On the existence of good divisors on Fano varieties of coindex 3

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Introduction

A normal projective variety $X$ is called Fano if some multiplicity $-nK_X$, $n \in \mathbb{N}$ of anticanonical (Weil) divisor $-K_X$ is an ample Cartier divisor. The number $r(X) := \sup\{t \in \mathbb{Q}|-K_X \equiv tH, \ H \text{ is an ample Cartier divisor}\}$ is called the index of a Fano variety $X$. A Fano variety with only log-terminal singularities (see [7]) we call briefly log-Fano variety, and a Fano variety with only $\mathbb{Q}$-factorial terminal singularities and Picard number $\rho = 1 - \mathbb{Q}$-Fano variety. If $X$ is a log-Fano variety, then $\text{Pic}(X)$ is a finitely generated torsion-free group. Therefore $r(X) \in \mathbb{Q}$, $r(X) > 0$. In that case there exists the ample Cartier divisor $H$, called a fundamental divisor of $X$, such that $-K_X \equiv r(X)H$. It is known that $0 < r(X) \leq \dim(X) + 1$ for any log-Fano variety $X$ (see e.g. [18]).

We say that there exists a good divisor on $X$ if the fundamental linear system $|H|$ is non-empty and contains a reduced irreducible divisor with singularities at worst the same as singularities of the variety $X$ (e.g. if $X$ is non-singular, or has terminal, canonical or log-terminal singularities, then $H$ is non-singular, or has terminal, canonical or log-terminal singularities, respectively).

For the first time the existence of good divisors was proved in three-dimensional non-singular case by Shokurov [17]. In his preprint [11] Reid used Kawamata’s technique for study of linear system $|H|$ and proved the existence of good divisors for Fano threefolds with canonical singularities. Later this technique was applied for Fano fourfolds of index 2 with Picard number 1 by Wilson [14][1] and for log-Fano varieties $X$ of indices $r > \dim(X) - 2$ by Alexeev [1].

In the present paper we study the case $r = \dim(X) - 2$. Mukai classified such non-singular Fano varieties of any dimension under the assumption of the existence of good divisors [9], [10] (see also [15]). We investigate five-dimensional case and prove the following result which is slightly weaker then Mukai’s classification claims [2].

**Theorem 1.** Let $X$ be a non-singular Fano fivefold of index 3 and let $H$ be a fundamental divisor on $X$. Then the linear system $|H|$ contains an irreducible divisor with only canonical Gorenstein singularities.

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1For any Picard number see [Prokhorov Yu. G. The existence of smooth divisors on Fano fourfolds of index 2, Russian Acad. Sci. Sb. Math. 83 (1995) no. 1, 119–131]. (Added in translation).

2In the paper [Prokhorov Yu. G. On the existence of good divisors on Fano varieties of coindex 3, II Contemporary Math. and its Appl. Plenum. 24 (1995) (to appear)] the author generalized this result and proved that $H$ is non-singular for $\dim X = 5$ and non-singular $X$. (Added in translation).
The main idea of the proof of the Theorem is to investigate the singular locus \( \text{Sing}(H) \) of a general divisor \( H \). Here the "bad" case is the case when \( \dim \text{Sing}(H) = 3 \). In this situation we study a three-dimensional component of \( \text{Sing}(H) \), applying the minimal model program. In fact using similar arguments one can prove more general result:

**Theorem 2.** Let \( X \) be a \( n \)-dimensional log-Fano variety of index \( n-2 \) and let \( H \) be a fundamental divisor on \( X \). Assume that in dimension \( n-2 \) flip-conjectures I and II hold (see [7]). Then if the linear system \( |H| \) is not empty and has no fixed components, then it contains an irreducible divisor with only log-terminal singularities.

**Remark.** If in notations of Theorem 0.2 \( X \) has only canonical Gorenstein singularities, then \( |H| \neq \emptyset \) (see 1.1).

**Notations and conventions:**

\( \text{Bs}\,|H| \): the scheme-theoretic base locus of a linear system \( |H| \);

\( \text{Sing}(X) \): the singular locus of a variety \( X \);

\( \equiv \): numerical equivalence of cycles;

The ground field is assumed to be the field of complex numbers \( \mathbb{C} \). We will use the basic definitions and concepts of the minimal model program (see [7]).

### 1. Preliminary results

From Kawamata-Viehweg vanishing theorem for varieties with log-terminal singularities (see [7]) we get

**1.1. Lemma [1].** Let \( X \) be a \( n \)-dimensional log-Fano variety of index \( r \), \( H \) be a fundamental divisor on \( X \) and \( H^n = d \). Then

(i) If \( r > n-2 \), then \( \dim |H| = n-2 + d(r-n+3)/2 > 0 \). Moreover \( r = n-3+2k/d \) for some \( k \in \mathbb{N} \), \( k > d/2 \);

(ii) If \( r = n-2 \) and \( X \) has only canonical Gorenstein singularities, then \( \dim |H| = g+n-2 > 0 \), where \( 2g-2 = d \), \( g \in \mathbb{Z} \), \( g \geq 2 \).

