Existence of a Stationary Navier–Stokes Flow Past a Rigid Body, with Application to Starting Problem in Higher Dimensions

Tomoki Takahashi

Communicated by G. P. Galdi

Abstract. We consider the large time behavior of the Navier–Stokes flow past a rigid body in \( \mathbb{R}^n \) with \( n \geq 3 \). We first construct a small stationary solution possessing the optimal summability at spatial infinity, which is the same as that of the Oseen fundamental solution. When the translational velocity of the body gradually increases and is maintained after a certain finite time, we then show that the nonstationary fluid motion converges to the stationary solution corresponding to a small terminal velocity of the body as \( t \to \infty \) in \( L^q \) with \( q \in [n, \infty] \). This is called Finn’s starting problem and the three-dimensional case was affirmatively solved by Galdi et al. (Arch Ration Mech Anal 138: 307–318, 1997). The present paper extends Galdi et al. (1997) to the case of higher dimensions. Even for the three-dimensional case, our theorem provides new convergence rate, that is determined by the summability of the stationary solution at infinity and seems to be sharp.

Mathematics Subject Classification. 35Q30, 76D05.

Keywords. Navier–Stokes flow, Oseen flow, Steady flow, Starting problem, Attainability.

1. Introduction and Main Results

We consider a viscous incompressible flow past a rigid body \( \mathcal{O} \subset \mathbb{R}^n \) (\( n \geq 3 \)). We suppose that \( \mathcal{O} \) is translating with a velocity \( -\psi(t)ae_1 \), where \( a > 0 \), \( e_1 = (1, 0, \cdots, 0)^\top \) and \( \psi \) is a function on \( \mathbb{R} \) describing the transition of the translational velocity in such a way that

\[
\psi \in C^1(\mathbb{R}; \mathbb{R}), \quad |\psi(t)| \leq 1 \quad \text{for } t \in \mathbb{R}, \quad \psi(t) = 0 \quad \text{for } t \leq 0, \quad \psi(t) = 1 \quad \text{for } t \geq 1. \tag{1.1}
\]

Here and hereafter, \((\cdot)^\top\) denotes the transpose. We take the frame attached to the body, then the fluid motion which occupies the exterior domain \( D = \mathbb{R}^n \setminus \mathcal{O} \) with \( C^2 \) boundary \( \partial D \) and is started from rest obeys

\[
\begin{aligned}
\frac{\partial u}{\partial t} + u \cdot \nabla u &= \Delta u - \psi(t)a \frac{\partial u}{\partial x_1} - \nabla p, & x \in D, \ t > 0, \\
\nabla \cdot u &= 0, & x \in D, t \geq 0, \quad t > 0, \\
|_{\partial D} &= -\psi(t)ae_1, & \text{as } |x| \to \infty, \\
\n u &= 0, \quad x \in D.
\end{aligned}
\tag{1.2}
\]

Here, \( u = (u_1(x, t), \cdots, u_n(x, t))^\top \) and \( p = p(x, t) \) denote unknown velocity and pressure of the fluid, respectively. Since \( \psi(t) = 1 \) for \( t \geq 1 \), the large time behavior of solutions is related to the stationary
problem

\[
\begin{align*}
    u_s \cdot \nabla u_s &= \Delta u_s - a \frac{\partial u_s}{\partial x_1} - \nabla p_s, \quad x \in D, \\
    \nabla \cdot u_s &= 0, \quad x \in D, \\
    u_s|_{\partial D} &= -ae_1, \\
    u_s &\to 0 \quad \text{as } |x| \to \infty.
\end{align*}
\]  

(1.3)

When \( n = 3 \), the pioneering work due to Leray [27] provided the existence theorem for weak solution to problem (1.3), what is called \( D \)-solution, having finite Dirichlet integral without smallness assumption on data. From the physical point of view, solutions of (1.3) should reflect the anisotropic decay structure caused by the translation, but his solution had little information about the behavior at large distances. To fill this gap, Finn [11–14] introduced the class of solutions with pointwise decay property

\[ u_s(x) = O(|x|^{-\frac{1}{4} - \varepsilon}) \quad \text{as } |x| \to \infty \]  

(1.4)

for some \( \varepsilon > 0 \) and proved that if \( a \) is small enough, (1.3) admits a unique solution satisfying (1.4) and exhibiting paraboloidal wake region behind the body \( \mathcal{G} \). He called the Navier–Stokes flows satisfying (1.4) physically reasonable solutions. It is remarkable that \( D \)-solutions become physically reasonable solutions no matter how large \( a \) would be, see Babenko [2], Galdi [18] and Farwig and Sohr [10]. Galdi developed the \( L^q \)-theory of the linearized system, that we call the Oseen system, to prove that every \( D \)-solution has the same summability as the one of the Oseen fundamental solution without any smallness assumption, see [19, Theorem X.6.4]. It is not straightforward to generalize his result to the case of higher dimensions and it remains open whether the same result holds true for \( n \geq 4 \). We also refer to Farwig [9] who gave another outlook on Finn’s results by using anisotropically weighted Sobolev spaces, and to Shibata [31] who developed the estimates of physically reasonable solutions and then proved their stability in the \( L^3 \) framework when \( a \) is small. There is less literature concerning the problem (1.3) for the case \( n \geq 4 \). When \( n \geq 3 \), Shibata and Yamazaki [32] constructed a solution \( u_s \), which is uniformly bounded with respect to small \( a \geq 0 \) in the Lorentz space \( L^{n,\infty} \), and investigated the behavior of \( u_s \) as \( a \to 0 \). If, in particular, \( n \geq 4 \) and if \( a \geq 0 \) is sufficiently small, they also derived

\[ u_s \in L^{\frac{n}{n+4}}(D) \cap L^{\frac{n}{n+2}}(D), \quad \nabla u_s \in L^{\frac{n}{n+4}}(D) \cap L^{\frac{n}{n+2}}(D) \]  

(1.5)

for some \( 0 < \rho_1, \rho_2 < 1 \), see [32, Remark 4.2].

Let us turn to the initial value problem. Finn [12] conjectured that (1.2) admits a solution which tends to a physically reasonable solution as \( t \to \infty \) if \( a \) is small enough. This is called Finn’s starting problem. It was first studied by Heywood [21], in which a stationary solution is said to be attainable if the fluid motion converges to this solution as \( t \to \infty \). Later on, by using Kato’s approach [24] (see also Fujita and Kato [15]) together with the \( L^q-L^r \) estimates for the Oseen initial value problem established by Kobayashi and Shibata [26], Finn’s starting problem was affirmatively solved by Galdi et al. [20]. After that, Hishida and Maremonti [23] constructed a sort of weak solution \( u \) that enjoys

\[ \|u(t) - u_s\|_{\infty} = O(t^{-\frac{1}{2}}) \quad \text{as } t \to \infty \]  

(1.6)

if \( a \) is small, but \( u(\cdot, 0) \in L^3(D) \) can be large. Here and hereafter, \( \| \cdot \|_q \) denotes the norm of \( L^q(D) \). Although we concentrate ourselves on attainability in this paper, stability of stationary solutions was also studied by Shibata [31], Enomoto and Shibata [8] and Koba [25] in the \( L^q \) framework. Those work except [8] studied the three-dimensional exterior problem, while [8] showed the stability of a stationary solution satisfying (1.5) for some \( 0 < \rho_1, \rho_2 < 1 \) in \( n \)-dimensional exterior domains with \( n \geq 3 \). Stability of physically reasonable solutions in 2D is much more involved for several reasons and it has been recently proved by Maekawa [29].

The aim of this paper is two-fold. The first one is to construct a small stationary solution possessing the optimal summability at spatial infinity, which is the same as that of the Oseen fundamental solution.
E:
\[
E \in L^q(\{x \in \mathbb{R}^n \mid |x| > 1\}), \quad q > \frac{n+1}{n-1}, \quad \nabla E \in L^r(\{x \in \mathbb{R}^n \mid |x| > 1\}), \quad r > \frac{n+1}{n},
\]
see Galdi [19, Section VII]. As already mentioned above, this result is well known in three-dimensional case even for large \(a > 0\), but it is not found in the literature for higher dimensional case \(n \geq 4\). Our theorem covers the three-dimensional case as well and the proof is considerably shorter than the one given by authors mentioned above since we focus our interest only on summability at infinity rather than anisotropic pointwise estimates. The second aim is to give an affirmative answer to the starting problem as long as \(a\) is small enough, that is, to show the attainability of the stationary solution obtained above. The result extends Galdi, Heywood and Shibata [20] to the case of higher dimensions. Even for the three-dimensional case, our theorem not only recovers [20] but also provides better decay properties, for instance,

\[
\|u(t) - u_s\|_{\infty} = O(t^{-\frac{n}{2} - \frac{2}{q}}) \quad \text{as} \quad t \to \infty
\]

for some \(\rho > 0\), that should be compared with (1.6). This is because the fluid is initially at rest and because the three-dimensional stationary solution \(u_s\) belongs to \(L^q(D)\) with \(q < 3\); to be precise, since \(q\) can be close to 2, one can take \(\rho\) close to 1/2 in (1.8). Due to the \(L^q\)-estimates of the Oseen semigroup established by Kobayashi and Shibata [26], Enomoto and Shibata [7,8], see Proposition 3.1, this decay rate is sharp in view of presence of \(u_s\), see (1.19), in forcing terms of the Eq. (1.18) for the perturbation. Our result can be also compared with [34] by the present author on the starting problem in which translation is replaced by rotation of the body \(\Theta \subset \mathbb{R}^3\). Under the circumstance of [34], the optimal spatial decay of stationary solutions observed in general is the scale-critical rate \(O(|x|^{-1})\), so that they cannot belong to \(L^q(D)\) with \(q \leq 3 = n\), and therefore, we have no chance to deduce (1.8). Another remark is that, in comparison with stability theorem due to [8] for \(n \geq 3\), more properties of stationary solutions are needed to establish the attainability theorem. Therefore, those properties must be deduced in constructing a solution of (1.3).

Let us state the first main theorem on the existence and summability of stationary solutions.

**Theorem 1.1.** Let \(n \geq 3\). For every \((\alpha_1, \alpha_2, \beta_1, \beta_2)\) satisfying
\[
\frac{n+1}{n-1} < \alpha_1 \leq n+1 \leq \alpha_2 < \frac{n(n+1)}{2}, \quad \frac{n+1}{n} < \beta_1 \leq \frac{n+1}{2} \leq \beta_2 < \frac{n(n+1)}{n+2},
\]
there exists a constant \(\delta = \delta(\alpha_1, \alpha_2, \beta_1, \beta_2, n, D) \in (0,1)\) such that if
\[
0 < a^{\frac{n+2}{n+1}} < \delta,
\]
problem (1.3) admits a unique solution \(u_s\) along with
\[
\|u_s\|_{\alpha_1} + \|u_s\|_{\alpha_2} \leq Ca^{\frac{n+2}{n+1}}, \quad \|\nabla u_s\|_{\beta_1} + \|\nabla u_s\|_{\beta_2} \leq Ca^{\frac{n}{n+1}},
\]
where \(C > 0\) is independent of \(a\).

The upper bounds of \(\alpha_2\) and \(\beta_2\) come from (2.2) with \(q < n/2\) in Proposition 2.1 on the \(L^q\)-theory of the Oseen system, whereas the lower bounds of \(\alpha_1\) and \(\beta_1\) are just (1.7).

For the proof of Theorem 1.1, we define a certain closed ball \(N\) and a contraction map \(\Psi : N \ni v \mapsto u \in N\) which provides the solution to the problem
\[
\begin{cases}
\Delta u - a \frac{\partial u}{\partial x_1} = \nabla p + v \cdot \nabla v, & x \in D, \\
\nabla \cdot u = 0, & x \in D, \\
u|_{\partial D} = -ae_1, & x \in D, \\
u \to 0 & \text{as} \quad |x| \to \infty.
\end{cases}
\]

In doing so, we rely on the \(L^q\)-theory of the Oseen system developed by Galdi [19, Theorem VII.7.1], see Proposition 2.1, which gives us sharp summability estimates of solutions at infinity together with explicit
dependence on $a > 0$. As long as we only use Proposition 2.1, the only space in which estimates of $\Psi$ are closed is

$$\{ u \in L^{n+1}(D) \mid \nabla u \in L^{n+1}(D) \}.$$  

From this, we can capture neither the optimal summability at infinity nor regularity required in the study of the starting problem. We thus use at least two spaces $L^{\alpha_i}(D)$ ($i = 1, 2$) for $u$ and $L^{\beta_i}(D)$ ($i = 1, 2$) for $\nabla u$, and intend to find a solution within a closed ball $N$ of

$$\{ u \in L^{\alpha_1}(D) \cap L^{\alpha_2}(D) \mid \nabla u \in L^{\beta_1}(D) \cap L^{\beta_2}(D) \}. \quad (1.12)$$

However, it is not possible to apply Proposition 2.1 to $f = v \cdot \nabla v$ with

$$v \in L^{\alpha_1}(D), \quad \nabla v \in L^{\beta_1}(D) \quad (1.13)$$

or

$$v \in L^{\alpha_2}(D), \quad \nabla v \in L^{\beta_2}(D) \quad (1.14)$$

if $\alpha_1$ and $\beta_1$ are simultaneously close to $(n+1)/(n-1)$ and $(n+1)/n$, or if $\alpha_2$ and $\beta_2$ are simultaneously close to $n(n+1)/2$ and $n(n+1)/(n+2)$, because the relation

$$\frac{2}{n} < \frac{1}{\alpha_2} + \frac{1}{\beta_2} < \frac{1}{\alpha_1} + \frac{1}{\beta_1} < 1$$

required in the linear theory, see Proposition 2.1, is not satisfied. In order to overcome this difficulty, given $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying (1.9), we choose auxiliary exponents $(q_1, q_2, r_1, r_2)$ fulfilling

$$\alpha_1 \leq q_1 \leq q_2 \leq \alpha_2, \quad \beta_1 \leq r_1 \leq r_2 \leq \beta_2, \quad \frac{2}{n} < \frac{1}{q_i} + \frac{1}{r_i} < 1, \quad i = 1, 2$$

such that the application of Proposition 2.1 to $f = v \cdot \nabla v$ with $v \in L^{q_1}(D)$ and $\nabla v \in L^{r_1}(D)$ (resp. $v \in L^{q_2}(D)$ and $\nabla v \in L^{r_2}(D)$) recovers (1.13) (resp. (1.14)) with $u$.

