Nonlinear quantum state transformation of spin-1/2

H. Bechmann-Pasquinucci, B. Huttner, N. Gisin
Group of Applied Physics, University of Geneva, CH-1211, Geneva 4, Switzerland
April 1, 2022

Abstract

A non-linear quantum state transformation is presented. The transformation, which operates on pairs of spin-1/2, can be used to distinguish optimally between two non-orthogonal states. Similar transformations applied locally on each component of an entangled pair of spin-1/2 can be used to transform a mixed nonlocal state into a quasi-pure maximally entangled singlet state. In both cases the transformation makes use of the basic building block of the quantum computer, namely the quantum-XOR gate.

1 Introduction

Consider the following transformation of a spin-1/2 density matrix:

\[ \rho^{\text{in}} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad \rightarrow \quad \rho^{\text{out}} = \begin{pmatrix} \frac{\rho_{11}^2}{\rho_{21}} & \frac{\rho_{12}^2}{\rho_{22}} \\ \frac{\rho_{21}}{\rho_{11}} & \frac{\rho_{22}}{\rho_{12}} \end{pmatrix} \]  

(1)

The question is if this transformation corresponds to a physically feasible quantum state transformation. At first one may be tempted to say: "of course not! This transformation is nonlinear and does not preserve the trace". But it turns out that this transformation is indeed possible to realize in the lab, it is only a question of technological difficulties and therefore of time. This transformation, which involves the fundamental building component of a quantum computer, can be used to differentiate optimally between non-orthogonal spin-1/2 states. A similar transformation applied to entangled pairs of spin-1/2 can be used to transform a mixed nonlocal state into a quasi-pure maximally entangled singlet state.

Before constructing such a state transformer (using unitary and other well accepted transformations), it should be stressed that the output \( \rho^{\text{out}} \) depends only on \( \rho^{\text{in}} \), not on any decomposition of \( \rho^{\text{in}} \) into pure states. Therefore, this kind of non-linear transformation does not lead to the
'arbitrary fast signalling' problem [1, 2]. That the trace is not preserved does not give rise to any problems either, since it merely reflects that some spins are lost during the transformation. If desirable, one could simply renormalize $\rho^{\text{out}}$.

In the next section it is shown how the transformation eq. (1) comes about. In Sec. 3, it will be shown how the non-linear transformation can be used to distinguish between two non-orthogonal states, provided two copies of the state are available. In Sec. 4 the Loss Induced Generalized (LIGe) quantum measurement is presented [3]. It is a special construction of Positive Operator Value Measure (POVM), and it is shown how this specific measurement can be applied in two different ways, when two copies of the state are provided. It turns out that the 3 different ways to distinguish between two non-orthogonal states which will be described here all leads to the same probability of successfully determining the input state. In Sec. 5 it is proven that the probability of successfully determining the state is indeed also the optimal solution to the state identification problem.

A generalized non-linear transformation, operating on entangled pairs of spin-1/2, can be used to transform a mixed nonlocal state into a quasipure maximally entangled singlet state. This will be described in Sec. 6.

2 How it works

To realize the transformation, one needs at least two identical independent copies of the same state $\rho^{\text{in}}$, and it will be assumed that this is possible.

The first step consists of considering the spins pairwise,

$$\rho^{\text{in}} \rightarrow \rho^{\text{in}} \otimes \rho^{\text{in}}$$

(2)

This is quite easy to do, since nothing needs to be done physically. It is however this crucial step which makes the final transformation nonlinear.

The second step consists of a 'controlled not gate' interaction between each spin in a pair. This is physically the hard part of the whole transformation. A controlled not gate (or quantum-XOR) flips the second spin (target spin) if and only if the first (source spin) is 'spin-up'. It is a unitary transformation, $U_{\text{XOR}}$, acting on pairs of spin-1/2:

$$|++\rangle \rightarrow |+-\rangle$$
$$|+\rangle \rightarrow |+\rangle$$
$$|--\rangle \rightarrow |++\rangle$$
$$| -\rangle \rightarrow |-\rangle$$

(3)

or when written in matrix notation:

$$U_{\text{XOR}} = \begin{pmatrix} \sigma_x & 0 \\ 0 & I \end{pmatrix}$$

(4)
where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the first Pauli matrix and $\mathbb{I}$ is the $2 \times 2$ identity matrix. Note that the XOR, is the basic building block for a quantum processor $\mathbb{I}$.

