Figure captions

Fig.1 $\alpha$ dependence of several quantities in the RS solution for $\delta = 1$ and $T = 1$. (a) $q$ and $R$ (dotted curve) (b) $S_{RS}$ (c) $\Lambda_1$ and $\Lambda_3$ (dotted curve)

Fig.2 $\alpha$ dependence of $q$ and $R$ (dotted curve) in the RS solution for $\delta = 0$. (a) $T = 5$ (b) $T = 0.5$

Fig.3 $\alpha$ dependence of the entropy in the RS solution for $\delta = 0$. Solid and dashed curves denote $S_{RS}^I$ and $S_{RS}^II$, respectively. (a) $T = 5$ (b) $T = 0.5$

Fig.4 $\alpha$ dependence of $\Lambda_1$ and $\Lambda_3$ (dotted curve) in the RS solution for $\delta = 0$. $\Lambda_1^I$ and $\Lambda_3^I$ correspond to curves starting from -1 for small $\alpha$. (a) $T = 5$ (b) $T = 0.5$

Fig.5 $T$ dependence of $q(\alpha)$ and $R(\alpha)$ (dotted curve) in the RS solution for $\delta = 0$. $\delta = 0.3$ (b) $\delta = 1.5$

Fig.6 $\alpha$ dependence of $T_c$ for $\delta = 1$. $\delta = 0.3$ (b) $\delta = 1.5$

Fig.7 ln $\alpha$ v.s. ln $\Delta \epsilon_g$ A line segment with estimated gradient is depicted together.

Fig.8 $\alpha$ dependence of $T_c$ for $\delta = 0$. Solid curve: 1RSB(I), dashed curve: 1RSB(II).

Fig.9 $\alpha$ dependence of Free energy. $f_{RSB}^I$ (solid curve) and $f_{RSB}^{II}$ (dashed curve)

Fig.10 $\alpha$ dependence of $R$ and $\delta R$ in the minimum-error algorithm for $\delta = 1$. Dotted curve: $N = 10$, Dashed curve: $N = 15$, Solid curve: $N = 17$. (a) $R$ (b) $\delta R$

Fig.11 $\alpha$ dependence of several quantities in the minimum-error algorithm for $\delta = 1$. +: numerical results for $N = 17$, bars indicate standard deviations, Dashed curve: RS solution, Solid curve: 1RSB solution

(a) $R$ (b) $q(RS)$ and $q_0(1RSB)$ (c) $\Delta \epsilon_g$

Fig.12 Asymptotic behavior of $R$ in the minimum-error algorithm for $\delta = 1$. +: numerical results for $N = 15$, bars indicate standard deviations, Dashed curve: RS solution, Solid curve: 1RSB solution

(a) $0 < \alpha < 15$ (b) $15 < \alpha < 50$ (c) $50 < \alpha < 100$

Fig.13 $\alpha$ dependence of $R$ and $\Delta \epsilon_g$ in the Gibbs algorithm for $\delta = 1$. +: numerical results for $N = 12$ with $T = 1$, dashed curve: RS solution with $T = 0$, dotted curve: RS solution with $T = 1$, solid curve: 1RSB solution

(a) $R$ (b) $\Delta \epsilon_g$

Fig.14 $\alpha$ dependence of $q$ and $q_0$ in the Gibbs algorithm for $\delta = 1$. +: numerical results with standard deviations for $N = 12$, dashed curve $q$: RS solution with $T = 0$, dotted curve $q$: RS solution with finite temperature, solid curve $q_0$: 1RSB solution

(a) $T = 0.15$, (b) $T = 0.5$ (c) $T = 5.0$

Fig.15 $T$ dependence of $P(q)$ in the Gibbs algorithm for $\delta = 1$. Histogram: numerical results for $N = 12$ and $p = 60$, solid line: 1RSB solution, dotted line: RS solution.

(a) $T = 0.15$ (b) $T = 0.5$ (c) $T = 5.0$

Fig.16 $\alpha$ dependence of several quantities in the minimum-error algorithm for $\delta = 0$. +: numerical results for $N = 17$, bars indicate standard deviations, Dashed curve: RS solution, Solid curve: 1RSB solution

(a) $0 < \alpha < 15$ (b) $15 < \alpha < 50$ (c) $50 < \alpha < 100$
+: numerical results for $N = 17$, bars indicate standard deviations, Dashed curve: RS($T = 0$), solid curve: 1RSB(I), dashed curve: 1RSB(II).

(a) $R$ (b) $q$(RS) and $q_0$(1RSB) (c) $\Delta \epsilon_g$

Fig.17 Asymptotic behavior of $R$ in the minimum-error algorithm for $\delta = 0$. +: numerical results for $N = 15$, bars indicate standard deviations, Dashed curve: RS solution, Solid curve: 1RSB solution

Fig.18 $\alpha$ dependence of $R$ and $\Delta \epsilon_g$ in the Gibbs algorithm for $\delta = 1$ and $T = 1.0$. +: numerical results for $N = 12$. dashed curve: RS solution with $T = 0$, dotted curve: RS solution for $T = 1.0$. solid curve: 1RSB solution

(a) $R$ (b) $\Delta \epsilon_g$

Fig.19 $\alpha$ dependence of $q$ and $q_0$ in the Gibbs algorithm for $\delta = 0$. +: numerical results with standard deviations for $N = 12$, dashed curve: RS solution with $T = 0$, dotted curve: RS solution with finite temperature, solid curve: 1RSB solution

(a) $T = 0.15$ (b) $T = 0.5$ (c) $T = 5.0$

Fig.20 $T$ dependence of $P(q)$ in the Gibbs algorithm for $\delta = 0$. $T_c \simeq 0.7$. Histgram: numerical results for $N = 12$ and $p = 60$, solid line: 1RSB solution, dotted line: RS solution.

(a) $T = 0.15$ (b) $T = 0.5$ (c) $T = 5.0$
Figure 1. Fig. 1 $\alpha$ dependence of several quantities in the RS solution for $\delta = 1$ and $T = 1$.

Figure 2. Fig. 1(a) $q$ and $R$ (dotted curve)

Figure 3. Fig. 1 (b) $S_{RS}$

Figure 4. Fig. 1 (c) $\Lambda_1$ and $\Lambda_3$ (dotted curve)

Figure 5. Fig. 2 $\alpha$ dependence of $q$ and $R$ (dotted curve) in the RS solution for $\delta = 0$.

Figure 6. Fig. 2(a) $T = 5$

Figure 7. Fig. 2(b) $T = 0.5$
**Figure 8.** Fig.3 $\alpha$ dependence of the entropy in the RS solution for $\delta = 0$. Solid and dashed curves denote $S_{RS}^I$ and $S_{RS}^{II}$, respectively.

![Figure 8](image1)

**Figure 9.** Fig.3(a) $T = 5$

![Figure 9](image2)

**Figure 10.** Fig.3(b) $T = 0.5$

![Figure 10](image3)

**Figure 11.** Fig.4 $\alpha$ dependence of $\Lambda_1$ and $\Lambda_3$ (dotted curve) in the RS solution for $\delta = 0$. $\Lambda_1^I$ and $\Lambda_3^I$ correspond to curves starting from -1 for small $\alpha$.

![Figure 11](image4)

**Figure 12.** Fig.4(a) $T = 5$

![Figure 12](image5)

**Figure 13.** Fig.4(b) $T = 0.5$

![Figure 13](image6)
Figure 14. Fig.5 $\alpha$ dependence of $q$ and $R$ (dotted curve) in the RS solution for $T = 5$.

![Figure 14](image)

Figure 15. Fig.5(a) $\delta = 0.3$

![Figure 15](image)

Figure 16. Fig.5(b) $\delta = 1.5$

![Figure 16](image)

Figure 17. Fig.6 $\alpha$ dependence of $T_c$ for $\delta = 1$.

![Figure 17](image)

Figure 18. Fig.7 $\ln \alpha$ v.s. $\ln \Delta \epsilon_g$ A line segment with estimated gradient is depicted together.

![Figure 18](image)

Figure 19. Fig.8 $\alpha$ dependence of $T_c$ for $\delta = 0$. Solid curve: 1RSB(I), dashed curve: 1RSB(II).

![Figure 19](image)
Figure 20. Fig.9 $\alpha$ dependence of Free energy. $f_{RSB}^I$ (solid curve) and $f_{RSB}^{II}$ (dashed curve).

Figure 21. Fig.10 $\alpha$ dependence of $R$ and $\delta R$ in the minimum-error algorithm for $\delta = 1$. Dotted curve: $N = 10$, Dashed curve: $N = 15$, Solid curve: $N = 17$.

Figure 22. Fig.10(a) $R$

Figure 23. Fig.10 (b) $\delta R$
Figure 24. Fig.11 $\alpha$ dependence of several quantities in the minimum-error algorithm for $\delta = 1$. +: numerical results for $N = 17$, bars indicate standard deviations, Dashed curve: RS solution, Solid curve: 1RSB solution

Figure 25. Fig.11(a) $R$

Figure 26. Fig.11 (b) $q(RS)$ and $q_0(1RSB)$

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**Figure 28.** Fig.12 Asymptotic behavior of $R$ in the minimum-error algorithm for $\delta = 1$. $+$: numerical results for $N = 15$, bars indicate standard deviations, Dashed curve: RS solution, Solid curve: 1RSB solution

**Figure 29.** Fig.12(a) $0 < \alpha < 15$

**Figure 30.** Fig.12(b) $15 < \alpha < 50$

**Figure 31.** Fig.12 (c) $50 < \alpha < 100$

**Figure 32.** Fig.13 $\alpha$ dependence of $R$ and $\Delta \epsilon_g$ in the Gibbs algorithm for $\delta = 1$. $+$: numerical results for $N = 12$ with $T = 1$, dashed curve: RS solution with $T = 0$, dotted curve: RS solution with $T = 1$, solid curve: 1RSB solution

**Figure 33.** Fig.13(a) $R$

**Figure 34.** Fig.13(b) $\Delta \epsilon_g$
Figure 35. Fig.14 $\alpha$ dependence of $q$ and $q_0$ in the Gibbs algorithm for $\delta = 1$. +: numerical results with standard deviations for $N = 12$, dashed curve $q$: RS solution with $T = 0$, dotted curve $q$: RS solution with finite temperature, solid curve $q_0$: 1RSB solution

Figure 36. Fig.14(a) $T = 0.15$

Figure 37. Fig.14(b) $T = 0.5$

Figure 38. Fig.14(c) $T = 5.0$
Figure 39. Fig.15 $T$ dependence of $P(q)$ in the Gibbs algorithm for $\delta = 1$. Histogram: numerical results for $N = 12$ and $p = 60$, solid line: 1RSB solution, dotted line: RS solution.

Figure 40. Fig.15(a) $T = 0.15$

Figure 41. Fig.15(b) $T = 0.5$

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Figure 1. Fig.16 α dependence of several quantities in the minimum-error algorithm for δ = 0. +: numerical results for N = 17, bars indicate standard deviations, Dashed curve: RS(T = 0), solid curve: 1RSB(I), dashed curve: 1RSB(II)

Figure 2. Fig.16(a) R

Figure 3. Fig.16 (b) q(RS) and q0(1RSB)

Figure 4. Fig.16 (c) Δεg

Figure 5. Fig.17 Asymptotic behavior of R in the minimum-error algorithm for δ = 0. +: numerical results for N = 15, bars indicate standard deviations, Dashed curve: RS solution, Solid curve: 1RSB solution
Figure 6. Fig.18 α dependence of R and ∆εg in the Gibbs algorithm for δ = 1 and T = 1.0. +: numerical results for N = 12, dashed curve: RS solution with T = 0, dotted curve: RS solution for T = 1.0, solid curve: 1RSB solution

Figure 7. Fig.18(a) R

Figure 8. Fig.18(b) ∆εg

Figure 9. Fig.19 α dependence of q and q0 in the Gibbs algorithm for δ = 0. +: numerical results with standard deviations for N = 12, dashed curve: RS solution with T = 0, dotted curve: RS solution with finite temperature, solid curve: 1RSB solution

Figure 10. Fig.19(a) T = 0.15

Figure 11. Fig.19(b) T = 0.5

Figure 12. Fig.19(c) T = 5.0
Figure 13. Fig.20 $T$ dependence of $P(q)$ in the Gibbs algorithm for $\delta = 0$. $T_c \simeq 0.7$. Histogram: numerical results for $N = 12$ and $p = 60$, solid line: 1RSB solution, dotted line: RS solution

Figure 14. Fig.20(a) $T = 0.15$

Figure 15. Fig.20(b) $T = 0.5$

Figure 16. Fig.20(c) $T = 5.0$
On the conditions for the existence of Perfect Learning and power law in learning from stochastic examples by Ising perceptrons

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Abstract.
In a previous letter, we studied learning from stochastic examples by perceptrons with Ising weights in the framework of statistical mechanics. Under the one-step replica symmetry breaking ansatz, the behaviours of learning curves were classified according to some local property of the rules by which examples were drawn. Further, the conditions for the existence of the Perfect Learning together with other behaviors of the learning curves were given. In this paper, we give the detailed derivation about these results and further argument about the Perfect Learning together with extensive numerical calculations.

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1. Introduction

In the problem of learning from examples by feed forward networks, learning curves of the generalization error $\epsilon_g$ have been calculated for various types of networks [1]. From these studies, it turned out that when the number of examples $p$ is large relative to the number of synaptic weights $N$, that is, when $\alpha = p/N$ is large, the learning curves exhibit only a few types of behaviours [2]-[9]. For example, learning curves of networks with continuous weights all exhibit power laws

$$ (\epsilon_g - \epsilon_{\text{min}}) \propto \alpha^{-\gamma}, $$

where $\gamma$ depends on architectures, types of weight vectors and so on.

