SOME SHARP SCHWARZ-PICK TYPE ESTIMATES AND THEIR APPLICATIONS OF HARMONIC AND PLURIHARMONIC FUNCTIONS

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Abstract. The purpose of this paper is to study the Schwarz-Pick type inequalities for harmonic or pluriharmonic functions. By analogy with the generalized Khavinson conjecture, we first give some sharp estimates of the norm of harmonic functions from the Euclidean unit ball in $\mathbb{R}^n$ into the unit ball of the real Minkowski space. Next, we give several sharp Schwarz-Pick type inequalities for pluriharmonic functions from the Euclidean unit ball in $\mathbb{C}^n$ or from the unit polydisc in $\mathbb{C}^n$ into the unit ball of the Minkowski space. Furthermore, we establish some sharp coefficient type Schwarz-Pick inequalities for pluriharmonic functions defined in the Minkowski space. Finally, we use the obtained Schwarz-Pick type inequalities to discuss the Lipschitz continuity, the Schwarz-Pick type lemmas of arbitrary order and the Bohr phenomenon of harmonic or pluriharmonic functions.

1. Introduction

Let $\mathbb{C}^n$ be the complex space of dimension $n$, where $n$ is a positive integer. We can also interpret $\mathbb{C}^n$ as the real $2n$-space $\mathbb{R}^{2n}$. We use $\ell^n_p$ to denote the Minkowski space defined by $\mathbb{C}^n$ together with the $p$-norm

$$
\|z\|_p := \left\{ \begin{array}{ll}
\left( \sum_{j=1}^{n} |z_j|^p \right)^{1/p}, & p \in [1, \infty), \\
\max_{1 \leq j \leq n} |z_j|, & p = \infty.
\end{array} \right.
$$

It is well known that $\ell^n_p$ is a Banach space. For $p \in [1, \infty]$, let $B_{\ell^n_p} := \{ z \in \mathbb{C}^n : \|z\|_p < 1 \}$ and $B_{\ell^n_p} := \{ x \in \mathbb{R}^n : \|x\|_p < 1 \}$.

The classical Schwarz-Pick lemma states that an analytic function $f$ of $\mathbb{D}$ into itself satisfies

$$
|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.
$$

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For complex-valued harmonic functions $f$ of $\mathbb{D}$ into itself, Colonna [13] proved the following sharp Schwarz-Pick lemma:

\begin{equation}
\left| \frac{\partial f(z)}{\partial z} \right| + \left| \frac{\partial f(z)}{\partial \overline{z}} \right| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.
\end{equation}

For real-valued harmonic functions $u$ of $\mathbb{D}$ into $(-1, 1)$, there is the following Schwarz-Pick lemma in [3, Theorem 6.26]:

\begin{equation}
\| \nabla u(0) \|_2 \leq \frac{4}{\pi}.
\end{equation}

Since the inequality (1.3) can be rewritten into the following form for an analytic function $f$ in $\mathbb{D}$ (cf. [29]):

\[ |f'(z)| \leq \frac{4}{\pi} \frac{1 - |z|^2}{1 - |z|^2} \sup_{w \in \mathbb{D}} |\text{Re}(f(w))|, \]

where $\text{Re}(f)$ means the real part of $f$, it may be considered as a version of the Schwarz-Pick lemma for real valued harmonic functions. Kalaj and Vuorinen [23] improved the classical inequality (1.3) into the following sharp form:

\begin{equation}
\| \nabla u(z) \|_2 \leq \frac{4}{\pi} \frac{1 - |u(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.
\end{equation}

In the book of Protter and Weinberger [37], there is the following estimate of the gradient of a harmonic function of a domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) into $\mathbb{R}$:

\begin{equation}
\| \nabla u(x) \|_2 \leq \frac{n \omega_{n-1}}{(n-1) \omega_n d_\Omega(x)} \text{osc}_\Omega(u),
\end{equation}

where $\omega_n$ is the volume of $B_{\mathbb{R}^2}$, $\omega_{n-1}$ is the area of $\partial B_{\mathbb{R}^2}$, $\text{osc}_\Omega(u)$ is the oscillation of $u$ in $\Omega$, and $d_\Omega(x)$ is the distance from $x$ to the boundary $\partial \Omega$ of $\Omega$. (1.5) is a consequence of the following inequality:

\[ \| \nabla u(0) \|_2 \leq \frac{2n \omega_{n-1}}{(n-1) \omega_n R} \sup_{\|y\| < R} |u(y)|. \]

We refer the reader to see [25] for more details. For any fixed $x \in B_{\ell_2^n}$, let $C(x)$ be the smallest number such that the following inequality

\[ \| \nabla u(x) \|_2 \leq C(x) \sup_{y \in B_{\ell_2^n}} |u(y)| \]

holds for all bounded harmonic functions $u$ of $B_{\ell_2^n}$ into $\mathbb{R}$. Similarly, for $x \in B_{\ell_2^n}$ and $\iota \in \partial B_{\ell_2^n}$, denote by $C(x, \iota)$ the smallest number such that the inequality

\[ |\langle \nabla u(x), \iota \rangle| \leq C(x, \iota) \sup_{y \in B_{\ell_2^n}} |u(y)| \]

holds for all bounded harmonic functions $u$ of $B_{\ell_2^n}$ into $\mathbb{R}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^n$. Since

\[ \| \nabla u(x) \|_2 = \sup_{\iota \in \partial B_{\ell_2^n}} |\langle \nabla u(x), \iota \rangle|, \]
we see that

\[ C(x) = \sup_{\iota \in \partial \mathbf{B}_2} C(x, \iota). \]

In [25, p. 171], Kresin and Maz’ya posed the generalized Khavinson problem for bounded harmonic functions of \( \mathbf{B}_2 \) into \( \mathbb{R} \) as follows (see also [24, p. 220], [31, Conjecture 1] and [27, Conjecture 1]).

**Conjecture 1.1.** For \( x \in \mathbf{B}_2 \backslash \{0\} \), we have

\[ C(x) = C(x, \mathbf{n}_x), \]

where \( \mathbf{n}_x := x/\|x\|_2 \) is the unit outward normal vector to the sphere \( \|x\|_2 \partial \mathbf{B}_2 := \{ y \in \mathbb{R}^n : \|y\|_2 = \|x\|_2 \} \) at \( x \).

In 2017, Marković [30] proved this conjecture when \( x \) is near the boundary of the unit ball. Kalaj [22] showed that the conjecture is true for \( n = 4 \). In 2019, Melentijević [31] proved a result, which confirmed the conjecture for \( n = 3 \). See [26, Chapter 6] for solutions of various Khavinson-type extremal problems for harmonic functions on \( \mathbf{B}_2 \) and on a half-space in \( \mathbb{R}^n \). Very recently, Liu [27] showed the following result, which confirmed the generalized Khavinson conjecture for \( n \geq 3 \).

**Theorem A.** ([27, Theorem 2]) For \( n \geq 3 \), if \( u \) is a bounded harmonic function of \( \mathbf{B}_2 \) into \( \mathbb{R} \), then we have the following sharp inequality:

\[
\|\nabla u(x)\|_2 \leq \frac{c_n}{1 - \|x\|_2^2} \left\{ \int_{-1}^{1} \left| t - \frac{n-2}{n} \|x\|_2 \right| (1 - t^2)^\frac{n-2}{2} dt \right\} \|u\|_{\infty}, \ x \in \mathbf{B}_2,
\]

where \( c_n = \frac{2\Gamma(\frac{n+2}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{2})} \) and \( \Gamma(x) \) is the Gamma function.

For mappings with values in higher dimensional spaces, Pavlović [35, 36] showed that the inequality (1.1) does not hold for analytic functions \( f \) of \( \mathbb{D} \) into \( \mathbf{B}_2 \), where \( k \geq 2 \) is an integer. For example, the function \( f(z) = (z, 1)/\sqrt{2} \) for \( z \in \mathbb{D} \) satisfies

\[ \|f'(0)\|_2 = \sqrt{1 - \|f(0)\|_2^2} > 1 - \|f(0)\|_2. \]

However, Pavlović proved the following Schwarz-Pick type lemma for analytic functions \( f \) of \( \mathbb{D} \) into \( \mathbf{B}_2 \):

\[ |\nabla \|f(z)\|_2| \leq \frac{1 - \|f(z)\|_2^2}{1 - |z|^2}, \ z \in \mathbb{D}, \]

where \( \nabla \|f(z)\|_2 \) denotes the gradient of \( \|f\|_2 \).

In general, for \( p \in [1, \infty] \), let \( f \) be a differentiable function of \( \mathbf{B}_2 \) into \( \mathbf{B}_k \), where \( k \) is a positive integer. Denote by

\[ |\nabla \|f(z)\|_p| := \limsup_{w \to z} \frac{\|\|f(z)\|_p - \|f(w)\|_p\|}{\|z - w\|_2} \]

the gradient of \( \|f\|_p \).
By analogy with the generalized Khavinson Conjecture, it is natural to ask the following question for mappings with values in higher dimensional spaces.

**Question 1.2.** For \( p \in (1, \infty) \) and \( n \geq 3 \), let \( u \) be a harmonic function of \( B_{\ell^n}^\nu \) into \( B_{\ell^n}^{\nu p} \), where \( \nu \) is a positive integer. Is there the sharp upper bound \( C^*(x) \) such that
\[
|\nabla \|u(x)\|_p| \leq C^*(x), \quad x \in B_{\ell^n}^\nu
\]
holds?

In [42], Zhu considered Question 1.2 for pluriharmonic functions (see Section 2 for the definition) and established a Schwarz-Pick type estimate for pluriharmonic functions \( f \) of \( B_{\ell^n}^\nu \) into itself as follows.

**Theorem B.** ([42, Theorem 1.1]) For \( n \geq 1 \), let \( f \) be a pluriharmonic function of \( B_{\ell^n}^\nu \) into itself. Then the following inequality
\[
|\nabla \|f(z)\|_2| \leq \frac{4\sqrt{n}}{\pi} \left( \frac{1}{1 - \|z\|_2^2} \right)
\]
holds for all \( z \in B_{\ell^n}^\nu \).

In [42], Zhu also obtained the following Schwarz-Pick type lemma for pluriharmonic functions of \( B_{\ell^n}^\nu \) into itself.

**Theorem C.** ([42, Theorem 1.2]) Let \( f \) be a pluriharmonic function of \( B_{\ell^n}^\nu \) into itself. Then the following inequality
\[
\sum_{j=1}^{n} \left( \left| \frac{\partial f_j(z)}{\partial z_k} \right|^2 + \left| \frac{\partial f_j(z)}{\partial \bar{z}_k} \right|^2 \right)^{1/2} \leq \frac{1 - \|f(z)\|_2^2}{(1 - \|z\|_2^2)^2}, \quad k = 1, \ldots, n
\]
holds for any \( z \in B_{\ell^n}^\nu \).

Let us recall the following classical result which we call the coefficient type Schwarz-Pick lemma of analytic functions (cf. [32]): If \( f \) is an analytic function of \( \mathbb{D} \) into itself with \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then for each \( n \geq 1 \),
\[
|a_n| \leq 1 - |a_0|^2.
\]
(1.6)

By (1.6), we have
\[
|f'(0)| \leq 1 - |f(0)|^2.
\]
(1.7)

For any fixed \( z \in \mathbb{D} \), let \( F(w) = f(\phi(w)) \), where \( \phi(w) = (z + w)/(1 + \bar{z}w) \) for \( w \in \mathbb{D} \). It follows from (1.7) that
\[
|f'(z)| (1 - |z|^2) = |F'(0)| \leq 1 - |F(0)|^2 = 1 - |f(z)|^2,
\]
which implies that the classical Schwarz-Pick lemma (1.1) is a special case of (1.6).

In [9, Lemma 1], the coefficient type Schwarz-Pick lemma of complex-valued harmonic functions was established as follows:
Theorem D. If \( f = h + \overline{g} \) is a complex-valued harmonic function of \( \mathbb{D} \) into itself with \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=1}^{\infty} b_n z^n \), then \( |a_0| \leq 1 \) and for all \( n \geq 1 \),

\[
|a_n| + |b_n| \leq \frac{4}{\pi}.
\]

Applying the Möbius transformation and Theorem D, we can easily obtain (1.2) (see [10, Theorem 4]).

In 1920, Szász [40] extended the inequality (1.1) to the following estimate involving higher order derivatives:

\[
|f^{(2k+1)}(z)| \leq \frac{(2k+1)!}{(1-|z|^2)^{2k+1}} \sum_{j=0}^{k} \binom{k}{j} |z|^{2j},
\]

where \( k \in \{1, 2, \ldots \} \). Later, Ruscheweyh (cf. [1, 2]) improved (1.8) to the following sharp form:

Theorem E. Let \( f \) be an analytic function of \( \mathbb{D} \) into itself. Then

\[
|f^{(k)}(z)| \leq \frac{k!(1-|f(z)|^2)}{(1-|z|)(1+|z|)}|z|, \quad z \in \mathbb{D},
\]

where \( k \in \{1, 2, \ldots \} \).

On the Schwarz-Pick type estimates for derivatives of arbitrary order on bounded analytic functions of \( \mathbb{B}_p \) into \( \mathbb{C} \), see [14, 28].

In 1914, Bohr [8] proved the following remarkable result on power series in one complex variable:

Theorem F. (Bohr) There exists \( \rho \in (0, 1) \) with the property that if a power series \( \sum_{k=0}^{\infty} a_k z^k \) converges in the unit disk and its sum has modulus less than 1, then

\[
\sum_{k=0}^{\infty} |a_k z^k| < 1, \quad \text{for all} \ |z| < \rho.
\]

Bohr’s paper [8], compiled by Hardy from correspondence, indicates that Bohr initially proved the radius \( \rho = 1/6 \), but this was quickly improved to the sharp constant \( \rho = 1/3 \) by Riesz, Schur, and Wiener, independently.

In the following, we write an \( n \)-variable power series \( \sum_{\alpha} a_{\alpha} z^{\alpha} \) using the standard multi-index notation: \( \alpha \) denotes an \( n \)-tuple \((\alpha_1, \ldots, \alpha_n)\) of nonnegative integers, \( |\alpha| \) denotes the sum \( \sum_{k=1}^{n} \alpha_k \) of its components, \( \alpha! \) denotes the product \( \prod_{k=1}^{n} \alpha_k! \) of the factorials of its components, \( z \) denotes an \( n \)-tuple \((z_1, \ldots, z_n)\) of complex numbers, and \( z^{\alpha} \) denotes the product \( \prod_{k=1}^{n} z_k^{\alpha_k} \).

For \( p \in [1, \infty] \), we denote by \( \mathcal{H}(\mathbb{B}_p) \) the set of all holomorphic functions of \( \mathbb{B}_p \) into \( \mathbb{C} \). Set \( \mathcal{H}_1(\mathbb{B}_p) = \{ f \in \mathcal{H}(\mathbb{B}_p) : \sup_{z \in \mathbb{B}_p} |f(z)| \leq 1 \} \). We use \( \mathcal{R}(\mathbb{B}_p) \) to denote the largest non-negative number \( \rho \) with the property that if \( f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha} \in \mathcal{H}_1(\mathbb{B}_p) \), then \( \sum_{\alpha} |a_{\alpha} z^{\alpha}| \leq 1 \) in \( \rho \mathbb{B}_p := \{ z \in \mathbb{C}^n : \|z\| < \rho \} \). We also call \( \mathcal{R}(\mathbb{B}_p) \) the \( n \)-dimensional Bohr radius.
When \( n > 1 \), the exact value of the Bohr radius \( R(\mathbb{B}_{\ell_p^n}) \) is still unknown. In the following result, Boas and Khavinson [6, Theorem 2] showed the upper estimate and Defant et al. in [16, Theorem 2] showed the lower estimate:

**Theorem G.** There exists a constant \( b_n \) such that
\[
R(\mathbb{B}_{\ell_p^n}) = b_n \sqrt{\frac{\log n}{n}}
\]
with \( 1/\sqrt{2} + o(1) \leq b_n \leq 2 \).