The third step is again easy: measure the spin component of the second particle along the z-direction and keep the pair only if the result is ’down’. Hence there is a probability of failure.

Altogether (see Fig.1) this procedure transforms the state of the source spin from $\rho_{\text{source}}^{\text{in}} \rightarrow \rho_{\text{source}}^{\text{out}}$, whereas the target spin is always left in the spin-down state $\rho_{\text{target}}^{\text{in}} \rightarrow \rho_{\text{target}}^{\text{out}} = | - \rangle \langle - | = \mathbb{P}_-$. The whole transformation can be written as

$$
\left( \mathbb{I} \otimes \mathbb{P}_- \left( U_{\text{XOR}} \left( \rho^{\text{in}} \otimes \rho^{\text{in}} \right) U_{\text{XOR}}^\dagger \right) \mathbb{I} \otimes \mathbb{P}_- \right) = \rho^{\text{out}} \otimes \mathbb{P}_- \quad (5)
$$

Where $\rho^{\text{out}}$ is the density matrix in eq. (1), where each matrix element has been squared by itself.

Other similar transformations can be built by using other components, which means substituting the control-not gate with other unitary interactions, see Fig. 2.

### 3 Using the transformation for state identification

The non-linear transformation which has just been described can be used to transform non-orthogonal states into orthogonal states, provided two copies of the state are available. By nature, non-orthogonal quantum states can not be distinguished with certainty, and even if this transformation can turn non-orthogonal states into orthogonal states, this does not in any way conflict with quantum mechanics, since there is a certain probability that the transformation fails.

A density matrix describing a spin-1/2 system can be represented on the Poincare sphere in terms of a polarization vector $P = (x, y, z)$ (also known as Bloch vector) and the Pauli-matrices $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ in the following way:

$$
\rho = \frac{1}{2}(\mathbb{I} + P \cdot \sigma) \quad (6)
$$

For two spin-1/2 states to be orthogonal their corresponding polarization vectors must point in opposite directions on the sphere, in other words two states $| a \rangle$ and $| b \rangle$ are orthogonal if their polarization vectors satisfy $P_a = -P_b$.

Consider the two pure spin-1/2 states $| \psi_{1}^{\text{in}} \rangle$ and $| \psi_{2}^{\text{in}} \rangle$ with the polar-
The polarization vectors $P_i^{in}$ and $P_i^{out}$ expressed in the usual spherical coordinates

\begin{align}
x_i^{in} &= \sin \theta_i \cos \phi_i \\
y_i^{in} &= \sin \theta_i \sin \phi_i \\
z_i^{in} &= \cos \theta_i
\end{align}

where $i = 1, 2$. When transforming these two states, which means taking two copies of the same state, performing the control-not gate interaction between them and the filtering measurement, the new state have the following polarization vectors $P_i^{out}$:

\begin{align}
x_i^{out} &= \frac{1}{2} \sin^2 \theta_i \cos 2\phi_i \\
y_i^{out} &= \frac{1}{2} \sin^2 \theta_i \sin 2\phi_i \\
z_i^{out} &= \cos \theta_i
\end{align}

For the corresponding spin states $|\psi_{1}^{out}\rangle$ and $|\psi_{2}^{out}\rangle$ to be orthogonal the requirement is that their corresponding polarization vectors must be opposite, i.e. $P_1^{out} = -P_2^{out}$. This can be fulfilled by imposing the following relations between the angles;

\begin{align}
\cos \theta_1 &= -\cos \theta_2 \implies \theta_2 = \theta_1 + \pi \quad \theta \equiv \theta_1 = \theta_2 - \pi \\
\cos 2\phi_1 &= -\cos 2\phi_2 \quad \text{and} \quad \sin 2\phi_1 = -\sin 2\phi_2 \quad \phi \equiv \phi_1 = \phi_2 - \frac{\pi}{2}
\end{align}

