No general relation between phase vortices and orbital angular momentum

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Abstract

Simple superpositions of Laguerre–Gauss beams illustrate, counterintuitively, the difference between two quantities that are commonly conflated: the component of orbital angular momentum \(\langle l \rangle\) in the propagation direction \(z\), and the total topological charge \(S\), which is the algebraic sum of the charges of vortices piercing any plane perpendicular to \(z\). The examples illustrate two contrasting situations: \(\langle l \rangle = 0, S \neq 0\), and \(\langle l \rangle \neq 0, S = 0\). In the second situation, not only is the total charge zero but also there are no vortices in the infinite half-space beyond the beam waist plane \(z = 0\).

Keywords: paraxial, singularities, Laguerre–Gauss beams, phase

(Some figures may appear in colour only in the online journal)

1. Introduction

Waves can possess geometrical features such as singularities, and also mechanical properties such as angular momentum. For complex scalar waves, the singularities are phase vortices (a.k.a. wave dislocations or nodes) [1–4], and the angular momentum is orbital (OAM) [5–7]. A widely-held assumption, often expressed casually, is that OAM and vortices are inextricably linked, with vortex beams being synonymous with beams carrying OAM, especially in the paraxial regime. Three quotations, from many similar, will suffice: ‘… if the phase structure rotates, the light has orbital angular momentum …’ [8]; ‘henceforth, the term ‘OAM beam’

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refers to all helically phased beams ...’ [9]; ‘... an optical vortex beam has a phase singularity with a certain topological charge, giving rise to a hollow intensity distribution. Such a beam with helical phase fronts and orbital angular momentum ...’ [10].

This is a misconception, probably arising from the familiar special beams that are eigenstates of OAM [e.g. individual Laguerre–Gauss (LG) beams]. Each of these contains a vortex core (except for the zero OAM fundamental modes) whose charge equals the OAM quantum number, and near which OAM can generate torques to rotate small particles (‘optical spanners’) [11, 12]. General beams can be represented by superpositions of eigenstates, for which there is no relation between the OAM in the beam and its vortices.

Our purpose in this note is to dispel the misconception with elementary counterexamples that are counterintuitive. These involve the OAM component in a well-defined propagation direction, and the total topological charge, which is the sum of 1/2π times the phase changes around the individual vortices, signed relative to the propagation direction (for subtleties applying the concept of topological charge to singular lines in 3D, see [4]).

The counterexamples are: a wave propagating with zero OAM while the total topological charge is nonzero; and a wave propagating with nonzero OAM while the total topological charge is zero, and manifesting no vortices at all in the infinite half-space beyond the beam waist plane.

For counterexamples, the simplest setting suffices. We choose complex scalar waves Ψ propagating in three dimensions, satisfying the paraxial wave equation (section 2), with solutions that are superpositions of LG beams [6]. The paraxial equation is identical in form to the time-dependent Schrödinger equation, so our counterexamples also apply to quantum waves evolving in two space dimensions.

The counterexample with zero OAM and finite S is in section 3. This is clearer than the counterexamples given previously for a special class of paraxial beams [13] and for finite superpositions of plane waves [14]. The counterexamples with finite OAM, S = 0, and in particular the absence of vortices in the infinite half-space, are presented in section 4 and the technical appendix.

The concluding section 5 outlines an alternative (nonparaxial) formulation, involving Bessel beams, and an extension to polarisation singularities.

2. Formulation

The paraxial wave equation is

\[ i \partial_z \Psi(r, z) = -\nabla^2 r \Psi(r, t), \quad \{r = \{x, y\} = r\{\cos \phi, \sin \phi\}\}. \]

(1)

For optical waves propagating in free space close to the z direction, with wavenumber k, \( z = \) (propagation distance)/k; for quantum particles with mass m, \( z = \) (time)\( \hbar / m \).