Another possibility to prove Theorem 1.1 is combining Proposition 2.1 with the Sobolev inequality. We then get a solution $(u_s, p_s) \in X_q(n)$ for all $q \in (1, \infty)$ with $n/3 \leq q \leq (n + 1)/3$, where $X_q(n)$ is defined in Proposition 2.1. The restriction $n/3 \leq q \leq (n + 1)/3$ is removed by applying a bootstrap argument to decrease the lower bound to $1$ and to increase the upper bound to $n/2$. As compared with this way, in our proof, we do not any use a bootstrap argument and directly construct a solution possessing the optimal summability at infinity as well as regularity required in the study of the starting problem.

Let us proceed to the starting problem. To study the attainability of the stationary solution $u_s$ of class (1.12) with $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying (1.9), it is convenient to set

$$\alpha_1 = \frac{n}{1 + \rho_1}, \quad \alpha_2 = \frac{n}{1 - \rho_2}, \quad \beta_1 = \frac{n}{2 + \rho_3}, \quad \beta_2 = \frac{n}{2 - \rho_4} \quad (1.15)$$

with $(\rho_1, \rho_2, \rho_3, \rho_4)$ satisfying

$$0 < \rho_1 < \frac{n^2 - 2n - 1}{n + 1}, \quad \frac{1}{n + 1} \leq \rho_2 < \frac{n - 1}{n + 1}, \quad 0 < \rho_3 < \frac{n^2 - 2n - 2}{n + 1}, \quad \frac{2}{n + 1} \leq \rho_4 < \frac{n}{n + 1} \quad (1.16)$$

and we need the additional condition

$$\rho_2 + \rho_4 > 1. \quad (1.17)$$

We note that the set of those parameters is nonvoid. It is reasonable to look for a solution to (1.2) of the form

$$u(x, t) = v(x, t) + \psi(t)u_s, \quad p(x, t) = \phi(x, t) + \psi(t)p_s.$$
Then the perturbation \((v, \phi)\) satisfies the following initial boundary value problem

\[
\begin{aligned}
\frac{\partial v}{\partial t} &= \Delta v - a \frac{\partial v}{\partial x_1} - v \cdot \nabla v - \psi(t)v \cdot \nabla u_s - \psi(t)u_s \cdot \nabla v + (1 - \psi(t))a \frac{\partial v}{\partial x_1} \\
&+ h_1(x, t) + h_2(x, t) - \nabla \phi, \quad x \in D, \ t > 0, \\
\nabla \cdot v &= 0, \quad x \in D, \ t \geq 0, \\
v|_{\partial D} &= 0, \quad t > 0, \\
v &\to 0 \text{ as } |x| \to \infty, \\
v(x, 0) &= 0, \quad x \in D,
\end{aligned}
\]  
(1.18)

where

\[
\begin{aligned}
h_1(x, t) &= -\psi'(t)u_s, \\
h_2(x, t) &= \psi(t)(1 - \psi(t)) \left( u_s \cdot \nabla u_s + a \frac{\partial u_s}{\partial x_1} \right).
\end{aligned}
\]  
(1.19) \hspace{1cm} (1.20)

In what follows, we study the problem (1.18) instead of (1.2). In fact, if we obtain the solution \(v\) of (1.18) which converges to 0 as \(t \to \infty\), the solution \(u\) of (1.2) converges to \(u_s\) as \(t \to \infty\). Problem (1.18) is converted into

\[
v(t) = \int_0^t e^{-(t-\tau)A_s} P \left[ -v \cdot \nabla v - \psi(\tau)v \cdot \nabla u_s - \psi(\tau)u_s \cdot \nabla v \\
+ (1 - \psi(\tau))a \frac{\partial v}{\partial x_1} + h_1(\tau) + h_2(\tau) \right] d\tau
\]  
(1.21)

by using the Oseen semigroup \(e^{-tA_s}\) (see Section 3) as well as the Fujita–Kato projection \(P\) from \(L^q(D)\) onto \(L^q_s(D)\) associated with the Helmholtz decomposition (see Fujiwara and Morimoto [16], Miyakawa [30] and Simader and Sohr [33]):

\[
L^q(D) = L^q_s(D) \oplus \{ \nabla p \in L^q(D) \mid p \in L^q_{loc}(\overline{D}) \} \quad (1 < q < \infty).
\]

Here,

\[
L^q_s(D) = \overline{C^\infty_0(D)}^||\cdot||_q, \quad C^\infty_0(D) = \{ u \in C^\infty_0(D)^n \mid \nabla \cdot u = 0 \}.
\]

We are now in a position to give the second main theorem on attainability of stationary solutions.

**Theorem 1.2.** Let \(n \geq 3\) and let \(\psi\) be a function on \(\mathbb{R}\) satisfying (1.1). We set \(M = \max_{t \in \mathbb{R}} |\psi'(t)|\). Suppose that \(\rho_1, \rho_2, \rho_3\) and \(\rho_4\) satisfy (1.16)–(1.17) and let \(\delta\) be the constant in Theorem 1.1 with (1.15). Then there exists a constant \(\varepsilon = \varepsilon(n, D) \in (0, \delta]\) such that if

\[
0 < (M + 1)a^{\frac{n-2}{2}} < \varepsilon,
\]

Equation (1.21) admits a unique solution \(v\) within the class

\[
Y_0 := \{ v \in BC((0, \infty) ; L^2_s(D)) \mid t^{\frac{1}{2}}v \in BC((0, \infty) ; L^\infty(D)), t^{\frac{1}{2}}\nabla v \in BC((0, \infty) ; L^n_s(D)), \\
\lim_{t \downarrow 0} t^{\frac{1}{2}}(||v(t)||_\infty + ||\nabla v(t)||_s) = 0 \}.
\]  
(1.22)

Moreover, we have the following.

1. (sharp decay) Let \(n = 3\). Then there exists a constant \(\varepsilon_* = \varepsilon_*(D) \in (0, \varepsilon]\) such that if \(0 < (M + 1)a^{1/4} < \varepsilon_*\), the solution \(v\) enjoys decay properties

\[
\begin{aligned}
||v(t)||_q &= O(t^{-\frac{1}{2} + \frac{2}{n} - \frac{a^2}{2}}), \quad 3 \leq q \leq \infty, \\
||\nabla v(t)||_3 &= O(t^{-\frac{1}{2} - \frac{a}{2}})
\end{aligned}
\]  
(1.23) \hspace{1cm} (1.24)

as \(t \to \infty\).

Let \(n \geq 4\) and suppose that \(\rho_3 > 1\) and \(1 < \rho_1 \leq 1 + \rho_3\) in addition to (1.16) (the set of those
parameters is nonvoid when \( n \geq 4 \). Then there exists a constant \( \varepsilon_* = \varepsilon_*(n, D) \in (0, \varepsilon] \) such that if 
\[ 0 < (M + 1)a^{(n-2)/(n+1)} < \varepsilon_* \], the solution \( v \) enjoys
\[
\|v(t)\|_q = O(t^{-\frac{1}{2} + \frac{n}{2q} - \frac{n}{q}}), \quad n \leq \forall q < \infty, \tag{1.25}
\]
and
\[
\|\nabla v(t)\|_n = O(t^{-\frac{1}{2} - \frac{n}{2q}}) \tag{1.26}
\]
as \( t \to \infty \).

2. (Uniqueness) There exists a constant \( \hat{\varepsilon} = \hat{\varepsilon}(n, D) \in (0, \varepsilon] \) such that if \( 0 < (M + 1)a^{(n-2)/(n+1)} < \hat{\varepsilon} \), 
the solution \( v \) obtained above is unique even within the class
\[
Y := \{ v \in BC((0, \infty); L^n_a(D)) \mid t^\frac{4}{3} v \in BC((0, \infty); L^\infty(D)), t^\frac{2}{3} \nabla v \in BC((0, \infty); L^n(D)) \}. \tag{1.27}
\]
For the sharp decay properties (1.23)-(1.26), the key step is to prove the \( L^n \)-decay of the solution, 
that is,
\[
\|v(t)\|_n = O(t^{-\frac{n}{2}}) \tag{1.28}
\]
as \( t \to \infty \). Once we have (1.28), the other decay properties can be derived by the similar argument to
[8]. Note that the condition \( \rho_1 \leq 1 + \rho_3 \) is always fulfilled and thus redundant for \( n = 3 \), since \( \rho_1 < 1/2 \) and \( \rho_3 < 1/4 \). On the other hand, it is enough for \( n \geq 4 \) to consider the case \( \rho_1, \rho_3 > 1 \). To prove (1.28), we first derive slower decay
\[
\|v(t)\|_n = O(t^{-\frac{n}{2}})
\]
with some \( \rho \in (0, 1) \) by making use of \( u_s \in L^{n/(1+\rho_1)}(D) \) and \( \nabla u_s \in L^{n/(2+\rho_3)}(D) \), see Lemma 3.6 in 
Section 3. When \( n = 3 \), one can take \( \rho := \min\{\rho_1, \rho_3\} \), yielding better decay properties of the other norms of the solution. With them at hand, we repeat improvement of the estimate of \( \|v(t)\|_n \) step by step to find (1.28). However, this procedure does not work for \( n \geq 4 \) because of \( \rho_1 > 1 \). In order to get around the difficulty, our idea is to deduce the \( L^{q_0} \)-decay of the solution with some \( q_0 < n \), that is appropriately chosen, see Lemma 3.8. We are then able to repeat improvement of estimates of several terms to arrive at
(1.28), where the argument is more involved than the three-dimensional case above. Finally, to prove the 
uniqueness within \( Y \), we employ the idea developed by Brezis [5], which shows that the solution \( v \in Y \) necessarily satisfies the behavior as \( t \to 0 \) in (1.22).

In the next section we introduce the \( L^q \)-theory of the Oseen system and then prove Theorem 1.1. The 
final section is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we first recall the result on the Oseen boundary value problem due to 
Galdi [19, Theorem VII.7.1], see also Galdi [17] for the first proof of this result.

**Proposition 2.1.** Let \( n \geq 3 \) and let \( D \subset \mathbb{R}^n \) be an exterior domain with \( C^2 \) boundary. Suppose \( a > 0 \) and \( 1 < q < (n+1)/2 \). Given \( f \in L^q(D) \) and \( u_s \in W^{-1/q,q}(\partial D) \), problem

\[
\begin{aligned}
\Delta u - a \frac{\partial u}{\partial x_1} &= \nabla p + f, & x &\in D, \\
\nabla \cdot u &= 0, & x &\in D, \\
u|_{\partial D} &= u_s, \\
u &\to 0 & \text{as } |x| &\to \infty
\end{aligned}
\tag{2.1}
\]

admits a unique (up to an additive constant for \( p \)) solution \((u, p)\) within the class
\[
X_q(n) := \left\{ (u, p) \in L^1_{loc}(D) \left| u \in L^{s_2}(D), \nabla u \in L^{s_1}(D), \nabla^2 u \in L^q(D), \frac{\partial u}{\partial x_1} \in L^q(D), \nabla p \in L^q(D) \right. \right\},
\]
where
\[
\frac{1}{s_1} = \frac{1}{q} - \frac{1}{n+1}, \quad \frac{1}{s_2} = \frac{1}{q} - \frac{2}{n+1}. \quad (2.2)
\]

Here, by $W^{2-1/q,q}(\partial D)$ we denote the trace space on $\partial D$ from the Sobolev space $W^{2,q}(D)$ (see, for instance, [1, 19]).

If, in particular, $a \in (0, 1]$ and $q < n/2$, then the solution $(u,p)$ obtained above satisfies
\[
a^{\frac{1}{q}+\frac{1}{2}}\|u\|_{s_2} + a\left\|\frac{\partial u}{\partial x_1}\right\|_{q} + a^{\frac{1}{q}+\frac{1}{2}}\|\nabla u\|_{s_1} + \|\nabla^2 u\|_{q} + \|\nabla p\|_{q} \leq C\left(\|f\|_2 + \|u_\ast\|_{W^{2-\frac{1}{q},q}(\partial D)}\right)
\]
with a constant $C > 0$ dependent on $q, n$ and $D$, however, independent of $a$.

