Dynamical observer-based fault detection and isolation for linear singular systems

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1. Introduction

This paper concerns the dynamical observer-based fault detection and isolation (FDI) for singular systems. Singular systems also known as descriptor or differential-algebraic systems can be considered as a generalization of dynamical systems. The singular system representation is a powerful modeling tool since they can describe processes governed by both, differential equations (dynamic) and algebraic equations (static). So that represents the physical phenomena that the model by ordinary differential equation cannot describe. These systems were introduced by Luenberger (1977) from a control theory point of view and since, great efforts have been made to investigate singular systems theory and its applications (Araujo, Barros, and Dorea, 2012, Bouklouarne, Darouach, Zasadzinski, Gillé, and Fiorelli, 2009, Darouach, 2009, 2012, Liu, Zhang, Yang, and Yang, 2008, Müller, 2005, Müller and Hou, 1993, Zhou and Lu, 2009).

The general goal of fault detection is to determine the fault presence into a system, whereas fault isolation is used to determine the location of the fault, after detection. During the last two decades, FDI has been of considerable interest (Hamdi, Rodriguez, Mechmeche, Theilliol, and Braiek, 2012, Li and Jaimoukha, 2009, Li and Yang, 2011). In Theilliol, Noura, and Ponsart (2002), the fault diagnosis for linear systems is treated. Rodrigues, Theilliol, Adam-Medina, and Sauter (2008) address the FDI for multimodels representation and in Bokor and Szabo (2009), the fault detection for nonlinear systems is presented.

In Liu and Si (1997), the problem of isolating multiples faults in linear systems, is presented by using an approach based on eigenstructure assignment to generate structured residuals, then the observer matrices are determined so that the \(i\)th residual represents the \(i\)th fault.

In Li and Jaimoukha (2009), an approach which generalizes the results of Liu and Si (1997) is presented. The authors construct an \(H_\infty\) FDI observer for linear systems, with the constraint that the transfer function from faults to residual of the error dynamics is equal to a preassigned diagonal transfer matrix. All these results use the proportional observer (PO).

By using a singular system representation, it is shown that the remainder kinds of FDI problems, that is, fault detection, FDI, and disturbance decoupled FDI are generally equivalent to the FDI problem for singular systems (Patton, Frank, and Clark, 2000). To our knowledge, few works have considered the approach of FDI based on an observer for singular systems, due to the structural complexity and strong constraints in designing procedure. Fault detection using unknown input observers were presented in Kim, Yeu, and Kawaji (2001), and in Duan, Howe, and Patton (2002) where the residuals are not affected by the unknown input. In Hamdi et al. (2012), Yeu, Kim, and Kawaji (2005), and Astorga, Theilliol, Ponsart, and Rodrigues (2012), fault diagnosis problem is treated by a PO.

In the estimation by a PO, there always exists a static error estimation. In order to deal with the inconveniences...
of PO, proportional-integral observers (PIO) were introduced with an integral gain of the output error in their structure. This change in the structure achieve steady-state accuracy in their estimations. Also a new structure of the observers was developed by Goodwin and Middleton (1989) and Marquez (2003), known as dynamic observers. This structure presents an alternative state estimation which can be considered as more general than PO and PIO. These last can be only considered as particular cases of this structure. The idea of including additional dynamics in the observer was presented by Goodwin and Middleton (1989).

In this paper, we consider FDI problem for singular systems with actuator faults. Residual signals are determined from properly weighted output errors between measurements and estimated outputs.

The main contribution of this paper is that the designed observer is a more general form than the existing dynamical observers: the PO and PIO which are only particular cases of the structure of our observer. This observer is used for actuator FDI in singular systems, which are a generalization of the standard systems. The proposed method is based on the directional residual generation in order to locate simultaneous faults. Finally, the effectiveness of this approach is shown through a numerical example simulation.

2. Preliminaries

In the present paper, the set of real matrices \( n \times m \) is denoted by \( \mathbb{R}^{n \times m} \). \( A^T \) denotes the transpose of the matrix \( A \), \( A^* \) is the generalized inverse of \( A \), i.e. \( AA^*A = A \). \( I_n \) denotes an \( n \times n \) identity matrix, \( I \) denotes an identity matrix with appropriate dimension, 0 denotes a zero scalar or matrix with appropriate dimension. The notation \( A = \text{diag}(a_1, \ldots, a_n) \) denotes that \( A \) is a diagonal matrix with elements \( (a_1, \ldots, a_n) \) in its diagonal, \( \text{ones}_{n,m} \) denotes an \( n \times m \) matrix with all elements one.

In Section 4.3, we use the following lemma to solve Linear matrix inequalities (LMIs).

