A REFINED COMBINATION THEOREM FOR HIERARCHICALLY HYPERBOLIC GROUPS

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Abstract. In this work, we are concerned with hierarchically hyperbolic spaces and hierarchically hyperbolic groups. Our main result is a wide generalization of a combination theorem of Behrstock, Hagen, and Sisto. In particular, as a consequence, we show that any finite graph product of hierarchically hyperbolic groups is again a hierarchically hyperbolic group, thereby answering [5, Question D] posed by Behrstock, Hagen, and Sisto. In order to operate in such a general setting, we establish a number of structural results for hierarchically hyperbolic spaces and hieromorphisms (that is, morphisms between such spaces), and we introduce two new notions for hierarchical hyperbolicity, that is concreteness and the intersection property, proving that they are satisfied in all known examples.

1. Introduction

Many seemingly different classes of groups, including hyperbolic 3-manifold groups, surface groups, small cancellation groups, share some common behavior and properties that were identified by Gromov (and others) and synthesized into the class of hyperbolic groups [15]. Gromov’s breakthrough consisted in considering groups purely as geometric objects, abstracting the properties shared by the above-mentioned classes. This geometric approach implies strong algebraic, asymptotic, and growth properties: hyperbolic groups are finitely presented, they have exponential growth (except the virtually cyclic ones), they satisfy a strong form of Tits’ alternative and a linear isoperimetric inequality. As already stressed by Gromov, some natural groups of geometric origin do not fit into this picture: fundamental groups of surfaces with cusps and mapping class groups are in general not hyperbolic. What is more, the class of hyperbolic groups is closed under taking free products, but not direct products. Therefore, as proved by Meier [24], a graph product of hyperbolic groups is again a hyperbolic group (if and) only if some strong conditions are met.

To overcome these limitations, several generalizations of hyperbolic groups have been introduced over the years. The notion of relative hyperbolicity [10, 27] recovers fundamental groups of surfaces with cusps, whereas mapping class groups are examples of acylindrically hyperbolic groups [28], and raags (that is right-angled Artin groups) are among the groups acting properly and cocompactly on CAT(0) cube complexes, that is cubulable groups [29, 33]. On the one hand, mapping class groups are not relatively hyperbolic (unless they are already hyperbolic). On the other one, the class of acylindrically hyperbolic groups is so wide that no general algebraic information can be deduced using only acylindricity. Therefore, one is brought to find a set of properties that would generalize hyperbolicity, include mapping class groups, be preserved by direct products, and still have strong algebraic consequences for groups satisfying them.

These conditions have been identified by Behrstock, Hagen, and Sisto, who isolated the notions of hierarchically hyperbolic spaces and of hierarchically hyperbolic groups [4, 5]. Again, the geometric approach that is undertaken reflects into strong algebraic and asymptotic properties: hierarchically hyperbolic groups are finitely presented [5, Corollary 7.5], they satisfy a quadratic isoperimetric inequality [5, Corollary 7.5], they are coarse median [5, Theorem 7.3], and they have finite asymptotic dimension [6].

The key insight was to axiomatize the Masur-Minsky machinery for mapping class groups, to be able to apply it to its full extent. Although not being in general hyperbolic, a mapping class group $\text{MCG}(S)$ (of a surface $S$ of finite complexity) can be studied, using the tools developed by Masur and Minsky [22, 23], through a family of
hyperbolic spaces, the curve complexes $CV$ associated to subsurfaces $V \subseteq S$. In a similar manner, hierarchically hyperbolic spaces and groups are the ones for which an analogous approach can be undertaken.

Hierarchically hyperbolic groups provide a common framework to work with hyperbolic groups, mapping class groups, and raags. Other examples comprise all known cubulable groups [18], toral relatively hyperbolic groups [5, Theorem 9.1], fundamental groups of many 3-manifolds [5, Theorem 10.1], free and direct products [5] of these. Hierarchically hyperbolic spaces include all hierarchically hyperbolic groups, the Teichmüller space with either the Thurston or the Weil-Petersson metric, almost all separating curve graphs of surfaces [32], universal covers of compact special cube complexes, and any space quasi-isometric to a hierarchically hyperbolic space.

Hierarchical hyperbolicity has been used to prove several new results, and to uniformize results previously known only for certain subclasses of hierarchically hyperbolic spaces. In [7], Behrstock, Hagen and Sisto show that, in a hierarchically hyperbolic space, any top-dimensional quasiflat is uniformly close to a union of standard orthants. This strengthened the known results [9, 20] in the cubulable setting, and resolved conjectures of Farb for mapping class groups and Brock for Teichmüller spaces. In [6], as already mentioned, the same authors show that every hierarchically hyperbolic space has finite asymptotic dimension, and obtain the sharpest known bound on the asymptotic dimension of mapping class groups.

The definition of hierarchical hyperbolicity is rather technical and we postpone it until Section 2.1. For the time being, it is enough to know that a hierarchically hyperbolic space $(X, S)$ is a metric space $(X, d_X)$ equipped with a collection of $\delta$-hyperbolic spaces $\{CV | V \in S\}$, and projections $\pi_V$ from $X$ onto the various hyperbolic spaces $CV$, for all $V \in S$. This index set $S$ is equipped with a partial order called nesting, a symmetric and anti-reflexive relation called orthogonality, and if $V, U \in S$ are neither nested nor orthogonal, then they are transversal. These relations are mutually exclusive, and in the mapping class group scenario their role is respectively taken by nesting, disjointness and overlapping of subsurfaces. The projections onto hyperbolic spaces and the nesting, orthogonality and transversality satisfy, in addition, several axioms (see Definition 2.3), which again are evocative of mapping class groups, and assure that the coarse geometry of the space $X$ can be reconstructed from the hierarchically hyperbolic structure.

This leads to one of the most salient features of hierarchical hyperbolicity: a distance formula that generalizes the celebrated distance formula for mapping class groups of Masur and Minsky [23]. In other words, distances in a hierarchically hyperbolic space $(X, S)$ can be (uniformly) coarsely computed by projecting onto the various hyperbolic spaces associated to $S$, determining distances there, and then sum. This is made precise by the following theorem:

**Distance Formula for hierarchically hyperbolic spaces** ([5, Theorem 4.5]). Let $(X, S)$ be a hierarchically hyperbolic space. There exists $s_0$ such that for all $s \geq s_0$ there exist constants $k, c > 0$ such that

$$d_X(x, y) = \sum_{V \in S} \{d_V(\pi_V(x), \pi_V(y))\} s,$$

where the symbol $\{a\}$, means that $a$ is added to the sum only if $a \geq s$, and $a = \{(k, c) \ b \}$ stands for $\frac{b}{k} - c \leq a \leq kb + c$.

For what concerns hierarchically hyperbolic groups, at this time let us just mention that there exist groups that are hierarchically hyperbolic spaces, but fail to be hierarchically hyperbolic groups, and therefore being a hierarchically hyperbolic group is a stronger condition than having a Cayley graph which is a hierarchically hyperbolic space (see Definition 2.12).

Given a class of groups $C$, it is natural to investigate under which group constructions the class is preserved. On the one hand, the fact that $C$ is closed under certain operations gives information on the nature of the class, and, on the other, it provides methods to construct new groups in the class from known examples.

A construction that generalizes free products (with amalgamation, and HNN extensions) is the fundamental group of a graph of groups, and results in this direction are usually referred to as combination theorems. The Bestvina-Feighn combination theorem [8] for hyperbolic groups is such an example: given a finite graph $G$ of hyperbolic groups satisfying certain conditions, the resulting fundamental group is again hyperbolic. Their
strategy of proof was to consider a metric space (more precisely, a tree of metric spaces obtained from the Bass-Serre tree of the graph and the vertex/edge groups of $G$) and study the action of the fundamental group on such space. This approach turned out to be very successful, and was later applied in several other related contexts. This is the case for the combination theorem of [26] in the class of strongly relatively hyperbolic groups, or for the Hsu-Wise combination theorem in the context of groups acting on cube complexes [19], or Alibegović’s combination theorem for relatively hyperbolic groups [2]. On the other hand, a more dynamical approach is undertaken by Dahmani [11] to obtain another combination theorem for relatively hyperbolic groups.

Also in the context of hierarchically hyperbolic groups and spaces, there have been efforts in establishing such combination theorems. In [5, Section 8], Behrstock, Hagen and Sisto impose strict conditions on a tree of hierarchically hyperbolic spaces (something completely analogous to the trees of hyperbolic groups considered by Bestvina and Feighn, and mentioned previously - see Definition 2.14) that ensure that the resulting space is again hierarchically hyperbolic. From this, they deduce [5, Corollary 8.24] the hierarchical hyperbolicity of fundamental groups of finite graph of groups satisfying related strict conditions. In [31, Theorem 4.17], Spriano shows that certain amalgamated products of hierarchically hyperbolic groups are hierarchically hyperbolic, building on results from his previous work [30].

In this work we provide a new combination theorem for hierarchically hyperbolic spaces (see Theorem A) and groups (see Corollary B). To do so, we introduce several new tools for the study of hierarchical hyperbolicity, which are of independent interest. The first one is the intersection property (see Definition 3.1, and the discussion after the statement of Theorem C), which in turn leads to the notion of concreteness. We introduce the latter notion to exclude artificial examples of hierarchically hyperbolic spaces that carry some undesirable features. As we will see in this Introduction, the intersection property has a very natural definition, and we conjecture that all hyperbolic spaces admit a hierarchically hyperbolic structure with the intersection property (see Question D).

These properties are of independent interest, and we expect them to be of further use. They allow us to assume much weaker hypotheses for our combination theorem than the ones used by Behrstock, Hagen, and Sisto [5, Theorem 8.6].

The first result of this paper is the following combination theorem. After having stated it, we will briefly comment on terminology and some concepts related to hierarchical hyperbolicity, relegating their full and precise introduction to Section 2.

**Theorem A.** Let $T$ be a tree of hierarchically hyperbolic spaces. Suppose that:

1. each edge-hieromorphism is hierarchically quasiconvex, uniformly coarsely lipschitz and full;
2. comparison maps are uniform quasi isometries;
3. the hierarchically hyperbolic spaces of $T$ have the intersection property and clean containers.

Then the metric space $X(T)$ associated to $T$ is a hierarchically hyperbolic space with clean containers and the intersection property.

As already mentioned, $X(T)$ is a metric space associated to $T$, and it is built from a tree, replacing vertices and edges with hierarchically hyperbolic spaces, with embeddings of edge spaces into vertex spaces (see Definition 2.14). These embeddings are given by hieromorphisms, which are morphisms between hierarchically hyperbolic spaces that agree with the hierarchical structure (see Definition 2.10 and Definition 2.11 for the notion of full hieromorphism). Theorem A then has three hypotheses: the first two are metric conditions, one of them imposing constraints on how the edge groups in $T$ are embedded into vertex groups, and the other one requiring certain natural maps (compare Definition 2.10) at the level of the hyperbolic spaces $CV$ to be uniform quasi isometries. These are the two fundamental hypotheses of the theorem, and, as we will see, neither of the two can be dropped or relaxed.

The third hypothesis invokes two conditions that, in view of the motivating examples, are very natural. These two properties are known to persist under all known operations that preserve the classes of hierarchically hyperbolic spaces and groups, and currently they are satisfied in all examples of hierarchically hyperbolic spaces.
As a consequence of Theorem (A) we obtain a combination theorem for hierarchically hyperbolic groups:

Corollary B. Let $G$ be a finite graph of hierarchically hyperbolic groups. Suppose that:

1. each edge-hieromorphism is hierarchically quasiconvex, uniformly coarsely lipschitz and full;
2. comparison maps are uniformly quasi isometries;
3. the hierarchically hyperbolic spaces of $G$ have the intersection property and clean containers.

Then the group associated to $G$ is itself a hierarchically hyperbolic group.

A more involved application of Theorem (A) and Corollary (B) is the following Theorem (C), which is concerned with permanence of hierarchical hyperbolicity under taking graph products. Theorem (C) answers in the positive a question posed by Behrstock, Hagen, and Sisto ([5, Question D]). As a byproduct of Theorem (C) we extend the results of [1] to show that clean containers are not only preserved by taking free and direct products, but also by graph products.

Theorem C. Let $\Gamma$ be a finite simplicial graph, $G = \{G_v\}_{v \in \Gamma}$ be a family of hierarchically hyperbolic groups with the intersection property and clean containers. Then the graph product $\Gamma G$ is a hierarchically hyperbolic group with the intersection property and clean containers.

Clean containers (see Remark (2.4)), a notion introduced originally by Abbott, Behrstock, and Durham ([1]), is a technical condition that in the mapping class group setting translates into the following: if $V \subseteq S$ is a subsurface of the surface $S$, then $V$ and $S \setminus V$ are disjoint, and any subsurface disjoint from $V$ is contained into $S \setminus V$. On the other hand, the intersection property is a condition that we introduce, and in the mapping class group setting means that, given two subsurfaces $V, U \subseteq S$, the subsurface $V \cap U$ is the biggest subsurface of $S$ that is contained in both $V$ and $U$. The intersection property gives to the index set $\mathcal{S}$ the structure of a lattice. At this point, it is instructive to notice that both $V \cap U$ and $S \setminus V$ could be non-connected subsurfaces of $S$, and indeed the hierarchically hyperbolic structure with clean containers and the intersection property of a mapping class group $\text{MCG}(S)$ is obtained considering all, possibly non-connected, subsurfaces of $S$.

Both properties are satisfied in all known examples of hierarchically hyperbolic spaces, in the sense that given a hierarchically hyperbolic space $X$ (respectively: hierarchically hyperbolic group $G$), there exists a hierarchical structure $\mathcal{S}$ such that $(X, \mathcal{S})$ is a hierarchically hyperbolic space (respectively: $(G, \mathcal{S})$ is a hierarchically hyperbolic group) with the intersection property and clean containers.

We are inclined to believe that any hierarchically hyperbolic space admits a hierarchically hyperbolic structure with the intersection property and clean containers:

Question D. Let $(X, d_X)$ be a hierarchically hyperbolic space. Does there exist a hierarchically hyperbolic structure $\mathcal{S}$ such that $(X, \mathcal{S})$ is a hierarchically hyperbolic space with the intersection property and clean containers?

All the stated theorems rely on the following fundamental result, Theorem (E), which is of independent interest. It provides equivalent conditions for a (full) hieromorphism $\phi: (X, \mathcal{S}) \to (X', \mathcal{S}')$ with hierarchically quasiconvex image, to be a coarsely lipschitz map.

An interesting feature of Theorem (E) is the following. On the one hand, its first two conditions are purely metric conditions on the hieromorphism, whereas the third is a metric condition on certain natural maps (that is gate maps, see Remark (2.8) between hierarchically quasiconvex subspaces of the hierarchically hyperbolic structure of $(X', \mathcal{S}')$). On the other hand, after the image of the hieromorphism $\phi$ is understood, the fourth and the fifth conditions can be detected in $(X', \mathcal{S}')$.

Therefore, Theorem (E) reveals that a seemingly mild condition on $\phi$ (being coarsely lipschitz) already guarantees that the hieromorphism is a quasi-isometric embedding, and has implications on the hierarchically hyperbolic structure of $(X', \mathcal{S}')$.

Theorem E. Let $\phi: (X, \mathcal{S}) \to (X', \mathcal{S}')$ be a full hieromorphism with hierarchically quasiconvex image, and let $S$ be the $\subseteq$-maximal element of $\mathcal{S}$. The following are equivalent:

1. $\phi$ is coarsely lipschitz;
(2) \( \phi \) is a quasi-isometric embedding;
(3) the maps \( g_{\phi(X)}: F_{\phi(S)} \to \phi(X) \) and \( g_{F_{\phi(S)}}: \phi(X) \to F_{\phi(S)} \) are quasi-inverses of each other, and in particular quasi isometries;
(4) the subspace \( \phi(X) \subseteq X' \), endowed with the subspace metric, admits a hierarchically hyperbolic structure obtained from the one of \( X \) by composition with the map \( \phi \) (and its induced maps at the level of hyperbolic spaces);
(5) \( \pi_W(\phi(X)) \) is uniformly bounded for every \( W \in S^0 \cap \phi(S) \).

Organisation of the paper. The paper is organized as follows. In Section 2 we fix the notation, and recall all the necessary definitions and facts concerning hierarchically hyperbolic spaces and groups. In Section 3 we introduce the notions of intersection property, of \( \varepsilon \)-support, and of concreteness of a hierarchically hyperbolic space (see Definition 3.1, Definition 3.6, and Definition 3.10). As already mentioned, we conjecture that all hierarchically hyperbolic spaces satisfy the intersection property. On the other hand, concreteness is a technical condition that will play a pivotal role in the proofs of Theorem 1.19 and of Theorem A. In Proposition 3.12 we prove that, without loss of generality, any hierarchically hyperbolic space with the intersection property can be assumed to be concrete (and this is why concreteness does not appear as an hypothesis in Theorem A).

In Section 4 we prove Theorem 2 of the Introduction, which is then used in the proofs of Theorem 1.19 and Lemma 4.10. These results will be applied repeatedly in Section 5, which is devoted to the proof of Theorem A.

In Subsection 5.1 we introduce a trick, which we call the decoration of a tree of hierarchically hyperbolic spaces \( T \), which is fundamental for our approach to prove Theorem A.

In Subsection 5.2, Subsection 5.3, and Subsection 5.4 we built the index set needed for Theorem A and describe projections onto the hyperbolic spaces associated to this index set. Finally, in Subsection 5.5 we prove Theorem A. We conclude with Section 6, where we discuss the connections of our result to the combination theorem of Bestvina and Feighn (see Subsection 6.2), and where the applications of Theorem A can be found, that is, where we prove Corollary B and Theorem C.

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2. Preliminaries

In this section, after setting the notation, we will recall the notions of hierarchically hyperbolic spaces and groups, and of morphisms - that is hieromorphisms - between these spaces, following [5].

Notation. For real-valued functions \( A \) and \( B \), we write \( A \simeq_{(K,C)} B \) if there exist constants \( C \) and \( K \) such that
\[
K^{-1}B(x) - C \leq A(x) \leq KB(x) + C
\]
for all \( x \) in the domain of the functions. With \( A = B \) we intend that there exist real numbers \( C \) and \( K \) such that \( A =_{(K,C)} B \).

Moreover, for real numbers \( a, b \) we define
\[
\{a \}_b := \begin{cases} a, & \text{if } a \geq b; \\ 0, & \text{if } a < b. \end{cases}
\]

Definition 2.1. A map \( \phi: (X, d_X) \to (Y, d_Y) \) between metric spaces is coarsely lipschitz if there exists constants \( K \geq 1 \) and \( C \geq 0 \) such that
\[
d_Y(\phi(x), \phi(y)) \leq Kd_X(x, y) + C, \quad \forall x, y \in X.
\]
In this case we call \( \phi \) a \((K,C)\)-coarsely lipschitz map.
The map $\phi$ is a quasi-isometric embedding if there exist constants $K \geq 1$ and $C \geq 0$ so that
$$K^{-1}d_{\mathcal{X}}(x,y) - C \leq d_{\mathcal{Y}}(\phi(x),\phi(y)) \leq Kd_{\mathcal{X}}(x,y) + C, \quad \forall x,y \in \mathcal{X}.$$  
In this case we call $\phi$ a $(K,C)$-quasi-isometric embedding. If, in addition, there exists a constant $M$ such that $\mathcal{Y} = \mathcal{N}_M(\phi(\mathcal{X}))$ then $\phi$ is a quasi-isometry. If $X$ is a connected subset of $\mathbb{R}$ then we call $\phi$ a $(K,C)$-quasiisometric.

**Definition 2.2 (Quasigeodesic metric space).** A metric space $(\mathcal{X},d_{\mathcal{X}})$ is $(K,C)$-quasigeodesic if there exist constants $K \geq 1$ and $C \geq 0$ such that $\mathcal{X} = \mathcal{N}_M(\phi(\mathcal{X}))$ then $\phi$ is a quasi-isometry. If $X$ is a connected subset of $\mathbb{R}$ then we call $\phi$ a $(K,C)$-quasiisometric.

2.1. Hierarchy hyperbolic spaces and groups. We start this subsection with the definition of hierarchy hyperbolic spaces.

**Definition 2.3 (Hierarchically hyperbolic spaces).** A $q$-quasigeodesic metric space $(\mathcal{X},d_{\mathcal{X}})$ is hierarchically hyperbolic if there exist $\delta \geq 0$, an index set $\mathcal{G}$, and a set $\{CW \mid W \in \mathcal{G}\}$ of $\delta$-hyperbolic spaces $(CU,d_{CU})$, such that the following conditions are satisfied:

1. **(Projections)** There is a set $\{\pi_W : \mathcal{X} \to 2^{CW} \mid W \in \mathcal{G}\}$ of projections that send points in $\mathcal{X}$ to sets of diameter bounded by some $\xi \geq 0$ in the hyperbolic spaces $CW \in \mathcal{G}$. Moreover, there exists $K$ so that all $W \in \mathcal{G}$, the coarse map $\pi_W$ is $(K,K)$-coarsely lipschitz and $\pi_W(\mathcal{X})$ is $K$-quasiconvex in $CW$.

2. **(Nesting)** The index set $\mathcal{G}$ is equipped with a partial order $\subseteq$ called nesting, and either $\mathcal{G}$ is empty or it contains a unique $\equiv$-maximal element. When $V \subseteq W$, $V$ is nested into $W$. For each $W \in \mathcal{G}$, $W \subseteq W$, and with $\mathcal{G}$ denote the set of all $V \in \mathcal{G}$ that are nested in $W$. For all $V,W \in \mathcal{G}$ such that $V \subseteq W$ there is a subset $\rho_W^V \subseteq CW$ with diameter at most $K$, and a map $\rho_W^V : CW \to 2^{CV}$.

3. **(Orthogonality)** The set $\mathcal{G}$ has a symmetric and antireflexive relation $\perp$ called orthogonality. Whenever $V' \subseteq W'$ and $V \perp W'$, then $V \perp W'$ as well. For each $Z \in \mathcal{G}$ and each $U \in \mathcal{G}$ for which $\{V \in \mathcal{G} \mid V \perp U \neq \emptyset\}$, there exists $X_V^Z \in \mathcal{G}_Z \setminus \{Z\}$ such that whenever $V \perp U$ and $V \subseteq Z$, then $V \subseteq X_V^Z \cup U$.

4. **(Transversality and Consistency)** If $V,W \in \mathcal{G}$ are not orthogonal and neither is nested into the other, then they are transverse: $V \perp W$. There exists $\kappa_0 \geq 0$ such that if $V \perp W$, then there are sets $\rho_{\perp W}^V \subseteq CW$ and $\rho_{\perp V}^W \subseteq CW$, each of diameter at most $\xi$, satisfying
$$\min\{d_W(\pi_W(x),\rho_{\perp W}^V),d_V(\pi_V(x),\rho_{\perp V}^W)\} \leq \kappa_0, \quad \forall x \in \mathcal{X}.$$  
Moreover, for $V \subseteq W$ and for all $x \in \mathcal{X}$ we have that
$$\min\{d_W(\pi_W(x),\rho_{\perp W}^V),\text{diam}_{CV}(\pi_W(x) \cup \rho_{\perp W}^V(\pi_W(x)))\} \leq \kappa_0.$$  
In the case of $V \subseteq W$, we have that $d_W(\rho_{\perp W}^V,\rho_{\perp V}^W) \leq \kappa_0$ whenever $U \in \mathcal{G}$ is such that either $W \subseteq U$, or $W \perp U$ and $U \not\subseteq V$.

5. **(Finite complexity)** There is a natural number $n \geq 0$, the complexity of $\mathcal{X}$ with respect to $\mathcal{G}$, such that any set of pairwise $\subseteq$-comparable elements of $\mathcal{G}$ has cardinality at most $n$.

6. **(Large links)** There exist $\lambda \geq 1$ and $\varepsilon \geq \max\{s,\kappa_0\}$ such that, given any $W \in \mathcal{G}$ and $x,x' \in \mathcal{X}$, there exists $\{T_i\}_{i=1,...,|W|} \subseteq \mathcal{G}_W$ such that for all $T \in \mathcal{G}_W \setminus \{W\}$ either $T \subseteq T_i$ for some $i$, or $d_T(\pi_T(x),\pi_T(x')) < \varepsilon$, where $N = \lambda d_W(\pi_W(x),\pi_W(x')) + \lambda$. Moreover, $d_W(\pi_W(x),\rho_{\perp W}^V) \leq N$ for all $i$.

7. **(Bounded geodesic image)** For all $W \in \mathcal{G}$, all $V \in \mathcal{G}_W \setminus \{W\}$ and all geodesics $\gamma$ of $CW$, either $\text{diam}_{CV}(\rho_{\perp V}^W(\gamma)) \leq \varepsilon$ or $\gamma \cap N_{E}(\rho_{\perp V}^W) \neq \emptyset$.

8. **(Partial realization)** There is a constant $\alpha$ satisfying: let $\{V_j\}$ be a family of pairwise orthogonal elements of $\mathcal{G}$, ad let $p_j \in \pi_{V_j}(\mathcal{X}) \subseteq CV_j$. Then there exists $x \in \mathcal{X}$ such that
- $d_{V_j}(\pi_{V_j}(x),p_j) \leq \alpha$ for all $j$.

1If $A \subseteq \mathcal{X}$, by $\pi_U(A)$ we mean $\bigcup_{a \in A} \pi_U(a)$.  


for each \( j \) and each \( V \in \mathcal{G} \) with \( V_j \subseteq V \), we have \( d_V(\pi_V(x), \rho_V^j) \leq \alpha \);

- if \( W \cap V_j \) for some \( j \), then \( d_W(\pi_W(x), \rho_W^j) \leq \alpha \).

(9) **(Uniqueness)** For each \( \kappa \geq 0 \) there exists \( \theta_\alpha = \theta_\alpha(\kappa) \) such that if \( x, y \in \mathcal{X} \) and \( d(x, y) \geq \theta_\alpha \), then there exists \( V \in \mathcal{G} \) such that \( d_V(x, y) \geq \kappa \).

The inequalities of the fourth axiom are called consistency inequalities.

Although most of the natural examples of hierarchically hyperbolic spaces are geodesic metric spaces, it is beneficial to work in the more general context of quasigeodesic metric spaces. This is because a quasiconvex subspace of a geodesic metric space might fail to be geodesic, but (hierarchically) quasiconvex subspaces of hierarchically hyperbolic spaces inherit naturally a hierarchically hyperbolic structure (compare Definition 2.6 and [5, Proposition 5.6]).

**Remark 2.4.** The element \( \text{cont}_U^\perp \) appearing in Axiom (3) of Definition 2.3 is called the orthogonal container (or the container of the orthogonal complement) of \( U \) in \( Z \). If \( Z \) is the \( \mathcal{G} \)-maximal element of \( \mathcal{G} \), then we might suppress it from the notation, write \( \text{cont}_U \) and call it higher container. If \( Z \) is not the \( \mathcal{G} \)-maximal, then we will talk about lower containers.

A hierarchically hyperbolic space has clean containers if \( U \perp \text{cont}_U^\perp \) for all \( U, Z \in \mathcal{G} \), as originally defined in [1, Definition 3.4].

For a hierarchically hyperbolic space \((\mathcal{X}, \mathcal{G})\) and a subset \( \mathcal{U} \subseteq \mathcal{G} \), we define

\[
\mathcal{U}^\perp := \{ V \in \mathcal{G} : V \perp U \text{ for every } U \in \mathcal{U} \}.
\]

**Remark 2.5.** By [5, Remark 1.3], the projections \( \pi_U \) of a hierarchically hyperbolic space \((\mathcal{X}, \mathcal{G})\) can always be assumed to be uniformly coarsely surjective. Without loss of generality, we will always assume this.

**Definition 2.6 (Hierarchical quasiconvexity).** Let \((\mathcal{X}, \mathcal{G})\) be a hierarchically hyperbolic space. A subspace \( \mathcal{Y} \subseteq \mathcal{X} \) is \( k \)-hierarchically quasiconvex for some function \( k: [0, +\infty) \rightarrow [0, +\infty) \), if:

1. for all \( U \in \mathcal{G} \) the image \( \pi_U(\mathcal{Y}) \) is a \( k(0) \)-quasiconvex subspace of the hyperbolic space \( \mathcal{C}U \);
2. for all \( \kappa \geq 0 \), if \( x \in \mathcal{X} \) is such that \( d_U(\pi_U(x), \pi_U(\mathcal{Y})) \leq \kappa \) for all \( U \in \mathcal{G} \), then \( d_{\mathcal{X}}(x, \mathcal{Y}) \leq k(\kappa) \).

**Remark 2.7.** It is extremely important to stress that, in [5], a hieromorphism \( \phi: (\mathcal{X}, \mathcal{G}) \rightarrow (\mathcal{X}', \mathcal{G}') \) is called \( k \)-hierarchically quasiconvex if \( \phi(\mathcal{X}) \) is a \( k \)-hierarchically quasiconvex subspace of \( \mathcal{X}' \) - in the sense of Definition 2.6 - and \( \phi \) is already a quasi-isometric embedding (compare [5, Definition 8.1]).

In this work, by \( k \)-hierarchically quasiconvex hieromorphism we just mean a hieromorphism whose image is a \( k \)-hierarchically quasiconvex subspace.

In practice, this will not produce diverging notions of hierarchical quasiconvexity: in this paper, whenever we consider a hierarchically quasiconvex hieromorphism \( \phi \), this map \( \phi \) is always also assumed to be coarsely lipschitz, and full. By what we will prove in Theorem 3, these hypotheses imply that \( \phi \) is a quasi-isometric embedding. Therefore, a \( k \)-hierarchically quasiconvex hieromorphism in the sense of [5] is equivalent to a \( k \)-hierarchically quasiconvex full, coarsely lipschitz hieromorphism in the sense of this paper.

We elected to do this because, in previous ArXiv-versions of [5], the assumption for \( \phi \) to be a quasi-isometric embedding was not included in the notion of hierarchically quasiconvex hieromorphism, and because, doing so, we remember the reader that the hieromorphisms we consider are always assumed to be coarsely lipschitz (and equivalently quasi-isometric embeddings), something which is now hidden in [5].

**Remark 2.8.** As for quasiconvexity in the hyperbolic setting, there exist coarse projections onto hierarchically quasiconvex subspaces. If \( \mathcal{Y} \subseteq \mathcal{X} \) is a hierarchically quasiconvex subspace, then there exists a coarsely lipschitz map \( \pi_Y : \mathcal{X} \rightarrow \mathcal{Y} \), called gate map [5, Section 5], with the following property: \( \pi_Y(x) \in \mathcal{Y} \) is such that for all \( V \in \mathcal{G} \) the set \( \pi_V(\pi_Y(x)) \) coarsely coincides (with uniform constants) with the projection of the element \( \pi_V(x) \in \mathcal{C}V \) to the quasiconvex subspace \( \pi_V(\mathcal{Y}) \) of the hyperbolic space \( \mathcal{C}V \).

