The Integrated Pulse Profiles of Fast Radio Bursts

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ABSTRACT

Some fast radio bursts manifest interference-like pulse profiles. The mechanism of pulse shaping is unknown or even beyond a usual physical concern. Here we show that these interference-like pulse profiles are mostly the instantaneous spectra near the source regions of fast radio bursts. The traditional approach to dedisperse fast radio bursts is incomplete in some circumstances. The original dynamic spectra are oblique for some bursts. The corresponding instantaneous spectra are coincident to the spectrum from a single electron passing through a tapered undulator. For this spectrum to be observed, the particles in the radiation beam have to be bunched. The bunching effect is probably due to the acceleration of particles in the plasma wakefield. This study also suggests a significant revision of the current data processing of radio pulsars.

Keywords: methods: analytical — radiation mechanisms: non-thermal — intergalactic medium — radio continuum: general

1. INTRODUCTION

The pulse profile of some fast radio bursts (hereafter FRBs) looks like an interference fringe, for example, the double-peaked FRB 121002 (Champion et al. 2016). The analogy stimulated us to find possible connection between FRBs and single-slit diffraction. We find that undulator radiation (Alferov et al. 1989) is a mechanism through which the connection can be built. However, to build the connection, the pulse profile needs to be a spectrum in a sense, which happens if only the original dynamic spectrum of FRBs is oblique.

Recently, some FRB bursts (Hessels et al. 2019) are found to have a downward drifting time-frequency structure, as shown by the schematic in Figure 1d. An alternative view of the feature is an oblique dynamic spectrum near the source region of FRBs, which coincides with our idea correctly.

In this report, we describe some FRBs as radiation from charged particles traveling in a quasi-periodic magnetic structure. An observational example is firstly presented to illustrate the basic idea. A detailed model based on the theory of radiation from a tapered undulator (Bosco & Colson 1983; Shih & Caponi 1982) is then outlined. In a subsequent section, an accurate simulation to fit the observed dynamic spectrum is introduced, and a comparison is made between the simulation and the observation of two FRB bursts. Finally, some discussions are made concerning the assumptions on the initial energy distribution of radiation particles and the low beam emittance required for the corresponding spectrum to be observed.

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2. OBSERVATION AND THEORY

2.1. An observational example

AO-06 (Hessels et al. 2019) is a triple-peaked burst from the repeating FRB 121102 (Spitler et al. 2016). Its dynamic spectrum shows a downward drifting time-frequency structure (Fig. 1d). Other FRB 121102 bursts like AO-02 and GB-01 also show such a feature. The cause of formation of the feature has been investigated by several authors (Metzger et al. 2019; Wang et al. 2019) Both studies are based on the dynamic spectrum dedispersed traditionally, like the one presented in Figure 1d. However, an alternative dynamic spectrum could also be considered, in which the emission time is continuous and frequency-dependent.

We suggest that the original dynamic spectrum should be the one illustrated in Figure 1c, which is an estimation spectrum of the burst AO-06 if it is dedispersed to 563.8 pc cm\(^{-3}\), instead of 560.5 pc cm\(^{-3}\). In this further dedispersed spectrum, the burst occurs in the lower frequency band first, reaches its maximum in the middle time of the burst on the middle-frequency, fades out from the higher frequency band at last. An essential feature of the newly conceived dynamic spectrum is that the instantaneous spectrum of it keeps being the same triple-peaked one throughout the radiation process. The instantaneous spectrum (Fig. 1a) is similar to the integrated pulse profile obtained in the usual way; both have three peaks. In this sense, we may say that the integrated pulse profiles of FRBs are the instantaneous spectra near the sources of FRBs.

2.2. Undulator radiation and the frequency-swept dynamic spectrum
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Figure 2. The magnetic field causing undulated motion of charged particles. (a) The schematic of an undulator. (b) A conceived scenario of radiation process of FRBs: charged beam passing through a quasi-periodic magnetic structure at a zero angle to the observer.

An eligible physical interpretation of the dynamic spectrum of AO-06 (Fig. 1c) needs to address two observational facts: [1] the triple-peaked instantaneous spectrum; [2] the drifting of central frequency from the low frequency to the high. Undulator radiation is a known mechanism which is capable of generating both effects at the same time.

