Bonus Symmetry and the Operator Product Expansion of $\mathcal{N} = 4$ Super-Yang-Mills

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The superconformal group of $\mathcal{N} = 4$ super-Yang-Mills has two types of operator representations: short and long. We conjecture that operator product expansions for which at least two of the three operators are short exactly respect a bonus $U(1)_Y$ R-symmetry, which acts as an automorphism of the superconformal group. This conjecture is for arbitrary gauge group $G$ and gauge coupling $g_{YM}$. A consequence is that $n \leq 4$-point functions involving only short operators exactly respect the $U(1)_Y$ symmetry, as has been previously conjectured based on AdS duality. This, in turn, would imply that all $n \leq 3$-point functions involving only short operators are not renormalized, as has also been previously conjectured and subjected to perturbative checks. It is argued that instantons are compatible with our conjecture. Some perturbative checks of the conjecture are presented and $SL(2, \mathbb{Z})$ modular transformation properties are discussed.
1. Introduction

The central objects which characterize conformal field theories in any dimension are the spectrum of operator dimensions and the operator product expansion (OPE) coefficients. These objects control the behavior of operator correlation functions, which are the observables of conformal field theories. Over the past several years, it has been appreciated that four dimensional gauge theories with enough matter generically lead to interacting conformal field theories, and it is an interesting avenue in field theory to consider correlation functions in any of these theories.

This avenue has not been much explored until recently, in the context of the maximally supersymmetric 4d conformal field theory, $\mathcal{N} = 4$ super-Yang-Mills. The motivation behind the recent work is the conjectured duality of [1-3] to gravity in anti-de Sitter space. There have been recent studies, from both the $\mathcal{N} = 4$ field theory and AdS gravity dual perspectives, of various $n$-point functions; see, e.g. [2-11] and references cited therein.

We will here conjecture that the OPE coefficients of $\mathcal{N} = 4$ supersymmetric Yang-Mills exactly obey certain selection rules. As will be discussed, a motivation for this conjecture comes from the conjectured AdS duality. Nevertheless, our conjecture itself is purely a statement about the $\mathcal{N} = 4$ field theory, and thus logically separate from the AdS duality conjecture; it should be possible to prove or disprove it purely in the context of field theory. Indeed, we believe that our conjecture is correct for $\mathcal{N} = 4$ super-Yang-Mills with arbitrary gauge group $G$. Unfortunately, we have not been able to prove the conjecture, so we will here only be able to present its motivation and some checks within the context of instantons and perturbative $\mathcal{N} = 4$ field theory.

As reviewed in the next section, operators form representations of the $\mathcal{N} = 4$ superconformal group, which has two types of representations: the generic “long” representations, and the special “short” representations. The short representations are the generalizations of chiral superfields and analogs of BPS states and satisfy special properties thanks to supersymmetry; for example, their dimensions are not renormalized. It is the short representations which are seen as single particle states in the $\text{AdS}$ supergravity dual. All operators in a short representation will be referred to as “short operators,” while those in a long representation will be referred to as “long operators.”

As emphasized in [12], the $\mathcal{N} = 4$ superconformal algebra admits a bonus $U(1)_Y$ symmetry, which acts on the supersymmetry generators as an $R$-symmetry. (See also [13,14] for earlier discussions of $U(1)_Y$.) Although $U(1)_Y$ is not a symmetry of the field
theory, all operators can be assigned a definite $U(1)_Y$ charge and, based on the $AdS$ duality, it was conjectured in \cite{[12]} that all correlation functions of short operators, for $G = SU(N)$ (and also $SO(N), Sp(N)$, and theories with less SUSY obtained by orbifolds) \textit{approximately} respect a $U(1)_Y$ conservation selection rule in the double limit where $g_{YM}^2 N \gg 1$ and $g_{YM}^{-2} N \gg 1$, where the supergravity dual is weakly coupled.

It was further conjectured in \cite{[12]} that the $U(1)_Y$ selection rule is actually \textit{exact} for correlation functions of $n \leq 4$ short operators. We believe that this statement applies for $\mathcal{N} = 4$ with arbitrary gauge group $G$ and gauge coupling $g_{YM}$. On the other hand, it is known that the $U(1)_Y$ selection rule is definitely violated for general $n \geq 5$ point functions of short operators.

The above conjecture, that a selection rule is exact for $n \leq 4$ point functions of operators, but generally violated for $n \geq 5$ point functions, prompts the question: “why should $n \leq 4$ point functions be so different from $n \geq 5$ point functions?” In fact, we point out that there is indeed a natural difference between $n \leq 4$ and $n \geq 5$ within the context of the OPE; this is the motivation for our conjecture. The statement of our conjecture is that the OPE coefficients involving either three short operators ($SSS$) or two short and one long operator ($SSL$) exactly respect the $U(1)_Y$ symmetry. On the other hand, OPE coefficients involving more than one long operator ($SLL$ or $LLL$) generally violate the $U(1)_Y$ symmetry. As we discuss, this conjecture has as a consequence that the $U(1)_Y$ symmetry is exact for $n \leq 4$ point functions but violated for $n \geq 5$ point functions of short operators. The exact $U(1)_Y$ selection rule for $n \leq 4$ point functions, in turn, implies \cite{[12]} the non-renormalization of 3-point functions of short operators which was conjectured in \cite{[13]} and checked in the weakly coupled field theory limit to leading order in perturbation theory in \cite{[14]}.

An outline of this paper is as follows. In the next section, we review the representation theory of the $\mathcal{N} = 4$ superconformal group and its $U(1)_Y$ outer automorphism. In sect. 3, we discuss the OPE and the motivation for our conjecture. In sect. 4 we discuss how the supercharges act on gauge-invariant, composite, operators and demonstrate in perturbation

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1 Generally, there can be contact term contributions to correlation functions, involving delta functions which vanish unless operators are at the same point, which violate these selection rules. For example, in $\mathcal{N} = 4$ supersymmetric $U(1)$ gauge theory we have $\langle F_{\alpha\beta}(x)F^{\alpha\beta}(y) \rangle \sim \delta^4(x - y)$, where the contact term violates the selection rule by 4 units. We will always take the operator insertion points to be separated and thus ignore contact terms.
theory that it can be necessary to include quantum corrections to descendent operators in the case of long representations of the superconformal group.

In sect. 5, we discuss instanton contributions to operator correlation functions. We present a heuristic argument for a simple relation between the $U(1)_Y$ charge of an operator and how many of the 16 exact fermion zero modes which it contains. Since instantons contribute to correlation functions only if the operators saturate all 16 of the exact fermion zero modes, the relation we present leads to selection rules for when instantons can contribute to correlation functions. For an $n$-point function of all short operators, instantons or anti-instantons can contribute only if $|q_T| = 4(n - 4)$, where $q_T$ is the net $U(1)_Y$ charge violation. This is nicely consistent with the conjectured non-renormalization of $n \leq 3$ point functions [3,4], and the conjecture [12] that the $U(1)_Y$ selection rule is exact for $n = 4$-point functions. For correlation functions of two short and one long operator, we argue that instantons can contribute only if $q_T = 0$, which is compatible with our conjecture that three-point functions with two short and one long operator exactly vanish if $q_T \neq 0$. Finally, for correlation functions of two long operators we argue that instantons only contribute if $q_T = 0$, which is compatible with our argument that all two-point functions of either short or long operators exactly respect the $U(1)_Y$ selection rule.

Sect. 6 presents a non-trivial, perturbative field theory check of the conjecture [12] that $n \leq 4$ point functions of short operators respect the $U(1)_Y$; leading order radiative corrections which could have violated $U(1)_Y$ “miraculously” sum to zero, much as in the 3-point functions analyzed in [3]. This result applies for any gauge group $G$.

