On Model Predictive Control of Hybrid Systems

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Abstract

Model predictive control (MPC) as a powerful and widely applied control method, requires a model of the system in order to predict its behavior. Therefore, when the dynamics of a system is of hybrid nature, formulation of the MPC problem in the hybrid framework is a crucial step towards a successful control implementation. The response of a hybrid system within the prediction horizon is composed of both discrete-valued sequences and continuous-valued time-trajectories. The continuous trajectories can be calculated given the discrete sequences by the means of the recent results on the hybrid maximum principle. It is shown that these calculations reduce to the solution of a system of algebraic equations in the case of affine hybrid systems. Then, an algorithm is proposed for hybrid MPC which calculates the control inputs by iterating on the discrete sequences and calculating the corresponding continuous trajectories. It is shown that the algorithm finds the correct solution in a finite number of steps if the selected cost functional satisfies some conditions. The results are validated during an example control problem.

1 Hybrid Systems

A hybrid system has a set of continuous state variables and a discrete state variable that interact with each other while evolving along with the time. There are various formal definitions of hybrid systems. The following definition is based on the notions of hybrid automaton in [1].

Definition 1. A hybrid system $\mathcal{H}$ is a tuple $(Q, \Sigma, f, D, G, R)$ where
• $Q$ is a finite set of discrete state values,
• $\Sigma$ is a finite set of discrete input values,
• $f : Q \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}$ is the vector field,
• $D : Q \rightarrow pw(\mathbb{R}^{n_x})$ is the domain function,
• $G : Q \times \Sigma \times Q \rightarrow pw(\mathbb{R}^{n_x})$ is the guard function,
• $R : Q \times \Sigma \times Q \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ is the reset function.

A change of the discrete state is referred to as a jump. There can be a countable number of jumps within a time interval. To formulate the time response of the hybrid system, we define a non-decreasing sequence of time instants $t^s_i \in \mathbb{R}^{[0..n]}$ with $n \in \mathbb{Z}_{>0}$ such that $t^s_0$ is the initial time, $t^s_n$ is the final time, and $t^s_i$ for $i \in [1..n-1]$ are the jump instants. Both $n$ and $t^s_n$ can tend to infinity. For an arbitrary time dependent variable $y : [t^s_0, t^s_n] \rightarrow \mathbb{R}^{n_y}$ with $n_y \in \mathbb{Z}_{>0}$ the following notations are used.

\[
\begin{align*}
y^+_i &= \lim_{t \uparrow t^s_i} y(t) & i \in [1..n] \\
y^-_i &= y(t^s_i) & i \in [0..n-1]
\end{align*}
\] (1a) (1b)

The time response of the hybrid system which is denoted as an execution can be defined as in the following.

**Definition 2.** An execution of a hybrid system $\mathcal{H} = (Q, \Sigma, f, D, G, R)$ is a tuple $E = (t^s, q, \sigma, x, u)$ where

• $q \in Q^{[1..n]}$ is the discrete state sequence,
• $\sigma \in \Sigma^{[1..n-1]}$ is the discrete input sequence,
• $t^s \in \mathbb{R}^{[0..n]}$ is the time sequence,
• $x : [t^s_0, t^s_n] \rightarrow \mathbb{R}^{n_x}$ is the continuous state trajectory,
• $u : [t^s_0, t^s_n] \rightarrow \mathbb{R}^{n_u}$ is the continuous input trajectory,
for some $n \in \mathbb{Z}_{>0}$, such that $t^*_i > t^*_{i-1}$ for $i \in [1..n-1]$, $t^*_n \geq t^*_{n-1}$, and the following relationships are satisfied.

\[
\dot{x}(t) = f(q_i, x(t), u(t)) \quad \forall \ i \in [1..n], \quad t \in (t_{i-1}, t_i) \tag{2a}
\]

\[
x^-_i \in G(q_i, \sigma_i, q_{i+1}) \quad \forall \ i \in [1..n-1] \tag{2b}
\]

\[
x^+_i = R(q_i, \sigma_i, q_{i+1}, x^-_i) \quad \forall \ i \in [1..n-1] \tag{2c}
\]

