LOCAL LI-YAU’S ESTIMATES ON $RCD^*(K,N)$ METRIC MEASURE SPACES

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ABSTRACT. In this paper, we will study the (linear) geometric analysis on metric measure spaces. We will establish a local Li-Yau’s estimate for weak solutions of the heat equation and prove a sharp Yau’s gradient estimate for harmonic functions on metric measure spaces, under the Riemannian curvature-dimension condition $RCD^*(K,N)$.

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1. Introduction

In the field of geometric analysis, one of the fundamental results is the following Li-Yau’s local gradient estimate for solutions of the heat equation on a complete Riemannian manifold.

Theorem 1.1 (Li-Yau [34]). Let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifold, and let $B_{2R}$ be a geodesic ball of radius $2R$ centered at $O \in M^n$. Assume that $\text{Ric}(M^n) \geq -k$ with $k \geq 0$. If $u(x,t)$ is a smooth positive solution of the heat equation $\Delta u = \partial_t u$ on $B_{2R} \times (0, \infty)$, then for any $\alpha > 1$, we have the following gradient estimate in $B_R$:

$$(1.1) \quad \sup_{x \in B_R} (|\nabla f|^2 - \alpha \cdot \partial_t f)(x,t) \leq \frac{C \alpha^2}{R^2} \left( \frac{\alpha^2}{\alpha^2 - 1} + \sqrt{\text{Vol}(B_R)} \right) + \frac{na^2k}{2(\alpha - 1)} + \frac{na^2}{2t}$$

where $f := \ln u$ and $C$ is a constant depending only on $n$.

By letting $R \to \infty$ in (1.1), one gets a global gradient estimate, for any $\alpha > 1$, that

$$(1.2) \quad |\nabla f|^2 - \alpha \cdot \partial_t f \leq \frac{na^2k}{2(\alpha - 1)} + \frac{na^2}{2t}.$$
for example, in the setting of Riemannian manifolds with Ricci curvature bounded below \([15, 9, 47, 33, 32]\), in the setting of weighted Riemannian manifolds with Bakry-Emery Ricci curvature bounded below \([12, 35, 43, 7]\) and some non-smooth setting \([10, 44]\), and so on.

Let \((X, d, \mu)\) be a complete, proper metric measure space with \(\text{supp}(\mu) = X\). The curvature-dimension condition on \((X, d, \mu)\) has been introduced by Sturm \([48]\) and Lott-Villani \([36]\). Given \(K \in \mathbb{R}\) and \(N \in [1, \infty]\), the curvature-dimension condition \(CD(K, N)\) is a synthetic notion for "generalized Ricci curvature \(\geq K\) and dimension \(\leq N\)" on \((X, d, \mu)\). Bacher-Sturm \([5]\) introduced the reduced curvature-dimension condition \(CD^*(K, N)\), which satisfies a local-to-global property. On the other hand, to rule out Finsler geometry, Ambrosio-Gigli-Savaré \([6]\) introduced the notion for "generalized Ricci curvature bounded below \([12, 35, 43, 7]\) and some non-smooth setting \([10, 44]\), and so on."

In this paper, we will study the \(\text{locally weak solution}\) of the heat equation on a metric measure space \((X, d, \mu)\). To seek an appropriate method to deal with the \(\text{locally weak solution}\) for the heat equation, let us recall what is the proof of Theorem \(1.1\) in the smooth context. There are two main ingredients: the Bochner formula and a maximum principle. The Bochner formula states that

\[
- \int_I \int_\Omega \langle \nabla u, \nabla \phi \rangle \, d\mu dt = \int_I \int_\Omega \frac{\partial u}{\partial t} \cdot \phi \, d\mu dt
\]

for all \(\text{Lipschitz functions} \ \phi\) with compact support in \(\Omega \times I\), where the inner product \(\langle \nabla u, \nabla \phi \rangle\) is given by polarization in \(W^{1,2}(\Omega)\).

Notice that the locally weak solutions \(u(x, t)\) do not form a semi-group in general. The method of \(\Gamma_2\)-calculus for the heat flow in the previous works \([17, 28, 31]\) is no longer be suitable for the problems on \(\text{locally weak solutions}\) of the heat equation.

To seek an appropriate method to deal with the \(\text{locally weak solution}\) for the heat equation, let us recall what is the proof of Theorem \(1.1\) in the smooth context. There are two main ingredients: the Bochner formula and a maximum principle. The Bochner formula states that

\[
\frac{1}{2} |\nabla f|^2 \geq \frac{(\Delta f)^2}{n} + \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2
\]

for any \(C^3\)-function \(f\) on \(M^n\) with Ricci curvature \(Ric(M^n) \geq K\) for some \(K \in \mathbb{R}\). The maximum principle states that if \(f(x)\) is of \(C^2\) on \(M^n\) and if it achieves its a local maximal value at point \(x_0 \in M^n\), then we have

\[
\nabla f(x_0) = 0 \quad \text{and} \quad \Delta f(x_0) \leq 0.
\]
For simplification, we only consider the special case that \( u(x,t) \) is a smooth positive solution for heat equation on a compact manifold \( M^n \) with \( \text{Ric}(M^n) \geq 0 \). By using the Bochner formula to \( \ln u \), one deduces a differential inequality

\[
\left( \Delta - \frac{\partial}{\partial t} \right) F \geq -2 \langle \nabla f, \nabla F \rangle + \frac{2}{nt} F^2 - \frac{F}{t},
\]

where \( f = \ln u \) and \( F = t(\langle \nabla f \rangle^2 - \partial_t f) \). Then by using the maximum principle to \( F \) at one of its maximum points \((x_0, t_0)\), one gets the desired Li-Yau’s estimate

\[
\max F = F(x_0, t_0) \leq \frac{n}{2}.
\]

In this paper, we want to extend these two main ingredients to non-smooth metric measure spaces. Firstly, let us consider the Bochner formula in non-smooth context. Let \( (X, d, \mu) \) be a metric measure space with \( \text{RCD}(K, N) \). Erbar-Kuwada-Sturm [16] and Ambrosio-Mondino-Savaré [5] proved that \( \text{RCD}(K, N) \) condition is equivalent to a Bakry-Emery’s Bochner inequality for the heat flow \((H_t)_t \geq 0\) on \( X \). This provides a global version of Bochner formula for the infinitesimal generator of the heat flow \((H_t)_t \geq 0\) (see Lemma 2.3). On the other hand, a good cut-off function has been obtained in [5, 40, 24]. By combining these two facts and an argument in [24], one can localize the global version of Bochner formula in [16, 5] to a local one.

To state the local version of Bochner formula, it is more convenient to work with a notion of the weak Laplacian, which is a slight modification from [18, 20]. Let \( \Omega \subset X \) be an open set. Denote by \( H^1(\Omega) := W^{1,2}(\Omega) \) and \( H^1_0(\Omega) := W^{1,2}_0(\Omega) \). The weak Laplacian on \( \Omega \) is an operator \( \mathcal{L} \) on \( H^1(\Omega) \) defined by: for each function \( f \in H^1(\Omega) \), \( \mathcal{L} f \) is a functional acting on \( H^1_0(\Omega) \cap L^{\infty}(\Omega) \) given by

\[
\mathcal{L} f(\phi) := -\int_{\Omega} \langle \nabla f, \nabla \phi \rangle d\mu \quad \forall \phi \in H^1_0(\Omega) \cap L^{\infty}(\Omega).
\]

In the case when it holds

\[
\mathcal{L} f(\phi) \geq \int_{\Omega} h \cdot \phi d\mu \quad \forall 0 \leq \phi \in H^1_0(\Omega) \cap L^{\infty}(\Omega)
\]

for some function \( h \in L^1_{\text{loc}}(\Omega) \), then it is well-known [23] that the weak Laplacian \( \mathcal{L} f \) can be extended to a signed Radon measure on \( \Omega \). In this case, we denote by

\[
\mathcal{L} f \geq h \cdot \mu
\]

on \( \Omega \) in the sense of distributions.

Now, the local version of Bochner formula is given as follows.

**Theorem 1.2** [5, 24]. Let \( (X, d, \mu) \) be a metric measure space with \( \text{RCD}(K, N) \) for some \( K \in \mathbb{R} \) and \( N \geq 1 \). Assume that \( f \in H^1(B_R) \) such that \( \mathcal{L} f \) is a signed measure on \( B_R \) with the density \( g \in H^1(B_R) \cap L^{\infty}(B_R) \). Then we have \( \langle \nabla f \rangle^2 \in H^1(B_{R/2}) \cap L^{\infty}(B_{R/2}) \) and that \( \mathcal{L}(\langle \nabla f \rangle^2) \) is a signed Radon measure on \( B_{R/2} \) such that

\[
\frac{1}{2} \mathcal{L}(\langle \nabla f \rangle^2) \geq \left[ \frac{g^2}{N} + \langle \nabla f, \nabla g \rangle + K|\nabla f|^2 \right] \cdot \mu
\]

on \( B_{R/2} \) in the sense of distributions.

Next, we consider to extend the maximum principle (1.5) from smooth Riemannian manifolds to non-smooth metric measure spaces \( (X, d, \mu) \). A simple observation is that the maximum principle (1.5) on a smooth manifold \( M^n \) has the following equivalent form:
Suppose that \( f(x) \) is of \( C^2 \) on \( M^n \) and that it achieves its a local maximal value at point \( x_0 \in M^n \). Given any \( w \in C^4(U) \) for some neighborhood \( U \) of \( x_0 \). Then we have
\[
\Delta f(x_0) + \langle \nabla f, \nabla w \rangle(x_0) \leq 0.
\]

In the following result, we will extend the observation to the non-smooth context. Technically, it is our main effort in the paper.

**Theorem 1.3.** Let \( \Omega \) be a bounded domain in a metric measure space \((X, d, \mu)\) with \( RCD^*(K, N) \) for some \( K \in \mathbb{R} \) and \( N \geq 1 \). Let \( f(x) \in H^1(\Omega) \cap L^\infty(\Omega) \) such that \( \mathcal{L} f \) is a signed Radon measure with \( \mathcal{L}^{\text{sing}} f \geq 0 \), where \( \mathcal{L}^{\text{sing}} f \) is the singular part with respect to \( \mu \). Suppose that \( f \) achieves one of its strict maximum in \( \Omega \) in the sense that: there exists a neighborhood \( U \subset \subset \Omega \) such that
\[
\sup_{U} f > \sup_{\Omega \setminus U} f.
\]

Then, given any \( w \in H^1(\Omega) \cap L^\infty(\Omega) \), there exists a sequence of points \( \{x_j\}_{j \in \mathbb{N}} \subset U \) such that they are the approximate continuity points of \( \mathcal{L}^{\text{ac}} f \) and \( \langle \nabla f, \nabla w \rangle \), and that
\[
f(x_j) \geq \sup_{\Omega} f - 1/j \quad \text{and} \quad \mathcal{L}^{\text{ac}} f(x_j) + \langle \nabla f, \nabla w \rangle(x_j) \leq 1/j.
\]

Here and in the sequel of this paper, \( \sup_{U} f \) means \( \text{ess sup}_{U} f \).

This result is close to the spirit of the Omori-Yau maximum principle \[41 \ [51\]. It has also some similarity with the approximate versions of the maximum principle developed, for instance by Jensen \[26\], in the theory of second order viscosity solutions.

A similar parabolic version of the maximum principle, Theorem \[1.2\] and Theorem \[1.4\], will be given in §4.

After obtaining the above Bochner formula and the maximum principle (Theorem \[1.3\] and Theorem \[1.4\]), we will show the following Li-Yau type gradient estimates for locally weak solutions of the heat equation, which is our main purpose in this paper.

**Theorem 1.4.** Let \( K \geq 0 \) and \( N \in [1, \infty) \), and let \((X, d, \mu)\) be a metric measure space satisfying \( RCD^*(-K, N) \). Let \( T_\ast \in (0, \infty) \) and let \( B_{2R} \) be a geodesic ball of radius \( 2R \) centered at \( p \in X \), and let \( u(x, t) \in W^{1,2}(B_{2R} \times (0, T_\ast)) \) be a positive locally weak solution of the heat equation on \( B_{2R} \times (0, T_\ast) \). Then, given any \( T \in (0, T_\ast) \), we have the following local gradient estimate
\[
\sup_{B_{2R} \times [\beta T, T]} \left( |\nabla f|^2 - \alpha \cdot \frac{\partial f}{\partial t} \right)(x, t) \leq \max \left\{ \frac{KT}{2(\alpha - 1)} \right\} \frac{Na^2}{2T} + \frac{1}{\beta^2} \cdot \left( \frac{\sqrt{K}}{R} + 1 \right) + \frac{C_N \cdot \alpha^4}{R^2(\alpha - 1)} + \frac{1}{(1 - \beta)\beta^2} + C_N \cdot \frac{\alpha^2}{\beta^2} \cdot \left( \frac{\sqrt{K}}{R} + 1 \right)
\]

for any \( \alpha > 1 \) and any \( \beta \in (0, 1) \), where \( f = \ln u \), and \( C_N \) is a constant depending only on \( N \). Here and in the sequel of this paper, \( \sup_{B_{R} \times [a, b]} g \) means \( \text{ess sup}_{B_{R} \times [a, b]} g \) for a function \( g(x, t) \).

The local boundedness and the Harnack inequality for locally weak solutions of the heat equation have been established by Sturm \[49 \ [50\] in the setting of abstract local Dirichlet form and by Marola-Masson \[39\] in the setting of metric measure with a standard volume doubling property and supporting a \( L^2\)-Poincare inequality. Of course, they are available on metric measure spaces \((X, d, \mu)\) satisfying \( RCD^*(K, N) \) for some \( K \in \mathbb{R} \) and \( N \in [1, \infty) \). In particular, any locally weak solutions for the heat equation must be locally H"older continuous.

As a consequence of Theorem \[1.4\] letting \( R \to \infty \) and \( \beta \to 1 \), we get the following global gradient estimates.
Corollary 1.5. Let $(X, d, \mu)$ and $K, N, T_*$ be as in the Theorem [14]. Let $u(x, t)$ be a positive solution of the heat equation on $X \times (0, T_*)$. Then, for almost all $T \in (0, T_*)$, the following gradient estimate holds

$$
\sup_{x \in X} \left( |\nabla f|^2 - \alpha \cdot \frac{\partial f}{\partial t} \right)(x, T) \leq \max \left\{ 1, \frac{1}{2} + \frac{KT}{(\alpha - 1)} \right\} \cdot \frac{N\alpha^2}{2T} \leq \left( 1 + \frac{KT}{(\alpha - 1)} \right) \cdot \frac{N\alpha^2}{2T}
$$

for any $\alpha > 1$, where $f = \ln u$.

