On The Hamiltonian Structures
and
The Reductions of The KP Hierarchy*

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Abstract: Recent work on a free field realization of the Hamiltonian structures of
the classical KP hierarchy and of its flows is reviewed. It is shown that it corresponds to
a reduction of KP to the NLS system.

Sep. 1992.

Talk given by D.A.D. at the NSERC-CAP Workshop on Quantum Groups, Integrable Models
and Statistical Systems, Kingston, Canada July 13-17 1992.
0. Introduction

It is well-known that it is possible to reduce the KP hierarchy to any of the SL(N) KdV hierarchies. The KP hierarchy has a bi-Hamiltonian structure \[1\] and its second Hamiltonian structure has been shown \[2\] to be related to \(\hat{W}_\infty\), a centerless, non-linear deformation of the \(W_\infty\) algebra of Pope, Romans and Shen \[3\]. This naturally leads to the conjecture that the \(W_N\) algebras of Zamolodchikov \[4\], which arise as the second Hamiltonian structure of the SL(N) KdV hierarchies, can be obtained via some sort of reduction of \(\hat{W}_\infty\) \[2\]. While the evidence for this conjecture is convincing, an explicit proof is still lacking.

On the other hand, inspired by the fact that the linear algebra \(W_\infty\) possesses a representation in terms of two bosons \[5\], Yu and Wu \[6\] presented a two boson representation of \(\hat{W}_\infty\). In \[7\], it was shown that this representation is related to the reduction of the KP hierarchy to a 1 + 1 dimensional integrable hierarchy. In this talk, we recall some of the results of \[6\] and of \[7\], and show further that the integrable hierarchy of \[7\] is related to the well-known nonlinear Schrödinger (NLS) hierarchy. It seems that the fact that KP admits such a reduction has been known to specialists in integrable systems for some time \[8\] \[9\].

Related results, albeit from a very different approach, were also presented by C.S.Xiong at this workshop \[10\].

We will be using differential \(\partial \equiv \partial_x\) and pseudo-differential \(\partial^{-1}\) operators, where \(\partial^{-1}\) is an integration symbol satisfying \(\partial\partial^{-1}a = \partial^{-1}\partial a = a\). We have \(\partial^{-1}a = a\partial^{-1} - a'\partial^{-2} + a''\partial^{-3} - \ldots\). To shorten the expressions of the Poisson brackets, we write \(\{a, b\} = c\delta'\) for \(\{a(x), b(x')\} = c(x)\partial_x\delta(x, x')\).

1. The two boson representation of \(\hat{W}_\infty\)

The KP hierarchy

The KP hierarchy is usually introduced following the approach of the Japanese school \[11\]; specifically, the \(r^{th}\) flow of the KP hierarchy is given by the Lax equation

\[
\frac{\partial}{\partial t_r}L = [(L^r)_+, L] , \quad r \in N \tag{1.1}
\]

where \(L\) is the pseudo-differential operator

\[
L = \partial + u_0\partial^{-1} + u_1\partial^{-2} + \ldots = \partial + \sum_{i=0}^\infty u_i\partial^{-i-1} , \quad \partial = \partial/\partial x . \tag{1.2}
\]
and \((L^r)⁺\) is the differential part of the \(r^{th}\) power of the KP operator \(L\). The fields \(u_i\) are implicitly understood to depend on \(x\) and on an infinite number of time coordinates \(t_i\). For instance, for \(r = 2\) we get \((L^2)_+ = \partial^2 + 2u_0\) and

\[
\begin{align*}
    r = 2 : & \quad u_{0,t_2} = 2u_{1,x} + u_{0,xx} \\
    & \quad u_{1,t_2} = 2u_{2,x} + u_{1,xx} + 2u_0u_{0,x} \\
    & \quad u_{2,t_2} = 2u_{3,x} + u_{2,xx} + 4u_{0,x}u_1 - 2u_0u_{0,xx} \\
    & \quad \vdots
\end{align*}
\]

(1.3)

