Group analysis of general Burgers–Korteweg–de Vries equations

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The complete group classification problem for the class of (1+1)-dimensional rth order general variable-coefficient Burgers–Korteweg–de Vries equations is solved for arbitrary values of r greater than or equal to two. We find the equivalence groupoids of this class and its various subclasses obtained by gauging equation coefficients with equivalence transformations. Showing that this class and certain gauged subclasses are normalized in the usual sense, we reduce the complete group classification problem for the entire class to that for the selected maximally gauged subclass, and it is the latter problem that is solved efficiently using the algebraic method of group classification. Similar studies are carried out for the two subclasses of equations with coefficients depending at most on the time or space variable, respectively. Applying an original technique, we classify Lie reductions of equations from the class under consideration with respect to its equivalence group. Studying alternative gauges for equation coefficients with equivalence transformations allows us not only to justify the choice of the most appropriate gauge for the group classification but also to construct for the first time classes of differential equations with nontrivial generalized equivalence group such that equivalence-transformation components corresponding to equation variables locally depend on nonconstant arbitrary elements of the class. For the subclass of equations with coefficients depending at most on the time variable, which is normalized in the extended generalized sense, we explicitly construct its extended generalized equivalence group in a rigorous way. The new notion of effective generalized equivalence group is introduced.

1 Introduction

The development of new powerful tools and techniques of group analysis of differential equations during the last decade has essentially extended the range of effectively solvable problems of this branch of mathematics. In particular, it became possible to study admissible transformations and Lie symmetries for systems of differential equations from complex classes parameterized by a few functions of several arguments.

A number of evolution equations that are important in mathematical physics are of the general form

\[ u_t + C(t,x)u u_x = \sum_{k=0}^{r} A^k(t,x)u_k + B(t,x). \] (1)

Here and in the following the integer parameter r is fixed, and \( r \geq 2 \). We require the condition \( CA^r \neq 0 \) guaranteeing that equations from the class (1) are nonlinear and of genuine order r. Throughout the paper we use the standard index derivative notation \( u_t = \partial u/\partial t, \ u_k = \partial^k u/\partial x^k \), and also \( u_0 = u, \ u_x = u_1, \ u_{xx} = u_2 \) and \( u_{xxx} = u_3 \).
The class \( \mathcal{H} \) contains a number of the prominent classical models of fluid mechanics, including

\[
\begin{align*}
&u_t + uu_x = a_2 u_{xx} & \text{Burgers equation,} \\
&u_t + uu_x = a_3 u_{xxx} & \text{Korteweg–de Vries (KdV) equation,} \\
&u_t + uu_x = a_1 u_4 + a_3 u_3 + a_2 u_2 & \text{Kuramoto–Sivashinsky equation,} \\
&u_t + uu_x = a_5 u_5 + a_3 u_3 & \text{Kawahara equation,} \\
&u_t + uu_x = a_r u_r & \text{generalized Burgers–KdV equation,}
\end{align*}
\]

where \( a's \) are constants, and the coefficient of the highest-order derivative is nonzero.

Due to the importance of equations from the class \( \mathcal{H} \), there exist already a lot of papers in which particular equations of the form \( \mathcal{H} \) have been considered in light of their symmetries, integrability, exact solutions, etc. Here we review only papers on admissible transformations and group classification of classes related to the class \( \mathcal{H} \).

Particular subclasses of the class \( \mathcal{H} \) with small values of \( r \) were the subject of a number of papers published over the past twenty-five years. Thus, the equivalence groupoids of the class of variable-coefficient Burgers equations of the form \( u_t + uu_x + f(t,x)u_{xx} = 0 \) with \( f \neq 0 \) and of its subclass of equations with \( f = f(t) \) were constructed by Kingston and Sophocleous in \( [14] \) as sets of point transformations in pairs of equations. In fact, it was implicitly proved there that these classes are normalized in the usual sense. Later such transformations were called allowed \( [10, 47] \) or form-preserving \( [15] \), and they had preceded the notion of admissible transformations, which is of central importance in group analysis of classes of differential equations. Solving of the group classification problem for the above subclass with \( f = f(t) \) was initiated in \( [7, 46] \) and completed in \( [33, 41] \). Most recently, an extended symmetry analysis of the class with \( f = f(t,x) \) considered in \( [14] \) was comprehensively carried out in \( [32] \). The partial preliminary group classification problem for the related class of Burgers equations with sources, which are of the form \( u_t + uu_x = u_{xx} + f(t,x,u) \), with respect to the maximal Lie symmetry group of the Burgers equation was considered in \( [23] \).

In \( [10, 47] \) allowed transformations were computed for the class of variable-coefficient KdV equations of the form \( u_t + f(t,x)u_x + g(t,x)u_{xxx} = 0 \) with \( fg \neq 0 \) and were then used in \( [10] \) to carry out the group classification of this class; see \( [45] \) for a modern interpretation of these results. An attempt to reproduce these results for the class of variable-coefficient Burgers equations of the form \( u_t + f(t,x)u_x + g(t,x)u_{xx} = 0 \) with \( fg \neq 0 \) was made in \( [38] \) supposing that admissible transformations of this class are similar to admissible transformations of its third-order counterpart but in fact the structure of the corresponding equivalence groupoid is totally different from that in \( [10] \). Transformational properties of the class of variable-coefficient KdV equations of the form

\[
u_t + f(t)u u_x + g(t)u_{xxx} + (g(t)x + p(t))u_x + h(t)u + k(t)x + l(t) = 0, \quad fg \neq 0,
\]

were studied in \( [37] \); see also \( [39] \). This class coincides with the class \( \mathcal{K}_1 \) (with the particular value \( r = 3 \)) arising in Section 9 of the present paper. The class \( \mathcal{K}_1^{-3} \) was proved to be normalized in the usual sense and mapped by a family of its equivalence transformations, e.g., to its subclass of equations of the form \( u_t + uu_x + g(t)u_{xxx} = 0 \) with \( g \neq 0 \), which is also normalized in the usual sense, and solving the group classification problem in the class \( \mathcal{K}_1^{-3} \) is equivalent to that in the subclass; cf. Proposition 7 and Section 9. Variable-coefficient generalizations of the Kawahara equation were studied in a similar way in \( [17, 18] \). The group classification of Galilei-invariant equations of the form \( u_t + uu_x = F(u_r) \) was carried out in \( [5, 8] \).

Unfortunately, the results obtained in many papers were incomplete or even faulty; for a partial listing of papers with such results on variable-coefficient Korteweg–de Vries equations, we refer to \( [37] \). It is thus appropriate to solve the group classification problem for the general class \( \mathcal{H} \) systematically for the first time.
More general classes of evolutions equations, which include the class (1), were also considered in the literature. Contact symmetries of (1+1)-dimensional evolution equations were studied by Magadeev [24]. More specifically, Magadeev proved that a (1+1)-dimensional evolution equation admits an infinite-dimensional Lie algebra of contact symmetries if and only if it is linearizable by a contact transformation. He also classified, up to contact equivalence transformations of evolution equations, realizations of finite-dimensional Lie algebras by contact vector fields with the independent variables \((t, x)\) and the dependent variable \(u\) each of which is the maximal contact invariance algebras of a non-linearizable evolution equation. At the same time, evolution equations admitting these realizations as maximal contact invariance algebra were not constructed. Moreover, singling out the classification of contact symmetries of equations from a particular subclass of the entire class of (1+1)-dimensional evolution equations is in general a more challenging problem than the direct classification of contact symmetries for equations from the subclass. A claim similar to the above one on results of [24] is also true for classifications of low-order evolution equations (with \(r = 2, 3, 4\)) considered, e.g., in [1, 11, 12, 19].

In the present paper we solve the complete group classification problem of the class (1) for any fixed value of \(r \geq 2\). An even more important result of the paper is the construction, in the course of the study of admissible transformations within subclasses of the class (1), of several examples of classes with specific properties, the existence of which was in doubt for a long time. These examples give us unexpected insights into the general theory of equivalence transformations within classes of differential equations. In particular, they justify introducing the notion of effective generalized equivalence group.

We also point out that the class of linear (in general, variable-coefficient) evolution equations is obtained from (1) by setting \(C = 0\). This class has transformational properties different from those of the class (1), and equations with \(C = 0\) and \(C \neq 0\) are not related to each other by point or contact transformations. Moreover, the class of linear equations was completely classified in [21] and [30, pp. 114–118] for \(r = 2\) and in [3] for \(r > 2\). (The last paper enhanced and extended results of [11, Section III] on \(r = 3\) and of [13] on \(r = 4\).) This is why it is appropriate to exclude linear equations from the present consideration.

The further structure of this paper is the following. In Section 2 we present sufficient background information on point transformations in classes of differential equations and on the algebraic method of group classification. Since the class (1) is normalized in the usual sense, particular emphasis is given to describing the concept of normalization of a class of differential equations. The notion of effective generalized equivalence group is introduced there for the first time. Using known results on admissible transformations of the entire class of \(r\)th order evolution equations and its wide subclasses, in Section 3 we compute the equivalence groupoids for the class (1) and its two subclasses singled out by the arbitrary-element gauges \(C = 1\) and \((C,A_1) = (1,0)\), which are realized by families of equivalence transformations. Due to each of the above classes being normalized in the usual sense, its equivalence groupoid coincides with the subgroupoid induced by the usual equivalence group of this class. The solution of the group classification problem for the class (1) is shown to reduce to that for its gauged subclass associated with the constraint \((C,A_1) = (1,0)\) and referred to in the paper as the class (5). Moreover, this subclass turns out to be the most convenient for carrying out the group classification since it is normalized in the usual sense and maximally gauged. The consideration of alternative arbitrary-element gauges in Section 4 additionally justifies the selection of the subclass with \((C,A_1) = (1,0)\) for group classification. This also allows us to construct for the first time examples of classes with nontrivial generalized equivalence groups, where transformation components for variables locally depend on nonconstant arbitrary elements. The determining equations for Lie symmetries of equations from the class (1) are derived and preliminarily studied in Section 5. These results are used in Section 6 for analyzing properties of appropriate (for group classification) subalgebras of the projection of the equivalence algebra of the class (5) to the space of equation variables. The group classification of the gauged subclass (5) and thus the entire class (1) is completed in Section 7 via classifying
the appropriate subalgebras and finding the corresponding values of the arbitrary-element tuple. In Section 8, we discuss the optimality of chosen inequivalent representatives for cases of Lie symmetry extensions. The subclass $\mathcal{K}_3$ of equations from the class $\mathcal{L}$ with coefficients depending at most on $t$ is the object of the study in Sections 9. We exhaustively describe the equivalence groupoid of this subclass, gauge its arbitrary elements by equivalence transformations and single out a complete list of inequivalent Lie symmetry extensions within this subclass from that for the class $\mathcal{L}$. In fact, we begin with the wider subclass $\mathcal{K}_1$, where $A^1$ and $B$ may affinely depend on $x$, since this subclass is normalized in the usual sense with respect to its nice usual equivalence group whereas the subclass $\mathcal{K}_2$, where only $B$ may affinely depend on $x$, and the subclass $\mathcal{K}_3$ give new nontrivial examples of classes normalized only in the generalized sense and only in the extended generalized sense, respectively. Similar results for the subclass $\mathcal{F}_1$ of equations from the class $\mathcal{L}$ with coefficients depending at most on $x$ are obtained in Section 10. Since this subclass is not normalized in any appropriate sense, in order to describe its equivalence groupoid $\mathcal{G}_{\mathcal{F}_1}$, it is necessary to solve a complicated classification problem for values of the arbitrary-element tuple that admit nontrivial admissible transformations, which are not generated by transformations from the corresponding usual equivalence group. It turns out that the equivalence groupoid $\mathcal{G}_{\mathcal{F}_1}$ has an interesting structure related to Lie symmetry extensions in the subclass $\mathcal{F}_1$ although the subclass $\mathcal{F}_1$ is far from even being semi-normalized. Lie reductions of equations from the class $\mathcal{L}$ are classified in Section 11 with respect to the usual equivalence group of this class using the fact that it is normalized in the usual sense. Other possibilities for finding exact solutions of these equations are also discussed. The conclusions of the paper are presented in the final Section 12.

2 Algebraic method of group classification

Basic notions and results underlying the algebraic method of group classification in its modern advanced form as will be used below were extensively discussed in [2] [3] [9] [34] [35] [36], to which we refer for further details. Examples of applying various versions of the algebraic method to particular group classification problems can be found in [1] [9] [10] [11] [13] [24] [27] [49]. In this section we not only review the needed part of the known theory on symmetry analysis in classes of differential equations but also present some new notions and results of this field.

Let $\mathcal{L}_g$ denote a system of differential equations of the form $L(x,u^{(r)},\theta(x,u^{(r)})) = 0$. Here, $x = (x_1, \ldots, x_n)$ are the $n$ independent variables, $u = (u^1, \ldots, u^m)$ are the $m$ dependent variables, and $L$ is a tuple of differential functions in $u$. We use the standard short-hand notation $u^{(r)}$ to denote the tuple of derivatives of $u$ with respect to $x$ up to order $r$, which also includes the $u$’s as the derivatives of order zero. The system $\mathcal{L}_g$ is parameterized by the tuple of functions $\theta = (\theta^1(x,u^{(r)}), \ldots, \theta^k(x,u^{(r)}))$, called the arbitrary elements, which runs through the solution set $\mathcal{S}$ of an auxiliary system of differential equations and inequalities in $\theta$, $S(x,u^{(r)},\theta(q)(x,u^{(r)})) = 0$ and, e.g., $\Sigma(x,u^{(r)},\theta(q)(x,u^{(r)})) \neq 0$. Here, the notation $\theta(q)$ encompasses the partial derivatives of the arbitrary elements $\theta$ up to order $q$ with respect to both $x$ and $u^{(r)}$. Thus, the class of (systems of) differential equations $\mathcal{L}|\mathcal{S}$ is the parameterized family of systems $\mathcal{L}_g$’s, such that $\theta$ lies in $\mathcal{S}$.

For the specific class of general Burgers–KdV equations $\mathcal{L}$ considered below, we have $n = 2$, $m = 1$, and, in the more traditional notation, $x_1 = t$ and $x_2 = x$. The tuple of arbitrary elements is $\theta = (A^0, \ldots, A^r, B, C)$, which runs through the solution set of the auxiliary system

$$A^k_{u^\alpha} = 0, \quad k = 0, \ldots, r, \quad B_{u^\alpha} = 0, \quad C_{u^\alpha} = 0, \quad |\alpha| \leq r, \quad CA^r \neq 0,$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \alpha_2$, and $u^\alpha = \partial^{(\alpha)}u/\partial t^{\alpha_1}\partial x^{\alpha_2}$. Satisfying the auxiliary differential equations is equivalent to the fact that the arbitrary elements do not depend on derivatives of $u$. The inequality $A^rC \neq 0$ ensures that equations from the class $\mathcal{L}$ are both nonlinear and of order $r$. 
Group classification of differential equations is based on studying how systems from a given class are mapped to each other. This study is formalized in the notion of admissible transformations, which constitute the equivalence groupoid of the class $\mathcal{L}|_S$. An admissible transformation is a triple $(\theta, \hat{\theta}, \varphi)$, where $\theta, \hat{\theta} \in S$ are arbitrary-element tuples associated with systems $\mathcal{L}_\theta$ and $\mathcal{L}_{\hat{\theta}}$ from the class $\mathcal{L}|_S$ that are similar to each other, and $\varphi$ is a point transformation in the space of $(x, u)$ that maps $\mathcal{L}_\theta$ to $\mathcal{L}_{\hat{\theta}}$.

A related notion of relevance in the group classification of differential equations is that of equivalence transformations. Usual equivalence transformations are point transformations in the joint space of independent variables, derivatives of $u$ up to order $r$ and arbitrary elements that are projectable to the space of $(x, u^{(r')})$ for each $r' = 0, \ldots, r$, respect the contact structure of the $r$th order jet space coordinatized by the $r$-jets $(x, u^{(r)})$ and map every system from the class $\mathcal{L}|_S$ to a system from the same class. The Lie (pseudo)group constituted by the equivalence transformations of $\mathcal{L}|_S$ is called the usual equivalence group of this class and denoted by $G^\sim$. If the arbitrary elements depend at most on derivatives of $u$ up to order $\hat{r} < r$, then one can assume that equivalence transformations act in the space of $(x, u^{(\hat{r})}, \theta)$ instead of the space of $(x, u^{(r)}, \theta)$.

The usual equivalence group $G^\sim$ gives rise to a subgroupoid of the equivalence groupoid $G^\sim$ since each equivalence transformation $T \in G^\sim$ generates a family of admissible transformations parameterized by $\theta$,

$$G^\sim \ni T \rightarrow \{ (\theta, T\theta, \pi_\ast T) \mid \theta \in S \} \subset G^\sim.$$  

Here $\pi$ denotes the projection of the space of $(x, u^{(r)}, \theta)$ to the space of equation variables only, $\pi(x, u^{(r)}, \theta) = (x, u)$. The pushforward $\pi_\ast T$ of $T$ by $\pi$ is then just the restriction of $T$ to the space of $(x, u)$.

The usual equivalence algebra $\mathfrak{g}^\sim$ of the class $\mathcal{L}|_S$ is an algebra associated with the usual equivalence group $G^\sim$ and constituted by the infinitesimal generators of one-parameter groups of equivalence transformations. These infinitesimal generators are vector fields in the joint space of $(x, u^{(r)}, \theta)$ that are projectable to $(x, u^{(r')})$ for each $r' = 0, \ldots, r$. Since equivalence transformations respect the contact structure on the $r$th order jet space, the vector fields from $\mathfrak{g}^\sim$ inherit this compatibility, meaning their projections to the space of $(x, u^{(\hat{r})})$ coincide with the $\hat{r}$th order prolongations of the associated projections to the space of $(x, u)$.

In the case when the arbitrary elements $\theta$’s are functions of $(x, u)$ only, we can assume that equivalence transformations of the class $\mathcal{L}|_S$ are point transformations of $(x, u, \theta)$ mapping every system from the class $\mathcal{L}|_S$ to a system from the same class. The projectability property for equivalence transformations is neglected here. Then these equivalence transformations constitute a Lie (pseudo)group $G^\sim$ called the generalized equivalence group of the class $\mathcal{L}|_S$. See the first discussion of this notion in [25] [26] with no relevant examples and the further development in [34] [36]. Often the generalized equivalence group coincides with the usual one; this situation is considered as trivial. Similar to usual equivalence transformations, each element of $G^\sim$ generates a family of admissible transformations parameterized by $\theta$,

$$G^\sim \ni T \rightarrow \{ (\theta', T\theta', \pi_\ast(T\mid_{\theta=\theta'(x,u)}) ) \mid \theta' \in S \} \subset G^\sim,$$

and thus the generalized equivalence groupoid $\mathcal{G}^\sim$ also generates a subgroupoid $\mathcal{H}$ of the equivalence groupoid $G^\sim$.

**Definition 1.** We call any minimal subgroup of $G^\sim$ that generates the same subgroupoid of $G^\sim$ as the entire group $G^\sim$ does an effective generalized equivalence group of the class $\mathcal{L}|_S$.

The uniqueness of an effective generalized equivalence group is obvious if the entire group $G^\sim$ is effective itself; cf. Remark 11 below. At the same time, there exist classes of differential equations, where effective generalized equivalence groups are proper subgroups of the corresponding
generalized equivalence groups that are even not normal. Hence each of these effective generalized equivalence groups is not unique since it differs from some of subgroups non-identically similar to it, and all of these subgroups are also effective generalized equivalence groups of the same class. See the discussion of particular examples in Remark 28 below.

Suppose that the class $\mathcal{L}_S$ possesses parameterized non-identity usual equivalence transformations and some of its arbitrary elements are constants. Then this class necessarily admits purely generalized equivalence transformations. Indeed, we can set all parameters of elements from the usual equivalence group $G^\sim$ depending on constant arbitrary elements, which gives generalized equivalence transformations. The set $\hat{G}_0^\sim$ of such transformations is a subgroup of the generalized equivalence group $G^\sim$. If $\hat{G}_0^\sim = G^\sim$, the usual equivalence group $G^\sim$ is an effective generalized equivalence group of the class $\mathcal{L}_S$.

The generalized equivalence algebra $\bar{g}^\sim$ and an effective generalized equivalence algebra of the class $\mathcal{L}_S$ are the algebras associated with the generalized equivalence group $\hat{G}^\sim$ and with an effective generalized equivalence group of this class and are constituted by the infinitesimal generators of one-parameter subgroups of these groups, respectively. These infinitesimal generators are vector fields in the joint space of $(x, u, \theta)$.

The property for equivalence transformations to be point transformations with respect to arbitrary elements can also be weakened. We formally extend the arbitrary-element tuple $\theta$ of the class $\mathcal{L}_S$ with virtual arbitrary elements that are related to initial arbitrary elements by differential equations and thus expressed via initial arbitrary elements in a nonlocal way. Denote the reparameterized class by $\hat{\mathcal{L}}_S$. Suppose that the usual (resp. generalized or effective generalized) equivalence group $\hat{G}^\sim$ of $\hat{\mathcal{L}}_S$ induces the maximal subgroupoid of the equivalence groupoid $G^\sim$ among the classes obtained from $\mathcal{L}_S$ by similar reparameterizations, and the extension of the arbitrary-element tuple $\theta$ for $\hat{\mathcal{L}}_S$ is minimal among the reparameterized classes giving the same subgroupoid of $G^\sim$ as $\hat{\mathcal{L}}_S$. Then we call the group $\hat{G}^\sim$ an extended equivalence group (resp. an extended generalized equivalence group) of the class $\mathcal{L}_S$.

A point symmetry transformation of a system $\mathcal{L}_\theta$ is a point transformation in the space of $(x, u)$ that preserves the solution set of $\mathcal{L}_\theta$. Each point symmetry transformation $\varphi$ of $\mathcal{L}_\theta$ gives rise to the single admissible transformation $(\theta, \theta, \varphi)$ of the class $\mathcal{L}_S$. The point symmetry transformations of the system $\mathcal{L}_\theta$ constitute the maximal point symmetry group $G_\theta$ of this system. The common part $G^\cap$ of all $G_\theta$ is called the kernel of maximal point symmetry groups of systems from the class $\mathcal{L}_S$, $G^\cap := \bigcap_{\theta \in S} G_\theta$. The infinitesimal counterparts of the maximal point symmetry group $G_\theta$ and the kernel $G^\cap$, which are called the maximal Lie invariance algebra $g_\theta$ of $\mathcal{L}_\theta$ and the kernel Lie invariance algebra $g^\cap$ of systems from $\mathcal{L}_S$, consist of the vector fields in the space of $(x, u)$ generating one-parameter subgroups of $G_\theta$ and $G^\cap$, respectively.