Important examples of hierarchically quasiconvex subspaces are standard product regions [5, Section 5]. To define them, we need the notion of consistent tuple [5, Definition 1.16].
Definition 2.9 (\(\kappa\)-consistent tuple). Fix \(\kappa \geq 0\), and consider a tuple \(\vec{b} = (b_U)_{U \in \mathcal{S}} \in \prod_{U \in \mathcal{S}} 2^{CU}\) such that for each coordinate \(U \in \mathcal{S}\) the coordinate \(b_U\) is a subset of \(CU\) with diameter bounded by \(\kappa\). The tuple \(\vec{b}\) is \(\kappa\)-consistent if whenever \(V \cap W\)

\[
\min\{d_W(b_W, \rho_W^V), d_V(b_V, \rho_V^W)\} \leq \kappa,
\]

and whenever \(V \subseteq W\)

\[
\min\{d_W(b_W, \rho_W^V), \text{diam}_{CW}(b_V \cup \rho_V^W(b_W))\} \leq \kappa.
\]

These inequalities generalize the consistency inequalities of the definition of hierarchically hyperbolic space.

Let \((\mathcal{X}, \mathcal{S})\) be a hierarchically hyperbolic space. For a given \(U \in \mathcal{S}\), let

\[
\mathcal{S}_U := \{V \in \mathcal{S} \mid V \subseteq U\}.
\]

Given \(\kappa \geq \kappa_0\), define \(F_U\) to be the set of \(\kappa\)-consistent tuples in \(\prod_{U \in \mathcal{S}_U} 2^{CU}\), and \(E_U\) to be the set of \(\kappa\)-consistent tuples in \(\prod_{V \in \mathcal{S}_U \setminus \{A\}} 2^{CU}\), where

\[
\mathcal{S}_U^A = \{V \in \mathcal{S} \mid V \subseteq U\} \cup \{A\}
\]

and \(A\) is a \(\subseteq\)-minimal element such that \(V \subseteq A\) for all \(V \subseteq A\).

These sets \(F_U\) and \(E_U\) can be canonically identified as subspaces of \(\mathcal{X}\). Indeed, by [3, Construction 5.10] there are coarsely well-defined maps \(\phi^\mathcal{S}_U : F_U \rightarrow \mathcal{X}\) and \(\phi^\mathcal{E}_U : E_U \rightarrow \mathcal{X}\) with hierarchically quasiconvex image, and by an abuse of notation we set that \(F_U = \text{im}\phi^\mathcal{S}_U\) and \(E_U = \text{im}\phi^\mathcal{E}_U\).

Then, if \(F_U\) and \(E_U\) are endowed with the subspace metric, the spaces \((F_U, \mathcal{S}_U)\) and \((E_U, \mathcal{S}_U^A)\) are hierarchically hyperbolic. The maps \(\phi^\mathcal{S}_U\) and \(\phi^\mathcal{E}_U\) extend to \(\phi_U : F_U \times E_U \rightarrow \mathcal{X}\). Call \(\phi_U = \text{im}\phi_U\) the standard product region in \(\mathcal{X}\) associated to \(U\) (compare [5, Definition 5.14]). This space is coarsely equal to \(F_U \times E_U\). We direct the interested reader to [3, Section 5] for more information on this.

2.2. Morphisms between hierarchically hyperbolic spaces, and groups. A hieromorphism is a morphism between hierarchically hyperbolic spaces that preserves the underlying structure. This statement is made precise by the following definition.

Definition 2.10 (Hieromorphism). Let \((\mathcal{X}, \mathcal{S})\) and \((\mathcal{X}', \mathcal{S}')\) be hierarchically hyperbolic spaces. A hieromorphism is a triple \(\phi = (\phi, \phi^\mathcal{S}, \phi^\mathcal{E}_{U \in \mathcal{S}})\), where \(\phi : \mathcal{X} \rightarrow \mathcal{X}'\) is a map, \(\phi^\mathcal{S} : \mathcal{S} \rightarrow \mathcal{S}'\) is an injective map that preserves nesting, transversality and orthogonality, and, for every \(U \in \mathcal{S}\), the maps \(\phi^\mathcal{S}_U : CU \rightarrow C\phi^\mathcal{S}(U)\) are quasi-isometric embeddings with uniform constants.

Moreover, the following two diagrams coarsely commute (again with uniform constants), for all non-orthogonal \(U, V \in \mathcal{S}\):

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi} & \mathcal{X}' \\
\pi_U \downarrow & & \downarrow \pi_{\phi^\mathcal{E}(U)} \\
CU & \xrightarrow{\phi_U} & C\phi^\mathcal{S}(U)
\end{array}
\quad
\begin{array}{ccc}
CU & \xrightarrow{\phi^\mathcal{E}_U} & C\phi^\mathcal{S}(U) \\
\rho_U \downarrow & & \downarrow \rho_{\phi^\mathcal{E}(U)} \\
CV & \xrightarrow{\phi^\mathcal{E}_V} & C\phi^\mathcal{S}(V)
\end{array}
\]

As hieromorphisms \(\phi : (\mathcal{X}, \mathcal{S}) \rightarrow (\mathcal{X}', \mathcal{S}')\) between hierarchically hyperbolic spaces induce injective maps \(\phi^\mathcal{S} : \mathcal{S} \rightarrow \mathcal{S}'\) at the level of indexing sets, with a slight abuse of notation one can think of \(\mathcal{S}\) as a subset of \(\mathcal{S}'\).

We will need the following strengthening of the notion of hieromorphism.

Definition 2.11. A hieromorphism \(\phi : (\mathcal{X}, \mathcal{S}) \rightarrow (\mathcal{X}', \mathcal{S}')\) is full if:

1. there exists \(\xi\) such that the maps \(\phi^\mathcal{S}_U : CU \rightarrow C\phi^\mathcal{S}(U)\) are \((\xi, \xi)\)-quasi-isometries, for all \(U \in \mathcal{S}\);
2. if \(S\) denotes the \(\subseteq\)-maximal element of \(\mathcal{S}\), then for all \(U' \in \mathcal{S}'\) nested into \(\phi^\mathcal{S}(S)\) there exists \(U \in \mathcal{S}\) such that \(U' = \phi^\mathcal{S}(U)\).
Such hieromorphism is called full because its image coincides with (and it is not only contained in) the sublattice of $\mathcal{S}'$ consisting of all $U'$ nested into $\phi^\circ(S)$.

Finally, we say that a hieromorphism $\phi: (X, \mathcal{S}) \to (X', \mathcal{S}')$ is $k$-hierarchically quasiconvex if $\phi(X)$ is a $k$-hierarchically quasiconvex subspace of $X'$, for some function $k: [0, +\infty) \to [0, +\infty)$.

An automorphism of a hierarchically hyperbolic space $(X, \mathcal{S})$ is a hieromorphism $\phi: (X, \mathcal{S}) \to (X, \mathcal{S})$ such that $\phi^\circ$ is bijective and each $\phi^\circ_{U'}$ is an isometry. The group of automorphisms of $(X, \mathcal{S})$ is denoted by $\text{Aut}(\mathcal{S})$.

**Definition 2.12 (Hierarchically hyperbolic group).** A finitely generated group $G$ is hierarchically hyperbolic if there exists an action $G \to \text{Aut}(\mathcal{S})$ on a hierarchically hyperbolic space $(X, \mathcal{S})$ such that the action of $G$ on $X$ is metrically proper, cobounded, and such that the induced action on $\mathcal{S}$ is cofinite. This, in particular, means that the metric space $(G, d)$, where $d$ is the word metric associated to any finite generating set of $G$, is hierarchically hyperbolic with respect to $\mathcal{S}$.

For each $g \in G$, we denote its image in $\text{Aut}(\mathcal{S})$ by $(f_g, f^g, \{f^g_{U'}\}_{U' \in \mathcal{S}})$.

Let $(G, \mathcal{S})$ and $(G', \mathcal{S}')$ be hierarchically hyperbolic groups. A hieromorphism $\phi: (G, \mathcal{S}) \to (G', \mathcal{S}')$ is a homomorphism of hierarchically hyperbolic groups if it is also a group-homomorphism $\phi: G \to G'$ that is $\phi$-equivariant, that is, for all $g \in G$ and $U \in \mathcal{S}$ we have that

$$\phi^\circ (f^g_{\phi(g)}(U)) = f^g_{\phi(g)}(\phi^\circ(U))$$

and the following diagram uniformly coarsely commutes:

$$
\begin{array}{ccc}
CU & \xrightarrow{\phi^\circ} & C\phi^\circ(U) \\
\downarrow f^g_{\phi(g)} & & \downarrow f^g_{\phi(g), \phi^\circ(U)} \\
Cf^g_{\phi(g)}(U) & \xleftarrow{\phi^\circ_{f^g_{\phi(g)}}} & C\phi^\circ(f^g_{\phi(g)}(U))
\end{array}
$$

As a particular example, we now describe the hierarchically hyperbolic structure of a direct product of two hierarchically hyperbolic groups. The hierarchical structure of the product of two hierarchically hyperbolic spaces would be completely similar.

**Example 2.13 (Direct product of hierarchically hyperbolic groups).** Let $(G_u, \mathcal{S}_u)$ and $(G_w, \mathcal{S}_w)$ be hierarchically hyperbolic groups. The direct product $G = G_u \times G_w$ is a hierarchically hyperbolic group [5 Proposition 8.25], and its hierarchical structure is described as follows.

The index set $\mathcal{S}$ for $G$ is defined to be the disjoint union of $\mathcal{S}_u$ with $\mathcal{S}_w$, inheriting the associated hyperbolic spaces, along with the following elements whose associated hyperbolic spaces are defined to be points. For each $U \in \mathcal{S}_u$ add an element $V_U$, into which every element of $\mathcal{S}_u$ orthogonal to $U$, and every element of $\mathcal{S}_w$, is nested. Analogously, for every $W \in \mathcal{S}_w$ include an element $V_W$ into which every element of $\mathcal{S}_w$ orthogonal to $W$, and every element of $\mathcal{S}_u$, is nested. Finally, include a $\subseteq$-maximal element $S$ into which each of the previous elements is nested.

Nesting, orthogonality, and transversality agree with the ones of $(G_u, \mathcal{S}_u)$ and $(G_w, \mathcal{S}_w)$ on the subsets $\mathcal{S}_u$ and $\mathcal{S}_w$ of $\mathcal{S}$, and any element of $\mathcal{S}_u$ is orthogonal to any element of $\mathcal{S}_w$. For any $A, B \in \mathcal{S}_u \cup \mathcal{S}_w$ we impose that

$$
\begin{aligned}
&\begin{cases}
A \subseteq V_B, & \text{whenever } A \perp B;
A \perp V_B, & \text{whenever } A \sqsubset B;
A \ll V_B, & \text{otherwise;}
\end{cases}
&\begin{cases}
V_B \subseteq V_A, & \text{whenever } A \sqsubset B;
V_A \ll V_B, & \text{otherwise.}
\end{cases}
\end{aligned}
$$

In particular, $A \perp V_A$ for any element $A \in \mathcal{S}_u \cup \mathcal{S}_w$. 

\[\text{\hspace{10cm}}\]
Projections to the hyperbolic spaces are either defined to be trivial, for elements with trivial hyperbolic space, or defined as the compositions $\pi_U \circ p_u$ (respectively $\pi_W \circ p_w$) for every $U \in \mathcal{G}_u$ (respectively for every $W \in \mathcal{G}_w$), where $p_u : G \to G_u$ is the canonical projection on the first direct factor, and $\pi_U : G_u \to \mathcal{G}_U$ is the projection given in $(G_u, \mathcal{G}_u)$.

It follows that for every $U \in \mathcal{G}_u$ the set $\pi_U(G_u)$ is uniformly bounded, and analogously for every $W \in \mathcal{G}_w$ the set $\pi_W(G_u)$ is uniformly bounded. Moreover, the inclusions of the subgroups $G_u$ and $G_w$ into $G$ are full, hierarchically quasiconvex hieromorphisms that induce isometries at the level of hyperbolic spaces.

### 2.3. Trees of hierarchically hyperbolic spaces

**Definition 2.14.** Let $T = (V, E)$ be a tree. A tree of hierarchically hyperbolic spaces is a quadruple $\mathcal{T} = (T, \{\mathcal{X}_e\}_{e \in V}, \{\mathcal{X}_e\}_{e \in E}, \{\phi_{e_\pm} : \mathcal{X}_e \to \mathcal{X}_{e_\pm}\}_{e \in E})$ such that

1. $\{\mathcal{X}_e\}$ and $\{\mathcal{X}_e\}$ are families of uniformly hierarchically hyperbolic spaces with index sets $\mathcal{G}_e$ and $\mathcal{G}_e$ respectively;
2. all $\phi_{e_\pm} : (\mathcal{X}_e, \mathcal{G}_e) \to (\mathcal{X}_{e_\pm}, \mathcal{G}_{e_\pm})$ and $\phi_{e_-} : (\mathcal{X}_e, \mathcal{G}_e) \to (\mathcal{X}_{e_-, \mathcal{G}_e})$ are hieromorphisms with all constants bounded uniformly by some $\xi \geq 0$.

To a tree of hierarchically hyperbolic spaces $\mathcal{T}$ we can associate the metric space $\mathcal{X}(\mathcal{T}) := \bigsqcup_{e \in V}(\mathcal{X}_e, d)$ in the following way. If $x \in \mathcal{X}_e$, then add an edge between $\phi_{e_0}(x)$ and $\phi_{e_+}(x)$. Given $x, x' \in \mathcal{X}$ in the same vertex space $\mathcal{X}_e$, then define $d'(x, x')$ to be $d_{\mathcal{X}}(x, x')$. Given $x, x' \in \mathcal{X}$ joined by an edge, define $d'(x, x') = 1$. If $x_0, x_1, \ldots, x_m \in \mathcal{X}$ is a sequence with consecutive points either joined by an edge or in a common vertex space, then define

$$d'(x_0, x_m) = \sum_{i=1}^{m} d'(x_{i-1}, x_i).$$

Finally, given $x, x' \in \mathcal{X}$, define

$$d(x, x') = \inf\{d'(x, x') \mid x = x_0, \ldots, x_m = x' \text{ a sequence}\}.$$ 

Following [5] Section 8], for each edge $e$ and each $W_{e_-} \in \mathcal{G}_{e_-}$ and $W_{e_+} \in \mathcal{G}_{e_+}$, we write $W_{e_-} \sim_d W_{e_+}$ if there exists $W_e \in \mathcal{G}_e$ such that $\phi_{e_+}^\circ(W_e) = W_{e_+}$ and $\phi_{e_-}^\circ(W_e) = W_{e_-}$. Then, the transitive closure of $\sim_d$ defines an equivalence relation in $\bigsqcup_{e \in V}\mathcal{G}_e$, denoted by $\sim$.

The support of an $\sim$-equivalence class $[V]$ is

$$T_{[V]} := \{v \in T \mid \text{there exists } V_v \in \mathcal{G}_v \text{ such that } [V] = [V_v]\}.$$ 

By definition of the equivalence $\sim$, supports are trees.

**Definition 2.15 (Gate maps in trees of hierarchically hyperbolic spaces).** Let $\mathcal{T}$ be a tree of hierarchically hyperbolic spaces and assume that the image of the hieromorphism $\phi_v : (\mathcal{X}_e, \mathcal{G}_e) \to (\mathcal{X}_v, \mathcal{G}_v)$ is hierarchically quasiconvex (recall Definition 2.10) for every $e \in E$ and $v \in V$ connected to $e$. The gate maps $\mathfrak{g}_v : \mathcal{X} \to \mathcal{X}_v$ is defined as follows. Let $x \in \mathcal{X}$ be an arbitrary element. If $x \notin \mathcal{X}_v$, then define $\mathfrak{g}_v(x) := \emptyset$. If $x \notin \mathcal{X}_v$, then we define $\mathfrak{g}_v(x)$ inductively. Let $w$ be the vertex such that $x \in \mathcal{X}_w$, suppose that $d_T(v, w) = n \geq 1$, and that $\mathfrak{g}_v(w)$ is defined on all vertex spaces that are at distance strictly less than $n$ from $v$. Let $\gamma$ be the geodesic in $T$ connecting $w$ to $v$, let $e$ be its first edge, with $e^- = e^+$. It follows that $d_T(e^+, v) = n - 1$. Then

$$\mathfrak{g}_v(x) := \phi_{e_+} \circ \phi_{e_-}(\mathfrak{g}_{\phi_{e_-}(\mathcal{X}_e})(x)),$$

where $\phi_{e_-} : \mathcal{X}_e \to \mathcal{X}_e$ is a quasi-inverse of $\phi_{e_+} : \mathcal{X}_e \to \mathcal{X}_e$.

**Definition 2.16 (Comparison maps).** Let $\mathcal{T}$ be a tree of hierarchically hyperbolic groups, $[V]$ be an equivalence class, and let $u \neq v$ be two vertices in the support of $[V]$. The comparison map $\mathfrak{c} : \mathcal{C}V_u \to \mathcal{C}V_v$ between the hyperbolic spaces associated to the representatives $V_u$ and $V_v$ of the class $[V]$ is defined as follows.

Assume first that $u$ and $v$ are vertices connected by a single edge $e$ such that $u = e^-$ and $v = e^+$. Then, the comparison map is defined as

$$\mathfrak{c} := \phi_{e^+}^\circ \circ \phi_{e^-}^\circ : \mathcal{C}V_u \to \mathcal{C}V_v.$$
Where the maps $\phi^+_e : CV_e \to CV_{e^+}$ and $\phi^-_e : CV_e \to CV_{e^-}$ are the quasi-isometries induced by the hieromorphisms $\phi^+_e : \mathcal{X}_e \to \mathcal{X}_{e^+}$ and $\phi^-_e : \mathcal{X}_e \to \mathcal{X}_{e^-}$ respectively and $\phi^{-\epsilon}_e$ denotes a quasi inverse of $\phi^+_e$.

For the general case, let $\gamma$ be the geodesic in $T$ connecting $u$ to $v$, let $u_i$ be the $i$-th vertex of this geodesic (so that $u = u_0$ and $v = u_n$ for some natural number $n > 0$), and let $e^i$ be the edge connecting $u_{i-1}$ to $u_i$. For all $i = 1, \ldots, n$ consider the hieromorphisms $\phi^+_e : \mathcal{X}_e \to \mathcal{X}_{e^+}$ and $\phi^-_e : \mathcal{X}_e \to \mathcal{X}_{e^-}$, and the induced quasi-isometries $\phi^+_i : CV_{e^i} \to CV_{u_{i-1}}$ and $\phi^-_i : CV_{e^i} \to CV_{u_i}$ from the hyperbolic space associated to the representative of $[V]$ in $\mathcal{G}_{e^i}$ to the hyperbolic spaces associated to $V_{u_{i-1}}$ and $V_{u_i}$ respectively. Finally, let $\phi^{-\epsilon}_i : CV_{u_{i-1}} \to CV_{e^i}$ be a quasi-inverse of the map $\phi^+_i$, for all $i$.

Then, the comparison map $\epsilon$ is defined to be the composition of the previous quasi-isometries:

\[(3) \quad \epsilon := \phi^{-\epsilon}_{e_n} \circ \phi^{-\epsilon}_{e_{n-1}} \circ \cdots \circ \phi^{-\epsilon}_{e_1} \circ \phi^{-\epsilon}_{e_0} : CV_{u_0} \to CV_{u_n}.
\]

**Remark 2.17.** It is a fact [5, Lemma 8.18] that if the cardinality of supports is uniformly bounded, then comparison maps are $(\xi, \xi)$-quasi-isometries, for some uniform (not depending on the two vertices $u$ and $v$) constant $\xi \geq 1$.

**Remark 2.18.** If the edge hieromorphisms $\{\phi_{e^\pm}\}_{e \in E}$ of the tree of hierarchically hyperbolic spaces $T$ induce isometries at the level of hyperbolic spaces, then we can choose inverse isometries for the maps $\phi_{e^\pm}$. Therefore, from Equation (3) it follows that comparison maps in this particular case are isometries.

We record now the following lemma, which is implicitly used in [5]. Its proof follows by applying repeatedly the (coarsely commutative) second diagram of Equation (2).

**Lemma 2.19.** Let $T$ be a tree of hierarchically hyperbolic spaces, and let $[U], [V]$ be two equivalence classes such that either $[U] \neq [V]$ or $[U] \subseteq [V]$. If comparison maps are uniform quasi-isometries, then for all vertices $u, v \in T_{[U]} \cap T_{[V]}$ the set $\rho^{U \cap V}_{U \cap V} \rho^{U \cap V}_{V \cap V}$ is coarsely equal to $\rho^{U \cap V}_{U \cap V}$.

**Definition 2.20 (Graph of hierarchically hyperbolic groups).** Let $\Gamma = (V, E)$ be a finite simplicial graph. A graph of hierarchically hyperbolic groups is given by the quadruple $(\Gamma, \{G_e\}_{e \in V}, \{G_e\}_{e \in E}, \{\phi_{e^\pm} : G_e \to G_{e^\pm}\}_{e \in E})$, where vertex and edge groups are hierarchically hyperbolic groups, and the $\phi_{e^\pm}$ are homomorphisms of hierarchically hyperbolic groups.

Let $F_E$ be the free group freely generated by the set $E$. The group $G$ associated to the graph of hierarchically hyperbolic groups is the quotient of $(\ast \in E G_e) \ast F_E$ obtained by adding the relations

- $e = G 1$, if $e \in E$ belongs to a fixed spanning tree of $\Gamma$;
- $\phi_{e+}(g) = e \phi_{e^-}(g) e^{-1}$, for all $e \in E$ and $g \in G_e$.

As described in [5, Section 8.2], a graph of hierarchically hyperbolic groups acts on a tree of hierarchically hyperbolic spaces whose associated tree is the Bass-Serre tree $\Gamma$ of the graph $\Gamma$.

3. **Intersection property and concrete hierarchically hyperbolic spaces**

We now introduce a notion that will play a pivotal role in the proof of Theorem A.

**Definition 3.1 (Intersection property).** A hierarchically hyperbolic space $(\mathcal{X}, \mathcal{G})$ has the intersection property if the index set admits an operation $\wedge : (\mathcal{G} \cup \{\emptyset\}) \times (\mathcal{G} \cup \{\emptyset\}) \to \mathcal{G} \cup \{\emptyset\}$ satisfying the following properties for all $U, V, W \in \mathcal{G}$:

- $(\wedge_1)$ $V \wedge \emptyset = \emptyset \wedge V = \emptyset$;
- $(\wedge_2)$ $U \wedge V = V \wedge U$;
- $(\wedge_3)$ $(U \wedge V) \wedge W = U \wedge (V \wedge W)$;
- $(\wedge_4)$ $U \wedge V \subseteq U$ and $U \wedge V \subseteq V$ whenever $U \wedge V \in \mathcal{G}$;
- $(\wedge_5)$ if $W \subseteq U$ and $W \subseteq V$, then $W \subseteq U \wedge V$. 

We call \( U \land V \) the *wedge* between \( U \) and \( V \). Notice that \( U \land V \in \mathcal{S}_U \cap \mathcal{S}_V \) as soon as \( U \land V \neq \emptyset \), by property \((\land_4)\). Therefore, whenever \( U \perp V \) it follows that \( U \land V = \emptyset \), as the intersection \( \mathcal{S}_U \cap \mathcal{S}_V \) is empty. Moreover, it follows that \( U \land V = V \) if and only if \( V \subseteq U \), and that for all \( U, V \in \mathcal{S} \) the set \( \mathcal{S}_U \cap \mathcal{S}_V \) either is empty or has a unique maximal element \( U \land V \).

Hyperbolic groups, mapping class groups (intersection of subsurfaces), raags (intersection of parabolic subgroups), and cubulable groups \(^{[18]}\) (gate maps) do have the intersection property. We stress that, for mapping class groups, *disconnected* subsurfaces are allowed, and in fact the intersection property and clean containers are not satisfied when one only considers connected ones.

Let \((X, \mathcal{S})\) be a hierarchically hyperbolic space with the intersection property, let \( U, V \in \mathcal{S} \), and define

\[
U \lor V := \bigwedge \{ W \in \mathcal{S} \mid U \subseteq W, \ V \subseteq W \}.
\]

We call \( U \lor V \) the *join* between \( U \) and \( V \). The operations \( \land \) and \( \lor \) give to the set \( \mathcal{S} \) a lattice structure.

Notice that the set \( W = \{ W \in \mathcal{S} \mid U \subseteq W, \ V \subseteq W \} \) appearing in Equation \((4)\) is never empty, because at least the \( \preceq \)-maximal element of \( \mathcal{S} \) belongs to it. Even if \( W \) is infinite, finite complexity of the hierarchically hyperbolic space implies that there exists a natural number \( n \), not greater than the complexity of the hierarchically hyperbolic space, such that \( U \lor V = W_1 \land \cdots \land W_n \), where \( W_i \in W \) for all \( i \). By definition, \( U \lor V \) is the \( \preceq \)-minimal element of \( \mathcal{S} \) in which both \( U \) and \( V \) are nested.

In raags, the join of two parabolic subgroups is the subgroup they generate, and in mapping class groups the join of two subsurfaces is their union (which might be disconnected).

In the following lemma we prove that direct product of hierarchically hyperbolic spaces/groups with the intersection property continues to satisfy the intersection property. As a consequence of Theorem \( \mathcal{Q} \) the intersection property is preserved also by graph products, and in particular by free products, when in presence of *clean containers*.

The intersection property for free products of hierarchically hyperbolic groups is preserved also *without* assuming clean containers, by deducing it from \(^{[5]}\) Theorem 8.6], but we elected not to write down the details, as clean containers is such a natural hypothesis to make.

**Lemma 3.2.** The intersection property is preserved by direct products. *If a group is hyperbolic relative to a finite collection of hierarchically hyperbolic spaces (respectively: groups) with the intersection property, then it is a hierarchically hyperbolic space (respectively: group) with the intersection property.*

Proof. The statement for groups hyperbolic relative to a collection of hierarchically hyperbolic spaces with the intersection property follows from the proof of \(^{[5]}\) Theorem 9.1].

Given two hierarchically hyperbolic spaces \((X_1, \mathcal{S}_1)\) and \((X_2, \mathcal{S}_2)\) with the intersection property, we endow the space \( X_1 \times X_2 \) with the hierarchically hyperbolic structure \( \mathcal{S} \) described in Example \( \mathcal{Z} \) (for hierarchically hyperbolic groups).

Let \( \land_1 \) and \( \land_2 \) be the wedge maps on \((X_1, \mathcal{S}_1)\) and \((X_2, \mathcal{S}_2)\), respectively, and let us define \( \land : (\mathcal{S} \cup \{\emptyset\}) \times (\mathcal{S} \cup \{\emptyset\}) \to \mathcal{S} \cup \{\emptyset\} \). If \( U \in \mathcal{S}_1, W \in \mathcal{S}_2 \) then \( U \perp W \) and therefore \( U \land W = \emptyset \). On the other hand, \( \land \) coincides with \( \land_1 \) or \( \land_2 \) if both arguments belong to \( \mathcal{S}_1 \) or \( \mathcal{S}_2 \) respectively. If \( W \in \mathcal{S}_1 \cup \mathcal{S}_2 \), \( U \in \mathcal{S}_1 \cup \mathcal{S}_2 \), and \( U \perp W \), for \( U \in \mathcal{S}_1 \cup \mathcal{S}_2 \), is an element with trivial associated hyperbolic space, as described in Example \( \mathcal{Z} \) then we have the following exhaustive disjoint cases: either \( W \subseteq U \), or \( W \subseteq U \) and \( U \subseteq W \), or \( W \subseteq U \) and \( U \subseteq W \). In the first case \( W \subseteq V \), and therefore \( W \land V = W \). In the other two cases, it must be that \( U \) and \( W \) belong to the same index factor, say \( \mathcal{S}_1 \). Therefore \( W \land V = W \land_1 \cont_1 U \), where \( \cont_1 U \) is the orthogonal container of \( U \) in \( \mathcal{S}_1 \).

Finally, if \( S \) is the \( \preceq \)-maximal element then \( S \land U = U \) for every \( U \in \mathcal{S}_1 \cup \mathcal{S}_2 \). \( \square \)

**Lemma 3.3.** Let \( \phi : (X, \mathcal{S}) \to (X', \mathcal{S}') \) be a full homomorphism between hierarchically hyperbolic spaces with the intersection property, and let \( U, V \in \mathcal{S} \). Then

\[
\phi^\mathcal{S} (U \land V) = \phi^\mathcal{S} (U) \land \phi^\mathcal{S} (V), \quad \phi^\mathcal{S} (U \lor V) = \phi^\mathcal{S} (U) \lor \phi^\mathcal{S} (V).
\]
Proof. We prove the lemma for the wedge $U \wedge V$. The proof for $U \vee V$ follows the same strategy. Let $U \wedge V = A$, and $\phi^\circ(A) = A' \in \mathfrak{S}$. We need to show that $\phi^\circ(U) \wedge \phi^\circ(V) = A'$. As $\phi^\circ$ preserves nesting, we have that $A' \subseteq \phi^\circ(U) \wedge \phi^\circ(V)$. As $\phi$ is full and $\phi^\circ(U) \wedge \phi^\circ(V)$ is nested into both $\phi^\circ(U)$ and $\phi^\circ(V)$, there exists $B \in \mathfrak{S}$ such that $\phi^\circ(B) = \phi^\circ(U) \wedge \phi^\circ(V)$ and $B$ is nested into both $U$ and $V$.

By maximality of $U \wedge V$, we conclude that $B = U \wedge V$, and it follows that $\phi^\circ(U \wedge V) = \phi^\circ(B) = \phi^\circ(U) \wedge \phi^\circ(V)$.

The next lemma is an example of why clean containers is a very natural property, and should be assumed without any hesitation. In the mapping class group setting the lemma just proves that if two subsurfaces $U$ and $V$ are disjoint from $W$, then $W$ is also disjoint from the subsurface $U \vee V$.

Lemma 3.4. Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with the intersection property and clean containers. If $U \perp W$ and $V \perp W$, then $(U \vee V) \perp W$.

Proof. Both the elements $U$ and $V$ are nested into the orthogonal container $\text{cont}_U W$, and by definition of join, it follows that $U \vee V \subseteq \text{cont}_U W$ as well. By clean containers we have that $W \perp \text{cont}_U W$, and therefore $(U \vee V) \perp W$.

Notice that we need the clean containers hypothesis for the case $U \vee V = \text{cont}_U W$.

Lemma 3.5. Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with the intersection property and clean containers. For all $U, V \in \mathfrak{S}$ we have that $\text{cont}_U^V = U \wedge \text{cont}_U V$.

Proof. If $\text{cont}_U V = \emptyset$, then also $\text{cont}_U^V$ is empty, and the equality is trivially satisfied.

If $\text{cont}_U V$ is not empty, but $\text{cont}_U^V = \emptyset$, then there does not exist an element nested into both $U$ and into $\text{cont}_U V$. Indeed, assume that there exists $W \in \mathfrak{S}$ such that $W \subseteq U$ and $W \subseteq \text{cont}_U V$. Then, $W \subseteq \text{cont}_U^V$ by definition of orthogonal containers, contradicting the assumption that $\text{cont}_U^V$ is empty. Therefore, also in this case the equality is trivially satisfied.

