A CATEGORY OF HYBRID SYSTEMS.

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Abstract. We propose a definition of the category of hybrid systems in which executions are special types of morphisms. Consequently morphisms of hybrid systems send executions to executions. We plan to use this result to define and study networks of hybrid systems.

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1. Introduction

In this paper propose a definition of a category of non-deterministic hybrid systems. Hybrid systems are dynamical systems that exhibit both continuous time evolution, which we model by vector fields on manifolds with corners, and abrupt transitions (“discrete transitions” or “jumps”).

Our basic philosophy is that of category theory — so rather than study dynamical systems one at a time we aim to study maps between all relevant dynamical systems at once. To quote Silverman [19]:

“A meta-mathematical principle is that one first studies (isomorphism classes of) objects, then one studies the maps between objects that preserve the objects’ properties, then the maps themselves become objects for study...”

Definitions of a hybrid dynamical systems varies widely in literature depending on the background and needs of the practitioners. They all include directed graphs, phase spaces attached to the nodes of the graph and partial maps or relations attached to arrows of the graph.

To get started we choose one definition of a directed graph. We recall a fairly standard definition of a hybrid dynamical system and its executions. We introduce the notion of a hybrid phase space so that we can think of a hybrid dynamical system as a hybrid phase space with a “hybrid vector field.” We propose a notion of a map between two hybrid systems. This turns hybrid systems into a category. We justify our definition by proving that maps of hybrid systems send executions to executions. We also explain why executions can be thought of as morphisms of hybrid dynamical systems. We are aware of one previous attempt to bring category theoretic methods to hybrid dynamical systems by Ames [1] Ames and Sastry [2]. Our construction is simpler and covers a larger class of systems. Readers who like category theory may be entertained by the appearance of pseudo-double categories.
We plan to use our approach to hybrid systems to define and study networks of hybrid systems. In particular our goal is to prove for hybrid systems analogues of results in [8], [9], [24], [15].

Acknowledgments: I thank Sayan Mitra for many hours of conversations. In a better world we would have written this paper together.

2. Background

In this section we review the definitions of directed graphs, manifolds with corners (which we call “regions”) and smooth maps between regions, set-theoretic relations and a traditions definition of a hybrid dynamical system essentially following [20].

Graphs. We start by fixing a notion of a graph and of a map of graphs.

Definition 2.1. A directed multigraph $A$ is a pair of collections $A_0$ (nodes, vertices) and $A_1$ (arrows, edges) together with two maps $s, t: A_1 \to A_0$ (source and target). We do not require that $A_1, A_0$ are sets in the sense of ZFC.

We depict an arrow $\gamma \in A_1$ with the source $a$ and target $b$ as $a \xrightarrow{\gamma} b$. We write $A = \{A_1 \Rightarrow A_0\}$ to remind ourselves that our graph $A$ consists of two collections and two maps.

A graph $A = \{A_1 \Rightarrow A_0\}$ is finite if the collections $A_1$ and $A_0$ are finite.

Remark 2.2. Every category has an underlying graph: forget the composition of morphisms. Since the collections of objects and morphisms in a given category may be too big to be sets, we choose to define graphs in such a way as to induce the underlying graphs of categories that are not necessarily small. This causes no problems.

Next we record our definition of a map of graphs:

Definition 2.3. A map of graphs $\varphi: A \to B$ from a graph $A$ to a graph $B$ is a pair of maps $\varphi_1: A_1 \to B_1$, $\varphi_0: A_0 \to B_0$ taking edges of $A$ to edges of $B$, nodes of $A$ to nodes of $B$ so that for any edge $\gamma$ of $A$ we have

\[ \varphi_0(s(\gamma)) = s(\varphi_1(\gamma)) \quad \text{and} \quad \varphi_0(t(\gamma)) = t(\varphi_1(\gamma)). \]

We will usually omit the indices 0 and 1 and write $\varphi(\gamma)$ for $\varphi_1(\gamma)$ and $\varphi(a)$ for $\varphi_0(a)$.

Note the maps of graphs can be composed and that the composition is associative. Hence graphs form a category. We now formally record its definition.

Definition 2.4. Directed multigraphs form a category Graph. Its objects are directed graphs (see Definition 2.1). Morphisms are maps of graphs (see Definition 2.3).

