A Note on a Problem of Erdős and Rothschild

Aaron Potechin
MIT
potechin@mit.edu

December 8, 2014

Abstract

A set of $q$ triangles sharing a common edge is called a book of size $q$. Letting $bk(G)$ denote the size of the largest book in a graph $G$, Erdős and Rothschild [6] asked what the minimal value of $bk(G)$ is for graphs $G$ with $n$ vertices and a set number of edges where every edge is contained in at least one triangle. In this paper, we show that for any graph $G$ with $n$ vertices and $\frac{n^2}{4} - nf(n)$ edges where every edge is contained in at least one triangle, $bk(G) \geq \Omega \left( \min \left\{ \frac{n}{\sqrt{f(n)}}, \frac{n^2}{f(n)^2} \right\} \right)$.

Acknowledgement:
This material is based on work supported by the National Science Foundation Graduate Research Fellowship under Grant No. 0645960.
1 Introduction

A set of \( q \) triangles sharing a common edge is called a book of size \( q \). Erdős [5] started the study of books in graphs and this study has since attracted a great deal of attention in extremal graph theory (see e.g. [2], [9], [10], [13] and graph Ramsey theory (see e.g. [11], [14], [15], [16], [17], [18], [20]).

Erdős and Rothschild [6] considered the problem of bounding \( h(n, c) \), which is defined as follows.

Definition 1.1.

1. Let \( bk(G) \) denote the size of the largest book in a graph \( G \).
2. Let \( h(n, c) \) denote the minimum value of \( bk(G) \) over all graphs on \( n \) vertices with more than \( cn^2 \) edges such that every edge is contained in at least one triangle.

This problem received considerable attention (see e.g. the Erdős problem papers [6], [7], [8] and the book [3]). In terms of lower bounds, Szemerédi used his regularity lemma to show that for all fixed \( c < \frac{1}{7} \), \( h(n, c) \to \infty \) as \( n \to \infty \). Ruzsa and Szemerédi [19] further showed that this implies Roth’s theorem, that every subset of \([1, n]\) without a 3-term arithmetic progression has size \( o(n) \), as well as the \((6,3)\)-theorem which states that every 3-uniform hypergraph on \( n \) vertices in which the union of the endpoints of any 3 edges has at least 6 vertices must have \( o(n^2) \) edges. Fox (see the end of the introduction of [12]) recently strengthened the quantitative bounds on \( h(n, c) \) to be \( 2^{\Omega \left( \log^* n \right)} \) rather than \( (\log^* n)^{\Omega(1)} \). For the case \( c \geq \frac{1}{4} \), Edwards [4] and Khadžiivanov, Nikiforov [13] independently showed that \( h(n, c) \geq \frac{n}{100} \) for all \( c \geq \frac{1}{7} \).

In terms of upper bounds, Alon and Trotter (see [8]) showed that for any \( c < \frac{1}{6} \), \( h(n, c) \leq O(\sqrt[n]{n}) \). Fox and Loh [12] recently strengthened this to show that for any fixed \( c < \frac{1}{7} \), \( h(n, c) \leq \sqrt[n]{n} \). Thus, there is a threshold for this problem at \( c = \frac{1}{4} \).

1.1 Previous Work and Our Results

In this paper we examine what happens at this threshold by posing the problem as follows.

Definition 1.2. Given a function \( f : \mathbb{Z}^+ \to \mathbb{R}^+ \), define \( \gamma(n, f) \) to be the minimal value of \( bk(G) \) over all graphs with \( n \) vertices and at least \( \left\lceil \frac{n^2}{4} - nf(n) \right\rceil \) edges such that every edge is contained in at least one triangle.

Bollobás and Nikiforov [2] showed the following bounds. For any \( \epsilon > 0 \) and \( 0 < c < \frac{2}{5} \), if \( f(n) \) is \( \Theta(n^c) \) for all \( n \) then for all sufficiently large \( n \), \((1 - \epsilon)\frac{n^2}{2\sqrt{2f(n)}} < \gamma(n, f) \prec (1 + \epsilon)\frac{n^2}{2\sqrt{2f(n)}} \). In fact, the upper bound comes from a graph described by Erdős [7] and applies whenever \( f(n) \) is \( \Theta(n^c) \) for any \( c \in (0, 1) \).

We extend the lower bounds of Bollobás and Nikiforov [2] by obtaining the following bounds on \( \gamma(n, f) \)

Theorem 1.3. If \( G \) is a graph with exactly \( \frac{n^2}{4} - nf(n) \) edges where each edge is contained in at least one triangle and \( f(n) \leq \frac{n}{1000} \) then either \( b(G) > \frac{n}{1000} \) or \( f(n)(f(n) + bk(G))bk(G) \geq \frac{n^2}{2500} \).

