Zeros of Gaussian power series, Hardy spaces and determinantal point processes

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Received: 6 July 2021 / Accepted: 19 November 2021 / Published online: 13 December 2021
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Abstract

Given a sequence \((\xi_n)\) of standard i.i.d complex Gaussian random variables, Peres and Virág (in the paper “Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process” Acta Math. (2005) 194, 1-35) discovered the striking fact that the zeros of the random power series \(f(z) = \sum_{n=1}^{\infty} \xi_n z^{n-1}\) in the complex unit disc \(\mathbb{D}\) constitute a determinantal point process. The study of the zeros of the general random series \(f(z)\), where the restriction of independence is relaxed upon the random variables \((\xi_n)\) is an important open problem. This paper proves that if \((\xi_n)\) is an infinite sequence of complex Gaussian random variables, such that their covariance matrix is invertible and its inverse is a Toeplitz matrix, then the zero set of \(f(z)\) constitutes a determinantal point process with the same distribution as the case of i.i.d variables studied by Peres and Virág. The arguments are based on some interplays between Hardy spaces and reproducing kernels. Illustrative examples are constructed from classical Toeplitz matrices and the classical fractional Gaussian noise.

Keywords Gaussian power series · Hardy spaces · Toeplitz matrices · Determinantal point process · Reproducing Hilbert spaces

Mathematics Subject Classification Primary 30B20 · 46E22 · Secondary 30H10

In the memory of President Dr. John Pombe Joseph Magufuli.

Communicated by Klaus Guerlebeck.

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1 Introduction

Given a sequence of independent and identically distributed standard complex Gaussian random variables \((\xi_n)\), consider the Gaussian power series 
\[ f(z) = \sum_{n=1}^{\infty} \xi_n z^{n-1}, \]
defined in the open unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and the zero set of \( f(z) \), that is
\[ \mathcal{Z} = \{ z \in \mathbb{D} : f(z) = 0 \}. \]
The zero set \( \mathcal{Z} \) constitutes a point process on \( \mathbb{D} \). The joint intensity \( p \) of the process \( \mathcal{Z} \) is defined as
\[ p(z_1, z_2, \ldots, z_n) = \lim_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(z_1, z_2, \ldots, z_n)}{\pi^n \epsilon^{2n}} \]
where \( \mathbb{P}_\epsilon(z_1, z_2, \ldots, z_n) \) is the probability that simultaneously for all \( i \in \{1, 2, \ldots, n\} \), the function \( f(z) \) has a zero in the disc of centre \( z_i \) and radius \( \epsilon > 0 \). Recently, Peres and Virág [16] obtained the striking fact that the point process \( \mathcal{Z} \) is a determinantal process. In fact, they proved that, for all \( z_1, z_2, \ldots, z_n \) in \( \mathbb{D} \):
\[ p(z_1, z_2, \ldots, z_n) = \det \left( \frac{1}{\pi(1 - z_k \overline{z_j})^2} \right)_{k,j=1}^n. \]
That is
\[ p(z_1, z_2, \ldots, z_n) = \det \left( K(z_k, z_j) \right)_{1 \leq k, j \leq n} \]
where \( K(z, w) = \pi^{-1}(1 - z \overline{w})^{-2} \) is the classical Bergman kernel in \( \mathbb{D} \). (A thorough discussion on determinantal point processes can be found in the book by Hough et al. [5].) The study of the zeros of the general random series \( f(z) = \sum_{n=1}^{\infty} \xi_n z^{n-1} \), where the restrictions of independence and identical distribution are relaxed upon the random variables \((\xi_n)\) is an important open problem. This paper considers this question in the following context: Determine sequences of dependent Gaussian random variables \((\xi_n)\), such that the zero set of the random series \( f(z) = \sum_{n=1}^{\infty} \xi_n z^{n-1} \) is a determinantal point process as in the case of i.i.d random variables. We recall that a matrix \((a_{k,j})\) is called a Toeplitz matrix if \( a_{k,j} \) depends only on the difference \( k - j \), that is, for all \( k, j \) and for any integer \( \ell \), such that \( a_{k+\ell,j+\ell} \) is defined, \( a_{k+\ell,j+\ell} = a_{k,j} \).
We consider a complex infinite Toeplitz matrix \( G \) that is hermitian and positive definite and we assume that \( G \) admits a classical inverse in the sense that there exists a hermitian positive definite matrix \( G^{-1} \), such that \( GG^{-1} = G^{-1}G = I \). Since \( G \) is positive definite and Toeplitz, then there exists a positive definite function \( \gamma \) on the integers, such that \( G_{k,j} = \gamma(k - j) = \gamma(j - k) \) for all \( k, j \). (We shall assume without loss of generality that \( G_{k,k} = \gamma(0) = 1 \) for all \( k \).) Then, by the classical Bochner theorem, one can associate to \( G \) a (unique) probability measure \( \mu \) on the unit circle \( \mathbb{T} \), such that
\[ \gamma(n) = \int_{\mathbb{T}} e^{2\pi in\theta} d\mu(\theta), \quad \text{for all } n \in \mathbb{Z}. \]

We shall assume throughout that the probability measure \( \mu \) satisfies the following condition:

**Condition (C):** The measure \( \mu \) is absolutely continuous and its density \( \varphi \) is strictly positive almost everywhere on \( \mathbb{T} \) with respect to the Lebesgue measure on \( \mathbb{T} \).

We now consider a discrete-time complex Gaussian process \((\xi_n)_{n \in \mathbb{N}}\) with zero mean, covariance matrix \( G^{-1} \) and zero pseudo-covariance matrix, that is, for all \( n, m \in \mathbb{N} \):

\[ \mathbb{E}(\xi_n) = 0, \quad \mathbb{E}(\xi_n \overline{\xi_m}) = (G^{-1})_{n,m} \text{ and } \mathbb{E}(\xi_n \xi_m) = 0. \]

The existence of such process \((\xi_n)\) is classical (see for example Miller [10] and references therein.) In the case where \( G \) (and hence \( G^{-1} \)) is a real matrix one can simply take a real Gaussian process \((\xi_n)\) of covariance matrix \( G^{-1} \) and write \( \xi_n = (\xi_n + i\xi_n')/\sqrt{2} \), where \( (\xi_n') \) is an independent copy of \((\xi_n)\). We shall consider the Gaussian analytic function:

\[ f(z) = \sum_{n=1}^{\infty} \xi_n z^{n-1}, \quad z \in \mathbb{D}. \]

Our main finding is that the zero set of the Gaussian analytic function \( f(z) \) is a determinantal point process governed by the Bergman kernel just as it is for the case of i.i.d random variables. The main result of this paper is the following.