Imposing these constrains imply that the initial states $|\psi_{1}^{in}\rangle$ and $|\psi_{2}^{in}\rangle$ had the following density matrices,

\begin{align}
\rho_1^{in} &= \frac{1}{2} \begin{pmatrix}
1 + \cos \theta & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & 1 - \cos \theta
\end{pmatrix} = |\psi_1^{in}\rangle \langle \psi_1^{in}| \\
\rho_2^{in} &= \frac{1}{2} \begin{pmatrix}
1 - \cos \theta & i \sin \theta e^{-i\phi} \\
-i \sin \theta e^{i\phi} & 1 + \cos \theta
\end{pmatrix} = |\psi_2^{in}\rangle \langle \psi_2^{in}|
\end{align}

Whereas the transformed states $|\psi_{1}^{out}\rangle$ and $|\psi_{2}^{out}\rangle$ have as density matrices the input matrices with each matrix element squared by itself, as seen in eq. (1).

The two initial states are not orthogonal, i.e. $\rho_1^{in} \rho_2^{in} \neq 0$ which means that they can not be distinguished with certainty. The overlap between the states, which can be obtained from $\rho_1^{in} \rho_2^{in} \rho_1^{in} = |\langle \psi_1^{in} | \psi_2^{in}\rangle|^2 \rho_1^{in}$, is found to be

\begin{equation}
|\langle \psi_1^{in} | \psi_2^{in}\rangle| = \frac{\sin \theta}{\sqrt{2}}
\end{equation}

The two outgoing states are orthogonal, i.e. $\rho_1^{out} \rho_2^{out} = 0$. This means that they can now be identified with certainty by performing a standard von Neumann measurement. If the state $|\psi_{1}^{out}\rangle$ is obtained the initial

\footnote{Notice that the outgoing polarization vectors $P_i^{out}$ are not normalized}
states was $|\psi_1^{\text{in}}\rangle$. Similarly, if the state $|\psi_2^{\text{out}}\rangle$ is found the initial state was $|\psi_2^{\text{in}}\rangle$. In other words the transformation makes it possible to distinguish with certainty between states which were originally not distinguishable. However, the transformation is not always successful. There is, in fact, only a certain probability that it will succeed, which is given by the trace of the $\rho_i^{\text{out}}$, where $i$ is either 1 or 2. It is important to realize that when the transformation is successful, the initial state is completely identified. Whereas if the transformation fails, no knowledge about the initial state can be obtained.

The total probability of successfully identifying the state is

$$\Pr_T(\text{success}) = \frac{1}{2} \left( \text{Tr}(\rho_1^{\text{out}}) + \text{Tr}(\rho_2^{\text{out}}) \right) = 1 - \frac{\sin^2 \theta}{2} \quad (11)$$

Notice that this is equal to $1 - |\langle \psi_1^{\text{in}} | \psi_2^{\text{in}} \rangle|^2$.

4 Another way to identify non-orthogonal states

There are other ways of distinguishing between two non-orthogonal states without making any errors. A well-known way is to use Positive Operator Value Measure (POVM), which are generalized quantum measurements. The Loss Induced Generalized (LIGe) quantum measurement is a special POVM [4, 5], which has even been performed experimentally [6]. In the case of LIGe the idea is that the two non-orthogonal spin-1/2 lie in a plane spanned by two orthogonal states $|\phi_1\rangle$ and $|\phi_2\rangle$, i.e.

