The inf-sup condition and error estimates of the Nitsche method for evolutionary diffusion-advection-reaction equations

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The Nitsche method is a method of “weak imposition” of the inhomogeneous Dirichlet boundary conditions for partial differential equations. This paper explains stability and convergence study of the Nitsche method applied to evolutionary diffusion-advection-reaction equations. We mainly discuss a general space semidiscrete scheme including not only the standard finite element method but also Isogeometric Analysis. Our method of analysis is a variational one that is a popular method for studying elliptic problems. The variational method enables us to obtain the best approximation property directly. Actually, results show that the scheme satisfies the inf-sup condition and Galerkin orthogonality. Consequently, the optimal order error estimates in some appropriate norms are proven under some regularity assumptions on the exact solution. We also consider a fully discretized scheme using the backward Euler method. Numerical example demonstrate the validity of those theoretical results.

Key words: diffusion-advection-reaction equation, inf-sup condition, IGA

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1 Introduction

The boundary condition is an indispensable component of the well-posed problem of partial differential equations. It is not merely a side condition. In computational mechanics, great attention should be paid the imposition of boundary conditions, although it is sometimes understood as a simple and unambiguous task.

The Neumann boundary condition or natural boundary condition is naturally considered in the variational equation so that it is handled directly in finite element method (FEM). By contrast, a specification of the Dirichlet boundary condition (DBC) has room for discussion. In traditional FEM including the continuous $p^k$ FEM for example, DBC is imposed by specifying the nodal values at boundary nodal points. Although it is simple, this “strong imposition” of DBC is based on the fact that unknown values of the resulting finite dimensional

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system agree with nodal values in traditional FEM. Therefore, it is difficult to apply this
technique to the iso-geometric analysis (IGA). Actually, IGA is a class of FEM using B-spline
or non-uniform rational B-spline (NURBS) basis functions. It has been widely applied in
many fields of computational mechanics, providing smooth approximate solutions of partial
differential equations using only a few degrees of freedom (DOF). Moreover, it provides a
more accurate representation of computational domains with complex shapes. That is, the
geometric representation of 3D computational domain generated by a CAD system is handled
directly. See [7] for more details. Unfortunately, unknown values in IGA do not generally
agree with nodal values. Furthermore, it has often been pointed out that strongly imposed
DBCs give rise to spurious oscillations, even for stable discretization methods. To resolve
those shortcomings, Bazilevs et al. [2, 3] proposed a method of “weak imposition” of DBC by
applying the methodology of the discontinuous Galerkin (DG) method and discussed its effi-

ciency by numerical experiments for non-stationary Navier–Stokes equations. Their method,
originally proposed by Nitsche [13], is commonly called the Nitsche method. Stability and
convergence of the Nitsche method for elliptic problems have been well studied to date.

This paper addresses the Nitsche method for evolutionary problems. In particular, we study
the stability and convergence of the FEM discretization including IGA. Earlier studies of the
Nitsche method were conducted by formulating the method as a one-step method, as in an
earlier report of the literature [16]. By contrast, we present a different perspective: we assess
the Nitsche method using a variational approach. Consequently, the analysis becomes greatly
simplified. Optimal order error estimates in some appropriate norms are established. Such
a variational approach was recently applied successfully to analysis of the DG time-stepping
method for a wide class of parabolic equations in an earlier paper [14]. To fix the idea, we
consider the following diffusion-advection-reaction equation for the function \( u = u(x, t) \), \( x \in \Omega \)
and \( t \in J \),

\[
\begin{align*}
\partial_t u + L(x, t)u &= f(x, t), \quad (x, t) \in \Omega \times J, \\
u(x, t) &= g(x, t), \quad (x, t) \in \Gamma \times J, \\
u(x, 0) &= u_0(x), \quad x \in \Omega.
\end{align*}
\]

Hereinafter, \( \Omega \) is a bounded polyhedral domain or NURBS domain (see Definition 3.3) in \( \mathbb{R}^d \)
with the boundary \( \Gamma = \partial \Omega \), \( J \) represents the time interval defined as \( J = (0, T) \) with \( T > 0 \),
and \( L(x, t) \) signifies the elliptic differential operator defined as

\[
L(x, t)w = -\nabla \cdot \mu(x, t) \nabla w + b(x, t) \cdot \nabla w + c(x, t)w.
\]

Moreover, \( \mu : \Omega \times J \to \mathbb{R}^{d \times d} \), \( b : \Omega \times J \to \mathbb{R}^d \), \( c, f : \Omega \times J \to \mathbb{R} \), \( g : \Gamma \times J \to \mathbb{R} \), and
\( u_0 : \Omega \to \mathbb{R} \) are given functions. The assumptions to these functions will be described later.

At this stage, we describe the idea of applying the Nitsche method for (1) to clarify the
novelty and motivation of this study. To avoid unimportant difficulties, we presume that
\( \mu = I \), \( b = 0 \) and \( c = 0 \) for the time being. In subsequent sections, we remove those
restrictions. By multiplying both sides of (1a) by a test function \( v \in H^1(\Omega) \), integrating over
\( \Omega \) and finally applying integration by parts, we obtain

\[
\int_{\Omega} (\partial_t u)v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\Gamma} (n \cdot \nabla u)v \, dS = \int_{\Omega} fv \, dx.
\]

Introducing a partition \( T_h \) of \( \Omega \), with \( h \) being the granularity parameter, and a finite dimen-
sional subspace \( V_h \) of \( H^1(\Omega) \), we consider the Galerkin approximation

\[
\int_{\Omega} (\partial_t u_h)v_h \, dx + \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx - \sum_{E \in \mathcal{E}_h} \int_E (n \cdot \nabla u_h)v_h \, dS = \int_{\Omega} fv_h \, dx \quad (v_h \in V_h),
\]
where $\mathcal{E}_h^e$ denotes the set of all boundary edges. (the definition of those notations will be stated in Section [2]) Then, the Nitsche method reads as shown below

$$
\int_{\Omega} (\partial_t u_h)v_h \, dx + \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx - \sum_{E \in \mathcal{E}_h^e} \int_E (n \cdot \nabla u_h)v_h \, dS
$$

$$
= I_s
$$

$$
- \sum_{E \in \mathcal{E}_h^e} \int_E (n \cdot \nabla v_h)(u_h - g) \, dS + \sum_{E \in \mathcal{E}_h^e} \int_E \frac{\varepsilon}{h}(u_h - g)v_h \, dS = \int_{\Omega} f v_h \, dx \quad (v_h \in V_h)
$$

for a.e. $t \in J$, where $\varepsilon > 0$ is a penalty parameter. This is written, equivalently, as

$$
\int_{\Omega} (\partial_t u_h)v_h \, dx + a_{\varepsilon,h}(u_h, v_h) = F_{\varepsilon,h}(t; v_h) \quad (v_h \in V_h, \ t \in J),
$$

where

$$
a_{\varepsilon,h}(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx - \sum_{E \in \mathcal{E}_h^e} \int_E (n \cdot \nabla u_h)v_h \, dS
$$

$$
- \sum_{E \in \mathcal{E}_h^e} \int_E (n \cdot \nabla v_h)u_h \, dS + \sum_{E \in \mathcal{E}_h^e} \int_E \frac{\varepsilon}{h}u_h v_h \, dS
$$

$$
F_{\varepsilon,h}(t; v_h) = \int_{\Omega} f v_h \, dx - \sum_{E \in \mathcal{E}_h^e} \int_E (n \cdot \nabla v_h)g \, dS + \sum_{E \in \mathcal{E}_h^e} \int_E \frac{\varepsilon}{h}g v_h \, dS.
$$

Term $I_s$ is added to symmetrize the equation. Term $I_p$ is called the penalty term. Letting $\varepsilon$ be sufficiently large, we expect that the boundary condition $u_h = g$ on $\Gamma$ is specified in a weak sense. An important advantage of (2) is that the “elliptic part”

$$
\int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx - \sum_{E \in \mathcal{E}_h^e} \int_E (n \cdot \nabla u_h)v_h \, dS - \sum_{E \in \mathcal{E}_h^e} \int_E (n \cdot \nabla v_h)u_h \, dS + \sum_{E \in \mathcal{E}_h^e} \int_E \frac{\varepsilon}{h}u_h v_h \, dS
$$

can be coercive in an appropriate norm by choosing suitably large $\varepsilon$. Moreover, the constant appearing in the coercive inequality is independent of the penalty parameter $\varepsilon$, which implies that the scheme can be stable in a certain sense. In fact, the classical penalty method has no such property. Another advantage is that the smooth solution $u$ of (1) exactly satisfies (2). Consequently, the “parabolic Galerkin orthogonality”

$$
\int_J \left[ \int_{\Omega} (\partial_t u - \partial_t u_h)y_h \, dx + a_{\varepsilon,h}(u - u_h, y_h) \right] \, dt = 0 \quad (y_h \in L^2(J; V_h))
$$

is available. This characteristic enables us to apply the variational method to study the Nitsche method (2): after having established the “inf-sup” condition, we can derive best approximation properties and optimal order error estimates directly by combining the “inf-sup” condition and (3). Therefore, our effort will be concentrated on the derivation of the “inf-sup” condition, which is the main result of this paper. Although such an approach is quite standard for elliptic problems, apparently little has been done for parabolic problems. The use of (3) is not originally our idea. Others have considered this identity before, but no report of the relevant literature describes systematic use of (3).

Before concluding this Introduction, we review earlier studies of the convergence of the Nitsche method applied to parabolic equations. Thomée [16] reported error estimates in the
This paper is organized as follows. Section 2 states the formulation of Nitsche method and the main results. Section 4, 5 and 6 provide proof of our main results. Analysis of the fully discretized problem is presented in Section 7. Finally, this report presents a numerical example in Section 8.

