Self-Averaging of Perturbation Hamiltonian Density in Perturbed Spin Systems

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Received: 25 August 2019 / Accepted: 15 October 2019 / Published online: 22 October 2019
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Abstract
It is shown that the variance of a perturbation Hamiltonian density vanishes in the infinite-volume limit of perturbed spin systems with quenched disorder. This is proven in a simpler way and under less assumptions than before. A corollary of this theorem indicates the impossibility of non-spontaneous replica symmetry-breaking in disordered spin systems. The commutativity between the infinite-volume limit and the switched-off limit of a replica symmetry-breaking perturbation implies that the variance of the spin overlap vanishes in the replica symmetric Gibbs state.

Keywords Perturbation Hamiltonian · Spontaneous symmetry-breaking · Replica symmetry-breaking · Disordered spin systems · Spin overlap · Self-averaging · The law of large numbers

1 Introduction

Recently, it was proven that the variance of the perturbation Hamiltonian density vanishes in the infinite-volume limit of disordered quantum spin systems [18]. In the present paper, we give a simpler proof of this theorem under less assumptions on the models. First, we give a definition of the model and the main theorem. We study quantum spin systems on a finite set \( V_N \) with \( |V_N| = N \). Spin operators \( S_j^p \) (\( p = x, y, z \)) at a site \( j \in V_N \) acting on a Hilbert space \( \mathcal{H} := \bigotimes_{j \in V_N} \mathcal{H}_j \) are defined by a tensor product of the spin matrix acting on \( \mathcal{H}_j \cong \mathbb{R}^{2S+1} \) and unit operators, where \( S \) is an arbitrarily fixed positive semi-definite half integer. These operators are self-adjoint and satisfy the commutation relations

\[
[S_j^x, S_k^z] = i \delta_{j,k} S_j^z, \quad [S_j^y, S_k^z] = i \delta_{j,k} S_j^y, \quad [S_j^z, S_k^x] = i \delta_{j,k} S_j^y,
\]

Communicated by Hal Tasaki.

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and the spin at each site \( i \in V_N \) has a fixed magnitude
\[
\sum_{p=x,y,z} (S^p_j)^2 = S(S+1) \mathbf{1}.
\]
We define an unperturbed Hamiltonian \( H_N(S) \) first. \( \mathcal{P}(V_N) \) denotes the collection of all subsets of \( V_N \). Let \( \mathcal{E}^p_N \subset \mathcal{P}(V_N) \) be a collection of interaction ranges which are bounded subsets of \( V_N \) and denote
\[
M := | \bigcup_{p=x,y,z} \mathcal{E}^p_N |.
\]
Let \( J = (J^p_X)_{X \in \mathcal{E}^p_N, p=x,y,z} \) be a sequence of real-valued independent random variables with finite expectations \( \mathbb{E} J^p_X \in \mathbb{R} \), where \( \mathbb{E} \) denotes the expectation over \( J \).

Assume the following bound on their variances by a positive constant \( \sigma \) independent of \( N \)
\[
\sum_{p=x,y,z} \sum_{X \in \mathcal{E}^p_N} \mathbb{E} (J^p_X - \mathbb{E} J^p_X)^2 \leq \sigma^2 N.
\]
Denote a sequence of spin operators \( S^p_X := (S^p_j)_{j \in X, p=x,y,z} \) on a subset \( X \) and let \( \varphi^p_X \) be a self-adjoint-operator-valued bounded function of \( S^p_X \), such that \( ||\varphi^p_X|| \leq C \varphi \), where the operator norm is defined by \( ||O|| := \sup_{(\phi,\phi)=1} (O\phi, O\phi) \) for an arbitrary linear operator \( O \) on \( \mathcal{H} \). We consider a model defined by the following Hamiltonian with \( \mathcal{E}^p_N, \varphi^p_X \) and \( J \)
\[
H_N(S, J) := \sum_{p=x,y,z} \sum_{X \in \mathcal{E}^p_N} J^p_X \varphi^p_X (S^p_X).
\]
One can assume a symmetry of the Hamiltonian \( H_N(S, J) \), if one is interested in symmetry-breaking phenomena. To detect a spontaneous symmetry-breaking, the long-range order of an order operator is utilized in the symmetric Gibbs state. Although the symmetric Gibbs state with long-range order is mathematically well defined, such a state is unstable due to strong fluctuations and it cannot be realized. On the other hand, it is believed that a perturbed Gibbs state with an infinitesimally symmetry-breaking term in the Hamiltonian is stable and realistic. Let \( \mathcal{O}_N \) be a symmetry-breaking self-adjoint operator acting on \( \mathcal{H} \) with a uniform bound
\[
||\mathcal{O}_N|| \leq C_o,
\]
where \( C_o \) is a positive constant independent of the system size \( N \). This operator \( \mathcal{O}_N \) can perturb the model as a symmetry-breaking perturbation. At the same time, it’s Gibbs expectation can measure the symmetry-breaking as an order parameter. Define a perturbed Hamiltonian by
\[
H_\lambda := H_N(S, J) - N \lambda \mathcal{O}_N,
\]
where \( \lambda \in \mathbb{R} \). To study spontaneous symmetry-breaking, one can regard \( \mathcal{O}_N \) as an order operator which breaks the symmetry. For instance, \( \mathcal{O}_N \) is a spin density
\[
\mathcal{O}_N = \frac{1}{N} \sum_{j \in V_N} S^z_j.
\]
Define the Gibbs state with the Hamiltonian (3). For \( \beta > 0 \), the partition function is defined by
\[
Z_N(\beta, \lambda, J) := \text{Tr} e^{-\beta H_N(S, J) + \beta N \lambda \mathcal{O}_N}
\]
where the trace is taken over the Hilbert space $\mathcal{H}$. Let $f$ be an arbitrary function of spin operators. The expectation of $f$ in the Gibbs state is given by
\[
\langle f(S) \rangle_{N,\lambda} = \frac{1}{Z_N(\beta, \lambda, J)} \text{Tr} f(S) e^{-\beta H_N(S, J) + \beta N \lambda \mathcal{O}_N}.
\]
(6)

