Categorically Closed Unipotent Semigroups

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Abstract: Let $C$ be a class of $T_1$ topological semigroups, containing all Hausdorff zero-dimensional topological semigroups. A semigroup $X$ is $C$-closed if $X$ is closed in any topological semigroup $Y \in C$ that contains $X$ as a discrete subsemigroup; $X$ is injectively $C$-closed if for any injective homomorphism $h : X \to Y$ to a topological semigroup $Y \in C$ the image $h[X]$ is closed in $Y$. A semigroup $X$ is unipotent if it contains a unique idempotent. It is proven that a unipotent commutative semigroup $X$ is (injectively) $C$-closed if and only if $X$ is bounded and nonsingular (and group-finite). This characterization implies that for every injectively $C$-closed unipotent semigroup $X$, the center $Z(X)$ is injectively $C$-closed.

Keywords: $C$-closed semigroup; unipotent semigroup

1. Introduction and Main Results

In many cases, the completeness properties of various objects of general topology or topological algebra can be characterized externally as closedness in ambient objects. For example, a metric space $X$ is complete if and only if $X$ is closed in any metric space containing $X$ as a subspace. A uniform space $X$ is complete if and only if $X$ is closed in any uniform space containing $X$ as a uniform subspace. A topological group $G$ is Raïkov complete if and only if it is closed in any topological group containing $G$ as a subgroup.

On the other hand, for topological semigroups there are no reasonable notions of (inner) completeness. Nonetheless, one can define many completeness properties of semigroups via their closedness in ambient topological semigroups.

A topological semigroup is a topological space $X$ endowed with a continuous associative binary operation $X \times X \to X$, $(x, y) \mapsto xy$.

Definition 1. Let $C$ be a class of topological semigroups. A topological semigroup $X$ is called

- $C$-closed if for any isomorphic topological embedding $h : X \to Y$ to a topological semigroup $Y \in C$, the image $h[X]$ is closed in $Y$;
- injectively $C$-closed if for any injective continuous homomorphism $h : X \to Y$ to a topological semigroup $Y \in C$, the image $h[X]$ is closed in $Y$;
- absolutely $C$-closed if for any continuous homomorphism $h : X \to Y$ to a topological semigroup $Y \in C$, the image $h[X]$ is closed in $Y$.

For any topological semigroup we have the implications:

absolutely $C$-closed $\Rightarrow$ injectively $C$-closed $\Rightarrow$ $C$-closed.

Definition 2. A semigroup $X$ is defined to be (injectively, absolutely) $C$-closed if it is $X$ endowed with the discrete topology.
In this paper, we are interested in the (absolute, injective) $C$-closedness for the classes:

- $T_1 S$ of topological semigroups satisfying the separation axiom $T_1$;
- $T_2 S$ of Hausdorff topological semigroups;
- $T_z S$ of Hausdorff zero-dimensional topological semigroups.

A topological space satisfies the separation axiom $T_1$ if all its finite subsets are closed. A topological space is zero-dimensional if it has a base of the topology consisting of clopen ($=\text{closed-and-open}$) sets.

Since $T_z S \subseteq T_2 S \subseteq T_1 S$, for every semigroup the following implications hold:

$$\begin{align*}
\text{absolutely } T_1 S\text{-closed} & \implies \text{absolutely } T_2 S\text{-closed} & \implies \text{absolutely } T_z S\text{-closed} \\
\text{injectively } T_1 S\text{-closed} & \implies \text{injectively } T_2 S\text{-closed} & \implies \text{injectively } T_z S\text{-closed} \\
T_1 S\text{-closed} & \implies T_2 S\text{-closed} & \implies T_z S\text{-closed}.
\end{align*}$$

From now on, we assume that $C$ is a class of topological semigroups such that $T_2 S \subseteq C \subseteq T_1 S$.

Semigroups having one of the above closedness properties are called categorically closed. Categorically closed topological groups and semilattices were investigated in [1–11] and [12–15], respectively. This paper is a continuation of the papers [3,15,16], which contain inner characterizations of semigroups possessing various categorically closed properties.