The OAM per unit length for fixed z is the transverse expectation value of the operator \( (r \times p)_z \), involving the momentum \( p = -i\hbar \nabla \); in units of \( \hbar \) [6, 13],

\[ \langle l \rangle = \frac{\text{Re} \left[ \int_{-\infty}^{\infty} dr \Psi^* (r, z) (r \times (-i\nabla)) \Psi (r, z) \right]}{\int_{-\infty}^{\infty} dr |\Psi (r, z)|^2} = \frac{\text{Im} \left[ \int_{-\infty}^{\infty} dr \Psi^* (r, z) \partial_z \Psi (r, z) \right]}{\int_{-\infty}^{\infty} dr |\Psi (r, z)|^2}. \]

(2)

This is a conserved quantity, i.e. independent of z. The total topological charge, also conserved unless vortices escape to infinity, is the algebraic sum of the charges of all vortices piercing each z plane. This can be expressed as an integral over the plane with \( \delta \) functions selecting the
(signed) vortices [15, 16], or as the total phase accumulated round a large circle enclosing all vortices in the $z$ plane, divided by $2\pi$:

$$S = \int_{\infty}^{\infty} dr \left( \frac{1}{2} \text{Im} \left[ (\nabla \Psi^*(r, z) \times \nabla \Psi(r, z))_z \right] \right)$$

$$\times \delta(\text{Re}[\Psi(r, z)])\delta(\text{Im}[\Psi(r, z)])$$

$$= \frac{1}{2\pi} \int d\phi (\partial_{\phi} \text{arg} \Psi(r, \phi, z))_{r=\infty} = \frac{1}{2\pi} \int d\phi \text{Im} \left( \frac{\partial_{\phi} \Psi(r, \phi, z)}{\Psi(r, \phi, z)} \right)_{r=\infty}. \quad (3)$$

As our counterexamples will confirm, the mechanical quantity $\langle l \rangle$ and the geometrical quantity $S$ are mathematically and physically different. This should not be surprising, because $S$ is the signed count of the zeros of $\Psi$, whereas $\langle l \rangle$ involves an integral of $\Psi$ over the whole $z$ plane.

For these counterexamples, we will choose superpositions of LG beams. There is a great deal of flexibility, which has been explored in detail [17, 18]: the beams need not have same waist radii, they need not be circular, their waists could be at different heights, they need not be coaxial, and they could be combinations of higher-order Gaussian beams of different types (e.g. LG and Hermite–Gauss beams). We choose the simplest superpositions: coaxial LG beams with the same waist radii, taken as unity [6, 17].

$$\psi_{l,p}(r, z) = N_{l,p} r^{|l|} L^{|l|}_p \left( \frac{r^2}{1+z^2} \right) \exp \left( -\frac{r^2}{2(1+iz)} \right) \frac{\exp(i(l\phi - 2p \arctan z))}{(1+iz)^{|l|+1}}, \quad (4)$$

in which $N_{l,p} = \sqrt{p!/(\pi(p+|l|)!)}$ and $L^{|l|}_p$ is the generalised Laguerre polynomial [19].

We consider superpositions of the form

$$\Psi(r, z) = \sum_{l,p} c_{l,p} \psi_{l,p}(r, z). \quad (5)$$

It follows from (2) that the angular momentum is the weighted sum

$$\langle l \rangle = \sum_{l,p} |l| |c_{l,p}|^2 / \sum_{l,p} |c_{l,p}|^2. \quad (6)$$

For the charge $S$, we note that the highest-degree term in the Laguerre polynomials $(r^{2p}/p!)$ [19] implies from (4) that $\psi_{l,p} \sim r^M$ as $r \to \infty$ (in addition to the exponential factor common to all $l, p$) where $M = |l| + 2p$. Therefore $S$ is determined by the coefficient(s) in the superposition with the largest value of $M$. If there is only one coefficient with the largest $M$, or several coefficients have the same largest $M$ (e.g. $c_{2,0}$ and $c_{3,1}$) but one of them dominates (its contribution $|c_{l,p}|N_{l,p}/p!$ is larger than the sum of all others), then $S$ is its $l$ value.

