THE HARTREE-FOCK EQUATIONS IN $L^p$ AND FOURIER-$L^p$ SPACES

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Abstract. We establish some local and global well-posedness for Hartree-Fock equations of $N$ particles (HFP) with Cauchy data in Lebesgue spaces $L^p \cap L^2$ for $1 \leq p \leq \infty$. Similar results are proven for fractional HFP in Fourier-Lebesgue spaces $\hat{L}^p \cap L^2$ $(1 \leq p \leq \infty)$. On the other hand, we show that Cauchy problem for HFP is ill-posed if we simply work in $\hat{L}^p$ $(2 < p \leq \infty)$. Analogue results hold for reduced HFP.

As a consequence, we get natural $L^p$ and $\hat{L}^p$ extension of classical well-posedness theories with Cauchy data in just $L^2$-based Sobolev spaces.

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1. Introduction

1.1. Motivation and physical context. The Hartree-Fock equation (HFE), defined in (1.1) below, is a key effective equation of quantum physics. It plays a role similar to that of the Boltzmann equation in classical physics. The HFE describes large systems of identical fermions by taking into account the self-interactions of charged fermions as well as an
exchange term resulting from Pauli’s principle. A semirelativistic version of the HFE was developed in [16] for modeling white dwarfs. The HFE model [26] leads to the Kohn-Sham equation underlying the density functional theory which is exceptionally effective in the computations in quantum chemistry and in particular, of the electronic structure of matter. The HFE is used for several applications in many-particle physics [29]. For detail background and recent development on HFE and beyond, we refer to excellent survey [14] and the references therein.

In [24] fractional Laplacians have been applied to model physical phenomena. It was formulated by Laskin [24] as a result of extending the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths. Specifically, when \( \alpha = 1 \), fractional Hartree equation, defined in (1.2) below, can be used to describe the dynamics of pseudo-relativistic boson stars in the mean-field limit, and when \( \alpha = 2 \) the Lévy motion becomes Brownian motion. The Hartree equation also arise in the nonlinear optics of nonlocal, nonlinear optical media [31].

1.2. Hartree-Fock equations. The Hartree-Fock equations of \( N \) particles is given by

\[
\begin{aligned}
\frac{i}{\hbar} \psi_k - (-\Delta)^{\alpha/2} \psi_k + \kappa \sum_{l=1}^{N} \left( \frac{e^{-a|x|}}{|x|^\gamma} \ast |\psi_l|^2 \right) \psi_k - \kappa \sum_{l=1}^{N} \psi_l \left( \frac{e^{-a|x|}}{|x|^\gamma} \ast \overline{\psi_l} \psi_k \right) &= 0 \\
\psi_k|_{t=0} &= \psi_{0,k}
\end{aligned}
\]

where \( a \geq 0, t \in \mathbb{R}, \psi_k : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}, k = 1, 2, ..., N, 0 < \gamma < d, \kappa \) is constant, and \( \ast \) denotes the convolution in \( \mathbb{R}^d \). The fractional Laplacian is defined as

\[
\mathcal{F}[-(\Delta)^{\alpha/2} u](\xi) = c|\xi|^\alpha \mathcal{F} u(\xi) \quad (0 < \alpha < \infty)
\]

where \( \mathcal{F} \) denotes the Fourier transform and \( c \) is some non zero constant. The Hartree factor

\[
H_\psi = \kappa \sum_{l=1}^{N} \left( \frac{e^{-a|x|}}{|x|^\gamma} \ast |\psi_l|^2 \right)
\]

describes the self-interaction between charged particles as a repulsive force if \( \kappa > 0 \), and an attractive force if \( \kappa < 0 \). In \( H_\psi \) the cases \( a = \gamma = 1 \) and \( a = 0, \gamma = 1 \) corresponds to, well-known, Yukawa and Coulomb potentials respectively. The last term on the left side of (1.1) is the so-called “exchange term (Fock term)”

\[
F_\psi(\psi_k) = \kappa \sum_{l=1}^{N} \psi_l \left( \frac{e^{-a|x|}}{|x|^\gamma} \ast \overline{\psi_l} \psi_k \right)
\]

which is a consequence of the Pauli principle and thus applies to fermions. In the mean-field limit \( (N \to \infty) \), this term is negligible compared to the Hartree factor. In this case, (1.1) is replaced by the \( N \) coupled equations, the so-called reduced Hartree-Fock equations:

\[
\begin{aligned}
\frac{i}{\hbar} \psi_k - (-\Delta)^{\alpha/2} \psi_k - \kappa \sum_{l=1}^{N} \left( \frac{e^{-a|x|}}{|x|^\gamma} \ast |\psi_l|^2 \right) \psi_k &= 0 \\
\psi_k|_{t=0} &= \psi_{0,k}
\end{aligned}
\]

In particular, when \( a = 0, N = 1, \) and \( \alpha = 2 \), (1.2) is the classical Hartree equation. We denote by (#) either (1.1) with \( N \geq 2 \) or (1.2) with \( N \geq 1 \), as most of our results work for
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both the equations. The rigorous time-dependent Hartree-Fock theory has been developed first by Chadam-Glassey [12] for (1.1) in $H^1(\mathbb{R}^3)$. In this setting, (1.2) is equivalent to the von Neumann equation

\begin{equation}
(1.3) \quad iK'(t) = [G(t), K(t)]
\end{equation}

for $K(t) = \sum_{i}^{N} |\psi_i(t)\rangle\langle \psi_i(t)|$ and $G(t) = (-\Delta)^{\alpha/2} + H_\psi(x, t)$, see, e.g., [23, 27, 28]. In (1.3) the bracket denotes a commutator and $|u\rangle\langle v|$ is the Dirac’s notation for the operator $f \mapsto \langle v, f \rangle u$. The von Neumann equation (1.3) can also be considered for more general class of density matrices $K(t)$. For example, one can consider the class of nonnegative self-adjoint trace class operators, for which $K(t)$ satisfies the following conditions:

$$K^*(t) = K(t), \quad K(t) \leq 1, \quad \text{tr}K = N,$$

where the condition $K(t) \leq 1$ corresponds to the Pauli exclusion principle, and $N$ is the “number of particles”.

The well-posedness for (1.3) was proved by Bove-Da Parto-Fano [6, 7] for a short-range pair-wisewise interaction potential $w$ instead of $e^{-a|x|}$ in $H_\psi$. The case of Coulomb potential was resolved by Chadam [13]. Lewin-Sabin [28] have established the well-posedness for (1.3) with density matrices of infinite trace for pair-wise interaction potentials $w \in L^1(\mathbb{R}^3)$. However, in [28], they did not include Coulomb and Yukawa type potential case. Moreover, they proved the asymptotic stability for the ground state in dimension $d = 2$.

Recently Fröhlich-Lenzmann [16, Theorem 2.1] proved that (#) with Coulomb type self-interactions is locally well-posed in $H^s(\mathbb{R}^3)$ ($s \geq 1/2$). Moreover, they [16, Theorem 2.2] proved global existence for sufficiently small initial data. Carles-Lucha-Moulay [8, Section IV] have studied global well-posedness of (1.1) for Coulomb type self-interactions and with an external potential, and obtained some $H^s(\mathbb{R}^3)$ regularity. Lenzmann [25, Theorems 1, 2 and 3] proved some local and global well-posedness for Hartree equation with Yukawa type self-interactions in $H^s(\mathbb{R}^3)$ ($s \geq 1/2$). Taulli-Venkov [33] have studied (1.1) in $H^1(\mathbb{R}^d)$ with more general nonlinearity (so called Choquard equation).

Thus most authors have studied well-posedness for the Cauchy problem of (#) in $L^2$-based Sobolev spaces. Perhaps the major reason behind this is the fact that free Schrödinger propagator $U(t) := e^{it\Delta} : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ if and only if $p = 2$. This raises a natural question: Can we expect well-posedness theory for (#) in function spaces—which are not just base on $L^2$-integrability? The fantastic progress has been made for this in the last decade. In fact, Zhou [35] proved some well-posedness for nonlinear Schrödiger equation (NLS) in some $L^p$-Sobolev spaces for $p < 2$. Wang et al. in [34, 32], Oh et al. in [30], and Bhimani et al. in [4], among others, obtained some well-posedness for NLS in modulation spaces. In [9, 19, 20, 21], authors have studied well-posedness for Hartree equation in fractional Bessel potential spaces. In [3], Bhimani-Grillakis-Okoudjou have studied well-posedness for (#) with Coulomb type self interaction in modulation spaces. However, we believe that yet we are far from a complete understanding in this direction.
Taking these considerations into account, we are inspired to study Cauchy problem for $(\#)$ with initial data in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $\tilde{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $1 \leq p \leq \infty$. We start with the following theorem:

**Theorem 1.1** (Local well-posedness in $L^p \cap L^2$). Let $\gamma$ satisfies one of the following

- $0 < \gamma < \min\{1, 2d\left(\frac{1}{p} - \frac{1}{2}\right)\}$ for $1 \leq p \leq \frac{4}{3}$
- $0 < \gamma < \min\{1, \frac{d}{2}\}$ for $\frac{4}{3} \leq p \leq \infty$.

Assume that $\psi_0 = (\psi_{0,1}, ..., \psi_{0,N}) \in \left(L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\right)^N$ and $\alpha = 2$. Then there exists $T > 0$ and a unique local solution $(\psi_1, ..., \psi_N)$ to $(\#)$ such that

$$(U(-t)\psi_1(t), ..., U(-t)\psi_N(t)) \in \left(C([0, T], L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))\right)^N.$$  

We note that the linear counterpart problem of $(\#)$ (free Schrödinger equation) is ill-posed in $L^p(\mathbb{R}^d)$ for $p \neq 2$. Though Theorem 1.1 reveals that after a linear transformation using the semigroup $U(-t)$ generated by the linear problem, $(\#)$ is locally well-posed in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $1 \leq p \leq \infty$. Recently Hyakuna [21, Theorems 1.1 and 1.3] used Zhou spaces (see (3.18) below) to get the local existence in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ ($1 \leq p \leq 2$) for the Hartree equation. The particular case of Theorem 1.1 extends this result for any $1 \leq p \leq \infty$.

We will give two different proofs of Theorem 1.1. In the first proof Zhou spaces will play no role, this contrasts with the proof given in [21] for Hartree equation. In the second proof we make use of Zhou spaces to get the local existence, in this case, local solution enjoys some Zhou spaces regularity.

**Remark 1.1.**

1. The proof of Theorem 1.1 rely on factorization formula for $U(t)$ associated to Hartree nonlinearity (Lemma 2.5), trilinear estimates (Subsection 2.2), and Strichartz estimates. For detail proof strategy, see Remark 3.1 below.
2. We do not know factorization formula for fractional Schrödinger propagator $e^{-it(-\Delta)^{\alpha/2}}$ ($\alpha \neq 2$), and so the analogue of Theorem 1.1 for $\alpha \neq 2$, remains an open question.
3. Zhou spaces [33] is similar to Bourgain-type spaces [3] but easier to handle Hartree-type nonlinearities in general space dimensions.

Local solution can be extended globally, under an additional assumption on $\gamma$. Specifically, we have the following theorem:

**Theorem 1.2** (Global well-posedness in $L^p \cap L^2$). Let $\alpha = 2$ and $0 < \gamma < \min\{1, \frac{d}{2}\}$. Then the local solution to $(\#)$ given by Theorem 1.1 extends to global one such that

$$(U(-t)\psi_1(t), ..., U(-t)\psi_N(t)) \in \left(C(\mathbb{R}, L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))\right)^N.$$  

Moreover, it follows that $(\psi_1(t), ..., \psi_N(t)) \in \left(C(\mathbb{R}, L^2(\mathbb{R}^d))\right)^N$ and that if $1 \leq p \leq 2$ global solution enjoys the following smoothing effect in terms of special integrability:

$$|_{(\mathbb{R} \setminus \{0\} \times \mathbb{R}^d)} \in \left(C(\mathbb{R} \setminus \{0\}, L^p(\mathbb{R}^d))\right)^N.$$  

We note that formally the solution of (#) satisfies (see e.g., [8]) the conservation of mass:
\[ \|\psi_k(t)\|_{L^2} = \|\psi_{0,k}\|_{L^2} \quad (t \in \mathbb{R}, k = 1, \ldots, N). \]

Exploiting this mass conservation law, Proposition 2.2, Strichartz estimates, and blow-up alternative, we prove global existence.

**Remark 1.2.** The sign of \( \kappa \) in Hartree and Fock terms determines the defocusing and focusing character of the nonlinearity. We shall see that this character will play no role in our analysis, as we do not use the conservation of energy of (#) to achieve global existence. This contrasts with well-posedness scenario in \( H^s(\mathbb{R}^3) \). For example, in [16, Theorem 2.4] it is proved that for radially symmetric data with negative energy lead to blow-up solutions in finite time.

We now turn our attention for the well-posedness of (#) in the Fourier–Lebesgue spaces \( \hat{L}^p(\mathbb{R}^d) \) (1 \( \leq \) \( p \) \( \leq \) \( \infty \)) defined by
\[ \hat{L}^p(\mathbb{R}^d) = \{ f \in S'(\mathbb{R}^d) : \|f\|_{\hat{L}^p} := \|\mathcal{F}f\|_{L^{p'}} < \infty \} \]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \). We note that by Hausdorff-Young inequality, \( L^p(\mathbb{R}^d) \subset \hat{L}^p(\mathbb{R}^d) \) for \( 1 \leq p \leq 2 \) and \( \hat{L}^p(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \) for \( 2 \leq p \leq \infty \). We denote by \( L^2_{sd}(\mathbb{R}^d) \), space of radial functions in \( L^2(\mathbb{R}^d) \). Now we state the following theorem:

**Theorem 1.3** (Local well-posedness in \( \hat{L}^p \cap L^2 \)). Let

\[ X = \begin{cases} \hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) & \text{if } a \geq 0, \quad \alpha = 2, \quad 0 < \gamma < \min\{2, \frac{d}{2}\}, \quad p \in [1, \infty] \\ \hat{L}^p(\mathbb{R}^d) \cap L^2_{sd}(\mathbb{R}^d) & \text{if } a \geq 0, \quad d \geq 2, \quad \frac{2d}{2d-1} < \alpha < 2, \quad 0 < \gamma < \min\{\alpha, \frac{d}{2}\}, \quad p \in [1, \infty] \\ \hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) & \text{if } a = 0, \quad 0 < \alpha < \infty, \quad 0 < \gamma < 2d\left(\frac{1}{2} - \frac{1}{p}\right), \quad p \in (2, \infty] \\ \hat{L}^p(\mathbb{R}^d) & \text{if } a > 0, \quad 0 < \alpha < \infty, \quad 0 < \gamma < 2d\left(\frac{1}{2} - \frac{1}{p}\right), \quad p \in (2, \infty]. \end{cases} \]

Assume that \( (\psi_{0,1}, \ldots, \psi_{0,N}) \in X^N \). Then there exist \( T > 0 \) and a unique local solution \( (\psi_1, \ldots, \psi_N) \) to (#) such that
\[ (\psi_1(t), \ldots, \psi_N(t)) \in (C([0,T],X))^N. \]

Moreover, the map \( (\psi_{0,1}, \ldots, \psi_{0,N}) \mapsto (\psi_1, \ldots, \psi_N) \) is locally Lipschitz from \( \hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) to \( \left(C([0,T],\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))\right)^N \).

We note that fractional Schrödinger propagator \( U_{\alpha}(t) := e^{-it(-\Delta)^{\alpha/2}} \) is bounded in \( \hat{L}^p(\mathbb{R}^d) \) for all \( 1 \leq p \leq \infty \) (see Lemma 3.4). Hence, we do not need to transfer (#) using the semigroup \( U_{\alpha}(-t) \) to establish local existence in \( \hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). This contrasts with local solutions of Theorem 1.1. In [9, Proposition 3.3], Carles-Mouzaoui proved that Hartree equation is locally well-posed in \( \hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) for \( p = \infty \) and Bhimani [2, Proposition 4.5] proved this result for fractional Hartree equation. Hykuna [21, Theorem 1.8] proved local well-posedness for the Hartree equation in \( \hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) for \( 2 \leq p \leq \infty \). In fact, he used Zhou spaces to construct local existence. The particular case of Theorem 1.3 establishes these result for any \( 1 \leq p \leq \infty \). Again, we note that Zhou spaces will play no role in
our proof. In [20] Theorem 1.2 it is proved that Hartree equation is locally well-posed in $\hat{L}^p(\mathbb{R}^d)$ $(2 \leq p < \infty)$ if $2d(\frac{1}{2} - \frac{1}{p}) \leq \gamma < \min\{2, d\}$. The particular case of Theorem 1.3 reveals that this result is true for any $2 \leq p \leq \infty$ and $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$ with Yukawa type self interactions.

Remark 1.3.

1. In Theorem 1.3 radial assumption for initial data comes due to use of fractional Strichartz estimates (Proposition 2.1 below) in the proof.
2. We have local existence result (Theorem 1.3) in $\hat{L}^p(\mathbb{R}^d)$ for $2 < p \leq \infty$, without any radial assumption on initial data, if $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$. We do not know the analogue of this result for $1 \leq p < 2$.

Theorem 1.4 (Global well-poisedness in $L^2 \cap \hat{L}^p$). Let

$$X = \begin{cases} \hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) & \text{if } a \geq 0, \quad \alpha = 2, 0 < \gamma < \min\{2, \frac{d}{2}\}, \quad p \in [1, \infty] \\ \hat{L}^p(\mathbb{R}^d) \cap L^2_{rad}(\mathbb{R}^d) & \text{if } a \geq 0, \quad d \geq 2, \quad \frac{2d}{d-1} < \alpha < 2, 0 < \gamma < \min\{\alpha, \frac{d}{2}\}, \quad p \in [1, \infty]. \end{cases}$$

Assume that $(\psi_{0,1}, ..., \psi_{0,N}) \in X^N$. Then the local solution to $(\#)$ given by Theorem 1.3 extends to a global one such that

$$(\psi_1(t), ..., \psi_N(t)) \in \left(C(\mathbb{R}, X) \cap L^{4\alpha/\gamma}_{loc}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d))\right)^N.$$

Carles-Mouzaoui [9, Theorem 1.1] proved that Hartree equation is globally well-posed in $L^2(\mathbb{R}^d) \cap \hat{L}^\infty(\mathbb{R}^d)$ and Hyakuna [21, Theorem 1.9] generalized this result in $L^2(\mathbb{R}^d) \cap \hat{L}^p(\mathbb{R}^d)$ for $2 \leq p < \infty$. On the other hand, Bhimani [2, Theorem 1.2] generalized Carles-Mouzaoui result for fractional Hartree equation in $L^2(\mathbb{R}^d) \cap \hat{L}^\infty(\mathbb{R}^d)$. The particular case of Theorem 1.4 establishes these results for any $1 \leq p \leq \infty$.

Remark 1.4.

1. To extend the local existence (proved in $\hat{L}^p(\mathbb{R}^d)$ for $\alpha \neq 2$) globally, first we prove $(\#)$ is globally well-posed (see Proposition 2.2 below) in $L^2_{rad}(\mathbb{R}^d)$ via Strichartz estimates for fractional Schrödinger equation (see Proposition 2.1 below) - where we need initial data to be a radial, $\alpha \in (2d/(2d-1), 2)$ and dimension $d \geq 2$ (see [17, p.26-27]). Invoking Proposition 2.2, we get the global existence in $\hat{L}^p(\mathbb{R}^d)$. Thus we notice in the proof that, to take the advantage of Proposition 2.2, the hypothesis, initial data to be radial of Theorem 1.4 is necessary.
2. The analogue for Theorem 1.4 without radial assumption on initial data remains interesting open question.
3. Due to lack of appropriate Strichartz estimates for fractional Schrödinger equation with $\alpha > 2$, we do not know whether $(\#)$ with $\alpha > 2$ is globally well-posed in $L^2(\mathbb{R}^d)$ and also whether the solution is in some mixed $L^{p(\gamma)}_{loc}(\mathbb{R}, L^{q(\gamma)}(\mathbb{R}^d))$ spaces (the analogue of Proposition 2.3). In view of this, the analogue of Theorem 1.4 for $\alpha > 2$ remains another interesting open question.
Theorem 1.5 (Improved well-posedness in 1D). Let $\alpha = 2, 0 < \gamma < 1$, and

$$X = \begin{cases} L^p(\mathbb{R}) \cap L^2(\mathbb{R}) & \text{if } p \in (4/3, 2] \\ \hat{L}^p(\mathbb{R}) \cap L^2(\mathbb{R}) & \text{if } p \in [2, 4). \end{cases}$$

Assume that $\psi_0 = (\psi_{0,1}, ..., \psi_{0,N}) \in X^N$. Then there exists a unique global solutions of (#) such that $(U(-t)\psi_1(t), ..., U(-t)\psi_N(t)) \in (C(\mathbb{R}, X))^N$ when $p \in (4/3, 2]$ and $(\psi_1(t), ..., \psi_N(t)) \in (C(\mathbb{R}, X))^N$ when $p \in [2, 4)$.