**Lemma 1** (Skelton, Iwasaki, and Grigoriadis, 1998). Let matrices \( B, C, D = D^T \) be given, then the following statements are equivalent:

1. There exists a matrix \( X \) satisfying
   \[
   BXC + (BXC)^T + D < 0.
   \]
2. The following two conditions hold:
   \[
   B^TXB^T < 0 \quad \text{or} \quad BB^T > 0,
   \]
   \[
   C^TXC^T < 0 \quad \text{or} \quad C^TC > 0.
   \]

Suppose that the statement 2 holds. Let \( r_B \) and \( r_C \) be the ranks of \( B \) and \( C \), respectively, and \( (B_1, B_r) \) and \( (C_1, C_r) \) be any full rank factors of \( B \) and \( C \), i.e. \( B = B_1B_r \), \( C = C_1C_r \). Then, the matrix \( X \) in statement 1 is given by
\[
X = B_1^T K C_r^T + Z - B_1^T B_r Z C_r C_r^T,
\]
where \( Z \) is an arbitrary matrix and
\[
K = -R^{-1} B_1^T \theta C_r^T (C_r \theta C_r^T)^{-1} + S^{1/2} L (C_r \theta C_r^T)^{-1/2},
\]
\[
S = R^{-1} - R^{-1} B_1^T [\theta - \theta C_r^T (C_r \theta C_r^T)^{-1} C_r] B_1 R^{-1},
\]
where \( L \) is an arbitrary matrix such that \( \|L\| < 1 \) and \( R \) is an arbitrary positive-definite matrix such that
\[
\theta = (B_r R^{-1} B_1^T - D)^{-1} > 0.
\]

3. Problem formulation

Consider the following singular system subject to actuator fault:
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Gf(t), \\
y(t) &= Cx(t),
\end{align*}
\]
where \( x(t) \in \mathbb{R}^n \) is the semi state vector, \( u(t) \in \mathbb{R}^m \) is the input, \( f(t) \in \mathbb{R}^n \) is the fault vector and \( y(t) \in \mathbb{R}^p \) represents the measured output vector. Matrices \( E \in \mathbb{R}^{n \times n}, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, G \in \mathbb{R}^{n \times p} \) and \( C \in \mathbb{R}_r^{n \times n} \).

Let \( \text{rank}(E) = q \leq n \) and \( E^2 = E \left\{ E \ G \right\} = 0 \), in this case \( q_1 = n - q \).

In the sequel, we assume that
\[
\text{rank} \left[ \begin{array}{c} E \\ E^2 A \\ C \end{array} \right] = n.
\]

**Remark 1** Assumption 1 is equivalent to the impulse observability for singular systems. This condition is more general than \( \text{rank} \left[ \begin{array}{c} E \\ C \end{array} \right] = n \), generally considered, see, for example, Verhaegen and Dooren (1986), Darouach and Boutayeb (1995), and Hou and Müller (1995).

Now, let us consider the following fault isolation dynamical observer for the system (1):
\[
\begin{align*}
\dot{\tilde{z}}(t) &= N \tilde{z}(t) + Hv(t) + F \left[ \begin{array}{c} -E^2 Bu(t) \\ y(t) \end{array} \right] + Ju(t), \\
\dot{v}(t) &= S \tilde{z}(t) + Lv(t) + M \left[ \begin{array}{c} -E^2 Bu(t) \\ y(t) \end{array} \right], \\
\dot{x}(t) &= P \tilde{z}(t) + Q \left[ \begin{array}{c} -E^2 Bu(t) \\ y(t) \end{array} \right], \\
r(t) &= W(C \tilde{x}(t) - y(t)),
\end{align*}
\]
where \( \tilde{z} \in \mathbb{R}^{n_1} \) represents the state vector of the observer, \( v(t) \in \mathbb{R}^q \) is an auxiliary vector, \( \hat{x}(t) \in \mathbb{R}^n \) is the estimate of \( x(t) \) and \( r(t) \in \mathbb{R}^{n_q} \) is the residual vector.
Now, the following lemma is considered.

**Lemma 2** There exist a fault isolation observer of the form Equations (2)–(5) for the system (1) if the following two statements hold.

I. There exist a matrix T of appropriate dimension such that the following conditions are satisfied:
   
   (a) $NTE + F \left[ \begin{array}{c} E^+ A \\ C \end{array} \right] - TA = 0$,
   (b) $J = TB$,
   (c) $STE + M \left[ \begin{array}{c} E^+ A \\ C \end{array} \right] = 0$, and
   (d) $[P \quad Q] \left[ \begin{array}{c} E^+ A \\ C \end{array} \right] = L_ε$.

II. The matrix $[S \quad H \quad L]$ is a stability matrix, when $f(t) = 0$.

**Proof** Let $T \in \mathbb{R}^{q_0 \times n}$ be a parameter matrix and define $ε(t) = \zeta(t) - TEx(t)$, then its dynamic is given by

$$\dot{ε}(t) = Nε(t) + Hv(t) + (J - TB)u(t) + \left( NTE + F \left[ \begin{array}{c} E^+ A \\ C \end{array} \right] - TA \right)x(t) - TGf(t)$$

by using the definition of $ε(t)$, Equations (3) and (4), can be written as

$$\dot{v}(t) = Se(t) + Lv(t) + \left( STE + M \left[ \begin{array}{c} E^+ A \\ C \end{array} \right] \right)x(t),$$

$$\dot{t}(t) = Pε(t) + [P \quad Q] \left[ \begin{array}{c} TE \\ E^+ A \end{array} \right] x(t).$$

Now, if the conditions (a)–(d) of Lemma 2 are satisfied, the following observer error dynamic is obtained from Equations (6) and (7)

$$\begin{bmatrix} \dot{ε}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N & H \\ S & L \end{bmatrix} \begin{bmatrix} ε(t) \\ v(t) \end{bmatrix} - TGf(t) \tag{9}$$

and from Equation (4)

$$\dot{t}(t) - x(t) = Pε(t), \tag{10}$$

$$\dot{t}(t) = Pε(t) \tag{11}$$

so, if $f(t) = 0$ and matrix $[S \quad H \quad L]$ is a stability matrix, then $\lim_{t \rightarrow \infty} ε(t) = 0$.

