Quantum Otto refrigerators in finite-time cycle period

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Keywords: quantum refrigerators, fluctuations, cooling rate, coefficient of performance, finite time

Abstract
Finite-time cycle period for a quantum Otto machine implies that either an adiabatic stroke or an isochoric process proceeds in finite time duration. The quantum Otto refrigerators under consideration consist of two adiabatic strokes, where the system (isolated from the heat reservoir) undergoes finite-time unitary transformation, and two isochoric steps, where the system may not reach thermal equilibrium even at the respective ends of the two stages due to finite-time interaction intervals. Using two-time projective measurement method, we find the probability distribution functions of both coefficient of performance and cooling load, which are dependent on the time duration along each process. With these distributions we find the analytical expressions for the performance parameters as well as their fluctuations. We then numerically determine the performance and fluctuations for the refrigerator operating with a two-level system employed in a recent experimental implementation. Our results clarify the role of finite-time durations of four processes on the performance and fluctuations of the quantum Otto refrigerators.

1. Introduction

A refrigerator as an inverse operation of a heat engine transfers energy from a cold thermal bath of temperature $T_c$ to a hot one with temperature $T_h$ by consuming work. Quantum refrigerators as well as quantum heat engines use quantum systems as their working substance and they can be classified either cyclic [1–9] or steady-state [10–15] models. The quantum Otto cycle of operation, as a typical example of cyclic machines, is controlled by the segments of time that the working system is coupled to a hot and a cold bath, and by the time interval required for driving the system by the external field. It was most studied [1, 2, 5–7, 9, 16, 17] as it is easier to analyze and realize. To describe the performance characteristics of a refrigerator, one introduces the coefficient of performance (COP) that is defined as the ratio of heat injection to work input. An upper bound on the COP imposed by the second law of thermodynamics is given by the Carnot COP: $\varepsilon_C = T_c / (T_h - T_c)$, which, however, requires infinitesimally slow transitions between thermodynamic states and thus produces vanishing cooling rate. Hence, the refrigerators actually operate far from the infinite long time limit in order for positive cooling rate to be produced [18–23]. The finite cooling rate for a cyclic refrigerator consisting of a sequence of thermodynamic processes indicates that each process must proceed in finite time. For an adequate description of an actual machine, the effects induced by finite-time duration along any thermodynamic stroke on heat and work have to be considered.

Practically the refrigerators can not reach the theoretical maximum COP due to irreversible entropy production, a thermodynamic quantifier of degree of irreversibility. The description and assessment of entropy production is one of the most interesting issues in non-equilibrium thermodynamics [24, 25]. The entropy production in the classical and quantum machines basically comes from two generic sources: finite-rate heat transfer and friction, where the friction can be classified into external friction and internal friction [26, 27]. The external friction is related to the exchange of energy through a mechanical linkage to the external surroundings, but the internal friction is associated with the dissipation of energy arising from the timescale disparity between the internal dynamics and finite-time operation [28]. A kind of friction is
attributed to a quantum phenomenon that the driving Hamiltonians at different times do not commutate \[2\], which induces possible transitions among the instantaneous energy eigenstates along a fast adiabatic process. Throughout this paper, the word ‘adiabatic’ merely indicates that the system is isolated from a heat reservoir and no heat is exchanged between the system and its surroundings. These transitions due to such nonadiabatic inner friction result in the irreversible entropy production which are responsible for the poor performance on the engines \[29, 30\]. The effects induced by this quantum inner friction were studied in quantum heat engines \[1, 2, 16, 17, 29, 30\]. However, the performance of quantum refrigerators with such kind of inner friction has not been analyzed so far.

While in a macroscopic system the work and heat are deterministic, for a microscopic quantum system (with a limited number of freedoms) these physical variables become random due to non-negligible thermal \[31, 32\] and quantum \[33, 34\] fluctuations. Theoretical and experimental investigation on the statistics of work \[35–44\], heat \[37, 45–52\], and even entropy \[53, 54\] has attracted much interest in the literature. On the other hand, for heat engines the statistics of power \[6, 55–57\] and efficiency \[30, 58–63\] has been analyzed, under the assumption that either system-bath interaction interval or unitary adiabatic process is quasistatic. Despite these achievements, no unified thermodynamic description of fluctuations in quantum cyclic refrigerators beyond the quasistatic limit is available.

In the present paper, we study the thermodynamics of quantum Otto refrigerators that consist of two finite-time isochoric and two finite-time adiabatic strokes, within a framework of stochastic thermodynamics. We determine the probability distribution functions for heat and work by adopting two-point projective measurement method. We then find general formulae of the COP and cooling rate as well as their variances, all of which are dependent on the time durations spent on the four processes. With these we then numerically determine the performance parameters (including cooling rate and COP) and the fluctuations for a quantum Otto cycle working with qubit system which is experimentally feasible. We discuss the effects induced by finite time durations of the isochoric and adiabatic strokes on the performance and fluctuations for the quantum refrigerator.

2. The probability distribution of stochastic COP for quantum Otto refrigerators

Consider the model of a quantum Otto refrigerator operating between a hot heat bath at inverse temperature \(\beta_h\) and a cold heat bath at inverse temperature \(\beta_c\) (see figure 1). This refrigerator consists of two isochoric branches, one with a cold and another with a hot heat reservoir where the system Hamiltonian \(H\) is kept constant, and two other strokes, where the system undergoes unitary transformation while isolated from the thermal reservoirs. Since quantum evolution is inherent nondeterministic and the initial state preparation is random, both quantum work and heat are stochastic along single realizations of a thermodynamic process. To determine the statistics of the quantum work and heat, we will quantify the work and heat probability distributions for each process in the Otto cycle by following a two-point projective measurement method \[30, 48, 50, 64\], one at the initial state and the other at the final instant of the process.

In the first branch \(A \rightarrow B\), the system undergoes a unitary expansion under the exclusive influence of the time-dependent Hamiltonian \(H_t\) which evolves from \(H_0 = \sum_n \delta^2_n \langle n | n \rangle\) at time \(t = 0\) to \(H_{\tau_{hc}} = \sum_n E_n \langle m | n \rangle \langle n | m \rangle\) at time \(t = \tau_{hc}\), where \(\langle n |\) (\(| m \rangle\) ) are normalized energy eigenstates, and \(E_n\) (\(E_m\) ) are the corresponding energy eigenvalues. At the weak-coupling limit \[48\] where the interaction energy between the system and the external field is much small in comparison to the system energy, the change in system energy can be interpreted as work in the adiabatic process where no heat is exchanged. Based on the two-point projective measurement approach, where the energy change in a quantum system during any single realization of a process is determined by the difference of eigenvalues of the quantum system at the beginning and at the end of the process, the probability distribution of stochastic work done by the system, \(w_{hc}\), can be given by

\[
p(w_{hc}) = \sum_{n, m} \delta(w_{hc} - (E_n - E_m)) p_{\tau_{hc} \rightarrow m} p_0^0,
\]

where \(p_0^0\) is the probability of the system being in state \(| n \rangle\) at time \(t = 0\), and \(p_{\tau_{hc} \rightarrow m} = |\langle n | U_{exp} | m \rangle|^2\) is the transition probability from eigenstate \(| n \rangle\) to \(| m \rangle\), with the unitary time evolution operator \(U_{exp}\). If the system Hamiltonian \(H\) evolves slowly enough wherein the quantum adiabatic condition \[65\] is satisfied in time interval \(\tau_{hc}\), the system remains in the same state and the transition probability therefore satisfies \(p_{\tau_{hc} \rightarrow m} = \delta_{nm}\) (with the Kronecker delta function \(\delta\) and thus the average population \(\langle n | t \rangle\) = \(\sum_n p_{\tau_{hc} \rightarrow n}^0\) \((0 \leq t \leq \tau_{ch})\) remains unchanged. However, when the expansion as a unitary evolution undergoes fast, the quantum adiabatic condition is violated, leading to the internal excitations as \(0 < p_{\tau_{hc} \rightarrow m}^{\tau_{hc}} < 1\).
second projective measurement. The mean population at instant along the isochore where no work is produced. The probability density of the stochastic heat isochore the external field is frozen and the system Hamiltonian is kept constant, the system must reach the incomplete thermalization stroke due to the finite time duration $\tau_c$ to the hot isochore. However, the system must be back to the instant $\tau_h$ to close the cycle when the hot

