Encoding 2D Range Maximum Queries

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Abstract. We consider the two-dimensional range maximum query (2D-RMQ) problem: given an array $A$ of ordered values, to pre-process it so that we can find the position of the largest element in a (user-specified) range of rows and range of columns. We focus on determining the effective entropy of 2D-RMQ, i.e., how many bits are needed to encode $A$ so that 2D-RMQ queries can be answered without access to $A$. We give tight upper and lower bounds on the expected effective entropy for the case when $A$ contains independent identically-distributed random values, and new upper and lower bounds for arbitrary $A$, for the case when $A$ contains few rows. The latter results improve upon upper and lower bounds by Brodal et al. (ESA 2010). We also give some efficient data structures for 2D-RMQ whose space usage is close to the effective entropy.

1 Introduction

In this paper, we study the two-dimensional range maximum query problem (2D-RMQ). The input to this problem is a two dimensional $m \times n$ array $A$ of $N = m \cdot n$ elements from a totally ordered set. We assume w.l.o.g. that $m \leq n$ and that all the entries of $A$ are distinct (identical entries of $A$ are ordered lexicographically by their index). We consider queries of the following types. A 1-sided query consists of the positions in the array in the range $q = [1 \cdot \cdot \cdot m] \times [1 \cdot \cdot \cdot j]$, where $1 \leq j \leq n$. (For the case $m = 1$ these may also be referred to as prefix maximum queries.) For a 2-sided query the range is $q = [1 \cdot \cdot \cdot i] \times [1 \cdot \cdot \cdot j]$, where $1 \leq i \leq m$ and $1 \leq j \leq n$; for a 3-sided query, $q = [1 \cdot \cdot \cdot i] \times [j_1 \cdot \cdot \cdot j_2]$, where $1 \leq i \leq m$ and $1 \leq j_1 \leq j_2 \leq n$ and for a 4-sided query, the query range is $q = [i_1 \cdot \cdot \cdot i_2] \times [j_1 \cdot \cdot \cdot j_2]$, where $1 \leq i_1 \leq i_2 \leq m$ and $1 \leq j_1 \leq j_2 \leq n$. In each case, the response to a query is the position of the maximum element in the query range, i.e., $\text{RMQ}(A, q) = \arg\max_{(i,j) \in q} A[i,j]$. If the number of sides is not specified we assume the query is 4-sided.

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We focus on the space requirements for answering this query in the encoding model [4], where the aim is to pre-process $A$ and produce a representation of $A$ which allows 2D-RMQ queries to be answered without accessing $A$ any further. We now briefly motivate this particular question. Lossless data compression is concerned with the information content of data, and how effectively to compress/decompress the data so that it uses space close to its information content. However, there has been an explosion of interest in operating directly (without decompression) on compressed data via succinct or compressed data structures [8, 12]. In such situations, a fundamental issue that needs to be considered is the “information content of the data structure,” formalized as follows. Given a set of objects $S$, and a set of queries $Q$, consider the equivalence class $C$ on $S$ induced by $Q$, where two objects from $S$ are equivalent if they provide the same answer to all queries in $Q$. Whereas traditional succinct data structures are focussed on storing a given $x \in S$ using at most $\lceil \log_2 |S| \rceil$ bits — the entropy of $S$, we consider the problem of storing $x$ in $\lceil \log_2 |C| \rceil$ bits — the effective entropy of $S$ with respect to $Q^6$ — while still answering queries from $Q$ correctly.

Although this term is new, the question is not: a classical result, using Cartesian trees [21], shows that given an array $A$ with $n$ values from \{1, \ldots, n\}, only $2n - O(\log n)$ bits are required to answer 1D-RMQ without access to $A$, as opposed to the $\Theta(n \log n)$ bits needed to represent $A$ itself. The low effective entropy of 1D-RMQ is useful in many applications, e.g. it is used to simulate access to LCP information in compressed suffix arrays (see e.g. [18]). This has motivated much research into data structures whose space usage is close to the $2n - O(\log n)$ lower bound and which can answer RMQ queries quickly (see [9] and references therein). In addition to being a natural generalization of the 1D-RMQ, the 2D-RMQ query is also a standard kind of range reporting query.

**Previous Work.** The 2D-RMQ problem, as stated here, was proposed by Amir et al. [1]$^7$. Building on work by Atallah and Yuan [2], Brodal et al. [4] gave a hybrid data structure that combined a compressed “index” of $O(N)$ bits along with the original array $A$. Queries were answered using the index along with $O(1)$ accesses to $A$. They showed that this is an optimal point on the trade-off between the number of accesses and the memory used. In contrast, Brodal et al. refined Demaine et al.’s [6] earlier lower bound to show that the effective entropy of 2D RMQ is $\Omega(N \log m)$ bits, thus resolving in the negative Amir et al.’s open question regarding the existence of an $O(N)$-bit encoding for the 2D-RMQ problem. Brodal et al. also gave an $O(N \min\{m, \log n\})$ bit encoding of $A$. Recalling that $m$ is the smaller of the two dimensions, it is clear that Brodal et al.’s encoding is non-optimal unless $m = n^{O(1)}$.

**Our Results.** We primarily consider two cases of the above problem: (a) the random case, where the input $A$ comprises $N$ independent, uniform (real) ran-

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6 We will abbreviate this as “the effective entropy of $Q$.”

7 The variant where elements are associated with a sparse set of points in 2D, introduced by [10], is fundamentally different and is not discussed further here.
dom numbers from \([0, 1]\), and (b) the case of fixed \(m\), where \(A\) is worst-case, but \(m\) is taken to be a (fairly small) constant. Random inputs are of interest in practical situations and provide insights into the lower bounds of \([4, 6]\) — for instance, we show that the 2D-RMQ can be encoded in \(O(N)\) expected bits as opposed to \(\Omega(N \log m)\) bits for the worst case — that could inform the design of adaptive data structures which could use significantly less space for practical inputs. For the case of fixed \(m\), we determine the precise constants in the effective entropy for particular values of \(m\) — applying the techniques of Brodal et al. directly yields significantly non-optimal lower and upper bounds. These results use ideas that may be relevant to solving the asymptotic version of the problem. The majority of our effort is directed towards determining the effective entropy and providing concrete encodings that match the effective entropy, but we also in some cases provide data structures that support range maximum queries space- and time-efficiently on the RAM model with logarithmic word size.