Define the following function from the partition function
\[
\psi_N(\beta, \lambda, J) := \frac{1}{N} \log Z_N(\beta, \lambda, J),
\]
(7)
and its expectation
\[
p_N(\beta, \lambda) := E \psi_N(\beta, \lambda, J).
\]
(8)

Assumption 1 has been proven for several spin models [7,14,17,19,21,25,31,32]. Any short-range Hamiltonian and a spin density as a perturbation operator $\mathcal{O}_N$ satisfy Assumption 2. In the present paper, we prove the following main theorem for an arbitrary spin model with the Hamiltonian (3) satisfying Assumptions 1 and 2. Though the proof in Reference [18] needs the assumption that the variance of $\psi_N(J)$ vanishes in the infinite-volume limit, we prove it as a lemma in the present paper.

Theorem 1 Consider a quantum spin model defined by the Hamiltonian (3) with perturbation operator $\mathcal{O}_N$ satisfying Assumptions 1 and 2. The expectation of the order operator
\[
\lim_{N \to \infty} E \langle \mathcal{O}_N \rangle_{N,\lambda}.
\]
exists in the infinite-volume limit for almost all $\lambda$ and its variance in the Gibbs state and the distribution of disorder vanishes
\[
\lim_{N \to \infty} E (\langle \mathcal{O}_N^2 \rangle - E \langle \mathcal{O}_N \rangle^2)_{N,\lambda} = 0,
\]
(12)
in the infinite-volume limit for almost all $\lambda \in \mathbb{R}$.

Theorem 1 implies also the existence of the following infinite-volume limit for almost all $\lambda \in \mathbb{R}$
\[
\lim_{N \to \infty} E (\mathcal{O}_N^2)_{N,\lambda} = \left( \lim_{N \to \infty} E \langle \mathcal{O}_N \rangle_{N,\lambda} \right)^2.
\]
(13)
The perturbation operator $\mathcal{O}_N$ is self-averaging in the perturbed model. Although Theorem 1 is physically natural and physicists believe it by their experiences supported by lots of examples, it has never been proven rigorously under Assumptions 1 and 2. In Sect. 2, we prove Theorem 1. In Sect. 3, we apply Theorem 1 to spontaneous symmetry-breaking phenomena in some examples. Theorem 1 indicates that replica symmetry-breaking in disordered spin systems should be a spontaneous symmetry-breaking.

2 Proof

To prove the following lemmas, here, we define the Duhamel product. Define an imaginary time $t \in [0, 1]$ and a time evolution of operators with the Hamiltonian. Let $O$ be an arbitrary self-adjoint operator, and we define an operator-valued function $O(t)$ of $t \in [0, 1]$ by

$$O(t) := e^{-tH} O e^{tH}. \quad (14)$$

The Duhamel product of time-independent operators $O_1, O_2, \ldots, O_k$ is defined by

$$(O_1, O_2, \ldots, O_k)_x := \int_{[0,1]^k} dt_1 \ldots dt_k \langle T[O_1(t_1)O_2(t_2) \ldots O_k(t_k)] \rangle_{N,x}, \quad (15)$$

where the symbol $T$ is a multilinear mapping of the chronological ordering. If we define a partition function with arbitrary self-adjoint operators $O_1, \ldots, O_k$ and real numbers $x_1, \ldots, x_k$

$$Z(x_1, \ldots, x_k) := \text{Tr} \exp \beta \left[ -H + \sum_{i=1}^k x_i O_i \right], \quad (16)$$

the Duhamel product of $k$ operators represents the $k$-th order derivative of the partition function $[8, 12, 26]$

$$\beta^k(O_1, \ldots, O_k)_x = \frac{1}{Z(x)} \frac{\partial^k Z(x)}{\partial x_1 \ldots \partial x_k}. \quad (17)$$

Furthermore, a truncated Duhamel product is defined by

$$\beta^k(O_1; \ldots; O_k)_x = \frac{\partial^k}{\partial x_1 \ldots \partial x_k} \log Z(x). \quad (18)$$

Note that the Duhamel product of a single operator is identical to its Gibbs expectation

$$\beta(O_1)_N, x = \beta(O_1)_x = \frac{1}{Z(x)} \frac{\partial Z(x)}{\partial x_1} = \frac{\partial}{\partial x_1} \log Z(x). \quad (19)$$

Lemma 1 There exists a positive number $K$ independent of the system size $N$, such that the variance of $\psi_N$ is bounded by

$$E \psi_N(\beta, \lambda, J)^2 - p_N(\beta, \lambda)^2 \leq \frac{K}{N}, \quad (20)$$

for any $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$.

