Approximability results for the $p$-centdian and the converse centdian problems∗

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Given an undirected graph $G = (V, E)$ with a nonnegative edge length function and an integer $p, 0 < p < |V|$, the $p$-centdian problem is to find $p$ vertices (called the centdian set) of $V$ such that the eccentricity plus median-distance is minimized, in which the eccentricity is the maximum (length) distance of all vertices to their nearest centdian set and the median-distance is the total (length) distance of all vertices to their nearest centdian set. The eccentricity plus median-distance is called the centdian-distance. The purpose of the $p$-centdian problem is to find $p$ open facilities (servers) which satisfy the quality-of-service of the minimum total distance (median-distance) and the maximum distance (eccentricity) to their service customers, simultaneously. If we converse the two criteria, that is given the bound of the centdian-distance and the objective function is to minimize the cardinality of the centdian set, this problem is called the converse centdian problem. In this paper, we prove the $p$-centdian problem is NP-Complete. Then we design the first non-trivial brute force exact algorithms for the $p$-centdian problem and the converse centdian problem, respectively. Finally, we design two approximation algorithms for both problems.

Keywords: combinatorial optimization, computational complexity, approximation algorithm, NP-Complete; network location, $p$-centdian problem, converse centdian problem

1 Introduction

The $p$-center problem [20, 30, 51] and $p$-median problem [20, 31, 51] are fundamental problems in graph theory and operations research. Let $G = (V, E, \ell)$ be an undirected graph with $\ell : E \rightarrow \mathbb{R}^+$ on the edges. Given a vertex set $V' \subset V$, for each vertex $v \in V$, we let $d(v, V')$ denote the shortest distance from $v$ to $V'$ (i.e., $d(v, V') = \min_{u \in V'} d(u, v)$, in which $d(u, v)$ is the length of the shortest path of $G$ from $u$ to $v$). The eccentricity of a vertex set $V'$ is defined as the maximum distance of $d(v, V')$ for all $v \in V$, denoted by $\varepsilon_C(V')$ (i.e., $\varepsilon_C(V') = \max_{v \in V} d(v, V')$). The median-distance $\varepsilon_M(V')$ of $V'$ denotes the total distance of $d(v, V')$ for all $v$ in $V$ (i.e., $\varepsilon_M(V') = \sum_{v \in V} d(v, V')$). Given an undirected complete graph $G = (V, E, \ell)$ with a nonnegative edge length function $\ell$ and an integer

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Given a set of customers on the network, the network location theory is concerned with the optimal locations of new facilities (servers) to minimize transportation distances (costs) of serving these customers and consider the population density area. The most fundamental problems of the network location theory are the $p$CP and the $p$MP, respectively. The $p$CP is suitable for emergency services where the objective is to have the farthest customers as close as possible to their facility centers. But this solution of the $p$CP may cause a substantial increase in total distance (cost), thus this result takes a huge loss of the spatial efficiency. The $p$MP is suitable for locating facilities providing a routine service, by minimizing the average distances from customers to these selected facilities. The solution of the $p$MP is beneficial in serving centrally located and high-population density areas but sacrifices the remote and low-population density areas [41, 42, 50]. Motivated by the application of finding $p$ open facilities (servers) which satisfy the quality-of-service of the minimum total distance (median-distance) and the maximum distance (eccentricity) to their service customers, simultaneously [21, 22, 25, 41, 42, 50], Halpern [21, 22] introduced a convex combination of the 1CP and the 1MP, which he called the 1-centdian problem. Hooker et al. [25] studied the generalization of the 1-centdian problem, called the $p$-centdian problem. Given an undirected complete graph $G = (V, E, \ell)$ with a nonnegative edge length function $\ell$, a real number $\lambda$, $0 \leq \lambda \leq 1$, and an integer $p$, $0 < p < |V|$, the $p$-centdian problem ($p$DP) is to find a vertex set $V'$ in $V$, $|V'| = p$, such that the $\lambda$eccentricity (respectively, the $\lambda$median-distance) of $V'$ is minimized [20, 30, 31, 51]. Both problems had been shown to be NP-Complete [16, 30, 31]. Hence, many approximation algorithms [3, 18, 19, 23, 43, 47] and inapproximability results [24, 26, 27] had been proposed for both problems. These two problems have many applications in the network location, clustering, and social networks [1, 8, 13, 14, 15, 20, 23, 30, 31, 38, 40, 45, 46, 48, 49, 51].
Proposed [4, 5]. However, the converse version of the \textit{p-centdian problem} is undefined. Hence, we present the converse version of the \textit{p-centdian problem}, called the \textit{converse centdian problem}. Given a graph $G = (V, E, \ell)$ with a nonnegative edge length function $\ell$ and two integers $\lambda$ and $U$, $0 \leq \lambda \leq 1$, $U > 0$, the \textit{converse centdian problem} (CDP) is to find a vertex set $V'$ in $V$ with minimum cardinality such that