**Effective Entropy on Random Inputs.** We first consider the 1D-RMQ problem for an array \(A[1 \cdots n]\) (i.e. \(m = 1\)) and show that, in contrast to the worst case lower bound of \(2n - O(\log n)\) bits, the expected effective entropy of RMQ is \(\leq cn\) bits for \(c = 1.9183\ldots < 1.92\). In the 2D case, \(A\) is an \(m \times n\) array with \(2 \leq m \leq n\). We show bounds on the expected effective entropy of RMQ as below:

|       | 1-sided | 2-sided | 3-sided | 4-sided |
|-------|---------|---------|---------|---------|
| bits  | \(\Theta((\log n)^2)\) | \(\Theta((\log n)^2 \log m)\) | \(\Theta(n(\log m)^2)\) | \(\Theta(nm)\) |

The 2D bounds are considerably lower than the known worst-case bounds of \(O(n \log m)\) for the 1-sided case, \(O(nm)\) for the 2-sided case, and known lower bounds of \(O(nm)\) and \(O(nm \log m)\) for the 3-sided and 4-sided cases respectively. The above results also hold in the weaker model where we assume all permutations of \(A\) are equally likely. We also give a data structure that supports (4-sided) RMQ queries in \(O(1)\) time using expected \(O(nm)\) bits of space.

**Effective Entropy for Small \(m\).** Our results for the 2D RMQ problem (4-sided queries) with worst-case inputs are as follows:

1. We give an encoding based on “merging” Cartesian trees\(^8\). While this encoding uses \(\Theta(nm^2)\) bits, the same as that of Brodal et al. [4], it has lower constant factors: e.g., it uses \(5n - O(\log n)\) bits when \(m = 2\) rather than \(7n - O(\log n)\) bits [4]. We also give a data structure for the case \(m = 2\) that uses \((5 + \epsilon)n\) bits and answers queries in \(O\left(\frac{\log(1/\epsilon)}{\epsilon}\right)\) time for any \(\epsilon > 0\).

2. We give a lower bound on the effective entropy based on “merging” Cartesian trees. This lower bound is not asymptotically superior to the lower bound of Brodal et al. [4], but for all fixed \(m < 2^{12}\) it gives a better lower bound than that of Brodal et al. For example, we show that for \(m = 2\), the effective entropy is \(5n - O(\log n)\) bits, exactly matching the upper bound, but the method of Brodal et al. yields only a lower bound of \(n/2\) bits.

\(^8\) This encoding has also been discovered by Brodal (personal communication).
3. For the case $m = 3$, we give an encoding that requires $(6+\log 5)n - O(\log n) \approx 8.32n$ bits\(^9\). Brodal et al.’s approach requires $(12+\log 5)n - O(\log n) \approx 14.32n$ bits and the method in (1) above would require about $9n$ bits. Our lower bound from (2) is $8n - O(\log n)$ for this case.

The paper is organized as follows: in Section 2 we give bounds on the expected entropy for random inputs, in Section 3 we consider the case of small $m$ and Section 4 gives the new data structures.

## 2 Random Input

In this section we consider the case of “random” inputs, where the array $A$ is populated with $N = m \cdot n$ independent uniform random real numbers in the range $[0, 1)$. We first consider the 1D-RMQ problem (Theorem 1), then the 2D cases, beginning with 1-sided queries (Theorem 2), 2-sided (Theorem 3) and finally 4- and 3-sided queries (Theorem 4).

**Theorem 1.** The expected effective entropy of 2-sided queries on a 1D array of size $n$ is at most $cn$ bits for $c = 1.9183 \ldots < 1.92$.

**Proof.** We study the distribution of different kinds of nodes in the Cartesian tree [21] of a random array. Given an array $A$ containing $n$ distinct numbers, its Cartesian tree is a binary tree in which each node corresponds to a unique position in the array, and is defined recursively as follows: the root of the tree corresponds to position $i$, where $A[i]$ is the maximum element in $A$, and the left and right subtrees of the root are the Cartesian trees for the sub-arrays $A[1 \ldots i-1]$ and $A[i+1 \ldots n]$ respectively (the Cartesian tree of a null array is the empty binary tree). A Cartesian tree for an array $A$ can be used to answer 1D-RMQ on $A$ via a lowest common ancestor query on the Cartesian tree.

Each node in a Cartesian tree can be of four types – it can have two children (type-2), only a left or right child (type-L/type-R), or it can be a leaf (type-0). Consider an element $A[i]$ for $1 \leq i \leq n$ and observe that the type of the $i$-th node in the Cartesian tree in inorder (which corresponds to $A[i]$) is determined by the relative values of $l = A[i-1]$, $m = A[i]$ and $r = A[i+1]$ (adding dummy random elements in $A[0]$ and $A[n+1]$). Specifically:

1. if $r > m > l$ then node $i$ is type-L;
2. if $l > m > r$ then node $i$ is type-R;
3. if $l > r > m$ or $r > l > m$ then node $i$ is type-0 and
4. if $m > l > r$ or $m > r > l$ then node $i$ is type-2.

In a random array, the probabilities of the alternatives above are clearly $1/6, 1/6, 1/3$ and $1/3$. By linearity of expectation, if $N_x$ is the random variable that denotes the number of type-$x$ nodes, we have that $E[N_0] = E[N_2] = n/3$ and $E[N_L] = E[N_R] = n/6$. The encoding consists in traversing the Cartesian tree in

\(^9\) All logarithms are to the base 2.
either level-order or in depth-first order (pre-order) and writing down the label of each node in the order it is visited: it is known that this suffices to reconstruct the Cartesian tree \[3, 14\]. The sequence of labels is encoded using arithmetic coding, choosing the probability of type-0 and type-2 to be 1/3 and that of type-L and type-R to be 1/6. The coded output would be of size \(\log_6(N_R + N_L) + \log_3(N_0 + N_2)\) (to within lower-order terms) \[13\]; plugging in the expected values of the random variables \(N_x\) gives the result. 

\[\square\]

**Theorem 2.** The expected effective entropy of 1-sided queries on an \(m \times n\) array is \(\Theta(\log^2 n)\) bits.

