Solution to a conjecture on edge rings with 2-linear resolutions

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Abstract

For a graph $G = (V, E)$ the edge ring $k[G]$ is $k[x_1, \ldots, x_n]/I(G)$, where $n = |V|$ and $I(G)$ is generated by $\{x_ix_j; \{i, j\} \in E\}$. The conjecture we treat is the following.

**Conjecture.** If $k[G]$ has a 2-linear resolution, then the projective dimension of $K[G]$, $\text{pd}(k[G])$, equals the maximal degree of a vertex in $G$.

As far as we know, this conjecture is first mentioned in a paper by Gitler and Valencia, [7, Conjecture 4.13], and there it is called the Eliahou-Villarreal conjecture. The conjecture is treated in a recent paper by Ahmed, Mafi, and Namiq, [1]. That there are counterexamples was noted already by Moradi and Kiani, [9]. By interpreting $k[G]$ as a Stanley-Reisner ring, we are able to characterize those graphs for which the conjecture holds.

1 Introduction

An edge ring $k[G]$ has a 2-linear resolution if and only if the complement graph $\overline{G}$ is chordal, [5, Theorem 1], i.e., every cycle of length $\geq 3$ in $\overline{G}$ has a chord. (The complement graph $\overline{G}$ has the same vertex set as $G$, and $\{i, j\}$ is an edge in $\overline{G}$ if and only if $\{i, j\}$ is not an edge in $G$.) There are other proofs of the theorem in [3, Theorem 2.1], [2, Theorem 3.4], [10, Corollary 3.3], and [8, Theorem 9.2.12].

However, we prefer to use the characterization of edge rings with 2-linear resolution in terms of Stanley-Reisner rings. The Stanley-Reisner ring of a simplicial complex $\Delta$ with vertex set $\{1, 2, \ldots, n\}$ is $k[x_1, \ldots, x_n]/I(\Delta)$, where $I(\Delta)$ is generated by those squarefree monomials $x_{i_1} \cdots x_{i_k}$ for which $\{i_1, \ldots, i_k\}$ is not a face in $\Delta$. Then $k[\Delta]$ has a 2-linear resolution if and only if $\Delta$ is a quasi-forest. [5, 6]. Quasi-forests are also named fat forests or generalized $d$-trees, but we have chosen the name quasi-forest here, because it seems to be the most common. A quasi-forest is defined recursively in the following way. A simplex $F_1$ is a quasi-forest. If $F_1 \cup F_2 \cup \cdots \cup F_{i-1}$, $F_j$ simplices for $1 \leq j \leq i - 1$, is a quasi-forest, then, if $F_i$ is a simplex, $F_1 \cup F_2 \cup \cdots \cup F_i$ is a quasi-forest if $G_i = (F_1 \cup F_2 \cup \cdots \cup F_{i-1}) \cap F_i$ is a simplex. If $F_1 \cup F_2 \cup \cdots \cup F_{i-1}$ and $F_i$ are disjoint, then $G_i = \emptyset$ of dimension -1.

For a simplicial complex $\Delta$, we let $G(\Delta)$ be its 1-skeleton. For a graph $G$ we let $\Delta(G)$ be the largest simplicial complex with $G$ as 1-skeleton. If $k[F] = k[x_1, \ldots, x_n]/I(F)$ is the Stanley-Reisner ring of the simplicial complex $F$, then $I(F)$ is generated in degree 2 if and
only if $\Delta(G(F)) = F$. We can go back and forth between Stanley-Reisner rings with 2-linear resolution and edge rings with 2-linear resolution since $k[G]$ has a 2-linear resolution if and only if $k[\Delta(G)]$ has a 2-linear resolution, and $k[\Delta]$ has a 2-linear resolution if and only if $k[\overline{G(\Delta)}]$ has a 2-linear resolution. Let $F = F_1 \cup \cdots \cup F_k$ be a fat forest, and $H$ the 1-skeleton of $F$. Then the edge ring of $\overline{H}$ is the Stanley-Reisner ring of $F$, and thus has a 2-linear resolution.

Suppose $\dim F_i = d_i - 1$ and $\dim G_i = r_i - 1$. If $d_1 \geq d_2 \geq \cdots \geq d_i$ and $d_j = r_j + 1$ for all $j$, then the 1-skeleton $H$ is what is called a $(d_1, d_2, \ldots, d_i)$-tree in [4].

A vertex $v$ in a simplicial complex $\Delta$ is called free if $v$ belongs to a unique facet (maximal face).

## 2 Hilbert series and Betti numbers of fat forests

If $k[x_1, \ldots, x_n]/I = S/I$ has a 2-linear resolution it looks like this:

$$0 \leftarrow S/I \leftarrow S[-2]^{b_1} \leftarrow S[-3]^{b_2} \leftarrow \cdots \leftarrow S[-p - 1]^{b_p} \leftarrow 0$$

where $S[-i]$ means that we have shifted degrees of $S$ $i$ steps.