In the next step $B \to C$, the quantum system with constant frequency $\omega = \omega_c$ is kept in contact with a cold thermal bath of inverse temperature $\beta_c$ during a period $\tau_c$. At time $t = \tau_{hc}$, the system is suddenly coupled to the heat reservoir, with which it is suddenly decoupled from the bath at time $t = \tau_{hc} + \tau_c$. When the system-reservoir interaction is weak, the eigenenergy change should be equal to the stochastic heat absorbed by the system, the probability $q_{c}$ can be determined by conditional distribution to arrive at

$$ p(q_c|w_{hc}) = \sum_{k,l} \delta[q_c - (E^k_c - E^l_c)]P^c_{k\to l}p^{hc}_k, $$

where $p^{hc}_k$ is the probability of finding the system in the eigenstate $|k\rangle$ with energy eigenvalue $E^k_c$ after the first projective measurement, and $P^c_{k\to l}$ is the probability of the system collapsing into another $|l\rangle$ after the second projective measurement. The mean population at instant $C$ with time $t = \tau_{hc} + \tau_c$ can thus be given by $\langle n_C\rangle = \sum_P P^c_{i\to j}p^{hc}_i$. As we are interested in the finite-time machine operation, this isochore is an incomplete thermalization stroke due to the finite time duration $\tau_c$ that is of the order $\tau_{c,relax}$, i.e., $\tau_c \ll \tau_{c,relax}$, where $\tau_{c,relax}$ is the relaxation time for the system with the cold reservoir. Since during an isochore the external field is frozen and the system Hamiltonian is kept constant, the system must reach the thermal equilibrium at the end of this isochore [66, 67] when the time duration $\tau_c$ satisfies the condition that $\tau_c \gg \tau_{c,relax}$ and then the probability becomes the canonical form: $p^{hc}_k = e^{-\beta_c E^k_c}/Z_c$ with partition function $Z_c = \sum_n e^{-\beta_c E^n_c}$. That is, when and only when $\tau_c \gg \tau_{c,relax}$, the mean population $\langle n_C\rangle$ at end of the isochoric process approaches the equilibrium value,

$$ \langle n_C\rangle_{\tau_c \gg \tau_{c,relax}} = \langle n_C\rangle^{eq} = Z_c^{-1} \sum_n e^{-\beta_c E^n_c}. $$

On the adiabatic compression $C \to D$, the system is isolated in a time duration $\tau_{ch}$, while the energy gap $\omega$ varies from $\omega_c$ to $\omega_h$. For given work output $w_{hc}$ and the heat $q_h$ absorbed by the system, the probability distribution of stochastic work input can be determined by using two-point projective measurement to arrive at

$$ p(w_{ch}|q_h, w_{hc}) = \sum_{ij} \delta[w_{ch} - (E^i_c - E^j_c)]P^c_{i\to j}P^{hc+\tau_c}_i, $$

where the occupation probability $P^{hc+\tau_c}_i = \delta_{il}$, and $P^{c\to i}_{l\to j} = |\langle i|U_{com}j\rangle|^2$ is transition probability from eigenstate $|i\rangle$ and $|j\rangle$, with the time evolution operator $U_{com}$ along the compression.

In the fourth step $D \to A$, the system is coupled to a hot reservoir of inverse temperature $\beta_h$ in time duration $\tau_h$ while keeping its frequency in a constant with $\omega = \omega_h$. The cycle period is then given by $\tau_{cycle} = \tau_{hc} + \tau_c + \tau_{ch} + \tau_h$. Here we assume [2] that the system returns to its initial state of the cycle even for incomplete thermalization where $\tau_h \ll \tau_{h,relax}$ with $\tau_{h,relax}$ being the relaxation time for the system along the hot isochore. However, the system must be back to the instant $A$ to close the cycle when the hot
thermalization becomes complete with \( \tau_h \gg \tau_{h, \text{relax}} \), without requiring this particular assumption [28]. We then will not derive the expression of the stochastic heat exchanged \( q_h \) along this stroke. If \( \tau_h \gg \tau_{h, \text{relax}} \), the system can reach the thermal equilibrium at the end of the isochore, which leads to

\[
\langle n_h \rangle_{\text{eq}} \equiv \langle n(\tau_h \gg \tau_{h, \text{relax}}) \rangle = Z_h^{-1} \sum_n n e^{-\beta_h E_n^h}
\]

with \( Z_h = \sum_n e^{-\beta_h E_n^h} \), and \( \gamma_h \) is the thermal conductivity between the system and the hot reservoir.

Using equations (1), (2) and (4), the probability distribution \( p(w_{ch}, q_c, w_{hc}) \) for the machine which has certain values of \( w_{ch}, q_c, w_{hc} \) can be calculated from the chain rule for condition probabilities

\[
p(w_{ch}, q_c, w_{hc}) = p(w_{ch}|q_c, w_{hc}) p(q_c| w_{hc}) p(w_{hc}) \quad [30]:
\]

\[
p(w_{ch}, q_c, w_{hc}) = \sum_{m,n,i,j} \delta \left[ q_c - (E_m^c - E_n^m) \right] \delta \left[ w_{hc} - (E_m^c - E_n^m) \right] \delta \left[ w_{hc} - (E_i^h - E_m^c) \right] \times \left| \langle n | U_{\text{exp}} | m \rangle \right|^2 \left| \langle i | U_{\text{com}} | j \rangle \right|^2 p_0^{p_{w,m}} p_{w_{hc}}.
\]

It is clear that this distribution function (6) depends on both transition probabilities along the two adiabatic strokes (\( \left| \langle n | U_{\text{exp}} | m \rangle \right|^2 \) and \( \left| \langle i | U_{\text{com}} | j \rangle \right|^2 \)) and the occupation probabilities at the beginning instants of these two adiabatic strokes (\( p_0^{p_{w,m}} \) and \( p_{w_{hc}}^{p_{w_{hc}}} \)), both of which are also respective ending points of the two isochoric strokes with fixed \( \omega_h \) and \( \omega_c \).

For the quantum Otto refrigerator, the stochastic COP reads

\[
\varepsilon = q_c/(w_{hc} + w_{hc}).
\]

It follows, integrating over all values of \( w_{ch}, q_c, w_{hc} \), that the probability distribution \( p(\varepsilon) \) becomes

\[
p(\varepsilon) = \sum_{m,n,i,j} \delta \left( \varepsilon - \frac{E_i^h - E_m^c}{E_m^c - E_n^m - E_i^h} \right) \left| \langle n | U_{\text{exp}} | m \rangle \right|^2 \left| \langle i | U_{\text{com}} | j \rangle \right|^2 p_0^{p_{w,m}} p_{w_{hc}}.
\]

While for quantum adiabatic driving, the system remains in the same state (\( n = m = i = j \)), in the nonadiabatic driving the transition probability \( \left| \langle n | U_{\text{exp}} | m \rangle \right|^2 \) (or \( \left| \langle i | U_{\text{com}} | j \rangle \right|^2 \)) is positive due to transitions between sates \( n \) and \( m \) (or \( i \) and \( j \)). For the quantum Otto refrigerator, the statistics of COP is fully determined by the unitary time evolution for the adiabatic expansion and compression \( U_{\text{exp}} \) and \( U_{\text{com}} \), and by the finite-time system dynamics along the thermalization processes, when the two temperatures of the two heat reservoirs (\( \beta_h \) and \( \beta_c \)) are given.

### 3. A quantum Otto refrigerator using a two-level system

#### 3.1. General description

Let us consider the Otto cycle in which a two level system as the working substance is weakly coupled to a hot and a cold bosonic thermal reservoir. The eigenenergies of the two level system are \( E_+ = \frac{1}{2} \hbar \omega \) and \( E_- = -\frac{1}{2} \hbar \omega \). It is shown in appendix A that, in the limit of weak coupling between system and bosonic heat reservoir, the dynamics of the system along the thermalization can be described by the motion equation of population, which gives

\[
\langle n_c \rangle = \langle n_c \rangle_{\text{eq}} + \left[ \langle n_h \rangle - \langle n_c \rangle_{\text{eq}} \right] e^{-\gamma \tau_c},
\]

and

\[
\langle n_h \rangle = \langle n_h \rangle_{\text{eq}} + \left[ \langle n_D \rangle - \langle n_h \rangle_{\text{eq}} \right] e^{-\gamma \tau_h},
\]

where \( \langle n_c \rangle = \langle n(\tau_{\text{cycle}}) \rangle, \langle n_D \rangle = \langle n(\tau_{\text{cycle}} - \tau_h) \rangle, \langle n_c \rangle = \langle n(\tau_{hc} + \tau_c) \rangle, \) and \( \langle n_D \rangle = \langle n(\tau_{hc}) \rangle \), with \( \gamma_c \) (\( \gamma_h \)) being the thermal conductivity between the system and the cold (hot) bath.

