MINIMAL RESOLUTIONS OF MONOMIAL IDEALS

JOHN EAGON, EZRA MILLER, AND ERIKA ORDOG

Abstract. An explicit combinatorial minimal free resolution of an arbitrary monomial ideal $I$ in a polynomial ring in $n$ variables over a field of characteristic 0 is defined canonically, without any choices, using higher-dimensional generalizations of combined spanning trees for cycles and cocycles (hedges) in the upper Koszul simplicial complexes of $I$ at lattice points in $\mathbb{Z}^n$. The differentials in these sylvan resolutions are expressed as matrices whose entries are sums over lattice paths of weights determined combinatorially by sequences of hedges (hedgerows) along each lattice path. This combinatorics enters via an explicit matroidal expression for the Moore–Penrose pseudoinverses of the differentials in any CW complex as weighted averages of splittings defined by hedges. This Hedge Formula also yields a projection formula from CW chains to boundaries. The translation from Moore–Penrose combinatorics to free resolutions relies on Wall complexes, which construct minimal free resolutions of graded ideals from vertical splittings of Koszul bicomplexes. The algebra of Wall complexes applied to individual hedgerows yields explicit but noncanonical combinatorial minimal free resolutions of arbitrary monomial ideals in any characteristic.

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Overview.
Irving Kaplansky had a habit of circulating to his students precise problem lists for potential dissertation topics. One such list contained a problem on ideals generated by subdeterminants of a matrix, which resulted in \[\text{EN62}\]. A later list, we speculate, concerned monomial ideals, resulting in Taylor’s thesis \[\text{Tay66}\], the first general construction of free resolutions for arbitrary monomial ideals. The problem of finding minimal free resolutions of monomial ideals in polynomial rings has been central to the combinatorial side of commutative algebra since then. The ultimate goal along these lines is a free resolution that is universal, canonical, combinatorial, and minimal. This means that the construction should work for any monomial ideal, involve no choices, and be explicit in terms of the discrete input that determines a monomial ideal, in addition to resulting in a resolution that has no redundancy in the algebraic or numerical senses. In the intervening decades, Kaplansky’s problem has stimulated an enormous amount of research on the algebraic, combinatorial, and homological structure of monomial ideals, including hundreds of research papers and several influential books.

Formulas for the Betti numbers—the ranks of the free modules in a minimal free resolution—based on the combinatorial topology of simplicial complexes have been known since the 1970s through work of Hochster \[\text{Hoc77}\] and others, but the differentials of the resolutions have remained elusive. All prior studies that produce differentials in resolutions of monomial ideals have dispensed with one or more of the desired properties. For example,

- nonminimal resolutions include those constructed by Taylor \[\text{Tay66}\] and Lyubeznik \[\text{Lyu88}\], as well as hull resolutions \[\text{BS98}\] and Buchberger resolutions \[\text{OW16}\], with the Taylor and hull resolutions being canonical; and
- nonuniversal resolutions have been constructed to minimally resolve many special classes of monomial ideals, including Eliahou–Kervaire resolutions of stable ideals \[\text{EK90}\], Scarf resolutions of generic monomial ideals \[\text{BPS98, MSY00}\] (these latter ones give rise as well to nonminimal resolutions of arbitrary monomial ideals by generic deformation), and planar graph resolutions of trivariate monomial ideals \[\text{Mil02}\].

The strongest structural result we know for minimal free resolutions of monomial ideals is that they all admit hcw-poset structures \[\text{CT19}\]—essentially poset-theoretic generalizations of the cellular structures in \[\text{BS98}\]. However, this poset framework does not construct resolutions but rather imposes structures a posteriori on a given resolution.

To date, no universal minimal construction has been available at all, regardless of canonicality or combinatoriality: it has not heretofore been known how to appropriately relate the various simplicial homology groups that categorify the multigraded Betti numbers. The sylvan resolutions introduced here use Moore–Penrose pseudoinverses of differentials in CW complexes, combinatorially characterized in terms of...
higher-dimensional analogues of spanning trees for cycles and cocycles, to produce a universal, canonical, combinatorial construction of minimal free resolutions of arbitrary monomial ideals over fields of characteristic 0. In any characteristic, the spanning-tree framework produces noncanonical but nonetheless universal combinatorial constructions of minimal free resolutions of monomial ideals.

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1.1. Koszul simplicial complexes and Hochster’s formula.

The combinatorics of minimal resolutions of monomial ideals is grounded in the local combinatorics near lattice points in the partially ordered set of exponent vectors.

**Definition 1.1.** For a monomial ideal $I$ and a nonnegative integer vector $b \in \mathbb{N}^n$ with $n$ entries, the (upper) **Koszul simplicial complex** of $I$ in degree $b$ is

$$K^b I = \{ \tau \in \{0, 1\}^n \mid x^{b-\tau} \in I \}. $$

That is, standing at the lattice point $b$, thought of as an exponent vector on a monomial in the ideal $I$, one looks backward to see which (combinations of distinct) coordinate directions one can move along to remain in $I$.

**Theorem 1.2** (Hochster’s formula). Fix a monomial ideal $I \subseteq k[x]$ and a degree vector $b \in \mathbb{N}^n$. There is a natural isomorphism of vector spaces

$$\text{Tor}_i(k, I)_b = \tilde{H}_{i-1}(K^b I; k). $$

Consequently, the Betti numbers of $I$ in degree $b$ can be expressed as

$$\beta_{i, b}(I) = \dim_k \tilde{H}_{i-1}(K^b I; k). $$

For an exposition and proof, see [MS05, Theorem 1.34] and surrounding material.

Theorem 1.2 is the sense in which simplicial homology categorifies monomial Betti numbers. At issue in Kaplansky’s problem is how to categorify the differentials in a minimal free resolution of $I$. More precisely, any attempt to produce general minimal free resolutions of arbitrary monomial ideals must reduce—explicitly or implicitly—to solving the following concrete problem.

**Problem 1.3.** For a monomial ideal $I \subseteq k[x]$, produce vector space homomorphisms

$$\tilde{H}_i(K^b I; k) \to \bigoplus_{a < b} \tilde{H}_{i-1}(K^a I; k) $$

for all $i \in \mathbb{N}$ and multigraded degrees $b \in \mathbb{N}^n$ whose induced $k[x]$-module homomorphisms

$$\bigoplus_{b \in \mathbb{N}^n} \tilde{H}_i(K^b I; k) \otimes_k k[x](-b) \to \bigoplus_{a \in \mathbb{N}^n} \tilde{H}_{i-1}(K^a I; k) \otimes_k k[x](-a) $$

constitute a free resolution of $I$. 
Such a free resolution would automatically be minimal, by the Betti number com-
putation in Theorem 1.2. To connect the vector space homomorphisms to the induced
module homomorphisms in more detail, first see the left-hand side of the vector space
homomorphism as $\text{Tor}_{i+1}(k, I)_b$. Thinking of it as (the $k$-linear span of) a basis for
the $(i+1)^{st}$ syzygies in degree $b$, the differential in a minimal free resolution preserves
the degree $b$ while taking this basis to homological stage $i$. The summands in homo-
logical stage $i$ that contribute nonzero components to degree $b$ have the natural form
$\text{Tor}_i(k, I)_a \otimes_k k[x]$ for some $a < b$. The degree $b$ component of this free summand is

$$(\text{Tor}_i(k, I)_a \otimes_k k[x])_b = x^{b-a} \text{Tor}_i(k, I)_a = \widetilde{H}_{i-1}(K^a I; k)$$

as an ungraded vector space. (To keep track of the grading, the right-hand side vector
space here would have to be shifted into multigraded degree $b$.) Taking the direct sum
over $a < b$ yields the right-hand side of the vector space homomorphism in Problem 1.3.

1.2. Sylvan combinatorics of the canonical differential.

Given a free resolution whose syzygy modules have specified bases, the differentials
can be expressed by matrices of scalars using monomial matrices [Mi00, Section 3].
However, one of the fundamental obstacles to overcome in expressing an explicit, closed-
form description of a minimal free resolution of an arbitrary monomial ideal is how to
present a homomorphism canonically between homology vector spaces, which do not
possess natural bases. Our approach is to specify a linear map $\widetilde{C}_i K^b I \to \widetilde{C}_{i-1} K^a I$
from $i$-chains to $(i-1)$-chains using their natural bases but then ensure that this linear map
induces a well defined homology homomorphism $\widetilde{H}_i K^b I \to \widetilde{H}_{i-1} K^a I$. (The field $k$
is fixed throughout this discussion and suppressed from the notation.) Thus, given
any cycle of dimension $i$, expressed in the basis of $i$-simplices in $K^b I$, the closed-form
description acts on each term in the cycle to produce a cycle expressed in the basis of
$(i-1)$-simplices in $K^a I$. Different input cycles can yield different output cycles, a priori,
even when they represent the same homology class, as long as homologous input cycles
yield homologous output cycles. That said, our formulation takes homologous cycles
to the same cycle, inducing a homomorphism $\widetilde{H}_i K^b I \to \widetilde{Z}_{i-1} K^a I$; see Remark 3.12.

The combinatorial description of the differential in our canonical sylvan resolution
in characteristic 0, which occupies Section 3, culminating in Definition 3.6 and Theo-
rem 3.7, is an explicit, combinatorially defined linear map

$$D : \widetilde{C}_i K^b I \to \widetilde{C}_{i-1} K^a I.$$ 

Specifying $D$ is the same as specifying its entries $D_{\sigma \tau}$ for $\sigma \in \widetilde{C}_i K^b I$ and $\tau \in \widetilde{C}_{i-1} K^a I$.
The combinatorics is matroidal, generalizing that of spanning trees in graphs (see
Section 1.3), applied to the upper Koszul simplicial complexes of $I$ at lattice points
in $\mathbb{Z}^n$. The entries $D_{\sigma \tau}$ are expressed as weighted sums over all saturated decreasing
lattice paths from $b$ to $a$, where the weights come from

- coefficients of faces in unique circuits or boundaries obtained by throwing one
  additional facet into a higher-dimensional analogue of a spanning tree, and
orders of torsion groups in homology generated by images of facets of higher-dimensional analogues of spanning trees.

These torsion numbers are unavoidable in higher dimension. They reflect the fact that boundaries of individual faces of can contribute to bases for sublattices of varying index.

1.3. Organizing homology: Koszul bicomplexes.

The homology homomorphisms in Problem 1.3 stratify naturally according to how far \( a \) is from \( b \) in lattice distance, namely \(|b| - |a|\), where \(|b| = b_1 + \cdots + b_n\) is the coordinate sum. Thus Problem 1.3 can be rephrased as looking for homology maps

\[
\tilde{H}_i K^b I \to \bigoplus_{|a|=|b|-j} \tilde{H}_{i-1} K^a I \quad \text{for} \quad j \geq 1.
\]

All of these homomorphisms are to be thought of as occurring in fixed \( \mathbb{N}^n \)-degree \( b \), as in the discussion at the end of Section 1.1.

Arranging the direct sums in an array, using standard choices of where to place each homology group, suggests that the required maps should be represented by arrows as if they were from a progression of pages in a spectral sequence. This observation is formalized in a two-step process (Section 6). First, express the reduced homology as Koszul homology, using differentials depicted (say) vertically, via Theorem 1.2. Although all of these vector spaces occur in \( \mathbb{N}^n \)-degree \( b \), they are the degree \( b \) components of multi-graded free modules whose generators lie naturally in various degrees \( a \prec b \); again see Section 1.1. Taking the arrangement of the homology groups and the standard grading into consideration immediately reveals how to connect these homology groups with a second, horizontal differential. The resulting fourth-quadrant spectral sequence \( E^r_{pq} \) of this bicomplex—that is, the one with vertical homology first, followed by horizontal homology—has its \( r = 1 \) page filled with reduced homology groups of Koszul simplicial complexes: \( E^1_{pq} = \bigoplus_{|a|=p} \tilde{H}_{p+q-1} K^a I \), so \( \tilde{H}_{i-1} K^a I \) sits at \( p = |a| \) and \( q = i - p \).

The differentials on the various pages of this spectral sequence automatically produce canonical homomorphisms from subquotients of \( \tilde{H}_i K^b I \) to subquotients of \( \tilde{H}_{i-1} K^a I \) for \( a \prec b \) (see Remark 6.11). The page on which this homomorphism occurs is \( E^r_{pq} \) for \( r = |b| - |a| \). Alas, there is no categorically natural way to lift these homomorphisms on subquotients to homomorphisms on the intact homology of the Koszul simplicial complexes; this can be viewed as the core reason why the problem of constructing minimal free resolutions of arbitrary monomial ideals has resisted solution so steadfastly. The remedy is to split the vertical differential, which forces the Koszul simplicial homology to split in such a way that the spectral sequence differentials collate into a compendium differential \cite{Eag90} (see Section 7 for an exposition) that solves Problem 1.3.

1.4. Algebra of vertically split bicomplexes.

Given any choice of splitting for the columns of a bicomplex, the differentials in its vertical-then-horizontal spectral sequence become the differentials in an associated Wall
complex [Eag90]. The modules in the Wall complex are the direct sums of the $E^1$ terms; for the Koszul bicomplex, these are the Koszul simplicial homology groups of interest. The Wall differentials are obtained essentially by projection to split summands that correspond to the relevant subquotients. Thus, in the context of Koszul bicomplexes, once a splitting has been selected, the spectral sequence homomorphisms on subquotients lift to homomorphisms on the intact $E^1$ homology terms.

When the Wall complex construction was introduced in the context of free resolutions [Eag90], exactness of Wall complexes was proved in general (except for a circular argument along the way, which we correct in the proof of our Proposition 7.7), and it was employed to construct minimal, noncanonical free resolutions of Stanley–Reisner rings of triangulations of spheres. In addition, it was suggested that Wall complexes can, in principle, be used to construct minimal free resolutions of arbitrary square-free monomial ideals—and hence, implicitly, all monomial ideals, by polarization—but that explicit constructions require explicit splittings [Eag90, Section 5]. A proof of this squarefree claim that works directly for arbitrary monomial ideals is presented in Theorem 5.3 including the crucial requirement that the constructed free resolutions resolve the ideal itself and not the associated graded module of a filtration of the ideal.

1.5. Combinatorics of Moore–Penrose pseudoinverses.

Splitting is a priori a choice. But as our chain complexes come with a canonical metric induced by topologically and combinatorially meaningful bases—namely the faces of the relevant Koszul simplicial complexes—there is a canonical splitting to use, in characteristic 0, at least: the Moore–Penrose pseudoinverse. It had already been suggested to use this splitting [Eag90, Proposition 3.1.4], but it was only applied in limited settings, as there was at the time no known formula for it useful in more generality.

One of our main contributions is an explicit combinatorial expression, the Hedge Formula (Theorem 5.5), for the Moore–Penrose pseudoinverses of the differentials in any CW complex as weighted averages of splittings defined by hedges (Definition 2.1), which generalize combined spanning trees for cycles and cocycles to arbitrary dimension. The Hedge Formula takes its cue from (and uses) the Higher Projection Formula of Catanzaro, Chernyak, and Klein [CCK15, Theorem A], which projects orthogonally from chains to cocycles. As an easy consequence, we derive a projection formula from chains to boundaries (Corollary 5.7).

The developments here, as in [CCK15], rest on the theory of higher-dimensional analogues of spanning trees initiated by Kalai [Kal83], as developed by Duval, Klivans, and Martin [DKM09, DKM11], Petersson [Pet09], and Lyons [Lyo09]. In particular, the weights fall naturally out of torsion subgroups that account for the transition between integer and rational homology, the point being that basis exchange over the integers incurs a penalty reflecting the orders of the images of the corresponding basis elements in homology (see Lemma 5.2).
1.6. Sylvan combinatorics of minimal resolutions.

The Hedge Formula for CW differential pseudoinverses drives our canonical sylvan combinatorial formula (Definition 3.6 and Theorem 3.7) for minimal free resolutions of monomial ideals in characteristic 0: the splitting moves up one dimension in the Koszul simplicial complex at a lattice point $b$, and the Wall differential subsequently returns to the original simplicial dimension, distributing the boundary among the neighboring lattice points beneath the starting point $b$. Iterating yields a sum over decreasing lattice paths, labeled by chain-link fences of faces to keep track of the up-down alternation of pseudoinverses and Wall differentials, with weights coming from torsion homology as in the Hedge Formula.

Wall complexes are flexible enough, in their abstract algebraic avatar, to use any vertical splitting. Arbitrary splittings of Koszul simplicial differentials therefore produce minimal free resolutions; this is Theorem 9.1, the master formula for Wall resolutions. Moore–Penrose pseudoinverses serve the purpose of manufacturing minimal free resolutions without choices, but the canonical choice is not the only choice: other combinatorial splittings, arising from individual hedgerows (Definition 3.1) contributing summands to the Moore–Penrose pseudoinverse formula, produce perfectly good combinatorial minimal resolutions. Better, these splittings—and hence the corresponding sylvan minimal resolutions—require no division and hence are defined over any field (Corollary 10.4). This raises the question: are extant families of minimal resolutions sylvan, with apt choices of hedges? That is, can they be constructed as Wall complexes for suitable splittings? The answer is yes for Eliahou–Kervaire resolutions of stable ideals [EK90] and planar graph resolutions of trivariate ideals [Mi02], but the question is open for all other cases, such as Scarf resolutions of generic ideals [BPS98, MSY00].

