About torsion of inhomogeneous rods made of ideal rigid plastic material under linearized condition of plasticity

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Abstract. The torsion of inhomogeneous rods made of ideal rigid plastic material is investigated in this work. The integrals determining the stress and strain states of a rod under linearized plasticity condition are obtained. The field of characteristics of the basic relations is constructed, the rupture lines of stresses are found.

1. Introduction
Torsion is one of the types of body deformation characterized by mutual rotation of its cross sections under the influence of moments acting in these sections. Torsion of the rods is quite common in engineering practice, especially in mechanical engineering. The theory of torsion of isotropic and anisotropic rods made of ideal rigid plastic material is described in [1-4]. Transition to the case of a rod of inhomogeneous material leads to difficulties: in general, the problem cannot be integrated. The individual cases of torsion of inhomogeneous and composite rods are considered in [5-7].

2. Materials and methods
The torsion of rods made of an ideal rigid-plastic material is considered in this work. It is assumed that the plastic properties of the rod material depend on the direction or coordinates of the point, that is, the rod material has inhomogeneity. The rod of inhomogeneous material is represented by a composite rod, when different plasticity conditions are true in different parts of the rod.

3. Results and discussion
The ratios of the theory of torsion of inhomogeneous rods made of ideal rigid plastic material can have the form of

\[\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0,\]
\[\tau_{xz} = \tau_{xz}(x, y), \tau_{yz} = \tau_{yz}(x, y)\]

is an equilibrium equation

\[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0\]  

is a plasticity condition
The plasticity condition (3) in the plane $\tau_{xz}, \tau_{yz}$ represents a circle of radius $k_0$ (figure 1) which center is at the point with coordinates $K_1, K_2$.

Suppose that the circle of the yield condition (3) is replaced by a closed polyline $M_1M_2M_3...M_nM_1$ (figure 1)

$$A_i(\tau_{xz} - K_1) + B_i(\tau_{yz} - K_2) = k_0$$

where $A_i, B_i = const, i = 1, 2, ..., n$.

Condition (5) represents a linearized plasticity condition (3) on a certain segment. Considering the condition (5) as a plastic potential, we obtain instead of (4) the ratio

$$\frac{\varepsilon_{xz}}{A_i} = \frac{\varepsilon_{yz}}{B_i}.$$  

Integrating the ratio (6) and a part of the ratios (4), and taking into account that at the initial moment of twisting the strain components $\varepsilon_{ij}$ are equal to 0, we obtain
From (7), follows

\begin{equation}
B_i e_{xz} - A_i e_{yz} = 0.
\end{equation}

Suppose that the displacement components \( u, v, w \) have the form

\begin{equation}
u = \theta yz, v = -\theta xz, w = w(x, y),
\end{equation}

where \( \theta \) is a twist, \( w \) is a deplanation.

Expressing the strain components by displacement components, from (8), (9) we obtain

\begin{equation}
-B_i \frac{\partial w}{\partial x} + A_i \frac{\partial w}{\partial y} = \theta (A_i x + B_i y).
\end{equation}

From (10), it follows that straight lines

\begin{equation}
A_i x + B_i y = C_{i1} (C_{i1} = \text{const})
\end{equation}

are characteristics. Along the characteristics (11), the ratios take place

\begin{equation}
B_i w + \theta C_{i1} x = C_{i2} \quad \text{or} \quad A_i w - \theta C_{i1} y = C_{i3}
\end{equation}

where \( C_{i2}, C_{i3} = \text{const} \) along the characteristic.

For further purposes, we need the expressions for the strain components more than the deplanation expressions.

Differentiating the ratio (10) by the variable \( x \), we get the equation

\begin{equation}
-B_i \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right) + A_i \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) = \theta A_i.
\end{equation}

From equation (13), it follows that along the characteristics (11) the ratios are valid

\begin{equation}
B_i \frac{\partial w}{\partial x} + \theta A_i x = C_{i4} \quad \text{or} \quad \frac{\partial w}{\partial x} - \theta y = C_{i5}
\end{equation}

where \( C_{i4}, C_{i5} = \text{const} \) along the characteristic.

Similarly, differentiating the ratio (10) by the variable \( y \), we get the equation

\begin{equation}
-B_i \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial y} \right) + A_i \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right) = \theta B_i.
\end{equation}

From equation (15), it follows that along the characteristics (11) the ratios are valid

\begin{equation}
\frac{\partial w}{\partial x} + \theta x = C_{i6} \quad \text{или} \quad A_i \frac{\partial w}{\partial y} - \theta B_i y = C_{i7}
\end{equation}

where \( C_{i6}, C_{i7} = \text{const} \) along the characteristic.