The residual equation is obtained from Equation (5)

$$r(t) = WCε(t). \tag{12}$$

**Remark 2**

- The observer (2)–(4) is in general form and generalizes the existing ones. In fact:

  - For $H = 0$, $S = 0$ and $M = 0$ with $L$ a stability matrix, the observer reduces to the PO for singular systems (see, for example, Darouach, 2012 and references therein).

  - For $S = 0$, $H = 0$, $P = I$, $M = 0$ and $L$ a stability matrix, let matrices $F$ and $Q$ be partitioned according to the partition of $\left[ \begin{array}{c} -E^+ Bu(t) \\ y(t) \end{array} \right]$ as $F = [0 \quad F_a]$ and $Q = [0 \quad Q_a]$, respectively, then we obtain the following observer:

    $$\dot{ε}(t) = Nε(t) + Hv(t) + F \left[ \begin{array}{c} -E^+ Bu(t) \\ y(t) \end{array} \right] + Ju(t),$$

    $$\dot{v}(t) = y(t) - Cε(t),$$

    $$\dot{t}(t) = ε(t) + Q \left[ \begin{array}{c} -E^+ Bu(t) \\ y(t) \end{array} \right].$$

Which is the form used for the unknown input PO for singular systems (Darouach, Zasadzinski, and Hayar, 1996).

- For $P = I$, $L = 0$ and let $S = -C$ and $M = -CQ + [0 \quad I]$, then we obtain the following observer:

    $$\dot{ε}(t) = Nε(t) + Hv(t) + F \left[ \begin{array}{c} -E^+ Bu(t) \\ y(t) \end{array} \right] + Ju(t),$$

    $$\dot{v}(t) = y(t) - Cε(t),$$

    $$\dot{t}(t) = ε(t) + Q \left[ \begin{array}{c} -E^+ Bu(t) \\ y(t) \end{array} \right].$$

Which is the form used for the unknown input PIO for singular systems.

- The order of the observer is $q_0 \leq n$, when $q_0 = n - p$, we obtain the reduced order observer and for $q_0 = n$, we obtain the full order one.

Now, the problem of the fault isolation observer is reduced to determine the matrices $N, H, F, S, L, M, P, Q, W$, and $T$ such that conditions (a)–(d) from Lemma 2 are satisfied.

### 4. Observed-based FDI design

#### 4.1. Parameterization of the observers matrices

Before giving the solution to the dynamical observer design, the parameterization of the solutions to the algebraic constraints (a)–(d) of Lemma 2 is presented. Let $R \in \mathbb{R}^{q_0 \times n}$ be a full row rank matrix such that the matrix $Σ = \left[ \begin{array}{c} R \\ E^+ A \end{array} \right]$ is of full column rank, and let $Ω = \left[ \begin{array}{c} E^+ A \\ C \end{array} \right]$, and define the following matrices: $T_1 = RΩ^+ \left[ \begin{array}{c} 1 \end{array} \right]$, $T_2 = (I_{n+p_1+n_γ} - ΩΩ^+) \left[ \begin{array}{c} t_0 \end{array} \right]$, $N_1 = T_1AΣ^+ \left[ \begin{array}{c} t_0 \end{array} \right]$, $N_2 = T_2AΣ^+ \left[ \begin{array}{c} t_0 \end{array} \right]$, $N_3 = (I_{q_0+p_1+n_γ} - ΣΣ^+) \left[ \begin{array}{c} t_0 \end{array} \right]$, and $P_1 = Σ^+ \left[ \begin{array}{c} t_0 \end{array} \right]$.

The following lemma gives the general form of matrices $T, S, M, P, Q, N,$ and $F$. 

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LEMMA 3  The general form of $T, S, M, P, Q, N,$ and $F$ are

\[
T = T_1 - Z_1 T_2, \quad S = -Y_1 N_3, \quad M = -Y_1 F_3, \quad P = P_1 - Y_2 N_3, \quad Q = Q_1 - Y_2 F_3, \quad N = N_1 - Z_1 N_2 - Y_3 N_3, \quad F = F_1 - Z_1 F_2 - Y_3 F_3,
\]

where $Z_1, Y_1, Y_2,$ and $Y_3$ are arbitrary matrices of appropriate dimensions.