Suppose now that both $\text{cont}_U V$ and $\text{cont}_U^V$ are non-empty. By definition, we have that $\text{cont}_U^V \subseteq U$. By clean containers $V \perp \text{cont}_U^V$, and thus $\text{cont}_U^V \subseteq \text{cont}_U V$. Therefore, $\text{cont}_U^V \subseteq U \wedge \text{cont}_U V$. On the other hand, as $V \perp \text{cont}_U V$ and $U \wedge \text{cont}_U V \subseteq U$, we conclude that $U \wedge \text{cont}_U V \subseteq \text{cont}_U^V$.

Definition 3.6 ($\varepsilon$-support). For $A \subseteq \mathcal{X}$ and a constant $\varepsilon > 0$, define the $\varepsilon$-support to be

$$\text{supp}_\varepsilon(A) := \{W \in \mathfrak{S} \mid \text{diam}_W(\pi_W(A)) > \varepsilon\}.$$ 

Notice that if $\text{supp}_\varepsilon(A) = \emptyset$, then $A \subseteq \mathcal{X}$ has uniformly bounded diameter: indeed, by the Uniqueness Axiom of Definition 2.8, it follows that $\text{diam}_\mathcal{X}(A) \leq \theta_\varepsilon(\varepsilon)$.

In the following lemma, we make use of a relevant feature of a given standard product region $\mathbf{P}_U$ associated to a given $U \in \mathfrak{S}$ as defined in Definition 2.3. For each $e \in \mathbf{E}_U$ we denote $\mathbf{F}_U \times \{e\}$ a parallel copy of $\mathbf{F}_U$ in $\mathcal{X}$. By construction of $\mathbf{P}_U$, there exists a constant $\alpha$ which depends only on $\mathcal{X}$ and $\mathfrak{S}$, such that for every $x \in \mathbf{P}_U$ we have that $d_V(\pi_V(x), \rho^U_V) \leq \alpha$ for all $U \in \mathfrak{S}$ satisfying either $U \cap V$ or $U \subseteq V$. Moreover, we can choose $\alpha$ so that, if $V \subseteq U$, then $\text{diam}_V(\pi_V(\mathbf{F}_U \times \{e\})) \leq \alpha$ (see [8, Definition 1.15] and [5, Section 5] for more information).

We recall that $\xi$ is the constant that uniformly bounds the sets $\rho^U_V$ for $U \cap V \subseteq \mathfrak{S}$ such that $U \cap V \subseteq \mathfrak{S}$.

Lemma 3.7. Let $\varepsilon > 3 \max\{\xi, \alpha\}$. If $W \in \text{supp}_\varepsilon(\mathbf{F}_U \times \{e\})$ then $W \subseteq U$, and therefore $\text{supp}_\varepsilon(\mathbf{F}_U \times \{e\}) \subseteq \mathbf{U}_U$.

Proof. If $U$ is either transversal to $V \in \mathfrak{S}$ or properly nested into $V$, then $d_V(\pi_V(x), \rho^U_V) \leq \alpha$ for every $x \in \mathbf{F}_U \times \{e\}$. As the diameter of the set $\rho^U_V$ is at most $\xi$, we obtain that

$$d_V(\pi_V(x), \pi_V(y)) \leq d_V(\pi_V(x), p(\pi_V(x))) + d_V(p(\pi_V(x)), p(\pi_V(y))) + d_V(p(\pi_V(y)), \pi_V(y)) \leq 2\alpha + \xi < \varepsilon,$$
where $p : CV \to \rho_U^\ell$ denotes the closest point projection. Therefore, we conclude that $V \not\subseteq \supp_\varepsilon(F_U \times \{e\})$. On the other hand, whenever $U \perp V$ we have that $\pi_U(F_U \times \{e\})$ is a set of diameter bounded by $\alpha$, and again $V \not\subseteq \supp_\varepsilon(F_U \times \{e\})$.

Therefore, by the choice of $\varepsilon$, we have that $\supp_\varepsilon(F_U \times \{e\}) \subseteq \mathcal{S}_U$. \hfill $\Box$

**Convention.** From now on, even if not explicitly stated, we assume that $\varepsilon > 3\max\{\xi, \alpha\}$.

**Remark 3.8.** For an element $U \in \mathcal{S}$, the set $\supp_\varepsilon(F_U \times \{e\})$ defined in Definition 3.6 is independent of the parallel copy of $F_U \times \{e\}$ that we consider, that is

$$\supp_\varepsilon(F_U \times \{e\}) = \supp_\varepsilon(F_U \times \{e'\})$$

for any two elements $e, e' \in \mathcal{E}_U$. Indeed, $\pi_W(F_U \times \{e\})$ uniformly coarsely coincides with $\rho_W^\ell$ when either $W \supseteq U$ or $W \cap U$, or its diameter is bounded by $\alpha$ if $W \perp U$. Therefore, for $\varepsilon > 3\max\{\xi, \alpha\}$, it follows that $W \in \supp_\varepsilon(F_U \times \{e\})$ if and only if $W \in \supp_\varepsilon(F_U \times \{e'\})$.

**Notation.** For every $\varepsilon > 3\max\{\xi, \alpha\}$ we denote by $\supp_\varepsilon(F_U)$ the set $\supp_\varepsilon(F_U \times \{e\})$ for any $e \in \mathcal{E}_U$.

**Lemma 3.9.** Let $\phi : (X, \mathcal{S}) \to (X', \mathcal{S}')$ be a full hieromorphism and let $\varepsilon > 0$. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon' \geq \varepsilon_0$

$$\phi(\supp_\varepsilon(X)) \subseteq \supp_\varepsilon(\phi(X)).$$

**Proof.** The hieromorphism $\phi$ is full, and the maps $\phi_\varepsilon^0 \circ \pi_U$ uniformly coarsely coincides with $\pi_U \circ \phi$ for all $U \in \mathcal{S}$ (here $U'$ denotes $\phi(U)$). Therefore, there exists $K > 0$ such that for all $x, y \in X$, for all $U \in \mathcal{S}$

$$K^{-1}d_U(\pi_U(x), \pi_U(y)) - K \leq d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))).$$

Let $\varepsilon_0 := K\varepsilon + K^2$. For $\varepsilon' \geq \varepsilon_0$, consider $W \in \supp_{\varepsilon'}(X)$: we prove that $\phi(\varepsilon)(W) \in \supp_{\varepsilon'}(\phi(X'))$. Indeed, let $x, y \in X$ be such that $d_W(\pi_W(x), \pi_W(y)) > \varepsilon'$. By Equation (5) and the definition of $\varepsilon_0$ we have that

$$d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))) > \varepsilon,$n

that is $W' = \phi(\varepsilon)(W) \in \supp_{\varepsilon'}(\phi(X'))$. \hfill $\Box$

**Definition 3.10 (Concreteness).** Let $(X, \mathcal{S})$ be a hierarchically hyperbolic space with the intersection property. We say that the hierarchically hyperbolic structure is $\varepsilon$-concrete if either the space $X$ is bounded, or the $\subseteq$-maximal element $S$ of $\mathcal{S}$ is equal to

$$\bigvee\{V \in \mathcal{S} \mid V \in \supp_{\varepsilon}(X)\}.\n$$

We say that the hierarchically hyperbolic space is concrete if it is $\varepsilon$-concrete for some $\varepsilon > 3\max\{\xi, \alpha\}$.

**Remark 3.11.** Given a hierarchically hyperbolic group $(X, \mathcal{S})$ with $\subseteq$-maximal element $S$, we have that $\supp_{\varepsilon}(F_S) \subseteq \supp_{\varepsilon}(X)$, because $F_S \subseteq X$.

Notice that the other inclusion is not guaranteed, in general. Nevertheless, if the hierarchical structure on $X$ is normalized [12, Definition 1.15], that is if the projections $\pi_U$ are uniformly coarsely surjective for all $U \in \mathcal{S}$, then it follows that $F_S = X$, and in particular that $\supp_{\varepsilon}(F_S) = \supp_{\varepsilon}(X)$. As specified in Remark 2.5 we are assuming this.

By [12, Proposition 1.16], any hierarchically hyperbolic space $(X, \mathcal{S})$ admits a normalized hierarchically hyperbolic structure $(X, \mathcal{S}')$ and a hieromorphism $\phi : (X, \mathcal{S}) \to (X, \mathcal{S}')$ where $\phi : X \to X$ is the identity and $\phi^0 : \mathcal{S} \to \mathcal{S}'$ is a bijection. Therefore, up to considering normalized hierarchically hyperbolic spaces, an unbounded hierarchically hyperbolic space $(X, \mathcal{S})$ is $\varepsilon$-concrete and its $\subseteq$-maximal element $S$ is equal to

$$\bigvee\{V \in \mathcal{S} \mid V \in \supp_{\varepsilon}(F_S)\}.\n$$

In Definition 3.10 we are not asking that the maximal element $S$ already belongs to $\supp_{\varepsilon}(X)$: for instance, this is not the case for direct products of hierarchically hyperbolic spaces and groups, where the hyperbolic space associated to this $\subseteq$-maximal element is bounded.

We are interested in concrete hierarchically hyperbolic spaces for the following proposition:
Proposition 3.12. Let \((\mathcal{X}, \mathcal{S})\) be an unbounded hierarchically hyperbolic space with the intersection property and let \(\varepsilon > 3 \max\{\xi, \alpha\}\). There exists \(\mathcal{S}_\varepsilon \subseteq \mathcal{S}\) such that \((\mathcal{X}, \mathcal{S}_\varepsilon)\) is an unbounded, \(\varepsilon\)-concrete hierarchically hyperbolic space with the intersection property.

Proof. Let \(S\) be the \(\subseteq\)-maximal element of \(\mathcal{S}\). If
\[
S = \bigvee \{ V \in \mathcal{S} \mid V \in \text{supp}_\varepsilon(\mathcal{X}) \},
\]
then \(\mathcal{S}_\varepsilon = \mathcal{S}\) and there is nothing to prove.

If the equality of Equation (6) is not satisfied, then \(\bigvee \{ V \in \mathcal{S} \mid V \in \text{supp}_\varepsilon(\mathcal{X}) \}\) is properly nested into the \(\subseteq\)-maximal element \(S\). Let \(S_{\varepsilon} := \bigvee \{ V \in \mathcal{S} \mid V \in \text{supp}_\varepsilon(\mathcal{X}) \}\) and \(\mathcal{S}_\varepsilon := \mathcal{S}_{s_{\varepsilon}}\).

We now claim that there exists \(C = C(\varepsilon)\) such that \(\mathcal{X} = \mathcal{N}_{C}(F_{S_{\varepsilon}})\). Let \(x \in \mathcal{X}\) and consider the tuple \(\tilde{c}\) defined as follows:

\[
c_V = \begin{cases} 
\pi_V(x), & \forall V \in \mathcal{S}_{\varepsilon}; \\
\pi_V(e), & \forall V \in \mathcal{S}_{\varepsilon}; \\
\rho_V^{s_{\varepsilon}}, & \forall V \supseteq \mathcal{S}_{\varepsilon} \text{ or } V \supseteq \mathcal{S}_{\varepsilon}.
\end{cases}
\]

where \(e \in \mathcal{E}_{S_{\varepsilon}}\) is a fixed, arbitrarily chosen element.

The tuple \(\tilde{c}\) is a \(\kappa\)-consistent tuple, where \(\kappa\) depends only on \(\varepsilon\) and the constants of the hierarchically hyperbolic space \((\mathcal{X}, \mathcal{S})\). By [5, Theorem 3.1], there exists \(z \in \mathcal{X}\) such that \(\pi_U(z) = \pi_U(\tilde{c})\) for every \(U \in \mathcal{S}\), and by Definition 2.9 the element \(z\) belongs to \(F_{S_{\varepsilon}} \times \{e\}\). Let \(s_0\) be the constant associated to the Distance Formula Theorem for the space \((\mathcal{X}, \mathcal{S})\), and consider \(s > \max\{\varepsilon, s_0\}\). There exist \(K, C > 0\) such that
\[
d(x, z) \leq K \sum_{U \in \mathcal{S}} \{d_U(\pi_U(x), \pi_U(z))\}_s + C.
\]

(7)
\[
= K \sum_{U \in \mathcal{S}} \{d_U(\pi_U(x), \pi_U(z))\}_s + C.
\]

Note that \(d_U(\pi_U(x), \pi_U(z)) \leq \varepsilon\) for every \(U \in \mathcal{S}\setminus \mathcal{S}_{\varepsilon}\). Since \(s > \varepsilon\), from Equation (7) we conclude that
\[
d(x, z) \leq C.
\]

To complete the proof, notice that \(F_{S_{\varepsilon}} \times \{e\}\) can be endowed with the hierarchical hyperbolic structure \(\mathcal{S}_{S_{\varepsilon}}\). Since \(\mathcal{X} = \mathcal{N}_{C}(F_{S_{\varepsilon}} \times \{e\})\), the space \((\mathcal{X}, \mathcal{S}_{\varepsilon})\) is hierarchically hyperbolic, being quasi isometric to \((F_{S_{\varepsilon}} \times \{e\}, \mathcal{S}_{\varepsilon})\), and it is concrete by construction.

The intersection property in \((\mathcal{X}, \mathcal{S}_{S_{\varepsilon}})\) follows from the intersection property in \((\mathcal{X}, \mathcal{S})\).

Concreteness will play an important role in Lemma 4.1 and Theorem 4.3 after the proof of Theorem 4.2.

4. General structure theorems for hierarchically hyperbolic spaces and groups

In this section we prove some general results for hierarchically hyperbolic spaces and for hieromorphisms, in particular we prove Theorem 4.1. All this machinery will be used in Section 4 to prove Theorem 4.4 and its corollaries.

The following lemma spells out a fact implicitly used in the proof of [5, Theorem 8.6].

Lemma 4.1. Let \(T\) be a tree of hierarchically hyperbolic spaces with full edge hieromorphisms. If \([U] \subseteq [V]\) then \(T_{[V]} \subseteq T_{[U]}\).

Proof. As \([U] \subseteq [V]\), there exist a vertex \(u \in T\) and representatives \(U_u, V_u \in \mathcal{S}_u\) of \([U]\) and \([V]\) respectively such that \(U_u \subseteq V_u\). Let \(v \in T_{[V]}\): we will prove that \(v \in T_{[U]}\).

Let \(\sigma\) be the geodesic connecting \(u\) to \(v\) in the tree \(T\), with consecutive edges \(e_1, \ldots, e_k\), so that \(e_1 = u\) and \(e_k = v\). Since \(u, v \in T_{[V]}\) and supports are connected, we conclude that \(e_i^{\pm} \in T_{[V]}\) for all \(i = 1, \ldots, k\). Therefore,
there exist representatives $V_{e_1}^-$ and $V_{e_1}^+$ of $[V]$ in each index set $\mathfrak{S}_{e_1}$, and there exist representatives $V_{e_1} \in \mathfrak{S}_{e_1}$ in each edge space on $\sigma$ such that $\phi_{e_1}^\circ (V_{e_1}) = V_{e_1}^\circ$.

Since $U_u \subseteq V_u = V_{e_1}^\circ$, by fullness of $\phi_{e_1}$ (compare Definition 2.11) we know that there exists some $U_{e_1} \in \mathfrak{S}_{e_1}$ such that $\phi_{e_1}^\circ (U_{e_1}) = U_u$ and $U_{e_1} \subseteq V_{e_1}$. Thus there exists a representative $U_{e_1}^\circ = \phi_{e_1}^\circ (U_{e_1})$ of $[U]$ in $\mathfrak{S}_{e_1}$.

As hieromorphisms respect nesting, we know that $U_{e_1}^\circ \subseteq V_{e_1}^\circ$. Applying the same argument to the other edges $e_i$ of $\sigma$, we conclude that there exists a representative $U_v$ of $[U]$ in $\mathfrak{S}_v$ such that $U_v \subseteq V_v$.

Therefore $T[V] \subseteq T[U]$. □

In general, the converse implication of Lemma 4.1 fails to be true. Nevertheless, in Subsection 5.1 we show that the tree $T$ of hierarchically hyperbolic spaces can always be enlarged in a way so that the converse implication holds in the bigger tree $\overline{T}$.

We now state a lemma that will be useful later.

**Lemma 4.2.** Given a full hieromorphism $\phi: (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$, there exist constants $K, \xi \geq 0$ and $s, s' > 0$ such that

$$
\sum_{U \in \mathfrak{S}} \{d_U(\pi_U(x), \pi_U(y))\}_s < K \sum_{U' \in \phi(\mathfrak{S})} \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_{s'} + C \quad \forall x, y \in \mathcal{X}.
$$

**Proof.** For $U \in \mathfrak{S}$, we denote $\phi^\circ(U)$ by $U'$. As the hieromorphism is full, there exists a uniform constant $\xi$ such that

$$(8) \quad d_U(\pi_U(x), \pi_U(y)) \leq \xi d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))) + \xi, \quad \forall U \in \mathfrak{S}, \forall x, y \in \mathcal{X}.$$  

Choose $s$ and $s'$ such that

$$s' := \frac{s - \xi}{\xi} > 1.$$  

Suppose that $s \leq d_U(\pi_U(x), \pi_U(y))$ for a given $U \in \mathfrak{S}$. Then, using Equation (8), we obtain that

$$(9) \quad 1 < s' \leq d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))) = \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_{s'}.$$

As $s \leq d_U(\pi_U(x), \pi_U(y))$ we have that $\{d_U(\pi_U(x), \pi_U(y))\}_s = d_U(\pi_U(x), \pi_U(y))$. It then follows that

$$(10) \quad \{d_U(\pi_U(x), \pi_U(y))\}_s = d_U(\pi_U(x), \pi_U(y)) \leq \xi d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))) + \xi 
\leq \xi \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_{s'} + \xi.$$

Therefore, using Equation (9) and Equation (11), we obtain

$$(11) \quad \{d_U(\pi_U(x), \pi_U(y))\}_s \leq \xi \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_{s'} + \xi
\leq \xi \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_{s'} + \xi \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_{s'}
= 2\xi \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_{s'}.$$  

On the other hand, if $s > d_U(\pi_U(x), \pi_U(y))$ then

$$(12) \quad \{d_U(\pi_U(x), \pi_U(y))\}_s = 0 \leq 2\xi \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_{s'},$$

so the inequality of Equation (11) is satisfied also in this case.
Concluding, we use Equation (11) and Equation (12) to obtain that
\[
\sum_{U \in \mathcal{G}} \{d_U(\pi_U(x), \pi_U(y))\}_s \leq \sum_{U \in \mathcal{G}} 2\xi \{d_U(\pi_U(\phi(x)), \pi_U(\phi(y)))\}_s
\]
\[
= 2\xi \sum_{U \in \mathcal{G}} \{d_U(\pi_U(\phi(x)), \pi_U(\phi(y)))\}_s
\]
\[
= 2\xi \sum_{U \in \mathcal{G}} \{d_U(\pi_U(\phi(x)), \pi_U(\phi(y)))\}_s,
\]
and therefore the lemma is satisfied with \( K = 2\xi \) and \( C = 0 \).

**Remark 4.3.** The argument of Lemma 4.2 can be used to show that there exist constants \( K, C \geq 0 \) and \( \delta, \delta' > 0 \) such that
\[
\sum_{U \in \mathcal{G}} \{d_U(\pi_U(\phi(x)), \pi_U(\phi(y)))\}_s \leq K \sum_{U \in \mathcal{G}} \{d_U(\pi_U(x), \pi_U(y))\}_s + C \quad \forall x, y \in \mathcal{X}.
\]

**Lemma 4.4.** Let \( \phi: (\mathcal{X}, \mathcal{G}) \to (\mathcal{X}', \mathcal{G}') \) be a full hieromorphism and \( S \) be the \( \subseteq \)-maximal element in \( \mathcal{G} \). If \( S' = \phi(S) \) and \( F_{S'} \times \{e\} \) is a parallel copy of \( F_{S'} \), then \( \pi_{V'}(F_{S'} \times \{e\}) \) is coarsely equal to \( \pi_{V'}(\phi(\mathcal{X})) \) for all \( V' \in \mathcal{G}'_{S'} \).

**Proof.** Let \( z \in F_{S'} \) and consider the tuple \( \bar{b} = (\pi_{V'}(z))_{V' \in \mathcal{G}'_{S'}} \). As \( z \in F_{S'} \), the tuple \( \bar{b} \) is \( \kappa \)-consistent. The hieromorphism \( \phi \) is full, therefore \( \mathcal{G}'_{S'} = \phi(\mathcal{G}) \) and
\[
(\pi_{V'}(z))_{V' \in \mathcal{G}'_{S'}} = (\pi_{V'}(\phi(z)))_{V' \in \mathcal{G}'_{S'}}.
\]
As the full hieromorphism \( \phi \) induces uniform quasi isometries \( \phi^\kappa: CV' \to CV \) at the level of hyperbolic spaces, we obtain a tuple \( \bar{a} = (a_{V'})_{V' \in \mathcal{G}} \), where \( a_{V'} := \phi^\kappa(\pi_{V'}(z)) \subseteq CV \).

The tuple \( \bar{a} \) is \( \kappa' \)-consistent, and therefore there exists \( z \in \mathcal{X} \) that realizes it, by [5, Theorem 3.1]. Exploiting the fact that the maps \( \phi^\kappa \circ \pi_{V} \) uniformly coarsely coincide with the \( \pi_{V'} \circ \phi \) (compare Definition 2.10 and in particular Equation (2)), we conclude that the element \( \phi(x) \) realizes the tuple \( \bar{b} \):
\[
(\pi_{V'}(z))_{V' \in \mathcal{G}'_{S'}} = (\pi_{V'}(\phi(x)))_{V' \in \mathcal{G}'_{S'}}.
\]
That is, there exists a constant \( T_1 \) depending only on the realization Theorem [5, Theorem 3.1] and the hieromorphism \( \phi \) such that \( d_{V'}(\pi_{V'}(z), \pi_{V'}(\phi(x))) \leq T_1 \) for every \( V' \in \mathcal{G}'_{S'} \).

Conversely, let \( \phi(x) \in \phi(\mathcal{X}) \) and consider the tuple \( \bar{c} \):
\[
c_{V'} = \begin{cases} 
\pi_{V'}(\phi(x)), & \forall V' \in \mathcal{G}_{S'}; \\
\pi_{V'}(e), & \forall V' \in \mathcal{G}_{S'}; \\
\rho_{V'}^{S'}, & \forall V' \cap S' \text{ or } V' \supseteq S'.
\end{cases}
\]
Since \( \bar{c} \) is a \( \kappa \)-consistent tuple, there exists \( z \in \mathcal{X} \) such that \( \pi_{V'}(z) = \pi_{V'}(\bar{c}) \), and \( z \) belongs to \( F_{S'} \times \{e\} \) by Definition 2.9. Therefore there exists \( T_2 \) such that \( d_{V'}(\pi_{V'}(z), \pi_{V'}(\phi(x))) \leq T_2 \) for every \( V' \in \mathcal{G}'_{S'} \).

**Proposition 4.5.** If \( \phi: (\mathcal{X}, \mathcal{G}) \to (\mathcal{X}', \mathcal{G}') \) is a full hieromorphism between hierarchically hyperbolic spaces, then the spaces \( \mathcal{X} \) and \( F_{S'} \) are quasi isometric, where \( S' \) is the image in \( \mathcal{G}' \) of the \( \subseteq \)-maximal element in \( \mathcal{G} \).

**Proof.** We define a map \( \psi: F_{S'} \to \mathcal{X} \) and we prove that it is a quasi isometry. Let \( z \in F_{S'} \), and consider the tuple \( \bar{b} = (\pi_{V'}(z))_{V' \in \mathcal{G}'_{S'}} \). As \( z \in F_{S'} \), the tuple \( \bar{b} \) is \( \kappa \)-consistent. The hieromorphism \( \phi \) is full, so that \( \mathcal{G}'_{S'} = \phi(\mathcal{G}) \) and
\[
(\pi_{V'}(z))_{V' \in \mathcal{G}'_{S'}} = (\pi_{V'}(z))_{V' \in \mathcal{G}'_{S'}}.
\]
As the full hieromorphism \( \phi \) induces uniform quasi isometries \( \phi^\kappa: CV' \to CV \) at the level of hyperbolic spaces, we obtain a tuple \( \bar{a} = (a_{V'})_{V' \in \mathcal{G}} \), where \( a_{V'} := \phi^\kappa(\pi_{V'}(z)) \subseteq CV \).
The tuple \( \vec{a} \) is \( \kappa' \)-consistent, and therefore there exists \( x \in \mathcal{X} \) that realizes it by [5, Theorem 3.1]. Exploiting the fact that the maps \( \phi^x \) uniformly coarsely commute with the projections \( \pi_V \) (compare Definition 2.10 and in particular Equation (2)), we conclude that the element \( \phi(x) \) realizes the tuple \( \vec{b} \):

\[
(\pi_V(z))_{V' \in \phi^x(\mathcal{E})} = (\pi_V(\phi(x)))_{V' \in \phi^x(\mathcal{E})}.
\]

Define \( \psi(z) := x \). The element \( x \) is not uniquely determined by the tuple \( \vec{b} \), but it is up to uniformly bounded error.

Let us prove that \( \psi \) is a quasi isometry. Indeed, let \( z_1, z_2 \in F_{S'} \). Using, in this order, the Distance Formula in \( \mathcal{X}' \), Remark 4.3 and the fact that \( \phi \) is a full hieromorphism combined with the Distance Formula in \( F_{S'} \), we have that

\[
d_{\mathcal{X}'}(\psi(z_1), \psi(z_2)) \leq K \sum_{U \in \mathcal{E}} \|d_U(\pi_U(\psi(z_1)), \pi_U(\psi(z_2)))\|_s + C
\]

\[
\leq K \left( K_1 \sum_{U' \in \phi^x(\mathcal{E})} \|d_{U'}(z_1, z_2)\|_s + C_1 \right) + C
\]

On the other hand, we have that

\[
d_{\mathcal{X}}(z_1, z_2) \leq K_3 \sum_{U' \in \phi^x(\mathcal{E})} \|d_{U'}(z_1, z_2)\|_{s'} + C_3
\]

\[
\leq K_3 \left( K_4 \sum_{U \in \mathcal{E}} \|d_U(\psi(z_1), \psi(z_2))\|_{s'} \right) + C_3
\]

\[
\leq K_3(K_5d_{\mathcal{X}}(\psi(z_1), \psi(z_2)) + C_5) + C_4 + C_3.
\]

Equation (15) and Equation (16) prove that \( \psi \) is a quasi-isometric embedding.

Moreover, the map \( \psi \) is coarsely surjective. Indeed, given an element \( x \in \mathcal{X} \), the tuple \( (\pi_{V'}(\phi(x)))_{V' \in \phi^x(\mathcal{E})} \) is consistent, and therefore there exists a point \( z \in F_{S'} \) coarsely realizing it, that is uniformly close to \( x \).

\[\square\]

**Example 4.6 (Hagen).** It very well may happen that a full hieromorphism between hierarchically hyperbolic spaces fails to be coarsely lipschitz.

We describe such a hieromorphism \( \phi: (\mathbb{R}, \{s\}) \to (X, \mathcal{E}) \) here, where \( X \) is the Cayley graph of the free group \( F_2 = \langle a, b \rangle \) with respect to the free generating set \( \{a, b\} \). The structure \( \mathcal{E} \) on \( X \) is given by the family \( \mathcal{E} \) of all axes of conjugates of \( a \) and of \( b \), and a \( \Sigma \)-maximal element \( M \):

\[\mathcal{E} := \bigcup_{g \in F_2} \text{Axis}(a^g) \cup \bigcup_{g \in F_2} \text{Axis}(b^g) \cup \{M\},\]

where the axis \( \text{Axis}(x) \) of an element \( x \) is defined to be the set of vertices of \( X \) with minimal displacement with respect to \( x \), that is \( \text{Axis}(x) := \{y \in F_2 \mid d_X(y, xy) \text{ is minimal}\} \).

In \( \mathcal{E} \) any two different axes are transversal, and everything is nested into \( M \). The hyperbolic spaces associated to the axes are their corresponding lines in \( X \), and \( CM \) is obtained from \( X \) by coning off all these axes.

The projections \( \pi_{\text{Axis}(x^g)}: F_2 \to 2^{\text{Axis}(x^g)} \) are given by closest-point projections, for all \( x = a, b \) and \( g \in F_2 \), as well as the \( \rho \) maps between two axes. The sets \( \rho_{M}^\text{Axis}(x^g) \) are the inclusion of the axis into the coned-off Cayley graph.

The map \( \phi \) is defined as follows. At the level of metric spaces, \( \phi \) maps \( \mathbb{R} \) homeomorphically to \( X \) in the following way. For \( n \in \mathbb{Z} \), the segment \( [n, n+1] \subset \mathbb{R} \) is mapped to the geodesic path that connects \( a^n b^n \) to \( a^{n+1} b^{n+1} \) in \( X \).

The map \( \phi^0: \{\mathbb{R}\} \to \mathcal{E} \) is defined as \( \phi^0(\mathbb{R}) = \text{Axis}(a) \), whilst the map \( \phi^0_x: \mathbb{R} \to \text{Axis}(a) \) is the isometry such that \( \phi^0_x(0) = e \) and \( \phi^0_x(1) = a \).
It can be checked that $\phi$ is a hieromorphism, and that it is full. Moreover, $\phi(\mathbb{R})$ is hierarchically quasiconvex inside $(X, \mathcal{S})$.

We now prove Theorem 5 from the Introduction.

**Theorem 5.** Let $\phi : (X, \mathcal{S}) \to (X', \mathcal{S}')$ be a full hieromorphism with hierarchically quasiconvex image, and let $S$ be the $\subseteq$-maximal element of $\mathcal{S}$. The following are equivalent:

1. $\phi$ is coarsely lipschitz;
2. $\phi$ is a quasi-isometric embedding;
3. the maps $g_{\phi(X)} : F_{\phi'(S)} \to \phi(X)$ and $g_{F_{\phi'(S)}} : \phi(X) \to F_{\phi'(S)}$ are quasi-inverses of each other, and in particular quasi isometries;
4. the subspace $\phi(X) \subseteq X'$, endowed with the subspace metric, admits a hierarchically hyperbolic structure obtained by from one of $X$ by composition with the map $\phi$ (and its induced maps at the level of hyperbolic spaces);
5. $\pi_W(\phi(X))$ is uniformly bounded for every $W \in \mathcal{S}' \setminus \phi'(\mathcal{S})$.

