CONCORDANCE INVARIANTS WITH APPLICATIONS TO THE
4-DIMENSIONAL CLASP NUMBER

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Abstract. Using knot Floer homology, we define a family of concordance
invariants which give lower bounds on the 4-dimensional clasp number. More
generally, our invariants obstruct a slice surface with genus $g$ and $c$
double points. We give examples where the clasp number is arbitrarily larger than
the 4-ball genus. We also prove that Hendricks and Manolescu’s involutive
correction terms of large surgery give a lower bound on the clasp number.

1. Introduction

If $K \subseteq S^3$ is a knot, the 4-dimensional clasp number $c_4(K)$ is the smallest
integer $c$ such that there is a normally immersed disk in $B^4$ bounding $K$ with $c$
double points. (A surface in a 4-manifold $W$ is normally immersed if it is the image
of an immersion $f : S \to W$ such that $f(S) \cap \partial W = f(\partial S)$, $f$ is transverse to $\partial W$,
and all self-intersections of $f$ are transverse double points in int($W$).) There are
refinements $c^+_4(K)$ (respectively $c^-_4(K)$), which are the minimal number of positive
(respectively negative) double points which appear in any such immersed disk. Of
course,

$$c^+_4(K) + c^-_4(K) \leq c_4(K).$$

Reversing the orientation of a normally immersed surface in $B^4$ does not change
the sign of a double point, and hence changing the orientation of a knot does not
affect $c^+_4$ or $c^-_4$. However, if $-K$ denotes the mirror of $K$, then

$$c^+_4(K) = c^-_4(-K).$$

Another basic inequality is

$$g_4(K) \leq c_4(K) \leq u(K),$$

where $g_4(K)$ denotes the 4-ball genus, and $u(K)$ the unknotting number.

More generally, we will study the set of pairs $(c, g)$ such that there is an oriented,
normally immersed surface in $B^4$ that bounds a given knot $K$, and has genus $g$ and $c$
double points.

Using the Heegaard Floer $d$-invariants of branched double covers, Owens and Strle
(OS16) gave lower bounds on the clasp number, and completed the computation of
the clasp number for all prime knots with ten or fewer crossings.

More recently, Kronheimer and Mrowka [KM19] constructed a 1-parameter family
of concordance invariants using instanton knot Floer homology, from which both
slice genus and clasp number bounds may be obtained. They asked if the clasp
number can be arbitrarily large, relative to the slice genus. Daemi and Scaduto [DS20] have recently announced a proof that
\[ c_4(\#^n7_4) - g_4(\#^n7_4) \to \infty \]
using instantons (cf. [DS19]).

The purpose of our present work is to investigate these questions from the perspective of Heegaard Floer homology.

1.1. A family of concordance invariants. In this paper, we introduce a family of concordance invariants
\[ Y_{c,g}(K) \in \mathbb{N}, \]
indexed over pairs of nonnegative integers \( c \) and \( g \). We prove the following:

**Theorem 1.1.** Suppose there is an oriented, normally immersed surface in \( B^4 \), bounding a knot \( K \) in \( S^3 \), which has genus \( \mathcal{G} \) and has \( \mathcal{C}^+ \) positive double points. If \( 0 \leq c \leq \mathcal{C}^+ \) and \( 0 \leq g \leq \mathcal{G} \), then
\[ Y_{c,g}(K) \leq \left\lfloor \frac{\mathcal{C}^+ - c}{2} \right\rfloor + \left\lfloor \frac{\mathcal{G} - g}{2} \right\rfloor. \]
In particular, if \( Y_{c,g}(K) \neq 0 \), then there is no such surface with genus \( g \) and \( c \) positive double points.

The invariants \( Y_{c,g} \) satisfy similar properties to Rasmussen’s correction terms of large surgeries [Ras03], which we denote by \( V_s(K) \). In comparison with Theorem 1.1, if there is a properly embedded surface of genus \( \mathcal{G} \) bounding \( K \), and \( 0 \leq s \leq \mathcal{G} \), then
\[ V_s(K) \leq \left\lfloor \frac{\mathcal{G} - s}{2} \right\rfloor. \] (1)
See [Ras04, Theorem 2.3].

We will prove that
\[ Y_{0,g}(K) = V_g(K) \quad \text{and} \quad Y_{c,g}(K) \geq V_{c+g}(K), \] (2)
whenever \( c \) and \( g \) are nonnegative integers. Further properties of \( Y_{c,g}(K) \) are considered in Section 4.

We recall that Hom and Wu [HW16] define an invariant \( \nu^+(K) \), which is the minimum integer \( s \) such that \( V_s(K) = 0 \). The invariant \( \nu^+(K) \) is a lower bound for the slice genus. We define the following analog of \( \nu^+ \):
\[ \omega^+(K) := \min \{ c \in \mathbb{N} : Y_{c,0}(K) = 0 \}. \]
Theorem 1.1 implies
\[ \omega^+(K) \leq c_4^+(K). \]
Clearly this also implies \( \omega^+(-K) \leq c_4^+(K) \), where \(-K\) denotes the mirror of \( K \). Summarizing, we have the following:

**Theorem 1.2.** If \( K \) is a knot in \( S^3 \), then
\[ \omega^+(K) + \omega^+(-K) \leq c_4(K). \]

Theorem 1.2 should be compared to a result of Bodnár, Celoria and Golla [BCG17, Proposition 1.5], which states that
\[ \nu^+(K) + \nu^+(-K) \leq c_4(K), \] (3)
for any knot \( K \subseteq S^3 \). Our Theorem 1.2 is always at least as strong, since \( \nu^+(K) \leq \omega^+(K) \), by (2).
In Section 7, we give some example computations where $\omega^+(K)$ is larger than $\nu^+(K)$. As an additional topological application, we prove the following:

**Theorem 1.3.** The knot $K = T_{2,11} \# - T_{1,5}$ satisfies $g_4(\#^nK) = n$ and $c_4(\#^nK) \geq 2n$. In particular

$$\lim_{n \to \infty} (c_4(\#^nK) - g_4(\#^nK)) = \infty.$$ 

Our proof of Theorem 1.3 uses only the bound in (3), as well a general relation between $\nu^+(K)$ and the invariant $\Upsilon_K(t)$ of Ozsváth, Stipsicz and Szabó [OSS17]. More generally, we prove

$$c_4(K) \geq \left( \max_{t \in [0,1]} \Upsilon_K(t)/t \right) - \left( \min_{t \in [0,1]} \Upsilon_K(t)/t \right). \quad (4)$$

See Proposition 2.1. The genus computation in Theorem 1.3 is due to Livingston and Van Cott [LVC18].

Our proofs make use of the link cobordism maps described by the first author [Juh16] and the second author [Zem19c]. The most basic strategy one can employ to understand double points is to smooth a double point by increasing the genus. A slightly more refined approach is to smooth only the positive double points, and blow up at the negative double points. This approach will lead to the bound shown in (3). Our approach is a refinement of this strategy, which builds off the observation that the cobordism maps for a surface obtained by smoothing have a particularly weak dependence on the dividing set. We encode this algebraically as the existence of an $\mathbb{F}[U]$-non-torsion cycle in a chain complex we call $S_{c,g}(K)$, whose algebraic $d$-invariant determines $Y_{c,g}(K)$. Since this is more restrictive algebraically, $\omega^+(K)$ gives a potentially better bound than $\nu^+(K)$.

1.2. **Involutive correction terms.** We additionally consider the involutive correction terms $V_0(K)$ and $\bar{V}_0(K)$ of large surgery due to Hendricks and Manolescu [HM17], which satisfy

$$V_0(K) \leq \bar{V}_0(K) \leq \bar{V}_0(K).$$

The involutive correction terms of large surgery are well known not to give a lower bound on the slice genus in the same manner as $V_0$ in equation (1). For example, $T_{2,3} \# T_{2,3}$ has slice genus 2, but $V_0 = 2$; see [HM17, Theorem 1.7]. In contrast, our techniques imply the following:

**Theorem 1.4.** If $K$ is a knot in $S^3$, then

$$-\left\lceil \frac{c_4^+(K) + 1}{2} \right\rceil \leq V_0(K) \leq \bar{V}_0(K) \leq \left\lceil \frac{c_4^+(K) + 1}{2} \right\rceil.$$

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1.4. **Organization.** In Section 2, we prove Theorem 1.3 and equation (4). In Section 3, we describe some background on knot Floer homology, and define the invariants $Y_{c,g}(K)$. In Section 4, we prove a few basic properties about $Y_{c,g}(K)$. In Section 5, we prove Theorem 1.1, the bound involving $Y_{c,g}(K)$. In Section 6, we consider involutive Heegaard Floer homology and prove Theorem 1.4. In Section 7 we perform a few computations.

