Separation of Variables in the BC-type Gaudin Magnet

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Abstract. The integrable system is introduced based on the Poisson $rs$-matrix structure. This is a generalization of the Gaudin magnet, and in SL(2) case isomorphic to the generalized Neumann model. The separation of variables is discussed for both classical and quantum case.

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1. Introduction

The classical integrable systems has been formulated in terms of the classical $r$-matrix \[1\]. In one sense, the system is proved to be integrable when we can perform the separation of variables; the reduction of a multi-dimensional system to a set of one-dimensional systems (see, for review, Ref. \[2\]). Although the separation of variables has been widely known as the Hamiltonian-Jacobi equation, Sklyanin proposed a new technique (functional Bethe ansatz), which is closely related with the (quantum) inverse scattering method. The functional Bethe ansatz method was first applied to classical top \[3\], and further applications, for the Toda lattice, Gaudin magnet, and Heisenberg spin chain, have been done. This technique is a new tool in studying integrable systems.

We briefly review the separation of variables for the Gaudin magnet \[4\] in classical case. The fundamental formulation of integrable system is based on the classical $r$-matrix structure,

\[
\{ \mathbf{L}(u), \mathbf{L}(v) \} = [r(u - v), \mathbf{L}(u)] + \mathbf{L}(v). \tag{1.1}
\]

Here we have used the standard notation, \[\mathbf{L}(u) = \mathbf{L}(u) \otimes 1\] and \[\mathbf{L}(v) = 1 \otimes \mathbf{L}(v)\]. As an example of the \[\mathbf{L}\]-matrix satisfying the linear Poisson structure (1.1), we can take

\[
\mathbf{L}(u) = \mathbf{Z} + \sum_{j=1}^{N} \frac{S_j}{u - z_j}. \tag{1.2}
\]

Here \[S_j (j = 1, \ldots, N)\] is a classical SL(n) spin matrix, and its elements \[S_{ab}^j (a, b = 1, \ldots, n)\] satisfy the Poisson relation,

\[
\{ S_{ab}^j, S_{cd}^k \} = \delta_{jk} \cdot (\delta^{bc} S_{ad}^j - \delta^{da} S_{cb}^j), \tag{1.3}
\]

and \[\text{Tr} S_j^2\] is the Casimir element. We suppose that matrix \[\mathbf{Z}\] is traceless, \[\text{Tr} \mathbf{Z} = 0\]. The \[\mathbf{L}\]-matrix (1.2) appears as a quasi-classical limit of the inhomogeneous Heisenberg XXX-spin chain \[5, 6\], and satisfies the linear Poisson relation (1.1) with the classical $r$-matrix,

\[
r(u) = \frac{\mathbf{P}}{u}. \tag{1.4}
\]

Matrix \(\mathbf{P}\) means a permutation matrix, \[P_{ab,cd} = \delta^{ad} \delta^{bc}\], which satisfies \[\mathbf{P} x \otimes y = y \otimes x\]. Note that \(r\)-matrix (1.4) is a rational solution of the classical Yang-Baxter equation \[7\],

\[
[r_{23}(v), r_{12}(u)] + [r_{23}(v), r_{13}(u + v)] + [r_{13}(u + v), r_{12}(u)] = 0. \tag{1.5}
\]

Once the \[\mathbf{L}\]-matrix satisfies the Poisson structure (1.1), the model can be proved to be integrable in the Liouville’s sense. If functions of the \[\mathbf{L}\]-matrix, \[\tau_m(u)\], are defined as

