On the Spatial Structure of Monopoles

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Abstract: We study the spatial structure of 1/4 BPS solitons in 4 dimensional $\mathcal{N} = 4$ gauge theory. A weak binding approximation is used where the soliton is made of several “ingredient” particles. Some spatial moduli are described which are not accounted for in the (p,q) web picture. These moduli are counted and their effect on the solutions is demonstrated. The potential for off BPS configurations is estimated by a simple expression and is found to agree with previous expressions. We discuss the fermionic zero modes of the solitons, and find agreement with web predictions.

Keywords: Supersymmetric Effective Theories, Solitons Monopoles and Instantons, D-branes.
1. Introduction

We continue the study of the BPS spectrum of the maximally supersymmetric gauge theory in 4 dimensions, namely $\mathcal{N} = 4$. In the introduction we begin with a review of our current knowledge, then we describe the open questions, and then the contribution of this paper.
1.1 Review of BPS particles in 4d $\mathcal{N} = 4$ gauge theory

The bosonic part of the Lagrangian of 4d $\mathcal{N} = 4$ gauge theory can be written as

$$S = -\frac{1}{16\pi} \text{Im} \int \tau \text{Tr}(F^2 - iF \wedge F) - \frac{1}{2g^2} \int \text{Tr}(|D\phi|^2 + \sum_{i<j} [\phi^i, \phi^j]^2)$$

(1.1)

where $g$ is the coupling, $\tau = \theta/2\pi + i 4\pi/g^2$ is the complex coupling that incorporates also the theta angle $\theta$, $F$ is the field strength and $\phi^i$, $i = 1..6$ are scalar fields. All fields are in the adjoint of some gauge group $G$. At a generic point in moduli space the scalars acquire a VEV $\vec{\phi}_r = \text{diag}(\vec{\phi}_{r1}, ..., \vec{\phi}_{rr})$ and the gauge group is broken to $U(1)^r$, where $r = \text{rank}(G)$. We consider states carrying various electric and magnetic charges $(p, q)_r$, under these $U(1)$'s, and we refer to all of them as ‘monopoles’.

Branes give a useful way to model this system. For gauge group $SU(N_c)$ one takes $N_c$ parallel D3 branes, and considers the scaling limit (“the field theory limit”) $M_s \to \infty$ ($M_s$ is the string scale) keeping all gauge theory energies fixed $E \sim \Delta x M_s^2$, where $\Delta x$ is any (shrinking) length scale perpendicular to the D3’s.

In the case $G = SU(2)$ the BPS spectrum is well known. It includes the $W$, the monopole and in general all $(p, q)$ dyons. All the states are 1/2 BPS and lie in a short (vector) multiplet. Actually, they are all $SL(2, \mathbb{Z})$ duals of each other.

When we take a bigger gauge group 1/4 BPS states become possible. We consider mainly $G = SU(3), SU(4)$. The mass of such states (when they exist) is given by the BPS formula

$$M[(p, q)_1] = |Z| = \sqrt{Q_e^2 + Q_m^2 + 2|Q_e \times Q_m|}$$

(1.2)

where $Q_e + iQ_m = \sum_{j=1}^r (p_j + \tau q_j)\vec{\phi}_j$.

It was shown that $(p, q)$ strings $[\underline{1}, \underline{2}, \underline{3}, \underline{4}]$ are important tools in analyzing 1/4 BPS monopoles $[\underline{1}, \underline{2}]$ (for related work see $[\underline{34}, \underline{35}]$). Recall that a string web is a planar collection of strings in the $(x, y)$ plane each carrying a $(p, q)$ label ($(p, q)$ are relatively prime integers) and satisfying

1. Slope. The slope of a $(p, q)$ string is given by $\Delta x + i \Delta y \parallel p + \tau q$

2. Junction. $(p, q)$ strings can meet at vertices as long as the $(p, q)$ charge is conserved: $\sum p_i = \sum q_i = 0$

Any $(p, q)$ web that can be drawn with external legs $(p, q)_j$ all ending on D3 branes is identified with a monopole carrying the electric and magnetic charges $(p, q)_j$ under the $U(1)$’s corresponding to each D3. It was shown that the mass of the web is the same as the BPS mass $[\underline{5}]$. Moreover, the web picture leads to predict that the

\footnote{The $Q$’s are six dimensional vectors, yet we find it convenient to use the $|Q_e \times Q_m|\sin \alpha$, as it is done in $[\underline{3}]$.}
monopole will reach marginal stability when a junction coincides with a D3 brane \[4\]. This prediction was verified by the classical solutions of \[5\] who found that the size of the solution diverges as marginal stability is approached.

 Knowing the mass of the monopoles, one would like to know the possible spins, namely the multiplet structure. Being 1/4 BPS it must contain the medium representation (with \(|j| \leq 3/2\)) as a factor. In \[6\] the maximum spin \(j\) in the multiplet was predicted by counting the number of fermionic zero modes (FZM) on the web:

\[
|j| \leq F + n_X / 2 \tag{1.3}
\]

where \(F = F(p, q)\) is the number of internal faces in the corresponding web and \(n_X\) is the number of external legs. So far little is known from field theory about the multiplets when internal faces are present. In a case with multiple external legs the known data \[7\] is consistent with the conjecture (1.3).

The growth of the degeneracy \(d = d(p, q)\) (multiplet size) for large charges was discussed in \[8\]. There it was “phenomenologically” found that the ground state entropy \(S = \log d\), behaves like

\[
S \sim \sqrt{F}. \tag{1.4}
\]

where \(F\), the number of internal faces, is quadratic in the charges.

Several other related studies appeared \[9, 10, 11, 12, 13, 14, 15, 16, 17, 18\], including studies of the potential energy of classical configurations of 1/4 BPS states.

1.2 Open questions

So far we discussed predictions for monopoles from the web model. It is natural to proceed in two directions: one is to test the web predictions in field theory and the other is to study monopole properties which are not modeled by webs.

In the first category we would like to know the exact multiplet structure, or at least to perform tests of the maximum spin prediction (1.3) and the ground state entropy (1.4).

Here we implicitly assume the existence of “large” or “accidental” BPS representations. In black hole physics a BPS black hole has a huge exact degeneracy, but in field theory this is unfamiliar. One would like to test whether the webs in \(SU(3)\) with many faces are indeed made of a sum of SUSY representations, and moreover that the planar \(SU(4)\) monopoles are BPS although they are in a large SUSY representation \[8\].

In the second category, we would like to study the spatial structure of the monopoles. The webs are thought to live in a point in the D3, and so do not give direct spatial information (Nevertheless, recall that Nahm’s equations for monopoles can be obtained from brane configurations \[19\]). In this paper we try to study this problem directly in the field theory, in a certain convenient limit.
We would like to mention an alternative geometric approach to the problem which involves the study of special Lagrangian submanifolds of $A_n \times T^2$. $A_n$ is a non-compact K3, a blow-up of an $A_n$ singularity and $n = 2, 3$ are of special interest. This formulation arises since compactifying type IIB on this manifold gives us the relevant field theory with gauge group $A_n = SU(n + 1)$, and D3 branes which wrap the special Lagrangian cycles give us the 1/4 BPS states. We would like to look at an arbitrary 3-homology class, and determine its moduli space with flat connections. (The 3-cycles of $A_n \times T^2$ are spanned by products of a 2-cycle in $A_n$ with a 1-cycle in $T^2$. Here we consider an arbitrary linear combination of these.) Actually, we are interested in the cohomologies of this moduli space, and each cohomology determines a state in the sought after multiplet. At the moment special Lagrangian submanifolds are an active field of research in mathematics [21, 22, 23] (see also [24]), and the results may be available soon.

1.3 In this paper

In this paper we will study the spatial structure of the 1/4 BPS states both for its own sake and in order to study the open questions described above from a new angle.

To analyze the spatial structure we restrict ourselves to limiting configurations - weakly bound monopoles - as explained in section 2. Moreover, we look at the effective low energy theory on one of the $U(1)$ factors, or equivalently on one of the D3 branes. Thus we will approximate the theory by 4d $\mathcal{N} = 4$ super Maxwell theory, and at times we will refer also to the full super Born-Infeld (BI) on the brane for comparison.

