SEMISIMPLICITY AND WEIGHT-MONODROMY FOR
FUNDAMENTAL GROUPS

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1. INTRODUCTION

The goal of this paper is to prove two structural results for Galois actions on pro-unipotent fundamental groups of smooth geometrically connected varieties $X$ over $p$-adic local fields $K$, and to give applications to the non-Abelian Chabauty–Kim program.

1.1. Main results.

1.1.1. Structural results. Our first main structural result is Theorem 3.2, which is an analogue of the weight-monodromy conjecture for the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of $X$. In the crystalline setting (that is, $\ell = p$), this result appears in work of Vologodsky [Vol03, Theorem 26], to whom this work owes a great debt; our proof is different and perhaps more direct.

Our second result is Theorem 3.3; it is a strengthening of [Lit18, Theorem 2.12], and a non-abelian generalization of the fact that Frobenius elements act semisimply on the $\ell$-adic Tate modules of Abelian varieties. Namely, we show that if $\ell \neq p$, any choice of Frobenius in $\text{Gal}(\overline{K}/K)$ acts semisimply on the Lie algebra $\mathfrak{g}$ of the $\mathbb{Q}_\ell$-pro-unipotent completion of $\pi_1^{et}(\overline{X}, \overline{x})$ (for a rational basepoint $x$); if $\ell = p$, we prove an analogous statement for a $K$-linear power of the crystalline Frobenius.

Both results are formulated in terms of Weil–Deligne representations, allowing us to give a uniform argument in the cases ($\ell = p$) and ($\ell \neq p$). In Section 2, we recall notation and general facts about Weil–Deligne representations, with a focus on the notion of mixedness (the correct analogue of the weight-monodromy conjecture for Galois representations arising from the cohomology of varieties which are not necessarily smooth or proper). The main observation of this section is in Definition 2.13, which gives a canonical splitting of the weight filtration on a mixed Weil–Deligne representation. This is a Galois-theoretic analogue of the canonical splitting of the weight filtration on a real mixed Hodge structure [CKS86, Proposition 2.20], and provides a unified explanation for the existence of certain canonical choices of pro-unipotent paths appearing in [Vol03, Proposition 29] in the case ($\ell = p$) and in [BD19, Remark 2.2.5] in the case ($\ell \neq p$).

1.1.2. Applications. In Section 4, we apply these results to prove structure theorems about the local Selmer varieties arising in the Chabauty–Kim program. These are three sub-presheaves $H^1_1(G_K, U) \subseteq H^1_2(G_K, U) \subseteq H^1_3(G_K, U)$ of the continuous Galois cohomology presheaf $H^1(G_K, U)$ associated to the $\mathbb{Q}_p$-pro-unipotent étale fundamental group $U = \pi_1^{et}(\overline{X}, \overline{x})$, and are all representable by affine schemes over $\mathbb{Q}_p$ [Kim05, Proposition 3][Kim09, Lemma 5]. The relevance of the local Selmer
varieties to Diophantine geometry and the Chabauty–Kim method comes via a certain non-abelian Kummer or higher Albanese map

\[ X(K) \to \mathbb{H}^1_G(U)(\mathbb{Q}_p). \]

When \( X \) is a smooth projective curve with good reduction, then the image of this map is contained in \( \mathbb{H}^1_G(U)(\mathbb{Q}_p) = \mathbb{H}^1_G(U)(\mathbb{Q}_p) \), which is known to be an affine space over \( \mathbb{Q}_p \) [Kim09]. However, without properness or good reduction assumptions, the image is not in general contained in \( \mathbb{H}^1_g \).

Our contribution in this paper is to describe the geometry of \( \mathbb{H}^1_g(U)(\mathbb{Q}_p) \) (for \( X \) not necessarily proper, of arbitrary dimension, and with no restrictions on the reduction type). In fact, we find that its geometry is as good as could be hoped: it is canonically isomorphic to the product of \( \mathbb{H}^1_e(U)(\mathbb{Q}_p) \) and an explicit vector space \( V_{st}^g/U_{st}^e \), and therefore also an affine space. As an illustration, we are able to write down explicit dimension formulae in the case that \( X \) is a curve.

1.2. Motivation and related work. Theorem 3.2 is evidently motivated by the weight-monodromy conjecture [Del71, Section 8]. Theorem 3.3 is motivated by the following naive observation: if \( X \) is a curve over \( K \) with semistable reduction, and the dual graph of the special fiber of the reduction of \( X \) is not simply-connected, then any eigenvalue of Frobenius which appears in \( g \) in fact appears with infinite multiplicity. So semi-simplicity of the Frobenius actions on \( g \) is far from clear in the case of varieties with bad reduction. Nonetheless, we show (using Theorem 3.2) that it holds.

2. Weil–Deligne representations

In this section, we fix a finite extension \( K \) of \( \mathbb{Q}_p \) with residue field \( k \), along with an algebraic closure \( \overline{K} \). We write \( W_K \) for the Weil group of \( K \), i.e. the subgroup of the absolute Galois group \( G_K \) consisting of elements acting on the residue field \( k \) via an integer power of the absolute Frobenius \( \sigma: x \mapsto x^p \), and we write \( \nu: W_K \to \mathbb{Z} \) for the unique homomorphism such that \( w \in W_K \) acts on \( k \) via \( \sigma^{\nu(w)} \). We fix a geometric Frobenius \( \phi_K \in W_K \), i.e. an element such that \( \nu(\phi_K) = -f(K/\mathbb{Q}_p) \). The following definition is standard.

**Definition 2.1.** Let \( E \) be a field of characteristic 0. A Weil representation with coefficients in \( E \) is a representation \( \rho: W_K \to \text{Aut}(V) \) of \( W_K \) on a finite dimensional \( E \)-vector space \( V \) such that the inertia group \( I_K \) acts through a finite quotient. A Weil–Deligne representation with coefficients in \( E \) consists of a Weil representation \( V \) endowed with an \( E \)-linear endomorphism \( N \in \text{End}(V) \) called the monodromy operator such that

\[ N \circ \rho(w) = p^{-\nu(w)} \cdot \rho(w) \circ N \]

for all \( w \in W_K \). It follows from this condition that \( N \) is necessarily nilpotent.

We denote the category of Weil–Deligne representations by \( \text{Rep}_E(W_K) \). The category \( \text{Rep}_E(W_K) \) has a canonical tensor product making it into a neutral Tannakian category, where the tensor product \( V_1 \otimes V_2 \) is endowed with the tensor product \( W_K \)-action, and with the endomorphism \( N \otimes 1 + 1 \otimes N \).

**Example 2.2.** The Weil–Deligne representation \( E(1) \) has underlying vector space \( E \), trivial monodromy operator \( N \), and the Weil group acts via \( w: x \mapsto p^{\nu(w)} x \).
As explained in [Fon94a], Weil–Deligne representations arise naturally from $\ell$-adic and $p$-adic Galois representations.

**Example 2.3.** Let $\ell$ be a prime distinct from $p$, and choose a generator $t \in \mathbb{Q}_\ell(1)$. Let $t_L: I_K \to \mathbb{Q}_\ell(1)$ denote the $\ell$-adic tame character $w \mapsto \left( \frac{w(t)^{1/n}}{p^{r/n}} \right)_{n \in \mathbb{N}}$. Then there is a fully faithful exact $\otimes$-functor

$$\text{Rep}_{\mathbb{Q}_\ell, cts}(G_K) \to \text{Rep}_{\mathbb{Q}_\ell}(W_K)$$

from the category of continuous $\mathbb{Q}_\ell$-linear representations of $G_K$ to the category of Weil–Deligne representations. This functor is defined as follows. If $(V, \rho_0)$ is a continuous $\mathbb{Q}_\ell$-linear representation of $G_K$, there is an open subgroup $I_L \leq I_K$ acting unipotently on $V$ by Grothendieck’s $\ell$-adic Monodromy Theorem. We let $N$ denote the endomorphism of $V$ such that

$$\rho_0(g) = \exp \left( t^{-1}t_L(g)N \right)$$

for all $g \in I_L$. We define an action $\rho$ of $W_K$ on $V$ by

$$\rho(\varphi_K g) = \rho_0(\varphi_K)^n \exp \left( -t^{-1}t_L(g)N \right)$$

for $n \in \mathbb{Z}$ and $g \in I_K$. The tuple $(V, \rho, N)$ is the Weil–Deligne representation associated to $V$.

There is an alternative construction of the Weil–Deligne representation due to Fontaine which avoids the choice of Frobenius [Fon94a, §2.2]. If we write $V$ for the $\mathbb{Q}_\ell$-linear Tate module of the Tate elliptic curve $\mathbb{G}_m/p^\mathbb{Z}$, then $V(-1)$ is an extension of $\mathbb{Q}_\ell(-1)$ by $\mathbb{Q}_\ell$ and hence the direct limit $\mathbb{B}_{\text{st}, \ell} := \varprojlim \text{Sym}^n(V(-1))$ has the structure of a commutative algebra over $\mathbb{Q}_\ell$, isomorphic to a polynomial algebra in one variable. The algebra $\mathbb{B}_{\text{st}, \ell}$ carries a natural action of $G_K$ and, after choosing a generator of $\mathbb{Q}_\ell(1)$, an $I_K$-equivariant derivation $N$. For a continuous $\ell$-adic representation $V$ of $G_K$, one sets

$$D_{\text{psT}}(V) := \varinjlim \mathbb{B}_{\text{st}, \ell}(\mathbb{B}_{\text{st}, \ell} \otimes V)^{I_L},$$

which is a Weil–Deligne representation with respect to the natural action of $W_K$ and the induced monodromy operator $N$. One can check that this yields an isomorphic Weil–Deligne representation to that constructed earlier.

**Example 2.4.** Let $L/K$ be a Galois extension, not necessarily finite. A *discrete* $(\varphi, N, G_{L|K})$-module [Fon94b, §4.2.1] consists of a finite-dimensional $L_0$-vector space $V$ endowed with a $\sigma$-linear Frobenius automorphism $\varphi: V \to V$, an $L_0$-linear endomorphism $N: V \to V$, and a semilinear action $\rho_0: G_{L|K} \to \text{Aut}_{\mathbb{Q}_p}(V)$ such that:

- the action of $G_{L|K}$ has open point-stabilisers and commutes with $\varphi$ and $N$;
- we have $N \circ \varphi = p \cdot \varphi \circ N$.

Given a discrete $(\varphi, N, G_{L|K})$-module $(V, \varphi, N, \rho_0)$, we obtain an $L_0$-linear Weil–Deligne representation whose underlying vector space and monodromy operator are $V$ and $N$, and whose representation of $W_K$ is given by

$$\rho(w) = \rho_0(w) \varphi^{-v(w)}.$$
We thus obtain a faithful, exact and conservative \( \otimes \)-functor \( \text{Mod}(\varphi, N, G_{L/K}) \to \text{Rep}_{\mathbb{Q}_p}(W_K) \) from the category of discrete \((\varphi, N, G_{L/K})\)-modules to the category of \(L_0\)-linear Weil–Deligne representations. Precomposing with the Dieudonné functor \( \text{D}_{st,L} : \text{Rep}_{\mathbb{Q}_p}(G_K) \to \text{Mod}(\varphi, N, G_{L/K}) \), we also obtain an exact \( \otimes \)-functor from the category of de Rham (=potentially semistable [Ber02, Théorème 0.7]) representations to the category of \(L_0\)-linear Weil–Deligne representations. In the particular case that \( L = K \) or \( L = \overline{\mathbb{K}} \), we denote the functor \( \text{D}_{st,L} \) simply by \( \text{D}_{st} \) or \( \text{D}_{pst} \) respectively; the latter functor is faithful and conservative.

If \( P \) is a property of (filtered) Weil–Deligne representations and \( \ell \) is a prime, we shall say that a \( \mathbb{Q}_\ell \)-linear (filtered) Galois representation \( V \) has property \( P \) just when its associated Weil–Deligne representation \( \text{D}_{pst}(V) \) has property \( P \); when \( \ell = p \) this means we assume that \( V \) is de Rham. For example, we say that a Weil–Deligne representation is semistable [Fon94a, §1.3.7] just when the action of \( I_K \) is trivial. This corresponds to the usual notions of semistability on \( \mathbb{Q}_\ell \)-linear representations, namely unipotence of the \( I_K \)-action when \( \ell \neq p \) and \( \text{B}_{st} \)-admissibility when \( \ell = p \).

The following two properties — being Frobenius-semisimple and mixed — will play a key role in this paper.

**Definition 2.5.** An \( E \)-linear Weil–Deligne representation \( V \) is said to be Frobenius-semisimple just when the action of the geometric Frobenius \( \varphi_K \) on \( V \) is semisimple, or equivalently just when every element of \( W_K \) acts semisimply. For a de Rham \( \mathbb{Q}_p \)-linear representation \( V \) of \( G_K \), this is the same as the action of crystalline Frobenius \( \varphi \) on \( \text{D}_{pst}(V) \) being semisimple (as a \( \mathbb{Q}_p \)-linear automorphism).

**Definition 2.6.** A \( q \)-Weil number of weight \( i \) in an algebraically closed field \( \overline{\mathbb{F}} \) of characteristic 0 is an element \( \alpha \in \overline{\mathbb{E}} \) which is algebraic over \( \mathbb{Q} \subseteq \overline{\mathbb{F}} \) and satisfies

\[
|\sigma(\alpha)| = q^{i/2}
\]

for every complex embedding \( \sigma : \overline{\mathbb{Q}} \to \mathbb{C} \).

Given a Weil–Deligne representation \( V \) over a characteristic 0 field \( E \), we write \( V^i \overline{\mathbb{F}} \) for the largest \( \varphi_K \)-stable subspace of \( V \overline{\mathbb{F}} \) such that all the eigenvalues of \( \varphi_K |_{V^i \overline{\mathbb{F}}} \) are \( q \)-Weil numbers of weight \( i \). By Galois descent, \( V^i \overline{\mathbb{F}} \) is the base change of a subspace \( V^i \mathbb{F} \) defined over \( E \). It follows from the definition that \( N(V^i) \subseteq V^{i-2} \).