If there are several such coefficients with the largest $M$, in the range $l_{\text{min}} \leq l \leq l_{\text{max}}$, without one of them dominating, $S$ could be any integer in the range $l_{\text{min}} \leq S \leq l_{\text{max}}$. In this case, a general formula for determining $S$ is obtained by defining $\zeta = \exp(i\phi)$ and regarding the integrals over $\phi$ in (3) as contour integrals in the $\zeta$ plane, around the unit circle $|\zeta| = 1$, with the integrand being a sum of powers $\zeta^l$. Then, by Cauchy’s theorem, $S$ is determined by counting: the zeros of the integrand that lie within the unit circle (positive) and the pole at $\zeta = 0$ (negative) if $l_{\text{min}} < 0$. There is no simpler general formula connecting $S$ with the coefficients $c_{l,p}$.

Since $S$ is an integer, any variation with coefficients must be discontinuous. Discontinuities correspond to vortex lines reaching infinity (when at least one zero of the aforementioned
integrand lies on the unit circle); these are excluded by our stipulation that the integration circle must enclose all vortices. Vortices reaching infinity are non-generic, i.e. unstable in the sense that they can be eliminated by infinitesimal perturbation (as in the example in [20]).

Each individual LG beam (4) is an eigenfunction of the OAM operator, and if all contributing LG beams share the same value \( l = l_0 \) then \( \langle l \rangle = l_0 = S \). Only in this special case, including the simplest and most familiar case of an individual LG beam, can we obtain \( S = \langle l \rangle \).

We are interested only in cases where the OAM is intrinsic, in the sense of not depending on the origin of \( r \). Under a shift of origin to \( r_0 \), OAM transforms to

\[
\langle l \rangle = \frac{\text{Im} \left[ \int \int_{-\infty}^{\infty} dr \Psi^*(r, z) (r - r_0) \times \nabla \Psi(r, z) \right]}{\int \int_{-\infty}^{\infty} dr |\Psi(r, z)|^2} + r_0 \times \frac{1}{\hbar} p.
\]

The origin-dependent (extrinsic) contribution involves the transverse linear momentum (also conserved)\[13, 21\]

\[
\langle p \rangle = \hbar \frac{\text{Im} \left[ \int \int_{-\infty}^{\infty} dr \Psi^*(r, z) \nabla \Psi(r, z) \right]}{\int \int_{-\infty}^{\infty} dr |\Psi(r, z)|^2}.
\]

Therefore OAM is intrinsic for all superpositions (5) for which \( \langle p \rangle \) is zero, and the beams we will consider will be of this type. We note that it follows from (8) that \( \langle p \rangle = 0 \) for each individual LG beam (4). When \( \langle p \rangle \neq 0 \), this is a consequence of interference between two or more contributions in the superposition (5); in such cases, the beam propagates obliquely with respect to the \( z \) direction of the individual contributing beams, and it has been suggested [13] that the paraxial propagation direction and transverse plane could be redefined so that \( \langle p \rangle = 0 \), guaranteeing that OAM is intrinsic.

3. Wave with zero orbital angular momentum and nonzero vortex strength

The condition for the superposition (5) to have zero OAM is (cf (6))

\[
\sum_{l,p} |c_{l,p}|^2 = 0.
\]

A simple example is

\[
c_{1,0} = \sqrt{2}, \quad c_{-2,0} = 1.
\]

Explicitly, the superposition is

\[
\Psi(r, z) = \frac{\exp\left(-\frac{r^2}{2(1 + iz)}\right)}{\sqrt{2\pi(1 + iz)^3}} (2(1 + iz)r \exp(i\phi) + r^2 \exp(-2i\phi)).
\]

Its linear expectation \( \langle p \rangle \) is zero, because the field intensity \( |\Psi(r, z)|^2 \) has threefold rotation symmetry.