In this paper we shall characterize (absolutely and injectively) $C$-closed unipotent semigroups.

A semigroup $X$ is called

- unipotent if $X$ contains a unique idempotent;
- chain-finite if any infinite set $I \subseteq X$ contains elements $x,y \in I$ such that $xy \notin \{x,y\}$;
- singular if there exists an infinite set $A \subseteq X$ such that $AA$ is a singleton;
- periodic if for every $x \in X$ there exists $n \in \mathbb{N}$ such that $x^n$ is an idempotent;
- bounded if there exists $n \in \mathbb{N}$ such that for every $x \in X$ the $n$-th power $x^n$ is an idempotent;
- group-finite if every subgroup of $X$ is finite;
- group-bounded if every subgroup of $X$ is bounded.

The following characterization of $C$-closed commutative semigroups was proved in the paper [16].

**Theorem 1.** A commutative semigroup is $C$-closed if and only if it is chain-finite, periodic, nonsingular and group-bounded.

For unipotent semigroups, this characterization can be simplified as follows:

**Theorem 2.** A unipotent semigroup $X$ is $C$-closed if and only if $X$ is bounded and nonsingular.

Another principal result of this paper is the following characterization of injectively $C$-closed unipotent semigroups.

**Theorem 3.** A unipotent commutative semigroup $X$ is injectively $C$-closed if and only if $X$ is bounded, nonsingular and group-finite.
Example 1. For an infinite cardinal $\kappa$, the Taimanov semigroup $T_\kappa$ is the set $\kappa$ endowed with the semigroup operation

$$xy = \begin{cases} 1, & \text{if } x \neq y \text{ and } x, y \in \kappa \setminus \{0, 1\}; \\ 0, & \text{otherwise.} \end{cases}$$

The semigroup $T_\kappa$ was introduced by Taimanov in [17]. Its algebraic and topological properties were investigated by Gutik [18] who proved that the semigroup $T_\kappa$ is injectively $T_1 S$-closed. The same also follows for Theorem 3 because the semigroup $T_\kappa$ is unipotent, bounded, nonsingular and group-finite. The Taimanov semigroups witness that there exist injectively $T_1 S$-closed unipotent semigroups of arbitrarily high cardinality.

For a semigroup $X$, let

$$Z(X) \overset{\text{def}}{=} \{z \in X : \forall x \in X \ (xz = zx)\}$$

be the center of $X$. The center of an (injectively) $C$-closed semigroup has the following properties, proven in Lemmas 5.1, 5.3, 5.4 of [16] (and Theorem 1.7 of [19]).

**Theorem 4.** The center $Z(X)$ of any (injectively) $C$-closed semigroup is chain-finite, periodic, nonsingular (and group-finite).

**Corollary 1.** The center $Z(X)$ of an injectively $C$-closed unipotent semigroup $X$ is injectively $C$-closed.

**Proof.** By Theorem 4, the semigroup $Z(X)$ is chain-finite, periodic, nonsingular, and group-finite. By Theorem 1, the semigroup $Z(X)$ is $C$-closed. By Theorem 2, $Z(X)$ is bounded. If $Z(X)$ is empty, then $Z(X)$ is injectively $C$-closed. So, we assume that $Z(X) \neq \emptyset$. Being bounded, the semigroup $Z(X)$ contains an idempotent. Being a subsemigroup of the unipotent semigroup $X$, the semigroup $Z(X)$ is unipotent. By Theorem 3, the unipotent bounded nonsingular group-finite semigroup $Z(X)$ is injectively $C$-closed. \(\blacksquare\)

Another corollary of Theorem 3 describes the center of an absolutely $C$-closed unipotent semigroup.

**Corollary 2.** The center $Z(X)$ of an absolutely $C$-closed unipotent semigroup $X$ is finite and hence absolutely $C$-closed.

