How to Lose with Least Probability

ROBERT W. CHEN\textsuperscript{1} AND BURTON ROSENBERG\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, University of Miami
\textsuperscript{2}Department of Computer Science, University of Miami

December 15, 2011

Abstract

Two players alternate tossing a biased coin where the probability of getting heads is $p$. The current player is awarded $\alpha$ points for tails and $\alpha + \beta$ for heads. The first player reaching $n$ points wins. For a completely unfair coin the player going first certainly wins. For other coin biases, the player going first has the advantage, but the advantage depends on the coin bias. We calculate the first player’s advantage and the coin bias minimizing this advantage.

Key words: probability game, stopping times.

1 Introduction

Alice and Bob alternate tossing a biased coin where the probability of getting heads is $p$. Players alternate turns, with Alice going first. The player tosses the coin and is awarded $\alpha$ points for tails and $\alpha + \beta$ for heads. The first player accumulating $n$ points wins the game. The values $\alpha, \beta$ and $n$ are positive constants.

In this paper we investigate $I(p \mid n, \alpha, \beta)$, the winning probability for Alice, the first player. We show that $I(p \mid n, \alpha, \beta)$ is a polynomial in $p$ of degree $2m - 2$, where $m$ is the smallest positive integer greater than or equal to $n/\alpha$.

It is easy to see that Alice has the advantage in this game since she goes first. As a trivial example, when $p$ is 0 or 1 then $I(p \mid n, \alpha, \beta) = 1$. We
consider $p^*_n$, the value of $p$ that minimizes $I(p \mid n, \alpha, \beta)$, and hence makes the
game most favorable to Bob,

$$I(p^*_n \mid n, \alpha, \beta) = \inf_{0 \leq p \leq 1} I(p \mid n, \alpha, \beta).$$

We provide examples of specific polynomials $I(p \mid n, \alpha \beta)$ and values of $p^*_n$, generated by Mathematica. Since these are cumbersome to calculate, and not closed form, we find a simple form for $p^*$, the limit of $p^*_n$ as $n$ goes to infinity,

$$p^* = \lim_{n \to \infty} p^*_n = 1 + t - \sqrt{1 + t + t^2},$$

where $t = \alpha/\beta$.

2 Theorems

Theorem 1 (Main Theorem) Let $\alpha, \beta > 0$ and the limit $n > 0$. Then the probability that the first player wins the game is,

$$I(p \mid n, \alpha, \beta) = \frac{1}{2} \left( 1 + \sum_k P(\tau_1 = k)^2 \right),$$

where the sum can range only over $[n/(\alpha + \beta)] \leq k \leq [n/\alpha]$ and,

$$P(\tau_1 = k) = \binom{k - 1}{i_k} p^{i_k+1} q^{k-i_k-1} + \sum_{j=i_k+1}^{i^*_k} \binom{k - 1}{j} p^j q^{k-j-1},$$

with $i_k = [(n - k\alpha)/\beta - 1]$ and $i^*_k = [(n - (k-1)\alpha)/\beta - 1]$.

Proof: Define stopping times $\tau_1$ and $\tau_2$, where $\tau_1$ is the first time that the first player has $n$ or more accumulated points and $\tau_2$ is the first time that the second player has $n$ or more accumulated points. Since the first player leads,

$$I(p \mid n, \alpha, \beta) = P(\tau_1 < \tau_2) + P(\tau_1 = \tau_2).$$

Since,

$$P(\tau_1 < \tau_2) + P(\tau_2 < \tau_1) + P(\tau_1 = \tau_2) = 1$$

and by symmetry,

$$P(\tau_1 < \tau_2) = P(\tau_2 < \tau_1),$$

$$P(\tau_1 < \tau_2) = \frac{1}{2},$$

$$P(\tau_1 = \tau_2) = \frac{1}{2}.$$
then,
\[ I(p \mid n, \alpha, \beta) = \frac{1 + P(\tau_1 = \tau_2)}{2}. \]

Note,
\[ P(\tau_1 = \tau_2) = \sum_k P(\tau_1 = k \land \tau_2 = k) \]
\[ = \sum_k P(\tau_1 = k)P(\tau_2 = k) = \sum_k P(\tau_1 = k)^2 \]
since \( \tau_1, \tau_2 \) are independent and they have the same distribution. So we need only to compute \( P(\tau_1 = k) \).

