THE ÉTALE COHOMOLOGY RING OF A PUNCTURED ARITHMETIC CURVE

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Abstract. We compute the cohomology ring $H^\ast(U, \mathbb{Z}/n\mathbb{Z})$ for $U = X \setminus S$ where $X$ is the spectrum of the ring of integers of a number field $K$ and $S$ is a finite set of finite primes.

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1. Introduction

Let $K$ be a number field with ring of integers $\mathcal{O}_K$ and let $X = \text{Spec } \mathcal{O}_K$. In this paper we compute the étale cohomology ring $H^\ast(U, \mathbb{Z}/n\mathbb{Z})$ where $U \subseteq X$ is an open subscheme. This generalizes the results in [AC18] not only by going from $X$ to an open subscheme $U$, but also in the sense that it covers the case when $K$ has a real place, $n$ is even, and $U = X$. Our results also generalize those of McCallum–Sharifi: in [MS03] they compute the cup product $H^1(U, \mu_n) \otimes H^1(U, \mu_n) \to H^2(U, \mu_n^{\otimes 2})$ when $K$ contains the $n$th roots of unity and $U = X \setminus S$ for $S$ a finite set of places containing the places above $n$ (note that since $S$ contains the places above $n$ and $K$ contains the $n$th roots of unity, $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$). The computations we make in this article gives the cup product for an arbitrary finite set of non-archimedean places $S$, and we do not assume that $K$ contains the $n$th roots of unity.

The ring $H^\ast(U, \mathbb{Z}/n\mathbb{Z})$ holds non-trivial arithmetic information about $K$. One illustration of this is the following example (see Section 4) first observed by Morishita: Let $U = \text{Spec } \mathbb{Z} \setminus \{p, q\}$ where $p$ and $q$ are primes which are both 1 (mod 4). Taking rings of integers of the quadratic extensions $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{-7})$, we get two elements $x_p, x_q \in H^1(U, \mathbb{Z}/2\mathbb{Z})$ and the element $x_p \sim x_q$ is completely determined by the Legendre symbol $\left(\frac{q}{p}\right)$ and vice versa. In particular, $x_p \sim x_q = 0$ if and only if $\left(\frac{q}{p}\right) = 1$.

We will now state the main results of this paper. Let $I^+_K$ be the group of non-complex idèles of $K$ and let $C_S(K)$ denote the $S$-idèle class group. We begin with stating the computation of the étale cohomology groups.

Theorem 2.5. Let $K_+ = \text{the set of totally positive elements in } K$ and $\text{Cl}^+(K)$ the narrow class group of $K$. Furthermore, let $r$ be the number of real places of $K$ and for any abelian group $A$ we denote by
\(A^\sim = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})\), the Pontryagin dual of \(A\). We then have
\[
H^i(X, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}/n\mathbb{Z} & i = 0, \\
(\text{Cl}^+(K)/n\text{Cl}^+(K))^\sim & i = 1, \\
(\mathbb{Z}^1/B^1)^\sim & i = 2, \\
(\mu_n(K_+) \oplus (\mathbb{R}^\times/n\mathbb{R}^\times)^\sim & i = 3, \\
((\mathbb{R}^\times/n\mathbb{R}^\times)^\sim & i \geq 4.
\end{cases}
\]

where
\[
Z^1 = \{(a, I) \in K^\times_+ \oplus \text{Div} U : \text{div}(a)I^n = 1\}, \\
B^1 = \{(b^{-n}, \text{div}(b)) : b \in K^\times\}.
\]

For a finite, non-empty set of finite places \(S\) and \(U = X \setminus S\), we have that
\[
H^i(U, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 
\mathbb{Z}/n\mathbb{Z} & i = 0, \\
(\text{Cl}_S(K)/n\text{Cl}_S(K))^\sim & i = 1, \\
(\text{Cl}_S(K)[n])^\sim & i = 2, \\
((\mathbb{R}^\times/n\mathbb{R}^\times)^\sim & i \geq 3.
\end{cases}
\]

The following theorem describes the cup product structure in \(H^*(U, \mathbb{Z}/n\mathbb{Z})\) when \(S\) contains at least one finite place. This is in some sense the only “interesting” cup product in this situation.

**Theorem 3.9** Suppose that \(S\) is a finite non-empty set of finite places and let \(U = X \setminus S\). Let \(y\) and \(z\) be elements in \(H^1(U, \mathbb{Z}/n\mathbb{Z})\) represented by cyclic extensions \(L/K\) and \(M/K\) unramified outside of \(S\) respectively and assume that \(L/K\) has degree \(d_n\). Then under the identifications \(H^1(U, \mathbb{Z}/n\mathbb{Z}) \cong (\text{Cl}_S(K)/n\text{Cl}_S(K))^\sim\) and \(H^2(U, \mathbb{Z}/n\mathbb{Z}) \cong (\text{Cl}_S(K)[n])^\sim\) we have that \(y \cup z \in (\text{Cl}_S(K)[n])^\sim\) satisfies the formula
\[
\langle y \cup z, \alpha \rangle = \langle z, \alpha^{n/2d}N_{L/K}(\beta)^{n/d} \rangle
\]
where \(\langle - , - \rangle\) is the evaluation map, \(\beta\) is an element in \(I_L\) such that \(\alpha^{n/d} = t\beta/\sigma(\beta)\), and \(t \in L^\times\) satisfies \(N_{L/K}(t) = \alpha^{-n}\).

For the case when \(S = \emptyset\) and \(K\) is totally imaginary, we have already computed the cohomology ring in \(\text{AC18}\), as previously noted. The following theorem is essential for determining the cohomology ring structure of \(X = \text{Spec} \mathcal{O}_K\) when \(K\) is not necessarily totally imaginary; in some sense, as in the situation when \(S \neq \emptyset\), when \(K\) has a real place, there is only one interesting class of cup products, and they are given by:

**Theorem 3.11** Let \(y\) and \(z\) be elements in \(H^1(X, \mathbb{Z}/n\mathbb{Z})\) represented by cyclic extensions \(L/K\) and \(M/K\), unramified at all finite places, respectively and assume that \(L/K\) has degree \(d_n\). Choose a generator \(\sigma \in \text{Gal}(L/K)\). Then under the identifications \(H^1(X, \mathbb{Z}/n\mathbb{Z}) \cong (\text{Cl}^+(K)/n\text{Cl}^+(K))^\sim\) and \(H^2(X, \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}^1/B^1)^\sim\) we have that \(y \cup z \in (\mathbb{Z}^1/B^1)^\sim\) satisfies the formula
\[
\langle y \cup z, (b, b) \rangle = \langle z, b^{n/2d}N_{L/K}(I)^{n/d} \rangle
\]
where \(I\) is an element in \(\text{Div}(L)\) such that \(b^{n/d} = \text{div}(t)I/\sigma(\sigma)\), where \(t \in L^\times\) satisfies \(N_{L/K}(t) = b^{-1}\).

The formulas we have found for the cup product in étale cohomology have already been applied to problems in number theory a number of times. We mention a few:

1. The inverse Galois problem \([CS16]\);
2. Verifying the unramified Fontaine–Mazur conjecture in some special cases \([Ma17]\);
3. Arithmetic Chern Simons theory \([AC18]\);
4. Mod 2 arithmetic Dijkgraaf-Witten invariants for certain real quadratic number fields \([Hir19]\).

The results of this paper gives a more complete picture of the étale cohomology of arithmetic curves and we hope that there are a lot more applications in number theory to be found in a not too distant future.
We will now start by recalling the definition of compactly supported fpfp cohomology, following [Mil06]. We have a canonical inclusion $C \rightarrow \prod_{\alpha} \mathbb{C}$ where $\mathbb{C}$ is the set of archimedean places of $K$.

Notation and conventions. Throughout the paper we use the following notation:

- $K$ is a number field,
- $X = \text{Spec } \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of $K$,
- $\Omega$ is the set of places of $K$,
- $\Omega_\infty$ is the set of archimedean places of $K$,
- $\Omega_\mathbb{C}$ is the set of complex places of $K$,
- $\Omega_\mathbb{R}$ is the set of real places of $K$,
- $S \subseteq \Omega$ is a finite non-empty set of places containing $\Omega_\mathbb{R}$,
- $S_I = S \setminus \Omega_\mathbb{R}$,
- $U = X \setminus S_I$,
- $j : \text{Spec } K \rightarrow X$ is the canonical inclusion,
- $U_{\text{fpf}}$ is the big fpf site of $U$,
- $\mu_n$ is the fpf-sheaf of $n$th roots of unity.

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2. The cohomology of a punctured arithmetic curve

The purpose of this section is to compute the étale cohomology groups $H^i(U, \mathbb{Z}/n\mathbb{Z})$. We compute these cohomology groups by using essentially the same techniques as in [ACTS], where we computed the cohomology groups $H^i(X, \mathbb{Z}/n\mathbb{Z})$ for $X = \text{Spec } \mathcal{O}_K$ with $K$ totally imaginary and $\mathcal{O}_K$ the ring of integers of $K$.

Before we compute the cohomology groups we will establish some notation. For a non-archimedean place $p$, $K_p$ denotes the usual completion of $K$ and at an archimedean place, $K_p = \mathbb{R}$ if $p$ is real, and $K_p = \mathbb{C}$ if $p$ is complex. We define $\mathcal{O}_p$ to be unit ball in $K_p$ if $p$ is non-archimedean, and we let $\mathcal{O}_p = K_p$ if $p$ is archimedean. Let us now define $I_K$ as the group of (non-complex) idèles

$$I_K = \prod_{p \in \Omega \setminus \Omega_\mathbb{C}} K_p^\times,$$

where $\prod'$ means that we take the restricted product with respect to the subgroup $\prod_{p \in \Omega \setminus \Omega_\mathbb{C}} \mathcal{O}_p^\times$. Define

$$C_K = I_K/K^\times,$$

$$C_S(K) = I_K/K^\times U_{K,S},$$

where $U_{K,S} = \prod_{p \in S} \{1\} \times \prod_{p \notin U} \mathcal{O}_p^\times$ and $K$ is embedded diagonally.

Let us also define $I_{K,S} = \prod_{p \in S} K_p^\times$, $\mathcal{O}_{K,S} = \{x \in K : |x|_p \leq 1 \text{ for all } p \notin S\}$, and $C_{K,S} = I_{K,S}/\mathcal{O}_{K,S}^\times$. We have a canonical inclusion $C_{K,S} \hookrightarrow C_S(K)$ induced by the map $I_{K,S} \rightarrow I_K$ which takes $(\alpha_p)_{p \in S}$ to the idèle which is $\alpha_p$ at places in $S$ and 1 outside of $S$. By [NSW05] Proposition 8.3.5] we have an exact sequence of topological groups

$$1 \rightarrow C_{K,S} \rightarrow C_S(K) \rightarrow \text{Cl}_S(K) \rightarrow 1,$$

where $\text{Cl}_S(K)$ is the $S$-ideal class group considered as a discrete group.
Flat cohomology with compact support. Following [DH19] and [GST8], we now proceed by defining flat cohomology with compact support.

