Time reflection and the dynamics of particles and antiparticles

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In this paper we continue our analysis of a formulation of electrodynamics fully covariant under the full Poincaré group. Transformations under the four different components of the group force on us the introduction of particles, either in the identification by Feynman or in the identification of Dirac.

Keywords: electrodynamics on space-time, parity and time inversion, particles and antiparticles

1. Introduction

The theory of relativity was presented by Einstein as a theory based on completely new concepts of space and time, these concepts had remained essentially the same from Newton’s time. Einstein arrived at this new formulation through a deep epistemological analysis of electromagnetic phenomena. In his view, Lorentz transformations were connecting different reference frames, they were “passive” transformations. It was Minkowski [1] who stressed the absolute nature of space-time and, in some sense, considered Lorentz transformations as “active” transformations. His viewpoint was clearly stated: “Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality”.

Accepting this point of view, we have recently [2] presented a formulation of Maxwell’s equations in intrinsic terms identifying two assignments of parities of electromagnetic objects matching the two concepts of time reflection formulated in the literature usually in terms distant from the Minkowski concept of unified space-time.

While in general relativity the motion of particles is fully determined by the Einstein field equations for the metric tensor [3] [4], this is not the case for the motion of charged particles subject to some external electromagnetic field. To put it differently, the Lorentz force cannot be derived from Maxwell’s equations, it has to be introduced separately. In particular, the concept of an antiparticle has to be defined. Two definitions are present in the literature. One due to Dirac [5] and one due to Feynman [6]. The first is known in its quantum version. The second is at present well formulated only as a classical theory [7] [8]. Continuing our analysis of parities in [2] we analyse the two concepts of an antiparticle again in terms of parities. It is shown that each of the two definitions matches one of the assignments of parities to electromagnetic objects. In order to make the presentation self-consistent we briefly restate the basic constructions of [2] with some improvements.

2. Orientation of vector spaces

Let $V$ be a vector space of dimension $m \neq 0$. We denote by $F(V)$ the space of linear isomorphisms from $V$ to $\mathbb{R}^m$ called frames. Let $G(V)$ be the group of linear automorphisms of $V$. There is a natural group action

$$G(V) \times F(V) \to F(V); (\rho, \xi) \mapsto \xi \circ \rho^{-1}$$

(1)
and $F(V)$ is a homogeneous space with respect to this action.

The sets

$$C^E(V) = \{ \rho \in G(V); \det(\rho) > 0 \}$$

and

$$C^P(V) = \{ \rho \in G(V); \det(\rho) < 0 \}$$

are the two connected components of the group $G(V)$. The set $G^E(V) = C^E(V)$ is the component of the unit element. It is a normal subgroup.

The set of orientations $O(V) = F(V)/G^E(V)$ has two elements. This set is a homogeneous space for the quotient group $H(V) = G(V)/G^E(V)$. The sets $C^E(V)$ and $C^P(V)$ are the elements of the quotient group. Symbols $E$ and $P$ will be used to denote these elements. The structure of the group $H(V)$ is simple. The element $E = C^E(V)$ is the unit and the element $P = C^P(V)$ is an involution.

There is an ordered base $(e_1, e_2, \ldots, e_m)$ of $V$ associated with each frame $\xi$. If $\xi(v) = \begin{pmatrix} v^1 \\ \vdots \\ v^m \end{pmatrix}$, then $v = e_\epsilon v^\epsilon$. For each $\rho \in G(V)$ the base $(\rho(e_1), \rho(e_2), \ldots, \rho(e_m))$ is associated with the frame $\xi \circ \rho^{-1}$ if $(e_1, e_2, \ldots, e_m)$ is the base associated with $\xi$.

3. Lorentz transformations, frames and orientation.

Let $V$ be a vector space of dimension 4 with a Minkowski metric $g: V \to V^*$ of signature $(1,3)$. The Lorentz group for $(V, g)$ is the group of linear automorphisms

$$G(V, g) = \{ \rho \in G(V); \rho^* \circ g \circ \rho = g \}.$$  \hspace{1cm} (5)

A linear automorphism

$$\eta: V \to \mathbb{R}^4$$

is called a Lorentz frame if

$$(\eta^* \circ g \circ \eta^{-1}) \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} = (v^0, -v^1, -v^2, -v^3)$$

for each vector

$$\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \in \mathbb{R}^4.$$  \hspace{1cm} (8)

We denote by $F(V, g)$ the space of Lorentz frames. This space is a homogeneous space for the group $G(V, g)$ with the natural group action

$$G(V, g) \times F(V, g) \to F(V, g): (\rho, \xi) \mapsto \xi \circ \rho^{-1}.$$  \hspace{1cm} (9)

The light cone

$$LC = \{ v \in V; \langle g(v), v \rangle = 0 \}$$

divides the space $V$ in three disjoint connected regions. There is the region

$$SP = \{ v \in V; \langle g(v), v \rangle < 0 \}$$

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of \textit{space-like} vectors and the region
\[ TI = \{ v \in V; \langle g(v), v \rangle > 0 \} \quad (12) \]
of \textit{time-like} vectors. This region is the union of two disjoint regions \( TI_1 \) and \( TI_2 \).

