The ‘corrected Durfee’s inequality’ for homogeneous complete intersections

Dmitry Kerner and András Némethi

Abstract. We address the conjecture of [Durfee1978], bounding the singularity genus \( p_g \) by a multiple of the Milnor number \( \mu \) for an \( n \)-dimensional isolated complete intersection singularity. We show that the original conjecture of Durfee, namely \( (n+1)! \cdot p_g \leq \mu \), fails whenever the codimension \( r \) is greater than one. Moreover, we propose a new inequality \( C_{n,r} \cdot p_g \leq \mu \), and we verify it for homogeneous complete intersections. In the homogeneous case the inequality is guided by a ‘combinatorial inequality’, that might have an independent interest.

1. Introduction

1.1. Let \((X,0) \subset (\mathbb{C}^N,0)\) be the analytic germ of an \( n \)-dimensional complex isolated complete intersection singularity (ICIS). One of the most important goals of the local singularity theory is the clarification of the subtle connections between the two basic numerical invariants, the Milnor number \( \mu \) and singularity (geometric) genus \( p_g \).

The surface case, \( n = 2 \), is already rather exotic and hard. In this case, if \( X_F \) is the Milnor fiber of \((X,0)\), and \((\mu_+, \mu_0, \mu_-)\) are the Sylvester invariants of the symmetric intersection form in the middle integral homology \( H_2(X_F, \mathbb{Z}) \), then \( 2p_g = \mu_0 + \mu_+ \) [Durfee1978], while, obviously, \( \mu = \mu_+ + \mu_0 + \mu_- \). Hence, numerical relations between \( \mu \) and \( p_g \) can be rewritten in terms of the Sylvester invariants too. In topology one also uses the signature \( \sigma := \mu_+ - \mu_- \) as well. In fact, for compact complex surfaces, the Euler number, Todd genus and the signature are the most important index–theoretical numerical invariants; their local analogs are the above integers \( \mu \), \( p_g \) and \( \sigma \). For more about these invariants see the monographs [Milnor-book, AGLV-book, Looijenga-book] or the articles [Lauefer77, Looijenga86, Saito81]. For various formulae regarding the Milnor number of weighted homogeneous complete intersections see [Greuel75, Greuel-Hamm78, Hamm86, Hamm11] and for the geometric genus see [Kovanskiil78, Morales85].

Examples show that for a local surface singularity \( \mu_- \) should be ‘large’ compared with the other Sylvester invariants, or equivalently, \( p_g \) small with respect to \( \mu \), or, \( \sigma \) rather negative.

This was formulated more precisely in Durfee’s Conjectures [Durfee1978] as follows:

(A) **Strong inequality**: if \((X,0)\) is an isolated complete intersection surface singularity, then \( 6p_g \leq \mu \).

(B) **Weak inequality**: if \((X,0)\) is a normal surface singularity (not necessarily ICIS) which admits a smoothing with Milnor number \((\text{second Betti number of the fiber}) \mu \), then \( 4p_g \leq \mu + \mu_0 \), or equivalently, \( \sigma \leq 0 \).

(C) **Semicontinuity of \( \sigma \)**: if \( \{(X_t,0)\}_{t \in (\mathbb{C},0)} \) is a family of isolated surface singularities then \( \sigma(X_{t=0}) \leq \sigma(X_{t \neq 0}) \).

Almost immediately a counterexample to the weak inequality was given in [Wahl81, page 240] providing a normal surface singularity (not ICIS) with \( \mu = 3 \), \( \mu_0 = 0 \) and \( p_g = 1 \).

A counterexample to the semicontinuity of the signature was found much later, in [Kerner-N÷methi09].

On the other hand, the strong inequality, valid for an ICIS, was believed to be true and was verified for many particular hypersurface singularities \((X_0)\) in \((\mathbb{C}^3,0)\):

- [Tomari93] proved \( 8p_g < \mu \) for \((X,0)\) of multiplicity 2.
- [Ashikaga92] proved \( 6p_g \leq \mu - 2 \) for \((X,0)\) of multiplicity 3.
- [Xu-Yau93] proved \( 6p_g \leq \mu - \text{mult}(X,0) + 1 \) for quasi-homogeneous singularities.
- [Némethi98, Némethi99] proved \( 6p_g \leq \mu \) for suspension type singularities \( \{g(x, y) + z^k = 0\} \subset (\mathbb{C}^3,0) \).
- [Melle-Hernández2000] proved \( 6p_g \leq \mu \) for absolutely isolated singularities.

Moreover, for arbitrary \( n \geq 2 \), [Yau-Zhang06] proved the inequality \((n+1)!p_g \leq \mu \) for isolated weighted-homogeneous hypersurface singularities in \((\mathbb{C}^{n+1},0)\). The natural expectation was that the same inequality holds for any ICIS of any dimension \( n \) and any codimension \( r := N - n \).

1.2. This paper is the continuation of [Kerner-Némethi.a]. The main results of the present article are the following:
I. For homogeneous ICIS of multidegree \((p_1, \ldots, p_r)\) we provide new formulae for \(\mu\) and \(p_g\); their special form allows us to compare them.

II. Using these formulae one sees rather easily that \((n+1)! \cdot p_g \leq \mu\) is not true whenever \(r \geq 2\) and \(n \geq 2\), already for \(p_1 = \cdots = p_r\) sufficiently large (that is, the strong inequality fails even asymptotically).

III. We propose a new set of conjectured inequalities with new bounds for \(\mu/p_g\). For any \(n \geq 1\) and \(r \geq 1\) consider the Stirling number of the second kind, cf. [Abramowitz-Stegun, §24.1.4], and the coefficient \(C_{n,r}\) defined by

\[
C_{n,r} := \frac{(n+r-1)(n+r)!}{\left\{ \begin{array}{c} n+r \\ r \end{array} \right\} r!} = \frac{|\mathcal{K}_{n,r}|}{\sum_{k \in \mathcal{K}_{n,r}} \prod_{i=1}^r (k_i+1)!},
\]

where \(|\mathcal{K}_{n,r}| = \binom{n+r-1}{n}\) is the cardinality of the set

\[
\mathcal{K}_{n,r} := \{ k = (k_1, \ldots, k_r) : k_i \geq 0 \text{ for all } i, \text{ and } \sum_i k_i = n \}.
\]

The second equality of (1) follows from [Jordan1965, pages 176-178]. One also shows, cf. Corollary 4.2, that (3)

\[
C_{n,1} > C_{n,2} > \cdots > C_{n,r} > \cdots > \lim_{r \to \infty} C_{n,r}.
\]

E.g., for small \(r\) and for \(r \to \infty\) one gets

\[
C_{n,1} = (n+1)! \quad C_{n,2} = \frac{(n+2)!(n+1)}{2^{n+2}}, \quad \lim_{r \to \infty} C_{n,r} = 2^n.
\]

The limit can be computed using the asymptotical growth of Stirling numbers of the second kind, [Abramowitz-Stegun, §24.1.4]: \(\left\{ \begin{array}{c} n+r \\ r \end{array} \right\} \sim \frac{r^n}{2^{n!}}\). This gives: \(C_{n,r} \sim 2^n \frac{(n+r-1)!}{r^{r+1}}\) with limit \(2^n\) as \(r \to \infty\).

**Conjecture.** Let \((X,0) \subset (\mathbb{C}^N,0)\) be an ICIS of dimension \(n\) and of codimension \(r = N - n\). Then

- for \(n = 2\) and \(r = 1\) one has \(6p_g \leq \mu\),
- for \(n = 2\) and arbitrary \(r\) one has \(4p_g < \mu\) (see last section too),
- for \(n \geq 3\) and arbitrary \(r\) one has \(C_{n,r} \cdot p_g \leq \mu\).