Remark 1.5. It can be observed from the proof that, the local result for data in $\hat{L}^p(\mathbb{R}) \cap L^2(\mathbb{R})$, in the above theorem, is valid for $0 < \gamma < 2$.

The proof of Theorem 1.5 relies on generalized Strichartz estimates (see Lemma 3.8 below) in 1D. Specifically, we shall see that this enables us to estimate the integral nonlinear part of (#) (see (3.7) and (3.35) below), and as a consequence we can improve the range of $\gamma$.

We next show that (#) with Coulomb type potential is not well-posed in the mere $\hat{L}^p(\mathbb{R}^d)^N (2 < p \leq \infty)$ for $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$. Specifically, we have the following result:

Theorem 1.6 (Ill-posedness in $\hat{L}^p$). Let $a = 0$ and $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$ for $2 < p \leq \infty$. Then (#) is locally well-posed in $\left( L^2(\mathbb{R}^d) \cap \hat{L}^p(\mathbb{R}^d) \right)^N$ but not in $\hat{L}^p(\mathbb{R}^d)^N$ : for any ball $B$ in $\hat{L}^p(\mathbb{R}^d)^N$, for all $T > 0$ the solution map $\psi_0 \in B \mapsto \psi \in \left( C([0,T], \hat{L}^p(\mathbb{R}^d)) \right)^N$ is not uniformly continuous.

The proof of Theorem 1.6 relies on the fact that the Fourier transform of Coulomb type potential is homogeneous. On the other hand, the Fourier transform of Yukawa type potential is not homogeneous. See Lemma 2.1 below. In fact, Theorem 1.3 says that (#) with Yukawa type potential is locally well-posed in $\hat{L}^p(\mathbb{R}^d)$ with the same range of $p$ and $\gamma$ as in Theorem 1.6. Thus, Theorems 1.3 and 1.6 reveal the contrast behavior of Coulomb and Yukawa type potentials in (#).

Remark 1.6. In [20, Theorem 1.2], it is proved that if $2d(\frac{1}{2} - \frac{1}{p}) \leq \gamma < \min\{d, 2\}, d \in \mathbb{N}$ or $\gamma = 2, d \geq 3$, then the Hartree equation is locally well-posed in $\hat{L}^p(\mathbb{R}^d) (2 \leq p \leq \frac{2d}{d-\gamma})$. Theorem 1.6 contrasts with this result.

We summarize our findings in Table 1. We write $x \wedge y = \min\{x,y\}$.

This paper is organized as follows. In Section 2, we introduce notations and preliminaries which will be used in the sequel. In Subsections 3.1, 3.2 we prove Theorems 1.1 and 1.3 respectively. In Subsections 3.3, 3.4 we prove Theorems 1.2 and 1.4 respectively. In Subsection 3.5 we prove Theorem 1.5. In Section 4 we prove Theorem 1.6.

2. Preliminaries and key ingredient

Notations and known results. The notation $A \lesssim B$ means $A \leq cB$ for some universal constant $c > 0$, whereas $A \simeq B$ means $c^{-1}A \leq B \leq cA$ for some $c \geq 1$. Also $A \gtrsim B$ means $B \lesssim A$. The characteristic function of a set $E \subset \mathbb{R}^d$ is $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if...
Let $I \subset \mathbb{R}$ be an interval and $X$ be a Banach space of functions. Then the norm of the space-time Lebesgue space $L^q(I, X)$ is defined by

$$
\|u\|_{L^q(I, X)} = \left( \int_I \|u(t)\|_X^q dt \right)^{1/q}
$$

and when $I = [0, T]$, $T > 0$ we denote $L^q(I, X)$ by $L^q_T(X)$. For $p \in [1, \infty]$, its Hölder conjugate, denoted by $p'$, is given by $\frac{1}{p} + \frac{1}{p'} = 1$. The norm on $N$-fold product $X^N$ of Banach space $(X, \| \cdot \|_X)$ is given by

$$
\|\psi\|_{X^N} = \max_{1 \leq j \leq N} \|\psi_j\|_X, \quad \psi = (\psi_1, \cdots, \psi_N) \in X^N.
$$

The Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^d)$ (with its usual topology), and the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. For two Banach spaces of functions $A, B$ in $\mathcal{S}'(\mathbb{R}^d)$ we note that $A \cap B$ is also a Banach space with the norm $\| \cdot \|_{A \cap B} = \max\{\| \cdot \|_A, \| \cdot \|_B\}$. For $x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d) \in \mathbb{R}^d$, we put $x \cdot y = \sum_{i=1}^d x_i y_i$. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be the Fourier transform defined by

$$
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^d.
$$

Then $\mathcal{F}$ is a bijection and the inverse Fourier transform is given by

$$
\mathcal{F}^{-1}f(x) = f^\vee(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^d,
$$

and this Fourier transform can be uniquely extended to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$.

**Lemma 2.1.** Let $d \geq 1$ and $0 < \gamma < d$. 

![Table 1. Results Summary](image)
Specifically, we have the following:

(1) For \( f(x) = e^{-2\pi|x|} \), we have

\[
\hat{f}(\xi) = \frac{c_d}{(1 + \xi^2)^{(d+1)/2}}
\]

with \( c_d = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \), \( \Gamma \) is the Gamma function and \( \hat{f} \in L^p(\mathbb{R}^d) \) for all \( 1 \leq p \leq \infty \).

(2) For \( f(x) = |x|^{-\gamma} \), we have

\[
\hat{f}(\xi) = \frac{C_{d,\gamma}}{\xi^{d-\gamma}}.
\]

For \( f \in S(\mathbb{R}^d) \), we define the fractional Schrödinger propagator \( e^{-it(-\Delta)^{\alpha/2}} \) for \( t \in \mathbb{R}, \alpha > 0 \) as follows:

(2.1) \[
[U_\alpha(t)f](x) = \left[ e^{-it(-\Delta)^{\alpha/2}} f \right](x) := \int_{\mathbb{R}^d} e^{-4\pi^2|\xi|^\alpha t} \hat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi.
\]

For \( \alpha = 2 \), we simply write \( U_2 = U \). In this case we have (see [10, Lemm 2.2.4])

(2.2) \[
[U(t)f](x) = \left[ e^{it\Delta} f \right](x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) \, dy.
\]

**Definition 2.1.** A pair \((q, r)\) is \(\alpha\)-fractional admissible if \( q \geq 2, r \geq 2 \) and

\[
\frac{\alpha}{q} = d \left( \frac{1}{2} - \frac{1}{r} \right).
\]

**Proposition 2.1** (Strichartz estimates). Denote

\[
DF(t, x) = U_\alpha \phi(x) + \int_0^t U_\alpha(t-s) F(s, x) \, ds.
\]

(1) Let \( \phi \in L^2(\mathbb{R}^d), d \in \mathbb{N} \) and \( \alpha = 2 \). Then for any time interval \( I \supseteq 0 \) and \( 2 \)-admissible pairs \((q_j, r_j), j = 1, 2\), there exists a constant \( C = C(|I|, r_1) \) such that

\[
\|D(F)\|_{L^{q_1}(I, L^{r_1})} \leq C\|\phi\|_{L^2} + C\|F\|_{L^{q_2}_{\hat{f}}(I, L^{r_2}_{\hat{f}})}, \quad \forall F \in L^{q_2}_{\hat{f}}(I, L^{r_2}_{\hat{f}}(\mathbb{R}^d))
\]

where \( q_j \) and \( r_j \) are Hölder conjugates of \( q_j \) and \( r_j \) respectively [22].

(2) Let \( \phi \in L^2_{\text{rad}}(\mathbb{R}^d), d \geq 2 \), and \( \frac{2d}{2d - 1} < \alpha < 2 \). Then for any time interval \( I \supseteq 0 \) and \( \alpha \)-fractional admissible pairs \((q_j, r_j), j = 1, 2\), there exists a constant \( C = C(|I|, r_1) \) such that

\[
\|D(F)\|_{L^{q_1}(I, L^{r_1})} \leq C\|\phi\|_{L^2} + C\|F\|_{L^{q_2}_{\hat{f}}(I, L^{r_2}_{\hat{f}})}, \quad \forall F \in L^{q_2}_{\hat{f}}(I, L^{r_2}_{\hat{f}}(\mathbb{R}^d))
\]

where \( q_j \) and \( r_j \) are Hölder conjugates of \( q_j \) and \( r_j \) respectively [17, Corollary 3.4].

For the sake of completeness, we recall the following standard existence result. We shall see that this result will play vital role to prove global existence (Theorems [12, 14] and [15]). Specifically, we have the following:

**Proposition 2.2.** Let \( \alpha > 0, 0 < \gamma < \min\{\alpha, d\} \) and

\[
X = \begin{cases} L^2(\mathbb{R}^d) & \text{if } \alpha = 2, d \geq 1 \\ L^2_{\text{rad}}(\mathbb{R}^d) & \text{if } \frac{2d}{2d - 1} < \alpha < 2, d \geq 2. \end{cases}
\]
If \((\psi_{0,1}, ..., \psi_{0,N}) \in X^N\) then (#) has a unique global solution
\[
(\psi_1, ..., \psi_N) \in \left( C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^{4\alpha/\gamma}(\mathbb{R}, L^{kd/(2d-\gamma)}(\mathbb{R}^d)) \right)^N.
\]
In addition, its \(L^2\)-norm is conserved,
\[
\|\psi_k(t)\|_{L^2} = \|\psi_{0,k}\|_{L^2}, \forall t \in \mathbb{R}, k = 1, 2, ..., N
\]
and for all \(\alpha\)-fractional admissible pairs \((q, r)\), \((\psi_1, ..., \psi_N) \in (L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}^d)))^N\).

**Proof.** For the proof of case \(a = 0\), that is, (#) with Coulomb type potential, see \cite{3} Propositions 4.2 and 4.3. The proof of case \(a > 0\), that is, (#) with Yukawa type potential is similar. Hence, we omit the details. See also \cite{9} Proposition 2.3 and \cite{8} Theorem 4.9. \(\square\)

### 2.1. Factorization formula for Schrödinger propagator

For \(t \neq 0\), we define multiplication, dilation and reflection operators (for functions \(w \) on \(\mathbb{R}^d\) and their inverses as follows:

- multiplication: \(M_t w(x) = e^{i|x|^2/4t}w(x), M_t^{-1}w(x) = e^{-i|x|^2/4t}w(x)\)
- dilation: \(D_t w(x) = \frac{1}{(4\pi it)^{d/2}}w\left(\frac{x}{4\pi t}\right)\) and \(D_t^{-1}w(x) = (4\pi it)^{d/2}w(4\pi tx)\)
- reflection: \(Rw(x) = w(-x)\) and \(R^{-1}w(x) = w(-x)\).

**Lemma 2.2.** Let \(0 \neq t \in \mathbb{R}\) and \(\varphi \in S(\mathbb{R}^d)\). Then we have
\[
U(t)\varphi = M_t D_t \mathcal{F} M_t \varphi \quad \text{and} \quad U(-t)\varphi = M_t^{-1} \mathcal{F}^{-1} D_t^{-1} M_t^{-1} \varphi.
\]

**Proof.** Taking formula (2.2) into account and using the above definitions, simple calculations gives the desired factorization for \(U(t)\). See for e.g., \cite{18} p.372. We omit the details. \(\square\)

For \(t \in \mathbb{R}\), we denote
\[
S_{a,t} = \begin{cases} 
\delta_0 & \text{if } a = 0 \\
\frac{a|t|}{(4\pi it)^{d/2}} & \text{if } a > 0
\end{cases}
\]
with \(\delta_0\) is the Dirac distribution with mass at origin in \(\mathbb{R}^d\).

Now, for \(f, g, h \in S(\mathbb{R}^d), a \geq 0, t \in \mathbb{R}\), and \(0 < \gamma < d\), we define **trilinear operators** associated to Hartree-type nonlinearity as follows
\[
\mathcal{H}_{a,\gamma}(f, g, h) = \left[ \frac{e^{-|x|}}{|x|^{\gamma}} \right] * (f \bar{g}) h \\
\mathcal{H}_{a,\gamma,t}(f, g, h) = \left[ (S_{a,t} * |\cdot|^{\gamma-d}) (f * \bar{g}) \right] * h.
\]
Now we decompose \(\mathcal{H}_{a,\gamma,t}\) in the following way
\[
\mathcal{H}^j_{a,\gamma,t}(f, g, h) := \left[ (S_{a,t} * k_j) (f * \bar{g}) \right] * h \quad (j = 1, 2),
\]
where \(k_1, k_2\) are given by
\[
k_j(x) = \begin{cases} 
\chi_{\{|x| \leq 1\}}(x)|x|^{\gamma-d} & \text{if } j = 1 \\
\chi_{\{|x| > 1\}}(x)|x|^{\gamma-d} & \text{if } j = 2.
\end{cases}
\]
We note that \(k_1 \in L^p(\mathbb{R}^d)\) for \(1 \leq p < \frac{d}{d-\gamma}\) and \(k_2 \in L^q(\mathbb{R}^d)\) for \(\frac{d}{d-\gamma} < q \leq \infty\).
Lemma 2.3. Let \( 0 \neq t \in \mathbb{R}, 0 < \gamma < d, a \geq 0, \) and \( v_j(t) = U(-t)u_j(t) \in S(\mathbb{R}^d) \) with \( j = 1,2,3. \) Then we have

\[
U(-t)\mathcal{H}_{a,\gamma}(u_1, u_2, u_3) \asymp |t|^{-\gamma}M_t^{-1}\tilde{\mathcal{H}}_{a,\gamma,t}(M_tv_1, RM_tv_2, M_tv_3).
\]

Proof. Note that \( D_t^{-1}(fg) = (4\pi it)^{-d/2}(D_t^{-1}f)(D_t^{-1}g), \mathcal{F}^{-1}D_t^{-1} = D_{-t}\mathcal{F} = cRD_t\mathcal{F} \) and \( U(t)\tilde{u} = U(-t)u. \) Using these equalities and performing the change of variable, we may rewrite

\[
D_t^{-1} \left( (| \cdot |^{-\gamma} e^{-a|\cdot|} ) * (fg) \right)(x) = (4\pi it)^{-d/2} \left( (| \cdot |^{-\gamma} e^{-a|\cdot|} ) * (fg) \right)(4\pi tx)
\]

\[
= (4\pi it)^{-d/2} \int_{\mathbb{R}^d} |y|^{-\gamma} e^{-a|y|}(fg)(4\pi tx - y) dy
\]

\[
= (4\pi it)^{-d/2}(4\pi t)^d \int_{\mathbb{R}^d} |4\pi ty|^{-\gamma} e^{-4\pi t|ty|}(fg)(4\pi t(x - y)) dy
\]

\[
= i^{-d/2}(4\pi t)^{d/2}(4\pi it)^d (4\pi t|t|)^{-\gamma} \int_{\mathbb{R}^d} |y|^{-\gamma} e^{-4\pi t|ty|}(fg)(4\pi t(x - y)) dy
\]

\[
= (-4\pi it)^{d/2} (4\pi t|t|)^{-\gamma} \int_{\mathbb{R}^d} |y|^{-\gamma} e^{-4\pi t|ty|} (D_t^{-1}f D_t^{-1}g)(x - y) dy
\]

\[
= (-4\pi it)^{d/2} (4\pi t|t|)^{-\gamma} \left( (| \cdot |^{-\gamma} e^{-4\pi|t|\cdot|} ) * (D_t^{-1}f D_t^{-1}g) \right)(x).
\]

Using the above equalities and Lemma 2.2, we obtain

\[
M_tU(-t)\mathcal{H}_{a,\gamma}(u_1, u_2, u_3)
= \mathcal{F}^{-1}D_t^{-1}M_t^{-1}\mathcal{H}_{a,\gamma}(u_1, u_2, u_3)
= \mathcal{F}^{-1}D_t^{-1} \left( (| \cdot |^{-\gamma} e^{-a|\cdot|} ) * (M_t^{-1}u_1)(M_t\tilde{u}_2) \right) M_t^{-1}u_3
\]

\[
\asymp t^{-d/2} \mathcal{F}^{-1} \left[ D_t^{-1} \left( (| \cdot |^{-\gamma} e^{-a|\cdot|} ) * (M_t^{-1}u_1)(M_t\tilde{u}_2) \right) D_t^{-1}M_t^{-1}u_3 \right]
\]

\[
\asymp |t|^{-\gamma} \mathcal{F}^{-1} \left[ \left( (| \cdot |^{-\gamma} e^{-4\pi|t|\cdot|} ) * (D_t^{-1}M_t^{-1}u_1)(D_t^{-1}M_t\tilde{u}_2) \right) D_t^{-1}M_t^{-1}u_3 \right]
\]

\[
\asymp |t|^{-\gamma} \left( (| \cdot |^{-d} \mathcal{F}^{-1} e^{-4\pi|t|\cdot|} ) ((\mathcal{F}^{-1}D_t^{-1}M_t^{-1}u_1) * (\mathcal{F}^{-1}D_t^{-1}M_t\tilde{u}_2)) \right) \mathcal{F}^{-1}D_t^{-1}M_t^{-1}u_3.
\]

Since, by Lemma 2.1

\[
(\mathcal{F}^{-1} e^{-4\pi|t|\cdot|} ) (\xi) = \frac{c_d}{(2\pi t)^d} \left( 1 + \frac{||\xi||^2}{4a^2t^2} \right)^{-(d+1)/2} \asymp \frac{a|t|}{(4a^2t^2 + ||\xi||^2)^{(d+1)/2}} =: S_{a,t} \ (a > 0),
\]
it follows that
\[
M_1U(-t)\mathcal{H}_{a,\gamma}(u_1, u_2, u_3) \approx |t|^{-\gamma} \left[ \left| \cdot \right|^{-\gamma} \ast S_{a,t} \right] \left( (M_1U(-t)u_1) \ast (RD_3FM_tu_2) \right) \\
\ast M_1U(-t)u_3 \\
\asymp |t|^{-\gamma} \left[ \left| \cdot \right|^{-\gamma} \ast S_{a,t} \right] \left( (M_1U(-t)u_1) \ast (RM_t^{-1}U(t)u_2) \right) \\
\ast M_1U(-t)u_3 \\
\asymp |t|^{-\gamma} \left[ \left| \cdot \right|^{-\gamma} \ast S_{a,t} \right] \left( (M_1U(-t)u_1) \ast (\overline{RM_tU(-t)u_2}) \right) \\
\ast M_1U(-t)u_3 \\
= |t|^{-\gamma} \hat{\mathcal{H}}_{a,\gamma,t}(M_1v_1, RM_tv_2, M_tv_3).
\]
This completes the proof for the case \( a > 0 \). For the proof for case \( a = 0 \), see \cite{21} Lemma 2.1.

\[\Box\]

2.2. Trilinear estimates. In this subsection we prove some useful trilinear estimates for \( \hat{\mathcal{H}}_{a,\gamma,t} \) and \( \mathcal{H}_{a,\gamma} \) (see (2.1)). We start with following

Lemma 2.4. Assume \( 0 < \gamma < d \). Let \( k_j \ (j = 1, 2) \) and \( S_{a,t} \) be given by (2.6) and (2.3) respectively. Then we have

\[
\|k_1 \ast S_{a,t}\|_{L^{r_1}} \lesssim \|k_1\|_{L^{r_1}} \quad \text{and} \quad \|k_2 \ast S_{a,t}\|_{L^{r_2}} \lesssim \|k_2\|_{L^{r_2}}
\]

for all \( r_1 \in [1, \frac{d}{d-\gamma}] \) and for all \( r_2 \in (\frac{d}{d-\gamma}, \infty] \).