The group \( G(V, g) \) has four connected components:
\[ C^E(V, g) = \{ \rho \in G(V, g); \det(\rho) = 1, \rho(TI_1) = TI_1 \} , \quad (13) \]
\[ C^T(V, g) = \{ \rho \in G(V, g); \det(\rho) = -1, \rho(TI_1) = TI_2 \} , \quad (14) \]
\[ C^S(V, g) = \{ \rho \in G(V, g); \det(\rho) = -1, \rho(TI_1) = TI_1 \} , \quad (15) \]
and
\[ C^{TS}(V, g) = \{ \rho \in G(V, g); \det(\rho) = 1, \rho(TI_1) = TI_2 \} . \quad (16) \]

The component of the unit element \( C^E(V, g) \) is a normal subgroup denoted by \( G^E(V, g) \).

The set of orientations
\[ O(V, g) = F(V, g)/G^E(V, g) \quad (17) \]
has four elements. This set is a homogeneous space for the quotient group \( H(V, g) = G(V, g)/G^E(V, g) \). The quotient group is commutative. Its elements are the four components \( C^E(V, g), C^T(V, g), C^S(V, g), \) and \( C^{TS}(V, g) \) denoted simply by \( E, T, S, \) and \( TS \) respectively. The element \( E = C^E(V, g) \) is the unit and all elements are involutions. The composition rule of \( T \) with \( S \) is incorporated in the notation used.

There is an ordered base \( (u_0, u_1, u_2, u_3) \) of \( V \) associated with each frame \( \eta \). If
\[ \eta(v) = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} , \quad (18) \]
then \( v = u_\kappa v^\kappa \). For each \( \rho \in G(V, g) \) the base \( (\rho(u_0), \rho(u_1), \rho(u_2), \rho(u_3)) \) is associated with the frame \( \eta \circ \rho^{-1} \) if \( (u_0, u_1, u_2, u_3) \) is the base associated with \( \eta \). Orthonormality relations
\[ \langle g(u_\kappa), u_\lambda \rangle = \begin{cases} 1, & \text{if } \lambda = \kappa = 0 \\ -1, & \text{if } \lambda = \kappa \neq 0 \\ 0, & \text{if } \lambda \neq \kappa \end{cases} \quad (19) \]
follow from (7).

\textbf{4. Multicovectors..}

\textit{A q-covector} in a vector space \( V \) is a mapping
\[ a : \times^q V \times O(V) \to \mathbb{R} . \quad (20) \]
This mapping is \textit{q-linear} and totally antisymmetric in its vector arguments. A \textit{q-covector} \( a \) is said to be \textit{even}, if
\[ a(v_1, v_2, \ldots, v_q, \rho o) = a(v_1, v_2, \ldots, v_q, o) . \quad (21) \]
It is said to be \textit{odd}, if
\[ a(v_1, v_2, \ldots, v_q, \rho o) = -a(v_1, v_2, \ldots, v_q, o) . \quad (22) \]
The vector space of \textit{even q-covectors} will be denoted by \( \wedge^q V^* \) and the space of \textit{odd q-covectors} will be denoted by \( \wedge^q V^* \). The symbol \( \wedge^q V^* \) will be used to denote either of the two spaces when the parity need not be explicitly specified. The index \( p \) with values \( e \) or \( o \) will be used in other constructions.
The algebra of multicovectors is included in the algebra of relativistic multicovectors introduced in the next section.

5. Multicovectors in the Minkowski space.

A relativistic \( q \)-covector in a the Minkowski vector space \( V \) is a mapping

\[
a : \times^q V \times O(V, g) \to \mathbb{R}.
\]

This mapping is \( q \)-linear and totally antisymmetric in its vector arguments. A \( q \)-covector \( a \) is said to have even temporal parity if

\[
a(v_1, \ldots, v_q, To) = a(v_1, \ldots, v_q, o).
\]

It is said to have odd temporal parity, if

\[
a(v_1, \ldots, v_q, To) = -a(v_1, \ldots, v_q, o).
\]

It is said to have even spatial parity, if

\[
a(v_1, \ldots, v_q, So) = a(v_1, \ldots, v_q, o).
\]

It is said to have odd spatial parity, if

\[
a(v_1, \ldots, v_q, So) = -a(v_1, \ldots, v_q, o).
\]

Relativistic \( q \)-covectors with different parities form four distinct vector spaces \( \Lambda^q_{t,e} V^* \), \( \Lambda^q_{o,e} V^* \), \( \Lambda^q_{e,o} V^* \), \( \Lambda^q_{o,p} V^* \). The first of the two subscripts identifies the temporal parity and the second identifies the spatial parity of the covectors. The symbol \( \Lambda^q_{t,s} V^* \) will be used to denote either of the four spaces. This use of indices \( t, s \) will be extended to other similar constructions.

Each of the relativistic orientations in \( O(V,g) \) is included in one of the elements of the set \( O(V) \). It follows that it makes sense to assign to each multicovector defined on orientations in \( O(V) \) a multicovector defined on relativistic orientations. The inclusions \( C^t(V,g) \subset C^p(V) \) and \( C^s(V,g) \subset C^p(V) \) imply that an even \( q \)-covector will be assigned an element in \( \Lambda^q_{t,e}(V) \) and an odd \( q \)-covector will be assigned an element in \( \Lambda^q_{o,o}(V) \). These assignments are one to one. With an orientation \( o \in O(V) \) we associate two orientations \( \overline{s}_1 \in O(V,g) \) and \( \overline{s}_2 \) is the covector \( \overline{a} \) of integers. We are using the following notational convention:

\[
\begin{align*}
\text{even} & = e, \\
\text{odd} & = o, \\
oo & = e.
\end{align*}
\]

If \( q \)-covectors \( a \) and \( a' \) with different parities form four distinct vector spaces.

