Quantum key distribution based on the single photon quantum eraser

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Quantum information and quantum foundations are becoming popular topics for advanced undergraduate courses. Many of the fundamental concepts and applications in these two fields, such as delayed choice experiments and quantum encryption, are comprehensible to undergraduates with basic knowledge of quantum mechanics. In this paper, we show that the single photon quantum eraser, usually used to study the duality between wave and particle properties, can also serve as a generic platform for quantum key distribution. We present a new algorithm to securely share random keys using the quantum eraser and propose its implementation using quantum circuits. The efficiency of the proposed algorithm is equivalent to the BB84 algorithm. The proposed scheme can serve as a useful pedagogical example in a typical quantum foundations course. We also show how to use the same algorithm to share three different keys among three parties in a pairwise manner.

I. INTRODUCTION

The fields of quantum foundations and quantum information are very good examples of themes that are being actively investigated at the research level and yet are accessible to undergraduate students who completed a first course on quantum mechanics $[1–3]$. Indeed, many undergraduate curricula include courses and seminars on topics such as quantum communication, quantum computing and algorithms, entanglement, quantum control, ... etc. The two fields are also strongly interrelated and advances in either one leads to breaking new grounds in the other one $[4]$. The purpose of this paper is to provide pedagogical example of a quantum computing application motivated by foundational aspects of quantum mechanics that illustrates the connections between the two subjects.

The wave-particle duality of quantum systems has been one of the cornerstone ideas in quantum mechanics since its inception in the 1920s. In short, this principle states that quantum system cannot exhibit full wave and particle properties in the same time. For example, in the double slit experiment, one cannot pinpoint the slit through which the quantum particle has passed (the path) and at the same time maintain the interference fringes on the screen. Most interestingly, delaying the choice of whether or not to measure the path until after the particle has passed through the double slits (delayed choice experiments $[5]$) or reversing the choice, i.e., erasing the which-path information (WPI) $[6]$, have been implemented in various experiments over the last two decades $[7–11]$.

On the other hand, quantum key distribution (QKD) has been one of the earliest applications of quantum information $[12,13]$. The most famous QKD protocol is the Bennett-Brassard 1984 (BB84) $[14]$. In the BB84 protocol, the sender encodes a classical bit by preparing a photon randomly in either one of two basis states. The state of the photon can be one of two orthogonal polarization states representing 1 and 0. The receiver, on the other hand, chooses randomly to measure the state of the photon in any of the two basis. The sender and receiver (usually called Alice and Bob) will communicate in public their basis choices and register the corresponding bits only when their choices agree. In doing so, they generate a shared random key that they can use later for encrypting a message sent on the public channel. By comparing a subset of the generated data, they can detect any attempt of eavesdropping on the communication since any intervention from a third party in the middle will eventually lead to some errors in the generated data. The security of this protocol has been analyzed by many authors (see e.g., $[15–17]$).

Recently, new protocols based on notions in the foundations of quantum mechanics such as counterfactual and interaction-free measurement have been proposed as platforms for quantum communications and quantum computing $[18–21]$. In this paper, we propose a new encryption scheme similar to the BB84 protocol and based on the single photon quantum eraser platform. We illustrate how to implement the proposed scheme using available quantum circuit components. Finally, we show how to generalize our routine to a 3-party network where Alice shares different keys with each of Bob and Careem, while Careem and Bob can generate a deterministic secure key unknown to Alice.

II. THE DIFFERENT FLAVORS OF THE SINGLE PHOTON QUANTUM ERASER

First, we give an overview of the different flavors of the single photon quantum eraser using the Mach-Zender interferometer (MZI). In MZI, a photon is sent through a beam splitter (BS), where its wavefunction is split into two parts traveling along different paths till they recombine again at another beam splitter before they are detected by either detector $D_1$ or $D_2$ (see Fig. 1). Inside the interferometer, the state of the photon is expressed as
\[ \psi = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle), \]

where \(|1\rangle\) and \(|2\rangle\) represent the states of the photon in the two paths. If we impose a phase difference \(\phi\) between the two paths, the probability of detecting the photon in \(D_1\) is \(P_1(\phi) = \cos^2(\phi/2)\) while the probability to detect the photon in \(D_2\) is \(P_2(\phi) = \sin^2(\phi/2)\).