Group analysis of differential equations becomes much simpler when working with infinitesimal counterparts of objects consisting of point transformations. Thus, the problem on Lie (i.e., continuous point) symmetries of a system $\mathcal{L}_\theta$ reduces to constructing the maximal Lie invariance algebra $g_\theta$ and, therefore, is linear in contrast to the similar problem on general point symmetries. Under certain quite natural conditions on the system $\mathcal{L}_\theta$, the infinitesimal invariance criterion states that a vector field $Q$ in the space of $(x, u)$ belongs to the maximal Lie invariance algebra $g_\theta$ if and only if the condition $Q^{(r)} L(x, u^{(r)}, \theta^{(r)}(x, u^{(r)})) = 0$ is identically satisfied on the manifold $\mathcal{L}_\theta$ defined by the system $\mathcal{L}_\theta$ and its differential consequences in the jet space $J^{(r)}$. Here $Q^{(r)}$ is the standard $r$th order prolongation of the vector field $Q$, see [29, 30] and Section 5.

The group classification problem for the class $\mathcal{L}_S$ is to list all $G^\sim$-inequivalent values for $\theta \in S$ such that the associated systems, $\mathcal{L}_\theta$, admit maximal Lie invariance algebras, $g_\theta$, that are wider than the kernel Lie invariance algebra $g^\cap$. Further taking into account additional point equivalences between obtained cases, provided such additional equivalences exist, one solves the group classification problem up to $G^\sim$-equivalence. Restricting the consideration to Lie symmetries is important for the group classification problem to be well-posed within the framework of classes of differential equations.
When applied to systems from a class $\mathcal{L}\upharpoonright_S$, the infinitesimal invariance criterion yields, after splitting with respect to the parametric derivatives of $u$, a system of the determining equations for the components of Lie symmetry generators of these systems, which is in general parameterized by the arbitrary-element tuple $\theta$. It is quite common that there is a subsystem of the determining equations that does not involve the tuple of arbitrary elements $\theta$ and hence may be integrated in a regular way. The remaining part of the determining equations that explicitly involve the arbitrary elements is referred to as the system of classifying equations. In this setting, the group classification problem reduces to the exhaustive investigation of the classifying equations. The direct integration of the classifying equations up to $G^r$-equivalence is usually only possible for classes of the simplest structure, e.g., classes involving only constants or functions of a single argument as arbitrary elements, see, e.g., examples in \[30\]. Since most classes of interest in applications are of more complicated structure, various methods have to be used, which at least enhance the direct method \[28\] \[42\] \[43\].

The most advanced classification techniques rest on the study of algebras of vector fields associated with systems from the class $\mathcal{L}\upharpoonright_S$ under consideration and constitute, in total, the algebraic method of group classification. For this method to be really effective, the class $\mathcal{L}\upharpoonright_S$ has to possess certain properties, which are conveniently formulated in terms of various notions of normalization. The class of differential equations $\mathcal{L}\upharpoonright_S$ is normalized in the usual (resp. generalized, extended, extended generalized) sense if the subgroupoid induced by its usual (resp. generalized, extended, extended generalized) equivalence group coincides with the entire equivalence groupoid $G^\sim$ of $\mathcal{L}\upharpoonright_S$.

The normalization of $\mathcal{L}\upharpoonright_S$ in the usual sense is equivalent to the following conditions. The transformational part $\varphi$ of each admissible transformation $(\theta', \theta'', \varphi) \in G^\sim$ does not depend on the fixed initial value $\theta'$ of the arbitrary-element tuple $\theta$ and, therefore, is appropriate for any initial value of $\theta$. Moreover, the prolongation of $\varphi$ to the space of $(x, u(x))$ and the further extension to the arbitrary elements according to the relation between $\theta'$ and $\theta''$ gives a point transformation in the joint space of $(x, u(x), \theta)$. Then $G_{\theta} \leq \pi_* G^\sim$ and $g_{\theta} \trianglelefteq \pi_* g^\sim$ for any $\theta \in S$, and hence the group classification of the class $\mathcal{L}\upharpoonright_S$ reduces to the classification of certain $G^\sim$-inequivalent subalgebras of the equivalence algebra $g^\sim$ or, equivalently, to the classification of certain $\pi_* G^\sim$-inequivalent subalgebras of the projection $\pi_* g^\sim$.

If the class $\mathcal{L}\upharpoonright_S$ is normalized in the generalized sense, the expression for transformational parts of admissible transformations may involve arbitrary elements but only in a quite specific way. The equivalence groupoid is partitioned into families of admissible transformations parameterized by the source arbitrary-element tuple, and the transformational parts of admissible transformations from each of these families jointly give, after the extension to the arbitrary elements according to the relation between the corresponding source and target arbitrary elements, a point transformation in the joint space of $(x, u, \theta)$. Then $G_{\theta'} \leq \pi_* (G^\sim|_{\theta=\theta'(x,u)})$ and $g_{\theta'} \leq \pi_* (g^\sim|_{\theta=\theta'(x,u)})$ for any $\theta' \in S$.

The class $\mathcal{L}\upharpoonright_S$ is called semi-normalized in the usual sense if for any $(\theta', \theta'', \varphi) \in G^\sim$ there exist $T \in G^\sim$, $\varphi' \in G_{\theta'}$ and $\varphi'' \in G_{\theta''}$ such that $\theta'' = T \theta'$ and $\varphi = \varphi''(\pi_* T)\varphi'$. One of the transformations $\varphi'$ and $\varphi''$ can always be assumed to coincide with the identity transformation. Semi-normalization in the generalized sense is defined in a similar way. Roughly speaking, a class is semi-normalized in a certain sense if its equivalence groupoid is generated by its relevant equivalence group jointly with point symmetry groups of systems from this class. Each normalized class is semi-normalized in the same sense, and the converse is not in general true. There are also more sophisticated notions, uniform semi-normalization and weak uniform semi-normalization, which mediate the notion of normalization and semi-normalization \[19\] \[20\].

To establish the normalization properties of the class $\mathcal{L}\upharpoonright_S$ one should compute its equivalence groupoid $G^\sim$, which is realized using the direct method. Here one fixes two arbitrary systems from the class, $L_0: L(x, u(x), \theta(x, u(x))) = 0$ and $L_0^\sim: L(\tilde{x}, \tilde{u}(\tilde{x}), \tilde{\theta}(\tilde{x}, \tilde{u}(\tilde{x}))) = 0$, and aims to find the (nondegenerate) point transformations, $\varphi$: $\tilde{x}_i = X^i(x, u)$, $\tilde{u}^a = U^a(x, u)$, $i = 1, \ldots, n,$
a = 1, \ldots, m, \text{ connecting them. For this, one changes the variables in the system } L_a \text{ by expressing the derivatives } \tilde{u}^{(r)} \text{ in terms of } u^{(r)} \text{ and derivatives of the functions } X^i \text{ and } U^a \text{ as well as by substituting } X^i \text{ and } U^a \text{ for } \tilde{x}_i \text{ and } \tilde{u}^a \text{, respectively. The requirement that the resulting transformed system has to be satisfied identically for solutions of } L_a \text{ leads to the system of determining equations for the components of the transformation } \varphi.

In the case of a single dependent variable (m = 1), all the above notions involving point transformations can be directly extended to contact transformations.

3 Equivalence groupoid

We now compute the equivalence groupoid and equivalence group of the class (1) using the direct method. Equivalence transformations will be used to find an appropriate gauged subclass of (1) that is suitable for carrying out the complete group classification. The presentation closely follows [3]. In particular, it is convenient to start with the wide superclass of general (1+1)-dimensional rth order (r ≥ 2) evolution equations of the form

\[ u_t = H(t, x, u_0, \ldots, u_r), \quad H_{u_r} \neq 0, \quad (2) \]

and sequentially narrowing it until the class (1) and its gauged subclasses are reached. The advantage of this method is that one can infer the normalization properties of the class (1) by keeping track of the normalization properties of the class (2) and its relevant subclasses. This not only gives restrictions on the transformational part of admissible transformations in the class (2) and its subclasses, but also leads to a more and more constrained relation between the initial and target arbitrary elements until this relation is sufficiently specified.

It was established in [24] that a contact transformation of the independent variables (t, x) and the dependent variable u connects two fixed equations from the class (2) if and only if its components are of the form \( \tilde{t} = T(t), \tilde{x} = X(t, x, u, u_x) \) and \( \tilde{u} = U(t, x, u, u_x) \) provided that the usual nondegeneracy assumption and contact condition hold,

\[ T_t \neq 0, \quad \text{rank} \left( \begin{array}{ccc} X_x & X_u & X_{u_x} \\ U_x & U_u & U_{u_x} \end{array} \right) = 2 \quad \text{and} \quad (U_x + U_u u_x)X_{u_x} = (X_x + X_u u_x)U_{u_x}. \]

The prolongation of the transformation to the first derivatives is given by

\[ \tilde{u}_{\tilde{x}} = V, \quad \tilde{u}_t = \frac{U_u - X_u V}{T_t} u_t + \frac{U_t - X_t V}{T_t}, \quad \text{where} \quad V = \frac{U_x + U_u u_x}{X_x + X_u u_x} \quad \text{or} \quad V = \frac{U_{u_x}}{X_{u_x}} \]

if \( X_x + X_u u_x \neq 0 \) or \( X_{u_x} \neq 0 \), respectively. Such transformations prolonged to the arbitrary element H according to

\[ \tilde{H} = \frac{U_u - X_u V}{T_t} H + \frac{U_t - X_t V}{T_t} \]

constitute the contact usual equivalence group of the class (2) and thus this class is normalized in the usual sense with respect to contact transformations. It is also normalized in the usual sense with respect to point transformations. The point equivalence groupoid and the point usual equivalence group are singled out from their contact counterparts by the condition \( X_{u_x} = U_{u_x} = 0 \).

Consider the subclass E of the class (2) singled out by the constraints

\[ H_{u_k u_k} = 0, \quad k = 1, \ldots, r, \quad H_{u_{r-l} u_l} = 0, \quad l = 1, \ldots, r - 1, \]

cf. [41]. Due to the constraints \( H_{u_k u_k} = 0, \) \( k = 2, \ldots, r \), it follows that contact admissible transformations in the subclass E are induced by point admissible transformations. In other
words, the contact equivalence groupoid of $\mathcal{E}$ coincides with the first prolongation of the point equivalence groupoid of $\mathcal{E}$. Then the constraints $H_{u_1} = 0$ and $H_{u_1} = 0, l = 1, \ldots, r - 1$, successively imply $X_u = 0$ and $U_u = 0$ for any admissible transformation in $\mathcal{E}$, i.e., its transformational part is of the form

$$\tilde{i} = T(t), \quad \tilde{x} = X(t, x), \quad \tilde{u} = U^1(t, x)u + U^0(t, x) \quad \text{with} \quad T_tX_uU^1 \neq 0. \quad (3)$$

The prolongations of these transformations to the arbitrary element $H$ constitute the usual equivalence group of class $\mathcal{E}$. Therefore, the class $\mathcal{E}$ is normalized in the usual sense.

To single out the class $[1]$, we should set more constraints for $H$. The complete system of these constraints is given by

$$H_{u_k} = 0, \quad 1 \leq k \leq l \leq r, \quad (k, l) \neq (0, 1), \quad H_{u_r} \neq 0, \quad H_{u_0u_1} \neq 0.$$ 

Then we should also reparameterize the obtained subclass, assuming $\theta = (A^0, \ldots, A^r, B, C)$ as the tuple of arbitrary elements instead of $H$. Using the direct method for computing the equivalence groupoid of the class $[1]$, we first fix two arbitrary equations $L_\theta$ and $L_\bar{\theta}$ from the class $[1]$ and require that they are connected through a point transformation $\varphi$ of the form $\tilde{\theta} = \tilde{\theta}(\theta)$. This particular form can be posed for admissible transformations since the class $[1]$ is a subclass of the normalized class $\mathcal{E}$. It is thus necessary to re-express the jet variables $(\tilde{i}, \tilde{x}, \tilde{u}(\tau))$ in terms of $(t, x, u(\tau))$. In view of $(3)$, the expressions for the transformed total derivative operators are

$$D_{\tilde{i}} = \frac{1}{T_t} \left( D_t - \frac{X_t}{X_x} D_x \right), \quad D_x = \frac{1}{X_x} D_x.$$ 

Substituting the expressions for the transformed values into $L_\bar{\theta}$ yields an intermediate equation $\tilde{\mathcal{L}}$. Because the equations $L_\theta$ and $L_\bar{\theta}$ are by assumption connected by $\varphi$, the equation $\tilde{\mathcal{L}}$ has to be satisfied by all solutions of $L_\theta$. We assume $u_l$ as the leading derivative in $L_\theta$ and substitute the expression for $u_l$ obtained from $L_\bar{\theta}$ into $\tilde{\mathcal{L}}$. This leads to an identity, which can be split with respect to the parametric derivatives $u_0, \ldots, u_r$. The condition that the coefficient of $u^2$ in $\tilde{\mathcal{L}}$ has to be zero requires $U^1_\tau = 0$. Collecting the other coefficients of powers of parametric derivatives, we derive the formulas pointwise relating $\theta$ and $\tilde{\theta}$ with no constraints for $T, X, U^1$ and $U^0$. These formulas are quite cumbersome (although obtainable using the Faà di Bruno’s formula). In addition, they are not needed at the present stage since we can first fix a suitable gauged subclass of the class $[1]$. To do this, we only need the transformation component for the arbitrary element $C = C(t, x)$, which is readily obtained without using Faà di Bruno’s formula,

$$\tilde{\mathcal{C}} = \frac{X_x}{T_tU^1}C.$$

**Proposition 2.** The class $[1]$ is normalized in the usual sense. Its usual equivalence group $G$ consists of the transformations in the joint space of $(t, x, u, \theta)$ whose $(t, x, u)$-components are of the form

$$\tilde{i} = T(t), \quad \tilde{x} = X(t, x), \quad \tilde{u} = U^1(t, x)u + U^0(t, x),$$

where $T = T(t), X = X(t, x), U^1 = U^1(t)$ and $U^0 = U^0(t, x)$ are arbitrary smooth functions of their arguments such that $T_tX_uU^1 \neq 0$. We can now use the family of equivalence transformations parameterized by the arbitrary element $C$, where

$$\tilde{i} = t, \quad \tilde{x} = \int_{x_0}^x dy \frac{C(t, y)}{C(t, \tilde{y})}, \quad \tilde{u} = u,$$

to map the class $[1]$ onto its subclass singled out by the constraint $C = 1$. Under this gauging we derive that $X_x = T_tU^1$ and thus $X_{xx} = 0$, i.e., $X = X^1(t)x + X^0(t)$ and $U^1 = X^1/T_t$. The gauged subclass is still normalized in the usual sense. Moreover, due to the most prominent equations from the class $[1]$ being of the form with $C = 1$, this gauge is quite natural.
Remark 3. Given a class of differential equations, if a subclass is singled from it by constraints with explicit expressions for some arbitrary elements, we reparameterize this subclass using the reduced tuple of arbitrary elements, which is obtained from the complete tuple by excluding the constrained arbitrary elements. For example, under the gauge $C = 1$ we can assume that the tuple of arbitrary elements for the corresponding subclass is $(A^0, \ldots, A^r, B)$. With the restrictions on $T, X$ and $U$ obtained thus far, we now complete the procedure for finding the equivalence groupoid of the subclass associated with gauge $C = 1$, which consists of equations of the form

$$
 u_t + uu_x = \sum_{k=0}^r A^k(t, x)u_k + B(t, x). \tag{4}
$$

Transforming the equations from the gauged subclass, we find

$$
\frac{1}{T_t} \left( U^1 u_t + U^t u + U^0_0 - \frac{X_t}{X^1} (U^1 u_x + U^0_U) \right) + \frac{1}{X^1} (U^1 u + U^0_0)(U^1 u_x + U^0_U)
$$

$$
= \sum_{k=0}^r \frac{A^k}{(X^1)^k} (U^1 u_k + U^0_U) + \hat{B},
$$

which can be written, after substituting for $u_t$ in view of the equation (4), as

$$
\sum_{k=0}^r A^k u_k + B + \left( \frac{U^0}{U^1} - \frac{X_t}{X^1} \right) u_x + \left( \frac{U^1}{U^1} + \frac{U^0}{U^1} \right) u + \frac{U^0}{X^1} - \frac{X^0}{X^1} U^1 + \frac{U^0 U^0}{(U^1)^2}
$$

$$
= \sum_{k=0}^r \left[ \frac{T_t}{(X^1)^k} \hat{A}^k u_k + \frac{T_t}{U^1} \left( \hat{B} + \frac{\hat{A}^k}{(X^1)^k} U^0_U \right) \right].
$$

Splitting this equation with respect to $u_k$ yields the transformation components for the arbitrary elements. We have thus proved the following theorem.

Theorem 4. The class (4) of reduced $(1+1)$-dimensional general $r$th order Burgers–KdV equations with $C = 1$ is normalized in the usual sense. Its usual equivalence group $G^r$ is constituted by the transformations of the form

$$
\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \quad \tilde{u} = \frac{X^1}{T_t} u + U^0(t, x),
$$

$$
\hat{A}^j = \frac{(X^1)^j}{T_t} A^j, \quad \hat{A}^0 = \frac{X^1}{T_t} A^1 + U^0 - \frac{X^0_x + X^0_t}{T_t}, \quad \hat{A}^0 = \frac{1}{T_t} \left( A^0 + \frac{X^1}{X^1} T_t + \frac{T_t}{X^1} U^0_x \right),
$$

$$
\hat{B} = \frac{X^1}{(T_t)^2} B + \frac{U^0}{T_t} + \frac{U^0_U}{X^1} \left( U^0 - \frac{X^0_x + X^0_t}{T_t} \right) - \sum_{k=0}^r \frac{U^0_U}{(X^1)^k} \hat{A}^k,
$$

where $j = 2, \ldots, r$, and $T = T(t), X^1 = X^1(t), X^0 = X^0(t)$ and $U^0 = U^0(t, x)$ are arbitrary smooth functions of their arguments such that $T_t X^1 \neq 0$.

At this stage, it is convenient to introduce one more gauge. In particular, the family of equivalence transformations parameterized by the arbitrary element $A^1$ with $T = t, X^1 = 1, X^0 = 0$ and $U^0 = -A^1$ maps the associated gauged subclass (4) with $C = 1$ to the subclass of (4) with $C = 1$ and $A^1 = 0$. This gauging implies that $U^0 = (X^1_x + X^0_t)/T_t$. The corresponding gauged subclass consisting of equations of the form

$$
L^k: \quad u_t + uu_x = \sum_{j=2}^r A^j(t, x)u_j + A^0(t, x)u + B(t, x), \tag{5}
$$
where \( A' \neq 0 \) and \( \kappa = (A^0, A^2, \ldots, A^r, B) \) is the reduced arbitrary-element tuple, is still normalized in the usual sense. This is the gauged subclass that is appropriate for solving the complete group classification problem for the class \((1)\). This leads to the following theorem.

**Theorem 5.** The class \((1)\) of reduced \((1+1)\)-dimensional general \( r \)th order Burgers–KdV equations, which is singled out from the class \((1)\) by the gauge \((C, A^1) = (1, 0)\), is normalized in the usual sense. Its usual equivalence group \( G^\sim \) consists of the transformations of the form

\[
\begin{align*}
\bar{t} &= T(t), \quad \bar{x} = X^1(t)x + X^0(t), \quad \bar{u} = \frac{X^1}{T^1}u + \frac{X^0}{T^1}x + \frac{X^0}{T^1} \tag{6a} \\
\tilde{A}_j &= \left(\frac{X^1}{T^1}\right)^2 A_j, \quad \tilde{A}^0 = \frac{1}{T^1} \left( A^0 + 2 \frac{X^1}{X^1} - \frac{T^0}{T^1} \right), \tag{6b} \\
\tilde{B} &= \frac{X^1}{(T^1)^2} B + \frac{1}{T^1} \left( \frac{X^1}{T^1} \right)_t x + \frac{1}{T^1} \left( \frac{X^0}{T^1} \right)_t - \left( \frac{X^1}{T^1} x + \frac{X^0}{T^1} \right) \tilde{A}^0, \tag{6c}
\end{align*}
\]

where \( j = 2, \ldots, r \), and \( T = T(t), X^1 = X^1(t) \) and \( X^0 = X^0(t) \) are arbitrary smooth functions of their arguments with \( T^1X^1 \neq 0 \).

**Corollary 6.** The equivalence algebra of the class \((1)\) of \((1+1)\)-dimensional general \( r \)th order Burgers–KdV equations is given by \( g^\sim = \langle \tilde{D}(\tau), \tilde{S}(\zeta), \tilde{P}(\chi) \rangle \), where \( \tau = \tau(t), \zeta = \zeta(t) \) and \( \chi = \chi(t) \) run through the set of smooth functions of \( t \), with

\[
\begin{align*}
\tilde{D}(\tau) &= \tau \partial_t - \tau u \partial_u - \tau \sum_{j=2}^{r} A^j \partial_{A^j} - (\tau A^0 + \tau u) \partial_{A^0} - 2 \tau B \partial_B, \\
\tilde{S}(\zeta) &= \zeta x \partial_x + (\zeta u + \zeta t) \partial_u + \zeta \sum_{j=2}^{r} A^j \partial_{A^j} + 2 \zeta \partial_{A^0} + (\zeta B + \zeta tx - \zeta t A^0) \partial_B, \\
\tilde{P}(\chi) &= \chi \partial_x + \chi t \partial_u + (\chi u - \chi t A^0) \partial_B.
\end{align*}
\]

**Proof.** Since we have already computed the usual equivalence group \( G^\sim \), the associated equivalence algebra \( g^\sim \) can be obtained in a straightforward deductive fashion. In particular, \( g^\sim \) is spanned by vector fields representing the infinitesimal generators of one-parameter subgroups of the usual equivalence group \( G^\sim \). Thus, successively assuming one of the parameter functions \( T, X^1 \) and \( X^0 \) to depend on a continuous group parameter \( \varepsilon \) (in such a manner that the identical transformation corresponds to the value \( \varepsilon = 0 \)), we can obtain the coefficients of the infinitesimal generators of the form \( \tilde{Q} = \tau \partial_t + \chi \partial_x + \eta \partial_u + \phi^0 \partial_{A^0} + \sum_{j=2}^{r} \phi^j \partial_{A^j} + \psi \partial_B \) by determining

\[
\tau = \left. \frac{d\bar{t}}{d\varepsilon} \right|_{\varepsilon=0}, \quad \xi = \left. \frac{d\bar{x}}{d\varepsilon} \right|_{\varepsilon=0}, \quad \eta = \left. \frac{d\bar{u}}{d\varepsilon} \right|_{\varepsilon=0}, \quad \phi^0 = \left. \frac{d\bar{A}^0}{d\varepsilon} \right|_{\varepsilon=0}, \quad \phi^j = \left. \frac{d\bar{A}^j}{d\varepsilon} \right|_{\varepsilon=0}, \quad \psi = \left. \frac{d\bar{B}}{d\varepsilon} \right|_{\varepsilon=0}.
\]

This results in the generating vector fields \( \tilde{D}(\tau), \tilde{S}(\zeta) \) and \( \tilde{P}(\chi) \), which are associated to the parameter functions \( T, X^1 \) and \( U^0 \), respectively. \( \square \)

Since the class \((1)\) is mapped onto the class \((1)\) by a family of equivalence transformations, and both the classes are normalized in the usual sense, the following assertion is obvious.