Bayart, Pellegrino and Seoane-Sepúlveda ([4]) proved the exact asymptotical behaviour of \( R(\mathbb{B}_{\ell_p^n}) \) as follows:

**Theorem H.**
\[
\lim_{n \to \infty} \frac{R(\mathbb{B}_{\ell_p^n})}{\sqrt{\log n}} = 1.
\]

In the case \( p \in [1, \infty) \), the results of Boas, Defant and Frerick from [5, 15] showed the following result:

**Theorem I.** There is a constant \( C \geq 1 \) such that for each \( p \in [1, \infty) \) and \( n \geq 2 \)
\[
\frac{1}{C} \left( \frac{\log n}{n} \right)^{1 - \frac{1}{\min\{p,2\}}} \leq R(\mathbb{B}_{\ell_p^n}) \leq C \left( \frac{\log n}{n} \right)^{1 - \frac{1}{\min\{p,2\}}}.
\]

The first aim of this paper is to establish some Schwarz-Pick type lemmas and discuss their applications. Based on Theorem A by Liu, we give an answer to Question 1.2 (see Theorem 2.1). In Theorem 2.2, we use a new proof method to improve and generalize Theorem B into the sharp form. In particular, in Theorem 2.3, we also give a sharp estimate for pluriharmonic functions with values in \( \mathbb{B}_{\ell_p^n} \). Moreover, we obtain the Schwarz-Pick type estimate for pluriharmonic functions which generalizes and improves Theorem C into the sharp form on \( \mathbb{B}_{\ell_p^n} \) (see Theorem 2.4) and we will use it to discuss the Lipschitz characteristic of harmonic functions on a domain in \( \mathbb{C} \) (see Proposition 2.5).

The second purpose of this paper is to investigate the coefficient type Schwarz-Pick lemmas and give their applications. In Theorems 2.6 and 2.7, we extend Theorem D to the \( n \)-dimensional case. Then, by using different proof techniques, we will use Theorem 2.6 to extend (1.2) and Theorem E to pluriharmonic functions on the unit ball \( \mathbb{B}_{\ell_p^n} \), and give an estimate for the partial derivatives of arbitrary order (see Theorem 2.8). Furthermore, by using Theorem 2.7, we will establish a sharp Schwarz-Pick type inequality of arbitrary order for some class of pluriharmonic functions on \( \mathbb{B}_{\ell_p^n} \) (see Theorem 2.10). At last, by applying Theorem 2.6, we obtain a Bohr type inequality which is an analogue of Theorems G and I (see Theorem 2.11). We also apply Theorem 2.7 to show that Theorems F, H and I also hold for a more general class of pluriharmonic functions on \( \mathbb{B}_{\ell_p^n} \) (see Theorem 2.15).
The paper is organized as follows. In section 2, we give the statements of our results. In sections 3 and 4, we give the proofs of our main results.

2. Preliminaries and main results

2.1. The Schwarz-Pick type lemmas and their applications. Based on Theorem A by Liu, we give an answer to Question 1.2 as follows.

Theorem 2.1. For \( p \in (1, \infty) \) and \( n \geq 3 \), let \( u \) be a harmonic function of \( B_{\ell_2} \) into \( B_{\ell_p^\nu} \), where \( \nu \) is a positive integer. Then, for \( x \in B_{\ell_2} \), we have the following sharp inequality:

\[
|\nabla\|u(x)\|_p| \leq \frac{c_n}{1 - \|x\|_2^2} \left\{ \int_{-1}^{1} t \left[ \frac{t^{n-2}}{n} \|x\|_2^{n-3} \right] \left[ 1 - \left( 1 - \left( 1 - 2t\|x\|_2 + \|x\|_2^2 \right) \frac{1}{\nu} \right) \right] dt \right\},
\]

where \( c_n \) is the same as in Theorem A.

A twice continuously differentiable complex-valued function \( f \) defined on a domain \( \Omega \subset \mathbb{C}^n \) is called a pluriharmonic function if for each fixed \( z \in \Omega \) and \( \theta \in \partial B_{\ell_2} \), the function \( f(z + \zeta\theta) \) is harmonic in \( \{ \zeta : |\zeta| < d_\Omega(z) \} \) (cf. [19, 39, 41]). Obviously, all pluriharmonic functions are harmonic. If \( \Omega \subset \mathbb{C}^n \) is a simply connected domain containing the origin, then a function \( f : \Omega \to \mathbb{C} \) is pluriharmonic if and only if \( f \) has a representation \( f = h + \overline{g} \), where \( h \) and \( g \) are holomorphic in \( \Omega \) with \( g(0) = 0 \) (see [41]). Furthermore, a twice continuously differentiable real-valued function in a simply connected domain \( \Omega \) is pluriharmonic if and only if it is the real part of some holomorphic function on \( \Omega \). In particular, if \( n = 1 \), then pluriharmonic functions are planar harmonic functions (cf. [18]).

In the following, we improve and generalize Theorem B into the sharp form.

Theorem 2.2. For \( n \geq 1 \) and \( p \in (1, \infty) \), let \( f \) be a pluriharmonic function of \( B_{\ell_2} \) into \( B_{p^\nu} \), where \( \nu \) is a positive integer. Then the following inequality holds:

\[
|\nabla\|f(z)\|_p| \leq \frac{4}{\pi} \frac{1}{1 - \|z\|_2^2}, \quad z \in B_{\ell_2}.
\]

Furthermore, (2.2) is sharp for each \( z \in B_{\ell_2} \).

In particular, if \( u \) is a pluriharmonic function of \( B_{\ell_2} \) into \( B_{p^\nu} \), then we have a better estimate as follows, where \( \nu \) is a positive integer.

Theorem 2.3. For \( n \geq 1 \) and \( p \in (1, \infty) \), let \( u \) be a pluriharmonic function of \( B_{\ell_2} \) into \( B_{p^\nu} \), where \( \nu \) is a positive integer. Then for \( z \in B_{\ell_2} \),

\[
|\nabla\|u(z)\|_p| \leq \frac{4}{\pi} \frac{1 - \|u(z)\|_p^2}{1 - \|z\|_2^2}.
\]

The inequality (2.3) is sharp for each \( z \).

We give the following Schwarz-Pick type estimate for pluriharmonic functions which generalizes and improves Theorem C into the sharp form on \( B_{\ell_\infty} \) and we will use it to discuss the Lipschitz characteristic of harmonic functions on a domain in \( \mathbb{C} \).
**Theorem 2.4.** For $n \geq 1$, let $f = (f_1, \ldots, f_\nu) : \mathbb{B}_{\ell^n} \to \mathbb{B}_{\ell_p}$ be a pluriharmonic function, where $\nu$ is a positive integer. Then, for $z \in \mathbb{B}_{\ell^n}$, we have

$$
(2.4) \quad \sum_{j=1}^\nu \sum_{k=1}^n \left( \frac{|\partial f_j(z)|^2}{\partial z_k} + \frac{|\partial f_j(z)|^2}{\partial \overline{z}_k} \right) (1 - |z_k|^2)^2 \leq 1 - \|f(z)\|^2.
$$

and

$$
(2.5) \quad \sum_{j=1}^\nu \sum_{k=1}^n \left( \frac{|\partial f_j(z)|^2}{\partial z_k} + \frac{|\partial f_j(z)|^2}{\partial \overline{z}_k} \right) \leq \frac{1 - \|f(z)\|^2}{(1 - |z|^2)^2}.
$$

Moreover, the inequality (2.4) is sharp for each $z \in \mathbb{B}_{\ell^n}$.

For a given constant $\alpha \in (0, \infty)$ and a given subset $\Omega$ of $\mathbb{C}$, a function $f : \Omega \to \mathbb{C}^n$ is said to belong to the Lipschitz space $\Lambda_\alpha(\Omega)$ if there is a constant $C \geq 1$ such that for all $z, w \in \Omega$,

$$
\|f(z) - f(w)\| \leq C|z - w|^\alpha.
$$

For the related investigation of the Lipschitz space of analytic functions, we refer to [20, 21, 34].

A domain $\Omega \subseteq \mathbb{C}$ is said to be linearly connected if there exists a constant $M > 0$ such that any two points $v_1, v_2 \in \Omega$ can be connected by a smooth curve $\gamma \subset \Omega$ with length $\ell(\gamma) \leq M|v_1 - v_2|$ (see [12]). By applying Theorem 2.4, we get the following result.

**Proposition 2.5.** Suppose that $\Omega$ is a linearly connected proper subdomain of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}^n$ be a harmonic function, where $n \geq 2$. If $\|f\|^2 \in \Lambda_2(\Omega)$, then $f \in \Lambda_1(\Omega)$.

### 2.2. The coefficient type Schwarz-Pick lemmas and their applications.

In the following, we extend Theorem D to the $n$-dimensional case.

**Theorem 2.6.** For $p \in [1, \infty]$, let $f(z) = \sum_\alpha a_\alpha z^\alpha + \sum_\alpha b_\alpha \overline{z}^\alpha$ be a pluriharmonic function of $\mathbb{B}_{\ell^p}$ into $\mathbb{D}$. Then for all $|\alpha| \geq 1$,

$$
(2.6) \quad |a_\alpha| + |b_\alpha| \leq \frac{4}{\pi} \left( \frac{|\alpha|^{||\alpha||}}{\alpha^{\alpha}} \right)^{\frac{1}{p}}.
$$

Moreover, the constant $4/\pi$ in (2.6) is sharp.

For $p \in [1, \infty]$, let $\mathcal{P}(\mathbb{B}_{\ell^p})$ denote the set of all pluriharmonic functions from $\mathbb{B}_{\ell^p}$ into $\mathbb{C}$. By analogy with Theorem D, we establish a coefficient type Schwarz-Pick lemma for a general case as follows.

**Theorem 2.7.** If $f(z) = \sum_\alpha a_\alpha z^\alpha + \sum_\alpha b_\alpha \overline{z}^\alpha \in \mathcal{P}(\mathbb{B}_{\ell^p})$ with $\sup_{z \in \mathbb{B}_{\ell^p}} \text{Re}(f(z)) \leq 1$, then for all $k \in \{1, 2, \ldots\}$,

$$
(2.7) \quad \sup_{z \in \mathbb{B}_{\ell^p}} \left| \sum_{|\alpha| = k} (a_\alpha + b_\alpha) z^\alpha \right| \leq 2 \left( 1 - \text{Re}(f(0)) \right),
$$

where $\text{Re}(z)$ denotes the real part of $z$. 
and for all $|\alpha| \geq 1$,

$$
(2.8) \quad |a_\alpha + b_\alpha| \leq 2 \left( \frac{|\alpha|}{\alpha^\alpha} \right)^{1/p} \left( 1 - \text{Re}(f(0)) \right).
$$

Moreover, the constant 2 in (2.7) and (2.8) are sharp.

In the following, we will use Theorem 2.6 to extend (1.2) and Theorem E to pluriharmonic functions on the unit ball $B_{\ell^2}$, and give an estimate for the partial derivatives of arbitrary order. One should note that the higher dimensional case is very different from the one dimensional situation and, because we are dealing with partial derivatives of arbitrary order, the method of proof for (1.2) from [13] cannot be used. The result is as follows.

**Theorem 2.8.** Suppose that $f$ is a pluriharmonic function of $B_{\ell^2}$ into $D$, where $n \geq 1$.

(i) If $n = 1$, then for $z \in B_{\ell^2} = D$,

$$
\left| \frac{\partial^{m} f(z)}{\partial z^{m}} \right| + \left| \frac{\partial^{m} f(z)}{\partial \overline{z}^{m}} \right| \leq \frac{4}{\pi} m! \left( 1 + |z| \right)^{m-1} \left( 1 - |z|^2 \right)^m,
$$

where $m \geq 1$. The constant $4/\pi$ in this inequality can not be improved.

(ii) If $n \geq 2$, then for $z \in B_{\ell^2}$,

$$
\left| \frac{\partial^{m|j|} f(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} \right| + \left| \frac{\partial^{m|j|} f(z)}{\partial \overline{z}_1^{m_1} \cdots \partial \overline{z}_n^{m_n}} \right| \leq \frac{4}{\pi} m! \left( n + |m| - 1 \right) \left( n - 1 \right) |m|! \prod_{j=1}^{n} (1 + |z_j|)^{m_j} \left( 1 - \|z\|^2 \right)^{|m|},
$$

where $m = (m_1, \ldots, m_n) \neq 0$ is a multi-index.

**Remark 2.9.** We remark that if $n = m = 1$, then Theorem 2.8 coincides with (1.2) (or [13, Theorem 3]). Moreover, the growth rate of the formula $\binom{n+|m|-1}{n-1}$ in Theorem 2.8 can be estimated. It follows from Stirling’s formula (cf. [33]) that there is an absolute constant $c \in [1, e]$ such that for any $n$ and $m$,

$$
(2.9) \quad \binom{n + |m| - 1}{n - 1} \leq c^{|m|} \left( 1 + \frac{n}{|m|} \right)^{|m|}.
$$

In particular, for the extension of (1.2) on the polydisc $B_{\ell^\infty}$, $n \geq 2$, see [11]. It is well known that there are no biholomorphic mappings between $B_{\ell^\infty}$ and $B_{\ell^2}$ in the case $n \geq 2$. Hence, the research methods to deal with these two situations are completely different (see [38, 39]).

By using Theorem 2.7, we will establish a sharp Schwarz-Pick type inequality of arbitrary order for $f \in \mathcal{P} \mathcal{H}(B_{\ell^\infty})$ with $\sup_{z \in B_{\ell^\infty}} \text{Re}(f(z)) \leq 1$.

**Theorem 2.10.** Let $f \in \mathcal{P} \mathcal{H}(B_{\ell^\infty})$ with $\sup_{z \in B_{\ell^\infty}} \text{Re}(f(z)) \leq 1$. Then

$$
(2.10) \quad \left| \frac{\partial^{m|j|} f(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} \right| + \left| \frac{\partial^{m|j|} f(z)}{\partial \overline{z}_1^{m_1} \cdots \partial \overline{z}_n^{m_n}} \right| \leq 2(m!) (1 - \text{Re}(f(z))) \left( 1 + \|z\|_{\infty} \right)^{|m| - N} \left( 1 - \|z\|^2 \right)^{|m|},
$$

where $m = (m_1, \ldots, m_n)$ is a multi-index.
where \( m = (m_1, \ldots, m_n) \neq 0 \) is a multi-index and \( N \) is the number of the indices \( j \) such that \( m_j \neq 0 \). Furthermore, the constant 2 in (2.10) cannot be improved.

We will use Theorems 2.6 and 2.7 to investigate the Bohr phenomenon of complex-valued pluriharmonic functions.

Let \( \mathcal{R}_P(\mathbb{B}_p^n) \) denote the \( n \)-dimensional Bohr radius: the largest number \( \rho \) such that if \( f(z) = \sum_\alpha a_\alpha z^\alpha + \sum_\alpha b_\alpha \overline{z}^\alpha \in \mathcal{P} \mathcal{H}(\mathbb{B}_p^n) \) with \( b_0 = 0 \) and \( \sup_{z \in \mathbb{B}_p^n} |f(z)| \leq 1 \), then

\[
\sum_{k=1}^{\infty} \sum_{|\alpha|=k} (|a_\alpha| + |b_\alpha|)|z^\alpha| \leq 1
\]

(2.11)

when \( z \in \rho \mathbb{B}_p^n \).

By applying Theorem 2.6, we obtain a Bohr type inequality which is an analogue of Theorems G and I as follows. Note that the proof of Theorem G depends on Wiener’s result (see [16]). However, the proof of Theorem 2.11 does not need Wiener’s result.

