FAST-GROWING SERIES ARE TRANSCENDENTAL

ROBERT J. MACG. DAWSON AND GRANT MOLNAR

Abstract. Let $R$ be a subring of $\mathbb{C}[[z]]$, and let $X \in \mathbb{C}[[z]]$. The Newton-Puiseux Theorem implies that if the coefficients of $X$ grow sufficiently rapidly relative to the coefficients of the series in $R$, then $X$ is transcendental over $R$. We prove an alternative proof of this result by establishing a relationship between the coefficients of $A(X)$ and $A'(X)$, where $A(T)$ is a polynomial over $\mathbb{C}[[z]]$.

1. Introduction

For $C(z) = C$ a power series in $\mathbb{C}[[z]]$, we write $(C)_n$ for the $n$th coefficient of $C$, or $C_n$ if no confusion arises. Throughout this paper, indices of series and sequences are always nonnegative unless otherwise noted.

Suppose we are given a ring $R \subseteq \mathbb{C}[[z]]$ and a power series $X \in \mathbb{C}[[z]]$. It is natural to ask what relationship $R$ bears to $X$. For instance, we may ask whether $X$ is algebraic or transcendental over $R$. (Recall that $X$ is algebraic over $R$ if there is a nonzero polynomial $F(T) \in R[T]$ such that $F(X) = 0$, and $X$ is transcendental over $R$ otherwise.)

Transcendental elements are useful building blocks in the theory of commutative rings. Indeed, $X$ is transcendental over $R$ precisely if the ring $R[X]$ satisfies the following universal property: for every $R$-algebra $S$, and every element $x \in S$, there is a unique $R$-algebra morphism $f : R[X] \to S$ such that $f(X) = x$. But if $X$ is not transcendental over $R$, our options for $f(X)$ are sharply curtailed. For instance, if $X \not\in R$ but $X^2 \in R$, then an $R$-algebra morphism $f : R[X] \to S$ must map $X$ to a square root of $X^2$ in $S$; any other series algebraic over $R$ is subject to similar restrictions.

However, transcendental elements are slippery. Indeed, many decades elapsed between Euler’s formulation of the concept for real numbers and Liouville’s 1844 construction of a transcendental number [5]; Hermite did not show $e$ to be transcendental until 1873 [3]. (For an historical overview of the history of transcendental numbers, see the first chapter of Baker [12] or Section 22.2.3 of Suzuki [7].)

The following example follows Liouville’s construction of a decimal expansion that must represent a transcendental number.

Example 1.1. Let $R = \mathbb{C}[z]$. Let $L(z) = \sum_{n\geq 0} z^{2^n}$; we note that the set of nonzero coefficients of $L^{p-1}$ is a proper subset of the set of nonzero coefficients of $L^p$. For $p < q$, define $c(p, q) := 2^q(1 - 2^{-p})$; then we have $(L^p)_{c(p, q)} = q!$, $(L^j)_{c(p, q)} = 0$ for all $j < p$, and $(L^j)_n = 0$ for all $j \leq p$ and all $n$ with $0 < c(p, q) - n < 2^{q-p}$.

Suppose now that $A(T) := \sum_{j \leq m} A_j T^j \in \mathbb{C}[z][T]$ is a degree $m$ polynomial. We select $n$ with $(A_m)_n \neq 0$, and let $d := \max \{ \deg(A_j) : j \leq m \}$, where we adopt the convention $\deg(0) := -\infty$. Then for any $q$ large enough that $2^{q-p} > d$, we have

$$(A(L))_{c(p, q)+n} = (L^m)_{c(p, q)} \cdot (A_m)_n \neq 0.$$

Thus $L$ is transcendental over $\mathbb{C}[z]$.

It is possible that the technique embodied in Example 1.1 can be extended to construct a series $X$ transcendental over (for instance) the ring of absolutely convergent series. But the
rate of growth of the coefficients of $X$, rather than of the gaps between them, may also be incompatible with the existence of a nonzero polynomial that takes $X$ to 0.