An undulator (Fig. 2a) is a periodic array of dipole magnets with alternating polarity. It is commonly used as an insertion device in the storage ring to produce monochromatic synchrotron radiation with relativistic particles. The frequency of radiation in the direction of angle $\theta$ is determined by equation (Hofmann 2007):

$$f = \frac{2\gamma^2}{1 + K^2 + \gamma^2 g^2 \left( \frac{c}{\lambda_u} \right)}$$  

Where $\gamma$ is Lorenz factor, $c$ is the speed of light, $\lambda_u$ is the period of the magnetic field structure, $K$ is the vector potential of magnetic field normalized to the rest energy of the radiation particle. The radiation produced in the weak magnetic field ($K < 1$) is the so-called undulator radiation. This equation is essentially describing a Doppler effect. The Doppler shift is angle-dependent; if you look on-axis, you get the highest frequency; if you go off-axis, the frequency gets lower and lower.

A frequency-swept dynamic spectrum can be produced by the process illustrated in Figure 2b, which is conceived mostly according to the circumstances between the Earth and the Sun. A beam of relativistic electrons is produced in an active source, might be a neutron star or double stars. The field in the acceleration site is strong. The radiation of AO-06 is not produced here; It is produced in a relatively weak field far away. After traveling about 1 AU or more, electrons meet with a quasi-periodic magnetic structure. The radiation of the burst AO-06 is produced here.

To describe the burst AO-06, an on-axis condition (where $\theta = 0$) suffices. In this case (Fig. 2b), the variation of $f$, the central frequency, is only determined by the variation of $\gamma$. $f$ is proportional to $\gamma^2$. For the burst AO-06, a raw estimation gives that the low energy electrons arrived 3 ms earlier (Fig. 1c). With a smaller $\gamma$, they produce radiation in the lower-frequency band. The central frequency increases as the electrons arriving later have larger $\gamma$. To offer an appreciation with magnitude, if a beam of electrons with $\gamma$ ranged from $2.07 \times 10^3$ to $2.37 \times 10^3$, sequentially passed through a periodic structure with a period of 2000 km and a weak magnetic field ($K < 1$). Then the central frequency received at a distance will drift from 1.28 to 1.68 GHz like the one shown in Figure 1c.

In addition to the central frequency drifting, the observed radiation intensity is also varying with frequency. A gaussian envelope has been used in our simulation to account for the energy distribution of radiation particles.

2.3. Peaky spectrum from a tapered undulator

The triple-peaked spectrum illustrated in Figure 1a is the squared Fourier transform of a sinusoidal function with 28 periods over which the wave vector ($k = 2\pi/\lambda$) is increased by 46.5% (hereafter the taper parameter $\eta$). This type of spectrum is known to the accelerator society as the radiation spectrum from a tapered undulator (Bosco & Colson 1983; Shih & Caponi 1982). The spectrum can be produced by passing mono-energetic electrons through a periodic magnetic structure with 28 periods over which the wave vector is increased by 46.5%.
The forming mechanism of the spectrum can be easily apprehended through the relationship between a driven motion and driven force. For simplicity, we use throughout this paper electrons as radiation particles, although they can be any charged particles. The undulated motion of electrons in periodic magnetic fields is a driven motion and a harmonic oscillation in the frame of reference moving with the electrons. It can be approximated by a wave function $e^{-i\omega t}$. Since the first-order approximation of the radiation spectrum of a single electron is the Fourier transform of the electron's apparent trajectory. The on-axis undulator radiation spectrum of a single electron is the Fourier transform of $e^{-i\omega t}$ with a Doppler shift due to the transformation between different frames. When the spatial frequency (driven frequency) of the magnetic structure is linearly increasing, the frequency of oscillation (driven motion) also increases linearly. The motion of particles can be approximated by $e^{-i(\omega t+bt^2)}$, which is merely the function $e^{-i\omega t}$ with $\omega$ substituted by a linear varying term $\omega_0 + bt$. The key point here is the $t^2$ term in the phasor. Because of this, the radiation spectrum, which can be approximated by the Fourier transform of this wave train, is a Fresnel integral. This process shares a similar mathematic formulism with near field (Fresnel) diffraction of visible light from narrow slits (Bosco & Dattoli 1983). In optics, the triple-peaked pulse profile of AO-06 corresponds to the diffraction pattern from a slit encompassing 1.5 Fresnel zones (Hecht 2017).

