Platonic polyhedra tune the three-sphere: III. Harmonic analysis on octahedral spherical three-manifolds

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Abstract
From the homotopy groups of three distinct octahedral spherical three-manifolds we construct the isomorphic groups $H$ of deck transformations acting on the three-sphere. The $H$-invariant polynomials on the three-sphere constructed by representation theory span the bases for the harmonic analysis on three spherical manifolds. Analysis of the Cosmic Microwave Background in terms of these new bases can reveal a non-simple topology of the space part of space–time.

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1. Introduction

We view a spherical topological three-manifold $M$, see [12, 14], as a prototile on its cover $\tilde{M} = S^3$. We studied in [7] the isometric actions of $O(4, R)$ on the three-sphere $S^3$ and gave its basis as well-known homogeneous Wigner polynomials in equation (37) of [5]. An algorithm due to Everitt in [3] generates the homotopies for all spherical three-manifolds $M$ from five Platonic polyhedra. Using intermediate Coxeter groups, we construct deck transformations acting on the three-sphere as isomorphic images [12] of homotopies and generate the groups $H = \text{deck}(M) \sim \pi_1(M)$. Following work on the Poincaré dodecahedral [5, 6], the tetrahedral [7] and two cubic spherical manifolds [8], we turn here to three octahedral spherical manifolds denoted in [3] as $N4, N5, N6$. We construct a basis for the harmonic analysis on each manifold from $H$-invariant polynomials on the three-sphere.

One field of applications for harmonic analysis is cosmic topology [10, 11]: The topology of a three-manifold $M$ is favoured if data from the Cosmic Microwave Background can be expanded in its harmonic basis. The present work provides three novel octahedral three-manifolds for this analysis. For the notions of homotopic boundary conditions and random point symmetry we refer to [9].

2. The Coxeter group $G$ and the 24-cell on $S^3$

The Cartesian coordinates $x = (x_0, x_1, x_2, x_3) \in E^4$ for $S^3$ we combine, as in [5, 7], in the matrix form

$$ u = \begin{bmatrix} z_1 \\ -z_2 \\ z_1 \\ z_2 \end{bmatrix}, \quad z_1 = x_0 + i x_3, $$
$$ z_2 = -x_2 - i x_1, \quad z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1. \quad (1) $$

For the group action, we start from the Coxeter group $G < O(4, R)$ ([4] and [3, p 254]) with the diagram

$$ G = : o - o - o - o. \quad (2) $$

For the Coxeter diagram equation (2), we give for the four Weyl reflections $W_s, s = 1, 2, 3, 4$ the Weyl vectors $a_s$ in table 1 and compute for each $a_s = (a_{s0}, a_{s1}, a_{s2}, a_{s3})$ the matrix

$$ v_s := \begin{bmatrix} a_{s0} - i a_{s3} & a_{s2} - i a_{s1} \\ -a_{s2} - i a_{s1} & a_{s0} + i a_{s3} \end{bmatrix} \in \text{SU}(2, C). \quad (3) $$

The matrices $v_s$ are used to relate, see [7], the Weyl reflections to $(\text{SU}(2, C) \times \text{SU}(2, C))$ acting by left and right multiplication on the coordinates equation (1). We include the (orientation preserving) inversion $J_s \in G$, and list the additional Weyl reflection $W_0$. The Coxeter group equation (2) is of order $|G| = 48 \times 24 = 1152$. The first three
Fourfold

are given between square brackets.

By a final rotation w.r.t. the centre of the prototile, the

An edge gluing scheme lists glued triples of oriented

in $\Theta := \exp(i\pi/4)$.

| $s$ | Weyl vector $a_s$ | Matrix $v_s$ |
|-----|------------------|-------------|
| 1   | $(0, \sqrt{2}/2, -\sqrt{2}/2, 0)$ | $\begin{bmatrix} 0 & \frac{\theta}{\sqrt{2}} \\ \frac{\theta}{\sqrt{2}} & 0 \end{bmatrix}$ |
| 2   | $(0, 0, 0, 1)$ | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| 3   | $(1, 0, 0, 0)$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| 4   | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$ | $\begin{bmatrix} \frac{\theta}{\sqrt{2}} & 0 \\ 0 & \frac{\theta}{\sqrt{2}} - \theta \end{bmatrix}$ |

Table 1. The Weyl vectors $a_s, s = 1, \ldots, 4$, and $a_0$ for the Coxeter group $G$ equation (2), and the $2 \times 2$ unitary matrices $v_s$ equation (3), in terms of $\theta := \exp(i\pi/4)$.