$$\lambda \mathcal{L}_C(V') + (1 - \lambda) \mathcal{L}_M(V') \leq U.$$ 

In this paper, we focus on a special case of the \textit{centdian-distance} for the pDP (respectively, CDP) : $\mathcal{L}_C(V') + \mathcal{L}_M(V')$ and discuss the complexity, the non-trivial brute force exact algorithms, and the approximation algorithms for the pDP and CDP, respectively. First, we prove that the pDP is NP-Complete even when the \textit{centdian-distance} is $\mathcal{L}_C(V') + \mathcal{L}_M(V')$. Then we present the first non-trivial brute force exact algorithms for the pDP and CDP, respectively. Finally, we design a $(1 + \epsilon)$-approximation algorithm for the pDP satisfying the cardinality of the centdian set is less than or equal to $(1 + 1/\epsilon)(\ln |V| + 1)p$ and a $(1 + 1/\epsilon)(\ln |V| + 1)$-approximation algorithm for the CDP satisfying the centdian-distance is less than or equal to $(1 + \epsilon)U$, in which $\epsilon > 0$, respectively.

The rest of this paper is organized as follows. In Section 2, some definitions and notations are given. In Section 3, we prove that the pDP is NP-Complete even when the centdian-distance is $\mathcal{L}_C(V') + \mathcal{L}_M(V')$. In Section 4, we present non-trivial brute force exact algorithms for the pDP and CDP, respectively. In Section 5, we design a $(1 + \epsilon)$-approximation algorithm for the pDP satisfying the cardinality of the centdian set is less than or equal to $(1 + 1/\epsilon)(\ln |V| + 1)p$. In Section 6, we design a $(1 + 1/\epsilon)(\ln |V| + 1)$-approximation algorithm for the CDP satisfying the centdian-distance is less than or equal to $(1 + \epsilon)U$, in which $\epsilon > 0$. Finally, we make a conclusion in Section 7.

2 Preliminaries

In this paper, a graph is simple, connected and undirected. By $G = (V, E, \ell)$, we denote a graph $G$ with vertex set $V$, edge set $E$, and edge length function $\ell$. The edge length function is assumed to be nonnegative. We use $|V|$ to denote the cardinality of vertex set $V$. Let $(v, v')$ denote an edge connecting two vertices $v$ and $v'$. For any vertex $v \in V$ is said to be adjacent to a vertex $v' \in V$ if vertices $v$ and $v'$ share a common edge $(v, v')$.

**Definition 1:** For $u, v \in V$, $SP(u,v)$ denotes a shortest path between $u$ and $v$ on $G$. The shortest path length is denoted by $d(u,v) = \sum_{e \in SP(u,v)} \ell(e)$.

**Definition 2:** Let $H$ be a vertex set of $V$. For a vertex $v \in V$, we let $d(v,H)$ denote the shortest distance from $v$ to $H$, i.e., $d(v,H) = \min_{h \in H} \{d(v,h)\}$.

