On the local structure of the Klein-Gordon field on curved spacetimes

Alexander Strohmaier

Universität Leipzig, Institut für theoretische Physik, Augustusplatz 10/11, D-04109 Leipzig, Germany
E-mail: alexander.strohmaier@itp.uni-leipzig.de
WWW: http://www.physik.uni-leipzig.de/~strohmaier/

Abstract

This paper investigates wave-equations on spacetimes with a metric which is locally analytic in the time. We use recent results in the theory of the non-characteristic Cauchy problem to show that a solution to a wave-equation vanishing in an open set vanishes in the “envelope” of this set, which may be considerably larger and in the case of timelike tubes may even coincide with the spacetime itself. We apply this result to the real scalar field on a globally hyperbolic spacetime and show that the field algebra of an open set and its envelope coincide. As an example there holds an analog of Borchers’ timelike tube theorem for such scalar fields and hence, algebras associated with world lines can be explicitly given. Our result applies to cosmologically relevant spacetimes.

Mathematics Subject Classification (2000): 81T05, 81T20, 35L05, 35L10, 34A12.

Keywords: Klein-Gordon field, curved spacetime, timelike curves, scalar field, unique continuation, quantum field theory.
1 Introduction

Since the canonical quantization procedure of fields requires a "frequency splitting" with respect to the time, the absence of a timelike symmetry group in general curved spacetimes causes severe problems (see [33, 19]) in the construction of quantum fields. The nature of these problems becomes particularly clear in the algebraic formulation of quantum field theory ([18]), where an algebra of observables \( \mathcal{A}(\mathcal{O}) \) is associated to each region \( \mathcal{O} \) in spacetime. Specializing to the Klein-Gordon field the local \( C^* \)-algebras \( \mathcal{A}(\mathcal{O}) \) can be constructed in a similar manner as in Minkowski spacetime (see [14]) and the problem reduces to finding the class of physical states. This is usually achieved by choosing a physically preferred state, the vacuum. The construction then gives a net of local von Neumann algebras the normal states being the physical ones. We favour the \( C^* \)-algebraic approach since our result is most easily stated using the local von Neumann algebras. It is also common to use \( * \)-algebras or the Borchers-Uhlmann-algebra of test functions ([9]) to define Quantum fields on curved spacetimes (e.g. [7, 25, 28, 31, 30, 11]). One may pass from these approaches to the \( C^* \)-algebraic in the same manner as in Minkowski spacetime.

For Wightman-fields in Minkowski spacetime it is known that the field algebra of a timelike tube is equal to the field algebra of the causal completion of this tube ([8]). This may be derived (see [8]) from a mean value theorem by Asgeirsson ([5, 6]) and one of its consequences, namely that a solution to the wave equation vanishing in such a tube, vanishes in the causal completion of this tube (see [12]). As far as more general spacetimes are concerned the author showed recently that a similar result holds for stationary spacetimes and quite general free fields ([32]). Namely the field algebra associated with a non-void open set which is invariant under the time translation is equal to the quasilocal algebra. This result was used there to prove the Reeh-Schlieder property of vacuum-like states. The question arises whether such a result holds for more general spacetimes. As we will see this question is strongly related to the timelike Cauchy problem for second order hyperbolic partial differential equations. In the case of analytic coefficients on an analytic manifold Holmgreens’ uniqueness theorem (see e.g. [21]) states that there is unique continuation of solutions across any non-characteristic hypersurface. Relaxing however the condition of analyticity, counterexamples show that the Cauchy problem for timelike surfaces is ill posed in the general case of partial differential equations with smooth coefficients (see e.g.
Even for the wave operator $\square = \frac{\partial^2}{\partial t^2} - \Delta$ in Minkowski spacetime it is known (see [2]) that there exists a smooth function $u$ such that there is a solution $\phi$ to the equation

$$\square \phi + u \phi = 0$$

which has support equal to a half-space with timelike boundary. It was quite recently shown by Tataru in [33] and further generalized by Hörmander in [22] (see also [29]) that under partial analyticity assumptions one still has unique continuation of solutions across non-characteristic surfaces. We give some global consequences of these unique continuation results. After defining an envelope of uniqueness $\mathcal{E}(\mathcal{O})$ of an open subset $\mathcal{O}$ of the spacetime we show that a solution to a wave equation vanishing in $\mathcal{O}$ vanishes in $\mathcal{E}(\mathcal{O})$. This envelope may be considerably larger than the original set and in particular in the case of small open neighbourhoods of inextendible timelike curves the envelope may coincide with the whole space. We investigate the consequences of these results for the real scalar field on curved spacetimes and show that the field algebra associated with the open set coincides with the field algebra associated with its envelope. The class of spacetimes considered here includes a large variety of physically relevant spacetimes, like cosmologically interesting Robertson-Walker spacetimes. We give an example of how the algebras associated with timelike curves can be explicitly given in curved spacetimes.