**Proposition 7.** The group classification of the class \((1)\) reduces to that of its subclass \((5)\). More specifically, any complete list of \( G^\sim \)-inequivalent Lie symmetry extensions in the class \((5)\) is a complete list of \( G^\sim \)-inequivalent Lie symmetry extensions in the class \((1)\).
4 Alternative gauges

We show that the gauge $C = 1$ is the best initial gauge for the class $\mathbb{I}$ and the gauge $(C, A^1) = (1, 0)$ is the best for singling out a subclass in order to carry out the group classification.

An obvious choice for an alternative gauge is $A^r = 1$. It was used in [3] as the basic gauge in the course of group classification of linear equations of the form $\mathbb{I}$, for which $C = 0$. The $A^r$-component of equivalence transformations in the class $\mathbb{I}$ is

$$\tilde{A}^r = \left(\frac{X_x}{T_x}\right)^r A^r.$$ 

If $A^r = 1$ and $\tilde{A}^r = 1$, then the parameters of the corresponding admissible transformations given in Proposition $2$ satisfy the constraint $\left(\frac{X_x}{T_x}\right)^r = T_i$, i.e., $X = X^1(t)x + X^0(t)$, where $\left(\frac{X_x}{T_x}\right)^r = T_i$, which makes the parameterization of the usual equivalence group of the corresponding subclass more complicated than using the gauge $C = 1$.

**Proposition 8.** The subclass of the class $\mathbb{I}$ singled out by the constraint $A^r = 1$ is normalized in the usual sense. Its usual equivalence group is constituted by the transformations of the form

$$\begin{align*}
\tilde{i} &= T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \quad \tilde{u} = U^1(t)u + U^0(t, x), \\
\tilde{A}^l &= \frac{(X^1)^l}{T_l} A^l, \quad \tilde{A}^1 = \frac{X^1}{T_l} \left(A^1 + \frac{U^0}{U^1} C - \frac{X^1_l x + X^0_l x^l}{X^1_x}\right), \quad \tilde{A}^0 = \frac{1}{T_l} \left(A^0 + \frac{U^0}{U^1} + \frac{U^0_0}{U^1^2} C\right), \\
\tilde{B} &= \frac{U^1}{T_l} B + \frac{U^0}{T_l} \left(\frac{U^0}{U^1} C - \frac{X^1_l x + X^0_l x^l}{X^1_x}\right) - \sum_{k=0}^{r-1} \frac{U^0_k}{(X^1)^{k+1}} \tilde{A}^k, \quad \tilde{C} = \frac{X^1}{T_l U^1} C,
\end{align*}$$

where $l = 2, \ldots, r - 1$, and $T = T(t)$, $X^0 = X^0(t)$, $U^1 = U^1(t)$ and $U^0 = U^0(t, x)$ are arbitrary smooth functions of their arguments such that $T_l U^1 \neq 0$, as well as $X^1 = (T_i)^{1/r}$ if $r$ is odd and $T_i > 0$, $X^1 = \varepsilon(T_i)^{1/r}$ with $\varepsilon = \pm 1$ if $r$ is even.

In contrast to the class $\mathbb{I}$, the additional gauge $A^1 = 0$ slightly worsens the normalization property. It leads to the appearance of the arbitrary element $C$ in the $u$-component of admissible transformations since then we have

$$U^0 = \frac{X^1_l x + X^0_l x^l}{X^1_x C} U^1.$$ 

Denote by $\theta'$ the arbitrary-element tuple reduced by the double gauge,

$$\theta' = (A^0, A^2, \ldots, A^r, B, C).$$

**Proposition 9.** The equivalence groupoid of the subclass $\mathbb{A}_1$ of the class $\mathbb{I}$ singled out by the constraints $A^1 = 1$ and $A^1 = 0$ consists of the triples $(\theta', \tilde{\theta}', \varphi)$’s, where the point transformation $\varphi$ is of the form

$$\begin{align*}
\tilde{i} &= T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \quad \tilde{u} = U^1(t)u + U^0, \quad U^0 := \frac{X^1_l x + X^0_l x^l}{X^1_x} U^1, \\
\tilde{A}^l &= \frac{(X^1)^l}{T_l} A^l, \quad \tilde{A}^0 = \frac{1}{T_l} \left(A^0 + \frac{U^1}{U^1} + \frac{U^0_0}{U^1^2} C\right), \quad \tilde{C} = \frac{X^1}{T_l U^1} C, \\
\tilde{B} &= \frac{U^1}{T_l} B + \frac{U^0}{T_l} \left(\frac{U^0}{U^1} C - \frac{X^1_l x + X^0_l x^l}{X^1_x}\right) - \sum_{k=0}^{r-1} \frac{U^0_k}{(X^1)^{k+1}} \tilde{A}^k - U^0 \tilde{A}^0,
\end{align*}$$

with $l = 2, \ldots, r - 1$, and $T = T(t)$, $X^0 = X^0(t)$ and $U^1 = U^1(t)$ being arbitrary smooth functions of $t$ such that $T_l U^1 \neq 0$, as well as $X^1 = (T_i)^{1/r}$ if $r$ is odd and $T_i > 0$, $X^1 = \varepsilon(T_i)^{1/r}$ with $\varepsilon = \pm 1$ if $r$ is even.
It is obvious that the subclass \( A_1 \) is not normalized in the usual sense. Its usual equivalence group is constituted by the point transformations of the form (7) in the joint space of the variables \((t, x, u)\) and the arbitrary elements \( \theta' \), where parameters satisfy more constraints, \( T_{tt} = X_t^0 = 0 \), and thus \( X^1_t = 0 \) and \( U^0_t = 0 \).

All the components of (7) locally depend on \( C \), and, moreover, the expressions for \( \tilde{A}^0 \) and \( \tilde{B} \) involve derivatives of \( C \) with respect to \( t \) and \( x \). This is why, to interpret (7) as generalized equivalence transformations, we need to formally extend the arbitrary-element tuple \( \theta' \) with the derivatives of \( C \) as new arbitrary elements, \( Z^0 := C_t \) and \( Z^k := C_k, \ k = 1, \ldots, r \), and prolong equivalence transformations to them,

\[
\tilde{Z}^0 = \frac{X^1_t}{T_t U^1} Z^0 + \left( \frac{X^1_t}{T_t U^1} \right)_t C_t, \quad \tilde{Z}^k = \frac{(X^1_t)^{1-k}}{T_t U^1} Z^k, \quad k = 1, \ldots, r.
\]  

The derivatives of \( U^0 \) in the expressions for \( \tilde{A}^0 \) and \( \tilde{B} \) should be expanded and then derivatives of \( C \) should be replaced by the corresponding \( Z \)'s.

We denote by \( \bar{A}_1 \) the class of equations of the form (11) with \( (A^r, A^1) = (1, 0) \) and the extended arbitrary-element tuple \( \bar{\theta} = (A^0, A^2, \ldots, A^{r-1}, B, C, Z^0, \ldots, Z^r) \), where the relations defining \( Z^0, \ldots, Z^r \) are assumed as additional auxiliary equations for arbitrary elements.

**Theorem 10.** The class \( \bar{A}_1 \) is normalized in the generalized sense. Its generalized equivalence group \( \bar{G}_{\bar{A}_1} \) coincides with its effective generalized equivalence group and consists of the point transformations in the joint space of the variables \((t, x, u)\) and the arbitrary elements \( \theta' \) with components of the form (7), (8) and the same constraints for parameters as in Proposition 9, where partial derivatives of \( U^0 \) are replaced by the corresponding restricted total derivatives with \( \bar{D}_t = \partial_t + Z^0 \partial_C \) and \( \bar{D}_x = \partial_x + Z^1 \partial_C + Z^2 \partial_{x1} + \cdots + Z^r \partial_{x_{r-1}} \).

**Proof.** The point transformations of the above form constitute a group \( G \), which generates the entire equivalence groupoid of the class \( \bar{A}_1 \) and is minimal among point-transformation groups in the joint space of \((t, x, u, \theta')\) that have this generation property. Therefore, \( G \) is an effective generalized equivalence group of the class \( \bar{A}_1 \). We are going to prove that the group \( G \) coincides with \( \bar{G}_{\bar{A}_1} \). Indeed, substituting every particular value of \( \theta' \) to any element of \( \bar{G}_{\bar{A}_1} \) gives an admissible transformation of the class \( \bar{A}_1 \). This implies that elements of \( \bar{G}_{\bar{A}_1} \) are of the form (7), (8), where the parameter functions \( T, X^0 \) and \( X^1 \) may depend on arbitrary elements, and the partial derivatives of these functions are replaced by the corresponding total derivatives prolonged to the arbitrary elements of the class \( \bar{A}_1 \). At the same time, these parameters satisfy the condition \( \bar{D}_x T = \bar{D}_x X^0 = \bar{D}_x X^1 = 0 \) with the prolonged total derivative operator \( \bar{D}_x \). This condition implies via splitting with respect to unconstrained derivatives of arbitrary elements that the parameters \( T, X^0 \) and \( X^1 \) are functions of \( t \) only. Hence \( \bar{G}_{\bar{A}_1} = G \).

Therefore, the gauge \((C, A^1) = (1, 0)\) is better than the gauge \((A^r, A^1) = (1, 0)\).

**Remark 11.** To the best of our knowledge, Theorem 10 provides the first example for a generalized equivalence group containing transformations whose components for equation variables depend on a nonconstant arbitrary element. This is also an example of a generalized equivalence group being effective itself, and thus the corresponding class of differential equations admits a unique effective generalized equivalence group.

A more complicated example of a generalized equivalence group is given by the subclass \( A_0 \) of the class (11) singled out by the mere constraint \( A^1 = 0 \). The \( A^1 \)-component of equivalence transformations of the class (11) takes the form

\[
\bar{A}^1 = \frac{X_x}{T_t} \bar{A}^1 + \frac{X_x}{T_t} U^0_t C - \frac{X_t}{T_t} - \sum_{j=2}^{r} \bar{A}^j X_x \left( \frac{1}{X_x} \partial_x \right)^{j-1} \frac{1}{X_x},
\]
where each $\tilde{A}^j$, $j = 2, \ldots, r$, is a combination of $A^i$, $i = j, \ldots, r$ with coefficients expressed via $T_i$ and derivatives of $X$ with respect to $x$. Substituting the expression for $U^0$ implied by the gauge $A^1 = 0$,

$$U^0 = \frac{X_t U^1}{X_x C} + \frac{T_i U^1}{C} \sum_{j=2}^r \tilde{A}^j \left( \frac{1}{X_x} \frac{\partial_x}{\partial t} \right)^{j-1} \frac{1}{X_x},$$

into the general form of admissible transformations of the class (1) and neglecting the relation between $A^1$ and $\tilde{A}^1$, we get the elements of the equivalence groupoid of the subclass $A_0$. Therefore, this subclass is not normalized in the usual sense, and its usual equivalence group is isomorphic to the subgroup of the group $G_\sim$, singled out by constraining group parameters with $X_{xx} = X_t = 0$. Similarly to Theorem 12 we can consider the counterpart $\tilde{A}_0$ of the subclass $A_0$, where the tuple of arbitrary elements $(A^0, A^2, \ldots, A^r, B, C)$ is formally extended with the derivatives $C_t$, $C_k$, $A^j_t$ and $A^j_k$, $j = 2, \ldots, r$, $k = 1, \ldots, r$. Then the expressions for transformational parts of admissible transformations of $A_0$ and their relations between initial and transformed arbitrary elements including the prolongation to the above derivatives give the components of the transformations constituting a group $G$, which is obviously an effective generalized equivalence group $G$ of the class $\tilde{A}_0$. Thus, the class $A_0$ is also normalized in the generalized sense.

Note that the entire generalized equivalence group $\tilde{G}_\sim$ of the class $\tilde{A}_0$ coincides with its effective generalized equivalence group $G$. Indeed, in view of the description of the equivalence groupoid of the subclass $A_0$, elements of $\tilde{G}_\sim$ are of the form similar to that of elements of $G$, where the group parameters $T$, $X$ and $U^1$ may also depend on arbitrary elements of the subclass $A_0$, and their partial derivatives in $t$ and $x$ are replaced by the corresponding total derivatives prolonged to the arbitrary elements of the class $\tilde{A}_0$. At the same time, the condition $D_x T = D_x U^1 = 0$ with the prolonged total derivative operator $D_x$ implies that the parameter functions $T$ and $U^1$ still depend at most on $t$. The corresponding expression for $U^0$ involves the derivative $D_x X$, and hence the transformation component for $B$ necessarily contains the derivative $D_x^2 X$. Splitting this component with respect to the $2$th order $x$-derivatives of all arbitrary elements, which are not constrained, we derive that in fact the parameter function $X$ also does not depend on arbitrary elements.

5 Preliminary analysis of Lie symmetries

We compute the maximal Lie invariance group of an equation $\mathcal{L}_\kappa$ from the class (1) using the infinitesimal method. For this, we define the generators of one-parameter point symmetry groups of $\mathcal{L}_\kappa$ through $Q = \tau \partial_t + \xi \partial_x + \eta \partial_u$ with the components $\tau$, $\xi$ and $\eta$ depending on $(t, x, u)$. The infinitesimal invariance criterion reads

$$Q^{(r)} \left( u_t + uu_x - \sum_{j=2}^r A^j u_j - A^0 u - B \right) = 0 \quad \text{for all solutions of } \mathcal{L}_\kappa.$$

The $r$th prolongation $Q^{(r)}$ of the vector field $Q$ is given by $Q^{(r)} = Q + \sum_{0<|\alpha|\leq r} \eta^\alpha \partial^\alpha u_a$. Recall that $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \alpha_2$, and $u_0 = \partial^{|\alpha|} u / \partial x^{\alpha_1} \partial t^{\alpha_2}$. The coefficients $\eta^\alpha$ in the prolonged vector fields $Q^{(r)}$ are obtainable from the general prolongation formula [29, Theorem 2.36],

$$\eta^\alpha = D^\alpha (\eta - \tau u_t - \xi u_x) + \tau u_{\alpha+\delta_1} + \xi u_{\alpha+\delta_2},$$

where $D^\alpha = D_t^\alpha D_x^\alpha$, $D_t = \partial_t + \sum_{\alpha} u_{\alpha+\delta_1} \partial_{u_\alpha}$ and $D_x = \partial_x + \sum_{\alpha} u_{\alpha+\delta_2} \partial_{u_\alpha}$ are the operators of total differentiation with respect to $t$ and $x$, respectively, and $\delta_1 = (1, 0)$ and $\delta_2 = (0, 1)$. 

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The maximal Lie invariance algebra 

\[ \eta^{(1,0)} + \omega \eta^{(0,1)} + \eta u_x = \sum_{j=2}^{r} \left[ (\tau A^j_t + \xi A^j_x)u_j + A^j \eta^{(0,j)} \right] + (\tau A^0_t + \xi A^0_x)u + A^0 \eta \]

\[ + \tau B_t + \xi B_x, \quad \text{wherever} \quad u_t + uu_x = \sum_{j=2}^{r} A^j u_j + A^0 u + B. \]  

(9)

We have shown in Section 5 that the class (5) is normalized in the usual sense. Therefore, we know that the restrictions derived in Corollary 6 for the \(t, x, u\)-components of vector fields from the equivalence algebra \(g\) also hold for the components of infinitesimal symmetry generators. It is thus true that \(\tau = \tau(t), \xi = \zeta(t)x + \chi(t), \eta = (\zeta(t) - \tau_l(t))u + \zeta_l(t)x + \chi_l(t)\).

Substituting this restricted form of the coefficients of \(Q\) into the infinitesimal invariance criterion yields

\[ \sum_{j=2}^{r} (\tau A^j_t + (\zeta x + \chi) A^j_x + (\tau_l - j \zeta) A^j) u_j + (\tau A^0_t + (\zeta x + \chi) A^0_x + \tau_l A^0) u \]

\[ + \tau B_t + (\zeta x + \chi) B_x - (\zeta - 2 \tau_l) B + (\zeta_l x + \chi_l) A^0 = (2 \zeta_l - \tau_l) u + \zeta_l u x + \chi_l u. \]

This equation can be split with respect to \(u\) and its spatial derivatives, resulting in the system

\[ \tau A^j_t + (\zeta x + \chi) A^j_x + (\tau_l - j \zeta) A^j = 0, \quad j = 2, \ldots, r, \quad \text{(10a)} \]

\[ \tau A^0_t + (\zeta x + \chi) A^0_x + \tau_l A^0 = 2 \zeta_l - \tau_l, \quad \text{(10b)} \]

\[ \tau B_t + (\zeta x + \chi) B_x - (\zeta - 2 \tau_l) B + (\zeta_l x + \chi_l) A^0 = \zeta_l u x + \chi_l u. \quad \text{(10c)} \]

Since all of the determining equations (10) essentially depend on the arbitrary elements \(\kappa\), they constitute the system of classifying equations for Lie symmetries of equations from the class (5). Thus, solving the group classification problem for the class (5) reduces to solving the classifying equations (10) up to the equivalence induced by \(G^\sim\). Due to the structure of the determining equations (10) we have proved the following proposition.

**Proposition 12.** The maximal Lie invariance algebra \(g_\kappa\) of the equation \(\mathcal{L}_\kappa\) from the class (5) is spanned by the vector fields of the form \(Q = D(\tau) + S(\zeta) + P(\chi)\), where the parameter functions \(\tau, \zeta\) and \(\chi\) run through the solution set of the determining equations (10), and

\[ D(\tau) = \tau \partial_t - \tau_l u \partial_u, \quad S(\zeta) = \zeta x \partial_x + (\zeta u + \zeta_l x) \partial_u, \quad P(\chi) = \chi \partial_x + \chi_l \partial_u. \]

**Proposition 13.** The kernel Lie invariance algebra \(g^\perp := \bigcap_\kappa g_\kappa\) of equations from the class (5) is trivial, that is, \(g^\perp = \{0\}\).

**Proof.** To derive the kernel of maximal Lie invariance algebras one assumes the arbitrary elements \(\kappa\) to vary. Then it is possible to split the determining equations (10) also with respect to the arbitrary elements \(\kappa\) and their derivatives. This immediately yields \(\tau = 0, \zeta x + \chi = 0\) and, since \(\zeta\) and \(\chi\) are functions of \(t\) only, it follows that \(\zeta = \chi = 0\).

At this stage, it is appropriate to introduce the linear span

\[ g(\tau) := (D(\tau), S(\zeta), P(\chi)), \]

where the parameter functions \(\tau, \zeta\) and \(\chi\) run through the set of smooth functions of \(t\). The nonzero commutation relations between vector fields spanning \(g(\tau)\) are given by

\[ [D(\tau), D(\tau')] = D(\tau' \tau_l - \tau_l \tau), \quad [D(\tau), S(\zeta)] = S(\tau \zeta_l), \quad [D(\tau), P(\chi)] = P(\tau \chi_l), \]

\[ [S(\zeta), P(\chi)] = -P(\zeta \chi). \]
In view of these commutation relations, it is obvious that $\mathfrak{g}_1$ is a Lie algebra. Moreover, it is true that $\mathfrak{g}_1 = \mathfrak{g}_\kappa$ since any of the vector fields $D(\tau), S(\zeta)$ and $P(\chi)$ lies in $\mathfrak{g}_\kappa$ for some particular value of the arbitrary element $\kappa$.

Let us here and in the following denote by $\pi$ the projection map from the joint space of $(t, x, u, \kappa)$ onto the space of the variables $(t, x, u)$ alone, i.e., $\pi(t, x, u, \kappa) = (t, x, u)$. This map properly pushforwards vector fields from $\mathfrak{g}^\sim$ and transformations from $G^\sim$, and $\pi_\ast \mathfrak{g}^\sim = \mathfrak{g}_1$. This displays that in fact the class (5) is strongly normalized in the usual sense [16].

Note that the normalization of the class (5) in the usual sense only implies that $\mathfrak{g}_1 \subseteq \pi_\ast \mathfrak{g}^\sim$. The spanning vector fields of $\mathfrak{g}^\sim$, $\tilde{D}(\tau)$, $\tilde{S}(\zeta)$ and $\tilde{P}(\chi)$, are pushforwarded by $\pi$ to the spanning vector fields $D(\tau)$, $S(\zeta)$ and $P(\chi)$ of $\mathfrak{g}_1$, respectively. By means of the pushforward by $\pi$, the adjoint action of $G^\sim$ on $\mathfrak{g}^\sim$ induces the action of $\pi_\ast G^\sim$ on $\mathfrak{g}_1$, and, therefore, on the set of subalgebras of $\mathfrak{g}_1$. Recall that a subalgebra $\mathfrak{s}$ of $\mathfrak{g}_1$ is called appropriate if there exists a $\kappa$ such that $\mathfrak{s} = \mathfrak{g}_\kappa$. For any value of the arbitrary element $\kappa$ and any transformation $T \in G^\sim$, the pushforward by $\pi_\ast T$ maps the maximal Lie invariance algebra $\mathfrak{g}_\kappa$ of the equation $L_\kappa$ onto the maximal Lie invariance algebra $\mathfrak{g}_{T\kappa}$ of the equation $L_{T\kappa}$, and both the algebras $\mathfrak{g}_\kappa$ and $\mathfrak{g}_{T\kappa}$ are included in $\mathfrak{g}_1$. In other words, the action of $\pi_\ast G^\sim$ on $\mathfrak{g}_1$ preserves the set of appropriate subalgebras of $\mathfrak{g}_1$, and hence the group $\pi_\ast G^\sim$ generates a well-defined equivalence relation on these subalgebras. As a result, we have proved the following proposition, which is the basis for the group classification of class (5).