**Proof.** By Theorem 4, the semigroup $Z(X)$ is chain-finite, periodic, nonsingular, and group-finite. If $Z(X)$ is empty, then $Z(X)$ is finite and hence absolutely $C$-closed. So, we assume that $Z(X)$ is not empty. Being periodic, the semigroup $Z(X)$ contains an idempotent $e$. Since $X$ is unipotent, $e$ is a unique idempotent of the semigroups $X$ and $Z(X)$. Let $H_e$ be the maximal subgroup of the semigroup $Z(X)$. The group $H_e$ is finite because $Z(X)$ is group-finite. By Theorem 1.7 of [19], the complement $Z(X) \setminus H_e$ is finite and hence the set $Z(X)$ is finite, too. \(\blacksquare\)

Corollaries 1 and 2 suggest the following open problems.

**Problem 1.**
1. Is the center of a $C$-closed semigroup $C$-closed?
2. Is the center of an injectively $C$-closed semigroup injectively $C$-closed?
3. Is the center of an absolutely $C$-closed semigroup absolutely $C$-closed?

2. Preliminaries

We denote by $\omega$ the set of finite ordinals and by $\mathbb{N} \overset{\text{def}}{=} \omega \setminus \{0\}$ the set of positive integer numbers.
For an element \( a \) of a semigroup \( X \) the set
\[
H_a \overset{\text{def}}{=} \{ x \in X : (xX^1 = aX^1) \land (X^1x = X^1a) \}
\]
is called the \( H \)-class of \( a \). Here \( X^1 \overset{\text{def}}{=} X \cup \{1\} \) where 1 is an element such that \( 1x = x = x1 \) for all \( x \in X^1 \).

By Corollary 2.2.6 [20], for every idempotent \( e \) of a semigroup \( X \) its \( H \)-class \( H_e \) coincides with the maximal subgroup of \( X \), containing the idempotent \( e \).

For a subset \( A \) of a semigroup \( X \) and a positive integer number \( n \), let
\[
\sqrt[n]{A} = \{ x \in X : x^n \in A \} \quad \text{and} \quad \overline{\sqrt[n]{A}} = \bigcup_{n \in \mathbb{N}} \sqrt[n]{A} = \{ x \in X : A \cap x^N \neq \emptyset \},
\]
where
\[
x^n = \{ x^n : n \in \mathbb{N} \}
\]
is the monogenic semigroup generated by \( x \).

The following lemma is proven in [16], 3.1.

**Lemma 1.** For any idempotent \( e \) of a semigroup, \( (\sqrt[n]{e} \cdot H_e) \cup (H_e \cdot \sqrt[n]{e}) \subseteq H_e \).

**3. Proof of Theorem 2**

Theorem 2 will be derived from the following lemmas.

**Lemma 2.** Let \( X \) be a periodic commutative semigroup with a unique idempotent \( e \) and trivial maximal subgroup \( H_e \). If \( X \) is not bounded, then there exists an infinite subset \( A \subseteq X \) such that \( AA = \{ e \} \).

**Proof.** To derive a contradiction, assume that \( X \) is not bounded but for every infinite set \( A \subseteq X \) we have \( AA \neq \{ e \} \). Taking into account that \( X \) is periodic and unipotent, we conclude that \( X = \sqrt[n]{e} \). By Lemma 1, the maximal subgroup \( H_e = \{ e \} \) is an ideal in \( X \).

Inductively we shall construct a sequence of points \( (x_k)_{k \in \omega} \) and a sequence of positive integer numbers \( (n_k)_{k \in \omega} \) such that for every \( k \in \omega \) the following conditions are satisfied:

(i) \( x_k^{n_k} \notin \{ e \} \cup \{ x_i^{n_i} : i < k \} \);
(ii) \( x_k^{n_k} = e \);
(iii) \( \max_{i < k} |x_i^{n_k}X| < n_k \).