\[
\tau_m(u) \equiv \frac{1}{m} \text{Tr} \mathbf{L}(u)^m, \tag{1.6}
\]
one can see that \( \tau_m(u) \)'s are spectral invariant,
\[
\{ \tau(u), \tau_m(v) \} = 0.
\] (1.7)
The identity (1.7) shows that \( \tau_m(u) \)'s are generating functions of the constants of motion. The first non-trivial invariant follows from \( \tau_2(u) \);
\[
\tau_2(u) = \frac{1}{2} \text{Tr} Z^2 + \sum_{j=1}^{N} \frac{H_j}{u - z_j} + \frac{1}{2} \sum_{j=1}^{N} \frac{\text{Tr} S_j S_j}{(u - z_j)^2},
\] (1.8)
where \( H_j \) is the hamiltonian of the SL(\( n \)) Gaudin magnet,
\[
H_j = \text{Tr} Z S_j + \sum_{k \neq j}^{N} \frac{\text{Tr} S_j S_k}{z_j - z_k}.
\] (1.9)
This model was introduced by Gaudin as an integrable spin system with long-range interaction [8]. Due to the involutiveness of \( \tau_m(u) \), one sees that the hamiltonian of the SL(\( n \)) Gaudin magnet is Poisson commutative,
\[
\{ H_j, H_k \} = 0, \quad \text{for } j, k = 1, \ldots, N.
\] (1.10)
The complete integrability of the model in Liouville’s sense can be proved directly from (1.7); when we introduce quantities \( \tau_{m,\alpha} \) by
\[
\tau_m(u) = \frac{1}{m} \text{Tr} Z^m + \sum_{j=1}^{N} \sum_{\alpha=1}^{m} \frac{\tau_{m,\alpha}}{(u - z_j)^\alpha},
\] (1.11)
we can see that the quantities \( \tau_{m,\alpha} \) (\( m = 2, \ldots, N; j = 1, \ldots, n; \alpha = 1, \ldots, m - 1 \)) form a commutative family of \( N n (n - 1)/2 \) independent hamiltonians.

For this type of the Gaudin magnet (1.9), the separation of variable (functional Bethe ansatz) has been widely studied in both classical and quantum cases [1, 2, 3, 4]. Let \( A(L) \) and \( B(L) \) be certain polynomials of degree \( n(n - 1)/2 \) in matrix elements \( L_{ab} \). When we define variables \( x_j \) and \( p_j \) by
\[
B(L(x_j)) = 0, \quad p_j = A(L(x_j)),
\] (1.12)
one sees that variables \( x_j \) and \( p_j \) are canonically Poisson conjugate [3, 16, 17],
\[
\{ x_j, x_k \} = 0, \quad \{ p_j, p_k \} = 0, \quad \{ p_j, x_k \} = \delta_{jk}.
\] (1.13)
This analysis, which is called the separation of variables, makes it possible to calculate the energy spectrum for the quantum Gaudin magnet.

By this way, we can perform the separation of variables for integrable systems formulated in the linear Poisson relation (1.1). Some of integrable systems, e.g. nonlinear integrable equations on finite segment, are formulated in terms of another Poisson structure; there exists ‘rs’-Poisson structure [18, 19],
\[
\{ L(u), \hat{L}(v) \} = [ r(u - v), \frac{1}{2} L(u) + \frac{1}{2} \hat{L}(v) ] + [ s(u + v), - \frac{1}{2} L(u) + \frac{1}{2} \hat{L}(v) ].
\] (1.14)
Here \( \mathbf{r} \) and \( \mathbf{s} \) are matrix structure constants. Remark that \( s \)-matrix depends on sum of spectral parameters while \( r \)-matrix on difference. The Poisson structure (1.14) can be viewed as a classical limit of the boundary Yang-Baxter equation [20], which is used to formulate the quantum spin chain with open boundary.

In this paper we shall study the separation of variables for BC-type integrable system formulated by the \( rs \)-Poisson structure (1.14). In section 2, we introduce the BC-type SL(\( n \)) Gaudin magnet. We give classical \( r \)- and \( s \)-matrices, and prove the integrability. We relate the hamiltonian of the BC-type SL(2) Gaudin magnet with the generalized Neumann model in section 3. The separation of variables is also studied. In section 4, we pay attention on quantum case. The energy spectrum is given based on the separation of variables. Section 5 is devoted to conclusion and discussion.