In sect. 7 we discuss perturbative field theory checks of our OPE conjecture involving two short and one long operator. In a variety of examples, we verify that leading contributions to correlation functions which would violate our conjectured selection rule indeed sum to zero. In many cases, this vanishing is a simple consequence of a sum over color indices $a, b$ of the form $f^{abc}d_{(ab);c_i}$, which vanishes because $f^{abc}$ is antisymmetric in $[ab]$ while $d_{(ab);c_i}$ is symmetric. In many other cases, the required vanishing of leading order radiative corrections is much more difficult to verify and it was beyond our patience to complete the task. We also present an example where the quantum correction found in sect. 4 for a long descendent is precisely correct to ensure that a possible violation to our selection rule is indeed canceled. On the whole, we find our checks presented in sect. 7 to be somewhat disappointing in that we did not find many tractable examples with the sorts of “miraculous” cancellations found in [1] and our sect. 6. On the other hand, at least all
tractable examples are indeed consistent with our conjectured OPE selection rule, even if the vanishing is not so impressive.

Finally, in sect. 8, we make some comments regarding \( SL(2, \mathbb{Z}) \) S-duality, the OPE, and our conjectures.

Of course, as with the conjectured non-renormalization of 3-point functions of short operators \[5,6\], instantons and leading order perturbative checks can provide some confidence that conjectured exact statements are indeed correct, but they are no substitute for a proof. It is quite possible that our conjecture concerning the OPE can actually be proven by making full use of the powerful constraints of the \( \mathcal{N} = 4 \) superconformal Ward identities. Indeed, because all superconformal primary operators are neutral under \( U(1)_Y \), the non-trivial content of our conjecture is a statement regarding correlation functions involving superconformal descendents. If, as naively expected (but it is not at all obvious if it’s true), all superconformal descendant correlation can be obtained via Ward identities from correlation functions involving only superconformal primary operators, it must be possible to prove or disprove our conjecture directly via superconformal Ward identities. We have not yet found such a direct proof and leave this as a problem for future investigation\[2\].

As mentioned above, we believe that our conjecture applies exactly for \( \mathcal{N} = 4 \) with any gauge group \( G \). This is in line with the expectation that it is actually a consequence of supersymmetric Ward-identities. If true, this implies that the exact \( U(1)_Y \) selection rule for \( n \leq 4 \) point functions and the non-renormalization of \( n \leq 3 \) point functions of short operators also apply for arbitrary gauge group \( G \) and gauge coupling \( g_{YM} \). It is indeed possible to verify that the cancellations of radiative corrections found in \[3\] in the context of \( G = SU(N) \) occur for arbitrary gauge group \( G \): the only group theory identity needed in \[3\] was \([T^a, T^p], T^p\] = \( NT^a \), which is a statement about the quadratic Casimir and generalizes to arbitrary group \( G \) as \([T^a, T^p], T^p\] = \( C_2(G)T^a \), with \( C_2(G) \) the quadratic Casimir of the adjoint representation.

\[2\] The \( \mathcal{N} = 4 \) harmonic superspace formalism of \[15\] is designed to efficiently make use of the superconformal Ward identities. However, we are presently wary of this formalism as it is purely on-shell and was shown in \[12\] to lead to the incorrect conclusion that all \( n \)-point functions of short operators exactly respect the \( U(1)_Y \) selection rule, for all \( n \). (See also \[16\].) As pointed out in \[12\], it is possible that the formalism can be salvaged by finding some missing superconformal invariants which violate \( U(1)_Y \), though this remains to be seen. In any case, it does not seem well suited for including operators in long representations.
2. $\mathcal{N} = 4$ superconformal reps. and the $U(1)_Y$ bonus symmetry

The 4d $\mathcal{N} = 4$ superconformal group $PSU(2, 2|4)$ has two types of representations, which we refer to as “short” and “long”. All representations are generated by a primary operator $O_P$, along with descendent operators, related to $O_P$ by supersymmetry, of the form $\delta^n \delta^m O_P$, and their conformal descendents. Here $\delta^n \delta^m O_P$ denotes a nested graded commutators with the sixteen supercharges $Q_\alpha^I$ and $\overline{Q}_{I, \dot{\alpha}}$, e.g. $\delta^2 \delta^2 O_P \equiv [Q, \{Q, [Q, O_P]\}]$. (The remaining 16 superconformal supercharges act as lowering operators.) For the generic, long, representation, the operators $\delta^n \delta^m O_P$ truncate at $n \leq 8$ and $m \leq 8$ by Fermi statistics. All of the operators $\delta^n \delta^m O_P$ in the long multiplet are referred to as “long operators.”

The short representations have the defining property that they instead truncate at $n \leq 4$ and $m \leq 4$; they are the analog of BPS objects of superconformal field theories. It turns out that such representations are completely characterized by an integer $p \geq 0$. In particular, the dimensions of such operators are fixed in terms of $p$ and thus not renormalized. The spectrum of short representations was found in [13] and their table can also be found reproduced in [12]. The primary operators which generate the short representations are $O_p \sim [\text{Tr}_G(\phi^p)]_{(0, p, 0)}$, where $\phi$ is the $\mathcal{N} = 4$ scalar in the adjoint of the gauge group $G$ and the $6 = (0, 1, 0)$ of the $SU(4)_R$ global symmetry and $(0, p, 0)$ are the Dynkin labels of the $SU(4)_R$ representation. There are $\text{rank}(G)$ independent short representations (in addition to the identity), labeled by $p$ which are the degrees of the independent Casimirs of $G$. We refer to all operators $\delta^n \delta^m O_P$ in short representation multiplets as “short operators.”

A simple example of a long representation primary operator is the $SU(4)_R$ singlet $O_K \sim [\text{Tr}_G(\phi^2)]_{(0, 0, 0)}$. As discussed in [17], the multiplet of long operators associated with $O_K$ includes the “Konishi current,” which is discussed extensively in [18]. More generally, we can obtain long, primary operators as $[\text{Tr}_G(\phi^{r+2s})]_{(0, r, 0)}$ via taking a completely symmetric combination of the $\phi$s and then taking any number $s > 0$ traces. Operators which are not completely symmetric in the $\phi$s are descendents since, in $N = 1$ SUSY notation, $\overline{D}^2 \Phi_i = g \epsilon^{ijk} [\Phi_j, \Phi_k]$, where $(\Phi_i)$ $\Phi_i$ are $N = 1$ (anti) chiral superfields and $i = 1, 2, 3$. Other examples of long representations are multi-trace operators, with more than a single trace over the gauge indices.

In [12] it was emphasized that $PSU(2, 2|4)$, admits an outer automorphism $U(1)_Y$, which acts as an $R$ symmetry, under which the supercharges transform with charge $\pm 1$. 


The $U(1)_Y$ charge assignment of the short representations was determined in the original analysis of \cite{[13]} in the context of the 5d, $\mathcal{N} = 8$, AdS$_5$ supergravity which is dual \cite{[1][3]} to the 4d $\mathcal{N} = 4$ field theory. In terms of the field theory, all operators can be assigned definite charges under $U(1)_Y$; we write each operator as $O_i^{(q_i)}$, indicating the $U(1)_Y$ charge $q_i$. The charge assignment is determined as follows \cite{[13],[12]}: the adjoint scalar $\phi$ of $\mathcal{N} = 4$ is assigned charge zero, while the supercharges $Q^I_\alpha$ and $\overline{Q}^{\dot{I},I}_{\dot{\alpha}}$ are assigned charges $-1$ and $+1$, respectively. Thus all primary representations $O_P(x)$, whether for a short or a long multiplet, which have the form of a trace or traces of symmetrized powers of $\phi$, are assigned charge zero. The superconformal descendents of the form $\delta^n \delta^m O_P$ have $U(1)_Y$ charge $m - n$. All operators in short representations thus have $U(1)_Y$ charges $|q| \leq 4$, while all operators in long representations have charges $|q| \leq 8$.