The flows defined by (1.1) commute \((\partial_i \partial_j u_k = \partial_j \partial_i u_k)\) and furthermore they are bi-Hamiltonian, i.e. (1.1) can also be written as

\[
\partial_t, u_i = \{u_i, \int \mathcal{H}_{r+1}\}_1 = \{u_i, \int \mathcal{H}_r\}_2
\]

(1.4)

where \(\{.,.\}_1\) and \(\{.,.\}_2\) are two Poisson brackets for the fields \(u_i\) and the \(\int \mathcal{H}_r\)'s are some Hamiltonians. The brackets \(\{.,.\}_1\) (historically the first set to be discovered [12]) were shown in [13] to correspond to the linear conformal algebra \(W_{1+\infty}\) with \(c = 0\). The second Hamiltonian structure [1] corresponds to \(\hat{W}_\infty\) [3]. The Hamiltonians are given explicitly by

\[
\mathcal{H}_r = \frac{1}{r} \text{res } L^r
\]

(1.5)

where \(\text{res } L^r\) denotes the coefficient of \(\partial^{-1}\) in \(L^r\).

We give the first commutators of these algebras, as we will need their expressions shortly. The relation between the \(u\) and \(W\) fields will be given later. For \(W_{1+\infty}\) we have

\[
\begin{align*}
    \{W_1, W_1\}_1 &= 0 \quad , \quad \{W_2, W_1\}_1 = W_1\delta', \\
    \{W_2, W_2\}_1 &= 2W_2\delta' + W'_2\delta \quad , \quad \{W_1, W_3\}_1 = 2W_2\delta' + 2W'_2\delta
\end{align*}
\]

(1.6)

and for \(\hat{W}_\infty\) we have

\[
\begin{align*}
    \{W_2, W_2\}_2 &= 2W_2\delta' + W'_2\delta \quad , \quad \{W_2, W_3\}_1 = 3W_3\delta' + 2W'_3\delta \\
    \{W_2, W_4\}_2 &= \frac{2}{3}W_2\delta'' + \frac{4}{3}W'_2\delta'' + (4W_4 + \frac{2}{3}W_2')\delta' + 3W'_4\delta \\
    \{W_3, W_3\}_2 &= \frac{1}{2}W_2\delta'' + \frac{3}{4}W'_2\delta'' + (4W_4 - \frac{1}{12}W_2' - 2W_2^2)\delta' + (2W_4 - \frac{1}{6}W_2' - W_2^2)'\delta \\
    \{W_4, W_4\}_2 &= \ldots + (6W_6 + \frac{1}{3}W_4' + \frac{1}{45}W_2^{(4)} - 6W_4W_2 - 6W_3^2 - 2W_2^3 - \frac{1}{2}W_2^2)\delta' \\
    &\quad + (3W_6 - W_4' + \frac{1}{15}W_2^{(4)} - 3W_4W_2 - 3W_3^2 - W_2^3 \\
    &\quad + \frac{1}{2}(W_2^2)' - \frac{1}{4}W_2^2)\delta
\end{align*}
\]

(1.7)
Towards a representation of $\hat{W}_\infty$

The algebra $W_\infty$ was found to possess a free field representation in terms of two real bosons $\tilde{j}$, so in [7] Yu and Wu proposed the existence of a similar representation of $\hat{W}_\infty$. To this end, they introduced the currents $j(x)$ and $\bar{j}(x)$ with the natural Poisson brackets

$$\{j(x), j(x')\}_2 = 0 \quad , \quad \{j(x), \bar{j}(x')\}_2 = \delta'(x, x') \quad , \quad \{\bar{j}(x), j(x')\}_2 = 0 \quad . \quad (1.8)$$