**Theorem 1.1** Let \( G \) be an infinite, invertible, hermitian and positive definite Toeplitz matrix, such that the associated probability measure \( \mu \) satisfies condition (C) and the inverse \( G^{-1} \) is such that

\[ \sup_{n,m} |(G^{-1})_{n,m}| < \infty. \quad (1) \]

If \((\xi_n)_{n \in \mathbb{Z}}\) is a centred complex Gaussian process with covariance matrix \( G^{-1} \) and zero pseudo-covariance matrix, then the zero set of the Gaussian analytic function:

\[ f(z) = \sum_{n=1}^{\infty} \xi_n z^{n-1}, \quad z \in \mathbb{D} \]

is a determinantal point process governed by the Bergman kernel. That is the joint intensity \( p \) of the zeros of \( f(z) \) is given by

\[ p(z_1, z_2, \ldots, z_n) = \det \left( \frac{1}{\pi (1 - z_k \overline{z_j})^2} \right)_{1 \leq k,j \leq n}, \quad z_1, z_2, \ldots, z_n \in \mathbb{D}. \]

If \((\xi_n)\) is a centred Gaussian process with covariance matrix \( G \) (that is a Toeplitz matrix), the zero set of \( f(z) = \sum_{n=1}^{\infty} \xi_n z^{n-1} \) does not necessarily have the same
distribution as $\sum_{n=1}^{\infty} x_n z^{n-1}$ for i.i.d Gaussian variables $(\chi_n)$. (This is discussed in [13]. An example is also given in Sect. 6.1.) However, if we take instead of $(\xi_n)$ a sequence $(\xi_n)$ with covariance matrix $G^{-1}$ then the corresponding zero set has the same distribution as for i.i.d. Gaussian variables. This implies that in terms of the corresponding zero sets sequences of Gaussian variables whose covariance matrix is the inverse of a Toeplitz matrix are more closer to the sequence of i.i.d Gaussian variables than sequences of variables whose covariance matrix is a Toeplitz matrix. This looks awkward but one should remember that the distribution of a Gaussian vector depends more directly on the inverse of its covariance matrix rather than the covariance matrix itself. Lemma 4.1 in Sect. 4 gives an interesting property of sequences of Gaussian variables whose covariance is an inverse of a Toeplitz matrix. It looks like such sequences are of some independent interest that requires further investigation.

The question of dependent random variables can be raised in connection with other determinantal point processes, for instance the point processes obtained by Krishnapur [8]. It is also relevant in the context of Pfaffian processes studied by Matsumoto and Shirai [9]. The rest of the paper is organised as follows. Section 2 contains some basic well-known facts about the zeros of the $f(z)$ and the Szegö kernel. Section 3 contains a connection between the covariance kernel of $f(z)$ and a Hardy space defined by the spectral measure of $(\xi_n)$. In Sect. 4 we obtain some properties of the sequence $(\xi_n)$. In Sect. 5 we provide an important connection between the classical Mobius transformation and the covariance kernel of $f(z)$ which is key to the proof of the main result. The last section contains some examples that illustrate the main result.

## 2 Szegö kernel and Hardy spaces

The starting point in the study of the zeros of any zero-mean Gaussian analytic function $f$ in a planar domain is the following general expression for its joint intensity function (Peres and Virág [16]):

$$
p(z_1, \ldots, z_n) = \frac{\mathbb{E} \left( |f'(z_1) \cdots f'(z_n)|^2 | f(z_1) = \cdots = f(z_n) = 0 \right)}{\pi^n \det(A)} \tag{2}
$$

or equivalently

$$
p(z_1, z_2, \ldots, z_n) = \frac{\text{perm} \left( C - BA^{-1}B^* \right)}{\pi^n \det(A)} \tag{3}
$$

where $A$, $B$ and $C$ are the $n \times n$ matrices:

$$
A = (\mathbb{E}(f(z_k)\overline{f(z_j)})), B = (\mathbb{E}(f'(z_k)\overline{f'(z_j)})) \text{ and } C = (\mathbb{E}(f'(z_k)\overline{f'(z_j)}))
$$

and perm denotes the permanent of a matrix. For the classical Gaussian power series $f(z) = \sum_{n=1}^{\infty} x_n z^{n-1}$ with independent and identically distributed (i.i.d) random variables $(\xi_n)$.
Zeros of Gaussian power series, Hardy spaces...

That is

$$\mathbb{E}(f(z)\overline{f(w)}) = \sum_{n=0}^{\infty} (zw)^{n-1} = \frac{1}{1-z\overline{w}}, \quad z, w \in \mathbb{D}. $$

where $\mathcal{K}$ is the classical Szegö kernel. This means that the covariance kernel of $f(z)$ is the Szegö kernel. The classical Hardy space $\mathbb{H}^2(\mathbb{D})$ is the class of holomorphic functions $f$ in the unit disc $\mathbb{D}$ for which

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}} \left| f(re^{2\pi i\theta}) \right|^2 d\theta = \lim_{r \to 1} \int_{\mathbb{T}} \left| f(re^{2\pi i\theta}) \right|^2 d\theta < \infty$$

where $\mathbb{T}$ is the unit circle $\mathbb{R}/\mathbb{Z}$. Equivalently $\mathbb{H}^2(\mathbb{D})$ is the class of holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \quad a_n \in \mathbb{C}$, such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. It is a Hilbert space with the inner product:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}, \quad f = \sum_{n=0}^{\infty} a_n z^n, \quad g = \sum_{n=0}^{\infty} b_n z^n. \quad (4)$$

Any function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{H}^2(\mathbb{D})$ is such that its radial limit

$$\tilde{f}(\theta) = \lim_{r \to 1} f(re^{2\pi i\theta}) = f(e^{2\pi i\theta}) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \theta}$$

exists almost everywhere in $\mathbb{T}$ and $\tilde{f} \in L^2(\mathbb{T})$. Moreover

$$\langle f, g \rangle = \int_{\mathbb{T}} \tilde{f}(\theta) \overline{\tilde{g}(\theta)} d\theta = \int_{\mathbb{T}} f(e^{2\pi i\theta}) \overline{g(e^{2\pi i\theta})} d\theta$$

and

$$\|f\|_{\mathbb{H}^2(\mathbb{D})}^2 = \|\tilde{f}\|_{L^2(\mathbb{T})}^2 = \sum_{n=0}^{\infty} |a_n|^2. \quad (\text{See Katznelson [7, p 98].})$$

It is well-known that $\mathbb{H}^2(\mathbb{D})$ is a reproducing kernel Hilbert space whose kernel is the Szegö kernel. This means that for each $y \in \mathbb{D}$ and for each $f \in \mathbb{H}^2(\mathbb{D})$, the function $\mathbb{K}(., y) : \mathbb{D} \to \mathbb{C}$ defined by $\mathbb{K}(., y)(x) = \mathbb{K}(x, y)$ is such that

$$f(y) = \langle f, \mathbb{K}(., y) \rangle. \quad (\text{We refer to the book by Paulsen [15] for a background on reproducing kernel Hilbert spaces.})$$

The main argument of Peres and Virág is to make use of these connections between the Hardy space $\mathbb{H}^2(\mathbb{D})$ and the Szegö kernel.
3 Inverse Toeplitz matrices and weighted Hardy spaces

In the general case, where the covariance matrix of the variables \((\xi_n)\) is the matrix \(G^{-1}\) (where \(G\) is an infinite hermitian and positive definite Toeplitz matrix), the covariance kernel of the function \(f(z)\) is given by

\[
\kappa_G(z, w) = \mathbb{E}\left( f(z) \overline{f(w)} \right) = \sum_{k,j=1}^{\infty} (G^{-1})_{k,j} z^k w^{j-1}, \quad z, w \in \mathbb{D}. \tag{5}
\]

For the convergence of the series in (5) to hold, it is enough to assume that \(\sup_{k,j} |(G^{-1})_{k,j}| < \infty\) (that is condition (1) in Theorem 1.1.) One can write

\[
\kappa_G(z, w) = Z^T G^{-1} W
\]

where \(Z^T = (1, z, z^2, \ldots)\) and \(W = (1, w, w^2, \ldots)^T\). Clearly in the particular case, where \(G\) is the identity matrix \(\kappa_G\) is the Szegö kernel. An important tool in the proof of the main result is the fact that there is a reproducing kernel Hilbert space whose kernel is the covariance kernel \(\kappa_G\). Assume that the Toeplitz matrix \(G\) is given by

\[
G_{k,j} = \gamma(k-j) = \gamma(j-k)
\]

for a function \(\gamma\) defined on the integers. We shall assume without loss of generality that \(G_{k,k} = \gamma(0) = 1\) for all \(k\). Since \(G\) is a Toeplitz matrix and it is positive definite then by the classical Bochner theorem, there exists a probability measure \(\mu\) on the unit circle \(\mathbb{T}\), such that

\[
\gamma(n) = \int_{\mathbb{T}} e^{2\pi i n \theta} d\mu(\theta), \quad \text{for all } n \in \mathbb{Z}. \tag{6}
\]