If we restrict the sequence to a certain degree, we get an exact sequence of vector spaces, thus the alternating sum of dimensions is 0. Using this we get that the Hilbert series $\sum_{i \geq 0} \dim_k k[\Delta_i]t^i$ of $k[\Delta]$ with 2-linear resolution equals

$$\frac{1 - \beta_{i,j} t^{2} + \beta_{i,j} t^{3} - \cdots - (-1)^{p} \beta_{p,p+1} t^{p+1}}{(1-t)^n},$$

where $\beta_{i,j}$ are the graded Betti numbers $\dim_k \text{Tor}_{i,j}^{S}(k[\Delta], k)$, and $n$ is the number of vertices in $\Delta$. If one is interested in the Betti numbers $\beta_{i,j} = \dim_k \text{Tor}_{i,j}^{S}(S/I, k)$ of Stanley-Reisner ring of a fat tree, since it has 2-linear resolutions, one could just as well study its Hilbert series, because this contains the same information as the set of Betti numbers. We now give the main result from [4]. To make this paper self contained we repeat the short proof.

**Theorem 1.** Let $F = F_1 \cup \cdots \cup F_k$ be a quasi-tree with $F_i$ a simplex of dimension $d_i$ and $(F_1 \cup \cdots \cup F_{j-1}) \cap F_j$ a simplex of dimension $r_j$. Then the Hilbert series of $k[F]$ is

$$\sum_{i=1}^{k} \frac{1}{(1-t)^{p_i+1}} - \sum_{i=2}^{k} \frac{1}{(1-t)^{r_i+1}}.$$

The projective dimension is $\sum_{i=1}^{k} d_i - \sum_{i=2}^{k} r_i + 1 - \min\{r_i\} - 2$. The depth of $k[F]$ is $\min\{r_i\} + 2$, and $F$ is CM (Cohen-Macaulay) if and only if there is a $d$ such that $d_i = d$ for all $i$ and $r_i = d - 1$ for all $i$.

**Proof** A $d$-simplex has Hilbert series $\frac{1}{(1-t)^{d+1}}$. We have to subtract the Hilbert series of the $G_i$’s from $\sum_{i=1}^{k} \frac{1}{(1-t)^{d_i+1}}$, because they are counted twice in $\sum_{i=1}^{k} \frac{1}{(1-t)^{d_i+1}}$. The number of vertices of $F$ is $\sum_{i=1}^{k} (d_i + 1) - \sum_{i=2}^{k} (r_i + 1) = \sum_{i=1}^{k} d_i - \sum_{i=2}^{k} r_i + 1 = n$, so the degree of the numerator $p(t)$ of the Hilbert series $\frac{p(t)}{(1-t)^n}$ of $k[F]$ is $n - \min\{r_i\} - 1$ so the projective dimension is $n - \min\{r_i\} - 2$, and the depth of $k[F]$ is $\min\{r_i\} + 2$ by the Auslander-Buchsbaum formula. We have $\dim k[F] = 1 + \max\{d_i\}$, depth $k[F] = \min\{r_i\} + 2$, and $d_i > r_i$ for all $i$. The only possibility for $\dim k[F] = \text{depth} k[F]$ is that there is a $d$ such that $d_i = r_i + 1 = d$ for all $i$. 

2
3 Main result

We now come to the main result, the characterization of edge rings with 2-linear resolution for which the conjecture holds. We first state it in terms of Stanley-Reisner rings $k[\Delta]$. If $\Delta$ is a simplex, the conjecture trivially holds. Thus we may suppose that $\Delta$ has at least two facets.

**Theorem 2.** Suppose $\Delta$ is a quasi-forest with at least two facets, in particular $k[\Delta]$ has a 2-linear resolution. Let $r = \min\{r_i\}$. Then the conjecture holds if and only if there is a free vertex $v$ in a facet $F$ of $\Delta$, $\dim F = r + 1$, $F$ connected to the remaining part of $\Delta$ in a simplex of dimension $r$.

**Proof** Let $v$ be a vertex in $\Delta$. Suppose $v$ belongs to $F_1, \ldots, F_k$. The dimension of $F_1$ is at least $r + 1$ so $F_1$ has at least $r + 2$ vertices. There must be at least one vertex in each $F_j$, $j \geq 2$, which is not in the union of the others. Thus there are $n - (r + 2 + k - 1)$ vertices which are not neighbours. (That $u$ is not a neighbour to $v$ means exactly that $u$ is a neighbour to $v$ in $G(\Delta)$.) The projective dimension is $n - r - 2$. If the numbers are equal, we get $k = 1$, and furthermore that $F_1$ has precisely $r + 2$ vertices, so $d_{i_1} = r + 1$.

