GRAVITATIONAL FORCES WITH STRONGLY LOCALIZED RETARDATION

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Abstract

We solve the linearized Einstein equations for a specific oscillating mass distribution and discuss the usual counterarguments against the existence of observable gravitational retardations in the "near zone", where \( \frac{d}{r} \ll 1 \) (\( d \) = oscillation amplitude of the source, \( r \) = distance from the source). We show that they do not apply in the region \( \frac{d}{r} \approx 1 \), and prove that gravitational forces are retarded in the immediate vicinity of the source. An experiment to measure this retardation is proposed, which may provide the first direct experimental observation of propagating gravitational fields.

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I  INTRODUCTION AND SURVEY OF RESULTS

Between the 1940s and the early 1960s, several researchers proposed the possibility of measuring the retardation of gravitational interaction using laboratory experiments [1, 2, 3, 4]. Because of the poor technology at that time and contradictory theoretical predictions about the retardation, interest in the subject declined. No experiments were performed during this period, and after 1962, the topic was never discussed again in the literature. At the 1996 Virgo conference, a new experimental set-up was described for the purpose of testing the retardation of gravitational interaction in the immediate vicinity of a vibrating mass distribution [5]. Discussions following the talk indicated that there is still ambiguity about the existence of such a retardation. In this paper, we will discuss the common arguments which lead to the conclusion that gravitational forces generated by oscillating mass distributions are instantaneous in the near-field. All of these arguments have the weakness that they are not valid at extremely small distances from the oscillating source. We will analyze the gravitational forces in the immediate vicinity of a specific oscillating mass distribution, for which the dominant part of the corresponding linearized Einstein equations can be solved. It shows that there is a possibly detectable retardation of the gravitational forces in the immediate vicinity of the source, which converges to zero with increasing distance. The dominant length scale for the decay of the retardation is given by the oscillation amplitude of the mass distribution and can be very small. Experimental verification of this local retardation effect would provide the first direct observation of propagating gravitational fields. We propose a laboratory experiment which requires a highly accurate phase measurement to test our result. Given the state of modern technology, this measurement may now be possible.

II  DEFINITION, DISCUSSION AND ANALYSIS OF THE MATHEMATICAL MODEL

We consider an idealized experiment in a laboratory environment, consisting of a mass-balanced, mechanical two-body oscillator, which generates disturbances in the space-time metric. The
oscillator consists of a point mass $m$ which is connected by way of a spring with negligible mass to a second point mass $M \gg m$. The resulting mass-spring system oscillates with eigen-frequency $\omega$ about its center of gravity, and does not interact dynamically with additional, external masses. The distance between the masses $m$ and $M$ is assumed to be much larger than the oscillation amplitude $d$ of the smaller mass $m$. The oscillation amplitude $D = \frac{m}{M}d$ of $M$, which is calculated with classical mechanics, is much smaller than $d$ and leads to negligible time-dependent gravitational effects in the immediate vicinity of $m$, as will be shown in section II.1. We will mainly consider the time-dependent gravitational forces generated by $m$, and will thus refer to $m$ as the ”source”, or the ”source mass”. Another mechanical oscillator, tuned to resonance to detect these gravitational forces, is placed on the oscillator axis at a distance of order $O(d)$ from $m$. The purpose of this paper is to show that the gravitational forces generated in this extreme near-field laboratory experiment are retarded, which may be verifiable with modern technology. Due to many objections against experiments of this kind in the past, we will first discuss the common counterarguments.

The occurrence of a measurable retardation in the proposed system may seem problematic for several reasons. One may suspect that the theorem about the non-existence of gravitational dipole radiation is violated by the mass-balanced, mechanical two-body oscillator described above. It states that isolated mechanical systems cannot generate any dipolar gravitational radiation, due to momentum and angular momentum conservation within the system [6]. However, this argument does not contradict the existence of retarded gravitational forces which are strongly localized around the source. We will refer to the immediate vicinity of $m$ as a region around $m$ with a diameter of order $O(d)$. In section II.1, we will prove that the time-dependent gravitational forces due to the larger mass $M$ can be neglected in the immediate vicinity of the source mass $m$, although the static field due to $M$ may dominate over the static field generated by $m$. The mass $m$, which is by itself not an isolated system, appears locally to be an isolated oscillator. Because the momentum of $m$ is by itself not conserved, $m$ is allowed to radiate gravitationally in its immediate vicinity without contradicting the overall conservation of momentum. At larger distances from the oscillator, the time-dependent gravitational forces
due to $M$ and $m$ have equal strength, and we will show in section II.2.1 that their radiative contributions cancel. Hence, there is no contradiction to overall momentum conservation.

Near-field set-ups similar to the proposed one are usually analyzed with the ”near zone” limit used in radiation theory, typically defined as a distance range comparable to the wavelength of the radiated signals, which is usually very large. At these large distances from the source, the vibrating mass distribution can be very accurately modelled as a series of multipole coefficients, which neglect all retardation effects within the source. But in the immediate vicinity of the source, i.e. at distances comparable to its vibration amplitude, it is not possible to make this simplification. Instead, one must take local retardation effects into account, even if they disappear at large distances.

Another common objection against the existence of observable retardations in the proposed set-up arises from an electrodynamic analogy. Retarded electromagnetic potentials may result in almost instantaneous electromagnetic fields, due to certain cancellation effects [7]. This phenomenon occurs in the ”near zone” introduced above, and is consistent with our results. We will show in section II.2.2 that the retardation of the gravitational forces decreases continuously to zero with increasing distance from the source, and that in the immediate vicinity of $m$, the retardation has possibly measurable values.

We will first show that our proposed system can be analyzed with the linearized Einstein equations. Let $c \approx 3 \cdot 10^8 \frac{m}{sec}$ denote the vacuum speed of light. We choose a local Minkowskian coordinate frame $\{ct, x, y, z\}$, in which $m$ oscillates about the origin with amplitude $d$, angular frequency $\omega$, and parallel to the $z$ axis

$$ \vec{z}(t) = \hat{z} \ d \ sin \omega t $$

(1)

$\hat{z}$ denotes the unit vector in the $z$ direction, and $\vec{z}(t)$ is the location of $m$ at time $t$. We assume mass $m \approx O(10^{-1}[kg])$, oscillation amplitude $d \approx O(10^{-2}[m])$, and angular frequency $\omega \approx O(10^{3}[rad/sec])$ as a set of reasonable parameters.

