Harmonic Analysis on Real Reductive Symmetric Spaces

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Abstract

Let $G$ be a reductive group in the Harish-Chandra class e.g. a connected semisimple Lie group with finite center, or the group of real points of a connected reductive algebraic group defined over $\mathbb{R}$. Let $\sigma$ be an involution of the Lie group $G$, $H$ an open subgroup of the subgroup of fixed points of $\sigma$. One decomposes the elements of $L^2(G/H)$ with the help of joint eigenfunctions under the algebra of left invariant differential operators under $G$ on $G/H$.

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1. Introduction

Let $G$ be a real reductive group in the Harish-Chandra class [H-C1], e.g. a connected semisimple Lie group with finite center, or the group of real points of a connected reductive algebraic group defined over $\mathbb{R}$. Let $\sigma$ be an involution of the Lie group $G$, $H$ an open subgroup of the subgroup of fixed points of $\sigma$.

Important problems of harmonic analysis on the so-called reductive symmetric space $G/H$ are:

(a) to make the simultaneous spectral decomposition of the elements of the algebra $\mathbb{D}(G/H)$ of left invariant differential operators under $G$ on $G/H$. In other words, one wants to write the elements of $L^2(G/H)$ with the help of joint eigenfunctions under $\mathbb{D}(G/H)$.

(b) to decompose the left regular representation of $G$ in $L^2(G/H)$ into an Hilbert integral of irreducible unitary representations of $G$ : this is essentially the Plancherel formula.

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(c) to decompose the Dirac measure at $eH$, where $e$ is the neutral element of $G$, into an integral of $H$-fixed distribution vectors: this is essentially the Fourier inversion formula.

These problems were solved for the "group case" (i.e. the group viewed as a symmetric space: $G = G_1 \times G_1$, $\sigma(x,y) = (y,x)$, $H$ is the diagonal of $G_1 \times G_1$) by Harish-Chandra in 1970s (see [H-C1, 2, 3]), the Riemannian case ($H$ maximal compact) had been treated before (see [He]). Later, there were deep results by T. Oshima [O1]. When $G$ is a complex group and $H$ is a real form, the Problems (a), (b), (c) were solved by P. Harinck, together with an inversion formula for orbital integrals ([Ha], see also [D3] for the link of her work with the work of A. Bouaziz on real reductive groups).

Then, E. van den Ban and H. Schlichtkrull, on the one hand, and I, on the other hand, obtained different solutions to problems (a), (b), (c). Moreover, they obtained a Paley-Wiener theorem (see [BS3] for a presentation of their work). I present here my point of view, with an emphasize on problem (a), because it simplifies the formulations of the results (nevertheless, the important aspect of representation theory is hidden). It includes several joint works, mainly with J. Carmona, and also with E. van den Ban and J.L. Brylinski. Several works of T. Oshima, linked to the the Flensted-Jensen duality, alone and with T. Matsuki are very important in my proof, as well as earlier results of E. van den Ban and H. Schlichtkrull.

I have to acknowledge the deep influence of Harish-Chandra’s work. The crucial role played by the work [Be] of J. Bernstein on the support of the Plancherel measure, and some part of Arthur’s article on the local trace formula [A] will be apparent in the main body of the article.

2. Temperedness of the spectrum

Let $\theta$ be a Cartan involution of $G$ commuting with $\sigma$, let $K$ be the fixed point set of $\theta$. Let $\mathfrak{g}$ be the Lie algebra of $G$, etc. Let $\mathfrak{s}$ (resp. $\mathfrak{q}$) be the space of elements in $\mathfrak{g}$ which are antiinvariant under the differential of $\theta$ (resp. $\sigma$). Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{s} \cap \mathfrak{q}$. If $P$ is a $\theta$-stable parabolic subgroup of $G$, containing $A_0 := \exp \mathfrak{a}_0$, we denote by $P = M_P A_P N_P$ its Langlands $\sigma$-decomposition. More precisely $A_P$ is the subgroup of the elements $a$ of the split component of the Levi factor $L_P = P \cap \theta(P)$ such that $\sigma(a) = a^{-1}$. Here $M_P$ is larger than that for the usual Langlands decomposition.

