Topological entropy, homological growth and zeta functions on graphs

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Abstract

In connection with the entropy conjecture it is known that the topological entropy of a continuous graph map is bounded from below by the logarithm of the spectral radius of the induced map on the first homology group. We show that in the case of a piecewise monotone graph map, its topological entropy is equal precisely to the maximum of the mentioned logarithm of the spectral radius and the exponential growth rate of the number of periodic points of negative type. This nontrivially extends a result of Milnor and Thurston on piecewise monotone interval maps. For this purpose we generalize the concept of Milnor–Thurston zeta function incorporating in the Lefschetz zeta function.

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1. Introduction and the main results

One of the most exciting problems in the theory of dynamical systems for the last three decades has been the so-called entropy conjecture which was first stated by Shub in [S] and which claims that for a smooth map on a compact differentiable manifold, its topological entropy is bounded from below by the logarithm of the spectral radius of the induced map in the corresponding full homology group (called also homological entropy). Many special cases of this conjecture have already been proved even in the nonsmooth case but it still remains open in general. In particular, Manning in [M] proved that the conjecture holds for all continuous maps on differentiable manifolds if we reduce our attention from that full homology group to the first homology group. Using the arguments from that paper it is possible to extend this result for
more general spaces including graphs (see [FM]). In this paper, we show that for piecewise monotone graph (PMG) maps, the topological entropy is not only bounded from below by its homological entropy but it is exactly equal to the maximum of its homological entropy and entropy given by the growth of its periodic points of negative type. (In fact, using this result it is possible to provide an alternative proof of Manning’s result for continuous graph maps.)

First, we recall some notions and definitions needed in the following. By the homological entropy of a map \( f : X \to X \) we mean a topological invariant \( h_{\text{hom}}(f) \) coming from considering the induced linear maps \( f_* \) on the homology groups \( H_i(X, \mathbb{R}) \) and defined by

\[
h_{\text{hom}}(f) = \log r(f),
\]

where \( r(f) = \max \{r(f_*^i) : i = 0, \ldots, \dim X \} \) and \( r(f_*^i) \) denotes the spectral radius of \( f_*^i \). The purpose of this paper is to establish a precise relationship between the topological entropy \( h_{\text{top}}(f) \) and the homological entropy \( h_{\text{hom}}(f) \) for a PMG map \( f \) (for the definition of topological entropy see any standard textbook on dynamical systems; a nice introduction to the topic can be found in [ALM]). The compact interval and the circle are the simplest examples of graphs. In general, a graph is a compact Hausdorff space which can be written as a union of finitely many homeomorphic copies of the closed interval \([0, 1]\) any two of which intersect at most at their endpoints. A point of a graph is called its vertex if it does not have any open neighbourhood homeomorphic to the open interval \([0, 1]\). The set of all vertices of \( G \) is denoted by \( \text{Ver}(G) \). Note that if \( G \) is a graph and \( f : G \to G \) is a continuous map, since \( H_i(G) = 0 \) for \( i \geq 2 \), and \( r(f_*^0) = 1 \), we obtain

\[
r(f) = \max\{1, r(f_*^1)\}.
\]

**Definition 1.** Let \( G \) be a graph. A continuous map \( f : G \to G \) is called a piecewise monotone graph (shortly PMG) map if there is a finite set \( C \subset G \) such that \( f \) is injective on each connected component of \( G \setminus C \).

As mentioned before, our goal is to study the relationship between \( h_{\text{top}}(f) \) and \( h_{\text{hom}}(f) \). To this end we define another topological invariant \( h_{\text{per}}(f) \). Let \( f : G \to G \) be a PMG map. By \( \text{Fix}(f) \) we denote the set of all fixed points of \( f \). A point \( x \in \text{Fix}(f) \setminus \text{Ver}(G) \) is called of negative type if \( f \) reverses orientation throughout a small neighbourhood of \( x \). (Since \( x \notin \text{Ver}(G) \), it has a neighbourhood homeomorphic to an open real interval on which we can consider \( f \) to be a selfmap of the real line.) We denote by \( \text{Fix}^-(f) \) the set of all fixed points of negative type of \( f \). Evidently, the set \( \text{Fix}(f) \) may be infinite but, since \( f \) is a PMG map, the set \( \text{Fix}^-(f) \) is always finite. Note that every iterate of a PMG map is again a PMG map. Hence the sets \( \text{Fix}^-(f^n) \) are always finite and therefore we can introduce another topological invariant

\[
h_{\text{per}}^- = \limsup_{n \to \infty} \frac{1}{\log^+ n} \log \# \text{Fix}^-(f^n),
\]

the exponential growth rate of the number of periodic points of negative type (we put \( \log^+ x = \log \max\{1, x\} \)).

Milnor and Thurston showed in [MTh] (see also theorem 4.11 of [MTr]) that

\[
h_{\text{top}}(f) = h_{\text{per}}^- (f) \tag{1}
\]

for any piecewise monotone interval map \( f \). The same result was established in [ASR2] for piecewise monotone tree maps. However, among graph maps this does not hold anymore—as an example consider the circle \( S^1 = \{x \in: |x| = 1\} \) and the map \( f : S^1 \to S^1 \) defined by \( f(x) = x^2 \). For this map one obtains \( h_{\text{top}}(f) = \log 2 \) and \( h_{\text{per}}^- (f) = 0 \). Anyway, it is still possible to establish an equality analogous to (1) if one changes the way of counting the
periodic points of \( f \). This was done by Baladi and Ruelle in [BR] for piecewise continuous, piecewise monotone interval maps. Instead of counting only the periodic points of negative type, they count all of them depending on expansivity of \( f^n \) at these points. Indeed, the heuristic reason for counting only the periodic points of negative type (‘what goes up, must go down’, see [MTh]) is not valid either in the noncontinuous graph cases. Nevertheless, periodic points of negative type play an important role in relating topological entropy and homological growth as our main result shows.

**Theorem 2.** Let \( f \) be a PMG map. Then

\[
h_{\text{top}}(f) = \max\{h_{\text{per}}^-(f), h_{\text{hom}}^+(f)\}.
\]

The spectral radius \( r(f) \) is an algebraic number for any PMG map \( f \). Using this, we get, as a consequence of the last theorem, the next result showing that the following entropies are equal for almost all values of topological entropy.

**Corollary 3.** Let \( f \) be a PMG map and suppose that \( \exp(h_{\text{top}}(f)) \) is a transcendental number. Then

\[
h_{\text{top}}(f) = h_{\text{per}}^-(f).
\]

Note that both results hold for PMG maps in general, even for those with \( \text{Fix}(f^n) \) infinite. In the case that \( \text{Fix}(f^n) \) is finite for every \( n \geq 1 \) then we can consider a topological invariant

\[
h_{\text{per}}(f) = \limsup_{n \to \infty} \frac{1}{n} \log^+ \#\text{Fix}(f^n).
\]

For many important cases of PMG maps (expanding maps and more generally maps with ‘few’ stable periodic orbits), topological entropy represents the exponential growth rate of the number of periodic orbits, that is \( h_{\text{top}}(f) = h_{\text{per}}(f) \). Moreover, if \( \exp(h_{\text{top}}(f)) \) is transcendental, we get from the last corollary, the following relation

\[
h_{\text{top}}(f) = h_{\text{per}}^-(f) = h_{\text{per}}(f).
\] (2)

Just stated identity shows that topological entropy in some sense describes the periodic structure of the system in both quantitative and qualitative ways—for an expanding piecewise monotone interval map, we have an obvious relationship between the number of fixed points of negative and positive types (the latter one defined analogously) because between any two consecutive fixed points of \( f^n \) of negative type there is exactly one of its fixed points of positive type and consequently \( h_{\text{per}}^-(f) = h_{\text{per}}(f) \). Indeed, we have no such relation between the number of the fixed points of a PMG map of negative and positive types even if the map is expanding.

