalent if one can be transformed to the other by applying a finite sequence of elementary transformations. One can enumerate the equivalence classes by finding suitable representatives of the classes. For that purpose we introduce the canonical form.

**Definition 3.1.** Let \(S = (A; B; C; D) \in BS(n + 1, n)\) and let \((2.2)\) respectively \((2.3)\) be the encoding of the pair \((A; B)\) respectively \((C; D)\). We say that \(S\) is in the canonical form if the following eleven conditions hold:

1. \(p_1 = 3', p_2 \in \{6, 8\}\) for \(n\) even and \(p_2 \in \{1, 6\}\) for \(n\) odd.
2. The first symmetric quad (if any) of \((A; B)\) is 1 or 8.
3. If \(n\) is even and \(p_i \in \{3, 4, 5, 6\}\) for \(2 \leq i \leq m\) then \(p_{m+1} \in \{0, 3\}\).
4. The first skew quad (if any) of \((A; B)\) is 3 or 6.
CLASSIFICATION OF BASE SEQUENCES $BS(n+1, n)$

(v) $q_1 = 1$ for $n$ even and $q_1 \in \{1, 6\}$ for $n$ odd.
(vi) The first symmetric quad (if any) of $(C; D)$ is 1.
(vii) The first skew quad (if any) of $(C; D)$ is 6.
(viii) If $i$ is the least index such that $q_i \in \{2, 7\}$ then $q_i = 2$.
(ix) If $i$ is the least index such that $q_i \in \{4, 5\}$ then $q_i = 4$.
(x) If $n$ is odd and $q_i \in \{1, 3, 6, 8\}$, $\forall i \leq m$, then $q_{m+1} \neq 2$.
(xi) If $n$ is odd and $q_i \in \{3, 4, 5, 6\}$, $\forall i \leq m$, then $q_{m+1} = 0$.

We can now prove that each equivalence class has a member which is in the canonical form. The uniqueness of this member will be proved in the next section.

**Proposition 3.2.** Each equivalence class $E \subseteq BS(n+1, n)$ has at least one member having the canonical form.

**Proof.** Let $S = (A; B; C; D) \in E$ be arbitrary and let (2.2) respectively (2.3) be the encoding of $(A; B)$ respectively $(C; D)$. By applying the first three types of elementary transformations and by Theorem 2.1 we can assume that $p_1 = 3'$ and $c_1 = d_1 = +1$. By Theorem 2.1, $q_1 \in \{1, 6\}$. If $n$ is even and $q_1 = 6$ we apply the elementary transformation (T5). Thus we may assume that $p_1 = 3'$ and that the condition (v) for the canonical form is satisfied. To satisfy the conditions (ii) and (iii), replace $B$ with $-B'$ (if necessary). To satisfy the condition (iv), replace $A$ with $A'$ (if necessary).

We now modify $S$ in order to satisfy the second part of condition (i). Note that $p_2$ is a BS-quad by Theorem 2.1. If $p_2 = 6$ there is nothing to do.

Assume that the quad $p_2$ is symmetric. By (ii), $p_2 \in \{1, 8\}$. From equation (2.1) for $i = n - 1$, we deduce that $p_2 = 8$ if $q_1 = 1$ and $p_2 = 1$ if $q_1 = 6$. Note that if $n$ is even, then $q_1 = 1$ by (v). If $n$ is odd and $q_1 = 1$, we switch $A$ and $B$ and apply the elementary transformation (T5). After this change we still have $p_1 = 3'$, $q_1 = 1$, the conditions (ii), (iii) and (iv) remain satisfied, and moreover $p_2 = 6$.

Now assume that $p_2$ is skew. In view of (iv), we may assume that $p_2 = 3$. Then the argument above based on the equation (2.1) shows that $q_1 = 6$, and so $n$ must be odd. After applying the elementary transformation (T5), we obtain that $p_2 = 1$. Hence the condition (i) is fully satisfied.

To satisfy (vi), in view of (v) we may assume that $n$ is odd and $q_1 = 6$. If the first symmetric quad in $(C; D)$ is 2 respectively 7, we reverse and negate $C$ respectively $D$. If it is 8, we reverse and negate both $C$ and $D$. Now the first symmetric quad will be 1.

To satisfy (vii), (if necessary) reverse $C$ or $D$, or both. To satisfy (viii), (if necessary) interchange $C$ and $D$. Note that in this process we
do not violate the previously established properties. To satisfy (ix), (if necessary) apply the elementary transformation (T4). To satisfy (x), switch \( C \) and \( D \) (if necessary). To satisfy (xi), (if necessary) replace \( C \) with \(-C'\) or \( D \) with \(-D'\), or both.

Hence \( S \) is now in the canonical form. \( \square \)

4. The symmetry group of \( BS(n + 1, n) \)

We shall construct a group \( G_{BS} \) of order \( 2^{12} \) which acts on \( BS(n + 1, n) \). Our (redundant) generating set for \( G_{BS} \) will consist of 12 involutions. Each of these generators is an elementary transformation, and we use this information to construct \( G_{BS} \), i.e., to impose the defining relations. We denote by \( S = (A; B; C; D) \) an arbitrary member of \( BS(n + 1, n) \).

To construct \( G_{BS} \), we start with an elementary abelian group \( E \) of order \( 2^8 \) with generators \( \nu_i, \rho_i, \ i \in \{1, 2, 3, 4\} \). It acts on \( BS(n + 1, n) \) as follows:

\[
\begin{align*}
\nu_1 S &= (-A; B; C; D), \quad \rho_1 S = (A'; B; C; D), \\
\nu_2 S &= (A; -B; C; D), \quad \rho_2 S = (A; B'; C; D), \\
\nu_3 S &= (A; B; -C; D), \quad \rho_3 S = (A; B; C'; D), \\
\nu_4 S &= (A; B; C; -D), \quad \rho_4 S = (A; B; C; D'), \\
\end{align*}
\]

i.e., \( \nu_i \) negates the \( i \)th sequence of \( S \) and \( \rho_i \) reverses it.

Next we introduce two commuting involutory generators \( \sigma_1 \) and \( \sigma_2 \). We declare that \( \sigma_1 \) commutes with \( \nu_3, \nu_4, \rho_3, \rho_4 \), and \( \sigma_2 \) commutes with \( \nu_1, \nu_2, \rho_1, \rho_2 \), and that

\[
\begin{align*}
\sigma_1 \nu_1 &= \nu_2 \sigma_1, \quad \sigma_1 \rho_1 = \rho_2 \sigma_1, \\
\sigma_2 \nu_3 &= \nu_4 \sigma_2, \quad \sigma_2 \rho_3 = \rho_4 \sigma_2.
\end{align*}
\]

The group \( H = \langle E, \sigma_1, \sigma_2 \rangle \) is the direct product of two isomorphic groups of order 32:

\[
H_1 = \langle \nu_1, \rho_1, \sigma_1 \rangle, \quad \text{and} \quad H_2 = \langle \nu_3, \rho_3, \sigma_2 \rangle.
\]

The action of \( E \) on \( BS(n + 1, n) \) extends to \( H \) by defining:

\[
\begin{align*}
\sigma_1 S &= (B; A; C; D), \quad \sigma_2 S = (A; B; D; C), \\
\end{align*}
\]

We add a new generator \( \theta \) which commutes elementwise with \( H_1 \), commutes with \( \nu_3 \rho_3, \nu_4 \rho_4 \) and \( \sigma_2 \), and satisfies \( \theta \rho_3 = \rho_4 \theta \). Let us denote this enlarged group by \( \tilde{H} \). It has the direct product decomposition

\[
\tilde{H} = \langle H, \theta \rangle = H_1 \times \tilde{H}_2.
\]
where the second factor is itself direct product of two copies of the dihedral group $D_8$ of order 8:

$$\tilde{H}_2 = \langle \rho_3, \rho_4, \theta \rangle \times \langle \nu_3 \rho_3, \nu_4 \rho_4, \theta \sigma_2 \rangle.$$  

The action of $H$ on $BS(n+1, n)$ extends to $\tilde{H}$ by letting $\theta$ act as the elementary transformation (T4).

Finally, we define $G_{BS}$ as the semidirect product of $\tilde{H}$ and the group of order 2 with generator $\alpha$. By definition, $\alpha$ commutes with each $\nu_i$ and satisfies:

$$\alpha \rho_i \alpha = \rho_i \nu_i^n, \ i = 1, 2;$$
$$\alpha \rho_j \alpha = \rho_j \nu_j^{n-1}, \ j = 3, 4;$$
$$\alpha \theta \alpha = \theta \sigma_2^{n-1}.$$  

The action of $\tilde{H}$ on $BS(n+1, n)$ extends to $G_{BS}$ by letting $\alpha$ act as the elementary transformation (T5), i.e., we have

$$\alpha S = (A^*; B^*; C^*; D^*).$$

We point out that the definition of the subgroup $\tilde{H}$ is independent of $n$ and its action on $BS(n+1, n)$ has a quad-wise character. By this we mean that the value of a particular quad, say $p_i$, of $S \in BS(n+1, n)$ and $h \in \tilde{H}$ determine uniquely the quad $p_i$ of $hS$. In other words $\tilde{H}$ acts on the Golay quads, the BS-quads and the set of central columns such that the encoding of $hS$ is given by the symbol sequences

$$h(p_1)h(p_2)\ldots h(p_{m+1}) \text{ and } h(q_1)h(q_2)\ldots.$$  

On the other hand the full group $G_{BS}$ has neither of these two properties.