In order to simplify the exposition we will make the following simplifying assumption:

**Hypothesis:** For any $P$ as above, $HP$ is the unique open $(H,P)$-double coset.

When $\sigma = \theta$ (the case of a riemannian symmetric space) or the “group case”, this hypothesis is satisfied.

To get the Plancherel formula, it is useful to use $K$-finite functions. They are often replaced by $\tau$-spherical functions. Here $(\tau, V_{\tau})$ is a finite dimensional unitary representation of $K$ and a $\tau$-spherical function on $G/H$ is a function $f : G/H \to V_{\tau}$ such that $f(kx) = \tau(k)f(x)$, $k \in K$, $x \in G/H$. 


Some spaces of $\tau$-spherical functions on $G/H$ play a crucial role in the theory, namely:

(a) $\mathcal{C}(G/H, \tau)$: the Schwartz space of $\tau$-spherical functions on $G/H$ which are rapidly decreasing as well as their derivatives by elements of the enveloping algebra $U(g)$ of $g$ (see [B2]).

(b) $\mathcal{A}(G/H, \tau)$: the space of smooth $\tau$-spherical functions on $G/H$ which are $D(G/H)$ finite. Here $\mathcal{A}$ is used to evoke automorphic forms.

(c) $\mathcal{A}_{\text{temp}}(G/H, \tau)$: the space of elements of $\mathcal{A}(G/H, \tau)$ which have tempered growth as well as their derivatives by elements of $U(g)$ ([D2]). Integration of functions on $G/H$ defines a pairing between $\mathcal{A}_{\text{temp}}(G/H, \tau)$ and $\mathcal{C}(G/H, \tau)$.

(d) $\mathcal{A}_2(G/H, \tau)$: the space of square integrable elements of $\mathcal{A}(G/H, \tau)$. This is a subspace of the three preceding spaces.

One has:

**Theorem 1** ([D2]): The space $\mathcal{A}_2(G/H, \tau)$ is finite dimensional.

This is deduced from the theory of discrete series for $G/H$ initiated by M. Flensted-Jensen [F-J] and achieved by T. Oshima and T. Matsuki, using the Flensted-Jensen duality [OM]. One has also to use the behaviour of the discrete series under certain translation functors, studied by D. Vogan [V] and a result of H. Schlichtkrull [S] on the minimal $K$-types of certain discrete series.

The next result follows from the work of J. Bernstein [Be] on the support of the Plancherel measure.

**Theorem 2** ([CD1], Appendix C): Every function in $\mathcal{C}(G/H, \tau)$ can be canonically desintegrated as an integral of elements of $\mathcal{A}_{\text{temp}}(G/H, \tau)$.

This information appeared to be crucial at the end of our proof.

### 3. The continuous spectrum: Eisenstein integrals

Let $P = MAN$ the Langlands $\sigma$-decomposition of a $\sigma\theta$-stable parabolic subgroup $P$ of $G$. Let $\rho_P$ be the half sum of the roots of $\mathfrak{a}$ in $\mathfrak{n}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be such that the real part of $\lambda - \rho_P$ is strictly dominant with respect to the roots of $\mathfrak{a}$ in $\mathfrak{n}$. Let $\tau_M$ be the restriction of $\tau$ to $M \cap K$. Then, if $x \in G/H$ and $\psi \in \mathcal{A}_2(M/M \cap H, \tau_M)$, the following integral is convergent:

$$E(P, \psi, \lambda)(x) := \int_K \tau(k^{-1})\Psi_\lambda(kx)dk,$$

where $\Psi_\lambda(x) = 0$ if $x \notin PH$, and $\Psi_\lambda(x) = a^{-\lambda+\rho_P}\psi(m)$ if $x = namH$ with $n \in N$, $a \in A$, $m \in M$. Moreover $E(P, \psi, \lambda)$ is an element of $\mathcal{A}(G/H, \tau)$. Eisenstein integrals are the $\tau$-spherical versions of $K$-finite functions of the form: $gH \mapsto <\pi'(g)\xi, v>$, where $\pi'$ is the contragredient of a generalized principal series $\pi$, $\xi$ is a certain $H$-fixed distribution vector of $\pi$, $v$ is a $K$-finite vector of $\pi$.