If the unitary expansion and compression during the Otto cycle (from \( \omega_h \) to \( \omega_c \) and vice versa) are such that there is a probability of state transitions due to quantum fluctuations, then there is a probability that population \( \langle n \rangle \) may change with varying time. After a simple calculation (see appendix B), we find that

\[
\langle n_h \rangle = (1 - 2\xi)\langle n_c \rangle, \quad \langle n_D \rangle = (1 - 2\xi)\langle n_h \rangle,
\]

where \( \xi = \left| \langle \pm | U_{\text{exp}} | \mp \rangle \right|^2 = \left| \langle \pm | U_{\text{com}} | \mp \rangle \right|^2 \) is called the adiabaticity parameter indicating the probability of transition between state \( |+\rangle \) and \( |-\rangle \) during the compression or expansion. As shown in figure 1, the mean populations at any instant along the cycle are negative, which means that \( \xi \) must be situated between \( 0 < \xi < 1/2 \). The probability of no state transition along either adiabatic branch is accordingly

\[
\left| \langle \pm | U_{\text{exp}} | \pm \rangle \right|^2 = \left| \langle \pm | U_{\text{com}} | \pm \rangle \right|^2 = 1 - \xi. \]

The adiabaticity parameter \( \xi \) depends on the speed at which the adiabatic stroke is performed [7, 9, 30], and it also depends on the form of the driving Hamiltonian \( H \) along compression and expansion strokes. When the time scale of the state change is much larger than that
of the dynamical one, the quantum adiabatic condition is satisfied and the population $\langle n \rangle$ remains unchanged in the adiabatic stage ($\xi = 0$). Rapid change in the control field $\omega$, however, leads to nonadiabatic behavior ($\xi > 0$) which can be understood as inner friction [1, 2, 16, 28, 68, 69] causing state transitions. Equation (10) shows that $\langle n_C \rangle > \langle n_H \rangle$ and $\langle n_H \rangle > \langle n_C \rangle$ for $\xi > 0$ (see also figure 1), as lies in the fact that the finite time duration of the expansion and compression accounts for the nonadiabatic inner friction related to the irreversible entropy production.

Using equations (8)–(10), it follows that the populations $\langle n_A \rangle$ and $\langle n_C \rangle$ can be expressed in terms of the equilibrium populations $\langle n_A \rangle^{eq}$ and $\langle n_C \rangle^{eq}$,

$$\langle n_A \rangle = \langle n_H \rangle^{eq} + \Delta_h, \quad \langle n_C \rangle = \langle n_C \rangle^{eq} + \Delta_c$$

where

$$\Delta_h = \frac{(2\xi - 1)[(2\xi - 1)\langle n_H \rangle^{eq} + \langle n_C \rangle^{eq}] - y[\langle n_H \rangle^{eq} + (2\xi - 1)\langle n_C \rangle^{eq}]}{xy - (2\xi - 1)^2},$$

$$\Delta_c = \frac{(2\xi - 1)[(2\xi - 1)\langle n_C \rangle^{eq} + \langle n_C \rangle^{eq}] - x[\langle n_C \rangle^{eq} + (2\xi - 1)\langle n_C \rangle^{eq}]}{xy - (2\xi - 1)^2},$$

with $x = e^{\hbar \omega h}$ and $y = e^{\hbar \omega c}$. In quasistatic limit, the system can reach thermal equilibrium at the end of the hot or cold process and $\Delta_{h,c} \to 0$. Therefore, $\Delta_c$ and $\Delta_h$ indicate how far the two isochoric processes deviate from the quasistatic limit. Hereafter we will refer $x$ and $y$ rather than $\tau_h$ and $\tau_c$ as the time durations along the hot and cold isochoric branches for simplicity. Here $\langle n_C \rangle^{eq}$ defined in equation (3) and $\langle n_H \rangle^{eq}$ in equation (5) can be obtained using the same approach as that described in appendix A for the derivation of equation (B.3) to arrive at

$$\langle n_C \rangle^{eq} = -\frac{1}{2} \tanh \left( \frac{\hbar \beta \omega_c}{2} \right), \quad \langle n_H \rangle^{eq} = -\frac{1}{2} \tanh \left( \frac{\hbar \beta \omega_h}{2} \right).$$

From equations (1) and (2), the average heat injection, $\langle q_c \rangle = \int \int q_c p(q_c, w_{hc}) p(w_{hc}) \, dw_{hc} \, dq_c$ can be obtained as

$$\langle q_c \rangle = \hbar \omega_c \left[ (\langle n_C \rangle^{eq} + \Delta_c) - (\langle n_H \rangle^{eq} + \Delta_h) \right] + 2\hbar \omega_c \xi \left( (\langle n_C \rangle^{eq} + \Delta_h) \right). \quad (15)$$

Note that the heat must be developed in the working system when work is done on the system along the adiabatic stroke, and it would be released into the reservoir when the system is coupled to this reservoir. The absolute value of the second term on the right-hand side of equation (15), $|2\hbar \omega_c \xi (\langle n_C \rangle^{eq} + \Delta_h)|$, denotes the amount of heat that is released to the cold reservoir and comes from the work against inner friction during the expansion. The adiabaticity parameter $\xi$ decreases with slowing of operation rate, although not monotonically. In order to operate as a refrigerator, the average heat injection should be positive, $\langle q_c \rangle > 0$, leading to

$$\xi > \xi^+ \equiv -\frac{\langle (n_C)^{eq} + \Delta_c \rangle - \langle (n_H)^{eq} + \Delta_h \rangle}{2 \langle (n_H)^{eq} + \Delta_h \rangle}, \quad (16)$$

which is thus dependent on the time allocations of four processes. The constraint that the cycle must cool the cold bath imposes the condition on the minimum time durations $\tau_h$ and $\tau_c$ (and thus $x$ and $y$) for given $\xi$. Substituting equations (12) and (13) into equation (15), we find that $\tau_c = 0$, and the time spent on the hot isochore must satisfy

$$x \geq \frac{(2\xi - 1)\langle n_H \rangle^{eq} + \langle n_C \rangle^{eq}(2\xi - 1)}{\langle n_C \rangle^{eq} + \langle n_H \rangle^{eq}(2\xi - 1)}, \quad \hbar \omega_c \geq \frac{1}{\hbar \omega_c} \ln \frac{(2\xi - 1)\langle n_H \rangle^{eq} + \langle n_C \rangle^{eq}(2\xi - 1)}{\langle n_C \rangle^{eq} + \langle n_H \rangle^{eq}(2\xi - 1)},$$

in order for positive cooling load to be produced. Integrals over the distribution function $p(q_c, w_{hc}) p(w_{hc}) \, dw_{hc} \, dq_c$, which reads

$$\langle q_c^2 \rangle = \hbar^2 \omega_c^2 \left[ 1/4 + (2\xi - 1) \langle (n_C)^{eq} + \Delta_h \rangle \right],$$

the fluctuations of heat $q_c$, $\langle \delta q_c^2 \rangle = \langle q_c^2 \rangle - \langle q_c \rangle^2$, then become

$$\langle \delta q_c^2 \rangle = \hbar^2 \omega_c^2 \left[ \frac{1}{2} - (2\xi - 1)^2 \langle (n_C)^{eq} + \Delta_h \rangle^2 - \langle (n_C)^{eq} + \Delta_c \rangle^2 \right]. \quad (17)$$

These fluctuations are upper limited by the value of $\hbar^2 \omega_c^2 \left[ \frac{1}{2} - \langle (n_C)^{eq} \rangle^2 \right]$ and the quasistatic limit leads to

$$\langle \delta q_c^2 \rangle = \hbar^2 \omega_c^2 \left[ \frac{1}{2} - \langle (n_H)^{eq} \rangle^2 - \langle (n_C)^{eq} \rangle^2 \right].$$