2. **AN APPLICATION OF $\Upsilon_K(t)$ TO THE CLASP NUMBER**

In this section, we give a family of knots where $c_4 - g_4$ increases without bound. Our proof is an application of the following general result:

**Proposition 2.1.** Suppose $K$ is a knot in $S^3$. Then

$$c_4(K) \geq \left( \max_{t \in [0,1]} \Upsilon_K(t) / t \right) - \left( \min_{t \in [0,1]} \Upsilon_K(t) / t \right).$$

**Proof.** According to [OSS17, Proposition 4.7], we have

$$-t\nu^+(K) \leq \Upsilon_K(t),$$

for $t \in [0,1]$ (cf. [OSS17, Proposition 2.13]). Hence

$$\nu^+(K) \geq \max_{t \in [0,1]} -\Upsilon_K(t) / t = -\min_{t \in [0,1]} \Upsilon_K(t) / t.$$  \hspace{1cm} (5)

Mirroring $K$, we obtain similarly

$$\nu^+(-K) \geq \max_{t \in [0,1]} \Upsilon_K(t) / t.$$  \hspace{1cm} (6)

Using Bodnár, Celoria and Golla’s bound in (3), we obtain

$$c_4(K) \geq \nu^+(K) + \nu^+(-K) \geq \max_{t \in [0,1]} \Upsilon_K(t) / t - \min_{t \in [0,1]} \Upsilon_K(t) / t,$$

completing the proof. \hspace{1cm} $\Box$

As a corollary, we prove the following, slightly more general version of Theorem 1.3:

**Corollary 2.2.** Let $K_r = T_{2,10r+1} - T_{4r+1}$ for $r \geq 1$. Then

$$g_4(\#^n K_r) = nr \quad \text{and} \quad c_4(\#^n K_r) \geq 2nr.$$  

**Proof.** Livingston and Van Cott [LVC18, Theorem 17] proved that the knot $K_r$ has slice genus $r$, and hence

$$g_4(\#^n K_r) \leq nr.$$  \hspace{1cm} (7)

On the other hand, they compute using [OSS17, Theorem 6.2] that

$$\tau(K_r) = -r \quad \text{and} \quad \Upsilon_{K_r}(1) = -r.$$  \hspace{1cm} (8)

In particular $\tau(\#^n K_r) = -rn$, so in light of (7), we conclude $g_4(\#^n K_r) = rn$. Since $\Upsilon_K(t) = -\tau \cdot t$ near $t = 0$ [OSS17, Proposition 1.6], we know that

$$\max_{t \in [0,1]} \Upsilon_{\#^n K_r}(t) / t = n \max_{t \in [0,1]} \Upsilon_{K_r}(t) / t \geq nr,$$

and (8) implies that

$$\min_{t \in [0,1]} \Upsilon_{\#^n K_r}(t) / t = n \min_{t \in [0,1]} \Upsilon_{K_r}(t) / t \leq -nr.$$  

Applying Proposition 2.1, the proof is complete. \hspace{1cm} $\Box$
Remark 2.3. The proof also shows that $\nu^+(\#^nK_r) = \nu^+(\#^nK_r) = nr$. The bounds on $\Upsilon$ imply that $\nu^+(\#^nK_r)$ and $\nu^+(-\#^nK_r)$ are both at least $nr$, while the fact that $g_4(\#^nK_r) = nr$ implies that they are at most $nr$, since $\nu^+(K) \leq g_4(K)$, for any knot $K$.

The graphs of $\Upsilon_{K_1}(t)$ and $\Upsilon_{K_1}(t)/t$ are shown in Figure 2.1.

![Figure 2.1. The graphs of $\Upsilon_{K_1}(t)$ (left) and $\Upsilon_{K_1}(t)/t$ (right). Here $K_1 = T_{2,11} \# T_{4,5}$.](image)

In this section, we review knot Floer homology [OS04] and define the concordance invariants $Y_{c,g}$.

3. The concordance invariants $Y_{c,g}$

Knot Floer homology. Let $\mathbb{F}[\mathcal{U}, \mathcal{V}]$ denote a 2-variable polynomial ring. Suppose that $K = (K, w, z)$ is a doubly pointed knot in a closed, oriented 3-manifold $Y$. If $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ is a Heegaard diagram of $(Y, K)$ and $s \in \text{Spin}^c(Y)$, we let $\mathcal{CFK}^-(Y, K, s)$ be the free $\mathbb{F}[U, V]$-module generated by intersection points $x \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta$ satisfying $s(w(x)) = s$. Here, $\mathcal{T}_\alpha := \alpha_1 \times \cdots \times \alpha_g$ and $\mathcal{T}_\beta := \beta_1 \times \cdots \times \beta_g$ are two half-dimensional tori in the $g$-fold symmetric product $\text{Sym}^g(\Sigma)$. We define a differential, which counts pseudo-holomorphic disks, via the formula

$$\partial x := \sum_{\substack{\phi \in \pi_2(x, y) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathcal{R}) \mathcal{U}^{n_w(\phi)} \mathcal{V}^{n_z(\phi)} \cdot y,$$

extended equivariantly over the action of $\mathbb{F}[\mathcal{U}, \mathcal{V}]$. Here, $\mathcal{M}(\phi)$ denotes the moduli space of pseudo-holomorphic disks in $\text{Sym}^g(\Sigma)$, with boundary on $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$, representing the class $\phi$. Also, $n_w(\phi)$ and $n_z(\phi)$ denote the multiplicities of the class $\phi$ on $w$ and $z$, respectively.

There is also an infinity version $\mathcal{CFK}^\infty(Y, K, s)$, which has the same generators, but is defined over the ring $\mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{U}^{-1}, \mathcal{V}^{-1}]$. When it causes no confusion, we will write $\mathcal{CFK}^-(K)$ instead of $\mathcal{CFK}^-(Y, K, s)$. We write $\mathcal{HFK}^-(K)$ for the homology.

Knot Floer homology admits a rich grading structure, which can be formulated in several ways. We will focus on the description which is considered in [Zem19b], since it is the most natural from the point of view of link cobordism maps. If $s$ is torsion and $K$ is null-homologous, then $\mathcal{CFK}^-(Y, K, s)$ admits two Maslov gradings.
gr\textsubscript{w} and gr\textsubscript{z}. The variables \( \mathcal{U} \) and \( \mathcal{V} \) have \((\text{gr}_w, \text{gr}_z)\)-bigradings \((-2, 0)\) and \((0, -2)\), respectively. The Alexander grading \( A \) satisfies

\[ A = \frac{1}{2}(\text{gr}_w - \text{gr}_z). \]

It is convenient also to consider the \( \delta \)-grading:

\[ \delta := \frac{1}{2}(\text{gr}_w + \text{gr}_z). \]

If \( K \) is a knot in \( S^3 \), we write \( \mathcal{A}_s(K) \) for the subset of \( \mathcal{CFK}^- (K) \) in Alexander grading \( s \). The complex \( \mathcal{A}_s(K) \) is a module over the polynomial ring \( \mathbb{F}[U] \), where \( U = \mathcal{U} \mathcal{V} \).

There is a more common formulation of the full knot Floer complex, denoted \( \mathcal{CFK}^\infty (K) \), which has generators \([x, i, j]\) where \( A(x) - j + i = 0 \). This is isomorphic to the subcomplex of \( \mathcal{CFK}^\infty (K) \) which lies in Alexander grading 0, via the map \( \mathcal{U}^i \mathcal{V}^j \cdot x \mapsto [x, -i, -j] \). The complex \( \mathcal{A}_s(K) \) is also canonically isomorphic to the more standard large surgery complex \( A_s(K) \), which is generated by \([x, i, j]\) satisfying \( A(x) + i - j = 0 \) and \( j \leq s \), via the map \( \mathcal{U}^i \mathcal{V}^j \cdot x \mapsto [x, -i, -j + s] \). As a \( \delta \)-graded complex, we have

\[ \mathcal{A}_s(K) \cong A_s(K)[s], \]

where \([-s]\) denotes shifting each \( \delta \)-grading by \(-s\).

It is easy to see that \( \mathcal{CFK}^\infty (K) \) and \( \mathcal{CFK}^- (K) \) contain equivalent information. Our reason for using \( \mathcal{CFK}^- (K) \) is that it is a more natural packaging from the perspective of the cobordism maps.

More generally, if \( L = (L, w, z) \) is an oriented, multi-pointed link in \( Y \) (cf. Definition 3.1), the above construction adapts to construct a link Floer complex over the ring \( \mathbb{F}[\mathcal{U}, \mathcal{V}] \). We denote this complex as \( CFL^- (Y, L, s) \). This is a variation on the original link Floer complex defined by Ozsváth and Szabó [OS08].

