Non-linear Model Predictive Control of Conically Shaped Liquid Storage Tanks

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Abstract

This paper deals with the analysis and synthesis of a model predictive control (MPC) strategy used in connection with level control in conically shaped industrial liquid storage tanks. The MPC is based on a dynamical non-linear model describing the changes of the liquid level with respect to changes in the inlet flow of the liquid. An Euler discretization of the dynamical system is exploited to transform the continuous time dynamics to its discrete time counterpart, used in the non-linear MPC (NMPC) synthesis. By means of a simulation case study will be shown, that NMPC better tracks the changes of the liquid level, hence provides increased control performance. This paper also compares the traditional approach of optimal control, the linear MPC, with the NMPC strategy.

1. Introduction

The model predictive control is a well-established control strategy in chemical process control. The main advantages stem from optimally shaping the trajectory of manipulated variables with respect to performance criteria and technological and safety constraints \cite{Mayne2000,Camacho2007}. The optimal control strategies have been systematically addressed in countless scientific works, including time optimal control \cite{Sharma2015}, or standard model predictive control \cite{Muske2002,Kvasnica2010,Bakosova2014}. All aforementioned works, however, focus on the standardized design of the model predictive control, which relies on linear...
state space models of the controlled plant. Such approaches, however, introduce an obstacle, which is called "model-mismatch", where the design model in the controller does not match the actual process.

To remedy the situation, researchers focus on non-linear model predictive control (NMPC), which improves given control strategies by incorporating the non-linear equations capturing the dynamics of the system (Allgöwer et al., 2004). This work focuses on the application of such a controller to the most common chemical process, which is the control of a level of the liquid inside storage tanks. Specifically, we focus on a conically-shaped liquid storage tank.

This paper is organized as follows. First, we introduce the non-linear mathematical model of conical tank. Second, we focus on the synthesis of two controllers, the linear MPC and the non-linear MPC. Lastly, we compare the performance of aforementioned controllers by the means of simulation case study.

2. Mathematical Modeling of Conically Shaped Tanks

The dynamical mathematical model of a tank with one inlet stream, denoted as \( q_{in}(t) \) and one outlet stream given by \( q_{out}(t) \), is given by a mass balance equation of following form

\[
q_{in}(t) = q_{out}(t) + \frac{dV(t)}{dt},
\]

where the \( V(t) \) stands for the volume of a liquid inside the tank. In this work, we consider the level of the liquid inside the tank as a process variable, hence we rewrite the model in (1) to

\[
q_{in}(t) = k_v \sqrt{h(t)} + \frac{V(t)}{h(t)} \frac{dV(t)}{dt},
\]

and we define

\[
F(h) = \frac{dV(t)}{dh}.
\]

For the purpose of performing simulations, we convert the model in (2) to a non-linear state space form

\[
\frac{d}{dt} \left( h(t) \right) = \frac{1}{F(h)} \left( q_{in}(t) - k_v \sqrt{h(t)} \right).
\]
Figure 1: Illustration of the conically-shaped tank.

The variable $k_v$ correspond to an output valve coefficient. The valve coefficient can be derived from Bernoulli equation, and it represents the friction of liquid movement in the outlet pipe (Miklč and Fikar 2007, ch. 2).

In this work we consider a controller synthesis, which is based on a discrete time model, hence the non-linear system model can obtained by Euler discretization of (4). Specifically,

$$h(t + T_s) = h(t) + T_s \cdot \left( \frac{1}{F(h)} \left( q_{in}(t) - k_v \sqrt{h(t)} \right) \right) , \quad (5)$$

where the variable $T_s$ represent the sampling time. Even though the Euler discretization process can be inexact, it is often used in controller design as suggested by Lawryńczuk (2017).