## 2 Causal structure of spacetimes

In the next two subsections we review the most important definitions and results about the causal structure of spacetimes. Details can be found in [19], [27] and references therein. In the last two subsections we define what we mean by the “envelope” of an open set. To do this we need to introduce the space of timelike curve-segments endowed with an appropriate topology.

### 2.1 Spacetimes and Causality

We assume we are given a smooth connected manifold $M$ (Hausdorff and second countable) of dimension $n \geq 2$ with a smooth metric $g$ of Lorentzian signature. We assume in addition that $M$ is oriented and time-oriented. In this case we say that $M$ is a spacetime. For a subset $\mathcal{S}$ we define the
chronological future/past \( I^\pm(S) \) of \( S \) to be the set of points in \( M \) that can be reached by future/past directed timelike curves \footnote{by curves we always mean maps from open intervals in \( \mathbb{R} \) into \( M \)}. We define the future/past domain of dependence \( D^\pm(S) \) of a subset \( S \subset M \) as the set of points \( p \) in \( M \) such that every past/future inextendible causal curve through \( p \) intersects \( S \). The domain of dependence \( D(S) \) is the union \( D^+(S) \cup D^-(S) \) (see [17] for a review). The causal complement \( O^\perp \) of an open set \( O \) is the set of points which cannot be reached by causal curves from \( O \). The causal completion of \( O \) is \( O^{\perp\perp} \).

### 2.2 Globally hyperbolic manifolds

A set \( S \) is called achronal if every timelike curve intersects \( S \) at most once. A spacelike hypersurface \( C \) is called Cauchy surface if it is achronal and \( D(C) = M \). In case there exists a Cauchy surface the manifold \( M \) is said to be globally hyperbolic. An open subset \( O \subset M \) is said to be globally hyperbolic if \((O, g|_O)\) is globally hyperbolic as a spacetime in its own right. Note that for an achronal set \( S \) the open set \( \text{int}(D(S)) \) (if non-empty) is globally hyperbolic.

### 2.3 Space of timelike curve-segments

We define curve-segments to be maps from compact intervals in \( \mathbb{R} \) into \( M \) which are restrictions of curves defined on a larger interval. We say a curve-segment is timelike if it is the restriction of a timelike curve. For two points \( p, q \in M \) let \( C(p, q) \) the set of all smooth timelike curve-segments \( \gamma : [0, 1] \to M \) with \( \gamma(0) = p \) and \( \gamma(1) = q \). These sets are commonly used to investigate properties of globally hyperbolic spacetimes (see e.g. [13, 19]). Curve-segments in different parametrizations are identified. We introduce on \( C(p, q) \) a \( C^1 \)-topology in the following way: First we endow \( M \) and \( TM \) with complete Riemannian metrics. We parameterize all curve-segments proportionally to their arc-length in the metric on \( M \). The derivation of a curve-segment \( \gamma \) with respect to this parameter gives a curve-segment \( \dot{\gamma} \) in \( TM \). We now define a metric on \( C(p, q) \) in the following way

\[
d(\gamma_1, \gamma_2) = \sup_{t \in [0, 1]} \text{dist}(\dot{\gamma}_1(t), \dot{\gamma}_2(t)),
\]

(1)
where dist(\(x, y\)) denotes the Riemannian distance between the points \(x\) and \(y\) in \(TM\). One checks that the corresponding topology is independent of the Riemannian metrics chosen on \(M\) and \(TM\).

Given a smooth timelike curve-segment \(\gamma\) joining \(p\) and \(q\) we denote by \(C_0(p, q, \gamma)\) the connected component of \(\gamma\) in \(C(p, q)\). We define the two sets

\[
I(p, q) := \bigcup_{\tilde{\gamma} \in C(p, q)} \tilde{\gamma}((0, 1)),
\]

\[
I_0(p, q, \gamma) := \bigcup_{\tilde{\gamma} \in C_0(p, q, \gamma)} \tilde{\gamma}((0, 1)),
\]

i.e. the set of points which are met by the curves \(\tilde{\gamma}|_{(0,1)}\) with \(\tilde{\gamma}\) in \(C(p, q)\) or \(C_0(p, q, \gamma)\) respectively. One has the following properties

**Proposition 2.1.**

1. \(I(p, q)\) coincides with \(I^+(q) \cap I^-(p)\) in case \(p\) is in the future of \(q\).

2. \(I(p, q)\) and \(I_0(p, q, \gamma)\) are open subsets of \(M\).

3. \(I(p, q)\) and \(I_0(p, q, \gamma)\) are invariant under conformal transformations of the metric.

