Connecting Gaits in Energetically Conservative Legged Systems

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Abstract—In this work, we present a nonlinear dynamics perspective on generating and connecting gaits for energetically conservative models of legged systems. In particular, we show that the set of conservative gaits constitutes a connected space of locally defined 1D submanifolds in the gait space. These manifolds are coordinate-free parameterized by energy level.

We present algorithms for identifying such families of gaits through the use of numerical continuation methods, generating sets and bifurcation points. To this end, we also introduce several details for the numerical implementation. Most importantly, we establish the necessary condition for the Delassus’ matrix to preserve energy across impacts.

An important application of our work is with simple models of legged locomotion that are often able to capture the complexity of legged locomotion with just a few degrees of freedom and a small number of physical parameters. We demonstrate the efficacy of our framework on a one-legged hopper with four degrees of freedom.

Index Terms—Energy conservation, passive gaits, legged robots, numerical continuation methods

I. INTRODUCTION

SIMPLISTIC conservative models of legged locomotion, in which no energy is lost during a stride, are a powerful tool for both the analysis of human and animal gaits in nature and the design and control of legged robots \cite{1,2,3,4}. With just a few degrees of freedom and a small number of physical parameters, these models can accurately predict the preferred locomotion patterns of humans \cite{5} and provide useful templates for energy-efficient robot motions \cite{6},

Despite the benefits of such models, the field is still lacking a unified approach that systematically takes advantage of the conservative nature of these models to identify and characterize the different types of periodic motions available. This becomes even more important given that the same model can exhibit multiple modes of locomotion (e.g., walking, hopping, and running). To the best of our knowledge, past works have only developed results for specific conservative models and gait type \cite{7,8,9,10} and not a class of energetically conservative systems with hybrid dynamics and multiple modes of locomotion. The goal of this paper is to create a mathematical framework rooted in the theory of hybrid dynamical systems and nonlinear dynamics to model, classify, and create periodic motions for energetically conservative models (ECMs) of legged systems.

To this end, we generalize the methodology introduced in \cite{7} and carefully embed it into a mathematical framework for general ECMs of legged systems. We prove that families of gaits exist for such systems and highlight the role of energy in providing a coordinate-free parameterization for these families. In order to make the approach practical, we present algorithms for identifying families of gaits through the use of numerical continuation methods and introduce a number of details for their implementation. Among others, these details include projecting the state space to the subspace of periodic motions, establishing the necessary condition for the Delassus’ matrix to preserve energy across impacts, introducing the use of additional (holonomic) constraints to avoid singular dynamics, embedding the conservative system in a one-parameter family of dissipative systems and transitioning from an event-driven formulation to a time-based formulation.

This paper can be considered to be a direct extension of \cite{7} which showed that a simple model exhibits all common bipedal gaits and that these form continuous families of gaits in the biped’s space of trajectories. These periodic motions all emerged from a one-dimensional (1D) family of hopping-in-place gaits. Other gaits, such as walking and running, were connected to these through a series of bifurcations. Furthermore, our work builds upon the one-parameter families of periodic orbits in smooth ECMs as they are the main subject in \cite{11} and \cite{12}. While \cite{11} provides conditions for the existence of this family, \cite{12} revisits concepts of so-called Nonlinear Normal Modes (NNMs) that aim to find analytic expressions of invariant lower-dimensional submanifolds. Herein, NNMs are explicitly parameterized representations of 1D manifolds that emanate from exploiting the system’s state dependencies inflicted by the conservation of energy.

In the remainder of this paper, we first introduce the mathematical theory for ECMs (Section \textbf{II}) before discussing a numerical algorithm for the automated search for gaits (Section \textbf{III}). The example application of a one-legged hopper then further illustrates these concepts (Section \textbf{IV}).