To start the inductive construction, take any \( x_0 \in X \setminus \{ e \} \) and let \( n_0 \) be the smallest number such that \( x_0^{n_0+1} = e \). Such number \( n_0 \) exists as \( X \) is periodic. Since \( \{ e \} = H_e \) is an ideal in \( X \), it follows from \( x_0^{n_0+1} = e \) and \( 2n_0 \geq n_0 + 1 \) that \( x_0^{2n_0} = e \). Assume that for some \( k \in \mathbb{N} \), we have chosen sequences \( (x_i)_{i < k} \) and \( (n_i)_{i < k} \). For every \( i < k \), consider the set \( x_i^{n_i}X \) and observe that for every \( a, b \in X \) the inductive condition (ii) implies
\[
(x_i^{n_i}a)(x_i^{n_i}b) = x_i^{2n_i}ab = eab = e.
\]

This means that \( (x_i^{n_i}X)^2 = \{ e \} \) and by our assumption, the set \( x_i^{n_i}X \) is finite. Since \( X \) is unbounded, there exists an element \( x_k \in X \) and a number \( m_k > k + \max_{i < k} |x_i^{n_i}X| \) such that \( x_k^{k+m_k} \neq e \) but \( x_k^{1+k+m_k} = e \). Since the set \( \{ x_k^j : m_k \leq j \leq m_k + k \} \) consists of \( k + 1 \) points, there exist a number \( n_k \in \{ m_k, \ldots, m_k + k \} \) such that \( x_k^{n_k} \notin \{ x_i^{n_i} : i < k \} \). It follows from \( x_k^{1+m_k} \neq e = x_k^{1+k+m_k} \) and \( 2n_k \geq m_k + m_k + 1 + k + m_k > n_k \) that \( x_k^{n_k} \neq e = x_k^{2n_k} \). This completes the inductive step.

After completing the inductive construction, consider the infinite set \( A = \{ x_i^{n_i} : k \in \omega \} \). We claim that \( x_i^{n_i}x_k^{n_k} = e \) for any \( i \leq k \). For \( i = k \) this follows from the inductive condition (ii). So, assume that \( i < k \). By the induction condition (iii) and the Pigeonhole
Principle, there exist two positive numbers \( j < j' \leq |x|^i_X| + 1 \leq n \) such that \( x^{n_i}x_j^i = x^{n_j}x_j^j \).

Let \( d = j' - j \leq n \) and observe that \( x^{n_i}x_k^j = x^{n_j}x_k^j = x^{n_i}x_k^{j+d} \). Then

\[
x^{n_i}x_k^j = x^{n_i}x_k^{j+d} x_{d}^j = x^{n_i}x_k^d = x^{n_i}x_k^{j+d} = x^{n_i}x_k^j.
\]

Proceeding by induction, we can prove that \( x^{n_i}x_k = x^{n_k}x_k^{j+pd} \) for every \( p \in \omega \). Since \( X \) is periodic and \( \{e\} = H_e \) is an ideal in \( X \), there exists \( p \in \mathbb{N} \) such that \( x_k^{j+pd} = e \) and hence \( x_k^j = x_k^{j+pd} = e \). Then, \( x_k^i = x_k = x_k^{j+pd} = e \).

Lemma 3. Let \( X \) be a periodic commutative semigroup with a unique idempotent \( e \) and bounded maximal subgroup \( H_e \). If \( X \) is not bounded, then there exists an infinite subset \( A \subseteq X \) such that \( AA = \{e\} \).

Proof. Since \( H_e \) is bounded, there exists a number \( p \in \mathbb{N} \) such that \( x^p = e \) for all \( x \in H_e \). Assuming that \( X \) is not bounded, we conclude that the subsemigroup \( P = \{x^p : x \in \mathbb{X}\} \) of \( X \) is not bounded. We claim that \( P \cap H_e = \{e\} \). Indeed, for every \( x \in X \) with \( x^p \in H_e \), we have \( xe \in H_e \) by Lemma 1 and hence \( x^p = x^p e = (xe)^p = e \). Since the maximal subgroup of \( P \) is trivial, one can apply Lemma 2 and find an infinite set \( A \subseteq P \subseteq X \) such that \( AA = \{e\} \).