2. Gaudin Magnet with Boundary

The quantum Gaudin magnet, whose hamiltonian has a form (1.9), was first introduced in Ref. [8] as an integrable spin system with long-range interaction, and solved by use of the coordinate Bethe ansatz. As was reviewed in the previous section, this original Gaudin magnet can be formulated with the linear Poisson structure in classical case (1.1). In this section, we consider the SL(\( n \)) Gaudin magnet with boundary (BC-type Gaudin magnet). The dynamical variables of this model are the classical SL(\( n \)) spin \( S_{ab}^j \) (\( a, b = 1, \ldots, n; j = 1, \ldots, N \)) satisfying the Poisson bracket (1.3). Consider the modified \( L \)-matrix,

\[
L(u) = \sum_{j=1}^{N} \left( \frac{S_j}{u - z_j} + \frac{\overline{S}_j}{u + z_j} \right),
\]

where the “reflected” classical spin \( \overline{S} \) is defined as

\[
\overline{S}^{ab} = (-)^{a+b} S^{ab}.
\]

See the difference from the usual \( L \)-operator (1.2). The second term in (2.1) is due to the effect of the reflection; the classical spin \( S_j \) is located at coordinate \( z_j \), while the “reflected” spin \( \overline{S}_j \) is at \(-z_j\). For this reason we say that the system has “boundary”. One can check easily that the modified \( L \)-operator (2.1) satisfies the linear Poisson structure (1.14) with the rational \( r \)-matrix (1.4) and \( s \)-matrix,

\[
s(u) = \overline{F} = \frac{P}{u},
\]

where we use the notation, \( F^{ab,cd} = (-)^{a+b} \delta^{ad} \delta^{bc} \).

Let us define functions \( \tau_m(u) \) of the matrix \( L(u) \) (2.1) as,

\[
\tau_m(u) = \frac{1}{m} \text{Tr} L(u)^m.
\]
From the Poisson structure (1.14), it is shown that functions $\tau_m(u)$ are the spectral invariants of dynamical system,

$$\{\tau(u), \tau(v)\} = 0. \quad (2.5)$$

The first nontrivial invariant is given from $\tau_2(u)$ as,

$$\tau_2(u) = \frac{1}{2} \text{Tr} L(u)^2 = \sum_{j=1}^{N} \frac{2z_j H_j}{(u-z_j)(u+z_j)} + \frac{1}{2} \sum_{j=1}^{N} \left( \frac{1}{(u-z_j)^2} + \frac{1}{(u+z_j)^2} \right) \text{Tr} S_j^2, \quad (2.6)$$

where $H_j$ has a form,

$$H_j = \frac{\text{Tr} (S_j \overline{S}_j)}{2z_j} + \sum_{k \neq j}^{N} \left( \frac{\text{Tr} (S_j S_k)}{z_j - z_k} + \frac{\text{Tr} (S_j \overline{S}_k)}{z_j + z_k} \right). \quad (2.7)$$

The involutiveness of the spectral invariants $\tau_m(u)$ indicates the Poisson commutativity of the hamiltonian $H_j$,

$$\{H_j, H_k\} = 0, \quad \text{for } j, k = 1, 2, \ldots, N. \quad (2.8)$$

We call $H_j$ the hamiltonian of the SL($n$) Gaudin magnet with boundary. Different from the original Gaudin magnet (1.9), the hamiltonian $H_j$ includes interacting terms between classical spins $S_j$ and “reflected” spins $\overline{S}_j$. By calculating the residues of $\tau_m(u)$ (2.4) as in the case of original Gaudin magnet (1.11), one can get “higher-order” hamiltonian $\tau_{m,j}^\alpha$ of the Gaudin magnet. The existence of commutative family $\{\tau_{m,j}^\alpha\}$ supports the complete integrability of model in Liouville’s sense.