**Sketch of Proof.** The definitions of \(I(p, q)\) and \(I_0(p, q, \gamma)\) depend only on the causal structure of the spacetime. Therefore, the sets are invariant under conformal transformations of the metric. The fact that \(I_0(p, q, \gamma)\) and \(I(p, q)\) are open follows from the causality properties of the tangent space and the properties of the exponential map (see in particular [27], Lemma 33). If \(p\) is in the future of \(q\) and \(x \in I^+(q) \cap I^-(p)\) then one can again use the exponential map \(\exp_x : T_x M \to \mathcal{O} \subset M\) to construct a smooth timelike curve through \(x\) joining \(p\) and \(q\). □

In Minkowski space \(I_0(p, q, \gamma)\) and \(I(p, q)\) coincide, this need not be the case however in general, even if one restricts the class of spacetimes to the globally hyperbolic ones. A simple example is given by the 2-dimensional cylinder \(\mathbb{R} \times S^1\) with metric \(dt^2 - d\phi^2\). The curves \(\gamma_1\) and \(\gamma_2\) (see figure [2]) cannot be deformed continuously into one another and the set \(I_0(p, q, \gamma_1)\) (the shaded region) is different from \(I(p, q)\) since it does not contain \(\gamma_2\).

For later considerations we need the following lemma.
Figure 1: Right and left edge are identified, $I(p,q,\gamma_1) \neq I(p,q)$

**Lemma 2.2.** Let $M$ be an $n$-dimensional spacetime and $p,q \in M$. Given a curve-segment $\gamma \in C(p,q)$ there is an open neighbourhood $\mathcal{O}$ of $\gamma([0,1])$ such that the following holds

- There is a surjective local diffeomorphism\footnote{We say a map $f : M \to N$ is a local diffeomorphism if for each point $x \in M$ there is a neighbourhood $\mathcal{O} \ni x$ such that $f|_{\mathcal{O}} : \mathcal{O} \to f(\mathcal{O})$ is a diffeomorphism.} $f : X \times B^{n-1} \to \mathcal{O}$, where $B^{n-1}$ is the open unit-ball in $\mathbb{R}^{n-1}$ and $X \subset \mathbb{R}$ a finite open interval.

- for each point $x \in B^{n-1}$ the curve $X \to M$, $t \to f(t,x)$ is timelike.

If $\gamma_1 \in C(p,q)$ is sufficiently close to $\gamma$ in the $C^1$-topology we can choose $\mathcal{O}$ and $f$ in such a way that the curve $X \to M$, $t \to f(t,0)$ is an extension of $\gamma_1$.

**Sketch of Proof.** We extend the curve-segment $\gamma$ to a timelike curve $\tilde{\gamma} : I \to M$, where $I \subset \mathbb{R}$ is an open interval containing $[0,1]$. We choose another open interval $X$ in $I$ which is relatively compact and still contains $[0,1]$. Note that $\tilde{\gamma}$ is an immersion. This means that locally $\tilde{\gamma}$ is an imbedding and we can form the conormal bundle $T^*_N I$ over $I$ together with a natural immersion $\zeta$ from $T^*_N I$ into $T^*M$. We use an arbitrary Riemannian metric on $M$ to identify $TM$ and $T^*M$. Then the exponential map defines a smooth map
where $T^* M \supset O_0 \to M$, where $O_0$ is an open neighbourhood of the zero section. The map $j = \exp \circ \zeta$ is well defined in a neighbourhood of the zero section in $T^*_N I \cong I \times \mathbb{R}^{n-1}$. Moreover it has full rank at the zero section and the curve $\mathbb{R} \ni t \to j(t, 0)$ is timelike. Hence, there exists a neighbourhood $O_1 \subset T^*_N I$ of the zero section such that $j|O_1$ is a local diffeomorphism and such that for all points $p \in O_1$ the push-forward $j_*(\partial_t)(p)$ is timelike, if $t$ is a coordinate for $I$. Since $X \subset I$ is relatively compact in $I$ there is an $\epsilon > 0$ such that the bundle of open balls in $T^*_N X \subset T^*_N I$ with radius $\epsilon$ is contained in $O_1$ and is relatively compact in $O_1$. Hence, $j$ restricts to a map $f$ from a set of the form $X \times B^{n-1}$ onto some open neighbourhood of $\gamma([0, 1])$. This map is a local diffeomorphism and moreover, by relative compactness of $X \times B^{n-1}$, the Lorentzian length of the tangent vectors of the curves $\gamma_x : t \to f(t, x)$ is uniformly bounded from below by some $\delta > 0$.

One shows that for a curve-segment $\alpha$ sufficiently close to $\tilde{\gamma}|X$ there is a unique curve-segment $\tilde{\alpha} : X \to \frac{1}{2}B^{n-1}$, such that $\alpha = f \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is the curve-segment $X \to X \times B^{n-1}$, $t \to (t, \tilde{\alpha}(t))$. Given $\tilde{\alpha}$ we can define a map

$$h : X \times B^{n-1} \to X \times B^{n-1}, \ (t, x) \to (t, \frac{1}{2}x + \tilde{\alpha}(t))$$

and it is clear that $f_1 := f \circ h$ is an immersion. It is easy to check that the differential $df_1|_p$ depends continuously on the choice of $\alpha$ and the point $p \in X \times B^{n-1}$. Remember that we had the length of the tangent vectors of the curves $\gamma_x$ uniformly bounded from below by a $\delta > 0$. Therefore, for $\alpha$ sufficiently close to $\gamma|X$, the curves $\gamma_{x, 1} : t \to f_1(t, x)$ are timelike for all $x \in B^{n-1}$. We can choose $\alpha$ as an extension of $\gamma_1$. One checks that the open set $O := \text{Im}(f_1)$ and the local diffeomorphism $f_1 : X \times B^{n-1} \to O$ have all the desired properties. \hfill $\Box$