Proof  Conditions (c) and (d) of Lemma 2 can be rewritten as

\[
\begin{bmatrix} S \\ P \\ M \\ Q \end{bmatrix} [TE \ E^+ A \ C] = \begin{bmatrix} 0 \\ K_0 \end{bmatrix}. \tag{20}
\]

The necessary and sufficient conditions for (20) to have a solution is

\[
\text{rank} \begin{bmatrix} TE \\ E^+ A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} TE \\ E^+ A \\ C \end{bmatrix} = n. \tag{21}
\]

Now, since \(\text{rank} \begin{bmatrix} TE \\ E^+ A \\ C \end{bmatrix} = \text{rank} \Omega = n\), there exist matrices $T \in \mathbb{R}^{q_0 \times n}$, and $K \in \mathbb{R}^{q_0 \times (q_1 + p)}$ such that

\[
TE + K \begin{bmatrix} E^+ A \\ C \end{bmatrix} = R. \tag{22}
\]

which can also be written as

\[
\begin{bmatrix} T & K \end{bmatrix} \Omega = R \tag{23}
\]

and since \(\text{rank} \begin{bmatrix} \Omega \end{bmatrix} = \text{rank} \Omega\), or equivalently

\[
R\Omega^+ \Omega = R \tag{24}
\]

the general solution to Equation (23) is given by

\[
\begin{bmatrix} T & K \end{bmatrix} = R\Omega^+ - Z_1(I_{n_1 + q_1 + p} - \Omega\Omega^+) \tag{25}
\]
or equivalently

\[
T = T_1 - Z_1 T_2, \quad K = K_1 - Z_1 K_2, \tag{26-27}
\]

where $K_1 = R\Omega^+ \begin{bmatrix} 0 \\ I_{q_1 + n_0} \end{bmatrix}$, $K_2 = (I_{n_1 + q_1 + n_0} - \Omega\Omega^+) \begin{bmatrix} 0 \\ I_{q_1 + n_0} \end{bmatrix}$, and $Z_1$ is an arbitrary matrix of appropriate dimension.

Now, define the following matrices:

\[
F_1 = T_1 A \Sigma^+ \begin{bmatrix} K \\ I_{q_1 + n_0} \end{bmatrix}, \quad F_2 = T_2 A \Sigma^+ \begin{bmatrix} K \\ I_{q_1 + n_0} \end{bmatrix}, \quad F_3 = (I_{q_0 + q_1 + p} - \Sigma \Sigma^+ \begin{bmatrix} 0 \\ I_{q_1 + n_0} \end{bmatrix}),
\]

\[
Q_1 = \Sigma \begin{bmatrix} K \\ I_{q_1 + n_0} \end{bmatrix}, \quad K_1 = T_1 A \Sigma^+ \begin{bmatrix} 0 \\ I_{q_1 + n_0} \end{bmatrix}, \quad K_2 = T_2 A \Sigma^+ \begin{bmatrix} 0 \\ I_{q_1 + n_0} \end{bmatrix}, \quad \text{and } K_3 = (I_{q_0 + q_1 + p} - \Sigma \Sigma^+ \begin{bmatrix} 0 \\ I_{q_1 + n_0} \end{bmatrix}).
\]

From Equation (22), we obtain

\[
\begin{bmatrix} TE \\ E^+ A \\ C \end{bmatrix} = \begin{bmatrix} I_{q_0} \\ -K \end{bmatrix} \Sigma \tag{28}
\]

by replacing Equation (28) into Equation (20), it leads to

\[
\begin{bmatrix} S \\ P \\ M \end{bmatrix} [I_{q_0} \ 0 \ -K] \Sigma = \begin{bmatrix} 0 \\ I_n \end{bmatrix}. \tag{29}
\]

Since $\Sigma$ is a full column rank matrix, and $\begin{bmatrix} I_{q_0} \\ 0 \ -K \end{bmatrix}^{-1} = \begin{bmatrix} I_{q_0} \ K \\ 0 \ I_p \end{bmatrix}$, the general solution of Equation (29) is given by

\[
\begin{bmatrix} S \\ P \\ M \end{bmatrix} = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \Sigma + \begin{bmatrix} Y_1 \\ Y_2 \\ \end{bmatrix} (I_{q_0 + q_1 + p} - \Sigma \Sigma^+) \begin{bmatrix} I_{q_0} \\ K \\ 0 \end{bmatrix}. \tag{30}
\]

where $Y_1$ and $Y_2$ are arbitrary matrices of appropriate dimensions. Then matrices $S, M, P,$ and $Q$ can be determined as

\[
S = -Y_1 N_3, \quad M = -Y_1 F_3, \quad P = P_1 - Y_2 N_3, \quad Q = Q_1 - Y_2 F_3. \tag{31-34}
\]

By inserting $TE$ from Equation (22) into condition (a) of Lemma 2 leads to

\[
N \left( R - K \begin{bmatrix} E^+ A \\ C \end{bmatrix} \right) + F \begin{bmatrix} E^+ A \\ C \end{bmatrix} = TA, \tag{35}
\]

\[
NR + \tilde{K} \begin{bmatrix} E^+ A \\ C \end{bmatrix} = TA. \tag{36}
\]

where $\tilde{K} = F - NK$. Equation (35) can be written as

\[
[N \quad \tilde{K} ] \Sigma = TA. \tag{37}
\]

The general solution of Equation (36) is given by

\[
[N \quad \tilde{K} ] = TA \Sigma^+ - Y_3 (I_{q_0 + q_1 + p} - \Sigma \Sigma^+). \tag{38}
\]

By replacing $T$ from Equation (26) into Equation (37) it gives

\[
N = N_1 - Z_1 N_2 - Y_3 N_3, \quad \tilde{K} = \tilde{K}_1 - Z_1 \tilde{K}_2 - Y_3 \tilde{K}_3, \tag{39}
\]

where $Y_3$ is an arbitrary matrix of appropriate dimension.
As $N, T, K$, and $\tilde{K}$ are known, we can deduce the form of $F$ as follows:

$$ F = F_1 - Z_1 F_2 - Y_3 F_3 $$  \hfill (40)

**Remark 3** From the above results, we can see that the determination of the matrices of the observer (2)-(4) can be done as follows: Matrices $N, T, K$, and $\tilde{K}$ are known matrices, matrix $J = TB$, and matrices $S, M, P$, and $Q$ can be deduced from Equations (31)-(34). On the other hand, parameter matrices $H, L, Z_1, Y_1$, and $Y_3$ can be obtained from the stability of Equation (9).