IV. We show that the third proposed inequality is asymptotically sharp, i.e. for any fixed \(n\) and \(r\) there exists a sequence of isolated complete intersections for which the ratio \(\frac{\mu}{p_g}\) tends to \(C_{n,r}\).

V. We support the above conjecture by its proof for any homogeneous ICIS with any multidegree \((p_1, \ldots, p_r)\).

Note that \(C_{n,r} \cdot p_g \leq \mu\) automatically implies

\[
\mu \geq \min_{k \in \mathcal{K}_{n,r}} \left\{ \prod_{i=1}^r (k_i + 1)! \right\} \cdot p_g \geq 2^n \cdot p_g.
\]

Some more comments are in order.

- The general definition of the singularity genus is the following. Let \((X,0)\) be a reduced isolated singularity of dimension \(n\), and let \(\tilde{X} \xrightarrow{\pi} (X,0)\) be a resolution. Let \(\mathcal{O}_{(X,0)}\) be the local ring of the singularity \((X,0)\), and let \(\mathcal{O}_{\tilde{X}}\) be the structure sheaf on \(\tilde{X}\). Then \(\pi_* \mathcal{O}_{\tilde{X}}\) is the normalization of \(\mathcal{O}_{(X,0)}\) and (cf. [Looijenga1986, §4.1]):

\[
(-1)^n p_g(X,0) := \sum_{i \geq 1} (-1)^{i-1} h^i (\mathcal{O}_{\tilde{X}}) - \dim_{\mathbb{C}} \pi_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_{(X,0)}.
\]

- For \(n = 1\) the preimage of the singular point is of dimension zero, hence the higher cohomologies vanish: \(h^i (\mathcal{O}_{\tilde{X}}) = 0\) for all \(i \geq 1\). Thus the singularity genus coincides with the classical delta invariant \(\delta\). It satisfies \(2\delta = \mu + b(X,0) - 1\), where \(b(X,0)\) is the number of locally irreducible components, [Milnor-book, Theorem 10.5]. Hence \(\mu \leq 2\delta\), and the analogue of the strong inequality fails (note that \(C_{1,1} = \frac{1}{2}\)). Moreover, for \(n = 2\) too, the inequality \(\mu(X,0) \geq C_{n,r} \cdot p_g(X,0)\), in general, is not satisfied, and the asymptotically sharp inequality of this form is impossible. For more comments regarding \(n = 2\) see section 5.

These facts also show that, in order to prove the conjectured inequalities, a ‘naive induction’ over \(n\) is impossible.

- Homogeneous singularities (considered in V) are rather particular ones. Nevertheless, it turns out that they are the building blocks for many other singularity types. In the forthcoming paper we use the statement of V to prove the
above conjectured inequalities from III for more general families of complete intersections (e.g., for absolutely isolated singularities), cf. [Kerner-Némethi.c].

• Even in this particular case of homogeneous germs the proof involves a non-trivial combinatorial inequality, which 'guides' the inequality $C_{n,r} \cdot p_g \leq \mu$, see §1.3 and §4. Although its proof in its current form is relatively short, we believe it is far from being straightforward.

1.3. Fix the integers $n \geq 1, r \geq 1$ and $\ell \geq 0$. We define $y_{k,\ell} := \prod_{i \in L_{k,\ell}}$ and for free positive variables $x_1, \ldots, x_r$ we set $z^r = x_1^{k_1} \cdots x_r^{k_r}$. By convention $0! = 1$.

**Combinatorial Inequality.** For any $n, r$ and $\ell$ as above, and for any positive $x_1, \ldots, x_r$ one has:

\[
(I_{n,r,\ell}) : \quad \frac{1}{|K_{n,r}|} \sum_{k \in K_{n,r}} y_{k,\ell} \cdot \sum_{j \leq 0} \prod_{i \in K_{n,r}} (x_i - 1)^k_j \geq \sum_{k \in K_{n,r}} y_{k,\ell} \cdot \prod_{i \in K_{n,r}} x_i^k.
\]

For our application we only need the $\ell = 1$ case, nevertheless its proof uses all the $\ell$ values (the induction over $n$ involves larger $\ell$'s for smaller $n$).

1.4. We wish to thank G.-M. Greuel, H. Hamm, A. Khovanskii, M. Leyenson, L. Lovász, P. Milman, E. Shustin for advices and important discussions.

2. The $\mu$ and $p_g$ Formulae

Let $(X, 0) = \{f_1 = \cdots = f_r = 0\}$ be a homogeneous ICIS of multidegree $(p_1, \ldots, p_r)$, that is deg($f_i$) = $p_i$.

**Proposition 2.1.**

1. $\mu = \left( \prod_{i=1}^{r} p_i \right) \sum_{j=0}^{n} \frac{(-1)^j}{\prod_{i \in K_{n,r}} (x_i + 1)^{k_j}} - (-1)^n$,

2. $p_g = \sum_{k \in K_{n,r}} \prod_{i \in K_{n,r}} p_i^{k_i}.$

The reader is invited to consult [Greuel1975, Greuel-Hamm1978, Hamm1986, Hamm2011] for $\mu$ and [Khovanskii1978, Morales1985, Hamm2011] for $p_g$ of a weighted homogeneous ICIS. These formulae usually are rather different than ours considered above; nevertheless, both formulae (1) and (2) can be derived from expressions already present in the literature (though we have found them in a different way).

**Proof.** (1) We determine the Euler characteristic $\chi = (-1)^n \mu + 1$ of the Milnor fiber. For a power series $Z := \sum_{i \geq 0} a_i x^i$ we write $[Z]_n$ for the coefficient $a_n$ of $x^n$. By 3.7(c) of [Greuel-Hamm1978]

\[
\chi = \prod_{i=1}^{r} p_i \cdot \left[ \frac{(1 + x)^N}{\prod_{i \in K_{n,r}} (1 + p_i x)} \right]_n.
\]

Rewrite $1 + p_i x$ as $(1 + x) (1 - \frac{1 - p_i x}{1 + x})$, hence

\[
\left[ \frac{(1 + x)^N}{\prod_{i \in K_{n,r}} (1 + p_i x)} \right]_n = \left[ (1 + x)^N \cdot \prod_{i \in K_{n,r}} \left( \frac{1 - p_i x}{1 + x} \right)^{k_i} \right]_n = \sum_{\prod_{i \in K_{n,r}} (1 - p_i x)^{k_i}} \prod_{i \in K_{n,r}} (1 - p_i x)^{k_i}.
\]

(2) By [Morales1985, Theorem 2.4] (and computation of the number of lattice points under the 'homogeneous Newton diagram')

\[
p_g = \left( \sum_{k=0}^{N} \frac{p_k N}{N} \right) - \sum_{1 \leq i \leq r} \left( \frac{\sum_{k=0}^{N} \frac{p_k}{N} - p_i}{N} \right) + \sum_{1 \leq i < j \leq r} \left( \frac{\sum_{k=0}^{N} \frac{p_k}{N} - p_i - p_j}{N} \right) - \ldots
\]

Using the Taylor expansion $\frac{1}{(1 - z)^{N+1}} = \sum_{l \geq 0} \left( \begin{array}{c} l \end{array} \right) z^l N^l$, the right hand side of (6) is $\left[ \prod_{i \in K_{n,r}} \left( \frac{1 - z p_i}{1 - z} \right) \right]_{\sum_{i \in K_{n,r}} p_i N}$. Thus

\[
p_g = \text{re} z_{z=0} F(z), \quad \text{where} \quad F(z) := \prod_{i \in K_{n,r}} \left( 1 - \frac{z p_i}{z} \right)^{N + l}.
\]