Upon imposing

$$L = \partial + \sum_{i=0}^{\infty} u_i \partial^{-i-1} = \partial + \bar{j} \frac{1}{\partial - (j + \bar{j})} \quad , \quad (1.9)$$

we get an expression for the fields $u_i$ in terms of the currents, as

$$u_i = \bar{j} [(-\partial + j + \bar{j})^i j] \quad . \quad (1.10)$$

If (1.10) were to provide a faithful representation of $\hat{W}_\infty$, one could just compute the $\{u_i, u_j\}$ commutators using their expression in (1.10) and the Poisson brackets (1.8). However, more careful examination [14] reveals that this is not the case. It is easy to check that (1.10) implies relations or constraints between the fields $u_i$; the simplest such relation is

$$u_2 u_0 - u_1^2 + u_1' u_0 - u_1 u_0' = 0 \quad . \quad (1.11)$$

Thus it is not possible to unambiguously translate an expression in $j \bar{j}$ into an expression in the fields $u_i$. The upshot of this is that the two boson representation is related to a reduction, as opposed to a realization of the KP hierarchy.

Let us show this more explicitly. The map between the fields $u_i$ and $W_i$ was derived in [7] and gives, for the $\hat{W}_\infty$ case

$$W_2 = u_0 \quad , \quad W_3 = u_1 + \frac{1}{2} u_0' \quad , \quad W_4 = u_2 + u_1' + \frac{1}{3} u_0'' + u_0^2 \quad ,$$

$$W_5 = u_3 + \frac{3}{2} u_2' + u_1'' + \frac{1}{4} u_0''' + 3 u_0 u_1 + \frac{3}{2} u_0 u_0' \quad . \quad (1.12)$$

(note the correction of a typographical error in the expression for $W_4$ in [7]). In terms of $j$ and $\bar{j}$, we find

$$W_2 = j \bar{j} \quad , \quad W_3 = \frac{1}{2} (j \bar{j}' - j' \bar{j}) + j \bar{j}^2 + j^2 \bar{j} \quad ,$$

$$W_4 = \frac{1}{3} (j \bar{j}'' - j' \bar{j}' + j'' \bar{j}) - j j' \bar{j} + j \bar{j} \bar{j}' - j' \bar{j}^2 + j^2 \bar{j}' + j^3 \bar{j} + 3 j^2 \bar{j}^2 + j^3 \bar{j} \quad ,$$

$$W_5 = \frac{1}{4} (j \bar{j}''' - j' \bar{j}'' + j'' \bar{j}' - j''' \bar{j}) + \frac{1}{2} (2 j^2 \bar{j}'' + 2 j'' \bar{j}' + 2 j \bar{j}''' + 3 j j'' \bar{j} + 3 j \bar{j} \bar{j}'$$

$$+ j^2 \bar{j} + j \bar{j}^2 - j j' \bar{j}' + j^2 \bar{j}' + 3 j^3 \bar{j}' - 3 j^2 \bar{j}^3 - 3 j^2 \bar{j} \bar{j}' + 3 j^2 \bar{j}^2 \bar{j}' - 9 j j' \bar{j}^2 + 9 j^2 \bar{j} \bar{j}' + j^4 \bar{j} + 6 j^3 \bar{j}^2 + 6 j^2 \bar{j}^3 + j \bar{j}^4) \quad . \quad (1.13)$$
so that under the interchange \( j \to -\bar{j}, \bar{j} \to -j \), we see that \( W_n \to (-)^n W_n \). Such a symmetry was already present for the bosonic representation of the linear \( W_\infty \), \cite{5}. We note that the linear part (in terms of \( u_i \) fields) of the field redefinitions \((1.12)\) are given by a formula analogous to \((1.10)\), namely

\[
W_{\text{lin}}^{n+2} = \{u_0, u_1 + \frac{1}{2} u'_0, u_2 + u'_1 + \frac{1}{3} u''_0, u_3 + \frac{3}{2} u'_2 + u''_1 + \frac{1}{4} u''''_0, \ldots \}
\]

\[
= \frac{1}{n + 1} \sum_{m=0}^{n} [(-\partial + j + \bar{j})^m j][\partial + j + \bar{j}]^{n-m} \bar{j} \] . \quad (1.14)

Given the relations \((1.12), (1.11)\) becomes a constraint on the \( W \) fields,

\[
W_4 W_2 = W_2^3 + W_3^2 - \frac{1}{4} W_2 W_3' + \frac{1}{3} W_2 W_2'' \] . \quad (1.15)

Note that the \( \{W_4, W_4\} \) commutator in \((1.7)\) involves a \( W_4 W_2 \) term, just like \((1.15)\).