**Proposition 14.** The complete group classification of the class (5) of gauged $(1+1)$-dimensional general Burgers–KdV equations of order $r$ is obtained by classifying all appropriate subalgebras of the Lie algebra $\mathfrak{g}_1$ under the equivalence relation generated by the action of $\pi_\ast G^\sim$.

To efficiently carry out the group classification using the algebraic method, it is necessary to compute the adjoint actions of the transformations from $\pi_\ast G^\sim$ on the vector fields $Q$ from $\mathfrak{g}_1$.

The adjoint actions of the transformations, $\varphi$, from $\pi_\ast G^\sim$ on the vector fields, $Q$, from $\mathfrak{g}_1$ are directly computable from the definition of the pushforward $\varphi_\ast$ of $Q$ by $\varphi$,

$$\varphi_\ast Q = Q(T)\partial_t + Q(x)\partial_x + Q(u)\partial_u,$$

see, e.g., [2, 3, 6, 10, 19]. Here, the coefficients of $\varphi_\ast Q$ are given in terms of the transformed variables obtained by substituting $(t, x, u) = \varphi^{-1}(\tilde{t}, \tilde{x}, \tilde{u})$ using the inverse transformation $\varphi^{-1}$ of $\varphi$.

In practice, this is done by considering the families of elementary transformations $D(T)$, $S(X^1)$ and $P(X^0)$ from $\pi_\ast G^\sim$, which follow from (1) by restricting all except one of the parameter functions $T$, $X^1$ and $X^0$ to trivial values (which are $t$ for $T$, one for $X^1$ and zero for $X^0$). The nontrivial pushforwards of the spanning vector fields of $\mathfrak{g}_1$ by these elementary transformations from $\pi_\ast G^\sim$ are given by

$$\mathcal{D}_\ast(T)D(\tau) = \tilde{D}(\tau T), \quad \mathcal{D}_\ast(T)S(\zeta) = \tilde{S}(\zeta), \quad \mathcal{D}_\ast(T)P(\chi) = \tilde{P}(\chi),$$

$$\mathcal{S}_\ast(X^1)D(\tau) = \tilde{D}(\tau) + \tilde{S}\left(\frac{X^1}{X^1}\right), \quad \mathcal{S}_\ast(X^1)P(\chi) = \tilde{P}(\chi X^1),$$

$$\mathcal{P}_\ast(X^0)D(\tau) = \tilde{D}(\tau) + \tilde{P}\left(\frac{X^0}{X^0}\right), \quad \mathcal{P}_\ast(X^0)S(\zeta) = \tilde{S}(\zeta) - \tilde{P}(\zeta X^0).$$

Here the tildes over the right-hand side operators indicate that the given vector fields are expressed using the transformed variables, which also includes substituting $t = T^{-1}(\tilde{t})$ for $t$, where $T^{-1}$ is the inverse function of $T$.

### 6 Properties of appropriate subalgebras

An important step for the group classification of class (5) is to study properties of appropriate subalgebras of the algebra $\mathfrak{g}_1$ to be classified. In particular, we determine the maximum dimension of admitted Lie invariance algebras of equations from this class. This is formulated in the following lemma.
Lemma 15. For any tuple of arbitrary elements \( \kappa \), \( \dim g_\kappa \leq 5 \).

Proof. This statement follows directly from analyzing the solution space of the linear system \((10)\) with respect to \( \tau \), \( \zeta \) and \( \chi \) for a fixed tuple \( \kappa \). Denote by \( \Omega_t \subseteq \mathbb{R} \) and \( \Omega_x \subseteq \mathbb{R} \) open intervals on the \( t \)- and \( x \)-axes, such that the equation \( L_\kappa \) is defined on the domain \( \Omega_t \times \Omega_x \). Since \( A^r \neq 0 \) by definition, we can resolve the equation \((10a)\) with \( j = r \) for \( \tau_t \) and fix \( x_1 \in \Omega_x \), yielding

\[
\begin{align*}
\tau_t &= r\zeta - \left( r \frac{A^r_t}{A^r} + (\zeta + \chi) \frac{A^r}{A^r} \right) \bigg|_{x=x_1} =: R^1. 
\end{align*}
\]

Evaluating the classifying condition \((10c)\) at the two distinct points \( x_2 \) and \( x_3 \) from \( \Omega_x \) and varying \( t \), we obtain

\[
\zeta_t x_2 - \chi_t = R^2, \quad \zeta_t x_3 - \chi_t = R^3,
\]

where \( R^2 \) and \( R^3 \) follow from substituting \( x_2 \) and \( x_3 \) into the left hand side of \((10c)\), respectively. Due to \( x_2 \) and \( x_3 \) being different points, the above system can be written as

\[
\zeta_t = \cdots, \quad \chi_t = \cdots. \tag{12b}
\]

If the arbitrary element \( \kappa \) is fixed, the system \((12)\) can be considered as a canonical system of linear ordinary differential equations in \( t \), \( \zeta \) and \( \chi \). Its solution space is thus five-dimensional. Further conditions derived from the classifying equations \((10)\) can only reduce this solution space and hence it follows that \( \dim g_\kappa \leq 5 \).

We introduce three integers related to the dimensions of certain subspaces of the maximal Lie invariance algebra \( g_\kappa \) of the equation \( L_\kappa \),

\[
\begin{align*}
k_1 &:= \dim (g_\kappa \cap \langle P(\chi) \rangle), \\
k_2 &:= \dim (g_\kappa \cap \langle S(\zeta), P(\chi) \rangle) - k_1, \\
k_3 &:= \dim g_\kappa - \dim (g_\kappa \cap \langle S(\zeta), P(\chi) \rangle) = \dim g_\kappa - k_1 - k_2.
\end{align*}
\]

Although these integers depend on \( \kappa \), in view of \((11)\) it is obvious that the dimensions of the subalgebras \( g_\kappa \cap \langle P(\chi) \rangle \) and \( g_\kappa \cap \langle S(\zeta), P(\chi) \rangle \) as well as of the entire algebra \( g_\kappa \) are \( G^\sim \)-invariant, and thus the integers \( k \)'s are also \( G^\sim \)-invariant.

Lemma 16. The integers \( k_1, k_2 \) and \( k_3 \) are \( G^\sim \)-invariant values, i.e., they are the same for all \( G^\sim \)-equivalent equations from the class \([5]\).

Proof. Let \( T \in G^\sim \) transform \( L_\kappa \) to \( L_\tilde{\kappa} \). The transformation \( \pi_* T \) pushforwards \( g_\kappa \) onto \( g_{\tilde{\kappa}} \). At the same time, it preserves the spans \( \langle S(\zeta), P(\chi) \rangle \) and \( \langle P(\chi) \rangle \). This is why \( \dim g_\kappa = \dim g_{\tilde{\kappa}}, \dim g_\kappa \cap \langle S(\zeta), P(\chi) \rangle = \dim g_{\tilde{\kappa}} \cap \langle S(\zeta), P(\chi) \rangle \) and \( \dim g_\kappa \cap \langle P(\chi) \rangle = \dim g_{\tilde{\kappa}} \cap \langle P(\chi) \rangle \).

We now proceed to find the upper bounds for values of these integers, which is proved in a way similar to Lemma \([15]\)

Lemma 17. For any \( \kappa \), we have \( (k_1, k_2) \in \{(0,0), (0,1), (2,0)\} \).

Proof. For any vector field \( Q = S(\zeta) + P(\chi) \) from the algebra \( g_\kappa \), the parameter functions \( \zeta \) and \( \chi \) satisfy the system of classifying equations \((10)\) for the chosen tuple of arbitrary elements \( \kappa = (A^0, A^2, \ldots, A^r, B) \) and \( \tau = 0 \).

In particular, if \( A^r \neq 0 \), we can solve the classifying equation \((10a)\) with \( j = r \) with respect to \( \chi \) to obtain \( \chi = (rA^r/A^r - x)\zeta \). After fixing a value \( x = x_0 \), this equation implies that there exists a function \( f = f(t) \) such that \( \chi = f\zeta \). Then the classifying equation \((10b)\) implies that \( 2\zeta_t = (x + f)A^0_2\zeta \). Again fixing \( x = x_0 \), we hence derive an equation \( \zeta_t = g\zeta \) with some function
The projection $\varpi$ from the space of $(t,x,u)$ on the space of $t$ alone properly pushforwards elements of $\mathfrak{g}_{(i)}$ according to $D(\tau) + S(\zeta) + P(\chi) \mapsto \tau \partial_t$, and hence $\varpi_* \mathfrak{g}_{(i)} = \{ \tau \partial_t \}$, where $\tau$ runs through the set of smooth functions of $t$. The pushforward $\varpi_* G^\sim$ of $G^\sim$ by $\varpi$ is also well defined.

**Lemma 18.** The projection $\varpi_* \mathfrak{g}_{\kappa}$ is a Lie algebra for any tuple of arbitrary elements $\kappa$, and $\dim \varpi_* \mathfrak{g}_{\kappa} = k_3 \leq 3$. It is true that $\varpi_* \mathfrak{g}_{\kappa} \in \{ 0, \langle \partial_t \rangle, \langle \partial_t, t \partial_t \rangle, \langle \partial_t, t \partial_t, t^2 \partial_t \rangle \}$ mod $\varpi_* G^\sim$.

**Proof.** We show that $\varpi_* \mathfrak{g}_{\kappa}$ is indeed a Lie algebra. Given $\tau^i \partial_t \in \varpi_* \mathfrak{g}_{\kappa}$, $i = 1, 2$, there exist $Q^i \in \mathfrak{g}_{\kappa}$ such that $\varpi_* Q^i = \tau^i \partial_t$. For any constants $c_1$ and $c_2$, we have that $c_1 Q^1 + c_2 Q^2 \in \mathfrak{g}_{\kappa}$ and thus $c_1 \tau^1 \partial_t + c_2 \tau^2 \partial_t = \varpi_* (c_1 Q^1 + c_2 Q^2) \in \varpi_* \mathfrak{g}_{\kappa}$. This means that $\varpi_* \mathfrak{g}_{\kappa}$ is indeed a linear space. This space is closed under the Lie bracket of vector fields and, therefore, is a Lie algebra, because $[\tau^1 \partial_t, \tau^2 \partial_t] = (\tau^1 \tau^2 \partial_t - \tau^2 \tau^1 \partial_t) = \varpi_* [Q^1, Q^2] \in \varpi_* \mathfrak{g}_{\kappa}$.

Since the pushforward $\varpi_* G^\sim$ of $G^\sim$ by the projection $\varpi$ coincides with the (pseudo)group of local diffeomorphisms in the space of $t$, Lie’s theorem can be invoked. It states that the maximum dimension of finite-dimensional Lie algebras of vector fields on the complex (resp. real) line is three. Up to local diffeomorphisms of the line, these algebras are given by $\{ 0 \}$, $\langle \partial_t \rangle$, $\langle \partial_t, t \partial_t \rangle$ and $\langle \partial_t, t \partial_t, t^2 \partial_t \rangle$.

It follows that $|k| := k_1 + k_2 + k_3 = \dim \mathfrak{g}_{\kappa} \leq 5$.

## 7 Group classification

The main result of the paper is given by the following assertion.

**Theorem 19.** A complete list of $G^\sim$-inequivalent (and, therefore, $G^\sim$-inequivalent) Lie symmetry extensions in the class (15) is exhausted by the cases given in Table 7.

**Proof.** Lemmas 17, 18 imply that any appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}_{(i)}$ has a basis consisting of

1. $k_1$ vector fields $Q^i = P(\chi^i)$, $i = 1, \ldots, k_1$, with linearly independent $\chi^i$’s,
2. $k_2$ vector fields $Q^i = S(\zeta^i) + P(\chi^i)$, $i = k_1 + 1, \ldots, k_1 + k_2$, with nonzero $\zeta^i$’s,
3. $k_3$ vector fields $Q^i = D(\tau^i) + S(\zeta^i) + P(\chi^i)$, $i = k_1 + k_2 + 1, \ldots, |k|$, with linearly independent $\tau^i$’s,

where $\langle k_1, k_2 \rangle \in \{ (0,0), (0,1), (2,0) \}$ and $k_3 \leq 3$. The proof proceeds by separately investigating the cases associated with the possible range of the tuple of invariant integers $(k_1, k_2, k_3)$. For each possible value of $(k_1, k_2, k_3)$, we start with the above form of basis vector fields $Q^i$’s of $\mathfrak{s}$ and simplify them as much as possible using the adjoint actions of equivalence transformations presented in (11) and linear recombination of $Q$’s. At the same time, we take into account the fact that $\mathfrak{s}$ is a Lie algebra, i.e., it is closed with respect to the Lie bracket of vector fields, $[Q^i, Q^{i''}] \in \{ Q^i, i = 1, \ldots, |k| \}$, $i', i'' = 1, \ldots, |k|$. This leads to constraints for the components of $Q$’s only if $k_2 + k_3 > 1$ or $k_2 + k_3 = 1$ and $k_1 = 2$. Substituting the components of each simplified $Q^i$ into the system of classifying equations (10) yields a system of equations in $\kappa$ for
Table 1. Complete group classification of the class $\Pi$ (resp. $\Pi'$).

| no. | $\kappa$ | Basis of $g_\kappa$ |
|-----|----------|-----------------|
| 0   | $A^0 = A^0(t, x)$, $A^0 = A^0(t, x)$, $B = B(t, x)$ | $-$ |
| 1   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $D(1)$ |
| 2   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $D(1), D(t) - P(1)$ |
| 3   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $D(1), D(t) - S(\nu^{-1}), \nu \neq 0$ |
| 4   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $D(1), D(t) - S(\nu^{-1}), \nu \neq 0$ |
| 5   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $D(1), D(t) - S(\nu^{-1}), \nu \neq 0$ |
| 6   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $S(1)$ |
| 7   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $S(1)$ |
| 8   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $S(1)$ |
| 9   | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $S(1)$ |
| 10  | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $S(1)$ |
| 11  | $A^1 = A^1(t, x)$, $A^0 = A^0(x)$, $B = B(t, x)$ | $S(1)$ |

Here $j = 2, \ldots, r$, $C = 1$, $A^t = 0$, $D(t) = \tau_\vartheta - \tau_\varphi$, $S(\varphi) = \zeta_\varphi \vartheta + (\zeta_\varphi + \zeta_\vartheta) \varphi$, and $P(\gamma) = \chi_\varphi \vartheta + \chi_\varphi \varphi$. The parameters $\alpha^t$, $\alpha^0$ and $\beta$ are arbitrary smooth functions of $t$ with $\alpha^t \neq 0$. The parameters $a_j$, $a_0$ and $b$ are arbitrary constants with $a_0 \neq 0$. In Case [1] the parameter functions $\chi^t$ and $\chi^t$ are linearly independent solutions of the equation $\chi^t - a_0 \chi^t - b \chi^t = 0$, and thus there are three cases for them depending on the sign of $\Delta := a_0^2 - 4b$.

(a) $\chi^t = e^{\lambda_1}t$, $\chi^t = e^{\lambda_2}t$ if $\Delta > 0$, where $\lambda_{1,2} = (a_0 \pm \sqrt{\Delta})/2$;
(b) $\chi^t = e^{\mu t}$, $\chi^t = e^{\mu t}$ if $\Delta = 0$, where $\mu = a_0/2$;
(c) $\chi^t = e^{\mu t} \cos \nu t$, $\chi^t = e^{\mu t} \cos \nu t$ if $\Delta < 0$, where $\mu = a_0/2$ and $\nu = \sqrt{-\Delta}/2$.

$a_0 = 1$ mod $G^\sim$ in Cases [2] [3] [4] [5] and [10]. Moreover, in Case [2] one of the constants $a_j$'s with $j < r$, $a_0$ or $b$, if it is nonzero, can be set to $\pm$ by shifts of $t$. See also Section [11] for the justification of gauging of parameters and Remark [22] for the conditions under which the presented vector fields really span the corresponding maximal Lie invariance algebra.

which $g_\kappa \supset s$. In total, we have $|k|$ such systems. We unite them and simultaneously solve for the arbitrary elements $\kappa$. The compatibility of the joint system with respect to $\kappa$ may imply additional constraints for the components of $Q$'s. The expression obtained for $\kappa$ can be simplified by equivalence transformations whose projected adjoint actions preserve $s$. Except Cases [1] and [1], the condition for $\kappa$ with $g_\kappa = s$ is obtained by the negation of the corresponding condition represented in Remark [22].

We have to consider the following cases:

$k_1 = k_2 = 0$. Here dim $s = k_3$. For $k_3 > 1$, in view of Lemma [18] we can use the simplified form of $Q^i$, $i = 1, \ldots, k_3 - 1$, derived in the subcase with the preceding value of $k_3$.

$k_3 = 0$. We obtain the general Case [1] where the algebra $s$ coincides with the kernel algebra $g^\sim = \{0\}$ and thus there are no constraints for $\kappa$.

$k_3 = 1$. A basis of $s$ consists of a single vector field $Q^1$ with $\tau^1 \neq 0$. Successively using the adjoint actions $D_\kappa(T)$, $S_\kappa(X^1)$ and $P_\kappa(Y^1)$ given in [11] for appropriate functions $X^0$, $X^1$ and $T$, this vector field can be mapped to $Q^1 = D(1)$. The system of classifying equations [10] with the components of $Q^1$ evidently is $A^1 = A^1 = B_t = 0$, which results in Case [1] of Table [1].
k_3 = 2. Modulo πG^~ -equivalence, basis elements of s take the form Q^1 = D(1) and Q^2 = D(t) + S(ζ^2) + P(χ^2). The condition [Q^1, Q^2] = D(1) + S(ζ^2) + P(χ^2) ∈ ⟨Q^1, Q^2⟩ requires that ζ^2 = χ^2 = 0 and thus ζ, χ = const. Substituting the components of Q^1 and then Q^2 into the classifying equations (10) and rearranging leads to the system

\[
\begin{align*}
A^i_0 &= 0, \quad (ζ^2 x + χ^2)A^i_0 + (1 - jζ^2)A^j = 0, \\
A^0_0 &= 0, \quad (ζ^2 x + χ^2)A^0_0 + A^0 = 0, \\
B^i &= 0, \quad (ζ^2 x + χ^2)B^i + (2 - ζ^2)B = 0.
\end{align*}
\]

(13)

If ζ^2 = 0, in view of the condition A^r = 0 the equation (ζ^2 x + χ^2)A^r + (1 - rζ^2)A^r = 0 implies that χ^2 = 0 and we can set χ = \pm 1 mod G^~; otherwise, we can set χ^2 = 0 using P_\ast(-νχ^2), where ν := -1/ζ^2, which preserves Q^1. Integrating the system (13) for each of the subcases, we respectively obtain Cases 2 and 3, where a_r = 0 and thus a_r = 1 mod G^~. In Case 2, one of the constants a_j’s with j < r, a_0 or b, if it is nonzero, can be set to ±1 by shifts of x.

k_3 = 3. Modulo πG^~ -equivalence and linearly combining basis elements, we can assume that Q^1 = D(1), Q^2 = D(t) + S(ζ^2) + P(χ^2) and Q^3 = D(t^2) + S(ζ^2) + P(χ^2). The completeness of the algebra s with respect to the Lie bracket of vector fields implies that [Q^1, Q^2] = Q^1, [Q^2, Q^3] = 2Q^2 and [Q^1, Q^3] = Q^3. As in the previous case, the first commutation relation holds only if ζ^2, χ^2 = const. The second commutation relation expands to

\[2D(t) + S(ζ^2) + P(χ^2) = 2D(t) + 2S(ζ^2) + 2P(χ^2),\]

and thus ζ^3 = 2ζ^2, χ^3 = 2χ^2. Integrating these two equations yields ζ^3 = 2ζ^2 t + ζ^30 and χ^3 = 2χ^2 t + χ^30, where ζ^30 and χ^30 are constants. Next, commuting Q^2 and Q^3 yields

\[Q^2, Q^3 = D(t^2) + S(2ζ^2 t + P(2χ^2 t + χ^30 - 2ζ^2 χ^30).\]

We thus have [Q^2, Q^3] = Q^3 if and only if ζ^30 = 0 and (1+ζ^2)χ^30 = 0. Considering the classifying conditions (10) for Q^1, Q^2 and Q^3, we obtain the system (13) supplemented by the equations χ^30 A^0 = 0, χ^30 A^0 = 4ζ^2 - 2 and χ^30 B^i + (2ζ^2 x + 3χ^2)A^0 = 0. If χ^30 ≠ 0, then ζ^2 = -1, A^j = 0, and the equations (1+j)A^j = 0 imply that A^j = 0 for any j, which contradicts the condition A^r ≠ 0. Therefore, χ^30 = 0, which requires that ζ^2 = 1/2, as well as A^0 = 0. We can then set χ = 0 using P_\ast(2χ^2). The remaining determining equations are A^2 = (j-2)A^2 and B^i = 3B, which are readily integrated to Case 4 with a_r ≠ 0, and hence a_r = 1 mod G^~.

k_1 = 0, k_2 = 1. Hence Q^1 = S(ζ^1) + P(χ^1), where ζ^1 ≠ 0.

k_3 = 0. We can use the adjoint action P_\ast(χ^1/ζ^1) to set χ = 0. Moreover, multiplying Q^1 by a nonzero constant and changing t if ζ^1 is not a constant, we can gauge ζ^1 to ζ^1 = εζ, where ε ∈ [0,1] mod G^~. The classifying conditions (10) with components of Q^1 then imply the system consisting of the equations xA^j = jA^j, xA^0 = 2ε and xB^i = B + εxA^0 = εx. The general solution of this system is

\[A^j = α^j(t)x^j, \quad A^0 = α^0(t) + 2ε ln |x|, \quad B = β(t)x - ε(α^0(t) - 1)x ln |x| - εx ln^2 |x|,\]

where α^j, α^0 and β are arbitrary smooth functions of t with α^r ≠ 0. The subgroups of the equivalence group G^~ whose projections to the space (t, x, u) preserve the appropriate subalgebras s = ⟨S(1)⟩ and s = ⟨S(ε^t)⟩ are singed out from G^~ by the constraints X^0 = 0 and (X^0, T^0) = (0,1), respectively. Elements from these subgroups allow us to set (α^0, α^r) = (0,1) if ε = 0 or α^0 = 1 if ε = 1, which corresponds to Case 5 or 6. In the latter case we have chosen the gauge α^0 = 1 instead of α^0 = 0 in order to make the corresponding values of the tuple of arbitrary elements κ simpler.

k_3 = 1. First we reduce, modulo πG^~ -equivalence, the basis element Q^2 with τ^2 ≠ 0 to the form Q^2 = D(1). The Lie bracket [Q^2, Q^1] = S(ζ^1) + P(χ^1) is in s provided that the tuples (ζ^1, χ^1) and
\((\zeta^1, \chi^1)\) are linearly dependent, i.e., \(\zeta^1 = c_1 e^{rt}, \chi^1 = c_0 e^{rt}\) for some constants \(c_0, c_1\) and \(r\), where \(c_1 \neq 0\). Multiplying \(Q^1\) by \(1/c_1\) and, if \(r \neq 0\), using \(D_r(\varepsilon t)\), we can set \(c_1 = 1\) and \(\varepsilon \in \{0, 1\}\). Since \(P_s(c_0)D(1) = \tilde{D}(1)\) and \(P_s(c_0)Q^1 = S(e^{rt})\), we can finally reduce \(Q^1\) to the form \(Q^1 = S(e^{rt})\). Therefore, the corresponding complete system for the arbitrary elements includes the system from the previous case as a subsystem supplemented by the equations \(A^j_t = A^0_t = B_t = 0\). The general solution of the complete system is of the same form as in the case \(k_3 = 0\), where the functions \(a^j, a^0\) and \(\beta\) should be replaced by the arbitrary constants \(a_j, a_0\) and \(b\) with \(a_r \neq 0\). Similarly to the case \(k_3 = 0\), the appropriate subalgebras \(s = \langle S(1), D(1) \rangle\) and \(s = \langle S(e^t), D(1) \rangle\) are preserved by projections of equivalence transformations with \((T_t, (X^1_t/X^1)_t) = (1, 0)\) and \((T_t, (e^{-t}X^1_t/X^1)_t) = (1, 0)\), which makes possible the gauges \((a_0, a_r) = (0, 1)\) of \(a_0 = 1\) if \(r = 0\) or \(\varepsilon = 1\) and obviously results in Cases 7 and 8 respectively.