Our path toward considering the weighted-average viewpoint on Moore–Penrose pseudoinverses came from the case of three variables, in which we attempted to construct minimal free resolutions by averaging all planar cellular minimal resolutions—those supported on planar graphs [Mi02]. Whether this can be made more than mere motivation is left as an open question: is the canonical sylvan resolution a (weighted) average of minimal resolutions from choices of hedges, in a global sense rather than merely locally in the Koszul simplicial complex at each lattice point?

1.7. Logical structure.

It seems worthwhile to provide a short, direct path to a rigorous statement of the main result—the combinatorial description of canonical minimal free resolutions of monomial ideals Theorem 3.7—so Sections 2 and 3 are self-contained introductions to the requisite simplicial notions and to the combinatorial assembly of these along descending lattice paths. The proof of Theorem 3.7, however, must wait until Section 9 as it relies on the Hedge Formula (Theorem 5.5), the Wall construction of minimal free resolutions of graded ideals via Koszul splittings (Theorem 8.3), and the Koszul simplicial formula for those (Theorem 9.1). No intervening result relies on the statement of Theorem 3.7.
The Hedge Formula for the Moore–Penrose pseudoinverses of the differentials in CW complexes (Theorem 5.5) is of independent interest—perhaps extending far beyond combinatorial commutative algebra—so a direct line to its statement and proof is provided immediately after the main theorem. The simplicial objects defined in Section 2 are fundamental to the Hedge Formula and hence cross-referenced repeatedly in the exposition of hedge splittings of CW complexes and Moore–Penrose pseudoinverses (Sections 4 and 5). In contrast, most of the objects in Section 3 are introduced solely for the purpose of stating the main theorem; hence they are only cross-referenced in the proof and applications of the main theorem (Sections 9 and 10).

The analysis of Koszul homology in Section 6 with an eye toward constructing minimal free resolutions of monomial ideals motivates the review of Wall complexes in Section 7 and the general construction of minimal resolutions of graded ideals from Wall complexes in Theorem 8.3. These three sections, which complete the line of thought initiated decades ago [Eag90], are logically independent from Sections 2–5.

1.8. Conventions.

**Convention 1.4** (Simplicial notions). A simplicial complex $K$ has its set $K_i$ of $i$-faces and integer reduced chain groups $\tilde{C}_i^Z K = \mathbb{Z}\{K_i\}$ with differential $\partial_i : \tilde{C}_i^Z K \to \tilde{C}_{i-1}^Z K$. Tensoring with any field $k$, such as the field $\mathbb{Q}$ of rational numbers, yields the reduced chain complex $\tilde{C}_i^k K = k \otimes \mathbb{Z} \tilde{C}_i K$ over $k$, with differential also denoted by $\partial$. The same conventions hold for the cycles $\tilde{Z}_i K = \ker \partial_i \subseteq \tilde{C}_i K$, boundaries $\tilde{B}_i K = \im \partial_{i+1} \subseteq \tilde{C}_i K$, and reduced homology $\tilde{H}_i K = \tilde{Z}_i K / \tilde{B}_i K$. To unclutter the notation, it helps to omit the superscript $Z$ or $k$; to that end, each section starts with an explicit assumption about which coefficient ring is the default. Also to unclutter, write $S \cup \sigma$ instead of $S \cup \{\sigma\}$ for the union of a subset $S \subseteq K_i$ with a singleton $\{\sigma\}$, and similarly write $S \setminus \sigma$ for the complement of $\{\sigma\}$ in $S$. Finally, for any subset $S \subseteq K_i$, let $\langle S \rangle$ be the smallest subcomplex of $K$ containing $S$ and all faces of dimension at most $(i - 1)$.

**Convention 1.5** (Polynomial and monomial notions). Fix, once and for all, an ideal $I$ in the polynomial ring $k[x]$ in $n$ variables $x = x_1, \ldots, x_n$ over a field $k$. Each section starts with an explicit assumption about which coefficient field is the default. Assume that $I$ is a monomial ideal unless otherwise explicitly stated. Monomials in $k[x]$ are denoted by $x^a$ for lattice points $a \in \mathbb{N}^n$. An unadorned tensor product $\otimes$ is to be understood as $\otimes_k$ unless otherwise stated.

2. Shrubs, stakes, and hedges in CW complexes

The default coefficient ring in this section is $\mathbb{Z}$, but also fix an arbitrary field $k$.

This section is presented in the generality of CW complexes, because it involves no additional complication, but the reader can safely restrict to the simplicial case, as that is the only case applied in this paper.
Definition 2.1. Fix a CW complex $K$.

1. A shrubbery in dimension $i$ is a subset $S_i \subseteq K_i$ whose set $\partial S_i = \{ \partial \tau \mid \tau \in S_i \}$ of boundaries is a $k$-basis for $\bar{B}_{i-1}^k K$.

2. A stake set is a subset $S_{i-1} \subseteq K_{i-1}$ whose complement $S_{i-1} = K_{i-1} \setminus S_{i-1}$ descends to a basis for $\bar{C}_{i-1}^k K/\bar{B}_{i-1}^k K$; equivalently, $\bar{C}_{i-1}^k K = k\{\bar{S}_{i-1}\} \oplus \bar{B}_{i-1}^k K$.

3. A hedge in $K$ of dimension $i$ is a choice of shrubbery in $K_i$ and stake set in $K_{i-1}$. A hedge of dimension $i$ may be expressed as $ST_i = (S_{i-1}, T_i)$. The set of all such hedges is denoted $ST_i(K)$. The set of all dimension $i$ shrubberies is denoted $T_i(K)$. The set of all stake sets $S_{i-1}$ is denoted $S_{i-1}(K)$. Faces in a stake set are called stakes.

Remark 2.2. In the literature shrubberies are often known as “spanning trees”, or “spanning forests”, or some variant; see [CCK15] and [DKM09], for example. We avoid these terms because they are deficient in certain ways—subcomplexes whose facets form shrubberies need not be connected in any appropriate sense (so should not be called trees) if the ambient CW complex is disconnected, and there could be forests that span in some appropriate sense but are nonetheless not spanning forests—and their precise definitions vary from paper to paper. But they explain our botanical terminology as well as our shrubbery symbol “$T$”, which classically stands for “tree”. Regardless of terminology or notation, a hedge is matroidal information, given by subsets of fixed bases, and is hence combinatorial in nature.

Remark 2.3. The notion of stake set is precisely dual to that of shrubbery. Indeed, $S_{i-1} \subseteq K_{i-1}$ is a stake set if and only if its set $\partial^\top S_{i-1} = \{ \partial^\top \sigma \mid \sigma \in S_{i-1} \}$ of coboundaries is a $k$-basis for the $i$-coboundaries $\bar{C}_i^k K$ under the transpose $\partial^\top$ of the differential $\partial$. Indeed, $V = U \oplus W \Leftrightarrow V^* = U^\perp \oplus W^\perp$ for any $k$-vector space $V$ of finite dimension; our situation has $V$ and $W$ being the chains and boundaries of dimension $i-1$ and $U = k\{\bar{S}_{i-1}\}$. Thus $V^* = V$ with its self-dual basis of faces, so $U^\perp = k\{S_{i-1}\}$ and $W^\perp = \bar{B}_{i-1}^k$.

Remark 2.4. Definition 2.12 implies that stakes are faces which, for the purpose of constructing bases, can be replaced by boundaries of shrubbery faces; see Lemma 2.7.

The usual property of a spanning tree in a graph is that every edge closes a unique circuit. The analogue for shrubberies is well known.

Lemma 2.5. Fix an $i$-face $\tau \in K_i$ and a shrubbery $T_i \subseteq K_i$. There is a unique chain $t \in k\{T_i\}$ such that $\tau - t$ is a cycle. Set $\zeta_{T_i}(\tau) = \tau - t \in \bar{Z}_i K$.

Proof. $\partial \tau$ is a boundary, so a unique chain $t \in k\{T_{i-1}\}$ has boundary $\partial t = \partial \tau$ by Definition 2.11 and $\tau - t$ is indeed a cycle because $\partial(\tau - t) = \partial \tau - \partial t = 0$. \qed

Remark 2.6. It can be helpful to think of Lemma 2.5 another way: every $i$-face $\tau \notin T_i$ lies in a unique “$T_i$-circuit” $\zeta_{T_i}(\tau) \in \tau + k\{T_i\}$ that is a cycle with coefficient 1 on $\tau$ in the CW complex with facets $\{\tau\} \cup T_i$. 
The next result extends the horticultural picture that goes with the terminology: each stake is tied to the tip of a unique shrub (see Definition 2.10).  

**Lemma 2.7.** Fix a hedge $ST_i$ in $K$ and a stake $\sigma \in S_{i-1} \subseteq K_{i-1}$. There is a unique chain $s \in k\{T_i\}$ whose boundary has coefficient 1 on $\sigma$ and 0 on all other stakes in $S_{i-1}$.

**Proof.** Definition 2.11 is set up precisely so that, as vector spaces over $k$,

$$\tilde{C}^k_{i-1}K = k\{S_{i-1}\} \oplus k\{\partial T_i\} = k\{S_{i-1}\} \oplus k\{S_{i-1}\}.$$  

That is, the vector subspace of $\tilde{C}^k_{i-1}K$ spanned by $S_{i-1}$ is just as much a complement to $k\{S_{i-1}\}$ as $k\{\partial T_i\}$ is. Hence projection to $k\{S_{i-1}\}$ modulo $k\{S_{i-1}\}$—that is, the projection that only keeps coefficients on stakes—induces an isomorphism

$$k\{\partial T_i\} \xrightarrow{\sim} k\{S_{i-1}\}.$$  

Each stake $\sigma \in S_{i-1}$ therefore corresponds to a unique boundary in $k\{\partial T_i\}$; by definition of the projection this boundary is the desired one having coefficient 1 on $\sigma$.  

**Lemma 2.8.** Fix an $i$-face $\rho \in K_i$ and a stake set $S_i \subseteq K_i$. There is a unique chain $r \in k\{S_i\}$ such that $\rho - r$ is a boundary in $K$. More precisely, if $\rho \neq r$ then $\rho - r$ is the generator of $k\{S_i \cup \rho\} \cap \tilde{B}^k_i K$ that has coefficient 1 on $\rho$. Set $b_{S_i}(\rho) = \rho - r$.

**Proof.** The intersection $k\{S_i \cup \rho\} \cap \tilde{B}^k_i K$ has dimension 1, and hence contains a unique element with coefficient 1 on $\rho$, because

$$\tilde{C}^k_i K = k\{S_i\} \oplus \tilde{B}^k_i K.$$  

**Remark 2.9.** An analogue of Remark 2.6 combines the two closely related Lemmas 2.7 and 2.8 with a similar geometric combinatorial interpretation: every $(i-1)$-stake $\rho \in S_{i-1}$ lies in a unique boundary with coefficient 1 on $\rho$, thought of as the “hedge rim” $b_{S_{i-1}}(\rho) \in \rho + k\{S_{i-1}\}$ because it is the boundary of the “shrub” $s$ from Lemma 2.8 (applied to $\sigma = \rho$) for any choice of shrubbery $T_i$; see Lemma 1.8 for the easy proof.

Circuits, shrubs, and hedge rims are the core simplicial combinatorial players.

**Definition 2.10.** Fix a CW complex $K$.

1. Fix a shrubbery $T_{i-1}$. An $(i-1)$-face $\sigma$ is cycle-linked to any $(i-1)$-face $\sigma' \in K_{i-1}$ with nonzero coefficient in the circuit of $\sigma$: the cycle $\zeta_{T_{i-1}}(\sigma)$ from Lemma 2.5.

2. Fix a hedge $ST_i$ in $K$. A stake $\sigma \in S_{i-1}$ is chain-linked to an $i$-face $\tau \in K_i$ if $\tau$ has nonzero coefficient in the shrub of $\sigma$: the chain $s$ from Lemma 2.7.

3. Fix a stake set $S_i$. An $i$-face $\rho \in K_i$ is boundary-linked to $\rho' \in K_i$ if $\rho'$ has nonzero coefficient in the hedge rim of $\rho$: the chain $r(\rho) = \rho - b_{S_i}(\rho)$ from Lemma 2.8.

Write $c_{\sigma}(\sigma', T_{i-1})$ and $c_{\sigma}(\tau, ST_i)$ and $c_{\rho}(\rho', S_i)$ for the coefficients on $\sigma'$ and $\tau$ and $\rho'$ in the circuit, shrub, and hedge rim of an $(i-1)$-face $\sigma$, an $(i-1)$-stake $\sigma$, and an $i$-face $\rho$.  

Example 2.11. In the following simplicial complex, choose the hedge $ST_2 = (S_1, T_2) = \{\{bc, cd\}, \{abc, bcd\}\}$ of dimension 2. The shrub and hedge rim of the stake $cd$ are

$$s(cd) = -abc + bcd$$

$$r(cd) = -ab + ac + bd.$$ 

Remark 2.12. Definition 2.10.2 allows $\tau \in K_i$, but in fact the shrub $s$ in Lemma 2.7 only has nonzero coefficients on faces in $T_i$, so a stake $\sigma \in S_{i-1}$ can only be chain-linked to a face $\tau \in T_i$. Similarly, Definition 2.10.3 allows $\rho' \in K - i$, but in fact the hedge rim $r(\rho)$ only has nonzero coefficients on non-stakes, so a face $\rho$ can only be boundary-linked to a non-stake $\rho' \in \overline{S}_i$. Note, furthermore, that if the input is already a non-stake $\rho \in \overline{S}_i$, then $r(\rho) = \rho$, so only $\rho' = \rho$ is possible, and $c_\rho(\rho, S_i) = 1$.

Hedges and shrubberies yield bases consisting of faces and boundaries of faces, but only over fields. Over the integers, the subgroups generated by these $\mathbb{Z}$-linearly independent sets need not be saturated. The resulting torsion quotients act as weights in combinatorial formulas for projections and splittings.

Definition 2.13. Fix a CW complex $K$. Write $A_{\text{tor}}$ for the torsion subgroup of any abelian group $A$. Recall from Convention 1.4 the notation $\langle X_i \rangle$ for any subset $X_i \subseteq K_i$ of $i$-faces in $K$.

1. The torsion group of a shrubbery $T_i$ is

$$\Theta_{T_i} = \tilde{H}_{i-1}(\langle T_i \rangle)_{\text{tor}}.$$ 

The order of the torsion group of a shrubbery $T_i$ is the torsion number

$$\theta_{T_i} = |\Theta_{T_i}|.$$ 

2. The torsion group of a stake set $S_{i-1}$ is

$$\Theta_{S_{i-1}} = \tilde{H}_{i-2}(\langle S_{i-1} \rangle)_{\text{tor}}.$$ 

The order of the torsion subgroup of a stake set $S_{i-1}$ is the torsion number

$$\theta_{S_{i-1}} = |\Theta_{S_{i-1}}|.$$ 

3. The torsion number of a hedge $ST_i$ is

$$\theta_{ST_i} = \theta_{S_{i-1}} \theta_{T_i}.$$ 

Abbreviate $\theta_i = \theta_{ST_i}$ for the torsion number of the ambient hedge when one has been unambiguously selected.
Even when the notation for the hedge is suppressed, torsion numbers are to be distinguished from the following universal integer invariants.

**Definition 2.14.** For a CW complex $K$, define

$$
\Delta_i^T K = \sum_{T_i \in T_i} \theta_{T_i}^2
$$

$$
\Delta_{i-1}^S K = \sum_{S_{i-1} \in S_{i-1}} \theta_{S_{i-1}}^2
$$

and

$$
\Delta_i^{ST} K = \sum_{ST_i} \theta_{ST_i}^2
$$

$$
= (\Delta_{i-1}^S K)(\Delta_i^T K)
$$

to take into account a contribution from every shrubbery, stake set, or hedge. These numbers may be abbreviated as $\Delta_i^T$ or $\Delta_{i-1}^S$ or $\Delta_i^{ST}$ or even simply $\Delta_i$ if the ambient complex $K$ and the nature of the torsion are unambiguous.

### 3. Canonical sylvan resolutions

The default coefficient ring in this section is a field $k$ of characteristic 0.

The differential in the canonical sylvan resolution is based on averaging over all hedge choices along lattice paths. This section develops notation for hedges along lattice paths. It contains no results—only definitions and notation—except for the main theorem at the end (Theorem 3.7).

**Definition 3.1.** Fix a monomial ideal $I$ and a saturated decreasing lattice path $\lambda$ from $b$ to $a$, meaning that adjacent nodes in the path differ by a standard basis vector of the integer lattice $\mathbb{Z}^n$. Write $\Lambda(a, b)$ for the set of such paths. A *hedgerow* on $\lambda$ is

- a stake set $S^b_i \subseteq K^b I$ at the upper endpoint $b$,
- a hedge $ST^c_i = (S^c_{i-1}, T^c_i)$ on the Koszul simplicial complex $K^c I$ for each interior (i.e., non-endpoint) vertex $c$ of $\lambda$, and
- a shrubbery $T^a_{i-1} \subseteq K^a I$ at the lower endpoint $a$.