### 3.2. Link Floer homology as a TQFT

Link Floer homology has a functorial framework which refines the original cobordism invariants of Ozsváth and Szabó [OS06]. We recall the relevant cobordism category from [Juh16]:

**Definition 3.1.**

1. The objects are pairs \((Y, L)\), where \( Y \) is a closed, oriented 3-manifold, and \( L = (L, w, z) \) is an oriented link with two collections of basepoints, which alternate as one traverses the link. Furthermore, each component of \( Y \) contains at least one component of \( L \), and each component of \( L \) contains at least two basepoints.
2. A morphism from \((Y_1, L_1)\) to \((Y_2, L_2)\) consists of a pair \((W, F)\), as follows: \( W \) is a compact, oriented 4-manifold such that \( \partial W = -Y_1 \cup Y_2 \). Furthermore, \( F \) consists of a properly embedded, oriented surface \( S \) such that \( \partial S = -L_1 \cup L_2 \), which is decorated by a properly embedded 1-manifold \( D \subseteq S \), such that \( S \setminus D \) consists of two subsurfaces, \( F_w \) and \( F_z \), which meet along \( D \). Furthermore, \( D \) is disjoint from the basepoints, each component of \( L_i \setminus D \) contains exactly one basepoint, and \( w_i \subseteq F_w \) and \( z_i \subseteq F_z \), for \( i \in \{1, 2\} \).

**Remark 3.2.** In figures, we follow the convention that the \( w \)-subregion is shaded, and the \( w \)-basepoints are solid dots. The \( z \)-subregion is unshaded, and the \( z \)-basepoints are open dots.
The first author associated functorial cobordism maps for decorated link cobordisms on the hat version of link Floer homology [Juh16] (the version obtained by setting all variables to zero). The construction used the contact gluing map of Honda, Kazez and Matić [HKM08]. The second author extended this to the full minus version, using maps associated to elementary cobordisms [Zem19c] (compare also [AE16]). The equivalence of the maps in [Juh16] and [Zem19c] is proven in [JZ20].

Following [Zem19c], if $(W, \mathcal{F})$ is a decorated link cobordism from $(Y_1, L_1)$ to $(Y_2, L_2)$ and $s \in \text{Spin}^c(W)$, there is an induced chain map

$$F_{W, x, s} : CFL^-(Y_1, L_1, s|_{Y_1}) \rightarrow CFL^-(Y_2, L_2, s|_{Y_2}).$$

which is equivariant with respect to the action of $\mathbb{F}[\mathcal{W}, \mathcal{V}]$. The grading changes of the cobordism maps are computed in [Zem19b, Theorem 1.4]. If $s|_{Y_1}$ and $s|_{Y_2}$ are torsion, then

$$\text{gr}_w(F_{W, x, s}(x)) - \text{gr}_w(x) = \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4} + \tilde{\chi}(\mathcal{F}_w),$$

where $\tilde{\chi}(\mathcal{F}_w) := \chi(\mathcal{F}_w) - \frac{1}{2}(|w_1| + |w_2|)$. Similarly, if $s|_{Y_1} = \partial PD[L_1]$ and $s|_{Y_2} = \partial PD[L_2]$ are torsion, then

$$\text{gr}_w(F_{W, x, s}(x)) - \text{gr}_w(x) = \frac{c_1(s - PD[\Sigma])^2 - 2\chi(W) - 3\sigma(W)}{4} + \tilde{\chi}(\mathcal{F}_x).$$

Similar to (9), if $L_1$ and $L_2$ are null-homologous, and $s$ has torsion restriction to $Y_1$ and $Y_2$, then

$$A(F_{W, x, s}(x)) - A(x) = \frac{\langle c_1(s), [\hat{S}] \rangle - [\hat{S}] \cdot [\hat{S}]}{2} + \frac{\chi(\mathcal{F}_w) - \chi(\mathcal{F}_x)}{2},$$

where $\hat{S}$ is obtained by capping $S$ with Seifert surfaces.

An additional relation which is helpful for our purposes is the bypass relation, which is inspired by contact geometry (cf. [HKM08, Lemma 7.4]). If $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are decorations of a fixed link cobordism $S \subseteq W$, which coincide outside of a disk $D$, and inside $D$ the dividing sets have the form shown in Figure 3.1, then

$$F_{W, \mathcal{F}_1, s} + F_{W, \mathcal{F}_2, s} + F_{W, \mathcal{F}_3, s} \cong 0.$$

See [Zem19a, Lemma 1.4].

![Figure 3.1. The bypass relation.](image)

### 3.3. The complex $\mathcal{S}_{c, g}$.

To define $Y_{c, g}(K)$, we construct an auxiliary chain complex $\mathcal{S}_{c, g}(K)$. It is the total complex of the diagram shown in Figure 3.2. By convention, $\mathcal{S}_{0, g}(K) = \mathcal{A}_g(K)$. In Figure 3.2, an arrow labeled with $\mathcal{W}$ or $\mathcal{V}$ denotes multiplication by $\mathcal{W}$ or $\mathcal{V}$. We view $\mathcal{S}_{c, g}$ as having homological grading given by the $\delta$-grading. It is straightforward to see that the differential lowers the $\delta$-grading by 1. We note that $\mathcal{S}_{c, g}$ is a free module over $\mathbb{F}[U]$, where $U = \mathcal{W} \mathcal{V}$. 
We can rewrite $\mathcal{S}_{c,g}(K)$ using the $\delta$-graded isomorphism in (10). Under these canonical isomorphisms, the map $V : A_s \to A_{s+1}$ is intertwined with the inclusion of $A_s$ into $A_{s+1}$, and the map $U : A_s \to A_{s-1}$ is intertwined with multiplication by $U$. Hence, we can alternatively view $\mathcal{S}_{c,g}(K)$ as the complex shown in Figure 3.3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.3.png}
\caption{A complex which is isomorphic to $\mathcal{S}_{c,g}(K)$.}
\end{figure}

The following is an easy algebra exercise:

**Lemma 3.3.**

1. The homology $H_\ast(\mathcal{S}_{c,g}(K))$ is isomorphic to $\mathbb{F}[U] \oplus \text{Tors}$, where Tors denotes a torsion $\mathbb{F}[U]$-module.
2. A chain $(x_{-c+g}, \eta_{-c+1+g}, x_{-c+2+g}, \ldots, x_{c-2+g}, \eta_{c-1+g}, x_{c+g}) \in \mathcal{S}_{c,g}(K)$ is $\mathbb{F}[U]$-non-torsion in $\mathcal{S}_{c,g}(K)$ if and only if each $x_i$ is a cycle which is $\mathbb{F}[U]$-non-torsion in $A_i(K)$, and
   \[ \partial \eta_i = V \cdot x_{i-1} + U \cdot x_{i+1} \]
   for each $i$ where $\eta_i$ is defined.

**Definition 3.4.** By Lemma 3.3, we can define the algebraic $d$-invariant of $\mathcal{S}_{c,g}$, which is the maximal $\delta$-grading of a homogeneously graded, $\mathbb{F}[U]$-non-torsion cycle in $\mathcal{S}_{c,g}$. We set
   \[ Y_{c,g}(K) := -\frac{d(\mathcal{S}_{c,g}(K)) + c + g}{2} \]  

Note that the quantity $d(\mathcal{S}_{c,g}(K)) + c + g$ is always even, due to the grading shifts appearing in Figure 3.3, and since $A_s(K)$ has $\mathbb{F}[U]$-non-torsion part supported in even gradings.

**Remark 3.5.** Lemma 3.3 gives an alternate description of $d(\mathcal{S}_{c,g}(K))$, which involves only $\mathcal{HFK}^-(K)$, and is hence sometimes easier for computations. Namely, $d(\mathcal{S}_{c,g})$ is the maximal integer $\delta_0$ such that there are elements
   \[ [x_{-c+g+2i}] \in H_\ast(A_{-c+g+2i}) \subseteq \mathcal{HFK}^-(K), \]
in $\delta$-grading $\delta_0$, for $0 \leq i \leq c$, such that $V \cdot [x_{-c+g+2i}] = U \cdot [x_{-c+g+2i+2}]$, for $0 \leq i \leq c - 1$.

Finally, we prove the following:
Lemma 3.6. The integer $Y_{c,g}(K)$ is a concordance invariant.

Proof. Suppose that $C \subseteq [0,1] \times S^3$ is a concordance from $K$ to $K'$. We let $C$ denote $C$, decorated with two arcs, running from $K$ to $K'$. We consider the cobordism map $F_{[0,1] \times S^3} : \text{CFK}^{-}(K) \to \text{CFK}^{-}(K')$. There is an induced map from $S_{c,g}(K)$ to $S_{c,g}(K')$, which preserves the $\delta$-grading by [Zem19b, Theorem 1.4]. By Lemma 3.3 and [Zem19c, Theorem 1.7], the induced map from $S_{c,g}(K)$ to $S_{c,g}(K')$ sends $\mathbb{F}[U]$-non-torsion elements to $\mathbb{F}[U]$-non-torsion elements. Hence $Y_{c,g}(K') \leq Y_{c,g}(K)$. Turning the concordance around and reversing orientation gives the inequality $Y_{c,g}(K) \leq Y_{c,g}(K')$. □

4. Basic properties of $Y_{c,g}$

In this section, we prove the following basic properties of the invariants $Y_{c,g}(K)$:

Proposition 4.1. Suppose $K$ is a knot in $S^3$. The invariants $Y_{c,g}(K)$ satisfy the following:

1. $Y_{c,g}(K) \geq 0$.
2. $Y_{c,g}(K) - 1 \leq Y_{c+1,g}(K) \leq Y_{c,g}(K)$.
3. $Y_{c,g}(K) - 1 \leq Y_{c,g+1}(K) \leq Y_{c,g}(K)$.
4. $Y_{c,g}(K) \geq Y_{c-1,g+1}(K)$.
5. $Y_{c,g}(K) \geq Y_{c+g}(K)$.