Proof. The case \( a = 0 \) being trivial assume that \( a > 0 \). Note that for \( d = 1 \), we have

\[
\|S_{a,t}\|_{L^1} = \int_{\mathbb{R}} \frac{a|t|}{4a^2t^2 + |\xi|^2} d\xi \asymp \frac{1}{a|t|} \int_0^\infty \frac{dr}{1 + (r/2a|t|)^2} \asymp \int_0^\infty \frac{ds}{1 + s^2} \asymp 1.
\]

For \( d \geq 2 \), we obtain

\[
\|S_{a,t}\|_{L^1} = \int_{\mathbb{R}^d} \frac{a|t|}{(4a^2t^2 + |\xi|^2)^{d+1/2}} d\xi \asymp a|t| \int_0^\infty \left( \frac{r^{d-1}}{(4a^2t^2 + r^2)^{(d+1)/2}} \right) dr \\
\asymp a|t| \int_0^\infty \left( \frac{s - 4a^2t^2}{s^{(d+1)/2}} \right) ds \\
\leq a|t| \int_0^\infty \frac{s^{(d-2)/2}}{s^{(d+1)/2}} ds \\
= a|t| \int_0^\infty s^{-3/2} ds = 1.
\]

Now Young inequality, gives the desired inequalities. \[\Box\]

Remark 2.1. Note that we separate the computation of \( L^1 \)-norm for \( S_{a,t} \) in two cases as the third step in the proof of case \( d \geq 2 \) does not work for \( d = 1 \).

Proposition 2.3 (\( L^p \)-estimates). Let \( 0 < \gamma < d \), \( f_j \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \ (j = 1, 2, 3) \) and \( \hat{\mathcal{H}}_{a,\gamma,t}, \hat{\mathcal{H}}_{a,\gamma,t}^j \) are given by (2.4) and (2.5) respectively.
(1) Assume that $1 \leq p < 2$ and $0 < \gamma < 2d(\frac{1}{p} - \frac{1}{2})$. Then we have
\[
\|\hat{\mathcal{H}}_{a,\gamma,t}(f_1, f_2, f_3)\|_{L^2} \lesssim \begin{cases} 
\|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} & \text{if } j = 1 \\
\|f_1\|_{L^p} \|f_2\|_{L^p} \|f_3\|_{L^2} & \text{if } j = 2
\end{cases}
\]
and
\[
\|\hat{\mathcal{H}}_{a,\gamma,t}(f_1, f_2, f_3)\|_{L^p} \lesssim \begin{cases} 
\|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^p} & \text{if } j = 1 \\
\|f_1\|_{L^p} \|f_2\|_{L^p} \|f_3\|_{L^p} & \text{if } j = 2.
\end{cases}
\]

As a consequence, we have
\[
\|\hat{\mathcal{H}}_{a,\gamma,t}(f_1, f_2, f_3)\|_{L^p \cap L^2} \lesssim \prod_{j=1}^{3} \|f_j\|_{L^p \cap L^2}.
\]

(2) Assume that $2 < p \leq \infty$ and $0 < \gamma < d(\frac{1}{2} - \frac{1}{p})$. Then we have
\[
\|\hat{\mathcal{H}}_{a,\gamma,t}(f_1, f_2, f_3)\|_{L^p} \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^p \cap L^2}.
\]

Proof. (1) By Young, Hölder, Hausdorff-Young inequalities and Lemma \[2.4\] for $1 \leq p \leq \infty$, we have
\[
\|[(k_1 * S_{a,t})(f_1 * f_2)] * f_3\|_{L^p} \leq \|(k_1 * S_{a,t})(f_1 * f_2)\|_{L^1} \|f_3\|_{L^p} \\
\leq \|k_1 * S_{a,t}\|_{L^1} \|f_1 * f_2\|_{L^\infty} \|f_3\|_{L^p} \\
\leq \|k_1 * S_{a,t}\|_{L^1} \left\|\hat{f}_1 \hat{f}_2\right\|_{L^1} \|f_3\|_{L^p} \\
\lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^p}.
\]

Similarly,
\[
\|[(k_1 * S_{a,t})(f_1 * f_2)] * f_3\|_{L^2} \leq \|(k_1 * S_{a,t})(f_1 * f_2)\|_{L^1} \|f_3\|_{L^2} \lesssim \prod_{j=1}^{3} \|f_j\|_{L^2}.
\]

Since
\[
\frac{1}{p/2(p-1)} + \frac{1}{p/(2-p)} = 1, \quad \frac{1}{p} + \frac{1}{p} = 1 + \frac{1}{p/(2-p)} \quad \text{and} \quad \frac{p}{2(p-1)} > \frac{d}{d-\gamma},
\]
Hölder and Young inequalities and Lemma \[2.4\] imply
\[
\|\hat{\mathcal{H}}_{a,\gamma,t}(f_1, f_2, f_3)\|_{L^p} = \|[(k_2 * S_{a,t})(f_1 * f_2)] * f_3\|_{L^p} \\
\leq \|(k_2 * S_{a,t})(f_1 * f_2)\|_{L^1} \|f_3\|_{L^p} \\
\leq \|k_2 * S_{a,t}\|_{L^{\frac{p}{2(p-1)}}} \|f_1 * f_2\|_{L^{\frac{2p}{p}}} \|f_3\|_{L^p} \\
\lesssim \prod_{i=1}^{3} \|f_i\|_{L^p}.
\]
Similarly,
\[
\left\| \mathcal{H}^2_{a,\gamma,t}(f_1, f_2, f_3) \right\|_{L^2} = \left\| (k_2 * S_{a,t})(f_1 * f_2) \right\|_{L^2} \times f_3 \right\|_{L^2} \\
\leq \left\| (k_2 * S_{a,t})(f_1 * f_2) \right\|_{L^1} \times f_3 \right\|_{L^2} \\
\lesssim \| f_1 \|_{L^p} \| f_2 \|_{L^p} \| f_3 \|_{L^2}.
\]

(2) Since
\[
\frac{1}{p} + 1 = \frac{1}{2} + \frac{1}{(2p)/(p+2)} \quad \text{and} \quad \frac{2p}{p+2} > \frac{d}{d-\gamma},
\]
Young inequality and Lemma 2.4 give
\[
\left\| \hat{H}^2_{a,\gamma,t}(f_1, f_2, f_3) \right\|_{L^p} = \left\| (k_2 * S_{a,t})(f_1 * f_2) \right\|_{L^p} \times f_3 \right\|_{L^p} \\
\leq \left\| (k_2 * S_{a,t})(f_1 * f_2) \right\|_{L^{2p/(p+2)}} \times f_3 \right\|_{L^2} \\
\leq \left\| k_2 * S_{a,t} \right\|_{L^{2p/(p+2)}} \times f_1 \times f_2 \right\|_{L^\infty} \times f_3 \right\|_{L^2} \\
\lesssim \prod_{l=1}^3 \| f_l \|_{L^2}.
\]
Combining the above inequality with (2.7), we get the desired estimate. \qed

Proposition 2.4 (\(\tilde{L}^p\)-estimates). Let 0 < \(\gamma < d\).

(1) Assume that 1 ≤ \(p \leq 2\) and 0 < \(\gamma < d(\frac{1}{p} - \frac{1}{2})\). Then we have
\[
\| \mathcal{H}_{a,\gamma}(f_1, f_2, f_3) \|_{\tilde{L}^p} \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_1 \|_{\tilde{L}^p \cap L^2}.
\]

(2) Assume that \(2 < p \leq \infty\) and 0 < \(\gamma < 2d(\frac{1}{2} - \frac{1}{p})\), and let
\[
\mathcal{X} = \begin{cases} 
L^2(\mathbb{R}^d) \cap \tilde{L}^p(\mathbb{R}^d) & \text{if } a \geq 0 \\
\tilde{L}^p(\mathbb{R}^d) & \text{if } a > 0.
\end{cases}
\]

Then we have
\[
\| \mathcal{H}_{a,\gamma}(f_1, f_2, f_3) \|_{\mathcal{X}} \lesssim \prod_{j=1}^3 \| f_j \|_{\mathcal{X}}.
\]

Proof. (1) Since
\[
\| \mathcal{H}_{a,\gamma}(f_1, f_2, f_3) \|_{\tilde{L}^p} = \| \mathcal{F} \mathcal{H}_{a,\gamma}(f_1, f_2, f_3) \|_{L^{p'}} \lesssim \left\| \hat{\mathcal{H}}_{a,\gamma}(\frac{1}{2r}(f_1, f_2, Rf_3)) \right\|_{L^{p'}}
\]
using Proposition 2.3 (2) we have
\[
\| \mathcal{H}_{a,\gamma}(f_1, f_2, f_3) \|_{\tilde{L}^p} \lesssim \| \hat{f}_1 \|_{L^2} \| \hat{f}_2 \|_{L^2} \| \hat{f}_3 \|_{L^2} \| f_1 \|_{\mathcal{X} \cap \tilde{L}^p} \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| f_1 \|_{\tilde{L}^p \cap L^2}.
\]

(2) Taking Proposition 2.3 (1) into account, and exploiting the proof of Proposition 2.4 (1), the assertion follows when \(X = L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\).

Next we assume that \(X = \tilde{L}^p(\mathbb{R}^d)\). Set \(K = e^{-|\cdot|} \cdot |^{-\gamma}\). Then \(\mathcal{F}K = k_1 * h_a + k_2 * h_a\) with \(h_a(\xi) = \mathcal{F}e^{-a|\cdot|} = \frac{c_{a,\alpha}}{(a^2 + 4\pi^2|\xi|^2(a + 1)^2)}\) (see (2.6) and Lemma 2.4) and so it follows that
\[ \mathcal{F}K \in L^r(\mathbb{R}^d) \text{ for all } \frac{d}{d - \gamma} < r \leq \infty. \]

Since
\[ \frac{1}{p/2} + \frac{1}{p'/(2 - p')} = 1, \quad \frac{1}{p'} + \frac{1}{p} = 1 + \frac{1}{p'/(2 - p')}, \quad p' \leq 2 \quad \text{and} \quad \frac{p}{2} > \frac{d}{d - \gamma}, \]
Hölder and Young inequalities imply
\[
\| \mathcal{H}_{a,\gamma}(f_1, f_2, f_3) \|_{L^{p'}} = \| \mathcal{F} \mathcal{H}_{a,\gamma}(f_1, f_2, f_3) \|_{L^{p'}} \\
= \| \mathcal{F} (K * (f_1 \overline{f}_2)) * \mathcal{F} f_3 \|_{L^{p'}} \\
\leq \| \mathcal{F} (K * (f_1 \overline{f}_2)) \|_{L^1} \| \mathcal{F} f_3 \|_{L^{p'}} \\
\leq \| \mathcal{F} K \mathcal{F} (f_1 \overline{f}_2) \|_{L^1} \| f_3 \|_{L^p} \\
\leq \| \mathcal{F} K \|_{L^{\frac{p}{2}}} \| \mathcal{F} (f_1 \overline{f}_2) \|_{L^{\frac{p'}{p'-2}}} \| f_3 \|_{L^p} \\
\leq \| \mathcal{F} K \|_{L^{\frac{p}{2}}} \| \mathcal{F} f_1 \|_{L^{p'}} \| \mathcal{F} \overline{f}_2 \|_{L^{p'}} \| f_3 \|_{L^p} \\
\leq \| \mathcal{F} K \|_{L^{\frac{p}{2}}} \| \mathcal{F} f_1 \|_{L^{p'}} \| \mathcal{F} \overline{f}_2 \|_{L^{p'}} \| f_3 \|_{L^p}. \]

But \[ \| \mathcal{F} \overline{f}_2 \|_{L^{p'}} = \| \mathcal{F} f_2 \|_{L^{p'}} = \| \mathcal{F} f_2 \|_{L^{p'}}. \] This completes the proof. \[ \square \]

3. Proofs of the main results

Remark 3.1 (Strategy of proof for local well-posedness). It is known that \( U(t) : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d) \) if and only if \( p = 2 \). For this reason, it is believed that, one cannot expect to solve NLS with initial data in \( L^p(\mathbb{R}^d) \) (\( p \neq 2 \)) as the linear counterpart of NLS is ill-posed in \( L^p(\mathbb{R}^d) \). However, we can overcome this difficulty via the following strategy:

(i) Apply \( U(-t) \) to the integral form of (\#), that is to (3.3), and search for solution \( \psi \) so that
\[ \phi(t) = U(-t)\psi(t) \in X_T = (C([0,T], L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)))^N. \]

Now notice that the linear counterpart of (3.3) is well-posed in \( L^p(\mathbb{R}^d) \). This idea is inspired by the work of Zhou [334] for the NLS in \( L^p(\mathbb{R}) \) (1 < p < 2).

(ii) Invoke factorization factorization formula (Lemma 2.3) to obtain transformed integral operator, say \( \Phi \) (see (3.5)).

(iii) Choose closed ball of radius \( b \), and centered at the origin, say \( V^T_b \), in \( X_T \) (we note that the choice of \( V^T_b \) vary as the Lebesgue space exponent \( p \) vary).

(iv) Apply trilinear (Subsection 2.2) and Strichartz estimates to obtain \( \Phi : V^T_b \rightarrow V^T_b \) is contraction, and hence the local existence.

Remark 3.2. We shall give the proof only for the Hartree-Fock equation (1.1). The proof for the reduced Hartree-Fock equation (1.12) can be proved similarly and hence we shall omit the details.

In this section we shall prove our main theorems (Theorem 1.1 to Theorem 1.5). To this end, we start with the following technical lemma.
Lemma 3.1. Let \( t \in \mathbb{R}, u_1, u_2 \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}), \) and
\[
(\Omega(u_1, u_2))(t) = (M_t U(-t) u_1(t)) * \left( R M_t U(-t) u_2(t) \right).
\]
Then for \( 0 < \rho < \infty \) we have
\[
\| \mathcal{F} ((\Omega(u_1, u_2))(t)) \|_{L^\rho} \lesssim |t|^{d(1/\rho)} \| u_1(t) \|_{L^2} \| u_2(t) \|_{L^2}.
\]

Proof. Note that \( \mathcal{FR} \varphi = \mathcal{F} \varphi \). Then
\[
\mathcal{F} ((\Omega(u_1, u_2))(t)) = (\mathcal{FM}_t U(-t) u_1(t)) \left( \mathcal{FR} M_t U(-t) u_2(t) \right)
= (\mathcal{FM}_t U(-t) u_1(t)) \left( \mathcal{FR} M_t U(-t) u_2(t) \right).
\]
Therefore by Hölder inequality, we have
\[
\| \mathcal{F} ((\Omega(u_1, u_2))(t)) \|_{L^\rho} \leq \| \mathcal{FM}_t U(-t) u_1(t) \|_{L^2} \| \mathcal{FM}_t U(-t) u_2(t) \|_{L^2}.
\]
By Lemma 2.2 we have
\[
\| \mathcal{FM}_t U(-t) u_1(t) \|_{L^2} = \left( \int_{\mathbb{R}^d} |\mathcal{FM}_t U(-t) u_1(t)|^{2\rho} dx \right)^{1/2\rho}
= \left( \int_{\mathbb{R}^d} |D_t^{-1} M_t^{-1} u_1(t)|^{2\rho} dx \right)^{1/2\rho}
= \left( (4\pi|t|)^d \int_{\mathbb{R}^d} |u_1(t)(4\pi tx)|^{2\rho} dx \right)^{1/2\rho}
= \left( (4\pi|t|)^d (4\pi|t|)^{-d} \int_{\mathbb{R}^d} |u_1(t)(x)|^{2\rho} dx \right)^{1/2\rho}
= (4\pi|t|)^{(1-1/\rho)d/2} \| u_1(t) \|_{L^2}.
\]
Using this in (3.2), we obtain the desired inequality. \( \square \)

3.1. Local well-posedness in \( L^p \cap L^2 \).

First proof of Theorem 1.1. By Duhamel’s formula, we rewrite (1.1) as
\[
\psi_k(t) = U(t) \psi_{0,k} + i \int_0^t U(t-s) (H_\psi \psi_k)(s) ds - i \int_0^t U(t-s) (F_\psi \psi_k)(s) ds.
\]
Writing \( \psi_k(t) = U(t) \phi_k(t) \), we have
\[
\phi_k(t) = \psi_{0,k} + i \int_0^t U(-s) (H_\psi \psi_k)(s) ds - i \int_0^t U(-s) (F_\psi \psi_k)(s) ds.
\]
Let \( \psi_0 = (\psi_{0,1}, \psi_{0,2}, \ldots, \psi_{0,N}) \). Using Lemma 2.3 we have
\[
\phi_k(t) = \psi_{0,k} + ci \sum_{l=1}^N \sum_{j=1}^2 \int_0^t s^{-\gamma} M_s^{-1} \mathcal{H}_{\alpha,j,s}^j (M_s \phi_l(s), R M_s \phi_l(s), M_s \phi_k(s)) ds
- ci \sum_{l=1}^N \sum_{j=1}^2 \int_0^t s^{-\gamma} M_s^{-1} \mathcal{H}_{\alpha,j,s}^j (M_s \phi_k(s), R M_s \phi_l(s), M_s \phi_l(s)) ds := \Phi_{\psi_{0,k}}(\phi)(t).
\]
• **Case I:** $0 < \gamma < \min\{1, \frac{d}{2}\} \ (1 \leq p \leq \infty)$.