Now, let $T_2 = T_2 G$ and $Z_1 = Z(\dot{X}_{n+n} - \dot{T}_2 T_2^T)$, where $Z$ is an arbitrary matrix of appropriate dimension. Then, matrices $T, K, N, \tilde{K}$, and $F$ can be expressed as

$$ T = T_1 - Z \bar{T}_2, $$  \hfill (41)

$$ K = K_1 - Z \bar{K}_2, $$  \hfill (42)

$$ N = N_1 - Z \bar{N}_2 - Y_3 N_3, $$  \hfill (43)

$$ \tilde{K} = \tilde{K}_1 - Z \bar{\tilde{K}}_2 - Y_3 \tilde{K}_3, $$  \hfill (44)

$$ F = F_1 - Z \bar{F}_2 - Y_3 F_3, $$  \hfill (45)

where $\bar{T}_2 = (I_{n+n} - \tilde{T}_2 T_2^T) T_2$, $\bar{K}_2 = (I_{n+n} - \tilde{T}_2 T_2^T) K_2$, $\bar{N}_2 = (I_{n+n} - \tilde{T}_2 T_2^T) N_2$, $\bar{\tilde{K}}_2 = (I_{n+n} - \tilde{T}_2 T_2^T) \bar{\tilde{K}}_2$, and $\bar{F}_2 = (I_{n+n} - \tilde{T}_2 T_2^T) F_2$.

The observer error dynamics (9) can be rewriting as

$$ \begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} N_1 - Z \bar{N}_2 & 0 \\ 0 & Y_3 H \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ Y_1 L \end{bmatrix} + \begin{bmatrix} N_3 & 0 \\ 0 & -I_{q_1} \end{bmatrix} \begin{bmatrix} \psi(t) \\ \varepsilon(t) \end{bmatrix}, $$  \hfill (46)

$$ r(t) = W \left[ CP_1 \right] \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix} $$  \hfill (47)

without lost of generality, $Y_2 = 0$ was taken for simplicity.

### 4.2. FDI design

The objective of the fault detection is to build a residual, which shows the presence of a fault into a system. The mathematical definition of a residual is

$$ \lim_{t \to \infty} r(t) = 0 \quad \text{for} \quad f(t) = 0, $$

$$ r(t) \neq 0 \quad \text{for} \quad f(t) \neq 0. $$

The fault isolation objective in this paper is to obtain a transfer function from faults to residual equal to a diagonal, in order to deal with faults that may occur simultaneously.

From Equations (46) and (47), the residual dynamics are given by

$$ \dot{\psi}(t) = (A_1 - Y \Lambda_2)\psi(t) + B_1 f(t), $$  \hfill (48)

$$ r(t) = WC_1 \psi(t). $$  \hfill (49)

Let $G_{fp}(s)$ be the transfer function from the fault $f(t)$ to the residual $r(t)$, such that

$$ G_{fp}(s) = \left[ \begin{array}{c} \Lambda_1 - Y \Lambda_2 \\ B_1 \\ W C_1 \end{array} \right]. $$  \hfill (50)

The objective is to render $G_{fp}(s)$ diagonal, i.e.

$$ G_{fp}(s) = \text{diag}(g_{11}(s), \ldots, g_{nf, nf}(s)), $$  \hfill (51)

while the stability of the observer is guaranteed. Since $G_{fp}(s)$ has a diagonal structure each residual are affected just by one fault. Considering this, it is possible to isolate simultaneous faults.

**Proposition 1** The transfer function (50) can be diagonalized if and only if $(C_1 B_1)$ has full column rank that $n_y \geq n_f$.

Proposition 1 is also called output separability condition (White and Speyer, 1987). To isolate $n_f$ faults in Equation (1) the rank of $(C_1 B_1)$ must be $n_f$, which in turn requires $n_y$ measured outputs.

The following theorem shows how to design an observer of the form Equations (2)-(4) to perform FDI.