As another application of the maximum principle, Theorem [14] and the Bochner formula, we will deduce a sharp Yau’s gradient estimate for harmonic functions on metric measure spaces satisfying $RCD^*(-K, N)$ for $K \geq 0$ and $N > 1$.

Let us recall the classical local Yau’s gradient estimate in geometric analysis (see [14, 51, 38]). Let $M^n$ be an $n(\geq 2)$-dimensional complete non-compact Riemannian manifold with $Ric(M^n) \geq -k$ for some $k > 0$. The local Yau’s gradient estimate asserts that for any positive harmonic function $u$ on $B_{2R}$, then

$$
\sup_{B_R} |\nabla \ln u| \leq \sqrt{(n-1)k + \frac{C(n)}{R}}.
$$

In particular, if $u$ is positive harmonic on $M^n$ and $\text{Ric} \geq -(n-1)$ on $M^n$ then it follows that $|\nabla \log u| \leq n - 1$ on $M^n$. This result is sharp, in fact the equality case was characterized in [38]. This means that for $k = n - 1$ in (1.7) the factor $\sqrt{n - 1}$ on the right hand side is sharp.

Let $(X, d, \mu)$ be a metric measure space satisfying $RCD^*(-K, N)$ for some $K \geq 0$ and $N \in (1, \infty)$. It was proved in [27] the following form of Yau’s gradient estimate that, for any positive harmonic function $u$ on $B_{2R} \subset X$, it holds

$$
\sup_{B_R} |\nabla \ln u| \leq C(N, K, R).
$$

In the setting of Alexandrov spaces, by using a Bochner formula and an argument of Nash-Moser iteration, it was proved in [33, 25] the following form of Yau’s gradient estimate holds: given an $n$-dimensional Alexandrov space $M$ and a positive harmonic function $u$ on $B_{2R} \subset M$, if the generalized Ricci curvature on $B_{2R} \subset M$ has a lower bound $\text{Ric} \geq -k$, $k \geq 0$, in the sense of [52], then

$$
\sup_{B_R} |\nabla \ln u| \leq C_1(n) \sqrt{k} + \frac{C_2(n)}{R}.
$$

Indeed, by applying Theorem [12] the same argument in [33, 25] implies this estimate still holds for harmonic function $u$ on a metric measure space $(X, d, \mu)$ with $RCD^*(-k, n)$. However, it seems hopeless to improve the fact $C_1(n)$ to the sharp $\sqrt{n - 1}$ in (1.7) via a Nash-Moser iteration argument.

The last result in this paper is to establish a sharp local Yau’s gradient estimate on metric measure spaces with Riemannian curvature-dimension condition.

Theorem 1.6. Let $K \geq 0$ and $N \in (1, \infty)$, and let $(X, d, \mu)$ be a metric measure space satisfying $RCD^*(-K, N)$. Let $B_{2R}$ be a geodesic ball of radius $2R$ centered at $p \in X$, and let $u(x)$ be a positive harmonic function on $B_{2R}$. Then the following local Yau’s gradient estimate holds

$$
\sup_{B_R} |\nabla \ln u| \leq \frac{1 + \beta}{1 - \beta} \cdot (N - 1)K + \frac{C(N)}{\sqrt{\beta(1 - \beta)} \cdot R}
$$

for any $\beta \in (0, 1)$.
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2. Preliminaries

Let \((X, d)\) be a complete metric space and \(\mu\) be a Radon measure on \(X\) with \(\text{supp}(\mu) = X\). Denote by \(B_r(x)\) the open ball centered at \(x\) and radius \(r\). Throughout the paper, we assume that \(X\) is proper (i.e., closed balls of finite radius are compact). Denote by \(L^p(\Omega) := L^p(\Omega, \mu)\) for any open set \(\Omega \subset X\) and any \(p \in [1, \infty]\).

2.1. Reduced and Riemannian curvature-dimension conditions.

Let \(\mathcal{P}_2(X, d)\) be the \(L^2\)-Wasserstein space over \((X, d)\), i.e., the set of all Borel probability measures \(\nu\) satisfying

\[
\int_X d^2(x_0, x) d\nu(x) < \infty
\]

for some (hence for all) \(x_0 \in X\). Given two measures \(\nu_1, \nu_2 \in \mathcal{P}_2(X, d)\), the \(L^2\)-Wasserstein distance between them is given by

\[
W^2(\nu_0, \nu_1) := \inf \int_{X \times X} d^2(x, y) dq(x, y)
\]

where the infimum is taken over all couplings \(q\) of \(\nu_1\) and \(\nu_2\), i.e., Borel probability measures \(q\) on \(X \times X\) with marginals \(\nu_1\) and \(\nu_0\). Such a coupling \(q\) realizes the \(L^2\)-Wasserstein distance is called an optimal coupling of \(\nu_0\) and \(\nu_1\). Let \(\mathcal{P}_2(X, d, \mu) \subset \mathcal{P}_2(X, d)\) be the subspace of all measures absolutely continuous w.r.t. \(\mu\). Denote by \(\mathcal{P}_\infty(X, d, \mu) \subset \mathcal{P}_2(X, d, \mu)\) the set of measures in \(\mathcal{P}_2(X, d, \mu)\) with bounded support.

**Definition 2.1.** Let \(K \in \mathbb{R}\) and \(N \in [1, \infty)\). A metric measure space \((X, d, \mu)\) is called to satisfy the reduced curvature-dimension condition \(CD(K, N)\) if any only if for each pair \(\nu_0 := \rho_0 \cdot \mu, \nu_1 := \rho_1 \cdot \mu \in \mathcal{P}_\infty(X, d, \mu)\) there exist an optimal coupling \(q\) of them and a geodesic \((\nu_t := \rho_t \cdot \mu)_{t \in [0, 1]}\) in \(\mathcal{P}_\infty(X, d, \mu)\) connecting them such that for all \(t \in [0, 1]\) and all \(N' > N\):

\[
\int_X \rho_t^{-1/N'} \, d\nu_t \geq \int_X \left[ \sigma_{K/N'}(d(x_0, x_1)) \rho_0^{-1/N'}(x_0) + \sigma_{K/N'}(d(x_0, x_1)) \rho_1^{-1/N'}(x_1) \right] dq(x_0, x_1),
\]

where the function

\[
\sigma_{K}(\theta) := \begin{cases} 
\frac{\sin(\sqrt{K} \theta)}{\sin(\sqrt{K} \theta)} & 0 < k\theta^2 < \pi^2, \\
\theta, & k\theta^2 = 0, \\
\frac{\sin(\sqrt{K} \theta)}{\sin(\sqrt{K} \theta)} & k\theta^2 \geq \pi^2.
\end{cases}
\]

Given a function \(f \in C(X)\), the pointwise Lipschitz constant \([13]\) of \(f\) at \(x\) is defined by

\[
\text{Lip}_f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)} = \limsup_{r \to 0} \sup_{d(x, y) < r} \frac{|f(y) - f(x)|}{r},
\]

where we put \(\text{Lip}_f(x) = 0\) if \(x\) is isolated. Clearly, \(\text{Lip}_f\) is a \(\mu\)-measurable function on \(X\). The Cheeger energy, denoted by \(\text{Ch} : L^2(X) \to [0, \infty]\), is defined \([14]\) by

\[
\text{Ch}(f) := \inf \left\{ \liminf_{j \to \infty} \frac{1}{2} \int_X (\text{Lip}_{f_j})^2 \, d\mu \right\},
\]

where the infimum is taken over all sequences of Lipschitz functions \((f_j)_{j \in \mathbb{N}}\) converging to \(f\) in \(L^2(X)\). In general, \(\text{Ch}\) is a convex and lower semi-continuous functional on \(L^2(X)\).
Definition 2.2. A metric measure space \((X,d,\mu)\) is called \textit{infinitesimally Hilbertian} if the associated Cheeger energy is quadratic. Moreover, \((X,d,\mu)\) is said to satisfy \textit{Riemannian curvature-dimension condition} \(RCD^*(K,N)\), for \(K \in \mathbb{R}\) and \(N \in [1,\infty)\), if it is infinitesimally Hilbertian and satisfies the \(CD^*(K,N)\) condition.

Let \((X,d,\mu)\) be a metric measure space satisfying \(RCD^*(K,N)\). For each \(f \in D(\text{Ch})\), i.e., \(f \in L^2(X)\) and \(\text{Ch}(f) < \infty\), it has

\[
\text{Ch}(f) = \frac{1}{2\pi} \int_X |\nabla f|^2 \, d\mu,
\]

where \(|\nabla f|\) is the so-called minimal relaxed gradient of \(f\) (see §4 in [4]). It was proved, according to [4, Lemma 4.3] and Mazur’s lemma, that Lipschitz functions are dense in \(D(\text{Ch})\), i.e., for each \(f \in D(\text{Ch})\), there exist a sequence of Lipschitz functions \((f_j)_{j \in \mathbb{N}}\) such that \(f_j \to f\) in \(L^2(X)\) and \(|\nabla (f_j - f)| \to 0\) in \(L^2(X)\). Since the Cheeger energy \(\text{Ch}\) is a quadratic form, the minimal relaxed gradients bring an inner product as following: given \(f, g \in D(\text{Ch})\), it was proved [18] that the limit

\[
\langle \nabla f, \nabla g \rangle := \lim_{\epsilon \to 0} \frac{|\nabla (f + \epsilon \cdot g)|^2 - |\nabla f|^2}{2\epsilon}
\]

exists in \(L^1(X)\). The inner product is bi-linear and satisfies Cauchy-Schwarz inequality, Chain rule and Leibniz rule (see Gigli [18]).

2.2. Canonical Dirichlet form and a global version of Bochner formula.

Given an infinitesimally Hilbertian metric measure space \((X,d,\mu)\), the energy \(\mathcal{E} := 2\text{Ch}\) gives a canonical Dirichlet form on \(L^2(X)\) with the domain \(\mathcal{V} := D(\text{Ch})\). Let \(K \in \mathbb{R}\) and \(N \in [1,\infty)\), and let \((X,d,\mu)\) be a metric measure space satisfying \(RCD^*(K,N)\). It has been shown [11, 13] that the canonical Dirichlet form \((\mathcal{E},\mathcal{V})\) is strongly local and admits a Carré du champ \(\Gamma\) with \(\Gamma(f) = |\nabla f|^2\) of \(f \in \mathcal{V}\). Namely, the energy measure of \(f \in \mathcal{V}\) is absolutely continuous w.r.t. \(\mu\) with the density \(|\nabla f|^2\). Moreover, the intrinsic distance \(d_\mathcal{E}\) induced by \((\mathcal{E},\mathcal{V})\) coincides with the original distance \(d\) on \(X\).

It is worth noticing that if a metric measure space \((X,d,\mu)\) satisfying \(RCD^*(K,N)\) then its associated Dirichlet form \((\mathcal{E},\mathcal{V})\) satisfies the standard assumptions: the local volume doubling property and supporting a local \(L^2\)-Poincare inequality (see [48, 49]).

Let \((\Delta_\mathcal{E}, D(\Delta_\mathcal{E}))\) and \((H_{\epsilon} f)_{\epsilon \geq 0}\) denote the infinitesimal generator and the heat flow induced from \((\mathcal{E},\mathcal{V})\). Let us recall the Bochner formula (also called the \textit{Bakry-Emery condition}) in [16] as following.

Lemma 2.3. Let \((X,d,\mu)\) be a metric measure space satisfying \(RCD^*(K,N)\) for \(K \in \mathbb{R}\) and \(N \in [1,\infty)\), and let \((\mathcal{E},\mathcal{V})\) be the associated canonical Dirichlet form. Then the following properties hold.

(i) ([16, Theorem 4.8]) If \(f \in D(\Delta_\mathcal{E})\) with \(\Delta_\mathcal{E} f \in \mathcal{V}\) and if \(\phi \in D(\Delta_\mathcal{E}) \cap L^\infty(X)\) with \(\phi \geq 0\) and \(\Delta_\mathcal{E} \phi \in L^\infty(X)\), then we have the Bochner formula:

\[
\frac{1}{2} \int_X \Delta_\mathcal{E} \phi \cdot |\nabla f|^2 \, d\mu + \frac{1}{N} \int_X \phi (\Delta_\mathcal{E} f)^2 \, d\mu + \int_X \phi (\nabla (\Delta_\mathcal{E} f), \nabla f) \, d\mu + K \int_X \phi |\nabla f|^2 \, d\mu.
\]
The dominated convergence theorem implies that
hand, since
χ
and
φ
are bounded on the
Ω
measure spaces (see, for example, [2]).

Proof. We say a function
f
∈
W
1,p
(Ω)
under the
\|f\|_{W^{1,p}(Ω)} := \|f\|_{L^p(Ω)} + \|\text{Lip } f\|_{L^p(Ω)}.
The Sobolev spaces
W^{1,p}(Ω)
are defined by the closure of the set
\{ f ∈ Lip_{loc}(Ω) \mid \|f\|_{W^{1,p}(Ω)} < +∞ \}
under the
W^{1,p}(Ω)-norm. Remark that
W^{1,p}(Ω)
is reflexive for any
1 < p < ∞
(see [13, Theorem 4.48]). Spaces
W^{1,p}_0(Ω)
is defined by the closure of
Lip_0(Ω)under the
W^{1,p}(Ω)-norm. We say a function
f ∈ W^{1,p}_0(Ω)
if
f ∈ W^{1,p}(Ω')
for every open subset
Ω' ⊂⊂ Ω.

The following two facts are well-known for experts. For the convenience of readers, we include a proof here.

Lemma 2.5. (i) For any
1 < p < ∞
we have
W^{1,p}(X) = W^{1,p}_0(χ(X)).
(ii) \(W^{1,2}(X) = D(\text{Ch})\).

Proof. If
X
is compact, the assertion (i) is clear. Without loss of generality, we can assume that
X
is non-compact. Given a function
f ∈ Lip(X) \cap W^{1,p}(X)
, in order to prove (i), it suffices to find a sequence \(\{f_j\}_{j∈\mathbb{N}}\) of Lipschitz functions with compact supports in
X
such that
f_j → f
in
W^{1,p}(X).