Note that $\text{re} z_{z=0} F(z) = 0$ since $F(1/z) / z^2$ is regular at zero. Hence, $p_g = - \text{re} z_{z=1} F(z)$ too. By the change of variables $z \mapsto 1/z$, this last expression transforms into

\[
p_g = \text{re} z_{z=1} \prod_{i=1}^{r} \left( \frac{z p_i - 1}{z - 1} \right) \left( z - 1 \right)^{N+1}.
\]
Since $z^n - 1 = \sum_{k \geq 0} (\frac{p}{k+1}) (z-1)^{k+1}$, we obtain

$$p_g = r \epsilon z^2 \sum_{k=1}^{r} \frac{1}{(z-1)^{n+1}} \prod_{i=1}^{r} \sum_{r_i \geq 0} \left( \frac{p_i}{k_i+1} \right) (z-1)^{k_i} = \sum_{k \in \mathbb{K}_{n,r}} \left( \frac{p_1}{k_1+1} \right) \cdots \left( \frac{p_r}{k_r+1} \right).$$

Consider the particular case $p_1 = \cdots = p_r = p$. Then the above formulae read as

$$\mu = (-1)^n \left( \frac{p^n}{n} \sum_{j=0}^{n} (1-p)^j \left( \frac{j!}{j^{r-1}} \right) - 1 \right),$$

(see also [Greuel-Hamm1978, 3.10(b)],

(7)

$$p_g = \sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} \left( \frac{p_i}{k_i+1} \right).$$

Therefore, for $p$ large, $\mu$ and $p_g$ asymptotically behave as follows:

$$\mu = p^n \left( \frac{n-1}{n} \right) + O(p^{n-1}) \quad \text{and} \quad p_g = p^n \cdot \sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} \left( \frac{1}{k_i+1} \right) + O(p^{n-1}).$$

Thus, asymptotically, $\frac{\mu}{p_g} = C_{n,r} + O(\frac{1}{p})$. Note that $C_{n,r} < C_{n,1} = (n+1)!$ for any $r \geq 2$, hence the strong Durfee’s inequality is violated for any $p$ sufficiently large whenever $r \geq 2$.

The inequality $C_{n,r} < (n+1)!$ is the consequence of Corollary 4.2, but it follows from the next elementary observation as well: $\frac{1}{(n+1)!}$ is the smallest of all elements of type $\prod_{i=1}^{r} \left( \frac{1}{(k_i+1)} \right)$, for $k \in \mathbb{K}_{n,r}$. Indeed, note that $(k_i+1)!/((k_i+2)!)$ is $C_{n,r}$, because $\left( \frac{2 \times \cdots \times (k_i+1)}{k_i} \right) \times \left( \frac{2 \times \cdots \times (k_i+2)}{k_i+1} \right) \leq \left( \frac{2 \times \cdots \times (k_1+k_2+1)}{k_1+k_2} \right)$, thus $(n+1)! \geq \prod_i (k_i+1)!$.

3. The inequality $C_{n,r} \cdot p_g \leq \mu$ in the homogeneous case

Note that if $r = 1$, then $C_{n,1} \cdot p_g = (n+1)! \cdot p_g \leq \mu$ for any isolated homogeneous germs and for any $n \geq 2$. This follows from $(n+1)!p_g = (n+1)!(\frac{1}{p}) \leq (p-1)^{n+1} = \mu$, where $p$ is the degree of the germ. If $r > 1$ then the $n = 2$ case is rather special, and it will be discussed in the last section. Hence, we start the case $n \geq 3$. We prove:

**Theorem 3.1.** Assume that $(X,0) \subset (\mathbb{C}^{n+r},0)$ is a homogeneous ICIS of dimension $n \geq 2$, codimension $r$ and multidegree $(p_1, \ldots, p_r)$. Then $C_{n,r} \cdot p_g \leq \mu$.

In the next discussions we assume $p_i \geq 2$ for all $i$; if $p_i = 1$ for some $i$ then one can reduce the setup to the smaller $(r-1)$-codimensional case. In addition, as all the formulas are symmetric in $\{p_i\}$, we will sometimes assume that $p_r$ is largest among all the $p_i$’s. Hence we wish to prove:

(8) \quad LHS := (\prod_{i=1}^{r} p_i) \sum_{j=0}^{n} (-1)^j \left( \sum_{k \in \mathbb{K}_{n-j,r}} \prod_{i=1}^{r} (p_i - 1)^{k_i} \right) - (-1)^n \geq C_{n,r} \cdot (\prod_{i=1}^{r} p_i) \cdot \sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} \left( \frac{p_i - 1}{k_i} \right) \frac{1}{k_i+1} =: RHS.$

The proof consists of several steps.

**Theorem 3.2.** (1) The inequality (8) is the consequence of the next inequality:

$$\sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} (p_i - 1)^{k_i} \geq C_{n,r} \left( \sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} \left( \frac{p_i - 1}{k_i} \right) \frac{1}{k_i+1} + \sum_{k \in \mathbb{K}_{n,r}} \left( \frac{p_r - 2}{k_r - 1} \right) \prod_{i=1}^{r-1} \left( \frac{p_i - 1}{k_i} \right) \frac{1}{k_i+1} \right).$$

This inequality is the consequence of two further inequalities, listed in part (2) and (3):

(2) For $n \geq 3$ the following inequality holds:

$$\sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} \frac{(p_i - 1)^{k_i}}{(k_i+1)!} \geq \sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} \left( \frac{p_i - 1}{k_i} \right) \frac{1}{k_i+1} + \sum_{k \in \mathbb{K}_{n,r}} \left( \frac{p_r - 2}{k_r - 1} \right) \prod_{i=1}^{r-1} \left( \frac{p_i - 1}{k_i} \right) \frac{1}{k_i+1}.$$

(3) For any $n \geq 2$ and $r \geq 1$ the following inequality holds:

$$\sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} (p_i - 1)^{k_i} \geq C_{n,r} \sum_{k \in \mathbb{K}_{n,r}} \prod_{i=1}^{r} \left( \frac{p_i - 1}{k_i} \right)^{k_i} \frac{1}{(k_i+1)!}.$$
Here the third statement (3) is the most complicated one, it follows from the general Combinatorial Inequality from the Introduction, and it is proved separately in the next section.

**Proof.** (1) Expand the LHS of (8) in terms with decreasing \( r \):

\[
(9) \quad \text{LHS} = (p_r - 1)(\prod_{i=1}^{r-1} p_i)C_n(p_1, \ldots, p_r) + (p_r - 1)(\prod_{i=1}^{r-2} p_i)C_n(p_1, \ldots, p_{r-1}) + \cdots + (p_1 - 1)(p_1 - 1)^n,
\]

where \( C_n(p_1, \ldots, p_s) := \sum_{k \in \mathcal{K}_{n,s}} \prod_{j=1}^n (p_i - 1)^{k_i} \), for \( 1 \leq s \leq r \). To prove this formula we observe:

\[
\sum_{j=0}^n (-1)^j \left( \sum_{k \in \mathcal{K}_{n-j,r}} \prod_{i=1}^n (p_i - 1)^{k_i} \right) = (-1)^n \sum_{\sum_{j=1}^r k_j \leq n} (1 - p_i)^{k_i} = (-1)^n \sum_{\sum_{j=1}^r k_j \leq n} \left( \frac{(1 - p_r)^n}{1 - p_r} - \sum_{j=1}^{r-1} \frac{\prod_{i=1}^{j=1} (1 - p_i)^{k_i}}{1 - p_i} \right) \prod_{i=1}^r (1 - p_i)^{k_i}
\]

Thus

\[
(\prod_{i=1}^r p_i)(-1)^n \sum_{\sum_{j=1}^r k_j \leq n} \prod_{i=1}^r (1 - p_i)^{k_i} = (p_r - 1)(\prod_{i=1}^{r-1} p_i)C_n(p_1, \ldots, p_r) + (p_r - 1)(\prod_{i=1}^{r-2} p_i)C_n(p_1, \ldots, p_{r-1}) + \cdots + (p_1 - 1)(p_1 - 1)^n \sum_{\sum_{j=1}^r k_j \leq n} \prod_{i=1}^r (1 - p_i)^{k_i}
\]

Iterating this gives equation (9).