It is convenient to write

$$
ST^\lambda_i = (S^\lambda_{i-1}, T^\lambda_i)
$$

for this hedgerow on $\lambda$, where—to emphasize—this notation carries an implicit stake set $S^b_i$ and shrubbery $T^a_{i-1}$ at the upper and lower endpoints. It is also convenient to write

$$
b = b_0, b_1, \ldots, b_{\ell-1}, b_\ell = a
$$

to denote the lattice points on the path $\lambda$, even if sometimes $c$ is written for an otherwise unnamed vertex $b_j$. The path $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ can be identified with the sequence

$$
\lambda_j = b_{j-1} - b_j \text{ for } j = 1, \ldots, \ell
$$

of steps in the path, so $x^{\lambda_j} \in \{x_1, \ldots, x_n\}$ is one of the $n$ variables.
Definition 3.2. Definitions 2.13 and 2.14 lead to the notations

\[
\begin{align*}
\theta_{i,b} & = \theta_{S_i^b}, \\
\theta_{i,b_j} & = \theta_{ST_i^{b_j}}, \\
\theta_{i,a} & = \theta_{T_{i-1}^a}, \\
\theta_{i,\lambda} & = \prod_{j=0}^{\ell} \theta_{i,b_j}
\end{align*}
\]

\[
\begin{align*}
\Delta_{i}^b & = \Delta_{i}^{S_i^b} K^b I, \\
\Delta_{i}^{b_j} & = \Delta_{i}^{ST_i^{b_j}} K^b_j I \quad \text{for } j = 1, \ldots, \ell - 1, \\
\Delta_{i-1}^a & = \Delta_{i-1}^{T_{i-1}^a} K^a I, \\
\Delta_{i,\lambda} & = \prod_{j=0}^{\ell} \Delta_{i}^{b_j}
\end{align*}
\]

for the integer invariants that take into account hedges in the sequence of Koszul simplicial complexes of \(I\) along the lattice path \(\lambda\).

Definition 3.3. Fix a lattice path \(\lambda \in \Lambda(a, b)\). A chain-link fence \(\varphi\) from an \(i\)-simplex \(\tau\) to an \((i - 1)\)-simplex \(\sigma\) along \(\lambda\) is a choice of hedgerow \(ST_i^\lambda\) and a sequence

\[
\sigma \rightarrow \sigma_{\ell} \rightarrow \sigma_{\ell-1} \rightarrow \ldots \rightarrow \sigma_{1} \rightarrow \tau
\]

in which \(\tau_j \in K_i^{b_j} I\) and \(\sigma_j \in K_{i-1}^{b_j} I\) and

- \(\tau_0 \rightarrow \tau\) the simplex \(\tau\) is boundary-linked to \(\tau_0\) via the stake set \(S_{i}^b\);
- \(\sigma \rightarrow \sigma_{\ell}\) the simplex \(\sigma_j \in S_{i-1}^{b_j}\) for \(j = 1, \ldots, \ell - 1\) is a stake chain-linked to \(\tau_j\);
- \(\sigma \rightarrow \sigma_{\ell}\) the simplex \(\sigma_j\) for \(j = 1, \ldots, \ell\) equals the facet \(\tau_{j-1} - \lambda_j\) of the simplex \(\tau_{j-1}\);
- \(\sigma \rightarrow \sigma_{\ell}\) the simplex \(\sigma_\ell \in K_{i-1}^a I\) is cycle-linked to \(\sigma\).

The set \(\Phi(\lambda)\) of chain-link fences along \(\lambda\) has the subset \(\Phi_{\sigma\tau}(\lambda)\) with posts \(\tau\) and \(\sigma\).

Definition 3.4. Each chain-link fence edge has a weight (an integer; see Lemma 9.4):

- the boundary-link \(\tau_0 \rightarrow \tau\) has weight \(\theta_{i,b}^2 c_\tau(\tau_0, S_i^b)\),
- the chain-link \(\tau_j \rightarrow \sigma_j\) has weight \(\theta_{i,b_j}^2 c_{\sigma_j}(\tau_j, ST_i^{b_j})\),
- the containment \(\sigma_j \rightarrow \tau_{j-1}\) has weight \((-1)^{\sigma_j \subset \tau_{j-1}}\), and
- the cycle-link \(\sigma \rightarrow \sigma_{\ell}\) has weight \(\theta_{i,a}^2 c_{\sigma_\ell}(\sigma, T_{i-1}^a)\).

The weight of the chain-link fence \(\varphi\) is the product \(w_\varphi\) of the weights on its edges.

Remark 3.5. The notion of chain-link fence \(\varphi\) includes the ambient hedgerow on \(\lambda\). It is convenient to define \(\varphi\) to be subordinate to the given hedgerow \(ST_i^\lambda\), which we express as \(\varphi = (\sigma, \sigma_{\ell}, \tau_{\ell-1}, \sigma_{\ell-1}, \ldots, \sigma_2, \tau_1, \sigma_1, \tau_0, \tau) \vdash ST_i^\lambda\).
Definition 3.6. Fix a monomial ideal $I$. The canonical sylvan homomorphism

$$D = D^{ab} : \tilde{C}_i K^b I \to \tilde{C}_{i-1} K^a I$$

is given by its sylvan matrix, whose entry $D_{\sigma \tau}$ for $\tau \in K^b I$ and $\sigma \in K^a_{i-1} I$ is the sum of the weights of all chain-link fences from $\tau$ to $\sigma$ along all lattice paths from $b$ to $a$:

$$D_{\sigma \tau} = \sum_{\lambda \in \Lambda(\alpha, \beta)} \frac{1}{\Delta_i, \lambda} \sum_{\varphi \in \Phi_{\sigma \tau}(\lambda)} w_{\varphi}. \tag{3.6}$$

Theorem 3.7. Fix a monomial ideal $I$. The canonical sylvan homomorphism for each comparable pair $b \succ a$ of lattice points induces a homomorphism $\tilde{Z}_i K^b I \to \tilde{Z}_{i-1} K^a I$ that vanishes on $\tilde{B}_i K^b I$, and hence it induces a well defined canonical sylvan homology morphism $\tilde{H}_i K^b I \to \tilde{H}_{i-1} K^a I$. The induced homomorphisms

$$\tilde{H}_i K^b I \otimes \mathbb{k}[x](-b) \to \tilde{H}_{i-1} K^a I \otimes \mathbb{k}[x](-a)$$

of $\mathbb{N}^n$-graded free $\mathbb{k}[x]$-modules constitute a minimal free resolution of $I$.

The proof is at the end of Section 9.

Example 3.8. One of the sticking points for any construction of canonical minimal free resolutions of monomial ideals is $I = \langle xy, yz, xz \rangle$, which has Betti number $\beta_{1,111}(I) = 2$ arising from the Koszul simplicial complex $K^{111} I$ whose facets are the three vertices

depicted as little blue dots. The two linearly independent syzygies in degree 111 split equally among the three generators, which is problematic for any construction that produces matrices for the differential, because any choice of basis breaks the symmetry. The sylvan method avoids this problem by defining the homomorphism on chains instead of on homology classes. Here is how it works for $I = \langle xy, yz, xz \rangle$.

Start by computing, say, the sylvan matrix $D^{110,111}$. There is only one lattice path $\lambda \in \Lambda(110, 111)$, since the length is 1. For this path, the stake set at the initial post 111 is forced to be $S_0^{111} = \emptyset$ because $K^{111} I$ has dimension 0 and hence no faces of dimension 1 to take the boundary of. The shrubbery at the terminal post 110 is forced to be the empty set of faces, which is written $T_{-1}^{110} = \emptyset$ so as not to confuse it with
the set \( \{ \emptyset \} \) consisting of the empty face. As there is only one hedgerow on \( \lambda \), and all of the torsion numbers equal 1 because the number of vertices is too small for torsion in homology, \( \Delta_{0,\lambda} = 1 \). To construct a chain-link fence along \( \lambda \), only one terminal post \( \sigma = \emptyset \) is available. Trying \( x \) as initial post yield no fences: the boundary link \( x - x \) is forced, because \( x \) is the unique non-stake \( \tau' \) in \( K^{111}I \) such that \( x - \tau' \) is a boundary, but then \( z = \lambda_1 \) is not a face of \( x \), so the fence has no continuation.

\[
\begin{array}{c}
  \emptyset & x & z \\
  0 & 1 & 0 \\
\end{array}
\]

Similarly, no fence has initial post \( y \). But the initial post \( z \) yields one fence, as depicted. The conclusion is that \( \Phi_{\emptyset,x} = \emptyset = \Phi_{\emptyset,y} \) and \( |\Phi_{\emptyset,z}| = 1 \), with all of the edge coefficients and torsion numbers equal to 1. This determines the top row of the matrix

\[
\begin{array}{c}
  \emptyset & x & y & z \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 1 \\
\end{array}
\]

in which the other two rows are determined by symmetry. The blocks in this matrix are the sylvan matrices \( D^{110,111}, D^{101,111}, \) and \( D^{011,111} \). Its rows and columns are labeled by the faces of the corresponding Koszul simplicial complexes. The cycles \( x - y, z - x, \) and \( y - z \) correspond to the column vectors

\[
\begin{bmatrix}
  1 \\
  -1 \\
  0
\end{bmatrix}, \quad \begin{bmatrix}
  -1 \\
  0 \\
  1
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  1 \\
  -1
\end{bmatrix}.
\]

The image of, say, \( x - y \) is the \( N^3 \)-degree 111 element \( x \cdot \emptyset - y \cdot \emptyset \) that is \( x \) times the free generator of \( H_{-1}K^{101} \otimes \langle yz \rangle \) minus \( y \) times the free generator of \( H_{-1}K^{101} \otimes \langle xz \rangle \). Looking back at the staircase drawn at the beginning of this Example, the face \( x \) of the cycle \( x - y \) has moved back parallel to the \( x \)-axis to become \( x \cdot \emptyset \), and \( -y \) has moved back parallel to the \( y \)-axis to become \( -y \cdot \emptyset \). This behavior is fundamental to the sylvan construction: faces move back in directions parallel to axes, picking up the corresponding monomial coefficients as they go.

**Convention 3.9.** The general block matrix notational device illustrated by Example 3.8 works as follows. Choose an order in which to list the \( N^n \)-degrees \( a \) and \( b \) of nonzero Betti numbers \( \beta_{i,a}(I) \) and \( \beta_{i+1,b}(I) \) in homological stages \( i \) and \( i + 1 \), say \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \). The \( pq \) block is the matrix \( D^{a_pb_q} \), which takes chains in \( \tilde{C}_iK^{b_q}I \), thought of as column vectors indexed by the \( i \)-simplices in \( K^{b_q}I \), to chains in
\[ \tilde{C}_{i-1}K^{a_p}I, \] thought of as column vectors indexed by the \((i - 1)\)-simplices in \(K^{a_p}_{i-1}I\). The orderings on the \(\mathbb{N}^a\)-degrees \(a_p\) and \(b_q\) are depicted by writing ordered direct sums vertically. The orderings on the simplices are depicted by labeling the rows and columns of each block.

**Example 3.10.** It might be helpful to see a more complicated canonical sylvan resolution, with multiple lattices paths from \(b\) to \(a\) and chain-link fences along a fixed lattice path. Alas, the torsion numbers are all 1 in this example, since nontrivial torsion numbers can only occur with enough vertices that writing down the full canonical sylvan resolution would be an ineffective use of space.

Let \(I = \langle xy, y^3, z \rangle\), whose staircase is depicted here.

![Staircase diagram](image)

The three large black dots are the generators, the three large blue dots are the first syzygies, and the large red dot is the second syzygy. The little dots and edges behind and below each of the large dots form its Koszul simplicial complex: a triangle (a nontrivial loop in \(\tilde{H}_1\)) for the second syzygy; a disconnected union of two vertices or an edge and a vertex (nontrivial \(\tilde{H}_0\)) for the first syzygies; and just the empty face (nontrivial \(\tilde{H}_{-1}\); not drawn) for the generators.

The canonical sylvan resolution of \(I\), notated as per Convention 3.9, is as follows.
Before getting to hedges and fences that compute the sylvan matrix entries, several features are to be noted. The column vector $(1, 1, 1)^\top$ generates $\tilde{H}_{1}K^{131}$ and maps to the sum $(1, 0, -1)^\top \oplus (-1, 1, 0)^\top \oplus (0, -1, 1)^\top$ in the $\mathbb{N}^3$-degree 131 component of the middle free module. (The relevant fractions magically either cancel or sum to 1.) This statement should be viewed geometrically on the staircase: the cycle that is the red triangle $zy + yx + xz$ maps to the difference of the two vertices closest to it in each of the blue simplicial complexes. At 111 this is $x - z$; at 130 this is $y - x$; at 031 this is $z - y$. The direct sum of these three blue cycles yields a vector of length 9 that lies in the kernel of the $3 \times 9$ matrix—that is, the $(3 \text{ block}) \times (3 \text{ block})$ matrix.

Let us start the sylvan matrix computations with $D^{001, 031}$. There is only one lattice path $\lambda \in \Lambda(001, 031)$, namely

$$
\begin{array}{c}
\lambda : & 001 \rightarrow 011 \rightarrow 021 \rightarrow 031 \\
ST_0^\lambda : & T_{-1} = \{\} & T_0 = \{y\} & T_1 = \{y\} & S_0 = \emptyset \\
& S_{-1} = \{\emptyset\} & S_1 = \{\emptyset\}
\end{array}
$$

whose unique dimension 0 hedgerow is indicated underneath. This uniqueness implies

$$\Delta_{0, \lambda} = 1$$

There is furthermore only one chain-link fence along $\lambda$: the initial post $x$ is barred because $x$ is not a face of $K^{031}I$; the initial post $z$ is boundary-linked only to $z$ (because $K^{031}I$ has no boundaries in dimension 0) and no continuation is possible because $y$ is not a face of $z$; and the initial post $y$ yields alternating faces $y$ and $\emptyset$.

$$
\begin{array}{cccc}
& z & y & y \\
\emptyset & \uparrow_1 & \uparrow_1 & \uparrow_1 \\
\emptyset & \emptyset
\end{array}
$$

Multiplying the coefficients in this fence to get $D^{001, 031}_{\emptyset, y} = 1$ completes the sylvan matrix $D^{001, 031} = [0 \ 1 \ 0]$ in the lower-right corner of $F_0 \leftarrow F_1$. Similar computations—but easier, because the lattice paths are shorter—yield the sylvan matrices

$$
\begin{align*}
D^{110, 111} &= [0 \ 0 \ 1] & D^{110, 130} &= [0 \ 1 \ 0] \\
D^{030, 130} &= [1 \ 0 \ 0] & D^{030, 031} &= [0 \ 0 \ 1]
\end{align*}
$$

in the top two rows of $F_0 \leftarrow F_1$. The only remaining $1 \times 3$ sylvan matrix not automatically $[0 \ 0 \ 0]$ for reasons of incomparability of the corresponding Betti degrees is $D^{001, 111}$ in the lower-left corner. That sylvan matrix involves two lattice paths in $\Lambda(001, 111)$:

$$
\begin{array}{c}
\lambda : & 001 \rightarrow 011 \rightarrow 111 \\
ST_0^\lambda : & T_{-1} = \{\} & T_0 = \{y\} & S_0 = \{x\} \\
& S_{-1} = \{\emptyset\} \quad \text{or} \quad \{y\}
\end{array}
$$
and
\[
\begin{align*}
\lambda' : & \quad 001 \quad 101 \quad 111 \\
ST_0^\lambda : & \quad T_{-1} = \{\} \quad T_0 = \{x\} \quad S_0 = \{x\} \\
& \quad S_{-1} = \{\emptyset\} \quad \text{or } \{y\}
\end{align*}
\]
both of which have two choices for the initial \( S_0 \) and hence have
\[
\Delta_{0,\lambda'} = \Delta_{0,\lambda} = 1 \cdot 1 \cdot 2 = 2.
\]

The chain-link fence computation for \( \lambda \) is a bit more interesting than previous ones. Regardless of whether \( S_0 = \{x\} \) or \( S_0 = \{y\} \), the initial post \( z \) is boundary-linked only to \( z \in S_0 \), which does not contain \( y \) and hence yields no fences along \( \lambda \). The initial post \( y \) is boundary-linked to \( y \) if \( S_0 = \{x\} \) and to \( x \) if \( S_0 = \{y\} \). However, the case of \( \tau_0 - \tau = y - y \) has no continuation because \( x \) is not a face of \( y \). In contrast, the case of \( \tau_0 - \tau = x - y \) yields a valid fence. The same logic shows that \( \tau_0 - \tau = y - x \) when \( S_0 = \{x\} \) has no continuation but \( \tau_0 - \tau = x - x \) when \( S_0 = \{y\} \) yields a valid fence. These possibilities for \( \lambda = 001 - 011 - 111 \) are summarized as follows.

\[
\begin{array}{cccc}
z & 1 & z \\
/0 & \emptyset & \emptyset & \emptyset \\
y & 1 & y \\
/0 & \emptyset & \emptyset & \emptyset \\
\end{array}
\]

\[
\begin{array}{ccc}
S_0 = \{x\} \\
or \{y\}
\end{array}
\]

The lattice path \( \lambda \) therefore contributes \( \frac{1}{2}[1 1 0] \) to \( D^{001,111} \). The calculation for the other path \( \lambda' = 001 - 101 - 111 \in \Lambda(001,111) \) is obtained by swapping the roles of \( x \) and \( y \) throughout; it also contributes \( \frac{1}{2}[1 1 0] \), so \( D^{001,111} = \frac{1}{2}[1 1 0] + \frac{1}{2}[1 1 0] = [1 1 0] \).

