GENERAL STATIC $N = 2$ BLACK HOLES

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ABSTRACT

We find general static BPS black hole solutions for general $N = 2$, $d = 4$ supergravity theories with an arbitrary number of vector multiplets. These solutions are completely specified by the Kähler potential of the underlying special Kähler geometry and a set of constrained harmonic functions.
Recently lots of activity have been devoted to the study of BPS black holes in ungauged $N = 2$ supergravity models coupled to vector multiplets. The underlying special geometry structure of the $N = 2$ theory has played an essential role in the analysis of the Bekenstein-Hawking entropy of the $N = 2$ black hole solutions. The special geometry of the couplings of $n$ vector multiplets in $N = 2$ supergravity is completely determined by the covariantly holomorphic symplectic sections $(L^I, M_I), (I = 0, \ldots, n)$, satisfying the symplectic constraint
\begin{equation}
  i(\bar{L}^I M_I - L^I \bar{M}_I) = 1
\end{equation}
and depend on the complex scalar fields of the vector multiplets $z^i (i = 1, \ldots, n)$. These scalars parametrise a special Kähler manifold with the metric $g_{ij} = \partial_i \partial_j K(z, \bar{z})$, where $K(z, \bar{z})$ is the Kähler potential expressed by
\begin{equation}
  e^{-K} = i(X^I F_I - X^I \bar{F}_I)
\end{equation}
and $(X^I, F_I)$ are holomorphic sections defined by
\begin{equation}
  X^I = e^{-K/2} L^I, \quad F_I = e^{-K/2} M_I.
\end{equation}
The relation between $F_I$ and $X^I$ depends very much on the particular embedding of the isometry group of the underlying special Kähler manifold into the symplectic group. In some cases, $F_I$ can be expressed as a first derivative (with respect to $X^I$) of a holomorphic prepotential.

In general $N = 2$ supersymmetric theories, the mass of a BPS state which breaks half of the supersymmetry, can be expressed in terms of the sections by
\begin{equation}
  M_{b_{BPS}}^2 = |Z|^2 = e^K |q_I X^I - p^I F_I|^2 = |q_I L^I - p^I M_I|^2
\end{equation}
where $q_I$ and $p^I$ are electric and magnetic charges corresponding to the gauge group $U(1)^{n+1}$, with $n$ the number of vector multiplets and the extra $U(1)$ factor is due to the graviphoton, and $Z$ is the central charge of the underlying $N = 2$ supersymmetry algebra.

The black hole ADM mass, which in the $N = 2$ theory depend on the electric and magnetic charges as well as the value of the scalar fields at spatial infinity, is given by
\begin{equation}
  M_{\text{ADM}}^2 = |Z_\infty|^2 = |Z(z^A_\infty, \bar{z}^A_\infty)|^2.
\end{equation}
Here $z^i_\infty (i = 1, \ldots, n)$, are the values of the moduli at spatial infinity.

A fundamental result has been obtained which provides an algorithm for the macroscopic determination of Bekenstein-Hawking entropy for extremal $N = 2$ black holes. One simply extremize the central charge at fixed charges, the extremum value then gives the Bekenstein-Hawking entropy $S$ via the relation
\begin{equation}
  S = \pi |Z_{\text{fix}}|^2.
\end{equation}
Moreover, it was found that the horizon acts as an attractor for the scalar fields, which means that the values of the moduli are fixed at the horizon and are independent of their values

\footnote{For a review on $N = 2$ supergravity and special geometry see for example.}
at spatial infinity. For purely magnetic case, the values where shown to depend on ratios of
the charges \([1]\). This was later shown to be a general phenomenon which extends to dyonic
solutions \([3, 4]\). In general the fixed values of the scalars are those which extremize the central
charge. Therefore, \(N = 2\) black holes lose all their scalar hair near the horizon and are uniquely
characterised by conserved charges corresponding to the gauge symmetries. For \(N=2\) supergravity
models obtained from Calabi-Yau three-folds compactification, the holomorphic prepotential and
hence the entropy depends on the the classical intersection numbers, Euler number and rational
instanton numbers of the underlying Calabi-Yau three-fold.

The above results come about by using the supersymmetry transformations laws for the
gravitino and gauginos in ungauged bosonic part of \(N = 2\) supergravity. Demanding unbroken
supersymmetry \([3]\) near the horizon, gives the result that the covariant derivative of the central
charge has to vanish. At this point, the moduli become functions of the charges only. The
resulting equations which define the moduli in terms of the charges are given by

\[
i(ZL^\Sigma - ZL^\Sigma) = p^\Sigma, \quad i(ZM^\Sigma - ZM^\Sigma) = q^\Sigma. \tag{7}
\]

These equations can also be obtained by solving the equations of motion for the double-extreme
black hole solutions, where the scalars are assumed to be constants everywhere \([4]\).