To keep track of order-of-magnitude estimates, we introduce dimensionless variables. We
define the dimensionless four-vector \((\tau, \vec{\xi}) := \frac{1}{d} (ct, \vec{x})\) and use the oscillation amplitude \(d\) to represent the length scale. The energy momentum tensor \(T_{\mu\nu}\) is related to the dimensionless \(T_{\mu\nu}\) by \(T_{\mu\nu} =: \frac{mc^2}{d^3} T_{\mu\nu}\), and the Einstein tensor \(G_{\mu\nu}\) which has the dimensions \([m^{-2}]\) introduces \(G_{\mu\nu} =: \frac{1}{d^2} G_{\mu\nu}\). The Einstein equations, written in terms of dimensionless variables, yield

\[
G_{\mu\nu} = \frac{8\pi mG}{c^2 d} T_{\mu\nu},
\]

in which \(G \approx 6.67 \cdot 10^{-11} \left[\frac{m^3}{sec^2 kg}\right]\) is the universal gravitational constant. The number \(\frac{8\pi Gm}{c^4 d}\) estimates the strength of gravitational signals due to \(m\). We bound the mass of \(M\) by \(M < O(10^2)\), such that all considerations about \(m\) also apply to \(M\). The RHS of Eq. (2) is very small if \(\| T_{\mu\nu} \| \ll 10^{-25}\), and the linearized Einstein equations can be used in this case. Consequently, one expands the space-time metric \(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\) around the flat Minkowski metric \(\eta_{\mu\nu}\), and introducing \(\gamma_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h_{\lambda\lambda}\), Eq. (2) reduces to

\[
\left( \partial^2 - \Delta_{\vec{\xi}} \right) \gamma_{\mu\nu}(\tau, \vec{\xi}) = -\frac{16\pi mG}{c^2 d} T_{\mu\nu}(\tau, \vec{\xi}),
\]

by use of the Hilbert gauge condition \(\gamma^{\nu}_{\mu,\nu} = 0\). \(\Delta_{\vec{\xi}}\) is the Laplace operator in the dimensionless spatial coordinates \(\vec{\xi}\). Eq. (3) is solved by the superposition of a particular solution with a homogenous solution. The full particular solution includes the influences of both \(M\) and \(m\). We impose the boundary condition \(\gamma_{\mu\nu} \to 0\) in the limit \(|\vec{\xi}| \to \infty\). Since the full particular solution is asymptotically zero, as will be shown in section II.2, this requires the homogenous solution, given by a superposition of plane waves with constant amplitudes, to be zero. Therefore, \(\gamma_{\mu\nu}\) is the particular solution of Eq. (3). Due to linearity, the contributions from \(M\) and \(m\) to \(\gamma_{\mu\nu}\) and \(T_{\mu\nu}\) are additive and can be discussed separately, thus \(\gamma_{\mu\nu} = \gamma^{(m)}_{\mu\nu} + \gamma^{(M)}_{\mu\nu}\) and \(T_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(M)}_{\mu\nu}\).

We will first analyze the influences due to \(m\), determined by \(\gamma^{(m)}_{\mu\nu}\) and \(T^{(m)}_{\mu\nu}\). We obtain

\[
\gamma^{(m)}_{\mu\nu}(\tau, \vec{\xi}) = -\frac{4mG}{c^2 d} \int d\tau' d^3\vec{\xi}' \frac{T^{(m)}_{\mu\nu}(\tau', \vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|} \delta(\tau' - \tau + |\vec{\xi} - \vec{\xi}'|) \Theta(\tau - \tau'),
\]

in which \(\delta\) is the Dirac-delta distribution and \(\Theta\) is the Heaviside function. The perturbation parameter of our analysis is the ratio \(\beta = \frac{\omega d}{c} \approx O(10^{-8})\) between the maximal velocity \(\omega d\) of the oscillating mass \(m\) and the speed of light \(c\). Our analysis will show that the retardation is
an effect of the order $O(\beta)$. Therefore, a relative accuracy of the order $O(\beta^2)$ is sufficient for our calculations. $T^{(m)}_{00}$ can be decomposed into

$$T^{(m)}_{00}(\tau', \xi') = T^{(m,0)}_{00}(\tau', \xi') + T^{(m,2)}_{00}(\tau', \xi'),$$

where $T^{(m,3)}_{00}$ is of order $O(\beta^3)$. Note that this decomposition does not correspond to the full Taylor expansion with respect to $\beta$. $T^{(m,0)}_{00}$ is the only contribution of order $O(1 = \beta^0)$ to $T_{\mu\nu}$, and is determined by the dimensionless rest energy density $\rho(\tau', \xi')$ of $m$, which satisfies $\int d\xi' \rho(\tau', \xi') = 1$. In all physical systems, $\rho(\tau', \xi')$ can be assumed to be a sharply-peaked, smooth function with $\sup_\xi \rho(\tau', \xi') \ll O(\theta_{25})$, such that Eq. (3) can be used. For mathematical convenience, we replace $\rho(\tau', \xi')$ by a Dirac delta distribution $\delta^{(3)}$, and consequently obtain

$$T^{(m,0)}_{00}(\tau', \xi') = \delta^{(3)}(\xi' - \hat{z} \sin \beta \tau'). \quad (5)$$

Note that the space and time dependent terms on the RHS of Eq. (5) cannot be separated, as opposed to the expression $\rho(\xi') \exp[\tilde{i} \omega \tau']$ commonly found in the literature which is suitable for multipole decomposition. $T^{(m,2)}_{00}$ is determined by the sum of the kinetic energy of $m$ and the potential energy of the spring, and is of order $O(\beta^2)$. The components $T^{(m)}_{0i}$, $i = 1, 2, 3$, are given by

$$T^{(m)}_{0i}(\tau', \xi') = \beta \hat{z}_i \cos \beta \tau' \delta^{(3)}(\xi' - \hat{z} \sin \beta \tau'),$$

and are of order $O(\beta)$. The components $T^{(m)}_{ij}$, $i, j = 1, 2, 3$, given by

$$T^{(m)}_{ij}(\tau', \xi') = \beta^2 \hat{z}_i \hat{z}_j \cos \beta \tau' \delta^{(3)}(\xi' - \hat{z} \sin \beta \tau'),$$

will only contribute to error terms of the order $O(\beta^2)$.