Proof. We begin with property (1). Let $B \subseteq \text{CFK}^{-}(K)$ denote the subcomplex generated over $\mathbb{F}[U]$ by tuples $[x,i,j]$, satisfying $A(x) - j + i = 0$, $i \leq 0$, and with no restriction on $j$. This is equivalent to the subset of $\text{CFK}^{-}(K)$ generated by $\mathcal{P}^i \mathcal{V}^j \cdot x$ satisfying $A(x) + j - i = 0$ and $i \geq 0$, but no restriction on $j$. Note that $B$ is homotopy equivalent to $\mathbb{F}[U]$, since $B \cong \text{CF}^{-}(S^3)$. We consider a complex $T_{c,g}$, defined similarly to the description of $S_{c,g}$ given in Figure 3.3, but with $B$ replacing each $A_x$. Since $B \cong \mathbb{F}[U]$, we may replace each $B$ with $\mathbb{F}[U]$, and the algebraic $d$-invariant is preserved. The generator of $H_*(T_{c,g})$ is given by the element

$$y = (U^c, 0, U^{c-1}, \ldots, 0, U, 0, 1),$$

which has $\delta$-grading $-c - g$. Note that $y$ is independent of $g$, except in the grading shifts, as in Figure 3.3. Since $S_{c,g}$ is a subcomplex of $T_{c,g}$, we obtain

$$d(S_{c,g}) \leq -c - g,$$

from which property (1) follows.

We now consider property (2). The inequality $Y_{c,g}(K) - 1 \leq Y_{c+1,g}(K)$ is equivalent to the inequality

$$d(S_{c+1,g}(K)) \leq d(S_{c,g}) + 1.$$  \quad (16)

In Figure 4.1, we construct a chain map

$$\Pi_{-1,0} : S_{c+1,g} \to S_{c,g},$$

which shifts the $\delta$-grading by $-1$, and sends $\mathbb{F}[U]$-non-torsion elements to $\mathbb{F}[U]$-non-torsion elements. Equation (16) follows.
We now consider the inequality $Y_{c+1,g}(K) \leq Y_{c,g}(K)$, which is equivalent to
\begin{equation}
\quad d(S_{c,g}) \leq d(S_{c+1,g}) + 1.
\end{equation}
(17)

In Figure 4.2, we construct a chain map
\[ \mathcal{J}_{1,0} : S_{c,g} \to S_{c+1,g}, \]
which has grading $-1$, and sends $F[U]$-non-torsion elements to $F[U]$-non-torsion elements. The existence of such a map implies (17). This completes the proof of property (2).

We now consider property (3). We construct chain maps $\mathcal{J}_{0,1} : S_{c,g} \to S_{c,g+1}$ and $\Pi_{0,-1} : S_{c,g+1} \to S_{c,g}$ which both have $\delta$-grading $-1$. The map $\mathcal{J}_{0,1}$ is induced by the inclusion map from $A_s$ to $A_{s+1}$ on each summand of $S_{c,g}$ and $S_{c,g+1}$. The map $\Pi_{0,-1}$ is induced by the map $U : A_{s+1} \to A_s$ on each summand. The maps $\mathcal{J}_{0,1}$ and $\Pi_{0,-1}$ clearly are chain maps, and map $F[U]$-non-torsion elements to $F[U]$-non-torsion elements, implying the statement.
Next, we consider property (4), which states that $Y_{c,g}(K) \geq Y_{c-1,g+1}(K)$. In Figure 4.3, we describe a map $\Pi_{-1,1} : S_{c,g}(K) \to S_{c-1,g+1}(K)$ which preserves the grading, and sends $F[U]$-non-torsion elements to $F[U]$-non-torsion elements. This easily gives the stated inequality.

Finally, we investigate property (5). Note that projection of $S_{c,g}$ onto the rightmost complex, $A_{c+g}[-c-g]$, gives a $\delta$-grading preserving chain map, which sends $F[U]$-non-torsion elements to $F[U]$-non-torsion elements. The claim follows since $V_s(K) = -d(A_s(K))/2$, by definition.

\[ \square \]

5. Obstructing genus and double points

In this section, we prove the main bound satisfied by $Y_{c,g}$, which is Theorem 1.1 from the introduction:

**Theorem 5.1.** Suppose there is an oriented, normally immersed surface in $B^4$ bounding a knot $K$ in $S^3$, which has genus $\mathscr{G}$ and $\mathscr{G}^+$ positive double points. If $0 \leq c \leq \mathscr{G}$ and $0 \leq g \leq \mathscr{G}$ then

\[ Y_{c,g}(K) \leq \left\lfloor \frac{\mathscr{G}^+ - c}{2} \right\rfloor + \left\lfloor \frac{\mathscr{G} - g}{2} \right\rfloor. \]

In particular, if $Y_{c,g}(K) \neq 0$, then there is no such surface with genus $g$ and $c$ positive double points.

5.1. Link Floer homology of Hopf links. In this section, we recall the link Floer homology of the Hopf link. The are two ways of orienting the components of the Hopf link which are not equivalent under an overall change of orientation. We refer to the orientation which results in two negative crossings of the standard diagram as the negative Hopf link, and we refer to the orientation which has two positive crossings as the positive Hopf link. A Hopf link of sign $\varepsilon \in \{+, -\}$ bounds a normally immersed $D^2 \sqcup D^2$ with a single double point, which has sign $\varepsilon$.

Thus far, we have focused mostly on the version of link Floer homology which has one variable $\mathscr{V}$ for all of the $w$-basepoints, and one variable $\mathscr{V}$ for all of the $z$-basepoints. It becomes convenient now to consider a refinement of this where
we have one \( \mathcal{U} \) variable for each \( w \)-basepoint, and one \( \mathcal{V} \) variable for each \( z \)-basepoint. A diagram for the Hopf link must have (at least) four basepoints, so we now consider the link Floer complex over the ring \( F[\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \mathcal{V}_2] \). The complexes for the positive and negative Hopf links are shown in Figure 5.1. The complexes may be computed by counting bigons on a simple Heegaard diagram (cf. [MO10, Figure 3]).

| \( \mathcal{U}_1 \) | \( \mathcal{U}_2 \) | \( \mathcal{V}_1 \) | \( \mathcal{V}_2 \) |
|---|---|---|---|
| \( a \leftarrow \mathcal{U}_2 \rightarrow b \) | \( a \rightarrow \mathcal{U}_2 \rightarrow b \) | \( c \leftarrow \mathcal{V}_2 \rightarrow d \) | \( c \rightarrow \mathcal{V}_2 \rightarrow d \) |

**Figure 5.1.** The link Floer complexes of the positive (left) and negative (right) Hopf links.

**Lemma 5.2.** Let \( (B^4, \mathcal{F}) \) denote a link cobordism from the empty set to a Hopf link \( H \) (of either sign), such that \( \mathcal{F} \) is topologically an annulus. Furthermore, suppose that the decoration of \( \mathcal{F} \) consists of a closed curve, which is a longitude of the annulus, as well as two bigons, which are each parallel to a boundary component. Then

\[
F_{B^4, \mathcal{F}} \simeq 0.
\]

**Proof.** Let \( L_1 \) denote the component of \( H \), parallel to which the dividing set bounds a bigon of type-\( z \). We isotope the dividing set as shown in Figure 5.2. It follows from [Zem19a, Lemma 4.1] that the cobordism map for \( \mathcal{F} \) factors through the composition \( \Psi_{z_1} \Phi_{w_1} \), where \( \Psi_{z_1} \) is the endomorphism of the Hopf link complex defined via the formula

\[
\Psi_{z_1}(x) := \sum_{\phi \in \pi_2(x, y)} \sum_{\mu(\phi) = 1} n_{z_1}(\phi) \#(M(\phi)/R) \mathcal{U}^{n_{w_1}(\phi)} \mathcal{V}^{n_{z_1}(\phi)+n_{z_2}(\phi)-1} y,
\]

and \( \Phi_{w_1} \) is defined similarly. The maps \( \Phi_{w_1} \) and \( \Psi_{z_1} \) can be computed just from knowledge of the version of the link Floer complex of the Hopf link over \( F[\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \mathcal{V}_2] \). Given the form of the complexes in Figure 5.1, the composition \( \Psi_{z_1} \Phi_{w_1} \) clearly vanishes. \( \square \)

**Figure 5.2.** Isotoping the dividing set of \( \mathcal{F} \) in Lemma 5.2.
5.2. Double points and knot Floer homology. Suppose \( S \) is an oriented, normally immersed surface in a 4-manifold \( W \), with boundary a knot \( K \). Then \( S \) is the image of a normal immersion

\[
\pi : \hat{S} \to S.
\]

Let \( \mathcal{D} \subseteq \hat{S} \) be a properly embedded arc, which cuts \( \hat{S} \) into two connected components, and avoids the double points. Let \( w \) and \( z \) be two basepoints in \( K \setminus \partial \mathcal{D} \). We think of \( w \) and \( z \) as determining a designation of one component of \( \hat{S} \setminus \mathcal{D} \) as type-\( w \), and the other as type-\( z \).