In [20] solutions representing a single string (of F or D type) emanating from a D3 were studied in the framework of the full Born-Infeld action of a D brane [23]. More elaborate configurations involving string webs were studied in [20, 27] and the BI BPS solutions were found to solve the Maxwell equations as well. For other related results see [36]. In solutions of Maxwell theory all fields are harmonic, that is, are linear combination of $1/|r - r_i|$ potentials. These solutions represent a collection of the ingredient particles with some of their relative distances being constrained. The locations of the singularities $r_i$ are moduli of the solutions (after accounting for the constraints), which are not evident in the web picture. In section 3 we compare these moduli with the moduli of the web. We find that for webs with 4 external legs or more there are essential spatial moduli which are not seen in the web picture. It is interesting to check the effect of these moduli on the web plane. We present graphs of this variation showing how it affects the thickness of junctions. On the other hand, we did not find in this approximation the moduli of the web which correspond to changing the size of an internal face.

In section 4 we study the potential for off BPS configurations or the restoring force when one tries to change a constrained relative distance of the solution. We use two methods, the first uses the super Maxwell picture and the other uses the
web picture. In both cases we find the same restoring force. We compare our results to the expression for the full potential found in [3] and find agreement. Whereas the latter expression is more general our expression is a useful simple approximation.

In section 6 we study the zero modes (especially the fermionic ones) of some solutions. First we analyze the solutions corresponding to 1/2 BPS states, an F string and a D string ending on a D3, and we write down the fermionic zero modes (FZM) explicitly. Then we discuss the FZM of a 1/4 BPS solution, where we find that the FZM counting coincides with the expectations of [6] for the case under study.

2. Weakly bound monopoles

One way to make the spatial structure tractable is to study weakly bound monopoles. In this case the bound state has a clear structure - it is made of well separated particles (which are more elementary as we will explain), and some of the relative geometry is fixed by the dynamics.

For concreteness let us consider our prime and simplest example - the weakly bound simple junction. The string web, fig.1 is made of a D string and an F string which intersect in a junction to produce a light (1,1) string. The fundamental string ends on the D3 brane denoted by A, the D string ends on B and the (1,1) on C. In field theoretic terms, it is the state in an $SU(3) \rightarrow U(1)^2$ gauge group, with charges $(1,0)_A$, $(0,1)_B$, $(-1,-1)_C$ under the three $U(1)$'s corresponding to the three D3's (of course there are only two independent $U(1)$'s because of the center of mass constraint. This redundancy is reflected in the constraint on the charges by having both the electric and magnetic ones sum to zero).

![Figure 1: The simple junction. The D3's are represented by circles. One of the legs is short.](image)

Assume for simplicity that $\tau = i$ (unit coupling), that the mass of the fundamental string segment is $M_1$, that the mass of the D string segment is $M_2$, and that the mass of the light (1,1) string is $2m$, $m \ll M_1, M_2$. This state has the same charges as a magnetic monopole in the $U(1)_{BC}$ (the $U(1)$ which corresponds to relative BC
motion), and an electric W particle in $U(1)_{AC}$, and hence could be thought to be a bound state of the two. The binding energy is

$$E_b = M_W + M_{mon} - M_{junc} = \sqrt{(M_1 + m)^2 + m^2 + \sqrt{(M_2 + m)^2 + m^2} - (M_1 + M_2 + 2m) = m^2 / 2\mu_M + O(m^3 / M^2) \quad (2.1)$$

where $1/\mu_M = 1/M_1 + 1/M_2$ is the reduced mass.

At low energies the effective action is that of a $U(1)^2$, $N = 4$ gauge theory. We choose to concentrate on the $U(1)$ which lives on the short leg (we can do that since the different $U(1)$’s are decoupled at low energies). We are interested in solutions in which two of the scalar fields $X, Y$ are excited, in addition to the gauge field.

As a general rule, short distances in the web plane are translated into large distance in the D3 world volume (the UV-IR relation). To see this in the case of the simple junction [7, 27], note that since the scalar fields must satisfy the Laplace equation (with sources) they should be of the form

$$X = \frac{1}{|\vec{r} - \vec{r}_1|}$$
$$Y = \frac{1}{|\vec{r} - \vec{r}_2|} \quad (2.2)$$

where $\vec{r}_1$ is the world-volume location of the electric charge, while $\vec{r}_2$ is the location of the magnetic one. We see that $|\vec{r}_1 - \vec{r}_2|$ is constrained by $m$ to be

$$m = Y(\vec{r}_1) = X(\vec{r}_2) = \frac{1}{|\vec{r}_1 - \vec{r}_2|} \quad (2.3)$$

thus establishing the inverse proportionality.

3. Soliton moduli from several approaches

The solutions found in [20, 27] are interpreted as representing strings and string junctions. To get a better understanding of this correspondence, we would like to compare the moduli of both. We will find that there are moduli which are found on the weak binding approximation side but not in the web picture and vice versa.

For the $SU(2)$ gauge group considered in [20] there are no moduli other than translations. The same is true in the web picture.

3.1 $SU(3)$

For $SU(3)$ the simplest web is the string junction (see figure [27]). In the weak binding approximation that we use here, we will choose the location of one of the D3’s at the origin, while the other two are at $(m, \infty)$ and $(\infty, m)$. These asymptotics
fix some of the moduli. Note that \( m \) is not a modulus of the solution, rather it is determined by the field theory VEVs, since we consider the space of all webs which terminate on a given configuration of D3’s.

To represent this configuration in the language of the effective action we have to consider solutions of the Laplace equation for both \( X \) and \( Y \) which obey the asymptotics. The general form of a solution with \( n \) singular points is:

\[
X = \sum_{i=1}^{n} \frac{p_i}{|\vec{r} - \vec{r}_i|} \quad \quad Y = \sum_{i=1}^{n} \frac{q_i}{|\vec{r} - \vec{r}_i|} \tag{3.1}
\]

where the \( \vec{r}_i \) are the spatial D3 coordinates. In addition to the scalar fields there is also a vector field, whose field strength is given by

\[
\vec{E} = \vec{\nabla} X \quad \quad \vec{B} = \vec{\nabla} Y \tag{3.2}
\]

We see that \( p_i, q_i \) are electric and magnetic charges, and as such must be quantized in the quantum theory. Quantization reduces the \( SL(2, \mathbb{R}) \) symmetry of these solutions to \( SL(2, \mathbb{Z}) \). To get two semi-infinite strings (In addition to the short one) we have to consider a two-centered solution. To get the desirable charges we take \( p_i = \{1, 0\} \), \( q_i = \{0, 1\} \). When \( X \to \infty \) we get \( Y \to m \equiv \frac{1}{|\vec{r}_1 - \vec{r}_2|} \), and for \( Y \to \infty \) we get \( X \to m \). By fixing the boundary condition \( m \), we fix the value of \( \frac{1}{|\vec{r}_1 - \vec{r}_2|} \) \[2.3\]. This is the only parameter we have. Classically the other parameters may be set to any value by affine transformations of the D3. In the quantum theory these moduli would be quantized and we assume that the S wave will be supersymmetric. We get that there are no essential moduli in this side either.

We show here the projection to the \( X - Y \) plane of this configuration, fig.(2). Note that the D3 brane surface though planar in this projection, is not planar in the ten dimensional sense, since \( X \to \infty \) for \( \vec{r} \to \vec{r}_1 \), whereas \( Y \to \infty \) for \( \vec{r} \to \vec{r}_2 \). Note also that the length scale in the \( X - Y \) plane is inversely proportional to the length scale in the D3 \[2.3\].