Consider a family of Lipschitz cut-off
χ_j
with, for each
j ∈ \mathbb{N}, \chi_j(x) = 1 for
x ∈ B_j(x_0)
and \(\chi_j(x) = 0\) for
x ∉ B_{j+1}(x_0)
, and \(0 ≤ \chi_j(x) ≤ 1, \|\nabla \chi_j\|(x) ≤ 1\) for all
x ∈ X.
Now
f · \chi_j ∈ Lip_0(Ω)
and \(f · \chi_j → f\)(\mu-almost all
x ∈ X).
Notice that \(|f · \chi_j| ≤ |f| ∈ L^p(Ω)\) for all
j.
The dominated convergence theorem implies that \(f · \chi_j → f\) in
L^p(Ω)
as \(j → ∞\). On the other hand, since
\(\|\nabla (f · \chi_j)\| ≤ |\nabla f| · \chi_j + |f| · |\nabla \chi_j| ≤ |\nabla f| + |f| ∈ L^p(Ω)\).
for all \( j \in \mathbb{N} \), we obtain that the sequence \( (f \cdot \chi_j)_{j \in \mathbb{N}} \) is bounded in \( W^{1,p}(X) \). By noticing that \( W^{1,p}(X) \) is reflexive (see [13, Theorem 4.48]), we can see that \( f \cdot \chi_j \) converges weakly to \( f \) in \( W^{1,p}(X) \) as \( j \to \infty \). Hence, by Mazurs lemma, we conclude that there exists a convex combination of \( f \cdot \chi_j \) converges strongly to \( f \) in \( W^{1,p}(X) \) as \( j \to \infty \). The proof of (i) is completed.

Let us prove (ii). It is obvious that \( W^{1,2}(X) \subset D(\text{Ch}) \), since \( \text{Lip}(X) \cap W^{1,2}(X) \subset D(\text{Ch}) \) and \( |\nabla f| \leq \text{Lip}(f) \). We need only to show \( D(\text{Ch}) \subset W^{1,2}(X) \). This follows immediately from the fact that Lipschitz functions are dense in \( D(\text{Ch}) \). The proof of (ii) is completed. \( \square \)

3. The weak Laplacian and a local version of Bochner formula

Let \((X,d,\mu)\) be a metric measure space satisfying \( RCD^*(K,N) \) for some \( K \in \mathbb{R} \) and \( N \in [1,\infty) \). Fix any open set \( \Omega \subset X \). We will denote by the Sobolev spaces \( H^1_0(\Omega) := W^{1,2}_0(\Omega) \), \( H^1(\Omega) := W^{1,2}(\Omega) \) and \( H^1_{\text{loc}}(\Omega) := W^{1,2}_{\text{loc}}(\Omega) \).

**Definition 3.1 (Weak Laplacian).** Let \( \Omega \subset X \) be an open set, the Laplacian \( \Delta \) on \( \Omega \) is an operator \( \mathcal{L} \) on \( H^1(\Omega) \) defined as the follows. For each function \( f \in H^1(\Omega) \), its Laplacian \( \mathcal{L} f \) is a functional acting on \( H^1_0(\Omega) \cap L^\infty(\Omega) \) given by

\[
\mathcal{L} f(\phi) := -\int_{\Omega} \langle \nabla f, \nabla \phi \rangle d\mu \quad \forall \ \phi \in H^1_0(\Omega) \cap L^\infty(\Omega).
\]

For any \( g \in H^1(\Omega) \cap L^\infty(\Omega) \), the distribution \( g \cdot \mathcal{L} f \) is a functional acting on \( H^1_0(\Omega) \cap L^\infty(\Omega) \) defined by

\[
(3.1) \quad g \cdot \mathcal{L} f(\phi) := \mathcal{L} f(g\phi) \quad \forall \ \phi \in H^1_0(\Omega) \cap L^\infty(\Omega).
\]

This Laplacian (on \( \Omega \)) is linear due to that the inner product \( \langle \nabla f, \nabla g \rangle \) is linear. The strongly local property of the inner product \( \int_{\Omega} \langle \nabla f, \nabla g \rangle d\mu \) implies that if \( f \in H^1(X) \) and \( f = \text{constant} \) on \( \Omega \) then \( \mathcal{L} f(\phi) = 0 \) for any \( \phi \in H^1_0(\Omega) \cap L^\infty(\Omega) \).

If, given \( f \in H^1(\Omega) \), there exists a function \( u_f \in L^1_{\text{loc}}(\Omega) \) such that

\[
(3.2) \quad \mathcal{L} f(\phi) = \int_{\Omega} u_f \cdot \phi d\mu \quad \forall \ \phi \in H^1_0(\Omega) \cap L^\infty(\Omega),
\]

then we say that “\( \mathcal{L} f \) is a function in \( L^1_{\text{loc}}(\Omega) \)” and write as “\( \mathcal{L} f = u_f \) in the sense of distributions”. It is similar to say that “\( \mathcal{L} f \) is a function in \( L^p_{\text{loc}}(\Omega) \) or \( W^{1,p}_{\text{loc}}(\Omega) \) for any \( p \in [1,\infty] \)”, and so on.

The operator \( \mathcal{L} \) satisfies the following Chain rule and Leibniz rule, which is essentially due to Gigli [18].

**Lemma 3.2.** Let \( \Omega \) be an open domain of a metric measure space \((X,d,\mu)\) satisfying \( RCD^*(K,N) \) for some \( K \in \mathbb{R} \) and \( N \in [1,\infty) \).

(i) (Chain rule) Let \( f \in H^1(\Omega) \cap L^\infty(\Omega) \) and \( \eta \in C^2(\mathbb{R}) \). Then we have

\[
(3.3) \quad \mathcal{L}[\eta(f)] = \eta'(f) \cdot \mathcal{L} f + \eta''(f) \cdot |\nabla f|^2.
\]

(ii) (Leibniz rule) Let \( f, g \in H^1(\Omega) \cap L^\infty(\Omega) \). Then we have

\[
(3.4) \quad \mathcal{L}(f \cdot g) = f \cdot \mathcal{L} g + g \cdot \mathcal{L} f + 2\langle \nabla f, \nabla g \rangle.
\]

**Proof.** The proof is given essentially in [18]. For the completeness, we sketch it. We prove only the Chain rule (3.3). The proof of Leibniz rule (3.4) is similar.
Given any $\phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, we have
\[
\mathcal{L}[\phi](f)(\Omega) = -\int_\Omega (\nabla[\phi](f), \nabla \phi)d\mu = -\int_\Omega \eta'(f) \cdot \langle \nabla f, \nabla \phi \rangle d\mu,
\]
where we have used that (see [18 §3.3]) the inner product $\langle \nabla f, \nabla \phi \rangle$ satisfies the Chain rule, i.e., $\langle \nabla[\phi](f), \nabla \phi \rangle = \eta'(f) \cdot \langle \nabla f, \nabla \phi \rangle$.

On the other hand, by (3.1), we obtain
\[
\int_\Omega \langle \nabla f, \nabla \phi \rangle d\mu = \int_\Omega \eta'(f) \cdot \langle \nabla f, \nabla \phi \rangle d\mu = -\int_\Omega \eta''(f) \cdot |\nabla f|^2 \cdot \phi d\mu.
\]

where we have used that $\eta'(f) \cdot \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and that (see [18 §3.3]) the inner product $\langle \nabla f, \nabla \phi \rangle$ satisfies the Chain rule and Leibniz rule, i.e.,
\[
\langle \nabla f, \nabla \eta'(f) \cdot \phi \rangle = \langle \nabla f, \nabla \phi \rangle \eta'(f) + \langle \nabla f, \nabla (\eta'(f)) \rangle \phi.
\]

The combination of the above two equations implies the Chain rule (3.3). The proof is completed.

To compare the above Laplace operator $\mathcal{L}$ on $X$ with the generator $\Delta_\mathcal{E}$ of the canonical Dirichlet form $(\mathcal{E}, \mathcal{V})$, it was shown [18] that the following compatibility result holds.

**Lemma 3.3** (Proposition 4.24 in [18]). The following two statements are equivalent:

i) $f \in H^1(X)$ and $\mathcal{L}f$ is a function in $L^2(X)$,

ii) $f \in D(\Delta_\mathcal{E})$.

In each of these cases, we have $\mathcal{L}f = \Delta_\mathcal{E}f$.

The following regularity result for the Poisson equation has been proved under a Bakry-Emery type heat semigroup curvature condition, which is implied by the Riemannian curvature-dimension condition $RCD^*(K, N)$ (see [16 Theorem 7] and [5 Theorem 7.5]).

**Lemma 3.4** ([27][29]). Let $(X, d, \mu)$ be a metric measure space satisfying $RCD^*(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $g \in L^\infty(B_R)$, where $B_R$ is a geodesic ball with radius $R$ and centered at a fixed point $x_0$. Assume $f \in H^1(B_R)$ and $\mathcal{L}f = g$ on $B_R$ in the sense of distributions. Then we have $|\nabla f| \in L^\infty_{\text{loc}}(B_R)$, and
\[
|\nabla f|_{L^\infty_{\text{loc}}(B_R)} = C(N, K, R) \cdot \left( \frac{1}{\mu(B_R)} ||f||_{L^1(B_R)} + ||g||_{L^\infty(B_R)} \right).
\]

**Proof.** In the case of $g = 0$, i.e., $f$ is harmonic on $B_R$, the assertion is proved in [27] Theorem 1.2 (see also [19] Theorem 3.9). In the general case $g \in L^\infty(\Omega)$, this is proved in [29] Theorem 3.1. The assertion of the constant $C(N, K, R)$ depending only on $N, K, R$ comes from the fact that both the doubling constant and $L^2$-Poincare constant on a ball $B_R$ of a $RCD^*(K, N)$-space depend on $N, K$ and $R$.

Now we will give a local version of the Bochner formula, Theorem [1.2] by combining the modified Bochner formula (2.2) and a similar argument in [24][28].
Lemma 3.3, we get

\[ \frac{1}{2} \mathcal{L}(|\nabla f|^2) \geq \left[ \frac{g^2}{N} + \langle \nabla f, \nabla g \rangle + K|\nabla f|^2 \right] \cdot \mu \quad \text{on } B_{R/2} \]

in the sense of distributions, i.e.,

\[ -\frac{1}{2} \int_{B_{R/2}} \langle \nabla |\nabla f|^2, \nabla \phi \rangle d\mu \geq \int_{B_{R/2}} \phi \cdot \left( \frac{g^2}{N} + \langle \nabla f, \nabla g \rangle + K|\nabla f|^2 \right) d\mu \]

for any \( 0 \leq \phi \in H^1_0(B_{R/2}) \cap L^\infty(B_{R/2}) \).

Proof. From Lemma 3.4 and that \( g \in L^\infty(B_R) \), we know \( |\nabla f| \in L^\infty_{\text{loc}}(B_R) \).

We take a cut-off \( \chi \) satisfying (i) and (ii) in Lemma 2.1. Let

\[ \tilde{f}(x) := \begin{cases} f \cdot \chi & \quad \text{if } x \in B_R \\ 0 & \quad \text{if } x \in X \setminus B_R. \end{cases} \]

Then we have \( \tilde{f} \in \text{Lip}_0(B_R) \). It is easy to check \( \text{supp}(\mathcal{L} \tilde{f}) \subset B_R \). In fact, given any \( \psi \in H^1_0(X) \) with \( \psi = 0 \) on \( B_R \), the strongly local property implies that \( \int_X \langle \nabla \tilde{f}, \nabla \psi \rangle d\mu = 0 \).

Now we want to calculate \( \mathcal{L} \tilde{f} \) on \( B_R \). By the Leibniz rule (3.4), we have, on \( B_R \),

\[ \mathcal{L} \tilde{f} = \mathcal{L}(f \cdot \chi) = \chi \cdot \mathcal{L} f + f \cdot \mathcal{L} \chi + 2 \langle \nabla f, \nabla \chi \rangle \]

where we have used \( g \in L^\infty(B_R) \) and \( |\nabla f| \in L^\infty_{\text{loc}}(B_R) \), and that \( \chi, |\nabla \chi|, \Delta_\varepsilon \chi \in L^\infty(X) \) in Lemma 2.1. Combining with \( \text{supp}(\mathcal{L} \tilde{f}) \subset B_R \), we have \( \mathcal{L} \tilde{f} \in L^2(X) \cap L^\infty(X) \). Therefore, by Lemma 3.4, we get \( \tilde{f} \in D(\Delta_\varepsilon) \) and

\[ L^2(X) \cap L^\infty(X) \ni \Delta_\varepsilon \tilde{f} = \mathcal{L} \tilde{f} = \begin{cases} \chi \cdot g + f \cdot \Delta_\varepsilon \chi + 2 \langle \nabla f, \nabla \chi \rangle & \quad \text{if } x \in B_R \\ 0 & \quad \text{if } x \in X \setminus B_R. \end{cases} \]

Since \( \tilde{f} = f \) on \( B_{R/2} \), we have \( |\nabla f| = |\nabla \tilde{f}| \) for \( \mu \)-a.e. on \( B_{R/2} \). Notice that \( |\nabla \tilde{f}|^2 \in \mathbb{V} \) implies that \( |\nabla f|^2 \in H^1(B_{R/2}) \). Then \( |\nabla f|^2 \in H^1(B_{R/2}) \), and \(|\nabla \tilde{f}|^2 = |\nabla \tilde{f}|^2 | \leq |\nabla \tilde{f}|^2 | \leq L^2(B_{R/2}) \). By (3.6) and that \( |\nabla \chi| = \Delta_\varepsilon \chi = 0 \) on \( B_{R/2} \) (since \( \chi = 1 \) on \( B_{2R/3} \)), we have \( \Delta_\varepsilon \tilde{f} = g \) on \( B_{R/2} \). Hence, we obtain

\[ \int_{B_{R/2}} \left( -\frac{1}{2} \langle \nabla |\nabla f|^2, \nabla \phi \rangle + g \cdot \langle \nabla f, \nabla \phi \rangle + \phi \cdot g^2 \right) d\mu \]

\[ \geq \frac{1}{N} \int_X \phi \cdot g^2 d\mu + K \int_{B_{R/2}} \phi |\nabla f|^2 d\mu. \]
Noticing that \( g \cdot \phi \in H^1_0(B_{R/2}) \cap L^\infty(B_{R/2}) \) and \( \mathcal{L} f = g \) on \( B_R \) in the sense of distributions, we have
\[
\int_{B_{R/2}} g \cdot \phi d\mu = \mathcal{L} f (g \phi) = -\int_{B_{R/2}} \langle \nabla f, \nabla (g \phi) \rangle d\mu \\
= -\int_{B_{R/2}} \langle \nabla f, \nabla g \rangle \cdot \phi d\mu - \int_{B_{R/2}} \langle \nabla f, \nabla \phi \rangle \cdot g d\mu.
\]
By combining this and (3.7), we get the desired inequality (3.5). The proof is finished. \( \Box \)

By using the same argument of [3], one can get an improvement of the above Bochner formula. One can also consult a detailed argument given in [31, Lemma 2.3].