Recall that a series of the form $C(z) := \sum_{n \gg -\infty} C_n z^{n/d}$ is a Puiseux series, and $\mathbb{C}((z^*)) := \bigcup_{d \geq 1} \mathbb{C}((z^{1/d}))$ is the field of Puiseux series. If for some $r > 0$, $|C_n| = O(r^{n/d})$ as $n \to \infty$, we say $C$ exhibits exponential growth. Otherwise, $C$ exhibits superexponential growth.

**Remark 1.2.** A power series $C(z) \in \mathbb{C}[[z]]$ exhibits exponential growth precisely if $C(z)$ converges for $z$ in a neighborhood of the origin. In other words, $C(z)$ exhibits exponential growth precisely if $C(z)$ defines an analytic function in a neighborhood of the origin.

**Theorem 1.3 (Newton-Puiseux Theorem, [6]).** The field $\mathbb{C}((z^*))$ is algebraically closed. Moreover, if a nonzero polynomial $A(T) \in \mathbb{C}((z^*))[T]$ has coefficients exhibiting exponential growth, then the roots of $A(T)$ exhibit exponential growth.

The following corollary is a straightforward application of the Newton-Puiseux Theorem (see [4]).

**Corollary 1.4.** If $X \in \mathbb{C}[[z]]$ exhibits superexponential growth, then $X$ is transcendental over the ring of series exhibiting exponential growth.

**Example 1.5.** The power series $\sum_{n \geq 0} n! z^n \in \mathbb{C}[[z]]$ is irrational over the ring of series exhibiting exponential growth. It is a fortiori irrational over the ring of Abel-summable series, the ring generated by the convergent series, and the ring of absolutely convergent series, since these are all subrings of the ring of series exhibiting exponential growth.

**Remark 1.6.** The proof of the Newton-Puiseux Theorem applies without modification if we replace $\mathbb{C}$ with any characteristic 0 algebraically closed field equipped with an absolute value $|\cdot|$. It may also be adapted to prove that fast-growing series are transcendental over other subrings of $\mathbb{C}[[z]]$ whose series exhibit modest growth.

The object of this note is to establish by purely elementary means that if every series in $R$ has coefficients exhibiting at most modest growth, then any series $X \in \mathbb{C}[[z]]$ with sufficiently fast-growing coefficients is transcendental over $R$. We attain this result by means of an elementary algebraic identity (Theorem 2.2), rather than by invoking the machinery of Puiseux series. We state our main theorem (Theorem 3.3) here.

**Theorem.** Fix $R$ a subring of $\mathbb{C}[[z]]$, and suppose we have a monotone increasing function $\rho : \mathbb{N} \to \mathbb{R}_{>0}$ such that for every $C \in R$, we have $C_n = O(\rho(n))$ as $n \to \infty$.

Let $X \in \mathbb{C}[[z]]$; suppose that $|X_0| \geq 1$, and that for every fixed $\lambda$ and $m$, the power series $X$ satisfies the following conditions as $n \to \infty$:

$$\rho(n) \left( \sum_{\ell \leq \frac{n}{4}} |X_\ell| \right)^m = o(|X_n - \lambda|);$$

$$\rho(n) |X_{n-\lambda-1}| \left( \sum_{\ell < \frac{n}{4}} |X_\ell| \right)^m = o(|X_n - \lambda|).$$

Then $X$ is transcendental over $R$. 
In particular, Theorem 3.3 applies to the ring of power series exhibiting exponential growth, for instance when we take $\rho(n) := n!$ (see Example 3.5 below). Although the content of Theorem 3.3 is implied by Corollary 1.4, at least when $R$ is the ring of series exhibiting exponential growth, we find merit in our approach and hope that Theorem 2.2 may find other applications in the future.

**Remark 1.7.** Our work applies verbatim to power series over any characteristic 0 field $K$ equipped with an absolutely value $|\cdot|$.