**Definition 3:** Let $H$ be a vertex set of $V$. The eccentricity of $H$, denoted by $\mathcal{L}_C(H)$, is the maximum distance of $d(v,H)$ for all $v \in V$, i.e., $\mathcal{L}_C(H) = \max_{v \in V} d(v,H)$.

**Definition 4:** Let $H$ be a vertex set of $V$. The median-distance of $H$, denoted by $\mathcal{L}_M(H)$, is the total distance of $d(v,H)$ for all $v \in V$, i.e., $\mathcal{L}_M(H) = \sum_{v \in V} d(v,H)$.

$pCP$ (p-center problem) [20, 30, 51]

**Instance:** A connected, undirected, complete graph $G = (V,E,\ell)$ and an integer $p > 0$.

**Question:** Find a vertex set $V'$, $|V'| = p$, such that the eccentricity of $V'$ is minimized.

$pMP$ (p-median problem) [20, 31, 51]
Fig. 1: An instance: complete graph \( G = (V, E, \ell) \), \( p = 2 \) and \( U = 117 \).

**Instance:** A connected, undirected, complete graph \( G = (V, E, \ell) \) and an integer \( p > 0 \).

**Question:** Find a vertex set \( V' \), \( |V'| = p \), such that the median-distance of \( V' \) is minimized.

\( p \)DP (\( p \)-centdian problem) [25]

**Instance:** A connected, undirected, complete graph \( G = (V, E, \ell) \) and an integer \( p > 0 \).

**Question:** Find a vertex set \( V' \), \( |V'| = p \), such that \( \mathcal{L}_C(V') + \mathcal{L}_M(V') \) of \( V' \) is minimized.

For the \( p \)DP, we have two criteria. The first criterion is the cardinality of the vertex set \( V' \) and the second is the \( \mathcal{L}_C(V') + \mathcal{L}_M(V') \). The vertex set \( V' \) is called the centdian set and \( \mathcal{L}_C(V') + \mathcal{L}_M(V') \) is called the centdian-distance. Hence, we can converse the two criteria, that is given the bound of the centdian-distance of the centdian set and the objective function is to minimize the cardinality of the centdian set.

\( CDP \) (converse centdian problem)

**Instance:** A connected, undirected graph \( G = (V, E, \ell) \) and an integer \( U > 0 \).

**Question:** Find a vertex set \( V' \) with \( \mathcal{L}_C(V') + \mathcal{L}_M(V') \leq U \) such that the cardinality of the \( V' \) is minimized.

The following examples illustrate the \( p \)DP and the \( CDP \). Consider the instance shown in Fig. 1, in which the graph \( G = (V, E, \ell) \) and integers \( p = 2 \) and \( U = 117 \). An optimal solution of \( G \) for the \( p \)DP is shown in Fig. 2, in which the centdian set is \{A, B, D\}.

In this paper, we will prove that the \( p \)DP is NP-Complete by a reduction from the dominating set problem [7, 11, 44, 52] to the \( p \)DP. Hence, we review the definition of the dominating set problem. A dominating set of \( G \), denoted by \( Z \), is a subset of \( V \) such that each vertex in \( V \setminus Z \) is adjacent to a vertex in \( Z \) [7, 11, 44, 52].
Fig. 2: The optimal solution \( \{B, C\} \) for the 2DP. (Note that \( L_C(\{B, C\}) + L_M(\{B, C\}) = 252 \))

Fig. 3: The optimal solution \( \{A, B, D\} \) for the CDP. (Note that \( L_C(\{A, B, D\}) + L_M(\{A, B, D\}) = 117 \))

DSP (dominating set problem) [7, 11, 44, 52]

**Instance:** A connected, undirected graph \( G = (V, E) \).

**Question:** Find a dominating set \( Z' \) with minimum cardinality.

Note that the DSP had been shown to be NP-Complete [16].

Since our approximation algorithm for the \( p \)DP is based on the set cover problem [10, 28, 36]. We also review the definition of the set cover problem. Given a finite set \( U \) of elements and a collection \( S \) of (non-empty) subsets of \( U \). A set cover [10, 28, 36] is to find a subset \( S' \subseteq S \) such that every element in \( U \) belongs to at least one element of \( S' \).