It is natural to expand the right hand side of (8) similarly. For this, define \( D_n(p_1, \ldots, p_s) := \sum_{k \in \mathcal{K}_{n,r}} \prod_{i=1}^r (p_i - 1)^{k_i} \). E.g., \( D_1(p_1) = (p_1 - 1)\frac{1}{p_1} \). Then, we write RHS/C\(n,r\) as

\[
\left( \prod_{i=1}^r p_i \right) D_n(p_1, \ldots, p_r) - \left( \prod_{i=1}^{r-1} p_i \right) D_n(p_1, \ldots, p_{r-1}) + \left( \prod_{i=1}^{r-2} p_i \right) D_n(p_1, \ldots, p_{r-2}) + \cdots + p_1 D_1(p_1).
\]

Thus, it is enough to prove the inequality for each pair of terms in these expansions, namely:

\[
(p_s - 1)(\prod_{i=1}^s p_i)C_n(p_1, \ldots, p_s) \geq C_{n,r} \left( \prod_{i=1}^s p_i \right) D_n(p_1, \ldots, p_s) - (\prod_{i=1}^{s-1} p_i) D_n(p_1, \ldots, p_{s-1}),
\]

for all \( 1 \leq s \leq r \). Since \( C_{n,r} \leq C_{n,s} \), cf (3), it is enough to prove the last inequality with coefficient \( C_{n,s} \) instead of \( C_{n,r} \), or equivalently,

\[
(p_r - 1)(\prod_{i=1}^r p_i)C_n(p_1, \ldots, p_r) \geq C_{n,r} \left( \prod_{i=1}^r p_i \right) D_n(p_1, \ldots, p_r) - (\prod_{i=1}^{r-1} p_i) D_n(p_1, \ldots, p_{r-1}),
\]

for all \( r \geq 1, n \geq 1 \). Further, split the right hand side of this last inequality into two parts:

\[
(p_r - 1)(\prod_{i=1}^r p_i)D_n(p_1, \ldots, p_r) + (\prod_{i=1}^r p_i) \left( D_n(p_1, \ldots, p_r) - D_n(p_1, \ldots, p_{r-1}) \right),
\]

and rewrite \( D_n(p_1, \ldots, p_r) - D_n(p_1, \ldots, p_{r-1}) \) into

\[
\sum_{k \in \mathcal{K}_{n,r}, k_i > 0} \prod_{k_i} (p_i - 1) \frac{1}{k_i + 1} = (p_r - 1) \sum_{k \in \mathcal{K}_{n,r}, k_r > 0} \prod_{k_i} (p_i - 1) \frac{1}{k_i + 1} - (p_r - 2) \sum_{k \in \mathcal{K}_{n,r}, k_{r-1} > 0} \prod_{k_i} (p_i - 2) \frac{1}{k_i + 1} \sum_{k \in \mathcal{K}_{n,r}, k_r > 0} \prod_{k_i} (p_i - 1) \frac{1}{k_i + 1}.
\]

This provides precisely the expression of part (1).

(2) We start to compare individually the particular summands indexed by \( k \in \mathcal{K}_{n,r} \) of both sides. First, we consider some \( k \in \mathcal{K}_{n,r} \) with \( k_r > 1 \). Then the corresponding individual inequality is true. Indeed,

\[
\frac{\prod_{i=1}^r (p_i - 1)^{k_i} \prod_{i=1}^r (p_i - 1)}{\prod_{i=1}^r (k_i + 1)} - \frac{(p_r - 1)^{k_r - 1}}{k_r(k_r + 1)} \prod_{i=1}^r (p_i - 1) \frac{1}{k_i + 1} \geq \frac{(p_r - 1)^{k_r - 1} - (p_r - 2)^{k_r - 2}}{k_r(k_r + 1)} \prod_{i=1}^r (p_i - 1) \frac{1}{k_i + 1} \geq 0.
\]
Here and in the sequel we constantly use \((p_i - 1)^{k_i} - (p_i - 1) \cdot (p_i - 2) \cdots (p_i - k_i) \geq (p_i - 1)^{k_i - 1}\), valid for \(k_i > 1\).

Next, we assume that \(k\) has the following properties: \(k_r = 1\), but \(k_j > 1\) for some \(j\). Then again

\[
\frac\prod_{i=1}^{r-1} \frac{(p_i - 1)^{k_i - 1}}{p_i - 1} \cdot \prod_{i=1}^{p_i - 1} \left( \frac{p_i - 1}{k_i + 1} \right) \geq \frac\prod_{i=1}^{r-1} \frac{(p_i - 1)^{k_i - 1}}{p_i - 1} \cdot \prod_{i=1}^{p_i - 1} \left( \frac{p_i - 1}{k_i + 1} \right) \geq 0.
\]

Here we used the initial assumption that \(p_r \geq p_i\) for any \(i\).

Now, we assume that \(k_r = 1\) and \(k_i \leq 1\) for all \(i\). This can happen only for \(n \leq r\). Let \(i_1, \ldots, i_{n-1}\) be those indices different than \(r\) for which \(k_i = 1\), that is, \(k_r = k_{i_1} = \cdots = k_{i_{n-1}} = 1\) and all the other \(k_i\)'s are zero.

In this case \(\prod_{i=1}^{p_i - 1} \left( \frac{p_i - 1}{k_i + 1} \right) = \prod_{i=1}^{r-1} \left( \frac{p_i - 1}{k_i + 1} \right)\), hence the individual inequality corresponding to \(k\) fails. Therefore, we will group this \(k\) together with terms which correspond to those \(k_i\)'s which satisfy \(k_r = 0\), \(k_j = 2\) for exactly one \(j \in \{1, \ldots, n-1\}\), \(k_i = 1\) if \(l \in \{1, \ldots, n-1\} \setminus \{j\}\), and all the other \(k_i\)'s are zero.

Note that if \(k_r = 0\) then there is no contribution from the sum \(\sum_{k \in K_{n,r}, k_r > 0} \). Therefore, the \(n\) individual inequalities corresponding to the above \(k\) altogether provide

\[
\sum_{i=1}^{n-1} \frac{(p_i - 1)}{2} \cdot \prod_{i=1}^{p_i - 1} \left( \frac{p_i - 1}{2} \right) = \frac{2n - 5}{6} \cdot \prod_{i=1}^{p_i - 1} \left( \frac{p_i - 1}{2} \right).
\]

For \(n \geq 3\) this is positive, hence the statement.

Any other remaining \(k \in K_{n,r}\) can again be treated individually: for all of them \(k_r = 0\) and \((p_i - 1)k_i \geq (p_i - 1)1\).