One extremely useful tool for studying the relation between topological entropy and the growth of the number of periodic points was introduced by Artin and Mazur in [AM]. Let \( X \) be an arbitrary set and \( f : X \to X \). The orbit of a point \( x \in X \) under the action of \( f \) is defined as the set \( \mathcal{O}_x = \{f^n(x) : n \geq 0\} \). An orbit \( \mathcal{O}_x \) is said to be periodic if there is a positive integer \( n \) such that \( f^n(x) = x \); the smallest such number we denote by \( p_x \) and call it period. The set of all periodic orbits of \( f \) is denoted by \( \mathcal{O} \). Suppose that each positive iterate \( f^n \) has only finitely many fixed points. Then we define the *Artin–Mazur zeta function* of \( f \), \( \zeta \), to be the formal power series

\[
\zeta(z) = \exp \sum_{n \geq 1} \frac{\#\text{Fix}(f^n)}{n} z^n.
\]
Recall that the Artin–Mazur zeta function of $f$ is a convenient way of enumerating the periodic orbits of $f$. Indeed, if each positive iterate of $f$ has only finitely many fixed points then the subset $\{o \in O : p(o) = k\}$ is for any $k$ always finite and the identity

$$\zeta(z) = \prod_{o \in O} (1 - z^{p(o)})^{-1}$$

holds in $\mathbb{Z}[[z]]$, the ring of all formal power series in $z$ over.

Later on, several variants of this notion were introduced by different authors (cf [MTh,BR]; for an extensive survey of the topic see [Bal, P]; cf also [R]). In particular, Milnor and Thurston in [MTh] modified the Artin–Mazur zeta function to obtain more information for a piecewise monotone interval map. Let us very briefly recall how they arrived at the identity (1). If $f : I \to I$ is a piecewise monotone interval map, we call the formal power series

$$\zeta_{MT}(z) = \exp \sum_{n \geq 1} \frac{2\#\text{Fix}^- (f^n) - 1}{n} z^n,$$

the Milnor–Thurston zeta function of an interval map $f$. Denote its radius of convergence by $\rho$. Starting from the main relation between $\zeta_{MT}(z)$ and the kneading determinant of $f$, Milnor and Thurston proved that

$$h_{\text{top}}(f) = \log^+ \frac{1}{\rho} = h_{\text{per}}(f).$$

Here we follow the same strategy to prove theorem 2. As the first step we generalize the concept of Milnor–Thurston zeta function. Let us begin by defining the Lefschetz and negative zeta functions of a PMG map. Let $f : G \to G$ be a PMG map. Recall that the formal power series

$$\zeta^L(z) = \exp \sum_{n \geq 1} \frac{\text{tr}(f_0^n) - \text{tr}(f_1^n)}{n} z^n$$

is called the Lefschetz zeta function of $f$. We define the negative zeta function of $f$ as

$$\zeta^-(z) = \exp \sum_{n \geq 1} \frac{2\#\text{Fix}^- (f^n)}{n} z^n.$$

Observe that if $f : I \to I$ is a piecewise monotone interval map then we have $\text{tr}(f_0) = 1$ and $\text{tr}(f_1) = 0$ for all $n \geq 1$ and therefore

$$\zeta^-(z) \zeta^L(z)^{-1} = \exp \sum_{n \geq 1} \frac{2\#\text{Fix}^- (f^n)}{n} z^n$$

holds in $[[z]]$. So, according to (3) it is natural to define the Milnor–Thurston zeta function of a PMG map $f$ as the formal power series

$$\zeta^\text{MT}(z) = \zeta^-(z) \zeta^L(z)^{-1}$$

and, as before, there is a close relation between $h_{\text{top}}(f)$ and the radius of convergence of $\zeta^\text{MT}(z)$. Theorem 2 is then an immediate consequence of the following theorem.

**Theorem 4.** Let $f$ be a PMG map and denote by $\rho$ the radius of convergence of $\zeta^\text{MT}(z)$. Then $\rho > 0$ and

$$h_{\text{top}}(f) = \log^+ \frac{1}{\rho} = \max\{h^-_{\text{per}}(f), h_{\text{hom}}(f)\}.$$
2. Proof of theorem 4

The rest of the paper is devoted to the proof of theorem 4. In order to simplify notation it is convenient to regard a PMG map \( f \) as a real map \( F \) with discontinuities defined on a subset of the real line. The proof of theorem 4 is given in two main steps. The first step is the construction of kneading determinant, \( D(z) \), associated with the map \( F \). In the second step, we set up the relationship between the kneading determinant and the zeta function \( \zeta^{MT}(z) \). Because it is not easy to establish a direct relation between \( D(z) \) and \( \zeta^{L}(z) \), we introduce another determinant, \( L(z) \), called the homological determinant of \( F \). These two determinants are defined in a very similar way following the techniques introduced in [ASR1]. With any \( F \) we associate two pairs of linear endomorphisms \((\epsilon F_{m0}, \epsilon F_{m1})\) and \((F_{m0}, F_{m1})\). Although these endomorphisms have in general infinite rank, we prove that their difference has always finite rank. This allows us to define \( D(z) \) and \( L(z) \) as the determinants of these pairs of linear endomorphisms.

Milnor and Thurston’s main result from [MTh] shows that the negative zeta function is reciprocal to the kneading determinant (modulo a polynomial factor). This was generalized by Baladi and Ruelle in [BR] for the case of piecewise continuous, piecewise monotone interval maps for their ‘reduced’ zeta function including weights. Similar results for tree maps were obtained by Baillif in [Bai]. Our theorem 17 can be considered as an analogue of the results mentioned, however, our approach is completely different, following the constructions introduced in [ASR1] and [ASR2]. For better readability, we present basic algebraic notions and constructions in the appendix.

In the remainder of the paper we use the symbol \( \Omega \) to denote a finite and disjoint union of compact intervals on the real line

\[
\Omega = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_m, b_m]
\]

with \( a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m \).

**Definition 5.** A piecewise monotone (shortly PM) map on \( \Omega \) is a map \( F : \Omega \setminus C_F \to \Omega \) where \( C_F \) is a finite subset of \( \Omega \) containing \( \partial \Omega = [a_1, b_1, \ldots, a_m, b_m] \) and such that \( F \) is continuous and strictly monotone on each connected component of \( \Omega \setminus C_F \).

Let \( F : \Omega \setminus C_F \to \Omega \) be a PM map and \( I = [x, y] \) (with \( x < y \)) be an interval. We say that \( F \) is monotone on \( I \) if \( |x, y| \subset \Omega \setminus C_F \). In this case, we define the sign function \( \epsilon([x, y]) = \pm 1 \) according to whether \( F \) is increasing or decreasing on \( |x, y| \). Moreover, for any \( x \in \Omega \setminus C_F \), put \( \epsilon(x) = \pm 1 \) according to whether \( F \) is increasing or decreasing on a neighbourhood of \( x \) and put \( \epsilon(x) = 0 \) for every \( x \in C_F \). By definition, a lap of \( F \) is a maximal interval of monotonicity of \( F \). That is to say, an interval \( I = [c, d] \subset \Omega \) (with \( c < d \)) is a lap of \( F \) if and only if \( [c, d] \cap C_F = \{c, d\} \). In what follows, we use the symbol \( L_F \) to denote the set of all laps of \( F \).

For a PM map \( F : \Omega \setminus C_F \to \Omega \) and \( n \) a positive integer we define its \( n \)th iterate as a map \( F^n : \Omega \setminus C_F \to \Omega \) inductively by \( F^n(x) = F(F^{n-1}(x)) \) for all \( x \in \Omega \setminus C_F \) where

\[
C_F = \{x \in \Omega : F^{k}(x) \in C_F \text{ for some } k = 0, \ldots, n-1\}.
\]

It can be easily seen that this map is PM as well.

Since it is easier to work with PM maps on the real line than with PMG maps, we want to replace the latter by the former. In fact, every PMG map is induced by some PM map on an appropriate set \( \Omega \) in the sense of the following definition.