Regions and continuous time dynamics. Continuous time dynamics takes place in regions. For most purposes of this paper one may take a region to be an open subset of some coordinate space $\mathbb{R}^n$. However, we also like to consider closed intervals $[a,b] \subset \mathbb{R}$ as regions. On the other hand, we do not want to consider arbitrary subsets of $\mathbb{R}^n$ as regions — those are too wild and we will not be able to make sense of differential equations on such sets. A subset $D$ of $\mathbb{R}^n$ (for some $n$) with smooth boundary should definitely be considered a region. For many purposes it is not very wrong to think of regions this way. However, later on we will need to take products of regions. The product of two regions with smooth boundary no longer has a smooth boundary. For example the product of two unit intervals is the unit square:

\[ [0,1] \times [0,1] = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}. \]

The boundary of the unit square is only piecewise smooth. This forces us to define a region to be a subset of $\mathbb{R}^n$ with smooth corners. (Recall that a corner of an $n$-dimensional region is smooth if
it is locally diffeomorphic to the standard orthant \([0, \infty)^n\). It is convenient for various purposes to treat regions as abstract manifolds with corners. However, a reader will not be too far wrong simply to think of a region as an open subset of some \(\mathbb{R}^n\)'s with a piecewise-smooth boundary. There are a number of textbooks and survey articles that deal with manifolds with corners. We recommend Joyce [13] and Michor [18]. Our notion of a map of manifolds with corners follows [18] and is much weaker than the one in [13]. Namely we only require that a smooth map between manifolds with corners pulls back smooth functions to a smooth functions.\(^1\) In particular we allow corners to be mapped into the interior.

**Definition 2.5.** A vector field \(X\) on a manifold with corners \(D\) is a section of its tangent bundle \(TD \to D\). We write \(X \in \Gamma(TD)\). A integral curve of the vector field \(X\) is a smooth map \(x : I \to D\) of \(X\) (where \(I\) is an interval) so that \(\frac{dx}{dt} = X(x(t))\) for all \(t \in I\).

**Definition 2.6.** A continuous time dynamical system is a pair \((D, X)\) where \(D\) is a manifold with corners (i.e., a region) and \(X\) is a vector field on \(D\).

Continuous time dynamical systems form a category.

Namely we define a map from a (continuous time) dynamical system \((D_1, X_1)\) to a dynamical system \((D_2, X_2)\) to be a map \(f : D_1 \to D_2\) of manifolds with corners with \(Tf \circ X_1 = X_2 \circ f\) (here and elsewhere in the paper \(Tf\) denotes the differential of \(f\)).

**Remark 2.7.** It is easy to see that if \(f : (D_1, X_1) \to (D_2, X_2)\) and \(g : (D_2, X_2) \to (X_3, D_3)\) are two maps of continuous time dynamical systems then so is their composite \(g \circ f\). Hence continuous time dynamical systems do form a category. We denote it by \(DS\).

**Definition 2.8 (the category of \(DS\) continuous time dynamical systems).** The objects of the category \(DS\) of continuous time dynamical systems are pairs \((D, X)\) where \(D\) is a manifold with corners and \(X\) is a vector field on \(D\). A morphism from \((D, X)\) to \((D', X')\) is a map \(f : D \to D'\) of manifolds with corners with

\[Tf \circ X = X' \circ f.\]

**Remark 2.9.** An integral curve \(x : [a, b] \to D\) of a dynamical system \((D, X)\) can be thought of a map of dynamical systems as follows. Recall that every interval \([a, b] \subset \mathbb{R}\) is equipped with the constant vector field \(\frac{dx}{dt}\). By definition the image of the vector field \(\frac{dx}{dt}\) by the map \(x\) is the derivative \(\frac{dx}{dt}\):

\[\left.\frac{dx}{dt}\right|_t := Tx\left(\frac{d}{dt}\right|_t\).\]

Since \(x\) is an integral curve of \(X\), \(\left.\frac{dx}{dt}\right|_t = X(x(t))\). Hence,

\[Tx \circ \frac{d}{dt} = X \circ x.\]

Thus a map of manifolds with corners \(x : [a, b] \to D\) is an integral curve of a vector field \(X\) on \(D\) if and only if \(x : ([a, b], \frac{dx}{dt}) \to (D, X)\) is a morphism in the category \(DS\).

**Remark 2.10.** It \(x : [a, b] \to D\) is a trajectory of a vector field \(X\) then for any \(b' < b\) the restriction \(x|_{[a, b']}\) is also a trajectory of \(X\). For this reason we regard maps of the form \(x : [a, a] \to D\) as integral curves of \(X\). Of course the closed interval \([a, a]\) is a single point, so the derivative of \(x\) in this case doesn’t quite make sense. None the less we will find this point of view convenient when we deal with executions of hybrid systems.

\(^1\)That is, for the purpose of defining smooth maps we think of manifolds with corners as diffeological spaces [12].
Relations.