We have assumed throughout that \(\mu\) is absolutely continuous and its density \(\varphi\) is called the spectral density function of the matrix \(G\). It is such that

\[
\gamma(n) = \int_{\mathbb{T}} e^{2\pi i n \theta} \varphi(\theta) d\theta, \quad \text{for all } n \in \mathbb{Z} \tag{7}
\]

which (under some conditions) implies in return that

\[
\varphi(\theta) = \sum_{n \in \mathbb{Z}} \gamma(n) e^{-2\pi i n \theta}, \quad \theta \in \mathbb{T}. \tag{8}
\]

Consider the sub-space \(H^2_G(\mathbb{D})\) of the Hardy space \(H^2(\mathbb{D})\) of functions \(f\) given by

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \quad a_n \in \mathbb{C},
\]

such that

\[
\|f(e^{2\pi i \theta})\|_{L^2(\mu)}^2 = \int_{\mathbb{T}} |f(e^{2\pi i \theta})|^2 d\mu(\theta) < \infty
\]

and set

\[
\|f\|_{H^2_G(\mathbb{D})} = \|f(e^{2\pi i \theta})\|_{L^2(\mu)},
\]

Clearly if
\[ \|f\|_{H^2_G(\mathbb{D})}^2 = \int_{\mathbb{T}} |f(e^{2\pi i \theta})|^2 \varphi(\theta) d\theta = 0, \]

then the fact that \( \varphi > 0 \) almost everywhere in \( \mathbb{T} \) yields

\[ \|f\|_{H^2(\mathbb{D})}^2 = \int_{\mathbb{T}} |f(e^{2\pi i \theta})|^2 d\theta = 0. \]

This implies that \( f = 0 \) everywhere in \( \mathbb{D} \). Moreover, the norm \( \| \cdot \|_{H^2_G(\mathbb{D})} \) is complete. Indeed, let \( (f_k) \) be a sequence of functions in \( H^2_G(\mathbb{D}) \), such that

\[ \lim_{k,j \to \infty} \|f_k - f_j\|_{H^2_G(\mathbb{D})}^2 = \lim_{k,j \to \infty} \|f_k(e^{2\pi i \theta}) - f_j(e^{2\pi i \theta})\|_{L^2(\mu)}^2 = 0. \]

Since \( L^2(\mu) \) is complete, then the sequence \( (f_k(e^{2\pi i \theta})) \) has a limit \( \ell : \mathbb{T} \to \mathbb{C} \) in \( L^2(\mu) \). That is

\[ \lim_{k \to \infty} \int_{\mathbb{T}} |f_k(e^{2\pi i \theta}) - \ell(\theta)|^2 \varphi(\theta) d\theta = 0. \]

Again, since \( \varphi > 0 \) almost everywhere, then

\[ \lim_{k \to \infty} \int_{\mathbb{T}} |f_k(e^{2\pi i \theta}) - \ell(\theta)|^2 d\theta = 0. \]  

(9)

This means that \( \ell \) is also the limit of the sequence \( (f_k(e^{2\pi i \theta})) \) in \( L^2(\mathbb{T}) \). It follows that \( \ell \in L^2(\mathbb{T}) \). We can consider the Fourier series of \( \ell \) and write \( \ell(\theta) = \sum_{n=-\infty}^{\infty} \hat{\ell}(n)e^{2\pi i n \theta} \) (this series converges in \( L^2(\mathbb{T}) \)) and

\[ \sum_{n=-\infty}^{\infty} |\hat{\ell}(n)|^2 = \|\ell\|_{L^2(\mathbb{T})}^2 < \infty. \]

Set

\[ f_{n,k}(z) = \sum_{n=0}^{\infty} a_{n,k} z^n. \]

Then, (9) implies

\[ \lim_{k \to \infty} \sum_{n \geq 0} |a_{n,k} - \hat{\ell}(n)|^2 + \sum_{n < 0} |\hat{\ell}(n)|^2 = 0. \]

Hence, \( \hat{\ell}(n) = 0 \) for all \( n < 0 \). Then, consider the function \( g(z) = \sum_{n=0}^{\infty} \hat{\ell}(n)z^n \). It is now clear that \( (f_k) \) converges to \( g \) both in \( H^2(\mathbb{D}) \) and \( H^2_G(\mathbb{D}) \).

We define the inner product on \( H^2_G(\mathbb{D}) \) by

\[ \langle f, g \rangle = \int_{\mathbb{T}} f(e^{2\pi i \theta}) \overline{g(e^{2\pi i \theta})} d\mu(\theta). \]  

(10)
Clearly $H^2_G(\mathbb{D})$ is a Hilbert space. We want to show that $H^2_G(\mathbb{D})$ is in fact a reproducing kernel Hilbert space whose kernel is $\mathcal{K}_G$.

**Theorem 3.1** The space $H^2_G(\mathbb{D})$ is a reproducing kernel Hilbert space whose kernel is $\mathcal{K}_G$ given by

$$\mathcal{K}_G(z, w) = Z^T G^{-1} \overline{W}, \text{ } z, w \in \mathbb{D}$$

with $Z = (z^n)_{n \in \mathbb{N}}$ and $W = (w^n)_{n \in \mathbb{N}}$.

**Proof** We shall first prove that the monomials $z^n$ are in the reproducing kernel Hilbert space associate to $\mathcal{K}_G$. That is

$$z^n = \int_{\mathbb{T}} e^{2\pi i n \theta} \mathcal{K}_G(e^{2\pi i \theta}, z) d\mu(\theta).$$

Since for all $w, y \in \mathbb{D}$

$$\mathcal{K}_G(w, y) = \sum_{k, j=1}^{\infty} (w)^{k-1} (\overline{y})^{j-1} (G^{-1})_{k,j} = \mathcal{K}_G(y, w),$$

then

$$\int_{\mathbb{T}} e^{2\pi i n \theta} \mathcal{K}_G(e^{2\pi i \theta}, z) d\mu(\theta) = \int_{\mathbb{T}} e^{2\pi i n \theta} \mathcal{K}_G(z, e^{2\pi i \theta}) d\mu(\theta) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (G^{-1})_{k,j} \int_{\mathbb{T}} e^{2\pi i (n-k+1) \theta} d\mu(\theta)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (G^{-1})_{k,j} \gamma(n - k + 1)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (G^{-1})_{k,j} G_{n+1,k} = z^n.$$

To complete of the proof it suffices to determine an orthonormal basis \{ $P_k(z)$ : $k = 1, 2, \ldots$ \} of $H^2_G(\mathbb{D})$ and prove that it is the case that

$$\sum_{k=1}^{\infty} P_k(z) \overline{P_k(w)} = \mathcal{K}_G(z, w), \text{ for all } z, w \in \mathbb{D}.$$

First, it is clear that in the Hilbert space $H^2_G(\mathbb{D})$, for all $k, j \in \mathbb{N}$:

$$\langle z^k, z^j \rangle = \int_{\mathbb{T}} e^{2\pi i (k-j) \theta} d\mu(\theta) = \gamma(k - j).$$
Next, we shall take \((P_k)\) to be the orthonormal basis deduced from the sequence of polynomials \((1, z, z^2, \ldots, z^k, \ldots)\) by the classical Gram–Schmidt process. Consider an infinite lower triangular matrix \(A = (a_{k,j})\) (that is \(a_{k,j} = 0\) for \(j > k\)), such that for each \(k\):

\[
z^{k-1} = a_{k,1}P_1(z) + a_{k,2}P_2(z) + \ldots + a_{k,k}P_k(z) \quad (11)
\]