The set of all executions of a hybrid system $\mathcal{H}$ is denoted by $\mathcal{E}(\mathcal{H})$. The set of executions $(t^*, q, \sigma, x, u) \in \mathcal{E}(\mathcal{H})$ that satisfy $t^*_0 = 0$, and $x(0) = x_{ic}$ for given value of $x_{ic} \in \mathbb{R}^n$ is denoted by $\mathcal{E}(\mathcal{H}, x_{ic})$. Also, we denote by $\mathcal{E}(\mathcal{H}, x_{ic}, T)$ the set of executions in $\mathcal{E}(\mathcal{H}, x_{ic})$ for which the final time is $T \in \mathbb{R}_{>0}$.

During the time interval $(t^*_{i-1}, t^*_i)$ with $i \in [1..n]$, the discrete state is constant and equal to $q_i$, and the vector of continuous states $x(t) \in \mathbb{R}^n$ evolves according to the differential equation in (2a) given the continuous input $u(t) \in \mathbb{R}^n$. This type of evolution of the state is denoted as a flow. The flow may continue as long as $x(t) \in D(q_i)$. The time instant for the $i$th jump $t^*_i$ for $i \in [1..n-1]$ is determined as the time at which $x(t)$ reaches the region specified by the guard function $G$ according to (2b). In this way, we avoid a kind of uncertainty when both flow and jump are possible at the same time by giving priority to jumps. At the time instant of jump, the continuous state changes instantaneously according to (2c) which is denoted as a reset.

There are conditions for existence and uniqueness of executions for hybrid systems. For example the Lipschitz continuity condition on the vector fields for existence and uniqueness of the flows. Another condition is to guarantee the possibility of either flow or jump at every time instant. More precisely, for every $(q, x) \in Q \times \mathbb{R}^n$ we have either $x \in D(q)$ or $x \in G(q, \sigma, q')$ for some $q' \in Q$ and $\sigma \in \Sigma$. In this paper, we assume that these conditions hold and an execution always exists from every initial conditions. The definition does not allow for jump at the initial time $t^*_0$ or multiple successive jumps at the same time (i.e. $t^*_i = t^*_{i-1}$ for some $i \in [1..n-1]$). To prevent from such conditions and to ensure that a flow is always possible after a jump, it is also assumed that the following relationships are satisfied for the initial continuous state $x_{ic}$, the initial discrete state $q_{ic}$, and for every $q, q' \in Q, \sigma \in \Sigma$. 

3
2 Hybrid MPC Algorithm

The basic idea of MPC, also known as the receding horizon optimal control, is to solve an optimal control problem at every time step over a finite horizon. In this section a continuous-time formulation of the MPC problem is considered for hybrid systems based on the Definition 1. Then, we make use of the hybrid maximum principle in order to develop an algorithm for hybrid MPC.

2.1 The Hybrid MPC Problem

Given the model of the system, the first step of formulating the MPC problem is to define a cost functional of the system variables over a fixed length time interval starting from the current time. The length of this time interval is referred to as the prediction horizon \( T_h \). The second step is to provide an algorithm for solving the optimal control problem at a given time step.

Considering an execution \( E = (t^s, q, \sigma, x, u) \), the cost functional \( J \) to be minimized is defined as in the following.

\[
J_m(E) = \sum_{i=1}^{n} \int_{t^i_{i-1}}^{t^i} l(q_i, u(t), x(t))dt + \sum_{i=1}^{n-1} h(i, q_i, \sigma_i, q_{i+1}, x_i^-) \\
J(E) = J_m(E) + h_f(q_n, x_n^-)
\]

In the above definition, \( l : Q \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0} \), \( h : \mathbb{Z}_{>0} \times Q \times \Sigma \times Q \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0} \), and \( h_f : Q \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0} \) are real-valued non-negative functions.