\(k_3 \geq 2\). This case cannot be realized. Indeed, otherwise we would additionally have, modulo \(\pi_s G^s\)-equivalence, the basis element \(Q^3 = D(t) + S(\zeta^3) + P(\chi^3)\) and would still be able to repeat the consideration of the case \(k_3 = 1\), deriving the expression \(A^r = a_r x^r\) with \(a_r \neq 0\). At the same time, the classifying equation (10a) for \(j = r\) with the components of \(Q^3\) and with the above \(A^r\) reads \(\chi^3 \neq 0\), which would imply \(a_r = 0\), giving a contradiction.

\(k_1 = 2, k_2 = 0\). Here we have \(Q^i = P(\chi^i), i = 1, 2\). The classifying equations (10) in this case imply that \(A^j_t = 0, A^0_t = 0\) and \(\chi^1 B_x + \chi^1 A^1_t = \chi^1 t\). Differentiating this last equation with respect to \(x\) leads to \(B_{xx} = t\), i.e., \(B = B^1(t)x + B^0(t)\), and therefore \(\chi^1 t = A^0 \chi^1 t + B^1 \chi^1 t\).

\(k_3 = 0\). We apply \(S_s(1/\chi^1)\) to \(s\) and set \(\chi^1 = 1\), which implies that \(\chi^2 \neq 0\). Then using the adjoint action \(D_r(\chi^2)\) we can set \(\chi^2 = t\). The equations \(\chi^1 t = A^0 \chi^1 t + B^1 \chi^1 t\), \(i = 1, 2\), then imply that \(A^0 = B^1 = 0\). The components of the transformation \(P(X^0)\) for the arbitrary elements are \(A^j = A^j_t, A^0 = A^0_t\) and \(B = B + X^0 t\). Thus, choosing \(X^0 t = -B^0\) allows us to gauge \(B = 0\), leading to Case 9.

\(k_3 = 1\). We start the simplification of basis elements of \(s\) from \(Q^3\) with \(r^3 \neq 0\), setting, modulo \(\pi_s G^s\)-equivalence, \(Q^3 = D(1)\). It then follows from the classifying equations (10) with the components of \(Q^3\) that \(A^j_t = A^0_t = B^1_t = B^0_t = 0\). We choose

\[
X^0 = \frac{B^0}{B^1}\quad \text{if} \quad B^1 \neq 0; \quad X^0 = \frac{B^0}{A^0 t}\quad \text{if} \quad B^1 = 0, A_0 \neq 0; \quad X^0 = -\frac{B^0}{2} t^2\quad \text{if} \quad B^1 = A_0 = 0.
\]

Since \(P_s(X^0)D(1) = \tilde{D}(1) + \tilde{P}(X^0)\), \(P_s(X^0)P(\chi^i) = \tilde{P}(\chi^i), i = 1, 2\), and for the chosen value of \(X^0\) we have \(X^0_t \in \langle \chi^1, \chi^2 \rangle\), the adjoint action \(P_s(X^0)\) preserves the algebra \(s\). At the same time, the component of \(P(X^0)\) for \(B\) is \(\tilde{B} = B + X^0 t - A^0 X^0 t\) and therefore \(\tilde{B} = B^0 + X^0 t - A^0 X^0 t - B^1 X^0 = 0\), which yields Case 10 where \(a_r = 1 \bmod G^s\).

\(k_3 \geq 2\). We have one more basis element whose simplified form is \(Q^4 = D(t) + S(\zeta^4) + P(\chi^4)\). Commuting \(Q^3\) and \(Q^4\) yields \(Q^4 = D(1) + S(\zeta^4) + P(\chi^4)\), which is in the algebra \(s\) provided that \(\zeta^4 = \text{const}\). The commutator of \(Q^4\) with \(Q^4\), \(i = 1, 2\), gives \([Q^4, Q^4] = P(t \chi^4 - \zeta^4 \chi^4), \) which is in the algebra \(s\) only if \(\chi^1 = 1\) and \(\chi^2 = u\) up to linearly combining \(P(\chi^1)\) and \(P(\chi^2)\). It then follows that \(A^0 = B^1 = 0\). We can let \(B^0 = 0 \bmod G^s\), and the classifying equation (10c) then implies that \(\chi^4 t = 0\). Upon linearly combining with \(P(1)\) and \(P(t)\) we can thus set \(\chi^4 = 0\). Moreover, the classifying equation (10a) for \(j = r\) with the components of \(Q^4\) implies that \(\zeta^2 = 1/r\) and thus, in view of the same equation for other \(j\)’s, \(A^j = 0\) if \(j \neq r\). We can gauge \(A^r = 1 \bmod G^s\). Now suppose that we also have a \(Q^5 = D(t^2) + S(\zeta^5) + P(\chi^5)\). Then, the classifying equations (10) for \(A^r\) and \(A^0\) require that \(2t - r\zeta^5 = 0\) and \(2\zeta^5 - 2 = 0\). This system is consistent only for \(r = 2\), where \(\zeta^5 = t\). This is why Case 11 splits depending on the value of \(r\). The equation (10c) with \(A^0 = B = 0\) implies that \(\chi^5 t = 0\) and thus we can set \(\chi^5 = 0\) upon linearly combining \(Q^5\) with \(Q^1\) and \(Q^2\).
Corollary 20. An rth order evolution equation of the form \( u_t + u u_x = u_r \) is reduced to the simplest form \( u_t + u u_x = u_r \) by a point transformation if and only if the dimension of its maximal Lie invariance algebra is greater than three.

Remark 21. Case 7 can be merged with Case 3 as the particular subcase with \( \nu = 0 \) if we choose another second basis element, \( \tilde{Q}^2 = -\nu Q^2 = S(1) - D(\nu t) \).

Remark 22. Each of the subalgebras of \( \mathfrak{g} \) whose bases are presented in the third column of Table 11 is really the maximal Lie invariance algebra for the general case of values of the arbitrary elements \( \kappa \) given in the same row. For most of the classification cases, it is not difficult to explicitly indicate the necessary and sufficient conditions for \( \kappa \) under which there is an additional Lie symmetry extension. They can be found by substituting the corresponding expressions for \( \kappa \)'s components into the system of classifying conditions (10). Thus, these conditions are related to the corresponding cases of Table 1 by the equivalence transformation with \( \kappa \) = 1 and \( \kappa \) = 0, respectively. For each value of the tuple \( \alpha \) if the parameter functions \( \zeta \)'s satisfy the system \( (\zeta^1 t^2 + \tau^1 t + \tau^0) \alpha^j + (2 \zeta^1 t + \tau^1 - j (\zeta^0 t + \zeta^0)) \alpha^j = 0 \) for some constants \( \zeta^0, \zeta^1, \tau^0 \) and \( \tau^1 \). Another form of this condition, which is convenient for checking, is that

\[
\left( \frac{\alpha^j}{\alpha^r} \right)_t = (r - j) f \frac{\alpha^j}{\alpha^r}, \quad r \alpha^r \alpha^j - j \alpha^j \alpha^j = (r - j) g \alpha^j \alpha^r,
\]

for some functions \( f = f(t) \) and \( g = g(t) \) constrained by \( g_{\tau t} + 3 g g_{\tau} + g^3 = 0, 2 f_{\tau t} + g f = g_\tau + g^2 \). See also the claim on the most complicated Case 11 in Remark 35.

Remark 23. The proof of Theorem 19 shows that the system of restrictions for appropriate subalgebras of \( \mathfrak{g}_{(1)} \) presented in Lemmas 12, 17 and 18 is not exhaustive. It can be completed by the conditions restricting the value set of \( k_3 \) depending on values of \( k_1 \) and \( k_2 \),

\[
k_3 \in \{0, 1\} \quad \text{if} \quad k_2 = 1;
\]

\[
\quad \text{if} \quad k_1 = 2, \quad \text{then} \quad k_3 \in \{0, 1, 2\} \quad \text{for} \quad r > 2 \quad \text{and} \quad k_3 \in \{0, 1, 3\} \quad \text{for} \quad r = 2.
\]

For each value of the tuple \( k = (k_1, k_2, k_3) \) satisfying the extended set of restrictions, there exists an appropriate subalgebra of \( \mathfrak{g}_{(1)} \) admitting this value of \( k \).

8 Alternative classification cases

There are various possibilities for choosing representatives in equivalence classes of pairs \( (\mathfrak{s}, \{ \mathcal{L}_\kappa \mid \mathfrak{g}_\kappa = \mathfrak{s} \}) \), where \( \mathfrak{s} \) is an appropriate subalgebra of \( \mathfrak{g} \). We have tried to simplify the representation of a pair by paying more attention to the pair's second entry. In most cases the optimal choice is obvious and coincides with the selection carried out for Table 11. At the same time, there are other options for the proof and representation of results in the cases \( (k_1, k_2, k_3) = (2, 0, 1) \) and \( (k_1, k_2) = (0, 1) \). We follow the proof of Theorem 19 but reduce the basis vector fields to another form. If \( k_3 = 0 \), we set, modulo \( G^\kappa \)-equivalence, \( Q^1 = S(t^\varepsilon) \) instead of \( Q^1 = S(\varepsilon^t) \), where still \( \varepsilon \in \{0, 1\} \). In the case \( k_3 = 1 \), for \( Q^2 \) we choose the form \( Q^2 = D(t) \) in order to be able to set \( Q^1 = S(t^\varepsilon) \) again. For \( \varepsilon = 1 \), this gives the following alternative cases, which are related to the corresponding cases of Table 11 by the equivalence transformation with \( T = e^{\varepsilon t}, X^1 = 1 \) and \( X^0 = 0 \):

\[
(k_1, k_2) = (0, 1).\]
Therefore, instead of the single Case 10 we have the following three cases:

\[ A^j = \alpha^j(t)x^j, \quad A^0 = \alpha^0(t) + 2t^{-1}\ln|x|, \]

\[ B = \beta(t)x + \alpha^0(t)t^{-1}x\ln|x| - xt^{-2}\ln^2|x| \quad \text{S}(t) \]

\[ A^j = a_jt^{-1}x^j, \quad A^0 = a_0t^{-1} + 2t^{-1}\ln|x|, \]

\[ B = bt^{-2}x - a_0t^{-2}x\ln|x| - xt^{-2}\ln^2|x| \quad \text{S}(t), \text{D}(t) \]

\((k_1, k_2, k_3) = (2, 0, 1)\). Instead of setting \(Q^3 = D(1)\) in the proof for this case, we can simplify the basis elements \(Q^1\) and \(Q^2\) to \(P(1)\) and \(P(t)\) as in the case \((k_1, k_2, k_3) = (2, 0, 0)\). Then readily \(A^j_2 = 0, A^0 = 0\) and, modulo \(G^\sim\) equivalence, \(B = 0\). Consider the subclass \(\mathcal{K}\) of equations from the class \(\{13\}\) with values of \(\kappa\) satisfying the above constraints,

\[ \mathcal{K} = \{ L_\kappa \mid A^j_2 = 0, A^0 = 0, B = 0, A^r \neq 0 \} \]

The subclass \(\mathcal{K}\) turns out to be normalized with respect to its usual equivalence group \(G^\sim_\mathcal{K}\), which is finite-dimensional and consists of the transformations in the space of \((t, x, u, A^2, \ldots, A^r)\) whose components are of the form \(\{13\\} - \{15\}\) with

\[ T = \frac{c_1t + c_2}{c_3t + c_4}, \quad X^1 = \frac{c_5}{c_3t + c_4}, \quad X^0 = \frac{c_6t + c_7}{c_3t + c_4}, \]

where \(c_1, \ldots, c_7\) are arbitrary constants with \((c_1c_4 - c_2c_3)c_5 \neq 0\) that are defined up to a nonzero multiplier. Hence the equivalence algebra \(g^\sim_{\mathcal{K}}\) of \(\mathcal{K}\) is spanned by \(\hat{D}(1), \hat{D}(t), \hat{D}(t^2) + \hat{S}(t), \hat{S}(1), \hat{P}(1), \hat{P}(t)\). The kernel Lie invariance algebra of \(\mathcal{K}\) is \(g^0_{\mathcal{K}} = \langle P(1), P(t) \rangle\). The restriction of \(G^\sim\) equivalence to the subclass \(\mathcal{K}\) coincides with \(G^\sim_{\mathcal{K}}\) equivalence. This is why, up to \(G^\sim\) equivalence, we can assume that in this case the appropriate subalgebra \(s\) is spanned by \(Q^1 = P(1), Q^2 = P(t)\) and \(Q^3 = D(\tau^3) + S(\hat{\tau}^3)\), and there are three cases for \((\tau^3, \hat{\tau}^3)\) and \((A^2, \ldots, A^r)\):

(a) \(\tau^3 = 1, \hat{\tau}^3 = \sigma \in \{0, 1\}, A^j = a_je^{i\sigma t}\);

(b) \(\tau^3 = t, \hat{\tau}^3 = \sigma = \text{const} \geq 0, A^j = a_jt^{-1}|t|^j\sigma\);

(c) \(\tau^3 = t^2 + 1, \hat{\tau}^3 = \sigma = \text{const} \geq 0, A^j = a_j(jt^2 + 1)^{j/2 - 1}e^{j\sigma\arctan t}\).

Therefore, instead of the single Case \(\{10\}\) we have the following three cases:

\[ A^j = a_je^{i\sigma t}, \quad A^0 = B = 0 \quad P(1), P(t), D(1) + S(\sigma) \]

\[ A^j = a_jt^{-1}\tau^j, \quad A^0 = B = 0 \quad P(1), P(t), D(t) + S(\sigma) \]

\[ A^j = a_j(t^j + 1)^{j/2 - 1}e^{j\sigma\arctan t}, \quad A^0 = B = 0 \quad P(1), P(t), D(t^2 + 1) + S(t + \sigma) \]

Modulo \(G^\sim\) equivalence, \(\sigma \in \{0, 1\}, \sigma \geq 0\) and \(\sigma \geq 0\) in Cases \(\{10\}\), \(\{10\}\) and \(\{10\}\), respectively.

The advantage of this form for Case \(\{10\}\) is that the vector fields \(Q^1\) and \(Q^2\) then have the evident interpretation as generators of translations with respect to the space variable \(x\) and Galilean boosts, respectively. Meanwhile the forms of \(Q^3\) and, especially, \(A^j\) become more complicated. This is why we chose the previous form of Case \(\{10\}\) for Table \(\{11\}\). Subcases of this case are related to subcases of the alternative Case \(\{10\}\) by equivalence transformations of the form \(\{13\}\), where \(X^0 = 0\) and

\[ \{10\} \rightarrow \{10\} : \quad T = e^{(\lambda_2 - \lambda_1)t}, \quad X^1 = e^{-\lambda_1 t}, \quad \sigma = -\lambda_1/(\lambda_2 - \lambda_1) \]

\[ \{10\} \rightarrow \{10\} : \quad T = t, \quad X^1 = e^{-\mu t}, \quad \sigma = -\mu \]

\[ \{10\} \rightarrow \{10\} : \quad T = \tan \nu t, \quad X^1 = e^{-\mu t}/\cos \nu t, \quad \sigma = -\mu/\nu \]

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9 Equations with time-dependent coefficients

To study Lie symmetries of equations from the class (I) with coefficients depending at most on \( t \), it is again convenient to start with a wider class, which is the subclass \( \mathcal{K}_0 \) of the class (I) singled out by the constraint \( C_x = 0 \) (resp. \( A^*_x = 0 \)) implying \( X_{xx} = 0 \) for admissible transformations.

**Proposition 24.** The class \( \mathcal{K}_0 \) is normalized in the usual sense. Its usual equivalence group is constituted by the transformations of the form

\[
\begin{align*}
\dot{i} &= T(t), \quad \dot{x} = X^1(t)x + X^0(t), \quad \dot{u} = U^1(t)u + U^0(t,x), \\
\dot{A}^j &= \left( \frac{X^1}{T_t} \right)^j A^j, \quad \dot{\bar{A}}^i = \frac{X^1}{T_t} \left( A^i + \frac{U^0}{U^1} C - \frac{X^0}{X^1} \right), \quad \dot{\bar{A}}^0 = \frac{1}{T_t} \left( \bar{A}^0 + \frac{U^1}{U^1} + \frac{U^0}{U^1} C \right), \\
\dot{B} &= \frac{U^1}{T_t} B + \frac{U^0}{T_t} A^1 + \frac{U^0}{T_t} \left( \frac{U^0}{U^1} C - \frac{X^0}{X^1} \right) - \sum_{k=0}^{r} \frac{U^0}{(X^1)^k} \bar{A}^k, \quad \dot{\bar{C}} = \frac{X^1}{T_t U^1} C,
\end{align*}
\]

where \( j = 2, \ldots, r \), and \( T = T(t) \), \( X^1 = X^1(t) \), \( X^0 = X^0(t) \), \( U^1 = U^1(t) \) and \( U^0 = U^0(t,x) \) are arbitrary smooth functions of their arguments with \( T_1 \langle U^1 \rangle \not\equiv 1 \).

Consider the subclass \( \mathcal{K}_1 \) obtained by attaching the constraints \( A^0_0 = 0, A^1_{xx} = 0, A^2_{x} = 0, \)

\( j = 2, \ldots, r, C_x = 0 \) and \( B_{xx} = 0 \) to the auxiliary system for arbitrary elements. It is also normalized in the usual sense and its usual equivalence group is the subgroup of the usual equivalence group \( G_{\mathcal{K}_0} \) of the class \( \mathcal{K}_0 \) that is associated with the constraint \( U^0_{xx} = 0 \), i.e., \( U^0 = U^{01}(t)x + U^{00}(t) \). Note that we can reparameterize the class \( \mathcal{K}_1 \) by representing \( B = B^1(t)x + B^0(t) \), \( A^1 = A^{11}(t)x + A^{10}(t) \) and assuming the coefficients \( B^1, B^0, A^{11} \) and \( A^{10} \) as arbitrary elements instead of \( B \) and \( A^1 \). The transformation component for \( B \) simplifies to

\[
\dot{B} = \frac{U^1}{T_t} \dot{B} + \frac{U^0}{T_t} \dot{A}^1 - \frac{U^0}{T_t} \dot{A}^0 - \sum_{k=0}^{r} \frac{U^0}{(X^1)^k} \bar{A}^k.
\]

The next intermediate subclass \( \mathcal{K}_2 \) is singled out by strengthening the constraint for \( A^1 \) to \( A^1_{xx} = 0 \). In fact, this can be realized by gauging \( A^1 \) in the class \( \mathcal{K}_0 \) up to \( G_{\mathcal{K}_0} \)-equivalence. The transformational properties of the subclass \( \mathcal{K}_2 \) are similar to those of the subclasses studied in Section [H] which is related to bringing the constraint for \( A^1 \) into the foreground among the constraints or gauges for other arbitrary elements; see further discussions below. Since the arbitrary element \( C \) is still not gauged to one, it parameterizes the \( \nu \)-component of admissible transformations in \( \mathcal{K}_2 \), \( U^{01} = X^1_{11}/(X^1 C) \), and this fact can again be interpreted in terms of generalized equivalence group.

**Theorem 25.** The equivalence groupoid of the subclass \( \mathcal{K}_2 \) of the class (I) singled out by the constraints \( A^k_x = 0, k = 0, \ldots, r, C_x = 0 \) and \( B_{xx} = 0 \) consists of the triples \((\theta, \bar{\theta}, \varphi)\)'s, where the point transformation \( \varphi \) is of the form

\[
\begin{align}
\bar{\dot{i}} &= T, \quad \bar{\dot{x}} = X^1 x + X^0, \quad \bar{\dot{u}} = U^1 u + \frac{X^1 U^1}{X^1 C} x + U^{00}, \quad (14a), \\
\text{the arbitrary-element tuples } \theta \text{ and } \bar{\theta} \text{ are related according to} \\
\bar{\dot{A}}^j &= \left( \frac{X^1}{T_t} \right)^j A^j, \quad \bar{\dot{A}}^i = \frac{X^1}{T_t} \left( A^i + \frac{U^{00}}{U^1} C - \frac{X^0}{X^1} \right), \quad \bar{\dot{\bar{A}}}^0 = \frac{1}{T_t} \left( \bar{A}^0 + \frac{U^1}{U^1} + \frac{X^1}{X^1} \right), \quad (14b), \\
\bar{\dot{B}} &= \frac{U^1}{T_t} \bar{\dot{B}} + \frac{X^1 U^1}{X^1 C} x + \frac{U^{00}}{T_t} A^1 \left( \frac{X^1 U^1}{X^1 C} x + U^{00} \right), \quad (14c) \\
\bar{\dot{\bar{C}}} &= \frac{X^1}{T_t U^1} \bar{\dot{C}}, \quad (14d)
\end{align}
\]

with \( j = 2, \ldots, r \), and \( T = T(t) \), \( X^1 = X^1(t) \), \( X^0 = X^0(t) \), \( U^1 = U^1(t) \) and \( U^{00} = U^{00}(t) \) are arbitrary smooth functions of \( t \) with \( T_1 \langle U^1 \rangle \not\equiv 1 \).
The usual equivalence group $\hat{G}_{K_2}$ of the subclass $K_2$ is constituted by the transformations (14a) – (14d) additionally satisfying the constraint $X^1_t = 0$. Hence it is clear that the subclass $K_2$ is not normalized in the usual sense.