For later use, we prepare the following lemma. The proof is essentially same as the one of Young’s inequality for convolution, thus we omit it.

**Lemma 2.2.** Let $R_0, d > 0$. Assume that $1 \leq q, s \leq \infty$ and $1/q + 1/s \geq 1$. Suppose $u \in L^q(\mathbb{R}^n)$ with supp $u \subset B_d := \{x \in \mathbb{R}^n \mid |x| < d\}$ and $p \in L^s(\mathbb{R}^n \setminus B_{R_0})$. Then for all $R \geq R_0 + d$, $\rho \ast u$ is well-defined as an element of $L^r(\mathbb{R}^n \setminus B_R)$ together with
\[
\|\rho \ast u\|_{L^r(\mathbb{R}^n \setminus B_R)} \leq \|\rho\|_{L^s(\mathbb{R}^n \setminus B_{R_0})}\|u\|_{L^q(B_d)},
\]
where $\ast$ denotes the convolution and $1/r := 1/q + 1/s - 1$.

When the external force $f$ is taken from $L^{q_1}(D) \cap L^{q_2}(D)$ with $1 < q_1, q_2 < (n+1)/2$ and $q_1 \neq q_2$, we can apply Proposition 2.1 to $f \in L^{q_i}(D)$ ($i = 1, 2$). The following tells us that the corresponding solutions coincide with each other.

**Lemma 2.3.** Suppose $n \geq 3, 1 < q_1, q_2 < (n+1)/2$ and $f \in L^{q_1}(D) \cap L^{q_2}(D)$. Let $(u_1, p_1)$ be a unique solution obtained in Proposition 2.1 with $f \in L^{q_i}(D)$ and $u_\ast = -ae_1$. Then $u_1 = u_2$.

**Proof.** We first show that $u_1 - u_2$ behaves like the Oseen fundamental solution $E$ at large distances. We fix $R_0 > 0$ satisfying $\mathbb{R}^n \setminus D \subset B_{R_0}$. Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a cut-off function such that $\zeta(x) = 0$ for $|x| \leq R_0$, $\zeta(x) = 1$ for $|x| \geq R_0 + 1$, and set
\[
u(x) := u_1(x) - u_2(x), \quad p(x) := p_1(x) - p_2(x),
\]
\[
u(x) := \zeta(x)u(x) - \mathbb{B}[u \cdot \nabla \zeta], \quad \pi(x) := \zeta(x)p(x).
\]
Here, $\mathbb{B}$ is the Bogovskiĭ operator defined on the domain $B_{R_0+1} \setminus B_{R_0}$, see Bogovskiĭ [3], Borchers and Sohr [4] and Galdi [19]. Then we have
\[
-\Delta \nu + a\frac{\partial \nu}{\partial x_1} + \nabla \pi = g(x), \quad \nabla \cdot \nu = 0 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n), \quad (2.3)
\]
where $\mathcal{S}'(\mathbb{R}^n)$ is the set of tempered distributions on $\mathbb{R}^n$ and
\[
g(x) = -\Delta \zeta + 2(\nabla \zeta \cdot \nabla + a\frac{\partial \zeta}{\partial x_1} + \tau \nabla \zeta + \left(\Delta - a\frac{\partial}{\partial x_1}\right)\mathbb{B}[u \cdot \nabla \zeta].
\]
For (2.3) with $g = 0$, we have supp $\hat{\nu} \subset \{0\}$ and supp $\hat{\pi} \subset \{0\}$, where $(\cdot)$ denotes the Fourier transform. We thus find
\[
\nu(x) = \int_{\mathbb{R}^n} E(x-y)g(y)\,dy + P(x), \quad \pi(x) = C(n)\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \cdot g(y)\,dy + Q(x)
\]
with some polynomials $P(x), Q(x)$ and some constant $C(n)$. In view of $v \in L^{(1/q_1-2/(n+1))^{-1}}(\mathbb{R}^n)$ and $\nabla \pi \in L^{q_1}(\mathbb{R}^n) + L^{q_2}(\mathbb{R}^n)$, we have $P(x) = 0$ and $Q(x) = 0$. Here, $\mathbb{P}$ is some constant. Then Lemma 2.2 with
\[
\rho = E, \quad \nabla E, \quad \frac{x-y}{|x-y|^n},
\]
For each $i$, then the conditions (2.7)–(2.10) are accomplished by

$\tilde{\varphi}_R(x) = \varphi(|x|/R)$ for $R \geq 2R_0 + 1, x \in \mathbb{R}^n$. We note that there exists a constant $C > 0$ independent of $R$ such that

$$
\|\nabla \varphi_R\|_n \leq C. 
$$

It follows from

$$
-\Delta u + a \frac{\partial u}{\partial x_1} + \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in} \quad D, \quad u|_{\partial D} = 0
$$

that

$$
0 = \int_D \left\{ -\Delta u + a \frac{\partial u}{\partial x_1} + \nabla (p - \tilde{p}) \right\} \cdot (\varphi_R u) \, dx
= \int_D |\nabla u|^2 \varphi_R \, dx + \int_{R \leq |x| \leq 2R} \left\{ (\nabla u \cdot \nabla \varphi_R) u - \frac{a}{2} \frac{\partial \varphi_R}{\partial x_1}|u|^2 - (p - \tilde{p}) \nabla \varphi_R \cdot u \right\} \, dx. 
$$

Since we can see

$$
|\nabla u|u|_1, |u|^2, (p - \tilde{p})|u| \in L^{n/(n-1)}(\mathbb{R}^n \setminus B_{2R_0+1})
$$

from (2.4), letting $R \to \infty$ in (2.6) yields $\|\nabla u\|_2^2 = 0$ because of (2.5). From this together with $u|_{\partial D} = 0$, we conclude $u_1 = u_2$. \hfill \Box

**Proof of Theorem 1.1.** Let $n \geq 3$ and let $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfy (1.9). We first choose parameters $(q_1, q_2, r_1, r_2)$ satisfying

$$
\begin{align*}
\frac{n+1}{n-1} &< \alpha_1 \leq q_1 \leq n + 1 \leq q_2 \leq \alpha_2 < \frac{n(n+1)}{2}, \\
\frac{n+1}{n} &< \beta_1 \leq r_1 \leq \frac{n+1}{2} \leq r_2 \leq \beta_2 < \frac{n(n+1)}{n+2}, \\
\max \left\{ \frac{1}{\alpha_1} + \frac{2}{\alpha_2}, \frac{1}{\beta_1} + \frac{1}{\beta_2} \right\} &\leq \frac{1}{q_1} + \frac{1}{r_1} < 1, \\
\frac{2}{n} &< \frac{1}{q_2} + \frac{1}{r_2} \leq \min \left\{ \frac{1}{\alpha_2} + \frac{2}{\alpha_1}, \frac{1}{\beta_2} + \frac{1}{\beta_1} \right\}.
\end{align*}
$$

(2.10)

It is actually possible to choose those parameters. In fact, we put

$$
\alpha_1 = \frac{n+1}{n-1 - \gamma_1}, \quad \alpha_2 = \frac{n(n+1)}{2 + \gamma_2}, \quad \beta_1 = \frac{n+1}{n - \eta_1}, \quad \beta_2 = \frac{n(n+1)}{n + 2 + \eta_2},
$$

with arbitrarily small $\gamma_i, \eta_i \in (0, n-2]$ and look for $(q_1, q_2, r_1, r_2)$ of the form

$$
q_1 = \frac{n+1}{n-1 - \tilde{\gamma}_i}, \quad q_2 = \frac{n(n+1)}{2 + \tilde{\gamma}_2}, \quad r_1 = \frac{n+1}{n - \tilde{\eta}_1}, \quad r_2 = \frac{n(n+1)}{n + 2 + \tilde{\eta}_2}.
$$

Then the conditions (2.7)–(2.10) are accomplished by

$$
\begin{align*}
n - 2 &< \tilde{\gamma}_1 + \tilde{\eta}_1 \leq n - 2 + \min \{ \gamma_1, \eta_1 \}, \quad n - 2 < \tilde{\gamma}_2 + \tilde{\eta}_2 \leq n - 2 + \min \{ \gamma_2, \eta_2 \}, \\
\gamma_i &\leq \tilde{\gamma}_i, \quad \eta_i \leq \tilde{\eta}_i, \quad i = 1, 2.
\end{align*}
$$

For each $i = 1, 2$, the set of $(\tilde{\gamma}_i, \tilde{\eta}_i)$ with those conditions is nonvoid for given $\gamma_i$ and $\eta_i$; for instance, we may take $\tilde{\gamma}_i = \gamma_i, \tilde{\eta}_i = n - 2$ when $\gamma_i \leq \eta_i$ and take $\tilde{\gamma}_i = n - 2, \tilde{\eta}_i = \eta_i$ when $\gamma_i \geq \eta_i$.

To obtain a small solution, we use the contraction mapping principle. We define

$$
B := \{ u \in L^{\alpha_1}(D) \cap L^{\alpha_2}(D) \mid \nabla u \in L^{\beta_1}(D) \cap L^{\beta_1}(D) \}.
$$
which is a Banach space endowed with the norm
\[ \|u\|_B := \sum_{i=1}^{2} (a^{\frac{n-2}{n+1}} \|u\|_{\alpha_i} + a^{\frac{1}{n+1}} \|\nabla u\|_{\beta_i}). \]

Given \( v \in B \), which satisfies
\[ v \cdot \nabla v \in \bigcap_{i=1}^{2} L^{\kappa_i}(D), \quad \frac{1}{\kappa_i} = \frac{1}{q_i} + \frac{1}{r_i}, \quad 1 < \kappa_i < \frac{n}{2} \]
for \( i = 1, 2 \), we can employ Proposition 2.1 with \( f = v \cdot \nabla v, \ q = \kappa_i \) \((i = 1, 2)\) and \( u_* = -ae_1 \). Then, due to Lemma 2.3, the problem (1.11) admits a unique solution \((u, p)\) such that
\[ a^{\frac{n-2}{n+1}} \|u\|_{\mu_i} + a \left( \frac{\partial u}{\partial x_1} \right)_{\kappa_i} + a^{\frac{1}{n+1}} \|\nabla u\|_{\lambda_i} + |\nabla^2 u|_{\kappa_i} + \|\nabla p\|_{\kappa_i} \]
\[ \leq C'(|v|_{\kappa_i} + a) \leq C'(|v|_{q_i} \|\nabla v\|_{r_i} + a) \leq C'(a^{-\frac{3}{n+1}} \|v\|_B^2 + a) \]
for \( i = 1, 2 \). Here, \( 1/\lambda_i = 1/\kappa_i - 1/(n + 1) \), \( 1/\mu_i = 1/\kappa_i - 2/(n + 1) \). Furthermore, because the conditions (2.9) and (2.10) ensure \( \mu_1 \leq \alpha_1 \leq \alpha_2 \leq \mu_2 \) and \( \lambda_1 \leq \beta_1 \leq \beta_2 \leq \lambda_2 \), we find \( u \in B \) with
\[ \|u\|_B \leq 4C'(a^{-\frac{3}{n+1}} \|v\|_B^2 + a). \]

Hence, we assume
\[ a^{\frac{n-2}{n+1}} < \frac{1}{64C'^2} =: \delta \quad (2.11) \]
and set
\[ N_a := \{u \in B \mid \|u\|_B \leq 8C'a\} \]
to see that the map \( \Psi : N_a \ni v \mapsto u \in N_a \) is well-defined. Moreover, for \( v_i \in N_a \) \((i = 1, 2)\), set \( u_i = \Psi(v_i) \) and let \( p_i \) be the pressure associated with \( u_i \). Then we have
\[ \begin{cases} 
\Delta (u_1 - u_2) - a \frac{\partial}{\partial x_1} (u_1 - u_2) = \nabla (p_1 - p_2) + (v_1 - v_2) \cdot \nabla v_1 + v_2 \cdot \nabla (v_2 - v_1), & x \in D, \\
\nabla \cdot (u_1 - u_2) = 0, & x \in D, \\
(u_1 - u_2)|_{\partial D} = 0, \\
u_1 - u_2 \to 0 & \text{as } |x| \to \infty. 
\end{cases} \]

By applying Proposition 2.1 again, we find
\[ \|u_1 - u_2\|_B \leq 4C'a^{-\frac{3}{n+1}} (\|v_1\|_B + \|v_2\|_B) \|v_1 - v_2\|_B \leq 64C'^2 a^{\frac{n-2}{n+1}} \|v_1 - v_2\|_B \]
and the map \( \Psi \) is contractive on account of (2.11). The proof is complete. \( \square \)

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We define the operator \( A_a : L^q_a(D) \to L^q_a(D) \) \((a > 0, 1 < q < \infty)\) by
\[ \mathcal{Q}(A_a) = W^{2,q}(D) \cap W^{1,q}_0(D) \cap L^q_a(D), \quad A_a u = -P \left[ \Delta u - a \frac{\partial u}{\partial x_1} \right]. \]
Here, \( W^{1,q}_0(D) \) denotes the completion of \( C_0^\infty(D) \) in the Sobolev space \( W^{1,q}(D) \). It is well known that \(-A_a\) generates an analytic \( C_0\)-semigroup \( e^{-tA_a} \) called the Oseen semigroup in \( L^q_a(D) \), see Miyakawa [30, Theorem 4.2], Enomoto and Shibata [7, Theorem 4.4]. The following \( L^q-L^r \) estimates of \( e^{-tA_a} \), which play an important role in the proof of Theorem 1.2, were established by Kobayashi and Shibata [26] in the three-dimensional case and further developed by Enomoto and Shibata [7, 8] for \( n \geq 3 \). We also note that
Proposition 3.1. \([7, 8, 26]\) Let \(n \geq 3, \sigma_0 > 0\) and assume \(|a| \leq \sigma_0\).