An important feature of the action of $\tilde{H}$ on the BS-quads is that it preserves the symmetry type of the quads.

The following proposition follows immediately from the construction of $G_{BS}$ and the description of its action on $BS(n+1, n)$.

**Proposition 4.1.** The orbits of $G_{BS}$ in $BS(n+1, n)$ are the same as the equivalence classes.

The main tool that we use to enumerate the equivalence classes of $BS(n+1, n)$ is the following theorem.

**Theorem 4.2.** For each equivalence class $E \subseteq BS(n+1, n)$ there is a unique $S = (A; B; C; D) \in E$ having the canonical form.

**Proof.** In view of Proposition 3.2 we just have to prove the uniqueness assertion. Let

$$S^{(k)} = (A^{(k)}; B^{(k)}; C^{(k)}; D^{(k)}) \in E, \ (k = 1, 2)$$
be in the canonical form. We have to prove that in fact \( S^{(1)} = S^{(2)} \).

By Proposition 4.1, we have \( gS^{(1)} = S^{(2)} \) for some \( g \in G_{BS} \). We can write \( g \) as \( g = \alpha^s h \) where \( s \in \{0, 1\} \) and \( h \in \tilde{H} \). Let \( p^{(k)}_1, p^{(k)}_2 \ldots p^{(k)}_{n+1} \) be the encoding of the pair \( (A^{(k)}; B^{(k)}) \) and \( q^{(k)}_1, q^{(k)}_2 \ldots q^{(k)}_n \) be the encoding of the pair \( (C^{(k)}; D^{(k)}) \). The symbols (i-xi) will refer to the corresponding conditions of Definition 3.1. Observe that \( p^{(1)}_1 = p^{(2)}_1 = 3' \) by (i).

We prove first preliminary claims (a-c).

(a): \( q^{(1)}_1 = q^{(2)}_1 \).

For \( n \) even see (v). Let \( n \) be odd. When we apply the generator \( \alpha \) to any \( S \in BS(n + 1, n) \), we do not change the first quad of \( (C; D) \). It follows that the quads \( q^{(1)}_1 \) and \( q^{(2)}_1 \) have the same symmetry type. The claim now follows from (v).

(b): \( g \in \tilde{H} \), i.e., \( s = 0 \).

Assume first that \( n \) is even. By (v), \( q^{(1)}_1 = q^{(2)}_1 = 1 \). For any \( S \in \mathcal{E} \) the first quad of \( (C; D) \) in \( S \) and in \( \alpha S \) have different symmetry types. As the quad \( h(1) \) is symmetric, the equality \( \alpha^s hS^{(1)} = S^{(2)} \) forces \( s \) to be 0. Assume now that \( n \) is odd. Then for any \( S \in \mathcal{E} \) the second quad of \( (A; B) \) in \( S \) and in \( \alpha S \) have different symmetry types. Recall that \( q^{(1)}_1 = q^{(2)}_1 = 1 \) for \( \alpha^s=1 \) and that (see (i)) \( p^{(1)}_2 \) and \( p^{(2)}_2 \) belong to \( \{1, 6\} \). From (2.1), with \( i = n - 1 \), we deduce that \( p^{(k)}_2 \neq q^{(k)}_1 \) for \( k = 1, 2 \). We conclude that \( p^{(1)}_2 = p^{(2)}_2 \). The claim now follows from the fact that \( h \) preserves while \( \alpha \) alters the symmetry type of the quad \( p_2 \).

As an immediate consequence of (b), we point out that a quad \( p^{(1)}_i \) is symmetric iff \( p^{(2)}_i \) is, and the same is true for the quads \( q^{(1)}_i \) and \( q^{(2)}_i \).

(c): \( p^{(1)}_2 = p^{(2)}_2 \).

This was already proved above in the case when \( n \) is odd. In general, the claim follows from (b) and the equality \( h(p^{(1)}_2) = p^{(2)}_2 \). Observe that each of the sets \( \{6, 8\} \) and \( \{1, 6\} \) consists of one symmetric and one skew quad and that \( h \) preserves the symmetry type of quads.

Recall that \( \tilde{H} = H_1 \times \tilde{H}_2 \). Since \( s = 0 \) we have \( g = h = h_1 h_2 \) with \( h_1 \in H_1 \) and \( h_2 \in \tilde{H}_2 \). Consequently, \( h_1(p^{(1)}_i) = p^{(2)}_i \) and \( h_2(q^{(1)}_i) = q^{(2)}_i \) for all \( i. \)

We shall now prove that \( A^{(1)} = A^{(2)} \) and \( B^{(1)} = B^{(2)} \). Since \( p^{(1)}_1 = p^{(2)}_1 = 3' \), the equality \( h_1(p^{(1)}_1) = p^{(2)}_1 \) implies that \( h_1(3') = 3' \). Thus \( h_1 = \rho^{e}_i (\nu_2 \rho_2)^{f} \) for some \( e, f \in \{0, 1\} \).

Assume first that \( p^{(1)}_2 \) is symmetric. By (ii), \( p^{(1)}_2 \in \{1, 8\} \). Then \( h_1(p^{(1)}_2) = p^{(2)}_2 = p^{(1)}_2 \) implies that \( f = 0 \). Hence, \( h_1 = \rho^{e}_1 \) and so \( B^{(1)} = B^{(2)} \). If all quads \( p^{(1)}_i, i \neq 1, \) are symmetric then also \( A^{(2)} = h_1 A^{(1)} = A^{(1)} \). Otherwise let \( i \) be the least index for which the quad...
$p_i^{(1)}$ is skew. Since $B^{(1)} = B^{(2)}$ and $p_i^{(1)}$ is 3 or 6 (see (iv)), we infer that $e = 0$. Hence $h_1 = 1$ and so $A^{(1)} = A^{(2)}$.

Now assume that $p_2^{(1)}$ is skew. By (ii), $p_2^{(1)} = 6$. Then $h_1(p_2^{(1)}) = p_2^{(2)} = p_2^{(1)}$ implies that $e = 0$. Thus $h_1 = (\nu_2\rho_2)^f$ and so $A^{(1)} = A^{(2)}$. If all quads $p_i^{(1)}$, $i \neq 1$, are skew then by invoking the condition (iii) we deduce that $f = 0$ and so $B^{(1)} = B^{(2)}$. Otherwise let $i$ be the least index for which the quad $p_i^{(1)}$ is symmetric. Since $A^{(1)} = A^{(2)}$ and $p_i^{(1)}$ is 1 or 8 (see (ii)), we infer that $f = 0$. Hence $h_1 = 1$ and so $B^{(1)} = B^{(2)}$.

It remains to prove that $C^{(1)} = C^{(2)}$ and $D^{(1)} = D^{(2)}$. We set $Q = \{q_i^{(1)} : 1 \leq i \leq m\}$. By (v) and the claim (a) we have $q_i^{(1)} = q_i^{(2)} \in \{1, 6\}$.

We first consider the case $q_1^{(1)} = q_1^{(2)} = 6$ which occurs only for $n$ odd. Then $h_2(6) = 6$ and so $h_2 \in \langle \nu_3\rho_3, \nu_4\rho_4, \theta, \sigma_2 \rangle$. It follows that $h_2(3) = 3$.

If some $q \in Q$ is symmetric, let $i$ be the least index such that $q_i^{(1)}$ is symmetric. Then (vi) implies that $q_i^{(1)} = q_i^{(2)} = 1$. Thus $h_2$ must fix the quad 1. As the stabilizer of the quad 1 in $\langle \nu_3\rho_3, \nu_4\rho_4, \theta, \sigma_2 \rangle$ is $\langle \theta, \sigma_2 \rangle$, we infer that $h_2$ must also fix the quad 8. Similarly, if $2 \in Q$ then (viii) implies that $h_2$ fixes 2 and 7. If $4 \in Q$ then (ix) implies that $h_2$ fixes 4 and 5. These facts imply that $h_2$ fixes all quads in $Q$, i.e., $q_i^{(1)} = q_i^{(2)}$ for all $i \leq m$. It remains to show that, for odd $n$, $q_{m+1}^{(1)} = q_{m+1}^{(2)}$. If $Q \subseteq \{3, 4, 5, 6\}$, this follows from (xi). Otherwise $Q$ contains a symmetric quad and so $h_2 \in \langle \theta, \sigma_2 \rangle$. If $Q \not\subseteq \{1, 3, 6, 8\}$ then $Q$ contains one of the quads 2, 4, 5 or 7. Since $h_2$ fixes all quads in $Q$, we infer that $h_2 \in \langle \theta \rangle$, and so $q_{m+1}^{(1)} = q_{m+1}^{(2)}$. If $Q \subseteq \{1, 3, 6, 8\}$, the equality $q_{m+1}^{(1)} = q_{m+1}^{(2)}$ follows from (x).

Finally, we consider the case $q_1^{(1)} = q_1^{(2)} = 1$. Since $h_2(q_1^{(1)}) = q_1^{(2)}$, $h_2 \in \langle \rho_3, \rho_4, \theta, \sigma_2 \rangle$. Hence $h_2$ fixes the quads 1 and 8.