**Theorem 2.11.** Let \( p \in [1, \infty] \) and \( n \geq 2 \). The \( n \)-dimensional Bohr radius \( \mathcal{R}_P(\mathbb{B}_p^n) \) satisfies

\[
C_1 \left( \frac{1}{n} \right)^{1 - \frac{1}{\min\{p, 2\}}} \leq \mathcal{R}_P(\mathbb{B}_p^n) \leq C_2 \left( \frac{\log n}{n} \right)^{1 - \frac{1}{\min\{p, 2\}}},
\]

(2.12)

where \( C_j > 0 \) \( (j = 1, 2) \) are absolute constants.

**Remark 2.12.** Is there the Bohr’s phenomenon if one replace (2.11) by \( \sum_\alpha (|a_\alpha| + |b_\alpha|)|z^\alpha| \leq 1 \)? The following example shows that the answer is no.

**Example 2.13.** For \( z \in \mathbb{B}_p^n \), let \( \mathcal{F}_k(z) = \Re(z_1) \sin \frac{1}{k} + i \cos \frac{1}{k} \), where \( k \in \{1, 2, \ldots \} \). Then \( \mathcal{F}_k \in \mathcal{P} \mathcal{H}(\mathbb{B}_p^n) \) and \( \sup_{z \in \mathbb{B}_p^n} |\mathcal{F}_k(z)| \leq 1 \) for all \( k \in \{1, 2, \ldots \} \). Suppose that there is \( \rho_0 > 0 \) such that

\[
\left| \sin \frac{1}{k} \right| |z_1| + \left| \cos \frac{1}{k} \right| \leq 1
\]

for \( \|z\|_p < \rho_0 \) and all \( k \in \{1, 2, \ldots \} \). This contradicts \( \lim_{k \to \infty} \cos \frac{1}{k} = 1 \).

For \( p \in [1, \infty] \), let

\[
\mathcal{P} \mathcal{H}_+(\mathbb{B}_p^n) = \{ f \in \mathcal{P} \mathcal{H}(\mathbb{B}_p^n) : \sup_{z \in \mathbb{B}_p^n} \Re(f(z)) \leq 1 \text{ and } f(0) \geq 0 \}.
\]

We use \( \mathcal{R}_P(\mathbb{B}_p^n) \) to denote the \( n \)-dimensional Bohr radius: the largest number \( \rho \) such that if \( f(z) = \sum_\alpha a_\alpha z^\alpha + \sum_\alpha b_\alpha \overline{z}^\alpha \in \mathcal{P} \mathcal{H}_+(\mathbb{B}_p^n) \) with \( b_0 = 0 \), then \( \sum_\alpha (|a_\alpha| + |b_\alpha|)|z^\alpha| \leq 1 \) when \( z \in \rho \mathbb{B}_p^n \). In fact, the assumption \( f(0) \geq 0 \) in \( \mathcal{P} \mathcal{H}_+(\mathbb{B}_p^n) \) is necessary. Without this condition, there is no Bohr’s phenomenon on the set

\[
\mathcal{P} \mathcal{H}_1(\mathbb{B}_p^n) = \{ f \in \mathcal{P} \mathcal{H}(\mathbb{B}_p^n) : \sup_{z \in \mathbb{B}_p^n} \Re(f(z)) \leq 1 \}.
\]

Here is an example.
Example 2.14. For $z = (z_1, \ldots, z_n) \in \mathbb{B}_p^k$, let $f(z) = \text{Re} \left( \frac{z}{1-z} \right) = -\sum_{j=1}^{\infty} \frac{z_j}{\bar{z}_1}$, and let $F_k(z) = (\sin \frac{1}{k}) f(z) + i \cos \frac{1}{k}$, where $k \in \{1, 2, \ldots\}$. Then $F_k \in \mathcal{PH}_1(\mathbb{B}_p^k)$. Suppose that there is $\rho_0 > 0$ such that

$$\sum_{j=1}^{\infty} 2 \left| \sin \frac{1}{k} \right| |z_j|^j + \left| \cos \frac{1}{k} \right| \leq 1$$

for $\|z\|_p < \rho_0$ and all $k \in \{1, 2, \ldots\}$. This contradicts $\lim_{k \to \infty} \cos \frac{1}{k} = 1$. Therefore, there is no Bohr's phenomenon on $\mathcal{PH}_1(\mathbb{B}_p^k)$.

In the following, by using Theorem 2.7, we shall show that we can go further: Theorems F, H and I also hold for a more general class $\mathcal{PH}_+(\mathbb{B}_p^k)$.

**Theorem 2.15.** Let $p \in [1, \infty]$. Then the $n$-dimensional Bohr radius $\mathcal{R}_p(\mathbb{B}_p^k)$ satisfies

$$\begin{cases} 
\mathcal{R}_p(\mathbb{B}_p^k) = \frac{1}{3}, & n = 1 \text{ and } p \in [1, \infty], \\
\frac{1}{C} \left( \frac{\log n}{n} \right)^{1-\min(p,2)} \leq \mathcal{R}_p(\mathbb{B}_p^k) \leq C \left( \frac{\log n}{n} \right)^{1-\min(p,2)}, & n \geq 2 \text{ and } p \in [1, \infty), \\
\lim_{n \to \infty} \frac{\mathcal{R}_p(\mathbb{B}_p^n)}{\frac{\log n}{n}} = 1, & n \geq 2 \text{ and } p = \infty,
\end{cases}$$

where $C \geq 1$ is an absolute constant. In particular, if $n = 1$, then the constant $1/3$ is sharp.

**Remark 2.16.** It is obvious that if $f \in \mathcal{H}_1(\mathbb{B}_p^\infty)$ with $f(0) = 0$, then $f \in \mathcal{PH}_+(\mathbb{B}_p^\infty)$. Furthermore, if $f(z) = \sum_{\alpha} a_\alpha z^\alpha \in \mathcal{H}_1(\mathbb{B}_p^\infty)$ with $f(0) \neq 0$, then $e^{-i \arg f(0)} f \in \mathcal{PH}_+(\mathbb{B}_p^\infty)$. Note that $\sum_{\alpha} |e^{-i \arg f(0)} a_\alpha z^\alpha| = \sum_{\alpha} |a_\alpha z^\alpha|$. Hence Theorem 2.15 is an improvement of Theorems H and I. In particular, the proof of Theorem H depends on Wiener's result (see [4]). However, the proof of Theorem 2.15 does not need Wiener's result.

## 3. The Schwarz-Pick type lemmas and their applications

For a differentiable mapping $f : \mathbb{B}_p^k \to \mathbb{R}^k$, let $Df(z)$ denote the Fréchet derivative of $f$ at $z$, where $k$ is a positive integer.

**Lemma 3.1.** For $p \in (1, \infty)$, let $f$ be a $C^1$ class function of $\mathbb{B}_p^k$ into $\mathbb{B}_p^k$, where $k$ is a positive integer. Then,

$$|\nabla\|f(z)\|_p| = \sup_{|\theta| = 1} \limsup_{\rho \to 0^+} \left| \frac{\|f(z + \rho \theta)\|_p - \|f(z)\|_p}{\rho} \right|, \quad z \in \mathbb{B}_p^k$$

holds.
**Proof.** We divide the proof into two cases.

**Case 3.1.** Let \( z \in \mathcal{P} := \{ \zeta \in B_{\ell_2^n} : f(\zeta) = 0 \} \) be fixed.

Elementary calculations yield

\[
\sup_{\|\theta\|_2 = 1} \limsup_{\rho \to 0^+} \frac{\|f(z + \rho \theta)\|_p - \|f(z)\|_p}{\rho} = \sup_{\|\theta\|_2 = 1} \limsup_{\rho \to 0^+} \frac{\|f(z + \rho \theta) - f(z)\|_p}{\rho}
\]

and

\[
\nabla \|f(z)\|_p = \limsup_{w \to z} \frac{\|f(w) - f(z)\|_p}{\|w - z\|_2} = \limsup_{w \to z} \left| Df(z) \frac{w - z}{\|w - z\|_2} \right|_p = \sup_{\|\theta\|_2 = 1} \|Df(z)\theta\|_p,
\]

which imply (3.1).

**Case 3.2.** Let \( z \in B_{\ell_2^n} \setminus \mathcal{P} \) be fixed.

In this case, let \( \Psi(w) = \|f(w)\|_p \) for \( w \in B_{\ell_2^n} \). It is not difficult to know that \( \Psi \) is \( C^1 \) on a neighbourhood of \( z \). Then (3.1) follows from the following two formulas:

\[
\nabla \|f(z)\|_p = \limsup_{w \to z} \frac{\|\Psi(w) - \Psi(z)\|_2}{\|w - z\|_2} = \limsup_{w \to z} \left| D\Psi(z) \frac{w - z}{\|w - z\|_2} \right|_p = \sup_{\|\theta\|_2 = 1} \|D\Psi(z)\theta\|
\]

and

\[
\sup_{\|\theta\|_2 = 1} \limsup_{\rho \to 0^+} \left( \frac{\|f(z + \rho \theta)\|_p - \|f(z)\|_p}{\rho} \right) = \sup_{\|\theta\|_2 = 1} \limsup_{\rho \to 0^+} \frac{\|\Psi(z + \rho \theta) - \Psi(z)\|_2}{\rho} = \sup_{\|\theta\|_2 = 1} \|D\Psi(z)\theta\|.
\]

This completes the proof. \( \square \)

**The proof of Theorem 2.1.** For \( p \in (1, \infty) \) and \( n \geq 3 \), let \( u = (u_1, \ldots, u_\nu) \) be a harmonic function of \( B_{\ell_2^n} \) into \( B_{\ell_p} \). We divide the proof of (2.1) into two steps.

**Step 3.1.** We first estimate \( \|\nabla \|u(x)\|_p\| \) for \( x \in \Omega := \{ y \in B_{\ell_2^n} : u(y) \neq 0 \} \).

By Lemma 3.1, we have

\[
(3.2) \quad \|\nabla \|u(x)\|_p\| = \max_{\theta \in \partial B_{\ell_2^n}} \left\{ \sum_{j=1}^n \left| \frac{\partial \|u(x)\|_p}{\partial x_j} \theta_j \right| \right\} = \left( \sum_{j=1}^n \left| \frac{\partial \|u(x)\|_p}{\partial x_j} \right|^2 \right)^{\frac{1}{2}}.
\]

Elementary calculations yield

\[
(3.3) \quad \frac{\partial \|u(x)\|_p}{\partial x_j} = \sum_{k=1}^{\nu} \frac{|u_k(x)|^{p-2} u_k(x) \partial u_k(x)}{\|u(x)\|_p^{p-1}} \frac{\partial x_j}{\partial x_j} = \left\langle \frac{\partial u(x)}{\partial x_j}, \eta(x) \right\rangle.
\]
for \( j \in \{1, \ldots, n\} \), where

\[
\frac{\partial u(x)}{\partial x_j} = \left( \frac{\partial u_1(x)}{\partial x_j}, \ldots, \frac{\partial u_\nu(x)}{\partial x_j} \right)
\]

and

\[
\eta(x) = \left( \frac{|u_1(x)|^{p-2}u_1(x)}{\|u(x)\|_{p-1}}, \ldots, \frac{|u_\nu(x)|^{p-2}u_\nu(x)}{\|u(x)\|_{p-1}} \right).
\]

It is not difficult to know that \( \eta \in \partial B_{\ell_{\nu}} \), where \( \frac{1}{q} + \frac{1}{p} = 1 \). By (3.2) and (3.3), we obtain

(3.4) \[ |\nabla \|u(x)\|_p| = \|(Du(x))^T(\eta(x))^T\|_2, \]

where "\( T \)" denotes the transpose of a matrix and

\[
Du(x) = \begin{pmatrix}
\frac{\partial u_1(x)}{\partial x_1} & \cdots & \frac{\partial u_1(x)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial u_\nu(x)}{\partial x_1} & \cdots & \frac{\partial u_\nu(x)}{\partial x_n}
\end{pmatrix}
\]

is the Fréchet derivative of \( u \) at \( x \).

For any fixed \( \alpha = (\alpha_1, \ldots, \alpha_\nu) \in \partial B_{\ell_{\nu}} \), let

(3.5) \[ u_\alpha(y) = \langle u(y), \alpha \rangle, \quad y \in B_{\ell_2}. \]

Then the Hölder inequality implies

\[
|u_\alpha| = |\langle u, \alpha \rangle| \leq \left( \sum_{j=1}^{\nu} |u_j|^p \right)^{1/p} \left( \sum_{j=1}^{\nu} |\alpha_j|^q \right)^{1/q} < 1.
\]

An application of Theorem A to the bounded harmonic function \( u_\alpha \) gives that, for all \( x \in B_{\ell_2} \),

\[
\|\nabla u_\alpha(x)\|_2 = \|((Du(x))^T \alpha^T)\|_2 \leq \frac{c_n}{1 - \|x\|_2^2} \left\{ \int_{-1}^{1} \frac{t^{n-2}}{n} \|x\|_2 \left( 1 - t^2 \right)^{\frac{n-4}{2}} dt \right\},
\]

which, together with (3.4), implies that (2.1) holds.

**Step 3.2.** Next, we estimate \( |\nabla \|u(x)\|_p| \) for \( x \in B_{\ell_2} \setminus \Omega \).
By Lemma 3.1, we have

\begin{equation}
\left| \nabla \|u(x)\|_p \right| = \limsup_{y \to x} \frac{\|u(x)\|_p - \|u(y)\|_p}{\|x - y\|_2}
= \max_{\varphi \in \partial \mathbf{B}_{\ell_2}^q} \limsup_{\rho \to 0^+} \frac{\|u(x + \rho \varphi) - u(x)\|_p}{\rho}
= \max_{\varphi \in \partial \mathbf{B}_{\ell_2}^q} \left( \sum_{k=1}^{\nu} \left| \langle \nabla u_k(x), \varphi \rangle \right|^p \right)^{\frac{2}{p}}.
\end{equation}

For any fixed \( \alpha = (\alpha_1, \ldots, \alpha_\nu) \in \partial \mathbf{B}_{\ell_2}^q \), where \( \frac{1}{q} + \frac{1}{p} = 1 \), let \( u_{\alpha} \) be the harmonic function defined in (3.5). It follows from Theorem A that

\begin{equation}
\left( 3.7 \right) \max_{\varphi \in \partial \mathbf{B}_{\ell_2}^q} \left| \sum_{j=1}^{\nu} \langle \nabla u_j(x), \varphi \rangle \alpha_j \right| = \max_{\varphi \in \partial \mathbf{B}_{\ell_2}^q} \lim_{\rho \to 0^+} \frac{|u_{\alpha}(x + \rho \varphi) - u_{\alpha}(x)|}{\rho}
\leq \left\{ \frac{c_n}{1 - \|x\|_2^2} \left[ \int_{-1}^{1} \left( 1 - \frac{n-2}{n} \|x\|_2^2 \right) \left( 1 - t^2 \right)^{\frac{n-3}{2}} dt \right] \right\} \left( \sum_{j=1}^{\nu} |\psi_j(x)|^\nu \right)^{\frac{2}{\nu}}.
\end{equation}

For \( j \in \{1, \ldots, \nu\} \), let \( \psi_j(x) = \langle \nabla u_j(x), \varphi \rangle \). Without loss of generality, we may assume that \( \sum_{j=1}^{\nu} |\psi_j(x)|^\nu \neq 0 \). Then let

\begin{equation}
\alpha_j := \alpha_j(\varphi) = \frac{|\psi_j(x)|^{\nu-2} \psi_j(x)}{\left( \sum_{j=1}^{\nu} |\psi_j(x)|^\nu \right)^{\frac{\nu-1}{\nu}}}.
\end{equation}

It is not difficult to know that \( \alpha := \alpha(\varphi) \in \partial \mathbf{B}_{\ell_2}^q \). Hence (2.1) follows from (3.6), (3.7) and (3.8).

