THE MULTI ELLIPTIC-BREATHER SOLUTIONS AND THEIR ASYMPTOTIC ANALYSIS FOR THE MKDV EQUATION

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ABSTRACT. We mainly construct and analyze the multi elliptic-breather solutions under the periodic solutions background for the focusing modified Korteweg-de Vries (mKdV) equation. Based on the Darboux-Bäcklund transformation, we provide a uniform expression for the multi elliptic-breather solutions by the Jacobi theta functions. Then, a sufficient condition of the elastic collision among the breathers is given through the symmetric analysis. As \( t \to \pm \infty \), the asymptotic behaviors of the multi elliptic-breather solutions are provided directly in two categories: along the line \( l_i^\pm \) with different velocity \( v_i \) and on the region \( R_i^\pm \) between the breathers. We also obtain that the multi elliptic-breather solutions could degenerate into the constant solution, solitons or breathers under the constant background, as \( k \to 0^+ \). Moreover, we provide some examples of the multi elliptic-breather solutions to visualize above analysis.

Keywords: modified Korteweg-de Vries equation, multi elliptic-breather solution, Darboux-Bäcklund transformation, Jacobi theta function, asymptotic analysis.

1 Introduction

In this work, we mainly study the exact expression for the multi elliptic-breather solutions under the elliptic functions backgrounds of the focusing modified Korteweg-de Vries (mKdV) equation

\[
 u_t + 6u^2 u_x + u_{xxx} = 0,
\]

where \( u(x,t) \) is a real-valued function with \( (x,t) \in \mathbb{R}^2 \). Based on the accurate expressions of the multi elliptic-breather solutions, we focus on studying the symmetric properties and the asymptotic analysis of them.

The mKdV equation is a well-known completely integrable model [1, 2], which admits a suitable Lax-pair formulation [30, 28], infinitely conserved quantities [35], Bi-Hamiltonian structure [33], and so on. Over the past few decades, many kinds of the nonlinear waves solutions on the plane waves background or the vanishing background, such as rational solutions [15], solitons [49], breather solutions [16, 26, 36, 45] and rogue waves [16, 25], have been obtained widely, through the inverse scattering transformation [37], the Darboux transformation [44], and so on.

Recently, some solutions in the context of elliptic functions have also been derived. Shin used the squared wave function method to obtain some solutions of the coupled NLS equation [40, 41, 42]. Hu, Lou and Chen [23] gained the explicit analytic interaction solutions between solitary waves and cnoidal waves of the KdV equation by utilizing the localization procedure of nonlocal symmetries. With the aid of the Darboux transformation method, Kedziora, Ankiewicz and Akhmediev [25] constructed the rogue waves of the nonlinear Schrödinger (NLS) equation under the cnoidal background. Based on the formal algebraic method from Cao et al. [6, 7, 21], Chen and Pelinovsky developed an algebraic method to gain the rogue wave solutions and the periodic traveling waves on the periodic waves background in some integrable nonlinear systems, such as the mKdV equation [10, 11], the NLS equation [9, 14], the Hirota equation [20, 39], the sine-Gordon equation [31, 38], the derivative NLS equation [12, 13, 50]. Those solutions are expressed by the Jacobi elliptic functions.

Using the Jacobi theta functions to express the solutions of the equation is also a powerful method to obtain them. As early as the 20th century, Its, Rybin and Sail [24] used the finite-gap expressions to gain smooth periodic solutions of the NLS equation in terms of the theta functions. In [4], Belokolos et al. investigated the finite-gap theta function formulas, which provide a theoretical basis in constructing the solutions in the theta function form. Fortunately, in recent years, expressing the solutions by the theta
functions have made some progresses. Shin [43] gained the solutions in the theta function form to study the soliton dynamics moving in phase-modulated lattices. Through utilizing the theta functions, Feng et al. [19] constructed the multi elliptic-breather solutions and rogue waves of the NLS equation successfully. Ling and Sun [32] provided breather solutions in the theta function forms to exhibit the unstable or stable dynamic behaviors vividly, after studying the spectral and orbital stability of the elliptic function solutions for the mKdV equation. However, to the best of our knowledge, there still lack the systematic analysis on the multi elliptic-breather solutions in theta function forms. Therefore, our main goal is to construct the multi elliptic-breather solutions through utilizing the Jacobi theta functions directly, analyze the symmetric properties and study asymptotic behaviors of those solutions.

In this paper, we use the Darboux-Bäcklund transformation to get the explicit expression of the multi elliptic-breather solutions under the elliptic function background. Unlike the breathers in the constant background, above multi elliptic-breather solutions are expressed by utilizing the theta function which could clearly and accurately describe the algebraic geometric solution of genus-1. The asymptotic analysis is a crucial way to study the behavior of multi elliptic-breather solution as \( t \to \pm \infty \) and analyze the relationship between the breathers under constant backgrounds as \( k \to 0^+ \). The innovations of this paper mainly include following aspects:

- We provide the exact expressions of multi elliptic-breather solutions uniformly expressed by the Jacobi theta function either in cn-type or dn-type solutions background. And then, we analyze some spectral properties of those solutions. A sufficient condition of the elastic collision among the breathers is given by the symmetric property. For the asymptotic analysis, we not only consider the dynamic behaviors of solutions as \( t \to \pm \infty \), but also focus on the degeneration of solutions as \( k \to 0^+ \).
- As modulus \( t \to \pm \infty \), the asymptotic analysis of the multi elliptic-breather solutions are revealed by the exact expressions directly in two categories: along the line \( I^\pm \) with velocity \( v_i \) and in the area \( R^\pm \) between two breathers. The asymptotic expression in region \( R^\pm \) can be regarded as a shift on the background solution. As \( k \to 0^+ \), the multi elliptic-breather solutions could be degenerated into many well-known solutions, such as constant solutions, solitons and breathers under the constant backgrounds.
- The relations between velocity \( v_i \) and the spectral parameter \( \lambda_i \) for the multi elliptic-breather solutions are obtained, which has never been reported before to the best of our knowledge. Analyzing the zeros and poles of meromorphic functions \( \mathcal{R}(I(z)) \) and \( \mathcal{R}(\Omega(z)) \) and studying the conformal map between spectral parameter \( \lambda \) and uniform parameter \( z \) are two indispensable steps to prove the relations between velocity \( v_i \) and spectral parameter \( \lambda_i \).

1.1 Main results

As we all know, the following two elliptic functions are the periodic solutions of mKdV equation (1):

\[
\begin{align*}
  u(x,t) &= k\text{cn}(a(x-st),k), \\
  u(x,t) &= a\text{dn}(a(x-st),k),
\end{align*}
\]

where \( \text{cn}(\cdot,k) \), \( \text{dn}(\cdot,k) \) denote the Jacobi elliptic functions with elliptic modulus \( k \) and \( s \) is the velocity between the time \( t \) and space \( x \), which depends on the modulus \( k \) and the parameter \( a \). If the solution of mKdV equation is \( \text{cn} \)-type, the value of \( s \) is \( a^2(2k^2-1) \). If it is \( \text{dn} \)-type solutions, \( s = a^2(2-k^2) \). The exact calculation process of above results was given in [32]. For convenience of the expression, we introduce the notation \( \xi := x - st \).

The mKdV equation (1) admits the Lax pair:

\[
\Phi_t(x,t;\lambda) = U(\lambda,u)\Phi(x,t;\lambda), \quad \Phi_x(x,t;\lambda) = V(\lambda,u)\Phi(x,t;\lambda),
\]

where \( \lambda \in \mathbb{C} \cup \{ \infty \} \) is called the spectral parameter and matrices \( U(\lambda,u) \) and \( V(\lambda,u) \) in equation (3) are defined as

\[
\begin{align*}
  U(\lambda,u) &\equiv -i\lambda\sigma_3 + Q, \\
  V(\lambda,u) &\equiv -4i\lambda^3\sigma_3 + 4\lambda^2Q + i\lambda\sigma_3\left(2Q_x - 2Q^2\right) + 2Q^3 - Q_{xx},
\end{align*}
\]

where \( \sigma_3 \equiv \text{diag}(1,-1) \), \( Q \equiv \begin{bmatrix} 0 & u \\ -u & 0 \end{bmatrix} \).
satisfying the symmetric properties

\[ U^t(\lambda^*; u) = -U(\lambda; u), \quad U^T(-\lambda; u) = -U(\lambda; u), \]
\[ V^t(\lambda^*; u) = -V(\lambda; u), \quad V^T(-\lambda; u) = -V(\lambda; u). \]

Furthermore, based on the algebraic calculation, we could verify that the mKdV equation (1) is equivalent to the compatibility conditions of Lax pair (3):

\[ U_x - V_t + [U, V] = 0, \quad [U, V] = UV - VU. \]

Considering the elliptic solutions (2) of mKdV equation (1), the fundamental solution \( \Phi(x, t; \lambda) \) of the Lax pair (3) could be written as

\[ \Phi(x, t; \lambda) = \frac{a_1 \vartheta_2(\eta_1)}{\vartheta_3(\eta_1)} \left[ \frac{\vartheta_1(\bar{\eta}_1 + \xi_1)}{\vartheta_2(\bar{\eta}_1)} E_1(x, t; z) - \frac{\vartheta_3(\bar{\eta}_1 + \xi_2)}{\vartheta_2(\bar{\eta}_1)} E_2(x, t; z) \right], \]

where \( \xi = x - st, l = 0 \) or \( \frac{K'}{2} \),

\[ E_1(z) \equiv E_1(x, t; z) = \exp \left( (aZ(i(z - l)) + i\lambda)(x - st) + 4i\lambda t \right), \]
\[ E_2(z) \equiv E_2(x, t; z) = \exp \left( \left( -i\frac{\alpha \pi}{2K} - aZ(i(z + l) + K + iK') + i\lambda \right)(x - st) - 4i\lambda t \right), \]

and

\[ \Omega(z) \equiv \Omega(x, t; \lambda) = -\alpha^3 \left( k^2 \text{dn}(i(z - l)) \text{sn}(i(z - l)) \text{cn}(i(z - l)) \right) + \frac{k^2 \text{dn}(i(z + l)) \text{sn}(i(z + l))}{\text{cn}^2(i(z + l))}, \]

with the parameter \( \lambda \) defined in (10), the Jacobi theta functions \( \vartheta_i(z), i = 1, 2, 3, 4 \) defined in (A.8), the Jacobi Zeta functions \( Z(z) \) defined in (A.13) and the complete elliptic integrals \( K = K(k), K' = K(k') \) defined in (A.1). The choosing of parameter \( l \) is closely related to the elliptic function solutions in (2). If the solution is cn-type, the corresponding fundamental solution (7) of Lax pair (3) is obtained by the parameter \( l = 0 \). If the solution is dn-type, the fundamental solution is obtained by the parameter \( l = \frac{K'}{2} \). The detailed analysis and calculation process of solution \( \Phi(x, t; \lambda) \) are given in [32]. Furthermore, we also obtain the expressions of spectral parameter \( \lambda \) by the uniform parameter \( z \) with different value of \( l \) in the following:

\[ \lambda(z) = i\frac{\alpha}{2} \frac{\text{dn}(i(z - l)) \text{sn}(i(z - l))}{\text{cn}(i(z - l))}, \quad l = 0, \]
\[ \lambda(z) = i\alpha k^2 \frac{\text{sn}(i(z - l)) \text{cn}(i(z - l))}{\text{dn}(i(z - l))}, \quad l = \frac{K'}{2}. \]

The conformal map \( \lambda(z) \) maps the rectangular area \( S \) onto the whole complex plane with two cuts, where the region \( S \) is defined as

\[ S := \left\{ z \in \mathbb{C} \left| -K' + l \leq \Re(z) \leq K' + l, -\frac{K}{2} \leq \Im(z) \leq \frac{K}{2}, \quad l = 0 \text{ or } \frac{K'}{2} \right. \right\}. \]

Therefore, the studies of the spectral parameter \( \lambda \) over the whole complex plane is converted to analyze the uniform parameter \( z \) in the rectangular region \( S \).

Based on the Darboux-Bäcklund transformation, Theorem 6 and fundamental solutions (7), the exact expression of \( N \) elliptic-breather solutions could be rewritten in the Jacobi theta function form. The form of the \( N \) elliptic-breather solutions has a lot relationship with the Darboux matrix \( T_k^P(\lambda; x, t) \) and \( T_k^C(\lambda; x, t), k = 1, 2, \cdots, N \), which is explained in detail in the section 2. Define the number of above two Darboux matrices as \( n_1 \) and \( n_2 \), respectively. And then, we obtain the exact expressions of the elliptic-breather solution \( u^{[N]}(x, t) \) in following Theorem.

**Theorem 1.** The multi elliptic-breather solution \( u^{[N]}(x, t) \) of the equation (1) could be written as

\[ u^{[N]}(x, t) = \frac{a_1 \vartheta_2(\eta_1)}{\vartheta_3(\eta_1)} \left( \frac{\vartheta_4(\bar{\eta}_1 + \xi_1)}{\vartheta_2(\bar{\eta}_1 + \xi_2)} \right)^{m-1} \frac{\det(\mathcal{M})}{\det(\mathcal{D})} e^{-a\xi Z(2l + K)}, \]
where $\xi = x - st$, the matrices $\mathcal{M}$ and $\mathcal{D}$ are $m \times m$, whose elements are given by

$$(M)_{i,j} = \left[ r_i^{-1}E_i^+(z_i) \quad r_i^*c_j^+E_j^+(z_i) \right]$$

$$=(\mathcal{M})_{i,j} = \left[ r_i^{-1}E_i^+(z_i) \quad r_i^*c_j^+E_j^+(z_i) \right]$$

$$= \left[ \frac{\partial_1}{\partial_4} \left( \frac{(\xi_{i,j} + z_i + 2\eta_i) + a_i}{2K} \right) \right]$$

$$= \left[ \frac{\partial_1}{\partial_4} \left( \frac{(\xi_{i,j} + z_i - 2\eta_i) - a_i}{2K} \right) \right]$$

and $m = n_1 + 2n_2$, the parameters $n_1$ and $n_2$ determined in equation (31). And the functions $E_1(z)$ and $E_2(z)$ are defined in equation (8) and $r_i$ are defined as

$$r_i \equiv r(z_i) := \frac{\partial_1}{\partial_4} \left( \frac{(\xi_{i,j} + 1) + a_i}{2K} \right)$$

$$= \frac{\partial_1}{\partial_4} \left( \frac{(\xi_{i,j} - 1) - a_i}{2K} \right)$$

Based on the above accurate $N$ elliptic-breather solutions, we conduct a series of analysis on them and get following conclusions.

**Theorem 2.** When $c_i = 1$, $i = 1, 2, 3, \ldots, m$, the solutions $u^{[N]}(x,t)$ in equation (12) have the symmetric relation:

$$u^{[N]}(x,t) = u^{[N]}(-x,-t).$$

As $t \to \pm \infty$, the asymptotic expression of $N$ elliptic-breather solutions along the line $l_k^\pm$ and on the region $R_k^\pm$ are divided into two categories. Take a rotation $\xi = x - st$ on the solution $u^{[N]}(x,t)$ and define function $\hat{u}^{[N]}(\xi,t)$ in equation (58). As $t \to \pm \infty$, we get that following asymptotic expressions.

**Theorem 3.** The asymptotic expression of the elliptic-breather solution $\hat{u}^{[N]}(\xi,t)$ along the line $l_k$ as $t \to \pm \infty$ could be rewritten as following two different forms:

(i) Along the line $l_k$, if there exists only one parameter $z_h$ satisfies $\eta_h = \text{const}$ defined in (52) as $t \to \pm \infty$, we obtain

$$\hat{u}^{[N]}(\xi,t;l_k) = \frac{\alpha \theta_2 \theta_4}{\theta_3 \theta_5} \left( \frac{\alpha \theta_2}{\theta_5} \right)^{m-1} \frac{\det \left( \sum_{i,j=1}^{2n} (i+j)/X_i[\eta_i,\zeta] \cdot \mathcal{M}[i,j] \cdot X_j[\eta,j] \right)}{\det \left( \sum_{i,j=1}^{2n} (i+j)/X_i[\eta_i,\zeta] \cdot \mathcal{D}[i,j] \cdot X_j[\eta,j] \right)} e^{\alpha \xi Z(2l+K)}$$

$$+ \mathcal{O} \left( \exp \left( - \min_{j \neq h} \Re (I_j) \left| v_h - v_j \right| \right) \right), \quad t \to \pm \infty$$

with matrices $Y_i[1], Y_i[2]$ defined in (51), $\mathcal{M}[i,j], \mathcal{D}[i,j]$ defined in (48) and

$$X_i^+[\eta,+] := \text{diag} \left( 0, \ldots, 0, \underbrace{1, \ldots, 1}_{m-h} \right), \quad X_2[\eta,+] := \text{diag} \left( \underbrace{1, \ldots, 1}_{h-1}, e^{\eta_h}, 0, \ldots, 0 \right),$$

$$X_1^-[\eta,-] := \text{diag} \left( \underbrace{1, \ldots, 1}_{h-1}, e^{-\eta_h}, 0, \ldots, 0 \right), \quad X_2[\eta,-] := \text{diag} \left( 0, \ldots, 0, \underbrace{1, \ldots, 1}_{m-h} \right).$$
(ii) Along the line $l_k$, if there exist two parameters $z_h, z_{h+1}$ satisfying $\eta_h = \eta_{h+1} = \text{const}$ as $t \to \pm \infty$, the asymptotic expression of elliptic-breather solution $\hat{u}^{[N]}(\xi, t; l_k)$ also could be written as (15), with the matrices

$$
X_1^{[h,+]} := \text{diag} \left( 0, \ldots, e^{\frac{m-h}{h-1}}, \ldots, 1 \right), \quad X_2^{[h,+]} := \text{diag} \left( e^{\frac{h-1}{h-1}}, \ldots, 1, e^{\eta_h+1}, 0, \ldots, 0 \right),
$$

(17)

$$
X_1^{[h,-]} := \text{diag} \left( e^{\frac{h-1}{h-1}}, \ldots, 1, e^{-\eta_h}, e^{-\eta_{h+1}}, 0, \ldots, 0 \right), \quad X_2^{[h,-]} := \text{diag} \left( 0, \ldots, 0, e^{\frac{m-h}{h-1}}, \ldots, 1 \right).
$$

Through utilizing the formulas of Jacobi theta functions in [47], we could simplify the first case of the solution $\hat{u}^{[N]}(\xi, t; l_k)$ (73) in the Remark 6. In addition, the asymptotic analysis on the region $R_{k}^\pm, k = 1, 2, \cdots, N$ between the line $l_k$ and $l_{k+1}$ could be obtained.

**Theorem 4.** The asymptotic analysis on the region $R_{k}^\pm, k = 1, 2, \cdots, N$, could be divided in following two types:

(i) Along the line $l_{k-1}$, if there exists only one parameter $z_h$ satisfies $\eta = \text{const}$ as $t \to \pm \infty$, the asymptotic expression on the region $R_{k}^\pm$ could be written as

$$
\hat{u}^{[N]}(\xi, t; R_{k}^\pm) \to (-1)^{2N-m-p} \text{akcn}(a \xi + s_{h,h+1}^{\pm \infty}), \quad t \to \pm \infty, \quad l = 0,
$$

or

$$
\hat{u}^{[N]}(\xi, t; R_{k}^\pm) \to (-1)^{2N-m-p} \text{adn}(a \xi + s_{h,h+1}^{\pm \infty}), \quad t \to \pm \infty, \quad l = \frac{K'}{2},
$$

in which $p$ is the number of $z_i$ such that $i(z_i - l) \in \mathbb{R}$ and

$$
s_{h,h+1}^{\pm \infty} = \pm \left( \sum_{j=1}^{h} 2 \Im(z_j) - \sum_{j=h+1}^{m} 2 \Im(z_j) \right).
$$

(ii) Along the line $l_{k-1}$, if there exist two parameters $z_h, z_{h+1}$ satisfies $\eta_h = \eta_{h+1} = \text{const}$ as $t \to \pm \infty$, the asymptotic expression on the region $R_{k}^\pm$ also could be written in (18) or (19) with parameters

$$
s_{h,h+1}^{\pm \infty} = \pm \left( \sum_{j=1}^{h+1} 2 \Im(z_j) - \sum_{j=h+2}^{m} 2 \Im(z_j) \right).
$$

**Theorem 5.** As $k \to 0^+$, the $N$ elliptic-breather solution $u^{[N]}(x, t)$ could be degenerated into the $N$-breather solution on the constant background. When $l = 0$, $\lim_{k \to 0^+} u^{[N]}(x, t)$ could be seen as breathers generated by the vanishing background solution. When $l = \frac{K'}{2}$, $\lim_{k \to 0^+} u^{[N]}(x, t)$ could be seen as solutions or breathers constructed by the constant $\alpha \in \mathbb{R} \setminus \{0\}$ background solution. In some special case, the solutions would be degenerated into a constant solution.