**Proof.** For the upper bound observe that we can recover the answers to the 1-sided queries by storing the positions of the prefix maxima, i.e., those positions, \((i, j)\), such that the value stored at \((i, j)\) is the maximum among those in positions \([1 \cdots m] \times [1 \cdots j]\). Since the position \((i, j)\) is a prefix maximum with probability \(1/jm\) and can be stored using \(\lceil \log(nm + 1) \rceil\) bits, the expected number of bits used is at most \(\sum_{i=1}^{m} \sum_{j=1}^{n} \lceil \log(nm + 1) \rceil / ij = O(\log^2 n)\) bits.

Consider a random source that generates \(n\) elements of \(\{0, 1, \ldots, m\}\) as follows: the \(i\)th element of the source is \(j\) if the answer to the query \([1 \cdots m] \times [1 \cdots i]\) is \((j, i)\) for some \(j\), and 0 otherwise. Clearly the entropy of this source is a lower bound on the expected size of the encoding. This source produces \(n\) independent elements of \(\{0, 1, \ldots, m\}\) with the \(i\)th equal to \(j\), \(j = 1, \ldots, m\), with probability \(1/im\) and is equal to 0 with probability \(1 - 1/i\). I.e., its entropy is \(\sum_{i=1}^{n} \left(1 - 1/i \right) \log \left(\frac{i+1}{i} \right) + \sum_{j=1}^{m} \log \left(\frac{im}{im} \right) = \Omega(\log^2 n)\) bits. 

\[\square\]

**Theorem 3.** The expected effective entropy of 2-sided queries on an \(m \times n\) array is \(\Theta(\log n \log m)\) bits.

**Proof.** As in the proof of Theorem 2, we store a list of the positions of the 2-sided prefix maxima sorted by their values. By 2-sided prefix maxima we mean those positions \((i, j)\) where the value in that position is maximum among all those in \([1 \cdots i] \times [1 \cdots j]\). The answer to any query is the position of the largest such 2-sided prefix maximum inside the query. This can be determined from the sorted list of positions. The expected number of bits in the encoding is at most \(\sum_{i=1}^{n} \sum_{j=1}^{m} \lceil \log(nm + 1) \rceil / ij = O(\log^2 n \log m)\) bits.

The lower bound is also similar. From an encoding for 2-sided queries of an \(m \times n\) array, we can create a source of \(nm\) independent bits with a bit being 1 if and only if the answer to the query \([1 \cdots i] \times [1 \cdots j]\) is \((i, j)\) (which occurs with probability \(1/ij\)). The entropy of this source is at least \(\sum_{i=1}^{n} \sum_{j=1}^{m} \log(ij) / ij = \Omega(\log^2 n \log m)\) bits. 

\[\square\]

**Theorem 4.** The expected effective entropies of 4-sided and 3-sided queries on an \(m \times n\) array are \(\Theta(nm)\) bits and \(\Theta(n(\log m)^2)\) respectively.

**Proof.** We begin with the 4-sided case\(^\text{10}\). For each position \((i, j)\) we store a region which has the property that for any query containing \((i, j)\) and lying entirely

\(^{10}\) NB: positions are given as row index then column index, not x-y coordinates.
within that region, \((i, j)\) is the answer to that query. This contiguous region is delimited by a monotone (along columns or rows) sequence of positions in each of the quadrants defined by \((i, j)\). A position \((k, l)\) delimits the boundary of the region of \((i, j)\) if the value in position \((k, l)\) is the largest and the value in position \((i, j)\) is the second largest, in the sub-array defined by \((i, j)\) and \((k, l)\), i.e., any query in this sub-array not including positions on row \(k\) or column \(l\) is answered with \((i, j)\) (dealing with boundary conditions appropriately). The answer to any query is the (unique) position inside the query whose region entirely contains the query. For each position, we store a clockwise ordered list of the positions delimiting the region (starting with the position above \(i\) in column \(j\)) by giving the position’s column and row offset from \((i, j)\).

By linearity of expectation, the expected number of bits stored is \(O(nm)\). The bound is tight as we can generate \(\lceil n/2 \rceil m\) equiprobable independent random bits (of entropy \(\Omega(nm)\)) from \(A\) by reporting 1 iff the answer to the query consisting of the two positions \((2i - 1, j)\) and \((2i, j)\) is \((2i, j)\), for \(i = 1, \ldots, \lceil n/2 \rceil\), \(j = 1, \ldots, m\).

For the 3-sided case, recall that we focus on queries that are open to the “top” side. We again define the region of position \((i, j)\) as the area such that \((i, j)\) is the answer to any query containing \((i, j)\) and lying entirely within that area. In contrast to the 4-sided case, the region of a point may be empty (if it is not a prefix maximum in its column). For points with non-empty regions, their region is delimited on the right by a monotone sequence of positions \((k, l)\) such that \(l > j\) for all positions, and \(k > i\) for all positions but one (see Fig. 1). The left delimiters

| 24 | 37 | 76 | 95 | 20 | 3 | 90 | 79 |
|----|----|----|----|----|---|----|----|
| 80 | 17 | 53 | 25 | 50 | 85 | 96 | 30 |
| 48 | 12 | 11 | 50 | 38 | 68 | 63 | 97 |
| 9  | 57 | 37 | 36 | 79 | 5 | 59 | 82 |

Fig. 1. Regions for the italicised values for 4-sided queries (left) and 3-sided queries (right), with region delimiters in blue.
are symmetric, and the region is obviously delimited from below by the next prefix maximum in the \( j \)-th column. To answer RMQs, we store all (ordered) pairs \((p, q)\) such that position \( q \) delimits the region of position \( p \) (assuming \( p \) has a non-empty region). The pairs are stored sorted by \( p \)'s column, and are represented as follows (all numbers are assumed stored in a self-delimiting manner, e.g. using Gamma codes). We store \( p \) and \( q \)'s row number using \( O(\log m) \) bits and the difference between the columns of \( p \) and \( q \) using \( O(1 + \log(|j - l| + 1)) \) bits.

