ON THE STRUCTURE OF COHOMOLOGY RINGS OF P-NILPOTENT LIE ALGEBRAS

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Abstract. In this paper the authors investigate the structure the restricted Lie algebra cohomology of p-nilpotent Lie algebras with trivial p-power operation. Our study is facilitated by a spectral sequence whose $E_2$-term is the tensor product of the symmetric algebra on the dual of the Lie algebra with the ordinary Lie algebra cohomology and converges to the restricted cohomology ring. In many cases this spectral sequence collapses, and thus, the restricted Lie algebra cohomology is Cohen-Macaulay. A stronger result involves the collapsing of the spectral sequence and the cohomology ring identifying as ring with the $E_2$-term. We present criteria for the collapsing of this spectral sequence and provide many examples where the ring isomorphism fails. Furthermore, we show that there are instances when the spectral sequence does not collapse and yields cohomology rings which are not Cohen-Macaulay.

1. Introduction

One of the major challenges in group cohomology is the computation of the cohomology of nilpotent groups. Such computations are important because general questions about modular group cohomology can often be reduced to questions that concern only the cohomology of its Sylow $p$-subgroup. The structure of the cohomology of $p$-groups can be quite complicated, but in the case when the cohomology ring is Cohen-Macaulay (i.e, when the depth equals the Krull dimension), the homological algebra of the representation theory has more orderly structural features. In the realm of Lie theory and modular representations of algebraic groups the nilpotent restricted $p$-Lie algebras play a similar role to that of the $p$-groups in group representations. In this paper we aim to address some basic questions on the structure of the cohomology rings for these algebras.

Suppose that $(\mathfrak{n}, [p])$ is a nilpotent restricted $p$-Lie algebra and that $k$ is an algebraically closed field of characteristic $p > 0$. The spectrum of the cohomology ring identifies with the restricted nullcone $N_1(\mathfrak{n}) = \{ x \in \mathfrak{n} : x^{[p]} = 0 \}$. If we assume that $p > 2$, then there is a spectral sequence $E_2^{i,j} = S^{2i}(\mathfrak{n}^*)^{(1)} \otimes H^j(\mathfrak{n}, k) \Rightarrow H^{2i+j}(u(\mathfrak{n}), k)$

where $S^{*}(\mathfrak{n}^*)^{(1)}$ is the Frobenius twist of the symmetric algebra on the dual of the underlying vector space of $\mathfrak{n}$, $H^{*}(\mathfrak{n}, k)$ is the ordinary Lie algebra of $\mathfrak{n}$, and $H^{*}(u(\mathfrak{n}), k)$, is the cohomology ring of the restricted enveloping algebra $u(\mathfrak{n})$ of $\mathfrak{n}$. There are many cases

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in which it is known that the spectral sequence collapse at the $E_2$ page, so that $E_2^{*,*}$ is isomorphic to the associated graded ring of $H^*(u(n), k)$. In this situation, the cohomology ring $H^*(u(n), k)$ is a free module over the symmetric algebra $S^*(n^*)^{(1)}$, and the cohomology ring is Cohen-Macaulay. In 1986, Friedlander and Parshall [FP] showed this happens when $n$ is the nilpotent radical of the Borel subalgebra of the restricted Lie algebra of an algebraic group, provided $p > h$ where $h$ is the Coxeter number of the associated root system. Twenty five years later, Drupieski, Ngo and the second author [DNN] showed that a stronger result holds when $p \geq 2(h - 1)$, that is, there is a ring isomorphism $H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k)$.

The investigations of this paper were originally inspired by a computer calculation by the first author which demonstrated that if $n$ is the Lie algebra of the nilpotent radical of a Borel subalgebra of a group of type $B_2$ and if $p = 5$ (which is larger than $h$ but not larger than $2(h - 1)$), then the isomorphism $H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k)$ holds as modules over the symmetric algebra $S^*(n^*)^{(1)}$ (so that the cohomology ring is Cohen-Macaulay), but it does not hold as rings. A search for the reason for this phenomenon led to the discovery of a one-dimensional central extension of $n$ (with trivial $p$th power) whose cohomology ring is easily proved to be not Cohen-Macaulay. Indeed, this illustrates a general situation. One of the theorems in the paper is that if $H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k)$ holds as an isomorphism of rings, then when $n$ is replaced by a one-dimensional extension, there is the same isomorphism, though perhaps only as an isomorphism of $S^*(n^*)^{(1)}$-modules. In the paper, we present numerous examples of this phenomenon.

Cohen-Macaulay rings are of general interest, in part, because they have very nice structural properties. For example, it can be shown that for any restricted Lie algebra, if its cohomology ring is Cohen-Macaulay, then the cohomology ring admits a formal Poincaré duality and its Poincaré series, as a rational polynomial, satisfies a functional equation [BC1, BC2]. In the case of a $p$-nilpotent Lie algebra with vanishing $p$-power operation, we show that the cohomology ring is Cohen-Macaulay if and only if the spectral sequence given above collapses at the $E_2$-page.

The paper is organized as follows. In the following section of the paper, we present preliminaries about cohomology rings and the definitions from commutative algebra. A proof that the cohomology ring is Cohen-Macaulay if and only if the spectral sequence collapses is given in Section 3. We also prove results which describe how the spectral sequence behaves under central extensions in that section. The next four sections are concerned with specific examples. In the case that if $n$ is the nilpotent radical of a Borel subalgebra of $\mathfrak{sl}_3$ (i.e., type $A_2$), and the field has characteristic 3, then the isomorphism $H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k)$ holds as modules over the symmetric algebra, but not as ring. Similar examples are given in type $B_2$ in characteristic 5 and in type $G_2$ in characteristic 7. In each example, there is a one-dimensional extension of the Lie algebra whose cohomology ring is not Cohen-Macaulay. Examples of Lie algebra whose cohomology rings are not Cohen-Macaulay are given in all characteristics. In Section 8, we outline which Lie algebra of dimension 5 have cohomology rings that are Cohen-Macaulay. In Section 9, we look at the special case of the nilpotent radical of the Borel subalgebra of $\mathfrak{sl}_4$ when $h < p < 2(h - 1)$ and show that the cohomology ring can be identified the symmetric algebra tensored with the ordinary Lie algebra cohomology as rings.
2. Preliminaries

Let \((\mathfrak{g}, [p])\) be a restricted Lie algebra over an algebraically closed field of characteristic \(p > 0\). Throughout this paper we will work with the added assumption that \(p \geq 3\). The restricted representations for \(\mathfrak{g}\) correspond to modules for the restricted enveloping algebra \(u(\mathfrak{g})\). Since \(u(\mathfrak{g})\) is a finite-dimensional cocommutative Hopf algebra the cohomology ring \(H^*(u(\mathfrak{g}), k)\) is a finitely generated graded-commutative \(k\)-algebra. For these types of rings, notions like Krull dimension and spectrum are well defined. The spectrum of the cohomology ring \(V_\mathfrak{g}\) is homeomorphic to the restricted nullcone:

\[
\mathcal{N}_1(\mathfrak{g}) := \{ x \in \mathfrak{g} : x^{[p]} = 0 \}.
\]

When \(p \geq 3\) there exists a the spectral sequence

\[
E_2^{2i,j} = S^j(\mathfrak{g}^*)^{(1)} \otimes H^i(\mathfrak{g}, k) \Rightarrow H^{2i+j}(u(\mathfrak{g}), k) \tag{2.0.1}
\]

where \(H^*(\mathfrak{g}, k)\) is the ordinary Lie algebra cohomology. In particular, the map of the universal enveloping algebra of \(\mathfrak{g}\) to the restricted enveloping algebra of \(\mathfrak{g}\) induces an edge homomorphism from the restricted cohomology to the ordinary Lie algebra cohomology. On the other hand, there is another edge homomorphism \(\Phi : S^*(\mathfrak{g}^*)^{(1)} \to H^*(u(\mathfrak{g}), k)\) which induces an inclusion of \(R = k[\mathcal{N}_1(\mathfrak{g})] \hookrightarrow H^*(u(\mathfrak{g}), k)\). Furthermore, \(H^*(u(\mathfrak{g}), k)\) is an integral extension of \(R\).

The following observation is a consequence of these facts in the event that the \(p\)-power operation vanishes on a nilpotent restricted \(p\)-Lie algebra \(n\). In this situation, \(\mathcal{N}_1(n) = n\) and its coordinate ring is \(S^*(n^*)^{(1)}\).

**Proposition 2.1.** Suppose that \(n\) is a \(p\)-nilpotent Lie algebra such that \(x^{[p]} = 0\) for all \(x \in n\). Then the edge homomorphism \(S^*(n^*)^{(1)} \to H^*(u(n), k)\) is an injection.

We require several ring theoretic notions which, though usually defined for commutative rings or commutative local rings, apply also to graded-commutative \(k\)-algebras.

Suppose that \(A = \sum_{i \geq 0} A_i\) is a graded-commutative \(k\)-algebra and that \(M = \sum_{i \geq 0} M_i\) is a graded \(A\)-module. A sequence \(x_1, \ldots, x_r\) of homogeneous elements of \(A\) is said to be a **regular sequence** for \(M\) if for every \(i = 1, \ldots, r\), we have that multiplication by \(x_i\) is an injective map from \(M/(x_1, \ldots, x_{i-1})M\) to itself. The **depth** of \(M\) is the length of the longest regular sequence for \(M\), and the depth of \(A\) is the depth of \(A\) as a module over itself.