The idea behind the proof of Theorem 3.3 is fairly straightforward: we suppose we have a polynomial $A(T) := \sum_{j \leq m} A_j T^j \in R[T]$ such that $A(X) = 0$, and deduce that $A(T) = 0$. To do so, we make a careful examination of $A(X)^n$ for $n$ large; if the coefficients of $X$ grow sufficiently rapidly, then the behavior of $X^n$ will dominate $A(X)^n$ unless the coefficient of $X^n$ is zero. But as $A(X) = 0$, the coefficient of $X^n$ must be zero, and so the behavior of $X^n$ will dominate $A(X)^n$ unless this coefficient is zero; proceeding inductively, for any $\ell$ small, we conclude the coefficient of $X^{n-\ell}$ is zero. It turns out that for $\ell$ small relative to $n$, the coefficient of $X^{n-\ell}$ in $A(X)^n$ is independent of $n$: in fact, the coefficient of $X^{n-\ell}$ is precisely $A'(X)_{\ell}$, where $A'(X) := \sum_{j \leq m} j A_j X^{j-1}$ is the formal derivative of $A(T)$ evaluated at $X$ (see Theorem 2.2 below). Consequently, if $A(X) = 0$ we would expect to have $A'(X) = 0$, and this is indeed the case; now an easy bit of algebra tells us that $A(T) = 0$ as desired. In the remainder of this note, we formalize the intuition outlined above.

**ACKNOWLEDGMENTS**

We thank Gary Walsh and John Voight for their helpful observations.

## 2. Algebraic Techniques

In this section, we give a relationship between the coefficients of $A(X)$ and $A'(X)$ which emphasizes the high-index coefficients of $X$.

**Lemma 2.1.** Fix a power series $X \in \mathbb{C}[[z]]$. For any $m$, we have
\[
X^m = X^{[m]} + mX^{(m)},
\]
where the power series $X^{[m]}$ and $X^{(m)}$ are defined by
\[
(X^{[m]})_n := \sum_{\sum k_j = n \atop k_1, \ldots, k_m \leq \frac{n}{\ell}} X_{k_1} \cdots X_{k_m},
\]
and
\[
(X^{(m)})_n := \sum_{\ell < \frac{n}{\ell}} (X^m)_{\ell} X_{n-\ell}.
\]

**Proof.** By definition, we have
\[
(X^m)_n = \sum_{\sum k_j = n \atop k_1, \ldots, k_m \leq \frac{n}{\ell}} X_{k_1} \cdots X_{k_m}
\]
\[
= \sum_{\sum k_j = n \atop k_1, \ldots, k_m \leq \frac{n}{\ell}} X_{k_1} \cdots X_{k_m} + \sum_{j=1}^m \sum_{\sum k_j = n \atop k_j > \frac{n}{\ell}} X_{k_1} \cdots X_{k_m}.
\]
If \( j = 1 \) and \( \ell := n - k_1 \), then \( \ell < n/2 \). By symmetry, the last sum is independent of \( j \), so
\[
(X^m)_n = (X^{[m]})_n + m \sum_{\ell < n/2} X_{n-\ell} \sum_{k_2 + \ldots + k_m = \ell} X_{k_2} \ldots X_{k_m}
\]
\[
= (X^{[m]})_n + m \sum_{\ell < n/2} X_{n-\ell}(X^{m-1})_{\ell},
\]
from which (1) follows. \( \square \)

Informally, Lemma 2.1 partitions the summands of \((X^m)_n\) into a central “core” in which every index is less than or equal to \( n/2 \), and \( m \) sets in which each summand has a (necessarily single) factor \( X_k z^k \) with \( k > n/2 \). We illustrate this with the summands of \((X^3)_4\), the coefficient of \( z^4 \) in \( X^3 = (X_0 + X_1 z + X_2 z^2 + \ldots)^3 \), arranged as

\[
X_0X_1X_3 \quad X_0X_0X_4 \quad X_1X_0X_3 \\
X_0X_2X_2 \quad X_1X_1X_2 \quad X_2X_0X_2 \\
X_0X_4X_0 \quad X_1X_3X_0 \quad X_2X_2X_0 \quad X_3X_1X_0 \quad X_4X_0X_0.
\]

The terms in any of the three boldfaced triangles sum to \((X^{(3)})_4\), while \((X^{[3]})_4\) is the sum over the central inverted triangle.