Then, since \(\{P_k : k = 1, 2, \ldots\}\) is orthonormal, then for \(j \leq k\):

\[
\langle z^{k-1}, z^{j-1}\rangle = a_{k,1}\overline{a}_{j,1} + a_{k,2}\overline{a}_{j,2} + \ldots + a_{k,j}\overline{a}_{j,j}.
\]

Moreover, using \(\langle z^{k-1}, z^{j-1}\rangle = \gamma(k - j)\), it follows that

\[
a_{k,1}\overline{a}_{j,1} + a_{k,2}\overline{a}_{j,2} + \ldots + a_{k,j}\overline{a}_{j,j} = \gamma(k - j) = G_{k,j}.
\]

This yields \(AA^* = G\), where \(A^*\) is the conjugate transpose of \(A\). It follows from (11) that

\[
\begin{pmatrix}
P_1(z) \\
P_2(z) \\
P_3(z) \\
\vdots \\
P_k(z) \\
\vdots
\end{pmatrix}
= A^{-1} \begin{pmatrix}
1 \\
z \\
z^2 \\
\vdots \\
z^{k-1} \\
\vdots
\end{pmatrix} = A^{-1}Z
\]

for a lower triangular matrix \(A\), such that \(AA^* = G\). It is clear that since \(A\) is a lower triangular matrix, then \(A^{-1}\) is also a lower triangular matrix, and moreover, \(P_k(z)\) is fully determined by the first \(k\) rows of \(A\). Now, clearly

\[
\sum_{k=1}^{\infty} P_k(z)\overline{P_k(w)} = \lim_{n \to \infty} \sum_{k=1}^{n} P_k(z)\overline{P_k(w)}
= \lim_{n \to \infty} \left( (A_n)^{-1}Z_n \right)^T (A_n)^{-1} W_n
= \lim_{n \to \infty} Z_n^T (G_n)^{-1} W_n
= Z^T G^{-1} \overline{W}
\]

where \(A_n\) (resp. \(G_n\)) is the block of \(A\) (resp. \(G\)) consisting of the first \(n\) rows and columns of \(A\) (resp. \(G\)) and \(Z_n = (1, z, z^2, \ldots, z^{n-1})\) and \(W_n = (1, w, w^2, \ldots, w^{n-1})\). It follows that

\[
\sum_{k=1}^{\infty} P_k(z)\overline{P_k(w)} = \|\xi_G(z, w)\|
\]

which concludes the proof.
Corollary 3.2 The Gaussian analytic function \( f(z) = \sum_{n=1}^{\infty} \xi_n z^{-n-1} \) (where \( (\xi_n) \) has covariance matrix \( G^{-1} \)) has the same distribution with the function \( g(z) = \sum_{n=1}^{\infty} \chi_n P_n(z) \), where \( (\chi_n) \) is a sequence of standard i.i.d. complex Gaussian random variables and \( (P_n(z)) \) are the polynomials defined by the matrix \( G \) as in the proof of Theorem 3.1.

It is so because the two random functions have the same covariance kernel:

\[
\mathbb{E}(f(z)f(w)) = \mathbb{E}(g(z)g(w)) = \sum_{k=1}^{\infty} P_n(z)\overline{P_n(w)} = Z^T G^{-1} \overline{W}.
\]

4 Some properties of the sequence \((\xi_n)\) of covariance \(G^{-1}\)

Here we obtain important properties of the sequence \((\xi_n)\) of covariance matrix \(G^{-1}\) that will be useful for the proof of main result.

Lemma 4.1 Assume that \((\xi_n)_{n \in \mathbb{N}}\) is a centred complex Gaussian process with zero pseudo-covariance and covariance matrix \(G^{-1}\), where \(G\) is an infinite hermitian positive definite Toeplitz matrix. Then, for each \(n \geq 2\), the conditional joint distribution of the sequence \((\xi_n, \xi_{n+1}, \xi_{n+2}, \ldots)\) under the condition \(\xi_1 = \xi_2 = \cdots = \xi_{n-1} = 0\) is equal to the unconditional joint distribution of \((\xi_1, \xi_2, \xi_3, \ldots)\). That is

\[
((\xi_n, \xi_{n+1}, \xi_{n+3}, \ldots) | \xi_1 = \xi_2 = \cdots = \xi_{n-1} = 0) \overset{d}{=} (\xi_1, \xi_2, \xi_3, \ldots)
\]

Proof Set

\[
S_1 = (\xi_1, \xi_2, \ldots), \quad S_n = (\xi_n, \xi_{n+1}, \ldots), \quad \text{for all } n \geq 1.
\]

Then, it is well-known that the covariance matrix of

\[
(S_n | \xi_1 = \xi_2 = \cdots = \xi_{n-1} = 0)
\]

is the Schur complement of Cov \((\xi_1, \xi_2, \ldots, \xi_{n-1})\) in the overall covariance matrix Cov \((S_1) = G^{-1}\). It is obtained by taking the matrix \(G^{-1}\), take its inverse, that is, \(G\), delete the rows and columns corresponding to the random variables \(\xi_1, \xi_2, \ldots, \xi_{n-1}\) and take the inverse of the resulting matrix. Now deleting the first \(n-1\) rows and columns of the infinite Toeplitz matrix \(G\) yields the very same matrix \(G\). It follows that the covariance matrix of \((S_n | \xi_1 = \xi_2 = \cdots = \xi_{n-1} = 0)\) is just \(G^{-1}\). Therefore, \((S_n | \xi_1 = \xi_2 = \cdots = \xi_{n-1} = 0)\) has the same distribution as \((\xi_1, \xi_2, \ldots)\).

An immediate consequence of this lemma is

Corollary 4.2 For any sequence \((\alpha_k)\) of complex numbers and for any integer \(n \geq 1\):
\[
\left( \sum_{k=1}^{\infty} \alpha_k \xi_{n+k-1} \right) \bigg|_{\xi_1 = \xi_2 = \cdots = \xi_{n-1} = 0}
\]

has the same distribution with
\[
\sum_{k=1}^{\infty} \alpha_k \xi_k
\]
provided the involved series converge almost surely. In particular
\[
(\xi_n | \xi_1 = \xi_2 = \cdots = \xi_k = 0) \overset{d}{=} \xi_{n-k} \text{ for all } 1 \leq k < n.
\]

5 Mobius transformation and the kernel \( \mathbb{K}_G \)

5.1 Mobius transformation

For \( w \in \mathbb{D} \), consider as in Peres and Virág [16], the Mobius transformation of the unit disc \( \mathbb{D} \):

\[
T_w(z) = \frac{z - w}{1 - z\bar{w}}, \quad z \in \mathbb{D}.
\]

In the case, where \( G \) is the identity matrix (or equivalently the random variables \( (\xi_k) \) are independent and identically distributed), Peres and Virág [16] proved the following lemma:

**Lemma 5.1** (Peres and Virág) Assume that \( (\xi_k) \) is the sequence of standard i.i.d complex Gaussian random variables. Then, for any \( w \) fixed in \( \mathbb{D} \), under the condition \( f(w) = 0 \), the random function \( f(z) \) has the same distribution with

\[
T_w(z)f(z) = \left( \frac{z - w}{1 - z\bar{w}} \right)f(z), \quad z \in \mathbb{D},
\]

that is

\[
(f(z)|f(w) = 0) \overset{d}{=} T_w(z)f(z)
\]

where \( \overset{d}{=} \) denotes equality in distribution. In general for \( w_1, w_2, \ldots, w_n \) fixed in \( \mathbb{D} \):

\[
(f(z)|f(w_1) = 0, \ldots, f(w_n) = 0) \overset{d}{=} T_{w_1}(z) \cdots T_{w_n}(z)f(z).
\]  \hfill (12)

A closer look at Peres and Virág’s proofs reveals that their main result (that is Theorem 1.1 in the case, where \( G \) is the identity matrix) is a consequence of Lemma 5.1. This implies that if we prove that Lemma 5.1 holds true in the general case of a Toeplitz matrix \( G \), then the same argument as in Peres and Virág [16] will complete the
proof of Theorem 1.1. In other words, to prove our main result, it is sufficient to prove that the following lemma holds.