In [2] Fujita defined \( \Delta \)-genus of a polarized \( n \)-dimensional variety \((X,H)\) as \( \Delta(X,H) = H^n - h^0(X,\mathcal{O}_X(H)) + n \).

**1.2. Corollary.** In notations (i) of Lemma 1.3 we have \( \Delta(X,H) = d - k + 1 \).

**1.3. Theorem [2].** \( \Delta(X,H) \geq \dim(\text{Bs}|H|) + 1 \geq 0 \) for any polarized variety \((X,H)\). Moreover if \( \Delta(X,H) = 0 \), then the divisor \( H \) is very ample.

The following is a consequence from the classification of polarized varieties of \( \Delta \)-genus zero [2]

**1.4. Corollary (Fujita).** Let \( X \) be a \( \mathbb{Q} \)-Fano threefold of index \( r > 2 \). Then either

\( X = \mathbb{P}^3 \),

\( X = Q \subset \mathbb{P}^4 \) is a smooth quadric, or
\(X = X_4 \subset \mathbb{P}^6\) is a projective cone over the Veronese surface.

1.5. **Theorem** [4], [12]. Let \(X\) be a three-dimensional Fano variety of index 2 with only canonical Gorenstein singularities, let \(E\) be a fundamental divisor on \(X\) and \(d := H^3\). Then

(i) if \(d \geq 3\), then \(H\) is very ample,

(ii) if \(d = 2\), then the linear system \(|H|\) defines a finite morphism \(X \to \mathbb{P}^3\) of degree 2.

2. Kawamata’s technique

We describe briefly the technique of resolution of base loci of linear systems on Fano varieties in connection with our situation (see [11], and also [1],[14]).

Let \(X\) be a non-singular Fano fivefold of index 3 and let \(H\) be a fundamental divisor on \(X\). By Lemma 1.1, \(|H| \neq \emptyset\). Then there exist a resolution \(f : Y \to X\) and a divisor \(\sum E_i\) on \(Y\) with only simple normal crossings such that

1) \(K_Y \equiv f^*K_X + \sum a_i E_i, \quad a_i \in \mathbb{Z}, \quad a_i \geq 0\),

where \(a_i \neq 0\) only if \(f_\ast E_i = 0\);

2) \(f^*|H| = |L| + \sum r_i E_i, \quad r_i \in \mathbb{Z}, \quad r_i \geq 0\),

where the linear system \(|L|\) is free;

3) \(\mathbb{Q}\)-divisor \(q f^*H - \sum p_i E_i\) is ample for some \(0 < p_i \ll 1, 0 < q < \min\{1/r_i | r_i \neq 0\}\).

Set \(c = \min\{(a_i + 1-p_i)/r_i | r_i \neq 0\}\). By changing coefficients \(p_i\) a little one can attain that the minimum will achieve for only one index, say for \(i = 0\). By Kleiman’s criterion for ampleness, the following \(\mathbb{Q}\)-divisor

\[N = N(t) = tf^*H + \sum (-cr_i + a_i - p_i) E_i - K_Y \equiv \]

\[= cL + f^*(t-c+3)H - \sum p_i E_i, \quad t \in \mathbb{Z}\]

is ample for \(t-c+3 \geq q > 0\). Since \(-cr_0 + a_0 - p_0 = -1\) and \(-cr_i + a_i - p_i > -1\) for \(i \neq 0\), for the rounding-up of \(N\) we have \(N = tfH - K + A - E\), where \(E = E_0, A \geq 0\) and \(A\) consists of exceptional for \(f\) components of \(\sum E_i\).

2.1. **Lemma** [1],[11]. If \(4 \geq c + q\), then \(H^0(E, \mathcal{O}_E(f^*H + A)) = 0\). In particular, \(H^0(E, \mathcal{O}_E(f^*H)) = 0\).

**Proof.** By the Kawamata-Viehweg vanishing theorem (see [6],[13]) the following sequence

\[0 \to H^0(Y, \mathcal{O}_Y(f^*H + A - E)) \to H^0(Y, \mathcal{O}_Y(f^*H + A)) \to H^0(E, \mathcal{O}_E(f^*H + A)) \to 0\]

is exact. If \(H^0(E, \mathcal{O}_E(f^*H + A)) \neq 0\), then we get a contradiction with the fact that \(E\) is a fixed component of the linear system \(|f^*H + A|\).

2.2. **Lemma** [1],[11]. For constants \(a_i\) and \(r_i\) in formulas (2.1) and (2.2) the inequality \(a_i + 1 \geq r_i\) holds for any \(i\).