Figure 1. The octahedron projected to the plane with faces $F_1, \ldots, F_8$ and directed edges $e_1, \ldots, e_{12}$ according to [3]. The products of Weyl reflections $(W_iW_j)$ and $(W_kW_l)$ generate right-handed three fold and four fold rotations, respectively.

Weyl reflections from table 1 generate, see [4], the cubic Coxeter subgroup

$$O := \langle \circ \circ \circ \circ \rangle,$$

isomorphic to the octahedral group $O \cong (C_2)^3 \times S(3)$ acting on $E^3 \in E^3$. The octahedral tiling of $S^3$ is the 24 cell discussed in [13, pp 171–2]. The centre positions of the 24 octahedra in the octahedral 24-cell tiling are the midpoints of the 24 square faces of the eight cubes in the eight-cell tiling shown in [8], figure 1. As shown in [13, pp 178–9] vertices of six octahedra are located at each centre of a cube from the eight-cell.

3. From homotopies to deck transformations

3.1. Generators

The spherical Coxeter group $G$ is generated by the Weyl reflections $W_i$ given in table 1. In the next section, we give for the three octahedral manifolds $N4, N5, N6$ the edge gluing schemes computed in [3], but include the corrections given in [1]. These corrections apply in particular to the manifold $N5$. The construction proceeds in the following steps:

(i) An edge gluing scheme lists glued triples of oriented edges for pairs of glued faces $Fi \cup Fj$ in its rows. The four generators of the first homotopy group each prescribe a gluing of three oriented chains of edges, bounding counterclockwise a preimage face $Fi$ and clockwise an image face $Fj$ of the prototile. These chains taken from figure 1 are given between square brackets.

(ii) Any deck transformation is constructed from a homotopy by first rotating the preimage face $Fi$ w.r.t. the centre $(1, 0, 0, 0)$ of the prototile to the position of face $F_4$, and then applying a rotation $(W_iW_j)^v, v = 0, 1, 2$ preserving the centre of $F4$. Inversion $J_4$ in the centre of the prototile then maps the preimage face from the position of face $F4$ to the one of $F6$. This inversion can be expressed as $J_3 = J_2W_0$. The total inversion $J_4$ preserves orientation and commutes with all rotations. Applying the Weyl reflection $W_4$, the preimage face now in position $F_6$ is mapped into itself, whereas the octahedral prototile is mapped into an image tile.

(iii) By a final rotation w.r.t. the centre of the prototile, the preimage face is mapped from the position of $F_6$ into the image position of face $F_j$. An appropriate choice of $v$ yields the edge mapping prescribed by the homotopy. By virtue of the Weyl reflection, the image face $F_j$ separates the prototile from a fixed octahedral image. The orientation of the chain of edges of the image face is now counterclockwise when referring to the centre of the image tile. The map from the prototile to the image tile in this position is the deck transformation isomorphic to the homotopy.

All the operations in (i)–(iii) are elements of the Coxeter group $G$ and, moreover of $SO(4, R)$. The rotations are generated from the threefold rotation $(W_iW_j)$ and the fourfold rotation $(W_2W_3)$, indicated in figure 1. Any Weyl reflection $W_i$ is associated with a $2 \times 2$ matrix $u_i$ given in table 1. Products of two Weyl reflections generate rotations. The conversion from these products to rotations $g = (u_1, u_2)$ is given from equation (60) of [7] by

$$(W_iW_j) \rightarrow T_g = T_{(u_1, u_2)}, \quad g = (u_1, u_2) = (u_1u_2^{-1}, u_1^{-1}u_2).$$

The operator $T_g$ acts on functions $f(u)$ on $S^3$ in coordinates $u$ from equation (1) as

$$(T_{(u_1, u_2)}f)(u) := f(u_1^{-1}u_2).$$

Any product of the generally five operations described under (i)–(iii) is a deck transformation, preserves orientation and is isomorphic to a homotopic gluing. We list them for the three manifolds. Finally, the deck transformations are converted by use of equation (5), table 1, and multiplication into pairs $(u_1, u_2) \in (SU^2(2, C) \times SU^2(2, C))$, given in the following tables.