### 1.2 Outline for this work

The organization of this work is as follows. In section 2, we obtain a uniform expression of the multi elliptic-breather solutions in the theta function forms and list some special examples to show some of them under the different times of the Darboux transformation. In section 3, we first prove a symmetric property of above solutions $u^{[N]}(x, t)$ with appropriate restrictions, which reflects the elastic collisions among the multi elliptic-breather solutions in this function. Then, we take an asymptotic analysis of the solutions along the line $l_k^\pm$ and on the region $R_{k}^\pm$ as $t \to \pm \infty$. Furthermore, in section 4, the asymptotic analysis of solutions as $k \to 0^+$ is obtained, which shows the relationship between multi elliptic-breather solutions and the breathers on constant background. The conclusions and discussions are involved in the section 5.

### 2 The multi elliptic-breather solutions of mKdV equation

In this section, our main goal is to provide the exact expressions of the multi elliptic-breather solutions. Firstly, we list some properties of the Darboux-Bäcklund transformation, which is useful in this section.
Secondly, we construct the multi elliptic-breather solutions and obtain the formula of them by using the Jacobi theta functions. At last, we exhibit some figures to the different types of elliptic-breather solutions.

2.1 The Darboux-Bäcklund transformation

There are many articles to study the Darboux-Bäcklund transformation [17, 22, 34]. These methods are well established, so we will not repeat them. Here, we just provide some conclusions, which are useful in the following analysis.

Based on the Darboux transformation, we know that the Darboux matrix $T(\lambda; x, t)$ could admit a new equation

\begin{equation}
T^{[1]}(x, t; \lambda) = U^{[1]}(\lambda; u^{[1]}) \Phi^{[1]}(x, t; \lambda), \quad \text{with} \quad \Phi^{[1]}(x, t; \lambda) := T(\lambda; x, t) \Phi(x, t; \lambda).
\end{equation}

And, the matrix $T(\lambda; x, t)$ satisfies the following properties:

**Proposition 1.** The Darboux matrix $T(\lambda; x, t)$ satisfies

\begin{equation}
T^{-1}(\lambda; x, t) = T^T(\lambda^*; x, t), \quad T^{-1}(\lambda; x, t) = T^T(-\lambda; x, t).
\end{equation}

*Proof.* From Lax pair (3) and its adjoint problem:

\begin{equation}
\begin{aligned}
-\Psi(x; t, \lambda) &= \Psi(x; t, \lambda) U(\lambda; u), \\
-\Psi^T(x; t, \lambda) &= \Psi(x; t, \lambda) V(\lambda; u),
\end{aligned}
\end{equation}

the fundamental solutions $\Phi(x; t, \lambda)$ and $\Psi(x; t, \lambda)$ satisfy $\Phi(0, 0; \lambda) = \Psi(0, 0; \lambda) = I$ and $\Psi(x; t, \lambda) \Phi(x; t, \lambda) = I$, i.e. $\Psi(x; t, \lambda) = \Phi^{-1}(x; t, \lambda)$. By the symmetric properties of the matrices $U(\lambda; u)$ and $V(\lambda; u)$ in equation (5), the following four equations $\Phi^{[1]}(x; t; \lambda^*) = -\Phi^T(x; t; \lambda^*) U(\lambda; u)$, $\Phi^{[1]}(x; t; \lambda^*) = -\Phi^T(x; t; \lambda^*) V(\lambda; u)$, $\Phi^{[1]}_x(x; t; -\lambda) = -\Phi^T(x; t; -\lambda) U(\lambda; u)$ and $\Phi^{[1]}_x(x; t; -\lambda) = -\Phi^T(x; t; -\lambda) V(\lambda; u)$ hold. Based on the existence and uniqueness theorem of the ordinary differential equation, we obtain

\begin{equation}
\Phi^{-1}(x, t; \lambda) = \Phi^T(x, t; -\lambda) \quad \text{and} \quad \Phi^{-1}(x, t; \lambda) = \Phi^T(x, t; \lambda^*).
\end{equation}

Similarly, the solution of equation $\Phi^{[1]}(x; t; \lambda) = U^{[1]}(\lambda; u^{[1]}) \Phi^{[1]}(x; t; \lambda)$ in equation (22) also satisfies $(\Phi^{[1]}(x; t; \lambda))^{-1} = (\Phi^{[1]}(x; t; -\lambda))^T$ and $(\Phi^{[1]}(x; t; \lambda))^{-1} = (\Phi^{[1]}(x; t; \lambda^*))^T$. Combining with the definition of $\Phi^{[1]}(x; t; \lambda)$ in equation (22), we get

\begin{equation}
T(\lambda; u) T^T(\lambda^*; u) = T(\lambda; u) \Phi(\lambda; u) \Phi^T(\lambda^*; u) T^T(\lambda^*; u) = \Phi^{[1]}(x; t; \lambda) \left(\Phi^{[1]}(x; t; \lambda^*)\right)^T = I,
\end{equation}

\begin{equation}
T(\lambda; u) T^T(-\lambda; u) = T(\lambda; u) \Phi(\lambda; u) \Phi^T(-\lambda; u) T^T(-\lambda; u) = \Phi^{[1]}(x; t; \lambda) \left(\Phi^{[1]}(x; t; -\lambda)\right)^T = I,
\end{equation}

which implies that the equation (23) holds.\qed

**Theorem 6.** To impose the symmetry (23), the Darboux matrix $T(\lambda; x, t)$ could be divided into following two types:

- If $\lambda_1 \in \mathbb{R}$ and $\Phi_1 \Phi_1^\dagger = (\Phi_1^\dagger \Phi_1)^\dagger$, the Darboux matrix could be written as

\begin{equation}
T^P(\lambda; x, t) = I - \frac{\lambda_1 - \lambda_1^*}{\lambda - \lambda_1^*} \Phi_1 \Phi_1^\dagger, \quad \Phi_1 := \Phi(x, t; \lambda_1) c_1 = \Phi(x, t; \lambda_1) [1, c_1]^T,
\end{equation}

with $c_1 \in \mathbb{C} \setminus \{0\}$;

- If $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$, the Darboux matrix could be written as

\begin{equation}
T^P(\lambda; x, t) = I - \begin{bmatrix} \Phi_1 & \Phi_1^* \end{bmatrix} M^{-1} D^{-1} \begin{bmatrix} \Phi_1^* \\ \Phi_1 \end{bmatrix}, \quad D = \text{diag} \left(\lambda - \lambda_1^*, \lambda + \lambda_1\right), \quad M = \begin{bmatrix} \Phi_1^\dagger \Phi_1 & \Phi_1^\dagger \Phi_1^* \\ \Phi_1^\dagger \Phi_1 & \Phi_1^\dagger \Phi_1^* \end{bmatrix},
\end{equation}

with $\Phi_1$ defined in (27).

The proof of Theorem 6 is given in [32], so we would not going to repeat it here.
Based on the elementary Darboux transformation $T_1^{p}(\lambda;x,t)$ or $T_1^{c}(\lambda;x,t)$, we can iterate it to obtain the multi-order ones, through iterating the Darboux transformation with different spectral parameters $\lambda_i$, $i=1, 2, \cdots, N$, called multifold Darboux matrix, which is a purely algebraic procedure. Combining above two conditions of the Darboux matrix, we could write the multifold Darboux matrix as a mixed product of $T_1^{p}(\lambda;x,t)$ and $T_1^{c}(\lambda;x,t)$:

$$
T^{[N]}(\lambda;x,t) = T_N^{p}(\lambda;x,t)T_{N-1}^{c}(\lambda;x,t) \cdots T_2^{p}(\lambda;x,t)T_1^{c}(\lambda;x,t), \quad J = P, C,
$$

where the matrices $T_i^{p}(\lambda;x,t), J = P, C, k = 1, 2, \cdots, N$ are defined in equations (27) and (28) through replacing spectral parameter $\lambda_1$ by $\lambda_k$. Define the numbers of $T_k^{p}(\lambda;x,t)$ in $T^{[N]}(\lambda;x,t)$ (31) as $n_1$ and the numbers of $T_k^{c}(\lambda;x,t)$ in $T^{[N]}(\lambda;x,t)$ (31) as $n_2$ with $N = n_1 + n_2$. Furthermore, combining with the Remark 2, the matrix $T^{[N]}(\lambda;x,t)$ could be seen as a $m$-fold Darboux matrix with $m = n_1 + 2n_2$. Then, we obtain the exact expression of multi elliptic-breather solutions as follows.

**Theorem 7.** Based on the Theorem 6, the $N$ elliptic-breather solutions could be expressed as

$$
u^{[N]}(x,t) = u(x,t) + 2iX_{m}M_{m}^{-1}(x,t)X_{m}^{\dagger},$$

where

$$M_m(x,t) = \left(\frac{\Phi_i^\dagger \Phi_j}{\lambda_j - \lambda_i}\right)_{1 \leq i,j \leq m}, \quad X_m = [\Phi_1, \Phi_2, \cdots, \Phi_m], \quad \Phi_i = \Phi(x,t;\lambda_i)[1, c_1]^\top.$$  

The dimension of $M_m(x,t)$ and $X_m$ depends on the transformation $T^{[N]}(\lambda;x,t)$ in (31) satisfying $N \leq m = n_1 + 2n_2 \leq 2N$, where $n_1, n_2$ are defined as the numbers of $T_k^{p}(\lambda;x,t)$ and $T_k^{c}(\lambda;x,t)$ in multi-Darboux matrix $T^{[N]}(\lambda;x,t)$ (31), respectively.

The proof of Theorem 7 is provided in Appendix B. In order to explain the dimension $m$ more clearly, we give an example on $u^{[2]}(x,t)$. By equation (31), it is easy to know that the Darboux matrix $T^{[2]}(\lambda;x,t)$ has three types:

- If $T^{[2]}(\lambda;x,t) = T_2^{p}(\lambda;x,t)T_1^{p}(\lambda;x,t)$, i.e., $n_1 = 2, n_2 = 0$, we know $N = n_1 + n_2 = 2$ and $m = n_1 + 2n_2 = 2$, which leads to $M_2(x,t)$ is $2 \times 2$ and $X_2$ is $2 \times 2$ with $\lambda_1 = \lambda(z_1) \in \mathbb{R}$, $\lambda_2 = \lambda(z_2) \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$.
- If $T^{[2]}(\lambda;x,t) = T_2^{c}(\lambda;x,t)T_1^{p}(\lambda;x,t)$ or $T^{[2]}(\lambda;x,t) = T_2^{p}(\lambda;x,t)T_1^{c}(\lambda;x,t)$, we obtain $n_1 = n_2 = 1$, $N = n_1 + n_2 = 2$ and $m = n_1 + 2n_2 = 3$, which leads to $M_3(x,t)$ is $3 \times 3$ and $X_3$ is $2 \times 3$ with $\lambda_1 = \lambda(z_1) \in \mathbb{R}$, $\lambda_2 = \lambda(z_2) \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_3 = \lambda(2I - z_2^*)$ or $\lambda_1 = \lambda(z_1) \in \mathbb{C} \setminus \mathbb{R}$, $\lambda_2 = \lambda(2I - z_1^*)$, $\lambda_3 = \lambda(z_2)$, $\lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$.
- If $T^{[2]}(\lambda;x,t) = T_2^{c}(\lambda;x,t)T_1^{c}(\lambda;x,t)$, we get that $M_4(x,t)$ is $4 \times 4$ and $X_4$ is $2 \times 4$ with $n = 2$, $m = 4$, $\lambda_1 = \lambda(z_1), -\lambda_1^* = \lambda(2I - z_1^*), \lambda_2 = \lambda(z_2), -\lambda_2^* = \lambda(2I - z_2^*)$, satisfying $\lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$.

Based on above fundamental analysis of the multi-breather solutions, we aim to calculate the explicit expression of them.
2.2 The explicit expression of solutions

Using the theta functions to express the exact multi elliptic-breather solution \( u^{[N]}(x,t) \) is our main purpose in this subsection. Firstly, we use the theta functions to express \( \lambda(z) \), which is useful in the studies of exact expressions for \( u^{[N]}(x,t) \).

Lemma 1. The function \( \lambda(z_j) - \lambda(z_j^*) \) defined in equation (10) with different value of \( l \) could rewrite by a uniform expression:

\[
\lambda(z_j) - \lambda(z_j^*) = \frac{2 \theta_3(\frac{2\pi i}{2\kappa})}{\theta_2(\frac{2\pi i}{2\kappa})} \left[ \frac{\theta_2(\frac{i(z_j-z_j^*)}{2\kappa})}{\theta_3(\frac{i(z_j+z_j^*)}{2\kappa})} \right]^{i(z_j-z_j^*)}, \quad l = 0, \quad l = \frac{K'}{2}.
\]

Proof. Firstly, we consider the condition \( l = 0 \). To simplify the notation and facilitate analysis, we set \( i(z_j - l) \) as a whole variable \( x \) and \( i(z_j^* - l) \) as a constant \( \beta \). Combining with equation (10a), we get

\[
\lambda(z_j) - \lambda(z_j^*) = \frac{2 \theta_3(\frac{2\pi i}{2\kappa})}{\theta_2(\frac{2\pi i}{2\kappa})} \frac{\theta_2(\frac{i(z_j-z_j^*)}{2\kappa})}{\theta_3(\frac{i(z_j+z_j^*)}{2\kappa})}, \quad l = 0.
\]

Plugging \( i(z_j - l) = x = 0 \) into (35) and (37), we get

\[
\frac{ia}{2} \frac{\text{sn}(x)\text{cn}(x)\text{cn}(\beta) - \text{dn}(\beta)\text{sn}(x)\text{cn}(\beta)}{\text{cn}(x)\text{cn}(\beta)} = C \frac{\theta_1(\frac{x-\beta}{2\kappa})\theta_3(\frac{x+\beta}{2\kappa})}{\theta_2(\frac{2\pi i}{2\kappa})\theta_4(\frac{2\pi i}{2\kappa})} \theta_4(\frac{i(z_j-z_j^*)}{2\kappa}) \theta_3(\frac{i(z_j+z_j^*)}{2\kappa}), \quad l = 0.
\]

which implies \( C = \frac{ia}{2} \frac{\text{sn}(z_j^*)\text{dn}(z_j^*)}{\text{cn}(z_j^*)} \), by the conversion formulas (A.10). Therefore, the equation (34) holds, at \( l = 0 \).

Similarly, when \( l = \frac{K'}{2} \), by equation (10b), we get

\[
\lambda(z_j) - \lambda(z_j^*) = \frac{ia}{2} \frac{\text{sn}(z_j^*)\text{dn}(z_j^*)}{\text{cn}(z_j^*)} = C \frac{\theta_1(\frac{z_j^*}{2\kappa})\theta_3(\frac{z_j^*}{2\kappa})}{\theta_2(\frac{2\pi i}{2\kappa})\theta_4(\frac{2\pi i}{2\kappa})} \theta_4(\frac{i(z_j-z_j^*)}{2\kappa}) \theta_3(\frac{i(z_j+z_j^*)}{2\kappa}), \quad l = \frac{K'}{2}.
\]

by the Liouville theorem. Substituting \( i(z_j - l) = x = 0 \) into equations (39) and (41), we obtain the equation (34).
Based on the Theorem 7 and Sherman-Morrison-Woodbury-type matrix identity, the elliptic-breather function

\[
\begin{align*}
&u^{[N]}(x, t) = u(x, t) + 2i \left[ \Phi_{1,1} \quad \Phi_{2,1} \quad \cdots \quad \Phi_{m,1} \right] M_m^{-1}(x, t) \left[ \Phi_{1,2}^* \quad \Phi_{2,2}^* \quad \cdots \quad \Phi_{m,2}^* \right], \\
&= \frac{u^{l-m}(x, t)}{\det(M_m(x, t))} \det \left( uM_m(x, t) - 2i \begin{bmatrix} \Phi_{1,2}^* & \Phi_{2,2}^* & \cdots & \Phi_{m,2}^* \end{bmatrix} \right).
\end{align*}
\]

is a solution of mKdV equation (1), where \( u(x, t) \) is the elliptic function solution of mKdV equation (1) and \( \Phi_l \) is defined in (27). Combining the matrix function \( \Phi(x, t; \lambda) \) in (7) and addition formulas of the theta functions in (A.12), we obtain the following functions directly:

\[
\Phi_{i,1}^* = -\frac{\alpha^2 \theta_2^2 \theta_4^2}{\theta_3^2 \theta_4^2} e^{-\alpha \xi Z (2l+K)} \left( \frac{\theta_3(\frac{i(z_i-l)+a\xi}{2K})}{\theta_2(\frac{i(z_i-l)+a\xi}{2K})} E_1^*(z_i) - \frac{\theta_1(\frac{i(z_i-l)-a\xi}{2K})}{\theta_4(\frac{i(z_i-l)-a\xi}{2K})} c_i^* E_2^*(z_i) \right) \cdot
\]

\[
\left( \frac{\theta_1(\frac{i(z_i-l)-a\xi}{2K})}{\theta_4(\frac{i(z_i-l)-a\xi}{2K})} E_1(z_i) + \frac{\theta_3(\frac{i(z_i+l)+a\xi}{2K})}{\theta_2(\frac{i(z_i+l)+a\xi}{2K})} c_j E_2(z_i) \right)
\]

\[
= -\frac{\alpha^2 \theta_2^2 \theta_4^2}{\theta_3^2 \theta_4^2} \left[ E_1^*(z_i) \quad c_i^* E_2^*(z_i) \right] \cdot
\]

\[
\left[ \begin{array}{cc}
\theta_1(\frac{i(z_i-l)+a\xi}{2K}) & \theta_2(\frac{i(z_i-l)+a\xi}{2K}) \\
\theta_3(\frac{i(z_i+l)+a\xi}{2K}) & \theta_4(\frac{i(z_i+l)+a\xi}{2K})
\end{array} \right] E_1(z_i) \quad c_j E_2(z_i),
\]

with \( A = e^{-\alpha \xi Z (2l+K)} \). Through using the addition and shift formulas of the Jacobi theta functions in (A.9) and (A.12), we obtain

\[
\Phi_{i,1}^* + \Phi_{i,2}^* = \frac{\alpha^2 \theta_2^2 \theta_4^2}{\theta_3^2 \theta_4^2} \left[ E_1^*(z_i) \quad c_i^* E_2^*(z_i) \right] \cdot
\]

\[
\left[ \begin{array}{cc}
\theta_1(\frac{i(z_i-l)+a\xi}{2K}) & \theta_2(\frac{i(z_i-l)+a\xi}{2K}) \\
\theta_3(\frac{i(z_i+l)+a\xi}{2K}) & \theta_4(\frac{i(z_i+l)+a\xi}{2K})
\end{array} \right] E_1(z_i) \quad c_j E_2(z_i),
\]

where \( B = \frac{\theta_2(\frac{i(z_i+l)+a\xi}{2K}) \theta_4(\frac{i(z_i+l)-a\xi}{2K})}{\theta_2(\frac{i(z_i+l)+a\xi}{2K}) \theta_3(\frac{i(z_i+l)-a\xi}{2K}) \theta_4(\frac{i(z_i+l)+a\xi}{2K})}. \) Therefore, combining with equation (34), we gain

\[
\frac{\Phi_m^*}{2(\lambda_j - \lambda_j^*)} = -\frac{i \alpha \theta_2 \theta_4}{\theta_3 \theta_4(\frac{a\xi}{2K})} \left[ E_1^*(z_i) \quad c_i^* E_2^*(z_i) \right] \cdot
\]

\[
\left[ \begin{array}{cc}
\theta_1(\frac{i(z_i-l)+a\xi}{2K}) & \theta_2(\frac{i(z_i-l)+a\xi}{2K}) \\
\theta_3(\frac{i(z_i+l)+a\xi}{2K}) & \theta_4(\frac{i(z_i+l)+a\xi}{2K})
\end{array} \right] E_1(z_i) \quad c_j E_2(z_i),
\]
Furthermore, by formulas (A.9) and (A.12), we also could obtain

\[ u \frac{\Phi^*_j \Phi_j}{2(\lambda_j - \lambda_i^*)} - i\Phi_{i,j}^* \Phi_{i,j} \]

\[ = -\frac{i \alpha^2 \theta_3^2 \theta_4^2}{\theta_3^2 \theta_4^2 (\frac{\alpha + 2l_1}{2K})} \left[ E_1^* (z_i) c_i^* E_2^* (z_i) \right] \]

\[ = \frac{\alpha^2 \theta_3^2 \theta_4^2 (\frac{2l}{2K})}{\theta_3^2 \theta_4^2 (\frac{\alpha + 2l_1}{2K})} \left[ r_i E_1^* (z_i) r_i^* c_i E_2^* (z_i) \right] \]

with \( A = e^{-\alpha^2 Z(2l + K)} \). Collecting equations (42), (45) and (46), the function (12) holds. Thus, the Theorem 1 is obtained.