II. THEORY

A. Dynamics of Legged Systems

In our work, we consider rigid body systems subject to contact without sliding, as they are commonly used to model legged robotic systems. An important restriction is that we...
limit ourselves to ECMs and periodic motions with a particular footfall sequence; for example, to either running or walking. The state of such a system is given by the vector \( x = (q, \dot{q}) \in TQ \subset \mathbb{R}^{2n_q} \), where \( n_q \) is the number of its degrees of freedom and \( TQ \) is the tangent bundle of the configuration space \( Q \subset \mathbb{R}^{n_q} \). In the following, we heavily rely on the concepts, assumptions, and notation from [13]. We refer to a motion within a persistent contact configuration as a phase \( i \). These phases are executed in a fixed, repeating order \( 1 \rightarrow 2 \rightarrow \cdots \rightarrow m \rightarrow 1 \). Adopting the notation of [13], the hybrid model is written as

\[
\begin{align*}
\mathcal{X} &= \{ X_i \}_{i=1}^m : X_i = \{ x \in TQ : g_i(q) = 0 \} \\
\mathcal{F} &= \{ f_i \}_{i=1}^m : x = f_i(x), x \in X_i \\
\mathcal{E} &= \{ E_i \}_{i=1}^m : E_i^{i+1} = \{ x \in X_i : e_i^{i+1}(x, \dot{x}) = 0 \} \\
\mathcal{D} &= \{ D_i \}_{i=1}^m : x^+ = \Delta_i^{i+1}(x^-), x^- \in E_i^{i+1}, x^+ \in X_{i+1}
\end{align*}
\]

where the codimension-one submanifold \( E_i^{i+1} \) determines a transition from phase \( i \) to phase \( i+1 \) with the reset map \( \Delta_i^{i+1} \). The representation of the autonomous flow \( f_i \) in phase \( i \) reflects the assumption of independent scleronomic constraints \( g_i : Q \rightarrow \mathbb{R}^{n_q} \) that allows us to uniquely solve for contact forces \( \lambda_i \in \mathbb{R}^{n_q} \) (Theorem 5.1 [14]). That is, the constraint Jacobian \( W_i(q) := \partial g_i / \partial q \) in the differential-algebraic equation

\[
\begin{align*}
M(q) \ddot{q} &= k(q, \dot{q}) + h(q, \dot{q}) + W_i(q) \lambda_i, \quad (1a) \\
g_i(q) &= 0, \quad (1b)
\end{align*}
\]

is full rank for all motions in phase \( i \). The mass matrix \( M \), elastic forces \( k \) and gravitational, centrifugal, and coriolis forces \( h \) are derived from the kinetic energy \( E_{\text{kin}} : TQ \rightarrow \mathbb{R} \) and potential energy \( E_{\text{pot}} : Q \rightarrow \mathbb{R} \) of the system. Note that we exclude non-potential forces in equation (1a), since \( \Sigma \) assumes state that for any \( x_0 \), we exclude non-potential forces in equation (1a), since \( \Sigma \) is an energetically conservative model (ECM). This definition of the hybrid flow enables us to directly relate to known properties of autonomous nonlinear dynamical systems. It will, however, require the construction of an additional event-like anchor constraint later on.

With the aforementioned assumptions from [13], the fundamental solution matrix

\[
\Phi(t, x_0) = \frac{\partial \varphi(t, x)}{\partial x} \bigg|_{x=x_0} \in \mathbb{R}^{2n_q \times 2n_q} \quad (5)
\]

is well-defined for any \( t \in I \) [16, 17].

B. Periodic Solutions in Energetically Conservative Hybrid Dynamical Systems

The total energy of the hybrid model \( \Sigma \) is given by \( E(x) = E_{\text{kin}}(q, \dot{q}) + E_{\text{pot}}(q) \).

**Definition (Energetically Conservative Model).** The hybrid system \( \Sigma \) is an energetically conservative model (ECM) if

\[
\text{Df1} \quad \text{all forces in the continuous dynamics of equation (1a) are conservative forces and}
\]

\[
\text{Df2} \quad \text{for all reset maps } x^+ = \Delta_i^{i+1}(x^-) \text{ it holds } E(x^+) = E(x^-). \text{ This implies } E_{\text{kin}}(x^+) = E_{\text{kin}}(x^-), \text{ since the discrete dynamics, with } q^+ = q^-, \text{ do not change the value of } E_{\text{pot}}, \text{ i.e., } E_{\text{pot}}(q^+) = E_{\text{pot}}(q^-).
\]

The definition of an ECM implies that for any \( x_0 \) its total energy \( E \) is invariant under the hybrid flow \( \varphi(t, x_0) \) for all times \( t \in I \).
To explore neighboring gaits, we perturb the initial state of the periodic solution (6) by an infinitesimal δ:

\[ \varphi(T, x_0) - x_0 = 0. \]  

Definition (Monodromy Matrix). The local linearization of a periodic solution \( \Phi_T := \Phi(T, x_0) \) is called the monodromy matrix.