2 Nitsche method and the main results

2.1 Weak formulation of (1)

We use the standard Lebesgue spaces $L^p(\Omega)$, $L^p(\Gamma)$ and Sobolev spaces $H^m(\Omega)$, $H^{m-1/2}(\Gamma)$, where $1 \leq p \leq \infty$ and $1 \leq m \in \mathbb{Z}$. The norms are denoted as $\| \cdot \|_{L^p(\Omega)}$, $\| \cdot \|_{L^p(\Gamma)}$, $\| \cdot \|_{H^m(\Omega)}$ and $\| \cdot \|_{H^{m-1/2}(\Gamma)}$ for example. Moreover, the $L^2$-inner product is denoted as $(\cdot, \cdot)_{L^2(\Omega)}$, and so on. The semi-norm $| \cdot |_{H^m(\Omega)}$ of $H^m(\Omega)$ is defined as

$$|v|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} \left\| \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} v \right\|^2_{L^2(\Omega)},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $0 \leq \alpha_1, \ldots, \alpha_d \in \mathbb{Z}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. In fact,

$$\|v\|^2_{H^m(\Omega)} = \|v\|^2_{L^2(\Omega)} + \sum_{1 \leq |\alpha| \leq m} |v|^2_{H^m(\Omega)}.$$

Let $\text{Tr} = \text{Tr}(\Omega, \Gamma)$ be a trace operator from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$, which is a linear and continuous operator. There exists a linear and continuous operator $\text{Tr}^{-1} = \text{Tr}^{-1}(\Omega, \Gamma)$ of $H^{1/2}(\Gamma) \to H^1(\Omega)$, which is called a lifting operator, such that $\text{Tr}(\text{Tr}^{-1} \eta) = \eta$ on $\Gamma$ for all $\eta \in H^{1/2}(\Gamma)$. Below, we write it as $v|\Gamma = \text{Tr} v$ if there is no fear of confusion.

As usual, we set $H^1_0(\Omega) = \{ v \in H^1(\Omega) \mid v|\Gamma = 0 \}$ and $H^{-1}(\Omega) = [H^1_0(\Omega)]'$ is the dual space of $H^1_0(\Omega)$.

Let $X$ be a Hilbert space. For $1 \leq r < \infty$ and $0 \leq t_0 < t_1$, the space $L^r(t_0, t_1; X)$ denotes a Bochner space equipped with the norm

$$\|v\|_{L^r(t_0, t_1; X)} = \left( \int_{t_0}^{t_1} \|v(t)\|_X^r \, dt \right)^{1/r}.$$

Let $Y$ be a (possibly another) Hilbert space. We also use the so-called Bochner–Sobolev space $H^1(t_0, t_1; X, Y)$ defined as

$$H^1(t_0, t_1; X, Y) = \left\{ v \in L^2(t_0, t_1; X) \mid \frac{dv}{dt} \in L^2(t_0, t_1; Y) \right\},$$
where \( \frac{d}{dt} \) denotes the weak derivative for \( t \). This is a Hilbert space equipped with the norm
\[
\|v\|_{H^1(t_0, t_1; X, Y)}^2 = \|v\|^2_{L^2(t_0, t_1; X)} + \left\| \frac{dv}{dt} \right\|^2_{L^2(t_0, t_1; Y)}.
\]

It is apparent that \( H^1(t_0, t_1; X, X) \subset C^0([t_0, t_1]; X) \) (see [11] theorem 2, Chapter 5.9 for example) is satisfied. Furthermore, letting \( H \) and \( \mathcal{V} \) be (real) Hilbert spaces such that \( \mathcal{V} \subset H \) is dense with continuous injection, we identify \( H \) with \( H' \) (\( H \simeq H' \)) as usual and consider the triple
\[
\mathcal{V} \subset H \subset \mathcal{V}'.
\]
Then we have \( H^1(t_0, t_1; \mathcal{V}, \mathcal{V'}) \subset C^0([t_0, t_1]; H) \) (see [8] Theorem 1, Chapter XVIII] for example).

Throughout this paper, we use the following assumptions:

**Assumption I.** Regularity of coefficients and data functions:

\[
\begin{align*}
\mu &\in C^0(\overline{\Omega} \times J)^{d \times d}, \quad \mu \text{ is symmetric; (5a)} \\
b &\in L^\infty(J; W^{1,\infty}(\Omega)^d); \quad (5b) \\
c &\in L^\infty(J; L^\infty(\Omega)); \quad (5c) \\
\exists \mu_1 > \mu_0 > 0, \quad \mu_0 |\xi|^2 &\leq \mu(x, t)\xi \cdot \xi \leq \mu_1 |\xi|^2 \quad (x \in \overline{\Omega}, t \in J, \xi \in \mathbb{R}^d); \quad (5d) \\
\exists c_0 > 0, \quad c - \frac{1}{2} \nabla \cdot b &\geq c_0 \quad (x \in \Gamma, t \in J); \quad (5e) \\
f &\in L^2(J; L^2(\Omega)), \quad u_0 \in L^2(\Omega), \quad g \in G_0. \quad (5f)
\end{align*}
\]

Therein,
\[
G_0 = \{ \eta = w|_r \in L^2(J; H^{1/2}(\Gamma)) \mid w \in L^2(J; H^1(\Omega)) \cap H^1(J; L^2(\Omega)) \}.
\]

Introducing the bilinear form on \( H^1(\Omega) \times H^1(\Omega) \) as
\[
a(t; w, v) = \int_\Omega [\mu \nabla w \cdot \nabla v + (b \cdot \nabla w)v + cwv] \, dx,
\]
we have
\[
\begin{align*}
|a(t; w, v)| &\leq M \|w\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad (w, v \in H^1(\Omega), t \in J); \quad (7a) \\
a(t; w, w) &\geq \alpha \|w\|^2_{H^1(\Omega)} \quad (w \in H^1_0(\Omega), t \in J); \quad (7b) \\
a(t; w, v) &= (L(.t), w, v)_{L^2(\Omega)} \quad (w \in H^2(\Omega), v \in H^1_0(\Omega), t \in J), \quad (7c)
\end{align*}
\]
where \( M = M(d, \mu, b, c) > 0 \) and \( \alpha = \min\{\mu_0, c_0\} \).

The following is the standard result (see [8] Chapter XVIII, [17] Chapter IV] for example).

**Lemma 2.1.** Presuming that Assumption I is satisfied, then there exists a unique
\[
u \in H^1(J; H^1(\Omega), H^{-1}(\Omega))
\]
such that
\[
u = g \quad \text{on } \Gamma, \ t \in J,
\]
and
\[
\int_0^T [(\partial_t \nu, y_1)_{H^{-1}, H^1_0} + a(t, u(t), y_1(t))] \, dt + (u(0), y_2)_{L^2(\Omega)}
= \int_0^T (f, y_1)_{L^2(\Omega)} \, dt + (u_0, y_2)_{L^2(\Omega)} \quad \forall (y_1, y_2) \in L^2(J; H^1_0(\Omega)) \times L^2(\Omega).
\]
Moreover, we have \( \partial_t u + L(t)u \in L^2(J; L^2(\Omega)) \) and it holds that [11] for \( x \in \Omega \) and \( t \in J \).
2.2 Finite dimensional subspaces

We introduce a finite dimensional subspace $V_h$ of $H^1(\Omega)$ in a somewhat abstract manner below. Concrete examples are given in Section 3. We also collect (finite dimensional) function spaces and norms used for this study.

Recall that $\Omega \subset \mathbb{R}^d$ is a polyhedral domain with the boundary $\Gamma$. We introduce a partition $\mathcal{T}_h$ of $\Omega$ such that each $K \in \mathcal{T}_h$ is a closed set in $\mathbb{R}^d$, the $\mathbb{R}^d$ Lebesgue measure of $K \cap K'$ vanishes for $K, K' \in \mathcal{T}_h$ with $K \neq K'$, and

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K.$$  

The diameter of $K \in \mathcal{T}_h$ is designated by $h_K$ and is set as $h = \max\{h_K \mid K \in \mathcal{T}_h\}$. Then, letting $\mathcal{E}_h$ be the set of edges and $\mathcal{E}_h^e = \{E \in \mathcal{E}_h \mid E \subset \Gamma\}$, we express $\Gamma$ as

$$\Gamma = \bigcup_{E \in \mathcal{E}_h^e} E.$$  

For $E \in \mathcal{E}_h$, the diameter of $E$ is designated by $h_E$. Moreover, for $E \in \mathcal{E}_h$, we write $K_E$ to express $K \in \mathcal{T}_h$ such that $E \subset \partial K$. In general, such $K_E$ is not unique. However, it is unique for any $E \in \mathcal{E}_h^e$.

**Assumption II.** There exists a positive constant $C$ such that

$$h_{K_E} \leq C h_E \quad (E \in \mathcal{E}_h^e, K_E \in \mathcal{T}_h).$$  

Below, we use the finite dimensional subspace

$$V_h \subset H^1(\Omega).$$  

We mention no specific definition, but we do make the following assumptions.