**Remark 3.** Combining the Theorem 1, solution (12) and the shift formula (A.9), we could rewrite it as

\[ u^{[N]}(x,t) = \frac{\alpha^2 \theta_3^2 \theta_4^2}{\theta_3^2 \theta_4^2 (\frac{\alpha + 2l_1}{2K})} \left( \frac{\theta_4^2 (\frac{\alpha + 2l_1}{2K})}{\theta_2^2 (\frac{\alpha + 2l_1}{2K})} \right)^{m-1} \left( \frac{\theta_3^2 (\frac{\alpha + 2l_1}{2K})}{\theta_3^2 (\frac{\alpha + 2l_1}{2K})} \right)^{m-1} \]

\[ = \frac{\alpha^2 \theta_3^2 \theta_4^2 (\frac{2l}{2K})}{\theta_3^2 \theta_4^2 (\frac{\alpha + 2l_1}{2K})} \left[ r_i E_1^* (z_i) r_i^* c_i E_2^* (z_i) \right] \]

where \( \xi = x - st \).

\[ D^{[h,n]} = \left( \frac{\theta_4 (\frac{\alpha + 2l_1}{2K})}{\theta_1 (\frac{\alpha + 2l_1}{2K})} \right), \quad M^{[h,n]} = \left( \frac{\theta_2 (\frac{\alpha + 2l_1}{2K})}{\theta_1 (\frac{\alpha + 2l_1}{2K})} \right) \]

\[ X_1 := \text{diag} \left( e^{-\eta_1}, e^{-\eta_2}, \ldots, e^{-\eta_{h-1}}, 1, 1 \right), \quad X_2 := \text{diag} \left( 1, 1, e^{\eta_h}, e^{\eta_{h+1}}, \ldots, e^{\eta_m} \right), \]

or

\[ X_1 := \text{diag} \left( 1, 1, e^{\eta_h}, e^{\eta_{h+1}}, \ldots, e^{\eta_m} \right), \quad X_2 := \text{diag} \left( e^{\eta_1}, e^{\eta_2}, \ldots, e^{\eta_{h-1}}, 1, 1 \right), \]

with

\[ Y^{[1]}_1 = Y, \quad Y^{[1]}_2 = Y^{-1}, \quad Y^{[2]}_1 = Y^{-1}, \quad Y^{[2]}_2 = Y, \]

\[ Y := \exp \left( \frac{l \pi}{2K} \right) \text{diag} \left[ r_1, r_2, \ldots, r_m \right], \]

and

\[ \eta_i := \gamma_i + l_i \xi - \Omega_i t, \quad l_i := - (\alpha Z(i(z_i + l) + K + K') + \alpha Z(i(z_i - l))), \quad \Omega_i := 8i \lambda_i y_i, \quad \gamma_i := \ln c_i. \]
Furthermore, we list the concrete sequence of $\zeta^{[i]}$ and $\mu^{[l]}$ as:

\[
\begin{align*}
\zeta^{[1]} &= \left( \zeta_1^{[1]}, \zeta_2^{[1]}, \ldots, \zeta_{m}^{[1]} \right) = (iZ_1, iZ_2, \ldots, iZ_n), \\
\zeta^{[2]} &= \left( \zeta_1^{[2]}, \zeta_2^{[2]}, \ldots, \zeta_{m}^{[2]} \right) = (-iZ_1 - (K + K') - iZ_2 - (K + K'), \ldots, -iZ_n - (K + K')), \\
\mu^{[1]} &= \left( \mu_1^{[1]}, \mu_2^{[1]}, \ldots, \mu_{m}^{[1]} \right) = (-iZ_1, -iZ_2, \ldots, -iZ_n), \\
\mu^{[2]} &= \left( \mu_1^{[2]}, \mu_2^{[2]}, \ldots, \mu_{m}^{[2]} \right) = (iZ_1 + K + K', iZ_2 + K + K', \ldots, iZ_n + K + K'),
\end{align*}
\]

(53)

where $Z_i = z_i$, if $\lambda(z_i) \in i\mathbb{R}, c_i \in \mathbb{R}$ or $Z_i = [z_i, 2l - z_i^*]$, if $\lambda(z_i) \in \mathbb{C}\setminus i\mathbb{R}$.

For any positive integer $N$, we could have infinitely many solutions through choosing the different value of uniform parameters $z_i \neq z_j, z_i \in \mathbb{S}, i = 1, 2, \ldots, N$. The exact expressions of $N$ elliptic-breather solutions are obtained by equation (12) or equation (47). Then, we just list some elliptic-breather solutions $u^{[N]}(x, t)$ under the different elliptic function backgrounds with different parameter $l$ in following subsection.

### 2.3 The dynamics for multi elliptic-breather solutions

The main difference of multi elliptic-breather solutions for the cn-type and dn-type background is in the choice of parameters $l = 0$ or $l = K' / 2$. Based on the Theorem 7, there are many different cases that can be obtained in equation (12) for the multi elliptic-breather solutions. Here, we mainly focus on the single breather solutions and two breather solutions.

Based on the expression of elliptic-breather solutions (32), we know that there are two cases to obtain the single elliptic-breather solution, which is divided by the parameter $\lambda(z)$.

**Case I-I:** If $\lambda_1 = \lambda(z_1) \in i\mathbb{R}$ and $c_1 \in \mathbb{R}$ with one-fold Darboux transformation $T^{[1]}(\lambda; x, t) = T_{\lambda}^{[1]}(\lambda; x, t)$, the single breather solution is

\[
u^{[1]}(x, t) = \frac{\alpha \theta_2 \theta_4 A}{\theta_3 \theta_5(2K)} \left[ -r_1 \theta_2 (\frac{2(\zeta_1 + 2l)}{2K}) - \frac{1}{r_1} \theta_4 (\frac{2(\zeta_1 + 2l)}{2K}) \right] E_1 + r_1 \theta_1 \theta_3 (\zeta_1) E_2 + \frac{\theta_4}{\theta_3 (\zeta_1)} |E_1|^2,
\]

(54)

where $A = \exp(-\alpha \xi Z(2l + K)), \xi = x - st, E_1 = \frac{\theta_2(c_1)}{\theta_3(c_1)}$. The functions $E_1(z)$ and $E_2(z)$ are defined in equation (8) and $r_1$ is defined in (14).

**Case I-II:** If $\lambda(z_1) \notin i\mathbb{R}$, the single breather solution could be obtained by two-fold Darboux transformation $T^{[1]}(\lambda; x, t) = T_{\lambda}^{[2]}(\lambda; x, t)$. By Theorem 7 and Remark 3, the matrices $\mathcal{M}$ and $\mathcal{D}$ are $2 \times 2$ with the parameters $z_1, -z_1^* + 2l$ in (12) or $[Z_1] = [z_1, -z_1^* + 2l]$ in (53).

Following, we give three exact examples on the single breather solutions under the cn-type background, i.e., $l = 0$. Firstly, taking $z_1 = K' + \frac{K'}{2} i, k = \frac{1}{2}, \alpha = 1$ and $l = 0$ into equation (10a), we get $\lambda_1 = \lambda(z_1) \approx -0.836 i \in i\mathbb{R}$. Therefore, plugging them into solution $u^{[1]}(x, t)$ in equation (54), we obtain a single elliptic-breather solution under the cn-type background in Figure 1(a). Through equation (32), we could obtain that the range of solution $u^{[1]}(x, t) = u(x, t) + 2i(\lambda_1 - \lambda_1)$ is in the interval $[\min(u(x, t)) - 2|\lambda_1|, \max(u(x, t)) + 2|\lambda_1|]$. And, only when $\frac{\Phi_1^1}{\Phi_1^2} = \pm 1$, solution $u^{[1]}(x, t)$ could get above maximum or minimum value. After numerical calculations, we know that the maximum value of the function $u^{[1]}(x, t)$ reaches at the origin point $(0, 0)$. Therefore, the maximum value of the function $u^{[1]}(x, t)$ is approximately equal to 2.172 with $\lambda_1 \approx -0.836 i$.

Then, we provide two conditions of the Case I-II. Selecting the parameters $l = 0, \alpha = 1, k = \frac{1}{2}, z_1 = 3K' + \frac{K'}{4} i$ and plugging them into equations (10a) and (12), we obtain the singe elliptic-breather solution $u^{[1]}(x, t)$ under a two-fold Darboux transformation with the spectral parameter $\lambda_1 = \lambda(z_1) \approx -0.504 - 0.429 i \notin i\mathbb{R}$ in Case I-II, and draw the 3d-plot figure in Figure 1(b). In addition, we also display a single elliptic-breather solution whose velocity is zero, through plugging $l = 0, \alpha = 1, k = \frac{1}{2}, c_1 = 1.375 - 1i$ and $z_1 = \frac{K}{2} + i\frac{K}{2}$.
into equation (10a) and (12). Therefore, this singe elliptic-breather solution \( u^{[1]}(x,t) \) is obtained under a two-fold Darboux transformation with the spectral parameter \( \lambda_1 \approx -0.368 - 0.257i \) in the Figure 1(c).

**Figure 1.** The 3d-plot of solutions \( u^{[1]}(x,t) \) for mKdV equation (1), under the cn-type background. (a): a single elliptic-breather solution with a one-fold Darboux transformation. (b), (c): single elliptic-breather solutions under the two-fold Darboux transformation.

Then, we list three exact examples of the single elliptic-breather solutions under the dn-type background, i.e., \( l = \frac{K'}{2} \). Substituting \( l = \frac{K'}{2}, \alpha = 1, k = \frac{9}{10}, z_1 = \frac{K'}{2} + \frac{2K}{3}i \) into equation (10b), we find \( \lambda(z_1) \approx -0.269i \in i\mathbb{R} \) and obtain a solution \( u^{[1]}(x,t) \) in equation (54). Thus, a single elliptic-breather solution \( u^{[1]}(x,t) \) under the dn-type background in Figure 2(a) reaches its maximum value approximately equal to 1.539 at the origin point \( (x,t) = (0,0) \). For \( z_1 = -\frac{K'}{2} + i \frac{K}{3}, \) i.e., \( \lambda(z_1) \approx -1.189i \), similarly as above, we obtain a single elliptic-breather solution \( u^{[1]}(x,t) \) drawn in Figure 2(b) with the maximum value approximately at 3.378 reaching at the origin point \( (x,t) = (0,0) \). Comparing the Figure 2(a) with Figure 2(b), we find that the parameter \( \lambda_1 \) is an important factor leading to the variation of peaks.

For the Case I-II under the dn-type background, we plug parameters \( l = \frac{K'}{2}, \alpha = 1, k = \frac{9}{10} \) and \( z_1 = -\frac{K'}{2} + \frac{2K}{3}i \) into equation (10b) and (12). It follows that the singe elliptic-breather solution \( u^{[1]}(x,t) \) under a two-fold Darboux transformation with the spectral parameter \( \lambda_1 \approx 0.103 - 0.535i \) can be obtained (see the Figure 1(c)), which shows the single elliptic-breather solution with the zero velocity.

**Figure 2.** The 3d-plot of solutions \( u^{[1]}(x,t) \) for mKdV equation (1) under the dn-type background. (a), (b): single elliptic-breather solutions under the one-fold Darboux transformation. (c): a single elliptic-breather solution under a two-fold Darboux transformation.

And then, we enumerate some two elliptic-breather solutions by (12). The conditions of two elliptic-breather solutions are more complicated than the single ones. By the expression of solution (32) and the Darboux matrix in equation (31), we find that the two breathers solution have three different types. Above three cases are divided by the different times fold of the Darboux transformation. Now, we show this three conditions as follows:
Case II-I: If \( \lambda(z_1), \lambda(z_2) \in i\mathbb{R} \) and \( c_1, c_2 \in \mathbb{R} \), the two elliptic-breather solution \( u^{(2)}(x,t) \) in (12) could be obtained by the two-fold Darboux transformation \( T^{(2)}(\lambda; x, t) = T^D_1(\lambda; x, t) T^D_2(\lambda; x, t) \), which reflects that matrices \( M \) and \( D \) are two by two with parameters \( |Z_1, Z_2| = |z_1, z_2| \) in equation (53).

Case II-II: If \( \lambda(z_1) \in i\mathbb{R}, \lambda(z_2) \notin i\mathbb{R} \) and \( c_1 \in \mathbb{R}, c_2 \in \mathbb{C} \), the two elliptic-breather solution \( u^{(2)}(x,t) \) could be obtained from the three-fold Darboux transformation \( T^{(2)}(\lambda; x, t) = T^D_1(\lambda; x, t) T^D_2(\lambda; x, t) T^C_1(\lambda; x, t) \). Changing the conditions of above two parameters, i.e. \( \lambda(z_1) \notin i\mathbb{R}, \lambda(z_2) \in i\mathbb{R} \) and \( c_1 \in \mathbb{C}, c_2 \in \mathbb{R} \), the two elliptic-breather solution could be obtained by \( T^{(2)}(\lambda; x, t) = T^D_1(\lambda; x, t) T^C_2(\lambda; x, t) T^C_1(\lambda; x, t) \). From (12), the matrices \( M \) and \( D \) in above two cases are both \( 3 \times 3 \), with parameters \( |Z_1, Z_2| = |z_1, z_2, -z_2^* + 2l i| \) or \( |Z_1, Z_2| = |z_1, -z_1^* + 2l, z_2| \) in (53).

Case II-III: If \( \lambda(z_1), \lambda(z_2) \notin i\mathbb{R} \) and \( c_1, c_2 \in \mathbb{C} \), the two elliptic-breather solution is obtained from the four-fold Darboux transformation \( T^{(2)}(\lambda; x, t) = T^D_1(\lambda; x, t) T^C_2(\lambda; x, t) T^C_1(\lambda; x, t) \), which implies that matrices \( M \) and \( D \) are \( 4 \times 4 \) with \( |Z_1, Z_2| = |z_1, -z_1^* + 2l, z_2, -z_2^* + 2l| \) in (53).

We would present some exact elliptic-breather solutions for above three cases in the context of cn-type and dn-type background. Under the cn-type background, i.e. \( l = 0 \), by parameters \( k = \frac{1}{2}, z_1 = K' + \frac{2K}{3} i \) and \( z_2 = K' + \frac{2K}{3} i, c_1 = c_2 = 1 \), the two elliptic-breather solution under the two-fold Darboux transformation is drawn in Figure 3(a), with \( \lambda_1 = \lambda(z_1) \approx -1.301 i \in i\mathbb{R} \) and \( \lambda_2 = \lambda(z_2) \approx -0.836 i \in i\mathbb{R} \) satisfying the Case II-I. If \( z_1 = \frac{3K'}{4} + \frac{K}{5} i, z_2 = K' + \frac{2K}{3} i, c_1 = c_2 = 1 \) and \( k = \frac{1}{2} \), a two elliptic-breather solution \( u^{(2)}(x,t) \) under the three-fold Darboux transformation is drawn in Figure 3(b), because \( \lambda_1(\lambda_2) \approx -0.504 - 0.429i \notin i\mathbb{R} \), \( \lambda_2(\lambda_1) \approx -0.673 i \in i\mathbb{R} \) and \( c_1 = c_2 = 1 \) satisfy the conditions of Case II-II. Plugging \( k = \frac{1}{2}, \alpha = 1, z_1 = -\frac{3K'}{4} + \frac{K}{5} i, z_2 = K' + \frac{2K}{3} i, c_1 = 3 + 4i \) and \( c_2 = 1.375 - i \) into equation (32), we get a two elliptic-breather solutions under a four-fold Darboux transformation in Figure 3(c) by \( \lambda_1 \approx 0.626 - 0.427i \notin i\mathbb{R} \) and \( \lambda_2 \approx -0.368 - 0.257i \notin i\mathbb{R} \), which satisfy the conditions of the Case II-III.

For the dn-type case, i.e. \( l = \frac{K'}{2} \), we choose \( z_1 = \frac{K'}{2} + \frac{2K}{3} i, z_2 = -\frac{K'}{2} + \frac{K}{5} i, c_1 = c_2 = 1 \) to gain a two elliptic-breather solution by the Case II-I, since \( \lambda_1 \approx 0.269i \notin i\mathbb{R} \) and \( \lambda_2 \approx -1.189i \notin i\mathbb{R} \) and \( c_1 = c_2 = 1 \in \mathbb{R} \). By choosing parameters \( k = \frac{9}{10}, \alpha = 1 \), we construct the two breather solution by a two-fold Darboux transformation and plot it in Figure 4(a). The different peaks of two breathers in Figure 4(a) are mainly determined by parameters \( \lambda_1 \) and \( \lambda_2 \). Letting \( z_1 = -\frac{K'}{8} + \frac{2K}{3} i, z_2 = -\frac{K'}{8} + \frac{K}{5} i, c_1 = c_2 = 1, \alpha = 1 \) with \( \lambda_1 \approx 0.103 - 0.535i \notin i\mathbb{R} \) and \( \lambda_2 \approx -1.189i \notin i\mathbb{R} \), we obtain a elliptic-breather solution \( u^{(2)}(x,t) \) in equation (54) and plot it in Figure 4(b). When \( z_1 = -\frac{K'}{8} + \frac{2K}{3} i, z_2 = -\frac{K'}{8} + \frac{K}{5} i, \lambda_1 \approx 0.143 - 0.737i, \lambda_2 \approx 0.103 - 0.535i \), the elliptic-breather solution \( u^{(2)}(x,t) \) could be obtained based on the Case II-III under the four-fold Darboux transformation in Figure 4(c) through utilizing the parameters \( l = \frac{K'}{2}, \alpha = 1, k = \frac{9}{10} \) and \( c_1 = c_2 = 1 \). The Figure 4(i), (ii) and (iii) show the corresponding spectral parameters \( \lambda_1 \) and \( \lambda_2 \) of the above two breather solutions in Figure 4(a), (b) and (c), respectively.

In this section, we firstly show the formulas for the multi elliptic-breather solution of mKdV equation under the elliptic function background. Then the single breathers and two-breathers are exhibited with the aid of the computer graph. In what follows, we would like to describe the asymptotic dynamics of multi elliptic-breather solutions systematically.

### 3 Asymptotic analysis for the multi elliptic-breather solutions

In the last section, we have obtained the general formulas for the multi elliptic-breather solutions. In this section, we would like to analyze a special type of multi elliptic-breather solutions by the asymptotic analysis, which can be regarded as the analog of multi-soliton.

#### 3.1 The symmetry of multi-breather solutions

In this subsection, we aim to introduce the symmetry of multi elliptic-breather solutions, which is an important property of the elliptic-breather solution \( u^{[N]}(x,t) \) satisfying \( u^{[N]}(x,t) = u^{[N]}(-x,-t) \), since the symmetry of functions can bring us great convenience in studying the dynamic behaviors. For the breathers under the elliptic function background, it is hard to discriminate the elastic interaction or not. But if the
multi elliptic-breather solutions has the symmetry \( u^{[N]}(x, t) = u^{[N]}(-x, -t) \), we can claim that the collisions among them are elastic.

Based on the expression of solution \( \Phi(x, t; \lambda) \) (7) of the Lax pair (3), the elliptic-breather solution \( u^{[N]}(x, t) \) has the symmetry \( u^{[N]}(x, t) = u^{[N]}(-x, -t) \), if it satisfies the conditions \( c_i = 1, i = 1, 2, \cdots, m \) given in the Theorem 2.