As $\langle \delta q_c \rangle = \langle q_c \rangle / \tau_{cycle}$ and $\sqrt{\langle \delta q_c^2 \rangle} = \langle \delta q_c \rangle / \tau_{cycle}$, the relative variance of cooling rare $f_{hc}$ can be obtained by using equations (15) and (17) to arrive at

$$f_{hc} = \frac{\sqrt{\langle \delta q_c^2 \rangle}}{\langle q_c \rangle} = \frac{\frac{1}{2} - (2\xi - 1)^2 \langle (n_C)^{eq} + \Delta_h \rangle^2 - \langle (n_C)^{eq} + \Delta_c \rangle^2}{(2\xi - 1) \langle (n_C)^{eq} + \Delta_h \rangle + \langle (n_C)^{eq} + \Delta_c \rangle}. \quad (18)$$
Since no work is produced in the two isochoric processes, the average total work done on the working system per cycle is \( \langle w_{hc} \rangle + \langle w_{ch} \rangle \), where \( \langle w_{hc} \rangle = \int w_{hc}p(w_{hc})dw_{hc} \) and \( \langle w_{ch} \rangle = \int w_{ch}p(w_{ch})dw_{ch} \). It follows, using equations (1) and (4), that the total work can be obtained,

\[
\langle w \rangle = h(\omega_h - \omega_c)\left[ \left( (n_c)^{eq} + \Delta_c \right) - \left( (n_h)^{eq} + \Delta_h \right) \right] - 2h\xi \left[ \omega_h \left( (n_h)^{eq} + \Delta_h \right) + \omega_h \left( (n_c)^{eq} + \Delta_c \right) \right].
\] (19)

The second term of the right-hand side denotes the work done on the system against the internal friction along the adiabatic expansion and compression. The thermodynamic COP, defined by \( \varepsilon_{th} = \langle q_h \rangle / \langle w \rangle \), is given by

\[
\varepsilon_{th} = \frac{\omega_c}{\omega_h + \frac{\xi}{\omega_h + \frac{\xi}{\omega_c}}}.
\] (20)

where \( \mathcal{F} = 2(\langle n_c^{eq} + \Delta_c \rangle - (\langle n_h^{eq} + \Delta_h \rangle) / (\langle n_c^{eq} + \Delta_c \rangle - (\langle n_h^{eq} + \Delta_h \rangle) \right) \) and \( \mathcal{G} = 2(\langle n_h^{eq} + \Delta_h \rangle - (\langle n_c^{eq} + \Delta_c \rangle - (\langle n_h^{eq} + \Delta_h \rangle) \). As \( \mathcal{F} \), \( \mathcal{G} < 0 \) and \( \xi \geq 0 \), the thermodynamic COP \( \varepsilon_{th} \) increases as the adiabaticity parameter \( \xi \) decreases, and it reaches its upper bound \( \varepsilon_{th}^{ad} = \omega_c / (\omega_h - \omega_c) \) in the ideal adiabatic case when \( \xi = 0 \). The fact that the additional heat is dissipated into the hot reservoir due to finite time realization of the compression or expansion, so that the additional work is input to overcome such heat loss, suggests that cycles consisting of nonadiabatic transformation along the expression and compression runs less efficiently than those with ideal adiabatic strokes. For quantum Otto refrigerators at weak coupling limit, the probability distribution \( p(\varepsilon) \) of the stochastic COP \( \varepsilon \) can be determined by

\[
p(\varepsilon) = \int \int du_{hc} dq_c du_{ch} p(w_{hc}, q_c, w_{hc}) \delta \left( \varepsilon - \frac{\omega_c}{\omega_h + \frac{\xi}{\omega_h + \frac{\xi}{\omega_c}}} \right)
\] to arrive at

\[
p(\varepsilon) = 2 \left\{ \frac{1}{2} \left( 1 - \xi \right) ^2 + \frac{1}{4} - (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \xi \left( \varepsilon - \varepsilon_{th}^{ad} \right)
\]

\[
+ 2 \left\{ \frac{1}{4} + (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \xi \left( \varepsilon - \frac{\varepsilon_{th}^{ad}}{2\varepsilon_{th}^{ad} + 1} \right) + (1 - \xi) \xi \left[ \delta (\varepsilon - 1) + (1 - \delta) \xi \right],
\] (21)

where we have used \( q_c / (w_{hc} + w_{ch}) = 0 / 0 = \varepsilon_{th}^{ad} \) (see appendix C for detailed description). While in the heat engine with fast adiabatic branches the quantum efficiency takes nonzero values and even such strange values as infinity \( [30] \), the stochastic COP for the quantum Otto refrigerator may be finite and zero (i.e., inverse of infinity). This follows from the fact that the refrigeration cycle is the reverse of the heat engine cycle. In the case of quantum adiabatic driving (\( \xi = 0 \)), the distribution (21) becomes a delta function: \( p(\varepsilon) = \delta(\varepsilon - \varepsilon_{th}^{ad}) \), indicating that the stochastic COP is equal to the adiabatic thermodynamic one. Using the distribution function (21), we find that the first two central moments are

\[
\langle \varepsilon \rangle = 2 \left\{ \frac{1}{4} - (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \xi \left( \varepsilon - \frac{\varepsilon_{th}^{ad}}{2\varepsilon_{th}^{ad} + 1} \right) \varepsilon_{th}^{ad}
\]

\[
- 2 \left\{ \frac{1}{4} + (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \xi \left( \varepsilon - \frac{\varepsilon_{th}^{ad}}{2\varepsilon_{th}^{ad} + 1} \right) + (1 - \xi) \xi
\] (22)

and \( \langle \varepsilon ^2 \rangle = 2 \left\{ \frac{1}{4} - (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \xi ^2 + \frac{1}{4} (1 - \xi ^2) \left( \varepsilon_{th}^{ad} \right)^2 +
\]

\[
2 \left\{ \frac{1}{4} + (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \xi ^2 \left( \varepsilon_{th}^{ad} / (2\varepsilon_{th}^{ad} + 1) \right) ^2 + (1 - \xi) \xi. \] This, combining with equation (22), gives rise to the fluctuations of stochastic COP, \( \langle \delta \varepsilon ^2 \rangle = \langle \varepsilon ^2 \rangle - \langle \varepsilon \rangle ^2 \), leading to

\[
\langle \delta \varepsilon ^2 \rangle = 2 \left\{ \frac{1}{4} - (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \xi ^2 + (1 - \xi ^2) \left( \varepsilon_{th}^{ad} \right)^2
\]

\[
+ 2 \left\{ \frac{1}{4} + (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \xi ^2 \left( \frac{\varepsilon_{th}^{ad}}{2\varepsilon_{th}^{ad} + 1} \right) ^2
\]

\[
- \left\{ \xi (\xi - 1) + \xi \frac{1}{2} - 2 (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) \right\} \varepsilon_{th}^{ad} + (1 - \xi ^2) \varepsilon_{th}^{ad}
\]

\[
- \xi ^2 \left[ \frac{2 (\langle n_h^{eq} + \Delta_h \rangle) (\langle n_c^{eq} + \Delta_c \rangle) + \frac{1}{2} \left( \varepsilon_{th}^{ad} \right)^2}{2\varepsilon_{th}^{ad} + 1} \right] ^2 + \xi (1 - \xi).
\] (23)

3.2. Numerical analysis
To obtain the adiabaticity parameter \( \xi \) in terms of time \( \tau_{hc} \) and \( \tau_{ch} \) of the expansion and compression Hamiltonian driving, we now analyze the statistics and thermodynamics of the Otto refrigerator by using a
single qubit as the working substance. Following a protocol design adopted in an experimental realization of a quantum Otto engine working with spin qubits [29], the system Hamiltonian along the unitary expansion $A \rightarrow B$ is chosen to be

$$H_{\text{exp}}(t) = \frac{\hbar \omega(t)}{2} \left[ \cos \left( \frac{\pi t}{2\tau_{\text{hc}}} \right) \sigma_x + \sin \left( \frac{\pi t}{2\tau_{\text{hc}}} \right) \sigma_y \right],$$

(24)

where $\omega(t) = \omega_h (1 - \frac{t}{\tau_{\text{hc}}}) + \omega_c \frac{t}{\tau_{\text{hc}}}$ (with $0 \leq t \leq \tau_{\text{hc}}$), and $\sigma_{x,y,z}$ are the Pauli operators. The expansion time evolution operator is then $U_{\text{exp}} = \mathcal{T}_\sigma \exp \left[ -i \int_0^{\tau_{\text{hc}}} dt H_{\text{exp}}(t) \right]$ where $\mathcal{T}_\sigma$ is the time-ordering operator. The compression can be realized by reversing the protocol used in the expansion, which implies that the compression Hamiltonian reads $H_{\text{com}}(t) = H_{\text{exp}}(\tau_{\text{hc}} + \tau_c + \tau_{\text{ch}} - t)$, where $\tau_{\text{hc}} + \tau_c \leq t \leq \tau_{\text{hc}} + \tau_c + \tau_{\text{ch}}$. The compression time evolution operator reads $U_{\text{com}} = \mathcal{T}_\sigma \exp \left[ i \int_t^{\tau_{\text{hc}}+\tau_{\text{ch}}} dt H_{\text{exp}}(t) \right]$.