Corollary 1.4. If \( \frac{n^2}{4} - nf(n) \) is an integer and \( f(n) \leq \frac{n}{1000} \) then \( \gamma(n, f) \geq \min\left\{ \frac{n}{50\sqrt{f(n)}}, \frac{n^2}{2500f(n)^2}, \frac{n}{1000} \right\} \).

Proof. For any graph \( G \), if \( bk(G) \leq \frac{n}{1000} \) then either \( bk(G) \geq f(n) \) or \( bk(G) \leq f(n) \). In the first case, \( 2f(n)bk(G)^2 \geq f(n)(f(n) + bk(G))bk(G) \geq \frac{n^2}{1250} \) so \( bk(G) \geq \frac{n}{50\sqrt{f(n)}} \). In the second case, \( 2f(n)^2bk(G) \geq f(n)(f(n) + bk(G))bk(G) \geq \frac{n^2}{1250} \) so \( bk(G) \geq \frac{n^2}{2500f(n)^2} \).

Corollary 1.5. For any \( c \in (0, 1) \), if \( f(n) = \Theta(n^c) \) for all \( n \) then \( \gamma(n, f) = \Theta(n^{1 - \frac{c}{2}}) \) if \( c \leq \frac{2}{5} \) and \( \gamma(n, f) \) is \( \Omega(n^{2 - 2c}) \) if \( c \geq \frac{2}{5} \).
1.2 Proof sketch

The idea behind our bound is as follows. If both \( f(n) \) and \( bk(G) \) are small then roughly speaking \( G \) will have the following structure. It will consist of a large set of vertices of high degree (degree at least \( \frac{r}{3} - O(f(n) + bk(G)) \)) and a set of at most \( O(f(n)) \) vertices with low degree (degree at most \( O(f(n) + bk(G)) \)) where the induced subgraph on the vertices of high degree is bipartite. The bound now follows from counting the number of triangles in the graph such that two of its vertices are high degree and one vertex is low degree.

On the other hand, each such triangle must contain two edges between a low degree vertex and a high degree vertex. The number of such edges is at most the number of low degree vertices (which is at most \( O(f(n)) \)) times the degree of these vertices (which is at most \( O(f(n) + bk(G)) \)). Each such edge appears in at most \( bk(G) \) triangles so we have that there are \( O(f(n)(f(n) + bk(G))bk(G)) \) such triangles and the result follows.

Unfortunately, the graph \( G \) may not quite have this structure. The main difficulty is in showing that the structure of \( G \) is close to the structure described above and that this is sufficient to prove our bounds.

2 Proof of Theorem 1.3

Throughout this section, we will assume that \( G \) is a graph with \( n \) vertices and \( \left( \frac{1}{4} - \frac{f(n)}{n} \right)n^2 \) edges such that every edge of \( G \) is in at least one triangle yet \( bk(G) \) is small. The structure of \( G \) is largely determined by the following lemma.

**Lemma 2.1.** If \( T \) be a triangle in \( G \) where the vertices have degrees \( d_1, d_2, d_3 \) then \( bk(G) \geq \frac{d_1 + d_2 + d_3 - n}{3} \)

**Proof.** If \( v \) is a vertex not in \( T \), let \( deg_T(v) \) be the number of vertices in \( T \) which are adjacent to \( v \). Now let \( x \) be the number of triangles excluding \( T \) which share an edge with \( T \).

\[
x = \sum_{v \not\in T} \left( \frac{\text{deg}_T(v)}{2} \right) \geq \sum_{v \not\in T} (\text{deg}_T(v) - 1) = (\sum_{v \not\in T} \text{deg}_T(v)) - (n - 3) = d_1 + d_2 + d_3 - 6 - (n - 3)
\]

By the pigeonhole principle, one edge of \( T \) must be contained in at least \( \frac{x}{3} \) triangles excluding \( T \), so \( bk(G) \geq \frac{x}{3} + 1 = \frac{d_1 + d_2 + d_3 - n}{3} \), as needed.

To analyze \( G \), we begin by splitting the vertices of \( G \) into two parts depending on their degree.

**Definition 2.2.**

1. Let \( V_H \) be the set of vertices with degree greater than \( \frac{2n}{3} \) and let \( n_h = |V_H| \)

2. Let \( V_L \) be the set of vertices with degree at most \( \frac{2n}{3} \) and let \( n_l = |V_L| \)

We now show that under our assumptions \( n_l \) is small with the following lemma.