The monodromy matrix is an important tool to study the stability and local existence of periodic flows (Chapter 7.1.1 [18]). For autonomous ECMs, it holds that:

\[ \Phi_T \cdot f_1(x_0) = f_1(x_0), \]  
\[ \nabla E(x_0)^T \Phi_T = \nabla E(x_0)^T. \]

Equation (7) is the well-known freedom of phase in autonomous systems, as any disturbance along the flow will remain on the same periodic motion in \( TQ \) (Theorem 2 [11]). Furthermore, since the total energy is flow-invariant: \( E(\varphi(t, x_0)) = \text{const.} = \bar{E} \), this yields the property in equation (8) (Chapter 2.4. [11]).

Lemma II.1. Outside of an equilibrium, where \( \nabla E(x_0) \) and \( f_1(x_0) \) are non-zero for a mechanical system, these vectors are also perpendicular.

Proof. Since the energy \( E(\varphi_1(t, x_0)) \) in phase \( i \) is constant for all \( t \in [0, t_{i,1}(x_0)] \), this implies:

\[ \frac{d}{dt} E(\varphi_1(t, x_0)) \bigg|_{t=0} = \nabla E(x_0)^T f_1(x_0) = 0. \]  

C. Connected Components of Energetically Conservative Gaits

The purpose of this work is to show connections between different periodic motions that we will refer to as different gaits. To eliminate the freedom-of-phase that is inherent to any autonomous system, we introduce an anchor constraint to further specify the solution that constitutes a specific gait:

Definition (Gait). A gait is a periodic solution that also fulfills the anchor constraint \( a(x_0) = 0 \), where \( a : \mathcal{X}_1 \to \mathbb{R} \) is a smooth function for which the transversality condition \( \nabla a(x_0)^T f_1(x_0) \neq 0 \) holds.

Theorem (Family of Gaits). In the vicinity of an energetically conservative gait there exist neighboring gaits.

Proof. Due to the periodicity, it must hold:

\[ a(x_0) = a(\varphi(T, x_0), x_0)) = 0, \]  

where we abuse the notation of the period \( T = T(x_0) \) to indicate its general dependency on \( x_0 \). Using the implicit function theorem, we get:

\[ \frac{\partial T}{\partial x_0} = -\frac{\nabla a(x_0)^T}{\nabla a(x_0)^T f_1(x_0)} \Phi_T. \]  

To explore neighboring gaits, we perturb the initial state of the periodic solution (6) by an infinitesimal \( \delta x \):

\[ \varphi(T(x_0 + \delta x), x_0 + \delta x) - (x_0 + \delta x) = 0. \]  

A first-order approximation of equation (12) yields

\[ \varphi(T(x_0), x_0) - x_0 + f_1(x_0) \frac{\partial T}{\partial x_0} \delta x + \Phi_T \delta x - \delta x = 0, \]  

which gives a more general coordinate-free parameterization for connected gaits by energy level \( E \). While other parameterizations are possible (e.g., using a state variable, such as speed [7]), \( \bar{E} \) gives a more general coordinate-free parameterization for ECMs, since gaits are inherently constrained to an equipotential surface (Lemma II.1). This parameterization is reflected in:

\[ r_E(x_0, T) := \begin{bmatrix} \varphi(T, x_0) - x_0 \\ a(x_0) \\ E(x_0) - \bar{E} \end{bmatrix} = 0, \]  

with its derivative

\[ \frac{\partial r_E}{\partial[T x_0^T]} = \begin{bmatrix} \Phi_T - I & f_1(x(T)) \\ \nabla a(x_0)^T & 0 \\ \nabla E(x_0)^T & 0 \end{bmatrix}. \]

The set of all solutions (with admissible flow) to equation (13) for all possible energy levels \( \bar{E} \) constitutes the gait space \( \mathcal{G} = \{ (x_0, T, \bar{E}) \in \mathcal{X}_1 \times \mathbb{R}^* \times \mathbb{R} : r_E(T, x_0) = 0 \} \).