**Proof.** The implications $5 \iff 5 \iff 1 \iff 2 \iff 4 \iff 1$ and $2 \iff 3$ are enough to prove the theorem.

By the Distance Formula applied in $(X', \mathcal{S}')$, there exists $s_0$ such that for every $s > s_0$ there exists $K', C' \geq 0$ for which

$$d(\phi(x), \phi(y)) \leq K' \sum_{V \in \mathcal{S}'} \{d_U(\pi_U(\phi(x)), \pi_U(\phi(y)))\}_s + C' \quad \forall x, y \in X'.$$

Also, the Distance Formula applied in $(X, \mathcal{S})$ implies that there exists $s_1$ such that for every $s > s_1$ there exist $K, C \geq 0$ for which

$$d(x, y) \geq K^{-1} \sum_{U \in \mathcal{S}} \{d_U(\pi_U(x), \pi_U(y))\}_s - C \quad \forall x, y \in X.$$

Now let $x, y \in X$. By hypothesis $\pi_W(\phi(X))$ is uniformly bounded for every $W \in \mathcal{S}' \setminus \phi'(\mathcal{S})$. Let $M$ be this uniform bound, and choose $s$ such that $s > \max\{M, s_0\}$. Therefore

$$\sum_{V \in \mathcal{S}'} \{d_U(\pi_U(\phi(x)), \pi_U(\phi(y)))\}_s = \sum_{U' \in \phi'(\mathcal{S})} \{d_U(\pi_U(\phi(x)), \pi_U(\phi(y)))\}_s$$

and Equation (17) implies that

$$d(\phi(x), \phi(y)) \leq K' \sum_{U' \in \phi'(\mathcal{S})} \{d_U(\pi_U(\phi(x)), \pi_U(\phi(y)))\}_s + C'.$$

Using Remark 4.3 we can choose $\bar{s}, \bar{s}' > s_1$ and $\bar{K}, \bar{C} > 0$ for which

$$\sum_{U' \in \phi'(\mathcal{S})} \{d_U(\pi_U(x), \pi_U(y))\}_s \leq \bar{K} \sum_{U \in \mathcal{S}} \{d_U(\pi_U(x), \pi_U(y))\}_{s'} + \bar{C}.$$
As \( s' > s_1 \), by the Distance Formula, Equation (17) and Equation (18) we obtain
\[
d(\phi(x), \phi(y)) \leq K' \sum_{U \in \phi^c(\mathcal{G})} \{d_U(\pi_U(x), \pi_U(y))\}_s + C'
\]
\[
\leq K' K \sum_{U \in \mathcal{G}} \{d_U(\pi_U(x), \pi_U(y))\}_x + K'C + C'
\]
\[
\leq K' K (Kd(x, y) + KC) + K'C + C' = Rd(x, y) + R'
\]
for appropriate constants \( R \) and \( R' \). Therefore, \( \phi \) is a coarsely lipschitz map.

\[1 \Leftrightarrow 2\] If \( \phi \) is a quasi-isometric embedding, then it is a coarsely lipschitz map.

Suppose now that \( \phi \) is a coarsely lipschitz map. To conclude that it is a quasi-isometric embedding, we need to prove that there exist constants \( K, C \geq 0 \) such that \( d(x, y) \leq Kd(\phi(x), \phi(y)) + C \) for every \( x, y \in \mathcal{X} \).

By the Distance Formula applied in \((\mathcal{X}, \mathcal{G})\), there exists \( s_0 \) so that for every \( s \geq s_0 \) there exist \( K_1, C_1 \geq 0 \) so that
\[
d(x, y) \leq K_1 \sum_{U \in \mathcal{G}} \{d_U(\pi_U(x), \pi_U(y))\}_s + C_1, \quad \forall x, y \in \mathcal{X}.
\]
Also by the Distance Formula applied to \((\mathcal{X}, \mathcal{G'})\), there exists \( s_1 \) so that for every \( s \geq s_1 \) there exist \( K_2, C_2 \geq 0 \) so that
\[
d(\phi(x), \phi(y)) \geq K_2^{-1} \sum_{W \in \mathcal{G'}} \{d_W(\pi_W(\phi(x)), \pi_W(\phi(y)))\}_s = C_2, \quad \forall x, y \in \mathcal{X}.
\]
By Lemma 12 we can choose \( \bar{s}, \bar{s}' > s_1 \) and \( \bar{K}, \bar{C} \geq 0 \) such that
\[
\sum_{U \in \mathcal{G}} \{d_U(\pi_U(x), \pi_U(y))\}_s \leq \bar{K} \sum_{U \in \phi^c(\mathcal{G})} \{d_U(\pi_U(x), \pi_U(y))\}_x + \bar{C}
\]
\[
\leq \bar{K} \sum_{W \in \mathcal{G'}} \{d_W(\pi_W(\phi(x)), \pi_W(\phi(y)))\}_x + \bar{C}, \quad \forall x, y \in \mathcal{X}.
\]
Let \( s = \max\{s_0, \bar{s}\} \). Since \( s \geq s_0 \) and \( s \geq \bar{s} \), for any \( x, y \in \mathcal{X} \) we obtain that
\[
d(x, y) \leq K_1 \sum_{U \in \mathcal{G}} \{d_U(\pi_U(x), \pi_U(y))\}_s + C_1 \leq K_1 \sum_{U \in \mathcal{G}} \{d_U(\pi_U(x), \pi_U(y))\}_s + C_1
\]
\[
\leq K_1 \left( \bar{K} \sum_{W \in \mathcal{G'}} \{d_W(\pi_W(\phi(x)), \pi_W(\phi(y)))\}_x + \bar{C} \right) + C_1
\]
\[
\leq K_1 \bar{K} (Kd(\phi(x), \phi(y)) + KC) + K_1 C + C_1
\]
\[
= Sd(\phi(x), \phi(y)) + S'
\]
for appropriate constants \( S \) and \( S' \). Therefore, \( \phi \) is a quasi-isometric embedding.

\[2 \Rightarrow 4\] If the map \( \phi \) is a quasi-isometric embedding then (4) is automatically satisfied, because hierarchical hyperbolicity is preserved under quasi isometries [4].

\[4 \Rightarrow 1\] As the hieromorphism is full, every induced map \( \phi_U^\# : \mathcal{U} \to \mathcal{C}(\phi(\mathcal{U})) \) is a \((\xi, \xi)\)-quasi isometry, where \( \xi \) is independent of \( U \in \mathcal{G} \), that is
\[
\xi^{-1}d_U(\pi_U(x), \pi_U(y)) - \xi \leq d_{\phi(\mathcal{U})}(\phi_U^\#(\pi_U(x)), \phi_U^\#(\pi_U(y))) \leq \xi d_U(\pi_U(x), \pi_U(y)) + \xi
\]
for all \( U \in \mathcal{G} \) and for all \( x, y \in \mathcal{X} \).

By the Distance Formula applied in \((\mathcal{X}, \mathcal{G})\), there exists \( s_0 \) such that for every \( s \geq s_0 \) there exist \( K_1, C_1 \geq 0 \) satisfying
\[
d(x, y) \geq K_1^{-1} \sum_{U \in \mathcal{G}} \{d_U(\pi_U(x), \pi_U(y))\}_s - C_1, \quad \forall x, y \in \mathcal{X}.
\]
We apply now the Distance Formula to the hierarchically hyperbolic space \((\phi(X), \phi^\circ (S))\). Therefore, there exists \(s_1\) such that for every \(s \geq s_1\) there exist \(K_2, C_2 \geq 0\) satisfying
\[
(20) \quad d(\phi(x), \phi(y)) \leq K_2 \sum_{U' \in \phi^\circ (S)} \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_s + C_2, \quad \forall x, y \in X.
\]
By Remark \([4,3]\) we can choose \(s, s' > s_0\) and \(K, C \geq 0\) for which
\[
(21) \quad \sum_{U' \in \phi^\circ (S)} \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_s \leq K \sum_{U \in S} \{d_U(\pi_U(x), \pi_U(y))\}_{s'} + C, \quad \forall x, y \in X.
\]
For \(s = \max\{s_1, s\}\), combining Equation \((19)\), Equation \((20)\), and Equation \((21)\), we obtain that
\[
d(\phi(x), \phi(y)) \leq K_2 \sum_{U' \in \phi^\circ (S)} \{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}_s + C_2
\]
\[
\leq K_2 K \left( K \sum_{U \in S} \{d_U(\pi_U(x), \pi_U(y))\}_{s'} + C \right) + C_2
\]
\[
\leq K_2 K d(\phi(x), y) + K_1 C_1 + K_2 C + C_2 = T d(x, y) + T',
\]
for appropriate constants \(T\) and \(T'\). Therefore, \(\phi\) is a coarsely lipschitz map.

\([3 \Rightarrow 5]\) By hypothesis, \(g_{F_{s'}} : \phi(X) \rightarrow F_{s'}\) and \(g_{\phi(X)} : F_{s'} \rightarrow \phi(X)\) are quasi inverses of each other, and by construction of gate maps they are also coarsely lipschitz. Therefore \(F_{s'}\) and \(\phi(X)\) are quasi-isometric, where the quasi-isometry is given by \(g_{F_{s'}}\), and in particular there exists \(C > 0\) such that
\[
\phi(X) \subseteq \mathcal{N}_C(g_{\phi(X)}(F_{s'})).
\]
Let \(W \in S \setminus \phi^\circ (S)\). By the previous inclusion, there exists \(C' > 0\), depending on \(C\) and on \(\pi_W\), such that
\[
(22) \quad \pi_W(\phi(X)) \subseteq \mathcal{N}_{C'}(\pi_W(g_{\phi(X)}(F_{s'}))).
\]
Since the hieromorphism \(\phi\) is full, \(\phi^\circ (S) = S'\). Moreover, by construction of gate maps, the set \(\pi_W(g_{\phi(X)}(F_{s'}))\) is uniformly coarsely equal to \(p_{\pi_W(\phi(X))}(\pi_W(F_{s'}))\). Since \(W \in S \setminus S'\), we have that \(\text{diam}(\pi_W(F_{s'})) \leq \alpha\) by \([5,\text{Construction 5.10}]\) and, as a consequence, that there exists \(\alpha'\) such that \(\text{diam}(\pi_W(g_{\phi(X)}(F_{s'}))) \leq \alpha'\). The first condition of the theorem follows from this, and Equation \((22)\).

\([5 \Rightarrow 3]\) We claim that there exists \(M > 0\) such that
\[
d(g_{F_{s'}} \circ g_{\phi(X)}(z), z) \leq M, \quad d(g_{\phi(X)} \circ g_{F_{s'}})(y, y) \leq M, \quad \forall z \in F_{s'}, \forall y \in (\phi(X)).
\]
By applying the Distance Formula to the space \((X', S')\), there exists \(s_0\) such that for every \(s \geq s_0\) there exist \(K_1, C_1 > 0\) such that
\[
d(g_{F_{s'}} \circ g_{\phi(X)}(z), z) \leq K_1 \sum_{U' \in S'} \{d_{U'}(\pi_{U'}(g_{F_{s'}} \circ g_{\phi(X)}(z)), \pi_{U'}(z))\}_s + C_1, \quad \forall z \in F_{s'}.
\]
By Lemma \([3,7]\) \(\text{diam}(\pi_W(F_{s'})) \leq \epsilon\) for every \(W \in S \setminus S'\) for an appropriate \(\epsilon > 0\). For \(s \geq \max\{s_0, \epsilon\}\) and the previous equation, it follows that
\[
d(g_{F_{s'}} \circ g_{\phi(X)}(z), z) \leq K_1 \sum_{U' \in S'} \{d_{U'}(\pi_{U'}(g_{F_{s'}} \circ g_{\phi(X)}(z)), \pi_{U'}(z))\}_s + C_1, \quad \forall z \in F_{s'}.
\]
For \(z \in F_{s'}\), using the fact that \(g_{F_{s'}}(z) = z\), we obtain
\[
d_{U'}(\pi_{U'}(g_{F_{s'}} \circ g_{\phi(X)}(z)), \pi_{U'}(z)) = d_{U'}(\pi_{U'}(g_{F_{s'}} \circ g_{\phi(X)}(z)), \pi_{U'}(g_{F_{s'}}(z))) \leq \]
\[
\leq d_{U'}(p(\pi_{U'} \circ g_{\phi(X)}(z)), p(\pi_{U'}(z))) + 2k
\]
\[
\leq k' d_{U'}(\pi_{U'}(g_{\phi(X)}(z)), \pi_{U'}(z)) + c' + 2k,
\]
where \( p : \mathcal{C}U' \to \pi_U'(\mathcal{F}_S) \) is the closest-point projection to the quasiconvex subspace \( \pi_U(\mathcal{F}_S) \subseteq \mathcal{C}U' \), and \( k', c' \) denote the multiplicative and additive constants associated to the coarsely lipschitz map \( p \), and \( k \) denotes the Hausdorff distance between the (uniformly) coarsely equal sets \( \pi_U(\mathcal{F}_S) \) and \( p(\pi_W(x)) \), for every \( x \in \mathcal{X}' \).

By Lemma \( [4, \text{Lemma} 1.3] \) there exists a constant \( T_0 > 0 \) such that for every \( z \in \mathcal{F}_S \), there exists \( \phi(x) \in \phi(\mathcal{X}) \) for which \( d^U_0(\phi(x)), \pi_U(z) \leq T \) for every \( U' \in \mathcal{S}_S' \). Since \( \pi_U(\phi(\mathcal{X})) \) coarsely equals \( \pi_U(\phi(\mathcal{X}))(\pi_U(z)) \), we obtain that

\[
d_U(\pi_U(\phi(\mathcal{X})), \pi_U(z)) \leq T' \quad \forall U' \in \mathcal{S}_S'.
\]

By choosing an adequate \( s \) in Equation \( (25) \), we conclude that

\[
d(\mathcal{F}_S') \circ \phi(\mathcal{X})(z), z \leq \mathcal{C}_1.
\]

In order to show that \( d(\mathcal{F}_S') \circ \phi(\mathcal{X})(y), y \) is uniformly bounded for every \( y \in \phi(\mathcal{X}) \) let \( \mu > 0 \) denote the constant such that \( \text{diam}(\pi_U(\phi(\mathcal{X}))) \leq \mu \) for every \( W \in \mathcal{S}' \). By the Distance Formula there exists \( s_0 > 0 \) such that for all \( s \geq s_0 \) there exists \( \mathcal{K}_2, \mathcal{C}_2 \) such that

\[
d(\mathcal{F}_S') \circ \phi(\mathcal{X})(y), y \leq \mathcal{K}_2 \sum_{U' \in \mathcal{S}_S'} \|d_U(\pi_U(\phi(\mathcal{X})) \circ \mathcal{F}_S', (y)), \pi_U(y)\|_s + \mathcal{C}_2, \quad \forall y \in \phi(\mathcal{X}).
\]

Since \( \pi_U \circ \phi(\mathcal{X}) = \pi_U(\phi(\mathcal{X})) \circ \pi_U, \) it follows that \( \pi_U(\phi(\mathcal{X})) \circ \mathcal{F}_S' = \pi_U(\mathcal{F}_S') \) for every \( U' \subseteq \mathcal{S}' \). Moreover, if \( U' \subseteq S' \), it follows that \( \pi_U(\phi(\mathcal{X})) \circ \mathcal{F}_S' = \pi_U(\mathcal{F}_S') \) for every \( U' \subseteq \mathcal{S}' \). Therefore, we conclude that \( \pi_U(\phi(\mathcal{X})) \circ \mathcal{F}_S' = \pi_U(\mathcal{F}_S') \) for every \( U' \subseteq \mathcal{S}' \). Therefore, we can define the hierarchically hyperbolic space \( \phi(\mathcal{X}, \mathcal{S}'_S) \) is a hierarchically hyperbolic space.

\[
\phi(\mathcal{X}, \mathcal{S}'_S) \colon \phi(\mathcal{X}) \to \mathcal{C}' \text{ in this latter hierarchically hyperbolic space are defined to be } \pi_U \circ \phi^{-1}, \text{ where } \phi^{-1} \text{ is a fixed quasi inverse of } \phi : \mathcal{X} \to \phi(\mathcal{X}), \text{ and } \pi_U \text{ are the projections in the space } (\mathcal{X}, \mathcal{S}).
\]

Moreover, we can define the hierarchically hyperbolic space \( \phi(\mathcal{X}, \mathcal{S}') \). For \( V' \in \phi(\mathcal{X}), \) that is for \( V' = \phi(\mathcal{X}) \) with \( \mathcal{S}' \subseteq \mathcal{S}_S' \), the projections \( \mathcal{P}_{V'} : \phi(\mathcal{X}) \to \mathcal{C}' \) are defined to be \( \phi_{V'} \circ \pi_U \circ \phi^{-1}, \) where \( \phi^{-1} \) and \( \pi_U \) as before, and \( \phi_{V'} : \mathcal{C}V' \to \mathcal{C}' \) the (uniform) quasi isometries provided by the hierarchically hyperbolic space \( (\mathcal{X}, \mathcal{S}). \)

By Definition \( [2, \text{Lemma} 2.2] \) we have that \( \phi_{V'} \circ \pi_U = \pi_U \circ \phi, \) where \( \pi_U \) is the projection in the space \( \phi(\mathcal{X}, \mathcal{S}'). \) Therefore \( \mathcal{P}_{V'} = \pi_U \circ \phi \circ \phi^{-1}, \) which uniformly coarsely coincides with \( \pi_U, \) being \( \phi \) and \( \phi^{-1} \) quasi inverses of each other. Thus \( \phi(\mathcal{X}, \mathcal{S}') \) is a hierarchically hyperbolic space, where we can take the projections to be \( \pi_U \) for all \( V' \in \phi(\mathcal{X}), \) instead of \( \mathcal{P}_{V'} \).

From this point, the argument to prove that there exists \( M > 0 \) such that

\[
d(\mathcal{F}_S') \circ \phi(\mathcal{X})(z), z \leq M, \quad d(\mathcal{F}_S') \circ \phi(\mathcal{X})(y), y \leq M \quad \forall z \in \mathcal{F}_S', y \in \phi(\mathcal{X})
\]

is exactly the same as the one used in the previous implication \( 1 \Rightarrow 5 \), and it is omitted.

Theorem \( [3] \) has several consequences. We start with the following, in the form of a remark:

**Remark 4.7.** The combination theorem of Behrstock, Hagen, and Sisto \( [5, \text{Theorem} 8.6] \) holds without the first part of their fourth hypothesis, that is if \( \epsilon \) is an edge of \( T \) and \( S_{\epsilon} \) is the \( \subseteq \)-maximal element of \( \mathcal{S}_{\epsilon} \), then for all \( V \in \mathcal{S}_{\epsilon} \), the elements \( V \) and \( \phi_{\epsilon \theta}(S_{\epsilon}) \) are not orthogonal in \( \mathcal{S}_{\epsilon} \).

Indeed, this hypothesis is used (compare \( [5, \text{Definition} 8.23] \)) to define the uniformly bounded sets \( \mathcal{P}^W_{\mathcal{V}} \) when \( W \) and \( V \) are transversal equivalence classes whose supports do not intersect. By Theorem \( [3] \) instead of
defining
\[ \rho_{[V]}^{[W]} = c_V \circ \phi_{e^+}^\phi(S) \]
as done in [5, Definition 8.23], we can impose that
\[ \rho_{[V]}^{[W]} = c_V(\pi V_+^e(\phi(X,e))) \]
where \( e \) is the last edge in the geodesic connecting \( T[V] \) to \( T[W] \), with \( e^+ \in T[V] \), and \( c_V \) is the comparison map from \( CV_+^e \) to the favorite representative of \( [V] \). We will exploit this fact in the proof of Theorem 4.9 (compare Subsection 5.4 and Equation (42)). The proof of [5, Theorem 8.6], after this modification, is not altered.

Lemma 4.8. Let \( \phi: (X, \mathcal{S}) \rightarrow (X', \mathcal{S}') \) be a full, coarsely lipschitz hieromorphism between hierarchically hyperbolic spaces such that \( \phi(X) \) is hierarchically quasiconvex in \( X' \), and let \( S \) be the \( \subseteq \)-maximal element of \( \mathcal{S} \).

There exist \( \varepsilon \) and \( \varepsilon_0 \) such that for all \( \varepsilon' \geq \varepsilon_0 \), if \( (X, \mathcal{S}) \) is \( \varepsilon' \)-concrete, with the intersection property and clean containers, then for any element \( W \in \mathcal{S}' \) we have that \( W \perp \supp_{\varepsilon'}(\phi(X)) \) if and only if \( W \perp \phi_0(S) \).

Proof. Let \( \varepsilon > \max\{3\alpha, 3\xi, \mu\} \), where \( \mu \) is the uniform bound given by Theorem 12 on the diameters of \( \pi_1'(\phi(X)) \) for all \( U \in \mathcal{S}' \backslash \phi_0(\mathcal{S}) \), and \( \varepsilon_0 \) and \( \varepsilon' \) be as in Lemma 3.9. Suppose that \( W \perp \phi_0(S) \), so that \( W \perp \mathcal{S}'_{\phi_0(\mathcal{S})} \). By the choice of \( \varepsilon \) and by Theorem 12 we have that \( \supp_{\varepsilon'}(\phi(X)) \subseteq \mathcal{S}'_{\phi_0(\mathcal{S})} \), because the hieromorphism if full, coarsely lipschitz, and with hierarchically quasiconvex image. Thus \( W \perp \supp_{\varepsilon'}(\phi(X)) \).

Assume now that \( W \perp \supp_{\varepsilon'}(\phi(X)) \). As the hierarchically hyperbolic space \( (X, \mathcal{S}) \) is \( \varepsilon' \)-concrete, we have that \( S = \bigvee \supp_{\varepsilon'}(\phi(X)) \) and therefore
\[ \phi_0(S) = \phi_0(\bigvee \supp_{\varepsilon'}(\phi(X))) \]
The hieromorphism \( \phi \) is full and \( (X, \mathcal{S}) \) satisfies the intersection property, therefore by Lemma 5.3 and Equation (26) we obtain that
\[ \phi_0(S) = \bigvee \phi_0(\supp_{\varepsilon'}(\phi(X))) \]
and by Lemma 5.3 we have that
\[ \phi_0(\supp_{\varepsilon'}(\phi(X))) \subseteq \supp_{\varepsilon'}(\phi(X)) \]
Combining Equation (27) and Equation (28), we conclude that \( \phi_0(S) \subseteq \bigvee \supp_{\varepsilon'}(\phi(X)) \). As \( W \perp \supp_{\varepsilon'}(\phi(X)) \), by clean containers and Lemma 5.3 it follows that \( W \perp \bigvee \supp_{\varepsilon'}(\phi(X)) \). Therefore \( W \perp \phi_0(S) \).

Theorem 4.9. Let \( \phi: (X, \mathcal{S}) \rightarrow (X', \mathcal{S}') \) be a full, coarsely lipschitz hieromorphism with hierarchically quasiconvex image, and assume that \( X \) is unbounded and concrete. There exists a constant \( \eta \geq 0 \), depending only on the hierarchical structures and the hieromorphism \( \phi \), such that \( d_{X'}(F_{S'}, \phi(X)) \leq \eta \), where \( S' = \phi_0(S) \) and \( S \) is the \( \subseteq \)-maximal element of \( \mathcal{S} \).

Proof. Let \( \kappa_0 \) and \( E \) be the constants coming from the hierarchically hyperbolic space \( X' \), and let \( \mu \) be the uniform constant on the diameters of the sets \( \pi W(\phi(X)) \), for all \( W \in \mathcal{S}' \backslash \phi_0(\mathcal{S}) \), provided by Theorem 12.

Let \( V' \in \supp(\phi(X)) \), take \( \kappa \) such that
\[ \kappa > \max\{2\kappa_0, 2E, E + \mu\} \]
and consider \( x, y \in X \) for which
\[ d_{V'}(\pi V'(\phi(x)), \pi V'(\phi(y))) > 2\kappa. \]
Let \( W \in \mathcal{S}' \) be such that either \( V' \cap W \) or \( V' \subseteq W \). We claim that either
\[ d_W(\pi W(\phi(x)), \rho W(\phi(x))) \leq \kappa_0 \quad \text{or} \quad d_W(\pi W(\phi(y)), \rho W(\phi(y))) \leq \kappa_0. \]
Indeed, assume that Equation (30) is not satisfied and that \( W \cap V' \). By consistency we have that
\[ d_{V'}(\pi V'(\phi(x)), \rho V'(\phi(x))) \leq \kappa_0 \quad \text{and} \quad d_{V'}(\pi V'(\phi(y)), \rho V'(\phi(y))) \leq \kappa_0. \]
This leads to a contradiction with Equation \((29)\).

Assume now that \(V' \subseteq W\). Again by consistency, we have that

\[
\text{diam}_{V'}(\pi_{V'}(\phi(x)) \cup \rho_{V'}^W(\pi_W(\phi(x)))) \leq \kappa_0 \quad \text{and} \quad \text{diam}_{V'}(\pi_{V'}(\phi(y)) \cup \rho_{V'}^W(\pi_W(\phi(y)))) \leq \kappa_0.
\]

Let \(\sigma\) be the geodesic in \(CW\) with endpoints \(\pi_W(\phi(x)) \) and \(\pi_W(\phi(y))\). By the Bounded Geodesic Axiom there are two possibilities:

1. \(\text{diam}_{V'}(\rho_{V'}^W(\sigma)) \leq E\), or
2. \(\sigma \cap N_E(\rho_{V'}^W) \neq \emptyset\).

In the first case, applying the triangle inequality we conclude that

\[
d_{V'}(\pi_{V'}(\phi(x))), \pi_{V'}(\phi(y)) \leq \kappa_0 + E + \kappa_0 = 2\kappa_0 + E \leq 2\kappa,
\]

which contradicts Equation \((29)\).

For the second case, since \(W \in G(\phi \circ \mathcal{S})\) we know that \(\pi_W(\phi(\mathcal{X}))\) is bounded by the uniform constant \(\mu\). This means that \(d_W(\pi_W(\phi(x)), \pi_W(\phi(y))) \leq \mu\). Furthermore, since there exists \(z \in \sigma\) such that \(d_W(z, \rho_{V'}^W) \leq E\), we have that

\[
d_W(\pi_W(\phi(x)), \rho_{V'}^W) \leq E + \mu \quad \text{and} \quad d_W(\pi_W(\phi(x)), \rho_{V'}^W) \leq E + \mu.
\]

Using the triangle inequality, we contradict Equation \((29)\). Therefore, Equation \((30)\) follows.

We have shown that for every \(V' \in \text{supp}_{\mathcal{S}}(\phi(\mathcal{X}))\) and every \(W \in G(\mathcal{S})\) such that \(W \supseteq V' \) or \(W \cap V'\) we have that \(d_W(\pi_W(\phi(\mathcal{X})), \rho_{V'}^W) \leq \kappa_0\). For \(S' = \phi(\mathcal{S})\), let \(U \in G(\mathcal{S})\) be such that \(U \supseteq S' \) or \(U \cap S'\). By Lemma \(13\) there exists \(V' \in \text{supp}_{\mathcal{S}}(\phi(\mathcal{X}))\) for which \(U \not\subseteq V'\). Since \(U \not\subseteq V'\) and \(V' \subseteq S'\), it follows that \(U \not\subseteq V'\). Therefore, either \(U \supseteq V'\) or \(U \not\subseteq V'\), and by the above argument \(d_U(\pi_U(\phi(\mathcal{X})), \rho_{V'}^W) \leq \kappa_0\). Since \(d_U(\rho_{V'}^W, \rho_{V'}^W) \leq \kappa_0\), it follows that \(d_U(\rho_{V'}^W, \pi_U(\phi(\mathcal{X}))) \leq 2\kappa_0\).

We now claim that there exists some constant \(\nu'\) such that \(d(F_{S'}, \phi(\mathcal{X})) \leq \nu'\). Fix \(x_0 \in \mathcal{X}\), and let \(z \in F_{S'}\) be the realization point of the consistent tuple

\[
\begin{align*}
\pi_U(\phi(x_0)), & \quad \forall U \in \mathcal{S}_Z; \\
\pi_U(\phi(x_0)), & \quad \forall U \in \mathcal{S}'_Z; \\
\rho_{U}^{S'}, & \quad \forall U \supseteq S' \text{ or } U \not\subseteq S'.
\end{align*}
\]

By the above argument and the choice of the realization point \(z\), it follows that the distance \(d_U(\pi_U(z), \pi_U(\phi(\mathcal{X})))\) is uniformly bounded, for all \(U \in \mathcal{S}'\). Since \(\phi(\mathcal{X})\) is a hierarchical quasi-convex subspace of \(\mathcal{X}'\), there exists a constant \(\nu'\) depending only on the hierarchically hyperbolic structure of \((\mathcal{X}', \mathcal{S}')\) for which \(d_{\mathcal{X}'}(z, \phi(\mathcal{X})) \leq \nu'\).

From the previous theorem, we obtain the following lemma:

**Lemma 4.10.** Let \(\phi : (\mathcal{X}, \mathcal{S}) \rightarrow (\mathcal{X}', \mathcal{S}')\) be a full, coarsely Lipschitz hieromorphism, assume that \(\mathcal{X}\) is unbounded and concrete, and let \(S' = \phi(\mathcal{S})\), where \(\mathcal{S} \in \mathcal{S}\) is the \(\subseteq\)-maximal element. For all \(U \in \mathcal{S}'\) such that either \(S' \subseteq U\) or \(S' \not\subseteq U\), the sets \(\rho_{U}^{S'}\) and \(\pi_U\) are uniformly bounded.

**Proof.** For any \(U \in \mathcal{S}'\) such that either \(S' \subseteq U\) or \(S' \not\subseteq U\), we have that \(\pi_U(F_{S'}) = \rho_{U}^{S'}\) by [5] Construction 5.10]. Moreover, the distance \(d_U(\pi_U(F_{S'}), \phi(\mathcal{X}))\) is at most \(K_\mu + K\) by Theorem [13].

Since \(\text{diam}_{\mathcal{U}}(\pi_U(\phi(\mathcal{X}))) \leq \mu\) and \(\text{diam}_{\mathcal{U}}(\rho_{U}^{S'}) \leq \xi\), any pair of elements in the sets \(\rho_{U}^{S'} = \pi_U(\phi(\mathcal{X}))\) and \(\pi_U(F_{S'})\) is at uniform bounded distance from each other.

**Lemma 4.11.** Let \(\phi : (\mathcal{X}, \mathcal{S}) \rightarrow (\mathcal{X}', \mathcal{S}')\) be a full, coarsely Lipschitz hieromorphism, assume that \(\mathcal{X}\) is unbounded and concrete, and let \(S \in \mathcal{S}\) be the \(\subseteq\)-maximal element. There exists a constant \(J\), depending only on the hierarchically hyperbolic structure \((\mathcal{X}', \mathcal{S}')\) and hieromorphism constants, such that \(d_{\text{Haus}}(\phi(S), F_{\phi(\mathcal{S})}) \leq J\).
Proof. By the third hypothesis of Theorem 1 there exist \( \nu, \nu' \), depending only on the hierarchically hyperbolic structures, such that \( d(\phi(x), g_{\phi(X)} \circ g_{F_{\phi(S)}(\phi(x))}) \leq \nu \) and \( d(z, g_{F_{\phi(S)}} \circ g_{\phi(X)}(z)) \leq \nu' \), for all \( \phi(x) \in \phi(X) \) and for all \( z \in F_{\phi(S)} \).