**Lemma 5.2** Let $G$ be an invertible infinite hermitian Toeplitz matrix such that its associated measure $\mu$ is absolutely continuous with density $\varphi > 0$ almost everywhere on $\mathbb{T}$. Assume that $(\xi_k)_{k \in \mathbb{N}}$ is a centred Gaussian process with covariance matrix $G^{-1}$ and zero pseudo-covariance. Let

$$f(z) = \sum_{k=1}^{\infty} \xi_k z^{k-1}, \quad z \in \mathbb{D}.$$  

Then, for any $w$ fixed in $\mathbb{D}$,

$$(f(z)|f(w) = 0)^d = T_w(z)f(z).$$

Moreover for $w_1, \ldots, w_n$ fixed in $\mathbb{D}$,

$$(f(z)|f(w_1) = 0, \ldots, f(w_n) = 0)^d = T_{w_1}(z) \ldots T_{w_n}(z)f(z). \quad (13)$$

It is now an easy matter to prove that Lemma 5.2 yields Theorem 1.1 based on Peres and Virág arguments.

### 5.2 Proof of Theorem 1.1.

For all fixed $z_1, z_2, \ldots, z_n, w_1, w_2, \ldots, w_n$ in $\mathbb{D}$, the conditional joint distribution of

$$(f(z_1), f(z_2), \ldots, f(z_n)|f(w_1) = f(w_2) = \ldots = f(w_n) = 0)$$

is equal to the non-conditional joint distribution of

$$\left( T_{w_1}(z_1)f(z_1), T_{w_2}(z_2)f(z_2), \ldots, T_{w_n}(z_n)f(z_n) \right).$$

Taking the derivatives, it follows as in [16, corollary 13] that the conditional joint distribution of

$$(f'(z_1), f'(z_2), \ldots, f'(z_n)|f(z_1) = f(z_2) = \ldots = f(z_n) = 0)$$

is the same as the unconditional joint distribution of

$$(\Upsilon'(z_1)f(z_1), \Upsilon'(z_2)f(z_2), \ldots, \Upsilon'(z_n)f(z_n))$$

where

$$\Upsilon(z) = T_{z_1}(z)T_{z_2}(z) \ldots T_{z_n}(z).$$

This follows from the fact that

$$T'_z(z) = \frac{1}{1 - |z|^2} \quad \text{and} \quad T_z(z) = 0, \quad z \in \mathbb{D}.$$
At this stage, we make use of relation (2) to obtain

\[ p_0(z_1, z_2, \ldots, z_n) = \frac{\mathbb{E} \left( \left| f'(z_1) \cdots f'(z_n) \right|^2 \mid f(z_1) = \cdots = f(z_n) = 0 \right)}{\pi^n \det(A)} = \frac{\mathbb{E} \left( \left| Y'(z_1)f(z_1)Y'(z_2)f(z_2) \cdots Y'(z_n)f(z_n) \right|^2 \right)}{\pi^n \det(A)} = \frac{\mathbb{E} \left( \left| f(z_1)f(z_2) \cdots f(z_n) \right|^2 \right) \prod_{k=1}^{n} \left| Y'(z_k) \right|^2}{\pi^n \det(A)}. \]

Using the classical Cauchy determinant formula, Peres and Virág [16] showed that

\[ \prod_{k=1}^{n} \left| Y'(z_k) \right| = \det(A_0) \]

where

\[ A_0 = \left( \frac{1}{1 - z_k z_j} \right)_{k,j=1}^{n}. \]

Moreover, since it is well-known that if \( X_1, X_2, \ldots, X_n \) are random variables with joint Gaussian distribution with mean 0 and covariance matrix \( \Sigma \):

\[ \mathbb{E} \left( \left| X_1 X_2 \cdots X_n \right|^2 \right) = \text{perm} \left( \Sigma \right), \]

it follows that

\[ p_0(z_1, z_2, \ldots, z_n) = \frac{\text{perm} \left( A_0 \right) \left( \det(A_0) \right)^2}{\pi^n \det(A)}. \quad (14) \]

Now elementary operations on the matrix \( A \) yields

\[ \text{perm} \left( A \right) = \text{perm} \left( A_0 \right) \prod_{k=1}^{n} \left( \frac{1}{\left| 1 - z_k \right|^2} \right) \]

\[ \det(A) = \det(A_0) \prod_{k=1}^{n} \left( \frac{1}{\left| 1 - z_k \right|^2} \right). \]

Hence, (14) yields

\[ p_0(z_1, z_2, \ldots, z_n) = \frac{\text{perm} \left( A_0 \right) \det(A_0)}{\pi^n} \]

and it is proven in Peres and Virág [16, rel. (27)] that

\[ \text{perm} \left( A_0 \right) \det(A_0) = \det \left( \frac{1}{\left( 1 - z_k z_j \right)^2} \right)_{k,j=1}^{n}. \]
This concludes the proof.

5.3 Proof of Lemma 5.2.

Peres and Virag’s proof is based on the invariance property of the Szegö kernel with respect to Mobius transformations that are conformal mappings. This property does not hold for the general kernel \( K_G \). Our proof is more general. (a) In the case, where \( w = 0 \), it is an immediate consequence of Corollary 4.2. Indeed, since \( f(0) = \xi_0 \) then

\[
(f(z)|f(w) = 0) = (f(z)|f(0) = 0) = \left( \sum_{k=2}^{\infty} \xi_k z^{k-1} \right) = 0
\]

where the equality in distribution follows from Corollary 4.2.

(b) For a general \( w \in \mathbb{D} \), set

\[
F(z) = (f(z)|f(w) = 0), \ z \in \mathbb{D}.
\]

Clearly, the covariance kernel of the random function \( F(z) \) is given by

\[
\mathbb{E}(F(z)\overline{F(y)}) = \mathbb{E}\left( f(z)\overline{f(y)} \right) - \frac{\mathbb{E}\left( f(z)\overline{f(w)} \right) \mathbb{E}\left( f(w)\overline{f(y)} \right)}{\mathbb{E}\left( f(w)\overline{f(w)} \right)} = K_G(z, y) - \frac{K_G(z, w)K_G(w, y)}{K_G(w, w)}
\]

where \( z, y \in \mathbb{D} \). This is clearly a kernel function and we shall denote it by \( \mathcal{K}_1 \), that is

\[
\mathcal{K}_1(z, y) = K_G(z, y) - \frac{K_G(z, w)K_G(w, y)}{K_G(w, w)}, \ z, y \in \mathbb{D}.
\]

We also denote by \( \mathcal{K}_2 \) the covariance kernel of the random function \( T_w(z)f(z) \), that is

\[
\mathcal{K}_2(z, y) = \mathbb{E}\left( T_w(z)\overline{f(z)}T_w(y)\overline{f(y)} \right) = T_w(z)K_G(z, y)T_w(y).
\]

To show that the (Gaussian) random functions \( F(z) \) and \( T_w(z)f(z) \) have the same distribution, it is sufficient to show that their covariance kernels \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are identical. So we shall prove the following important identity: For all \( w, z, y \in \mathbb{D} \):