Since unless \( \lceil n/(\alpha + \beta) \rceil \) turns are taken, player one cannot win, and if more than \( \lceil n/\alpha \rceil \) turns are taken player one must have already won, we have that \( P(\tau_1 = k) = 0 \) unless \( \lceil n/(\alpha + \beta) \rceil \leq k \leq \lceil n/\alpha \rceil \), hence the above sums can be restricted to that range.

Consider the case where the player has not won after \( k-1 \) tosses, and that \( i \) heads are tossed in the first \( k-1 \) tosses, that the player wins on the \( k \)-th toss, and that the \( k \)-th toss is a tail. This is possible for all \( i \) such that,
\[ (i\beta + (k-1)\alpha < n) \land (i\beta + k\alpha \geq n) \land i \in [0, k-1]. \]

Since it does not matter the placement of the heads in the first \( k-1 \) tosses, each \( i \) contributes to the probability \( P(\tau_1 = k) \) the amount,
\[ \binom{k-1}{i} p^i q^{k-1-i}. \]
Consider the similar situation, but where the \( k \)-th toss is a head. The constraint on \( i \) is,
\[ (i\beta + (k-1)\alpha < n) \land ((i+1)\beta + k\alpha \geq n) \land i \in [0, k-1], \]
and for each \( i \) the contribution to \( P(\tau_1 = k) \) is the amount,
\[ \binom{k-1}{i} p^i q^{k-1-i}. \]
Since these two cases share the range of \( i \) such that,
\[ n - k\alpha \leq i\beta < n - (k-1)\alpha \]
such $i$ contribute,

$$\binom{k - 1}{i} p^i q^{k-1-i}(p + q) = \binom{k - 1}{i} p^i q^{k-1-i}. $$

The remaining $i$ must have a head on the $k$-th toss, and must satisfy,

$$n - k\alpha - \beta \leq i\beta < n - k\alpha$$

and contribute,

$$\binom{k - 1}{i} p^{i+1} q^{k-1-i}. $$

The only possible such $i$ is,

$$i_k = \lceil (n - k\alpha)/\beta - 1 \rceil$$

when $i_k$ it is in the range $[0, k - 1]$. However, the binomial coefficient is zero outside this range, so we can drop this requirement in the writing of the formula for $P(\tau_1 = k)$.

Returning to the case where either a heads or tails is possible on the $k$-th toss, the inequalities rearrange to,

$$[ (n - k\alpha)/\beta ] \leq i \leq \lceil (n - (k - 1)\alpha)/\beta \rceil - 1 = i_k^*.$$

Again, we can drop the requirement that such $i$ be in the range $[0, k - 1]$ in writing the formula for $P(\tau_1 = k)$ since the binomial coefficient is zero for $i$ outside this range. \hfill \Box

Note: It is possible that $i_k^* = i_k$, and therefore the large summation can be empty.

**Corollary 2** When $\lceil n/(\alpha + \beta) \rceil = \lceil n/\alpha \rceil$ then $I(p \mid n, \alpha, \beta) = 1$ for all coin bias $p$.

**Proof:** In the previous proof, the range of $k$ to sum is only the single value,

$$k = \lceil n/(\alpha + \beta) \rceil = \lceil n/\alpha \rceil.$$ 

From the calculation of $k$ we have $k\alpha \geq n$ and $k(\alpha + \beta) < n + \alpha + \beta$. Hence $i_k < 0$ and for any $i \in [0, k - 1]$, $i$ heads among the first $k - 1$ tosses does not
win, but any \( k \) tosses does win. Therefore the sum for \( P(\tau_1 = k) \) reduces to,

\[
P(\tau_1 = k) = \left( \frac{k-1}{i_k} \right) p^{i_k+1} q^{k-i_k-1} + \sum_{j=i_k+1}^{i_k^*} \left( \frac{k-1}{j} \right) p^j q^{k-j-1}
\]

\[
= 0 + \sum_{j=0}^{k-1} \left( \frac{k-1}{j} \right) p^j q^{k-j-1} = 1.
\]

\[\square\]

**Corollary 3** \( I(p \mid n, \alpha, \beta) \) is either a polynomial in \( p \) of degree \( 2 \lceil n/\alpha \rceil - 2 \), or is 1, in the case \( \lceil n/(\alpha + \beta) \rceil = \lceil n/\alpha \rceil \).