We say that a Weil–Deligne representation \( V \) is pure of weight \( i \) just when \( V = \bigoplus_j V^j \) (i.e. all the eigenvalues of \( \varphi_K \) are \( q \)-Weil numbers) and the map \( N^j : V^{i+j} \to V^{i-j} \) is an isomorphism for all \( j \geq 0 \). We say that a Weil–Deligne representation \( V \) endowed with an increasing filtration

\[
\cdots \subseteq W_i V \subseteq W_{i+1} V \subseteq \ldots
\]

by Weil–Deligne subrepresentations is mixed just when \( W_* \) is exhaustive and separated and \( gr^i W \) is pure of weight \( i \) for all \( i \). The filtration \( W_* V \) is called the weight filtration of a mixed Weil–Deligne representation \( V \); its set of weights is the set

\[
\text{wt}(V) := \{ i \in \mathbb{Z} : gr^i_W V \neq 0 \}.
\]

The collection of mixed Weil–Deligne representations naturally forms a symmetric monoidal category \( \text{Rep}^\text{mix}_E(W_K) \), whose morphisms are filtered maps of Weil–Deligne representations, and whose tensor product \( \otimes \) and tensor unit \( 1 \) are defined in the usual way.
Remark 2.7. The subspaces $V^i \subseteq V$ defined in Definition 2.6 are stable under the action of $W_K$ and do not depend on the choice of geometric Frobenius $\varphi_K$. Indeed, for any other geometric Frobenius $\varphi'_K$ there is an $n \in \mathbb{N}$ such that $\rho(\varphi_K)^n = \rho(\varphi'_K)^n$, and we can equivalently describe $V^i_E$ as the largest subspace of $V^i_E$ on which all the eigenvalues of $\rho(\varphi_K)^n$ are $q$-Weil numbers of weight $ni$.

Remark 2.8. Let $V$ be a filtered de Rham representation of $G_K$ on a finite dimensional $\mathbb{Q}_p$-vector space, and suppose that $V$ is mixed, which for us means that the $\mathbb{Q}_p$-linear Weil–Deligne representation associated to $D_{pst}(V)$ is mixed. Then the $K_0$-linear Weil–Deligne representation associated to $D_{st}(V)$ is also mixed. Indeed, $\mathbb{Q}_p \otimes_K D_{st}(V)$ is the inertia-invariant subspace of the Weil–Deligne representation associated to $D_{pst}(V)$, and hence is mixed since taking invariants under actions of finite groups is exact in characteristic 0 vector spaces.

Remark 2.9. In what follows, our Weil–Deligne representations will not necessarily be finite dimensional. Instead, they will usually be pro-finite dimensional Weil–Deligne representations, i.e. pro-objects of the category of finite dimensional Weil–Deligne representations. When we say, for example, that a filtered pro-finite-dimensional Weil–Deligne representation is mixed, we mean that it is an inverse limit of mixed Weil–Deligne representations, endowed with the inverse limit filtration. All of the constructions in this section are functorial, so extend naturally to the setting of pro-finite-dimensional Weil–Deligne representations, with the proviso that certain direct sums may change into products in the pro-finite-dimensional case.

The following result is well-known.

Theorem 2.10 (cf. [Vol03, Proposition 20]). The category $\text{Rep}_{E}^{\text{mix}}(W_K)$ is a neutral Tannakian category over $E$, and the forgetful functor $\text{Rep}_{E}^{\text{mix}}(W_K) \rightarrow \text{Rep}_{E}(W_K)$ is exact, conservative and compatible with the tensor structure. Morphisms in $\text{Rep}_{E}^{\text{mix}}(W_K)$ are strict for the weight filtration.

Proof (sketch). Compatibility with the tensor structure is easy to check, and conservativity will be a consequence of exactness, since the forgetful functor reflects zero objects. For the remainder, is suffices to prove that any morphism

$$f : V_1 \rightarrow V_0$$

of mixed representations is strict and that its kernel and cokernel are again mixed when endowed with the subspace and quotient filtrations, respectively.

We begin by proving this in the case that $V_0$ and $V_1$ are pure\(^2\) of weights $i_0$ and $i_1$, respectively. In this case, the claim amounts to showing that $f = 0$ if $i_0 \neq i_1$, and that $\ker(f)$ and $\coker(f)$ are pure of weight $i_0 = i_1$ otherwise. If $i_1 < i_0$, then compatibility with the weight filtration ensures that $f = 0$; we suppose henceforth that $i_1 \geq i_0$.

Now the functor $V \mapsto V^j$ picking out the weight $j$ generalised eigenspace is exact for all $j$, and hence for all $j \geq 0$ we have a commuting diagram

\(^2\)When we refer to a filtered Weil–Deligne representation $V$ as being pure of weight $i$, we mean that its underlying representation is pure and the filtration on $V$ is supported in degree $i$. 

with exact rows. The middle and right-hand vertical maps are an isomorphism and injective, respectively, by purity of \( V_0 \) and \( V_1 \), and hence \( N^j \): \( \ker(f)^{i_1+j} \rightarrow \ker(f)^{i_1-j} \) is an isomorphism. Thus \( \ker(f) \) is pure of weight \( i_1 \); the dual argument establishes that \( \coker(f) \) is pure of weight \( i_0 \) and we are done in the case \( i_0 = i_1 \).

Finally, in the case \( i_1 > i_0 \), we see from the equal-weight case that the image of \( f \) is pure of weight \( i_0 \), while its coimage is pure of weight \( i_1 \). But these have the same underlying representation, which is only possible if this is zero (e.g. since the weight of a non-zero pure representation is the average weight of its generalised \( \varphi_K^{-1} \)-eigenvalues). Hence \( f = 0 \) in this case too.

Now we deal with the general case. We view \( V_1 \xrightarrow{f} V_0 \) as a filtered chain complex in the category of Weil–Deligne representations, with \( V_0 \) in degree 0. The associated (homological) spectral sequence [ML95, Theorem XI.3.1] has first page given by

\[
E^1_{i,j} = \begin{cases} 
\coker(\text{gr}^W_i f) & \text{if } i + j = 0, \\
\ker(\text{gr}^W_i f) & \text{if } i + j = 1, \\
0 & \text{else,}
\end{cases}
\]

and degenerates to

\[
E^\infty_{i,j} = \begin{cases} 
\text{gr}^W_i (\coker(f)) & \text{if } i + j = 0, \\
\text{gr}^W_i (\ker(f)) & \text{if } i + j = 1, \\
0 & \text{else.}
\end{cases}
\]

The differentials on the first page all vanish, since they are morphisms of pure Weil–Deligne representations whose domain has strictly higher weight than the codomain. The same argument establishes that all differentials on higher pages also vanish, and hence we have \( E^1_{i,j} = E^\infty_{i,j} \). In particular, \( \text{gr}^W_i (\ker(f)) \) and \( \text{gr}^W_i (\coker(f)) \) are both pure of weight \( i \) for all \( i \), so that \( \ker(f) \) and \( \coker(f) \) are mixed. Strictness of \( f \) also follows from degeneration at the first page, since this ensures that the natural maps \( \text{gr}^W_i \ker(f) \rightarrow \ker(\text{gr}^W_i f) \) and \( \text{gr}^W_i \coker(f) \rightarrow \coker(\text{gr}^W_i f) \) are isomorphisms.

**Proposition 2.11** (cf. [Vol03, Lemma 21]). In a \( W \)-strict short exact sequence

\[
0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0
\]

of filtered Weil–Deligne representations, if \( V_1 \) and \( V_2 \) are mixed, so too is \( V \).

**Proof.** Taking \( W \)-graded pieces, it suffices to prove that any extension of Weil–Deligne representations \( V_1, V_2 \) which are both pure of weight \( i \) is again of weight \( i \). It is easy to see that the sequences

\[
0 \rightarrow V_1^j \rightarrow V^j \rightarrow V_2^j \rightarrow 0
\]

are exact for each \( j \), and hence we are done by the five-lemma applied to \( N^j \). \( \square \)
2.1. Structure theory of mixed Weil–Deligne representations. In what follows, we will need several basic facts about the structure of mixed Weil–Deligne representations, most notably that their weight filtrations have canonical splittings compatible with their Weil group actions. This will be an immediate consequence of the following lemma describing to what extent one can lift maps between associated graded of mixed Weil–Deligne representations.

**Lemma 2.12.** Let $V_1$ and $V_2$ be mixed Weil–Deligne representations, and let $\gr_V^i: \gr_V^i V_1 \to \gr_V^i V_2$ be a morphism of graded Weil–Deligne representations. Then there exists a unique linear map $f: V_1 \to V_2$ satisfying the following properties:

1. $f$ is $W_K$-equivariant and preserves the $W$-filtration;
2. the associated $W$-graded of $f$ is the map $\gr_V^i f$; and
3. for every $r > 0$, the map

$$
\sum_{s=0}^{r} \binom{r}{s} (-1)^s N^{r-s} \circ f \circ N^s
$$

is $W$-filtered of degree $-r - 1$, i.e. takes $W_i V_1$ into $W_{r-i} V_2$ for every $i$. Moreover, the assignment $\gr_V^i f \mapsto f$ is linear, and compatible with composition and tensor products.

**Proof.** Let us say that a linear map $f: V_1 \to V_2$ is a weak morphism just when it satisfies conditions (1) and (3) above. In other words, a weak morphism is an element $f \in W_0 \Hom(V_1, V_2)^{W_K}$ such that $N^r(f) \in W_{-r-1} \Hom(V_1, V_2)$ for all $r > 0$, where $N$ denotes the monodromy operator on $\Hom(V_1, V_2) = V_1^* \otimes V_2$. It follows from this description that composites and tensor products of weak morphisms are weak morphisms, so it suffices to prove that every morphism $\gr_V^i f: \gr_V^i V_1 \to \gr_V^i V_2$ is induced by a unique weak morphism $f: V_1 \to V_2$.

To prove this, it suffices to prove that for every mixed Weil–Deligne representation $V$ and every element $\overrightarrow{f} \in \gr_V^i V^{W_K, N=0}$, there is a unique $f \in W_0 V^{W_K}$ lifting $\overrightarrow{f}$ such that $N^r(f) \in W_{-r-1} V$ for all $r > 0$; applying this to $V = \Hom(V_1, V_2)$ yields the desired result. Let $-i$ denote the lowest weight of $V$ — if $i \leq 0$ then all the weights of $V$ are non-negative and the result is trivial. In general, we proceed by induction on $i$, and write $V$ as an extension

$$
0 \to \gr_{-i} V \to V \to \overline{V} \to 0
$$

where the weights of $\overline{V}$ are all $> -i$. It follows from the inductive hypothesis that $\overrightarrow{f} \in \gr_V^i V = \gr_V^i (\overline{V})$ has a unique lift to an element $\overrightarrow{f} \in W_0 \overline{V}^{W_K}$ such that $N^r(f) \in W_{-r-1} \overline{V}$ for all $r > 0$. We have $\overrightarrow{f} \in \overline{V}^0$, and any lift of $\overrightarrow{f}$ to $f \in V^0$ lies in $W_0$ and satisfies $N^r(f) \in W_{-r-1} V$ for all $r \neq i$. Since $N^i(f) \in W_{-i} V = V_i^{-2i}$ (by assumption on $\overrightarrow{f}$ if $i = 1$ and by inductive assumption if $i > 1$), it follows from purity of $\gr_{-i} V$ that there is a unique choice of lift $f$ of $\overrightarrow{f}$ satisfying also $N^i(f) = 0$. Unicity implies that $f$ in $W_K$-fixed, and hence $f$ is the unique lift of $\overrightarrow{f}$ we sought.

**Definition 2.13.** Let $V$ be a mixed Weil–Deligne representation. The canonical splitting of the weight filtration is the $W_K$-equivariant linear isomorphism

$$
\gr_V^i V \cong V
$$
obtained by applying Lemma 2.12 to the evident isomorphism $\text{gr}_W^W \text{gr}_W^W V \cong \text{gr}_W^W V$. In other words, it is the $W_K$-equivariant map $f$ uniquely characterised by the fact that it takes $\text{gr}_i^W V$ into $W_i V$, and that for any $v_i \in \text{gr}_i^W V$ we have

$$\sum_{s=0}^{r} \binom{r}{s} (-1)^s N^{r-s} (f ((\text{gr}_i^W N)^s(v_i))) \in W_{i-r-1} V$$

for all $r > 0$, where $\text{gr}_W^W N$ denotes the induced monodromy operator on $\text{gr}_i^W V$.

It follows from Lemma 2.12 that this splitting is functorial and compatible with tensor products.

**Example 2.14.** Suppose that $V$ is an extension of $E$ by $E(1)$ in the category of Weil–Deligne representations. We endow $V$ with the filtration such that $W_0 V = V$, $W_{-1} V = W_{-2} V = E(1)$ and $W_{-3} V = 0$, so that $V$ is a mixed Weil–Deligne representation. $V$ admits a canonical choice of basis $v_0, v_2$, where $v_2 \in E(1)$ is the canonical generator and $v_0$ is the unique $\varphi_K$-invariant lift of the canonical generator of $E$. With respect to this basis, the actions of $\varphi_K$ and $N$ are given by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix},$$

respectively, for some $\lambda \in E$.

It follows from this description that the linear isomorphism $E \oplus E(1) \xrightarrow{\sim} V$ defined by the basis $v_0, v_2$ satisfies the conditions of Lemma 2.12, and hence is the canonical splitting of the weight filtration. Note that this splitting is not a splitting in the category of Weil–Deligne representations when $\lambda \neq 0$ above.