Corollary 3.6. Let \((X, d, \mu)\) be a metric measure space satisfying \( RCD^*(K, N) \) for \( K \in \mathbb{R} \) and \( N \in [1, \infty) \). Let \( B_R \) be a geodesic ball with radius \( R \) and centered at a fixed point \( x_0 \).

Assume that \( f \in H^1(B_R) \) satisfies \( \mathcal{L} f = g \) on \( B_R \) in the sense of distributions with the function \( g \in H^1(B_R) \cap L^\infty(B_R) \). Then we have \( |\nabla f|^2 \in H^1(B_{R/2}) \cap L^\infty(B_{R/2}) \) and that the distribution \( \mathcal{L}^2(|\nabla f|^2) \) is a signed Radon measure on \( B_{R/2} \). If its Radon-Nikodym decomposition w.r.t. \( \mu \) is denoted by
\[
\mathcal{L}(|\nabla f|^2) = \mathcal{L}^{\text{ac}}(|\nabla f|^2) \cdot \mu + \mathcal{L}^{\text{sing}}(|\nabla f|^2),
\]
then we have \( \mathcal{L}^{\text{sing}}(|\nabla f|^2) \geq 0 \) and, for \( \mu \)-a.e. \( x \in B_{R/2} \),
\[
\frac{1}{2} \mathcal{L}^{\text{ac}}(|\nabla f|^2) \geq \frac{g^2}{N} + \langle \nabla f, \nabla g \rangle + K |\nabla f|^2.
\]
Furthermore, if \( N > 1 \), for \( \mu \)-a.e. \( x \in B_{R/2} \cap \{y : |\nabla f(y)| \neq 0\} \),
\[
(3.8) \quad \frac{1}{2} \mathcal{L}^{\text{ac}}(|\nabla f|^2) \geq \frac{g^2}{N} + \langle \nabla f, \nabla g \rangle + K |\nabla f|^2 + \frac{N}{N-1} \left( \frac{\langle \nabla f, \nabla |\nabla f|^2 \rangle}{2|\nabla f|^2} - \frac{g^2}{N^2} \right).
\]

4. The maximum principle

Let \( K \in \mathbb{R} \) and \( N \in [1, \infty) \) and let \((X, d, \mu)\) be a metric measure space satisfying \( RCD^*(K, N) \). In this section, we will study the maximum principle on \((X, d, \mu)\). Let us begin from the Kato’s inequality for weighted measures.

4.1. The Kato’s inequality.

Let \( \Omega \) be a bounded open set of \((X, d, \mu)\). Fix any \( w \in H^1(\Omega) \cap L^\infty(\Omega) \), we consider the weighted measure
\[
\mu_w := e^w \cdot \mu \quad \text{on} \quad \Omega.
\]
Since, the density \( e^{-\|w\|_{L^\infty(\Omega)}} \leq e^w \leq e^{\|w\|_{L^\infty(\Omega)}} \) on \( \Omega \), we know that the associated the Lebesgue space \( L^p(\Omega, \mu_w) \) and the Sobolev spaces \( W^{1,p}(\Omega, \mu_w) \) are equivalent to the original \( L^p(\Omega) \) and \( W^{1,p}(\Omega) \), respectively, for all \( p \geq 1 \). Both the measure doubling property and the \( L^2 \)-Poincare inequality still hold with respect to this measure \( \mu_w \) (the constants, of course, depend on \( \|w\|_{L^\infty(\Omega)} \)).

For this measure \( \mu_w \), we defined the associated Laplacian \( \mathcal{L}_w \) on \( f \in H^1(\Omega) \) by
\[
\mathcal{L}_w f (\phi) := -\int_\Omega \langle \nabla f, \nabla \phi \rangle d\mu_w = -\int_\Omega \langle \nabla f, \nabla \phi \rangle e^w d\mu
\]
for any \( \phi \in H^1_0(\Omega) \cap L^\infty(\Omega) \). It is easy to check that
\[
\mathcal{L}_w f = e^w \cdot \mathcal{L} f + e^w \cdot \langle \nabla w, \nabla f \rangle
\]
in the sense of distributions, i.e., \( \mathcal{L}_w f(\phi) = \mathcal{L}_w f(e^\phi - \phi) + \int_{\Omega} (\nabla w, \nabla f)e^\phi \cdot \phi \, d\mu. \)

When \( \Omega \) be a domain of the Euclidean space \( \mathbb{R}^N \) with dimension \( N \geq 1 \), the classical Kato’s inequality states that given any function \( f \in L^1_{\text{loc}}(\Omega) \) such that \( \Delta f \in L^1_{\text{loc}}(\Omega) \), then \( \Delta f \) is a signed Radon measure and the following holds:

\[
\Delta f_x \geq \chi[f \geq 0] \cdot \Delta f
\]

in the sense of distributions, where \( f_x := \max\{f, 0\} \). Here, \( \chi[f \geq 0](x) = 1 \) for \( x \) such that \( f(x) \geq 0 \) and \( \chi[f \geq 0](x) = 0 \) for \( x \) such that \( f(x) < 0 \). In [11], the result was extended to the case when \( \Delta f \) is a signed Radon measure.

In the following, we will extend the Kato’s inequality to the metric measure spaces \((X, d, \mu_w)\), under assumption \( f \in H^1(\Omega) \).

**Proposition 4.1** (Kato’s inequality). Let \( \Omega \) be a bounded open set of \((X, d, \mu)\) and let \( w \in H^1(\Omega) \cap L^\infty(\Omega) \). Assume that \( f \in H^1(\Omega) \) such that \( \mathcal{L}_w f \) is a signed Radon measure. Then \( \mathcal{L}_w f_x \) is a signed Radon measure and the following holds:

\[
\mathcal{L}_w f_x \geq \chi[f > 0] \cdot \mathcal{L}_w f \quad \text{on } \Omega,
\]

in the sense of distributions. In the sequel, we denote the Radon-Nikodym decomposition \( \mathcal{L}_w f = \mathcal{L}_w^{\text{loc}} f \cdot \mu_w + \mathcal{L}_w^{\text{sing}} f \).

**Proof.** It suffices to prove the following equivalent property:

\[
\mathcal{L}_w |f| \geq \text{sgn}(f) \cdot \mathcal{L}_w f,
\]

where \( \text{sgn}(t) = 1 \) for \( t > 0 \), \( \text{sgn}(t) = -1 \) for \( t < 0 \), and \( \text{sgn}(t) = 0 \) for \( t = 0 \).

Fix any \( \epsilon > 0 \) and let \( f_\epsilon(x) := (f^2 + \epsilon^2)^{1/2} \geq \epsilon \).

We have \( f_\epsilon^2 = f^2 + \epsilon^2 \),

\[
|\nabla f_\epsilon| = \frac{|f|}{f_\epsilon} |\nabla f| \leq |\nabla f|
\]

and

\[
2f_\epsilon \cdot \mathcal{L}_w f_\epsilon + 2|\nabla f_\epsilon|^2 = \mathcal{L}_w f_\epsilon^2 = \mathcal{L}_w f^2 = 2f \cdot \mathcal{L}_w f + 2|\nabla f|^2.
\]

Thus,

\[
\mathcal{L}_w f_\epsilon \geq \frac{f}{f_\epsilon} \cdot \mathcal{L}_w f,
\]

Notice that \( |\nabla f_\epsilon| \leq |\nabla f| \) and \( f_\epsilon \to |f| \) in \( L^2(\Omega) \) implies that \( f_\epsilon \) is bounded in \( H^1(\Omega) \) and, hence, there exists a subsequence \( f_{\epsilon_j} \) converging weakly to \( |f| \) in \( H^1(\Omega) \). Thus, the measures \( \mathcal{L}_w(f_{\epsilon_j}) \) converges weakly to \( \mathcal{L}_w |f| \). On the other hand, notice that \( f_{\epsilon_j}(x) \to |f(x)| \) for each \( x \in \Omega \) and that \( |f/f_{\epsilon_j}| \leq 1 \) on \( \Omega \). Letting \( \epsilon := \epsilon_j \to 0 \) in (4.4), we conclude that

\[
\mathcal{L}_w(|f|) \geq \frac{f}{|f|} \cdot \mathcal{L}_w f.
\]

This is (4.2), and the proof is completed. \( \Box \)
4.2. Maximum principles.

The above Kato’s inequality implies the maximum principle Theorem 1.3. Precisely, we have the following.

**Theorem 4.2.** Let $\Omega$ be a bounded domain. Let $f(x) \in H^1(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ such that $L^w f$ is a signed Radon measure with $L^\text{sing} f \geq 0$. Suppose that $f$ achieves one of its strict maximum in $\Omega$ in the sense that: there exists a neighborhood $U \subset \Omega$ such that

$$\sup_{U} f > \sup_{\Omega \setminus U} f.$$  

(4.5)

Here and in the sequel of the paper, the notion $\sup_{U} f$ means always ess sup$_U f$. Then, given any $w \in H^1(\Omega) \cap L^{\infty}(\Omega)$, for any $\epsilon > 0$, we have

$$\mu \{ x : f(x) \geq \sup_{\Omega} f - \epsilon \text{ and } L^w f(x) + \langle \nabla f, \nabla w \rangle(x) \leq \epsilon \} > 0.$$  

(4.6)

In particular, there exists a sequence of points $\{x_j\}_{j \in \mathbb{N}} \subset U$ such that they are the approximate continuity points of $L^w f$ and $\langle \nabla f, \nabla w \rangle$, and that

$$f(x_j) \geq \sup_{\Omega} f - 1/j \quad \text{and} \quad L^w f(x_j) + \langle \nabla f, \nabla w \rangle(x_j) \leq 1/j.$$  

Proof. Suppose the first assertion (4.6) fails for some sufficiently small $\epsilon_0 > 0$. Then we have

$$(f - (\sup_{\Omega} f - \epsilon_0))_+ \in H^1_0(\Omega) \text{ (by the maximal property (4.5))} \quad \text{and} \quad \mu \{ x : f(x) \geq \sup_{\Omega} f - \epsilon_0 \} = 0.$$  

Then for almost $x \in \{y : f(y) \geq \sup_{\Omega} f - \epsilon_0\}$ we have

$$L^w f(x) \cdot \mu_w = e^{w(x)} \cdot (L^w f + \langle \nabla f, \nabla w \rangle)(x) \cdot \mu > e^{-\|w\|_{L^\infty}} \cdot \epsilon_0 \cdot \mu > 0.$$  

The assumption $L^\text{sing} f \geq 0$ implies that $L^w f \geq 0$. By applying the Proposition 4.1 to the function $f - (\sup_{\Omega} f - \epsilon_0)$, we have

$$L^w (f - (\sup_{\Omega} f - \epsilon_0))_+ \geq \chi \{ f \geq \sup_{\Omega} f - \epsilon_0 \} \cdot L^w f \cdot \mu_w \geq 0$$  

on $\Omega$, in the sense of distributions. Recall that the metric measure space $(X, d, \mu_w)$ satisfies a doubling property and supports a $L^2$-Poincare inequality. Now the weak maximum principle [13, Theorem 7.17] implies that $(f - (\sup_{\Omega} f - \epsilon_0))_+ = 0$ on $\Omega$. Thus, sup$_\Omega f \leq \sup_{\Omega} f - \epsilon_0$ on $\Omega$. This is a contradiction, and proves the first assertion (4.6).

The second assertion follows from the first one by taking $\epsilon = 1/j$. 

Next, let us consider the parabolic version of the maximum principle. We need the following parabolic weak maximum principle.

**Lemma 4.3.** Let $\Omega$ be a bounded open subset and let $T > 0$. Let $w \in H^1(\Omega_T) \cap L^\infty(\Omega_T)$ with $d_w(x, t) \leq C$ for some constant $C > 0$, for almost all $(x, t) \in \Omega_T$. Suppose that $f(x, t) \in H^1(\Omega_T) \cap L^\infty(\Omega_T)$ with $\lim_{t \to 0} \|f(\cdot, t)\|_{L^2(\Omega_T)} = 0$ and, for almost all $t \in (0, T)$, that the functions $f(\cdot, t) \in H^1_0(\Omega)$. Assume that, for almost every $t \in (0, T)$, the function $f(\cdot, t)$ satisfies

$$L^w f(\cdot, t) - \frac{\partial}{\partial t} f(\cdot, t) \cdot \mu_w(\cdot, t) \geq 0 \quad \text{on } \Omega.$$  

(4.7)

Then we have

$$\sup_{\Omega \times (0, T)} f(x, t) \leq 0.$$
Proof. The proof is standard via a Gaffney-Davies’ method (see also [49, Lemma 1.7]). We include a proof here for the completeness. Since \( f_* \) meets all of conditions in this lemma, by replacing \( f \) by \( f_* \), we can assume that \( f \geq 0 \).

Put
\[
\bar{\xi}(t) := \int_{\Omega} f^2(\cdot, t) d\mu_{w(\cdot, t)}.
\]
Since \( \mu_{w(\cdot, t)} = e^w \cdot \mu \leq e^{\|w\|_{L^\infty}} \cdot \mu \) and \( f \in H^1(\Omega_T) \), we have, for almost all \( t \in (0, T) \),
\[
\bar{\xi}'(t) = \int_{\Omega} \partial_t f^2 d\mu_{w(\cdot, t)} + \int_{\Omega} f^2 \cdot \partial_t w \cdot d\mu_{w(\cdot, t)} \\
\leq -2 \int_{\Omega} |\nabla f|^2 d\mu_{w(\cdot, t)} + C \cdot \bar{\xi}(t) \leq C \cdot \bar{\xi}(t),
\]
where we have used \( \partial_t w \leq C \) and that the functions \( f(\cdot, t) \in H^1_0(\Omega) \cap L^\infty(\Omega) \) for almost all \( t \in (0, T) \). By using \( \lim_{t \to 0} \bar{\xi}(t) = 0 \) (since \( \bar{\xi}(t) \leq e^{\|w\|_{L^\infty}} \cdot \|f(\cdot, t)\|_{L^2(\Omega)} \) and the assumption \( \lim_{t \to 0} \|f(\cdot, t)\|_{L^2(\Omega)} = 0 \), one can obtain that \( \bar{\xi}(t) \leq 0 \). This implies \( f = 0 \) almost all in \( \Omega_T \). The proof is finished. \( \square \)

By using the same argument as in Theorem \[4.2\] the combination of the Kato’s inequality and Lemma \[4,3\] implies the following parabolic maximum principle.