Using the second ratio (14) and the first ratio (16), we obtain that along the characteristics (11) the ratios are valid

\begin{equation}
e_{xz} - \theta y = \frac{1}{2} C_{i5}, \quad e_{yz} + \theta x = \frac{1}{2} C_{i6}.
\end{equation}
It should be noted that from the ratios (17), (11), (8), it follows that
\[ A_1 C_{i6} - B_1 C_{i5} = 2 \theta C_{i1}. \] (18)

Differentiating the ratio (5) by the variable \( y \), we obtain
\[ A_1 \frac{\partial \tau_{xz}}{\partial y} + B_1 \frac{\partial \tau_{yz}}{\partial y} = A_1 \frac{\partial K_1}{\partial y} + B_1 \frac{\partial K_2}{\partial y}. \] (19)

Taking into account (2) from the equation (19), we have
\[ A_1 \frac{\partial \tau_{xz}}{\partial y} - B_1 \frac{\partial \tau_{yz}}{\partial y} = A_1 \frac{\partial K_1}{\partial y} + B_1 \frac{\partial K_2}{\partial y}. \] (20)

From (20), it follows that along the characteristics (11) the following ratios for strain components are valid
\[ A_1 \tau_{xz} = A_1 K_{11} + B_1 K_{12}, B_1 \tau_{yz} = k_0 + A_1(K_1 - K_{11}) + B_1(K_2 - K_{12}) \] (21)

where
\[ K_{1s} = \int \frac{\partial K_s}{\partial y}(\alpha, y) dy, \alpha = \frac{1}{A_i}(C_{i1} - B_i y) \quad (s = 1, 2). \] (22)

Similarly, differentiating the ratio (5) by the variable \( x \), taking into account (2) we obtain that along the characteristics (11) the following ratios for strain components are valid
\[ B_1 \tau_{yz} = A_1 K_{21} + B_1 K_{22}, A_1 \tau_{xz} = k_0 + A_1(K_1 - K_{21}) + B_1(K_2 - K_{22}) \] (23)

where
\[ K_{2s} = \int \frac{\partial K_s}{\partial x}(x, \beta) dx, \beta = \frac{1}{B_i}(C_{i1} - A_i x) \quad (s = 1, 2). \] (24)

Consider the torsion of a rod of rectangular cross section \( m_1 m_2 m_3 m_4 \) with sides \( 2a \) and \( 2b \) (figure 2). On the contour of cross section, the tangent stress vector \( \vec{\tau} = (\tau_{xz}, \tau_{yz}) \) is parallel to the contour.

![Figure 2](image_url)

**Figure 2.** The torsion of a rod of rectangular cross section.
In the case of isotropic ideal rigid plastic material, the characteristics are directed perpendicular to the contour. In this case, the characteristics (11) are fixed, so it is always possible to choose a linearized plasticity condition (5) for a given contour of cross section of the rod so that the characteristics remain perpendicular to the contour. For this, it is necessary to choose $A_i, B_i$ in condition (5) so that the vector $\vec{n}_i = (A_i, B_i)$ is parallel to the segment $m_i m_{i+1}$ of the contour (figure 2).

Here we have four groups of characteristics

\begin{align*}
A_1x + B_1y &= C_{11}, \\
A_2x + B_2y &= C_{21}, \\
A_3x + B_3y &= C_{31}, \\
A_4x + B_4y &= C_{41}.
\end{align*}

In order to have the characteristics (25) orthogonal to the segment $m_1 m_2$ of cross section contour of the rod, we should set $A_1 = 0, B_1 = 1$. The plasticity condition (5) takes the form

\begin{equation}
\tau_{yz} - K_2 = k_0.
\end{equation}

The characteristics (25) have the form of

\begin{equation}
y = C_{11}.
\end{equation}

From (18), (21) and (23), it follows

\begin{equation}
C_{15} = -2\theta C_{11}, K_{12} = 0.
\end{equation}

Then from (29) and (2), we have

\begin{equation}
\tau_{yz} = K_2 + k_0, \tau_{xz} = k_{11}(x, y)
\end{equation}

where $k_{11} = -\int \frac{\partial K_2}{\partial y}(x, y)dx$, $k_{11}(a, y) = 0$.

In order to have the characteristics (26) orthogonal to the segment $m_2 m_3$ of cross section contour of the rod, we should set $A_2 = 1, B_2 = 0$. The plasticity condition (5) takes the form

\begin{equation}
\tau_{xz} - K_1 = k_0.
\end{equation}

The characteristics (26) have the form of

\begin{equation}
x = C_{21}.
\end{equation}

From (18), (21) and (23), it follows

\begin{equation}
C_{26} = 2\theta C_{21}, K_{21} = 0.
\end{equation}

Then from (33) and (2), we have

\begin{equation}
\tau_{xz} = K_1 + k_0, \tau_{yz} = k_{21}(x, y)
\end{equation}
where \( k_{21} = -\int \frac{\partial K_1}{\partial x}(x,y)\,dy, \quad k_{21}(x,-b) = 0. \)