On the other hand, in the learning behaviors for the case of discrete weights, in addition to the power laws it was shown that there exists the Perfect Learning(PL) for deterministic and realizable cases [10, 11]. That is, learners’ weight vectors coincide to the teacher’s weight vector at a finite $\alpha$. Then, it is very interesting to clarify the existence conditions for the PL in the case of discrete weights and under the presence of external noise.

In the previous paper [12], we reported about these conditions. The results are similar to those by Seung [13] who classified the learning behaviours of Ising networks by introducing two exponents $y$ and $z$. We gave the other meaning of $y$ and $z$ and obtained the relation $y = 2z$. Further, asymptotic behaviours of learning curve were also investigated.

The purpose of this paper is to give the detailed derivation on the conditions for the existence of the Perfect Learning and the asymptotic behaviours of learning curves in the problem of learning from stochastic examples by perceptrons with Ising weights by using the replica method.

In the following section, we formulate the problem. In §3, we analyse the replica symmetric(RS) solution. The conditions for the existence of the PL is given in §4. The one-step replica symmetry breaking(1RSB) solution is studied in §5. The results of numerical calculations are given in §6. §7 is devoted to summary and discussion.

2. Formulation

We consider a stochastic target relation between $N$-dimensional input vector $\mathbf{x}$ and binary output $r \in \{1, -1\}$ which is represented by a conditional probability $p_r(r|\mathbf{x})$. It is assumed that an input vector $\mathbf{x}$ is normalized as $|\mathbf{x}| = \sqrt{N}$ and $p_r(r|\mathbf{x})$ is a function of the inner product between the input $\mathbf{x}$ and the optimal Ising weight $\mathbf{w}^o$ as

$$ p_r(+1|\mathbf{x}) = P(u^o) = \frac{1 + P(u^o)}{2}, $$

$$ u^o \equiv (\mathbf{x} \cdot \mathbf{w}^o)/\sqrt{N}. $$

We further assume that the function $P(u)$ is not decreasing w.r.t. $u$ and behaves as

$$ P(u) \simeq a \text{sgn}(u)|u|^\delta, \quad (\delta \geq 0), $$

(2)
near $u = 0$. Further, $P(-u) = -P(u)$ is assumed for brevity. The case of $\delta = 0$ corresponds to the output noise model \[\text{[3]}\] in which the output of the target perceptron is reversed to the opposite by noise with a probability. On the other hand, $\delta = 1$ corresponds to the input noise model \[\text{[10]}\] in which the input of the target perceptron is corrupted by Gaussian noise with mean zero.

We assume that a set of $p$ examples $\xi_p = \{(x^1, r^o_1), (x^2, r^o_2), \ldots, (x^p, r^o_p)\}$ is obtained as follows. $x^\mu$ is independently and uniformly drawn from a hyper sphere of radius $\sqrt{N}$ at the origin in the $N$-dimensional space and $r^o_\mu$ is obtained with the conditional probability $p_r(r^o_\mu|x^\mu)$ for each $x^\mu$. For the given realization of examples $\xi_p$, the number of false predictions is given as

$$E[w, \xi_p] = \sum_{\mu=1}^{p} \Theta(-r^o_\mu u_\mu), \quad u_\mu \equiv (x^\mu \cdot w)/\sqrt{N},$$

where $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$. The performance of the learning is evaluated by the generalization error $\epsilon_g$. This is expressed as

$$\epsilon_g = \langle \mathcal{P}(u^o)(1 - \Theta(u)) + (1 - \mathcal{P}(u^o))\Theta(u) \rangle$$

$$= \epsilon_{min} + 2 \int_{0}^{\infty} Dy P(y) H\left(\frac{Ry}{\sqrt{1 - R^2}}\right),$$

where $\langle \cdots \rangle$ represents the average over a novel example and $\epsilon_{min}$ is the minimum value of the generalization error obtained by the optimal weight $w^o$. $R$ is the overlap between the optimal weight vector and a weight vector of a learner, $R = (w^o \cdot w)/N$. Further, as usual, $Dy = \exp(-y^2/2)dy/\sqrt{2\pi}$ and $H(x) = \int_{x}^{\infty} Dy$. In particular, when $\Delta R = 1 - R$ is small, we obtain the relation

$$\Delta \epsilon_g \equiv (\epsilon_g - \epsilon_{min}) \approx \frac{2s}{(1 + \delta)\sqrt{2\pi}}(2\Delta R)^{1+\delta},$$

where $s \equiv \int_{0}^{\infty} Dy y^{1+\delta}$.

In this paper, we adopt the Gibbs algorithm with temperature $T$ as a learning algorithm. The minimum-error algorithm, which minimizes the number of false predictions on the presented examples, is obtained by taking $T \to +0$ limit.

From the energy defined by the equation \[\text{[3]}\] the partition function $Z$ with the inverse temperature $\beta$ is given by

$$Z = \text{Tr}_w e^{-\beta E[w, \xi_p]} = \text{Tr}_w \prod_{\mu=1}^{p} [e^{-\beta} + (1 - e^{-\beta})\Theta(r_\mu u_\mu)],$$

where $\text{Tr}_w$ implies the summation over all configurations of $w$. The average free energy $f$ per weight is calculated by the standard recipe

$$-\beta Nf = \langle \ln Z \rangle_{\xi_p, w^o} = \lim_{n \to 0} \frac{1}{n} \langle Z^n \rangle_{\xi_p, w^o} - 1,$$

where $\langle \cdots \rangle_{\xi_p, w^o}$ denotes the average over quenched variables.

$$\langle Z^n \rangle_{\xi_p, w^o}$$ becomes a function of several replica order parameters, namely the overlap between weight vectors of learners $q^{a\beta} = (w^{a\beta} \cdot w^o)/N$, its conjugate $\tilde{q}^{a\beta}$, the overlap
between the weight vector of a learner and the optimal weight vector \( R^\alpha = \frac{w^o w^o}{N} \), and its conjugate \( \hat{R}^\alpha \). See Appendix A for a derivation of the free energy.

3. RS solution

Let us consider the RS solution. For the RS solution, any quantity does not depend on the replica indices and we put \( q^{\alpha\beta} = q, \hat{q}^{\alpha\beta} = \hat{q}, R^\alpha = R \) and \( \hat{R}^\alpha = \hat{R} \). Then, the RS free energy \( f_{RS} \) becomes

\[
- \beta f_{RS}(q, \hat{q}, R, \hat{R}, \beta) = -\frac{\hat{q}}{2}(1 - q) - R\hat{R} + \alpha K + I,
\]

\[
K \equiv \int Dy2P(y) \int Du \ln \tilde{H}\left(\frac{\sqrt{q - R^2}u - Ry}{\sqrt{1 - q}}\right) = \int Du \ln \tilde{H}(u/Q)E(u/Q),
\]

\[
I \equiv \int Dt \ln[2\cosh(\sqrt{q}t + \hat{R})],
\]

\[
E(u) = \int Dy2P(-\Lambda) = 1 - e^{-v/2} \int_0^\infty Dy\left(e^{-vy} - e^{vy}\right)P(\xi y),
\]

\[
\tilde{H}(u) = e^{-\beta} + (1 - e^{-\beta})H(u), \quad \Lambda = \xi y + \sqrt{1 - \xi^2}Q u = \xi(y - v),
\]

\[
\xi = \sqrt{1 - \frac{R^2}{q}}, \quad Q = \sqrt{\frac{1 - q}{q}}, \quad v = -\frac{R}{\sqrt{4\chi}}, \quad \chi = \frac{R}{Q}.
\]

3.1. Saddle point equations (S.P.E.)

The saddle point equations are given by

\[
q = \int Du \tanh^2(\sqrt{q}u + \hat{R}),
\]

\[
R = \int Du \tanh(\sqrt{q}u + \hat{R}),
\]

\[
\hat{q} = \frac{\alpha Q}{1 - q} \int \tilde{D}u(\tilde{\varphi}(u))^2 E(u),
\]

\[
\hat{R} = -\frac{\alpha}{\sqrt{q - R^2}} \int \tilde{D}u \tilde{\varphi}(u) \int Dyy2P(\Lambda) = -\frac{\alpha}{\sqrt{q - R^2}} \int \tilde{D}u \tilde{\varphi}(u) w(u),
\]

\[
w(u) \equiv \int Dyy P(\Lambda) = e^{-v^2/2} \int_0^\infty DyP(\xi y)\left[(y + v)e^{-vy} + (y - v)e^{vy}\right].
\]

\[
\tilde{D}u = \frac{du}{\sqrt{2\pi}} e^{-Q^2u^2/2}, \quad \tilde{\varphi}(u) = \frac{\tilde{H}'(u)}{\tilde{H}(u)}.
\]

For later use, we give the expression of the entropy \( S_{RS} \),

\[
S_{RS} = -\beta f_{RS} - \alpha \beta e^{-\beta} J,
\]

\[
J = \int Dy2P(y) \int Du \frac{H\left(\sqrt{q - R^2}u - Ry\right)}{\sqrt{1 - q}} - 1.
\]

Defining \( L = K - \beta e^{-\beta} J \), \( S_{RS} \) becomes

\[
S_{RS} = -\frac{\hat{q}}{2}(1 - q) - R\hat{R} + I + \alpha L.
\]
where $L$ is expressed as

$$L = \int DuE(u/Q)\{\ln[1 + (e^\beta - 1)H(u/Q)] - \beta \frac{H(u/Q)}{H(u/Q)}\}. \quad (17)$$

Also, the energy (training error) per weight is expressed as

$$e_t = -\alpha e^{-\beta} J. \quad (18)$$

3.2. Numerical calculations of S.P.E. for the RS solution

Here, we show the results of numerical calculations for the RS solution.

(I) $\delta = 1$

We treated $P(y) = 1 - 2H(y)$, in which $\epsilon_{\text{min}} = \frac{1}{4}$. In Figure 1, for $T = 1$ we depict $\alpha$ dependence of $q$, $R$, $S_{RS}$, and $\Lambda_1$ and $\Lambda_3$ which are indicators of AT-stability. The RS solution is stable only when both $\Lambda_1$ and $\Lambda_3$ are negative. From the numerical results, it seems that as $\alpha \to \infty$, $q$ and $R$ tend to 1. In this case, the entropy $S_{RS}$ becomes zero at some value of $\alpha$, $\alpha_s(T)$, and $\Lambda_3$ becomes zero at another value of $\alpha$, $\alpha_{AT}(T)$, for any $T$.

(II) $\delta = 0$

For the numerical calculations, we treated $P(y) = \frac{1}{2} \operatorname{sgn}(y)$, in which $\epsilon_{\text{min}} = \frac{1}{4}$. In Figures 2-4, for several temperatures we depict $\alpha$ dependence of $q$, $R$, $S_{RS}$, $\Lambda_1$ and $\Lambda_3$. The most interesting feature is that there is no solutions in which $q$ and $R$ tend to 1. There are two branches of solutions. We call them the branch I and the branch II. Each solution is characterized by the behaviour in the limit of $\alpha \to 0$. In the branch I, $q$ and $R$ tend to 0. On the other hand, in the branch II, $q$ and $R$ tend to 1. We attach the superscript I or II to any quantity estimated in the branch I or II, respectively. From these figures, we note that when $T$ is greater than some temperature, say $T_s$, solutions in the both branches are AT-stable and their entropies are positive. When $T < T_s$, the entropy of the branch II, $S^H_{RS}$, becomes negative for any $\alpha$. Thus, $T_s$ is determined by the condition that $S^H_{RS}$ changes its sign at small value of $\alpha$. There exists a critical value of $\alpha = \alpha_s(T)$ at which the entropy of the branch I, $S^I_{RS}$, becomes 0. Then, $S^I_{RS} > 0$ for $\alpha < \alpha_s(T)$ and $S^I_{RS} < 0$ for $\alpha > \alpha_s(T)$. Also, we note that for the branch I, the AT-instability takes place at $\alpha = \alpha_{AT}(T)$ when $T$ is smaller than some temperature, say, $T_{AT}(< T_s)$. For $T < T_{AT}$, $\Lambda^I_3$ is positive for $\alpha > \alpha_{AT}(T)$, whereas $\Lambda^H_3$ is positive for any $\alpha$. On the other hand, $\Lambda^I_1$ and $\Lambda^H_1$ are always negative for any $T$ and $\alpha$.

(III) general $\delta$

The numerical calculations were performed for several values of $\delta$ and $T$. For example, when $T = 5$ and $\delta = 0.3$, $q$ and $R$ tend to 1 as $\alpha \to 0$. See Figure 5(a). In this case, the RS solution is AT stable and its entropy is positive. There exists the other case in which $q$ and $R$ tend to 1 as $\alpha \to \infty$. See Figure 5(b). In this case, $S_{RS}$ decreases and becomes 0 at finite $\alpha$, $\alpha_s(T)$, for any $T$. $\Lambda_3$ also becomes 0 at finite $\alpha$, $\alpha_{AT}(T)$. 

On the conditions for the existence of Perfect Learning

Within our numerical calculations, for any value of \( \delta \) and for \( T < T_{AT} \), we obtain the relation \( \alpha_{AT}(T) > \alpha_s(T) \), and \( \alpha_{AT}(T) \) and \( \alpha_s(T) \) are increasing as \( T \) increases, as long as these quantities are defined. Further, except for the case of \( \delta = 0 \), \( \alpha_{AT}(T) \) and \( \alpha_s(T) \) increase as \( \delta \) increases.

For any value of \( \delta \), the entropy becomes negative for small \( T \). Thus, in this case we have to consider the replica symmetry breaking ansatz.