Let $F$ be a bounded complex of abelian sheaves on $U_{\text{fppf}}$. Define $Z' = \Pi_{p \in X \setminus \text{Spec } K_p} K_p$ and write $\gamma: Z' \to U$ for the canonical morphism. By [Sta21, Tag 06VX] the big fppf site has enough points. According to [Joh02, C.2.2.11] one can choose a set of points such that the points are jointly conservative. We now let $F \to G(F)$ be the Godement resolution with respect to a fixed choice of a set of points which are jointly conservative. We thus resolve $F$ degreewise so that $G(F)$ is a double complex. Then we have the unit map

$$\Gamma(U, G(F)) \to \Gamma(U, \gamma^*G(F)) = \Gamma(Z', \gamma^*G(F)).$$

For each $\nu \in \Omega_k$ we let $a_{\nu}^*: (\text{Sch}/\text{Spec } \mathbb{R})_{\text{fppf}} \to (\text{Spec } \mathbb{R})_{\text{fppf}}$ be the canonical morphism of sites. Then $a_{\nu}^*$ is exact and hence $a_{\nu}^*G(F)_\nu$ is a resolution of $a_{\nu}^*F_\nu$ into acyclics. As explained in [GST8] §2, there is a resolution $D^\bullet(a_{\nu}^*F_\nu)$ of $a_{\nu}^*F_\nu$ which is pointwise acyclic, and splicing $D^\bullet(a_{\nu}^*F_\nu)$ and $a_{\nu}^*G(F)_\nu$ together, one gets a functorial complete (pointwise) acyclic resolution $\hat{G}(F_\nu)$ of $a_{\nu}^*F_\nu$. The resolution $\hat{G}(F_\nu)$ computes the Tate-hypercohomology of $a_{\nu}^*F_\nu$ and there is a canonical map $a_{\nu}^*G(F)_\nu \to \hat{G}(F_\nu)$.

We define

$$R\hat{\Gamma}_c(U_{\text{fppf}}, F) = \text{Tot}(C(\Gamma(U, G(F)) \to \Gamma(Z', \gamma^*G(F)) \oplus \bigoplus_{\nu \in \Omega_k} \Gamma(K_\nu, \hat{G}(F_\nu))[1]),$$

and

$$R\Gamma_c(U_{\text{fppf}}, F) = \text{Tot}(C(\Gamma(U, G(F)) \to \Gamma(Z', \gamma^*G(F)) \oplus \bigoplus_{\nu \in \Omega_k} \Gamma(K_\nu, a_{\nu}^*G(F)_\nu))[1]),$$

where $C(-)$ denotes the mapping cone. Then we have a canonical natural transformation

$$R\Gamma_c(U_{\text{fppf}}, F) \to R\hat{\Gamma}_c(U_{\text{fppf}}, F).$$

We will also at times view $R\Gamma_c(U_{\text{fppf}}, F)$ and $R\hat{\Gamma}_c(U_{\text{fppf}}, F)$ as objects of $D(\text{Ab})$.

Definition 2.1. For a bounded complex $F$ of abelian sheaves on $U_{\text{fppf}}$, we define

$$H^i_c(U_{\text{fppf}}, F) = H^i(R\hat{\Gamma}_c(U_{\text{fppf}}, F)).$$

The following two definitions will be useful in Section 3. Given a bounded complex $F$ of abelian sheaves on $U_{\text{fppf}}$, we now define, functorially in $F$, an object $\Gamma_c(U_{\text{fppf}}, F)$, which will be a complex of abelian groups.

Definition 2.2. For a bounded complex $F$ of abelian sheaves on $U_{\text{fppf}}$, we define $\Gamma_c(U_{\text{fppf}}, F)$ to be the complex given by

$$H^0(C(\Gamma(U, G(F)) \to \Gamma(Z', \gamma^*G(F)) \oplus \bigoplus_{\mu \in \Omega_k} \Gamma(K_\nu, a_{\nu}^*G(F)_\nu))[1]),$$

where we take cohomology “vertically”.

Hence we have canonical natural transformations

$$\Gamma_c(U_{\text{fppf}}, F) \to R\Gamma_c(U_{\text{fppf}}, F) \to R\hat{\Gamma}_c(U_{\text{fppf}}, F).$$

Cohomology of $U$. We will now resolve $\mu_n$. We have $\mu_n \cong \mathbb{H}\text{om}(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$, where we are taking the internal hom in the category of fppf sheaves. We define resolutions $E \to \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{G}_{m,U} \to \mathcal{C}$ by

$$\mathcal{E} = (\mathbb{Z} \xrightarrow{n} \mathbb{Z})$$

and

$$\mathcal{C} = (j_\ast \mathbb{G}_{m,K} \xrightarrow{\text{div}} \mathbb{Div}_U),$$

where $\mathcal{E}$ is concentrated in degree $-1, 0$, $\mathcal{C}$ is concentrated in degree $0, 1$, and

$$\mathbb{Div}_U = \bigoplus_{p \in U} (i_p)_\ast \mathbb{Z}$$
is the sheaf of Weil divisors, where \( i_p : \text{Spec} \, \kappa(p) \to U \) is the inclusion of the residue field (for a proof that \( C \) is a resolution of \( G_m \), see [MHS09 II.3.9]). The complex \( \mathcal{H}om(E, C) \) is a resolution of \( \mu_n \) for the following reasons. First, since the map \( n : G_m \to G_m \) is an epimorphism of sheaves, \( \mathcal{H}om(E, G_m) \) is a resolution of \( \mu_n \). Then, since \( E \) is a complex of locally free sheaves, \( \mathcal{H}om(E, -) \) is exact, so \( \mathcal{H}om(E, -) \) preserves the quasi-isomorphism \( G_m \to C \), which gives that \( \mathcal{H}om(E, C) \) is a resolution of \( \mu_n \).

There is a horizontal filtration on \( R\hat{\Gamma}_c(U_{fppf}, \mathcal{H}om(E, C)) \) coming from the columns of the corresponding double complex

\[
L = C(\Gamma(U, G(\mathcal{H}om(E, C)))) \to \Gamma(Z', \gamma^* G(\mathcal{H}om(E, C))) \oplus \bigoplus_{p \in \Omega_k} \Gamma(K_{p'}, \hat{G}(\mathcal{H}om(E, C)_{p'}))[−1],
\]

so one obtains a spectral sequence

\[
E^{s,t}_2 = H^s(\Gamma(L)) \Rightarrow H^{s+t}(R\hat{\Gamma}_c(U_{fppf}, \mathcal{H}om(E, C))) \cong H^{s+t}_c(U_{fppf}, \mu_n)
\]

with \( E^{s,t}_1 = H^t(L^{s,*}) \). We will now use this spectral sequence to calculate \( H^*_c(U_{fppf}, \mu_n) \), and proceed by calculating the \( E_2 \)-page. The following well-known lemma is crucial for computing the \( E_2 \)-page.

**Lemma 2.3.** For any \( \nu \in \Omega_k \), we have

\[
H^*_c(\text{Gal}(\mathcal{K}_k/K_{fppf}), G_m \mu_n) \cong \begin{cases} K^\times_p / 2K^\times_p & \text{if } i \text{ is even}, \\ 0 & \text{if } i \text{ is odd}. \end{cases}
\]

**Proof.** Since \( H^*_c(\text{Gal}(\mathbb{C}/\mathbb{R}), G_m \mu_n) \) is periodic in \( i \) with period 2 [Neu13 Theorem I.6.1], it is enough to compute for instance \( H^*_c(\text{Gal}(\mathbb{C}/\mathbb{R}), G_m \mu_n) \) and \( H^*_c(\text{Gal}(\mathbb{C}/\mathbb{R}), G_m \mu_n) \). This is left to the reader. \( \square \)

For \( 0 \leq t \leq 2 \) we have that \( E^{s,t}_1 = H^t(L^{s,*}) \) is

\[
\begin{array}{cccccc}
\text{Br}(K) & \to & \text{Br}(K) \oplus \prod_{p \in S_j} \mathbb{Q}/\mathbb{Z} & \oplus & \prod_{\nu \in \Omega_k} K^\times_p / 2K^\times_p & \to & \prod_{p \in S_j} \mathbb{Q}/\mathbb{Z} \oplus \prod_{\nu \in \Omega_k} K^\times_p / 2K^\times_p \\
0 & \to & 0 & \to & 0 & \to & 0
\end{array}
\]

\[
\begin{array}{cccccc}
K^\times & \to & K^\times \oplus \text{Div } U \oplus \prod_{p \in S_j} K^\times_p \oplus \prod_{\nu \in \Omega_k} K^\times_p / 2K^\times_p & \to & \text{Div } U \oplus \prod_{p \in S_j} K^\times_p \oplus \prod_{\nu \in \Omega_k} K^\times_p / 2K^\times_p \\
\text{d}^{0,2} & & & & & \\
\text{d}^{1,0} & & & & & 
\end{array}
\]

where \( \text{d}^{0,2} \) is injective since it is the invariant map into the second factor. Hence the \( E_2 \)-page may be pictured as

\[
\begin{array}{cccc}
2 & \ast & \ast & \\
0 & \ast & \ast & \ast \\
-2 & \ast & \ast & \ast \\
0 & 2 & 4 \\
\end{array}
\]

in vertical degrees \(-2 \leq t \leq 2\). We see that \( E^{1,−2}_2, E^{1,2}_2, \) and \( E^{2,2}_2 \) will not contribute to \( H^*_c(U_{fppf}, \mu_n) \) for \( 1 \leq i \leq 2 \). Hence we conclude that for \( 1 \leq i \leq 2 \), \( H^*_c(U_{fppf}, \mu_n) \) is the \( i \)th cohomology of the complex

\[
K^\times \xrightarrow{\text{d}^{0,0}} K^\times \oplus \text{Div } U \oplus \prod_{p \in S_j} K^\times_p \oplus \prod_{\nu \in \Omega_k} K^\times_p / 2K^\times_p \xrightarrow{\text{d}^{1,0}} \text{Div } U \oplus \prod_{p \in S_j} K^\times_p \oplus \prod_{\nu \in \Omega_k} K^\times_p / 2K^\times_p
\]

where

\[
\text{d}^{0,0} = \left( \begin{array}{ccc}
-n \\
\eta \\
\eta \\
\eta \gamma_p
\end{array} \right) \quad \text{and} \quad \text{d}^{1,0} = \left( \begin{array}{ccc}
\text{div} & n & 0 & 0 \\
0 & n & 0 & 0 \\
0 & 0 & n & 0 \\
(\gamma_p)_v
\end{array} \right).
\]
Lemma 2.4. Let $K_+$ be the set of totally positive elements in $K$ and $\text{Cl}^+(K)$ the narrow class group of $K$. Furthermore, let $r$ be the number of real places of $K$. For $S = \Omega_R$ we have

\[
H_c^i(X_{\text{fppt}}, \mu_n) = \begin{cases} 
(\mathbb{R}^\times / n\mathbb{R}^\times)^r & i < 0, \\
\mu_n(K_+) \oplus (\mathbb{R}^\times / n\mathbb{R}^\times)^r & i = 0, \\
\text{Z}/B^1 & i = 1, \\
\text{Cl}^+(K) / n\text{Cl}^+(K) & i = 2, \\
\mathbb{Z}/n\mathbb{Z} & i = 3, \\
0 & i \geq 4,
\end{cases}
\]

where

\[Z^1 = \{(a, I) \in K_+^\times \oplus \text{Div} U : \text{div}(a)I^n = 1\},\]
\[B^1 = \{(b^{-n}, \text{div}(b)) : b \in K^\times\}.
\]

For $S \neq \Omega_R$ we have

\[
H_c^i(U_{\text{fppt}}, \mu_n) = \begin{cases} 
(\mathbb{R}^\times / n\mathbb{R}^\times)^r & i < 0, \\
\mu_n(K_+) \oplus (\mathbb{R}^\times / n\mathbb{R}^\times)^r & i = 0, \\
\text{C}_S(K)[n] & i = 1, \\
\text{Cl}^+(K) / n\text{C}_S(K) & i = 2, \\
\mathbb{Z}/n\mathbb{Z} & i = 3, \\
0 & i \geq 4.
\end{cases}
\]

Note in the above that $K^+ = K$ when $K$ is totally imaginary. By flat Artin–Verdier duality (see [DH19, Thm 1.1]) we get:

Theorem 2.5. Let $K_+$ be the set of totally positive elements in $K$ and $\text{Cl}^+(K)$ the narrow class group of $K$. Furthermore, let $r$ be the number of real places of $K$ and for any abelian group $A$ we denote by $A^\sim = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$, the Pontryagin dual of $A$. For $S = \Omega_R$ we have

\[
H^i(X, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}/n\mathbb{Z} & i = 0, \\
(\text{Cl}^+(K) / n\text{Cl}^+(K))^\sim & i = 1, \\
(Z^1 / B^1)^\sim & i = 2, \\
(\mu_n(K_+) \oplus (\mathbb{R}^\times / n\mathbb{R}^\times)^r)^\sim & i = 3, \\
((\mathbb{R}^\times / n\mathbb{R}^\times)^r)^\sim & i \geq 4.
\end{cases}
\]

where

\[Z^1 = \{(a, I) \in K_+^\times \oplus \text{Div} U : \text{div}(a)I^n = 1\},\]
\[B^1 = \{(b^{-n}, \text{div}(b)) : b \in K^\times\}.
\]

For $S \neq \Omega_R$ we have

\[
H^i(U, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 
\mathbb{Z}/n\mathbb{Z} & i = 0, \\
(\text{C}_S(K) / n\text{C}_S(K))^\sim & i = 1, \\
(\text{C}_S(K)[n])^\sim & i = 2, \\
((\mathbb{R}^\times / n\mathbb{R}^\times)^r)^\sim & i \geq 3.
\end{cases}
\]

Proof of Lemma 2.4. We saw that, for $1 \leq i \leq 2$, $H_c^i(U_{\text{fppt}}, \mu_n)$ is the $i$th cohomology of the complex

\[
K^\times \xrightarrow{d^0} K^\times \oplus \text{Div} U \oplus \prod_{p \in \mathcal{S}_f} K_p^\times \oplus \prod_{\nu \in \Omega_h} K_\nu^\times / 2K_\nu^\times \xrightarrow{d^1} \text{Div} U \oplus \prod_{p \in \mathcal{S}_f} K_p^\times \oplus \prod_{\nu \in \Omega_h} K_\nu^\times / 2K_\nu^\times
\]
where

\[
d^l = \begin{pmatrix} -n \\ \text{div} \\ \eta \end{pmatrix} \quad \text{and} \quad d^l = \begin{pmatrix} \text{div} & n & 0 \\ \eta & 0 & n \end{pmatrix}.
\]

We first consider the case \( S = \Omega \). The cases \( i = 1 \) and \( i = 2 \) follows by the approximation theorem \([\text{Neu99} \ §3, (3.4)]\), which implies that the map \( K^\times \to \prod_{\nu \in \Omega} K^\times_{\nu}/2K^\times_{\nu} \) is surjective. In the negative degrees, we just get the Tate cohomology of \( \mu_n \) at the real places.

Next we consider the case \( S \neq \Omega \). The map \( \eta \) is injective and hence \( H^i_c(U_{\text{fppf}}, \mu_n) = 0 \).

Note that if \( (b, b, \alpha, \beta) \in \ker d^l \), then \( \eta(b) = \alpha^{-n} \) and since \( \eta \) is injective, we get that \( b \) is determined by \( \alpha \) and in particular, if \( \alpha = 1 \) then \( b = 1 \). This means that the projection

\[
K^\times \oplus \text{Div} U \oplus \prod_{p \in S} K^\times_p \oplus \prod_{\nu \in \Omega} K^\times_{\nu}/2K^\times_{\nu} \to \text{Div} U \oplus \prod_{p \in S} K^\times_p \oplus \prod_{\nu \in \Omega} K^\times_{\nu}/2K^\times_{\nu}
\]

gives an isomorphism

\[
\ker d^l / \text{im} d^l \cong \left((\text{Div} U \oplus \prod_{p \in S} K^\times_p \oplus \prod_{\nu \in \Omega} K^\times_{\nu}/2K^\times_{\nu})/K^\times\right)\left[\eta\right].
\]

We have a canonical surjection

\[
C_S(K) \to (\text{Div} U \oplus \prod_{p \in S} K^\times_p \oplus \prod_{\nu \in \Omega} K^\times_{\nu}/2K^\times_{\nu})/K^\times.
\]

with kernel \((1, 1, \prod_{\nu \in \Omega} (K^\times_{\nu})^2)/K^\times/K^\times\). The kernel is thus isomorphic to \( \prod_{\nu \in \Omega} (K^\times_{\nu})^2 \) since \( K^\times \to K^\times_p \) is injective for a non-archimedean prime. But \( \prod_{\nu \in \Omega} (K^\times_{\nu})^2 \) is uniquely divisible, so upon applying the \( n \)-torsion functor, we get an isomorphism

\[
C_S(K)[n] \cong \left((\text{Div} U \oplus \prod_{p \in S} K^\times_p \oplus \prod_{\nu \in \Omega} K^\times_{\nu}/2K^\times_{\nu})/K^\times\right)[n].
\]

Thus our claim for \( H^1_c(U_{\text{fppf}}, \mu_n) \) follows, as does the claim for \( H^2_c(U_{\text{fppf}}, \mu_n) \). To see that \( H^0_c(U_{\text{fppf}}, \mu_n) = \mu_n(K_+) \oplus (R^\times/nR^\times)^\ast \) note first that if \( K \) has a real place, then the right hand side is isomorphic to \((R^\times/nR^\times)^\ast \) since \( \mu_n(K_+) \) vanishes. If on the other hand \( K \) is totally imaginary, the right hand side is isomorphic to \( \mu_n(K) \). By using the long exact sequence

\[
\cdots \to H^i_c(U_{\text{fppf}}, G_m) \to H^i(U_{\text{fppf}}, G_m) \to \bigoplus_{p \in S} H^i(K_p, G_m) \to \cdots
\]

together with the fact that \( H^{-1}(K_p, G_m) = 0 \) for \( p \in S \) (where we interpret \( H^{-1}(K_p, G_m) \) as Tate cohomology in the archimedean situation), we get that \( H^0_c(U_{\text{fppf}}, G_m) = O_{U, +} \), i.e., the totally positive units of \( U \). By considering the long exact sequence in compactly supported cohomology associated to the Kummer sequence

\[
0 \to \mu_n \to G_m \to G_m \to 0
\]

we then get that \( H^0_c(U_{\text{fppf}}, \mu_n) \) sits in a short exact sequence

\[
0 \to (R^\times/nR^\times)^\ast \to H^0_c(U_{\text{fppf}}, G_m) \to \mu_n(K_+) \to 0.
\]

By our previous remarks on when the leftmost term and the rightmost term are zero, our claim follows. To find the value of \( H^i_c(U_{\text{fppf}}, \mu_n) \) for \( i = 3 \), one proceeds with the Kummer sequence once again and uses that \( H^2_c(U_{\text{fppf}}, \mu_n) = 0 \) (see \([\text{HM06} \ II, \text{Proposition 2.6}]\)) is divisible and that \( H^2(U_{\text{fppf}}, G_m) = \mathbb{Q}/\mathbb{Z} \). To lastly see that \( H^i_c(U_{\text{fppf}}, \mu_n) = 0 \) for \( i > 3 \) one can use flat Artin–Verdier duality to see that they then coincide with the Pontryagin dual of \( H^{3-i}(U_{\text{fppf}}, \mathbb{Z}/n\mathbb{Z}) \), and these vanish since cohomology groups in negative degrees always are zero. \( \square \)
Remark 2.6. When $K$ is totally imaginary, a slightly more geometric approach to computing the groups $H_c^2(U_{fppf}, \mu_n)$ and $H_c^2(U_{fppf}, \mu_n)$ without using resolutions is as follows:

Let us first consider the case $S = \emptyset$. The group $H^1(X_{\text{fppf}}, \mu_n)$ classifies $\mu_n$-torsors up to isomorphism. Let $B\mu_n(X)$ be the groupoid of $\mu_n$-torsors. Any $\mu_n$-torsor is isomorphic to one on the form $\text{Spec } A \to X$ where $A$ is the $O_X$-algebra with module structure

$$A \cong \bigoplus_{i=0}^{n-1} L^\otimes i$$

for $L$ a line bundle on $X$ and with multiplication determined by an isomorphism $L^\otimes - \to O_X$ with we think of as an invertible global section $u \in \Gamma(X, L^\otimes n)$. Hence we see that $B\mu_n(X)$ is equivalent to the groupoid of pairs $(L, u)$ and with morphisms $(L', u') \to (L, u)$ given by isomorphisms $\varphi: L' \to L$ such that $\varphi^om u' = u$. A pair $(L, u)$ is trivial if there is an isomorphism $\varphi: L \cong O_X$ such that $\varphi^om u = 1$, i.e., $u$ is an $n$th power. Writing a line bundle $L \cong O_X(1)$ for some fractional ideal $I$ we may interpret $u \in \Gamma(X, L^\otimes n) \subset K^\times$ as a generator of the ideal $I^n$. This gives

$$\pi_0 B\mu_n(X) \cong Z_1/B_1 \cong H^1(X_{\text{fppf}}, \mu_n)$$

with

$$Z_1 = \{(b, b) \in \text{Div}(X) \otimes K^\times : \text{div}(b) b^n = 1\},$$

$$B_1 = \{(\text{div}(b), b^{-n}) : b \in K^\times\}.$$ 

Now to the case $S \neq \emptyset$: Write $i_p: \text{Spec } O_p \to U$ for the canonical map where $O_p$ is the complete local ring at the prime $p$. The group $H^1(U_{\text{fppf}}, \mu_n)$ classifies objects in the groupoid $B\mu_n(U)$ up to isomorphism. That is, the groupoid with objects that are triples $(L, u, (\alpha_p)_{p \in S})$ where

1. $L$ is a line bundle on $U$,
2. $u \in \Gamma(U, L^\otimes n)$, and
3. $\alpha_p: i_p^*L \to O_p$ is an isomorphism

such that $\alpha_p^o u_p = 1$. A morphism $(L', u', (\alpha_p')_{p \in S}) \to (L, u, (\alpha_p)_{p \in S})$ is given by an isomorphism $\varphi: L' \to L$ such that $\varphi^om u' = u$ and $\alpha_p^o \varphi = \alpha_p'$. An object $(L, u, (\alpha_p)_{p \in S})$ is trivial if there is an isomorphism $\varphi: O_X \to L$ such that $u^o = 1$ and $(i_p^* u) \alpha_p^o = 1$. As before, by interpreting $L$ via fractional ideals we may think of $\alpha_p$ as a choice of generator in $K_p^\times$ of the invertible module $i_p^*L$. We conclude that

$$\pi_0 B\mu_n(U) \cong Z^2_1/B_1 \cong H^2(U_{\text{fppf}}, \mu_n)$$

where

$$Z^2_1 = \{(b, b, a) \in \text{Div}(X) \otimes K^\times \prod_{p \in S} K_p^\times : \text{div}(b) b^n = 1 \text{ and } \eta(b) a^n = 1\},$$

$$B_1 = \{(\text{div}(b), b^{-n}, \eta(b)) : b \in K^\times\},$$

as in the proof of Lemma 2.4.

To compute $H^2(U_{\text{fppf}}, \mu_n)$ we use the Kummer sequence

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

which is exact since we are working in the fppf topology. Hence we obtain an exact sequence

$$H^2(U_{\text{fppf}}, \mathbb{G}_m) \xrightarrow{\delta} H^1(U_{\text{fppf}}, \mu_n) \to H^2(U_{\text{fppf}}, \mathbb{G}_m) = 0$$

where the last equality follows from the exact sequence

$$0 = \bigoplus_{p \in S} H^1(O_p, \mathbb{G}_m) \to H^2(U_{\text{fppf}}, \mathbb{G}_m) \to H^2(X_{\text{fppf}}, \mathbb{G}_m) = 0$$

of [MHH] Proposition 3.0.4(c), Remark 3.0.6].