**Remark 2.15.** We’ve seen above that the canonical splitting from Definition 2.13 need not be a splitting in the category of Weil–Deligne representations. However, when the weights of $V$ are contained in $\{i-1, i\}$ for some $i$, then the splitting is automatically $N$-equivariant; one sees from this that any extension of a pure Weil–Deligne representation of weight $i$ by a pure representation of weight $i-1$ splits.

The following corollary plays a crucial role in this paper.

**Corollary 2.16 (to Definition 2.13).** Let $V$ be a mixed Weil–Deligne representation. Then we have that $V$ is Frobenius-semisimple if and only if $\text{gr}_i^W V$ is Frobenius-semisimple.

In fact, this canonical splitting of the weight filtration is part of a more refined structure theorem for mixed Weil–Deligne representations. To make this explicit, for $j \geq 0$ let us define the $j$th standard Weil–Deligne representation $\text{std}_j$ to be the Weil–Deligne representation on basis $1, \zeta, \ldots, \zeta^j$ on which the Weil group acts via $w: \zeta^r \mapsto p^{\varphi(w)} \zeta^r$ and monodromy acts via $N(\zeta^r) = (j-r) \zeta^{r+1}$. This is a pure Weil–Deligne representation of weight $-j$. The tensor product of two standard representations has a Clebsch–Gordan decomposition

$$\text{std}_{j_1} \otimes \text{std}_{j_2} = \bigoplus_{r \leq \min\{j_1, j_2\}} \text{std}_{j_1+j_2-2r}(r), \quad (2.17)$$

where $\text{std}_{j_1+j_2-2r}(r)$ is the subrepresentation generated by the element

$$\sum_{r_1 + r_2 = r} (-1)^{r_2} \binom{j_1 - r_2}{r_1} \binom{j_2 - r_1}{r_2} \zeta^{r_1} \otimes \zeta^{r_2}.$$

(This follows, for instance, from the fact that the Weil–Deligne action on $\text{std}_j$ factors through a certain action of $\text{GL}_2$, as described in §2.1.1.)
Definition 2.18. Let $V$ be a mixed Weil–Deligne representation. For any integers $i \in \mathbb{Z}$ and $j \in \mathbb{N}_0$, we define $$V^{i,j} = \left\{ x \in W_i V^{i,j} : N^{i+j}(x) \in W_{i-j-2r} V^{i-j-2r} \text{ for all } r > 0 \right\}.$$ It is easy to see that $V^{i,j}$ is a Weil subrepresentation of $V$ which is pure of weight $i+j$ (when viewed as a Weil–Deligne representation with trivial monodromy operator).

Theorem 2.19 (Structure theorem for mixed Weil–Deligne representations). Let $V$ be a mixed Weil–Deligne representation. Then the maps $V^{i,j} \hookrightarrow V$ extend canonically to a $W_K$-equivariant linear isomorphism $$\bigoplus_{i,j} V^{i,j} \otimes \text{std}_j \cong V$$ which is functorial in $V$ and compatible with tensor products with respect to the Clebsch–Gordan decomposition (2.17). The canonical splitting of the weight filtration from Definition 2.13 corresponds to the direct sum in the $i$ variable.

Proof. It follows from the definition that the composite maps $V^{i,j} \hookrightarrow W_i V \to \text{gr}^W_k V$ extend uniquely to morphisms $V^{i,j} \otimes \text{std}_j \to \text{gr}^W_k V$ of Weil–Deligne representations, which by Lemma 2.12 lift uniquely to weak morphisms $V^{i,j} \otimes \text{std}_j \to V$. It is easy to check that the construction $V \mapsto V^{i,j}$ is functorial with respect to weak morphisms, and hence this map is the identity on $V^{i,j}$.

We have thus constructed a map $\bigoplus_{i,j} V^{i,j} \otimes \text{std}_j \to V$, functorial with respect to weak morphisms. To show that this map is an isomorphism, it suffices by Definition 2.13 to treat the case that $V$ is pure of weight $i$. If $V$ is of the form $V_0 \otimes \text{std}_j$ with $V_0$ a pure Weil representation of weight $i$, then this is obvious (we have $V^{i,j} = V_0$ and all other $V^{i,j'} = 0$), so it suffices to prove that all pure Weil–Deligne representations of weight $i$ are direct sums of Weil–Deligne representations of this form.

Let $j$ be greatest such that $V^{i+j} \neq 0$. There is a morphism of Weil–Deligne representations $V^{i+j} \otimes \text{std}_j \to V$ sending $v \otimes \zeta^r \mapsto \left( \sum_{j' \leq j} N^{i+j'} (v) \right) \otimes \zeta^{i-j}$ if $r-j$ is even, and to 0 otherwise. Thus $V = V^{i+j} \otimes \text{std}_j \oplus V'$ for some pure Weil–Deligne representation $V'$ of weight $i$, and hence we are done by induction.

Lastly, if $V_1$ and $V_2$ are mixed Weil–Deligne representations, then tensoring together the decompositions of $V_1$ and $V_2$ gives a weak isomorphism $$\bigoplus_{i,j} \left( \bigoplus_{i_1+i_2=i, j_1+j_2=j, r \geq 0} V_1^{i_1,j_1+r} \otimes V_2^{i_2,j_2+r} (r) \right) \otimes \text{std}_j \cong V_1 \otimes V_2$$ using the Clebsch–Gordan decomposition (2.17). Since this is a weak isomorphism, it is necessarily the decomposition of $V_1 \otimes V_2$. \qed

Remark 2.20. If $V$ is a $W$-filtered $(\varphi, N, G_{L/K})$-module which is mixed in the sense that its associated Weil–Deligne representation is mixed, then all of the above constructions are compatible with the crystalline Frobenius $\varphi$. The subspace $V^i$ from Definition 2.6 is the largest $\varphi$-stable $\mathbb{Q}_p$-subspace such that the eigenvalues of $\varphi |_{V^i}$ (as a $\mathbb{Q}_p$-linear endomorphism) are $p$-Weil numbers of weight $i$. The decompositions in Definition 2.13 and Theorem 2.19 are $\varphi$-stable, where in the latter the crystalline Frobenius on $\text{std}_j$ acts via $\varphi(\zeta^r) = p^{-r} \zeta^r$. 

2.1.1. The metalinear action. The Structure Theorem 2.19 has the following curious consequence. One can endow the standard representation $\text{std}_j$ with an action of the general linear group $GL_2 = GL_{2,E}$, given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \zeta^r \mapsto \frac{1}{\det(M)^j}(a + c\zeta)^{j-r}(b + d\zeta)^r$$

(this formula defines a $\Lambda$-linear action of $GL_2(\Lambda)$ on $\Lambda \otimes \text{std}_j$ for every $E$-algebra $\Lambda$, and hence an action of the algebraic group $GL_2$). In particular, the matrices $X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ act unipotently on $\text{std}_j$, and their logarithms are the nilpotent endomorphisms given by $\log(X)(\zeta^r) = (j - r)\zeta^{r+1}$ (so equal to the monodromy operator on $\text{std}_j$) and $\log(Y) = r\zeta^{-1}$.

Now let $ML_2$ denote the metalinear group, i.e. the unique non-split extension of $GL_2$ by $C_2$. The elements of $ML_2$ are elements $M$ of $GL_2$ together with a choice of square root of $\det(M)$. Via the decomposition in Theorem 2.19, one can endow any mixed Weil–Deligne representation with an action of $ML_2$ (on its underlying vector space), by letting $ML_2$ act on each term $\text{std}_j$ as above and act via the $\frac{1}{2}$th power of the determinant on each $V^{i,j}$.

This action is functorial and compatible with tensor products, hence defines a morphism $ML_2 \to \mathcal{G}_{\text{mix}}$ into the Tannaka group $\mathcal{G}_{\text{mix}}$ of the category $\text{Rep}_E^{\text{mix}}(W_K)$. This map almost completely determines the structure of the Tannaka group $\mathcal{G}_{\text{mix}}$.

**Corollary 2.21** (to Theorem 2.19). The group $\mathcal{G}_{\text{mix}}$ is an amalgamated product of $U_{\text{mix}} \times ML_2$ and $\mathbb{G}_a \times W_K^{\text{mix}}$ over $\mathbb{G}_a \times \mathbb{G}_m$, where $U_{\text{mix}}$ is a pro-unipotent group and $W_K^{\text{mix}}$ is the Tannaka group of Weil representations all of whose Frobenius eigenvalues are $q$-Weil numbers.

**Proof (sketch).** Let $\mathcal{C}$ denote the category of finite dimensional graded vector spaces $V = \bigoplus_{i \in \mathbb{Z}} V^i$ endowed with a graded endomorphism $N$ of degree $-2$. The category $\mathcal{C}$ is Tannakian, canonically equivalent to the category of representations of $\mathbb{G}_a \times \mathbb{G}_m$ where the action of $\lambda \in \mathbb{G}_m$ on $\mathbb{G}_a$ is multiplication by $\lambda^{-2}$. We let $\mathcal{C}^{\text{mix}}$ denote the category of mixed objects of $\mathcal{C}$, defined analogously to Definition 2.6. It follows by the same argument as that in Theorem 2.10 that $\mathcal{C}^{\text{mix}}$ is Tannakian — we write $\mathcal{G}_{\text{mix}}'$ for its Tannaka group. One can prove that any object of $\mathcal{C}^{\text{mix}}$ admits a canonical $\mathbb{G}_m$-equivariant decomposition as in Theorem 2.19. It follows as above that there is a canonical map $ML_2 \to \mathcal{G}_{\text{mix}}'$.

On the other hand, any representation $V$ of $ML_2$ can be endowed with the structure of an object of $\mathcal{C}^{\text{mix}}$: the grading $V^\bullet$ is the one corresponding to the torus $\mathbb{G}_m \leq ML_2$ consisting of elements of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$; the monodromy operator $N$ is the logarithm of the action of $X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$; and the $W$-filtration is the filtration underlying the grading corresponding to the torus consisting of elements of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. This construction is functorial and compatible with tensor products, hence induces a map $\mathcal{G}_{\text{mix}}' \to ML_2$ which is easily checked to be a retraction of the map $ML_2 \to \mathcal{G}_{\text{mix}}'$ constructed above.

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To specify an element of $ML_2$, we need to specify an element $M$ of $GL_2$ together with a choice of square root of $\det(M)$. We suppress
Hence $\mathcal{G}_{\text{mix}}' = U_{\text{mix}} \times \text{ML}_2$ for some affine group-scheme $U_{\text{mix}}$. Since the essential image of the functor $\text{Rep}_G(\text{ML}_2) \to \mathcal{E}_{\text{mix}}$ contains all pure objects, it follows that the action of $U_{\text{mix}}$ on any $V \in \mathcal{E}_{\text{mix}}$ acts trivially on $\text{gr}^W V$, from which it follows that $U_{\text{mix}}$ is pro-unipotent.

Now to conclude, note that specifying the structure of a mixed Weil–Deligne representation on a vector space $V$ is equivalent to giving it the structure of an object of $\mathcal{E}_{\text{mix}}$ and the structure of a Weil–Deligne representation all of whose Frobenius-eigenvalues are $q$-Weil numbers such that the underlying gradings and monodromy operators agree. This gives the desired amalgamated product description of $\mathcal{G}_{\text{mix}} = \mathcal{G}_{\text{mix}}' * G_a \rtimes G_m \left( G_a \rtimes W_{\text{mix}}^\ell \right)$.

\begin{remark}
If $U$ is the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of a smooth curve $X/K$ (with $\ell \neq p$), then we will see in the next section that $\mathcal{O}(U)$ is ind-mixed with respect to a certain natural weight filtration (cf. also [BD19, Lemma 2.3.5]). [BD19, §5] gives a description of the splitting of the weight filtration guaranteed by Defnition 2.13 in terms of a certain graph of surface groups associated to $X$.
\end{remark}

\section{Results on semisimplicity and weight-monodromy}

As before, we fix $K$ a finite extension of $\mathbb{Q}_p$, with residue field $k$ and ring of integers $\mathcal{O}_K$. Let $X$ be a geometrically connected $K$-variety, and let $x \in X(K)$ be a rational point. Fixing an algebraic closure $\overline{K}$ of $K$, we let $\overline{x}$ be the geometric point of $X$ associated to $x$.

\subsection{The étale fundamental group.}

Let $\ell$ be a prime. We let $\pi_1^\ell(X_{\overline{K}}, \overline{x})$ be the pro-$\ell$ completion of the geometric étale fundamental group $\pi_1^1(X_{\overline{K}}, \overline{x})$. As $x$ was a rational point of $X$, there is a natural action of $\text{Gal}(\overline{K}/K)$ on $\pi_1^1(X_{\overline{K}}, \overline{x})$.

We let

$$\mathbb{Z}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]] := \lim_{\pi_1^1(X_{\overline{K}}, \overline{x}) \to H} \mathbb{Z}_\ell[H]$$

be the group ring of $\pi_1^1(X_{\overline{K}}, \overline{x})$, where the inverse limit is taken over all finite $\ell$-groups arising as continuous quotients of $\pi_1^1(X_{\overline{K}}, \overline{x})$. There is a natural augmentation map

$$\epsilon: \mathbb{Z}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]] \to \mathbb{Z}_\ell$$

(induced by the map $g \mapsto 1$, for $g \in \pi_1^1(X_{\overline{K}}, \overline{x})$), and we let $\mathcal{I}$ be the kernel of this map — the augmentation ideal. As with any group algebra, there is a natural comultiplication map

$$\Delta: \mathbb{Z}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]] \to \mathbb{Z}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]] \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]]$$

sending a group element $g$ to $g \otimes g$. (Here $\otimes$ denotes the completed tensor product.)