### 2. The \( j, \bar{j} \) hierarchy

In \cite{7} a reduction similar to the one given in the last section was considered at the level of the KP flows themselves. The second Hamiltonian structure of this \( j, \bar{j} \) hierarchy is clearly given by \((1.8)\), so we can immediately write down the flows as

\[
\bar{j}, t_r = \left( \frac{j}{\bar{j}} \right)_{t_r} = P_2 \nabla_{\bar{j}} \int \mathcal{H}_r \] , \quad (2.1)

where \( \nabla_{\bar{j}} = (\delta/\delta j, \delta/\delta \bar{j}) \) and \( P_2 \) is the Hamiltonian structure corresponding to \((1.8)\),

\[
P_2 = \frac{j}{\bar{j}} \begin{pmatrix} \bar{j} & j \\ \partial & 0 \end{pmatrix} \] , \quad (2.2)

and \( \mathcal{H}_r \) is obtained by taking the expression for the KP Hamiltonian \((1.5)\) and writing it in terms of the \( j, \bar{j} \) fields through \((1.10)\).

Let us consider the second flow in more detail. It is written explicitly as

\[
r = 2 : \mathcal{H}_2 = -j' \bar{j} + j^2 \bar{j} + j \bar{j}^2 \] , \quad (2.3)

\[
\begin{align*}
  j, t_2 &= (-j' + j^2 + 2 j \bar{j})' \\
  \bar{j}, t_2 &= (\bar{j}' + \bar{j}^2 + 2 \bar{j} j)' \end{align*}
\]

The \( j, \bar{j} \) hierarchy turns out to be bi-Hamiltonian, with the first structure \( P_1 \) (corresponding to \( W_{1+\infty} \)) being non local, but the third Hamiltonian structure \( P_3 = P_2 P_1^{-1} P_2 \) being local. \( P_3 \) and \( P_1 \) are given explicitly in \cite{7}.
Let us try to manipulate (2.3) to see if it can be made to correspond to a known integrable system. We notice that upon setting \( \bar{j} \) to 0, (2.3) becomes the “derivative” of the Burgers equation, \( h_t = h'' + 2hh' \), which can be linearized by the Cole–Hopf transformation \( h = u'/u \) into the heat equation \( u_t = u'' \) [15]. Guided by this analogy, let us first “integrate” the flows (2.3) by introducing \( h \) and \( \bar{h} \) defined by \( h' = -\bar{j} \) and \( \bar{h}' = \bar{j} \). We get:

\[
\begin{align*}
\bar{h},t_2 &= -h'' - h'^2 - 2h'h', \\
h, t_2 &= \bar{h}'' + h'^2 - 2h'h'.
\end{align*}
\tag{2.4}
\]

Using these equations, we find that \( \psi = h'e^{h-\bar{h}} \) and \( \bar{\psi} = \bar{h}'e^{\bar{h}-h} \) satisfy

\[
\begin{align*}
\psi_{t_2} &= -\psi'' + 2\psi^2 \bar{\psi}, \\
\bar{\psi}_{t_2} &= \bar{\psi}'' - 2\bar{\psi}^2 \psi.
\end{align*}
\tag{2.5}
\]

which is the second flow of the NLS system. A more careful treatment of the relation between our \( j, \bar{j} \) system and the NLS one in standard form can be found in [16]. In fact it can be checked that the entire \( j, \bar{j} \) hierarchy can be mapped in this manner to the NLS hierarchy. The simplest way to do this is to exploit the powerful concept of Fréchet derivatives (see for instance [15]) to map the different Hamiltonian structures of the \( j, \bar{j} \) hierarchy to those of the NLS hierarchy. From \( h = -\partial^{-1}j \) and \( \bar{h} = \partial^{-1}\bar{j} \), we find that the Fréchet derivative of \((h, \bar{h})^T\) with respect to \((j, \bar{j})^T\) is