A sequence of homogeneous elements \(x_1, \ldots, x_r\) is a homogeneous set of parameters for \(A\), provided \(x_1, \ldots, x_r\) generate a polynomial subring \(R = k[x_1, \ldots, x_r] \subseteq A\) and that \(A\) is finitely generated as a module over \(R\). In this case, the number \(r\) must be the Krull dimension of \(A\). The module \(M\) is **Cohen-Macaulay** if its depth is equal to the Krull dimension of \(A\). The algebra \(A\) is Cohen-Macaulay if it is Cohen-Macaulay as a module over itself. This is equivalent to the condition that there be a homogeneous set of parameters \(x_1, \ldots, x_r\) for \(A\) such that \(A\) is a finitely generated free module over the polynomial subring \(k[x_1, \ldots, x_r]\). It is a theorem that if there is a homogeneous set of parameters \(x_1, \ldots, x_r\) such that \(A\) is a finitely generated free module over \(k[x_1, \ldots, x_r]\), then \(A\) is a finitely generated free module over \(k[y_1, \ldots, y_r]\) for any homogeneous set of parameters \(y_1, \ldots, y_r\). For a reference, see Proposition 3.1 of [St] or Theorem 2, page
IV-20 of [Se]. Note that in the book of Serre, the proof is given only for commutative local rings, but can be easily adapted to the graded-commutative case.

With these preliminaries, we can extract the results that we need for this paper.

**Theorem 2.2.** Suppose that \( n \) is a \( p \)-nilpotent Lie algebra with trivial \( p \)-th power operation (i.e., \( x[p] = 0 \) for all \( x \in n \)). Then the cohomology ring \( H^*(u(n), k) \) is Cohen-Macaulay if and only if it is a free module over the polynomial subring \( S^*(n^*)^{(1)} \). In particular, no nonzero element of \( S^*(n^*)^{(1)} \) can be a divisor of zero if the cohomology ring is Cohen-Macaulay.

**Proof.** The last statement clearly follows from the first part of the theorem. Suppose that \( x_1, \ldots, x_r \) is a basis for \( n^* \). By Proposition 2.1

\[
S^*(n^*)^{(1)} = k[x_1, \ldots, x_r] \subseteq H^*(u(n), k).
\]

Because the ordinary Lie algebra cohomology \( H^*(n, k) \) is finitely generated, we have that \( x_1, \ldots, x_r \) is a homogeneous set of parameters for \( H^*(u(n), k) \).

If \( H^*(u(n), k) \) is a finitely generated free module over \( S(n^*)^{(1)} \), then it is Cohen-Macaulay. On the other hand, if \( H^*(u(n), k) \) is Cohen-Macaulay, then it must be free as a module over \( S(n^*)^{(1)} \), by the results cited above. \( \square \)

3. **Consequences of the Lie cohomology spectral sequence**

Let \( n \) be a nilpotent restricted Lie algebra with trivial \( p \)-restriction. We consider the first quadrant spectral sequence given as

\[
E_2^{i,j} = S^i(n^*)^{(1)} \otimes H^j(n, k) \Rightarrow H^{i+j}(u(n), j)
\]

One of important fact to note about the spectral sequence is that \( E_2^{i,j} = \{0\} \) if \( j > \dim(n) \). This is simply because the ordinary Lie algebra cohomology of \( n \) has the property that \( H^j(n, k) = \{0\} \) for \( j > \dim(n) \).

This spectral sequence can be used to show that the cohomology of \( u(\mathfrak{g}) \) is Cohen-Macaulay.

**Proposition 3.1.** If the spectral sequence \( E_2^* \) collapses at the \( E_2 \) page, then \( H^*(u(n), k) \) is free as module over the polynomial subalgebra \( S = S^*(n^*)^{(1)} \). In particular, it is Cohen-Macaulay.

**Proof.** The spectral sequence is a filtered version of the cohomology \( H^*(u(n), k) \). That is, if \( \zeta \in E_2^{i,j} \) and \( \eta \in E_2^{r,s} \) then \( \zeta \eta \in \sum_{\ell \geq 0} E_2^{i+r+\ell, j+s-\ell} \). For this reason, for any \( m \) the collection of the lowest \( m \) rows, \( (U_m = \sum E_2^{i,j} \text{ with } i \geq 0 \text{ and } 0 \leq j \leq m) \) is a module over \( S \), which lies in the bottom row. Because the spectral sequence collapses, \( E_2 = E_\infty \) and the quotients \( U_m/U_{m-1} \) are free \( S \)-modules. Hence, the proposition follows from the fact that every one of the quotient maps \( U_m \to U_m/U_{m-1} \) must split as a map of \( S \)-modules. \( \square \)

Many of the results of [BC1] apply in the case that the polynomial ring is Cohen-Macaulay. In particular, we have the following adaptation of [BC1, Theorem 1.1]. We refer the reader to that paper for the proof which carries over from group cohomology to restricted Lie algebra cohomology with only minimal changes.
Theorem 3.2. Suppose that \( n \) is a restricted p-Lie algebra with trivial p-th power operation. Let \( d \) denote the dimension of \( u \). If the cohomology ring \( H^*(u(n), k) \) is Cohen-Macaulay, then any complete linearly-independent set of degree-two generators \( X_1, \ldots, X_d \) for the symmetric algebra \( S^*(n^*)^{(1)} \) is a homogeneous set of parameters for \( H^*(u(n), k) \) and the quotient

\[
H^*(u(n), k)/(X_1, \ldots, X_d)
\]
satisfies Poincaré duality in formal dimension \( d \). Moreover, the Poincaré series \( P_k(t) = \sum_{i \geq 0} \dim H^i(u(u), k) t^i \), regarded as a rational function of \( t \) satisfies the functional equation

\[
P_k(1/t) = (-t)^d P_k(t).
\]

A consequence of the preceding result is the following.

Corollary 3.3. Suppose that \( n \) is a restricted p-Lie algebra of dimension \( d \) whose \( p \)-power operation is trivial. Then the spectral sequence (3.0.1) collapses at the \( E_2 \) page if and only if the cohomology ring \( H^*(u(n), k) \) is Cohen-Macaulay. In this case, the edge homomorphism \( H^*(u(n), k) \to H^*(n, k) \) is surjective and the Poincaré series, as a rational function has the form

\[
P_k(t) = \frac{f_k(t)}{(1 - t^2)^d},
\]

where \( f_k(t) \) is the Poincaré polynomial for the ordinary Lie algebra cohomology \( H^*(n, k) \).

Proof. For convenience of notation let \( S = S^*(n^*)^{(1)} \) and let \( H^* = H^*(u(n).k) \). By Proposition 3.1 if the spectral sequence (3.0.1) collapses at the \( E_2 \) page then \( H^* \) is Cohen-Macaulay. We need to prove the converse. So assume that \( H^* \) is Cohen-Macaulay. For \( t \geq 0 \), let \( M_t = S - \sum_{j=0}^{t} H^j \), the \( S \)-submodule of \( H^* \) generated by elements of degree at most \( t \). By Theorem 3.2, this is a free \( S \)-module. That is, let \( I = (X_1, \ldots, X_d) \) (in the notation of the theorem). Then \( H^* \) is a free \( S \)-module on a set of homogeneous elements \( \zeta_1, \ldots, \zeta_m \) whose classes form a basis of \( H^*/I \). If \( \zeta_1, \ldots, \zeta_d \) are all of those element of degree at most \( t \), then it is easily seen that \( M_t \) is a free module on these elements.

Now we proceed by induction on \( t \) to prove that

\[
M_t/I \cdot M_t \cong \sum_{i=0}^{t} E_2^{0,i},
\]
as vector spaces, and that no differential \( d_r \) on the \( r \)th page of the spectral sequence has a nonzero image on any of the lines \( E_{r}^{*,0}, \ldots, E_{r}^{*,t} \). This is true if \( t = 0 \), by Proposition 2.1. That is, the differential \( d_r \) on the \( r \)th page cannot have a nonzero image \( d_r : E_r^{j,r-1} \to E_r^{j+r,0} \) as otherwise the edge homomorphism onto the bottom row would not be injective.

So assume that the statement is true for a certain value of \( t \). Then the differential \( d_r \) on the \( r \)th page of the spectral sequence must vanish on \( E_r^{*,t+1} \), as otherwise there would be a nonzero image on one of the lower lines. Therefore, \( E_2^{0,j} = E_\infty^{0,j} \) for all \( j \) with \( 0 \leq j \leq t+1 \). As a consequence, \( M_{t+1} \) is generated as an \( S \)-module by elements representing a \( k \)-basis for \( \sum_{i=0}^{t+1} E_\infty^{0,i} \cong \sum_{i=0}^{t+1} E_2^{0,i} \). That is, \( M_{t+1} \cong \sum_{i=0}^{t+1} E_\infty^{*,i} \). Because, this is a free \( S \)-module,
it must be that
\[ \sum_{i=1}^{t+1} E_{\infty}^{s,i} \cong \sum_{i=1}^{t+1} E_2^{s,i} \]
since these are both free modules on the same number of generators. It follows that no differential of the spectral sequence can have a nonzero image on row \( t + 1 \). The proves the corollary. \( \square \)

We record the following lemma which will be later useful in comparing spectral sequences.

**Lemma 3.4.** Suppose that \( \text{Dim } H^0(\mu(n), k) = \sum_{2i+j=m} \text{Dim}(S_{2i}(n^*) \otimes H^j(n, k)) \) for all \( m \geq 0 \). Then the spectral sequence \((3.0.1)\) collapses at the \( E_2 \) page and the cohomology ring \( H^*(\mu(n), k) \) is Cohen-Macaulay.

**Proof.** The hypotheses of the lemma asserts that \( \text{Dim } H^0(\mu(n), k) = \sum_{m=2i+j} \text{Dim } E_2^{2i+j} \). The condition forces the spectral sequence to collapse at the \( E_2 \) page, because otherwise there would be some further nontrivial differential that would reduce the dimension. \( \square \)

Nilpotent Lie algebras can be built up from central extensions. The next theorem provides conditions on when the spectral sequence will collapse at \( E_2 \) under a central extension and yield an isomorphism of \( S^*(n^*)^{(1)} \)-modules.