If some $q \in Q$ is skew, then (vii) implies that $h_2$ fixes the quads 3 and 6. If $2 \in Q$ then (viii) implies that $h_2$ fixes the quads 2 and 7. If $4 \in Q$ then (ix) implies that $h_2$ fixes the quads 4 and 5. These facts imply that $h_2$ fixes all quads in $Q$. If $n$ is odd, then we invoke the conditions (x) and (xi) to conclude that $h_2$ also fixes the central column of $(C^{(1)}; D^{(1)})$. Hence $C^{(1)} = C^{(2)}$ and $D^{(1)} = D^{(2)}$ also in this case.

5. Representatives of the equivalence classes

We have computed a set of representatives for the equivalence classes of base sequences $BS(n+1, n)$ for all $n \leq 30$. Due to their excessive
size, we tabulate these sets only for \( n \leq 13 \). Each representative is given in the canonical form which is made compact by using our standard encoding. The encoding is explained in detail in Section 2.

As an example, the base sequences

\[
A = +, +, +, +, +, +, +, -; \\
B = +, +, +, +, +, +, +, -; \\
C = +, +, -; \\
D = +, +, +, +, +, -; \\
\]

are encoded as 3′6142; 1675. In the tables we write 0 instead of 3′. This convention was used in our previous papers on this and related topics.

This compact notation is used primarily in order to save space, but also to avoid introducing errors during decoding. For each \( n \), the representatives are listed in the lexicographic order of the symbol sequences (2.2) and (2.3).

In Table 2 we list the codes for the representatives of the equivalence classes of \( BS(n + 1, n) \) for \( n \leq 8 \). This table also records the values \( N_X(k) \) of the nonperiodic autocorrelation functions for \( X \in \{A, B, C, D\} \) and \( k \geq 0 \). For instance let us consider the first item in the list of base sequences \( BS(8, 7) \) given in Table 2. The base sequences are encoded in the first column as 0165; 6123. The first part 0165 encodes the pair \((A; B)\), and the second part 6123 the pair \((C; D)\). The function \( N_A \), at the points \( 0, 1, \ldots, 7 \), takes the values \( 8, -1, 2, -1, 0, 1, 2, 1 \) listed in the second column. Just below these values one finds the values of \( N_B \) at the same points. In the third column we list likewise the values of \( N_C \) and \( N_D \) at the points \( i = 0, 1, \ldots, 6 \).

Tables 3-7 contain only the list of codes for the representatives of the equivalence classes of \( BS(n + 1, n) \) for \( 9 \leq n \leq 13 \).

Let us say that the base sequences \( S = (A; B; C; D) \in BS(n + 1, n) \) are \textit{normal} respectively \textit{near-normal} if \( b_i = a_i \) respectively \( b_i = (-1)^i a_i \) for all \( i \leq n \). We denote by \( NS(n) \) respectively \( NN(n) \) the set of all normal respectively near-normal sequences in \( BS(n + 1, n) \). Let us say also that an equivalence class \( \mathcal{E} \subseteq BS(n + 1, n) \) is \textit{normal} respectively \textit{near-normal} if \( \mathcal{E} \cap NS(n) \) respectively \( \mathcal{E} \cap NN(n) \) is not void. Our canonical form has been designed so that if \( \mathcal{E} \) is normal then its canonical representative \( S \) belongs to \( NS(n) \). The analogous statement for near-normal classes is false. It is not hard to recognize which representatives \( S \) in our tables are normal sequences. Let (2.2) be the encoding of the pair \((A; B)\). Then \( S \in NS(n) \) iff all the quads
$p_i, i \neq 1$, belong to $\{1, 3, 6, 8\}$ and, in the case when $n = 2m$ is even, the central column symbol $p_{m+1}$ is 0 or 3.

It is an interesting question to find the necessary and sufficient conditions for two binary sequences to have the same norm. The group of order four generated by the negation and reversal operations acts on binary sequences. We say that two binary sequences are equivalent if they belong to the same orbit. Note that the equivalent binary sequences have the same norm. However the converse is false. Here is a counter-example which occurs in Table 2 for the case $n = 8$. The base sequences 15 and 16 differ only in their first sequences, which we denote here by $U$ and $V$ respectively:

\[
U = + + - - - + - + , \quad V = + + - + + - - + .
\]

Their associated polynomials are

\[
U(x) = 1 + x - x^2 - x^3 - x^4 + x^5 - x^6 - x^7 + x^8 ,
\]
\[
V(x) = 1 + x - x^2 + x^3 + x^4 - x^5 - x^6 - x^7 + x^8 .
\]

It is obvious that $U$ and $V$ are not equivalent in the above sense. On the other hand, from the factorizations $U(x) = p(x)q(x)$ and $V(x) = p(x)r(x)$, where $p(x) = 1 + x - x^2$, $q(x) = 1 - x^3 - x^6$ and $r(x) = 1 + x^3 - x^6 = -x^6 q(x^{-1})$, we deduce immediately that $N_U(x) = N_V(x)$.

This counter-example can be easily generalized. Let us define binary polynomials as polynomials associated to binary sequences. If $f(x)$ is a polynomial of degree $d$ with $f(0) \neq 0$, we define its dual polynomial $f^*$ by $f^*(x) = x^d f(x^{-1})$. Then for any positive integer $k$ we have $f^*(x^k) = f(x^k)^*$, i.e., $f^*(x^k) = g^*(x)$ where $g^*$ is the dual of the polynomial $f(x^k)$. In general we can start with any number of binary sequences, but here we take only three of them: $A; B; C$ of lengths $m, n, k$ respectively. From the associated binary polynomials $A(x), B(x), C(x)$ we can form several binary polynomials of degree $mnk - 1$. The basic one is $A(x)B(x^m)C(x^{mn})$. The other are obtained from this one by replacing one or more of the three factors by their duals. It is immediate that the binary sequences corresponding to these binary polynomials all have the same norm. In general many of these sequences will not be equivalent. However note that if we replace all three factors with their duals, we will obtain a binary sequence equivalent to the basic one.
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Table 2: Equivalence classes of $BS(n+1, n)$, $n \leq 8$