### 2.4 The envelope of an open set

Suppose now we are given an open subset $O$ of $M$. We define the envelope $\mathcal{E}(O)$ of $O$ to be the smallest set containing $O$ with the following properties:

1. If an open subset $U \subset M$ is globally hyperbolic and $S$ is a Cauchy surface for $U$ which is contained in $\mathcal{E}(O)$ then $U \subset \mathcal{E}(O)$.

2. For each timelike curve-segment $\gamma \in C(p, q)$ such that $\gamma([0, 1])$ is contained in $\mathcal{E}(O)$ the set $I_0(p, q, \gamma)$ is contained in $\mathcal{E}(O)$. 

7
Our definition follows [34] and [3], where such an envelope was defined for open sets in Minkowski spacetime, instead of lines we use however arbitrary timelike curves. Note that the envelope is well defined, since the intersection of two sets with the listed properties has again these properties. One checks that the causal completion $\mathcal{O}^{\perp\perp}$ always fulfills 1 and 2 and hence, $\mathcal{E}(\mathcal{O}) \subset \mathcal{O}^{\perp\perp}$. Furthermore $\mathcal{E}(\mathcal{O})$ is open, since if a set satisfies 1 and 2 this is true for the interior of this set as well.

3 Unique continuation and the envelope of uniqueness

In this section $(M, g)$ will be an $n$-dimensional spacetime.

**Definition 3.1.** We say a family of tensors $(f_i)$ is locally analytic in the time if for each point $p \in M$ there is an open neighbourhood $\mathcal{O}$ and a chart $u : \mathcal{O} \to U \subset \mathbb{R}^n$ with coordinates $(x_0, x_1, \ldots, x_{n-1})$ mapping $p$ to 0 such that

- in coordinates the local vectorfield $\partial_{x_0}$ is timelike.

- the components of the tensors $f_i$ in the coordinate basis viewed as functions on $U$ are analytic in $x_0$ in a neighbourhood of 0. Hence, they are defined on $\{(z, x) \in \mathbb{C} \times \mathbb{R}^{n-1}; |z| < r, |x| < r\}$ for some $r > 0$.

If $H$ is a hypersurface defined as the zero set of a smooth local function $h : M \supset U \to \mathbb{R}$ we say there is unique continuation for a class of distributions $\mathcal{T}$ across that hypersurface $H$ if the following is true for each point $x \in H$: if there is a neighbourhood $\mathcal{O}$ of $x$ in $U$ such that $g \in \mathcal{T}$ is zero in the open set $\{y \in \mathcal{O}; h(y) < 0\}$ then $x$ is not in the support of $g$. Specializing the results in [33] and in particular in [22] (the theorem and the remark in section 5) to the case of second order differential equation with metric principal part, we get that as soon as the coefficients of such a differential equation are locally analytic in the time there is unique continuation of distributional solutions across any non-characteristic hypersurface. This result is local and we will study now the global consequences.

Denoting by $\nabla$ the Levi-Civita connection, the wave operator $\Box_g$ is given by $g^{jk}\nabla_j \nabla_k$. The Klein-Gordon equation with mass $m \geq 0$ and coupling $\kappa \in \mathbb{R}$ reads

$$ (\Box_g + m^2 + \kappa R)\psi = 0, \quad (4) $$
where $R$ is the scalar curvature. We treat a more general form of equation and show the following.

**Theorem 3.2.** Let $\psi \in \mathcal{D}'(M)$ be a solution to the wave equation

$$((\square_g + a^k(x)\partial_k + V(x))\psi = 0,$$

where $a$ is a smooth vector field and $V$ a smooth potential. Assume that the family of tensors $(g, a, V)$ is locally analytic in the time. If $\psi$ vanishes in an open set $\mathcal{O}$, then it vanishes in the envelope $E(\mathcal{O})$ of this set.