**Definition 6.** Let \( f : G \to G \) be a PMG map, \( F : \Omega \setminus C_F \to \Omega \) a PM map and \( \pi : \Omega \to G \) a continuous map such that \( \text{Ver}(G) \subset \pi(\partial \Omega) \) and \( \pi \) maps \( \Omega \setminus \partial \Omega \) homeomorphically into
Let $F$ be a topological space. Denote by $S_0(X; \mathbb{R})$ the $\mathbb{R}$-vector space whose basis consists of the formal symbols $x \in X$, and by $S_1(X; \mathbb{R})$ its subspace generated by the vectors $y - x$ where $x$ and $y$ are points lying in the same connected component of $X$. If $Y$ is a subset of $X$ and $F : X \setminus Y \to X$ is a map, we denote by $F_{y_0} : S_0(X; \mathbb{R}) \to S_0(X; \mathbb{R})$ the unique linear endomorphism verifying $F_{y_0}(x) = F(x)$ for $x \in X \setminus Y$, and $F_{y_0}(x) = 0$ for $x \in Y$.

Let $F : \Omega \setminus C_F \to \Omega$ be a PM map. According to the previous definitions, we have then a vector space $S_0(\Omega; \mathbb{R})$, a subspace $S_1(\Omega; \mathbb{R})$ of $S_0(\Omega; \mathbb{R})$, and a linear endomorphism $F_{y_0} : S_0(\Omega; \mathbb{R}) \to S_0(\Omega; \mathbb{R})$. Notice that both spaces $S_0(\Omega; \mathbb{R})$ and $S_1(\Omega; \mathbb{R})$ are infinite dimensional but the quotient space $S_0(\Omega; \mathbb{R})/S_1(\Omega; \mathbb{R})$ is finite dimensional with the dimension equal to the number of connected components of $\Omega$.

Starting from $F_{y_0}$ we define another linear endomorphism $\epsilon F_{y_0} : S_0(\Omega; \mathbb{R}) \to S_0(\Omega; \mathbb{R})$, putting $\epsilon F_{y_0}(x) = \epsilon(x) F_{y_0}(x)$ for all $x \in \Omega$. Next, we define the linear endomorphisms $F_{y_1} : S_1(\Omega; \mathbb{R}) \to S_1(\Omega; \mathbb{R})$ and $\epsilon F_{y_1} : S_1(\Omega; \mathbb{R}) \to S_1(\Omega; \mathbb{R})$. Note that, since $F$ is a PM map, the subset of $S_1(\Omega; \mathbb{R})$ spans $S_1(\Omega; \mathbb{R})$. Furthermore, if $F$ is monotone on $[x, y]$ then $F(y^-)$ and $F(x^+)$ lie in the same connected component of $\Omega$ and therefore $(F(y^-) - F(x^+)) \in S_1(\Omega; \mathbb{R})$ where $F(y^-)$ and $F(x^+)$ denote the corresponding one-sided limits. So we can define $F_{y_1}$ and $\epsilon F_{y_1}$ as the unique linear endomorphisms of $S_1(\Omega; \mathbb{R})$ such that

$$F_{y_1}(y - x) = F(y^-) - F(x^+) \quad \text{and} \quad \epsilon F_{y_1}(y - x) = \epsilon([x, y]) F_{y_1}(y - x)$$

for all $y - x \in I_F$.

As mentioned above, if $F : \Omega \setminus C_F \to \Omega$ is a PM map then $F^n : \Omega \setminus C_F \to \Omega$ is also a PM map and therefore the linear endomorphisms $F_{y_0}^n$, $F_{y_1}^n$, $\epsilon F_{y_0}^n$ and $\epsilon F_{y_1}^n$ are defined as well. The next lemma is a simple consequence of the definitions and shows that the correspondences $(\cdot)_{y_0}$, $(\cdot)_{y_1}$, $\epsilon(\cdot)_{y_0}$ and $\epsilon(\cdot)_{y_1}$ behave nicely under iteration.

**Lemma 7.** Let $F : \Omega \setminus C_F \to \Omega$ be a PM map. Then we have $(F_{y_0})^n = F_{y_0}^n$, $(F_{y_1})^n = F_{y_1}^n$, $(\epsilon F_{y_0})^n = \epsilon F_{y_0}^n$ and $(\epsilon F_{y_1})^n = \epsilon F_{y_1}^n$ for all $n \geq 1$.

Thus for each PM map $F : \Omega \setminus C_F \to \Omega$ we have two pairs of linear endomorphisms on $S_0(\Omega; \mathbb{R})$, $(F_{y_0}, F_{y_1})$ and $(\epsilon F_{y_0}, \epsilon F_{y_1})$. Next we prove that these pairs have both finite ranks (see definition 21). For this we need first to define extensions of $F_{y_1}$ and $\epsilon F_{y_1}$ to the common superspace $S_0(\Omega; \mathbb{R})$.

For each $c \in [a_i, b_i] \subset \Omega$, let $\alpha_{c}^- : \Omega \to \mathbb{R}$ and $\alpha_{c}^+ : \Omega \to \mathbb{R}$ be step functions defined by

$$\alpha_{c}^-(x) = \begin{cases} 1 & \text{for } x \in [c, b_i], \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \alpha_{c}^+(x) = \begin{cases} -1 & \text{for } x \in [c, b_i], \\ 0 & \text{otherwise.} \end{cases}$$

These step functions induce the linear forms $\omega_c^- : S_0(\Omega; \mathbb{R}) \to \mathbb{R}$ and $\omega_c^+ : S_0(\Omega; \mathbb{R}) \to \mathbb{R}$ defined by $\omega_c^-(x) = \alpha_c^-(x)$ and $\omega_c^+(x) = \alpha_c^+(x)$ for all $x \in \Omega$. We also introduce the following
notation for special vectors from \( S_0(\Omega; \mathbb{R}) \): \( v_c^- = F(c-) \), \( \epsilon v_c^- = \epsilon F(c-) v_c^- \), \( v_c^+ = F(c+) \) and \( \epsilon v_c^+ = \epsilon F(c+) v_c^+ \) putting \( v_c^- = 0 \) if \( c = a_i \) and \( v_c^+ = 0 \) if \( c = b_i \).

**Lemma 8.** Let \( F \) be a PM map on \( \Omega \). Then the linear endomorphism \( \psi : S_0(\Omega; \mathbb{R}) \rightarrow S_0(\Omega; \mathbb{R}) \) defined by \( \psi(x) = \psi(x) = F_{\psi}(x - a_i) \) for all \( x \in [a_i, b_i] \). Evidently, \( \psi \) is an extension of \( F_{\psi} \) to \( S_0(\Omega; \mathbb{R}) \). Furthermore, since the image of \( \psi \) is, by definition, contained in the image of \( F_{\psi} \) which is contained in \( S_1(\Omega; \mathbb{R}) \), it follows that \( \psi(S_0(\Omega; \mathbb{R})) \subset S_1(\Omega; \mathbb{R}) \). Consequently, the pair of linear endomorphisms \((F_{\psi}, F_{\psi})\) has finite rank and

\[
\text{tr}(F_{\psi}, F_{\psi}) = \sum_{c \in C_T} \omega_c^-(c) v_c^- + \omega_c^+(c) v_c^+.
\]

**Proof.** Let \( \varphi : S_0(\Omega; \mathbb{R}) \rightarrow S_0(\Omega; \mathbb{R}) \) be the linear endomorphism defined by \( \varphi(x) = F_{\psi}(x - a_i) \) for all \( x \in [a_i, b_i] \). Evidently, \( \varphi \) is an extension of \( F_{\psi} \) to \( S_0(\Omega; \mathbb{R}) \). Furthermore, since the image of \( \varphi \) is, by definition, contained in the image of \( F_{\psi} \) which is contained in \( S_1(\Omega; \mathbb{R}) \), it follows that \( \varphi(S_0(\Omega; \mathbb{R})) \subset S_1(\Omega; \mathbb{R}) \). On the other hand, from the definitions of \( F_{\psi} \) and \( F_{\psi} \) we may write

\[
F_{\psi}(x - a_i) = F_{\psi}(x) - F_{\psi}(a_i) + \sum_{c \in [a_i, b_i] \cap C_T} \omega^c_-(c) v_c^- - \sum_{c \in [a_i, b_i] \cap C_T} \omega^c_+(c) v_c^+ = F_{\psi}(x) - \sum_{c \in C_T} \omega^c_-(c) v_c^- + \sum_{c \in C_T} \omega^c_+(c) v_c^+
\]

for all \( x \in [a_i, b_i] \) and thus

\[
\varphi(x) = F_{\psi}(x) - \sum_{c \in C_T} \omega^c_-(c) v_c^- + \sum_{c \in C_T} \omega^c_+(c) v_c^+
\]

for all \( x \in \Omega \).