**Theorem 4.4.** Let \( \Omega \) be a bounded domain and let \( T > 0 \). Let \( f(x, t) \in H^1(\Omega_T) \cap L^\infty(\Omega_T) \) and suppose that \( f \) achieves one of its strict maximum in \( \Omega \times (0, T] \) in the sense that: there exists a neighborhood \( U \subset \subset \Omega \) and an interval \( (\delta, T] \subset (0, T] \) for some \( \delta > 0 \) such that
\[
\sup_{U \times (\delta, T]} f > \sup_{\Omega \setminus (U \times (\delta, T])} f.
\]
Here \( \sup_{U \times (\delta, T]} f \) means \( \text{ess sup}_{U \times (\delta, T]} f \). Assume that, for almost every \( t \in (0, T) \), \( \mathcal{L} f(\cdot, t) \) is a signed Radon measure with \( \mathcal{L}^{\text{sing}} f(\cdot, t) \geq 0 \). Let \( w \in H^1(\Omega_T) \cap L^\infty(\Omega_T) \) with \( \partial_t w(x, t) \leq C \) for some constant \( C > 0 \), for almost all \( (x, t) \in \Omega_T \). Then, for any \( \varepsilon > 0 \), we have
\[
(\mu \times \mathcal{L}^1)((x, t) : f(x, t) > \text{ess sup}_{\Omega_T} f - \varepsilon) \leq 0,
\]
where \( \mathcal{L}^1 \) is the 1-dimensional Lebesgue’s measure on \((\delta, T]\).

In particular, there exists a sequence of points \( \{(x_j, t_j)\}_{j \in \mathbb{N}} \subset U \times (\delta, T] \) such that every \( x_j \) is an approximate continuity point of \( \mathcal{L}^{\text{ac}} f(\cdot, t_j) \) and \( \langle \nabla f, \nabla w \rangle(x_j, t_j) \), and that
\[
f(x_j, t_j) \geq \text{ess sup}_{\Omega_T} f - 1/j \quad \text{and} \quad \mathcal{L}^{\text{ac}} f(x_j, t_j) + \langle \nabla f, \nabla w \rangle(x_j, t_j) - \frac{\partial}{\partial t} f(x_j, t_j) \leq 1/j.
\]

**Proof.** We will argue by contradiction, which is similar to the proof of Theorem \[4.2\]. Suppose the assertion fails for some small \( \varepsilon_0 > 0 \). Then, for almost all \( (x, t) \in \{(y, s) : f(y, s) \geq \text{ess sup}_{\Omega_T} f - \varepsilon_0\} \), we have
\[
\mathcal{L}^{\text{ac}} f(x, t) + \langle \nabla f, \nabla w \rangle(x, t) - \frac{\partial}{\partial t} f(x, t) \geq \varepsilon_0.
\]
Thus, at such \((x, t)\),
\[
\left[ \mathcal{L}^{\text{ac}} f(x, t) - \frac{\partial}{\partial t} f(x, t) \right] \cdot \mu_w \\
\geq \left[ \mathcal{L}^{\text{ac}} f(x, t) + \langle \nabla f, \nabla w \rangle(x, t) - \frac{\partial}{\partial t} f(x, t) \right] \cdot e^w \cdot \mu \geq \varepsilon_0 \cdot e^w \cdot \mu \geq 0.
\]
The strictly maximal property of $f$ gives that $f_{x_0} := (f - (\sup_{\Omega_T} f - \epsilon_0))_+ \in H^1(\Omega_T)$ with $
abla f_{x_0}(\cdot, t) ||_{L^2(\Omega)} = 0$ and, for almost all $t \in (0, T)$, that the functions $f_{x_0}(\cdot, t) \in H^1(\Omega)$. Notice that $L^\mathrm{sing}_w f(\cdot, t) \geq 0$ by $L^\mathrm{sing}_w f(\cdot, t) \geq 0$. By using the Kato’s inequality, we have that, for almost every $t \in (0, T)$,

$$L_w(f - (\sup_{\Omega_T} f - \epsilon_0))_+ \geq \chi[f \geq (\sup_{\Omega_T} f - \epsilon_0)] \cdot L^\mathrm{ac}_w f$$

$$\geq \chi[f \geq (\sup_{\Omega_T} f - \epsilon_0)] \cdot \frac{\partial f}{\partial t} \cdot \mu_w = \frac{\partial}{\partial t}(f - (\sup_{\Omega_T} f - \epsilon_0))_+ \cdot \mu_w.$$  

Then Lemma 4.3 implies that $(f - (\sup_{\Omega_T} f - \epsilon_0))_+ = 0$ for almost all $(x, t) \in \Omega_T$. This is a contradiction.

\section{Local Li-Yau’s Gradient Estimates}

Let $K \in \mathbb{R}$ and $N \in [1, \infty]$ and let $(X, d, \mu)$ be a metric measure space satisfying $\text{RCD}^*(K, N)$. In this section, we will prove the local Li-Yau’s gradient estimates--Theorem 1.3.

Let $\Omega \subset X$ be a domain. Given $T > 0$, let us still denote

$$\Omega_T := \Omega \times (0, T]$$

the space-time domain, with lateral boundary $\Sigma$ and parabolic boundary $\partial_\Sigma \Omega_T$:

$$\Sigma := \partial \Omega \times (0, T) \quad \text{and} \quad \partial_\Sigma \Omega_T := \Sigma \cup (\Omega \times \{0\}).$$

We adopt the following precise definition of \textit{locally weak solution} for the heat equation.

\textbf{Definition 5.1}. Let $T \in (0, \infty]$ and let $\Omega$ be a domain. A function $u(x, t)$ is called a \textit{locally weak solution} of the heat equation on $\Omega_T$ if $u(x, t) \in H^1(\Omega_T)$ ($= W^{1, 2}(\Omega_T)$) and if for any subinterval $[t_1, t_2] \subset (0, T)$ and any geodesic ball $B_R \subset \subset \Omega$, it holds

$$\int_{t_1}^{t_2} \int_{B_R} (\partial_t u \cdot \phi + \langle \nabla u, \nabla \phi \rangle) d\mu dt = 0$$

for all test functions $\phi(x, t) \in Lip_0(B_R \times (t_1, t_2))$. Here and in the sequel, we denote always $\partial_t u := \frac{\partial u}{\partial t}$.

\textbf{Remark 5.2}. The test functions $\phi$ in this definition can be chosen such that it has to vanish only on the lateral boundary $\partial B_R \times (0, T)$. That is, $\phi \in Lip(B_R)$ with $\phi(\cdot, t) \in Lip_0(B_R)$ for all $t \in (0, T)$.

The local boundedness and the Harnack inequality for locally weak solutions of the heat equation have been established by Sturm \cite{50} and Marola-Masson \cite{39}. In particular, any locally weak solutions for the heat equation in Definition 5.1 must be locally Hölder continuous.

Let $u(x, t)$ be a locally weak solution of the heat equation on $\Omega \times (0, T)$. Fubini Theorem implies, for a.e. $t \in [0, T)$, that the function $u(\cdot, t) \in H^1(\Omega)$ and $\partial_t u \in L^2(\Omega)$. Hence, for a.e. $t \in (0, T)$, the function $u(\cdot, t)$ satisfies, in the distributional sense,

$$\mathcal{L} u = \partial_t u \quad \text{on} \quad \Omega.$$  

Conversely, if a function $u(x, t) \in H^1(\Omega_T)$ and \text{5.2} holds for a.e. $t \in [0, T]$, then it was shown \cite{54} Lemma 6.12] that $u(x, t)$ is a locally weak solution of the heat equation on $\Omega_T$.

In the case that $u(x, t)$ is a (globally) weak solution of heat equation on $X \times (0, \infty)$ with initial value in $L^2(X)$, the theory of analytic semigroups asserts that the function $t \mapsto ||u||_{W^{1, 2}(X)}$
is analytic. However, for a locally weak solution of the heat equation on $\Omega_T$, we have not sufficient regularity for the time derivative $\partial_t u$: in general, $\partial_t u$ is only in $L^2$. This is not enough to use Bochner formula in Theorem 3.5 to (5.2). For overcoming this difficulty, we recall the so-called Steklov average.

Definition 5.3. Given a geodesic ball $B_R$ and a function $u(x, t) \in L^1(B_{R,T})$, where $B_{R,T} := B_R \times (0, T)$, the Steklov average of $u$ is defined, for every $\varepsilon \in (0, T)$ and any $h \in (0, \varepsilon)$, by

$$u_h(x, t) := \frac{1}{h} \int_0^h u(x, t + \tau)d\tau, \quad t \in (0, T - \varepsilon].$$

From the general theory of $L^p$ spaces, we know that if $u \in L^p(B_{R,T})$, then the Steklov average $u_h$ converges to $u$ in $L^p(B_{R,T-\varepsilon})$ as $h \to 0$, for every $\varepsilon \in (0, T)$.

Lemma 5.4. If $u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T})$, then we have, for every $\varepsilon \in (0, T)$, that

$$u_h \in H^1(B_{R,T-\varepsilon}) \cap L^\infty(B_{R,T-\varepsilon}) \quad \text{and} \quad \partial_t u_h \in H^1(B_{R,T-\varepsilon}) \cap L^\infty(B_{R,T-\varepsilon})$$

for every $h \in (0, \varepsilon)$, and that $\|u_h\|_{H^1(B_{R,T-\varepsilon})}$ is bounded uniformly with respect to $h \in (0, \varepsilon)$.

Proof. Since $u \in H^1(B_{R,T})$, according to [22], there exists a function $g(x, t) \in L^2(B_{R,T})$ such that

$$|u(x, t) - u(y, s)| \leq d_P((x, t), (y, s)) \cdot \left(g(x, t) + g(y, s)\right),$$

for almost all $(x, t), (y, s) \in B_{R,T}$ with respect to the product measure $d\mu \times dt$, where $d_P$ is the product metric on $B_{R,T}$ defined by

$$d_P^2((x, t), (y, s)) := d^2(x, y) + |t - s|^2.$$  

Such a function $g$ is called a Hajłasz-gradient of $u$ on $B_{R,T}$ (see [21] §8). By the definition of the Steklov average $u_h$, we have

$$|u_h(x, t) - u_h(y, s)| \leq \frac{1}{h} \int_0^h \left(g(x, t + \tau) + g(y, s + \tau)\right) \cdot d_P((x, t + \tau), (y, s + \tau))d\tau$$

$$= \frac{1}{h} \int_0^h \left(g(x, t + \tau) + g(y, s + \tau)\right)d\tau \cdot d_P((x, t), (y, s))$$

for almost all $(x, t), (y, s) \in B_{R,T}$. The fact $g(x, t) \in L^2(B_{R,T})$ implies that $g_h(x, t) \in L^2(B_{R,T-\varepsilon})$ for each $h \in (0, \varepsilon)$ and that the functions $g_h$ converges to $g$ in $L^2(B_{R,T-\varepsilon})$ as $h \to 0$. Then the previous inequality implies that $g_h$ is a Hajłasz-gradient of $u_h$ on $B_{R,T-\varepsilon}$ for all $h \in (0, \varepsilon)$ (see [21]). According to [21] Theorem 8.6], $2g_h$ is a 2-weak upper gradient of $u_h$. Thus we conclude that $u_h \in W^{1,2}(B_{R,T-\varepsilon})$ and

$$\limsup_{h \to 0} \int_{B_{R,T-\varepsilon}} (|\nabla u_h|^2 + |\partial_t u_h|^2)d\mu dt \leq \limsup_{h \to 0} \int_{B_{R,T-\varepsilon}} (2g_h)^2d\mu dt \leq 4 \int_{B_{R,T-\varepsilon}} g^2d\mu dt.$$

Therefore, we get that $\|u_h\|_{H^1(B_{R,T-\varepsilon})}$ is bounded uniformly with respect to $h \in (0, \varepsilon)$ (by combining with $u_h \to u$ in $L^2(B_{R,T-\varepsilon})$ as $h \to 0$).

Lastly, the assertion $u_h \in L^\infty(B_{R,T-\varepsilon})$ follows directly from the definition of $u_h$ and $u \in L^\infty(B_{R,T})$. The assertion of $\partial_t u$ follows from that

$$\partial_t u_h = \frac{u(x, t + h) - u(x, t)}{h}.$$  

The proof is completed. \qed
For a locally weak solution \( u \) for the heat equation, we have the following property of \( u_h \).

**Lemma 5.5.** Let \( u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T}) \) be a locally weak solution for the heat equation, and fix any two constants \( \varepsilon, h \) such that \( \varepsilon \in (0,T) \) and \( h \in (0,\varepsilon) \). Then for almost all \( t \in (0,T-\varepsilon) \)

\[
\mathcal{L} u_h = \partial_t u_h
\]
on \( B_R \), in the sense of distributions.

**Proof.** The proof is standard. In fact, one can show the assertion for locally Lipschitz function \( u \), and then use an approximating argument to prove the lemma. \( \square \)

With the aid of the above two lemmas, we will consider firstly the case when a locally weak solution \( u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T}) \) with \( \partial_t u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T}) \).

**Lemma 5.6.** Given \( K \in \mathbb{R} \) and \( N \in [1,\infty) \), let \( (X,d,\mu) \) be a metric measure space satisfying \( \text{RCD}^* (K,N) \). Let \( u(x,t) \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T}) \) be a locally weak solution of the heat equation on \( B_{2R,T} \). Assume that \( \partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T}) \). Then we have \( |\nabla u|^2 \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T}) \).