Let $q_1 = \frac{8}{\gamma}$, $r = \frac{4d}{2d - \gamma}$, and introduce the space

$$V_b^T = \left\{ v \in L^\infty_T(L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)) : \|v\|_{L^\infty_T(L^p)} \leq b, \|U(t)v(t)\|_{L^p(L^2)} \leq b \right\},$$

where $q_2, \rho$ to be chosen later. We set $V_b^T = (V_b^T)^N$ and the distance on it by

$$d(u, v) = \max \left\{ \|u_j - v_j\|_{L^\infty_T(L^2)}, \|U(t)[u_j(t) - v_j(t)]\|_{L^p(L^2)}, \|U(t)[u_j(t) - v_j(t)]\|_{L^p(L^2)} : j = 1, 2, \ldots, N \right\}.$$

Then $(V_b^T, d)$ is a complete metric space. Next, we show that the mapping $\Phi_{\psi_0} := (\Phi_{\psi_0,1}, \ldots, \Phi_{\psi_0,N})$, defined by (3.5), takes $V_b^T$ into itself for suitable choice of $b$ and small $T > 0$. Let $\phi = (\phi_1, ..., \phi_N) \in V_b^T$. Denote

$$I_{k,l,m}^1(t) = \int_0^t s^{-\gamma} M_s^{-1} \mathcal{F}_{a,\gamma,s}(M_s \phi_k(s), RM_s \phi_l(s), M_s \phi_m(s)) ds \quad (j = 1, 2)$$

and

$$(3.7) \quad I_{k,l,m}(t) = I_{k,l,m}^1(t) + I_{k,l,m}^2(t).$$

By Proposition 2.3 we have

$$\left\| I_{k,l,m}^1(t) \right\|_{L^p} = \left\| \int_0^t s^{-\gamma} M_s^{-1} \mathcal{F}_{a,\gamma,s}(M_s \phi_k(s), RM_s \phi_l(s), M_s \phi_m(s)) ds \right\|_{L^p} \lesssim \int_0^t s^{-\gamma} \|\phi_k(s)\|_{L^2} \|\phi_l(s)\|_{L^2} \|\phi_m(s)\|_{L^p} ds \lesssim b^2 T^{1-\gamma}.$$

In view of (2.5) and (3.1), we note that

$$\mathcal{H}_{a,\gamma}(M_s \phi_l(s), RM_s \phi_l(s), M_s \phi_k(s)) = [(k_2 * S_{a,s})(\Omega(\psi_l, \psi_l))(s)] * (M_s \phi_k(s)).$$

By Young and Hölder inequalities, we have

$$\left\| I_{k,l,m}^2(t) \right\|_{L^p} \lesssim \int_0^t s^{-\gamma} \| (k_2 * S_{a,s})(\Omega(\psi_k, \psi_l))(s) \|_{L^1} \|\phi_m(s)\|_{L^p} ds \lesssim \int_0^t s^{-\gamma} \| (k_2 * S_{a,s}) \|_{L^p} \|\Omega(\psi_k, \psi_l)(s) \|_{L^{p'}} \|\phi_m(s)\|_{L^p} ds.$$

Here we choose $\rho$ such that

$$\frac{d}{d - \gamma} < \rho < 2.$$

Note that we are able to choose such $\rho$ as $\gamma < d/2$. By Lemmas 2.3 and 3.1 and Hausdorff-Young inequality, we have

$$\left\| I_{k,l,m}^2(t) \right\|_{L^p} \lesssim \int_0^t s^{-\gamma} \|\mathcal{F}(\psi_k, \psi_l)(s)\|_{L^{p'}} \|\phi_m(s)\|_{L^p} ds \lesssim \int_0^t s^{d - \gamma - d/\rho} \|\psi_k(s)\|_{L^{2p'}} \|\psi_l(s)\|_{L^{2p'}} \|\phi_m(s)\|_{L^p} ds.$$
Note that $d - \gamma - \frac{d}{\rho} > 0$. Choose $q_2, q_3$ so that

$$\frac{1}{q_2} = \frac{d}{4} \left( 1 - \frac{1}{\rho} \right) \quad \text{and} \quad \frac{1}{q_3} = 1 - \frac{d}{2} \left( 1 - \frac{1}{\rho} \right).$$

By Hölder inequality, we have

$$\| I_{k,l,m}(t) \|_{L^p} \lesssim T^{d-\gamma-d/p} \int_0^t \| \psi_k(s) \|_{L^{2p}} \| \psi_l(s) \|_{L^{2p}} \| \phi_m(s) \|_{L^p} ds \lesssim T^{d-\gamma-d/p} \left( \| \psi_k \|_{L^{2p}(L^{2p})} \| \psi_l \|_{L^{2p}(L^{2p})} \| \phi_m \|_{L^{2p}(L^{2p})} \right)$$

$$\lesssim T^{d-\gamma-\frac{d}{p} + \frac{1}{2} \rho} \left( \| \psi_k \|_{L^{2p}(L^{2p})} \| \psi_l \|_{L^{2p}(L^{2p})} \| \phi_m \|_{L^\infty(L^p)} \right) \lesssim T^{d-\gamma-\frac{d}{p} + \frac{1}{2} \rho} \left( \| U(t) \phi_k \|_{L^{2p}(L^{2p})} \| U(t) \phi_l \|_{L^{2p}(L^{2p})} \| \phi_m \|_{L^\infty(L^p)} \right) \lesssim T^{d-\gamma-\frac{d}{2} + \frac{1}{3} b^3}. \quad (3.11)$$

Combining (3.5), (3.8) and the above inequality, we have

$$\| \Phi_{\psi_0,k}(\phi) \|_{L^2(L^p)} \lesssim \| \psi_0,k \|_{L^p} + N b^3 (T^{1-\gamma} + T^{d-\gamma-\frac{d}{2} + \frac{1}{3} b^3}). \quad (3.12)$$

For $(q,r) \in \{(q_1,r),(q_2,2\rho),(\infty,2)\}$ and $K = \frac{\gamma}{\rho - 1}$, by Proposition 2.1 we have

$$\| U(t) I_{k,l,m} \|_{L^2(L^2)} \lesssim \| (K \ast (\psi_k \overline{\psi_l})) \psi_m \|_{L^{q_1}(L^{r_1})}. \quad (3.13)$$

Note that

$$\frac{1}{q_1'} = \frac{4 - \gamma}{4} + \frac{1}{q_1}, \quad \frac{1}{r'} = \frac{\gamma}{2d} + \frac{1}{r}, \quad \frac{4 - \gamma}{4} = \frac{2 - \gamma}{2}. \quad (3.14)$$

By Hölder and Hardy-Littlewood-Sobolev inequalities, we have

$$\| (K \ast (\psi_k \overline{\psi_l})) \psi_m \|_{L^{q_1'}(L^{r_1'})} = \| (K \ast (\psi_k \overline{\psi_l})) \psi_m \|_{L^{q_1}(L^{r_1})} \leq \| (K \ast (\psi_k \overline{\psi_l})) \|_{L^{2d}} \| \psi_m \|_{L^{r_1}} \| \psi_m \|_{L^{q_1}(L^{r_1})} \leq \| (\psi_k \overline{\psi_l}) \|_{L^{2d}} \| \psi_m \|_{L^{r_1}} \| \psi_m \|_{L^{q_1'}(L^{r_1'})} \leq T^{1-\frac{\gamma}{2}} \| \psi_k \|_{L^{q_1'}(L^{r_1'})} \| \psi_l \|_{L^{q_1'}(L^{r_1'})} \| \psi_m \|_{L^{q_1'}(L^{r_1'})} \quad \text{and hence}$$

$$\| U(t) I_{k,l,m}(t) \|_{L^2(L^2)} \lesssim T^{1-\gamma/2} \| \psi_k \|_{L^{q_1'}(L^{r_1'})} \| \psi_l \|_{L^{q_1'}(L^{r_1'})} \| \psi_m \|_{L^{q_1'}(L^{r_1'})}. \quad (3.15)$$

Therefore by (3.5) and Proposition 2.1 we have

$$\| U(t) \Phi_{\psi_0,k}(\phi) \|_{L^2(L^2)} \lesssim \| \psi_0,k \|_{L^2} + N b^2 T^{1-\gamma/2}. \quad (3.16)$$
Choose \( b = 2c\|\psi_0\|_{(L^2\cap L^p)^N} \) and \( T > 0 \) small enough so that (3.12), (3.13) imply \( \Phi_\psi(\phi) \in \mathcal{V}_b^T \).

Note that by tri-linearity of \( \hat{H}_{a,\gamma,t} \), we have

\[
\hat{H}_{a,\gamma,s}(f_1, f_2, f_3) - \hat{H}_{a,\gamma,s}(g_1, g_2, g_3) = \hat{H}_{a,\gamma,s}(f_1 - g_1, f_2, f_3) + \hat{H}_{a,\gamma,s}(g_1, f_2 - g_2, f_3) + \hat{H}_{a,\gamma,s}(g_1, g_2, f_3 - g_3).
\]

Using (3.14), for \( u, v \in \mathcal{V}_b^T \), we have

\[
\Phi_{\psi_0,k}(u)(t) - \Phi_{\psi_0,k}(v)(t) \approx \sum_{l=1}^N \int_0^t s^{-\gamma} M_{s-1}^l \hat{H}_{a,\gamma,s}(M_s u_l(s) - v_l(s)), RM_s u_l(s), M_s u_k(s)) ds
\]

\[
+ \int_0^t s^{-\gamma} M_{s-1}^l \hat{H}_{a,\gamma,s}(M_s v_l(s), RM_s u_l(s) - v_l(s)), M_s u_k(s)) ds
\]

\[
+ \int_0^t s^{-\gamma} M_{s-1}^l \hat{H}_{a,\gamma,s}(M_s v_l(s), RM_s v_l(s), M_s u_k(s) - v_k(s))) ds
\]

\[
+ \sum_{l=1}^N \int_0^t s^{-\gamma} M_{s-1}^l \hat{H}_{a,\gamma,s}(M_s u_k(s) - v_k(s)), RM_s u_l(s), M_s u_l(s)) ds
\]

\[
+ \int_0^t s^{-\gamma} M_{s-1}^l \hat{H}_{a,\gamma,s}(M_s v_k(s), RM_s v_l(s), M_s u_l(s) - v_l(s))) ds.
\]

So arguing as above, we have

\[
\left\| \Phi_{\psi_0,k}(u) - \Phi_{\psi_0,k}(v) \right\|_{L^\infty_t(L^p)} \lesssim Nb^2(T^{1-\gamma} + T^{d-\gamma - \frac{d}{m} + 1}) d(u, v)
\]

and

\[
\left\| U(t)[\Phi_{\psi_0,k}(u)(t) - \Phi_{\psi_0,k}(v)(t)] \right\|_{L^2(L^\infty)} \leq \sum_{l=1}^N \left\| (K \ast |\tilde{u}_l|^2) \tilde{u}_k - (K \ast |\tilde{v}_l|^2) \tilde{v}_k \right\|_{L^q_t(L^{r')}}
\]

\[
+ \sum_{l=1}^N \left\| (K \ast (\tilde{u}_k \tilde{u}_l) \tilde{u}_l - (K \ast (\tilde{v}_k \tilde{v}_l) \tilde{v}_l) \right\|_{L^q_t(L^{r'})},
\]

where \( \tilde{u}(t) = U(t) u(t) \) and \( \tilde{v}(t) = U(t) v(t) \). Now

\[
\left\| (K \ast (\tilde{u}_k \tilde{u}_l) \tilde{u}_m - (K \ast (\tilde{v}_k \tilde{v}_l) \tilde{v}_m \right\|_{L^q_t(L^{r'})} \leq \left\| (K \ast ((\tilde{u}_k - \tilde{v}_k) \tilde{u}_l) \tilde{u}_m \right\|_{L^q_t(L^{r'})}
\]

\[
+ \left\| (K \ast (\tilde{v}_k \tilde{u}_l - \tilde{v}_l)) \tilde{u}_m \right\|_{L^q_t(L^{r'})}
\]

\[
+ \left\| (K \ast (\tilde{v}_k \tilde{v}_l)) (\tilde{u}_m - \tilde{v}_m) \right\|_{L^q_t(L^{r'})}
\]

and hence

\[
\left\| U(t)[\Phi_{\psi_0,k}(u)(t) - \Phi_{\psi_0,k}(v)(t)] \right\|_{L^2(L^\infty)} \lesssim T^{1-\gamma/2} Nb^2 d(u, v).
\]

Using (3.16) and (3.17), we may conclude that \( \Phi \psi : \mathcal{V}_b^T \rightarrow \mathcal{V}_b^T \) is a contraction provided \( T \) is sufficiently small(depending on \( \|\psi_0\|_{L^{p} \cap L^2}, \ldots, \|\psi_{0,N}\|_{L^{p} \cap L^2}, d, \gamma, N \)). Then, by Banach
contraction principle, there exists a unique \((\phi_1, \ldots, \phi_N) \in \mathcal{V}_b^T\) solving (3.5).

**Case II:** \(0 < \gamma < \min \{1, 2d(\frac{1}{p} - \frac{1}{q})\}\) and \(1 \leq p < 2\) (improves Case I when \(1 \leq p < \frac{4}{3}\)).

For \(b, T > 0\), let

\[ V_b^T = \{ v \in L^\infty((0, T), L^p \cap L^2(\mathbb{R}^d)) : \|v\|_{L^\infty(L^p \cap L^2)} \leq b \}. \]

We set \(\mathcal{V}_b^T = (V_b^T)^N\) and the distance on it by

\[ d(u, v) = \max \left\{ \|u_j - v_j\|_{L^\infty(L^p \cap L^2)} : j = 1, 2, \ldots, N \right\}, \]

where \(u = (u_1, u_2, \ldots, u_N), v = (v_1, v_2, \ldots, v_N) \in \mathcal{V}_b^T\). Then \((\mathcal{V}_b^T, d)\) is a complete metric space. Next, we show that the mapping \(\Phi_{\psi_0}\), defined by (3.5), takes \(\mathcal{V}_b^T\) into itself for a suitable choice of \(b\) and small \(T > 0\). Let \(\phi = (\phi_1, \ldots, \phi_N) \in \mathcal{V}_b^T\). Choose \(b\) so that \(\|\psi_0\|(L^p \cap L^2)^N = b/2\). Then, by Proposition 2.3 [11], for \(0 < t < T\), we obtain

\[
\|\Phi_{\psi_0,k}(\phi)(t)\|_{L^p \cap L^2} \leq \frac{b}{2} + cN \int_0^t s^{-\gamma} \|M_s \phi_l(s)\|_{L^p \cap L^2}^2 \|M_s \phi_k(s)\|_{L^p \cap L^2} ds
\]

\[
\leq \frac{b}{2} + 2cNb^3 \int_0^t s^{-\gamma} ds = \frac{b}{2} + \frac{2cNb^3}{1 - \gamma} T^{1-\gamma}
\]

Now we choose \(T > 0\) small enough so that

\[
\frac{2cNb^2}{1 - \gamma} T^{1-\gamma} \leq \frac{1}{2}
\]

to achieve

\[
\|\Phi_{\psi_0,k}(\phi)(t)\|_{L^p \cap L^2} \leq b
\]

for all \(k = 1, \ldots, N\). Hence \(\Phi_{\psi_0}\) is a map from \(\mathcal{V}_b^T\) to itself with the above choices of \(b\) and \(T\).

For \(u, v \in \mathcal{V}_b^T\), by Proposition 2.3 [11], (3.15), and arguing as above, we obtain

\[
d(\Phi_{\psi_0}(u), \Phi_{\psi_0}(v)) \leq Nb^2 \int_0^t s^{-\gamma} \|u_l(s) - v_l(s)\|_{L^p \cap L^2} ds \leq Nb^2 T^{1-\gamma} d(u, v).
\]

Thus \(\Phi_{\psi_0} : \mathcal{V}_b^T \to \mathcal{V}_b^T\) is a contraction map provided that \(T\) is sufficiently small (depending on \(\|\psi_{0,1}\|_{L^p \cap L^2}, \ldots, \|\psi_{0,N}\|_{L^p \cap L^2}, d, \gamma, N\)). Then, by Banach contraction principle, there exists a unique \((\phi_1, \ldots, \phi_N) \in \mathcal{V}_b^T\) solving (3.5). \(\square\)

In [33], Zhou proved local existence for cubic NLS in \(L^p(\mathbb{R})\) by introducing a function space (to be defined below) based on the fundamental theorem of calculus and the Schrödinger propagator. Specifically, for \(T > 0, 1 \leq p, q \leq \infty\) and \(\theta \geq 0\), Zhou spaces \(\tilde{X}^p_{q,\theta}(T), \tilde{Y}^p_{q,\theta}(T)\), and \(Y^p_{q,\theta}(T)\) are given by

\[
\tilde{X}^p_{q,\theta}(T) = \left\{ \left. v : [0, T] \times \mathbb{R}^d \to \mathbb{C} : \|v\|_{\tilde{X}^p_{q,\theta}(T)} := \left( \int_0^T s^{q\theta} \|\partial_s v(s, \cdot)(s, \cdot)\|_{L^p}^q ds \right)^{1/q} \right\} < \infty \right\}
\]

for \(1 \leq q < \infty\) and

\[
\tilde{X}^p_{\infty,\theta}(T) = \left\{ \left. v : [0, T] \times \mathbb{R}^d \to \mathbb{C} : \|v\|_{\tilde{X}^p_{\infty,\theta}(T)} := \sup_{s \in [0, T]} s^\theta \|\partial_s v(s, \cdot)\|_{L^p} < \infty \right\}
\],
\[ \tilde{Y}_{q, \theta}^p(T) = \left\{ v \in \tilde{X}_{q, \theta}^p(T) : \|v\|_{\tilde{Y}_{q, \theta}^p(T)} = \|v(0)\|_{L^p} + \|v\|_{\tilde{X}_{q, \theta}^p(T)} < \infty \right\} \]

and
\[ Y_{q, \theta}^p(T) = \left\{ u : [0, T] \times \mathbb{R}^d \to \mathbb{C} \mid U(-t)u(t) \in \tilde{Y}_{q, \theta}^p(T) \right\}. \tag{3.18} \]

Later Hyakuna [21] used Zhou spaces to get the local existence in \( L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) for the Hartree equation. We note that, in the above proof of Theorem 1.1, we do not use Zhou spaces, to get the local existence \( L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \).

Now we will briefly give different proof of local existence in \( L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) (\( p \in [1, \infty) \setminus \{2\} \)) for (\#) using Zhou spaces for certain range of \( \gamma \). Strictly speaking, in this case, we will prove the local existence in the space \( \tilde{Y}_{\infty, \gamma}^p(T) \) which is continuously embedded in \( C([0, T], L^p(\mathbb{R}^d)) \) (and so local solution enjoys \( \tilde{Y}_{\infty, \gamma}^p(T) \)-regularity). For the sake of completeness, we give the proof of following embedding result:

**Lemma 3.2.** Let \( T > 0, p \geq 1 \) and \( 0 < \gamma < 1 \). Then
\[ \tilde{Y}_{\infty, \gamma}^p(T) \hookrightarrow C([0, T], L^p(\mathbb{R}^d)). \]

**Proof.** Let \( v \in \tilde{Y}_{\infty, \gamma}^p(T) \). Notice that
\[ \tag{3.19} v(t) = v(0) + \int_0^t \partial_s v(s) ds. \]

Using (3.19), we obtain
\[
\|v(t)\|_{L^p} \leq \|v(0)\|_{L^p} + \int_0^t s^\gamma s^{-\gamma} \|\partial_s v(s)\|_{L^p} ds \\
\lesssim \|v(0)\|_{L^p} + T^{1-\gamma} \sup_{s \in [0, T]} s^\gamma \|\partial_s v(s)\|_{L^p} \\
\leq \max\{1, T^{1-\gamma}\} \|v\|_{\tilde{Y}_{\infty, \gamma}^p(T)}.
\]

Hence we get that \( v \in L^\infty_T(\mathbb{R}^p) \) and similarly we get for \( t_1, t_2 \in [0, T] \) (with \( t_1 < t_2 \))
\[
\|v(t_1) - v(t_2)\|_{L^p} \leq \int_{t_1}^{t_2} s^\gamma s^{-\gamma} \|\partial_s v(s)\|_{L^p} ds \\
\lesssim |t_2^{1-\gamma} - t_1^{1-\gamma}| \|v\|_{\tilde{Y}_{\infty, \gamma}^p(T)} \lesssim |t_1 - t_2|^{1-\gamma} \|v\|_{\tilde{Y}_{\infty, \gamma}^p(T)}
\]
resulting \( v \in C([0, T], L^p(\mathbb{R}^d)) \). \( \square \)

The key estimates for \( \tilde{Y}_{\infty, \gamma}^p \)-regularity of the local existence is:

**Lemma 3.3.** Denote the Duhamel type operator by
\[
D_{a, \gamma}(v_1, v_2, v_3)(t) = \int_0^t M_s^{-1} s^{-\gamma} \tilde{H}_{a, \gamma, s}(M_s v_1(s), R M_s v_2(s), M_s v_3(s)) ds.
\]

1. Assume \( 0 < \gamma < 1 \). Then
\[
\|D_{a, \gamma}(v_1, v_2, v_3)\|_{\tilde{X}^{2/\gamma, 0}_{q, \theta}(T)} \lesssim \prod_{l=1}^3 \|v_l\|_{\tilde{Y}^0_{q, \theta}(T)}.
\]
(2) Assume $0 < \gamma < \min\{1, 2d(\frac{1}{p} - \frac{1}{2})\}$ when $1 \leq p < 2$ and $0 < \gamma < \min\{1, d(\frac{1}{2} - \frac{1}{p})\}$ when $2 < p \leq \infty$. Then
\[
\|D_{a,\gamma}(v_1, v_2, v_3)\|_{\tilde{X}_{\xi,\gamma}(T)} \lesssim \prod_{j=1}^{3} \|v_j\|_{\tilde{Y}_{\xi,\gamma}^{\epsilon}(T)}.
\]