| $ABCD$ | $N_A & N_B$ | $N_C & N_D$ |
|--------|-------------|-------------|
| $n = 1$ |
| 1 0    | 2, 1        | 1           |
| 0 1    | 2, −1       | 1           |
| $n = 2$ |
| 1 03   | 3, −2, 1    | 2, 1        |
| 1 07   | 3, 0, −1    | 2, 1        |
| $n = 3$ |
| 1 06   | 4, −1, 0, 1 | 3, 2, 1    |
| 11 06  | 4, 1, −2, −1| 3, −2, 1   |
| $n = 4$ |
| 1 060  | 5, 0, 1, 0, 1| 4, −1, 0, 1|
| 16 08  | 5, 2, −1, −2, −1| 4, −1, 0, 1|
| 2 082  | 5, 0, −1, −2, 1| 4, 3, 2, 1|
| 12 083 | 5, −2, 1, 0, −1| 4, −1, −2, 1|
| $n = 5$ |
| 1 016  | 6, 1, 0, 1, 2, 1| 5, 2, −1, −2, −1|
| 640 016| 6, −1, 2, −1, 0, −1| 5, −2, −1, 2, −1|
| 2 017  | 6, 1, −4, −1, 2, 1| 5, −2, 1, 0, −1|
| 613 017| 6, 3, 2, 1, 0, −1| 5, −2, 1, 0, −1|
| 3 064  | 6, 1, −2, −1, 0, 1| 5, 0, 1, 0, 1|
| 160 064| 6, −1, 0, 1, −2, −1| 5, 0, 1, 0, 1|
| $n = 6$ |
| 1 061  | 7, −2, 3, −2, 1, 0, 1| 6, 1, −4, −1, 2, 1|
| 127 061| 7, 4, 1, 0, −1, −2, −1| 6, −3, 0, 3, −2, 1|
| 2 0820 | 7, −2, 1, 0, 3, −2, 1| 6, 1, 0, −1, −2, 1|
| 188 0820| 7, 0, −1, 2, 1, 0, −1| 6, 1, 0, −1, −2, 1|
| 3 0861 | 7, −2, −3, 2, 1, −2, 1| 6, 1, 2, 1, 0, 1|
| 162 0861| 7, 0, 3, 0, −1, 0, −1| 6, 1, −2, −3, 0, 1|
| $n = 7$ |
| 4 0872 | 7, 2, 1, 0, −1, −2, 1| 6, 1, 0, 1, 2, 1|
| 126 0872| 7, 0, −1, −2, 1, 0, −1| 6, −3, 0, 1, −2, 1|
| 5 0882 | 7, 2, 1, 0, −1, −2, 1| 6, 1, −2, −1, 0, 1|
| 164 0882| 7, 0, −1, 2, 1, 0, −1| 6, −3, 2, −1, 0, 1|
| $n = 8$ |
| 1 0165 | 8, −1, 2, −1, 0, 1, 2, 1| 7, 0, −1, 2, 1, 0, −1|
| 6123 0165| 8, 1, 0, 1, −2, −1, 0, −1| 7, 0, −1, −2, 1, 0, −1|
| 2 0165 | 8, −1, 2, −1, 0, 1, 2, 1| 7, 0, 3, 0, −1, 0, −1|
| 6141 0165| 8, 1, 0, 1, −2, −1, 0, −1| 7, 0, −5, 0, 3, 0, −1|
| $ABCD$ | $N_A$ & $N_B$ | $N_C$ & $N_D$ |
|--------|--------------|--------------|
| $n = 7$ |              |              |
| 3 0166 | 8, 3, −2, −1, 0, 1, 2, 1 | 7, 0, −1, 2, 1, 0, −1 |
| 6122  | 8, 1, 0, 1, −2, −1, 0, −1 | 7, −4, 3, −2, 1, 0, −1 |
| 4 0173 | 8, −1, −2, 1, −2, −1, 2, 1 | 7, 0, 3, 0, −1, 0, −1 |
| 6161  | 8, 1, 0, −1, 4, 1, 0, −1 | 7, 0, −1, 0, −1, 0, −1 |
| 5 0173 | 8, −1, −2, 1, −2, −1, 2, 1 | 7, 4, 1, 0, −1, −2, −1 |
| 6411  | 8, 1, 0, −1, 4, 1, 0, −1 | 7, −4, 1, 0, −1, 2, −1 |
| 6 0183 | 8, −1, −2, 1, −2, −1, 2, 1 | 7, 4, 3, 2, 1, 0, −1 |
| 6121  | 8, −3, 0, −1, 0, 1, 0, −1 | 7, 0, −1, −2, 1, 0, −1 |
| 7 0613 | 8, −1, 0, 3, 0, 1, 0, 1 | 7, −2, 3, −2, 1, 0, 1 |
| 1623  | 8, 1, −2, 1, 2, −1, −2, −1 | 7, 2, −1, −2, −3, 0, 1 |
| 8 0614 | 8, −1, 4, −1, 0, 1, 0, 1 | 7, 2, −1, 0, −1, 0, 1 |
| 1641  | 8, 1, −2, 1, 2, −1, −2, −1 | 7, −2, −1, 0, −1, 0, 1 |
| 9 0615 | 8, −1, 0, 3, 0, 1, 0, 1 | 7, 2, −3, −2, 1, 2, 1 |
| 1263  | 8, 1, 2, 1, −2, −1, −2, −1 | 7, −2, 1, −2, −1, −2, 1 |
| 10 0615 | 8, −1, 0, 3, 0, 1, 0, 1 | 7, 2, −3, −4, −1, 2, 1 |
| 1272  | 8, 1, 2, 1, −2, −1, −2, −1 | 7, −2, 1, 0, 3, −2, 1 |
| 11 0616 | 8, −1, 4, −1, 0, 1, 0, 1 | 7, 2, −3, −2, 1, 2, 1 |
| 1262  | 8, 1, 2, 1, −2, −1, −2, −1 | 7, −2, −3, 2, 1, −2, 1 |
| 12 0618 | 8, −1, 0, −1, −2, 1, 0, 1 | 7, 2, 1, 2, 1, 2, 1 |
| 1261  | 8, −2, 1, 0, −1, −2, −1 | 7, −2, 1, −2, 1, −2, 1 |
| 13 0635 | 8, −1, −4, 1, 2, −1, 0, 1 | 7, 2, 3, 2, 1, 0, 1 |
| 1621  | 8, −3, 2, −1, 0, 1, −2, −1 | 7, 2, −1, −2, −3, 0, 1 |
| 14 0638 | 8, −1, 0, −3, 0, −1, 0, 1 | 7, 2, 3, 2, 1, 0, 1 |
| 1620  | 8, 1, −2, −1, 2, 1, −2, −1 | 7, −2, −1, 2, −3, 0, 1 |
| 15 0641 | 8, 3, 0, 1, 0, −1, 0, 1 | 7, −2, 3, −2, 1, 0, 1 |
| 1622  | 8, 1, −2, −1, 2, 1, −2, −1 | 7, −2, −1, 2, −3, 0, 1 |
| 16 0646 | 8, 3, 0, −3, −2, −1, 0, 1 | 7, 2, 1, 0, 3, 2, 1 |
| 1222  | 8, −3, 2, −1, 0, 1, −2, −1 | 7, −2, −3, 4, −1, −2, 1 |
| 17 0648 | 8, 3, 0, −3, −2, −1, 0, 1 | 7, 2, 1, 2, 1, 2, 1 |
| 1260  | 8, −3, 2, −1, 0, 1, −2, −1 | 7, −2, −3, 2, 1, −2, 1 |

$n = 8$

| 1 06113 | 9, 0, 1, 4, −1, 2, 1, 0, 1 | 8, −1, 0, −3, 0, −1, 0, 1 |
| 1638  | 9, 2, −1, 2, 1, 0, −1, −2, −1 | 8, −1, 0, −3, 0, −1, 0, 1 |
| 2 06122 | 9, 0, 1, 4, −1, 2, 1, 0, 1 | 8, 3, 0, −3, −2, 1, 0, 1 |
| 1644  | 9, −2, 3, −2, 1, 0, −1, −2, −1 | 8, −1, −4, 1, 2, 1, 0, 1 |
| 3 06141 | 9, 0, 5, 0, 1, 0, 1, 0, 1 | 8, −1, 0, 1, −2, 1, 0, 1 |
| 1663  | 9, 2, −5, −2, 3, 2, −1, −2, −1 | 8, −1, 0, 1, −2, −1, 0, 1 |
| 4 06142 | 9, 0, 1, 0, −3, 0, 1, 0, 1 | 8, −1, −4, 3, 0, −3, 0, 1 |
| 1624  | 9, 2, −1, −2, 3, 2, −1, −2, −1 | 8, −1, −4, 3, 0, −3, 0, 1 |
### Table 2 (continued)

| $ABCD$ | $N_A$ & $N_B$ | $N_C$ & $N_D$ |
|--------|--------------|--------------|
| $n = 8$ |              |              |
| 5      | $06151$      | $9, 0, 1, 0, 5, 0, 1, 0, 1$ | $8, -1, 0, -1, -2, 1, 0, 1$ |
|        | $1618$       | $9, 2, -1, 2, -1, -2, -1, -2, -1$ | $8, -1, 0, -1, -2, 1, 0, 1$ |
| 6      | $06152$      | $9, 0, -3, 0, 1, 0, 1, 0, 1$ | $8, 3, -2, -1, 0, 1, 2, 1$ |
|        | $1264$       | $9, 2, 3, 2, -1, -2, -1, -2, -1$ | $8, -5, 2, -1, 0, 1, -2, 1$ |
| 7      | $06183$      | $9, 0, 1, -4, -1, -2, 1, 0, 1$ | $8, -1, -2, 5, 0, -1, 2, 1$ |
|        | $1271$       | $9, 2, -1, -2, 1, 0, -1, -2, -1$ | $8, -1, 2, 1, 0, 3, -2, 1$ |
| 8      | $06183$      | $9, 0, 1, -4, -1, -2, 1, 0, 1$ | $8, -1, 0, 3, 0, 1, 0, 1$ |
|        | $1613$       | $9, 2, -1, -2, 1, 0, -1, -2, -1$ | $8, -1, 0, 3, 0, 1, 0, 1$ |
| 9      | $06310$      | $9, 0, 1, 2, 1, 4, -1, 0, 1$ | $8, -1, 0, 1, -3, 0, 1, 0$ |
|        | $1686$       | $9, 2, -1, 0, -1, 2, 1, -2, -1$ | $8, -1, 0, -1, -3, 0, 1, 0$ |
| 10     | $06380$      | $9, -4, 1, -2, 1, 0, -1, 0, 1$ | $8, 3, 0, 1, 0, -1, 0, 1$ |
|        | $1661$       | $9, -2, -1, 0, -1, 2, 1, -2, -1$ | $8, 3, 0, 1, 0, -1, 0, 1$ |
| 11     | $06382$      | $9, 0, 1, -2, -3, 0, -1, 0, 1$ | $8, 3, 0, 1, 0, -1, 0, 1$ |
|        | $1641$       | $9, -2, -1, 0, -1, 2, 1, -2, -1$ | $8, -1, 0, 1, 4, -1, 0, 1$ |
| 12     | $06412$      | $9, 0, 1, 2, -3, 0, -1, 0, 1$ | $8, -1, 0, 1, 4, -1, 0, 1$ |
|        | $1632$       | $9, 2, -1, 0, -1, 2, 1, -2, -1$ | $8, -1, 0, -3, 0, -1, 0, 1$ |
| 13     | $06580$      | $9, -4, 1, -2, 1, 0, -1, 0, 1$ | $8, 3, -2, -3, 0, 3, 2, 1$ |
|        | $1127$       | $9, 2, 3, 0, -1, -2, -3, -2, -1$ | $8, -1, -2, 5, 0, -1, 2, 1$ |
| 14     | $06633$      | $9, 0, -3, 2, -1, -2, -1, 0, 1$ | $8, -1, 2, -1, 0, 1, 2, 1$ |
|        | $1163$       | $9, 2, -1, 0, 1, 0, -3, -2, -1$ | $8, -1, -2, -1, 0, 1, 2, 1$ |
| 15     | $06852$      | $9, 0, -3, 0, 1, 0, -3, 0, 1$ | $8, -1, 2, -1, 0, 1, 2, 1$ |
|        | $1163$       | $9, 2, -1, 2, -1, -2, -1, -2, -1$ | $8, -1, -2, -1, 0, 1, 2, 1$ |
| 16     | $06860$      | $9, 0, -3, 0, 1, 0, -3, 0, 1$ | $8, -1, 2, -1, 0, 1, 2, 1$ |
|        | $1163$       | $9, 2, -1, 2, -1, -2, -1, -2, -1$ | $8, -1, -2, -1, 0, 1, 2, 1$ |
| 17     | $08110$      | $9, 0, 3, 2, 1, 0, 3, -2, 1$ | $8, -1, -2, -1, 0, 1, -2, 1$ |
|        | $1866$       | $9, 2, 1, 0, -1, -2, 1, 0, -1$ | $8, -1, -2, -1, 0, 1, -2, 1$ |
| 18     | $08350$      | $9, 0, -1, -4, 1, 0, 1, -2, 1$ | $8, -1, 2, 1, 0, 3, -2, 1$ |
|        | $1822$       | $9, -2, -3, 2, -1, -2, 3, 0, -1$ | $8, 3, 2, 1, 0, -1, -2, 1$ |
| 19     | $08383$      | $9, 0, 3, 0, -1, -2, 1, -2, 1$ | $8, -1, -2, -1, 0, 1, -2, 1$ |
|        | $1866$       | $9, 2, 1, 2, 1, 0, 3, 0, -1$ | $8, -1, -2, -1, 0, 1, -2, 1$ |
| 20     | $08630$      | $9, -4, -1, 0, 1, 0, 1, -2, 1$ | $8, 3, 0, 1, 0, -1, 0, 1$ |
|        | $1661$       | $9, -2, 1, -1, -2, 1, -1, 0, -1$ | $8, 3, 0, 1, 0, -1, 0, 1$ |
| 21     | $08640$      | $9, 0, -1, -4, 1, 0, 1, -2, 1$ | $8, -1, -2, 5, 0, -1, 2, 1$ |
|        | $1282$       | $9, -2, 1, -2, -1, 2, -1, 0, -1$ | $8, 3, 2, 1, 0, -1, -2, 1$ |
| 22     | $08642$      | $9, 0, -1, 0, -3, 0, 1, -2, 1$ | $8, 3, 0, 1, 0, -1, 0, 1$ |
|        | $1641$       | $9, -2, 1, -2, -1, 2, -1, 0, -1$ | $8, -1, 0, 1, 4, -1, 0, 1$ |
Table 2 (continued)