SCP (Set cover problem) [10, 28, 36]

**Instance:** A finite set \( U \) of elements, a collection \( S \) of (non-empty) subsets of \( U \).

**Question:** Find a set cover \( S'' \) such that the number of sets in \( S'' \) is minimized.

3 Hardness Result for the \( p \)DP

In this section, we prove that the \( p \)DP is NP-Complete. We transform the DSP to the \( p \)DP by the reduction. Hence we need to define \( p \)DP and DSP decision problems.

**pDP Decision Problem**
Instance: A connected, undirected complete graph \( G = (V, E, \ell) \) and two integers \( p > 0 \) and \( U > 0 \).

Question: Does there exist a vertex set \( V' \), \( |V'| = p \), such that \( L_C(V') + L_M(V') \leq U \)?

DSP Decision Problem

Instance: A connected, undirected graph \( G = (V, E) \), and a positive integer \( \kappa \).

Question: Does there exist a dominated set \( Z \) such that \( |Z| \) is less than or equal to \( \kappa \)?

Theorem 1: The \( p \)DP decision problem is NP-Complete.

Proof: First, it is easy to see that the \( p \)DP decision problem is in NP. Then we show the reduction: the transformation from the DSP decision problem to the \( p \)DP decision problem.

Let a graph \( G = (V, E) \) and a positive integer \( \kappa \) be an instance of the DSP decision problem. We transform it into an instance of the \( p \)DP decision problem, say \( \overline{G} = (\overline{V}, \overline{E}, \ell) \) and two positive integers \( p \) and \( U \), as follows.

\[
\overline{V} = V.
\]
\[
\overline{E} = E.
\]

For each edge \( (u, v) \in \overline{E} \), \[
\ell(u, v) = \begin{cases} 
1, & \text{if } (u, v) \in E \\ 
d(u, v), & \text{otherwise} 
\end{cases}
\] (1)

\( U = |V| - \kappa + 1 \) and \( p = \kappa \).

Now, we show that there is a dominating set \( Z \) such that \( |Z| \) is \( \kappa \) if and only if there is a vertex set \( \overline{V'} \) in \( \overline{G} \) such that \( |\overline{V'}| \) is \( p \) and \( L_C(\overline{V'}) + L_M(\overline{V'}) \) is \( U \).

(Only if) If there exists a dominating set \( Z \) in \( G \) and the cardinality of \( Z \) is at most \( \kappa \). Then we choose the corresponding vertex set \( \overline{V'} \) in \( \overline{G} \) of the dominating set \( Z \) in \( G \). Hence, we have \( L_C(\overline{V'}) = 1 \) and \( L_M(\overline{V'}) = |\overline{V'}| - \kappa \). (If) If there exists a vertex set \( \overline{V'} \) in \( \overline{G} \) such that \( |\overline{V'}| \) is \( p \) and \( L_C(\overline{V'}) + L_M(\overline{V'}) \) is \( U \). Clearly, each vertex \( v \) in \( V \setminus \overline{V'} \), \( d(v, \overline{V'}) = 1 \), otherwise \( L_C(\overline{V'}) + L_M(\overline{V'}) > U = |V| - p + 1 \). Hence, we choose the corresponding vertex set \( Z \) in \( G \) of the vertex set \( \overline{V'} \) in \( \overline{G} \) and \( Z \) is a dominating set in \( G \) with \( |Z| = p \).

\[ \square \]

4 Exact Algorithms for the \( p \)DP and CDP

In this section, we show integer programings to solve the \( p \)DP and CDP, respectively. We combine the integer programings for the \( pMP \) and \( pCP \) by [13]. Given an undirected complete graph \( G = (V, E, \ell) \) with a nonnegative edge length function \( \ell \), the \( p \)DP can be formulated as an integer programing (I) as follows.
minimize $\sum_{i \in V} \sum_{j \in V} d(i, j)x_{i,j} + C$  \hspace{1cm} (2)

subject to

$\sum_{j \in V} x_{i,j} = 1, \forall i \in V$  \hspace{1cm} (3)