The exterior product of a covector \( a \in \Lambda^q_{t,s} V^* \) and a covector \( a' \in \Lambda^{q'}_{t',s'} V^* \) is the covector \( a \wedge a' \in \Lambda^{q+q'}_{t+t',s+s'} V^* \) defined by

\[
a \wedge a' : \times^{q+q'} V \times O(V, g) \to \mathbb{R} : (v_1, \ldots, v_{q+q'}, o) \\
\mapsto \sum_{\sigma \in S(q+q')} \frac{\text{sgn}(\sigma)}{q!q'!} a(v_{\sigma(1)}, \ldots, v_{\sigma(q)}, o) a'(v_{\sigma(q+1)}, \ldots, v_{\sigma(q+q')}, o),
\]

where \( S(q+q') \) is the group of permutations of the set \( \{1, \ldots, q+q'\} \) of integers. We are using the following notational convention:
The exterior product is commutative in the graded sense. If \(a\) is a \(q\)-covector and \(a'\) is a \(q'\)-covector, then

\[ a' \wedge a = (-1)^{qq'} a \wedge a'. \]  

(31)

The exterior product is associative. The relation

\[ a \wedge (a' \wedge a'') = (a \wedge a') \wedge a'' \]  

(32)

holds for any three multivectors \(a\), \(a'\) and \(a''\).

6. Multivectors.

Spaces \(\Lambda^q_p V\) of multivectors of parity \(p\) are defined as quotients

\[ \Lambda^q_p V = K(\times^q V \times O(V))/A^p_q(V) \]  

(33)

of the space \(K(\times^q V \times O(V))\) of formal linear combinations of sequences

\[(v_1, \ldots, v_q, o) \in \times^q V \times O(V)\]  

(34)

by the spaces

\[ A^p_q(V) = \{ \sum_{i=1}^n \lambda_i(v^{1}_{i}, \ldots, v^{q}_{i}, o^i) \in K(\times^q V \times O(V)); \sum_{i=1}^n \lambda_i(v^{1}_{i}, \ldots, v^{q}_{i}, o^i) = 0 \text{ for each } a \in \Lambda^q_p V^* \}. \]  

(35)

The algebra of multivectors will be covered as a part of the algebra of relativistic multivectors.

7. Multivectors in the Minkowski space.

We consider the space \(K(\times^q V \times O(V, g))\) of formal linear combinations of sequences

\[(v_1, \ldots, v_q, o) \in \times^q V \times O(V, g)\]  

(36)

In the space \(K(\times^q V \times O(V, g))\) we introduce subspaces

\[ A^t_s q(V) = \{ \sum_{i=1}^n \lambda_i(v^{1}_{i}, \ldots, v^{q}_{i}, o^i) \in K(\times^q V \times O(V, g)); \sum_{i=1}^n \lambda_i(v^{1}_{i}, \ldots, v^{q}_{i}, o^i) = 0 \text{ for each } a \in \Lambda^q_p V^* \}. \]  

(37)

We then define quotient spaces

\[ \Lambda^q_{t,s} V = K(\times^q V \times O(V, g))/A^t_s q(V). \]  

(38)

Elements of spaces \(\Lambda^q_{t,s} V\) \(q\)-vectors of parity \(t, s\). The identification of the space \(\Lambda^q_{p,p} V\) with \(\Lambda^q_p V\) is made.

The *exterior product* of multivectors

\[ w_1 = [\sum_{i=1}^{n_1} \lambda^1_i(v^{1 1}_{1}, v^{1 2}_{2}, \ldots, v^{1 1}_{q_1}, o)]_{p_1} \]  

(39)

and

\[ w_2 = [\sum_{j=1}^{n_2} \lambda^2_j(v^{2 1}_{1}, v^{2 2}_{2}, \ldots, v^{2 1}_{q_2}, o)]_{p_2} \]  

(40)

is the multivector

\[ w_1 \wedge w_2 = [\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \lambda^1_i \lambda^2_j(v^{1 1}_{1}, v^{1 2}_{2}, \ldots, v^{1 1}_{q_1}, v^{2 1}_{1}, v^{2 2}_{2}, \ldots, v^{2 1}_{q_2}, o)]_{p_1,p_2}. \]  

(41)

Note that we are using a fixed orientation \(o\) in all representatives. The parity of the exterior product is odd if the parity of one of the factors is odd. It is even otherwise.
Evaluation of $q$-covectors on sequences $(v_1, \ldots, v_q, o) \in \times^q V \times O(V, g)$ extends to linear combinations and their equivalence classes. If $w$ is a $q$-vector represented by the linear combination
\[ \sum_{i=1}^{n} \lambda_i(v_i^1, \ldots, v_i^q, o_i) \] (42)
and $a$ is a $q$-covector of the same parity as $w$, then
\[ \langle a, w \rangle = \sum_{i=1}^{n} \lambda_i a(v_i^1, \ldots, v_i^q, o_i) \] (43)
is the evaluation of $a$ on $w$. We have constructed pairings
\[ \langle \cdot, \cdot \rangle : \wedge^q_{t,s} V^* \times \wedge^q_{t',s'} V^* \to \mathbb{R}. \] (44)
The right interior multiplication [Sternberg] is an operation
\[ \wedge^q_{t,s} V \times \wedge^{q'}_{t',s'} V^* \to \wedge^{q'-q}_{t',s,s'} V^* : (w, a) \mapsto w \cdot a \] (45)
defined for $q' > q$. The covector $w \cdot a$ is characterized by
\[ \langle w \cdot a, u \rangle = \langle a, w \wedge u \rangle \] (46)
for $u \in \wedge^q_{t',s,s'} V$.