3. Separation of Variables

In this section we first show the isomorphism of the $N$-site SL(2) Gaudin magnet and the $N$-dimensional generalized Neumann model, and then study the separation of variables based on the technique of Sklyanin. In SL(2) case we can define the classical spin matrices $S_j$ and $\overline{S}_j$ in the $L$-operator (2.1) as,

$$S_j = \begin{pmatrix} S_j^+ & S_j^- \\ S_j^+ & -S_j^- \end{pmatrix}, \quad \overline{S}_j = \sigma^z S_j \sigma^z, \quad (3.1)$$

where $\sigma^z$ is denoted as Pauli spin matrix. These spin variables satisfy the following Poisson relations;

$$\{S_j^\epsilon, S_k^\delta\} = \pm \delta_{jk} S_j^\epsilon, \quad \{S_j^+, S_k^-\} = 2\delta_{jk} S_j^z. \quad (3.2)$$

Above Poisson structures for spin variables can be realized with new variables $x_j$ and $p_j$ as,

$$S_j^+ = \frac{i}{2} x_j^2, \quad S_j^- = \frac{i}{2} p_j^2, \quad S_j^z = -\frac{1}{2} x_j p_j, \quad (3.3)$$
where \( \{x_j, p_j \mid j = 1, \ldots, N\} \) are canonical variables satisfying the Poisson relations; \( \{x_j, x_k\} = \{p_j, p_k\} = 0 \), and \( \{x_j, p_k\} = \delta_{jk} \). In this case the Casimir element is set to be zero, \( \text{Tr} \mathbf{S}_j^2 = 0 \). In terms of canonical variables, one can obtain the hamiltonian from the spectral invariant \( \tau_2(u) \),

\[
\tau_2(u) = \frac{1}{2} \text{Tr} \mathbf{L}(u)^2 = \sum_{j=1}^{N} \frac{z_j}{(u - z_j)(u + z_j)} H_j,
\]

(3.4)

where \( H_j \) is calculated as,

\[
H_j = \frac{x_j^2 p_j^2}{z_j} - \frac{1}{2} \sum_{k \neq j}^{N} \left( \frac{(p_j x_k - x_j p_k)^2}{z_j - z_k} - \frac{(p_j x_k + x_j p_k)^2}{z_j + z_k} \right).
\]

(3.5)

This hamiltonian can be viewed as generalization of the Neumann model [9, 11]. This proves the fact that the \( N \)-dimensional generalized Neumann model (3.5) is isomorphic to the \( N \)-site BC-type SL(2) Gaudin magnet (2.7).

Now we study the separation of variables for the generalized Neumann model (3.5). Define \( A(u) \) and \( B(u) \) as

\[
A(u) = L_{11}(u), \quad B(u) = L_{12}(u).
\]

(3.6)

The linear Poisson structure (1.14) includes relations between functions \( A(u) \) and \( B(u) \);

\[
\{A(u), A(v)\} = 0, \quad \{B(u), B(v)\} = 0,
\]

\[
\{A(u), B(v)\} = \frac{2u}{(u - v)(u + v)} (B(u) - B(v)).
\]

(3.7)

Both matrix elements, \( A(u) \) and \( B(u) \), Poisson-commute among themselves.

We choose separable coordinates as zeros of the off-diagonal element of \( L \)-operator,

\[
B(\pm u_\alpha) = 0, \quad \text{for } \alpha = 1, 2, \ldots, N - 1.
\]

(3.8)

Note that \( u = -u_\alpha \) is solution if \( u = u_\alpha \) solves equation \( B(u) = 0 \). By use of a set of variables \( u_\alpha \), we further introduce the canonical variables by

\[
v_\alpha \equiv A(u_\alpha).
\]

(3.9)

From the Poisson relations (3.7) one can see that variables \( u_\alpha \) and \( v_\alpha \) are canonically conjugate,

\[
\{u_\alpha, u_\beta\} = 0, \quad \{v_\alpha, v_\beta\} = 0, \quad \{u_\alpha, v_\beta\} = \delta_{\alpha\beta}.
\]

(3.10)

The first two Poisson relations are easy to be proved. The third relation follows from

\[
\{u_\alpha, v_\beta\} = \lim_{u \to u_\alpha} \frac{1}{B'(u)} \left( \frac{1}{u_\beta - u} + \frac{1}{u_\beta + u} \right) B(u).
\]
The Poisson relations (3.10) show that \( u_\alpha \) and \( v_\alpha \) are canonically conjugate variables, and that the generalized Neumann model (3.5) is separated by transforming the dynamical variables as

\[
\{ x_j, p_j | j = 1, \ldots, N \} \to \{ u_\alpha, v_\alpha | j = 1, \ldots, N - 1 \}.
\]