**Theorem 3** ([BrD]): The function $\lambda \mapsto E(P, \psi, \lambda)$ admits a meromorphic continuation in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. This meromorphic continuation, denoted in the same way, multiplied by a suitable product, $p_{\psi}$, of functions of type $\lambda \mapsto (\alpha, \lambda) + c$, where $\alpha$ is a root of $\mathfrak{a}$ and $c \in \mathbb{C}$, is holomorphic around $i\mathfrak{a}_{\mathbb{C}}^*$.
This meromorphic continuation is an interesting feature of the theory. For the the "group case", it comes down to the meromorphic continuation of Knapp-Stein intertwining integrals. My proof with Brylinski uses $D$-modules arguments.

The case where $P$ is minimal had been treated separately by E. van den Ban [B1] and G. Olafsson [Ol]. One has also to mention the earlier work of T. Oshima and J. Sekiguchi [OSe] on the spaces of type $G/K$.

The proof which gives the best results uses a method of tensoring by finite dimensional representations. It is a joint work with J. Carmona. It was initiated by D. Vogan and N. Wallach (see [W], chapter 10) for the meromorphic continuation of the Knapp-Stein intertwining integrals. For symmetric spaces and the most continuous spectrum, the proof is due to E. van den Ban [B2]. This proof uses Bruhat’s thesis and tensoring by finite dimensional modules. This implies rough estimates for Eisenstein integrals, which generalize those obtained by E. van den Ban when $P$ is minimal [B2].

**Theorem 4 ([D1]):** If $\lambda \in i\mathfrak{a}^*$ is such that $E(P, \psi, \lambda)$ is well defined, then it is tempered, i.e. is an element of $A_{tem}(G/H, \tau)$.

This is a natural result but the proof is quite long. It uses the behaviour under translation functors of $H$-fixed distribution vectors of discrete series and of generalized principal series, and also of the Poisson transform. Moreover the duality of M. Flensted-Jensen, [F-J], and a criteria of temperedness due to Oshima [O2] play a crucial role (apparently, J. Carmona has a way to avoid boundary values).

With the help of this theorem and by using techniques due to E. van den Ban [B2], the rough estimates for Eisenstein integrals can be improved to get uniform sharp estimates for $p_\psi(\lambda)E(P, \psi, \lambda)$, $\lambda \in i\mathfrak{a}^*$ (cf. [D1]).

### 4. C-functions

Let $P$ be as above and let $L$ be equal to $MA$. The theory of the constant term, due to J. Carmona [C1] (Harish-Chandra for the group case, [H-C1]), gives a linear map from $A_{tem}(G/H, \tau)$ into $A_{tem}(L/L \cap H, \tau_L)$, $\varphi \mapsto \varphi_P$, characterized by:

$$\lim_{t \to +\infty} \frac{1}{\delta_P^{1/2}}((\exp tlX)\varphi((\exp tlX)l) - \varphi((\exp tlX)l)) = 0,$$

where $l \in L/L \cap H$, $X \in \mathfrak{a}_P$ is $P$-dominant and $\delta_P$ is the modular function of $P$.

Let $Q$ be a $\sigma\theta$-stable parabolic subgroup of $G$ with the same $\theta$-stable Levi subgroup $L$ other than $P$. Let $W(\mathfrak{a})$ be the group of automorphisms of $\mathfrak{a}$ induced by an element of $Ad(G)$. One defines meromorphic functions on $\mathfrak{a}^*_C$, $\lambda \mapsto C_{Q,P}(s, \lambda)$ with values in $End(A_2(M/M \cap H, \tau_M))$ such that:

$$E(P, \psi, \lambda)_Q(ma) = \sum_{s \in W(\mathfrak{a})} (C_{Q,P}(s, \lambda)\psi)(m)a^{-s\lambda}, \quad m \in M, a \in A, \lambda \in i\mathfrak{a}^*,$$

or rather for $\lambda$ in an open dense subset of $i\mathfrak{a}^*$.