While supersymmetric solutions of pure \(N = 2\) supergravity are known \([12]\), supersymmetric
solutions in \(N = 2\) supergravity coupled to vector multiplets and hypermultiplets are not fully
understood. For the latter theories, only simplified purely magnetic black hole solutions have
been constructed \([1]\). However, in \([1]\) it was realized that the axion-dilaton black hole solutions
constructed in \([16]\) can be reinterpreted as solutions of \(N = 2\) supergravity coupled to one vector
multiplet, namely the \(SU(1,1) \times U(1)\) model. One simply sets the holomorphic coordinates to complex
harmonic functions and then the metric solution can be expressed in terms of the Kähler potential.
Later it was realised in \([9]\) that the conditions imposed on the harmonic functions for the axion-
dilaton black hole solution in \([16]\) correspond in the \(N = 2\) language to the vanishing of the Kähler
connection. A non-vanishing Kähler connection leads to the supersymmetric Israel-Wilson-Preješ
like solution \([10]\) for the dilaton-axion supergravity \([9]\). Moreover in \([6]\) black hole solutions were
found to be expressed in terms of harmonic functions.

All the above observations provide an insight for the study of static black hole and stationary
solutions for the more general \(N = 2\) supergravity theory with arbitrary number of vector mul-
tiplets. In this work and as a starting point in this direction, we will focus on the study of static
black hole solutions and report on the stationary solutions in a future publication. Our analysis
is completely independent of the existence of holomorphic prepotential and formulated entirely in
terms of the holomorphic sections \((X^I, F_I)\). The main result is that the Kähler potential provides
the expression for the metric and that the imaginary part of the holomorphic sections are given
by a set of harmonic functions which depend on the electric and magnetic charges of the model.
Details of this work will be presented elsewhere.

We will ignore the hypermultiplets and assume that they are set to constant values. In this
case, the bosonic \(N = 2\) supergravity action includes one gravitational and \(n\) vector multiplets

\[
3
\]
and is given by
\[
\mathcal{L}_{N=2} = \int \sqrt{-g} \, d^4x \left( -\frac{1}{2} R + g_{ij} \nabla^\mu z^i \nabla_\mu z^j + i \left( \mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{\Lambda}_{\mu\nu} \mathcal{F}^{-\Sigma}_{\mu\nu} - \mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{+\Sigma}_{\mu\nu} \mathcal{F}^{\Lambda}_{\mu\nu} \right) \right)
\] (8)
where \(\mathcal{N}_{\Lambda\Sigma}\) is the symmetric period matrix whose imaginary and real part are, respectively, related to the Lagrangian kinetic and topological terms of the vector fields, and \(\mathcal{F}^{\pm\Sigma}_{\mu\nu}, \mathcal{F}^{\Lambda}_{\mu\nu}\) are respectively, the self-dual and anti-self-dual vector field strengths.

It is known \([12]\) in general that for \(N = 2\) theories, a static metric admitting supersymmetries can be put in the Majumdar-Papapetrou metric form \([11]\),
\[
ds^2 = e^{2U} dt^2 - e^{-2U} d\vec{x}^2,\]
(9)
Here we consider spherically symmetric solutions, i.e., \(U\) is only a function of \(r \equiv \sqrt{\vec{x}.\vec{x}}\).

To find the explicit BPS black hole solution it is more convenient to solve the supersymmetry transformations since these transformation rules depend linearly on the first derivatives of the bosonic fields and thus provide first order differential equations. The supersymmetry transformation rules for the gravitino and gauginos in a bosonic background of \(N = 2\) supergravity are given by \([8]\)
\[
\begin{align*}
\delta \psi_{\alpha\mu} &= \nabla_\mu \epsilon_\alpha - \frac{1}{4} T^{-}_{\rho\sigma} \gamma^\rho_{\alpha} \gamma^\sigma_{\mu} \epsilon_{\alpha\beta} \epsilon^\beta, \\
\delta \lambda^{i\alpha} &= i \nabla_\mu z^i \gamma^\mu \epsilon^\alpha + G^{-i}_{\rho\sigma} \epsilon^\alpha \epsilon^\beta.
\end{align*}
\] (10) (11)
where \(\psi_{\alpha\mu}\) and \(\lambda^{i\alpha}\) are the chiral gravitino and gauginos fields, \(\epsilon_\beta, \epsilon^\beta\) are the chiral and antichiral supersymmetry parameters respectively and \(\epsilon^{\alpha\beta}\) is the \(SO(2)\) Ricci tensor. The quantities \(T^{-}_{\rho\sigma}\) and \(G^{-i}_{\rho\sigma}\) are the field strengths of the graviphoton and matter-photon, respectively.