$\sum_{j \in V} y_j = p$  \hspace{1cm} (4)

$x_{i,j} \leq y_j, \forall i, j \in V$  \hspace{1cm} (5)

$\sum_{j \in V} d(i,j)x_{i,j} \leq C, \forall i \in V$  \hspace{1cm} (6)

$x_{i,j}, y_j \in \{0, 1\}$  \hspace{1cm} (7)

$C \geq 0$  \hspace{1cm} (8)

where the variable $y_j = 1$ if and only if vertex $j$ is chosen as a centdian, and the variable $x_{i,j} = 1$ if and only if $y_j = 1$ and vertex $i$ is assigned to vertex $j$, and $C$ is a feasible eccentricity. For completeness, we list the exact algorithm for the pDP as follows.

**Algorithm OPT-pDP**

**Input:** A connected, undirected complete graph $G = (V, E, \ell)$ with a nonnegative length function $\ell$ on edges and an integer $p > 0$.

**Output:** A vertex set $P_{opt}$ with $|P_{opt}| = p$.

1. Use the integer programming (I) to find all $y_j = 1$ and put the corresponding vertex $j$ of $y_j$ to $P_{opt}$.

2. Return $P_{opt}$.

It is easy to show that Algorithm OPT-pDP is an exact algorithm for the pDP. However, to solve an integer programming is NP-hard [12, 16]. Hence, next section we show $(1 + \epsilon)$-approximation algorithm for the pDP satisfying the cardinality of centdian set is less than or equal to $(1 + 1/\epsilon)(ln|V| + 1)p$, $\epsilon > 0$.

Next, we modify integer programming (I) to design another integer programming (II) for the CDP with an integer $U$ as follows.

minimize $\sum_{j \in V} y_j$  \hspace{1cm} (9)

subject to

$\sum_{j \in V} x_{i,j} = 1, \forall i \in V$  \hspace{1cm} (10)
\[
\sum_{i \in V} \sum_{j \in V} d(i, j)x_{i,j} + C \leq U \quad (11)
\]

\[
x_{i,j} \leq y_j, \forall i, j \in V \quad (12)
\]

\[
\sum_{j \in V} d(i, j)x_{i,j} \leq C, \forall i \in V \quad (13)
\]

\[
x_{i,j}, y_j \in \{0, 1\} \quad (14)
\]

\[
C \geq 0. \quad (15)
\]

For completeness, we list the exact algorithm for the CDP as follows.

**Algorithm OPT-CDP**

**Input:** A connected, undirected complete graph \( G = (V, E, \ell) \) with a nonnegative length function \( \ell \) on edges and an integer \( U > 0 \).

**Output:** A vertex set \( P_{\text{opt}} \) with \( \ell_C(P_{\text{opt}}) + \ell_M(P_{\text{opt}}) \leq U \).

1. Use the integer programming (II) to find all \( y_j = 1 \) and put the corresponding vertex \( j \) of \( y_j \) to \( P_{\text{opt}} \).

2. Return \( P_{\text{opt}} \).

5 An Approximation Algorithm for the pDP

In this section, we show \((1+\epsilon)-approximation algorithm for the pDP satisfying the cardinality of centdian set is less than or equal to \((1+1/\epsilon)(\ln|V|+1)p, \epsilon > 0 \). First, we relax the integer programming (I) for the pDP to the linear programming (IL) to solve the pDP called the fractional pDP as follows.

\[
\text{minimize} \quad \sum_{i \in V} \sum_{j \in V} d(i, j)x_{i,j} + C \quad (16)
\]

subject to

\[
\sum_{j \in V} x_{i,j} = 1, \forall i \in V \quad (17)
\]

\[
\sum_{j \in V} y_j = p \quad (18)
\]

\[
x_{i,j} \leq y_j, \forall i, j \in V \quad (19)
\]

\[
\sum_{j \in V} d(i, j)x_{i,j} \leq C, \forall i \in V \quad (20)
\]

\[
0 \leq x_{i,j}, y_j \leq 1 \quad (21)
\]

\[
C \geq 0. \quad (22)
\]
Let use linear programming (LP). For each vertex \( i \), compute the fractional solutions \( \tilde{y} \) and \( \tilde{x} \) of the optimal solution for the fractional \( p \)DP. The main difference between \( I_L \) and \( I \) is that \( y_j \) and \( x_{i,j} \) can take rational values between 0 and 1 for \( I_L \).