The pair \((p, q)\) is stored iff the value in position \( q = (k, l) \) is the largest and the value in position \( p = (i, j) \) is the second largest in the sub-array defined by \( p \) and \( q \), i.e. \( A[1 \cdots \max\{i, k\}; j \cdots l] \) (assuming that \( l > j \)). For a fixed pair \((p, q), p \neq q\), the probability that \((p, q)\) is stored is \( \frac{1}{(ab)(ab - 1)} \leq \frac{2}{(ab)^2} \), where \( a = \max\{i, k\} \) and \( b = |j - l| + 1 \). We now calculate the expected cost of storing \((p, q)\) over all pairs \((p, q)\), taking \( j = 1 \) to simplify the summation (arbitrary \( j \) will have a summation at most twice that of \( j = 1 \)).

\[
2 \cdot \sum_{i=1}^{m} \sum_{l=1}^{n} \sum_{k=1}^{m} \frac{1}{\max\{i, k\}^2} \cdot O(\log m + \log l) = O \left( \sum_{i=1}^{m} \sum_{l=1}^{n} \frac{(\log m + \log l)}{il^2} \right)
\]

\[
= O \left( \sum_{i=1}^{m} (\log m)/i \right) = O((\log m)^2).
\]

The summation uses the observation that, for any fixed \( i \), \( \sum_{k=1}^{m} \frac{1}{\max\{k, i\}^2} = \sum_{k=1}^{i} \frac{1}{k^2} + \sum_{k=i+1}^{m} \frac{1}{k^2} = O(1/i) \). Summing over all \( j \), we get that the expected effective entropy is \( O(n(\log m)^2) \). For the lower bound, the \( n \) columns can be considered independent \( 1 \times m \) prefix maxima problems each requiring expected \( \Omega(\log^2 m) \) bits by Theorem 2.

\[\Box\]

**3 Small \( m \)**

Brodal et al. [4] gave a 2D-RMQ encoding of size essentially \( \left( \frac{m(m+3)}{2} - O(\log m) \right) \).

\( 2n \approx n \cdot m(m+3) \) bits for a \( m \times n \) array. In order that precise comparisons can be made for fixed values of \( m \), we outline their approach. For each of the \( m \) rows of the matrix, they store a Cartesian tree for that row, and for each of the \((m)(m-1)/2\) possible subrange of rows, they store a Cartesian tree for the maximum value in each column that lies within that set of rows. Since we consider \( m \) fixed in this section, the space bound for these Cartesian trees is \((m)(m+1)/2)(2n - O(\log n)) \) bits which is essentially \( 2n \cdot m(m+1) \) bits. Given any query spanning a subrange of rows, the Cartesian tree for that subrange tells us which column the range maximum lies in. However, to find which row the maximum lies, in Brodal et al. also store a Cartesian tree for each column of the matrix. The space used by these column-wise Cartesian trees needs to be calculated more carefully since \( m \) is small, and is taken to be \( n \cdot \log \left( \frac{1}{m+1} \left( \frac{2m}{m} \right) \right) \) (we do not take the ceiling of the log since the Cartesian trees for all columns could be encoded together). Specifically this gives:
Furthermore, Brodal et al. showed that the effective entropy of 2D-RMQ is at least \( \log((\frac{m!}{\pi^m} n^{-\frac{m}{2+1}})) \) bits. For \( m = 2 \), their techniques give a lower bound of \( n/2 \), but this is worse than the obvious lower bound of \( 4n - O(\log n) \) obtained by considering each row independently.

In this section we improve upon these results for small \( m \). Our main tool is the following lemma:

**Lemma 1.** Let \( A \) be an arbitrary \( m \times n \) array, \( m \geq 2 \). Given an encoding capable of answering range maximum queries of the form \([1 \cdots (m-1)] \times [j_1 \cdots j_2]\) \((1 \leq j_1 \leq j_2 \leq n)\) and an encoding answering range maximum queries on the last row of \( A \), \( n \) additional bits are necessary and sufficient to construct an encoding answering queries of the form \([1 \cdots m] \times [j_1 \cdots j_2]\) \((1 \leq j_1 \leq j_2 \leq n)\) on \( A \).

**Proof.** The proof has two parts, one showing sufficiency (upper bound) and the other necessity (lower bound).

**Upper Bound.** We construct a joint Cartesian tree that can be used in answering queries of the form \([1 \cdots m] \times [j_1 \cdots j_2]\) for \( 1 \leq j_1 \leq j_2 \leq n \), using an additional \( n \) bits. The root of the joint Cartesian tree is either the answer to the query \([1 \cdots (m-1)] \times [1 \cdots n]\) or \([m] \times [1 \cdots n]\). We store a single bit indicating the larger of these two values. We now recurse on the portions of the array to the left and right of the column with the maximum storing a single bit, which indicates which sub-problem the winner comes from, at each level of the recursion. Following this procedure, using the \( n \) additional bits it created, we can construct the joint Cartesian tree. To answer queries of the form \([1 \cdots m] \times [i \cdots j]\), the lowest common ancestor \( x \) (in the joint Cartesian tree) of \( i \) and \( j \) gives us the column in which the maximum lies. However, the comparison that placed \( x \) at the root of its subtree also tells us if the maximum lies in the \( m \)-th row or in rows \( 1 \cdots m - 1 \); in the latter case, query the given data structure for rows \( 1 \cdots m - 1 \).

**Lower Bound.** For simplicity we consider the case \( m = 2 \) — it is easy to see that by considering the maxima of the first \( m - 1 \) elements of each column the general problem can be reduced to that of an array with two rows. Let the elements of the top and bottom rows be \( t_1, t_2, \ldots, t_n \) and \( b_1, b_2, \ldots, b_n \). Given two arbitrary Cartesian trees \( T \) and \( B \) that describe the answers to the top and bottom rows, the procedure described in the upper bound for constructing the Cartesian tree for the \( 2 \times n \) array from \( T \) and \( B \) makes exactly \( n \) comparisons between some \( t_i \) and \( b_j \). Let \( c_1, \ldots, c_n \) be a bit string that describes the outcomes of these comparisons in the order they are made. We now show how to assign values to the top and bottom rows that are consistent with any given \( T \), \( B \), and comparison string \( c_1, \ldots, c_n \). Notice this is different from (and stronger than) the trivial observation that there exists a \( 2 \times n \) array \( A \) such that merging \( T \)
and $B$ must use $n$ comparisons: we show that $T$, $B$ and the $n$ bits to merge the two rows are independent components of the $2 \times n$ problem.