\[
D = \begin{pmatrix} -\partial^{-1} & 0 \\ 0 & \partial^{-1} \end{pmatrix}
\tag{2.6}
\]

so that the Hamiltonian structure of the flows (2.4) is

\[
DP_2D^\dagger = \begin{pmatrix} -\partial^{-1} & 0 \\ 0 & \partial^{-1} \end{pmatrix} \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \partial^{-1} & 0 \\ 0 & -\partial^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \partial^{-1} \\ \partial^{-1} & 0 \end{pmatrix}
\tag{2.7}
\]

(Here we are assuming the possibility of defining \( \partial^{-1} \) as an anti-self-adjoint operator). The Fréchet derivative of \((\psi, \bar{\psi})^T\) with respect to \((h, \bar{h})^T\) is

\[
\tilde{D} = \begin{pmatrix} e^{h-\bar{h}}(\partial + h') & -e^{h-\bar{h}}h' \\ -e^{h-\bar{h}}h' & e^{\bar{h}-h}(\partial + \bar{h}') \end{pmatrix}
\tag{2.8}
\]

and using this we find

\[
\tilde{D}DP_2D^\dagger \tilde{D}^\dagger = \begin{pmatrix} -2\psi \partial^{-1}\psi & -\partial + 2\psi \partial^{-1} \bar{\psi} \\ -\partial + 2\bar{\psi} \partial^{-1}\psi & -2\bar{\psi} \partial^{-1} \bar{\psi} \end{pmatrix}
\tag{2.9}
\]

This is the second Hamiltonian structure of the NLS system. Similarly one can show explicitly that the \( P_1 \) induces the first Hamiltonian structure of NLS.
We can formulate the connection between the KP and NLS hierarchies directly, without going through the \( j, \bar{j} \) hierarchy. It is straightforward to show that the Lax operator (1.9) can be written
\[
L = \partial + j(\partial - j - \bar{j})^{-1} j = \partial - \psi \partial^{-1} \bar{\psi} \tag{2.10}
\]
by simply using the map from the \( j \)'s to the \( \psi \)'s we just presented. So the reduction from KP to NLS is given by the constraint \( L_{KP} = \partial - \psi \partial^{-1} \bar{\psi} \), or, equivalently, by constraining the KP fields as \( u_i = (-)^{i+1} \psi^{(i)} \). A natural question is to ask what kind of hierarchies one gets by considering the reduction \( L^n_{KP} = \partial^n + n \partial^{n-2} + \ldots + \psi \partial^{-1} \bar{\psi} \). This has been considered in some details in [8].

Finally the relation between the NLS hierarchy and the \( j, \bar{j} \) system allows us to answer the question that was left open in [7], of finding a zero-curvature formulation of the \( j, \bar{j} \) hierarchy, and of how to obtain it by reduction from the self–dual Yang–Mills system [17]. Since we use slightly different notation than in [7], we recall how such a reduction is introduced. The self–dual Yang–Mills equations in four dimensions are usually written as
\[
F_{12} = F_{34} , \quad F_{13} = F_{42} , \quad F_{14} = F_{23} , \tag{2.11}
\]
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \). Introducing \( w = x^1 + ix^2, z = x^3 - ix^4 \) and their complex conjugates \( \bar{w} \) and \( \bar{z} \), eqs. (2.11) become
\[
F_{\bar{z}\bar{w}} = 0 , \quad F_{z\bar{z}} + F_{w\bar{w}} = 0 , \quad F_{zw} = 0 , \tag{2.12}
\]
where now \( F_{zw} = \partial_z A_w - \partial_w A_z + [A_z, A_w] \). The gauge freedom in (2.12) is expressed through the invariance of (2.12) under the transformations \( A_\mu \to A'_\mu = g A_\mu g^{-1} - \partial_\mu g g^{-1} \).

