ON A CLASS OF TWO-INDEX REAL HERMITE POLYNOMIALS†

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Abstract We discuss some basic properties of a class of doubly indexed real Hermite polynomials including recurrence formulae, Runge’s addition formula, generating function and Nielsen’s identity.

1 Introduction

The Burchnall’s operational formula ([2])

$$\left(-\frac{d}{dx} + 2x\right)^m (f) = m! \sum_{k=0}^{m} \frac{(-1)^k H_{m-k}(x)}{k! (m-k)!} \frac{d^k}{dx^k} (f),$$

(1.1)

where $H_m(x)$ denotes the usual Hermite polynomial ([5, 10])

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} \left(e^{-x^2}\right),$$

(1.2)

enjoy a number of remarkable properties. It is used by Burchnall [2] to give a direct proof of Nielsen’s identity ([8])

$$H_{m+n}(x) = m! n! \sum_{k=0}^{\text{min}(m,n)} \frac{(-2)^k H_{m-k}(x) H_{n-k}(x)}{k! (m-n)! (n-k)!}.$$

(1.3)

The special case of (1.1) where $f = 1$, i.e.,

$$H_m(x) = \left(-\frac{d}{dx} + 2x\right)^m \cdot (1),$$

(1.4)

can be employed to recover in a easier way the generating function

$$\sum_{m=0}^{+\infty} H_m(x) \frac{t^m}{m!} = \exp(2xt - t^2)$$

(1.5)

as well as the Runge addition formula ([9, 7])

$$H_m(x+y) = \left(\frac{1}{2}\right)^{m/2} m! \sum_{k=0}^{n} \frac{H_k(\sqrt{2x})}{k!} \frac{H_{m-k}(\sqrt{2y})}{(m-k)!}.$$  

(1.6)

Many generalizations of such Hermite polynomials can be found in the literature including multi-index ones [11, 6, 1, 3]. In this paper, we consider the following class of two-index Hermite polynomials of single real variable:

$$H_{m,n}(x) = \left(-\frac{d}{dx} + 2x\right)^m \cdot (x^n),$$

(1.7)

and we derive some of their useful properties. More essentially, we discuss the associated recurrence formulae, Runge’s addition formula, generating function and Nielsen’s identity.

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2 Doubly indexed real Hermite polynomials \( H_{m,n}(x) \)

By taking \( f(x) = x^n \) in (1.1), we obtain

\[
H_{m,n}(x) := \left(-\frac{d}{dx} + 2x\right)^m (x^n) = m! n! \sum_{k=0}^{\min(m,n)} (-1)^k \frac{x^{n-k}}{k!} \frac{H_{m-k}(x)}{(n-k)! (m-k)!} .
\]

(2.1)

It follows that \( H_{m,n}(x) \) is a polynomial of degree \( m + n \), since

\[
Q(x) := H_{m,n}(x) - x^n H_m(x)
\]

is a polynomial of degree \( \text{deg}(Q) \leq n + m - 2 \). For the unity of the formulations, we shall define trivially

\[
H_{m,n}(x) = 0
\]

whenever \( m < 0 \) or \( n < 0 \). We call them doubly indexed real Hermite polynomials. Note that \( H_{m,0}(x) = H_m(x) \), \( H_{0,n}(x) = x^n \) and

\[
H_{m,n}(0) = \begin{cases} 0 & m < n \\ (-1)^n \frac{m!}{(m-n)!} H_{m-n}(0) & m \geq n \end{cases} .
\]

(2.3)

A direct computation using (2.1) gives rise to

\[
H_{1,n}(x) = -nx^{n-1} + 2x^{n+1}
\]

for every integer \( n \geq 1 \). Note also that, since \( H_1(x) = 2x \), it follows

\[
H_{m+1}(x) = \left(-\frac{d}{dx} + 2x\right)^m (H_1(x)) = \left(-\frac{d}{dx} + 2x\right)^m (2x) = 2H_{m,1}(x).
\]

(2.4)

The first few values of \( H_{m,n} \) are given by

| \( m \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) |
|---|---|---|---|
| 1 | 1 + 2x^2 = H_2(x) | -2x + 2x^3 | -3x^2 + 2x^4 |
| 2 | -6x + 4x^3 = H_3(x) | 2 - 10x^2 + 4x^4 | 6x - 14x^3 + 4x^5 |
| 3 | 6 - 24x^2 + 8x^3 = H_4(x) | 24x - 36x^2 + 8x^3 | -6 + 54x^2 - 48x^3 + 8x^5 |