**Proof:** The \( P(\tau_1 = k) \) are polynomials in \( p \) of highest degree \( \lceil n/\alpha \rceil - 1 \), since \( i_k < i_k^* \), and \( i_k^* \) is the maximum number of heads among the first \( k-1 \) tosses yet the player does not win, and this is bounded by \( \lceil n/\alpha \rceil - 1 \) which is the maximum number of toss of all tails such that the player does not win. \[\square\]

**Theorem 4 (Optimal coin bias)** Assumptions as in the previous theorem, let \( p_n^* \) be the coin bias that minimizes the probability of the first player winning. Then,

\[
p^* = \lim_{n \to \infty} p_n^* = 1 + t - \sqrt{1 + t + t^2},
\]

where \( t = \alpha/\beta \).

**Proof:** In order to find the limit \( p^* \) of \( p_n^* \) as \( n \to \infty \), let \( X_1, X_2, \ldots \) be i.i.d. random variables such that \( P(X_i = 1) = p \) and \( P(X_i = 0) = 1 - p \). For each \( k \), let \( S_k = X_1 + \ldots + X_k \) and \( P(\tau_1 > k) = P(k\alpha + \beta S_k < n) \).

For the positive integer \( k \) choose \( x \) such that \( k = n/(\alpha + p\beta) + \sqrt{nx} \). Then,

\[
k\alpha + \beta S_k \approx k\alpha + \beta \left( kp + \sqrt{kp(1-p)}Z \right)
\]

\[
\approx n + (\alpha + \beta p)\sqrt{nx} + \beta \sqrt{\frac{np(1-p)}{\alpha + \beta p}}Z.
\]

where \( Z \) is the standard normal random variable. Therefore,

\[
P(k\alpha + \beta S_k < n) \approx P(Z < -\sigma x),
\]

5
where
\[ \sigma^2 = \frac{(\alpha + \beta p)^3}{(\beta^2 p(1 - p))}. \]

Since,
\[ P(Z < -\sigma) \approx \int_{-\infty}^{-\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, \]
we have,
\[ P(\tau_1 = \tau_2) = \sum_k P^2(\tau_1 = k) \approx \sigma^2 \int_{-\infty}^{\infty} e^{-\sigma^2 u^2} du. \]

It is easy to see that to minimize \( I(p \mid n, \alpha, \beta) \) is the same to minimize \( P(\tau_1 = \tau_2) \). Since the integral on the right hand side of the above equation is increasing in \( \sigma^2 \), to minimize \( I(p \mid n, \alpha, \beta) \) we minimize \( \sigma^2 \) with respect to \( p \).

Solving,
\[ \frac{d(\ln(\sigma^2))}{dp} = \frac{3\beta}{\alpha + \beta p} - \frac{1}{p} + \frac{1}{1 - p} = 0 \]
so,
\[ p^* = \left( \beta + \alpha - \sqrt{\beta^2 + \beta \alpha + \alpha^2} \right) / \beta = 1 + t - \sqrt{1 + t + t^2}, \]
where \( 0 < t = \alpha/\beta < \infty. \]

**Remarks:** Since \( \sqrt{1 + t + t^2} > 1/2 + t \) for all \( 0 < t < \infty \), therefore \( p^* < 1/2 \). For example, if \( \alpha = \beta > 0 \), \( p^* = 2 - \sqrt{3} \approx 0.267949192 \), if \( \alpha/\beta = 2 \), \( p^* = 3 - \sqrt{7} \approx 0.354248688 \), and if \( \alpha/\beta = 0.5 \), \( p^* = (3 - \sqrt{7})/2 \approx 0.177124344 \). When \( t \to \infty \), \( p^* \to 0.5 \) and when \( t \to 0 \), \( p^* \to 0.5 \).

## 3 Computer experiments

### 3.1 Mathematica Code

In Figure 1 we give Mathematica code which generates \( I(p \mid n, \alpha, \beta) \).

In Figure 2 we give Mathematica code which generates \( I(p^* \mid n, \alpha, \beta) \), the coin bias that minimizes player one’s advantage.