**Theorem 3.5.** Let \( n \) be a nilpotent restricted \( p \)-Lie algebra with trivial \( p \)-power operation. Assume that \( \mathfrak{z} \) is a central ideal of dimension one. Suppose that we have an isomorphism of rings
\[ H^*(\mu(\mathfrak{z}/\mathfrak{z}), k) \cong S^*((\mathfrak{z}/\mathfrak{z})^*)^{(1)} \otimes H^*(\mathfrak{z}/\mathfrak{z}, k). \]
Then
\[ H^*(\mu(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k). \]
as modules over \( S^*(n^*)^{(1)} \). In particular, \( H^*(\mu(n), k) \) is Cohen-Macaulay.

**Proof.** We use the Lyndon-Hochschild-Serre (LHS) spectral sequence
\[ E_2^{i,j} = H^i(\mu(\mathfrak{z}/\mathfrak{z}), H^j(\mathfrak{z}, k)) \Rightarrow H^{i+j}(\mu(n), k). \]
Since \( \mathfrak{z} \) is central, we have \( E_2^{i,j} \cong H^i(\mu(\mathfrak{z}/\mathfrak{z}), k) \otimes H^j(\mathfrak{z}, k). \)

Next note that \( E_2^{0,1} \) has dimension one, and is spanned by an element \( x \). Then \( d_2(x) = s + v \), where \( s \) is in \( S^1((\mathfrak{z}/\mathfrak{z})^*)^{(1)} \otimes 1 \) and \( v \) is in \( 1 \otimes H^2(\mathfrak{z}/\mathfrak{z}, k) \) by the hypothesis. Also by the hypothesis, we have that \( v^n = 0 \) for \( n \) sufficiently large, since \( H^*(\mathfrak{z}/\mathfrak{z}, k) \) has finite dimension. Consequently, for \( r \) sufficiently large we have that \( 0 = (s+v)^r = s^r + v^r = s^r \) on the \( E_3 \) page of the spectral sequence. However, if \( s \) is not zero, then we have a contradiction to the fact that \( S^*(n^*)^{(1)} \) injects into the cohomology ring \( H^*(\mu(n), k) \). So \( d_2(x) \in 1 \otimes H^2(\mathfrak{z}/\mathfrak{z}, k) \).

Next we see that the rows \( E_2^{0,0} \) and \( E_2^{1,1} \) are both isomorphic to \( H^*(\mu(\mathfrak{z}/\mathfrak{z}), k) \) as modules over the symmetric algebra \( S^*(\mathfrak{z}/\mathfrak{z})^*)^{(1)} \). Moreover, the differential \( d_2 : E_2^{*,1} \rightarrow E_2^{*,2,0} \) is a homomorphism of \( S^*((\mathfrak{z}/\mathfrak{z})^*)^{(1)} \)-modules. More specifically, we have that the differential
\[ d_2 : S^*(n^*)^{(1)} \otimes H^*(n, k) \cong E_2^{*,1} \rightarrow E_2^{*,0} \cong S^*(n^*)^{(1)} \otimes H^*(n, k) \]
The hypothesis, we have that $E^{s,1}_3 \cong S(n^*)^{(1)} \otimes K$ and $E^{s,1}_3 \cong S(n^*)^{(1)} \otimes C$, where $K$ and $C$ are respectively the kernel and cokernel of multiplication by $w$ on $H^*(n/3, k)$.

We now observe that the element $w \in H^2(n/3, k)$ is the extension class associated to the extension $3 \to n \to n/3$. In the LHS spectral sequence $E^{s,0}_2 = H^*(n/3, H^j(3, k)) \Rightarrow H^{i+j}(n, k)$, of ordinary Lie algebra cohomology associated to that sequence, the differential $d_2 : E^{s,1}_2 \to E^{s,0}_2$ is multiplication by $w$. In addition, there can be no further differentials in that spectral sequence, since the sequence has only two nonzero rows. It follows that $E^{s,0}_2 \oplus E^{s,1}_2 \cong S^*((n/3)^*)^{(1)} \otimes H^*(n, k)$ as (free) modules over $S^*((n/3)^*)^{(1)}$.

Finally, we note that $E_3 = E_\infty$. The reason is that the $E_3$ page is generated, as a ring by the elements on the bottom two rows ($E^{s,0}_3$ and $E^{s,1}_3$) and an element $X$ in $E^{0,2}_3$ that represents the class of a generator in $S^*(3^*)^{(1)} \subseteq S^*(n^*)^{(1)}$. The class $X$ must survive until the $E_\infty$ page of the spectral sequence. So $d_2(X) = 0$, and we must have that $XE_3^{i,j} = E_3^{i,j+2}$ for all $i, j \geq 0$. Consequently, the differential $d_3$ must vanish, because it vanishes on a collection of ring generators. The same holds for all further differentials in the spectral sequence.

We have verified the hypothesis of Lemma 3.4 and the theorem is proved. \hfill \qed

As an initial application we offer the following.

**Corollary 3.6.** Suppose that $\text{Dim}([n,n]) = 1$. Then $H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k)$, as $S^*(n^*)^{(1)}$-modules, and $H^*(u(n), k)$ is Cohen-Macaulay.

*Proof.* Observe that the quotient algebra $v = n/[n,n]$ is commutative and hence we have that $H^*(u(v), k) \cong S^*(v^*)^{(1)} \otimes H^*(v, k)$ as rings. Thus, Theorem 3.5 implies the corollary. \hfill \qed

The next theorem provides stronger conditions which will show when one can identify $H^*(u(n), k)$ with $S^*(n^*)^{(1)} \otimes H^*(n, k)$ as rings. Together this theorem in conjunction with Theorem 3.5 can be applied to inductively compute cohomology rings.

**Theorem 3.7.** Let $n$ be a $p$-nilpotent Lie algebra and suppose that there is an isomorphism of $S^*(n^*)^{(1)}$-modules,

$$H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k).$$

Moreover, assume that there exists a subalgebra $B$ in $H^*(u(n), k)$ such that $B \cong H^*(n, k)$ under the map $\phi : H^*(u(n), k) \to H^*(n, k)$. Then $H^*(u(n), k) \cong S^*(n^*)^{[1]} \otimes H^*(n, k)$ as rings.

*Proof.* Let $A$ be the subalgebra in $H^*(u(n), k)$ isomorphic to $S^*(n^*)^{(1)}$. We have an algebra homomorphism $\Gamma$ defined by

$$S^*(n^*)^{(1)} \otimes H^*(n, k) \to A \otimes B \to H^*(u(n), k) \otimes H^*(u(n), k) \to H^*(u(n), k).$$

The last map is given by the cup product. This map is bijective because

$$H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k).$$

as $S^*(n^*)^{(1)}$-modules. \hfill \qed
4. SOME EXAMPLES OF TYPE A

We begin with the example of the nilpotent radical \(\mathfrak{n}\) of a Borel subalgebra of \(\mathfrak{sl}_3\). The relations for the cohomology in characteristic 3 were calculated by computer using the system Magma \([BoCa]\). Specifically, we use the package for basic algebras written by the first author. Two of the three unusual relation can be derived from the second example in this section.

Note that \(\mathfrak{n}\) has an action of a two dimensional torus. Let \(\alpha\) and \(\beta\) be the simple roots. By convention the weights of \(\mathfrak{n}\) consist of sums of negative roots so that its cohomology has weights in the positive cone of roots. For convenience, we subscript elements by their weights whenever this causes no problems.

**Lemma 4.1.** Let \(\mathfrak{n}\) be the nilpotent radical of a Borel subalgebra of \(\mathfrak{sl}_3\) over a field of characteristic at least three. Then the ordinary Lie algebra cohomology of \(\mathfrak{n}\) is given as

\[ H^*(\mathfrak{n}, k) = k[\eta_{\alpha}, \eta_{\beta}, \eta_{2\alpha+\beta}, \eta_{\alpha+2\beta}]/I \]

where \(I\) is the ideal generated by

\[ \eta_{\alpha}^2, \eta_{\alpha}\eta_{\beta}, \eta_{\beta}^2, \eta_{\alpha}\eta_{2\alpha+\beta}, \eta_{\beta}\eta_{2\alpha+\beta}, \eta_{\beta}\eta_{\alpha+2\beta} + \eta_{\alpha}\eta_{2\alpha+\beta}, \eta_{2\alpha+\beta}^2, \eta_{\alpha+2\beta}^2, \eta_{2\alpha+\beta}\eta_{\alpha+2\beta} \]

**Proof.** The result follows easily from the LHS spectral sequence \(E^{2}_{2,j} = H^j(\mathfrak{n}/\mathfrak{h}, H^j(\mathfrak{h}, k)) \Rightarrow H^{i+j}(\mathfrak{n}, k)\). Note that the torus \(T\) acts on the spectral sequence. We let \(\eta_\alpha\) and \(\eta_\beta\) be the generators of \(E^{1,0}_2\) having weights \(\alpha\) and \(\beta\) and let \(\eta_{\alpha+\beta}\) be the generator of \(E^{0,1}_2\) with weight \(\alpha + \beta\). It is easy to check that \(d_2(\eta_{\alpha+\beta}) = \eta_\alpha \eta_\beta\), which is the extension class. The elements \(\eta_\alpha \eta_{\alpha+\beta}\) and \(\eta_\beta \eta_{\alpha+\beta}\) survive to the \(E_3 = E_{\infty}\) page, and are ring generators - which we call \(\eta_{2\alpha+\beta}\) and \(\eta_{\alpha+2\beta}\), respectively. The relations follow from easily from the relations in the exterior algebra of \(\mathfrak{n}^*\).