In the same way we can prove the following lemma.

**Lemma 9.** Let \( F \) be a PM map on \( \Omega \). Then the linear endomorphism \( \psi : S_0(\Omega; \mathbb{R}) \rightarrow S_0(\Omega; \mathbb{R}) \) defined by \( \psi(x) = \psi(x) = \epsilon F_{\psi} + \sum_{c \in C_T} \omega^c_-(c) v_c^- + \sum_{c \in C_T} \omega^c_+(c) v_c^+ \) is an extension of \( \epsilon F_{\psi} \) to \( S_0(\Omega; \mathbb{R}) \) that verifies \( \psi(S_0(\Omega; \mathbb{R})) \subset S_1(\Omega; \mathbb{R}) \). Consequently, the pair of linear endomorphisms \((\epsilon F_{\psi}, \epsilon F_{\psi})\) has finite rank and

\[
\text{tr}(\epsilon F_{\psi}, \epsilon F_{\psi}) = \sum_{c \in C_T} \omega^c_-(\epsilon v_c^-) + \omega^c_+(\epsilon v_c^+).
\]

The previous lemma shows that the determinants of the pairs \((\epsilon F_{\psi}, \epsilon F_{\psi})\) and \((F_{\psi}, F_{\psi})\) (see definition 23) are well defined. We define the kneading determinant of \( F \), \( D(z) \), and the homological determinant of \( F \), \( L(z) \), by

\[
D(z) = D_{(\epsilon F_{\psi}, \epsilon F_{\psi})}(z) = \exp - \sum_{n \geq 1} \text{tr}((\epsilon F_{\psi})^n, (\epsilon F_{\psi})^n) \frac{z^n}{n}
\]

and

\[
L(z) = D_{(F_{\psi}, F_{\psi})}(z) = \exp - \sum_{n \geq 1} \text{tr}((F_{\psi})^n, (F_{\psi})^n) \frac{z^n}{n}.
\]

Due to lemmas 8, 9 and proposition 24 there are vectors \( u_1, \ldots, u_p, v_1, \ldots, v_p \in S_0(\Omega; \mathbb{R})^* \) and linear forms \( v_1, \ldots, v_p \in S_0(\Omega; \mathbb{R})^* \) such that

\[
D(z) = \det(\text{Id} - z M(z)) \quad \text{and} \quad L(z) = \det(\text{Id} - z N(z)),
\]
where \( M(z) = [m_{ij}(z)] \) and \( N(z) = [n_{ij}(z)] \) are \( p \times p \) matrices with entries from \( \mathbb{Z}[[z]] \) defined by
\[
m_{ij}(z) = \sum_{n \geq 0} \mu_i \circ (F_{00})^n(u_j)z^n \quad \text{and} \quad n_{ij}(z) = \sum_{n \geq 0} \mu_i \circ (F_{01})^n(v_j)z^n.
\]
\tag{7}

Remark that as a consequence of the definitions, the entries of \( M(z) \) and \( N(z) \) are formal power series that can be computed in terms of the orbits of the points of \( CF \) and whose coefficients are from \( \{-1, 0, 1\} \). Therefore, the entries of \( M(z) \) and \( N(z) \) and the corresponding determinants \( D(z) \) and \( L(z) \) converge for all \( |z| < 1 \).

At first glance, it is not clear which kind of relationship can hold between the traces of \( (F_{00}, F_{01}) \) and \( (\epsilon F_{00}, \epsilon F_{01}) \) and the number of fixed points of \( F \). For convenience, we introduce the following notation. Let the symbols \( L^+ \) and \( L^- \) denote the set of all laps on which \( F \) is increasing and decreasing, respectively. We have then \( L_F = L^- \cup L^+ \). For each \( I = [c, d] \in L_F \), define the number \( \sigma(I) = \omega^+(v_c^+ + v_c^-) + \omega^-(v_d^-) \).

Note that from lemmas 8 and 9 we have
\[
\text{tr}(F_{00}, F_{01}) = \sum_{I \in L_F} \sigma(I) \quad \text{and} \quad \text{tr}(\epsilon F_{00}, \epsilon F_{01}) = \sum_{I \in L_F} \epsilon_F(I) \sigma(I).
\]
\tag{8}
\tag{9}

On the other hand, it is easy to check the following lemma.

**Lemma 10.** Let \( F : \Omega \setminus CF \to \Omega \) be a PM map and \( I = [c, d] \in L_F \). Then \( \sigma(I) \in \{-1, 0, 1\} \) and
\[
\sigma(I) = \begin{cases} 
1 & \text{if and only if } F(c+) \leq c \text{ and } d \leq F(d-), \\
-1 & \text{if and only if } c < F(c+) \text{ and } F(d-) < d.
\end{cases}
\]

We use this result to prove the following main relation between the traces \( \text{tr}(\epsilon F_{00}, \epsilon F_{01}) \), \( \text{tr}(F_{00}, F_{01}) \) and the number \( \#\text{Fix}^-(F) \).

**Lemma 11.** Let \( F : \Omega \setminus CF \to \Omega \) be a PM map. Then we have
\[
\text{tr}(\epsilon F_{00}, \epsilon F_{01}) - \text{tr}(F_{00}, F_{01}) = 2\#\text{Fix}^-(F).
\]

**Proof.** If \( I = [c, d] \in L^- \) then there is at most one fixed point of \( F \) lying in \([c, d]\) because \( F \) is decreasing on \( I \) and from lemma 10 we have that \( \sigma(I) = -1 \) if there exists such a fixed point; \( \sigma(I) = 0 \) otherwise. Therefore, from (8) and (9), we obtain
\[
\text{tr}(\epsilon F_{00}, \epsilon F_{01}) - \text{tr}(F_{00}, F_{01}) = -2 \sum_{I \in L^-} \sigma(I) = 2\#\text{Fix}^-(F)
\]
as desired. \( \square \)

Notice that from lemmas 7 and 11 we have
\[
\text{tr}((\epsilon F_{00})^n, (\epsilon F_{01})^n) = \text{tr}((F_{00})^n, (F_{01})^n) = \text{tr}(\epsilon F_{00}, \epsilon F_{01}) - \text{tr}(F_{00}, F_{01}) = 2\#\text{Fix}^-(F^n)
\]
and this proves the main theorem of this subsection.

**Theorem 12.** Let \( F : \Omega \setminus CF \to \Omega \) be a PM map. Then
\[
\exp \sum_{n \geq 1} \frac{2\#\text{Fix}^-(F^n)}{n} z^n = L(z)D(z)^{-1}
\]
holds in \( \mathbb{Z}[[z]] \).
Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. We have then two zeta functions, $\zeta^-(z)$ and $\zeta^L(z)$, and two determinants, $D(z)$ and $L(z)$. By $P$ we denote the union of all periodic orbits of $f$ that intersect $\pi(C_F)$ which is always a finite set. Notice that the numbers $\#\text{Fix}^-(F^n)$ and $\#\text{Fix}^-(f^n)$ do not need to coincide because there may exist periodic orbits of $f$ which intersect simultaneously $\pi(C_F)$ and $\text{Fix}^-(f^n)$ for some $n \geq 1$. Nevertheless, we have

$$\#\text{Fix}^-(f^n) - \#\text{Fix}^-(F^n) = \#P \cap \text{Fix}^-(f^n)$$

for all $n \geq 1$, and consequently

$$\max \left\{ 1, \limsup_{n \to \infty} \#\text{Fix}^-(F^n)^{1/n} \right\} = \max \left\{ 1, \limsup_{n \to \infty} \#\text{Fix}^-(f^n)^{1/n} \right\}. \tag{11}$$

As an immediate consequence of (10) and theorem 12 we also have the following corollary.