**Proof of Theorem 2.** By the formula (7) and \( E^{-1}_2(x, t) = \exp(a\xi Z(2il + K))E_1(x, t) \) (obtained in [32]), it is easy to verify that

\[
\Phi(-x, -t; \lambda) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \Phi(x, t; \lambda) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

which implies

\[
\Phi_i(-x, -t) \equiv \begin{bmatrix} \Phi_{i,1}(-x, -t) \\ \Phi_{i,2}(-x, -t) \end{bmatrix} = \begin{bmatrix} -\Phi_{i,2}(x, t) \\ -\Phi_{i,1}(x, t) \end{bmatrix}, \quad \Phi_i(x, t) \equiv \Phi(x, t; \lambda_i) \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]

and

\[
(\Phi_i^t(-x, -t)\Phi_j(-x, -t))^t = \Phi_i^t(-x, -t)\Phi_i(-x, -t) = \Phi_j^t(x, t)\Phi_i(x, t).
\]

Considering the formula \( u^{[N]}(x, t) \) in equation (42) and combining equations (56) with (57), we get that the symmetry \( u^{[N]}(-x, -t) = u^{[N]}(x, t) \) holds.

From the Theorem 2, we know that the dynamic behavior is consistent at time \( t \) and \(-t\), because of \( u^{[N]}(x, t) = u^{[N]}(-x, -t) \). Thus, the collision dynamics between the breathers are elastic, which means that the shape of breathers do not change after the collision. Two typical examples of the above elastic collisions.
Figure 4. The 3d-plot of two-breather solutions $u^{[2]}(x,t)$ for mKdV equation (1) and the corresponding spectral parameters $\lambda_1$, $\lambda_2$, under the dn-type background. The solution $u^{[2]}(x,t)$ in figures (a), (b) and (c) are obtained by two-fold, three-fold and four-fold Darboux transformations, respectively. The red curves in figure (i), (ii), (iii) are the cuts in $\lambda$-plane and the green and blue points are spectral points $\lambda_1$ and $\lambda_2$, respectively.

are shown in Figure 5 and Figure 6. In addition to elastic collisions, there are many solutions that are not easily distinguishable from elastic collisions, as shown in Figure 7. Then, we will describe each of the above situations.

Plugging $k = \frac{1}{2}$, $l = 0$, $z_1 = K' + i\frac{K}{2}$, $z_2 = K' + i\frac{2K}{3}$, $c_1 = c_2 = 1$ into equation (42), two elliptic-breather solution $u^{[2]}(x,t)$ obtained by two-fold Darboux transformation satisfies the Case II-I under the cn-type background. The Figure 5(a) shows the density evolution of solution $u^{[2]}(x,t)$ and the Figure 3(a) is the 3d-plot of this solution. Two breathers of $u^{[2]}(x,t)$ in Figure 5(a) collide at the moment $t = 0$. By the symmetry of solution $u^{[2]}(x,t)$ proved in the Theorem 2, we know that the behaviors of above two breathers in solution $u^{[2]}(x,t)$ are same at the corresponding moments before and after the collision. Here, $t = 5$, for example, we plot the sectional view of functions $u^{[2]}(x, \pm 5)$, which is shown in Figure 5(b). The graphs $u^{[2]}(x, 5)$ and $u^{[2]}(x, -5)$ reflect the variation before the collision ($t = -5$) and after the collision ($t = 5$) in the Figure 5(b). Above variation clearly depicts the same breathers in above two positions. Furthermore, in the downmost figure of Figure 5(b), we know that the above two functions are completely coincident, which provide a more vivid image of the elastic collision.

Similarly, the Figure 6 also reflects an elastic collision, which is obtained by three-fold Darboux transformation under the dn-type background with parameters $k = \frac{9}{10}$, $l = \frac{K'}{2}$, $z_1 = -\frac{K'}{3} + i\frac{2K}{3}$, $z_2 = -\frac{K'}{3} + i\frac{2K}{3}$, $c_1 = c_2 = 1$. The Figure 6(a) is a density evolution of solution $u^{[2]}(x,t)$ showing the collision between two breathers at the time $t = 0$. And the 3d-plot of solution $u^{[2]}(x,t)$ is shown in Figure 4(b). In the Figure 6(b), we draw two plane graphs of functions $u^{[2]}(x, 6)$ and $u^{[2]}(x, -6)$ which reflect the variation before ($t = -6$)
and after \((t = 6)\) the collision. The lowest figure of Figure 6(b) shows a completely consistence between 
\(u^{[2]}(x, 6)\) and \(u^{[2]}(-x, -6)\).

Figure 5. The solution \(u^{[2]}(x, t)\) with two-fold Darboux transformation under the \(cn\)-type background, describes an elastic collision between two bound-state breathers.

Figure 6. The solution \(u^{[2]}(x, t)\) with three-fold Darboux transformation under the \(dn\)-type background, describes an elastic collision between two bound-state breathers.

In addition to above elastic collision solutions, there are many multi elliptic-breather solutions that do not satisfy the condition \(c_i = 1, i = 1, 2, \cdots, m\), which are not easy to discriminate whether it is an elastic collision or not (see in Figure 7). The solution \(u^{[2]}(x, t)\) under a four-fold Darboux transformation is obtained by parameters \(k = \frac{1}{3}, l = 0, z_1 = -\frac{3K'}{4} - i\frac{K}{4}, z_2 = \frac{K'}{4} + i\frac{K}{4}, c_1 = 3 + 4i\) and \(c_2 = 1.3754 - i\). The 3d-plot of this solution is shown in Figure 3(c). After calculating, we know that this two breathers collide with each other at the time \(t = 0\). The Figure 7(a) shows a density evolution of solution \(u^{[2]}(x, t)\). By Figure 7(a), we can clearly see the collision between two breathers. Through choosing two moments before \((t = -15)\) and after \((t = 15)\) the collisions and drawing the plane graphs in Figure 7(b), we intuitively get the variation before and after the collision for functions \(u^{[2]}(x, \pm 15)\). The Figure 7(b) clearly depicts the significant difference between the before collision \(u^{[2]}(x, -15)\) and after collision \(u^{[2]}(x, 15)\) of breathers, which is hard to discriminate whether it is elastic or not.
3.2 The variation of the velocity

From the figures and expressions of solution \( u^{[N]}(x,t) \) in sections 2, we found that the breathers on solution \( u^{[N]}(x,t) \) always have different velocities. Therefore, we want to consider the relationship between the breathers and the velocities. For the velocity of soliton solution, it is easy to obtain the expressions of velocity in a polynomial form. But for the breathers on the background of elliptic functions, the velocity of expressions are given by the harmonic functions with respect to the spectral parameter. In this subsection, we first study the relationship between velocity \( v \) and uniform parameter \( z \), and then get the relationship between the velocity \( v \) and spectral parameter \( \lambda \) based on the conformal mapping \( \lambda(z) \).

For the convenience of analysis, we will make a rotation \( \tilde{\zeta} = x - st \) on the solution \( u^{[N]}(x,t) \) and define it as

\[
\hat{u}^{[N]}(\tilde{\zeta}, t) = \frac{u^{[N]}(x+st,t)-u^{[N]}(x-t,0)}{u^{[N]}(x+st,t)-u^{[N]}(x-t,0)}.
\]

In this section, we mainly consider the function \( \hat{u}^{[N]}(\tilde{\zeta}, t) \) under the \( \tilde{\zeta} \) and \( t \) axis.

**Definition 1.** Define the line \( l_i \) as

\[
l_i := \mathbb{R}(\gamma_i + \mathbb{R}(I_i) \tilde{\zeta} - \mathbb{R}(\Omega_i) t) = \mathbb{R}(I_i) (\tilde{\zeta} - v_i t + \xi_i) \equiv C, \quad i = 1, 2, \cdots, N,
\]

where \( C \) is a constant and \( \gamma_i, I_i = I(z_i), \Omega_i = \Omega(z_i) \) are defined in equation (52).

For an elliptic-breather solutions \( u^{[N]}(x,t) \), there are \( N \) lines \( l_i, i = 1, 2, \cdots, N \). However, \( u^{[N]}(x,t) \) is constructed by choosing \( m \) different parameters \( z_i \)'s. Therefore, the relationship between \( z_i \) and \( l_i \) is not a one-to-one correspondence. Sometimes, different parameters \( z_i \neq z_{i+1} \) represent the same \( l_i \).

The multi elliptic-breather solution \( u^{[N]}(x,t) \) divides the whole space into \( 2N \) region, shown in Figure 8. In order to have a clearer understanding of the above solution, we mainly consider the following aspects: (1) Study the relation between velocity \( v \) and parameter \( z \) in this subsection. (2) Analyze the asymptotic behavior of solutions along the line \( l_i \), \( i = 1, 2, \cdots, N \), and the region \( R_i \), \( i = 1, 2, \cdots, N \), as \( t \to \pm \infty \) in the next subsection.

For the definition of functions \( I(z) \) and \( \Omega(z) \) in equation (52), the velocity \( v(z) \) is mainly determined by following two conditions, one is \( \mathbb{R}(I) = \mathbb{R}(I(z)) = \mathbb{R}((-aZ(i(z+l) + K) + ik') - aZ(i(z-l))) \) and other is \( \mathbb{R}(\Omega) = \mathbb{R}(\Omega(z)) = \mathbb{R}(8\lambda(z)y(z)) \). Both of them are the meromorphic functions of variable \( z \in S \). To study the meromorphic function \( v(z) \) clearly, we need to obtain all possibilities of \( z \in S \) satisfying \( \mathbb{R}(\Omega) \equiv 0, \mathbb{R}(I) \equiv 0, \mathbb{R}(\Omega) \equiv \infty \) and \( \mathbb{R}(I) \equiv \infty \). The condition \( \mathbb{R}(I) \equiv 0 \) is studied in [32]. Therefore, we just analyze the case of \( \mathbb{R}(\Omega) \equiv 0, z \in S \). For ease of representation, we introduce the notation \( S_1 \) defined as the first quadrants of \( S \).

**Figure 7.** The solution \( u^{[2]}(x,t) \) with a four-fold Darboux transformation under the cn-type background by setting \( c_1 = 3 + 4i \) and \( c_2 = 1.3754 - i \).
Lemma 2. On the boundary of \( S_1 \), marked as \( \partial S_1 \), the points satisfying \( \Re(\Omega) \equiv 0 \) could be classified into following conditions:

- If \( l = 0 \) and \( k \in \left( 0, \frac{\sqrt{3}}{2} \right) \), we get that when \( z \in \partial S_1 \cap \Re, \Re(\Omega) = 0 \). On the upper boundary of \( S_1 \), there exist three points \( z = z_{R1} + i\frac{k}{2}, \frac{k}{2} + i\frac{k}{2} \) and \( z_{R2} + i\frac{k}{2}, z_{R1} \). On the remaining left and right boundary of \( S_1 \), there not exist any points such that \( \Re(\Omega) = 0 \).
- If \( l = 0 \) and \( k \in \left( \frac{\sqrt{3}}{2}, 1 \right) \), we get that when \( z \in \partial S_1 \cap \Re, \Re(\Omega) = 0 \). On the upper boundary of \( S_1 \), i.e., \( z \in \partial S_1 \cap \left\{ z \mid z = z_{R} + i\frac{k}{2} \right\} \), only exist one point \( (\Re(z), \Im(z)) = \left( \frac{k'}{2}, \frac{k'}{2} \right) \) such that \( \Re(\Omega) = 0 \). The remaining left and right boundary of \( S_1 \) do not exist any points satisfying \( \Re(\Omega) = 0 \).

Proof. When \( l = 0 \), the function \( \Omega(z) \) could be simplified as

\[
\Omega(z) = -a^3 \left( k^2 \text{dn}(iz)\text{sn}(iz)\text{cn}(iz) + k'^2 \frac{\text{dn}(iz)\text{sn}(iz)}{\text{cn}(iz)} \right).
\]

And then, we consider the value of \( \Re(\Omega(z)) \) for \( z \in \partial S_1 \). Since \( \text{dn}(iz), \text{cn}(iz) \in \Re, z \in \Re \) and \( \text{sn}(iz) \in \Re, z \in \Re \), we get that \( \Omega(z) \in \Re \), as \( z \in \Re \), which implies that the equation \( \Re(\Omega(z)) \equiv 0 \) holds on the real line.

Consider the left boundary of set \( S_1 \). On the left boundary of set \( S_1 \), i.e., \( z \in \Re \cap \partial S_1 \), we could set \( z = iz_l, z_l \in \left( 0, \frac{a}{2} \right) \), i.e., \( z \in \Re \). Substituting \( z = iz_l \) into equation (60), we obtain \( \Re(\Omega(z)) \neq 0 \) with \( z_l \neq 0 \), which implies \( \Re(\Omega(z)) \neq 0, z \in \Re \cap (\partial S_1 \setminus \{0\}) \).

If parameter \( z \) in the right boundary of set \( \partial S_1 \), we plug \( z = K' + iz_l \) into equation (60) and get

\[
\Omega(z) = -a^3 \left( k^2 \text{dn}(-z_l + iK')\text{sn}(-z_l + iK')\text{cn}(-z_l + iK') + k'^2 \frac{\text{dn}(-z_l + iK')\text{sn}(-z_l + iK')}{\text{cn}^3(-z_l + iK')} \right)
\]

\[
= -a^3 \left( \frac{\text{cn}(z_l)\text{dn}(z_l)}{\text{sn}^3(z_l)} + k'^2 \frac{\text{cn}(z_l)\text{sn}(z_l)}{\text{dn}^3(z_l)} \right),
\]

which implies that for any \( z_l \in \left( 0, \frac{a}{2} \right), \Re(\Omega(z)) \neq 0 \).
Then, we study the upper boundary of set $\partial S_1$. Through utilizing the half arguments formulas of the elliptic functions in (A.5) and (A.6), we obtain
\[
\Omega(z) = -\alpha^3 \left( k^2 \text{dn}(iz) \text{sn}(iz) \text{cn}(iz) + k'^2 \frac{\text{dn}(iz) \text{sn}(iz)}{\text{cn}(iz)} \right),
\]
\[
= -\alpha^3 k^2 \left( \frac{\text{cn}(2iz) + \text{dn}(2iz)}{1 + \text{dn}(2iz)} - \frac{\text{cn}(2iz) - \text{dn}(2iz)}{1 - \text{dn}(2iz)} \right) \frac{\text{sn}(2iz)}{1 + \text{cn}(2iz)}.
\]

(62)

Plugging $z = z_R + i \frac{k}{2}$ into (62) and using the translation formula (A.3), imaginary arguments formual (A.4), we get
\[
\Omega(z) = -\alpha^3 \frac{2\text{dn}(2iz_R - K)(1 - \text{cn}(2iz_R - K))^2}{\text{sn}^2(2iz_R - K)}
\]
\[
= \frac{2\alpha^3 k' \left( \text{dn}(2iz_R - K') - k' \text{sn}(2iz_R - K') \right)^2}{\text{cn}^2(2iz_R - K)}
\]
\[
= \frac{2\alpha^3 k' \text{cn}(2iz_R - K') \left( 1 - 2k'^2 \text{sn}^2(2iz_R - K') - 2ik' \text{dn}(2iz_R - K') \text{sn}(2iz_R - K') \right)}{\text{cn}^2(2iz_R - K)}.
\]

(63)

Imposing that $\Re(\Omega(z)) = 0$, we have $\text{cn}(2iz_R, k') = 0$ or $1 - 2k'^2 \text{sn}^2(2iz_R, k') = 0$. It is easy to verify that if $z_R = \frac{k'}{2}$, equation $\text{cn}(2iz_R, k') = 0$ holds. Solving $1 - 2k'^2 \text{sn}^2(2iz_R, k') = 0$, we know that only if $k \in \left(0, \frac{\sqrt{2}}{2}\right)$, equation $1 - 2k'^2 \text{sn}^2(2iz_R, k') = 0$ have two different roots $z_{R1}, z_{R2}$ satisfying $z_{R1} < \frac{k'}{2} < z_{R2} < K'$. When $k = \frac{\sqrt{2}}{2}$, equation $1 - 2k'^2 \text{sn}^2(2iz_R, k') = 0$ has the multiple root $z_{R1} = z_{R2} = \frac{k'}{2}$. Otherwise, when $k \in \left(\frac{\sqrt{2}}{2}, 1\right)$, $1 - 2k'^2 \text{sn}^2(2iz_R, k') > 0, z_R \in [0, K']$.

\textbf{Remark 4.} Substituting $z = \frac{k'}{2} + iz_i$ into equation (60), we obtain
\[
\Omega(z) = -\alpha^3 \frac{2\text{dn}(-2z_i + iK') \left( 1 - \text{cn}(-2z_i + iK') \right)}{\text{sn}^2(-2z_i + iK')}
\]
\[
= -\alpha^3 k\text{cn}(2z_i) \left( 2k\text{sn}(2z_i) \text{dn}(2z_i) + i(2k^2 \text{sn}^2(2z_i) - 1) \right),
\]
which implies that on the line $\{z \mid z = \frac{k'}{2} + iz_i\} \cap S_1$, only when $z = \frac{k'}{2}, \frac{k'}{2} + i\frac{k}{2}$, the equation $\Re(\Omega(z)) = 0$ holds.

For the different value of $l$, we study the function $\Omega(z)$ to obtain the curve $\Re(\Omega(z)) \equiv 0$. Based on the properties of elliptic functions and the equation (60), it is easy to verify that $\Omega(-z) = -\Omega(z)$ and $\Omega(z^*) = -\Omega^*(z)$ in equation (60), which implies that the function $\Re(\Omega)$ is symmetric about the real axis $\Re(z) = 0$ and the line $\Re(z) = 0$. Therefore, when $l = 0$, we just need to study the region $S_1$.

\textbf{Proposition 2.} The curve $\Re(\Omega(z)) \equiv 0$ could be divided into the following cases:

(i) If $l = 0$ and $k \in \left(0, \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)$, there are three line segments in $S_1$ satisfying $\Re(\Omega(z)) \equiv 0$. Those three line segments start with points $z = z_{R1} + i\frac{k}{2}, z = \frac{k'}{2} + i\frac{k}{2}$ and $z = z_{R2} + i\frac{k}{2}, z_{R1} < \frac{k'}{2} < z_{R2} < K'$, on the upper boundary of $S_1$ and end with points $z = z_1, z = z_2$ and $z = K', z_1 < z_2 < K'$ on the real axis. Furthermore, above curves do not intersect with each other.

(ii) If $l = 0$ and $k \in \left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}\right)$, there are two line segments in $S_1$ satisfying $\Re(\Omega(z)) \equiv 0$. One of those line segments starts with $z = z_{R1} + i\frac{k}{2}$ and ends with $z = \frac{k'}{2} + i\frac{k}{2}$. The other curve starts with $z = z_{R2} + i\frac{k}{2}$ and ends with $z = K'$. Furthermore, above curves do not intersect with each other.

(iii) If $l = 0$ and $k \in \left(\frac{\sqrt{2}}{4}, 1\right)$, there is only one curve in $S_1$ meeting $\Re(\Omega(z)) \equiv 0$, which connect points $\frac{k'}{2} + i\frac{k}{2}$ and $K'$. 

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Proof. By the definition of $\Omega(z)$ in equation (9), it is easy to obtain that function $\Omega(z)$ is a meromorphic function on the whole complex plane. Considering the derivative of $\Omega(z)$, we get the curve $\Re(\Omega(z)) = 0$, through choosing an initial point $z_0$ satisfying $\Re(\Omega(z_0)) = 0$ and taking the derivatives with respect to $z_R$ and $z_I$ generated the curve $\Re(\Omega(z)) = 0$ by the tangent vector

$$
\left( -\frac{d\Re(\Omega(z))}{dz_I}, \frac{d\Re(\Omega(z))}{dz_R} \right) = \Re \left( \frac{d\Omega(z)}{dz} \right) .
$$

Based on above iteration, the curve $\Re(\Omega(z)) = 0$ could be divided into following two categories:

1. The curve $\Re(\Omega(z)) = 0$ forms a closed circle. According to the maximum value principle of harmonic function, if there is not any singularity in the area, all the values of $\Omega(z)$ in the closed area satisfying $\Re(\Omega(z)) = 0$. Otherwise, there must exist at least one pole in this closed region.

2. Without any closed loops, the curve must end up at the point $z_0$ satisfying $\Omega'(z_0) = \infty$ or $\Omega'(z_0) = 0$

When $z_0$ satisfyin $\Omega''(z_0) = 0$, the Taylor expanding at this point could be written as $\Omega(z) = \Omega(z_0) + \Omega''(z_0)(z - z_0)^2 + O((z - z_0)^3)$, with $\Omega''(z_0) \neq 0$.