### 3.2 Advantage polynomials

Table 1 through Table 5 give example \( I(p \mid n, \alpha, \beta) \) for various values of \( n, \alpha, \) and \( \beta \).
Mathematica Code

```mathematica
PlayerOne[n_, alpha_, beta_] := Module[{l, m, k, expr},
  l = Ceiling[n/(alpha + beta)];
  m = Ceiling[n/alpha];
  For[k = l, k <= m, k++, Module[{h, i, j, k = k},
    h = Max[0, Ceiling[(n - k*alpha)/beta - 1]];
    i = Max[0, Ceiling[(n - k*alpha + alpha)/beta - 1]];
    If[h == i, p[k, p_] :=
      Binomial[k - 1, h]*p^(h + 1)*(1 - p)^(k - h - 1)];
    If[h < i, p[k, p_] :=
      Binomial[k-1,h]*p^(h + 1)*(1 - p)^(k - h - 1)
      + Sum[Binomial[k-1,j]*p^j*(1 - p)^(k - j - 1), {j, h + 1, i}]];
    If[k == m, p[m, p_] :=
      (1 - p)^(m - 1)
      + Sum[Binomial[m - 1, j]*p^j*(1 - p)^(m - j - 1), {j, 1, i}]];
    expr = Expand[((1/2)*(1 + Sum[p[k, p]^2, {k, l, m}]))];
    If[l == m, expr = 1];
    Return[expr]];
];
```

Figure 1: Polynomial expression for the advantage of player one.

Mathematica Code

```mathematica
PlayerOneMin[n_, alpha_, beta_] :=
  N[Minimize[Evaluate[PlayerOne[n, alpha, beta]], p]]
```

Figure 2: Minimizing the advantage of player one.

### 3.3 Example minimized advantages

There is an online computer program at:

[http://epaper-live.appspot.com/2011-CGZ](http://epaper-live.appspot.com/2011-CGZ)

using a randomized simulation to estimate $I(p^*_n | n, \alpha, \beta)$.

The following are some values of $I(p^*_n | n, \alpha, \beta)$ and $I(p^* | n, \alpha, \beta)$ for $m = 5, 10, 15$ and various values of $\alpha$ and $\beta$.

Notice that when $m$ is moderate or large, it is very cumbersome to write down the polynomial of $I(p | n, \alpha, \beta)$. Since $I(p^* | n, \alpha, \beta)$ is a good approximation for $I(p^*_n | n, \alpha, \beta)$, we may use $p^*$ for the probability for heads and use simulation to estimate $I(p^*_n | n, \alpha, \beta)$. 

7
| n  | \( I(p \mid n, \alpha = 1, \beta = 1) \)                                                                 |
|----|-------------------------------------------------------------------------------------------------|
| 1  | 1                                                                                              |
| 2  | 1 - \( p + p^2 \)                                                                             |
| 3  | 1 - 2\( p + 5p^2 - 4p^3 + p^4 \)                                                               |
| 4  | 1 - 3\( p + 12p^2 - 22p^3 + 19p^4 - 7p^5 + p^6 \)                                             |
| 5  | 1 - 4\( p + 22p^2 - 64p^3 + 104p^4 - 92p^5 + 43p^6 - 10p^7 + p^8 \)                          |
| 6  | 1 - 5\( p + 35p^2 - 140p^3 + 341p^4 - 508p^5 + 459p^6 - 247p^7 + 77p^8 - 13p^9 + p^{10} \)   |
| 7  | 1 - 6\( p + 51p^2 - 260p^3 + 850p^4 - 1,816p^5 + 2,548p^6 - 2,336p^7 + 1,385p^8 - 522p^9 + 121p^{10} - 16p^{11} + p^{12} \) |
| 8  | 1 - 7\( p + 70p^2 - 434p^3 + 1,786p^4 - 5,011p^5 + 9,709p^6 - 13,030p^7 + 12,079p^8 - 7,683p^9 + 3,316p^{10} - 952p^{11} + 175p^{12} - 19p^{13} + p^{14} \) |
| 9  | 1 - 8\( p + 92p^2 - 672p^3 + 3,339p^4 - 11,648p^5 + 29,037p^6 - 52,154p^7 + 67,644p^8 - 63,248p^9 + 42,440p^{10} - 20,280p^{11} + 6,812p^{12} - 1,572p^{13} + 239p^{14} - 22p^{15} + p^{16} \) |
| 10 | 1 - 9\( p + 117p^2 - 984p^3 + 5,734p^4 - 23,968p^5 + 73,381p^6 - 166,489p^7 + 281,524p^8 - 355,393p^9 + 334,585p^{10} - 234,160p^{11} + 121,147p^{12} - 45,916p^{13} + 12,559p^{14} - 2,417p^{15} + 313p^{16} - 25p^{17} + p^{18} \) |
| 11 | 1 - 10\( p + 145p^2 - 1,380p^3 + 9,231p^4 - 45,024p^5 + 163,864p^6 - 451,312p^7 + 948.352p^8 - 1,526,710p^9 + 1,885,453p^{10} - 1,785,028p^{11} + 1,292,464p^{12} - 712,744p^{13} + 297,382p^{14} - 92,900p^{15} + 21,369p^{16} - 3,522p^{17} + 397p^{18} - 28p^{19} + p^{20} \) |
| 12 | 1 - 11\( p + 176p^2 - 1,870p^3 + 14,125p^4 - 78,807p^5 + 332,865p^6 - 1,081,308p^7 + 2,728,525p^8 - 5,379,992p^9 + 8,314,959p^{10} - 10,083,869p^{11} + 9,591,305p^{12} - 7,142,250p^{13} + 4,150,664p^{14} - 1,873,073p^{15} + 651,365p^{16} - 172,523p^{17} + 34,180p^{18} - 4,922p^{19} + 491p^{20} - 31p^{21} + p^{22} \) |