Next, we prove the sharpness part of (2.1). Let \( u = (u_1, 0, \ldots, 0) \) be a harmonic function of \( \mathbf{B}_{\ell_2}^q \) into \( \mathbf{B}_{\ell_2}^q \), where \( u_1 \) is an extremal function of Theorem A. Then \( \|u\|_p = |u_1| \). We split the remaining proof into two cases

**Case 3.3.** Let \( x \in \mathcal{P}^* := \{ y \in \mathbf{B}_{\ell_2}^q : u_1(y) = 0 \} \).

In this case, by (3.6), we have

\[ \left| \nabla \|u(x)\|_p \right| = \limsup_{y \to x} \frac{\|u(x)\|_p - \|u(y)\|_p}{\|x - y\|_2} = \limsup_{y \to x} \frac{|u_1(x) - u_1(y)|}{\|x - y\|_2} = \|\nabla u_1(x)\|_2. \]

**Case 3.4.** Let \( x \in \mathbf{B}_{\ell_2}^q \setminus \mathcal{P}^* \).

By (3.2), we see that

\[ \left| \nabla \|u(x)\|_p \right| = \max_{\varphi \in \partial \mathbf{B}_{\ell_2}^q} \left| \sum_{j=1}^{n} \frac{\partial u_1(x)}{\partial x_j} \varphi_j \right| = \max_{\varphi \in \partial \mathbf{B}_{\ell_2}^q} \left| \sum_{j=1}^{n} \frac{\partial u_1(x)}{\partial x_j} \frac{u_1(x)}{u_1(x)} \varphi_j \right| = \|\nabla u_1(x)\|_2. \]
The proof of this theorem is completed.

The proof of Theorem 2.2. Let \( f = (f_1, \ldots, f_\nu) \) be a pluriharmonic function of \( \mathbb{B}_q^2 \) into \( \mathbb{B}_q^\nu \). We divide the proof of (2.2) into two cases.

Case 3.5. \( n = 1 \).

We split the proof of this case into two steps.

Step 3.3. We first estimate \( |\nabla f(z)|_p \) for \( z = x + iy \in \Omega := \{ \mathcal{Z} \in \mathbb{D} : \|f(\mathcal{Z})\|_p \neq 0 \} \).

Since
\[
\max_{\alpha \in [0,2\pi]} \left| \frac{\partial f(z)}{\partial x} \cos \alpha + \frac{\partial f(z)}{\partial y} \sin \alpha \right| = \frac{1}{2} \left( \left| \frac{\partial f(z)}{\partial x} \right| + \left| \frac{\partial f(z)}{\partial y} \right| \right),
\]
by Lemma 3.1, we see that
\[
|\nabla f(z)|_p = \max_{\alpha \in [0,2\pi]} \limsup_{\rho \to 0^+} \frac{\|f(z + \rho e^{i\alpha}) - f(z)\|_p}{\rho} = \max_{\alpha \in [0,2\pi]} \left| \frac{\partial f(z)}{\partial x} \cos \alpha + \frac{\partial f(z)}{\partial y} \sin \alpha \right| = \left| \frac{\partial f(z)}{\partial z} \right| + \left| \frac{\partial f(z)}{\partial \overline{z}} \right|.
\]

Next, we estimate \( \frac{\partial f(z)}{\partial z} \) and \( \frac{\partial f(z)}{\partial \overline{z}} \). Elementary computations give that
\[
\frac{\partial f(z)}{\partial z} = \frac{1}{2} \|f(z)\|_p^{1-p} \sum_{k=1}^\nu |f_k(z)|^{p-2} \left( \frac{\partial f_k(z)}{\partial z} f_k(z) + \frac{\partial f_k(z)}{\partial \overline{z}} \right) f_k(z) \quad \text{and} \quad \frac{\partial f(z)}{\partial \overline{z}} = \frac{1}{2} \|f(z)\|_p^{1-p} \sum_{k=1}^\nu |f_k(z)|^{p-2} \left( \frac{\partial f_k(z)}{\partial z} f_k(z) + \frac{\partial f_k(z)}{\partial \overline{z}} \right) f_k(z).
\]

Let
\[
\xi(z) = \left( \frac{|f_1(z)|^{p-2} f_1(z)}{\|f(z)\|_p^{p-1}}, \ldots, \frac{|f_\nu(z)|^{p-2} f_\nu(z)}{\|f(z)\|_p^{p-1}} \right).
\]
Then \( \xi(z) \in \partial \mathbb{B}_q^\nu \), where \( \frac{1}{q} + \frac{1}{p} = 1 \). It follows from (3.10), (3.11) and (3.12) that
\[
\frac{\partial f(z)}{\partial z} = \frac{1}{2} \left( \langle \partial f(z), \xi(z) \rangle + \langle \overline{\partial f(z)}, \xi(z) \rangle \right) \quad \text{and} \quad \frac{\partial f(z)}{\partial \overline{z}} = \frac{1}{2} \left( \langle \partial f(z), \xi(z) \rangle + \langle \overline{\partial f(z)}, \xi(z) \rangle \right),
\]
where $\partial f(z) = \left( \frac{\partial f_1(z)}{\partial z}, \ldots, \frac{\partial f_n(z)}{\partial z} \right)$, $\overline{\partial f}(z) = \left( \frac{\partial f_1(z)}{\partial \bar{z}}, \ldots, \frac{\partial f_n(z)}{\partial \bar{z}} \right)$ and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{C}^n$. Hence we obtain the following inequality

\[(3.13) \quad \left| \frac{\partial \| f(z) \|_p}{\partial z} \right| + \left| \frac{\partial \| f(z) \|_p}{\partial \bar{z}} \right| \leq |\langle \partial f(z), \xi(z) \rangle| + |\langle \overline{\partial f}(z), \xi(z) \rangle|.
\]

Now we estimate $|\langle \partial f(z), \xi(z) \rangle| + |\langle \overline{\partial f}(z), \xi(z) \rangle|$. For any fixed $\theta = (\theta_1, \ldots, \theta_\nu) \in \partial \mathbb{B}_{\nu}$, let $F_\theta(w) = \langle f(w), \theta \rangle$, $w \in \mathbb{D}$. Then by Hölder’s inequality, we have

\[|F_\theta(w)| \leq |\langle f(w), \theta \rangle| \leq \left( \sum_{k=1}^\nu |f_k(w)|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^\nu |\theta_k|^q \right)^{\frac{1}{q}} < 1.
\]

This implies that $F_\theta$ is a harmonic function of $\mathbb{D}$ into itself. For any fixed $\theta \in \partial \mathbb{B}_{\nu}$, an application of $(1.2)$ to $F_\theta$ gives

\[(3.14) \quad \left| \frac{\partial F_\theta(z)}{\partial z} \right| + \left| \frac{\partial F_\theta(z)}{\partial \bar{z}} \right| = |\langle \partial f(z), \theta \rangle| + |\langle \overline{\partial f}(z), \theta \rangle| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}.
\]

Combining (3.9), (3.13) and (3.14), we conclude that

\[|\nabla \| f(z) \|_p| = \left| \frac{\partial \| f(z) \|_p}{\partial z} \right| + \left| \frac{\partial \| f(z) \|_p}{\partial \bar{z}} \right| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}.
\]

**Step 3.4.** Next, we estimate $|\nabla \| f(z) \|_p|$ for $z = x + iy \in \mathbb{D}\setminus \Omega$.

By Lemma 3.1, we see that

\[(3.15) \quad |\nabla \| f(z) \|_p| = \max_{\alpha \in [0,2\pi]} \left\{ \lim_{\rho \to 0^+} \frac{\| f(z + \rho e^{i\alpha}) \|_p - \| f(z) \|_p}{\rho} \right\}
\]

\[= \max_{\alpha \in [0,2\pi]} \left\{ \lim_{\rho \to 0^+} \frac{\| f(z + \rho e^{i\alpha}) - f(z) \|_p}{\rho} \right\}
\]

\[= \max_{\alpha \in [0,2\pi]} \left( \sum_{k=1}^\nu \left| \frac{\partial f_k(z)}{\partial x} \cos \alpha + \frac{\partial f_k(z)}{\partial y} \sin \alpha \right| \right)^{\frac{1}{p}}.
\]

For any fixed $\theta \in \partial \mathbb{B}_{\nu}$, let $F_\theta(w) = \langle f(w), \theta \rangle$, $w \in \mathbb{D}$. Then $F_\theta$ is a harmonic function of $\mathbb{D}$ into itself. For all $\alpha \in [0,2\pi]$, we have

\[\lim_{\rho \to 0^+} \frac{|F_\theta(z + \rho e^{i\alpha}) - F_\theta(z)|}{\rho} = \left| \frac{\partial F_\theta(z)}{\partial x} \cos \alpha + \frac{\partial F_\theta(z)}{\partial y} \sin \alpha \right|
\]

\[= \left| e^{i\alpha} \frac{\partial F_\theta(z)}{\partial z} + e^{-i\alpha} \frac{\partial F_\theta(z)}{\partial \bar{z}} \right|
\]

\[\leq \left| \frac{\partial F_\theta(z)}{\partial z} \right| + \left| \frac{\partial F_\theta(z)}{\partial \bar{z}} \right|.
\]
Some sharp Schwarz-Pick type estimates and their applications

\[ \left| \sum_{k=1}^{\nu} \left( \frac{\partial f_k(z)}{\partial x} \cos \alpha + \frac{\partial f_k(z)}{\partial y} \sin \alpha \right) \bar{\theta}_k \right| = \lim_{\rho \to 0^+} \left| \sum_{k=1}^{\nu} \frac{(f_k(z + \rho e^{i\alpha}) - f_k(z))}{\rho} \bar{\theta}_k \right| = \lim_{\rho \to 0^+} \frac{|F_\theta(z + \rho e^{i\alpha}) - F_\theta(z)|}{\rho}, \]

which, together with (1.2), imply that

\[ (3.16) \left| \sum_{k=1}^{\nu} \left( \frac{\partial f_k(z)}{\partial x} \cos \alpha + \frac{\partial f_k(z)}{\partial y} \sin \alpha \right) \bar{\theta}_k \right| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}. \]

For \( k \in \{1, 2, \ldots, \nu\} \), let \( \mu_k(z) = \frac{\partial f_k(z)}{\partial x} \cos \alpha + \frac{\partial f_k(z)}{\partial y} \sin \alpha \). Without loss of generality, we assume \( \sum_{k=1}^{\nu} |\mu_k(z)|^p \neq 0 \). Then we let

\[ \theta_k := \theta_k(\alpha) = \frac{|\mu_k(z)|^{p-2} \mu_k(z)}{(\sum_{k=1}^{\nu} |\mu_k(z)|^p)^{\frac{1}{p}}}. \]

It is not difficult to know that \( \theta(\alpha) = (\theta_1(\alpha), \ldots, \theta_\nu(\alpha)) \in \partial \mathbb{B}_{\ell^p} \). From (3.15) and (3.16), we conclude that for \( z \in \mathbb{D} \setminus \Omega \),

\[ |\nabla \|f(z)\|_{p}| = \max_{\alpha \in [0, 2\pi]} \left( \sum_{k=1}^{\nu} \left| \frac{\partial f_k(z)}{\partial x} \cos \alpha + \frac{\partial f_k(z)}{\partial y} \sin \alpha \right|^p \right)^{\frac{1}{p}} \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}. \]

**Case 3.6.** \( n \geq 2 \).

We also split the proof of this case into two steps.

**Step 3.5.** We first estimate \( |\nabla \|f(0)\|_{p}| \).

For any fixed \( \theta \in \partial \mathbb{B}_{\ell^p} \), let

\[ \Phi_\theta(\zeta) = f(\zeta \theta), \quad \zeta \in \mathbb{D}. \]

Then \( \Phi_\theta \) is a harmonic function of \( \mathbb{D} \) into \( \mathbb{B}_{\ell^p} \). By the case 3.5, we have

\[ |\nabla \|\Phi_\theta(0)\|_{p}| \leq \frac{4}{\pi}. \]

Since \( \theta \in \partial \mathbb{B}_{\ell^p} \) is arbitrary, in view of Lemma 3.1, we have

\[ (3.17) \quad |\nabla \|f(0)\|_{p}| \leq \frac{4}{\pi}. \]

**Step 3.6.** Next, we estimate \( |\nabla \|f(z)\|_{p}| \) for \( z \in \mathbb{B}_{\ell^p} \setminus \{0\} \).
In this case, by [39, Theorem 2.2.2], there exists an automorphism \( \varphi_z \) of \( \mathbb{B}_n \) such that \( \varphi_z(0) = z \) and
\[
D(\varphi_z^{-1})(z)(w) = -\frac{\langle w, z \rangle z}{\|z\|_2^2 (1 - \|z\|_2^2)} - \frac{\|z\|_2^2 w - \langle w, z \rangle z}{\|z\|_2^2 \sqrt{1 - \|z\|_2^2}},
\]
where \( D(\varphi_z^{-1})(z) \) is the Fréchet derivative of \( \varphi_z^{-1} \) at \( z \). Since \( f \circ \varphi_z \) is a pluriharmonic function of \( \mathbb{B}_n \) into \( \mathbb{B}_n \), and
\[
\|D(\varphi_z^{-1})(z)\| = \sup_{\|w\|_2 = 1} \|D(\varphi_z^{-1})(z)(w)\|_2 = \frac{1}{1 - \|z\|_2^2},
\]
by using (3.17), we have
\[
\|\nabla f(z)\|_p = \limsup_{w \to z} \frac{\|f(z) - f(w)\|_p}{\|z - w\|_2} = \limsup_{w \to z} \frac{\|f \circ \varphi_z(0)\|_p}{\|\varphi_z^{-1}(w)\|_2} \cdot \|\varphi_z^{-1}(w) - \varphi_z^{-1}(z)\|_2 \frac{\|\varphi_z^{-1}(w)\|_2}{\|z - w\|_2} \leq \frac{4}{\pi} \frac{1}{1 - \|z\|_2^2} \cdot \|D(\varphi_z^{-1})(z)\|.
\]

Therefore, combining the cases 3.5 and 3.6 yields the final estimate (2.2).

Next, we prove the sharpness part. Let \( a \in \mathbb{B}_n \) be arbitrarily fixed. There exists a unitary transformation \( U \) such that \( Ua = (b_1, 0, \ldots, 0) \) for some \( b_1 \in \mathbb{R} \) with \( b_1 \in [0, 1) \). Obviously,
\[
(3.18) \quad \frac{1}{1 - |b_1|^2} = \frac{1}{1 - \|a\|_2^2}.
\]
Let \( f : \mathbb{B}_n \to \mathbb{B}_n \) be such that
\[
\tilde{f}(z) = (f \circ U^{-1})(z) = ((g \circ \phi_{-b_1})(z_1), 0, \ldots, 0) = (\tilde{f}_1(z_1), 0, \ldots, 0),
\]
where \( g(\zeta) = \frac{2}{\pi} \arctan \frac{b_1 - \zeta}{1 - b_1 \zeta} \) and \( \phi_{-b_1}(\zeta) = \frac{b_1 + \zeta}{1 - b_1 \zeta} \) is a conformal automorphism of \( \mathbb{D} \). Since \( f(a) = 0 \) and
\[
\limsup_{w \to a} \frac{\|\tilde{f}(Uw)\|_p - \|\tilde{f}(Ua)\|_p}{\|Uw - Ua\|_2} = \sup_{|\theta_1| = 1} \limsup_{\rho \to 0^+} \frac{\|\tilde{f}_1(b_1 + \rho \theta_1) - \tilde{f}_1(b_1)\|_p}{\rho} = \left| \frac{\partial \tilde{f}_1(b_1)}{\partial z_1} \right| + \left| \frac{\partial \tilde{f}_1(b_1)}{\partial \bar{z}_1} \right| = \frac{4}{\pi} \frac{1}{1 - |b_1|^2},
\]
by (3.18), we conclude that
\[
|\nabla \| f(a) \|_p | = \limsup_{w \to a} \frac{\| f(w) \|_p - \| f(a) \|_p}{\| w - a \|_2} = \limsup_{w \to a} \frac{\| f(Uw) \|_p - \| f(Ua) \|_p}{\| Uw - Ua \|_2}
\]
\[
= \frac{4}{\pi (1 - \| a \|_2^2)}.
\]

The proof of this theorem is completed. \qed

**The proof of Theorem 2.3.** Let \( u = (u_1, \ldots, u_\nu) \) be a pluriharmonic function of \( \mathbb{B}_{\ell^2_\nu} \) into \( \mathbb{B}_{\ell^p} \). We also divide the proof of (2.3) into two cases.

