Stueckelberg breaking of Weyl conformal geometry
and applications to gravity

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Abstract

Weyl conformal geometry may play a role in early cosmology where effective theory at short distances becomes conformal. Weyl conformal geometry also has a built-in geometric Stueckelberg mechanism: it is broken spontaneously to Riemannian geometry after a particular Weyl gauge transformation (of “gauge fixing”) while Stueckelberg mechanism re-arranges the degrees of freedom (conserving their number). The Weyl gauge field of local scale transformations acquires a mass after absorbing a compensator (dilaton), decouples, and Weyl connection becomes Riemannian. In applications, a “gauge fixing” transformation of the original Weyl’s quadratic gravity gives the Einstein-Proca action for the Weyl gauge field and a positive cosmological constant (plus matter action, if present). The mass of the Weyl gauge field, setting the non-metricity scale, can be much smaller than $M_{\text{Planck}}$, for ultraweak values of the coupling ($q$), with implications for phenomenology. If matter is present, a positive contribution to the Planck scale from a scalar field ($\phi_1$) vev generates a negative (mass)$^2$ term for $\phi_1$, and vice-versa. These results are immediate when using Weyl-covariant (invariant) scalar (tensor) curvatures, respectively, instead of their Riemannian form. The results suggest that Weyl gauge symmetry is physically relevant and its role in high scale physics should be reconsidered.

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1 Weyl gauge transformations and Stueckelberg mechanism

In 1918 Weyl introduced his vector-tensor theory of quadratic gravity [1–3] built on what is now known as Weyl conformal geometry. Weyl’s idea was that the action should be invariant under a most general symmetry: a (Weyl) scaling gauge symmetry [4]. Weyl also thought of identifying this gauge field ($\omega_\mu$) with electromagnetism, which inevitably failed since electromagnetic gauge transformations are “internal” symmetry (not spacetime geometry) transformations. Weyl quadratic gravity was disregarded after Einstein’s early criticism [1] that the spacing of atomic spectral lines changes in such theory, in contrast with experience. This happens because in Weyl geometry a vector parallel transported around a curve changes not only the direction (as in Riemannian geometry) but also its length. Then clock’s rates and rod’s lengths depend on their path history. This is caused by the massless Weyl gauge field $\omega_\mu$ responsible for the non-metric connection of Weyl geometry $\tilde{\nabla}_\mu g_{\alpha\beta} = -\omega_\mu g_{\alpha\beta}$. This is in contrast to (pseudo)Riemannian case (Einstein gravity) where $\nabla_\mu g_{\alpha\beta} = 0$, with $\nabla_\mu$ the Levi-Civita connection. Eventually, (gauged) local scale transformations were abandoned and replaced by phase transformations [5], setting the foundation of modern gauge theories.

After Dirac [6] a different version of Weyl gravity was introduced, linear in Weyl scalar curvature ($\tilde{R}$), of the form $\phi^2 \tilde{R}$, with an additional matter scalar $\phi$ [7–18]. This term easily recovers Einstein gravity, the Weyl gauge field becomes massive (mass $\sim qM_{\text{Planck}}$), decouples ($\omega_\mu = 0$), the geometry becomes Riemannian and Einstein’s criticism is avoided. Recently it was shown [19] that even the original Weyl quadratic gravity without matter [1–3] avoids this criticism since the field $\omega_\mu$ acquires a similar mass and decouples.

Weyl conformal geometry may play a role in early cosmology where effective field theory at short distances becomes conformal. But we argue that Weyl geometry also has a natural built-in geometric Stueckelberg mass mechanism [20]. This is easily seen by using the curvature scalar and tensors of Weyl geometry (instead of their Riemannian expression), which simplifies calculations dramatically. Then a simple Weyl “gauge fixing” symmetry transformation (not fields redefinition) takes one from an action with Weyl geometry symmetry to an action in Riemannian geometry, after Stueckelberg mechanism.

For example, the original Weyl quadratic gravity action is easily “gauge transformed” into Einstein-Proca action for $\omega_\mu$ and a positive cosmological constant (plus matter action, if present). This shows the results of [19] in a new perspective. In the Stueckelberg mechanism the dilaton (spin-zero mode in $\tilde{R}^2$ term) is absorbed by the Weyl gauge field which becomes massive. This conserves the number of dynamical degrees of freedom ($n_{df}$) as expected for a spontaneous breaking of Weyl gauge symmetry. In the absence of $\omega_\mu$, in local conformal symmetry models (e.g. [21]) this mechanism is not available, so $n_{df}$ is not conserved.