3.3. Asymptotic relations for \( \hat{q} \) and \( \hat{R} \) when \( q \to 1 \) and \( R \to 1 \)

In this section, in order to derive the asymptotic learning curves and to discuss the conditions for the existence of the PL, we summarize asymptotic relations for \( \hat{q} \) and \( \hat{R} \) when \( q \to 1 \) and \( R \to 1 \) for any values of \( \alpha, \beta \) and \( \chi \) by evaluating equations (12) and (13) under these limits. See Appendix B for the derivation.

\[ \hat{q} \approx \frac{\alpha}{\sqrt{\Delta q}} g_{1,\delta}(\chi, \beta) \text{ for } \delta \geq 0, \]
\[ \hat{R} \approx \frac{\alpha}{\sqrt{\Delta q}} g_{2,\delta}(\chi, \beta) \text{ for } \delta \geq 0. \]

For \( \beta = \infty \),

\[ \hat{q} \approx \frac{\alpha}{(\Delta q)^2} g_3 \text{ for } \delta > 0, \text{ or for } \delta = 0 \text{ and } k < 1, \]
\[ \approx \frac{\alpha}{\Delta q} \text{ for deterministic case,} \]
\[ \hat{R} \approx \frac{\alpha}{\Delta q} g_4 \text{ when } P(y) \text{ is not constant for } y > 0, \]
\[ \approx \frac{\alpha}{\Delta q} \frac{2k\xi^3}{\pi} \text{ when } P(y) \equiv k \text{ for } y > 0 \text{ and } k < 1, \]
\[ = \hat{q} \approx \frac{\alpha}{\sqrt{\Delta q}} g_{3,D} \text{ for deterministic case.} \]

Expressions for \( g \) are given as follows.

\[ g_{1,\delta}(\chi, \beta) \equiv \frac{1}{\sqrt{2\pi}} \int du \tilde{\varphi}(u)^2 E_\delta(u, \chi), \]
\[ E_\delta(u, \chi) = 1 \text{ for } \delta > 0, \quad E_0(u, \chi) = 1 - k + 2kH(u/\chi), \]
\[ g_{2,\delta}(\chi, \beta) \equiv \frac{a\delta}{\sqrt{2\pi}} \frac{(1 - e^{-\beta})}{\chi} \frac{1}{(1 + \chi^{-2})^{(\delta-1)/2}} \]
\[ \times \int_0^\infty Dz \int_{-\infty}^{\infty} Dt \left[ \frac{1}{H\left(\frac{\chi t}{\sqrt{1 + \chi^2}}\right)} + \frac{1}{H\left(-\frac{\chi t}{\sqrt{1 + \chi^2}}\right)} \right], \text{ for } \delta > 0, \]
\[ g_{2,0}(\chi, \beta) = \frac{k(1 - e^{-\beta})}{\pi} \frac{1}{\sqrt{1 + \chi^2}} \int_{-\infty}^{\infty} Dx \frac{1}{H\left(\frac{\chi x}{\sqrt{1 + \chi^2}}\right)}, \]
\[ g_3 \equiv \int_0^\infty Dyy^2[1 - P(y)], \quad g_{3,D} \equiv \frac{2}{\sqrt{2\pi}} \int Du \frac{h(u)}{H(u)}, \]
\[ g_4 \equiv \int_0^\infty DyP(y)(y^2 - 1). \]
Now, let us see the behaviours of $g_s$ for later use. $g_{1,\delta}(\chi, \beta)$ is finite for $0 < \beta < \infty$ and for $\delta \geq 0$ for any $\chi$, since $H(x)$ is bounded. As for $g_{2,\delta}(\chi, \beta)$, we obtain for $0 < \beta < \infty$ and for $\delta > 0$,

$$g_{2,\delta}(\chi, \beta) \sim \nu_0 \chi^{-\delta}, \text{ for } \chi \ll 1,$$

$$\sim \nu_1 \chi^{-1}, \text{ for } \chi \gg 1,$$

$$\sim \text{ finite for finite } \chi,$$

where $\nu_0 = \frac{\alpha \delta (1-e^{-\beta})}{\sqrt{2\pi}} \int_0^\infty D z \chi^{\delta-1} \left\{ \frac{1}{H(z)} + \frac{1}{H(-z)} \right\}$ and $\nu_1 = \frac{2s \beta}{\sqrt{2\pi}}$. In the case of $\delta = 0$, for $0 < \beta < \infty$, $g_{2,0}$ is finite except for $\chi \gg 1$, that is,

$$g_{2,0} \sim \frac{2k}{\pi} \tanh(\beta/2) \text{ for } \chi \ll 1,$$

$$\sim \frac{k \beta}{\pi} \chi^{-1} \text{ for } \chi \gg 1,$$

$$\sim \text{ finite for finite } \chi.$$

On the other hand, quantities for $\beta = \infty$, $g_3$, $g_{3,D}$ and $g_4$ are all finite.

Here, for later use, we give the asymptotic form of the entropy $S_{RS}$. $L$ is evaluated as

$$L = \sqrt{\Delta q} \, r(\chi, \beta),$$

$$r(\chi, \beta) \equiv \frac{1}{\sqrt{2\pi}} \int du E_\delta(u, \chi) \left\{ \ln[1 + (e^\beta - 1)H(u)] - \beta \frac{H(u)}{H(0)} \right\}.$$  \(\text{(32)}\)

$r(\chi, \beta)$ is finite for $0 < \beta < \infty$ and for any $\delta$ and any $\chi$. Then, as $q \to 1$ and $R \to 1$, for $0 < \beta < \infty$ and for any $\delta$ and any $\chi$, $S_{RS}$ is expressed as follows.

$$S_{RS} = -\frac{\dot{q} \Delta q}{2} - (1 - \Delta R)\dot{R} + \alpha \sqrt{\Delta q} \, r(\chi, \beta) + I.$$  \(\text{(34)}\)

3.4. Asymptotic solutions of S.P.E. when $q \to 1$ and $R \to 1$.

The equations for $\Delta q = 1 - q$ and $R$ are obtained by

$$\Delta q = 2 \frac{\partial I}{\partial q}, \quad R = \frac{\partial I}{\partial R}.$$  \(\text{(35)}\)

Thus, we have to estimate $I$ in the asymptotic region. $I$ behaves differently according to the values of $\mu \equiv \frac{\dot{R}}{2q}$. We give the expressions for $I$ in Appendix C.

After several algebra, we obtain the following asymptotic behaviours.

(i) $\delta > \frac{1}{3}$.

For $\alpha \gg 1$,

$$\mu \sim \mu_0 \left( \frac{\ln \alpha}{\alpha} \right)^{\frac{2\delta}{3\delta - 1}}, \quad \mu_0 = \left[ \frac{\delta}{q_0(3\delta - 1)} \right]^{\frac{1}{3\delta - 1}},$$

$$\Delta q \sim q_0 \mu_0^\delta \left( \frac{\ln \alpha}{\alpha} \right)^{\frac{2(1+\delta)}{3\delta - 1}}, \quad \Delta R \sim R_0 \mu_0^{1/\delta} \left( \frac{\ln \alpha}{\alpha} \right)^{\frac{2}{3\delta - 1}},$$

$$\dot{q} \sim \dot{q}_0 \mu_0^{1-\delta} \alpha^{\frac{4\delta}{3\delta - 1}} (\ln \alpha)^{-\frac{1+\delta}{3\delta - 1}}, \quad \dot{R} \sim \dot{R}_0 \mu_0^{1-\delta} \alpha^{\frac{2\delta}{3\delta - 1}} (\ln \alpha)^{-\frac{1-\delta}{3\delta - 1}}.$$
(ii) $\delta = \frac{1}{3}$.

For $\alpha \gg 1$,
\[
\mu \simeq \mu_0 \alpha^{-\frac{1}{3}} e^{\frac{2}{3} \hat{q}_0 \alpha} , \quad \mu_0 = \left[ \frac{1}{2\sqrt{q_0}} \right]^\frac{1}{3} ,
\]
\[
\Delta q \simeq q_0 \mu_0^4 \alpha^{-\frac{2}{3}} e^{\frac{5}{3} \hat{q}_0 \alpha} , \quad \Delta R \simeq R_0 \mu_0^3 \alpha^{-\frac{2}{3}} e^{-2\hat{q}_0 \alpha} ,
\]
\[
\hat{q} \simeq \hat{q}_0 \mu_0^{-2} \alpha^\frac{1}{3} e^{\frac{2}{3} \hat{q}_0 \alpha} , \quad \hat{R} \simeq \hat{R}_0 \mu_0^{-1} \alpha^{\frac{1}{6}} e^{\frac{2}{3} \hat{q}_0 \alpha} .
\]

(iii) $0 < \delta < \frac{1}{3}$.

For $\alpha \ll 1$,
\[
\mu \simeq \mu_0 \left( \frac{1}{\alpha} \right)^{-\frac{1}{2}} , \quad \mu_0 = \left[ \frac{\delta}{q_0 (1 - 3\delta)} \right]^{-\frac{1}{1 - 3\delta}} ,
\]
\[
\Delta q \simeq q_0 \mu_0 \left( \frac{1}{\alpha} \right)^{-\frac{1}{2}} \left( \frac{1}{\alpha} \right)^{\frac{1}{4}} , \quad \Delta R \simeq R_0 \mu_0 \left( \frac{1}{\alpha} \right)^{-\frac{1}{2}} \left( \frac{1}{\alpha} \right)^{\frac{1}{4}} ,
\]
\[
\hat{q} \simeq \hat{q}_0 \mu_0^{-1} \alpha^{-\frac{1}{3}} e^{\frac{4}{3} \hat{q}_0 \alpha} , \quad \hat{R} \simeq \hat{R}_0 \mu_0^{-1} \alpha^{-\frac{1}{6}} e^{\frac{2}{3} \hat{q}_0 \alpha} .
\]

(iv) $\delta = 0$.

(a) $0 \ll \mu < 1$.

For $\alpha \ll 1$,
\[
\Delta R \simeq \frac{1}{\sqrt{2\pi}} \psi(\mu) \alpha^2 \left( \frac{1}{\alpha} \right)^{-2} , \quad \Delta q = 2\mu \Delta R ,
\]
\[
\hat{R} \simeq \frac{2}{\mu} \ln \left( \frac{1}{\alpha} \right)^{\frac{1}{4}} , \quad \hat{q} = \frac{\hat{R}}{2\mu} .
\]

$\chi$ and $\mu$ are determined by the following equations.
\[
g_{2,0}(\chi, \beta) = \frac{1}{2g_{1,0}(\chi)} = \frac{1}{1 + \chi^2} ,
\]
\[
\mu = \frac{1}{1 + \chi^2} .
\]  \hspace{1cm} (35)  \hspace{1cm} (36)

(b) $1 \leq \mu$.

For $\alpha \ll 1$,
\[
\Delta R \simeq 2\mu \left( \frac{1}{\alpha} \ln \frac{1}{\alpha} \right)^{-2} , \quad \Delta q = 2\Delta R ,
\]
\[
\hat{R} \simeq \frac{2\mu}{2\mu - 1} \ln \left( \frac{1}{\alpha} \ln \frac{1}{\alpha} \right) , \quad \hat{q} = \frac{\hat{R}}{2\mu} .
\]

$\mu$ is determined by
\[
\mu = \frac{g_{2,0}(\chi = 0, \beta)}{2g_{1,0}(\beta)} .
\]  \hspace{1cm} (37)

Now, let us check the validity of the above solutions. First, let us see the case of $\delta > 0$. We only have to see the conditions for $\mu \ll 1, \Delta R \ll 1$ and $\hat{R} \gg 1$. For $\delta > 1/3$,
On the conditions for the existence of Perfect Learning

The conditions are
\[ g_{1,\delta}^{(\delta - 1)/\delta} \beta^{1/\delta} \gg \frac{\ln \alpha}{\alpha} \quad (\mu \ll 1), \]
\[ g_{1,\delta}^{-2} \beta^{3} \gg \frac{\ln \alpha}{\alpha} \quad (\Delta R \ll 1), \]
\[ g_{1,\delta}^{2(\delta - 1)} \beta^{2} \gg \left( \frac{\ln \alpha}{\alpha} \right)^{1-\delta} \quad (\hat{R} \gg 1). \]

These are satisfied if \( \alpha \gg 1 \). For \( \delta = 1/3 \), the condition is \( \hat{q}_0 > 0 \) and automatically satisfied for \( \beta > 0 \). For \( 0 < \delta < 1/3 \), the conditions are
\[ g_{1,\delta}^{(\delta - 1)/\delta} \beta^{1/\delta} \ll \frac{1}{\alpha} \quad (\mu \ll 1), \]
\[ g_{1,\delta}^{-2} \beta^{3} \ll \frac{1}{\alpha} \quad (\Delta R \ll 1), \]
\[ g_{1,\delta}^{2(\delta - 1)} \beta^{2} \ll \left( \frac{1}{\alpha} \right)^{1-\delta} \quad (\hat{R} \gg 1). \]

These are satisfied if \( \alpha \ll 1 \). Therefore, no extra condition is necessary for the case of \( \delta > 0 \). On the other hand, for \( \delta = 0 \), in the case of (a) the condition is that there is a positive solution \( \chi \) of equation (35), and in the case of (b) the condition is \( \mu \geq 1 \), where \( \mu \) is defined by equation (37). When \( \beta \ll 1 \), \( \mu \) is estimated as
\[ \mu \sim \frac{k \sqrt{2}}{\beta \sqrt{1 + \xi^2}} \quad \text{for the case (a)} \]
\[ \sim \frac{k \sqrt{2}}{\beta} \quad \text{for the case (b)}. \]

Thus, the case (a) is impossible for high temperatures. On the other hand, when \( \beta \gg 1 \), since \( g_{1,0} \) becomes very large and \( g_{2,0} \) remains bounded in the both cases, we obtain \( \mu \ll 1 \). Therefore, the case (b) is impossible for low temperatures.