Table 1: \( \alpha = 1, \beta = 1 \)
| $n$ | $I(p \mid n, \alpha = 1, \beta = 2)$ |
|-----|----------------------------------|
| 1   | 1                                |
| 2   | 1 $- p + p^2$                    |
| 3   | 1 $- 2p + 4p^2 - 3p^3 + p^4$    |
| 4   | 1 $- 3p + 10p^2 - 14p^3 + 11p^4 - 5p^5 + p^6$ |
| 5   | 1 $- 4p + 19p^2 - 43p^3 + 54p^4 - 42p^5 + 22p^6 - 7p^7 + p^8$ |
| 6   | 1 $- 5p + 31p^2 - 100p^3 + 186p^4 - 216p^5 + 169p^6 - 94p^7 + 37p^8 + 9p^9 + p^{10}$ |
| 7   | 1 $- 6p + 46p^2 - 195p^3 + 497p^4 - 808p^5 + 891p^6 - 700p^7 + 407p^8 - 178p^9 + 56p^{10} - 11p^{11} + p^{12}$ |
| 8   | 1 $- 7p + 64p^2 - 338p^3 + 1,112p^4 - 2,393p^5 + 3,538p^6 - 3,755p^7 + 2,963p^8 - 1,785p^9 + 836p^{10} - 302p^{11} - 79p^{12} - 13p^{13} + p^{14}$ |
| 9   | 1 $- 8p + 85p^2 - 539p^3 + 2,191p^4 - 5,966p^5 + 11,335p^6 - 15,606p^7 + 16,088p^8 - 12,744p^9 + 7,911p^{10} - 3,905p^{11} + 1,540p^{12} - 474p^{13} + 106p^{14} - 15p^{15} + p^{16}$ |
| 10  | 1 $- 9p + 109p^2 - 808p^3 + 3,929p^4 - 13,068p^5 + 30,811p^6 - 53,204p^7 + 69,302p^8 - 69,820p^9 + 55,500p^{10} - 35,338p^{11} + 18,228p^{12} - 7,670p^{13} + 2,619p^{14} - 702p^{15} + 137p^{16} - 17p^{17} + p^{18}$ |
| 11  | 1 $- 10p + 136p^2 - 1,153p^3 - 25,912p^4 - 73,689p^5 - 155,186p^6 + 248,468p^7 - 309,592p^8 + 306,164p^9 - 244,118p^{10} + 158,834p^{12} - 85,084p^{13} + 37,759p^{14} - 13,898p^{15} + 4,189p^{16} - 994p^{17} + 172p^{18} - 19p^{19} + p^{20}$ |
| 12  | 1 $- 11p + 166p^2 - 1,590p^3 + 10,337p^4 - 47,509p^5 + 159,238p^6 - 399,557p^7 + 768,438p^8 - 1,157,198p^9 + 1,390,348p^{10} - 1,354,014p^{11} + 1,082,537p^{12} - 717,534p^{13} + 397,153p^{14} - 184,505p^{15} + 72,133p^{16} - 23,647p^{17} + 6,382p^{18} - 1,358p^{19} + 211p^{20} - 21p^{21} + p^{22}$ |