Consider the derivatives of function $\Omega(z)$:

$$
\Omega'(z) = -\alpha^3 \left( k^2 d\text{dn}(iz) \text{sn}(iz) + k^2 d\text{dn}(iz) \text{sn}(iz) \right)'
$$

$$
= -\frac{ik^3}{\text{cn}^4(iz)} \left( 3k^4 \text{sn}^8(iz) - (8k^4 + 2k^2) \text{sn}^6(iz) + (8k^4 + 4k^2) \text{sn}^4(iz) - (8k^2 - 2) \text{sn}^2(iz) + 1 \right) .
$$

Based on $\Omega(z)$ in (60), we know that the periods of $\Omega(z)$ and $\Omega'(z)$ are both $2iK$ and $iK + K'$. By the zeros and poles of the Jacobi elliptic functions in equation (A.2), we know that only when $z = (2m_1 + 1)K' + (2n_1 + 1)iK$ and $2n_2K' + (2m_2 + 1)iK, n_1, m_1, n_2, m_2 \in \mathbb{Z}$ the derivative function reaches infinity, i.e. $\Omega'(z) = \infty$, and both of them are four order poles. Consider a period parallelogram starting from $(0,0)$ and taking $(0,0), (K', iK), (K', 3iK), (0, 2iK)$ as vertices. Based on the above studies of the poles of $\Omega'(z)$, we claim that there are four poles in this region including the multiple numbers.

And then, we consider the zeros of $\Omega'(z)$ in this region. It is easy to know that only if the numerator of $\Omega'(z)$ in (66) is zero, the derivative function $\Omega'(z)$ is zero. Seeing $q := \text{sn}^2(iz)$ as a whole, we convert to consider the solution of equation $f(q) = 0, f(q) := 3k^4q^4 - (8k^4 + 2k^2)q^3 + (8k^4 + 4k^2)q^2 - (8k^2 - 2)q + 1$. When $z \in \mathbb{R} \cap S_1$, the range of $q = \text{sn}^2(iz)$ is $(-\infty, 0]$. We get $f(0) = 1, f(-\frac{1}{k}) = 8(k + 1)^2 > 0$ and $f(-\infty) = +\infty$. Differentiating for function $f(q)$ with respect to variable $q$, we get

$$
f'(q) = 12k^4 \left( \frac{q - 1}{2k^2} \right) \left( q - 1 + \sqrt{-\frac{3k^2 + 3}{3k}} \right) \left( q - 1 + \sqrt{-\frac{3k^2 + 3}{3k}} \right).
$$

Analyzing the zero points of function $f'(q) = 0$, it is easy to obtain that only when $k \in \left( 0, \frac{1}{2} \right)$, there exist a negative root $q_0 = 1 - \sqrt{-\frac{3k^2 + 3}{3k}}$ satisfying $f'(q_0) = 0$. Plugging $q_0$ into function $f(q)$ and studying the case $f(q_0) < 0, f(q_0) = -\frac{4(k - 1)(k + 1)(\sqrt{-\frac{3k^2 + 3}{3k}(2k^2 - 1) + 6k^2 - 1})}{3k}$, we obtain that only when $k \in \left( 0, \sqrt{\frac{k}{k}} - \sqrt{\frac{2}{k}} \right)$, equation $f(q) = 0$ get two negative roots $q_1, q_2$ satisfying $q_2 < q_0 < q_1 < 0$. Since $-\frac{1}{k} < q_0 = -\frac{3k^2 + 3}{3k} < 1 < 0$ and $f(-\frac{1}{k}) = 8(k + 1)^2 > 0$, we know that above two negative roots satisfy $-\frac{1}{k} < q_2 < q_1 < 0$. When $k = \sqrt{\frac{k}{k}} - \sqrt{\frac{2}{k}}$, above two roots are the same, i.e. $q_1 = q_2 = q_0$. By equation (A.4), we get two roots $0 < z_1 < z_2 < K'$ satisfying $q_1 = \text{sn}^2(iz_i, k)$. For the period of function $\Omega'(z)$, we know that $z = z_1 + 2iK$ and $z = z_2 + 2iK$ are also the zero points of $\Omega'(z)$ with order two. Therefore, we get two second-order zero points in the above periodic parallelogram. Since in a periodic parallelogram the elliptic function $\Omega'(z)$ has the same number of zeros and poles including the multiple points and we have obtained four zero points including multiples, we could claim that all zero points are gained, when $k \in \left( 0, \sqrt{\frac{k}{k}} - \sqrt{\frac{2}{k}} \right)$. If $k \in \left( \sqrt{\frac{k}{k}} - \sqrt{\frac{2}{k}}, 1 \right)$, $f(q)$ does not exist any negative roots $q$ such that $f(q) = 0$. 

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Combining with the Lemma 2 and Remark 4, we divide modules $k$ into following three cases in the region $S_1$.

(i) When $k \in (0, \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4})$, we consider three points $z = z_{R1} + i\frac{k}{2}$, $z = \frac{k}{2} + i\frac{K}{2}$ and $z = z_{R2} + i\frac{K}{2}$ in the upper region of $\partial S_1$. Along the tangent vector, we want to show that they would end at points $z = z_1$, $z = z_2$ and $z = K'$. If not, there must exist two curves intersecting with others at point $z_0 \in S_1$ such that $\Omega(z_0) = 0$, which contradicts with the fact that all zero points could be written as $z = z_1 + m_1K' + 2n_1ik$ or $z = z_2 + m_2K' + 2n_2ik$. Thus, we obtain that excepting the curve $\mathbb{R} \cap \partial S_1$, there exist three curves in the region $S_1$ satisfying $\Re(\Omega(z)) = 0$. They start at the points $z = z_{R1} + i\frac{k}{2}$, $z = \frac{k}{2} + i\frac{k}{2}$ and $z = z_{R2} + i\frac{k}{2}$, along the tangent vector (65) and end at points $z = z_1$, $z = z_2$ and $z = K'$.

(ii) When $k \in (\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2})$, we consider three points $z = z_{R1} + i\frac{k}{2}$, $z = \frac{k}{2} + i\frac{k}{2}$ and $z = z_{R2} + i\frac{K}{2}$ in the upper region of $\partial S_1$. Firstly, we consider the point $z = z_{R1} + i\frac{k}{2}$. From Remark 4, we know that this curve does not cross the line $\Re(z) = \frac{k}{2}$. Furthermore, on the real line there do not exist any points such that $\Omega(z) = 0$. Thus, it must end up at the boundary point $z = \frac{k}{2} + i\frac{k}{2}$, by Lemma 2. Similarly, the point $z = z_{R2} + i\frac{k}{2}$ must end up at point $z = K'$. Therefore, excepting the boundary of $S_1$, there exist two curves in the region $S_1$. They start at the points $(\frac{z_{R2} + k}{2}, \frac{K'}{2})$, $(\frac{k}{2}, \frac{k}{2})$ and end at the points $(\frac{z_{R1} + k}{2}, \frac{K'}{2})$ and $(K', 0)$. Since there is not any point on the line $z = \frac{k}{2} + iz_1, z_1 \in (0, \frac{k}{2})$, satisfying $\Re(\Omega(z)) = 0$, above two curves do not intersect with each other. The difference from the above condition is that there do not exist points on line $z \in \mathbb{R}$ such that $\Omega(z) = 0$.

(iii) When $k \in (\frac{\sqrt{2}}{2}, 1)$, there exists only one curve in the region $S_1$. It will start at point $z = \frac{K'}{2} + i\frac{K}{2}$ and end at point $z = K'$. Since on the upper boundary of $\partial S_1$, there only exist one point satisfying $\Re(\Omega(z)) = 0$ and on the line $z \in \mathbb{R}, \Re(z) \neq 0$.

Then, we consider the velocity $v(z) = \frac{\Re(\Omega(z))}{\Re(I(z))}$. In our previous studies [32], we have proved that in the region $S_1$, there is only one curve satisfying $\Re(I(z)) = 0$. When $k \in (0, 0.9089)$ this curve intersects with the real axis and when $k \in (0.9089, 1)$ this curve intersects with the imaginary axis. We also proved that the curves $\Re(I(z)) = 0$ and $\Re(\Omega) = 0$ do not intersect with each others. Moreover, only when $z = \pm K' + i$, $\Re(I(z)) = \infty$ and $\Re(\Omega(z)) = \infty$. Together with the curves satisfying $\Re(\Omega(z)) = 0$, $\Re(I(z)) = 0$ and the points satisfying $\Omega(z) = \infty$ and $I(z) = \infty$, we could obtain that the region of velocity $v$ can be classified into four different cases of parameter $k$ in different regions.

- When $k \in (0, \sqrt{\frac{6}{4} - \sqrt{\frac{2}{2}}})$, the value of $v = v(z)$ in region $S_1$ is divided into five areas. The above regions are separated by three curves $\Re(\Omega(z)) = 0$ satisfying the case (i) of Proposition 2 and one curve $\Re(I(z)) = 0$ intersecting with real line (See in Figure 9(i));
- When $k \in (\sqrt{\frac{6}{4} - \sqrt{\frac{2}{2}}}, \sqrt{\frac{2}{2}})$, the value of $v = v(z), z \in S_1$ is divided into four areas. The above regions are separated by two curves $\Re(\Omega(z)) = 0$ satisfying the case (ii) of Proposition 2 and one curve $\Re(I(z)) = 0$ intersecting with real line (See in Figure 9(ii));
- When $k \in (\sqrt{\frac{2}{2}}, 0.9089)$, the value of $v = v(z), z \in S_1$ is divided into three regions, which are separated by a curve $\Re(\Omega(z)) = 0$ satisfying the case (iii) of Proposition 2 and a curve $\Re(I(z)) = 0$ intersecting with real line (See in Figure 9(iii));
- When $k \in (0.9089, 1)$, the value of $v = v(z), z \in S_1$ is divided into three regions. However, unlike the above, the division of regions is different. The above regions are separated by a curve $\Re(\Omega(z)) = 0$ satisfying the case (iii) of Proposition 2 and a curve $\Re(I(z)) = 0$ intersecting with imaginary axis (See in Figure 9(iv)).

For the above four cases, we give corresponding values to depict them clearly in Figure 9. We choose the modulus $k$ as 0.25, 0.35, 0.85, 0.95, which satisfies above four cases, respectively. The first four figures (i), (ii), (iii), (iv) show the velocity $v$ in $z$-plane and the rest four figures (a), (b), (c), (d) describe the velocity $v$ in $\lambda$-plane with cuts in yellow. The green regions in them represent the condition $v > 0$ and the white ones represent $v < 0$. In Figure 9, we could find that above regions with different sign of velocity $v$ are
separated by curves $\Re(\Omega(z)) = 0$ in red and curves $\Re(I(z)) = 0$ in blue. It is worth noting that we do not use the same method as the $z$-plane above to obtain the variation of $v$ in the $\lambda$-plane. We get it based on the conformal map between $\lambda$-plane and $z$-plane.

**Figure 9.** The first four images (i), (ii), (iii), (iv) draws $v > 0$ in green and $v < 0$ in white on the $z$-plane. The red curves describe the condition $\Re(\Omega(z)) = 0$ and the blue curves describe the condition $\Re(I(z)) = 0$. The rest four images (a), (b), (c), (d) draws $v > 0$ in green and $v < 0$ in white on the $\lambda$-plane. The red curves describe the condition $\Re(\Omega) = 0$ and the blue curves describe the condition $\Re(I) = 0$. The figures (a), (b), (c) and (d) are obtained from the $z$-plane through the conformal map $\lambda(z)$.

**Remark 5.** Consider the condition $l = K^2/2$. The function $\Omega(z)$ could be written as

$$\Omega(z) = -\alpha^3 \left( k^2 \text{dn}(i(z-l))\text{sn}(i(z-l))\text{cn}(i(z-l)) + k^2 \frac{\text{dn}(i(z+l))\text{sn}(i(z+l))}{\text{cn}^3(i(z+l))} \right).$$

From the work [32], we know that when $z \in \mathbb{R}$ and $z = z_R \pm i \frac{K}{2}$, the equation $\Re(\Omega(z)) = 0$ holds. The same as above, through studying the derivative of function $\Omega(z)$, we obtain that there only exist points $z = K' + i \frac{K}{2}$ and $z = i \frac{K}{2}$ such that $\Omega'(z) = 0$. Therefore, we obtain that excepting above two lines, there are only two curves in the upper half plane of $S$, which connect the points $z = K' + i \frac{K}{2}, z = \frac{3K'}{2}$ and the points $z = i \frac{K}{2}, z = -K^2/2$.

We also describe the case $l = K^2/2$ with $k = \frac{9}{10}$ in Figure 10. The Figure 10(i) shows the velocity $v$ in $z$-plane and Figure 10(a) describes the velocity $v$ in $\lambda$-plane with cuts in yellow. The green areas in them represent the condition $v > 0$ and the white represent $v < 0$. We could find that above regions with different sign of velocity $v$ are separated by curves satisfying $\Re(\Omega(z)) = 0$ in red. The result of Figure 10(a) in the $\lambda$-plane depends on the conformal map between $\lambda$-plane and $z$-plane.

Looking back at the variation between $(\zeta, t)$ and $(x, t)$, we know that the above velocity $v$ describes the variation between $\zeta$ and $t$. Combining the rotation $\zeta = x - st$, the line $l_i$ in equation (59) could be convert into $x = (s + v(z_i))t + \zeta_i$.

### 3.3 Asymptotic analysis for the $N$ elliptic-breather solutions with distinct velocities

Considering the $N$ elliptic-breather solutions (47), we know that there are only $N$ different velocities. However, for the solution $u^{[N]}(x, t)$, the matrix $M$ and $D$ are $m \times m$, $N \leq m \leq 2N$, i.e. the number of
parameters $z_i$ and the corresponding spectral parameters $\lambda_i$ is $m$, which implies that there must exist some $z_i, z_{i+1}$ with the same velocity $v(z_i) = v(z_{i+1})$. The main reason for above conditions lies in the different kinds of the Darboux matrix $T_i^C(\lambda; x, t)$ and $T_i^P(\lambda; x, t)$ we choose. Either of these two matrices will only add the breather in one direction. But, the Darboux matrix $T_i^C(\lambda; x, t)$ is obtained by two points $z_i$ and $-z_i^* + 2t$.

Here, we just choose the line $l_i$ with different velocity $v_i$. Then, we study the asymptotically analysis along the line $l_k$. As $t \to +\infty$, we define the line as $l_i^+, i = 1, 2, \ldots, N$. When $t \to -\infty$, we define the lines $l_i^-, i = 1, 2, \ldots, N$. Those $2N$ lines divided the $(\xi, t)$ plane into $2N$ different pieces of the region $R_i^\pm, i = 1, 2, \ldots, N$. The sketch map of above conditions is shown in Figure 8. It should be noted that the direction depends only on the number $N$ and not the dimension $m$.

Proof of Theorem 3. For the different Darboux matrix $T_i^C(\lambda; x, t)$ and $T_i^P(\lambda; x, t)$, we divide the asymptotic analysis into two conditions. By the definition of $\eta_i$ in equation (52), it is easy to obtain that

$$\exp(\eta_i) = \exp(\gamma_i + \lambda_i^0 - \Omega t)$$
$$= \exp(\gamma_i + i\Im(\lambda_i)\xi - i\Im(\Omega)t + \Re(I_i)(\xi - v_it))$$
$$= \exp(\gamma_i + i\Im(\lambda_i)\xi - i\Im(\Omega)t + \Re(I_i)(\xi - v_ht) + \Re(I_i)(v_{(h-v_i)t}), \quad i = 1, 2, \cdots, m.$$ (69)

Without losing the generality, we can assume $v_i = v(z_i) \leq v_{i+1} = v(z_{i+1}), i = 1, 2, \cdots, m - 1$ and $l_i = \Pi(z_i) > 0, i = 1, 2, \cdots, m$. Along the line $l_k^\pm$, if there is only one parameter $z_h$ such that $\eta_h = \const$ in (52) as $t \to \pm\infty$, we could verify that as $t \to +\infty$, the value $\eta_i \to -\infty$, $e^{\eta_i} = O(e^{\Re(I)(v_{(h-v_i)t})}, i = h + 1, h + 2, \cdots, m$, and $-\eta_i \to -\infty$, $e^{-\eta_i} = O(e^{-\Re(I)(v_{(h-v_i)t})}, i = 1, 2, \cdots, h - 1$. Similarly, as $t \to -\infty$, the value $-\eta_i \to -\infty$, $e^{-\eta_i} = O(e^{-\Re(I)(v_{(h-v_i)t})}, i = h + 1, h + 2, \cdots, m$ and $\eta_i \to -\infty$, $e^{\eta_i} = O(e^{\Re(I)(v_{(h-v_i)t})}, i = 1, 2, \cdots, h - 1$. Thus, combining equations (49) and (50), we prove the first condition (i) in Theorem 3.

Considering the second case that there are two parameters $z_h$ and $z_{h+1}$ such that $\eta_h = \eta_{h+1} = \const$ as $t \to \pm\infty$, along the line $l_k^\pm$, which occurs at the point $Z_k = [z_k, 2l - z_k^*]$. The difference from the previous case is that in addition to $\eta_h$, the parameter $\eta_{h+1} = \const$. And then, we get that as $t \to +\infty$, $\eta_i \to -\infty$, $e^{\eta_i} = O(e^{\Re(I)(v_{(h-v_i)t})}, i = h + 1, h + 2, \cdots, m$, $-\eta_i \to -\infty$, $e^{-\eta_i} = O(e^{-\Re(I)(v_{(h-v_i)t})}, i = 1, 2, \cdots, h - 1$ and as $t \to -\infty$, $-\eta_i \to -\infty$, $e^{-\eta_i} = O(e^{-\Re(I)(v_{(h-v_i)t})}, i = h + 1, h + 2, \cdots, m$, $\eta_i \to -\infty$, $e^{\eta_i} = O(e^{\Re(I)(v_{(h-v_i)t})}, i = 1, 2, \cdots, h - 1$. Plugging them into equation (47), we prove the condition (ii) in Theorem 3.