1. Let \(1 < q \leq r \leq \infty\) (\(q \neq \infty\)). Then we have

\[
\|e^{-tA_a}f\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{r})}\|f\|_q
\]

for \(t > 0\) and \(f \in L^q(D)\), where \(C = C(n, \sigma_0, q, r, D) > 0\) is independent of \(a\).

2. Let \(1 < q \leq r \leq n\). Then we have

\[
\|\nabla e^{-tA_a}f\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{r})} - \frac{1}{2}\|f\|_q
\]

for \(t > 0\) and \(f \in L^q(D)\), where \(C = C(n, \sigma_0, q, r, D) > 0\) is independent of \(a\).

3. Let \(n/(n - 1) \leq q \leq r \leq \infty\) (\(q \neq \infty\)). Then we have

\[
\|e^{-tA_a}P\nabla \cdot F\|_r \leq Ct^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{r})} - \frac{1}{2}\|F\|_q
\]

for \(t > 0\) and \(F \in L^q(D)\), where \(C = C(n, \sigma_0, q, r, D) > 0\) is independent of \(a\).

The proof of the assertion 3 is simply based on duality argument together with semigroup property especially for the case \(r = \infty\).

We also prepare the following lemma, which plays a role to prove the uniqueness within \(Y\) defined by (1.27).

Lemma 3.2. Let \(n \geq 3\) and \(a > 0\). For each precompact set \(K \subset L^n_\sigma(D)\), we have

\[
\lim_{t \to 0} \sup_{f \in K} t^{\frac{1}{2}}(\|e^{-tA_a}f\|_\infty + \|\nabla e^{-tA_a}f\|_n) = 0. \tag{3.4}
\]

Proof. By applying Proposition 3.1 and approximating \(f \in L^n_\sigma(D)\) by a sequence in \(C^n_\sigma(D)\), we have

\[
\lim_{t \to 0} t^{\frac{1}{2}}(\|e^{-tA_a}f\|_\infty + \|\nabla e^{-tA_a}f\|_n) = 0 \tag{3.5}
\]

for all \(f \in L^n_\sigma(D)\). Given \(\eta > 0\), let \(f_1, \cdots, f_m \in K\) fulfill \(K \subset \bigcup_{j=1}^{m} B(f_j; \eta)\), where \(B(f_j; \eta) := \{g \in L^n_\sigma(D) \mid \|g - f_j\|_n < \eta\}\). For each \(f \in K\), we choose \(f_i \in K\) such that \(f \in B(f_i; \eta)\). Then it follows from (3.1) that

\[
t^{\frac{1}{2}}\|e^{-tA_a}f\|_\infty \leq t^{\frac{1}{2}}\|e^{-tA_a}f_i\|_\infty + t^{\frac{1}{2}}\|e^{-tA_a}(f - f_i)\|_\infty \leq t^{\frac{1}{2}}\|e^{-tA_a}f_i\|_\infty + C\|f - f_i\|_n \leq \sum_{j=1}^{m} t^{\frac{1}{2}}\|e^{-tA_a}f_j\|_\infty + C\eta.
\]

Since the right-hand side is independent of \(f \in K\) and since \(\eta\) is arbitrary, (3.5) yields

\[
\lim_{t \to 0} \sup_{f \in K} t^{\frac{1}{2}}\|e^{-tA_a}f\|_\infty = 0.
\]

We can discuss the \(L^n\) norm of the first derivative similarly and thus conclude (3.4). \(\square\)

We recall a function space \(Y_0\) defined by (1.22), which is a Banach space equipped with norm \(\|\cdot\|_{Y_0} = \|\cdot\|_{Y_\infty}\), where

\[
\|v\|_{Y_0} := \|v\|_{Y_\infty} + \|v\|_{Y_\infty,t} + \|\nabla v\|_{Y_\infty,t};
\]

\[
|v|_{q,t} := \sup_{0 < \tau < t} \tau^{\frac{1}{2} - \frac{1}{q}}\|v(\tau)\|_q, \quad q = n, \infty;
\]

\[
[\nabla v]_{n,t} := \sup_{0 < \tau < t} \tau^{\frac{1}{2}}\|\nabla v(\tau)\|_n
\]

for \(t \in (0, \infty]\). Construction of the solution is based on the following.
Lemma 3.3. Suppose $0 < a^{(n-2)/(n+1)} < \delta$, where $\delta$ is a constant in Theorem 1.1 with (1.15)–(1.17). Let $\psi$ be a function on $\mathbb{R}$ satisfying (1.1) and set $M = \max_{t \in \mathbb{R}} |\psi'(t)|$. Suppose that $u_s$ is the stationary solution obtained in Theorem 1.1. For $u, v \in Y_0$, we set

$$G_1(u, v)(t) = \int_0^t e^{-(t-\tau)A_n} P[u \cdot \nabla v] d\tau, \quad G_2(v)(t) = \int_0^t e^{-(t-\tau)A_n} P[\psi(\tau)v \cdot \nabla u_s] d\tau,$$

$$G_3(v)(t) = \int_0^t e^{-(t-\tau)A_n} P[\psi(\tau)u_s \cdot \nabla v] d\tau,$$

$$G_4(v)(t) = \int_0^t e^{-(t-\tau)A_n} P[(1 - \psi(\tau))a \frac{\partial v}{\partial x_1}(\tau)] d\tau,$$

$$H_1(t) = \int_0^t e^{-(t-\tau)A_n} P h_1(\tau) d\tau, \quad H_2(t) = \int_0^t e^{-(t-\tau)A_n} P h_2(\tau) d\tau,$$

where $h_1$ and $h_2$ are defined by (1.19) and (1.20), respectively. Then we have $G_1(u, v), G_i(v), H_j \in Y_0$ $(i = 2, 3, 4, j = 1, 2)$ along with

$$\|G_1(u, v)\|_{Y, t} \leq C[u]^\frac{1}{2}_{n,t}[u]^\frac{1}{2}_{\infty,t}\|\nabla v\|_{n,t}, \quad (3.6)$$

$$\|G_2(v)\|_{Y, t} \leq C(\|\nabla u_s\|_{2^+} + \|\nabla u_s\|_{2^+} + \|\nabla u_s\|_{n, t} + \|u_s\|_{n, t})\|v\|_{\infty, t}, \quad (3.7)$$

$$\|G_3(v)\|_{Y, t} \leq C(\|u_s\|_{2^+} + \|u_s\|_{n} + \|u_s\|_{n, t})\|v\|_{n, t}, \quad (3.8)$$

$$\|G_4(v)\|_{Y, t} \leq C(\|\nabla u_s\|_{2^+} + \|\nabla u_s\|_{n, t} + \|u_s\|_{n, t}), \quad (3.9)$$

$$\|H_1\|_{Y, t} \leq CM\|u_s\|_{n}, \quad (3.10)$$

$$\|H_2\|_{Y, t} \leq C(\|\nabla u_s\|_{2^+} + \|\nabla u_s\|_{n, t} + a\|\nabla u_s\|_{n, t}), \quad (3.11)$$

for all $t \in (0, \infty)$ and

$$\lim_{t \to 0} \|H_j(t)\|_{Y, t} = 0 \quad (3.12)$$

for $j = 1, 2$. Here, $C$ is a positive constant independent of $u, v, \psi, a$ and $t$.

**Proof.** The continuity of those functions in $t$ is deduced by use of properties of analytic semigroups together with Proposition 3.1 in the same way as in Fujita and Kato [15]. Since $L^\infty$ estimate is always the same as $L^n$ estimate of the first derivative, the estimate of $[\cdot]_{\infty, t}$ may be omitted. Although (3.6)–(3.8) are discussed in Enomoto and Shibata [8, Lemma 3.1] we briefly give the proof for completeness. We find that $u \in Y_0$ satisfies $u(t) \in L^{2n}(D)$ and

$$\|u(t)\|_{2n} \leq t^{-\frac{1}{4}}[u]^\frac{1}{2}_{n,t}[u]^\frac{1}{2}_{\infty,t},$$

for all $t > 0$, which together with Proposition 3.1 implies

$$\int_0^t \|e^{-(t-\tau)A_n} P[u \cdot \nabla v](\tau)\| d\tau \leq C \int_0^t (t - \tau)^{-\frac{1}{4}}\|u(\tau)\|_{2n}\|\nabla v(\tau)\|_{n} d\tau \leq C[u]^\frac{1}{2}_{n,t}[u]^\frac{1}{2}_{\infty,t}\|\nabla v\|_{n,t}$$

and

$$\int_0^t \|\nabla e^{-(t-\tau)A_n} P[u \cdot \nabla v](\tau)\| d\tau \leq C \int_0^t (t - \tau)^{-\frac{1}{4}}\|u(\tau)\|_{2n}\|\nabla v(\tau)\|_{n} d\tau \leq C[t^{-\frac{1}{4}}[u]^\frac{1}{2}_{n,t}[u]^\frac{1}{2}_{\infty,t}\|\nabla v\|_{n,t}.$$}

We thus conclude (3.6). It follows from Proposition 3.1 that

$$\int_0^t \|e^{-(t-\tau)A_n} P[\psi(\tau)v \cdot \nabla u_s]\| d\tau \leq C \int_0^t (t - \tau)^{-\frac{1}{4}}\|v(\tau)\|_{\infty}\|\nabla u_s\|_{2^+} d\tau \leq C[v]_{\infty,t}\|\nabla u_s\|_{2^+} \quad (3.13)$$
and that
\[ \int_0^t \| \nabla e^{-(t-\tau)A_n} P[\psi(\tau)v \cdot \nabla u_s] \|_n \, d\tau \leq C \int_0^t (t-\tau)^{-1+\frac{2\delta}{\alpha}} \| \psi(\tau) \|_\infty \| \nabla u_s \|_{\frac{2n}{2n+2}} \, d\tau \]
\[ \leq C t^{-\frac{1}{2}+\frac{2\delta}{\alpha}} \| v \|_\infty \| \nabla u_s \|_{\frac{2n}{2n+2}} \]  
(3.14)
for \( t > 0 \). Furthermore, for \( t \geq 2 \), we split the integral into
\[ \int_0^t \| \nabla e^{-(t-\tau)A_n} P[\psi(\tau)v \cdot \nabla u_s] \|_n \, d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t \]  
(3.15)
as in [6, 8]. By applying (3.2), we have
\[ \int_0^{\frac{t}{2}} \leq C \int_0^{\frac{t}{2}} (t-\tau)^{-1+\frac{2\delta}{\alpha}} \| \psi(\tau) \|_\infty \| \nabla u_s \|_{\frac{2n}{2n+2}} \, d\tau \leq C t^{-\frac{1}{2}} \| v \|_\infty \| \nabla u_s \|_{\frac{2n}{2n+2}}, \]  
(3.16)
\[ \int_{\frac{t}{2}}^{t-1} \leq C \int_{\frac{t}{2}}^{t-1} (t-\tau)^{-1+\frac{2\delta}{\alpha}} \| \psi(\tau) \|_\infty \| \nabla u_s \|_{\frac{2n}{2n+2}} \, d\tau \leq C t^{-\frac{1}{2}} \| v \|_\infty \| \nabla u_s \|_{\frac{2n}{2n+2}}, \]  
(3.17)
\[ \int_{t-1}^t \leq C \int_{t-1}^t (t-\tau)^{-1+\frac{2\delta}{\alpha}} \| \psi(\tau) \|_\infty \| \nabla u_s \|_{\frac{2n}{2n+2}} \, d\tau \leq C t^{-\frac{1}{2}} \| v \|_\infty \| \nabla u_s \|_{\frac{2n}{2n+2}}. \]  
(3.18)
Combining (3.13)–(3.18) yields (3.7). By the same manner, we obtain (3.8). We use Proposition 3.1 to find
\[ \int_0^t \left\| \nabla^k e^{-(t-\tau)A_n} P \left[ (1-\psi(\tau))a \frac{\partial v}{\partial x_1} \right] \right\|_n \, d\tau \leq C a \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{k}{2}} \| \nabla \psi(\tau) \|_n \, d\tau \]
\[ \leq C a [\nabla v]_{n,t} \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{k}{2}} \tau^{-\frac{1}{2}} \, d\tau \]
for \( k = 0, 1 \), which lead us to (3.9). We see (3.10) from
\[ \int_0^t \left\| \nabla^k e^{-(t-\tau)A_n} P[\psi'(\tau)u_s] \right\|_n \, d\tau \leq C M \| u_s \|_n \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{k}{2}} \, d\tau \]  
(3.19)
for \( k = 0, 1 \) and (3.11) from
\[ \int_0^t \left\| \nabla^k e^{-(t-\tau)A_n} P \left[ \psi(\tau)(1-\psi(\tau)) \left( u_s \cdot \nabla u_s + a \frac{\partial u_s}{\partial x_1} \right) \right] \right\|_n \, d\tau \]
\[ \leq C \| u_s \|_{\frac{n}{2n+2}} \| \nabla u_s \|_{\frac{2n}{2n+2}} \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{k}{2}+\frac{2\delta}{\alpha}-\frac{1}{2}} \, d\tau \]
\[ + C a \| u_s \|_{\frac{n}{2n+2}} \int_0^{\min\{1,t\}} (t-\tau)^{-\frac{k}{2}+\frac{2\delta}{\alpha}-\frac{1}{2}} \, d\tau \]  
(3.20)
for \( k = 0, 1 \), where the condition (1.17) is used. The behavior of \( G_1(u,v)(t) \) and \( G_i(v)(t) \) as well as the one of \( H_j(t) \), see (3.12), as \( t \to 0 \) follows from (3.6)–(3.9) and (3.19)–(3.20) with \( t < 1 \), so that \( G_1(u,v), G_i(v), H_j \in Y_0 \) and \( \| G_1(u,v)(t) \|_n + \| G_i(v)(t) \|_n + \| H_j(t) \|_n \to 0 \) as \( t \to 0 \). The proof is complete. \( \square \)

Let us construct a solution of (1.21) by applying Lemma 3.3.