**Assumption III.**

$$v_h|_K \in H^2(K) \quad (v_h \in V_h, K \in \mathcal{T}_h).$$  

**Assumption IV.**  

(i) **Trace inequality.** There exists a positive constant $C_{\text{Tr}}$ such that

$$\|v\|^2_{L^2(E)} \leq C_{\text{Tr}} h_E^{-1} \left( \|v\|^2_{L^2(K_E)} + h_{K_E}^2 \|v\|^2_{H^1(K_E)} \right) \quad (v \in H^1(K_E), E \in \mathcal{E}_h^e).$$  

(ii) **Inverse inequality.**

$$\|v_h\|_{H^1(K)} \leq C h_K^{-1} \|v_h\|_{L^2(K)} \quad (v_h \in V_h, K \in \mathcal{T}_h).$$  

(iii) **Interpolation error estimates.** There exists a positive integer $k$ and a projection $\Pi_h : H^k(\Omega) \rightarrow V_h$ such that, for $2 \leq l \leq k + 1$,

$$\|w - \Pi_h w\|_{H^l(\Omega)} \leq C h^{l-j} \|w\|_{H^j(\Omega)} \quad (w \in H^l(\Omega), j = 0, 1, 2).$$  

Assumptions [II] and [IV] imply that there exists a positive constant $C^*$ such that

$$\|v_h\|^2_{L^2(E)} \leq C^* h_E^{-1} \|v_h\|^2_{L^2(K_E)} \quad (v_h \in V_h, E \in \mathcal{E}_h^e).$$  

Moreover, Assumption [III] gives that the same constant $C^*$ satisfies

$$\|n \cdot \nabla v_h\|^2_{L^2(E)} \leq C^* h_E^{-1} \|v_h\|^2_{H^1(K_E)} \quad (v_h \in V_h, E \in \mathcal{E}_h^e).$$
Setting \( V = \{ v \in H^1(\Omega) \mid v|_K \in H^2(K) \ (K \in \mathcal{T}_h) \} \), then \( V_h \subset V \). Furthermore, we define
\[
\|v_h\|_{V_h}^2 = \|v_h\|_{H^1(\Omega)}^2 + \sum_{E \in \mathcal{E}^h} h^{s_1}_E \|v_h\|_{L^2(E)}^2,
\]
\[
\|v\|_{V}^2 = \|v\|_{H^1(\Omega)}^2 + \sum_{K \in \mathcal{T}_h} h^{s_2}_K \|v\|_{H^2(K)}^2 + \sum_{E \in \mathcal{E}^h} h^{s_1}_E \|v\|_{L^2(E)}^2.
\]
This definition implies that \( \|v_h\|_{V_h} \leq \|v_h\|_{V} \leq C \|v_h\|_{V_h} \) for all \( v_h \in V_h \), where \( C \) is a positive constant. Moreover, for \( \phi \in L^2(\Omega) \), we write that
\[
\|\phi\|_{V_h'} = \sup_{v_h \in V_h} \frac{\langle \phi, v_h \rangle_{L^2(\Omega)}}{\|v_h\|_{V_h}},
\]
(17)
It is apparent that \( \|\phi\|_{V_h'} \leq \|\phi\|_{L^2(\Omega)} \) for every \( \phi \in L^2(\Omega) \).
Furthermore, we define the space of trial function and test function in the Nitsche method. Let
\[
X_h = H^1(J; V_h, V_h), \quad Y_h = L^2(J; V_h) \times V_h.
\]
(18)
They are Hilbert spaces equipped with the norms
\[
\|z_h\|_{X_h}^2 = \int_J (\|z_h\|_{V_h}^2 + \|\partial_t z_h\|_{V_h'}^2) \, dt + \|z_h(0)\|_{L^2(\Omega)}^2,
\]
\[
\|(y_h, \tilde{y}_h)\|_{Y_h}^2 = \int_J \|y_h\|_{V_h}^2 \, dt + \|\tilde{y}_h\|_{L^2(\Omega)}^2,
\]
respectively. In fact, \( X_h \subset C^0(J; V_h) \). We also define the space
\[
X_V = \{ z \in H^1(J; H^1(\Omega), L^2(\Omega)) \mid z(t) \in V \ (t \in J) \},
\]
and norm
\[
\|z\|_{X_V}^2 = \int_J (\|z\|_{V}^2 + \|\partial_t z\|_{V_h'}^2) \, dt + \|z(0)\|_{L^2(\Omega)}^2,
\]
which satisfies \( X_h \subset X_V \subset C^0(J; L^2(\Omega)) \) and \( \|z_h\|_{X_h} \leq \|z_h\|_{X_V} \leq C\|z_h\|_{X_h} \) for all \( z_h \in X_h \), where \( C \) is a positive constant.

2.3 Formulation of the Nitsche method

The Nitsche method for parabolic problems is presented below.

(\( P_{\varepsilon,h} \)) Find \( u_{\varepsilon,h} \in X_h \) such that
\[
\int_{\Omega} (\partial_t u_{\varepsilon,h}) v_h \, dx + a_{\varepsilon,h}(t; u_{\varepsilon,h}(t), v_h) = F_{\varepsilon,h}(t; v_h) \quad (v_h \in V_h, \ t \in J),
\]
(19a)
\[
\int_{\Omega} u_{\varepsilon,h}(0) \tilde{v}_h \, dx = \int_{\Omega} u_0 \tilde{v}_h \, dx \quad (\tilde{v}_h \in V_h).
\]
(19b)
Therein, we set
\[
a_{\varepsilon,h}(t; w, v_h) = a(t; w, v_h) - \sum_{E \in \mathcal{E}^h} \int_{E} (n \cdot \mu \nabla w) v_h \, dS
\]
\[
- \sum_{E \in \mathcal{E}^h} \int_{E} (n \cdot \mu \nabla v_h) w \, dS - \int_{\Gamma_{in}} b \cdot n w v_h \, dS + \sum_{E \in \mathcal{E}^h} \int_{E} \varepsilon h^{s_1}_E w v_h \, dS
\]
\[
F_{\varepsilon,h}(t; v_h) = \int_{\Omega} f v_h \, dx - \sum_{E \in \mathcal{E}^h} \int_{E} (n \cdot \mu \nabla v_h) g \, dS - \int_{\Gamma_{in}} b \cdot n g v_h \, dS + \sum_{E \in \mathcal{E}^h} \int_{E} \varepsilon h^{s_1}_E g v_h \, dS
\]
Theorem 2

(Coercivity of Assumption V is not necessary for these inequalities to hold. Hereinafter, we write a where B

In particular, there exists a positive constant \( \hat{\alpha} \) such that

\[ a_{\varepsilon,h}(t; v_h, v_h) \geq \hat{\alpha} \| v_h \|^2_{V_h} \quad (v_h \in V_h, \ t \in J). \]  

(25)
\textbf{Theorem 3} (Continuity of $B_{\varepsilon,h}$). \textit{There exists a positive constant $C$ such that}
\[ B_{\varepsilon,h}(z, y_h) \leq C \| z \|_{X_V} \| y_h \|_{Y_h} \quad (z \in X_V, \ y_h \in Y_h). \] 
\textit{Particularly, we have}
\[ B_{\varepsilon,h}(z_h, y_h) \leq C \| z_h \|_{X_h} \| y_h \|_{Y_h} \quad (z_h \in X_h, \ y_h \in Y_h). \] \hspace{1cm} (27)
\textit{Assumption $\square$ is not necessary for these inequalities to hold.}

\textbf{Theorem 4} (Inf-sup condition of $B_{\varepsilon,h}$). \textit{There exists a positive constant $\beta$ such that}
\[ \inf_{z \neq z_h \in X_h} \sup_{0 \neq y_h \in Y_h} \frac{B_{\varepsilon,h}(z_h, y_h)}{\| z_h \|_{X_h} \| y_h \|_{Y_h}} \geq \beta. \] \hspace{1cm} (28)

\textbf{Theorem 5.} \textit{The problem $(P_{\varepsilon,h})$ has a unique solution $u_{\varepsilon,h} \in X_h$.}

\textbf{Proof.} It is sufficient to prove that
\[ B_{\varepsilon,h}(z_h, y_h) = 0 \quad (\forall z_h \in X_h) \Rightarrow y_h = (y_h, \bar{y}_h) = 0. \] \hspace{1cm} (29)
Actually, using (28) and (29), we can apply the Banach–Nečas–Babuška theorem (see [9, theorem2.6] for example) to deduce the conclusion. First, presuming that $z_h \in X_h$ satisfies
\[ z_h(0) = \bar{y}_h, \quad \text{and} \quad z_h(t) = 0 \quad (t \geq \delta) \]
for all $\delta > 0$, then we have $\bar{y}_h = 0$.
To prove $y_h = 0$, we take the basis functions $\{ \phi_i \}_{i=1}^N$ of $V_h$, where $N := \dim V_h$ and let
\[ y_h(t) := \sum_{i=1}^N a_i(t) \phi_i. \]
Then, $B_{\varepsilon,h}(z_h, (y_h, 0)) = 0$ implies
\[ A_{\varepsilon,h}(t)a(t) = 0, \]
where $\mathbf{A}_{\varepsilon,h}(t_{i,j}) := a_{\varepsilon,h}(t; \phi_i, \phi_j)$ and $\mathbf{a}(t) := (a_1(t), \ldots, a_N(t))^T$. In view of the coercivity of $a_{\varepsilon,h}(t; \cdot, \cdot)$ (Theorem 2), we obtain $\mathbf{a}(t) = 0$ for $t \in J$. This implies $y_h = 0$; (29) is proved. \hfill \Box

\textbf{Theorem 6} (Galerkin orthogonality). \textit{Letting $u_{\varepsilon,h} \in X_h$ be the solution of $(P_{\varepsilon,h})$, then if the solution $u$ of $(1)$ satisfies $u \in X_V$, we have}
\[ B_{\varepsilon,h}(u - u_{\varepsilon,h}, y_h) = 0 \quad (y_h \in Y_h). \] \hspace{1cm} (30)

\textbf{Proof.} Noting Lemma 2.1 we have
\[ B_{\varepsilon,h}(u - u_{\varepsilon,h}, y_h) = \int_J \left[ (\partial_t u, y_h)_{L^2(\Omega)} + a_{\varepsilon,h}(t; u, y_h) \right] dt + (u(0), \bar{y}_h)_{L^2(\Omega)} \\
- \int_J F_{\varepsilon,h}(t; y_h) dt + (u_0, \bar{y}_h)_{L^2(\Omega)} \\
= \int_J \left[ (\partial_t u + L(t)u - f, y_h)_{L^2(\Omega)} - \sum_{E \in \mathcal{E}_h} \int_E (n \cdot \nabla y_h)(u - g) dS \\
- \int_{\Gamma_{in}} b \cdot n(u - g)y_h dS + \sum_{E \in \mathcal{E}_h} \int_E \frac{\varepsilon}{h_E} (u - g)y_h dS \right] dt \\
= 0 \]
for any $y_h = (y_h, \bar{y}_h) \in Y_h$. \hfill \Box
Theorem 7 (stability). Let $u_{\varepsilon,h} \in X_h$ be the solution of $(P_{\varepsilon,h})$. If the solution $u$ of (1) satisfies $u \in X_V$, then we have
\[ \|u_{\varepsilon,h}\|_{X_h} \leq C\|u\|_{X_V}. \] (31)

Proof. Combining Theorems 3, 4 and 6, we have
\[ \|u_{\varepsilon,h}\|_{X_h} \leq \frac{1}{\beta} \sup_{y_h \in Y_h} \frac{B_{\varepsilon,h}(u_{\varepsilon,h}, y_h)}{\|y_h\|_{Y_h}} \leq \frac{1}{\beta} C \|u\|_{X_V}. \]

Theorem 8 (best approximation property). If the solution $u$ of (1) satisfies $u \in X_V$, then there exists a positive constant $C$ such that
\[ \|u - u_{\varepsilon,h}\|_{X_h} \leq C \inf_{z_h \in X_h} \|u - z_h\|_{X_V}. \] (32)

Proof. In exactly the same way as the proof of Theorem 7, we have for any $z_h \in V_h$
\[ \|z_h - u_{\varepsilon,h}\|_{X_h} \leq C \|z_h - u\|_{X_V}. \]
This, together with the triangle inequality, implies the desired estimate.