We consider an inverted frustum of a right cone as an open conical tank process. The geometrical representation of the conical tank is shown in the Fig. 2. The model of such a process is based on findings by King (2010), and it is derived by expressing the volume of the frustum as a function of the level of the liquid. The tank is characterized by variables $R_1$, $R_2$, which are radii of the
bottom and upper base, respectively and by the height $h_{\text{max}}$ (cf. Fig. 2). The volume of the liquid inside the frustum is given by

$$V_f(h(t)) = \frac{\pi h(t)}{3} \left( r_f^2(h(t)) + R_2 r_l(h(t)) + R_2^2 \right),$$

(6)

where the variable $r_l(h(t))$ is the radius of a disc representing the surface of the liquid at level $h(t)$. The radius $r_l(h(t))$ is explicit function of the liquid level, expressed as

$$r_l(h(t)) = R_2 + \frac{R_1 - R_2}{h_{\text{max}}} h(t).$$

(7)

By substituting the expression in (7) to (6) we obtain

$$V_f(h(t)) = \frac{\pi h(t)}{3} \left( 3R_2^2 + 3R_2 \frac{R_1 - R_2}{h_{\text{max}}} h(t) + \left( \frac{R_1 - R_2}{h_{\text{max}}} \right)^2 h^2(t) \right).$$

(8)

Next, we combine the expression for the volume in (8) and the general mass balance model in (2), which results in

$$q_{\text{in}}(t) = k_v \sqrt{h(t)} + \pi \left( R_2 + h(t) \frac{R_1 - R_2}{h_{\text{max}}} \right)^2 \frac{d}{dt} h(t).$$

(9)

Symbols, physical quantities and parameters are reported in the table 1. The non-linear mathematical model reported in (9) is used in the synthesis of the NMPC strategy, addressed in the next section.

3. Synthesis of Controllers

In this work, we consider the synthesis of the non-linear model predictive control strategy, which exploits the non-linear nature of the dynamical model. In order to demonstrate the benefits of the non-linear controller, we compare this approach with the standardized linear version of the MPC. Both of these controllers are implemented in scheme depicted on the Fig. 2.

The closed-loop control is realized also with an estimator, which purpose is to estimate possible mismatch between the design model and the actual process. Such a control strategy has been adopted from works by Rawlings and Mayne (2009) and Muske (1997).
Table 1: Parameters of the conical tank system and quantities related to system dynamics.

| Physical quantity          | Symbol | Value       |
|----------------------------|--------|-------------|
| Height steady state        | $h_L$  | 0.4000 m    |
| Inlet steady state         | $q_{in, L}$ | 0.0474 m$^3$s$^{-1}$ |
| Valve coefficient          | $k_v$  | 0.0750 m$^2$s$^{-1}$ |
| Maximum height             | $h_{max}$ | 2.0000 m    |
| Upper radius               | $R_1$  | 1.0000 m    |
| Bottom radius              | $R_2$  | 0.4000 m    |
| Minimum flow               | $q_{in, min}$ | 0.0000 m$^3$s$^{-1}$ |
| Maximum flow               | $q_{in, max}$ | 0.1000 m$^3$s$^{-1}$ |
| Sampling time              | $T_s$  | 2.0000 s    |

Figure 2: General model predictive control strategy scheme. The $r(t)$ stands for the reference signal, i.e., the desired level of the liquid, next the $u^*(t)$ is the optimal control action, i.e., the inlet flow of liquid. The actual measurement of the liquid level is depicted by $h_m(t)$, while the estimate of the level is denoted by $\hat{h}(t)$. 
The synthesis and implementation of model predictive control follow the principles receding horizon policy established by [Mayne et al., 2000]. It optimizes control actions over a prediction horizon $N$ based on predictions of the future trajectory of the process variable.

Specifically, the non-linear model predictive controller is casted as an optimization problem with a quadratic cost function and nonlinear equality constraints,

\[
\min_{u_0, \ldots, u_{N-1}} \sum_{k=0}^{N-1} \left( ||(x_k - r_k)||_Q^2 - ||(u_k - u_{k-1})||_Q^2 \right) \quad \text{(10a)}
\]

s.t. $x_{k+1} = x_k + T_s \cdot h(x_k, u_k)$, \quad \text{(10b)}
\[
x_k \in [h_{\min}, h_{\max}], \quad \text{(10c)}
\]
\[
u_k \in [q_{\text{in}, \min}, q_{\text{in}, \max}], \quad \text{(10d)}
\]
\[
(u_k - u_{k-1}) \in [\Delta q_{\text{in}, \min}, \Delta q_{\text{in}, \max}], \quad \text{(10e)}
\]
\[
x_0 = h(t), \quad u_{-1} = u(t - T_s). \quad \text{(10f)}
\]