**Proposition 3.4.** Let \( \delta \) be the constant in Theorem 1.1 with (1.15)–(1.17). Let \( \psi \) be a function on \( \mathbb{R} \) satisfying (1.1) and set \( M = \max_{t \in \mathbb{R}} |\psi'(t)| \). Then there exists a constant \( \varepsilon = \varepsilon(n, D) \in (0, \delta) \) such that if \( 0 < (M+1)a^{(n-2)/(n+1)} < \varepsilon \), (1.21) admits a solution \( v \in Y_0 \) with
\[ \| v \|_Y \leq C(M+1)a^{\frac{n}{n+2}} \]  
(3.21)
and
\[ \lim_{t \to 0} \|v(t)\|_n = 0. \tag{3.22} \]

Proof. We set
\[ v_0(t) = 0, \]
\[ v_{m+1}(t) = \int_0^t e^{-(t-\tau)A_s}P \left[ -v_m \cdot \nabla v_m - \psi(\tau)v_m \cdot \nabla u_s - \psi(\tau)u_s \cdot \nabla v_m + (1 - \psi(\tau))a \frac{\partial v_m}{\partial x_1} \right. \]
\[ + h_1(\tau) + h_2(\tau) \big] d\tau \tag{3.23} \]
for \( m \geq 0 \). It follows from Theorem 1.1, Lemma 3.3 and \( a \in (0, 1) \) that \( v_m \in Y_0 \) together with
\[ \|v_m\|_{Y,t} \leq \|G_1(v_{m-1}, v_{m-1})\|_{Y,t} + \sum_{i=2}^4 \|G_i(v_{m-1})\|_{Y,t} + \|H_1\|_{Y,t} + \|H_2\|_{Y,t}, \tag{3.24} \]
\[ \|v_m\|_Y \leq C_1\|v_{m-1}\|^2_Y + C_2a^{\frac{n-2}{n+1}}\|v_{m-1}\|_Y + C_3(M+1)a^{\frac{n-2}{n+1}}, \]
\[ \|v_{m+1} - v_m\|_Y \leq \{C_1(\|v_m\|_Y + \|v_{m-1}\|_Y) + C_2a^{\frac{n-2}{n+1}}\}\|v_m - v_{m-1}\|_Y \tag{3.25} \]
for all \( m \geq 1 \). Hence, if we assume
\[ (M+1)a^{\frac{n-2}{n+1}} < \min \left\{ \delta, \frac{1}{2C_2}, \frac{1}{16C_1C_3} \right\} =: \varepsilon, \tag{3.26} \]
it holds that
\[ \|v_m\|_Y \leq \frac{1 - C_2a^{\frac{n-2}{n+1}} - \sqrt{(1 - C_2a^{\frac{n-2}{n+1}})^2 - 4C_1C_3(M+1)a^{\frac{n-2}{n+1}}}}{2C_1} \leq 4C_3(M+1)a^{\frac{n-2}{n+1}}, \]
\[ \|v_{m+1} - v_m\|_Y \leq \{8C_1C_3(M+1)a^{\frac{n-2}{n+1}} + C_2a^{\frac{n-2}{n+1}}\}\|v_m - v_{m-1}\|_Y \tag{3.27} \]
for all \( m \geq 1 \) and that
\[ 8C_1C_3(M+1)a^{\frac{n-2}{n+1}} + C_2a^{\frac{n-2}{n+1}} < 1. \]
Therefore, we obtain a solution \( v \in Y_0 \) satisfying (3.21) with \( C = 4C_3 \). Moreover, by letting \( m \to \infty \) in (3.24) and by using (3.6)–(3.9) and (3.12), we have (3.22), which completes the proof. \( \square \)

Remark 3.5. Let \( b \in L^q(D) \). By the same procedure, we can also construct a solution \( T(t)b := v(t) \in Y_0 \) for the integral equation
\[ v(t) = e^{-tA_s}b + \int_0^t e^{-(t-\tau)A_s}P \left[ -v \cdot \nabla v - \psi(\tau)v \cdot \nabla u_s - \psi(\tau)u_s \cdot \nabla v \right. \]
\[ + (1 - \psi(\tau))a \frac{\partial v}{\partial x_1} \big] \bigg|_{\tau=\tau(t)} d\tau \tag{3.28} \]
whenever
\[ \|b\|_n + (M+1)a^{\frac{n-2}{n+1}} < \min \left\{ \delta, \frac{1}{2C_2}, \frac{1}{16C_1C_0}, \frac{1}{16C_1C_3} \right\} \]
is satisfied. Here, the constant \( C_0 \) is determined by the following three estimates:
\[ \|e^{-tA_s}b\|_q \leq C_0t^{-\frac{q}{2}+\frac{2}{q}}\|b\|_n, \quad q = n, \infty; \quad \|\nabla e^{-tA_s}b\|_n \leq C_0t^{-\frac{1}{2}}\|b\|_n. \]
Moreover, we find that the solution \( T(t)b \) is estimated by
\[ \|T(\cdot)b\|_Y \leq 4(C_0\|b\|_n + C_3(M+1)a^{\frac{n-2}{n+1}}). \]
This will be used in the proof of uniqueness of solutions within \( Y \), see (1.27).
We further derive sharp decay properties of the solution $v(t)$ obtained above. To this end, the first step is the following. In what follows, for simplicity of notation, we write

$$G_i(t) = G_i(v(t)), \quad G_i(t) = G_i(v(t))$$

for $i = 2, 3, 4$, which are defined in Lemma 3.3.

**Lemma 3.6.** Let $\varepsilon$ be the constant in Proposition 3.4. Given $\rho \in (0, 1)$ satisfying $\rho \leq \min\{\rho_1, \rho_3\}$, there exists a constant $\varepsilon' = \varepsilon'(\rho, n, D) \in (0, \varepsilon]$ such that if $0 < (M + 1)a^{(n-2)/(n+1)} < \varepsilon'$, then the solution $v(t)$ obtained in Proposition 3.4 satisfies

$$\|v(t)\|_q = O(t^{-\frac{1}{2} + \frac{n-2}{2q - n}}), \quad n \leq \forall q \leq \infty, \quad (3.29)$$

as $t \to \infty$.

**Proof.** We start with the case $q = n$, that is,

$$\|v(t)\|_n = O(t^{-\frac{n}{2}}) \quad (3.31)$$

as $t \to \infty$. By using (3.1), we have

$$\|G_1(t)\|_n \leq Ct^{-\frac{n}{2}} \left( \sup_{0 < \tau < t} \tau^\frac{n}{2} \|v(\tau)\|_n \right) \left( \sup_{0 < \tau < t} \tau^\frac{n}{2} \|v(\tau)\|_n \right) \leq Ct^{-\frac{n}{2}} \|v\|_Y \sup_{0 < \tau < t} \tau^\frac{n}{2} \|v(\tau)\|_n, \quad (3.32)$$

$$\|G_2(t)\|_n \leq Ct^{-\frac{n}{2}} \|\nabla u_s\|_n \sup_{0 < \tau < t} \tau^\frac{n}{2} \|v(\tau)\|_\infty \leq Ct^{-\frac{n}{2}} \|\nabla u_s\|_n \sup_{0 < \tau < t} \tau^\frac{n}{2} \|v(\tau)\|_Y \quad (3.33)$$

and

$$\|G_3(t)\|_n \leq Ct^{-\frac{n}{2}} \|u_s\|_{\frac{n}{1+\rho_3}} \|v\|_Y \quad (3.34)$$

for all $t > 0$. Moreover, we obtain

$$\|G_4(t)\|_n \leq Ca \int_0^{\min\{1, t\}} (t - \tau)^{-\frac{1}{2}} \|v(\tau)\|_n d\tau \leq Cat^{-\frac{1}{2}} \|v\|_Y \quad (3.35)$$

for all $t > 0$ by use of (3.3). From (3.1) we see that

$$\|H_1(t)\|_n \leq C M t^{-\frac{n}{2}} \|u_s\|_{\frac{n}{1+\rho_1}} \quad (3.36)$$

and that

$$\|H_2(t)\|_n \leq Ct^{-\frac{n}{2} - \rho_2} \|u_s\|_{\frac{n}{1+\rho_3}} \|\nabla u_s\|_{\frac{n}{2}} + Cat^{-\frac{n}{2} - \rho_2} \|\nabla u_s\|_{\frac{n}{1+\rho_3}} \quad (3.37)$$

for $t > 0$. Note that $\rho_2 < 1$, see (1.16). Collecting (3.32)–(3.37) for $t > 1$ and (2.21) with $C = 4C_3$ yields

$$\sup_{0 < \tau < t} \tau^\frac{n}{2} \|v(\tau)\|_n \leq C_4 \|v\|_Y \sup_{0 < \tau < t} \tau^\frac{n}{2} \|v(\tau)\|_n + C_5$$

$$\leq 4C_3C_4(M + 1)a^{\frac{n-2}{n+2}} \sup_{0 < \tau < t} \tau^\frac{n}{2} \|v(\tau)\|_n + C_5$$

with some constants $C_4 = C_4(\rho) > 0$ and $C_5 = C_5(\|v\|_Y, u_s, a, M, \rho_1, \rho_2, \rho_3) > 0$ independent of $t$, where $C_3$ comes from estimates of $H_3(t)$ ($j = 1, 2$) in (2.25). Therefore, if we assume

$$(M + 1)a^{\frac{n-2}{n+2}} < \min \left\{ \varepsilon, \frac{1}{4C_3C_4} \right\} =: \varepsilon',$$

we have $\|v(t)\|_n \leq Ct^{-\rho/2}$ for all $t > 0$, which implies (3.31).

We next show that

$$\|v(t)\|_\infty + \|\nabla v(t)\|_n = O(t^{-\frac{1}{2} - \frac{n}{2}})$$

as $t \to \infty$, which together with (3.31) implies (3.29) and (3.30). It suffices to show that

$$t^{\frac{1}{2}} \|v(t)\|_\infty + t^{\frac{n}{2}} \|\nabla v(t)\|_n \leq C \left\|v \left( \frac{t}{2} \right) \right\|_n \quad (3.38)$$
for all $t \geq 2$. The following argument is similar to Enomoto and Shibata [8]. When $t \geq T > 1$, we have

$$v(t) = e^{-(t-T)A_n}v(T) - \int_T^t e^{-(t-\tau)A_n}P \left[ v \cdot \nabla v + v \cdot \nabla u_s + u_s \cdot \nabla v \right] d\tau. \quad (3.39)$$

By the same argument as in the proof of Lemma 3.3 and by (1.10), (3.26) as well as (3.21) with $C = 4C_3$, the integral of (3.39) is estimated as

$$\begin{align*}
\int_T^t \| e^{-(t-\tau)A_n}P[\cdots] \|_\infty d\tau + \int_T^t \| \nabla e^{-(t-\tau)A_n}P[\cdots] \|_n d\tau \\
\leq C_1(t-T)^{-\frac{1}{2}} \left( \sup_{T \leq \tau \leq t} \| v(\tau) \|_n \right)^{\frac{1}{2}} \left( \sup_{T \leq \tau \leq t} (\tau - T)^{-\frac{3}{2}} \| v(\tau) \|_\infty \right)^{\frac{1}{2}} \left( \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| \nabla v(\tau) \|_n \right) \\
+ C_2 a^{\frac{n-2}{n+1}}(t-T)^{-\frac{3}{2}} \left\{ \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| v(\tau) \|_\infty + \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| \nabla v(\tau) \|_n \right\} \\
\leq C_1(t-T)^{-\frac{1}{2}} \| v \|_n \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| \nabla v(\tau) \|_n \\
+ \frac{1}{2} (t-T)^{-\frac{1}{2}} \left\{ \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| v(\tau) \|_\infty + \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| \nabla v(\tau) \|_n \right\} \\
\leq \frac{3}{4} (t-T)^{-\frac{1}{2}} \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| \nabla v(\tau) \|_n \| v \|_n + \frac{1}{2} (t-T)^{-\frac{1}{2}} \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| v(\tau) \|_\infty.
\end{align*}$$

Therefore, we have

$$\sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| \nabla v(\tau) \|_n + \sup_{T \leq \tau \leq t} (\tau - T)^{\frac{1}{2}} \| v(\tau) \|_\infty \leq C \| v(T) \|_n$$

for all $t \geq T$. This combined with $t^{1/2} \leq \sqrt{2}(t-T)^{1/2}$ for $t \geq 2T$ asserts that

$$t^{\frac{1}{2}} \| \nabla v(t) \|_n + t^{\frac{1}{2}} \| v(t) \|_\infty \leq C \| v(T) \|_n$$

for all $t \geq 2T$. We then put $T = t/2$ ($t \geq 2$) to conclude (3.38).