The results obtained in this section suggest that for \( \delta < 1/3 \) the PL exists and the solution with \( q < 1 \) exists only in the finite region of \( \alpha \), and for \( \delta \geq 1/3 \) the PL does not exist and the solution with \( q < 1 \) exists for any \( \alpha \). However, as is shown in the next two sections, this is not correct. One reason is that the entropy of the RS solution becomes negative for \( T \to 0 \) when \( \alpha \) can be large enough. The other reason is that the condition \( \delta < 1/3 \) is different from the existence condition for the PL.

In the next section, we investigate the necessary and sufficient conditions for the existence of the Perfect Learning.

### 4. Perfect Learning

In the Perfect Learning, the weight vector of a student coincides with the optimal weight vector at a finite value of \( \alpha \), \( \bm{w} = \bm{w}^o \). In this case, \( q = 1 \) and \( R = 1 \). From equation (10) and (11) the necessary and sufficient conditions for \( q = 1 \) and \( R = 1 \) at finite value
of $\alpha$ are
\[ \hat{R} \to \infty \text{ and } \tau \equiv \frac{\hat{R}}{\sqrt{\hat{q}}} \to \infty. \] (38)

In the case of the PL, we impose the further condition $q = R$ when the limits $q \to 1$ and $R \to 1$ are taken, because in the Perfect Learning, the teacher and a student coincide. Therefore, we have $\chi = \sqrt{\frac{2-\delta}{\delta}} = \sqrt{q} = 1$ and then $\xi = Q = \sqrt{\Delta q}$. Thus, for $0 < \beta < \infty$ we obtain from equations (19) and (20),
\[ \hat{q} = \frac{\alpha}{\sqrt{\Delta q}} g_{1,\delta}(\chi = 1, \beta), \tag{39} \]
\[ \hat{R} = \alpha (\Delta q)^{\delta - 1/2} g_{2,\delta}(\chi = 1, \beta). \tag{40} \]

For $\beta = \infty$,
\[ \hat{q} \simeq \frac{\alpha}{(\Delta q)^{2}} g_{3} \text{ for } \delta > 0 \text{ or } \delta = 0 \text{ and } k < 1, \tag{41} \]
\[ \simeq \frac{\alpha}{\sqrt{\Delta q}} g_{3,D} \text{ for deterministic case}, \tag{42} \]
\[ \hat{R} \simeq \frac{\alpha}{\Delta q} g_{4} \text{ when } P(y) \text{ is not constant for } y > 0, \tag{43} \]
\[ \simeq \frac{\alpha}{\Delta q} \frac{2k\xi^{3}}{\pi} \propto \alpha \sqrt{\Delta q} \text{ when } P(y) \equiv k \text{ for } y > 0 \text{ and } k < 1, \tag{44} \]
\[ = \hat{q} \simeq \frac{\alpha}{\sqrt{\Delta q}} g_{3,D} \text{ for deterministic case}. \tag{45} \]

Let us see what are derived from these conditions. First, let us consider the case of $0 < \beta < \infty$. In this case, both of $g_{1,\delta}(\chi = 1, \beta)$ and $g_{2,\delta}(\chi = 1, \beta)$ are finite for $\delta \geq 0$. See subsection 3.3. Thus, from (39) and (40), the conditions for $\hat{q} \to \infty, \hat{R} \to \infty$ and $\tau \to \infty$ as $q \to 1$ for any $\alpha$ are derived as
\[ \hat{q} \to \infty \text{ for any } \delta \geq 0 \text{ and } 0 < \beta < \infty, \tag{46} \]
\[ \hat{R} \to \infty \text{ for } 0 \leq \delta < 1 \text{ and } 0 < \beta < \infty, \tag{47} \]
\[ \tau \simeq \alpha^{1/2} (\Delta q)^{2\delta - 1/4} \frac{g_{2,\delta}}{\sqrt{g_{1,\delta}}} \]
\[ \to \infty \text{ for } 0 \leq \delta < 1/2 \text{ and } 0 < \beta < \infty. \tag{48} \]

Hence, the condition for the PL is $0 \leq \delta < 1/2$. Next, let us consider the case of $\beta = \infty$. When $P(y)$ is not constant for $y > 0$, $g_{s}$ are finite. Then, from (41) and (42), we obtain that both $\hat{q}$ and $\hat{R}$ tend to infinity and $\tau$ becomes $\tau \simeq \sqrt{\frac{\alpha}{g_{1,\delta}}} g_{4}$ and is finite. Thus, in this case the PL does not exist. On the other hand, when $P(y) \equiv k < 1$ for $y > 0$, $\hat{R} \propto \alpha \sqrt{\Delta q}$ tends to 0. Thus, the PL does not exist. Finally, in the deterministic case, from (43) and (44), $\hat{q} = \hat{R}$ and $\tau = \sqrt{q}$ tend to infinity since $g_{3,D}$ is finite. Hence, the PL exists.

Therefore, summarizing the above results, we conclude that the PL exists for $0 \leq \delta < 1/2$ and $0 < \beta < \infty$, and for the deterministic case.

As for the entropy $S_{PL}$ and the free energy $f_{PL}$ for the PL, we obtain the following reasonable results,
\[ S_{PL} = 0, \quad f_{PL} = \alpha \epsilon_{\text{min}}. \]
5. 1RSB solution

Although we adopt the Gibbs algorithm as the learning strategy, we also have interest in the minimum-error algorithm. In the minimum-error algorithm we have to choose weights with minimum errors, and on that account we only have to take the limit \( T \to +0 \). However, as is shown in §3 by numerical calculations, in the RS solution, the entropy becomes negative for small \( T \). Thus, we have to consider the breaking of the replica symmetry [14]. In the one-step RSB solution, the matrix \( q^{\alpha \beta} \) is divided into \((n/m)^2\) small matrices with the dimension \( m \times m \). The components of each off-diagonal matrix are all \( q_0 \) and the components of each diagonal matrix are \( q_1 \) except for diagonal components with the value 0. Likewise, \( \hat{q}_0 \) and \( \hat{q}_1 \) are defined for the matrix \( \hat{q}^{\alpha \beta} \). Further, \( R^\alpha = R \) and \( \hat{R}^\alpha = \hat{R} \) are assumed. Then, the one-step RSB free energy \( f_{1RSB} \) is derived to be

\[
-\beta f_{1RSB}(q_0, \hat{q}_0, q_1, \hat{q}_1, R, \hat{R}, m, \beta) = -\frac{\hat{q}_1}{2}(1 - q_1) + \frac{m}{2}(\hat{q}_0q_0 - \hat{q}_1q_1) - R\hat{R} + \frac{\alpha}{m}\int Dy2P(y) \int Dz_0 \ln \int Dz_1[H(\sqrt{q_0 - R^2z_0} + \sqrt{q_1 - q_0z_1} - Ry)]^m
+ \frac{1}{m}\int Dz_0 \ln \int Dz_1[2\cosh(\sqrt{q_0z_0} + \sqrt{q_1 - q_0z_1} + \hat{R})]^m.
\]

Further, according to Krauth-Mézard [15], we take the limits \( q_1 \to 1 \) and \( \hat{q}_1 \to \infty \). Then, we obtain

\[
f_{1RSB}(q_0, \hat{q}_0, q_1 = 1, \hat{q}_1 = \infty, R, \hat{R}, m, \beta) = f_{RS}(q_0, m^2\hat{q}_0, R, m\hat{R}, \beta m).
\]

From this relation, the equations for \( q_0, \hat{q}_0, R, \hat{R} \) and \( m \) become the coupled equations of the saddle point equations for the RS solution and the equation of \( S_{RS} = 0 \), where \( S_{RS} \) is the entropy for the RS solution. Let us denote the solutions of these coupled equations by \( q = q_e, \hat{q} = \hat{q}_e, R = R_e, \hat{R} = \hat{R}_e \) and \( \beta = \beta_e \). Then, the one-step RSB solutions are expressed by \( q_0 = q_e, \hat{q}_0 = (\frac{\beta}{\beta_e})^2\hat{q}_e, R = R_e, \hat{R} = \frac{\beta}{\beta_e}\hat{R}_e \) and \( m = \frac{\beta}{\beta_e} \). Thus, to obtain the \( T \to +0 \) limit we only have to know the solution at \( T = T_e = \beta_e^{-1} \).

5.1. Numerical calculation of S.P.E. for the 1RSB solution

(I) \( \delta > 0 \)

As a special case, we treated \( P(y) = 1 - 2H(y) \), that is the case of \( \delta = 1 \). This is the same function as that calculated for the RS solution. In Figure 6, we show \( \alpha \) dependence of \( T_e \). See Figure 11 for \( \alpha \) dependence of \( q_0, R \) and \( \Delta\epsilon_g \). As is seen from these figures, 1RSB solution seems to extend to \( \alpha = \infty \). To study asymptotic behaviors, assuming the following relations for several quantities, we estimated the coefficients \( a_i \) and exponents \( b_i \) by the least square methods. That is, for \( \Delta q, \Delta R, \Delta\epsilon_g \) and \( T_e \) we assume

\[
\ln A = a_1 + b_1 \ln \alpha,
\]
Table 1. Coefficient $a_i$ and exponent $b_i$ estimated for $200 \leq \alpha \leq 381$ and theoretical value $b_{2,th}$.

|   | $\Delta q$ | $\Delta R$ | $T_c$ | $\Delta \epsilon_g$ | $\hat{R}$ | $\hat{q}$ |
|---|---|---|---|---|---|---|
| $a_1$ | 2.5 | 2.2 | -1.8 | 1.1 | -0.49 | 0.89 |
| $b_1$ | -1.5 | -1.5 | 0.81 | -1.5 | 1.1 | 0.28 |
| $a_2$ | 1.1 | 0.88 | -1.1 | -0.27 | 0.50 | 1.1 |
| $b_2$ | -1.8 | -1.9 | 0.99 | -1.9 | 1.4 | 0.35 |
| $b_{2,th}$ | -2 | -2 | 1 | -2 | 1 | 1 |

$$\ln A = a_2 + b_2 \ln(\frac{\alpha}{\ln \alpha}),$$

and for $\hat{q}$ and $\hat{R}$

$$A = a_1 + b_1 \ln \alpha,$$

$$A = a_2 + b_2 \ln(\frac{\alpha}{\ln \alpha}).$$

In the Table I, we give the list of $a_i$ and $b_i$ for these quantities. In particular, we note that $T_c \to \infty$ as $\alpha \to \infty$. Further, we obtained $\frac{\Delta q}{\Delta R} \sim 1.7$ and $\frac{\hat{R}}{\hat{q}} \sim 2.5$. As an example, we show the asymptotic behaviour of $\Delta \epsilon_g$ in Figure 7.

(II) $\delta = 0$

We treated the same function $P(y)$ as in the calculation for the RS solution, $P(y) = \frac{1}{2} \operatorname{sgn}(y)$. We depict $\alpha$ dependence of $T_c$ in Figure 8. See Figure 16 for $\alpha$ dependence of $q_0$, $R$ and $\Delta \epsilon_g$. In the 1RSB solution, there exist two branches I and II. In the branch I the quantities agree to those in the RS solution with $T = 0$ as $\alpha \to \alpha_s(T = 0)$. On the other hand, $q$ and $R$ tend to 1 as $\alpha$ tends to 0 in the branch II.

Within our calculation, it is difficult to determine which case of (a) $0 \lesssim \mu < 1$ or (b) $\mu \geq 1$ takes place in the branch II, since we could obtain solutions only for $\alpha \gtrsim 0.45$. As for $T_c$ it seems that $T_c \to \text{finite}$ as $\alpha \to 0$. In both branches, $\Lambda_1$ and $\Lambda_3$ are negative, that is, AT stable. As for the free energy, $f^I_{RSB} < f^II_{RSB}$ holds. See Figure 9.

5.2. Asymptotic behaviors

As is suggested from the above numerical results and will be shown later, asymptotic behaviors of $\beta_c$ are different in the cases of $\delta = 0$ and $\delta > 0$. Thus, we discuss these cases separately.