Proof Fix a bijection $\mathcal{U}_{p=x,y,z} \mathcal{E}_N \rightarrow [1, M] \cap \mathbb{Z}$ arbitrarily for identification $\mathcal{U}_{p=x,y,z} \mathcal{E}_N = [1, M] \cap \mathbb{Z}$ for $M := | \cup_{p=x,y,z} \mathcal{E}_N |$, such that this bijection numbers the sequence $J$ as $(J_X^p)_{X \in \mathcal{E}_N, p=x,y,z} = (J_j)_{j=1,2,\ldots,M}$.

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For an integer $1 \leq m \leq M$, define a symbol $E_m$ which denotes the expectation over random variables $(J_j)_{j=m}$. Note that $E_0 = E$ is the expectation over all random variables $(J_j)_{j=1,2,\ldots,M}$, and $E_M$ is the identity. Here, we represent $\psi_N(J)$ as a function of a sequence of random variables $J = (J_j)_{j=1,\ldots,M}$ for lighter notation.

\[
E\psi_N(J)^2 - (E\psi_N(J))^2 = E(E_M\psi_N(J))^2 - (E_0\psi_N(J))^2 = \sum_{m=1}^M E[(E_m\psi_N(J))^2 - (E_{m-1}\psi_N(J))^2].
\]

In the $m$-th term, regard $\psi_N(J_m)$ as a function of $J_m$ for simplicity. Let $J_m'$ be an independent random variable satisfying the same distribution as that of $J_m$, and $E'$ denotes an expectation over only $J_m'$. In this notation,

\[
E_{m-1}\psi_N(J_m) = E_{m-1}\psi_N(J) = E_{m-1}\psi_N(J_1, \ldots, J_{m-1}, J_m, J_{m+1}, \ldots, J_M) = E_mE'(\psi_N(J_1, \ldots, J_{m-1}, J_m', J_{m+1}, \ldots, J_M).
\]

Therefore, a bound on the $m$-th term is given by

\[
E[(E_m\psi_N(J_m))^2 - (E_{m-1}\psi_N(J_m))^2] = E[E_mE'(\psi_N(J_m) - \psi_N(J_m'))^2]
\]

\[
= \frac{1}{N^2} E[E'E' \int_{J_m'}^J dJ \beta \partial_J \psi_N(J)]^2
\]

\[
\leq \frac{1}{N^2} \beta^2 C_\psi^2 \frac{(J_m - J_m')^2}{J_m - J_m'} \int_{J_m'}^J dJ \beta \psi_N(S_m) \leq \frac{\beta^2 C_\psi^2}{N^2} \int_{J_m'}^J dJ \beta \psi_N(S_m).
\]

where we have used Jensen’s inequality. Note that there exist $p = x, y, z$ and $X \in \mathcal{C}_N^\beta$ and $S_m = S^\beta_X$.

Therefore

\[
E\psi_N(J)^2 - (E\psi_N(J))^2 \leq \sum_{m=1}^M \frac{2\beta^2 C_\psi^2 \sigma^2}{N^2} \leq \frac{2\beta^2 C_\psi^2 \sigma^2}{N}.
\]

This completes the proof, if $K$ denotes $2\beta^2 C_\psi^2 \sigma^2$. \qed
The following lemma can be shown by the standard convexity argument to imply the Ghirlanda-Guerra identities \([1,3,6,11,17,18,23,29]\) in classical and quantum disordered systems. The proof can be done on the basis of convexity of functions \(\psi_N, p_N, p\) and their almost everywhere differentiability and Assumptions 1 and 2.

**Lemma 2** For almost all \(\lambda \in \mathbb{R}\), the infinite-volume limit of the expectation of \(\mathcal{O}_N\)

\[
\lim_{N \to \infty} \mathbb{E}\langle \mathcal{O}_N \rangle_{N, \lambda} = \frac{1}{\beta} \frac{\partial p}{\partial \lambda}(\beta, \lambda)
\]

exists and the following variance vanishes

\[
\lim_{N \to \infty} [\mathbb{E}\langle \mathcal{O}_N \rangle_{N, \lambda} - (\mathbb{E}\langle \mathcal{O}_N \rangle_{N, \lambda})^2] = 0,
\]

for each \(\beta \in (0, \infty)\).

**Proof** First, let us regard \(p_N(\lambda), p(\lambda)\) and \(\psi_N(\lambda)\) as functions of \(\lambda\) for lighter notation. Define the following functions of \(\epsilon > 0\).

\[
\begin{align*}
w_N(\epsilon) &:= \frac{1}{\epsilon} [\psi_N(\lambda + \epsilon) - p_N(\lambda + \epsilon) + |\psi_N(\lambda - \epsilon) - p_N(\lambda - \epsilon)| + |\psi_N(\lambda) - p_N(\lambda)|], \\
e_N(\epsilon) &:= \frac{1}{\epsilon} [p_N(\lambda + \epsilon) - p(\lambda + \epsilon) + |p_N(\lambda - \epsilon) - p(\lambda - \epsilon)| + |p_N(\lambda) - p(\lambda)|].
\end{align*}
\]