$U(1)_Y$ is generally not a symmetry of the $\mathcal{N} = 4$ field theory. Nevertheless, it was argued in \cite{[12]} (see also \cite{[14]}) that $U(1)_Y$ is an approximate “bonus symmetry” of general correlation functions of small representation operators in an appropriate limit ($g^2_{YM} N \gg 1$ with also $g^{-2}_{YM} N \gg 1$). It was further conjectured to be an exact (i.e. valid for all gauge groups and all $g_{YM}$) symmetry of all $n \leq 4$ point functions of operators in small representations. The argument of \cite{[12]} relied on the AdS duality: $U(1)_Y$ is an approximate symmetry of IIB string theory in its classical supergravity limit. However, by arguments similar to those of \cite{[19]}, it was suggested in \cite{[12]} that the stringy and quantum corrections to supergravity, which generally violate $U(1)_Y$, in fact vanish for all $n \leq 4$ point functions of short operators. While $U(1)_Y$ is conjectured to be an exact symmetry of $n \leq 4$ point functions, it definitely can not be an exact symmetry of general $n \geq 5$ point functions of operators in small representations. This is seen directly \cite{[1][12]} both in the field theory, as will be reviewed below, and via the AdS duality.

3. Bonus symmetry and the OPE

Our interest is in characterizing the extent to which the operator product expansions respect the $U(1)_Y$ bonus symmetry. We will show that the OPE is indeed compatible with $U(1)_Y$ being an exact symmetry of $n \leq 4$ point functions of operators in short representations, but generally violated for $n \geq 5$ point functions.

We will be interested in operator product expansions of the general form (for simplicity we write expressions for scalar operators)

$$O_i(x)O_j(y) \sim \sum_k C_{ij}^k \frac{1}{(x-y)^{\Delta_i + \Delta_j - \Delta_k}} O_k(y),$$

(3.1)
with some OPE coefficients $C^k_{ij}$ which can in general depend on the choice of gauge group $G$, the gauge coupling $g_{YM}$, and $\theta_{YM}$. The OPE coefficients appearing in (3.1) also appear in the three-point functions

$$\langle O_i(x)O_j(y)O_k(z) \rangle = \frac{C_{ijk}}{(x-y)^{\Delta_i+\Delta_j-\Delta_k}(y-z)^{\Delta_j+\Delta_k-x-z}x^{\Delta_i+\Delta_k-\Delta_j}}.$$  (3.2)

Indices are lowered as $C_{ijk} = \sum_m C_{ijm}^m \eta_{km}$, with the metric $\eta_{ij}$ given by

$$\langle O_i(x)O_j(y) \rangle = \frac{\eta_{ij}}{(x-y)^{2\Delta_i}},$$  (3.3)

which, by conformal invariance, satisfies $\eta_{ij} = 0$ unless $\Delta_i = \Delta_j$.

Any of the operators appearing in (3.1), (3.2), and (3.3) can be either of short or long type. Unless indicated otherwise, the indices $i$, $j$, and $k$ above run over all operators of both types. When we want to restrict attention to an operator of a given type, we use superscripts to denote the type, e.g. $C_{ijk}^{SS}$ when $i$, $j$, and $k$ are each taken to run only over short operators, $C_{ijk}^{SL}$ when $i$ and $j$ are short but $k$ is long, etc.

Consider first the metric (3.3). The correlation function is non-zero only if the operators $O_i$ and $O_j$ have the same anomalous dimension, conjugate $SO(4)$ Lorentz spins, and conjugate $SU(4)_R$ representations. These requirements prohibit a non-zero two-point function with one operator in a short representation and the other in a long representation,

$$\eta_{ij}^{SL} = 0.$$  (3.4)

The reason is that operators are in a short representation if and only if their dimensions are correlated with their $SU(4)_R$ transformation properties; if $O_i$ is such a short representation and $O_j$ must have the same dimension and conjugate $SU(4)_R$ representation as $O_i$, then $O_j$ must also be in a short representation. It was proven in [12] that the two-point function of operators in short representations exactly obey the $U(1)_Y$ selection rule. The argument of [12] did not rely on the operators being in short representations and we expect that the selection rule applies for long representations as well:

$$\eta_{ij}^{SS} = 0 \quad \text{and} \quad \eta_{ij}^{LL} = 0 \quad \text{unless} \quad q_i + q_j = 0.$$  (3.5)

In short, the metric $\eta_{ij}$ exactly respects the $U(1)_Y$ selection rule for all operators.

We now consider three point functions and $U(1)_Y$ bonus symmetry. There are generally non-zero three-point functions involving any combinations of short and long operators:
Our main conjecture is that all three-point functions involving at least two short representations exactly respect the $U(1)_Y$ selection rule:

\begin{align}
C_{ijk}^{(SSS)} &= 0 \quad \text{unless } q_i + q_j + q_k = 0; \\
C_{ijk}^{(SSL)} &= 0 \quad \text{unless } q_i + q_j + q_k = 0.
\end{align}

On the other hand, as discussed below, we know that there are some $C_{ijk}^{(LLS)} \neq 0$ with $q_i + q_j + q_k \neq 0$; i.e. the case $(LLS)$ of two longs and a short generally does not respect the $U(1)_Y$ selection rule. Similarly, the case $(LLL)$ of three longs is generally not expected to obey the $U(1)_Y$ selection rule.

As emphasized in [12] there is a special short representation operator: the exactly marginal operator $O^{(-4)}_{\tau}$, corresponding to changing the gauge coupling $\tau = \frac{\theta_Y M}{2\pi} + 4\pi i g Y_M^{-2}$; as indicated, this operator carries $U(1)_Y$ charge $q = -4$. There is a conjugate operator $O_{\overline{\tau}}^{(4)}$, corresponding to changing $\overline{\tau}$. The metric $\eta_{\tau \overline{\tau}} \neq 0$ (it’s proportional to $|G|$) and $\eta_{\tau \tau} = \eta_{\overline{\tau} \overline{\tau}} = 0$ thanks to (3.5). The variation of a general $n$-point correlation function with respect to the gauge coupling $\tau$ is given in terms of the $n+1$-point function with an insertion of $\int O^{(-4)}_{\tau}$:

\begin{equation}
\partial_\tau \langle \prod_{i=1}^n O_i^{(q_i)}(x_i) \rangle = \tau_2^{-1} \int d^4z (O_{\tau}^{(-4)}(z) \prod_{i=1}^n O_i^{(q_i)}(x_i)).
\end{equation}

Note that the correlation function on the left side has total $U(1)_Y$ charge $\sum_{i=1}^n q_i$, while that on the right side has total $U(1)_Y$ charge $-4 + \sum_{i=1}^n q_i$. If both sides were required to respect the $U(1)_Y$ selection rule, both sides would have to vanish: the $n$-point function would be independent of $\tau$ for all $\tau$ – i.e. be not renormalized.

A consequence of the $U(1)_Y$ selection rule for two-point and three-point functions involving all short operators is thus that two-point functions of operators in short representations are not renormalized. It’s known that the dimensions of short representations can not be renormalized, so the content of this statement is that the metric $\eta_{ij}^{(SS)}$ in (3.3) is also not renormalized, i.e. it is independent of $g_Y M$ and $\theta_Y M$. This agrees with the vanishing of the leading order, radiative corrections found in [3].