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \xi_j z^{j-1} = \mathcal{K}_1(z, y).
\]
\[ |\kappa_G(z, y) - \frac{\kappa_G(z, w)\kappa_G(w, y)}{\kappa_G(w, w)}| = T_w(z)|\kappa_G(z, y)|T_w(y). \]

We shall prove that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are both reproducing kernels of a certain Hilbert space, namely, the subspace \( \mathcal{H} \) of \( H^2_G(\mathbb{D}) \) of the functions that vanish at the point \( w \). That is

\[ \mathcal{H} = \{ g \in H^2_G(\mathbb{D}) : g(w) = 0 \}. \]

(To see that \( \mathcal{H} \) is closed in \( H^2_G(\mathbb{D}) \), assume that \( g_k = \sum_{n=0}^{\infty} a_{n,k} z^n \) converges to \( g = \sum_{n=0}^{\infty} b_n z^n \). Then, as discussed earlier, this convergence also holds with respect to the \( H^2(\mathbb{D}) \)-norm and hence \( \lim_{k \to \infty} \sum_{n=0}^{\infty} |a_{n,k} - b_n|^2 = 0 \). Then, for each \( z \in \mathbb{D} \) (by the Cauchy–Schwarz inequality):

\[ \lim_{k \to \infty} |g_k(z) - g(z)|^2 = \lim_{k \to \infty} |\sum_{n=0}^{\infty} (a_{n,k} - b_n)z^n|^2 \leq \lim_{k \to \infty} \sum_{n=0}^{\infty} |a_{n,k} - b_n|^2 \sum_{n=0}^{\infty} |z|^{2n} = 0. \]

In particular, since \( g_k(w) = 0 \) for all \( n \), then \( g(w) = 0 \) and hence \( g \in \mathcal{H} \).

To prove the claim that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are reproducing kernels \( \mathcal{H} \), we shall prove that for any function \( g \in \mathcal{H} \) and any \( y \in \mathbb{D} \):

\[ g(y) = \langle g, \mathcal{H}_1(., y) \rangle = \langle g, \mathcal{H}_2(., y) \rangle \]

(15)

where \( \mathcal{H}_1(., y) \) is the function defined by

\[ \mathcal{H}_1(., y) : \mathbb{D} \to \mathbb{C}, \ z \mapsto \kappa_1(z, y) \]

and similarly for \( \mathcal{H}_2 \). Clearly

\[ \langle g, \mathcal{H}_1(., y) \rangle = \langle g, \kappa_1(., y) \rangle - \frac{\kappa_1(., y)\kappa_G(w, y)}{\kappa_G(w, w)} \]

\[ = \langle g, \kappa(., y) \rangle - \left( \frac{\kappa_G(., y)}{\kappa_G(., w)} \right) \langle g, \kappa_G(., w) \rangle \]

\[ = g(y) - \left( \frac{\kappa_G(., y)}{\kappa_G(., w)} \right) g(w) \]

\[ = g(y) \]

since \( g(w) = 0 \).

For the kernel \( \mathcal{H}_2 \), we shall make use of the explicit inner product in \( H^2_G(\mathbb{D}) \) in relation (10), and show that

\[ g(y) = \int_T g(e^{2\pi i \theta})\overline{\mathcal{H}_2(e^{2\pi i \theta}, y)}d\mu(\theta). \]

(16)
Clearly, by definition of the kernel $\mathcal{K}_2$, 
\[
\int_T g(e^{2\pi i \theta}) \mathcal{K}_2(e^{2\pi i \theta}, y) d\mu(\theta) = T_w(y) \int_T g(e^{2\pi i \theta}) T_w(e^{2\pi i \theta}) \times \mathcal{K}_G(e^{2\pi i \theta}, y) d\mu(\theta). \tag{17}
\]

Now, note that 
\[
T_w(e^{2\pi i \theta}) = \frac{e^{-2\pi i \theta} - \overline{w}}{1 - we^{-2\pi i \theta}} = 1 - \frac{\overline{w}e^{2\pi i \theta}}{e^{2\pi i \theta} - w}.
\]

Therefore 
\[
\langle g, \mathcal{K}_2(., y) \rangle = T_w(y) \int_T g(e^{2\pi i \theta}) \left( \frac{1 - \overline{w}}{e^{2\pi i \theta} - w} \right) \mathcal{K}_G(e^{2\pi i \theta}, y) d\mu(\theta)
\]

\[
= T_w(y) \langle g, \mathcal{K}_G(., y) \rangle
\]

where $p$ is the function defined by 
\[
p(z) = \frac{1 - \overline{w}z}{z - w} = \frac{1}{T_w(z)}, \quad z \in \mathbb{D}.
\]

Since $g \in H^2_G(\mathbb{D})$, then the product $gp$ is also in $H^2_G(\mathbb{D})$. Indeed 
\[
\int_T |g(e^{2\pi i \theta})|^2 |p(e^{2\pi i \theta})|^2 d\mu(\theta) \leq C \int_T |g(e^{2\pi i \theta})|^2 d\mu(\theta) < \infty
\]

where 
\[
C = \sup_{\theta \in \mathbb{T}} \left| p(e^{2\pi i \theta}) \right|^2 = \sup_{\theta \in \mathbb{T}} \left| \frac{1 - \overline{w}e^{2\pi i \theta}}{e^{2\pi i \theta} - w} \right|^2 < \infty
\]

(because $w \in \mathbb{D}$ and hence $e^{2\pi i \theta} - w \neq 0$). It follows that 
\[
\langle gp, \mathcal{K}_G(., y) \rangle = g(y)p(y) = g(y)(T_w(y))^{-1}.
\]

Hence 
\[
\langle g, \mathcal{K}_2(., y) \rangle = T_w(y)g(y)(T_w(y))^{-1} = g(y).
\]

This yields (15) and concludes the proof of 
\[
F(z) = (f(z)|f(w) = 0) = T_w(z)f(z).
\]

Now, the general case that 
\[
(f(z)|f(w_1) = 0, f(w_2) = 0, \ldots, f(w_n) = 0) = T_{w_1}(z)T_{w_2}(z) \ldots T_{w_n}(z)f(z)
\]
follows immediately by an induction argument. This concludes the proof of Lemma 5.2.

6 Illustrating examples

6.1 Explicit inverse of tridiagonal Toeplitz matrices

Given a real number $q$, such that $|q| < 1/2$, consider the Toeplitz matrix

$$G = (\gamma(k-j))_{k,j=1}^{\infty}$$

where

$$\gamma(k) = \begin{cases} 1 & \text{if } k = 0 \\ q & \text{if } |k| = 1 \\ 0 & \text{otherwise}. \end{cases}$$

The spectral density function $\varphi$ of $G$ (i.e., the density of the corresponding measure $\mu$) is given by

$$\varphi(\theta) = 1 + qe^{2\pi i \theta} + qe^{-2\pi i \theta} = 1 + 2q \cos(2\pi \theta), \quad \theta \in \mathbb{T}. $$

Let $G_n$ be the submatrix of $G$ formed by its first $n$ rows and first $n$ columns. Then, the inverse $G_n^{-1}$ of $G_n$ is the symmetric matrix given by (see da Fonseca and Petronilho [1]):

$$ (G_n^{-1})_{k,j} = (-1)^{k+j} \frac{q^{j-k}}{|q|^{j-k+1}} \frac{U_{k-1}(\alpha)U_{n-j}(\alpha)}{U_n(\alpha)}, \quad 1 \leq k \leq j \leq n $$

where

$$\alpha = \frac{1}{2|q|}$$

and $(U_k)$ is the sequence of Chebyshev polynomials of second kind given by

$$U_0 = 1$$

$$U_1(x) = 2x$$

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x), \quad k = 1, 2, \ldots$$