Assumption 1 and Lemma 1 give

\[
\lim_{N \to \infty} \mathbb{E}w_N(\epsilon) = 0, \quad \lim_{N \to \infty} e_N(\epsilon) = 0,
\]

for any \(\epsilon > 0\). Since \(\psi_N, p_N\) and \(p\) are convex functions of \(\lambda\), we have

\[
\begin{align*}
\frac{\partial \psi_N}{\partial \lambda} (\lambda) - \frac{\partial p}{\partial \lambda} (\lambda) &\leq \frac{1}{\epsilon} [\psi_N(\lambda + \epsilon) - \psi_N(\lambda)] - \frac{\partial p}{\partial \lambda} (\lambda) \\
&\leq \frac{1}{\epsilon} [\psi_N(\lambda + \epsilon) - p_N(\lambda + \epsilon) + p_N(\lambda + \epsilon) - p_N(\lambda) + p_N(\lambda) - \psi_N(\lambda) \\
&\quad - p(\lambda + \epsilon) + p(\lambda + \epsilon) + p(\lambda) - p(\lambda)] - \frac{\partial p}{\partial \lambda} (\lambda) \\
&\leq \frac{1}{\epsilon} [||\psi_N(\lambda + \epsilon) - p_N(\lambda + \epsilon)|| + |p_N(\lambda) - \psi_N(\lambda)| + |p_N(\lambda + \epsilon) - p(\lambda)|] \\
&\quad + |p_N(\lambda) - p(\lambda)|] + \frac{1}{\epsilon} [p(\lambda + \epsilon) - p(\lambda)] - \frac{\partial p}{\partial \lambda} (\lambda) \\
&\leq w_N(\epsilon) + e_N(\epsilon) + \frac{\partial p}{\partial \lambda} (\lambda + \epsilon) - \frac{\partial p}{\partial \lambda} (\lambda).
\end{align*}
\]

As in the same calculation, we have

\[
\begin{align*}
\frac{\partial \psi_N}{\partial \lambda} (\lambda) - \frac{\partial p}{\partial \lambda} (\lambda) &\geq \frac{1}{\epsilon} [\psi_N(\lambda) - \psi_N(\lambda - \epsilon)] - \frac{\partial p}{\partial \lambda} (\lambda) \\
&\geq -w_N(\epsilon) - e_N(\epsilon) + \frac{\partial p}{\partial \lambda} (\lambda - \epsilon) - \frac{\partial p}{\partial \lambda} (\lambda).
\end{align*}
\]

Then,

\[
\mathbb{E} \left| \frac{\partial \psi_N}{\partial \lambda} (\lambda) - \frac{\partial p}{\partial \lambda} (\lambda) \right| \leq \mathbb{E}w_N(\epsilon) + e_N(\epsilon) + \frac{\partial p}{\partial \lambda} (\lambda + \epsilon) - \frac{\partial p}{\partial \lambda} (\lambda - \epsilon).
\]
Convergence of $p_N$ in the infinite-volume limit implies
\[
\lim_{N \to \infty} E \left| \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda} - \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda + \epsilon} - \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda - \epsilon} \right| \leq \left| \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda + \epsilon} - \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda - \epsilon} \right|.
\] (46)
The right-hand side vanishes, since the convex function $p(\lambda)$ is continuously differentiable almost everywhere and $\epsilon > 0$ is arbitrary. Therefore
\[
\lim_{N \to \infty} \left| \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda} - \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda + \epsilon} - \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda - \epsilon} \right| = 0.
\] (47)
for almost all $\lambda$. Jensen’s inequality gives
\[
\lim_{N \to \infty} \left| \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda} - \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda + \epsilon} - \frac{\partial}{\partial \lambda} \beta \langle \mathcal{O}_N \rangle_{N, \lambda - \epsilon} \right| = 0.
\] (48)
This implies the first equality (35). Since the $p(\lambda)$ is continuously differentiable almost everywhere in $\mathbb{R}$, these equalities imply also
\[
\lim_{N \to \infty} E \left| \langle \mathcal{O}_N \rangle_{N, \lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda} \right| = 0.
\] (49)
The bound on $\mathcal{O}_N$ implies the following limit
\[
\lim_{N \to \infty} E \left[ \left( \langle \mathcal{O}_N \rangle_{N, \lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda} \right)^2 \right] \leq 2C_0 \lim_{N \to \infty} \left| \langle \mathcal{O}_N \rangle_{N, \lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda} \right| = 0.
\] (50)
This completes the proof. \(\square\)

Note that Lemma 2 guarantees the existence of the following infinite-volume limit for almost all $\lambda \in \mathbb{R}$
\[
\lim_{N \to \infty} E \langle \mathcal{O}_N \rangle_{N, \lambda}^2 = \left( \lim_{N \to \infty} E \langle \mathcal{O}_N \rangle_{N, \lambda} \right)^2.
\] (51)

**Lemma 3** For almost all $\lambda \in \mathbb{R}$, the following variance of $\mathcal{O}_N$ vanishes in the infinite-volume limit
\[
\lim_{N \to \infty} E \left[ \left( \langle \mathcal{O}_N \rangle_{N, \lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda} \right)^2 \right]_{N, \lambda} = 0.
\] (52)