It remains to compute \( F_1 \leftarrow F_2 \). Start with \( D^{130,131} \). There is only one lattice path \( \lambda \in \Lambda(130,131) \), namely \( \lambda = 130 - 131 \). Since \( K^{131} \) has dimension 1, it has no boundaries in dimension 1 and hence \( S_{131} = \emptyset \) is forced; this remains true for the entire computation of \( D^{130,131} \). On the other hand, every vertex of \( K^{130} \) has boundary \( \emptyset \) and hence is a valid choice of \( T_0^{130} \); this reasoning holds more generally for \( T_0 \) in any nonempty CW complex. Consequently
\[
\Delta_{0,130-131} = 2 \cdot 1 = 2.
\]

For chain-link fences, the initial boundary-link \( yx - yx \) has no continuation because \( z \) is not a face of \( yx \); this explains the column of zeros in the middle block of \( F_1 \leftarrow F_2 \). The initial boundary-link \( xz - xz \) yields \( x \) with coefficient \(-1\); when \( T_0 = \{x\} \) the \( T_0 \)-circuit of \( x \) is \( \zeta_{T_0}(x) = x - x = 0 \); but when \( T_0 = \{y\} \) the \( T_0 \)-circuit of \( x \) is \( \zeta_{T_0}(x) = x - y \), so two chain-link fences result, one in \( \Phi_{x,zx} \) with coefficient \(-1 \cdot -1 \cdot 1 = -1 \) and one in \( \Phi_{y,zx} \) with coefficient \(-1 \cdot -1 \cdot 1 = 1 \). This explains the right-hand column in the middle block of \( F_1 \leftarrow F_2 \). The left-hand column is similar, with \( \varphi \in \Phi_{x,zy} \) having
coefficient \( w_\varphi = -1 \cdot -1 \cdot 1 = 1 \) \( \varphi \in \Phi_{y,x} \) having coefficient \( w_\varphi = 1 \cdot 1 \cdot 1 = 1 \).

\[
\begin{array}{cccc}
yx & 1 & yx & /0 \\
xz & 1 & xz & /-1 \\
xz & 1 & xz & /-1 \\
xz & 1 & xz & /-1 \\
\end{array}
\]

\[ T_0 = \{x\} \quad \text{or} \quad \{y\} \]

\[ T_0 = \{x\} \quad T_0 = \{y\} \quad T_0 = \{y\} \quad T_0 = \{y\} \]

\[ \begin{array}{cccc}
zy & 1 & zy & /0 \\
zx & 1 & zx & /-1 \\
zx & 1 & zx & /-1 \\
zx & 1 & zx & /-1 \\
\end{array}
\]

\[ T_0 = \{y\} \quad T_0 = \{x\} \quad T_0 = \{x\} \quad T_0 = \{x\} \quad T_0 = \{x\} \]

The calculation of \( D^{031,131} \) is the same as \( D^{130,131} \) after cyclic permutation of the variables \( x \mapsto y \mapsto z \). Note: this symmetry argument applies to the relative locations of the Koszul simplicial complexes at \( a \) and \( b \), not to the absolute positions of \( a \) and \( b \).

Finally, compute \( D^{111,131} \) using the sole lattice path \( \lambda \in \Lambda(111,131) \), namely

\[ \lambda : \quad 111 \quad 121 \quad 131 \]

\[ ST_0^\lambda : \quad T_0 = \{x\} \quad T_1 = \{zy, yx\} \quad S_1 = \emptyset \]

\[ \text{or} \{y\} \quad S_0 = \{y, z\} \quad \text{or} \{x, z\} \quad \text{or} \{x, y\} \]

where the only new type of observation is that the vector space of homological stage 0 boundaries in \( K^{121} \) has dimension 2, so any pair of vertices is a stake set because no single vertex is a boundary. Thus

\[ \Delta_{1,111-131} = 3 \cdot 3 \cdot 1 = 9. \]

As in all of the other examples, the initial boundary-link \( xz \rightarrow xz \) has no continuation because \( y \) is not a face of \( xz \); this explains the zero column in the top block of \( F_1 \leftarrow F_2 \).

The initial boundary-link \( yx \rightarrow yx \) yields a much more interesting computation. When the stake set \( S_0 = \{y, z\} \) is selected at the interior lattice point 121, the facet \( x \) of \( yx \) is forced by Definition 3.3, but \( x \) is not a stake, so no continuation is possible. In contrast, when \( S_0 = \{x, z\} \) is selected, four chain-link fences result: two from the choice of \( T_0 = \{y\} \), because the circuit of \( x \) with respect to the shrubbery \( \{y\} \) is \( x - y \), so each of the terms \( x \) and \( -y \) contributes a cycle-link; and two from \( T_0 = \{z\} \) for a similar reason with \( z \) in place of \( y \). To save space, these four fences are drawn as a single fork.
When $S_0 = \{x, y\}$ is selected, eight chain-link fences result, by reasoning as in the four fences just constructed. (In fact, the first four fences, depicted in the left-hand diagram here, have exactly the same sequences of simplices and coefficients as the four fences just produced. They count separately because they are subordinate to a different hedgerow.) The reason why there are twice as many is that the shrub of $x$ for the hedge $ST_1 = (\{x, y\}, \{zy, yx\})$ is $zy + yx$, the unique path in $K^{121}$ that joins the non-stake $z$ to the stake $x$. Thus the fence with $S_0 = \{x, y\}$ bifurcates at $x$ into $yx$ and $zy$.

Now count weighted fences as a sum of three terms, one from each fork:

$$
D_{x, yx} = 2 + 2 + 1 = 5
$$
$$
D_{y, yx} = -1 - 1 + 1 = -1
$$
$$
D_{z, yx} = -2 - 2 - 1 = -5
$$

This explains the middle column in the top block of $F_1 \leftarrow F_2$. The left column, starting from the initial boundary-link $zy \rightarrow y$, is similar—in fact, the transposition $x \leftrightarrow z$ of the computation for $yx$ just completed, with different signs to compensate. In particular, $S_0 = \{x, y\}$ yields no fences because $z$ is not a stake, and the remaining fences can be drawn with three forks.

$$
T_0 = \{y\} : \xymatrix{x \ar@{-}[r]^1 & y \ar@{-}[l]_{-1} \ar@{-}[r] & z \ar@{-}[l]_1 \ar@{-}[r] & y} \quad T_0 = \{x\} : \xymatrix{z \ar@{-}[r]_{-1} & y \ar@{-}[l]_1 \ar@{-}[r] & x \ar@{-}[l]_1 \ar@{-}[r] & z}
$$

Counting weighted fences as before yields

$$
D_{x, yz} = 1 + 1 + 2 = 4
$$
$$
D_{y, yz} = 1 + 1 - 1 = 1
$$
$$
D_{z, yz} = -2 - 2 - 1 = -5
$$

to explain the left column of the top block of $F_1 \leftarrow F_2$. 
Example 3.11. This example illustrates that lattice paths can cross and overlap without harm. Let \( I = \langle yz, xz, x^2y, xy \rangle \), whose staircase is depicted here.

The canonical sylvan resolution of \( I \), notated as per Convention 3.9, is as follows.

Several features are to be noted. As in Example 3.10, the column \((1, 1, 1)\) generates \( \tilde{H}_1K^{221} \), this time mapping to the sum \((-1, 0, 1) \oplus (0, -1, 1) \oplus (1, 0, -1) \oplus (1/2, -1/2, 0)\) in the \( \mathbb{N}^3 \)-degree 221 component of the middle free module. This vector of length 12 lies in the kernel of the \( (4 \text{ block}) \times (4 \text{ block}) \) matrix for \( F_0 \leftarrow F_1 \); again, the relevant fractions magically all cancel. Even in \( F_0 \leftarrow F_1 \) the coefficients are not all \( \pm 1 \), so the first syzygies do not simply align one whole generator with positive sign against another with negative sign.
The chain-link fence calculations mostly follow those in Example 3.10. Exception:

\[
\begin{align*}
\lambda & : \quad 011 \quad 111 \quad 121 \\
ST_0^\lambda & : \quad T_{-1} = \{\} \quad T_0 = \{x\} \quad S_0 = \{x\} \\
& \quad \text{or} \{y\} \quad \text{or} \{y\} \\
S_{-1} & = \{\emptyset\}
\end{align*}
\]

yields two chain-link fences which, together, contribute \(\frac{1}{4}[1 1 0]\) to \(D^{011,121}\).

\[
\begin{align*}
x & \quad y \quad 1 \quad x \\
\emptyset & \quad \emptyset \quad / \quad \emptyset \\
\emptyset & \quad \emptyset \quad / \quad \emptyset \\
T_0 & = \{x\} \quad S_0 = \{x\} \quad T_0 = \{x\} \quad S_0 = \{y\}
\end{align*}
\]

The other lattice path \(\lambda' = 011 - 021 - 121 \in \Lambda(011,121)\) contributes \(\frac{1}{2}[1 1 0]\), as in the calculation of \(D^{001,111}\) in Example 3.10 to yield a total of \(D^{011,121} = [\frac{3}{4} \frac{3}{4} 0]\) in the top row of \(F_0 \leftarrow F_1\). Another exception, the calculation of \(D^{101,121}\), does not follow from Example 3.10 but is almost exactly the same as \(\lambda = 011 - 021 - 121\) just computed; the only difference is that \(y\) must replace the leftmost \(x\) in both fences.

The fences comprising the map \(F_1 \leftarrow F_2\) roughly follow the same outline as those in Example 3.10. For the reader attempting these calculations, it is useful as a check to note that \(\Delta_{1,111-221} = 3 \cdot 2 \cdot 1 = 6\), that each of the two paths in \(\Lambda(111,221)\) yields four fences, with \(111 - 121 - 221\) contributing \(\frac{3}{4} - \frac{1}{2} - \frac{1}{2}\) to the middle column and \(111 - 211 - 221\) contributing \(\frac{1}{2} - \frac{1}{2} - \frac{1}{2}\). Finally, given that \(|221 - 121| = 1\) and that \(\Delta_{1,121-221} = 3 \cdot 1 = 3\), the numbers of chain-link fences and their signs are readily deduced from the top middle sylvan matrix in \(F_1 \leftarrow F_2\).

**Remark 3.12.** The proof in Section 9 that the canonical sylvan homomorphism induces a well defined homology morphism is a formality—and hence quite simple—using the Hedge Formula and Wall complexes. However, it admits an elementary proof that sheds a little light on the structure of the sylvan formulation.

That \(D\) takes cycles in \(\tilde{Z}_iK^bI\) to cycles in \(\tilde{Z}_{i-1}K^aI\) can be seen by fixing \(\lambda \in \Lambda(a,b)\) and a hedgerow \(ST_\lambda^i\). In fact \(D\) takes all of \(\tilde{C}_iK^bI\) to cycles in \(\tilde{Z}_{i-1}K^aI\). Indeed, the image of \(\tau\) under \(D\) is \(\sum_\sigma D_{\sigma\tau}\sigma\), corresponding to the column of \(D\) indexed by \(\tau\). Restrict to the subsum over fences \(\varphi \vdash ST_\lambda^i\) such that \(\tau\) through \(\sigma_\ell\) are fixed. Since the only part of the fence allowed to vary is the terminal post \(\sigma\), Definition 3.4 implies that \(w\varphi/c_{\sigma_\ell}(\sigma,T_{i-1}^a)\) is a constant \(w\) for this set of fences \(\varphi\); note that \(\theta_{i,a} = \theta_{i,T_{i-1}^a}\) is constant because \(T_{i-1}^a\) is part of \(ST_\lambda^i\). The subsum yields the scalar multiple

\[
\frac{w}{\Delta_{i,\lambda}} \sum_{\sigma_\ell = \sigma} c_{\sigma_\ell}(\sigma,T_{i-1}^a)\sigma
\]

of the \(T_{i-1}^a\)-circuit of \(\sigma_\ell\) from Definition 2.10.1 which is of course a cycle.
Similarly, that $D$ takes boundaries to boundaries can be seen by fixing $\lambda \in \Lambda(a,b)$ and a hedgerow $ST^\lambda$. In fact, $D$ takes every boundary to 0. Indeed, restrict to the set $\Phi$ of fences $\varphi$ to $ST^\lambda$ such that $\varphi$ has all but the initial post $\tau$ fixed. This time Definition 3.4 implies that $w_\varphi/c_\varphi(\tau_0,S^\lambda_b)$ is a constant $w$ for this set of fences. Applying $D$ to a chain $u \in \tilde{C}_i K^b I$, the sum over $\Phi$ yields $w/\Delta_i \lambda I$ times the coefficient on $\tau_0$ in $u - b_S(u)$. When $u = b_S(\rho)$ for a stake $\rho \in S^\lambda_b$, this coefficient on $\tau_0$ vanishes because in that case $b_S(u) = b_S(\rho - r) = b_S(\rho)$, using that $b_S(r) = 0$ for all faces $\rho' \in S^b_i$ (see Remark 2.12). But the boundaries $b_S(\rho)$ for stakes $\rho \in S^\lambda_b$ constitute a basis for $B_i K^b I$, so all boundaries go to 0. This is not an accident: the hedge rim is a linear map defined precisely to be the identity minus a projection onto the boundaries.

4. Hedge splittings of CW chain complexes

The default coefficient ring in this section is $\mathbb{Z}$, but also fix an arbitrary field $\mathbb{k}$.

**Definition 4.1.** For any dimension $i$ hedge $ST_i$ in $K$, define the hedge splitting

$$\partial_{ST_i}^+: C_{i-1}^k \to \tilde{C}_i^k K$$

over $\mathbb{k}$ by its action on the basis $\overline{S}_{i-1} \cup \partial T_i$:

1. $\partial_{ST_i}^+(\sigma) = 0$ for any non-stake $\sigma \in \overline{S}_{i-1}$ and
2. $\partial_{ST_i}^+(\tau) = \tau$ for any face $\tau \in T_i$.

**Proposition 4.2.** If $ST_i$ is any hedge over $\mathbb{k}$, then for any face $\tau \in K_i$,

$$\partial_{ST_i}^+(\partial \tau) = 1 - \zeta_{ST_i}(\tau),$$

where $1$ is the identity on $\tilde{C}_i^k K$ and $\zeta_{ST_i}$ is from Lemma 2.5 (see also Remark 2.6).

**Proof.** Since $\tau - \zeta_{ST_i}(\tau)$ involves only faces in $T_i$, it is fixed by $\partial_{ST_i}^+ \partial$. Therefore

$$\tau - \zeta_{ST_i}(\tau) = \partial_{ST_i}^+ \partial (\tau - \zeta_{ST_i}(\tau)) = \partial_{ST_i}^+ \partial (\tau)$$

for all $\tau \in K_i$ because $\zeta_{ST_i}(\tau)$ is a cycle. $\square$

**Definition 4.3.** Fix a CW complex $K$. A community in $K$ is a sequence

$$ST_* = (ST_0, ST_1, ST_2, \ldots)$$

with $T_i \cap S_i = \emptyset$ for all $i$.

**Proposition 4.4.** Any community $ST_*$ induces a differential $\partial_{ST_*}^+$ over $\mathbb{k}$ such that

1. $\partial_i \partial_{ST_*}^+ \partial_i = \partial_i$ and
2. $\partial_{ST_*}^+ \partial_i \partial_{ST_*}^+ = \partial_{ST_*}^+$.

**Proof.** The disjointness of $T_{i-1}$ and $S_{i-1}$ means that $T_{i-1} \subseteq \overline{S}_{i-1}$, so $\partial_{ST_{i+1}}^+ \partial_{ST_i}^+ = 0$ by Definition 4.1. Property 1 follows from Proposition 4.2 because $\zeta_{ST_i}(\tau)$ is a cycle—that is, $\partial_i \zeta_{ST_i}(\tau) = 0$. Property 2 is immediate from Definition 4.1 with both sides of the equation being 0 for non-stakes and $\tau$ when applied to any boundary $\partial \tau$. $\square$
Remark 4.5. Constructing the hedge splitting $\partial^+_{ST_i}$ in Definition 4.1 and the sylvan matrix in Definition 3.6 uses hedges in a fixed homological stage. The existence of hedges in other homological stages to ensure that $\partial^+_{ST_i}$ belongs to a community is important to make the theory surrounding Wall complexes (Sections 7 and 8) apply to noncanonical sylvan resolutions in examples.

The remainder of this section contains facts about shrubs and stakes that are crucial for the combinatorics of Moore–Penrose pseudoinverses of CW complex differentials.

Recall Lemma 2.5 and Definition 2.10.

Lemma 4.6. Fix a shrubbery $T_i \subseteq K_i$ over $\mathbb{Q}$ and a face $\tau \in T_i$. The smallest positive integer $\nu = \nu_{T_i}(\tau)$ such that $\nu \gamma_{T_i}(\gamma) \in \tilde{Z}_i K \subseteq \tilde{Z}_i^1 K$ is an integer cycle, and not merely a rational cycle, is the order of $\partial \tau$ in $\tilde{H}_{i-1}(T_i)$.

Proof. By Definition 2.11, every integer multiple $\gamma \partial \tau$ is the boundary of a unique rational chain $\gamma t \in \mathbb{Q}\{T_i\}$, and $t$ has integer coefficients if and only if $\nu$ divides $\gamma$. □

Remark 4.7. It may help to paraphrase Lemma 4.6 verbally: what the circuit of $\gamma$ must be multiplied by to get an integer cycle is what $\partial \tau$ must be multiplied by to get an integer boundary, and that is the order of $\partial \tau$ in integer homology.

Lemma 4.8. Fix a stake set $S_{i-1} \subseteq K_{i-1}$ over $\mathbb{k}$ and a stake $\sigma \in S_{i-1}$. The boundary of the shrub $\partial^+_{ST_i}(\sigma)$ of $\sigma$ in the hedge $ST_i$ over $\mathbb{k}$ depends only on $S_{i-1}$, not on the shrubbery $T_i$. More precisely $\partial^+_{ST_i}(\sigma) = b_{S_{i-1}}(\sigma)$ is the generator $\sigma - r$ from Lemma 2.8.