Finally, we set

$$\mathbb{Q}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]] = \lim_{\ell} (\mathbb{Z}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]]/\mathcal{I}^n \otimes \mathbb{Q}_\ell).$$

The comultiplication map $\Delta$ induces a comultiplication on $\mathbb{Q}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]]$. We abuse notation to denote the augmentation ideal of $\mathbb{Q}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]]$ by $\mathcal{I}$. The category of topological $\mathbb{Q}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]]$-modules which are finite-dimensional as $\mathbb{Q}_\ell$-vector spaces is equivalent to the category of continuous unipotent $\pi_1^1(X_{\overline{K}}, \overline{x})$-representations on $\mathbb{Q}_\ell$-vector spaces. The ring $\mathbb{Q}_\ell[[\pi_1^1(X_{\overline{K}}, \overline{x})]]$ is thus the (topological) opposite Hopf algebra to the ring of functions on the $\mathbb{Q}_\ell$-pro-unipotent fundamental group of $X$.
If $\bar{x}_1, \bar{x}_2$ are two geometric points of $X$, we let $\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)$ be the pro-finite set of “étale paths” from $\bar{x}_1$ to $\bar{x}_2$ (that is, the set of isomorphisms between the fiber functors associated to $\bar{x}_1, \bar{x}_2$). This is a (right) torsor for the group $\pi^\ell_1(X_K; \bar{x}_1)$; let $\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)$ be the associated (rightt) torsor for $\pi^\ell_1(X_K, \bar{x}_1)$. It is easy to check that the natural left action of $\pi^\ell_1(X_K, \bar{x}_2)$ on $\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)$ descends to a left action of $\pi^\ell_1(X_K; \bar{x}_1)$ on $\pi^\ell_1(X_K; \bar{x}_2)$. Let

$$Z_\ell[\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)] := \lim_{\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2) \to H} Z_\ell[H],$$

where the inverse limit is taken over all finite sets with a continuous surjection from $\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)$. This is a free right module of rank one over $\mathbb{Z}_\ell[\pi^\ell_1(X_K, \bar{x}_1)]$, and thus inherits an $\mathcal{I}$-adic filtration; we let

$$\mathbb{Q}_\ell[\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)] := \lim_{n} (\lim_{\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2) \to H} \mathbb{Z}_\ell[\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)]/\mathcal{I}^n \otimes \mathbb{Q}_\ell).$$

This vector space also has a natural filtration, which we call the $\mathcal{I}$-adic filtration by an abuse of notation, defined by

$$\mathcal{I}^n = \ker(\mathbb{Q}_\ell[\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)] \to \mathbb{Z}_\ell[\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)]/\mathcal{I}^n \otimes \mathbb{Q}_\ell).$$

We define a rational tangential basepoint of $X$ to be a $K((t))$-point of $X$; the inclusion $K \hookrightarrow K((t))$ allows one to view any $K$-point of $X$ as a rational tangential basepoint. We let $\overline{K((t))}$ be the usual algebraic closure of $K((t))$, namely the field of Puiseux series $\overline{K((t)^{\ell})}$, which we fix for the rest of this paper.

Now if $x_1, x_2$ are rational tangential basepoints of $X$, the group $\text{Gal}(\overline{K((t))/K((t))})$ acts on the triple $(X_{\overline{K}}, \bar{x}_1, \bar{x}_2)$, and hence by functoriality of $\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)$, on $\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)$. In particular, if $\bar{x}_1 = \bar{x}_2$ arise from a rational point $x \in X(K)$, we obtain an action of $\text{Gal}(\overline{K}/K)$ on $\pi^\ell_1(X_K, \bar{x}).$

Suppose $\ell \neq p$. Then we set

$$\Pi^\ell(X_K; \bar{x}_1, \bar{x}_2) := \mathbb{Q}_\ell[\pi^\ell_1(X_K; \bar{x}_1, \bar{x}_2)].$$

We give $\Pi^\ell(X_K; \bar{x}_1, \bar{x}_2)$ the structure of a Weil–Deligne representation as in Example 2.3. Explicitly, for each $n$, $\Pi^\ell(X_K; \bar{x}_1, \bar{x}_2)/\mathcal{I}^n$ is a finite-dimensional vector space, and thus naturally admits the structure of a Weil–Deligne representation as in Example 2.3. The construction is functorial, giving $\Pi^\ell(X_K; \bar{x}_1, \bar{x}_2)$ the structure of a pro-finite-dimensional Weil–Deligne representation.

3.2. The crystalline setting. Suppose $\ell = p$ (the residue characteristic of $K$), and let $x_1, x_2$ be rational tangential basepoints of $X$. Then we set

$$\Pi^p(X_K; \bar{x}_1, \bar{x}_2) := \lim_n \mathcal{D}_\text{pst}(\mathbb{Z}_p[\pi^p_1(X_K; \bar{x}_1, \bar{x}_2)]/\mathcal{I}^n \otimes \mathbb{Q}_p).$$

**Remark 3.1.** The object $\Pi^p(X_K; \bar{x}_1, \bar{x}_2)$ may be interpreted in terms of the log-crystalline cohomology of $(X, D)$, but we will not need this interpretation here.

We abuse notation and set

$$\mathcal{I}^n := \ker(\Pi^p(X_K; \bar{x}_1, \bar{x}_2) \to \mathcal{D}_\text{pst}(\mathbb{Z}_p[\pi^p_1(X_K; \bar{x}_1, \bar{x}_2)]/\mathcal{I}^n \otimes \mathbb{Q}_p)).$$

As in Example 2.4, $\Pi^p(X_K; \bar{x}_1, \bar{x}_2)$ has the structure of a (pro-finite-dimensional) Weil–Deligne representation.
3.3. The main theorems. Let \( X \) be a smooth geometrically connected variety over \( K \), and \( x_1, x_2 \) rational tangential basepoints of \( X \). Let \( \ell \) be a prime.

The main theorems of this section are:

**Theorem 3.2** (Weight-monodromy). The Weil–Deligne representation \( \Pi^\ell(X_\bar K; \bar x_1, \bar x_2) \), with the canonical weight filtration (Definition 3.8 below), is mixed.

**Theorem 3.3** (Semisimplicity). Each element of \( W_K \) acts semisimply on the Weil–Deligne representation \( \Pi^\ell(X_\bar K; \bar x_1, \bar x_2) \).

Theorem 3.3 above admits the following down-to-earth reformulation. If \( \ell \neq p \), the theorem says that every Frobenius element of \( \text{Gal}(\bar K/K) \) acts semisimply on \( \mathbb{Z}_\ell[[\pi^\ell_1(X_\bar K, \bar x_1)]] \), or equivalently that every element of \( W_K \) acts semisimply on \( \Pi^\ell(X_\bar K; \bar x_1, \bar x_2)/\mathcal{I}^n \) for all \( n \) (with the structure of a Weil–Deligne representation given by Example 2.3). If \( \ell = p \), the theorem is the analogous statement for a \( K \)-linear power of the crystalline Frobenius. In both cases, the statement is equivalent to the semi-simplicity of the geometric Frobenius \( \varphi_K \) fixed at the beginning of Section 2.

As an immediate corollary, we have

**Corollary 3.4.** Let \( x \) be a rational tangential basepoint of \( X \). Then the Lie algebra \( g_X \) of the \( \mathbb{Q}_\ell \) pro-unipotent completion of \( \pi^\ell_1(X_\bar K, \bar x) \) is a mixed Weil-Deligne representation (with respect to the weight filtration) and each element of \( W_K \) acts semisimply on it.

**Proof.** The Lie algebra \( g_K \) may be identified as the set of primitive elements in \( \Pi^\ell(X_\bar K; \bar x, \bar x) \), i.e. the kernel of the map
\[
\Delta - \text{id} \otimes 1 - 1 \otimes \text{id}.
\]

The result is immediate. \( \square \)

3.3.1. Preliminaries. Before giving the proof, we will need to recall some lemmas, most of which are likely well-known to experts.

**Proposition 3.5.** There is a canonical (Galois-equivariant) isomorphism
\[
\pi^\ell_1(X_\bar K, \bar x_1)^{ab} \cong \mathcal{S}/\mathcal{S}^2.
\]
Moreover \( \pi^\ell_1(X_\bar K, \bar x_1)^{ab}/\pi^\ell_1(X_\bar K, \bar x_1)^{ab}[\ell^\infty] \cong H^1(X_{\bar K, \text{ét}}, \mathbb{Z}_\ell) \) canonically (in particular, as Galois modules).

**Proof.** See [Lit18, Proposition 2.4]. \( \square \)

We will also need the following part of the Weight–Monodromy Conjecture.

**Proposition 3.6.** Let \( Y \) be any smooth \( K \)-variety with \( \dim(Y) \leq 2 \). Let \( i \in \mathbb{Z} \) and let \( W_* \) be the weight filtration on \( H^i(Y_{\bar K, \text{ét}}, \mathbb{Q}_\ell) \) [Del71]. Then the Weil–Deligne representation \( H^i(Y_{\bar K, \text{ét}}, \mathbb{Q}_\ell) \) is mixed with positive weights.

**Proof.** Let \( Y \) be a simple normal crossings compactification of \( Y \), which exists by resolution of singularities.

For \( \ell \neq p \), the case of smooth proper \( Y \) is proven for \( \ell \neq p \) in [RZ82, Satz 2.13]; the case \( \ell = p \) is proven by Mokrane [Mok93, Corollaire 6.2.3] (Mokrane proves the log-crystalline statement; the statement here follows by applying the \( p \)-adic comparison theorem [Tsu99, Theorem 0.2].) The general case follows immediately...
from the Deligne spectral sequence, i.e. the Leray spectral sequence associated to the embedding $Y \hookrightarrow \overline{Y}$ (see e.g. [Jan10, pg. 2]), using Theorem 2.10 and Proposition 2.11.

**Proposition 3.7.** Let $Y$ be any smooth $K$-variety.

1. $(\ell \neq p)$: $\varphi_K$ acts semi-simply on $H^1(Y_{\overline{K}, \text{ét}}, \mathbb{Q}_\ell)$.
2. $(\ell = p)$: $\varphi_K$ acts semi-simply on $D_{\text{pers}}(H^1(Y_{\overline{K}, \text{ét}}, \mathbb{Q}_p))$.

**Proof.** We may reduce to the case $Y$ is quasi-projective by replacing $Y$ with an affine open. By the Lefschetz hyperplane theorem, it suffices to prove this for $Y$ a curve. Let $\overline{Y}$ by the smooth compactification of $Y$; after extending $K$, we may assume $\overline{Y}$ has semistable reduction, and that $\overline{Y} \setminus Y$ is a disjoint union of rational points of $Y$.

Then the result for $\overline{Y}$ smooth proper is immediate from the Rapoport–Zink spectral sequence (see [RZ82, Satz 2.10] for the case $\ell \neq p$ and [Mok93, 3.23] for the case $\ell = p$, again using the $p$-adic comparison theorem [Tsu99, Theorem 0.2] to apply the statement) and the analogous fact for abelian varieties. To deduce the result for $Y$, note that

$$H^1(\overline{Y}_{\overline{K}}, \mathbb{Q}_\ell) \cong W_1H^1(Y_{\overline{K}}, \mathbb{Q}_\ell) \rightarrow H^1(Y_{\overline{K}}, \mathbb{Q}_\ell)$$

splits $\varphi_K$-equivariantly by Definition 2.13; but $H^1(Y_{\overline{K}}, \mathbb{Q}_\ell)/W_1H^1(Y_{\overline{K}}, \mathbb{Q}_\ell)$ is isomorphic to a direct sum of copies of $\mathbb{Q}_\ell(1)$, so the result follows.

**3.4. Weight-monodromy for $\pi_1$.** We are now ready to prove Theorem 3.2.

Let $Y$ be any smooth geometrically connected $K$-variety with simple normal crossings compactification $\overline{Y}$; let $x_1, x_2$ be rational tangential basepoints of $Y$.

**Definition 3.8 (Weight filtration on $\Pi^f$).** We define the weight filtration on $\Pi^f(Y_{\overline{K}}; \overline{x}_1, \overline{x}_2)$.

Let

$$\mathcal{W} = \ker(\Pi^f(Y_{\overline{K}}; \overline{x}_1, \overline{x}_2) \rightarrow \Pi^f(\overline{Y}_{\overline{K}}; \overline{x}_1, \overline{x}_2))$$

$$W_{-1} = \mathcal{W}$$

$$W_{-2} = \mathcal{W}^2 + \mathcal{W}$$

and in general

$$W_{-i} = \sum_{p+q=i, p,q > 0} W_{-p} \cdot W_{-q} \text{ for } i > 2.$$

**Remark 3.9.** As with any weight filtration arising in algebraic geometry, we claim the filtration defined above is uniquely characterized as follows. Let $R \subset K$ be a finitely-generated $\mathbb{Z}$-algebra, $\mathcal{F}$ an $R$-model of $Y$ (that is, a flat $R$-scheme equipped with an isomorphism $\mathcal{F}_K \cong Y$). After possibly enlarging $R$, we may let $y_1, y_2$ be $R$-points of $Y$ so that the fiber functors associated to $y_i, \overline{y}_i$ are Galois-equivariantly isomorphic to those associated to those associated to $\overline{x}_i$ (this is possible by [Del89, 15.13–15.27]), which shows that these fiber functors are determined by a finite amount of data, though a $K((t))$-point of $Y$ is not). Then there exists an open subset $U$ of $\text{Spec}(R[1/\ell])$ such that for any closed point $p$ of $U$, the associated Frobenius element acts on $\text{gr}^W_0{\Pi^f(Y_{\overline{K}(p)}; y_1, \overline{y}(p), y_2, \overline{y}(p))}$ with eigenvalues #$(k(p))$-Weil numbers of weight $i$.

In particular, the induced filtration on $\mathcal{F}/\mathcal{F}^2$ agrees with the usual weight filtration coming from the identification with $H^1(Y_{\overline{K}}, \mathbb{Q}_\ell)$ in Proposition 3.5 by construction; then the claim above follows by the multiplicativity of the weight filtration.
We will use below the resulting compatibility with another description of the weight filtration arising from work of Deligne and Goncharov [DG05].