The (isomorphic) groups $H$ of homotopies and of deck transformations, distinct for different manifolds, all have order 24 equal to the number of octahedral tiles. These groups if not Abelian must appear in the table of Coxeter and Moser [2, pp 134–5].
3.2. Centre positions under deck transformations

From the coordinates equation (1) of $S^3$, the centre $u = e$ of the octahedral prototile is transformed by $g = (w_1, w_i)$ into an image centre
$$g = (w_3, w_i) \in H : e \rightarrow u' = w_1^{-1}w_2.$$  
(7)

Since all three groups of deck transformations must produce the same 24 cell, it follows that their lists of octahedral centres must coincide up to permutations. For the manifold $N6$, we shall find that its group of deck transformations is the binary tetrahedral group $T$ with all elements of the form $g = (w_1, e)$. From equation (7) it then follows that the list of the 24 octahedral centres $u'$ in the 24 cell can be written as
$$N6 : g = (w_1, e) \in T, u' = w_1^{-1}. 
(8)$$

with $w_1$ running over all group elements in table 8. The elements $g = (w_1, w_i)$ of the groups $H \neq T$ for the manifolds $N4, N5$ must therefore reproduce by the products $u'$ in equation (7) these 24 centre positions. For the manifold $N5$, we display this relation in table 6.

3.3. Harmonic analysis on octahedral-three manifolds

Once we have derived the explicit matrix form of the three groups $H$ of deck transformations, we have all the tools for the harmonic analysis. From any $g = (w_1, w_i) \in \text{SO}(4,R)$ we can, as outlined in general in equation (44) of [7], pass to its representations $D^{(j)}(g) = D^{(j)}(w_1) \times D^{(j)}(w_i)$ by use of Wigner representation matrices $D^{(j)}$ of $\text{SU}(2,\mathbb{C})$. From these representations we can construct the general projection and Young operators equation (82) of [7] to $H$-invariant polynomials of fixed $j$ and degree 2$j$. The projection yields linear combinations of spherical harmonics or Wigner polynomials $D_{\alpha_l,\nu}(u)$ of degree $2j$. For the octahedral manifold $N4$, we give the final result of this projection in table 4. The characters follow from equation (45) of [7] and allow us to derive by equation (62) of [7] the multiplicities for any degree $2j$ and group $H$.

4. Manifold $N4$

Edge gluing scheme:

$$
\begin{bmatrix}
1 & 4 & 9 \\
2 & 7 & 12 \\
3 & 6 & 10 \\
5 & 8 & 11
\end{bmatrix}. 
(9)
$$

Edge and face gluing generators of $\pi_1(N4)$:

$$
g_1 : 6 \cup 2, \begin{bmatrix} 10 & 7 \\ 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 10 \end{bmatrix},
$$

$$
g_2 : 5 \cup 3, \begin{bmatrix} 5 & 9 \\ 6 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 4 \\ 3 & 4 \end{bmatrix},
$$

$$
g_3 : 1 \cup 4, \begin{bmatrix} 9 & 1 \\ 2 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 12 \\ 4 \end{bmatrix},
$$

$$
g_4 : 7 \cup 8, \begin{bmatrix} 7 & 11 \\ 8 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 12 \\ 8 \end{bmatrix}. 
(10)
$$

Table 2. Generators $g = (w_1, w_i)$ of deck($N4$) in the scheme equations (5) and (6). We use the short-hand notation of table 8.

| $g$ | $w_1$ | $w_i$ |
|-----|-------|-------|
| $g_1$ | $-\alpha_2$ | $\mu$ |
| $g_2$ | $-\alpha_2$ | $-\epsilon$ |
| $g_3$ | $\epsilon_2$ | $\nu$ |
| $g_4$ | $\epsilon_2$ | $\omega$ |

Table 3. The elements $g = (w_1, w_i)$ of the group deck($N4$) = $C^1 \times Q^1$ in the notation of table 8.