From the energy perspective of view, the radiation spectrum of electrons reflects how electrons lose energy in the magnetic vector potential field through radiation. The spectrum is determined by the magnetic vector potential's spatial distribution along the traveling path of electrons. Assuming the magnetic structure to have plane polarization and the magnetic field is weak, then the spontaneous radiation spectrum is proportional to the squared Fourier transform of the transverse magnetic vector potential (the Appendix). The vector potential $A_1(z)$ of a tapered magnetic field and the spectrum $S(\omega)$ measured at the radio telescope can be written as:

$$A_1 = A(z) \sin(k_0[z + bz^2])$$

$$S(\omega) \propto |\omega \cdot \text{Fourier}(A_1)|^2$$

Where $A(z)$ is the envelope of the magnetic field's amplitude along the electrons’ path: $z$; it is approximated by a Gaussian function in our simulation and affects the symmetry of the spectrum; $k_0$ is the wave vector at $z = 0$; $b$ is a constant and describes the linear variation of the wave vector along $z$; $\omega$ is the radiation frequency. Assuming a constant velocity along the $z$-direction, then the variable $z$ in Equation 2 can be substituted with the time variable $t$, that is the time representation discussed in the previous paragraph.

The multi-peak feature of the spectrum comes from the Fourier component of Equation 3. A parameter $W$ can be defined as $\frac{\log(1+\eta) \times N}{2}$ (the Appendix), where $N$ is the number of undulator periods, $\eta$ is the taper parameter. $W$ is a measure of the phase error introduced into the radiation. When $W = 1$, there is a $\pi/2$ phase error among the radiation (Walker 1988). If the phase error vary systematically, the peak of a spectrum will oscillate, because the interference will alternate between destructive and constructive at different frequencies. The number of peaks appearing on a spectrum is about $W/2$ (illustrated by Figure 3). For example, the double-peaked FRB 121002 (Champion et al. 2016) should have a $W$ value around four. Figure 3 shows the spectrum for a general case expressed in terms of the $W$ parameters, calculated through Equation 3 with a symmetric envelope $A(z)$. This plot is the equivalence of Figure 4 of the paper (Walker 1988). The code generating Figure 3 is available online.

At first, it was the analogy between the spectrum ($W = 10$ in Figure 3), and the integrated pulse profile of the burst 7# and 5# of FRB 121102 (Spitler et al. 2016) stimulated the analysis presented in this section. However, further analysis shows that the fringe in the observation data is more complicated than what we have anticipated. So even if there is an analogy between them, the connection is not clear and a subject for further study.

3. SIMULATION

3.1. the procedure of simulation

We will explain the simulation procedure through the simulation of the burst AO-06, which is a case with $W \sim 5.44$. The objective of the simulation is twofold. We need to generate a dynamic spectrum whose shape correlates with the observation approximately. Meanwhile, its integrated pulse profile should be similar to the observed one. We can only strike a balance between them by adjusting eleven parameters manually iteratively. In the following, we will skip the iteration process and directly use the known value of the relevant parameters.

The main steps of the simulation are:

1. A wave train (in Figure 4) is firstly generated to represent the magnetic field potential $A_1$ in Equation 2. Two parameters determine the sine component of the Equation 2: $W = 5.44$ and $\eta = 0.46$ from which the b parameter can
be calculated. It is equivalent to know how many periods the magnetic structure has and how it is varying. The unit length of the undulator is defined by the periodic length at $z = 0$, hence $k_0 = 2\pi$. The envelope $A(z)$ is approximated by a Gaussian function defined by the variable $MuWave$ and $SigmaWave$ in the code. The values of $MuWave$ and $SigmaWave$ affect the relative heights of the peaks of the final integrated pulse profile. The length of $A(z)$ is normalized to the length of the undulator ($L_{\text{eta}}$ in the code).

(2) The instantaneous spectrum in Figure 1a is computed according to the Equation 3 by doing a Fourier transform to the wave train, multiplying circular frequency, and squaring itself. The red dashed line in Figure 1a represents the background noise level, which is a free parameter, defined by the variable $bg$ in the code.

**Figure 3.** Spectral distribution around the central radiation frequency $\omega_c$ from a tapered undulator with $\eta = 0.12$.