The equation (14e) hints that the proper treatment of the related generalized equivalence group within the framework of point transformations needs considering the derivative $C_t$ as an additional arbitrary element $Z^0$ and prolonging the relation (14d) to $Z^0$ as a derivative of $C$,

$$\dot{Z}^0 = \frac{X^1}{T_tU_1}Z^0 + \left(\frac{X^1}{T_tU_1}\right)_t C \frac{T_t}{T_t}. \quad (14e)$$

We denote by $\bar{K}_2$ the class $K_2$ in which the tuple of arbitrary elements $\theta$ is formally extended to $\bar{\theta} = (A^0, \ldots, A^r, B, C, Z^0)$ with $Z^0 := C_t$.

**Corollary 26.** The class $\bar{K}_2$ is normalized in the generalized sense. The group $\hat{G}_{\bar{K}_2}$ constituted by the transformations of the form (14) is an effective generalized equivalence group of this class.

**Proof.** The set of the transformations of the form (14), which is temporally denoted by $M$, is closed with respect to the transformation composition and contains the identity transformation. Each transformation from $M$ is invertible by definition. So, $M$ is a group. The components of transformations from $M$ are of the same form as the components of admissible transformations and the formulas relating the initial and target arbitrary elements. This is why the group $M$ generates the equivalence groupoid $\bar{K}_2$ and, moreover, it is minimal among subgroups with such property. Therefore, $M$ is an effective generalized equivalence group of the class $\bar{K}_2$.

The entire generalized equivalence group $\hat{G}_{\bar{K}_2}$ of the class $\bar{K}_2$ is much wider than its effective part $\hat{G}_{K_2}$.

**Corollary 27.** The generalized equivalence group $\hat{G}_{\bar{K}_2}$ of the class $\bar{K}_2$ consists of the transformations of the modified form (14), where $T = T(t)$, $X^1 = X^1(t)$, $X^0 = X^0(t, C)$, $U^1 = U^1(t, C)$ and $U^{00} = U^{00}(t, C)$ are arbitrary smooth functions of their arguments with $T_tX^1(CU_C^{-1} - U^1) \neq 0$, and the partial derivatives of $X^0$, $U^1$ and $U^{00}$ in $t$ should be replaced by the corresponding restricted total derivatives in $t$ with $\bar{D}_t = \partial_t + Z^0\partial_C$.

**Proof.** Theorem 25 implies that elements of $\hat{G}_{\bar{K}_2}$ are of the modified form (14), where the group parameters $T$, $X^1$, $X^0$, $U^1$ and $U^{00}$ may depend on $t$ and the arbitrary elements $\bar{\theta}$. Hence partial derivatives of these parameter functions should be replaced by the corresponding total derivatives in $t$ with

$$D_t = \partial_t + \sum_\alpha u_\alpha + \delta_i w_\alpha + \sum_{k=0}^r A^k_t\partial A^k + B_t \partial B + C_t \partial C + Z^0_\alpha \partial Z^0 + \cdots.$$ 

After substituting $Z^0$ for the derivative $C_t$, the transformation components can be split with respect to the other derivatives of arbitrary elements in $t$. The splitting implies that in fact the group parameters do not depend on $A$’s, $B$ and $Z^0$, and, moreover, the parameters $T$ and $X^1$ do not depend on $C$. The nondegeneracy condition for elements of $\hat{G}_{\bar{K}_2}$ is modified in comparison with that for elements of the effective part $\hat{G}_{K_2}$ in view of the parameter function $U^1$ becoming dependent on $C$. This condition takes the form $T_tX^1U^1(C/U^1)_C \neq 0$ and reduces to the condition given in the statement of the theorem.

**Remark 28.** Given a class of differential equations with nontrivial effective generalized equivalence group, this group is in general not defined in a unique way. Indeed, consider the class $\bar{K}_2$. The effective generalized equivalence group $\hat{G}_{\bar{K}_2}$ defined in Corollary 26 is not a normal subgroup of the entire generalized equivalence group $\hat{G}_{\bar{K}_2}$ of the class $\bar{K}_2$. Each subgroup of $\hat{G}_{\bar{K}_2}$ that is conjugate to $\hat{G}_{\bar{K}_2}$ is an effective generalized equivalence group of the class $\bar{K}_2$. In other words, the class $\bar{K}_2$ possesses a wide family of conjugate effective generalized equivalence groups. The similar fact is even more obvious for the class $\bar{K}_3$ studied below.
To have the required subclass $\mathcal{K}_3$ of equations from the class $\mathcal{K}$ whose coefficients depend at most on $t$, we now only need to impose a more restrictive constraint on $B$, replacing the additional auxiliary equation $B_{xx} = 0$ by $B_x = 0$, which can be implemented by gauging $B$ within the class $\mathcal{K}_2$ using its equivalence transformations. Unfortunately, this deteriorates the normalization property since then the function $X^1$ parameterizing elements of the equivalence groupoid $\mathcal{G}_{\mathcal{K}_3}$ of the class $\mathcal{K}_3$ depends on the initial arbitrary elements $C$ and $A^0$ in a nonlocal way via the equation

$$\left( \frac{X^1_t}{C(X^1)^2} \right)_t = A^0 \frac{X^1_t}{C(X^1)^2}. \quad (15)$$

At the same time, the usual equivalence group $G_{\mathcal{K}_3}$ of the subclass $\mathcal{K}_3$ coincides with the group $\mathcal{K}_3$. The computation of the generalized equivalence group of the subclass $\mathcal{K}_3$ gives the same group, which is a trivial situation from the point of view of generalized equivalence. As a result, the class $\mathcal{K}_3$ is definitely not normalized in both the usual and the generalized senses. This is why we construct the extended generalized equivalence group of the subclass $\mathcal{K}_3$ in a rigorous way. In fact, this is the first construction of such kind in the literature.

We extend the arbitrary-element tuple $\theta$ to $\tilde{\theta} = (A^0, \ldots, A^r, B, C, Y^1, Y^2)$ with two more arbitrary elements, $Y^1$ and $Y^2$, which are functions of $t$ only and satisfy the auxiliary equations

$$Y^1_t = A^0, \quad Y^2_t = Ce^{Y^1}. \quad (16)$$

Thus, we also implicitly impose the auxiliary equations $Y^i_{uu} = Y^i_x = 0, \ |\alpha| \leq r, \ i = 1, 2$. Each value of $\tilde{\theta}$ satisfying all auxiliary equations of the class $\mathcal{K}_3$ as well as the above equations for $Y^1$ and $Y^2$ is associated with an equation of the form $|\bar{\theta}|$ with the corresponding value of $\theta$. We formally denote this equation by $\mathcal{L}_{\bar{\theta}}$ and the class of such equations by $\mathcal{K}_3$. It is obvious that the equations $\mathcal{L}_{\bar{\theta}_1}$ and $\mathcal{L}_{\bar{\theta}_2}$ coincide if $\theta^1 = \theta^2$. This defines a gauge equivalence relation on the value set of arbitrary-element tuple $\theta$. We show below that this gauge equivalence gives rise to a nontrivial gauge equivalence group of the class $\mathcal{K}_3$. (See Sections 2.1 and 2.5 of [33] for notions related to gauge equivalence, which is called trivial equivalence in [22].) Since the set of point transformations from $\mathcal{L}_{\bar{\theta}_1}$ to $\mathcal{L}_{\bar{\theta}_2}$ coincides with that from $\mathcal{L}_{\bar{\theta}_1}$ to $\mathcal{L}_{\bar{\theta}_2}$, the equivalence groupoid of $\mathcal{K}_3$ is isomorphic to the equivalence groupoid of $\mathcal{K}_3$ factorized with respect to the gauge equivalence. In the class $\mathcal{K}_3$, the constraint (15) can be solved with respect to $X^1$ in terms of $Y^2$,

$$X^1 = \frac{1}{\varepsilon_1 Y^2 + \varepsilon_0}, \quad (17)$$

where $\varepsilon_1$ and $\varepsilon_0$ are arbitrary constants with $(\varepsilon_1, \varepsilon_0) \neq (0, 0)$. Using this solution and the auxiliary equations (16), we prolong the relation (14b)–(14d) between initial and transformed arbitrary elements to $Y^1$ and $Y^2$. Thus, the equality chain

$$\tilde{Y}^1_t = \tilde{Y}^1_t T_t = \tilde{A}^0 T_t = A^0 + \frac{U^1_t}{U^1} + \frac{X^1_t}{X^1} = Y^1_t + \frac{U^1_t}{U^1} + \frac{X^1_t}{X^1}$$

implies $\tilde{Y}^1 = Y^1 + \ln |U^1 X^1| + \delta'$ for some constant $\delta'$. Considering the equality chain

$$\tilde{Y}^2_t = \tilde{Y}^2_t T_t = \tilde{C} e^{\tilde{Y}^1} T_t = \frac{X^1}{T_t U^1} C e^{Y^1} U^1 X^1 \delta T_t = \frac{\delta Y^2_t}{(\varepsilon_1 Y^2 + \varepsilon_0)^2},$$

where $\delta = e^{\delta'} \text{sgn}(U^1 X^1) \neq 0$, we derive for some constants $\varepsilon'_1$ and $\varepsilon'_0$ with $\varepsilon_0 \varepsilon'_1 - \varepsilon'_0 \varepsilon_1 = \delta$ that

$$\tilde{Y}^2 = \frac{\varepsilon'_1 Y^2 + \varepsilon'_0}{\varepsilon_1 Y^2 + \varepsilon_0}, \quad \text{and hence} \quad \tilde{Y}^1 = Y^1 + \ln(\delta U^1 X^1). \quad (18)$$

We use parentheses instead of vertical bars in the logarithm since $\delta U^1 X^1 > 0$. This completes the description of the equivalence groupoid $\mathcal{G}_{\mathcal{K}_3}$. Note that here

$$U^{01} = \frac{X^1 U^1}{X^1 C} = -\varepsilon_1 U^1 X^1 e^{Y^1}, \quad U^{01} = U^{01} \left( A^0 + \frac{U^1}{U^1} - \varepsilon_1 C X^1 e^{Y^1} \right) = T_t U^{01} A^0.$$
Theorem 29. Let $K_3$ be the subclass of equations from the class $\mathcal{K}$ with coefficients depending at most on $t$, which is singled out from the class $\mathcal{K}$ by the constraints $A_{k}^0 = C_x = B_x = 0$, $k = 0, \ldots, r$. The class $\tilde{K}_3$ of the same equations, where the arbitrary-element tuple is formally extended with the virtual arbitrary elements $Y^1$ and $Y^2$ defined by (16), is normalized in the generalized sense. Its generalized equivalence group $\tilde{G}_{\tilde{K}_3}$ consists of the transformations of the form

\[
\begin{align*}
\tilde{t} &= \tilde{T}(t, Y^1, Y^2), \quad \tilde{x} = \tilde{X}^1 x + \tilde{X}^0(t, Y^1, Y^2), \quad \tilde{X}^1 := \frac{1}{\varepsilon_1 Y^2 + \varepsilon_0}, \\
\tilde{u} &= \tilde{U}^1(t, Y^1, Y^2)(u - \varepsilon_1 \tilde{X}^1 e^{Y^1} x) + \tilde{U}^{00}(t, Y^1, Y^2), \\
\tilde{A}^j &= \left(\frac{X^1}{D_t} + A^j\right) \tilde{A}^0, \quad \tilde{A}^1 = \frac{\tilde{X}^1}{D_t} \left(A^1 + \frac{U^{00}}{U^1} C - \frac{D_t \tilde{X}^0}{X^1}\right), \\
\tilde{A}^0 &= \frac{1}{D_t} \left(A^0 + \frac{D_t \tilde{U}^0}{U^1} - \varepsilon_1 C \tilde{X}^1 e^{Y^1}\right), \\
\tilde{B} &= \tilde{U}^1 \frac{D_t^0}{D_t^0} B + \tilde{D_t} U^{00} + \varepsilon_1 U^1 \tilde{X}^1 e^{Y^1} \frac{D_t^0}{D_t^0} - U^{00} \tilde{A}^0, \quad \tilde{C} = \frac{\tilde{X}^1}{U^1} C, \\
\tilde{Y}^1 &= Y^1 + \ln(\delta U^1 \tilde{X}^1), \quad \tilde{Y}^2 = \frac{\varepsilon_1^2 Y^2 + \varepsilon_0^2}{\varepsilon_1 Y^2 + \varepsilon_0},
\end{align*}
\]

where \( j = 2, \ldots, r; \tilde{T}, \tilde{U}^0, \tilde{U}^{00} \) and \( \tilde{U}^1 \) are smooth functions of \( t, Y^1 \) and \( Y^2 \) with \( \tilde{T}, \tilde{U}^0 \neq 0; \varepsilon_0, \varepsilon_1, \varepsilon_0^1 \) and \( \varepsilon_1^1 \) are arbitrary constants with \( \delta := \varepsilon_0 \varepsilon_1^1 - \varepsilon_0^1 \varepsilon_1 \neq 0 \) and, moreover, \( \delta U^1 \tilde{X}^1 > 0; D_t = \partial_t + A^0 \partial_{Y^1} + C e^{Y^1} \partial_{Y^2} \) is the restricted total derivative operator with respect to \( t \).

**Proof.** In view of the above description of the equivalence groupoid $G_{\tilde{K}_3}$ of the class $\tilde{K}_3$, elements of $G_{\tilde{K}_3}$ have the general form

\[
\begin{align*}
\hat{t} &= \hat{T}(t, \theta), \quad \hat{x} = \hat{X}^1(t, \theta)x + \hat{X}^0(t, \theta), \quad \hat{u} = \hat{U}^1(t, \theta)u + \hat{U}^{01}(t, \theta)x + \hat{U}^{00}(t, \theta), \\
\hat{\theta} &= \Theta(t, x, u, \theta).
\end{align*}
\]

The computation of $G_{\tilde{K}_3}$ by the direct method is quite similar to the computation of $G_{\tilde{K}_3}$ and, after splitting with respect to $x$ and parametric derivatives of $u$, gives similar expressions for transformation components for the variables \((t, x, u)\) and similar constraints for parameter functions. The relations between the initial and target arbitrary elements in the equivalence groupoid just convert to the transformation components for arbitrary elements in the equivalence group. But there are several differences, which we are going to discuss.

In particular, the total derivative operators should be prolonged to the arbitrary elements. Since the arbitrary elements of the class $\tilde{K}_3$ depend at most on \( t \), the prolongation is essential only for $D_t$,

\[
D_t = \partial_t + \sum_{\alpha} u_{\alpha+\delta t} \partial_{u_{\alpha}} + \sum_{k=0}^r A_k^0 \partial_{A^k} + B_0 \partial_B + C_1 \partial_C + Y^1_\theta \partial_{Y^1} + Y^2_\theta \partial_{Y^2} + \cdots.
\]

The expression for $D_x$ is formally preserved, \( D_x = \partial_x + \sum_{\alpha} u_{\alpha+\delta x} \partial_{u_{\alpha}} \). As a result, all partial derivatives with respect to \( t \) in the expressions derived after splitting with respect to \( x \) and parametric derivatives of \( u \) are converted to the total derivatives with respect to \( t \).

The second difference is the possibility of splitting with respect to arbitrary elements and their derivatives. After substituting for the constrained derivatives $Y^1_\theta$ and $Y^2_\theta$ in view of (16) into the constraint for $\tilde{X}^1$,

\[
D_t^2 \frac{1}{\tilde{X}^1} = \left(\frac{C_t}{C} + A^0\right) D_t \frac{1}{\tilde{X}^1},
\]

we can split the resulting equation with respect to $A^{00}_0, \ldots, A^{00}_r, B_0, C_1, A^0_t$ and $C_t$. This leads to the system $\tilde{X}^1_A^0 = \cdots = \tilde{X}^1_A^r = 0, \tilde{X}^1_B = 0, \tilde{X}^1_C = 0, \tilde{X}^1_{Y^1} = 0, \tilde{X}^1_{Y^2} = 0$ and $(1/\tilde{X}^1)_{Y^2 Y^2} = 0$.
whose general solution is of the form (17). The expressions for the transformed arbitrary elements $\bar{A}^0, \ldots, \bar{A}^r, B$ and $\bar{C}$ can also be split with respect to unconstrained derivatives of arbitrary elements in $t$, implying that the derivatives of $\bar{T}, \bar{X}^0, \bar{U}^1$ and $\bar{U}^{00}$ with respect to $A^0, \ldots, A^r, B$ and $C$ are zero. Hence the operator $D_t$ can be replaced by the restricted total derivative operator $D_t$. In particular, the parameter function $U^{01}$ is defined by $U^{01} = (U^1D_t\bar{X}^1)/(\bar{X}^1C)$.

The additional auxiliary equations (16) are also treated in a different way. We substitute the expressions for $Y^1_t$ and $Y^2_t$ given by these equations into their expanded version for transformed arbitrary elements. Splitting the resulting equations with respect to the other derivatives of arbitrary elements leads to the system of determining equations for the $(Y^1, Y^2)$-components of equivalence transformations

$$\bar{Y}^2_1 = \bar{Y}^2_2 = \bar{Y}^2_3 = \bar{Y}^2_4 = 0, \quad i = 1, 2, \quad k = 0, \ldots, r,$$

$$\bar{Y}^1_1 = \frac{\bar{U}_1^1}{\bar{U}^1}, \quad \bar{Y}^1_2 = \frac{\bar{U}_1^2}{\bar{U}^1} - \frac{\varepsilon_1}{\varepsilon_1Y^2 + \varepsilon_0}, \quad \bar{Y}^2_2 = \frac{\bar{X}^1}{\bar{U}^1}e^{\bar{Y}^1_3-Y^1},$$

whose general solution is of the form presented in the statement of the theorem.

**Remark 30.** Each element of the generalized equivalence group $\bar{G}_{\bar{K}_3}$ generates a family of admissible transformations of the class $\bar{K}_3$ with sources at those values of $\bar{\theta}$ where the evaluation of $D_tT$ does not vanish,

$$\bar{G}_{\bar{K}_3} \ni T \mapsto \{(\bar{\theta}^1, \bar{\theta}^2, \bar{v}) \mid \bar{\theta}^1 \in \bar{S}_3, (D_tT)_{\bar{\theta}=\bar{\theta}^1} \neq 0, \bar{\theta}^2 = T\bar{\theta}^1, \bar{v} = (T|_{\bar{\theta}=\bar{\theta}^1})_{(t,x,u)} \} \subset \bar{G}_{\bar{K}_3}.$$

Here $\bar{S}_3$ is the value set of the arbitrary-element tuple $\bar{\theta}$ of the class $\bar{K}_3$.

The gauge equivalence group of the class $\bar{K}_3$ is the subgroup of $\bar{G}_{\bar{K}_3}$ that is singled out by the constraints $\varepsilon_0 = 1, \varepsilon_1 = 0, \bar{T} = t, \bar{X}^0 = 0, \bar{U}^1 = 1, \bar{U}^{00} = 0$. In other words, all the components of gauge equivalence transformations are identities, except the components for $Y^1$ and $Y^2$, for which we get $\bar{Y}^1 = Y^1 + \ln \varepsilon'_1, \bar{Y}^2 = \varepsilon'_1Y^2 + \varepsilon'_0$ with $\varepsilon'_1 > 0$. The usual equivalence group of the class $\bar{K}_3$ is singled out from $\bar{G}_{\bar{K}_3}$ by the constraints

$$\varepsilon_1 = 0, \quad \bar{T}Y^i = \bar{X}^0Y^i = \bar{U}_1^1Y^i = \bar{U}_1^{00}Y^i = 0, \quad i = 1, 2,$$

and its quotient group with respect the gauge equivalence group of the class $\bar{K}_3$ is isomorphic to the usual equivalence group of the class $K_3$.

It is obvious that the generalized equivalence group $\bar{G}_{\bar{K}_3}$ of the class $\bar{K}_3$ generates the whole equivalence groupoid of this class. At the same time, functions parameterizing the group depend on two more arguments, $Y^1$ and $Y^2$, than functions parameterizing the groupoid do. If we omit the arguments $Y^1$ and $Y^2$ in the parameter functions, the corresponding set of transformations still generates the equivalence groupoid but it is not a group with respect to the transformation composition. This shows that the class $\bar{K}_3$ may possess an effective generalized equivalence group being a proper subgroup of $\bar{G}_{\bar{K}_3}$, and its construction needs a more delicate consideration than, e.g., for the class $K_2$.

**Corollary 31.** The class $K_3$ is normalized in the extended generalized sense. Its extended generalized equivalence group $\bar{G}_{\bar{K}_3}$ can be identified with the effective generalized equivalence group of the class $\bar{K}_3$ that consists of the transformations of the form

$$\begin{align*}
\bar{t} &= T(t), \quad \bar{x} = X^1(x + X^{01}(t)Y^2 + X^{00}(t)), \quad X^1 := \frac{1}{\varepsilon_1Y^2 + \varepsilon_0}, \\
\bar{u} &= V(t)\left(\frac{u}{X^1} - e^{\bar{Y}^1}(\varepsilon_1x - \varepsilon_0X^{01} + \varepsilon_1X^{00})\right), \\
\bar{A}^i &= \frac{(X^1)^j}{T}A^i, \quad \bar{A}^1 = \frac{X^1}{T}(A^1 - X^{01}Y^2 - X^{00}), \quad \bar{A}^0 = \frac{1}{T} \left(A^0 + \frac{V^i}{V}\right),
\end{align*}$$

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The group \( G \) of the arbitrary element \( Y \) entire equivalence groupoid of the class \( \bar{M} \) by the equivalence transformation from \( M \) values of \( X \) smooth functions \( T \), \( X^{00} \), \( X^{01} \) and \( V \) are arbitrary smooth functions of \( t \) with \( T_i V \neq 0 \); \( \varepsilon_0, \varepsilon_1, \varepsilon_0' \) and \( \varepsilon_1' \) are arbitrary constants with \( \delta := \varepsilon_0 \varepsilon_1' - \varepsilon_0' \varepsilon_1 \neq 0 \) and, moreover, \( \delta V > 0 \).