In our numerical calculation, we consider the energy scales compatible with experiments in nuclear magnetic resonance setups. We choose the frequency gaps of the cold and hot isochores (24) as $\omega_c/2\pi = 2.0$ kHz and $\omega_h/2\pi = 3.6$ kHz, and use $\tau_{\text{hc}} = \tau_c = 5.6$ ms to denote the time of expansion or compression. The strength of the interaction between the system and the hot (cold) reservoir is assumed as $\gamma_h = \gamma_c = 10$ Hz. The system-bath interaction interval has a time scale of larger than 100 ms, while the scale of $\tau_{\text{cli}}$ is smaller than 1 ms, which implies that the cycle period $\tau_{\text{cy}}$ for the machine under consideration is dominated by the time durations (\(\tau_1\) and $\tau_c$) along the hot and cold isochores.

In figures 2(a) and (b), we plot the relative fluctuations of cooling rate $f_{\text{cl}}$ as a function of effective time durations (x and y), both for adiabatic and nonadiabatic driving in the physical region where ($q_c > 0$). If the isochoric cold (hot) branch is completed in a finite time $\tau_c$ ($\tau_h$), with finite value of $y$ ($x$), this isochoric step is out of equilibrium and the fluctuations are inevitable. When we slow down the adiabatic processes, we find that the relative fluctuations $f_{\text{cl}}$ are increasing. Comparison between figures 2(a) and (b) also shows that the relative fluctuations are larger in nonadiabatic driving with finite time ($\tau_{\text{cli}} = 0.276$ ms) than in quasistatic, adiabatic evolution ($\tau_{\text{cli}} = 0.702$ ms).

The statistics of stochastic COP is examined in figure 3 at different values of $\tau_{\text{cli}}$ for given time durations ($\tau_r = \tau_h = 69.315$ ms). For adiabatic driving with $\tau_{\text{cli}} = 0.702$ ms (red squares), the distribution $p(\varepsilon)$ has only one peak at $\varepsilon_{\text{ad}} = 1$, and thus the stochastic COP takes only one value equal to the adiabatic thermodynamic COP $\varepsilon_{\text{ad}}$, as also noted from equation (21) with $\xi = 0$. By contrast, for nonadiabatic driving with $\tau_{\text{cli}} = 0.222$ ms (blue dots), the probabilities of negative values $(-\varepsilon_{\text{ad}}/(2\varepsilon_{\text{ad}} + 1)$ and $-1$) are also visible due to quantum fluctuations. In this case, the stochastic COP is different from the thermodynamic COP, as lies in the fact that $\varepsilon_{\text{ad}} = (\langle q_c \rangle/\langle w \rangle)$, where $\langle q_c \rangle$ and $\langle w \rangle$ are associated with change of average system energy, is determined by ensemble average of single realizations of a process, and $\varepsilon = q_c/w$, where $q_c$ and $w$ are related to the change of eigenenergy, is dependent on a particular single realization of a process (such a single realization with probability $p(\varepsilon)$). We note that the probability distribution $p(\varepsilon)$ is normalized to 1 for both fast and slow driving rates as it should. Unlike in quantum heat engines of fast driving rate [30] where the average efficiency cannot be defined due to divergent stochastic efficiency, the stochastic COP is observed to be not divergent at the fast driving rate, which implies that the average COP, $\langle \varepsilon \rangle = \int p(\varepsilon)\varepsilon d\varepsilon$, is well defined even in the nonadiabatic driving. The
stochastic COP has the probabilities of taking the values of $\pm x = x_0 = \pm \varepsilon$, as shown in equation (21). As emphasized, our calculation shows that, in contrast to the fact that the probability distribution $p(\varepsilon)$ depends sensitively on the time duration of adiabatic branch $\tau_{dri}$, the influence of time durations $x$ and $y$ (i.e., $\tau_c$ and $\tau_h$) on the distribution $p(\varepsilon)$ is extremely small (and here it is therefore not plotted in a figure). To illustrate this, we note the special case of the quantum adiabatic driving with $\tau_{dri} = 0.702$ ms. In such a case, the plot of $p(\varepsilon)$ as a function of $\varepsilon$ is independent of $\tau_h$ and $\tau_c$ ($x$ and $y$), since $p(\varepsilon) = \delta(\varepsilon - \varepsilon_{th}^{ad})$ is only dependent on the frequencies $\omega_c$ and $\omega_h$ (see figure 3).

While in the adiabatic driving case the COP is deterministic and its fluctuation is vanishing, for nonadiabatic driving the COP is random and its fluctuations are thus finite. For a cycle with nonadiabatic driving branches, the average COP $\langle \varepsilon \rangle$ slightly increases as time duration $y = e^{\gamma \tau_c}$ increases, but it slightly decreases as time duration $x = e^{\gamma \tau_h}$ increases, see figures 4(a) and (b). This follows from the fact that, while the heat absorbed by the system along the cold isochoric stroke slightly increases as $y$ increases, the heat released to the hot reservoir slightly increases as $x$ increases. The fluctuations $\langle \delta \varepsilon^2 \rangle$ as a function of $x$ and $y$ are shown in figures 4(c) and (d), where $\langle \delta \varepsilon^2 \rangle$ are also observed to depend very slightly on the time durations $x$ and $y$. Finally, it is important to note that the slower the expansion and compression strokes with larger driving time duration $\tau_{dri}$, the greater the machine COP and the smaller COP fluctuations.

Now let us consider fixed time durations of two isochoric strokes. Given the time allocations ($x$ and $y$) to the two isochores, we can analyze the statistics of the stochastic cooling rate and COP, as shown in figures 5(a)–(d). In these figures, we consider the incomplete thermalization with the cold reservoir by adopting $y = 2$ which corresponds to $\tau_c = 69.315$ ms. The hot isochoric thermalization is incomplete with $x = 2$ ($\tau_h = 69.315$ ms) or complete with $x = 20$ ($\tau_h = 299.573$ ms and $\Delta h$ in equation (12) becomes vanishing). The cycle time period $\tau_{cycle} = \tau_h + \tau_c + 2\tau_{dri}$ is dominated by the total system-bath interaction interval which is much larger than the time duration for the two adiabatic strokes. The positive cooling rate imposes a lower bound for the time duration of adiabatic branch, as can be seen in figure 5(a), and also was anticipated by condition (16). If the machine operates at a too-fast driving speed, the internal friction is so large that the average cooling load $\langle q_\gamma \rangle$ and cooling rate $\langle \dot{q}_\gamma \rangle$ becomes non-positive. While the condition of the complete thermalization with the hot bath leads to an increase of average cooling load, the cooling rate in the complete isochoric thermalization might be smaller than that in the partial isochoric thermalization.

The average COP $\langle \varepsilon \rangle$ as a function of driving time duration $\tau_{dri}$ is demonstrated in figure 5(b), comparing with thermodynamic COP $\varepsilon_{th}$. It is clear that an increase in the driving time leads to increase of both $\varepsilon_{th}$ and $\langle \varepsilon \rangle$, although not monotonically. While $\langle \varepsilon \rangle$ is always positive, $\varepsilon_{th}$ may be positive or negative. We note that there is a relation: $0 \leq \langle \varepsilon \rangle - \varepsilon_{th}$, where equality holds if and only if the two adiabatic branches satisfy quantum adiabatic condition. At not-too-fast driving rate ($\tau_{dri} \geq 300 \mu$s), both $\langle \varepsilon \rangle$ and $\varepsilon_{th}$ oscillate.