| Subgroup | Elements |
|----------|----------|
| $C_1^1$ | $-\alpha_2, e, (\epsilon_2^2, e), ((-\epsilon_2^2), e) = (e, e)$ |
| $Q^1$ | $(e, \pm e), (e, \pm \mu), (e, \pm \nu), (e, \pm \omega)$ |

Isomorphic generators of deck($N4$):

$$
g_1 = (W_3W_5)^2(W_1W_2)(W_4W_0)J_4,
$$

$$
g_2 = (W_3W_5)(W_1W_2)(W_4W_0)(W_2W_5)J_4,
$$

$$
g_3 = (W_1W_2)(W_4W_0)(W_2W_5)^3(W_1W_4)J_4,
$$

$$
g_4 = (W_1W_2)(W_5W_2)(W_3W_2)(W_4W_0)(W_3W_1)J_4. 
(11)
$$

See table 2. The generators $g = (w_1, w_i)$ have for $w_1$ the order 6 or 3 and $w_i^3 = \pm e$, for $w_i$ the order 4 and $w_i^2 = -e$. From this it follows that $g_3^4 = (\pm e, -w_i) \sim (e, \pm w_i)$. It is easy to see that the four elements $g_1^2$ generate the quaternion group by right action, which we denote by $Q^1$. Similarly, the powers 4 of the generators fulfill $g_2^4 = (w_1^4, -e) \sim (w_1^{-4}, e)$ and so act from the left. Inspection of these elements shows that they can be written as powers of $(-\epsilon_2, e)$. The group generated by them is a cyclic group of order 3, which we denote as $C_3^1$. Now it is easy to conclude that the two subgroups generate the direct product group $C_3^1 \times Q^1$ of order 24, compare Coxeter and Moser [2, pp 134–5], as the group of homotopies and of deck transformations for the three-manifold $N4$.

See table 3. The 24 centre positions $u' = w_1^{-1}w_i$ of $C_3^1 \times Q^1$ reproduce the elements of the binary tetrahedral group table 8.

For the projection to an $H$-invariant basis we first diagonalize the generator $-\alpha_2 \in C_3^1$.

$$
-\alpha_2 = c \left[ \begin{array}{cc} \exp \left( \frac{2\pi i}{3} \right) & 0 \\
0 & \exp \left( -\frac{2\pi i}{3} \right) \end{array} \right] c^t, 
(12)
$$

$$
c = \left[ \begin{array}{cc} (1 - i) \frac{-1 + \sqrt{3}}{2\sqrt{3 + \sqrt{3}}} & -(1 - i) \frac{1 + \sqrt{3}}{2\sqrt{3 + \sqrt{3}}} \\
\frac{1}{\sqrt{3 + \sqrt{3}}} & \frac{1}{\sqrt{3 + \sqrt{3}}} \end{array} \right]. 
$$

Upon the coordinate transform from $u$ to $c^t u$, we can replace the matrix $-\alpha_2$ by its diagonal representative. Now the projection to the identity representations of $C_3^1$ simply requires $m_1 \rightarrow \rho \equiv 0 \mod 3$ and excludes any other value of $m_1$. Next, we consider the group $Q^1$ acting from the right. We simply transcribe the result on the group $Q$ from [8, table 10] from left to right action. Combining left and right action into $C_3^1 \times Q^1$, we arrive at the $H$-invariant basis of the harmonic analysis on $N4$ given in table 4.
Table 4. The \( C^3 \times Q^0 \)-invariant basis for the manifold \( N4 \) in terms of Wigner polynomials \( D^j \). Only integer values of \( j \) appear. The coordinate transform \( u \to u' = c^j u \) in \( D^j(u) \) follows with \( c \) from equation (12).

\[
\begin{align*}
  j = \text{odd}, \; j \geq 3, \; m_2 &= \text{even}, \; 0 < m_2 \leq j, \; m_1 = \rho \equiv 0 \bmod 3: \\
  \phi_{w_2}^{j, \text{odd}} &= [D_{j, m_2}(u') - D_{j, m_2}(u')] \\
  j = \text{even}, \; m_2 = 0, \; m_1 = \rho \equiv 0 \bmod 3: \\
  \phi_{w_2}^{j, \text{even}} &= D_{j, \rho, 0}(u') \\
  j \geq 2, \; \text{even}, \; 0 < m_2 \leq j, \; m_2 = \text{even}, \; m_1 = \rho \equiv 0 \bmod 3: \\
  \phi_{w_2}^{j, \text{even}} &= [D_{j, m_2}(u') + D_{j, m_2}(u')] \\
\end{align*}
\]