With the coefficients (10), \( \langle l \rangle = 0 \). For large \( r \), the dominant term, growing as \( r^2 \), has \( l = -2 \), so the total charge is \( S = -2 \). There are four vortex lines: one of charge +1, at \( r = 0 \), and three of charge −1, spiralling around the \( r = 0 \) axis for all \( z \), at \( r = 2\sqrt{1 + z^2} \) and \( \phi = -\frac{1}{3} \arctan z + \{+\frac{1}{3}\pi, -\frac{1}{3}\pi, \pi\} \). The wave (11) is illustrated in figures 1(a) and (b).
Figure 1. (a) and (b) illustrate the wave (11), with four vortices and \( \langle l \rangle = 0, z = 0 \); (a) is the phase (colour-coded by HUE), and (b) shows the local OAM density (integrand of (7) for \( r_0 = 0 \)). (c) and (d): as (a) and (b), but illustrating the wave (14), with no vortices and nonzero \( \langle l \rangle \), for \( a = \frac{3}{4}, b = \frac{1}{2}, z = 1, r_0 = 0 \).

4. Wave with no vortices and finite orbital angular momentum

We choose the family of superpositions

\[
e_{0,0} = 1, \quad e_{0,1} = ia, \quad e_{2,0} = \sqrt{2}b,
\]

with parameters \( a, b \) real. From (6), the OAM is

\[
\langle l \rangle = \frac{4b^2}{1 + a^2 + 2b^2}.
\]

The wave thus represented is, explicitly,

\[
\Psi(r, z; a, b) = \frac{\exp\left(\frac{-r^2}{2(1+iz)}\right)}{\sqrt{\pi(1+iz)^3}} \left(1 + iz\right)^2 - ia\left(1 - r^2 + z^2\right) + br^2 \exp(2i\phi). \tag{14}
\]

For this wave, the linear expectation \( \langle p \rangle \) is zero, because the field intensity \( |\Psi(r, z)|^2 \) has twofold rotation symmetry. For \( r \to \infty \), there are two leading terms, growing as \( r^2 \), and the total charge is \( S = 0 \) if the \( a^2 \) term dominates over the \( \phi \)-dependent term \( br^2 \), i.e. if \( |b| < |a| \). This wave can simultaneously...
exhibit nonzero OAM when $|b| \neq 0 \ (13)$, refuting the misconception that finite OAM requires finite total charge. Nevertheless, $S = 0$ does not secure the absence of vortices. Such vortices can be born in charge-cancelling pairs in a particular $z$ plane, to which a curved vortex line is momentarily tangent as it loops up and down $[4,22,23]$. Stronger conditions on $a$ and $b$ have to be imposed to eliminate the vortices throughout the infinite half-space $0 \leq z < \infty$.

The condition for no vortices in the plane $z$ is that (14) never vanishes for any $r$ and $\phi$. To analyse this, we define

$$\xi = \frac{1 + z^2}{r^2}, \quad s = \frac{2z}{1 + z^2},$$

separate the term containing $b$ in (14), and take the absolute square. We seek the condition that there are no vortices for $0 \leq z < \infty$. A short calculation leads to the requirement that

$$F(\xi, s) \equiv \xi^2 + a^2(\xi - 1)^2 - 2as\xi(\xi - 1) = b^2$$

has no solutions for $s \geq 0$ and any $\xi \geq 0$. In the appendix, we show that the condition is

$$0 < a < 1, \quad |b| < \frac{a}{\sqrt{1 + a^2}}.$$  \hspace{1cm} (17)

In this range, the beam has finite OAM while propagating vortex-free throughout the half-space from the waist plane $z = 0$ to the far field $z = \infty$. For $z < 0$, charge-cancelling pairs of vortices emerge, constrained by the invariant total charge $S = 0$ throughout the whole space. Figures 1(c) and (d), with coefficients satisfying (17), illustrate the wave (14) in the vortex-free region.

It is mathematically interesting to ask whether such beams can be made vortex-free in the full space, not just in the half-space, but in practice beams are always generated at finite locations and then propagate to infinity.