In order to know through which path the photon has traveled, the two paths should be given two different labels through a new degree of freedom of the photon. This labeling process would make the two paths distinguishable and the interference pattern would collapse making the probability to detect the photon at any of the two detectors equals 0.5. The extra degree of freedom can be, for example, the polarization of the photon (i.e., vertical or horizontal polarization; see Fig. 1-b), or the time of arrival of the photon (i.e., by introducing a time delay \(T\) in one path; see Fig. 1-c) as in time-bin qubits [22]. The former case can be implemented using a source of unpolarized photons, and absorbing polarizers in each path, or a source of linearly polarized photons with a half-wavelength plate in one path only that rotates the plane of polarization by 90 degrees. In the later case, the time delay should be adjusted in a way that does not alter the phase difference between the two paths.  

The quantum eraser setup aims at erasing the which-path information after the photon has passed through \(BS_2\), thus restoring the sinusoidal interference fringes at \(D_1\) and \(D_2\). The ability to erase WPI and restore the interference pattern after the photon has left the interferometer has invited an interesting discussion in the foundations of quantum mechanics that still stimulates new models and experiments [23, 24]. When exactly does the photon make the choice to follow a single path (i.e., behave as a particle) or traverse the two paths simultaneously in superposition (i.e., behave as a wave)? It seems upfront that the delayed choice experiment hints at a retrocausal behavior incompatible with orthodox quantum mechanics. Nevertheless, orthodox quantum mechanics faces no dilemma here since it is concerned only with observable effects and the question posed earlier has no observable consequences. The question may even have a natural answer in alternative interpretations such as the de-Broglie Bohm pilot wave interpretation [25, 26] that employs particles following deterministic trajectories in configuration space or the two-state vector formalism of quantum mechanics [27] that uses forward and backward propagating waves. We shall not be concerned any further with this issue in the rest of this paper.

Assuming that the two paths are distinguished by \(H\) and \(V\) polarization and the two output channels of \(BS_2\) are denoted by \(A\) and \(B\), the state of the photon at the output of \(BS_2\) is the entangled state \(\frac{1}{\sqrt{2}} |A\rangle \otimes \{ |H\rangle + e^{i\phi} |V\rangle \} + \frac{1}{\sqrt{2}} |B\rangle \otimes \{ |H\rangle - e^{i\phi} |V\rangle \}\). The ideal quantum eraser would utilize a unitary circuit that takes the outputs of the MZI, erases the WPI by disentangling the two degrees of freedom and recovers the interference pattern as shown in Fig. 2(a). It is very likely that this ideal scheme cannot be achieved in a reversible and deterministic manner by a single unitary circuit for an arbitrary \(\phi\). We will show in the next section an example for a unitary eraser that works for a single value of \(\phi\). Nevertheless, we list below three different nonideal schemes, depicted in Figs. 2-b, 2-c and 2-d, that erase WPI in a probabilistic manner. The idea is to project the photon onto a state that conceals the path information or selectively measure it at times when the path information is hidden.

1. A unitary quantum circuit is connected with the MZI that erases the WPI at certain times only in a probabilistic manner. When the WPI is encoded through the photon time of arrival by including a delay element \(T\) in one path, the photons will arrive at \(BS_2\) at \(t = 0\) or \(t = T\) depending on the path taken by the photon. Adding another MZI with the same delay element, see Fig. 2-b, will let the photons be detected at \(t = 0\), \(t = T\) and \(t = 2T\).
(a) The ideal quantum eraser: A unitary quantum circuit takes the outputs of the second beamsplitter in a Mach-Zender Interferometer that encodes the which-path information and erases it with a unitary transformation, thus restoring the interference pattern at D₁ and D₂ in a deterministic manner. We are not sure whether this ideal scheme is feasible. (b,c,d) Different realizations of the nonideal quantum eraser. (b) Which-path information is encoded by the photon time of arrival. Photons arriving at the detectors at $t = T$ are in superposition of the two paths with phase difference $\phi$ and thus exhibit interference fringes. (c) Which-path information is encoded by inserting horizontal and vertical polarizers in the two paths. Beam-splitting 45° polarizers are inserted at the outputs of BS₂. Photons passing or reflected from these polarizers will be in superposition of the two paths and will be detected by D₁ (D₂) or D₃ (D₄) and exhibit interference fringes. (d) Same as in (c), but using absorbing 45° polarizers after BS₂ that will allow the photons from either paths to pass through with probability 50%, thus eliminating WPI and restoring the interference pattern at 50% of the time.