**Figure 4.** The magnetic field breeding the burst AO-06. It is a sinusoidal function with 28.5 periods over which the wave vector increases 46.5%.
(3) The dynamic spectrum in Figure 1c is created by subtracting the background noise $b_q$ from the spectrum in Figure 1a, then moving it along the white line in Figure 1c, and applying a Gaussian function to modulate the radiation intensity. The white line is a dispersion curve defined by $t = \frac{DM \cdot \mu}{f_{\text{frequency}}}$ with the dispersion measure (hereafter $DM$) $\sim -3.1$ pc cm$^{-3}$. The value of the DM is defined by the variable $dm$ in the code. The Gaussian function is defined by the variables $sigma$ and $mu$. The unit length of the Gaussian function is defined by the frequency range of the white line. The high and low-frequency boundaries of the line determine the frequency range. They are defined by the variables $fh$ and $fl$ in the code. The output is not sensitive to $fh$ and $fl$ because we can adjust the width of the Gaussian envelope through the variable $sigma$. However, sufficient values should be assigned to them to guarantee reasonable time-frequency coverage. The parameters set in this step are corresponding to the effects related to the distribution of the radiation particles. Since the distribution is unknown, we have to guess the radiation spectrum in the time-frequency domain. The reasonability of the conjecture made here will be discussed in Section 4.

(4) The dynamic spectrum in Figure 1d is obtained by dispersing the dynamic spectrum in Figure 1c by 3.3 pc cm$^{-3}$ further. The variable $dm1$ in the code is account for this DM lag.

A physical interpretation of the simulation above is that the original dynamic spectrum is oblique. It is produced by a beam of charged particles passing through a quasi-periodic magnetic structure. The Fourier transform of the potential of the magnetic field has three peaks. The space-time distribution of the particles results in a dispersive-like frequency drift. However, the current data processing method is incomplete; that the dynamic spectrum is presented vertically like the one in Figure 1d. The dynamic spectrum in Figure 1d is supposed to be observed in a position far away from the source region of FRB, to which the DM from the source of the burst is about 3.3 pc cm$^{-3}$.

The above approach can be easily generalized to simulate other triple-peaked bursts like the burst GB-01, AO-02, and AO-11 of FRB 121102 (Hessels et al. 2019), the September 17 burst of FRB 180814.J0422+73 (Amiri et al. 2019) and FRB 181017 (Farah et al. 2019). They are just the burst AO-06 with slightly different magnetic field configuration and the energy distribution of radiation particles. Naturally, the pulse of triple-peaked radio pulsars, like PSR J1757-2421 (Yuan et al. 2017), can also be reproduced with the same approach. The simulation codes of the burst AO-06, GB-01, AO-02, AO-11, and AO-13 written in Matlab language can be downloaded online$^1$. The output pictures of the codes are also available online for none Matlab user’s references.

3.2. Comparison with the observation

The burst AO-02 is another triple-peaked burst of FRB 121102 (Hessels et al. 2019). A comparison between the simulation and the observation of the burst AO-02 is in Figure 5. Its simulation can be obtained by dispersing a $DM \sim -4.5$ pc cm$^{-3}$ dynamic spectrum by 5.5 pc cm$^{-3}$. The blue and yellow image in Figure 5a is the simulated time-frequency structure. The gray-colored part of the image is created from a 3D printable data$^2$ of the burst. It can be seen that the the two parts correlate to each other very well, except that the simulated separation between the two strips on the right-side does not match that of the observation. The reproduction of the integrated pulse profile (Figure 5b) is not bad.

In the simulation of AO-06, we cannot keep the strip in the top-left corner of Figure 1d strictly vertical like the observation. Otherwise, the envelope of the other two strips cannot match the observation very well.

The discrepancies between the simulation and observation may stem from two main factors: (1) Our simulation can only reflect the radiation process in some degree but not the whole process, including propagation and data collection. (2) The slope of the original dynamic spectrum can not be described by a dispersion relationship strictly as what we have assumed.

A comparison between the simulation and the observation of the burst AO-13 (Hessels et al. 2019) bring an inconspicuous feature into the attention. Although the pulse of the burst has two peaks, it is probably a triple-peaked burst with the high-frequency part buried under the background noise level. The fact is revealed by the residual radiation in the high-frequency range, marked by the black arrow in Figure 6. A small amount of observed radiation(gray colored) locates in the position where the third strip is supposed to appear. We intentionally lower the background noise level of the image to let the leftmost strip appeared.

4. DISCUSSION

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$^1$ The Matlab codes can be downloaded from: https://github.com/huangyu7991/frb.git

$^2$ The 3D printable data is downloaded from: https://www.thingiverse.com/thing:2723399/files
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Figure 5. A comparison between the simulation and the observation of the burst AO-02. a) the simulated dynamic spectrum (blue and yellow) of the burst AO-02, blended with the observation (gray). b) the integrated pulse of the dynamic spectrum in A.

Figure 6. A picture with the observation (gray colored) and the simulation (blue-yellow colored) of the burst AO-13 blended.