These definitions extend to full power series. For instance, we have
\[
X^{[3]}(z) = X_0 + 3X_0^2X_1 z + 3X_0X_1^2 z^2 + X_1^3 z^3 + (3X_0X_2 + 3X_1^2X_2)z^4 + \ldots
\]
and
\[
X^{(3)}(z) = X_0^2X_2 z^2 + (X_0^2X_3 + 2X_0X_1X_2)z^3 + (X_0^2X_4 + 2X_0X_1X_3)z^4 + \ldots.
\]

But observe that for \( m = 0, 1, 2 \) and \( n \) arbitrary, the set
\[
\{ (k_1, \ldots, k_m) \in \mathbb{Z}^m : k_1 + \ldots + k_m = n \text{ and } 0 \leq k_1, \ldots, k_m \leq n/2 \}
\]
may be empty or singleton, so there are a few irregularities for low powers: we have \( X^{[0]} = 1 \), \( X^{[1]} = X_0 \), and \( X^{[2]} = 1 + X_1^2 z^2 + X_2^2 z^4 + \ldots \).

Let
\[
\delta(A, X) := \sum_{|j| \leq m} A_j X^{[j]},
\]
and write \( \delta_n(A, X) := \delta(A, X)_n \). We may think of \( \delta(A, X) \) as a “pseudopolynomial” version of \( A(X) \) using only the cores of the powers \( X^m \). Let also
\[
\epsilon_n(A, X) := \sum_{j \leq m} \sum_{k+p+q = m \atop q < p \leq m/2} j(A_j)_k (X^{j-1})_q X_p
\]
and, for \( \lambda < n/2 \), let
\[
\gamma_{n,\lambda}(A, X) := \sum_{\lambda \leq \ell < \frac{n}{2}} A'(X)_\ell X_{n-\ell}.
\]

While innocuous on its face, Lemma 2.1 is instrumental in the proof of the following algebraic identity.

**Theorem 2.2.** Fix a power series \( X \in \mathbb{C}[[z]] \) and a polynomial \( A(T) \in \mathbb{C}[[z]][T] \). For any \( n \) and any \( \lambda < n/2 \), we have
\[
(2) \quad A(X)_n = \sum_{\ell < \lambda} A'(X)_\ell X_{n-\ell} + \gamma_{n,\lambda}(A, X) + \delta_n(A, X) + \epsilon_n(A, X),
\]
Proof. Write $A(T) = \sum_{j \leq m} A_j T^j$, where $m$ is the degree of $A$. By definition,

\begin{equation}
A(X)_n = \sum_{j \leq m} (A_j X^j)_n = \sum_{j \leq m} \sum_{k \leq n} (A_j)_k (X^j)_{n-k}.
\end{equation}

Substituting (1) into (3), and recalling the definition of $X^{[j]}$, we obtain

\[
A(X)_n = \sum_{j \leq m} \sum_{k \leq n} (A_j)_k ((X^{[j]})_{n-k} + j(X^{[j]})_{n-k})
\]

\[
= \sum_{j \leq m} \left( (A_j X^{[j]})_n + j \sum_{k \leq n} (A_j)_k \sum_{q < \frac{n-k}{2}} (X^{j-1})_q X_{n-k-q} \right)
\]

\[
= \delta_n(A, X) + \sum_{j \leq m} j \sum_{k \leq n} (A_j)_k \sum_{q < \frac{n-k}{2}} (X^{j-1})_q X_{n-k-q}
\]

Let $p := n - k - q$, so $q < (n - k)/2$ if and only if $q < p$. Thus

\[
A(X)_n - \delta_n(A, X) = \sum_{k+p+q=n, q<p} \left( \sum_{j \leq m} j A_j \right)_k (X^{j-1})_q X_p
\]

We break this sum into two cases, depending on whether $p \leq n/2$ or $p > n/2$.

In the first case, we get

\[
\sum_{k+p+q=n} j (A_j)_k (X^{j-1})_q X_p = \epsilon_n(A, X).
\]

In the second case, $k$ and $q$ have no restriction other than that $k + p + q = n$, and

\[
\sum_{k+p+q=n, \frac{n}{2} < p \leq n} \left( \sum_{j \leq m} j A_j \right)_k (X^{j-1})_q X_p = \sum_{\frac{n}{2} < p \leq n} \left( \sum_{j \leq m} j A_j X^{j-1} \right)_{n-p} X_p
\]

\[
= \sum_{\frac{n}{2} < p \leq n} A'(X)_{n-p} X_p.
\]

Setting $\ell := n - p$, (2) follows immediately. \qed

Figure 1 may make what we have just done clearer. The coefficient $A(X)_n$ is the sum of a number of terms of the form $(A_j)_k X_{\ell_1} X_{\ell_2} \cdots X_{\ell_j}$, where $k + \ell_1 + \cdots + \ell_j = n$. If none of the indices $\ell_i$ exceeds $(n-k)/2$, the summand is part of the core. Otherwise, let $p$ be some maximal $\ell_i$, and let $q$ the sum of the others. For fixed $n$, the triplets $(k, p, q)$ may be considered as barycentric coordinates.