2.3. **Lemma.** Assume that \(a_i + 1 = r_i\) for some \(i\). Then then there are the following possibilities for the divisor \(E = E_0\):
(i) $a_0 = 0, r_0 = 1, \dim(f(E)) = 4$, and $f(E)$ is a fixed component of $|H|$ of multiplicity 1;

(ii) $a_0 = 1, r_0 = 2, \dim(f(E)) = 3$, and a general divisor $|H|$ has only double singularities along $f(E)$.

Proof. Since $a_i + 1 = r_i$, we have $c \leq 1$. Set $d := \dim(f(E))$. Consider the following polynomial of degree $\leq d$: $p(t) = \chi(O_E(tf^*H + A))$. For $t > q - 2$ $\mathbb{Q}$-divisor $N(t)$ is ample and by the Kawamata-Viehweg vanishing theorem (see [6],[13]), $p(t) = h^0(E, O_E(tf^*H + A))$, i.e. the polynomial $p(t)$ has zero for $t = -1$, and also, by Lemma 2.1, one more zero for $t = 1$. On the other hand, $p(0) = h^0(E, O_E(A)) = 1$. Hence, if $d \leq 2$, then $p(t) \leq 0$ for $t \gg 0$, a contradiction with ampleness of $H$. Therefore, $d = 4$ and $d = 3$.

2.4. Lemma. Let $E_k$ be a component of divisor $\sum E_i$ such that $\dim f(E_k) = 3$ and $x \in f(E_k)$ be a general point. Then a general divisor $H$ is non-singular or has only double normal crossings at $x$.

Proof. A general surface $X' = D_1 \cap D_2 \cap D_3$, where $D_i \in |mH|$, $m \gg 0$ is non-singular. Moreover for the corresponding resolution

$Y' \twoheadrightarrow Y$
$\downarrow f' \quad \downarrow f$
$X' \twoheadrightarrow X$

we have

$K_Y = f^*K_{X'} + \sum a_i E'_{ij}$,
$f^*|H'| = |L'| + \sum r_i E'_{ij}$

where $E'_{ij}$ is a component of $Y' \cap E_i$. Assume that a general divisor $H$ has worse singularity at $x$ than double normal crossings. Then the curve $H'$ has worse singularity at the point $x$ than ordinary double. If $x \in H'$ is not simple cusp, then for the "first" blow-up at $x$ we have $a_1 = 1, r_1 \geq 2$, and for the "second" blow-up $a_2 = 2, r_2 \geq 4$, a contradiction with Lemma 2.2. But if $x \in H'$ is a simple cusp, then similar we have $a_1 = 1, r_1 = 2, a_2 = 2, r_2 = 3, a_3 = 4, r_4 = 6$, again it contradicts to Lemma 2.2. Thus $H'$ has at $x$ only ordinary double point. This proves our lemma.

3. Some corollaries from the minimal model program

3.1. Theorem [8]. Let $S$ be a projective three-dimensional variety. Then there exist birational morphisms $\mu : W' \to S, \nu : W \to W'$ and $\tau = \mu \circ \nu : W \to S$ with the following properties:

(i) $W$ has only terminal $\mathbb{Q}$-factorial singularities, $W'$ has only canonical singularities;

(ii) $K_W$ is $\tau$-numerically effective, $K_{W'}$ is $\mu$-ample;

(iii) morphism $\nu$ is crepant, i.e. $K_W = \nu^*K_{W'}$;

(iv) if $\alpha : V \to S$ is any birational morphism, $V$ has only terminal $\mathbb{Q}$-factorial singularities, then the map $\tau^{-1} \circ \alpha : V \to W$ is a composition of contractions of extremal rays and flips.

\footnote{i.e. $x \in H$ is analytically isomorphic to $(0 \in \{x_1x_2 = 0\}) \subset (0 \in \mathbb{C}^3)$. (Added in translation).}
Definition. Varieties $W$ and $W'$ from Theorem 3.1 are called by terminal and canonical modifications of $S$, respectively.

3.2. Theorem [3, §3-4]. Let $W$ be a three-dimensional projective variety with only $\mathbb{Q}$-factorial terminal singularities and $N$ be a numerically effective big Cartier divisor on $W$. Assume that there exists an extremal ray on $W$ is generated by a class of a curve $\ell$ such that $(K_W + 2N) \cdot \ell < 0$ and let $\varphi : W \to C$ be its contraction. Then one of the following holds:

(i) $\varphi : W \to C$ is a birational morphism and $\ell \cdot N = 0$;

(ii) $W$ and $C$ are smooth, $\dim C = 1$, $W = \mathbb{P}_C(\mathcal{E})$ and $N = \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$, where $\mathcal{E}$ is a locally free sheaf of rank 3 on $C$;

(iii) $C$ is a point and $W$ is a $\mathbb{Q}$-Fano variety of index $r > 2$.