(assuming the delay associated with the other path without delay to be negligible). The photons detected at $t = T$ will be in a coherent superposition of the two paths with phase difference $\phi$ and thus exhibit interference. Here, we conditionally select the cases where the photon emerges in a state that conceals the path information.

2. Each of the outputs of BS₂ is fed into a circuit, whose two outputs exhibit the interference pattern in a complementary way, i.e., the probability of detection at the output of the circuit are $P₁(\phi)$ and $0.5 - P₁(\phi)$. As an example, when WPI is encoded through the polarization of the photon by adding orthogonal horizontal and vertical polarizers to the two paths, we can restore the interference by adding 45° Beam-splitting polarizers to the outputs of BS₂ (see Fig. 2-c). The photons emerging from each 45° polarizer have either 45° or 135° polarization and exhibit interference fringes.

3. Each of the outputs of BS₂ is fed into a dissipative circuit which absorbs the incoming photon or reemits it in a random fashion. The emerging photons will have a sinusoidal dependence on $\phi$ akin to the interference fringes without WPI. For example, considering again the MZI in case 2, we encode WPI through the polarization of the photon and add an absorbing 45° polarizer to each of the outputs of BS₂ (see Fig. 2-d). These polarizers let the photon pass through only if it is polarized at 45°. Those photons will be in superposition of the two paths and will exhibit interference fringes but with lower visibility [28]. In both 2 and 3, the role of the 45° polarizer is to scramble the WPI encoded in the photon polarization.

We note that in case 1, the second interferometer has mixed the two outputs of BS₂ to conditionally

create a new interference that occurs when the time delay no longer distinguishes the two paths. On the other hand, cases 2 and 3 can be viewed as doing projective measurements in a basis that does not distinguish the two paths.

III. A QKD ALGORITHM BASED ON THE QUANTUM ERASER

We are now ready to present a scheme for securely encrypting information using the quantum eraser. Alice wants to generate a secret key consisting of bits 1 and 0 shared with Bob in order to use it for encrypting messages. The key should be communicated such that any eavesdropping attempt by Evan, an evil agent trying to intercept the message, can be detected by Alice and Bob.

Alice uses an MZI to encode the bits by the state of the photon. She has two degrees of freedom, the phase difference between the two paths and whether to include WPI or not. Let $\phi = \pi$ corresponds to bit 1 and $\phi = 0$ corresponds to bit 0. The two outputs of BS$_2$ are sent to Bob. Bob on his part has the freedom to insert a quantum eraser or not. In order to share a random secret key, Alice sends a stream of photons to Bob, with randomly setting $\phi = \pi$ or $\phi = 0$ and randomly choosing to insert WPI or not. Bob, on the other side, randomly chooses to insert the eraser setup or not and records through which detector the photon was detected. At the end of the transmission, Alice and Bob announce over the public channel their choices of WPI and the eraser setup for each transmission event. Bob also announces whether his eraser setup successfully erased the WPI, i.e., the photon was detected at the right time in Fig. 2(b) or was actually detected in either D$_1$ or D$_2$ in Fig. 2(d).

A bit will be securely shared between Alice and Bob when Alice inserts WPI and Bob utilizes the quantum eraser and successfully registers the photon or when Alice does not insert WPI and Bob does not insert the quantum eraser. In these two cases, Bob knows with certainty the value of $\phi$ used by Alice through the measurement outcomes of his detectors. The bits corresponding to these cases are kept by Alice and Bob as parts of the key. According to the no-cloning theorem [29], Evan can not measure the phase relationship between the two channels and the state of WPI (i.e., whether it was inserted or not and what it was) simultaneously without altering the state of the photon in the channels, thus causing discrepancy between the values of the bits registered by Alice and Bob. Therefore, Alice and Bob can detect the existence of an eavesdropper by comparing over the public channel a subset of their data that will be discarded later.