We evaluate $\gamma^{(m)}_{\mu\nu}(\tau, \xi)$ at the point $\xi = \frac{\xi}{d} \hat{z}$ on the positive $z$ axis, such that $\xi_3 = \frac{\xi}{d} > 1$. Inserting Eq. (5) into Eq. (4), we first calculate $\gamma^{(m,0)}_{00}(\tau, \xi_3 \hat{z})$, defined by $\gamma^{(m)}_{00} = \gamma^{(m,0)}_{00} + \gamma^{(m,2)}_{00}$. The $\delta$-integration in $\xi_1$ and $\xi_2$ can be readily carried out, yielding

$$\gamma^{(m,0)}_{00}(\tau, \xi_3 \hat{z}) = \gamma^{(m,0)}_{00}(\tau, \xi_3 \hat{z}) = -\frac{4mG}{c^2d} \int d\xi'_3 d\tau' \frac{\delta(\xi'_3 - \sin \beta \tau')}{\xi_3 - \xi'_3} \delta(\tau' + \xi_3 - \xi'_3 - \tau) \Theta(\tau - \tau')$$

$$= -\frac{4mG}{c^2d} \int d\tau' \frac{\delta(\tau' + \xi_3 - \sin \beta \tau' - \tau)}{\xi_3 - \sin \beta \tau'} \Theta(\tau - \tau'). \quad (6)$$
Further reduction is possible using the formula
\[ \int d\tau' \delta(f(\tau')) \ g(\tau') = \sum_{\theta \in \{\theta|f(\theta) = 0\}} \frac{1}{f'(\theta)} \ g(\theta), \]
in which we identify the zeros of the argument in the delta distribution as the roots of \( \theta \) in the equation
\[ \theta(\tau, \xi_3) = \sin \beta \theta(\tau, \xi_3) + \tau - \xi_3. \] (7)
Note that \( \theta(\tau, \xi_3) \) is a function of the combination \( \tau - \xi_3 \). The derivative of the argument of the delta in Eq. (6) yields \( 1 - \beta \cos \beta \theta(\tau, \xi_3) \) at \( \theta(\tau, \xi_3) \). We obtain for \( \gamma^{(m,0)}_{00}(\tau, \xi_3 \tilde{z}) \)
\[ \gamma^{(m,0)}_{00}(\tau, \xi_3 \tilde{z}) = -\frac{4mG}{c^2d} \frac{1}{1 - \beta \cos \beta \theta(\tau, \xi_3)} \frac{1}{\xi_3 - \sin \beta \theta(\tau, \xi_3)}. \] (8)
In Eq. (8), \( \beta \) is not only a small perturbation parameter, but also a "dimensionless angular frequency". In contrast to \( \beta \), the arguments \( \beta \theta(\tau, \xi_3) \) of the trigonometric functions need not be small for arbitrary \( \tau \) or \( \xi_3 \), and cannot be used as perturbation parameters. Since \( T^{(m,2)}_{00} \) is of order \( O(\beta^2) \), it follows that \( \gamma^{(m,2)}_{00} = O(\beta^2) \) is an error term, and that \( \gamma^{(m)}_{00}(\tau, \xi_3 \tilde{z}) = \gamma^{(m,0)}_{00}(\tau, \xi_3 \tilde{z}) + O(\beta^2) \). Hence it is sufficient to analyze \( \gamma^{(m,0)}_{00} \) to the order of \( O(1 = \beta^0) \) and \( O(\beta) \), to determine \( \gamma^{(m)}_{00} \) with a relative error of order \( O(\beta^2) \). The solutions for \( \gamma^{(m)}_{0i}(\tau, \xi_3 \tilde{z}) \), \( i = 1, 2, 3 \), can be derived in a similar manner, yielding
\[ \gamma^{(m)}_{0i}(\tau, \xi_3 \tilde{z}) = -\frac{4mG}{c^2d} \frac{ \dot{z}_i \cos \beta \theta(\tau, \xi_3) }{1 - \beta \cos \beta \theta(\tau, \xi_3)} \frac{1}{\xi_3 - \sin \beta \theta(\tau, \xi_3)}. \]
The solutions for \( \gamma^{(m)}_{ij}(\tau, \xi_3 \tilde{z}) \), \( i, j = 1, 2, 3 \), which are given by
\[ \gamma^{(m)}_{ij}(\tau, \xi_3 \tilde{z}) = -\frac{4mG}{c^2d} \frac{ \dot{z}_i \dot{z}_j \beta^2 \cos^2 \beta \theta(\tau, \xi_3) }{1 - \beta \cos \beta \theta(\tau, \xi_3)} \frac{1}{\xi_3 - \sin \beta \theta(\tau, \xi_3)}, \]
contribute to error terms of order \( O(\beta^2) \). The contributions \( \gamma^{(M)}_{\mu \nu}(\tau, \xi_3 \tilde{z}) \) of the mass \( M \) can be obtained by substituting \( d \rightarrow D, r \rightarrow R, m \rightarrow M, \beta \rightarrow -\frac{D}{d} \beta, \) \( \theta \rightarrow \frac{D}{d} \theta, \) \( (\tau, \xi) \rightarrow \frac{D}{d}(\tau, \xi), \) \( \xi_3 = \frac{r}{d} \rightarrow \frac{R}{D} \) in \( \gamma^{(m)}_{\mu \nu}(\tau, \xi_3 \tilde{z}) \), where \( D = \frac{m}{M} d \) is the oscillation amplitude of \( M \), and \( R \) is its average distance to the point of measurement. This is because both masses oscillate sinusoidally with the same frequency \( \omega \) and opposite phases along the \( z \) axis.
It follows that \( \partial_\tau \gamma_{0i}^{(m,M)}(\tau, \xi_3 \hat{z}) \) is of order \( O(\beta^2) \) for both \( M \) and \( m \). We will show below that the only Levi-Civita connection coefficients needed for our analysis are

\[
\Gamma_{00}^\mu(\tau, \xi_3 \hat{z}) = -\frac{1}{2}(2h_{0\mu,0}(\tau, \xi_3 \hat{z}) - h_{00,\mu}(\tau, \xi_3 \hat{z}))
\]

\[
= \frac{1}{4} \gamma_{00,\mu}(\tau, \xi_3 \hat{z}) - \gamma_{0\mu,0}(\tau, \xi_3 \hat{z}) + \frac{1}{4} \gamma_{k,\mu}^k(\tau, \xi_3 \hat{z}) .
\]

Thus, \( \Gamma_{00}^i = \frac{1}{4} \gamma_{00,0}^{(0)} + O(\beta^2) \) for \( i = 1, 2, 3 \). Here, we have used that \( h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \gamma_{\lambda\lambda} \) and \( \Gamma_{\rho\sigma}^\mu = \frac{1}{2} \eta^{\mu\nu}(h_{\rho\nu,\sigma} + h_{\sigma\nu,\rho} - h_{\sigma\rho,\nu}) \), which include the influences of both \( M \) and \( m \).