With variables \( u_\alpha \) and \( v_\alpha \), the action \( W \) of the generalized Neumann model (3.5) is written as separated form,

\[
W = \sum_{\alpha=1}^{N-1} \int v_\alpha du_\alpha. \tag{3.11}
\]

The separated variables \( u_\alpha \) and \( v_\alpha \) can be written explicitly in terms of \( x_j \) and \( p_j \). By definition (3.8) we can solve the coordinates \( p_j \) as

\[
p_j^2 = \frac{z}{z_j} \left( \prod_{\alpha=1}^{N-1} (z_j - u_\alpha)(z_j + u_\alpha) \right) \left( \prod_{k \neq j}^N (z_j - z_k)(z_j + z_k) \right), \tag{3.12}
\]

with \( z \equiv \sum_j z_j p_j^2 \). The canonically conjugate variables \( v_\alpha \) are also solved as

\[
v_\alpha \equiv A(u_\alpha) = -\sum_{j=1}^N \frac{u_\alpha x_j p_j}{(u_\alpha - z_j)(u_\alpha + z_j)}. \tag{3.13}
\]

Remark that the relation between \( \{ u_\alpha, v_\alpha \} \) and the Hamiltonian \( H_j \) is given from the spectral invariants \( \tau_2(u = u_\alpha) \),

\[
v_\alpha^2 = \sum_{j=1}^N \frac{z_j H_j}{(u_\alpha - z_j)(u_\alpha + z_j)}. \tag{3.14}
\]

4. Quantum Case

In this section we consider to quantize the BC-type Gaudin magnet (2.7). For brevity we study SL(2) case. We set the \( L \)-operator as

\[
L(u) = \sum_{j=1}^N \left( \frac{S_j}{u - z_j} + \frac{\overline{S}_j}{u + z_j} \right), \tag{4.1}
\]

where the spin operator \( S_j \) and reflected spin operator \( \overline{S}_j \) are defined as

\[
S_j = \left( \begin{array}{cc} S_j^+ & S_j^- \\ S_j^- & -S_j^+ \end{array} \right), \quad \overline{S}_j = \sigma^z S_j \sigma^z = \left( \begin{array}{cc} S_j^z & -S_j^- \\ -S_j^+ & -S_j^z \end{array} \right). \tag{4.2}
\]
Here operators $S_j^z$ and $S_j^{\pm}$ denote bases of the $su(2)$ Lie algebra, and they satisfy the commutation relations,

$$\left[ S_j^z, S_k^\pm \right] = \pm S_j^\pm \delta_{jk}, \quad \left[ S_j^+, S_k^- \right] = 2 S_j^z \delta_{jk},$$

$$\left( S_j^z \right)^2 + \frac{1}{2} \left( S_j^+ S_j^- + S_j^- S_j^+ \right) = \ell_j (\ell_j + 1), \quad \ell_j \in \mathbb{Z}_+/2. \quad (4.3)$$

We have set $\ell_j$ as spin of the $j$-th site. One can check from direct calculations that the $L$-operator (4.1) satisfies the quantum analogue of the linear Poisson structure (1.14),

$$\left[ \hat{1} L(u), \hat{2} L(v) \right] = \left[ r(u-v), \frac{1}{2} L(u) + \frac{1}{2} L(v) \right] + \left[ s(u+v), -\frac{1}{2} L(u) + \frac{1}{2} L(v) \right]. \quad (4.4)$$

In this case the constant $r$- and $s$-matrices are defined as

$$r(u) = -\frac{P}{u}, \quad s(u) = \frac{1}{2} \sigma^z r(u) \frac{1}{2} \sigma^z. \quad (4.5)$$

The conserved operators are generated by the same way in the classical case; trace of the $L$-operator (2.4). We get the first non-trivial operator from $\hat{\tau}_2(u)$,

$$\hat{\tau}_2(u) = \frac{1}{2} \text{Tr} L(u)^2$$

$$= \sum_{j=1}^{N} \frac{2 z_j}{(u-z_j)(u+z_j)} \hat{H}_j + \sum_{j=1}^{N} \frac{4 z_j^2 \ell_j (\ell_j + 1)}{(u-z_j)^2 (u+z_j)^2}. \quad (4.6)$$