□
Furthermore, we list the concrete sequence of $\zeta^{[a, +\infty]}$ and $\mu^{[a, +\infty]}, a = 1, 2$:

\[
\begin{align*}
\zeta^{[1, +\infty]} &= (\xi_1^{[1, +\infty]}, \xi_2^{[1, +\infty]}, \ldots, \xi_h^{[1, +\infty]}, \xi_{h+1}^{[1, +\infty]}, \ldots, \xi_m^{[1, +\infty])}
&= (i z_1^* - (K + i K'), i z_2^* - (K + i K'), \ldots, i z_h^* - (K + i K'), i z_{h+1}^* , \ldots, i z_m^*) \\
\zeta^{[2, +\infty]} &= (\xi_1^{[2, +\infty]}, \xi_2^{[2, +\infty]}, \ldots, \xi_h^{[2, +\infty]}, \xi_{h+1}^{[2, +\infty]}, \ldots, \xi_m^{[2, +\infty])}
&= (i z_1^* + (K + i K'), i z_2^* + (K + i K'), \ldots, i z_h^* + (K + i K'), i z_{h+1}^* , \ldots, i z_m^*) \\
\mu^{[1, +\infty]} &= (\mu_1^{[1, +\infty]}, \mu_2^{[1, +\infty]}, \ldots, \mu_h^{[1, +\infty]}, \mu_{h+1}^{[1, +\infty]}, \ldots, \mu_m^{[1, +\infty])}
&= (iz_1 + K + i K', iz_2 + K + i K', \ldots, iz_h - i K, -iz_{h+1}, \ldots, iz_m) \\
\mu^{[2, +\infty]} &= (\mu_1^{[2, +\infty]}, \mu_2^{[2, +\infty]}, \ldots, \mu_h^{[2, +\infty]}, \mu_{h+1}^{[2, +\infty]}, \ldots, \mu_m^{[2, +\infty])}
&= (iz_1 + K + i K', iz_2 + K + i K', \ldots, iz_h - i K, iz_{h+1}, \ldots, -iz_m)
\end{align*}
\]

**Remark 6.** Considering the condition (i) of the asymptotic analysis of function $\hat{u}^{[N]}(\xi, t; I_k)$ in Theorem 3 and combining the formulas of Jacobi theta functions and the result in Appendix of [47], we could rewrite the solution $\hat{u}^{[N]}(\xi, t; I_k)$ as $t \to +\infty$ in equation (70) as follows. Moreover, as $t \to -\infty$, we also could obtain the similar expression. In summary, we gain

\[
\hat{u}^{[N]}(\xi, t; I_k) \sim \frac{\alpha}{\theta_2} \frac{\theta_4}{\theta_2} \left( \frac{\alpha \zeta^{[a, +\infty]} + \mu^{[b, +\infty]}}{2k} \right)^{m-1} \prod_{j=1}^{m} \frac{r_j}{r_j - \eta_h} \prod_{j=h+1}^{m} \frac{r_j}{r_j + \eta_h}, \quad t \to +\infty
\]
where

\begin{align}
\mathbf{r}_h^+ &= \prod_{j=1}^{h-1} r_j^+ \prod_{j=h+1}^{n} r_j^+,
\mathbf{r}_h^- &= \prod_{j=1}^{h-1} r_j^- \prod_{j=h+1}^{n} r_j^-,
\mathbf{\Delta}_h^\pm &= \left[ \frac{1}{r_h} \mathbf{r}_h^\pm \mathbf{\Delta}_h^\pm \right] \left[ \begin{array}{c}
\theta_0 \left( \frac{a_1^2 + 2d + i^\pm + s_h^\pm + i_2 + 2K'}{2K} \right) \\
\theta_0 \left( \frac{a_1^2 + 2d + i^\pm + s_h^\pm + i_2 + 2K'}{2K} \right)
\end{array} \right],
\mathbf{D}_h^\pm &= \left[ \mathbf{1} \right] \mathbf{\Delta}_h^\pm \mathbf{\eta}_h^\pm.
\end{align}

Furthermore,

\begin{align}
\mathbf{s}_h^\pm &= \pm \left( \sum_{j=1}^{h-1} (iz_j^+ - iz_j^-) + \sum_{j=h+1}^{n} (-iz_j^+ + iz_j^-) \right),
\mathbf{\Delta}_h^\pm &= \prod_{j=1}^{h-1} \theta_1 \left( \frac{i^\pm - i_2}{2K} \right) \theta_1 \left( \frac{i^\pm + i_2 + K'}{2K} \right) \prod_{j=h+1}^{n} \theta_1 \left( \frac{i^\pm - i_2}{2K} \right) \theta_1 \left( \frac{i^\pm + i_2 + K'}{2K} \right),
\mathbf{\Delta}_h^\pm &= \prod_{j=1}^{h-1} \theta_1 \left( \frac{i^\pm - i_2}{2K} \right) \theta_1 \left( \frac{i^\pm + i_2 + K'}{2K} \right) \prod_{j=h+1}^{n} \theta_1 \left( \frac{i^\pm - i_2}{2K} \right) \theta_1 \left( \frac{i^\pm + i_2 + K'}{2K} \right).
\end{align}

**Lemma 3.** The asymptotically analysis on the region \( R_k^\pm, k = 1, 2, \cdots, n \) could also be classified into following two cases:

(i) Along the line \( l_{k-1} \), if there exist only one parameter \( z_h \) satisfying \( \eta_h = \text{const} \) in equation (52) as \( t \to \pm \infty \), the asymptotic expression on the region \( R_k^\pm \) could be written as

\begin{align}
\mathbf{u}[N](z_h^\pm, t, R_k^\pm) &\rightarrow \frac{\alpha_0^\pm}{A \theta_3^\pm} \frac{1}{\theta_3^\pm} \mathbf{r}_h^\pm \mathbf{d}_h^\pm \mathbf{\Delta}_h^\pm \mathbf{\eta}_h^\pm, \quad t \to \pm \infty,
\end{align}

where

\begin{align}
\mathbf{r}_h^\pm &= \prod_{j=1}^{h-1} r_j^\pm \prod_{j=h+1}^{n} r_j^\pm, \quad \mathbf{r}_h^- &= \prod_{j=1}^{h-1} r_j^- \prod_{j=h+1}^{n} r_j^-,
\mathbf{s}_h^\pm &= \pm \left( \sum_{j=1}^{h-1} 23(z_j) - \sum_{j=h+1}^{n} 23(z_j) \right).
\end{align}

(ii) Along the line \( l_{k-1} \), if there exist two parameters \( z_h, z_{h+1} \) satisfies \( \eta_h = \eta_{h+1} = \text{const} \) as \( t \to \pm \infty \), the asymptotic expression on the region \( R_k^\pm \) could be written in (76) with parameters

\begin{align}
\mathbf{r}_h^\pm &= \prod_{j=1}^{h-1} r_j^\pm \prod_{j=h+2}^{n} r_j^\pm, \quad \mathbf{r}_h^- &= \prod_{j=1}^{h-1} r_j^- \prod_{j=h+2}^{n} r_j^-,
\mathbf{s}_h^\pm &= \pm \left( \sum_{j=1}^{h-1} 23(z_j) - \sum_{j=h+2}^{n} 23(z_j) \right).
\end{align}

**Proof.** To consider the asymptotic analysis of function \( u[N](x, t) \) on the region \( R_k^\pm, k = 1, 2, \cdots, N \), as \( t \to \pm \infty \). Similarly, we also divide it into two conditions. One is that along the line \( l_{k-1} \), if there is only one parameter \( z_h \) such that \( \eta_h = \text{const} \), as \( t \to \pm \infty \). Then, we consider that on the region \( R_k^\pm \), the value \( \eta_j \) satisfying \( \eta_j \to \pm \infty, j = h + 1, h + 2, \cdots, m \) and \( -\eta_j \to \pm \infty, j = 1, 2, \cdots, h \), as \( t \to \pm \infty \). Similarly, as \( t \to -\infty \), the value \( -\eta_j \to \pm \infty, j = h + 1, h + 2, \cdots, m \) and \( \eta_j \to \pm \infty, j = 1, 2, \cdots, h \). Thus, combining with equations (49) and (50), we prove the condition (i) in Lemma 3. Similarly, we also could obtain the condition (ii) in Lemma 3. \( \square \)
Lemma 4. The function $r_i$ have the following properties:

- For $i(z_i - l) = \pm iK' - z_{il}$, $z_{il} = \Im(z_i)$ i.e., $\lambda(z_i) \in i\mathbb{R}$, we get

\[
(79) \quad \frac{r_i}{r_i^*} = -1, \quad l = 0, \quad \text{and} \quad \frac{r_i}{r_i^*} = -e^{\frac{2i(z_i)\pi}{2K}}, \quad l = \frac{K'}{2};
\]

- For $i(z_i - l) = -z_{il}$, $z_{il} = \Im(z_i)$, i.e., $\lambda(z_i) \in i\mathbb{R}$, we get

\[
(80) \quad \frac{r_i}{r_i^*} = 1, \quad l = 0, \quad \text{and} \quad \frac{r_i}{r_i^*} = e^{\frac{2i(z_i)\pi}{2K}}, \quad l = \frac{K'}{2};
\]

- For $i(z_i - l) \neq \pm iK' - z_{il}$, $z_{il} = \Im(z_i)$ and $z_{i+1} = -z_i^* + 2l$, i.e., $\lambda(z_i) \in \mathbb{C} \setminus i\mathbb{R}$, we get

\[
(81) \quad \frac{r_i r_{i+1}}{r_i^* r_{i+1}^*} = 1, \quad l = 0, \quad \frac{r_i r_{i+1}}{r_i^* r_{i+1}^*} = \frac{4i(z_i)\pi}{2K}, \quad l = \frac{K'}{2}.
\]

Proof. By the function $r_i$ defined in equation (14) and the shift formulas of the Jacobi theta function (A.9), we know that if $i(z_i - l) = \pm iK' - z_{il}$,

\[
(82) \quad \frac{r_i}{r_i^*} = \frac{\vartheta_2(\frac{i(z_i+l)}{2K})}{\vartheta_4(\frac{i(z_i-l)}{2K})} = -i \frac{\vartheta_2(\frac{z_i}{2K})}{\vartheta_4(\frac{z_i}{2K})}, \quad l = 0,
\]

\[
(83) \quad \frac{r_i}{r_i^*} = \frac{\vartheta_2(\frac{i(z_i+l)}{2K})}{\vartheta_4(\frac{i(z_i-l)}{2K})} = \frac{\vartheta_2(\frac{z_i}{2K})}{\vartheta_4(\frac{z_i}{2K})}, \quad l = \frac{K'}{2},
\]

which implies that $\frac{r_i}{r_i^*} = -1$ when $l = 0$ and $\frac{r_i}{r_i^*} = -e^{\frac{2i(z_i)\pi}{2K}}$ when $l = \frac{K'}{2}$. Thus, the equation (79) holds.

When $i(z_i - l) = -z_{il}$, plugging them into equation (14), we obtain that

\[
(84) \quad \frac{r_i}{r_i^*} = \frac{\vartheta_2(\frac{i(z_i+l)}{2K})}{\vartheta_4(\frac{i(z_i-l)}{2K})} = \frac{\vartheta_2(\frac{z_i}{2K})}{\vartheta_4(\frac{z_i}{2K})} = \frac{\vartheta_2(-i(z_i^*+l)+4i)}{\vartheta_4(-i(z_i^*+l)-2K)}, \quad l = 0,
\]

\[
(85) \quad \frac{r_i}{r_i^*} = \frac{\vartheta_2(\frac{i(z_i+l)}{2K})}{\vartheta_4(\frac{i(z_i-l)}{2K})} = \frac{\vartheta_2(-i(z_i^*+l)+4i)}{\vartheta_4(-i(z_i^*+l)+2K)} \exp \left( -\frac{2i(z_{il})^2 + K'}{4K} \pi \right), \quad l = \frac{K'}{2},
\]

which implies that the equation (81) holds.

Proof of Theorem 4. We set that the number of $z_i, i = 1, 2, \cdots, m$ satisfying $i(z_i - l) \in \mathbb{R}$ is $p$. Combining with the Lemma 3 and Lemma 4, we obtain that when $l = 0$, equation (76) could be rewritten as

\[
(85) \quad u^{[N]}(z, t; R_k^+) \rightarrow \frac{\vartheta_2 \vartheta_4}{\vartheta_3 \vartheta_3} \frac{r_i^* r_{i+1}^*}{r_i r_{i+1}} \frac{\vartheta_2(\frac{z_i^*+s_{h,b+1}^+}{2K})}{\vartheta_4(\frac{z_i^*+s_{h,b+1}^+}{2K})} = (-1)^{2m-p} \exp(\alpha a^* + s_{h,b+1}^+).\]
If \( l = \frac{K'}{2} \), by formulas (A.9), equation (76) could be rewritten as

\[
\hat{u}^N(\xi, \pm \infty; R^\pm_k) \rightarrow \frac{\alpha \theta_2 \theta_4}{A \theta_3 \theta_3 \left( \frac{iK'}{2K} \right)} \theta_4 \left( \frac{\alpha \xi + \frac{\pi}{2} + \frac{s^\pm_{h,h+1}}{2K}}{2K} \right) \\
= (-1)^{2N-m-p-k} \theta_4 \left( \frac{\alpha \xi + \frac{s^\pm_{h,h+1}}{2K}}{2K} \right) \\
= (-1)^{2N-m-p} \theta_4 \left( \alpha \xi + s^\pm_{h,h+1} \right).
\]

Then, the Theorem 4 holds.

Now, we perform the exact asymptotic analysis based on above-mentioned elliptic-breather solution as an example to show the asymptotic analysis vividly. For the Type-I, we consider the elliptic-breather solution \( u^{[2]}(x,t) \) in the Case I. Setting \( l = 0, k = \frac{1}{2}, \alpha = 1, c_1 = c_2 = 1, z_1 = K' + \frac{iK'}{2} \) and \( z_2 = K' + \frac{iK'}{2} \), we obtain the 3d-figure of function \( u^{[2]}(x,t) \) under the \((x,t)\)-axis in Figure 3(a). Then, we consider the corresponding asymptotic analysis under the \((\xi,t)\)-axis. Plugging above parameters into equations (15) and (16) or substituting them into equation (73), we get the asymptotic expressions and draw their graphs in Figure 11. The blue curves in the Figure 11(a) and Figure 11(b) describe the function \( \hat{u}^{[2]}(\xi, \pm 6) \), respectively. The red curve describes \( \hat{u}^{[2]}(\xi, 6; l_1) \) and \( \hat{u}^{[2]}(\xi, 6; l_2) \) in Figure 11(a). The purple curve describes \( \hat{u}^{[2]}(\xi, -6; l_1) \) and \( \hat{u}^{[2]}(\xi, -6; l_2) \) in Figure 11(b).

![Figure 11](image-url)

(a) The asymptotic analysis for the breathers at \( t = 6 \).

(b) The asymptotic analysis for the breathers at \( t = -6 \).

**FIGURE 11.** The asymptotic analysis for breathers of solution \( \hat{u}^{[2]}(\xi, t) \) at \( t = \pm 6 \). The blue curve describes the solution \( \hat{u}^{[2]}(\xi, \pm 6) \). The red and purple curves correspond to the asymptotic expression in equations (15) and (16) with \( l = 0, k = \frac{1}{2}, \alpha = 1, c_1 = c_2 = 0, z_1 = K' + \frac{2K}{2} \) and \( z_2 = K' + \frac{K}{2} \).

Then, we consider the asymptotic analysis form of solution on the regions \( R^+_1, R^+_2, R^-_1 \). Combining with equation (18), we obtain the following three asymptotic solutions as \( t \rightarrow +\infty \):

\[
\hat{u}^{[2]}(\xi, t; R^+_1) \rightarrow -k \text{cn}(a \xi - 2\Im(z_1) - 2\Im(z_2)) = -k \text{cn}(a \xi - \frac{10K}{9}) ,
\]

\[
\hat{u}^{[2]}(\xi, t; R^+_2) \rightarrow -k \text{cn}(a \xi + 2\Im(z_1) - 2\Im(z_2)) = -k \text{cn}(a \xi - \frac{2K}{9}) ,
\]

\[
\hat{u}^{[2]}(\xi, t; R^-_1) \rightarrow -k \text{cn}(a \xi + 2\Im(z_1) + 2\Im(z_2)) = -k \text{cn}(a \xi + \frac{10K}{9}) .
\]
Similarly, as $t \to -\infty$, we get
\[
\hat{u}^{[2]}(\xi, t; R^-_1) \to -k\text{cn}(\alpha \xi + 2\Im(z_1) + 2\Im(z_2)) = -k\text{cn}\left(\alpha \xi + \frac{10K}{9}\right),
\]
(88)
\[
\hat{u}^{[2]}(\xi, t; R^-_2) \to -k\text{cn}(\alpha \xi - 2\Im(z_1) + 2\Im(z_2)) = -k\text{cn}\left(\alpha \xi + \frac{2K}{9}\right),
\]
\[
\hat{u}^{[2]}(\xi, t; R^+_1) \to -k\text{cn}(\alpha \xi - 2\Im(z_1) - 2\Im(z_2)) = -k\text{cn}\left(\alpha \xi - \frac{10K}{9}\right).
\]
Above six functions in equations (87) and (88) are plotted in Figures 12(a) and 12(b), respectively.

![Figure 12](image)

**Figure 12.** The asymptotic analysis for breathers of solution $\hat{u}^{[2]}(\xi, t)$ at $t = \pm 6$. The blue curve describes the solution $\hat{u}^{[2]}(\xi, \pm 6)$. The red and purple curves draw the asymptotic expression on the region $R^\pm_{12}$ in equations (87) and (88) with $l = 0, k = \frac{1}{2}, \alpha = 1, c_1 = c_2 = 0, z_1 = K' + \frac{2K}{3}i$ and $z_2 = K' + \frac{K}{3}i$.

4 The relationship between the multi elliptic-breather solutions and the constant background multi-breather solutions

Considering elliptic function solutions $u(x, t)$ (2) of equation (1), we could prove that as $k \to 0^+$ above solutions are degenerated into constant solutions, which link the periodic solutions and constant solutions together. Then, we want to consider that whether there exist the relationship between the multi elliptic-breather and multi breather solutions on the zero or constant background.

Through utilizing the approximation formulas (A.7), we can easily get the results
\[
\lim_{k \to 0^+} \text{adn}(\alpha(x - st), k) = \alpha, \quad \lim_{k \to 0^+} \text{acn}(\alpha(x - st), k) = 0.
\]

Accordingly, we consider the limitation of multi elliptic-breather solutions $u^{[N]}(x, t)$ in equation (42) as $k \to 0^+$. Here, we mainly study the function $\Phi(x, t; \lambda)$ in equation (7). If the fundamental solution $\Phi(x, t; \lambda)$ of Lax pair (3) could be degenerated into the fundamental solution of corresponding Lax pair with constant-valued function $u(x, t)$, we could know that the multi elliptic-breather solutions could be degenerated into the solitons or the breathers under the zero or constant background.

Reviewing the results in our previous work [32] (Theorem 6), we obtain that the fundamental solution of Lax pair (3) could also be rewritten as
\[
\Phi(x, t; \lambda) = \begin{bmatrix} \sqrt{u^2(x, t) - \beta_1 \exp(\theta_1)} & \sqrt{u^2(x, t) - \beta_2 \exp(\theta_2)} \\ -\sqrt{u^2(x, t) - \beta_2 \exp(-\theta_2)} & -\sqrt{u^2(x, t) - \beta_1 \exp(-\theta_1)} \end{bmatrix},
\]
where
\[
\beta_1 = 2\lambda^2 + \frac{s}{2} - 2y, \quad \beta_2 = 2\lambda^2 + \frac{s}{2} + 2y, \quad \theta_i = \int_0^\xi \frac{2i\lambda \beta_i}{u^2(x) - \beta_i} dx + i\lambda \xi \pm 4i\lambda yt, \quad i = 1, 2,
\]
s is defined in equation (2) and y satisfies the algebraic curve

\[ y^2 = (\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2)(\lambda - \hat{\lambda}_3)(\lambda - \hat{\lambda}_4), \]

and the value of \( \hat{\lambda}_i \) are shown in Appendix C.

**Proposition 3.** As \( k \to 0^+ \), we analyze the value of \( s \) and the expression of \( y^2 \) as follows:

- Under the cn-type background, i.e., \( l = 0 \), we get \( \lim_{k \to 0^+} s = -\alpha^2 \) and \( \lim_{k \to 0^+} y^2 = \left( \lambda^2 - \frac{\alpha^2}{4} \right)^2 \);
- Under the dn-type background, i.e., \( l = \frac{K'}{2} \), we get \( \lim_{k \to 0^+} s = 2\alpha^2 \) and \( \lim_{k \to 0^+} y^2 = \lambda^2 \left( \lambda^2 + \alpha^2 \right) \).

**Proof.** From the solution (2), we know that the parameter \( s \) in the different background are different, so the limitation of \( s \) are

\[ \lim_{k \to 0^+} s = \lim_{k \to 0^+} \alpha^2(2k^2 - 1) = -\alpha^2, \quad l = 0 \quad \text{and} \quad \lim_{k \to 0^+} s = \lim_{k \to 0^+} \alpha^2(2 - k^2) = 2\alpha^2, \quad l = \frac{K'}{2}. \]

Then, we consider the value of \( y \) in equation (92). The parameters \( \hat{\lambda}_i = \lambda(\pm i) \) could be obtained by following four points \( \pm \frac{K'}{2} \pm i \frac{K}{2} \). Combining with equations (10), (C.11) and (C.13), we obtain

\[ \lim_{k \to 0^+} \hat{\lambda}_1 = \lim_{k \to 0^+} \hat{\lambda}_2 = \frac{\alpha}{2}, \quad \lim_{k \to 0^+} \hat{\lambda}_3 = \lim_{k \to 0^+} \hat{\lambda}_4 = -\frac{\alpha}{2}, \quad l = 0, \]

\[ \lim_{k \to 0^+} \hat{\lambda}_1 = \lim_{k \to 0^+} \hat{\lambda}_2 = 0, \quad \lim_{k \to 0^+} \hat{\lambda}_3 = i\alpha, \quad \lim_{k \to 0^+} \hat{\lambda}_4 = -i\alpha, \quad l = \frac{K'}{2}. \]

The above calculation process is given in equations (C.11) and (C.13) of Appendix C. Combining above results, we get

\[ \lim_{k \to 0^+} y^2 = \left( \lambda^2 - \frac{\alpha^2}{4} \right)^2, \quad l = 0, \quad \lim_{k \to 0^+} y^2 = \lambda^2 \left( \lambda^2 + \alpha^2 \right), \quad l = \frac{K'}{2}. \]

\[ \square \]

**Remark 7.** Considering the expression of the fundamental solution \( \Phi(x, t; \lambda) \) and definition of parameters \( \beta_i, \theta_i \) in equation (91), we could obtain that if we change the sign of \( y \), i.e., from \( \sqrt{\prod_{i=1}^{4}(\lambda - \lambda_i)} \) to \( -\sqrt{\prod_{i=1}^{4}(\lambda - \lambda_i)} \), the solution \( \Phi(x, t; \lambda) \) is just taking a column transformation. Thus, the fundamental solutions \( \Phi(x, t; \lambda) \) are equivalent to each other whatever we take positive or negative sign of parameter \( y \).