If the $i$-th comparison compares the maximum in $t_i, \ldots, t_r$, with the maximum in $b_k, \ldots, b_r$, say that the range $[l, r]$ is associated with the $i$-th comparison. The following properties of ranges follow directly from the algorithm for constructing the Cartesian tree for both rows:

(a) for a fixed $T$ and $B$, $[l, r]$ is uniquely determined by $c_1, \ldots, c_{i-1}$;
(b) if $j > i$ then the range associated with $j$ is either contained in the range associated with $i$, or is disjoint from the range associated with $i$.

By (a), given distinct bit strings $c_1, \ldots, c_n$ and $c_1', \ldots, c_n'$ that differ for the first time in position $i$, the $i$-th comparison would be associated with the same interval $[l, r]$ in both cases, and the query $[1..2] \times [l, r]$ then gives different answers for the two bit strings. Thus, each bitstring gives distinguishable $2 \times n$ arrays, and we now show that each bitstring gives valid $2 \times n$ arrays.

First note that if the $i$-th bit in a given bit string $c_1, \ldots, c_n$ is associated with the interval $[l, r]$, then it enforces the condition that $t_j > b_k$, where $j = \arg\max_{l \leq j \leq r} \{t_i\}$ and $k = \arg\max_{l \leq k \leq r} \{b_i\}$ or vice-versa. Construct a digraph $G$ with vertex set $\{t_1, \ldots, t_n\} \cup \{b_1, \ldots, b_n\}$ which contains all edges in $T$ and $B$, as well as edges for conditions $t_j > b_k$ (or vice versa) enforced by the bit string. All arcs are directed from the larger value to the smaller. We show that $G$ is a DAG and therefore there is a partial order of the elements satisfying $T$ and $B$ as well as the constraints enforced by the bit string.

Suppose that for some value of $c_1, \ldots, c_n$, $G$ is not a DAG. Pick any cycle in $G$: there must be some node $t \in \{t_1, \ldots, t_n\}$ that is explicitly enforced (i.e. by a comparison) to be greater than some $b \in \{b_1, \ldots, b_n\}$, such that some descendant $b'$ of $b$ in $B$ has been explicitly enforced to be greater than an ancestor $t'$ of $t$ (or the symmetric case with $T$ and $B$ interchanged must hold). Let the interval associated with the $b$-$t$ comparison be $[l, r]$. First consider the case that $b = b'$. Since an element that wins a comparison is never compared again, the comparison between $b$ and $t'$ must have occurred after the comparison with $t$, in which case the interval associated with the $b$-$t'$ comparison is a sub-interval of $[l, r]$ by property (b). This means that $t'$ must be a descendant of $t$. Therefore $b \neq b'$ and $b'$ must be a proper descendant of $b$. If $b'$ belongs to $[l, r]$, it will never have been compared prior to the $b$-$t$ comparison, and will subsequently only be compared (if at all) to a descendant of $t$. If $b'$ does not belong to $[l, r]$, then there must have been a comparison between $b$, or one of $b$'s ancestors in $B$, that was won by a proper ancestor $t''$ of $t$, such that the range $[l', r']$ associated with that comparison was split into two parts, one containing $[l, r]$ and one containing $b$. Clearly, $b$ could not have been compared prior to this comparison, and subsequently, can only be compared to elements from $T$ that are in a different subtree of $T$ than $t$.

Using Lemma 1 we show by induction:

**Theorem 5.** There exists an encoding solving the $2$D-RMQ problem on a $m \times n$ array requiring at most $n \cdot \frac{m(m+1)}{2}$ bits.
Proof. The theorem follows by induction from Lemma 1 and the fact that $2n$ bits are sufficient to store a Cartesian tree of a $1 \times n$ array (the base case). Given an encoding solving the RMQ problem for a $(m - 1) \times n$ array (using $(m - 1)(m + 2)n/2$ bits by induction) and a Cartesian tree for a 1D array (using $2n$ bits) we construct a solution to the 2D-RMQ problem on a $m \times n$ array combining the two (with the 1D array as the last row of the combined array) by using Lemma 1 to construct $m - 1$ Cartesian trees answering queries of the form $[i \cdots m] \times [j_1 \cdots j_2]$ for $1 \leq i \leq m - 1$ using $(m - 1)n$ additional bits. \(\square\)

**Theorem 6.** The minimum space required for any encoding for the 2D-RMQ problem on a $m \times n$ array is at least $n \cdot (3m - 1) - O(m \log n)$ bits.

Proof. The result follows by induction from Lemma 1 and the fact that $2n - O(\log n)$ bits are required to solve the RMQ problem for a $1 \times n$ array (the base case). Note that any encoding that solves the 2D RMQ problem for a $m \times n$ array must be able to solve the 1D RMQ problem on its last row, the 2D RMQ problem on the array consisting of the first $m - 1$ rows as well as queries of the form $[1 \cdots m] \times [j_1 \cdots j_2]$ ($1 \leq j_1 \leq j_2 \leq n$). The first two problems are entirely independent, i.e., answers to queries from one provide no information about the answers to the other. The 1D problem requires $2n - O(\log n)$ bits and the additional queries require at least $n$ bits by Lemma 1, i.e., $3n - O(\log n)$ bits are needed on top of those required for solving the problem on the first $m - 1$ rows. \(\square\)

For fixed $m < 2^{12}$ this lower bound is better (in $n$) than that of Brodal et al. [4]. For the case $m = 2$ our bounds are tight:

**Corollary 1.** $5n - O(\log n)$ bits are necessary and sufficient for an encoding answering range maximum queries on a $2 \times n$ array.

For the case $m = 3$, our upper bound is $9n$ bits and our lower bound is $8n - O(\log n)$ bits. We can improve the upper bound slightly:

**Theorem 7.** The 2D-RMQ problem can be solved using at most $(6 + \log 5)n + o(n) \approx 8.322n$ bits on a $3 \times n$ array.