**Proof.** A distributional solution on a globally hyperbolic manifold $U$ vanishing in a neighbourhood of a Cauchy surface $S$ vanishes in $U$. This is a general property of hyperbolic wave equations (see e.g. [19, 13, 26]). We will show, that if a solution vanishes in a neighbourhood $\mathcal{O}$ of $\gamma([0, 1])$ for some $\gamma \in C(p, q)$, $p, q \in M$ then it vanishes in the set $I_0(p, q, \gamma)$. Let $\mathcal{V}$ be the open subset of $C_0(p, q, \gamma)$ consisting of curves $\tilde{\gamma}$ such that supp($\psi$) is disjoint from some open neighbourhood of $\tilde{\gamma}([0, 1])$. We show that the boundary $\partial \mathcal{V}$ of $\mathcal{V}$ is empty. Thus $\mathcal{V}$ is open and closed and therefore coincides with $C_0(p, q, \gamma)$. Suppose the boundary $\partial \mathcal{V}$ of $\mathcal{V}$ were non-empty and let $\gamma_1$ be a curve-segment in $\partial \mathcal{V}$. This implies that there is a point $x \in \text{supp}(\psi)$ which is met by $\gamma_1$ (see also figure 3). Since $\gamma_1 \in \partial \mathcal{V}$ there is a curve-segment $\gamma_2 \in \mathcal{V}$ which is sufficiently close to $\gamma_1$, so that we can choose an open neighbourhood $\mathcal{O}_1$ of $\gamma_1([0, 1])$ and a local diffeomorphism $f : \mathbb{R} \supset X \times B^{n-1} \to \mathcal{O}_1$ with the properties listed in lemma 2.2, in particular such that $\gamma_2([0, 1]) \subset f(X \times \{0\})$.  

Figure 2: Geometrical idea underlying the proof of theorem 3.2.
We may even choose \( f \) and \( O_1 \) such that \( f \) has a continuous extension \( \overline{f} \) to \( X \times B^{n-1} \) with \( \overline{f}(\partial X \times B^{n-1}) \subset \hat{O} \). Then there exists an open ball \( B^{n-1}_r \) with radius \( 0 < r < 1 \) such that \( X \times B^{n-1}_r \cap f^{-1}(\text{supp}(\psi)) = \emptyset \) and \( Y := S_r \cap f^{-1}(\text{supp}(\psi)) \neq \emptyset \) with \( S_r := X \times \partial B^{n-1}_r \). We take a point \( y \in Y \) and a neighbourhood \( \hat{O} \) of \( y \) such that \( f|_{\hat{O}} \) is a diffeomorphism onto an open neighbourhood of \( f(y) \). Because \( f \) has the properties listed in lemma 2.2 (in particular the second property), the smooth hypersurface \( f(S_r \cap \hat{O}) \) is time-like and hence non-characteristic. Therefore there is unique continuation for solutions across this surface. The solution \( \psi \) vanishes on \( f((X \times B^{n-1}_r) \cap \hat{O}) \) and by unique continuation the point \( f(y) \) cannot be in \( \text{supp}(\psi) \) which is a contradiction. Hence, \( \partial V \) is empty.

Since the envelope \( \mathcal{E}(O) \) is the smallest set containing \( O \) with the properties that have now been shown for the complement of \( \text{supp}(\psi) \), it follows that \( \psi \) vanishes on \( \mathcal{E}(O) \).

In case the metric tensor is locally analytic in the time the Klein-Gordon equation for mass \( m \) and coupling \( \kappa \) satisfies the assumptions of the above theorem.

**Example 3.3.** If \((S, h)\) is a complete connected Riemannian manifold and \( f : I \to \mathbb{R}^+ \) is a smooth positive function on an interval \( I \subset \mathbb{R} \) then the manifold \( M := I \times S \) with metric \( g := dt \otimes dt - f(t)h \) is a globally hyperbolic spacetime. In case \( S \) has constant curvature such spacetimes are called Robertson-Walker spacetimes. If \( f \) is analytic the metric tensor is locally analytic in the time. Examples are the Friedmann models (see e.g. [27] for details).

## 4 The real scalar field on a globally hyperbolic spacetime

In this section we recall the construction of the real scalar field on a globally hyperbolic spacetime \( M \). The Klein-Gordon operator for mass \( m \geq 0 \) and coupling \( \kappa \) is:

\[
P := \Box_g + m^2 + \kappa R.
\]

This operator acts on the real-valued smooth functions with compact support \( C_0^\infty(M, \mathbb{R}) \). It has unique advanced and retarded fundamental solutions (see
\( \Delta^\pm : C^\infty_0(M, \mathbb{R}) \to C^\infty_0(M, \mathbb{R}) \) satisfying
\[
P\Delta^\pm = \Delta^\pm P = \text{id} \quad \text{on} \quad C^\infty_0(M, \mathbb{R}),
\]
\[
\text{supp}(\Delta^\pm f) \subset J^\pm(\text{supp}(f)).
\]
With \( \Delta := \Delta^+ - \Delta^- \), \( \hat{\sigma}(f_1, f_2) := \int_M f_1 \Delta(f_2)w \) defines an antisymmetric bilinear form on \( C^\infty_0(M, \mathbb{R}) \times C^\infty_0(M, \mathbb{R}) \), where \( w \) is the pseudo-Riemannian volume form on \( M \). Defining \( W := C^\infty_0(M, \mathbb{R})/\ker(\Delta) \) with quotient map \( \eta \), the bilinear form \( \sigma(\eta(f_1), \eta(f_2)) := \hat{\sigma}(f_1, f_2) \) on \( W \) is symplectic. The field algebra \( \mathcal{F} \) is defined to be the CCR-algebra \( \text{CCR}(W, \sigma) \) (see \([23, 24, 10]\)). This is the \( C^* \)-algebra generated by symbols \( W(v) \) with \( v \in W \) and the relations
\[
W(-v) = W(v)^*,
\]
\[
W(v_1)W(v_2) = e^{-i\sigma(v_1, v_2)/2}W(v_1 + v_2).
\]
We define for each open subset \( \mathcal{O} \subset M \) the local field algebra \( \mathcal{F}(\mathcal{O}) \subset \mathcal{F} \) to be the closed \(*\)-subalgebra generated by the symbols \( W(\eta(f)) \) with \( f \in C^\infty_0(\mathcal{O}, \mathbb{R}) \).