**Proof** The derivative of the Gibbs expectation is represented in the Duhamel product
\[
\frac{\partial}{\partial \lambda} \langle \mathcal{O}_N \rangle_{N, \lambda} = N \beta \mathbb{E}[(\mathcal{O}_N, \mathcal{O}_N)_{N, \lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda}^2].
\] (53)
The integration of both sides in (53) over an arbitrary interval $(\lambda', \lambda'')$ with $\lambda' < \lambda''$ gives
\[
\mathbb{E} \langle \mathcal{O}_N \rangle_{N, \lambda''} - \mathbb{E} \langle \mathcal{O}_N \rangle_{N, \lambda'} = N \beta \int_{\lambda'}^{\lambda''} d\lambda \mathbb{E}[(\mathcal{O}_N, \mathcal{O}_N)_{N, \lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda}^2].
\] (54)
Since $\mathcal{O}_N$ has a uniform bound, we have
\[
\int_{\lambda'}^{\lambda''} d\lambda \mathbb{E}[(\mathcal{O}_N, \mathcal{O}_N)_{\lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda}^2] = \frac{1}{N \beta} \mathbb{E}[(\mathcal{O}_N, \mathcal{O}_N)_{\lambda''} - \langle \mathcal{O}_N \rangle_{N, \lambda'}] \leq \frac{2C_0}{N \beta}.
\] (55)
The positive semi-definiteness of the integrand gives
\[
\lim_{N \to \infty} \int_{\lambda'}^{\lambda''} d\lambda \mathbb{E}[(\mathcal{O}_N, \mathcal{O}_N)_{N, \lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda}^2] = 0.
\] (56)
The integrand has uniform bounds
\[ 0 \leq \mathbb{E}[\theta_N, \theta_N, \lambda - \langle \theta_N \rangle_N, \lambda, \lambda^2] \leq \mathbb{E}[\theta_N^2]_{N, \lambda} \leq C_0^2, \]
thus Lebesgue’s dominated convergence theorem gives
\[
\int_{\lambda'}^{\lambda''} d\lambda \limsup_{N \to \infty} \mathbb{E}[\theta_N, \theta_N, \lambda - \langle \theta_N \rangle_N, \lambda, \lambda^2] = \lim_{N \to \infty} \int_{\lambda'}^{\lambda''} d\lambda \mathbb{E}[\theta_N, \theta_N, \lambda - \langle \theta_N \rangle_N, \lambda, \lambda^2] = 0.
\]
(58)
Since the integration interval \((\lambda', \lambda'')\) is arbitrary, we have
\[
\lim_{N \to \infty} \mathbb{E}[\theta_N, \theta_N, \lambda - \langle \theta_N \rangle_N, \lambda, \lambda^2] = 0,
\]
(59)
for almost all \(\lambda \in \mathbb{R}\). Harris’ inequality of the Bogolyubov type between the Duhamel function and the Gibbs expectation of the square of arbitrary self-adjoint operator \(\theta_N\) [2,16] reads
\[
\langle \theta_N^2 \rangle_{\lambda} - \frac{\beta}{12} \langle [\theta_N, [H_\lambda, \theta_N]] \rangle_{\lambda} \leq \langle \theta_N, \theta_N, \lambda, \lambda \rangle \leq \langle \theta_N^2 \rangle_{\lambda}.
\]
(60)
These inequalities, Assumption 2 and the limit (59) imply
\[
\lim_{N \to \infty} \mathbb{E}[\langle (\theta_N - \langle \theta_N \rangle_N, \lambda, \lambda \rangle^2 \rangle_{\lambda} = \lim_{N \to \infty} \mathbb{E}[\langle \theta_N^2 \rangle_{\lambda} - \langle \theta_N \rangle_{\lambda, \lambda}^2] = 0.
\]
(61)
This completes the proof.

**Proof of Theorem 1** Lemmas 2 and 3 imply
\[
\lim_{N \to \infty} \mathbb{E}[\langle \theta_N^2 \rangle_{\lambda} - \langle \mathbb{E}[\theta_N] \rangle_{\lambda}^2] = \lim_{N \to \infty} \mathbb{E}[\langle \theta_N^2 \rangle_{\lambda} - \langle \mathbb{E}[\theta_N] \rangle_{\lambda}^2 + \langle \theta_N \rangle_{\lambda}^2 - \langle \mathbb{E}[\theta_N] \rangle_{\lambda}^2] = 0.
\]
(62)
This completes the proof.

## 3 Applications

### 3.1 Spontaneous Symmetry-Breaking in Quantum Spin Systems

First, we remark the general properties of spontaneous symmetry-breaking in quantum spin systems without disorder. Spontaneous symmetry-breaking phenomena are observed generally in many-body systems possessing some symmetries. It is well-known that the quantum Heisenberg model gives a simple example of spontaneous symmetry-breaking. The SU(2) symmetry-breaking occurs in the ferromagnetic (conjectured) or antiferromagnetic (proven) phases in the Heisenberg model. First, we give a brief review of the relation between the long-range order and the spontaneous symmetry-breaking. Then, we describe the property of the symmetric and the symmetry-breaking Gibbs states from the point of view of Theorem 1 and the law of large numbers which plays a fundamental role in statistical physics.

Define a Hamiltonian \(H_N(S)\) of the Heisenberg model with a sequence of coupling constants \(J = (J_{i,j})_{(i,j) \in B_N; p=x,y,z}\)
\[
H_N(S, J) := \sum_{(i,j) \in B_N} J_{i,j} \sum_{p=x,y,z} S_i^p S_j^p,
\]
(63)
where $B_N$ is a set of bonds which consists of $i, j \in V_N$ under a certain condition, such as nearest-neighbor interaction $|i - j| = 1$. The Hamiltonian is invariant under an arbitrary SU(2) transformation $U = \bigotimes_{j \in V_N} U_j$

$$H_N(U SU^{-1}, \mathbf{J}) = H_N(\mathbf{S}, \mathbf{J}),$$

where $U_j$ is in SU(2) acting on $H_j$. The partition function is defined on the basis of this Hamiltonian

$$Z_N(\beta, \mathbf{J}) := \text{Tr} e^{-\beta H(\mathbf{S}, \mathbf{J})}.$$  

The SU(2) symmetric Gibbs expectation for an arbitrary function $f(\mathbf{S})$ of spin operators is

$$\langle f(\mathbf{S}) \rangle_N = \frac{1}{Z_N(\beta, \mathbf{J})} \text{Tr} f(\mathbf{S}) e^{-\beta H(\mathbf{S}, \mathbf{J})}.$$  