3.3. Proposition. Let $\alpha : \hat{S} \to S$ be a birational morphism of three-dimensional normal projective varieties, where $\hat{S}$ has only $\mathbb{Q}$-factorial terminal singularities, and let $M$ be an ample Cartier divisor on $S$. Assume that $K_\hat{S} = -2\hat{M} - \hat{R} + \hat{D}$, where $\hat{M} = \alpha^*M$, $\hat{R}$ is an effective $\mathbb{Q}$-divisor, $\hat{D}$ is a $\mathbb{Q}$-divisor such that all the components of $\hat{D}$ are contracted by the morphism $\alpha$. Then for terminal and canonical modifications $\tau : W \to S$ and $\mu : W' \to S$, and also for Cartier divisors $N = \tau^*M$ and $N' = \mu^*M$ we have only one of the following possibilities:

(i) $W = \mathbb{P}_C(\mathcal{E})$, $N = \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$, where $C$ is a smooth curve, $\mathcal{E}$ is a locally free sheaf on $C$ rank 3;

(ii) $S = W = W' = \mathbb{P}^3$, $M = N = N' = \mathcal{O}_{\mathbb{P}^3}(1)$;

(iii) $S = W = W' = Q \subset \mathbb{P}^4$ is a non-singular quadric, $M = N = N' = \mathcal{O}_Q(1)$;

(iv) $S = W = W' = W_4 \subset \mathbb{P}^6$ is a cone over the Veronese surface, $M = N = N' = \mathcal{O}_W(1)$;

(v) $S = W'$ is a Fano variety of index 2 with only canonical Gornstein singularities, $-K_{W'} = 2N' = 2M$, $W$ has only isolated cDV-points, $-K_W = 2N$ is a numerically effective big divisor.

(vi) $S = W = W' = \mathbb{P}^3$, $M = N = N' = \mathcal{O}_{\mathbb{P}^3}(2)$.

Proof. By Theorem 3.1, the birational map $\beta := \tau^{-1} \circ \alpha : \hat{S} \dashrightarrow W$ is a composition of contractions of extremal rays and flips. In particular the inverse map $W \dashrightarrow \hat{S}$ doesn’t contract divisors. Therefore

$$K_W = -2N - B + G,$$

where $N := \tau^*M$, $B$ is an effective $\mathbb{Q}$-divisor on $W$, and $G$ is a $\mathbb{Q}$-divisor such that all its components are contracted by the morphism $\tau$. From (3.1) we have

$$K_{W'} = -2N' - B' + G',$$

where $N' := \tau^*M$, $B' = v_*B$ is an effective $\mathbb{Q}$-Weil divisor on $W$ and $G' = v_*G$ is $\mathbb{Q}$-Weil divisor such that all the components of $G$ are contracted by the morphism $\mu$. First we
assume that \( B = G = 0 \). Then \( B' = G' = 0 \) and \(-K_W' = 2N'\). If \( c \) is a curve in a fiber of \( \mu \), then \( K_W \cdot c = -2N' \cdot c = 0 \), this contradicts to \( \mu \)-ampleness of \( K_W' \). So \( \mu \) is a finite birational morphism on the normal variety \( S \). Hence \( \mu \) is an isomorphism and \(-K_W = 2N\) is an ample divisor. We obtain cases (v), (vi). Now let \( B \neq 0 \) or \( G \neq 0 \). We claim that \( K_W + 2N \) is not numerically effective. Indeed, in the opposite case \(-B + G = K_W + 2N\) is numerically effective and for any irreducible curve on \( S \) we have 
\[
0 \leq (-B + G) \cdot \tau^*c = -B \cdot \tau^*c.
\]
Thus \( \tau^*B = 0 \), i.e. we may assume that \( B = 0 \). But then by the Base Point Free Theorem (see e. g. [18],[7]), the linear system \([kG]\) is free for \( k \gg 0 \), it contradicts to contractedness of the divisor \( G \). Further by the Cone Theorem (see [7]), there exists an extremal ray on \( W \) generated by a class of a curve \( \ell \) such that \((K_W + 2N) \cdot \ell < 0\). Let \( \varphi : W \to C \) be the contraction of this extremal ray. Note that if \( \ell \cdot N = 0 \), then \( \ell \) is contained in a fiber of morphism \( \tau \). It contradicts \( \tau \)-numerical effectiveness of the divisor \( K_W \). Using Theorem 3.2 and Corollary 1.7, we obtain cases (i)-(iv).