Proof. If $\sigma \in S_{i-1}$ is a stake, then $\partial^+_{ST_i}(\sigma) = \partial^+_{ST_i}(b_{S_{i-1}}(\sigma))$ by Definition 4.11, since $b_{S_{i-1}}(\sigma) \in \sigma + \mathbb{k}\{\overline{S}_{i-1}\}$. But $b_{S_{i-1}}(\sigma)$ is fixed by $\partial \sigma$ because it is a boundary. □

Definition 4.9. The shrub boundary $b_{S_{i-1}}(\sigma) \in \tilde{B}^k_{i-1} K$ is the boundary in Lemma 4.8.

Lemma 4.10. Fix a stake set $S_{i-1} \subseteq K_{i-1}$ over $\mathbb{Q}$ and a stake $\sigma \in S_{i-1}$. The smallest positive integer $\mu = \mu_{S_{i-1}}(\sigma)$ such that $\mu b_{S_{i-1}}(\sigma) \in \tilde{B}_{i-1} K \subseteq \tilde{B}^\mathbb{Q}_{i-1} K$ is an integer boundary, and not merely a rational boundary, is the order of $\partial \sigma$ in $\tilde{H}_{i-1}(\overline{S}_{i-1})$.

Proof. The class of $\partial \sigma$ is torsion in the integer homology $\tilde{H}_{i-1}(\overline{S}_{i-1})$ because, although $\partial \sigma$ need not be a boundary in $\tilde{B}_{i-1}(\overline{S}_{i-1})$ over the integers, it becomes a boundary over the rationals because $\tilde{B}^\mathbb{Q}_{i-1}(\overline{S}_{i-1}) = \tilde{B}_{i-2}^\mathbb{Q} K$. The rest of the proof is not quite as immediate as that of Lemma 4.6. It is a consequence of Proposition 4.11. □

Proposition 4.11. If $ST_i$ is a hedge over $\mathbb{Q}$ and $Y_{\sigma} = \langle \overline{S}_{i-1} \cup \sigma \rangle$ for $\sigma \in S_{i-1}$, then $Y_{\sigma}$ has a boundary in $\tilde{B}_{i-1} K$ with coefficient $\mu$ on $\sigma$ if and only if the class $[\mu \partial(\sigma)]$ vanishes in $\tilde{H}_{i-1}(\overline{S}_{i-1})$. 
Lemma 4.13. Fix a stake set 
\[\langle \tilde{w} \rangle \]
which happens precisely when there is a chain \( w \in \mathbb{Q}\{\mathcal{S}_{i-1}\} \) such that
\[
\mu \partial(\sigma) = \partial w \iff \mu \partial(\sigma) - \partial w = 0 \\
\iff \partial(\mu \sigma) - w = 0.
\]
Since \( \tilde{C}_{i-1}^Q K = \mathbb{Q}\{\mathcal{S}_{i-1}\} \oplus \tilde{B}_{i-1}^Q K \), there is an (unique) expression \( \mu \sigma - w = w' + b \) with \( w' \in \mathbb{Q}\{\mathcal{S}_{i-1}\} \) and \( b \in \tilde{B}_{i-1}^Q K \). Since \( \sigma \in S_{i-1} \) the coefficient on \( \sigma \) in \( b \) is \( \mu \). Finally, note that \( b \in \tilde{C}_{i-1} Y_\sigma \) because \( b = \mu \sigma - w - w' \) and the right-hand side lies in \( \tilde{C}_{i-1} Y_\sigma \). \( \square \)

Proposition 4.12. If \( ST_i \) and \( S'T'_i \) are two hedges in \( K \) in dimension \( i \) over \( k \), then
\[
\partial^+_{S'T_i} \partial^+_{ST_i} = \partial^+_{ST_i}
\]
uses the stake set from \( ST_i \) and the shrubbery from \( S'T'_i \).

Proof. For any stake \( \sigma \in S_{i-1} \), already \( \partial^+_{ST_i}(\sigma) = 0 \), so certainly \( \partial^+_{S'T_i} \partial^+_{ST_i}(\sigma) = 0 \). On the other hand, \( \partial^+_{ST_i} \partial^+_{ST_i} \) fixes every boundary \( b \in \tilde{B}_{i-1}^k K \) because \( \partial^+_{ST_i} \partial = \partial \) by Proposition 4.11; indeed, writing \( b = \partial c \) yields \( \partial^+_{ST_i}(b) = \partial^+_{ST_i}(\partial(c)) = \partial(c) = b \). Consequently, \( \partial^+_{S'T_i} \partial^+_{ST_i}(b) = \partial^+_{S'T_i}(b) \) for \( b \in \tilde{B}_{i-1}^k K \). Thus \( \partial^+_{S'T_i} \partial^+_{ST_i} \) satisfies the defining properties of \( \partial^+_{ST_i} \) via its actions on \( k\{\mathcal{S}_i\} \) and on \( \tilde{B}_{i-1}^k K \) (Definition 4.11). \( \square \)

Lemma 4.13. Fix a stake set \( S_{i-1} \subseteq K_{i-1} \) over \( k \) and a stake \( \sigma \in S_{i-1} \). Consider the coefficient \( \langle b_{S_{i-1}}(\sigma), \sigma' \rangle \) on a face \( \sigma' \in K_{i-1} \) in the shrub boundary of \( \sigma \).

1. \( \langle b_{S_{i-1}}(\sigma), \sigma' \rangle = 0 \) unless \( \sigma' \in \mathcal{S}_{i-1} \cup \sigma \).
2. If \( \langle b_{S_{i-1}}(\sigma), \sigma' \rangle \neq 0 \) then \( S'_{i-1} = (S_{i-1} \setminus \sigma) \cup \sigma' \) is a stake set.
3. \( \langle b_{S_{i-1}}(\sigma), \sigma' \rangle \neq 0 \iff \langle \sigma, b_{S_{i-1}}(\sigma') \rangle \neq 0 \).
4. When \( k = \mathbb{Q} \), this nonvanishing occurs \( \iff \mu_{S_{i-1}}(\sigma)b_{S_{i-1}}(\sigma) = \mu_{S_{i-1}}(\sigma')b_{S_{i-1}}(\sigma') \).

Proof. The first and second claims follow from Lemma 2.7; the coefficients in \( b_{S_{i-1}}(\sigma) \) are 0 on stakes other than \( \sigma \), and the nonvanishing of the coefficient on \( \sigma' \) allows the basis exchange. The two subsequent claims follow from Lemma 4.8. \( \square \)

5. Moore–Penrose pseudoinverses of CW differentials

The default coefficient ring in this section is \( \mathbb{Z} \), but also fix a real subfield \( k \subseteq \mathbb{R} \).

Definition 5.1. Fix a homomorphism \( C \leftarrow^d C' \) of \( k \)-vector spaces with basis. The Moore–Penrose pseudoinverse of \( d \) is the unique homomorphism \( C \rightarrow^{d^+} C' \) that satisfies

1. \( dd^+ d = d \)
2. \( d^+ dd^+ = d^+ \)
3. \( (dd^+)^\top = dd^+ \)
4. \( (d^+ d)^\top = d^+ d \).
When \( d \) is the differential of a CW complex \( K \), the indices are such that \( d \) and \( d^+ \) pass between two fixed homological positions, so \( d \) might always mean \( C_{i-1} \xrightarrow{d} C_i \) and then \( d^+ \) would always mean \( C_{i-1} \xrightarrow{d^+} C_i \).

**Lemma 5.2.** Fix a stake set \( S_{i-1} \). For any subset \( S \subseteq S_{i-1} \), Definition 2.13 works verbatim with \( S \) instead of \( S_{i-1} \). As such, for any stake \( \sigma \in S_{i-1} \),

\[
\mu_{S_{i-1}}(\sigma)\theta_{S_{i-1}\setminus\sigma} = \theta_{S_{i-1}}.
\]

**Proof.** The long exact sequence of homology for the inclusion \( \langle S_{i-1} \setminus \sigma \rangle \hookrightarrow \langle S_{i-1} \rangle \)
yields a short exact sequence

\[
0 \to A \to \Theta_{S_{i-1}\setminus\sigma} \to \Theta_{S_{i-1}} \to 0,
\]

where \( A \) is the subgroup of \( \Theta_{S_{i-1}\setminus\sigma} \) generated by \( \partial \sigma \). The result follows because this subgroup \( A \) has order \( \mu_{S_{i-1}}(\sigma) \) by Lemma 4.10.

**Proposition 5.3.** For any hedge \( ST_i \) in dimension \( i \) and any \((i-1)\)-faces \( \sigma \) and \( \sigma' \),

\[
\theta_{S_{i-1}}^2 \langle \partial \partial_{ST_i}^+(\sigma), \sigma' \rangle = \mu_{S_{i-1}}(\sigma) \theta_{S_{i-1}\setminus\sigma} b(\sigma) \langle b, \sigma' \rangle
\]

where \( b \in \tilde{B}_{i-1}K \) is the generator of \( \mathbb{Z}\{S_{i-1} \cup \sigma \} \cap \tilde{B}_{i-1}K \).

**Proof.** By Lemmas 4.8 and 2.8, the boundary \( b \) is a nonzero scalar multiple of the shrub boundary of \( \sigma \). If the left-hand side is 0 then so is the right-hand side, because of the factor of \( \langle b, \sigma' \rangle \). So assume that the left-hand side is nonzero. Then

\[
\theta_{S_{i-1}}^2 \langle \partial \partial_{ST_i}^+(\sigma), \sigma' \rangle = \theta_{S_{i-1}}^2 \langle b_{S_{i-1}}(\sigma), \sigma' \rangle \quad \text{by Definition 4.9}
\]

\[
= \theta_{S_{i-1}}^2 \langle b, \mu_{S_{i-1}}(\sigma), \sigma' \rangle \quad \text{by definition of } \mu_{S_{i-1}}(\sigma) \text{ in Lemma 4.10}
\]

\[
= \theta_{S_{i-1}\setminus\sigma}^2 \langle b, \sigma \rangle \langle b, \sigma' \rangle \quad \text{by Lemma 5.2}
\]

\[
= \theta_{S_{i-1}\setminus\sigma}^2 \langle b, \sigma \rangle \langle b, \sigma' \rangle \quad \text{because } b = \mu_{S_{i-1}}(\sigma)b_{S_{i-1}}(\sigma).
\]

**Corollary 5.4.** For any hedge \( ST_i \) in dimension \( i \) and any \((i-1)\)-faces \( \sigma \) and \( \sigma' \),

\[
\theta_{S_{i-1}}^2 \langle \partial \partial_{ST_i}^+(\sigma), \sigma' \rangle = \theta_{S'_{i-1}}^2 \langle \partial \partial_{ST_i}^+(\sigma), \partial \partial_{ST_i}^+(\sigma') \rangle,
\]

where \( S'_{i-1} = (S_{i-1} \setminus \sigma) \cup \sigma' \) as in Lemma 2.8.

**Proof.** The display in Proposition 5.3 is symmetric in \( \sigma \) and \( \sigma' \), since \( S_{i-1} \cup \sigma = S'_{i-1} \cup \sigma' \). Now multiply by \( \theta_{T_i}^2 \), noting that the shrubbery \( T_i \) is the same on both sides.
Theorem 5.5 (Hedge Formula). The Moore–Penrose pseudoinverse of the $i$th differential $\partial_i$ of a CW complex $K$ is a sum over hedges $ST_i \in ST_i(K)$:

$$\partial_i^+ = \frac{1}{\Delta_i^{ST}} \sum_{ST_i} \theta_{ST_i}^2 \partial_{ST_i}^+.$$ 

Proof. The four properties in Definition 5.1 are proved in turn.

1. This property holds because summing $\partial(\theta_{ST_i}^2 \partial_{ST_i}^+) \partial = \theta_{ST_i}^2 \partial \partial_{ST_i}^+ \partial = \theta_{ST_i}^2 \partial$ over $ST_i$ yields $\Delta_i \partial$ by definition of $\Delta_i = \Delta_i^{ST} K = \sum_{ST_i} \theta_{ST_i}^2$ (Definition 2.14).

2. The sum here is over all pairs of hedges $S'T_i$ and $ST_i$, each summand being

$$\left( \frac{1}{\Delta_i} \theta_{S'T_i}^2 \theta_{ST_i}^+ \right) \left( \frac{1}{\Delta_i} \theta_{ST_i}^2 \theta_{ST_i}^+ \right) = \left( \frac{1}{\Delta_i} \theta_{S_i-1}^2 \theta_{T_i} \right) \left( \frac{1}{\Delta_i} \theta_{S_i-1}^2 \theta_{T_i} \theta_{ST_i}^+ \right)$$

by Proposition 4.12 and Definition 2.13. Since the choices of a stake set and an arbitrary shrubbery yield a hedge—the sum can be taken over $S'T_i$ and $ST_i$. What results is $\left( \frac{1}{\Delta_i} \right) \left( \sum_{ST_i} \theta_{ST_i}^2 \theta_{ST_i}^+ \right)$, as desired.

3. Summing $\langle \theta_{S_i-1}^2 \partial \partial_{ST_i}^+ (\sigma), \sigma' \rangle$ and $\langle \sigma, \theta_{S_i-1}^2 \partial \partial_{ST_i}^+ (\sigma') \rangle$ over $ST_i$ yield the same sum, term by term, under the bijection $\langle \theta_{S_i-1}^2 \partial \partial_{ST_i}^+ (\sigma), \sigma' \rangle = \langle \sigma, \theta_{S_i-1}^2 \partial \partial_{ST_i}^+ (\sigma') \rangle$ for $S_i-1 = (S_i \setminus \sigma) \cup \sigma'$ afforded by Corollary 5.4.

4. The Higher Projection Formula [CCK15, Theorem A], with all resistances $r_b = 1$, is precisely the statement that (in our notation)

$$\pi_{Z_i} = \frac{1}{\Delta_i} \sum_{T_i \in T_i} \theta_{T_i}^2 \tilde{\zeta}_{T_i}$$

is the orthogonal projection $\tilde{C}_i K \to \tilde{Z}_i K$. This map is symmetric, as all orthogonal projections are. So, too, is $1 - \pi_{Z_i}$. Summing $\theta_{T_i}^2 / \Delta_i^T$ times Proposition 4.2 over $T_i$ therefore yields the symmetric linear map

$$\frac{1}{\Delta_i} \sum_{T_i \in T_i} \theta_{T_i}^2 \partial \partial_{ST_i}^+ = \frac{1}{\Delta_i} \sum_{T_i \in T_i} \theta_{T_i}^2 (1 - \zeta_{T_i}).$$

Multiplying this by $\theta_{S_i-1}^2 / \Delta_i^{S_i-1}$ and summing over $S_i-1 \in S_i-1$ yields the desired symmetry result, after noting that $\Delta_i^{S_i-1} \Delta_i^T = \Delta_i^{ST}$ by Definition 2.14. 

□

Remark 5.6. The formal similarity of Theorem 5.5 to the Higher Projection Formula [CCK15, Theorem A] led us to try reducing the entire CW differential Moore–Penrose problem to a problem of projection onto cycles of a CW complex. And indeed, the differentials in the algebraic chain complex of the cone over $K$ has cycle groups that are the graphs of the differentials of $K$ itself. Orthogonal projection to these cone cycles therefore yields a candidate pseudoinverse $C_{i-1} \to C_{i-1} \times C_i \to \Gamma_i \to C_i$. This candidate even has kernel $B_{i-1}^\perp$. But alas, it does not appear that this composite
enacts $d_{B_i - 1}$ on $B_i$. On the other hand, the composite need only enact $d_{B_i - 1}$ on $B_i$ up to rescaling, using the resistances in the Higher Projection Formula, but it remains unclear to us whether that is the case.

**Corollary 5.7** (Boundary Projection Formula). The orthogonal projection $\pi_{B_i}$ from $i$-chains to $i$-boundaries in a CW complex $K$ is a sum over stake sets $S_i \subseteq K_i$:

$$\pi_{B_i} = \frac{1}{\Delta^i} \sum_{S_i} \theta^2_{S_i} b_{S_i},$$

where $b_{S_i}$ is the shrub boundary homomorphism from Definition 4.9.

**Proof.** Multiply the Hedge Formula at homological stage $i + 1$ on the left by $\partial_{i+1}$. The result, on the left-hand side, becomes $\pi_{B_i} = \partial_{i+1} \partial_{i+1}^+$. The right-hand side becomes

$$\frac{1}{\Delta^i_{ST} \Delta^T_{i+1}} \sum_{S_i} \theta^2_{ST_i+1} \partial_{i+1} \partial^+_{ST_i+1} = \frac{1}{\Delta^i_{ST} \Delta^T_{i+1}} \sum_{S_i} \theta^2_{ST_i+1} b_{S_i},$$

$$= \frac{1}{\Delta^i S \Delta^T_{i+1}} \sum_{S_i} \sum_{T_{i+1}} \theta^2_{S_i} \theta^2_{T_{i+1}} b_{S_i},$$

$$= \left( \frac{1}{\Delta^i S} \sum_{S_i} \theta^2_{S_i} b_{S_i} \right) \left( \frac{1}{\Delta^T_{i+1}} \sum_{T_{i+1}} \theta^2_{T_{i+1}} \right),$$

$$= \frac{1}{\Delta^i} \sum_{S_i} \theta^2_{S_i} b_{S_i},$$

where the first equality is by Lemma 4.8, the second is by Definitions 2.13 and 2.14 and the others are straightforward. $\square$

6. **Koszul bicomplex**

The default coefficient ring in this section is an arbitrary field $k$.