**Lemma 2.3.** If \( f(n) \leq \frac{n}{10000} \) and \( bk(G) \leq \frac{n}{10000} \) then \( n_l \leq 20f(n) \).

**Proof.** Applying the following structural graph theorem proved by Andrasfai, Erdos, and Sas with \( r = 2 \), the induced subgraph of \( G \) on the set of vertices \( V_H \) must either contain a triangle (in which case \( bk(G) > \frac{n}{15} \)) by Lemma 2.1 or it must be bipartite:

**Theorem 2.4.** If \( G \) is a \( K_{r+1} \)-free graph with \( n \) vertices and minimal degree greater than \( (1 - \frac{3}{3r - 1})n \) then \( G \) is \( r \)-colorable.
Now let’s count up the total number of possible edges in $G$. There are at most \( \frac{(m_1)^2}{4} = \frac{(n - n_l)^2}{4} \) edges between vertices in $V_H$ and there are at most \( \frac{2n}{5} n_l \) edges containing a vertex in $V_L$. This gives a total of \( \frac{n^2}{4} - \frac{n}{10} n_l + \frac{(m_1)^2}{4} \) possible edges. However, by definition $G$ has exactly \( \frac{1}{4} - \frac{f(n)}{n} \) $n^2$ edges.

Thus we have that \( \frac{n^2}{4} - \frac{n}{10} n_l + \frac{(m_1)^2}{4} \geq \left( \frac{1}{4} - \frac{f(n)}{n} \right) n^2 \). Multiplying this equation by \(-\frac{10}{n}\) and rearranging the terms we have that $n_l - \frac{5(n_l)^2}{2n} \leq 10f(n)$. If $n_l < \frac{5}{8}$ then the result follows, so we just need to prove that $n_l < \frac{5}{8}$. For this, we will use the following lemma, which is a weaker but simpler version of Theorem 1 of Bollobas and Nikiforov [2] which is sufficient for our purposes.

**Lemma 2.5.** For any graph $G$, \( \sum_{v \in V(G)} d(v)^2 \leq |E(G)|(n + bk(G)) \)

**Proof.** Note that \[
\sum_{v \in V(G)} d(v)^2 = \sum_{v \in V(G)} \sum_{e \in V(G) \setminus \{v_1, v_2\}} d(v) = \sum_{e \in V(G) \setminus \{v_1, v_2\}} d(v_1) + d(v_2) \]

We bound this with the following proposition.

**Proposition 2.6.** For any edge $e = \{v_1, v_2\} \in E(G)$, \( d(v_1) + d(v_2) \leq n + bk(G) \)

**Proof.** Given an edge $e = \{v_1, v_2\}$, for each $i \in \{0, 1, 2\}$ let $m_i$ be the number of vertices in $V(G) \setminus \{v_1, v_2\}$ which are adjacent to $i$ vertices in $\{v_1, v_2\}$. We have the following equations

1. $m_0 + m_1 + m_2 = n - 2$
2. $d(v_1) + d(v_2) - 2 = m_1 + 2m_2$

Subtracting the first equation from the second equation gives $m_2 - m_0 = d(v_1) + d(v_2) - n \leq m_2 \leq bk(G)$. Thus, $d(v_1) + d(v_2) \leq n + bk(G)$. \hfill \( \square \)

Using this proposition, \( \sum_{v \in V(G)} d(v)^2 \leq |E(G)|(n + bk(G)) \), as needed. \hfill \( \square \)

We can now prove that $n_l < \frac{5}{8}$. Consider the quantity \( \sum_{v \in V(G)} (d(v) - \frac{2|E(G)|}{n})^2 \). On one hand,

\[
\sum_{v \in V(G)} \left( d(v) - \frac{2|E(G)|}{n} \right)^2 = \sum_{v \in V(G)} \left( d(v)^2 - 4d(v) \frac{|E(G)|}{n} + \frac{4|E(G)|^2}{n^2} \right) = \sum_{v \in V(G)} d(v)^2 - \frac{4|E(G)|^2}{n} \\
\leq (n + bk(G) - \frac{4|E(G)|}{n})|E(G)| = (bk(G) + 4f(n))|E(G)| \\
\leq \frac{n^2}{4} (bk(G) + 4f(n)) \leq \frac{n^3}{800}
\]

On the other hand, if $n_l \geq \frac{n}{5}$ then we already have $\frac{2}{5}$ vertices with degree less than $\frac{2n}{5}$ so

\[
\sum_{v \in V(G)} (d(v) - \frac{2|E(G)|}{n})^2 \geq \frac{n}{5} \left( \frac{2n}{5} - \frac{n}{2} + 2f(n) \right)^2 \\
= \frac{n}{5} \left( \frac{n}{10} - 2f(n) \right)^2 \\
> \frac{n^3}{800}
\]

This is a contradiction, which completes the proof that $n_l < \frac{5}{8}$ and thus the proof of the lemma. \hfill \( \square \)
Now that we have shown Lemma 2.3, we are ready to prove our bound.