Given the metric \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), we intend to study the motion of a test mass point \( \tilde{m} \ll m \) in the immediate vicinity of \( m \), that is part of another mechanical two-body oscillator on the \( z \) axis. The dimensionless space-time coordinates of \( \tilde{m} \) are scaled with \( d \), yielding \((\xi_0(\tau), \vec{\xi}(\tau))\). We are only interested in oscillatory solutions with small amplitudes of order \( \approx d \cdot O(\beta) \), which are induced by the oscillations of the space-time metric. Therefore, we require that the dimensionless velocity components \( \xi_i, 0, i = 1, 2, 3 \), of \( \tilde{m} \), are of order \( O(\beta^2) \). The subscript ”,0” denotes the derivative with respect to \( \tau \). The proper time of the test mass satisfies \( \xi_0(\tau) \approx \tau \) and \( \xi_0 = O(1) \), since the deviations from flat Minkowskian space-time induced by \( M \) and \( m \) are very small. The action which determines the dynamics of the mass \( \tilde{m} \) in our coordinate system is given by

\[
S = \tilde{m} cd \int_{\tau_0}^{\tau_1} d\tau \left\{ -\sqrt{g_{\mu\nu}\xi_\mu,0\xi_\nu,0} - U(\xi) \right\},
\]

where \( U(\xi) \) is the dimensionless potential energy due to the spring. For the moment, we assume that the second mass of the pick-up system is ”infinitely” larger than \( \tilde{m} \), and static. The space-time trajectory of the test mass \( \tilde{m} \) is determined by the Euler-Lagrange equation

\[
-\frac{\xi^\mu_{,00} + \Gamma_{\rho\sigma}^\mu \xi^\rho_{,0} \xi^\sigma_{,0}}{\sqrt{g_{\mu\nu}\xi^\mu_{,0} \xi^\nu_{,0}}} + U_{,\mu} = 0 ,
\]

where \( \Gamma_{\rho\sigma}^\mu \) are linearized Levi-Civita connection coefficients. The dominant terms in the sum \( \Gamma_{\rho\sigma}^\mu \xi^\rho_{,0} \xi^\sigma_{,0} \) arise from \( \Gamma_{00}^\mu \xi_{,0}^0 \xi_{,0}^0 \). All terms \( \Gamma_{\rho\sigma}^\mu \xi^\rho_{,0} \xi^\sigma_{,0} \) which involve \( \xi_j^i, j = 1, 2, 3 \), are at least a factor of order \( O(\beta^2) \) smaller, and can be neglected. From the equation of motion for \( \xi^0 \) follows that \( \frac{\xi^0_{,00}}{(\xi_0^0)^2} + \Gamma^0_{00} = 0 \) with a relative error of order \( O(\beta^2) \). Using the fact that \( \Gamma^0_{00} \) is a
total derivative with respect to $\tau$, and that $\| \gamma_{\mu\nu} \| \ll O(\beta^2)$, one obtains $\xi_{00} = 1 + O(\beta^2)$ and $\sqrt{g_{\mu\nu} \xi_{00} \xi_{00}} = 1 + O(\beta^2)$. Consequently, the motion of $\tilde{m}$ is determined by

$$\xi_{i,00} + U_{,i} + \frac{1}{4} \gamma_{00,i}^{(0)} = 0,$$

with a relative error of the order $O(\beta^2)$. The contributions of $M$ and $m$ to the "gravitational force" $\frac{1}{4} \gamma_{00,i}^{(0)}$ are additive, and have the same functional form. This weak-curvature limit is usually referred to as the post-Newtonian approximation, or the Newtonian limit. The pick-up oscillator is typically driven at resonance, but has a finite $Q$ factor. In all realistic systems, the condition that the velocity is of order $\approx O(\beta^2)$ is satisfied, because the gravitational excitation of the pick-up system is very weak, and Eq. (10) can be used.

The Levi-Civita connection coefficients Eq. (9) depend on our specific choice of coordinates, or gauge. It is not a priori clear that these coefficients are the correct expression for the physical gravitational force, since they could be influenced by meaningless coordinate effects. The physically measurable force is determined by the change of these coefficients in the absence of the source, i.e. for $T_{\mu\nu} = 0$, calculated in the same gauge, and using the same boundary conditions.

In the Hilbert gauge, $T_{\mu\nu} = 0$ results in the vacuum field equations $\left( \partial_\tau^2 - \Delta \xi \right) \gamma_{\mu\nu}(\tau, \xi) = 0$. The solution is a superposition of plane waves with constant amplitudes, and must be zero due to the boundary condition $\gamma_{\mu\nu} \to 0$ for $|\xi| \to \infty$. Consequently, the corresponding Levi-Civita connection coefficients are zero, which shows that the terms $\Gamma_{i00}^i$ in Eq. (9), $i = 1, 2, 3$, correctly describe the physical force.