For the purpose of visualizing the homomorphisms in Section 1.3, it is useful to organize the data graphically in an array. To determine how the homology groups are arranged, assume that, for a fixed $\mathbb{N}^n$-degree $a$,

$$\tilde{H}_{-1} K^a I, \; \tilde{H}_0 K^a I, \; \tilde{H}_1 K^a I, \; \tilde{H}_2 K^a I, \ldots$$

are placed in order, upward along a single column, which may as well be column $|a|$. These assumptions are arbitrary, but (i) any other choices would yield the same result, up to symmetry; and (ii) our choices follow standard homological practice in the context of bicomplexes, where chain complexes (as opposed to cochain complexes) are depicted with lowered indices and arrows pointing downward or to the left. There remains only the issue of alignment: where in each column to place the lowest-indexed group. It is convenient to place it on the diagonal, to get the simplicial homology array.
because the homology homomorphisms from Problem 1.3 can then be drawn as arrows whose sources and targets are located as they would be in pages of a spectral sequence:

\[
\begin{array}{cccccc}
\oplus \tilde{H}_{-1}K^a I & \oplus \tilde{H}_0K^a I & \oplus \tilde{H}_1K^a I & \oplus \tilde{H}_2K^a I & \oplus \tilde{H}_3K^a I \\
|a|=0 & |a|=1 & |a|=2 & |a|=3 & |a|=4 \\
\oplus \tilde{H}_{-1}K^a I & \oplus \tilde{H}_0K^a I & \oplus \tilde{H}_1K^a I & \oplus \tilde{H}_2K^a I & \cdots \\
|a|=1 & |a|=2 & |a|=3 & |a|=4 \\
\oplus \tilde{H}_{-1}K^a I & \oplus \tilde{H}_0K^a I & \oplus \tilde{H}_1K^a I \\
|a|=2 & |a|=3 & |a|=4 \\
\cdots & \cdots & \cdots \\
\end{array}
\]

For the moment, the vector spaces and homomorphisms in these arrays are to be thought of as occurring in fixed \( \mathbb{N}^n \)-degree \( b \), as in the discussion after Problem 1.3, so implicitly \( a \preceq b \) throughout. That said, since \( b \) is arbitrary, every nonzero homology \( \tilde{H}_{i-1}K^a I \) can be made to appear by taking \( b \) so big that every generator of \( I \) divides \( x^b \).

To make the transition to \( \mathbb{N}^n \)-graded modules over the polynomial ring \( k[x] \), the vector spaces \( \tilde{H}_{i-1}K^a I \) in the Koszul simplicial array \( HE^1 \) should be thought of as modules \( \text{Tor}_i(k, I)_a \), but that entails a shift in \( \mathbb{N}^n \)-grading.

**Definition 6.1.** The *Koszul array* is the array \( KE^1 \) of \( \mathbb{N}^n \)-graded vector spaces over \( k \) obtained from the simplicial homology \( HE^1 \) by placing \( \tilde{H}_{i-1}K^a I \) in \( \mathbb{N}^n \)-degree \( a \).

The picture of \( KE^1 \) is no different from that of \( HE^1 \); the distinction is merely the category in which the depicted vector spaces lie and, for \( KE^1 \), their \( \mathbb{N}^n \)-degrees. It is useful sometimes to think of \( KE^1 \) as the array of Tor modules of \( I \), by Hochster’s formula.
Lemma 6.2. Tensoring the Koszul array with $k[x]$ yields the fourth-quadrant array $KE^1 \otimes k[x]$ with entries

$$KE^1_{pq} \otimes k[x] = \bigoplus_{|a|=p} \text{Tor}_{p+q}(k, I)_{a} \otimes k[x],$$

in which each summand $\text{Tor}_{i}(k, I)_{a} \otimes k[x]$ is a free module over $k[x]$ that has rank $\dim_k \tilde{H}_i K^n I$ and is generated in $N^n$-degree $a$. □

Remark 6.3. The arrows from $(*), superimposed onto $KE^1 \otimes k[x], point appropriately to represent the $k[x]$-module homomorphisms in Problem 1.3, with $N^n$-degree $b$ component $HE^1$.

To summarize the discussion leading to this point, here is the precise relation between the simplicial homology array $HE^1$ and Koszul array $KE^1$.

Lemma 6.4. $HE^1 = (KE^1 \otimes k[x])_b$. □

The Koszul array $KE^1$ is the Koszul homology of $I$, by Hochster’s formula (Theorem 1.2), but the manner in which the relevant Koszul complexes are spread out on the page is perhaps unexpected, so it is worth describing in detail. To avoid confusion with the polynomial ring in Lemma 6.2, write this Koszul homology over a polynomial ring in variables $y = y_1, \ldots, y_n$.

Convention 6.5 (Koszul complex notation). Let $V$ be a vector space over $k$ of dimension $n$ that is $N^n$-graded to have one basis vector $z_1, \ldots, z_n$ in each of the degrees $e_1, \ldots, e_n$ of the variables $x_1, \ldots, x_n$. The Koszul complexes on the variables $x$ and on the variables $y$ are denoted by

$$K^x_\bigwedge = \bigwedge^* V \otimes k[x] \quad \text{and} \quad K^y_\bigwedge = \bigwedge^* V \otimes k[y]$$

with their usual $N^n$-graded differentials. Thus, for example, the degree $b$ differential

$$(K^y_{i-1})_b = \bigoplus_{|\sigma| = i - 1} z^\sigma \otimes y^{b-\sigma} \leftarrow \bigoplus_{|\sigma| = i} z^\sigma \otimes y^{b-\sigma} = (K^y_i)_b$$
is induced by the $\mathbb{N}^n$-graded $k[y]$-linear map $k[y] \leftarrow V \otimes k[y]$ that sends $y_j \leftrightarrow z_j \otimes 1$: 

$$\sum_{j \in \sigma} \pm z^{\sigma-e_j} \otimes y^{b+e_j-\sigma} \leftrightarrow z^{\sigma} \otimes y^{b-\sigma}.$$ 

The symbol $z^{\sigma}$ denotes the exterior product of the basis vectors of $V$ indexed by the simplex $\sigma \subseteq \{1, \ldots, n\}$, which is also identified with its characteristic vector in $\{0, 1\}^n$. The monomial $z^{\sigma}$ is a $k[x]$-basis for a rank 1 free $\mathbb{N}^n$-graded summand $k_x^\sigma$ of $k_x$, and similarly for $k_y^\sigma$. Write $I^y \subseteq k[y]$ for the copy of the monomial ideal $I$ inside of $k[y]$.

**Lemma 6.6.** The fourth-quadrant array $KE^0$ with entries 

$$KE^0_{pq} = \bigoplus_{|a|=q} k_y^0 \otimes I_a$$

has a vertical differential that is the direct sum of Koszul morphisms 

$$(k_y^0 \otimes k[y]) I^y \rightarrow (k_y^0 \otimes k[y]) I^y$$

for $i = p+q$ and $|b| = p$. Consequently, gathering all entries along the diagonal $p+q = i$ into one module $KE^0_i = k_y^0 \otimes k[y] I^y$, the total complex $KE^0_i = k_y^0 \otimes k[y] I^y$ with decreasing homological index $i$ is the usual Koszul complex of $I^y$ over $k[y]$.

**Proof.** Comparing the numbering in 

$$KE^0 = \bigoplus_{|a|=0} k_y^0 \otimes I_a \bigoplus_{|a|=0} k_y^1 \otimes I_a \bigoplus_{|a|=0} k_y^2 \otimes I_a \bigoplus_{|a|=0} k_y^3 \otimes I_a \bigoplus_{|a|=0} k_y^4 \otimes I_a$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\bigoplus_{|a|=1} k_y^0 \otimes I_a \bigoplus_{|a|=1} k_y^1 \otimes I_a \bigoplus_{|a|=1} k_y^2 \otimes I_a \bigoplus_{|a|=1} k_y^3 \otimes I_a \bigoplus_{|a|=1} k_y^4 \otimes I_a \cdots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\bigoplus_{|a|=2} k_y^0 \otimes I_a \bigoplus_{|a|=2} k_y^1 \otimes I_a \bigoplus_{|a|=2} k_y^2 \otimes I_a \bigoplus_{|a|=2} k_y^3 \otimes I_a$$

$$\cdots$$

$$\cdots$$

to the numbering in the earlier diagrams of $HE^1$ and $KE^1$, the content of the lemma is that $(k_y^0 \otimes k[y] I^y)_{|b|} = k_y^0 \otimes I_{|b|-|\sigma|}$, so $|\sigma| = p+q$ and $|b| = p \Rightarrow |a| = |b-\sigma| = -q$. □

Given that Lemma 6.2 tensors $KE^1$ with $k[x]$ over $k$, it is now an easy step to a master bicomplex that has a horizontal differential, in addition the vertical one, to induce the desired spectral sequence homomorphisms.

**Proposition 6.7.** Tensoring $KE^0$ with $k[x]$ over $k$ yields an array $k_*(I)$ that has a

- horizontal differential induced by $k_x^*$
- vertical differential induced by $k_y^*$
• total differential induced by $\mathbb{K}^{x+y}$

where $x + y = x_1 + y_1, \ldots, x_n + y_n$ lies in the polynomial ring $k[x, y] = k[x] \otimes k[y].$

That is, $\mathbb{K}_{\ast \ast}$ is a fourth-quadrant bicomplex all of whose differentials are Koszul; it is concentrated in a strip consisting of the $p = -q$ diagonal and $n$ superdiagonals.

**Proof.** Using equality signs for the natural isomorphisms of modules, tensor

$$K^y \otimes k[x] = \wedge^IV \otimes k[x] \otimes k[y] = K^x \otimes k[y]$$

\[\|\]

over $k[y]$ with $I^y$. The vertical differential is as claimed by the second half of Lemma 6.6 with a further tensor product over $k$ with $k[x].$

The horizontal differential is as claimed by the first description of $KE^0$ in the same Lemma 6.6 which characterizes row $-q$ of $KE^0 \otimes k[k] \otimes \bigoplus_{|a| = q} I_a,$ whose tensor products are again all over $k.$

The total differential is as claimed because, for any $k[x, y]$-module $M,$ such as $K^{x+y},$

$$M \otimes_{k[y]} I^y \cong M \otimes_{k[x, y]} k[x, y] \otimes_{k[y]} I^y \cong M \otimes_{k[x, y]} I^y[k[x, y]],$$

where the second isomorphism is by flatness of $k[x, y]$ over $k[y].$ \qed

**Remark 6.8.** The statements and proofs of Lemma 6.6 and Proposition 6.7 work verbatim for standard $\mathbb{N}$-graded ideals $I,$ noting that $a = |a|$ when $a \in \mathbb{N}.$ This extra generality is worth noting, when it incurs no additional difficulty.

**Definition 6.9.** The array $K_{\ast \ast}(I)$ in Proposition 6.7 is the Koszul bicomplex of $I$ for any standard $\mathbb{N}$-graded ideal $I.$

**Corollary 6.10.** The vertical-then-horizontal spectral sequence of $K_{\ast \ast}(I)$ is

$$KE^1_{pq} \otimes k[x] \Rightarrow H_{p+q}K_{\ast \ast}^{x+y}(I)$$

for any standard $\mathbb{N}$-graded ideal $I.$

**Proof.** Follows immediately from Proposition 6.7 given that the vertical homology of $KE^0$ is $KE^1$ by Lemma 6.6 \qed

**Remark 6.11.** The spectral sequence in Corollary 6.10 is, at last, the one that produces arrows as in Remark 6.3. However, by the nature of spectral sequences, these arrows only represent homomorphisms between subquotients of $KE^1 \otimes k[x]$ and therefore cannot directly be differentials in a free resolution of $I.$ The next section is the remedy.

The fundamental role of the spectral sequence in Corollary 6.10 makes it crucial to understand its limit.

**Proposition 6.12.** The total complex $K_{\ast \ast}^{x+y}(I)$ is a free resolution of any standard $\mathbb{N}$-graded ideal $I$ as a module over $k[x, y],$ where $y_j$ acts as $-x_j$ on $I$ for $j = 1, \ldots, n.$
**Proof.** The action of $y_1, \ldots, y_n$ via $-x_1, \ldots, -x_n$ makes every module over $k[x]$ into a module over $k[x,y]$ in which the action of $k[x,y]$ factors through the quotient morphism $k[x,y] \to k[x,y]/(x+y) \cong k[x]$. Under this action, $x+y$ is a regular sequence on every $k[x]$-module, including $I$, so the Koszul complex $K_{x+y}(I)$ is acyclic. $\square$

7. **Wall complexes**

The default coefficient ring in this section is an arbitrary ring $R$.

Having produced homomorphisms between subquotients of the relevant $k[x]$-modules $\text{Tor}_{p+q}(k, I)_a \otimes k[x]$ (see Remark 6.11), the goal is to lift these to homomorphisms between the free modules themselves. The observation that drives this lifting is quite general: any splitting of the differential of a chain complex induces a direct sum decomposition with the homology as a summand, so homomorphisms among homology modules can be lifted to homomorphisms among the original modules. In the current context, this observation is applied to the homomorphisms in the spectral sequence of the Koszul bicomplex from Corollary 6.10 to get a Wall complex, as in [Eag90].

The formula looks complicated, but all it does is formally express the composite maps arising from the spectral sequence and from projection to homology, and furthermore every term in the formula can be made explicitly combinatorial, in the canonical case of the Moore–Penrose splitting as well as in various noncanonical cases, such as monomial ideals that have only three variables or are Borel-fixed.

This section is mainly a summary of the relevant constructions and main results from [Eag90], largely without repeating the proof. The exceptions are Remark 7.8 which is new, and that we fix a circular error in the proof of [Eag90, Proposition 2.3].

**Definition 7.1.** Fix a ring $R$ and a doubly indexed array $W_{..}$ of $R$-modules with maps $\omega_j: W_{pq} \to W_{p-j,q+j-1}$ for $j \in \mathbb{N}$ (the index $pq$ on $\omega_j$ is suppressed). Assume that for each element $w \in W_{pq}$, only finitely many images $\omega_j(w)$ are nonzero. Set

$$W_i = \bigoplus_{p+q=i} W_{pq} \text{ and } D_i = \sum_{j=0}^{\infty} \omega_j: W_i \to W_{i-1}.$$ 

These data constitute a **Wall complex** if $D^2 = 0$, and $W_{..}$ with the differential $D$ is the **total complex** of $W_{..}$.

**Definition 7.2.** Fix a bicomplex $C_{..}$ of $R$-modules with vertical differential $d = d_0$ and horizontal differential $d_1$. A **vertical splitting** of $C_{..}$ consists of a differential

$$d^+: d^+_{pq}: C_{pq} \to C_{p,q+1}$$

with $dd^+d = d$ and $d^+dd^+ = d^+$. The condition of being a differential means $d^+d^+ = 0$.

Thus $d^+$ is a vertical cohomological differential, going up the columns in the direction opposite to the homological vertical differential $d$. The following is elementary to check [Eag90, Proposition 1.1].
Proposition 7.3. A vertical splitting of $C_{\bullet\bullet}$ is equivalent to a direct sum decomposition

$$C_{pq} = B'_{p,q-1} \oplus H_{pq} \oplus B_{pq} \oplus Z_{pq}$$

in which, for all indices $p$ and $q$,

- $H_{pq} \oplus B_{pq} = Z_{pq} = \ker d_{pq}$ and
- $d_{p,q-1} : B'_{p,q-1} \cong B_{p,q-1} = \text{im } d_{pq}$, where $d_{p,q-1}$ is the restriction of $d_{pq}$ to $B'_{p,q-1}$.

More precisely, a vertical splitting is constructed from this direct sum decomposition by

$$d_{pq}^+ = \iota_{pq} \circ d_{pq}^{-1} \circ \pi_{pq},$$

where $\pi_{pq} : C_{pq} \twoheadrightarrow B_{pq}$ projects to the summand $B_{pq}$ and $\iota_{pq} : B'_{pq} \hookrightarrow C_{p,q+1}$ is inclusion.

The homomorphisms whose composites define Wall complexes from bicomplexes are elementary to isolate.

Lemma 7.4. Fix a vertical splitting of $C_{\bullet\bullet}$, with notation as in Proposition 7.3.

1. The homology projection $C_{\bullet\bullet} \rightarrow H_{\bullet\bullet}$ is

$$P = 1 - dd^+ - d^+d.$$

2. The composite of the upward and leftward differentials induces homomorphisms

$$C_{p-1,q+1} \xleftarrow{d^+d_1} C_{pq} \xrightarrow{d_{pq}}$$

Together, these homomorphisms induce morphisms $\omega_j : H_{pq} \rightarrow H_{p-j,q+j-1}$ for $j \geq 1$ via

$$\omega_j = P(d_1d^+)j^{-1}d_1.$$

Remark 7.5. Pictorially, the homomorphisms $\omega_1, \omega_2, \omega_3, \omega_4, \ldots$ are drawn with the arrows in (∗) before Definition 6.1.

Definition 7.6. The derived Wall complex of a bicomplex $C_{\bullet\bullet}$ with vertical differential $d$ split by $d^+$ is $H_{\bullet\bullet}$ with the differentials $\omega_0 = 0$ and $\omega_i$ from Lemma 7.4 for $i \geq 1$.

Proposition 7.7 ([Eag90, Theorem 1.2]). The derived Wall complex $H_{\bullet\bullet}$ of a vertically split bicomplex $C_{\bullet\bullet}$ is a Wall complex as long as the local finiteness of $\omega$ is satisfied. The total complex of $H_{\bullet\bullet}$ has a filtration by taking successively more columns, starting from the left. The spectral sequence $HE^*$ for this filtration of $H_{\bullet\bullet}$ is the same as the vertical-then-horizontal spectral sequence $E^*$ of $C_{\bullet\bullet}$, in the sense that $HE^r_{pq} \cong E^r_{pq}$ for $r \geq 1$. 