With this lemma, we can compute the cohomology of the restricted Lie algebra. The following calculation is computer generated in part. However, as we see in the next example all but one of the relations can be derived by hand.

**Proposition 4.2.** Let \(k\) be a field of characteristic 3. Let \(u(\mathfrak{n})\) be the restricted enveloping algebra of the Lie algebra \(\mathfrak{n}\) which is the nilpotent radical of the Borel subalgebra of \(\mathfrak{sl}_3\). Then the cohomology ring \(H^*(u(\mathfrak{n}), k) \cong S^* (\mathfrak{n}^*)^{(1)} \otimes H^*(\mathfrak{n}, k)\) is a free \(S^*(\mathfrak{n}^*)^{(1)}\)-module with basis consisting of the images of a basis of the ordinary Lie algebra cohomology of \(\mathfrak{n}\) as in Lemma 4.1. The ring \(S^*(\mathfrak{n}^*)^{(1)}\) is a polynomial ring in variables \(X_\alpha, X_\beta, X_{\alpha+\beta}\) having weights \(3\alpha, 3\beta, 3(\alpha + \beta)\) under the action of the torus. The multiplicative relations are given by

\[ \eta_{\alpha}^2, \eta_{\alpha} \eta_{\beta}, \eta_{\beta}^2, \eta_{\alpha} \eta_{2\alpha+\beta} + \eta_{\alpha} \eta_{\alpha+2\beta}, \eta_{2\alpha+\beta}^2, \eta_{\alpha+2\beta}^2, \eta_{2\alpha+\beta} \eta_{\alpha+2\beta} \]

\[ \eta_{\alpha} \eta_{2\alpha+\beta} - \eta_{\beta} X_{\alpha}, \eta_{\beta} \eta_{\alpha+2\beta} - \eta_{\alpha} X_{\beta}, \eta_{2\alpha+\beta} \eta_{\alpha+2\beta} - X_{\alpha} X_{\beta} \]

**Proof.** First note that the isomorphism \(H^*(u(\mathfrak{n}), k) \cong S^*(\mathfrak{n}^*)^{(1)} \otimes H^*(\mathfrak{n}, k)\) is a consequence of Corollary 3.6. Consider that LHS spectral sequence

\[ E^{i,j}_2 = H^j(u(\mathfrak{n}/\mathfrak{h}, H^j(\mathfrak{h}, k)) \Rightarrow H^{i+j}(u(\mathfrak{n}, k)) \]

We have elements \(a, b\) in \(E^{1,0}_2\) and \(u\) in \(E^{0,1}_2\). The differential on the \(E_2\) page has the form \(d_2(u) = ab\) as in the proof of Lemma 4.1. A representative of \(X_{\alpha+\beta}\) is in \(E^{2,2}_2\), and
this element must live until the $E_\infty$ page. Because the resulting ring at the $E_3$ page is generated in degrees one and two, we conclude that all further differentials must vanish and $E_3 = E_\infty$. The problem is to ungrade the spectral sequence. The first six relations are forced by the grading on the spectral sequence and by the action of the torus. For example, the relation $\eta_\beta \eta_{2\alpha+\beta} + \eta_\alpha \eta_{\alpha+2\beta}$ holds on the (graded) $E_3$ page and must hold in the ungrading because there is no nonzero element of that weight $(2\alpha + 2\beta)$ in $E_3^{3,0}$.

For the last three relations, we rely on the computer. For example, $\eta_\alpha \eta_{2\alpha+\beta} = 0$ in the graded $E_3$ page. This element lies in $E_3^{2,1}$ in the ungrading it is equal to $\eta_\beta X_\alpha$. Note that both elements have weight $3\alpha + \beta$. The other two relations are similar. \qed

Next we extend the example slightly to get a nilpotent Lie algebra (with trivial $p$th power) where the restricted cohomology fails to be Cohen-Macaulay. We consider the algebra $\mathfrak{n}$, labeled as $L_{5,9}$ in the list of de Graaf \cite{deG}. This algebra is isomorphic to the quotient of the Lie algebra of all upper triangular $4 \times 4$ matrices by its center. The algebra can also be represented as the algebra of all strictly upper triangular matrices such that all entries in the third (and fourth) row are zero. Notice that in this representation, the algebra has trivial $p$-power operation. The algebra has a basis consisting of $u_1, \ldots, u_5$, where $u_4$ and $u_5$ are central, and with the additional relations:

$$[u_1, u_2] = u_4, [u_2, u_3] = u_5, [u_1, u_3] = 0.$$  

The reader should be aware that this is a minor change from the presentation in \cite{deG}.

**Proposition 4.3.** Suppose that the characteristic of $k$ is 3, and let $\mathfrak{n}$ be as above. The cohomology ring $H^*(u(\mathfrak{n}), k)$ is not Cohen-Macaulay. In particular, it has an associated prime $\mathfrak{P}$ such that $H^*(u(\mathfrak{n}), k)/\mathfrak{P}$ has Krull dimension four.

**Proof.** First we observe that the algebra has an action of a three dimensional torus, $T$ (in the representation as upper triangular $4 \times 4$ matrices modulo the center of that algebra). With this action, the basis elements $u_1, \ldots, u_5$ have weights $-\alpha, -\beta, -\gamma, -\alpha - \beta$ and $-\beta - \gamma$, respectively. The torus also acts on the cohomology.

Let $\mathfrak{v}$ be the commutative subalgebra of $\mathfrak{n}$ spanned by $u_1, u_3, u_4, u_5$ and consider the spectral sequence. Its Lie algebra cohomology is an exterior algebra generated by elements $\eta_\alpha, \eta_{\alpha+\beta}, \eta_{\beta+\gamma}, \eta_\gamma$ all in degree one and having weights as indicated by the subscripts. The element $u_2$ acts on $\mathfrak{v}$ and on its cohomology. After possible rescaling, we have that $u_2 \eta_{\alpha+\beta} = \eta_\alpha$ and $u_2 \eta_{\beta+\gamma} = \eta_\gamma$. Recall that the action on the cohomology is dual to the action on the algebra, so multiplying by $u_2$ on an element of cohomology subtracts the root $\beta$.

Now we consider the spectral sequence

$$E_2^{i,j} = H^i(u(\mathfrak{n}/\mathfrak{v}), H^j(u(\mathfrak{v}), k) \Rightarrow H^{i+j}(u(\mathfrak{n}), k).$$

As a module over $u(\mathfrak{n}/\mathfrak{v})$,

$$H^1(u(\mathfrak{v})) \cong M_1 \oplus M_2$$

where $M_1$ is the span of $\{\eta_\alpha, \eta_{\alpha+\beta}\}$, $M_2$ is the span of $\{\eta_\gamma, \eta_{\beta+\gamma}\}$ and the action of the class of $u_2$ is given as above. In degree 2, we have that

$$H^2(u(\mathfrak{n}), k) \cong \Lambda^2(M_1) \oplus M_1 \otimes M_2 \oplus \Lambda^2(M_2) \oplus k^4$$
where the last factor is spanned by the generators $X_\alpha, X_\gamma, X_{\alpha+\beta}, X_{\beta+\gamma}$ (each having weight equal to three times its index) of $S^*(u)^{(1)}$ that are fixed by the action of $u_2$. The interesting part of this is the tensor product $M_1 \otimes M_2$ which is the direct sum of a trivial module spanned by $\eta_\alpha \wedge \eta_{\beta+\gamma} - \eta_{\alpha+\beta} \wedge \eta_\gamma$ and an indecomposable three dimensional module generated by $\eta_{\alpha+\beta} \wedge \eta_{\beta+\gamma}$ and with socle spanned by $w = \eta_\alpha \wedge \eta_\gamma$. Because the characteristic is 3, this is a free module over $u(\mathfrak{n}/\mathfrak{v})$. Thus there is an element $E_2^{3,2} \cong H^2(u(\mathfrak{v}), k)^{n/5}$ that is determined by $w$. Write $M_1 \otimes M_2 \cong k \oplus N$, where $N$ is the free submodule with socle spanned by $w$.

Let $X_\beta \in E_2^{2,0}$ be the generator of $S^1((\mathfrak{n}/\mathfrak{v})^*)^{(1)}$. This elements survives to the $E_\infty$ page. Moreover, every element in $E_2^{i,j}$ for $j \geq 2$ is a multiple of this element. We claim that this element must be contained in an associated prime of $H^*(u(\mathfrak{n}), k)$. Specifically, the element $\hat{w} \in H^0(u(\mathfrak{n}/\mathfrak{v}), N)$ determined by $w$ in $E_2^{2,0}$ has the property that $X_\beta \hat{w} = 0$ on the $E_2$ page because $N$ is free over $u(\mathfrak{n}/\mathfrak{v})$. So we only need to show that $\hat{w}$ survives to the $E_\infty$ page and that the product $X_\beta \hat{w}$ does not ungrade to something that is nonzero. Both of these statements can be deduced from looking at the action of the torus. That is, $w$ has weight $\alpha + \gamma$ and there is no element of that weight in either $E_2^{2,1}$ or $E_2^{3,1}$. Hence both $d_2$ and $d_3$ must both vanish on the class of $w$. Likewise, $X_\beta \hat{w}$ has weight $\alpha + 3\beta + \gamma$ and there is no element of that weight in $E_\infty^{3,1}$ or $E_\infty^{4,0}$. So we must have that $X_\beta$ annihilates the class of $w$ in $H^*(u(\mathfrak{n}), k)$.