Figure 3. The probability distribution $p(\varepsilon)$ of the quantum stochastic COP for both adiabatic (red squares) and nonadiabatic (blue dots) driving. The values of the parameters are $\beta_h = \frac{\hbar}{\omega_h}$, $\beta_c = \frac{1}{\pi \omega_c}$, $\omega_c = 4\pi$ kHz, $\omega_h = 7.2\pi$ kHz, and $x = y = 2$. We observe the appearance of peaks at negative COP in the nonadiabatic case.
as a function $\tau_{\text{dri}}$, but the oscillation amplitude of $\langle \varepsilon \rangle$ is much less than that of $\varepsilon_{\text{th}}$. The fluctuations of stochastic cooling rate and COP are shown in figures 5(c) and (d), where it is clear that fast driving rate leads to the larger fluctuations of both COP and cooling rate, though not monotonically. Figure 5(d) shows that the time duration $x(y)$ along the hot (cold) isochoric stroke has extremely small influence on the fluctuations $\langle \delta \varepsilon^2 \rangle$, as also noted from figures 4(c) and (d). In contrast, in figure 5(c) the cooling rate fluctuations in the complete hot thermalization (upper panel) are much smaller than corresponding ones in the incomplete hot thermalization (lower panel), and thus these fluctuations depend sensitively on the time duration $x$.

To see the effects of temperature on the statistics of COP, in figure 6 we plot the variance of the stochastic COP as a function of the inverse temperature $\beta_c$ of cold reservoir. For adiabatic branches (with $\tau_{\text{dri}} = 0.702$ ms), $\langle \varepsilon \rangle$ is found to be $\varepsilon = \varepsilon_{\text{th}}$ and $\langle \delta \varepsilon^2 \rangle = 0$. In the nonadiabatic driving case, as the temperature decreases, $\langle \varepsilon \rangle$ decreases but $\langle \delta \varepsilon^2 \rangle$ increases. The decrease in the average COP with decreasing temperature can be understood that the probabilities $p(\varepsilon)$ of negative (positive) values of stochastic COP increases (decreases) with decreasing temperature due to quantum fluctuations that dominate the low temperature domain. The quantum fluctuations are also responsible for the result that the fluctuations $\langle \delta \varepsilon^2 \rangle$ gets increased while the temperature is lowered.

We finally consider the difference between the thermodynamic COP and average COP by evaluating the covariance between the total stochastic work ($\langle w \rangle$) and stochastic COP ($\varepsilon = q_c / \langle w \rangle$) that is defined by [70]

$$\text{Cov} \left( \frac{q_c}{\langle w \rangle} , w \right) = \langle \varepsilon_{\text{th}} - \langle \varepsilon \rangle \rangle / \langle w \rangle.$$ (25)

Substitution of equations (20) and (22) into equation (25) allows us to determine the ratio $\text{Cov} \left( \frac{q_c}{\langle w \rangle} , w \right) / \langle w \rangle$. The ratio $\text{Cov} \left( \frac{q_c}{\langle w \rangle} , w \right) / \langle w \rangle$ as a function of inverse temperature $\beta_c$ is plotted in figure 7, where the time durations along two isochores are $x = 20$ and $x = 2$ with fixed $y = 2$, respectively. In the quantum adiabatic driving case, the COP is delta distributed due to absence of quantum fluctuations and there is an equality of $\langle \varepsilon \rangle = \varepsilon_{\text{th}}$. The covariance in equation (25) is zero, which means that stochastic COP $\varepsilon$ and total work input $w$ are uncorrelated. By contrast, this covariance is always negative for nonadiabatic driving, as also shown in figure 5(b). As the temperature is lowered, the ratio $\text{Cov} \left( \frac{q_c}{\langle w \rangle} , w \right) / \langle w \rangle$
Figure 5. Performance and fluctuations for given time duration spent on the two isochores: (a) the average absorbed heat \( \langle q_c \rangle \) and cooling rate \( \langle \dot{q}_c \rangle \), (b) the average COP (\( \epsilon \)) and the thermodynamic COP \( \epsilon_{\text{th}} \), (c) the cooling rate fluctuations \( \langle \delta \dot{q}_c^2 \rangle \), and (d) the COP fluctuations \( \langle \delta \epsilon^2 \rangle \) as a function of \( \tau_{\text{dri}} \). Some parameters are \( \beta_h = \frac{5}{6} \hbar \omega_h \), \( \beta_c = \frac{1}{2} \hbar \omega_c \), \( \omega_c = 4\pi \text{ kHz} \), and \( \omega_h = 7.2\pi \text{ kHz} \).

Figure 6. The variance of stochastic COP \( \langle \delta \epsilon^2 \rangle \), and average COP \( \langle \epsilon \rangle \) (inset) as a function of inverse temperature of cold reservoir \( \beta_c \omega_c = \frac{1}{\hbar} \) for \( \tau_{\text{dri}} = 0.234 \text{ ms} \) and \( \tau_{\text{dri}} = 0.222 \text{ ms} \). The parameters are \( \omega_c = 4\pi \text{ kHz} \), \( \omega_h = 7.2\pi \text{ kHz} \), and \( x = y = 20 \). When \( \tau_{\text{dri}} = 0.702 \text{ ms} \), \( \langle \epsilon \rangle = 1.25 \), and \( \langle \delta \epsilon^2 \rangle = 0 \).

monotonically decreases, and thus its deviation from the value 0 increases. Physically, this follows the fact that correlation between stochastic COP \( \epsilon \) and stochastic total work \( w \) becomes stronger with decreasing temperature due to quantum fluctuations.
remove the simplifying assumption that
of
photons. Supposing that the system size is small with respect to the radiation length2
interactions
not take into account effects induced by the dynamical parameters that enter in the system-bath
entering in the jump operators
and adiabatic branches proceed with finite-time durations, under the simplifying assumption that we did
4. Discussions and conclusions
parameters involved in Kraus operators
strokes. Therefore, a quite interesting extension of our work would be consideration of the dynamical
core conductivity between the system and the environment defined in equation (A.8) becomes
\[ \beta \gamma \]

\[ \beta \gamma \] that, in such a case, \( c \) in equation (A.3) can be given by \( c \) = \( d \) = \( m \) = \( n \) = \( 0 \) as expected. Therefore, the thermal conductivity between the system and the environment defined in equation (A.8) becomes
\[ \gamma = |v_{++}|^2 + |v_{--}|^2 = c_{++} (1 + e^{-\beta \omega /\hbar}) \]. We therefore conclude that, the thermal conductivity \( \gamma \) depends on both the system (e.g., particle number \( N \)) and the reservoir (e.g. temperature \( \beta_r \)) in the weak coupling limit, though the stochastic heat exchange along the hot or cold isochore is expected to be always \( q_r = \pm \beta_r \omega /2 \), 0 based on the limited internal energy of the qubit [71].
As an illustration, we consider how \( |v_{++}|^2 \) and \( |v_{--}|^2 \) in the dissipation operators \( V \) affect the probability \( P_{k+}^z \) (2) along the cold isochore in our model. Note that, \( P_{k+}^z \) is equivalent to \( \rho_{mm}(\tau + \tau_{hc}) \) which is determined according to equation (A.4). Solving the time dependence of the master equation equation (A.4), we can obtain the \( \gamma \), dependent expression of \( \rho_{mm}(\tau_{hc} + \tau_r) \) which is thus dependent on the system-environment interaction. Moreover, for a quantum Otto cycle, \( c_{++} \) entering the dissipative operator \( V \) in the hot isochore should be different from corresponding that in the cold isochore [28], since \( c_{++} \) depends on the system (e.g. size and energy gap) and environment (e.g., the temperature). If we remove the simplifying assumption that \( \gamma = \gamma_r \), without consideration the specific form of \( c_{++} \) in \( |v_{++}|^2 \) entering in the jump operators \( V \) in the master equation (A.1), the optimal time durations (\( \tau_0 \) and \( \tau_c \)) of the two isochores could be found to decrease the statistical moments associated with energy fluctuations even under the constraint of fixed \( \tau_{cyc} \). There would be the similar case for propagators along adiabatic strokes. Therefore, a quite interesting extension of our work would be consideration of the dynamical parameters involved in Kraus operators \( V \) (A.1) and even in the propagators \( U_{exp} (U_{com}) \).
In summary, we have developed a general scheme allowing to determine statistics of cooling rate and COP for a quantum Otto refrigerator by analyzing the time evolution of the two isochores and two adiabats.