\section{4. Proof of the Combinatorial Inequality}

We use the notations of §1.3 and introduce more objects. We will consider the following partition of \(K_{n,r}\): for any \(s \in \{0, \ldots, r - 1\}\) we define

\[(10)\]

\[K_{n,r}^s := \{k \in K_{n,r} : |\{i : k_i = 0\}| = s\}.
\]

Note that for \(s < r - n\) one has \(K_{n,r}^s = \emptyset\). Corresponding to these sets, we consider the arithmetical mean \(X_{n,r}\) and \(X_{n,r}^s\) of the elements \(\underline{y}_{n,r}\) indexed by the sets \(K_{n,r}\) and \(K_{n,r}^s\) respectively. In parallel, \(Y_{n,r,k}\) and \(Y_{n,r,k}^s\) are the arithmetical mean of elements \(\underline{y}_{n,r,k}\) indexed by the same sets \(K_{n,r}\) and \(K_{n,r}^s\) respectively.

\begin{theorem}
With the above notation one has
\begin{enumerate}
\item \(X_{n,r}^s \leq X_{n,r} \leq \cdots \leq X_{n,r}^{r-1}\)
\item \(Y_{n,r,k}^s \geq Y_{n,r,k} \geq \cdots \geq Y_{n,r,k}^{r-1}\)
\end{enumerate}
\end{theorem}

\begin{proof}
\begin{enumerate}
\item \(\underline{X}_{n,r}^s (x_1, \ldots, x_r) = X_{n,r}^s \left( \frac{\underline{X}_{n,r} (x_1, \ldots, x_r)}{s_r} \right)\)
\end{enumerate}
\end{proof}

\section{4. Proof of the Combinatorial Inequality}

We use the notations of §1.3 and introduce more objects. We will consider the following partition of \(K_{n,r}\): for any \(s \in \{0, \ldots, r - 1\}\) we define

\[(10)\]

\[K_{n,r}^s := \{k \in K_{n,r} : |\{i : k_i = 0\}| = s\}.
\]

Note that for \(s < r - n\) one has \(K_{n,r}^s = \emptyset\). Corresponding to these sets, we consider the arithmetical mean \(X_{n,r}\) and \(X_{n,r}^s\) of the elements \(\underline{y}_{n,r}\) indexed by the sets \(K_{n,r}\) and \(K_{n,r}^s\) respectively. In parallel, \(Y_{n,r,k}\) and \(Y_{n,r,k}^s\) are the arithmetical mean of elements \(\underline{y}_{n,r,k}\) indexed by the same sets \(K_{n,r}\) and \(K_{n,r}^s\) respectively.

\begin{theorem}
With the above notation one has
\begin{enumerate}
\item \(X_{n,r}^s \leq X_{n,r} \leq \cdots \leq X_{n,r}^{r-1}\)
\item \(Y_{n,r,k}^s \geq Y_{n,r,k} \geq \cdots \geq Y_{n,r,k}^{r-1}\)
\end{enumerate}
\end{theorem}

\begin{proof}
\begin{enumerate}
\item \(\underline{X}_{n,r}^s (x_1, \ldots, x_r) = X_{n,r}^s \left( \frac{\underline{X}_{n,r} (x_1, \ldots, x_r)}{s_r} \right)\)
\end{enumerate}
\end{proof}

\section{4. Proof of the Combinatorial Inequality}

We use the notations of §1.3 and introduce more objects. We will consider the following partition of \(K_{n,r}\): for any \(s \in \{0, \ldots, r - 1\}\) we define

\[(10)\]

\[K_{n,r}^s := \{k \in K_{n,r} : |\{i : k_i = 0\}| = s\}.
\]

Note that for \(s < r - n\) one has \(K_{n,r}^s = \emptyset\). Corresponding to these sets, we consider the arithmetical mean \(X_{n,r}\) and \(X_{n,r}^s\) of the elements \(\underline{y}_{n,r}\) indexed by the sets \(K_{n,r}\) and \(K_{n,r}^s\) respectively. In parallel, \(Y_{n,r,k}\) and \(Y_{n,r,k}^s\) are the arithmetical mean of elements \(\underline{y}_{n,r,k}\) indexed by the same sets \(K_{n,r}\) and \(K_{n,r}^s\) respectively.

\begin{theorem}
With the above notation one has
\begin{enumerate}
\item \(X_{n,r}^s \leq X_{n,r} \leq \cdots \leq X_{n,r}^{r-1}\)
\item \(Y_{n,r,k}^s \geq Y_{n,r,k} \geq \cdots \geq Y_{n,r,k}^{r-1}\)
\end{enumerate}
\end{theorem}

\begin{proof}
\begin{enumerate}
\item \(\underline{X}_{n,r}^s (x_1, \ldots, x_r) = X_{n,r}^s \left( \frac{\underline{X}_{n,r} (x_1, \ldots, x_r)}{s_r} \right)\)
\end{enumerate}
\end{proof}

\section{4. Proof of the Combinatorial Inequality}

We use the notations of §1.3 and introduce more objects. We will consider the following partition of \(K_{n,r}\): for any \(s \in \{0, \ldots, r - 1\}\) we define

\[(10)\]

\[K_{n,r}^s := \{k \in K_{n,r} : |\{i : k_i = 0\}| = s\}.
\]

Note that for \(s < r - n\) one has \(K_{n,r}^s = \emptyset\). Corresponding to these sets, we consider the arithmetical mean \(X_{n,r}\) and \(X_{n,r}^s\) of the elements \(\underline{y}_{n,r}\) indexed by the sets \(K_{n,r}\) and \(K_{n,r}^s\) respectively. In parallel, \(Y_{n,r,k}\) and \(Y_{n,r,k}^s\) are the arithmetical mean of elements \(\underline{y}_{n,r,k}\) indexed by the same sets \(K_{n,r}\) and \(K_{n,r}^s\) respectively.

\begin{theorem}
With the above notation one has
\begin{enumerate}
\item \(X_{n,r}^s \leq X_{n,r} \leq \cdots \leq X_{n,r}^{r-1}\)
\item \(Y_{n,r,k}^s \geq Y_{n,r,k} \geq \cdots \geq Y_{n,r,k}^{r-1}\)
\end{enumerate}
\end{theorem}

\begin{proof}
\begin{enumerate}
\item \(\underline{X}_{n,r}^s (x_1, \ldots, x_r) = X_{n,r}^s \left( \frac{\underline{X}_{n,r} (x_1, \ldots, x_r)}{s_r} \right)\)
\end{enumerate}
\end{proof}
These two identities and \( s \left( \binom{s}{r-s} \right) \binom{n-1}{r-s} = \binom{r-s}{n-1} \binom{n-1}{r-s} \) provide
\[
X_{n,r}^s(x_1, \ldots, x_r) = \sum_{\{i_1, \ldots, i_{r+s}\} \subseteq \{1, \ldots, r\}} \frac{1}{s!} \cdot X_{n,r+s}(x_{i_1}, \ldots, x_{i_{r+s}}).
\]

Using the first line of (11), by similar comparison we get
\[
X_{n,r}^{s-1}(x_1, \ldots, x_r) = \sum_{\{i_1, \ldots, i_{r+s}\} \subseteq \{1, \ldots, r\}} \frac{1}{s!} \cdot X_{n,r+s}(x_{i_1}, \ldots, x_{i_{r+s}}).
\]

Therefore,
\[
X_{n,r}^s - X_{n,r}^{s-1} = \sum_{\{i_1, \ldots, i_{r+s}\} \subseteq \{1, \ldots, r\}} \frac{1}{s!} \left( X_{n,r+s}(x_{i_1}, \ldots, x_{i_{r+s}}) - X_{n,r+s-1}(x_{i_1}, \ldots, x_{i_{r+s-1}}) \right).
\]

Thus, if the inequality \( X_{n,r}^s \geq X_{n,r}^{s-1} \) is satisfied for any \( n' \leq n \) and \( r' \leq r \), then \( X_{n',r'}^s \geq X_{n',r'}^{s-1} \) is also satisfied for any \( n' \leq n, r' \leq r \) and \( 0 \leq s' < r' - 1 \).

**Step 2.** We prove \( X_{n,r}^s \geq X_{n,r}^{s-1} \), or equivalently, \( \frac{X_{n,r}^s}{X_{n,r}^{s-1}} \geq \frac{X_{n,r}^{s-1}}{X_{n,r}^{s-2}} \).

Note that \( \tilde{X}_{n,r}^s = \left( \prod_{i=1}^r x_i \right) \tilde{X}_{n-r,s} \) and similarly \( \tilde{X}_{n,r}^{s-1} = \left( \prod_{i=1}^r x_i \right) \tilde{X}_{n-r,s}^{s-1} \). Both \( \tilde{X}_{n-r,s} \) and \( \tilde{X}_{n-r,s}^{s-1} \) can be decomposed further according to the \( s \)-types: \( \tilde{X}_{n-r,s} = \sum_{s=0}^{r-1} \tilde{X}_{n-r,s}^s \) and \( \tilde{X}_{n-r,s}^{s-1} = \sum_{s=0}^{r-2} \tilde{X}_{n-r,s}^s \).