The $C$-functions allow to normalize Eisenstein integrals as follows:

$$E^0(P, \psi, \lambda) := E(P, C_{P,P}(1, \lambda)^{-1}\psi, \lambda).$$
5. Truncation, Maass-Selberg relations and the regularity of normalized Eisenstein integrals

Let $P$ be as above and let $P' = M'A'N'$ be the Langlands $\sigma$-decomposition of another $\sigma\theta$-stable parabolic subgroup of $G$. Let $\psi$ (resp. $\psi'$) be an element of $A_2(M/M \cap H, \tau_M)$ (resp. $A_2(M'/M' \cap H, \tau_{M'})$). One chooses $p_\psi$ as in Theorem 3, such that the product of $p_\psi$ with the $C$-functions are holomorphic in a neighbourhood of $i\alpha^*$ which is a product of $i\alpha^*$ with a neighbourhood of 0 in $\alpha^*$. We do the same for $\psi'$. One defines $F(\lambda) := p_\psi(\lambda)E(P, \psi, \lambda)$. One defines similarly $F'$. We assume, for the rest of the article, that $G$ is semisimple. This is just to simplify the exposition. One chooses $T \in \mathfrak{a}_0$, regular with respect to the roots of $\mathfrak{a}_0$ in $\mathfrak{g}$. Let $C_T^1$ be the convex hull of $W(\mathfrak{a}_0)T$ and let $C_T$ be equal to the subset $K(exp C_T^1)H$ of $G/H$.

**Theorem 5 ([D2]):**

(i) One gets an explicit expression $\omega^T(\lambda, \lambda')$, involving the $C$-functions (see an example below) and vanishing when $A$ and $A'$ are not conjugate under $\mathcal{K}$, which is asymptotic to

$$\Omega^T(\lambda, \lambda') := \int_{C_T} (F(\lambda)(x), F'(\lambda')(x))dx,$$

when $T$ goes to infinity and $\lambda \in i\alpha^*, \lambda' \in i\alpha'^*$. More precisely for $\delta > 0$ there exists $C > 0$, $k \in \mathbb{N}$ and $\varepsilon > 0$, such that for all $T$ satisfying $|T| \geq \delta$, for every root $\alpha$ of $\mathfrak{a}_0$ in $\mathfrak{g}$, one has:

$$|\Omega^T(\lambda, \lambda') - \omega^T(\lambda, \lambda')| \leq C(1+||\lambda||)k(1+||\lambda'||)k e^{-\varepsilon|T|}.$$

(ii) Moreover $\omega^T$ is analytic in $(\lambda, \lambda') \in i\alpha^* \times i\alpha'^*$.

This generalizes a result of J. Arthur for the group case [A], Theorem 8.1. My proof is quite similar, but I was able to avoid his use of the Plancherel formula.

(iii) is an easy consequence of (i). In fact, the explicit form of $\omega^T$ implies that it is meromorphic around $i\alpha^* \times i\alpha'^*$. Moreover $\Omega^T$ is holomorphic, hence locally bounded, around $i\alpha^* \times i\alpha'^*$. From the inequality in (i), one deduces that $\omega^T$ is locally bounded, hence holomorphic, around $i\alpha^* \times i\alpha'^*$.

We will now show, by an example, how the explicit form of $\omega^T$ and its analyticity in $(\lambda, \lambda') \in i\alpha^* \times i\alpha'^*$ imply the Maass-Selberg relations.

Let $\sigma$ be equal to $\theta$, $H = K$, and $\tau$ be the trivial representation. Let $P, P'$ be minimal parabolic subgroups of $G$. Then $dim A_2(M/M \cap H, \tau_M) = 1$ and the $C$-functions are scalar valued. One assumes $\mathfrak{g}$ to be semisimple and that the dimension of $A$ is one. Then $W(\mathfrak{a})$ has 2 elements, $\pm 1$, and one has the following explicit expression of $\omega^T$:

$$\omega^T(\lambda, \lambda') = p_\psi(\lambda)p_{\psi'}(\lambda') \sum_{s=\pm 1, s'=\pm 1} e^{s\lambda_T - s'\lambda'_T}C_{P|P}(s, \lambda)C_{P'|P}(s', \lambda')(s\lambda - s'\lambda')^{-1}.$$