**Definition 2.11 (Relation).** Given two sets \( X \) and \( Y \) we call a subset \( R \) of the product \( Y \times X \) a relation and think of it as a “generalized map” from \( X \) to \( Y \) (note the order!).

**Remark 2.12.** The reason from why we think of \( R \subset Y \times X \) as a map from \( X \) to \( Y \) and not from \( Y \) to \( X \) has to do with compositions of relations and of functions. Namely, given two relations \( S \subset Z \times Y \) and \( R \subset Y \times X \) their composition \( S \circ R \) is defined by

\[
S \circ R := \{(z,x) \in Z \times X \mid \text{there is } (z,y) \in S, (y',x) \in R \text{ with } y = y'\}.
\]

If \( f : X \to Y \) is a function, its graph is the relation

\[
\text{graph}(f) := \{(y,x) \in Y \times X \mid y = f(x)\}.
\]

With these definitions, given a function \( g : Y \to Z \) we have

\[
\text{graph}(g \circ f) = \text{graph}(g) \circ \text{graph}(f).
\]

**Remark 2.13.** Note that if \( R \subset Y \times X \) is a relation so that the intersection \( R \cap (Y \times \{x\}) \) consists of exactly one point for each \( x \in X \) then \( R \) is a graph of a function from \( X \) to \( Y \). If the intersection \( R \cap (Y \times \{x\}) \) consists of at most one point for each \( x \in X \) then \( R \) is a graph of a partial function from \( X \) to \( Y \) whose domain of definition is the set \( \{x \in X \mid R \cap (Y \times \{x\}) \neq \emptyset\} \). We will refer to the image of a relation \( R \subset Y \times X \) under the projection \( \pi_X : Y \times X \to X \) as the domain of the relation \( R \). In the hybrid dynamical systems literature domains of relations and/or partial maps are sometimes referred to as guards and relations as resets.

Hybrid dynamical systems. We next recall the traditional definition of a hybrid dynamical system. It is a slight variant of [20, Definition 2.1]). Note that in [20] what we call manifolds with corners/regions are called domains. Since in mathematics and computer science literature the word “domain” has other meanings we prefer to use the word “region.” Another name for what we call “regions” is invariants. But in mathematics an “invariant” has too many other meanings (e.g., invariant submanifolds, invariant functions, invariant vectors etc.).

**Definition 2.14 (Hybrid dynamical system).** A hybrid dynamical system (HDS) consists of

1. **(2.14.i)** A directed graph \( A = \{A_1 \Rightarrow A_0\} \);
2. **(2.14.ii)** For each node \( a \in A_0 \) a dynamical system \((R_a, X_a)\) where \( X_a \) is a vector field on the manifolds with corners \( R_a \);
3. **(2.14.iii)** For each arrow \( a \xrightarrow{\gamma} b \) of \( A \) a reset relation \( R_\gamma \subset R_b \times R_a \).

Thus a hybrid dynamical system is a tuple \((A = \{A_1 \Rightarrow A_0\}, \{(R_a, X_a)\}_{a \in A_0}, \{R_\gamma\}_{\gamma \in A_1})\).

**Example 2.15.** Here is an example of a very simple hybrid dynamical system. We take \( A \) to be the graph with one node \( * \) and one arrow \( * \xrightarrow{\gamma} * \) (formally \( A_1 = \{\gamma\}, A_0 = \{*\} \) and \( s(\gamma) = * = t(\gamma) \)). We assign to \( * \) the constant vector field \( \frac{d}{dx} \) on the unit interval \([0,1]\). We take \( R_\gamma \) to be the one element set

\[
R_\gamma := \{(0,1)\}.
\]

By our convention (Definition 2.11) it is the graph of a map that takes the endpoint \( 1 \) of the closed interval \([0,1]\) to the endpoint \( 0 \) (and not \( 0 \) to \( 1 \)). Thus

\[
H := \left\{ \{\gamma \Rightarrow *\}, ([0,1], \frac{d}{dx}), \{(0,1)\} \right\}.
\]