From the conditions of vanishing supersymmetry transformation, i.e., \(\delta \psi_{\alpha\mu} = \delta \lambda^{i\alpha} = 0\), one obtain for a particular choice of the supersymmetry parameter, the following two equations \([13]\)
\[
\begin{align*}
\frac{dU}{dr} &= \frac{Ze^U}{r^2}, \\
(\partial_i + \frac{1}{2} \partial_i K) L^i \frac{dz^i}{dr} &= -\frac{e^U}{r^2} \left( ZL^\Sigma - \frac{1}{2} (ip^\Sigma -(ImN^{-1})^\Sigma q_\Sigma + (ImN^{-1})^\Sigma \text{Re} \mathcal{N}_\Gamma p_\Delta) \right).
\end{align*}
\] (12)

The solution of the above equations of course depends on the particular model one is considering, i.e., the choice of the special Kähler manifold. In what follows we will solve the above differential equations in a model independent way. Our ansatz for the solution is to take
\[
e^{-2U} = e^{-K} = i(\bar{X}^I F_I - X^I \bar{F}_I)\]
(13)
Then the first differential equation in (12) gives
\[
\frac{d}{dr} e^{-2U} = -2 \frac{(X^I q_I - F_I p^I)}{r^2}
\] (14)

2These equations were also derived very recently in \([14]\) using a different approach.
If we further demand that our solution must satisfy the following relation

$$\tilde{X}^I \frac{dF_I}{dr} - \frac{d\tilde{X}^I}{dr} \tilde{F}_I = \frac{d\tilde{X}^I}{dr} F_I - \frac{dF_I}{dr} \tilde{X}^I,$$

then from (13) we obtain

$$\frac{d}{dr} e^{-2U} = 2i(\frac{d\tilde{X}^I}{dr} F_I - \tilde{X}^I \frac{d\tilde{F}_I}{dr}).$$

If we write

$$i(X^I - \tilde{X}^I) = f^I, \quad i(F_I - \tilde{F}_I) = g_I$$

where $f^I$ and $g_I$ are real functions which only depend on the radial distance $r$, eq. (16) can then be rewritten as

$$\frac{d}{dr} e^{-2U} = 2 \left( X^I \frac{dg_I}{dr} - F_I \frac{df^I}{dr} \right)$$

where we made use of the relation

$$\frac{dX^I}{dr} F_I - X^I \frac{dF_I}{dr} = 0.$$

which follows from the underlying special geometry structure.

Comparing the equation (18) with (14), one immediately arrive at the following solution

$$f^I = \tilde{h}^I + \frac{p^I}{r}, \quad g_I = h_I + \frac{q_I}{r}$$

where $h_I$ and $\tilde{h}^I$ are constants related to the values of the holomorphic sections (scalars) at infinity. These constants are restricted by demanding that the metric is asymptotically flat. One can also demonstrate that our solution solves the second differential equation in (12).

In deriving our solutions, we had to impose the condition (15), which is nothing but the condition of the vanishing of the the Kähler connection which can be expressed in the following form

$$A_\mu = \frac{-i}{2} \left( \partial_i \bar{K} \partial_{\bar{\mu}} z^i - \partial_{\bar{\mu}} \bar{K} \partial_i z^i \right).$$

Using (17), the vanishing of $A_\mu$ corresponds to

$$f^I \frac{dg_I}{dr} - g_I \frac{df^I}{dr} = 0.$$

This non-trivial condition does not impose any constraints on the electric and magnetic charges and only puts an additional constraint on $h_I$ and $\tilde{h}^I$, and ensures that the black hole saturates the BPS bound.

In conclusion, we have found general supersymmetric static black hole solutions for $N = 2$ supergravity theories coupled to an arbitrary number of vector multiplets. These solutions are completely determined in terms of the underlying Kähler geometry of the moduli space of the
scalars of the vector-multiplets. The metric is expressed in terms of the Kähler potential of the theory where the imaginary part of the holomorphic sections are given by harmonic functions. In summary

\[ ds^2 = e^{2U} dt^2 - e^{-2U} d\vec{x} d\vec{x}, \]

\[ e^{-2U} = e^{-K} = i(X^I F_I - \bar{X}^I \bar{F}_I) \]

\[ \text{Im} \left( \frac{X^I}{F_I} \right) = -\frac{1}{2} \left( \bar{h}_I + \frac{\bar{q}_I}{r} \right) \]

\[ \bar{h}_I q_I - h_I p_I = 0, \quad e^{-K_\infty} = 1. \]

We believe that our results are essential to the study of perturbative and nonperturbative corrections for $N = 2$ black hole solutions as well as the study of massless black holes.

**Acknowledgements**

This work is supported by DFG and in part by DESY-Zeuthen. I would like to thank K. Behrndt and T. Mohaupt for useful discussions.
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