8. Differential forms in the Minkowski space-time.

A relativistic differential $q$-form on the Minkowski space-time $M$ is a differentiable function
\[ A : M \times (\times^q V) \times O(V, g) \to \mathbb{R} \] (47)
multilinear and totally antisymmetric in its vector arguments. A differential form $A$ is said to have even temporal parity if
\[ A(x, v_1, \ldots, v_q, T o) = A(x, v_1, \ldots, v_q, o). \] (48)
It is said to have odd temporal parity, if
\[ A(x, v_1, \ldots, v_q, T o) = -A(x, v_1, \ldots, v_q, o). \] (49)
It is said to have even spatial parity, if
\[ A(x, v_1, \ldots, v_q, S o) = A(x, v_1, \ldots, v_q, o). \] (50)
It is said to have odd spatial parity, if
\[ A(x, v_1, \ldots, v_q, S o) = -A(x, v_1, \ldots, v_q, o). \] (51)

For each degree $q$ there are four spaces $\Phi^q_{e,c}(M)$, $\Phi^q_{o,c}(M)$, $\Phi^q_{e,o}(M)$, and $\Phi^q_{o,o}(M)$ of forms of different parities. As in the case of $q$-covectors the first of the two subscripts identifies the temporal parity and the second identifies the spatial parity of the forms.

The exterior product of a form $A \in \Phi^q_{e,s}(M)$ with a form $A' \in \Phi^{q'}_{e',s'}(M)$ is the form $A \wedge A' \in \Phi^{q+q'}_{e',s'}(M)$ defined by
\[ A \wedge A' : M \times V^{q+q'} \times O(V) \to \mathbb{R}: (x, v_1, \ldots, v_{q+q'}, o) \mapsto \sum_{\sigma \in S(q+q')} \frac{\text{sgn}(\sigma)}{q! q'!} A(x, v_{\sigma(1)}, \ldots, v_{\sigma(q)}, o) A'(x, v_{\sigma(q+1)}, \ldots, v_{\sigma(q+q')}, o), \] (52)
The exterior product is commutative in the graded sense. If $A$ is a $q$-form and $A'$ is a $q'$-form, then

$$A' \wedge A = (-1)^{qq'} A \wedge A'.$$  \hfill (53)

The exterior product is associative. The relation

$$A \wedge (A' \wedge A'') = (A \wedge A') \wedge A''$$  \hfill (54)

holds for any three forms $A$, $A'$ and $A''$.

The *exterior differential* of a $q$-form $A$ is the $(q+1)$-form

$$dA: \mathbb{M} \times \mathbb{R}^{q+1} \times O(V,g) \to \mathbb{R}: (x, v_1, \ldots, v_{q+1}, o) \mapsto -\sum_{i=1}^{q+1} (-1)^i d \left. A(x + sv_i, v_1, \ldots, \hat{v}_i, \ldots, v_{q+1}, o) \right|_{s=0}.$$  \hfill (55)

The parity of the differential $dA$ is the same as the parity of the original form $A$. The operator $d$ is a *differential* in the sense that $ddA = 0$ for each form $A$.

A form $A$ is said to be *closed* if $dA = 0$. It is said to be *exact* if there is a form $B$ such that $a = dB$. The *Poincaré lemma* implies that in an affine space each closed form is exact.

A form $A \in A_{t,s}(M)$ can be interpreted as a mapping

$$\tilde{A}: \mathbb{M} \to \wedge^q_{t,s} V^*.$$  \hfill (56)

The relation between the form $A$ and the mapping $\tilde{A}$ is expressed by

$$\tilde{A}(x)(v_1, \ldots, v_q, o) = A(x, v_1, \ldots, v_q, o).$$  \hfill (57)

## 9. Multivector fields..

A *$q$-vector field* of parity $t, s$ is a mapping

$$W: \mathbb{M} \to \wedge^q_{t,s} V.$$  \hfill (58)

Operations such as the exterior product and the right interior multiplication extend to multivector fields. The exterior product of a multivector field

$$W_1: \mathbb{M} \to \wedge^{q_1}_{t_1,s_1} V$$  \hfill (59)

and

$$W_2: \mathbb{M} \to \wedge^{q_2}_{t_2,s_2} V$$  \hfill (60)

is the multivector field

$$W_1 \wedge W_2: \mathbb{M} \to \wedge^{q_1+q_2}_{t_1+t_2,s_1+s_2} V: x \mapsto W_1(x) \wedge W_2(x).$$  \hfill (61)

The right interior multiplication of a form $A \in A_{t',s'}(M)$ represented by a mapping

$$\tilde{A}: \mathbb{M} \to \wedge^{q'}_{t',s'} V^*$$  \hfill (62)

by a multivector field

$$W: \mathbb{M} \to \wedge^q_{t,s} V$$  \hfill (63)

is the form $W \mathcal{J} A \in A_{t',s'}(M)$ represented by

$$W \mathcal{J} \tilde{A}: \mathbb{M} \to \wedge^{q'-q}_{t',s'} V^*: x \mapsto W(x) \mathcal{J} \tilde{A}(x).$$  \hfill (64)
10. **The metric volume form.**

Let \( g: V \to V^* \) be the Minkowski metric tensor. We define an odd 4-form

\[ \sqrt{|g|}: M \times \mathbb{R}^4 V \times O(V) \to \mathbb{R} \tag{65} \]

by the formula

\[ \sqrt{|g|(x, v_1, \ldots, v_4, o)} \mapsto \pm \sqrt{|\det((g(v_\kappa), v_\lambda))|}. \tag{66} \]

If vectors \((v_1, \ldots, v_4)\) are dependent, then \( \det((g(v_\kappa), v_\lambda)) = 0 \). If the vectors are independent, then they determine an orientation \( o' \in O(V) \). The sign \( + \) in the formula is chosen if the orientations \( o \) and \( o' \) agree. Otherwise the sign \( - \) is chosen. It follows from elementary properties of determinants that the formula defines \( \sqrt{|g|} \) as an element of \( \Phi^4_0(M) \) or an element of \( \Phi^4_{o,o}(M) \).