It was shown in \([14]\) that there is a canonical representation \( \tau \) of the group of isometries \( G \) of \( M \) by Bogoliubov automorphisms of \( \mathcal{F} \) and the net \( \mathcal{O} \to \mathcal{F}(\mathcal{O}) \) has the following properties:

1. Isotony: \( \mathcal{O}_1 \subset \mathcal{O}_2 \) implies \( \mathcal{F}(\mathcal{O}_1) \subset \mathcal{F}(\mathcal{O}_2) \).
2. Causality: if \( \mathcal{O}_1 \subset \mathcal{O}_2^\perp \), then \( [\mathcal{F}(\mathcal{O}_1), \mathcal{F}(\mathcal{O}_2)] = \{0\} \).
3. Covariance: \( \tau(q)\mathcal{F}(\mathcal{O}) = \mathcal{F}(q\mathcal{O}) \quad \forall q \in G \).

Moreover, \( \mathcal{F} \) is the quasilocal algebra of the net \( \mathcal{O} \to \mathcal{F}(\mathcal{O}) \).

Assume that we are given a scalar product \( \mu \) on \( W \) which dominates \( \sigma \), i.e. satisfies the estimate
\[
|\sigma(v_1, v_2)|^2 \leq 4\mu(v_1, v_1)\mu(v_2, v_2) \quad v_1, v_2 \in W.
\]
In this case the linear functional \( \omega_\mu : \mathcal{F} \to \mathbb{C} \), defined by
\[
\omega_\mu(W(v)) := e^{-\mu(v,v)/2} \quad v \in W,
\]
is a state (see \([23, 24, 10]\)). The states over \( \mathcal{F} \) which can be realized in this way are called quasifree states. A quasifree state \( \omega_\mu \) gives rise to a one particle structure (Proposition 3.1 in \([24]\)), that is a map \( K_\mu : W \to H_\mu \) to some complex Hilbert space \( H_\mu \), such that
1. the complexified range of $K_\mu$, (i.e. $K_\mu W + iK_\mu W$), is dense in $H_\mu$.

2. $\langle K_\mu v_1, K_\mu v_2 \rangle = \mu(v_1, v_2) + \frac{i}{2} \sigma(v_1, v_2)$.

This structure is unique up to equivalence. A one particle structure $(K_\mu, H_\mu)$ for a quasifree state allows one to construct the GNS-triple $(\pi_{\omega_\mu}, H_{\omega_\mu}, \Omega_{\omega_\mu})$ explicitly (see [24, 23, 10]). Namely, one takes $H_{\omega_\mu}$ to be the bosonic Fock space over $H_\mu$ with Fock vacuum $\Omega_{\omega_\mu}$, and defines the representation by $\pi_{\omega_\mu}(W(v)) = \exp(-\hat{a}^*(K_\mu v) - \hat{a}(K_\mu v))$, where $\hat{a}^*(\cdot)$ and $\hat{a}(\cdot)$ are the usual creation and annihilation operators. One has the following (see e.g. [4], Proposition 3.4 (iii)):

**Proposition 4.1.** Let $\omega_\mu$ be a quasifree state over the $C^*$-algebra $F = CCR(W, \sigma)$ and let $(\pi_{\omega_\mu}, H_{\omega_\mu}, \Omega_{\omega_\mu})$ be its GNS-triple. If $V \subset W$ is a subspace which is dense in $W$ in the topology defined by $\mu$, then the $*$-algebra generated by the set

$$\{\pi_{\omega_\mu}(W(v)), v \in V\} \subset \pi_{\omega_\mu}(F)$$

is strongly dense in the von Neumann algebra $\pi_{\omega_\mu}(F)''.

**Definition 4.2.** Let $F$ be the field algebra of the real scalar field with mass $m \geq 0$ and coupling $\kappa$. We call a quasifree state $\omega_\mu$ over $F$ continuous if the 2-point function $w_2(\cdot, \cdot) := \langle K_\mu \eta(\cdot), K_\mu \eta(\cdot) \rangle$ is a distribution in $D'(M \times M)$.