The non-metricity scale is set by the mass of the Weyl field ($\sim qM_{\text{Planck}}$), naively expected to be large. Interestingly, small values of this mass are allowed (demanding ultraweak values of coupling $q$) because the lower bound on non-metricity scale is $\mathcal{O}(\text{TeV})$ [22]. Then the Weyl field could even be a (TeV) dark matter candidate [24]. The phenomenology of Standard Model (SM) endowed with Weyl gauge symmetry [14, 17] deserves careful study.
1.1 Weyl gauge transformations

Consider a Weyl scaling gauge transformation $\Omega(x)$ of the metric $g_{\mu\nu}$ and of scalar field $\phi$

$$\hat{g}_{\mu\nu} = \Omega g_{\mu\nu}, \quad \hat{\phi} = \frac{1}{\sqrt{\Omega}} \phi, \quad \hat{\omega}_\mu = \omega_\mu - \frac{1}{q} \partial_\mu \ln \Omega. \quad (1)$$

Here $\omega_\mu$ is the Weyl gauge field, $q$ is the coupling to $\phi$; we also have $\sqrt{g} = \Omega^2 \sqrt{\hat{g}}$, $g \equiv |\det g_{\mu\nu}|$ and metric $(+,−,−,−)$ and conventions as in [25]. The Weyl-covariant derivative of $\phi$ is

$$\hat{D}_\mu \phi = (\partial_\mu - q/2 \omega_\mu) \phi \quad (2)$$

$$= (-q/2) \phi \left[ \omega_\mu - (1/q) \partial_\mu \ln \phi^2 \right]. \quad (3)$$

$\Omega(x)$ is real, there is no complex factor “i” in (1) or in $\hat{D}_\mu \phi$. The gauge symmetry is a dilatation group which is isomorphic to $R^+$. $\hat{D}_\mu \phi$ transforms under (1) like a scalar field $\hat{D}_\mu \hat{\phi} = (1/\sqrt{\Omega}) \hat{D}_\mu \phi$. Given (1), $\omega_\mu$ has geometric origin while eq.(3) has an obvious resemblance to the Stueckelberg mechanism, see later.

In Weyl geometry ($\tilde{\nabla}_\mu + q \omega_\mu) g_{\alpha\beta} = 0$, with $\tilde{\nabla}_\mu$ defined by the Weyl connection coefficients, denoted $\tilde{\Gamma}^\rho_{\mu\nu}$. This differs from Riemannian geometry where $\nabla_\mu g_{\alpha\beta} = 0$ with $\nabla_\mu$ defined by Levi-Civita connection and $\Gamma^\rho_{\mu\nu} = (1/2) g^{\rho\beta}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu})$. $\tilde{\Gamma}^\rho_{\mu\nu}$ can be found from $\Gamma^\rho_{\mu\nu}$ by replacing $\partial_\mu \to \partial_\mu - q \omega_\mu$ giving $\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + (q/2) [\delta^\rho_\mu \omega_\nu + \delta^\rho_\nu \omega_\mu - g_{\mu\nu} \omega^\rho]$. $\tilde{\Gamma}^\rho_{\mu\nu}$ are symmetric $\tilde{\Gamma}^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\nu\mu}$ (no torsion) and are invariant under (1) since their variation induced by the metric is compensated by that of $\omega_\mu$. The Riemann and Ricci tensors in Weyl geometry are defined as in Riemannian geometry but in terms of new $\tilde{\Gamma}^\rho_{\mu\nu}$, and are also invariant under (1). One can then show that the Weyl scalar curvature ($\tilde{R}$)

$$\tilde{R} = R - 3 q D_\mu \omega^\mu - \frac{3}{2} q^2 \omega^\mu \omega_\mu, \quad (4)$$

where $R$ is the Riemannian scalar curvature and $D_\mu \omega^\mu$ is defined by Levi-Civita connection. Using the curvature tensors and scalar of Weyl geometry has an advantage: unlike in Riemannian case, $\tilde{R}$ transforms covariantly under (1) similar to $g^{\mu\nu}$ entering its definition:

$$\hat{R} = \frac{1}{\Omega} \tilde{R}. \quad (5)$$

This simplifies calculations and helps one build Weyl gauge invariant individual operators. Then this symmetry and internal gauge symmetries are on an equal footing in the action.

The criticisms of Weyl gravity based on Weyl geometry (such as the change of a vector length under parallel displacement or of atomic spectral lines spacing) are avoided if $\omega_\mu = 0$. In such case $\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$, Weyl connection becomes Levi-Civita, the geometry becomes Riemannian, so these criticisms do not apply. This happens if we do not introduce $\omega_\mu$ in (1) i.e. we go back to Riemannian geometry gravity (e.g. Brans-Dicke). Alternatively, $\omega_\mu = 0$ if this field decouples after acquiring a high mass. This is the idea we study below.

1 These are $\tilde{R}^\lambda_{\mu\nu\sigma} = \partial_\sigma \tilde{\Gamma}^\lambda_{\mu\nu} - \partial_\nu \tilde{\Gamma}^\lambda_{\sigma\mu} + \tilde{\Gamma}^\lambda_{\rho\nu} \partial_\sigma \tilde{\Gamma}^\rho_{\mu\lambda} - \tilde{\Gamma}^\lambda_{\rho\mu} \tilde{\Gamma}^\rho_{\nu\lambda}$, and $\hat{R}_{\mu\nu\sigma} = \tilde{R}^\lambda_{\mu\nu\sigma}$. Also we have: $\tilde{R} = g^{\mu\nu} \hat{R}_{\mu\nu}$. 2
1.2 Weyl gauge transformation and Proca action