Definition (Regular Point). We call a solution \( z^* \) of an implicit function \( F : \mathbb{R}^j \to \mathbb{R}^k \) with \( F(z^*) = 0 \) a regular point if \( \left( \partial F/\partial z \right)_{z=z^*} \) has maximum rank.

\[ \text{These so-called first integrals are considered in [19] for smooth systems.} \]
In the general case, energy conservation would only be possible if the inverse Delassus’ matrix \( G_{i+1}^{-1} \) were zero. Loosely speaking, this is because inertia and masses involved in the projection need to vanish to conserve energy. This is problematic, as this requirement leads to singularities in the systems mass matrix \( M \).

Instead, we consider vanishing masses and inertias only as a limiting case. That is, with some abuse of notation, we define a parameterized mass matrix \( \bar{M}(q, \varepsilon) = M_{\varepsilon} \), with parameter \( \varepsilon \) such that the Delassus’ matrix reads as \( G_{i+1}(q, \varepsilon) \). This parameterization must yield

\[
\lim_{\varepsilon \to 0} G_{i+1}(q, \varepsilon)^{-1} = 0.
\]

Considering equation (16), the mechanical system is only energetically conservative in the limit of \( \varepsilon \to 0 \). As pointed out in chapter 2.3 of [15], massless appendages of a robot possibly yield an inconsistent relationship between accelerations and net forces in equation (1a).

In practice, however, this can cause issues, as fluctuations of \( \varepsilon \) in \( h, k \) and \( \lambda_i \) such that equation (1a) can be stated as

\[
\ddot{q} = M_{\varepsilon}^{-1} h(x, \varepsilon) + M_{\varepsilon}^{-1} (k(q, \varepsilon) + W_i(q) \lambda_i(\varepsilon)).
\]

The resulting conservative vector field, defined by equation (18), is \( C^1 \) and complete in the analytic limit of \( \varepsilon \to 0 \). In other words: while \( M_{\varepsilon} \) can become singular in the limit of \( \varepsilon \to 0 \), the products \( M_{\varepsilon}^{-1} h \) and \( M_{\varepsilon}^{-1} (k + W_i) \) remain finite.

**Remark III.1.** The vector field properties are similar to A7 in [15]. However, we do not need to require \( [M_{\varepsilon}, W_i, \lambda_i] \) to be invertible in the limit and do not explicitly change the topology of the robot whenever a massless limb is unconstrained to the ground (A6 in [15]).

**B. Numerical Exploration**

The goal of our implementation is to solve the implicit function (14) in a systematic fashion to obtain the connected component \( \mathcal{V} \). Our primary tool for the computation of generators are numerical continuation methods [21].

The issue with numerically solving equation (14) is that it has \( 2n_q + 2 \) constraints but only \( 2n_q + 1 \) decision variables in \( x_0 \) and \( T \). In theory, this is no problem, as the equations in (14) are not independent due to the energetically conservative nature of the dynamics [22], as was shown above. In practice, however, this can cause issues, as fluctuations in energy can be introduced during numerical integration. When this is the case, equation (14) may not be solvable with only \( 2n_q + 1 \) decision variables. To tackle this issue, we use the approach reported in [11] and add a parameter \( \xi \) to the continuous dynamics (1):

\[
\mathcal{F}_\xi = \{ \tilde{f}_i \}_{i=1}^m : \tilde{f}_i := f_i(x) + \xi \cdot \nabla E(x).
\]
With the new representation \(19\), the conservative system \(\Sigma\) is embedded in a one-parameter family of dissipative dynamics \(\Sigma_\xi = (\mathcal{X}, \Sigma, D, \mathcal{F}_\xi)\). Analytically, a periodic orbit only exists for a vanishing perturbation \(\xi\) (Lemma 1 \(11\)). Hence, solutions \(\varphi(t, x_0, \xi)\) of \(\Sigma_\xi\) with \(\xi = 0\) are periodic solutions of the underlying conservative system. In the numerical computation of gaits, however, we might obtain solutions with a small \(\xi\) to compensate for small energy losses caused by numerical damping in the integration schemes.