Sharp decay properties (1.23)–(1.24) for the case $n = 3$ are established in the following proposition.

**Proposition 3.7.** Let $n = 3$ and set $\varepsilon_\ast := \varepsilon'(\rho, 3, D)$ which is the constant in Lemma 3.6 with $\rho := \min\{\rho_1, \rho_3\}$ (recall that $0 < \rho_1 < 1/2$, $0 < \rho_3 < 1/4$ for $n = 3$). If $0 < (M + 1)a^{1/4} < \varepsilon_\ast$, then the solution $v(t)$ obtained in Proposition 3.4 enjoys (1.23) and (1.24).

**Proof.** The case $\rho_1 \leq \rho_3$ directly follows from Lemma 3.6. To discuss the other case $\rho_3 < \rho_1$, we show by induction that if $0 < (M + 1)a^{1/4} < \varepsilon_\ast$, then

$$\| v(t) \|_3 = O(t^{-\sigma_k}), \quad \sigma_k := \min \left\{ \frac{k}{2} \rho_3, \frac{\rho_1}{2} \right\} \quad (3.40)$$

as $t \to \infty$ for all $k \geq 1$. We already know (3.40) with $k = 1$ from Lemma 3.6.

Let $k \geq 2$ and suppose (3.40) with $k - 1$. By taking (3.21) (near $t = 0$) and (3.38) into account, we have

$$J_{k-1}(v) := \sup_{\tau > 0} (1 + \tau)^{\sigma_k - 1} \| v(\tau) \|_3 + \sup_{\tau > 0} \tau^{\frac{1}{2}} (1 + \tau)^{\sigma_k - 1} \left( \| v(\tau) \|_\infty + \| \nabla v(\tau) \|_3 \right) < \infty.$$ 

We use this to see that

$$\| G_1(t) \|_3 \leq C \int_0^t (t - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} (1 + \tau)^{-2\sigma_k - 1} d\tau \times \left( \sup_{\tau > 0} (1 + \tau)^{\sigma_k - 1} \| v(\tau) \|_3 \right) \left( \sup_{\tau > 0} \tau^{\frac{1}{2}} (1 + \tau)^{\sigma_k - 1} \| \nabla v(\tau) \|_3 \right) \leq Ct^{-2\sigma_k - 1} J_{k-1}(v)^2,$$
and that
\[ \|G_2(t)\|_3 \leq C \int_0^t (t - \tau)^{-\frac{1 + \rho_1}{2}} \tau^{-\frac{1}{2}} (1 + \tau)^{-\sigma_k - 1} d\tau \sup_{\tau > 0} \tau^{\frac{3}{2}} (1 + \tau)^\sigma ||v(t)||_\infty \]
\[ \leq Ct^{-\frac{\rho_2}{2} - \sigma_k - 1} \|\nabla u\|_{2v/\tau} J_{k-1}(v) \]
for \( t > 0 \) due to \( \sigma_{k-1} \leq \rho_1/2 < 1/4 \). From these and (3.34)–(3.37), we obtain (3.40) with \( k \). We thus conclude (1.23) with \( q = 3 \), which together with (3.38) completes the proof.

To derive even more rapid decay properties of the solution \( v(t) \) for \( n \geq 4 \), we need the following lemma, which gives the \( L^q \)-decay of \( v(t) \) with a specific \( q_0 \), see (3.43).

**Lemma 3.8.** Let \( n \geq 4 \). Suppose \( 1 < \rho_1 \leq 1 + \rho_3 \) in addition to (1.16) (the set of those parameters is nonvoid when \( n \geq 4 \)). Let \( \varepsilon \) be the constant in Proposition 3.4 and \( v(t) \) the solution obtained there. Given \( \gamma \) satisfying
\[ \max\left\{ 0, \frac{\rho_1 + 3 - n}{2}\right\} < \gamma < \frac{1}{2} \]  
(3.41)
(note that (1.16) yields \( \rho_1 < n - 2 \)), there exists a constant \( \varepsilon'' = \varepsilon''(\gamma, n, D) \in (0, \varepsilon] \) such that if \( 0 < (M + 1) a^{(n-2)/(n+1)} < \varepsilon'' \), then \( v(t) \in L^{q_0}(D) \) for all \( t > 0 \) and
\[ \sup_{\tau > 0} (1 + \tau)^\gamma \|v(t)\|_{q_0} < \infty, \]
(3.42)
where
\[ q_0 := \frac{n}{1 + \rho_1 - 2\gamma} (< n). \]  
(3.43)

**Proof.** We show that there exists a constant \( \varepsilon''(\gamma, n, D) \in (0, \varepsilon] \) such that if \( 0 < (M + 1) a^{(n-2)/(n+1)} < \varepsilon'' \), then \( v_m(t) \in L^{q_0}(D) \) for all \( t > 0 \) along with
\[ K_m := \sup_{\tau > 0} (1 + \tau)^\gamma \|v_m(\tau)\|_{q_0} < \infty, \]  
\[ K_m \leq \frac{1}{2} K_{m-1} + C (M + 1) a^{\frac{n-1}{n+1}} \]  
(3.44)
for all \( m \geq 1 \), where \( v_m(t) \) is the approximate solution defined by (3.23) and \( C \) is a positive constant independent of \( a \) and \( m \). We use (3.1) to see that
\[ \int_0^t \|e^{-(t-\tau)A_n} Ph_1(\tau)\|_{q_0} d\tau \leq CM \|u\|_{\frac{1}{1+\rho_1}} \int_0^{\min\{1,t\}} (t - \tau)^{-\gamma} d\tau \leq CM \|u\|_{\frac{n}{1+\rho_1}} (1 + t)^{-\gamma} \]  
(3.45)
for \( t > 0 \). Moreover, it holds that
\[ \int_0^t \|e^{-(t-\tau)A_n} P[\psi(\tau)(1 - \psi(\tau)) a \frac{\partial u}{\partial x_1}]\|_{q_0} d\tau \leq Ca \|\nabla u\|_r \]  
for \( t \leq 2 \), where \( r := \min\{n/(2 - \rho_4), q_0\} \) and that
\[ \int_0^t \|e^{-(t-\tau)A_n} P[\psi(\tau)(1 - \psi(\tau)) a \frac{\partial u}{\partial x_1}]\|_{q_0} d\tau \leq Ca \|\nabla u\|_{\frac{n}{2+\rho_3}} \int_0^1 (t - \tau)^{-\gamma} (1 + \tau)^{-\frac{1 + \rho_3 - \rho_1}{2}} d\tau \]
\[ \leq Ca \|\nabla u\|_{\frac{n}{2+\rho_3}} t^{-\gamma} \]  
for \( t > 2 \) as well as that
\[ \int_0^t \|e^{-(t-\tau)A_n} P[\psi(\tau)(1 - \psi(\tau)) u \cdot \nabla u\|_{q_0} d\tau \leq C \|u\|_{\frac{n}{1+\rho_1}} \|\nabla u\|_{\frac{2}{2+\rho_3}} \int_0^{\min\{1,t\}} (t - \tau)^{-1 - \gamma + \frac{\rho_1 - \rho_3}{2}} d\tau \]
\[ \leq C \|u\|_{\frac{n}{1+\rho_1}} \|\nabla u\|_{\frac{2}{2+\rho_3}} (1 + t)^{-\gamma} \]
for \( t > 0 \), where \( \max\{0, \rho_1 - 2\} < \kappa < \min\{n - 3, \rho_1 - 2\gamma\} \) (note that (1.16) yields \( \rho_1 < n - 1 \)). These estimates imply
\[ \int_0^t \|e^{-(t-\tau)A_n} Ph_2(\tau)\|_{q_0} d\tau \leq C(a \|\nabla u\|_r + a \|\nabla u\|_{\frac{n}{2+\rho_3}} + \|u\|_{\frac{n}{1+\rho_1}} \|\nabla u\|_{\frac{2}{2+\rho_3}} (1 + t)^{-\gamma} \]
for $t > 0$, which together with (3.45) and (1.10) leads us to $v_1(t) \in L^{q_0}(D)$ for all $t > 0$ with
\[ K_1 \leq C(M + 1)a^{\frac{n+1}{n}}. \]  
(3.46)

This proves (3.44) with $m = 1$ since $K_0 = 0$.

Let $m \geq 2$ and suppose that $v_{m-1}(t) \in L^{q_0}(D)$ for all $t > 0$ and (3.44) with $m - 1$. Then we have $G_1(v_{m-1}, v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with
\[ \sup_{\tau > 0} (1 + \tau)\gamma \|G_1(v_{m-1}, v_{m-1})(\tau)\|_{q_0} \leq CK_{m-1} \sup_{\tau > 0} \tau^{\frac{1}{2}} \|\nabla v_{m-1}(\tau)\|_n. \]  
(3.47)

Let $t \geq 2$ and split the integral into
\[ \int_0^t \|e^{-(t-\tau)A_s}P[\psi(\tau)u_s \cdot \nabla v_{m-1}]\|_{q_0} d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t. \]

Let $\lambda \in (0, \rho_1]$ satisfy $\lambda < n - 3 + 2\gamma - \rho_1$; in fact, we can take such $\lambda$ due to (3.41). Then (3.3) with $F = v_{m-1} \otimes u_s$ implies
\[
\begin{align*}
\int_0^{\frac{t}{2}} \leq C \int_0^{\frac{t}{2}} (t-\tau)^{-1} \|u_s\|_n \|v_{m-1}(\tau)\|_{q_0} d\tau &\leq Ct^{-\gamma} \|u_s\|_n K_{m-1}, \\
\int_{\frac{t}{2}}^{t-1} \leq C \int_{\frac{t}{2}}^{t-1} (t - \tau)^{-1 - \frac{1}{n}} \|u_s\|_{\frac{n+1}{n-1}} \|v_{m-1}(\tau)\|_{q_0} d\tau &\leq Ct^{-\gamma} \|u_s\|_{\frac{n+1}{n-1}} K_{m-1}, \\
\int_{t-1}^t \leq C \int_{t-1}^t (t - \tau)^{-1 + \frac{2}{n-2}} \|u_s\|_{\frac{n-2}{n}} \|v_{m-1}(\tau)\|_{q_0} d\tau &\leq Ct^{-\gamma} \|u_s\|_{\frac{n-2}{n}} K_{m-1}
\end{align*}
\]
for $t \geq 2$. Moreover, we use (3.3) again to see that
\[
\begin{align*}
\int_0^t \|e^{-(t-\tau)A_s}P[\psi(\tau)u_s \cdot \nabla v_{m-1}]\|_{q_0} d\tau &\leq C \int_0^t (t - \tau)^{-1 + \frac{2}{n-2}} \|u_s\|_{\frac{n-2}{n}} \|v_{m-1}(\tau)\|_{q_0} d\tau \\
&\leq C \|u_s\|_{\frac{n-2}{n}} K_{m-1}
\end{align*}
\]
for $t \leq 2$. We thus conclude $G_3(v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with
\[ \sup_{\tau > 0} (1 + \tau)\gamma \|G_3(v_{m-1})(\tau)\|_{q_0} \leq C(\|u_s\|_n + \|u_s\|_{\frac{n+1}{n-1}} + \|u_s\|_{\frac{n-2}{n}})K_{m-1}. \]  
(3.48)

By the same calculation, we have $G_2(v_{m-1})(t) \in L^{q_0}(D)$ for $t > 0$ with
\[ \sup_{\tau > 0} (1 + \tau)\gamma \|G_2(v_{m-1})(\tau)\|_{q_0} \leq C(\|u_s\|_n + \|u_s\|_{\frac{n+1}{n-1}} + \|u_s\|_{\frac{n-2}{n}})K_{m-1}. \]  
(3.49)