Table 2: $\alpha = 1, \beta = 2$
| $n$ | $I(p \mid n, \alpha = 2, \beta = 1)$ |
|-----|----------------------------------|
| 2   | 1                                |
| 4   | 1                                |
| 6   | $1 - p^2 + p^4$                  |
| 8   | $1 - 3p^2 + 2p^3 + 9p^4 - 12p^5 + 4p^7$ |
| 10  | $1 - 6p^2 + 8p^3 + 33p^4 - 96p^5 + 100p^6 - 48p^7 + 9p^8$ |
| 12  | $1 - 10p^2 + 20p^3 + 85p^4 - 396p^5 + 690p^6 - 660p^7 + 371p^8 - 116p^9 + 16p^{10}$ |
| 14  | $1 - 15p^2 + 40p^3 + 180p^4 - 1,176p^5 + 2,870p^6 - 4,060p^7 + 3,735p^8 - 2,300p^9 + 921p^{10} - 220p^{11} + 25p^{12}$ |
| 16  | $1 - 21p^2 + 70p^3 + 336p^4 - 2,856p^5 + 8,960p^6 - 16,668p^7 + 21,015p^8 - 19,032p^9 + 12,531p^{10} - 5,850p^{11} + 1,839p^{12} - 360p^{13} + 36p^{14}$ |
| 18  | $1 - 28p^2 + 112p^3 + 574p^4 - 6,048p^5 + 23,184p^6 - 53,264p^7 + 84,616p^8 - 100,128p^9 + 91,385p^{10} - 64,204p^{11} + 33,678p^{12} - 12,608p^{13} + 3,214p^{14} - 532p^{15} + 49p^{16}$ |
| 20  | $1 - 36p^2 + 168p^3 + 918p^4 - 11,592p^5 + 52,500p^6 - 143,256p^7 + 273,133p^8 - 396,296p^9 + 461,528p^{10} - 437,572p^{11} + 331,905p^{12} - 193,564p^{13} + 82,979p^{14} - 25,046p^{15} + 5,165p^{16} - 728p^{17} + 64p^{18}$ |
| 22  | $1 - 45p^2 + 240p^3 + 1,395p^4 - 20,592p^5 + 107,580p^6 - 339,480p^7 + 752,661p^8 - 1,287,280p^9 + 1,818,666p^{10} - 2,192,040p^{11} + 2,229,076p^{12} - 1,842,576p^{13} + 1,187,376p^{14} - 573,496p^{15} + 199,623p^{16} - 48,144p^{17} + 7,891p^{18} - 936p^{19} + 81p^{20}$ |
| 24  | $1 - 55p^2 + 330p^3 + 2,035p^4 - 34,452p^5 + 203,940p^6 - 729,960p^7 + 1,840,785p^8 - 3,613,360p^9 + 5,997,570p^{10} - 8,828,788p^{11} + 11,452,248p^{12} - 12,571,976p^{13} + 11,200,776p^{14} - 7,840,824p^{15} + 4,195,879p^{16} - 1,666,048p^{17} + 472,645p^{18} - 91,446p^{19} + 11,741p^{20} - 1,140p^{21} + 100p^{22}$ |