**Corollary 13.** Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. Then

$$\zeta^-(z) = L(z)D(z)^{-1} \exp \sum_{n \geq 1} \frac{2\# P \cap \text{Fix}^-(f^n)}{n} z^n$$

holds in $\mathbb{Z}[[z]]$.

The next result, together with corollary 13, allow us to establish the main relationship between $\zeta^M(z)$ and $D(z)$.

**Theorem 14.** Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. Then

$$\zeta^L(z) = L(z) \exp \sum_{n \geq 1} \frac{\# P \cap \text{Fix}(f^n)}{n} z^n$$

holds in $[[z]]$.

In order to prove theorem 14, we define two auxiliary pairs of linear endomorphisms. Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. The continuous map $\pi : \Omega \to G$ induces linear endomorphisms

$$\pi_0 : S_0(\Omega ; \mathbb{R}) \to S_0(G ; \mathbb{R}) \quad \text{and} \quad \pi_1 : S_1(\Omega ; \mathbb{R}) \to S_1(G ; \mathbb{R}),$$

where $\pi_0$ is the unique linear map that verifies $\pi_0(x) = \pi(x)$ for all $x \in \Omega$ and $\pi_1$ is the restriction of $\pi_0$ to $S_1(\Omega ; \mathbb{R})$ (since $\pi$ is continuous, $\pi_0$ maps $S_1(\Omega ; \mathbb{R})$ into $S_1(G ; \mathbb{R})$).

Now define the linear endomorphisms $\beta_0 : S_0(G ; \mathbb{R}) \to S_0(G ; \mathbb{R})$ and $\beta_1 : S_1(G ; \mathbb{R}) \to S_1(G ; \mathbb{R})$ by

$$\beta_0(x) = \begin{cases} f(x) & x \in G \setminus \pi(C_F), \\ 0 & x \in \pi(C_F) \end{cases}$$

and

$$\beta_1(y - x) = f(y) - f(x) \quad \text{for} \ x, y \ \text{from the same component of} \ G.$$
holds in commute therefore we have the two following commutative diagrams with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ker } \pi_0 & \longrightarrow & S_0(\Omega; \mathbb{R}) & \longrightarrow & \tau_0 & \longrightarrow & S_0(G; \mathbb{R}) & \longrightarrow & 0 \\
\downarrow{\alpha_0} & & \downarrow{\varphi_0} & & \downarrow{\beta_0} & & \downarrow{\beta_0} & & \downarrow{\beta_0} & & \downarrow{\beta_0} \\
0 & \longrightarrow & \text{Ker } \pi_0 & \longrightarrow & S_0(\Omega; \mathbb{R}) & \longrightarrow & \tau_0 & \longrightarrow & S_0(G; \mathbb{R}) & \longrightarrow & 0
\end{array}
\]

(12)

and

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ker } \pi_1 & \longrightarrow & S_1(\Omega; \mathbb{R}) & \longrightarrow & \tau_1 & \longrightarrow & S_1(G; \mathbb{R}) & \longrightarrow & 0 \\
\downarrow{\alpha_1} & & \downarrow{\varphi_1} & & \downarrow{\beta_1} & & \downarrow{\beta_1} & & \downarrow{\beta_1} & & \downarrow{\beta_1} \\
0 & \longrightarrow & \text{Ker } \pi_1 & \longrightarrow & S_1(\Omega; \mathbb{R}) & \longrightarrow & \tau_1 & \longrightarrow & S_1(G; \mathbb{R}) & \longrightarrow & 0
\end{array}
\]

(13)

where \(\alpha_i\) is the restriction of \(\varphi_i\) to \(\text{Ker } \pi_i\) for \(i = 0, 1\). Notice that

\[\dim \text{Ker } \pi_0 = 2 \dim H_0(\Omega; \mathbb{R}) - \dim H_0(\pi(\partial \Omega); \mathbb{R})\]

and

\[\dim \text{Ker } \pi_1 = \dim \text{Ker } \pi_0 \cap S_1(\Omega; \mathbb{R})\]

\[= \dim H_0(\pi(\partial \Omega); \mathbb{R}) + \dim H_0(\Omega; \mathbb{R}) = \dim H_1(\Omega; \mathbb{R}).\]

The first equality shows, in particular, that \(\text{Ker } \pi_0\) is finitely dimensional and therefore the pair \((\alpha_0, \alpha_1)\) of linear endomorphisms on \(\text{Ker } \pi_0\) has a finite rank. On the other hand, the pair \((\beta_0, \beta_1)\) of linear endomorphisms on \(\text{Ker } \pi_0\) has also finite rank. Indeed, the endomorphism

\[\beta : S_0(G; \mathbb{R}) \rightarrow S_0(G; \mathbb{R}),\]

defined by \(\beta(x) = f(x)\) for all \(x \in G\) is evidently an extension of \(\beta_1\) to \(S_0(G; \mathbb{R})\). Since \(\pi(C_F)\) is finite, \(\beta = \beta_1\) has finite rank.

We have three pairs of linear endomorphisms with finite rank, \((\alpha_0, \alpha_1)\), \((\varphi_0, \varphi_1)\) and \((\beta_0, \beta_1)\). From the commutative diagrams (12) and (13) and proposition 25 we obtain

\[L(z) = D(\varphi_0, \varphi_1)(z) = D(\alpha_0, \alpha_1)(z)D(\beta_0, \beta_1)(z).\]

Since, by (21)

\[\xi^L(z) = \exp \sum_{n \geq 1} \frac{\text{tr}(f_0)^n - \text{tr}(f_1)^n}{n}z^n = \frac{D_{f_1}(z)}{D_{f_0}(z)},\]

Theorem 14 is an immediate consequence of the two following lemmas.

**Lemma 15.** Let \(f : G \rightarrow G\) be a PMG map induced by \(F : \Omega \setminus C_F \rightarrow \Omega\). Then

\[D(\beta_0, \beta_1)(z)D_{f_0}(z) = \exp \sum_{n \geq 1} \frac{# P \cap \text{Fix}(f^n)}{n}z^n\]

holds in \(\mathbb{Z}[[z]]\).

**Proof.** Because

\[\frac{S_0(G; \mathbb{R})}{S_1(G; \mathbb{R})} = H_0(G; \mathbb{R}),\]

we have the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & S_1(G; \mathbb{R}) & \longrightarrow & S_0(G; \mathbb{R}) & \longrightarrow & \text{pr} & \longrightarrow & H_0(G; \mathbb{R}) & \longrightarrow & 0 \\
\downarrow{\beta} & & \downarrow{\beta} & & \downarrow{\beta} & & \downarrow{\beta} & & \downarrow{\beta} & & \downarrow{\beta} \\
0 & \longrightarrow & S_1(G; \mathbb{R}) & \longrightarrow & S_0(G; \mathbb{R}) & \longrightarrow & \text{pr} & \longrightarrow & H_0(G; \mathbb{R}) & \longrightarrow & 0.
\end{array}
\]
Thus, from definition 21 we have
\[ \text{tr}(\beta^n_0, \beta^n_1) + \text{tr}(f_n^*0) = \text{tr}(\beta^n - \beta^n_0) \]
for all \( n \geq 1 \), and the proof follows because as an immediate consequence of the definitions one has \( \text{tr}(\beta^n - \beta^n_0) = \#P \cap \text{Fix}(f^n) \) for all \( n \geq 1 \). \( \square \)

**Lemma 16.** Let \( f : G \to G \) be a PMG map induced by \( F : \Omega \setminus C_F \to \Omega \). Then
\[ D_{(0, \alpha_1)}(z) = D_{f, 1}(z) \]
holds in \( \mathbb{Z}[[z]] \).