Theorem 9 (optimal order error estimate). Letting $l$ and $m$ be integers with $2 \leq l, m \leq k + 1$ and letting $u \in X_{l,m} := H^1(J; H^l(\Omega), H^m(\Omega))$ be the solution of (1), we have
\[ \|u - u_{\varepsilon,h}\|_{X_h}^2 \leq C \left( h^{2(l-1)} \|u\|_{L^2(J; H^l(\Omega))}^2 + h^{2m} \|\partial_t u\|_{L^2(J; H^m(\Omega))}^2 + h^{2j} \|u(0)\|_{H^j(\Omega)}^2 \right), \] (33)
where $j := \min\{l, m\}$.

3 Concrete examples of finite dimensional subspace

In this section, we give two concrete examples of the finite dimensional subspace $V_h$ of $H^1(\Omega)$.

3.1 Finite element method

Letting $\Omega \subset \mathbb{R}^d$ be a polygonal domain and introducing the triangulation $T_h$ of $\Omega$, we consider the standard continuous $P_k$ finite element space
\[ V_h = \{ v_h \in C^0(\Omega) : v_h|_K \in \mathbb{P}_k \ (K \in T_h) \}. \]
It is readily apparent that $V_h \subset H^1(\Omega)$ and that Assumptions I and III are satisfied.

Assuming that the family of triangulations $\{T_h\}_h$ is regular ([5, (4.4.16)] for example) with satisfaction of the inverse assumption ([5, (4.4.15)] for example), then Assumption IV is satisfied. In summary, our results are applicable to standard finite element method.

3.2 Iso-Geometric Analysis

Isogeometric analysis describes a computational domain by the so-called NURBS geometry. Furthermore, the finite dimensional subspace in the Galerkin method is introduced directly using the NURBS mesh. Here, we will review the definition and properties of NURBS.
Univariate B-spline basis functions on $[0,1]$ We designate a vector $\Xi := \{\xi_1, \xi_2, \ldots, \xi_r\}$ the knot vector if
\[ \xi_1 \leq \xi_2 \leq \cdots \leq \xi_r. \] (34)
It is noteworthy that repetition of the knots is allowed. Without loss of generality, we let $\xi_1 = 0$ and $\xi_r = 1$. Let $k$ be a given positive integer. Then, the univariate B-spline functions of degree $k$ associated with the knot vector $\Xi$ are defined by the Cox – de Boor algorithm.

Definition 3.1. Let $\Xi = \{\xi_1, \ldots, \xi_r\}$ be a knot vector. Then the $k$-th degree B-spline basis functions $\hat{B}_{i,k}$ are defined as
\[
\hat{B}_{i,0}(\hat{x}) = \begin{cases} 
1 & \text{if } \xi_i \leq \hat{x} \leq \xi_{i+1} \quad (k = 0), \\
0 & \text{otherwise} 
\end{cases}
(35)
\]
\[
\hat{B}_{i,k}(\hat{x}) = \frac{\hat{x} - \xi_i}{\xi_{i+k} - \xi_i} \hat{B}_{i,k-1}(\hat{x}) + \frac{\xi_{i+k+1} - \hat{x}}{\xi_{i+k+1} - \xi_{i+1}} \hat{B}_{i+1,k-1}(\hat{x}) \quad (k \geq 1),
(36)
\]
with $i = 1, \ldots, r - k - 1$. Here, $0/0 = 0$ should be replaced by 0 in this definition.

We state some properties of the B-spline basis functions of degree $k$. They are non-negative $k$-th degree piecewise polynomials such that $\hat{B}_{i,k}(\hat{x}) = 0$ for $\hat{x} \notin [\xi_i, \xi_{i+k+1}]$. Now we introduce an alternative representation of $\Xi$ to state the other properties. Let
\[
\Xi = \{\zeta_0, \ldots, \zeta_0, \zeta_1, \ldots, \zeta_{m_1}, \ldots, \zeta_N, \ldots, \zeta_N\},
(37)
\]
where $\zeta_0 < \zeta_1 < \cdots < \zeta_N$. Therein we designate the multiplicity of $\zeta_n$ by $m_n$. Assume that $m_n \leq k + 1$ for all knots, then $\hat{B}_{i,k}$ has $k - m_n$ continuous derivatives at internal node $\zeta_n$. Furthermore, one can say that the knot vector $\Xi$ is $k$-open if $m_0 = m_N = k + 1$. Letting $\Xi$ be a $k$-open knot vector, then $\hat{B}_{i,k}$ form the partition of unity. They also form the basis of spline space, i.e., the space of piecewise polynomials of degree $k$ with $k - m_n$ continuous derivatives at $\zeta_n$ for $n = 1, \ldots, N - 1$.

Henceforth, we assume the knot vector $\Xi$ is $k$-open. We define the univariate spline $S_k(\Xi)$ as
\[
S_k(\Xi) = \text{span}\{\hat{B}_{i,k} \mid i = 1, \ldots, r - k - 1\}.
(38)
\]
For partition size $h_n = \zeta_n - \zeta_{n-1}$, the following assumption is needed.

Assumption VI (Local quasi-uniform). The knot vector $\Xi$ is locally quasi-uniform, i.e., there exists a constant $\theta \geq 1$ such that
\[
\frac{1}{\theta} \leq \theta_n = \frac{h_n}{h_{n+1}} \leq \theta \quad (n = 1, \cdots, N - 1).
(39)
\]
Here we mention that the quasi-interpolant operator $\Pi_{k,\Xi} : L^\infty(I) \to S_k(\Xi)$ satisfies the error estimate (as described in greater detail in [15] Chapter 4). Letting
\[
I_n = (\zeta_{n-1}, \zeta_n), \quad h_n = |I_n|, \quad \tilde{I}_n = \bigcup \{ \supp \hat{B}_{i,p} \mid \hat{B}_{i,p} |_{I_n} \neq 0 \}, \quad \tilde{h}_n = |\tilde{I}_n|,
(40)
\]
then the following estimate holds.
Lemma 3.2 (Error estimate). Let \( s \) be a positive integer, \( p \in [1, \infty] \) and \( l = \min\{k + 1, s\} \). Then there exists a positive constant \( C \) such that
\[
\| f - \Pi_{k,\Xi}(f) \|_{L^p(I_n)} \leq C h_n^l \| f \|_{W^{s,p}(I_n)} \quad (f \in W^{s,p}(I), \ n = 1, \cdots, N). \tag{41}
\]
Moreover, let Assumption \( \text{[7]} \) be satisfied and let \( m \) be an integer with \( 0 \leq m \leq l \). Then there exists a constant \( C \) such that
\[
| f - \Pi_{k,\Xi}(f) |_{W^{m,p}(I_n)} \leq C h_n^{l-m} \| f \|_{W^{s,p}(I_n)} \quad (f \in W^{s,p}(I), \ n = 1, \cdots, N). \tag{42}
\]

The proof can be found in \[4, Proposition\] for \( p = 2 \). We refer the reader to \[15, Theorem\ 4.41\] for general \( p \).

Multivariate B-spline basis functions and NURBS basis functions Let \( \tilde{d} \) be the space dimension with \( \tilde{d} \leq d \). For \( j = 1, \ldots, \tilde{d} \), given degree \( k_j \) and \( k_j \)-open and locally quasi-uniform knot vector
\[
\Xi_j = \{ \xi_{j,1}, \ldots, \xi_{j,r_j} \} = \{ (\xi_{j,0}, \ldots, \xi_{j,0}, \xi_{j,2}, \ldots, \xi_{j,2}, \ldots, \xi_{j,N_j}, \ldots, \xi_{j,N_j}) \}
\]
we can define the \( k_j \)-th degree univariate B-spline basis functions
\[
\tilde{B}_{i_j} \mid h \tilde{\xi}_j, \quad i_j = 1, 2, \ldots, r_j - k_j - 1. \tag{43}
\]
Furthermore, the knots without repetition provide the mesh on a parametric domain \( \tilde{\Omega} = (0, 1)^{\tilde{d}} \), which is denoted as \( \tilde{\mathcal{T}}_h \):
\[
\tilde{\mathcal{T}}_h = \{ I_1, n_1 \times \cdots \times I_{\tilde{d}, n_{\tilde{d}}} \mid I_{j,n_j} := (\zeta_{j,n_j-1}, \zeta_{j,n_j}), n_j = 1, \ldots, N_j \}. \tag{44}
\]
Then we define the multivariate B-spline basis functions as
\[
\tilde{B}_{i_1}^\ldots \tilde{B}_{i_{\tilde{d}}} \mid h \tilde{\xi}, \quad i = (i_1, \ldots, i_{\tilde{d}}), \text{ where } \mathbf{k} = (k_1, \ldots, k_{\tilde{d}}) \text{ and } \tilde{\Xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_{\tilde{d}}) \in \tilde{\Omega}. \text{ We define the multivariate spline } S_k(\tilde{\Xi}) \text{ as }
\]
\[
S_k(\Xi) = S_k(\Xi_1) \times \cdots \times S_k(\Xi_{\tilde{d}}) = \text{span}\{ \tilde{B}_{i_1}^\ldots \tilde{B}_{i_{\tilde{d}}} \mid i \in I \}, \tag{46}
\]
where \( \Xi = \Xi_1 \times \cdots \times \Xi_{\tilde{d}} \) and \( I = \{ i = (i_1, \ldots, i_{\tilde{d}}) \mid i_j = 1, \ldots, r_j - k_j - 1 \} \). The quasi-interpolation for multivariate B-spline is defined also by the tensor product:
\[
\Pi_{k,\Xi} = \Pi_{k_1,\Xi_1} \times \cdots \times \Pi_{k_{\tilde{d}},\Xi_{\tilde{d}}} : L^\infty(\tilde{\Omega}) \rightarrow S_k(\Xi). \tag{47}
\]
Here, the definition of NURBS basis functions for given weight
\[
W(\tilde{\xi}) = \sum_{j \in I} u_{j} \tilde{B}_{j,k}(\tilde{\xi}) \tag{48}
\]
is described as
\[
\tilde{N}_{i,k}(\tilde{\xi}) = \frac{u_{i} \tilde{B}_{j,k}(\tilde{\xi})}{W(\tilde{\xi})} \quad (i \in I), \tag{49}
\]
where positive constants \( w_j > 0 \) \((j \in I)\) are called weights. Furthermore, a NURBS parametrization is given by a linear combination of NURBS basis functions. Letting \( P_i \in \mathbb{R}^d \) be control points, then a NURBS parametrization \( F(\mathbf{x}) \) is given as

\[
F(\mathbf{x}) = \sum_{i \in I} P_i \hat{N}_{i,k}(\mathbf{x}).
\]  