11. **Maxwell’s equations.**

Let \( M \) be the affine Minkowski space-time of special relativity with the model space \( V \) and the metric tensor \( g: V \to V^* \) of signature \((1,3)\).

Electrodynamic phenomena in the affine Minkowski space-time of special relativity are described in terms of the following geometric quantities:

1. a 1-form \( A \) called the potential,
2. a 2-form \( F \) called the electromagnetic field,
3. a 2-form \( G \) called the electromagnetic induction,
4. a 3-form \( J \) called the current.

These quantities satisfy the Maxwell’s equations

\[ dF = 0 \tag{67} \]

and

\[ dG = \frac{4\pi}{c} J. \tag{68} \]

There is also the constitutive relation

\[ G = \left( \wedge^2 g^{-1} \circ \tilde{F} \right) J \sqrt{|g|}. \tag{69} \]

We are using the mapping

\[ \wedge^2 g^{-1}: \wedge^2 V^* \to \wedge^2 V \tag{70} \]

characterized by the equality

\[ \wedge^2 g^{-1}(a_1 \wedge a_2) = g^{-1}(a_1) \wedge g^{-1}(a_2) \tag{71} \]

for simple even bivectors.

The equality

\[ F = dA \tag{72} \]

is a consequence of the Poincaré lemma.

We will discuss the possible assignments of parities of electromagnetic objects after we have introduced the dynamics of particles.

12. **Dynamics of test particles.**

12.1 **Free particles.**

We examine the dynamics of a relativistic test particle. The configuration space of the particle is the affine Minkowski space-time \( M \) and the phase space is the product \( M \times \wedge^1_{t,s} V^* \). A phase space trajectory of the particle is the image of a differentiable mapping

\[ (\xi, \pi): \mathbb{R} \to M \times \wedge^1_{t,s} V^*: s \mapsto (\xi(s), \pi(s)). \tag{73} \]
At each value of the parameter $s$ the tangent vector $D\xi(s, 1)$ of the space-time component

$$\xi: \mathbb{R} \to M$$

is time-like in the sense that

$$\langle g(D\xi(s, 1)), D\xi(s, 1) \rangle > 0.$$  \hfill (75)

The image of the space-time component is the *world line* of the particle. The image of a curve such as the mapping $\xi$ is best represented as an equivalence class of curves. Two curves $\xi$ and $\xi'$ are equivalent if there is a *reparameterization diffeomorphism* $\gamma: \mathbb{R} \to \mathbb{R}$ such that $\xi' = \xi \circ \gamma$. The same constructions are applied to phase space trajectories. The representatives of equivalence classes are called *parameterizations*. Reparameterizations with positive derivatives are called *direction preserving reparameterizations*. Direction preserving reparameterizations establish a narrower equivalence relation. Equivalence classes are the *directed world lines* and the *directed phase space trajectories*. Two directed world lines correspond to each world line.

The *space-time velocity* and the *space-time acceleration* of the particle are mappings

$$\xi': \mathbb{R} \to V: s \mapsto D\xi(s, 1)$$

and

$$\xi'': \mathbb{R} \to V: s \mapsto D\xi'(s, 1)$$

derived from a parameterization $\xi$. The rate of change

$$\pi': \mathbb{R} \to \wedge^1_T V^*: s \mapsto D\pi(s, 1)$$

of space-time momentum is also used. Equations of motion are formulated in terms of parameterizations. Direction preserving reparameterization invariance must be assured.

Equations of motion

$$\pi(s)(\delta\xi'(s), o) = \frac{m}{\sqrt{\langle g(\xi'(s)), \xi'(s) \rangle}} \langle g(\xi'(s)), \delta\xi'(s) \rangle$$

and

$$\pi'(s)(\delta\xi(s), o) = 0$$

of a free particle of mass $m$ are derived from the Lagrangian

$$L: M \times V \to \mathbb{R}: (x, x') \mapsto m\sqrt{\langle g(x'), x' \rangle}.$$  \hfill (81)

The variational principle

$$\frac{d}{du}L(\xi(s) + u\delta\xi(s), \xi'(s) + u\delta\xi'(s))\big|_{u=0} = \pi'(s)(\delta\xi(s), o) + \pi(s)(\delta\xi'(s), o)$$

with an arbitrary variation

$$\delta\xi: \mathbb{R} \to V$$

of the world line is used. Note that at a fixed value of the parameter $s$ the variation $\delta\xi(s)$ and its derivative $\delta\xi'(s)$ are independent. The Lagrangian is homogeneous in the sense that

$$L(x, x') = |\lambda|L(x, x')$$

for each $\lambda \in \mathbb{R}$. Parameterizations of world lines are solutions of the *Euler-Lagrange equation*

$$\frac{m}{\sqrt{\langle g(\xi'(s)), \xi'(s) \rangle}} \begin{pmatrix} \langle g(\xi''(s)), \delta\xi(s) \rangle - \langle g(\xi'(s)), \xi''(s) \rangle \langle g(\xi'(s)), \delta\xi(s) \rangle \\ \langle g(\xi'(s)), \delta\xi(s) \rangle \end{pmatrix} = 0$$

(85)
derived from (79) and (80). This equation is identically satisfied with $\delta\eta(s) = \eta'(s)$.