Whenever a grid diagram \[3\], which corresponds to a given web has an inner point, there exists a modulus that corresponds to “blowing up a hidden face” \[3\]. The simplest such configuration for the \( SU(3) \) case is represented in fig.(3) (the grid), and fig.(4) (the web). Note that the length parameter \( a \) which appears in the web figure is not coded in the grid. \( a \) is a modulus of the configuration since changing it does not change the mass of the web.

\[Figure 2: \] The basic \( SU(3) \) solution.
We would like to argue that such an internal face cannot exist in the $U(1)$ field theoretical description. The scalars map the $\mathbb{R}^3 - \{\text{points}\}$ worldvolume into $\mathbb{R}^2$, the $X - Y$ plane. By a hidden face we actually mean not only an incontractible loop in the target space, but an incontractible loop in the graph of the map in $(\mathbb{R}^3 - \{\text{points}\}) \times \mathbb{R}^2$. However, since $\pi_1(\mathbb{R}^3 - \{\text{points}\}) = 0$ that would be impossible. It may still be possible, though, to represent this modulus in field theory with higher gauge groups.

Figure 3: The grid diagram of an $SU(3)$ solution. (A) before, and (B) after the blowup.

Figure 4: The web diagram of an $SU(3)$ solution. (A) before, and (B) after the blowup.

3.2 $SU(4)$

We have seen a case where a modulus exists in the web picture, but not in the “weak binding” approximation, and a mild example to the opposite (mild in the sense that the modulus did not affect the $X - Y$ projection of the configuration). To see an example where the weak binding approximation yields an essential modulus overlooked by the web we have to go to $SU(4)$. The simplest $SU(4)$ configuration is given by $(1, 0)$ and $(0, 1)$ strings joining to form a $(1, 1)$ or a $(1, -1)$ string, which then splits again. It is easy to see that there are no web moduli in this case. For the field theory we have to choose the short leg first. We choose it to be the left side $(1, 0)$ leg. The solution is:

$$X = \frac{1}{|\vec{r} - \vec{r}_1|} \quad Y = \frac{1}{|\vec{r} - \vec{r}_2|} - \frac{1}{|\vec{r} - \vec{r}_3|} \quad (3.3)$$

We see that when $Y \to \infty$, $X \to a \equiv \frac{1}{|\vec{r}_1 - \vec{r}_3|}$, when $Y \to -\infty$, $X \to b \equiv \frac{1}{|\vec{r}_1 - \vec{r}_2|}$, and when $X \to \infty Y \to a - b$. When $a - b$ changes sign, there is a transition from a $(1, 1)$ internal leg to a $(1, -1)$ internal leg in the corresponding web. In the grid
diagram this transition is represented by going from the grid of fig.(5A) to that of fig.(5B).

In the D3 the three singular points define a plane, in which \(a\) and \(b\) are two edges of a (possibly singular) triangle. The location and orientation moduli do not change the shape of the \(X−Y\) projection. Since \(a\) and \(b\) are fixed by the boundary conditions there is only one modulus left, the angle between these two edges. It is, nevertheless, one modulus more then in the web picture. To see what is the meaning of this modulus we simply show the projection of the configuration on the \(X−Y\) plane for several values of the angle \(\alpha\) between the two edges, fig.(6). It is clear why this modulus is not visible in the web as our web “has no width”.

![Figure 5: The grid diagram of the SU(4) solution.](image)

![Figure 6: The projection of the field theory solution for different values of \(\alpha\). In A, \(\alpha = 0\), in B, \(\alpha = \frac{\pi}{8}\) and in C, \(\alpha = \pi\).](image)

### 3.3 Analogy with smooth membranes

We will find some analogy between the moduli we find here and moduli of complex curves describing a smooth M2 configuration. Such a description via a smooth M2 appears after a compactification of the theory on a circle of radius \(L\). By doing that, though, we can no longer discuss the four dimensional field theory which we had on the D3. Therefore this analogy is only qualitative. We continue the discussion with this in mind.

Following the analogous case of \((p, q)\) five branes in [3] (based on [31]) we take the coordinates to be

\[
s = \exp((X + ix_t)/L_t)
\]

\[
t = \exp((Y + iy_t)/L_t)
\]

(3.4)

where \(x_t\) and \(y_t\) are the coordinates on the M-torus of length \(L_t\) \((\tau = i)\). The equation

\[
F(s, t) = 0
\]

(3.5)
where $F$ is holomorphic, defines a surface $S$ in the space $M = \mathbb{R}^2 \times T^2$ parameterized by $(X, Y, x_t, y_t)$.

For the $SU(4)$ case above we read from the grid that $F$ should be the sum of four monomials: $1, s, t$ and $st$. We can divide by the coefficient of 1. The coefficients of $s$ and $t$ determine the origin of the axes, and can be scaled to 1 as well. We are left with the curve $F(s, t) = 1 + s + t + Ast$, where $A$ is a complex coefficient. With this choice of coefficients the asymptotic behavior of two out of the four legs is determined to be $(X, Y) \to (0, -\infty), (-\infty, 0)$. Note that this is possible, since in the M-theory picture we do not use the one short leg approximation, but rather we consider all legs to be semi-infinite with no D3's present. We are left with one constrained parameter $a$, which describes the asymptotic values of the other two legs $(X, Y) \to (a, \infty), (\infty, a)$. It is easy to see, by considering the asymptotic behavior of the other two legs that

$$|A| = \exp(-a/L_t)$$

(3.6)

However, the argument of $A$ is not fixed. When, say, $X \to \infty, x_t \to Const$. This constant is represented by the argument of $A$. We see that like the field theory, the M-theory representation has one modulus. Projections of the curve on the $X-Y$ plane are represented in fig.7 for several values of this modulus.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7}
\caption{The projection of the M-theory solution for different values of $\alpha$. In $A$, $\alpha = 0$, in $B$, $\alpha = \frac{\pi}{8}$ and in $C$, $\alpha = \pi$.}
\end{figure}

There are some similarities between the field configuration fig.6 and the M-theory solution fig.7. In both cases the modulus is an angle. Note also that in both cases the center of the configuration expands fast, as figures (6B,7B) correspond to $\alpha = \frac{\pi}{8}$, rather than $\frac{\pi}{2}$.

It is possible to represent in the M-theory language all of the web moduli. It would have been nice if all of the weak binding moduli could be represented as well. The simplest check one can perform is to compare the dimension of the moduli space. We consider now webs which can be represented by the field theory. These webs have no “hidden faces”, and therefore the associated grid diagram has no inner points. The total number of points in this diagram, which is also the number of monomials in $F$, is equal to the number of external legs. The absolute values of all the coefficients
is fixed by the boundary condition. As for the arguments, we can again scale away three of them. The number of moduli is

$$n_{\text{M-moduli}} = n_{\text{Legs}} - 3$$  \hspace{1cm} (3.7)

This number makes no sense for $n_{\text{Legs}} < 3$, where the number of moduli is just zero as there are no monomials to scale.

On the weak binding side each leg, except the short one, is produced by a singular point. Each such point can be located anywhere on the D3, which gives $3(n_{\text{Legs}} - 1)$ parameters. From this number we have to subtract the number of constraints, which is $n_{\text{Legs}} - 2$, since the short leg is not counted, and the location of the last leg is determined by the others. Then there is also the group of affine transformation of the D3 which we should divide by. We are left with

$$n_{\text{Field-moduli}} = 2n_{\text{Legs}} - 7$$  \hspace{1cm} (3.8)

The last expression is valid only for $n_{\text{Legs}} \geq 4$, while for $n_{\text{Legs}} = 2, 3$ we have $n_{\text{Field-moduli}} = 0$. This happens because not all of the 6 affine parameters are relevant. For 2 legs there is one singular point, so the rotations are immaterial, while for 3 legs, there are two points, so one rotation is irrelevant. For $n_{\text{Legs}} = 4$ both equations (3.7,3.8) give us one modulus, as shown above. For $n_{\text{Legs}} > 4$ however, the equations differ. Not only that, but the weak binding side has more moduli then the M-theory side.

This difference can be formally accounted for by recalling that instead of a D3, we have now a M2 which has one fewer dimension. If we repeat the counting that we did for a brane of dimension $p$, we get

$$n_{\text{Field-moduli}} = p(n_{\text{Legs}} - 1) - (n_{\text{Legs}} - 2) - \frac{p(p + 1)}{2}$$  \hspace{1cm} (3.9)

For $p = 2$, we get exactly the same number as in (3.7).