By [7] Lemma 1.26, there exists a constant \( J \), depending on the hierarchically hyperbolic structure, such that

\[
d(\phi(X), F_{\phi(S)}) = \min(J, \bar{J}) d\left(g_{F_{\phi(S)}}(\phi(x)), g_{\phi(X)} \circ g_{F_{\phi(S)}}(\phi(x))\right)
\]

for all \( \phi(x) \in \phi(X) \). Furthermore, if \( \eta \) denotes the bound of Theorem 4.9 using the previous equation we obtain that

\[
d(\phi(x), g_{F_{\phi(S)}}(\phi(x))) \leq d(\phi(x), g_{\phi(X)} \circ g_{F_{\phi(S)}}(\phi(x))) + d(g_{\phi(X)} \circ g_{F_{\phi(S)}}(\phi(x)), g_{F_{\phi(S)}}(\phi(x))) \\
\leq \nu + Jd(\phi(X), F_{\phi(S)}) + J \leq \nu + J\eta + J = J
\]

for all \( \phi(x) \in \phi(X) \).

In an analogous manner, we obtain that \( d(z, g_{\phi(X)}(z)) \leq J \) for all \( z \in F_{\phi(S)} \). Thus, the bound on the Hausdorff distance is proved. \( \square \)

5. Combination Theorem

In this section we prove Theorem A of the Introduction:

**Theorem A.** Let \( T \) be a tree of hierarchically hyperbolic spaces. Suppose that:

1. each edge-hieromorphism is hierarchically quasiconvex, uniformly coarsely lipschitz and full;
2. comparison maps are uniform quasi isometries;
3. the hierarchically hyperbolic spaces of \( T \) have the intersection property and clean containers.

Then the metric space \( X(T) \) associated to \( T \) is a hierarchically hyperbolic space with clean containers and the intersection property.

We recall that \( X(T) \) denotes the metric space defined in Definition 2.13.

**Remark 5.1.** Comparison maps being uniform quasi isometries is, in particular, a consequence of uniformly bounded supports (see [5] Lemma 8.20), which is a hypothesis in the original Combination Theorem of Behrstock, Hagen, and Sisto [3] Theorem 8.6.

Hypothesis 2 of Theorem A cannot be further relaxed, or dropped. A counterexample of Theorem A without the second hypothesis is given by Baumslag-Solitar groups. Indeed, non-abelian Baumslag-Solitar groups are HNN extensions \( \mathbb{Z} \ast \mathbb{Z} \) and their Dehn function is not quadratic [13] Theorem B. Therefore, they are not hierarchically hyperbolic, because hierarchically hyperbolic groups have quadratic Dehn function [5] Corollary 7.5.

The second part of the fourth hypothesis of [5] Theorem 8.6

there exists \( K \geq 0 \) such that for all vertices \( v \) of \( T \) and edges \( e \) incident to \( v \), we have

\[
d_{\text{Haus}}(\phi_v(X_v), F_{\phi_v(S_v)}(\bullet)) \leq K, \text{ where } S_v \in \mathcal{S}_v \text{ is the unique maximal element and } \bullet \in E_{\phi_v(S_v)}
\]

is a consequence of Theorem 4.9 through Lemma 4.11 and therefore is automatically satisfied in our setting.

**Remark 5.2 (Comparison between index sets).** Before proving Theorem A and therefore before describing the index set \( \mathcal{S} \) for the tree of hierarchically hyperbolic spaces \( T \), compare Section 5.2, we wish to compare the index set that we are using with the index set of [5], for the familiar readers.

Although our approach might appear more complicated than the one of Behrstock, Hagen, and Sisto [5], we want to emphasize that the index set we consider and use is more natural than (and a generalization of) the one constructed in [5]. Indeed, in [5] Definition 8.11, the index set of Behrstock, Hagen, and Sisto is defined as the inductive closure of the set \( \mathcal{S}_0 = \{ T \} \cup \left( \bigsqcup_{v \in V} \mathcal{S}_v \right) / \sim \), with respect of adding orthogonal containers (compare the sets \( K_\alpha \) in that definition).
On the other hand, our index set is defined (compare Equation (32)) as
\[
\{T\} \cup \left( \bigsqcup_{v \in V} \mathcal{G}_v / \sim \right) \cup \{T[V] \mid [V] \in \left( \bigsqcup_{v \in V} \mathcal{G}_v \right) / \sim \},
\]
because, allowing infinitely-supported equivalence classes, we are forced to add their supports to our index set. This turns out to be a very reasonable addition because, among other things, it “fixes” orthogonal containers, which no longer need to be added manually.

Conversely, if the support trees were uniformly bounded (as assumed in [5]), then the support trees we are considering would correspond to the artificial containers defined in [5, Definition 8.11] (the ones with uniformly bounded attached hyperbolic spaces, and the others with trivial hyperbolic spaces), and therefore the two constructions would coincide.

Before going into the proof of Theorem A, we state an immediate consequence of it:

**Corollary 5.3.** Let \( M \) be a closed 3-manifold that does not have a Sol or Nil component in its prime decomposition. Then \( \pi_1(M) \) is a hierarchically hyperbolic space with the intersection property and clean containers.

**Proof.** The fact that \( \pi_1(M) \) is a hierarchically hyperbolic space is proved in [5, Theorem 10.1], and it has clean containers by [1, Proposition 3.5]. Hierarchical hyperbolicity is proved, for closed non-geometric irreducible 3-manifolds, by constructing a tree of hierarchically hyperbolic spaces where supports are uniformly bounded (therefore, by [5, Lemma 8.20], comparison maps are uniform quasi isometries), and where vertex spaces are direct products \( \mathbb{R}_v \times \Sigma_v \). Here, \( \mathbb{R}_v \) is a copy of the real line, and \( \Sigma_v \) is the universal cover of a hyperbolic surface with totally geodesic boundary, whose hierarchically hyperbolic structure originates from the fact that \( \Sigma_v \) is hyperbolic relative to its boundary components. Edge spaces are \( \mathbb{R}_v \times \partial_0 \Sigma_v \), where \( \partial_0 \Sigma_v \) is a particular boundary component of \( \Sigma_v \).

By Lemma 3.2, the spaces \( \Sigma_v, \mathbb{R}_v \times \partial_0 \Sigma_v \) and \( \mathbb{R}_v \times \Sigma_v \) are hierarchically hyperbolic spaces with the intersection property, and they all have clean containers by [1, Section 3]. Moreover, as seen in [5, Theorem 10.1], the hieromorphisms are coarsely lipschitz, full, and with hierarchically quasiconvex images. Therefore all the hypotheses of Theorem A are satisfied, thus proving that \( \pi_1(M) \) is a hierarchically hyperbolic space with the intersection property and clean containers.

Geometric irreducible 3-manifolds have fundamental groups that are quasi isometric to direct products of hyperbolic groups, and therefore they have the intersection property by Lemma 5.2. Finally, the fundamental group of any reducible 3-manifold is the free product of the fundamental groups of irreducible 3-manifolds, and therefore it has the intersection property by what just proved, and Lemma 5.2.

5.1. Trees with decorations. Recall that a tree of hierarchically hyperbolic spaces (as defined in Definition 4.11) is a tuple
\[
(31) \quad \mathcal{T} = \left( T, \left\{ (X_v, \mathcal{G}_v) \right\}_{v \in V}, \left\{ (X_e, \mathcal{G}_e) \right\}_{e \in E}, \left\{ \phi_{e^+} : (X_{e^+}, \mathcal{G}_{e^+}) \rightarrow (X_{e^-}, \mathcal{G}_{e^-}) \right\} \right).
\]
where \( T = (V, E) \) is a tree, \( \left\{ (X_v, \mathcal{G}_v) \right\}_{v \in V} \) and \( \left\{ (X_e, \mathcal{G}_e) \right\}_{e \in E} \) are families of uniformly hierarchically hyperbolic spaces, and \( \phi_{e^+} : (X_{e^+}, \mathcal{G}_{e^+}) \rightarrow (X_{e^-}, \mathcal{G}_{e^-}) \) and \( \phi_{e^-} : (X_{e^-}, \mathcal{G}_{e^-}) \rightarrow (X_{e^+}, \mathcal{G}_{e^+}) \) are hieromorphisms with constants all bounded uniformly.

On \( \bigsqcup_{v \in V} \mathcal{G}_v \) one defines the following equivalence class: given an edge \( e = \{v, w\} \in E \) and \( U \in \mathcal{G}_v \), impose \( \phi_e(U) \) to be equivalent to \( \phi_{e^-}(U) \), and take the transitive closure of this to obtain the desired equivalence relation. Given \( U \in \bigsqcup_{v \in V} \mathcal{G}_v \), its equivalence class is denoted by \([U]\).

In general, in a tree of hierarchically hyperbolic spaces \( T \) it might happen that two distinct equivalence classes \([U] \neq [V]\) are supported on exactly the same vertices of the tree \( T \), that is \( T[U] = T[V] \). This is not desirable, and in this subsection we describe a slight modification of the tree \( T \) (and therefore of the metric space \( \mathcal{X}(T) \) associated to it) that ensures that \([U] \neq [V]\) if and only if \( T[U] = T[V] \). We achieve this by attaching to each vertex \( v \) of \( T \) a tree of uniformly bounded diameter, and refer to these attached trees as "decorations."
A Refined Combination Theorem for Hierarchically Hyperbolic Groups

We denote the tree that is obtained with this process by $\widehat{T}$. As a consequence, the new support trees $\hat{T}_{[U]}$ will become larger than the original ones (i.e. $T_{[U]} \subseteq \hat{T}_{[U]}$ for each equivalence class $[U]$).

All the hypotheses of Theorem $A$ are preserved by adding these decorated trees (furthermore, the metric spaces associated to the two trees of hierarchically hyperbolic spaces are quasi-isometric), and therefore for the proof of the theorem we will assume without loss of generality that equivalence classes are discriminated by their supports.

We now describe how to decorate the tree $T$ of hierarchically hyperbolic spaces of Equation (51), to ensure that $[U] = [V]$ if and only if $T_{[U]} = T_{[V]}$.

For any vertex $v \in T$, let $S_v$ be the $\Xi$-maximal element in $G_v$, let $U$ be any $\Xi$-maximal element of $G_v\setminus\{S_v\}$ and let $F_{U} \times \{f\}$ be a parallel copy of the $F_{U}$ inside of $X_v$. For any such choice, we add a new vertex $\tilde{v} = v$ and a new edge $\tilde{e}$ connecting $v$ and $\tilde{v}$. The metric spaces $X_{\tilde{e}}$ and $X_{\tilde{v}}$ are defined to be $F_U \times \{f\}$, with the induced metric.

It follows from [3] Proposition 5.11 that $(X_{\tilde{e}}, G_U)$ and $(X_{\tilde{v}}, G_U)$ are hierarchically hyperbolic spaces, of complexity strictly lower than $(X_v, G_v)$. We refer to these index sets as $G_{U,\tilde{e}}$ and $G_{U,\tilde{v}}$ respectively, where the exponent is added to keep track of the choices of the $\Xi$-maximal element $U \in G_v \setminus \{S_v\}$, and of the parallel copy $F_{U} \times \{f\}$.

The hieromorphisms $\phi_{\tilde{e}}^+$ and $\phi_{\tilde{v}}^-$ are defined as follows. At the level of metric spaces, $\phi_{\tilde{e}}^+ : X_{\tilde{v}} \to X_{\tilde{e}}$ is the identity map and $\phi_{\tilde{e}}^- : X_{\tilde{e}} \to X_{\tilde{v}}$ is the subspace inclusion. The map $\phi_{\tilde{e}}^0 : G_{U,\tilde{e}} \to G_{U,\tilde{v}}$ is the identity of the set $G_{U}$, and $\phi_{\tilde{v}}^e : G_{U,\tilde{v}} \to G_{U}$ is the inclusion. At the level of hyperbolic spaces, the maps $\phi_{\tilde{e}}^+, \phi_{\tilde{v}}^- : CW \to CW$ are the identity for each $W \in G_{U,\tilde{v}}$. It is straightforward to check that the commutative diagrams of Definition 2.10 are satisfied. Furthermore, since $\phi_{\tilde{e}}^0$, $\phi_{\tilde{v}}^e$, and $\phi_{\tilde{v}}^- \phi_{\tilde{e}}^+$ are identity maps or inclusions, it follows that $\phi_{\tilde{e}}^+$ and $\phi_{\tilde{v}}^-$ are full hieromorphisms. Moreover, they are quasiconvex.

We repeat this process for any newly produced vertex, until the complexity of the resulting hierarchically hyperbolic spaces is one. In particular, given a new vertex $\tilde{v}$ with associated hierarchically hyperbolic space $(F_{U} \times \{f\}, G_U)$ not of complexity one, consider a $\Xi$-maximal element $V \in G_U \setminus \{U\}$. Consider moreover a parallel copy $F_V \times \{f'\}$ of $F_V$ in $F_{U} \times \{f\}$, and repeat the process to construct a new vertex with associated hierarchically hyperbolic space $(F_V \times \{f'\}, G_V)$. We stress that $F_V$ is defined in the hierarchically hyperbolic space $(F_{U} \times \{f\}, G_U)$, and not in the space $(X_v, G_v)$ for which $U \in \mathcal{S}_v$.

We denote by $\mathcal{X}(T)$ the tree of hierarchically hyperbolic spaces obtained from $T$ following this process. Notice that $\mathcal{X}(T)$ can be naturally seen as a subspace of $\mathcal{X}(\widehat{T})$, that is $\mathcal{X}(T) \subseteq \mathcal{X}(\widehat{T})$. Moreover, as the complexity of the hierarchically hyperbolic spaces of $T$ is uniformly bounded and each step of the described process reduces the complexity by one, there exists a uniform constant $C$ such that $\mathcal{N}_C(\mathcal{X}(T)) = \mathcal{X}(\widehat{T})$. In particular, the inclusion map $\iota : \mathcal{X}(T) \hookrightarrow \mathcal{X}(\widehat{T})$ is a quasi isometry, and therefore the two spaces $\mathcal{X}(T)$ and $\mathcal{X}(\widehat{T})$ are quasi-isometric.

In $\mathcal{X}(\widehat{T})$, we denote by $\sim$, the equivalence relation described in Subsection 2.3, by $[U]$, the equivalence class of $U \in \bigsqcup_{v \in \mathcal{V}} G_v$ with respect to $\sim$, and by $\hat{T}_{[U]}$, the support of $[U]$. Notice that $\hat{T}_{[U]} \cap T = T_{[U]}$ for all $U \in \bigsqcup_{v \in \mathcal{V}} G_v$, and that for all $V \in \bigsqcup_{v \in \mathcal{V}} G_v$ there exists $V \in \bigsqcup_{v \in \mathcal{V}} G_v$ such that $\hat{T} \sim V$.

Remark 5.4. In the context of hierarchically hyperbolic groups, decorating a tree $T$ amounts to the following. Let $v$ be a vertex in $T$ with associated group $G$, and consider the Bass-Serre tree of $G \ast_H H$, where $H$ is a hierarchically quasiconvex subgroup of $G$ of maximal, strictly smaller complexity, and the two edge-embeddings are given by the identity map $\text{id}_H : H \to H$ and by the inclusion $\iota : H \to G$. This Bass-Serre tree has one vertex $v_0$ with associated group $G$, and $[G : H]$ vertices $v_i$ whose associated groups are the $G$-cosets of the subgroup $H$, and edges $e_i$, connecting $v_0$ to $e_i$.

In the tree $T$, we replace the vertex $v$ by $v_0$, and we add new vertices $v_i$ and edges $e_i$ connecting $v_0$ to $v_i$. To these new vertices $v_0$ and $v_i$, we associate the groups given by the Bass-Serre tree of the splitting $G \ast_H H$.

For any new vertex $v_i$ added in such a way, we repeat the process unless the vertex group $H$ has complexity one.
Lemma 5.5. In the tree of hierarchically hyperbolic spaces $\hat{T}$ we have that $[U]_* = [V]_*$ if and only if $\hat{T}_{[U]} = \hat{T}_{[V]}$.

Proof. One implication is trivial. Assume now that $\hat{T}_{[U]} = \hat{T}_{[V]}$. If the complexity of the two equivalence classes $[V]_*$ and $[U]_*$ is different, then the decorations added to the tree $T$ are trees of different diameter, and therefore we cannot have that $\hat{T}_{[U]} = \hat{T}_{[V]}$. Thus, the equivalence classes have the same complexity, so neither cannot be properly nested into the other.

By construction, in the tree $\hat{T}$ there are vertices $\tilde{u}$ and $\tilde{v}$ such that $U$ and $V$ are $\subseteq$-maximal elements of $\mathcal{S}_{\tilde{u}}$ and $\mathcal{S}_{\tilde{v}}$, respectively. As $\hat{T}_{[U]} = \hat{T}_{[V]}$, the equivalence class $[U]_*$ must have a representative in $\mathcal{S}_{\tilde{u}}$, and $[V]_*$ must have a representative in $\mathcal{S}_{\tilde{v}}$. As neither equivalence class can be properly nested into the other, it must then be that $[U]_* = [V]_*$. □

If the tree $T$ satisfies the hypotheses of Theorem $\textbf{A}$ then also $\hat{T}$ does. We prove this in the following lemmas.

Lemma 5.6. In the tree of hierarchically hyperbolic spaces $\hat{T}$ the edge hieromorphisms are full, coarsely lipschitz, and hierarchically quasiconvex.

Proof. Let $e$ be an edge in $\hat{T}$. Two cases can occur: either $e$ is an edge already in the tree $T$, or it was added with the decoration of $T$.

If $e$ was already an edge in $T$, then the edge hieromorphisms are full, coarsely lipschitz, and hierarchically quasiconvex by the hypotheses of Theorem $\textbf{A}$. On the other hand, if $e$ is a new edge then the two maps $\phi_{e-}$ and $\phi_{e+}$ are full, hierarchically quasiconvex isometric embeddings (one is actually an isometry), by construction. □

Lemma 5.7. The hierarchically hyperbolic spaces of $\hat{T}$ have the intersection property and clean containers.

Proof. Let $\tilde{v}$ be a vertex of $\hat{T}$. If $\tilde{v} \in T$ then $\mathcal{S}_{\tilde{v}}$ has the intersection property and clean containers, by the hypotheses of Theorem $\textbf{A}$. If $\tilde{v} \in \hat{T} \setminus T$, then $\mathcal{S}_{\tilde{v}} = \mathcal{S}_U^{U,f}$ coincides with $\mathcal{S}_U$, for some $U \in \bigsqcup_{v \in V} \mathcal{S}_v$. Therefore, $\mathcal{S}_U$ has in intersection property. Let $v \in T$ be the vertex such that $U \in \mathcal{S}_v$.

Suppose that $\mathcal{S}_{\tilde{v}} = \mathcal{S}_U^{U,f} = \mathcal{S}_U$ does not have clean containers. Therefore, there exists $W \in \mathcal{S}_U \setminus \{U\}$ such that the set $\{Z \in \mathcal{S}_U \mid Z \not\perp W\}$ is not empty, and $W \not\perp \text{cont}_U W$. By Lemma 3.3 we know that $\text{cont}_U W = U \wedge \text{cont}_U W$, where $\text{cont}_U W$ is the orthogonal container of $W$ in $\mathcal{S}_U$. Moreover $W \not\perp \text{cont}_U W$ by clean containers in $\mathcal{S}_U$, and therefore we reach a contradiction, as $\text{cont}_U W \subseteq \text{cont}_U U$. Thus, $\mathcal{S}_U^{U,f}$ has clean containers.

The argument for edge spaces is similar. □

Lemma 5.8. Comparison maps in $\hat{T}$ are uniformly quasi-isometries.

Proof. Let $v, w$ be two vertices in $\hat{T}$ and let $[V]_*$ be an equivalence class supported on both vertices, with representatives $V_v$ and $V_w$ respectively. Consider the comparison map $\epsilon : CV_v \to CV_w$, as defined in Equation 3. If both vertices already belong to $T \subseteq \hat{T}$, then the map $\epsilon$ is a uniform quasi-isometry by the hypotheses of Theorem $\textbf{A}$.

If one vertex, say $w$, belongs in $\hat{T} \setminus T$, and $v \in T$, consider the geodesic $\sigma$ in $\hat{T}$ connecting $v$ to $w$. Let $v = v_0, \ldots, v_n = w$ be the vertices of $\sigma$, such that $v_i$ is joined by an edge to $v_{i+1}$ for all $i = 0, \ldots, n - 1$. Then, there exists a maximal index $i$, such that $v_i \in T$ and $v_{i+1} \in \hat{T} \setminus T$; let $V_*$ be the representative of $[V]$ in $\mathcal{S}_{v_{i+1}}$. From Equation 3 we see that $\epsilon$ is the composition of $\epsilon_1 : CV_v \to CV_{v_i}$ with $\epsilon_2 : CV_{v_i} \to CV_w$. As noticed in the previous case, the map $\epsilon_1$ is a uniform quasi-isometry. Moreover, by construction, the map $\epsilon_2$ is an isometry, and therefore $\epsilon$ is a uniform quasi-isometry, being the composition of these two maps.

The last case to consider is when both vertices belong to $\hat{T} \setminus T$. Depending on whether the geodesic $\sigma$ does not intersect $T$, or does intersect it, the map $\epsilon$ will be an isometry, or a composition of three maps, two of which isometries and the remaining a uniform quasi isometry.

Therefore, all comparison maps are uniform quasi isometries. □
In view of this, for the whole proof of Theorem A we assume without loss of generality that equivalence classes are differentiated by their supports already in the tree of hierarchically hyperbolic space $\mathcal{T}$, that is $[U] = [V]$ if and only if $T_{[U]} = T_{[V]}$.

On the other hand, for the proof of Corollary B that is the application of Theorem A to hierarchically hyperbolic groups, we will not decorate the tree $\mathcal{T}$. This is because, even if a hierarchically hyperbolic group $(G, \mathcal{G})$ acts on the index set $\mathcal{G}$, the set of product regions $\{F_U \times \{f\} \mid U \in \mathcal{G}, f \in E_U\}$ might not be $G$-invariant. Therefore, it might happen that the hierarchically hyperbolic space $(\mathcal{X}(\mathcal{T}), \hat{\mathcal{G}})$, where $\hat{\mathcal{G}}$ denotes the index set associated to the decorated tree $\mathcal{T}$, does not admit a non-trivial action of $G$ onto $\hat{\mathcal{G}}$. We refer to Section 6.1 for the complete treatment of this delicate point.

We now define the hierarchically hyperbolic structure on this tree of hierarchically hyperbolic spaces.

5.2. Index set, nesting, orthogonality, and transversality.

Remark 5.9 (Concreteness of the edge spaces). In the proof of Theorem A we will need to exploit concreteness of the edge spaces, which is not an hypothesis of the theorem. We now explain why we can suppose, without loss of generality, that all the hierarchically hyperbolic edge-spaces of $\mathcal{T}$ are $\varepsilon$-concrete.

Let $\varepsilon \geq 3 \max(\alpha, \xi)$ as in Lemma 5.7. If the edge spaces are not all $\varepsilon$-concrete, then we apply Proposition 3.12 to each edge space $\mathcal{G}_e$ of $\mathcal{T}$ to obtain a sub-index set $\mathcal{G}_{e,\varepsilon} \subseteq \mathcal{G}_e$ such that $(\mathcal{X}_e, \mathcal{G}_{e,\varepsilon})$ is $\varepsilon$-concrete. Notice that if $\mathcal{G}_e$ is already $\varepsilon$-concrete, then $\mathcal{G}_{e,\varepsilon} = \mathcal{G}_e$.

Similarly to what defined in Subsection 2.3, define $\sim \varepsilon$ to be the transitive closure of $\sim_{e,\varepsilon}$: for any edge $e$ and any $U \in \mathcal{G}_{e,\varepsilon}$, we have that $d_{e,\varepsilon}(U) \sim d_{e,\varepsilon}^{-1}(U)$.

Doing so (and not defining equivalence classes with respect to the equivalence class $\sim$ of Subsection 2.3) will be crucial to be able to apply Lemma 1.10 during the proof of Theorem A. Moreover, this does not affect the hypotheses of the theorem, that continue to be satisfied. Indeed, edge spaces continue to be uniformly quasiconvex in vertex spaces, with edge hieromorphisms being full and uniformly coarsely lipschitz. Comparison maps are not affected by this change (but there might be fewer of them, as we are considering possibly smaller edge-space index sets). Finally, the intersection property is preserved by Proposition 3.12 and clean containers are preserved by Lemma 5.8.

In view of Remark 5.9, from now on we assume without loss of generality that all edge spaces are $\varepsilon$-concrete for some appropriate $\varepsilon$, that is that the equivalence relations $\sim_{e,\varepsilon}$ and $\sim$ are the same.

We define the index set $\mathcal{G}$ associated to the tree of hierarchically hyperbolic spaces $\mathcal{T}$ as

$$\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \{\hat{T}\}.$$  

The set $\mathcal{G}_1$ is

$$\mathcal{G}_1 := \left( \bigcup_{v \in V} \mathcal{G}_v \right) / \sim,$$

as defined in Subsection 2.3.

Elements of $\mathcal{G}_2$ correspond to supports of elements in $\mathcal{G}_1$, up to equality of sets:

$$\mathcal{G}_2 := \{T_{[V]} \mid [V] \in \mathcal{G}_1\} / =.$$

We stress that all these elements are subtrees of the tree $\mathcal{T}$, the tree attached to the tree of hierarchically hyperbolic spaces $\mathcal{T}$. By the following lemma, the set $\mathcal{G}_2$ is closed under intersections.

Lemma 5.10. Suppose that $T_{[V]} \cap T_{[V]}$ is not empty. Then there exists $[A] \in \mathcal{G}_1$ for which $T_{[A]} = T_{[U]} \cap T_{[V]}$ and $[U], [V] \subseteq [A]$.

Proof. Let $V_v$ and $U_v$ be the representatives of $[V]$ and $[U]$ in the index set $\mathcal{G}_v$, for all $v \in T_{[U]} \cap T_{[V]}$.

For all $v \in T_{[U]} \cap T_{[V]}$, consider the set

$$\Lambda_v = \{W \in \mathcal{G}_v \mid V_v, U_v \subseteq W\},$$
which is non-empty since it contains the maximal element of $S$.

Since $V_e \vee W_e$ is, by definition, the $\sqsubseteq$-minimal element of $S_e$ containing both $V_e$ and $W_e$, it is the unique $\sqsubseteq$-minimal element of $A_e$, which we denote also by $A_e$. If $T_{[U]} \cap T_{[V]}$ consists of just one vertex $v$, then $[A] = [V_e \vee U_e]$ is the desired equivalence class: as $[V_e]$ and $[U_e]$ are nested into $[A]$, it follows that $T_{[A]} \subseteq T_{[V]} \cap T_{[U]}$.

Therefore $T_{[A]} = T_{[V]} \cap T_{[U]}$.

If $T_{[V]} \cap T_{[U]}$ has more than one vertex, analogously to what constructed in the index sets of the vertices, there is a unique $\sqsubseteq$-minimal element in the edge-index set $S_e$ that we denote by $A_e$, where $e$ is any edge that contains representatives of both $[U]$ and $[V]$.

Assume now that $v, w \in T_{[U]} \cap T_{[V]}$ and that there is an edge $e$ that connects these two vertices. Then $\phi_v^e(A_v) = A_v$ and $\phi_w^e(A_w) = A_w$. Therefore

$$\phi_v^e(A_v) = \phi_v^e(V_e \vee U_e) = \phi_v^e(V_e) \vee \phi_v^e(U_e) = V_e \vee U_e = A_v$$

by Lemma 3.3.

Thus $A_v \sim A_w$ for all $v, w \in T_{[U]} \cap T_{[V]}$, and we denote by $[A]$ the equivalence class of (any of the) $[A_e]$. By construction, $[A]$ has a representative where both $[V]$ and $[U]$ have, and hence $T_{[V]} \cap T_{[U]} \subseteq T_{[A]}$.

On the other hand we have that $[V]$ and $[U]$ are nested in $[V_e \vee U_e] = [A]$, and therefore $T_{[A]} \subseteq T_{[U]} \cap T_{[V]}$ by Lemma 4.1. Thus, the lemma is proved. \hfill \Box

**Corollary 5.11.** Let $[V], [W]$ be equivalence classes. Then, $[V] \subseteq [W]$ if and only if $T_{[W]} \subseteq T_{[V]}$.

*Proof.* If $[V] \subseteq [W]$ then $T_{[W]} \subseteq T_{[V]}$, by Lemma 4.1. On the other hand, if $T_{[W]} \subseteq T_{[V]}$ we can see that $T_{[W]} = T_{[W]} \cap T_{[V]}$. By Lemma 5.10 there exists $[A] \in S_1$ for which $T_{[A]} = T_{[W]} \cap T_{[V]}$ and $[V], [W] \subseteq [A]$. It follows that $T_{[W]} = T_{[A]}$, and therefore that $[W] = [A]$, because we are assuming that the tree $T$ is decorated (compare Lemma 5.5). Thus $[V] \subseteq [W]$. \hfill \Box

To define nesting, orthogonality, and transversality, we proceed as follow. The element $\hat{T}$ is the $\sqsubseteq$-maximal element.

Relations in $S_1$ are as in 5: two $\sim$-equivalence classes $[V]$ and $[W]$ are nested (respectively orthogonal), $[V] \subseteq [W]$ (respectively $[V] \perp [W]$), if there exist a vertex $v \in T$ and representatives $V_e, W_e \in S_e$ such that $[V] = [V_e]$, $[W] = [W_e]$ and $V_e \subseteq W_e$ (respectively $V_e \perp W_e$) in $S_e$. If $[V]$ and $[W]$ are not orthogonal and neither is nested into the other, then they are transversal: $[V] \perp [W]$.

Relations in $S_2$ are as follows. For two elements $T_{[V]}, T_{[U]} \in S_2$, if $T_{[V]}$ is contained as a set in $T_{[U]}$ then $T_{[V]} \subseteq T_{[U]}$, and vice versa. Otherwise they are transverse, $T_{[V]} \perp T_{[U]}$.

Relations between an equivalence class $[W]$ and an element $T_{[V]} \in S_2$ are as follows:

1. if $[W] \subseteq [V]$ we declare $[W] \perp T_{[V]}$;
2. if $[W] \perp [V]$ we declare $[W] \subseteq T_{[V]}$;
3. otherwise, we declare $[W] \perp T_{[V]}$.

Notice that $[W] \perp T_{[V]}$ if and only if $T_{[V]} \subseteq T_{[W]}$, by Corollary 5.11.

**5.3. Hyperbolic spaces associated to elements of the index set, and projections onto them.** Let $C\hat{T} = \hat{T}$, which is produced from the tree $T$ by coning-off each subtree $T_{[W]} \in S_2$.

