Gravito-magnetic gyroscope precession in Palatini $f(R)$ gravity

Matteo Luca Ruggiero
Dipartimento di Fisica, Politecnico di Torino, Corso Duca degli Abruzzi 23, Torino, Italy
INFN, Sezione di Torino, Via Pietro Giuria 1, Torino, Italy
(Dated: March 18, 2009)

We study gravito-magnetic effects in the Palatini formalism of $f(R)$ gravity. On using the Kerr-de Sitter metric, which is a solution of $f(R)$ field equations, we calculate the impact of $f(R)$ gravity on the gravito-magnetic precession of an orbiting gyroscope. We show that, even though an $f(R)$ contribution is present in principle, its magnitude is negligibly small and far to be detectable in the present (like GP-B) and foreseeable space missions or observational tests around the Earth.

I. INTRODUCTION

Among the theories that have been proposed to explain the present acceleration of the Universe [1, 2, 3] without requiring the existence of a dark energy, $f(R)$ theories of gravity received much attention in recent years. In these theories the gravitational Lagrangian depends on an arbitrary function $f$ of the scalar curvature $R$; they are also referred to as “extended theories of gravity”, since they naturally generalize, on a geometric ground, General Relativity (GR): namely, when $f(R) = R$ the action reduces to the usual Einstein-Hilbert action, and Einstein’s theory is obtained. Extended theories of gravity can be studied in different (non equivalent) formalisms (see Capozziello and Francaviglia [4] and references therein); in order to obtain the field equations in the metric formalism the action is varied with respect to the metric tensor only; in the Palatini formalism the action is varied with respect to the metric and the affine connection, which are supposed to be independent from one another. Actually, $f(R)$ provide cosmologically viable models, where both the inflation phase and the accelerated expansion are reproduced (see Nojiri and Odintsov [5] and references therein) and, furthermore, they have been used to reproduce the rotation curves of galaxies without need for dark matter [6, 7]. It is worthwhile noticing that, because of the excellent agreement of GR with Solar System and binary pulsar observations, every modified theory of gravity should have the correct Newtonian and post-Newtonian limits, in order to agree with GR tests (see e.g. Will [8]), and this is an important issue for Newtonian limits, in order to agree with GR tests (see e.g. Ciufolini and Pavlis [13], Iorio [14] and references therein); in April 2004 Gravity Probe B was launched to accurately measure the frame dragging (and the geodetic precession) of an orbiting gyroscope: the final results are going to be published [15].

Here, working on an exact solution of the field equations in the Palatini formalism (GM effects and other Post-Newtonian effects were obtained in metric $f(R)$ gravity by Clifton [16]), we want to evaluate the impact of $f(R)$ theories on GM effects, in order to see if there are corrections to the GR predictions that can be detected (at least in principle) by GP-B or other foreseeable experiments around the Earth.

II. VACUUM FIELD EQUATIONS OF PALATINI $f(R)$ GRAVITY

The equations of motion of $f(R)$ extended theories of gravity can be obtained starting from the action:

$$A = A_{\text{grav}} + A_{\text{mat}} = \int \left[ \sqrt{g} f(R) + 2 \chi L_{\text{mat}}(\psi, \nabla \psi) \right] \, d^4x.$$  \hfill (1)

The gravitational part of the Lagrangian is represented by a function $f(R)$ of the scalar curvature $R$. The total Lagrangian contains also a first order matter part $L_{\text{mat}}$ functionally depending on matter fields $\Psi$, together with their first derivatives, equipped with a gravitational coupling constant $\chi = \frac{8\pi G}{c^4}$. In the Palatini formalism the metric $g$ and the affine connection $\Gamma$ are supposed to be independent, so that the scalar curvature $R$ has to be intended as $R \equiv R(g, \Gamma) = g^{\alpha\beta} R_{\alpha\beta}(\Gamma)$, where $R_{\alpha\beta}(\Gamma)$ is the Ricci-like tensor of the connection $\Gamma$.

By independent variations with respect to the metric $g$ and the connection $\Gamma$, we obtain the following equations of motion:

$$f'(R) R_{\mu\nu}(\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = \chi T_{\mu\nu},$$  \hfill (2)

$$\nabla_{\alpha} \left[ \sqrt{g} f'(R) g^{\mu\nu} \right] = 0,$$  \hfill (3)

where $f'(R) = df(R)/dR$, $T_{\mu\nu}$ is the matter source stress-energy tensor and $\nabla^I$ means covariant derivative with...
respect to the connection $\Gamma$. The equation of motion (2) can be supplemented by the scalar-valued equation obtained by taking the contraction of (2) with the metric tensor:

$$f'(R)R - 2f(R) = \chi T,$$

(4)

where $T$ is the trace of the energy-momentum tensor. Equation (4) is an algebraic equation for the scalar curvature $R$, it is called the structural equation and it controls the solutions of equation (2).