(1) A significant assumption made in our simulation is that the energy distribution of the radiation particles will result in a dispersive-like frequency drift, as shown by the white line in Figure 1c. The validity of such an assumption comes from the selection effect that has been inevitably introduced into the FRB observation through the current FRB searching strategy. In practice, the FRB searching algorithm dedisperse time series to find a single pulse. Such kind of algorithm cannot recognize any burst without a dispersive-like frequency drift. The searching strategy itself might be the reason why there are FRBs non-repeatable. It is because the initial time-frequency drift is not always dispersive-like.

(2) The other implicit assumption in our work is that the spectrum from an electron can approximate the radiation spectrum of the whole electron beam. This assumption is only valid when the emittance of the beam is much smaller than the radiation cone (∼ 1/γ) of an electron. Otherwise, a large number of electrons traveling at different angles with the magnetic field will emit a range of lower frequency emission due to a non-zero θ in Equation 1. That emission will smooth-off the spectrum.

How can a charged particle beam achieve a low emittance? One possibility is that the electron beam is bunched. A plasma wakefield acceleration (PWFA) process may have occurred when the charged beam propagates in plasmas (Chen et al. 1985). The PWFA process is self-modulated (Kumar et al. 2010; Pukhov et al. 2011). The beam will split into a series of bunches due to two-stream instability. This effect may also be an answer to the question of why the
radiation of FRBs is so intense. Bunching effect will increase the radiation strength by order of the electron number in a bunch.

(3) The quasi-periodic sinusoidal magnetic structure is a strong assumption on the boundary condition of the problem. It is natural to doubt where to find a tapered undulator in the universe. Our opinion is that the accuracy required here is far less than that of the accelerator. The relative bandwidth is $\sim 46\%$. Hence the nominal accuracy of the periodic length required for 28 periods is about 1.6%, and the real requirement on accuracy can be one order less (Walker 2013). Since the Fourier transform is additive, the periodic structure in practice needs not to increase linearly strictly. The position of the different periods is commutative. What is more, the variation of the field is not necessarily sinusoidal; it can be any periodic function and even intermittent.

From the observational point of view, this quasi-periodic structure comes up with the triple-peaked spectrum. If the oblique spectrum exists, the quasi-periodic structure is a possible source of the triple-peaked spectrum.

(4) Some FRB properties inferred here can be related to the behaviors of radio pulsars, especially the Crab pulsar. The Crab pulsar is known to have a highly variable DM. While the average value of its DM is $56.8 \text{ pc cm}^{-3}$, the variation of the DM is about $50 \text{ pc cm}^{-3}$ (Cordes & McLaughlin 2003). This behavior can be explained by an oblique dynamic spectrum, whose slope is variable from pulse to pulse. The intensity of some giant pulses from the Crab pulsar can exceed 2MJy (Hankins & Eilek 2007). Such an increase in the radiation intensity by orders can also be a manifestation of bunching effect.

The dedispersion method of FRBs is inherited from that of radio pulsars. It is reasonable to doubt that some radio pulsars also have an oblique original dynamic spectrum, and the data processing of radio pulsars should be subject to a reexamination.

(5) The instantaneous spectra of some FRB 121102 bursts, like the burst 11O and 11H observed in the frequency range $5\sim8 \text{ GHz}$ (Gajar et al. 2018), are already multi-peaked. Our theory can be applied to them directly, without further dedispersing and dispersing. They are bursts produced with highly mono-energetic beam, the energy of the radiation particles of the whole beam is single-valued.

(6) Besides the ideal FRB samples studied in this work, there are FRBs showing complex multi-peaked integrated pulse profiles, for example, FRB 170827 (Farah et al. 2018) and the burst 10 of FRB 121102 (Spitler et al. 2016). It seems to us that these bursts can be described by a process with compound magnetic field structure or energy distribution of radiation particles. Their production is a subject of further study.

5. SUMMARY

The provocation of this study is the similarity between a diffraction pattern and the integrated pulse profile of some FRBs and radio pulsars. In this paper, we build a mathematical connection between them through the mechanism of undulator radiation. Although behind the same pattern, there might be a different process, the general formulation of the pattern is probably similar. The contribution of this work to the FRB study is more on an abstract layer.

It is important to have the ability to change our perspective on the time lag in the observation of both FRBs and radio pulsars. We should avoid from investing only in propagation effect or dispersion effect to explain the observed time lag among different frequencies, and ignoring the possibility that the radiation process itself may bring a millisecond time lag into the observation. The oblique dynamic spectrum proposed here is intended to detangle them.