**Remark 4.4.** As mentioned earlier in the paper, two of the computer generated relations in Proposition 4.3 are indicated by the calculation above. For this we require the spectral sequence

$$E_2^{i,j} = \cdots \Rightarrow H^i(u(\mathfrak{n}/\mathfrak{j}), H^j(u(\mathfrak{j}), k))$$

where $\mathfrak{n}$ is the 5-dimensional Lie algebra as above and $\mathfrak{j}$ is the one-dimensional subalgebra spanned by $u_5$. Note that $\mathfrak{n}/\mathfrak{j} \cong \mathfrak{a} \oplus \mathfrak{b}$ where $\mathfrak{a}$ (generated by the classes of $u_1, u_2, u_4$) is the nilpotent radical of a Borel subalgebra of $\mathfrak{sl}_3$, and $\mathfrak{b}$ (generated by $u_3$) is a one dimensional Lie algebra. The bottom row of the spectral sequence is generated by elements $\eta_\alpha, \eta_\beta, \eta_{2\alpha+\beta}, \eta_{\alpha+2\beta}, X_\alpha, X_\beta, X_{\alpha+\beta}$, generating $H^*(u(\mathfrak{a}), k)$, and $\eta_\gamma, X_\gamma$, generating $H^*(u(\mathfrak{b}), k)$. The weights are as indicated by the subscripts. The term $E_2^{1,0}$ is spanned by an element $\eta_{\beta+\gamma}$. The image of this under the differential $d_2$ is $\eta_{\beta+\gamma}$. What we know from the proof of Proposition 4.3 is that $\eta_\alpha \eta_\gamma X_\beta = 0$. This element can only be zero if it is in the image of $d_2$. Then by an examination of weights we see that (up to some nonzero scalar multiple) $d_2(\eta_{\beta+\gamma} \eta_{\alpha+2\beta}) = \eta_\beta \eta_\gamma \eta_{\alpha+2\beta} = \eta_\alpha \eta_\gamma X_\beta$. Note that this relation occurs in the ring $E_2^{*,0} \cong H^*(u(\mathfrak{a}), k) \otimes H^*(u(\mathfrak{b}), k)$. Consequently, we must have that $\eta_\gamma \eta_{\alpha+2\beta} = \eta_\alpha X_\beta$ in $H^*(u(\mathfrak{a}), k)$, as asserted. The relation $\eta_\alpha \eta_{2\alpha+\beta} = \eta_\beta X_\alpha$ follows by symmetry (interchanging $u_1$ with $u_3$ and $u_4$ with $u_5$).

We shall see in Section 7 that the example in Proposition 4.3 can be generalized, giving a metabelian Lie algebra whose cohomology ring is not Cohen-Macaulay for any prime $p$.

5. Some examples of type B

In this section, we consider the nilpotent radical $\mathfrak{n}$ of the Borel subalgebra of a Lie algebra of type $B_2$ and some extensions thereof. We show that in characteristic 5, the
cohomology of \( \mathfrak{n} \) has the form \( H^*(u(\mathfrak{n}), k) \cong S(\mathfrak{n}^*)^{(1)} \otimes H^*(\mathfrak{n}, k) \) as a module over \( S(\mathfrak{n}^*)^{(1)} \), but not as a ring. Moreover, there is a one-dimensional extension of this Lie algebra whose cohomology is not Cohen-Macaulay. The basic idea of the construction applies other non-simply laced cases in other characteristics.

Note that if the characteristic of \( k \) is greater than \( 6 = 2(h - 1) \) then the isomorphism \( H^*(u(\mathfrak{n}), k) \cong S(\mathfrak{n}^*)^{(1)} \otimes H^*(\mathfrak{n}, k) \) is an isomorphism of rings by [DNN] Theorem 3.1.1.

The Lie algebra \( \mathfrak{n} \) has a basis \( u_1, u_2, u_3, u_4 \) and the Lie bracket is given by \([u_1, u_2] = u_3, [u_1, u_3] = u_4, [u_2, u_3] = 0\) with \( u_4 \) being central. There is an action of a two-dimensional torus, \( T \), relative to which the basis elements have weights \(-\alpha, -\beta, -\alpha - \beta, -2\alpha - \beta\) respectively (\( \alpha \) being the short simple root).

We begin with a calculation of the ordinary Lie algebra cohomology of \( \mathfrak{sl}_3 \). Its image under the \( d_2 \) differential is the extension class \( \eta_{\alpha+\beta} \). The product structure is derived from the product on the \( E_2 \) page as well as weight considerations.  

**Lemma 5.1.** The cohomology ring \( H^*(\mathfrak{n}, k) \) is generated by elements which we denote \( \eta_\alpha, \eta_\beta, \eta_{\alpha+\beta}, \eta_{3\alpha+\beta}, \eta_{\alpha+2\beta}, \eta_{3\alpha+3\beta}, \eta_{\alpha+3\beta} \) in degrees \( 1, 1, 2, 2, 3, 3 \). All products of two of the given generators are zero except for the products

\[
\eta_\alpha \eta_{\alpha+3\beta} = -\eta_\beta \eta_{\alpha+2\beta} = \eta_{3\alpha+\beta} \eta_{\alpha+2\beta}
\]

**Proof.** Let \( \mathfrak{z} \) be the subalgebra spanned by \( u_4 \). Then we have a spectral sequence given as

\[
E_2^{i,j} = H^i(\mathfrak{n}/\mathfrak{z}, H^j(\mathfrak{z}, k)) \Rightarrow H^{i+j}(\mathfrak{n}, k).
\]

The bottom row is the cohomology of the nilpotent radical of a Borel subalgebra of \( \mathfrak{sl}_3 \), which is provided in Lemma [4.1]. We adopt the notation for the elements as given in that lemma. The \( E_2 \) term of the spectral sequence is generated as a ring by one additional element \( \zeta = \zeta_{2\alpha+\beta} \in E_2^{0,1} \). Its image under the \( d_2 \) differential is the extension class \( \eta_{2\alpha+\beta} \) (having the same weight). The differential vanishes on every product of \( \zeta \) with an element on the bottom row except \( \eta_\beta \zeta \), and there \( d_2(\eta_\alpha \zeta) = \eta_\beta \eta_{2\alpha+\beta} = -\eta_\alpha \eta_{\alpha+2\beta} \). The product structure is derived from the product on the \( E_2 \) page as well as weight considerations.  

Now we extend this to the restricted Lie algebra cohomology. Some of the relations in the proposition given below were calculated using the basic algebra package in Magma.

**Proposition 5.2.** Suppose that \( k \) is a field of characteristic 5. Let \( \mathfrak{n} \) be the nilpotent radical of the Borel subalgebra of a Lie algebra of type \( B_2 \). The cohomology ring \( H^*(u(\mathfrak{n}), k) \cong S^*(\mathfrak{n}^*)^{(1)} \otimes H^*(\mathfrak{n}, k) \) is a free \( S^*(\mathfrak{n}^*)^{(1)} \)-module with basis consisting of the images of the basis of the ordinary Lie algebra cohomology of \( \mathfrak{n} \) as in Lemma [4.4]. The ring \( S^*(\mathfrak{n}^*)^{(1)} \) is a polynomial ring in variables \( X_\alpha, X_\beta, X_{\alpha+\beta} \) and \( X_{2\alpha+\beta} \) having weights \( 5\alpha, 5\beta, 5(\alpha + \beta) \) and \( 5(2\alpha + \beta) \) under the action of the torus. The multiplicative relations among the generators of \( 1 \otimes H^*(\mathfrak{n}, k) \subseteq H^*(u(\mathfrak{n}), k) \) are exactly as given in Lemma [7.4], except that \( \eta_{3\alpha+\beta}^2 = \eta_{\alpha+2\beta} X_\alpha \). In particular, the isomorphism \( H^*(u(\mathfrak{n}), k) \cong S^*(\mathfrak{n}^*)^{(1)} \otimes H^*(\mathfrak{n}, k) \) does not hold as rings.

**Proof.** The first statement follows from [DNN] Theorem 3.1.1. A large part of the remainder of the proof can be derived from the LHS spectral sequence and weight considerations. The unusual relation was verified by the computer.
Next we consider an extension of the algebra, whose cohomology ring is not Cohen-Macaulay. As in the case of Remark 5.4, we can derive the unusual relation in the above proposition from the calculations of the cohomology of the extension.

Let $\mathfrak{n}$ be the restricted Lie algebra of dimension 5, with basis $u_1, \ldots, u_5$ and Lie bracket given by $[u_1, u_2] = u_3, [u_1, u_3] = u_4, [u_1, u_4] = u_5$, with $u_2, \ldots, u_5$ forming a commutative subalgebra that we denote $\mathfrak{v}$. We continue to assume that the characteristic of $k$ is 5. The algebra $\mathfrak{n}$ has an action of a 2-dimensional torus, so that the elements $u_1, \ldots, u_5$ have weights $-\alpha, -\beta, -\alpha - \beta, -2\alpha - \beta$ and $-3\alpha - \beta$.

**Proposition 5.3.** The cohomology ring $H^*(\mathfrak{u}(\mathfrak{n}), k)$ is not Cohen-Macaulay. In particular, there is an element in $H^2(\mathfrak{u}(\mathfrak{n}), k)$ whose annihilator $\mathfrak{P}$ has the property that $H^*(\mathfrak{u}(\mathfrak{n}), k)/\mathfrak{P}$ has Krull dimension four.