(50)

This parametrization can define the NURBS geometry in \( \mathbb{R}^d \). In this paper, we only consider when \( \hat{d} = d \).

The requirement on the map \( F \) is that it satisfy the following regularity.

**Assumption VII.** The map \( F : \hat{\Omega} \to \Omega \) is bijective Lipschitz function whose inverse function is also Lipschitz. Moreover, \( F|_Q \in C^\infty(Q) \) and \( F^{-1}|_K \in C^\infty(K) \) for all \( Q \in \hat{T}_h, K \in T_h \).

**Definition 3.3.** Under Assumption VII the domain \( \Omega \) defined as \( \Omega = F(\hat{\Omega}) \) is called the NURBS domain.

A mesh on \( \Omega \) is provided as the image of the parametric mesh as

\[
T_h = \{ K = F(Q) \mid Q \in \hat{T}_h \}. \tag{51}
\]

Under Assumption VII we define

\[
V_h = \text{span}\{ \mathbf{N}_{i,k}(\mathbf{x}) = \hat{N}_{i,k} \circ F^{-1}(\mathbf{x}) \mid i \in I \},
\]

(52)

where \( h \) is the mesh size \( h = \max\{ h_Q = \text{diam}(Q) \mid Q \in \hat{T}_h \} \). Furthermore, we define

\[
h_K = \| \nabla F \|_{L^\infty(Q)} h_Q \quad (K \in T_h),
\]

(53)

where \( Q = F^{-1}(K) \). For the NURBS mesh, we define the regularity of the family of mesh \( \{ T_h \}_h \) using \( \{ \hat{T}_h \} \).

**Assumption VIII.** The family of the mesh \( \{ T_h \}_h \) is regular, meaning that there exists a positive constant \( \sigma \) such that

\[
\frac{h_Q}{h_{Q,\text{min}}} \leq \sigma \left( Q \in \bigcup_h \hat{T}_h \right), \tag{54}
\]

where \( h_{Q,\text{min}} \) represents the length of the smallest edge of hypercube \( Q \).
We always assume that Assumptions [VI], [VII] and [VIII] are satisfied. Now we review some results obtained in earlier studies.

Lemma 3.4 (Trace inequality, Theorem 3.2 of [10]). Letting $K \in \mathcal{T}_h$ and $Q = F^{-1}(K)$, then
\[
\|f\|_{L^2(\partial K)}^2 \leq C \lambda_Q \lambda_K \left( h_K^{-1} \|f\|^2_{L^2(K)} + h_K |f|_{H^1(K)}^2 \right) \quad (f \in H^1(K)),
\] (55)
where $\lambda_Q$ and $\lambda_K$ respectively represent the local shape regularity constants of $Q$ and $K$. They are independent of $h_K$.

Lemma 3.5 (Inverse inequality, Theorem 4.2. of [1]). Letting $l, m$ be integers with $0 \leq m \leq l$, we have
\[
\|v_h\|_{H^1(K)} \leq C_{\text{shape}} h_K^{m-l} \sum_{i=0}^{m} \|\nabla F\|_{L^\infty(F^{-1}(K))}\|v_h\|_{H^i(K)}, \quad (K \in \mathcal{T}_h, v_h \in V_h).
\] (56)
Especially, we have
\[
|v_h|_{H^1(K)} \leq \|v_h\|_{H^2(K)} \leq C_{\text{shape}} h_K^{-1} \|v_h\|_{L^2(K)} \quad (K \in \mathcal{T}_h, v_h \in V_h).
\] (57)

Lemma 3.6 (Quasi-interpolation error estimate, Corollary 4.21 of [4]). Let the projection $\Pi_{V_h} : L^2(\Omega) \to V_h$ be
\[
\Pi_{V_h} f = \frac{\Pi_{k,\Xi}(W(f \circ F))}{W} \circ F^{-1} \quad (f \in L^2(\Omega)).
\] (58)
Furthermore, letting $s$ be an integer, $l = \min\{k_1 + 1, \cdots, k_d + 1, s\}$ and $0 \leq m \leq l$, then there exists a positive constant $C$ such that
\[
\|v - \Pi_{V_h} v\|_{H^m(K)} \leq C h_K^{l-m} \|v\|_{H^s(\tilde{K})} \quad (K \in \mathcal{T}_h, v \in H^s(\Omega)),
\] (59)
where $\tilde{K} = F(\tilde{Q})$ for $Q = F^{-1}(K)$ and
\[
\tilde{Q} = \bigcup \{ \text{supp} \, \tilde{N}_{i,k} : \tilde{N}_{i,k} \, |Q \neq 0 \}.
\] (60)

In summary, Assumptions [II] – [IV] are satisfied under Assumptions [VI] – [VIII].

4 Proof of Theorem 1 and 2

This section is devoted to the proof of theorems for the “elliptic part” $a_{\varepsilon,h}$. We start with the following auxiliary lemma: the estimate (62) itself is well known (see for instance [16, Lemma 2.1]); the estimate (61) is apparently unfamiliar.

Lemma 4.1. There exists a positive constant $C$ such that
\[
\sum_{E \in \mathcal{E}_h^{\varepsilon}} h_E \|n \cdot \mu \nabla v\|_{L^2(E)}^2 \leq C \left( |v|_{H^1(\Omega)}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{H^2(K)}^2 \right) \quad (v \in V).
\] (61)
Particularly, there exists a positive constant $C_1$ such that
\[
\sum_{E \in \mathcal{E}_h^{\varepsilon}} h_E \|n \cdot \mu \nabla v_h\|_{L^2(E)}^2 \leq C_1 |v_h|_{H^1(\Omega)}^2 \quad (v_h \in V_h).
\] (62)
Proof. Because $\mu \in \mathbb{R}^{d \times d}$ is symmetric, we have

$$\sup_{\xi \in \mathbb{R}^d} \frac{|\mu \xi|}{|\xi|} = \sup_{\xi \in \mathbb{R}^d} \frac{\mu \xi \cdot \xi}{|\xi|^2} \leq \mu_1.$$  

(63)

This result implies $|n \cdot \mu \nabla v| \leq \mu_1 |\nabla v|$. Therefore

$$\|n \cdot \mu \nabla v\|^2_{L^2(E)} \leq \mu_1^2 \sum_{i=1}^d \left\| \text{Tr} \left( \frac{\partial v}{\partial x_i} \right) \right\|_{L^2(E)}^2$$  

(64)

for all $E \in \mathcal{E}_h^\epsilon$. Here Assumption IV (i) yields

$$\mu_1^2 \sum_{i=1}^d \left\| \text{Tr} \left( \frac{\partial v}{\partial x_i} \right) \right\|_{L^2(E)}^2 \leq C \text{Tr} \mu_1^2 h_E^{-1} \left( |v|_{H^1(K_E)}^2 + h_E^2 |v|_{H^2(K_E)}^2 \right) \quad (v \in V, \ E \in \mathcal{E}_h^\epsilon),$$  

(65)

which implies (61) with $C = C_1 \mu_1^2$. Furthermore, using equation (16) leads to the following:

$$\|n \cdot \mu \nabla v_h\|^2_{L^2(E)} \leq \mu_1^2 \sum_{i=1}^d \left\| \text{Tr} \left( \frac{\partial v_h}{\partial x_i} \right) \right\|_{L^2(E)}^2 \leq C^* \mu_1^2 h_E^{-1} |v_h|_{H^1(K_E)}^2 \quad (v_h \in V_h, \ E \in \mathcal{E}_h^\epsilon).$$

Then we have the estimate (62) with $C_I = C^* \mu_1^2$. \hfill $\square$

Next we state the following proofs.