\[
\begin{array}{cccc}
\text{ABCD} & N_A & N_B & N_C & N_D \\
n = 8 & & & & \\
23 & 08660 & 9, 0, -1, -4, 1, 0, 1, -2, 1 & 8, -1, -2, 5, 0, -1, 2, 1 \\
1271 & 9, 2, 1, -2, -1, -2, -1, 0, -1 & 8, -1, 2, 1, 0, 3, -2, 1 \\
24 & 08660 & 9, 0, -1, -4, 1, 0, 1, -2, 1 & 8, -1, 0, 3, 0, 1, 0, 1 \\
1613 & 9, 2, 1, -2, -1, -2, -1, 0, -1 & 8, -1, 0, 3, 0, 1, 0, 1 \\
25 & 08833 & 9, 0, -1, 2, -1, 2, -1, -2, 1 & 8, -1, 0, -1, 0, -3, 0, 1 \\
1686 & 9, 2, 1, 0, 1, 4, 1, 0, -1 & 8, -1, 0, -1, 0, -3, 0, 1 \\
26 & 08862 & 9, 0, -1, 2, -1, 2, -1, -2, 1 & 8, -1, 4, -1, 0, 1, 0, 1 \\
1626 & 9, 2, -3, 0, 1, 0, 1, 0, -1 & 8, -1, 0, -1, 0, -3, 0, 1 \\
27 & 08863 & 9, 0, -1, 2, -1, 2, -1, -2, 1 & 8, -1, 0, -3, 0, -1, 0, 1 \\
1638 & 9, 2, 1, 4, 1, 0, 1, 0, -1 & 8, -1, 0, -3, 0, -1, 0, 1 \\
\end{array}
\]

Table 3: Equivalence classes of \( BS(10, 9) \)

\[
\begin{array}{cccc}
\text{AB} & \text{CD} & \text{AB} & \text{CD} & \text{AB} & \text{CD} & \text{AB} & \text{CD} \\
1 & 01235 66450 & 2 & 01324 66181 & 3 & 01618 64150 & 4 & 01624 64183 \\
5 & 01627 64130 & 6 & 01633 64140 & 7 & 01642 64560 & 8 & 01652 61453 \\
9 & 01652 64313 & 10 & 01654 61163 & 11 & 01655 61180 & 12 & 01672 61281 \\
13 & 01675 61430 & 14 & 01682 61180 & 15 & 01684 61122 & 16 & 01734 64160 \\
17 & 01735 61640 & 18 & 01764 61821 & 19 & 01765 61281 & 20 & 01767 61831 \\
21 & 01783 61411 & 22 & 01867 61311 & 23 & 06124 16282 & 24 & 06136 16640 \\
25 & 06147 16450 & 26 & 06152 12763 & 27 & 06164 16133 & 28 & 06172 12681 \\
29 & 06175 12670 & 30 & 06175 16143 & 31 & 06187 16131 & 32 & 06351 16460 \\
33 & 06382 16460 & 34 & 06388 16340 & 35 & 06412 16273 & 36 & 06412 16381 \\
37 & 06413 16460 & 38 & 06451 16163 & 39 & 06458 12612 & 40 & 06481 12623 \\
41 & 06481 16161 & 42 & 06581 11622 & 43 & 06583 11631 & 44 & 06875 11622 \\
\end{array}
\]
### Table 4: Equivalence classes of $BS(11, 10)$

| $AB$ | $CD$ | $AB$ | $CD$ | $AB$ | $CD$ |
|------|------|------|------|------|------|
| 1    | 061173 16456 | 2    | 061253 16246 | 3    | 061350 16645 |
| 4    | 061450 16267 | 5    | 061450 16443 | 6    | 061460 16434 |
| 7    | 061463 12826 | 8    | 061463 16271 | 9    | 061553 12716 |
| 10   | 061563 16134 | 11   | 061582 12631 | 12   | 061633 12671 |
| 13   | 061740 12684 | 14   | 061870 12286 | 15   | 061870 16144 |
| 16   | 063140 16862 | 17   | 063413 16822 | 18   | 063510 16382 |
| 19   | 063513 16441 | 20   | 063550 16414 | 21   | 063583 16341 |
| 22   | 063810 16616 | 23   | 063821 16445 | 24   | 063833 16613 |
| 25   | 063840 16262 | 26   | 063842 16242 | 27   | 063870 16322 |
| 28   | 063873 16217 | 29   | 063881 16342 | 30   | 064122 16277 |
| 31   | 064130 16465 | 32   | 064141 16423 | 33   | 064141 16452 |
| 34   | 064170 16344 | 35   | 064313 16282 | 36   | 064413 12826 |
| 37   | 064413 16271 | 38   | 064480 16213 | 39   | 064480 16321 |
| 40   | 064510 12638 | 41   | 064510 12676 | 42   | 064870 12616 |
| 43   | 065843 11276 | 44   | 068110 11863 | 45   | 068383 11863 |
| 46   | 068560 11263 | 47   | 068571 11632 | 48   | 068580 11627 |
| 49   | 068580 11643 | 50   | 068611 11634 | 51   | 068632 11632 |
| 52   | 068641 11634 | 53   | 068752 11276 | 54   | 068771 11645 |
| 55   | 082661 18642 | 56   | 083110 18863 | 57   | 083383 18863 |
| 58   | 083510 18226 | 59   | 083521 18642 | 60   | 083850 18622 |
| 61   | 086231 16248 | 62   | 086243 16277 | 63   | 086263 16332 |
| 64   | 086310 16613 | 65   | 086333 16616 | 66   | 086343 16228 |
| 67   | 086421 16427 | 68   | 086432 16242 | 69   | 086463 16217 |
| 70   | 086473 16217 | 71   | 086483 16344 | 72   | 086532 16142 |
| 73   | 086640 12631 | 74   | 086643 16134 | 75   | 086740 12642 |
| 76   | 086840 12682 | 77   | 086860 12671 | 78   | 086870 12671 |
| 79   | 087110 12863 | 80   | 087110 16273 | 81   | 087120 16461 |
| 82   | 087130 16461 | 83   | 087131 16262 | 84   | 087221 16284 |
| 85   | 087323 16282 | 86   | 087343 16282 | 87   | 087361 16422 |
| 88   | 087372 12864 | 89   | 087383 16382 | 90   | 087663 16146 |
| 91   | 087683 12637 | 92   | 087732 12684 | 93   | 088651 16264 |
| 94   | 088651 16424 | 95   | 088673 16434 | 96   | 088762 16246 |
| 97   | 088771 16264 | 98   | 088771 16424 |
Table 5: Equivalence classes of $BS(12, 11)$