3.4. Lemma. In the case (i) of Proposition 3.3 the curve \( C \) can be rational or elliptic.

Proof. Denote by \( F \) the class of a fiber of the projection \( \varphi : \mathbb{P}(\mathcal{E}) \to C \). We have the standard formula
\[
K_W = -3N + \varphi(\det(\mathcal{E}) + K_C).
\]
Set \( d := \deg \mathcal{E} \). Then 
\[
K_W \equiv -3N + (2g(C) - 2 + d)F.
\]
(3.3)
Comparing (3.1) and (3.3) we obtain 
\[
B \equiv N \cdot (2g(C) - 2 + d)F + G
\]
(3.4)
We may assume that the morphism \( \tau \) does not contract components of \( B \). In the Chow ring \( A(W) \) the following conditions are satisfied
\[
N^3 = d, \quad N^2 \cdot F = 1, \quad F^2 = 0
\]
(3.5)
Moreover \( d = N^3 = M^3 \geq 1 \) because \( N = \tau^*M \).

Assume that an irreducible divisor \( P \) is contracted by the morphism \( \tau \). Then \( P \equiv aN + bF \), \( a, b \in \mathbb{Z} \), and from (3.5) one can see \( 0 = N^2 \cdot P = da + b \) and \( 0 \leq N \cdot F \cdot P = da \), i.e. \( P \equiv a(N - dF) \), \( a \in \mathbb{N} \). If \( \tau \) contracts one more irreducible divisor \( P' \), then \( P' \equiv a'(N - dF) \), \( a' \in \mathbb{N} \) and \( 0 \leq P \cdot P' \cdot N = -aa'd < 0 \). The contradiction shows that \( \tau \) may contracts at most one irreducible divisor on \( W \), i.e. \( G = 0 \) or \( G = pP \), where \( p \in \mathbb{Q} \) and \( P \equiv a(N - dF) \) is an irreducible exceptional divisor. First assume that \( B = 0 \). Then from (3.4) and (3.5) we have 
\[
N \equiv (2g(C) - 2 + d)F - G, \quad d = N^3 = (2g(C) - 2 + d)N^2, \quad F = 2g(C) - 2 + d, \quad \text{i.e.} \quad g(C) = 1.
\]
But if \( B \neq 0 \), then \( 0 < N^2 \cdot B = 2 - 2g(C) \), i.e. \( g(C) = 0 \).

3.5. Corollary. In conditions of Proposition 3.3 we have \( H^0(S, \mathcal{O}_S(M)) \neq 0 \).

Proof. Since \( S \) is normal, then it is sufficient to prove only that \( H^0(W, \mathcal{O}_W(N)) \neq 0 \) or \( H(W', \mathcal{O}_{W'}(N')) \neq 0 \). The last two inequalities are easy in cases (ii), (iii), (iv) and (vi), and in the case (v) they follow from Lemma 1.1. Consider the case (i). Then \( H^0(W, \mathcal{O}(N)) = H^0(C, \mathcal{E}) \) and by Riemann-Roch on \( C \) we have \( h^0(C, \mathcal{E}) = 3(1 - g) + d + h^1(C, \mathcal{E}) \geq 1 \).
4. Proof of Theorem 1

Let $X$ be a non-singular Fano fivefold of index 3 and let $H$ be a fundamental divisor on $X$. Then by Lemma 1.1, $\dim |H| \geq 6$. It follows from results of [16] that $\text{Pic}(X) \simeq \mathbb{Z} \cdot H$, except the following three cases: $X = \mathbb{P}^2 \times \mathbb{Q}^3$, $X$ is a divisor of bidegree $(1,1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$ or $X$ is the blow-up of $\mathbb{P}^5$ along $\mathbb{P}^1$. In all of these cases it is easy to check directly that the linear system $|H|$ contains a smooth divisor. Thus we will suppose that $\text{Pic}(X) \simeq \mathbb{Z} \cdot H$. In particular we suppose that the linear system $|H|$ has no fixed components. We will use all the notations of Section 2. The following proposition is the main step in the proof of Theorem 1.

4.1. Proposition. $r_i < a_i + 1, \forall i$.

Proof. According to Lemma 2.2, $r_i \leq a_i + 1$. Assume that $r_i = a_i + 1$ for some $i$. Then $c := \min\{(a_i + 1 - p_i)/r_i | r_i \neq 0\} \leq 1$. Set $Z := f(E)$ (with reduced structure). By Lemma 2.3 a general divisor $H$ has only double normal crossings singularities in a general point $x \in Z$. We may suppose that the resolution $f : Y \to X$ is a composition $h : Y \to \tilde{X}$ and $g : \tilde{X} \to X$, where $g : \tilde{X} \to X$ is a resolution of singularities of a general divisor $H$ (i.e. we fix some general divisor $H$ and suppose that $g$ is a composition of blow-ups with centers in subvarieties contained in singular locus of $H$). We have