Similarly as before, $H^2(U_{\text{fppf}}, \mu_n)$ classifies $\mu_n$-gerbes with certain trivializations over the complete local rings at the primes in $S$ up to equivalence. We translate $\delta$ to the language of groupoids as follows.
With notation as before, $H^1(U_{\log}, \mathbb{G}_m)$ corresponds to the groupoid $B_1 \mathbb{G}_m(U)$ of pairs $(\mathcal{L}, (\alpha_p)_{p \in S})$ where $\mathcal{L}$ is a line bundle on $U$ and the $\alpha_p$’s are trivializations as before over the complete local rings of the primes in $S$. The map $\delta$ sends such a pair $(\mathcal{L}, (\alpha_p)_{p \in S})$ to the pair $(U(\sqrt{\mathcal{D}}), (\alpha_p)_{p \in S})$, where $U(\sqrt{\mathcal{D}})$ is the $\mu_n$-gerbe classifying nth roots of the line bundle $\mathcal{L}$ and
\[
\delta(\alpha_p) \colon \text{Spec } \mathcal{O}_p \times_U U(\sqrt{\mathcal{D}}) \simeq \text{Spec } \mathcal{O}_p(\sqrt{\mathcal{O}_p}) = B\mu_n
\]
is the trivialization induced by $\alpha_p$. $\mu_n$ is unique up to isomorphism. If $Y$ is not connected, then
\[
\mathcal{L} = \text{Ind}_{\mathcal{L}}(\mathcal{L}^\times) \simeq \mathcal{O}_p
\]
is the trivialization obtained by the choice of generator $\alpha_p$ for $I$ over $\mathcal{O}_p$. The pair $(U(\sqrt{\mathcal{D}}), (\alpha_p)_{p \in S})$ will be trivial in the appropriate sense exactly when $I$ is in the image of the map
\[
\left( \begin{array}{ccc}
\text{div} & n & 0 \\
\eta & 0 & n
\end{array} \right) : K^\times_p \oplus \text{Div } U \oplus \prod_{p \in S} K^\times_p \to \text{Div } U \oplus \prod_{p \in S} K^\times_p.
\]

3. The Cup Product

In this section we compute the cohomology ring of an arithmetic curve. We start by giving some heuristics for how we will compute the cup product map. Assume first that $S_f \neq \emptyset$ and that $K$ has no real places. By analogy we may think of $U$ as the complement of a knot embedded in a 3-manifold. In particular, (here we use the assumption that $K$ has no real places), the étale cohomological dimension is 2. The cup product for $H^*(U, \mathbb{Z}/n\mathbb{Z})$ is graded commutative and by considering the cohomological dimension, the only non-trivial task is to compute the map
\[
cy_y := y \smile - : H^1(U, \mathbb{Z}/n\mathbb{Z}) \to H^2(U, \mathbb{Z}/n\mathbb{Z})
\]
for any $y \in H^1(U, \mathbb{Z}/n\mathbb{Z})$. We will compute the map $c_y$ as follows: we have a canonical isomorphism
\[
H^1(U, \mathbb{Z}/n\mathbb{Z}) \cong \text{Ext}^1_U(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})
\]
and the group $\text{Ext}^1_U(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ classifies extensions
\[
0 \to \mathbb{Z}/n\mathbb{Z} \to E \to \mathbb{Z}/n\mathbb{Z} \to 0,
\]
of abelian étale sheaves on $U$. By choosing an extension $E$ representing the class $y \in H^1(U, \mathbb{Z}/n\mathbb{Z})$ we get that $y \smile - = \delta_E$ where
\[
\delta_E : H^1(U, \mathbb{Z}/n\mathbb{Z}) \to H^{1+1}(U, \mathbb{Z}/n\mathbb{Z})
\]
is the connecting homomorphism coming from the extension $E$. We will then compute the map which is dual to $\delta_E$ under Artin–Verdier duality by taking resolutions of the sheaves occurring in the dual exact sequence, and then, dualize once again, to get a formula for $\delta_E$.

S-torsors. There is a simple description of $\mathbb{Z}/n\mathbb{Z}$-torsors $Y \to U$. Recall that if $d/n$ and $Z \to U$ is a $\mathbb{Z}/d\mathbb{Z}$-torsor, then we may form the induced $\mathbb{Z}/n\mathbb{Z}$-torsor $\text{Ind}_{\mathbb{Z}/d\mathbb{Z}}^{\mathbb{Z}/n\mathbb{Z}}(Z)$ as in [AC13] p. 11. Every $\mathbb{Z}/n\mathbb{Z}$-torsor is then of the form $Y = \text{Spec } R$, and if $Y$ is connected, then $L = \text{Frac}(R)$ is a degree $n$ extension of $K$ which is unramified outside of $S_f$, and the canonical morphism $Y \to \text{Spec } \mathcal{O}_L \times_X U$ is an isomorphism. If $Y$ is not connected, then $Y$ is induced from a connected degree $d$ torsor $Z \to U$ where $d/n$. We summarize the above discussion in the following lemma, and the proof of the lemma is entirely analogous to the proof of [AC13] Lemma 2.20.

Lemma 3.1. Let $Y \to U$ be a $\mathbb{Z}/n\mathbb{Z}$-torsor. Then there exists a degree $d/n$ extension $L/K$ which is unramified outside of $S_f$ and an isomorphism
\[
Y \cong \text{Ind}_{\mathbb{Z}/d\mathbb{Z}}^{\mathbb{Z}/n\mathbb{Z}}(\text{Spec } \mathcal{O}_L \times_X U).
\]
Further, $L/K$ is unique up to isomorphism.
The extension associated to a torsor. The method is the same as in \cite{ACTS}. For $y \in H^1(U, \mathbb{Z}/n\mathbb{Z})$, choose a $\mathbb{Z}/n\mathbb{Z}$-torsor $\pi: Y \to U$ representing $y$. Then $Y$ is the restriction to $U$ of a ramified $\mathbb{Z}/n\mathbb{Z}$-torsor $Y' \to X$, where $Y'$ is of the form

$$Y' = \text{Ind}_{\mathbb{Z}/n\mathbb{Z}}^\mathbb{Z}(Z)$$

where $Z$ is the spectrum of the ring of integers of a degree $d|n$ extension Galois $L/K$ which is unramified outside $S_f$. Since $\pi$ is finite étale we get that $\pi_* = \pi_1$ and hence $\pi_*$ is left adjoint to $\pi^*$. The counit $N: \pi_*\pi^*\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is called the norm. In algebraic geometry literature $N$ is usually called the trace, but we use the name norm since it is standard number theoretic nomenclature.

Remark 3.2. To clarify, we have two adjunctions

$$\pi^*: \text{Sh}(\mathbb{Z}_\text{et}) \cong \text{Sh}(Y_\text{et}) : \pi_* \quad \text{and} \quad \pi_*: \text{Sh}(Y_\text{et}) \cong \text{Sh}(U_\text{et}) : \pi^*$$

where the functor on the left is left adjoint to the functor on the right and vice versa.

We have a short exact sequence of abelian étale sheaves

$$0 \to \ker N \to \pi_*\pi^*\mathbb{Z}/n\mathbb{Z} \xrightarrow{N} \mathbb{Z}/n\mathbb{Z} \to 0 \quad (3)$$

which we refer to as the norm sequence. There is an equivalence of abelian categories between the category of locally constant sheaves split by $\pi$ and the category of $C_n$-modules, where $C_n$ is the cyclic group with $n$ elements. Under this equivalence the norm sequence corresponds to the short exact sequence of $C_n$-modules

$$0 \to \ker \epsilon \to \mathbb{Z}/n\mathbb{Z}[C_n] \xrightarrow{s} \mathbb{Z}/n\mathbb{Z} \to 0 \quad (4)$$

where $C_n$ acts trivially on $\mathbb{Z}/n\mathbb{Z}$ and acts on $\mathbb{Z}/n\mathbb{Z}[C_n]$ by translation on the generators. The morphism $\epsilon: \mathbb{Z}/n\mathbb{Z}[C_n] \to \mathbb{Z}/n\mathbb{Z}$ sends $g \in C_n$ to $1$. The $C_n$-module $\ker \epsilon$ is free as an $\mathbb{Z}/n\mathbb{Z}$-module on $\{g-1\}_{g \in C_n}$. Choose a generator $e$ of the cyclic group $C_n$. We define a map $s: \ker \epsilon \to \mathbb{Z}/n\mathbb{Z}$ by sending $e^l - 1$ to $l$. If we take the pushout of the sequence $\{4\}$ along $s$, we get a diagrams

$$
\begin{array}{ccc}
0 & \to & \ker \epsilon \\
\downarrow s & & \downarrow \text{id} \\
\mathbb{Z}/n\mathbb{Z}[C_n] & \to & \mathbb{Z}/n\mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z}/n\mathbb{Z} & \to & P \\
\downarrow & & \downarrow \\
0 & \to & \mathbb{Z}/n\mathbb{Z} \\
\end{array}
$$

where $P = \mathbb{Z}[G]/(\ker \epsilon)^2$, and the short exact sequence of $C_n$-modules on the bottom line corresponds to a short exact sequence of locally constant $\pi$-split sheaves

$$0 \to \mathbb{Z}/n\mathbb{Z} \to E \to \mathbb{Z}/n\mathbb{Z} \to 0 \quad (5)$$

which is called the transfer sequence associated to the $\mathbb{Z}/n\mathbb{Z}$-torsor $Y \to U$.

Lemma 3.3. The connecting homomorphism $H^i(U, \mathbb{Z}/n\mathbb{Z}) \to H^{i+1}(U, \mathbb{Z}/n\mathbb{Z})$ associated to the transfer sequence $\{5\}$ is given by cup product with $y \in H^1(U, \mathbb{Z}/n\mathbb{Z})$.

Proof. The proof of \cite{ACTS} Lemma 3.1] goes through with $\tilde{X}_\text{et}$ replaced by $U_\text{et}$. \hfill $\Box$

Denote by $\delta_E$ the connecting homomorphism corresponding to $\{5\}$ and $\delta_y$ the connecting homomorphism of $\{3\}$. Then $\delta_E = f_* \circ \delta_y$ where $f_*: H^*(U, \ker N) \to H^*(U, \mathbb{Z}/n\mathbb{Z})$ is the morphism on cohomology induced by $f$. 

**Duality and functoriality.** Let $A$ be a finite flat group scheme and denote by $D(A) = \mathcal{H}om(A, \mathbb{G}_m)$ its Cartier dual. As a consequence of [DH19, Lemma 4.1], we have a canonical pairing in the derived category of abelian groups

$$R\hat{\Gamma}_c(U_{\text{fppf}}, A) \otimes^L R\Gamma(U_{\text{fppf}}, D(A)) \to R\hat{\Gamma}_c(U_{\text{fppf}}, \mathbb{G}_m)$$

which is functorial in $A$. But $R\hat{\Gamma}_c(U_{\text{fppf}}, \mathbb{G}_m)$ is isomorphic via the trace map to $\mathbb{Q}/\mathbb{Z}$ concentrated in degree $3$. This gives an isomorphism

$$R\hat{\Gamma}_c(U_{\text{fppf}}, A) \cong (R\Gamma(U_{\text{fppf}}, D(A)))^3$$

(see [DH19 Theorem 1.1]) in $D(Ab)$. We then have an induced pairing in the category of abelian groups

$$H_c^{3-i}(U_{\text{fppf}}, A) \times H^i(U_{\text{fppf}}, D(A)) \to H_c^3(U_{\text{fppf}}, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

which again by [Mil06 Coro. III.3.2] and [DH19 Theorem 1.1] is perfect and induces a duality between the group $H_c^{3-i}(U_{\text{fppf}}, A)$ and the group $H^i(U_{\text{fppf}}, D(A))$.