Consider $L$ of a (real) scalar field $\phi$ and Weyl gauge field $\omega$, invariant under (1)

$$L = \sqrt{g} \left[ -\frac{1}{4} F^2_{\mu\nu} + \frac{1}{2} (\tilde{D}_\mu \phi)^2 \right].$$

(6)

To simplify notation, we do not show appropriate indices contractions which are implicit, e.g.: $F^2_{\mu\nu} = g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$ and $(\tilde{D}_\mu \phi)^2 = g^{\mu\nu} \tilde{D}_\mu \phi \tilde{D}_\nu \phi$, etc. Since there is no torsion, the field strength $F_{\mu\nu}$ does not feel the connection. From $F_{\mu\nu} = \tilde{D}_\mu \omega_\nu - \tilde{D}_\nu \omega_\mu$ with $\tilde{D}_\mu \omega_\nu = \partial_\mu \omega_\nu - \tilde{\Gamma}_\rho^{\mu\nu} \omega_\rho$ then $F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$ which coincides with its Riemannian expression and is invariant under (1). A gauge transformation (1) with $\Omega = \phi^2 / M^2$ gives

$$L = \sqrt{\hat{g}} \left[ -\frac{1}{4} \hat{F}^2_{\mu\nu} + \frac{q^2}{8} M^2 \hat{\omega}_\mu \hat{\omega}^\mu \right].$$

(7)

where $M$ is an arbitrary scale and all indices contractions are made with new metric $(\hat{g})$.

The Weyl “photon” has become massive and no trace of $\phi$ is left. This is a geometric version of Stueckelberg mechanism [20] which is naturally built-in Weyl conformal geometry due to the definition of Weyl-covariant derivative $\tilde{D}_\mu$. The presence of $\sqrt{g}$ is essential as it ensures invariance of $L$. The (charged) scalar and Weyl kinetic term in Weyl geometry are gauge transformed into an equivalent Proca action with spontaneous breaking of Weyl gauge symmetry. If we do the inverse gauge transformation, Proca action (of a massive theory) can be written in a Weyl gauge invariant way as a sum of kinetic terms.

The gauge transformation we did is essentially “gauge fixing” $\phi = M$ (constant) but what is most important here is the conservation of the number of dynamical degrees of freedom, $n_{df}$ ($n_{df} = 3$): initially we had a massless scalar and a massless vector field and finally a massive vector field. $M$ is regarded as the scale where Weyl gauge symmetry is broken.

If in (6) there are more ($n$) scalar fields kinetic terms, consider a gauge transformation $\Omega = \rho^2 / M^2$, with $\rho$ the radial direction $\rho^2 = \sum_j \phi_j^2$, absorbed by the only vector field present $\hat{\omega}_\mu = \omega_\mu - (1/q) \partial_\mu \ln \rho^2$ under (1). One recovers (7) with $n - 1$ additional kinetic terms for the angular variables fields. To conclude, the Weyl boson is massive and can decouple.

1.3 Weyl linear gravity as Einstein-Proca action

Consider a linear version of Weyl gravity [6] coupled to a scalar $\phi_1$, invariant under (1)

$$\mathcal{L} = \sqrt{g} \left[ -\frac{\xi_1}{12} \phi_1^2 \tilde{R} + \frac{1}{2} g^\mu\nu \tilde{D}_\mu \phi_1 \tilde{D}_\nu \phi_1 - \frac{\lambda_1}{4!} \phi_1^4 - \frac{1}{4} F^2_{\mu\nu} \right].$$

(8)

where $\tilde{R}$ is the scalar curvature in Weyl geometry, eq.(1), and $\tilde{D}_\mu \phi_1 = (\partial_\mu - q / 2 \omega_\mu) \phi_1$.

After a Weyl gauge transformation (1), (3), with $\Omega = \xi_1 \phi_1^2 / (6M^2)$, then using eq.(1)
\[\mathcal{L} = \sqrt{g} \left[ -\frac{1}{2} M^2 \hat{R} - \frac{1}{4} \hat{F}_{\mu\nu}^2 + \frac{3}{4} q^2 M^2 (1 + 1/\xi) \hat{\omega}_\mu \hat{\omega}^\mu - \frac{3\lambda_1 M^4}{2 \xi^2} \right], \] (9)

up to a total derivative term. Here Riemannian scalar curvature \( \hat{R} \) and indices contractions are computed with new \( \hat{g}_{\mu\nu} \), as indicated by the presence of \( \sqrt{\hat{g}} \).

The gauge transformation considered sets \( \hat{\phi}_1 \) to a constant \( (6M^2/\xi) \), and the Einstein frame results from “gauge fixing” Weyl gauge symmetry. The Stueckelberg mechanism ensures the number of dynamical degrees of freedom \( n_{df} \) is conserved when going from (8) to (9) as expected for spontaneous breaking (and which does not require a potential for \( \phi_1 \)). Here \( \phi_1 \) was “eaten” by the Weyl gauge field which is now massive. What survives of the scalar kinetic term is the \( \xi \)-dependent mass term for \( \hat{\omega}_\mu \), but there is an additional mass correction to \( \hat{\omega}_\mu \) beyond (7), due to the \( \hat{R} \)-dependent term.