To detect the symmetry-breaking, define an order operator with a sequence $a_j \in \mathbb{R}$ for $p = x, y, z$

$$\mathcal{O}_N := \frac{1}{N} \sum_{j \in V_N} a_j S_j^p.$$  

For example, the sequence $a_j \in \mathbb{R}$ is defined by $a_j = 1$ for the ferromagnetic order, and by $a_j = (-1)^{j_1 + \cdots + j_d}$ for antiferromagnetic order at $j = (j_1, \ldots, j_d) \in V_N$ in $d$ dimensional cubic lattice $V_N = [1, L]^d \cap \mathbb{Z}^d$. The order operator transforms $U \mathcal{O}_N U^{-1} \neq \mathcal{O}_N$ under an arbitrary SU(2) transformation $U$ and it has a uniform bound $\| \mathcal{O}_N \| \leq C_0$ with a positive constant independent of the system size $N$. Then, the SU(2) symmetric Gibbs expectation of the order operator vanishes

$$\langle \mathcal{O}_N \rangle_N = 0.$$  

Dyson, Lieb and Simon proved that the symmetric Gibbs expectation of the square of the order operator does not vanishes for $d \geq 3$ with the antiferromagnetic nearest-neighbor interactions and for sufficiently large $\beta > 0$

$$\lim_{N \to \infty} \langle \mathcal{O}_N^2 \rangle_N > 0,$$

in the infinite-volume limit [9]. This phenomenon is long-range order. For the SU(2) invariant ferromagnetic interaction, the existence of the long-range order is believed, nonetheless it has never been proven at finite temperature. The long-range order and the vanishing Gibbs expectation of the order operator imply that the variance of $\mathcal{O}_N$ in the symmetric Gibbs state does not vanish in the infinite-volume limit. It is believed that the expectation value calculated in the Gibbs state can be identical to its observed value, if its variance vanishes. This is guaranteed by the law of large numbers, which is proven by the Chebyshev inequality. If the variance becomes finite, the Paley-Zygmund inequality rules out its identification between the expectation value and the observed value. It is believed that the symmetry-breaking occurs instead of the violation of the law of large numbers. To consider the spontaneous symmetry-breaking, apply a symmetry-breaking perturbation in the Hamiltonian

$$H_\lambda := H(\mathbf{S}, \mathbf{J}) - N \lambda \mathcal{O}_N,$$

and define an expectation of the function $f(\mathbf{S})$ in a symmetry-breaking Gibbs state

$$\langle f(\mathbf{S}) \rangle_{N, \lambda} := \frac{1}{Z_N(\beta, \lambda, \mathbf{J})} \text{Tr} f(\mathbf{S}) e^{-\beta H_\lambda}.$$  

@article{springer, title={Self-Averaging of Perturbation Hamiltonian Density}, volume={1071}, pages={353-24123}, }
where the partition function is
\[ Z_N(\beta, \lambda, J) := \text{Tr}_f(S) e^{-\beta H_f}. \]
If spontaneous symmetry-breaking occurs, the following expectation of an order operator
\[ \lim_{\lambda \downarrow 0} \lim_{N \to \infty} \langle \mathcal{O}_N \rangle_{N, \lambda} \neq 0, \tag{70} \]
even exists as a non-zero value in the infinite-volume and switched-off limits. In this case, the two limiting procedures do not commute
\[ 0 \neq \lim_{\lambda \downarrow 0} \lim_{N \to \infty} \langle \mathcal{O}_N \rangle_{N, \lambda} \neq \lim_{N \to \infty} \lim_{\lambda \downarrow 0} \langle \mathcal{O}_N \rangle_{N, \lambda} = 0. \tag{71} \]

The Griffiths–Koma–Tasaki theorem indicates the inequality between the long-range order and the spontaneous symmetry-breaking \([13,20]\). The standard deviation of an order operator \( \mathcal{O}_N \) in the symmetric Gibbs state is bounded by its expectation in the switched-off limit of the symmetry-breaking perturbation after the infinite-volume limit.
\[ \lim_{N \to \infty} \sqrt{\langle \mathcal{O}_N^2 \rangle_{N, 0}} \leq \lim_{\lambda \downarrow 0} \lim_{N \to \infty} \langle \mathcal{O}_N \rangle_{N, \lambda}. \tag{72} \]

The existence of the long-range order in the symmetric Gibbs state implies the existence of corresponding spontaneous symmetry-breaking. Theorem 1 shows that the variance of the order operator vanishes always in the symmetry-breaking Gibbs state
\[ \lim_{N \to \infty} \langle \mathcal{O}_N^2 \rangle_{N, \lambda} - \langle \mathcal{O}_N \rangle_{N, \lambda}^2 = 0, \tag{73} \]
since the concerned model satisfies Assumptions 1 and 2. Theorem 1 and the existence of long-range order in the symmetric Gibbs state imply the non-commutativity between the infinite-volume limit \( N \to \infty \) and switched-off limit \( \lambda \downarrow 0 \), since the above infinite-volume limit is valid also in the switched off limit of the uniform field \( \lambda \downarrow 0 \) or \( \lambda \uparrow 0 \). In addition to this fact, the expectation \( \langle \mathcal{O}_N \rangle_{N, \pm 0} \) in the symmetry-breaking Gibbs state should be identical to the corresponding observed value of \( \mathcal{O} \) according to the law of large numbers. The symmetry-breaking should occur instead of the violation of the law of large numbers, because of the limit (73).