Remark 3.10. We briefly explain why the Galois representation \( \mathbb{Z}_p[\pi_1^p(X_{\bar{K}}; \bar{x}_1, \bar{x}_2)]/\mathcal{F}^n \otimes \mathbb{Q}_p \) is de Rham — this is proven in [Bet19, Lemma 7.1] if \( \bar{x}_1, \bar{x}_2 \) arise from rational points of \( X \), but does not appear in the literature if these geometric points arise from rational tangential basepoints. We will require this for the proof of Theorem 3.2.

Deligne and Goncharov [DG05, Proposition 3.4] construct a local system on \( X \times X \) whose fiber at a point \( (\bar{x}_1, \bar{x}_2) \) is \( (\mathbb{Z}_p[\pi_1^p(X_{\bar{K}}; \bar{x}_1, \bar{x}_2)]/\mathcal{F}^n \otimes \mathbb{Q}_p) \), as a higher direct image of a sheaf on a diagram of schemes over \( X \times X \) (strictly speaking, Deligne and Goncharov work in the Betti setting, but an identical construction works in the étale setting). The fiber of this local system at a \( \overline{K} \)-point of \( X \times X \) is de Rham by e.g. [Bet19, Lemma 7.1] (or by [AIK15, Theorem 1.4] in the case this point lies on the diagonal of \( X \times X \)).

Hence this local system is de Rham in the sense of [LZ17] by [LZ17, Theorem 1.3]. Now the result at tangential basepoints follows from [DLLZ18, Corollary 5.60], for example.

Proof of Theorem 3.2. We explain how to deduce the theorem from Proposition 3.6. By Chow’s lemma, we may assume \( X \) is quasi-projective.

First, note that by the Lefschetz hyperplane theorem [GM88, pg. 195] for fundamental groups, we may reduce to the case where \( \dim(X) \leq 2 \).

In the case \( \ell \neq p \), recall from [DG05, Proposition 3.4] that \( (\Pi^*(X_{\bar{K}}; \bar{x}_1, \bar{x}_2)/\mathcal{F}^n)^\vee \) may be computed as the hypercohomology of a complex of sheaves on \( X^{n-1} \); each of these sheaves is a direct sum of sheaves of the form \( j_! \mathbb{Q}_\ell \), where \( j \): \( X^m \to X^{n-1} \) is a closed embedding. Thus, by Proposition 3.6, there is a spectral sequence whose \( E^1 \) term consists of mixed Weil–Deligne representations of the form \( H^i(X^m, \mathbb{Z}_\ell) \), where the weight filtration comes from the usual weight filtration on cohomology [Del71] (using the Künneth formula), and whose \( E^\infty \) page has on it \( gr_{\mathcal{F}^*}(\Pi^*(X_{\bar{K}}; \bar{x}_1, \bar{x}_2)/\mathcal{F}^n)^\vee \) for some filtration \( \mathcal{F}^* \). Now we may conclude the result by Theorem 2.10 and Proposition 2.11.

In the case \( \ell = p \), we may conclude once we know that the spectral sequence indeed converges after applying \( D_{\text{pst}} \), which follows as \( \mathbb{Z}_p[\pi_1^p(X_{\bar{K}}; \bar{x}_1, \bar{x}_2)]/\mathcal{F}^n \otimes \mathbb{Q}_p \) is de Rham (hence potentially semistable) by Remark 3.10. \( \square \)

3.5. Semisimplicity. We now begin preparations for the proof of Theorem 3.3. The canonical splitting of the weight filtration from Definition 2.13 induces a \( W_K \)-equivariant splitting of the natural quotient map

\[
\Pi^*(Y_{\bar{K}}; \bar{x}_1, \bar{x}_2) \to \Pi^*(Y_{\bar{K}}; \bar{x}_1, \bar{x}_2)/\mathcal{F}.
\]

Definition 3.11 (Canonical Paths). We denote the image of 1 under this splitting by \( p(x_1, x_2) \). This is \( W_K \)-invariant by construction – in the case \( \ell = p \) it is moreover invariant under the crystalline Frobenius.

Proposition 3.12. Let \( x_1, x_2, x_3 \) be rational points or rational tangential basepoints of \( X \). Then

1. \( p(x_1, x_1) = 1 \), and
2. \( p(x_2, x_3) \circ p(x_1, x_2) = p(x_1, x_3) \).
Proof: (1) is immediate from the definition; (2) follows from compatibility with tensor products. 

Remark 3.13. In the case \( \ell = p \), the paths \( p(x_1, x_2) \) are Vologodsky’s canonical \( p \)-adic paths [Vol03, Proposition 29]. In the case \( \ell \neq p \), the paths \( p(x_1, x_2) \) are the canonical \( \ell \)-adic paths \( \gamma_{x_1, x_2}^{\text{can}} \) from [BD19, Remark 2.2.5].

Proof of Theorem 3.3. This is a more involved variant of [Lit18, Theorem 2.12].

By the Lefschetz hyperplane theorem for fundamental groups, we may without loss of generality assume \( \dim(X) \leq 1 \). Indeed, there exists a smooth curve \( C \hookrightarrow X \) such that the induced map on fundamental groups is a surjection; it suffices to prove the theorem for \( C \). So we assume \( \dim(X) = 1 \) and let \( \overline{X} \) be the connected smooth proper curve compactifying \( X \). Let \( D = \overline{X} \setminus X \).

Without loss of generality (by replacing \( K \) with a finite extension) we may assume \( D = \{x_1, \ldots, x_n \} \), with the \( x_i \in \overline{X}(K) \) rational points of \( \overline{X} \).

Recall that if \( \ell \neq p \), we have fixed a Frobenius element \( \varphi_K \in \text{Gal}(\overline{K}/K) \); if \( \ell = p \), we let \( \varphi_K = \varphi^{(K/Q_p)}(\ell) \) be the smallest power of the crystalline Frobenius which is \( K \)-linear (so the geometric Frobenius of the underlying Weil–Deligne representation).

We first claim that it suffices to prove the theorem when \( x_1 = x_2 \). Indeed, suppose we know the theorem for \( x_1 \). Then composition with \( p(x_1, x_2) \) is a \( \varphi_K \)-equivariant isomorphism

\[
\Pi^\ell(X_K, \overline{x}_1) \iso \Pi^\ell(X_K; \overline{x}_1, \overline{x}_2).
\]

So \( \varphi \) acts semisimply on \( \Pi^\ell(X_K; \overline{x}_1, \overline{x}_2) \). Hence we may and do assume \( x_1 = x_2 = x \) for the rest of the proof.

We now claim it suffices to show that the quotient map

\[
\mathcal{I} \to \mathcal{I}/\mathcal{I}^2
\]

splits \( \varphi_K \)-equivariantly. Indeed, let \( s: \mathcal{I}/\mathcal{I}^2 \to \mathcal{I} \) be such a splitting; then the map

\[
\bigoplus_n (\mathcal{I}/\mathcal{I}^2)^{\otimes n} \oplus \mathcal{I}^{\otimes n} \to \Pi^\ell(X_K; \overline{x}_1, \overline{x}_2)
\]

has dense image. But \( \varphi_K \) acts semi-simply on \( (\mathcal{I}/\mathcal{I}^2) \) by Propositions 3.5 and 3.7, so we may conclude the theorem.

We now construct such a \( \varphi_K \)-equivariant splitting \( s \).

Step 1. We first construct a splitting of the quotient map

\[
\mathcal{I} \to \mathcal{I}/W_{-2}.
\]

But this map splits \( \varphi_K \)-equivariantly by the formula in Definition 2.13; choose any \( \varphi_K \)-equivariant splitting \( s_1 \).

Step 2. We now construct a splitting of the map

\[
W_{-2} \Pi^\ell(X_K, \overline{x}) \to W_{-2}/\mathcal{I}^2.
\]

For each \( i = 1, \ldots, n \), we choose a rational tangential basepoint \( y_i \in X(K((t))) \) so that the associated \( K[[t]] \)-point of \( \overline{X} \) (obtained via the valuative criterion for properness) specializes to \( x_i \). Let \( p_i = p(x, y_i) \) be the canonical path arising from Definition 3.11.

\( (\ell \neq p) \): Let \( \gamma \) be a topological generator of

\[
\text{Gal}(K((t))/\overline{K((t)))}^\ell = \pi_1^\ell(\text{Spec}(\overline{K((t)))}, \text{Spec}(K((t)))) = \mathbb{Z}_\ell(1).
\]
Then the maps
\[ \iota_i: y_i \rightarrow X \]
induce maps
\[ \iota_i: \pi_1^e(\text{Spec}(K(t)), \text{Spec}(K(t))) \rightarrow \pi_1^e(X_K, \bar{y}_i). \]
Let \( \gamma_i \) be the image of \( \iota_i(\gamma) - 1 \) in \( W_{-2}/\mathcal{G}^2 \). Then the map
\[ \iota_*: \mathbb{Q}_\ell(1)^{\{x_1, \cdots, x_n\}} \rightarrow W_{-2}/\mathcal{G}^2 \]
\[ (a_1, \cdots, a_n) \gamma \mapsto \sum a_i \gamma_i \]
is surjective and \( \varphi_K \)-equivariant; as \( \varphi_K \) acts semisimply on \( \mathbb{Q}_\ell(1)^{\{x_1, \cdots, x_n\}} \), \( \iota_* \) splits \( \varphi_K \)-equivariantly, so it suffices to construct a \( \varphi_K \)-equivariant map
\[ \tilde{s}_2: \mathbb{Q}_\ell(1)^{\{x_1, \cdots, x_n\}} \rightarrow \mathcal{G} \]
such that the diagram
\[ \begin{array}{ccc}
\mathbb{Q}_\ell(1)^{\{x_1, \cdots, x_n\}} & \xrightarrow{\tilde{s}_2} & \mathcal{G} \\
\downarrow{\iota_*} & & \downarrow{\iota_*} \\
W_{-2}/\mathcal{G}^2 & \xrightarrow{\iota_*} & W_{-2}/\mathcal{G}^2
\end{array} \]
commutes.

Set \( \tilde{s}_2 \) to be the map
\[ \tilde{s}_2: (a_1, \cdots, a_n) \gamma \mapsto \sum_{i=1}^n a_i p_i \cdot \log(\iota_* \gamma) \cdot \bar{p}_i^{-1} \]

A direct computation shows that this gives the desired section; see the proof of [Lit18, Theorem 2.12] for an identical computation.

Finally, let \( p \) be any section to \( \iota_*; \) then we set \( s_2 = \tilde{s}_2 \circ p \).

(\( \ell = p \)): Let \( \beta \) be a topological generator of \( \mathbb{Z}_p(1) \) and let
\[ \Pi^p(y_i) := \varprojlim_n \text{D}_\text{st}(\mathbb{Z}_p[\mathbb{Z}_p(1)])/(\beta - 1)^n \otimes \mathbb{Q}_p). \]

Now the Weil–Deligne representation \( K(1) := \text{D}_\text{st}(\mathbb{Q}_p(1)) \) has \( \varphi_K \)-action given by multiplication by \( q^{-1} = (\# k)^{-1} \) and \( N \equiv 0 \). As \( (\beta - 1)/(\beta - 1)^2 \simeq \mathbb{Z}_p(1) \), there is a \( \varphi \)-equivariant isomorphism
\[ \Pi^p(y_i) \simeq \prod_{i \geq 0} K(i), \]
where \( K(i) \) is the \( \varphi_K \)-module \( K(1)^{\otimes i} \). Let \( \gamma \) be an element of \( \Pi^p(y_i) \) such that \( \varphi_K(\gamma) = q \gamma \).

The map \( \iota_i \) induces a map
\[ \iota_i: \Pi^p(y_i) \rightarrow \Pi^p(X, \bar{y}_i); \]
let \( \gamma_i = \iota_i(\gamma) \).

Now the map
\[ \iota_*: K(1)^{\{x_1, \cdots, x_n\}} \rightarrow W_{-2}/\mathcal{G}^2 \]
\[ (a_1, \cdots, a_n) \mapsto \sum a_i \gamma_i \]
is surjective and \( \varphi_K \)-equivariant; as \( \varphi_K \) acts semisimply on \( K(1)^{\{x_1, \cdots, x_n\}} \), \( \iota_* \) splits \( \varphi_K \)-equivariantly, so it suffices to construct a \( \varphi_K \)-equivariant map
\[ \tilde{s}_2: K(1)^{\{x_1, \cdots, x_n\}} \rightarrow \mathcal{G} \]
such that the diagram
\[
\begin{array}{ccc}
K(1)\{x_1, \cdots, x_n\} & \xrightarrow{\tilde{s}_2} & W_{-2}/\mathcal{I}^2 \\
\downarrow_{\iota_*} & & \downarrow \\
W_{-2} & \xrightarrow{\iota_*} & W_{-2}/\mathcal{I}^2
\end{array}
\]
commutes.

Set \(\tilde{s}_2\) to be the map
\[
\tilde{s}_2: (a_1, \cdots, a_n) \gamma \mapsto \sum_{i=1}^n a_i p_i \cdot \gamma_i \cdot p_i^{-1}
\]
Again, this gives the desired section by an argument identical to the proof of [Lit18, Theorem 2.12].

Finally, let \(p\) be any section to \(\iota_*\); then we set \(s_2 = \tilde{s}_2 \circ p\).