**Proof.** Since \( \pi \) maps \( \Omega \setminus \partial \Omega \) homeomorphically into \( G \setminus \pi(\partial \Omega) \), we have \( \text{Ker}_0 \subset S_0(\partial \Omega; \mathbb{R}) \subset \text{Ker}_F \) thus \( \alpha_0 = 0 \) and, consequently
\[ D_{(0, \alpha_1)}(z) = D_{0, \alpha_1}(z) = D_{\alpha_1}(z) \]
So, it remains to prove that
\[ D_{\alpha_1}(z) = D_{f, 1}(z) \]
holds in \( \mathbb{Z}[[z]] \). To prove this, we construct an isomorphism \( \Phi : H_1(G; \mathbb{R}) \to \text{Ker}_1 \). Let \( s : [0, 1] \to G \) be a closed path on \( G \) and \( 0 = t_0 < t_1 < \cdots < t_p = 1 \) points of \([0, 1]\) such that for all \( i = 0, \ldots, p - 1 \) there exists a \( j \) verifying \( f([t_i, t_{i+1}]) \subset \pi([a_j, b_j]) \). Notice that, since \( s(0) = s(1) \), the vector
\[ \Phi(s) = \sum_{i=0}^{p-1} (\pi_{j}^{-1}(f(t_{i+1})) - \pi_{j}^{-1}(f(t_i))) \]
lies in \( \text{Ker}_1 \) (we denote by \( \pi_{j}^{-1} \) the inverse of \( \pi_{|[a_j, b_j]} : [a_j, b_j] \to \pi([a_j, b_j]) \)). With this, we obtain a correspondence \( s \mapsto \Phi(s) \) which induces a monomorphism
\[ \Phi : H_1(G; \mathbb{R}) \to \text{Ker}_1, \]
\[ [s] \mapsto \Phi(s), \]
which is actually an isomorphism by (14). Commutativity of the diagram
\[
\begin{array}{ccc}
H_1(G; \mathbb{R}) & \xrightarrow{\Phi} & \text{Ker}_1 \\
\downarrow f_* & & \downarrow \alpha_1 \\
H_1(G; \mathbb{R}) & \xrightarrow{\Phi} & \text{Ker}_1
\end{array}
\]
finishes the proof. \( \square \)

From corollary 13 and theorem 14 we obtain the following theorem.

**Theorem 17.** Let \( f : G \to G \) be a PMG map induced by \( F : \Omega \setminus C_F \to \Omega \). Then there exists a formal power series \( H(z) \) such that \( H(z) \) converges and is nonzero for all \( |z| < 1 \) and
\[ \zeta^{MT}(z) = (H(z)D(z))^{-1} \]
holds in \( \mathbb{Z}[[z]] \).

**Proof.** We have
\[ \zeta^{MT}(z) = \zeta^{-1}(z)\zeta^L(z) \]
with
\[ a(z) = \sum_{n \geq 1} \frac{\#P \cap \text{Fix}(f^n) - \#P \cap \text{Fix}(f^n)}{n} z^n. \]
Thus, because $P$ is a finite set, it follows immediately that $a(z)$ converges for all $|z| < 1$ and consequently $H(z) = \exp a(z) \neq 0$ for all $|z| < 1$. $\square$

As mentioned before, the kneading determinant $D(z)$ converges for all $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, so, as an immediate consequence of theorem 17, we obtain the following corollary.

**Corollary 18.** Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$, $\rho$ the radius of convergence of $\xi_M(z)$, and $z_0$ a zero of $D(z)$ lying in $\mathbb{D}$. Then we have $\rho \leq |z_0|$.

Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$. Next we discuss the relationship between topological entropy of $f$ and the zeros of $D(z)$. From now on we use the symbol $\ell(F^n)$ to denote the number of laps of the iterate $F^n : \Omega \setminus C_F \to \Omega$, in other words $\ell(F^n) = |\mathcal{L}(F^n)|$. If $I = [c, d] \in \mathcal{L}(F^n)$ we define the variation of $F^n$ on $I$ by $\text{Var}_f(F^n) = |F^n(d) - F^n(c)|$. The variation of $F^n$ is defined by

$$\text{Var}(F^n) = \sum_{I \in \mathcal{L}(F^n)} \text{Var}_f(F^n).$$

Recall that, for interval and circle maps Misiurewicz and Szlenk proved in [MSz] that

$$h_{\text{top}}(f) = \log \lim_{n \to \infty} \frac{\ell(F^n)}{n} = \log^+ \lim_{n \to \infty} \text{Var}(F^n)^{1/n} \tag{15}$$

holds for any piecewise monotone circle map $f$ induced by $F : \Omega \setminus C_F \to \Omega$ and the same arguments can be adapted to show that (15) holds for any PMG map.

Let us begin with the following theorem.

**Theorem 19.** Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$ and $h_{\text{top}}(f) > 0$. Then $D(z) = 0$ for some $z$ with $|z| = \lim_{n \to \infty} \text{Var}(F^n)^{-1/n}$.

**Proof.** Let $\nu = (b_1 - a_1) + \cdots + (b_n - a_n) \in S_1(\Omega; \mathbb{R})$ and $\xi : S_1(\Omega; \mathbb{R}) \to \mathbb{R}$ be the linear form defined by $\xi(y - x) = y - x$ for all $x$ and $y$ lying in the same connected component of $\Omega$. We have then a pair $(\epsilon F_{a_1}, \epsilon F_{b_1} + \xi \otimes \nu)$ of endomorphisms on $S_1(\Omega; \mathbb{R})$ with finite rank. As mentioned before, the kneading determinant $D(z) = D(\epsilon F_{a_1}, \epsilon F_{b_1})(z)$ converges for all $|z| < 1$. Using the same argument, it is easy to show that $D(\epsilon F_{a_1}, \epsilon F_{b_1} + \xi \otimes \nu)(z)$ also converges for all $|z| < 1$.

Notice that if $I = [c, d]$ is a lap of $F^n$, we have $\text{Var}_f(F^n) = \xi \circ \epsilon F^n_{a_1}(d - c)$. Thus, from the linearity of $\epsilon F^n_{a_1}$, and by lemma 7, we arrive at

$$\text{Var}(F^n) = \xi \circ \epsilon F^n_{a_1}(v) = \xi \circ (\epsilon F^n_{a_1})^\nu(v),$$

since $h_{\text{top}}(f) > 0$, we have then

$$\limsup_{n \to \infty} |\xi \circ (\epsilon F^n_{a_1})^\nu(v)|^{1/n} = \lim_{n \to \infty} \text{Var}(F^n)^{1/n} > 1$$

and from proposition 26, $D(z) = 0$ for some $z$ with $|z| = \lim_{n \to \infty} \text{Var}(F^n)^{-1/n}$. $\square$

We have now everything we needed to prove theorem 4. Let $f : G \to G$ be a PMG map induced by $F : \Omega \setminus C_F \to \Omega$, and denote by $\rho$ the radius of convergence of $\xi_M(z)$. From the definition of $\xi_M(z)$, we see at once that

$$\frac{1}{\rho} \leq \max \{1, \limsup_{n \to \infty} \# \text{Fix}^-(f^n)^{1/n}, r(f_{a_1})\}$$

and thus

$$\log^+ \frac{1}{\rho} \leq \max \{h_{\text{per}}(f), h_{\text{hom}}(f)\}. \tag{16}$$
Let us now prove that
\[
    h_{\text{top}}(f) \leq \log^+ \frac{1}{\rho}.
\]  
(17)

Since this is trivial when \( h_{\text{top}}(f) = 0 \), we can assume that \( h_{\text{top}}(f) > 0 \). In this case, from (15) and theorem 19, we have \( D(z) = 0 \), for some \( z \) such that \( |z| = \lim_{n \to \infty} \text{Var}(F^n)^{-1/n} < 1 \).

Thus from corollary 18, we have \( \rho \leq \lim_{n \to \infty} \text{Var}(F^n)^{-1/n} \). This proves (17).