**Case 3.7.** \( n = 1 \).

We split the proof of this case into two steps.

**Step 3.7.** *We first estimate \( |\nabla \| u(z) \|_p | \) for \( z = x + iy \in \Omega := \{ \zeta \in \mathbb{D} : u(\zeta) \neq 0 \} \).*

As in the proof of Theorem 2.2, we have
\[
|\nabla \| u(z) \|_p | = \left| \frac{\partial \| u(z) \|_p}{\partial z} \right| + \left| \frac{\partial \| u(z) \|_p}{\partial \bar{z}} \right|.
\]
By calculations, we obtain
\[
\frac{\partial \| u(z) \|_p}{\partial z} = \sum_{j=1}^\nu \frac{|u_j(z)|^{p-2} u_j(z) \partial u_j(z)}{\| u(z) \|_p^{p-1}} \frac{\partial u(z)}{\partial z}
\]
\[
= \frac{1}{2} \left< \tau(z), \frac{\partial u(z)}{\partial x} \right> - \frac{i}{2} \left< \tau(z), \frac{\partial u(z)}{\partial y} \right>
\]
and
\[
\frac{\partial \| u(z) \|_p}{\partial \bar{z}} = \frac{1}{2} \left< \tau(z), \frac{\partial u(z)}{\partial x} \right> + \frac{i}{2} \left< \tau(z), \frac{\partial u(z)}{\partial y} \right>,
\]
which, together with (3.19), implies that
\[
|\nabla \| u(z) \|_p | = \sqrt{\left< \frac{\partial u(z)}{\partial x}, \tau(z) \right>^2 + \left< \frac{\partial u(z)}{\partial y}, \tau(z) \right>^2},
\]
where
\[
\tau(z) = \left( \frac{|u_1(z)|^{p-2} u_1(z)}{\| u(z) \|_p^{p-1}}, \ldots, \frac{|u_\nu(z)|^{p-2} u_\nu(z)}{\| u(z) \|_p^{p-1}} \right),
\]
\[
\frac{\partial u(z)}{\partial x} = \left( \frac{\partial u_1(z)}{\partial x}, \ldots, \frac{\partial u_\nu(z)}{\partial x} \right) \quad \text{and} \quad \frac{\partial u(z)}{\partial y} = \left( \frac{\partial u_1(z)}{\partial y}, \ldots, \frac{\partial u_\nu(z)}{\partial y} \right). \]

Note that \( \tau(z) \in \partial \mathbb{B}_{\ell^q} \), where \( \frac{1}{q} + \frac{1}{p} = 1 \). For any fixed \( \theta \in \mathbb{B}_{\ell^q} \), let \( U_\theta(z) = \langle u(z), \theta \rangle, \ z \in \mathbb{D} \). Then we infer from Hölder’s inequality that \( U_\theta \) is a real harmonic function of \( \mathbb{D} \) into \((-1, 1)\). Hence, by (1.4), we have
\[
\| \nabla U_\theta(z) \|_2 = \sqrt{\left< \frac{\partial u(z)}{\partial x}, \theta \right>^2 + \left< \frac{\partial u(z)}{\partial y}, \theta \right>^2} \leq \frac{4}{\pi} \frac{1 - |U_\theta(z)|^2}{1 - |z|^2},
\]
which, together with (3.20), gives that
\[
|\nabla\|u(z)\|_p| \leq \frac{4}{\pi} \frac{1 - |U_\tau(z)|^2}{1 - |z|^2} = \frac{4}{\pi} \frac{1 - \|u(z)\|^2_p}{1 - |z|^2}.
\]

**Step 3.8.** Next, we estimate $|\nabla\|u(z)\|_p|$ for $z \in \mathbb{D} \setminus \Omega$.

This step follows from Theorem 2.2.

**Case 3.8.** $n \geq 2$.

We also divide the proof of this case into two steps.

**Step 3.9.** We first estimate $|\nabla\|u(0)\|_p|$.

For any fixed $\theta \in \partial \mathbb{B}_{\mathbb{E}_2}$, let $\psi_\theta(\xi) = u(\xi \theta)$, $\xi \in \mathbb{D}$. Obviously, $\psi_\theta$ is a harmonic function of $\mathbb{D}$ into $\mathbb{B}_{\mathbb{E}_p}$. By Case 3.7, we have
\[
|\nabla\|\psi_\theta(0)\|_p| \leq \frac{4}{\pi} \left(1 - \|\psi_\theta(0)\|^2_p\right) = \frac{4}{\pi} \left(1 - \|u(0)\|^2_p\right).
\]

It follows from Lemma 3.1 and the arbitrariness of $\theta$ that

(3.21)
\[
|\nabla\|u(0)\|_p| \leq \frac{4}{\pi} \left(1 - \|u(0)\|^2_p\right).
\]

**Step 3.10.** Next, we estimate $|\nabla\|u(z)\|_p|$ for $z \in \mathbb{B}_{\mathbb{E}_2} \setminus \{0\}$.

By using (3.21) and the similar reasoning as in the proof of Step 3.6 of Theorem 2.2, we obtain that
\[
|\nabla\|u(z)\|_p| \leq \frac{4}{\pi} \frac{1 - \|u(z)\|^2_p}{1 - |z|^2}.
\]

Combining Cases 3.7 and 3.8 yields the final estimate (2.3).

To show that the inequality of (2.3) is sharp, let $a \in \mathbb{B}_{\mathbb{E}_2}$ be arbitrarily fixed. There exists a unitary transformation $U$ such that $Ua = (a_1^*, 0, \ldots, 0)$ for some $a_1^* \in \mathbb{R}$ with $a_1^* \in [0, 1)$. Let $u : \mathbb{B}_{\mathbb{E}_2} \to \mathbb{B}_{\mathbb{E}_p}$ be a harmonic function such that
\[
\tilde{u}(z) := (u \circ U^{-1})(z) = ((g \circ \phi_{-a_1^*})(z_1), 0, \ldots, 0) = (\tilde{u}_1(z_1), 0, \ldots, 0),
\]

where $g(\zeta) = \frac{2}{\pi} \arctan \frac{i(\zeta - \zeta_1)}{1 - |\zeta|^2}$ and $\phi_{-a_1^*}(\zeta) = \frac{-a_1^* + \zeta}{1 - a_1^* \zeta}$ is a conformal automorphism of $\mathbb{D}$. Since $u(a) = 0$, as in the proof of Theorem 2.2, we have
\[
|\nabla\|u(a)\|_p| = \frac{4}{\pi} \frac{1 - \|u(a)\|^2_p}{1 - |a|^2}.
\]

The proof of this theorem is complete. \qed
The proof of Theorem 2.4. Since (2.5) easily follows from (2.4), we only need to prove (2.4). Let $\beta = (\beta_1, \ldots, \beta_n)$ be a multi-index consisting of $n$ nonnegative integers $\beta_k$, where $k \in \{1, \ldots, n\}$. For $z = (z_1, \ldots, z_n) \in \mathbb{B}_{\ell_\infty^n}$, we write $f$ in the following form

$$f(z) = \sum_\beta a_\beta z^\beta + \sum_\beta b_\beta \overline{z}^{\beta},$$

where $a_\beta = (a_{1,\beta}, \ldots, a_{n,\beta})$, $b_\beta = (b_{1,\beta}, \ldots, b_{n,\beta})$ and $f_j(z) = \sum_\beta a_{j,\beta} z^\beta + \sum_\beta b_{j,\beta} \overline{z}^\beta$ for $j \in \{1, \ldots, \nu\}$. For any $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{B}_{\ell_\infty^n}$, let $\zeta \otimes \theta := (\zeta_1 e^{i \theta_1}, \ldots, \zeta_n e^{i \theta_n})$, where $\theta_k \in [0, 2\pi]$ for $k \in \{1, \ldots, n\}$. Then

$$\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \|f(\zeta \otimes \theta)\|_2^2 \, d\theta_1 \cdots d\theta_n = \|f(0)\|_2^2 + \sum_{|\beta| = 1} \left( \|a_\beta\|_2 + \|b_\beta\|_2 \right) |\zeta^\beta|^2 \leq 1.$$

By letting $\zeta \to \partial \mathbb{B}_{\ell_\infty^n}$, we get

$$\sum_{|\beta| = 1} \left( \|a_\beta\|_2^2 + \|b_\beta\|_2^2 \right) \leq \sum_{|\beta| = 1} \left( \|a_\beta\|_2^2 + \|b_\beta\|_2^2 \right) \leq 1 - \|f(0)\|_2^2,$

which implies that

$$\sum_{j=1}^\nu \sum_{k=1}^n \left( \left| \frac{\partial f_j(0)}{\partial z_k} \right|^2 + \left| \frac{\partial f_j(0)}{\partial \overline{z}_k} \right|^2 \right) \leq 1 - \|f(0)\|_2^2. \tag{3.22}$$

For any fixed $a = (a_1, \ldots, a_n) \in \mathbb{B}_{\ell_\infty^n}$, let

$$F(\zeta) = (F_1(\zeta), \ldots, F_\nu(\zeta)) = f(\phi_a(\zeta)),$$

where $\phi_a(\zeta) = (\phi_a(\zeta_1), \ldots, \phi_a(\zeta_n))$ is a holomorphic automorphism of $\mathbb{B}_{\ell_\infty^n}$ with $\phi_a(\zeta_k) = a_{\zeta_k} + \zeta_k$ for $k \in \{1, \ldots, n\}$. Elementary calculations show that

$$\frac{\partial F_j(\zeta)}{\partial \zeta_k} = \frac{\partial f_j(\phi_a(\zeta))}{\partial w_k} \frac{\phi'_a(\zeta_k)}{\partial w_k} = \frac{\partial f_j(\phi_a(\zeta))}{\partial w_k} \frac{1 - |a_k|^2}{(1 + \overline{a_k} \zeta_k)^2},$$

and

$$\frac{\partial F_j(\zeta)}{\partial \overline{\zeta}_k} = \frac{\partial f_j(\phi_a(\zeta))}{\partial \overline{w}_k} \frac{\phi'_a(\zeta_k)}{\partial \overline{w}_k} = \frac{\partial f_j(\phi_a(\zeta))}{\partial \overline{w}_k} \frac{1 - |a_k|^2}{(1 + a_k \overline{\zeta}_k)^2},$$

which, together with (3.22), imply

$$\sum_{j=1}^\nu \sum_{k=1}^n \left( \left| \frac{\partial f_j(a)}{\partial w_k} \right|^2 + \left| \frac{\partial f_j(a)}{\partial \overline{w}_k} \right|^2 \right) (1 - |a_k|^2)^2 = \sum_{j=1}^\nu \sum_{k=1}^n \left( \left| \frac{\partial F_j(0)}{\partial \zeta_k} \right|^2 + \left| \frac{\partial F_j(0)}{\partial \overline{\zeta}_k} \right|^2 \right) \leq 1 - \|F(0)\|_2^2 = 1 - \|f(a)\|_2^2,$$

where $w_k = \phi_a(\zeta_k)$. 
It follows from Theorem 2.4 that
\[ f(z) = \frac{a_1 + z\overline{a_1}}{1 - az}, \]
which implies
\[ \|f(z)\|_2 = 1 - \|f(a)\|^2. \]
Finally, since \( \Omega \) is linearly connected, there exists a constant \( L > 0 \) such that given any two points \( z_1, z_2 \in \Omega \), there exists a smooth curve \( \gamma \subset \Omega \) with the endpoints \( z_1 \) and \( z_2 \) such that \( \ell(\gamma) \leq L|z_1 - z_2| \). Then, by (3.25) and the Cauchy-Schwarz inequality, we have
\[ \|f(z_1) - f(z_2)\|_2 \leq \int_{\gamma} (\|f(z)\|_2 + \|\overline{f(z)}\|_2) |dz| \leq (2C)^{\frac{1}{2}} L|z_1 - z_2|. \]

The proof of this proposition is completed. \( \square \)

**The proof of Proposition 2.5.** For any fixed point \( z \in \Omega \), we consider the function
\[ F(\epsilon) = f(z + d(z)\epsilon)/M, \quad \epsilon \in \mathbb{D}, \]
where \( d(z) := d_\Omega(z) \) and \( M := \sup\{\|f(\lambda)\|_2 : |\lambda - z| < d(z)\} \). Then \( F \) is a harmonic mapping of \( \mathbb{D} \) into \( \mathbb{B}_{\ell_2^\infty} \),
\[ \partial F(\epsilon) = d(z)\partial f(z + d(z)\epsilon)/M, \]
and
\[ \overline{\partial} F(\epsilon) = d(z)\overline{\partial} f(z + d(z)\epsilon)/M. \]
It follows from Theorem 2.4 that
\[ \left(\|\partial F(0)\|_2^2 + \|\overline{\partial} F(0)\|_2^2\right)^{\frac{1}{2}} \leq (1 - \|F(0)\|_2^2)^{\frac{1}{2}}, \]
which implies
\[ d(z) \left(\|\partial f(z)\|_2^2 + \|\overline{\partial} f(z)\|_2^2\right)^{\frac{1}{2}} \leq \left(M^2 - \|f(z)\|_2^2\right)^{\frac{1}{2}}. \]
(3.23)

Let \( \eta \in \mathbb{D} \in (d(z)) := \{\lambda \in \mathbb{C} : |\lambda - z| < d(z)\} \). It follows from the assumption \( \|f\|_2^2 \in \Lambda_2(\Omega) \) that there is a positive constant \( C \) such that
\[ \|f(\eta)\|_2^2 - \|f(z)\|_2^2 \leq C|\eta - z|^2 \leq C(d(z))^2, \]
which yields
\[ M^2 - \|f(z)\|_2^2 = \sup_{\eta \in \mathbb{D} \in (d(z))} (\|f(\eta)\|_2^2 - \|f(z)\|_2^2) \leq C(d(z))^2. \]
(3.24)

Combining (3.23) and (3.24) gives the following estimate
\[ \left(\|\partial f(z)\|_2^2 + \|\overline{\partial} f(z)\|_2^2\right)^{\frac{1}{2}} \leq C^{\frac{1}{2}}. \]
(3.25)
Finally, since \( \Omega \) is linearly connected, there exists a constant \( L > 0 \) such that given any two points \( z_1, z_2 \in \Omega \), there exists a smooth curve \( \gamma \subset \Omega \) with the endpoints \( z_1 \) and \( z_2 \) such that \( \ell(\gamma) \leq L|z_1 - z_2| \). Then, by (3.25) and the Cauchy-Schwarz inequality, we have
\[ \|f(z_1) - f(z_2)\|_2 \leq \int_{\gamma} (\|\partial f(z)\|_2 + \|\overline{\partial} f(z)\|_2) |dz| \leq (2C)^{\frac{1}{2}} L|z_1 - z_2|. \]
The proof of this proposition is completed. \( \square \)
4. The coefficient type Schwarz-Pick lemmas and their applications

We begin this part with the following useful lemma.