### 3.2 Replica Symmetry-Breaking in Disordered Spin Systems

Replica symmetry-breaking (RSB) phenomena have been studied in extensive fields in science in addition to statistical physics, since Parisi found the replica symmetry-breaking formula for the Sherrington-Kirkpatrick (SK) model [27] which gave a great breakthrough in the theory of disordered spin systems [24]. In the low-temperature phase of the SK model, the distribution of the spin overlap becomes broadened, which shows RSB. Since Talagrand proved the Parisi formula rigorously [28,29], mathematicians have been studying RSB phenomena in many mathematical systems. Recently, Chatterjee has proven that there is no RSB phase in the random field Ising model [3]. He defines RSB in terms of the replica symmetric Gibbs state and the sample distribution, then he has proven that the variance of the spin overlap vanishes in the replica symmetric Gibbs state and the sample expectation. Note that Chatterjee’s definition does not exclude a possibility of RSB without spontaneous symmetry-breaking. Here, we give an extension of Chatterjee’s definition of RSB and prove that RSB should occur as a spontaneous symmetry-breaking phenomenon in disordered quantum spin systems.
systems. To study spontaneous RSB, we apply a RSB perturbation to the system, as discussed in the random energy model [15,22]. Then, we prove that the commutativity between the infinite-volume limit and the replica symmetric limit implies the absence of RSB in Chatterjee’s definition.

Consider \( n \) replicated spin operators \((S^p, \alpha_i)_{i \in V N; p = x, y, z; \alpha = 1, \ldots, n}\) and replica symmetric Hamiltonian

\[
H(S^1, \ldots, S^n, J) := \sum_{\alpha=1}^{n} H_N(S^\alpha, J). \tag{74}
\]

The replica symmetry is a permutation symmetry among these replicated spins. The replica symmetry is the invariance of the Hamiltonian under an arbitrary permutation \( \sigma \) on \( \{1, \ldots, n\} \)

\[
H(S^1, \ldots, S^n, J) = H(S^{\sigma 1}, \ldots, S^{\sigma n}, J).
\]

Define a spin overlap by

\[
R^p_{\alpha, \beta} := \frac{1}{|D^p_N|} \sum_{X \in D^p_N} S^\alpha_X S^\beta_X. \tag{75}
\]

where \( D^p_N \) is a certain collection of subsets \( X \subset V_N \). Note that the replica symmetric Gibbs expectation of the overlap becomes

\[
\langle R^p_{\alpha, \beta} \rangle_N = \frac{1}{|D^p_N|} \sum_{X \in D^p_N} \langle S^\alpha_X S^\beta_X \rangle_N = \frac{1}{|D^p_N|} \sum_{X \in D^p_N} \langle S^p_X \rangle_N^2. \tag{76}
\]

For \( D^p_N = V_N \) with \( p = z \) in the Ising model, this Gibbs expectation is the Edwards-Anderson spin glass order parameter [10], as mathematically studied by van Enter and Griffiths [30]. Define a replica symmetric Gibbs state with the Hamiltonian (74). For \( \beta > 0 \), the partition function is defined by

\[
Z_{N, n}(\beta, J) := \text{Tr} e^{-\beta H(S^1, \ldots, S^n, J)}. \tag{77}
\]

where the trace is taken over the Hilbert space \( \bigotimes_{\alpha=1}^{n} \mathcal{H} \). Let \( f \) be an arbitrary function of \( n \) replicated spin operators. The expectation of \( f \) in the Gibbs state is given by

\[
\langle f(S^1, \ldots, S^n) \rangle_N = \frac{1}{Z_{N, n}(\beta, J)} \text{Tr} f(S^1, \ldots, S^n) e^{-\beta H(S^1, \ldots, S^n, J)}. \tag{78}
\]

Let us define RSB in an extension of Chatterjee’s definition [3]. For simplicity, we assume that the Gibbs state has only replica symmetry. If the Hamiltonian has other symmetry, we remove it from the Gibbs state by some suitable perturbation or boundary condition.

**Definition 1** Consider the replica symmetric Gibbs state of a disordered quantum spin model defined by the replica symmetric Hamiltonian (74) and assume that the Gibbs state has no other symmetry. We say that replica symmetry-breaking (RSB) does not occur in Chatterjee’s sense, if the variance of any RSB order operator \( R \) whose expectation value \( E \langle R \rangle_N \), exists in the infinite-volume limit, vanishes in the infinite-volume limit

\[
\lim_{N \to \infty} E((R - E \langle R \rangle_N)^2)_N = 0. \tag{79}
\]
This variance is decomposed into the following two terms
\[ E\langle (\mathcal{R} - E(\mathcal{R})_N, )^2 \rangle_N = E\langle (\mathcal{R} - E(\mathcal{R})_N, )^2 \rangle_N + E\langle (\mathcal{R})_N, -E(\mathcal{R})_N, )^2 \rangle_N. \] (80)