Table 3: $\alpha = 2, \beta = 1$
\[
\begin{array}{|c|c|}
\hline
n & I(p \mid n, \alpha = 2, \beta = 3) \\
\hline
2 & 1 \\
4 & 1 - p + p^2 \\
6 & 1 - 2p + 5p^2 - 4p^3 + p^4 \\
8 & 1 - 3p + 12p^2 - 22p^3 + 19p^4 - 7p^5 + p^6 \\
10 & 1 - 4p + 22p^2 - 64p^3 + 102p^4 - 90p^5 + 43p^6 - 10p^7 + p^8 \\
12 & 1 - 5p + 35p^2 - 140p^3 + 332p^4 - 478p^5 + 429p^6 - 238p^7 + 77p^8 - 13p^9 + p^{10} \\
14 & 1 - 6p + 51p^2 - 260p^3 + 826p^4 - 1,678p^5 + 2,251p^6 - 2,039p^7 + 1,247p^8 - 498p^9 + 121p^{10} - 16p^{11} + p^{12} \\
16 & 1 - 7p + 70p^2 - 434p^3 + 1,736p^4 - 4,601p^5 + 8,335p^6 - 10,604p^7 + 9,653p^8 - 6,309p^9 + 2,906p^{10} - 902p^{11} + 175p^{12} - 19p^{13} + p^{14} \\
18 & 1 - 8p + 92p^2 - 672p^3 + 3,249p^4 - 10,688p^5 + 24,647p^6 - 40,904p^7 + 49,930p^8 - 45,534p^9 + 31,190p^{10} - 15,890p^{11} + 5,852p^{12} - 1,482p^{13} + 239p^{14} - 22p^{15} + p^{16} \\
20 & 1 - 9p + 117p^2 - 984p^3 + 5,587p^4 - 22,036p^5 + 62,161p^6 - 128,594p^7 + 199,200p^8 - 235,130p^9 + 214,326p^{10} - 151,840p^{11} + 83,252p^{12} - 34,696p^{13} + 10,727p^{14} - 2,270p^{15} + 313p^{16} - 25p^{17} + p^{18} \\
22 & 1 - 10p + 145p^2 - 1,380p^3 + 9,007p^4 - 41,524p^5 + 139,189p^6 - 347,593p^7 + 659462p^8 - 966,212p^9 + 1,108,918p^{10} - 1,008,598p^{11} + 732,121p^{12} - 423,904p^{13} + 193,663p^{14} - 68,225p^{15} + 17,869p^{16} - 3,298p^{17} + 397p^{18} - 28p^{19} + p^{20} \\
24 & 1 - 11p + 176p^2 - 1,870p^3 + 13,801p^4 - 72,939p^5 + 284,173p^6 - 836,056p^7 + 1,892,572p^8 - 3,346,345p^9 + 4,682,293p^{10} - 5,246,194p^{11} + 4,755,285p^{12} - 3,512,514p^{13} + 2,118,592p^{14} - 1,037,420p^{15} + 406,113p^{16} - 123,831p^{17} + 28,312p^{18} - 4,598p^{19} + 491p^{20} - 31p^{21} + p^{22} \\
\hline
\end{array}
\]

Table 4: $\alpha = 2, \beta = 3$
| n   | $I(p \mid n, \alpha = 3, \beta = 2)$ |
|-----|----------------------------------|
| 3   | 1                                |
| 6   | 1                                |
| 9   | $1 - p^2 + p^4$                  |
| 12  | $1 - 3p^2 + 2p^3 + 9p^4 - 12p^5 + 4p^6$ |
| 15  | $1 - 6p^2 + 8p^3 + 33p^4 - 102p^5 + 109p^6 - 51p^7 + 9p^8$ |
| 18  | $1 - 10p^2 + 20p^3 + 85p^4 - 436p^5 + 826p^6 - 824p^7 + 455p^8 - 132p^9 + 16p^{10}$ |
| 21  | $1 - 15p^2 + 40p^3 + 180p^4 - 1,326p^5 + 3,670p^6 - 5,760p^7 + 5,595p^8 - 3,420p^9 + 1,281p^{10} - 270p^{11} + 25p^{12}$ |
| 24  | $1 - 21p^2 + 70p^3 + 336p^4 - 3,276p^5 + 12,020p^6 - 26,028p^7 + 36,835p^8 - 35,347p^9 + 23,176p^{10} - 10,220p^{11} + 2,899p^{12} - 480p^{13} + 36p^{14}$ |
| 27  | $1 - 28p^2 + 112p^3 + 574p^4 - 7,028p^5 + 32,249p^6 - 89,524p^7 + 167,706p^8 - 221,823p^9 + 211,485p^{10} - 146,003p^{11} + 72,252p^{12} - 24,943p^{13} + 5,699p^{14} - 777p^{15} + 49p^{16}$ |
| 30  | $1 - 36p^2 + 168p^3 + 918p^4 - 13,608p^5 + 75,124p^6 - 255,144p^7 + 597,093p^8 - 1,011,526p^9 + 1,274,175p^{10} - 1,210,029p^{11} + 869,107p^{12} - 468,840p^{13} + 186,558p^{14} - 52,997p^{15} + 10,149p^{16} - 1,176p^{17} + 64p^{18}$ |
| 33  | $1 - 45p^2 + 240p^3 + 1,395p^4 - 24,372p^5 + 157,476p^6 - 633,564p^7 + 1,782,165p^8 - 3,689,932p^9 + 5,794,834p^{10} - 7,033,324p^{11} + 6,664,633p^{12} - 4,941,244p^{13} + 2,850,905p^{14} - 1,263,598p^{15} + 420,991p^{16} - 101,764p^{17} + 16,795p^{18} - 1,692p^{19} + 81p^{20}$ |
| 36  | $1 - 55p^2 + 330p^3 + 2,035p^4 - 41,052p^5 + 304,140p^6 - 1,415,760p^7 + 4,657,305p^8 - 11,410,660p^9 + 21,494,814p^{10} - 31,814,548p^{11} + 37,533,952p^{12} - 35,564,264p^{13} + 27,090,180p^{14} - 16,505,584p^{15} + 7,962,291p^{16} - 2,994,900p^{17} + 858,625p^{18} - 80,870p^{19} + 26,261p^{20} - 2,340p^{21} + 100p^{22}$ |