We’ll describe the dynamics of the system once we define executions.
**Executions.** Having defined hybrid dynamical systems we now define the corresponding dynamics. The notion of an execution (that is of an “integral curve” or of a “hybrid trajectory”) of a hybrid dynamical system is supposed to captures the following idea. Given a hybrid dynamical system

\[(A, \{(R_a, X_a)\}_{a \in A_0}, \{R_{\gamma}\}_{\gamma \in A_1})\]

a hybrid trajectory would start at a point in some region \(R_{a(1)}\). For an interval of time \([t_0, t_1]\) it would follow an integral curve \(\sigma_{a(1)}\) of the vector field \(X_{a(1)}\) until it reaches a point \(\sigma_{a(1)}(t_1)\) inside the domain of a relation \(R_{a(2)} : R_{a(1)} \rightarrow R_{a(2)}\). Now the trajectory is allowed to jump to a point \(y\) in some region \(R_{a(2)}\) with \((y, \sigma_{a(1)}(t_1)) \in R_{a(2)}\) and follow the integral curve \(\sigma_{a(2)}\) through the point \(y\) of the vector field \(X_{a(2)}\) for an interval of time \([t_1, t_2]\). And so on for an increasing sequence of times \(\{t_0, t_1, t_2, \ldots\}\), which may be finite or infinite. This leads to the following definition, which is fairly standard. We will revisit the definition: see Definition 4.7 below.

**Definition 2.16** (An execution with jump times indexed by the natural numbers \(\mathbb{N}\)). Let \(H = (A = \{A_1 \Rightarrow A_0\}, \{(R_a, X_a)\}_{a \in A_0}, \{R_{\gamma}\}_{\gamma \in A_1})\) be a hybrid dynamical system. An execution of \(H\) is

1. (2.16.i) an nondecreasing sequence \(\{t_i\}_{i \geq 0}\) of real numbers
2. (2.16.ii) a function \(\varphi_0 : \mathbb{N} \rightarrow A_0\);
3. (2.16.iii) a function \(\varphi_1 : \mathbb{N} \rightarrow A_1\) compatible with \(\varphi_0\): we require that \(s(\varphi_1(i)) = \varphi_0(i)\) and \(t(\varphi_1(i)) = \varphi_0(i+1)\);
4. (2.16.iv) an integral curve \(\sigma_i : [t_{i-1}, t_i] \rightarrow R_{\varphi_0(i)}\) of \(X_{\varphi_0(i)}\) (with \(t_{-1}\) being some number less than \(t_0\));
5. (2.16.v) the terminal end point of \(\sigma_i\) and the initial end point of \(\sigma_{i+1}\) are related by the reset relation \(R_{\varphi_1(i)}\):

\[(\sigma_{i+1}(t_i), \sigma_i(t_i)) \in R_{\varphi_1(i)}\].

**Example 2.17.** Consider the hybrid system \(H\) of Example 2.15. What would an execution of such a system look like? We have no choice in defining the functions \(\varphi_0\) and \(\varphi_1\) since \(A_0\) and \(A_1\) are one point sets: we set \(\varphi_0(n) = *\) and \(\varphi_1(n) = \gamma\) for all \(n \in \mathbb{Z}\). If we take \(t_i = i\), then \(\sigma_i : [i-1, i] \rightarrow [0, 1]\) is given by \(\sigma_i(t) = t - i + 1\). Therefore

\[\{(t_i)_{i \in \mathbb{N}}, \varphi_0, \varphi_1, \{\sigma_i\}_{i \in \mathbb{N}}\}\]

is an execution of \(H\).

**Remark 2.18.** If \(t_i = t_{i-1}\) the condition in Definition 2.16 that \(\sigma_i : [t_{i-1}, t_i] \rightarrow R_{\varphi_0(i)}\) is an integral curve of \(X_{\varphi_0(i)}\) should be interpreted in the sense of Remark 2.10. This amounts to saying that \(\sigma_i(t_i) = \sigma_i(t_{i-1})\) is a point of \(R_{\varphi_0(i)}\). Note that the next conditions forces \((\sigma_{i+1}(t_i), \sigma_i(t_i)) \in R_{\varphi_1(i)}\). In other words if \(t_{i-1} = t_i\) the execution jumps.

**Remark 2.19.** More generally jump times of an execution may be indexed by a subset \(S\) of the integers \(\mathbb{Z}\) of the form \(S = [n, m]\), \(n \leq m\), \(n, m \in \mathbb{Z}\), or by \(S = (-\infty, N]\) by \(S = [N, +\infty)\) for some \(N \in \mathbb{Z}\). Note that \(S = \emptyset\) also makes sense: this is an execution that is simply an integral curve of a vector field. We will give a different definition of an execution that includes all of these cases, see Definition 4.7 below.

3. **Hybrid phase spaces**

If we forget the vector field of a continuous time dynamical system \((D, X)\) we get a manifold with corners \(D\), which we think of as the phase space of our dynamical system. Therefore it makes sense to define a hybrid phase space to be a “hybrid dynamical system without the vector fields.” Formally we record the following definition, which we think is new.