This situation is in contrast with the (ungauged) local conformal symmetry case recovered from (8) for \( \omega_\mu = 0 \); then there is no gauge field to “absorb” the scalar “compensator” and the action would be invariant under the first two transformations in (1) only if \( \xi_1 = -1 \).

To conclude, Weyl photon again became massive by “absorbing” a “compensator” field \( \phi_1 \). But what happens in Weyl quadratic gravity with no matter fields present?

1.4 Weyl quadratic gravity as Einstein-Proca action

The original action of Weyl (quadratic) gravity without matter [2] invariant under (1) is

\[ L_1 = \sqrt{g} \left[ \frac{\xi_0}{4!} \hat{R}^2 - \frac{1}{4} F_{\mu\nu}^2 \right], \quad \xi_0 > 0. \] (10)

Each term is Weyl gauge invariant (\( \hat{R} \) transforms covariantly, eq. (5)). We can replace \( \hat{R}^2 \to -2\phi_0^2 \hat{R} - \phi_0^4 \), since integrating the auxiliary field \( \phi_0 \) via its equation of motion, of solution \( \phi_0^2 = -\hat{R} \), recovers the \( \hat{R}^2 \) term in the action; so \( \phi_0 \) transforms like any scalar field and \( \ln \phi_0 \) is the Goldstone of the scale symmetry (1), \( \ln \phi_0^2 \to \ln \phi_0^2 - \Omega \). Then

\[ L_1 = \sqrt{g} \left[ \frac{\xi_0}{4!} \left(-2\phi_0^2 \hat{R} - \phi_0^4\right) - \frac{1}{4} F_{\mu\nu}^2 \right]. \] (11)

Using gauge transformation (1), (5) with \( \Omega = \xi_0 \phi_0^2/(6M^2) \), then using relation (4) we find

\[ L_1 = \sqrt{g} \left\{ -\frac{1}{2} M^2 \hat{R} - \frac{3 M^4}{2 \xi_0} + \frac{3}{4} q^2 M^2 \hat{\omega}_\mu \hat{\omega}^\mu - \frac{1}{4} F_{\mu\nu}^2 \right\}. \] (12)

which is in the Einstein frame. Here we chose \( M = M_{\text{Planck}} \); \( \hat{R} \) is the Riemannian scalar curvature evaluated from new metric \( \hat{g}_{\mu\nu} \) also used for index contractions).

These simple steps show an interesting result: a Weyl “gauge fixing” symmetry transformation (not fields redefinition) applied to the original Weyl quadratic gravity without matter eq. (10) gives the Einstein-Proca action for the Weyl gauge field; this became massive via
Stueckelberg mechanism (spontaneous breaking). There is also a positive cosmological constant. Conversely, the inverse gauge transformation of Einstein-Proca action takes one to Weyl quadratic gravity action. Note again the conservation of the number of degrees of freedom, impossible in the absence of Weyl gauge field.

To illustrate better the Stueckelberg mechanism, write first eq. (11) in a Riemannian language using eq. (4) followed by an integration by parts, which gives:

\[ L_1 = \sqrt{g} \left\{ -\frac{\xi_0}{2} \left[ \frac{1}{6} \phi_0^2 R + (\partial_\mu \phi_0)^2 \right] - \frac{\xi_0}{4!} \phi_0^4 + \frac{q^2}{8} \xi_0 \phi_0^2 \left( \omega_\mu - 1/q \partial_\mu \ln \phi_0 \right)^2 - \frac{1}{4} F_{\mu\nu}^2 \right\}. \]  

(13)

where we used that \( \sqrt{g} D_\mu \omega^\mu = \partial_\mu (\sqrt{g} \omega^\mu) \). Then using gauge transformation (1) with \( \Omega = \xi_0 \phi_0^2 / (6 M^2) \), one finds again eq. (12). It is then obvious how the first term becomes the Einstein term in (12) and how the term \( \propto q^2 \) gives the mass term for \( \tilde{\omega}_\mu \) (Stueckelberg mechanism) in (12). There is no negative kinetic term (ghost) in eq. (13).

The mass of Weyl gauge boson is near the Planck scale \( \sqrt{3}/2 q M \) for a coupling \( q \) not too small, and comes from the \( \tilde{R}^2 \) term alone. Below this mass scale this field decouples, Weyl connection becomes Riemannian and Weyl quadratic action becomes Einstein-Hilbert action. So Einstein gravity is just a “low energy” limit of Weyl gravity. Then previous, long held criticisms of Weyl (quadratic) gravity are avoided; the effects mentioned earlier, associated with Weyl geometry, are suppressed by a large value of the Weyl “photon” mass. Then any change of the spacing of the atomic spectral lines is suppressed by a high scale and can be ignored.