$$
K_{\tilde{X}} = g^* K_X + \sum a'_i E'_i, \quad (4.1)
$$

$$
g^* H = \tilde{H} + \sum r'_i E'_i, \quad r_i \geq 2, \quad (4.2)
$$

(here $\tilde{H}$ is a proper transform of $H$, $\sum E'_i$ is an exceptional divisor). For corresponding $E_i, E'_i, r_i, r'_i$ and $a_i, a'_i$ the following conditions are satisfied:

$$
h(E_i) = E'_i, \quad r_i \leq r'_i, \quad a_i = a'_i.
$$

Denote by $E'$ the component of the divisor $\sum E'_i$, corresponding $E$. According to Lemma 2.4, there are two possibilities:

$$
(I) \quad \tilde{H} \cap E' = \tilde{Z} + \tilde{Z}_1 + \tilde{Z}_0, \quad (4.3)
$$

where restrictions $g : \tilde{Z} \to Z$ and $g : \tilde{Z}_1 \to Z$ are birational, $g_* \tilde{Z}_0 = 0$;

$$
(II) \quad \tilde{H} \cap E' = \tilde{Z} + \tilde{Z}_0, \quad (4.4)
$$

where the restriction $g : \tilde{Z} \to Z$ is generically finite of degree 2, $g_* \tilde{Z}_0 = 0$.

We study the variety $\tilde{Z}$ and corresponding morphism $\psi := g|_{\tilde{Z}} : \tilde{Z} \to Z$. Denote by $T$ the Cartier divisor $H|_Z$ on $Z$.

4.2. Lemma. $K_{\tilde{Z}} = -2\tilde{T} - \tilde{B} + \tilde{G}$, where $\tilde{T}, \tilde{B}, \tilde{G}$ Cartier divisors on $\tilde{Z}$, $\tilde{G}$ is contracted by the morphism $\psi$, $\tilde{B}$ is effective and $\tilde{T} := \psi^* T$.

Proof. Consider for example the case I. The variety $\tilde{Z}$ is Gorenstein because it is divisor on a non-singular variety. By the adjunction formula we have

$$
K_{\tilde{Z}} = (K_{\tilde{H}} + \tilde{Z})|_{\tilde{Z}} = (K_{\tilde{X}}|_{\tilde{H}} + \tilde{H}|_{\tilde{H}} + \tilde{Z})|_{\tilde{Z}}
$$

$$
= -2\tilde{T} + \left( \sum (a'_i + 1 - r'_i) E'_i|_{\tilde{H}} + E'|_{\tilde{H}} - \tilde{Z}_1 - \tilde{Z}_0 \right)|_{\tilde{Z}} =
$$
(because \(a'_0 = 1, r'_0 = 2\)). It follows from 2.6 that \(\tilde{Z}\) is not contained in \(E'_i\) for \(i \neq 0\). It is sufficient to prove that in the last formula \(a'_i \leq r'_i\), if the corresponding divisor \(E'_i|\tilde{Z}\) is not contracted by the morphism \(\psi\). If \(\dim(g(E'_i)) \leq 1\), then for every component \(F'\) of the divisor \(E'_i|\tilde{Z}\) we have \(\dim(\psi(F')) \leq \dim(g(E'_i)) \leq 1 < \dim(F')\), i.e. \(F'\) is contracted by the morphism \(\psi\) is this case. But if \(\dim(g(E'_i)) = 3\), then by Lemma 2.4, \(r'_i \geq a'_i\). It remains to prove only that the case \(\dim(g(E'_i)) = 2\) and \(a'_i > r'_i\) is impossible. Suppose that \(a'_i > r'_i\) for some \(i\) and consider the following partial ordering of the set \(\{E'_j\}\): \(E'_j \succ E'_k\) (or simply \(j \succ k\), if and only if the center of the \(k\)-th blow-up is contained in the \(j\)-th exceptional divisor. Then the divisor \(E'_i\) cannot be maximal, otherwise \(a'_i = 2, r'_i \geq 2\), that contradicts our assumption. Let \(\sigma : X_i \to X_{i-1}\) be the blow-up of a smooth surface corresponding to the divisor \(E'_i\). Then \(K_{X_i} = \sigma^*_i K_{X_{i-1}} + 2E_i, \sigma^*_i H_{i-1} = H_i + mE_i\), \(m \geq 2\), where \(E_i\) is the exceptional divisor of the blow-up \(\sigma_i\) (its proper transform on \(\tilde{X}\) is the divisor \(E'_i\)), \(H_{i-1}\) and \(H_i\) are proper transforms of \(H\) on \(X_{i-1}\) and \(X_i\), respectively. We obtain

\[
a'_i = 2 + \sum_{j \succ i} a'_j, \quad r'_i = m + \sum_{j \succ i} r'_j, \quad m \geq 2.
\]

Hence \(\sum_{j \succ i} a'_j > \sum_{j \succ i} r'_j\) (we assume, that \(a'_j > r'_j\)). Therefore there exists \(j\) such that \(j \succ i, \dim(g(E'_j)) \geq 2\) and \(a'_j > r'_j\). Thus we may obtain the maximal element \(E'_j\). According the above for that maximal element we have \(\dim(g(E'_j)) \neq 2\), i.e. \(\dim(g(E'_j)) = 3\), but then by Lemma 2.4 \(a'_j = 1, r'_j \geq 2\) and the inequality \(a'_j > r'_j\) is impossible. Lemma is proved.