The optimal control problem that should be solved at a given time \( t \) is as the Problem \([1]\) in the following.
Problem 1. Given a hybrid system $H = (Q, \Sigma, f, D, G, R)$ according to the Definition 1, a cost functional $J$ as in (4), prediction horizon $T_h \in \mathbb{R} > 0$, initial states $q_{ic} \in Q$ and $x_{ic} \in \mathbb{R}^{n_x}$, find an execution $E = (t^*, q, \sigma, x, u) \in \mathcal{E}(H, x_{ic}, T_h)$ with $q_1 = q_{ic}$ which minimizes $J$.

Due to the time invariance of the hybrid dynamics in (2), the current time is shifted to the origin for simplicity by assuming that $E \in \mathcal{E}(H, x_{ic}, T_h)$. The values of $x_{ic}$ and $q_{ic}$ must be respectively set to the values of continuous and discrete states at the current time $t$. After solving the problem and obtaining $E$, the continuous input $u(0)$ and the discrete input $\sigma_1$ must be applied to the hybrid system $H$ as the plant.

2.2 Calculating the continuous trajectories given the discrete sequences

By fixing the discrete sequences of the executions in Problem 1 we arrive at the following problem which can be solved using the hybrid maximum principle (the extension of the the Pontryagin’s maximum principle to the hybrid systems).

Problem 2. Given a hybrid system $H = (Q, \Sigma, f, D, G, R)$ according to the Definition 1, a cost functional $J$ as in (4), prediction horizon $T_h \in \mathbb{R} > 0$, initial continuous state $x_{ic} \in \mathbb{R}^{n_x}$, and sequences $q \in Q^{[1..n]}$, $\sigma \in \Sigma^{[1..n-1]}$ for some $n \in \mathbb{Z}_{>0}$, find an execution $E \in \mathcal{E}(H, x_{ic}, T_h)$ with discrete state sequence $q$ and discrete input sequence $\sigma$ which minimizes $J$.

It would be more convenient to present the solution of the Problem 2 under the following assumption.

Assumption 1. Considering a hybrid system $H = (Q, \Sigma, f, D, G, R)$ according to the Definition 1, it is assumed that the functions $f$, $R$, $l$, $h$, and $h_f$ are differentiable with respect to their continuous arguments. Also, for every $(q, \sigma, q') \in \Theta$ there exist a differentiable function $g_{q,\sigma,q'}: \mathbb{R}^{n_x} \to \mathbb{R}$ that satisfies

$$\partial G(q, \sigma, q') = \{x \in \mathbb{R}^{n_x} : g_{q,\sigma,q'}(x) = 0\}$$

Since jump is assumed to have priority with respect to flow, a jump may occur if the continuous state lies on the boundary of the corresponding guard.
set. Hence, (2b) together with (5) results in
\[ g_{q_i, \sigma_i, q_{i+1}}(x_i^-) = 0 \quad i \in [1..n-1] \] (6)

The hybrid maximum principle has been presented in various forms in the previous works. The one which is more useful in here is from [2] based on which the set of conditions that are necessary for an execution in order to solve the Problem 2 is as in the following.

**Proposition 1.** Given a hybrid system \( \mathcal{H} = (Q, \Sigma, f, D, G, R) \) which satisfies the Assumption [1] if an execution of \( \mathcal{H} \) denoted by \( E = (t^s, q, \sigma, x, u) \) solves the Problem 2 for the sequences \( q \) and \( \sigma \), then there exist \( \alpha_i \in \mathbb{R} \) for \( i \in [1..n-1] \) with \( n = |q| \) and \( \lambda : [t_0^s, t_n^s] \rightarrow \mathbb{R}^{n_u} \) denoted as the costate such that the set of equations in (7) for \( i \in [1..n] \), (8) for \( i \in [1..n-1] \), and (9) are satisfied with the Hamiltonian function \( H \) defined in (10).