Proof of Theorem 7. First, the Cauchy–Schwarz inequality gives

$$a_{\epsilon,h}(t; w, v_h) \leq M \|w\|_{H^1(\Omega)} \|v_h\|_{H^1(\Omega)} + \left( \sum_{E \in \mathcal{E}_h^\epsilon} h_E \|n \cdot \mu \nabla w\|^2_{L^2(E)} \right)^{1/2} \left( \sum_{E \in \mathcal{E}_h^\epsilon} h_E^{-1} \|v_h\|^2_{L^2(E)} \right)^{1/2}$$

$$+ \left( \sum_{E \in \mathcal{E}_h^\epsilon} h_E \|n \cdot \mu \nabla v_h\|^2_{L^2(E)} \right)^{1/2} \left( \sum_{E \in \mathcal{E}_h^\epsilon} h_E^{-1} \|w\|^2_{L^2(E)} \right)^{1/2} + C \|w\|_{L^2(\Gamma)} \|v_h\|_{L^2(\Gamma)}$$

$$+ \varepsilon \left( \sum_{E \in \mathcal{E}_h^\epsilon} h_E^{-1} \|w\|^2_{L^2(E)} \right)^{1/2} \left( \sum_{E \in \mathcal{E}_h^\epsilon} h_E^{-1} \|v_h\|^2_{L^2(E)} \right)^{1/2}$$

for all $w \in V$, $v_h \in V_h$ and $t \in J$. By applying the (standard) trace inequality and Lemma 11 we have

$$a_{\epsilon,h}(t; w, v_h) \leq C \|w\|_V \|v_h\|_{V_h} \quad (w \in V, \ v_h \in V_h, \ t \in J).$$  

(66)

Moreover, the definition of norm $\| \cdot \|_V$ and $\| \cdot \|_{V_h}$ yields that there exists a positive constant $\tilde{M}$ such that

$$a_{\epsilon,h}(t; w, v_h) \leq C \|w_h\|_V \|v_h\|_{V_h} \leq \tilde{M} \|w_h\|_{V_h} \|v_h\|_{V_h} \quad (w_h, v_h \in V_h, \ t \in J),$$  

(67)

which is the desired inequality. \hfill $\square$
Proof of Theorem 3. Fix \( v_h \in V_h \) and \( t \in J \) arbitrarily. First, we mention that \( V_h \not\subset H^1_0(\Omega) \). Therefore
\[
a(t; v_h, v_h) \geq \alpha \|v_h\|_{H^1(\Omega)}^2 + \frac{1}{2} (b \cdot n v_h, v_h)_{L^2(\Gamma_{in})},
\]
and \((b \cdot n v_h, v_h)_{\Gamma_{in}} < 0\). Now we have the following:
\[
a_{\varepsilon,h}(t; v_h, v_h) \geq \alpha \|v_h\|_{H^1(\Omega)}^2 + \frac{1}{2} (b \cdot n v_h, v_h)_{L^2(\Gamma_{in})} - 2 \sum_{E \in E_h^c} (n \cdot \mu \nabla v_h, v_h)_{L^2(E)}
\]
\[
- (b \cdot n v_h, v_h)_{L^2(\Gamma_{in})} + \varepsilon \sum_{E \in E_h^c} h_E^{-1} \|v_h\|_{L^2(E)}^2
\]
\[
\geq \alpha \|v_h\|_{H^1(\Omega)}^2 - \frac{1}{2} (b \cdot n v_h, v_h)_{L^2(\Gamma_{in})} + \varepsilon \sum_{E \in E_h^c} h_E^{-1} \|v_h\|_{L^2(E)}^2
\]
\[
- 2 \left( \sum_{E \in E_h^c} h_E \|n \cdot \mu \nabla v_h\|_{L^2(E)}^2 \right)^{1/2} \left( \sum_{E \in E_h^c} h_E^{-1} \|v_h\|_{L^2(E)}^2 \right)^{1/2}.
\]
Recalling \(-(b \cdot n v_h, v_h)_{L^2(\Gamma_{in})} > 0\) and Lemma 4.1, then we have
\[
a_{\varepsilon,h}(t; v_h, v_h) \geq \alpha \|v_h\|_{H^1(\Omega)}^2 + \varepsilon \sum_{E \in E_h^c} h_E^{-1} \|v_h\|_{L^2(E)}^2 - 2C_i^{1/2} \|v_h\|_{H^1(\Omega)} \left( \sum_{E \in E_h^c} h_E^{-1} \|v_h\|_{L^2(E)}^2 \right)^{1/2}.
\]
Here we apply Young’s inequality to obtain
\[
2C_i^{1/2} \|v_h\|_{H^1(\Omega)} \left( \sum_{E \in E_h^c} h_E^{-1} \|v_h\|_{L^2(E)}^2 \right)^{1/2} \leq \frac{\alpha}{2} \|v_h\|_{H^1(\Omega)}^2 + 2\alpha C_i \sum_{E \in E_h^c} h_E^{-1} \|v_h\|_{L^2(E)}^2.
\]
Then we have
\[
a_{\varepsilon,h}(t; v_h, v_h) \geq \frac{\alpha}{2} \|v_h\|_{H^1(\Omega)}^2 + (\varepsilon - 2\alpha C_i) \sum_{E \in E_h^c} h_E^{-1} \|v_h\|_{L^2(E)}^2.
\]
Letting \( \tilde{\alpha} = \min\{\frac{\alpha}{2}, \varepsilon - 2\alpha C_i\} \) yields the desired conclusion.

5 Proof of Theorem 3 and 4

This section presents that \( B_{\varepsilon,h} \) is continuous and satisfies the inf-sup condition. First, we show Theorem 3.

Proof of Theorem 3. Recalling that every \( z \in X_V \) and \( z_h \in X_h \subset C^0(J; V_h) \) satisfy \( z(t) \in V \), \( z_h(t) \in V_h \) for any \( t \in J \), we can apply Theorem 1 to obtain
\[
B_{\varepsilon,h}(z, y_h) = \int_J \left( (\partial_t z, y_h)_{L^2(\Omega)} + a_{\varepsilon,h}(t; z, y_h) \right) dt + (z(0), \tilde{y}_h)_{L^2(\Omega)}
\]
\[
\leq \int_J \left( \|\partial_t z\|_{L^2(\Omega)} \|y_h\|_{L^2(\Omega)} + C \|z\|_{L^2(\Omega)} \|y_h\|_{L^2(\Omega)} \right) dt + \|z(0)\|_{L^2(\Omega)} \|\tilde{y}_h\|_{L^2(\Omega)}
\]
\[
\leq C \|z\|_{X_V} \|y_h\|_{V_h} \quad (z \in X_V, \ y_h = (y_h, \tilde{y}_h) \in Y_h),
\]
which is the first desired estimate. Moreover, \( \|z_h\|_{X_V} \leq C \|z_h\|_{X_h} \) leads to the second inequality.
Stating the proof of Theorem 4 requires an auxiliary operator $A_{\varepsilon, h}(t) : V_h \rightarrow V_h$ defined by setting
\[
(A_{\varepsilon, h}(t)w_h, v_h)_{L^2(\Omega)} = a_{\varepsilon, h}(t; w_h, v_h) \quad (w_h, v_h \in V_h, \; t \in J).
\]
We recall that the bilinear form $a_{\varepsilon, h}(t; \cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R}$ is continuous and coercive. Therefore, the Lax–Milgram theorem shows that the operator $A_{\varepsilon, h}(t)$ is invertible for $t \in J$. Now we have the following lemma.

**Lemma 5.1.** Operator $A_{\varepsilon, h}(t)^{-1} : V_h \rightarrow V_h$ satisfies
\[
\|A_{\varepsilon, h}(t)^{-1}v_h\|_{V_h} \leq \alpha^{-1}\|v_h\|_{V_h'}, \quad (v_h \in V_h, \; t \in J).
\]
\[
(v_h, A_{\varepsilon, h}(t)^{-1}v_h)_{L^2(\Omega)} \geq \alpha\hat{M}^{-2}\|v_h\|^2_{V_h'}
\]
for $v_h \in V_h$ and $t \in J$.

**Proof.** First, we have
\[
\hat{\alpha}\|A_{\varepsilon, h}(t)^{-1}v_h\|^2_{V_h} \leq (A_{\varepsilon, h}(t)(A_{\varepsilon, h}(t)^{-1}v_h), A_{\varepsilon, h}(t)^{-1}v_h)_{L^2(\Omega)}
\]
\[
= (v_h, A_{\varepsilon, h}(t)^{-1}v_h)_{L^2(\Omega)} \leq \|v_h\|_{V_h'}\|A_{\varepsilon, h}(t)^{-1}v_h\|_{V_h}
\]
for all $v_h \in V_h$. Therefore we have
\[
\|A_{\varepsilon, h}(t)^{-1}v_h\|_{V_h} \leq \alpha^{-1}\|v_h\|_{V_h'} \quad (v_h \in V_h, \; t \in J).
\]

Next, it is noteworthy that the constant $\hat{M}$ satisfies
\[
\sup_{0 \neq w_h \in V_h} \frac{(A_{\varepsilon, h}(t)w_h, v_h)_{V_h}}{\|w_h\|_{V_h} \cdot \|v_h\|_{V_h'}} \leq \hat{M}.
\]
Therefore,
\[
\|v_h\|_{V_h'} = \|A_{\varepsilon, h}(t)(A_{\varepsilon, h}(t)^{-1}v_h)\|_{V_h'} \leq \hat{M}\|A_{\varepsilon, h}(t)^{-1}v_h\|_{V_h}
\]
for all $v_h \in V_h$. This yields
\[
(v_h, A_{\varepsilon, h}(t)^{-1}v_h)_{L^2(\Omega)} = (A_{\varepsilon, h}(t)(A_{\varepsilon, h}(t)^{-1}v_h), A_{\varepsilon, h}(t)^{-1}v_h)_{L^2(\Omega)}
\]
\[
\geq \alpha\|A_{\varepsilon, h}(t)^{-1}v_h\|^2_{V_h} \geq \alpha\hat{M}^{-2}\|v_h\|^2_{V_h'} \quad (v_h \in V_h, \; t \in J).
\]

\[\square\]