4. Energetics

We have seen that in the weak binding limit a 1/4 BPS monopole is composed of a number of constituents which are arranged spatially such that they are at equilibrium. We shall now compute the restoring potential - the potential for configurations which are close to equilibrium.

The computation is carried out for the case of the simple junction. As in fig. (1) we have a short $(1,1)$ leg of mass $2m$, and two long legs: one electric oriented along the $x$ axis with mass $M_1$, and the other magnetic oriented along the $y$ axis with mass $M_2$. It is a bound state of an electric particle and a magnetic particle at a distance $1/m$. As will be shown later, the relative potential we get at large separations is

$$V(r) = \frac{1}{2\mu M r} \left( \frac{1}{r^2} - 2m \right),$$  \hspace{1cm} (4.1)
where \( r \) is the relative separation, \( 1/\mu_M = 1/M_1 + 1/M_2 \) is the reduced mass and terms subleading in \( m/\mu_M \) were neglected. It satisfies that the equilibrium is at \( r = 1/m \) with binding energy \( E_b = m^2/(2\mu_M) \), as expected (2.1). In addition we find that the frequency of small oscillations is

\[
\omega^2 = \frac{m^4}{\mu^2_M}.
\]

(4.2)

Note that this frequency is of the same order as the binding energy.

We use two different methods to derive (4.1). One uses field theory and the other uses webs. Then we successfully test it against the results of [9], which gives an expression for the complete potential. While our expression for the potential holds only close to equilibrium it has the advantage of being simple in form and derivation.

4.1 Field theory computation

The (static) force between the two particles is determined by their gauge charges and scalar charges according to

\[
F(r) = -\frac{p_1 p_{2i} + q_1 q_{2i}}{r^2} + \frac{\Lambda_{1j} \Lambda_{2j}}{r^2},
\]

(4.3)

where \((p, q)\) are electric and magnetic charges, \(i\) runs over the different gauge fields, \(j\) runs over the different scalars, and \(r = |\vec{r}_1 - \vec{r}_2|\). The gauge charges are conserved and cannot depend on \(r\), while the scalar charges \(\Lambda(r)\) change with \(r\).

In order to find \(\Lambda(r)\) recall that (by definition) we have

\[
X = \frac{\Lambda_{X1}}{|\vec{r} - \vec{r}_1|} + \frac{\Lambda_{X2}}{|\vec{r} - \vec{r}_2|},
\]

(4.4)

\[
Y = \frac{\Lambda_{Y1}}{|\vec{r} - \vec{r}_1|} + \frac{\Lambda_{Y2}}{|\vec{r} - \vec{r}_2|},
\]

(4.5)

At equilibrium \(\Lambda_{X1} = \Lambda_{Y2} = 1\) and \(\Lambda_{X2} = \Lambda_{Y1} = 0\). In order to compute the lowest order contribution to the force equation (4.3) it is enough to assume \(\Lambda_{X1} = \Lambda_{Y2} = 1\) for all \(r\). We get two constraints by looking at the scalar fields near the singularities at \(r_1, r_2\) and requiring that they pass through the D3 branes. Near \(r_1\)

\[
M_1 + m = X \simeq \frac{1}{|\vec{r} - \vec{r}_1|} + \frac{\Lambda_{X2}}{r} \quad (4.6)
\]

\[
m = Y \simeq \frac{\Lambda_{Y1}}{|\vec{r} - \vec{r}_1|} + \frac{1}{r} \quad (4.7)
\]

from which we can solve for \(\Lambda_{Y1}\). A similar argument near \(r_2\) solves for \(\Lambda_{X2}\)

\[
\Lambda_{X2} = (m - \frac{1}{r})/M_2 \quad (4.8)
\]

\[
\Lambda_{Y1} = (m - \frac{1}{r})/M_1 \quad (4.9)
\]

Now we substitute back in the force equation (4.3) (there are no gauge forces) and then integrate and find the potential to be exactly (4.1).
4.2 Web computation

It is interesting to note that a simple computation within the web model can reproduce the result (4.1) as well. We know that for separation $|\vec{r}_2 - \vec{r}_1| = 1/m$ the junction is at equilibrium in coordinates $(X, Y) = (m, m)$. We shall calculate the potential of a the radial mode, that is the potential of the configurations for which the junction is at $(X, Y) = (1/r, 1/r)$.

Let us compute the mass of the web by summing the masses of all strings and neglecting any interactions between them. After subtracting the masses of the two particles we find a potential

$$V(r) = \sqrt{(M_1 + \delta)^2 + \delta^2 + (M_2 + \delta)^2 + \delta^2 + \frac{2}{r} - V_\infty} \quad (4.10)$$

where $\delta = m - 1/r$, $V_\infty = \sqrt{(M_1 + m)^2 + m^2 + (M_2 + m)^2 + m^2}$. This coincides with (4.1) to second order in $\delta$.

We can calculate also the frequency of another mode of oscillations, that is, $\delta X = -\delta Y$ rather then $\delta X = \delta Y$. In this case, however, $\omega^2_{\delta X = -\delta Y} \propto \frac{\mu M}{m} \omega^2_{\delta X = \delta Y}$ which is a much higher frequency then the binding energy (2.1), (4.2). So at the quantum level there would be no fluctuations in this direction.

4.3 Another test

Let us test our expression (4.1) against the result of [9]. We will make a test which relies only on the functional form of their result, so we will not need to compare the various constants which they use, the sole exception to this is the identification of our parameter $\mu M$ with their parameter $\mu$, since both are supposed to represent the reduced mass.

The functional form is

$$V_{BLLY}(r) = A^2 f(r) + B^2 / f(r) - V_\infty$$

$$f(r) = 1 + \frac{1}{2\mu r}, \quad V_\infty = A^2 + B^2 \quad (4.11)$$

By comparing the location of the minimum $r_{\text{min}} = 1/m$ and the binding energy eq.(2.1) we determine $A, B$

$$A^2 = 2\mu \quad (4.12)$$
$$B^2 = 2\mu (1 + \frac{m}{2\mu})^2 \quad (4.13)$$

With this identification of the constants, and neglecting higher order terms of the small parameters $\frac{m}{\mu}$ and $mr - 1$, the two potentials coincide.
5. Zero modes

In this section we find some of the zero modes of the solutions. Both bosonic zero modes (BZMs) and fermionic zero modes (FZMs) have a geometric interpretation (though it is much more transparent in the BZM case). The FZMs are relevant to the multiplet structure and spin of the solutions. This link is carried out by quantizing the FZM and BZM into a quantum mechanics on moduli space, and looking for the degeneracy and spin of the ground states. Such an analysis would allow us to test the existence of “accidental long representations” and other predictions in [6, 8]. Here we will take some steps towards finding these zero modes. We shall concentrate on counting the FZMs, but before that we shall discuss some BZMs for completeness.

5.1 A comment on BI action and BPS states

Monopoles are usually described by a field theory. In order to compare the field theoretical results to the brane picture one needs to consider the BI action, as was done in [21]. Note that in general, the effective action for several D3’s is a non-Abelian Born-Infeld action [28, 29, 30].

By taking the scaling limit, as we do here, the nonlinearities can be neglected, and the theory becomes SYM. A different limit is to consider all legs in the web except for one to be infinite, in which case the non-Abelian part may be neglected, and one gets the S-BI action (equations (85)-(88) of [25])

\[
S = - \int \sqrt{-\det (\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu \phi_\alpha \partial_\nu \phi^\alpha - 2\bar{\lambda}(\Gamma_\mu + \Gamma_\alpha \partial_\mu \phi^\alpha) \partial_\nu \lambda + \lambda \Gamma^m \partial_\mu \lambda \lambda \Gamma_m \partial_\nu \lambda)}
\]

where \( \alpha = 4..9, \) and \( m = 0..9^2 \). In this work we took both limits, thereby getting the S-Maxwell action which we used throughout this paper.