A morphism $\varphi: F \to G$ of finite flat group schemes induces a dual morphism $D(\varphi): D(G) \to D(F)$ and hence we obtain maps

$$\varphi*: R\Gamma(U_{\text{fppf}}, F) \to R\Gamma(U_{\text{fppf}}, G),$$

$$D(\varphi)_*: R\hat{\Gamma}_c(U_{\text{fppf}}, D(G)) \to R\hat{\Gamma}_c(U_{\text{fppf}}, D(F)),$$

and by functoriality of the above pairing, the diagram

$$\begin{array}{cccc}
R\hat{\Gamma}_c(U_{\text{fppf}}, F) & \xrightarrow{\varphi*} & R\Gamma(U_{\text{fppf}}, G) \\
\downarrow \cong & & \downarrow \cong \\
R\hat{\Gamma}_c(U_{\text{fppf}}, D(F)) & \xrightarrow{D(\varphi)_* \sim} & R\hat{\Gamma}_c(U_{\text{fppf}}, D(G)) \sim
\end{array}$$

commutes, where $\sim$ denotes the functor $R\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$.

We will also need the following lemma which is a direct consequence of [DH19 Lemma 4.3]:

**Lemma 3.4.** Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finite flat group schemes and let $0 \to D(C) \to D(B) \to D(A) \to 0$ be the short exact sequence obtained by Cartier duality. Then, for all $i, j \geq 0$, we have connecting homomorphisms $\delta_i: H^i(U, C) \to H^{i+1}(U, A)$ and $\delta'_j: H^j(U_{\text{fppf}}, D(A)) \to H^{j+1}(U_{\text{fppf}}, D(C))$, and the diagram

$$\begin{array}{ccc}
H^i_c(U_{\text{fppf}}, D(A)) \times H^{i+1}(U, A) & \xrightarrow{(\cdot, \cdot)} & H^{i+j+1}_c(U_{\text{fppf}}, \mathbb{G}_m, U) \\
\downarrow \delta_j & & \downarrow \\
H^{i+j+1}(U_{\text{fppf}}, D(C)) \times H^j(U, C) & \xrightarrow{(\cdot, \cdot)} & H^{i+j+1}_c(U_{\text{fppf}}, \mathbb{G}_m, U)
\end{array}$$

commutes, in the sense that for all $a \in H^i(U, C)$ and $b \in H^j(U_{\text{fppf}}, D(A))$ we have

$$\langle b, \delta_i(a) \rangle = \langle \delta'_j(b), a \rangle.$$ 

As a consequence we get that

$$\delta_i = \text{Hom}(\delta_{i-1}', \mathbb{Q}/\mathbb{Z})$$

when identifying $H^i(U, \mathbb{Z}/n\mathbb{Z}) \cong \text{Hom}(H^{3-i}_c(U_{\text{fppf}}, \mathbb{G}_m), \mathbb{Q}/\mathbb{Z})$.

**Computing the cup product.** For $y \in H^1(U, \mathbb{Z}/n\mathbb{Z})$ we will compute the dual

$$c_y^*: H^2_c(U_{\text{fppf}}, \mathbb{G}_m) \to H^2_c(U_{\text{fppf}}, \mathbb{G}_m)$$

of $c_y: H^1(U, \mathbb{Z}/n\mathbb{Z}) \to H^2(U, \mathbb{Z}/n\mathbb{Z})$. We saw that $c_y = f_* \circ \delta_y$ and from the argument above we get the following description of the dual $c_y^*$:
Lemma 3.5. The map $c_\sim: R\hat{\Gamma}_c(U_{fppf}, \mu_n) \to R\hat{\Gamma}_c(U_{fppf}, \mu_n)[1]$ is given by $c_\sim = \delta_\sim \circ D(f)_*$, where 

$$\delta_\sim = R\text{Hom}(\delta_y, \mathbb{Q}/\mathbb{Z}[-3])$$

and $\delta_y: \mathbb{Z}/n\mathbb{Z} \to \text{ker } N[1]$ is the connecting homomorphism.

Consider the diagram

$$
\begin{array}{cccccccccc}
0 & \to & \text{ker } N & \to & \pi_*\pi^*\mathbb{Z}/n\mathbb{Z} & \to & \mathbb{Z}/n\mathbb{Z} & \to & 0 \\
\downarrow f & & \downarrow u & & & & \downarrow 1 \\
 & & \mathbb{Z}/n\mathbb{Z} & & & & & & \\
\end{array}
$$

(7)

In addition to the resolution $E \to \mathbb{Z}/n\mathbb{Z}$, where $E$ is as in equality (1), and $\pi_*\pi^*E \to \pi_*\pi^*\mathbb{Z}/n\mathbb{Z}$, we define a resolution $K \to \text{ker } N$ by

$$K = \mathbb{Z} \xrightarrow{(n)} \mathbb{Z} \oplus \pi_*\pi^*\mathbb{Z} \xrightarrow{(\Delta \ n)} \pi_*\pi^*\mathbb{Z}$$

where $\Delta: \mathbb{Z} \to \pi_*\pi^*\mathbb{Z}$ is the unit for the adjunction

$$\pi^*: \text{Sh}(U_{\acute{e}t}) \xhookrightarrow{\dashv} \text{Sh}(Y_{\acute{e}t}) : \pi_*.$$

To see that this really gives a resolution one may once again consider the sheaves as $C_n$-modules.

The diagram (7) is then quasi-isomorphic to a diagram

$$
\begin{array}{cccccccccc}
0 & \to & K & \xrightarrow{\hat{u}} & \pi_*\pi^*E & \xrightarrow{\hat{N}} & E & \to & 0 \\
\downarrow f & & \downarrow \hat{f} & & \downarrow \hat{N} & & \downarrow 1 \\
 & & \mathcal{E} & & & & & & \\
\end{array}
$$

(8)

The short exact sequence of (8) gives a distinguished triangle

$$K \to \text{Cyl}(\hat{u}) \to C(\hat{u}) \xrightarrow{pr} K[1]$$

where $\text{Cyl}$ is the mapping cylinder, and the connecting homomorphism $\delta_y$ is represented by $\text{pr}: C(\hat{u}) \to K[1]$. Hence we have a map

$$\text{pr} \circ \hat{f}[1] : C(\hat{u}) \to K[1] \to C[1].$$

The map $q(\hat{u}) := (0, \hat{N}): C(\hat{u}) \to \mathcal{E}$ is a quasi-isomorphism (see e.g. [GM03, III.3.5]) and we get the zig-zag

$$\mathcal{E} \xrightarrow{q(\hat{u})} C(\hat{u}) \xrightarrow{\text{pr}} K[1] \xrightarrow{\hat{f}[1]} \mathcal{E}[1].$$

Now we apply $\mathcal{H}\text{om}(\mathcal{E}, \mathcal{C})$, where $\mathcal{C}$ is the complex as in equality (2), to obtain the zig-zag

$$\mathcal{H}\text{om}(\mathcal{E}, \mathcal{C}) \xrightarrow{q(\hat{u})^\ast} \mathcal{H}\text{om}(C(\hat{u}), \mathcal{C}) \xleftarrow{\text{pr}^\ast} \mathcal{H}\text{om}(K[1], \mathcal{C}) \xrightarrow{\hat{f}[1]^\ast} \mathcal{H}\text{om}(\mathcal{E}[1], \mathcal{C}).$$

The map $q(\hat{u})^\ast$ is a quasi-isomorphism since $q(\hat{u})$ is a quasi-isomorphism of locally free sheaves.

Using flat cohomology with compact support. Now by applying $\Gamma_c(U_{fppf}, -)$ (see Definition 2.2) to the above zig-zag we get (where we write $\Gamma_c$ for legibility)

$$\Gamma_c(\mathcal{H}\text{om}(\mathcal{E}, \mathcal{C})) \xrightarrow{\Gamma_c(q(\hat{u})^\ast)} \Gamma_c(\mathcal{H}\text{om}(C(\hat{u}), \mathcal{C})) \xleftarrow{\text{pr}^\ast} \Gamma_c(\mathcal{H}\text{om}(K[1], \mathcal{C})) \xrightarrow{\hat{f}[1]^\ast} \Gamma_c(\mathcal{H}\text{om}(\mathcal{E}[1], \mathcal{C})).$$
We will show that the above zig-zag represents the map \( c_y^*: H^1_\mathcal{E}(U_{\text{fppf}}, \mu_n) \to H^2_\mathcal{E}(U_{\text{fppf}}, \mu_n) \) in cohomology. Note that we have a commutative diagram

\[ \begin{array}{ccc}
\Gamma_\mathcal{E}(\mathcal{Hom}(\mathcal{E}, \mathcal{C})) & \xrightarrow{\Gamma_\mathcal{E}(q(\hat{u}))^*} & \Gamma_\mathcal{E}(\mathcal{Hom}(\mathcal{C}(\hat{u}), \mathcal{C})) \\
\downarrow \alpha & & \downarrow \beta \\
\mathcal{R}\Gamma_\mathcal{E}(\mathcal{Hom}(\mathcal{E}, \mathcal{C})) & \xrightarrow{\mathcal{R}\Gamma_\mathcal{E}(q(\hat{u}))^*} & \mathcal{R}\Gamma_\mathcal{E}(\mathcal{Hom}(\mathcal{C}(\hat{u}), \mathcal{C})) \\
\downarrow \gamma & & \downarrow \delta \\
\mathcal{R}\mathcal{R}\Gamma_\mathcal{E}(\mathcal{Hom}(\mathcal{E}, \mathcal{C})) & \xrightarrow{\mathcal{R}\mathcal{R}\Gamma_\mathcal{E}(q(\hat{u}))^*} & \mathcal{R}\mathcal{R}\Gamma_\mathcal{E}(\mathcal{Hom}(\mathcal{C}(\hat{u}), \mathcal{C})) \\
\end{array} \]

The bottom zig-zag, from right to left, represents the Pontryagin dual \( \hat{\Gamma} \) of cup product with \( y \). The idea is to compute \( c_y^* \), by first moving up via the rightmost zig-zag, then moving right via the uppermost zig-zag, and finally moving down via the leftmost zig-zag.

**Lemma 3.6.** The maps \( s, t, t', \) and \( \Gamma_\mathcal{E}(C(q(\hat{u}))) \) of \( \mathcal{A} \) give isomorphisms on \( H^2 \). Furthermore, the leftmost and rightmost vertical zig-zags gives isomorphisms on cohomology in degree 2.

**Proof.** See Appendix \( \mathcal{A} \). \( \square \)

**Corollary 3.7.** The map \( c_y^*: H^1_\mathcal{E}(U_{\text{fppf}}, \mu_n) \to H^2_\mathcal{E}(U_{\text{fppf}}, \mu_n) \) is given by

\[ H^1(\Gamma_\mathcal{E}(q(\hat{u}))^* \circ H^1(\Gamma_\mathcal{E}(pr^*)) \circ H^1(\hat{f}[1]^*)). \]

We are now in a position to compute the dual \( c_y^* \) of the cup product

\[ c_y = y \sim - : H^1(U, \mathbb{Z}/n\mathbb{Z}) \to H^2(U, \mathbb{Z}/n\mathbb{Z}). \]

Recall that any element \( y \in H^1(U, \mathbb{Z}/n\mathbb{Z}) \) may be represented by a \( \mathbb{Z}/n\mathbb{Z} \)-torsor of the form \( \text{Ind}_{\mathbb{Z}/d\mathbb{Z}}^{\mathbb{Z}/n\mathbb{Z}}(Y \times_X U) \to U \) where \( Y = \text{Spec} \mathcal{O}_L \) and \( \mathcal{O}_L \) is the ring of integers of a cyclic degree \( d/n \) extension \( L/K \) which is unramified outside of \( S \). The map \( c_y^\# \) will be computed up to a choice of generator \( \sigma \in \text{Gal}(L/K) \) and any two such choices will give isomorphic versions of \( c_y^\# \).