Table 5. Generators \( g = (u_1, u_i) \) of deck(\(N5\)) with partial use of table 8. Note that the matrices \((w_1, u_i)\) for the generators \(g_1, g_3\) do not occur in table 8 and so do not belong to the binary tetrahedral group.

| \( g \) | \( u_1 \) | \( u_i \) |
|-----|-----|-----|
| \( g_1 \) | \( \frac{\sqrt{2}}{2} [1 -1 i -1] \) | \( \frac{\sqrt{2}}{2} [1 -1 i -1] \) |
| \( g_2 \) | \( \alpha_2 \) | \( v \) |
| \( g_3 \) | \( [0 \; \theta \; 0] \) | \( \sqrt{\frac{1}{2}} [1 1 i -1] \) |
| \( g_4 \) | \( \alpha_2 \) | \( e \) |

5. Manifold \( N5 \)

Edge gluing scheme:

\[
\begin{pmatrix}
1 & 4 & 9 \\
2 & 7 & 12 \\
3 & 6 & 8 \\
5 & 10 & 11
\end{pmatrix}
\]

(13)

Edge and face gluing generators of \( \pi_1(N5) \):

\[
g_1: 6 \cup 8, \begin{bmatrix} 10 & 6 \\ 2 & 7 \\ 3 & 6 \\ 5 & 10 \end{bmatrix} \to \begin{bmatrix} 5 & 8 \\ 12 \end{bmatrix},
\]

\[
g_2: 1 \cup 4, \begin{bmatrix} 9 & 2 \\ 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \end{bmatrix},
\]

\[
g_3: 2 \cup 7, \begin{bmatrix} 2 & 3 \\ 10 & 11 \end{bmatrix} \to \begin{bmatrix} 7 & 8 \end{bmatrix},
\]

\[
g_4: 3 \cup 5, \begin{bmatrix} 11 & 4 \\ 3 & 6 \end{bmatrix} \to \begin{bmatrix} 5 & 6 \end{bmatrix},
\]

Isomorphic generators of deck(\(N5\)):

\[
g_1 = (W_1 W_2)(W_3 W_2)(W_4 W_0) J_4,
\]

\[
g_2 = (W_2 W_1)(W_4 W_0)(W_2 W_3) J_4,
\]

\[
g_3 = (W_1 W_2)^2 (W_2 W_1)(W_4 W_0)(W_2 W_3) J_4,
\]

\[
g_4 = (W_2 W_1)(W_2 W_3)^2 (W_1 W_2)(W_4 W_0)(W_2 W_3)^2 (W_1 W_2) J_4.
\]

See tables 5 and 6.

Table 6. Elements \( g_j = (u_1, u_i) \): \( j = \pm 1, \ldots, 12 \) of the group deck(\(N5\)), enumerated according to the 24 octahedral centre positions \( u' = u_1^{-1} u_i \in S^3 \), in the order and notation of table 8.