Let us finally show that
\[
    \max\{h_{\text{per}}(f), h_{\text{hom}}(f)\} \leq h_{\text{top}}(f).
\]  
(18)

Together with (16) and (17), it will imply theorem 4. Since in each lap of \( F^n \) there is at most one fixed point of negative type, we have
\[
    \lim_{n \to \infty} \ell(F^n)^{1/n} \geq \max \left\{ 1, \limsup_{n \to \infty} \#\text{Fix}^-(f^n)^{1/n} \right\}
\]  
and from (11)
\[
    \lim_{n \to \infty} \ell(F^n)^{1/n} \geq \max \left\{ 1, \limsup_{n \to \infty} \#\text{Fix}^-(f^n)^{1/n} \right\}.
\]

Thus, by (15)
\[
    h_{\text{top}}(f) \geq h_{\text{per}}(f).
\]  
(19)

From (8) and lemma 7, we also have
\[
    \ell(F^n) \geq \left| \sum_{I \in \mathcal{L}_{\mathcal{S}}} \sigma(I) \right| = |\text{tr}(F^n_{\#0}, F^n_{\#1})| = |\text{tr}(F^n)_{\#0}, (F^n)_{\#1})|
\]  
for all \( n \geq 1 \). But from theorem 14
\[
    |\text{tr}(F^n)_{\#0}, (F^n)_{\#1})| = |\#P \cap \text{Fix}(f^n) + \text{tr}(f_{\#1})^n - \text{tr}(f_{\#0})^n|
\]  
for all \( n \geq 1 \). Thus, since \( P \) is a finite set, it follows
\[
    \lim_{n \to \infty} \ell(F^n)^{1/n} \geq \max \left\{ 1, \limsup_{n \to \infty} |\text{tr}(f_{\#1})^n|^{1/n} \right\} = r(f)
\]
and, once again from (15), \( h_{\text{top}}(f) \geq h_{\text{hom}}(f) \). This together with (19) proves (18).

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Appendix (pairs of linear endomorphisms)

Let \( V \) be a vector space over \( \mathbb{R} \) and let \( \varphi : V \to V \) be a linear map with finite rank. As usual, we define the trace of \( \varphi \) by
\[
    \text{tr} \varphi = \text{tr} \varphi|_{\text{Im} \varphi}.
\]

If \( \varphi \) has finite rank then there are vectors \( v_1, \ldots, v_k \in V \) and linear forms \( \omega_1, \ldots, \omega_k \in V^* \) such that
\[
    \varphi = \sum_{i=1}^k \omega_i \otimes v_i.
\]
Considering the matrix
\[ M = \begin{pmatrix} \omega_1(v_1) & \ldots & \omega_1(v_k) \\ \vdots & \ddots & \vdots \\ \omega_k(v_1) & \ldots & \omega_k(v_k) \end{pmatrix} \]  
(20)
we have
\[ \text{tr } \varphi = \text{tr } M. \]

More generally, if \( \varphi \) has finite rank then \( \varphi^n, n \geq 1 \), also has finite rank and
\[ \text{tr } \varphi^n = \text{tr } M^n. \]

The following proposition is well known and gives an explicit method for computing the numbers \( \text{tr } \varphi^n \) for \( n \geq 1 \). Defining the determinant of \( \varphi \) to be the following formal power series
\[ D_\varphi(z) = \exp \sum_{n \geq 1} -\text{tr } \varphi^n \frac{z^n}{n}, \]
(21)
we have the following proposition.

**Proposition 20.** Let \( \varphi \) be an endomorphism with finite rank. Then we have
\[ D_\varphi(z) = \det(\text{Id} - zM) \quad \text{in } \mathbb{R}[[z]]. \]

Now we consider a more general situation. By a pair of endomorphisms \((\varphi_0, \varphi_1)\) on \( V \) we mean two linear maps \( \varphi_0 : V_0 \to V_0 \) and \( \varphi_1 : V_1 \to V_1 \) defined on two finite-codimensional subspaces \( V_0 \) and \( V_1 \) of the same \( \mathbb{R} \)-vector space \( V \).

**Definition 21.** We say that the pair of endomorphisms \((\varphi_0, \varphi_1)\) on \( V \) has a finite rank if there exist linear maps \( \widetilde{\varphi}_0, \widetilde{\varphi}_0, \widetilde{\varphi}_1 \) and \( \widetilde{\varphi}_1 \) such that the following diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & V_j & \overset{\psi_j}{\longrightarrow} & V & \overset{\text{pr}}{\longrightarrow} & V/V_j & \longrightarrow & 0 \\
0 & \longrightarrow & V_j & \overset{\widetilde{\psi}_j}{\longrightarrow} & V & \overset{\text{pr}}{\longrightarrow} & V/V_j & \longrightarrow & 0
\end{array}
\]
commutes for \( j = 0, 1 \) and the linear map \( \widetilde{\varphi}_1 - \widetilde{\varphi}_0 \) has finite rank. The trace of a pair \((\varphi_0, \varphi_1)\) with finite rank is defined by \( \text{tr } (\varphi_0, \varphi_1) = \text{tr } (\varphi_1 - \varphi_0) - \text{tr } \widetilde{\varphi}_1 + \text{tr } \widetilde{\varphi}_0 \).

It is easy to see that the definition does not depend on \( \varphi_0, \varphi_0, \varphi_1 \) and \( \varphi_1 \). As an immediate consequence of the definition we get the following proposition.

**Proposition 22.** Let \((\varphi_0, \varphi_1)\) be a pair of endomorphisms on \( V \), and \( \varphi_i : V \to V \) an extension of \( \varphi_i \) such that \( \varphi_i(V) \subset V_i \), for \( i = 0, 1 \). Then \((\varphi_0, \varphi_1)\) has finite rank if and only if \( \varphi_1 - \varphi_0 \) has finite rank. Furthermore, if \((\varphi_0, \varphi_1)\) has finite rank then \( \text{tr } (\varphi_0, \varphi_1) = \text{tr } (\varphi_1 - \varphi_0) \).

Let \((\varphi_0, \varphi_1)\) be a pair of endomorphisms on \( V \) having finite rank, and consider endomorphisms \( \varphi_0 \) and \( \varphi_1 \) as in proposition 22. Since \( \varphi_1 - \varphi_0 \) has finite rank, we can consider vectors \( v_1, \ldots, v_k \in V \) and linear forms \( \omega_1, \ldots, \omega_k \in V^* \) such that
\[ \varphi_1 - \varphi_0 = \sum_{i=1}^{k} \omega_i \otimes v_i. \]
(22)

More generally, we have
\[ \varphi^n_1 - \varphi^n_0 = \sum_{i=1}^{k} \sum_{j=0}^{n} (\omega_i \circ \varphi^{n-j}_1) \otimes \varphi^{j-1}_0(v_i) \]
for each \( n \geq 1 \). This shows that \( \phi_n^0 - \phi_n^1 \) has finite rank for each \( n \geq 1 \). Thus, once more from proposition 22, we conclude that the pair \((\phi_n^0, \phi_n^1)\) has finite rank and
\[
\text{tr}(\phi_n^0, \phi_n^1) = \text{tr}(\phi_n^1 - \phi_n^0) \quad \text{for each } n \geq 1.
\]

**Definition 23.** Let \((\phi_0, \phi_1)\) be a pair of endomorphisms having finite rank. We define the determinant of \((\phi_0, \phi_1)\) to be the following element of \(\mathbb{R}[z]\)
\[
D_{(\phi_0, \phi_1)}(z) = \exp\left(-\sum_{n \geq 1} \text{tr}(\phi_n^0, \phi_n^1) z^n \right).
\]

Observe that if \( \phi \) has finite rank, then
\[
D_{(0, \phi)}(z) = D_\phi(z).
\]
If \( \phi_0 \) and \( \phi_1 \) both have finite rank, then
\[
D_{(\phi_0, \phi_1)}(z) = D_{\phi_1}(z) D_{\phi_0}(z) - 1.
\]
So, in these cases, we can use proposition 20 for computing \( D_{(\phi_0, \phi_1)}(z) \). Obviously, in the general case, proposition 20 does not allow us to compute \( D_{(\phi_0, \phi_1)}(z) \) since \( D_{\phi_0}(z) \) and \( D_{\phi_1}(z) \) are not defined in general.