The result in (12) is in the Einstein gauge of constant \( \phi_0^2 = 6 M^2 / \xi_0 \) which coincides with the Weyl gauge (of constant Weyl scalar curvature) since we saw \( \langle \phi_0 \rangle^2 = -\tilde{R} \), so on the ground state \( \phi_0^2 = (-\tilde{R}) = 6 M^2 / \xi_0 \). Actually, for a Friedmann-Robertson-Walker universe, the scalar field naturally evolves in time to \( \phi_0 = \)constant because of a conserved current \( K_\mu = \phi_0 \partial_\mu \phi_0 \) [26]. The Planck scale acquires a physical meaning as the scale where Weyl gauge symmetry is broken.

1.5 A more general case

In a most general case, Weyl quadratic gravity can contain another independent term \( ^3 \)

\[ L'_1 = \sqrt{g} \left\{ \frac{1}{\eta} \tilde{C}_{\mu\nu\rho\sigma} \tilde{C}^{\mu\nu\rho\sigma} + \frac{\xi_0}{4!} \tilde{R}^2 \right\} \]

\[ = \sqrt{g} \left\{ \frac{1}{\eta} \left[ C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{3}{2} q^2 F_{\mu\nu}^2 \right] + \frac{\xi_0}{4!} \tilde{R}^2 \right\}. \]

(14)

where \( \tilde{C}_{\mu\nu\rho\sigma} \) and \( (C_{\mu\nu\rho\sigma}) \) is the Weyl tensor in Weyl geometry (Riemannian geometry), respectively; these tensors are related as shown above [16] with \( F_{\mu\nu} \) the field strength of Weyl gauge boson. Notice that in this case \( F_{\mu\nu}^2 \) term is automatically present, so there

\(^2\)A Gauss-Bonnet (total derivative) term of Weyl geometry can also be present [10], not relevant here.

\(^3\)The Weyl tensor squared term we included here is usually required at the quantum level.
is no need to add it “by hand” (on symmetry grounds as in (10)); however, for canonical normalization one has in this case \( q^2 = -\eta/6 \) \((\eta < 0)\). The result of eq. (12) is still valid since both Weyl tensors are invariant under Weyl gauge transformations; then the final Lagrangian contains an additional term \( C_{\mu\nu\rho\sigma}^2 \); this is needed anyway at the quantum level when trying to renormalize SM in the presence of gravity in (ungauged) local conformal models [21]. In this most general case the mass of the Weyl gauge field \( m_2^2 \sim q^2 M^2 \sim (-\eta) M^2 \) is thus related to the mass of the spin-two ghost contained in \( C_{\mu\nu\rho\sigma}^2 \). At this scale the non-metricity of Weyl geometry steps in to modify the Levi-Civita connection.

1.6 Adding matter

Consider now Weyl quadratic gravity, eq. (10), coupled to a matter scalar \( \phi_1 \):

\[
L_2 = \sqrt{g} \left[ \frac{\xi_0}{4!} \tilde{R}^2 - \frac{1}{4} F_{\mu\nu}^2 \right] - \frac{\sqrt{g}}{12} \xi_1 \phi_1^2 \tilde{R} + \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu} \tilde{D}_\mu \phi_1 \tilde{D}_\nu \phi_1 - \frac{\lambda_1}{4!} \phi_1^4 \right],
\]

(15)

which is invariant under (1) and the potential for \( \phi_1 \) is the only allowed by this symmetry.

As in eq. (11), replace \( \tilde{R}^2 \rightarrow -2 \phi_0^2 \tilde{R} - \phi_0^2 \), to obtain a classically equivalent action. Then \( L_2 \) contains a term linear in the Weyl scalar curvature: \((-1/2) \rho^2 \tilde{R} \) and a positive correction \( \xi_0 \phi_0^2/4! \) \((\xi_0 > 0)\) to the potential which becomes

\[
\mathcal{V}(\phi_1, \rho) = \frac{1}{4!} \left[ \frac{1}{\xi_0} \left( 6 \rho^2 - \xi_1 \phi_1^2 \right)^2 + \phi_1^4 \right], \quad \text{with} \quad \rho^2 = \frac{1}{6} (\xi_1 \phi_1^2 + \xi_0 \phi_0^2),
\]

(16)

where we replaced \( \phi_0 \) by field combination \( \rho \). Using eq. (6), a Weyl gauge transformation (1) with \( \Omega = \rho^2/M^2 \) followed by (1) that introduces Riemannian \( \tilde{R} \), gives

\[
L_2 = \sqrt{g} \left\{ - \frac{1}{2} M^2 \left[ \tilde{R} - \frac{3}{2} \tilde{g}^{\mu\nu} \tilde{\omega}_\mu \tilde{\omega}_\nu \right] - \frac{\sqrt{g}}{4!} \tilde{F}_{\mu\nu}^{\rho\sigma} \tilde{D}_\mu \phi_1 \tilde{D}_\nu \phi_1 - \mathcal{V}(\phi_1, M) \right\},
\]

(17)

with \( \tilde{D}_\mu \phi_1 = (\partial_\mu - g/2 \tilde{\omega}_\mu) \phi_1 \). As in the case without matter, we obtain the Einstein-Proca action with a gauge field that became massive, of mass \( m_2^2 = (3/2)q^2 M^2 \), after Stueckelberg mechanism of “absorbing” the dilaton \( \ln \rho \). The kinetic term of \( \phi_1 \) remains present, since only one degree of freedom (radial direction \( \rho \)) is “eaten” by the vector field.