Let \( \tilde{y} \) and \( \tilde{x} \) be the output values of the linear programming \( I_L \). Then it is clear that the centdian-distance of the optimal solution for the fractional \( p \)DP is a lower bound on the centdian-distance of the optimal solution for the \( p \)DP. Moreover, the linear programming can be solved in polynomial time \([32, 33]\).

**Lemma 2:** Given a solution \( \tilde{y} = \{ \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{|V|} \} \) for the fractional \( p \)DP, we can determine the optimal fractional values for \( \tilde{x}_{i,j} \).

**Proof:** Similar with \([2]\), for each \( i \in V \), we sort \( d(i, j), j \in V \), so that \( d(i, j_1(i)) \leq d(i, j_2(i)) \leq \ldots \leq d(i, j_{|V|}(i)) \) and let \( s \) be a value such that \( \sum_{k=1}^{s-1} \tilde{y}_{j_k(i)} \leq 1 < \sum_{k=1}^{s} \tilde{y}_{j_k(i)} \). Then let \( \tilde{x}_{i,j} = \tilde{y}_j \) for each \( j = j_1(i), j_2(i), \ldots, j_{s-1}(i) \), \( \tilde{x}_{i,j_s(i)} = 1 - \sum_{k=1}^{s-1} \tilde{y}_{j_k(i)} \), and otherwise \( \tilde{x}_{i,j} = 0 \).

Given a fractional solution \( \tilde{x}_{i,j} \), for each \( i \in V \), let \( \tilde{D}(i) = \sum_{j \in V} d(i, j) \tilde{x}_{i,j} \) be the distance of assigning vertex \( i \) to its fractional centdian. Given \( \epsilon > 0 \), we also let the neighborhood set \( N(i) \) of vertex \( i \) be \( N(i) = \{ j \in V | d(i, j) \leq (1 + \epsilon)\tilde{D}(i) \} \).

**Lemma 3:** \([35]\) For each \( i \in V \) and \( \epsilon > 0 \), we have \( \sum_{j \in N(i)} \tilde{y}_j \geq \sum_{j \in N(i)} \tilde{x}_{i,j} > \epsilon/(1 + \epsilon) \).

Then we transform the \( p \)DP to the SCP. An instance of SCP contains a finite set \( U \) of elements, a collection \( S \) of (non-empty) subsets of \( U \). We let each vertex \( i \in V \) correspond to each element in \( U \), and each vertex \( j \in E \) with \( y_j > 0 \) correspond to each set in \( S \), respectively. Then for each vertex \( i \in V \), if \( j \in N(i) \), then the corresponding element of \( i \) in \( U \) belongs to the corresponding set of \( j \) in \( S \).

Then we use the greedy approximation algorithm for the SCP whose approximation ratio is \((\ln |U| + 1)/|U| \) \([10, 28, 36]\) to find a set cover of \( U \) and \( S \). Let \( A_{SCP} \) be the greedy approximation algorithm for the SCP. Finally, output the corresponding vertex set for the output set by \( A_{SCP} \). Given a graph \( G = (V, E, \ell) \), let \( P_{APX} \) be a vertex set in \( G \). Initially, \( P_{APX} \) is empty. Now, for clarification, we describe the \((1 + \epsilon)\)-approximation algorithm for the \( p \)DP as follows.