We write \( d \subseteq S \) for the set of double points. We partition \( d \) into two collections, \( d_w \) and \( d_z \). We do not yet require that \( d_w \) and \( d_z \) be compatible with \( \mathcal{D} \). On the surface \( \hat{S} \) we pick a collection of pairwise disjoint arcs \( a \subseteq \hat{S} \), as follows. For each \( d \in \pi^{-1}(d_w) \) which is in the type-\( z \) subregion of \( \hat{S} \setminus \mathcal{D} \), we pick an arc from \( d \) to \( \mathcal{D} \), whose interior is disjoint from \( \mathcal{D} \) and the double points. Similarly, for each \( d \in \pi^{-1}(d_z) \) which is in the type-\( w \) subregion, we pick an analogous arc which connects a point of \( d \) to \( \mathcal{D} \). We write \( \mathcal{D}_a \) for the dividing set obtained by isotoping \( \mathcal{D} \) along the arcs \( a \). By construction, \( \pi^{-1}(d_w) \) lies in the \( w \)-subregion of \( \hat{S} \setminus \mathcal{D}_a \), and \( \pi^{-1}(d_z) \) lies in the \( z \)-subregion.

Let \( S' \) be the surface obtained by smoothing \( S \), i.e. removing each double point by increasing the genus by 1. If \( \mathcal{D}_a \) is a dividing set, constructed using the above procedure, then \( \mathcal{D}_a \) induces a well-defined dividing set on \( S' \), which we also denote by \( \mathcal{D}_a \).

It is natural to conjecture that the cobordism maps induced by the above procedure only depend on \( S, \mathcal{D}, s \) and the quantities \(|d_w|\) and \(|d_z|\), however this does not seem to be the case. However, we will show that this holds after one stabilization:

**Proposition 5.3.** Suppose that \( S \subseteq W \) is a normally immersed surface, \( \mathcal{D} \) is a dividing arc arc on \( \hat{S} \), \( d = d_w \cup d_z \) is a partition of the double points of \( S \), as above. If \( i \) and \( j \) are nonnegative integers satisfying \( i + j = 1 \), then the homotopy type of the map

\[
\Upsilon^i \Upsilon^j : F_{W,(S',\mathcal{D}_a),s}: \mathbb{F}[\Upsilon, \Upsilon] \to CFK^-(K)
\]

depends only on \( W, \mathcal{D}, s \) and the two quantities

\[
i + |d_w| \quad \text{and} \quad j + |d_z|.
\]

We prove Proposition 5.3 in several steps. The following is similar to [JZ18, Proposition 6.12]:

**Lemma 5.4.** The cobordism map

\[
F_{W,(S',\mathcal{D}_a),s}: \mathbb{F}[\Upsilon, \Upsilon] \to CFK^-(K)
\]

is independent of the arcs \( a \), and hence depends only on \( W, S', \mathcal{D}, s \), and the choice of partition \( d = d_w \cup d_z \).

**Proof:** Suppose that \( a_1 \) and \( a_2 \) are two choices of arcs for a partition of \( d \). We prove the claim by induction on the number of intersections of the interiors of the arcs in \( a_1 \) and \( a_2 \). Suppose first that the interior of each arc of \( a_1 \) is disjoint from each arc of \( a_2 \). We may switch the arcs of \( a_1 \) to the arcs of \( a_2 \) one-by-one by using the bypass relation and Lemma 5.2, as shown in Figure 5.3.
Next, we claim that if $a_1$ and $a_2$ have arcs which intersect each other away from the double points, then we may modify $a_2$ to obtain a set of arcs $a_3$ which has one fewer intersection with $a_1$, such that further

$$FW,(S',D_{a_2}),\sigma \simeq FW,(S',D_{a_3}),\sigma.$$

Orient all arcs of $a_1$ and $a_2$ to go from $D$ to their double point. Suppose that $a_1 \in a_1$, and $a_1$ intersects the interior of some arc of $a_2$. Let $d$ denote the double point that $a_1$ is connected to. Let $a_2 \in a_2$ be the arc which intersects int($a_1$) closest to $d$. Let $a_2'$ be the arc of $a_2$ which connects to $d$. (We allow either $a_2 = a_2'$ or $a_2 \neq a_2'$). We perform a bypass relation to $D_{a_2}$ along an arc which runs parallel to a portion of $a_1$. See Figure 5.4. This gives two dividing sets, one of which induces the zero map by Lemma 5.2. The remaining dividing set is $D_{a_3}$ for a set of arcs $a_3$ which has one fewer intersection with $a_1$ than $a_2$ did. By induction, the proof is complete.

Next, we perform a model computation involving the Hopf link:

**Lemma 5.5.** Let $S$ denote the standard, annular surface bounding a Hopf link $\mathbb{H}$ (of either sign). Let $W$ denote the decoration of $S$ which has $w$-subsurface equal to an annulus, and $z$-subsurface equal to two bigons. Let $Z$ denote the decoration of $S$ which has $w$-subsurface equal to two bigons, and $z$-subsurface equal to an annulus. Then

$$\nabla \cdot FB^4,W \simeq \nabla \cdot FB^4,Z.$$

**Proof.** The map $\nabla \cdot FB^4,W$ is equal to the link cobordism map for the surface obtained by stabilizing $W$ once in the $z$-subsurface. Similarly $\nabla \cdot FB^4,Z$ is equal to the map for the link cobordism obtained by stabilizing $Z$ once in the $w$-subsurface. The surfaces $W$ and $Z$ are illustrated in Figure 5.5. Ignoring the decorations, these two surfaces are equal. It is an easy exercise to see that the surfaces are isotopic as decorated surfaces, so the link cobordism maps are homotopic.

Figure 5.3. A bypass relation showing that $FW,(S',D_{a_1}),\sigma \simeq FW,(S',D_{a_2}),\sigma$ when $a_1$ and $a_2$ are disjoint. The solid dots on top indicate double points. The open circles on the bottom indicate where double points have been smoothed. The map associated to the middle dividing set is zero by Lemma 5.2.
Lemma 5.5 may be understood concretely, as follows. We focus on the case of a negative double point, as the case of a positive double point is analogous. For a negative double point, the associated Hopf link is the negative Hopf link, whose complex is shown in Figure 5.1. The \((\text{gr}_w, \text{gr}_z)\)-bigradings for \(a, b, c\) and \(d\) are \(\left(\frac{1}{2}, -\frac{3}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right)\) and \(\left(-\frac{3}{2}, \frac{1}{2}\right)\), respectively. The bigrading of the element \(F_{B^+, W}(1)\) is \(\left(-\frac{3}{2}, \frac{1}{2}\right)\), and the bigrading of \(F_{B^+, Z}(1)\) is \(\left(\frac{1}{2}, -\frac{3}{2}\right)\), using the grading formula from [Zem19b, Theorem 1.4]. Since \(F_{B^+, W}\) and \(F_{B^+, Z}\) are non-zero, it follows that

\[ F_{B^+, W}(1) = d, \quad \text{and} \quad F_{B^+, Z}(1) = a. \]

Lemma 5.5 now becomes equivalent to the relation

\[ \mathcal{Y} \cdot [d] = \mathcal{Y} \cdot [a], \]

where the brackets denote the induced elements of homology. Equation (18) is immediate from the complex displayed in Figure 5.1.

We can now prove Proposition 5.3:
Proof of Proposition 5.3. Suppose that \( d_w \sqcup d_z \) and \( d'_w \sqcup d'_z \) are two partitions of the double points of \( S \), and let \( i, j, i' \) and \( j' \) be nonnegative integers so that \( i + j = 1 \) and \( i' + j' = 1 \). Suppose further that
\[
|d_w| + i = |d'_w| + i' \quad \text{and} \quad |d_z| + j = |d'_z| + j'.
\] (19)
According to Lemma 5.4, the cobordism map obtained by picking a set of arcs on \( S \) which are adapted to the partition \( P = \{d_w, d_z\} \) depends only on the partition \( P \) (and is independent of the choice of arcs).