\[
\dot{\lambda}(t) = -D^T_x H(q_i, x, u, \lambda) \quad t \in (t_{i-1}, t_i) \tag{7a}
\]
\[
H(q_i, x, \lambda, u) \leq H(q_i, x, \lambda, \bar{u}) \quad \forall \bar{u} \in \mathbb{R}^{n_u} \tag{7b}
\]

\[
\lambda_i^- = D^T_x R(q_i, \sigma_i, q_{i+1}, x_i^-) \lambda_i^+ + D^T_x h(i, q_i, \sigma_i, q_{i+1}, x_i^-) + \alpha_i D^T_x g_{q_i, \sigma_i, q_{i+1}}(x_i^-) \tag{8a}
\]
\[
H(q_i, x_i^-, \lambda_i^-, u_i^-) = H(q_{i+1}, x_i^+, \lambda_i^+, u_i^+) \tag{8b}
\]

\[
\lambda_n^- = D^T_x h_f(q_n, x_n^-) \tag{9}
\]
\[
H(q, x, u, \lambda) = l(q, u, x) + \lambda^T f(q, x, u) \tag{10}
\]

In the above equations, \( D^T_x \) denotes the transpose of the Jacobian matrix with respect to \( x \) which becomes the gradient column vector for scalar-valued functions. It is mentioned that the discrete input is not included explicitly in the version of the hybrid maximum principle in [2]. However, insertion of the discrete input in the conditions is trivial since the discrete input affects the conditions indirectly through the discrete state which is assumed to be known and fixed in the Problem 2.
The conditions given in the Proposition 1 together with (2a), (2c), and (6) constitute a differential-algebraic system of equations that can be solved for \(x\), \(u\), and \(t_i^*\) for \(i \in [1..n-1]\) (in order to solve the Problem 2). In general, the solution can be obtained by using the numerical methods developed for hybrid optimal control based on the maximum principle. However, the solution process becomes considerably easier for an important class of hybrid systems as explained in the next part.

### 2.3 The case of affine hybrid systems

In this part, solution of the Problem 2 for the class of affine hybrid systems is studied in which the vector fields, reset functions, and the functions \(g_{q,\sigma,q'}\) in (5) take the following affine forms for every \(q, \sigma, q' \in \Theta\) (the matrix and vector coefficients on the right hand sides have the appropriate dimensions).

\[
\begin{align*}
f(q, x, u) &= A_q x + B^u_q u + B^c_q \\
g_{q,\sigma,q'}(x) &= M^x_{q,\sigma,q'} x + M^e_{q,\sigma,q'} \\
R(q, \sigma, q', x) &= L^x_{q,\sigma,q'} x + L^e_{q,\sigma,q'}
\end{align*}
\]

Many of the practical hybrid systems are affine in the above sense. Additionally, affine functions can approximate some more general classes of functions over restricted parts of their domains. In the case of hybrid systems, one can decompose a discrete state \(q \in Q\) with a nonlinear vector field \(f\) over the domain \(D(q)\) into several discrete states with approximately affine vector fields over smaller partitions of \(D(q)\). In the same way, the functions \(R\) and \(g\) can be approximated by affine functions when they are restricted to smaller regions. The Problem 2 can be solved much more efficiently in the case of affine hybrid systems with the cost functional in (4) which has quadratic elements as in the following (the matrix and vector coefficients have the appropriate dimensions).

\[
\begin{align*}
l(q, u, x) &= \frac{1}{2}(x - \bar{x}_q)^TW^x_q (x - \bar{x}_q) \\
&\quad + \frac{1}{2}(u - \bar{u}_q)^TW^u_q (u - \bar{u}_q) + W^c_q \\
h(i, q, \sigma, q', x) &= W^j_{i,q,\sigma,q'} \\
h_f(q, x) &= \frac{1}{2}(x - \bar{x}_q)^TW^f_q (x - \bar{x}_q)
\end{align*}
\]
Minimization of $H$ according to (7b) gives

$$u_i = \bar{u}_{q_i} - W^{u_{-1}}_{q_i} B^{u_T}_{q_i} \lambda_i \quad \forall \ i \in [1..n]$$  (13)