For the different background of elliptic function solutions, we divide them into following two conditions to study the limitations of \( \Phi(x, t; \lambda) \), as \( k \to 0^+ \):

**Theorem 8.** When \( l = 0 \), as \( k \to 0^+ \), the limit of matrix \( \Phi(x, t; \lambda) \) is

\[ \lim_{k \to 0^+} \frac{\Phi(x, t; \lambda)\sigma_3}{\sqrt{u^2(0, 0)} - \beta_1} = \begin{bmatrix} e^{-i\lambda x - 4i\lambda t} & 0 \\ 0 & e^{i\lambda x + 4i\lambda t} \end{bmatrix}. \]

When \( l = \frac{K'}{2} \), define \( v = \sqrt{-\lambda^2 - \alpha^2} \), \( \lambda \in (-i\infty, -i\infty) \cup (i\alpha, i\infty) \), we have

\[ \lim_{k \to 0^+} \frac{\Phi(x, t; \lambda)\sigma_3}{\sqrt{u^2(0, 0)} - \beta_1} = \Psi(x, t; \lambda), \quad \Psi(x, t; \lambda) := \begin{bmatrix} e^{-v(x + 2(\lambda^2 - \alpha^2))} & \frac{ae^{v(x + 2(\lambda^2 - \alpha^2))}}{v + i\lambda} \\ \frac{ae^{-v(x + 2(\lambda^2 - \alpha^2))}}{v + i\lambda} & e^{v(x + 2(\lambda^2 - \alpha^2))} \end{bmatrix}. \]

**Proof.** When \( l = 0 \), combining with the Proposition 3 and equation (A.7) and letting \( y = -\sqrt{\prod_{i=1}^{4}(\lambda - \lambda_i)} \), we get

\[ \lim_{k \to 0^+} \theta_1 = \lim_{k \to 0^+} \int_0^\xi \frac{2i\lambda \beta_1}{u^2(s) - \beta_1} ds = i\lambda \xi + 4i\lambda yt = \int_0^\xi -2i\lambda ds + i\lambda \xi + 4i\lambda yt = -i\lambda x - 4i\lambda t, \]
and
\[
\lim_{k \to 0^+} u^2(x, t) - \beta_2 = \lim_{k \to 0^+} (\text{akcn}(x - st))^2 - \left(2\lambda^2 + \frac{s}{2} + 2y\right) = -2\lambda^2 + \frac{a^2}{2} + 2\left(\lambda^2 - \frac{\alpha^2}{4}\right) = 0,
\]
(99)
\[
\lim_{k \to 0^+} u^2(x, t) - \beta_1 = \lim_{k \to 0^+} (\text{akcn}(x - st))^2 - \left(2\lambda^2 + \frac{s}{2} - 2y\right) = \alpha^2 - 4\lambda^2.
\]

Furthermore, since \(\lim_{k \to 0^+} \text{akcn}(x - st) = \lim_{k \to 0^+} \text{akcn}(0) = 0\), it is easy to obtain that
\[
\lim_{k \to 0^+} u^2(0, 0) - \beta_2 = \lim_{k \to 0^+} u^2(x, t) - \beta_2 = 0, \quad u^2(0, 0) - \beta_1 = \lim_{k \to 0^+} u^2(x, t) - \beta_1 = \alpha^2 - 4\lambda^2.
\]
(100)

Thus, combining with equations (98), (99), (100), we get the equation (96).

Similarly, we consider the case \(l = \frac{\alpha}{2}\) and let \(y = \sqrt{\prod_{i=1}^{4} (\lambda - \lambda_i)}\) which means \(y_0 = \lim_{k \to 0^+} y = i\lambda v\). It is easy to verify that
\[
\int_0^x \frac{2i\lambda (2\lambda^2 + \alpha^2 + 2y_0)}{\alpha^2 - (2\lambda^2 + \alpha^2 + 2y_0)} ds + i\lambda x = -i\lambda \frac{\lambda^2 + y_0 + \alpha^2}{\lambda^2 + y_0} x = -i\lambda \frac{\lambda^4 - y_0^2 + \alpha^2(\lambda^2 + y_0)}{\lambda^4 - y_0^2} x = \pm i y_0 x.
\]
(101)

Based on the definition of \(\beta_{1,2}\) in equation (91), the Proposition 3, equations (A.7) and (101), we obtain
\[
\lim_{k \to 0^+} \theta_{1,2} = \lim_{k \to 0^+} \int_0^s \frac{2i\lambda \beta_{1,2}}{u^2(s) - \beta_{1,2}} ds + i\lambda x = \pm i \frac{y_0}{\lambda} (x - 2\alpha^2 t) \pm 4i\lambda y_0 t
\]
(102)

Furthermore, we also could get
\[
\lim_{k \to 0^+} \frac{u^2(x, t) - \beta_2}{u^2(0, 0) - \beta_1} = \frac{\alpha^2 - (2\lambda^2 + \alpha^2 + 2y_0)}{\lambda^4 - y_0^2} = 1.
\]

Thus the equation (97) holds.

**Remark 8.** Expanding the right side of the matrix function (97) in the small neighborhood of \(\alpha = 0\), we obtain
\[
\begin{bmatrix}
\text{e}^{-i\lambda(x + 2(\lambda^2 - \alpha^2)t)} & \text{e}^{(x + 2(\lambda^2 - \alpha^2)t)} \\
\text{e}^{-i\lambda(x + 2(\lambda^2 - \alpha^2)t)} & \text{e}^{(x + 2(\lambda^2 - \alpha^2)t)}
\end{bmatrix}
= \begin{bmatrix}
\text{e}^{-i\lambda(x + 4\lambda^2 t)} + \mathcal{O}(\alpha^2) & \frac{\text{e}^{i\lambda(x + 4\lambda^2 t)}}{2\lambda} \alpha + \mathcal{O}(\alpha^2) \\
\frac{\text{e}^{-i\lambda(x + 4\lambda^2 t)}}{2\lambda} \alpha + \mathcal{O}(\alpha^2) & \text{e}^{i\lambda(x + 4\lambda^2 t)} + \mathcal{O}(\alpha^2)
\end{bmatrix}.
\]
(104)

Comparing with the right side of equation (96), we get that the equation (96) could be seen as the degeneration of equation (97) as \(\alpha \to 0\). So, in the following analysis, we will just consider the equation (97).

Define
\[
A(\lambda) := \lim_{k \to 0^+} \frac{\sigma_3}{\sqrt{u^2(0, 0) - \beta_1}} = a(\lambda)\sigma_3.
\]
(105)

Thus, the limitation of solution \(\Phi(x, t; \lambda)\) as \(k \to 0^+\) is
\[
\lim_{k \to 0^+} \Phi(x, t; \lambda) = \Psi(x, t; \lambda) A(\lambda),
\]
and
\[
\lim_{k \to 0^+} \Phi_t := \lim_{k \to 0^+} \Phi(x, t; \lambda_i) \left[ \frac{1}{c_i} \right] = \Psi(x, t; \lambda_i) \lim_{k \to 0^+} A(\lambda_i) \left[ \frac{1}{c_i} \right] = \Psi(x, t; \lambda_i) a(\lambda_i) \left[ \frac{1}{c_i} \right] = \Psi_i.
\]
(107)
Based on the Darboux transformation, the $N$-breather solution $v^{[N]}(x,t)$ under the constant background is obtained as

$$v^{[N]}(x,t) = \frac{a^{1-m}}{\det(D_m(x,t))} \det \left( aD_m(x,t) - 2i \left[ \begin{array}{c} \Psi_{1,2}^+ \\ \Psi_{2,2}^+ \\ \vdots \\ \Psi_{m,2}^+ \\ \Psi_{1,1}^- \Psi_{2,1}^- \cdots \Psi_{m,1}^- \end{array} \right] \right),$$

with

$$D_m(x,t) = \left( \frac{\Psi_i^+ \Psi_j^-}{\lambda_i - \lambda_j^z} \right)_{1 \leq i,j \leq m}.$$  

Since removing both the numerator and denominator of equation (108) from $a(\lambda_i), i = 1, 2, \cdots, m$ would not affect the result of solution $v^{[N]}(x,t)$, we could redefine $\Psi_i$ as

$$\Psi_i \equiv (\Psi(x,t;\lambda_i)^{-1}.$$

To sum up the above analysis, we get that as $k \to 0^+$, the elliptic-breather solution $u^{[N]}(x,t)$ could be degenerated to the constant-breather solutions $v^{[N]}(x,t)$ or solitons, because of $\lim_{k \to 0^+} u^{[N]}(x,t) = v^{[N]}(x,t)$. Furthermore, the vanishing background breather solutions [46] or solitons [8] could be obtained by studying the function $v^{[N]}(x,t)$ as $a \to 0$.

Then, we want to describe above degradation of solutions for the choosing of different spectral parameters $\lambda$. Here, we take the soliton as an example. The soliton could be seen as the constant solution obtained by a one-fold Darboux transformation, which is written as:

$$v^{[1]}(x,t) = -a + \frac{2(a^2 - \lambda_1^2)}{a - \lambda_1 \cosh \left[ \frac{2}{\sqrt{\lambda_1^2 - a^2}} (x - 2(2\lambda_1^2 + a^2)t) + \ln(c_1) \right]},$$

with $\lambda_1 = i\lambda_{11}, \lambda_{11}^2 > a^2$. The variation on region $R^\pm_1$ and the maximum of solution $v^{[1]}(x,t)$ are two important things of solitons under the constant background, since we could get the dynamic behavior of the function after getting above information. So we will study them in the following.

The solution $u^{[1]}(x,t)$ is obtained by the parameter $\lambda_1 \in i\mathbb{R}$, i.e., $z_1 = mK' + l + iz_l, m = -1,0,1$ gained from Appendix C directly. Combining the Theorem 4, it is easy to know that

$$v^{[1]}(x,t) = \lim_{k \to 0^+} u^{[1]}(\xi, t; R^\pm_1) \to 0, \quad t \to \pm \infty, \quad l = 0 \quad \text{and} \quad z_1 = mK' + iz_l, m = -1,0,1,$$

$$v^{[1]}(x,t) = \lim_{k \to 0^+} u^{[1]}(\xi, t; R^\pm_1) \to a, \quad t \to \pm \infty, \quad l = K' \frac{2}{2} \quad \text{and} \quad z_1 = l + iz_l,$$

$$v^{[1]}(x,t) = \lim_{k \to 0^+} u^{[1]}(\xi, t; R^\pm_1) \to -a, \quad t \to \pm \infty, \quad l = K' \frac{2}{2} \quad \text{and} \quad z_1 = \pm K' + l + iz_l.$$

Then we study the variation of peaks as $k \to 0^+$. In Section 2, we have known that the range of $u^{[1]}(x,t)$ is $\min(u(x,t)) - 2|/3(\alpha_1)|, \max(u(x,t)) + 2|/3(\alpha_1)|$]. Because the limitation $\lim_{k \to 0^+} u(x,t)$ is obtained in equation (89), we just study the value of $\zeta_1 = \lambda(z_1)$. From Appendix C, we consider the conformal map $\lambda(z)$ and obtain following cases:

(i) For any fixed $z_1 \in i\mathbb{R}$ or $z_1 = \pm K' + iz_l \in S$ with $l = 0$, we get $\Im(\lambda(z_1)) \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, +\infty)$ and $\lambda(z_1) \in i\mathbb{R}$. As $k \to 0^+$, $\lim_{k \to 0^+} \lambda(z_1)$ always satisfies $\Im(\lambda(z_1)) \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, +\infty)$ and $\lambda(z_1) \in i\mathbb{R}$ shown in Appendix C, which implies that the region of parameter $\lambda_1$ is independent of modulus $k$. Thus, the peak could not vanish and the maximum value of $\lim_{k \to 0^+} u^{[1]}(x,t)$ is $2|/3|\Im(\lim_{k \to 0^+} \lambda_1)| > 0$.

(ii) For any fixed $z_1 - l \in i\mathbb{R}$ and $l = K' \frac{2}{2}$, we get $\Im(\lambda(z_1)) \in \left(0, \frac{a(1-K')}{2}\right)$ and $\lambda(z_1) \in i\mathbb{R}$. Because of $\lim_{k \to 0^+} \frac{a(1-K')}{2} = 0$ proved in Appendix C, we get $\lim_{k \to 0^+} \lambda(z_1) = 0$. Combining with the maximum value analysis, we obtain that the peak would vanish as $k \to 0^+$. 

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(iii) For any fixed \( z_1 = \pm K' + l + iz \), we get \( \Im(\lambda(1 + K')) \in \left( \frac{a(1 + K')}{2}, \infty \right) \) and \( \lambda(z_1) \in i\mathbb{R} \).

Since \( \lim_{k \to 0^+} \frac{a(1 + K')}{2} = a \), we get \( \lim_{k \to 0^+} \lambda(z_1) \in (a, \infty) \). Thus, the peak could not vanish and \( \Im(\lim_{k \to 0^+} \lambda(z_1)) > \alpha \).

To illustrate the above situation more clearly, we provide some special examples. Firstly, we consider the cn-type background case with \( l = 0 \). For choosing \( z_1 = K' + \xi \), it is easy to verify that \( z_1 \) satisfies the case (i) and equation (112a). Plugging \( l = 0, \alpha = 1, K = \frac{1}{100}, c = 1 \) into equation (54), we obtain a soliton in Figure 13(a). Comparing Figure 13(a) with Figure 1(a) drawn by the modulus \( k = \frac{1}{2} \) with the same parameter \( z_1 = K' + \xi \), we know that when the modulus \( k \) changes from \( \frac{1}{2} \) to \( \frac{1}{100} \), the value of \( u^1(x, t) \) in equation (112a) tends to zero but the peak would not vanish, which provide a vividly description for the above analysis in case (i) and equation (112a).

Then, we consider the dn-type background case. Letting \( l = \frac{K'}{2} + i\xi \), we could verify that it satisfies the case (ii) and equation (112b). Plugging \( l = \frac{K'}{2} + i\xi \), \( \alpha = 1, k = \frac{2}{100}, c = 1 \) into equation (54), we draw this solution in Figure 13(b). Comparing with Figure 2(a) drawn by the modulus \( k = \frac{9}{10} \), we find that the small peak in Figure 2(a) gradually disappear in Figure 13(b), when the modulus \( k \) changes from \( \frac{9}{10} \) to \( \frac{2}{100} \). Furthermore, the amplitude is getting smaller and smaller with parameter \( \lambda_1 \) turning to zero. Thus, as \( k \) changes from \( \frac{9}{10} \) to \( \frac{2}{100} \), the function \( u^1(x, t) \) tends to be constant \( \alpha = 1 \) with the peak disappearing. This phenomenon confirms the above-mentioned analysis of condition (ii) and equation (112b).

When \( z_1 = -\frac{K'}{2} + i\xi \) and \( l = \frac{K'}{2} \), it falls into the case (iii) and equation (112c). Plugging \( l = \frac{K'}{2} + i\xi \), \( \alpha = 1, k = \frac{2}{100}, c = 1 \) into equation (54), we draw this soliton in Figure 13(c). Comparing with Figure 2(b) drawn by the modulus \( k = \frac{9}{10} \), we find that as the modulus \( k \) changing from \( \frac{9}{10} \) to \( \frac{2}{100} \), the peak in Figure 2(b) does not disappear in Figure 13(c) and the function \( u^1(x, t) \) in (112c) tends to \( -\alpha = -1 \). This phenomenon confirms the above-mentioned analysis of condition (iii) and equation (112c).

![Figure 13](image)

**Figure 13.** The 3d-plot of single breather solutions \( u^1(x, t) \) under one-fold Darboux transformation, as \( k \to 0^+ \). (a): The solution \( u^1(x, t) \) constructed by \( k = \frac{1}{100}, \alpha = 1, z_1 = K' + \xi \) under cn-type background. (b): The solutions \( u^1(x, t) \) constructed by \( k = \frac{2}{100}, c = 1 \) into equation (54), we draw this soliton in Figure 13(b). Comparing with Figure 2(a) drawn by the modulus \( k = \frac{9}{10} \), we find that the small peak in Figure 2(a) gradually disappear in Figure 13(b), when the modulus \( k \) changes from \( \frac{9}{10} \) to \( \frac{2}{100} \). Furthermore, the amplitude is getting smaller and smaller with parameter \( \lambda_1 \) turning to zero. Thus, as \( k \) changes from \( \frac{9}{10} \) to \( \frac{2}{100} \), the function \( u^1(x, t) \) tends to be constant \( \alpha = 1 \) with the peak disappearing. This phenomenon confirms the above-mentioned analysis of condition (ii) and equation (112b).

In addition to the solitons, the multi elliptic-breather solutions could also be degenerated into the constant-breather solutions as \( k \to 0^+ \) (in Figure 14). The single breather solutions \( u^1(x, t) \) in Figure 14(a) is obtained by \( k = \frac{1}{10}, l = 0, z_1 = -\frac{3K'}{2} + i\xi, \alpha = 1, c_1 = 3 + 4i, \lambda_1 \approx 0.696 - 0.612i \). Consider the function \( u^1(x, t) \) in region \( R^2 \). From the Theorem 4, we know that as \( k \to 0^+ \), the function \( u^1(x, t; R^2) \) tends to zero by equation (18), if the solution \( u^1(x, t) \) is constructed under the cn-type background. The Figure 14(b) is obtained by the dn-type background with \( k = \frac{1}{100} \). Choosing \( z_1 = -\frac{K'}{2} + i\xi, l = \frac{K'}{2}, \lambda_1 \approx 0.244 - 0.502i, c_1 = 1, \alpha = 1 \), we get a single breather solution \( u^1(x, t) \) in Figure 14(b). As \( k \) changing to \( \frac{1}{100} \), the function \( u^1(x, t; R^2) \) is approximating to the constant \( \alpha = 1 \) by equation (19) in Theorem 4. The breather
solution \( u^{(2)}(x,t) \) in Figure 14(c) is obtained by a four-fold Darboux transformation with \( k = \frac{1}{20}, \lambda_1 = 0.706 - 0.504i, \lambda_2 = -0.519 - 0.243i \).

The same as the case in Figure 14(a), the function \( u^{(1)}(x,t; R^+_{1}) \to 0 \), as \( k \to 0^+ \).

![Figure 14](image)

**Figure 14.** The 3d-plot of solutions \( u^{(1)}(x,t) \) or \( u^{(2)}(x,t) \) for the mKdV equation (1), as \( k \to 0^+ \). (a): The single breather solutions \( u^{(1)}(x,t) \) with \( k = \frac{1}{10} \) under the cn-type background. (b): The single breather solutions \( u^{(1)}(x,t) \) with \( k = \frac{1}{10} \), under the cn-type background. (c): A two breather solutions \( u^{(2)}(x,t) \) with \( k = \frac{1}{20} \) under the cn-type background.

Combining the above analysis, we get that as \( k \to 0^+ \), the multi elliptic-breather solutions could be degenerated into constant solutions, soliton or breather solutions under the zero or constant background, which reflect that the modulus \( k \) is an important bridge connecting those two groups of solutions together. Furthermore, it also could be used in describing some stable or unstable dynamic behaviors.

The breather solutions on the zero and constant background are well known in the previous literature [8, 46]. Through studying the maximum value of breathers solutions and soliton, it is easy to obtain that the imaginary part of the spectral parameter \( \lambda \) depends on the height of the peak for the solutions. In section 4, we know that as the modulus \( k \) decreasing, the periodic background generally going to constant background. Under the elliptic solution background, as the modulus \( k \) increases, the dynamic behavior of the nonlinearly superimposed soliton interacting with the spatial oscillations is more complex than the solitons or breathers under the constant backgrounds. Considering the peaks of solutions, the periodic background could influence the shape of the peaks but it do not influence the height of the peaks. Thus, when we choose a small enough modulus \( k \), the periodic background could be seen as the constant background adding a small perturbed. Fixing an initial time and changing the value of \( k \), the variation shapes of functions before and after changing could be seen as a perturbation on the initial function. Therefore, when we choose a suitable modulus \( k \) such that the error between the cn background breather solutions [25] and the zero background breather solutions [8, 46] sufficiently small, the elliptic-breather solution could be seen as the constant-breather solution by adding a small perturbation. As time changes, if this small perturbation has made a huge difference, we claim that this solution reflects an unstable dynamic behavior; if this small perturbation barely changes at any one moment, we deem that this solution reflects a stable dynamic behavior. Thus, through changing of the modulus \( k \), we could describe some stable and unstable dynamic behaviors vividly.