The reason why we know that the $U(1)_Y$ selection rule must be violated for general $C_{ijk}^{(LLS)}$ (3.8) is that we know that operators in long representations generally do receive
quantum corrections to their anomalous dimensions. For example, as shown in [18], the “Konishi current” $J^K_\mu$, which is a descendent of the long primary operator $O_K$ mentioned above, gets a non-zero radiative correction to its anomalous dimension. The operator $O_K$ and the current $J^K_\mu$ both carry $U(1)_Y$ charge zero. Since $\partial_\tau \langle J^K_\mu(x) J^K_\nu(y) \rangle \neq 0$, (3.9) gives $\langle O^{(-4)}(z) J^K_\mu(x) J^K_\nu(y) \rangle \neq 0$, which is a $C^{(SSL)}_{ijk}$ which violates the $U(1)_Y$ selection rule.

As shown in [18], the operator product expansion of two stress tensors, which are short operators, includes the Konishi current, which is a long operator. Since the stress tensor and the Konishi current both have vanishing $U(1)_Y$ charge, this is compatible with our conjecture that all non-zero $C^{(SSL)}_{ijk}$ exactly respect the $U(1)_Y$ selection rule.

Consider now four-point functions of operators in short representations. We assume that there is an expansion of the four point function in terms of the OPEs of the form

$$\langle \prod_{i=1}^{4} O_i(x_i)^{(q_i)} \rangle = \sum_j C^{(SSX)}_{12} j C^{(XSS)}_{34} F_{\{i\};j}(x_i),$$

(3.10)

where $X$ denotes that $j$ should be summed over all representations, both short and long, and we will not be concerned with the form of the functions $F_{\{i\};j}(x_i)$. A consequence of (3.6), (3.7), and (3.8) is that the right side of (3.10) exactly vanishes unless the charges of the operators satisfy the $U(1)_Y$ selection rule. Our OPE conjectures thus implies the conjecture of [12] that, for a general 4-point function of operators in short representations,

$$\langle \prod_{i=1}^{4} O_i^{(q_i)}(x_i) \rangle = 0 \quad \text{unless} \quad \sum_{i=1}^{4} q_i = 0.$$  

(3.11)

A consequence of this exact selection rule for 4-point functions is that all three-point functions of operators in short representations are not renormalized, as explained above; i.e. the $C^{(SSS)}_{ijk}$ are constants, independent of $g_{YM}$ and $\theta_{YM}$.

We now turn to five-point functions of operators in short representations. Again, assuming that an OPE expansion is valid, these will be of the form

$$\langle \prod_{i=1}^{5} O_i^{(q_i)}(x_i) \rangle = \sum_{j,k} C^{(SSX)}_{12} j C^{(XSY)}_{3j} k C^{(YSS)}_{45} F_{\{i\};j;k}(x_i).$$

(3.12)

Unlike the above case of four-point functions, the OPE for two longs and a short representation enters as $C^{(LSL)}_{ijk}$ in the expansion (3.12). Because these OPE violate the $U(1)_Y$
selection rule, the 5-point function (3.12) of short representations does not satisfy an exact selection rule, i.e. it is generally possible to have
\[
\langle \prod_{i=1}^{5} O_i^{(q_i)}(x_i) \rangle \neq 0 \quad \text{with} \quad \sum_{i=1}^{5} q_i \neq 0.
\] (3.13)

This situation clearly generalizes for higher \( n \geq 5 \) point functions.

The violations (3.13) of \( U(1)_Y \) for \( n \geq 5 \) point functions can be seen in the context of the field theory, for example in instanton contributions to correlation functions \([10,11]\). Instantons will be discussed further in sect. 5. There are also contributions to (3.13) which violate the \( U(1)_Y \) for \( n \geq 5 \) point functions which are visible in perturbation theory. For example, the perturbative renormalization of 4-point functions demonstrated in \([7,8]\) implies via (3.9) a perturbative violation of the \( U(1)_Y \) selection rule for the 5-point function with an additional insertion of \( O_{-4} \). Violations of \( U(1)_Y \) for \( n \geq 5 \) point functions is also compatible with \( AdS \) duality, where it is associated with the stringy corrections to supergravity.

4. The form of descendent operators

Because superconformal primary operators are neutral under \( U(1)_Y \), the non-trivial content of the \( U(1)_Y \) selection rule conjectures is for correlation functions involving at least one superconformal descendent operator. It is thus important to determine the correct form of the descendent operators \( \delta^n \bar{\tau}^m O_P \). The on-shell supersymmetry transformations of the \( \mathcal{N} = 4 \) Yang-Mills theory are given by
\[
\begin{align*}
DA_{\alpha \dot{\alpha}} &= \eta^I \gamma^\beta \epsilon_{\dot{\alpha} \alpha} \phi_I + \eta^{\dot{\alpha}} \epsilon_{\alpha \beta} \bar{\psi}_I, \\
D\phi_{IJ} &= \eta^J \bar{\psi}_J \bar{\phi}_I + \epsilon_{IJK} \bar{\bar{F}}^{K \dot{\alpha}} \bar{\psi}_I, \\
D\bar{\psi}_I &= \eta^J \partial^\alpha F_{\alpha \beta} + \bar{F}^{J \dot{\alpha}} \partial^\beta \bar{\phi}_{IJ} + g \eta^J \epsilon_{\alpha \beta} [\phi_{IK}, \phi^{JK}] \\
DF_{(\alpha \beta)} &= \eta^I \gamma_\gamma \partial_{(\alpha} \phi_{\beta)J},
\end{align*}
\] (4.1)

where \( \eta^I \) and \( \bar{\eta}^{I \dot{\alpha}} \) are Grassmann parameters to keep track of the action of \( Q^I_\alpha \) and \( \bar{Q}_{I \dot{\alpha}} \), there are similar transformations for \( \bar{\bar{\psi}}_I \) and \( \bar{\bar{F}}_{\alpha \beta} \), and we have left out numerical constants for simplicity. The variations under the other 16 superconformal supersymmetries \( S_{\alpha I} \) and \( \bar{S}^{I \dot{\alpha}} \) can also be easily written, roughly by simply replacing \( \eta^I \) by \( \eta^I + x^{\alpha \dot{\alpha}} \bar{\zeta}_{\alpha I}, \) and similarly for \( \bar{\eta}^{I \dot{\alpha}} \) in (4.1), but we will not need these transformations here, as it is the action of \( Q^I_\alpha \) and \( \bar{Q}_{I \dot{\alpha}} \) which generate descendents.
The issue now is how the supersymmetry generators $Q^I_\alpha$ and $\overline{Q}^{I\dot{\alpha}}$ act on the gauge invariant operators, which are traces of products of the fields in (4.1). Classically this is given simply by the acting with the transformations in (4.1) on each of the fields in the operator. For example, classical expressions for some of the descendents of the short primary operator $O_2 = [\text{Tr}_G(\phi^2)]_{(0,2,0)}$ are

\begin{align*}
\delta O_2 &= \text{Tr}(2\phi_{IJ}\psi_{K\alpha} + \phi_{KI}\psi_{J\alpha} - \phi_{KI}\psi_{J\alpha}) \\
\delta^2 O_2 &= \text{Tr}(\psi_{\alpha I}\psi_{\beta J}e^{\alpha\beta} + g[\phi_{IK},\phi_{JL}]\phi^{KL}) \\
\delta^2 O_2 &= \text{Tr}(\phi_{IJ}F_{\alpha\beta} + \psi_{I(\alpha}\psi_{\beta)}J) \\
\text{operator} & \quad SO(4) \quad SU(4)_R \quad U(1)_Y \\
\delta O_2 & \quad (\frac{1}{2}, 0) \quad (0, 1, 1) \quad -1 \\
\delta^2 O_2 & \quad (0, 0) \quad (0, 0, 2) \quad -2 \\
\delta^2 O_2 & \quad (1, 0) \quad (0, 1, 0) \quad -2. \\
\end{align*} 