**Proof.** We temporarily denote by \( M \) the set of the transformations of the above form. This set is a subset of the group \( G_{\bar{K}_3} \). It is singled out from \( G_{\bar{K}_3} \) by setting the following values for group parameters:

\[
\tilde{T} = T(t), \quad \tilde{X}^0 = X^1(X^{01}(t)Y^2 + X^{00}(t)) \quad \tilde{U}^0 = V(t)X^1, \quad \tilde{U}^{01} = -\varepsilon_1 V(t)e^{Y^1}, \quad \tilde{U}^{00} = V(t)e^{Y^1}(\varepsilon_0 X^{01}(t) - \varepsilon_1 X^{00}(t)).
\]

The set \( M \) is closed with respect to the transformation composition, i.e., \( M \) is a subgroup of the group \( G_{\bar{K}_3} \).

The subgroup \( M \) generates the entire equivalence groupoid \( G_{\bar{K}_3} \) of the class \( \bar{K}_3 \) and thus the entire equivalence groupoid of the class \( K_3 \). Indeed, let us fix any equation \( \bar{L}_\theta \) from the class \( \bar{K}_3 \).

The set \( T_{\bar{L}} \) of all admissible transformations with source at \( \bar{L}_\theta \) is parameterized by the arbitrary smooth functions \( T, \ X^0, \ U^0 \) and \( U^{00} \) of \( t \) and the arbitrary constants \( \varepsilon_0, \varepsilon_1, \varepsilon_0' \) and \( \varepsilon_1' \) with \( T_i U^1 \neq 0 \), \( \delta := \varepsilon_0 \varepsilon_1' - \varepsilon_0' \varepsilon_1 \neq 0 \) and \( \delta U^1 X^1 > 0 \), where \( X^1 \) is defined by (17) for the fixed value of the arbitrary element \( Y^1 \), \( Y^2 = Y^2(t) \). Each admissible transformation from \( T_{\bar{L}} \) is generated by the equivalence transformation from \( M \) with the same values of \( T, \varepsilon_0, \varepsilon_1, \varepsilon_0' \) and \( \varepsilon_1' \), and the values of \( X^{00}, \ X^{01} \) and \( V \) defined by

\[
X^{00} = \varepsilon_0 X^0(t) - Y^2(t)U^{00}(t)U^{11}(t)e^{Y^1(t)}, \quad X^{01} = \varepsilon_1 X^0(t) + U^{00}(t)U^{11}(t)e^{Y^1(t)}, \quad V = \frac{U^1(t)}{\varepsilon_1 Y^2(t) + \varepsilon_0}.
\]

This establishes a one-to-one correspondence between \( M \) and \( T_{\bar{L}} \), and thus the subgroup \( M \) is minimal among the subgroups of \( G_{\bar{K}_3} \) that generate the groupoid \( G_{\bar{K}_3} \).

Therefore, \( M \) is the effective generalized equivalence group of the class \( \bar{K}_3 \). \( \square \)

Nevertheless, we can further gauge arbitrary elements of the class \( K_3 \) in such a way that the corresponding subclasses of \( K_3 \) have better normalization properties. Up to \( G_{\bar{K}_3} \)-equivalence we can set, as above, \( C = 1 \) and \( A^1 = 0 \). Moreover, since the class \( K_3 \) is parameterized by functions depending at most on \( t \), \( G_{\bar{K}_3} \)-equivalence allows us to further gauge arbitrary elements by setting \( A^0 = 0 \) and \( B = 0 \), which results in the class \( K \) normalized in the usual sense and whose usual equivalence group \( G_K \) is finite-dimensional, see Section S. At the same time, gauging arbitrary elements by usual equivalence transformations in a class that is not normalized in the usual sense does not allow one to easily control the changing of the corresponding equivalence groupoid. This is why we revise the order of gauging in order to construct a subclass normalized in the usual sense for each step of gauging.

**Remark 32.** In general, if a multiple gauge is realized step-by-step, then the order of steps is also important and intermediate subclasses may have inferior normalization properties than the original class and the final gauged subclass.

We use the following order of gauging for singling out the subclass \( \mathcal{K} \) from the class \( \mathcal{K}_1 \): \( C = 1, \ A^1 = 0, \ A^0 = 0, \ B^1 = 0 \) and \( B^0 = 0 \), which successively constrains the group(oid) parameters by

\[
U^1 = \frac{X^1}{T_i}, \quad U^0 = \frac{X^1 x + X^0}{T_i}, \quad \frac{T_i U}{T_t} = 2 \frac{X^1}{X^1}, \quad \left( \frac{X^1}{(X^1)^2} \right)_t = 0, \quad \left( \frac{X^0}{T_t} \right)_t = 0.
\]
Note that, e.g., the ordering of the last two gauges is not essential but if we try to carry out at least one of them before the gauge $A^0 = 0$, then we obtain a subclass normalized in the generalized extended sense; cf. the derivation of Corollary 31. Thus, the parameter function $X^1$ of the equivalence groupoid for the subclass gauged with $C = 1$, $A^1 = 0$ and $B^1 = 0$ is related to the initial value of the arbitrary element $A^0$ via the equation (15) with $C = 1$.

Finally, the solution of the group classification problem for the class $K_3$ up to the generalized extended $G\hat{\sim}$-equivalence reduces to that for the class $K$ up to the usual $G\hat{\sim}$-equivalence. Combining results of Sections 7 and 8 we readily obtain the following assertion.

**Theorem 33.** A complete list of $G\hat{\sim}$-inequivalent Lie symmetry extensions in the class $K_3$ (resp. $G\hat{\sim}$-inequivalent Lie symmetry extensions in the class $K$) is exhausted by Cases 2, 10a, 10b, 10c and 11 given in Table 4 and Section 8.

Whereas each equation from the class $K_3$ admits the two-dimensional algebra $(P(\chi^1), P(\chi^2))$ with $\chi^1$ and $\chi^2$ constituting a fundamental set of solutions of the linear ordinary differential equation $\chi^1_t = A^0 \chi^1_t + B^1 \chi^1$, the kernel Lie invariance algebra of this class is zero since the functions $\chi^1$ and $\chi^2$ depend on the arbitrary elements $A^0$ and $B^1$. The situation with the class $K$ is different: its kernel Lie invariance algebra is $\langle P(1), P(t) \rangle$.

## 10 Equations with space-dependent coefficients

The subclass $F_1$ of equations with space-dependent coefficients is singled out from the class $P$, which is normalized in the usual sense, by the constraints $A^k_t = 0$, $k = 0, \ldots, r$, $B_t = 0$ and $C_t = 0$. This is why the usual equivalence group $G\hat{\sim}$ of the subclass $F_1$ is a subgroup of the equivalence group $G\hat{\sim}$ of the class $P$ that consists of transformations preserving the above constraints. In view of Proposition 2 the $(t, x, u, A', C, A^1)$-components of transformations from $G\hat{\sim}F_1$ are of the form

$$\tilde{t} = T(t), \quad \tilde{x} = X(t, x), \quad \tilde{u} = U^1(t)u + U^0(t, x), \quad A' = \frac{T_t}{(X)^r} \tilde{A}', \quad C = \frac{T_t U^1}{X} \tilde{C},$$

$$A^1 = \frac{T_t}{X} A^1 + T_t \sum_{j=2}^{r} \bar{A}^j \left( \frac{1}{X} D_x \right)^j \frac{1}{X} - \frac{T_t U^0}{X} \tilde{C} + \frac{X}{X} X_t \tilde{C},$$

where $T = T(t), X = X(t, x), U^1 = U^1(t)$ and $U^0 = U^0(t, x)$ are smooth functions of their arguments such that $T_t X_x U^1 \neq 0$. (It is convenient here to conversely express source arbitrary elements via target ones.) Differentiating the $A'$- and $C$-components with respect to $t$ and taking into account the additional constraints for arbitrary elements of the subclass $F_1$, we derive classifying equations for admissible transformations in $F_1$,

$$\frac{T_t}{(X)^r} X_t \tilde{A}' + \left( \frac{T_t}{(X)^r} \right)_t \tilde{A}' = 0, \quad \frac{T_t U^1}{X} X_t \tilde{C} + \left( \frac{T_t U^1}{X} \right)_t \tilde{C} = 0,$$

which should be split with respect to derivatives of arbitrary elements when constructing usual equivalence transformations of the subclass $F_1$. Hence $X_t = U^0_t = 0$ for these transformations. Then treating the $A^1$-component in the same way gives one more condition for group parameters, $U^0_t = 0$. In fact, the obtained conditions for group parameters constitute the complete system of such conditions that singles out the group $G\hat{\sim}F_1$ as a subgroup of the group $G\hat{\sim}P$.

**Proposition 34.** The usual equivalence group $G\hat{\sim}F_1$ of the class $F_1$ of general Burgers–Korteweg–de Vries equations with space-dependent coefficients consists of the transformations in the joint space of $(t, x, u, \theta)$ whose $(t, x, u)$-components are of the form

$$\tilde{t} = c_1 t + c_2, \quad \tilde{x} = X(x), \quad \tilde{u} = \tilde{c}_3 u + U^0(x),$$
where \(c_1, c_2\) and \(c_3\) are arbitrary constants and \(X = X(x)\) and \(U^0 = U^0(x)\) are arbitrary smooth functions of \(x\) such that \(c_1 X_x c_3 \neq 0\).

The existence of classifying conditions for admissible transformations means that the class \(F_1\) is definitely not normalized in any sense. This is why a proper starting point for computing the equivalence group of a subclass \(F'\) of \(F_1\) is not the equivalence group of \(F_1\) but the equivalence group of the superclass \((1)\) or of a normalized subclass of \((1)\) that contains \(F'\). Two parameters of the group of the class \(F_1\) are of the same arbitrariness as the arbitrary elements of this class. Therefore, we can set two gauges for the arbitrary elements of \(F_1\) using parameterized families of transformations from \(G_{F_1}^{-}\). Following the consideration of the superclass \((1)\) in Section 3.1 we successively gauge \(C\) to 1 and \(A^1\) to 0.

In view of the expression for the \(C\)-component of transformations from \(G_{F_1}^{-}\), the gauge \(C = 1\) can really be implemented. We denote the corresponding subclass of \(F_1\) by \(F_2\). The usual equivalence group \(G_{F_2}^{-}\) of the subclass \(F_2\) turns out to be the subgroup of \(G_{F_1}^{-}\) singled out by the constraint \(T_1 U^1/X_x = 1\), i.e., \(X_xx = 0\) and \(U^1 = X_x/T_1\). For transformations from \(G_{F_2}^{-}\), the \(C\)-component can be neglected and the expression for the \(A^1\)-component simplifies to \(\tilde{A}^1 = X_x A^1/T_1 + U^0\). It becomes clear that the further gauge \(A = 0\) is also realizable, which gives a subclass of \(F_2\), and we denote it by \(F_3\). The usual equivalence group \(G_{F_3}^{-}\) of this subclass turns out to be the subgroup of the group \(G_{F_2}^{-}\) singled out by the constraint \(U^0 = 0\). In other words, the group \(G_{F_3}^{-}\) is just four-dimensional and consists of the transformations of the simple form

\[
\tilde{t} = c_1 t + c_2, \quad \tilde{x} = c_3 x + c_4, \quad \tilde{u} = \frac{c_3}{c_1} u, \quad \tilde{A}' = \frac{(c_3)^2}{c_1} A', \quad \tilde{A}^0 = \frac{1}{c_1} A^0, \quad \tilde{B} = \frac{c_3}{(c_1)^2} B,
\]

where \(j = 2, \ldots, r\), and \(c_1, \ldots, c_4\) are arbitrary constants with \(c_1 c_3 \neq 0\). The \(A^1\)-component can be neglected.

Since the subclasses \(F_2\) and \(F_3\) are constructed via gauging the arbitrary elements of the class \(F_1\) by usual equivalence transformations, and \(G_{F_1}^{-} \supset G_{F_2}^{-} \supset G_{F_3}^{-}\), then any classification problem for the class \(F_1\) up to \(G_{F_1}^{-}\)-equivalence (like the group classification problem or the description of the corresponding equivalence groupoid) reduces to the respective classification problem for the class \(F_3\) up to \(G_{F_3}^{-}\)-equivalence, cf. Proposition 7.

We begin with the description of the equivalence groupoid \(G_{F_3}^{-}\) of the class \(F_3\). Since the class \(F_3\) is not normalized and neither are its superclasses \(F_1\) and \(F_2\), this problem should be considered as the classification of admissible transformations up to \(G_{F_3}^{-}\)-equivalence. See Section 2.6 and 3.4 in [36]. Since the class \(F_3\) is a subclass of the class \((1)\), the equivalence groupoid \(G_{F_3}^{-}\) is a subgroupoid of the equivalence groupoid of the class \((1)\). Therefore, all the restrictions from Theorem 3 on the form of admissible transformations are relevant here. We solve the relations \((19a)\)–\((19c)\) with respect to the source arbitrary elements, and differentiate the result with respect to \(t\). This gives the classifying conditions for admissible transformations in terms of target arbitrary elements,

\[
(X_x^1 x + X_0^0) \tilde{A}'_x + \left(\frac{T_1}{T_1} - j \frac{X_0^1}{X_1^1}\right) \tilde{A}' = 0, \tag{19a}
\]

\[
(X_x^1 x + X_0^0) \tilde{A}_x^0 + \frac{T_1}{T_1} \tilde{A}_x^0 = \frac{1}{T_1} \left(\frac{2X_0^1}{X_1^1} - \frac{T_1}{T_1}\right), \tag{19b}
\]

\[
(X_x^1 x + X_0^0) \tilde{B}_x + \left(\frac{2T_1}{T_1} \frac{X_0^1}{X_1^1}\right) \tilde{B} = -\frac{T_1}{X_1^1} (X_x^1 x + X_0^0)^2 \tilde{A}_x^0
\]

\[
- \frac{X_1^1}{T_1} \left(\frac{T_1}{T_1} \frac{X_x^1 x + X_0^0}{X_1^1} \right)_t \tilde{A}_x^0 + \frac{X_1^1}{T_2^2} \left(\frac{T_1}{X_1^1} \left(\frac{X_x^1 x + X_0^0}{T_1}\right)_t\right)_t, \tag{19c}
\]

where the initial space variable \(x\) should be substituted, after expanding all derivatives, by its expression via \(\tilde{x}\), \(x = (\tilde{x} - X_0^0)/X_1^1\). Since admissible transformations with \(T_1 t = X_0^0 = X_1^1 = 0\)
are generated by the usual equivalence group $G_{F_3}^{\sim}$, the problem on classifying of admissible transformations can be stated in the following way: To find $G_{F_3}^{\sim}$-inequivalent values of the arbitrary-element tuple $\tilde{\kappa} = (\tilde{A}^0, \tilde{A}^2, \ldots, \tilde{A}^\tau, \tilde{B})$ for which the system (19) with respect to $(T, X^0, X^1)$ has solutions with $(T_0, X^0_0, X^1_0) \neq (0, 0, 0)$.

For each fixed $t$, the system (19) implies a system of ordinary differential equations with respect to the arbitrary-element tuple $\tilde{\kappa}$ of the form

$$(\tilde{v}_1 \tilde{x} + \tilde{v}_2)\tilde{A}_2^\nu + (\tilde{v}_3 - \tilde{v}_1)\tilde{A}_j^\nu = 0, \quad j = 2, \ldots, n,$$

$$(\tilde{v}_1 \tilde{x} + \tilde{v}_2)\tilde{A}_2^0 + \tilde{v}_3 \tilde{A}^0 = \tilde{v}_4,$$

$$(\tilde{v}_1 \tilde{x} + \tilde{v}_2)B_2^\nu + (2\tilde{v}_3 - \tilde{v}_1)\tilde{B} = \tilde{v}_0(\tilde{v}_1 \tilde{x} + \tilde{v}_2)^2\tilde{A}_2^\nu - (\tilde{v}_7 \tilde{x} + \tilde{v}_8)\tilde{A}^0 + \tilde{v}_5 \tilde{x} + \tilde{v}_6,$$

where $\tilde{v}_1, \ldots, \tilde{v}_0$ are constants. For each value of the arbitrary-element tuple $\tilde{\kappa}$, we denote by $N_{\tilde{\kappa}}$ the span of the set through which $\tilde{\nu} = (\tilde{v}_1, \ldots, \tilde{v}_0)$ runs when the tuple $(T, X^0, X^1)$ takes values possible for elements of $G_{F_3}^{\sim}$ with the target $\tilde{\kappa}$ and the variable $t$ is varied. It is obvious that $k_{\tilde{\kappa}} := \dim\{(\tilde{v}_1, \tilde{v}_2) \mid \tilde{\nu} \in N_{\tilde{\kappa}}\} \in \{0, 1, 2\}$. The consistency of system (19) with the condition $\tilde{A}^\tau \neq 0$ implies that $\dim\{(\tilde{v}_1, \tilde{v}_2) \mid \tilde{\nu} \in N_{\tilde{\kappa}}\} = k_{\tilde{\kappa}}$. Therefore, the equality $k_{\tilde{\kappa}} = 0$ means that $T_0 = X^0_0 = X^1_0 = 0$ for all admissible transformations in $F_3$ with target at $\tilde{\kappa}$. The further consideration partitions into five principal cases, depending on the value $k_{\tilde{\kappa}}$ and certain constraints for components of nonzero $\tilde{\nu} \in N_{\tilde{\kappa}}$. I) $\tilde{\nu} \tilde{\nu} = 0$, II) $\tilde{\kappa} = 0$, III) $\tilde{\kappa} = 0$, IV) $\tilde{\kappa} = 0$, V) $\tilde{\kappa} = 2$. For each of these cases, we integrate the corresponding set of independent copies of the system (20) and thus derive the respective form of $\tilde{\kappa}$ parameterized by arbitrary constants. The relations (6b)–(6c) between $\kappa$ and $\tilde{\kappa}$ imply that the arbitrary-element tuple $\kappa$ is of the same form as $\tilde{\kappa}$, but maybe with other values of parameter constants. In other words, subclasses associated with the above classification cases are invariant under the action of the equivalence groupoid $G_{F_3}^{\sim}$, which additionally justifies the chosen partition into cases. We substitute the derived expressions for $\kappa$ and $\tilde{\kappa}$ into the relations (6b)–(6c) and, assuming $\tilde{x} = X^1 x + X^0$, split the resulting equations with respect to $x$. When integrating the obtained system of determining equations for the parameter functions $T$, $X^0$ and $X^1$ of point transformations mapping the equation $L_\kappa$ to the equation $L_{\tilde{\kappa}}$, we check for the existence of solutions with $(T_0, X^0_0, X^1_0) \neq (0, 0, 0)$. This implies constraints for constants parameterizing the expressions for $\kappa$, which is in fact identical to the similar constraints for $\tilde{\kappa}$. Finally, we gauge constant parameters of arbitrary elements with transformations from $G_{F_3}^{\sim}$, where this is possible, and show that the lists of $G^{\sim}$-inequivalent cases of admissible-transformation extensions and of Lie symmetry extensions in the class $F_3$ coincide.

Note that a complete list of $G_{F_3}^{\sim}$-inequivalent Lie symmetry extensions in the class $F_3$ (resp. $G_{F_3}^{\sim}$-inequivalent Lie symmetry extensions in the class $F_1$) is exhausted by Cases II 2 8 10 and 11 of Table I. This follows from the facts that the integer $k_{3}$, which is defined in Section 3 is $G_{F_3}^{\sim}$-invariant when considered for equations from the class $F_3$ and is greater than or equal to 1 for these equations. Case I of Table I is the generic case for the class $F_3$. In other words, the kernel Lie invariance algebra of the class $F_3$ is $\mathfrak{g}_{F_3} = \langle D(1) \rangle$.

Now we list the subclasses of $F_3$ with admissible-transformation extensions. For each of the subclasses, we present the corresponding form of arbitrary elements, the description of admissible transformations and the equivalent case of Table I. Below $j = 2, \ldots, r; a$’s and $b$’s are constants parameterizing the arbitrary-element tuple within the subclass under consideration, and $c$’s are constants parameterizing admissible transformations.

I. $A^j = a_j x^j |x|^{-\nu_3}, A^0 = a_{00} + a_{01} |x|^{-\nu_3}$, $B = x (b_0 + b_1 |x|^{-\nu_3} + b_2 |x|^{-2\nu_3})$ modulo shifts of $x$, where $\nu_2, \nu_3 \neq 0$; $a_{00} = 0$ if $\nu_3 = 2$; $b_0 = 0$ if $a_{00} = 0$; $a_{00} a_{01} = (\nu_3 - 2) b_1, (\nu_3 - 2)^2 b_0 = (\nu_3 - 1) a_{00}^2$, and $a_{00} b_1 = (\nu_3 - 1) b_1^2$. Then $X^0 = 0$ (due to the above shifts of $x$), $|X^1|^{\nu_3}/T_1$ =: $c_0$ = const, $c_0 \neq 0, \nu_3, a_j = c_0 a_j, a_0_1 = c_0 a_0_1$ and $b_2 = c_0^2 b_2$. $a_r = 1$ mod $G_{F_3}^{\sim}$. 

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a) If \((\nu_3, a_{01}) \neq (2, 0)\), then the expression for \(T\) is given by the general solution of the equation \(T_{tt}/T_t = \lambda - \bar{\lambda} T_t\) with the constants \(\lambda\) and \(\bar{\lambda}\) defined by

\[
\left( \frac{a_{00}}{\bar{a}_{00}} \right) = \frac{\nu_3 - 2}{\nu_3} \left( \frac{\lambda}{\bar{\lambda}} \right), \quad \left( \frac{b_1}{\bar{b}_1} \right) = \nu_3 \left( \frac{\nu_3 a_{01}}{\nu_3 \bar{a}_{01}} \right).
\]

(See \[30\] for a proper representation of this solution.) Since \(a_{00}, b_0, b_1 = 0 \mod G_{\mathbb{F}_3^*}\), this case reduces to Case 3 of Table I

b) If \((\nu_3, a_{01}) = (2, 0)\), then \(a_{00} = b_1 = 0\), and the expression for \(T\) is given by the general solution of the equation

\[
\left( \frac{T_{tt}}{T_t} \right)_t - \frac{\nu_3 - 1}{\nu_3} \left( \frac{T_{tt}}{T_t} \right)^2 = \nu_3 \bar{b}_0 T_t^2 - \nu_3 b_0.
\]

Since \(b_0 = 0 \mod G_{\mathbb{F}_3^*}\), this case reduces to Case 4 of Table I

II. \(A^j = a_j x^j, A^0 = a_0 \ln |x| + a_{00}, B = x(b_0 + b_1 \ln |x| + b_2 \ln^2 |x|)\) modulo shifts of \(x\) with \(a_r \neq 0, b_0 = \nu_3 a_{00}, b_2 = b_0 = -b_0 b_0 a_{00}\). Then \(X^0 = 0\) (due to the above shifts of \(x\)). \(T = c_1 t^2x^2c_2\) with \(c_1 \neq 0, \bar{a}_j = c_1^{-1} a_j, \bar{a}_{00} = c_1^{-1} a_{00}, \bar{b}_0 = b_0 + b_0 b_1 c_2 (a_1 - c_1 a_{00} - a_{00})(c_1 \bar{a}_{00} - a_{00})\), and \(X^1\) runs through the set of (nonzero) solutions of the equation

\[
2 \frac{X^1}{X^0} = a_0 \ln |X^1| + c_1 \bar{a}_{00} - a_{00}.
\]

If \(a_{00} = 0\), then \(a_r = 1 \mod G_{\mathbb{F}_3^*}\) and \(a_{00} = 1 \mod G_{\mathbb{F}_3^*}\), giving Case 7 of Table I. Otherwise, \(a_0 = 1 \mod G_{\mathbb{F}_3^*}\) and \(a_{00} = 1 \mod G_{\mathbb{F}_3^*}\), i.e., we obtain Case 8 of Table I.