Given a continuous quasifree state $\omega_\mu$ we can construct the net of von Neumann algebras $O \to \hat{F}(O) := \pi_{\omega_\mu}(F(O))''$. This assignment is isotone, causal and covariant, and there is a unique $\sigma$-weakly-continuous representation $\hat{\tau}$ of $G$ on $\hat{F} = \pi_{\omega_\mu}(F)'''$ which extends $\tau$. The net gives rise to a quantum field theory on $M$.

## 5 The local structure of the real scalar field

**Theorem 5.1.** Let $M$ be a globally hyperbolic spacetime with a metric which is locally analytic in the time. Let $O \to F(O)$ be the net of $C^*$-algebras for the real scalar field on $M$ with mass $m \geq 0$ and coupling $\kappa$. If $\omega$ is a continuous quasifree state over the quasi-local algebra $F$, then for each open set $O$ the von Neumann algebra $\hat{F}(O) = \pi_\omega(F(O))''$ is equal to the von Neumann algebra $\hat{F}(E(O)) = \pi_\omega(F(E(O)))''$, i.e. the local field algebras of an open set and its envelope coincide.
Proof. Let $\mu$ be the scalar product on $W$ inducing the state $\omega$. By proposition 4.1 it is sufficient to show that $\eta(C^\infty_0(\mathcal{O}, \mathbb{R}))$ is $\mu$-dense in $\eta(C^\infty_0(\mathcal{E}(\mathcal{O}), \mathbb{R}))$. We show that a $\mu$-continuous linear form $\hat{\psi} \mid_W$ vanishing on $\eta(C^\infty_0(\mathcal{O}, \mathbb{R}))$ vanishes on the set $\eta(C^\infty_0(\mathcal{E}(\mathcal{O}), \mathbb{R}))$. Note that $\hat{\psi} := \hat{\psi}(\eta(\cdot))$ is a real-valued distribution in $\mathcal{D}'$ and a solution to the Klein-Gordon equation. By assumption $\psi$ vanishes in $\mathcal{O}$ and by theorem 3.2 it vanishes in $\mathcal{E}(\mathcal{O})$. Hence, the theorem is proved.

Remark 5.2. Of course the same conclusion holds on general spacetimes with metric locally analytic in the time as soon as the field operator satisfies the Klein-Gordon equation. Here we specialized to the case of a globally hyperbolic spacetime since we gave the construction of the field only in this case.

Given a local net of von Neumann algebras $\mathcal{O} \to \hat{\mathcal{F}}(\mathcal{O})$ on a spacetime $M$ one may define the local algebras associated to curves (see [36]). For a curve $\gamma : I \to M$ which is contained in a compact subset $\mathcal{K} \subset M$ we define the algebra of the curve to be the von Neumann algebra

$$\hat{\mathcal{F}}(\gamma) := \bigcap_{\mathcal{O} \supset \gamma(I)} \hat{\mathcal{F}}(\mathcal{O}).$$

(10)

An immediate consequence of our theorem is

Corollary 5.3. Let the assumptions of theorem 5.1 be fulfilled. Let $\mathcal{O} \to \hat{\mathcal{F}}(\mathcal{O})$ be the corresponding net of von Neumann algebras (see the end of the previous section). If $\gamma : (0,1) \to M$ is a timelike curve which can be continued to a curve segment $\tilde{\gamma} : [0,1] \to M$ with endpoints $p$ and $q$. Then the algebra $\hat{\mathcal{F}}(\gamma)$ coincides with $\hat{\mathcal{F}}(I_0(p,q,\tilde{\gamma}))$.

This shows that the algebra of a timelike curve coincides with the algebra of a neighbourhood of this curve. This was conjectured in [36], p.239 to hold for globally hyperbolic spacetimes and our result may be seen as a partial positive answer. Another application is

Example 5.4. If $(S,h)$ is a complete connected Riemannian manifold the manifold $M := \mathbb{R} \times S$ with metric $g := dt \otimes dt - h$ is a globally hyperbolic Lorentzian manifold. Such manifolds are called ultrastatic. If $\mathcal{O} \subset S$ is non-void, then one can show that the envelope of the set $\mathbb{R} \times \mathcal{O}$ coincides with
M. The same holds for any metric that is conformally equivalent to g. If \( f \) is a positive function on \( M \) which is locally analytic in \( t \) and \((M, \tilde{g})\) is the Lorentzian manifold with metric \( \tilde{g} := f \cdot g \) then the local algebra \( \hat{\mathcal{F}}(\mathbb{R} \times O) \) of the real scalar field on \((M, \tilde{g})\) coincides with the quasilocal algebra \( \hat{\mathcal{F}} \) whenever \( O \) is non-void.

6 Acknowledgements

The author would like to thank Prof. M. Wollenberg, Dr. R. Verch and Prof. H.J. Borchers for useful discussions and comments. This work was supported by the Deutsche Forschungsgemeinschaft within the scope of the postgraduate scholarship programme “Graduiertenkolleg Quantenfeldtheorie” at the University of Leipzig.

References

[1] S. Alinhac. Uniqueness and non-uniqueness in the Cauchy problem. Contemp. Math., 27:1–22, 1984.