(Irrelevant overall normalization factors are suppressed.) Likewise, classical expressions for some of the descendents of the long operator $O_K = \text{Tr}_G(\phi_{IJ}\phi^{IJ})$ are

\begin{align*}
\delta O_K &= \text{Tr}(\phi^{IJ}\psi_{\alpha I}) \\
\delta^2 O_K &= \text{Tr}(g[\phi^{IK},\phi^{JL}]\phi^{KL}) \\
\delta^2 O_K &= \text{Tr}(\phi_{IJ}F_{\alpha\beta} + 2\psi_{I(\alpha}\psi_{\beta)}J) \\
\text{operator} & \quad SO(4) \quad SU(4)_R \quad U(1)_Y \\
\delta O_K & \quad (\frac{1}{2}, 0) \quad (1, 0, 0) \quad -1 \\
\delta^2 O_K & \quad (0, 0) \quad (2, 0, 0) \quad -2 \\
\delta^2 O_K & \quad (1, 0) \quad (0, 1, 0) \quad -2. \\
\end{align*} 

Generally, however, we must expect that the superconformal generators have quantum corrections when acting on gauge invariant composite operators. For example, the dilatation generator $D$, which acts on primary operators as $D = (-i)x^\mu\partial_\mu + \Delta$, clearly gets quantum contributions when acting on composite operators because $\Delta$, which gives the dimension of the operator, gets quantum contributions. Because $D$ appears in $\{Q^I_\alpha, S^\alpha_I\}$, the action of the supersymmetry generators on composite operators clearly must also generally have additional quantum contributions.

Because the short operators do not have quantum corrections to their operator dimensions, it is also natural to expect that the classical expressions for their operator descendents are, in fact, exact. This is compatible with (3.5) and Ward identities such as that discussed in [12] applied to 2-point functions. On the other hand, we should generally expect that descendents of long operators, such as (4.3), do receive quantum corrections. Indeed, this is the resolution to the following “puzzle”:

4.1. A “puzzle” and comments about operator mixings

Consider the two-point function

$$\langle (\delta^2 O_2)(x)(\delta^2 O_K)(y) \rangle,$$  (4.4)
involving the Lorentz spin \((0,0)\) operators in the \((0,0,2)\) and \((2,0,0)\) of \(SU(4)_R\), respectively; the first operator is given in the second line in (4.2), while the classical expression for the second operator is given in the second line in (4.3). If this two-point function were non-zero, it would violate the \(U(1)_Y\) selection rule; however, as indicated generally in (3.4), (4.4) must vanish. This follows from the conformal group because \(\delta^2 O_2\) is a conformal primary with exact dimension \(\Delta = 3\), while \(\delta^2 O_K\) classically has dimension 3 but gets non-zero quantum corrections, corresponding to the non-zero anomalous dimension of \(O_K\). Since, for general non-zero \(g_{YM}\), the two operators have different dimensions, the conformal group requires that the two-point function vanishes. Indeed, the only operator which can have a non-zero two-point function with \(\delta^2 O_2\) is the conjugate short operator \(\overline{\delta^2 O_2}\), for which
\[
\langle (\delta^2 O_2)(x)(\delta^2 O_2)(y) \rangle = -\frac{2|G|}{(2\pi)^4|x - y|^6}; \tag{4.5}
\]
the result (4.5) is known to be exact since it is related by supersymmetry to a non-renormalized current two-point function.

However, using the expressions in the second lines in (4.2) and (4.3), we find a non-zero result for (4.4) at order \(g_{YM}^2\) coming from:
\[
\langle (g_{YM}\phi^3)_{IJ}(x)(g_{YM}\phi^3)_{IJ}(y) \rangle = C_2(G)|G|g_{YM}^2 \frac{1}{(2\pi)^6|x - y|^6} + O(g_{YM}^4), \tag{4.6}
\]
where \(C_2(G)\) is the quadratic Casimir of gauge group \(G\), normalized to be \(N\) for \(SU(N)\), and the factor of \(C_2(G)|G|\) comes from \(f^{abc}f_{abc}\).

The resolution to this apparent puzzle is that there must be a quantum correction to the second line in (4.3) which compensates for (4.6), preserving the vanishing of (4.4). To order \(g_{YM}^2\), we must have
\[
\delta^2 O_K \equiv L^{IJ} = \text{Tr}(g_{YM}[\phi^{IK}, \phi^{JL}]\phi_{KL}) + \frac{1}{2}g_{YM}^2 \frac{C_2(G)}{(2\pi)^2} S^{IJ}, \tag{4.7}
\]
where \(S^{IJ} = \delta^2 O_2\) is the conjugate operator to \(\delta^2 O_2\) in (4.3). Using (4.5), the \(g_{YM}^2\) correction term in (4.7) cancels the contribution to (4.4) from (4.6). At higher orders in \(g_{YM}\) there can be additional quantum corrections to (4.7).

Finally, we would like to comment on the issue of operator mixing. Generally long primary operators need not be "pure" primaries, in the sense that they need not be eigenvectors of the anomalous dimensions matrix, which arises in the OPE with the dilatation
operator $D$. It is the eigenvectors of this anomalous dimension matrix, with differing eigenvalues, which are orthogonal in that their two-point functions vanish. Operators which are not eigenvectors have non-zero two-point function mixings among themselves; diagonalizing the two-point functions is a practical way to obtain the pure primary eigenvectors.

One might thus be tempted to interpret the above puzzle differently: rather than correcting the action of $\delta$ as in (4.7) to make (4.4) vanish, perhaps (4.4) is actually non-zero and simply expresses that $\delta^2O_K$ is not an eigenvector of the anomalous dimension matrix but, instead, mixes with other operators such as $S^{IJ}$? This latter interpretation requires that

$$\langle O_K(x)O^{(-4)}(y) \rangle \quad (4.8)$$

is also non-zero, as (4.8) is related to (4.4) by supersymmetry. The interpretation of the non-zero result for (4.8) would, similarly, be that $O_K = \text{Tr}(\phi^{IJ}\phi_{IJ})$ itself is not a pure primary operator but mixes with other operators, including the operator $C^{(4)}$. In other words, this interpretation would require that $O_K$ is actually superposition $O_K = \tilde{O}_K + cg^{2}_{YM}O^{(4)} + \ldots$, where $\tilde{O}_K$ and the other terms are eigenvectors of the anomalous dimension matrix. If this were the case, the 2-point function (4.8) would be given by

$$\langle O_K(x)O^{(-4)}(y) \rangle \quad (4.8)$$

However, it is easily seen that this can not happen in perturbation theory: because (4.8) has classical scaling dimension 6, perturbation theory can only lead to terms scaling as $1/|x-y|^6$ up to additional perturbative corrections depending on $\log(x-y)$. Resumming the logs can lead to perturbative expressions such as $f(y)/|x-y|^6+O(g^2_{YM})$, but we do not expect to be able to get the $1/|x-y|^8$ dependence above in perturbation theory. Briefly put: we expect that, in perturbation theory, there can be operator mixing only among operators with the same classical scaling dimensions.

Since there is no other $SU(4)$ singlet with classical scaling dimension 2, we do not expect that the above $O_K$ can have any operator mixing in perturbation theory and, in particular, (4.8) must vanish in perturbation theory. Consequently, we believe that (4.4) really must vanish and the correct interpretation of the above puzzle is the one given above: that the action of $\delta$ on long operators such as $O_K$ gets quantum corrections.