We set
\[
\tilde{X}_{n,r}^s := \left( \prod_{i=1}^r x_i \right) \tilde{X}_{n,r,s}^s \quad \text{and} \quad \tilde{X}_{n,r}^{s-1} := \left( \prod_{i=1}^r x_i \right) \tilde{X}_{n,r,s}^{s-1}.
\]

We claim that for any \( 0 \leq s \leq r - 2 \) one has:
\[
(12) \quad (r-s-1) \cdot \tilde{X}_{n,r}^{s+1} \geq (s+1)(s+2) \cdot \tilde{X}_{n,r}^{s+2}.
\]

This follows from the ‘elementary’ inequality \( (P_{i_1,i_2}^k) := x_{i_1}^{k-1} + x_{i_2}^{k-1} \geq x_{i_1}^{k-1} x_{i_2}^{k-1} + x_{i_1} x_{i_2} \), where \( \{i_1, i_2\} \subseteq \{1, \ldots, r\} \) and \( k \geq 2 \). Indeed, for any fixed pair \( (i_1, i_2) \) consider all the monomials of type \( M = \prod_{i=1}^r x_i^{m_i} / (x_{i_1}^{m_{i_1}} x_{i_2}^{m_{i_2}}) \) with \( m_i > 0 \) and of degree \( n - k \). Then \( (x_{i_1}^{k-1} + x_{i_2}^{k-1}) \cdot M \in \tilde{X}_{n,r,s} \). Moreover, each monomial \( \tilde{X}_{n,r,s}^k \) in \( \tilde{X}_{n,r,s} \) can be realized in exactly \( r-1-s \) ways, where \( (r-1-s) \) stays for the number of \( k \)'s which \( k \geq 2 \).

Consider next the same monomial \( M \) as before. Then \( \overline{M} := (x_{i_1}^{k-1} + x_{i_2}^{k-1}) \cdot M \in \tilde{X}_{n,r,s}^{s+1} \). If \( k > 2 \) then the number of exponents in \( \overline{M} \) which equal 1 is \( s+1 \), hence \( \overline{M} \in \tilde{X}_{n,r,s}^{s+1} \). If \( k = 2 \) then \( \overline{M} \in \tilde{X}_{n,r,s}^{s+2} \). Any monomial \( \tilde{X}_{n,r,s}^k \) in \( \tilde{X}_{n,r,s}^{s+1} \) can be realized in exactly \( (r-1-s)(s+1) \) ways, which is \( \{i : k_i \geq 2\} : \{i : k_i = 1\} \), the number of possible candidates for the pair \( (i_1, i_2) \) for \( (P_{i_1,i_2}^k) \). Furthermore, any monomial \( \tilde{X}_{n,r,s}^k \) in \( \tilde{X}_{n,r,s}^{s+2} \) can be realized in exactly \( (s+1)(s+2) \) ways, the possible ordered pairs of \( \{i : k_i = 1\} \).

Hence all the possible inequalities \( (P_{i_1,i_2}^k) \) multiplied by the possible monomials provide exactly (12). Note that the above monomial counting is compatible with the identity obtained from \( \left| K_{n,r}^s \right| = \left( \binom{s-1}{r-s} \right) \) and
\[
r \cdot \left| K_{n,r+s-1,r-1} \right| = (s+1) \cdot \left| K_{n,r,s} \right| + \frac{(s+1)(s+2)}{r-s-1} \cdot \left| K_{n-r,s} \right|.
\]

Now, by taking the sum over \( s \) in (12), and by reorganizing the right hand side we obtain
\[
\tilde{X}_{n,r}^1 \geq \left( \prod_{i=1}^r x_i \right) \sum_{s=0}^{r-2} \left( (s+1) \tilde{X}_{n-r,s}^{s+1} + \frac{(s+1)(s+2)}{r-s-1} \tilde{X}_{n-r,s}^{s+2} \right) = \left( \prod_{i=1}^r x_i \right) \sum_{s=1}^{r-1} \frac{sr}{r-s+1} \tilde{X}_{n-r,s}^s
\]

Note that in the last sum the term with \( s = 0 \) can also be included, as its coefficient vanishes. Thus (for some \( c > 0 \)):
\[
(13) \quad c(X_{n,r}^1 - X_{n,r}^0) = \frac{n-r+1}{r(r-1)} \cdot \tilde{X}_{n,r} - \tilde{X}_{n,r}^0 \geq \left( \prod_{i=1}^r x_i \right) \sum_{s=0}^{r-1} \frac{s(n-r+1)}{(r-s-1)(r-1)} \left( \frac{n-r}{s} \right)^{r-1} \cdot X_{n-r,s}.
\]

Next, the right hand side of (13) is non-negative by the following generalization of the Chebyshev’s sum inequality (which basically is the summation \( \sum_{s,t} (\alpha_s \alpha_t \beta_s - \alpha_s \alpha_t \beta_t)(x_s - x_t) \geq 0 \)), see [Hardy-Littlewood-Pólya, p. 43],
\[
(14) \quad \left( \sum_{s} \alpha_s \beta_s \right) \left( \sum_{s} \alpha_s x_s \right) \leq \left( \sum_{s} \alpha_s \right) \left( \sum_{s} \alpha_s x_s \right)
\]
whenever $x_s$ and $\beta_s$ are both decreasing (or both increasing) sequences and $\alpha_s > 0$. In the present case take $\alpha_s := \binom{r}{s}^{n-r-1}$, $\beta_s := \frac{s(n-r+1)}{(r-s+1)(r-1)}$ and $x_s := X_{n-r,r}^s$. Clearly $\beta_s$ is increasing, $x_s$ is increasing by induction, and $\sum_s \alpha_s = \sum_s \beta_s \alpha_s$ by a computation based on $\sum_s \binom{r}{s}^{n-r-1} = \binom{n}{r}^s$. Hence, via (13), $X_{n,r}^1 \geq X_{n,r}^0$.

(b) As in part (a), we first reduce the general statement to $Y_{n,r,\ell}^0 > Y_{n,r,\ell}^1$, and then we prove this particular case too. As in the previous case, set $\tilde{Y}_{n,r,\ell}^s := \sum_{k \in K_{n,r}^s} \gamma_{k,\ell} = \binom{r}{s}^{n-s-1}Y_{n,r,\ell}^s$. We wish to prove:

$$Y_{n,r,\ell}^s \frac{(s+1)(n-r+s+1)}{(r-s)(r-s-1)} > \tilde{Y}_{n,r,\ell}^s.$$

**Step 1.** Use the decomposition $K_{n,r}^s = \prod K_{n,r-s}^i$, the disjoint union of $\binom{r}{s}$ copies, to get:

$$\tilde{Y}_{n,r,\ell}^s = \sum_{\sum_{i=1}^{\ell} (k_i + \ell) = s} \frac{1}{(k_i + \ell)!} = \frac{1}{(\ell)!} \tilde{Y}_{n,r-s,\ell}^0.$$

Similarly, $\tilde{Y}_{n,r,\ell}^s = \binom{r}{s}^{n-s-1}Y_{n,r-s,\ell}^s$. Here we used $\tilde{Y}_{n,r-s,\ell}^s = \sum_{k \in K_{n,r-s,\ell}^s} \gamma_{k,\ell} = \frac{1}{(s)!} \tilde{Y}_{n,r-s,\ell}^0$, cf. (16).

Therefore, the inequality (15) for any $s$ is equivalent to (15) for $s = 0$.