Proof. We refer to the three rows of the array as T (top), M (middle) and B (bottom). We store Cartesian trees for each of the three rows (using $6n$ bits). We now show how to construct data structures for answering queries for TM (the array consisting of the top and middle rows), MB (the middle and bottom rows) and TMB (all three rows) using an additional $n \log 5 + o(n)$ bits. Let $0 \leq x \leq 1$ be the fraction of nodes in the Cartesian tree for TMB such that the maximum lies in the middle row. Given the trees for each row, and a sequence indicating for each node in the Cartesian tree for TMB in in-order, which row contains the maximum for that node, we can construct a data structure for TMB. The sequence of row maxima is coded using arithmetic coding, taking $\Pr[M] = x$ and $\Pr[B] = \Pr[T] = (1 - x)/2$; the output takes $(-x \log x - (1 - x) \log((1 - x)/2))n + o(n)$ bits [13].
We now apply the same procedure as in Lemma 1 to construct the Cartesian
trees for TM and MB, storing a bit to answer whether the maximum is in the top
or middle row for TM (middle or bottom row for MB) for each query made in the
construction of the tree starting with the root. However, before comparing the
maxima in T and M in some range, we check the answer in TMB for that range; if
TMB reports the answer is in either T or M, we do not need to store a bit for that
range for TM. It is easy to see that every maximum in TMB that comes from T
or B saves one bit in TM or MB, and every maximum in TMB that comes from M
saves one bit in both TM and MB. Thus, a total of 
\[2n - (1 - x)n - 2xn = (1 - x)n - 2xn\]
= 
\[(1 - x)n\]
bits are needed for TM and MB. The total number of bits needed for all three
trees (excluding the \(o(n)\) term) is 
\[2(1 - x)n - 2xn = (1 - x)n\]
This takes a maximum at 
\[x = 1/5\]

4 Data Structures for 2D-RMQ

In this section, we give efficient data structures for 2D-RMQ. We begin by a
recap of rank and select operations on bit strings. Given a bit string \(S[1..t]\) of
length \(t\), define the following operations, for \(x \in \{0, 1\}\):

- \(\text{rank}_x(S, i)\) returns the number of occurrences of \(x\) in the prefix \(S[1..i]\).
- \(\text{select}_x(S, i)\) returns the position of the \(i\)th occurrence of \(x\) in \(S\).

Such a data structure is called a fully indexable dictionary (FID) by Raman et
al. [16], who show that:

**Theorem 8.** There is a FID for a bit string \(S\) of size \(t\) using at most \(\log_2(t)\) bits, that supports all operations in \(O(1)\) time on the RAM model with wordsize \(O(\log t)\) bits, where \(r\) is the number of 1s in the bit string.

Since \(\log_2(t) \leq t\), the space used by the FID is always \(t + o(t)\) bits.

**Theorem 9.** There is a data structure for 2D-RMQ on a random \(m \times n\) array
\(A\) which answers queries in \(O(1)\) time using \(O(mn)\) expected bits of storage.

**Proof.** Take \(N = mn\) and \(\lambda = \lceil 2 \log \log N \rceil + 1\), and define the label of an element \(z = A[i, j]\) as \(\min\{\lfloor \log(1/(1 - z)) \rfloor, \lambda\}\) if \(z \neq 0\). The labels bucket the elements into \(\lambda\) buckets of exponentially decreasing width. We store the following:

(a) An \(m \times n\) array \(L\), where \(L[i, j]\) stores the label of \(A[i, j]\).
(b) For labels \(x = 1, 2, \ldots, \lambda - 1\), take \(r = 2^{2x}\) and partition \(A\) into \(r \times r\) submatrices (called grid boxes or grid sub-boxes below) using a regular grid
with lines \(r\) apart. Partition \(A\) four times, with the origin of the grid once
each at \((0, 0), (0, r/2), (r/2, 0)\) and \((r/2, r/2)\). For each grid box, and for all
elements labelled \(x\) in it, store their relative ranks within the grid box.
(c) For all values with label \(\lambda\), store their global ranks in the entire array.

The query algorithm is as follows:

1. Find an element with the largest label in the query rectangle.
2. If the query contains an element with label $\lambda$, or if the maximum label is $x$ and the query fits into a grid box at the granularity associated with label $x$, then use (c) or (b) respectively to answer the query.

A query fails if the maximal label in the query rectangle is $x < \lambda$ but the rectangle does not fit into any grid box associated with label $x$. For this case:

(d) Explicitly store the answer for all queries that fail.

We now give an efficient implementation of the above, and prove the stated space bounds.

The data structures to support steps (1) and (2) also store (a) and (b) in $O(N)$ bits. Firstly, $L$ is represented as a bit string that represents the concatenation of all elements of $L$ in row-major order, with a $x$ encoded in unary as $1^x 0$.

Since the expected number of nodes with label $\geq x$ is $O(N/2^x)$, it follows that the expected size of this encoding is $O(N)$ bits. To access $L[i, j]$ in $O(1)$ time, we store this bit string as an FID (Theorem 8) and use the select$_0$ operation on this bit string. Furthermore, we use Brodal et al.’s “hybrid” 2D-RMQ indexing structure [4, Theorem 3] over the array $L$: this data structure stores an index of $O(N)$ bits, and answers queries in $O(1)$ time, using $O(1)$ comparisons between elements of $L$. The comparisons are implemented by accessing $L$ and breaking ties arbitrarily, implementing step (1).

However, we cannot use the above approach to implement step (2), since the problem now focuses on a sparse set of points with label $x$. Hence, we use a data structure to solve the following problem: for each label value $x < \lambda$, answer range maximum queries which lie entirely in a grid box of size $r \times r$ where $r = 2^{2x}$. We allow the data structure for a given grid box to use $O((t+1)x)$ bits of space, where $t$ is the number of elements in the box that have label $x$. Note, however, that $x \leq 2 \log \log N + 1$ so $r = O((\log N)^4)$. For any grid box where $r^2 \leq c \log N/\log \log N$ for some sufficiently small constant $c > 0$, this can be done by table lookup, since we can write down all the coordinates as well as the relative priorities in as a bit string of fewer than $(\log N)/2$ bits: the required table will be of size at most $O(\sqrt{N})$ words, or $o(N)$ bits. The space used per grid block is clearly $O(t x)$ since the bit string comprises just the positions of the points and their relative priorities, all of which take $O(x)$ bits. The expected space used across all grid blocks for label $x$ is at most $O(x N/2^x)$ bits, summing up to $O(N)$ expected bits overall.