By replacing $u_i$ from the above equation in (2a) and (7a), we have two coupled differential equations that can be solved together as in the following ($I$ is the identity matrix and $0$ is a column vector of zero elements with the appropriate dimensions).

$$[x_i^- \lambda_i^-] = \Psi_{q_i}(t_i - t_{i-1}) \begin{bmatrix} x_{i-1}^+ \\ \lambda_{i-1}^+ \\ 1 \end{bmatrix} \quad \forall \ i \in [1..n]$$  (14a)

$$\Psi_q(\alpha) = [I \ 0] \exp \left( A^c_q(\alpha) \right) \quad \forall q \in Q, \alpha \in \mathbb{R}$$  (14b)

$$A^c_q = \begin{bmatrix} A_q & -B^u_q W^{u^{-1}}_q B^{u_T}_q & B^c_q + B^u_q \bar{u}_q \\ -W^x_q & A^T_q & W^x_q \bar{x}_q \\ 0 & 0^T & 0 \end{bmatrix}$$  (14c)

Also, equation (6) is written as

$$M^{x}_{q_i, \sigma, q_{i+1}} x_i^- + M^{c}_{q_i, \sigma, q_{i+1}} = 0 \quad \forall \ i \in [1..n - 1]$$  (15)

The equations (14) and (15) together with $x(t^0_i) = x_{ic}, \ (2a), \ (8a), \ (9)$ with the special forms of the functions in (11) and (12), constitute a system of linear equations in terms of the elements of $\mathcal{Y}$ defined in the following as the set of unknowns.

$$\mathcal{Y} = (x_1^-, \cdots, x_n^-, \lambda_1^-, \cdots, \lambda_n^-, \ x_0^+, \cdots, x_{n-1}^+, \lambda_0^+, \cdots, \lambda_{n-1}^+; \alpha_1, \cdots, \alpha_{n-1})$$  (16)

The set of linear equations can be solved by a matrix inversion and the closed form solution can be written in terms of $t_i$ for $i \in [1..n - 1]$ as in (17) for some $F_s : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{4nn_x+n-1}$.

$$\mathcal{Y} = F_s(t^s_1, \cdots, t^s_{n-1})$$  (17)
Considering that \( u_i^+ \) and \( u_i^- \) for \( i \in [1..n-1] \) are obtained from \( \lambda_i^+ \) and \( \lambda_i^- \) respectively according to (13), the equation (8b) for \( i \in [1..n-1] \) can be represented as (18a) with \( F_h : \mathbb{R}^{4n_x+n-1} \rightarrow \mathbb{R}^{n-1} \) defined in (18b).

\[
F_h(Y) = 0 \quad \text{(18a)}
\]

\[
[F_h(Y)]_i = H(q_i, x_i^-, \lambda_i^-, u_i^-) - H(q_{i+1}, x_i^+, \lambda_i^+, u_i^+) \quad \text{(18b)}
\]

The function \( F_h \) is defined in terms of the whole \( Y \) for simplicity of the equations, despite of the fact that the right hand side of (18b) does not depend on all of the elements of \( Y \). Replacing \( Y \) in (18a) from (17) we arrive at the following set of nonlinear algebraic equations with \( F_t = F_h \circ F_s \) that can be solved for \( t_i, i \in [1..n-1] \).

\[
F_t(t_1^s, \ldots, t_{n-1}^s) = 0 \quad \text{(19)}
\]

Then, \( Y \) is obtained from (17) which gives the remaining elements of the execution.