**Proof.** We consider the spectral sequence: $E_2^{i,j} = H^i(\mathfrak{u}(\mathfrak{n}/\mathfrak{v}), H^j(\mathfrak{u}(\mathfrak{v}), k)) \Rightarrow H^{i+j}(\mathfrak{u}(\mathfrak{n}), k)$. As a module over $u(\mathfrak{n}/\mathfrak{v})$, $M = H^1(\mathfrak{u}(\mathfrak{v}), k)$ is uniserial of length 4, generated by an element $\eta_{\alpha+\beta}$. Then $H^2(\mathfrak{u}(\mathfrak{v}), k)$ is the exterior square of $M$ and it has a free $u(\mathfrak{n}/\mathfrak{v})$-summand generated by $\eta_{\alpha+\beta} \wedge u_\alpha \eta_{\alpha+\beta}$ and having socle generated by $\eta_{\alpha+\beta} \wedge \eta_\beta$. Hence there is a class $\zeta$ in $E_2^{2,2}$ represented by a $u(\mathfrak{n}/\mathfrak{v})$-homomorphism of $k$ onto the socle of this summand. This class is annihilated on the $E_2$ page of the spectral sequence by the class $X_\alpha$ in $E_2^{0,2}$. As in the case of Proposition 4.3, we can argue by weights that $\zeta$ is represented by a nonzero class $\zeta \in H^2(\mathfrak{u}(\mathfrak{n}), k)$ such that $X_\alpha \zeta = 0$, and the annihilator of $\zeta$ is contained in a prime $\mathfrak{P}$ having the asserted properties.

**Remark 5.4.** We should note that the action of the torus is not required to show that the example is not Cohen-Macaulay. If it were the case that one of the differentials $d_2$ or $d_3$ failed to vanish on the class $\zeta$, then we would have that $d_2(\zeta) = X_\alpha \mu$ for $\mu$ in $E_2^{0,1}$ or that $d_3(\zeta) = X_\alpha \mu$ for $\mu$ in $E_3^{0,0}$. In either case we would have a class in $H^1(\mathfrak{u}(\mathfrak{n}), k)$ that was annihilated by $X_\alpha$. Similarly, there is no way to ungrade the spectral sequence to avoid having a large associate prime in the cohomology ring.

**Remark 5.5.** We remark, as in 4.4, that the unusual relation $\eta_{\alpha+\beta}^2 = \eta_{\alpha+2\beta}X_\alpha$ in Proposition 5.2 can be derived from the last Proposition. The proof is almost exactly the same as in Remark 4.4 and we leave the details to the interested reader.

**Remark 5.6.** The situation in Proposition 5.3 can be extended to give examples for other primes. For suppose that $p > 5$, and let $\mathfrak{n}$ be the restricted $p$-Lie algebra of dimension $n + 1$ for some $n$ with $(p+3)/2 < n < p$, defined as follows. A basis for $\mathfrak{n}$ consists of the elements $v, u_1, \ldots, u_n$, where $u_1, \ldots, u_n$ span a commutative subalgebra, which we denote $\mathfrak{v}$. Then the product is given by $[v, u_i] = u_{i+1}$ for $i = 1, \ldots, n-1$, and $[v, u_n] = 0$. The $p$th-power operation is zero on $\mathfrak{n}$. The algebra $\mathfrak{n}$ has an action of a two-dimensional torus such that the basis elements $v, u_1, \ldots, u_n$ have weights $-\alpha, -\beta, -\alpha - \beta, \ldots, -(n-1)\alpha - \beta$, respectively. Then $M = H^1(\mathfrak{u}(\mathfrak{v}), k)$ is an indecomposable uniserial module of dimension $n$ over the algebra $u(\mathfrak{n}/\mathfrak{v})$. Because $n \geq (p+3)/2$, its exterior square $\Lambda^2(M)$ has a free summand. Hence, considering the spectral sequence with $E_2$ term $E_2^{i,j} = H^i(\mathfrak{u}(\mathfrak{n}/\mathfrak{v}), H^j(\mathfrak{u}(\mathfrak{v}), k)) \Rightarrow H^{i+j}(\mathfrak{u}(\mathfrak{n}), k)$ and arguing exactly as in the proof of Proposition 5.3 we get that $H^*(\mathfrak{u}(\mathfrak{n}), k)$ has an associated prime $\mathfrak{P}$ such that $H^*(\mathfrak{u}(\mathfrak{n}), k)/\mathfrak{P}$ has Krull dimension at most $n$. 
6. An example of type $G_2$ in characteristic 7

In this section we consider the nilpotent radical of a Borel subalgebra of the restricted Lie algebra of type $G_2$. We obtain a similar result to that in Proposition 4.2 and Proposition 5.2. Because the methods are also very similar to those in the aforementioned propositions, we give only a sketch.

**Proposition 6.1.** Suppose that $n$ is the nilpotent radical of a Borel subalgebra of the restricted Lie algebra of type $G_2$. Then as a module over the symmetric algebra $S^{*}(n^{*})^{(1)}$, we have that $H^{i}(u(n), k) \cong S^{*}(n^{*})^{(1)} \otimes H^{*}(n, k)$. However, this is not an isomorphism as rings.

**Proof.** The characteristic of the field $k$ is larger than the Coxeter number and hence the first statement is a consequence of [FP] (3.5) Prop]. Our task is to prove the second statement. Suppose that $\alpha$ and $\beta$ are the simple roots for the root system of type $G_2$. Assume that $\alpha$ is the short root. The other positive roots are $\alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$. We construct the central extension $\varepsilon$

$$0 \longrightarrow a \longrightarrow g \longrightarrow n \longrightarrow 0$$

where $a$ has dimension one, and $g$ has an action by the two dimensional torus so that an element of $a$ has weight $-4\alpha - \beta$. Thus $g$ has basis $u_{\alpha}, u_{\beta}, u_{\alpha + \beta}, u_{2\alpha + \beta}, u_{3\alpha + \beta}, u_{4\alpha + \beta}, u_{3\alpha + 2\beta}$ where the subscript on each element indicates the negative of its weight.

Let $v$ be the subalgebra generated by $u_{\beta}, u_{\alpha + \beta}, u_{2\alpha + \beta}, u_{3\alpha + \beta}, u_{4\alpha + \beta}, u_{3\alpha + 2\beta}$, and let $j$ be the subalgebra generated by $u_{3\alpha + 2\beta}$. The cohomology of $v$ can be computed from the spectral sequence $E_{2}^{ij} = H^{i}(v/j, H^{j}(j, k)) \Rightarrow H^{i+j}(v, k)$. Let $\eta_{i}$ denote an element of weight $\gamma$ on this $E_{2}$ page. The differential must take the element $\eta_{3\alpha + 2\beta} \in E_{2}^{0,1}$ to the extension class $d_{2}(\eta_{3\alpha + 2\beta}) = \eta_{\beta} \wedge \eta_{3\alpha + \beta} - \eta_{\alpha + \beta} \wedge \eta_{2\alpha + \beta} \in E_{2}^{2,0}$. Note that this element is annihilated by the action of $u_{\alpha}$, as is $\eta_{2\alpha + \beta}$. The point of this calculation is that $H^{2}(v, k)$ contains a free module under the action of $u(g/v)$. That is the element $\eta_{\alpha + \beta} \wedge \eta_{3\alpha + \beta}$ generates a uniserial module of dimension 7 over $u(g/v)$, whose socle (the submodule annihilated by the action of $u_{\alpha}$) is spanned by $\eta_{\alpha + \beta} \wedge \eta_{\beta}$, a class that survives to the $E_{\infty}$ page of the spectral sequence. This implies that $H^{*}(u(g), k)$ is not Cohen-Macaulay. That is, we see from the spectral sequence $E_{2}^{ij} = H^{i}(u(g/v), H^{j}(u(v), k) \Rightarrow H^{i+j}(u(g), k)$, that the element $X_{\alpha} \in E_{2}^{0,2}$ in the symmetric algebra annihilates the element corresponding to $\eta_{\alpha + \beta} \wedge \eta_{\beta} \in E_{2}^{0,2}$. This spectral sequence collapses at the $E_{2}$ page, and hence the relation exists in $H^{*}(u(g), k)$.

The proposition is a consequence of Theorem 3.3. More specifically, by following the arguments in Proposition 4.2 and using the weight information, we see that in $H^{4}(u(n), k)$ there must be a relation having roughly the form $(\eta_{\alpha} \wedge \eta_{3\alpha + \beta})^{2} = (\eta_{\alpha + \beta} \wedge \eta_{\beta})X_{\alpha}$. Note that both are elements of weight $8\alpha + 2\beta$.

$\square$

7. A metabelian example

In this section we present an example of a metabelian restricted Lie algebra with the property that its cohomology ring is not Cohen-Macaulay. Such an example can be constructed for any value of $p \geq 3$, except that the dimension of the example depends on the prime $p$. 

Let \( \mathfrak{n} \) be the nilpotent restricted Lie algebra with basis consisting of the elements \( u, v_i, w_i \) for \( i = 1, \ldots, n \), and Lie bracket defined by the rule

\[
[u, v_i] = w_i, \quad [u, w_i] = 0 = [v_i, v_j] = [v_i, w_j] = [w_i, w_j]
\]

for all \( i, j \) such that \( 1 \leq i, j \leq n \). Note that \( \mathfrak{n} \) is isomorphic to a subalgebra of \( \mathfrak{sl}_{n+2} \). That is, we can define a homomorphism \( \varphi : \mathfrak{n} \to \mathfrak{sl}_{n+2} \) as follows. Let \( E_{i,j} \) be the matrix with \( 1 \) in the \( (i,j) \) position and 0 elsewhere. Then define \( \varphi \) by \( \varphi(u) = E_{1,2} \), \( \varphi(v_i) = E_{2,i+2} \) and \( \varphi(w_i) = E_{1,i+2} \). The image of \( \varphi \) has the property that the \( p^{\text{th}} \)-power of any element in the algebra is zero, since \( p \geq 3 \). Also, we note that the algebra has an action of the diagonal torus of \( \mathfrak{sl}_{n+2} \) of dimension \( n + 1 \).