By this same argument, we expect that the perturbative quantum corrections to the action of the supersymmetry generators on long operators must also respect the classical scaling dimensions of operators. For example, $\delta^2O_K$ has classical scaling dimension 3 so there can be a quantum correction in perturbation theory by an operator in the same $SU(4)_R$ representation which also has classical scaling dimension 3; this is compatible
with (4.7). Consider, on the other hand, $\delta O_K$, which is in the 4 of $SU(4)_R$, with Lorentz spin $(\frac{1}{2}, 0)$ and classical scaling dimension $5/2$. Because there is no other operator with the same classical scaling dimension and Lorentz and $SU(4)_R$ representations, we do not expect to find a quantum correction to $\delta O_K$ in perturbation theory. Thus, for example, we expect that

$$\langle (\delta O_K)(x)(\delta^3 O_2)(y) \rangle = 0 \quad (4.9)$$

in perturbation theory, though we have not completed the task of explicitly verifying this. Again, the only way (4.9) could be non-zero is if $\delta O_K$ mixes with $\overline{\sigma}^3 O_2$, in which case (4.3) would be proportional to $1/|x - y|^7$ – but in perturbation theory (4.9) would go as $1/|x - y|^6$ up to $g_{YM}$ corrections in $\log(x - y)$.

5. Comments on instanton contributions to correlation functions

An instanton of $\mathcal{N} = 4$ super Yang-Mills, with arbitrary gauge group $G$, has $8C^2(G)$ fermion zero modes, where $C^2(G)$ is the Casimir of the adjoint representation, normalized to be $N$ for $SU(N)$. More generally, an instanton number $k$ configuration has $8kC^2(G)$ fermion zero modes. Of these fermion zero modes, 16 are special: they are 8 zero modes generated by acting on the instanton configuration with the 8 supercharges $Q^I_{\alpha}$, and 8 generated by the superconformal supercharges $\overline{S}^I_{\dot{\alpha}}$. The remaining 16 supercharges annihilate the instanton (they generate the 16 fermion zero modes of the anti-instanton). We will denote these 16 special zero modes by $\lambda(x)$ and the remaining $(8kC^2(G) - 16)$ zero modes by $\chi(x)$.

The 16 $\lambda$ are exact zero modes, while the $\chi$ can generally be lifted. In particular, at the origin of the moduli space of vacua, which is the vacuum of interest for conformal invariance, the $\chi$ zero modes can be lifted in multiples of 4 by a term in the instanton action $S_{\text{inst}} = \ldots + \chi^4$; this is discussed in [11] and references therein. An instanton contributes to a correlation function only if the 16 zero modes $\lambda$ are soaked up by the operators involved in a correlation function.

The general procedure is to replace every operator in the correlation function with its instanton background version, $O_i \rightarrow O_{i\text{inst}}$ by the prescription $\phi \rightarrow \phi^{\text{inst}}, \psi \rightarrow \psi^{\text{inst}}$ and $F \rightarrow F^{\text{inst}}$, where $\phi^{\text{inst}}, \psi^{\text{inst}},$ and $F^{\text{inst}}$ are the adjoint scalars, fermions, and self-dual field strength solutions in the instanton backgrounds. Expressions for these solutions for the general $SU(N)$ instanton background are quite complicated and can be found in [11]. (This uses the ADHM construction, which is not known for exceptional groups.)
The instanton can then contribute to \( \langle \prod_i O_i \rangle \) if all 16 fermion zero modes \( \lambda \) appear in \( \prod_i O_i^{\text{inst}} \). If the 16 \( \lambda \) zero modes are indeed soaked up, it will always be possible to soak up the remaining \( \chi \) zero modes by bringing down powers of \( S_{\text{inst}} \). We can thus just focus on the \( \lambda \) zero modes.

We will give a heuristic argument for a relation between the \( U(1)_Y \) charge of an operator and how many \( \lambda \) fermion zero modes it contains. Consider the supersymmetry relations (4.1) in an instanton background, where we replace \( F \to F^{\text{inst}} \). The solution \( \psi^{\text{inst}} \) satisfying \( \delta \psi^{\text{inst}} = F^{\text{inst}} \) is \( \psi^{\text{inst}} = \lambda F^{\text{inst}} \), and the solition \( \phi^{\text{inst}} \) satisfying \( \delta^2 \phi^{\text{inst}} = F^{\text{inst}} \) is \( \phi^{\text{inst}} = \lambda \lambda F^{\text{inst}} \).

Our basic observation is that the fermion zero mode \( \lambda \sim \delta^{-1} \). Thus, if an operator \( O_{\text{top}} \) satisfies \( \delta O_{\text{top}} = 0 \), then \( O_{\text{top}} \) contains no \( \lambda \) fermion zero modes – it can only depend on \( F^{\text{inst}} \), and possibly also any of the \((8kC_2(G) - 16) \chi \) zero modes. If an operator \( O_r \) has \( \delta^r O_r = O_{\text{top}} \), with \( \delta O_{\text{top}} = 0 \), then \( O_r \) has \( r \lambda \) fermion zero modes.

The operator \( \bar{\delta} \) annihilates the fields in the instanton background; in an anti-instanton background the roles of \( \delta \) and \( \bar{\delta} \) are reversed. Assigning \( \delta \) charge \(-1\) under \( U(1)_Y \) as in [12] (the sign is to agree with the supergravity convention for the charges of the conjugate sources), all operators with \( U(1)_Y \) charge \( q > 0 \) vanish in an instanton background, as they are obtained with \( \delta s \) on a \( U(1)_Y \) neutral primary. Similarly, only those operators with \( U(1)_Y \) charge \( q > 0 \) are non-vanishing in an anti-instanton background.

 Operators in short representations have \( U(1)_Y \) charge \( q \) with \(|q| \leq 4\). In particular, the operator \( \delta^4 O_p \) has \( U(1)_Y \) charge \(-4\) and thus must be annihilated if acted on by another power of \( \delta \), \( \delta^5 O_p = 0 \). Thus \( \delta^4 O_p \) can contain no \( \lambda \) fermion zero modes, it can only depend on \( F^{\text{inst}} \) and the \( \chi \). More generally, a short operator with \( U(1)_Y \) charge \( q \) has

\[
O_S^{(q)} \sim \lambda^{4-|q|},
\]

in an instanton background, where the \( \sim \) includes some polynomial in \( F^{\text{inst}} \) and \( \chi \).

Similarly, operators in long representations have \( U(1)_Y \) charge \(|q| \leq 8\). An operator of the form \( \delta^8 O \) can thus have no \( \lambda \) zero modes in an instanton background, as it is annihilated by \( \delta \). Thus, for a generic operator in a long representation,

\[
O_L^{(q)} \sim \lambda^{8-|q|},
\]

in an instanton background, where again the \( \sim \) includes some polynomial in \( F^{\text{inst}} \) and \( \chi \). Of course there are some long multiplets, obtained from \( \text{Tr} \phi^p \) with \( p < 4 \), which truncate...
earlier on, i.e. the operator $O_{\text{top}}$ with $\delta O_{\text{top}} = 0$ has $U(1)_Y$ charge $|q|_{\text{max}} < 8$. An example is the Konishi operator $O_K = \text{Tr}(\phi^I J^J \phi_{IJ})$, for which $|q|_{\text{max}} = 4$. As always, $O_{\text{top}} \sim \lambda^0$ and thus the generalization of (5.2) to the other operators in the multiplet is $O_L^{(q)} \sim \lambda |q|_{\text{max}} - |q|$. Consider a correlation function of $n$ operators in short representations, $\langle \prod_{i=1}^n O_i^{(q_i)} \rangle$. Using (5.1), the condition for instantons to contribute to the correlation function is

$$\sum_{i=1}^n (4 - |q_i|) = 16,$$

in which case the 16 $\lambda$ zero modes can be saturated. It thus follows that instantons (or anti-instantons) can contribute to a given $n$-point function of operators in short representations if and only if

$$n = 4 + \frac{|q_T|}{4},$$

where $q_T = \sum_{i=1}^n q_i$ is the total $U(1)_Y$ charge. It thus immediately follows that instantons can never contribute to $n < 4$-point functions of short operators; this is compatible with the conjectured non-renormalization of [5,6] for $n \leq 3$ point functions. We also see that instantons can contribute to a $n = 4$-point function only if $q_T = 0$; this is consistent with the conjectured selection rule [12] that $n \leq 4$ point functions with $q_T \neq 0$ exactly vanish.