We also have
\[
\begin{align*}
\int_0^t \|e^{-(t-\tau)A_s}P[(1 - \psi(\tau))a\partial_{v_{m-1}}]\|_{q_0} &\leq Ca \int_0^{\min\{1,t\}} (t - \tau)^{-\frac{1}{2}} \|v_{m-1}(\tau)\|_{q_0} d\tau \\
&\leq CaK_{m-1} (1 + t)^{-\frac{1}{2}} \leq CaK_{m-1} (1 + t)^{-\gamma}
\end{align*}
\]
for $t > 0$ by (3.3). This together with (3.46)–(3.49), (1.10) and (3.27) yields $v_m(t) \in L^{q_0}(D)$ for $t > 0$ and
\[
K_m \leq C(M + 1)a^{\frac{n+1}{n+2}} + \tilde{C} \left\{ \left( \sup_{\tau > 0} \tau^{\frac{1}{2}} \|\nabla v_{m-1}(\tau)\|_n \right) + \|u_s\|_n + \|u_s\|_{\frac{n+1}{n-1}} + \|u_s\|_{\frac{n-2}{n}} + a \right\}K_{m-1}
\leq C(M + 1)a^{\frac{n+1}{n+2}} + \tilde{C} \left( 4C_3 + \tilde{C}_2 \right)(M + 1)a^{\frac{n-2}{n+1}}K_{m-1}.
\]

Suppose
\[
(M + 1)a^{\frac{n-2}{n+1}} < \min \left\{ \varepsilon, \frac{1}{2\tilde{C}_1(4C_3 + \tilde{C}_2)} \right\} =: \varepsilon''.
\]
then we get (3.44) with $m$ and, thereby, conclude
\[
K_m \leq 2C(M + 1)a^{\frac{n+1}{n+2}}.
\]
for all \( m \geq 1 \). Since we know that \( \|v_m(t) - v(t)\|_n \to 0 \) as \( m \to \infty \) for each \( t > 0 \), we obtain \( v(t) \in L^{q_0}(D) \) for \( t > 0 \) with
\[
\sup_{\tau > 0} (1 + \tau)^{\gamma} \|v(\tau)\|_{q_0} \leq 2C(M + 1)a^{\frac{n-1}{n+1}} < \infty,
\]
which completes the proof. \( \square \)

In view of Lemmas 3.6 and 3.8, we prove sharp decay properties (1.25)–(1.26) for \( n \geq 4 \).

**Proposition 3.9.** Let \( n \geq 4 \). Suppose \( \rho_3 > 1 \) and \( 1 < \rho_1 \leq 1 + \rho_3 \) in addition to (1.16) (the set of those parameters is nonvoid when \( n \geq 4 \)). Let \( \varepsilon \) be the constant in Proposition 3.4. There exists a constant \( \varepsilon_* = \varepsilon_*(n, D) \in (0, \varepsilon] \) such that if \( 0 < (M + 1)a^{(n-2)/(n+1)} < \varepsilon_* \), then the solution \( v(t) \) obtained in Proposition 3.4 enjoys (1.25) and (1.26).

**Proof.** Fix \( 1/2 < \rho < 1 \) and \( \gamma > 0 \) such that
\[
\max \left\{ \frac{1}{2} - \frac{\rho}{2}, \frac{\rho + 3 - n}{2} \right\} < \gamma < \frac{1}{2}, \tag{3.50}
\]
Let \( \varepsilon'(\rho, n, D) \) and \( \varepsilon''(\gamma, n, D) \) be the constants in Lemmas 3.6 and 3.8, respectively. We show by induction that if
\[
(M + 1)a^{\frac{n-2}{n+1}} < \min\{\varepsilon'(\rho, n, D), \varepsilon''(\gamma, n, D)\} =: \varepsilon_*(n, D),
\]
then \( v(t) \) satisfies
\[
\|v(t)\|_n = O(t^{-\sigma_k}), \quad \sigma_k := \min\left\{ \frac{k}{2}, \frac{\rho_1}{2} \right\} \tag{3.51}
\]
as \( t \to \infty \) for all \( k \geq 1 \). This implies (1.25) with \( q = n \), which together with (3.38) completes the proof. Since \( \rho < \rho_1 \), (3.51) with \( k = 1 \) follows from Lemma 3.6. We note that \( \sigma_1 < 1/2 \) and \( \sigma_k > 1/2 \) for \( k \geq 2 \).

Let \( k \geq 2 \) and suppose (3.51) with \( k - 1 \). Then
\[
L_k-1(v) := \sup_{\tau > 0} (1 + \tau)^{\sigma_k-1} \|v(\tau)\|_n + \sup_{\tau > 0} \frac{1}{2}(1 + \tau)^{\sigma_k-1} (\|v(\tau)\|_{q_0} + \|\nabla v(\tau)\|_n) < \infty
\]
holds due to (3.21) (near \( t = 0 \)) as well as (3.38). In what follows, we always assume \( t \geq 2 \). From (3.42), it follows that
\[
\|G_1(t)\|_n \leq \int_0^\frac{t}{2} (t - \tau)^{-\frac{\rho_0}{n}} \|v(\tau)\|_{q_0} \|\nabla v(\tau)\|_n d\tau + \int_{\frac{t}{2}}^t (t - \tau)^{-\frac{1}{2}} \|v(\tau)\|_n \|\nabla v(\tau)\|_n d\tau =: I + II \tag{3.52}
\]
with
\[
I \leq Ct^{-\frac{\rho_0}{n}} (\sup_{\tau > 0} (1 + \tau)^\gamma \|v(\tau)\|_{q_0}) L_{k-1}(v) \leq Ct^{-\frac{\rho_1}{2}} (\sup_{\tau > 0} (1 + \tau)^\gamma \|v(\tau)\|_{q_0}) L_{k-1}(v), \tag{3.53}
\]
where (3.43) and (3.50) are taken into account and
\[
II \leq Ct^{-2\sigma_k-1} L_{k-1}(v)^2. \tag{3.54}
\]
For \( G_2(t) \), we split the integral into
\[
\int_0^t \|e^{-(t-\tau)A_N P[\psi(\tau)v \cdot \nabla u_s]}\|_n d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t.
\]
Then we find
\[
\int_0^{\frac{t}{2}} \leq C \int_0^{\frac{t}{2}} (t - \tau)^{-\frac{1+k_3}{2} - \frac{1}{2}} (1 + \tau)^{-\sigma_k-1} d\tau \|\nabla u_s\|_{q_0} (\sup_{\tau > 0} (1 + \tau)^{\sigma_k-1} \|v(\tau)\|_{q_0}) \leq \begin{cases} Ct^{-\frac{\rho_1}{2}} \|\nabla u_s\|_{q_0} L_{k-1}(v) \leq Ct^{-\frac{\rho_1}{2}} \|\nabla u_s\|_{q_0} L_{k-1}(v) \quad &\text{if } k = 2, \\ Ct^{-\frac{1+k_3}{2}} \|\nabla u_s\|_{q_0} L_{k-1}(v) \leq Ct^{-\frac{\rho_1}{2}} \|\nabla u_s\|_{q_0} L_{k-1}(v) \quad &\text{if } k \geq 3 \end{cases}
\]
and
\[
\int_{\frac{t}{2}}^{t} + \int_{t-1}^{t} \leq C t^{-\sigma_k-\frac{1}{2}} \left( \|\nabla u_s\|_{1} + \|\nabla u_s\|_{2} \right) L_{k-1}(v),
\]
where we have used \( \rho_3 > 1 \) and \( \rho_1 \leq 1 + \rho_3 \). Estimates above imply that
\[
\|G_2(t)\|_n \leq C t^{-\sigma_k} \left( \|\nabla u_s\|_{1} + \|\nabla u_s\|_{2} \right) L_{k-1}(v).
\]
Similarly, we observe
\[
\|G_3(t)\|_n \leq C t^{-\sigma_k} \left( \|\nabla u_s\|_{1} + \|\nabla u_s\|_{n} \right) L_{k-1}(v).
\]
Moreover, by the same manner as in the proof of Lemma 3.6, we obtain
\[
\|G_4(t)\|_n \leq C t^{-\frac{\rho_1\rho_3}{\sigma_0}} \sup_{\tau > 0} (1 + \tau)^{\gamma} \|v(\tau)\|_{q_0} \leq C t^{-\frac{\rho_1}{\rho_3}} \sup_{\tau > 0} (1 + \tau)^{\gamma} \|v(\tau)\|_{q_0},
\]
\[
\|H_1(t)\|_n \leq C M t^{-\frac{\rho_1}{\rho_3}} \|u_s\|_{1},
\]
\[
\|H_2(t)\|_n \leq C t^{-\frac{\rho_1}{\rho_3}} \|u_s\|_{1} \left\| \nabla u_s \right\|_{2} + C t^{-\frac{\rho_1}{\rho_3}} \left\| \nabla u_s \right\|_{2} + \|\nabla u_s\|_{n}
\]
\[
\leq C t^{-\frac{\rho_1}{\rho_3}} \left( \|u_s\|_{1} \left\| \nabla u_s \right\|_{2} + \|\nabla u_s\|_{n} \right)
\]
for all \( t \geq 2 \), where \( \kappa \) is chosen such that \( \max\{0, \rho_1 - 2\} < \kappa < \min\{n - 3, \rho_1\} \). Collecting (3.52)–(3.59), we conclude (3.51) with \( k \). The proof is complete. \( \square \)

We next consider the uniqueness. We begin with the classical result on the uniqueness of solutions within \( Y_0 \) as in Fujita and Kato [15].

**Lemma 3.10.** Let \( \psi \) be a function on \( \mathbb{R} \) satisfying (1.1) and let \( \delta \) be the constant in Theorem 1.1 with (1.15)–(1.17). Then there exists a constant \( \bar{\varepsilon} = \bar{\varepsilon}(n, D) \in (0, \delta) \) such that if \( 0 < a^{(n-2)/(n+1)} < \bar{\varepsilon} \), (1.21) admits at most one solution within \( Y_0 \).

**Proof.** The following argument is based on [15]. Suppose that \( v, \bar{v} \in Y_0 \) are solutions. Then we have
\[
\|v - \bar{v}\|_{Y, t} \leq \left\{ C_1 \left( \|\nabla v\|_{n, t} + \|\bar{v}\|_{n, t} \right) + C_2 a^{\frac{n-2}{n+1}} \right\} \|v - \bar{v}\|_{Y, t}, \quad t > 0
\]
by applying (1.10) and Lemma 3.3. If we assume
\[
a^{\frac{n-2}{n+1}} < \min \left\{ \delta, \frac{1}{2C_2} \right\} =: \bar{\varepsilon}
\]
and choose \( t_0 > 0 \) such that
\[
C_1 \left\{ \|\nabla v\|_{n, t_0} + \left( \sup_{0 < \tau < \infty} \|\bar{v}(\tau)\|_n \right)^{\frac{1}{2}} \left( \bar{v}\right)_{\infty, t_0} \right\} < \frac{1}{2},
\]
then (3.60) yields \( \|v - \bar{v}\|_{Y, t_0} = 0 \). Hence, we conclude \( v = \bar{v} \) on \( (0, t_0) \) and obtain
\[
v(t) - \bar{v}(t) = \int_{t_0}^{t} e^{-\left( t - \tau \right) a^{\frac{n-2}{n+1}}} \left[ - (v - \bar{v}) \cdot \nabla v - \bar{v} \cdot \nabla (v - \bar{v}) - \psi(\tau)(v - \bar{v}) \cdot \nabla u_s - \psi(\tau) u_s \cdot \nabla (v - \bar{v}) + (1 - \psi(\tau)) a \frac{\partial}{\partial x_1} (v - \bar{v}) \right] d\tau.
\]
By the same argument as in the proof of Lemma 3.3 together with (1.10), we see that
\[
\|v - \bar{v}\|_{Y, t_0, t} \leq C_3 \|v - \bar{v}\|_{Y, t_0, t}
\]
(3.62)
for all $t > t_0$, where
\begin{equation}
\|v\|_{Y,t_0,t} := \sup_{t_0 \leq \tau \leq t} \|v(\tau)\|_n + \sup_{t_0 \leq \tau \leq t} \|v(\tau)\|_\infty + \sup_{t_0 \leq \tau \leq t} \|\nabla v(\tau)\|_n,
\end{equation}
\begin{equation}
C_* = C\left[\left(t_0^{-\frac{1}{2}}\|v\|_Y + t_0^{-\frac{1}{4}}\|\tilde{v}\|_Y\right)\left\{(t-t_0)^{\frac{3}{4}} + (t-t_0)^{\frac{1}{4}}\right\}
+ a\frac{\eta^{\frac{1}{4}}}{t}\left\{(t-t_0)^{\frac{1}{4}} + (t-t_0)^{\frac{3}{4}}\right\} + a\left\{(t-t_0)^{\frac{3}{4}} + (t-t_0)^{\frac{1}{4}}\right\}\right]
\end{equation}
and the constant $C$ is independent of $v$, $\tilde{v}$, $t$ and $t_0$. We choose $\eta > 0$ such that
\[\eta := C\left[\left(t_0^{-\frac{1}{2}}\|v\|_Y + t_0^{-\frac{1}{4}}\|\tilde{v}\|_Y\right)\left\{(t-t_0)^{\frac{3}{4}} + (t-t_0)^{\frac{1}{4}}\right\}
+ a\frac{\eta^{\frac{1}{4}}}{t}\left\{(t-t_0)^{\frac{1}{4}} + (t-t_0)^{\frac{3}{4}}\right\} + a\left\{(t-t_0)^{\frac{3}{4}} + (t-t_0)^{\frac{1}{4}}\right\}\right] < 1.\]
On account of (3.62), we have $\|v - \tilde{v}\|_{Y,t_0,t_0+\eta} \leq \eta\|v - \tilde{v}\|_{Y,t_0,t_0+\eta}$, which leads us to $v = \tilde{v}$ on $[t_0, t_0 + \eta]$. By the same calculation, we can obtain (3.62)–(3.64), in which $t_0$ should be replaced by $t_0 + \eta$ and hence
\begin{equation}
\|v - \tilde{v}\|_{Y,t_0+\eta,t_0+2\eta} \leq C\left[\left\{(t_0 + \eta)^{\frac{1}{2}}\|v\|_Y + (t_0 + \eta)^{\frac{1}{4}}\|\tilde{v}\|_Y\right\}\left(\eta^{\frac{3}{4}} + \eta^{\frac{1}{4}}\right)
+ a\frac{\eta^{\frac{1}{4}}}{\eta}\left\{(\eta^{\frac{3}{4}} + \eta^{\frac{1}{4}}) + a\left(\eta + \eta^{\frac{1}{4}}\right)\right\}\|v - \tilde{v}\|_{Y,t_0+\eta,t_0+2\eta}
< \eta\|v - \tilde{v}\|_{Y,t_0+\eta,t_0+2\eta}\right]
\end{equation}
holds. This implies $v = \tilde{v}$ on $[t_0 + \eta, t_0 + 2\eta]$. Repeating this procedure, we conclude $v = \tilde{v}$. \hfill \Box