Table 5: $\alpha = 3, \beta = 2$
| $n$  | $\alpha$ | $\beta$ | $I(p^*_n|n,\alpha,\beta)$ | $I(p^*|n,\alpha,\beta)$ |
|------|----------|---------|----------------------------|--------------------------|
| 5    | 1        | 1       | 0.700                      | 0.700                    |
| 10   | 1        | 1       | 0.643                      | 0.643                    |
| 15   | 1        | 1       | 0.617                      | 0.617                    |
| 5    | 1        | 2       | 0.689                      | 0.695                    |
| 10   | 1        | 2       | 0.635                      | 0.641                    |
| 15   | 1        | 2       | 0.610                      | 0.616                    |
| 5    | 1        | 3       | 0.660                      | 0.663                    |
| 10   | 1        | 3       | 0.624                      | 0.630                    |
| 15   | 1        | 3       | 0.604                      | 0.609                    |
| 10   | 2        | 1       | 0.750                      | 0.753                    |
| 20   | 2        | 1       | 0.714                      | 0.718                    |
| 30   | 2        | 1       | 0.683                      | 0.686                    |
| 10   | 2        | 3       | 0.669                      | 0.701                    |
| 20   | 2        | 3       | 0.714                      | 0.719                    |
| 30   | 2        | 3       | 0.570                      | 0.612                    |
| 15   | 3        | 2       | 0.716                      | 0.752                    |
| 30   | 3        | 2       | 0.665                      | 0.676                    |
| 45   | 3        | 2       | 0.649                      | 0.666                    |

Table 6: Sample approximate and exact advantage values
```python
import random
import math
import os

HEADS = 1
TAILS = -1
MAX_TRIALS = 10000

def usage():
    print "Usage: game -a alpha - b beta -n limit - t trials"

def myrand(p):
    r = random.random()
    if r < p:
        return HEADS
    return TAILS

def f(alpha, beta, n, p):
    na = nb = 0
    turns = 0
    while 1:
        turns += 1
        na += alpha
        a_toss = myrand(p)
        if a_toss == HEADS:
            na += beta
        if na >= n:
            return turns
        nb += alpha
        b_toss = myrand(p)
        if b_toss == HEADS:
            nb += beta
        if nb >= n:
            return -turns
        print "never gets here"
    return
```

Figure 3: Python program for minimized advantage, online at URL in text.
References

[1] Chen, Lester H., Chen, Robert W. and Shepp Larry A. (1994) A Note on a Probabilistic Game, Soochow Journal of Mathematics, Vol.20, pp.369–371.

[2] Victor Addona, Herb Wilf, and Stan Wagon, How to lose as little as possible, Ars Mathematica Contemporanea, 4:1 (Spring/Summer, 2011) 2962. Electronic version: Nov 2010, arXiv:1002.1763.v2.

[3] Chen, Robert W. and Burton Rosenberg, (2011) http://epaper-live.appspot.com/2011-CGZ