The objective function (10a) penalizes the difference between prediction of the liquid level $x_k$ and height reference $r_k$, followed by a second term which penalizes the increments of control actions. Such a structure of the objective function enforces offset-free control performance [Muske and Badgwell, 2002]. Note, that the term $||z||_M^2 = z^\top M z$ represents a squared Euclidean norm. The prediction equation (10b) is represented by the non-linear dynamical model from (5). Constraints (10c) and (10d) ensure, that technological limits on the process variable, and on the manipulated variable are satisfied. Namely, the constraint (10c) represent physical dimension of the tank, the constraint (10d) defines the range of inlet flow, while the equation (10e) bounds how fast the inlet flow can change, i.e., how fast can the control valve by open or closed. Lastly, the optimization problem is initialized by the current measurement of the height and by previous control action, as in (10f), and constraints (10b)-(10e) are enforced for $k = 0, \ldots, N - 1$.

The optimization problem given by (10) can be solved by off-the-shelve tools like \textit{fmincon} in Matlab, which exploits procedures like interior-point method or
trust-region method [Nocedal and Wright 2006].

The linear version of the MPC has the same form as the non-linear version given by (10), except the constraint in (10b) which represent the prediction equation. Here, the non-linear dynamical equation is linearized by Taylor first-order expansion around an operating point (cf. Remark 3.1) denoted as \((h_{L}, q_{in,L})\). The resulting prediction equation has the form of a linear state space model, which is subsequently discretized by a sampling time \(T_{s}\), specifically

\[
x(t + T_{s}) = Ax(t) + Bu(t),
\]

where the state vector \(x(t)\) and control input \(u(t)\) is define as a deviation from respective steady states values. The linear MPC is then casted as quadratic optimization problem (QP) with linear constraints. This QP problem can be solved by \texttt{quadprog} function in Matlab, or by GUROBI solver. Note, that the synthesis of individual controllers is a general procedure, however, we used parameters from the table I to construct the optimization problems.

\textbf{Remark 3.1.} The operating point, often called a steady state, can be explicitly calculated from the non-linear mode in (4) by solving

\[
\frac{1}{F(h_{L})} (q_{in,L} - k_{v} \sqrt{h_{L}}) = 0.
\]

Note, that the choice of operating point affects the performance of linear-based control strategies. Note, that the linearisation point should be chosen with respect to technological properties of the plant.

\section*{4. Comparisons and Results}

The performance of proposed control strategies has been tested on a simulation scenario involving a single conical tank, described by equation (4) and parameters reported in the table I. We consider a simulation window of 400 s, where a reference change, i.e. the desired level of the liquid changes, occurs at times \(t_{up} = 50\) s and at \(t_{down} = 350\) s. Specific time profiles of process and manipulated variables can be viewed on the Fig. 3.

Both presented approaches have a couple of advantages, which includes constraint satisfaction as well as their enforce optimal behavior. Furthermore,
Figure 3: Comparison of control performance under authorities of linear-based MPC and non-linear model predictive control.
the nature of predictive control can see around the times $t_{up}$ and $t_{down}$, where the controller reacts in an anticipation of the reference changes. Naturally, the non-linear predictive controller handles the changes in the level control better, that the linear-based control. The advantage can be seen mainly in operation towards lower levels of liquid, there the linear-based control undershoots the reference significantly. Note, that such a controller cannot be used when considering liquid levels close to the bottom of the tank.

5. Conclusions

This paper covered the design and comparison of two predictive control strategies for the most important chemical process, the liquid storage tank. Specifically, a conically-shaped storage device was considered. Both controllers have enforced constraint satisfaction, which is one of the most important tasks in the process control. Moreover, we have shown, that by considering a non-linear prediction equation in the controller, we have achieved better tracking of the desired liquid level when considering step-down reference change. Compared to the linear-based MPC, the NMPC is capable of regulating the liquid level even near the bottom of the storage tank.

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