Proof. This is the main result of [Eag90], so the proof is largely omitted. However, as there is a circular argument in the proof of Eq. (2.3.1) in [Eag90, Proposition 2.3], we include here a correct argument for that one point. For this purpose, let us assume the notation and context directly from the proof of [Eag90, Proposition 2.3].

The statement in question claims the following. Let $x \in \mathbb{Z}^r_{p,q}$. Define $y \in \text{Tot}(E)$ inductively downward by

$$y_i = (-1)^i x_i - d^+ d_1 y_{i+1} \quad \text{for all } i.$$ 

All terms are zero for $i > p$. Then

$$d(y_i) + d_1(y_{i+1}) = 0 \quad \text{for } p \geq i > p - r. \quad (2.3.1)$$

The first step in the proof of the equation is to show that

$$y_i = (-1)^i \sum_{j=0}^{p-i} (d^+ d_1)^j x_{i+j}$$

by downward induction, as follows. The base case, where $i = p$, is $y_p = (-1)^p x_p$ by [Eag90, Eq. (2.3.3)]. Now assume $y_{i+1} = (-1)^{i+1} \sum_{j=0}^{p-i-1} (d^+ d_1)^j x_{i+j+1}$. Then

$$y_i = (-1)^i x_i - d^+ d_1 y_{i+1}$$

$$= (-1)^i x_i - (-1)^{i+1} \sum_{j=0}^{p-i-1} (d^+ d_1)^{j+1} x_{i+j+1}$$

$$= (-1)^i \left( x_i + \sum_{j=1}^{p-i} (d^+ d_1)^j x_{i+j} \right)$$

$$= (-1)^i \left( \sum_{j=0}^{p-i} (d^+ d_1)^j x_{i+j} \right).$$

The next step is to show that if $d(y_i) + d_1(y_{i+1}) = 0$, then

$$d^+ dd_1 d^+ d_1 (y_{i+1}) = -d^+ d_1 dd^+ d_1 (y_{i+1})$$

$$= d^+ d_1 dd^+ d(y_i)$$

$$= d^+ d_1 d(y_i)$$

$$= -d^+ d_1 d_1 (y_{i+1})$$

$$= 0.$$ 

Now the proof of Eq. (2.3.1) proceeds by downward induction. In the base case $i = p$,

$$d(y_p) + d_1(y_{p+1}) = (-1)^p d(x_p) = 0$$
by definition of $\mathbb{Z}_{p,q}^r$. Now assume by induction that $d(y_i) + d_1(y_{i+1}) = 0$. Then
\[
d(y_{i-1}) + d_1(y_i) = (-1)^{i-1}d(x_{i-1}) - dd^+d_1(y_i) + d_1(y_i)
\]
\[
= (1 - dd^+)d_1(y_i)
\]
\[
= (1 - dd^+)d_1(-1)^i(x_i) - (1 - dd^+)d_1d_1(y_{i+1})
\]
\[
= (-1)^i(1 - dd^+ - d^+d)d_1(x_i) - (1 - dd^+ - d^+d)d_1d_1(y_{i+1})
\]
\[
= (-1)^i(1 - dd^+ - d^+d)\left(d_1(x_i) + d_1\sum_{j=0}^{p-i-1}(d^+d_1)^{j+1}x_{i+j+1}\right)
\]
\[
= (-1)^i(1 - dd^+ - d^+d)\left((d_1(x_i) + d_1\sum_{j=1}^{p-i}(d^+d_1)^jx_{i+j}\right)
\]
\[
= (-1)^i(1 - dd^+ - d^+d)\left(d_1\sum_{j=0}^{p-i}(d^+d_1)^jx_{i+j}\right)
\]
\[
= 0 \quad \text{by definition of } \mathbb{Z}_{p,q}^r. \quad \square
\]

**Remark 7.8.** Let $C_{\ast,\ast}$ be a vertically split bicomplex. The derived Wall complex selects a split submodule $H_{pq} \subseteq Z_{pq}$ inside the vertical cycles of $C_{pq}$ that maps isomorphically to the vertical homology—naturally defined as the quotient of these same cycles modulo the boundary submodule—which we denote by $\bar{H}_{pq} = Z_{pq}/B_{pq}$ so as not to confuse it with the submodule $H_{pq}$. The Wall differential $\omega_j = P(d_1d^+)^{j-1}d_1$ assumes that its input is an element of the split homology submodule $H_{pq}$. In applications such as to Problem 1.3 where one wishes to specify homomorphisms
\[
\tilde{\omega}_j : \bar{H}_{pq} \rightarrow \bar{H}_{p-j,q+j-1}
\]
on natural homology, the input should be a homology class—specified as a cycle that is well defined only up to adding a boundary element, rather than specified as an element of the split submodule $H_{pq}$—but at a cost: $\tilde{\omega}_j$ must first project $Z_{pq}$ to $H_{pq}$ to ensure that the Wall differential acts indistinguishably on different cycles representing the same homology class. This projection is
\[
1 - dd^+ : Z_{pq} \rightarrow H_{pq}.
\]
(For readers comparing Remark 3.12, this is the formality that proves $D$ annihilates boundaries.) On the other hand, $\tilde{\omega}_j$ need not be forced to produce output that lies in the split submodule $H_{p-j,q+j-1}$; it need only produce a cycle in $Z_{p-j,q+j-1}$, since the output of $\tilde{\omega}_j$ is to be understood modulo $B_{pq}$. That means $\tilde{\omega}_j$ can use the simpler projection
\[
1 - d^+d : C_{pq} \rightarrow Z_{pq}
\]
from chains to cycles instead of the split homology projection $P = 1 - dd^+ - d^+d$ from Lemma 7.4.1. (For readers comparing Remark 3.12 this is the formality that proves $D$
tj takes chains to cycles.) In total, then, the projection \( dd^+ \) moves from the left end of the expression defining \( \omega_j = (1 - dd^+ - d^+d)(d_1d^+)^{j-1}d_1 \) to the right end of the expression

\[
\tilde{\omega}_j = (1 - d^+d)(d_1d^+)^{j-1}d_1(1 - dd^+): Z_{pq} \to Z_{p-j,q+j-1},
\]

which defines a differential—the same differential as \( \omega_j \) defines—because

\[
(1 - dd^+)(1 - d^+d) = 1 - dd^+ - d^+d + dd^+d^+d
\]

occurs between the factors of \((d_1d^+)^{j-1}d_1\) and \((d_1d^+)^{j'-1}d_1\) in the square of the Wall differential either way.

It is these differentials \( \tilde{\omega}_j \), rather than \( \omega_j \) from Lemma \ref{lem:omega_j}, that solve Problem \ref{prob:omega_j} and give rise to the combinatorics in Section \ref{sec:omega_j}. We therefore record this shift from split homology \( H_{\cdot \cdot} \) to natural homology \( \tilde{H}_{\cdot \cdot} \) formally, the proof being in Remark \ref{rem:shift}.

**Definition 7.9.** The natural Wall complex of a bicomplex \( C_{\cdot \cdot} \) with vertical differential \( d \) split by \( d^+ \) is \( \tilde{H}_{\cdot \cdot} \) with the differentials \( \tilde{\omega}_0 = 0 \) and \( \tilde{\omega}_i \) from Remark \ref{rem:shift} for \( i \geq 1 \).

**Proposition 7.10.** Using the natural Wall complex \( \tilde{H}_{\cdot \cdot} \) in place of \( H_{\cdot \cdot} \) and \( \tilde{\omega}_\cdot \) in place of \( \omega_\cdot \) in Proposition \ref{prop:omega_j}, its conclusions hold verbatim. \( \Box \)

8. Minimal free resolutions from Wall complexes

The default coefficient ring in this section is an arbitrary field \( k \).

The goal of this section is to prove that derived Wall complexes of Koszul bicomplexes are minimal free resolutions (Theorem \ref{thm:Koszul}). The key point is that the Wall complex should resolve the given ideal \( I \), and not some associated graded module \( \text{gr} I \).

In the course of the proof, and indeed throughout the combinatorial elucidation in later sections, it is useful to have notation in which to make concrete computations.

**Lemma 8.1.** \( \mathbb{K}_{\cdot \cdot}(I) \cong \bigwedge^* V \otimes I^y \otimes k[x] \) has a \( k \)-linear basis \( z^\tau \otimes y^b \otimes x^a \) for

- \( z^\tau \in \bigwedge^{|\tau|} V \),
- \( y^b \in I^y \), and
- \( x^a \in k[x] \).

The \( \mathbb{N}^n \)-degree of \( z^\tau \otimes y^b \otimes x^a \) is \( \tau + b + a \). The differentials of \( \mathbb{K}_{\cdot \cdot} \) in this basis are

\[
\sum_{k \in \tau} (-1)^{k \in \tau} z^{\tau - e_k} \otimes y^b \otimes x^{a + e_k} \leftarrow z^\tau \otimes y^b \otimes x^a \leftarrow d \sum_{k \in \tau} (-1)^{k \in \tau} z^{\tau - e_k} \otimes y^{b + e_k} \otimes x^a.
\]

**Proof.** Lemma \ref{lem:K starving} implies that

\[
\tilde{K}E^0_{\cdot \cdot} = \bigwedge^* V \otimes k[y] \otimes k[x] I^y = \bigwedge^* V \otimes I^y,
\]

so the basis result follows by tensoring with \( k[x] \). The differentials are those from Proposition \ref{prop:omega_j} expressed explicitly in coordinates. \( \Box \)
Remark 8.2. If $I$ is an arbitrary standard $\mathbb{N}$-graded ideal, then the same argument shows that $\mathbb{K}_\ast(I)$ is spanned by tensor products $z^r \otimes g(y) \otimes x^a$, where $g(y) \in I^y$. It is of course possible to choose elements $g(y)$ so that these tensor products constitute a $k$-linear standard $\mathbb{N}$-graded basis, but there is no canonical choice, in general, because the homogeneous components of $I^y$ generally do not possess distinguished bases.

Theorem 8.3. Fix an arbitrary standard $\mathbb{N}$-graded ideal $I$. The total complex $W_\ast(I)$ of the derived or natural Wall complex $W_\ast(I)$ for any vertical splitting of the Koszul bicomplex $\mathbb{K}_\ast(I)$ is a minimal free resolution of $I$.

Proof. Remark 7.8 explains why the derived and natural Wall complexes have the same homology, so we present the proof only for the derived Wall case.

First observe that the Wall differentials are locally finite by the strip containment in Proposition 6.12 so $W_\ast(I)$ is indeed a Wall complex by Proposition 7.7. The homology of $W_\ast(I)$ vanishes except for $H_0W_\ast(I)$, which is an associated graded module of $I$; these are by Proposition 6.12 and the spectral sequence part of Proposition 7.6. What needs to be checked is that $H_0W_\ast$ is $I$ itself and not $\text{gr} I$ for some filtration.

By the direct sum decomposition in Proposition 7.3, the isomorphism of $W_0(I)$ with $\bigoplus_{a \in \mathbb{N}} \text{Tor}_0(k, I)_a \otimes k[x]$ afforded by Lemma 6.6 (see Remark 6.8) is realized by selecting a vector subspace $G_a^I$ of $I^y$ spanned by a complete set of minimal degree $a$ generators of $I^y$ and tensoring $G_a^I$ with $k[x]$. Therefore $W_0(I)$ has a canonical map $\gamma$ to $k[x]$ whose image is $I$, with $\text{Tor}_0(k, I)_a \otimes 1$ mapping to the vector subspace $G_a \subseteq I_a$. The theorem reduces to verifying two claims: first, the composition $k[x] \xrightarrow{\gamma} W_0(I) \xleftarrow{\text{gr} I} W_1(I)$ is zero, so $\gamma$ induces a surjection $I \twoheadrightarrow H_0W_\ast(I)$; and second, this homomorphism is injective.

Start with the vanishing composite. In the notation of Lemma 8.1, $\gamma$ acts simply by

$$
\begin{align*}
\mathbb{K}[x] &\xrightarrow{\gamma} \bigwedge V \otimes I^y \otimes k[x] \\
x^{a+b} &\leftarrow 1 \otimes y^b \otimes x^a.
\end{align*}
$$

(In the more general standard $\mathbb{N}$-graded setting of Remark 8.2, $g(x)x^a \xleftarrow{\gamma} 1 \otimes g(y) \otimes x^a$.) Colloquially, $\gamma$ sets $y = x$, and composing with $\gamma$ makes the horizontal and vertical differentials of $\mathbb{K}_\ast$ become equal; the precise version is immediate from Lemma 8.1:

$$
\mathbb{K}[x] \xrightarrow{\gamma} \mathbb{K}[y] \quad \text{when } p + q = 1.
$$

Observe that $W_1(I)$ is spanned by elements $z_k \otimes g(y) \otimes 1$ for which $z_k \otimes g(y)$ is a Koszul 1-cycle in $\bigwedge V \otimes I^y = \mathbb{K}_1^y \otimes k[y] I^y$. That means $d(z_k \otimes g(y) \otimes 1) = 0$, so

$$
0 = \gamma d(z_k \otimes g(y) \otimes 1) = \gamma d_1(z_k \otimes g(y) \otimes 1).
$$

Thus $\gamma d_1 = 0$ on $W_1(I)$. Next observe that $P = 1 - dd^+$ on $\mathbb{K}_{p,-p}(I)$, because the term $-d^+d$ of $P$ vanishes on $\mathbb{K}_{p,-p}(I)$; indeed already $d = 0$ on $\mathbb{K}_{p,-p}(I)$. Now compute:

$$
D_1 = P d_1 + P d_1 d^+ d_1 + P d_1 d^+ d_1 d^+ d_1 + \cdots
\begin{aligned}
= (d_1 - dd^+ d_1) + (d_1 d^+ d_1 - dd^+ d_1 d^+ d_1) + (d_1 d^+ d_1 d^+ d_1 - dd^+ d_1 d^+ d_1 d^+ d_1) + \cdots.
\end{aligned}
$$
Since $\gamma d = \gamma d_1$, upon multiplying both sides by $\gamma$ this sum telescopes, as desired, to

$$\gamma D_1 = \gamma d_1 = 0.$$ 

Finally, to check that $I \hookrightarrow H_0 W_*(I)$ is injective, simply note that $I$ and $H_0 W_*(I)$ have the same $\mathbb{N}$-graded Hilbert function: $\dim_k I_a = \dim_k H_0 W_*(I)_a$ for all $a \in \mathbb{N}$. □

9. Resolutions from splittings

The default coefficient ring in this section is a field $k$ of characteristic 0.

The first result in this section accomplishes the non-combinatorial part of the proof of Theorem 3.7. It is separated out from the rest of the proof because it applies in much more generality than the canonical sylvan setting, which is the choice to use the Moore–Penrose pseudoinverse as the Koszul simplicial splitting $s$. The notation for saturated decreasing lattice paths is as in Definition 3.1: $\lambda \in \Lambda(a, b)$ is equivalently described as $b = b_0, b_1, \ldots, b_{\ell-1}, b_\ell = a$ or as its successive differences $(\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_j = b_{j-1} - b_j$, each of which is a standard basis vector of $\mathbb{Z}^n$.

**Theorem 9.1.** Fix a monomial ideal $I$. Any splittings $\partial^{b+}$ of the differentials $\partial^b$ of the Koszul simplicial complexes $K^b I$ for $b \in \mathbb{N}^n$ that are themselves differentials satisfying

1. $\partial^b \partial^{b+} \partial^b = \partial^b$ and
2. $\partial^{b+} \partial^b \partial^{b+} = \partial^{b+}$

yield a minimal free resolution of $I$ whose differential from homological stage $i + 1$ to stage $i$ has its component $\tilde{H}_i K^b I \otimes_k [x](-b) \rightarrow \tilde{H}_{i-1} K^a I \otimes_k [x](-a)$ induced by the map

$$D : \tilde{H}_i K^b I \rightarrow \tilde{H}_{i-1} K^a I$$

in $\mathbb{N}^n$-degree $b$ that acts on any $i$-cycle in $\tilde{Z}_i K^b I$ via

$$D = \sum_{\lambda \in \Lambda(a,b)} (I^a - \partial_i^{a+} \partial_i^a) d_1^{\lambda_1} \left( \prod_{j=1}^{\ell-1} \partial_j^{b_j+} d_1^{\lambda_j} \right) (I^b - \partial_i^{b_i+} \partial_i^{b_{i+1}}),$$

where $d_1^{\lambda_j}$ takes $\tau \subseteq \{1, \ldots, n\}$ to 0 if $\lambda_j$ is not a vertex of $\tau$ and to $(-1)^{(\tau - \lambda_j, \tau)}$ if it is.

**Remark 9.2.** To visualize the formula for $D$, read the “$\partial$” and “$d$” maps in the following diagram from right to left, ignoring height, but note that the products of the
rightmost up-down and leftmost down-up maps must be subtracted from the identity.

\[
\begin{align*}
\tilde{C}_{i-1}K^aI & \xleftarrow{d_1^b} \\
\partial_i^a & \uparrow \quad \partial_i^a & \quad \partial_i^{b_{i-1}} & \quad \partial_i^{b_{i-1}} & \quad \partial_i^{b_{i-1}} & \quad \partial_i^{b_{i-1}} & \quad \partial_i^{b_{i-1}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\vdots & \uparrow \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \\
\tilde{C}_{i-1}K^{b_{i-1}}I & \xleftarrow{d_1^{b_{i-1}}} \quad \tilde{C}_{i-1}K^{b_{i-1}}I \\
\end{align*}
\]

This diagram, when rotated counterclockwise by $\pi/4$, is a zoomed-in, labeled version of the chain-link fence in Definition 3.3.