(I) The case of $\delta > 0$

Suggested by numerical results we consider the limit $\beta_c \ll 1$. For $\beta \ll 1$, $f_{RS}$ and $S_{RS}$ are estimated in the asymptotic region as follows. See Appendix E.

$$-\beta f_{RS} = -\frac{\hat{q}}{2}(1-q) - R\hat{R} + I$$

$$-\alpha \beta [\epsilon_{min} + \frac{2s}{(1+\delta)\sqrt{2\pi}} (2\Delta R)^{\frac{1+\delta}{2}} - \beta \sqrt{\Delta q} - \frac{\beta \sqrt{\Delta q}}{2\pi \sqrt{2}}],$$

(51)
On the conditions for the existence of Perfect Learning

\[ S_{RS} = -\frac{\hat{q}}{2}\Delta q - \hat{R}R + I - \alpha\beta^2 \sqrt{\Delta q} \]  
(52)

First, let us show that for 0 ≲ \mu < 1, no consistent 1RSB solution exists in the present situation. The saddle point equations are

\[ \Delta q \simeq \frac{2h(\tau)\psi(\mu)}{\sqrt{q_c}}, \]  
(53)

\[ \Delta R \simeq \frac{\Delta q}{2\mu}, \]  
(54)

\[ \hat{q}_c = \hat{q}_0\alpha\beta^2/\sqrt{\Delta q}, \]  
(55)

\[ \hat{R}_c = \hat{R}_0\alpha\beta(\Delta R)^{(\delta-1)/2}, \]  
(56)

\[ \hat{q}_0 = \frac{1}{2\pi\sqrt{2}}, \quad \hat{R}_0 = \frac{s}{\sqrt{\pi}}\delta^{\delta/2}, \]

where \( \Delta q = 1 - q_c \) and \( \Delta R = 1 - R_c \). Then, the condition that the entropy is zero becomes

\[ \Delta q \simeq \frac{4\psi(\mu)h(\tau)}{R\tau}. \]  
(57)

From equations (53) and (57), we obtain \( \tau^2 = 2 \). Since we consider the case of \( R \simeq 1 \) and \( q \simeq 1 \), \( \tau \) should be large. Thus, this case is inadequate. Therefore, in the below, we consider the case of \( 1 \leq \mu \). In this case, \( I \simeq \hat{R} + a_\mu e^{-2(\hat{R} - \hat{q})} \). Then, \( f_{RS} \) and \( S_{RS} \) become

\[ -\beta_c f_{RS} \simeq -\frac{\hat{q}_c}{2} \Delta q + \hat{R}_c\Delta R + a_\mu e^{-2(\hat{R}_c - \hat{q}_c)} - \alpha\beta_c(\epsilon_g - \frac{\beta_c}{2\pi\sqrt{2}}\sqrt{\Delta q}), \]

\[ S_{RS} \simeq -\frac{\hat{q}_c}{2} \Delta q + \hat{R}_c\Delta R - \frac{\alpha\beta_c^2}{2\pi\sqrt{2}}\sqrt{\Delta q} + a_\mu e^{-2(\hat{R}_c - \hat{q}_c)}. \]

The saddle point equations for \( \hat{q} \) and \( \hat{R} \) are the same as in the case (a), and those for \( q, R \) and zero entropy condition are

\[ \Delta R \simeq 2a_\mu e^{-2(\hat{R}_c - \hat{q}_c)}, \]  
(58)

\[ \Delta q \simeq 2\Delta R, \]  
(59)

\[ S_{RS} \simeq -\frac{\hat{q}_c}{2} \Delta q + \hat{R}_c\Delta R - \frac{\alpha\beta_c^2}{2\pi\sqrt{2}}\sqrt{\Delta q} + a_\mu e^{-2(\hat{R}_c - \hat{q}_c)} = 0. \]  
(60)

Since \( \hat{R}_c \gg 1 \) and \( \hat{q}_c \gg 1 \), using equations (53), (58) and (59), we obtain from equation (60) the following relation,

\[ \hat{R}_c = 3\hat{q}_c. \]  
(61)

That is, \( \mu = 3/2 \) and \( a_\mu = 1 \). Thus, from equations (53), (60) and (61), we obtain

\[ \hat{q}_c = F_0\alpha e^{-2(2\delta-1)\hat{q}_c}, \]

that is,

\[ \ln \hat{q}_c = \ln \alpha - 2(2\delta - 1)\hat{q}_c + \ln F_0, \]
where \( F_0 = 4s^20^{2q}/9\sqrt{2} \). Thus,
\[
\dot{q}_c \simeq \frac{\ln \alpha}{2(2\delta - 1)}
\]
for \( 2\delta - 1 \neq 0 \). This implies that \( \dot{q}_c \) tends to infinity under the condition that \( \alpha \) tends to infinity for \( \delta > 1/2 \) or \( \alpha \) tends to 0 for \( \delta < 1/2 \). For \( \delta = 1/2 \), \( \dot{q}_c = F_0\alpha \). Therefore, we obtain the following results.

(i) In the case of \( \delta > \frac{1}{2} \), as \( \alpha \to \infty \),
\[
\Delta R \simeq 2\left(\frac{\ln \alpha}{\alpha}\right)^{\frac{2}{2\delta - 1}}, \quad \Delta q \simeq 2\Delta R,
\]
\[
\dot{R}_c \simeq \frac{3}{2(2\delta - 1)} \ln\left(\frac{\alpha}{\ln \alpha}\right), \quad \dot{q}_c \simeq \dot{R}_c/3, \quad \beta_c \simeq 4\sqrt{\pi s} \cdot 2^{\delta} \left(\frac{\ln \alpha}{\alpha}\right)^{\frac{2}{2\delta - 1}},
\]
\[
\Delta \epsilon_g \simeq \epsilon_0(\Delta R)^{\frac{\delta + 1}{4}} \simeq \epsilon_0 2^{\delta + 1} \left(\frac{\ln \alpha}{\alpha}\right)^{\frac{\delta + 1}{2\delta - 1}},
\]
\( \epsilon_0 = \frac{2s}{(1 + \delta)^{\sqrt{2\pi}} 2^{\frac{\delta + 4}{4}}} \).

(ii) In the case of \( \delta = \frac{1}{2} \), as \( \alpha \to \infty \),
\[
\Delta R \simeq 2e^{-4F_0\alpha}, \quad \Delta q \simeq 2\Delta R,
\]
\[
\dot{R}_c \simeq 3F_0\alpha, \quad \dot{q}_c \simeq \dot{R}_c/3, \quad \beta_c \simeq 4\sqrt{\pi s} \cdot 2^{\delta} \left(\frac{\alpha}{\ln \alpha}\right)^{\frac{2}{2\delta - 1}},
\]
\[
\Delta \epsilon_g \simeq \epsilon_0 2^{\frac{\delta}{4}} e^{-3F_0\alpha}.
\]

(iii) In the case of \( 0 < \delta < \frac{1}{2} \), as \( \alpha \to 0 \),
\[
\Delta R \simeq 2\left(\frac{\alpha}{\ln \alpha}\right)^{\frac{2}{2\delta}}, \quad \Delta q \simeq 2\Delta R,
\]
\[
\dot{R}_c \simeq \frac{3}{2(1 - 2\delta)} \ln\left(\frac{1}{\ln \alpha}\right), \quad \dot{q}_c \simeq \dot{R}_c/3, \quad \beta_c \simeq 4\sqrt{\pi s} \cdot 2^{\delta} \left(\frac{\alpha}{\ln \alpha}\right)^{\frac{2}{2\delta - 1}},
\]
\[
\Delta \epsilon_g \simeq \epsilon_0 2^{\delta + 1} \left(\frac{\alpha}{\ln \alpha}\right)^{\frac{\delta + 4}{2\delta - 1}}.
\]

Thus, when \( 0 < \delta < 1/2 \), for large \( \alpha \), there is no solution such that \( q_c \to 1 \) and \( R_c \to 1 \). This implies that there is a value of \( \alpha = \alpha_{max} \) such that for \( \alpha > \alpha_{max} \) there is no solution except for the PL when \( 0 < \delta < 1/2 \).

Here, let us compare the theoretical results with numerical ones on asymptotic behaviours for \( \delta = 1 \). As is shown in the Table I, we note that numerically obtained exponents \( b_2 \) and theoretical ones \( b_{2, th} \) agree fairly well except for those of \( \dot{q} \) and \( \dot{R} \). \( \dot{q} \) and \( \dot{R} \) are proportional to \( \ln(\frac{\alpha}{\ln \alpha}) \) and it is difficult to estimate a logarithmic dependence directly. Instead, we can check the theoretical result of the relation \( \dot{R}/\dot{q} = 3 \). Numerically, this value is 2.5. Further, as for the relation \( \Delta \dot{R}/\Delta \dot{q} = 2 \), we obtained numerically 1.7. Therefore, we can conclude that the agreement between theoretical and numerical results are fairly well.

Now, let us examine the case \( \delta = 0 \). In this case, substituting \( \delta = 0 \) into the above expressions for the case of \( 0 < \delta < 1/2 \), \( \beta_c \) becomes constant. That is, the assumption \( \beta_c \ll 1 \) is not satisfied. This is the reason why we treat the cases \( \delta > 0 \) and \( \delta = 0 \).
separately.

(II) The case of $\delta = 0$.

In §3, we examined the RS solution with $\beta$ fixed. Now, let us investigate the 1RSB solution by imposing the condition $S_{RS} = 0$.

(a) $0 \lesssim \mu < 1$

In this case, $S_{RS}$ becomes

$$S_{RS} \simeq \frac{\hat{q}\Delta q}{2} + \alpha\sqrt{\Delta q} r.$$  

Then, from the condition $S_{RS} = 0$ we obtain

$$g_{1,0}(\chi, \beta_c) = -2r(\chi, \beta_c).$$  

From equations (35) and (71), $\chi$ and $\beta_c$ are determined. The 1RSB solution appears for $\beta > \beta_c$.

(b) $\mu \geq 1$

$S_{RS}$ is

$$S_{RS} \simeq -\frac{\hat{q}\Delta q}{2} + \hat{R}\Delta q + \alpha\sqrt{\Delta q} r.$$  

Then, the condition $S_{RS} = 0$ becomes

$$g_{1,0}(\chi = 0, \beta_c) = g_{2,0}(\chi = 0, \beta_c) + 2r(\chi = 0, \beta_c).$$  

If the solution $\beta_c$ of equation (72) exists, 1RSB solution appears for $\beta > \beta_c$. The condition of the existence of this type of solution is $g_{2,0}/g_{1,0} \geq 2$.

Numerical calculations for $\delta = 0$ indicate that $\chi$ tends to constant and then the case (a) appears. Thus, for $\alpha \ll 1$

$$\Delta R \simeq \frac{1}{\sqrt{2\pi}}\psi(\mu_c)\alpha^2(\ln \frac{1}{\alpha})^{-2}, \quad \Delta q = 2\mu_c\Delta R,$$  

$$\hat{R}_c \simeq \frac{2}{\mu_c}\ln[\frac{1}{\alpha}(\ln \frac{1}{\alpha})^{3/4}], \quad \hat{q}_c = \frac{\hat{R}_c}{2\mu_c},$$  

$$\Delta \epsilon_g = \epsilon_0\frac{\psi(\mu_c)}{2\pi}\alpha(\ln \frac{1}{\alpha})^{-1}.$$  

This implies that for $\delta = 0$, the PL takes place.

As for the validity of the above asymptotic solutions for any $\delta \geq 0$, since the coefficient of any quantity does not contain $\beta$, there is no condition for the range of $\beta$.

Thus, in the case of $0 \leq \delta < \frac{1}{2}$, there exists no solution for $\alpha > \alpha_{\text{max}}$. This is consistent with the result derived in §4 that the PL exists for $0 \leq \delta < \frac{1}{2}$ when $0 < \beta < \infty$.

Putting together all results obtained in this paper, we get the following behaviours of learning.

When $T$ is small enough, there is a critical value of $\alpha$, $\alpha_s(T)$ above which the entropy of the RS solution becomes negative. Thus, for $\alpha > \alpha_s(T)$, the 1RSB solution appears. Within the 1RSB ansatz, we found that the behaviour of the generalization error $\epsilon_g$ is classified into the following three categories according to the value of $\delta$. 


(i) If $0 \leq \delta < \frac{1}{2}$, solutions with $R < 1$ exist only for a finite range of $\alpha$, $[0, \alpha_{\text{max}}]$. There is a critical temperature $T_c$. When $T > T_c$, the entropy of the RS solution is positive and this solution is AT stable. When $T < T_c$, for $\alpha > \alpha_s(T)$, the 1RSB solution appears. In both cases, at $\alpha = \alpha_{\text{max}}$, a first-order phase transition from the RS solution with positive entropy or from the 1RSB solution to the Perfect Learning takes place.  

(ii) If $\delta = \frac{1}{2}$, $\alpha_s(T)$ is defined for any temperature $T$, and 1RSB solution appears for $\alpha > \alpha_s(T)$. $\epsilon_g$ for the 1RSB solution decays exponentially,$$
abla\epsilon_g \sim e^{-3F_0\alpha},$$where $F_0$ is a constant.

(iii) If $\delta > \frac{1}{2}$, for any temperature $T$, $\alpha_s(T)$ is defined and 1RSB solution appears for $\alpha > \alpha_s(T)$. $\epsilon_g$ for the 1RSB solution decays as a power law with a logarithmic correction,$$
abla\epsilon_g \propto \left(\frac{\ln \alpha}{\alpha}\right)^{\frac{1+\delta}{2}},$$

To check these theoretical results, we performed numerical calculations. In the next section, we give the results of the calculations.

6. Numerical Calculations by exhaustive method

We performed numerical calculations by the exhaustive method for $\delta = 0$ and $\delta = 1$. We used the minimum-error algorithm and the Gibbs algorithm for several temperatures. We calculated several quantities such as $q, R, \epsilon_g$, etc. For example, $q$ and its standard deviation $\delta q$ are calculated by the following formulas.

\begin{equation}
q = \frac{1}{N_\xi} \sum_\xi q_\xi, \quad q_\xi = \sum_{a<\beta} q^{a,\beta} P_a P_\beta / \sum_{a<\beta} P_a P_\beta, \tag{76}
\end{equation}

\begin{equation}
\delta q = \sqrt{\sum_\xi q^2_\xi - (\sum_\xi q_\xi)^2 / N_\xi} / (N_\xi - 1), \tag{77}
\end{equation}

\begin{equation}
P_\alpha = e^{-\beta E_\alpha} / \sum_\alpha e^{-\beta E_\alpha}, \tag{78}
\end{equation}

where $\alpha$ denotes one of $2^N$ configurations of weight vectors and $E_\alpha$ is its energy, $q_\xi$ is the thermal average for a given example $\xi$ and $N_\xi$ is the number of samples. The calculations were performed for $N$ up to 20 with $N_\xi = 200$. First, let us show the results for $\delta = 1$.