**Step 2.** Here we prove $\tilde{Y}_{n-r,0,\ell}^0 > \tilde{Y}_{n-r,1,\ell}^1$. We run induction on $n$: we assume the stated inequalities, indexed by $(n,r,\ell)$, is true for any $(n',r,\ell')$ with $n' < n$ (but $\ell'$ can be larger than $\ell$). In fact, we use $(n-r,r,\ell+1) \Rightarrow (n,r,\ell)$.

We consider exactly the same combinatorial set–decomposition as in Step 2 of (a), the only difference is that we replace the inequality $\binom{k}{i_1,i_2}$, written for $k \geq 2$, by

$$\binom{k}{i_1,i_2} : \frac{2}{(k+\ell)!} < \frac{2}{(k-1+\ell)!}.\quad (17)$$

After a computation, we obtain the analogue of (13) (for the same positive constant $c$), namely

$$c(Y_{n,r,\ell}^1 - Y_{n,r,\ell}^0) = \frac{n-r+1}{r(r-1)} \cdot \tilde{Y}_{n,r,\ell}^1 - Y_{n,r,\ell}^0 < \sum_{s=0}^{r-1} \frac{s(n-r+1)}{(r-s+1)(r-1)} \binom{r}{s} \frac{n-r-1}{r-s-1} \cdot Y_{n-r,r,\ell+1}^s.$$  

The right hand side of (17) is non-positive by (14) with reversed inequality, valid for $\alpha_s > 0$, $\beta_s$ and $x_s$ oppositely ordered. Indeed, $\alpha_s$ and $\beta_s$ are the same as before with $\beta_s$ increasing, while $x_s := Y_{n-r,r,\ell+1}^s$ is a decreasing by induction. Hence, via (17), $Y_{n,r,\ell}^1 < Y_{n,r,\ell}^0$.

(c) The proof is double induction over $r$ and $n$. We assume that for any fixed $r$ and $n$ the inequality $(I_{n,r,\ell})$ is true for any $n$ and $\ell$ and $r' < r$, and $(I_{n',r,\ell})$ is true for any $n' < n$ and any $\ell$. We wish to prove $(I_{n,r,\ell})$.

First we write the left hand side of the inequality as a sum

$$\sum_{k \in K_{n,r}^s} \gamma_{k,\ell} \cdot 2^k = \sum_{s=0}^{r-1} \sum_{k \in K_{n,r}^s} \gamma_{k,\ell} \cdot 2^k.$$  

Note that corresponding to $s = 0$, after we factor out $x_1 \cdots x_r$, the sum over $K_{n,r}^0$ can be identified with the left hand side of the inequality $(I_{n-r,r,\ell+1})$ (multiplied by $x_1 \cdots x_r$). Hence, by the inductive assumption,

$$\sum_{k \in K_{n,r}^0} \gamma_{k,\ell} \cdot 2^k \leq Y_{n,r,\ell}^0 \cdot \sum_{k \in K_{n,r}^0} 2^k.$$  

For $s = 1$, the sum over $K_{n,r}^1$ is a sum of $r$ sums corresponding to the ‘missing’ coordinate $x_1$, and each of them can be identified (after factorization of a monomial) with the inequality $(I_{n-(r-1),r-1,\ell+1})$. For an arbitrary $s \leq r - 2$ one can apply in the similar way the inequality $(I_{n-(r-s),r-s,\ell+1})$. In the case of $s = r - 1$ all coefficients $\gamma_{k,\ell}$ equal $[(r')^{r-1}(n+l)]^{-1}$. Therefore, by induction, we get

$$\sum_{s=0}^{r-1} \sum_{k \in K_{n,r}^s} \gamma_{k,\ell} \cdot 2^k \leq \sum_{s=0}^{r-1} |K_{n,r}^s| \cdot Y_{n,r,\ell}^s \cdot X_{n,r}^s.$$  

But, using parts (a) and (b), by Chebyshev’s sum inequality ((14) with $\alpha_s = 1$):

$$\sum_{s=0}^{r-1} |K_{n,r}^s| \cdot Y_{n,r,\ell}^s \cdot X_{n,r}^s \leq |K_{n,r}| \cdot Y_{n,r,\ell}^s \cdot X_{n,r}.$$  

Therefore, the inequality (15) holds for any $s$ and is equivalent to (15) for $s = 0$.
whose right hand side is the left hand side of Combinatorial Inequality. This ends the proof of (c). ■

The above discussion and the statement of Theorem 4.1(b) imply the inequality (3) from the introduction as well. This is a proof of (3) in the spirit of the Combinatorial Inequality (based on Chebyshev’s type inequalities), for a different proof see [Kerner-Némethi.a].

Corollary 4.2. \( C_{n,r} > C_{n,r+1} \) for any \( n \geq 1 \) and \( r \geq 1 \).

Proof. By (1) the inequality \( C_{n,r} > C_{n,r+1} \) is equivalent to \( Y_{n,r} < Y_{n,r+1} \). We drop the index \( \ell \) = 1 from the notations (hence, we write e.g. \( Y_{n,r} := Y_{n,r,1} \)), and we set \( Y_{n,r}^{\geq 1} := \cup_{s \geq 1} Y_{n,r,s}^s \), and similarly \( Y_{n,r}^{\geq 1} \) and \( K_{n,r}^{\geq 1} \). By 4.1, part (b), the mean \( Y_{n,r+1}^{0} \) is the largest among \( \{ Y_{n,r+1}^{s} \}_{s} \), hence \( Y_{n,r+1}^{s} > Y_{n,r}^{s} \). Hence, we need to prove \( Y_{n,r}^{s} \geq Y_{n,r} \). They can be decomposed as sums over the same index set. Indeed, by similar arguments as in the previous proof part (b)

\[
\tilde{Y}_{n,r+1}^{\geq 1} = \sum_{s=0}^{r-1} \binom{r+1}{s+1} Y_{n,r-s} Y_{n,r-s} \quad \text{and} \quad Y_{n,r} = \sum_{s=0}^{r-1} \binom{r+1}{s+1} Y_{n,r-s} Y_{n,r-s}.
\]

This can be rewritten as

\[
Y_{n,r}^{s} = \frac{1}{\sum_{s=0}^{r-1} \binom{r+1}{s+1}} \sum_{s=0}^{r-1} \binom{r}{s} Y_{n,r-s} Y_{n,r-s}.
\]

Similarly,

\[
Y_{n,r} = \frac{1}{\sum_{s=0}^{r-1} \binom{r}{s}} \sum_{s=0}^{r-1} \binom{r}{s} Y_{n,r-s} Y_{n,r-s}.
\]

Hence, via (18) and (19), the inequality \( Y_{n,r} \leq Y_{n,r+1}^{s} \) follows from (14) applied for \( \alpha_s = \binom{r}{s}(n-r-s) \), \( \beta_s = \frac{r+1}{s+1} \), and \( x_s = Y_{n,r}^{s} \). Indeed, \( \alpha_s > 0 \) while \( x_s \) and \( \beta_s \) are both decreasing sequences. For \( x_s \) this follows from Theorem 4.1(b). ■

5. THE HOMOGENEOUS CASE FOR \( n = 2 \)

Assume that \((X,0)\) is a 2-dimensional ICIS with \( p_i \geq 2 \) for all \( i \). Then \( C_{2,r} = \frac{r+1}{r+\frac{1}{2}} \). Set

\[
P := \prod_i p_i, \quad \text{the multiplicity of } (X,0).
\]

Then using (1) one obtains

\[
\frac{p_i}{P} = \frac{1}{6} \sum_i (p_i - 1)(p_i - 2) + \sum_{i<j} (p_i - 1)(p_j - 1)
\]

and

\[
\frac{\mu + 1 - P}{P} = \frac{1}{6} \sum_i ((1-p_i) + (1-p_i)^2) + \sum_{i<j} (1-p_i)(1-p_j).
\]

By a computation \( \mu + P \cdot E + 1 = C_{2,r} \cdot p_g \), where \( E := \frac{r+1}{r+\frac{1}{2}} \sum_{i=1}^r (p_i + 1) - \sum_{i<j} (p_i - p_j)^2 \). If \( r = 1 \) then \( E = -1 \), but for \( r \geq 2 \) and for some choices of \( p_i \)'s (e.g. whenever they are all equal) \( E \) might be positive, providing \( C_{2,r} \cdot p_g \geq \mu + 1 \). We collect here the precise statements:

Theorem 5.1. (a) If \( r = 1 \) then \( 6 p_g = \mu + 1 - P \).