$$|\psi_1^{\text{in}}\rangle = \cos \frac{\alpha}{2} |\phi_1\rangle + \sin \frac{\alpha}{2} |\phi_2\rangle \quad (12)$$

$$|\psi_2^{\text{in}}\rangle = \cos \frac{\alpha}{2} |\phi_1\rangle - \sin \frac{\alpha}{2} |\phi_2\rangle$$

with the overlap $|\langle \psi_1^{\text{in}} | \psi_2^{\text{in}} \rangle| = \cos \alpha$. The procedure is to add one dimension $|\phi_0\rangle$ orthogonal to $|\phi_1\rangle$ and $|\phi_2\rangle$, and then perform a rotation around $|u\rangle \equiv |\phi_1\rangle - |\phi_2\rangle$, with an angle $\cos \theta = \tan \frac{\alpha}{2}$. After the rotation the two states can be written as

$$|\psi_1^{\text{out}}\rangle = \sqrt{2} \sin \frac{\alpha}{2} |\phi_1\rangle + \sqrt{\cos \alpha} |\phi_0\rangle \quad (13)$$

$$|\psi_2^{\text{out}}\rangle = \sqrt{2} \sin \frac{\alpha}{2} |\phi_2\rangle + \sqrt{\cos \alpha} |\phi_0\rangle$$

Since the three states $|\phi_0\rangle$, $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal they can be separated deterministically with a standard measurement.

If the initial state was $|\psi_1^{\text{in}}\rangle$ the result of a measurement will either be $|\phi_1\rangle$ with the probability $2\sin^2 \frac{\alpha}{2}$ or $|\phi_0\rangle$ with probability $\cos \alpha$. Where as if the initial state was $|\psi_2^{\text{in}}\rangle$ the result of a measurement will either be $|\phi_2\rangle$ with the probability $2\sin^2 \frac{\alpha}{2}$ or $|\phi_0\rangle$ with probability $\cos \alpha$. 
Therefore when the state obtained is either $|\phi_1\rangle$ or $|\phi_2\rangle$, we can conclude that the initial state was $|\psi_{1n}\rangle$ and $|\psi_{2n}\rangle$ respectively. Whereas if the obtained state is $|\phi_0\rangle$ the initial state could have been either of the two and in order not to introduce any errors these results are discarded. The probability of identifying the state is $1 - |\langle \psi_{1n} | \psi_{2n} \rangle|^2$

In order to compare the results obtained using the LIGe with the results obtained when using the nonlinear transformation, one has to think about how the LIGe can be used when two copies of the initial state are available. It turns out that there are two possibilities: (1) perform two independent LIGe measurements, one on each copy of the state or (2) perform one single measurement on the product state $|\psi_{1n}\rangle \otimes |\psi_{2n}\rangle$.

In the first case the probability of successfully identifying the state is the probability of success in the first measurement plus the probability of failure in the first times success in the second, i.e.

$$
Pr_{T}^{2xLIGe}(success) = Pr(success) + Pr(failure) \times Pr(success) = 1 - |\langle \psi_{1n} | \psi_{2n} \rangle|^2
$$

(14)

In the second case there one single LIGe measurement is performed on the product state $|\psi_{1n}\rangle \otimes |\psi_{2n}\rangle$, the probability of successfully identifying the state is

$$
Pr_{T}^{1xLIGe}(success) = 1 - |\langle \psi_{1n} | \psi_{2n} \rangle|^2
$$

(15)

since the overlap between the two product states $|\psi_{1n}\rangle \otimes |\psi_{1n}\rangle$ and $|\psi_{2n}\rangle \otimes |\psi_{2n}\rangle$ is equal to $|\langle \psi_{1n} | \psi_{2n} \rangle|^2$.

Notice that the three methods for state identification which have been presented so far, all have the the same probability of success. In the next section it is proven that this is indeed also the optimal solution.

5 The optimal solution to the state identification problem

In general when one wishes to distinguish deterministically between two non-orthogonal states and no errors are accepted, one is forced to introduce inconclusive answers. This means that there are three possible outcomes, namely; the state was $|\psi_{1n}\rangle$, the state was $|\psi_{2n}\rangle$ or ”don’t know”. A ”don’t know” means that that state was not successfully identified and the result is discarded in order not to introduce any errors. This kind of measurement is realized by what is called a Positive-Operator Value Measure (POVM) [[7]].