Under the same gauge transformation (“gauge fixing”) the initial potential \( \phi_1^4 \) becomes

\[
\mathcal{V} = \frac{3 M^4}{2 \xi_0} \left[ 1 - \frac{\xi_1 \phi_1^2}{6 M^2} \right]^2 + \frac{\lambda_1}{4!} \phi_1^4.
\]

(18)

We have a negative mass term \((m_2^2 = -\xi_1 M^2/\xi_0)\) if \( \xi_1 > 0 \). This originates in (16) due to the initial dilaton contribution to the potential \( \propto \phi_0^4 \) (coming from \( \tilde{R}^2 \)), with \( \phi_0 \) replaced by \((6 \rho^2 - \xi_1 \phi_1^2)\) and \( \rho \) “gauge fixed” to \( M \). Then, if massless \( \phi_1 \) gives a positive contribution \( \xi_1 \phi_1^2 \) to the Planck scale \( M \) \((\xi_1 > 0)\) this effect is “compensated” by a negative contribution.
to its mass term in the potential (and vice-versa) in (16). The original dilaton (in \( R^2 \)) plays a mediator role in bringing this negative contribution. It is then interesting that both mass scales of the theory, Planck scale and the scale \( \langle \phi_1 \rangle \) are simultaneously generated by the same “gauge fixing” transformation (11).

If \( \phi_1 \) is the higgs field, \( \langle \phi_1 \rangle \) is the electroweak (EW) scale, then Stueckelberg mechanism also triggers EW breaking. This discussion remains valid for more matter fields \( \phi_j \), in eqs. (15) to (18) simply replace \( \xi_1 \phi_1^2 \rightarrow \sum_j \xi_j \phi_j^2 \) and \( (\hat{D}_\mu \phi_1)^2 \rightarrow \sum_j (\hat{D}_\mu \phi_j)^2 \).

### 1.7 Additional effects

The Weyl-covariant derivative acting on \( \hat{\phi}_1 \) in (17) is a remnant of Weyl gauge symmetry, now broken; to have a “standard” kinetic term for \( \phi_1 \), one can now do a field redefinition

\[
\hat{\omega}' = \hat{\omega} - \frac{1}{q} \partial_\mu \ln (\hat{\phi}_1^2 + 6M^2), \quad \hat{\phi}_1 = M\sqrt{6} \sinh \left[ \frac{\sigma}{M\sqrt{6}} \right]
\]

(19) to find

\[
L_2 = \sqrt{q} \left\{ -\frac{1}{2} M^2 \hat{R} + \frac{3}{4} q^2 M^2 \cosh \left[ \frac{\sigma}{M\sqrt{6}} \right] \hat{\omega}' \hat{\omega}' - \frac{3}{4} \hat{\omega}' \hat{\mu} \hat{\nu} - \frac{3}{4} \hat{\omega}' \hat{\mu} \hat{\nu} - \frac{3}{2} \hat{F}_\mu \hat{F}_\nu \right\}
\]

(20)

with

\[
\hat{\nu} = \frac{3}{2} M^4 \xi \left[ 1 - \xi_1 \sinh^2 \left( \frac{\sigma}{M\sqrt{6}} \right) \right]^2 + \frac{3}{2} M^4 \lambda_1 \sinh^4 \left( \frac{\sigma}{M\sqrt{6}} \right)
\]

(21)

Taylor expanding the mass term of the Weyl gauge field for small \( \sigma < M \) shows there are additional corrections to this mass beyond those due to Stueckelberg mechanism, since \( \langle \sigma \rangle \neq 0 \). Note there is no restriction in the action regarding the relative values of \( \sigma \) versus \( M \). For \( \sigma > M \), \( \hat{\nu} \) is always positive if \( \xi_1 \xi_0 + \lambda_1 > 0 \) which can be true even for \( \lambda_1 < 0 \). For more matter fields present, replace in (19) \( \phi_1^2 \rightarrow \sum_j \phi_j^2 \).

This potential is relevant for models of inflation, assuming \( \sigma \) is the inflaton. For \( \lambda_1 \) and \( \xi_1 \) very small, e.g. \( \lambda_1 \xi_1 \approx 5 \times 10^{-10} \) and \( \xi_1 \approx 10^{-5} \), the potential is nearly flat and one has inflation similar to Starobinsky model (27) for suitable \( \xi_0 \approx 10^{10} \), see (28) for

\[\text{For two scalar fields with initial potential } V(\phi_1, \phi_2) \text{ one obtains, with the notation } m^2(\sigma) = (3/4)q^2 M^2 \cosh(\sigma/(M\sqrt{6})): \]

\[
L_2 = \sqrt{q} \left\{ -\frac{1}{2} M^2 \hat{R} - \frac{1}{4} F_\mu F^\mu + \frac{1}{2} m^2(\sigma) \hat{\omega}' + \frac{1}{2} \left[ \sin^2 \phi \right] \right\} \hat{\omega}' + \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \hat{\mu} \right] \hat{\omega}' + \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \hat{\nu} \right] \hat{\omega}' - \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\omega}' - \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\nu} \hat{\omega}' - \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\omega}' \hat{\nu} - \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\nu} \hat{\omega}' + \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\nu} \hat{\omega}' + \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\omega}' \hat{\nu} + \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\nu} \hat{\omega}' + \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\omega}' \hat{\nu} + \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\nu} \hat{\omega}' + \frac{1}{2} \left[ \sin^2 \phi \hat{\omega}' \right] \hat{\omega}' 
\]