In order to have the characteristics (27) orthogonal to the segment \( m_3m_4 \) of cross section contour of the rod, we should set \( A_1 = 0, B_1 = -1. \) The plasticity condition (5) takes the form

\[
\tau_{yz} - K_2 = -k_0. \tag{37}
\]

The characteristics (27) have the form of

\[
y = C_{31}. \tag{38}
\]

From (18), (21) and (23), it follows

\[
C_{35} = 2\theta C_{31}, K_{12} = 0. \tag{39}
\]

Then from (37) and (2), we have

\[
\tau_{yz} = K_2 - k_0, \tau_{xz} = k_{12}(x,y) \tag{40}
\]

where \( k_{12} = -\int \frac{\partial K_2}{\partial y}(x,y)\,dx, \quad k_{12}(-a,y) = 0. \)

In order to have the characteristics (28) orthogonal to the segment \( m_4m_1 \) of cross section contour of the rod, we should set \( A_1 = -1, B_1 = 0. \) The plasticity condition (5) takes the form

\[
\tau_{xz} - K_1 = -k_0. \tag{41}
\]

The characteristics (25) have the form of

\[
x = C_{41}. \tag{42}
\]

From (18), (21) and (23), it follows

\[
C_{46} = -2\theta C_{41}, K_{21} = 0. \tag{43}
\]

Then from (31) and (2), we have

\[
\tau_{xz} = K_1 - k_0, \tau_{yz} = k_{22}(x,y) \tag{44}
\]

where \( k_{22} = -\int \frac{\partial K_2}{\partial x}(x,y)\,dy, \quad k_{22}(x,b) = 0. \)

It is necessary to pay particular attention to the lines of strain rupture (lines \( m_1L, m_2L, m_3N, m_4N, NL \) in figure 2), which occur when two or more characteristics pass through a given point of cross section.

The lines of strain rupture are a trace of vanishing hard areas. They always satisfy the ratios

\[
e_{xz} = e_{yz} = 0. \tag{45}
\]

The curve \( m_1L \) is a line of strain rupture coming out of the vertex \( m_1 \) of cross section contour of the rod and formed by intersection of the group of characteristics (30) and (42). According to (17), from (45) it follows that on \( m_1L \) the ratios are valid

\[
C_{15} = C_{45}, C_{16} = C_{46} \tag{46}
\]
From (32) and (44), we have the equation for the line of strain rupture $m_1L$

$$\frac{dx}{k_1-k_0-k_{11}} = \frac{dy}{k_{22}-k_2-k_0}. \tag{47}$$

The curve $m_2L$ is a line of strain rupture coming out of the vertex $m_2$ of cross section contour of the rod and formed by intersection of the group of characteristics (30) and (34). According to (17), from (45) it follows that on $m_2L$ the ratios are valid

$$C_{15} = C_{25}, \quad C_{16} = C_{26}. \tag{48}$$

From (32) and (36), we have the equation for the line of strain rupture $m_2L$

$$\frac{dx}{k_1+k_0-k_{11}} = \frac{dy}{k_{21}-k_2-k_0}. \tag{49}$$

The curve $m_3N$ is a line of strain rupture coming out of the vertex $m_3$ of cross section contour of the rod and formed by intersection of the group of characteristics (38) and (34). According to (17), from (45) it follows that on $m_3N$ the ratios are valid

$$C_{35} = C_{25}, \quad C_{36} = C_{26}. \tag{50}$$

From (40) and (36), we have the equation for the line of strain rupture $m_3N$

$$\frac{dx}{k_1+k_0-k_{12}} = \frac{dy}{k_{21}-k_2+k_0}. \tag{51}$$

The curve $m_4N$ is a line of strain rupture coming out of the vertex $m_4$ of cross section contour of the rod and formed by intersection of the group of characteristics (38) and (42). According to (17), from (45) it follows that on $m_3N$ the ratios are valid

$$C_{35} = C_{45}, \quad C_{36} = C_{46}. \tag{52}$$

From (40) and (44), we have the equation for the line of strain rupture $m_4N$

$$\frac{dx}{k_1-k_0-k_{12}} = \frac{dy}{k_{22}-k_2+k_0}. \tag{53}$$

The curve $NL$ is a line of strain rupture formed by intersection of the group of characteristics (34) and (42). According to (17), from (45) it follows that on $m_3N$ the ratios are valid

$$C_{25} = C_{45}, \quad C_{26} = C_{46}. \tag{54}$$

From (36) and (44), we have the equation for the line of strain rupture $NL$

$$\frac{dx}{2k_0} = \frac{dy}{k_{21}-k_{22}}. \tag{55}$$

4. Conclusion

Thus, in this work:

- the integrals determining the stress and strain states of inhomogeneous ideal rigid plastic rod at torsion are obtained for a linearized plasticity condition, the characteristics of the basic relations are found;
• the limit state of an inhomogeneous ideal rigid plastic rod with a rectangular cross section is investigated: the field of characteristics of the basic relations is constructed, the ratios along the characteristics and the lines of strain rupture are found.

References
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