**Step 3.** We now construct the desired \(\varphi_K\)-equivariant section
\[
s: \mathcal{I}/\mathcal{I}^2 \to \mathcal{I}.
\]

This is \(\varphi_K\)-equivariant because the same is true for \(s_1, s_2\), and is a section by direct computation. This completes the proof. \(\square\)

4. Structure of local Selmer schemes

We now turn to our main application of these results. As before, let \(K\) be a finite extension of \(\mathbb{Q}_p\), and let \(U/\mathbb{Q}_p\) be a topologically finitely generated pro-unipotent group with a continuous action of \(G_K\); assume moreover that \(U\) is de Rham in the sense of [Bet19, Definition–Lemma 4.2.2]. We have in mind that \(U\) is the \(\mathbb{Q}_p\)-pro-unipotent étale fundamental group of a smooth geometrically connected variety, but will not assume this in what follows. For the sake of simplicity, we will assume throughout this section that \(U\) is unipotent (i.e. finite-dimensional); the arguments are easily extended to the finitely generated pro-unipotent case.

One can associate to \(U\) a continuous Galois cohomology presheaf \(H^1(G_K, U)\) of pointed sets on the category \(\text{Aff}_{\mathbb{Q}_p}\) of affine \(\mathbb{Q}_p\)-schemes, namely the presheaf whose sections over some \(\text{Spec}(\Lambda)\) is \(H^1(G_K, U(\Lambda))\). Here \(H^1(G_K, -)\) denotes continuous Galois cohomology, and the topology on \(U(\Lambda)\) is the natural one arising from endowing \(\Lambda\) with the inductive limit topology, viewing it as a direct limit of finite dimensional \(\mathbb{Q}_p\)-vector spaces. The local Bloch–Kato Selmer presheaves (cf. [Kim05, §2] & [Bet19, Definition 1.2.1]) are three sub-presheaves

\[
H^1_e(G_K, U) \subseteq H^1_f(G_K, U) \subseteq H^1_g(G_K, U) \subseteq H^1(G_K, U) \quad (4.1)
\]
whose sections over some \(\text{Spec}(\Lambda)\) are given by
\[
H^1_e(G_K, U)(\Lambda) = \ker \left( H^1(G_K, U(\Lambda)) \to H^1(G_K, U(B^{\text{cris}}_\Lambda \otimes \Lambda)) \right),
\]
\[
H^1_f(G_K, U)(\Lambda) = \ker \left( H^1(G_K, U(\Lambda)) \to H^1(G_K, U(B_{\text{cris}} \otimes \Lambda)) \right),
\]
\[
H^1_g(G_K, U)(\Lambda) = \ker \left( H^1(G_K, U(\Lambda)) \to H^1(G_K, U(B_{\text{-stat}} \otimes \Lambda)) \right).
\]
On the right-hand side of these equations, the topology on, for example, $\mathcal{B}_{\text{cris}} \otimes \Lambda$ is the natural one arising from writing $\Lambda$ as a direct limit of finite dimensional $\mathbb{Q}_p$-subspaces.

An argument of Kim establishes that the Bloch–Kato Selmer presheaf $H^1_d(G_K, U)$ is representable when $U$ is the maximal $n$-step unipotent quotient of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of a smooth projective curve. In fact, the argument only uses a condition on the weights of $U$.

**Lemma 4.2** ([Kim09, Lemma 5]). Let $U/\mathbb{Q}_p$ be a de Rham representation of $G_K$ on a unipotent group, and assume that $U$ is mixed\(^4\) with negative weights. Then $H^1_d(G_K, U)$ is representable by an affine scheme over $\mathbb{Q}_p$.

In fact, the argument gives a precise description of the representing scheme. For a de Rham representation of $G_K$ on a unipotent group $U/\mathbb{Q}_p$, let $D_{\text{dR}}(U)$ be the unipotent group over $K$ representing the presheaf

$$\text{Spec}(\Lambda) \mapsto \text{D}_{\text{dR}}(U)(\Lambda) := U(\text{D}_{\text{dR}} \otimes_K \Lambda)^{G_K}$$

where $\text{D}_{\text{dR}}$ is the de Rham period ring (see [Bet19, Lemma 4.2.1]). We define in the same way unipotent groups $D^+_{\text{dR}}(U)$, $D^{\varphi=1}_{\text{cris}}(U)$, $D_{\text{cris}}(U)$ and $D_{\text{st}}(U)$ over $K$, $\mathbb{Q}_p$, $K_0$ and $K_0$, respectively, using the rings $\mathcal{B}_{\text{dR}}$, $\mathcal{B}_{\text{cris}}^{\varphi=1}$, $\mathcal{B}_{\text{cris}}$ and $\mathcal{B}_{\text{st}}$, respectively, in place of $\mathcal{B}_{\text{dR}}$. The argument of Kim establishes that in the setup of Lemma 4.2 there is an isomorphism

$$H^1_d(G_K, U) \cong \text{Res}^K_{\mathbb{Q}_p} (\text{D}_{\text{dR}}(U)/\text{D}^+_{\text{dR}}(U))$$

of presheaves on $\text{Aff}_{\mathbb{Q}_p}$ (where the right-hand side denotes the presheaf quotient). In particular, the representing variety is an affine space: if one chooses a splitting $D_{\text{dR}}(\text{Lie}(U)) = V \oplus D^+_{\text{dR}}(\text{Lie}(U))$ of the Hodge filtration on the Lie algebra of $U$, then there is an isomorphism

$$\text{Res}^K_{\mathbb{Q}_p} V \cong \text{Res}^K_{\mathbb{Q}_p} (\text{D}_{\text{dR}}(U)/\text{D}^+_{\text{dR}}(U))$$

of presheaves on $\text{Aff}_{\mathbb{Q}_p}$, where by abuse of notation we also denote by $V$ the associated affine space $\text{Spec}(\Lambda) \mapsto \Lambda \otimes V$.

Our aim in this section is to extend this to descriptions of all three Bloch–Kato Selmer presheaves, and in particular to show that under the same assumptions on the weights, they are all also represented by affine spaces. In fact, by imitating the arguments in [BD19] we will obtain descriptions of the Bloch–Kato Selmer presheaves as (the affine spaces underlying) vector spaces; these descriptions are all canonical, up to a choice of the splitting of the Hodge filtration.

**Theorem 4.3.** Let $U$ be a filtered de Rham representation of $G_K$ on a unipotent group over $\mathbb{Q}_p$, which is mixed with negative weights. Then there are canonical natural isomorphisms

$$H^1_d(G_K, U) \cong \text{Res}^K_{\mathbb{Q}_p} (\text{D}_{\text{dR}}(U)/\text{D}^+_{\text{dR}}(U))$$

$$H^1_d(G_K, U) \cong \text{Res}^K_{\mathbb{Q}_p} (\text{D}_{\text{dR}}(U)/\text{D}^+_{\text{dR}}(U))$$

$$H^1_d(G_K, U) \cong \text{Res}^K_{\mathbb{Q}_p} (\text{D}_{\text{dR}}(U)/\text{D}^+_{\text{dR}}(U)) \times \mathbb{V}_{\text{st}}^{p=1}(U)^{p=1}$$

\(^4\)Whenever we say that a representation of $G_K$ on a unipotent group $U$ is mixed, we will always mean that $\text{Lie}(U)$ is a mixed representation, or equivalently that $\vartheta(U)$ is an ind-mixed representation. When we refer to the weights of $U$, we mean the weights of $\text{Lie}(U)$. 
of presheaves for a certain \( \varphi \)-module \( V_{g/e}^\ast (U) \) functorially assigned to \( U \) (for a precise description, see below). These descriptions are compatible with the inclusions (4.1).

In particular all three presheaves are representable by affine spaces, and the dimension of these spaces is given by

\[
\dim \mathbb{Q}_p H^1_c(G_K, U) = [K : \mathbb{Q}_p] \sum_{i > 0} (\dim_K D_{\text{dr}}(\text{gr}^W_{-i} U) - \dim_K D^+_{\text{dr}}(\text{gr}^W_{-i} U))
\]

\[
\dim \mathbb{Q}_p H^1_t(G_K, U) = [K : \mathbb{Q}_p] \sum_{i > 0} (\dim_K D_{\text{dr}}(\text{gr}^W_{-i} U) - \dim_K D^+_{\text{dr}}(\text{gr}^W_{-i} U))
\]

\[
\dim \mathbb{Q}_p H^1_g(G_K, U) = [K : \mathbb{Q}_p] \sum_{i > 0} (\dim_K D_{\text{dr}}(\text{gr}^W_{-i} U) - \dim_K D^+_{\text{dr}}(\text{gr}^W_{-i} U))
\]

\[+ \sum_{i > 0} \dim \mathbb{Q}_p D^\ast_{\text{cris}}(1) (\text{gr}^W_{-1} U)^\ast (1)\]

where the right-hand side is the sum of the dimensions of the Bloch–Kato Selmer groups. If \( \ast \in \{ e, f \} \), or if \( \ast = g \) and \( U \) is Frobenius-semisimple, the same holds for the descending central series in place of the weight filtration.

Example 4.4. Suppose that \( X/K \) is a smooth projective curve of genus \( g \) with semistable reduction, and that all irreducible components of the geometric special fibre of the minimal regular model of \( X \) are defined over the residue field \( k \). Let \( Y = X \setminus \{ x \} \) for a point \( x \in X(K) \) and let \( U_n/\mathbb{Q}_p \) denote the maximal \( n \)-step unipotent quotient of the \( \varphi \)-pro-unipotent étale fundamental group of \( Y_{\overline{K}} \) (at a basepoint \( b \in Y(K) \)). Then we have

\[
\dim \mathbb{Q}_p H^1_c(G_K, U_n) = [K : \mathbb{Q}_p] \cdot (L_{\leq n}(2g) - L_{\leq n}(g)) - L_{\leq n}(g_0) + \frac{g_0^{n+1} - g_0}{g_0 - 1} + \frac{(n - 1)g_0^n - n g_0^{n-1} + 1}{2(g_0 - 1)^2} \sum_{\lambda} \nu_{\lambda}^2 - 2(\nu_0^2 + \nu_{\sqrt{-1}}^2 + \nu_{-\sqrt{-1}}^2)\]

where \( g_0 \) is the genus of the reduction graph of \( X \), \( \nu_{\lambda} \) are the multiplicities of the weight 1 eigenvalues of \( \varphi_K \) acting on \( H^1_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p) \), and \( L_{\leq n}(T) := \sum_{1 \leq i \leq n} \sum_{d_i} \nu(d_i) T^{i/d} \) is the summed necklace polynomial (the number of Lyndon words of length \( \leq n \) in an alphabet of \( T \) letters).

Proof (sketch). We will calculate the \( \mathbb{Q}_p \)-dimension of \( D^\ast_{\text{cris}}(1) (\text{gr}^W_{-1} U_n)^\ast (1) \) for \( i \leq n \), leaving the remainder of the calculation to the reader. This \( \mathbb{Q}_p \)-dimension is equal to the \( K_0 \)-dimension of the subspace of \( D_{\text{et}}(\text{gr}^W_{-1} U_n^\ast) \) on which \( \varphi_K \) acts via \( q \) and \( N \) acts by 0. Since \( U_n \) is a free \( n \)-step unipotent group, \( D_{\text{et}}(\text{gr}^W_{-1} U_n^\ast) \) has a basis parametrised by Lyndon words in a basis of \( D_{\text{et}}(H^1_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)) \).

By semisimplicity, we may pick a basis of \( \overline{K} \otimes_{K_0} D_{\text{et}}(H^1_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)) \) consisting of \( \varphi_K \)-eigenvectors. Exactly \( g_0 \) of the corresponding eigenvalues are equal to 1 and \( g_0 \) are equal to \( q \); all the remaining eigenvalues are \( q \)-Weil numbers of weight 1. The monodromy operator \( N \) maps the \( q \) eigenspace isomorphically onto the 1 eigenspace, and acts as 0 on all other eigenspaces. The corresponding Lyndon basis of \( D_{\text{et}}(\text{gr}^W_{-1} U_n^\ast) \) is also a basis of \( \varphi_K \)-eigenvectors, with the eigenvalue of (the basis element corresponding to) a Lyndon word \( w \) being the product of the eigenvalues of its letters.
follow immediately from the description that the monodromy operator on $\text{gr}^{W_i}U^*$ maps the $q$ eigenspace surjectively onto the 1 eigenspace, and so the desired dimension is equal to the number of Lyndon words of length $i$ and eigenvalue $q$, minus the number of Lyndon words of length $i$ and eigenvalue 1. This latter quantity is simply the number of Lyndon words in the $g_0$ eigenspace of eigenvalue 1, and hence equal to $\sum_{d|i} \frac{\mu(d)}{q/d} g_0^d$ [Reu93, Theorem 7.1]. Summed over $1 \leq i \leq n$, this yields the term $-L_{\leq n}(g_0)$ in the claimed formula.

Now there are two types of Lyndon words $w$ of eigenvalue $q$: the letters of $w$ with eigenvalue not equal to 1 are either a single eigenvector with eigenvalue $q$, or two eigenvectors with eigenvalues $\lambda, q/\lambda$ with $\lambda$ of weight 1. To count words of the former type, suppose that $x$ is an eigenvector with eigenvalue $q$. There are $g_0^{i-1}$ Lyndon words of length $i$ containing $x$ and $i-1$ vectors of eigenvalue 0 by [Reu93, Theorem 7.1(7.1.2)]. Summed over $x$ and over $1 \leq i \leq n$, this yields the term $\sum_{i=0}^{n-1} g_0^{i-1}$ in the claimed formula.