Triplets with $p \leq q$ (shown in white) are the summands of the core, $\delta_n(A, X)$. (Due to the definition of $p$, we cannot have $p = 0$ and $q > 0$ simultaneously; these positions on the graph are left empty.) Coordinates with $q < p \leq n/2$ (shown in light grey) are summands of $\epsilon_n(A, X)$. Finally, coordinates with $p > n/2$ correspond to the region in which the coefficient of $X_p$ in $A(X)_n$ is $A'(X)_{n-p}$. We subdivide this into the summands of $\gamma_{n,\lambda}(X, A)$ (dark grey) and the set of triplets (shown in black) for which $l < \lambda$. Unlike the others, this last set of triplets does not become more numerous with increasing $n$. We shall see that if the coefficients of $X$ grow fast enough, each of $\gamma_{n,\lambda}(X, A), \delta_n(X, A)$, and $\epsilon_n(X, A)$ is insignificant compared to the terms with $l < \lambda$. 
3. CRITERIA AND CONSTRUCTIONS FOR TRANSCENDENTAL POWER SERIES

In this section, we prove our main theorem and furnish some related results.

Lemma 3.1. Fix a power series \( X \in \mathbb{C}[[z]] \) and a polynomial \( A(T) \in \mathbb{C}[[z]][T] \), and suppose that \( A(X) = 0 \). Suppose moreover that for each \( \lambda \), we have, as \( n \to \infty \):

\[
|\gamma_{n,\lambda}(A, X)| = o(|X_{n-\lambda}|); \tag{4}
|\delta_n(A, X)| = o(|X_{n-\lambda}|); \tag{5}
|\epsilon_n(A, X)| = o(|X_{n-\lambda}|). \tag{6}
\]

Then \( A'(X) = 0 \).

**Proof.** Let \( n \geq 0 \) be arbitrary. By Theorem 2.2 we have

\[
A(X)_n = \sum_{\ell<\lambda} A'(X)_{\ell} X_{n-\ell} + \gamma_{n,\lambda}(A, X) + \delta_n(A, X) + \epsilon_n(A, X),
\]

We claim \( A'(X)_\ell = 0 \) for each \( \ell \). If not, then let \( \lambda \) be minimal such that \( A'(X)_\lambda \neq 0 \). Then

\[
A(X)_n = A'(X)_\lambda X_{n-\lambda} + \gamma_{n,\lambda}(A, X) + \delta_n(A, X) + \epsilon_n(A, X),
\]

By conditions (4), (5) and (6) we observe

\[
|\gamma_{n,\lambda}(A, X)|, |\delta_n(A, X)|, |\epsilon_n(A, X)| < \frac{|A'(X)_\lambda X_{n-\lambda}|}{3}
\]

for \( n \) sufficiently large. Thus

\[
0 = |A(X)_n| \geq |A'(X)_\lambda X_{n-\lambda}| - |\delta_n(A, X)| - |\epsilon_n(A, X)| - |\gamma_{n,\lambda}(A, X)| > 0,
\]

and we have obtained a contradiction. Then \( A'(X)_\ell = 0 \) for all \( \ell \), and thus \( A'(X) = 0 \) as desired. \( \square \)

Lemma 3.2. Let \( R \) be a subring of \( \mathbb{C}[[z]] \) and let \( X \in \mathbb{C}[[z]] \). Suppose that conditions (4) through (6) of Lemma 3.1 hold for all series polynomials \( A(T) \in R[T] \) such that \( A(X) = 0 \). Then \( X \) is transcendental over \( \mathbb{C} \).