Remark 3.11. It is clear that the Eq. (3.28) admits at most one solution within $Y_0$ under the same condition as in Lemma 3.10.

Let us close the paper with completion of the proof of Theorem 1.2.

Proof of Theorem 1.2. Since we know $\varepsilon \leq \tilde{\varepsilon}$ from (3.26) and (3.61), Proposition 3.4 and Lemma 3.10 yield the unique existence of solutions in $Y_0$ when $(M + 1)a^{(n-2)/(n+1)} < \varepsilon$. Moreover, Propositions 3.7 and 3.9 give us sharp decay properties of the solution provided $a$ is still smaller. We finally show the uniqueness of the solution constructed above within $Y$ by following the argument due to Brezis [5]. It suffices to show that if $v \in Y$ is a solution, it necessarily satisfies
\begin{equation}
\lim_{t \to 0} [v]_t = 0,
\end{equation}
where
\[\[v\]_t := \sup_{0 < \tau < t} \tau^{\frac{1}{2}}(\|v(\tau)\|_\infty + \|\nabla v(\tau)\|_n).\]
We assume
\begin{equation}
(M + 1)a^{\frac{n-2}{n+1}} < \min\left\{\delta, \frac{1}{2C_2}, \frac{1}{16C_1C_0}, \frac{1}{16C_1C_3}\right\} =: \hat{\varepsilon}(n,D) \leq \varepsilon\end{equation}
and let $v \in Y$ be a solution. Here, the constants $C_i$ are as in Remark 3.5 as well as the proof of Proposition 3.4. Since $v \in BC([0, \infty); L_0^\alpha(D))$ with $v(0) = 0$, there exists $s_0 > 0$ such that
\[\|v(s)\|_n + (M + 1)a^{\frac{n-2}{n}} < \hat{\varepsilon}\]
for all $0 < s \leq s_0$. Hence by Remark 3.5, the integral equation (3.28) with $b = v(s)$ admits a solution $T(t)v(s) \in Y_0$ along with
\begin{equation}
\|T(s)v(s)\|_Y \leq 4\left(C_0 \|v(s)\|_n + C_3(M + 1)a^{\frac{n-2}{n+1}}\right) < 4(C_0 + C_3)\hat{\varepsilon} \leq \frac{1}{2C_1}.
\end{equation}
On the other hand, given $s \in (0, s_0]$, we can see that $z_s(t) := v(t + s)$ for $t \geq 0$ also satisfies (3.28) with $b = v(s)$ and $z_s \in Y_0$. In view of Remark 3.11, we have $z_s(t) = T(t)v(s)$ for $s \in (0, s_0]$, which implies
\[t^{\frac{1}{2}}(\|v(t + s)\|_\infty + \|\nabla v(t + s)\|_n) \leq \sup_{f \in K} |T(\cdot)f|_t, \quad K = v((0, s_0]) := \{v(t) \in L_0^\alpha(D) \mid t \in (0, s_0]\}$
for all \( s \in (0, s_0] \) and \( t > 0 \). Passing to the limit \( s \to 0 \), we find
\[
[v(\cdot)]_t \leq \sup_{f \in K} [T(\cdot)f]_t.
\] (3.68)

Furthermore, applying Lemma 3.3 to (3.28) with \( b = f \in v((0, s_0]) \) as well as Proposition 3.1 and (1.10), we have
\[
[T(\cdot)f]_t \leq C_0[S(\cdot)f]_t + \left( C_1 \sup_{f \in K} ||T(\cdot)f||_Y + C_2a^{\frac{n-2}{n+2}} \right) [T(\cdot)f]_t + ||H_1||_{Y, t} + ||H_2||_{Y, t},
\]
where \( S(t)f := e^{-tA_h}f \), and deduce from (3.66) and (3.67) that
\[
[T(\cdot)f]_t \leq \frac{C_0[S(\cdot)f]_t + ||H_1||_{Y, t} + ||H_2||_{Y, t}}{1 - \left( C_1 \sup_{f \in K} ||T(\cdot)f||_Y + C_2a^{\frac{n-2}{n+2}} \right)}
\] (3.69)
for all \( f \in K \) and \( t > 0 \). Collecting (3.69), (3.4), (3.68) and \( H_1, H_2 \in Y_0 \) leads to (3.65). The proof is complete. \( \square \)

Acknowledgements. The author would like to thank Professor Toshiaki Hishida for valuable comments and constant encouragement. The author is also grateful to the referee for suggesting the another way to prove Theorem 1.1, which is mentioned in Sect. 1.

Compliance with Ethical Standards
Conflict of interest
The author declares that there is no conflict of interest.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
[2] Babenko, K.I.: On stationary solutions of the problem of flow past a body of viscous incompressible fluid. Math. Sb. 91, 3–27 (1973). English Translation: Math. USSR Sbornik 20, 1–25 (1973)
[3] Bogovski˘ı, M.E.: Solution of the first boundary value problem for the equation of continuity of an incompressible medium. Sov. Math. Dokl. 20, 1094–1098 (1979)
[4] Borchers, W., Sohr, H.: On the equations rotv=g and divu=f with zero boundary conditons. Hokkaido Math. J. 19, 67–87 (1990)
[5] Brezis, H.: Remarks on the preceding paper by M. Ben-Artzi, “Global solutions of two dimensional Navier–Stokes and Euler equations”. Arch. Ration. Mech. Anal. 128, 359–360 (1994)
[6] Chen, Z.M.: Solutions of the stationary and nonstationary Navier–Stokes equations in exterior domains. North-Holland Math. 53, 1291–1330 (2004)
[7] Enomoto, Y., Shibata, Y.: Local energy decay of solutions to the oseen equation in the exterior domains. Indiana Univ. Math. J. 53, 339–367 (2004)
[8] Enomoto, Y., Shibata, Y.: On the rate of decay of the oseen semigroup in the exterior domains and its application to Navier–Stokes equation. J. Math. Fluid Mech. 7, 339–367 (2005)
[9] Farwig, R.: The stationary exterior 3D-problem of Oseen and Navier–Stokes equations in anisotropically weighted Sobolev spaces. Math. Z. 211, 409–447 (1992)
[10] Farwig, R., Sohr, H.: Weighted estimates for the Oseen equations and the Navier–Stokes equations in exterior domains. In: Heywood, J.G., et al. (eds.) Theory of the Navier–Stokes Equations, Series on Advances in Mathematics for Applied Sciences, vol. 47, pp. 11–30. World Scientific Publishing, River Edge (1998)
[11] Finn, R.: Estimate at infinity for stationary solutions of the Navier–Stokes equations. Bull. Math. Soc. Sci. Math. Phys. R. P. Roum. (N.S.) 3, 387–418 (1959)
[12] Finn, R.: Stationary solutions of the Navier–Stokes equations. Proc. Symp. Appl. Math. 17, 121–153 (1965)
[13] Finn, R.: On the exterior stationary problem for the Navier–Stokes equations, and associated perturbation problems. Arch. Ration. Mech. Anal. 19, 363–406 (1965)
[14] Fujita, H.: Mathematical questions relating to viscous fluid flow in an exterior domain. Rocky Mt. J. Math. 3, 107–140 (1973)
[15] Fujita, H., Kato, T.: On the Navier–Stokes initial value problem. I. Arch. Ration. Mech. Anal. 16, 269–315 (1964)
[16] Fujiwara, D., Morimoto, H.: An $L_r$-theorem of the Helmholtz decomposition of vector fields. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24, 685–700 (1977)

[17] Galdi, G.P.: On the Oseen boundary value problem in exterior domains. In: The Navier–Stokes Equations II —Theory and Numerical Methods (Oberwolfach, 1991), vol. 1530 of Lecture Notes in Mathematics, pp. 111–131. Springer, Berlin (1992)

[18] Galdi, G.P.: On the asymptotic structure of $D$-solutions to steady Navier–Stokes equations in exterior domains. In: Galdi, G.P. (ed.) Mathematical Problems Related to the Navier–Stokes Equation, Series on Advances in Mathematics for Applied Sciences, vol. 11, pp. 81–104. World Scientific Publishing, River Edge (1992)

[19] Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady-State Problems, 2nd edn. Springer, New York (2011)

[20] Galdi, G.P., Heywood, J.G., Shibata, Y.: On the global existence and convergence to steady state of Navier–Stokes flow past an obstacle that is started from rest. Arch. Ration. Mech. Anal. 138, 307–318 (1997)

[21] Heywood, J.G.: The exterior nonstationary problem for the Navier–Stokes equations. Acta Math. 129, 11–34 (1972)

[22] Hishida, T.: $L^q$-$L^r$ estimate of Oseen flow in plane exterior domains. J. Math. Soc. Jpn. 68, 295–346 (2016)

[23] Hishida, T., Maremonti, P.: Navier–Stokes flow past a rigid body: attainability of steady solutions as limits of unsteady weak solutions, starting and landing cases. J. Math. Fluid Mech. 20, 771–800 (2018)

[24] Kato, T.: Strong $L^p$ solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. Math. Z. 187, 471–480 (1984)

[25] Koba, H.: On $L^{3,\infty}$-stability of the Navier–Stokes system in exterior domains. J. Differ. Equ. 262, 2618–2683 (2017)

[26] Kobayashi, T., Shibata, Y.: On the Oseen equation in the three dimensional exterior domains. Math. Ann. 310, 1–45 (1998)

[27] Leray, J.: Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l’hydrodynamique. J. Math. Pures Appl. 12, 1–82 (1933)

[28] Maekawa, Y.: On local energy decay estimate of the Oseen semigroup in two dimensions and its application. J. Inst. Math. Jussieu (2019). https://doi.org/10.1017/s1474748019000355

[29] Maekawa, Y.: On stability of physically reasonable solutions to the two-dimensional Navier–Stokes equations. J. Inst. Math. Jussieu (2019). https://doi.org/10.1017/s1474748019000240

[30] Miyakawa, T.: On nonstationary solutions of the Navier–Stokes equations in an exterior domain. Hiroshima Math. J. 12, 115–140 (1982)

[31] Shibata, Y.: On an exterior initial boundary value problem for Navier–Stokes equations. Quart. Appl. Math. LVII, 117–155 (1999)

[32] Shibata, Y., Yamazaki, M.: Uniform estimates in the velocity at infinity for stationary solutions to the Navier–Stokes exterior problem. Jpn. J. Math. (N.S.) 31, 225–279 (2005)

[33] Simader, C.G., Sohr, H.: A new approach to the Helmholtz decomposition and the Neumann problem in $L^q$-spaces for bounded and exterior domains. In: Galdi, G.P. (ed.) Mathematical Problems Relating to the Navier–Stokes Equations, Series on Advances in Mathematics for Applied Sciences, vol. 11, pp. 1–35. World Scientific Publishing, River Edge (1992)

[34] Takahashi, T.: Attainability of a stationary Navier–Stokes flow around a rigid body rotating from rest. arXiv:2004.00781 (to appear in Funkcial. Ekvac.)

Tomoki Takahashi
Graduate School of Mathematics
Nagoya University
Furo-cho,Chikusa-ku
Nagoya 464-8602
Japan
e-mail: m17023c@math.nagoya-u.ac.jp

(accepted: November 26, 2020; published online: March 1, 2021)