Moreover, as far as BPS states are concerned, there is no need to consider the BI action anyway. In [20, 27] it was shown that some BPS solutions satisfy both the Maxwell and the BI equations of motion. This is a general property. In addition to \( \vec{E}, \vec{B} \) one can consider in the BI theory the fields \( \vec{D}, \vec{H} \), which are defined by

\[
\vec{D} = \frac{\partial L}{\partial \vec{E}} \quad \quad \vec{H} = - \frac{\partial L}{\partial \vec{B}}
\]

Discarding the fermions in eq. (5.1), eq.(3.1,3.2) together with

\[
\vec{D} = \vec{E} \quad \quad \vec{H} = \vec{H}
\]

Our conventions are the same as those of [24]. We do not decompose the ten dimensional spinors to four dimensional ones. \( \{\Gamma^m, \Gamma^n\} = 2\eta^{mn}, \) where \( \eta = (- + ... +) \). Both type IIB spinors have positive chirality \( \langle Q = \Gamma^{11}Q \rangle \). The gauge is such that \( r_\mu = \phi_\mu \) for \( \mu = 0..3 \) leaving the usual six scalars. In the following we shall denote \( X = \phi_4 \) and \( Y = \phi_5 \). The other four scalars would not be excited.
can be shown to be BPS solutions, and so the BPS states of Maxwell and BI theory
indeed coincide. Evaluation of the Lagrangian (5.1) at the BPS states gives

\[ L_{BPS} = -(1 + B^2) \]  

while the energy density is

\[ H_{BPS} = 1 + E^2 + B^2 \]  

Note that the energy density simplifies to a sum of three terms - the tension of the
D3, the electric and the magnetic energies.

On the other hand, to study non-BPS states, or linear waves on a given BPS
background, one has to use the nonlinearities. The quadratic Maxwell action looks
the same evaluated at any background, thus suggesting that all configurations have
the same zero modes, once the locations of the singular points are prescribed.

By using the S-BI action we would see different properties of the D3 theory than
the ones grasped by the SYM action. Although the conclusions of this section would
not necessarily be relevant to the description of the FZMs of the SYM monopoles,
they would be relevant to the description of the D3.

We must remember that as we are using approximations to the full theory, such
as S-Maxwell or BI, a FZM of the approximate theory will be a good approximation
only away from the soliton, and may diverge in its vicinity even if the full FZM does
not. In addition these theories may contain spurious solutions as well.

### 5.2 Bosonic zero modes

Before we turn to the more complicated task of finding general FZMs, we discuss
briefly some BZMs of the F-string solution, namely BZMs associated with transverse
scalars - scalars which are not excited in the background. In this case the equations
(3.1, 3.2) defining the background reduce to

\[ X = \frac{1}{|\vec{r} - \vec{r}_0|} \quad \vec{E} = \vec{\nabla} X \]  

We exclusively examine the BZMs of the transverse scalars. In [20] the linearized
equation of the radial mode of a transverse scalar field was found to be

\[-(1 + \frac{1}{r^4})\partial_t^2 \phi + r^{-2} \partial_r (r^2 \partial_r \phi) = 0\]  

(5.7)

to get the radial zero mode equation one has to drop the time dependence out of this
equation. One gets the radial part of the Laplace equation, and it can be checked that
the angular dependence of the BZM equation is restored by using the full Laplacian

\[ \nabla^2 \phi = 0 \]  

(5.8)
One would like to find a normalizable solution to this equation, but there are none. Solutions of the Laplace equation are characterized by their angular momentum. Each value of angular momentum \( l \) has two types of solutions. One is proportional to \( r^l \), and the other to \( r^{-l-1} \). For \( l = 0 \) we have the constant solution which represents the motion of the brane as a whole in the \( \phi \) direction (a VEV for the \( \phi \) field), and the \( \phi = \frac{1}{r} \) solution with the geometric interpretation of a rotation in the \( X - \phi \) plane.

We would be interested in the localized modes \( (r^{-l-1}) \), which represent a zero mode of the soliton rather than a zero mode of the D3 brane. However, even the localized \( l = 0 \) mode is too singular. It is not a “length normalizable” mode - a mode which has a chance to become a normalizable zero mode in the full theory. This term will be defined later. The modes with higher \( l \) are even more singular.

### 5.3 Fermionic zero modes

For any soliton FZMs can be found by operating on them with broken supersymmetries

\[
Q_b(\text{Soliton}) = \text{FZM} \tag{5.9}
\]

We can produce FZMs from BZMs by acting on them with preserved supersymmetries

\[
Q_p(\text{BZM}) = \text{FZM} \tag{5.10}
\]

and vice versa

\[
Q_p(\text{FZM}) = \text{BZM} \tag{5.11}
\]

There is, however, no guarantee that all the FZMs will be found in any of these ways. In the case where more FZMs are present there are “accidental” large BPS representations, as in the case of the planar \( SU(4) \) web [6].

The general FZMs can be found by solving the FZM equation, that is, by solving the linearized equation of motion of the fermions with time derivatives set to zero. In order to get this linearized equation we have to expand the Lagrangian up to the second order with respect to the fermions around the solution. Neglecting the last term in the square root of eq.(5.1), we notice that it depends on the fermions only through the expression

\[
c_{\mu\nu} = \bar{\lambda} (\Gamma_\mu + \Gamma_\alpha \partial_\mu \phi^\alpha) \partial_\nu \lambda \tag{5.12}
\]

The linearized Lagrangian is therefore

\[
L_l = \frac{\partial L}{\partial c_{\mu\nu}} \bigg|_{\text{background}} c_{\mu\nu} = L_s + L_t \tag{5.13}
\]

where \( L_s \) is the part of the Lagrangian which contains the spatial dependence, and thus is relevant for finding the FZMs, and \( L_t \) is the part with the time dependence.
A direct calculation shows that (recall our conventions from footnote 2)

\[ L_s = \bar{\lambda} (\bar{\Gamma} + (\bar{E} + \bar{B} \times \bar{E}) \Gamma^4 - \Gamma^0) + \bar{B} \Gamma^5 + \bar{B} \times \bar{\Gamma} \cdot \bar{\nabla} \lambda \]  
\[ L_t = \bar{\lambda} (\bar{\Gamma} \cdot (\bar{E} + \bar{E} \times \bar{B}) + E^2 \Gamma^4 + \bar{E} \cdot \bar{B} \Gamma^5 - (1 + E^2 + B^2) \Gamma^0) \dot{\lambda} \]  

(5.14)  
(5.15)

We shall use \( L_t \) when we would discuss the normalizability of the FZMs.

The FZM equation is just the spatial part of the equation of motion derived from \( L_t \), that is, it is the equation one would get by variation of \( L_s \) (recall eq.(3.1,3.2))

\[ (\bar{\Gamma} + (\bar{E} + \bar{B} \times \bar{E}) (\Gamma^4 - \Gamma^0) + \bar{B} \Gamma^5 + \bar{B} \times \bar{\Gamma}) \cdot \bar{\nabla} \lambda = 0 \]  

(5.16)

### 5.3.1 FZM of the F-string

We start with the F-string solution eq.(5.6), after which the more complicated configurations will be dealt. It was shown in [32] that this solution is supersymmetric. It breaks eight out of the sixteen supersymmetries which are present in the D3 world volume theory. The preserved/broken supersymmetries of this solution were found to coincide with those of a F-string, as it should be. The broken ones are given by

\[ Q_b = -\Gamma^{04} Q_b \]  

(5.17)

We shall first find the eight FZMs generated by broken supersymmetries, and then we shall solve the FZM equation. We recognize these FZMs by using (5.9) on the F-string background (5.6).

\[ \lambda = \bar{E} \cdot \bar{\Gamma} \epsilon = \frac{(r - r_0)_i}{|r - r_0|^3} \Gamma^i \epsilon \]  

(5.18)

where \( \epsilon \) is a constant spinor obeying eq.(5.17). The standard S-Maxwell SUSY variation gives the same result in this case as the S-BI one ³.