The finiteness of the first term in the right-hand side implies that the observed value of RSB order operator differs from its replica symmetric Gibbs state in some samples, then this means spontaneous RSB. This seems like the long-range order in the SU(2) invariant Gibbs state in the Heisenberg model, when spontaneous SU(2) symmetry-breaking occurs [13,20]. On the other hand, the finiteness of the second term implies that the replica symmetric Gibbs expectation of the RSB order operator in an arbitrary sample differs from its sample expectation. In classical Ising systems with Gaussian disorder and with \( D_p N = C_p N \), such as the SK model, the Ghirlanda–Guerra identities [1,3–6,11] give the relation between two terms [3]
\[ E\langle (R^c_{1,2} - \langle R^c_{1,2} \rangle_N, )^2 \rangle_N = \frac{2}{3} E\langle (R^c_{1,2} - E\langle R^c_{1,2} \rangle_N, )^2 \rangle_N. \] (81)

This identity implies that a non-zero value of the right-hand side is equivalent to a non-zero value of the left hand side. Therefore, replica symmetry-breaking occurs always as a spontaneous symmetry-breaking in this case. Next, we consider general cases, for example \( D_p N \neq C_p N \) or disordered quantum spin systems on the basis of Theorem 1.

To study spontaneous replica symmetry-breaking, we apply a RSB order operator as a perturbation to the replica symmetric Hamiltonian. Let \( \mathcal{R} \) be a RSB perturbation operator which is self-adjoint operator with a uniform bound \( \| \mathcal{R} \| \leq C_R \). Define a perturbed Hamiltonian by
\[ H_\lambda(S_1, \ldots, S_n, J) = H(S^1, \ldots, S^n, J) - N\lambda R, \] (82)
with coupling constants \( \lambda \in \mathbb{R} \). For example, \( \mathcal{R} \) is given by a linear combination of functions of spin overlaps
\[ \mathcal{R} = \sum_{a \in A} c_a (R^p_{1,2})^a. \] (83)

Define the Gibbs state with the Hamiltonian (82). For \( \beta > 0 \), the partition function is defined by
\[ Z_{N,n}(\beta, \lambda, J) := \text{Tr} e^{-\beta H_\lambda(S_1, \ldots, S_n, J)}. \] (84)

Let \( f \) be an arbitrary function of \( n \) replicated spin operators. The expectation of \( f \) in the Gibbs state is given by
\[ \langle f(S^1, \cdots, S^n) \rangle_{N,\lambda} = \frac{1}{Z_{N}(\beta, \lambda, J)} \text{Tr} f(S^1, \cdots, S^n) e^{-\beta H_\lambda(S_1, \cdots, S_n, J)}. \] (85)

Define the following function from the partition function
\[ \psi_{N,n}(\beta, \lambda, J) := \frac{1}{N} \log Z_{N,n}(\beta, \lambda, J). \]
and its expectation
\[ p_{N,n}(\beta, \lambda) := E\psi_{N,n}(\beta, \lambda, J). \]

Next we remark a property of spontaneous replica symmetry-breaking.
**Note 1** Consider a RSB Gibbs state of a disordered quantum spin model defined by the Hamiltonian (82), and assume that the Gibbs state at $\lambda = 0$ has no other symmetry. We say that a spontaneous RSB does not occur if the following two limiting procedures commute for any RSB perturbation operator $R$ whose expectation exists in the infinite-volume limit for almost all coupling constant $\lambda \in \mathbb{R}$,

$$\lim_{N \to \infty} \lim_{\lambda \to 0} E\langle R \rangle_{N,\lambda} = \lim_{\lambda \to 0} \lim_{N \to \infty} E\langle R \rangle_{N,\lambda}. \tag{86}$$

Then, the following corollary is obtained from Theorem 1.

**Corollary 1** Consider a disordered quantum spin model defined by the Hamiltonian (82) with a perturbation operator $\Theta_N$ defined by a RSB order operator $R =: O_N$ satisfying Assumptions 1 and 2, and assume that the Hamiltonian (82) has no other symmetry at $\lambda = 0$. If spontaneous replica symmetry-breaking (RSB) does not occur, then RSB does not occur in Chatterjee’s sense either.

**Proof** We prove the contrapositive. In the Hamiltonian (82), Theorem 1 gives that the infinite-volume limit of expectation value $E\langle R \rangle_{N,\lambda}$ exists, and the variance of the perturbation operator $R$ vanishes

$$\lim_{N \to \infty} E\langle (R - E\langle R \rangle_{N,\lambda})^2 \rangle_{N,\lambda} = 0, \tag{87}$$

for almost all $\lambda \in \mathbb{R}$. If spontaneous RSB does not occur, then the two limiting procedures for $E\langle R \rangle_{N,\lambda}$ and $E\langle R^2 \rangle_{N,\lambda}$ commute, as remarked in Note 1. Therefore,

$$\lim_{N \to \infty} [E\langle R^2 \rangle_{N,\lambda} - (E\langle R \rangle_{N,\lambda})^2] = \lim_{\lambda \to 0} \lim_{N \to \infty} [E\langle R^2 \rangle_{N,\lambda} - (E\langle R \rangle_{N,\lambda})^2] = 0, \tag{88}$$

for almost all $\lambda \in \mathbb{R}$. Since $R$ is arbitrary, RSB does not occurs in Chatterjee’s sense. \qed

Note that Assumption 1 can be proven in several short-range interacting spin models for only $A := \{1\}$ in the definition (83) of $\mathcal{R}$. Corollary 1 is valid under Assumption 1. A quantum version of the Sherrington-Kirkpatrick model is worth studying from the view point of this paper still, since such a quantum spin model with a long-range interaction is not covered by the analysis in the present paper.

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