**Proof.** Notice that, for almost all \( t \in (0,T) \), we have \( u(\cdot,t), \partial_t u(\cdot,t) \in H^1(B_{2R}) \cap L^\infty(B_{2R}) \) and that \( \mathcal{L} u = \partial_t u \) on \( B_{2R} \). By Lemma 3.4 we get

\[
|||\nabla u(\cdot,t)|||^2_{L^\infty(B_{3R/2})} \leq C(N,K,R) \cdot (|u(\cdot,t)|_{L^\infty(B_{2R})} + |\partial_t u(\cdot,t)|_{L^\infty(B_{2R})}).
\]

This implies \( |\nabla u|^2 \in L^\infty(B_{3R/2},T) \) and

\[
|||\nabla u(\cdot,\cdot)|||^2_{L^\infty(B_{3R/2},T)} \leq C(N,K,R) \cdot (|u|_{L^\infty(B_{2R},T)} + |\partial_t u|_{L^\infty(B_{2R},T)}) := C_*.
\]

On the other hand, for almost all \( t \in (0,T) \), by applying the Bochner formula (3.5) to \( \mathcal{L} u = \partial_t u \) on \( B_{2R} \), we conclude that \( |\nabla u(\cdot,t)|^2 \in H^1(B_{3R/2}) \cap L^\infty(B_{3R/2}) \) and

\[
\mathcal{L}^2(|\nabla u|^2) \geq \left[ 2 \frac{(|\partial_t u|^2)}{N} + 2(|\nabla u, \nabla \partial_t u| + 2K|\nabla u|^2) \right] \cdot \mu
\]

\[
\geq -2|\nabla u| \cdot |\nabla \partial_t u| \cdot \mu + 2K|\nabla u|^2 \cdot \mu
\]

\[
\geq -2 \left[ C_* \cdot |\nabla \partial_t u| + 2K|C_*^2| \cdot \mu,
\right.
\]
on \( B_{3R/2} \) in the sense of distributions. By using the Caccioppoli inequality, we conclude that, for almost all \( t \in (0,T) \),

\[
||||\nabla u|^2(\cdot,t)|||_{L^2(B_{R})} \leq C_{N,K,R} \cdot (2C_* \cdot |||\nabla \partial_t u|||_{L^2(B_{3R/2})} + 2K \cdot C_*^2 + |||\nabla u|^2|||_{L^2(B_{3R/2})}).
\]
The integration on \( (0,T) \) implies that

\[
||||\nabla u|^2|||_{L^2(B_{R,T})} \leq C_{**} \cdot (|||\nabla \partial_t u|||_{L^2(B_{3R/2},T)} + |||\nabla u|^2|||_{L^2(B_{3R/2},T)} + 1),
\]

for the constants \( C_{**} \) depending on \( N, K, R, T \) and \( C_* \). Thus, \( |\nabla \nabla u|^2 \in L^2(B_{R,T}) \).

Lastly, noting that, for almost all \( (x,t) \in B_{R,T} \),

\[
|\partial_t |\nabla u|^2| = |\partial_t (\nabla u, \nabla u)|^2 \leq 4|\nabla \partial_t u|^2 \cdot |\nabla u|^2.
\]

Then, by using \( |\nabla u|^2 \in L^\infty(B_{3R/2},T) \) and \( \partial_t u \in H^1(B_{R,T}) \), we get \( |\partial_t |\nabla u|^2| \in L^2(B_{3R/2},T) \). By combining with \( |\nabla \nabla u|^2 \in L^2(B_{R,T}) \), we conclude \( |\nabla u|^2 \in H^1(B_{R,T}) \). Now we finish the proof. \( \square \)
Lemma 5.7. Given $K \geq 0$ and $N \in [1, \infty)$, let $(X, d, \mu)$ be a metric measure space satisfying $RCD^\ast(-K,N)$. Let $u(x, t) \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ be the locally weak solution of the heat equation on $B_{2R,T}$. Assume that $u \geq \delta > 0$ and $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$. We put

$$F(x, t) = t \cdot [\nabla f]^2 - \alpha \cdot \partial_t f]_+,\$$

where $f = \log u$ and $\alpha > 1$. Then, we have

$$\frac{F}{t} \in H^1(B_{R,T}) \cap L^\infty(B_{R,T}),$$

and that, for almost every $t \in (0, T)$, the function $F(\cdot, t)$ satisfies

$$(5.4) \quad \mathcal{L} F - \partial_t F \cdot \mu \geq -2 \langle \nabla f, \nabla F \rangle \cdot \mu \frac{F}{t} + \mu + 2 \left[ \frac{1}{N} (\nabla f)^2 - \partial_t f)^2 - K|\nabla f|^2 \right] \cdot \mu$$

on $B_R$, in the sense of distributions.

Proof. From Lemma 5.6, we have $|\nabla u|^2 \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$. By combining with that $\partial_t u \in L^\infty(B_{2R,T}) \cap H^1(B_{2R,T})$ and that $u \geq \delta > 0$, we get that

$$|\nabla f|^2 - \alpha \partial_t f = \frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T}).$$

This implies $F/t = |\nabla f|^2 - \alpha \partial_t f \cdot \mu \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$ and proves the first assertion.

By $\partial_t u \in H^1(B_{2R,T})$, we see that $\partial_t u \in L^2(B_{2R,T})$ and that, for almost all $t \in (0, T)$,

$$\mathcal{L}(\partial_t u) = \partial_t u$$

in the sense of distributions. Since $u, \partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ and $u \geq \delta > 0$, by using the chain rule in Lemma 3.2(i) to both $u$ and $\partial_t u$, we have, for almost all $t \in (0, T)$, that the functions $f(\cdot, t), \partial_t f(\cdot, t) \in H^1(B_{2R})$ and

$$(5.5) \quad \mathcal{L} f = \partial_t f - |\nabla f|^2, \quad \mathcal{L}(\partial_t f) = \partial_t f - 2 \langle \nabla f, \nabla \partial_t f \rangle$$

on $B_{2R}$ in the sense of distributions.

Consider $F_1(x, t) := t \cdot \partial_t f$. We have, for almost all $t \in (0, T)$, the function $F_1(\cdot, t) \in H^1(B_{2R})$ with

$$\mathcal{L} F_1 = t \mathcal{L} \partial_t f = t \cdot (\partial_t f - 2 \langle \nabla f, \nabla \partial_t f \rangle).$$

Noting that

$$\partial_t F_1 = \partial_t f + t \partial_{tt} f \quad \text{and} \quad \langle \nabla f, \nabla F_1 \rangle = t \langle \nabla f, \nabla \partial_t f \rangle,$$

we conclude that

$$(5.6) \quad \mathcal{L} F_1 - \partial_t F_1 = -2 \langle \nabla f, \nabla F_1 \rangle + \frac{F_1}{t}$$

on $B_{2R}$ in the sense of distributions.

Consider $F_2 := t |\nabla f|^2$. Recall that, for almost all $t \in (0, T)$, the function $f(\cdot, t) \in H^1(B_{2R})$ and

$$\partial_t f - |\nabla f|^2 = \partial_t u - \frac{|\nabla u|^2}{u^2} \in L^\infty(B_{3R/2}) \cap H^1(B_{3R/2}).$$

Recalling that $(X, d, \mu)$ satisfies $RCD^\ast(-K,N)$, we can apply the Bochner formula (3.5) to

$$\mathcal{L} f = \partial_t f - |\nabla f|^2$$

to conclude that $|\nabla f|^2 \in H^1(B_{2R})$ and

$$\mathcal{L}(|\nabla f|^2) \geq 2 \left[ \frac{1}{N} (\partial_t f - |\nabla f|^2) + \langle \nabla f, \nabla (\partial_t f - |\nabla f|^2) \rangle - K|\nabla f|^2 \right] \cdot \mu$$
on $B_R$, in the sense of distributions. Therefore, for almost all $t \in (0, T)$, we get the function $F_2(\cdot, t)$ satisfies

\begin{equation}
\mathcal{L} F_2 - \partial_t F_2 \cdot \mu \geq 2t \cdot \left[ \frac{1}{N} \left( \nabla f - |\nabla f|^2 \right)^2 - K |\nabla f|^2 \right] \cdot \mu - 2 \langle \nabla f, \nabla F_2 \rangle \cdot \mu - \frac{F_2}{t} \cdot \mu
\end{equation}

on $B_R$, in the sense of distributions. By combining (5.6) and (5.7), we conclude, for almost all $t \in (0, T)$, that we have, for $\tilde{F} := F_2 - \alpha \cdot F_1$,

\begin{equation}
\mathcal{L} \tilde{F} - \partial_t \tilde{F} \cdot \mu \geq -2 \langle \nabla f, \nabla \tilde{F} \rangle \cdot \mu - \frac{\tilde{F}}{t} \cdot \mu + 2t \left[ \frac{1}{N} \left( |\nabla f|^2 - \partial_t f \right)^2 - K |\nabla f|^2 \right] \cdot \mu.
\end{equation}

Now, by using the Kato’s inequality to $F = \tilde{F} + \phi$, we have the desired estimate (5.4). The proof of this lemma is finished.

We are ready to prove the following local Li-Yau’s estimate under some additional assumptions.

**Lemma 5.8.** Given $K \geq 0$ and $N \in [1, \infty)$, let $(X, d, \mu)$ be a metric measure space satisfying $RCD^*(-K, N)$. Let $T \in (0, \infty)$ and let $u(x, t) \in H^1(B_{2R, T}) \cap L^\infty(B_{2R, T})$ be a locally weak solution of the heat equation on $B_{2R, T}$. Assume that $u \geq \delta > 0$ and $\partial_t u \in H^1(B_{2R, T}) \cap L^\infty(B_{2R, T})$.

Then, for any $\alpha > 1$ and any $\beta, \gamma \in (0, 1)$, the following local gradient estimate holds

\begin{equation}
\sup_{B_{R(\gamma T, T)}} \left( |\nabla f|^2 - \alpha \cdot \frac{\partial}{\partial t} f \right)(x, t) \leq \max \left\{ \frac{1}{2}, \frac{KT}{2(\alpha - 1)} \right\} \cdot \frac{N \alpha^2}{2T} \cdot \frac{1}{(1 - \beta) \gamma} + C_N \cdot \frac{\alpha^4}{R^2(\alpha - 1)} \cdot \frac{1}{(1 - \beta) \gamma} + \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \cdot \frac{C_N \cdot \alpha^2}{(1 - \beta) \gamma} \cdot \sup_{B_{R(\gamma T, T)}} |\nabla f|^2
\end{equation}

where $f = \ln u$, and $C_N$ is a constant depending only on $N$.

**Proof.** From the previous Lemma 5.7, we have $F := t \cdot |\nabla f|^2 - \alpha \cdot \frac{\partial}{\partial t} f |_{+} \in L^\infty(B_{3R/2, T})$. Put $M_1 := \sup_{B_{R, T}} F$ and $M_2 := \sup_{B_{3R/2, T}} F$.

We can assume $M_1 > 0$. If not, we are done.

Now let us choose $\phi(x) = \phi(\rho(x))$ to be a function of the distance $\rho$ to the fixed point $x_0$ with the following property that

$$\frac{M_1}{2M_2} \leq \phi \leq 1 \text{ on } B_{3R/2}, \quad \phi = 1 \text{ on } B_R, \quad \phi = \frac{M_1}{2M_2} \text{ on } B_{3R/2} \setminus B_{5R/4},$$

and

$$-C \frac{\rho^2}{R} \phi \leq \phi'(r) \leq 0 \quad \text{and} \quad |\phi''(r)| \leq \frac{C}{R^2} \quad \forall \ r \in (0, 3R/2)$$

for some universal constant $C$ (which is independent of $N, K, R$). Then we have

\begin{equation}
\frac{|\nabla \phi|^2}{\phi} = \frac{|\phi'|^2 |\nabla r|^2}{\phi} \leq \frac{C^2}{R^2} \cdot \frac{C_1}{R^2} \text{ on } B_{3R/2},
\end{equation}

and, by the Laplacian comparison theorem [18 Corollary 5.15] for $RCD^*(-K, N)$ with $N > 1$ and $K > 0$, that

\begin{equation}
\mathcal{L} \phi = \phi' \mathcal{L} r + \phi'' |\nabla r|^2 \geq -C \left( \frac{\sqrt{(N - 1)} K }{2} \right) \text{ coth} \left( \sqrt{ \frac{K}{N - 1} } r \right) - \frac{C}{R^2}
\end{equation}

\begin{equation}
\geq -C \left( \frac{\sqrt{(N - 1)} K }{2} + \frac{N - 1}{R} \right) - \frac{C}{R^2} \geq -C_2 \left( \sqrt{K} + \frac{1}{R^2} \right)
\end{equation}
on $B_{3R/2}$, in the sense of distributions, where we have used that
\[
\coth \left( r \sqrt{\frac{K}{N-1}} \right) \leq \coth \left( R \sqrt{\frac{K}{N-1}} \right) \leq 1 + \frac{1}{R \sqrt{K/(N-1)}}.
\]
We claim that the estimate (5.10) still holds for $RCD^*(-K,N)$ with $N \geq 1$ and $K \geq 0$. Indeed, in the case when $K = 0$ and $N > 1$, the Laplacian comparison theorem states $\mathcal{L}r \leq (N-1)/r$. Then (5.10) still holds. In the case when $N = 1$, since that $(X,d,\mu)$ satisfies $RCD^*(-K,N)$ implies that it satisfies $RCD^*(-K,N+1)$, we can use the Laplacian comparison theorem for $RCD^*(-K,N+1)$ to conclude that (5.10) still holds in this case. Therefore, the claim is proved.

Here and in the sequel of this proof, we denote $C_1, C_2, C_3, \cdots$ the various constants which depend only on $N$. (5.10) implies that the distribution $\mathcal{L}\phi$ is a signed Radon measure (since $\mathcal{L}\phi + C_2(\sqrt{K}/R + 1/R^2)$ is a positive distribution). Then its absolutely continuous part $(\mathcal{L}\phi)^{ac} \geq -C_2(\sqrt{K}/R + 1/R^2)$ a.e. $x \in B_{3R/2}$ and its singular part $(\mathcal{L}\phi)^{sing} \geq 0$.