**Definition 3.1** (Hybrid phase space). A hybrid phase space consists of
(3.1.i) A directed graph $A = \{A_1 \Rightarrow A_0\}$;
(3.1.ii) For each node $a \in A_0$ a manifold with corners $R_a$;
(3.1.iii) For each arrow $\gamma \Rightarrow a'$ of $A$ a reset relation $R_\gamma \subset R_{a'} \times R_a$.
Thus a hybrid phase space is a tuple $(A = \{A_1 \Rightarrow A_0\}, \{R_a\}_{a \in A_0}, \{R_\gamma\}_{\gamma \in A_1})$.

**Example 3.2.** The underlying hybrid phase space of the hybrid dynamical system of Example 2.15 consists of the following data:

(3.2.i) the directed graph $A = \{\gamma \Rightarrow \ast\}$;
(3.2.ii) the region $R_\ast = [0, 1]$;
(3.2.iii) the reset relation $R_\gamma = \{(0, 1)\} \subset [0, 1] \times [0, 1]$.

**Remark 3.3.** The two collections $\{R_a\}_{a \in A_0}, \{R_\gamma\}_{\gamma \in A_1}$ in the definition of a hybrid phase space above look like the components of a map of directed graphs and they are. To make this precise we need a definition.

**Definition 3.4.** We define the graph RR of regions and relations as follows: the collection of nodes of RR is the collection of all regions (i.e., manifolds with corners); the collection of all arrows of RR is the collection of all (set-theoretic) relations between the regions.

**Remark 3.5.** The graph RR is the underlying graph of a category whose objects are manifolds with corners and morphisms are relations between the underlying sets of manifolds with corners.

With this definition and notational convention we can restate the definition of a hybrid phase space as follows:

**Definition 3.6** (Hybrid phase space, version 2). A hybrid phase space is a map of directed graphs $R : A \rightarrow RR$.

**Example 3.7.** The underlying hybrid phase space of the hybrid dynamical system of Example 2.15 is a map of graphs $R : \{\gamma \Rightarrow \ast\} \rightarrow RR$ with $R(\ast) = [0, 1]$ and $R(\gamma) = \{(0, 1)\}$.

The following example will be important when we discuss executions as maps of hybrid dynamical systems and when we re-define our notion of an execution.

**Example 3.8.** [Hybrid phase space associated with a nondecreasing sequence $\{t_i\}_{i \in \mathbb{N}}$.] Define $Z$ to be the graph with the set of edges $Z_1 := \mathbb{N}$, the set of nodes $Z_0 := \mathbb{N}$ and the source and target maps given by $s(i) = i, t(i) = i + 1$:

$$0 \xrightarrow{0} 1 \xrightarrow{i} \cdots \xrightarrow{i - 1} i - 1 \xrightarrow{i} i \xrightarrow{i + 1} i + 2 \xrightarrow{i + 1} \cdots .$$

Let $T : Z \rightarrow RR$ be the map of graphs defined on vertices by $T(i) = [t_{i-1}, t_i]$ and on arrows by $T(i \xrightarrow{i} i + 1) = T_i$

where $T_i : [t_{i-1}, t_i] \rightarrow [t_i, t_{i+1}]$ is the relation consisting of one point $\{(t_i, t_i)\} \subset [t_i, t_{i+1}] \times [t_{i-1}, t_i]$.

Our definition of hybrid phase spaces as maps of graphs from arbitrary graphs to RR suggests the category of hybrid phase spaces could be the slice category Graph/RR, but this is too strict. Note that RR has more structure: in addition to the set-theoretic relations as morphisms we also have smooth maps as morphisms between regions. This suggests that we should think of RR as a double category [7,21,22]. Recall that double categories have two types of 1-arrows ("vertical" and
“horizontal”) and, in addition, 2-cells that are shaped like rectangles. Composition is defined by pasting the rectangles—vertically and horizontally.

**Definition 3.9** (The double category \( \mathbb{RR} \) of manifolds with corners, smooth maps and set-theoretic relations). The double category \( \mathbb{RR} \) is defined as follows. Its objects are manifolds with corners. The vertical 1-arrows are smooth maps. The horizontal 1-arrow are set-theoretic relations. The 2-cells are diagrams of the form

\[
\begin{array}{ccc}
X & \overset{R}{\rightarrow} & X' \\
\downarrow & & \downarrow \\
Y & \overset{S}{\rightarrow} & Y'
\end{array}
\]

where \( f, f' \) are smooth maps, \( R \subset X' \times X, S \subset Y' \times Y \) are relations satisfying \((f', f)(R) \subset S\).