In the Newtonian limit, $\frac{c^2}{4}$ times (8) is interpreted as the classical gravitational potential

$$V(\tau, \xi_3 \hat{z}) = - \frac{mG}{d} \left( \frac{1}{1 - \beta \cos \beta \theta(\tau, \xi_3)} - \frac{1}{\xi_3 - \sin \beta \theta(\tau, \xi_3)} \right)$$

generated by $m$, which acts on the external mass point $\tilde{m}$ by a gravitational force that is proportional to the gradient of $V$, according to Eq. (10). The physical meaning of the solution Eq. (11) becomes more apparent if one expands the first fraction in a geometric series, and the second fraction in a Fourier series

$$V(t, r \hat{z}) = - \frac{mG}{r} \left\{ 1 + \beta \cos \beta \theta(\frac{\xi_3}{d} t, \frac{r}{d}) + O(\beta^2) \right\}.$$
\[
\left\{ \frac{1}{\sqrt{1 - \left(\frac{d}{r}\right)^2}} + \left(\frac{r}{d}\right) \left[ \frac{2}{\sqrt{1 - \left(\frac{d}{r}\right)^2}} - 2 \right] \sin \beta \theta \left(\frac{c}{d} t, \frac{r}{d}\right) + \right.
\]
\[\sum_{n>2, \ n \ odd} \left(\frac{d}{r}\right)^n 2^{1-n} \left[ 1 + c_n \right] \sin n \beta \theta \left(\frac{c}{d} t, \frac{r}{d}\right) + \]
\[\sum_{n>1, \ n \ even} \left(\frac{d}{r}\right)^n 2^{1-n} \left[ 1 + d_n \right] \cos n \beta \theta \left(\frac{c}{d} t, \frac{r}{d}\right) \right\},
\]
(12)
in which all coefficients \(c_n\) and \(d_n\) are of order \(O(\left(\frac{d}{r}\right)^2)\). The leading terms of order \(O(1 = \beta^0)\) can be obtained by setting \(\beta\) equal to zero in the first bracket. The time-independent contribution \(\frac{mg}{r_1}\left(1 - \left(\frac{d}{r}\right)^2\right)^{-1/2}\) in the Fourier expansion is the static potential of a resting mass distribution centered at the origin, and converges to the classical Newtonian potential \(\frac{mg}{r}\) for large \(r\). The next, first harmonic term has the asymptotic form \(\frac{mgd}{r_1^2}(1 + O(\left(\frac{d}{r}\right)^2)) \sin \beta \theta (\frac{c}{d} t, \frac{r}{d})\). For large \(r\), it describes a propagating, sinusoidal oscillation proportional to \(\frac{1}{r}\), which decays more rapidly than the usual radiative \(\frac{1}{r}\) solutions. All following terms are higher harmonics that decay even faster with increasing distance. The contributions of \(O(\beta)\) exhibit the same asymptotic behaviour with respect to \(\frac{d}{r}\).

II.1 GRAVITATIONAL EFFECTS OF THE MASS \(M\)

In this section, we will discuss the gravitational effects of the mass \(M\) in the immediate vicinity of \(m\). The average distance \(R\) between \(M\) and the point of measurement, lying in the immediate vicinity of \(m\), is much larger than \(d\), and also much larger than \(r\). Due to \(M \gg m\), the oscillation amplitude \(D = \frac{m}{M}d\) of \(M\), which is calculated from classical mechanics, is much smaller than \(d\). Both masses oscillate at frequency \(\omega\) along the \(z\) axis, and the gravitational potential due to \(M\) at \(\xi = \frac{r}{d}\hat{\xi}\) can be obtained by inserting \(d \to D, r \to R, m \to M, \beta \to \frac{-D}{d} \beta, \theta \to \frac{\theta}{d}, (\tau, \vec{\xi}) \to \frac{d}{D}(\tau, \vec{\xi}), \xi_3 = \frac{r}{d} \to \frac{R}{D}\) in Eq. (12). The static term is approximately \(-\frac{MG}{R}\) and can dominate over the static field stemming from \(m\). However, one obtains approximately \(\frac{MG \cdot \frac{D}{R}}{\frac{R}{D}} = \frac{mgd}{r_1^2}(\frac{r}{R})^2\) for the amplitude of the first time-dependent term in Eq. (12), which is a factor of \((\frac{r}{R})^2\) smaller than the corresponding term due to \(m\). The amplitude of the first harmonic term due to \(M\) has the opposite sign as compared to corresponding term due to \(m\),
because $M$ and $m$ oscillate with opposite phases.

In addition, the amplitudes of the higher harmonic terms, approximately given by $\frac{MG}{R} \left( \frac{R}{r} \right)^{n} = \frac{mG}{r} \left( \frac{r}{R} \right)^{n+1} \left( \frac{R}{r} \right)^{n-1}$, are a factor of $\left( \frac{r}{R} \right)^{n+1} \left( \frac{R}{r} \right)^{n-1}$ smaller than the corresponding terms due to $m$. One can therefore neglect the time-dependent gravitational influences due to $M$ if $\frac{r}{R}$ is small, which is the case in the immediate vicinity of $m$. At larger distances from $m$, the number $\frac{r}{R}$ approaches 1, and all radiative effects cancel, which is required by momentum conservation.

II.2 RETARDATION OF GRAVITATIONAL FORCES

It will first be shown that the forces derived from the gravitational potential Eq. (11) are not retarded in the region known as the "near zone" in the literature, which is in accordance to [7]. Then, it will be proved that these gravitational forces are retarded in the immediate vicinity of $m$. As discussed in the derivation of Eq. (11), we place a small test mass $\tilde{m}$ which is part of a resonant mechanical oscillator at position $\tilde{r} = r\hat{z}$ on the $z$ axis to detect the gravitational forces generated by the source $m$. From $F(\tau, \frac{\hat{r}}{d} \hat{z}) = -\tilde{m} \partial_{r} V(\tau, \frac{\hat{r}}{d} \hat{z})$ it follows that the gravitational force acting on $\tilde{m}$ is

$$F(\tau, \frac{r}{d} \hat{z}) = -\tilde{m} mG \left\{ \left( \frac{1}{1 - \beta \cos \theta(\tau, \frac{\hat{r}}{d})} \right)^{2} + O(\beta^{2}) \right\} . \quad (13)$$