### 2.4 Calculating the discrete elements

In order to apply the MPC technique to a continuous-time hybrid system, the Problem 1 is solved by iterating on the discrete state and input sequences and solving a number of subproblems in which the discrete sequences are fixed. An algorithm is presented in this part which finds the optimal solution in a few number of iterations compared to the number of all possible discrete sequences. The subproblems are either in the form of the Problem 2 in the previous part or the Problem 3 defined in the following.

**Problem 3.** Given a hybrid system \( \mathcal{H} = (Q, \Sigma, f, D, G, R) \) according to the Definition 7, a cost functional \( J_m \) as in (4a), initial continuous state \( x_{ic} \in \mathbb{R}^{n_x} \), and sequences \( q \in Q^{[1..n]} \) and \( \sigma \in \Sigma^{[1..n-1]} \) with \( n \in \mathbb{Z}_{>0} \), find an execution \( E \in \mathcal{E}(\mathcal{H}, x_{ic}) \) with discrete state sequence \( q \) and discrete input sequence \( \sigma \) which satisfies the Equation (20) in the following and minimizes \( J_m \).

\[
t_n^s = t_{n-1}^s \quad \text{(20)}
\]
The above problem is different from the Problem 2 in that the terminal cost given by $h_f$ in (4b) is eliminated and the constraint $t^s_n = T_h$ is replaced with (20). The solution of the Problem 3 can be obtained by repeating the variational procedure for proof of the Proposition 1 and including (20) as a constraint. However, in this work the solution is obtained from the Proposition 1 for sufficiently large value of $T_h$ as in the following.

**Proposition 2.** Given a hybrid system $H = (Q, \Sigma, f, D, G, R)$ which satisfies the Assumption 1, if an execution of $H$ denoted by $E = (t^s, q, \sigma, x, u)$ solves the Problem 3 for the sequences $q$ and $\sigma$, then there exist $\alpha_i \in \mathbb{R}$ for $i \in [1..n-1]$ with $|q| = n$ and $\lambda : [t^s_0, t^s_n] \rightarrow \mathbb{R}^{n_x}$ such that the set of equations (7) for $i \in [1..n-1]$, (8) for $i \in [1..n-1]$, and (21) are satisfied.

$$\lambda_{n-1}^+ = 0 \quad (21)$$

**Proof.** The proof is omitted. \qed

Solution of the Problem 3 for affine hybrid systems is obtained by modifying the set of linear equations in the part 2.3 that must be solved in order to obtain (17). The modification includes removing (14a) for $i = n$ from the set of equations, and correspondingly removing $x_{ic}^{-}$ and $\lambda_{n}^{-}$ from the set of unknowns in (16). Also, the equations $t^s_n = T_h$ and (9) are replaced with (20) and (21).

Applications of the propositions 1 and 2 for solving the problems 2 and 3 for the general hybrid system, are respectively represented as the functions JPMPa and JPMPb in the following.

1 function JPMPa($x_{ic}, \sigma, q$)  
2 given $t^s_0 = 0$, $t^s_n = T_h$, and $x_{ic}^+ = x_{ic}$, solve the set of equations: (7) and (2a) for $i \in [1..n]$; (8), (2c), and (6) for $i \in [1..n-1]$; together with (9) to obtain $t^s_{1,n-1}$, $x$, $u$, and $\lambda$;  
3 calculate the right-hand side of (4b) and assign the result to $J$;  
4 return $(u(t_0), J)$;  
5 end

Using these functions, an implementation of the hybrid MPC is as the Algorithm 1 in the following which is based on the branch and bound method. The algorithm gets the current discrete and continuous states $q_{ic} \in Q$, $x_{ic} \in \mathbb{R}^{n_x}$.
function JPMPb(xic, σ, q)

given \( t^* = 0 \) and \( x^+_0 = x_{ic} \), solve the set of equations: (7), (8), (2a), (2c), and (6) for \( i \in [1..n-1] \); (20), and (21) to obtain \( t^*_1..n, x, u, \)
and \( \lambda \);
calculate the right-hand side of (4a) and assign the result to \( J_m \);
return \((u(t_0), J_m)\);
end