We want to check whether the solution we found is normalizable. For that we first remind briefly the way in which zero modes should be dealt (see [33] for details). The zero modes should be elevated to the status of collective coordinates by giving them time dependence. In the action we should set

\[ \lambda \to \sum_{a=1}^{n} \lambda_a(\vec{r}) b_a(t) \]  

(5.19)

where \( \lambda_a(\vec{r}) \) is the \( a \)th FZM, \( b_a(t) \) are the new collective coordinates, and \( n \) counts all the zero modes (\( n = 8 \) here). Solutions of the FZM equation (5.16) nullify \( L_s \), the spatial part of the action eq.(5.14). From eq.(5.13) we see that what remains is a quantum mechanics of the collective coordinates

\[ S = \int b_a^\dagger M_{ab} \dot{b}_b dt \]  

(5.20)

³The S-BI variation is given here and in what follows by eq.(84) of [25] after \( \zeta^{(3)} \) is calculated in our background. Note that the number of supersymmetries in these equations is twice what we have. This is so because it contains also the supersymmetries which are broken by the D3.
where the mass matrix $M$ is defined by

$$M_{ab} = \int \tilde{L}_t[\tilde{\lambda}_a, \lambda_b]d^3r$$

where $\tilde{L}_t$ stands for $L_t$ with $\dot{\lambda} \rightarrow \lambda$. We call a set of solutions normalizable if all the entries of the mass matrix are finite.

In the F-string case the mass matrix is given by

$$M_{ab} = \int \tilde{\epsilon}_a(\tilde{\Gamma} \cdot \tilde{E})(\Gamma^0 - \tilde{\Gamma} \cdot \tilde{E})(\tilde{\Gamma} \cdot \tilde{E})\epsilon_b d^3r$$

where the $\tilde{B}$ dependent terms were discarded, and the $E^2(\Gamma^4 - \Gamma^0)$ term drops since $\epsilon$ obeys eq.(5.17). This can be simplified to

$$M_{ab} = \int (\epsilon^\dagger_a \epsilon_b - \bar{\epsilon}_a \Gamma^i \epsilon_b E^i_E)E^2 d^3r$$

The second term in this expression vanishes. To show that we use eq.(5.17) again.

$$\bar{\epsilon}_a \Gamma^i \epsilon_b = \bar{\epsilon}_a \Gamma^{04} \Gamma^i \epsilon_b = \bar{\epsilon}_a \Gamma^i \Gamma^{04} \epsilon_b = -\bar{\epsilon}_a \Gamma^i \epsilon_b$$

We are left now with

$$M_{ab} = \epsilon^\dagger_a \epsilon_b \int E^2 d^3r = \epsilon^\dagger_a \epsilon_b \int \frac{2\pi r^2 dr}{r^4}$$

which diverges. However, this divergence can be understood when we change coordinates from $r$ on the D3 to $X$ on the string by

$$X = \frac{1}{r}$$

The divergent integral is proportional to

$$\int dX$$

This is a constant (smooth) density, and the divergence comes only from the infinite length of the string. The same divergence in fact is present in the energy integral of our background [20]. In the full theory, where we expect BPS solutions that represent finite strings, modes similar to this one should be present, and these modes would be normalizable. We shall call modes with this degree of divergence “length-normalizable” (LN for short). We shall discard modes with higher degree of divergence.

We found the FZMs which originate from the broken SUSY. We now turn to solve the FZM equation (5.16). In the F-string background it reduces to

$$(\tilde{\Gamma} \cdot \tilde{\nabla} - (\Gamma_4 - \Gamma_0)\frac{1}{r^2} \partial_r)\lambda = 0$$
where we have set \( r_0 = 0 \) for simplicity. Note that had we used the Maxwell theory to obtain an FZM equation the last term would be absent.

To find the solutions we define

\[
P = \frac{1}{2}(1 + \Gamma_0 \Gamma_4) \quad P' = \frac{1}{2}(1 - \Gamma_0 \Gamma_4)
\]

and decompose \( \lambda \)

\[
\lambda = \lambda_1 + \lambda_2 \quad \lambda_1 = P \lambda \quad \lambda_2 = P' \lambda
\]

The equation (5.28) becomes

\[
\vec{\Gamma} \cdot \vec{\nabla} \lambda_1 = 0 \quad \vec{\Gamma} \cdot \vec{\nabla} \lambda_2 = \frac{2}{r^2} \partial_r \Gamma_0 \lambda_1
\]

In the case \( \lambda_1 = 0 \) we get the equation

\[
\vec{\Gamma} \cdot \vec{\nabla} \lambda_2 = 0
\]

Squaring the differential operator shows that \( \lambda_2 \) should be a solution of the Laplace equation. We are interested in localized solutions with singularity at the origin (at \( \vec{r}_0 \)). We can write

\[
\lambda_2 = \sum Y_{m}^{l}(r) \lambda_{m}^{l}
\]

with \( \lambda_{m}^{l} \) constant spinors, and check which conditions should these spinors obey. It is easy to check that \( l = 0 \) has no solution. For \( l = 1 \) the solutions are exactly what we have found above (5.18).

For the case \( \lambda_1 \neq 0 \) we will get for \( \lambda_1 \) the same equation that we have got for \( \lambda_2 \), namely the three (spatial) dimensional free Dirac equation. The only solution that we will consider is \( l = 1 \), for which \( \lambda \propto \frac{1}{r} \). But now we have to take this solution and substitute it in the equation of \( \lambda_2 \). A solution of this equation, if exist at all, would have to behave as \( \frac{1}{r^4} \), which is too singular. Thus we conclude that the SUSY generated FZMs are the only LN FZMs in this case.

### 5.3.2 FZM of the D-string

For the D-string solution equations (3.1,3.2) reduce to

\[
Y = \frac{1}{|\vec{r} - \vec{r}_1|} \quad \vec{B} = \vec{\nabla} Y
\]

Now the S-Maxwell and S-BI SUSY variations give different expressions. The S-Maxwell gives a result very similar to the electric one

\[
\lambda = \vec{B} \cdot \vec{\Gamma} \epsilon = \frac{(r - r_1)i}{|\vec{r} - \vec{r}_1|^3} \Gamma^i \epsilon
\]
where $\epsilon$ is a constant spinor obeying the equation of a broken SUSY in the magnetic case

$$\epsilon = -\Gamma^{1235} \epsilon$$

(5.36)

The S-BI SUSY variation on the other hand gives

$$\lambda = \frac{\vec{B} \cdot \vec{\Gamma} + B^2 \Gamma^{123}}{1 + B^2} \epsilon$$

(5.37)

At a neighborhood of $\vec{r}_1$ this solution has a finite limit. Far from the core the BI mode reduces to the Maxwell one.

This finite behavior, and the fact that the most singular term in $L_t$ goes like $B^2$ implies that these solutions are LN. A direct calculation shows that they are non-normalizable. In fact, the mass matrix here is identical in form to that of the F-string

$$M_{ab} = \epsilon^a \epsilon_b \int B^2 d^3r$$

(5.38)

Note also, that while the expression we got by using the S-Maxwell SUSY variation (5.35) does not solve the S-BI FZM equation (5.16), the same expression is a solution after reversing the “chirality”, $\epsilon = \Gamma^{1235} \epsilon$. However, these solutions are non-LN. In the planar case we shall meet similar solutions which would be LN.