4.3. Lemma. There exists a resolution \(\sigma : \mathcal{Z} \to \tilde{Z}\) such that

\[
K_{\mathcal{Z}} = -2\hat{T} - \hat{B} + \hat{G},
\]

where \(\hat{T}, \hat{B}, \hat{G}\) are \(\mathbb{Q}\)-divisors on \(\mathcal{Z}\), \(\hat{G}\) is contracted by the morphism \(\hat{\psi} := \psi \circ \sigma\), \(\hat{B}\) is effective and \(\hat{T} = \hat{\psi}^*\hat{T}\).

Proof. It is sufficient to prove that formulas (4.5), (4.6) are preserved under one blow-up \(\sigma_1 : \hat{H} \to \hat{H}\) of non-singular \(k\)-dimensional subvariety in \(\hat{Z}\) \((k \leq 2)\). Let \(\hat{E}\) be an exceptional divisor of \(\sigma_1\) and \(\hat{Z}\) be the proper transform of \(\hat{Z}\). Then \(K_{\hat{H}} = \sigma_1^* K_{\hat{H}} + (3-k)\hat{E}\), \(\hat{Z} \sim \sigma_1^* \hat{Z} - p\hat{E}\), \(p \geq 2, p \in \mathbb{Z}\) (remind that we assumed that \(\hat{Z}\) is singular along our suvartiy). Thus by the adjunction formula \(K_{\mathcal{Z}} = \sigma_1^* K_{\mathcal{Z}} + (3 - k - p)\hat{E}|_{\mathcal{Z}}\), where \(\hat{E}\) is contracted by the morphism \(\sigma_1\) (the case \(k \leq 1\)) or \(k = 2, 3 - k - p = 1 - p < 0\), i.e. formulas (4.5), (4.6) are preserved.

Now applying Proposition 3.3 we obtain the diagram below.

\[
\begin{array}{ccc}
\hat{Z} & \xrightarrow{\beta} & W' \\
\downarrow \alpha & & \downarrow \nu \\
\hat{S} & \xrightarrow{\pi} & S' \\
\downarrow \tau & & \downarrow \nu \\
\hat{Z} & & \end{array}
\]

(4.8)
where \( \nu : Z' \to Z \) is the normalization of \( Z \), \( \pi : S \to Z' \) is the normalization of \( Z' \) in the function field of \( Z \). In the case I \( \pi \) is an isomorphism. In the case II \( \pi \) is a finite morphism of degree 2, and \( Z' \simeq S/\Gamma \), where \( \Gamma \) is the group of order 2. The other notations we fix the same as in Theorem 3.1. For \( W \) and \( W' \) we have only possibilities (i)-(vi) from Proposition 3.4. Set \( M := \pi^*T' \).

The divisor \( E \) is a proper transform of the exceptional divisor of some blow-up of a non-singular model \( Z'' \) of the variety \( Z \) on \( X'' \). Let \( Z'' \to Z \) be a corresponding birational morphism. Clearly that it factors through the normalization \( Z'' \nu'\to Z' \nu \to Z \).

If \( r_i \geq a_i + 1 \) for some \( i \), then by Lemma 2.1 \( H^0(E, \mathcal{O}_E(f^*H)) = 0 \). Therefore to prove Proposition 4.1 it is sufficient to show that

\[
H^0(Z'', \nu'^*\mathcal{O}_{Z'}(T')) \neq 0
\]

or

\[
H^0(Z', \mathcal{O}_{Z'}(T')) \neq 0, \tag{4.8}
\]

where \( T' := \nu^*T \), \( T := H|_Z \). In the case I the inequality (4.8) follows from 3.5. Consider the case II. Then \( Z' = S/G \) and \( H^0(Z', \mathcal{O}_{Z'}(T')) = H^0(Z', \pi_*\mathcal{O}_S \otimes \mathcal{O}_{Z'}(T'))^\Gamma = H^0(S, \mathcal{O}_S(M))^\Gamma \). Further we assume that \( H^0(S, \mathcal{O}_S(M))^\Gamma = 0 \).

**4.4. Lemma.** If \( H^0(S, \mathcal{O}_S(M))^\Gamma = 0 \), then the rational map \( S-\to S_0 \subset \mathbb{P}^{\dim |M|} \) associated with the linear system \( |M| \) is not birational on its image.