We are interested into solutions of the field equation in vacuum, in particular outside a rotating source of matter: so the field equations become

$$f'(R)R_{\mu\nu}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = 0,$$

(5)

$$\nabla^\Gamma_{\alpha}[\sqrt{f'}(R)g^{\mu\nu}] = 0,$$

(6)

and they are, again, supplemented by the scalar equation

$$f'(R)R - 2f(R) = 0.$$  

(7)

The trace equation (7) is an algebraic equation for $R$ which admits constant solutions $R = c_i$. Then, it is possible to show that, under suitable conditions (see Ferraris et al. [17], Allemandi et al. [18]), the field equations (5,6) reduce to

$$R_{\mu\nu}(g) = kg_{\mu\nu},$$

(8)

1. The space-time metric has signature $(-1,1,1,1)$, we use geometrized units such that $G = c = 1$, greek indices run from 0 to 3, and latin ones run from 1 to 3, boldface letters like $x$

Consequently, the vacuum solutions of GR with a cosmological constant can be used in Palatini $f(R)$ gravity: the role of the $f(R)$ function is determining the solutions of the structural equation (7). It is useful to point out that, for a given $f(R)$ function, in vacuum case the solutions of the field equations of Palatini $f(R)$ gravity are a subset of the solutions of the field equations of metric $f(R)$ gravity (see e.g. Magnano [19]). So every solution of eqs. (8) is also a solution of the field equations of metric $f(R)$ gravity with constant scalar curvature $R$.

The Kerr-de Sitter metric, which is an exact solution of the field equations in the form (8), describes a rotating black-hole in a space-time with a cosmological constant [20, 21, 22] and can be used to investigate GM effects in extended theories of gravity.

The Kerr-de Sitter metric in the standard Boyer-Lindquist coordinates $x^\mu = (t, r, \theta, \phi)$ has the form (1)

$$ds^2 = -\left[1 - \frac{2Mr}{\Sigma} - \frac{k}{3}(r^2 + a^2 \sin^2 \theta)\right]dt^2 - 2a\left[\frac{2Mr}{\Sigma} + \frac{k}{3}(r^2 + a^2)\right] \sin^2 \theta dtd\phi$$

$$+ \frac{\Sigma}{\Delta}dr^2 + \frac{\Sigma}{\chi}d\theta^2 + \left[\frac{2Mr}{\Sigma}a^2 \sin^2 \theta + (1 + \frac{k}{3}a^2)(r^2 + a^2)\right] \sin^2 \theta d\phi^2,$$

(9)

where

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \chi = 1 + \frac{k}{3}a^2 \cos^2 \theta,$$

(10)

$$\Delta = r^2 - 2Mr + a^2 - \frac{k}{3}r^2(r^2 + a^2).$$

(11)

The mass of the source is $M$, while $J = Ma$ is its angular momentum (which is perpendicular to the $\theta = \pi/2$ plane). When $k = 0$ the Kerr-de Sitter metric (9) reduces to the Kerr metric. Other limiting cases can be checked: for instance, when $a = 0$, we obtain the Schwarzschild-de Sitter solution, and when $M = a = 0$ we have the de Sitter space-time.

It is interesting to consider the “weak-field” approximation of the metric (9); in other words, we expand it up to linear terms in $M/r, Ma/r^2, kr^2, kar, kMr$. What we get is
We notice that on using Boyer-Lindquist coordinates the weak field metric (12) does not contain terms in the form $kMr$. However, these coordinates are not directly related to physical lengths. We can express the metric (12) in the more familiar isotropic coordinates, which is necessary to properly deal with gravito-magnetic effects [12]. To this end we introduce a new radial coordinate $\rho$.

\[
\begin{align*}
\rho &= r \left(1 - \frac{M}{r} - \frac{kr^2}{12}\right), \quad \text{(13)}
\end{align*}
\]

and, up to the required approximation order, the metric turns out to be:

\[
\begin{align*}
\frac{\mathrm{d}s^2}{\rho^2} &= - \left(1 - \frac{2M}{\rho} - \frac{k\rho^2}{3}\right) \mathrm{d}t^2 + \left(1 + \frac{2M}{\rho} - \frac{k\rho^2}{6}\right) \left(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2 + \rho^2 \sin^2 \theta \mathrm{d}\phi^2\right) + 2a \left(\frac{2M}{\rho} + \frac{k\rho^2}{3} + \frac{5}{6} Mk\rho\right) \sin^2 \theta \mathrm{d}\phi \mathrm{d}t. \quad \text{(14)}
\end{align*}
\]

On using these coordinates we see that terms in the form $kMr$ are present: as a consequence, their absence in the metric (14) was due to the use of Boyer-Lindquist coordinates.