**Corollary 1.** Let $G$ be a graph and suppose that $k[G]$ has a 2-linear resolution. Then the conjecture holds for $G$ if and only $G$ is the 1-skeleton of a simplicial complex as in Theorem 2.

**Corollary 2.** The smallest countereexample to the conjecture is the the edge ring of a 4-cycle, noted already in [7]. This corresponds to the Stanley-Reisner ring on four vertices $\{1, 2, 3, 4\}$ with facets $\{1, 2\}, \{3, 4\}$.

**Proof** If $G = C_4$, then $\Delta(G)$ is the complex $F$ on four vertices $\{1, 2, 3, 4\}$ with facets $\{1, 2\}, \{3, 4\}$. We have $pd(k[C_4]) = pd(k[F]) = 4 - (-1) - 2 = 3$ and $\max\deg\{v; v \in G\} = 2$.

**Corollary 3.** If the quasi-forest $\Delta$ has an isolated vertex $v$, then the conjecture holds.

**Proof** We have $\dim\{v\} = 0$ and $r = -1$.

**Remark 1.** This is equivalent to [4, Theorem 4.11], which considers the case when there is a vertex in the graph with all other vertices a neighbours (a full vertex).

**Theorem 3.** [4, Theorem 4.21] Let $G$ be a graph such that $\overline{G}$ is a $(d_1, d_2, \ldots, d_q)$-tree. Then the conjecture holds.

**Proof** This follows from Corollary 1.

**Corollary 4.** If $k[G]$ is a Cohen-Macaulay ring with 2-linear resolution, then the conjecture holds for $k[G]$.
Proof If \( k[G] \) is Cohen-Macaulay, then \( k[G] = k[\Delta] \), and the complement of the 1-skeleton of \( \Delta \) is a \((d,d,\ldots,d)\)-tree for some \( d \) by definition of \((d_1,\ldots,d_q)\)-trees and by Theorem 1. Thus the result follows Theorem 3.

Remark 2. The authors of [1] also give an example to show that the difference \( \text{pd}(k[G]) - \max \{ \deg v; v \in G \} \) can be arbitrarily large. The example is the complement of the \( r \)-barbell, i.e., the complement to the graph consisting of two complete graphs \( K_r \) with a bridge. We give a slightly smaller example.

Corollary 5. Let \( K_{r,r} \) be the complete bipartite graph. Then \( \text{pd}(k[K_{r,r}]) - \max \{ \deg v; v \in K_{r,r} \} = r \).

Proof \( K_{r,r} \) is the complement to the 1-skeleton of the disjoint union \( S_{r-1} \cup S_{r-1} \) of two \((r-1)\)-simplices \( S_{r-1} \). Now the edge ring \( k[K_{r,r}] \) equals the Stanley-reisner ring \( k[S_{r-1} \cup S_{r-1}] \) and \( \text{pd}(k[S_{r-1} \cup S_{r-1}]) - \max \{ \deg v; v \in K_{r,r} \} = 2r - 1 - 2 - r = r - 1 \).

Remark 3. In [1] it is claimed that the conjecture is stated in [4]. This seems to be a mistake, the conjecture is not mentioned in that paper, and the name of the conjecture is a bit of the mystery.

References

[1] Ahmed, C., Mafi, A., Namiq, M.R., On the Eliahou and Villarreal conjecture about the projective dimension of co-chordal graphs, arXiv:2205.07059v1

[2] Dochtermann, A., Engström, A., Algebraic properties of edge ideals via combinatorial topology, Electr. J. Combin. 16(2) 24pp. (2009).

[3] Eisenbud, D., Green, M., Hulek, K., Popescu, S., Restricting linear syzygies: algebra and geometry, Compos. Math. 141(6), 1460–1478 (2005).

[4] Eliahou, S., Villarreal, R.H., The second Betti number of an edge ideal, XXXI national Congress of the Mexican mathematical Society (Hermosillo, 1998), vol. 25, 115–119. Sociedad Matematica Mexicana, 1999.

[5] Fröberg, R., On Stanley-Reisner rings Topics in algebra, Banach Center Publications 26(2), 57–70 (1990).

[6] Fröberg R., Betti numbers of fat forests and their Alexander dual, on line in J. Algebraic combinatorics, https://10.1007/s10801-022-01143-0 (2022).

[7] Gitler, I., Valencia, C.E., Bounds for invariants of edge-rings, Comm. in Algebra 33(5), 1603–1616 (2005).

[8] Herzog, J., Hibi, Monomial ideals Graduate texts in Math. 200, Springer 2011.
[9] Moradi, S., Kiani, D., *Bounds for the regularity of edge ideal of vertex decomposable and shellable graphs*, Bull. Iranian Math. Soc. **36**(2), 267–277 (2010).

[10] Nero, E., *Regularity of edge ideals of $C_4$-free graphs via the topology of the lcm-lattice*, J. Combin. Theory Ser. **A118**(2), 491–501 (2011).