| \( \pm j \) | \( u_1 \) | \( u_i \) | \( u_1^{-1} u_i \) |
|-----|-----|-----|-----|
| \( \pm 1 \) | \( \alpha_{-1} \) | \( \mp v \) | \( \pm \alpha_1 \) |
| \( \pm 2 \) | \( \alpha_2 \) | \( \pm e \) | \( \pm \alpha_2 \) |
| \( \pm 3 \) | \( \alpha_3 \) | \( \pm v \) | \( \pm \alpha_3 \) |
| \( \pm 4 \) | \( \sqrt{\frac{1}{2}} [1 -1 i -1] \) | \( \pm \sqrt{\frac{1}{2}} [1 -1] \) | \( \pm \alpha_4 \) |
| \( \pm 5 \) | \( \sqrt{\frac{1}{2}} [1 -1 i -1] \) | \( \pm \sqrt{\frac{1}{2}} [-1 -1] \) | \( \pm \alpha_5 \) |
| \( \pm 6 \) | \( \alpha_2 \) | \( \pm e \) | \( \pm \alpha_2 \) |
| \( \pm 7 \) | \( [0 \; \theta \; 0] \) | \( \pm \sqrt{\frac{1}{2}} [1 -1] \) | \( \pm \alpha_3 \) |
| \( \pm 8 \) | \( [0 \; \theta \; 0] \) | \( \pm \sqrt{\frac{1}{2}} [-1 -1] \) | \( \pm \alpha_4 \) |
| \( \pm 9 \) | \( e \) | \( \pm e \) | \( \pm e \) |
| \( \pm 10 \) | \( \sqrt{\frac{1}{2}} [1 i -1] \) | \( \pm \sqrt{\frac{1}{2}} [-1 1] \) | \( \pm \mu \) |
| \( \pm 11 \) | \( e \) | \( \pm v \) | \( \pm v \) |
| \( \pm 12 \) | \( \sqrt{\frac{1}{2}} [1 i -1] \) | \( \pm \sqrt{\frac{1}{2}} [-1 1] \) | \( \pm \omega \) |

Table 7. Generators \( g = (u_1, u_i) \) of deck(\(N6\)); compare with table 8.

| \( g \) | \( u_1 \) | \( u_i \) |
|-----|-----|-----|
| \( g_1 \) | \( \sqrt{\frac{1}{2}} [\theta \theta \theta \theta] \) | \( : = \alpha_1 \; e \) |
| \( g_2 \) | \( \sqrt{\frac{1}{2}} [\theta \theta \theta \theta] \) | \( : = \alpha_2 \; e \) |
| \( g_3 \) | \( \sqrt{\frac{1}{2}} [\theta \theta \theta \theta] \) | \( : = \alpha_3 \; e \) |
| \( g_4 \) | \( \sqrt{\frac{1}{2}} [\theta \theta \theta \theta] \) | \( : = \alpha_4 \; e \) |

6. Manifold \( N6 \)

Edge gluing scheme:

\[
\begin{pmatrix}
1 & 8 & 10 \\
2 & 5 & 11 \\
3 & 6 & 12 \\
4 & 7 & 9
\end{pmatrix}
\]

(16)

Edge and face gluing generators of \( \pi_1(N6) \):

\[
g_1: 6 \cup 4, \begin{bmatrix} 10 & 6 \\ 2 & 7 \\ 3 & 6 \\ 4 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \\ 12 \end{bmatrix},
\]

\[
g_2: 5 \cup 3, \begin{bmatrix} 5 & 9 \\ 6 & 3 \end{bmatrix} \to \begin{bmatrix} 11 & 4 \end{bmatrix},
\]

\[
g_3: 8 \cup 2, \begin{bmatrix} 12 & 3 \\ 8 & 10 \end{bmatrix} \to \begin{bmatrix} 3 & 2 \end{bmatrix},
\]

\[
g_4: 7 \cup 1, \begin{bmatrix} 7 & 11 \\ 8 & 1 \end{bmatrix} \to \begin{bmatrix} 9 & 2 \end{bmatrix}.
\]

(17)
Table 8. The binary tetrahedral group \( T \sim \text{deck}(N6) \) has 16 elements \( \pm \alpha_j, \pm \alpha_j^{-1} \) and eight elements \( \pm e, \pm \mu, \pm \nu, \pm \omega \), with
\( \theta = \exp(i\pi/4), \bar{\theta} = \exp(-i\pi/4) \). It acts from the left on \( u \in S^3 \).

| \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \alpha_4 \) |
| --- | --- | --- | --- |
| \( \sqrt{\frac{1}{2}} \left[ \theta \bar{\theta} \right] \) | \( \alpha_5^{-1} \) | \( \alpha_6^{-1} \) | \( \alpha_7^{-1} \) |
| \( \sqrt{\frac{1}{2}} \left[ \bar{\theta} \bar{\theta} \right] \) | \( \alpha_8^{-1} \) | \( \alpha_9^{-1} \) | \( \alpha_{10}^{-1} \) |
| \( \sqrt{\frac{1}{2}} \left[ \bar{\theta} \theta \right] \) | \( \alpha_{11}^{-1} \) | \( \alpha_{12}^{-1} \) | \( \alpha_{13}^{-1} \) |
| \( \sqrt{\frac{1}{2}} \left[ \theta \theta \right] \) | \( \alpha_{14}^{-1} \) | \( \alpha_{15}^{-1} \) | \( \alpha_{16}^{-1} \) |