(I) $\delta = 1$

The $i$-th component of an example $x_i$ is corrupted by a gaussian noise $\eta_i$ with mean 0 and standard deviation 1. This corresponds to $P(y) = 1 - 2H(y)$. First, we show the results for the minimum-error algorithm. In Figure 10, to see the system size ($N$) dependence of quantities, we show the $\alpha$ dependence of $R$ and its standard deviation $\delta R$ for $N = 10, 15$ and 17. From these results, it seems that the calculations in $N = 15$ is sufficient to obtain $N = \infty$ results at least for $\alpha$ up to 15. In Figure 11, we compare
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numerical results with theoretical ones. The agreement between the theoretical and numerical results is fairly well except for \( q_0 \). As for \( q_0 \), we calculate it in each sample by the formula (76). In general, \( q_0 \) exhibits a large finite size effect, because it is calculated for pair of states. Thus, when the number of states with the minimum energy becomes small, the fluctuation of \( q_0 \) becomes very large. This is the reason why the agreement between the numerical and the theoretical results for \( q_0 \) is worse than other quantities.

A suitable quantity is the distribution of \( q \), \( P(q) \). We calculate this in the case of the Gibbs Algorithm. In Figure 12 we show the \( \alpha \) dependence of \( R \) for larger value of \( \alpha \) in the case of \( N = 15 \). At \( \alpha \gtrsim 85 \), there exists only one state. Since theoretically \( R \) tends to 1 as \( \alpha \) goes to \( \infty \), to see whether this is a finite size effect or not we estimated the value of \( \alpha_{\text{max}} \) by the condition that \( R \) exceeds \( 1 - 1/N \) for the first time as \( \alpha \) increases. Then, we found that as \( N \) increases \( \alpha_{\text{max}} \) increases. Thus, it seems that we observed the finite size effect.

Next, we show the results for the Gibbs algorithm. In this algorithm, we calculated for \( N = 10 \) and 12 and for several temperatures, and we took into account the all states. We confirmed that the results for \( N = 10 \) and 12 are almost same. We show both numerical and theoretical results in Figure 13 for \( R \) and \( \Delta \epsilon_g \) and in Figure 14 for \( q \) and \( q_0 \). The distribution of \( q \), \( P(q) \), is also shown for several temperatures together with theoretical results in Figure 15. \( P(q) \) is calculated by

\[
P(q) = \left< \sum_{\alpha,\beta} \delta(q, q^{\alpha\beta}) P_\alpha P_\beta \right>,
\]

where \( \delta(q, q^{\alpha\beta}) \) is the Kronecker’s delta and \( \left< \cdot \right> \) means the sample average. From these, we see the agreement between theoretical and numerical results is fairly well.

(II) \( \delta = 0 \)

The output by a teacher is reversed with the probability \( (1 - k)/2 \) with \( k = \frac{1}{2} \), that is we treat the same case as before. First, we show the results for the minimum-error algorithm. We investigated the \( N \) dependence of \( R(\alpha) \) and \( \delta R(\alpha) \) for \( N = 10, 15, 17 \) and 20 and found that the results for \( N = 15, 17 \) and 20 are almost same. In Figure 16, \( \alpha \) dependence of several quantities are depicted for \( N = 17 \) together with theoretical results. In the figure of \( q \) and \( q_0 \)(Figure 16(b)), we note that \( q_0 \) takes values at above the theoretical upper bound of \( \alpha \), \( \alpha_{\text{max}} \). This is due to a finite size effect mentioned in the above. In Figure 17 we show the behaviour of \( R \) for larger value of \( \alpha \) in the case of \( N = 15 \). For \( \alpha > 22 \), there exists only one state. In the case of \( N = 10 \), it occurs for \( \alpha > 25 \). To investigate whether the PL exists or not, we numerically estimated \( \alpha_{\text{max}} \) by the same method as that in the case of \( \delta = 1 \). Contrary to the case of \( \delta = 1 \), \( \alpha_{\text{max}} \) decreases as \( N \) increases. Thus, we conclude the PL takes place even when \( N = \infty \).

Theoretically, \( \alpha_{\text{max}} \) is about 9.13.

Next, we show the results for the Gibbs algorithm. In this algorithm, we calculated for \( N = 10 \) and 12, and for several temperatures. We took into account the all states. We confirmed that \( N = 12 \) is sufficient for convergence for any \( T \). In Figure 18, \( \alpha \) dependence of \( R \) and \( \Delta \epsilon_g \) are shown for \( T = 1.0 \). Also, \( \alpha \) dependence of \( q \) and \( q_0 \) are depicted in Figure 19 for \( T = 0.15, 0.5 \) and 5.0. As is shown in these figures,
the agreement between the theoretical values and numerical ones are fairly well except for the case of $q_0$. Concerning $q_0$, we further calculated $P(q)$ at $\alpha = 5$ for several temperatures. In Figure 20, we show the numerical results of $P(q)$ together with theoretical ones. The positions of peak values agree in the theoretical and numerical results.

In conclusion, in both cases of $\delta = 1$ and 0, although there exist finite size effects as is seen for $q_0$, as a whole theoretical results and numerical ones agree fairly well within the 1RSB ansatz.

7. Summary and Discussion

In this paper, we studied the learning from stochastic examples by perceptrons with Ising weights. By using the replica method, we obtained the condition for the existence of the Perfect learning and power law of learning curves in the asymptotic region, in terms of $\delta$ which represents the local property of the rules by which examples are drawn. First, let us summarize the results in more details.

Our assumptions are as follows.

When an input vector $x$ is given, the probability $p_r(+1|x)$ that a teacher returns an output +1 is a function of the inner product between the input $x$ and the teacher’s weight $w^o$ and take the following form,

$$p_r(+1|x) = \mathcal{P}(u^o) = \frac{1 + P(u^o)}{2},$$

$$u^o \equiv (x \cdot w^o)/\sqrt{N}, \quad |x| = \sqrt{N}, \quad |w^o| = \sqrt{N}.$$ 

Further, we assume $P(y)$ is non-decreasing and near $y = 0$ it behaves as $P(y) \simeq a \operatorname{sgn}(y)|y|^{\delta}$, $(\delta \geq 0)$. For simplicity, we assumed that $P(y)$ is an odd function.

Under the above assumptions, we obtained the following results.

**Conditions for the PL**

The the necessary and sufficient conditions for the existence of the Perfect Learning are

(a) $0 \leq \delta < 1/2$ when $0 < \beta < \infty$,

(b) deterministic case.

**Behaviour of learning curves**

Within the 1RSB ansatz, we found that the behaviour of the generalization error $\epsilon_g$ is classified into the following three categories according to the value of $\delta$.

(i) If $0 \leq \delta < \frac{1}{2}$, at $\alpha = \alpha_{\text{max}}$, a first-order phase transition from the RS solution with positive entropy or from the 1RSB solution to the Perfect Learning takes place.

(ii) If $\delta = \frac{1}{2}$, for large $\alpha$ 1RSB solution appears and $\epsilon_g$ for the 1RSB solution decays exponentially,

$$\Delta \epsilon_g \sim e^{-3F_0\alpha},$$

where $F_0$ is a constant.
(iii) If $\delta > \frac{1}{2}$, for large $\alpha$ 1RSB solution appears and $\epsilon_g$ for the 1RSB solution decays as a power law with a logarithmic correction,
\[ \Delta \epsilon_g \propto \left( \frac{\ln \alpha}{\alpha} \right)^{\frac{1+\delta}{2\delta-1}}. \]

To check these results, we performed several numerical calculations. Those are, the calculations of the saddle point equations for the RS and the 1RSB solutions and the direct calculations of concerned quantities by enumeration methods. The numerical and theoretical results showed fairly well agreement. As is mentioned in the introduction, Seung also investigated the existence of the Perfect Learning when the weights are Ising and a rule to be learnt is stochastic [13] by the annealed approximation. He classified learning behaviour of Ising networks by introducing the following two exponents $y$ and $z$. The first exponent $y$ is associated with $\rho(\epsilon_g)$ which is the logarithm of the number of weight vectors whose generalization errors have a value $\epsilon_g$. He assumed that when $\Delta \epsilon_g = \epsilon_g - \epsilon_{\text{min}}$ is small, $\rho(\epsilon_g)$ increases as $\rho(\epsilon_g) \sim O((\Delta \epsilon_g)^y)$, where $\epsilon_{\text{min}}$ is the minimum value of the generalization error obtained by the unique optimal weight vector $w^o$. The second exponent $z$ is introduced to characterize $e_d(w, w^o)$ which is the probability that the output for the weight vector $w$ differs from that for the optimal weight vector $w^o$. He also assumed that $e_d(w, w^o)$ is scaled as $e_d(w, w^o) \sim O((\Delta \epsilon_g)^z)$. He estimated the upper bounds for the generalization errors and found that the behavior of learning curves varies according to the values of indices $y$ and $z$. His results are summarized as follows.

(i) If $y + z > 2$, there is a first-order transition.

(ii) If $y + z < 2$, the generalization error decays as a power law, $\Delta \epsilon_g \sim \alpha^{\frac{1}{y+z}}$.

(iii) If $y + z = 2$, there is a second-order transition or the generalization error decays exponentially.

In our model, the exponents $y$ and $z$ are expressed as $y = \frac{2}{1+\delta}$, $z = \frac{1}{1+\delta} = \frac{2}{2}$ respectively, and then $v = \frac{3}{1+\delta}$. Therefore, $\delta = \frac{3}{1+\delta}$ follows and it is found that our results on the typical learning behavior agrees with Seung’s results which are the upper bounds of the learning curves.

As for the condition of the existence of the PL, we note that for $\beta = \infty$, i.e. $T = 0$, the PL does not exist in the learning from stochastic examples. The reason is that for $T = 0$ and for large $\alpha$ there exists no student whose outputs are the same as the teacher’s, since the teacher makes mistakes. Thus, the volume of weight vectors whose energies are 0 vanishes for large $\alpha$. On the other hand, for the case of $T \to +0$, we consider the weight vectors of the minimum energy, and there is at least one solution of $w = w^o$ when $\alpha$ is large enough. Thus, the PL is possible in the limit $T \to 0$.

As the learning advances, $w$ tends to $w^o$. The examples which give the crucial influence on the learning are such that $u_0 = (x \cdot w^o)/\sqrt{N} \sim 0$. The more slowly the probability $P(u)$ varies around $u = 0$ for larger $\delta$, the more difficult students tune the optimal vector $w^o$. This is the reason why the larger $\delta$ is, the Perfect Learning becomes more difficult to take place.
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Appendix A. Derivation of free energy

Here, we derive the free energy by the replica method. Introducing $n$ replicas, the partition function $Z^n$ becomes

$$Z^n = \text{Tr} \prod_{\alpha=1}^n \left[ e^{-\beta \int_{-\infty}^0 d\lambda_{\alpha} + \int_0^\infty d\lambda_{\alpha}^2 \int_{-\infty}^\infty \frac{dy_{\alpha}}{2\pi} \exp[-iy_{\alpha} (r_{\alpha} u_{\alpha} - \lambda_{\alpha})] \right],$$

where $u_{\alpha} = (x_{\alpha} \cdot w_{\alpha})/\sqrt{N}$ and Tr implies the summation over all configurations of $w_{\alpha}, \alpha = 1, \cdots \cdot n$. Defining the overlap between the weight vector of a learner and the optimal weight vector, $R_{\alpha} = \frac{1}{N} \sum_{j=1}^N w_{\alpha,j} w_{\beta,j}$, and the overlap between the weight vectors of learners, $q_{\alpha\beta} = \frac{1}{N} \sum_{j=1}^N w_{\alpha,j} w_{\beta,j}$, and using the relations

$$\delta(\sum_{j=1}^N (x_{\mu,j})^2 - N) = \int_{-i\infty}^{i\infty} \frac{dK_{\mu}}{2\pi i} \exp[-K_{\mu}(\sum_{j=1}^N (x_{\mu,j})^2 - N)],$$

$$1 = \prod_{\alpha} \int dR_{\alpha} \int_{-i\infty}^{i\infty} \frac{N d\hat{R}_{\alpha}}{2\pi i} \exp[-N\hat{R}_{\alpha}(R_{\alpha} - \frac{1}{N} \sum_{j=1}^N w_{\alpha,j} w_{\beta,j})],$$

$$1 = \prod_{\alpha<\beta} \int_{-i\infty}^{i\infty} \frac{N dq_{\alpha\beta} d\hat{q}_{\alpha\beta}}{2\pi i} \exp[-N\hat{q}_{\alpha\beta}(q_{\alpha\beta} - \frac{1}{N} \sum_{j=1}^N w_{\alpha,j} w_{\beta,j})],$$

we take the average over $r_{\mu,\alpha}$ and $x_{\mu}$, and obtain the expression for $< Z^n >_{\xi_{\mu,\alpha}}$.

$$< Z^n >_{\xi_{\mu,\alpha}} = \int \left[ \prod_{\alpha<\beta} \frac{N dq_{\alpha\beta} d\hat{q}_{\alpha\beta}}{2\pi i} \right] \prod_{\alpha} \frac{N dR_{\alpha}}{2\pi i} e^{NG},$$

$$G = \frac{p}{N} G_1(\{q_{\alpha\beta}\}, \{R_{\alpha}\}) + G_2(\{\hat{q}_{\alpha\beta}\}, \{\hat{R}_{\alpha}\}) - \sum_{\alpha} \hat{R}_{\alpha} R_{\alpha} - \sum_{\alpha<\beta} \hat{q}_{\alpha\beta} q_{\alpha\beta},$$

$$e^{G_1} = \left[ \prod_{\alpha} (e^{-\beta \int_{-\infty}^0 d\lambda_{\alpha} + \int_0^\infty d\lambda_{\alpha}^2 \int_{-\infty}^\infty \frac{dy_{\alpha}}{2\pi}}) \times \exp[-\frac{1}{2} \sum_{\alpha} (y_{\alpha})^2 - \sum_{\alpha<\beta} q_{\alpha\beta} y_{\alpha} y_{\beta} + i \sum_{\alpha} y_{\alpha} \lambda_{\alpha}] \Psi(\sum_{\alpha} y_{\alpha} R_{\alpha}) \right],$$

$$e^{G_2} = \text{Tr} \exp[\sum_{\alpha} \hat{R}_{\alpha} w_{\alpha} + \sum_{\alpha<\beta} \hat{q}_{\alpha\beta} w_{\alpha} w_{\beta}],$$

$$\Psi(y) \equiv \frac{1}{\sqrt{2\pi}} \int d\xi e^{-\frac{1}{2}(\xi-i\gamma)^2} \{1 - \frac{1}{2} [P(\xi) - P(-\xi)] \}.$$
The general form of the free energy per weight is given by
\[
f = -\frac{\langle \ln Z \rangle^{\inp_{p,w^o}}}{N\beta} = -\frac{G}{n\beta}.
\] (A.7)

Appendix B. Derivation of asymptotic relations for \( \hat{q} \) and \( \hat{R} \) when \( q \rightarrow 1 \) and \( R \rightarrow 1 \)

In this appendix, we briefly derive the asymptotic relations for \( \hat{q} \) and \( \hat{R} \).