If
\[ \bar{\xi} = \xi \circ \gamma \]  
and
\[ \bar{\pi} = \pi \circ \gamma, \]
then
\[ \bar{\xi}' = (\xi' \circ \gamma) \gamma', \]  
\[ \bar{\pi}' = (\pi' \circ \gamma) \gamma', \]
and
\[ \bar{\xi}'' = (\xi'' \circ \gamma) (\gamma')^2 + (\xi' \circ \gamma) \gamma'' . \]

For the reparameterized mappings $\bar{\xi}$ and $\bar{\pi}$ we obtain the equalities
\[ \bar{\pi}(s)(\delta\xi'(\tilde{s}), o) = \pi(\tilde{s})(\delta\xi'(\tilde{s}), o), \]  
\[ \bar{\pi}'(s)(\delta\xi(\tilde{s}), o) = \gamma'(s)\pi'(s)(\delta\xi(\tilde{s}), o), \]
and
\[ \frac{m}{\sqrt{|g(\xi'(s), \xi'(s))|}} \langle g(\xi''(s)), \delta\xi'(\tilde{s}) \rangle = \frac{m}{\sqrt{|g(\xi'(s), \xi'(s))|}} \langle g(\xi'(s)), \delta\xi'(\tilde{s}) \rangle \]
in terms of a parameter $\tilde{s}$ related to the original parameter $s$ by $\tilde{s} = \gamma(s)$. If $\gamma'(s) > 0$, then the mappings $\bar{\xi}$ and $\bar{\pi}$ satisfy the equation (79). If $\gamma'(s) < 0$, then the equation (79) is satisfied by mappings $\bar{\xi}$ and $-\bar{\pi}$. The equation (80) has the total reparameterization invariance property. From
\[ \frac{m}{\sqrt{|g(\xi'(s), \xi'(s))|}} \left( \langle g(\xi''(s)), \delta\xi(\tilde{s}) \rangle - \langle g(\xi'(s), \xi''(s)) \rangle \langle g(\xi'(s), \xi(\tilde{s})) \rangle \right) \]
\[ = \frac{m|\gamma'(s)|}{\sqrt{|g(\xi'(s), \xi'(s))|}} \left( \langle g(\xi''(s)), \delta\xi(\tilde{s}) \rangle - \langle g(\xi'(s), \xi''(s)) \rangle \langle g(\xi'(s), \xi(\tilde{s})) \rangle \right) \]
it follows that also the equation (85) is totally reparameterization invariant.

There are distinguished natural parameterizations of world lines in the Minkowski space-time characterized by the condition
\[ \langle g(\xi'(s), \xi'(s)) \rangle = 1. \]
The parameter in a natural parameterization is the proper time calculated from some initial event. Fixing a parameterization interferes with a variational derivation of the equations of motion. Once the equations are derived they can be simplified by using the natural parameterization. The normalization condition (95) and the derived equality
\[ \langle g(\xi'(s), \xi''(s)) \rangle = 0 \]
reduce the equations of motion to the simpler forms
\[ \pi(s)(\delta\xi'(s), o) = m\langle g(\xi'(s)), \delta\xi'(s) \rangle, \]  
\[ \pi'(s)(\delta\xi(s), o) = 0, \]
and
\[ m\langle g(\xi''(s)), \delta\xi(s) \rangle = 0. \]
12.2 Charged particles.. 

The Lagrangian of a particle of charge $e$ in an electromagnetic field $F = dA$ is the function 

$$L: M \times V \to \mathbb{R}: (x, x') \mapsto m \sqrt{(g(x'), x') - e(o) A(x, x', o)}, \quad (100)$$

with an arbitrary choice of the orientation $o$. This Lagrangian is positive homogeneous. The equality 

$$L(x, \lambda x') = \lambda L(x, x') \quad (101)$$

holds for each $\lambda \geq 0$. The equations of motion assume the form 

$$\pi(s)(\delta \xi'(s), o) = \frac{m}{\sqrt{g(\xi''(s)), \xi'(s)}} \langle g(\xi'(s)), \delta \xi'(s) \rangle \quad (102)$$

and 

$$\pi'(s)(\delta \xi(s), o) = e(o) F(\xi(s), \xi'(s), \delta \xi(s), o). \quad (103)$$

Following Landau and Lifshitz [16] we are using for charged particles the velocity-momentum relation (102) known also as the Legendre relation derived for free particles. The momentum $\pi(s)$ in (102) and (103) is not canonical. The variational principle 

$$\frac{d}{du}L(\xi(s) + u\delta \xi(s), \xi'(s) + u\delta \xi'(s))\big|_{u=0} = (\pi'(s) - e(o) D\tilde{A}(\xi(s), \xi'(s)))(\delta \xi(s), o) + (\pi(s) - e(o) \tilde{A}(\xi(s)))(\delta \xi'(s), o) \quad (104)$$

is used to obtain this result. 

The equation 

$$\frac{m}{\sqrt{g(\xi''(s)), \xi'(s)}} \langle g(\xi''(s)), \delta \xi(s) \rangle - \langle g(\xi'(s)), \xi''(s) \rangle \langle g(\xi'(s)), \delta \xi(s) \rangle = e(o) F(\xi(s), \xi'(s), \delta \xi(s), o). \quad (105)$$

has parameterizations of world lines of charged particles as solutions. This equation is again satisfied identically with $\delta \xi(s) = \xi'(s)$.