**Proof.** Suppose \( A(T) \in R[T] \) is chosen with \( A(X) = 0 \). Repeated applications of Lemma 3.1 show that every derivative of \( A(X) \) vanishes. Then \( A(T) = A(X) = 0 \). Thus \( X \) is transcendental over \( R \) as desired. \( \square \)

At this point, we are ready to prove our main theorem.
Theorem 3.3. Fix $R$ a subring of $\mathbb{C}[[z]]$, and suppose we have a monotone increasing function $\rho : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that for every $C \in R$, we have $|C| = O(\rho(n))$ as $n \to \infty$.

Suppose that $|X_0| \geq 1$, and that for every fixed $\lambda$ and $m$, the power series $X$ satisfies the following conditions as $n \to \infty$:

$$\rho(n) \left( \sum_{\ell \leq \frac{n}{\lambda}} |X_\ell| \right)^m = o(|X_{n-\lambda}|);$$

$$\rho(n) |X_{n-\lambda-1}| \left( \sum_{\ell < \frac{n}{\lambda}} |X_\ell| \right)^m = o(|X_{n-\lambda}|).$$

Then $X$ is transcendental over $R$.

Proof. Note that the last condition of Theorem 3.3 gives us $|X_n| = o(|X_{n+1}|)$ as $n \to \infty$, and a fortiori $|X_n|$ is eventually increasing. We will prove that conditions 4, 5, and 6 of Lemma 3.1 hold for every polynomial $A(T) \in R[T]$ with $A(X) = 0$, so Lemma 3.2 will give us our desired result.

Fix a polynomial $A(T) = \sum_{j \leq m} A_j T^j \in R[T]$ for which $A(X) = 0$; and fix $\lambda \geq 0$. We compute (as $n \to \infty$):

$$|\gamma_{n,\lambda}(A, X)| \leq \sum_{\lambda < \ell \leq \frac{n}{\lambda}} \sum_{j \leq m} \sum_{k \leq \ell} |j(A_j)_k (X^{j-1})_{\ell-k} X_{n-\ell}|$$

$$= O \left( \rho(n) \sum_{\lambda < \ell < \frac{n}{\lambda}} \sum_{k \leq \ell} \left| (X^{m-1})_{\ell-k} X_{n-\ell} \right| \right)$$

$$= O \left( n \rho(n) |X_{n-\lambda-1}| \sum_{k_1, \ldots, k_{m-1} < \frac{n}{\lambda}} |X_{k_1} \cdots X_{k_{m-1}}| \right)$$

$$= O \left( \rho(n) |X_{n-\lambda-1}| \left( \sum_{\ell < \frac{n}{\lambda}} |X_\ell| \right)^m \right)$$

$$= o(|X_{n-\lambda}|),$$

where the second-to-last asymptotic holds because $n = O \left( \sum_{\ell < \frac{n}{\lambda}} |X_\ell| \right)$. Thus condition 4 holds.

Next we compute

$$|\delta_n(A, X)| \leq \sum_{j \leq m} \sum_{k \leq n} |(A_j)_k (X^{[j]}_{n-k})|$$

$$= O \left( \rho(n) \sum_{k \leq n} \sum_{k_1, \ldots, k_m = n-k \atop k_1, \ldots, k_m \leq \frac{n}{\lambda}} |X_{k_1} \cdots X_{k_m}| \right)$$

$$= O \left( n \rho(n) \left( \sum_{\ell < \frac{n}{\lambda}} |X_\ell| \right)^m \right)$$
\[ = O \left( \rho(n) \left( \sum_{\ell \leq \frac{n}{2}} |X_\ell| \right)^{m+1} \right) \]
\[ = o(|X_{n-\lambda}|), \]
where the second-to-last asymptotic holds because \( n = O \left( \sum_{\ell \leq \frac{n}{2}} |X_\ell| \right) \). Thus condition 5 holds.