$F(\tau, \frac{\hat{r}}{d} \hat{z})$ points along the $z$ axis, whereas the other force components vanish due to symmetry reasons. Eq. (13) is accurate to orders $O(1 = \beta^{0})$ and $O(\beta)$, but not to the order $O(\beta^{2})$, since $T_{00}^{(m,2)}$ has not been considered in $\gamma_{00}^{(m,0)}$. For our analysis it suffices that the results derived from Eq. (13) are conclusive to $O(1 = \beta^{0})$ and $O(\beta)$. To obtain Eq. (13), we have used the fact that the derivative of $\theta(\tau, \frac{\hat{r}}{d})$ with respect to $r$ is given by

$$\partial_{r} \theta(\tau, \frac{r}{d}) = -\frac{1}{d} \frac{1}{1 - \beta \cos \theta(\tau, \frac{\hat{r}}{d})} \quad (14)$$

which follows from

$$\partial_{r} \theta(\tau, \frac{r}{d}) = \partial_{r} \sin \beta \theta(\tau, \frac{r}{d}) - \frac{1}{d} = \beta \cos \beta \theta(\tau, \frac{r}{d}) \partial_{r} \theta(\tau, \frac{r}{d}) - \frac{1}{d} . \quad (15)$$
Note that the leading term in $F(\tau, \frac{r}{d})$ is proportional to $V(\tau, \frac{r}{d} \hat{z})^2$. The expression for the gravitational force generated by $M$ is similar to Eq. (13). It has a static component which is much stronger than the static term due to $m$. However, the time-dependent gravitational forces generated by $M$ are negligible in the immediate vicinity of $m$, because $\frac{d}{r} \approx 1$ and therefore $\frac{r}{R} \ll 1$. At large distances from $m$, $M$ and $m$ produce time-dependent gravitational forces of the same strength, since $\frac{d}{r} \to 0$ and $\frac{r}{R} \to 1$.

II.2.1 GRAVITATIONAL FORCES IN THE "NEAR ZONE"

The spherical region $\{r = |\vec{x}|; \beta \ll \frac{d}{r} \ll 1\}$ is commonly defined as the "near zone" in the literature. For this region, the approximation $\theta(\tau, \frac{r}{d}) \approx \tau - \frac{r}{d}$ can be used, which is familiar from radiation theory. In addition, we have used the identity $\frac{1}{1+y} = 1 - y + O(|y|^2)$, with $|y| < 1$, to simplify the fractions in $V(\tau, \frac{r}{d} \hat{z})$, and to reduce the potential to

$$ V(\tau, \frac{r}{d} \hat{z}) = -\frac{mG}{d} \left\{ 1 + \beta \cos \beta (\tau - \frac{r}{d}) \right\} \left\{ \frac{d}{r} \sin \beta (\tau - \frac{r}{d}) \right\} + O(\beta^2). \tag{16} $$

The trigonometric functions can be approximated by their Taylor expansions to first order in the variable $\beta \frac{d}{r} \ll 1$. Keeping terms only to the orders $O(\beta)$ and $O(\frac{d}{r})$, one obtains

$$ F(\tau, \frac{r}{d} \hat{z}) = -\frac{\tilde{m}mG}{r^2} \left\{ 1 + 2\beta \cos \beta \tau \right\} \left\{ 1 + 2\frac{d}{r} \sin \beta \tau - 2\beta \cos \beta \tau \right\} + O(\beta^2) $$

$$ = -\frac{\tilde{m}mG}{r^2} \left\{ 1 + 2\frac{d}{r} \sin \beta \tau + 4\beta \frac{d}{r} \cos \beta \tau \sin \beta \tau \right\} + O(\beta^2) \tag{17} $$

for the gravitational force acting on the test mass $\tilde{m}$. The term of order $O(\beta \frac{d}{r}) \ll O(\beta)$ is a negligible contribution to the second harmonics. Hence the contributions of order $O(\beta)$ cancel.

The time-dependent term of order $O(1 = \beta^0)$ in the force acts instantaneously, thus there is no measurable retardation in the "near zone" where $\frac{d}{r} \ll 1$. The electromagnetic version of this result can be found in [7]. Historically, this result has convinced the physics community that there exists no region at all where any retardation can be observed.

The additional constraint $\frac{r}{R} \ll 1$ allows the influence of the second mass $M$ to be neglected. As $r$ increases, this condition can not be satisfied, and $\frac{r}{R} \to 1$. The complete gravitational potential in the "near zone" limit, which includes the contributions of both $m$ and $M$ is given
by
\[ V_{\text{tot}}(\tau, \frac{r}{d} \hat{z}) \approx -\frac{mG}{r} \left\{ 1 + \frac{M}{m} \frac{r}{R} \right\} - \frac{mGd}{r^2} \left\{ \sin \beta \tau - \left( \frac{r}{R} \right)^2 \sin \beta \tau \right\}, \]

which follows from the discussion in section II.1. The time-dependent part of this solution vanishes in the limit \( \frac{r}{R} \to 1 \), and there remains a static solution which is dominated by the larger mass \( M \). Note that \( \frac{(m+M)G}{r} \) is the rotation symmetric limit solution for \( r \to \infty \), which corresponds to the asymptotically vanishing particular solution of Eq. (3). We conclude that our result is consistent with [7], and that it is in accordance with the conservation of momentum.

II.2.2 GRAVITATIONAL FORCES IN THE IMMEDIATE VICINITY OF THE SOURCE

The immediate vicinity of the source is characterized by \( \frac{d}{r} \approx 1 \), in contrast to \( \frac{d}{r} \ll 1 \) which is a necessary condition for radiation theory. Consequently, radiation theory breaks down. At this distance range, \( \frac{r}{R} \ll 1 \) holds, and time-dependent influences of \( M \) are negligible, as shown in section II.1. As opposed to the situation in the "near zone", it is not possible to approximate functions of \( \frac{d}{r} \) by the leading terms in their Taylor series. For this reason, we will have to explicitly calculate the dominant Fourier components of the lowest harmonics in Eq. (13). To order \( O(\beta) \), we find
\[ F(\tau, \frac{r}{d} \hat{z}) = -\frac{\tilde{m}mG}{d^2} \left( 1 + 2\beta \cos \theta(\tau, \frac{r}{d}) \left( \frac{1}{\frac{r}{d} - \sin \beta \theta(\tau, \frac{r}{d})} \right)^2 + O(\beta^2) \right). \quad (18) \]