For larger values of $x$ we use a data structure that takes $o(N/\log \log N)$ expected bits for each value of $x$, but since there are $O(\log \log N)$ values of $x$ this is still $o(N)$ bits overall. We divide each grid box into grid sub-boxes each of side $r'$, where $r' = 2^{2y}$ for the largest $y$ such that $(r')^2 \leq c \log N/\log \log N$. The number of sub-boxes is $N/(r')^2$, or $O(N \log \log N/\log N)$. For this grid box, we store a matrix $R$ which is $(r/r') \times r$ where each entry corresponds to a row of a sub-box, and contains the largest relative rank in this row of this sub-box. The space used by $R$ is $O((N/r') \log \log N) = o(N/\log \log N)$. Using Brodal et al’s data structure we can do 2D-RMQ queries on $R$. We also create a $r \times (r/r')$ matrix $C$ where each entry corresponds to a column of a sub-box, and similarly
we can do 2D-RMQ queries on \( C \). A general query can either be decomposed into \( O(1) \) queries on \( R \) and \( C \), plus \( O(1) \) 2-sided or 3-sided queries each on one sub-block, or is a 4-sided query completely contained in a sub-block, each of which is done by table lookup. This implements step (2) in \( O(1) \) time using \( O(N) \) bits. A very similar data structure is used to handle elements with label \( \lambda \). We divide the input matrix \( A \) into sub-boxes of size \( \log N \times \log N \), and create matrices \( R' \) and \( C' \) which represent elements with label \( \lambda \) in each row/column respectively. We store \( R' \) and \( C' \) explicitly using \( O(N) \) bits each and store a 2D-RMQ indexing structure on \( R' \) and \( C' \). To answer queries inside a sub-block, we use an algorithm quite similar to that used for the smaller labels.

We finally need to bound the space usage in (d), and also to describe how to represent the solutions. We classify queries based upon their area, i.e. the number of positions they contain. For a given value of the area \( A \), there are at most \( A \) (in fact, considerably fewer) different aspect ratios that give rise to that area, and at most \( N \) queries with that aspect ratio, or \( NA \) queries in all. To encode the maximum in a query with area \( A \) requires \( O(\log A) \) bits. The smallest grid granularity that will contain all queries of area \( A \) is the granularity associated with label \( x = \lceil (\log A)/2 \rceil \). If a query of area \( A \) contains no positions with labels \( \geq x \), it may fail. The probability of this happening is at most \( 1 - 2^{-x-1}A = 2^{-O(\sqrt{A})} \), so the expected number of failing queries of area \( A \) is \( O(N/A^3) \). For each area \( A \leq (\log N)^4 \), we store a minimal perfect hash function from \([1 \cdots N] \times [1 \cdots A] \rightarrow [1..NA] \), where \( NA \) is the number of failing queries of area \( A \) (the domain of the hash function specifies the top left corner and, say, the width of the query). Such hash functions can be stored in \( O(NA + \log \log N) \) bits and evaluated in \( O(1) \) time \([20]\), and are used to index into an array of length \( NA \) that contains the answer to that query. The space used is \( \sum_{A=1}^{(\log N)^4} O(NA \cdot \log A + \log \log N) \) bits, and since \( E(NA) = O(N/A^3) \), by linearity of expectation, the expected space used is \( O(N) \).

We now show how to support 2D-RMQ queries efficiently on a \( 2 \times n \) array, using space close to that of Corollary 1. In the following data structure, we use a Cartesian tree augmented with additional leaves, which we call as an augmented Cartesian tree or ACT for short. The ACT of a given array \( A \) is the Cartesian tree in which every node is augmented with a leaf in between its left and right children. If a node has only a left child then we add the leaf as its second child, and if it only has a right child, then we add the leaf as its first child. Finally, if a node (in the Cartesian tree) is a leaf, then we add the new leaf as its child. This structure was used by Sadakane [17] to obtain a space-efficient data structure supporting RMQ queries. The indices of the array \( A \) correspond to the inorder numbers of the nodes in the Cartesian tree, and to the preorder numbers of the leaves in the ACT.

**Theorem 10.** There is a data structure for 2D-RMQ on an arbitrary \( 2 \times n \) array \( A \), which answers queries in \( O(k) \) time using \( 5n + O(n \log k/k) \) bits of space, for any parameter \( k = (\log n)^{O(1)} \).
Proof. For an integer array of length \( n \), the 2D-maxHeap\(^{11} \) of Fischer and Heun [9] uses \( 2n + O(n \log \log n / \log n) \) bits and supports RMQ queries on the array in \( O(1) \) time. The lower order term can be further reduced to \( O(n/(\log n)^{O(1)}) \) using the tree representation of Sadakane and Navarro [19]. We use this 2D-maxHeap structure (which is essentially a space- and query-efficient representation of a Cartesian tree) to support queries on each of the individual rows, using a total of \( 4n + o(n) \) bits. The upper bound of Lemma 1 shows how to combine these two Cartesian trees (2D-maxHeaps) using \( n \) bits, to answer queries involving both the rows. We refer to these \( n \) bits as the bit vector \( M \) (that merges the two Cartesian trees). Each of these \( n \) bits in \( M \) corresponds to a unique column in \( A \), and the bits in \( M \) are written in the order of the inorder numbers of the nodes in the joint Cartesian tree. These bits can be decoded in that order in \( O(1) \) time per bit, using the Cartesian trees for the individual rows, to reconstruct the joint Cartesian tree. A query involving both the rows can be answered by decoding the first bit that falls within the query range. Thus the worst-case query complexity is \( O(n) \) for this representation.

If we represent the joint Cartesian tree for both the rows as a 2D-maxHeap, and an additional \( n \) bits to indicate the column maxima, we can answer any RMQ query in \( O(1) \) time using a total of \( 7n + o(n) \) bits. We now describe how to reduce the space to achieve the trade-off described in the statement of the theorem.

The main idea is to represent the ACT of the joint Cartesian tree using a succinct tree representation based on tree decomposition. The representation decomposes the ACT into \( O(n/k) \) microtrees, each of size at most \( k \), and represents the microtrees using a total of \( 4n + o(n) \) bits (as the ACT has \( 2n \) nodes), and stores several auxiliary structures of total size \( O(n \log k/k) \) bits. It supports various queries (such as LCA) in constant time by accessing a constant number of microtrees and reading a constant number of words from the auxiliary structures. Instead of storing the representations of microtrees, we show how reconstruct any microtree in \( O(k) \) time (in fact, time proportional to its size) using the bit vector \( M \) (together with additional auxiliary structures of size \( O(n \log k/k) \) bits). The new representation consists of the 2D-maxHeap for both the rows (\( 4n + o(n) \) bits) and the bit vector \( M \) that ‘merges’ these two trees (\( n \) bits) in addition to these auxiliary structures. Thus the representation uses \( 5n + O(n \log k/k) \) bits overall, and supports RMQ queries in \( O(k) \) time. We now describe this in detail.