Here the quantum operator $\hat{H}_j$ is the hamiltonian of the quantum BC-type Gaudin magnet,

$$\hat{H}_j = \left( \frac{S_j^z}{2} \right)^2 + \sum_{k \neq j} \frac{\text{Tr} S_j S_k}{z_j - z_k} + \frac{\text{Tr} \bar{S}_j \bar{S}_k}{z_j + z_k}. \quad (4.7)$$

From the commutativity of the generating function $\hat{\tau}_2(u)$, we can see that the operators $\hat{H}_j$ are commutative,

$$[\hat{H}_j, \hat{H}_k] = 0, \quad \text{for } j, k = 1, \ldots, N, \quad (4.8)$$

which proves the quantum integrability of the BC-type Gaudin magnet. Note that the operator $\hat{H}_j$ [4.7] has been appeared in recent studies of the generalized Knizhnik-Zamolodchikov (KZ) equation [21, 22].

The separation of variables for quantum case can be performed as follows. When we define operator $A(u)$ and $B(u)$ as

$$A(u) = L_{11}(u), \quad B(u) = L_{12}(u),$$

we obtain from the quantum $rs$-structure (4.4) that the commutation relations among operators $A(u)$ and $B(u)$ can be written as,

$$[A(u), A(v)] = 0, \quad [B(u), B(v)] = 0,$$

$$[A(u), B(v)] = \frac{2 u}{(u-v)(u+v)} \left( B(u) - B(v) \right). \quad (4.9)$$
All the calculation is essentially same with the classical case. In the quantum case, we can also introduce the “canonical operators” $u_\alpha$ and $v_\alpha$ by,

$$B(\pm u_\alpha) = 0,$$

$$v_\alpha = A(u_\alpha).$$  \hfill (4.10)

These operators $u_\alpha$ and $v_\alpha$ satisfy the commutation relations,

$$[u_\alpha, u_\beta] = 0, \quad [v_\alpha, v_\beta] = 0, \quad [u_\alpha, v_\beta] = \delta_{\alpha\beta}. \quad [4.11]$$

To perform the separation of variables (4.10) for the quantum BC-type Gaudin magnet, we use the realization of the spin operators, $S_z^j$ and $S_\pm^j$, as

$$S_z^j = -x_j \frac{\partial}{\partial x_j} + \ell_j, \quad S_-^j = x_j, \quad S_+^j = -x_j \frac{\partial^2}{\partial x_j^2} + 2\ell_j \frac{\partial}{\partial x_j}. \quad (4.12)$$

With this realization, the functional equation, $B(\pm u_\alpha) = 0$, does not include differential operator, and can be solved easily. The result is

$$x_j = \frac{z}{2z_j} \frac{\prod_{\alpha=1}^{N-1} (z_j - u_\alpha)(z_j + u_\alpha)}{\prod_{k\neq j} N (z_j - z_k)(z_j + z_k)}, \quad \hfill (4.13)$$

where we set $z \equiv \sum_k z_k x_k$. When we change the variables from $\{x_j\}$ to $\{u_\alpha\}$, we can see that operator $v_\alpha$ is represented in terms of $u_\alpha$ as

$$v_\alpha = -\frac{\partial}{\partial u_\alpha} + \Lambda(u_\alpha), \quad \hfill (4.14)$$

where we use function $\Lambda(u)$ as

$$\Lambda(u) = \sum_{j=1}^N \frac{2u \ell_j}{(u - z_j)(u + z_j)}. \quad \hfill (4.15)$$

Remark that identification of operator $v_\alpha$ (4.14) is consistent with the commutation relations (4.11). A set of operators $\{u_\alpha, v_\alpha\}$ is called the separated operator.