Thus $\omega^T(\lambda, \lambda')$ is the sum of a product of $(\lambda - \lambda')^{-1}$ by an analytic function with a product of $(\lambda + \lambda')^{-1}$ by an analytic function. The analyticity at $(\lambda, \lambda)$ implies
easily that the factor in front of \((\lambda - \lambda')^{-1}\) vanishes for \(\lambda = \lambda'\). Hence we get 
\[
| C_{P_{1}}P(1, \lambda) |^2 = | C_{P_{1}}P(-1, \lambda) |^2, \ \lambda \in \mathfrak{i}a^*.
\]
This is one of the Maass-Selberg relations (cf. [D2], Theorem 2, and the work with J. Carmona [CD2], Theorem 2 for the general case, see [B1], [B2] for the case where \(P\) is minimal). These relations imply that the \(C\)-functions attached to normalized Eisenstein integrals are unitary, when defined, for \(\lambda\) purely imaginary. Hence they are locally bounded. This implies that they are holomorphic around the imaginary axis. This implies in particular some holomorphy property of the constant term of normalized Eisenstein integrals. From this, with the help of [BCD], one deduces:

**Theorem 6** (Regularity theorem for normalized Eisenstein integrals, [CD2], [BS1] for \(P\) minimal): The normalized Eisenstein integrals are holomorphic in a neighbourhood of the imaginary axis.

6. Fourier transform and wave packets

**Theorem 7** ([CD2], [BS1] for \(P\) minimal): For \(f \in \mathcal{C}(G/H, \tau)\), one has \(\mathcal{F}_0^0 P f \in \mathcal{S}(\mathfrak{i}a^*) \otimes A_2(M/M \cap H, \tau_M)\), where \(\mathcal{F}_0^0 P f\) is characterized by:

\[
((\mathcal{F}_0^0 P f)(\lambda), \psi) = \int_{G/H} f(x), E^0(P, \psi, \lambda)(x) dx, \ \lambda \in \mathfrak{i}a^*, \ \psi \in A_2(M/M \cap H, \tau_M),
\]

here \(\mathcal{S}(\mathfrak{i}a^*)\) is the usual Schwartz space.

This theorem follows from the sharp estimates of Eisenstein integrals.

**Theorem 8** ([BCD]): If \(\Psi\) is an element of \(\mathcal{S}(\mathfrak{i}a^*) \otimes A_2(M/M \cap H, \tau_M)\), one has \(\mathcal{F}_p P \in \mathcal{C}(G/H, \tau)\), where :

\[
\mathcal{F}_p P(x) := \int_{\mathfrak{i}a^*} E^0(P, \Psi(\lambda), \lambda) d\lambda, \ \ x \in G/H.
\]

This theorem follows from the regularity theorem and from the joint work with E. van den Ban and J. Carmona, [BCD].

Now we want to compute \(\mathcal{F}_p \mathcal{F}_p P\). For this purpose one has to study the integral:

\[
I := \int_{G/H} \int_{\mathfrak{i}a^*} \alpha(\lambda) E^0(P, \psi, \lambda)(x) d\lambda, E^0(P', \psi', \lambda')(x) dx.
\]

One truncates the integral on \(G/H\) to \(C_T\) and let \(T\) goes to infinity (far from the walls). Let us denote the truncated inner product of the normalized Eisenstein integrals by \(\Omega_T^0(\lambda, \lambda')\). Using Fubini’s theorem one has:

\[
I = lim_{T \to \infty} \int_{\mathfrak{i}a^*} \alpha(\lambda) \Omega_T^0(\lambda, \lambda') d\lambda.
\]

As for unnormalized Eisenstein integrals, one has an asymptotic evaluation of \(\Omega_T^0(\lambda, \lambda')\) by an explicit expression \(\omega_T(\lambda, \lambda')\). One can replace \(\Omega_T^0\) by \(\omega_T\) in the previous formula. By using an expression of \(\omega_T(\lambda, \lambda')\), viewed as a distribution in \(\lambda\), for fixed \(\lambda'\), involving Fourier transforms of cones ([D2], Theorem 3) one gets:
Theorem 9 ([CD2]): Let \( \mathcal{F} \) be a set of representative of \( \sigma \)-association classes of \( \sigma\theta \)-stable parabolic subgroups. Here \( \sigma \)-association means that the \( a \) are conjugated under \( K \). Define:

\[
P_{\tau} = \sum_{P \in \mathcal{F}} (W(a_P))^{-1} \eta_P^0 \eta_P^0.
\]

Then \( P_{\tau} \) is an orthogonal projection operator in \( \mathcal{C}(G/H, \tau) \) endowed with the \( L^2 \) scalar product.