**Lemma 4.1.** ([11, Lemma 1]) Let $m$ be a positive integer and $\gamma$ be a real constant. Then

$$\int_0^{2\pi} |\cos(m\theta + \gamma)|d\theta = 4.$$  

**The proof of Theorem 2.6.** We first prove (2.6). Let $f = h + \overline{g} \in \mathcal{P}(\mathbb{B}^n_r)$ with $\sup_{z \in \mathbb{B}^n_r} |f(z)| \leq 1$, where $h(z) = \sum_\alpha a_\alpha z^\alpha$ and $g(z) = \sum_\alpha b_\alpha z^\alpha$. For any $\theta = (\xi_1, \ldots, \xi_n) = (\rho_1 e^{i\mu_1}, \ldots, \rho_n e^{i\mu_n}) \in \mathbb{B}^n_r$, let $\xi \otimes \theta := (\xi_1 e^{i\mu_1}, \ldots, \xi_n e^{i\mu_n})$, where $\mu_k, \theta_k \in \mathbb{R}$ and $\rho_k = |\xi_k|$ for all $k \in \{1, \ldots, n\}$. Then $\xi \otimes \theta \in \mathbb{B}^n_r$. Let $\alpha$ with $|\alpha| \geq 1$ be fixed. Without loss of generality, we may assume that $|a_\alpha||b_\alpha| \neq 0$. By the orthogonality, we have

$$|a_\alpha| |\xi^\alpha| = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{-i (\arg a_\alpha + \sum_{k=1}^n \alpha_k (\mu_k + \theta_k))} f(\xi \otimes \theta) d\theta_1 \cdots d\theta_n$$
and

$$|b_\alpha| |\xi^\alpha| = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{i (\arg b_\alpha + \sum_{k=1}^n \alpha_k (\mu_k + \theta_k))} f(\xi \otimes \theta) d\theta_1 \cdots d\theta_n,$$

which give that

$$
\begin{aligned}
(|a_\alpha| + |b_\alpha|) |\xi^\alpha| &= \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( e^{-i (\arg a_\alpha + \sum_{k=1}^n \alpha_k (\mu_k + \theta_k))} \\
&\quad + e^{i (\arg b_\alpha + \sum_{k=1}^n \alpha_k (\mu_k + \theta_k))} \right) f(\xi \otimes \theta) d\theta_1 \cdots d\theta_n \\
&\leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| 1 + e^{i (\arg a_\alpha + \arg b_\alpha + 2 \sum_{k=1}^n \alpha_k (\mu_k + \theta_k))} \right| f(\xi \otimes \theta) d\theta_1 \cdots d\theta_n \\
&\leq \frac{2}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \cos \left( \sum_{k=1}^n \alpha_k (\mu_k + \theta_k) \right) + \frac{\arg a_\alpha + \arg b_\alpha}{2} \right| d\theta_1 \cdots d\theta_n.
\end{aligned}
$$

(4.1)

Since $|\alpha| \geq 1$, without loss of generality, we may assume that $\alpha \neq 0$. It follows from Lemma 4.1 that

$$\int_0^{2\pi} \left| \cos \left( \sum_{k=1}^n \alpha_k (\mu_k + \theta_k) + \frac{\arg a_\alpha + \arg b_\alpha}{2} \right) \right| d\theta_1 = 4,$$

which, together (4.1), implies that
\begin{equation}
|a_\alpha| + |b_\alpha| \leq \frac{4}{\pi} \inf_{\xi \in \mathbb{B}_{L^p}} \frac{1}{|\xi|}. \tag{4.2}
\end{equation}

From [17, p.43], we see that

\begin{equation}
\sup_{\xi \in \mathbb{B}_{L^p}} |\xi^\alpha| = \left( \frac{|\alpha|}{|\alpha||\alpha|} \right)^{1/p}. \tag{4.3}
\end{equation}

Therefore, combining (4.2) and (4.3) yields the final estimates

\begin{equation*}
|a_\alpha| + |b_\alpha| \leq \frac{4}{\pi} \inf_{\xi \in \mathbb{B}_{L^p}} \frac{1}{|\xi|} = \frac{4}{\pi} \left( \frac{|\alpha|}{|\alpha|} \right)^{1/p}.
\end{equation*}

Next, we prove the sharpness part. For \( z \in \mathbb{B}_{L^p} \) and some \( k \in \{1, \ldots, n\} \), let

\[ f(z) = \frac{2}{\pi} \text{arg} \left( \frac{1 + z_{\alpha}^{(k)}}{1 - z_{\alpha}^{(k)}} \right). \]

Then

\[ f(z) = \frac{2}{i\pi} \left( \sum_{j=1}^{\infty} \frac{1}{2j - 1} z_{\alpha}^{(2j-1)} - \sum_{j=1}^{\infty} \frac{1}{2j - 1} z_{\alpha}^{(2j-1)} \right), \]

which implies that

\[ |a_{(0,\ldots,\alpha_{k},0\ldots,0)}| + |b_{(0,\ldots,\alpha_{k},0\ldots,0)}| = \frac{4}{\pi}, \]

where \( |\alpha| \geq 1 \) and \( \alpha_{k} = |\alpha| \). The proof of this theorem is finished.\( \square \)

\textbf{The proof of Theorem 2.7.} We first prove (2.7). For \( z \in \mathbb{B}_{L^p} \), let \( h(z) = \sum_{\alpha} a_{\alpha} z^\alpha \) and \( g(z) = \sum_{\alpha} b_{\alpha} z^\alpha \). Then, by the orthogonality, we have

\[ \sum_{|\alpha|=k} a_{\alpha} z^\alpha = \frac{1}{2\pi} \int_0^{2\pi} h(ze^{i\tau}) e^{-ik\tau} d\tau \]

and

\[ 0 = \frac{1}{2\pi} \int_0^{2\pi} g(ze^{i\tau}) e^{-ik\tau} d\tau, \]

which give that

\begin{equation}
\sum_{|\alpha|=k} a_{\alpha} z^\alpha = \frac{1}{\pi} \int_0^{2\pi} \text{Re}(h(ze^{i\tau})) e^{-ik\tau} d\tau, \tag{4.4}
\end{equation}

where \( k \geq 1 \). By a similar proof process of (4.4), we get

\begin{equation}
\sum_{|\alpha|=k} b_{\alpha} z^\alpha = \frac{1}{\pi} \int_0^{2\pi} \text{Re}(g(ze^{i\tau})) e^{-ik\tau} d\tau. \tag{4.5}
\end{equation}

It follows from (4.4) and (4.5) that
By using a similar reasoning as in the proof of (4.6), we obtain
\[
- \sum_{|\alpha|=k} (a_\alpha + b_\alpha) z^\alpha = \frac{1}{\pi} \int_0^{2\pi} (1 - \text{Re}(f(ze^{i\tau}))) e^{-ik\tau} d\tau,
\]
and consequently,
\[
\left| \sum_{|\alpha|=k} (a_\alpha + b_\alpha) z^\alpha \right| \leq \frac{1}{\pi} \int_0^{2\pi} \left| 1 - \text{Re}(f(ze^{i\tau})) \right| d\tau = \frac{1}{\pi} \int_0^{2\pi} \left( 1 - \text{Re}(f(ze^{i\tau})) \right) d\tau = 2 \left( 1 - \text{Re}(f(0)) \right).
\]

Now we prove (2.8). For any \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{B}_p^n \), let \( \xi \otimes \theta := (\xi_1 e^{i\theta_1}, \ldots, \xi_n e^{i\theta_n}) \), where \( \theta_k \in [0, 2\pi] \) for all \( k \in \{1, \ldots, n\} \). Then \( \xi \otimes \theta \in \mathbb{B}_p \). It follows from the orthogonality that
\[
a_\alpha \xi^\alpha = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{-i\sum_{k=1}^n \alpha_k \theta_k} h(\xi \otimes \theta) d\theta_1 \cdots d\theta_n
\]
and
\[
\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{-i\sum_{k=1}^n \alpha_k \theta_k} \overline{h(\xi \otimes \theta)} d\theta_1 \cdots d\theta_n = 0,
\]
which yield that
\[
(4.6) \quad a_\alpha \xi^\alpha = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{-i\sum_{k=1}^n \alpha_k \theta_k} (h(\xi \otimes \theta) + \overline{h(\xi \otimes \theta)}) d\theta_1 \cdots d\theta_n.
\]
By using a similar reasoning as in the proof of (4.6), we obtain
\[
(4.7) \quad b_\alpha \xi^\alpha = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} e^{-i\sum_{k=1}^n \alpha_k \theta_k} (g(\xi \otimes \theta) + \overline{g(\xi \otimes \theta)}) d\theta_1 \cdots d\theta_n.
\]

We infer from (4.6) and (4.7) that
\[
-(a_\alpha + b_\alpha) = \frac{1}{2^{n-1}\pi^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{e^{-i\sum_{k=1}^n \alpha_k \theta_k}}{\xi^\alpha} (1 - \text{Re}(f(\xi \otimes \theta))) d\theta_1 \cdots d\theta_n,
\]
and consequently,
\[
|a_\alpha + b_\alpha| \leq \frac{1}{2^{n-1}\pi^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{(1 - \text{Re}(f(\xi \otimes \theta)))}{|\xi^\alpha|} d\theta_1 \cdots d\theta_n = \frac{2(1 - \text{Re}(f(0)))}{|\xi^\alpha|}.
\]
Hence combining (4.3) and (4.8) gives the final estimate
\[
|a_\alpha + b_\alpha| \leq 2(1 - \text{Re}(f(0))) \inf_{\xi \in \mathbb{B}_p^n} \frac{1}{|\xi^\alpha|} = 2(1 - \text{Re}(f(0))) \left( \frac{|\alpha||\alpha|}{\alpha^n} \right)^{1/p}.
\]
At last, we prove the sharpness part. For \( z = (z_1, \ldots, z_n) \in \mathbb{B}_{\ell_p^n} \) and some \( k \in \{1, \ldots, n\} \), let
\[
 f(z) = \frac{-2z_k}{1-z_k} = -2 \sum_{j=1}^{\infty} \frac{z_k^j}{j} \left( \text{or } f(z) = \frac{-2z_k}{1-z_k} \right).
\]
Then \( f(0) = 0 \), \( \text{Re}(f) < 1 \) and the modulus of all nonzero coefficients of \( f \) is 2, which shows that the constant 2 in (2.7) and (2.8) cannot be improved. The proof of this theorem is completed.

**Lemma 4.2.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( m = (m_1, \ldots, m_n) \) be multi-indices satisfying \( |\alpha| \geq 1 \) and \( m_k \geq \alpha_k \) for all \( k \in \{1, \ldots, n\} \). Then
\[
 \frac{|\alpha|^{|\alpha|}}{\alpha^\alpha} \leq n^{|m|}.
\]
Furthermore, the above equality holds if and only if \( \alpha = m \) and \( m_1 = \cdots = m_n \).

**Proof.** Since
\[
 \left( \frac{|\alpha|}{\alpha_k} \right)^{\alpha_k} = \left( 1 + \frac{|\alpha| - \alpha_k}{\alpha_k} \right)^{\alpha_k} \leq \left( 1 + \frac{|\alpha| - \alpha_k}{m_k} \right)^{m_k} \\
 \leq \left( 1 + \frac{|m| - m_k}{m_k} \right)^{m_k} = \left( \frac{|m|}{m_k} \right)^{m_k},
\]
in the case \( \alpha_k \geq 1 \), we see that
\[
(4.9) \quad \frac{|\alpha|^{|\alpha|}}{\alpha^\alpha} = \prod_{k=1}^{n} \left( \frac{|\alpha|}{\alpha_k} \right)^{\alpha_k} \leq \prod_{k=1}^{n} \left( \frac{|m|}{m_k} \right)^{m_k} = \frac{|m|^{|m|}}{m^m}.
\]
Next, we show that
\[
\frac{|m|^{|m|}}{m^m} \leq n^{|m|}.
\]
For any fixed \( \epsilon > 0 \), let \( \mu(x) = (x+\epsilon) \log(x+\epsilon) \), \( x \geq 0 \). Then \( \mu \) is strictly convex in \([0, \infty)\). It follows from Jensen’s inequality that
\[
\frac{\sum_{k=1}^{n} \mu(m_k)}{n} \geq \mu \left( \frac{\sum_{k=1}^{n} m_k}{n} \right).
\]
Consequently,
\[
\lim_{\epsilon \to 0^+} \frac{\sum_{k=1}^{n} \mu(m_k)}{n} \geq \lim_{\epsilon \to 0^+} \mu \left( \frac{\sum_{k=1}^{n} m_k}{n} \right),
\]
which, together with (4.9), implies that
\[
\frac{|\alpha|^{|\alpha|}}{\alpha^\alpha} \leq \frac{|m|^{|m|}}{m^m} \leq n^{|m|}.
\]
The proof of this lemma is finished.
The proof of Theorem 2.8. We first give a proof for (ii). It suffices to show for $z \in \mathbb{B}_{\ell_2^2} \setminus \{0\}$. Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{B}_{\ell_2^2} \setminus \{0\}$ be fixed. Then there exists a unitary matrix $U_{\xi}$ such that $U_{\xi}\xi^T = (||\xi||_2, 0, \ldots, 0)^T$. Let

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & s_\xi & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_\xi \end{pmatrix} U_{\xi},$$

where $s_\xi = (1 - ||\xi||_2^2)^{\frac{1}{2}}$. Let

$$\varphi^T(z) = \frac{P(z^T - \xi^T)}{1 - \langle z, \xi \rangle} = \left( \frac{P_1(z^T - \xi^T)}{1 - \langle z, \xi \rangle}, \ldots, \frac{P_n(z^T - \xi^T)}{1 - \langle z, \xi \rangle} \right)^T,$$

where $P = (P_1^T, \ldots, P_n^T)^T$. Since $\overline{U_{\xi}^T} P \varphi^T = \xi^T$ and $\overline{U_{\xi}^T} P Q_{\xi} = s_\xi Q_{\xi}$, $\overline{U_{\xi}^T} \varphi^T(z)$ can be written as follows:

$$\overline{U_{\xi}^T} \varphi^T(z) = -\frac{\xi^T - P(z^T - s_\xi Q_{\xi}^T)}{1 - \langle z, \xi \rangle},$$

where $P_{\xi}$ is the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[\xi]$ generated by $\xi$, and $Q_{\xi} = I - P_{\xi}$ is the projection onto the orthogonal complement of $[\xi]$. According to the representation of automorphism of $\mathbb{B}_{\ell_2^2}$ in [39, Chapter 2] (or [17, 28]), we obtain that $\varphi \in \text{Aut}(\mathbb{B}_{\ell_2^2})$, where $\text{Aut}(\mathbb{B}_{\ell_2^2})$ is the automorphism group of $\mathbb{B}_{\ell_2^2}$.

Let $f$ be a pluriharmonic function of $\mathbb{B}_{\ell_2^2}$ into $\mathbb{C}$ satisfying $\sup_{z \in \mathbb{B}_{\ell_2^2}} |f(z)| \leq 1$. Then, by [41], we see that $f$ has a representation $f = h + \overline{g}$, where $h$ and $g$ are holomorphic in $\mathbb{B}_{\ell_2^2}$ with $g(0) = 0$. Let

$$\mathcal{F}(z) := f(\varphi^{-1}(z)) = H(z) + \overline{G(z)},$$

where

$$H(z) := h(\varphi^{-1}(z)) = c_0 + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} c_\alpha z^\alpha$$

and

$$G(z) := g(\varphi^{-1}(z)) = d_0 + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} d_\alpha z^\alpha.$$

Then

$$f(z) = \mathcal{F}(\varphi(z)) = c_0 + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} c_\alpha u_{\alpha}(z)v_{\alpha}(z) + d_0 + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} d_\alpha u_{\alpha}(z)v_{\alpha}(z),$$

where $u_{\alpha}(z) = \prod_{j=1}^{n} (P_j(z^T - \xi^T))^{\alpha_j}$ and $v_{\alpha}(z) = (1 - \langle z, \xi \rangle)^{-|\alpha|}$. Then for the multi-index $m = (m_1, \ldots, m_n) \neq 0$, we have

$$\frac{\partial^{|m|} f(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} c_\alpha \frac{\partial^{|m|} (u_{\alpha}(z)v_{\alpha}(z))}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}$$

(4.10)
Then we have
\begin{equation}
\frac{\partial |m| f(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} d_{\alpha} \frac{\partial |m| (u_\alpha(z)v_\alpha(z))}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}.
\end{equation}

By elementary calculations, we see that for $|m| \geq |\alpha|$,
\begin{equation}
\frac{\partial |m| (u_\alpha(z)v_\alpha(z))}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} = \sum_{|\beta|=|\alpha|, m_j \geq j} \frac{\partial |\beta| u_\alpha(z)}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}} \frac{\partial |m| - |\beta| v_\alpha(z)}{\partial z_1^{m_1-\beta_1} \cdots \partial z_n^{m_n-\beta_n}} \prod_{j=1}^{n} (m_j - \beta_j),
\end{equation}
where $\beta = (\beta_1, \ldots, \beta_n)$ is a multi-index.