Gaits of legged systems are not necessarily periodic in all states. In particular, the horizontal position is aperiodic to allow for forward motion. Hence, to relax the periodicity constraint \(6\), we split the state \(x\) into a periodic part \(x_p := A_p x\) and a non-periodic part \(x_{np} := A_{np} x\) by introducing the constant orthonormal selection matrix \(A_s = [A_p, A_{np}] \in \mathbb{R}^{2n_q \times 2n_q}\).

In the following, we do not implement the time-to-impact function and thus, decouple the time duration \(t_i\) of each phase \(i\) from the initial conditions \(x_{0,i}\). This allows us to move away from an event-driven evaluation of \(\Sigma_\xi\). In this approach, the event constraints \(E_i^{t_i} = 0\) become explicit components of the root function \(\tilde{r}_E\), rather than being implicitly stated in the set \(\mathcal{E}_i^{t_i+1}\). This change greatly facilitates the computation of the derivatives in \(\Phi_T\). Hence, a periodic solution for a given \(\tilde{E}\) can be obtained numerically by solving the root-finding problem \(\tilde{r}_E : \mathbb{R}^{2n_q + m + 2} \to \mathbb{R}^{2n_q + m + 2}\):

\[
\tilde{r}_E(x_0, t, \xi) = \begin{bmatrix}
    A_p \cdot (\varphi(t_{m+1}, x_{0,m+1}; \xi) - x_0) \\
    A_{np} \cdot x_0 \\
    A_{np} \cdot a(x_0) \\
    E(x_0) - \tilde{E} \\
    e_m^1 (\varphi_m(t_m, x_{0,m}; \xi)) \\
    \vdots \\
    e_m^2 (\varphi_m(t_1, x_{0,1}; \xi)) \\
  \end{bmatrix} = 0,
\]

where \(t = [t_1 \ldots t_{m+1}]^T\) and the initial states \(x_{0,i}\) of each mode are defined recursively as in equations \(4\), substituting the function \(t_{i+1}\) by the variable \(t_i\). With \(z^T = [x_0^T \ x^T \ \xi]\), we refer to the Jacobian of \(\tilde{r}_E\) as \(\tilde{R}_E : \partial \tilde{r}_E / \partial z\).

In addition to the implicit equation \(20\), we define an extended root function \(\tilde{r} : \mathbb{R}^{2n_q + m + 3} \to \mathbb{R}^{2n_q + m + 2}\) that also includes \(\tilde{E}\) as a free variable:

\[
\tilde{r}(z, \tilde{E}) := \tilde{r}_E(z), \quad \tilde{R}(u) := \frac{\partial \tilde{r}}{\partial u} = \begin{bmatrix}
    \tilde{R}_E(z) \\
    \tilde{E}(z) \\
    \varphi_m(t_m, x_{0,m}; \xi) \\
    \vdots \\
    \varphi_m(t_1, x_{0,1}; \xi) \\
  \end{bmatrix},
\]

If \(z^*\) is a regular point of \(\tilde{r}_E\), then \(\tilde{r}(u) = 0\) characterizes a locally defined 1D solution manifold. The function \(\tilde{r}\) is well suited for a pseudo-arclength continuation which is utilized to compute generators. This approach employs a predictor-corrector (PC) method with a variable step size \(h\) (Chapter 6.1 \(21\)), which takes small iterative steps in the tangent space of \(\tilde{r}(u) = 0\) to locally trace the solution curve of regular points.