7. The Plancherel formula

Essentially, the solution to problem (a) is contained in the following:

Theorem 10 ([D4]): The projection \( P_{\tau} \) is the identity operator on \( \mathcal{C}(G/H, \tau) \).

Actually, this gives an expression of every element in \( \mathcal{C}(G/H, \tau) \) as a wave packet of normalized Eisenstein integrals. The proof goes as follows. If \( P_{\tau} \) was not the identity, using Theorem 1 on the temperedness of the spectrum, one could find a non zero element of \( A_{\text{temp}}(G/H, \tau) \) which is orthogonal to the image of \( P_{\tau} \). Then, generalizing Theorem 5 to the truncated inner product of an Eisenstein integral with a general element of \( A_{\text{temp}}(G/H, \tau) \), this orthogonality can be explicitly described (cf. the evaluation of \( I \) before Theorem 9). As a result, this function has to be zero, a contradiction which proves the theorem.

The theorem translates to \( K \)-finite functions, involving representations and \( H \)-fixed distribution vectors.

The theorem can also be expressed with the unnormalized Eisenstein integrals. Then there are certain Plancherel factors involved. They are linked, as in the group case, to the intertwining integrals. Following the approach of A. Knapp and G. Zuckerman, [KZ], their computation is reduced to find an embedding of discrete series into principal series attached to minimal parabolic subgroups. For connected groups, this has been done by J. Carmona [C2].

8. Applications and open problems

Schwartz space for the hypergeometric Fourier transform

The image of a natural Schwartz space by the hypergeometric Fourier transform is characterized [D5]. The work uses the Plancherel formula of E. Opdam [Op], and the techniques mentioned above: theory of the constant term, \( C \)-functions, truncation ...

Generalized Schur orthogonality relations

Using the Plancherel formula for reductive symmetric spaces groups, K. Ankabout, [An], has proved generalized Schur orthogonality relations for generalized coefficients related to real reductive symmetric spaces. In particular, at least if we assume multiplicity one in the Plancherel formula, it implies the following:

There exists an explicit positive function \( d \) on \( G/H \), such that, for almost all representations \( \pi \) occurring in the Plancherel formula, for \( \xi \) an \( H \)-fixed distribution vector of \( \pi \) occurring in the decomposition of the Dirac measure, there exists an
explicit non zero constant $C_\pi$ such that, for all $v, v'$, $K$-finite vectors in the space of $\pi$:

$$\lim_{\varepsilon \to 0^+} \varepsilon^{\pi_\pi} \int_{G/H} e^{-\varepsilon d(x)} < \pi'(g) \xi, v > \overline{< \pi'(g) \xi, v'>} dx = C_\pi (v, v').$$

Here $n_\pi$ is the dimension of the support of the Plancherel decomposition, around $\pi$. This refines and generalizes a work of Mirodikawa. It suggests to look for such type of relations in other situations.

**$\mathbb{D}(G/H)$-finite $\tau$-spherical functions on reductive symmetric spaces**

S. Souaifi [So] showed how these functions appear as linear combinations of derivatives along the complex parameter $\lambda$, of Eisenstein integrals. For $K$-finite functions, filtrations are introduced, whose subquotients are described in terms of induced representations. The starting point is an adaptation of ideas used by J. Franke to study spaces of automorphic forms. The use of the spectral decomposition by Langlands is replaced here by the use of the Plancherel formula. For reductive $p$-adic groups, and for the group case, I got similar results.

**Invariant harmonic analysis on real reductive symmetric spaces**

The goal is to study the $H$-invariant eigendistributions under $\mathbb{D}(G/H)$ on $G/H$ and to express invariant measures on certain $H$-orbits in terms of these distributions (cf [D3] for the work of A. Bouaziz and P. Harinck for the group case and $G(\mathbb{C})/G(\mathbb{R})$, see also [OSe]).

**Harmonic analysis on $p$-adic reductive symmetric spaces**

For the group case, the Problems (b) and (c) of the Introduction have been solved by Harish-Chandra, up to the explicit description of the discrete series. In general, the problems are open (see [HH] for interesting structural results).

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Harmonic Analysis on Real Reductive Symmetric Spaces

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