The optimal POVM which answers these question is constructed in the following way; the two projection operators $P_{-|\psi_{1n}\rangle} = 1 - |\psi_{1n}\rangle \langle \psi_{1n}|$ $P_{-|\psi_{2n}\rangle} = 1 - |\psi_{2n}\rangle \langle \psi_{2n}|$ projects onto states orthogonal to $|\psi_{1n}\rangle$ and
ψin\sub{2}}, respectively. The three positive-operators which are needed (one of each possible answer) are formed using these two projection operators,

\begin{align*}
A_1 |\psi_1^\text{in}\rangle &= x (\mathbb{1} - |\psi_2^\text{in}\rangle \langle \psi_2^\text{in}|) \\
A_2 |\psi_2^\text{in}\rangle &= x (\mathbb{1} - |\psi_1^\text{in}\rangle \langle \psi_1^\text{in}|) \\
A_3 &= \mathbb{1} - A_1 |\psi_1^\text{in}\rangle - A_2 |\psi_2^\text{in}\rangle
\end{align*}

where the coefficient \(x\) now is to be optimized. The first two operators have the same coefficient because the initial states are equi-probable. The requirement is now that the probability of an inconclusive answer should be as low as possible, and that all three operators must be positive. These requirements leads to the following value,

\begin{equation}
x = \frac{1}{1 + |\langle \psi_1^\text{in} | \psi_2^\text{in} \rangle|^2}
\end{equation}

and gives probability \(|\langle \psi_1^\text{in} | \psi_2^\text{in} \rangle|^2\) of obtaining an inconclusive answer. Hence the probability of successfully determining the state is

\begin{equation}
\text{Pr}(\text{success}) = 1 - |\langle \psi_1^\text{in} | \psi_2^\text{in} \rangle|^2
\end{equation}

Suppose now that two copies of the initial state is available, then the probability of successfully determining the state is

\begin{equation}
\text{Pr}(\text{success}) = 1 - |\langle \psi_1^\text{in} | \psi_2^\text{in} \rangle|^2
\end{equation}

since the two copies can be thought of as the state \(|\psi_1^\text{in}\rangle \otimes |\psi_1^\text{in}\rangle\) or the state \(|\psi_2^\text{in}\rangle \otimes |\psi_2^\text{in}\rangle\), and these two state have overlap \(|\langle \psi_1^\text{in} | \psi_2^\text{in} \rangle|^2\).

This shows that the methods for state identification which have been presented in Sec. 3 and Sec. 4 are indeed optimal.

### 6 Purification of mixed states of spin-1/2

It is straightforward to generalize the nonlinear transformations described in Sec. 2, to apply them to mixed states of entangled pairs of spin-1/2. The transformations can then be used to construct a purification scheme. As previously, the idea is to have two physical systems in the same state \(\rho^\text{in}\), where \(\rho^\text{in}\) now represents an entangled pair of spin-1/2, i.e. it is a 4 \(\times\) 4 density matrix. For concreteness it is assumed that within each pair, one spin is carried by a particle flying towards the left, while the other one is carried by a particle flying towards the right. The generalization consists in performing independently similar operations as in (20) to the two spins on the left hand side (known as Alice) and to the two on the right hand side (known as Bob). The operation is nearly identical to the XOR defined in (3), with a sign change on Bob\’s side:

\begin{equation}
\mathbb{U}_A = 
\begin{pmatrix}
-\sigma_y & 0 \\
0 & \mathbb{I}
\end{pmatrix}
\quad \mathbb{U}_B = 
\begin{pmatrix}
\sigma_y & 0 \\
0 & \mathbb{I}
\end{pmatrix}
\end{equation}
The filtering is done, as in the single spin case, by selecting the spin "down" state of each of the spins in the target pair. This leads to the following transformation:

\[
(\mathbb{1} \otimes \mathbb{P}_-)_A (\mathbb{1} \otimes \mathbb{P}_-)_B U_A U_B \left( \rho^{in} \otimes \rho^{in} \right) U_B^\dagger U_A^\dagger (\mathbb{1} \otimes \mathbb{P}_-)_B (\mathbb{1} \otimes \mathbb{P}_-)_A = \rho^{out} \otimes \mathbb{P}_- \tag{21}
\]

and gives rise to the following outgoing density matrix shared between Alice and Bob

\[
\rho^{in} \rightarrow \rho^{out} = \begin{pmatrix}
(\rho_{11})^2 & -(\rho_{12})^2 & (\rho_{13})^2 & -(\rho_{14})^2 \\
-(\rho_{21})^2 & (\rho_{22})^2 & -(\rho_{23})^2 & (\rho_{24})^2 \\
(\rho_{31})^2 & -(\rho_{32})^2 & (\rho_{33})^2 & -(\rho_{34})^2 \\
-(\rho_{41})^2 & (\rho_{42})^2 & -(\rho_{43})^2 & (\rho_{44})^2
\end{pmatrix} \tag{22}
\]

This transformation preserves the the singlet state \(\psi^-\). After the transformation Alice and Bob both perform a bilateral (i.e. on both sides) \(\pi/2\) rotation around the \(x\)-axis of their remaining spin. This rotation interchanges the \(\psi^+\) and the \(\phi^+\) Bell states without affecting the other two Bell states. What they have hereby obtained is a quantum state purification scheme which purifies towards the singlet state, \(\psi^-\).

A purification scheme works in the following way: suppose the initial state \(\rho^{in}\) had fidelity \(F^{in} = \langle \psi^- | \rho^{in} | \psi^- \rangle \) with respect to the singlet state. Taking two copies of \(\rho^{in}\), performing the transformation and afterwards the rotation, the fidelity \(F^{out}_{rot}\) of the new state \(\rho^{out}_{rot}\) is bigger than \(F^{in}\), i.e. \(F^{out}_{rot} > F^{in}\). When the fidelity is equal to 1 it means that the state is a pure state and therefore not entangled with the environment (or an eavesdropper) \[8, 9\].

Repeating the above operations, including the bilateral rotation, on a state with fidelity \(F^{in} > \frac{1}{2}\) selects a subensemble with larger fidelity \(F^{out} > F^{in}\). For example, suppose the initial state had fidelity \(F^{in} = 0.51\), after 10 iterations the fidelity is \(F^{(10)} = 0.809\) and after 15 iterations the fidelity is \(F^{(15)} = 0.99997\). It should be mentioned that, depending on the input fidelity, \(F^{in}\), the fidelity after the purification, \(F^{out}\), may actually decrease for the first few iterations, but it will increase afterwards.

In order to have a higher efficiency (keep more pairs), Alice and Bob can also keep the source pair when the outcome of their measurement on the target pair gave them ++, in other words Alice and Bob can keep their source pair if they both find \(-\) or they both find \(+\) when measuring their target pair.

This purification scheme is, up to a phase, identical to the one developed by Deutsch et al. \[9\]. The only difference being that in their
7 Concluding remarks

It has been shown that a non-linear quantum state transformation which operates on pairs of spin-1/2, can be used to distinguish deterministically between two non-orthogonal states, provided two copies of the initial states are available. The transformation, which involves only a unitary operation (here the quantum XOR was used) and a filtering process (a measurement), can actually transform non-orthogonal states into orthogonal states. These states can now be separated deterministically with a standard von Neuman measurement. This transformation does not conflict with the basic laws for quantum mechanics, which tells us that non-orthogonal states can not be distinguished with certainty, since it only has a certain probability of success, which then becomes the probability of successfully determining the state.

The result obtained when applying the non-linear transformation to the state identification problem was compared with a specific POVM measurement known as the LIGe (Loss Induced Generalized) quantum measurement. When two copies of the initial state are provided the LIGe can be applied in two different ways: either two independent LIGe measurements (one on each copy) or a single measurement on the product state of the two copies. Both procedures lead to the same probability for successfully determining the initial state. The same probability was obtained when using the non-linear transformation. This is in fact the optimal solution to the state identification problem, when inconclusive results — but no errors — are accepted.