**Theorem 2.7.** If \( f(n) \leq \frac{n}{1000} \) and \( bk(G) \leq \frac{n}{1000} \), then \( f(n)(f(n) + bk(G))bk(G) \geq \frac{n^2}{2500} \).

**Proof.** The idea of the proof is to count the number of triangles satisfying the following property:

**Definition 2.8.** We call a triangle of \( G \) well-behaved if it contains two vertices in \( V_H \) and one vertex in \( V_L \) of degree at most \( 5(f(n) + bk(G)) \).

To lower bound the number of such triangles, we count the number of edges between vertices of \( V_H \) satisfying the following property

**Definition 2.9.** Call an edge \( e = \{v_1, v_2\} \) between two vertices of \( V_H \) well-behaved if \( d(v_1) + d(v_2) \geq n - 2bk(G) - 5f(n) \).

**Proposition 2.10.** There are at least as many well-behaved triangles as there are well-behaved edges.

**Proof.** By our assumptions about \( G \), every well-behaved edge of \( G \) must be in a triangle and by Lemma 2.1, this triangle must be well-behaved. \( \square \)

**Lemma 2.11.** There are at least \( \frac{n^2}{25} \) well-behaved edges.

**Proof.** Let \( x \) be the number of well-behaved edges and consider the expression

\[
\sum_{e = \{v_1, v_2\}} (d(v_1) + d(v_2) - n + 4f(n) + bk(G)) = \sum_{v \in V(G)} d(v)^2 - (n - 4f(n) - bk(G)) \cdot |E(G)|
\]

\[
\geq \frac{1}{n} \left( \sum_{v \in V(G)} d(v) \right)^2 - (n - 4f(n) - bk(G)) \cdot |E(G)|
\]

\[
= \frac{|E(G)|^2}{n} - (n - 4f(n) - bk(G)) \cdot |E(G)|
\]

\[
= (n - 4f(n)) \cdot |E(G)| - (n - 4f(n) - bk(G)) \cdot |E(G)|
\]

\[
\geq 0
\]

On the other hand, since the maximum degree of a vertex in \( V_L \) is \( \frac{n}{5} \), there are at most \( \frac{2n}{5} \leq 8nf(n) \leq \frac{n^2}{125} \) edges which are not between vertices of \( V_H \). Now for any edge between vertices in \( V_H \) which is not well-behaved, it will make a negative contribution to this sum of at least \( f(n) + bk(G) \), which gives a total negative contribution of at least \( \left( \frac{n^2}{4} - nf(n) - x - \frac{n^2}{125} \right)(f(n) + bk(G)) \). For any edge \( e = \{v_1, v_2\} \), \( d(v_1) + d(v_2) - n \leq bk(G) \) so the contribution to this sum from \( e \) is at most \( 4(f(n) + bk(G)) \). Thus, the total positive contribution to the sum is at most \( 4 \left( x + \frac{n^2}{125} \right)(f(n) + bk(G)) \). Putting everything together we have that

\[
\sum_{e = \{v_1, v_2\}} (d(v_1) + d(v_2) - n + 4f(n) + bk(G))
\]

\[
\leq - \left( \frac{n^2}{4} - nf(n) - x - \frac{n^2}{125} \right)(f(n) + bk(G)) + 4 \left( x + \frac{n^2}{125} \right)(f(n) + bk(G))
\]

\[
\leq \left( 5x + nf(n) - \frac{n^2}{4} + \frac{n^2}{25} \right)(f(n) + bk(G))
\]

From before, this must be nonnegative so we have that \( x \geq \frac{n^2}{5} \left( \frac{1}{4} - \frac{1}{25} - \frac{1}{1000} \right) > \frac{n^2}{25} \), as needed. \( \square \)
Since there are at least as many well-behaved triangles as there are edges, $G$ must contain at least $\frac{n^2}{25}$ well-behaved triangles. We now complete the proof of Theorem 1.3 by upper bounding the number of well-behaved triangles. Note that every well-behaved triangle must contain two edges incident with a vertex of degree at most $5(f(n) + bk(G))$. However, there are at most $n_i \leq 20f(n)$ such vertices, so there can be at most $100f(n)(f(n) + bk(G))$ such edges. By definition, each such edge can only appear in $bk(G)$ triangles so there can be at most $100f(n)(f(n) + bk(G))bk(G) = 50f(n)(f(n) + bk(G))$ well-behaved triangles.