[2] S. Alinhac and M.S. Baouendi. A non uniqueness result for operators of principal type. Math. Z., 220:561–568, 1995.

[3] H. Araki. A generalization of Borchers theorem. Helv. Phys. Acta, 36:132–139, 1963.

[4] H. Araki and S. Yamagami. On quasi-equivalence of quasifree states of the canonical commutation relations. Publ. RIMS, Kyoto Univ., 18:283–338, 1982.

[5] L. Asgeirsson. Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten (German). Math. Ann., 113:321–346, 1936.

[6] L. Asgeirsson. Über Mittelwertgleichungen, die mehreren partiellen Differentialgleichungen 2. Ordnung zugeordnet sind (German). Studies and Essays, Interscience, New York. 7-20, 1948.
[7] S. J. Avis and C. J. Isham. Quantum field theory and fiber bundles in a general space-time. Lectures given at Cargese summer school on Recent Advances in Gravitation, France, Jul 10-29, 1978.

[8] H.J. Borchers. Über die Vollständigkeit lorentzinvanter Felder in einer zeitartigen Röhre. Nuovo Cim., 19:787, 1961.

[9] H.J. Borchers. On the structure of the algebra of field operators. Nuovo Cimento, 24:214, 1962.

[10] O. Bratteli and D.W. Robinson. Operator Algebras and Quantum Statistical Mechanics 2. Springer, 1996.

[11] R. Brunetti and K. Fredenhagen. Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. Commun. Math. Phys., 208:623, 2000.

[12] R. Courant and D. Hilbert. Methods of mathematical physics II. Interscience Publishers, 1962.

[13] Y. Choquet-Bruhat. Hyperbolic partial differential equations on a manifold. Battelle Rencontres, 1967 Lectures Math. Phys:84–106, 1968.

[14] J. Dimock. Algebras of local observables on a manifold. Commun. Math. Phys., 77:219–228, 1980.

[15] F.G. Friedlander. The wave equation on a curved space-time. Cambridge university press, 1975.

[16] S. A. Fulling. Aspects of Quantum Field Theory in Curved Space-Time. Univ. Pr., Cambridge, UK, 1989.

[17] R. Geroch. Domain of dependence. J. Math. Phys., 11:437–449, 1970.

[18] R. Haag. Local quantum physics: Fields, particles, algebras. Springer, Berlin, Germany, 1992.

[19] S.W. Hawking and G.F.R. Ellis. The large scale structure of space-time. Cambridge university press, 1973.

[20] L. Hörmander. Non-uniqueness for the Cauchy problem. Lect. Notes Math., 459:36–72, 1975.
[21] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer, 1990.

[22] L. Hörmander. On the uniqueness of the Cauchy problem under partial analyticity assumptions. Colombini, Ferruccio (ed.) et al., Geometrical optics and related topics. Selected papers of the meeting, Cortona, Italy, September 1996. Boston, MA: Birkhaeuser. Prog. Nonlinear Differ. Equ. Appl., 32:179–219, 1997.

[23] B. S. Kay. Sufficient conditions for quasifree states and an improved uniqueness theorem for quantum fields on space-times with horizons. *J. Math. Phys.*, 34:4519–4539, 1993.

[24] B. S. Kay and R. M. Wald. Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on space-times with a bifurcate Killing horizon. *Phys. Rept.*, 207:49–136, 1991.

[25] Michael Keyl. Quantum field theory and the geometric structure of Kaluza-Klein space-time. *Class. Quant. Grav.*, 14:629–652, 1997.

[26] J. Leray. *Hyperbolic differential equations*. Institute of advanced study, 1955.

[27] B. O’Neill. *Semi-Riemannian Geometry*. Academic Press, 1983.

[28] M. J. Radzikowski. The Hadamard condition and Kay’s conjecture in (axiomatic) quantum field theory on curved space-times. PhD-Thesis, 1992.

[29] L. Robbiano and C. Zuily. Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients. *Invent. math.*, 131:493–539, 1998.

[30] H. Sahlmann and R. Verch. Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime. math-ph/0008029.

[31] H. Sahlmann and R. Verch. Passivity and microlocal spectrum condition. math-ph/0002021, to appear in Commun. Math. Phys.

[32] A. Strohmaier. The Reeh-Schlieder property for quantum fields on stationary spacetimes. math-ph/0002054, to appear in Commun. Math. Phys.
[33] D. Tataru. Unique continuation for solutions to PDE’s; between Hörmander’s theorem and Holmgren’s theorem. Comm. Partial. Differential Equations, 20:855–884, 1995.

[34] L. J. Thomas and E. H. Wichmann. On the causal structure of Minkowski space-time. J. Math. Phys., 38:5044, 1997.

[35] R. M. Wald. Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics. Chicago Lectures in Physics, 1994.

[36] M. Wollenberg. On conformal structure in space-time and nets of local algebras of observables. Math. Nachr., 193:235–242, 1998.