**Claim 1.**
\[ |\frac{\partial |\beta| u_\alpha(z)}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}}| \leq |\alpha|! \]
for all $\beta$ with $|\beta| = |\alpha|$.

Now we prove Claim 1. Set
\[ P = (p_{jk})_{1 \leq j, k \leq n}. \]

Then, we have
\[ u_\alpha(z) = \prod_{k=1}^{n} \left( \sum_{j=1}^{n} p_{kj}(z_j - \xi_j) \right)^{\alpha_k}. \]

By calculating the partial derivative directly from the above formula and using $|p_{jk}| \leq 1$ ($1 \leq j, k \leq n$), we can prove that
\[ \left| \frac{\partial |\beta| u_\alpha(z)}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}} \right| \leq \left| \frac{\partial |\beta| \tilde{u}_\alpha(z)}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}} \right|, \]
where
\[ \tilde{u}_\alpha(z) = \prod_{k=1}^{n} \left( \sum_{j=1}^{n} (z_j - \xi_j) \right)^{\alpha_k} = \left( \sum_{j=1}^{n} (z_j - \xi_j) \right)^{|\alpha|}. \]

Since
\[ \frac{\partial |\beta| \tilde{u}_\alpha(z)}{\partial z_1^{\beta_1} \cdots \partial z_n^{\beta_n}} = |\alpha|!, \]
we obtain the desired result.

**Claim 2.** For $|\beta| = |\alpha| \geq 1$ and $m_j \geq \beta_j$ for $j = 1, \ldots, n$,
\[ \left| \frac{\partial |m| - |\beta| v_\alpha(z)}{\partial z_1^{m_1-\beta_1} \cdots \partial z_n^{m_n-\beta_n}} \right| \leq \frac{|m| - 1)! \prod_{j=1}^{n} |\xi_j|^{m_j-\beta_j}}{(n! |(\alpha| - 1)! (1 - \langle z, \xi \rangle)^{|m|}}}. \]

Next, we prove Claim 2. Elementary computations show that
\[ \frac{\partial |m| - |\beta| v_\alpha(z)}{\partial z_1^{m_1-\beta_1} \cdots \partial z_n^{m_n-\beta_n}} = |\alpha|(|\alpha| + 1) \cdots (|m| - 1) \frac{\prod_{j=1}^{n} \xi_j^{m_j-\beta_j}}{(1 - \langle z, \xi \rangle)^{|m|}}. \]

Then replacing $z$ by $\xi$, implies that Claim 2 is true. The proof of this claim is finished.
Let
\[
\partial^m \omega_\alpha(\xi) := \frac{\partial^{|m|} \omega_\alpha(\xi)}{\partial z_1^{|m_1|} \cdots \partial z_n^{|m_n|}},
\]
where \( \omega_\alpha = u_\alpha v_\alpha \). Then combining (4.12), Claims 1 and 2 gives
\[
|\partial^m \omega_\alpha(\xi)| \leq \sum_{|\beta| = |\alpha|, m_j \geq \beta_j} |\alpha|! \frac{(|m|! - 1)!}{(|\alpha| - 1)! (1 - \|\xi\|_2^2)^{|m|}} \prod_{j=1}^n |\xi_j|^{m_j - \beta_j} \binom{m_j}{\beta_j}
\]
which, together with (4.10), (4.11), Lemmas 2.6 and 4.2, implies that
\[
|d^m \omega_\alpha(\xi)| \leq \sum_{|\beta| = |\alpha|, m_j \geq \beta_j} |\alpha|! \frac{(|m|! - 1)!}{(|\alpha| - 1)! (1 - \|\xi\|_2^2)^{|m|}} \prod_{j=1}^n |\xi_j|^{m_j - \beta_j} \binom{m_j}{\beta_j},
\]
where \( \Psi(\xi) = \sum_{k=1}^{|m|} \sum_{|\alpha| = k} \sum_{|\beta| = |\alpha|, m_j \geq \beta_j} |m|! \prod_{j=1}^n |\xi_j|^{m_j - \beta_j} \binom{m_j}{\beta_j} \).

In the following, we begin to estimate \( \Psi(\xi) \). Since the dimension of the space of \( k \)-homogeneous polynomials in \( \mathbb{C}^n \) is \( \binom{n+k-1}{n-1} \), we see that
\[
\Psi(\xi) = |m|! \sum_{k=1}^{|m|} \sum_{|\beta| = k} \prod_{j=1}^n |\xi_j|^{m_j - \beta_j} \binom{m_j}{\beta_j} \left( \binom{n+k-1}{n-1} \right)
\]
Therefore, substituting (4.14) into (4.13) and replacing \( \xi \) by \( z \), we can get the desired result.
Next, we give a proof for (i). As in the proof for (ii), we have
\[
\left| \frac{\partial^m f(\xi)}{\partial z^m} \right| + \left| \frac{\partial^m f(\xi)}{\partial \bar{z}^m} \right| \leq \sum_{\alpha=1}^{\infty} (|c_\alpha| + |d_\alpha|) \left| \partial^m \omega_\alpha(\xi) \right|
\]
\[\leq \frac{1}{\pi (1 - |\xi|^2)^{m}} \sum_{\alpha=1}^{m} \alpha (m - 1)! |\xi|^{m-\alpha} \binom{m}{\alpha}
\]
\[= \frac{1}{\pi (1 - |\xi|^2)^{m}} m! (1 + |\xi|)^{m-1}.
\]
Now we prove the sharpness part for \( n = 1 \). For \( z \in \mathbb{D} \), let
\[
f_m(z) = \frac{2}{\pi} \arg \left( \frac{1 + z^m}{1 - z^m} \right) = \frac{2}{i\pi} \left( \sum_{j=1}^{\infty} \frac{1}{2j-1} z^{(2j-1)m} - \sum_{j=1}^{\infty} \frac{1}{2j-1} z^{(2j-1)m} \right).
\]
Then \( f_m \) is harmonic on \( \mathbb{D} \), \( |f_m(z)| < 1 \) for \( z \in \mathbb{D} \) and
\[
\left| \frac{\partial^m f_m(0)}{\partial z^m} \right| + \left| \frac{\partial^m f_m(0)}{\partial \bar{z}^m} \right| = \frac{4}{\pi} m!.
\]
The proof of this theorem is completed. \( \square \)

The proof of Theorem 2.10. From [41], we know that \( f \in \mathcal{PH}(B_{f_\infty}) \) has a representation \( f = \overline{h + g} \), where \( h \) and \( g \) are holomorphic in \( B_{f_\infty} \) with \( g(0) = 0 \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a multi-index. Then \( f \) can be expressed as a power series in \( B_{f_\infty} \) as follows:
\[
f(z) = h(z) + \overline{g(z)} = \sum_{\alpha} a_\alpha z^\alpha + \sum_{\alpha} \overline{a_\alpha} \overline{z}^\alpha.
\]
It follows from (2.8) that, for all \( |\alpha| \geq 1 \),
\[
|a_\alpha + \overline{a_\alpha}| \leq 2 (1 - \text{Re}(f(0))).
\]
For \( k \in \{1, \ldots, n\} \) and \( z = (z_1, \ldots, z_n) \in B_{f_\infty} \), let
\[
\phi(\zeta) = (\phi_1(\zeta_1), \ldots, \phi_n(\zeta_n)),
\]
where \( \zeta = (\zeta_1, \ldots, \zeta_n) \in B_{f_\infty} \) and
\[
\phi_k(\zeta_k) = \frac{z_k + \zeta_k}{1 + \overline{z_k} \zeta_k}.
\]
Then \( f \circ \phi \) can be written in the following form:
\[
\chi(\zeta) = f(\phi(\zeta)) = H(\zeta) + \overline{G(\zeta)} = \sum_{\alpha} c_\alpha \zeta^\alpha + \sum_{\alpha} \overline{d_\alpha} \overline{\zeta}^\alpha,
\]
where \( H = h \circ \phi \) and \( G = g \circ \phi \). By (4.15), we see that
\[
|c_\alpha + d_\alpha| \leq 2 (1 - \text{Re}(\chi(0))) = 2 (1 - \text{Re}(f(z))).
\]
Let \( z \in \mathbb{B}^n_\infty \). We may assume that \( m_k \geq 1 \) for \( k \in \{1, \ldots, n\} \). Then as in the proof of [11, Theorem 1], by using the Cauchy integral formula (cf. [38]) and by changing the integral variable \( \eta_k = \phi_k(\zeta_k) \), we have

\[
\left| \frac{\partial^{m_1} f(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} + \frac{\partial^{m_1} f(z)}{\partial \zeta_1^{m_1} \cdots \partial \zeta_n^{m_n}} \right|
\]

(4.17)

\[
\leq \frac{m!}{\prod_{k=1}^n (1 - |z_k|^2)^{m_k}} \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} \left( \frac{m_1 - 1}{j_1} \right) \cdots \left( \frac{m_n - 1}{j_n} \right) \prod_{k=1}^n |z_k|^{j_k}.
\]

Combining (4.16) and (4.17), we conclude that

\[
\left| \frac{\partial^{m_1} f(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} + \frac{\partial^{m_1} f(z)}{\partial \zeta_1^{m_1} \cdots \partial \zeta_n^{m_n}} \right|
\]

\[
\leq 2 \left( 1 - \text{Re}(f(z)) \right) \frac{m!}{\prod_{k=1}^n (1 - |z_k|^2)^{m_k}} \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} \left( \frac{m_1 - 1}{j_1} \right) \cdots \left( \frac{m_n - 1}{j_n} \right) \prod_{k=1}^n |z_k|^{j_k}
\]

\[
\leq 2 \left( 1 - \text{Re}(f(z)) \right) \frac{m!}{\prod_{k=1}^n (1 - |z_k|^2)^{m_k}} \prod_{k=1}^n (1 + |z_k|)^{m_k-1}
\]

\[
= m! 2 \left( 1 - \text{Re}(f(z)) \right) \prod_{k=1}^n \frac{(1 + |z_k|)^{m_k-1}}{(1 - |z_k|^2)^{m_k}} \leq m! 2 \left( 1 - \text{Re}(f(z)) \right) \frac{(1 + \|z\|_\infty)^{m-n}}{(1 - \|z\|_\infty)^{m}}.
\]

Now we show the sharpness part. For \( z = (z_1, \ldots, z_n) \in \mathbb{B}^n_\infty \) and some \( k \in \{1, \ldots, n\} \), let

\[
f(z) = \frac{-2z_k}{1 - z_k} = -2 \sum_{j=1}^{\infty} \frac{z_j^k}{1 - z_k^j} \quad \text{or} \quad f(z) = \frac{-2\bar{z}_k}{1 - \bar{z}_k^j}.
\]

Then \( f(0) = 0 \), \( \text{Re}(f) < 1 \) and

\[
\left| \frac{\partial^{m_k} f(0)}{\partial z_k^{m_k}} \right| = 2(m_k !)
\]

which implies that the constant 2 in (2.10) is sharp. The proof of this theorem is completed.

\[\square\]

**Lemma 4.3.** For \( p \in [1, \infty] \), let \( f(z) = \sum_\alpha a_\alpha z^\alpha + \sum_\alpha \bar{b}_\alpha z^\alpha \) be a pluriharmonic function of \( \mathbb{B}^n_p \) into \( \mathbb{D} \). Then \( k \geq 1 \),

\[
\sup_{z \in \mathbb{B}^n_p} \left( \sum_{|\alpha| = k} (|a_\alpha|^2 + |b_\alpha|^2) |z_\alpha|^2 \right) \leq \frac{16}{\pi^2}.
\]
Proof. For \( z \in \mathbb{B}_{\ell^p} \), \( \vartheta \in [0, 2\pi] \) and \( k \geq 1 \), it follows from the orthogonality that
\[
\sum_{|\alpha| = k} a_\alpha z^\alpha = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\vartheta} z) e^{-ik\vartheta} \, d\vartheta
\]
and
\[
\sum_{|\alpha| = k} b_\alpha z^\alpha = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\vartheta} z) e^{ik\vartheta} \, d\vartheta,
\]
which, together with Lemma 4.1, implies that
\[
\left| \sum_{|\alpha| = k} a_\alpha z^\alpha + \sum_{|\alpha| = k} b_\alpha z^\alpha \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\vartheta} z)| |e^{-ik\vartheta} + e^{ik\vartheta}| \, d\vartheta \leq \frac{1}{\pi} \int_0^{2\pi} |\cos k\vartheta| \, d\vartheta = \frac{4}{\pi}.
\]

Let \( z = (r_1 e^{i\vartheta_1}, \ldots, r_n e^{i\vartheta_n}) \in \mathbb{B}_{\ell^p} \), where \( \vartheta_j \in [0, 2\pi] \) and \( r_j \geq 0 \) for all \( j \in \{1, \ldots, n\} \). Then, by (4.18), we have
\[
\frac{16}{\pi^2} \geq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \sum_{|\alpha| = k} a_\alpha z^\alpha + \sum_{|\alpha| = k} b_\alpha z^\alpha \right|^2 \, d\vartheta_1 \cdots d\vartheta_n \geq \sum_{|\alpha| = k} (|a_\alpha|^2 + |b_\alpha|^2) |z^\alpha|^2,
\]
which completes the proof.

The proof of Theorem 2.11. We first prove the left hand side of the inequality (2.12). Let \( f(z) = \sum_{|\alpha| = k} a_\alpha z^\alpha + \sum_{|\alpha| = k} b_\alpha z^\alpha \in \mathcal{P}(\mathbb{B}_{\ell^p}) \) with \( b_0 = 0 \) and \( \sup_{z \in \mathbb{B}_{\ell^p}} |f(z)| \leq 1 \). We split the remaining proof into two cases.

Case 4.1. \( p \in [2, \infty] \).