By inspection of the metric (14) we see that for $a = 0$, we obtain the weak field limit of the Schwarzschild-de Sitter solution, for $k = 0$ we obtain the weak field limit of the Kerr solution.

\section{III. Gravito-Magnetic Field Effects in Palatini $f(R)$ Gravity}

According to what we have seen, the terms containing $k$ in the metric (9) are due to the non-linearity of the gravity Lagrangian in the framework of Palatini $f(R)$ gravity, or to the presence of a cosmological constant in the framework of GR. Some effects of these terms have been already investigated in the literature. For instance Kerr et al. [20], Sereno and Jetzer [23] showed that, due to the cosmological constant, the mean motion for circular geodesics is modified according to

\[
\begin{align*}
\omega_k &= \frac{M}{\rho^3} - \frac{k}{3}, \quad \text{(15)}
\end{align*}
\]

and that the precession of pericenter (in the $\theta = \pi/2$ plane) gains an additional contribution:

\[
\begin{align*}
\Delta \phi_k &= \frac{\pi kA^3}{M} \sqrt{1 - e^2}, \quad \text{(16)}
\end{align*}
\]

where $A$ is the semi-major axis of the (unperturbed) orbit, and $e$ is its eccentricity. The effects of the cosmological constant on gravitational lensing was recently studied by Rindler and Ishak [24], Sereno [25] and by Ruggiero [26] in connection with Palatini $f(R)$ gravity. Actually, the effects (15,16) of the $k$-term (whose expected order of magnitude is comparable to the cosmological constant $k \simeq \Lambda \simeq 10^{-52} m^{-2}$), though present in principle, are too small to be detected.

Here, we would like to study the GM precession of an orbiting gyroscope, focusing on the corrections due to the $k$-term.

We remember that such an orbiting gyroscope undergoes also a geodetic precession with an angular frequency (averaged over a revolution) that can be written in the form

\[
\Omega_{S-O} = \Omega_{S-O}^M + \Omega_{S-O}^k \quad \text{(17)}
\]

where

\[
\Omega_{S-O}^M = \frac{3M}{2A^3} \frac{L}{(1 - e^2)^{3/2}} \quad \text{and} \quad \Omega_{S-O}^k = -\frac{1}{2} kL \quad \text{(18)}
\]

is the classical GR geodetic (or de Sitter) precession, where $L$ is the specific orbit angular momentum of the gyroscope, moving along the orbit with semi-major axis $A$ and eccentricity $e$, and

is the correction due to the $k$-term (see e.g. Sereno and Jetzer [23]).

The angular velocity $\Omega_{S-O}$ can be referred to as a spin-orbit term, because it is due to the coupling of the gyroscope’s spin $\mathbf{S}$ with the orbit angular momentum $\mathbf{L}$. 
As for the GM precession, it can be calculated, using the standard approach (see e.g. Ciufolini and Wheeler [28]), starting from the GM potential

\[ g_{0\phi} = -\frac{2Ma}{\rho} \sin^2 \theta - \frac{ka}{3} \rho^2 \sin^2 \theta - \frac{5}{6} Mak \rho \sin^2 \theta \]  

(20)

of the metric [14]. The first term in eq. (20) is the “classical” GM potential arising in GR, while the other two terms are proportional to \( k \). As a consequence, we can write the angular frequency of GM precession in the form

\[ \Omega_{S-S} = \Omega_{S-S}^J + \Omega_{S-S}^{ka} + \Omega_{S-S}^{jk} \]  

(21)

where

\[ \Omega_{S-S}^J = - \left[ \frac{J}{|\rho|^3} - 3 \left( \frac{J \cdot \rho}{|\rho|^3} \right) \right], \]  

(22)

is the GR gravito-magnetic precession (see e.g. Ciufolini and Wheeler [28]), and

\[ \Omega_{S-S}^{ka} = \frac{ka}{3} J, \]  

(23)

\[ \Omega_{S-S}^{jk} = \frac{5Jk}{12\rho} \left[ J + \left( J \cdot \rho \right) \rho \right]. \]  

(24)

are new contributions due to the presence of the \( k \)-term.