We start by computing \( c_y^\# \) in the case where \( S \neq \Omega_R \). By Lemma \( \mathcal{Z} \), we may view \( c_y^\#: H^1_\mathcal{E}(U_{\text{fppf}}, \mu_n) \to H^2_\mathcal{E}(U_{\text{fppf}}, \mu_n) \) as a map \( c_y^\#: C_S(K)[n] \to C_S(K)/nC_S(K) \) which we now describe:

**Lemma 3.8.** Suppose that \( S \neq \Omega_R \). Let \( y \in H^1(U, \mathbb{Z}/n\mathbb{Z}) \) be represented by a cyclic extension \( L/K \) of degree \( d/n \). Choose a generator \( \sigma \in \text{Gal}(L/K) \). Then in view of Lemma \( \mathcal{Z} \), we have that

\[ c_y^\#: C_S(K)[n] \to C_S(K)/nC_S(K) \]

is the map taking

\[ \alpha \mapsto \alpha^{n^2/2d}N_L|K(\beta)^{n/d}, \]

where \( \beta \) is an element in \( I_L \) such that \( \alpha^{n/d} = t\beta/\sigma(\beta) \) in \( I_L \), where \( t \in L^\times \) satisfies \( N_L|K(t) = \alpha^{-n} \) in \( I_K \). Here \( K^\times \) and \( L^\times \) are embedded diagonally in \( I_K \) and \( I_L \), respectively.

Before proving this, let us state the immediate consequence:

**Theorem 3.9.** Suppose that \( S \neq \Omega_R \). Let \( y \) and \( z \) be elements in \( H^1(U, \mathbb{Z}/n\mathbb{Z}) \) represented by cyclic extensions \( L/K \) and \( M/K \) respectively and assume that \( L/K \) has degree \( d/n \). Choose a generator \( \sigma \in \text{Gal}(L/K) \). Then under the identifications \( H^1(U, \mathbb{Z}/n\mathbb{Z}) \cong (C_S(K)/nC_S(K))^\sim \) and \( H^2(U, \mathbb{Z}/n\mathbb{Z}) \cong (C_S(K)[n])^\sim \) we have that \( y \sim z \) in \( (C_S(K)[n])^\sim \) satisfies the formula

\[ (y \sim z, \alpha) = \langle z, \alpha^{n^2/2d}N_L|K(\beta)^{n/d} \rangle \]

where \( (\sim, \sim) \) is the evaluation map and where \( \beta \) is an element in \( I_L \) such that \( \alpha^{n/d} = t\beta/\sigma(\beta) \) in \( I_L \) where \( t \in L^\times \) satisfies \( N_L|K(t) = \alpha^{-n} \) in \( I_K \). Here \( K^\times \) and \( L^\times \) are embedded diagonally in \( I_K \) and \( I_L \), respectively.
Proof of Lemma 3.8. First recall that if \( K \) is totally imaginary or \( n \) is odd, we need not take the infinite places into account.

We will use Corollary 3.7 and thus compute \( H^1(\Gamma_c(q(u)^*)^{-1} \circ H^1(\Gamma_c(pr^*) \circ H^1(\Gamma_c(f[1]^*) \circ H^1(\Gamma_c(f[1]^*)))) \). Let \( \alpha \in C_S(K)[n] \) be represented by \( (b, b, \alpha_S) \in \ker d^0 \subseteq K^\times \oplus \text{Div}(U) \oplus (\prod_{p \in S} K_p^\times \oplus \prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2) \) where \( d^0 \) is as in the proof of Lemma 2.4. Let \( S_f^0 \) be the set of finite places in \( L \) lying above \( S_f \) and let \( \Omega^\prime \) be the real places of \( L \). We apply \( \Gamma_c \) to the zig-zag (3) and we obtain:

\[
\begin{align*}
K^\times \oplus \text{Div}(U) \oplus & \prod_{p \in S_f} K_p^\times \oplus \prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 \\
\text{Div}(U) \oplus & \prod_{p \in S_f} K_p^\times \oplus \prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 \\
\prod_{p \in S_f} K_p^\times \oplus & \prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 \\
\prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 & \\
K^\times \oplus & \text{Div}(U) \oplus \prod_{p \in S_f} K_p^\times \oplus \prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 \\
\prod_{p \in S_f} K_p^\times & \oplus \prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 \\
\prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 & \\
K^\times \oplus & \text{Div}(U) \\
\prod_{p \in S_f} K_p^\times & \oplus \prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 \\
\prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 & \\
K^\times \oplus & \text{Div}(U) \\
\prod_{p \in S_f} K_p^\times & \oplus \prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 \\
\prod_{\nu \in \Omega^\prime} \Gamma_{c}(\nu)^2 & \end{align*}
\]
where

\[ \Gamma_c(f^*)^1[1](b, i(b), i(a)) \]
\[ \Gamma_c(pr^*)^1(b', b'', I, a') = (b', b'', I, 0, a', 1) \]
\[ \Gamma_c(q(\hat{u})^*)^2(J, a'') = (1, 1, 0, i(J), 1, i(a'')) \]

\[ d_{I_c}^1 = \begin{pmatrix}
    N_{L/K} & 0 & 0 & 0  \\
    n & 1 - \sigma' & 0 & 0  \\
    -\text{div} & 0 & 1 - \sigma' & 0  \\
    0 & -\text{div} & -n & 0  \\
    -\eta & 0 & 0 & 1 - \sigma'  \\
    0 & -\eta & 0 & -n
\end{pmatrix}. \]

Here \( a \) is an element in \( \prod_{p \in S_I} K_p^\times \otimes \prod_{\nu \in \Omega_k} K^\times_{\nu} \) and \( i \) is the canonical map
\[ i: \prod_{p \in S_I} K_p^\times \otimes \prod_{\nu \in \Omega_k} K^\times_{\nu} \to \prod_{p \in S_I} (L_p^\times)^{n/d} \oplus \prod_{\nu \in \Omega_k} (L_{\nu}^\times)^{n/d}. \]

By \( \Omega'_R \) we mean the set of places over \( \Omega_k \), including the ones that ramify. Note that in the matrix description of \( d_{I_c}^1 \), every summand of the form

\[ \prod_{p \in S_I} (L_p^\times)^{n/d} \oplus \prod_{\nu \in \Omega_k} (L_{\nu}^\times)^{n/d} \]

is treated as a single summand. Furthermore, by abuse of notation, \( \eta \) is the canonical map
\[ \eta: (L^\times)^{n/d} \to \prod_{p \in S_I} (L_p^\times)^{n/d} \oplus \prod_{\nu \in \Omega_k} (L_{\nu}^\times)^{n/d}, \]

\[ (\sigma' - 1)(a) = (\sigma(a_{n/d})/a_1 \ a_1/a_2 \ \ldots \ a_{(n/d)-1}/a_{n/d}) \]

and
\[ N_{Y(U)}(a) = \prod_{i=1}^{n/d-1} \prod_{j=0}^{d-1} \sigma(a_i)^j. \]

Here \( (a_1, \ldots, a_n) \) is an element in \( (L^\times)^{n/d} \), \( \text{Div}(Y)^{n/d} \), or \( (\prod_{p \in S_I} L_p^\times \oplus \prod_{\nu \in \Omega_k} L_{\nu}^\times)^{n/d} \).

The element \((b, b, \alpha_S) \in \ker d^0 \subseteq K^\times \oplus \text{Div}(U) \oplus (\prod_{p \in S_I} K_p^\times \oplus \prod_{\nu \in \Omega_k} K_{\nu}^\times)\) representing \( \alpha \in C_S(K)[n] \) is sent to
\[ (b \ i(b) \ i(b) \ 0 \ 0 \ i(\alpha_S) \ 1) \]

via \( \Gamma_c(pr^*)^1 \circ \Gamma_c(f^*)^1 \). We need to reduce this modulo the image of \( d_{I_c}^1 \) to get an element of the form
\[ \Gamma_c(q(\hat{u})^*)^2(J, a') = (1 \ 1 \ 0 \ i(J) \ 1 \ i(a')) \]

The image of \( \alpha \) under the connecting homomorphism is then given by the class of \((J, a')\) in \( C_S(K)/n \). If \( n \) is odd, we do not need to consider the archimedean places. Assume that \( n \) is even. We have seen in Section 2 that the canonical surjection
\[ C_S(K) \to (\text{Div } U \oplus \prod_{p \in S} K_p^\times \oplus \prod_{\nu \in \Omega_k} K_{\nu}^\times/2K_{\nu}^\times)/K^\times \]
induces an isomorphism on \( n \)-torsion.

We have that \( N_{L/K}(i(\alpha_{n/d})) = (\alpha_{n/d})^d = 1 \) in \( C_S(K) \), since \( \alpha \) is \( n \)-torsion. But
\[ H^1(G_{L/K}, C_S(L)) = \ker N_{L/K}/(\sigma - 1)C_S(L) = 0 \]
(see e.g. [NSW08, p. 620]) and hence there exists a $\beta = (I, \beta) \in \text{Div}(Y) \oplus (\prod_{p \in S_I} L_p^\times \oplus \prod_{\nu \in \Omega_k} L_\nu^\times)$ and a $t \in L^\times$ such that

$$b^{n/d} \Omega_t = \text{div}(t)I/\sigma(I),$$

$$i(\alpha_S^{n/d}) = \eta(t)\beta/\sigma(\beta).$$

Note here that $\Omega_k^\times$ includes complex places over $\Omega_k$. Taking the norm on both sides of the latter equality we get $\alpha_S^{n} = N_{L/K}(\eta(t))$ and since $\alpha_S^{n} = \eta(b^{-1})$ we get that

$$N_{L/K}(t) = b^{-1}.$$

We now put

$$\left(\frac{1}{\beta}\right) = \left(I \ bI \ b^2I \ldots b^{n/d-1}I\right) \in \text{Div}(L)^{n/d} \oplus \left(\prod_{p \in S_I} L_p^\times \oplus \prod_{\nu \in \Omega_k} L_\nu^\times\right)^{n/d}$$

which satisfies

$$(\sigma' - 1)(L) = (\text{div}(t)b^{-1} b^{-1} \ldots b^{-1})$$

$$(\sigma' - 1)(\beta) = (\eta(t)\alpha_S^{-1} \alpha_S^{-1} \ldots) .$$

If we reduce (10) modulo

$$\begin{pmatrix}
N_{L/K} & 0 & 0 & 0 \\
n & 1 - \sigma' & 0 & 0 \\
-\text{div} & 0 & 1 - \sigma' & 0 \\
0 & -\text{div} & -n & 0 \\
-\eta & 0 & 0 & 1 - \sigma' \\
0 & -\eta & 0 & -n
\end{pmatrix}
\begin{pmatrix}
L^{-1} \\
1 \\
\beta \\
-\sigma'
\end{pmatrix}
= 
\begin{pmatrix}
N_{L/K}(L^{-1}) \\
L^{-1} \\
\text{div}(t)I/\sigma'(L) \\
\eta(L)\beta/\sigma'(\beta)
\end{pmatrix}
\begin{pmatrix}
b \\
b^{n/d-1} \\
\alpha_S^{-1} \\
\beta^{-n}
\end{pmatrix},$$

then we get

$$\begin{pmatrix}
b \\
b \\
b^{n/d-1} \\
\alpha_S^{-1}
\end{pmatrix}
+ 
\begin{pmatrix}
b^{-1} \\
L^{-1} \\
-b \\
L^{-1}
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
\eta^{-1} \\
1 \\
1
\end{pmatrix},$$

$$\begin{pmatrix}
N_{L/K} & 0 & 0 & 0 \\
n & 1 - \sigma' & 0 & 0 \\
-\text{div} & 0 & 1 - \sigma' & 0 \\
0 & -\text{div} & -n & 0 \\
-\eta & 0 & 0 & 1 - \sigma' \\
0 & -\eta & 0 & -n
\end{pmatrix}
\begin{pmatrix}
w^{-1} \\
1 \\
\eta(w)
\end{pmatrix},$$

since $bN_{L/K}(t) = b\alpha_S^n = 1$.