Our final lemma implies Theorem 2.

Lemma 4. For a unipotent commutative semigroup \( X \), the following conditions are equivalent:
1. \( X \) is C-closed;
2. \( X \) is periodic, nonsingular and group-bounded;
3. \( X \) is bounded and not singular.

Proof. The equivalence \((1) \Leftrightarrow (2)\) follows from Theorem 1, and \((3) \Rightarrow (2)\) is trivial. The implication \((2) \Rightarrow (3)\) follows from Lemma 3.

4. Proof of Theorem 3

In this section we prove Lemmas 5 and 6 implying the “only if” and “if” parts of the characterization Theorem 3, respectively.

Lemma 5. If a unipotent semigroup \( X \) is injectively \( C \)-closed, then its center \( Z(X) \) is bounded, nonsingular, and group-finite.

Proof. By Theorem 4, the semigroup \( Z(X) \) is periodic, nonsingular and group-finite. If \( Z(X) \) is not empty, then \( Z(X) \) is bounded. If \( Z(X) \) is empty, then by the periodicity, \( Z(X) \) contains an idempotent and hence is unipotent, being a subsemigroup of the unipotent semigroup \( X \). By Lemma 4, \( Z(X) \) is bounded.

Lemma 6. Every bounded nonsingular group-finite unipotent commutative subsemigroup \( X \) of a \( T_1 \) topological semigroup \( Y \) is closed and discrete in \( Y \).

Proof. Replacing \( Y \) by the closure of \( X \), we can assume that \( X \) is dense in \( Y \).

Claim 1. For every \( x \in X \) and \( y \in Y \) there exists a neighborhood \( U \subseteq Y \) of \( y \) such that the set \( x(U \cap X) \) is finite.

Proof. To derive a contradiction, assume that there exists \( a \in X \) and \( y \in Y \) such that for every neighborhood \( U \subseteq Y \) of \( y \) the set \( a(U \cap X) \) is infinite. The periodicity of \( X \) ensures
that \( X = \bigcup_{e \in \mathbb{N}} \sqrt[\ell]{H_e} \) where \( \sqrt[\ell]{H_e} = \{ x \in X : x^\ell \in H_e \} \subseteq \ell^{1/\sqrt[\ell]{H_e}} \), see Lemma 1. This lemma also implies that the set
\[ \sqrt[\ell]{H_e} \cdot Y = H_eY = H_eX \subseteq H_eX = H_e \]
is finite.

Let \( k \) be the largest number such that for every \( x \in \sqrt[\ell]{H_e} \) there exists a neighborhood \( U \subseteq Y \) of \( y \) such that the set \( x(U \cap X) \) is finite. Since \( 1 \leq k < \min\{ \ell : a \in \sqrt[\ell]{H_e} \} \), the number \( k \) is well-defined.

**Subclaim 1.** For every \( x \in \sqrt[\ell]{H_e} \) there exists a neighborhood \( V \subseteq Y \) of \( y \) such that the set \( xV \) is a singleton in \( X \).

**Proof.** By the choice of \( k \), for every \( x \in \sqrt[\ell]{H_e} \) there exists a neighborhood \( U \subseteq Y \) of \( y \) such that the set \( x(U \cap X) \) is finite and hence closed in the \( T_1 \)-space \( Y \). Then
\[ xy \in xU \subseteq x(U \cap X) \subseteq x(U \cap X) = x(U \cap X) \subseteq X. \]

Since the space \( Y \) is \( T_1 \) there exists an open neighborhood \( W \) of \( xy \) such that \( W \cap x(U \cap X) = \{ xy \} \). By the continuity of the semigroup operation, the point \( y \) has an open neighborhood \( V \subseteq U \) such that \( xV \subseteq W \). Then,
\[ xV \subseteq xV \cap X \subseteq x(V \cap X) \subseteq xV \cap x(U \cap X) = \{ xy \} = \{ xy \} \subseteq X. \]

\[ \square \]