We have already seen that the equation (102) is direction preserving reparameterization invariant. The equality (92) together with the equality 

$$e(o) F(\xi(s), \xi'(s), \delta \xi(s), o) = e(o) F'(\xi(s), \xi'(s), \delta \xi(s), o) \quad (106)$$

show that the equation (103) has the total reparameterization invariance property. The equalities (94) and (106) prove direction preserving reparameterization invariance of (105). Total reparameterization invariance is no longer present.

Equations 

$$\pi(s)(\delta \xi'(s), o) = m \langle g(\xi'(s)), \delta \xi'(s) \rangle \quad (107)$$

$$\pi'(s)(\delta \xi(s), o) = e(o) F(\xi(s), \xi'(s), \delta \xi(s), o). \quad (108)$$

and 

$$m \langle g(\xi''(s)), \delta \xi(s) \rangle = e(o) F(\xi(s), \xi'(s), \delta \xi(s), o) \quad (109)$$

are obtained if a natural parameterization is used.

13. Transformation properties.. 

13.1 Transformation of multico vectors..
The group $G(V, g)$ has natural representations in the spaces $\Lambda^q_t s V^*$. A Lorentz transformation $\rho \in G(V, g)$ applied to a $q$-covector $a$ produces the $q$-covector

$$(\rho^{-1})^* a: \times^q V \times O(V, g) \to \mathbb{R}$$

$$(v_1, v_2, \ldots, v_q, o) \mapsto a(\rho^{-1}(v_1), \rho^{-1}(v_2), \ldots, \rho^{-1}(v_q), [\rho](o)),$$  

(110)

where $[\rho]$ is the class of $\rho$ in the quotient group $H(V, g) = G(V, g)/G_E(V, g)$. We introduce the index $\text{id}_{t,s}(\rho)$ of a Lorentz transformation $\rho$ in terms of the result

$$(\rho^{-1})^* a: O(V, g) \to \mathbb{R}: o \mapsto a([\rho](o)) = \text{id}_{t,s} a(o)$$

(111)

of the transformation applied to a 0-covector $a: O(V, g) \to \mathbb{R}$.

of parity $t, s$. The values of the index are listed in the following table.

| $\rho \in E$ | $\rho \in T$ | $\rho \in S$ | $\rho \in TS$ |
|------------|------------|------------|------------|
| $\text{id}_{e,e}(\rho)$ | 1 | 1 | 1 |
| $\text{id}_{o,e}(\rho)$ | 1 | -1 | 1 |
| $\text{id}_{e,o}(\rho)$ | 1 | 1 | -1 |
| $\text{id}_{o,o}(\rho)$ | 1 | -1 | -1 |

In terms of the index the result of the application of a Lorentz transformation $\rho$ to a $q$-covector $a$ of parity $t, s$ is the $q$-covector

$$(\rho^{-1})^* a: \times^q V \times O(V, g) \to \mathbb{R}$$

$$(v_1, \ldots, v_q, o) \mapsto \text{id}_{t,s} a(\rho^{-1}(v_1), \ldots, \rho^{-1}(v_q), o).$$

(113)

13.2 Transformation of multivectors.

Let $w$ be a $q$-vector represented by the linear combination

$$\sum_{i=1}^n \lambda_i(v_1^i, \ldots, v_q^i, o^i).$$

(114)

The image of this multivector by an automorphism $\rho \in G(V, g)$ is the multivector $\rho_w w$ represented by the combination

$$\sum_{i=1}^n \lambda_i(\rho(v_1^i), \ldots, \rho(v_q^i), [\rho](o^i)),$$

(115)

where $[\rho]$ is again the class of $\rho$ in the quotient group $H(V, g) = G(V, g)/G_E(V, g)$. If $w \in \Lambda^q_t s V$, then the multivector $\rho_w w$ is represented by

$$\text{id}_{t,s} \sum_{i=1}^n \lambda_i(\rho(v_1^i), \ldots, \rho(v_q^i), o^i).$$

(116)

The equality

$$\rho_w (w_1 \land w_2) = \rho_w (w_1) \land \rho_w (w_2)$$

(117)

holds for multivectors $w_1$ and $w_2$ of any parity.
It follows from the definition (43) that the pairings (44) are invariant in the sense that
\[ \langle (\rho^{-1})^*a, \rho_*w \rangle = \langle a, w \rangle. \] (118)

13.3 The Poincaré group..

Let \( M \) be an affine space modelled on a vector space \( V \). A mapping \( \varphi: M \to M \) is said to be affine if there is a linear mapping \( \chi: V \to V \) such that
\[ \varphi(x') - \varphi(x) = \chi(x' - x). \] (119)
If the mapping \( \chi: V \to V \) satisfying the above condition exists, then it is unique. It is called the linear part of the affine mapping \( \varphi \) and is denoted by \( V \).

An affine mapping \( \varphi \) is invertible if and only if its linear part \( V \) is invertible. If \( \varphi \) is invertible then \( \varphi^{-1} \) is an affine mapping and \( \varphi^{-1} = (\varphi)^{-1} \).

An affine mapping \( \varphi \) is differentiable. Its differential and its linear part are related by
\[ D\varphi(x, v) = V(v). \] (120)

Let \( M \) be the Minkowski space-time of special relativity. A Poincaré transformation is an affine mapping \( \varphi: M \to M \) such that \( V \) is a Lorentz transformation. Poincaré transformations form a group denoted by \( P(M, g) \).