**Proof.** In view of (2.31), we have
\[
\hat{H}_{a,\gamma,t}(M_t v_1(t), RM_t v_2(t), M_t v_3(t)) = \left( (S_{a,t} * |\cdot|^\gamma)(M_t v_1(t) * RM_t v_2(t)) \right) * M_t v_3(t).
\]
Then using Lemma 2.3, we get
\[
I := \|\partial_t D_{a,\gamma}(v_1, v_2, v_3)\|_{L^{2/\gamma}(L^2)} \simeq \left\| t^{-\gamma} \hat{H}_{a,\gamma,t}(M_t v_1(t), RM_t v_2(t), M_t v_3(t)) \right\|_{L^{2/\gamma}(L^2)} \simeq \|U(-t)H_{a,\gamma}(U(t)v_1(t), U(t)v_2(t), U(t)v_3(t))\|_{L^{2/\gamma}(L^2)} = \|H_{a,\gamma}(U(t)v_1(t), U(t)v_2(t), U(t)v_3(t))\|_{L^{2/\gamma}(L^2)}.
\]
By Hölder and Hardy-Littlewood-Sobolev inequalities, we have
\[
I \lesssim \left\| \left( |\cdot|^{-\gamma} e^{-a|\cdot|} \right)^* \left( \frac{(U(t)v_1(t))(U(t)v_2(t))}{U(t)} \right) \right\|_{L^{\frac{2d}{\gamma}}(L^{\frac{6d}{3d-2\gamma}})} \left\| U(t)v_3(t) \right\|_{L^{\frac{6d}{3d-2\gamma}}} \left\| \frac{a}{L^2} \right\|_{L^{\frac{6d}{3d-2\gamma}}}.
\]
By Hölder and Hardy-Littlewood-Sobolev inequalities, we have
\[
I \lesssim \left\| \left( |\cdot|^{-\gamma} e^{-a|\cdot|} \right)^* \left( \frac{(U(t)v_1(t))(U(t)v_2(t))}{U(t)} \right) \right\|_{L^{\frac{2d}{\gamma}}(L^{\frac{6d}{3d-2\gamma}})} \left\| U(t)v_3(t) \right\|_{L^{\frac{6d}{3d-2\gamma}}} \left\| \frac{a}{L^2} \right\|_{L^{\frac{6d}{3d-2\gamma}}}.
\]
In view of (3.19), we have
\[
U(t)v_1(t) = U(t)v_1(0) + \int_0^t U(t)\partial_s v_1(s)ds.
\]
Put $q = \frac{2}{\gamma}$ and $r = \frac{6d}{3d-2\gamma}$. Then by the above equality we have
\[
\|U(t)v_1(t)\|_{L^{q}(L^r)} \leq \|U(t)v_1(0)\|_{L^{q}(L^r)} + \left\| \int_0^t U(t)\partial_s v_1(s)ds \right\|_{L^{q}(L^r)} \leq \|U(t)v_1(0)\|_{L^{q}(L^r)} + \left\| \int_0^t \|U(t)\partial_s v_1(s)\|_{L^r}ds \right\|_{L^{q}}.
\]
By Minkowski inequality, we have
\[
\left\| \int_0^t \| U(t) \partial_s v_l(s) \|_{L^r} ds \right\|_{L^{3q}} \geq \left( \int_0^T \left( \int_0^t \| U(t) \partial_s v_l(s) \|_{L^r} ds \right)^{3q} dt \right)^{1/3q} \]
\[
\leq \int_0^T \left( \int_0^t \| U(t) \partial_s v_l(s) \|_{L^r}^{3q} dt \right)^{1/3q} ds \]
\[
= \int_0^T \| U(t) \partial_s v_l(s) \|_{L^{3q}(L^r)} ds.
\]
Therefore \((3q, r)\) being 2-admissible pair we get
\[
\| U(t) v_l(t) \|_{L^{3q}_T(L^r)} \lesssim \| v_l(0) \|_{L^2} + \int_0^T \| \partial_s v_l(s) \|_{L^2} ds = \| v_l \|_{Y_{1,0}(T)}.
\]

\[\square\] In view of Proposition 2.3, we obtain
\[
\| D_{a, \gamma}(v_1, v_2, v_3) \|_{\tilde{X}_{b, \gamma}^p(T)} = \sup_{s \in [0, T]} s^\gamma \| \partial_s D_{a, \gamma}(v_1, v_2, v_3)(s) \|_{L^p} \]
\[
= \sup_{s \in [0, T]} \| \tilde{H}_{a, s}(M_s v_1(s), R M_s v_2(s), M_s v_3(s)) \|_{L^p} \]
\[
\lesssim \sup_{s \in [0, T]} \prod_{l=1}^3 \| v_l(s) \|_{L^2 \cap L^p} \]
\[
\lesssim \prod_{l=1}^3 \| v_l \|_{\tilde{Y}_{1,0}^p(T) \cap \tilde{Y}_{1,0}^p(T)},
\]
where the last inequality follows from (3.19) by taking \(L^2, L^p\)-norms on both sides of it.

**Second proof of Theorem 1.1** Here we assume \(0 < \gamma < \min\{1, 2d(\frac{1}{2} - \frac{1}{2})\}\) when \(1 \leq p < 2\) and \(0 < \gamma < \min\{1, d(\frac{1}{2} - \frac{1}{2})\}\) when \(2 < p \leq \infty\).

For \(b, T > 0\), let
\[
V_b^T(v_0) = \left\{ v \in \tilde{Y}_{2/\gamma, 0}^p(T) \cap \tilde{Y}_{\infty, \gamma}^p(T) : \| v \|_{\tilde{X}_{2/\gamma, 0}^p(T) \cap \tilde{X}_{\infty, \gamma}^p(T)} \leq b, v(0) = v_0 \right\}.
\]

For \(\psi_0 = (\psi_{0,1}, \ldots, \psi_{0,N}) \in (L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N\), introduce the space
\[
V_b^T(\psi_0) = V_b^T(\psi_{0,1}) \times V_b^T(\psi_{0,2}) \times \cdots \times V_b^T(\psi_{0,N})
\]
and the distance on it by
\[
d(u, v) = \max \left\{ \| u_j - v_j \|_{\tilde{X}_{2/\gamma, 0}^p(T) \cap \tilde{X}_{\infty, \gamma}^p(T)} : j = 1, 2, \ldots, N \right\}.
\]
Next, we show that the mapping \(\Phi_{\psi_0}\), defined by (3.5), takes \(V_b^T(\psi_0)\) into itself for suitable choice of \(b\) and small \(T > 0\). Let \(\phi = (\phi_1, \ldots, \phi_N) \in V_b^T(\psi_0)\). Since \(\| \psi_{k,0} \|_{\tilde{X}_{2/\gamma, 0}^p(T) \cap \tilde{X}_{\infty, \gamma}^p(T)} = 0\),
by Lemma 3.3, we have

\[
\|\Phi_{\psi_0,k}(\phi)\|_{\tilde{X}^p_{2,0,\gamma}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} \lesssim \sum_{l=1}^{N} \|D_{a,\gamma}(\phi_l, \phi_l, \phi_k)\|_{\tilde{X}^2_{2,0,\gamma}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} + \sum_{l=1}^{N} \|D_{a,\gamma}(\phi_k, \phi_l, \phi_l)\|_{\tilde{X}^2_{2,0,\gamma}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)}
\]

\[
\lesssim \|\phi_k\|_{\tilde{Y}^2_{1,0}(T) \cap \tilde{Y}^p_{1,0}(T)} \sum_{l=1}^{N} \|\phi_l\|^2_{\tilde{Y}^2_{1,0}(T) \cap \tilde{Y}^p_{1,0}(T)}.
\]

By Hölder inequality, we have

\[
\|\phi_l\|_{\tilde{X}^2_{1,0}(T)} \leq T^{-1/2}\|\phi_l\|_{\tilde{X}^2_{2,0,\gamma}(T)} \quad \text{and} \quad \|\phi_l\|_{\tilde{X}^p_{1,0}(T)} \leq T^{-1-\gamma}\|\phi_l\|_{\tilde{X}^p_{\infty,\gamma}(T)}.
\]

Therefore, we have

\[
\|\phi_l\|_{\tilde{Y}^2_{1,0}} \lesssim \|\phi_l,0\|_{L^2} + T^{-1/2}\|\phi_l\|_{\tilde{X}^2_{2,0,\gamma}(T)} \quad \text{and} \quad \|\phi_l\|_{\tilde{Y}^p_{1,0}} \lesssim \|\phi_l,0\|_{L^p} + T^{-1-\gamma}\|\phi_l\|_{\tilde{X}^p_{\infty,\gamma}(T)}.
\]

Hence, by taking \(0 < T < 1\), we obtain

\[
\|\Phi_{\psi_0,k}(\phi)\|_{\tilde{X}^2_{2,0,\gamma}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} \lesssim N \left(\|\psi_0\|^3_{L^2 \cap L^p} + T^{3(1-\gamma)b^3}\right).
\]

We set

\[
b = 2cN\|\psi_0\|^3_{(L^p \cap L^2)^N}
\]

and \(T > 0\) small enough so that we have

\[
\|\Phi_{\psi_0,k}(\phi)\|_{\tilde{X}^2_{2,0,\gamma}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} \leq b.
\]

Consequently, we have \(\Phi_{\psi_0}(\phi) \in \mathcal{V}^T_b(\psi_0)\). Next, we show that \(\Phi_{\psi_0} : \mathcal{V}^T_b(\psi_0) \to \mathcal{V}^T_b(\psi_0)\) is a contraction. Let \(u, v \in \mathcal{V}^T_b(\psi_0)\). Then

\[
\|\Phi_{\psi_0,k}(u) - \Phi_{\psi_0,k}(v)\|_{\tilde{X}^2_{2,0,\gamma}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} \lesssim \sum_{l=1}^{N} \|D_{a,\gamma}(u_l, u_l, u_k) - D_{a,\gamma}(v_l, v_l, v_k)\|_{\tilde{X}^2_{2,0,\gamma}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} + \sum_{l=1}^{N} \|D_{a,\gamma}(u_k, u_l, u_l) - D_{a,\gamma}(v_k, v_l, v_l)\|_{\tilde{X}^2_{2,0,\gamma}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)}.
\]
Using (3.14) together with Lemma 3.3 we have
\[ \| \Phi_{\psi_0,k}(u) - \Phi_{\psi_0,k}(v) \|_{\tilde{L}^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)} \leq \sum_{i=1}^{N} \left( \| u_i - v_i \|_{\tilde{L}^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)} \| u_i \|_{\tilde{L}^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)} \right) + \| u_i \|_{\tilde{L}^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)} \| u_i \|_{\tilde{L}^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)} \| u_i \|_{\tilde{L}^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)} \].

Therefore we get
\[ d(\Phi_{\psi_0}(u), \Phi_{\psi_0}(v)) \leq T^{1-\gamma}(\| \psi_0 \|_{L^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)})^2 \sum_{i=1}^{N} \| u_i - v_i \|_{\tilde{L}^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)} \leq NT^{1-\gamma}(\| \psi_0 \|_{L^2_{2/\gamma,0}(T) \cap \tilde{X}_{\gamma,0}(T)})^2 d(u, v). \]

Thus \( \Phi_{\psi_0} : \mathcal{V}^T_b(\psi_0) \to \mathcal{V}^T_b(\psi_0) \) is a contraction provided \( T > 0 \) is small enough. Finally, we note that since \( \psi_0 \in L^2(\mathbb{R}^d)^N \), by Proposition 2.2, we have \( \psi \in C([0, T], L^2(\mathbb{R}^d))^N \), and consequently \( \phi \in C([0, T], L^p(\mathbb{R}^d))^N \). On the other hand by Lemma 3.2 it follows that \( \phi \in C([0, T], L^p(\mathbb{R}^d))^N \). □

3.2. Local well-posedness in \( \mathcal{L}_T^p \cap L^2 \).

**Lemma 3.4.** For all \( t \in \mathbb{R} \) and \( 0 < \alpha < \infty \) the fractional Schrödinger propagator \( U_\alpha(t) = e^{-it(-\Delta)^{\alpha/2}} \) is unitary on \( \mathcal{L}_T^p(\mathbb{R}^d) \) \((1 \leq p \leq \infty)\).

**Proof.** Note that \( \| e^{-it(-\Delta)^{\alpha/2}} f \|_{\mathcal{L}_T^p} = \| \mathcal{F} e^{-it(-\Delta)^{\alpha/2}} f \|_{L^p} = \| e^{-4\pi^2 t |\xi|^\alpha} \hat{f} \|_{L^p} = \| \hat{f} \|_{L^p} = \| f \|_{\mathcal{L}_T^p}. \)

**First proof of Theorem 1.3** By Duhamel’s formula, we rewrite (1.1) as
\[ (3.20) \quad \psi_k(t) = U_\alpha(t) \psi_0,k + i \int_0^t U_\alpha(t - s) (H \psi_k)(s) ds - i \int_0^t U_\alpha(t - s) (F \psi_k)(s) ds = \Psi_{\psi_0,k}(\psi)(t). \]

**Case I:** \( 0 < \gamma < \min\{\alpha, 2d\} \) \((1 \leq p \leq \infty)\).

Hereafter, for \( \alpha \in (2d/2d - 1, 2) \), we assume initial data is radial and \( d \geq 2 \). In fact, in this case, the members of \( U^T_b \), to be defined below, are radial functions. For the notational convenience, we omit mentioning this explicitly in the proof below. Let \( s = \alpha/2 \) and \( q_1 = \frac{8s}{\gamma}, r = \frac{4d}{2d - \gamma} \), and for \( T, b > 0 \), introduce the space
\[ U^T_b = \{ v \in L^\infty_T(L^2(\mathbb{R}^d) \cap \mathcal{L}_T^p(\mathbb{R}^d)) : \| v \|_{L^2_T(\mathcal{L}_T^p(\mathbb{R}^d) \cap \mathcal{L}_T^p(\mathbb{R}^d))} \leq b, \| v \|_{L^q_T(L^p)} \leq b, \| v \|_{L^{2q_T}(L^{2p})} \leq b \}. \]
where $q_2, \rho$ to be chosen later. We set $\mathcal{U}_b^T = (U_b^T)^N$ and the distance on it by

$$d(u,v) = \max \left\{ \|u_j - v_j\|_{L^p_{\mathcal{T}}(L^2 \cap \mathcal{H}_p)}, \|u_j - v_j\|_{L^{2q_1}_p(L')}, \|u_j - v_j\|_{L^{2q_2}_p(L^2)} : j = 1, 2, \ldots, N \right\},$$

where $u = (u_1, u_2, \ldots, u_N), v = (v_1, v_2, \ldots, v_N) \in \mathcal{U}_b^T$. Next, we show that the mapping

$$\Psi_{\psi_0} = (\Psi_{\psi_0,1}, \ldots, \Psi_{\psi_0,N}),$$

defined by (3.20), takes $\mathcal{U}_b^T$ into itself for suitable choice of $b$ and small $T > 0$. Let $\psi = (\psi_1, \ldots, \psi_N) \in \mathcal{U}_b^T$. Denote

$$J_{k,l,m}(t) = \int_0^t U_a(t-s)\mathcal{H}_{a,\gamma}(\psi_k(s), \psi_l(s), \psi_m(s))ds.$$

Let $\frac{d}{\alpha - \gamma} < \rho \leq 2$, $h_a(\xi) = \frac{c_2|a|}{(a^2 + 4\pi^2|\xi|^2)^{d+1/2}}$ (see (2.6) and Lemma 2.1) and $K = \frac{e^{s\gamma}}{|t|}$. We have

$$\|\mathcal{H}_{a,\gamma}(f, g, h)\|_{L^p} = \|\mathcal{F}[(K * (f\varphi)) h]\|_{L^{p'}} \leq \|\mathcal{F}[K * (f\varphi)]\|_{L^1} \|\mathcal{F}h\|_{L^{p'}} = \|\mathcal{F}K\mathcal{F}(f\varphi)\|_{L^1} \|h\|_{L^{p'}} \leq \|f\|_{L^2} \|g\|_{L^2} + \|f\|_{L^{2p}} \|f\|_{L^{2p'}} \|h\|_{L^{p'}} \leq (\|f\|_{L^2} + \|f\|_{L^{2p}}) \|h\|_{L^{p'}}.$$ 

Choose $q_2$ as

$$\frac{\alpha}{2q_2} = \frac{d}{2} \left( \frac{1}{2} - \frac{1}{2\rho} \right)$$

so that $(2q_2, 2\rho)$ is an $\alpha$-fractional admissible pair. Then we have

$$\|J_{k,l,m}(t)\|_{L^{p'}} \lesssim \int_0^t \left( \|\psi_k(s)\|_{L^2} \|\psi_l(s)\|_{L^2} + \|\psi_k(s)\|_{L^{2p}} \|\psi_l(s)\|_{L^{2p'}} \right) \|\psi_m(s)\|_{L^{p'}} ds \lesssim t \|\psi_k\|_{L^\infty(L^2)} \|\psi_l\|_{L^\infty(L^2)} \|\psi_m\|_{L^\infty(L^p)} + \|\psi_k\|_{L^{2q_2}(L^2)} \|\psi_l\|_{L^{2q_2}(L^2)} \|\psi_m\|_{L^{2q_2}(L^p)}$$

using Hölder inequality. Therefore

$$\|\Psi_{\psi_0,k}(\psi)(t)\|_{L^{p'}} \lesssim \|\psi_0,k\|_{L^{p'}} + Nb^{3}(T + T^{1/d}).$$

For $(q,r) \in \{(q_1,r), (2q_2,2p), (\infty,2)\}$, by Proposition 2.1 we have

$$\|\Psi\psi_{0,k}(\psi)(t)\|_{L^q(L^r)} \lesssim \|\psi_0,k\|_{L^2} + \sum_{l=1}^{N} \|(K * |\psi_l|^2)\psi_k\|_{L^q_t(L^{r'})} + \sum_{l=1}^{N} \|(K * (\psi_k\overline{\psi_l}))\psi_l\|_{L^q_t(L^{r'})}.$$

Now we have

$$\frac{1}{q} = \frac{4s - \gamma}{4s} + \frac{1}{q_1}, \quad \frac{1}{r'} = \frac{\gamma}{2d} + \frac{1}{r} \quad \text{and} \quad \frac{4s - \gamma}{4s} = \frac{2}{q_1} + \frac{2s - \gamma}{2s}.$$
By Hölder and Hardy-Littlewood-Sobolev inequalities, we have
\[
\| (K \ast (\psi_k \psi_t) \psi_m) \|_{L^{q_1}_{\mathbb{R}} (L^{r})} = \| (K \ast (\psi_k \psi_t) \psi_m) \|_{L^{q_1}_{\mathbb{R}} (L^{r})} \\
\leq \| (K \ast (\psi_k \psi_t)) \|_{L^{\frac{2d}{2d-\gamma}}_{\mathbb{R}}} \| \psi_m \|_{L^{r}} \| (\psi_k \psi_t) \|_{L^{\frac{2d}{2d-\gamma}}(L^{r})} \\
\leq \| \| \psi_k \|_{L^{r}} \| \psi_t \|_{L^{r}} \| \psi_m \|_{L^{q_1}(L^{r})} \| (\psi_k \psi_t) \|_{L^{\frac{2d}{2d-\gamma}}(L^{r})} \\
\leq T^{1-\frac{2d}{r}} \| \psi_k \|_{L^{q_1}(L^{r})} \| \psi_t \|_{L^{q_1}(L^{r})} \| \psi_m \|_{L^{q_1}(L^{r})}.
\]
Combining the above two inequalities, we obtain
\[
\| \Psi_{\psi_0,k} (\psi) (t) \|_{L^2(\mathbb{R})} \lesssim \| \psi_0,k \|_{L^2} + T^{1-\frac{2d}{r}} \sum_{l=1}^N \| \psi_t \|_{L^{q_1}(L^{r})} \| \psi_k \|_{L^{q_1}(L^{r})}
\]
(3.24) \[ \lesssim \| \psi_0,k \|_{L^2} + T^{1-\frac{2d}{r}} Nb^3. \]
Choose \( b = 2c \| \psi_0 \|_{L^{2\gamma}(L^{p})}^N \) and \( T > 0 \) small enough so that (3.23) and (3.24) imply \( \Psi_{\psi_0}(\psi) \in \mathcal{U}^T_b \). On the other hand for \( u, v \in \mathcal{V}^T_b \), using trilinearity of \( \mathcal{H}_{a,\gamma} \), we have
\[
\| \Psi_{\psi_0,k}(u)(t) - \Psi_{\psi_0,k}(v)(t) \|_{L^p} \leq \sum_{l=1}^N \int_0^t \| \mathcal{H}_{a,\gamma}(u_l, u_t, u_k) - \mathcal{H}_{a,\gamma}(v_l, v_t, v_k) \|_{L^p} \\
+ \sum_{l=1}^N \int_0^t \| \mathcal{H}_{a,\gamma}(u_k, u_t, u_k) - \mathcal{H}_{a,\gamma}(v_k, v_t, v_k) \|_{L^p}
\]
(3.25) \[ \lesssim T (1 + T^{\frac{1}{\gamma_2}}) b^2 d(u, v). \]
By Proposition 2.3 we have
\[
\| \Psi_{\psi_0,k}(u)(t) - \Psi_{\psi_0,k}(v)(t) \|_{L^2(\mathbb{R})} \leq \sum_{l=1}^N \| (K \ast |u_t|^2) u_k - (K \ast |v_t|^2) v_k \|_{L^{q_1}(L^{r})} \\
+ \sum_{l=1}^N \| (K \ast (u_k \overline{v_t}) u_t - (K \ast (v_k \overline{v_t}) v_t) \|_{L^{q_1}(L^{r})}.
\]
Note that
\[
\| (K \ast (u_k \overline{v_t}) u_m - (K \ast (v_k \overline{v_t}) v_m) \|_{L^{q_1}(L^{r})} \leq \| (K \ast ((u_k - v_k) \overline{v_t}) u_m \|_{L^{q_1}(L^{r})} \\
+ \| (K \ast (v_k (u_l - v_l))) u_m \|_{L^{q_1}(L^{r})} \\
+ \| (K \ast (v_k \overline{v_t}) (u_m - v_m) \|_{L^{q_1}(L^{r})}
\]
and hence
\[
(3.26) \quad \| \Psi_{\psi_0,k}(u)(t) - \Psi_{\psi_0,k}(v)(t) \|_{L^2(\mathbb{R})} \lesssim T^{1-\frac{2d}{r}} Nb^2 d(u, v).
\]
Choose $T > 0$ further small so that (3.25) and (3.26) imply $\Psi_{\psi_0}$ is a contraction to complete the proof.