We were not able to find a formula along the lines of (5.1) in [11], but expect that it must be possible to prove, at least for $SU(N)$, using the complete (and complicated) analysis presented there. Demonstrating (5.1) would provide further support for the matching between instantons and $AdS_5 \times S^5$ supergravity results found in [10,11]. Indeed, the resulting relation (5.4) nicely agrees with results from IIB string theory. See, for example, sect. 3 of [20], where the Grassman coordinates $\theta^A, A = 1, \ldots, 16$, correspond to the 16 zero modes $\lambda$ of the (D)-instanton. The supersymmetry generators $Q_A = \partial/\partial \theta^A$, which matches with our basic observation that $\lambda \sim \delta^{-1}$. Relation (3.7) of [20] nicely corresponds to our (5.1) and eqn. (3.14) of [20] corresponds to our (5.4). (This latter correspondence uses the fact that stringy interactions involving $n$ fields only contribute to $n$-point functions; their contribution to correlation functions with fewer operators vanishes because the fields involved vanish when evaluated in the $AdS_5 \times S^5$ vacuum, as in [19].) While much of the agreement found in [11] between multi-instanton collective coordinates and $AdS_5 \times S^5$ relied on large $N$ $SU(N)$, we expect that (5.1) and (5.2) apply for any gauge group.

Using (5.1) and (5.2) we also see that instantons can contribute to a SSL three-point function, involving two short and one long operator, only if $q_T = 0$. (More generally, for
a $n$-point function involving a generic long operator and $n - 1$ short operators, instantons contribute only if $n = 3 + \frac{1}{4}|q_T|$. Therefore instantons are nicely compatible with our conjectured selection rule (3.7). We also see that instantons can contribute to the two-point function of two long operators only if $q_T = 0$; this agrees with (3.3). (More generally, instantons contribute to a $n$-point function involving two generic long operators and $n - 2$ short operators only if $n = 2 + \frac{1}{4}|q_T|$.)

6. Perturbative checks of the selection rule for $n \leq 4$ point functions.

A non-trivial perturbative check of (3.6) appears in [6], where the leading order radiative corrections to a descendent correlation function, which would violate (3.6) if non-zero, was found to vanish. If one believes the conjectured [5,6] non-renormalization of all 3-point functions of short operators, the selection rule (3.6) for 3 short operators would follow because all correlation functions respect $U(1)_Y$ in the $g_{YM} \rightarrow 0$ limit [12]. Because checks of (3.6) for other (SSS) descendent 3-point functions are similar to the example considered in [6], we will not present any additional examples. Instead, in this section, we will present a non-trivial perturbative check of the selection rule (3.11) for 4-point functions of short operators. In the next section, we present checks of the selection rule (3.7) involving two short and one long operator.

We consider the 4-point function of short operators:

$$\langle O_2^{(-2)}(x_1)O_2^{(-2)}(x_2)O_2(x_3)O_2(x_4)\rangle,$$

(6.1)

where $O_2(x) = [\text{Tr}(\phi^2)]_{(0,2,0)}$ is the primary operator with $U(1)_Y$ charge zero and $O_2^{(-2)} \equiv S_{(IJ)}$ is its second descendent, which appears in (1.2) and is a Lorentz scalar in the $(0,0,2)$ representation of $SU(4)_R$. According to (3.11), (6.1) must exactly vanish, for any gauge group $G$, as it violates the $U(1)_Y$ selection rule. Note that the $SU(4)_R$ group theory does allow for a non-zero result for (6.1), so the vanishing is non-trivial.

We carried out the perturbative calculation in $N = 1$ component language, where only $SU(3)$ subgroup of $SU(4)$ R-symmetry is manifest. Specifically, for the operators $O_2^{(-2)}$ we took flavor combinations $S^{11}$ and $S^{44}$ defined in Ref. [6], while $O_2(x_3) = (\bar{z}_1)^2$ and
$O_2(x_4) = \bar{z}tz$. We denote the $N = 1$ scalar fields by $z$, while $t$ is a traceless $SU(3)$ flavor generator.

There are 4 diagrams contributing to the 4-point function (6.1), they are depicted in figure 1. The diagrams with exchanges of only scalar fields are straightforward to evaluate and give

\[(a) = g^2 C_2(G) |G| t_{11} G(x_1, x_2) G(x_1, x_3) G(x_2, x_4) G(x_3, x_4)
\]
\[(b) = g^2 C_2(G) |G| t_{11} G(x_1, x_2) G(x_1, x_4) G(x_2, x_3) G(x_3, x_4) \quad (6.2)
\]
\[(c) = g^2 C_2(G) |G| (t_{22} + t_{33}) G(x_1, x_2) G(x_1, x_3) G(x_1, x_4) G(x_2, x_3) G(x_2, x_4),
\]

where $G(x, y) = 1/(4\pi^2|x-y|^2)$ is the free scalar propagator and the factors of $C_2(G)|G|$ arise from factors of $f^{abc}f_{abc}$.

Evaluating the diagram in fig. 1(d) involves an eight-dimensional integral. We found it easiest to perform such integrals in coordinate space using the technique of conformal inversion \[21\]

\[(d) = (-) g^2 C_2(G) |G| [G(x_1, x_2)^2 G(x_3, x_4) (G(x_1, x_3) G(x_2, x_4) + G(x_1, x_4) G(x_2, x_3))
\]
\[- G(x_1, x_2) G(x_1, x_3) G(x_1, x_4) G(x_2, x_3) G(x_2, x_4)]. \quad (6.3)
\]

The minus sign in front of the fermion contribution is the usual fermion loop factor. Contributions from diagrams (a) and (b) cancel against the first two terms of the fermionic diagram. The remaining part of diagram (d) and diagram (c) are proportional to

\[g^2 C_2(G) |G| (t_{11} + t_{22} + t_{33}), \quad (6.4)\]

which vanishes, for any gauge group $G$, since the $SU(3)_F$ generator $t$ is traceless.
7. Perturbative checks of the \( (SSL) \) selection rule.

We now turn to some checks of the conjectured selection rule (3.7). As discussed in sect. 4, there can be quantum corrections to descendents of long operators. To avoid this subtlety, we first consider the situation where the long operator \( O_L \) in (3.7) is primary and the short operators are descendents.