By the maximality of \( k \), there exists \( b \in \sqrt[k+1]{\sqrt[\ell]{H_e}} \) such that for every neighborhood \( V \subseteq Y \) of \( y \) the set \( b(V \cap X) \) is infinite. It follows from \( b \in \sqrt[k+1]{\sqrt[\ell]{H_e}} \) that \( b^{k+1} \in H_e \) and hence \( (b^2)^k = b^{2k} = b^{k+1}b^{k-1} \in H_eX^1 \subseteq H_e \) and hence \( b^2 \in \sqrt[k+1]{\sqrt[\ell]{H_e}} \). By Subclaim 1, there exists a neighborhood \( U \subseteq Y \) of \( y \) such that the set \( b^2U \) is a singleton in \( X \). Choose any \( u \in U \cap X \). By Lemma 1, \( (b^2u)^k = b^{2k}u^k \in H_eX \subseteq H_e \) and \( b^2Y = b^2u \in \sqrt[k+1]{\sqrt[\ell]{H_e}} \). By Subclaim 1, there exists a neighborhood \( V \subseteq U \) such that \( (b^2Y)V \) is a singleton in \( X \). Then, the set \( A = b(V \cap X) \) is infinite but
\[ AA \subseteq b^2VV \subseteq b^2UV = (b^2Y)V \]
is a singleton. However, this contradicts the nonsingularity of \( X \). \[ \square \]

**Claim 2.** For every \( x \in X \) and \( y \in Y \) there exists a neighborhood \( V \subseteq Y \) of \( y \) such that \( xV \) is a singleton in \( X \).

**Proof.** By Claim 1, there exists a neighborhood \( U \subseteq Y \) of \( y \) such that the set \( x(U \cap X) \) is finite and hence closed in the \( T_1 \)-space \( Y \). Then,
\[ xy \in xU \subseteq x(U \cap X) \subseteq x(U \cap X) = x(U \cap X) \subseteq X. \]

Since the space \( Y \) is \( T_1 \), there exists an open neighborhood \( W \) of \( xy \) such that \( W \cap x(U \cap X) = \{ xy \} \). By the continuity of the semigroup operation, the point \( y \) has an open neighborhood \( V \subseteq U \) such that \( xV \subseteq W \). Then,
\[ xV \subseteq xV \cap X \subseteq x(V \cap X) \subseteq xV \cap x(U \cap X) = \{ xy \} = \{ xy \} \subseteq X. \]

\[ \square \]

**Claim 3.** For every \( n \in \mathbb{N}, x \in X \) and \( y \in Y \), there exists a neighborhood \( V \subseteq Y \) of \( y \) such that \( xV^n \) is a singleton in \( X \).
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Proof. For \( n = 1 \) the statement follows from Claim 2. Assume that for some \( n \in \mathbb{N} \) we know that for every \( x \in X \) and \( y \in Y \) there exists a neighborhood \( U \subseteq Y \) of \( y \) such that \( xU^n \) is a singleton \( \{a\} \) in \( X \). By Claim 2, there exists a neighborhood \( V \subseteq U \) of \( y \) such that \( aV \) is a singleton in \( X \). Then \( xV^{n+1} \subseteq xV^nV = aV \) is a singleton in \( X \).

Claim 4. For every \( k \in \mathbb{N} \) the subspace \( \sqrt[k]{Y} = \{x \in X : x^k \in H_e\} \) of \( Y \) is discrete.

Proof. To derive a contradiction, assume that for some \( k \in \mathbb{N} \) the subspace \( \sqrt[k]{Y} \) is not discrete and let \( k \) be the smallest number with this property. Since \( \sqrt[k]{Y} = H_e \) is finite, \( k > 1 \). Let \( y \) be a non-isolated point of \( \sqrt[k]{Y} \). It follows that \( y^k \in H_e \) and \( (y^2)^{k-1} = y^2y^{k-2} \in H_e \) and hence \( y^2 \in \sqrt[k-1]{Y} \). By the minimality of \( k \), the space \( \sqrt[k-1]{Y} \) is discrete. By the continuity of the semigroup operation, there exists a neighborhood \( V_0 \subseteq Y \) of \( y \) such that \( V_0V_0 \cap \sqrt[k-1]{Y} = \{y^2\} \). By Claim 2, we can additionally assume that \( V_0g = \{y^2\} \).