**• Case II:** $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$, $2 < p \leq \infty$.

Let $X$ be as in Proposition 2.4. For $b, T > 0$, let

$$U_b^T = \{ v \in L^\infty_T(X) : \|v\|_{L^\infty_T(X)} \leq b \}.$$ 

We set $\mathcal{U}_b^T = (U_b^T)^N$ and the distance on it by

$$d(u, v) = \max \left\{ \|u_j - v_j\|_{L^\infty_T(X)} : j = 1, 2, \ldots, N \right\},$$

where $u, v \in \mathcal{U}_b^T$. Next, we show that the mapping $\Psi_{\psi_0}$, defined by (3.20), takes $\mathcal{U}_b^T$ into itself for suitable choice of $b$ and small $T > 0$. Let $\psi = (\psi_1, \ldots, \psi_N) \in \mathcal{U}_b^T$. We note that

$$\|\Psi_{\psi_0,k}(\psi)(t)\|_X \lesssim \|\psi_0,k\|_X + \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(\psi_l(s), \psi_l(s), \psi_k(s))\|_X \, ds + \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(\psi_k(s), \psi_l(s), \psi_l(s))\|_X \, ds.$$ 

Therefore taking $b = 2 \|\psi_0\|_{X^N}$ and using Proposition 2.4 (2), we have

$$\|\Psi_{\psi_0,k}(\psi)(t)\|_X \lesssim \frac{b}{2} + \sum_{l=1}^N \int_0^t \|\psi_l\|^2 \, ds \lesssim \frac{b}{2} + NTb^3.$$ 

For $u, v \in \mathcal{U}_b^T$, we have

$$\|\Psi_{\psi_0,k}(u)(t) - \Psi_{\psi_0,k}(v)(t)\|_X \lesssim \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(u_l, u_l, u_k) - \mathcal{H}_{a,\gamma}(v_l, v_l, v_k)\|_X \, ds + \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(u_k, u_l, u_l) - \mathcal{H}_{a,\gamma}(v_k, v_l, v_l)\|_X.$$ 

By tri-linearity of $\mathcal{H}_{a,\gamma}$, it follows that

$$\|\Psi_{\psi_0,k}(u)(t) - \Psi_{\psi_0,k}(v)(t)\|_X \lesssim Nb^2Td(u, v).$$

Thus $\Psi : \mathcal{U}_b^T \to \mathcal{U}_b^T$ is a contraction provided $T > 0$ is small enough. \hfill \Box

We introduce function space $\hat{W}^p_{q,\theta}(T)$ $(1 \leq p \leq \infty)$ which is similar to $\tilde{Y}^p_{q,\theta}(T)$ to get local well-posedness. Specifically, we define

$$\hat{W}^p_{q,\theta}(T) = \left\{ v : [0, T] \times \mathbb{R}^d \to \mathbb{C} : \|v\|_{\hat{W}^p_{q,\theta}(T)} = \left( \int_0^T s^q \| (\partial_s v)(s, \cdot) \|_{L^p_{s,\theta}}^q \, ds \right)^{1/q} < \infty \right\}$$

for $1 \leq q < \infty$,

$$\hat{W}^p_{\infty,\theta}(T) = \left\{ v : [0, T] \times \mathbb{R}^d \to \mathbb{C} : \|v\|_{\hat{W}^p_{\infty,\theta}(T)} = \sup_{s \in [0,T]} s^q \| (\partial_s v)(s, \cdot) \|_{L^p_{s,\theta}} < \infty \right\}.$$
and
\[ \tilde{Z}^p_{q,0}(T) = \left\{ v \in \tilde{W}^p_q(T) : \|v\|_{\tilde{Z}^p_{q,0}(T)} := \|v(0)\|_{\tilde{L}^p} + \|v\|_{\tilde{W}^p_q(T)} < \infty \right\}. \]

Now we state the required inclusion result.

**Lemma 3.5.** Let \( T > 0, p \geq 1 \) and \( 0 < \gamma < 1 \). Then
\[ \tilde{W}^p_{\infty,0}(T) \hookrightarrow C([0,T], \tilde{L}^p(\mathbb{R}^d)). \]

**Proof.** The proof is similar to the proof of Lemma 3.2 and so we omit the details. \( \square \)

**Lemma 3.6.** Denote
\[ \mathcal{D}^\alpha_{a,\gamma}(v_1, v_2, v_3)(t) = \int_0^t U_\alpha(-s)\mathcal{H}_{a,\gamma}(U_\alpha(s)v_1(s), U_\alpha(s)v_2(s), U_\alpha(s)v_3(s))ds. \]

1. Assume that \( 0 < \gamma < \min\{\alpha, d(\frac{1}{p} - \frac{1}{2})\} \) when \( 1 \leq p < 2 \) and \( 0 < \gamma < \min\{\alpha, 2d(\frac{1}{2} - \frac{1}{p})\} \) when \( 2 < p \leq \infty \). Then
\[ \|\mathcal{D}^\alpha_{a,\gamma}(v_1, v_2, v_3)\|_{\tilde{W}^2_{\infty,0}(T) \cap \tilde{W}^p_{\infty,0}} \lesssim \sum_{l=1}^3 \|v_l\|_{\tilde{Z}^p_{1,0}(T) \cap \tilde{Z}^2_{1,0}(T)}. \]

2. Assume that \( 2 < p \leq \infty \) and \( 0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p}) \). Then
\[ \|\mathcal{D}^\alpha_{a,\gamma}(v_1, v_2, v_3)\|_{\tilde{W}^2_{\infty,0}(T) \cap \tilde{W}^p_{\infty,0}} \lesssim \sum_{l=1}^3 \|v_l\|_{\tilde{Z}^2_{1,0}(T) \cap \tilde{Z}^p_{1,0}(T)}. \]

**Proof.** Set \( q = \frac{6d}{6d - 2\gamma} \) so that \((3q, r)\) becomes an \( \alpha\)-fractional admissible pair. By similar argument as in the proof of Lemma 3.3, we obtain
\[ \| \partial_t \mathcal{D}^\alpha_{a,\gamma}(v_1, v_2, v_3) \|_{L^p_2(L^2)} \lesssim \| U_\alpha(-t)\mathcal{H}_{a,\gamma}(U_\alpha(t)v_1(t), U_\alpha(t)v_2(t), U_\alpha(t)v_3(t)) \|_{L^p_2(L^2)} \]
\[ = \| \mathcal{H}_{a,\gamma}(U_\alpha(t)v_1(t), U_\alpha(t)v_2(t), U_\alpha(t)v_3(t)) \|_{L^p_2(L^2)} \]
\[ \lesssim \prod_{l=1}^3 \| U(t)v_l \|_{L^p_2(\mathbb{R}^d)} \]
\[ \lesssim \prod_{l=1}^3 \| v_l \|_{\tilde{Z}^2_{1,0}(T)}. \]

In view of Proposition 2.4 and Lemma 3.1, we obtain
\[ \| \mathcal{D}^\alpha_{a,\gamma}(v_1, v_2, v_3) \|_{\tilde{W}^p_{\infty,0}(T)} \lesssim \sup_{t \in [0,T]} \| \partial_t \mathcal{D}^\alpha_{a,\gamma}(v_1, v_2, v_3)(t) \|_{\tilde{L}^p} \]
\[ \lesssim \sup_{t \in [0,T]} \| \mathcal{H}_{a,\gamma}(U_\alpha(t)v_1(t), U_\alpha(t)v_2(t), U_\alpha(t)v_3(t)) \|_{\tilde{L}^p} \]
\[ \lesssim \sup_{t \in [0,T]} \prod_{l=1}^3 \| v_l(t) \|_{L^2 \cap \tilde{L}^p} \]
\[ \lesssim \prod_{l=1}^3 \| v_l \|_{\tilde{Z}^2_{1,0}(T) \cap \tilde{Z}^p_{1,0}(T)}. \]
Here it remains to estimate the $\widetilde{W}^2_{\infty,0}(T)$-semi norm which follows in a similar way as $\widetilde{W}^p_{\infty,0}(T)$ estimate above. $\square$

**Second proof of Theorem 1.3**

For $\alpha \in \left(\frac{2d}{2d-1}, 2\right)$, we assume $d \geq 2$ and initial data to be radial.

- **Case I:** $0 < \gamma < \min\{\alpha, d\left(\frac{1}{p} - \frac{1}{2}\right)\}$ when $1 \leq p < 2$ and $0 < \gamma < \min\{\alpha, 2d\left(\frac{1}{2} - \frac{1}{p}\right)\}$ when $2 < p \leq \infty$.

Applying $U_\alpha(-t)$ to the Duhamel’s formula, we rewrite (1.1) as

$$
\phi_k(t) = \psi_{0,k} + i \int_0^t U_\alpha(-s)(H_\psi \psi_k)(s)ds - i \int_0^t U_\alpha(-s)(F_\psi \psi_k)(s)ds =: \Phi_{\psi_0}(\phi).
$$

For $b > b, T > 0$, let

$$
V_b^T(v_0) = \left\{ v \in \widetilde{Z}^2_{\alpha/\gamma,0}(T) \cap \widetilde{Z}^p_{\infty,0}(T) : \|v\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T) \cap \widetilde{W}^p_{\infty,0}(T)} \leq b, v(0) = v_0 \right\}.
$$

We set

$$
V_b^T(\psi_0) = V_b^T(\psi_{0,1}) \times V_b^T(\psi_{0,2}) \times \cdots \times V_b^T(\psi_{0,N}),
$$

and the distance on it by

$$
d(u, v) = \max \left\{ \|u_j - v_j\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T) \cap \widetilde{W}^p_{\infty,0}(T)} : j = 1, 2, \cdots, N \right\}.
$$

Next, we show that the mapping $\Phi_{\psi_0}$ defined by (3.27) takes $V_b^T(\psi_0)$ into itself for suitable choice of $b > 0$ and small $T > 0$. In fact, taking $0 < T < 1$ and by Lemma 3.6 we obtain

$$
\|\Phi_{\psi_0,k}(\phi)\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T) \cap \widetilde{W}^p_{\infty,0}(T)} \lesssim \sum_{l=1}^N \|D_{\alpha,\gamma}^l(\phi_l, \phi_l, \phi_k)\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T) \cap \widetilde{W}^p_{\infty,0}(T)} + \sum_{l=1}^N \|D_{\alpha,\gamma}^l(\phi_k, \phi_l, \phi_l)\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T) \cap \widetilde{W}^p_{\infty,0}(T)}.
$$

By Hölder inequality, we have

$$
\|v_l\|_{\widetilde{Z}^2_{\alpha,0}} \leq T^{1-\frac{\alpha}{2}} \|v_l\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T)} \quad \text{and} \quad \|v_l\|_{\widetilde{W}^p_{\infty,0}(T)} \leq T \|v_l\|_{\widetilde{W}^p_{\infty,0}(T)}.
$$

Therefore, for $0 < T < 1$, we have

$$
(3.28) \quad \|v_l\|_{\widetilde{Z}^2_{\alpha,0} \cap \widetilde{Z}^p_{\infty}} \lesssim \|v_l(0)\|_{L^p \cap L^2} + T^{1-\frac{\alpha}{2}} \|v_l\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T) \cap \widetilde{W}^p_{\infty,0}(T)}.
$$

Hence

$$
\|\Phi_{\psi_0,k}(\phi)\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T) \cap \widetilde{W}^p_{\infty,0}(T)} \lesssim N \|\psi_0\|_{L^2 \cap L^p}^3 + T^{3(1-\frac{\alpha}{2})} \sum_{l=1}^N \|\phi_l\|_{\widetilde{W}^2_{\alpha/\gamma,0}(T) \cap \widetilde{W}^p_{\infty,0}(T)}^3.
$$
Set $b = 2cN\|\psi_0\|_{L^p(\mathbb{R}^d)^N}^2$ then choose $T > 0$ small enough to get $\|\Phi_{\psi_0,k}(\phi)\|_{\tilde{W}^{2,\gamma,0}(T) \cap W^{p,\gamma}_\infty(T)} \leq b$. It follows that $\Phi(\phi) \in V^T_0(\psi_0)$. Let $u,v \in V^T_0(\psi_0)$. Then

$$d(\Phi_{\psi_0,k}(u),\Phi_{\psi_0,k}(v)) \lesssim \sum_{l=1}^N \|D_{a,\gamma}^\alpha(u_l, u_l, u_k) - D_{a,\gamma}^\alpha(v_l, v_l, v_k)\|_{\tilde{W}^{2,\gamma,0}(T) \cap W^{p,\gamma}_\infty(T)}$$

$$+ \sum_{l=1}^N \|D_{a,\gamma}^\alpha(u_k, u_l, u_l) - D_{a,\gamma}^\alpha(v_k, v_l, v_l)\|_{\tilde{W}^{2,\gamma,0}(T) \cap W^{p,\gamma}_\infty(T)}.$$

But recalling (3.14) by triangular inequality and Lemma 3.6 and (3.28)

$$d(\Phi_{\psi_0,k}(u),\Phi_{\psi_0,k}(v)) \lesssim T^{1-\frac{2}{\alpha}} \left(\|\psi_0\|_{L^p(\mathbb{R}^d)}^2 + T^{2(1-\frac{2}{\alpha})}b^2\right) \sum_{l=1}^N \|u_l - v_l\|_{\tilde{W}^{2,\gamma,0}(T) \cap W^{p,\gamma}_\infty(T)}$$

$$\lesssim T^{1-\frac{2}{\alpha}} N \left(\|\psi_0\|_{L^p(\mathbb{R}^d)}^2 + T^{2(1-\frac{2}{\alpha})}b^2\right) d(u,v).$$

Then $\Phi_{\psi_0} : V^T_0(\psi_0) \to V^T_0(\psi_0)$ is a contraction for further small enough $T > 0$ if needed.

- Case II: $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$ with $2 < p \leq \infty$.

Lemma 3.6 (2) and similar argument as in Case I give the solution $\phi \in (\tilde{Z}^2_{\infty,0}(T) \cap \tilde{Z}^p_{\infty,0}(T))^N$ to (3.27). \(\square\)

**Remark 3.3.** Note that the second proof gives the existence of solution in Zhou spaces. So it adds Zhou space regularity to the solutions given by the first proof. For $0 < \gamma < 1$ such solutions are in $C([0,T], \tilde{L}^p(\mathbb{R}^d))$ (see Lemma 3.5).

### 3.3. Global well-posedness in $L^p \cap L^2$. We extend the local solution established in Theorem 1.1 globally. Let $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$ be the local solution (given by Theorem 1.1) to (3.4) which is in $C([0,T], L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N$ for any $0 < T < T_0$. We start with the following lemma.

**Lemma 3.7.** On the time interval $[0, T_0)$, the local solution (given by Theorem 1.1) $\psi(t) = (U(t)\phi_1(t), \cdots, U(t)\phi_N(t))$ coincides with the global $L^2$-solution for the initial datum $\psi_0 = \psi(0)$ given by Proposition 2.2.

**Proof.** The assertion follows from uniqueness of local solution given by Theorem 1.1. \(\square\)

**Proposition 3.1.** Assume $0 < \gamma < d/2$. Let $T_0 > 0$ be such that for any $0 < T < T_0$ the local solution $\phi$ of (3.3) exists in $C([0,T], L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N$. Then

$$\sup_{t \in [0,T_0)} \|\phi(t)\|_{L^p} < \infty.$$ 

**Proof.** We fix $T \in (0, T_0)$ and $t \in [0, T]$. Taking (3.5) into account, we obtain

$$\|\phi_k(t)\|_{L^p} \lesssim \|\psi_{0,k}\|_{L^p} + \sum_{l=1}^N \sum_{j=1}^2 \|I_{l,l,k}(t)\|_{L^p} + \sum_{l=1}^N \sum_{j=1}^2 \|I_{l,l,k}(t)\|_{L^p},$$

$$\|\phi_k(t)\|_{L^p} \leq c_0 + c_1 \sum_{l=1}^N \sum_{j=1}^2 \|I_{l,l,k}(t)\|_{L^p},$$

where $c_0$ and $c_1$ are constants.

$$\|\phi_k(t)\|_{L^p} \leq \sum_{l=1}^N \sum_{j=1}^2 \|I_{l,l,k}(t)\|_{L^p},$$

and

$$\|\phi_k(t)\|_{L^p} \leq \sum_{l=1}^N \sum_{j=1}^2 \|I_{l,l,k}(t)\|_{L^p},$$

where $c_0$ and $c_1$ are constants.
where $I_{k,l}^j$ are given by (3.6). Using Proposition 2.3[1], we have

$$\|I_{k,l,m}(t)\|_{L^p} \lesssim \int_0^t s^{-\gamma} \|\phi_k(s)\|_{L^2} \|\phi_l(s)\|_{L^2} \|\phi_m(s)\|_{L^p} ds$$

(3.30)

where $q_4$ is chosen so that $q_4 > \frac{1}{1-\gamma}$. Let $q_2$ and $\rho$ are given as in (3.9) and (3.10) respectively. By (3.11), we have

$$\|I_{k,l,m}(t)\|_{L^p} \lesssim T_0^{d-\gamma-d/\rho} \|\psi_j\|_{L^q_0(L^2)} \|\psi_k\|_{L^q_0(L^2)} \|\psi_l\|_{L^q_0(L^2)} \|\phi_m\|_{L^q_3(L^p)}.$$ Note that $d - \gamma - \frac{d}{\rho} > 0$. Since $(q_2, 2\rho)$ is admissible pair, in view of Lemma 3.7 and Proposition 2.2 we have

$$\|\psi_j\|_{L^q_0(L^2)} < \infty.$$

It follows that

$$\|I_{k,l,m}(t)\|_{L^p} \leq C_{T_0} \|\phi_m\|_{L^q_3(L^p)}.$$ (3.31)

Thus we have from (3.28), (3.30) and (3.31) that

$$\|\phi(t)\|_{(L^p)^N} \lesssim C_{\psi_0,T_0} + NC_{\psi_0,T_0} \|\phi\|_{L^q_3((L^p)^N)},$$

where $q = \max\{q_3, q_4\}$. Therefore

$$\|\phi(t)\|_{(L^p)^N} \lesssim C_{\psi_0,T_0}^q + N^q C_{\psi_0,T_0}^q \int_0^t \|\phi(t)\|_{(L^p)^N} dt.$$ Now by Gronwall’s lemma it follows that

$$\|\phi(t)\|_{(L^p)^N} \lesssim C_{\psi_0,T_0}^q \left(1 + N^q C_{\psi_0,T_0}^q t e^{C_{T_0} \cdot N \cdot 4t}\right).$$

Hence the result follows. \qed

Let $\psi(t) = (\psi_1(t), \ldots, \psi_N(t))$ be a global $L^2$-solution given by Proposition 2.2. We define

$$T_+(\psi_0) = \sup \left\{T > 0 : U(-t)\psi(t)|_{[0,T]} \in C([0, T], L^p(\mathbb{R}^d))^N\right\}$$

where $U(-t)\psi(t) = (U(-t)\psi_1(t), \ldots, U(-t)\psi_N(t))$. By Theorem 1.1 we have $T_+(\psi_0) > 0$.