**Theorem 1** Consider that $n_y \geq n_f$ and let

$$ \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{n_y}), \quad \lambda_i < 0, $$  \hfill (52)

$$ \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_{n_f}), \quad |\gamma| > 0, $$  \hfill (53)

where $i = 1, \ldots, n_f$. Be given. Then, there exist matrices $\Psi$ and $W$ such that

$$ (A_1 - Y \Lambda_2) B_1 = B_1 \Lambda, $$  \hfill (54)

$$ W C_1 B_1 = \Gamma. $$  \hfill (55)

If $(A_2 B_1)$ has full column rank, then matrices $\Psi$ and $W$ are given by

$$ \Psi = (A_1 B_1 - B_1 \Lambda)(A_2 B_1)^+, $$

$$ W = \Gamma(C_1 B_1)^+, $$

where $\bar{Z}$ is an arbitrary matrix of appropriate dimension. Finally, if there exist matrices $\Psi$ and $W$ satisfying Equations (54) and (55), then

$$ G_{fp}(s) = \left[ \begin{array}{c} \Lambda \\ \Gamma \end{array} \right] $$

$$ = \text{diag} \left( \frac{\gamma_1}{s - \lambda_1}, \ldots, \frac{\gamma_{n_f}}{s - \lambda_{n_y}} \right). $$  \hfill (58)
Proof Since $\mathbb{B}_1$ has full column rank there exist a matrix completion $\mathbb{B}_1^+ \in \mathbb{R}^{(q_0+n_q)\times(q_0+n_q-n_1)}$ such that $\tilde{B} = [\mathbb{B}_1 \ \mathbb{B}_1^+] \in \mathbb{R}^{(q_0+n_q)\times(q_0+n_q)}$ is nonsingular. Let $\tilde{B}^{-1} = [\tilde{B}_1 \ \tilde{B}_2]^T$ with $\tilde{B}_1 \in \mathbb{R}^{(q_0+n_q)\times n_1}$. Then, we obtain

$$G_P(s) = \begin{bmatrix} \tilde{B}^{-1}(A_1 - YA_2)\tilde{B} & 0 \\ WC_1\tilde{B} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{B}_1^T(A_1 - YA_2)\tilde{B}_1 & 0 \\ WC_1[\tilde{B}_1 \ \tilde{B}_2]^T & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{B}_1^T(A_1 - YA_2)\tilde{B}_1 & 0 \\ \tilde{B}_2^T(A_1 - YA_2)\tilde{B}_1 & 0 \end{bmatrix}$$

consider $[\tilde{B}_1 \ \tilde{B}_2]^T[\tilde{B}_1 \ \tilde{B}_2]^T = I$, then we have

$$G_P(s) = \begin{bmatrix} \Lambda & 0 \\ \Gamma & 0 \end{bmatrix}$$

diag \left( \frac{\gamma_1}{s - \lambda_1}, \ldots, \frac{\gamma_{nf}}{s - \lambda_{nf}} \right). \quad (59)

From Equation (59), we found that

$$(A_1 - YA_2)\mathbb{B}_1 = \mathbb{B}_1 \Lambda,$$ \quad (60)

$$(A_1 - YA_2)\mathbb{B}_1 = \Gamma,$$ \quad (61)

the general form of $Y$ from Equation (60) is given by

$$Y = (A_1\mathbb{B}_1 - \mathbb{B}_1 \Lambda)(A_2\mathbb{B}_1)^+$$

$$- \tilde{Z}(I - (A_2\mathbb{B}_1)(A_2\mathbb{B}_1)^+),$$ \quad (62)

where $\tilde{Z}$ is an arbitrary matrix of appropriate dimension. And the particular form of $W$ in Equation (61) is

$$W = \Gamma(\mathcal{C}_1\mathbb{B}_1)^+,$$ \quad (63)

Replacing Equations (62) and (63) in Equations (48) and (49), we obtain

$$\psi(t) = \left[ A_1 - (A_1\mathbb{B}_1 - \mathbb{B}_1 \Lambda)(A_2\mathbb{B}_1)^+A_2 \right] \dot{\psi}_1$$

$$+ \tilde{Z}(I - (A_2\mathbb{B}_1)(A_2\mathbb{B}_1)^+) \psi(t) + \mathbb{B}_1 f(t), \quad \dot{\psi}_2$$

$$r(t) = \Gamma(\mathcal{C}_1\mathbb{B}_1)^+\mathcal{C}_1 \psi(t).$$ \quad (64)

Now, is necessary to study the stability of the observer and determine the remainder of matrices of the observer.

### 4.3. Observer design

The following theorem gives the LMI conditions that allow the determination of the dynamical observer matrices.

**Theorem 2** There exist matrices $\hat{Z}$ and $Z$ such that system (64)–(65) is asymptotically stable if and only if there exist a matrix $X = X^T > 0$ such that the following LMIs are satisfied:

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0,$$ \quad (66)

where $X_1 = X_1^T > 0$, $X_3 = X_3^T > 0$ and $X_3 - X_2^TX_1^{-1}X_2 > 0$.

$$(N_3 - N_3T_rG(N_3T_rG)^+N_3)^{T\perp}\Pi, \quad (N_3 - N_3T_rG(N_3T_rG)^+N_3)^{T\perp} < 0,$$ \quad (67)

where

$$\Pi = \Pi_1^TX_1 + X_1 \Pi_1 - \Pi_1^T W_1 - W_1 \Pi_2, \quad (68)$$

$$\Pi_1 = N_1 + (T_r G \Lambda - N_r T_r G)^+ N_3, \quad (69)$$

$$\Pi_2 = N_2 - N_r T_r G(N_3 T_r G)^+ N_3. \quad (70)$$

In this case matrix, $Z = X_1^{-1}W_1$ and the matrix $\hat{Z}$ is parameterized as follows:

$$\hat{Z} = X^{-1}(K\mathcal{C}_1^+ + Z(I - \mathcal{C}_r\mathcal{C}_1^+)), \quad (71)$$