|   | $AB$   | $CD$   |   | $AB$   | $CD$   |   | $AB$   | $CD$   |
|---|--------|--------|---|--------|--------|---|--------|--------|
| 1 | 011823 | 661422 | 2 | 012356 | 661422 | 3 | 013682 | 663120 |
| 4 | 013753 | 663120 | 5 | 016426 | 641272 | 6 | 016445 | 616230 |
| 7 | 016472 | 645121 | 8 | 016525 | 614232 | 9 | 016525 | 643511 |
| 10| 016535 | 612770 | 11| 016542 | 612463 | 12| 016542 | 612843 |
| 13| 016542 | 614242 | 14| 016546 | 612440 | 15| 016572 | 614320 |
| 16| 016634 | 614123 | 17| 016634 | 614231 | 18| 016643 | 612271 |
| 19| 016653 | 612313 | 20| 016724 | 643121 | 21| 016727 | 612440 |
| 22| 016756 | 611280 | 23| 016774 | 612363 | 24| 016817 | 614320 |
| 25| 017262 | 641243 | 26| 017356 | 616123 | 27| 017374 | 616242 |
| 28| 017375 | 641243 | 29| 017632 | 614620 | 30| 017646 | 612640 |
| 31| 017664 | 614520 | 32| 017674 | 612462 | 33| 017674 | 614272 |
| 34| 018265 | 612772 | 35| 018342 | 614620 | 36| 018382 | 612770 |
| 37| 018767 | 613123 | 38| 018767 | 613230 | 39| 061186 | 164231 |
| 40| 061246 | 166323 | 41| 061256 | 164341 | 42| 061264 | 162273 |
| 43| 061264 | 162433 | 44| 061284 | 164231 | 45| 061462 | 162452 |
| 46| 061462 | 164322 | 47| 061463 | 162262 | 48| 061472 | 126861 |
| 49| 061473 | 164251 | 50| 061476 | 162431 | 51| 061476 | 164320 |
| 52| 061547 | 127621 | 53| 061575 | 126232 | 54| 061618 | 126232 |
| 55| 061624 | 126332 | 56| 061644 | 126341 | 57| 061764 | 126242 |
| 58| 061774 | 126343 | 59| 063144 | 168281 | 60| 063412 | 168481 |
| 61| 063512 | 162642 | 62| 063515 | 162441 | 63| 063515 | 164321 |
| 64| 063541 | 162660 | 65| 063541 | 164420 | 66| 063551 | 163241 |
| 67| 063811 | 164261 | 68| 063824 | 162660 | 69| 063824 | 164420 |
| 70| 063825 | 162772 | 71| 063828 | 162273 | 72| 063828 | 162433 |
| 73| 063842 | 164261 | 74| 063858 | 163320 | 75| 063877 | 128160 |
| 76| 063882 | 163423 | 77| 063884 | 163421 | 78| 064143 | 164251 |
| 79| 064146 | 162431 | 80| 064146 | 164320 | 81| 064615 | 122632 |
| 82| 064826 | 126262 | 83| 064838 | 126451 | 84| 064842 | 126451 |
### Classification of Base Sequences $BS(n+1,n)$

#### Table 6: Equivalence classes of $BS(13,12)$

| AB | CD | AB | CD | AB | CD |
|----|----|----|----|----|----|
| 1  | 0611863 164521 | 2  | 0611871 166143 | 3  | 0612360 164844 |
| 4  | 0612573 164215 | 5  | 0612760 162827 | 6  | 0612760 162443 |
| 7  | 0612870 162424 | 8  | 0614230 164864 | 9  | 0614553 164226 |
| 10 | 0614670 162327 | 11 | 0614671 128266 | 12 | 0614830 162624 |
| 13 | 0615230 162867 | 14 | 0615230 161883 | 15 | 0615643 161234 |
| 16 | 0615733 126452 | 17 | 0616433 126452 | 18 | 0616533 126143 |
| 19 | 0616733 126143 | 20 | 0617212 126847 | 21 | 0617220 126876 |
| 22 | 0617220 126883 | 23 | 0617230 126813 | 24 | 0617651 122818 |
| 25 | 0617811 126428 | 26 | 0617820 126428 | 27 | 0618333 126413 |
| 28 | 0631412 168644 | 29 | 0634170 168382 | 30 | 0635120 164558 |
| 31 | 0635152 164341 | 32 | 0635340 164621 | 33 | 0635440 164242 |
| 34 | 0635483 162246 | 35 | 0635483 162432 | 36 | 0635513 164132 |
| 37 | 0635500 163214 | 38 | 0635810 163422 | 39 | 0635810 164142 |
| 40 | 0635872 162134 | 41 | 0638112 163822 | 42 | 0638121 164423 |
| 43 | 0638212 164242 | 44 | 0638222 164278 | 45 | 0638241 164423 |
| 46 | 0638781 162167 | 47 | 0641282 162424 | 48 | 0641363 164522 |
| 49 | 0641370 164265 | 50 | 0641470 162732 | 51 | 0641471 128642 |
| 52 | 0641471 126143 | 53 | 0641481 164215 | 54 | 0643513 164242 |
| 55 | 0643822 164522 | 56 | 0643880 162424 | 57 | 0644833 163214 |
| 58 | 0646430 126143 | 59 | 0648112 162623 | 60 | 0648212 161624 |
| 61 | 0648213 161644 | 62 | 0648363 162461 | 63 | 0648821 126716 |
| 64 | 0648433 126452 | 65 | 0658112 116273 | 66 | 0658363 116342 |
| 67 | 0658463 112645 | 68 | 0661363 116245 | 69 | 0661453 112634 |
| 70 | 0663640 116342 | 71 | 0663853 116324 | 72 | 0663880 116245 |
| 73 | 0664360 112645 | 74 | 0685733 116245 | 75 | 0685860 116245 |
| 76 | 0685871 112766 | 77 | 0686130 116245 | 78 | 0686230 116636 |
| 79 | 0686240 116273 | 80 | 0686433 116245 | 81 | 0687623 116245 |
| 82 | 0688613 116245 | 83 | 0688671 112764 | 84 | 0811283 182788 |
| 85 | 0812661 182668 | 86 | 0812883 182647 | 87 | 0816621 181128 |
| 88 | 0817883 181128 | 89 | 0826620 182647 | 90 | 0826782 182266 |
| 91 | 0826783 182641 | 92 | 0826851 186444 | 93 | 0835382 182266 |
| 94 | 0836121 186621 | 95 | 0837383 182278 | 96 | 0837383 182771 |
| 97 | 0838521 186627 | 98 | 0838652 182777 | 99 | 0838673 182668 |
| 100| 0838751 182777 | 101| 0838871 186644 | 102| 0862230 164278 |
| 103| 0862270 164413 | 104| 0862312 164226 | 105| 0862322 166128 |
| 106| 0862383 162748 | 107| 0862441 164265 | 108| 0862483 128674 |
| 109| 0862650 163341 | 110| 0862651 162138 | 111| 0863220 164287 |
| 112| 0863353 162642 | 113| 0863353 164612 | 114| 0863482 166144 |
| 115| 0864121 164324 | 116| 0864311 164226 | 117| 0864343 164226 |
| 118| 0864382 164413 | 119| 0864463 162451 | 120| 0864781 162167 |
|   | AB     | CD     |   | AB     | CD     |   | AB     | CD     |
|---|--------|--------|---|--------|--------|---|--------|--------|
| 121 | 0865261 | 126237 | 122 | 0865310 | 126452 | 123 | 0865343 | 161642 |
| 124 | 0865382 | 126238 | 125 | 0865382 | 161267 | 126 | 0866310 | 126413 |
| 127 | 0866443 | 126314 | 128 | 0867740 | 126174 | 129 | 0868761 | 126238 |
| 130 | 0868761 | 161267 | 131 | 0871130 | 162747 | 132 | 0871332 | 162624 |
| 133 | 0872231 | 162838 | 134 | 0872833 | 164432 | 135 | 0873111 | 166361 |
| 136 | 0873470 | 162842 | 137 | 0873470 | 166413 | 138 | 0873481 | 162862 |
| 139 | 0873581 | 164413 | 140 | 0873583 | 162445 | 141 | 0873670 | 162248 |
| 142 | 0873681 | 162268 | 143 | 0873711 | 164226 | 144 | 0873750 | 164413 |
| 145 | 0876221 | 127763 | 146 | 0876510 | 126342 | 147 | 0876510 | 126451 |
| 148 | 0876570 | 122864 | 149 | 0876570 | 161247 | 150 | 0876581 | 126238 |
| 151 | 0876581 | 161267 | 152 | 0876612 | 126341 | 153 | 0876663 | 126314 |
| 154 | 0876750 | 126238 | 155 | 0876750 | 161267 | 156 | 0877382 | 126876 |
| 157 | 0877382 | 161883 | 158 | 0877871 | 126748 | 159 | 0878631 | 127778 |
| 160 | 0878663 | 127647 | 161 | 0878861 | 161886 | 162 | 0878870 | 161884 |
| 163 | 0881211 | 168382 | 164 | 0882762 | 168242 | 165 | 0883571 | 168242 |
| 166 | 0883671 | 168422 | 167 | 0886261 | 162874 | 168 | 0886280 | 162867 |
| 169 | 0886471 | 162842 | 170 | 0886471 | 166413 | 171 | 0886560 | 164215 |
| 172 | 0886613 | 164521 | 173 | 0886671 | 162741 | 174 | 0886760 | 164215 |
| 175 | 0887780 | 162748 |
Table 7: Equivalence classes of $BS(14, 13)$