### Algorithm 1: Compute Generator \(\mathcal{M}_j\)

**Input:** Regular point \(u^*: \) Initial step size \(h > 0\)
- Maximal number of generated points \(N_{\text{max}}\)

**Output:** Generator \(\mathcal{M}_j\), BP, TP, IP

1. \(u^0 \leftarrow u^*\)
2. add \(u^0\) to \(\mathcal{M}_j\)
3. \(d_1 = +1\)
4. While \(k = 0 \ldots N_{\text{max}}\)
   - \(PC\)-step \((u^k, d_k)\):
     - Predictor Step (Explicit-Euler Step)
     - Corrector Step (Newton’s Method)
     - If \(u^{k+1}\) is inadmissible then
       - If \(p^k \cdot p^{k+1} < 0\) then
         - Search for simple BP between \(u^k\) and \(u^{k+1}\)
       - Else if \(\text{det}(\tilde{R}_E(z^{k+1})) < 0\) then
         - Search for TP between \(u^k\) and \(u^{k+1}\)
     - Else
       - Search for IP between \(u^k\) and \(u^{k+1}\)
     - \(d_k = +1\)
6. return \(\mathcal{M}_j\), BP, TP, IP

### Algorithm 2: Compute Connected Component \(\mathcal{V}\)

**Input:** Starting point \(u_0;\)

- Maximal number of generators \(N_{\text{max}}\)

**Output:** Connected Component \(\mathcal{V}\)

1. push \(u_0\) to queue \(Q\)
2. While \(k = 1 \ldots N_{\text{max}}\) and \(Q\) is not empty
   - pull \(u^*\) from \(Q\)
   - Algorithm 1 \((u^*);\)
   - Add \(\mathcal{M}_j\), TP, BP to \(\mathcal{V}\)
   - Find regular points \(u^*_n\) in \(\text{nbhd}\) of TP, BP
   - For each \(u^*_n\) not in \(\mathcal{V}\) do push \(u^*_n\) to \(Q\)
3. return \(\mathcal{V}\)

This tangent space is equivalent to the kernel of \(\tilde{R}\) at a regular point \(u^*\) of equation \(21\), with the tangent vector \(p\):

\[
\tilde{R}(u^*)p = 0, \quad \|p\| = 1, \quad \text{det} \left( \begin{bmatrix}
    \tilde{R}(u^*) \\
    p^T
  \end{bmatrix} \right) < 0. \quad \text{(23)}
\]

As the curve can be locally pursued in two directions, \(\text{det}(J) > 0\) defines positive orientation \(21\).

This process, the crossing of simple (codimension-one) bifurcations are detected by a flip in direction of the tangent vector \(p\) (i.e., \(p^k \cdot p^{k+1} < 0\) \(21\)). The detection of turning points (TP) follows from a change in sign of \(\text{det}(\tilde{R}_E(z))\) (i.e.,

\footnote{A simple or codimension-one bifurcation point \(u_s\) is defined by a loss of rank in \(\tilde{R}\), i.e., \(\text{rank}(\tilde{R}(u_s)) = 2n_q + n\).}
configuration of the robot. During the
stance phase, the constraint forces in these phases are

\[ F_n(q) = k_f(l_o - l), \]

where \( F_n(q) \) are the result of a PC-step that has crossed a
foot contact point on the ground. The algorithm returns the new generator \( M_j \) and its associated
TPs and BPs. The curve \( M_j \) has at most 2 limiting special
points. As mentioned previously, TPs and BPs are singular
points that connect to different generators \( M_j \). Algorithm 2
constructs a subset of the space of connected components. It
utilizes a breadth-first-search to explore different generators
given the location of connected TPs and BPs. Locations of
regular points \( \hat{u}_i \) in the neighborhood of simple bifurcations
are found by the bifurcation equation (Chapter 3.2). As
indicated above, it is essential to have a problem specific
starting point \( m_0 \) that solves equation (20) and is regular.

We note that Algorithm 1 is only able to detect TPs and
simple BPs. Bifurcations of codimension-two and higher are
overlooked or wrongly classified as simple bifurcations. Test
functions for their detection are described in [23].