Note that multiplication by \( U \) or \( V \) may be realized by adding a stabilization to the type-\( w \) or type-\( z \) subsurface, respectively, according to [JZ18, Lemma 5.3]. We formally adjoin the letter \( s \) to the set \( d_w \) if we perform a stabilization in the \( w \)-subsurface, and similarly we adjoin \( s \) to \( d_z \) if instead we stabilize the \( z \)-subregion.

We say the resulting sets form an enhanced partition of the double points. By Lemma 5.5, if \( s \in d_w \), we may trade \( s \) with any double point in \( d_z \) without changing the cobordism map. The analogous claim holds if \( s \in d_z \) instead.

Note that a genuine partition of \( d \) into \( d_w \sqcup d_z \) as well as a choice of integers \( i \) and \( j \) satisfying \( i + j = 1 \) is equivalent to an enhanced partition. Furthermore, it is clear that two enhanced partitions can be related by the above trading move if and only if (19) holds. The proof is complete. \( \square \)

5.3. Proof of Theorem 1.1. We are now in place to prove Theorem 1.1:

Proof of Theorem 1.1. Let \( S \subseteq B^4 \) be a normally immersed surface with genus \( \mathcal{G} \), \( \mathcal{G}^- \) negative double points, and \( \mathcal{G}^+ \) positive double points. We blow up at the negative double points, to obtain a normally immersed surface \( S' \) in \( W := \#^n \mathbb{CP}^2 \# B^4 \) which represents the trivial homology class (see [GS99, Proposition 2.3.5]). Furthermore, \( S' \) has genus \( \mathcal{G} \), \( \mathcal{G}^+ \) positive double points, and no negative double points. Let \( s \in \text{Spin}^c(W) \) be a Spin\(^c \) structure of maximal square.

Assume that \( c \) and \( g \) are nonnegative integers such that \( c \leq \mathcal{G}^+ \) and \( g \leq \mathcal{G} \). We may assume that \( \mathcal{G}^+ - c \) and \( \mathcal{G} - g \) are both even, since, if not, we may trivially stabilize to increase the genus, or perform a twist move (see Figure 5.6) to add a positive double point if necessary, without changing the quantity appearing in the right-hand side of the desired inequality.

Let \( \tilde{S}' \) denote the underlying source surface of \( S' \). We pick a dividing set \( \mathcal{D} \) on \( \tilde{S}' \) so that
\[
g(\tilde{S}'_w) = \frac{\mathcal{G} - g}{2} \quad \text{and} \quad g(\tilde{S}'_z) = \frac{\mathcal{G} + g}{2}.
\]
For each integer \( i \) satisfying \( 0 \leq i \leq c \), we construct dividing sets \( \mathcal{D}_i \), by picking arcs to guide an isotopy, as in Section 5.2, so that exactly \( (\mathcal{G}^+ - c)/2 + i \) double points occur in the \( w \)-subsurface of the smoothed surface, and \( (\mathcal{G}^+ + c)/2 - i \) occur in the \( z \)-subsurface. We write \( \mathcal{F}_i \) for the decorated surface obtained by smoothing \( \tilde{S}' \), and using the dividing set induced by \( \mathcal{D}_i \).
Using the grading change formulas described in Section 3.2, we compute that $F_{W,F,s}$ has the following grading change:

\[ \Delta \text{gr}_w = -(\mathcal{G} + \mathcal{C}^+ - c - g) - 2i, \quad \Delta \text{gr}_z = -(\mathcal{G} + \mathcal{C}^+ + c + g) + 2i, \]
\[ \Delta \delta = -(\mathcal{G} + \mathcal{C}^+) \quad \text{and} \quad \Delta A = c + g - 2i. \]

We put the elements $F_{W,F,s}(1)$ along the bottom row of $S_{c,g}$ to obtain an element $x_0$. It follows from Proposition 5.3 that we may pick homogeneously graded chains for the top row of $S_{c,g}$ which give a cycle when added to $x_0$. Call the resulting cycle $x$. Then $[x]$ is an $F[U]$-non-torsion element in $H_*(S_{c,g}(K))$, by Lemma 3.3 and [Zem19b, Theorem 1.7]. Further, $x$ has $\delta$-grading $-(\mathcal{G} + \mathcal{C}^+)$. Hence,

\[ d(S_{c,g}(K)) \geq -(\mathcal{G} + \mathcal{C}^+), \]

which immediately gives

\[ Y_{c,g}(K) \leq \frac{\mathcal{G}^+ - c}{2} + \frac{\mathcal{G} - g}{2}, \]

completing the proof. \hfill \Box

6. Involutive Heegaard Floer homology and the clasp number

In this Section, we review Hendricks and Manolescu’s involutive Heegaard Floer homology, and prove Theorem 1.4 from the introduction:

**Theorem 6.1.** If $K$ is a knot in $S^3$, then

\[ - \left\lfloor \frac{c_4^{-}(K) + 1}{2} \right\rfloor \leq V_0(K) \leq V_0(K) \leq \left\lceil \frac{c_4^{+}(K) + 1}{2} \right\rceil. \]

6.1. Background on involutive Heegaard Floer homology. Involutive Heegaard Floer homology is a refinement of Heegaard Floer homology described by Hendricks and Manolescu [HM17]. If $Y$ is a 3-manifold equipped with a self-conjugate Spin$^c$ structure $s$, they constructed an $F[U,Q]/Q^2$-module $\text{HFI}^{-}(Y,s)$. Their construction gives two involutive correction terms, $\partial(Y,s)$ and $d(Y,s)$, which satisfy

\[ d(Y,s) \leq \partial(Y,s) \leq \partial(Y,s), \]

where $d$ is the ordinary correction term of Ozsváth and Szabó [OS03].

For a knot $K$ in $S^3$, Hendricks and Manolescu defined two integer invariants $V_0(K)$ and $V_0(K)$, which satisfy

\[ \nabla_0(K) \leq V_0(K) \leq V_0(K). \]

The invariants $\nabla_0(K)$ and $\nabla_0(K)$ may be computed directly from the knot Floer complex, as we now describe. Hendricks and Manolescu describe a knot involution $\iota_K : CFK^\infty(K) \to CFK^\infty(K)$.

For our present purposes, it is helpful to actually consider the knot involution $\iota_K$ as an endomorphism of the complex $CFK^\infty(K)$. On $CFK^\infty(K)$, the map $\iota_K$ satisfies

\[ \iota_K \mathcal{V} = \mathcal{V} \iota_K \quad \text{and} \quad \mathcal{U} \iota_K = \iota_K \mathcal{U}. \quad (20) \]

More generally, we say a map which satisfies (20) is skew-equivariant. Furthermore, the map $\iota_K$ interchanges the gradings $\text{gr}_w$ and $\text{gr}_z$. 
The subcomplex $A_0(K) \subseteq CFK^\infty(K)$ is preserved by $\iota_K$. Hendricks and Manolescu define

$$AI_0(K) := \text{Cone}(A_0(K) \xrightarrow{Q(1+\iota_K)} QA_0(K)).$$

We give $AI_0(K)$ the grading induced by $CFK^\infty(K)$ and by setting $\deg(Q) = -1$. The invariants $V_0(K)$ and $\overline{V}_0(K)$ are defined as

$$\overline{V}_0(K) := -\frac{1}{2}d(AI_0(K)) \quad \text{and} \quad V_0(K) := -\frac{1}{2}d(AI_0(K)).$$

6.2. More on the Hopf link. The Hopf link of sign $\varepsilon \in \{-, +\}$ bounds an immersed $D^2 \sqcup D^2$ in $B^4$ with a single transverse intersection point, whose sign is $\varepsilon$.

If the sign of this intersection point is negative, we may blow up at the transverse double point, and obtain a properly embedded surface in $B^4 \# \mathbb{CP}^2$ that has relative homology class

$$0 \in H_2(B^4 \# \mathbb{CP}^2, S^3),$$

and which topologically consists of two disks. The set of Spin$^c$ structures on $\mathbb{CP}^2$ can be identified with the odd integers, and the conjugation action sends $n$ to $-n$. Since $\mathbb{CP}^2$ is negative definite, there are two Spin$^c$ structures whose Chern classes have maximal square, and these classes are switched by conjugation. Let $S$ denote this surface in $B^4 \# \mathbb{CP}^2$, decorated with two arcs which each divide a disk into two bigons.