First, let us consider the case of \( 0 < \beta < \infty \). The equation (12) for \( \hat{q} \) is rewritten as follows.
\[
\hat{q} = \frac{\alpha Q (1 - e^{-\beta})^2}{1 - q \sqrt{2\pi}} (A - B),
\] (B.1)

\[
A = \int du e^{-Q^2u^2/2} \frac{h(u)^2}{H(u)^2},
\] (B.2)

\[
B = \frac{1}{\sqrt{2 + Q^2}} \frac{1}{\sqrt{2\pi}} \int_0^\infty Dz P(\varepsilon z) \int_{-\infty}^{\infty} Dt H_2[k(t + \frac{1}{\chi}\sqrt{\frac{1 - \xi^2}{2 + Q^2}}z)](B.3)
\]
where \( H_2(u) = H(u^2) - H(-u^2) \), \( \kappa = \frac{\chi}{\sqrt{1 + 2\chi^2}} \) and \( \varepsilon = \sqrt{\frac{Q^2 + 2\xi^2}{2Q^2}} \). It follows that \( H_2(u) \) is strictly increasing odd function and \( 0 < |H_2(u)| < e^{2\beta} - 1 \) for \( u \neq 0 \). Thus, for \( \delta > 0 \), \( P(\varepsilon z) \) can be replaced by \( a(\varepsilon z)^\delta \) in the equation (B.3).

\[
B \simeq \frac{1}{2\sqrt{\pi}} \int_0^{1/\varepsilon} Dz a(\varepsilon z)^\delta \int_{-\infty}^{\infty} Dt H_2[k(t + \frac{1}{\sqrt{2}\chi}z)] + O(H(1/\varepsilon)).
\]
As \( q \) and \( R \) tend to 1, \( \varepsilon \) tends to 0 and then \( B \rightarrow 0 \). On the other hand, for \( T > 0 \) \( A \) is finite for these limits. Thus, \( A - B \simeq A \). Therefore, we obtain for \( \delta > 0 \),
\[
\hat{q} \simeq \frac{\alpha}{\sqrt{\Delta q}} g_{1,\delta}(\chi, \beta),
\] (B.4)

\[
g_{1,\delta}(\chi, \beta) = \frac{1}{\sqrt{2\pi}} \int du \tilde{\varphi}(u)^2,
\] (B.5)
where \( \Delta q \equiv 1 - q \). For the case of \( \delta = 0 \), from equation (B.3), we obtain
\[
B \simeq \frac{1}{2\sqrt{\pi}} \int_0^\infty Dz k \int_{-\infty}^{\infty} Dt H_2[k(t + \frac{1}{\sqrt{2}\chi}z)] = k \int du \frac{h(u)^2}{H(u)^2}[1 - 2H(u/\chi)],
\]
where \( k = \lim_{y \rightarrow +0} P(y) \). Then, we obtain
\[
\hat{q} \simeq \frac{\alpha}{\sqrt{\Delta q}} g_{1,0}(\chi, \beta),
\] (B.6)

\[
g_{1,0}(\chi, \beta) = \frac{1}{2\sqrt{\pi}} \int du \tilde{\varphi}(u)^2[1 - k + 2kH(u/\chi)].
\] (B.7)

Now, let us estimate \( \hat{R} \). The equation (13) is rewritten as
\[
\hat{R} = \frac{\alpha}{q^x} \frac{(1 - e^{-\beta})^2}{\sqrt{2\pi}} D,
\] (B.8)
Thus, we obtain
\[
\delta
\]
We assume that
\[
|\delta| > 1
\]
For \(\delta > 0\), \(D\) is calculated as
\[
D = \frac{\xi}{\sqrt{1 + Q^2}} \int_{0}^{\infty} D_{y}P'(\zeta z)\psi\left(\frac{1}{\sqrt{1 + \chi^2}} \sqrt{1 - \frac{\xi^2}{1 + Q^2}} z\right),
\]
where \(\zeta = \sqrt{\frac{Q^2 + \xi^2}{1 + Q^2}}\) and \(\psi(z) = \int_{-\infty}^{\infty} D_{t}H_{1}(\nu t - z)\). Since \(\psi(z)\) is bounded, \(D\) is evaluated as follows.
\[
D \simeq \xi \int_{0}^{1/\zeta} D_{z}a\delta(\zeta z)^{\delta-1}\psi\left(\frac{z}{\sqrt{1 + \chi^2}}\right) + O\{H(1/\zeta)\} \simeq \xi a\delta^{\delta-1} \int_{0}^{\infty} D_{z}z^{\delta-1}\psi\left(\frac{z}{\sqrt{1 + \chi^2}}\right).
\]
That is, we obtain
\[
\hat{R} \simeq \alpha \frac{\xi \delta}{\sqrt{\Delta q}} g_{2,\delta}(\chi, \beta),
\]
\[
g_{2,\delta}(\chi, \beta) \equiv \frac{a\delta}{\sqrt{2\pi}} (1 - e^{-\beta}) \frac{1}{\chi} (1 + \chi^{-2})^{(\delta-1)/2} \int_{0}^{\infty} D_{z}z^{\delta-1}\psi\left(\frac{z}{\sqrt{1 + \chi^2}}\right).
\]
For \(\delta = 0\), from equation (B.9), we obtain
\[
D = \nu \int_{-\infty}^{\infty} D_{x}H_{1}(\nu x)\left[\frac{k}{\sqrt{2\pi}} + \xi \int_{0}^{\infty} D_{y}e^{-\eta xy}P'(\xi y)\right].
\]
We assume that \(|P'(y)|\) is bounded. Then the second term in the parenthesis is evaluated as \(O(\xi)\) and is neglected.
Then,
\[
D \simeq \frac{kr}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D_{x}H_{1}(\nu x) = \frac{2k}{\sqrt{2\pi}} \frac{\chi}{\sqrt{1 + \chi^2}} \int_{-\infty}^{\infty} D_{x} \frac{1}{H\left(\frac{\chi x}{\sqrt{1 + \chi^2}}\right)}.
\]
Thus, we obtain
\[
\hat{R} \simeq \alpha \frac{1}{\sqrt{\Delta q}} g_{2,\delta},
\]
\[
g_{2,\delta} = \frac{k(1 - e^{-\beta})}{\pi} \frac{1}{\sqrt{1 + \chi^2}} \int_{-\infty}^{\infty} D_{x} \frac{1}{H\left(\frac{\chi x}{\sqrt{1 + \chi^2}}\right)}.
\]
Now, let us consider the case of \(\beta = \infty\). In this case, \(\hat{\varphi}(u)\) becomes \(\varphi(u) \equiv \frac{H'(u)}{H(u)}\).
Therefore, from the equation (12) we obtain
\[
\hat{q} = \frac{\alpha}{\Delta q} \int D_{u}\varphi(u/Q)^{2} E(u/Q) \simeq \frac{\alpha}{\Delta q} (A' - B'),\]
\[
A' = \int_{0}^{\infty} D_{u}u(u/Q)^{2} = \frac{1}{2Q^{2}},
\]
\[
B' = \frac{\xi}{\sqrt{2\pi Q^{2}}} \int_{0}^{\infty} D_{y}P(\xi y)
\]
\[
\times \left\{2\sqrt{1 - \xi^2 \xi^2 y} + \sqrt{2\pi}[\xi^2 + (1 - \xi^2)\xi^2 y^2]e^{\frac{1 - \xi^2}{2}\xi^2 y^2}[1 - 2H(\sqrt{1 - \xi^2 y})]\right\}.
\]
\footnote{Here, we assumed the boundedness of \(|P'(y)|\). However, we can obtain the same results as those obtained here without using \(P'(y)\).}
In the expression (B.14), the first term in the parenthesis is $O(\xi^3 + \delta/Q^2)$ and the second term is evaluated as $\frac{1}{Q^2} \int_0^\infty Dy(1 - P(y))y^2$. Then,

$$\hat{q} \simeq \frac{\alpha}{(\Delta q)^2} \int_0^\infty Dy y^2 [1 - P(y)] = \frac{\alpha}{(\Delta q)^2} g_3. \quad (B.15)$$

If $P(u) = 1$, that is in the deterministic case, this gives 0. Thus, we discuss this case later. Similarly, from equation (13) for $\hat{R}$ we obtain

$$\hat{R} \simeq \frac{\alpha}{\xi Q^2} \left\{ \frac{2 \xi^4}{\sqrt{2\pi}} \int_0^\infty Dy P(\xi y - \xi \sqrt{1 - \xi^2}) \int_0^\infty Dy P(y)(1 - y^2)[1 - 2H(\frac{\sqrt{1 - \xi^2}}{\xi}y)] \right\}$$

$$\simeq \frac{\alpha}{\Delta q} \int_0^\infty DyP(y)(y^2 - 1) \equiv \frac{\alpha}{\Delta q} g_4. \quad (B.16)$$

If $P(y)$ is not constant for $y > 0$, the integration is positive. If $P(y)$ is constant for $y > 0$, which can happen when $\delta = 0$, the integration is 0. The latter case, we can perform the exact calculation and obtain

$$\int_0^\infty Dy P(\xi y)\tilde{F}(y) = \frac{k\xi^4}{\pi} (2 - \xi^2) \simeq \frac{2k\xi^4}{\pi},$$

where $k = P(y)$ for $y > 0$. Therefore,

$$\hat{R} \simeq \frac{\alpha}{\Delta q} \frac{2k\xi^3}{\pi}. \quad (B.17)$$

Finally, let us consider the deterministic case. In this case, $\delta = 0$ and $k = 1$ and $q = R$ and $\hat{q} = \hat{R}$ hold. Then, we obtain $\frac{\sqrt{1 - \xi^2}}{\xi} = \frac{1}{q}$ and $E(u/Q) = 2H(u/Q)$. Thus, equation (12) becomes

$$\hat{q} = \frac{2\alpha}{1 - q} \int D_u \frac{h(u/Q)^2}{H(u/Q)} \simeq \frac{\alpha}{\sqrt{\Delta q} \sqrt{2\pi}} \int D_u \frac{h(u)}{H(u)} \equiv \frac{\alpha}{\sqrt{\Delta q}} g_{3,D}. \quad (B.18)$$

As for $\hat{R}$, since $v = -u$, from equation (12) we obtain exactly

$$\hat{R} = \frac{2\alpha}{1 - q} \int D_u \frac{h(u/Q)^2}{H(u/Q)} = \hat{q}. \quad (B.19)$$