**Algorithm APX-\( p \)DP**

**Input:** A connected, undirected complete graph \( G = (V, E, \ell) \) with a nonnegative length function \( \ell \) on edges, an integer \( p > 0 \), and a real number \( \epsilon, 0 < \epsilon < 1 \).

**Output:** A vertex set \( P_{APX} \) with \(|P_{APX}| \leq (1 + 1/\epsilon)(\ln |V| + 1)p \).

1. Let \( P_{APX} \leftarrow \emptyset \).
2. Use linear programming (PL) to solve the fractional \( p \)DP and find the fractional solutions \( \tilde{y} \) and \( \tilde{x} \).
3. For each \( i \in V \), compute \( \tilde{D}(i) \) and find its neighborhood set \( N(i) = \{ j \in V | d(i, j) \leq (1 + \epsilon)\tilde{D}(i) \} \).
4. For each \( i \in V \) do
   a. For each \( j \in V \) with \( y_j > 0 \) do
      i. create a subset \( S_j = \{ u_i | j \in N(i) \} \) of \( U \) in \( S \).
Use the greedy approximation algorithm \( A_{SCP} \) for the SCP to find a set cover \( S' \) of the instance \( \mathcal{U} \) and \( S \). Let \( y_j = 1 \) if \( S_j \in S' \), and then \( x_{i,j} = 1 \) if set \( S_j \in S' \) and \( u_i \) is covered by \( S_j \), and otherwise is 0.

7. Let \( P_{APX} \) be the corresponding vertex set of \( S' \).

The result of this section is summarized in the following theorem.

**Theorem 4:** Algorithm APX-\( p \)-DP is a \((1 + \epsilon)\)-approximation algorithm for the \( p \)-DP satisfying \(|P_{APX}| \leq (1 + 1/\epsilon)(\ln|V| + 1)p\), in which \( \epsilon > 0 \).

**Proof:** Let \( P_{OPT} \) be the optimal solution for the \( p \)-DP. Clearly, by Step 5 and Step 6, a subset \( S_j \) contains the element \( u_i \) in \( \mathcal{U} \) if \( d(i,j) \leq (1 + \epsilon)\hat{D}(i) \), where \( i \) is the corresponding vertex of \( u_i \) and \( j \) is the corresponding vertex of \( S_j \), and each \( i \in V \), \( \sum_{j \in V} d(i,j)x_{i,j} \leq (1 + \epsilon)\hat{D}(i) \). Hence, we have

\[
\mathcal{L}_M(P_{APX}) + \mathcal{L}_C(P_{APX}) \leq \sum_{i \in V} \sum_{j \in V} d(i,j)x_{i,j} + \max_{i \in V} \sum_{j \in V} d(i,j)x_{i,j} \\
\leq \sum_{i \in V} (1 + \epsilon)\hat{D}(i) + \max_{i \in V} (1 + \epsilon)\hat{D}(i) \\
\leq (1 + \epsilon)\mathcal{L}_M(P_{OPT}) + (1 + \epsilon)\mathcal{L}_C(P_{OPT}),
\]

since the center-distance of the fractional \( p \)-DP is a lower bound on the center-distance of the optimal solution for the \( p \)-DP.

Then we show \(|P_{APX}| \leq (1 + 1/\epsilon)(\ln|V| + 1)p\). By [35] and Lemma 3, we have the cardinality of set for the optimal fractional cover is less than \((1 + 1/\epsilon)p\) and the cardinality of set by the output of the greedy algorithm is at most \((\ln|\mathcal{U}| + 1)\) [10, 36] of the cardinality of set for the optimal fractional cover. Immediately, we have \(|P_{APX}| \leq (1 + 1/\epsilon)(\ln|V| + 1)p\). \( \square \)

### 6 An Approximation Algorithm for the CDP

In this section, we show a \((1 + 1/\epsilon)(\ln|V| + 1)\)-approximation algorithm for the CDP satisfying the center-distance is less than or equal to \((1 + \epsilon)U\), \( \epsilon > 0 \). We only run Algorithm APX-\( p \)-DP for the \( p \)-DP, for \( p = 1 \) to \(|V| \) and find the first center set such its center-distance is less than or equal to \((1 + \epsilon)U\).