**Remark 5.12.** As soon as there exists a vertex space $(X_e, S_e)$ and two orthogonal elements $U \perp V$ in $S_e$, then the decoration trick of Section 5.1 implies that all supports trees $T_{[W]} \in S_2$ are properly contained into the tree $T$. Indeed, if $T_{[W]} = T_{[V]}$ for some equivalence class, it must then be that $T_{[U]}$ and $T_{[V]}$ are properly nested into $T_{[W]}$, and thus $[W] \subseteq [U]$ and $[W] \subseteq [V]$ by Lemma 5.4. This contradicts the fact that $[U] \perp [V]$, and in particular that there is no equivalence class nested into both.

To each equivalence class $[V]$ we associate a favorite vertex $v \in T_{[V]}$ and the favorite representative $V_e \in S_e$, so that $[V] = [V_e]$. Then, define $C[V]$ to be $CV_e$. By assumption, there exists a uniform constant $\xi \geq 1$ such that for all vertices $w$ such that there exists $W \in S_w$ with $W \sim V_e$, the comparison map $c: V_e \to W$ is a $(\xi, \xi)$-quasi-isometry.
For $T[W] \in \mathcal{S}_2$, let $\mathcal{C}T[W] := \hat{T}[W]$ be the hyperbolic space obtained from the tree $T[W]$ by coning-off each subtree $T[V] \in \mathcal{S}_2$ properly contained in $T[W]$, that is $T[V] \subseteq T[W]$.

Define $\pi_{\hat{T}} : \mathcal{X} \to \hat{T}$ as follows: for $x \in \mathcal{X}$, define $\pi_{\hat{T}}(x) = v$. Notice that $\pi_{\hat{T}}$ is the composition of the projection $\mathcal{X} \to T$ of $\mathcal{X}$ on its Bass-Serre tree with the inclusion of the tree $T$ into $\hat{T}$. For all $T[W] \in \mathcal{S}_2$ the projection $\pi_{T[W]}$ is defined analogously: for $x \in \mathcal{X}$, consider the closest-point projection of the vertex $v$ onto the subtree $T[V]$ in the tree $T$. The image of this point under the inclusion map $T \hookrightarrow \hat{T}$ is $\pi_{T[W]}(x) \in \mathcal{C}T[W]$. These projection maps $\pi_{T[W]}$ and the projection map $\pi_{\hat{T}}$ are uniformly coarsely surjective, being surjective on the set of non-cone points.

Given $[V] \in \mathcal{S}$ with favorite representative $V_v \in \mathcal{S}_v$, we define $\pi_{[V]} : \mathcal{X} \to \mathcal{C}[V]$ as follows. If $\pi_{\hat{T}}(x) = v$ is a vertex in the support of $[V]$, then there exists a representative $V_v \in \mathcal{S}_v$ of the class $[V]$, and $\pi_{[V]}(x)$ is defined to be

$$\pi_{[V]}(x) := \phi_{c_+}(\chi_c) \subseteq \mathcal{C}_{\hat{T}} = \mathcal{C}[V],$$

where $\phi : \mathcal{C}_{\hat{T}} \to \mathcal{C}[V]$ is the comparison map.

If $\pi_{\hat{T}}(x) = v$ is not in the support of $[V]$, let $e$ be the last edge in the geodesic connecting $v$ to $T[V]$, so that $e^+ \in T[V]$. Define

$$\pi_{[V]}(x) := \phi_{e^+}(\chi_{e^+}) \subseteq \mathcal{C}_{\hat{T}} = \mathcal{C}[V],$$

where $\phi : \mathcal{C}_{\hat{T}} \to \mathcal{C}[V]$ is the comparison map.

**Lemma 5.13.** The projections defined in Equation (35) and Equation (36) are uniform coarsely lipschitz maps. Moreover, they are uniformly coarsely surjective.

**Proof.** In Equation (35) the projections are defined as a composition of a uniform quasi isometry with a uniform coarsely lipschitz map. Therefore, it suffices to show that the projections in Equation (36) are uniformly coarsely lipschitz too.

To prove so, notice that the edge $e$ connects the vertex $e^-$, which lies outside of $T[V]$, with the vertex $e^+ \in T[V]$, and notice that there exists a representative $V_{e^+} \in \mathcal{S}_{e^+}$ of $[V]$. This means that $V_{e^+} \neq U$ for any $U \in \mathcal{S}_{e^-}$, that is $V_{e^+} \in \mathcal{S}_{e^+} - \phi_{e^+}(\mathcal{X}_e)$.

As all hieromorphisms are full and coarsely lipschitz, invoking Theorem 13 we know that the set $\pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e))$ are uniformly bounded. Therefore the projections as defined in Equation (36) are uniformly coarsely lipschitz, because the comparison maps $\phi$ are uniform quasi-isometries and the sets on which they are applied to are uniformly bounded.

These projections are uniformly coarsely surjective, because the projections of the vertex spaces are, following the assumption of Remark 25.

---

### 5.4. Projections between hyperbolic spaces.

Given an equivalence class $[V]$, define $\rho_{\hat{T}}^{[V]}$ to be the support $T[V]$ of the equivalence class $[V]$, which is uniformly bounded in $\hat{T}$ because it is coned-off. Define $\rho_{\hat{T}}^{[V]} : \hat{T} \to \mathcal{C}[V]$ as follows. For $w \in T[V]$, consider the geodesic connecting $w$ to $T[V]$, and let $e$ be its last edge, so that $e^+ \in T[V]$. Define

$$\rho_{\hat{T}}^{[V]}(w) := \phi_{c}(\phi_{e^+}(\chi_{e^+})) \subseteq \mathcal{C}_{\hat{T}} = \mathcal{C}[V],$$

where $\phi : \mathcal{C}_{\hat{T}} \to \mathcal{C}[V]$ is the comparison map. If $w \in T[V]$, then $\rho_{\hat{T}}^{[V]}(w)$ can be chosen arbitrarily. On the other hand, if $w \in \hat{T} \setminus T$, that is $w$ is a cone point, then define $\rho_{\hat{T}}^{[V]}(w) = \rho_{\hat{T}}^{[V]}(w')$, where $w'$ is an arbitrarily chosen vertex in the support tree associated to the cone-point $w$.

For an element $T[W] \in \mathcal{S}_2$, define $\rho_{\hat{T}}^{[W]}$ to be $T[W]$, and $\rho_{\hat{T}}^{[W]} : \hat{T} \to \mathcal{C}[V]$ as follows. For $v \in T$, let $\rho_{\hat{T}}^{[W]}(v)$ be the closest-point projection (in the tree $T$) of $v$ onto $T[W]$. On the other hand, if $v \in \hat{T} \setminus T$, that is $v$ is a
cone point, then define \( \rho_{\mathcal{W}}^{[V]}(v) = \rho_{\mathcal{W}}^{[V]}(v'), \) where \( v' \) is any of the points in the support tree associated to the cone-point \( v. \)

To define the projections \( \rho_{\mathcal{W}}^{[V]} \) between \( (\sim\text{-classes}) \) of hyperbolic spaces, we proceed as follows. If \( [V] \subseteq [W] \) or \( [V] \cap [W] \), then we define the projections as in [5, Theorem 8.6]. In particular, if \( [V] \subseteq [W] \) there exist vertices \( v, w, v' \) such that \( V_v, W_w \) are the favorite representatives of \([V]\) and \([W]\) respectively, \( V_v' \) and \( W_w' \) are representatives of \([V]\) and \([W]\) (possibly different from the favorite ones), and \( V_v \subseteq W_w \). Moreover, let \( \ell_v: CV_v \to CW_v \) and \( \ell_W: CW_v' \to CW_w \) be comparison maps. Define

\[
\rho_{\mathcal{W}}^{[W]} = \ell_W \left( \rho_{\mathcal{W}}^{[V]}(v') \right) \subseteq CW_w = C[W],
\]

which is a uniformly bounded set in \( C[W] \), and define \( \rho_{[V]}: C[W] \to C[V] \) as

\[
\rho_{[V]} = \ell_V \circ \rho_{\mathcal{W}}^{[V]} \circ \ell_W,
\]

where \( \ell_W \) is a quasi inverse of \( \ell_V \) and \( \rho_{\mathcal{W}}^{[V]}: CW_v' \to CW_v \) is the projection provided by the hierarchical hyperbolicity of the vertex space \( (X_v, \mathcal{S}_v). \)

Analogously, if \( [V] \cap [W] \) and there exists a vertex \( w' \in T \) such that \( \mathcal{S}_w \) contains representatives \( V_w \cap W_w' \) of \([V]\) and \([W]\), then define

\[
\rho_{[V]}^{[V]} = \ell_W \left( \rho_{[V]}^{[V]}(v') \right) \subseteq CW_w = C[W]
\]

and

\[
\rho_{[V]}^{[V]} = \ell_V \circ \rho_{\mathcal{W}}^{[V]} \circ \ell_W.
\]

If there is no common vertex for the supports of \([V]\) and \([W]\), let \( v, w \) be the closest pair of vertices such that \( \mathcal{S}_v, \mathcal{S}_w \) contain representatives \( V_v \) of \([V]\) and \( W_w \) of \([W]\) respectively, and let \( e \) be the last edge of the geodesic starting at \( w \) and ending at \( v \). Define

\[
\rho_{[V]} = e \circ \pi_{V_v} \left( \phi_{v+}(X_v) \right),
\]

where \( e: CV_v \to CW_v \) is the comparison map to the favorite representative. In a completely symmetrical way we also define \( \rho_{[V]}^{[W]} \).

For two elements \( T_{[V]} \) and \( T_{[V']} \) of \( \mathcal{S}_2 \), if \( T_{[V]} \subseteq T_{[V']} \) then define \( \rho_{T_{[V]}}^{T_{[V']}} \) to be \( \rho_{T_{[V]}} \), which is uniformly bounded in \( \hat{T}_{[V']} \) since it is coned-off. Define \( \rho_{T_{[V]}}^{T_{[V']}}: \hat{T}_{[V]} \to \hat{T}_{[V]} \) as the closest-point projection.

If \( T_{[V]} \cap T_{[V']} \) then \( \rho_{T_{[V]}}^{T_{[V]}} \) and \( \rho_{T_{[V']}^{V}} \) are either the closest-point projections (if \( T_{[V]} \) and \( T_{[V']} \) do not intersect), or are defined to be \( \hat{T}_{[V]} \cap \hat{T}_{[V']} \), which by (the proof of) Lemma [5,10] is equal to \( \hat{T}_{[V] \cup [V']} \), where \( V_v \) and \( V_v' \) are representatives of \([V]\) and \([V']\) in a vertex \( v \in T_{[V]} \cap T_{[V']} \). Notice that if \( T_{[V]} \cap T_{[V']} \) is not empty, then it is properly contained in both \( T_{[V]} \) and \( T_{[V']} \), and therefore will be coned-off in both \( \hat{T}_{[V]} \) and \( \hat{T}_{[V']} \).

Finally, we define projections between an equivalence class \([W]\) and an element \( T_{[V]} \in \mathcal{S}_2 \) as follows. The items of the list refer to the mutually exclusive cases described at the end of Subsection [5,2]

(c1) in this case \([W]\) and \( T_{[V]} \) are orthogonal, and no projection needs to be defined;

(c2) define the set \( \rho_{T_{[V]}}^{[W]} \) to be \( T_{[V]} \cap T_{[W]} \), which is uniformly bounded in \( \hat{T}_{[V]} \) because it is coned off, being properly contained in \( T_{[V]} \). Define the map \( \rho_{T_{[V]}^{[V]}}: \hat{T}_{[V]} \to 2^{\mathcal{S}^+} \) as follows. For \( x \in \hat{T}_{[V]} \setminus \hat{T}_{[W]} \), define \( \rho_{T_{[V]}^{[V]}}(x) = \pi_{V_v} \left( \phi_{v+}(X_v) \right) \), where the edge \( e \) is the last edge on the geodesic connecting \( x \) to the support \( T_{[W]} \), the vertex \( e' \) is in \( T_{[W]} \), the element \( W_{e+} \in \mathcal{S}_{e+} \) is the representative of \([W]\), and \( \pi_{V_v} \) is the comparison map to the favorite representative of \([W]\). For \( x \in \hat{T}_{[W]} \), define \( \rho_{T_{[V]}^{[V]}}(x) \) arbitrarily;
Lemma 5.14. All the maps and sets \( \rho^* \) between hyperbolic spaces defined in this subsection are uniformly bounded sets and well-defined maps, for all \( \bullet, \ast \in \mathcal{S} \).

**Proof.** The case when \( T[V] \subsetneq T[W] \) is immediate.

For any equivalence class \([W]\), the set \( \rho_T^{[W]} = T[W] \) is uniformly bounded because it is coned off in \( \hat{T} \), and the map \( \rho_T^{[W]} \) is well defined: if \( w \in T \setminus T[W] \), then \( \rho_T^{[W]}(w) \) is defined in terms of the closest-point projection in the tree \( T \) of \( w \) onto \( T[W] \). Suppose now that \( w \) is a cone point of a support which is not \( T[W] \), nor contained in \( T[W] \). By definition \( \rho_T^{[W]}(w) = \rho_T^{[W]}(w') \), where \( w' \) is a chosen vertex in the support whose cone point is \( w \).

Analogously, for a support \( T[V] \), the set \( \rho_T^{[V]} \) is uniformly bounded and the map \( \rho_T^{[V]} \) is well defined.

The sets and maps \( \rho_T^{[W]} \) between two equivalence classes are uniformly bounded sets and well-defined maps because comparison maps are quasi isometries, and by Theorem E (compare also Remark 4.7). For instance, the set \( \rho_T^{[W]} \) of Equation 12 is uniformly bounded, because comparison maps are uniform quasi isometries by hypothesis, and because the set \( \pi_{W^e} \left( \phi_{c+} (X_e) \right) \) appearing in the equation is uniformly bounded by Theorem E.

The set \( \rho_T^{[W]} = T[V] \cap T[W] \) defined in item \((c_2)\) is uniformly bounded, because \( T[V] \cap T[W] \) is properly contained in \( T[W] \), and therefore it is coned off, and analogous argument proves that the sets defined in item \((c_3)\) are uniformly bounded. The map \( \rho_T^{[W]} \) of item \((c_2)\) is also well defined.

We are now ready to prove Theorem A.

5.5. **Proof of Theorem A.** We verify that the axioms for hierarchically hyperbolic spaces hold for \((X, \mathcal{S})\).

The set of uniform hyperbolic spaces is described in Subsection 5.3 along with the projections from \(X\) onto these hyperbolic spaces. These are uniformly coarsely lipschitz maps, as proved in Lemma 5.13. The projections \( \rho^* \) between hyperbolic spaces are uniformly bounded sets, and well defined maps, by Lemma 5.14.

Nesting, orthogonality, and transversality are defined in Subsection 5.2.

(Nesting) The only non-immediate condition to check is the transitivity of the nesting we defined, and in particular that if \([U] \subseteq [V] \) and \([V] \subseteq [W] \), then \([U] \subseteq [W] \). Furthermore, since \([U] \subseteq [V] \) then \([W] \perp [U] \), which implies that \([U] \subseteq [W] \).

Assume now that \([U] \subseteq [V] \) and \([V] \subseteq [W] \). By Corollary 5.11 it follows that \([W] \subseteq [V] \). By definition we get \([U] \perp [W] \). Therefore \([W] \perp [U] \), which implies that \([U] \subseteq [W] \).

(Intersection property) We construct the wedges between elements of \( \mathcal{S} \), for all possible cases.

Let \([V]\) and \([W]\) be two equivalence classes. If \(T[V] \cap T[W]\) is non-empty, then there exists a vertex \(v\) and representatives \(V_v\) and \(W_v\) of the two classes in \(\mathcal{S}_v\). We have that

\[
[V] \cap [W] = [V_v \cap W_v],
\]

where we define \([V_v \cap W_v] = \emptyset\) if \(V_v \cap W_v = \emptyset\).
If the supports $T[V]$ and $T[W]$ do not intersect, then $[V]$ and $[W]$ are transversal. Suppose that $\mathcal{S}[V] \cap \mathcal{S}[W]$ is non-empty, and suppose that it has more than one $\sqsupseteq$-maximal. Call these maximals $[U_i]$, for $i \in I$. As $[U_i] \subseteq [V]$ and $[U_i] \subseteq [W]$, the supports $T[V]$ and $T[W]$ are both contained into $T[U_i]$, for all $i$. As supports are connected, each $T[U_i]$ contains the geodesic $\sigma$ that connects $T[V]$ to $T[W]$. Therefore, each $[U_i]$ has representatives in all edge-spaces in the geodesic $\sigma$, which by abuse of notation we also denote by $U_i$.

Let $U_\nu := \bigvee_{i \in I} U_i$. Notice that $U_\nu$ is nested into each $\sqsupseteq$-maximal element of each edge-space on $\sigma$. Moreover, $[U_i] \subseteq [U_\nu]$ for all $i \in I$, which leads to a contradiction if $|I| > 1$. Therefore, there is only one $\sqsupseteq$-maximal element $[U_1]$ in $\mathcal{S}[V] \cap \mathcal{S}[W]$, and $[V] \cap [W] = [U_1]$.

Let $[V]$ be an equivalence class and $T[W]$ be a support. We have that
\[
[V] \cap T[W] = \bigvee \{ [U] \mid [U] \subseteq [V] \text{ and } [U] \subseteq T[W] \} = [V \cap \text{cont}_\perp V_e]
\]
The only non-immediate point of Equation (43) is to check that if two equivalence classes $[U]$ and $[U']$ are nested into $T[W]$, then so is their join $[U] \cup [U']$. This is indeed the case, by clean containers, as proved in Lemma 3.3.

Therefore, $[V] \cap T[W]$ is nested into both $[V]$ and $T[W]$, and by construction is the $\sqsupseteq$-maximal of such elements.

Let $T[V]$ and $T[W]$ be two distinct supports. If $T[V] \cap T[W] \neq \emptyset$, then the support $T[V] \cap T[W]$ is nested in both $T[V]$ and $T[W]$. We prove that
\[
T[V] \cap T[W] = T[V] \cap T[W].
\]

To prove that Equation (43) defines the wedge between $T[V]$ and $T[W]$, it needs to be shown that if $[U]$ is nested into both $T[V]$ and $T[W]$, then it is also nested into $T[V] \cap T[W]$.

By definition of nesting, we have that $[U] \perp [V]$ and $[U] \perp [W]$, and therefore, by Lemma 3.4 we have that $[U] \perp ([V] \cap [W]) = [V \cap W]$, that is $[U] \subseteq T[V \cap W] = T[V] \cap T[W]$.

If $T[V] \cap T[W] = \emptyset$, then there is no element $S \in \mathcal{S}_2$ that is nested in both $T[V]$ and $T[W]$. The wedge between these two elements of the index set is
\[
T[V] \cap T[W] = \bigvee \{ [U] \mid [U] \subseteq T[V] \text{ and } [U] \subseteq T[W] \} = [\text{cont}_\perp V_e \cap \text{cont}_\perp W_e]
\]

Notice that any $[U]$ as in Equation (45) will be supported on the geodesic $\sigma$ connecting $T[V]$ to $T[W]$.

**Orthogonality** We first prove that if $T[V] \subseteq T[U]$ and $T[W] \perp [U]$, then $T[V] \perp [U]$. As $[U] \perp T[W]$, we have that $T[W] \subseteq T[U]$. Therefore $T[V] \subseteq T[U]$, that is $[U] \perp T[V]$. The analogous case of three equivalence classes satisfying the relations $[V] \subseteq [W]$ and $[W] \perp [U]$ is proved in [5, Lemma 8.9].

We now construct the (upper) orthogonal containers for elements of $\mathcal{S}$. Consider $T[V] \in \mathcal{S}_2$. By definition, there is no orthogonality between elements of $\mathcal{S}_2$. We have that $\text{cont}_\perp T[V] = [V]$. This follows from the definition of orthogonality between equivalence classes and supports.

We claim that $\text{cont}_\perp [V] = T[V]$. To prove this claim, first notice that a support $T[W]$ is orthogonal to $[V]$ if and only if $T[W] \subseteq T[V]$. Consider now an equivalence class $[W]$ orthogonal to $[V]$. By definition, $[W] \subseteq T[V]$. The lower orthogonal containers are constructed using Lemma 3.5.

**Consistency** We verify the various cases for this Axiom.

Choose a vertex $z \notin T[W]$ and let $x \in X_z$. Let $e$ be the last edge in the geodesic connecting the vertex $z$ to $T[W]$, so that $e^+ = w \in T[W]$.
As \( \pi_T(x) = z \), we have that \( \rho^T_W(\pi_T(x)) = \epsilon_W \circ \pi_W \circ (\phi_u(\mathcal{X}_c)) \), where \( \epsilon_W \) is the comparison map from \( \mathcal{C}W \) to the favorite representative of \( \{W\} \). On the other hand, \( \pi_W(x) = \epsilon_W \circ \pi_W \circ (\phi_u(\mathcal{X}_c)) \). This means that

\[
\rho^T_W(\pi_T(x)) = \pi_W(x) = \epsilon_W \circ \pi_W \circ (\phi_u(\mathcal{X}_c))
\]

is a uniformly bounded set by Theorem 2 and therefore

\[
diam_{C[W]} \left( \pi_W(x) \cup \rho^T_W(\pi_T(x)) \right) = diam(\epsilon_W \circ \pi_W \circ (\phi_u(\mathcal{X}_c)))
\]

is uniformly bounded.

If \( z \in T_W \), then

\[
d_f(\pi_T(x), \rho^T_W) = d_f(z, T_W) = 0.
\]

Therefore, there exists a uniform bound \( N \) such that

\[
\min \left\{ d_f(\pi_T(x), \rho^T_W) \right\} \leq N
\]

for all \( x \in \mathcal{X} \) and for all \( \{W\} \in \mathcal{S} \).

Let \( T_W \in \mathcal{S}_2 \) and \( x \in \mathcal{X} \). If \( x \in T_W \), then \( \pi_T(x) \in \rho^T_W \), and therefore \( d_f(\pi_T(x), \rho^T_W) = 0 \). On the other hand, if \( d_f(\pi_T(x), \rho^T_W) > 1 \), and in particular \( x \in \mathcal{X}_c \), where \( v \notin T_W \), then \( \pi_{\{W\}}(x) = \rho^T_W(\pi_T(x)) \), and therefore

\[
diam_{\{W\}}(\pi_{\{W\}}(x) \cup \rho^T_W(\pi_T(x))) = diam_{\{W\}}(\pi_{\{W\}}(x)) = 0.
\]

This concludes consistency for this case.

Let \( U, V \in \mathcal{S} \) and assume that \( U \cap V = \emptyset \). We need to prove that there exists some uniform constant \( \kappa \) such that either

\[
d_U(\pi_U(x), \rho^U_U) \leq \kappa \quad \text{or} \quad d_V(\pi_V(x), \rho^V_U) \leq \kappa
\]

for each \( x \in \mathcal{X} \). We proceed by induction on \( d_T(T_U, T_V) \).

If \( d_T(T_U, T_V) = 0 \), then these two finite sets intersect. Therefore, there exists a vertex \( w \) such that \( \mathcal{S}_w \) contains representatives \( U_w \cap V_w \) of \( \{U\} \) and \( \{V\} \) respectively. Since consistency holds in each hyperbolic space, it follows that there exists \( \kappa_0 \) that satisfies Equation (46).

Suppose now that \( d_T(T_U, T_V) > 0 \), and consider the geodesic \( \gamma \) in \( T \) connecting \( T_U \) to \( T_V \), with initial vertex \( u \) and final vertex \( v \), so that \( u \in T_U \) and \( v \in T_V \). Let \( x \in \mathcal{X} \) be so that \( x \in \mathcal{X}_c \) for some vertex \( z \in T \).

There are three possible configurations: either \( d_T(u, v) < d_T(x, z) \), or \( d_T(u, v) > d_T(x, z) \), or \( d_T(u, v) = d_T(x, z) \).

If one of the geodesics in \( T \) connecting \( x \) to \( T_U \) or to \( T_V \) has a vertex that lies in \( T_U \) or \( T_V \), then Equation (46) trivially satisfied. Indeed, suppose that the geodesic connecting the vertices \( u \) through \( T_U \) to \( T_V \).

In this case, it follows from the definitions that \( \pi_{\{W\}}(x) \in \rho^U_{\{W\}} \), and thus \( d_{\{W\}}(\pi_{\{W\}}(x), \rho^U_{\{W\}}) = 0 \).

Therefore, it remains to check the case in which the geodesics and \( \gamma \) connecting \( z \) to \( T_U \) and to \( T_V \) respectively have that \( \gamma \cap \sigma \neq \emptyset \) and \( \gamma \cap \sigma' \neq \emptyset \), but \( \gamma \cap \sigma \neq \gamma \cap \sigma' \). Let \( e \) and \( e' \) be the first and the last edges (possibly equal) of \( \gamma \), so that \( e = u \in T_U \) and \( e' = v \in T_V \).

The first two cases are symmetric, so suppose that \( d_T(u, z) < d_T(v, z) \). Let \( w \in T_{\{V\}} \) be the favorite vertex of the class \( \{V\} \), and \( V_w \in \mathcal{S}_w \) be its the favorite representative. By definition

\[
\pi_{\{V\}}(x) = \epsilon_V \circ \pi_{\{V\}}(\phi_u(\mathcal{X}_c)),
\]

where \( \epsilon_V : C_{\{V\}} \to C_{\{V\}} \) is the comparison map. We obtain that

\[
d_{\{V\}}(\pi_{\{V\}}(x), \rho^U_{\{V\}}) = d_{\{V\}}(\epsilon_V \circ \pi_{\{V\}}(\phi_u(\mathcal{X}_c)), \epsilon_V \circ \pi_{\{V\}}(\phi_u(\mathcal{X}_c))) = 0.
\]

If \( d_T(u, z) < d_T(v, z) \), the same argument shows that

\[
d_{\{V\}}(\pi_{\{V\}}(x), \rho^U_{\{V\}}) = 0.
\]
We consider now the case $d_T(u, z) = d_T(v, z)$. As $z \not\in T[U] \cup T[V]$, we have that
\[ \pi_V(x) = c_v \circ \pi_{V_e}(\phi_e(\mathcal{X}_e)) \quad \text{and} \quad \pi_U(x) = c_u \circ \pi_{U_v}(\phi_v(\mathcal{X}_v)). \]
It follows that
\[ d_{[V]}(\rho_{[V]}^U, \pi_V(x)) = 0 \quad \text{and} \quad d_{[U]}(\rho_{[U]}^V, \pi_U(x)) = 0. \]
Therefore, consistency holds for every $[U] \cap [V] \in \mathcal{S}$.

Consistency for the pair $[U] \in [V]$ is immediate: by definition there exist a vertex $v$ and representatives $U_v \subseteq V_v$ of $[U]$ and $[V]$ respectively. As Consistency holds in all vertex spaces, the statement follows.

Suppose now that $[W]$ is such that either
1. $[V] \subseteq [W]$, or
2. $[V] \cap [W]$ and $[U] \not\subseteq [W]$.
We claim that $d_{[W]}(\rho_{[W]}^V, \rho_{[W]}^U)$ is uniformly bounded.

As $[U] \subseteq [V]$, let $U_u, V_v \in \mathcal{S}_u$ be representatives of $[U]$ and $[V]$ such that $U_u \subseteq V_u$. We now check all the possible cases.

Suppose that $T[U] \cap T[W] \neq \emptyset$ and $T[V] \cap T[W] \neq \emptyset$: this can happen either if $[V] \subseteq [W]$ or if $[V] \cap [W]$ and there exist transversal representatives of $[U]$ and $[V]$. Let $v, w \in T$ be such that there exist representatives $V_v, W_w \in \mathcal{S}_v$ satisfying $V_v \subseteq W_w$ (respectively $V_v \cap W_w$), and representatives $U_v, W_v \in \mathcal{S}_v$ such that $U_v \subseteq W_v$ (respectively $U_v \cap W_v$).

Let $m \in T$ be the median of $u, v, w$. As $u, w$ belong to the support of $[U]$ and $[W]$, then so does $m$, since supports are connected trees. Likewise, $m$ lies in the support of $[V]$. Let $U_m, V_m$ and $W_m$ be representatives of $[U], [V]$ and $[W]$ in $\mathcal{S}_m$. Since edge-hieromorphisms are full, we have that $U_m \subseteq V_m$, and $U_m \not\subseteq W_m$, and $V_m \subseteq W_m$ (respectively $V_m \cap W_m$). Because consistency holds in each vertex space, and in particular in $(\mathcal{X}_m, \mathcal{S}_m)$, we conclude that $d_{[W]}(\rho_{[W]}^U_{[U]}, \rho_{[W]}^V^W)$ is uniformly bounded. Applying the appropriate comparison maps (that are uniform quasi isometries), it follows that $d_{[W]}(\rho_{[W]}^V^U, \rho_{[W]}^U)$ is uniformly bounded.

If $T[U] \cap T[W] \neq \emptyset$ and $T[V] \cap T[W] = \emptyset$, let $w$ be a vertex such that there are transversal representatives $U_w \cap W_w$, of $[U]$ and $[W]$. Moreover, let $e$ be the edge separating $T[V]$ from $T[W]$, so that $e^+ \in T[W]$. We have that $\rho_{[W]}^V = \pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e))$ and $\rho_{[W]}^U = \pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e))$.

Let $S_e$ denote the $\sqsubseteq$-maximal element of the index set $\mathcal{S}_e$ and $S_e' = \phi_{ee}(S_e)$. Recall that the constant $\kappa_0$ denotes the constant coming from the consistency axiom of Definition 2.3 and $\xi$ denotes the constant which uniformly bounds the multiplicative and additive constant of comparison maps (see Definition 2.14 and the second hypothesis of Theorem A).

Therefore,
\begin{align*}
\quad & d_{[W]}(\rho_{[W]}^U, \rho_{[W]}^V) = d_{W_w}(\pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e)), \pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e))) \\
& \leq d_{W_w}(\pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e)), \pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e))) + d_{W_w}(\pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e)), \pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e))) \\
& \leq d_{W_w}(\pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e)), \pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e))) + \\
& \quad + d_{W_w}(\pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e)), \pi_W \circ \pi_{W_e}(\phi_e(\mathcal{X}_e))) \quad (47) \\
& \leq \kappa_0 + \xi \kappa_0 + \xi.
\end{align*}

Notice that
\begin{align*}
\quad & d_{W_e}(\pi_{W_e}(\phi_e(\mathcal{X}_e)), \pi_{W_e}(\phi_e(\mathcal{X}_e))) = d_{W_e}(\pi_{W_e}(\phi_e(\mathcal{X}_e)), \pi_{W_e}(\phi_e(\mathcal{X}_e))) \\
& \leq Kd(\phi_e(\mathcal{X}_e), F_{S_e}) + K,
\end{align*}
and so, by Theorem 4.9 we have that

\[
d_{W_+} \left( \pi_{W_+}(\phi_{\epsilon_+}(X_\epsilon)), \rho_{W_+}^{S_\epsilon} \right) \leq K\eta + K.
\]

Combining Equation (47) and Equation (48) we obtain that \(d_{[W]}([\rho_{[W]}], [\rho_{[W]}])\) is uniformly bounded.