Finally, we compute
\[
|\epsilon_n(A, X)| \leq \sum_{j \leq m} \sum_{k+p+q=n, q < p \leq \frac{n}{2}} |j(A_j)k (X^{j-1})_q X_p| \\
= O \left( n\rho(n) \left| X_{\left| \frac{n}{2} \right|} \right| \sum_{k+q < n, q < \frac{n}{2}} \left| (X^{m-1})_{q} \right| \right) \\
= O \left( n^2 \rho(n) \left| X_{\left| \frac{n}{2} \right|} \right| \left( \sum_{q < \frac{n}{2}} |X_q| \right)^{m-1} \right) \\
= O \left( \rho(n) \left( \sum_{\ell \leq \frac{n}{2}} |X_\ell| \right)^{m+2} \right) \\
= o(|X_{n-\lambda}|),
\]
where the second-to-last asymptotic holds because \( n, \sum_{q < \frac{n}{2}} |X_q| \), and \( |X_{\left| \frac{n}{2} \right|}| \) are each \( O \left( \sum_{\ell < \frac{n}{2}} |X_\ell| \right) \). Thus condition 6 holds, and \( X \) is transcendental over \( R \). \( \square \)

**Corollary 3.4.** Assume the notation of Theorem 3.3. Suppose that \( |X_0| \geq 1 \), and that for every fixed \( \lambda \) and \( m \), the power series \( X \) satisfies the following condition as \( n \to \infty \):
\[
\rho(n) |X_{n-\lambda-1}| \left( \sum_{\ell \leq \frac{n}{2}} |X_\ell| \right)^{m} = o(|X_{n-\lambda}|).
\]
Then \( X \) is transcendental over \( R \).

**Example 3.5.** Let \( R \subseteq \mathbb{C}[[z]] \) comprise the power series which exhibit exponential growth. Then we may take \( \rho(n) = n! \), since \( n! \) exhibits superexponential growth. Let \( X = \sum_{n \geq 0} 2^{n!} z^n \), so \( X_n = 2^{n!} \). Then \( |X_0| = 2 > 1 \). Moreover, for every fixed \( \lambda \) and \( m \), we have (as \( n \to \infty \)):
\[
n! \left( \sum_{\ell \leq \frac{n}{2}} 2^\ell \right)^{m} \leq n! \left( \sum_{\ell \leq \left\lfloor \frac{n}{2} \right\rfloor} 2^\ell \right)^{m} = O \left( 2^\left\lfloor \frac{n+4}{2} \right\rfloor! \right) = o \left( 2^{(n-\lambda)!} \right),
\]
so \( X \) is transcendental over \( R \) by Corollary 3.4. The series \( X \) is a fortiori transcendental over the ring of Abel-summable series, the ring generated by the convergent series, and the ring of absolutely convergent series, since these are all subrings of \( R \).
References

[1] Baker, A. Transcendental Number Theory, Cambridge, C.U.P., 1975
[2] Cantor, G. Über eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen, Jour. Reine Angew. Math. 77 (1874), 258-62
[3] Hermite, C., Sur la fonction exponentielle, Comptes Rendus 77 (1874), 18-24, 74-9, 226-33, 285-93; Oeuvres III, 150-81
[4] Leason, Todd (https://math.stackexchange.com/users/173354/todd leason), Roots of polynomials those coefficients are analytic functions, Mathematics Stack Exchange.
[5] Liouville, J., Sur des classes très-étendues de quantités dont la valeur n’est ni algébrique, ni même reductible à des irrationnelles algébriques, Comtes Rendus 18 (1844), 883-5, 910-11; J. Math. pures appl. 16 (1851), 133-42.
[6] Krzysztof Jan Nowak, Some elementary proofs of Puiseux’s theorems, Univ. Iagel. Acta Math. (2000), no. 38, 279–282. MR 1812118
[7] Suzuki, J, A History of Mathematics, Prentice-Hall, Upper Saddle River, (2002)
[8] Weierstrass, K. Mathematische Werke. II. Abhandlungen 2, Georg Olms Verlagsbuchhandlung, Hildesheim; Johnson Reprint Corp., New York (1967), 135-142

Email address: rdawson@cs.smu.ca

DEPT. OF MATHEMATICS AND COMPUTING SCIENCE, SAINT MARY’S UNIVERSITY, HALIFAX, NS, CANADA, B3H 3C3

Email address: Grant.S.Molnar.GR@dartmouth.edu

MATHEMATICS DEPARTMENT, DARTMOUTH COLLEGE, HANOVER, NH, USA, 245 KEMENY HALL