where

$$K = -\mathcal{R}^{-1}\partial \mathcal{C}_r^T(\mathcal{C}_r\partial \mathcal{C}_r^T)^{-1} + S^{-1/2} \mathcal{L}(\partial \mathcal{C}_r^T)^{-1/2}, \quad (72)$$

$$\partial = (\mathcal{R}^{-1} - \mathcal{D})^{-1} > 0, \quad (73)$$

$$S = \mathcal{R}^{-1} - \mathcal{R}^{-1}[\partial - \partial \mathcal{C}_r^T(\mathcal{C}_r\partial \mathcal{C}_r^T)^{-1}\partial] - \mathcal{R}^{-1} \mathcal{D} \mathcal{R}^{-1} \mathcal{L}.$$ \quad (74)

with

$$C = \left[ \begin{array}{cc} N_3 - N_3 T_r G(N_3 T_r G)^+ N_1 & 0 \\ 0 & 1 \end{array} \right]$$

and

$$D = \left[ \begin{array}{c} n_1 \Pi_1^X N_1 - \Pi_1^X N_2 \Pi_2 \end{array} \right]$$

where $\Pi_1, \Pi_2$, and $\Pi_2$ are defined in Equations (68)–(70), respectively, and $W_2 = X_1^T Z$.

Matrices $\mathcal{R}, \mathcal{L},$ and $Z$ are arbitrary matrices of appropriate dimensions satisfying $\mathcal{R} > 0$ and $\|\mathcal{L}\| < 1$. Matrices $\mathcal{C}_r$ and $\mathcal{C}_1$ are full rank matrices such that $C = \mathcal{C}_1 C_r$.

**Proof** Consider a matrix $X = X^T > 0$ such that

$$(\hat{A}_1 + \tilde{Z}\hat{A}_2)^TX + X(\hat{A}_1 + \tilde{Z}\hat{A}_2) < 0.$$ \quad (75)

This last inequality can be rewritten as

$$B \mathcal{X} C + (B \mathcal{X} C)^T + D < 0,$$ \quad (76)

where $\mathcal{X} = X \tilde{Z}$, $D = \hat{A}_1^T X + X \hat{A}_1$, its equivalence is defined in Theorem 2. Also matrix $B$ is taken as $B = I$, then $E_l = I, B_r = I$ and $B^\perp = 0.$
The solvability conditions of Lemma 1 applied to Equation (76) are reduced to
\[ C^T L C^T T < 0 \]  
(77)
with \( C^T = [(N_3 - N_2 G N_2^T G)^{-} N_2^T 0] \). By using the definition of \( D \) and \( W_1 \) we obtain Equation (67). From Theorem 2 if condition (67) is satisfied, the matrix \( \hat{Z} \) is obtained as in Equations (71)–(74).

**Remark 4** The dynamical observer application to standard systems can be obtained directly form our results by setting \( E = I \), then we have \( E^T = 0, \Sigma = \left[ \frac{E}{\hat{E}} \right] \), and \( \Omega = \left[ \frac{\hat{E}}{\hat{E}} \right] \).

The following algorithm summarize the procedure to compute all the observer matrices.

**ALGORITHM 1**

**Step 1.** Select the observer order \( q_0 \) and a matrix \( R \in \mathbb{R}^{q_0 \times q_0} \) such that \( \text{rank}(\Sigma) = n \).

**Step 2.** Compute the matrices \( T_1, T_2, K_1, K_2, N_1, N_2, \ldots, N_b, \hat{K}_1, \hat{K}_2, \hat{K}_3, \text{ and } P_1 \) defined in Section 4.

**Step 3.** Select the matrices \( \Lambda \) and \( \Gamma \) as defined in Theorem 1.

**Step 4.** Compute matrix \( W \) as in Equation (63).

**Step 5.** Find \( R > 0 \) such that Equation (73) be positive definite.

**Step 6.** Find the matrices \( L \) and \( Z \) such that \( \| L \| < 1 \) to solve Equations (66) and (67), then obtain the matrix \( \hat{Z} \) as in (71).

**Step 7.** Compute the matrices of the dynamical observer (2)–(3): \( N, H, F, J, S, L, M, P \) and \( Q \), by using (43) to compute \( N \), (56) to compute \( H \) and \( L \), (31)–(34) to compute \( S, M, P \) and \( Q \). \( F \) is defined in (45) and \( J \) is defined by Lemma 2.

5. **Illustrative example**

In order to illustrate our results, let us consider the following singular system:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\dot{x}(t) =
\begin{bmatrix}
-2.7 & 0 & 0.3 \\
-0.2 & -3 & 0 \\
-0.11 & 1.74 & -1 \\
\end{bmatrix}
x(t) +
\begin{bmatrix}
1 \\
0.5 \\
1 \\
\end{bmatrix}
u(t)
+ \begin{bmatrix}
1 & 1 \ 
0 & 1 \\
0 & 0 \\
\end{bmatrix}f(t),
\]

\[ y(t) = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}x(t). \]