(22)

with polar coordinates fields, tan \( \theta = \phi_1/\phi_2 \) and \( \phi_1^2 + \phi_2^2 = 6M^2 \sin^2 \frac{\phi}{\sqrt{2}} \) and \( \hat{\theta} = M\sqrt{6} \theta \).

Finally

\[
\hat{\nu} = \frac{3}{2} M^4 \xi \left[ \left[ 1 - \xi_1 \sin^2 \phi \right] \sin^2 \frac{\phi}{\sqrt{2}} \right] + 24 \xi_0 \sin^2 \phi \cos \phi + \frac{24 \xi_0 \sin^2 \phi \cos \phi}{M\sqrt{6}} \]

(23)

where \( \xi_1 \) is \( \xi_1 \sin^2 \theta + \xi_2 \cos^2 \theta \) and \( s_0 = \sin \theta, c_0 = \cos \theta \). For an \( O(2) \) symmetry, \( \theta \) dependence in the potential disappears, so one can introduce \( V(s_0, c_0) = \lambda/4! \) but kinetic mixing still exists.
the analysis of this potential. The larger quoted values of $\xi_1$ mark a departure from the Starobinsky inflation. But unlike in that work, here there is no flat direction left - this was “absorbed” by the gauge field.

The situation here is also different from the common models of inflation of no matter field present with inflation driven by $\sqrt{g}(R^2 + M^2 R)$. Here the “scalaron” mode is actually a compensator eaten by $\omega_\mu$, while the matter field $\sigma$ is the inflaton.

Further, if there is no Weyl gauge field in (15), (set $\omega_\mu = 0$), inflation is still possible and was already studied in [29]. The scenario is again similar to that in Starobinsky models. Finally, in the absence of the quadratic term, with only a linear term in scalar curvature and global scale invariance, inflation is again possible and was discussed in [26, 30].

1.8 Related models

These results make it attractive to consider the Weyl gauge symmetry for model building beyond SM. With Einstein gravity a “low energy” limit of Weyl quadratic gravity, below the Stueckelberg breaking scale, Weyl gravity and gauge symmetry are “freed” from past criticisms (based on the presence of a massless Weyl gauge field). One can consider the SM with a higgs mass parameter set to zero and extend it with Weyl gauge symmetry. In such case only the SM Higgs couples to the Weyl gauge boson, as above, while the SM fermions do not [14, 17]. This has implications for phenomenology.

There is a difference between the Weyl gauge invariant model discussed here and the case of local conformal extensions of the Standard Model. As we saw, a Weyl gauge invariant model conserves the number of degrees of freedom during the breaking of this symmetry. Moreover, there is no ghost field in Section 1.4 when “gauge fixing” the Planck scale in eqs. (12), (13). This is to be compared to local conformal extensions of the SM where the Einstein term $(-\sqrt{g}/2)M^2 R$ is written in a local conformal invariant way as

$$L_E = \sqrt{g} \left\{ -\frac{\xi_0}{2} \left[ \frac{1}{6} \phi_0^2 R + (\partial_\mu \phi_0)^2 \right] \right\},$$

(24)

to be compared to (13). We see here that trying to generate Planck scale as a vev of $\phi_0$, by “gauge fixing” $\phi_0^2 = 6M^2/\xi_0$, demands a negative kinetic term (ghost) be present (see [17] for a discussion). Also, in this local conformal case, when “gauge fixing” $\phi_0$ to a constant (“unitary gauge”) and this symmetry is broken, there is no gauge field to “absorb” this scalar (dilaton) mode. This indicates that $n_{df}$ is not conserved and suggests the need for Weyl gauge symmetry.

A quantum analysis of the models with Weyl gauge symmetry, similar to that for local conformal SM [21, 31] would be interesting. At the quantum level the local conformal symmetry demands the presence of the Weyl curvature-squared term, the first term on the rhs of (14). But one should also include $\bar{R}^2$ and the gauge kinetic term in (14). An ultraviolet regularization that preserves the Weyl symmetry of the theory may be preferable [32, 33].

The analysis presented here can be extended to other non-metric theories, with torsion,
etc. Our result is in agreement with more general approaches [34] that consider that at a fundamental level gravity is a theory of connections as dynamical objects. Some of these connections become massive (via Stueckelberg mechanism), as we saw for the Weyl connection, while Levi-Civita connection remains massless. In our case Weyl connection departed from the (fixed) Levi-Civita by a correction $\omega_\mu$, which was a dynamical field. In fact one can write any dynamical connection as a Levi-Civita connection plus a tensor field contribution which is a sum of a non-metricity tensor (here due to $\omega_\mu$) and a contorsion tensor, and then re-do this analysis. From a particle physics perspective, this tensor field, being massive, decouples, to leave in the “low energy” limit only the Levi-Civita connection. It is then obvious why in Einstein gravity it is imposed that the connection be metric and is not allowed to have dynamics.