From (2.2), one can deduce easily the symmetry formula

\[
H_{m,n}(-x) = (-1)^{n+m} H_{n,m}(x),
\]

(2.5)

so that the \( H_{m,n}(x) \) is odd (resp. even) if and only if \( n + m \) is odd (resp. even). Furthermore, the Rodrigues formula for \( H_{m,n}(x) \) is

\[
H_{m,n}(x) = (1)^{m} e^{z^2} \frac{d^m}{dz^m} \left(z^n e^{-z^2}\right) .
\]

(2.6)

Indeed, this can be proved easily making use of

\[
\left(\frac{d}{dx} + 2x\right)^m f(x) = (-1)^m e^{z^2} \frac{d^m}{dz^m} \left(e^{-z^2} f(x)\right) .
\]

(2.7)

Therefore, these polynomials constitute a subclass of the generalized Hermite polynomials

\[
H_{m,n}(x, \alpha, \beta) := (-1)^m x^{-\alpha} e^{\beta x^2} \frac{d^m}{dx^m} \left(x^n e^{-\beta x^2}\right) .
\]

(2.8)

considered by Gould and Hopper in [4]. In fact, we have \( H_{m,n}(x) = x^n H_{m,n}(x, n, 1) \).

Proposition 2.1. The polynomials \( H_{m,n} \), \( m, n \geq 1 \), satisfy the following recurrence formulae

\[
H_{m,n}(x) + H_{m+1,n}(x) - 2x H_{m,n}(x) = 0,
\]

(2.9)

\[
H_{m,n}(x) + n H_{m-1,n-1}(x) - 2 H_{m-1,n+1}(x) = 0,
\]

(2.10)

\[
H_{m,n}(x) + m H_{m-1,n-1}(x) - x H_{m,n-1}(x) = 0,
\]

(2.11)

\[
(m-n) H_{m-1,n-1}(x) + 2 H_{m-1,n+1}(x) + x H_{m,n-1}(x) = 0.
\]

(2.12)
Proof. The first one follows by writing the derivation operator as

\[ \frac{d}{dx} = - \left( - \frac{d}{dx} + 2x \right) + 2x. \]

Indeed, we get

\[ \frac{d}{dx} (H_{m,n}(x)) = - \left( - \frac{d}{dx} + 2x \right) H_{m,n}(x) + 2x H_{m,n}(x) \]
\[ = -H_{m+1, n}(x) + 2x H_{m,n}(x). \]

For the second one, write \( H_{m,n}(x) \) as

\[ H_{m,n}(x) = \left( - \frac{d}{dx} + 2x \right)^{m-1} (H_{1,n}(x)) \]
\[ = \left( - \frac{d}{dx} + 2x \right)^{m-1} (-nx^{n-1} + 2x^{n+1}) \]
\[ = -nH_{m-1, n-1}(x) + 2H_{m-1, n+1}(x). \]

To prove (2.11), we use (2.6) combined with Leibnitz formula. Indeed,

\[ H_{m,n}(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} \left(x \cdot x^{n-1} e^{-x^2}\right) \]
\[ = (-1)^m e^{x^2} \left[ x \frac{d^m}{dx^m} \left(x^{n-1} e^{-x^2}\right) + m \frac{d^{m-1}}{dx^{m-1}} \left(x^{n-1} e^{-x^2}\right) \right] \]
\[ = xe^{x^2}H_{m,n-1}(x) - mH_{m-1,n-1}(x). \]

Finally, (2.12) follows from (2.10) and (2.11) by subtractions.

Remark 2.2. According to (2.4), the (2.11) (corresponding to \( n = 1 \)) leads to the well known recurrence formula \( H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x) \) for \( H_m(x) \). Note also that (2.9) reduces further to \( H_m'(x) + H_{m+1}(x) - 2x H_m(x) = 0 \) by taking \( n = 0 \), so that we recover the known result that \( H_m'(x) = 2mH_{m-1}(x) \).

Proposition 2.3. We have the following addition formula

\[ H_{m,n}(x+y) = m!n! \left( \frac{1}{\sqrt{2}} \right)^{m+n} \sum_{k=0}^{m} \sum_{j=0}^{n} \frac{H_{k,j}(\sqrt{2}x) H_{m-k,n-j}(\sqrt{2}y)}{k!j!(m-k)!(n-j)!}. \quad (2.13) \]