IV. EXAMPLE: ONE-LEGGED HOPPER

A. Model Description

In this section, we highlight the application of our method
to a SLIP-like one-legged hopper introduced in [7] with
passive swing leg dynamics that are created by a torsional
hip spring (Fig. 2). Here, however, it is derived in a more
formal manner including a rigorous treatment of the previously
unsolved issue of the spring leg dynamics during flight. This
motion, which becomes singular for vanishing foot-masses,
was simply ignored in [7] and is treated here by the inclusion
of additional holonomic constraints.

The model consists of a torso with mass \( m_t \) which is
constrained to purely linear motions as defined in [7]. Thus,
the torso’s configuration is given by the hip position \( (x, y) \).
The leg is connected to the hip via a rotational joint (with
joint angle \( \alpha \)) that includes a torsional spring (with stiffness \( k_\alpha \)
and no damping). We model the legs as massless linear springs
with leg length \( l_o \), natural spring length \( l_o \), spring stiffness \( k_l \),
no damping, and a point mass \( m_l \) at the foot. The total
mass of the model is \( m_o = m_l + m_f \). We use generalized
coordinates \( q = [x \ y \ \alpha \ l]^T \) (i.e., \( n_q = 4 \)) to represent
the configuration of the robot.

The model has two phases: stance \( S \) and flight \( F \). The
corresponding constraint forces in these phases are \( \lambda_S = [\lambda_F \ \lambda_N]^T \)
and \( \lambda_F \). These forces satisfy the constraints

\[ g_F(q) = \begin{bmatrix} l - l_o \\ x + l \sin(\alpha) - x_c \\ y - l \cos(\alpha) \end{bmatrix} = 0, \]

\[ g_S(q) = \begin{bmatrix} \sin(\alpha) - x_c \\ y - l \cos(\alpha) \end{bmatrix} = 0, \]

during flight and stance, respectively. The constraint (24)
fixes the leg length to \( l_o \) during flight, whereas equation (25)
implies the assumption of no sliding during stance (with
a horizontal contact point position \( x_c \)). For the continuous
dynamics in equations (1), we have

\[ k = \begin{bmatrix} 0 & 0 & F_\alpha & F_1 \end{bmatrix}, \]

This implies that \( \omega_{\text{swing}} \) remains a finite constant value when
the foot mass \( m_f \) is brought to zero and thus \( k_\alpha \rightarrow 0 \).
With the modifications in equations (29), (30) and taking the
limit \( \varepsilon \rightarrow 0 \), we arrive at the same finite dimensional dynamics
reported in [7]. To allow for horizontal displacement in equation (20), the matrix \( A_{np} \) selects the initial state \( x_0 = A_{np} x_0 \).
The remaining periodic states are selected by its orthogonal complement $\mathbf{A}_p$. In this energetically conservative model, all state and parameter values are normalized with respect to $m_o$, $g$ and $l_o$. To allow a comparison with [7], we set the leg stiffness to $k_1 = 40 m_o g l_o$ (which is equivalent to hopping with 2 legs of stiffness $20 m_o g l_o$) and the swing frequency to $\omega_{\text{swing}} = \sqrt{5} g/l_o$.

**B. Results**

Using this model, Algorithm 2 was initialized with a vertical hopping motion at energy level $E = 1.001 m_o g l_o$ (that is, with initial apex height of $y_0 = 1.001 l_o$). Here, the motion in $y$ and $\alpha$ simply follows a parabolic trajectory during flight and a linear oscillation during stance. There is no movement in $x$ and $\theta$. This hopping motion constitutes a regular point $\mathbf{u}_0$ that solves equation (20). This initial point is connected to a locally defined 1D manifold $\mathcal{M}_0$ (Fig. 3) of hopping in place motions. Towards lower energies, hopping height is reduced and this generator is bounded by a point that corresponds to a vanishing time $t_f$ in flight at energy level $E = 1 m_o g l_o$. Periodic solutions of $\Sigma$ with even lower energy do exist, yet they correspond to an oscillating in-place motion. Since there is no lift-off in this motion, going beyond this point leads to a change in phase sequence. This is an inadmissible point. However, there exists a locally defined manifold with this different contact sequence $S \rightarrow \mathbb{S}$ It can be independently computed by Algorithm 11; however, it is not in the connected component $\mathcal{V}$ of generators with contact sequence $F \rightarrow S \rightarrow F$.