\( \mathbb{R}^{nx} \) and returns the discrete and continuous inputs \( \sigma_{ap} \in \Sigma, \ u_{ap} \in \mathbb{R}^{nu} \) that
should be applied to the hybrid system. Some analytical investigation of the
algorithm that show its correctness and some of its properties are provided
in the next section.

input : Continuous state \( x_{ic} \), discrete state \( q_{ic} \)
output: Continuous input \( u_{ap} \), discrete input \( \sigma_{ap} \)

(\( \hat{\nu}, \hat{\sigma}, \hat{q}, \hat{u}_0, \hat{J} \)) ← (0, \{\}, q_{ic}, 0, 0) ;
\( S \leftarrow \{(\hat{\nu}, \hat{\sigma}, \hat{q}, \hat{u}_0, \hat{J})\} ;
\textbf{while} \( \hat{\nu} = 0 \) \textbf{do}
\( (u^*_0, J_c) \leftarrow \text{JPMPa}(x_{ic}, \hat{\sigma}, \hat{q}) ;
\)\( S \leftarrow S \cup \{(1, \hat{\sigma}, \hat{q}, u^*_0, J_c)\} ;
\( T \leftarrow \{(\sigma, q) \in \Sigma \times Q \mid G(\hat{q}|\hat{q}_0, \sigma, q) \neq \emptyset \} ;
\textbf{for} \( (\sigma, q) \in T \) \textbf{do}
\( (u^*_0, J_c) \leftarrow \text{JPMPb}(x_{ic}, \hat{\sigma}\sigma, \hat{q}\sigma) ;
\)\( S \leftarrow S \cup \{(0, \hat{\sigma}\sigma, \hat{q}\sigma, u^*_0, J_c)\} ;
\textbf{end}
\( S \leftarrow S \setminus \{(\hat{\nu}, \hat{\sigma}, \hat{q}, \hat{u}_0, \hat{J})\} ;
\( (\hat{\nu}, \hat{\sigma}, \hat{q}, \hat{u}_0, \hat{J}) \leftarrow (\nu, \sigma, q, u_0, J) \in S \text{ such that } J \leq J' \text{ for all }
(\nu', \sigma', q', u_0', J') \in S ;
\textbf{end}
\( u_{ap} = \hat{u}_0 ;
\)\( \sigma_{ap} = \hat{\sigma}_1 ;

Algorithm 1: MPC Algorithm
3 Correctness and finiteness of the MPC algorithm

Theorem 1. If the Algorithm I is applied to a hybrid system $\mathcal{H}$ with the cost functional $J$ in (4), the initial continuous state $x_{ic} \in \mathbb{R}^{n_x}$, and the initial discrete state $q_{ic} \in Q$, then after the termination of the algorithm, the execution $\hat{E} = E_{\text{opt}}(1, \hat{q}, \hat{\sigma}, x_{ic})$ solves the Problem I.

Proof. The proof is omitted. \hfill \Box

Theorem 2. Considering a hybrid system $\mathcal{H}$ and the execution $E$ which solves the Problem I for $\mathcal{H}$ with the cost functional $J$ in (4), if the function $h$ is lower bounded by $h_{\text{min}}$, then the number of iterations of the Algorithm I will be less than $N(m, q_{ic})$ given in (22a) with $m = 1 + J(E)/h_{\text{min}}$.

$$N(m, q_{ic}) = \begin{cases} \frac{m^{n_a-1}}{n_a-1} & n_a > 1 \\ m & n_a = 1 \end{cases} \quad (22a)$$

$$n_a = |\Sigma|(|Q| - 1) \quad (22b)$$

Proof. The proof is omitted. \hfill \Box

References

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[2] A. Pakniyat and P. Caines. On the relation between the minimum principle and dynamic programming for classical and hybrid control systems. IEEE Transactions on Automatic Control, 62(9):4347–4362, 2017.