5.3.3 FZM of a planar configuration

In the planar case both electric and magnetic charges are present. SUSY is preserved by supercharges obeying

$$\frac{1}{2}(1 + \Gamma^{04}) Q_{pp} = Q_{pp} \quad \frac{1}{2}(1 + \Gamma^{1235}) Q_{pp} = Q_{pp}$$

(5.39)

There are three sectors of broken SUSY which we label by $Q_{pb}$, $Q_{bp}$, $Q_{bb}$, according to the sector which breaks SUSY ($[\Gamma^{04}, \Gamma^{1235}] = 0$). For example $Q_{bp}$ breaks the electric and preserve the magnetic SUSY, that is

$$\frac{1}{2}(1 - \Gamma^{04}) Q_{bp} = Q_{bp} \quad \frac{1}{2}(1 + \Gamma^{1235}) Q_{bp} = Q_{bp}$$

(5.40)

Using eq.(3.2) and the formulas of [25] (recall footnote (3)) we get for these three sectors

$$\lambda_{pb} = \frac{\Gamma^0 (\vec{\Gamma} \cdot \vec{B} + B^2 \Gamma^{123})}{1 + B^2} \epsilon_{pb}$$

(5.41)

$$\lambda_{bp} = \frac{\vec{E} \cdot ((\vec{\Gamma} \times \vec{B} - \vec{\Gamma}) \Gamma^{123} + \vec{B})}{1 + B^2} \epsilon_{bp}$$

(5.42)

$$\lambda_{bb} = \frac{\Gamma^0 (\vec{\Gamma} \cdot \vec{B} + B^2 \Gamma^{123}) + \vec{E} \cdot ((\vec{\Gamma} \times \vec{B} + \vec{\Gamma}) \Gamma^{123} - \vec{B})}{1 + B^2} \epsilon_{bb}$$

(5.43)
The $\epsilon$’s are constant spinors. Each sector has four independent spinors, which amounts to twelve FZMs. Note that $\lambda_{\alpha\beta}$ does not have the same eigenvalues with respect to $\Gamma^{04}$, $\Gamma^{1235}$ as $\epsilon_{\alpha\beta}$. For example $\Gamma^{04}\epsilon_{pb} = \epsilon_{pb}$, but $\Gamma^{04}\lambda_{pb} = -\lambda_{pb}$, and it is not an eigenvector of $\Gamma^{1235}$ at all.

One can verify that these modes are indeed solutions of the FZM equation (5.16). To check that they are LN we note that at the vicinity of magnetic singularities, or more generally any $(p, q)$ charge for $q \neq 0$, these solutions approach a constant. These singularities will not cause a problem. We have to check the behavior of the modes in the vicinity of the purely electric singularities. The four modes (5.41) are independent of $E$ and so not only the other eight modes (5.42, 5.43) are potentially problematic. Computing their mass matrix and using eq.(5.17) as in eq.(5.24), one can see that these modes are LN.

These are not all the LN solutions of the FZM equation (5.16). Consider

$$\frac{(r - r_a)_i}{|r - r_a|^3} \Gamma^i \epsilon_{bp}$$

(5.44)

where $r_a$ is any singular point with no magnetic charge, we see that it is a LN solution. The number of these solutions is $4n_E$ where $n_E$ is the number of electric singular points. However, since these modes exist only for configurations with this kind of singular points, they are probably artifacts of our approximation, since they are not symmetric with respect to electric-magnetic duality.

Next we note that (5.42) is linear with respect to $E$. Replacing $E_i$ in (5.42) by $(r - r_a)_{i} |r - r_a|^3$, where now $r_a$ is any singular point, with or without magnetic charge, we get another LN solution to the FZM equation (5.16). A web with $n_X$ external legs will have together with the modes of (5.41, 5.43) $4n_X$ FZMs. This exactly coincides with

$$n_{FZM} = 8F + 4n_X$$

(5.45)

of [3] for the number of web FZMs, for the case $F = 0$ of no internal faces. As we mentioned in section [3] we can not describe configurations with internal faces in the effective action language. In particular we see that in the $SU(4)$ case there are indeed 16 FZMs, in agreement with [3].

We did not show that the solutions we found are all the LN solutions. However, we considered an ansatz, similar in form to the solutions we found $\lambda = \frac{\text{numerator}}{1 + B^2} \epsilon$, where the numerator is a sum of terms at most linear with respect to $(r - r_a)_{i} |r - r_a|^3$ for any singular point times factors at most quadratic with respect to the magnetic field.

For the $SU(3)$ and $SU(4)$ cases the solutions we have found are the only ones of this form.

4Recall that each such ‘mode’ actually represents four modes, and note that by replacing $\vec{E}$ by $\vec{B}$, (5.42) is reduced to (5.41).
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References

[1] O. Aharony and A. Hanany, “Branes, superpotentials and superconformal fixed points,” Nucl. Phys. B504, 239 (1997) [hep-th/9704170].

[2] M. R. Gaberdiel and B. Zwiebach, “Exceptional groups from open strings,” Nucl. Phys. B518, 151 (1998) [hep-th/9709013].

[3] O. Aharony, A. Hanany and B. Kol, “Webs of (p,q) 5-branes, five dimensional field theories and grid diagrams,” JHEP 9801, 002 (1998) [hep-th/9710116].

[4] A. Sen, “String network,” JHEP 9803, 005 (1998) [hep-th/9711130].

[5] O. Bergman, “Three-pronged strings and 1/4 BPS states in N = 4 super-Yang-Mills theory,” Nucl. Phys. B525, 104 (1998) [hep-th/9712211].

[6] O. Bergman and B. Kol, “String webs and 1/4 BPS monopoles,” Nucl. Phys. B536, 149 (1998) [hep-th/9804160].

[7] K. Lee and P. Yi, “Dyons in N = 4 supersymmetric theories and three-pronged strings,” Phys. Rev. D58, 066005 (1998) [hep-th/9804174].

[8] B. Kol, “Thermal monopoles,” [hep-th/9812021].

[9] D. Bak, C. Lee, K. Lee and P. Yi, “Low energy dynamics for 1/4 BPS dyons,” Phys. Rev. D61, 025001 (2000) [hep-th/9906119].

[10] B. Kol and J. Rahmfeld, “BPS spectrum of 5 dimensional field theories, (p,q) webs and curve counting,” JHEP 9808, 006 (1998) [hep-th/9801067].

[11] B. Kol, “5d field theories and M theory,” JHEP 9911, 026 (1999) [hep-th/9705031].

[12] D. Bak, K. Lee and P. Yi, “Complete supersymmetric quantum mechanics of magnetic monopoles in N = 4 SYM theory,” [hep-th/9912083].

[13] J. P. Gauntlett, N. Kim, J. Park and P. Yi, “Monopole dynamics and BPS dyons N = 2 super-Yang-Mills theories,” [hep-th/9912082].
[14] K. Lee and P. Yi, “Quantum spectrum of instanton solitons in five dimensional non-commutative U(N) theories,” [hep-th/9911180].

[15] D. Bak, K. Lee and P. Yi, “Quantum 1/4 BPS dyons,” Phys. Rev. D61, 045003 (2000) [hep-th/9907090].

[16] D. Tong, “A note on 1/4-BPS states,” Phys. Lett. B460, 295 (1999) [hep-th/9902003].

[17] K. Lee, “Massless monopoles and multi-pronged strings,” Phys. Lett. B458, 53 (1999) [hep-th/9903095].

[18] D. Bak, K. Hashimoto, B. Lee, H. Min and N. Sasakura, “Moduli space dimensions of multi-pronged strings,” Phys. Rev. D60, 046005 (1999) [hep-th/9901107].

[19] D. Diacosencu, “D-branes, Monopoles and Nahm Equations”, Nucl. Phys. B503, 220 (1997) [hep-th/9608163].

[20] C. Callan and J. Maldacena, “Brane Dynamics From the Born-Infeld Action”, Nucl. Phys. B513, 198 (1998) [hep-th/9708147].

[21] N.J. Hitchin, “Lectures given at the ICTP School on Differential Geometry April 1999,” math/9907034. N.J. Hitchin, “The moduli space of complex Lagrangian submanifolds,” math/9901069. N.J. Hitchin, “The moduli space of special Lagrangian submanifolds,” dg-ga/9711002.

[22] D. Joyce, “On counting special Lagrangian homology 3-spheres,” [hep-th/9907013].

[23] R.C. McLean, ”Deformations and moduli of calibrated submanifolds,” PhD thesis, Duke University, 1990.

[24] E. Scheidegger, [hep-th/9912188]. S. Kachru, S. Katz, A. Lawrence and J. McGreevy, [hep-th/9912151]. R. Hernandez, [hep-th/9912022]. S. Gukov, [hep-th/9911011]. A. D. Shapere and C. Vafa, [hep-th/9910182]. D. Diaconescu and C. Romelsberger, [hep-th/9910172]. M. R. Douglas, [hep-th/9910171]. S. Kachru and J. McGreevy, Phys. Rev. D61, 026001 (2000) [hep-th/9908135].