Let \( \mathfrak{v} \) be the subalgebra with basis consisting of all of the elements \( v_i, w_i \) for \( i = 1, \ldots, n \). This subalgebra is commutative. We consider the spectral sequence \( E^r_{2,s} = H^r(u(\mathfrak{n}/\mathfrak{v}), H^s(u(\mathfrak{v}), k)) = H^r(u(\mathfrak{v}), k) \). The cohomology group \( H^1(u(\mathfrak{v}), k) = H^1(\mathfrak{v}, k) \) has dimension \( 2n \) and is spanned by elements \( \gamma_i \) (of weight \( \alpha_2 + \cdots + \alpha_{i+1} \)) and \( \eta_i \) (of weight \( \alpha_1 + \cdots + \alpha_{i+1} \), for \( i = 1, \ldots, n \). The action of the element \( u \in \mathfrak{n}/\mathfrak{v} \) on \( H^1(u(\mathfrak{v}), k) \) is given by \( u \cdot \eta_i = \gamma_i \) and \( u \cdot \gamma_i = 0 \). Thus, \( H^1(u(\mathfrak{v}), k) \) is a direct sum of \( n \) uniserial \( u(\mathfrak{n}/\mathfrak{v}) \)-modules of dimension 2.

With this information, we can prove the following.

**Proposition 7.1.** Let \( n = p - 1 \). Then the cohomology ring \( H^*(u(\mathfrak{n}), k) \) is not Cohen-Macaulay.

**Proof.** Let \( M \) denote the uniserial \( u(\mathfrak{n}/\mathfrak{v}) \)-module of dimension 2. As noted \( E^0_{2,1} = H^1(u(\mathfrak{v}), k) \) is a direct sum of \( p - 1 \) copies of \( M \). Hence, \( E^0_{2,p-1} \) contains the \( p - 1 \) exterior power of \( H^1(u(\mathfrak{v}), k) \) which includes the \( p - 1 \) tensor power of \( M \). The \( p - 1 \) tensor power of \( M \) has a projective \( u(\mathfrak{n}/\mathfrak{v}) \)-module, the uniserial module of dimension \( p \) generated by \( \eta_1 \wedge \cdots \wedge \eta_{p-1} \), and having socle spanned by \( \gamma_1 \wedge \cdots \wedge \gamma_{p-1} \). Consequently, \( E^0_{2,p-1} \) has an element \( y \) of weight \( \alpha_2 + \cdots + \alpha_p \). This element must survive to the \( E_{\infty} \) page of the spectral sequence, because there is no element of the same weight in \( E^r_{p-1-r,r+2} \) for any value of \( r > 0 \). On the other hand, because \( \gamma_1 \wedge \cdots \wedge \gamma_{p-1} \) is in the socle of a projective \( u(\mathfrak{n}/\mathfrak{v}) \)-module, we must have that \( X.y = 0 \) for \( X \) the generator of the symmetric algebra in \( E_{2,0} \). Again, by a weight argument we see that the product \( X.y \) cannot upgrade to an element that is not zero. Consequently, this relation must exist in \( H^*(u(\mathfrak{n}), k) \). This implies that \( X \) is not a regular element and that the depth of \( H^*(u(\mathfrak{n}), k) \) is less than the Krull dimension. \[ \square \]

**8. Nilpotent Lie algebras of dimension \( \leq 5 \)**

In this section we will use deGraaf’s classification of indecomposable nilpotent Lie algebras over fields of characteristic not equal to 2. Again our interest is in whether there is an isomorphism

\[
H^*(u(\mathfrak{n}), k) \cong S^*(\mathfrak{n}^*)^{(1)} \otimes \Lambda^*(\mathfrak{n}^*) \cong S^*(\mathfrak{n}^*)^{(1)} \otimes H^*(\mathfrak{n}, k)
\]

as rings or as modules over the symmetric algebra. If the algebra is commutative, then the isomorphism holds as rings. On the other hand if the \( p \)-power operation on the Lie algebra fails to vanish \( (x^{[p]} \neq 0 \) for some \( x \) in \( \mathfrak{n} \)\), then the above isomorphism can not hold.
Any finite dimensional nilpotent Lie algebras may be considered to be restricted. That is, there is some $p$-power operation that can be imposed on the algebra that makes it a restricted Lie algebra. However, it not always possible to impose a trivial $p$-power operation. We have listed below the indecomposable non-abelian nilpotent Lie algebras of dimension less than or equal to 5 along with the restrictions on the prime $p$ that are necessary to impose a trivial $p$-power operation. In this case, our standing assumption that $N_1(n) \equiv n$ holds. The notation for the Lie algebras is taken from DeGraaf’s list [deG].

Dimension 3: $L_{3,2}$ ($p \geq 3$)

Dimension 4: $L_{4,3}$ ($p \geq 5$)

The Lie algebras $L_{3,2}$ (reps. $L_{4,3}$) arise naturally as the unipotent radicals of the Borel subalgebras of simple Lie algebras of type $A_2$ (resp. $B_2$). We have $L_{3,2} = \langle x_{-\alpha}, x_{-\beta}, x_{-\alpha-\beta} \rangle$ and $L_{4,3} = \langle x_{-\alpha}, x_{-\beta}, x_{-\alpha-\beta}, x_{-2\alpha-\beta} \rangle$. The restricted cohomology rings of these algebras are given in Propositions 4.2 and 5.2.

Dimension 5: $L_{5,4}$ ($p \geq 3$), $L_{5,5}$ ($p \geq 5$), $L_{5,6}$ ($p \geq 5$), $L_{5,7}$ ($p \geq 5$), $L_{5,8}$ ($p \geq 3$), $L_{5,9}$ ($p \geq 5$).

We now use deGraaf’s description of the five dimensional nilpotent Lie algebra (cf. [deG, Section 4]) and describe natural gradings on these Lie algebras. The natural gradings are induced by toral actions given by outer automorphisms. When we compute the cohomology of the algebras, differentials in the spectral sequences respect the actions of these tori.

$L_{5,4}$: This nilpotent Lie algebra arises as a subalgebra of the nilpotent radical of a simple Lie algebra of type $A_3$. Let $\alpha_1, \alpha_2, \alpha_3$ denote the simple roots. The $L_{5,4}$ consists of the span of the root vectors $\{x_{-\alpha_1}, x_{-\alpha_2}, x_{-\alpha_1-\alpha_2}, x_{-\alpha_2-\alpha_3}, x_{-\alpha_1-\alpha_2-\alpha_3} \}$.

$L_{5,5}$: This Lie algebra has a double grading with basis $\langle x_{-\alpha}, x_{-\beta}, x_{-\alpha-\beta}, x_{-2\alpha-\beta}, x_{-2\alpha} \rangle$.

$L_{5,6}$: Let $W(1) = \langle e_i : i \in \mathbb{Z} \rangle$ be the Wittt algebra defined over $\mathbb{Z}$ with Lie bracket $[e_i, e_j] = (i + j)e_{i+j}$. One can consider the subalgebra $\mathfrak{a} = \langle e_i : i < 0 \rangle$ and factor this out by the ideal $\mathfrak{z} = \langle e_i : i \leq -6 \rangle$. The Lie algebra $L_{5,6}$ is the Lie algebra $\mathfrak{a}/\mathfrak{z}$ tensored by $k$. This has a natural $\mathbb{Z}$ grading, thus an action of a one-dimensional torus on $L_{5,6}$. This Lie algebra can also be viewed as a non-graded central extension of the unipotent radical of type $B_2$.

$L_{5,7}$: The Lie algebra $L_{5,7}$ is a graded central extension of $L_{4,3}$. The Lie algebra can be graded by a two-dimensional torus and has basis given by $\langle x_{-\alpha}, x_{-\beta}, x_{\alpha-\beta}, x_{-2\alpha-\beta}, x_{-3\alpha-\beta} \rangle$.

$L_{5,8}$: Let $\mathfrak{a}$ be the nilpotent radical for the Borel subalgebra of a Lie algebra of type $A_3$, and $\mathfrak{z}$ be the center of this Lie algebra. The Lie algebra $L_{5,8}$ can be realized as $\mathfrak{a}/\mathfrak{z}$ and has basis (with an action of a three dimensional torus) given by $\langle x_{-\alpha_1}, x_{-\alpha_2}, x_{-\alpha_3}, x_{-\alpha_1-\alpha_2}, x_{-\alpha_2-\alpha_3} \rangle$.

$L_{5,9}$: One can realize this Lie algebra as another graded central extension of $L_{4,3}$. This Lie algebra has a double grading with basis $\langle x_{-\alpha}, x_{-\beta}, x_{-\alpha-\beta}, x_{-2\alpha-\beta}, x_{-\alpha-2\beta} \rangle$. 




The ordinary Lie algebra cohomology can be computed recursively using central extensions and the LHS spectral sequence. For example, if \( \mathfrak{a} \) is a nilpotent Lie algebra and \( \mathfrak{z} \) is a one-dimensional central subalgebra then the LHS spectral sequence:

\[
E^{i,j}_2 = H^i(\mathfrak{a}/\mathfrak{z}, k) \otimes H^j(\mathfrak{z}, k) \Rightarrow H^{i+j}(\mathfrak{a}, k)
\]

will converge after the second page (i.e., \( E_3 \cong E_\infty \)). We have

\[
H^2(\mathfrak{a}, k) \cong H^2(\mathfrak{a}/\mathfrak{z}, k)/\langle \text{Im} \, \delta_2 \rangle \oplus \langle \text{Ker} \, \hat{\delta}_2 \rangle.
\]

where \( \delta_2 : E^{0,1}_2 \to E^{2,0}_2 \) and \( \hat{\delta}_2 : E^{1,1}_2 \to E^{3,0}_2 \). With appropriate choices of central subalgebras one can guarantee that the differentials respect the gradings above. This allows us to compute the differentials \( \delta_2 \) and \( \hat{\delta}_2 \) inductively. The weight spaces for the ordinary Lie algebra cohomology for these Lie algebras are multiplicity free (i.e., one-dimensional) and given in the following tables.