Assume now that \(T_{[U]} \cap T_{[W]} = \emptyset\) in particular \([U] \triangleleft [W]\). By Lemma 4.11 we know that \(T_{[V]} \subseteq T_{[U]}\).

Therefore, there exists an edge \(e\) separating \(T_{[V]} \) (and \(T_{[U]}\)) from \(T_{[W]}\), so that \(e^+ \in T_{[V]}\).

As defined in Equation (42), we have that

\[
\rho_{[W]}^{[V]} = \epsilon_W \circ \pi_{W_+}(\phi_{\epsilon_+}(X_\epsilon)) = \rho_{[W]}^{[U]}.
\]

Therefore \(\rho_{[W]}^{[V]} = \rho_{[W]}^{[U]}\) and \(d_{[W]}([\rho_{[W]}^{[V]}], [\rho_{[W]}^{[U]}]) = 0\) is uniformly bounded.

Let \(T_{[W_1]}, T_{[W_2]} \in \mathcal{S}_2\) satisfying \(T_{[W_1]} \triangleleft T_{[W_2]}\), and let \(x \in \mathcal{X}\). In this case, we always have that

\[
\min\{d_{T_{[W_1]}(\pi_{T_{[W_1]}(X)}), \rho_{T_{[W_1]}(X)}^{T_{[W_2]}(x)}), d_{T_{[W_2]}(\pi_{T_{[W_2]}(X)}, \rho_{T_{[W_2]}(X)}^{T_{[W_1]}(x)})\} = 0,
\]

because \(T_{[W_1]}\) and \(T_{[W_2]}\) are defined as closest-point projections if \(T_{[W_1]} \cap T_{[W_2]} = \emptyset\), or as the (coned-off) intersection, if it is non-empty.

Let \(T_{[W_1]} \subseteq T_{[W_2]}\) Let \(T_{[W_1]}, T_{[W_2]} \in \mathcal{S}_2\) satisfying \(T_{[W_1]} \subseteq T_{[W_2]}\). Consistency follows, because for all \(x \in \mathcal{X}\) we have that

\[
\pi_{T_{[W_1]}(X)} = \rho_{T_{[W_2]}(X)}^{T_{[W_1]}(x)}.
\]

Therefore \(\text{diam}_{T_{[W_1]}(\pi_{T_{[W_1]}(X)} \cup \rho_{T_{[W_2]}(x)}^{T_{[W_1]}(x)})} = 0\) and the consistency inequality is satisfied.

Let \(T_{[W_1]} \in \mathcal{S}_2\) be such that either

1. \(T_{[W_1]} \subseteq T_{[W_2]} \subseteq T_{[W_1]}\), or
2. \(T_{[W_2]} \triangleleft T_{[W_1]}\).

In either case we have that \(\rho_{T_{[W_1]}(X)} = \rho_{T_{[W_2]}(x)}^{T_{[W_1]}(x)}\) and therefore \(d_{T_{[W_1]}(\rho_{[W]}^{T_{[W_1]}}, \rho_{[W]}^{T_{[W_2]}}) = 0\).

Let \(V \in \mathcal{S}_1\) be such that \([V] \triangleleft T_{[W_1]}\) and \([V] \not\in T_{[W_1]}\). We want to prove that \(d_{[V]}([\rho_{[V]}^{T_{[W_1]}}, \rho_{[V]}^{T_{[W_2]}}) = 0\) uniformly bounded. We now check every possible case. If the support of \([V]\) does not intersect \(T_{[W_2]}\) (and therefore, does not intersect \(T_{[W_1]} \subseteq T_{[W_2]}\)), then \(\rho_{[W]}^{T_{[W]}(x)} = \rho_{[W]}^{T_{[W_2]}(x)}\) and the claim is satisfied. If the support of \([V]\) intersects both \(T_{[W_1]}\) and \(T_{[W_2]}\), then also in this case we have that \(\rho_{[W]}^{T_{[W]}(x)} = \rho_{[W]}^{T_{[W_2]}(x)}\). Finally, if \([V]\) intersects \(T_{[W]}\) but not \(T_{[W_1]}\), then \(\rho_{[W]}^{T_{[W]}(x)} = \epsilon \circ \pi_{V_+}(\phi_{\epsilon_+}(X_\epsilon))\), where \(\epsilon\) is the last edge in the geodesic connecting \(T_{[W_1]}\) to \(T_{[V]}\), the vertex \(e^+\) lies in \([V]\), and \(V_+\) is the representative of \([V]\) in \(\mathcal{S}_e\). On the other hand, \(\rho_{[W]}^{T_{[W]}(x)} = \rho_{[W]}^{T_{[W_2]}(x)}\) and \([W_2] \triangleleft [V]\). As both classes \([V]\) and \([W_2]\) are supported on the vertex \(e^+\), we have that \(\rho_{[V]}^{T_{[W]}(x)} = \epsilon \circ \rho_{[W]}^{T_{[W_2]}(x)}\), where \(W_{2e^+}\) is the representative of \([W_2]\) in that vertex.

By Lemma 4.11 we have that \(\pi_{V_+}(\phi_{\epsilon_+}(X_\epsilon))\) is coarsely equal to \(\hat{S}_e^\epsilon(\epsilon e)\), where \(\hat{S}_e^\epsilon = \phi_\epsilon(\epsilon e)\) and \(S_\epsilon\) is the \(\subseteq\)-maximal element of \(\mathcal{S}_e\). Therefore \(d_{[V]}([\rho_{[V]}^{T_{[W]}(x)}], \rho_{[V]}^{T_{[W_2]}(x)}) = 0\) uniformly bounded.

Let \(T_{[W]} \in \mathcal{S}_2\). If \([V] \cap T_{[W]} = \emptyset\), then

\[
\min\{d_{[V]}(\pi_{[V]}(x), \rho_{[V]}^{T_{[W]}(x)}), d_{[W]}(\pi_{[W]}(x), \rho_{[W]}^{[V]}(x))\} = 0, \quad \forall x \in \mathcal{X}.
\]

Thus, suppose that the intersection is non-empty. Since \([V] \triangleleft T_{[W]}\) it follows that \([V] \triangleleft [W]\). Suppose that \(d_{[W]}(\pi_{[W]}(x), \rho_{[W]}^{[V]}(x))\) is big, so that in particular \(x \notin [V] \cap T_{[W]} = \rho_{[W]}^{[V]}(x)\) and the geodesic connecting \(x\) to \(T_{[V]}\) passes through the set \([V]\).
By definition, \( \pi_{[V]}(x) = c \circ \pi_{V_{e^+}}(\phi_{e^+}(x)) \), and \( \rho_{[V]}^t(w) = \rho_{[V]}^{[W]} = c(\rho_{V_{e^+}}^{[W]}) \), where \( e^+ \) is the vertex of the edge \( e \) that belongs to \( T_{[V]} \cap T_{[W]} \), while \( e^- \in T_{[W]} \setminus T_{[V]} \), and \( V_{e^+} \) and \( W_{e^+} \) are the representatives of \([V]\) and \([W]\) respectively at the vertex \( e^+ \).

Let \( S_e \) be the \( \sqcap \)-maximal element of \( \mathcal{S}_e \). As the equivalence class \([V]\) is not supported in the vertex \( e^- \), it follows that \( V_{e^+} \) is not nested into \( \phi_{e^+}^0(S_e) = \tilde{S}_e \). On the other hand \( W_{e^+} \subseteq \tilde{S}_e \). Therefore, \( \rho_{V_{e^+}}^{[W]} \) and \( \tilde{S}_e \) coarsely coincide by Definition 2.3.4, and by Lemma 4.10 we obtain that

\[
\pi_{V_{e^+}}(\phi_{e^+}(x)) = \tilde{S}_e = \rho_{V_{e^+}}^{[W]},
\]

that is, \( \pi_{[V]}(x) \) and \( \tilde{T}_{[V]}^{[W]} \) coarsely coincide. Thus, \( d_{[V]}(\pi_{[V]}(x), \tilde{T}_{[V]}^{[W]}) \) is uniformly bounded.

If the distance \( d_{T_{[W]}}(\pi_{T_{[W]}}(x), \tilde{T}_{[W]}^{[V]}) \geq \kappa_0 \), it follows in particular that \( \pi_{T}(x) \neq \tilde{T}_{[V]}^{[W]} = T_{[V]} \cap T_{[W]} \), and that the geodesic in \( \hat{T} \) connecting \( x \) to \( \tilde{T}_{[V]}^{[W]} \) passes through the set \( T_{[V]} \setminus T_{[W]} \). In this case, we have that \( \pi_{[V]}(x) = \pi_{[V]}(\pi_{T_{[W]}}(x)) \) is equal to \( \tilde{T}_{[V]}^{[W]}(\pi_{T_{[W]}}(x)) \). Therefore the consistency inequality is satisfied also in this case.

(Finite complexity) It is enough to show finite complexity in \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) independently.

Finite complexity in \( \mathcal{S}_1 \) follows from [5, Lemma 8.11]. For \( \mathcal{S}_2 \), notice that any chain of proper nestings

\[
T_{[U_1]} \supseteq T_{[U_2]} \supseteq \cdots \supseteq T_{[U_n]}
\]

induces the corresponding chain of proper nestings \( [U_1] \subseteq [U_2] \subseteq \cdots \subseteq [U_n] \) in \( \mathcal{S}_1 \), by Corollary 5.11.

As only equivalence classes are allowed to be nested into an intersection of supports, and not vice versa, finite complexity is proved.

In particular, it follows that the complexity of \( (\mathcal{X}(T), \mathcal{S}) \) is twice the complexity of \( \mathcal{S}_1 \) plus one, and the complexity of \( \mathcal{S}_2 \) is \( \max_0 \chi_e + 1 \), where \( \chi_e \) is the complexity of the vertex space \( (\mathcal{X}_e, \mathcal{S}_e) \).

(Large links) Let \( [W] \in \mathcal{S}_1 \) and \( x, x' \in \mathcal{X} \). Suppose that \( x \in X_e \) and \( x' \in X_{e'} \) for some \( v, v' \in T \), and let \( w \) be the favorite vertex for \( [W] \). Let \( E \) denote the maximal of the constants \( E_0 \) of the Boundod Geodesic Axiom of the hierarchically hyperbolic space \( (\mathcal{X}_e, \mathcal{S}_e) \).

Suppose that, for some \( [V] \subseteq [W] \), we have \( d_{[V]}(\pi_{[V]}(x), \pi_{[V]}(x')) \geq E' \), where \( E' \) depends on \( E \) and on the quasi-isometry constants of the edge isomorphisms. Then \( d_{V_e}(\zeta \circ \pi_{V_e}(x), \zeta \circ \pi_{V_e}(x')) \geq E \), for a representative \( V_w \in \mathcal{S}_w \) of \([V]\). As the large links axiom holds in \( \mathcal{S}_w \), we have that \( V_w \subseteq T_i \), where \( \{T_i \in \mathcal{S}_w \} \) is a set of \( N \) elements in \( \mathcal{S}_w \), where \( N = \lceil d_{[W]}(\pi_{[W]}(x), \pi_{[W]}(x')) \rceil \) and each \( T_i \) satisfies \( T_i \subseteq W_w \). Moreover, the Large Links Axiom in \( \mathcal{S}_w \) implies that \( d_{[W]}(\pi_{[W]}(x), \pi_{[W]}(x)) = d_{W_w}(w \circ \pi_{W_w}(x), \pi_{W_w}(x)) \leq N \) for all \( i = 1, \ldots, N \). Thus the large links axiom for elements \([V] \in \mathcal{S}_1 \) and \([U] \in \mathcal{S}_2 \) follows.

We now consider the case of \( T_{[W]} \in \mathcal{S}_2 \), and \( \mathcal{X} \in \mathcal{S}(T_{[W]}) \). This can happen both when \( X \) is an equivalence class, or when \( X \in \mathcal{S}_2 \). We deal with the case \( X \in \mathcal{S}_2 \) in the following lemma, whilst the case \( X = [V] \in \mathcal{S}_1 \) is considered after the lemma.

**Lemma 5.15.** Let \( x, x' \in \mathcal{X} \) and \( T_{[W]} \in \mathcal{S}_2 \). The set

\[
Y = \{ X \in \mathcal{S}_2 \mid X \subseteq T_{[W]}, d_X(\pi_X(x), \pi_X(x')) > 4 \}
\]

is finite. Moreover, the set of \( \sqcap \)-maximal elements in \( Y \) has cardinality bounded linearly in terms of the distance \( d_{T_{[W]}}(\pi_{T_{[W]}}(x), \pi_{T_{[W]}}(x')) \).

**Proof.** Let \( \sigma \) be the geodesic in \( T \) connecting \( v = \pi_T(x) \) to \( v' = \pi_T(x') \). We begin by noticing that, if \( X \cap \sigma = \emptyset \), then \( d_X(\pi_X(x), \pi_X(x')) = 0 \) because these two sets coincide, and therefore \( X \not\in Y \). In particular, as nesting between equivalence class is inclusion, if \( \sigma \) does not intersect \( T_{[W]} \) then \( Y \) will be empty, and the lemma is trivially satisfied.

Suppose now that \( \sigma \) intersects \( T_{[W]} \), and consider the map \( \varphi : Y \to \mathcal{P}(\sigma) \) defined as \( \varphi(X) = X \cap \sigma \), where \( \mathcal{P}(\sigma) \) is the set of subpaths of \( \sigma \). We first prove that \( \varphi \) is an injective map. Let \( X, X' \in Y \) be such that
$X \neq X'$ and, looking for a contradiction, suppose that $\varphi(X) = \varphi(X')$, so that $X \cap \sigma = X' \cap \sigma$ and therefore $X \cap \sigma \subseteq X \cap X' \cap \sigma$.

Since $X$ intersects $\sigma$, we have that $\pi_X(x)$ and $\pi_X(x')$ are vertices of $\sigma$. Therefore $\pi_X(x)$ and $\pi_X(x')$ lie in $X \cap X' \subseteq X \cap X' \cap \sigma$. Since $X \cap X'$ is properly contained in both $X$ and $X'$, it will be coned-off in both $CX$ and $CX'$ by construction. Therefore $d_X(\pi_X(x), \pi_X(x')) \leq 2$, which contradicts the definition of the set $Y$. Therefore the map $\varphi$ is injective, and the set $Y$ is finite.

We now claim that, for elements $X, X' \in Y$, we have that $\varphi(X) \subseteq \varphi(X')$ if and only if $X \subseteq X'$. Indeed, if $X \subseteq X'$, that is $X \subseteq X' \cap \sigma$, then $\varphi(X) \subseteq \varphi(X')$. On the other hand, suppose that $\varphi(X) \subseteq \varphi(X')$, and let $X = T[V]$ and $X' = T[V']$, for some equivalence classes $[V]$ and $[V']$. Since $\varphi(X) = X \cap \sigma \subseteq \varphi(X') = X' \cap \sigma$, we have that

\begin{equation}
X \cap \sigma = X \cap X' \cap \sigma.
\end{equation}

Moreover, as $X \cap X' = T[V] \cap T[V'] = T[V \vee V']$, from Equation 49 we obtain that

\begin{equation}
T[V] \cap \sigma = T[V \vee V'] \cap \sigma.
\end{equation}

As $[V] \subseteq [V \vee V']$, Lemma 4.1 implies that $T[V \vee V'] \subseteq T[V]$. If $T[V \vee V']$ is properly nested into $T[V]$, then $T[V \vee V']$ is coned off in $CT[V]$. Equation 50 implies that $d_{T[V]}(\pi_{T[V]}(x), \pi_{T[V]}(y)) = 2$, which is a contradiction since $T[V] \subseteq Y$ by hypothesis. Therefore, $T[V \vee V'] = T[V]$, which implies that $T[V \subseteq T[V\vee V]$, as desired.

We now show that $Y_{max} = \{X_1, \ldots, X_n\} \subseteq Y$, the set of $\subseteq$-maximal elements in $Y$, has cardinality at most $d_{T[U]}(\pi_{T[U]}(x), \pi_{T[U]}(y)) = 2$. Since every element of $Y_{max}$ is properly nested into $T[U]$, it follows that its support is coned off in $CT[U]$. Let $\tilde{\sigma}$ be a geodesic path in $CT[U]$ connecting $\pi_{T[U]}(x)$ to $\pi_{T[U]}(x')$, and let $T[U_1], \ldots, T[U_k]$ denote the supports properly nested in $T[U]$ that $\tilde{\sigma}$ intersects. If $d_{T[U]}(\pi_{T[U]}(x), \pi_{T[U]}(x')) > 4$, then $T[U_j] \subseteq Y$ and there exists $j$ such that $T[U_j] \subseteq X_j$.

We now prove that $X_j \cap \sigma \subseteq (X_k \cup \cdots \cup X_k) \cap \sigma$, for any $j = 1, \ldots, n$, any $r \geq 2$, and $Y_{max}$-elements $X_j, X_{k_1}, \ldots, X_{k_r}$.

Indeed, suppose that $X_j \cap \sigma \subseteq (X_k \cup X_k) \cap \sigma$, and let $T[U_1], T[U_{k_1}]$, and $T[U_{k_2}]$ denote $X_j, X_{k_1}$, and $X_{k_2}$ respectively. In this case, there exists a path in $C_{X_j}$ from $\pi_{X_j}(x)$ to $\pi_{X_j}(x')$ that passes through the cone points of $T[U_{k_1} \cup U_{k_2}]$ and $T[U_{k_1} \cup U_{k_2}]$, which are properly nested into $X_j$. Then, $d_{X_j}(\pi_{X_j}(x), \pi_{X_j}(x')) \leq 4$, contradicting the assumption that $X_j \in Y_{max}$.

On the other hand, assume that $X_j \cap \sigma \subseteq (X_k \cup X_k \cup \cdots \cup X_k) \cap \sigma$, where $r > 2$, $k_i \neq k_j$ for all $i, k_i \neq k_j$ for all $a \neq b$, and there does not exist $k_i \neq k_j$ such that $X_j \cap \sigma \subseteq (X_k \cup X_k) \cap \sigma$. We claim that there exists $s$ such that $X_{k_s} \cap \sigma \subseteq X_j \cap \sigma$.

Indeed, assume without loss of generality that the endpoints of $X_j \cap \sigma$ are contained in $X_{k_s} \cap \sigma$ and $X_{k_s} \cap \sigma$ respectively. By hypothesis, $X_j \cap \sigma$ cannot be entirely contained in $(X_{k_1} \cup X_{k_2}) \cap \sigma$. Therefore, there exists $v \in X_j \cap \sigma \cap (X_{k_1} \cup X_{k_2}) \cap \sigma$, that is $v \in X_{k_s} \cap \sigma$ for $1 < s < r$. Note that $X_{k_s} \cap \sigma$ cannot contain either of the endpoints of $X_j \cap \sigma$, since that would imply that $X_j \cap \sigma$ is contained in either $(X_{k_1} \cup X_{k_2}) \cap \sigma$ or $(X_{k_1} \cup X_{k_2}) \cap \sigma$. As a consequence we obtain that $X_{k_s} \cap \sigma \subseteq X_j \cap \sigma$, which is a contradiction, since $X_{k_s}$ is maximal with respect to nesting.

From here we can conclude that $|Y_{max}| \leq d_{T[U]}(\pi_{T[U]}(x), \pi_{T[U]}(x'))$. Indeed, given any $\subseteq$-maximal element $X_i \in Y_{max}$ and its cone point $v_i$, the following dichotomy holds: either $v_i$ is a vertex in the geodesic path $\tilde{\sigma}$, or not. In the latter case, it must be that $\tilde{\sigma}$ contains one or two edges of the support $X_i$. Therefore, the bound is proved.

Therefore, if $d_{T[U]}(\pi_{T[U]}(x), \pi_{T[U]}(x')) \leq 4$ for some $T[U] \in \Theta_{T[W] \setminus \{T[W]\}}$, that is $T[U] \subseteq Y$, then $T[U] \subseteq Y$ for some $\subseteq$-maximal element $X$ of the set $Y$.

We now address the case when $X$ is an equivalence classes $X = [V] \in \Theta_{T[W]}$. By definition, $[V] \subseteq T[W]$ if and only if $[V]$ is orthogonal to $[W]$. In particular, it follows that $T[W] \cap T[W] \neq \emptyset$.

If $T[W]$ does not intersect the geodesic $\sigma$ then the distance $d_{T[W]}(\pi_{T[W]}(x), \pi_{T[W]}(x'))$ is equal to zero by Equation 39, because the edge $e$ appearing in the cited equation will be the same for both $x$ and $x'$.
Now assume that $T_{|V|} \cap \sigma \neq \emptyset$. As a fist sub-case, suppose that $\sigma \cap T_{|W|}$ is empty, let
\begin{equation}
\mathcal{I} := \{\{V\} \subseteq T_{|W|} \mid T_{|V|} \cap \sigma \neq \emptyset\},
\end{equation}
and notice that $\mathcal{I}$ could be infinite. Consider the geodesic $\alpha$ connecting $T_{|W|}$ to $\sigma$ in the tree $T$, and notice that $\alpha$ has at least one edge, being $T_{|W|}$ and $\sigma$ disjoint. For $[V] \in \mathcal{I}$, we have that $T_{|V|}$ intersects both $T_{|W|}$ and $\sigma$, and therefore $\alpha$ is contained in $T_{|V|}$, being $T$ is a tree. Thus the set $T_{|W|} \cap \bigcap_{\{V\} \in \mathcal{I}} T_{|V|}$ is not empty, because (at least) the initial vertex on the geodesic $\alpha$ belongs to this intersection.

Let the set $I$ index $\mathcal{I}$, that is $\mathcal{I} = \{\{[V]\}\subseteq [I]\}$. Without loss of generality, we can suppose that each $V_i$ is the representative of $[V_i]$ in the vertex space $(\mathcal{X}_v, \mathcal{G}_v)$. Let $S_v \in \mathcal{G}_v$ be the $\Xi$-maximal element, and notice that $[V_i] \subseteq [S_v]$ for all $i \in I$. Furthermore, note that $[V] \in \bigcup_{i \in I} [V_i]$ for all $[V] \in \mathcal{I}$ and let $[V_v]$ denote $[V_{i \in I} V_i]$. Therefore, in this first sub-case, Large Links is satisfied by the family $Y \cup \{[V_v]\}$ for the elements $T_{|W|} \in \mathcal{S}$ and $x, x' \in X$.

For the second sub-case, suppose that $\sigma \cap T_{|W|}$ is not empty, and let $\{v_1, \ldots, v_n\}$ be the finitely many vertices of $\sigma \cap T_{|W|}$ (there can be only finitely many such vertices because $\sigma$ is a geodesic). Analogously to Equation (51), for all $v_i \in \sigma \cap T_{|W|}$ define
\begin{equation}
\mathcal{I}_v := \{\{V\} \subseteq T_{|W|} \mid v_i \in T_{|V|} \cap \sigma\},
\end{equation}
and notice that $\mathcal{I} = \bigcup_{i \in [I]} \mathcal{I}_v$. As in the previous case, for each $\mathcal{I}_v$, consider $[S_{v_i}]$, and notice that $[V] \subseteq [S_{v_i}]$ for all $[V] \in \mathcal{I}_v$, for all $i = 1, \ldots, n$. Therefore, Large Links for an element $T_{|W|} \in \mathcal{S}_2$ is satisfied considering the set $Y \cup \{[V_{1}]^{(0)}, \ldots, [V_{n}]^{(0)}\}$

Notice that, in both sub-cases, we bounded the cardinality of the sets $Y \cup \{[V_v]\}$ and $Y \cup \{[V_{1}^{(0)}], \ldots, [V_{n}^{(0)}]\}$ in terms of $\sigma$, that is in terms of $d_T(x, x')$. As $d_{T_{|W|}}(\pi_{T_{|W|}}(x), \pi_{T_{|W|}}(x'))$ is bounded from above by $d_T(x, x')$, we obtained the desired bound on the cardinality of these sets.

Combining these bounds with Lemma 5.12, we conclude the proof of Large Links for the case $X \subseteq T_{|W|}$.

Finally, we prove Large Links for the $\Xi$-maximal element $\hat{T}$. From Lemma 5.13 applied with $S = \hat{T}$, there are only finitely many (and the number depends only on the distance in $\hat{T}$ from $x$ to $x'$) elements $X \in \mathcal{S}_2$ such that $d_X(\pi_X(x), \pi_X(x'))$ is big. On the other hand, for an equivalence class $[V] \subseteq \hat{T}$, the distance $d_{[V]}(\pi_{[V]}(x), \pi_{[V]}(x'))$ can be big only if the support $T_{|V|}$ intersects the geodesic $\sigma$ connecting $v$ to $v'$ (otherwise, it would be zero). Let $S_1, \ldots, S_n$ be the $\Xi$-maximal elements of all the finitely many edges in $\sigma \cap T_{|V|}$. We have that $[V] \subseteq [S_i]$ for all $i = 1, \ldots, n$. Therefore, the set $Y \cup \{S_1, \ldots, S_n\}$ is the set of significant elements for the Axion.

Let $E'$ be the constant that satisfies the Large Links Axiom of the (uniformly) hierarchically hyperbolic vertex spaces (see Definition 2.3), and let $E > \max(2, E')$. Then Large Links is satisfied with this constant $E$.

(Bounded geodesic image) Consider $[W] \subseteq \hat{T}$, and let $\gamma$ be a geodesic in $\hat{T}$. If $\gamma \cap T_{|V|} = \emptyset$, let $e$ be the last edge in the geodesic connecting $\gamma$ to $T_{|V|}$, and suppose $e^+ \in T_{|V|}$. Then $\rho^{\hat{T}}_{T_{|V|}}(\gamma) = \rho^w \circ \rho_{\pi_{T_{|V|}^+}}(\phi_{e^+}([X]))$ is a uniformly bounded set. If not, then $\gamma$ intersects $\rho_T^{[V]}$. The cases $[V] \subseteq T_{|W_1|}, T_{|W_1|} \subseteq T_{|W_2|}$, and $T_{|W_1|} \subseteq \hat{T}$, where $T_{|W_1|}, T_{|W_2|} \in \mathcal{S}_2$, are analogous.

Let $[W] \in \mathcal{S}$, let $[V] \subseteq [W]$, and let $\gamma$ be a geodesic in $\mathcal{C}(W) = \mathcal{C}W_w$ (where $w$ is the favorite vertex of $[W]$ and $W_w \in \mathcal{G}_w$ is the favorite representative). Let $V_w$ be the representative of $[V]$ supported in the vertex $w$, so that $\rho_T^{[V]} = \rho^w_{\pi_{V_w}}$. The Bounded Geodesic Image Axiom in this case follows because it holds in the vertex space $(\mathcal{X}_w, \mathcal{G}_w)$ (notice that the constant $E$ changes according to the quasi-isometry constant of the comparison maps).

(Partial realization) Notice that two elements $T_{|W_1|}$ and $T_{|W_2|}$ of $\mathcal{S}_2$ are never orthogonal. Consider $k + 1$ pairwise orthogonal elements $[V_1], \ldots, [V_k], T_{|W|} \in \mathcal{S}$, and let $p_i \in \pi_{[V_i]}(\mathcal{X}) \subseteq \mathcal{C}(V_i)$, for $i = 1, \ldots, k$, and $\mathcal{V} \subseteq \mathcal{T}_{|W|}$.

By definition of orthogonality, $T_{|V_i|} \cap T_{|V_j|} \neq \emptyset$ for all $i \neq j, T_{|V_i|} \subseteq T_{|V_j|}$ for all $i = 1, \ldots, n$, and in particular $T_{|W|} \subseteq \bigcap_{i=1}^{n} T_{|V_i|}$. Consider a vertex $v \in T_{|W|}$ that is not a cone point and has distance at most one
Comparison maps are uniform quasi-isometries, and $p_i \in \pi[V_i](X)$, therefore the element $\epsilon_i(p_i)$ is uniformly close to the set $\pi[V_i](X)$ for all $i = 1, \ldots, k$, where $\epsilon_i : C[V_i] \to CV_i$ is the comparison map. For $i = 1, \ldots, k$, let $p_i' \in \pi[V_i](X)$ be a point such that $d_{V_i}(p_i', \epsilon_i(p_i))$ is uniformly bounded.

By Partial realization in the vertex space $(X_v, \mathcal{S}_v)$, there exists $x \in X_v$ such that $d_{V_i}(\pi[V_i](x), p_i')$ is uniformly bounded for all $i$. As comparison maps are uniform quasi-isometries, we obtain that $d_{V_i}[1](\pi[V_i](x), p_i)$ is uniformly bounded for all $i$. Moreover, $d_{T[W]}(\pi[T[W](x), v_S]) = d_{T[W]}(v, v_S) \leq 1$.

If $[V_i] \subseteq [U]$, then $[U]$ has a representative $U_v \in \mathcal{S}_v$ such that $V_i \subseteq U_v$. Therefore $d_{U}[1](\pi[U_i](x), \rho[U_i]^{[V_i]})$ is uniformly bounded, because $x$ is a realization point for $\{V_i\}_{i=1}^k$, and comparison maps are uniform quasi-isometries.

If $[V_i] \subseteq [T[U]]$, then $\rho[T[U]]^{[V_i]} = T[V_i] \cap T[U]$ and $\pi[T[U]](x) \in \rho[T[U]]^{[V_i]}$. Therefore, $d_{T[U]}(\pi[T[U]](x), \rho[T[U]]^{[V_i]}) = 0$. Analogously, for $[T[W]] \subseteq [T[U]]$ we have that $d_{T[U]}(\pi[T[U]](x), \rho[T[U]]^{[V_i]}) = 0$.

Let now $[V_i] \cap [W]$. Either $[T[W]] \cap [T[V_i]] = \emptyset$, in which case the distance $d_{T[W]}(\pi[T[W](x), \rho[T[W]]^{[V_i]})$ is uniformly bounded, or $T[W] \cap [T[V_i]] \neq \emptyset$, in which case $[W]$ has a representative $W_v \in \mathcal{S}_v$ that is transitive to $V_i$. Therefore, in the latter case the distance $d_{T[W]}(\pi[T[W](x), \rho[T[W]]^{[V_i]})$ is again uniformly bounded, because it is in the vertex space $X_v$, and comparison maps are uniform quasi-isometries.

If $[V_i] \cap [T[U]]$ then $\pi[T[U]](x) \in \rho[T[U]]^{[V_i]}$ and therefore $d_{T[U]}(\pi[T[U]](x), \rho[T[U]]^{[V_i]}) = 0$. For the last case, suppose that $T[W] \cap [U]$ for some $[U] \in \mathcal{S}_1$. If the support of $[U]$ does not intersect $T[W]$, then $\pi[U](x) \in \rho[U]^{[T[W]]}$. So, suppose that $T[W]$ intersects $T[U]$. Again using Lemma 8.10 we can conclude.