The angular velocity \( \Omega_{S-S} \) can be referred to as a spin-spin term, because it is due to the coupling of the gyroscope’s spin \( S \) with the spin angular momentum \( J \) of the source of the gravitational field. We point out that all contributions in (21) do not depend on the velocity of the gyroscope; furthermore, the term \( \Omega_{S-S}^{ka} \) is a constant contribution over the whole orbit.

It is interesting now evaluate the magnitude of the new contributions (23-24). To this end, it is useful to remember that the GP-B mission is expected to measure the precession (geodetic plus frame-dragging) of the orbiting gyroscope with an accuracy of 0.1 milliarcseconds/year, which is a very hard task, as the long story of this mission teaches [15]. Taking into account that the geodetic precession has a magnitude of about 6.6 arcseconds/year, and the frame-dragging effect of 0.041 arcseconds/year, to give an idea of the magnitude of the effects of the non linearity of the gravity Lagrangian, we may calculate the ratio between the two angular frequencies (23) and (24) and the GR one (22), at a distance \( R = |x| = \rho \) from the source

\[ \frac{\Omega_{S-S}^{ka}}{\Omega_{S-S}^J} = \frac{kR^3}{M^3}; \]  

(25)

\[ \frac{\Omega_{S-S}^{jk}}{\Omega_{S-S}^J} = \frac{kR^2}{M^2}. \]  

(26)

Using \( k = 10^{-52} m^{-2} \), i.e. the current estimate of the cosmological constant, \( M \) equal to the Earth mass, \( R \approx 650 \) \( Km \), i.e. order of magnitude of the GP-B orbit, we get

\[ \frac{\Omega_{S-S}^{ka}}{\Omega_{S-S}^J} \approx 10^{-28}. \]  

(27)

\[ \frac{\Omega_{S-S}^{jk}}{\Omega_{S-S}^J} \approx 10^{-39}. \]  

(28)

Accordingly, we may conclude that the impact of \( f(R) \) gravity on GM gyroscope precession is very small, and completely negligible for a mission like GP-B. For similar reasons, we can say that \( f(R) \) gravity is not relevant for other experiments aimed at the measurement of the gravito-magnetic field of the Earth, such as those performed in the past with LAGEOS satellites (see Ciufolini and Pavlis [13]), or those that are planned in the next months such as LARES [29].

On the other hand, if GP-B will confirm the GR predictions for the orbiting gyroscope precession and no additional terms will be seen, from the expected accuracy of 0.1 milliarcseconds/year, we might deduce an estimate for the \( k \)-term in (24): \( |k| \leq 10^{-28} m^{-2} \), which is considerably greater than the current best estimates of the cosmological constant.

IV. CONCLUSIONS

We have studied gravito-magnetic effects in the framework of \( f(R) \) gravity. Namely, thanks to the analogy, in the Palatini formalism, between general relativistic vacuum field equations with cosmological constant and vacuum \( f(R) \) field equations, we have considered the Kerr-de Sitter metric as a solution of \( f(R) \) field equations. Since this metric describes a rotating black-hole, it is suitable to evaluate the gravito-magnetic effects. In particular, we have considered the weak-field approximation of the Kerr-de Sitter metric (which, as far as we know, has never been studied before) and then we have calculated the contribution to the gravito-magnetic precession of an orbiting gyroscope due to the non linearity of the gravity Lagrangian. We have shown that, though present in principle, this contribution is very small, and far to be detectable by a mission like GP-B and, probably, also by other foreseeable tests around the Earth. This confirms that the non-linearities appearing in \( f(R) \) become important on length scales much larger than the Solar System (e.g. on the cosmological scale) and their effects on local physics are probably negligible.

ACKNOWLEDGMENTS

The author would like to thank Prof. B. Mashhoon and Dr. M. Sereno for useful discussions. The author
una nuova era della ricerca sulle pulsar.