**Definition 3.10** (A category of hybrid phase spaces \( \text{HyPh} \)). The objects of the category \( \text{HyPh} \) are maps of graphs \( \mathcal{R} : A \rightarrow \mathbb{RR} \) with target \( \mathbb{RR} \). A morphisms from \( \mathcal{R} : A \rightarrow \mathbb{RR} \) to \( \mathcal{Q} : B \rightarrow \mathbb{RR} \) “2-commuting” triangle of the form

\[
\begin{array}{ccc}
A & \overset{\varphi}{\rightarrow} & B \\
\downarrow & \nearrow & \downarrow \\
\mathcal{R} & \overset{\alpha}{\rightarrow} & \mathcal{Q} \\
\downarrow & \nearrow & \downarrow \\
\mathbb{RR} & \overset{\beta}{\rightarrow} & \mathbb{RR} \\
\downarrow & \nearrow & \downarrow \\
\psi & \overset{\delta}{\rightarrow} & \varphi
\end{array}
\]

That is, \( \varphi : A \rightarrow B \) is a map of graphs and \( \alpha \) assigns to each node \( a \) of the graph \( A \) a map of manifolds with corners

\( \alpha_a : \mathcal{R}(a) \rightarrow \mathcal{Q}(\varphi(a)) \)

so that for each arrow \( a_1 \xrightarrow{\gamma} a_2 \) in the graph \( A \), we have a 2-cell

\[
\begin{array}{ccc}
\mathcal{R}(a_1) & \overset{\mathcal{R}(\gamma)}{\rightarrow} & \mathcal{R}(a_2) \\
\downarrow & \downarrow & \downarrow \\
\mathcal{Q}(\varphi(a_1)) & \overset{\mathcal{Q}(\varphi(\gamma))}{\rightarrow} & \mathcal{Q}(\varphi(a_2))
\end{array}
\]

in \( \mathbb{RR} \). Note that the latter condition amounts to the inclusion

\( (\alpha_{a_1}, \alpha_{a_2})(\mathcal{R}(\gamma)) \hookrightarrow \mathcal{Q}(\varphi(\gamma)) \).

The composition of morphisms is given by pasting of triangles:

\[
\begin{array}{ccc}
C & \overset{\psi}{\rightarrow} & B \\
\downarrow & \nearrow & \downarrow \\
\mathbb{RR} & \overset{\mathcal{R}}{\rightarrow} & \mathbb{RR} \\
\downarrow & \nearrow & \downarrow \\
\mathcal{Q} & \overset{\mathcal{Q}}{\rightarrow} & \mathbb{RR} \\
\downarrow & \nearrow & \downarrow \\
A & \overset{\delta}{\rightarrow} & \mathbb{RR} \\
\downarrow & \nearrow & \downarrow \\
B & \overset{\varphi}{\rightarrow} & A \\
\downarrow & \nearrow & \downarrow \\
\mathbb{RR} & \overset{\mathcal{R}}{\rightarrow} & \mathbb{RR}
\end{array}
\]

where

\( \delta(a) := \beta_{\varphi(a)} \circ \alpha_a : \mathcal{R}(a) \rightarrow \mathcal{S}(\psi(\varphi(a))) \)

for all nodes \( a \) of \( A \).

**Remark 3.11.** Equation (3.3) strongly suggests that we should view \( \mathbb{RR} \) as having more structure than just a double category. Namely the horizontal category of \( \mathbb{RR} \) should really be view as the strict 2-category of regions, set-theoretic relations and inclusions of relations.
2.17 is a map of hybrid phase spaces. This can be seen as follows. The source hybrid phase space is the map $T: Z \rightarrow RR$ with $T(i) = [i-1,i]$ for all $i \in \mathbb{N}$ (q.v. Example 3.8). The target hybrid phase space is the phase space $\mathcal{R}: \{\{\gamma\} \Rightarrow \{\ast\}\} \rightarrow RR$ of Examples 2.15, 3.7. The desired map of hybrid phase spaces consists of the map of graphs

$$\varphi: Z \rightarrow \{\{\gamma\} \Rightarrow \{\ast\}\}, \quad \varphi(i - 1 \xrightarrow{i-1} i) = \ast \xrightarrow{\gamma} \ast \quad \text{for all } i,$$

and of the collection of smooth maps of closed intervals

$$\{\sigma_i : T(i) = [i-1,i] \rightarrow [0,1] | \sigma_i(t) = t - i + 1\}.$$

4. HYBRID DYNAMICAL SYSTEMS

We are now in position to redefine a hybrid dynamical system as follows.