**Proof of Theorem 4.** After fixing $z_h \in X_h$ arbitrarily, let $y_h := (A_{\varepsilon, h}(t)^{-1}(\partial_t z_h) + \delta z_h, \delta z_h(0)) \in Y_h$, where $\delta := \alpha^{-4}M^4$. Then we have
\[
\|y_h\|^2_{X_h} = \int_J \|A_{\varepsilon}(t)^{-1}(\partial_t z_h) + \delta z_h\|^2_{V_h} \, dt + \|\delta z_h(0)\|^2_{L^2(\Omega)}
\]
\[
\leq \max\{\alpha^{-1}, \delta\} \int_J (\partial_t z_h|_{V_h'}^2 + \|z_h\|^2_{V_h'}) \, dt + \delta\|z_h(0)\|^2_{L^2(\Omega)} \leq \max\{\alpha^{-1}, \delta\}\|z_h\|^2_{X_h},
\]

\[17\]
and
\[
B_{\varepsilon,h}(z_h, y_h) = \int_J \left( (\partial_t z_h, A_{\varepsilon,h}(t)^{-1}(\partial_t z_h) + \delta z_h)_{L^2(\Omega)} + a_{\varepsilon,h}(t; z_h, A_{\varepsilon,h}(t)^{-1}(\partial_t z_h) + \delta z_h) \right) dt \\
+ (z_h(0), \delta z_h(0))_{L^2(\Omega)}
\]
\[
\geq \int_J \left( \alpha \overline{M}^{-2} \|\partial_t z_h\|_{V_h'}^2 - \gamma_1 \overline{M} \|z_h\|_{V_h} \|\partial_t z_h\|_{V'_h} + \delta \gamma \|z_h\|_{V_h'}^2 \right) dt \\
+ \delta/2 (\|z_h(T)\|_{L^2(\Omega)}^2 - \|z_h(0)\|_{L^2(\Omega)}^2) + \delta \|z_h(0)\|_{L^2(\Omega)}^2
\]
\[
\geq \int_J \left( \alpha \overline{M}^{-2} \|\partial_t z_h\|_{V_h'}^2 - \alpha^{-1} \overline{M} \|z_h\|_{V_h} \|\partial_t z_h\|_{V'_h} + \delta \alpha \|z_h\|_{V_h'} \right) dt \\
+ \delta/2 \|z_h(0)\|_{L^2(\Omega)}^2
\]
\[
\geq \frac{1}{2} \min \left\{ \alpha \overline{M}^{-2}, \alpha^{-3} \overline{M}^4, \alpha^{-4} \overline{M}^4 \right\} \left( \int_J \left( \|\partial_t z_h\|_{V_h'}^2 + \|z_h\|_{V_h'}^2 \right) dt + \|z_h(0)\|_{L^2(\Omega)}^2 \right)
\]
\[
= \frac{1}{2} \min \left\{ \alpha \overline{M}^{-2}, \alpha^{-3} \overline{M}^4, \alpha^{-4} \overline{M}^4 \right\} \|z_h\|_{V_h'}^2.
\]

The two inequalities above imply that
\[
a_{\varepsilon,h}(z_h, y_h) \geq \beta \|z_h\|_{X_h} \|y_h\|_{Y_h}, \tag{76}
\]
where \( \beta = 1/2 \min \left\{ \alpha \overline{M}^{-2}, \alpha^{-3} \overline{M}^4, \alpha^{-4} \overline{M}^4 \right\} \min \left\{ \alpha^{1/2}, \alpha^{-2} \overline{M}^{-2} \right\} \). Therefore we have
\[
\inf_{0 \neq z_h \in X_h} \sup_{0 \neq y_h \in Y_h} \frac{a_{\varepsilon,h}(z_h, y_h)}{\|z_h\|_{X_h} \|y_h\|_{Y_h}} \geq \beta \tag{77}
\]

The inf-sup condition follows.

\[\square\]

6 Proof of Theorem 9

We need the following lemma, which is a direct consequence of Assumptions II and IV (see [16] Lemma 2.3 for example).

Lemma 6.1. Letting \( l \) be an integer with \( 2 \leq l \leq k + 1 \), then there exist two positive constants \( C_1 \) and \( C_2 \) such that
\[
\|v - \Pi_h v\|_{V_h} \leq C_1 h^{l-1} \|v\|_{H^l(\Omega)}, \tag{78a}
\]
\[
\|v - \Pi_h v\|_{V} \leq C_2 h^{l-1} \|v\|_{H^l(\Omega)} \tag{78b}
\]

for all \( v \in H^l(\Omega) \).

We recall that the exact solution is \( u \) assumed to belong to \( X_{l,m} = H^1(J; H^l(\Omega), H^m(\Omega)) \) for \( 2 \leq l, m \leq k + 1 \). Then \( \Pi_h \) defines a projection from \( X_{l,m} \) to \( X_h \). We designate by \( \Pi_h \) again, i.e.,
\[
(\Pi_h u)(t) = \Pi_h u(t) \quad (t \in J), \quad \partial_t (\Pi_h u)(t) = \Pi_h (\partial_t u(t)) \quad (t \in J).
\]

The estimate in Theorem 8 is valid for \( z_h = \Pi_h u \), which gives the following proof.
\textbf{Proof of Theorem 8.} It is readily apparent that \( X_{\ell,m} \subset X_V \). Therefore Theorem 3 implies
\[
\|u - u_{\varepsilon,h}\|^2_{X_h} \leq \inf_{z_h \in X_h} C\|u - z_h\|^2_{X_V} \leq \|u - \Pi_h u\|^2_{X_V} = \int_j \left( \|u - \Pi_h u\|^2_{V^n} + \|\partial_t(u - \Pi_h u)\|^2_{V^n} \right) dt + \|u(0) - \Pi_h u(0)\|^2_{L^2(\Omega)}.
\]
We can estimate that
\[
\|u - \Pi_h u\|^2_{V^n} \leq C h^{2(l-1)}\|u\|_{H^l(\Omega)},
\]
\[
\|\partial_t(u - \Pi_h u)\|^2_{V^n} \leq \|\partial_t u - \Pi_h (\partial_t u)\|_{L^2(\Omega)} \leq C h^{2m}\|\partial_t u\|_{H^m(\Omega)}.
\]
Moreover, because
\[
X_{\ell,m} = H^1(J; H^l(\Omega), H^m(\Omega)) \subset H^1(J; H^l(\Omega), H^l(\Omega)) \subset C^0(\mathcal{J}; H^l(\Omega)),
\]
where \( j = \min\{l, m\} \), we have \( u(0) \in H^j(\Omega) \), and
\[
\|u(0) - \Pi_h u(0)\|^2_{L^2(\Omega)} \leq C h^{2j}\|u(0)\|^2_{H^j(\Omega)}.
\]
Summing up those estimates, we complete the proof. \( \square \)

\section{7 Full discretization}

Let \( N \in \mathbb{N} \) be the number of time steps, \( \tau = T/N \) and \( t_n = n\tau \). We now consider the temporal discretization with implicit Euler (backward Euler) method.

\( \mathbf{P}_{\varepsilon,h,\tau} \) Find \( \{u_{\varepsilon,h,\tau}^n\}^{N}_{n=0} \in (V_h)^{N+1} \) such that
\[
\frac{1}{\tau}(u_{\varepsilon,h,\tau}^n - u_{\varepsilon,h,\tau}^{n-1}, v_h)_{L^2(\Omega)} + a_{\varepsilon,h}(t_n; u_{\varepsilon,h,\tau}^n, v_h) = F_{\varepsilon,h}(t_n; v_h) \quad (v_h \in V_h),
\]
\[
(u_{\varepsilon,h,\tau}^0 - u_0, \overline{v}_h)_{L^2(\Omega)} = 0 \quad (\overline{v}_h \in V_h)
\]

Clearly, \( u_{\varepsilon,h,\tau}^0 = u_{\varepsilon,h}(0) \), where \( u_{\varepsilon,h} \in X_h \) represents the solution of \( \mathbf{P}_{\varepsilon,h} \).

\textbf{Lemma 7.1.} The problem \( \mathbf{P}_{\varepsilon,h,\tau} \) has a unique solution \( \{u_{\varepsilon,h,\tau}^n\}^{N}_{n=0} \in (V_h)^{N+1} \).

\textbf{Proof.} It is readily apparent that \( u_{\varepsilon,h,\tau}^0 \in V_h \). We take the basis functions \( \{\phi_i\}_{i=1}^N \) of \( V_h \), where \( N = \dim V_h \), and
\[
u_{\varepsilon,h,\tau}^n := \sum_{i=1}^N u_{\varepsilon,h,\tau}^n \phi_i.
\]

Then, the equation (83a) implies the following for \( n = 1, \cdots, N \),
\[
(M + \Delta t A^n) u_{\varepsilon,h,\tau}^n = Mu_{\varepsilon,h,\tau}^{n-1} + \Delta t F^n,
\]
where \( (M)_{ij} := (\phi_i, \phi_j)_{L^2(\Omega)}, (A^n)_{ij} := a_{\varepsilon,h}(t_n; \phi_i, \phi_j), (F^n)_i := F_{\varepsilon,h}(t_n; \phi_i) \) and \( u_{\varepsilon,h,\tau}^n := (u_{\varepsilon,h,\tau,1}^n, \cdots, u_{\varepsilon,h,\tau,N})^T \). Theorem 2 gives \( M + \Delta t A^n \) as a positive definite matrix. Therefore, there exists a unique \( u_{\varepsilon,h,\tau}^n \) for any \( u_{\varepsilon,h,\tau}^{n-1} \). \( \square \)
Let
\[ S_r = \left\{ w_h : [0, T] \rightarrow V_h \mid \text{For all } n = 1, \ldots, N, \text{ there exists } v_h \in V_h \text{ such that } w_h|_{[t_{n-1}, t_n]} = v_h, \text{ and } w_h(0) \in V_h \right\}. \] (86)
and \( v_{h, \tau}(t_0^+) = \lim_{t \rightarrow t_0^+} v_{h, \tau}(t) \) for \( v_{h, \tau} \in S_r \). It is noteworthy that \( S_r \subset L^2(J; V_h) \). The solution of \( (P_{\varepsilon, h, \tau}) \) can be extended to an element of \( S_r \) as
\[
u_{\varepsilon, h, \tau}(t) = \begin{cases} u_0 & \text{if } t = 0, \\ u_{\varepsilon, h, \tau}^{n+1} & \text{if } t \in (t_n, t_{n+1}] \quad (n = 0, \ldots, N - 1). \end{cases} \] (87)

One can show that \( u_{\varepsilon, h, \tau} \in S_r \) satisfies the following estimate.