We have completed in separating variables for quantum SL(2) Gaudin magnet. In the rest of this section, we show that the energy spectrum for the Gaudin magnet can be calculated from the functional Bethe ansatz. By definition of the generating function, $\hat{\tau}_2(u) = \frac{1}{2} \text{Tr} L(u)^2$, one can see that

$$v_\alpha^2 - \hat{\tau}_2(u_\alpha) = 0, \quad \hfill (4.16)$$

which corresponds, in classical case, to (3.14). With the operator realization of $v_\alpha$ obtained in (4.14), we can read off the identity (4.16) as the differential operator for the separated spectral problem,

$$\psi''(u) - 2\Lambda(u)\psi'(u) + \left(\Lambda^2(u) - \Lambda'(u)\right)\psi(u) = \tau_2(u)\psi(u), \quad \hfill (4.17)$$
where $\tau_2(u)$ is an eigenvalue of operator $\hat{\tau}_2(u)$ (4.6). This equation can be seen as a generalized Lamé equation [10]. To solve this second order differential equation (4.17), we assume that the wave function $\psi(u)$ is a polynomial of $u$, and that zeros of $\psi(u)$ are denoted as $\pm \lambda_\alpha$ [4],

$$\psi(u) = \prod_{\alpha=1}^{M} (u - \lambda_\alpha)(u + \lambda_\alpha).$$  \hspace{1cm} (4.18)

Substituting the wave function $\psi(u)$ into differential equation (4.17), we can see that the eigenvalue $E_j$ of $\hat{H}_j$ is given by,

$$E_j = -2 \chi(z_j) \ell_j - \frac{\ell_j}{z_j} + \sum_{k \neq j}^{N} \frac{4 z_j \ell_j \ell_k}{(z_j - z_k)(z_j + z_k)}.$$  \hspace{1cm} (4.19)

where function $\chi(u)$ is defined from the wave function $\psi(u)$ as

$$\chi(u) = \frac{d}{du} \log \psi(u) = \sum_{\alpha=1}^{M} \frac{2 u}{(u - \lambda_\alpha)(u + \lambda_\alpha)}.$$  \hspace{1cm} (4.20)

Notice that the zeros of the wave function $\psi(u)$ should be fixed to satisfy a set of equations,

$$\Lambda(\lambda_\alpha) = \sum_{\beta \neq \alpha}^{M} \frac{2 \lambda_\alpha}{(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha + \lambda_\beta)} + \frac{1}{2} \frac{2 \lambda_\alpha}{\lambda_\alpha}, \quad \text{for } \alpha = 1, \ldots, M,$$  \hspace{1cm} (4.21)

which follows from the conditions in cancellation of the residues at $u = \lambda_\alpha$ in (4.17). This equation is a quasi-classical limit of the Bethe ansatz equation for open boundary spin chain, and plays a crucial role in construction of the integral solution for the generalized KZ equation [22].

5. Discussion

We have introduced the generalized Gaudin magnet in this paper. The hamiltonian is written as

$$H_j = \frac{\text{Tr} (S_j \overline{S}_j)}{2 z_j} + \sum_{k \neq j}^{N} \left( \frac{\text{Tr} (S_j S_k)}{z_j - z_k} + \frac{\text{Tr} (S_j \overline{S}_k)}{z_j + z_k} \right).$$

This model can be regarded as the Gaudin magnet with boundary, and be formulated in terms of “classical” reflection equation (1.14). As in the case of the original Gaudin magnet, this model has many interesting aspects; in particular for SL(2) case, the model is proved to be isomorphic to the generalized Neumann model. In both classical and quantum cases we have performed the separation of variables. In this analysis the eigenvalue problem can be reduced to the second-order differential equation (Lamé equation). We can obtain the so-called quasi-classical Bethe ansatz equation from this
differential equation. The XXZ-Gaudin magnet with boundary will be analysed in the same method.

The point of the separation of variables (functional Bethe ansatz) is to take zeros of the wave-function $\psi(u)$; the zeros may be identified with the “rapidities” of the spin-wave from the viewpoint of the inverse scattering method. This kind of analysis was used in recent studies of the Asbel-Hofstadter problem [23]. From the viewpoint of the $q$-polynomial theory the Askey-Wilson polynomial may be related with quantum $rs$-structure [24].

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