For the completeness, we describe the approximation algorithm for the CDP and obtain the center set \( P_\gamma \) as follows.

**Algorithm APX-CDP**

**Input:** A connected, undirected complete graph \( G = (V, E, \ell) \) with a nonnegative length function \( \ell \) on edges, an integer \( U > 0 \) and a real number \( \epsilon, 0 < \epsilon < 1 \).

**Output:** A vertex set \( P_\gamma \) with \( \mathcal{L}_C(P_\gamma) + \mathcal{L}_M(P_\gamma) \leq (1 + \epsilon)U \).
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1. Let $p = 1$ and $P_\gamma \leftarrow \emptyset$.

2. Use Algorithm APX-$p$DP to find a vertex set $P_p$ that satisfies Theorem 4.

3. If $\mathcal{L}_C(P_p) + \mathcal{L}_M(P_p) > (1 + \epsilon)U$ then
   
   Let $p = p + 1$ and go to step 2.

4. Let $P_\gamma \leftarrow P_p$.

Theorem 5: Algorithm APX-CDP is a $(1 + 1/\epsilon)(\ln|V| + 1)$-approximation algorithm for the CDP satisfying the centdian-distance is less than or equal to $(1 + \epsilon)U$, in which $\epsilon > 0$.

Proof:

Let $P'$ be the centdian set of optimal solutions for the CDP with an integer $U$. We have $\mathcal{L}_C(P') + \mathcal{L}_M(P') \leq U$. Let $P''$ (respectively, $P'''$) be the centdian set of optimal solutions for the $p$DP with $p = |P'|$ (respectively, $\gamma = \gamma$). Clearly, $\mathcal{L}_C(P'') + \mathcal{L}_M(P'') \leq \mathcal{L}_C(P') + \mathcal{L}_M(P') \leq U$. If $p = |P''|$, Algorithm APX-CDP returns a centdian set $P_{P''}$ such that $\mathcal{L}_C(P_{P''}) + \mathcal{L}_M(P_{P''}) \leq (1 + \epsilon)\mathcal{L}_C(P'') + \mathcal{L}_M(P'') \leq (1 + \epsilon)U$. Since Algorithm APX-CDP returns the first centdian such its centdian-distance is less than or equal to $(1 + \epsilon)U$, we have that $\gamma$ is less than or equal to $|P''|$. By Theorem 4, we have

$$|P_\gamma| \leq (1 + 1/\epsilon)(\ln|V| + 1)\gamma \leq (1 + 1/\epsilon)(\ln|V| + 1)|P''| = (1 + 1/\epsilon)(\ln|V| + 1)|P'|,$$

and

$$\mathcal{L}_C(P_\gamma) + \mathcal{L}_M(P_\gamma) \leq (1 + \epsilon)(\mathcal{L}_C(P') + \mathcal{L}_M(P')) \leq (1 + \epsilon)(\mathcal{L}_C(P'') + \mathcal{L}_M(P'')) \leq (1 + \epsilon)U.$$

\[\square\]

7 Conclusion

In this paper, we have investigated the $p$DP and the CDP and prove that these problems are NP-Complete even when the centdian-distance is $\mathcal{L}_C(V') + \mathcal{L}_M(V')$. Then we have presented non-trivial brute force exact algorithms for the $p$DP and the CDP, respectively. Moreover, we have designed a $(1 + \epsilon)$-approximation algorithm for the $p$DP satisfying the cardinality of the centdian set is less than or equal to $(1 + 1/\epsilon)(\ln|V| + 1)p$ and a $(1 + 1/\epsilon)(\ln|V| + 1)$-approximation algorithm for the CDP satisfying the centdian-distance is less than or equal to $(1 + \epsilon)U$, in which $\epsilon > 0$. It would be interesting to find approximation complexities for the $p$DP and the CDP. Another direction for future research is whether the $p$DP has a polynomial time exact algorithm for some special graphs.
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