By induction we shall construct a sequence of points \( (x_n)_{n \in \omega} \) in \( \sqrt[k]{Y} \) and a decreasing sequence \( (V_n)_{n \in \omega} \) of open sets in \( Y \) such that for every \( n \in \omega \) the following conditions are satisfied:

(i) \( x_n \in V_n \cap \sqrt[k]{Y} \setminus \{x_i\}_{i<n} \);
(ii) \( y \in V_{n+1} \subseteq V_n \) and \( x_nV_{n+1} = \{y^2\} \).

Assume that for some \( n \in \omega \) we have chosen a neighborhood \( V_n \) of \( y \) and a sequence of points \( \{x_i\}_{i<n} \). Since \( y \) is a non-isolated point of \( \sqrt[k]{Y} \), there exists a point \( x_n \) satisfying the inductive condition (i). Observe that \( x_ny \in V_0Y = \{y^2\} \). By Claim 2, there exists a neighborhood \( V_{n+1} \subseteq V_n \) of \( y \) such that \( x_nV_{n+1} = \{x_ny\} = \{y^2\} \). This completes the inductive step.

After completing the inductive construction, we obtain the infinite set \( A = \{x_n\}_{n \in \omega} \subseteq X_k \) such that \( AA = \{y^2\} \). However, this contradicts the nonsingularity of \( X \).

Claim 5. For every \( k \in \mathbb{N} \) the set \( \sqrt[k]{Y} \) is closed in \( Y \).

Proof. To derive a contradiction, assume that for some \( k \) the set \( \sqrt[k]{Y} \) is not closed in \( Y \). We can assume that \( k \) is the smallest number with this property. Since \( \sqrt[k]{Y} = H_e \) is finite, \( k > 1 \) and hence \( \{x^2 : x \in \sqrt[k]{Y}\} \subseteq \sqrt[k]{Y} \). Fix any point \( y \in \sqrt[k]{Y} \setminus \sqrt[k]{Y} \) and observe that \( y^2 \in \{x^2 : x \in \sqrt[k]{Y}\} \subseteq \sqrt[k]{Y} \). See Lemma 1. By Claim 4, the space \( \sqrt[k]{Y} \) is discrete. Consequently, there exists a neighborhood \( U \subseteq Y \) of \( y \) such that \( UU \cap \sqrt[k]{Y} = \{y^2\} \). Since \( y \in \sqrt[k]{Y} \setminus \sqrt[k]{Y} \), the set \( A = U \cap \sqrt[k]{Y} \) is infinite and \( AA \subseteq UU \cap \sqrt[k]{Y} = \{y^2\} \), which contradicts the nonsingularity of \( X \).

The boundedness of \( X \) implies that \( X = \sqrt[k]{Y} \) for some \( k \in \mathbb{N} \). By Claims 4 and 5, the set \( \sqrt[k]{Y} = X \) is closed and discrete in \( Y \).

5. Conclusions

This paper studies the categorical closedness properties of unipotent commutative semigroups. The main results of our work are Theorem 2, characterizing \( C \)-closed unipotent commutative semigroups, and Theorem 3, characterizing injectively \( C \)-closed unipotent semigroups. Theorem 3 implies Corollary 1 (and Corollary 2), in that the center of an injective (and absolutely) \( C \)-closed unipotent semigroup remains injectively (and absolutely) \( C \)-closed. Since each periodic commutative semigroup decomposes into a disjoint union of unipotent subsemigroups, the information on the categorical closedness properties of unipotent semigroups can shed some light on the categorical closedness properties of commutative semigroups and will be applied in our subsequent papers devoted to studying categorically closed semigroups.
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