Table 9. Multiplication table for 12 elements \( g \) of the binary tetrahedral group \( \text{deck}(N6) \) given in table 8. The 12 elements \( g \) have been suppressed.

| \( \alpha_1 \) | \( \alpha_2 \) | \( \alpha_3 \) | \( \alpha_4 \) | \( \alpha_5 \) | \( \alpha_6 \) | \( \alpha_7 \) | \( \alpha_8 \) | \( \alpha_9 \) | \( \alpha_{10} \) | \( \alpha_{11} \) | \( \alpha_{12} \) | \( \alpha_{13} \) | \( \alpha_{14} \) | \( \alpha_{15} \) | \( \alpha_{16} \) | \( \nu \) | \( \mu \) | \( \omega \) | \( e \) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( \alpha_1 \) | \( -\alpha_1^{-1} \) | \( -\alpha_2^{-1} \) | \( -\omega \) | \( -\nu \) | \( e \) | \( \mu \) | \( \alpha_1^{-1} \) | \( \alpha_2^{-1} \) | \( \alpha_3^{-1} \) | \( \alpha_4^{-1} \) | \( \alpha_5^{-1} \) | \( \alpha_6^{-1} \) | \( \alpha_7^{-1} \) | \( \alpha_8^{-1} \) | \( \alpha_9^{-1} \) | \( \alpha_{10}^{-1} \) | \( \alpha_{11}^{-1} \) | \( \alpha_{12}^{-1} \) | \( \alpha_{13}^{-1} \) | \( \alpha_{14}^{-1} \) | \( \alpha_{15}^{-1} \) | \( \alpha_{16}^{-1} \) | \( \nu \) | \( \mu \) | \( \omega \) | \( e \) |

Isomorphic generators of deck(\( N6 \)):

\( g_1 = (W_1 W_2)(W_4 W_0) J_4 \),
\( g_2 = (W_3 W_2)(W_1 W_7)(W_4 W_0)(W_2 W_1) J_4 \),
\( g_3 = (W_2 W_3)^2 (W_1 W_2)(W_4 W_0)(W_2 W_3)^2 J_4 \),
\( g_4 = (W_2 W_3)(W_1 W_2)(W_4 W_0)(W_2 W_3) J_4 \).

See table 7. Using the equivalence \( (g_1, g_2) \sim (-g_1, -g_2) \), we can write \( H \) entirely in terms of left actions. The group \( H \) of homotopies and deck transformations of the three-manifold \( N6 \) then turns out to be the binary tetrahedral group \( (2, 3, 3) \) of order 24 in the notation of Coxeter and Moser [2, pp 134–5]. The elements and multiplication rules are given in tables 8 and 9.

The elements in this table obey

\( (\alpha_j)^3 = (\alpha_j^{-1})^3 = -e, \quad \frac{1}{2} \text{Tr}(\alpha_j) = \frac{1}{2} \text{Tr}(\alpha_j^{-1}) = \frac{1}{2} \),
\( j = 1, \ldots, 4 \), \( \mu^2 = \nu^2 = \omega^2 = -e \).

The last four elements generate as subgroup the quaternion group \( Q \) of order 8 with \( i = -\omega, j = -\nu \) and \( k = \mu \).

7. Conclusion

In the present work, we extend the study of the harmonic analysis on Platonic three-manifolds beyond the dodecahedral, the tetrahedral and the two cubic spherical manifolds. From homotopy, we construct and identify three groups \( H, |H| = 24 \) of deck transformations for three octahedral spherical three-manifolds and give their action on the three-sphere. Representation theory of \( SO(4, R) \) \( H \) provides the tools for the multiplicity and projection of \( H \)-invariant polynomial bases of the harmonic analysis on the octahedral three-manifolds.

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