It is evident that this algorithm very much resembles the BB84 algorithm. Therefore, the security of our scheme can be arbitrarily improved using privacy amplification methods [30, 31]. On the other hand, it suffers from the main drawback of the BB84 [17], namely that it requires very low-noise detectors and single photon sources since inefficient single-photon generation will send multiple photons in the same quantum state. Recently, a new protocol called “quantum erasure cryptography” [32], was proposed that involves a single MZI loop divided among the sender and the receiver. We do not think that this scheme should be called quantum eraser cryptography as the author maintains since it does not really involve erasing the WPI after the photon has passed the second beam splitter, but rather reversing the choice of the polarization rotators at the sender’s side.

IV. QUANTUM CIRCUIT IMPLEMENTATION

In this section, we introduce a quantum circuit implementation of the proposed scheme using basic quantum logic gates. Alice sends two qubits, Q and F, to Bob as shown in Fig. 3 that are initialized to $|0\rangle$. Q represents the photon in the quantum eraser while the flag qubit F represents the which-path information. Note that in the MZI of section 2, WPI is encoded through additional degrees of freedom of the same photon, while here we encode WPI through another qubit. The two Hadamard (H) gates in Fig. 3 represent the two beam splitters while the Z gate represents the phase element $\phi = \pi$. When Alice chooses to insert WPI, she inserts a controlled-NOT (CNOT) gate between the two Hadamard gates that entangles the two qubits. On the other side, Bob performs two measurements in the Z-basis on the two qubits received from Alice. If he chooses to erase WPI, he can insert an H gate in F before measurement as in Fig. 3-a, thus effectively measuring in the X-basis, i.e., the information about the WPI encoded in the Z-basis is scrambled. Alternatively, as in Fig. 3-b, he can insert a CNOT gate to F followed by a Hadamard gate to Q before doing the measurements in the Z-basis.

Let us analyze the operation of these two circuits. The simplest scenario occurs when Alice chooses not to insert WPI and Bob chooses not to insert the eraser. In this case, Bob can detect directly whether Z-gate was inserted at Alice’s side or not by measuring Q alone since $H^2|0\rangle = |0\rangle$ and $HZH|0\rangle = |1\rangle$. The F qubit is redundant in this case and it can be set to a random value. The other scenario corresponding to the valid generation of a secretly shared bit occurs when Alice chooses to insert WPI and Bob chooses to insert the quantum eraser. Starting with $|QF\rangle = |00\rangle$, it is straightforward to see that with the Z-gate inserted, the the state of the two qubits at Bob’s side directly before the measurement in Fig. 3-a is $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and without the Z-gate, the state is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, while in Fig. 3-b is $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ with the Z-gate inserted and $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ without the Z-gate. Bob can then detect whether the Z-gate is inserted or not while using any of the two circuits by detecting whether the measurements of Q and F yield similar or different results. If Z-gate is inserted, bit 1 is recorded.
FIG. 3. A quantum circuit implementation for the QKD scheme based on the quantum eraser. Alice sends two qubits $Q$ and $F$ to Bob and chooses randomly to insert the CNOT and $Z$ gates or not. Bob measures the state of the two qubits after choosing randomly to insert the eraser circuit or not. (a,b) depicts two different implementation for the eraser circuit.

FIG. 4. A similar circuit to Fig. 3-b with a twist to enable three-party secure communication. This circuit enables one to generate 3 secret keys shared between (A,B), (A,C) and (B,C) respectively. See text for the detailed algorithm.

and if $Z$-gate is not inserted, bit 0 is recorded. As in BB84, Alice and Bob will communicate their choices over the classical channel to determine the cases where their choices comply with each other.