Put $G(x,t) := \phi F$. According to Lemma 5.7 and the Leibniz rule 5.2(ii), we have $G \in H^1(B_{3R/2},T)$ and, for almost every $t \in (0, T)$, that the function $G(\cdot,t)$ satisfies that
\[
\mathcal{L}G = F\mathcal{L}\phi + \phi \mathcal{L}F + 2\langle \nabla \phi, \nabla F \rangle
\]
in the sense of distributions. Fix arbitrarily a such $t \in (0, T)$. Then $\mathcal{L}G$ is a signed Radon measure on $B_{3R/2}$ with
\[
(\mathcal{L}G)^{sing} = F(\mathcal{L}\phi)^{sing} + \phi(\mathcal{L}F)^{sing} \geq 0
\]
and $(\mathcal{L}G)^{ac} = F(\mathcal{L}\phi)^{ac} + \phi(\mathcal{L}F)^{ac} + 2\langle \nabla \phi, \nabla F \rangle$ a.e. $x \in B_{3R/2}$. We have, for almost all $x \in B_{3R/2}$,
\[
(\mathcal{L}G)^{ac} - \partial_t G + 2\langle \nabla f, \nabla G \rangle = \phi(\mathcal{L}F)^{ac} - \partial_t F + 2\langle \nabla f, \nabla F \rangle + F(\mathcal{L}\phi)^{ac} + 2\langle \nabla \phi, \nabla F \rangle + 2\langle \nabla f, \nabla \phi \rangle F.
\]
By (5.4) and $G = \phi F$, we have, for almost all $x \in B_{3R/2}$, that, for any fixed $\epsilon > 0$,
\[
\text{RHS of (5.12)} \geq \phi \left[ -\frac{F}{t} + 2t \left( \frac{1}{N}(|\nabla f|^2 - \partial_t f)^2 - K|\nabla f|^2 \right) \right] + G \left( \frac{(\mathcal{L}\phi)^{ac}}{\phi} + 2\langle \nabla \phi, \nabla (G/\phi) \rangle + 2\langle \nabla f, \nabla \phi \rangle \frac{G}{\phi} \right)
\geq \frac{G}{t} \left[ -\frac{1}{N}(|\nabla f|^2 - \partial_t f)^2 - K|\nabla f|^2 \right]
+ G \left[ \frac{\sqrt{K}}{R} + \frac{1}{R^2} - \frac{2C_1}{R^2} \right] + 2\langle \nabla \phi, \nabla G \rangle / \phi - 2|\nabla f| |\nabla \phi| / \phi \cdot G
\geq -\frac{G}{t} \left[ -\frac{1}{N}(|\nabla f|^2 - \partial_t f)^2 - K|\nabla f|^2 \right]
+ G \left[ \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right] + 2\langle \nabla \phi, \nabla G \rangle / \phi - \epsilon \frac{G^2}{\phi} \frac{C_1}{R^2} - |\nabla f|^2 \frac{1}{\epsilon},
\]
where we have used (5.9), (5.10) and that, for any $\epsilon > 0$, the following
\[
2|\nabla f| \cdot G \frac{|\nabla \phi|}{\phi} \leq \epsilon G^2 \frac{|\nabla \phi|^2}{\phi^2} + |\nabla f|^2 \frac{1}{\epsilon} \leq \epsilon \frac{G^2}{\phi} \frac{C_1}{R^2} + |\nabla f|^2 \frac{1}{\epsilon}.
\]
If we put 
\[ v = \frac{\|\nabla f\|^2}{F} \]
then we get \( \|\nabla f\|^2 = F \cdot v \) and 
\[ F = t(\|\nabla f\|^2 - \alpha \cdot \partial_t f) = t(F \cdot v - \alpha \cdot \partial_t f). \]
So 
\[ \partial_t f = \frac{F(vt - 1)}{\alpha t}. \]
Therefore we obtain 
\[ \frac{-G}{t} + 2t\phi \left[ \frac{1}{N}(\|\nabla f\|^2 - \partial_t f)^2 - K\|\nabla f\|^2 \right] - \epsilon^{-1}\|\nabla f\|^2 \]
(5.14) 
\[ = \frac{-G}{t} + \phi \frac{2F^2}{\alpha^2 t^3}(\alpha - 1)vt + 1)^2 - 2tKvG - \epsilon^{-1}vF \]
\[ \geq \frac{-G}{t} + \frac{2G^2}{\alpha^2 t^2}(\alpha - 1)vt + 1)^2 - 2tKvG - \epsilon^{-1}vG \]
where we have used that \( 0 < \phi \leq 1 \) and \( KvG > 0 \). Denoting by 
\[ z := (\alpha - 1)vt \quad \text{and} \quad A_\epsilon := \frac{2Kt + \epsilon^{-1}}{\alpha - 1}, \]
we have 
\[ \text{RHS of (5.14)} = \frac{1}{\phi} \cdot \left( \frac{2G^2}{N\alpha^2 t}(z + 1)^2 - \frac{G}{t}(1 + A_\epsilon z) \right) \]
Finally \( z \geq 0 \) implies that 
\[ \frac{1 + A_\epsilon z}{(1 + z)^2} \leq \max \left\{ 1, \frac{1}{2} + \frac{A_\epsilon}{4} \right\} \leq \max \left\{ 1, \frac{1}{2} + \frac{Kt}{2(\alpha - 1)} \right\} + \frac{\epsilon^{-1}}{4(\alpha - 1)}. \]
Denote by 
\[ B_0 := \max \left\{ 1, \frac{1}{2} + \frac{Kt}{2(\alpha - 1)} \right\}, \]
we have \( \frac{1 + A_\epsilon z}{(1 + z)^2} \leq B_0 + \frac{\epsilon^{-1}}{4(\alpha - 1)}, \) (since \( K \geq 0 \) and \( t \leq T \)) so 
\[ \text{RHS of (5.14)} \geq \frac{1}{\phi} \cdot \frac{G}{t} \cdot \left( \frac{2G^2}{N\alpha^2} - B_0 - \frac{\epsilon^{-1}}{4(\alpha - 1)} \right) \cdot (z + 1)^2. \]
By combining this with (5.12), (5.13) and (5.14), we obtain that 
\[ (\mathcal{L}G + \partial_t G + 2\langle \nabla f, \nabla G \rangle - 2\langle \nabla \phi, \nabla G \rangle) / \phi \]
(5.15) 
\[ \geq \frac{1}{\phi} \cdot \frac{G}{t} \cdot \left( \frac{2G^2}{N\alpha^2} - B_0 - \frac{\epsilon^{-1}}{4(\alpha - 1)} \right) \cdot (z + 1)^2 - C_3 \left\{ \frac{\sqrt{K}}{R} + \frac{1}{R^2} - \epsilon \frac{2C_1}{\phi R^2} \right\}. \]
From the definition of \( \phi \) and \( F/t \in L^{\infty}(B_{3R/2,T}) \) (by Lemma 5.7), we see that \( G \) achieves one of its strict maximum in \( B_{3R/2,T} \) in the sense of Theorem 4.4. By (5.11), we know that \( \mathcal{L}^{\text{sing}} G \geq 0 \). Notice also \( \partial_t f \in L^{\infty}(B_{2R,T}) \) since \( u \geq \delta > 0 \) and \( \partial_t u \in H^1(B_{2R,T}) \cap L^{\infty}(B_{2R,T}) \) Hence, by using Theorem 4.4 with \( w := 2f - 2\ln \phi \in H^1(B_{3R/2,T}) \cap L^{\infty}(B_{3R/2,T}) \), and combining with (5.15), we conclude that there exit a sequence \( \{x_j, t_j\}_{j \in \mathbb{N}} \) such that, for each \( j \in \mathbb{N}, \)
\[ G_j := G(x_j, t_j) \geq \sup_{B_{3R/2,T}} G - 1/j. \]
(5.16)
and that
\[
\frac{G_j}{t_j} \left( \frac{2G_j}{Na^2} - B_0 - \frac{e^{-1}}{4(\alpha - 1)} \right) \cdot (z(x_j, t_j) + 1)^2 - C_3G_j \cdot \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) - \epsilon G_j^2 \cdot \frac{C_1}{R^2} \\
\leq \phi(x_j, t_j) \cdot \frac{1}{j} \leq \frac{1}{j}
\]

We consider firstly the case when
\[
\tilde{G} := \sup_{B_{3R/2, T}} G > \frac{Na^2}{2} \left( B_0 + \frac{e^{-1}}{4(\alpha - 1)} \right).
\]

In this case, the equation (5.16) tells us \( G_j \geq \frac{Na^2}{2}(B_0 + \frac{e^{-1}}{4(\alpha - 1)}) \) for all sufficiently large \( j \).

Thus, from (5.17), we have
\[
\frac{G_j}{t_j} \left( \frac{2G_j}{Na^2} - B_0 - \frac{e^{-1}}{4(\alpha - 1)} \right) - C_3G_j \cdot \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) - \epsilon G_j^2 \cdot \frac{C_1}{R^2} \leq \frac{1}{j}.
\]

Letting \( j \to \infty \), we have
\[
\frac{\tilde{G}}{T} \left( \frac{2\tilde{G}}{Na^2} - B_0 - \frac{e^{-1}}{4(\alpha - 1)} \right) \leq C_3\tilde{G} \cdot \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) + \epsilon \tilde{G}^2 \cdot \frac{C_1}{R^2},
\]

where we have used \( t_j \leq T \) for all \( j \in \mathbb{N} \). Thus, we have
\[
(5.18)
\]

\[
\tilde{G} \leq \frac{B_0 + \frac{e^{-1}}{4(\alpha - 1)} + C_3T \cdot \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right)}{\frac{2Na^2}{N} - \epsilon T \cdot \frac{C_1}{R^2}}.
\]

In the case when \( \tilde{G} \leq \frac{Na^2}{2}(B_0 + \frac{e^{-1}}{4(\alpha - 1)}) \), it is clear that (5.18) still holds.

Fix any \( \beta \in (0, 1) \). By choosing \( \epsilon = 2\beta R^2/(C_1 \cdot Na^2T) \). Then we conclude, by (5.18), that
\[
\tilde{G} \leq \frac{B_0 + \frac{e^{-1}}{4(\alpha - 1)} + C_3T \cdot \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right)}{\frac{2Na^2}{N} - \epsilon T \cdot \frac{C_1}{R^2}}
\]

\[
(5.19)
\]

\[
\tilde{G} \leq B_0 \cdot \frac{Na^2}{2} \cdot \frac{1}{1 - \beta} + \left( \frac{C_1 \cdot N^2 \alpha^4 \cdot T}{16(\alpha - 1) \cdot \beta R^2} + C_3T \cdot \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \cdot \frac{Na^2}{2} \right) \cdot \frac{1}{1 - \beta}
\]

\[
\leq B_0 \cdot \frac{Na^2}{2} \cdot \frac{1}{1 - \beta} + \frac{C_4 \cdot \alpha^4 \cdot T}{(\alpha - 1)R^2} \cdot \frac{1}{1 - \beta} + C_5T \cdot \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \cdot \frac{\alpha^2}{1 - \beta}.
\]

Therefore, we have
\[
\sup_{B_{3R/2, T}} F \leq \sup_{B_{3R/2, T}} F \leq \sup_{B_{3R/2, T}} G
\]

\[
\leq B_0 \cdot \frac{Na^2}{2} \cdot \frac{1}{1 - \beta} + \frac{C_4 \cdot \alpha^4 \cdot T}{(\alpha - 1)R^2} \cdot \frac{1}{1 - \beta} + C_5T \cdot \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \cdot \frac{\alpha^2}{1 - \beta}.
\]

By recalling \( F = t(\nabla f)^2 - \alpha \cdot \partial_t f^+ \) and \( B_0 = \max \{ 1, \frac{1}{2} + \frac{\sqrt{T}}{2(\alpha - 1)} \} \), we conclude that the local gradient estimate (5.5) holds, since \( t > \gamma \cdot T \). This completes the proof. \( \square \)

Now, let us remove the additional assumption \( \partial_t u \in H^1(B_{2R, T}) \cap L^\infty(B_{2R, T}) \) and prove Theorem 1.4.
Proof of Theorem 1.4: Let $\alpha > 1$ and $\beta \in (0, 1)$. Without loss of generality, we can assume that $T_\varepsilon < \infty$. Given any $\delta > 0$, from [50, Theorem 2.2], we have $u + \delta \in L^\infty_{\text{loc}}(B_{2R_\varepsilon})$. Without loss the generality, we can assume that $u + \delta \in L^\infty(B_{2R_\varepsilon})$, since the desired result is a local estimate.

Given any $\varepsilon > 0$, according to Lemma 5.4 and Lemma 5.5, we can use Lemma 5.8 to the Steklov averages $(u + \delta)_h$. Then, by an approximating argument (and taking $\gamma = 1 - \beta$), we have

$$
\sup_{B_{R\times((1-\beta)T,T)}} \left( \frac{|\nabla u|^2}{(u+\delta)^2} - \alpha \cdot \frac{\partial_t u}{u+\delta} \right)(x, t) \leq \max \left\{ 1, \frac{1}{2} + \frac{KT}{2(\alpha-1)} \right\} \cdot \frac{Na^2}{2T} \cdot \frac{1}{(1-\beta)^2} + \frac{C_N \cdot \alpha^2}{(R^2(\alpha-1))} \cdot \frac{1}{(1-\beta)^2}.
$$

Letting $\varepsilon(\in \mathbb{Q})$ tend to $0^+$ and replacing $1 - \beta$ by $\beta$, we have the desired (1.6). By combining with the arbitrariness of $\varepsilon$, we complete the proof of Theorem 1.4.

6. A sharp local Yau’s gradient estimate

Let $K > 0$, $N \in (1, \infty)$ and let $(X, d, \mu)$ be a metric measure space satisfying $RCD^*(K, N)$. Suppose that $\Omega$ is a domain in $X$. In this section, we will prove a sharp local Yau’s gradient estimate—Theorem 1.6.