Putting everything together, $50f(n)(f(n) + bk(G))bk(G) \geq \frac{n^2}{1250}$ so we obtain that $f(n)(f(n) + bk(G))bk(G) \geq \frac{n^2}{25}$, as claimed. 

3 Conclusion

In this paper, we improved the lower bounds on $\gamma(n, f)$ whenever $f$ is $\Theta(nc)$ for all $c \in (\frac{2}{3}, 1)$. With these results, we now know $\gamma(n, f)$ to within a constant factor whenever $f$ is $\Theta(nc)$ and $c \in [0, \frac{2}{3}]$. However, this raises several new questions. First, can we merge the upper bounds of Fox, Loh [12] and Bollobás, Nikiforov [2] to obtain improved upper bounds when $f$ is $\Theta(nc)$ and $c \in (\frac{2}{3}, 1)$? Second, can we merge our new lower bounds with the lower bounds of Fox [12]? We have a long way to go before we fully understand $h(n, c)$ and $\gamma(n, f)$.

References

[1] B. Andrásfai, P. Erdős, V.T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, Discrete Math. 8 (1974) p. 205-218.

[2] B. Bollobás and V. Nikiforov, Books in graphs, European J. Combin. 26 (2005), p. 259-270.

[3] F. Chung and R. Graham, Erdős on graphs. His legacy of unsolved problems. A K Peters, Ltd., Wellesley, MA, 1998.

[4] C.S. Edwards, A lower bound for the largest number of triangles with a common edge, 1977 (unpublished manuscript)

[5] P. Erdős, On a theorem of Rademacher-Turán, Illinois J. Math. 6 (1962), p. 122-127.

[6] P. Erdős, Some problems on finite and infinite graphs. Logic and combinatorics (Arcata, Calif., 1985), p. 223-228, Contemp. Math., 65, Amer. Math. Soc., Providence, RI, 1987.

[7] P. Erdős, Problems and results in combinatorial analysis and graph theory, Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986), Discrete Math. 72 (1988), p. 81-92.

[8] P. Erdős, Some of my favourite problems in various branches of combinatorics. Combinatorics 92 (Catania, 1992). Matematiche (Catania) 47 (1992), no. 2, p. 231-240

[9] P. Erdős, R. Faudree and E. Győri, On the book size of graphs with large minimum degree, Studia Sci. Math. Hungar. 30 (1995), p. 25-46.

[10] P. Erdős, R. Faudree, C. Rousseau, Extremal problems and generalized degrees, Graph Theory and Applications (Hakone, 1990), Discrete Math. 127 (1994), p. 139-152.

[11] R.J. Faudree, C.C. Rousseau, J. Sheehan, More from the good book, Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Florida Atlantic Univ., Boca Raton, Fla., 1978, p. 289-299. Congress. Numer. XXI, Utilitas Math., Winnipeg, Man., 1978.
[12] J. Fox and P.S. Loh, On a problem of Erdős and Rothschild on edges in triangles. Combinatorica Vol. 32 Issue 6. (2012) p. 619-628

[13] N. Khadžiivanov and V. Nikiforov, Solution of a problem of P. Erdős about the maximum number of triangles with a common edge in a graph, C. R. Acad. Bulgare Sci. 32 (1979) p. 1315-1318

[14] V. Nikiforov and C. C. Rousseau, Large generalized books are p-good, J. Combin. Theory Ser. B 92 (2004), p. 85-97.

[15] V. Nikiforov and C. C. Rousseau, A note on Ramsey numbers for books, J. Graph Theory 49 (2005), p. 168-176.

[16] V. Nikiforov and C. C. Rousseau, Book Ramsey numbers, I. Random Structures Algorithms 27 (2005), p. 379-400.

[17] V. Nikiforov, C. C. Rousseau, and R. H. Schelp, Book Ramsey numbers and quasi-randomness, Combin. Probab. Comput. 14 (2005), p. 851-860.

[18] C. C. Rousseau and J. Sheehan, On Ramsey numbers for books, J. Graph Theory 2 (1978), p. 77-87.

[19] I. Z. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles. Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, p. 939-945, Colloq. Math. Soc. Jnos Bolyai, 18, North-Holland, Amsterdam-New York, 1978.

[20] B. Sudakov, Large Kr-free subgraphs in Ks-free graphs and some other Ramsey-type problems, Random Structures Algorithms 26 (2005), p. 253-265.