Explicitly, for $|x| > 1$

$$U_k(x) = \frac{(x + \sqrt{x^2 - 1})^{k+1} - (x - \sqrt{x^2 - 1})^{k+1}}{2\sqrt{x^2 - 1}}.$$
Taking the limit of $G_n^{-1}$ as $n \to \infty$, it can be easily checked that the infinite matrix $G$ is indeed invertible and its inverse is the infinite symmetric matrix given for $k \leq j$ by

$$
(G^{-1})_{k,j} = \lim_{n \to \infty} (G_n^{-1})_{k,j} = \frac{(-2q)^{j-k}(C_1(q))^{-j}((C_1(q))^k - (C_2(q))^k)}{\sqrt{1 - 4|q|^2}},
$$

where

$$C_1(q) = 1 + \sqrt{1 - 4|q|^2} \text{ and } C_2(q) = 1 - \sqrt{1 - 4|q|^2}.$$ 

For example, if $q = -\frac{1}{3}$, the inverse of the infinite matrix $G$ is given by

$$
(G^{-1})_{k,j} = \left(\frac{3}{\sqrt{5}}\right)\left(\frac{3 - \sqrt{5}}{2}\right)^j\left(\frac{3 + \sqrt{5}}{2}\right)^k - \left(\frac{3 - \sqrt{5}}{2}\right)^k, \text{ for } j \geq k.
$$

Now, some involved (but elementary) calculations yield that the kernel function $\mathbb{K}_G$ defined by the matrix $G$ is given by

$$
\mathbb{K}_G(z, w) = Z^T G^{-1} W = \frac{\psi(z)\psi(w)}{1 - z\bar{w}}, \quad (18)
$$

where $\psi$ is the function defined in the unit disc $\mathbb{D}$ by

$$
\psi(z) = \left(\frac{2}{|q|}\right)^{1/2}\left(\frac{1}{a + bz}\right)^{1/2}
$$

with

$$
a = \sqrt{|q|^{-1} + \sqrt{q^{-2} - 4}}
$$

$$
b = (2/a) \text{ sign}(q).$$

For example for $q = -1/3$,

$$\psi(z) = 5^{1/4}(3 + \sqrt{5})\left(\frac{3}{2(5 + 3\sqrt{5})}\right)^{1/2}z - \frac{3 + \sqrt{5}}{2}.$$

Note that relation (18) means that

$$
\mathbb{K}_G(z, w) = \psi(z)\mathbb{K}_G(z, w)\psi(w)
$$

where $\mathbb{K}$ is the classical Szegö kernel (which is the covariance kernel associated to $\sum_{n=1}^{\infty} \xi_n z^{n-1}$ for i.i.d $(\xi_n)$). Then, for a sequence $(\xi_n)$ of Gaussian random variables
with covariance matrix $G^{-1}$, this implies that the random functions $\sum_{n=1}^{\infty} \xi_n z^n$ and $\psi(z) \sum_{n=1}^{\infty} \zeta_n z^n$ have the same distribution, that is
\[
\sum_{n=1}^{\infty} \xi_n z^n \overset{d}{=} \psi(z) \sum_{n=1}^{\infty} \zeta_n z^n.
\]
(for i.i.d $(\zeta_n)$). Since clearly $\psi(z) \neq 0$ everywhere in $\mathbb{D}$, it follows that the zeros of $\sum_{n=1}^{\infty} \xi_n z^n$ have the same distribution as the zeros of $\sum_{n=1}^{\infty} \zeta_n z^n$, and therefore, they constitute a determinantal point process as predicted by Theorem 1.

In particular, the intensity of the zeros of $f(z)$ is also given as it is the case for i.i.d variables. To emphasize that we need to consider the $G^{-1}$ as the covariance matrix instead of $G$, consider a sequence of Gaussian variables $(\xi_n)$ with covariance matrix $G$ and the function $g(z) = \sum_{n=1}^{\infty} \tau_n z^n$. Using relation (3), it is easy to derive that the intensity of the corresponding zero set is
\[
p(z) = \frac{1}{\pi |1 - |z|^2|^2}, \quad z \in \mathbb{D},
\]
as it is the case for i.i.d variables. To emphasize that we need to consider the $G^{-1}$ as the covariance matrix instead of $G$, consider a sequence of Gaussian variables $(\tau_n)$ with covariance matrix $G$ and the function $g(z) = \sum_{n=1}^{\infty} \tau_n z^n$. Using relation (3), it is easy to derive that the intensity of the corresponding zero set is
\[
p(z) = \frac{1}{\pi |1 - |z|^2|^2} \left(1 - \frac{q^2(1 - |z|^2)^2}{(1 + qz + q\overline{z})^2}\right), \quad z \in \mathbb{D}.
\]
Hence, clearly the zeros of $g(z)$ do not have the same distribution as the zeros of $f(z) = \sum_{n=1}^{\infty} \zeta_n z^n$ for i.i.d $(\zeta_n)$, since the corresponding intensity is $p(z) = \pi^{-1}(1 - |z|^2)^{-2}$.

Finally, since the function $\phi(\theta) = 1 + 2q \cos(2\pi \theta)$ is bounded on $\mathbb{T}$, then the set $H^2_G(\mathbb{D})$ is equal to $H^2(\mathbb{D})$ but with a different norm:
\[
\|g\|^2_{H^2_G(\mathbb{D})} = \int_{\mathbb{T}} |g(\theta)|^2 \phi(\theta) d\theta = \int_{\mathbb{T}} |g(\theta)|^2 (1 + 2q \cos(2\pi \theta)) d\theta.
\]
Therefore, with this norm, $H^2(\mathbb{D})$ is the reproducing kernel Hilbert space given by the kernel:
\[
K_G(z, w) = \frac{\psi(z) \overline{\psi(w)}}{1 - z \overline{w}},
\]
where the function $\psi(z)$ is given above.

### 6.2 Inverse of the Kac–Murdock–Szegö matrix

The same property is also observed for the classical Kac–Murdock–Szegö matrix. It is the Toeplitz matrix $G$ defined for a complex number $q$ by
\[
(G)_{k,j} = q^{|k-j|}.
\]
For $q$ real with $|q| < 1$, the spectral density function $\phi$ of $G$ is given by
\[ \varphi(\theta) = \sum_{n=-\infty}^{\infty} q^n e^{2\pi i n \theta} = \frac{1 - q^2}{1 - 2q \cos(2\pi \theta) + q^2}. \]

(Here, \( \varphi(\theta) \) can be seen as the classical Poisson kernel in the unit disc.) Under these conditions (\( q \) real and \( |q| < 1 \)), \( G \) is symmetrical and invertible and its inverse is well-known and given by

\[
(G^{-1})_{k,j} = \begin{cases} 
(1 - q^2)^{-1} & \text{if } k = j = 1 \\
(1 + q^2)(1 - q^2)^{-1} & \text{if } k = j \geq 2 \\
-q(1 - q^2)^{-1} & \text{if } |k - j| = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

(More details on the spectral properties of Kac–Murdock–Szegö matrix for a complex parameter are given in Fikioris [4].) Then, clearly, the corresponding kernel is

\[
\kappa_G(z, w) = \frac{(1 - qz)(1 - \overline{qw})}{(1 - q^2)(1 - zw)} = \frac{\psi(z)\overline{\psi(w)}}{1 - zw}
\]

where

\[
\psi(z) = \frac{1 - qz}{\sqrt{1 - q^2}}.
\]

In general, for \( q \in \mathbb{C} \) with \( |q| < 1 \), let

\[
G_{k,j} = \begin{cases} 
q^{k-j} & \text{if } k \geq j \\
(\overline{q})^{k-j} & \text{otherwise.}
\end{cases}
\]