**Proposition 3.2.** Assume $T_+(\psi_0) < \infty$. Then

$$\lim_{t \nearrow T_+(\psi_0)} \|U(-t)\psi(t)\|_{(L^p)^N} = \infty.$$ \textit{Proof.} We point out that the assertion relies on the fact that local existence time $T$, from Theorem 1.1, depend only on $\|\psi_0\|_{(L^2 \cap L^p)^N}, \gamma, d, N$. Now the proof is standard, see e.g. [21] Lemma 5.4 for the Hartree equation, and so we omit the details. \qed

**Proof of Theorem 1.2** It is enough to prove that $T_+(\psi_0) = \infty$. If not, Proposition 3.2 implies

$$\lim_{t \nearrow T_+(\psi_0)} \|U(-t)\psi(t)\|_{(L^p)^N} = \infty.$$
contradicting Proposition 3.1 as $T_+(\psi_0) > 0$.

The last assertion of the theorem follows from Proposition 2.2 and Hausdorff-Young inequality.

3.4. Global well-posedness in $\hat{L}^p \cap L^2$.

Proof of Theorem 1.4: The proof strategy is similar to the proof of Theorem 1.2. Specifically, taking Theorem 1.3 and Proposition 2.2 into account, to prove Theorem 1.4 it is enough to show that the $(\hat{L}^p)^N$-norm of the solution remains bounded in finite time. Let $t \in [0, T]$.

- Case I: $0 < \gamma < \min\{\alpha, \frac{d}{2}\}$, $1 \leq p < 2$.

  By (3.20), we have

  \[
  \|\psi_k(t)\|_{\hat{L}^p} \leq \|\psi_{0,k}\|_{\hat{L}^p} + \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(\psi_l(s), \psi_l(s), \psi_k(s))\|_{\hat{L}^p} ds
  + \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(\psi_k(s), \psi_l(s), \psi_l(s))\|_{\hat{L}^p} ds.
  \]

  By Propositions 2.4 and 2.2, we have

  \[
  \int_0^t \|\mathcal{H}_{a,\gamma}(\psi_k(s), \psi_l(s), \psi_m(s))\|_{\hat{L}^p} ds \lesssim \int_0^t \|\psi_k(s)\|_{L^2} \|\psi_l(s)\|_{L^2} \|\psi_m(s)\|_{\hat{L}^p \cap L^2} ds
  \leq \int_0^t \|\psi_m(s)\|_{\hat{L}^p \cap L^2} ds
  = T\|\psi_{0,m}\|_{L^2} + \int_0^t \|\psi_m(s)\|_{\hat{L}^p} ds.
  \]

  Using this and (3.33), we have

  \[
  \|\psi(t)\|_{(\hat{L}^p)^N} \lesssim \|\psi_0\|_{(\hat{L}^p)^N} + NT\|\psi_0\|_{(L^2)^N} + N \int_0^t \|\psi(s)\|_{(\hat{L}^p)^N} ds.
  \]

  Now the result follows by Gronwall’s lemma.

- Case II: $0 < \gamma < \min\{\alpha, \frac{d}{2}\}$ ($1 \leq p \leq \infty$).

  For $\alpha \in \left(\frac{2d}{2d-1}, 2\right)$, we assume $d \geq 2$ and initial data is radial. By (3.21) and (3.22), we have

  \[
  \|J_{k,l,m}(t)\|_{\hat{L}^p} \lesssim T^{\frac{1}{q_2}} \|\psi_k\|_{L^\infty_T(L^2)} \|\psi_l\|_{L^\infty_T(L^2)} \|\psi_m\|_{L^{q_2}_T(\hat{L}^p)}
  + \|\psi_k\|_{L^{q_2}_T(L^{q_2})} \|\psi_l\|_{L^{q_2}_T(L^{q_2})} \|\psi_m\|_{L^{q_2}_T(\hat{L}^p)}.
  \]

  Therefore by (3.33) and Strichartz estimates, we have

  \[
  \|\psi_k(t)\|_{\hat{L}^p} \lesssim \|\psi_{0,k}\|_{\hat{L}^p} + (1 + T^{\frac{1}{q_2}}) \sum_{l=1}^N \|\psi_l\|_{L^{q_2}_T(\hat{L}^p)}
  \]

  and hence

  \[
  \|\psi(t)\|_{(\hat{L}^p)^N} \lesssim C_{\psi_0,T} + NC_{\psi_0,T}\|\psi\|_{L^{q_2}_T(\hat{L}^p)^N},
  \]

  completing the proof.
3.5. Improved well-posedness in 1D. We have proved the result (local and global) if $0 < \gamma < \frac{1}{2}$ for $d = 1$. See Theorems 1.4, 1.4. Now we improve it to $0 < \gamma < 1$ for global existence. The extra ingredient we use here is below Lemma 3.8.

Lemma 3.8 (Generalized Strichartz estimate, see [15, 11]). Assume $\frac{4}{3} < p \leq 2$. Then

\[ \|U(t)\phi\|_{L^p(\mathbb{R}^+ \times \mathbb{R})} \lesssim \|\phi\|_{\dot{L}^p(\mathbb{R})}. \]

As a consequence, by the duality argument, for $2 \leq p < 4$ we have

\[ \sup_{I \subset J} \left\| \int_I U(-s)F(s)ds \right\|_{L^p} \leq \|F\|_{L^{(3p')}(J \times \mathbb{R})}. \]

Proof of Theorem 3.5 As an application of Lemma 3.8 we shall obtain some improved estimate for $\|I_{k,l,m}\|_{L^p}$ (see (3.7)). Specifically, observe estimates (3.8) and (3.35). We shall see that this will play a vital role to improve the range of exponent $\gamma$ of Hartree factor.

• Step A I: Improving the local result for $L^p$ space.

Note that

\[ \|\varphi\|_{L^p} = \|F^{-1}\varphi\|_{L^{p'}} = \|\mathcal{F}\varphi\|_{L^{p'}} = \|\mathcal{F}\varphi\|_{L^{p'}} \]

and using (2.1) $\mathcal{F}M_s\mathcal{F}^{-1} = U(-1/16\pi^2 s)$ as

\[ \mathcal{F}M_s\mathcal{F}^{-1}\varphi(x) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} M_s\mathcal{F}^{-1}\varphi(x)d\xi = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} e^{i|x|^2/4s} \mathcal{F}^{-1}\varphi(x)d\xi \]

\[ = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} e^{-i|x|^2/4s} \mathcal{F}\varphi(x)d\xi \]

\[ = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} e^{-4\pi^2|x|^2(-1/16\pi^2 s)t}) \mathcal{F}\varphi(x)d\xi = [U(-1/16\pi^2 s)\varphi](\xi). \]

In view of these and (3.7), we obtain

\[ \|I_{k,l,m}(t)\|_{L^p} \lesssim \left\| \int_0^t s^{-\gamma} M_s^{-1} \mathcal{H}_{a,\gamma,s}(M_s\phi_k(s), R M_s\phi_l(s), M_s\phi_m(s))ds \right\|_{L^p} \]

\[ = \left\| \int_0^t s^{-\gamma} F M_s^{-1} \mathcal{H}_{a,\gamma,s}(M_s\phi_k(s), R M_s\phi_l(s), M_s\phi_m(s))ds \right\|_{L^{p'}} \]

\[ = \left\| \int_0^t s^{-\gamma} F M_s \mathcal{H}_{a,\gamma,s}(M_s\phi_k(s), R M_s\phi_l(s), M_s\phi_m(s))ds \right\|_{L^{p'}} \]

\[ = \left\| \int_0^t s^{-\gamma} U(-1/16\pi^2 s) F \mathcal{H}_{a,\gamma,s}(M_s\phi_k(s), R M_s\phi_l(s), M_s\phi_m(s))ds \right\|_{L^{p'}} \]

\[ = \left\| \int_1^{\infty} s^{-2\gamma} U \left( \frac{s}{16\pi^2} \right) F \mathcal{H}_{a,\gamma,1/s}(M_{1/s}\phi_k(1/s), R M_{1/s}\phi_l(1/s), M_{1/s}\phi_m(1/s))ds \right\|_{L^{p'}}. \]
Using Lemma 3.8 and changing the $s$-variable, we get
\[
\|I_{k,l,m}(t)\|_{L^p} \lesssim \left\| s^{\gamma - 2} \mathcal{F} \tilde{H}_{a,\gamma/s}(M_{1/s} \phi_k(1/s), RM_{1/s} \phi_l(1/s), M_{1/s} \phi_m(1/s)) \right\|_{L^p([16\pi^2/t, \infty) \times \mathbb{R})}
\leq \left\| s^{\gamma - 2} \mathcal{F} \tilde{H}_{a,\gamma/s}(M_{1/s} \phi_k(1/s), RM_{1/s} \phi_l(1/s), M_{1/s} \phi_m(1/s)) \right\|_{L^p((1/t, \infty) \times \mathbb{R})}
= \left\| s^{2 - \gamma/2 \tilde{r}} \mathcal{F} \tilde{H}_{a,\gamma,s}(M_s \phi_k(s), RM_s \phi_l(s), M_s \phi_m(s)) \right\|_{L^p((0,t) \times \mathbb{R})},
\]
where $\tilde{r} = (3p)'. $ Since $U(-t) \psi_k(t) = \phi_k(t)$, by (2.4) and (3.1), we have
\[
\tilde{H}_{a,\gamma,s}(M_s \phi_k(s), RM_s \phi_l(s), M_s \phi_m(s)) = \left[ (| \cdot |^{\gamma - 1} * S_{a,s}) \Omega(\psi_k(s), \psi_l(s)) \right] * M_s \phi_m(s).
\]
In view of this, we may obtain
\[
\left\| \mathcal{F} \tilde{H}_{a,\gamma,s}(M_s \phi_l(s), RM_s \phi_l(s), M_s \phi_k(s)) \right\|_{L^p}
\leq \left\| \mathcal{F} \left[ (| \cdot |^{\gamma - 1} * S_{a,s}) \Omega(\psi_k(s), \psi_l(s)) \right] \right\|_{L^p/2} \left\| \mathcal{F} M_s \phi_m(s) \right\|_{L^{p'}}
\lesssim \left\| \left| \cdot \right|^{-\gamma} * \left\{ \mathcal{F} \left( \Omega(\psi_k(s), \psi_l(s)) \right) \right\} \right\|_{L^{3p/2}} \left\| \phi_m(s) \right\|_{L^p}
\lesssim \left\| \mathcal{F} \left( \Omega(\psi_k(s), \psi_l(s)) \right) \right\|_{L^\tilde{R}} \left\| \phi_k(s) \right\|_{L^p},
\]
where
\[
\tilde{R} = \left( 1 + \frac{2}{3p} - \gamma \right)^{-1}.
\]
Using Lemma 3.1, we have
\[
\left\| \mathcal{F} \tilde{H}_{a,\gamma,s}(M_s \phi_l(s), RM_s \phi_l(s), M_s \phi_k(s)) \right\|_{L^p} \lesssim \left\| s^{1 - 1/\tilde{R}} \left\| \psi_k(s) \right\|_{L^2 \tilde{R}} \left\| \psi_l(s) \right\|_{L^2 \tilde{R}} \left\| \phi_m(s) \right\|_{L^p} \right\| \left\| \tilde{H}_{a,\gamma,s}(M_s \phi_l(s), RM_s \phi_l(s), M_s \phi_k(s)) \right\|_{L^p}.
\]
Note that $3 - \gamma - 2/\tilde{r} - 1/\tilde{R} = 0$ and hence by Hölder’s inequality
\[
\|I_{k,l,m}(t)\|_{L^p} \lesssim \|\psi_k\|_{L^\tilde{Q}(L^2 \tilde{R})} \|\psi_l\|_{L^\tilde{Q}(L^2 \tilde{R})} \|\phi_m\|_{L^\tilde{Q}(L^2 \tilde{R})} \leq T^{1 - \gamma} \|\psi_k\|_{L^\tilde{Q}(L^2 \tilde{R})} \|\psi_l\|_{L^\tilde{Q}(L^2 \tilde{R})} \|\phi_m\|_{L^\tilde{Q}(L^p)},
\]
where
\[
\tilde{Q} = \left( \frac{\gamma}{4} - \frac{1}{6p} \right)^{-1}.
\]
Let $q_1 = \frac{8}{\gamma}$ and $r = \frac{4d}{2d - \gamma}$. We define
\[
V_b^T = \left\{ v \in L^\infty_T (L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)) : \|v\|_{L^\infty_T(L^p \cap L^2)} \leq b, \|U(t)v(t)\|_{L^\tilde{Q}(L^2 \tilde{R})} \leq b, \|U(t)v(t)\|_{L^\tilde{Q}(L^r)} \leq b \right\}
\]
and
\[
V_b^T = (V_b^T)^N.
\]
Now arguing as in Case I of the proof of Theorem 1.1, we can establish the local well-posedness of (1.1) with $0 < \gamma < 2$ in $L^p(\mathbb{R}) \cap L^2(\mathbb{R})$.

**Step A II:** Improving the global result for $0 < \gamma < 1$ for $L^p$ space.
Note that from (3.5) and (3.35) we have

\[ \| \phi_k(t) \|_{L^p} \lesssim \| \psi_{0,k} \|_{L^p} + \sum_{i=1}^{N} \| \psi_i \|_{L_t^q L_x^r(L^2 \mathbb{R}^n)} \| \hat{\phi}_k \|_{L_t^{2/(2-\gamma)}(L^p)} \]

\[ + \sum_{i=1}^{N} \| \psi_k \|_{L_t^q L_x^r(L^2 \mathbb{R}^n)} \| \psi_i \|_{L_t^q L_x^r(L^2 \mathbb{R}^n)} \| \hat{\phi}_k \|_{L_t^{2/(2-\gamma)}(L^p)}. \]

Now \((\tilde{Q}, 2 \tilde{R})\) being admissible, Strichartz estimate gives

\[ \| \phi_k(t) \|_{L^p} \lesssim \| \psi_{0,k} \|_{L^p} + \sum_{i=1}^{N} \| \phi_i \|_{L_t^{2/(2-\gamma)}(L^p)}. \]

Now we can proceed as before in Subsection 3.3.

- **Step B I:** Improving the local result for \(\hat{L}^p\)-space.
  Using Lemma 3.8 we have that

\[ \left\| \int_0^t U(t-s) H_{\gamma, \psi}(\psi_k)(s) ds \right\|_{\hat{L}^p} \lesssim \sum_{i=1}^{N} \| \mathcal{H}_{a, \gamma}(\psi_1, \psi, \psi_k) \|_{L_t^r([0,t] \times \mathbb{R})}. \]

Now using Hölder, Hausdorff-Young and Hardy-Littlewood-Sobolev

\[ \| \mathcal{H}_{a, \gamma}(\psi_k(s), \psi_{\ell}(s), \psi_{m}(s)) \|_{L^r} \leq \left\| | \cdot |^{-\gamma} \ast | \psi_k(s) \psi_{\ell}(s) | \right\|_{\hat{L}^\tilde{R}} \| \psi_{m} \|_{\hat{L}^\tilde{p}} \]

\[ \lesssim \left\| \psi_k(s) \psi_{\ell}(s) \right\|_{L^\tilde{R}} \| \psi_{m} \|_{\hat{L}^\tilde{p}} \]

\[ \leq \left\| \psi_k(s) \right\|_{L^2} \| \psi_{\ell}(s) \|_{L^2} \| \psi_{m} \|_{\hat{L}^p}, \]

where

\[ \tilde{R} = \frac{3p'}{2}, \quad R = \left( \frac{5}{3} - \gamma - \frac{2}{3p} \right)^{-1}. \]

Therefore Hölder’s inequality in \(t\)-variable we have (recall \(J_{k,\ell,m}\) from (3.21), we have

\[ (3.36) \quad \| J_{k,\ell,m}(t) \|_{\hat{L}_p} \lesssim \| \psi_k \|_{L^Q([0,T_0], L^2 \mathbb{R}^n)} \| \psi_{\ell} \|_{L^Q([0,T_0], L^2 \mathbb{R}^n)} \| \psi_{m} \|_{L_t^{2/(2-\gamma)}([0,T], \hat{L}^p)} \]

\[ \lesssim T^{1-\frac{\gamma}{4}} \| \psi_k \|_{L^Q([0,T_0], L^2 \mathbb{R}^n)} \| \psi_{\ell} \|_{L^Q([0,T_0], L^2 \mathbb{R}^n)} \| \psi_{m} \|_{L^Q_{t, \gamma}(L^2 \mathbb{R})}, \]

where

\[ Q = \left( \frac{\gamma}{4} + \frac{1}{6p} - \frac{1}{6} \right)^{-1}. \]

Let \( q_1 = \frac{8}{\gamma} \) and \( r = \frac{4d}{2d-\gamma} \), and for \( T, b > 0 \), introduce the space

\[ U^T_b = \{ v \in L^\infty_t (L^2(\mathbb{R}^d) \cap \hat{L}^p(\mathbb{R}^d)) : \| v \|_{L^\infty_t (L^2 \cap \hat{L}^p)} \leq b, \| v \|_{L_t^{q_1} (L^r)} \leq b, \| v \|_{L_t^{q_2} (L^2 \mathbb{R})} \leq b \}. \]

Now we proceed as case I in subsection 3.2.

- **Step B II:** Improving the global result for \(\hat{L}^p\)-space.
By \(3.33\) and \(3.36\) we have
\[
\|\psi_k(t)\|_{L^p} \lesssim \|\psi_{0,k}\|_{L^p} + \sum_{l=1}^{N} \|\psi_l\|_{L^Q([0,T],L^{2R})} \|\psi_k\|_{L^{2/(2-\gamma)}([0,t],\hat{L}^p)} + \sum_{l=1}^{N} \|\psi_l\|_{L^Q([0,T],L^{2R})} \|\psi_l\|_{L^{2/(2-\gamma)}([0,t],\hat{L}^p)}
\]
Now \((Q,2R)\) being admissible, Strichartz estimate gives
\[
\|\psi_k(t)\|_{L^p} \lesssim \|\psi_{0,k}\|_{L^p} + \sum_{l=1}^{N} \|\psi_l\|_{L^{2/(2-\gamma)}([0,t],\hat{L}^p)}.
\]

Now we can proceed as before in Subsection 3.4.

4. ILL-POSEDNESS IN THE MERE \(\hat{L}^p\)

We recall the definition of well-posedness for the problem \((\#)\).

**Definition 4.1.** Let \((D, \| \cdot \|_D)\) be a Banach space of initial data, and \((S, \| \cdot \|_S)\) be a Banach space of space-time functions. We say that \((\#)\) is well-posed from \(D\) to \(S\) if, for all bounded subset \(B \subset D\), there exist \(T > 0\) and a Banach space \(X_T \hookrightarrow C([0,T],D)\) such that:

1. For all \(\varphi \in B\), \((\#)\) has a unique solution \(u \in X_T\) with \(u_{t=0} = \varphi\).
2. The mapping \(B \ni \varphi \mapsto u \in C([0,T],D)\) is uniformly continuous.

**Proof of Theorem 1.6** In view of \([1]\), it suffices to prove that one term in the Picard iteration of \(\Psi\) define in \((3.20)\) does not verify the Definition 4.1. We argue by contradiction and assume that \((\#)\) is well-posed in \(\hat{L}^p(\mathbb{R}^d)\). Then, let \(T > 0\) be the local existence of the solution. We recall that for \(a = 0, 0 < \alpha < \infty, 0 < \gamma < 2d(\frac{1}{p} - \frac{1}{p})\), and \(p \in (2,\infty]\), \((1.1)\) is well-posed from \(\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\) to \(\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)\) (see Theorem 1.3).