As in the proof of [ACT18 Lemma 3.7] we now consider the element

$$w = \prod_{j=0}^{d-1} \sigma^j(t^{n-jn/d})$$

and put

$$\bar{w} = (w \ b w \ b^2w \ldots b^{n/d-1}w).$$

We have

$$\begin{pmatrix}
N_{L/K} & 0 & 0 & 0 \\
n & 1 - \sigma' & 0 & 0 \\
-\text{div} & 0 & 1 - \sigma' & 0 \\
0 & -\text{div} & -n & 0 \\
-\eta & 0 & 0 & 1 - \sigma' \\
0 & -\eta & 0 & -n
\end{pmatrix}
\begin{pmatrix}
1 \\
\bar{w} \\
\text{div}(\bar{w})
\end{pmatrix} = 
\begin{pmatrix}
1 \\
b^{-1}L^{-n} \\
1 \\
1
\end{pmatrix}.$$
since

\[(\sigma' - 1)(w) = (\sigma(wb^{n/d-1})w^{-1} b^{-1} b^{-1} \ldots b^{-1})
= (b^{-1} b^{-1} b^{-1} \ldots b^{-1})\]

Hence we get in cohomology

\[
\begin{pmatrix}
1 & b^n \\
1 & b^n \\
1 & b^n \\
\beta^n
\end{pmatrix}
\sim
\begin{pmatrix}
1 & b^{-1} b^{-n} \\
1 & \text{div}(w) \\
1 & \eta(w) \\
\beta^n \eta(w)
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
\beta^n \eta(w)
\end{pmatrix}
\]

We have

\[
\begin{pmatrix}
I^n \text{div}(w) \\
\beta^n \eta(w)
\end{pmatrix}
= \begin{pmatrix}
I^n \text{div}(w) \\
\beta^n \eta(w)
\end{pmatrix},
\]

and since

\[
I^n \text{div}(w) = I^n \prod_{j=0}^{d-1} \sigma^j(\text{div}(t))^{n-jn/d}
= I^n \prod_{j=0}^{d-1} \sigma^j(b^{n/d}I^{-1} \sigma(I))^{n-jn/d}
= I^n b^{n/d(d+1)/2} I^{-n}(I \sigma(I) \ldots \sigma^{d-1}(I))^{\frac{n}{d}}
= b^{n/d(d+1)/2} N_{L|K}(I)^{\frac{n}{d}}
\]

and similarly

\[
\beta^n \eta(w) = \beta^n \prod_{j=0}^{d-1} \sigma^j(\eta(t))^{n-jn/d}
= \beta^n \prod_{j=0}^{d-1} \sigma^j(\alpha_S \sigma(\beta)/\beta)^{n-jn/d}
= \beta^n \alpha_S^{\frac{n(d+1)}{d+1}} \beta^{-n} N_{L|K}(\beta)^{n/d}
= \alpha_S^{\frac{n(d+1)}{d+1}} N_{L|K}(\beta)^{n/d},
\]

we get that

\[
\begin{pmatrix}
I^n \text{div}(w) \\
\beta^n \eta(w)
\end{pmatrix}
\sim
\begin{pmatrix}
1 \\
\beta^n \eta(w)
\end{pmatrix}
\begin{pmatrix}
1 \\
\frac{n}{d(d+1)}
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
1 \\
\beta^n \eta(w)
\end{pmatrix}
\begin{pmatrix}
N_{L|K}(I)^{\frac{n}{d}} \\
N_{L|K}(\beta)^{n/d}
\end{pmatrix}
\]

This completes the proof. \(\square\)

Finally we are ready to compute the cup product for the case \(S = \Omega_2\). We proceed by computing the cup product \(H^1(X, \mathbb{Z}/n\mathbb{Z}) \times H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^2(X, \mathbb{Z}/n\mathbb{Z})\) and then sketch how to compute the higher cup products.
Lemma 3.10. Suppose that \( S = \Omega_S \). Let \( y \in H^1(X, \mathbb{Z}/n\mathbb{Z}) \) be represented by a cyclic extension \( L/K \) of degree \( d|n \). Choose a generator \( \sigma \in \text{Gal}(L/K) \). Then in view of Lemma 2.4 we have that
\[
c^* : Z^1/B^1 \to \text{Cl}^+(K)/n\text{Cl}^+(K)
\]
is the map
\[
(b, b) \mapsto b^{n/2d}N_{L/K}(I)^{n/d},
\]
where \( I \) is an element in \( \text{Div}(L) \) such that \( b^{n/d} = \text{div}(t)I/\sigma(I) \), where \( t \in L^\times \) satisfies \( N_{L/K}(t) = b^{-1} \).

The following theorem is an immediate consequence:

Theorem 3.11. Suppose that \( S = \Omega_S \). Let \( y \) and \( z \) be elements in \( H^1(X, \mathbb{Z}/n\mathbb{Z}) \) represented by cyclic extensions \( L/K \) and \( M/K \) respectively and assume that \( L/K \) has degree \( d|n \). Choose a generator \( \sigma \in \text{Gal}(L/K) \). Then under the identifications \( H^1(X, \mathbb{Z}/n\mathbb{Z}) \cong (\text{Cl}^+(K)/n\text{Cl}^+(K))^\wedge \) and \( H^2(X, \mathbb{Z}/n\mathbb{Z}) \cong (Z^1/B^1)^\wedge \) we have that \( y \sim z \in (Z^1/B^1)^\wedge \) satisfies the formula
\[
\langle y \sim z, (b, b) \rangle = \langle z, b^{n/2d}N_{L/K}(I)^{n/d} \rangle
\]
where \( I \) is an element in \( \text{Div}(L) \) such that \( b^{n/d} = \text{div}(t)I/\sigma(I) \), where \( t \in L^\times \) satisfies \( N_{L/K}(t) = b^{-1} \).

Proof of Lemma 3.10. Since the case where \( K \) is totally imaginary or \( n \) is odd is covered in \[AC18\], we may assume that \( K \) is not totally imaginary and that \( n \) is even. We have \( H^1_S(X_{\text{fppf}}, \mu_n) \cong Z^1/B^1 \) where
\[
Z^1 = \langle (b, b) \in K^+ \oplus \text{Div} : \text{div}(b)b^n = 1 \rangle, \quad B^1 = \langle (a^{-n}, \text{div}(a)) : a \in K^\times \rangle.
\]
The proof is very similar to the proof of \[AC18\] Lemma 3.7. We will again consider the zig-zag represented by the big diagram\[3\] Again, \( (b, b, 1) \in \ker d^0 \subseteq K^+ \oplus \text{Div}(K) \oplus \prod_{\nu \in \Omega_S} K_\nu^\times \) is sent to
\[
(b \quad i(b) \quad i(b) \quad 0 \quad 1 \quad 1)
\]
via \( \Gamma_c(pr^*)^1 \circ \Gamma_c(\hat{f}^*)^1 \). We need to reduce this modulo the image of \( d^1_{\hat{c}_z} \) to get an element of the form
\[
\Gamma_c(q(\hat{u})^2)(J, a') = (1 \quad 1 \quad 0 \quad i(J) \quad 1 \quad i(a')).
\]
The pair \( (b^{n/d}O_L, i(\eta(b^{-1/d}))) \) defines a class in \( C_S(L) \) (this time including complex places) which lies in the kernel of \( N_{L/K} : C_S(L) \to C_S(K) \) since \( i(\eta(b^{-1/d})) \) has norm \( \eta(b^{-1}) \). But
\[
H^1(G_{L/K}, C_S(L)) = \ker N_{L/K}/(\sigma - 1)C_S(L) = 0
\]
and hence there exists a pair \( (b', \tilde{\beta}) \in \text{Div}(L) \oplus \prod_{\nu \in \Omega_L} L_\nu^\times \) and \( a \in L^\times \) such that
\[
b^{n/d}O_L = \text{div}(a)b'/\sigma(b'), \quad i(\eta(b^{-1/d})) = \eta(a)\tilde{\beta}/\sigma(\tilde{\beta}).
\]
Taking the norm on both sides we get \( \text{div}(b^{-1}) = b^n = \text{div}(N_{L/K}(a)) \) and hence \( N_{L/K}(a) = b^{-1}u \) for some unit \( u \in K^\times \). From the second equation we get \( \eta(b^{-1}) = N_{L/K}(\eta(a)) \) and hence \( N_{L/K}(a) \) is totally positive. Since \( b^{-1} \) is totally positive we get that \( u \) is totally positive.

Units are always norms in unramified extensions of local fields and \( u \) is totally positive. Hence Hasse’s norm theorem \[Has1\] implies that there is a \( v \in L^\times \) such that \( u^{-1} = N_{L/K}(v) \). Since \( N_{L/K}(\text{div}(v)) \) is the unit ideal, Hilbert 90 for ideals implies that there is a \( J \in \text{Div}(L) \) such that \( \text{div}(v) = J/\sigma(J) \). We now put \( t = av \) and \( I = b'/J \). Then
\[
b^{n/d}O_L = \text{div}(av)(b'/J)/(\sigma(b'/J)) = \text{div}(t)I/\sigma(I).
\]
Note that \( N_{L/K}(\eta(i(b^{-1/d})(t^{-1}))) = 1 \). By Hilbert’s theorem 90 for idèles, we have
\[
H^1(G_{L/K}, I_L) \cong \ker N_{L/K}/(1 - \sigma)I_L = 1.
\]
and hence there exists a $\xi \in I_L$ (with a one in every non-archimedean component) such that $\eta(i(b^{-1/d})t^{-1}) = \xi/\sigma(\xi)$. Put $\beta' = \beta \xi$, where $\beta \in \prod_{v \in \Omega} L_v^\times$ is the class represented by $\beta$.

We now put

$$\left( \frac{L}{\beta} \right) = (i_{\beta'} b_{j1} b_{j2}^2 \cdots b_{j-1}^t) \in \text{Div}(L)^{n/d} \oplus \prod_{v \in \Omega} L_v^\times)^{n/d}$$

and the rest follows word by word as the remainder of the proof of Lemma starting from page 16. □

Let us summarize what we know so far about the ring structure of $H^*(U, \mathbb{Z}/n\mathbb{Z})$. First note that if $K$ is totally imaginary or $n$ is odd, then the picture is complete since the case $U = X$ was treated in \cite{AC18} and the case $U \neq X$ is completely determined by Theorem 3.9.