**Appendix C. Asymptotic form of $I$ for $\tau \gg 1$ and $\hat{R} \gg 1$**

$I$ is expressed as

$$I = \int Dt \ln[2\cosh(\sqrt{\hat{q}t + \hat{R}})] \simeq \hat{R} + \frac{h(\tau)}{\tau \mu} + I_1, \quad (C.1)$$

$$I_1 = I_1^+ + I_1^-,$$

$$I_1^+ = \int_{\pm \tau}^\infty Dt \ln[1 + e^{-2\sqrt{\hat{q}t + \hat{R}}}] = \sqrt{2\pi}h(\tau) \int_0^\infty Dxe^{\pm x\tau} \ln(1 + e^{-2\sqrt{x}}),$$

where $\tau = \hat{R}/\sqrt{\hat{q}}$ and $\mu = \frac{\hat{R}}{\tau \hat{q}}$. The following relations are proved mathematically exactly,

$$I_1^+ = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{2\hat{q}n^2 \pm 2\hat{R}n} H[\tau(n\mu \pm 1)]. \quad (C.2)$$
For $0 < \mu < 1$, in equations (C.2), $H(\tau(\frac{\mu}{\mu} - 1))$ and $H(\tau(\frac{\mu}{\mu} + 1))$ are approximated by \( \frac{h(\tau(\frac{\mu}{\mu} - 1))}{\tau(\frac{\mu}{\mu} - 1)} \) and \( \frac{h(\tau(\frac{\mu}{\mu} + 1))}{\tau(\frac{\mu}{\mu} + 1)} \), respectively. Then, \( I_1 \) is estimated as
\[
I_1 \simeq \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} h(\tau) \left[ \frac{1}{\tau(\frac{\mu}{\mu} - 1)} + \frac{1}{\tau(\frac{\mu}{\mu} + 1)} \right] = \frac{2\mu}{\tau} h(\tau) c(\mu),
\]
where \( c(\mu) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \frac{1}{1 - (\mu/n)^2} \). When \( \mu \) is not an integer,
\[
c(\mu) = \frac{\pi}{2\mu \sin(\mu \pi)} - \frac{1}{2\mu^2}.
\]
Thus,
\[
I \simeq \hat{R} + \frac{h(\tau)}{\tau \mu} \left[ 1 + 2\mu^2 c(\mu) \right] = \hat{R} + \frac{\psi(\mu) h(\tau)}{\tau \mu} \quad \text{for} \quad 0 < \mu < 1,
\]
\[
\psi(\mu) \equiv 1 + 2\mu^2 c(\mu) = \frac{\pi \mu}{\sin(\pi \mu)}.
\]
When \( \mu = 0 \), \( c(0) = \pi^2/12 \) and \( \psi(0) = 1 \). Then,
\[
I \simeq \hat{R} + \frac{h(\tau)}{\tau \mu} \quad \text{for} \quad \mu \simeq 0.
\]
For \( \mu \geq 1 \), \( I_1^- \) is expressed as
\[
I_1^- \simeq \sum_{n=1}^{n_0} \frac{(-1)^{n-1}}{n} e^{2\hat{q}n^2 - 2\hat{R}n} \{ 1 - H[-\tau(\frac{n}{\mu} - 1)] \} + \frac{h(\tau)}{\tau} \sum_{n=n_0+1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1}{\frac{n}{\mu} - 1} \quad \text{(C.3)}
\]
where \( n_0 = [\mu] \), i.e., \( n_0 \) is the largest integer which does not exceed the value of \( \mu \). Let us compare terms in equation (C.3). Let us assume \( 1 \leq n_1 < n_2 \leq n_0 \). Then,
\[
e^{2\hat{q}n_1^2 - 2\hat{R}n_1}/e^{2\hat{q}n_2^2 - 2\hat{R}n_2} = e^{4\hat{q}(n_2-n_1)(\mu-(n_2-n_1)/2)} > e^{4\hat{q}(n_2-n_1)(\mu-n_1)}.
\]
Thus, we obtain \( e^{2\hat{q}n_1^2 - 2\hat{R}n_1} \gg e^{2\hat{q}n_2^2 - 2\hat{R}n_2} \) for \( \hat{q} \gg 1 \).
\[
\hat{q} \gg 1 \quad \text{is satisfied when} \quad \hat{R} \gg 1 \quad \text{as long as} \quad \mu = \frac{\hat{R}}{2\hat{q}} \quad \text{is bounded from the above. Further, since} \quad e^{-\tau^2/2}/e^{2\hat{q}n^2 - 2\hat{R}n} = e^{-2\hat{q}(n-\mu)^2}, \quad \text{each term in the first summation in equation (C.3)} \quad \text{is lower order than} \quad h(\tau)/\tau \quad \text{for} \quad \tau \gg 1. \quad \text{Thus,} \quad I_1^- \quad \text{and} \quad \text{the second term in} \quad I_1^- \quad \text{are higher order than terms in the first term in} \quad I_1^- \quad \text{. Then, for} \quad 1 < \mu < 2,
\]
\[
I_1^- \simeq e^{-2(\hat{R}-\hat{q})} + \frac{2h(\tau)\mu}{\tau} c^-(\mu), \quad I_1^+ \simeq \frac{2h(\tau)\mu}{\tau} c^+(\mu),
\]
\[
c^\pm(\mu) \equiv \frac{1}{2\mu} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1}{\frac{n}{\mu} \pm 1}.
\]
Thus,
\[
I_1 \simeq e^{-2(\hat{R}-\hat{q})} + \frac{2h(\tau)\mu}{\tau} c(\mu),
\]
where \( c(\mu) = c^-(\mu) + c^+(\mu) \). Therefore,
\[
I \simeq \hat{R} + e^{-2(\hat{R}-\hat{q})} + \frac{h(\tau)\psi(\mu)}{\tau \mu} \quad \text{for} \quad 1 < \mu < 2.
\]
For $\mu = 1$, we obtain
\[ I_1 \simeq e^{-2(\hat{R} - \hat{q})}/2 + \frac{2h(\tau)\mu}{\tau}c_2(\mu), \]
where $c_2(\mu)$ is defined as
\[ c_2(\mu) \equiv \frac{1}{2\mu} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} \frac{1}{\mu} + c^+(\mu) = c(\mu) - \frac{1}{2(1-\mu)}. \]
Thus,
\[ I \simeq \hat{R} + e^{-2(\hat{R} - \hat{q})}/2 \text{ for } \mu = 1. \]

We use the facts that $c_2(\mu)$ is analytic at $\mu = 1$ and $c_2(1) = 0$.

For $\mu \geq 2$,
\[ I_1 - I_1 \simeq e^{-2(\hat{R} - \hat{q})} - \frac{1}{2}b_\mu e^{-4(\hat{R} - 2\hat{q})}, \]
then,
\[ I_1 \simeq I_1 - I_1 \simeq e^{-2(\hat{R} - \hat{q})} - \frac{1}{2}b_\mu e^{-4(\hat{R} - 2\hat{q})}, \]
\[ b_\mu = 1 \text{ for } \mu > 2, \quad b_2 = 1/2. \]
Thus,
\[ I \simeq \hat{R} + e^{-2(\hat{R} - \hat{q})}/2 \text{ for } \mu \geq 2. \]

In summary, up to the second order term in $I$, we obtain
\[ I \simeq \hat{R} + \frac{h(\tau)\psi(\mu)}{\tau\mu} \text{ for } 0 \leq \mu < 1, \quad (C.4) \]
\[ I \simeq \hat{R} + a_\mu e^{-2(\hat{R} - \hat{q})} \text{ for } 1 \leq \mu, \quad (C.5) \]
\[ \psi(\mu) \equiv 1 + 2\mu^2c(\mu) = \frac{\pi\mu}{\sin(\pi\mu)}, \quad a_1 = 1/2, \quad a_\mu = 1 \text{ for } \mu > 1. \quad (C.6) \]

Appendix D. $S_{PL} = 0$ and $f_{PL} = \alpha\epsilon_{\min}$

As is shown in equation (3.84), when $q \to 1$ and $R \to 1$ for $0 < \beta < \infty$ and for any $\delta$ and any $\chi$, $S_{RS}$ is expressed as
\[ S_{RS} = -\frac{\hat{q}\Delta q}{2} - (1 - \Delta R) \hat{R} + \alpha\sqrt{\Delta q} r(\chi, \beta) + I. \quad (D.1) \]

We consider the case of $0 < \beta < \infty$ and $0 \leq \delta < 1/2$.

For the PL, $R = q = 1, \chi = 1$ and $\xi = Q = \sqrt{\Delta q}$. Then,
\[ \hat{q} = \frac{\alpha}{\sqrt{\Delta q}} g_{1,\delta}(\chi = 1, \beta), \quad (D.2) \]
\[ \hat{R} = \alpha(\Delta q)^{(\delta - 1)/2} g_{2,\delta}(\chi = 1, \beta). \quad (D.3) \]

For $0 < \beta < \infty$ and $\delta \geq 0, g_{1,\delta}(1, \beta)$ and $g_{2,\delta}(1, \beta)$ are finite. Then, $\mu = \frac{\hat{R}}{2\hat{q}} = \frac{g_{2,\delta}}{2g_{1,\delta}}(\Delta q)^{\delta/2}$ becomes
\[ \mu \simeq 0 \text{ for } \delta > 0, \]
\[ \mu = \frac{g_{2,\delta}}{2g_{1,\delta}} \text{ finite for } \delta = 0. \]
Let us estimate $I$. In the case of $\delta > 0$, since $\mu = 0$ we obtain from (C.4)
\[
I \simeq \hat{R} + \frac{h(\tau)}{\tau \mu} = \hat{R} + \frac{2\hat{R} h(\tau)}{\tau^3}.
\]
Then,
\[
S = -\frac{1}{2} g_1,\alpha \sqrt{\Delta q} + g_2,\alpha (\Delta q)^{\frac{\delta}{1+\delta}} + \frac{2\hat{R} h(\tau)}{\tau^3} + \alpha r \sqrt{\Delta q}.
\]
From the equations (D.2) and (D.3), we obtain
\[
\hat{q}^{\delta-1} \hat{R} = (\alpha g_1,\alpha)^{\delta-1} \alpha g_2,\alpha \equiv C.
\]
Then,
\[
\hat{R} = C \hat{q}^{1-\delta}, \quad \frac{\hat{R}}{\tau^3} = C^{-\frac{1}{1+\delta}}. \tau^{\frac{1}{1+\delta}}.
\]
Since $\Delta q = 0, \tau = \infty$ and $g_1,\delta, g_2,\delta, r$ and $C$ are finite, we obtain $S = 0$. For the case of $\delta = 0$, we have to determine the value of $\mu$. As is discussed in §3, when $\chi$ is finite, $\mu = \frac{1}{1+\chi^2}$ from equation (36). In our case, $\chi = 1$ and then $\mu = 1/2$. Thus,
\[
I \simeq \hat{R} + \frac{\pi h(\tau)}{\tau}.
\]
By a similar argument to the case of $\delta > 0$, we obtain $S = 0$.

Now, let us estimate $<e_t> = -\alpha e^{-\beta} J$. $J$ is estimated as
\[
J = \int D\mu \frac{H(u/Q) - 1}{H(u/Q)} \int D\mu [1 - P(\xi u + \sqrt{1 - \xi^2} u)]
\[
= \int D\mu \frac{H(u/Q) - 1}{H(u/Q)} [1 - P(u)] = -e^\beta \int_0^\infty D\mu [1 - P(u)] = -e^\beta \epsilon_{\text{min}}.
\]
Thus, $<e_t> = -\alpha e^{-\beta} J = \alpha \epsilon_{\text{min}}$. Then, we obtain $f_{PL} = <e_t> - TS_{PL} = \alpha \epsilon_{\text{min}}$.

**Appendix E. Asymptotic form of $f_{RS}$ and $S_{RS}$ for $\beta \ll 1$.**

In this appendix, we derive the asymptotic forms of the free energy and the entropy for the RS solution for $\beta \ll 1$.

$f_{RS}$ is expressed as
\[
-\beta f_{RS}(q, \hat{q}, R, \hat{R}, \beta) = -\frac{\hat{q}}{2} (1 - q) - R\hat{R} + \alpha K + I,
\]
where
\[
K \equiv \int D\mu \frac{H(u/Q)}{H(u/Q)} \int D\mu \ln H(Y), \quad I \equiv \int D\mu \frac{\epsilon_{\text{min}}}{2 \cosh(\sqrt{\mu} t - \hat{R})},
\]

By defining $K_a$ and $K_b$ as follows,
\[
K_a \equiv \int D\mu \frac{H(u/Q)}{H(u/Q)} \int D\mu H(-Y) = \epsilon_{\text{min}} + 2 \int_0^\infty D\mu [H(Y) + H(-Y)] = \epsilon_a,
\]
\[
K_b \equiv \int D\mu \frac{H(u/Q)}{H(u/Q)} \int D\mu [H(Y) - H(-Y)] = Q \int D\mu H(u) H(-u),
\]
$K$ and $f_{RS}$ are expressed as

$$K = -\beta (K_a - \frac{\beta}{2} K_b) + O(\beta^3),$$  

$$-\beta f_{RS} = -\frac{\hat{q}}{2} (1 - q) - R \hat{R} + I - \alpha \beta (K_a - \frac{\beta}{2} K_b) + O(\beta^3)$$

$$\simeq -\frac{\hat{q}}{2} (1 - q) - R \hat{R} + I - \alpha \beta \epsilon_g + \frac{\alpha \beta^2}{2} K_b.$$

The entropy $S_{RS}$ is expressed as

$$S_{RS} = -\frac{\hat{q}}{2} (1 - q) - R \hat{R} + I + \alpha K - \alpha \beta e^{-\beta} J,$$

$$J = \int D_y 2 \mathcal{P}(y) \int D_u \frac{H(Y) - 1}{H(Y)}.$$

Then, defining $L$ as $L = K - \beta e^{-\beta} J$, we get

$$L = -\frac{\beta^2}{2} K_b + O(\beta^3).$$

Thus, we obtain

$$S_{RS} = -\frac{\hat{q}}{2} \Delta q - R \hat{R} + I - \alpha \beta^2 K_b + O(\beta^3).$$

For $\Delta q \ll 1$ and $\Delta R \ll 1$,

$$K_a = \epsilon_g \simeq \epsilon_{\text{min}} + \frac{2s}{(1 + \delta) \sqrt{2\pi}} (2\Delta R)^{\frac{3}{2}}, \quad K_b \simeq \frac{\sqrt{\Delta q}}{\pi \sqrt{2}}.$$

Then, $K$, $f_{RS}$ and $S_{RS}$ are expressed as

$$K = -\beta [\epsilon_{\text{min}} + \frac{2s}{(1 + \delta) \sqrt{2\pi}} (2\Delta R)^{\frac{3}{2}} - \frac{\beta \sqrt{\Delta q}}{2\pi \sqrt{2}}],$$

$$-\beta f_{RS} = -\frac{\hat{q}}{2} (1 - q) - R \hat{R} + I - \alpha \beta [\epsilon_{\text{min}} + \frac{2s}{(1 + \delta) \sqrt{2\pi}} (2\Delta R)^{\frac{3}{2}} - \frac{\beta \sqrt{\Delta q}}{2\pi \sqrt{2}}],$$

$$S_{RS} = -\frac{\hat{q}}{2} \Delta q - R \hat{R} + I - \alpha \beta^2 \frac{\sqrt{\Delta q}}{2\pi \sqrt{2}}.$$
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