For \(\psi_0 = (\psi_{0,1}, \ldots, \psi_{0,N})\), we define the operator \(D = (D_1, \ldots, D_N)\) associated to second Picard iterate, specifically:
\[
D_k(\psi_0)(t) = -i \sum_{l=1}^{N} \int_0^t U_\alpha(t-s) \left[ (K \ast |U_\alpha(s)\psi_{0,l}|^2) U_\alpha(s)\psi_0,k \right] ds + i \sum_{l=1}^{N} \int_0^t U_\alpha(t-s) \left[ \left( K \ast \left( (U_\alpha(s)\psi_{0,k}U_\alpha(s)\psi_0,l) \right) \right) U_\alpha(s)\psi_0,l \right] ds.
\]

By \([1]\) Proposition 1] the operator \(D_k\) is continuous from \(\hat{L}^p(\mathbb{R}^d)\) to \(\hat{L}^p(\mathbb{R}^d)\), that is,
\[
\|D_k(\psi_0)(t)\|_{\hat{L}^p} \lesssim \|\psi_0\|_{(\hat{L}^p)^N}^3 \text{ for all } t \in [0,T], k = 1, \ldots, N.
\]
Let \(\psi_0 = (\psi_{0,1}, \psi_{0,2}, 0, \ldots, 0) \in \mathcal{S}(\mathbb{R}^d)^N\) be such that \(0 \neq \psi_{0,1} \neq \psi_{0,2} \neq 0\). We define a family \(\{\psi_0^h\}_{h>0}\) of functions by
\[
\psi_0^h(x) = h^\lambda \psi_0(h^\mu x), \quad (x \in \mathbb{R}^d, \mu, \lambda > 0).
\]
For all \( h > 0, \|\psi^h_{0,k}\|_{L^2} = \frac{1}{\|\psi_0\|_{L^2}} \|\psi_{0,k}\|_{L^2}. \) So for \( h > 0 \) close to 0 and \( d\mu/2 - \lambda > 0 \) the family \( \{\psi^h_{0,k}\}_{h>0} \) leaves any compact subset of \((L^2(\mathbb{R}^d))^N\). We note that
\[
\|\psi^h_{0,k}\|_{L^p} = \|\mathcal{F}\psi^h_{0,k}\|_{L^{p'}} = h^{\lambda-d\mu} \left( \int_{\mathbb{R}^d} \|\hat{\psi}_{0,k}(\xi)\|^{p'} d\xi \right)^{1/p'} = h^{\lambda-d\mu} \left( \int_{\mathbb{R}^d} \|\hat{\psi}_{0,k}(\xi)\|^{p'} d\xi \right)^{1/p'} = h^{\lambda-d\mu(1-1/p')} \|\mathcal{F}\psi^h_{0,k}\|_{L^{p'}} = h^{\lambda-d\mu/p} \|\psi^h_{0,k}\|_{L^p}.
\]

Let \( \lambda = d\mu/p \) to get \( \|\psi^h_{0,k}\|_{L^p} = \|\psi^h_{0,k}\|_{L^p} \) for all \( h > 0 \). Note that this choice of \( \lambda \) is compatible with the condition \( d\mu/2 - \lambda > 0 \) as \( 2 < p \leq \infty \). Now by (4.1) we have
\[
(4.2) \quad \|D_1(\psi^h_0)(t)\|_{L^p} \lesssim \|\psi^h_0\|^3_{(L^p)^N}, \quad \forall \ h > 0.
\]

Next, we develop the expression of \( \|D_1(\psi^h_0)(t)\|_{L^p} \). We note that
\[
\mathcal{F}\left[ \left( \hat{K} \ast \left( U_\alpha (s) \psi^h_{0,k}(s) \overline{U_\alpha (s) \psi^h_{0,l}} \right) \right) U_\alpha (s) \psi^h_{0,m} \right](\xi) \\
\lesssim \mathcal{F}\left[ \left( \hat{K} \ast \left( U_\alpha (s) \psi^h_{0,k}(s) \overline{U_\alpha (s) \psi^h_{0,l}} \right) \right) \right] \ast \mathcal{F}\left[ U_\alpha (s) \psi^h_{0,m} \right](\xi) \\
= \int_{\mathbb{R}^d} \mathcal{F}\left[ \hat{K}(\xi - y) \right] \mathcal{F}\left[ U_\alpha (s) \psi^h_{0,k}(s) \overline{U_\alpha (s) \psi^h_{0,l}} \right](\xi - y) \mathcal{F}\left[ U_\alpha (s) \psi^h_{0,m} \right](y) dy \\
\lesssim \int_{\mathbb{R}^d} \hat{K}(\xi - y) \mathcal{F}\left[ U_\alpha (s) \psi^h_{0,k}(s) \overline{U_\alpha (s) \psi^h_{0,l}} \right](\xi - y) \mathcal{F}\left[ U_\alpha (s) \psi^h_{0,m} \right](y) dy \\
\lesssim \int_{\mathbb{R}^d} \hat{K}(\xi - y) \left( \mathcal{F}\left[ U_\alpha (s) \psi^h_{0,k}(s) \overline{U_\alpha (s) \psi^h_{0,l}} \right] \right) (\xi - y) \mathcal{F}\left[ U_\alpha (s) \psi^h_{0,m} \right](y) dy \\
(4.3) \\
= \int_{\mathbb{R}^{2d}} e^{-4\pi^2 is|y|^\alpha} e^{-4\pi^2 is|\xi - y - z|^\alpha} e^{4\pi^2 is|z|^\alpha} \hat{K}(\xi - y) \mathcal{F}\psi^h_{0,k}(\xi - y - z) \mathcal{F}\psi^h_{0,l}(z) \mathcal{F}\psi^h_{0,m}(y) dydz \\
= h^2(\lambda - d\mu) \int_{\mathbb{R}^{2d}} e^{-4\pi^2 is|y|^\alpha} e^{-4\pi^2 is|\xi - y - z|^\alpha} e^{4\pi^2 is|z|^\alpha} \hat{K}(\xi - y) \psi^h_{0,k}(\xi - y - z) \psi^h_{0,l}(z) \psi^h_{0,m}(y) dydz.
\]

We may rewrite that
\[
D_1(\psi^h_0)(t) = -i \sum_{l=1}^{2} \int_0^t U_\alpha (t - s) \left[ \left( \hat{K} \ast |U_\alpha (s) \psi^h_{0,l}|^2 \right) \overline{U_\alpha (s) \psi^h_{0,1}} \right] ds \\
+ i \sum_{l=1}^{2} \int_0^t U_\alpha (t - s) \left[ \left( \hat{K} \ast \left( U_\alpha (s) \psi^h_{0,1} \overline{U_\alpha (s) \psi^h_{0,l}} \right) \right) \overline{U_\alpha (s) \psi^h_{0,l}} \right] ds \\
= -i \int_0^t U_\alpha (t - s) \left[ \left( \hat{K} \ast |U_\alpha (s) \psi^h_{0,2}|^2 \right) \overline{U_\alpha (s) \psi^h_{0,1}} \right] ds \\
+ i \int_0^t U_\alpha (t - s) \left[ \left( \hat{K} \ast \left( U_\alpha (s) \psi^h_{0,1} \overline{U_\alpha (s) \psi^h_{0,2}} \right) \right) \overline{U_\alpha (s) \psi^h_{0,2}} \right] ds.
\]
Taking the $L^p$–norm gives

\[
\|D_1(\psi^h_0(t))\|^{p}_L = \|\mathcal{F}D_1(\psi^h_0(t))\|^{p}_L
\]

\[
= \int_{\mathbb{R}^d} \left| \int_0^t \mathcal{F} \left[ U_\alpha(t-s) \left[ (K \ast |U_\alpha(s)\psi^h_{0,1}|^2) U_\alpha(s)\psi^h_{0,1} \right] \right] (\xi) ds \right|^{p'} d\xi
\]

\[
= \int_{\mathbb{R}^d} \left| \int_0^t e^{-4\pi^2i(t-s)\xi^\alpha} \mathcal{F} \left[ (K \ast |U_\alpha(s)\psi^h_{0,1}|^2) U_\alpha(s)\psi^h_{0,1} \right] (\xi) ds \right|^{p'} d\xi
\]

\[
= :I.
\]

Performing change of variables ($\xi \mapsto \xi/h, y \mapsto y/h, z \mapsto z/h, s \mapsto s/h^{\alpha\mu}$), we obtain

\[
\int_0^t e^{-4\pi^2i(t-s)\xi^\alpha} \mathcal{F} \left[ (K \ast \left( U_\alpha(s)\psi^h_{0,k} U_\alpha(s)\psi^h_{0,l} \right) \right] U_\alpha(s)\psi^h_{0,m} (\xi) ds
\]

\[
= \int_0^t e^{-4\pi^2i(t-s)} h^{3(\lambda-d\mu)} \int_{\mathbb{R}^{2d}} e^{-4\pi^2i|y|^\alpha} e^{-4\pi^2is|\xi-u-y-z|^\alpha} e^{4\pi^2is|z|^\alpha} \tilde{K}(\xi-y)
\]

\[
\tilde{\psi}_{0,k} \left( \frac{\xi}{h^\mu} \right) \tilde{\psi}_{0,l} \left( \frac{\xi}{h^\mu} \right) \tilde{\psi}_{0,m} \left( \frac{y}{h^\mu} \right) dy dz ds
\]

\[
= \int_0^t e^{-4\pi^2i(t-s)} h^{3\lambda-d\mu} \int_{\mathbb{R}^{2d}} e^{-4\pi^2i|y|^\alpha} e^{-4\pi^2is|\xi-h^\mu(y+z)|^\alpha} e^{4\pi^2is|z|^\alpha} \tilde{K}(\xi-h^\mu y)
\]

\[
\tilde{\psi}_{0,k} \left( \frac{\xi}{h^\mu} - y - z \right) \tilde{\psi}_{0,l} \left( z \right) \tilde{\psi}_{0,m} \left( y \right) dy dz ds
\]

\[
= h^{3\lambda-d\mu-\alpha\mu} \int_0^t e^{-4\pi^2i(t-h^{-\alpha\mu}s)} \mathcal{F} \left[ (K \ast e^{-4\pi^2ish^{-\alpha\mu}|\xi-h^{-\alpha\mu}(y+z)|^\alpha} e^{4\pi^2is|z|^\alpha} \tilde{K}(\xi-h^\mu y) \right]
\]

\[
\tilde{\psi}_{0,k} \left( \frac{\xi}{h^\mu} - y - z \right) \tilde{\psi}_{0,l} \left( z \right) \tilde{\psi}_{0,m} \left( y \right) dy dz ds.
\]
In view of this and since the kernel is homogeneous in the Hartree factor (as $\alpha = 0$), we may rewrite

$$I = \int_{\mathbb{R}^d} |h^{3\lambda-\alpha} - \alpha\mu| \int_{0}^{\theta_{0}} e^{-4\pi^2 i (t - h^{-\alpha}_{\mu}s) \xi} |\xi|^\alpha \int_{\mathbb{R}^{2d}} e^{-4\pi^2 i s y |\xi|} e^{-4\pi^2 i s h^{-\alpha}_{\mu}(\xi - h^{-\alpha}_{\mu}(y + z)) |\xi|} e^{4\pi^2 i s z |\xi|}$$

$$\tilde{K}(\xi - h^{-\alpha}_{\mu} y) \left( \tilde{\psi}_{0,2} \left( \frac{\xi}{h^{-\alpha}_{\mu} - y - z} \tilde{\psi}_{0,1}(y) - \tilde{\psi}_{0,1} \left( \frac{\xi}{h^{-\alpha}_{\mu} - y - z} \tilde{\psi}_{0,2}(y) \right) \right) \tilde{\psi}_{0,2}(z) dydzds \right)^{p'} d\xi$$

$$= h^{d\mu + (3\lambda - d\mu - \alpha\mu)p'} \int_{\mathbb{R}^d} \int_{0}^{\theta_{0}} e^{-4\pi^2 i (th^{-\alpha}_{\mu}s) \xi} |\xi|^\alpha \int_{\mathbb{R}^{2d}} e^{-4\pi^2 i s y |\xi|} e^{-4\pi^2 i s |\xi|} e^{4\pi^2 i s z |\xi|}$$

$$\tilde{K}(h^{-\alpha}_{\mu} - y - z) \tilde{\psi}_{0,2}(y) - \tilde{\psi}_{0,1} \left( \frac{\xi}{h^{-\alpha}_{\mu} - y - z} \tilde{\psi}_{0,2}(y) \right) \tilde{\psi}_{0,2}(z) dydzds \right)^{p'} d\xi.$$ 

Using (4.3), we have

$$I = h^{d\mu + (3\lambda - 2d\mu - \alpha\mu + \alpha\gamma)p'} \int_{\mathbb{R}^d} \int_{0}^{\theta_{0}} e^{-4\pi^2 i (th^{-\alpha}_{\mu}s) \xi} |\xi|^\alpha \mathcal{F} \left[ (K * |U_{0,2}(s)|^2) U_{0,2}(s) \psi_{0,2}(s) \right] (\xi) ds -$$

$$- \int_{0}^{\theta_{0}} e^{-4\pi^2 i (th^{-\alpha}_{\mu}s) \xi} |\xi|^\alpha \mathcal{F} \left[ (K * \left( U_{0,2}(s) \overline{U_{0,2}(s)} \psi_{0,2}(s) \right) U_{0,2}(s) \right] (\xi) ds \right)^{p'} d\xi$$

$$= h^{d\mu + (3\lambda - 2d\mu - \alpha\mu + \alpha\gamma)p'} ||D(\psi_0)(th^{-\alpha}_{\mu})||_{L^p}^{p'}.$$ 

Since $\lambda = d\mu/p$, combining the above equalities, we obtain

$$||D_1(\psi_0^h(t)) ||_{L^p} \approx h^{2\lambda p/d\mu - \alpha\mu + \alpha\gamma} ||D(\psi_0)(th^{-\alpha}_{\mu})||_{L^p}.$$ 

Next we investigate more closely the term

$$F(t) = \int_{0}^{t} U_{0,1}(t - s) g(s) ds,$$

where

$$g(s) = -i \sum_{l=1}^{2} \left[ (K * |U_{0,1}(s)|^2) U_{0,1}(s) \right]$$

$$+ i \sum_{l=1}^{2} \left[ (K * \left( U_{0,1}(s) \overline{U_{0,1}(s)} \right) U_{0,1}(s) \right]$$

$$= -i (K * |U_{0,2}(s)|^2) U_{0,2}(s) + i \left( (K * \left( U_{0,2}(s) \overline{U_{0,2}(s)} \right) U_{0,2}(s) \right).$$ 

Taylor formula gives

$$F(t) = F(0) + F'(0) t + \frac{t^2}{2} \int_{0}^{1} (1 - \theta) F''(t\theta) d\theta.$$
Note that $F(0) = 0$ and hence for $0 \leq t \leq 1$, we have

\begin{equation}
(4.5) \quad \|F(t) - F'(0)t\|_{\tilde{L}^p} \leq t^2 \int_0^1 \|F''(s\theta)\|_{\tilde{L}^p} d\theta \leq t^2 \|F''\|_{L^\infty([0,1],\tilde{L}^p)}.
\end{equation}

By Leibniz integral rule and below Lemma 4.1, the first derivative of $F$ is given by

\[ F'(t) = U_\alpha(0)g(t) + \int_0^t \frac{\partial}{\partial t} U_\alpha(t-s)g(s)ds \]

and similarly the second derivative of $F$ is given by

\[ F''(t) = g'(t) - i(-\Delta)^{\alpha/2}g(t) - \int_0^t U_\alpha(t-s)(-\Delta)^{\alpha}g(s)ds. \]

Hence, we have

\begin{equation}
(4.6) \quad \|F''\|_{L^\infty([0,1],\tilde{L}^p)} \leq \|g'\|_{L^\infty([0,1],\tilde{L}^p)} + \|(-\Delta)^{\alpha/2}g\|_{L^\infty([0,1],\tilde{L}^p)} + \|(-\Delta)^{\alpha}g\|_{L^\infty([0,1],\tilde{L}^p)} < \infty,
\end{equation}

as $\psi_0 \in \mathcal{S}(\mathbb{R}^d)^N$. Using (4.5) and (4.6), we have

\[ \|F(t) - F'(0)t\|_{\tilde{L}^p} \lesssim t^2 \]

Using this and $F'(0) = g(0)$, we obtain

\[ t\|g(0)\|_{\tilde{L}^p} \lesssim \|F(t)\|_{\tilde{L}^p} + t^2. \]

Hence in particular

\[ th^{\alpha\mu}\|g(0)\|_{\tilde{L}^p} \lesssim \|D_1(\psi_0)(th^{\alpha\mu})\|_{\tilde{L}^p} + t^2h^{2\alpha\mu} \]

and so by (4.4)

\[ \|D_1(\psi_0^h)(t)\|_{\tilde{L}^p} \asymp h^{2d\mu/p-d\mu-\alpha\mu+\alpha\gamma}\|D_1(\psi_0)(th^{\alpha\mu})\|_{\tilde{L}^p} \]

\[ \gtrsim th^{2d\mu/p-d\mu+\alpha\gamma}\|g(0)\|_{\tilde{L}^p} - t^2h^{2d\mu/p-d\mu+\alpha\mu+\alpha\gamma}. \]

Putting $\mu = 1$, we have

\[ 2d\mu/p - d\mu + \mu\gamma = 2d/p - d + \gamma. \]

Note that the above quantity is negative if $\gamma < d - 2d/p = 2d(\frac{1}{2} - \frac{1}{p})$. Now since $\alpha > 0$, we have

\[ \|D_1(\psi_0^h)(t)\|_{\tilde{L}^p} \gtrsim th^{2d/p-d+\gamma}\|g(0)\|_{\tilde{L}^p} - t^2h^{2d/p-d+\gamma+\alpha} \rightarrow \infty \]

as $h \to 0$ and this contracts (4.2). This completes the proof. \qed

**Lemma 4.1.** Let $f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R})$. Then for all $s,t \in \mathbb{R}$, we have

\[ \frac{\partial}{\partial t} U_\alpha(t-s)f(s) = -iU_\alpha(t-s)(-\Delta)^{\alpha/2}f(s). \]
Proof. Let $v_s(t) = U_\alpha(t) f(s)$ ($s \in \mathbb{R}$). Then $v_s$ solves
\[
\begin{cases}
  \partial_t v_s + i(-\Delta)^{\alpha/2} v_s = 0 \\
  v_s|_{t=0} = f(s).
\end{cases}
\]
Hence $\partial_t v_s(t) = -i(-\Delta)^{\alpha/2} v_s(t)$ for all $t \in \mathbb{R}$. Therefore
\[
\frac{\partial}{\partial t} U_\alpha(t - s) f(s) = \partial_t v_s(t - s) = -i(-\Delta)^{\alpha/2} v_s(t - s) = -i(-\Delta)^{\alpha/2} U_\alpha(t - s) f(s).
\]
Note that operators $(-\Delta)^{\alpha/2}$ and $U_\alpha$ commute. Indeed, for $h \in \mathcal{S}(\mathbb{R}^d)$, we have
\[
\mathcal{F} \left[ (-\Delta)^{\alpha/2} U_\alpha(t) h \right] = c |\xi|^{\alpha} \mathcal{F} \left[ U_\alpha(t) h \right]
= c |\xi|^{\alpha} e^{-4\pi^2 t |\xi|^2} \hat{h}
= e^{-4\pi^2 t |\xi|^2} \mathcal{F} \left[ (-\Delta)^{\alpha/2} h \right]
= \mathcal{F} \left[ U_\alpha(t)(-\Delta)^{\alpha/2} h \right].
\]
It follows that
\[
\frac{\partial}{\partial t} U_\alpha(t - s) f(s) = -i U_\alpha(t - s)(-\Delta)^{\alpha/2} f(s).
\]
This completes the proof. \qed

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