13.4 Transformation properties of forms..

A Poincaré transformation \( \varphi \) applied to a form \( A \in \Phi^q_{t,s} \) produces the form
\[ (\varphi^{-1})^*A: M \times (\times^q V) \times O(V, g)o \to \mathbb{R} \]
\[ : (x, v_1, \ldots, v_q, o) \mapsto \text{id}_{t,s} A(\varphi^{-1}(x), V^{-1}(x, v_1), \ldots, V^{-1}(x, v_q), o). \] (121)
The pull back \( (\varphi^{-1})^*A \) of a \( q \)-form \( A \) represented by the mapping \( A \) is represented by
\[ (\varphi^{-1})^*A: M \to \wedge^q V^* : x \mapsto (V^{-1})^*A(\varphi^{-1}(x)). \] (122)

14. Transformation properties of Maxwell’s equations..

If objects \( F, A, G, \) and \( J \) satisfy equations (67), (68), (69), and (72), then the pullbacks of these objects by a Poincaré transformation again satisfy the equations on the condition that \( F \) and \( A \) have the same parities and \( G \) and \( J \) have the same parities opposed to those of \( F \) and \( A \). The exact transformations of these objects depend on the choice of parities.

Two choices of parities are known:

1. The parity of \( F \) and \( A \) is \( e, e \) and that of \( G \) and \( J \) is \( o, o \).
2. The parity of \( F \) and \( A \) is \( o, e \) and that of \( G \) and \( J \) is \( e, o \).

The first choice is the natural one. It can be made in terms of ordinary even and odd orientations and extends easily to curved space-times. The second choice is an intrinsic formulation of choices found in physics literature [15], [16].

Different choices of parity result in different responses to time reversing transformations. If \( \overline{\varphi} \in T \) or \( \overline{\varphi} \in TS \), then according to the first choice we have
\[ \varphi^* F(x, v_1, v_2, o) = F(\varphi(x), \overline{\varphi}(v_1), \overline{\varphi}(v_2), o), \] (123)
\[ \varphi^* A(x, v, o) = A(\varphi(x), \overline{\varphi}(v), o), \] (124)
\[ \varphi^* G(x, v_1, v_2, o) = -G(\varphi(x), \overline{\varphi}(v_1), \overline{\varphi}(v_2), o), \] (125)
The transformation rules according to the second choice are

\[ \varphi^* J(x, v_1, v_2, v_3, o) = -J(\varphi(x), \overline{\varphi}(v_1), \overline{\varphi}(v_2), \overline{\varphi}(v_3), o). \] (126)

15. Particles and antiparticles..

15.1 Antiparticles according to Stückelberg and Feynman..

Following Stückelberg and Feynman we recognize particles as antiparticles as different states of motion of the same physical object. If a world line with one direction is recognized as describing the motion of a particle, then the world line with the opposite direction describes the motion of an antiparticle. It is impossible to tell which of the two possible directions should qualify a directed world line as that of a particle rather than an antiparticle. According to this interpretation both the electron and the positron are objects of the same negative charge. Reparameterizations with negative derivatives are called direction inverting reparameterizations. Equations of motion are preserved by such reparameterizations if the sign of momentum is inverted. Particle and antiparticle states are interchanged.

This theory of antiparticles fits well the natural choice of parities of electromagnetic objects. The potential \( A \) and the field \( F \) are even forms, the charge \( e \) and the momentum \( \pi \) are even objects. The equations of motion (102) and (103) as well as the equation (105) for world lines are Poincaré invariant. Time reflecting transformations interchange particle and antiparticle states.

15.2 A classical version of Dirac’s theory of antiparticles..

We present an attempt to formulate a classical mechanics of antiparticles reflecting the features of Dirac’s theory of positrons. This theory is not fully relativistic since it distinguishes between the future and the past.

Vectors in one of the two disjoint regions \( TI_1 \) or \( TI_2 \) inside the light cone are declared as pointing towards the future. Let \( TI_1 \) be this region. Particles have charge \( e \) and antiparticles have the opposite charge \( -e \) both charges are of parity \( o, e \) equal to the parity of the potential \( A \). World lines of particles and antiparticles are directed towards the future. World lines of observers are also directed into the future and energies of particles and antiparticles observed by such observers are positive.

In order to achieve Poincaré invariance of this scheme the parity of the momentum is set to \( o, e \) and time reflecting transformations are accompanied by direction inverting reparameterizations.

15.3 Closing remarks..

When pair creations were observed in the Wilson chamber placed in a magnetic field positive charge had to be assigned to positrons in order to obtain the observed trajectory curvatures from the Lorentz force formula with observed velocities. The concept of an antiparticle as a particle of opposite charge fits well the observed situations. The antiparticle is described as it appears to an observer.

Stückelberg [7] and Feynman [6] considered trajectories of particles and particles as world lines with opposite orientations. Using in the Lorentz force formula vectors tangent to the world lines compatible with their orientations correct curvatures for particle and antiparticle trajectories with the same charge is obtained. Minkowski [1] considered physical reality as existing in space-time rather than being a succession of spatial configurations evolving in time. We find the Stückelberg and Feynman interpretation of antiparticle states very much in line with the Minkowski concept of physical reality.

Feynman proposed a quantum theory of antiparticles based on the classical mechanics of antiparticles formulated by Stückelberg. A complete implementation of this theory was never reached. Only the Feynman diagrams and the Feynman propagator survived. Dirac formulated his theory of antiparticles as a quantum
We have not found in the literature a complete classical description of antiparticles compatible with Dirac theory. The description we propose is probably not the only possible.

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