The condition \( \beta \ll \frac{d}{r} \) which has defined the "near zone" is also valid for the immediate neighborhood of the oscillating mass point. Therefore, it is still possible to expand the trigonometric functions with respect to \( \beta \frac{r}{d} \approx O(\beta) \). From the defining equation Eq. (4) for \( \theta(\tau, \frac{r}{d}) \), it can be seen that \( \beta \theta(\tau, \frac{r}{d}) \) is a function of the variable \( \beta \frac{r}{d} \). From Eq. (14), one finds
\[ \beta \theta(\tau, \frac{r}{d}) = \beta \theta(\tau, 0) - \beta \frac{r}{d} + O(\beta^2) \quad (19) \]
for the leading terms of the Taylor series in \( \beta \frac{r}{d} \). We expand the trigonometric functions in Eq. (18) with respect to \( \beta \theta(\tau, \frac{r}{d}) \approx \beta \theta(\tau, 0) - \beta \frac{r}{d} \), and perform the Taylor series to first order
in β in the second bracket. Consequently, the RHS of Eq. (18) yields

\[- \frac{\ddot{m}mG}{r^2} \left(1 + 2\beta \cos \beta \theta(\tau, 0)\right) \left(\frac{1}{1 - \frac{d}{r} \sin \beta \theta(\tau, 0)} - \frac{\beta \cos \beta \theta(\tau, 0)}{(1 - \frac{d}{r} \sin \beta \theta(\tau, 0))^2}\right)^2 + O(\beta^2).\]

The expansion of this product to order \(O(\beta)\) gives

\[- \frac{\ddot{m}mG}{r^2} \left(\frac{1}{1 - \frac{d}{r} \sin \beta \theta(\tau, 0)}\right)^2 - \frac{\beta}{(1 - \frac{d}{r} \sin \beta \theta(\tau, 0))^3} + O(\beta^2).\]

Calculation of the Fourier components of the first harmonics to order \(O(\beta)\) results in

\[- \frac{\ddot{m}mG}{r^2} \left\{1 + 2\beta \sin 2\beta \theta(\tau, 0)\cos \beta \theta(\tau, 0)\right\} + h.h. + O(\beta^2),\] (20)

where the abbreviation \(h.h.\) stands for "higher harmonics". The function \(\Phi(x)\) is defined by

\[\Phi(x) = \frac{2}{x^2} \left((1 - x^2)^{\frac{3}{2}} - 1 + \frac{3}{2} x^2\right)\] (21)

for \(x = \frac{d}{r} \in [0, 1] \). \(\Phi(x)\) is a monotonically increasing function on the unit interval with values \(\Phi(0) = 0\) and \(\Phi(1) = 1\) at the boundaries of its domain, see Fig. 1. We have only calculated the first harmonic terms of maximal order \(O(\beta)\) in Eq. (20), because we assume that the measurement in our proposed experiment is only sensitive to the first harmonics. Using the relationship

\[\beta \theta(\tau, \Phi(\frac{d}{r})\frac{r}{d}) = \beta \theta(\tau, 0) - \beta \Phi(\frac{d}{r})\frac{r}{d} + O(\beta^2),\] (22)

which is a generalization of Eq. (19), we finally obtain

\[F(\tau, \frac{r}{d} \dot{z}) = - \frac{\ddot{m}mG}{r^2} \frac{1}{\left(1 - \left(\frac{d}{r}\right)^2\right)^{\frac{3}{2}}} \left\{1 + 2 \frac{d}{r} \sin \beta \theta(\tau, \Phi(\frac{d}{r})\frac{r}{d})\right\} + h.h. + O(\beta^2).\] (23)

This result shows that the first harmonic term in the gravitational force of order \(O(\beta)\) can be absorbed into the contribution of order \(O(1 = \beta^0)\). Moreover, it is obvious from Eq. (7) that \(\theta(\tau, \Phi(\frac{d}{r})\frac{r}{d})\) is a function of the combination \(\tau - \Phi(\frac{d}{r})\frac{r}{d}\), which shows that the gravitational force is retarded. For two points \(r_1 \dot{z}\) and \(r_2 \dot{z}\) on the oscillator axis, the relative phase difference \(\Delta \phi\) of the gravitational force is given by

\[\Delta \phi = \beta \left(\Phi(\frac{d}{r_2})\frac{r_1}{d} - \Phi(\frac{d}{r_2})\frac{r_2}{d}\right) = \frac{\omega}{c} \left(\Phi(\frac{d}{r_1})r_1 - \Phi(\frac{d}{r_2})r_2\right),\] (24)
and is of order $O(\beta)$. At very small distances $r$ from the source, where $\frac{d}{r} \to 1$ and $\Phi(\frac{d}{r}) \to 1$, the value of the phase shift approximates the result which would occur for a wave that travels at speed of light. At larger distances in the ”near zone”, we have $\frac{d}{r} \to 0$ and $\frac{\partial}{\partial d} \Phi(\frac{d}{r}) \to 0$. Hence there is no measurable phase shift, and the gravitational forces seem to act instantaneously, as was shown above. In the intermediate region where $\frac{d}{r} \approx 1$, the phase shift can be calculated from Eq. (24). Note that in the limit $\frac{d}{r} \to 0$, (23) converges to (17), as it should be. The absolute phase shift $\phi(r) := \frac{\omega}{c} \Phi(\frac{d}{r}) r$ converges rapidly to zero for $r \to \infty$, see Fig. 2.

We conclude that there is a possibly measurable retardation of the gravitational forces in the immediate vicinity of the oscillating source in our proposed system. A discussion of extended gravitating sources with the linearized Einstein equations would be a straightforward generalization of the presented analysis. Due to the superposition principle, expressions similar to Eq. (23) would then have to be integrated over the spatial spread of the oscillating continuum. However, we have assumed cylindrical symmetry in our mathematical model, and thus could find a solution for the dominating gravitational force term. For systems without cylindrical symmetry, the analysis will be much more difficult. From the results in this section, we will now discuss how they could be experimentally verified.

III EXPERIMENTAL POSSIBILITIES

Eq. (23) suggests that it may be possible to experimentally verify the retardation of gravitational forces in the immediate vicinity of an oscillating mass distribution. This would require both the amplitude and phase detection of gravitationally transmitted vibrations. With the previous mathematical model in mind, we propose the following experiment:

A mass $m$ within a mass-balanced apparatus is oscillated with a very stable phase and frequency $\omega$, as described in the mathematical model. If a mechanical resonator with eigenfrequency $\omega$, mass $\tilde{m}$, and quality factor $Q$ is placed at position $r \hat{z}$, it will be driven into resonance by the first harmonic term in the gravitational force. Given suitably chosen parameters, the amplitude of the sinusoidally vibrating resonator, typically of the order $O(10^{-9}[m])$, can be
detected with modern technology [5].