In order to compute \( D_{(\phi_0, \phi_1)}(z) \) in the general case, we generalize the proposition 20. Let \( \phi_0 \) and \( \phi_1 \) be endomorphisms as in proposition 22. Considering vectors \( v_1, \ldots, v_k \in V \) and linear forms \( \omega_1, \ldots, \omega_k \in V^* \) as in (22), we define the matrix
\[
M(z) = \begin{pmatrix}
\sum_{n \geq 0} \omega_1(\phi_0^n(v_1)) z^n & \cdots & \sum_{n \geq 0} \omega_k(\phi_0^n(v_1)) z^n \\
\vdots & \ddots & \vdots \\
\sum_{n \geq 0} \omega_1(\phi_0^n(v_k)) z^n & \cdots & \sum_{n \geq 0} \omega_k(\phi_0^n(v_k)) z^n
\end{pmatrix} \quad \text{(23)}
\]
with coefficients in \( \mathbb{R}[z] \). Observe that if we identify an endomorphism with finite rank \( \phi : V \to V \) with the corresponding pair of finite rank \((0, \phi)\) then the matrix \( M(z) \) defined in (23) coincides with the matrix \( \mathbf{M} \) defined in (20).Thus the next proposition which gives an explicit method for computing \( D_{(\phi_0, \phi_1)}(z) \), can be regarded as a generalization of proposition 20 (for a proof see [A]).

**Proposition 24.** Let \((\phi_0, \phi_1)\) be a pair of endomorphisms having finite rank. Then
\[
D_{(\phi_0, \phi_1)}(z) = \det(\mathbf{I} - z M(z)).
\]

Let \( 0 \to U \to V \to W \to 0 \) be an exact sequence of \( \mathbb{R} \)-vector spaces. Recall that if the diagram
\[
\begin{array}{ccccccccc}
0 & \to & U & \xrightarrow{i} & V & \xrightarrow{p} & W & \to & 0 \\
\downarrow{\chi} & & \downarrow{\psi} & & \downarrow{\psi} & & \downarrow{\psi} & & \\
0 & \to & U & \xrightarrow{\psi} & V & \xrightarrow{\psi} & W & \to & 0
\end{array}
\]
commutes and the endomorphisms \( \chi, \psi \) and \( \psi \) have finite rank, then we have
\[
\text{tr} \psi^n = \text{tr} \chi^n + \text{tr} \psi^n
\]
for all \( n \geq 1 \), and therefore
\[
D_{\phi}(z) = D_{\chi}(z) D_{\psi}(z).
\]
The next proposition can be regarded as a generalization of the last formula. Let \((\chi_0, \chi_1), (\varphi_0, \varphi_1)\) and \((\psi_0, \psi_1)\) be pairs of endomorphisms in \(U, V\) and \(W\), respectively, such that the following diagram

\[
\begin{array}{c}
0 \longrightarrow U_j \xrightarrow{\chi_j} V_j \xrightarrow{\text{pr}} W_j \longrightarrow 0 \\
0 \longrightarrow U_j \xrightarrow{\psi_j} V_j \xrightarrow{\text{pr}} W_j \longrightarrow 0
\end{array}
\]

commutes for \(j = 0, 1\). Then we have the following proposition.

**Proposition 25.** Let \((\chi_0, \chi_1), (\varphi_0, \varphi_1)\) and \((\psi_0, \psi_1)\) be pairs of endomorphisms with finite rank such that the last diagram above commutes. Then

\[
\text{tr}(\varphi_n^0, \varphi_n^1) = \text{tr}(\chi_n^0, \chi_n^1) + \text{tr}(\psi_n^0, \psi_n^1)
\]

and

\[
D_{(\varphi_n, \varphi_1)}(z) = D_{(\chi_n, \chi_1)}(z)D_{(\psi_n, \psi_1)}(z).
\]

Let us consider, for the last time, a linear endomorphism \(\varphi : V \rightarrow V\) with finite rank. Recall that, if \(v \in V\) and \(\xi \in V^\ast\), then there exists an eigenvalue, \(\lambda\), of \(\varphi\) such that

\[
\limsup_{n \to \infty} |\xi \circ \varphi^n(v)|^{1/n} = |\lambda|
\]

and consequently

\[
D_\varphi(z) = 0 \text{ for some } |z| = \frac{1}{\limsup_{n \to \infty} |\xi \circ \varphi^n(v)|^{1/n}}. \quad (24)
\]

We will finish this appendix with a generalization of (24). Let \((\varphi_0, \varphi_1)\) be a pair of endomorphisms on \(V\) with finite rank, \(v \in V_1\), \(\xi \in V_1^\ast\). Then the pair \((\varphi_0, \varphi_1 + \xi \otimes v)\) of endomorphisms on \(V\) also has finite rank. Note that, since \(D_{(\varphi_0, \varphi_1)}(z)\) and \(D_{(\varphi_0, \varphi_1 + \xi \otimes v)}(z)\) are not (in general) polynomials, we have to assume that there exists \(r > 0\) such that \(D_{(\varphi_0, \varphi_1)}(z)\) and \(D_{(\varphi_0, \varphi_1 + \xi \otimes v)}(z)\) converge for all \(|z| < r\). If we consider the pair \((\varphi_1, \varphi_1 + \xi \otimes v)\) of endomorphisms on \(V_1\), this pair has evidently finite rank, and from proposition 24 we see that

\[
D_{(\varphi_1, \varphi_1 + \xi \otimes v)}(z) = 1 - \sum_{n \geq 0} \xi \circ \varphi_1^n(v)z^{n+1}
\]

holds in \(\mathbb{R}[z]\). On the other hand, regarding \((\varphi_1, \varphi_1 + \xi \otimes v)\) as a a pair of endomorphisms on \(V\), we have the decomposition

\[
D_{(\varphi_1, \varphi_1 + \xi \otimes v)}(z) = D_{(\varphi_1, \varphi_0)}(z)D_{(\varphi_0, \varphi_1 + \xi \otimes v)}(z) = \frac{D_{(\varphi_0, \varphi_1 + \xi \otimes v)}(z)}{D_{(\varphi_0, \varphi_1)}(z)}
\]

and therefore

\[
1 - \sum_{n \geq 0} \xi \circ \varphi_1^n(v)z^{n+1} = \frac{D_{(\varphi_0, \varphi_1 + \xi \otimes v)}(z)}{D_{(\varphi_0, \varphi_1)}(z)}
\]

holds in \(\mathbb{R}[z]\). Thus, since the radius of convergence of

\[
1 - \sum_{n \geq 0} \xi \circ \varphi_1^n(v)z^{n+1}
\]

is

\[
\rho = \frac{1}{\limsup_{n \to \infty} |\xi \circ \varphi_1^n(v)|^{1/n}}
\]
and the function
\[ \gamma(z) = \frac{D(\phi_0, \phi_1 + t \otimes v)(z)}{D(\phi_0, \phi_1)(z)} \]
is meromorphic on \( |z| < r \), we can conclude: if \( \rho < r \) then \( \gamma(z) \) has a pole lying in \( |z| = \rho \). So, because the poles of \( \gamma(z) \) are zeros of \( D(\phi_0, \phi_1)(z) \), we may write the following proposition.

**Proposition 26.** Let \((\phi_0, \phi_1)\) be a pair of endomorphisms on \( V \) with finite rank, \( v \in V_1 \), \( \xi \in V^\ast \) and \( r > 0 \) such that \( D(\phi_0, \phi_1)(z) \) and \( D(\phi_0, \phi_1 + t \otimes v)(z) \) converge for all \( |z| < r \), and \( \limsup_{n \to \infty} |\xi \circ \phi^n_1(v)|^{1/n} > r^{-1} \). Then we have
\[ D(\phi_0, \phi_1)(z) = 0, \quad \text{for some } |z| = \frac{1}{\limsup_{n \to \infty} |\xi \circ \phi^n_1(v)|^{1/n}}. \]

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