[25] M. Aganagic, C. Popescu and J. H. Schwarz, “Gauge-Invariant and Gauge-Fixed D-Brane Actions”, Nucl. Phys. B495, 99-126 (1997) [hep-th/9612080].

[26] J. Gutowski and G. Papadopoulos, “The dynamics of D-3-brane dyons and toric hyper-Kaehler manifolds,” Nucl. Phys. B551, 650 (1999) [hep-th/9811207].

[27] J.P. Gauntlett, C. Koehl, D. Mateos, P.K. Townsend and M. Zamaklar, “Finite energy Dirac-Born-Infeld monopoles and string junctions”, Phys. Rev. D60, 045004 (1999) [hep-th/9903156].

[28] A. A. Tseytlin, “On Non-Abelian Generalisation of Born-Infeld Action in String Theory”, Nucl. Phys. B501, 41-52 (1997) [hep-th/9701123].
[29] D. Brecher, “BPS States of the Non-Abelian Born-Infeld Action”, Phys. Lett. B442, 117-124 (1998) hep-th/9804180.

[30] R. C. Myers, “Dielectric-Branes”, JHEP 9912, 022 (1999) hep-th/9910053.

[31] E. Witten, “Solutions of Four-Dimensional Field Theories via M Theory”, Nucl. Phys. B500, 3-42 (1997) hep-th/9703166.

[32] S. Lee, A. Peet and L. Thorlacius, “Brane-Waves and Strings”, Nucl. Phys. B514, 161-176 (1998) hep-th/9710097.

[33] R. Rajaraman, “Solutions of Four-Dimensional Field Theories via M Theory”, North Holland Publishing Company (1982).

[34] A. Kumar, JHEP 9912, 001 (1999) hep-th/9911090. A. Hashimoto and K. Hashimoto, JHEP 9911, 005 (1999) hep-th/9909202. Y. Yamada and S. Yang, hep-th/9907134. N. D. Lambert and D. Tong, Phys. Lett. B462, 89 (1999) hep-th/9907014. P. Ramadevi, hep-th/9906247. T. Muto, hep-th/9905230. A. Gorsky and K. Selivanov, hep-th/9904041. O. DeWolfe, A. Hanany, A. Iqbal and E. Katz, JHEP 9903, 006 (1999) hep-th/9902179. A. Hashimoto, JHEP 9901, 018 (1999) hep-th/9812159. O. DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, hep-th/9812028. N. Sasakura and S. Sugimoto, Prog. Theor. Phys. 101, 749 (1999) hep-th/9811087. R. Argurio, hep-th/9807171. D. Diaconescu and R. Entin, Nucl. Phys. B538, 451 (1999) hep-th/9807170. A. Kumar and S. Mukhopadhyay, hep-th/9806126. C. Ahn, K. Oh and R. Tatar, JHEP 9811, 024 (1998) hep-th/9806041. K. Okuyama and Y. Sugawara, JHEP 9808, 002 (1998) hep-th/9806001. T. Hauer, Nucl. Phys. B538, 117 (1999) hep-th/9805076. O. DeWolfe and B. Zwiebach, Nucl. Phys. B541, 509 (1999) hep-th/9804210. K. Hashimoto, H. Hata and N. Sasakura, Nucl. Phys. B535, 83 (1998) hep-th/9804164. A. Sen, JHEP 9806, 007 (1998) hep-th/9803194. A. Armoni and A. Brandhuber, Phys. Lett. B438, 261 (1998) hep-th/9803186. A. Mikhailov, N. Nekrasov and S. Sethi, Nucl. Phys. B531, 345 (1998) hep-th/9803142. C. S. Chu, P. S. Howe, E. Sezgin and P. C. West, Phys. Lett. B429, 273 (1998) hep-th/9803041. J. E. G. Gimon and M. Premnath, Phys. Lett. B433, 318 (1998) hep-th/9803033. S. Rey and J. Yee, hep-th/9803001. A. Giveon and D. Kutasov, Rev. Mod. Phys. 71, 983 (1999) hep-th/9802067. M. R. Gaberdiel, T. Hauer and B. Zwiebach, Nucl. Phys. B525, 117 (1998) hep-th/9801205. S. Bhattacharyya, A. Kumar and S. Mukhopadhyay, Phys. Rev. Lett. 81, 754 (1998) hep-th/9801141. A. Hanany and A. Zaffaroni, JHEP 9805, 001 (1998) hep-th/9804134. I. Kishimoto and N. Sasakura, Phys. Lett. B432, 305 (1998) hep-th/9712181. C. Ahn, Phys. Lett. B426, 306 (1998) hep-th/9712149. Y. Matsuo and K. Okuyama, Phys. Lett. B426, 294 (1998) hep-th/9712070. C. Ahn, K. Oh and R. Tatar, J. Geom. Phys. 28, 163 (1998) hep-th/9712003. B. Kol, JHEP 9911, 017 (1999) hep-th/9711017. N. C. Leung and C. Vafa, Adv. Theor. Math. Phys. 2, 91 (1998) hep-th/9711013. A. Gorsky, S. Gukov and A. Mironov, Nucl. Phys. B518, 689 (1998) hep-th/9710239.
[35] J. P. Gauntlett, N. Kim, J. Park and P. Yi, hep-th/9912082. A. Kumar, JHEP 9912, 001 (1999) [hep-th/9911090]. O. J. Ganor, hep-th/9910236. D. Bak and K. Lee, Phys. Lett. B468, 76 (1999) [hep-th/9909035]. S. Ferrara and A. Zaffaroni, hep-th/9908163. T. Ioannidou and P. M. Sutcliffe, Phys. Lett. B467, 54 (1999) [hep-th/9907157]. P. Claus, M. Gunaydin, R. Kallosh, J. Rahmfeld and Y. Zunger, JHEP 9905, 019 (1999) [hep-th/9905112]. O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, hep-th/9905111. A. Sen, hep-th/9904207. G. W. Semenoff and K. Zarembo, Nucl. Phys. B556, 247 (1999) [hep-th/9903140]. K. Lee, Phys. Lett. B458, 53 (1999) [hep-th/9903095]. D. Minic, hep-th/9903079. J. H. Schwarz, hep-th/9812037. O. Bergman, JHEP 9905, 004 (1999) [hep-th/9811064]. M. Gunaydin, D. Minic and M. Zagermann, Nucl. Phys. B544, 737 (1999) [hep-th/9810226]. K. Hashimoto, Prog. Theor. Phys. 101, 1353 (1999) [hep-th/9808185]. J. H. Schwarz, Phys. Rept. 315, 107 (1999) [hep-th/9807132]. A. Iqbal, JHEP 9910, 032 (1999) [hep-th/9807117]. M. Gunaydin, D. Minic and M. Zagermann, Nucl. Phys. B534, 96 (1998) [hep-th/9806043]. O. Bergman and A. Fayyazuddin, Nucl. Phys. B535, 139 (1998) [hep-th/9806011]. O. DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, Nucl. Phys. B534, 261 (1998) [hep-th/9805220]. A. Sen, JHEP 9808, 010 (1998) [hep-th/9805019].

[36] K. Hashimoto, H. Hata and N. Sasakura, Phys. Lett. B431, 303 (1998) [hep-th/9803127]. T. Kawano and K. Okuyama, Phys. Lett. B432, 338 (1998) [hep-th/9804139]. J. H. Schwarz, Nucl. Phys. Proc. Suppl. 55B, 1 (1997) [hep-th/9607201]. K. Dasgupta and S. Mukhi, Phys. Lett. B423, 261 (1998) [hep-th/9711094]. J. P. Gauntlett, J. Gomis and P. K. Townsend, JHEP 9901, 003 (1998) [hep-th/9711205]. G. W. Gibbons, Nucl. Phys. B514, 603 (1998) [hep-th/9709027]. A. Hashimoto, Phys. Rev. D57, 6441 (1998) [hep-th/9711097].