We propose to measure the retardation of the gravitational force in the following way. Place the mechanical resonator at the location \( r_1 \hat{z} \) on the oscillator axis, such that \( \frac{r_1}{\hat{z}} \approx 1 \). From the observed gravitationally induced sinusoidal vibration, the phase difference \( \Delta \phi_1 \) between the phase of the resonator and the phase of the vibrating source mass can be measured. If the mechanical resonator is placed at another location \( r_2 \hat{z} \) on the oscillator axis, a different result \( \Delta \phi_2 \) will be observed. By subtracting \( \Delta \phi_1 \) from \( \Delta \phi_2 \), all internal influences from the experimental measuring instruments can be eliminated, resulting in the relative phase shift \( \Delta \phi_1 - \Delta \phi_2 \). The value of \( c \) in Eq. (24) can then be calculated from the measured relative phase shift \( \Delta \phi_2 - \Delta \phi_1 \), which would be the speed of light according to General Relativity. The expected values for the relative phase shift are shown in Fig. 3. Since linearized General Relativity is formally a special relativistic field theory, measurement of \( c \) with the proposed system would provide a new test of its Lorentz covariance. Moreover, our proposed laboratory experiment would provide the first direct observation of propagating gravitational fields. In addition, the scalar component of order \( O(1 = \beta^0) \) in Eq. (3) also determines the Coulomb potential of an oscillating point charge after a substitution of parameters. Therefore, our solution Eq. (23) also applies to the retarded Coulomb force generated by an oscillating point charge. Measurement of the retardation in the immediate vicinity of the charge would thus be another interesting experimental possibility.

Since the 1960s, several researchers have detected the amplitudes of gravitationally induced vibrations in laboratory experiments [5, 9, 10, 11]. It may be possible that these experiments can be analyzed in a similar way as the model in this paper, which would provide information about the magnitude of the retardation, if present. The detection of the retardation of gravitational interaction in laboratory experiments has not been discussed much recently because the usual "near zone" calculations predict instantaneous interactions, as was shown in section II. However, this result does not apply in the immediate vicinity of the oscillating source, as was also shown in section II. Note that our experiment is not related to the gravitational waves of helicity two, which are far more difficult to detect, since they are of order \( O(\beta^2) \). Unfortunately,
several technical problems must be solved before measurements to test these retardations can be realized. In 1958, Q. Kerns demonstrated an analog electronic circuit capable of measuring $10^{-6}$ degree phase shifts [4]. We have made several modifications to Kerns’ original design, and we are currently able to measure $10^{-7}$ degree relative phase shifts over a 30 second measurement period [13]. Currently, the technology is thermally limited, but thermal control of the system and application of noise reduction techniques may soon enable us to increase the phase sensitivity to the $10^{-9}$ degree phase accuracy required, to measure the retardation to 1% [5]. It is not yet known if the phase stability of the mechanical systems used for the experiments can be maintained to this accuracy over the measurement time.

IV CONCLUSIONS

Using the linearized Einstein equations, we have proved that the gravitational force generated by a specific oscillating mass distribution is retarded in its immediate vicinity. We have proposed an experiment to measure this retardation which may enable the first direct experimental observation of propagating gravitational fields. The amplitudes of gravitationally induced vibrations in systems similar to ours have been successfully detected since the 1960s. The detection of the retardation requires a very sensitive phase measurement, which may now be possible given the state of modern technology.

REFERENCES

[1] J. Cook, ”An analysis of methods for measuring the velocity of gravitational disturbances”, thesis on file at Pennsylvania State University (1947).

[2] J. Cook, ”On measuring the phase velocity of an oscillating gravitational field”, J. of the Franklin Inst., Vol. 273, No. 6, June (1962).

[3] I. Bershtein and M. Gertsenshtein, ”Possibility of measuring the velocity of propagation of gravitation in the laboratory”, J. Expt. Theort. Phys. U.S.S.R., 37, pp. 1832-1833 (1959).
[4] Q. Kerns, ”Proposed laboratory measurement of the propagation velocity of gravitational interaction”, UCRL - 8438, Berkeley, CA (1958).

[5] W. D. Walker and J. Dual, ”Experiment to measure the propagation speed of gravitational interaction”, Talk presented at Virgo International Conference on Gravitational Waves: Sources and Detectors, March (1996). To be published in Virgo conference proceedings.

[6] W. Misner, K. Thorne, J. Wheeler, ”Gravitation”, W. H. Freeman and Company New York, Ch. 36.1 - 36.3, (1973).

[7] R. P. Feynman, ”Feynman Lectures in Physics”, Vol. 2, Addison Wesley, Ch. 21, (1989)

[8] N. Straumann, ”General Relativity and Relativistic Astrophysics”, texts and monographs in physics, Springer Verlag Berlin Heidelberg, pp. 218-219 (1991).

[9] J. Sinsky and J. Weber, ”New source for dynamical gravitational fields”, Physical Review Letters, Vol. 18, No. 19 (1967).

[10] Y. Ogawa, K. Tsubono, H. Hirakawa, ”Experimental test of the law of gravitation”, Phys. Rev. D, Vol. 26, No.4, August (1982).

[11] P. Astone et al., ”Evaluation and preliminary measurement of the interaction of a dynamical gravitational near field with a cryogenic gravitational wave antenna”, Z. Phys. C 50, pp. 21-29 (1991).

[12] Y. T. Chen, A. Cook, ”Gravitational Experiments in the Laboratory”, Cambridge University Press, pp. 187-195 (1993).

[13] W. D. Walker and J. Dual, ”Sub-Microdegree Phase Measurement Technique”, submitted for publication in Review of Scientific Instruments, American Institute of Physics.
V Figure captions

Fig. 1: The function $\Phi(x)$ versus $x$, where $x = \frac{d}{r}$.

Fig. 2: The absolute phase shift $\phi(r)$ for $r$ between $d$ and $1[m]$, at fixed parameter values $d = 0.01[m]$, $\omega = 251.3[rad/sec]$, and $c = 3 \cdot 10^8[m/sec]$.

Fig. 3: The change in phase shift $\Delta \phi$ for $r_2$ between $r_1$ and $0.1[m]$, and for fixed parameter values $r_1 = 0.03[m]$, $d = 0.01[m]$, $\omega = 251.3[rad/sec]$, $c = 3 \cdot 10^8[m/sec]$. Note that the value $r_1 = 0.03[m]$ corresponds to the smallest possible separation between the centers of the source mass and the detector mass in our experiment, due to the finite size of the physical systems being used.
$\Delta \phi [10^{-8} \text{deg}]$