(b) If \( r \geq 2 \) then \( C_{2,r} \cdot p_g = \frac{r+1}{r+\frac{1}{2}} \cdot p_g \). In general the bound \( C_{2,r} \cdot p_g \leq \mu + 1 \) does not hold, although asymptotically \( \frac{\mu}{p_g} \) tends to \( \frac{r+1}{r+\frac{1}{2}} \).

(c) For any \( r \) the inequality \( 4 p_g \leq \mu + 1 - P \) is valid, and if \( p_i = 2 \) is allowed then \( 4 \) is the sharpest bound whenever \( r > 1 \).

(d) If \( p_i \geq d + 1 \) for all \( i \) then

\[
4 \frac{d(r-1) + 2(d-1)}{d(r-1) + \frac{3}{4}(d-1)} \cdot p_g \leq \mu + 1 - P.
\]

Proof. Using the above explicit formulae, all the statements are elementary. Let us give some hint for (d). Note that \( (4+c)p_g \leq \mu + 1 - P \) reads as

\[
(4+c) \sum_i \frac{(p_i - 1)(p_i - 2)}{6} + c \sum_{i<j} \frac{(p_i - 1)(p_j - 1)}{6} \leq \sum_i (1-p_i) + \sum_i (1-p_i)^2.
\]
Using \( \sum_{i=1}^{r} a_i^2 \geq \frac{2}{r-1} \sum_{i<j} a_i a_j \), the inequality (20) follows from
\[
(4 + c) \sum_{i} \left( \frac{p_i - 1}{2} \right) + c \cdot \frac{r - 1}{8} \sum_{i} (p_i - 1)^2 \leq \sum_{i} (1 - p_i) + \sum_{i} (1 - p_i)^2.
\]
This is a sum over \( i \) of elementary quadratic inequalities whose discussion is left to the reader. ■

REFERENCES

[Abramowitz-Stegun] M. Abramowitz, I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards Applied Mathematics Series, 55, 1964.

[AGLV-book] V.I. Arnol’d, V.V. Goryunov, O.V. Lyashko, V.A. Vasil’ev, Singularity theory. I. Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [Dynamical systems. VI, Encyclopaedia Math. Sci., 6, Springer, Berlin, 1993]. Springer-Verlag, Berlin, 1998.

[Ashikaga] T. Ashikaga, Normal two-dimensional hypersurface triple points and the Horskawa type resolution. Tohoku Math. J. (2) 44 (1992), no. 2, 177–200.

[Buchweitz-Greuel1980] R.-O. Buchweitz, G.-M. Greuel, The Milnor number and deformations of complex curve singularities. Invent. Math. 88 (1980), no. 3, 241–281.

[Durfee1978] A.H. Durfee, The signature of smoothings of complex surface singularities. Math. Ann. 232 (1978), no. 1, 85–98.

[Greuel1975] G.M. Greuel, Der Gauss-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten. Math. Ann. 214 (1975), 235-266.

[Greuel-Hamm1978] G.M. Greuel, H.A. Hamm, Invarianten quasihomogener vollständiger Durchschnitte., Invent. Math. 49 (1978), no. 1, 67-88.

[Hamm1986] H.A. Hamm, Invariants of weighted homogeneous singularities. Journées Complexes 85 (Nancy, 1985), 613, Inst. Élie Cartan, 10, Univ. Nancy, Nancy, 1986.

[Hamm2011] H.A. Hamm, Differential forms and Hodge numbers for toric complete intersections, arXiv:1106.1826.

[Hardy-Littlewood-Pólya] G.H.Hardy, J.E.Littlewood, G.Pólya, Inequalities. Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

[Jordan1965] Ch.Jordan, Calculus of finite differences. Third Edition. Introduction by Harry C. Carver Chelsea Publishing Co., New York 1965 xxi+655 pp

[Kerner-Némethi2009] D. Kerner and A. Némethi, The Milnor fibre signature is not semi-continuous, Proc. of the Conference in Honor of the 60th Birthday of A. Libgober, Topology of Algebraic Varieties, Jaca (Spain), June 2009; Contemporary Math. 538 (2011), 369-376.

[Kerner-Némethi.a] D. Kerner and A. Némethi, A counterexample to Durfee conjecture, Comptes Rendus Mathématiques de l’Académie des Sciences, vol.34 (2012), no.2. arXiv:1109.4869

[Kerner-Némethi.c] D. Kerner, A. Némethi, On Milnor number and singularity genus, in preparation.

[Khovanskii1978] A.G. Khovanskii, Newton polyhedra, and the genus of complete intersections. (Russian) Funktsional. Anal. i Prilozhen. 12 (1978), no. 1, 51-61.

[Lauffer1977] H.B. Lauffer, On μ for surface singularities, Proceedings of Symposia in Pure Math. 30, 45-49, 1977.

[Looijenga-book] E. Looijenga, Isolated Singular Points on Complete Intersections. London Math. Soc. LNS 77, CUP, 1984.

[Looijenga1986] E. Looijenga, Riemann-Roch and smoothings of singularities. Topology 25 (1986), no. 3, 293-302.

[Morales1985] M. Morales, Fonctions de Hilbert, genre géométrique d’une singularité quasi homogène Cohen-Macaulay. C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), no. 14, 699–702.

[Melle-Hernández2000] A. Melle-Hernández, Milnor numbers for surface singularities. Israel J. Math. 115 (2000), 29–50.

[Milnor-book] J. Milnor, Singular points of complex hypersurfaces, Annals of Math. Studies 61, Princeton University Press 1968.

[Némethi98] A. Némethi, Dedekind sums and the signature of f(x,y) + zN. Selecta Math. (N.S.) 4 (1998), no. 2, 361–376.

[Némethi99] A. Némethi, Dedekind sums and the signature of f(x,y) + zN. II. Selecta Math. (N.S.) 5 (1999), 161–179.

[Saito1981] M. Saito, On the exponents and the geometric genus of an isolated hypersurface singularity. Singularities, Part 2 (Arcata, Calif., 1981), 465–472, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.

[Seade-book] J. Seade, On the Topology of Isolated Singularities in Analytic Spaces. Progress in Mathematics 241, Birkhäuser 2006.

[Tomari1993] M. Tomari, The inequality 8\(p_0 < \mu \) for hypersurface two-dimensional isolated double points. Math. Nachr. 164 (1993), 37–48.

[Wahl1981] J. Wahl, Smoothings of normal surface singularities, Topology 20 (1981), 219–246.

[Xu-Yau1993] Y.-J. Xu, S.S.-T. Yau, Durfee conjecture and coordinate free characterization of homogeneous singularities. J. Differential Geom. 37 (1993), no. 2, 375–396.

[Yau-Zhang2006] S.-T.Yau, L.Zhang, An upper estimate of integral points in real simplices with an application to singularity theory. Math. Res. Lett. 13 (2006), no. 5-6, 911–921.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE STREET, TORONTO, CANADA
E-mail address: dmitry.kerner@gmail.com

RÉNYI INSTITUTE OF MATHEMATICS, BUDAPEST, REÁLTANODA U. 13–15, 1053, HUNGARY
E-mail address: nemethi@renyi.hu