**Proof.** Lemma 8.1 implies that the vertical differentials of the Koszul bicomplex $\mathbb{K}_{*,*}(I)$ (Definition 6.9) are obtained from the chain complexes of Koszul simplicial complexes of $I$ by tensoring with $k[x]$ over $k$ (or use Lemma 6.6 and Proposition 6.7, in view of Convention 6.5). The given splittings $\partial^b$ thus induce a vertical splitting $d^+$ of $\mathbb{K}_{*,*}(I)$.

The natural Wall complex (Definition 7.9) of this vertically split Koszul bicomplex minimally resolves $I$ by Theorem 8.3. The differentials in this resolution are, by Definition 7.1, $D = \sum_j \tilde{\omega}_j$ for the homomorphisms $\tilde{\omega}_j = (1 - d^+d)(d_1d^+)j^{-1}d_1(1 - dd^+)$ from Remark 7.8. The goal is to determine the action of $d^+$ on $\mathbb{N}$-degree $b$ Koszul cycles in $\tilde{Z}_iK^bI \otimes k[x] \subseteq (\mathbb{K}^b)[x]$ (see the proof of Proposition 6.7), and then the action of $d$ on the output of this $d^+$, and the action of $d_1$ on the output of this $d$, and so on.

The reason why this requires care is that the action of $d^+$ on an $\mathbb{N}$-degree $b$ element of $\mathbb{K}_{*,*}(I)$ depends on how the element decomposes in the basis from Lemma 8.1: the vertical splitting is $\partial^b$ only on basis vectors the form $z^\tau \otimes y^{b-\tau} \otimes x^n$. It is therefore crucial that the $\mathbb{N}$-degree $b$ elements in $(\mathbb{K}^b)[x] \otimes k[x]$ all have the form $z^\tau \otimes y^{b-\tau} \otimes 1$ and are not mixtures in which the $x$-factors have nonzero $\mathbb{N}$-degree.

In contrast to $d^+$, the actions of $d_1$ and $d$ do not depend on the tensor decomposition. Let us start with $d$. The isomorphism $(\mathbb{K}^b(P))_b \cong \tilde{C}_bK^bI$ of the $\mathbb{N}$-graded components of the columns of $\mathbb{K}_{*,*}(I)$ with chain complexes of Koszul simplicial complexes identifies $z^\tau \otimes y^{b-\tau}$ with the face $\tau$. As such, Lemma 8.1 identifies $d$ with the simplicial boundary operator $\partial^b$ of $\tilde{C}_bK^bI$. This is true regardless of the $x$-factor and, indeed, regardless of the $\mathbb{N}$-degree, although of course for different $\mathbb{N}$-degrees the differential occurs in a different Koszul simplicial complex. Importantly, the $zy$-degree, for purposes of the splitting $d^+$ does not change under $d$, as is visible from Lemma 8.1.
Similarly, $d_1$ acts on simplices $\tau = z^\tau y^{c-\tau} x^{b-c}$ of $K^c I$ (thought of as residing in $\mathbb{N}^n$-degree $b$ of $\tilde{C}_r K^c I \otimes \mathbb{k}[x]$) as the boundary operator, but in this case the $zy$-degree of the boundary face $\tau - e_k$ has $zy$-degree $c - e_k$. Therefore $d_1 = d_1^{\tau_1} + \cdots + d_1^{\tau_n}$ decomposes into the components that alter the $zy$-degree by $e_1, \ldots, e_n$. Substituting this decomposition of $d_1$ back into the formula for $\tilde{\omega}_j$ and introducing the $\mathbb{N}^n$-degree indices $b, b_1, b_2, \ldots, b_{t-1}, b_t = a$ on the upward and downward differentials yields the sum over saturated decreasing lattice paths $\lambda \in \Lambda(a, b)$, as desired. \hfill \Box

Remark 9.3. The proof of Theorem 9.1 shows that the only summand $\tilde{\omega}_j$ contributing to the component $\tilde{H}_i K^b I \otimes \mathbb{k}[x](-b) \to \tilde{H}_{i-1} K^a I \otimes \mathbb{k}[x](-a)$ induced by the homomorphism $D : \tilde{H}_i K^b I \to \tilde{H}_{i-1} K^a I$ is $\tilde{\omega}_\ell$, where $\ell = |b| - |a|$. Let us tie up a couple of loose ends before getting to the proof of Theorem 3.7.

Lemma 9.4. The weights in Definition 3.4 are integers.

Proof. The coefficients $c$ need not be integers, but each of their denominators divides the relevant $\theta$ (let alone $\theta^2$) because these denominators are orders of torsion elements in the relevant homology groups by Lemma 4.6, Lemma 4.10 and Proposition 4.11. \hfill \Box

Proposition 9.5. Fix a hedge $ST_i$ in $K$. The shrub of any stake $\tau \in S_{i-1}$ (Definition 2.7) equals $\partial^{ST_i}_i(\tau)$ (Definition 4.7).

Proof. Let $s$ be the shrub of $\tau$. Then $s = \partial^{ST_i}_i \partial_i(s)$ by Definition 4.1. But $\tau - \partial_i s$ is a linear combination of non-stakes by Lemma 2.7 so $\partial^{ST_i}_i \partial_i(s) = \partial^{ST_i}_i(\tau)$. \hfill \Box

For the reader’s convenience, here is a restatement of the main result to be proved.

Theorem 3.7. Fix a monomial ideal $I$. The canonical sylvan homomorphism for each comparable pair $b \triangleright a$ of lattice points induces a homomorphism $\tilde{Z}_r K^b I \to \tilde{Z}_{r-1} K^a I$ that vanishes on $\tilde{B}_r K^b I$, and hence it induces a well defined canonical sylvan homology morphism $\tilde{H}_i K^b I \to \tilde{H}_{i-1} K^a I$. The induced homomorphisms

$$\tilde{H}_i K^b I \otimes \mathbb{k}[x](-b) \to \tilde{H}_{i-1} K^a I \otimes \mathbb{k}[x](-a)$$

of $\mathbb{N}^n$-graded free $\mathbb{k}[x]$-modules constitute a minimal free resolution of $I$.

Proof. Given Theorem 9.1 it remains only to prove that the formula for $D$ in Theorem 9.1 specializes to the canonical sylvan homomorphism whose matrix has entries

$$D_{\sigma \tau} = \sum_{\lambda \in \Lambda(a, b)} \frac{1}{\Delta_{i, \lambda} I} \sum_{\varphi \in \Phi_{\sigma \tau}(\lambda)} w_\varphi$$

from Definition 3.6 when all of the splittings $\partial^{b_j+}$ are the Moore–Penrose pseudoinverses of the differentials $\partial^{b_j}$. The proof is lattice path by lattice path, so fix henceforth a saturated decreasing lattice path $\lambda$ from $b$ to $a$ of length $\ell = |b| - |a|$. Fix as well the simplices $\tau \in K^b I$ and $\sigma \in K^a_{i-1} I$. 

The Boundary Projection Formula (Corollary [5.7]) for \( \pi_{B_i} = \partial_i^{+} \partial_i^{+} \) shows that

\[
1 - \partial_i^{+} \partial_i^{+} = \frac{1}{\Delta_i^b} \left( \Delta_i^s 1 - \sum_{S_i} \theta_{S_i}^2 b_{S_i} \right)
= \frac{1}{\Delta_i^s} \sum_{S_i} \theta_{S_i}^2 (1 - b_{S_i})
\]

In the image of \( \tau \) under this homomorphism at \( b \in \mathbb{N}^n \), the coefficient on \( \tau_0 \) in the summand for the stake set \( S_i^b \) is the weight of the boundary-link \( \tau_0 - \tau \) by Definition [2.10.3].

Summing over stake sets and dividing by \( \Delta_i^s K^b I \) yields the \( \tau_0 \tau \) matrix entry in \( \partial_i^{+} \partial_i^{+} \).

Let \( j \geq 1 \). In the image of any \( i \)-simplex \( \tau_{j-1} \) under \( d_1 = d_{e_1}^s + \cdots + d_{e_i}^s \), where \( d_{e_i}^s \) alters the \( \text{zy} \)-degree by \( e_k \) (see the end of the proof of Theorem [9.1]), the coefficient on \( \tau_j \) is 0 unless \( \sigma_j = \tau_{j-1} - \lambda_j \) as in Definition [3.3], in which case the coefficient output by Theorem [9.1] is a sign—the correct one for the containment \( \sigma_j \setminus \tau_{j-1} \) by Definition [3.4].

In the image of any \((i-1)\)-simplex \( \sigma_j \) under the Moore–Penrose pseudoinverse \( \partial_i^{b,\dagger} \) of the boundary \( \partial_i^b \), the coefficient on \( \tau_j \) in the summand indexed by the hedge \( ST_i^b \) in the Hedge Formula (Theorem [5.5]) is the weight of the chain-link \( \tau_j^{\dagger} / \sigma_j \) by Proposition [9.5].

Summing over hedges and dividing by \( \Delta_i^s ST_i^b I \) yields the \( \sigma_j \tau_j \) matrix entry in \( \partial_i^{b,\dagger} \).

In the image of any \((i-1)\)-simplex \( \sigma_j \) under orthogonal projection \( 1 - \partial_i^a + \partial_i^a \) to the cycles \( \tilde{Z}_{i-1} K^a I \), the coefficient on \( \sigma_j \) in the summand indexed by the shrubbery \( T_{i-1}^a \) in the Higher Projection Formula [CCK15, Theorem A] (stated with our notation in the proof of Theorem [5.5], part [4]) is the weight of the cycle-link \( \sigma_j - \sigma \) Definition [2.10.1].

Summing over shrubberies and dividing by \( \Delta_i^s T_{i-1}^a I \) yields the \( \sigma_j \sigma \) matrix entry in \( 1 - \partial_i^a + \partial_i^a \).

Summing the products of these three kinds of sums and the sign over all chain-link fences from \( \tau \) to \( \sigma \) yields the matrix entry \( D_{\sigma \tau} \) by matrix multiplication from elementary linear algebra. Definition [3.6] expresses this product of sums as a sum of products. \( \square \)

10. NONCANONICAL SYLVAN RESOLUTIONS

The default coefficient ring in this section is an arbitrary field \( k \).

**Remark 10.1.** The construction of a canonical sylvan resolution of a monomial ideal \( I \) in \( n \) variables always works in characteristic 0, and it works in any finite characteristic that does not divide any of the constants \( \Delta_{i,\lambda} I \) (Definition [3.2]) for \( i \in \{0, \ldots, n\} \) and lattice paths \( \lambda \in \Lambda(a, b) \) ending at nonzero Betti numbers \( \beta_{i,a}(I) \neq 0 \) and \( \beta_{i+1,b}(I) \neq 0 \). At present we are unaware of any construction of free resolutions for monomial ideals in positive characteristic that is universal, canonical, combinatorial, and minimal.

That said, the master formula for Wall resolutions from Koszul simplicial splittings (Theorem [9.1]) has the consequence that once the canonical requirement is dropped, our constructions work universally, combinatorially, and minimally. The format is basically the same as Theorem [3.7] but there is no division and the weights are simpler.
Definition 10.2. A chain-link fence over $k$ is defined verbatim as in Definition 3.3 (with the $-$links over the arbitrary field $k$ instead of a field of characteristic 0). Each chain-link fence edge has a simple weight over $k$:

- the boundary-link $\tau_0 - \tau$ has simple weight $c_{\tau}(\tau_0, S^b)$,
- the chain-link $\tau_j \setminus \sigma_j$ has simple weight $c_{\sigma_j}(\tau_j, ST^b_{i})$,
- the containment $\sigma_j \cap \tau_j - 1$ has simple weight $(-1)^{\sigma_j \subset \tau_j - 1}$, and
- the cycle-link $\sigma - \sigma_\ell$ has simple weight $c_{\sigma_\ell}(\sigma, T^a_{i-1})$.

The simple weight of the fence $\varphi$ is the product $w^\varphi_k$ of the simple weights on its edges.

Definition 10.3. Fix a monomial ideal $I$ and a community (Definition 4.3 and Proposition 4.4) for each Koszul simplicial complex $K^b_I$. These data endow each lattice path $\lambda \in \Lambda(a, b)$ with a fixed hedgerow $ST^\lambda_i$ for $i = 0, \ldots, n$. The sylvan homomorphism

$$D = D^{ab} : \widetilde{C}_i K^b I \to \widetilde{C}_{i-1} K^a I$$

for these data is given by its sylvan matrix, whose entry $D_{\sigma\tau}$ for $\tau \in K^b_i$ and $\sigma \in K^a_{i-1}$ is the sum, over all lattice paths from $b$ to $a$, of the weights of all chain-link fences from $\tau$ to $\sigma$ that are subordinate (Remark 3.5) to the relevant hedgerow $ST^\lambda_i$:

$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(a, b)} \sum_{\varphi \in \Phi^\lambda(\lambda)} \sum_{\varphi \notin ST^\lambda_i} w^\varphi_k.$$

Corollary 10.4. Fix a monomial ideal $I$ and a community for each Koszul simplicial complex $K^b_I$. The sylvan homomorphism for these data on each comparable pair $b > a$ of lattice points induces a homomorphism $\widetilde{Z}_i K^b I \to \widetilde{Z}_{i-1} K^a I$ that vanishes on $\widetilde{B}_i K^b I$, and hence it induces a well defined sylvan homology morphism $\widetilde{H}_i K^b I \to \widetilde{H}_{i-1} K^a I$. The induced homomorphisms

$$\widetilde{H}_i K^b I \otimes k[x](-b) \to \widetilde{H}_{i-1} K^a I \otimes k[x](-a)$$

of $\mathbb{N}$-graded free $k[x]$-modules constitute a minimal free resolution of $I$.

Proof. The proof of Theorem 3.7 in Section 9, which is already done lattice path by lattice path, works mutatis mutandis in this setting but simplifies because the fixed hedgerows eliminate the summations over stake sets, hedges, and shrubbery.

Remark 10.5. Since Corollary 10.4 occurs at the end of this paper, it is worth taking precise account of the relatively meager prerequisites—beyond standard constructions like $K^b I$—on which its statement (but not its proof) relies. It requires the notions of

- shrubbery, stake, hedge (Definition 2.1 and their coefficients (Definition 2.10);
- hedgerow (Definition 3.1) to assemble this combinatorics along lattice paths;
- chain-link fence (Definition 3.3) with simple weights (Definition 10.2); and
- community (Definition 4.3) and hedge splitting (Definition 4.4) for the differential.
Remark 10.6. In general, a minimal free resolution should be called *sylvan* if its differentials are expressed as linear combinations of those for individual choices of hedges. Thus the canonical resolutions in Theorem 3.7 are sylvan because its differentials are weighted averages of differentials from hedges, and the resolutions in Corollary 10.4 are sylvan because each fixes single choices of hedges. Of course, all minimal free resolutions of a given graded ideal are isomorphic; the question is how the resolution is expressed. Usually in commutative algebra the differentials are expressed by selecting bases for the syzygies. In contrast the sylvan method avoids choosing such bases, even in Corollary 10.4 because the syzygies are naturally homology vector spaces. Instead, the sylvan method selects bases for chains in a manner that descends to homology.

Remark 10.7. Canonical sylvan resolutions in Theorem 3.7 are not suited to efficient algorithms, as they require storage, manipulation, and sums over bases for chains in simplicial complexes. In contrast, noncanonical sylvan resolutions could potentially lead to efficient algorithmic computation of free resolutions, since they select bases not for chains but for homology and cohomology (see Remark 2.3) in each $N^n$-degree.

Example 10.8. Noncanonical sylvan resolutions provide combinatorial minimal free resolutions of monomial ideals whose Betti numbers vary with the characteristic of the field, such as the Stanley–Reisner ideal of the six-vertex triangulation of the real projective plane. In any characteristic other than 2, this ideal has a minimal cellular free resolution of length 2; see [MS05, Section 4.3.5], for instance. But in characteristic 2, the top Betti number is at homological stage 3, namely $\beta_{3,1}(I) = 1$, where here $1 = (1,1,1,1,1,1)$. A sylvan resolution compensates by selecting hedges that respect the dependencies in characteristic 2. In particular, although the stake set $S_2^3 = \emptyset$ at 1 is forced, because $K^3 I = \mathbb{RP}^2$ has dimension 2, the stake sets at degrees $1 - e_i$ that differ from 1 by a standard basis vector $e_i$ have cardinality 1 in characteristic 2, each consisting of any single edge in the relevant Koszul simplicial complex, thereby allowing the construction of chain-link fences to get started.

Remark 10.9. Judicious choices of communities in Corollary 10.4 can recover known special classes of resolutions of monomial ideals, such as the Eliahou–Kervaire resolution of any Borel-fixed or stable ideal [EK90] (see also [MS05, Chapter 2]) or any planar graph resolution of a trivariate ideal [Mil02] (see also [MS05, Chapter 3]). These assertions require proof; they are planned for a subsequent paper.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455

Mathematics Department, Duke University, Durham, NC 27708

Website: [http://math.duke.edu/people/ezra-miller](http://math.duke.edu/people/ezra-miller)

Mathematics Department, Duke University, Durham, NC 27708

Website: [https://fds.duke.edu/db/aas/math/grad/ordog](https://fds.duke.edu/db/aas/math/grad/ordog)