By following Algorithm 1, an observer with order \( q_0 = 3 \) was selected and \( R = I_3 \), such that \( \text{rank}(\Sigma) = 3 \). Matrices \( R, L \) and \( Z \) were selected as \( R = I_b, L = \text{ones}_{6,4} \times 0.1 \), and

\[
Z = \begin{bmatrix}
9 & 3 & 2 & 1 & 8 & 9 & 0 & 3 \\
9 & 4 & 1 & 3 & 8 & 7 & 8 & 2 \\
9 & 2 & 8 & 4 & 1 & 7 & 7 & 4 \\
9 & 4 & 4 & 5 & 7 & 8 & 3 & 8 \\
9 & 1 & 8 & 4 & 7 & 2 & 8 & 4 \\
9 & 4 & 8 & 2 & 8 & 4 & 9 & 3 \\
\end{bmatrix}
\]

By using the LMI toolbox of MATLAB, we solved the inequalities (66) and (67). The observer gains are constructed using Theorem 2.

\[
\dot{\zeta}(t) =
\begin{bmatrix}
-3.11 & -2.89 & -3.67 \\
-1.10 & -2.04 & -2.08 \\
0.21 & 0.36 & -0.66 \\
\end{bmatrix}
\zeta(t)
+ \begin{bmatrix}
0.87 & 0.87 & 0.87 \\
0.36 & 0.36 & 0.36 \\
0.20 & 0.20 & 0.20 \\
\end{bmatrix}
\times 0.01v(t)
+ \begin{bmatrix}
14.56 & -0.97 & 2.23 \\
6.13 & 0.10 & -0.10 \\
2.55 & 0.42 & -0.88 \\
\end{bmatrix}
\begin{bmatrix}
-E^\perp Bu(t) \\
y(t) \\
\end{bmatrix}
+ \begin{bmatrix}
14.09 \\
6.17 \\
2.75 \\
\end{bmatrix} u(t),
\]

\[
\dot{\nu}(t) =
\begin{bmatrix}
0.44 & -0.69 & 1.18 \\
0.44 & -0.69 & 1.18 \\
0.44 & -0.69 & 1.18 \\
\end{bmatrix}
\times 0.01\zeta(t),
+ \begin{bmatrix}
-2.03 & 0.22 & 0.22 \\
0.22 & -2.03 & 0.22 \\
0.22 & 0.22 & -2.03 \\
\end{bmatrix}
\times 0.01v(t)
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
-E^\perp Bu(t) \\
y(t) \\
\end{bmatrix}
,\]

\[
\begin{bmatrix}
0.59 & -0.05 & -0.19 \\
-0.05 & 0.31 & 0.15 \\
-0.19 & 0.15 & 0.37 \\
\end{bmatrix}
\zeta(t)
+ \begin{bmatrix}
0.07 & 0.56 & -0.49 \\
0.49 & 0.16 & 0.33 \\
-0.09 & 0.24 & 0.67 \\
\end{bmatrix}
\begin{bmatrix}
-E^\perp Bu(t) \\
y(t) \\
\end{bmatrix}
,\]

\[
r(t) =
\begin{bmatrix}
-2.20 & -5.85 \\
6.05 & 4.97 \\
\end{bmatrix}(C\hat{x}(t) - y(t)),
\]

In order to evaluate the observer performance a measurement noise \( n(t) \) was considered in the measured output, then the noise-corrupted outputs become \( y_1(t) = x_1(t) + x_3(t) + n(t), \) and \( y_2(t) = x_3(t) + n(t) \).

The results are depicted in Figure 1–5, which show two cases of simulation. The first case shows step faults with different time of apparition, and the second case shows simultaneous faults, and one of these is time variant.
Figure 1 shows the measurement noise, the system input \( u(t) \) was considered as a constant \( u(t) = 2 \).

Case 1 Step actuator faults: In this case, the actuators faults were considered as a step, each one applied at different time, see Figure 2.

Figure 3 gives the residual where each fault can be readily distinguished from the other, which illustrates that the proposed observer satisfies the requirement of FDI.

Once the residuals were generated, the next step is the evaluation of the residuals by assigning a symptom.

| symptom | if residue > threshold, | if residue < threshold. |
|---------|-------------------------|-------------------------|
| 1       |                         |                         |
| 0       |                         |                         |

With these symptoms we can generate the following signature table.

From Table 1, we observe that the signature to represent the presence of the fault \( f_1(t) \) is different from the signature representing the fault \( f_2(t) \), so that we can isolate each fault.

Case 2 Simultaneous faults: Figure 4 shows the faults, where \( f_1(t) \) is time variant with a ramp behavior. Between 30 and 35 both faults are present into the system.

From Table 2, we observer that the signature in the case of simultaneous faults is different from the other cases, so that even simultaneous faults can be isolated.

The example makes clear that the actual residual response reflects the fault presence in the system. With a good choice of threshold, the observer designed satisfies the performance requirements of FDI.
6. Conclusion

In this paper, a dynamical observer-based FDI for singular systems has been presented. The conditions for the existence of this observer were given in terms of a set of LMIs. The obtained observer satisfies the constraint to obtain a diagonal transfer function between the fault \( f(t) \) and the residual \( r(t) \) to isolate faults. The approach presented here permits to parameterize the others dynamical observers for fault detection (Remark 2). The case of standard systems can be directly obtained from our results (Remark 4).

Disclosure statement

No potential conflict of interest was reported by the author(s).

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