**Definition 4.1** (Hybrid dynamical system, version 2). A hybrid dynamical system is a hybrid phase space

$$\mathcal{R} : A \rightarrow RR,$$

together with a family of vector field $\{X_a \in \Gamma(T\mathcal{R}(a))\}_{a \in A_0}$, one for each region $\mathcal{R}(a)$. Thus a hybrid dynamical system is a pair $(\mathcal{R} : A \rightarrow RR, X = \{X_a \in \Gamma(T\mathcal{R}(a))\}_{a \in A_0})$.

**Remark 4.2.** Let $\{t_i\}_{i \in \mathbb{N}}$ be a nondecreasing sequence and $T : Z \rightarrow RR$ the corresponding hybrid phase space as in Example 3.8. On each interval $T(i) = [t_{i-1}, t_i]$ choose the constant vector field

$$\frac{d}{dt}|_{[t_{i-1}, t_i]}.$$ Then $(T : Z \rightarrow RR, \{\frac{d}{dt}|_{[t_{i-1}, t_i]}\}_{i \in \mathbb{N}})$ is a hybrid dynamical system.

**Definition 4.3** (Maps of hybrid dynamical systems). A map from a hybrid dynamical system $(Q : A \rightarrow RR, X)$ to a hybrid dynamical system $(\mathcal{R} : B \rightarrow RR, Y)$ is

- (4.3.i) a map of hybrid phase spaces $(\varphi, \{\alpha_a\}) : Q \rightarrow \mathcal{R}$ so that
- (4.3.ii)

$$Y_{\varphi(a)} \circ \alpha_a = T\alpha_a \circ X_a$$

for all nodes $a \in A_0$.

**Remark 4.4.** It is not hard to check that the composition of two maps of hybrid dynamical systems is a map of hybrid dynamical systems and that the composition is associative. Hence hybrid dynamical systems form a category, which we denote by HDS.

We are now in position to interpret executions as maps of hybrid dynamical systems. This reinterpretation allows us to broaden the notion of an execution and to give a short proof of the main result of this part of the paper: maps of hybrid dynamical systems take executions to executions.

**Proposition 4.5.** An execution of a hybrid dynamical system $(\mathcal{R} : A \rightarrow RR, X)$ in the sense of Definition 2.16 is a map $(\varphi, \{\sigma_i\}_{i \in \mathbb{N}} : (T : Z \rightarrow RR, \{\mathcal{R} : A \rightarrow RR, X\} \rightarrow (T : Z \rightarrow RR, \{\frac{d}{dt}|_{[t_{i-1}, t_i]}\}_{i \in \mathbb{N}})$ is a hybrid dynamical system defined in Remark 4.2.

**Proof.** Compare Definitions 2.16 and 4.3. □

We now extend the notion of an execution to allow for indexing of jump times by various subsets of the integers (q.v. Remark 2.19). We start by defining an appropriate generalization of the hybrid dynamical system $(T : Z \rightarrow RR, \{\frac{d}{dt}|_{[t_{i-1}, t_i]}\}_{i \in \mathbb{N}})$ of Remark 4.2.
Definition 4.6 (hybrid time dynamical system). Let $Z$ be a directed tree with countably many vertices and no branching. That is, $Z$ is a directed graph such that for any two distinct nodes $x, y$ of $Z$ there exists a unique directed path in $Z$ either from $x$ to $y$ or from $y$ to $x$ (but not both).

Let $T : Z \to \mathbb{R}$ be a map of graphs with the following properties:

(4.6.i) For any node $i$ of $Z$

$$T(i) = [t^+_i, t^-_i]$$

for some $t^+_i, t^-_i \in \mathbb{R}$ with $t^-_i \leq t^+_i$.

(4.6.ii) For any edge $i \xrightarrow{j} j$ of $Z$ we have $t^+_j = t^-_i$ and

$$T(\gamma) = \{(t^+_j, t^-_i) : [t^-_i, t^+_i] \to [t^-_j, t^+_j]\},$$

is a relation from $T(i)$ to $T(j)$.

On each interval $T(i)$ choose the constant vector field $X_i = \frac{d}{dt}$. We define a hybrid dynamical system of the form

$$(T : Z \to \mathbb{R}, \partial_t := \left\{ \frac{d}{dt} \big|_{T(i)} \right\}_{i \in \mathbb{N}})$$

to be a hybrid time dynamical system. We think of such a system as being analogous to the system \((a, b), \frac{d}{dt}\) in continuous time dynamics.

Definition 4.7 (An execution of a hybrid dynamical system). We define an execution of a hybrid system \((Q : A \to \mathbb{R}, X)\) to be a map of hybrid dynamical systems

$$(\varphi, \{\sigma_i\}_{i \in Z_0}) : (T : Z \to \mathbb{R}, \partial_t) \to (Q : A \to \mathbb{R}, X),$$

where \((T : Z \to \mathbb{R}, \partial_t)\) is a hybrid time dynamical system.