|   | $AB$   | $CD$   |   | $AB$   | $CD$   |   | $AB$   | $CD$   |
|---|--------|--------|---|--------|--------|---|--------|--------|
| 1 | 0116455| 6616380| 2 | 0116536| 6645183| 3 | 0116546| 6645153|
| 4 | 0116734| 6641822| 5 | 0117653| 6618273| 6 | 0117653| 6618422|
| 7 | 0117663| 6618450| 8 | 0118176| 6618441| 9 | 0118323| 6618222|
| 10| 0118324| 6618241| 11| 0118327| 6612712| 12| 0118327| 6614122|
| 13| 0118345| 6614121| 14| 0123628| 6645153| 15| 0123644| 6641450|
| 16| 0123672| 6614413| 17| 0123827| 6611811| 18| 0131554| 6618272|
| 19| 0131657| 6618273| 20| 0131657| 6618422| 21| 0131676| 6612763|
| 22| 0131755| 6614185| 23| 0131743| 6644513| 24| 0131745| 6645560|
| 25| 0131842| 6618222| 26| 0132157| 6641822| 27| 0132284| 6614580|
| 28| 0132354| 6614153| 29| 0132427| 6641810| 30| 0132463| 6618843|
| 31| 0133154| 6612743| 32| 0133414| 6618451| 33| 0133425| 6611813|
| 34| 0134174| 6614185| 35| 0134174| 6644513| 36| 0134234| 6641513|
| 37| 0134246| 6645113| 38| 0134273| 6621412| 39| 0134274| 6612280|
| 40| 0134314| 6641851| 41| 0134416| 6614182| 42| 0136167| 6636440|
| 43| 0136764| 6634111| 44| 0137536| 6631223| 45| 0138324| 6631411|
| 46| 0161327| 6418636| 47| 0161533| 6418643| 48| 0161633| 6414853|
| 49| 0161655| 6456330| 50| 0161742| 6418643| 51| 0161755| 6412743|
| 52| 0161762| 6418430| 53| 0162173| 6162842| 54| 0162283| 6415680|
| 55| 0162328| 6418650| 56| 0162556| 6451340| 57| 0162582| 6451343|
| 58| 0162617| 6451282| 59| 0162766| 6413151| 60| 0163157| 6618433|
| 61| 0163174| 6168441| 62| 0163255| 6164143| 63| 0163287| 6162131|
| 64| 0163325| 6168241| 65| 0163532| 6162760| 66| 0163372| 6141416|
| 67| 0163375| 6412612| 68| 0163417| 6414653| 69| 0163462| 6412681|
| 70| 0163562| 6451883| 71| 0163822| 6455630| 72| 0163844| 6451220|
| 73| 0164133| 6456613| 74| 0164143| 6168242| 75| 0164156| 6456550|
| 76| 0164234| 6164450| 77| 0164314| 6162862| 78| 0164317| 6162871|
| 79| 0164327| 6142623| 80| 0164367| 6164311| 81| 0164471| 6418541|
| 82| 0164481| 6141811| 83| 0164624| 6451133| 84| 0165173| 6124832|
| 85| 0165174| 6142582| 86| 0165243| 6141643| 87| 0165274| 6122463|
| 88| 0165327| 6142412| 89| 0165347| 6141622| 90| 0165367| 6142312|
| 91| 0165413| 6144831| 92| 0165428| 6142152| 93| 0165523| 6143513|
| 94| 0165716| 6143581| 95| 0165816| 6114424| 96| 0165826| 6114531|
| 97| 0165826| 6123142| 98| 0166137| 6128182| 99| 0166153| 6127433|
| 100| 0166173| 6128371| 101| 0166317| 6127481| 102| 0166324| 6124841|
| 103| 0166413| 6127832| 104| 0166423| 6141830| 105| 0166543| 6112471|
| 106| 0167125| 6142682| 107| 0167134| 6148463| 108| 0167156| 6434350|
| 109| 0167162| 6127643| 110| 0167162| 6142763| 111| 0167238| 6128222|
| 112| 0167286| 6122162| 113| 0167345| 6144161| 114| 0167356| 6141461|
| 115| 0167356| 6183881| 116| 0167365| 6144311| 117| 0167385| 6116270|
| 118| 0167416| 6144581| 119| 0167426| 6142280| 120| 0167455| 6431311|
| \( AB \) | \( CD \) | \( AB \) | \( CD \) |
|---|---|---|---|
| 121 | 0167457 | 6435150 | 122 | 0167465 | 6141272 |
| 123 | 0167466 | 6124512 | 124 | 0167583 | 6123240 |
| 125 | 0167584 | 6123240 | 126 | 0167585 | 6112740 |
| 127 | 0167586 | 6112740 | 128 | 0167817 | 6118282 |
| 129 | 0168171 | 6143841 | 130 | 0168171 | 6143841 |
| 131 | 0168276 | 6114521 | 132 | 0168286 | 6114531 |
| 133 | 0168425 | 6143411 | 134 | 0168465 | 6112422 |
| 135 | 0168476 | 6123420 | 136 | 0171655 | 6416273 |
| 137 | 0171656 | 6416851 | 138 | 0171831 | 6416481 |
| 139 | 0172647 | 6411422 | 140 | 0172656 | 6415223 |
| 141 | 0173413 | 6441863 | 142 | 0173474 | 6445160 |
| 143 | 0173523 | 6411422 | 144 | 0173524 | 6414151 |
| 145 | 0173554 | 6161242 | 146 | 0173557 | 6164880 |
| 147 | 0173612 | 6414863 | 148 | 0173612 | 6414863 |
| 149 | 0173612 | 6414863 | 150 | 0173612 | 6414863 |
| 151 | 0173843 | 6412141 | 152 | 0173843 | 6412781 |
| 153 | 0173843 | 6412781 | 154 | 0173843 | 6412781 |
| 155 | 0173843 | 6412781 | 156 | 0173843 | 6412781 |
| 157 | 0176164 | 6416273 | 158 | 0176164 | 6416273 |
| 159 | 0176164 | 6416273 | 160 | 0176164 | 6416273 |
| 161 | 0176164 | 6416273 | 162 | 0176164 | 6416273 |
| 163 | 0176164 | 6416273 | 164 | 0176164 | 6416273 |
| 165 | 0176164 | 6416273 | 166 | 0176164 | 6416273 |
| 167 | 0176164 | 6416273 | 168 | 0176164 | 6416273 |
| 169 | 0176164 | 6416273 | 170 | 0176164 | 6416273 |
| 171 | 0176164 | 6416273 | 172 | 0176164 | 6416273 |
| 173 | 0176164 | 6416273 | 174 | 0176164 | 6416273 |
| 175 | 0176164 | 6416273 | 176 | 0176164 | 6416273 |
| 177 | 0176164 | 6416273 | 178 | 0176164 | 6416273 |
| 179 | 0176164 | 6416273 | 180 | 0176164 | 6416273 |
| 181 | 0176164 | 6416273 | 182 | 0176164 | 6416273 |
| 183 | 0176164 | 6416273 | 184 | 0176164 | 6416273 |
| 185 | 0176164 | 6416273 | 186 | 0176164 | 6416273 |
| 187 | 0176164 | 6416273 | 188 | 0176164 | 6416273 |
| 189 | 0176164 | 6416273 | 190 | 0176164 | 6416273 |
| 191 | 0176164 | 6416273 | 192 | 0176164 | 6416273 |
| 193 | 0176164 | 6416273 | 194 | 0176164 | 6416273 |
| 195 | 0176164 | 6416273 | 196 | 0176164 | 6416273 |
| 197 | 0176164 | 6416273 | 198 | 0176164 | 6416273 |
| 199 | 0176164 | 6416273 | 200 | 0176164 | 6416273 |
| 201 | 0176164 | 6416273 | 202 | 0176164 | 6416273 |
| 203 | 0176164 | 6416273 | 204 | 0176164 | 6416273 |
| 205 | 0176164 | 6416273 | 206 | 0176164 | 6416273 |
| 207 | 0176164 | 6416273 | 208 | 0176164 | 6416273 |
| 209 | 0176164 | 6416273 | 210 | 0176164 | 6416273 |
| 211 | 0176164 | 6416273 | 212 | 0176164 | 6416273 |
| 213 | 0176164 | 6416273 | 214 | 0176164 | 6416273 |
| 215 | 0176164 | 6416273 | 216 | 0176164 | 6416273 |
| 217 | 0176164 | 6416273 | 218 | 0176164 | 6416273 |
| 219 | 0176164 | 6416273 | 220 | 0176164 | 6416273 |
| 221 | 0176164 | 6416273 | 222 | 0176164 | 6416273 |
| 223 | 0176164 | 6416273 | 224 | 0176164 | 6416273 |
| 225 | 0176164 | 6416273 | 226 | 0176164 | 6416273 |
| 227 | 0176164 | 6416273 | 228 | 0176164 | 6416273 |
| 229 | 0176164 | 6416273 | 230 | 0176164 | 6416273 |
| 231 | 0176164 | 6416273 | 232 | 0176164 | 6416273 |
| 233 | 0176164 | 6416273 | 234 | 0176164 | 6416273 |
| 235 | 0176164 | 6416273 | 236 | 0176164 | 6416273 |
| 237 | 0176164 | 6416273 | 238 | 0176164 | 6416273 |
### Table 7: (continued)