**Lemma 6.2.** View $(B^4 \# \mathbb{CP}^2, S)$ as a link cobordism from the empty set to the negative Hopf link, and let $s$ be a Spin$^c$ structure on $B^4 \# \mathbb{CP}^2$, with maximal square. Then

$$\mathcal{U} \cdot F^\pm_{B^4 \# \mathbb{CP}^2, S, s} \simeq \mathcal{U} \cdot F^\pm_{B^4 \# \mathbb{CP}^2, S, \overline{s}} \quad \text{and} \quad \mathcal{V} \cdot F^\pm_{B^4 \# \mathbb{CP}^2, S, s} \simeq \mathcal{V} \cdot F^\pm_{B^4 \# \mathbb{CP}^2, S, \overline{s}}.$$  

**Proof.** Two $\mathbb{F}[\mathcal{U}, \mathcal{V}]$-equivariant maps from $\mathbb{F}[\mathcal{U}, \mathcal{V}]$ to $CFK^\infty(\mathbb{H})$ are homotopic if and only if their evaluations on 1 are homologous. The link Floer complex of the negative Hopf link is shown below:

$$\begin{array}{cccc}
a & \xrightarrow{\mathcal{U}} & b \\
\downarrow \mathcal{V} & \uparrow & \\
c & \xleftarrow{\mathcal{V}} & d.
\end{array}$$  

Using the grading formulas from [Zem19b, Theorem 1.4], we compute that

$$\text{gr}_w(F^+_{B^4 \# \mathbb{CP}^2, S, s}(1)) = \text{gr}_x(F^+_{B^4 \# \mathbb{CP}^2, S, s}(1)) = \frac{1}{2}, \quad \text{and} \quad \text{gr}_w(F^+_{B^4 \# \mathbb{CP}^2, S, s}(1)) = \text{gr}_x(F^+_{B^4 \# \mathbb{CP}^2, S, s}(1)) = \frac{1}{2}.$$  

The $(\text{gr}_w, \text{gr}_x)$-bigradings of $a$, $b$, $c$ and $d$ are $(-\frac{1}{2}, \frac{3}{2})$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{3}{2}, -\frac{1}{2})$, respectively. Hence $F^+_{B^4 \# \mathbb{CP}^2, S, s}(1)$ and $F^+_{B^4 \# \mathbb{CP}^2, \overline{s}}(1)$ are in the $\mathbb{F}$-span of $b$ and $c$. Furthermore, neither can be $b + c$, because $\mathcal{U} \cdot [b + c] = 0$ and both $F^+_{B^4 \# \mathbb{CP}^2, S, s}(1)$ and $F^+_{B^4 \# \mathbb{CP}^2, S, s}(1)$ are non-torsion, by a simple adaptation of [Zem19b, Theorem 1.7]. Hence, each map sends 1 to either $b$ or $c$. Since

$$\mathcal{U} \cdot [b] = \mathcal{U} \cdot [c] \quad \text{and} \quad \mathcal{V} \cdot [b] = \mathcal{V} \cdot [c],$$
the ambiguity disappears after multiplying by either \( W \) or \( \mathcal{W} \), and the main claim follows.

\[ \square \]

6.3. Involutive correction terms and the clasp number. We now prove the main result of the section, Theorem 6.1.

Proof of Theorem 6.1. We focus first on the inequality involving \( W_0(K) \). Suppose that \( S \) is a normally immersed disk in \( B^4 \), which bounds \( K \). Let \( \mathcal{C}^+ \) denote the number of positive double points, and \( \mathcal{C}^- \) the number of negative double points. We blow up at the negative double points, and smooth the positive double points to obtain a surface \( S' \) in \( \#^n S^2 \# B^4 \), which bounds \( K \). Here \( n = \mathcal{C}^- \). Note \( g(S') = \mathcal{C}^+ \).

Suppose first that \( g(S') = 2k + 1 \), for an integer \( k \geq 0 \). Let \( \hat{S} \) denote the source of the normally immersed surface \( S \) (which is a disk, by assumption). We choose an arbitrary dividing arc \( D \) on \( \hat{S} \), and then modify it by picking a collection of arcs to guide an isotopy, as in Section 5.2, so that the \( w \)-subsurface contains the preimages of \( k \) double points, and the \( z \)-subsurface contains the preimage of \( k + 1 \) double points. Let \( \mathcal{F} \) denote the induced decoration on \( S' \).

Let \( s \) be a maximal \( \text{Spin}^+ \) structure on \( W := \#^n S^2 \# B^4 \) and we consider the map 

\[ W \cdot F_{W,F,s} : \mathbb{F}[W, \mathcal{V}] \to CFK^-(K). \]

The map \( W \cdot F_{W,F,s} \) preserves the Alexander grading by (13), since \( g(F_W) + 1 = g(F_s) \).

It follows from [Zem19a, Theorem 1.3] that 

\[ \iota_K \circ (W \cdot F_{W,F,s}) \simeq (F \cdot F_{W,F,s}) \circ \iota_0, \tag{22} \]

where \( \simeq \) denotes skew-equivariant chain homotopy. Here \( F \) denotes the surface obtained by reversing the designation of \( W \) and \( z \)-subsurfaces, and also adding a positive half-twist to the dividing set near \( K \). Also, \( \iota_0 : \mathbb{F}[W, \mathcal{V}] \to \mathbb{F}[W, \mathcal{V}] \) denotes the map which sends \( W \cdot F_{W,F,s} \) to \( W \cdot F_{W,F,s} \).

We can view the dividing set \( F \) as also being induced by modifying an arc \( D' \) by picking a collection of arcs which connect \( D' \) to the preimages of double points on \( \hat{S} \). Since \( \hat{S} \) is a disk, we can view \( F \) and \( F \) as both being constructed from the same arc, but with a different partition of the double points and guiding arcs on \( \hat{S} \).

Hence, by applying Proposition 5.3 and Lemma 6.2, we conclude 

\[ \mathcal{V} \cdot F_{W,F,s} \simeq W \cdot F_{W,F,s} \simeq W \cdot F_{W,F,s}. \]

Equation (22) now implies that 

\[ \iota_K \circ (W \cdot F_{W,F,s}) \simeq (W \cdot F_{W,F,s}) \circ \iota_0. \tag{23} \]

Using a homotopy \( H \) between \( \iota_K \circ (W \cdot F_{W,F,s}) \) and \( (W \cdot F_{W,F,s}) \circ \iota_0 \), we obtain a chain map 

\[ F : CFK^-(0) \to AI_0(K), \]

which lowers the Maslov grading by \( 2(k + 1) \). Concretely, if we view \( AI_0(K) \) as consisting of the direct sum of two copies of \( A_0(K) \), then the matrix for \( G \) is given by the formula 

\[ F = \begin{pmatrix} W \cdot F_{W,F,s} & H \\ 0 & W \cdot F_{W,F,s} \end{pmatrix}. \tag{24} \]
It follows from [Zem19b, Theorem 1.7] that the map $F$ becomes an isomorphism after localizing at $U$. Hence, the odd and even towers of $A_I^0(K)$ contain a non-torsion element of grading $2(k+1)$ less than the generators of the odd and even towers of $HFI^-(\emptyset) \cong \mathbb{F}[U,Q]/Q^2$. Hence

$$d(A_I^0(K)) \geq -2(k+1),$$

and hence,

$$\nabla_0(K) \leq k + 1 = \left\lceil \frac{\mathcal{C}^- + 1}{2} \right\rceil.$$ 

This proves the bound when there are an odd number of positive double points.

If instead there are an even number of double points, we may perform a twist move as in Figure 5.6 to add a new positive double point, without changing the quantity $\left\lceil (\mathcal{C}^+ + 1)/2 \right\rceil$. Applying the proof in the odd case completes the proof of the inequality involving $\nabla_0(K)$.

For the inequality involving $\nabla_0(K)$ and $c_4^-(K)$, the proof is similar, except we now turn around and reverse the orientation of $B^4$ to obtain a cobordism from $(S^3, K)$ to the empty set. This changes the sign of all double points. Let us write $\mathcal{C}^-$ for the original number of negative double points (which are now positive).

Assume first that $\mathcal{C}^- = 2k + 1$, for an integer $k \geq 0$, and let $D$ be a dividing arc on $\hat{S}$ as before. We isotope $D$ using a collection of arcs on $\hat{S}$, so that the w-subsurface has $k$ double points, and the z-subsurface has $k + 1$. Let $G$ denote the induced decorated surface in $\#^n \mathbb{C}P^2 \# B^4$, obtained by smoothing the now positive double points, and blowing up at the now negative double point. Here, $n$ is the original number of positive double points. Let $s$ be a Spin$^c$ structure on $W$ with maximal square.

Similar to the previous case, we consider

$$\mathcal{U} \cdot F_{W,G,s} : CFK^-(K) \to \mathbb{F}[\mathcal{U}, \nu].$$

The same proof as for (23) shows that

$$(\mathcal{U} \cdot F_{W,G,s} \circ \iota_K) \cong \iota_0 \circ (\mathcal{U} \cdot F_{W,G,s}).$$  

Let $H$ be a homogeneously graded, skew-equivariant homotopy between $(\mathcal{U} \cdot F_{W,G,s}) \circ \iota_K$ and $\iota_0 \circ (\mathcal{U} \cdot F_{W,G,s})$.