(Uniformity) Suppose $x, y \in X$ are such that $d_R(\pi_R(x), \pi_R(y)) \leq K$, for all $R \in \mathcal{S}$. In particular, we have that $d_2(\pi_2(x), \pi_2(y)) \leq K$, that $d_3(\pi_3(x), \pi_3(y)) \leq K$ for all $S \in \mathcal{S}_2$, and that $d_{[V]}(\pi[V](x), \pi[V](y)) \leq K$ for all $[V] \in \mathcal{S}_1$.

Suppose that the distance in $\hat{T}$ from $\pi_2(x)$ to $\pi_2(y)$ is realized by a path only consisting of vertices of $T \subseteq \hat{T}$, and let

$$v_0 = \pi_T(x), v_1, \ldots, v_{k-1}, \pi_T(y) = v_k,$$

be these vertices, where $k \leq K$. In particular, no four consecutive vertices can belong to the same support tree, because this would produce a shorter path in $\hat{T}$ joining $x$ to $y$.

We have that $d_\chi(x, y) \leq \sum_{i=0}^{k} d_{X_v}(\phi_v(x, \phi_v(y)) + k$. Moreover, for all $i = 0, \ldots, k$ we have that the distance $d_{X_v}(\phi_v(x, \phi_v(y))$ is uniformly bounded. Indeed, if this is not the case, by Uniformity in the hierarchically hyperbolic space $(X_v, \mathcal{S}_v)$, there exists $V \in \mathcal{S}_v$ such that $d_\chi(\pi[V](x), \pi[V](y))$ is not bounded. By [5] Lemma 8.18 and Theorem [5] we have that $d_\chi(\pi[V](x), \phi_v(x, \phi_v(y))$ and $d_{[V]}(\pi[V](x), \phi_v(x, \phi_v(y))$ coarsely coincide, and therefore the latter is not bounded. This contradicts the fact that $d_{[V]}(\pi[V](x), \pi[V](y)) \leq K$, and thus $d(X_v(\phi_v(x, \phi_v(y)) \leq \zeta = \zeta(K)$ is uniformly bounded, as claimed. Therefore, $d_\chi(x, y) \leq \zeta(K)$, for some uniform bound $\zeta(K)$.

Suppose now that in the geodesic $\sigma \in \hat{T}$ connecting $\pi_2(x)$ to $\pi_2(y)$ there is a cone point. Therefore, there exists an element $T[W]$ containing two points $x_1$ and $y_1$ in this geodesic (that, therefore, have distance two in $\hat{T}$ since $T[W]$ is coned-off in $\hat{T}$). As $T[W] \in \mathcal{S}_2$, we have that $d_{T[W]}(\pi[T[W](x_1), \pi[T[W](y_1)] = d_{T[W]}(x_1, y_1) \leq K$. Either the geodesic $\sigma_1$ in $S_1$ connecting these two points only consists of vertices of $T$, or there are cone points, and therefore an element $T[W] \in \mathcal{S}_2$ containing two elements $x_2, y_2$ of the geodesic $\sigma_1$.

As complexity in $\mathcal{S}_2$ is finite and nesting coincides with inclusion, this process must end after a finite number of steps (that depends only on $K$). Therefore, there exists a geodesic in $T$ connecting $\pi_2(x)$ to $\pi_2(y)$, whose length is bounded from above by a function in $K$. Repeating the argument given before, we conclude that $d_\chi(x, y)$ is uniformly bounded.
This concludes the proof of hierarchical hyperbolicity of the space \((\mathcal{X}(T), \mathcal{G})\).

6. Applications of Theorem \[\text{A}\]

In this concluding section, we collect two applications of Theorem \[\text{A}\] that is Corollary \[\text{B}\] and Theorem \[\text{C}\].

6.1. Finite graphs of hierarchically hyperbolic groups. In this subsection we apply Theorem \[\text{A}\] to prove Corollary \[\text{B}\].

**Corollary \[\text{B}\].** Let \(\mathcal{G} = (\Gamma, \{G_v\}_{v \in V}, \{G_e\}_{e \in E}, \{\phi_{e^+} : G_e \to G_{e^+}\}_{e \in E})\) be a finite graph of hierarchically hyperbolic groups. Suppose that:

1. each edge-homomorphism is hierarchically quasiconvex, uniformly coarsely lipschitz and full;
2. comparison maps are uniformly quasi isometries;
3. the hierarchically hyperbolic spaces of \(\mathcal{G}\) have the intersection property and clean containers.

Then the group associated to \(\mathcal{G}\) is itself a hierarchically hyperbolic group.

We begin with the following lemma, in which we use the notation of Section 5.1.

**Lemma 6.1.** Let \(T\) be a tree of hierarchically hyperbolic spaces and \(\tilde{T}\) be the corresponding decorated tree. Then

1. \(\pi_{\tilde{T}|V}(\mathcal{X}(T))\) is isometric to \(C \mathcal{T}|V|\), and quasi-isometric to \(C \mathcal{T}|V|\), for all support trees \(T|V| \in \mathcal{G}_2\);
2. \(\pi_{\tilde{T}|V}(\mathcal{X}(T))\) is isometric to \(\pi_{V}(\mathcal{X}(T))\), and quasi-isometric to \(\pi_{V}(\mathcal{X}(\tilde{T}))\), for all equivalence classes \(\mathcal{X}(T)\).
3. \(\mathcal{X}(T)\) is hierarchically quasiconvex in \(X(\tilde{T})\).

**Proof.**

1. The first assertion of this item follows from the fact that the projections to hyperbolic spaces for elements in \(\mathcal{X}(T)\) are not modified by decorating the tree \(T\). Furthermore, by the construction of Section 5.1, there exists a constant \(C > 0\) such that \(C \mathcal{T}|V| = \mathcal{N}_C(\pi_{\tilde{T}|V}(\mathcal{X}(T)))\), and therefore \(\pi_{\tilde{T}|V}(\mathcal{X}(T))\) is quasi-isometric to \(\mathcal{T}|V|\).
2. As the favorite representative of the equivalence class \([V]\), it is the same as of the class \([V]\), it follows that \(\pi_{\tilde{T}|V}(\mathcal{X}(T))\) is isometric to \(\pi_{V}(\mathcal{X}(\tilde{T}))\). The second assertion of this item follows from the equality \(\mathcal{X}(\tilde{T}) = \mathcal{N}_C(\mathcal{X}(T))\).
3. By what was just proved in the previous points, \(\pi_{U}(\mathcal{X}(T))\) is \(k(0)\)-quasiconvex in \(\pi_{U}(\mathcal{X}(\tilde{T}))\), for all \(U \in \mathcal{G}\), for some fixed number \(k(0)\).

Moreover, let \(b\) be a \(k\)-consistent tuple such that \(b_X \in \pi_X(\mathcal{X}(T))\) for every \(X \in \mathcal{G}\) and let \(x \in \mathcal{X}(\tilde{T})\) be a realization point of \(b\). Since \(\mathcal{X}(\tilde{T}) = \mathcal{N}_C(\mathcal{X}(T))\) there exists \(x' \in \mathcal{X}(T)\) such that \(d_{\mathcal{X}(T)}(x, x') \leq C\), and therefore the proof is complete.

\[\square\]

As already mentioned in Section 5.1 to construct the hierarchically hyperbolic structure of the graph of hierarchically hyperbolic groups \(\mathcal{G}\) of Corollary \[\text{B}\] we do not consider directly a decorated tree, because there might not be a non-trivial action of the fundamental group of \(\mathcal{G}\) on that hierarchically hyperbolic space. Instead, we proceed as follows. Let

\[\mathcal{T} = \left( T, \{H_w\}_{w \in V}, \{H_f\}_{f \in E}, \{\phi_{f}^{+}\} \right)\]

be the tree of hierarchically hyperbolic groups associated to \(\mathcal{G}\), as described in [5 Section 8.2]. In particular, \(T = (V, E)\) is the Bass-Serre tree associated to the finite graph \(\Gamma\), each \(H_w\) is conjugated in the total group \(G_e\), where \(w\) maps to \(v\) via the quotient map \(T \to \Gamma\), analogously \(H_f\) is conjugated to \(G_e\), and the edge maps \(\phi_{f}^{+}\) agree with these conjugations of edge and vertex groups to give the embeddings in the tree of hierarchically hyperbolic groups. Let \(\mathcal{X}(\mathcal{T})\) be the metric space associated, and let \(\mathcal{G}\) denote the index set associated to \(\mathcal{X}(\mathcal{T})\), as described in Section 4.
Associated to this, we consider the decorated tree $\tilde{T}$ of hierarchically hyperbolic groups, as described in Section 5.1. By Theorem A, the metric space $X(T)$ admits a hierarchically hyperbolic space structure, that we denote by $\tilde{\mathcal{G}}$. By Lemma 4.4 the metric space $X(T)$ is hierarchically quasiconvex in $X(T)$, and therefore $(X(T), \tilde{\mathcal{G}})$ is a hierarchically hyperbolic space by [5] Proposition 5.5, where the hyperbolic spaces associated to an element $U \in \tilde{\mathcal{G}}$ is defined as $\pi_U(X(T)) \subseteq \mathcal{C}U$. That is to say, $(X(T), \tilde{\mathcal{G}})$ is a hierarchically hyperbolic space.

As shown in [5] Section 8.1, the hierarchically hyperbolic structure associated to $\mathcal{G}$ can be made equivariant, if the starting hierarchically hyperbolic spaces are hierarchically hyperbolic groups. We recall the construction here, extend it to cover the bigger index set we are using, and use it to prove Corollary B.

We recall here the notion of $\mathcal{T}$-coherent bijections, where $\mathcal{T}$ is the tree of hierarchically hyperbolic spaces. A bijection of the index set $\mathcal{G}$ given in Equation (52) is said to be $\mathcal{T}$-coherent if:

- it induces bijections on the sets $\mathcal{G}_1$ and $\mathcal{G}_2$;
- it preserves the relation $\sim$ on $\mathcal{G}_1$;
- it induces a bijection $b$ of the underlying tree $T$ that commutes with $f$: $\bigsqcup_{v \in V} \mathcal{G}_v \to T$, where $f$ sends each $V \in \mathcal{G}_v$ to the vertex $v$. That is, $fb = bf$.

Notice that the composition of $\mathcal{T}$-coherent bijections is $\mathcal{T}$-coherent. Therefore, let $\mathcal{P}_T \leq \text{Aut}(\mathcal{G})$ be the group of $\mathcal{T}$-coherent bijections.

To produce the index set $\mathcal{G}$ in a $\mathcal{T}$-equivariant manner, we proceed as follows. For each $\mathcal{P}_T$-orbit in $\mathcal{G}_1$ choose a representative $[V]$ of the orbit, a favorite vertex $v$ for $[V]$, and a favorite representative $V_0 \in \mathcal{G}_v$ for $[V]$. Then, declare $gV_0 \in \mathcal{G}_v$ to be the favorite representative of $g[V]$, and $g.v$ its favorite vertex, for all $g \in G$.

From the definition of the action of $G$ on $\mathcal{G}_2$, it follows that $Cg.T[v] = CT[gv]$.

We are now ready to prove Corollary B.

**Proof of Corollary B** Let $\mathcal{T}$ be the tree of hierarchically hyperbolic spaces constructed from the finite graph of hierarchically hyperbolic groups, as done in Equation (52). By Theorem A the metric space $X(T)$ associated to $\mathcal{T}$ admits a hierarchical hyperbolic structure $\mathcal{G}$. We choose $\mathcal{G}$ following the constraints of Subsection 6.1.

The group $G$ acts on $X(T)$ in the following way. At the level of the metric space $g.x = gx \in X(T)$ for all $x \in X(T)$. The action at the level of the index set $\mathcal{G}$ is defined by $g.[V] = [gV] \in \mathcal{G}_1$ for all $[V] \in \mathcal{G}_1$, and $g.T[v] = T[gv] \in \mathcal{G}_2$ for all $T[v] \in \mathcal{G}_2$. At the level of hyperbolic spaces, the action of $G$ is completely determined by the actions of the hierarchically hyperbolic groups $G_v$ on the hyperbolic spaces associated to elements of $\mathcal{G}_1$, and by the action of $G$ on the Bass-Serre tree for hyperbolic spaces associated to elements of $\mathcal{G}_2$.

Therefore $(G, \mathcal{G})$ is a hierarchically hyperbolic space. Moreover, $G \leq \mathcal{P}_G$, because the action is given by $\mathcal{T}$-coherent automorphisms. As in [5] Corollary 8.22, this action is oocompact and proper. The action of $G$ on $\mathcal{G}$ is cofinite if and only if the induced actions on $\mathcal{G}_1$ and $\mathcal{G}_2$ are cofinite, and this is indeed the case. The action on $\mathcal{G}_1$ coincides with the action considered in [5] Corollary 8.22 and therefore is cofinite, and the action on

$$\mathcal{G}_2 = \{T[v] \mid [V] \in \mathcal{G}_1\}$$

is cofinite because the action on $\mathcal{G}_1$ is.

This proves that $G$ is a hierarchically hyperbolic group. It has the intersection property and clean containers because $(X(T), \mathcal{G})$ has these properties.

**6.2. Finite graph of hyperbolic groups.** Let us briefly comment on some Bestvina-Feighn flavored applications of Corollary B concerning graphs of hyperbolic groups.

First, let us stress that with Corollary B we cannot hope to recover the full combination theorem of Bestvina-Feighn. Indeed, consider the graph of groups associated to the HNN extension where vertex and edge groups are the same free group $F$, one embedding is the identity map $id_F$, and the other is a hyperbolic automorphism $\phi$.

$$F \rtimes_\phi \langle F, t \mid tft^{-1} = \phi(f) \quad \forall f \in F \rangle.$$ 

This group is hyperbolic, by means of [5].
As the vertex and the edge groups are hyperbolic, they admit the hierarchically hyperbolic structure \((F, \{F\})\) with intersection property and clean containers, and the embeddings \(\text{id}_F\) and \(\phi\) extend to full, hierarchically quasiconvex coarsely lipschitz hieromorphisms. The only equivalence class in the index set is \([F]\), and its support tree \(T[F]\) is equal to the whole Bass-Serre tree associated to the HNN extension.

It can be seen that the comparison maps \(c: CF_v \to CF_u\) are not uniform quasi isometries, where \(v, u\) are vertices in the support \(T[F]\) and \(F_u, F_v\) are representatives of \([F]\), because the automorphism \(\phi\) is hyperbolic. Therefore, the hypotheses of Corollary \(B\) are not met, and in particular the projection \(\pi_V\) (as defined in Equation (33)) would not be a \((K,K)\)-coarsely lipschitz map for any \(K\), as required in the definition of hierarchically hyperbolic space.

On the other hand, if also the automorphism \(\phi\) is the identity of \(F\), that is the HNN extension is the direct product \(F \times Z\), then all the hypotheses of Corollary \(B\) are met, and the index set produced for the group by Corollary \(B\) is \([F], T[F], \tilde{T}\), where \(\tilde{T}\) is the \(\subseteq\)-maximal element and has a bounded associated hyperbolic space, \([F]\) is the free group \(F\), and \(CT[F]\) is the Bass-Serre tree of the HNN extension, which is isometric to a line. That is, in this case we recover the usual index set for the direct product of the two hyperbolic groups \(F\) and \(Z\).

Let us now suppose that the groups appearing in Corollary \(B\) are hyperbolic, and that edge groups are (hierarchically quasiconvex in vertex groups. For the sake of simplicity, let us also suppose that the finite graph of hyperbolic groups \(G\) has two vertices and an edge, that is, we are considering an amalgamated free product. To construct the hierarchically hyperbolic structures for the vertex groups \(G_v\) and \(G_u\), we proceed as follows. Let \(G_e\) be the edge group, and let \(\phi_v(G_e)\) and \(\phi_w(G_e)\) be its (hierarchically quasiconvex) images into the vertex groups. By [30, Theorem 1], the subgroup \(\phi_v(G_e)\) induces a hierarchically hyperbolic structure \(G^{0}_{\phi_v(G_e)}\) on \(G_v\), given by cosets of certain quasiconvex subgroups (up to finite Hausdorff distance). To obtain a full hieromorphisms \(\phi_w\), we are forced to induce on the edge group \(G_e\) the hierarchical structure \(G^{0}_{\phi_w(G_e)}\). On the other hand, \(G^{0}_{\phi_v(G_e)}\) induces on the other vertex group \(G_w\) a new hierarchical structure \(G^1_{\phi_v(G_e)}\), and to make the hieromorphism \(\phi_w\) full, we need to enrich the structure of the edge group \(G_e\) with all the (possibly new) cosets that appear in \(G^1_{\phi_v(G_e)}\), and so on.

If this process stabilizes after a finite number of times, then the groups can be given hierarchically hyperbolic structures that induce a full, coarsely lipschitz hieromorphism with hierarchically quasiconvex images. This construction always produces structures with the intersection property and clean containers, but it is unclear whether there is a simpler way to articulate the necessary hypothesis in this case, than just requiring the comparison maps to be uniformly quasi isometries.

6.3. Graph products of hierarchically hyperbolic groups. In this subsection, we prove Theorem \(C\) of the Introduction:

**Theorem \(C\).** Let \(\Gamma\) be a finite simplicial graph, \(G = \{G_v\}_{v \in V}\) be a family of hierarchically hyperbolic groups with the intersection property and clean containers. Then the graph product \(G = \Gamma G\) is a hierarchically hyperbolic group with the intersection property and clean containers.

**Proof.** Throughout the proof, if \(G\) denotes the graph product \(\Gamma G\) and \(\Delta\) is a subgraph of \(\Gamma\), we denote with \(G_{\Delta}\) the subgroup of \(G\) generated by the family of subgroups \(\{G_v \mid v \in \Delta\}\). This is canonically isomorphic to the graph product \(\Delta G_{\Delta}\), where \(G_{\Delta}\) is the subfamily of \(G\) indexed by elements in \(\Delta\). Given vertex groups \(\{G_v\}_{v \in V}\), we fix once and for all word metrics on them, and we always consider the graph product metric on \(\Gamma G\), so that the (infinite) generating set of the graph product \(\Gamma G\) consists of all vertex-groups elements. In particular, for a full subgroup \(H\) of the graph product \(G\), that is a subgroup conjugated to \(G_{\Delta}\) as above, the inclusion map \(H \to G\) is an isometric embedding.

We show by induction on the number of vertices that every graph product \(G\) of hierarchically hyperbolic groups with the intersection property and clean containers is again a hierarchically hyperbolic group with the intersection property and clean containers, and that for any full subgroup \(H\) of \(G\), hierarchically hyperbolic
group structures (with intersection property and clean containers) can be given to \( H \) and \( G \) so that the canonical inclusion \( H \hookrightarrow G \) is a full, hierarchically quasiconvex hierarchy, inducing isometries at the level of hyperbolic spaces.

The case \( n = 1 \) is trivial, so let us suppose that \( V = \{ v, w \} \). If the vertices are connected by an edge, then the graph product is the direct product of the two vertex groups, its hierarchically hyperbolic structure is described in Example 2.16 and it satisfies the inductive statement we want to prove.

On the other hand, if the two vertices are not connected by an edge, then the graph product is the free product of the two vertex groups, and also in this case the inductive statement is satisfied.

Let us suppose that the graph \( \Gamma \) has \( n \) vertices, that is \( |V| = n \), and that the lemma is satisfied by graph products on at most \( n - 1 \) vertices. If the graph product splits non-trivially as a direct or free product, then either \( G = G_\Delta \times G_\Theta \) or \( G = G_\Delta \ast G_\Theta \), where \( \Delta \) and \( \Theta \) are proper non-trivial subgraphs of \( \Gamma \). In both cases the inductive statement is satisfied, by induction and by either invoking Example 2.16 or the free product case (as done for graph products on two vertices). Therefore, suppose that \( G \) does not split non-trivially as a direct nor as a free product. Consider any (non-central and non-isolated) vertex \( v \in V \) and the splitting

\[
G \cong G_{\Gamma \setminus \{v\}} \ast G_{\text{link}(v)} (G_{\text{link}(v)} \times G_v).
\]

We now check that all the hypotheses of Corollary 3.2 are satisfied.

By the inductive hypotheses the groups \( G_{\Gamma \setminus \{v\}} \) and \( G_{\text{link}(v)} \) admit a hierarchically hyperbolic group structures with the intersection property and clean containers, and we call \( S_{\Gamma \setminus \{v\}} \) and \( S_{\text{link}(v)} \) their index sets, respectively. By Lemma 3.6 the direct product \( G_{\text{link}(v)} \times G_v \) is a hierarchically hyperbolic group with the intersection property, and it also satisfies clean containers by Lemma 3.6. Moreover, also by inductive hypotheses, the inclusions \( \iota_1: G_{\text{link}(v)} \hookrightarrow G_{\Gamma \setminus \{v\}} \) and \( \iota_2: G_{\text{link}(v)} \hookrightarrow G_{\text{link}(v)} \times G_v \) are full, hierarchically quasiconvex hieromorphisms, and \( \iota_{1*}^{U} \) are isometries for \( i = 1, 2 \) and for all \( U \in S_{\text{link}(v)} \). Moreover, \( \iota_1 \) and \( \iota_2 \) are isometric embeddings. By choosing inverse isometries for the maps \( \iota_{1*}^U \) for \( i = 1, 2 \) and all \( U \in S_{\text{link}(v)} \), we conclude that the comparison maps, as defined in Definition 2.16, are again isometries. Therefore, all of the hypotheses of Corollary 3.2 are satisfied, and we apply it to the graph of groups appearing in Equation (53). Thus, the group \( G \) admits a hierarchically hyperbolic group structure with the intersection property and clean containers. To conclude the proof, it is enough to prove that the embedding \( G_\Delta \hookrightarrow G \) is a full, hierarchically quasiconvex hierarchy, and that induces isometries at the level of hyperbolic spaces, where \( \Delta \) is any proper subgraph of \( \Gamma \).

Let us first consider the case \( \Delta = \Gamma \setminus \{ v \} \), and let us show that \( G_{\Gamma \setminus \{v\}} \) is hierarchically quasiconvex in \( G \). Recall that the index set \( S \) constructed in Corollary 3.2 for \( G_{\Gamma} \) is \( S_1 \cup S_2 \cup \{ \bar{T} \} \), as fully described in Equation (33) and Equation (34).

Any element of \( S_1 \) is an equivalence class \([V]\), equipped with a favourite representative \( V_v \) in the Bass-Serre tree \( T \) for which \( C[V] = \Gamma V_v \). On the other, any element of \( S_2 \) is a support tree \( T[V] \), and the metric space \( CT[V] \) is the tree \( T[V] \) in which all properly contained support trees \( T[W] \) are coned-off.

For each \( [V] \in S_1 \), the projection \( \pi_{[V]} \), as defined in Equation (33) and Equation (36), is

\[
\pi_{[V]}(x) = \begin{cases} 
\epsilon_v \circ \pi_{V_v}(x), & \forall x \in X_v, \ v \in T[V]; \\
\epsilon_+ \circ \pi_{V_+}(\phi_+(X_+)), & \forall x \in X_v, \ v \notin T[V],
\end{cases}
\]

where \( e = e(v) \) is the last edge in the geodesic connecting \( v \) to \( T[V] \) such that \( e^+ \in T[V] \), and the maps \( \epsilon_v \) and \( \epsilon_+ \) denote the appropriate comparison maps to the favorite representative of \([V]\).

Let \( x \in X_v \subseteq X \) and let \( T[V] \in S_2 \). Then, \( \pi_{T[V]}(x) \) is defined as the composition of the closest point projection of \( v \) to \( T[V] \) in the Bass-Serre tree \( T \), with the inclusion of \( T[V] \) into the coned-off \( CT[V] \).

To prove that \( G_{\Gamma \setminus \{v\}} \) is hierarchically quasiconvex in \( G_\Gamma \), we need to check the two conditions of Definition 2.6. For each element \( T[V] \in S_2 \) we have that \( \pi_{T[V]}(G_{\Gamma \setminus \{v\}}) \) is a point in \( CT[V] \) and, therefore, it is quasiconvex in \( CT[V] \).
Suppose that \( [V] \in \mathcal{S}_1 \), and assume that \([V]\) has a representative in \( g.\mathcal{S}_v \), where \( \mathcal{S}_v \) is the index set associated to the vertex group \( G_v \). In particular \([V] = \{V\}\), and \( \pi_{[V]}(G_{\Gamma(\{V\})}) \subseteq \pi_{V}(g.G_{\text{link}(v)}) \). Since \( V \notin g.\mathcal{S}_{\text{link}(v)} \), the set \( \pi_{[V]}(g.G_{\text{link}(v)}) \) is uniformly bounded, and therefore \( \pi_{[V]}(G_{\Gamma(\{V\})}) \) is quasiconvex in \( \mathcal{C}_{[V]} \).

On the other hand, assume that the group orbit \( G.[V] \) intersects \( \mathcal{S}_{\Gamma(\{V\})} \). Without loss of generality, as the group acts isometrically on the hyperbolic spaces, we can assume that \([V]\) has a representative \( \tilde{V} \in \mathcal{S}_{\Gamma(\{V\})} \). By definition \( \pi_{[V]}(G_{\Gamma(\{V\})}) = e \circ \pi_{\tilde{V}}(G_{\Gamma(\{V\})}) \), where \( e \) is the comparison map from \( \tilde{V} \) to the favourite representative of \([V]\). By Axiom (1) of Definition 2.3, the set \( \pi_{\tilde{V}}(G_{\Gamma(\{V\})}) \) is quasiconvex in \( \mathcal{C}_{\tilde{V}} \), and therefore \( \pi_{[V]}(G_{\Gamma(\{V\})}) \) is quasiconvex in \( \mathcal{C}_{[V]} \), being \( e \) an isometry. It follows that for every element \([V] \in \mathcal{S}_1 \), the set \( \pi_{[V]}(G_{\Gamma(\{V\})}) \) is quasiconvex in \( \mathcal{C}_{[V]} \).

To conclude the proof of hierarchical quasiconvexity, consider a consistent tuple \( \tilde{b} \) in \((G,\mathcal{S})\) such that \( b_{[V]} \in \pi_{[V]}(G_{\Gamma(\{V\})}) \) and \( b_{T[V]} \in \pi_{T[V]}(G_{\Gamma(\{V\})}) \) for every \([V] \in \mathcal{S}_1 \). The sets \( \pi_{[V]}(G_{\Gamma(\{V\})}) \) are uniformly bounded, being points, for all \( T[V] \in \mathcal{S}_2 \). Moreover, \( \pi_{[V]}(G_{\Gamma(\{V\})}) \) are uniformly bounded for every equivalence class \([V] \in \mathcal{S}_1 \) which has a representative in \( g.\mathcal{S}_v \).

Let \( \alpha \) denote the vertex of the Bass-Serre tree in which the subgroup \( G_{\Gamma(\{V\})} \) is supported. Let \( i : G_{\Gamma(\{V\})} \to G_\Gamma \) be the hierarchymorphism as defined above. At the metric-space level define it to be the natural inclusion. At the level of index sets \( i^0(U) = [U] \) and, at the level of hyperbolic spaces, \( i^1_U : \mathcal{C}U \to \mathcal{C}[U] \) is the comparison map \( e : \mathcal{C}U_\alpha \to \mathcal{C}[U] \), which is an isometry.

For each \([V] \in \mathcal{S}_1 \), we have that

\[
\pi_{[V]}(G_{\Gamma(\{V\})}) = \begin{cases} 
\epsilon_\alpha \circ \pi_{V_\alpha}(G_{\Gamma(\{V\})}), & \text{if } \alpha \in T[V]; \\
\epsilon_\alpha^+ \circ \pi_{V_\alpha^+}(\phi_{\alpha^+}(X_\alpha)), & \text{if } \alpha \notin T[V]. 
\end{cases}
\]

By Theorem E the set \( \pi_{V_\alpha^+}(\phi_{\alpha^+}(X_\alpha)) \) is uniformly bounded, and thus \( \epsilon_\alpha^+ \circ \pi_{V_\alpha^+}(\phi_{\alpha^+}(X_\alpha)) \) is uniformly bounded. For each \([V] \in \mathcal{S}_1 \) such that \( \alpha \in T[V] \), let \( c_{[V]} \) denote \( \epsilon(b_{[V]}) \), where the maps \( \epsilon \) denote the comparison maps (which are isometries) from the favourite representative of \([V]\) to the representative \( V_\alpha \) (therefore, the maps \( \epsilon \) change with respect to different equivalence classes). Consider the consistent tuple

\[
\tilde{c} = \prod_{[V] \in \mathcal{S}_1, \alpha \in T[V]} c_{[V]}
\]

By induction hypothesis, \( G_{\Gamma(\{V\})} \) is a hierarchically hyperbolic group. Therefore, the consistent tuple \( \tilde{c} \) admits a realization point \( z \in G_{\Gamma(\{V\})} \), and thus we obtain that \( \pi_{[V]}(z) = b_{[V]} \) for every \([V] \in \mathcal{S}_1 \). Furthermore, since \( \pi_{T[V]}(G_{\Gamma(\{V\})}) \) is a point, we also have that \( \pi_{T[V]}(z) = b_{T[V]} = \pi_{T[V]}(G_{\Gamma(\{V\})}) \) for every \( T[V] \in \mathcal{S}_2 \). That is, the second condition of hierarchical quasiconvexity is proved, and the inclusion \( G_{\Gamma(\{V\})} \hookrightarrow G_\Gamma \) is a hierarchically quasiconvex hieromorphism.

Moreover, for each \( V \in \mathcal{S}_{\Gamma(\{V\})} \) the map \( \mathcal{C}V \to \mathcal{C}[V] \) is an isometry. Note that, if an element \([V] \subseteq i^0(U) = [U] \), where \( U \in \mathcal{S}_{\Gamma(\{V\})} \), then \( T[U] \subseteq T[V] \). By assumption \( \alpha \in T[U] \), and therefore \( \alpha \in T[V] \) and there exists \( V \in \mathcal{S}_{\Gamma(\{V\})} \) such that \( i^0(V) = [V] \).

Thus, we proved that all induction hypotheses are satisfied by the inclusion \( G_{\Gamma(\{V\})} \hookrightarrow G \), that is that the embedding is a full, hierarchically quasiconvex hieromorphism, which induces isometries at the level of hyperbolic spaces.

To deduce the same for an arbitrary \( G_\Delta \), we proceed as follows. If \( \Delta = \Gamma \setminus \{u\} \) for some (other) vertex \( u \in V \), then the above argument, where in Equation 53 we consider the splitting over the subgroup \( G_{\text{link}(u)} \), proves that the inclusion \( G_\Delta \hookrightarrow G \) satisfies the desired properties. If not, then \( \Delta \) is a proper subgraph of \( \Gamma \setminus \{u\} \), for some \( u \in V \). Induction proves that the embedding \( G_\Delta \hookrightarrow G_{\Gamma(\{u\})} \) satisfies said properties, and again the above argument proves the claim for the inclusion \( G_{\Gamma(\{u\})} \hookrightarrow G \). As fullness, hierarchical quasiconvexity, and inducing isometries at the level of hyperbolic spaces, are all properties preserved by composition of hieromorphisms, we conclude that the inclusion \( G_\Delta \hookrightarrow G \) satisfies the inductive statement, and the proof is thus complete. \( \square \)
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