Now, we know that to get the NLS equations by reduction of the self–dual Yang–Mills system, we just need to choose
\[
A_z = \begin{pmatrix} 0 & \psi \\ \bar{\psi} & 0 \end{pmatrix} , \quad A_{\bar{w}} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{2.13}
\]
and reduce the system (2.12) first with respect to \( \partial_{\bar{w}} \) and then \( \partial_z - \partial_{\bar{z}} \) [17]. The potentials \( A_z \) and \( A_{\bar{w}} \) are determined by (2.12). Here, since we know that the \( j, \bar{j} \) system is related to the \( \psi, \bar{\psi} \) system by the map given above, we consider the effect of a \( \bar{w} \)-independent gauge transformation on \( A_z \) which leaves \( A_{\bar{w}} \) invariant. Under a transformation by
\[
g = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} , \tag{2.14}
\]

we get
\[ A'_z = \begin{pmatrix} \alpha^{-1} \alpha_z & \psi \alpha^{-2} z \\ \bar{\psi} \alpha^2 & -\alpha^{-1} \alpha_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(j + \bar{j}) & -j \\ \frac{1}{2}(j + \bar{j}) & \end{pmatrix}, \quad (2.15) \]
where we have taken \( \alpha^2 = e^{\hbar - \h}. \)

If we take the new form of \( A_z \) from (2.15), and the form of \( A_{\bar{w}} \) from (2.13) and plug into (2.12), imposing \( \partial_{\bar{w}} = 0 \) but not \( \partial_z = \partial_{\bar{z}} \), we find
\[ A_{\bar{z}} = \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \beta_{\bar{z}} = -\frac{1}{2}(j + \bar{j})_{\bar{z}} \]
\[ \begin{pmatrix} j \\ \bar{j} \end{pmatrix}_z = \begin{pmatrix} -\partial + j z \partial^{-1} - (j + \bar{j}) & j z \partial^{-1} + 2j \\ j \bar{z} \partial^{-1} + 2\bar{j} & \partial + \bar{j} \bar{z} \partial^{-1} - (j + \bar{j}) \end{pmatrix} \begin{pmatrix} j \\ \bar{j} \end{pmatrix}_{\bar{z}}. \quad (2.16) \]
The matrix operator in the second equation above is equal to \( P_3 P_{-1} \) in [4]. The method of reduction used here, not imposing immediately \( \partial_z = \partial_{\bar{z}} \), gives us not only the equations of the hierarchy we are looking for, but also its recursion operator (of course, the \( r = 2 \) flow (2.3) is obtained after setting \( \partial_z = \partial_{\bar{z}} \)). The same applies to the reduction to the KdV equation, where one gets not only the usual KdV equation, but also its recursion operator.

3. Some open questions

The results presented here open up some potentially interesting avenues of research. The \( \hat{W}_\infty \) algebra, which was originally introduced as an algebra that “contains” all the \( W_N \) algebras, apparently contains even more. It is clearly of some interest to try to find all reductions of the KP system and to understand the physical significance of the associated reductions of \( \hat{W}_\infty \). Work on the algebras associated with the class of reductions of \([8]\), generalizing the result here, is currently in progress.

Another question that arises is as follows: we know that the KdV equation arises by reduction (i.e. constraining) the KP hierarchy. On the other hand, we also know how to reduce the self-dual Yang-Mills system to the KdV equation \([17]\). In the work presented here, we see that the KP hierarchy can also be reduced to the NLS hierarchy, and again we know how to reduce the self–dual Yang–Mills equations to NLS. It is believed that self–dual Yang–Mills is a universal integrable system, and in particular KP can be obtained by reduction from it, for a suitable infinite dimensional gauge group. It would be interesting to understand better why integrable systems can appear both as direct reductions of self–dual Yang–Mills with a finite dimensional gauge group, and as indirect reductions via KP. Work on this is also in progress.

Acknowledgements

We wish to thank Pierre Mathieu and Walter Oevel for discussions.

D.A.D was supported by NSERC (Canada), FCAR (Québec), and BSR (Université Laval), and J.S. was supported by a grant in aid from the U.S. Department of Energy, # DE-FG02-90ER40542.
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