Then, \( G \) is hermitian and invertible and its inverse is given by

\[
(G^{-1})_{k,j} = \begin{cases} 
(1 - |q|^2)^{-1} & \text{if } k = j = 1 \\
(1 + |q|^2)(1 - |q|^2)^{-1} & \text{if } k = j \geq 2 \\
-q(1 - |q|^2)^{-1} & \text{if } k - j = 1 \\
-\overline{q}(1 - |q|^2)^{-1} & \text{if } k - j = -1 \\
0 & \text{otherwise.}
\end{cases}
\]

This yields

\[
\kappa_G(z, w) = ZG^{-1}\overline{W} = \frac{(1 - qz)(1 - \overline{qw})}{(1 - |q|^2)(1 - zw)} = \frac{\psi(z)\overline{\psi(w)}}{1 - zw}
\]

where

\[
\psi(z) = \frac{1 - qz}{\sqrt{1 - |q|^2}}.
\]
6.3 Inverse fractional Gaussian noise

Given $0 < h < 1$, the classical complex fractional Gaussian noise of Hurst index $h$ is a sequence $(\Delta_n)_{n=1}^\infty$ of centred Gaussian random variables, such that $\mathbb{E}(\Delta_n \Delta_m) = 0$ and with covariance structure:

$$\gamma(k) := \mathbb{E}(\Delta_n \Delta_{n+k}) = \frac{1}{2}|k|^{2h} + \frac{1}{2}|k-1|^{2h} - |k|^{2h}, \ k, n \in \mathbb{N}. \quad (19)$$

The covariance matrix of $(\Delta_n)_{n=0}^\infty$ is the Toeplitz matrix $G$ given by

$$G = (\gamma(k-j))_{k,j=1}^\infty. \quad (20)$$

(The particular case $h = 1/2$ corresponds i.i.d random variables.) It is well-known that the matrix $G$ is invertible (see for example [2].) Unfortunately an explicit inverse of $G$ is not known. Consider its inverse matrix $G^{-1}$. A sequence of Gaussian random variables with covariance matrix $G^{-1}$ shall be called the inverse fractional Gaussian noise of index $h$. Some properties of the zeros of the random polynomial $\sum_{k=0}^n \Delta_k x^k$ and the power series $\sum_{k=0}^\infty \Delta_k x^n$, where $(\Delta_n)$ is the fractional Gaussian noise are given in [11] and [13]. Here, we are interested in the function $f(z) = \sum_{n=1}^\infty \xi_n z^{n-1}$, where $(\xi_n)$ is the inverse fractional Gaussian noise. It is well-known (using an argument by Sinai [17, Theorem 2.1]) that the matrix $G$ admits a spectral density function $\varphi_h$ given by

$$\varphi_h(\theta) = C(h) e^{\pi i \theta} - 1 \left( \sum_{n=-\infty}^{\infty} \frac{1}{|\theta + n|^{2h+1}} \right), \ \theta \in \mathbb{T}, \ \theta \neq 0 \quad (20)$$

where $C(h)$ is a normalising constant given by

$$C(h) = \frac{\zeta(-2h)}{2\zeta(1+2h)}$$

where $\zeta(.)$ is the Riemann zeta function. Clearly

$$\varphi_h(\theta) = 4C(h) (\sin^2 \pi \theta) \sum_{n=0}^{\infty} \left( \frac{1}{(n+\theta)^{2h+1}} + \frac{1}{(n+1-\theta)^{2h+1}} \right) \quad (21)$$

where $\zeta(.,.)$ is the classical Hurwitz zeta function.

It is not difficult to see that the function $\varphi_h$ is continuous on $(0, 1)$ and satisfies

$$\varphi_h(t) = O(t^{1-2h}(1-t)^{-2h}), \ \text{for } t \text{ near 0 or 1.}$$

This implies that both functions $\varphi(t)$ and $1/\varphi(t)$ are integrable on the unit circle. The inverse matrix $G^{-1}$ is, therefore, such that

$$(G^{-1})_{kj} = \int_{\mathbb{T}} \frac{e^{2\pi i (k-j)t}}{\varphi_h(t)} dt, \ \text{for } j + k \to \infty.$$
(See D’Ambrogi-Ola [2].) This yields that $G^{-1}$ is asymptotically a Toeplitz matrix in the sense that for each $k, j$ fixed:

$$\lim_{n \to \infty} (G^{-1})_{k+n,j+n} = \int_{\mathbb{T}} e^{-2\pi i (k-j)t} \varphi_h(t) \, dt = (1/\varphi_h)(k-j).$$

This implies in particular that $(G^{-1})_{k,j} \to 0$ for $k + j \to \infty$ and hence $\sup_{k,j} |(G^{-1})_{k,j}| < \infty$ which guarantees that for each $z, w \in \mathbb{D}$ the series $Z^T G^{-1} W$ converges (for $Z = (1, z, z^2, \ldots)$ and $W = (1, w, w^2, \ldots)$). The corresponding space $H^2_G(\mathbb{D})$ is the class of functions $g \in H^2(\mathbb{D})$, such that

$$\int_{\mathbb{T}} \left| g(e^{2\pi i \theta}) \right|^2 \varphi_h(\theta) \, d\theta < \infty.$$ 

For $h$ varying in $(0, 1)$, this yields a family of sub-spaces of the Hardy space $H^2(\mathbb{D})$. The exact entries of the matrix $G^{-1}$ are not known, and therefore, we do not have an explicit representation of the kernel $\mathbb{K}_G(z, w)$ as in the first two examples. However, Theorem 1.1 yields that if $(\xi_n)_{n \in \mathbb{N}}$ is a zero-mean complex Gaussian sequence of covariance matrix $G^{-1}$ and zero pseudo-covariance, then the zeros of $f(z) = \sum_{n=1}^{\infty} \xi_n z^{n-1}$ constitute a determinantal point process.

As in the general case one can compute from the sequence of polynomials $1, z, z^2, \ldots$ a sequence of orthonormal polynomials $\{P_n(z) : n = 1, 2, \ldots\}$ and deduce that if $(\chi_n)$ is a sequence of i.i.d standard Gaussian random variables, then the zeros of $f(z) = \sum_{n=1}^{\infty} \chi_n P_n(z)$ constitute a determinantal point process.

In the limit case, where $h = 0$, the fractional Gaussian noise with index $h$ is such that the covariance matrix $G$ is given by

$$G_{k,k} = 1, G_{k,k+1} = G_{k+1,k} = -1/2 \text{ and } G_{k,j} = 0 \text{ for } |k-j| \geq 2,$$

and it is not invertible. However, it still determines a determinantal point process. The spectral density function of $G$ is

$$\varphi_0(\theta) = 1 - \cos(2\pi \theta), \quad \theta \in \mathbb{T}.$$ 

From the sequence of polynomials $(1, z, z^2, z^3, \ldots)$, we derive the orthonormal sequence:

$$P_n(z) = \left( \frac{2}{n(n+1)} \right)^{1/2} (1 + 2z + 3z^2 + \ldots + nz^{n-1}), \quad n = 1, 2, \ldots$$

In this case, the kernel of $H^2_G(\mathbb{D})$ is explicitly given by

$$\mathbb{K}_0(z, w) = \sum_{n=1}^{\infty} P_n(z) \overline{P_n(w)} = \frac{2}{(1-z)(1-w)(1-z\overline{w})}, \quad z, w \in \mathbb{D}.$$ 

This is exactly the limit case of the kernel given in Example 6.1 when the parameter $q$ approaches $-1/2$. Then, for a sequence $(\chi_n)$ of i.i.d standard Gaussian variables, the zeros of
constitute a determinantal point process.

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