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APPENDIX
A. THE RADIATION SPECTRUM AS THE SPATIAL SPECTRUM OF MAGNETIC FIELD.

The content here is a physical interpretation of the formula in Section 2 of Shih & Caponi (1982) in an on-axis case. The focus is on a fundamental physical understanding, but not math strictness. The vector potential of a plane-polarized magnetic field inside the undulator with a linear taper can be written as:

$$\overrightarrow{A}_\perp(z) = A(z) \cos[\int_0^z k(z)dz] \overrightarrow{e}_x = A(z) \cos(1 + \frac{\eta}{2L(\eta)} z |k_0 z|) \overrightarrow{e}_x$$  \hspace{1cm} (A1)

Where \( \eta \) is the taper parameter, \( L(\eta) \) is the undulator’s length, \( k_0 \) is the undulator’s wave vector at \( z = 0 \). \( A(z) \) is the amplitude. A similar representation can be found in the Equation (1.2) of Bosco & Colson (1983). It can be seen that the \( b \) term in Equation 2 is \( \frac{\eta}{2L(\eta)} \).

Far away from the undulator, the radiation spectrum of a relativistic electron is given as follows (Hofmann 2007):

$$\frac{dW}{d\Omega d\omega} = \frac{e^2 \omega^2}{4\pi^2 c} | \int_0^L \overrightarrow{n} \times [\overrightarrow{n} \times \overrightarrow{\beta}(z)] e^i \beta z \{1 - \overrightarrow{n} \cdot \overrightarrow{\beta}(z) dz \}|^2$$ \hspace{1cm} (A2)

Where \( \overrightarrow{n} \) is the direction of observation, \( L \) is the length of the undulator, \( \overrightarrow{\beta}(z) \) is the electron velocity at position \( z \) normalized to light speed in a vacuum.

Consider the on-axis case where \( \overrightarrow{n} = \hat{z} \), and ignore the small difference between the longitudinal velocity \( \beta_z \) and \( \beta \), then the phasor can be simplified as:

$$i \frac{\omega}{c} \int_0^Z [1 - \beta_z(z)] dz = i [\frac{k}{2\gamma^2 z}]$$ \hspace{1cm} (A3)

The spectrum of Equation A2 can be expressed in this form:

$$\frac{dW}{d\Omega d\omega} = \frac{e^2 \omega^2}{4\pi^2 c} \omega^2 \cdot [\int_0^L \beta_\perp(z) e^{i \frac{\omega}{2\gamma^2 z}} dz]^2$$ \hspace{1cm} (A4)

It can be seen that the spectral shape can be approximated by the Fourier transform of the driven motion \( \beta_\perp(z) \). The \( 2\gamma^2 \) in the denominator is according to the Doppler shift. The \( \omega^2 \) term will cause asymmetry to the spectrum. For narrow band spectrum, it can be neglected.

$$\beta_\perp(z) = - \frac{eA_\perp(z)}{\gamma mc^2}$$ \hspace{1cm} (A5)

Inserting the above expressing of \( \beta_\perp(z) \) into Equation A4, we may see that the radiation spectrum represents the spatial spectrum of the magnetic potential along the electron’s path. The complete form of the spectrum is:

$$\frac{dW}{d\Omega d\omega} = \frac{e^3}{4\pi^2 \gamma mc^2} \omega^2 \cdot [\int_0^L A(z) \cos(1 + \frac{\eta}{2L(\eta)} z |k_0 z|) e^{i \frac{\omega}{2\gamma^2 z}} dz]^2$$ \hspace{1cm} (A6)

B. THE DEFINITION OF THE PARAMETER W

This parameter is equivalent to the near field parameter defined by Walker (1988), where \( \sqrt{W} \) can be thought of dimensionless slit width.

Combining Equation (13) of Walker (1988), Equation (1.4) of Bosco & Colson (1983):

$$aL^2 = 4\pi W$$ \hspace{1cm} (B7)

$$L \approx N \lambda_0 \log(1 + \eta)$$ \hspace{1cm} (B8)

Where \( L \) is the length of the undulator, \( N \) is the number of undulator periods, \( \alpha \) is the coefficient before the \( z \) square term in Equation 2 (hence \( \alpha = \frac{2\eta}{\lambda_0 L(\eta)} \)). We may get:

$$W = \frac{aL^2}{4\pi} = \frac{N \log(1 + \eta)}{2 \lambda_0 L(\eta)}$$ \hspace{1cm} (B9)

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