**Proof.** If \( H^0(S, \mathcal{O}_S(M))^\Gamma = 0 \), then the action of \( \Gamma \) on the linear system \( |M| \) is trivial and if \( |M| \) defines the rational map \( S-\to S_0 \), then it factors through quotient morphism by \( \Gamma \): \( S \overset{\pi}{\to} S/\Gamma = Z'-\to S_0 \).

Lemma 4.4 excludes cases (ii),(iii),(iv),(vi) of Proposition 3.3. In the case (v) of Proposition 3.3 by Theorem 1.5, possibility \( M^3 \geq 3 \) is also impossible. On the other hand, we have \( M^3 = (\pi^*T')^3 = 2(T')^3 \geq 2 \). If \( M^3 = 2 \), then again by Theorem 1.5 \( S_0 = \mathbb{P}^3 \) and \( S-\to S_0 \) is a finite morphism of degree 2, so \( S/\Gamma = S_0 = \mathbb{P}^3 \). It gives us inequality (4.8).

Finally let \( S \) and \( W \) be as in (i) of Proposition 3.3. By Lemma 3.4, \( g(C) = 1 \) or \( g(C) = 0 \). If \( g(C) = 0 \), then the sheaf \( \mathcal{E} \) on \( C = \mathbb{P}^1 \) is decomposable: \( \mathcal{E} = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \oplus \mathcal{O}(d_3) \), where \( d_i \geq 0 \). Then the linear system \( |N| = |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)| \) defines a birational map (see e. g. [5]), again we have a contradiction with Lemma 4.4. The last case 3.3 (i) and \( g(C) = 1 \) is treated similar to 3.3 (v). Here insted Theorem 1.5 we use the following

**4.5. Lemma.** Let \( W \) and \( N \) be as in (i) of Proposition 3.3. Assume also that \( g(C) = 1 \) and \( d := N^3 \geq 2 \). Then the linear system \( |N| \) on \( W \) defines a morphism \( W \to W_0 \subset \mathbb{P}^{d+1} \). Moreover this morphism is finite of degree 2 if \( d = 2 \) and birational if \( d \geq 3 \).

**Proof.** It follows from proof of Lemma 3.4 that the morphism \( \tau \) contracts a surface \( P \equiv a(N - dF) \), \( a \in \mathbb{N} \) to a curve. We claim that \( a = 1 \). Indeed in the opposite case every divisor from \( |N| \) is irreducible. Then \( N \) is a smooth geometrically ruled surface over \( C \) and \( P|_N \) is an effective divisor with negative self-intersection number. It is known in this case that \( P|_N \) is a section of the ruled surface \( N \), hence \( a = 1 \). Therefore \( P \) is also a non-singular geometrically ruled surface over \( C \). The morphism \( \tau \) maps \( P \) onto curve and fibers of \( P \to C \) are not contracted. It gives us that \( P \simeq C \times \mathbb{P}^1 \). Consider the exact sequence

\[
0 \to H^0(\varphi^*\mathcal{F}) \to H^0(\mathcal{O}_W(N)) \to H^0(\mathcal{O}_P(N)) \to 0, \tag{4.9}
\]
where $\mathcal{F}$ is a sheaf of degree $d$ on $C$ such that $\mathcal{O}_W(N - P) = \varphi^*\mathcal{F}$. It is easy to see that the sheaf $\mathcal{O}_P(N)$ on $P \simeq C \times \mathbb{P}^1$ has bidegree $(0,1)$. So $h^0(\mathcal{O}_P(N)) = 2$, $h^0(\mathcal{O}_W(N)) = d + 2$ and $P$ is not a fixed component in $|N|$. From (4.9) and $d \geq 2$ we get, that the linear system $|N|$ is free and defines a morphism $\tau_0 : W \to W_0 \subset \mathbb{P}^{d+1}$. Moreover $\text{deg}\, \tau_0 \cdot \text{deg}\, W_0 = N^3 = d$. Applying to variety $W_0 \subset \mathbb{P}^{d+1}$ the inequality $\text{deg}\, W_0 \geq \text{codim}\, W_0 + 1$ we obtain $\text{deg}\, \tau_0 = 2$ for $d = 2$, and $\text{deg}\, \tau_0 = 1$ for $d \geq 3$.

Thus Proposition 4.1 is proved.

Now we prove that Proposition 4.1 implies Theorem 1. By our assumption every divisor $H$ is irreducible and for general $H$ we have $\dim \text{Sing}(H) < 3$ (see 2.4, 4.1). Therefore a general divisor $H$ is normal. For such $H$ by the adjunction formula and by Proposition 4.1 we have $K_L = f|_L^*(K_H) + \sum (a_i - r_i) E_i|_L$ and $a_i - r_i \geq 0$, where $f|_L : L \to H$ is the corresponding resolution of singularities of $H$. It is equivalent the fact that $H$ has only canonical singularities. This proves our theorem.

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