5. Concluding remarks

Our two counterexamples establish that there is no general relation between the OAM in a beam and the total topological charge of its vortices. We further demonstrate a family of beams with finite OAM that is vortex-free in the infinite half-space. These examples are in no way special. It is easy to create other superpositions exemplifying different aspects of the OAM/charge discordance. For example, by superposing two LG modes with different waist radii, it is possible to create a beam with $\langle l \rangle = 0$, $\langle p \rangle = 0$, $S = +1$, possessing a single charge $+1$ vortex along a region $z_{\text{min}} < z < \infty$ of the $z$ axis.

We chose our examples as elementary superpositions of paraxial LG beams. But non-paraxial formulations are possible, and we outline one of them. This involves diffraction-free Bessel beams [24]. The simplest such superpositions—solutions of the Helmholtz equation with wavenumber $k$ and coefficients $c_l$—are

$$\Psi(r, z) = \exp\left(iz\sqrt{k^2 - q^2}\right) \sum_l c_l J_l(qr) \exp(il\phi). \hspace{1cm} (18)$$

(More general superpositions contain more than one value of the parameter $q$.) The individual Bessel beams in the superpositions (18) are orthogonal but not normalisable. Nevertheless, the
OAM in the beam can be calculated by a simple limiting process, and is
\[ \langle l \rangle = \frac{\sum_i |c_i|^2}{\sum_i |c_i|^2}. \] (19)

All Bessel functions have the same large r decay (oscillatory/√r). Therefore the total charge S (equation (3)) is the l that corresponds to the dominant |c_l|. The quantities ⟨l⟩ and S are usually different, and it is easy to construct examples where either is zero and the other is not.

Finally, we note a superficial polarisation analogy to our main result: for paraxial transverse electric fields \( E(r, z) \), there is no general relation between the spin angular momentum (SAM) and the total charge of the lines of circular polarisation singularity (C lines [25]) piercing any transverse plane, regarded as vortices of the complex scalar field \( \Psi = E \cdot E \) [26–28]. As in the scalar case, the two quantities are mathematically different. The SAM per unit length, in units of \( \bar{\hbar} \), is [13]
\[ \langle s \rangle = \frac{\Im \left[ \int \int_{-\infty}^{\infty} \int dr \left( E^*(r, z) \times E(r, z) \right) \right]}{\int \int_{-\infty}^{\infty} dr E^*(r, z) \cdot E(r, z)}, \] (20)
and the total charge of the vortices representing the C lines is given by (3).

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Data availability statement

No new data were created or analysed in this study.

Appendix. Proof of vortex boundary formula (17)

If the function \( F(\xi, s) \) in (16) has a positive minimum \( F_{\text{min}}(a) \), the required condition is
\[ b^2 < F_{\text{min}}(a). \] (A.1)
Minimising the quadratic form in \( \xi \) gives
\[ \xi_{\text{min}} = \frac{a(a - s)}{1 + a^2 - 2as}, \] (A.2)
and the minimum value
\[ F_{\text{min}} = \min_s \left( \frac{a^2(1 - s^2)}{1 + a^2 - 2as} \right). \] (A.3)

We need consider only \( s \geq 0 (z \geq 0) \), because, from (16), changing the sign of \( a \) is equivalent to changing the sign of \( s \) and we are considering only \( s \geq 0 \) i.e. \( z \geq 0 \).
If \(a > 1\) or \(a < 0\), \(\xi_{\text{min}}\) in (A.2) is positive for all \(s\), and the minimum in (A.3) corresponds to \(s = 1\), giving \(F_{\text{min}} = 0\), so the condition (A.1) does not apply: \(b\) is unrestricted and vortices exist. If \(0 < a < 1\), \(s < a\) ensures \(\xi > 0\), and the minimum is at \(s = 0\), and

\[
\xi_{\text{min}} = \frac{a^2}{1 + a^2}, \quad F_{\text{min}} = \frac{a^2}{1 + a^2},
\]

so (A.3) implies the claimed condition (17). The confirmation that \(F_{\text{min}}\) is indeed a minimum follows from

\[
F_{\text{min}} < F(\xi \to 0) = a^2 \quad \text{and} \quad F_{\text{min}} < F(\xi \to \infty) = 1.
\]

\[\text{(A.5)}\]

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