To count words of the latter type, suppose that $x$ and $y$ are eigenvectors with eigenvalues $\lambda$ and $q/\lambda$ respectively. Using [Reu93, Theorem 7.1(7.1.2)] again again, we find that if $x \neq y$, then the number of Lyndon words containing $x$, $y$ and $i-2$ eigenvectors of eigenvalue 1 is $(i-1)g_0^{i-2}$. If $x = y$, then the number of Lyndon words containing two copies of $x, y$ and $i-2$ eigenvectors of eigenvalue 1 is $\frac{i-1}{2} g_0^{i-2}$ if $i$ is odd, and $\frac{i-1}{2} g_0^{i-2} - \frac{1}{2} g_0^{i/2-1}$ if $i$ is even. Summed over $x, y$ and over $1 \leq i \leq n$, this yields the quantity in the final line of the claimed formula. Here, we are using that $\nu_\lambda = \nu_{q/\lambda}$ by Poincaré duality, so that there are $\frac{k}{2} (\sum_\lambda \nu_\lambda^2 - \nu_\lambda \sqrt{\tau} - \nu_\lambda - \sqrt{\tau})$ unordered pairs with $x \neq y$, and $\nu_\lambda + \nu_\lambda + \nu_{q/\lambda} + \nu_{q/\lambda}$ pairs with $x = y$. \hfill \Box

**Definition 4.5** (cf. [BD19, Definition 2.2.2]). The $\varphi$-module $\mathcal{V}_{g/e}^{st}(U)$ appearing in Theorem 4.3 is defined as follows. The $K_0$-vector space $\text{Lie}(D_{st}(U)) = D_{st}(\text{Lie}(U))$ carries the structure of a filtered $(\varphi, N)$-module whose underlying Weil–Deligne representation is mixed with negative weights. We define

$$\mathcal{V}_{g/e}^{st}(U) := \{ x \in \text{Lie}(D_{st}(U))^{-2} : N^r(x) \in W_{-r-2}\text{Lie}(D_{st}(U))^{-2-r} \text{ for all } r \geq 0 \},$$

where as usual $\text{Lie}(D_{st}(U))^{-2}$ denotes the largest $\varphi$-invariant $\mathbb{Q}_p$-subspace on which all the eigenvalues of $\varphi$ are $p$-Weil numbers of weight $-2$ (this is a $K_0$-subspace).

In other words, with respect to the decomposition

$$\text{Lie}(D_{st}(U)) = \bigoplus_{i>0} \bigoplus_{j \geq 0} \text{Lie}(D_{st}(U))^{-i,j} \otimes \text{std}_j$$

from Theorem 2.19, we have

$$\mathcal{V}_{g/e}^{st}(U) = \bigoplus_{i-j=2} \text{Lie}(D_{st}(U))^{-i,j}.$$

**Remark 4.6.** The dimension formulae in Theorem 4.3 follow immediately from the explicit descriptions. Indeed, $\dim_K(D_{\text{dR}}(U)/D_{\text{dR}}^+(U)) = \dim_K(D_{\text{dR}}(\text{Lie}(U)) - \dim_K(D_{\text{dR}}(\text{Lie}(U)))$ is additive in short exact sequences of $\text{dR}$ representations by [Fon94b, Théorème 5.3.5(iv)], which establishes the dimension formulae for $H^1_\varphi$ and $H^1_\varphi$. It also follows from the proof of [Ber02, Lemme 6.2] that the functor $U \mapsto \mathcal{V}_{g/e}^{st}(U)$ takes exact sequences of mixed representations to exact sequences, and hence $\dim_{\mathbb{Q}_p}(\mathcal{V}_{g/e}^{st}(U))^{\rho = 1}$ is additive in short exact sequences

$$1 \to U_2 \to U \to U_1 \to 1.$$
of mixed representations, provided that $U$ is Frobenius-semisimple or that $U_1$ and $U_2$ have no weights in common. It thus suffices to establish the dimension formula for $H^1_{\text{c}}$ in the case that $U$ is abelian and mixed with negative weights, when this is well-known.

4.1. Bloch–Kato Selmer presheaves in general. We will ultimately deduce Theorem 4.3 from an explicit description of the Bloch–Kato Selmer presheaves $H^1_{\text{c}}(G_K, U)$ for a general de Rham representation of $G_K$ on a $\mathbb{Q}_p$-unipotent group $U$. The description we will give arises from a certain “non-abelian filtered $(\varphi, N)$-module structure” on the $K_0$-unipotent group $D_{\text{st}}(U)$, corresponding to the filtered $(\varphi, N)$-module structure on the Lie algebra $\text{Lie}(D_{\text{st}}(U)) = D_{\text{st}}(\text{Lie}(U))$ via the exponential isomorphism $\text{Lie}(D_{\text{st}}(U)) \cong D_{\text{st}}(U)$.

Explicitly, the $\sigma$-semilinear crystalline Frobenius $\varphi$ on $\text{Lie}(D_{\text{st}}(U))$ induces a scheme isomorphism

$$\varphi: D_{\text{st}}(U) \xrightarrow{\sim} D_{\text{st}}(U).$$

Both sides of this isomorphism are $K_0$-schemes, and $\varphi$ is $\sigma$-semilinear in the sense that it fits into a commuting square

$$\begin{array}{ccc}
D_{\text{st}}(U) & \xrightarrow{\varphi} & D_{\text{st}}(U) \\
\downarrow & & \downarrow \\
\text{Spec}(K_0) & \xrightarrow{\sigma} & \text{Spec}(K_0)
\end{array}$$

with $\sigma: \text{Spec}(K_0) \xrightarrow{\sim} \text{Spec}(K_0)$ the map induced by the arithmetic Frobenius $\sigma$. Since the crystalline Frobenius on $\text{Lie}(D_{\text{st}}(U))$ is compatible with the Lie bracket, it follows that the map $\sigma^*D_{\text{st}}(U) \xrightarrow{\sim} D_{\text{st}}(U)$ induced by crystalline Frobenius is an isomorphism of $K_0$-unipotent groups.

Additionally, the monodromy operator $N$ on $\text{Lie}(D_{\text{st}}(U))$ is a Lie derivation, so can be viewed as a section $s_N = 1 + \epsilon N$ of the natural morphism $K_0[\epsilon]/(\epsilon^2) \otimes_{K_0} \text{Lie}(D_{\text{st}}(U)) \to \text{Lie}(D_{\text{st}}(U))$ of Lie algebras. Transposing across the exponential isomorphism we obtain a section $s_N$ of the tangent bundle $TD_{\text{st}}(U) = \text{Res}_{K_0/[\epsilon]/(\epsilon^2)} \text{D}_{\text{st}}(U) \otimes_{K_0} \text{D}_{\text{st}}(U) \to D_{\text{st}}(U)$ which is a homomorphism when $TD_{\text{st}}(U) = \text{Lie}(D_{\text{st}}(U)) \times D_{\text{st}}(U)$ is given its natural structure of a unipotent group over $K_0$. From the identity $N \varphi = p \cdot \varphi N$, we find that the crystalline Frobenius $\varphi$ and monodromy vector field $s_N$ on $D_{\text{st}}(U)$ satisfy the relation

$$s_N \circ \varphi = p \cdot d\varphi \circ s_N$$

(4.7)

where $d\varphi: TD_{\text{st}}(U) \to TD_{\text{st}}(U)$ denotes the induced map on tangent bundles and $\cdot$ denotes the usual multiplication action on vector fields. Note that the scheme-theoretic vanishing locus of $s_N$ is exactly $D_{\text{cris}}(U)$. We write $\xi_U: D_{\text{st}}(U) \to \text{Lie}(D_{\text{st}}(U))$ for the map $u \mapsto s_N(u)s_0(u)^{-1}$ where $s_0: D_{\text{st}}(U) \to TD_{\text{st}}(U)$ is the zero-section.

There is also an inclusion $D_{\text{st}}(U) \to \text{Res}_{K_0}^K D_{\text{dR}}(U)$ arising from the inclusion $B_{\text{st}} \hookrightarrow B_{\text{dR}}$ (depending on a choice of $p$-adic logarithm). The Bloch–Kato Selmer presheaves can then be described in terms of the crystalline Frobenius $\varphi$ and the monodromy cocycle $\xi_U$ on $D_{\text{st}}(U)$, and the Hodge filtration on $D_{\text{dR}}(U)$.

**Theorem 4.8.** Let $U$ be a de Rham representation of $G_K$ on a unipotent group.

- There is a canonical isomorphism

$$H^1_{\text{c}}(G_K, U) \cong \text{Res}_{\mathbb{Q}_p}^K D_{\text{dR}}(U)/\left(\text{Res}_{\mathbb{Q}_p}^K D_{\text{dR}}^+(U) \times D_{\text{cris}}^{\varphi=1}(U)\right),$$

where $D_{\text{dR}}^+(U)$ denotes the $\varphi$-semisimple part of $D_{\text{dR}}(U)$.
of presheaves over $\text{Aff}_{Q_p}$, where the quotient is taken with respect to the right action given by 
\[ x \cdot (z, w) = w^{-1}xz. \]

- There is a canonical isomorphism 
\[ H^1_f(G_K, U) \cong (\text{Res}_{Q_p}^K \text{D}_{\text{dR}}(U) \times \text{Res}_{Q_p}^{K_0} \text{D}_{\text{cris}}(U))/(\text{Res}_{Q_p}^K \text{D}_{\text{dR}}^+(U) \times \text{Res}_{Q_p}^{K_0} \text{D}_{\text{cris}}(U)), \]
  of presheaves over $\text{Aff}_{Q_p}$, where the quotient is taken with respect to the right action given by 
\[ (x, u) \cdot (z, w) = (w^{-1}xz, w^{-1}u\varphi(w)). \]

- There is a canonical isomorphism 
\[ H^1_g(G_K, U) \cong Z^1_g(G_K, U)/(\text{Res}_{Q_p}^K \text{D}_{\text{dR}}(U) \times \text{Res}_{Q_p}^{K_0} \text{D}_{\text{st}}(U)), \]
  of presheaves over $\text{Aff}_{Q_p}$. Here $Z^1_g(G_K, U)$ denotes the subscheme of the product $\text{Res}_{Q_p}^K \text{D}_{\text{dR}}(U) \times \text{Res}_{Q_p}^{K_0} \text{Lie}(\text{D}_{\text{st}}(U)) \times \text{Res}_{Q_p}^{K_0} \text{D}_{\text{st}}(U)$ consisting of elements $(x, v, u)$ such that 
\[ v + \xi_N(u) = p\text{Ad}_u(\varphi(v)), \]
and the quotient is taken with respect to the right action given by 
\[ (x, v, u) \cdot (z, w) = (w^{-1}xz, \text{Ad}_{w^{-1}}(v + \xi_N(w)), w^{-1}u\varphi(w)). \]

Before we come to the proof of this theorem, let us use it to deduce Theorem 4.3. We do this via three preparatory propositions, corresponding to the three cases of $H^1_c$, $H^1_f$ and $H^1_g$ respectively. In what follows, we will abuse notation slightly and write $\text{D}_{\text{cris}}(U)$, $\text{D}_{\text{st}}(U)$ and $\text{D}_{\text{dR}}^+(U)$ for the Weil restrictions $\text{Res}_{Q_p}^K \text{D}_{\text{cris}}(U)$, $\text{Res}_{Q_p}^{K_0} \text{D}_{\text{st}}(U)$ and $\text{Res}_{Q_p}^K \text{D}_{\text{dR}}^+(U)$, so that these all denote unipotent groups over $Q_p$.

**Proposition 4.9.** Let $U/Q_p$ be a de Rham representation of $G_K$ on a unipotent group which is mixed with negative weights. Then $\text{D}_{\text{cris}}^{\varphi=1}(U) = 1$ is the trivial group-scheme.

**Proof.** $\text{D}_{\text{cris}}^{\varphi=1}(U)$ is a pro-unipotent group over $Q_p$, whose corresponding Lie algebra is the $\varphi$-invariant subspace of $\text{D}_{\text{st}}(\text{Lie}(U))^{N=0}$. Since $\text{D}_{\text{st}}(\text{Lie}(U))$ is mixed with negative weights, this is the zero subspace. \( \square \)

**Proposition 4.10.** Let $U/Q_p$ be a de Rham representation of $G_K$ on a unipotent group which is mixed with negative weights. Then for every $Q_p$-algebra $\Lambda$, the twisted right-conjugation action of $\text{D}_{\text{cris}}(U)(\Lambda)$ on itself given by 
\[ u \cdot w = w^{-1}u\varphi(w) \]
is free and transitive.

**Proof.** Our assumptions ensure that $\text{D}_{\text{cris}}(U)$ is an iterated central extension of vector groups on which the endomorphism $\varphi - 1$ is invertible. The result follows by an easy induction. \( \square \)
Proposition 4.11. Let $U/{\mathbb Q}_p$ be a de Rham representation of $G_\mathbb K$ on a unipotent group which is mixed with negative weights. For a $\mathbb Q_p$-algebra $\Lambda$, we write $Z^1_{g/e}(G_\mathbb K, U)(\Lambda)$ for the set of pairs $(v, u) \in (\Lambda \otimes \text{Lie}(D_{st}(U))) \times D_{st}(U)(\Lambda)$ satisfying
\[ v + \xi_N(u) = p\text{Ad}_u(\varphi(v)). \] (\ast)
Then the right action of $D_{st}(U)(\Lambda)$ on $Z^1_{g/e}(G_\mathbb K, U)(\Lambda)$ given by
\[(v, u) \cdot w = (\text{Ad}_{w^{-1}}(v + \xi_N(w)), w^{-1}w\varphi(w))\]
is free, and a fundamental set is given by $(\Lambda \otimes Z^1_{g/e}(U)_{pe=1}) \times \{1\}$.

Proof. Note that $Z^1_{g/e}(U)_{pe=1}$ is the $\mathbb Q_p$-subspace of $\text{Lie}(D_{st}(U))$ consisting of elements $v$ such that $p\varphi(v) = v$ and $Y(v) = v$ where $Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{SL}_2$ acts as in §2.1.1. We thus want to prove that for every $(v, u) \in Z^1_{g/e}(G_\mathbb K, U)(\Lambda)$ there is a unique $w \in D_{st}(U)(\Lambda)$ such that $(v', u') = (v, u) \cdot w$ satisfies $u' = 1$, $(p\varphi - 1)(v') = 0$ and $\log(Y)(v') = 0$. Note that the second of these identities is in fact a consequence of the first via (\ast).