When $K$ has real places and $n$ is even, we have that $H^*(U, \mathbb{Z}/n\mathbb{Z}) \neq 0$ for $i \geq 3$. Hence it remains to compute the cup products $H^*(U, \mathbb{Z}/n\mathbb{Z}) \times H^*(U, \mathbb{Z}/n\mathbb{Z}) \to H^{i+j}(U, \mathbb{Z}/n\mathbb{Z})$ for $i + j \geq 3$. For $i \geq 3$, the restriction map $H^i(U, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v \in \Omega} H^i(SK_v, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism and the right hand side is the group cohomology of $\mathbb{Z}/n\mathbb{Z}$ viewed as a $\text{Gal}(\mathbb{C}/\mathbb{R})$-module. The cup product can then be determined as follows: For $i$ and $j$ such that $i + j \geq 3$, let $(x, y) \in H^i(U, \mathbb{Z}/n\mathbb{Z}) \times H^j(U, \mathbb{Z}/n\mathbb{Z})$ and let $p: H^i(U, \mathbb{Z}/n\mathbb{Z}) \to \prod_{v \in \Omega} H^i(SK_v, \mathbb{Z}/n\mathbb{Z})$ be the restriction. Then we have that $x \sim y$ is the unique element in $H^{i+j}(U, \mathbb{Z}/n\mathbb{Z})$ restricting to $r(x) \sim r(y) \in \prod_{v \in \Omega} H^{i+j}(SK_v, \mathbb{Z}/n\mathbb{Z})$. We know that $H^1(U, \mathbb{Z}/n\mathbb{Z})$ classifies torsors and $H^2(U, \mathbb{Z}/n\mathbb{Z})$ classifies gerbes. The restriction to a real place then simply tests if the torsor or gerbe is trivial or not over that place. Hence we have now determined the whole ring structure of $H^*(U, \mathbb{Z}/n\mathbb{Z})$, by essentially arguing that it reduces to group cohomology in the degrees we have not discussed in detail in this section.

4. SOME COMPUTATIONS OF CUP PRODUCTS

Comparison with the Legendre symbol. We will now show how to view the classical Legendre symbol $(\frac{q}{p})$ for primes $p, q \equiv 1 \pmod{4}$ as a cup product.

**Proposition 4.1.** Let $p$ and $q$ be distinct odd primes both equal to 1 (mod 4) and let $U = \text{Spec } \mathbb{Z} \setminus \{p, q\}$. Let $x_p$ and $x_q$ be the elements in $H^1(U, \mathbb{Z}/2\mathbb{Z})$ corresponding to the extensions $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{q})$ respectively. Then $x_p \sim x_q$ is completely determined by the Legendre symbol $(\frac{q}{p})$ and vice versa. In particular, $x_p \sim x_q = 0$ if and only if $(\frac{q}{p}) = 1$.

**Proof.** We have that $x_p \sim x_q = 0$ is zero if and only if $\langle x_p \sim x_q, \alpha \rangle = 0$ for all $\alpha \in H^1_c(U_{\text{ppf}}, \mu_2)$, where $\langle -, -, \rangle : H^2(U, \mathbb{Z}/2\mathbb{Z}) \times H^1_c(U_{\text{ppf}}, \mu_2) \to \mathbb{Q}/\mathbb{Z}$ is the pairing coming from Artin–Verdier duality. Let $S = \{p, q\} \subset \text{Spec } \mathbb{Z}$. Since $\mathbb{Q}$ has trivial class group we have

$$H^1_c(U_{\text{ppf}}, \mu_2) \cong C_{\mathbb{Q}, S}[2] \cong (\mathbb{Z}_p^\times \times \mathbb{Z}_q^\times)[2],$$

which is generated by the elements $(-1, 1)$ and $(1, -1)$. First consider the case when $\alpha$ is the class represented by $(1, -1, 1, 1) \in I_{\mathbb{Q}} = \mathbb{R}^\times \times \mathbb{Q}_p^\times \times \mathbb{Q}_q^\times \times \prod_{v \neq p, q, \infty} \mathbb{Q}_v$. By Theorem 3.9 we have

$$\langle x_p \sim x_q, \alpha \rangle = \langle x_q, \alpha N(\beta) \rangle$$

where $\beta$ is an element in $I_{\mathbb{Q}(\sqrt{p})}$ such that $\alpha = \beta/\sigma(\beta)$, where $t \in \mathbb{Q}(\sqrt{p})^\times$ satisfies $N(t) = \alpha^{-n}$. Here $\sigma$ is the generator of $\text{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q})$ and $N$ denotes the norm on idèles $N: I_{\mathbb{Q}(\sqrt{p})} \to I_{\mathbb{Q}}$. The group $I_{\mathbb{Q}(\sqrt{p})}$ will have one component above the prime $p$ and one or two components above $q$ depending on whether $q$ splits in $\mathbb{Q}(\sqrt{p})$ or not. Let $\beta \in I_{\mathbb{Q}(\sqrt{p})}$ be the element with component over $p$ equal to $\sqrt{p}$ and with a 1 in all other components. Then $\beta/\sigma(\beta)$ has component $-1$ at the prime above $p$ and is equal to 1 in all other components. We further have that $\alpha N(\beta) \in I_{\mathbb{Q}}$ is the idèle which is $p$ in the component corresponding to $p$ and is otherwise equal to 1. We thus have that

$$\langle x_p \sim x_q, \alpha \rangle = \langle x_q, \alpha N(\beta) \rangle$$
is zero if and only if $p$ is in the image of the norm map $N: \mathbb{Q}[\sqrt{q}] \to \mathbb{Q}$, which is to say, if and only if $p$ splits in $\mathbb{Q}(\sqrt{q})$.

The case when $\alpha$ equals $(1, 1, -1, 1) \in I_{\mathbb{Q}, S} = \mathbb{R}^\times \times \mathbb{Q}_p^\times \times \mathbb{Q}_q^\times \times \Pi_{v \neq p, q, \infty} \mathbb{Q}_v$ follows analogously by graded commutativity of the cup product. □

Remark 4.2. We note that as a consequence of Proposition 4.1, the quadratic reciprocity law $(\frac{p}{q}) = (\frac{q}{p})$ then follows from graded commutativity of the cup product.
Appendix A. Computations

Consider the big diagram (9). The differential $d_1^{c_0}$ is given by

$$d_1^{c_0} = \begin{pmatrix} \sigma - 1 \\ \eta \\ - \text{div} \\ -\eta \end{pmatrix}$$

where

$$(\sigma' - 1)(a) = (\sigma(a_{n/d})/a_1, a_1/a_2, \ldots, a_{(n/d)-1}/a_{n/d}).$$

**Proof of Lemma 3.6.** The case of $s$ was shown in Section 2. To prove that statement for $t$ we consider the spectral sequence of $R\Gamma_c(U_{\text{fppf}}, \mathcal{H}om(C(\hat{u}), \mathcal{C}))$ obtained by filtering by columns of the corresponding double complex. This spectral sequence has page $E_1$ given by the double complex

$$H^q(C(\Gamma(U, G(\mathcal{H}om(C(\hat{u}), \mathcal{C})))) \rightarrow \Gamma(Z', \gamma^*G(\mathcal{H}om(C(\hat{u}), \mathcal{C})))) \oplus \bigoplus_{v \in \Omega} \Gamma(K_v, a_v^*G(\mathcal{H}om(C(\hat{u}), \mathcal{C})_v))[-1])^p$$

for $q \geq 0$, which looks like

![Diagram](image)

from which it is obvious that the spectral sequence converges. The $\mathbb{Z}/n\mathbb{Z}$-torsor $Y$ is of the form $Y = \text{Ind}^{C_{y'}}_{C_y}(Y')$ for $Y' = \text{Spec} \mathcal{O}_L$ with $\mathcal{O}_L$ the ring of integers of an extension $L/K$ unramified over $U$. We write $U_{Y'} = U \times_X Y'$. The differential $d_1^{0,2}: E_1^{0,2} \rightarrow E_1^{1,2}$ is given by

$$\text{Br}(L)_{n/d} \oplus \text{Br}(L)_{n/d} \oplus \text{Br}(L)_{n/d} \rightarrow \bigoplus_{p \in U_{y'}} \text{Br}(L_p)_{n/d} \oplus \bigoplus_{w \in S'} \text{Br}(L_w)_{n/d} \oplus \bigoplus_{v \in \Omega} \text{Br}(L_v)_{n/d}$$

which is injective since

$$\text{Br}(L)_{n/d} \rightarrow \bigoplus_{v \in U_{y'}} \text{Br}(L_v)_{n/d} \oplus \bigoplus_{w \in S'} \text{Br}(L_w)_{n/d} \cong \bigoplus_{v \in Y'} \text{Br}(L_v)_{n/d}$$
is just \((\mathrm{inv}_v)^{n/d}\) which we know is injective (see Appendix B). Hence the \(E_2\)-page will look like

and if we look at the filtration associated to this spectral sequence, the index \((j,2)\) does not contribute before level 3 if \(j \geq 1\). This shows that \(t\) is an isomorphism.

The fact that the leftmost and right most vertical zig-zags of (3) gives isomorphisms on cohomology in degree 2 follows from the spectral sequence argument on page 5, together with the fact that the morphism of complexes

\[
\begin{align*}
K^\times & \xrightarrow{d^0} K^\times \oplus \text{Div} U \oplus \prod_{p \in S_f} K_p^\times \oplus \prod_{v \in \Omega_K} K_v^\times \\
& \xrightarrow{d^1} \text{Div} U \oplus \prod_{p \in S_f} K_p^\times \oplus \prod_{v \in \Omega_K} K_v^\times
\end{align*}
\]

where

\[
d^0 = \begin{pmatrix}
-n \\
\eta \\
(\iota_v)^* \nu
\end{pmatrix}
\quad \text{and} \quad
d^1 = \begin{pmatrix}
\text{div} \\
\eta \\
0 \\
0
\end{pmatrix},
\]

gives isomorphisms on \(H^1\) and \(H^2\).

The statement about \(t'\) and \(\Gamma_v(q(\bar{u})^*)\) follows by the 2-out-of 3 property of isomorphisms since the functors \(R\Gamma_v(U_{\mathrm{fppf}},-)\) and \(\hat{R}\Gamma_v(U_{\mathrm{fppf}},-)\) preserve quasi-isomorphisms. \(\square\)

### Appendix B. Cohomology of Idèles

Most results of the following section are standard and can be found in most books on class field theory. Our main references is \([\text{Cas67}]\). Let \(L/K\) be a Galois extension of number fields with Galois group \(G\). For every prime \(v\) in \(K\) we let \(L_v^r = L_w^r\) for some (any) prime \(w\) over \(v\) and we write \(G_v^r = \text{Gal}(L_v^r/K_v^r)\).

**Lemma B.1** (\([\text{Cas67} \text{ VII.7.3.(b)}]\)). We have an isomorphism

\[H^i(G, I_L) \cong \bigoplus_v H^i(G_v^r, (L_v^r)^\times)\]

for every \(i \in \mathbb{Z}\).

If we consider the cohomology sequence associated to the short exact sequence

\[0 \to L^r \to I_L \to C_L \to 0\]

and using the fact that \(H^1(G, C_L) = 0\) \([\text{Cas67} \text{ VII.9.1.(2)}]\) we get an injection

\[H^2(G, L^r) \to \bigoplus_v H^i(G_v^r, (L_v^r)^\times).\]

If we consider the limit over all finite sub-extensions \(K \subseteq L \subseteq K^{\mathrm{sep}}\), we get the following result:

**Proposition B.2.** We have a canonical injection

\[
\text{Br}(K) \to \bigoplus_v \text{Br}(K_v).
\]
Proof. Taking the limit over all finite sub-extensions $K \subseteq L \subseteq K^{sep}$, we get an injection

$$H^2(\text{Gal}(K^{sep}/K), (K^{sep})^\times) \to H^2(\text{Gal}(K^{sep}/K), I_{K^{sep}})$$

and the result follows since $\text{Br}(K) \cong H^2(\text{Gal}(K^{sep}/K), (K^{sep})^\times)$ and

$$H^2(\text{Gal}(K^{sep}/K), I_{K^{sep}}) \cong \bigoplus_v H^1(\text{Gal}((K^{sep})^v/K_v), ((K^{sep})^v)^\times)$$

$$\cong \bigoplus_v \text{Br}(K_v). \quad \square$$

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