Finally it was shown how similar transformations applied locally on each component of an entangled pair of spin-1/2 can be used to transform a mixed nonlocal state into a quasi-pure maximally entangled singlet state.

It should be mentioned that it is not only possible to square each component of the density matrix as was seen in eq. (1), but it can be raised to any power $n + 1$. This is done by taking $n + 1$ copies of $\rho^{in}$, where $n$ of the copies act as target spins. A generalized XOR is then applied, which flips the target spins if and only if the source spin is spin-up. This is followed by a projection onto the spin-down of all the target spins. This operation results in a density matrix $\rho^{out}$ of the source spin where each component has been raised to the power $n + 1$.

It is also possible to extent the ‘squaring’ of the components of the density matrix to higher dimensions. Suppose the initial pure state has dimension $n$, i.e. $|\psi\rangle = (\psi_1, ..., \psi_n)$. Making the tensor product of two identical states gives the new state, $|\psi\rangle \otimes |\psi\rangle = (\psi_{11}, \psi_{12}, ..., \psi_{nn})$ with
the elements $\psi_{11} = (\psi_1)^2$, $\psi_{12} = \psi_1\psi_2$, ..., $\psi_{nn} = (\psi_n)^2$. In order to select the squared elements, i.e. $\psi_{ii} = (\psi_i)^2$ the product state is rotated so that $\psi_{11} \rightarrow \psi_{1n}$, $\psi_{22} \rightarrow \psi_{2n}$, ..., $\psi_{nn} \rightarrow \psi_{nn}$. These are the elements of interest. The other elements can be rotated in an arbitrary way, as long as they are not transformed into $\psi_{jn}$ for all $j$. A projection onto the spin-$n$ component of the target state leaves the target spin in the spin-$n$ state, where as the source spin is left in the state where each component has been squared by itself.

**Acknowledgement**

We would like to thank Asher Peres for useful comments. H.B.-P. is supported by the Danish National Science Research Council (grant no. 9601645), B.H. is supported by the TMR network on Physics of Quantum Information. This work is supported by the Swiss Fonds National de Recherche Scientifique.

**References**

[1] N. Gisin, Phys. Lett. A **143**, (1990) 1
[2] N. Gisin, Helv. Phys. Acta, **62** (1989) 363
[3] A. Barenco, D. Deutsch, A. Ekert and R. Jozsa, Phys. Rev. Lett. **74**, (1995) 4083
[4] I. D. Ivanović, Phys. Lett. A **123**, (1987) 257
[5] A. Peres, Phys. Lett. A **128**, (1988) 19
[6] B. Huttner, A. Muller, J. D. Gautier, H. Zbinden and N. Gisin, Phys. Rev. A, **54**, (1996) 3783
[7] A. Peres, ”Quantum Theory: Concepts and Methods” (Kluwer, Dordrecht, 1993)
[8] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin and W. Wootters, Phys. Rev. Lett, **76**, (1996) 722
[9] D. Deutsch, A Ekert, R. Jozsa, C. Macchiavello, S. Popescu and A. Sanpera, Phys. Rev. Lett, **77**, (1996) 2818
Figure 1: Here the process is schematically outlined. The two copies of the same spin state first undergo an unitary interaction, and afterwards a filtering process. The source spin $\rho^{\text{in}}$ will afterwards be in a new state $\rho^{\text{out}}$, whereas the state of the target spin always is reduced to a spin 'down' state.
Figure 2: The transformation of the sphere. The upper figure shows the spin-1/2 states represented on the sphere in terms of their polarization or Bloch vector (see Sec. 3). A fully painted sphere corresponds to all spin states. The lower figure shows the transformed spin states. Here the unitary operator used is not the XOR, but the operator $U = \exp(i\frac{\pi}{8}\sigma_z \otimes \sigma_z)$. 