We proceed by induction, writing $U$ as a central extension
\[ 1 \to U_1 \to U \to U_2 \to 1 \]
where $U_1$ is pure of weight $-i < 0$, the weights of $U_2$ are all $> -i$, and where we assume the result for $U_2$. Since the sequence
\[ 1 \to D_{st}(U_1)(\Lambda) \to D_{st}(U)(\Lambda) \to D_{st}(U_2)(\Lambda) \to 1 \]
is still exact [Bet19, Lemma 4.2.5], it suffices to consider the case that $w \in D_{st}(U_1)(\Lambda)$ and $(p\varphi - 1)(v, \log(Y)(v)) \in \Lambda \otimes \text{Lie}(D_{st}(U_1))$, where we need only consider the action of elements $w \in D_{st}(U_1)(\Lambda)$. In this case, the identity (\ast) reads
\[ v + N(\log(u)) = p\varphi(v) \]
and the action of $w \in \Lambda \otimes \text{Lie}(D_{st}(U_1))$ is given by
\[(v, u) \cdot w = (v + N(\log(w)), u \cdot w^{-1}w\varphi(w)). \]

Now consider the decomposition
\[ \text{Lie}(D_{st}(U)) = \bigoplus_{i' > 0} \bigoplus_{j \geq 0} \text{Lie}(D_{st}(U))^{-i', j} \otimes \text{std}_j \]
from Theorem 2.19, and write
\[ \log(u) = \sum_{j \geq r \geq 0} \log(u_{j, r}) \otimes \zeta^r \]
\[ v = \sum_{i' > 0} \sum_{j \geq r \geq 0} v_{i', j, r} \otimes \zeta^r \]
with $\log(u_{j, r}) \in \Lambda \otimes \text{Lie}(D_{st}(U))^{-i, j}$ and $v_{i', j, r} \in \Lambda \otimes \text{Lie}(D_{st}(U))^{-i', j}$. From (\ast) we obtain that
\[ \log(u_{j, r}) = \frac{1}{j - r}(p^{-r}\varphi - 1)(v_{i, j, r + 1}) \]
(\ast\ast)
for all $j > r \geq 0$. By definition, we also have
\[ \log(Y)(v) = \sum_{i' > 0} \sum_{j \geq r \geq 0} r \cdot v_{i', j, r} \otimes \zeta^{r-1} = \sum_{j \geq r \geq 0} r \cdot v_{i, j, r} \otimes \zeta^{r-1}. \]
(\dagger)
From (**) and (†) we see that there is a unique \( w \in D_{st}(U_1)(\Lambda) \) such that 
\[
\log(v) + (\phi - 1)(\log(w)) = 0 \quad \text{and} \quad \log(Y)(v + N(\log(w))) = 0,
\]
namely the element with 
\[
-\log(w) = \sum_{j > r \geq 0} \frac{1}{j-r} v_{i,j,r+1} \otimes \zeta^r + \sum_{j \geq 0} (p^{-j}\phi - 1)^{-1}(\log(u_{j,j})) \otimes \zeta^j,
\]
where we use the fact that \( p^{-j}\phi - 1 \) acts invertibly on \( \text{Lie}(D_{st}(U)) \) for weight reasons. This is what we wanted to prove. \( \square \)

Equipped with these three propositions, we now prove Theorem 4.3 from Theorem 4.8.

Proof of Theorem 4.3. For \( H^1_g \), Theorem 4.8 tells us that \( H^1_g(G_K, U) \) is the presheaf quotient of \( Z^1_g(G_K, U) \) by \( D^+_{\text{dR}}(U) \times D_{st}(U) \). Proposition 4.11 tells us that every orbit of this action contains an element in \( D_{\text{dR}}(U) \times V_{st}^e(G_K, U) \), and that this element is unique up to the action of \( D^+_{\text{dR}}(U) \) (by right-multiplication on \( D_{\text{dR}}(U) \)). This gives the desired description.

The descriptions of \( H^1_f \) and \( H^1_e \) follow in a similar (and simpler) manner from Propositions 4.10 and 4.9, respectively. \( \square \)

4.2. Cosimplicial models of local Selmer presheaves. To conclude this section, we turn to the proof of Theorem 4.8, which is ultimately an explicit spelling-out of the cosimplicial models for local Bloch–Kato Selmer sets studied in [Bet19]. Recall that in [Bet19, Definition 6.1.2], three cosimplicial period rings \( B^\bullet_e \subseteq B^\bullet_f \subseteq B^\bullet_g \) were defined. For a de Rham representation \( U \) of \( G_K \) on a unipotent group, these give rise to cosimplicial unipotent groups \( D^\bullet_e(U) \subseteq D^\bullet_f(U) \subseteq D^\bullet_g(U) \) over \( \mathbb{Q}_p \), representing the presheaves
\[
D^\bullet_*: \text{Spec}(\Lambda) \mapsto U(B^\bullet_* \otimes \Lambda)^{G_K}
\]
for \(* \in \{e, f, g\} \). Each \( D^\bullet_*(U) \), viewed as a presheaf of cosimplicial groups, gives rise to 0th and 1st cohomotopy presheaves \( \pi^0(D^\bullet_* (U)) \) and \( \pi^1(D^\bullet_* (U)) \), given sectionwise by [Bet19, Definition 5.1.4]. These are presheaves of groups and pointed sets respectively; if \( U \) is abelian then they are both presheaves of abelian groups and there are higher cohomotopy presheaves \( \pi^i(D^\bullet_* (U)) \) for \( i > 1 \). These cohomotopy presheaves are given explicitly as follows.

\footnote{In fact, \( D^\bullet_* (U) \) as defined here is canonically the Weil restriction of a cosimplicial unipotent group over the cosimplicial ring \( (B^\bullet_*)^{G_K} \), for a suitable interpretation of this assertion. We shall ignore this extra structure in what follows.}
Theorem 4.12. Let $U$ be a de Rham representation of $G_K$ on a unipotent group over $\mathbb{Q}_p$. Then there are canonical isomorphisms of presheaves

$$
\pi^i(D^\bullet_\ast(U)) = \begin{cases} 
U^{G_K} & \text{if } i = 0, \\
H^1_G(K, U) & \text{if } i = 1, \\
1 & \text{if } i > 1 \text{ and } U \text{ abelian}; 
\end{cases}
$$

$$
\pi^i(D^\bullet_f(U)) = \begin{cases} 
U^{G_K} & \text{if } i = 0, \\
H^1_f(K, U) & \text{if } i = 1, \\
1 & \text{if } i > 1 \text{ and } U \text{ abelian}; 
\end{cases}
$$

$$
\pi^i(D^\bullet_\ast(U)) = \begin{cases} 
U^{G_K} & \text{if } i = 0, \\
H^1_f(K, U) & \text{if } i = 1, \\
D^\text{cris}_{\ast=1}(U^\ast(1))^s & \text{if } i = 2 \text{ and } U \text{ abelian,} \\
1 & \text{if } i > 2 \text{ and } U \text{ abelian.} 
\end{cases}
$$

Proof. This is proved in [Bet19, Theorem 6.2.3] for $\mathbb{Q}_p$-points. The point here is that the theory set up in [Bet19, §6] has a natural (and functorial) generalisation if we replace $\mathbb{Q}_p$ with Qp-algebra $\Lambda$ throughout.

More precisely, the statements and proofs of Lemma 6.1.4, Proposition 6.2.1, Corollary 6.2.2 and then Theorem 6.2.3 of [Bet19] hold essentially verbatim if instead of taking points of unipotent groups in $\mathbb{Q}_p$ (or some $\mathbb{Q}_p$-algebra $B$), we take points in $\Lambda$ (or $B \otimes \Lambda$), where the action on $\Lambda$ is trivial and the topology on $\Lambda$ (or $B \otimes \Lambda$) is the natural one coming from writing $\Lambda$ as a direct limit of finite-dimensional $\mathbb{Q}_p$-subspaces.

Indeed, the only part of any of the proofs which isn’t immediate is the base case of the induction in Proposition 6.2.1 with $U$ abelian, for which we quoted [BK90, Lemma 3.8.1]. To prove this base case, we may simply note that the map $H^1(G, U \otimes B^{\text{dr}}_G) \to H^1(G, U \otimes B^{\text{dr}}_G)$ is the directed colimit of the maps $H^1(G, U \otimes B^{\text{dr}}_G \otimes H) \to H^1(G, U \otimes B^{\text{dr}}_G \otimes H)$ for $H_i$ ranging over the finite-dimensional $\mathbb{Q}_p$-subspaces of $\Lambda$. These maps are all injective by [BK90, Lemma 3.8.1] applied to $U \otimes H_i$, so by the exactness of directed colimits, the map $H^1(G, U \otimes B^{\text{cris}}_G) \to H^1(G, U \otimes B^{\text{cris}}_G)$ is injective too. This completes the base case of the induction in Proposition 6.2.1, and the inductive step proceeds exactly as before. □

Theorem 4.8 is then just the $i = 1$ part of Theorem 4.12. We will spell this out explicitly for $* = g$, leaving the other cases to the reader (these also directly follow from [Bet19, Remark 5.1.7]).

We begin by recalling the construction of $B^\ast_g$. Let $B^\ast_N$ denote the cosimplicial algebra where

$$
B^\ast_N := \frac{B_m[\varepsilon_1, \ldots, \varepsilon_m]}{(\varepsilon_i \varepsilon_j \text{ for } 1 \leq i, j \leq m)},
$$

and whose coface maps $d^k : B^{m-1}_N \to B^m_N$ are given by

$$
d^k \left( u + \sum_{i=1}^{m-1} v_i \varepsilon_i \right) = \begin{cases} 
N(u) \varepsilon_1 + \sum_{i=2}^m v_i \varepsilon_i & \text{if } k = 0, \\
\sum_{i=1}^m v_i \varepsilon_i & \text{if } 0 < k < m, \\
u + \sum_{i=1}^{m-1} v_i \varepsilon_i & \text{if } k = m,
\end{cases}
$$
where \( s^k(i) = i \) if \( i \leq k \) and \( s^k(i) = i - 1 \) if \( i > k \). We write \( \varphi \) for the natural Frobenius on \( B^*_N \), given by \( \varphi(u + \sum_{i=1}^m v_i \varepsilon_i) = \varphi(u) + p \sum_{i=1}^m v_i \varepsilon_i \). The cosimplicial period ring \( B^*_g \) is the diagonal in the bicosimplicial ring \( B^{*,*}_g \) whose entries are

\[
B^{m,n}_g := B^*_N \times B^*_N \times (B^*_N)^{n+1},
\]

whose horizontal coface maps are induced from those on \( B^*_N \), and whose vertical coface maps \( d^k : B^{m,n-1}_g \to B^{m,n}_g \) are given by

\[
d^k(x_0, \ldots, x_{n-1}, w_0, \ldots, w_{n-1}) = \begin{cases} (x_0, x_0, x_1, \ldots, x_{n-1}, w_0, \varphi(w_0), w_1, \ldots, w_{n-1}) & \text{if } k = 0, \\
(x_0, x_0, x_1, \ldots, x_{k(1)}, \ldots, x_{k(n)}, w_0, w_1, \ldots, w_{n-1}) & \text{if } 0 < k < n, \\
(x_0, x_1, x_2, \ldots, x_{n-1}, w_0, w_1, w_2, \ldots, w_{n-1}, w_0) & \text{if } k = n,
\end{cases}
\]

where \( \overline{w}_0 \) denotes the image of \( w_0 \) under the composite map \( B_{st} \to B_{D^*_N} \to B_{st} \).

It follows from this description that \( D^*_g(U) \) is the cosimplicial unipotent group with entries

\[
D^*_g(U) = D^*_d(U) \times D^*_d(U)^n \times (\text{Lie}(D_{st}(U))^n \times D_{st}(U))^{n+1}
\]

and whose first few coface maps are given by

\[
d^k(x_0; u_0) = \begin{cases} (x_0; x_0; \xi(u_0), u_0; p\varphi \xi(u_0), \varphi(u_0)) & \text{if } k = 0, \\
(x_0; u_0; 0, u_0; 0, v_0) & \text{if } k = 1
\end{cases}
\]

\[
d^k(x_0; x_1; v_0; u_0; v_1; u_1) = \begin{cases} (x_0; x_0; x_1; \xi(u_0), v_0, u_0; p\varphi \xi(u_0), p\varphi(v_0), \varphi(u_0); \xi(u_1), v_1, u_1) & \text{if } k = 0, \\
(x_0; x_1; v_0; u_0; v_1; u_1; v_1; u_1; v_0; u_0; 0, u_0; 0, v_0; u_0; v_0; u_0; v_0; 0, u_0; v_0) & \text{if } k = 2
\end{cases}
\]

where we write, for example, elements of \( \text{Lie}(D_{st}(U))^2 \times D_{st}(U) \) in the form \( (v, v', u) \) in the usual way.

Now the presheaf of 1-cocycles is the sub-presheaf of \( D^*_d(U) \) cut out by the equation \( d^1 = d^2 \cdot d^0 \), i.e. cut out by the equations

\[
x_0 = 1,
\]

\[
u_0 = 1,
\]

\[
v_1 = p\text{Ad}_{u_1}(\varphi(v_0)),
\]

\[
v_1 = v_0 + \xi(u_1).
\]

In other words, \( Z^1(D^*_d(U)) \) is the presheaf \( Z^1_d(G_K, U) \) described explicitly in Theorem 4.8, where the variables \( x, v, u \) correspond to \( x_1, v_0, u_1 \) respectively. It is easy to check that the coboundary action of \( D^*_d(U) = D^*_d(U) \times D_{st}(U) \) on \( Z^1(D^*_d(U)) \) is the action described in Theorem 4.8. Thus Theorem 4.12 provides the description of \( H^1_d \) from Theorem 4.8.

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