Proof. We begin by writing have \( H_{m,n}(x+y) \) as

\[ H_{m,n}(x+y) = \left( - \frac{d}{d(x+y)} + 2(x+y) \right)^m \cdot (x+y)^n \]
\[ = \left( - \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 2(x+y) \right)^m \cdot (x+y)^n \]
\[ = \left( \frac{1}{\sqrt{2}} \right)^m (A_x + A_y)^m \cdot (x+y)^n \]
\[ = \left( \frac{1}{\sqrt{2}} \right)^m \sum_{j=0}^{n} \binom{n}{j} (A_x + A_y)^m \cdot (x^j y^{n-j}), \]

where \( A_t \) stands for \( A_t = -\partial / (\partial \sqrt{2}t) + 2\sqrt{2}t \). Thus, since \( A_x \) and \( A_y \) commute, we can make use of the binomial formula to get

\[ H_{m,n}(x+y) = \left( \frac{1}{\sqrt{2}} \right)^m \sum_{k=0}^{m} \binom{m}{k} \frac{1}{k!} \binom{n}{j} A_x^k A_y^{n-k} \cdot (x^j y^{n-j}), \]

whence, we obtain the asserted result according to the fact that

\[ A_t^m (t^*) = 2^{-s/2} H_{s,s}(\sqrt{2}t). \]

\[ \square \]
Therefore, the desired result follows since we get

\[ \text{making use of the Weyl identity which reads for the operators } A \]

\[ \text{holds by taking } x = 0 \text{ and setting } t = y \text{ in (2.13), keeping in mind (2.3). We get also} \]

\[ H_{m,n}(t) = m!n! \left( \frac{1}{\sqrt{2}} \right)^{m+n} \sum_{k=0}^{m} \sum_{j=0}^{n} (-1)^{j} H_{k-j}(0) \frac{H_{m-k,n-j}(\sqrt{2}t)}{(m-k)!(n-j)!} \]

by setting \( x = y = t/2 \) in (2.13). While for \( t = -\sqrt{2}x = \sqrt{2}y \), we obtain

\[ \sum_{k=0}^{m} \sum_{j=0}^{n} (-1)^{k} H_{k,j}(t) \frac{H_{m-k,n-j}(t)}{k!j!} \frac{2}{(m-k)!(n-j)!} = 0 \]

whenever \( m + n \) is odd or \( m > n \).

Next, we state the following

**Proposition 2.6.** The generating function of \( H_{m,n} \) is given by

\[ \sum_{m,n=0}^{\infty} H_{m,n}(x) \frac{u^{m} v^{n}}{m! n!} = \exp \left( -u^{2} + (2u + v)x - uv \right). \]  

(2.14)

**Proof.** According to the definition of \( H_{m,n} \), we can write

\[ \sum_{m,n=0}^{\infty} H_{m,n}(x) \frac{u^{m} v^{n}}{m! n!} = \left[ \sum_{m=0}^{\infty} \frac{1}{m!} \left( -u \frac{d}{dx} + 2ux \right)^{m} \right] \cdot \left( \sum_{n=0}^{\infty} \frac{v^{n}}{n!} x^{n} \right) \]

\[ = \exp \left( -u \frac{d}{dx} + 2ux \right) (e^{vx}). \]

Making use of the Weyl identity which reads for the operators \( A = 2x1d \) et \( B = -d/dx \) as

\[ \exp(uA + uB) = \exp(uA) \exp(uB) \exp \left( -u^{2}Id \right); \quad u \in \mathbb{R}, \]

we get

\[ \sum_{m,n=0}^{\infty} H_{m,n}(x) \frac{u^{m} v^{n}}{m! n!} = e^{2ux-u^{2}} \exp \left( -u \frac{d}{dx} \right) (e^{vx}). \]

Therefore, the desired result follows since

\[ \exp \left( -u \frac{d}{dx} \right) (e^{vx}) = \sum_{k=0}^{\infty} \frac{(-u)^{k}}{k!} \left( \frac{d}{dx} \right)^{k} (e^{vx}) = e^{-uv} e^{vx}. \]

\[ \blacksquare \]

**Remark 2.7.** The special case of \( v = 0 \) in (2.14) infers the generating function (1.5) of the standard real Hermite polynomials \( H_{m,v} \). Furthermore, for \( y = u = -v \), we get

\[ e^{xy} = \sum_{m,n=0}^{\infty} (-1)^{n} H_{m,n}(x) \frac{y^{m+n}}{m!n!}. \]  

(2.15)

**Proposition 2.8.** We have the recurrence formula

\[ H'_{m,n}(x) = 2mH_{m-1,n}(x) + nH_{m,n-1}(x). \]  

(2.16)
Proof. Differentiating the both sides of (2.14) and making appropriate changes of indices yield (2.16). □

Corollary 2.9. We have

\[
\frac{d^n}{dx^n}(H_{r,n}(x)) = r!n! \sum_{j=0}^{\nu} \alpha_{j,\nu} \frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!},
\]