**Lemma 7.2.** Assuming that \( u_{\varepsilon, h} \in C^2(J; V_h) \), then let \( \Pi_r u_{\varepsilon, h} \in S_r \) be
\[
\Pi_r u_{\varepsilon, h}(t) = \begin{cases} u_0 & \text{if } t = 0, \\ u_{\varepsilon, h}(t_{n+1}) & \text{if } t \in (t_n, t_{n+1}] \quad (n = 0, \ldots, N - 1). \end{cases} \] (88)
and \( \rho = u_{\varepsilon, h, \tau} - \Pi_r u_{\varepsilon, h} \in S_r \), where \( u_{\varepsilon, h} \in X_h \) is the solution of \( (P_{\varepsilon, h}) \). Based on the relations presented above,
\[
\max_{n=0, \ldots, N-1} \| \rho(t_n^+) \|_{L^2(\Omega)} \leq \frac{\sqrt{T}}{8\alpha} \| \partial_t^2 u_{\varepsilon, h} \|_{L^\infty(J; L^2(\Omega))},
\] (89a)
\[
\| \rho \|_{L^2(J; V_h)} \leq \frac{\sqrt{T}}{2\alpha} \| \partial_t^2 u_{\varepsilon, h} \|_{L^\infty(J; L^2(\Omega))}.
\] (89b)

**Proof.** Clearly, \( \rho(t_0) = 0 \). In fact, \( u_{\varepsilon, h} \in C^2(J; V_h) \) and the definition of \( u_{\varepsilon, h, \tau} \) yield
\[
u_{\varepsilon, h}^{n+1} = u_{\varepsilon, h, \tau}(t_n^+) = u_{\varepsilon, h, \tau}(t_{n+1}),
\] (90a)
\[
u_{\varepsilon, h}(t_{n+1}) = \Pi_r u_{\varepsilon, h}(t_n^+) = \Pi_r u_{\varepsilon, h}(t_{n+1}).
\] (90b)

Using (90a), equation (83a) is written, equivalently, as
\[
(u_{\varepsilon, h, \tau}(t_n^+) - u_{\varepsilon, h, \tau}(t_n); v_{h, \tau}(t_n^+))_{L^2(\Omega)} + \tau a_{\varepsilon, h}(t_{n+1}; u_{\varepsilon, h, \tau}(t_n^+), v_{h, \tau}(t_n^+)) = \tau F_{\varepsilon, h}(t_{n+1}; v_{h, \tau}(t_n^+)) \quad (v_{h, \tau} \in S_r). \] (91)

Furthermore, equations (90b) and (19) yield
\[
(\Pi_r u_{\varepsilon, h}(t_n^+) - \Pi_r u_{\varepsilon, h}(t_n) + \tau^2/2 \partial_t^2 u_{\varepsilon, h}(c_n); v_{h, \tau}(t_n^+))_{L^2(\Omega)} + \tau a_{\varepsilon, h}(t_{n+1}; \Pi_r u_{\varepsilon, h}(t_n^+), v_{h, \tau}(t_n^+)) = \tau F_{\varepsilon, h}(t_{n+1}; v_{h, \tau}(t_n^+)) \quad (v_{h, \tau} \in S_r) \] (92)
for some \( c_n \in (t_n, t_{n+1}] \). By the two equations presented above, setting \( v_{h, \tau} = \rho \in S_r \) leads to
\[
(\rho(t_n^+) - \rho(t_n))_{L^2(\Omega)} - \tau^2/2 \| \partial_t^2 u_{\varepsilon, h}(c_n) \|_{L^2(\Omega)} + \| a_{\varepsilon, h}(t_{n+1}; \rho, \rho) \|_{L^2(\Omega)} = 0. \] (93)

Now we can estimate
\[
(\rho(t_n^+) - \rho(t_n))_{L^2(\Omega)} = \frac{1}{2} \left( \| \rho(t_n^+) - \rho(t_n) \|_{L^2(\Omega)}^2 + \| \rho(t_n^+) \|_{L^2(\Omega)}^2 - \| \rho(t_n) \|_{L^2(\Omega)}^2 \right)
\geq \frac{1}{2} \left( \| \rho(t_n^+) \|_{L^2(\Omega)}^2 - \| \rho(t_n) \|_{L^2(\Omega)}^2 \right),
\]
\[-\tau^2/2(\partial_t^2 u_{\varepsilon,h}(t_n))_{L^2(\Omega)} \geq -\tau^{3/2}/2\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}\|\rho\|_{L^2(t_n,t_{n+1};V_h)} \geq -\left(\frac{\tau^3}{16q}\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}^2 + q\|\rho\|_{L^2(t_n,t_{n+1};V_h)}^2\right)\]

for any \( q \in \mathbb{R} \), and

\[\tau a_{\varepsilon,h}(t_{n+1};\rho,\rho) \geq \hat{\alpha}\|\rho\|_{L^2(t_n,t_{n+1};V_h)}^2.\]

Therefore, letting \( q = \frac{\hat{\alpha}}{2} \) yields

\[\|\rho(t_n^+)\|_{L^2(\Omega)}^2 \leq \|\rho(t_{n-1}^-)\|_{L^2(\Omega)}^2 + \frac{\tau^3}{8\hat{\alpha}}\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}^2\]

\[\leq \|\rho(t_0)\|_{L^2(\Omega)}^2 + \frac{\tau^3(n+1)}{8\hat{\alpha}}\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}^2\]

\[\leq \frac{T}{8\hat{\alpha}}\tau^2\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}^2\]

for all \( n = 0, \ldots, N = 1 \). Furthermore, if we set \( q = \frac{\hat{\alpha}}{2} \), then we have

\[\hat{\alpha}\|\rho\|_{L^2(t_n,t_{n+1};V_h)}^2 + \|\rho(t_n^+)\|_{L^2(\Omega)}^2 - \|\rho(t_n^-)\|_{L^2(\Omega)}^2 \leq \frac{\tau^3}{4\hat{\alpha}}\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}^2.\] (95)

Summing this from \( n = 0 \) to \( n = N - 1 \) gives

\[\|\rho\|_{L^2(J;V_h)}^2 \leq \frac{1}{\hat{\alpha}}\|\rho(t_0)\|_{L^2(\Omega)}^2 + \frac{\tau^3N}{4\hat{\alpha}^2}\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}^2\]

\[\leq \frac{T}{4\hat{\alpha}^2}\tau^2\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}^2.\]

\[\square\]

**Theorem 10** (Error estimate). Let \( l, m \) be two integers satisfying \( 2 \leq l, m \leq k+1 \). Assuming that \( u \in X_{l,m} \) and \( u_{\varepsilon,h} \in C^2(J;V_h) \), then there exists a positive constant \( C \) such that

\[\|u - u_{\varepsilon,h}\|_{L^2(J;H^1(\Omega))} \leq C\left(\int_J (h^{2l-1}\|u\|_{H^1(\Omega)} + h^{2m}\|\partial_t u\|_{H^m(\Omega)}) \, dt + h^{2j}\|u(0)\|_{H^j(\Omega)} + \tau^2\|\partial_t u_{\varepsilon,h}\|_{L^2(J;H^1(\Omega))} + \frac{T}{4\hat{\alpha}^2}\tau^2\|\partial_t^2 u_{\varepsilon,h}\|_{L^\infty(J;L^2(\Omega))}^2\right),\]

where \( j := \min\{l, m\} \).

**Proof.** We have

\[\|u - u_{\varepsilon,h}\|_{L^2(J;H^1(\Omega))} \leq \|u - u_{\varepsilon,h}\|_{X_h} + \|u_{\varepsilon,h} - \Pi_r u_{\varepsilon,h}\|_{L^2(J;H^1(\Omega))} + \|\Pi_r u_{\varepsilon,h} - u_{\varepsilon,h}\|_{L^2(J;V_h)}.\]

Here Theorem 9, Lemma 7.2, and the approximation error estimate for piecewise constant yield the result. \(\square\)

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That is, we let problem has a unique solution for any \( \varepsilon, h, \tau \) and \( \mu (x, t) = I, a(x, t) = (1, 1)^T \) and \( c(x, t) = 1 \). One can readily check that this problem has a unique solution for any \( f \in L^2(J; H^{-1}(\Omega)) \). We let

\[
    f(x, y, t) := \left( (x+y+2t-2t^2+2\pi^2) \sin(\pi x) \sin(\pi y) + (\pi - 2\pi t) \cos(\pi x) \sin(\pi y) + (\pi - 2\pi t) \sin(\pi x) \cos(\pi y) \right) e^{(x+y-1)t},
\]

Then \( f \in L^2(J; H^1(\Omega)) \), and

\[
    u(x, y, t) := \sin(\pi x) \sin(\pi y) e^{(x+y-1)t}
\]

is the unique solution. In Figure 2, we show the exact solution at different time steps.

We use the \((k, k)\)-th degree B-spline basis functions for spatial discretization using the uniform mesh and implicit Euler scheme for temporal discretization, where \( k = 1, 2 \): we consider the approximate problem \((83a)\). We let \( \tau = O(h^k) \), where \( h \) is the mesh size for uniform mesh. Then we know from Theorem \( 10 \) that

\[
    \| u - u_{\varepsilon, h, \tau} \|_{L^2(J; H^1(\Omega))} \leq C h^k.
\]

Figure 2: Exact solutions at different time steps.

Figure 3: Numerical solutions of the Nitsche method at different time steps by consideration of a uniform mesh with mesh size \( h = 1/30 \).

8 Numerical examples

Our example is given as \( \Omega := (0, 1)^2, J = (0, 4) \), and

\[
    \begin{aligned}
    \partial_t u - \Delta u + (1, 1)^T \cdot \nabla u + u &= f(x, t) \quad \text{in } \Omega \times J, \\
    u &= 0 \quad \text{on } \Gamma \times J, \\
    u(x, 0) &= \sin(\pi x) \sin(\pi y) \quad \text{for } x = (x, y) \in \Omega.
    \end{aligned}
\]

8 Numerical examples
Figure 4: Numerical solutions of the Nitsche method on boundary Γ, at different time steps by consideration of a uniform mesh with mesh size $h = 1/30$.

Figure 5: $L^2(J;H^1(\Omega))$ error on a uniform mesh.

As shown in Figure 3, this report describes the numerical solution shape. Then we show the boundary value of the numerical solution in Figure 4. The weak imposition of the Dirichlet boundary condition is actually observed because the boundary value does not vanish.

Furthermore, this report describes the error for uniform mesh in Figure 5, which shows that the rate of convergence is approximately equal to $k$, which is expected by the theory.

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