[1] Riess, A.G. et al., Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, Astron. J., 116, 1009-1038, 1998
[2] Perlmutter, S., et al., Measurements of Omega and Lambda from 42 High-Redshift Supernovae, Astrophys. J., 517, 565-586, 1999
[3] Bennet, C.L., et al., First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Basic Results, Astrophys. J. Suppl. 148, 1-27, 2003
[4] Capozziello, S., and Francaviglia, M., Extended Theories of Gravity and their Cosmological and Astrophysical Applications, arXiv:0706.1146 [astro-ph], 2007
[5] Nojiri, S., Odintsov, S.D., Modified $f(R)$ gravity unifying $R^n$ inflation with $\Lambda$CDM epoch, arXiv:0710.1738 [astro-ph], 2007
[6] Capozziello, S., Cardone, V.F., and Troisi, A., Low surface brightness galaxies rotation curves in the low energy limit of $R^n$ gravity: no need for dark matter?, Mon. Not. Roy. Astron. Soc., 375, 1423-1440, 2007a
[7] F. Martins, C., and Salucci, P., Analysis of rotation curves in the framework of $R^n$ gravity, Mon. Not. Roy. Astron. Soc., 381, 1103-1108, 2007
[8] Will, C.M., The Confrontation between General Relativity and Experiment, Living Rev. Relativity, 9, 2006, http://www.livingreviews.org/lrr-2006-3
[9] Sotiriou, T., Faraoni, V., $f(R)$ Theories of Gravity, arXiv:0805.1726 [gr-qc], 2008
[10] Mashhoon, B., Gronwald, F., Lichtenegger, H.I.M., Gravitomagnetism and the Clock Effect, Lect. NotesPhys. 562, 83-108, 2001
[11] Ruggiero, M.L., Tartaglia, A., Gravitomagnetic Effects, Il Nuovo Cimento B 117, 743-768, 2002
[12] Mashhoon, B., Gravitoelectromagnetism: A Brief Review, in The Measurement of Gravitomagnetism: A Challenging Enterprise, Iorio, L., Editor, Nova Science, New York, arXiv:gr-qc/0311030 2007
[13] Ciufolini, I., and Pavlis, E.C., A confirmation of the general relativistic prediction of the Lense/Thirring effect, Nature 431, 958, 2004
[14] Iorio, L., On the reliability of the so-far performed tests for measuring the Lense/Thirring effect with the LAGEOS satellites, New Astronomy 10, 603-615, 2005
[15] C.W.F., Everitt et al., Gravity Probe B: Countdown to Launch, Lect. Notes Phys. 562, 52-82, 2001; see also the web site http://einstein.stanford.edu
[16] Clifton, T., The Parameterised Post-Newtonian Limit of Fourth-Order Theories of Gravity, Phys. Rev. D 77, 024041, 2008
[17] Ferraris, M., Francaviglia, M., Volovich, I., The Universality of Einstein Equations, Nuovo Cim. B 108, 1313, 1993
[18] Allemandi, G., Francaviglia, M., Ruggiero, M.L., Tartaglia, A., Post-Newtonian Parameters from Alternative Theories of Gravity, Gen. Rel. Grav. 37, 1891-1904, 2005
[19] Magnano G., Kerr, A.W., Are there metric theories of gravity other than general relativity?, arXiv:gr-qc/9511027, 1994
[20] Kerr, A.W., Hauck, J.C., Mashhoon B., Standard clocks, orbital precession and the cosmological constant, Class. Quantum Grav. 20, 2727-2736, 2003
[21] Demianski M., Some New Solutions of the Einstein Equations of Astrophysical Interest, Acta Astronomica 23, 197-232, 1973
[22] Carter, B., Black hole equilibrium states, in Black Holes (Les Houches 1972), Gordon and Breach, London, 1973
[23] Sereno, M., Jetzer, Solar and stellar system tests of the cosmological constant, Phys. Rev. D 73, 063004, 2006
[24] Rindler, W., Ishak, M., The Contribution of the Cosmological Constant to the Relativistic Bending of Light Revisited, Phys. Rev. D 76, 043006, 2007
[25] Sereno, M., On the influence of the cosmological constant on gravitational lensing in small systems., Phys. Rev. D 77, 043004, 2008
[26] Ruggiero, M.L., Gravitational Lensing and $f(R)$ theories in the Palatini approach, Gen. Rel. Grav. to appear, arXiv:0712.3218 [astro-ph], 2007
[27] Misner, C.W., Thorne, K.S., Wheeler, J.A., Gravitation, W. H. Freeman and Company, San Francisco, 1973
[28] Ciufolini, I., Wheeler J.A., Gravitation and Inertia, Princeton University Press, Princeton, 1995
[29] Ciufolini, I., LARES/WEBER-SAT, frame-dragging and fundamental physics, arXiv:gr-qc/0412001 2004