Proof of Theorem 1.6: Fix $\beta \in (0, 1)$. Let $u$ be a positive harmonic function on $B_{2R} := B_{2R}(p)$ and let $f = \log u$. Without loss of generality, we can assume that $u \geq \delta$ for some $\delta > 0$. By the chain rule 3.2(ii), a direct computation shows that

$$
\mathcal{L} f = -|\nabla f|^2 \quad \text{on} \quad B_{2R}.
$$

Since $|\nabla f| \in L^\infty_{\text{loc}}(B_{2R})$, by setting $g := |\nabla f|^2$ and using Corollary 3.6 (noticing that $N > 1$) we know that $g \in H^1(B_{3R/2}) \cap L^\infty(B_{3R/2})$ and $\mathcal{L}^\text{sing} g \geq 0$ and, for $\mu$-a.e. $x \in \{ y : g(y) \neq 0 \} \cap B_{3R/2}$,

$$
\frac{1}{2} \mathcal{L}^\text{loc} g \geq \frac{g^2}{N} - \langle \nabla g, \nabla f \rangle - Kg + \frac{N}{N-1} \cdot \left( \frac{\langle \nabla f, \nabla g \rangle}{2g} \right)^2 \geq \frac{g^2}{N} - \langle \nabla g, \nabla f \rangle - Kg + \frac{N}{N-1} \cdot \left( \frac{\langle \nabla f, \nabla g \rangle}{2g} \right)^2 \geq \frac{g^2}{N-1} - \frac{N-2}{N-1} \cdot \langle \nabla g, \nabla f \rangle - Kg.
$$

(6.1)

Since $g \in L^\infty(B_{3R/2})$, we define

$$
M_1 := \sup_{B_R} g \quad \text{and} \quad M_2 := \sup_{B_{3R/2}} g.
$$

We assume that $M_1 > 0$ (otherwise, we are done). Now let us choose $\phi(x) = \phi(r(x))$ as above. That is, $\phi(x)$ is a function of the distance $r$ to the fixed point $x_0$ with the following property that

$$
\frac{M_1}{2M_2} \leq \phi \leq 1 \quad \text{on} \quad B_{3R/2}, \quad \phi = 1 \quad \text{on} \quad B_R, \quad \phi = \frac{M_1}{2M_2} \quad \text{on} \quad B_{3R/2} \setminus B_{5R/4},
$$

and

$$
-\frac{C}{R^2} \phi^\frac{1}{2} \leq \phi'(r) \leq 0 \quad \text{and} \quad |\phi''(r)| \leq \frac{C}{R^2} \quad \forall \ r \in (0, 3R/2)
$$
for some universal constant $C$ (which is independent of $N, K, R$). Then we have, from (5.9)-(5.10), that

$$\frac{|\nabla \phi|^2}{\phi} \leq \frac{C_1}{R^2} \quad \text{and} \quad \mathcal{L} \phi \geq -C_2(\frac{\sqrt{K}}{R} + \frac{1}{R^2})$$

on $B_{3R/2}$. Then the distribution $\mathcal{L} \phi$ is a signed Radon measure and its absolutely continuous part $(\mathcal{L} \phi)^{ac} \geq -C_2(\frac{\sqrt{K}}{R} + \frac{1}{R^2})$ a.e. $x \in B_{3R/2}$, and its singular part $(\mathcal{L} \phi)^{sing} \geq 0$. Here and in the sequel of this proof, we denote $C_1, C_2, C_3, \ldots$ the various constants which depend only on $N$.

Put $G(x) := \phi \cdot g$. According to the Leibniz rule (3.2(ii)), we have $G \in H^1(B_{3R/2})$ and

$$\mathcal{L} G = g \mathcal{L} \phi + \phi \mathcal{L} g + 2(\nabla \phi, \nabla g)$$

in the sense of distributions. Then, by $\mathcal{L}^{sing} g \geq 0$ and $\mathcal{L}^{sing} \phi \geq 0$, we get $\mathcal{L}^{sing} G \geq 0$. The combination of (6.1) and (6.2) implies that

$$\mathcal{L}^{ac} G \geq \phi \mathcal{L}^{ac} g + 2(\nabla \phi, \nabla (G/\phi)) + G(\mathcal{L} \phi)^{ac}$$

and that, for any $\epsilon > 0$, we have used $g = |\nabla f|^2 = G/\phi$, $2KG \leq 2K G/\phi$ and that, for any $\epsilon > 0$, the following

$$-G(\nabla \phi, \nabla f)/\phi \leq 2|\nabla f| \cdot \frac{|\nabla \phi|}{\phi} \leq \epsilon G\frac{|\nabla \phi|^2}{\phi^2} + |\nabla f| \cdot \frac{1}{\epsilon} \leq \epsilon G\frac{C_1}{R^2} + |\nabla f|^2 \cdot \frac{1}{\epsilon}.$$

From the definition of $\phi$, we know that $G$ achieves one of its strict maximum in $B_{3R/2}$ in the sense of Theorem 4.2. Notice that $\mathcal{L}^{sing} G \geq 0$. Hence, according to Theorem 4.2 for $w := 2\frac{N-2}{N-1} f - 2 \ln \phi \in H^1(B_{3R/2}) \cap L^\infty(B_{3R/2})$ (since $u \geq \delta > 0$), and by combining with (6.3), we conclude that there exit a sequence $(x_j)_{j \in \mathbb{N}}$ such that, for each $j \in \mathbb{N}$,

$$G_j := G(x_j) \geq \sup_{B_{3R/2}} G - 1/j$$

and that (noticing that $\phi \in (0, 1]$)

$$2\frac{G_j^2}{N-1} - 2\frac{(N-2)}{N-1} \cdot \left(\epsilon G_j^2 \cdot \frac{C_1}{R^2} + G_j \frac{1}{\epsilon}\right) - 2KG_j - C_3 \cdot G_j (\frac{\sqrt{K}}{R} + \frac{1}{R^2})$$

$$\leq \phi(x_j) \cdot \frac{j}{f} \leq \frac{1}{j}.$$
for any $\epsilon > 0$. Letting $j \to \infty$ and denoting $\bar{G} := \sup_{B_{3R}/2} G = \lim_j G_j$, we obtain
\begin{equation}
\frac{1}{N-1} \cdot \frac{(N-2)\epsilon \cdot C_1}{(N-1)R^2} \cdot \bar{G} \leq K + \frac{N-2}{(N-1)\epsilon} + \frac{C_3}{2} \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right)
\end{equation}
for any $\epsilon > 0$.

In the case when $N > 2$, by choosing $\epsilon = \frac{\beta R^2}{(N-2) C_1}$, we obtain from (6.6) that
\begin{align*}
\frac{1}{N-1} \cdot \bar{G} &\leq K + \frac{C_1 \cdot (N-2)^2}{\beta R^2} + \frac{C_3}{2} \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \\
&\leq K + \frac{C_1 \cdot (N-2)^2}{\beta R^2} + \beta K + \frac{C_3^2}{16 \beta R^2} + \frac{C_3}{2R^2},
\end{align*}
where we have used
\[ \frac{C_3}{2} \frac{\sqrt{K}}{R} = 2 \sqrt{K} \cdot \frac{C_3}{4R} \leq \beta K + \frac{C_3^2}{4R^2}. \]

Then, we get
\begin{equation}
\bar{G} \leq \frac{1 + \beta}{1 - \beta} \left( (N-1)K + \frac{N-1}{1-\beta} \cdot \frac{1}{\beta R^2} \left( C_1 \cdot (N-2)^2 + \frac{C_3^2}{16} + \frac{C_3 \beta}{2} \right) \right)
\end{equation}
where we have used $\beta < 1$.

In the case when $N \in (1, 2]$, from (6.6), we have
\begin{align*}
\frac{1}{N-1} \cdot \bar{G} &\leq K + \frac{C_3}{2} \left( \frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \leq K + \beta K + \frac{C_3^2}{16 \beta R^2} + \frac{C_3}{2R^2}.
\end{align*}

Thus, the estimate (6.7) still holds in this case.

Therefore, the equation (6.7) shows that, for any $\beta \in (0, 1)$,
\[ \sup_{B_R} g \leq \frac{1 + \beta}{1 - \beta} \left( (N-1)K + \frac{C_4}{\beta(1-\beta)} \cdot R^2 \right). \]

Now the proof is finished. \qed

References

[1] L. Ambrosio, N. Gigli, G. Savaré, Bakry–Emery curvature-dimension condition and Riemannian Ricci curvature bounds, Ann. Probab., 43 (2015), 339–404.
[2] L. Ambrosio, N. Gigli, G. Savaré, Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces, Rev. Mat. Iberoam., 29 (2013), 969–996.
[3] L. Ambrosio, N. Gigli, G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below, Duke Math. J., 163(7) (2014), 1405–1490.
[4] L. Ambrosio, N. Gigli, G. Savaré, Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below, Invent. Math., 195(2) (2014), 289–391.
[5] L. Ambrosio, A. Mondino, G. Savaré, On the Bakry-Emery condition, the gradient estimates and the local-to-global property of $\text{RCD}^*(K,N)$ metric measure spaces, J. Geom. Anal., 26(1) (2016), 24–56.
[6] K. Bacher & K. Sturm, Localization and tensorization properties of the curvature-dimension for metric measure spaces, J. Funct. Anal., 259(1) (2010) 28–56.
[7] D. Bakry, F. Bolley & I. Gentil, The Li-Yau inequality and applications under a curvature-dimension condition, available at arxiv.org/abs/1412.5165.
[8] D. Bakry, Z. Qian, Some new results on eigenvectors via dimension, diameter, and Ricci curvature, Adv. Math. 155(1) (2000), 98–153.
[9] D. Bakry & M. Ledoux, *A logarithmic Sobolev form of the Li-Yau parabolic inequality*, Rev. Mat. Iberoam., 22(2): 683–702, 2006.

[10] F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi & S-T. Yau, *Li-Yau inequality on graphs*, J. Differ. Geom., 99 (2015), 359–405.

[11] H. Brezis & A. C. Ponce, *Kato’s inequality when $\Delta u$ is a measure*, C. R. Acad. Sci. Paris, Ser. I 338, 599–604, 2004.

[12] D. Bakry & Z. Qian, *Harnack inequalities on a manifold with positive or negative Ricci curvature*. Rev. Mat. Iberoam., 15(1): 143–179, 1999.

[13] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*. Geom. Funct. Anal. 9, (1999), 428–517.

[14] S. Y. Cheng, S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (1975), 333–354.

[15] E. B. Davies, *Heat kernels and spectral theory*, volume 92 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1989.

[16] M. Erbar, K. Kuwada, K. Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner inequality on metric measure spaces*, Invent. Math., 201 (2015), 993–1071.

[17] N. Gigli, A. Mondino, *Li-Yau and Harnack type inequalities in metric measure spaces*, Nonlinear Anal., 95(2014), 721–734.

[18] N. Gigli, *On the differential structure of metric measure spaces and applications*, Mem. Amer. Math. Soc. 236 (1113) (2015).

[21] P. Hajłasz, *Sobolev spaces on metric-measure spaces, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 173–218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[22] P. Hajłasz, P. Koskela, *Sobolev met Poincaré*, Mem. Am. Math. Soc. 145(688), (2000), x–101.

[23] L. Hörmander, *The analysis of linear partial differential operators I*, 2th edition, in ‘Grundlehren der mathematischen Wissenschaften’ 256, Springer-Verlag, Berlin, 1989.

[25] B. Hua, M. Kell & C. Xia, *Harmonic functions on metric measure spaces, available at http://arxiv.org/abs/1308.3607*

[26] B. Hua, & C. Xia, *A note on local gradient estimate on Alexandrov spaces*, Tohoku Math. J. 66(2) (2014), 259–267.

[27] R. Jiang, *Cheeger-harmonic functions in metric measure spaces revisited*, J. Funct. Anal., 266 (2014) 1373–1394.

[28] R. Jiang, *The Li-Yau inequality and heat kernels on metric measure spaces*, J. Math. Pures Appl. 104 (9) (2015) 29–57.

[29] R. Jiang, P. Koskela & D. Yang, *Isoperimetric inequality via Lipschitz regularity of Cheeger-harmonic functions*, J. Math. Pures Appl., 101(2014), 583–598.

[30] R. Jiang & H. C. Zhang, *Hamiltons gradient estimates and a monotonicity formula for heat flows on metric measure spaces*, Nonlinear Anal., 131(2016), 32–47.

[31] Y. Jiang & H. C. Zhang, *Sharp spectral gaps on metric measure spaces*, Calc. Var. PDE. 55 (2016), no. 1, Art. 14, 14 pp.

[32] P. W. Y. Lee, *Generalized Li-Yau estimates and Huisken’s monotonicity formula, available at http://arxiv.org/abs/1211.5559*

[33] J. Li & X. Xu, *Differential Harnack inequalities on Riemannian manifolds I: linear heat equation*. Adv. Math., 226(5): 4456–4491, 2011.

[34] P. Li & S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*. Acta Math., 156(3-4):153–201, 1986.

[35] X.-D. Li, *Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds*, J. Math. Pures Appl. 54, 1295–1361, 2005.

[36] J. Lott, C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. 169 (2009), 903–991.
[37] J. Lott, C. Villani, *Weak curvature bounds and functional inequalities*, J. Funct. Anal. 245(1) (2007), 311–333.
[38] P. Li, J. Wang, *Complete manifolds with positive spectrum. II*, J. Differential Geom. 62(1) (2002), 143–162.
[39] N. Marola, & M. Masson, *On the Harnack inequality for parabolic minimizers in metric measure spaces*, Tohoku Math. J., 65 (2013), 569–589.
[40] A. Mondino, A. Naber, *Structure theory of metric measure spaces with lower Ricci curvature bounds I*, available at http://arxiv.org/abs/1405.2222.
[41] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan 19 (1967) 205–214.
[42] A. Petrunin, *Alexandrov meets Lott–Villani–Sturm*, Münster J. of Math. 4 (2011), 53–64.
[43] B. Qian, *Remarks on differential Harnack inequalities*, J. Math. Anal. Appl., 409(1): 556–566, 2014.
[44] Z. Qian, H.-C. Zhang & X.-P. Zhu, *Sharp spectral gap and Li-Yau’s estimate on Alexandrov spaces*, Math. Z., 273(3-4) (2013) 1175–1195.
[45] T. Rajala, *Local Poincaré inequalities from stable curvature conditions on metric spaces*, Calc. Var. PDE 44(3-4) (2012), 477–494.
[46] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoam. 16,(2000), 243–279.
[47] P. Souplet & Q. S. Zhang, *Sharp gradient estimate and Yau’s Liouville theorem for the heat equation on noncompact manifolds*, Bull. London Math. Soc., 38(6) (2006) 1045–1053.
[48] K. Sturm, *On the geometry of metric measure spaces. I, II*, Acta Math. 196(1) (2006), 65–131, 133–177.
[49] K. Sturm, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math., 32(2) (1995), 275–312.
[50] K. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl., 75 (1996), 273–297.
[51] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201–228.
[52] H. C. Zhang, X. P. Zhu, *Ricci curvature on Alexandrov spaces and rigidity theorems*, Comm. Anal. Geom. 18(3) (2010), 503–554.
[53] H. C. Zhang, X. P. Zhu, *Yau’s gradient estimates on Alexandrov spaces*, J. Differ. Geom., 91(3) (2012), 445–522.
[54] H. C. Zhang, X. P. Zhu, *Lipschitz continuity of harmonic maps between Alexandrov spaces*, available at http://arxiv.org/abs/1311.1331.

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