Similar to (24), we construct a map

$$G : A_I^0(K) \to \mathbb{F}[U,Q]/Q^2,$$

which maps $\mathbb{F}[U]$-non-torsion elements to $\mathbb{F}[U]$-non-torsion elements.

Furthermore, $G$ drops the Maslov grading by $2(k+1)$. We conclude that

$$\overline{d}(A_I^0(K)) \leq 2(k+1).$$

Rearranging, we obtain

$$\nabla_0(K) \geq -(k + 1) = -\left\lceil \frac{\mathcal{C}^- + 1}{2} \right\rceil.$$ 

This gives the stated inequality when $\mathcal{C}^-$ is odd. When $\mathcal{C}^-$ is even, we apply a twist move to add one negative double point, then apply the previous argument. □
7. Computations and examples

7.1. Topological examples. We ran code in SageMath [SD18] and Macaulay2 [GS] to compute several examples. The code is available on the second author’s webpage.

In Figure 7.1, we display the computations of $Y_{c,g}$ for the knots $T_{2,3}#T_{5,9}#-T_{6,7}$, $T_{2,3}#T_{4,7}#-T_{5,6}$ and the mirror of $T_{2,3}#T_{4,7}#-T_{5,6}$. We remark that computer experimentation suggests that the family

$$K_n := T_{2,2n+1}#T_{4,7}#-T_{5,6}$$

has $\omega^+(K_n) = \nu^+(K_n) + 1$, for each $n \geq 1$. (Recall that $\nu^+(K)$ is the minimum $s$ such that $V_s(K) = 0$, and $\omega^+(K)$ is the minimum $c$ such that $Y_{c,0}(K) = 0$).

![Figure 7.1](image)

Figure 7.1. The invariants $Y_{c,g}(T_{2,3}#T_{5,9}#-T_{6,7})$ (left), $Y_{c,g}(T_{2,3}#+T_{4,7}#-T_{5,6})$ (middle) and $Y_{c,g}(-T_{2,3}#-T_{4,7}#T_{5,6})$ (right). The invariants are zero except at the labeled points.

We focus on $K = T_{2,3}#T_{4,7}#-T_{5,6}$. The computations of $Y_{c,g}(K)$ and $Y_{c,g}(-K)$ summarized in Figure 7.1 imply that $c_4(K) \geq 2$ and $c_4^-(K) \geq 1$. Hence

$$c_4(K) \geq 3.$$ (26)

Note that the bound from (3) only implies that $c_4(K) \geq 2$.

Remark 7.1. The knot $K = T_{2,3}#T_{4,7}#-T_{5,6}$ has involutive correction terms $V_0(K) = 1$ and $V_0(-K) = 2$. The mirror has $V_0(-K) = -1$ and $V_0(-K) = 1$. The fact that $V_0(K) = 2$ gives an alternate proof via Theorem 1.4 that $c_4^+(K) \geq 2$.

On the other hand, via the following direct computation, we can see that the slice genus is at most 2:

**Proposition 7.2.** We have $g_4(T_{2,3}#T_{4,7}#-T_{5,6}) \leq 2$.

**Proof.** We will construct an embedded slice surface with genus 2. Consider the standard presentation

$$(\sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } i-j \geq 2)$$

of the braid group $B_n$ on $n$ strands, where $\sigma_i$ corresponds to a right-handed half-twist between strands $i$ and $i+1$.

We will use a pictorial description of the braid group. We denote a positive crossing by a bar connecting two vertical lines, as in Figure 7.2.
We construct a genus 2 cobordism from $T_{2,3} \# T_{4,7}$ to $T_{5,6}$. This is shown in Figure 7.3. We work in the braid group $B_5$, and start from the braid $\sigma_2^7(\sigma_2\sigma_3\sigma_4)^7$, whose closure is $T_{2,3} \# T_{4,7}$. In the figure, we write $S$ for a saddle move, $C$ for a cyclic permutation, $B$ for the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, and $R$ for a relation in (27), which we describe presently.

Define
\[ \delta_n := \sigma_{n-1} \cdots \sigma_1. \]

We claim
\[ \sigma_{i-k} \delta_n^k = \delta_n^k \sigma_i, \] (27)
whenever \( i \neq k \). Here, we take \( i - k \) modulo \( n \). Furthermore, \( \delta_n^k \) is central.

For general \( k \), (27) follows from the \( k = 1 \) and \( k = 2 \) cases. The \( k = 1 \) case is proven by noting that

\[
(\sigma_{n-1} \ldots \sigma_1)\sigma_i = (\sigma_{n-1} \ldots \sigma_i \sigma_{i-1} \ldots \sigma_1) = (\sigma_{n-1} \ldots \sigma_{i-1} \sigma_i \sigma_{i-1} \ldots \sigma_1) = \sigma_i \sigma_{n-1}(\sigma_{n-1} \ldots \sigma_1),
\]

as long as \( i \neq 1 \). For the \( k = 2 \) case, the claim follows from the \( k = 1 \) case, except when \( i = 1 \). We compute

\[
\delta_n^2 = (\sigma_{n-1} \ldots \sigma_1)(\sigma_{n-1} \ldots \sigma_1) = \sigma_{n-1} \sigma_{n-2} \sigma_{n-3} \sigma_{n-2} \ldots \sigma_4 \sigma_5 \sigma_3 \sigma_4 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1 = \sigma_{n-1} \delta_n^2.
\]

In more detail, to go from the first to the second line, we repeatedly apply \( \sigma_i \sigma_j = \sigma_j \sigma_i \) when \( |i - j| > 1 \). To go from the second to the third line, we repeatedly apply \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \). From (28), it is clear that \( \delta_n^2 \sigma_1 = \sigma_{n-1} \delta_n^2 \).

Using cyclic permutations and the relation \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for non-adjacent \( i \) and \( j \), we obtain the braid for \( T_{2,3} \# T_{4,7} \) shown in the top left of Figure 7.3. We add a band, which is achieved by removing the generator \( \sigma_4 \) in red. We commute the \( \sigma_3 \) preceding \( \sigma_3^4 \) past \( \sigma_3^4 \), and then add a band via adding the generator \( \sigma_2 \) in green to obtain the third braid in the top row.

To obtain the fourth braid, we apply (27), which in this case reads

\[
(\sigma_4 \ldots \sigma_1)^2 \sigma_1 = \sigma_4 (\sigma_4 \ldots \sigma_1)^2,
\]

and then we cyclically move \( \sigma_4 \) to the beginning of the braid word.

To obtain the fifth braid, we apply the relation \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) to the subword \( \sigma_3 \sigma_5 \sigma_5 \sigma_3 \) in blue. We then apply equation (27) to \( \sigma_3 \) preceding \( \sigma_4 \) in the top row. The last braid in the top row is obtained by cyclically moving the first \( \sigma_2 \) to the end of the braid word, then applying the relation \( \sigma_1 \sigma_3 \sigma_4 = \sigma_3 \sigma_4 \sigma_1 \), and finally cyclically moving \( \sigma_3 \sigma_4 \) to the end.

The first braid in the bottom row of Figure 7.3 is the result of applying the relations \( \sigma_3 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_2 \) and \( \sigma_3 \sigma_4 \sigma_3 = \sigma_4 \sigma_3 \sigma_4 \) to the subwords in blue.

The second braid in the bottom is obtained by using relation (27), which trades the \( \sigma_4 \) with a \( \sigma_1 \), and then cyclically permuting the the subword \( \sigma_2 \sigma_3 \sigma_4 \).

To obtain the third braid on the bottom, we cyclically permute \( \sigma_2 \) to the top, and apply relation (27) to trade it with a \( \sigma_1 \) preceding \( \sigma_4 \ldots \sigma_1 \).

Via a band move, we add \( \sigma_1 \) to the beginning of the word, and then perform a cyclic permutation to obtain the fifth braid.

The sixth braid in the bottom is obtained by commuting \( \sigma_1 \) with the central element \( \delta_5^2 \), moving it to the beginning of the word, and then moving the first \( \sigma_4 \) to the end of the word, and commuting it past \( \delta_5^2 \).

Finally, to obtain the final braid, we delete the \( \sigma_2 \) marked in red via a saddle. We are left with \( \delta_5^2 \), whose closure is \( T_{5,6} \). Since we have used 4 bands, the above cobordism has genus 2.

\[ \square \]

**Question 7.3.** We have shown that \( g_4(T_{2,3} \# T_{4,7} \# - T_{5,6}) \in \{1, 2\} \). What is the exact value of the genus?
7.2. **Further questions.** Here are a few interesting questions:

1. Are there knots such that \( c^+_4(K) - g_4(K) \) is arbitrarily large?
2. Are there knots where \( c^+_4(K) > c^+_4(K) + c^-_4(K) \)? Can the difference be arbitrarily large?
3. Are there knots where \( \omega(K) > \nu(K) + 1 \).
4. Is there an invariant, defined using Khovanov homology, which also gives a lower bound on the 4-dimensional clasp number?

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