Figure 7. The ratio \( \text{Cov}(\langle \omega \rangle /\omega) \) as a function of the inverse temperature of cold reservoir \( \beta = \omega /h \) for given time durations \( y = 2 \) and \( x = 2, 20 \) at different values of \( \tau_{dri} \). The other parameters are \( \omega = 4\pi \) kHz, \( \omega_c = 7.2\pi \) kHz.
The analytical expressions of these performance parameters as well as their statistics were obtained, in which the finite time durations required for completing the two adiabatic strokes and two isochoric branches enter. When treating the machine working with qubits realized in a recent experiment, we find that stochastic COP may be negative due to quantum fluctuations, and but its average value converges and can thus be well defined. Our numerical calculations show that the effects induced by finite time durations of two isochoric and two adiabatic branches on the performance characteristics and fluctuations in the quantum Otto refrigerators. We have additionally compared the average COP and the conventional thermodynamic COP, and we found that they are correlated for nonadiabatic driving, but their correlation becomes vanishing for adiabatic branches.

Acknowledgments

This work is supported by National Science Foundation of China (Grant Nos. 11875034 and 11505091), and by the State Key Program of China under Grant No. 2017YFA0304204. We also acknowledge the financial support from the Major Program of Jiangxi Provincial Natural Science Foundation (Grant No. 20161ACB21006).

Appendix A. Time evolution of average population along an isochoric process

This system under consideration is subjected to an external field, and it is weakly coupled to a thermal bosonic bath of the Hamiltonian \( H_b = \sum \Omega_k b_k b_k^\dagger \) by the interaction Hamiltonian

\[
H_{\text{int}} = \sum_k g_k (a_k b_k^\dagger a_k^\dagger b_k b_k + a_k^\dagger b_k a_k^\dagger b_k^\dagger) + a_k b_k^\dagger b_k a_k^\dagger b_k + a_k^\dagger b_k^\dagger a_k b_k^\dagger b_k, \tag{A.1}
\]

where \( a_k \) and \( b_k \) are the respective bosonic creation and annihilation operators for the system (the heat bath), with the interaction strength \( g_k \) (bath frequency \( \Omega_k \)) at mode \( k \) \cite{57}. An isochore means that the system is kept in contact with the bath but the external field is frozen. When the system with frozen external field is put in contact with a heat reservoir, the dynamics on the hot (cold) isochore, along the isochoric process is an equilibration process of the open quantum system, though the equilibration may not be complete due to finite time duration for this process. The dynamics on the hot (cold) isochore, along which the system is coupled to a bath at inverse temperature \( \beta \), and its Hamiltonian \( H = H(\omega) \) is static, can be described (at the weak-coupling limit) by a master equation of Lindblad form \cite{2, 72}:

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}_D(\rho). \tag{A.1}
\]

On the right-hand side of this equation, the first term \(-\frac{i}{\hbar}[H, \rho]\) denotes the conservative part, and the second term can be expressed in terms of system density matrix \( \rho \): \( \mathcal{L}_D(\rho) = \sum_{\alpha} V_{\alpha} \rho V_{\alpha}^\dagger - \frac{\beta}{2}[V_{\alpha}^\dagger V_{\alpha}, \rho] \) \), with \([\_, \_]\) being the anticommutator and \( \rho \) being the density operator. This superoperator \( \mathcal{L}_D(\rho) \) denotes the system-bath interaction that is responsible for driving the system to reach thermal equilibrium. The Kraus operators \( V_{\alpha} \) are the quantum jump operators induced by the thermal interaction between the system and the heat reservoir. The system evolving under such a Lindbladian dynamics would reach a thermal equilibrium state if the time duration is sufficiently long. Conditions under which the Lindbladian dynamics \( (A.1) \) can give rise to a thermal equilibrium state have attracted much interest in the literature. Following the framework adopted in references \cite{51, 52}, the Kraus operators \( V_{\alpha} \) can be chosen as the jump operators among all different energy levels, namely,

\[
V_{mn} = v_{mn}|m\rangle\langle n|, \quad (m \neq n) \tag{A.2}
\]

where \( v_{mn} \) fulfill the detailed balance condition

\[
|v_{mn}|^2 = \epsilon_{mn} e^{-\frac{\hbar}{\beta}(E_m - E_n)} \tag{A.3}
\]

in order for the thermal equilibrium to be reached with sufficiently long time. Here \( \epsilon_{mn} = \epsilon_{nm} > 0 \) and \( \epsilon_{mn} \) depend on the interaction of the system and the heat reservoir. Let us denote \( \langle m|\rho|n \rangle = \rho_{mn} \) and \( \langle m|H|n \rangle = H_{mn} \), with \( H|n \rangle = E_n|n \rangle \). It follows that the evolutions of the diagonal components \( \rho_{mm} \) and the off-diagonal elements \( \rho_{m,n} \) take the respective forms \cite{51}:

\[
\frac{d\rho_{mm}}{dt} = -\sum_k |v_{km}|^2 \rho_{mm} + \sum_k |v_{mk}|^2 \rho_{k,k}, \tag{A.4}
\]

\[
\frac{d\rho_{mn}}{dt} = \left[ -\frac{i}{\hbar}(E_m - E_n) - \frac{1}{2} \sum_k (|v_{km}|^2 + |v_{kn}|^2) \right] \rho_{mn}, \quad (m \neq n). \tag{A.5}
\]

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The expectation of an operator $\hat{X}$ is described by $\langle \hat{X} \rangle = \text{Tr}(\hat{X} \rho)$. The change in time of an operator $\hat{X}$ for a system with Hamiltonian constant $H$ is determined according to the Heisenberg equation \[ 1, 2, 57]\:
\[
\frac{d\hat{X}}{dt} = -\frac{i}{\hbar}[H, \hat{X}] + \mathcal{L}_D(\hat{X}),
\]
where the Liouville dissipative generator can be written as $\mathcal{L}_D(\hat{X}) = \sum_n k_n \left( V_n^\dagger [\hat{X}, V_n] + [V_n^\dagger, \hat{X}] V_n \right)$.

Substituting $\hat{X} = H$ into equation (A.6) leads to
\[
\frac{d\langle H \rangle}{dt} = \frac{\partial \langle H \rangle}{\partial t} + \langle \mathcal{L}_D(H) \rangle.
\]

This reproduces the time derivative of quantum version of the first law of thermodynamics $d\langle W \rangle / dt = d\langle \dot{q} \rangle / dt$, where the instantaneous average power and the average heat current are identified as, $d\langle w \rangle / dt = (\langle H \rangle / \partial t)$ and $d\langle q \rangle / dt = (\mathcal{L}_D(H))$, respectively.

For a two-level system with $E_+ = -\hbar \omega/2$ and $E_+ = \hbar \omega/2$, the operators $V^\dagger$ and $V$ corresponding to the creation operator $\hat{a}^\dagger$ and annihilation operator $\hat{a}$. Inserting the system Hamiltonian $H = \hbar \omega \hat{a}^\dagger \hat{a}$ into equation (A.6) and taking the expectation value, the motion of the population $\langle n \rangle = \langle \hat{a}^\dagger \hat{a} \rangle$ along an isochoric process with constant $\omega$ can be obtained as
\[
\frac{d\langle n \rangle}{dt} = -\gamma (\langle n \rangle - \langle n \rangle^\text{eq}),
\]
where $\gamma = |v_+|^2 + |v_-|^2$ denotes the thermal conductivity for the two-level system, with $|v_+|^2$ and $|v_-|^2$ being defined in equation (A.2). As emphasized, the detailed balance $|v_+|^2 = |v_-|^2 e^{-\beta \hbar \omega}$ is satisfied to guarantee the existence of a unique stationary solution of a thermal equilibrium. Here $\langle n \rangle^\text{eq} = \frac{1}{2}[|v_+|^2 + |v_-|^2]$ is the asymptotic value of $\langle n(t) \rangle$ with $t \to \infty$, and it corresponds to the equilibrium population: $\langle n \rangle = -\frac{1}{\gamma} \tanh(\beta \hbar \omega/2)$. From equation (A.8), we find that instantaneous population $\langle n(t) \rangle$ along the thermalization process (staring at initial time $t = 0$) can be written in terms of the population $\langle n(0) \rangle$,

\[
\langle n(t) \rangle = \langle n \rangle^\text{eq} + [\langle n(0) \rangle - \langle n \rangle^\text{eq}] e^{-\gamma t}.
\]