Unlike the BB84, which can be implemented with a single qubit, we need here two qubits to implement this protocol, one of them is redundant half the time. The redundancy of the flag qubit can be reduced by several methods. For example, by using the flag bit to send another bit that Bob will record as it is during the No-WPI-No-Eraser mode, therefore sharing two bits instead of one bit during that mode. If Evan decides to intercept this bit, he will eventually introduce errors during the WPI-Eraser mode since he does not know a priori when each mode occurs. Another example is to send two qubits $Q_1$ and $Q_2$ served by the same $F$ bit, thus doubling the transmission rate. In this case, Alice and Bob use WPI-Eraser mode with one of the two bits (selected randomly by each of them every time) and the other mode with the other bit. As before, two secret bits are shared when the choices of Alice and Bob agree.

The circuits in Fig. 3 can serve as a good exercise to look for novel communication protocols. For example, we can devise a scheme based on the circuit in Fig. 3-b that can be used to generate three different keys shared among a group of three agents, A, B and C such that each key is shared among one pair and is not known to the third party (see Fig. 4). When A desires to share a secret key with B (C), B will remove the $Z$ gate in the dashed box and the protocol will be similar to the one in Fig. 3-b. C (B) announces in public the outcome of his measurement. B (C) can determine whether the two bits are the same or different, i.e., whether A chose to insert $Z$ or not since he knows the outcome of the measurement of his own bit, and the bit of C (B). This will be one bit shared secretly between A and B (C) and unknown to C (B). Note that, in this mode, the eraser setup is divided between B and C, each having the freedom to insert his part or not. A valid bit will be shared when the choices of WPI and the eraser setup are consistent as before.

On the other hand, when B and C want to share a secret key unknown to A, they instruct A to insert its $Z$ gate. Now, B has the freedom to insert a $Z$ gate or not. As we will see, this represents an interesting scheme of de-
termistic quantum communication that was introduced in [33]. Without the Z-gate inserted, the state of the two qubits before measurement is $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ and with the Z-gate, the state is $\frac{1}{\sqrt{2}}(-|01\rangle + |10\rangle)$. Therefore, if B announces his measurement outcome in public, C will know whether B chose to insert his Z gate or not. This choice represents a secretly shared bit between B and C unknown to A. One crucial difference between this mode of operation and the BB84 is that while in BB84 the generated key is random, here it is deterministic. Note that, in this mode, the eraser setup divided between B and C is always in place. An intervention attempt between B and C will obviously affect the entangled state of the two qubits and thus spoil some of the shared data that can be detected by some sort of redundancy check as before [33]. A rigorous security proof may be interesting to investigate for this scheme.

V. DISCUSSION AND CONCLUSION

Let us analyze the most famous attack strategy, namely the intercept and resend attack, and compute the quantum bit error rate (QBER) for the quantum circuits in Fig. 3. In this attack, Evan poses as Bob, intercepts the two qubits, makes measurements with or without an eraser setup, and sends a new version of the two qubits forward to Bob. Let us first consider the simple No-WPI-No-Eraser mode. In this case, the two qubits are not entangled, and the flag qubit has a random value. The smartest thing Evan can do is to measure Q and resends it to Bob, thus always obtaining a valid bit in this mode which occurs 50% of the time. Since Evan does not know beforehand when this mode occurs, Evan will inadvertently corrupt the two-qubit state as well during the WPI-Eraser mode, causing Bob to get the wrong bit. Therefore, unsurprisingly like the BB84 algorithm, Evan will corrupt on average 25% of the bits of the shared key, i.e., QBER=25%. Similarly, if Evan chooses to intercept Q and F with an eraser circuit, posing as Bob, and then generates an entangled state and sends it to Bob, posing as Alice, he will inadvertently corrupt 25% of all the shared bits.

While the algorithm presented in this paper may not be efficient from the practical point of view, it serves as a pedagogical approach to quantum key encryption that has the same spirit as the BB84 algorithm using the quantum eraser platform. It also shows how closely the fundamental notions of quantum mechanics and the emerging field of quantum algorithms are tied together. We have shown how to map the quantum eraser setup to a quantum circuit and how variants of this circuit can lead to new quantum communication protocols.

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