Carrying on with $F \rightarrow S \rightarrow F$, we traverse the generator $\mathcal{M}_0$ towards higher energies. $\mathcal{M}_0$ is bounded by a simple bifurcation point at energy level $E_{\text{BP1}} \approx 1.247 m_o g l_o$. At this point, we find three nearby generators for which the last tangent direction $\mathbf{p}$ of $\mathcal{M}_0$ points into the new generator $\mathcal{M}_1$.

The computation of $\mathcal{M}_1$ leads to another simple bifurcation point at $E_{\text{BP2}} \approx 1.614 m_o g l_o$. We find three new connected generators $\mathcal{M}_4$–$\mathcal{M}_6$. The vertical motions in $\mathcal{M}_4$ are similar to gaits in $\mathcal{M}_0$ and $\mathcal{M}_1$. The generators $\mathcal{M}_5$ and $\mathcal{M}_6$ correspond to forward and backwards hopping motions, respectively.

**V. DISCUSSION & CONCLUSION**

In this paper, we introduced a formal framework and a generalized methodology for the computation of connected gaits in energetically conservative legged systems. This work extends and clarifies the methodology introduced in [7] to apply not only to the gaits of legged models but to a broader class of ECMs. In terms of theory, our work extends the results generated by Algorithm 2 and generalizes the methodology introduced in [7] to apply not only to the gaits of legged models but to a broader class of ECMs. In terms of theory, our work extends the results of [7].
in \( \mathcal{G} \) to hybrid dynamical systems and clarifies the connected structure of the gait space \( G \) of energetically conservative legged systems.

Our contributions further relate the study of passive gaits to established and emerging concepts in the field of nonlinear dynamics. Similar to the generators in \([12]\), we (locally) define 1D manifolds in which there is a unique relation between motion and energy. However, our definition of these generators is different in that these 1D manifolds do not include equilibria and they are defined for hybrid dynamical systems. As a consequence, the direct connection to linear oscillations, that occur in the linearized system at equilibrium and that is a characteristic of today’s NNMs definitions, is lost. This loss is caused by two required assumptions. The first is the transversality condition of the anchor constraint that is violated so does the associated tangent space. This is no longer true for both, robotics and biology.

As shown in Fig. 3, the linear modes of the 1D oscillator correspond to bouncing in place. In future work, it may be possible to formally link them to the hopping gaits characterized in this paper. To make this possible, we need to relax the assumption that the phase sequence is fixed. This assumption constitutes the primary limitation of our work. It is necessary, as the core results in this paper follow from the monodromy matrix \( \Phi_T \). For a fixed phase sequence, \( \Phi_T \) changes differentiably in neighboring periodic solutions and so does the associated tangent space. This is no longer true when certain assumptions from \([13]\), e.g., no grazing contacts, do not hold. With a vanishing phase duration, a Saltation matrix \( \Phi_T \) may become discontinuous \([17]\), which directly propagates to discontinuities in \( \Phi_T \) and the associated tangent space. For legged systems, continuity in the Saltation matrix can be ensured under certain conditions \([24]\). Turning these conditions into systematic modeling guidelines, or finding ways to connect gaits despite these discontinuities are avenues for future work.

While the focus of this paper is on energetically conservative systems, real robot systems are not energetically conservative and sources of energy loss (heat, impacts, batteries, vibrations) cannot be completely eliminated. The benefit of our approach is in utilizing the explanatory power of ECMs. While these systems do not exist in the real world, these simple models often form the core model dynamics for trajectory generation, motion planning, and control algorithms in the field. Mapping trajectories from ECMs to more realistic models with energy loss would be an interesting extension of our work, as their passivity makes them ideal candidates for the use as templates to develop energetically economical motions for legged robotic systems.

Beyond this very practical significance, the identified passive motions are a key characteristic of a given ECM. Their study, not only in simple models of legged systems, will thus allow us to better understand the fundamental nature of gait for both, robotics and biology.

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