## 5 Conclusion and discussion

In this work, we systematically constructed the multi elliptic-breather solutions for the focusing mKdV through using the Jacobi theta functions. And then, we provide the exact expressions to the multi elliptic-breather solutions under the different elliptic function solution background and perform uniform processing.

There are some interesting properties proved in this paper. The asymptotic analysis of breathers with different velocity and the regions. To verify above two asymptotic analysis, we plot some figures of interaction between two breathers. The collision among multi elliptic-breathers are classified into the elastic and...
Jacobi elliptic function

To know that the stability of the breather solution. For the above works, we will analyze them in the near of the focusing NLS equation and focusing mKdV equation are obtained in [18, 32]. Naturally, we want addition, the breathers are also appeared in the wave field dynamics [47].

That the shaped molecular breathing light waves propagate in an almost conservative fiber optic system. In generated under several initial disturbance. Through experimental observations, Xu et al.[48] confirmed and so on.

Elliptic-breather solutions obtained here could motivate more studies in theories and experiments. In internal wave records in symmetrically stratified fluids, Lamb et al. [29] revealed that the breather were generated under several initial disturbance. Through experimental observations, Xu et al.[48] confirmed that the shaped molecular breathing light waves propagate in an almost conservative fiber optic system. In addition, the breathers are also appeared in the wave field dynamics [47].

The degenerated analysis among the branch points have not been studied. The rogue wave of the NLS equation in Jacobi theta function form is obtained by Feng et al. [19], but the rogue waves of the mKdV equation in theta function form are not provided. The stability analysis of the elliptic function solutions of the focusing NLS equation and focusing mKdV equation are obtained in [18, 32]. Naturally, we want to know that the stability of the breather solution. For the above works, we will analyze them in the near future.

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Appendix A. The definitions and properties of elliptic functions

Jacobi elliptic function

The function $K(k)$ and $E(k)$ in above equations are called the first and second complete elliptic integrals [5], which are defined as

\begin{equation}
K \equiv K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad \text{and} \quad E \equiv E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta.
\end{equation}

In addition to above two integrals, we usually use an associated complete elliptic integrals $K' = K(k'), k' = \sqrt{1 - k^2}$. Then, we list some useful formulas in [5] as follows:

- Zeros and poles of Jacobi elliptic functions:
  \begin{align}
  \text{sn}(2mK + (2n + 1)iK') &= \infty, \quad \text{sn}(2mK + 2niK') = 0, \\
  \text{cn}(2mK + (2n + 1)iK') &= \infty, \quad \text{cn}((2m + 1)K + 2niK') = 0, \\
  \text{dn}(2mK + (2n + 1)iK') &= \infty, \quad \text{dn}((2m + 1)K + (2n + 1)iK') = 0,
  \end{align}

where $n$ and $m$ are any integers including zero;

- Shift formulas:
  \begin{align}
  \text{sn}(u + K) &= \text{cd}(u), \quad \text{sn}(u + iK') = \text{ns}(u)/k, \quad \text{sn}(u + K + iK') = \text{dc}(u)/k, \\
  \text{cn}(u + K) &= -k'\text{sd}(u), \quad \text{cn}(u + iK') = -i\text{ds}(u)/k, \quad \text{cn}(u + K + iK') = -ik'\text{nc}(u)/k, \\
  \text{dn}(u + K) &= k'\text{nd}(u), \quad \text{dn}(u + iK') = -i\text{cs}(u), \quad \text{dn}(u + K + iK') = ik'\text{tn}(u);
  \end{align}

- Imaginary arguments:
  \begin{align}
  \text{sn}(iu, k) &= i\text{tn}(u, k'), \quad \text{cn}(iu, k) = \text{nc}(u, k'), \quad \text{dn}(iu, k) = \text{dc}(u, k'), \quad k^2 + k'^2 = 1;
  \end{align}

- Half arguments:
  \begin{align}
  \text{sn}^2\left(\frac{u}{2}\right) &= \frac{1 - \text{cn}(u)}{1 + \text{dn}(u)}, \quad \text{cn}^2\left(\frac{u}{2}\right) &= \frac{\text{dn}(u) + \text{cn}(u)}{1 + \text{dn}(u)}, \quad \text{dn}^2\left(\frac{u}{2}\right) &= \frac{\text{dn}(u) + \text{cn}(u)}{1 + \text{cn}(u)};
  \end{align}
where \( q = e^{i\pi\tau}, \tau = \frac{ik'}{K} \).

There are many relationships among the above four theta functions. Here, we just provide some common formulas:

- **Shift formulas among four theta functions in [3]:**
  \[
  \begin{align*}
  \vartheta_1(z) &= -\vartheta_2 \left( z + \frac{1}{2} \right) = -iM\vartheta_3 \left( z + \frac{1}{2} + \frac{\tau}{2} \right) = -iM\vartheta_4 \left( z + \frac{\tau}{2} \right), \\
  \vartheta_2(z) &= \vartheta_1 \left( z + \frac{1}{2} \right) = M\vartheta_3 \left( z + \frac{1}{2} + \frac{\tau}{2} \right) = M\vartheta_3 \left( z + \frac{\tau}{2} \right), \\
  \vartheta_3(z) &= \vartheta_4 \left( z + \frac{1}{2} \right) = M\vartheta_1 \left( z + \frac{1}{2} + \frac{\tau}{2} \right) = M\vartheta_2 \left( z + \frac{\tau}{2} \right), \\
  \vartheta_4(z) &= \vartheta_3 \left( z + \frac{1}{2} \right) = iM\vartheta_2 \left( z + \frac{1}{2} + \frac{\tau}{2} \right) = -iM\vartheta_1 \left( z + \frac{\tau}{2} \right),
  \end{align*}
\]

where \( M = q^{1/4}e^{i\pi\tau} \).

- **Conversion formulas between Jacobi theta functions and elliptic functions in [5]:**
  \[
  \begin{align*}
  \text{sn}(u) &= \frac{\vartheta_3\vartheta_1(z)}{\vartheta_2\vartheta_4(z)}, & \text{cn}(u) &= \frac{\vartheta_4\vartheta_2(z)}{\vartheta_3\vartheta_4(z)}, & \text{dn}(u) &= \frac{\vartheta_4\vartheta_3(z)}{\vartheta_3\vartheta_4(z)},
  \end{align*}
\]

where \( z = \frac{u}{2k} \) and \( k^{1/2} = \frac{k}{\vartheta_3} \).

- **Weierstrass addition formulas (or Fay’s identities) are given by [27]:**
  1. **Complimentary system:**
     \[
     \begin{align*}
     \vartheta_k(u + v)\vartheta_k(u - v)\vartheta_l(x + y)\vartheta_l(x - y)
     &= \vartheta_l(v + x)\vartheta_l(v - x)\vartheta_l(u + y)\vartheta_l(u - y) - \vartheta_l(u + x)\vartheta_l(u - x)\vartheta_l(v + y)\vartheta_l(v - y),
     \end{align*}
     \]
     where the combinations of \([k, l, i, j]\) are \([1, 4, 2, 3] \), \([1, 3, 2, 4] \) and \([1, 2, 3, 4] \);
  2. **Mixed identities:**
     \[
     \begin{align*}
     \vartheta_1(u + x)\vartheta_2(u - x)\vartheta_3(v - y)\vartheta_4(v + y) - \vartheta_1(u - y)\vartheta_2(u + y)\vartheta_3(v + x)\vartheta_4(v - x)
     &= \vartheta_1(x + y)\vartheta_2(x - y)\vartheta_3(u + v)\vartheta_4(u - v),
     \end{align*}
     \]
Jacobi Zeta function

The definition of the Jacobi Zeta function in [5] is as follows:

**Definition A.2.** The Jacobi Zeta function is defined by

\[
Z(u) \equiv \int_0^u \left( \frac{dn^2(v)}{\lambda} - \frac{E}{K} \right) dv,
\]

where \( E \equiv E(k), K \equiv K(k) \) is the complete elliptic integrals, defined in (A.1).

**Appendix B.** Darboux-Bäcklund transformation for mKdV equation

In this Appendix, we give some fundamental properties about Darboux-Bäcklund transformation.

**Proof of Theorem 7.** Based on the Darboux transformation, the \( N \)-fold Darboux transformation lead to the \( N \)-fold Darboux matrix (31), and it also could be rewritten as

\[
T^{[N]}(\lambda; x, t) = \mathbb{I} - X_m(x,t)M_m(x,t)^{-1}D_m(\lambda)^{-1}X_m(x,t)^t,
\]

where the dimension of matrix \( M_m(x,t) \) and \( D_m(\lambda) \) are \( m \) satisfying \( N \leq m = n_1 + 2n_2 \leq 2N \). It should be noted that \( n_1 \) and \( n_2 \) are the number of the Darboux matrix \( T^P_1(\lambda; x, t) \) (27) and \( T^C_1(\lambda; x, t) \) (28) in Darboux matrix \( T^{[N]}(\lambda; x, t) \), respectively.

Firstly, we verify

\[
T^{[N]}(\lambda; x, t) = T^P_1(\lambda; x, t) \cdots T^P_{i-1}(\lambda; x, t)T^P_i(\lambda; x, t)T^{[N-i]}(\lambda; x, t) = \mathbb{I} - iX_N(x,t)M_N(x,t)^{-1}D_N(\lambda)^{-1}X_N(x,t)^t,
\]

where \( T^P_i(\lambda; x, t) = \mathbb{I} - \frac{\lambda - \lambda^+_i}{\lambda - \lambda^-_i} \Phi_i \Phi_i^t \), \( T^{[N]}(\lambda; x, t) = \mathbb{I} - \frac{\lambda - \lambda^+_i}{\lambda - \lambda^-_i} \Phi_i \Phi_i^t \Phi_i \), \( \Phi_i = T^P_1(\lambda; x, t) \Phi_i \),

\[
M_N(x,t) = \left( \frac{\Phi_i \Phi_j^t}{\lambda_j - \lambda_i^t} \right)_{1 \leq i,j \leq N}, \quad D_N(\lambda) = \text{diag}(\lambda - \lambda_1^t, \lambda - \lambda_2^t, \ldots, \lambda - \lambda_N^t)
\]

Combing equation (B.2), we know

\[
T^{[N]}(\lambda; x, t) = \mathbb{I} + \sum_{i=1}^{N} \frac{|x_i^t \langle y_i|}{\lambda - \lambda_i^t}, \quad T^{[N]}(\lambda^*; x, t) = \mathbb{I} + \sum_{i=1}^{N} \frac{|y_i \langle x_i|}{\lambda - \lambda_i^t},
\]

where \( |y_i \rangle = (|y_i|)^t \) and \( |x_i \rangle \) is a vector with two rows and one column. Since \( T^{[N]}(\lambda; x, t) \) satisfies the symmetry (23),

\[
\text{Res}_{\lambda = \lambda_i} (T^{[N]}(\lambda; x, t)T^{[N]}(\lambda; x, t)^{-1}) = \text{Res}_{\lambda = \lambda_i} (T^{[N]}(\lambda; x, t)T^{[N]}(\lambda^*; x, t)) = T^{[N]}(\lambda; x, t)|y_i \rangle \langle x_i| = 0,
\]

which means \( T^{[N]}(\lambda; x, t) |y_i \rangle = 0, i = 1,2, \cdots, N \). Because \( T^{[N]}(\lambda; x, t) \) is a non-zero two-dimensional matrix, without loss of generality, we could set \( |y_i \rangle = \Phi_i \). Plugging \( |y_i \rangle = \Phi_i \) into equation (B.5), we gain

\[
(\Phi_1, \Phi_2, \ldots, \Phi_N) = - (|x_1\rangle, |x_2\rangle, \ldots, |x_N\rangle) M_N(x,t),
\]
which means
\[(B.8) \quad (|x_1\rangle, |x_2\rangle, \cdots, |x_N\rangle) = - (\Phi_1, \Phi_2, \cdots, \Phi_N) M_N^{-1}(x,t).\]

Therefore, we rewritten equation (B.5) as
\[(B.9) \quad T^{[N]}(\lambda; x, t) = I - (\Phi_1, \Phi_2, \cdots, \Phi_N) M_N^{-1}(x,t) D_N^{-1}(\lambda) \begin{pmatrix} \Phi_1^\dagger \\ \Phi_2^\dagger \\ \vdots \\ \Phi_N^\dagger \end{pmatrix},\]

where \(M_N(x, t)\) and \(D_N(\lambda)\) are defined in (B.3).

Then, we consider the fixed conditions including both the \(T^P_i(\lambda; x, t)\) and \(T^C_i(\lambda; x, t)\). For the Darboux matrix \(T^C_i(\lambda; x, t)\), we could divide it into two transformation as
\[(B.10) \quad T^C_i(\lambda; x, t) = \begin{pmatrix} \lambda - \Phi_1^\dagger \Phi_1 & 1 \\ 1 & \lambda - \Phi_2^\dagger \Phi_2 \end{pmatrix} \begin{pmatrix} \lambda - \Phi_1^\dagger \Phi_1 & 1 \\ 1 & \lambda - \Phi_2^\dagger \Phi_2 \end{pmatrix}.\]

Noticing the matrix \(T^C_i(\lambda; x, t)\) as two-fold Darboux transformation, we obtain equation (B.1) with the dimension of matrices \(M_m(x, t)\) and \(D_m(\lambda)\) satisfying \(N \leq m = n_1 + 2n_2 \leq 2N\). Based on the B"acklund transformation, the solution \(u^{[N]}(x,t)\) with elliptic-breather solutions could be obtained by (32).  \(\square\)

**Appendix C. The conformal map and the asymptotic analysis**

In this section, we briefly review the conformal mapping which is studied in [32], because this helps us a lot with the asymptotic analysis as \(k \to 0^+\). And then, we provide some preliminary results before the asymptotic analysis.

The conformal map \(\lambda(z)\) maps a rectangular region \(S\) to the entire complex plane \(\lambda\)-plane with two cuts. More details are given in reference [32]. Here, we consider some specific points \(\hat{z} = \pm \frac{K}{2} \pm i \frac{K}{2}\), which play an important role in the cuts in this conformal map. The schematic of this conformal map is given in Figure C.1 when \(l = 0\) and in Figure C.2 when \(l = \frac{K}{2}\).

![Figure C.1](image-url)  
*Figure C.1. The conformal map \(\lambda(z)\) in equation (10a) with \(l = 0\).*
We consider the four points $\hat{z}_i = \pm K/2 \pm i k, i = 1, 2, 3, 4$ that determine the endpoint of cuts in $\lambda$-plane. Combining with equation (10a), we obtain that when $l = 0$ those four points satisfy the equations

\[
\lambda_1 = \lambda(\hat{z}_1) = \lambda \left( \frac{K'}{2} + \frac{K}{2} \right) = \frac{ia}{2} (-k + i k'), \quad \lim_{k \to 0^+} \lambda_1 = \lim_{k \to 0^+} \frac{ia}{2} (-k + i k') = -\frac{\alpha}{2},
\]
\[
\lambda_2 = \lambda(\hat{z}_2) = \lambda \left( \frac{K'}{2} - \frac{K}{2} \right) = \frac{ia}{2} (k + i k'), \quad \lim_{k \to 0^+} \lambda_2 = \lim_{k \to 0^+} \frac{ia}{2} (k + i k') = -\frac{\alpha}{2},
\]
\[
\lambda_3 = \lambda(\hat{z}_3) = \lambda \left( \frac{-K'}{2} + \frac{i K}{2} \right) = \frac{ia}{2} (-k - i k'), \quad \lim_{k \to 0^+} \lambda_3 = \lim_{k \to 0^+} \frac{ia}{2} (-k - i k') = \frac{\alpha}{2},
\]
\[
\lambda_4 = \lambda(\hat{z}_4) = \lambda \left( \frac{-K'}{2} - \frac{i K}{2} \right) = \frac{ia}{2} (k - i k'), \quad \lim_{k \to 0^+} \lambda_4 = \lim_{k \to 0^+} \frac{ia}{2} (k - i k') = \frac{\alpha}{2}.
\]

Then, we analyze the changes in cuts, as $k \to 0^+$. For any point on the cut in $\lambda$-plane, there must exist two points $z_1 \neq z_2$ on the lines $z \in \left\{ z \in S \mid z = z_R \pm i \frac{K}{2} \right\}$ satisfying $\lambda(z_1) = \lambda(z_2)$. Therefore, we mainly consider the above two lines and obtain the Lemma C.5.

**Lemma C.5.** When $l = 0$, for any $k \in (0, 1)$ and $z \in \left\{ z \in S \mid z = z_R \pm i \frac{K}{2} \right\}$, the value of function $\lambda(z)$ is on the circle centered at the origin with radius $\frac{\alpha}{2}$.

**Proof.** For any $z \in \left\{ z \in S \mid z = z_R \pm i \frac{K}{2} \right\}$, through utilizing the formulas (A.5), we know that

\[
\lambda \left( z_R \pm i \frac{K}{2} \right) = \frac{ia}{2} \frac{\text{sn}(\mp \frac{K}{2} + iz_R) \text{dn}(\mp \frac{K}{2} + iz_R)}{\text{cn}(\mp \frac{K}{2} + iz_R)}
\]
\[
= \frac{ia}{2} \frac{1 - \text{cn}(\mp K + 2iz_R)}{\text{sn}(\mp K + 2iz_R)}
\]
\[
= \frac{ia}{2} \frac{\text{dn}(2iz_R) \pm k' \text{sn}(2iz_R)}{\pm \text{cn}(2iz_R)}
\]
\[
= \pm \frac{ia}{2} \left( \text{dn}(2z_R, k') \mp i k' \text{sn}(2z_R, k') \right),
\]

which implies that for any $k \in (0, 1)$ and $z \in \left\{ z \in S \mid z = z_R \pm i \frac{K}{2} \right\}$, the value of $\lambda(z)$ must satisfy $|\lambda(z)| = \frac{\alpha}{2}$. Thus, $\lambda(z)$ is on the circle centered at the origin with radius $\frac{\alpha}{2}$. \hfill \Box

Then, we consider the case $l = \frac{K'}{2}$. By the definition of $\lambda(z)$ in equation (10b), it is easy to verify that

\[
\lambda_1 = \lambda(\hat{z}_1) = \lambda \left( \frac{K'}{2} + \frac{i K}{2} \right) = -\frac{ia}{2} (1 - k'), \quad \lim_{k \to 0^+} \lambda_1 = \lim_{k \to 0^+} -\frac{ia}{2} (1 - k') = 0,
\]
\[
\lambda_2 = \lambda(\hat{z}_2) = \lambda \left( \frac{K'}{2} - \frac{i K}{2} \right) = \frac{ia}{2} (1 - k'), \quad \lim_{k \to 0^+} \lambda_2 = \lim_{k \to 0^+} \frac{ia}{2} (1 - k') = 0,
\]
\[
\lambda_3 = \lambda(\hat{z}_3) = \lambda \left( \frac{-K'}{2} + \frac{i K}{2} \right) = \frac{ia}{2} (1 + k'), \quad \lim_{k \to 0^+} \lambda_3 = \lim_{k \to 0^+} \frac{ia}{2} (1 + k') = -ia,
\]
\[
\lambda_4 = \lambda(\hat{z}_4) = \lambda \left( \frac{-K'}{2} - \frac{i K}{2} \right) = \frac{ia}{2} (1 + k'), \quad \lim_{k \to 0^+} \lambda_4 = \lim_{k \to 0^+} \frac{ia}{2} (1 + k') = ia.
\]

**Lemma C.6.** When $l = \frac{K'}{2}$, for any $k \in (0, 1)$ and $z \in \left\{ z \in S \mid z = z_R \pm i \frac{K}{2} \right\}$, the value of function $\lambda(z)$ is on the imaginary axis.

\[38\]
which means that for any $z$, through utilizing the formulas (A.5), we know that

$$
\lambda^* \left( z_R \pm \frac{iK}{2} \right) = \left( \frac{\text{sn}(\mp \frac{K}{2} + iz_R) \text{cn}(\mp \frac{K}{2} + iz_R)}{\text{dn}(\mp \frac{K}{2} + iz_R)} \right) \ast
$$

(C.14)

which means that for any $z \in \{ z \in S \mid z = z_R \pm i \frac{K}{2} \}$, $\lambda(z) \in i\mathbb{R}$. Thus, the value of $\lambda(z)$ must on the imaginary axis for any $k \in (0, 1)$. 

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