For \( \|z\|_p \leq \rho_n := \pi/((\pi + 4\sqrt{2}) \sqrt{n}) \) and \( \zeta = z/\rho_n \), it follows from \( \sum_{|\alpha| = k} 1 \leq n^k \) and the Cauchy-Schwarz inequality that
\[
\sum_{|\alpha| = k} (|a_\alpha| + |b_\alpha|) |z^\alpha| \leq \left( \sum_{|\alpha| = k} (|a_\alpha| + |b_\alpha|)^2 |\zeta^\alpha|^2 \right)^{1/2} \left( \sum_{|\alpha| = k} \rho_n^{2k} \right)^{1/2} \leq \sqrt{2} \left( \sum_{|\alpha| = k} (|a_\alpha|^2 + |b_\alpha|^2) |\zeta^\alpha|^2 \right)^{1/2} n^{k} \rho_n^k,
\]
which, together with Lemma 4.3, yields that
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\[ \sum_{|\alpha| = k} (|a_\alpha| + |b_\alpha|)|z^\alpha| \leq \frac{4\sqrt{2}}{\pi} \frac{1}{n^2 \rho_n^k} \leq \frac{4\sqrt{2}}{\pi} \left( \frac{1}{1 + \frac{4\sqrt{2}}{\pi}} \right)^k. \]  

Consequently, by (4.19), we have

\[ \sum_{k=1}^{\infty} \sum_{|\alpha| = k} (|a_\alpha| + |b_\alpha|)|z^\alpha| \leq \frac{4\sqrt{2}}{\pi} \sum_{k=1}^{\infty} \frac{1}{\left(1 + \frac{4\sqrt{2}}{\pi}\right)^k} = 1. \]

Therefore, \( R^*_p(\mathbb{B}_{\ell_p}) \geq \frac{\pi}{(\sqrt{\pi} + 4\sqrt{2})\sqrt{n}}. \)

**Case 4.2.** \( p \in [1, 2) \).

By Theorem 2.6, we have

\[ \sum_{|\alpha| = k} (|a_\alpha| + |b_\alpha|)|z^\alpha| \leq \frac{4\pi}{\pi} \sum_{|\alpha| = k} \left( \frac{|\alpha|^{1/p}}{\alpha} \right) |z^\alpha| \leq \frac{4}{\pi} \sum_{|\alpha| = k} \frac{|\alpha|^{|\alpha|}}{\alpha^{|\alpha|}} |z^\alpha| \]

\[ = \frac{4}{\pi} \sum_{|\alpha| = k} \frac{k!}{|\alpha|^{|\alpha|}} \cdot \cdot \alpha^{|\alpha|} |z^\alpha|. \]  

By applying Hölder’s inequality in the case \( p \in (1, 2) \), we see that

\[ \sum_{|\alpha| = k} \frac{k!}{\alpha^{|\alpha|}} |z^\alpha| = \|z\|^k_1 \leq n^{k(1-\frac{1}{p})} \|z\|^k_p. \]  

Combining (4.20) and (4.21), and using \( \alpha^{|\alpha|}/(\alpha!) \geq 1 \), we have

\[ \sum_{k=1}^{\infty} \sum_{|\alpha| = k} (|a_\alpha| + |b_\alpha|)|z^\alpha| \leq \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{k^k}{k!} \left( n^{1-\frac{1}{p}} \|z\|_p \right)^k. \]

Hence \( R^*_p(\mathbb{B}_{\ell_p}) \geq x_0/n^{1-1/p} \), where \( x_0 \) is the unique positive solution to the following equation

\[ \sum_{k=1}^{\infty} \frac{k^k}{k!} x^k = \frac{\pi}{4}. \]  

By [5, p. 328], we have

\[ \sum_{k=1}^{\infty} \frac{k^k}{k!} \left( \frac{1}{3\sqrt{e}} \right)^k = \frac{1}{2}, \]  

which implies that the unique positive solution \( x_0 \) to the equation (4.22) is bigger than \( 1/(3\sqrt{e}) \).

Next, we prove the right hand side of the inequality (2.12). By the Kahane-Salem-Zygmund inequality (see [5, Corollary] or [6]), for \( n \geq 2 \) and \( k \geq 2 \), there exist coefficients \( (c_\alpha)_{|\alpha| = k} \) with \( |c_\alpha| = k!/\alpha! \) for all \( \alpha \) such that
\[
\sup_{z\in \mathcal{B}_{\ell^p}} \left| \sum_{|\alpha|=k} c_\alpha z^\alpha \right| \leq \sqrt{32k \log(6k) n^{\frac{k}{2} + \left(\frac{1}{2} - \frac{1}{\max(2,p)}\right)k(1 - \frac{1}{\min(p,2)})}}.
\]

Since
\[
\sum_{|\alpha|=k} \frac{k!}{\alpha!} = n^k,
\]
we see from the definition of \( \mathcal{R}_p^*(\mathcal{B}_{\ell^p}) \) that
\[
\left( \frac{\mathcal{R}_p^*(\mathcal{B}_{\ell^p})}{n^{1/p}} \right)^k n^k = \sum_{|\alpha|=k} |c_\alpha| \left( \frac{\mathcal{R}_p^*(\mathcal{B}_{\ell^p})}{n^{1/p}} \right)^k \leq \sqrt{32k \log(6k) n^{\frac{k}{2} + \left(\frac{1}{2} - \frac{1}{\max(2,p)}\right)k(1 - \frac{1}{\min(p,2)})}}.
\]

Consequently,
\[
\mathcal{R}_p^*(\mathcal{B}_{\ell^p}) \leq \left( \frac{(k)!^{\frac{1}{k}}}{n} \right)^{\frac{1}{\min(p,2)}} (32kn \log(6k))^{\frac{1}{k}}.
\]

Choosing \( k \) to be an integer close to \( \log n \), we get the desired result. The proof of this theorem is completed. \( \square \)

Let
\[
P(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha
\]
be a \( k \)-homogeneous polynomial in \( \mathcal{B}_{\ell^p} \). Using polarization, Bohnenblust and Hille [7] obtained the following inequality for \( k \)-homogeneous polynomials on \( (\mathbb{C}^n, \| \cdot \|_\infty) \): for any \( k \geq 1 \), there exists a constant \( D_k \geq 1 \) such that, for any complex \( k \)-homogeneous polynomial \( P(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha \) on \( (\mathbb{C}^n, \| \cdot \|_\infty) \), we have
\[
(4.23) \quad \left( \sum_{|\alpha|=k} |a_\alpha|^{\frac{k+1}{k+1}} \right)^{\frac{k+1}{2k}} \leq D_k \|P\|_\infty,
\]
where \( \|P\|_\infty = \sup_{z\in \mathcal{B}_{\ell^p}} |P(z)| \). In the following, the best constant \( D_k \) in (4.2.3) will be denoted by \( \mathcal{B}_{\ell^p}^{\text{pol}}(\mathbb{C}^k) \) (see [4]). Recently, Bayart, Pellegrino and Seoane-Sepúlveda ([4, Theorem 1.1]) obtained the following estimate: For any \( \varepsilon > 0 \), there exists \( \kappa > 0 \) such that, for any \( k \geq 1 \),
\[
(4.24) \quad \mathcal{B}_{\ell^p}^{\text{pol}}(\mathbb{C}^k) \leq \kappa(1 + \varepsilon)^k.
\]

A Schauder basis \( (x_n) \) of a Banach space \( X \) is said to be unconditional if there is a constant \( C \geq 0 \) such that \( \| \sum_{k=1}^n \varepsilon_k \beta_k x_k \| \leq C \| \sum_{k=1}^n \beta_k x_k \| \) for all \( n \) and \( \beta_1, \ldots, \beta_n, \varepsilon_1, \ldots, \varepsilon_n \in \mathbb{C} \) with \( |\varepsilon_k| \leq 1 \). In particular, the best constant \( C \) is called the unconditional basis constant of \( (x_n) \). If \( X = (\mathbb{C}^n, \| \cdot \|) \) is a Banach space and \( k \in \{1, 2, \ldots \} \), then \( \mathcal{P}(k)X \) stands for the Banach space of all \( k \)-homogeneous polynomials \( P(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha \), \( z \in \mathbb{C}^n \), together with the norm \( \|P\|_{\mathcal{P}(k)X} := \)
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The unconditional basis constant of all monomials $z^\alpha$ with $|\alpha| = k$ is denoted by $\chi_{\text{mon}}(\mathcal{P}(kX))$. For more details on this topic, we refer to [15].

The proof of Theorem 2.15. We divide the proof of this theorem into three cases.

Case 4.3. $n = 1$.

Let $f(z) = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=0}^{\infty} b_j \overline{z}^j \in \mathcal{P} \mathcal{H}_+ \left( \mathbb{B}_{\ell_p^n} \right)$ with $b_0 = 0$. For $z \in \mathbb{B}_{\ell_p^n}$, by (2.7), we have

$$\sum_{j=0}^{\infty} |(a_j + b_j) z^j| = f(0) + \sum_{j=1}^{\infty} |(a_j + b_j) z^j| \leq f(0) + 2(1 - f(0)) \frac{|z|}{1 - |z|},$$

and consequently,

$$\sum_{j=0}^{\infty} \left| (a_j + b_j) \frac{1}{3^j} \right| \leq 1.$$

Next, we prove the sharpness part. For $z \in \mathbb{B}_{\ell_p^n}$, let

$$f(z) = -\frac{2z}{1 - z} = -2 \sum_{k=1}^{\infty} z^k \quad \text{or} \quad f(z) = \frac{-2z}{1 - z}.$$

It is not difficult to know that $\text{Re}(f) \leq 1$ and

$$M(|z|) := \sum_{k=1}^{\infty} 2|z|^k = \frac{2|z|}{1 - |z|},$$

which gives that $M\left(\frac{1}{3}\right) = 1$.

Case 4.4. $n \geq 2$ and $p \in [1, \infty)$.

Step 4.1. We first prove that there is an absolute constant $C$ such that

$$\mathcal{R}_P(\mathbb{B}_{\ell_p^n}) \leq C \left( \frac{\log n}{n} \right)^{1 - \frac{1}{\min\{p, 2\}}}.$$

This step easily follows from the proof of Theorem 2.11 by replacing $\mathcal{R}_{P}^*(\mathbb{B}_{\ell_p^n})$ by $\mathcal{R}_P(\mathbb{B}_{\ell_p^n})$.

Step 4.2. Next, we prove that there is an absolute constant $C$ such that

$$\mathcal{R}_P(\mathbb{B}_{\ell_p^n}) \geq \frac{1}{C} \left( \frac{\log n}{n} \right)^{1 - \frac{1}{\min\{p, 2\}}}.$$

Let $f(z) = \sum a_\alpha z^\alpha + \sum b_\alpha \overline{z}^\alpha \in \mathcal{P} \mathcal{H}_+ \left( \mathbb{B}_{\ell_p^n} \right)$ with $b_0 = 0$. By Theorem 2.7, for all $k \in \{1, 2, \ldots \}$, we have

$$\sum_{|\alpha| = k} |a_\alpha + b_\alpha| |z^\alpha| \leq 2(1 - \text{Re}(f(0))) \chi_{\text{mon}}(\mathcal{P}(kX_p^n)) \|z\|_p^k.$$
Since there exists an absolute constant $C > 0$ such that (see [15, p.144])

$$\chi_{\text{mon}}(\mathcal{P}(k \ell_p^n)) \leq C^k \left(1 + \frac{n}{k}\right)^{(k-1)(1-\frac{1}{\min(p,2)})},$$

we obtain

$$\sum_{\alpha} |a_\alpha + b_\alpha| |z^\alpha| \leq f(0) + 2(1 - f(0)) \sum_{k=1}^{\infty} C^k \left(1 + \frac{n}{k}\right)^{(k-1)(1-\frac{1}{\min(p,2)})} \|z\|_p^k.$$

At last, by using a similar proof process of [15, Theorem 1.1], we obtain the desired result.

**Case 4.5.** $n \geq 2$ and $p = \infty$.

We divide the proof of this situation into two parts (see Claims 1 and 2).

**Claim 1.** $\limsup_{n \to \infty} \frac{\mathcal{R}(\mathcal{P}_n)}{\sqrt{\log n}} \geq 1$.

Let $f(z) = \sum_{\alpha} a_\alpha z^\alpha + \sum_{\alpha} b_\alpha z^\alpha \in \mathcal{P} H^+_{\infty}(\mathbb{B}_\ell_\infty^n)$ with $b_0 = 0$. Elementary computations show that, for $z \in \mathbb{B}_\ell_\infty^n$,

$$\sum_{\alpha} |(a_\alpha + b_\alpha)z^\alpha| \leq f(0) + \sum_{k=1}^{\infty} \|z\|_\infty^k \sum_{|\alpha|=k} |a_\alpha + b_\alpha|.$$

By (4.24) and (2.7), for any $\varepsilon > 0$, there exists $\kappa > 0$ such that, for any $k \geq 1$,

$$\sum_{|\alpha|=k} \left(|a_\alpha + b_\alpha| \frac{2^k}{2^k + 1}\right)^{\frac{k+1}{2k}} \leq \kappa (1 + \varepsilon)^k \sup_{z \in \mathbb{B}_\ell_\infty^n} \left|\sum_{|\alpha|=k} (a_\alpha + b_\alpha)z^\alpha\right|$$

$$\leq 2(1 - f(0))\kappa (1 + \varepsilon)^k.$$

Since the dimension of the space of $k$-homogeneous polynomials in $\mathbb{C}^n$ is $\binom{n+k-1}{k}$, an application of (2.7), (4.26) and Hölder’s inequality in the case $k \geq 2$ to the sum $\sum_{|\alpha|=k} |a_\alpha + b_\alpha|$ gives

$$\sum_{|\alpha|=k} |a_\alpha + b_\alpha| \leq \binom{n+k-1}{k} \left(\sum_{|\alpha|=k} \left(|a_\alpha + b_\alpha| \frac{2^k}{2^k + 1}\right)^{\frac{k+1}{2k}}\right)^{\frac{k+1}{2k}}$$

$$\leq 2(1 - f(0))\kappa (1 + \varepsilon)^k \binom{n+k-1}{k} \left(\frac{n+k-1}{k}\right)^{\frac{k+1}{2k}}.$$

It follows from (2.9) that

$$\binom{n+k-1}{k} \leq e^k \left(1 + \frac{n}{k}\right)^k,$$

which, together with (4.25) and (4.27), implies that
∑ \alpha |(a_\alpha + b_\alpha)z^\alpha| \leq f(0) + 2(1 - f(0)) \sum_{k=1}^{\infty} \kappa \|z\|_\infty^k (1 + \varepsilon)^k (1 + \frac{n}{k})^{\frac{k-1}{k}}.

Let \varepsilon \in (0, 1/2). Set \|z\|_\infty = (1 - 2\varepsilon)^{\sqrt{{\log n}/n}}. Then, as in the proof of [4, Section 6], we obtain that

\sum_{k=1}^{\infty} \kappa \|z\|_\infty \sqrt{e(1 + \varepsilon)}^k (1 + \frac{n}{k})^{\frac{k-1}{k}} \leq \frac{1}{2}

for large enough n.

Hence we conclude from (4.28) and (4.29) that for large enough n, we have

\sum_\alpha |(a_\alpha + b_\alpha)z^\alpha| \leq f(0) + (1 - f(0)) = 1,

which yields that

\mathcal{R}_P(\mathbb{B}_\ell^\infty) \geq (1 - 2\varepsilon)^{\sqrt{{\log n}/n}}

for large enough n.

Claim 2. \limsup_{n \to \infty} \frac{\mathcal{R}_P(\mathbb{B}_\ell^\infty)}{\sqrt{\frac{\log n}{n}}} \leq 1.

It is easy to know that if f \in \mathcal{H}_1(\mathbb{B}_\ell^\infty) with f(0) = 0, then f \in \mathcal{P} \mathcal{H}_+(\mathbb{B}_\ell^\infty). Moreover, we observe that if f \in \mathcal{H}_1(\mathbb{B}_\ell^\infty) with f(0) \neq 0, then e^{-i \arg f(0)} f \in \mathcal{P} \mathcal{H}_+(\mathbb{B}_\ell^\infty). Thus, we can use a similar proof method as in [6, Remark 1] to show that Claim 2 is true. For the sake of completeness, we recall the proof process.

It follows from the Kahane-Salem-Zygmund inequality (see [4, 6]) that there exist a positive constant C and coefficients (c_\alpha)_{|\alpha|=k} with |c_\alpha| = k!/\alpha! for all \alpha such that

\sup_{z \in \mathbb{B}_\ell^\infty} \left| \sum_{|\alpha|=k} c_\alpha z^\alpha \right| \leq C \sqrt{k \log k} (k!)^{\frac{k+1}{2}} n^{\frac{k+1}{2}},

and consequently,

\mathcal{R}_P^k(\mathbb{B}_\ell^\infty) = \sum_{|\alpha|=k} |c_\alpha| \mathcal{R}_P^k(\mathbb{B}_\ell^\infty) \leq \sup_{z \in \mathbb{B}_\ell^\infty} \left| \sum_{|\alpha|=k} c_\alpha z^\alpha \right| \leq C \sqrt{k \log k} (k!)^{\frac{k+1}{2}} n^{\frac{k+1}{2}}.

Hence

\mathcal{R}_P(\mathbb{B}_\ell^\infty) \leq C^{\frac{k}{2}} \left( \sqrt{k \log k} \right)^{\frac{k}{2}} \frac{1}{\sqrt{n}} n^{\frac{k}{2}} (k!)^{\frac{k}{2}}.

If we take k = \lfloor \log n \rfloor (n \geq 3) and use Stirling’s formula, then we can get the desired result.
Therefore, this case follows from Claims 1 and 2. The proof of this theorem is completed.

\[ \Box \]

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