For high scale physics and early cosmology, non-metricity effects cannot be ignored. In fact current lower bounds on non-metricity scale, which is set by the mass of the Weyl field, are very low, in the region of few TeV [22]. With the mass of Weyl field $\omega_\mu$ of $\sqrt{3/2} \, q \, M_{\text{Planck}}$, this region would correspond to ultraweak values of the coupling $q$. One may also explore the possibility that $\omega_\mu$ is a dark matter candidate. In Weyl invariant models of vector dark matter (DM), the mass of the DM vector field is again in the region of few TeV [24] (also [23]). This is interesting for phenomenology and deserves careful study.

The aforementioned separation of the connection into metric and non-metric contributions is also useful for studies of asymptotic safety. These are using the metric formalism (with Levi-Civita connection), so they do not take into account non-metricity effects, etc. Their results could however be extended by simply taking into account the new degrees of freedom (fields) which are corrections to the Levi-Civita connection. Then asymptotic safety in a non-metric theory is that of a theory with Levi-Civita connection plus the dynamical effects of these fields.

\section{Conclusions}

Weyl conformal geometry is of interest to theorists since it may play a role in early cosmology or at high scales when effective field theory becomes nearly conformal. To take advantage of its symmetry we used the action expressed in terms of tensors and scalar curvatures of Weyl geometry (instead of their Riemannian expressions) since these are Weyl-invariant and covariant, respectively. Individual operators in the action are then invariant under Weyl gauge symmetry. Then this symmetry and internal gauge symmetries are on an equal footing in the action and calculations are simplified.

Weyl conformal geometry has a built-in geometric Stueckelberg mass mechanism. In a Weyl gauge invariant action there is a “gauge fixing” symmetry transformation that realizes a Stueckelberg mechanism and re-arranges the degrees of freedom: the initial massless Weyl gauge field (defining the Weyl connection) absorbs the dilaton (compensator), becomes massive and decouples. Then Weyl geometry (connection) is broken spontaneously to Riemannian geometry (Levi-Civita connection). The latter is then a result of a particular Weyl
gauge symmetry transformation and not a field redefinition, while conserving the number of dynamical degrees of freedom of the theory, as required for spontaneous breaking. In (ungauged) local conformal models, a similar “gauge fixing” (of the dilaton to a constant vev), does not conserve this number when the symmetry breaking takes place, since there is no vector field to “absorb” the Goldstone mode of the symmetry.

Applying this idea to the original Weyl quadratic gravity with/without matter present, one finds that this action is immediately transformed after a “gauge fixing” symmetry transformation, into Einstein-Proca action for the Weyl gauge field plus a (positive) cosmological constant and matter (if initially present); the Weyl gauge field undergoes a Stueckelberg mechanism. Below its mass (∼ qM_{\text{Planck}}) the Weyl gauge field decouples and Einstein gravity is a “low energy” limit of Weyl quadratic gravity.

Previous criticisms of Weyl gravity are avoided since effects induced by the Weyl gauge field, long thought to be massless, are actually strongly suppressed by its mass expected to be rather high. However, note that current lower bounds of the non-metricity scale (set by the mass of ω_μ) are low (TeV region); this suggests that the Weyl field could in principle be lighter, if one considers ultraweak values of the coupling q, and even act as a dark matter candidate. This is interesting and deserves careful study.

Following the same “gauge fixing” transformation, there exists a “compensating” mechanism for matter scalars with non-minimal couplings to ˜R: if a massless scalar gives a positive (negative) contribution to the generation of the Planck scale, this is “compensated” by a simultaneous negative (positive) mass squared term, i.e. a spontaneous breaking of the symmetry under which it is charged. This is due to a dilaton term in the potential induced when “linearising” the quadratic Weyl scalar curvature term.

The results are of interest for asymptotic safety theories; these are using the metric formalism (Levi-Civita connection) and miss the effects discussed in this work. However, these studies can be extended to apply here by taking into account the dynamics of the new fields (ω_μ) that are corrections to the Levi-Civita connection. So asymptotic safety in non-metric case is that for Levi-Civita connection plus the additional fields dynamics.

These results indicate that the original Weyl quadratic gravity is physically relevant and its role should be reconsidered, together with its implications for other areas: SM extended with Weyl gauge symmetry/gravity, black-hole physics, cosmology, supersymmetric version. As stated by Weyl [2]: “The action […] that was implemented in the previous sections is constituted as […] a linear combination of ˜R^2 and F_{μν}^2. I believe that one can assert that this action principle implies everything that Einstein’s theory has implied up to now, but in the more far-reaching questions of cosmology and the constitution of matter, it exhibits a clear superiority. Nevertheless, I do not believe that the laws of nature that are exactly applicable in reality are resolved by it.”
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