|   |   | AB | CD |   |   | AB | CD |
|---|---|----|----|---|---|----|----|
| 241 | 0612356 | 1648241 | 242 | 0612457 | 1662171 | 243 | 0612517 | 1642753 |
| 244 | 0612528 | 1638242 | 245 | 0612536 | 1287762 | 246 | 0612547 | 1624820 |
| 247 | 0612548 | 1644511 | 248 | 0612556 | 1286422 | 249 | 0612585 | 1623820 |
| 250 | 0612586 | 1286420 | 251 | 0612645 | 1286441 | 252 | 0612646 | 1624730 |
| 253 | 0612744 | 1642280 | 254 | 0612752 | 1624282 | 255 | 0612752 | 1624670 |
| 256 | 0612753 | 1644131 | 257 | 0612753 | 1661181 | 258 | 0612758 | 1624250 |
| 259 | 0612764 | 1286420 | 260 | 0612844 | 1661271 | 261 | 0612853 | 1626142 |
| 262 | 0613515 | 1663843 | 263 | 0613554 | 1628243 | 264 | 0613647 | 1628323 |
| 265 | 0613753 | 1628263 | 266 | 0614236 | 1664450 | 267 | 0614274 | 1628213 |
| 268 | 0614271 | 1663481 | 269 | 0614274 | 1662142 | 270 | 0614475 | 1628213 |
| 271 | 0614527 | 1622843 | 272 | 0614571 | 1286372 | 273 | 0614571 | 1268462 |
| 274 | 0614577 | 1634513 | 275 | 0614638 | 1286321 | 276 | 0614642 | 1268341 |
| 277 | 0614675 | 1623412 | 278 | 0614681 | 1236233 | 279 | 0614773 | 1643450 |
| 280 | 0614774 | 1643270 | 281 | 0614784 | 1264250 | 282 | 0614873 | 1643211 |
| 283 | 0614884 | 1645130 | 284 | 0615118 | 1268632 | 285 | 0615146 | 1628462 |
| 286 | 0615147 | 1268452 | 287 | 0615471 | 1276443 | 288 | 0615487 | 1628111 |
| 289 | 0615627 | 1612342 | 290 | 0615741 | 1263482 | 291 | 0615743 | 1616242 |
| 292 | 0615783 | 1228631 | 293 | 0615784 | 1263621 | 294 | 0615785 | 1614350 |
| 295 | 0615874 | 1612341 | 296 | 0615883 | 1262333 | 297 | 0616234 | 1624760 |
| 298 | 0616237 | 1264280 | 299 | 0616281 | 1228632 | 300 | 0616358 | 1267113 |
| 301 | 0616472 | 1614551 | 302 | 0616481 | 1262343 | 303 | 0616481 | 1614232 |
| 304 | 0616534 | 1261732 | 305 | 0616841 | 1611822 | 306 | 0617413 | 1618861 |
| 307 | 0617426 | 1268423 | 308 | 0617526 | 1616431 | 309 | 0617556 | 1271642 |
| 310 | 0617556 | 1612741 | 311 | 0617572 | 1614651 | 312 | 0617586 | 1271660 |
| 313 | 0617625 | 1624333 | 314 | 0617626 | 1262343 | 315 | 0617626 | 1614232 |
| 316 | 0617655 | 1613422 | 317 | 0617682 | 1613640 | 318 | 0617786 | 1271642 |
| 319 | 0617786 | 1612741 | 320 | 0617824 | 1264620 | 321 | 0617824 | 1616231 |
| 322 | 0617844 | 1616450 | 323 | 0617845 | 1266212 | 324 | 0617853 | 1612461 |
| 325 | 0617854 | 1614222 | 326 | 0617874 | 1271662 | 327 | 0617884 | 1262450 |
| 328 | 0618516 | 1262443 | 329 | 0618517 | 1262433 | 330 | 0618527 | 1612441 |
| 331 | 0618557 | 1613450 | 332 | 0618616 | 1613441 | 333 | 0618625 | 1613441 |
| 334 | 0618714 | 1612662 | 335 | 0618748 | 1228631 | 336 | 0618824 | 1264142 |
| 337 | 0618825 | 1263172 | 338 | 0618847 | 1612361 | 339 | 0618874 | 1613241 |
| 340 | 0618883 | 1613451 | 341 | 06191147 | 1686422 | 342 | 0631352 | 1686222 |
| 343 | 0631458 | 1682421 | 344 | 0631557 | 1681411 | 345 | 0631778 | 1681411 |
| 346 | 0634135 | 1682202 | 347 | 0634187 | 1682321 | 348 | 0635124 | 1644340 |
| 349 | 0635132 | 1644270 | 350 | 0635137 | 1638220 | 351 | 0635138 | 1626240 |
| 352 | 0635152 | 1624740 | 353 | 0635311 | 1645860 | 354 | 0635381 | 1626710 |
| 355 | 0635441 | 1646161 | 356 | 0635857 | 1621341 | 357 | 0636178 | 1634410 |
| 358 | 0636178 | 1641321 | 359 | 0636882 | 1631443 | 360 | 0638171 | 1644131 |
Table 7: (continued)

|    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|
| 361 | 0638171 | 1661181 | 362 | 0638177 | 1286420 | 363 | 0638188 | 1286620 |
| 364 | 0638221 | 1645681 | 365 | 0638237 | 1642620 | 366 | 0638248 | 1642470 |
| 367 | 0638281 | 1644141 | 368 | 0638285 | 1643440 | 369 | 0638372 | 1624821 |
| 370 | 0638384 | 1661242 | 371 | 0638427 | 1638221 | 372 | 0638428 | 1638421 |
| 373 | 0638457 | 1624420 | 374 | 0638487 | 1626132 | 375 | 0638522 | 1633822 |
| 376 | 0638744 | 1632281 | 377 | 0638747 | 1641272 | 378 | 0641147 | 1638422 |
| 379 | 0641163 | 1286372 | 380 | 0641163 | 1286671 | 381 | 0641264 | 1643430 |
| 382 | 0641278 | 1624260 | 383 | 0641361 | 1624842 | 384 | 0641361 | 1643833 |
| 385 | 0641451 | 1286372 | 386 | 0641451 | 1286671 | 387 | 0641457 | 1622453 |
| 388 | 0641517 | 1632482 | 389 | 0641712 | 1632783 | 390 | 0641754 | 1621662 |
| 391 | 0641867 | 1621370 | 392 | 0643113 | 1648422 | 393 | 0643412 | 1662713 |
| 394 | 0643536 | 1286881 | 395 | 0643545 | 1661421 | 396 | 0643611 | 1624573 |
| 397 | 0643814 | 1638221 | 398 | 0643836 | 1661222 | 399 | 0643842 | 1661271 |
| 400 | 0643843 | 1646150 | 401 | 0644124 | 1624570 | 402 | 0644134 | 1627611 |
| 403 | 0644145 | 1286240 | 404 | 0644381 | 1626122 | 405 | 0644813 | 1621263 |
| 406 | 0644813 | 1634121 | 407 | 0644823 | 1634141 | 408 | 0645172 | 1616441 |
| 409 | 0645387 | 1263821 | 410 | 0645651 | 1226343 | 411 | 0645857 | 1261243 |
| 412 | 0645871 | 1261623 | 413 | 0646123 | 1261373 | 414 | 0646138 | 1261622 |
| 415 | 0646387 | 1261232 | 416 | 0648121 | 1264760 | 417 | 0648175 | 1262282 |
| 418 | 0648176 | 1614430 | 419 | 0648225 | 1263482 | 420 | 0648272 | 1612662 |
| 421 | 0648275 | 1614460 | 422 | 0648276 | 1262282 | 423 | 0648376 | 1624620 |
| 424 | 0648376 | 1616231 | 425 | 0648412 | 1264523 | 426 | 0648426 | 1623323 |
| 427 | 0648472 | 1264141 | 428 | 0648481 | 1264161 | 429 | 0648586 | 1226480 |
| 430 | 0648587 | 1226380 | 431 | 0648635 | 1226481 | 432 | 0648863 | 1226453 |
| 433 | 0655416 | 1126471 | 434 | 0655416 | 1162182 | 435 | 0655417 | 1126371 |
| 436 | 0658125 | 1164341 | 437 | 0658135 | 1166450 | 438 | 0658146 | 1162461 |
| 439 | 0658147 | 1162451 | 440 | 0658173 | 1127621 | 441 | 0658275 | 1126760 |
| 442 | 0658364 | 1126461 | 443 | 0658463 | 1127762 | 444 | 0658487 | 1127631 |
| 445 | 0661274 | 1127641 | 446 | 0663582 | 1164522 | 447 | 0663614 | 1163441 |
| 448 | 0663875 | 1163241 | 449 | 0663885 | 1163422 | 450 | 0682412 | 1186822 |
| 451 | 0685236 | 1128263 | 452 | 0685424 | 1128623 | 453 | 0685536 | 1162740 |
| 454 | 0685637 | 1126370 | 455 | 0685724 | 1162323 | 456 | 0685753 | 1126341 |
| 457 | 0685827 | 1162742 | 458 | 0685846 | 1162762 | 459 | 0685863 | 1126432 |
| 460 | 0686142 | 1163422 | 461 | 0686154 | 1122671 | 462 | 0686213 | 1162471 |
| 463 | 0686215 | 1163322 | 464 | 0686253 | 1126452 | 465 | 0686273 | 1126760 |
| 466 | 0686357 | 1163222 | 467 | 0686374 | 1164650 | 468 | 0686413 | 1164560 |
| 469 | 0686424 | 1162731 | 470 | 0686451 | 1126341 | 471 | 0687515 | 1162451 |
| 472 | 0687525 | 1162631 | 473 | 0687561 | 1126361 | 474 | 0687645 | 1126760 |
| 475 | 0688763 | 1164561 |
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