We take the ACT of the joint Cartesian tree and partition it into \( O(n/k) \) microtrees, each of size at most \( k \), for some parameter \( k \geq 2 \), using the tree decomposition algorithm of Farzan and Munro [7]. The microtrees produced by the decomposition algorithm have the property that for each microtree, there is at most one node such that one of its children is the root of another microtree. In addition, two microtrees can share a common root. We modify the decomposition so that whenever a node \( x \) is the root of two microtrees (a node cannot be the

\(^{11}\) Fischer and Heun [9] use the term 2D-minHeap as they consider the problem of answering range minimum queries
root of more than two microtrees, as we have a ternary tree in which one of
the children of every internal node is a leaf), we remove the node \(x\) from both
the microtrees, and make another microtree containing \(x\) and its second child
(which is a leaf). One can show that the number of microtrees produced by the
modified decomposition is still \(O(n/k)\). Now, no two microtrees share a node,
and thus we obtain a partition of the ACT into microtrees.

Each leaf in the ACT corresponds to a column in the array \(A\), and as men-
tioned earlier, the leaves of the ACT in preorder correspond to the columns of
the array \(A\) from left to right. The above partitioning procedure splits the ACT
such that the columns corresponding the all the nodes in a microtree are in at
most two consecutive chunks in the array \(A\) (as all the nodes in a microtree
have consecutive preorder numbers, except when a node has a child outside the
microtree; and since there is at most one such node, the claim follows).

We store the auxiliary structures to support LCA and rank/select on leaves in
preorder, using \(O(n \log k/k)\) bits. As the ACT has \(2n\) nodes, we need \(4n + o(n)\)
bits to store the representations of all the microtrees. Instead of storing the
representations of the microtrees, we show how to reconstruct the representation
of any microtree in time proportional to its size by simply storing the bit vector
\(M\). In addition, we also store auxiliary structures to represent the corre-
spondence between the microtrees and the positions in the array, as explained below.

We construct two bit vectors \(B_1\) and \(B_2\) of length \(n + O(n/k)\) as follows: we
initialize both arrays with \(n\) zeroes each. We insert a 1 after the \(i\)-th zero in \(B_1\)
\((B_2)\) if position \(i\) \((i - 1)\) is the starting (ending) position of a chunk in \(A\). We
store these two bit vectors using the FID structure of Theorem 8, which takes
\(O(n \log k/k)\) bits of space. Now, considering the 1s in \(B_1\) as open parentheses
and the 1s in \(B_2\) as close parentheses, we ‘merge’ the two parenthesis sequences
to obtain a balanced parenthesis sequence that represents the tree structure
of the microtrees. We store this sequence along with an auxiliary structure to
support parenthesis operations, such as find-open, find-close, excess, rank-open,
rank-close \([11]\). Using all these data structures, we can support the following op-
erations: (i) number the microtrees in the sorted order of their preorder number
of the leftmost leaf, (ii) find the microtree that contains the leaf corresponding
to a given column index in \(A\), and (iii) find the chunk corresponding to a given
microtree.

For each microtree \(\mu\), we store the bits of \(M\) corresponding to the leaves of
\(\mu\) in the order of their preorder numbers. Given these bits, we now show how
to reconstruct \(\mu\) in time linear in the size of \(\mu\). Suppose \(C_\ell\) and \(C_r\) are the
two chunks corresponding to \(\mu\) (where \(C_r\) is empty if the nodes in \(\mu\) correspond
to a single chunk). We first observe that all the elements that lie between the
chunks \(C_\ell\) and \(C_r\) in \(A\) are strictly smaller than the maximum element in the
last column of \(C_\ell\) as well as the the maximum element in the first column of \(C_r\).
Thus any RMQ query whose one end point lies in \(C_\ell\) and the other end point
lies in \(C_r\) has its answer in one these two chunks and never in between these
chunks.
Let \( i \) be the first column of \( C_\ell \) and \( j \) be the last column of \( C_r \). We first find \( t = \text{RMQ}(A, [1] \times [i..j]) \) and \( b = \text{RMQ}(A, [2] \times [i..j]) \) which return the positions of the maximum elements in the range \([i..j]\) in the top and bottom rows respectively. The first bit of \( M \) that we store for the microtree is a 0 or 1 depending on whether \( A[1, t] \) is greater or less than \( A[2, b] \). Suppose that \( A[1, t] \) is the larger of the two (the other case is similar). Then the root of \( \mu \) has a leaf child which corresponds to the position \( t \). The leftmost (first) and rightmost (third) subtrees of the root correspond to all the positions in the chunks \( C_\ell \) and \( C_r \) that are to the left and right of the position \( t \) respectively, and can be constructed recursively using the same procedure. Since each node of \( \mu \) can be ‘constructed’ in \( O(1) \) time, the overall time to reconstruct \( \mu \) is \( O(|\mu|) \), where \( |\mu| \) denotes the size of \( \mu \).

To answer an RMQ query that spans both the rows, we use an algorithm that answers a query by storing the tree decomposition representation. Whenever we need to access a microtree, we reconstruct it using the above procedure, in \( O(k) \) time; all the auxiliary structures are stored explicitly, and hence can be accessed in \( O(1) \) time. Since the query algorithm accesses \( O(1) \) microtrees, the overall running time is \( O(k) \). Finally, the lower order terms that arise in various substructures above, such as FIDs and auxiliary structures for tree representations, which are independent of \( k \) can be made \( O(n/(\log n)^{O(1)}) \) using the ideas from [15]. Thus the overall space used is \( 5n + O(n \log k / k) \) bits, for any parameter \( k = (\log n)^{O(1)} \).

5 Conclusions and Open Problems

We have given new explicit encodings for the RMQ problem, as well as (in some cases) efficient data structures whose space usage is close to the sizes of the encodings. We have focused on the cases of random matrices (which may have relevance to practical applications such as OLAP cubes [5]) and the case of small values of \( m \). Obviously, the problem of determining the asymptotic complexity of encoding RMQ for general \( m \) remains open, as does the problem of determining the precise effective entropy of 1D-RMQ for a random array.

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