(2.17)

where

\[
\alpha_{j,\nu} = \begin{cases} 
2^\nu & \text{for } j = 0 \\
2\alpha_{j-1,\nu-1} + \alpha_{j-1,\nu-1} & \text{for } 1 \leq j < \nu \\
1 & \text{for } j = \nu
\end{cases}
\]

Proof. This can be handled by mathematical induction using (2.16). □

Remark 2.10. The \(\alpha_{j,\nu}\) are even positive numbers and their first values are

\[
\begin{array}{c|cccccc}
\nu & j = 0 & j = 1 & j = 2 & j = 3 & j = 4 & j = 5 \\
\hline
\nu = 0 & 1 \\
\nu = 1 & \frac{2^2}{1} & 1 \\
\nu = 2 & \frac{2^4}{2} & \frac{4}{1} & 1 \\
\nu = 3 & \frac{2^6}{3} & \frac{12}{6} & \frac{6}{1} \\
\nu = 4 & \frac{2^8}{4} & \frac{32}{8} & \frac{24}{1} \\
\nu = 5 & \frac{2^{10}}{5} & \frac{80}{10} & \frac{80}{1} & 1
\end{array}
\]

We conclude this paper by giving a formula for the two-index Hermite polynomial \(H_{m,n}(x)\) expressing it as a weighted sum of a product of the same polynomials. Namely, we state the following

Proposition 2.11. Keep notation as above. Then the Nielsen identity for \(H_{m,n}; \ n \geq 1\), reads

\[
H_{m+r,n}(x) = m!r!mn! \sum_{k,\nu,j=0}^{m,k,\nu} \alpha_{j,\nu} \frac{\Gamma(n+k-\nu)}{(k-\nu)!}\frac{(-x)\nu}{x} \frac{H_{m-k,n}(x)}{(m-k)!} \frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!}.
\]

Proof. Recall first that \(H_m^{\gamma}(x, \alpha, p)\), the polynomials given through (2.8), can be rewritten in the following equivalent form ([4])

\[
H_m^{\gamma}(x, \alpha, p) := \left(-\frac{d}{dx} + p \gamma x^{\gamma-1} - \frac{\alpha}{x}\right)^m (1).
\]

Now, since for the special values \(p = 1, \gamma = 2\) and \(\alpha = n\), we have

\[
H_{m+r,n}(x) = x^n H_{m+r,n}^2(x, n, 1)
\]

\[
= x^n \left(-\frac{d}{dx} + 2x - \frac{n}{x}\right)^m (H_r^2(x, n, 1))
\]

\[
= x^n \left(-\frac{d}{dx} + 2x - \frac{n}{x}\right)^m (x^{-n}H_r(x))
\]

we can make use of the Burchnall’s formula extension proved by Gould and Hopper [4], to wit

\[
\left(-\frac{d}{dx} + p \gamma x^{\gamma-1} - \frac{\alpha}{x}\right)^m (f) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{H_{m-k}(x, \alpha, p)}{(m-k)!} \frac{d^k}{dx^k} (f).
\]

Thus, for \(f = x^{-n}H_r(x)\), we obtain

\[
H_{m+r,n}(x) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{H_{m-k,n}(x)}{(m-k)!} \frac{d^k}{dx^k} (x^{-n}H_r(x)).
\]
Therefore, by applying the Leibnitz formula and appealing the result of Corollary 2.9, we get

\[
H_{m+r,n}(x) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} H_{m-k,n}(x) \sum_{\nu=0}^{k} \frac{k!}{(m-k)!} \frac{d^{k-\nu}}{dx^{k-\nu}} \left( x^{-n} \right) \frac{d^{\nu}}{dx^{\nu}} (H_r(x))
\]

\[
= m! \Gamma(m+n) \sum_{k,\nu,\mu=0}^{m,\nu\mu} \frac{\alpha_k \Gamma(n+k-\nu)}{(k-\nu)!} \frac{(-x)^{\nu}}{\nu!} \frac{H_{m-k,n}(x)}{x^{n+k}} \frac{H_{r-\nu,j,n-j}(x)}{(m-k)!} \frac{1}{(r-\nu+j)!(n-j)!}
\]

for every integer \( n \geq 1 \). Note that for \( n = 0 \), (2.18) reads simply

\[
H_{m+r}(x) = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!} H_{m-k}(x) \frac{d^k}{dx^k} (H_r(x)).
\]

In this case, we recover the usual Nielsen formula (1.3) for the real Hermite polynomials \( H_m \).

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