To aid in the computation we use some facts about the ordinary Lie algebra cohomology. For example, that if \( \mathfrak{g} \) has dimension \( d \), then the ordinary Lie algebra cohomology vanishes in degrees greater than \( d \). In addition there is a Poincaré duality that is also respected by the action of the tori. So for example, if \( d \) is the dimension of the algebra \( \mathfrak{n} \) and if the element in \( H^d(\mathfrak{n}, k) \) has weight \( \gamma \), then the weights of the cohomology element in degrees \( d - 1 \) will be \( \gamma - \zeta_1, \gamma - \zeta_2, \ldots \) where \( \zeta_1, \zeta_2, \ldots \) are the weights of the cohomology elements in degree 1.

| Degree | Weights      | Degree | Weights      |
|--------|--------------|--------|--------------|
| 0      | 0            | 0      | 0            |
| 1      | \( \alpha, \beta \) | 1      | \( \alpha, \beta \) |
| 2      | \( \alpha + 2\beta, 2\alpha + \beta \) | 2      | \( \alpha + 2\beta, 3\alpha + \beta \) |
| 3      | \( 2\alpha + 2\beta \) | 3      | \( 3\alpha + 3\beta, 4\alpha + 2\beta \) |
| 4      |              | 4      | \( 4\alpha + 3\beta \) |

| Degree | Weights                      |
|--------|------------------------------|
| 0      |                             |
| 1      | \( \alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3 \) |
| 2      | \( \alpha_1 + \alpha_3, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3 \) |
| 3      | \( 2\alpha_1 + 3\alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 3\alpha_3, 2\alpha_1 + 2\alpha_2 + 2\alpha_3, 3\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + 2\alpha_3 \) |
| 4      | \( 2\alpha_1 + 3\alpha_2 + 3\alpha_3, 3\alpha_1 + 3\alpha_2 + 2\alpha_3, 2\alpha_1 + 2\alpha_2 + 3\alpha_3, 3\alpha_1 + 2\alpha_2 + 2\alpha_3 \) |
| 5      | \( 3\alpha_1 + 3\alpha_2 + 3\alpha_3 \) |
With this information about the ordinary Lie algebra cohomology $H^*(n,k)$ we can deduce structural properties about the restricted Lie algebra cohomology.

**Theorem 8.1.** Let $n$ be a five-dimensional nilpotent Lie algebra. Then

(a) $H^*(u(n),k) \cong S^*(n^*)^{(1)} \otimes H^*(n,k)$ as $S^*(n^*)^{(1)}$ module for $L_{5,4}$ ($p \geq 3$), $L_{5,5}$ ($p \geq 5$), $L_{5,6}$ ($p \geq 7$), $L_{5,7}$ ($p \geq 9$), $L_{5,8}$ ($p \geq 5$), and $L_{5,9}$ ($p \geq 5$). In these cases the cohomology ring $H^*(u(n),k)$ is Cohen-Macaulay.

(b) In addition, the above isomorphism holds as rings for the algebras $L_{5,4}$ ($p \geq 5$), $L_{5,5}$ ($p \geq 7$), $L_{5,6}$ ($p \geq 13$), $L_{5,7}$ ($p \geq 11$), $L_{5,8}$ ($p \geq 5$), and $L_{5,9}$ ($p \geq 5$).

**Proof.** (a) There exist a one-dimensional central ideal $\mathfrak{j}$ such that $n/\mathfrak{j}$ is isomorphic to (i) the nilpotent radical for a simple Lie algebra of Type $B_2$ for $L_{5,5}$, $L_{5,6}$, $L_{5,7}$ and $L_{5,9}$, (ii) an abelian Lie algebra for $L_{5,4}$ and (iii) the nilpotent radical of type $A_2 \times A_1$ for $L_{5,8}$. Now assume that $p \geq 7$ in case (i), $p \geq 3$ in case (ii), and $p \geq 5$ in case (iii). Then $H^*(u(n/\mathfrak{j}),k) \cong S^*((n/\mathfrak{j})^*)^{(1)} \otimes H^*(n/\mathfrak{j},k)$ as rings. Hence, by Theorem 3.3, $H^*(u(n),k) \cong S^*(n^*)^{(1)} \otimes H^*(n,k)$ as $S^*(n^*)^{(1)}$ module.
The remaining cases when $L_{5,8}$ ($p = 3$), $L_{5,5}$ ($p = 5$) and $L_{5,9}$ ($p = 5$) can be verified directly by showing that the spectral sequence (3.0.1) collapses.

(b) We construct a subalgebra $B$ in $H^*(u(n), k)$ which is isomorphic to $H^*(n, k)$. This is accomplished using the gradings to show that the following conditions cannot simultaneously hold. First,
\[ \gamma_1 + \gamma_2 = \gamma_3 + p\sigma \]
where $\gamma_j$ is a weight of $H^a_j(n, k)$ for $j = 1, 2, 3$ and $\sigma \neq 0$ is a weight of $S^*(n^*)^{(1)}$. Second,
\[ a_1 + a_2 = a_3 + \deg(\sigma) \]
were $\deg(\sigma)$ is the cohomological degree of the element that $\sigma$ represents.

In all the cases listed in the statement this was verified, and proves there do not exist two elements of weights $\gamma_1$ and $\gamma_2$ whose product is the product of an element of the symmetric algebra with an element of weight $\gamma_3$. Furthermore, this shows that a basis of weight vectors in $H^*(n, k)$ form a subalgebra of $H^*(u(n), k)$.□

**Theorem 8.2.** Let $n$ be a five-dimensional nilpotent Lie algebra. Then $H^*(u(n), k)$ is not Cohen-Macaulay in the cases that $n$ is $L_{5,7}$ for $p = 5$ and $L_{5,8}$ for $p = 3$. In these cases, the depth of the cohomology ring is one less than the dimension.

**Proof.** The results are proved in Propositions 4.3 and 5.3.□

We remark that the work in [BC1, BC2] can be adapted for restricted Lie algebra cohomology to show that in the case when the cohomology ring has depth at least one less than the Krull dimension, the cohomology ring is Cohen-Macaulay if and only if it satisfies the functional equation in Theorem 3.2. We can conclude that the cohomology rings for $L_{5,7}$ ($p = 5$) and $L_{5,8}$ ($p = 3$) do not satisfy the functional equation.

9. **Type $A_3$, $p > h$**

From the Examples 4.2, 5.2, 6.1 one might get the impression that if $n$ is a unipotent radical of a Borel subalgebra of a simple Lie algebra and $h < p < 2(h - 1)$, then the restricted cohomology is not isomorphic as rings to the tensor of the symmetric algebra and the ordinary Lie algebra cohomology. However, this is not true. In this section we analyze the smallest example of this sort where the ring isomorphism exists. In this case, the root system $\Phi$ is of type $A_3$ and $n$ is the six dimensional Lie algebra of strictly upper triangular $4 \times 4$ matrices over a field of characteristic 5. In the DeGraaf notation this is the Lie algebra $L_{6,19}(\epsilon)$.

Since $p > h$, the ordinary Lie algebra cohomology is given by Kostant’s theorem. A proof in characteristic $p$ with $p > h$ is found in [UGA] Theorem 4.1.1. As a module for $T$ we have
\[ H^a(n, k) \cong \bigoplus_{w \in W; l(w) = n} -w \cdot 0 \]  
(9.0.1)
As before we are using the convention that $n$ consists of negative root vectors. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ denote the simple root vectors. Then (9.0.1) can be used to produce the following table which describes the weights in the cohomology groups $H^a(n, k)$. 
Theorem 9.1. Let $n$ be the unipotent radical corresponding to the simple group with root system $A_3$ (i.e., $4 \times 4$ upper triangular matrices) with $p > h$. Then

$$H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k)$$

as rings.

Proof. For $p > 2(h - 1)$ we can apply [DNN, Theorem 3.1.1]. This leaves us with the case when $p = 5$. In order to invoke Theorem 3.7 we need to analyze the equation

$$-w_1 \cdot 0 - w_2 \cdot 0 = p\sigma - w_3 \cdot 0 \quad (9.1.1)$$

where $-w_j \cdot 0$ is a weight of $H^j(n, k)$ for $j = 1, 2, 3$ and $\sigma$ is a weight of $S^*(n^*)^{(1)}$. Furthermore, by consideration of cohomological degrees we must have that

$$l(w_1) + l(w_2) - l(w_3) = 2\deg(\sigma) \quad (9.1.2)$$

where $\deg(\sigma)$ is the cohomological degree corresponding to the element of weight $\sigma$.

The first equation (9.1.1) does have solutions. For example,

$$(2\alpha_1 + 4\alpha_2 + 3\alpha_3) + (3\alpha_1 + 3\alpha_2 + 3\alpha_3) = (2\alpha_2 + \alpha_3) + 5(\alpha_1 + \alpha_2 + \alpha_3).$$

Here $l(w_1) = l(w_2) = 5$ and $l(w_3) = 2$. However, $\deg(\sigma) \leq 6$. Thus, the second equation (9.1.2) cannot hold. A careful, case by case analysis rules out the possibility that both equations could simultaneously be satisfied. □

We end the paper with the following intriguing question.

Question 9.2. Suppose that $n$ is the nilpotent radical of a Borel subalgebra of a Lie algebra arising from a reductive algebraic group. In the case that the root system $\Phi$ is simply laced and that $p > h$, is there always a ring isomorphism

$$H^*(u(n), k) \cong S^*(n^*)^{(1)} \otimes H^*(n, k)?$$

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