Notation 4.8. We abbreviate a hybrid dynamical system \((Q : A \to \mathbb{R}, X)\) as \((Q, A, X)\).

We now obtain the following useful theorem:

Theorem 4.9. Let \((\psi, \{\alpha_a\}) : (R, A, X) \to (Q, B, Y)\) be a map of hybrid dynamical systems and \((\varphi, \{\sigma_i\}_{i \in Z_0}) : (T, \partial_t) \to (Q, X)\) be an execution of the first system. Then the composite morphism

$$(\psi, \{\alpha_a\}) \circ (\varphi, \sigma_i) : (T, \partial_t) \to (Q, B, Y)$$

is an execution of the second system. In other words morphisms of hybrid dynamical systems send executions to executions.

Proof. By Definition 4.7 the composite morphism \((\psi, \{\alpha_a\}) \circ (\varphi, \sigma_i)\) is an execution of the system \((Q, B, Y)\). □

Example 4.10. Let $A$ be the graph with one node and one arrow: $A = \{\alpha, \Rightarrow \ast\}$. Let $Q : A \to \mathbb{R}$ be the map of graphs with

$$Q(\ast) = [0, 1]^2 \quad \text{and} \quad Q(\alpha) = \{(0, 0), (1, 1)\}. $$

We think of the relation $Q(\alpha)$ as a partial map sending $(1, 1)$ to $(0, 0)$. Let $Y : [0, 1]^2 \to \mathbb{R}^2$ be a vector field of the form

$$Y(x, y) = (f(x, y), f(y, x))$$

for some smooth function

$$f : [0, 1]^2 \to \mathbb{R}. $$

Let $R : \{\ast \Rightarrow \ast\} \to \mathbb{R}$ be the hybrid phase space of Example 3.7 and $X : [0, 1] \to \mathbb{R}$ be the vector field of the form

$$X(x) = f(x, x),$$

for the same function $f$. The map

$$\psi : \{\ast \Rightarrow \ast\} \to A, \quad \varphi(\ast \Rightarrow \ast) = \ast \Rightarrow \ast \in A$$
is a map of graphs. Define \( \tau_* : \mathcal{R}(\ast) = [0,1] \to \mathcal{Q}(\ast) \) by
\[ \tau_*(x) = (x,x). \]
It is easy to see that
\[ (\psi, \{\tau_*\}) : (\mathcal{R}, X) \to (\mathcal{Q}, Y) \] (4.1)
is a map of hybrid dynamical systems. It is also not hard to check that for any execution
\[ (\varphi, \{\sigma_i\}_{i \in \mathbb{Z}_0}) : (T : \mathcal{Z} \to \mathcal{R}, \partial_i) \to (\mathcal{R}, X) \] the composite \( (\psi, \tau_*) \circ (\varphi, \{\sigma_i\}_{i \in \mathbb{Z}_0}) \) is an execution of \((\mathcal{Q}, Y)\).

Remark 4.11. Both hybrid dynamical systems in Example 4.10 are built out of one open hybrid system. The larger system \((\mathcal{Q}, Y)\) is built by interconnecting two copies of an open system. The smaller system \((\mathcal{R}, X)\) is built by interconnecting inputs and outputs of the same open system. The existence of the map \((\psi, \{\tau_*\})\) can be seen as being induced by a map of finite sets. It is the one map from the two element set \(\{1,2\}\) to the one element set \(\{0\}\). We plan to address open hybrid systems, their interconnections and, more generally, networks of hybrid systems in future work.

5. Discussion

In this brief paper we introduced a new version of the notion of a hybrid dynamical system. We made it more compact. We also introduced the notion of a morphism (“map”) of hybrid dynamical systems. This allowed us to turn hybrid dynamical systems into a category. Our notion of morphism also allowed us to view executions as a particular kind of morphisms. Consequently morphisms of hybrid dynamical systems take executions to executions even if the hybrid systems in question are not deterministic!

In the present paper we viewed hybrid systems as generalizations of continuous time dynamical systems. There is another approach that views hybrid systems as generalizations of automata and of labeled transition systems (see, for example [14]). One important aspect of labeled transition systems is that of parallel composition that allows one to synchronize parallel transitions by way of label sharing. Since our category of hybrid dynamical systems is built on directed graphs, we fail to properly account for parallel composition of hybrid systems. However there is a fix to this problem: replace directed graphs in Definition 3.10 by labeled transition systems. We plan to address this elsewhere.

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