For our long primary operator, we take \( O_L = \text{Tr}_G(\phi^r + 2s)_{(0,r,0)} \) where \( s > 0 \) for this to be a long operator and the subscript gives the Dynkin indices of the \( SU(4)_R \) representation. We can consider, for example, the 3-point function \( \langle O_L \delta O_p \delta O_q \rangle \), for which \( SU(4)_R \) allows a non-zero result provided \( r = p + q - 1 \pmod{2} \) in the range \( p + q - 1 \geq r \geq |p - q| + 1 \). The leading contribution to this 3-point function, which would violate \( U(1)_Y \) conservation if non-zero, occurs at order \( g_{YM} \) and is associated with a single diagram, of the form of diagram (a) in fig. 2. Fortunately this diagram vanishes because it involves a color contraction of the form: \( f^{abc}d_{e_i(bc)} \), where \( f^{abc} \) is associated with the Yukawa interaction and \( d_{e_i(bc)} \) is associated with \( O_L \), which appears in the diagram at the vertex without a fermion line. The sum vanishes due to the antisymmetry of \( f \) and symmetry of \( d \) in the summed color indices \( ab \). We refer to such vanishing as the \( d \cdot f = 0 \) rule.

![Figure 2. Examples of contributions to the three-point function (3.7). All these diagrams vanish due to contractions of the color indices.](image)

We have thus verified that the leading radiative corrections to \( \langle O_L \delta O_p \delta O_q \rangle \) vanish, in agreement with (3.7). The task of verifying that radiative corrections continue to vanish to higher orders in \( g_{YM} \) appears to be quite complicated and tedious, and we have not carried it out.

We note that \( SU(4)_R \) does not allow for a non-zero 3-point function of the form \( \langle O_L \delta^2 O_p \delta^2 O_q \rangle \), with primary \( O_L \). So, for our next examples, we consider \( \langle O_L \delta^2 O_p \delta^2 O_q \rangle \), where both \( \delta^2 O_{p,q} \) are either the \((0,0)\) Lorentz or \((1,0)\) Lorentz spin descendents, as in (4.2). Consider first the case where both are the Lorentz spin \((0,0)\) descendents. Then \( SU(4)_R \) allows for a non-zero result if \( p + q - 2 \geq r \geq |p - q| + 2 \), with \( r = p + q \pmod{2} \).
The leading contribution to this 3-point function, which would violate $U(1)_Y$ if non-zero, occurs at order $g^2_{YM}$ and is associated with the diagrams (b) and (c) in fig. 2. These indeed vanish by the $d \cdot f = 0$ rule, where again the $d$ color factor is associated with the $O_L$ vertex.

The leading order contribution to $\langle O_L \delta^2 O_p \delta^2 O_q \rangle$, where both $\delta^2 O_{p,q}$ are the $(1,0)$ Lorentz spin descendents, is at order $g^2_{YM}$. The relevant diagrams involve gauge field propagators, which seem to complicate matters, and we did not complete the task of evaluating the diagrams and verifying that, as required by (3.7), they indeed sum to zero.

Other cases for which the leading radiative contributions can be easily verified to vanish are $\langle \delta O_L \delta O_p \delta O_q \rangle$ and $\langle \delta O_L \delta^3 O_p \delta O_q \rangle$. The relevant diagrams are again of the types shown in fig. 2 and, in both cases, they vanish by the $d \cdot f = 0$ rule. Evaluating the leading radiative correction for $\langle \delta O_L \delta^3 O_p \delta O_q \rangle$ is more difficult, as the diagrams involve gauge field propagators and, again, have not completed this task.

Finally, we consider an example involving a second descendents of a long operator, where the quantum corrections found in (4.7) will prove crucial. Consider $\langle \delta^2 O_K \delta O_p \delta O_q \rangle$, which has the $SU(4)_R$ flavor structure $L^{IJ} S_{AB,1} S_{CD,0} \epsilon^{ABCD}$. We consider $L^{44} S_{12,4} S_{34,4}$ in $N = 1$ component fields where, using (4.7), $L^{44} = \text{Tr}(g_{YM} \langle x \rangle \langle x \rangle + \frac{1}{8\pi^2} g_{YM} C_2(G) \langle x \rangle + \ldots)$, $S_{12,4} = 2z_3 \lambda - z_2 \psi_1 + \chi \psi_2$, and $S_{34,4} = 3z_3 \lambda$.

The leading contribution to the correlation function is at order $g^2_{YM}$ and includes a term $g_{YM}^2 C_2(G) G F |x-y| G F |x-z| G B |y-z|$ at Born level, coming from the order $g^2_{YM}$ correction in (4.7). The other order $g_{YM}^2$ terms come from the order $g_{YM}$ term in $L^{IJ}$ along with one interaction vertex. There are two identical contributions coming from the $-z_2 \psi_1$ and the $\chi \psi_2$ terms in $S_{12,4}$. The sum of these two contributions precisely cancel the above additional term associated with the correction in (4.7). This is a non-trivial check of our conjecture, as the coefficient of the correction term in (4.7), which was precisely right to cancel the radiative corrections found here, was independently determined in sect. 4.

8. Comments on SL(2, Z) S-duality and the OPE

It was conjectured in [12] that an arbitrary $n$-point function of short operators transforms under SL(2, Z) modular transformations as

$$\langle \prod_i O^{(q_i)}(x_i) \rangle_{\frac{\sigma \tau + d}{\sigma \tau + d}} = \left( \frac{\sigma \tau + d}{\sigma \tau + d} \right)^{g \tau / 4} \langle \prod_i O^{(q_i)}(x_i) \rangle_{\tau}, \quad (8.1)$$

20
where \( \tau \equiv \frac{\theta Y M}{2\pi} + 4\pi i g Y M^{-2} \) and \( q_T = \sum_i q_i \), the net \( U(1)_Y \) charge of the correlation function. This conjecture was motivated by AdS duality, but could apply generally for arbitrary gauge groups. It is natural to expect that (8.1) applies for any operator correlation function, including correlation functions involving long operators.

The conjecture (8.1), applied for arbitrary long or short operators, is compatible with an OPE expansion of correlation functions. Since \( U(1)_Y \) charge is additive, it is consistent to associate the modular transformation properties in (8.1) entirely with the modular transformation properties of the OPE coefficients and metric. By our \( U(1)_Y \) selection rule, the metrics \( \eta_{ij} \) and \( C_{SSS}^{ijk} \) and \( C_{SSL}^{ijk} \) OPE coefficients are expected to be modular invariant under \( SL(2, \mathbb{Z}) \) transformations of \( \tau \). As discussed above, according to our conjectures \( \eta_{ij}^{SS} \) and \( C_{SSS}^{ijk} \) are actually constants independent of \( \tau \), while \( \eta_{ij}^{LL} \) and \( C_{SSL}^{ijk} \) are non-trivial functions of \( \tau \), which should nevertheless be modular invariant. In order to satisfy (8.1) the anomalous dimensions \( \Delta_i \) of all operators must be modular invariant; for short operators they are constant, while for long operators they should be non-trivial, modular invariant, functions of \( \tau \).

The \( U(1)_Y \) charge violation of general correlation functions is associated entirely with the charge violation \( (q_T)_{ijk} \) of the OPE vertices \( C_{ijkl}^{SSL} \) and \( C_{ijkl}^{LLL} \). Correspondingly, the non-trivial modular transformation properties (8.1) of a general correlation function is associated entirely with the modular transformation properties of \( C_{ijkl}^{SSL}(\tau) \) and \( C_{ijkl}^{LLL}(\tau) \),

\[
C_{ijkl}^{XLL}(a\tau + b \over c\tau + d) = (c\tau + d)^{(q_T)_{ijk}/4} C_{ijkl}^{XLL}(\tau). \tag{8.2}
\]

The general correlation function (8.1) involves products of the factors in (8.2).

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3 In the case of \( Sp(n) \) and \( SO(2n + 1) \), which are exchanged by \( \tau \to -1/\tau \), the correlation functions on the two sides of (8.1) would be for these two dual groups.
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