III. \(A^j = a_j e^{-r j x}, A^0 = a_0 e^{-r j x} + a_{00}, B = b_0 + b_1 e^{-r j x} + b_2 e^{-r j x}, a_r \neq 0, b_0 = \nu_3^{-1} a_{00}\) and \(b_1 = \nu_3^{-1} a_{00} a_{00}\). Then \(X^1 = c_1 \neq 0, X^0 = \nu_3^{-1} \ln (c_0 T_t), c_0 T_t > 0, \nu_3 = c_1^{-1} \nu_3, \bar{a}_j = c_0 c_1^{-1} a_j, \bar{a}_{00} = c_0 a_{00}, b_2 = c_0^2 c_1 b_2\), the expression for \(T\) is given by the general solution of the equation \(T_{tt}/T_t = \tilde{a}_{00} T_t - a_{00}\). We can gauge \((\nu_3, a_r) = (-1, 1) \mod G_{\mathbb{F}_3^*}\) and \(a_{00} = 0 \mod G_{\mathbb{F}_3^*}\), which leads to Case 2 of Table I. One of the constants \(a_{00}\)'s with \(j < r, a_0\) or \(b_2\), if it is nonzero, can be set to \(\pm 1\) by shifts of \(x\).

IV. \(A^j = a_j, A^0 = a_0, B = b_1 x + b_0\) with \(a_r \neq 0\), where also \((r - 2)^2 b_1 \neq (r - 1) a_0^2\) or \(a_j \neq 0\) for some \(j < r\). Then \(T = c_1 t^2x^2c_2\) with \(c_1 \neq 0, X^1 = c_3 \neq 0, \bar{a}_j = c_1^{-1} c_3 a_j, \bar{a}_{00} = c_1^{-1} a_{00}, \bar{b}_1 = c_1^{-2} b_1\), and \(X^0\) runs through the solution set of the equation \(X_0^0 - a_0 X_0^0 - b_1 X^0 = c_1^2 b_0 - c_3 b_0\). Gauging \(b_0 = 0 \mod G_{\mathbb{F}_3^*}\) and \(a_r = 1 \mod G_{\mathbb{F}_3^*}\), we reduce this case to Case 10 of Table I.

V. \(A^r = a_r \neq 0, A^j = 0, j = 2, \ldots, r - 1, A^0 = a_0, B = b_1 x + b_0\) with \((r - 2)^2 b_1 = (r - 1) a_0^2\). We have \((X^1)^r/T_t = c_1 \neq 0, \bar{a}_r = c_1 a_r\), and the parameter function \(T\) runs through the solution set of either the equation

\[
\frac{r - 2}{r} \frac{T_{tt}}{T_t} = \tilde{a}_0 T_t - a_{00} \quad \text{or} \quad \left( \frac{T_{tt}}{T_t} \right)_t - \frac{r - 1}{r} \left( \frac{T_{tt}}{T_t} \right)^2 = \tilde{a}_0 T_{tt} + r \tilde{b}_0 T_t^2 - r b_0
\]

if \(r > 2\) or \(r = 2\), respectively. The expression for \(X^0\) is found from the equation

\[
\frac{1}{T_t} \left( \frac{X_0^0}{T_t} \right)_t - \tilde{a}_0 X_0^0 - \tilde{b}_1 X^0 = \tilde{b}_0 - b_0 \frac{X^1}{T_t^2}.
\]

For any allowed value of \(r\), we have \(a_0, b_0, b_1 = 0 \mod G_{\mathbb{F}_3^*}\) and thus get Case 11 of Table I.

There are no additional \(G_{\mathbb{F}_3^*}\)-gauges of constants parameterizing the arbitrary-element tuple \(\kappa\).

By \(F_3^i, i\) we denote the subclass of equations from the class \(F_3\) whose maximal Lie invariance algebras are similar to the algebra presented in Case \(i\) of Table I \(i \in \{1, 2, 3, 4, 7, 8, 10, 11\}\).
In this way, we partition the class \( F_1 \) into eight subclasses, that are associated with cases of Lie symmetry extensions in this class. The seven subclasses \( F_{3,i}, i = 2, 3, 4, 7, 8, 10, 11 \) can be shown to be normalized in the generalized sense, and the complement to their union, which is the subclasses \( F_3 \) is normalized in the usual sense with respect to the usual equivalence group \( G_{F_3} \) of the entire class \( F_3 \). There are no point transformations between equations from the subclasses \( F_{3,i} \) and \( F_{3,i'} \) if \( i \neq i' \). In other words, the equivalence groupoid \( G_{F_3} \) of the class \( F_3 \) is the disjoint union of the equivalence groupoids of the normalized subclasses \( F_{3,i} \)’s.

**Remark 35.** The above consideration implies that the Lie symmetry extension given in Case 11 of Table 1 is maximal if and only if the arbitrary-element tuple \( \kappa = (A^0, A^2, \ldots, A^r, B) \) is not of one of the forms listed in Cases I–V.

### 11 Lie reductions and exact solutions

Since the class (1) is mapped onto its subclass (5) by a family of equivalence transformations and both these classes are normalized, for studying Lie reductions of equations from the class (1) we can apply the technique suggested in [32] to the subclass (5). Within the framework of the standard approach to Lie reduction, one considers each case of Lie symmetry extensions separately, lists subalgebras of the corresponding maximal Lie invariance algebra that are inequivalent with respect to internal automorphisms of this algebra, constructs ansatzes with them and then uses these ansatzes to reduce relevant original equations to equations with fewer independent variables. For a normalized class of differential equations, one can classify Lie reductions of equations from the class with respect to its equivalence group of respective kind, taking subalgebras in the projection of its equivalence algebra [32]. Restrictions for appropriate subalgebras are relevant here except those restrictions related to the property to be maximal for some equations from the class.

A complete list of \( G^\sim \)-inequivalent one-dimensional subalgebras of the algebra \( g \) is exhausted by four subalgebras, \( \langle D(1) \rangle, \langle S(1) \rangle, \langle S(e^t) \rangle \) and \( \langle P(1) \rangle \), cf. Table 1. The subalgebra \( \langle P(1) \rangle \) does not appear in Table 1 since, in view of Lemma 17, there is no equation from the class (5) for which this algebra is the maximal Lie invariance algebra. This subalgebra is associated with Case 9 of Table 1. We construct an ansatz for \( u \) with each of the listed one-dimensional subalgebras, select \( G^\sim \)-inequivalent equations from the subclass (5) that are invariant with respect to this subalgebra and reduce them to ordinary differential equations. In the course of selecting invariant equations, we use Table 1 but neglect the conditions of maximality of invariance algebras, which are collected in Remark 22. As a result, we obtain the following \( G^\sim \)-inequivalent reductions to ordinary differential equations:

**Case 1.** \( \langle D(1) \rangle \):
- \( u = \varphi(\omega), \omega = x, \varphi \varphi_\omega = \sum_{j=2}^r A^j(\omega)\varphi^{(j)} + A^0(\omega)\varphi + B(\omega); \)

**Case 5.** \( \langle S(1) \rangle \):
- \( u = \varphi(\omega)x, \omega = t, \varphi_\omega + \varphi^2 = \beta(\omega); \)

**Case 6.** \( \langle S(e^t) \rangle \):
- \( u = \varphi(\omega)x + x \ln |x|, \omega = t, \varphi_\omega + \varphi^2 = \sum_{j=2}^r (-1)^j(j-2)!\alpha^{(j)}(\omega) + \beta(\omega); \)

**Case 9.** \( \langle P(1) \rangle \):
- \( u = \varphi(\omega), \omega = t, \varphi_\omega = 0. \)

Now we classify two-dimensional subalgebras of \( g \) that are subalgebras of appropriate subalgebras of \( g \) and carry out corresponding Lie reductions of equations from the class (5) to algebraic equations. To obtain a complete list of such subalgebras up to \( G^\sim \)-equivalence, we realize the following algorithm: take two operators of the most general form, verify that they satisfy necessary conditions and then simplify successively their form until all cases are listed. Computations
from the proof of Theorem 19 can be used here. At first, let two vector fields \( Q^1 \) and \( Q^2 \) be of the form \( Q^i = D(t^i) + S(\zeta^i) + P(\chi^i), i = 1, 2 \), where the parameter functions \( \tau^1 \) and \( \tau^2 \) are linearly independent. As has already been shown in the case \((k_1, k_2, k_3) = (0, 0, 2)\) of the above proof, this case is exhausted by the subalgebras \( \langle D(1), D(t) - P(1) \rangle \) and \( \langle D(1), D(t) - S(\nu^{-1}) \rangle \), where \( \nu \neq 0 \). Suppose that \( \tau^1 \) and \( \tau^2 \) are linearly dependent but are not simultaneously zero. Then, up to linearly re-combining \( Q^1 \) and \( Q^2 \), we can assume that \( \tau^1 \neq 0, \tau^2 = 0 \) and, moreover, the first basis vector field can be transformed to \( Q^1 = D(1) \). If \( \zeta^2 \neq 0 \) in the new \( Q^2 \), the consideration is entirely reduced to the case \((k_1, k_2, k_3) = (0, 1, 1)\) of the proof of Theorem 19 which gives the subalgebras \( \langle D(1), S(1) \rangle \) and \( \langle D(1), S(\epsilon^1) \rangle \). Otherwise we have two other inequivalent possibilities for \( Q^2 \). \( Q^2 = P(1) \) and \( Q^2 = P(\epsilon^1) \). The basis elements cannot be of the form \( Q^i = S(\zeta^i) + P(\chi^i), i = 1, 2 \), with \((\zeta^1, \zeta^2) \neq (0, 0)\) in view of Lemma 17. And finally, the subalgebra spanned by \( Q^i = P(\chi^i), i = 1, 2 \), with linearly independent parameter functions \( \chi^1 \) and \( \chi^2 \) does not provide a Lie ansatz for \( u \). Therefore, \( G^\sim \)-inequivalent reductions of equations from the class \( \mathbb{L} \) to algebraic equations are exhausted by the following reductions:

- **Case 2**: \( \langle D(1), D(t) - P(1) \rangle \): \( u = \varphi e^x; \varphi^2 = (a_0 + \sum_{j=2}^r a_j)\varphi + b; \)

- **Case 3**: \( \langle D(1), D(t) - S(\frac{1}{2}) \rangle \): \( u = \varphi x|\nu|^r; (\nu + 1)\varphi^2 = (a_0 + \sum_{j=2}^r (\nu + 1) j! a_j)\varphi + b; \)

- **Case 4**: \( \langle D(1), S(1) \rangle \): \( u = \varphi x; \varphi^2 = b; \)

- **Case 5**: \( \langle D(1), S(\epsilon^1) \rangle \): \( u = \varphi x + x \ln|x|; \varphi^2 = \sum_{j=2}^r (-1)^j (j-2)! a_j + b; \)

- **Case 6**: \( \langle D(1), P(1) \rangle \): \( u = \varphi; a_0\varphi = 0; \)

- **Case 7**: \( \langle D(1), P(1) \rangle \): \( u = \varphi x; b\varphi = 0. \)

Here \((\nu + 1)j! = (\nu + 1)\nu \cdots (\nu + j - 2)(\nu + j - 1)\nu \cdots (\nu + j - 2)(\nu + j - 1)\nu \cdots (\nu - 1)(\nu)\) is the falling factorial.

It is obvious that the Burgers, Korteweg–de Vries, Kuramoto–Sivashinsky and Kawahara equations admit solutions affine with respect to \( x \), which are of the form \( u = (x + c_1)/(t + c_2) \) or \( u = c_0 \) with arbitrary constants \( c_0 \) and \( c_1 \). This is also true for generalized Burgers equations of the form \( u_t + uu_x = f(t, x)u_{xx} \), which is related to the fact that each of these equations is conditionally invariant with respect to the vector field \( \partial_t + u\partial_x \) \([31, 32]\) or, equivalently, with respect to the generalized vector field \( u_{xx}\partial_u \). The above Lie reductions with respect to the algebras \( (S(1)) \) and \( (P(1)) \) also solely give solutions affine in \( x \). This is why we can look for such solutions of equations from a wide subclass of the class \( \mathbb{L} \). More specifically, each equation \( L_\theta \) of the form \( \mathbb{L} \) with \( A_0^0 = B_0 = C_0 = 0 \), i.e., \( A^0 = \alpha^0(t), B = \beta^1(t)x + \beta^0(t) \) and \( C = \gamma(t) \), is reduced by the ansatz \( u = \varphi^1(t) + \varphi^0(t) \) to a system of two ordinary differential equations for \( \varphi^1 \) and \( \varphi^0 \),

\[
\begin{align*}
\varphi^1_t + \gamma\varphi^1\varphi^1 &= \alpha^0\varphi^1 + \beta^1, \\
\varphi^0_t + \gamma\varphi^1\varphi^0 &= \alpha^0\varphi^0 + \beta^0.
\end{align*}
\]

This means that the equation \( L_\theta \) possesses the generalized conditional symmetry \( u_{xx}\partial_u \), and the above ansatz just represents the general solution of the corresponding invariant surface condition \( u_{xx} = 0 \); cf. \([38]\) as well as \([16]\). If \( A^0 = B = 0 \) and \( C = 1 \), then solutions constructed with the ansatz affine in \( x \) are of the same form as for the Burgers and Korteweg–de Vries equations.

All the solutions constructed here are quite simple. Perhaps the most direct way to construct nontrivial solutions for variable-coefficient equations from the class \( \mathbb{L} \) is to generate them by equivalence transformations from wide families of solutions known for famous (constant-coefficient) equations from the same class, like the Korteweg–de Vries equation. This way is often ignored in the literature, which is comprehensively discussed in \([37]\).
12 Conclusion

The class (1) is much wider than the classes of variable-coefficient Korteweg–de Vries and Burgers equations, which were considered in [10] and [38] and are the subclasses of (1) for \( r = 3 \) and \( r = 2 \) with the constraints \( A^0 = A^1 = A^2 = B = 0 \) and \( A^0 = A^1 = B = 0 \), respectively. At the same time, we have carried out the complete group classification of the class (1) with less efforts than the ones made in [10, 38] for the group classification of the above particular subclasses, and the classification list is much more compact for the class (1), although the equation order \( r \) is not fixed here. This became possible due to using several ingredients.

The first ingredient was the proper choice of the class to be studied. In general, the choice should be realized in such a way for the selected class to be convenient for group classification, with the main criterion for this being the property of normalization. In contrast to the classes considered in [10] and [38], the class (1) is normalized in the usual sense.

Since there are two arbitrary smooth functions of \((t, x)\) among the parameters of the usual equivalence group of the class (1) and arbitrary elements of this class also depend on \((t, x)\) only, two of the arbitrary elements can be gauged by equivalence transformations. It seems at first sight that there are several equally appropriate gauges for arbitrary elements of the class (1) but this is not the case. It has been shown in Section 4 that although the gauge \( A^r = 1 \) is quite natural and gives a subclass normalized in the usual sense, the structure of the corresponding usual equivalence group is more complicated than for the gauge \( C = 1 \), and the further possible gauge \( A^1 = 0 \) leads to a subclass that is normalized solely in the generalized sense. The equivalence groupoid of the subclass associated with the other secondary gauge \( A^1 = 0 \), which can also be realized after gauging \( A^r = 1 \), has even much more involved normalization properties. This is why it is essential to select the gauge of arbitrary elements that is optimal for group classification of the class (1). The careful analysis has resulted in the selection of the initial gauge \( C = 1 \) and the secondary gauge \( A^1 = 0 \), and this selection of, especially, the secondary gauge is quite unforeseen without having the complete description of the equivalence groupoid of the class (1). Both the subclasses (4) and (5), which are associated with the gauges \( C = 1 \) and \((C, A^1) = (1, 0)\), respectively, are normalized in the usual sense. Due to the normalization, we easily control the changes of equivalence groups in the course of gauging. The group classifications of the class (1) and of its subclass (5) are equivalent, cf. Proposition 7. Moreover, the usual equivalence group of the maximally gauged subclass (5) is parameterized by just three functions of the only argument \( t \). This fact jointly with the normalization property makes it convenient to solve the group classification problem for the subclass (5) using the algebraic method.

The final ingredient for the efficient solution of the group classification problem was the accurate study of properties of the maximal Lie invariance algebras of equations from the subclass (5). As a result, we have found quite strong restrictions on appropriate subalgebras of the equivalence algebra \( g^\sim \) (more precisely, its projection \( \hat{g}^\sim \)) of the subclass (5), which has simplified the classification of such subalgebras. In particular, we have proved in Lemma 15 that the dimension of maximal Lie invariance algebras of equations from the subclass (5) is less than or equal to five. We have furthermore introduced three invariant integer parameters \( k_1, k_2 \) and \( k_3 \) characterizing the dimensions of relevant subspaces of such algebras and found essential low-dimensional restrictions for the values that the parameters \( k \)'s can assume. The conditions for subspaces related to time transformations that are presented in Lemma 18 are highly common for evolution equations and their systems, cf. [3, 19]. In contrast to this, the restriction for values of the pair of parameters \((k_1, k_2)\) that is given in Lemma 17 is specific for the subclass (5) but is still important for effectively classifying appropriate subalgebras of \( \hat{g}^\sim \). The restrictions for \( k \)'s collected in Section 6 are in total not complete, i.e., there are values of \( k \)'s among selected ones that are associated with no maximal Lie invariance algebras of equations from the subclass (5): see Remark 23. At the same time, finding the exhaustive set of restrictions for \( k \)'s in fact needs
carrying out the complete group classification of the subclass (5). In this sense, the present classification is more similar to the group classification of linear Schrödinger equations in [19] than the group classification of linear evolution equations in [3], where all selected potential values for analogous invariant integer parameters are admitted by some appropriate subalgebras.

Due to the normalization of the class (1) in the usual sense (resp. the subclass (5)), we were able to classify Lie reductions of equations from the class with respect to its usual equivalence group using the technique proposed in [32] and modified in [3]. We have constructed solutions that are at most affine in \(x\) to equations from a quite wide subclass of the class (1) as well as have discussed generating new solutions from known ones by equivalence transformations.

In spite of the comprehensive symmetry analysis of the class (1) and a number of its subclasses in the present paper, it is pertinent to carry out group classification of other subclasses of the class (1) even for particular small values of \(n\). We plan to solve the group classification problem for the class of variable-coefficient Burgers equations of the form

\[
\frac{du}{dt} + f(t, x)uu_x + g(t, x)u_{xx} = 0
\]

with \(fg \neq 0\), which was considered in [38] (see a discussion in the introduction of the present paper). Although this class looks simple, its group classification is quite tricky and is supposed to involve partition into subclasses jointly with successively arranging and classifying each part separately.

Although initially the main purpose of the paper had been the exhaustive solution of the group classification problem for the class (1) of general variable-coefficient Burgers–KdV equations of arbitrary fixed order, the study of the equivalence groupoids of subclasses of this class led to more important results, which are worth recalling. For a long time after the first discussion of the notion of generalized equivalence groups in [25, 26], no examples of nontrivial generalized equivalence groups were known in the literature, except classes for which some of arbitrary elements are constants and thus some of components of equivalence transformations associated with system variables depend on such arbitrary elements; see, e.g., [36] Section 6.4], [40] Section 2] and [43] Section 3]. Note that in all these papers, effective generalized equivalence groups were given instead of the corresponding generalized equivalence groups. This is why certain doubts started to circulate in the symmetry community whether this notion is valuable at all. In the present paper we have happened to construct for the first time several examples of nontrivial generalized equivalence groups such that equivalence-transformation components corresponding to equation variables locally depend on nonconstant arbitrary elements of the corresponding classes. All related classes are (reparameterized) subclasses of the class (1). The most significant consequence of the construction of these examples is that they make evident the necessity of introducing the notion of effective generalized equivalence group. Moreover, they also answer, just by their existence, some theoretical questions, which leads to properly posing further questions. In particular, the entire generalized equivalence group of a class may be effective itself and thus it is a unique effective generalized equivalence group of this class, see Remark 11.

Nevertheless, there are classes of differential equations admitting multiple effective generalized equivalence groups. As discussed in Remark 28 this claim is exemplified by classes \(\bar{K}_2\) and \(\bar{K}_3\), for which we have constructed effective generalized equivalence groups that are proper but not normal subgroups of the corresponding generalized equivalence groups. All known examples of generalized equivalence groups that are related to constant arbitrary elements have the same property. Then the natural question is whether there exists a class of differential equations with effective generalized equivalence group being a proper normal subgroup of the corresponding generalized equivalence group. By the way, Corollary 31 shows that even merely singling out an effective generalized equivalence group from the already known generalized equivalence group of a class may be a nontrivial problem.

For the class \(K_3\) of general Burgers–KdV equations with coefficients depending at most on the time variable, which is normalized in the extended generalized sense, we have explicitly constructed its extended generalized equivalence group in a rigorous way. We have reparameterized the class \(K_3\) introducing virtual arbitrary elements that are nonlocally related to the native
arbitrary elements of this class. This is the first example of such a construction for partial differential equations in the literature. Similar results were earlier obtained only for classes of linear ordinary differential equations in the preprint version of [4]. The reparameterization technique developed in the present paper gives hope to us that such construction will be realized soon for many classes of differential equations.

The equivalence groupoid of the class $F_1$ of general Burgers–KdV equations with coefficients depending at most on $x$ has an elegant structure but this structure seems uncommon in comparison with various kinds of normalization. Two equations from the class $F_1$ are related by a point transformation $\varphi$ that is not the projection of a transformation from the corresponding usual equivalence group if and only if these equations belong to a case of Lie symmetry extension for this class. At the same time, the transformation $\varphi$ is in general not decomposed into the projection of an equivalence transformation and a Lie symmetry transformation, and thus the class $F_1$ is even not semi-normalized. Hence no wonder that this example is unique.

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