### Appendix B. Relation between average populations at the beginning and the end of a unitary adiabatic process

We consider the unitary time evolution along the expansion $A \to B$ to identify the explicit relation of the populations at $A$ and $B$. The eigenenergies of a two level system are $E_+ = \frac{1}{\hbar} \omega$ and $E_- = -\frac{1}{\hbar} \omega$, respectively. The average population at instant $B$ can then be determined according to
\[
\langle n_B \rangle = \sum_{n,m} m p_{n,m}^B \rho_{n,m}^B = \sum_{n,m} m |\langle n | U_{\text{exp}} | m \rangle|^2 \rho_{n,m}^B = \rho_{n,m}^B \left( (|\langle n | U_{\text{exp}} | + \rangle|^2 - |\langle n | U_{\text{exp}} | - \rangle|^2) + \rho_{n,m}^B \left( (|\langle n | U_{\text{exp}} | + \rangle|^2 - |\langle n | U_{\text{exp}} | - \rangle|^2) \right) \right) \]

where $\xi = |\langle \pm | U_{\text{exp}} | \mp \rangle|^2$ and $\langle n \rangle_A = \sum_{n=\pm} n \rho_{n,m}^A$ with $n = \pm 1/2$ have been used. As the mean population for the two-level system, $\langle n \rangle = \sum_{n=\pm} \frac{1}{2} \rho_{n,m}$, must be negative, $\xi$ must be upper limited by $1/2$. Note that, when the hot thermalization of the quantum Otto cycle is complete, the occupation probabilities at instant $A$ (which is also the ending instant of hot isochore $D \to A$) can be canonical form:
\[
\rho_{n,m}^A \big|_{\tau_{\text{hot}}} = \rho_{n,m}^\text{eq} \equiv \frac{1}{Z_{\text{eq}}} e^{-\beta \hbar \omega n},
\]
where the partition function $Z_{\text{eq}} = \sum_{n=\pm} e^{-\beta \hbar \omega n} = 2 \cosh \left( \frac{\beta \hbar \omega}{2} \right)$. It then follows, using equation (B.2) and $\langle n \rangle = \sum_{n=\pm} \frac{1}{2} \rho_{n,m}^A$, that the mean population of the system at instant $A$ in the thermal equilibrium with the hot bath reads
\[
\langle n_A \rangle \big|_{\tau_{\text{hot}}} = \langle n \rangle^\text{eq} = -\frac{1}{2} \tanh \left( \frac{\beta \hbar \omega}{2} \right).
\]

Similarly, for the unitary compression $C \to D$, we can obtain by using $\xi = |\langle \pm | U_{\text{com}} | \mp \rangle|^2$,
\[
\langle n_D \rangle = (1 - 2\xi) \langle n_C \rangle.
\]
If the cold isochore $B \rightarrow C$ is complete thermalization due to $\tau_c \gg \tau_{c, \text{relax}}$, the mean population of the system at instant $C$ becomes

$$\langle n_c \rangle_{\tau_c \gg \tau_{c, \text{relax}}} = \langle n_c \rangle_{\text{eq}} = -\frac{1}{2} \tanh \left( \frac{\beta \hbar \omega_c}{2} \right).$$

**Appendix C. Physical meaning of zero-over-zero COP**

In deriving equation (21) of the main text, we have identified the expression of the form 0/0 (which corresponds to no heat injection and no work input) as the adiabatic thermodynamic COP $\varepsilon_{\text{ad}}^0$, following the approach adopted in the heat engine [30]. Physically, this can be understood by identifying the term $0/0$ in the delta function as the adiabatic thermodynamic COP, though the term 0/0 is ill-defined mathematically. Without loss of generality, we consider the machine working with the two-level system as adopted in the main text, by assuming $\xi = 0$. In such a case, the efficiency distribution in equation (7) of the main text becomes

$$p(\varepsilon) = 2 \left[ \frac{1}{4} + \left( \langle n_h \rangle_{\text{eq}} + \Delta_h \right) \left( \langle n_c \rangle_{\text{eq}} + \Delta_c \right) \right] \delta \left( \varepsilon - 0 \right)$$

$$+ 2 \left[ \frac{1}{4} - \left( \langle n_h \rangle_{\text{eq}} + \Delta_h \right) \left( \langle n_c \rangle_{\text{eq}} + \Delta_c \right) \right] \delta \left( \varepsilon - \varepsilon_{\text{th}}^{\text{ad}} \right).$$

This distribution has two peaks at the two values 0/0 and $\varepsilon_{\text{th}}^{\text{ad}}$, respectively, and thus the COP is a random quantity. To proceed with analysis on the physical meaning of the term 0/0, we now analyze the correlation between the work input and heat injection per cycle by considering the Pearson correlation coefficient [73],

$$\theta_{w,q_c} = \frac{\text{Cov}(w, q_c)}{\delta q_c \delta w},$$

where $\text{Cov}(w, q_c) = \langle w q_c \rangle - \langle w \rangle \langle q_c \rangle$, and $\delta w (\delta q_c)$ is standard deviation of work input (absorbed heat). The value of the coefficient $\theta_{w,q_c}$ ranges from $-1$ to $1$. While the value $1$ indicates maximal correlation between these two quantities, $-1$ corresponds to maximal anticorrelation. The joint probability distribution for certain values of total work input $w$ and heat injection $q_c$ can be determined according to

$$p(w, q_c) = \int dw_{ch} dw_{hc} \delta[w - (w_{ch} + w_{hc})] p(w_{hc}, q_c, w_{ch}),$$

where $p(w_{hc}, q_c, w_{ch})$ was given by equation (6) of the main text, leading to

$$p(w, q_c) = \left[ \frac{1}{2} - \left( \langle n_c \rangle_{\text{eq}} + \Delta_c \right) \right] \left[ \frac{1}{2} + \left( \langle n_h \rangle_{\text{eq}} + \Delta_h \right) \right] \delta(q_c + \hbar \omega_c) \delta[w + \hbar(\omega_h - \omega_c)]$$

$$+ \left[ \frac{1}{2} + \left( \langle n_h \rangle_{\text{eq}} + \Delta_h \right) \right] \left[ \frac{1}{2} - \left( \langle n_c \rangle_{\text{eq}} + \Delta_c \right) \right] \delta(q_c - \hbar \omega_c) \delta[w - \hbar(\omega_h - \omega_c)]$$

$$+ \left[ \frac{1}{2} + 2 \left( \langle n_c \rangle_{\text{eq}} + \Delta_c \right) \right] \left( \langle n_h \rangle_{\text{eq}} + \Delta_h \right) \delta(q_c) \delta(w).$$

The probability distribution of total work input, $p(w) = \int dq_c p(w, q_c)$, can be obtained as

$$p(w) = \left[ \frac{1}{2} - \left( \langle n_c \rangle_{\text{eq}} + \Delta_c \right) \right] \left[ \frac{1}{2} + \left( \langle n_h \rangle_{\text{eq}} + \Delta_h \right) \right] \delta[w + \hbar(\omega_h - \omega_c)]$$

$$+ \left[ \frac{1}{2} + \left( \langle n_h \rangle_{\text{eq}} + \Delta_h \right) \right] \left[ \frac{1}{2} - \left( \langle n_c \rangle_{\text{eq}} + \Delta_c \right) \right] \delta[w - \hbar(\omega_h - \omega_c)]$$

$$+ \left[ \frac{1}{2} + 2 \left( \langle n_c \rangle_{\text{eq}} + \Delta_c \right) \right] \left( \langle n_h \rangle_{\text{eq}} + \Delta_h \right) \delta(w).$$

Using $\langle w_q \rangle = \int p(w, q_c) w q_c dw dq_c$, and $\langle w^n \rangle = \int w^n p(w) dw (n = 1, 2)$, with consideration of $\langle w \rangle \geq 0$ and $\langle q_c \rangle \geq 0$, we find that the Pearson coefficient turns to be $\theta_{w,q_c} = -1$, and that the total work input and heat injection are perfectly anticorrelated for quantum adiabatic driving. In such a case, the COP is a fixed, deterministic value.

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