On the dual minimum distance and minimum weight of codes from a quotient of the Hermitian curve

Edoardo Ballico · Alberto Ravagnani

Abstract In this paper we study evaluation codes arising from plane quotients of the Hermitian curve, defined by affine equations of the form \( y^q + y = x^m \), \( q \) being a prime power and \( m \) a positive integer which divides \( q + 1 \). The dual minimum distance and minimum weight of such codes are studied from a geometric point of view. In many cases we completely describe the minimum-weight codewords of their dual codes through a geometric characterization of the supports, and provide their number. Finally, we apply our results to describe Goppa codes of classical interest on such curves.

Keywords quotient of Hermitian curve · Goppa code · minimum distance · minimum-weight codeword · evaluation code

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0 Introduction

Let \( q \) be a prime power, and let \( X \) be a smooth projective curve defined over the finite field \( \mathbb{F}_q \) with \( q \) elements. Consider a divisor \( D \) on \( X \), and take points \( P_1, \ldots, P_n \in X(\mathbb{F}_q) \) avoiding the support of \( D \). Set \( \mathcal{D} := \sum_{i=1}^n P_i \). The Goppa code \( \mathcal{C}(\mathcal{D}, D) \) is defined as the linear code obtained evaluating the Riemann-Roch space \( L(D) \) at the points \( P_1, \ldots, P_n \) (see [17] for a geometric introduction to Goppa codes). If \( D \) is supported...
by \( s \) points, then \( \mathcal{C}(D,D) \) is said to be an \( s \)-point code. Goppa codes yield good parameters (see [12] for a useful analysis) and their minimum distance can be lower-bounded thanks to their geometric structure ([17], Chapter 10).

The parameters of a Goppa code strictly depend on the curve chosen in the construction: different curves give different codes. The most studied Goppa codes are probably those arising from the Hermitian curve (see the references in Subsection 0.1). In this paper we focus on plane quotients of the Hermitian curve, and study a wide class of evaluation codes on such singular curves by means of geometric techniques. We also show that our family includes many Goppa codes of classical interest, such as one-point and two-point codes arising from these curves.

0.1 Main references on codes from the Hermitian curve and its quotients

One-point codes on a Hermitian curve (see [18], Example VI.3.6) are well-studied in the literature, and efficient methods to decode them are known (see for instance [18], [19] and [20]). The minimum distance of Hermitian two-point codes has been first determined by M. Homma and S. J. Kim ([7], [8], [9], [10]). S. Park gave explicit formulas for the dual minimum distance of such codes (see [15]). More recently, Hermitian codes from higher-degree places have been considered in [11]. The dual minimum distance of many three-point codes on the Hermitian curve is computed in [1], by extending a recent and powerful approach by A. Couvreur (see [5]).

Two-point codes arising from quotients of the Hermitian curve are deeply studied in [3] and [4], computing, in particular, their dimensions. See [6] and [13] for the explicit computation of several Weierstrass semigroups on these curves.

0.2 Layout of the paper

Let us briefly discuss the structure of the paper. We introduce plane quotients of the Hermitian curve in Section 1 summarizing their projective geometry. In Section 2 we define two families of evaluation codes arising from these curves, and state some preliminary geometric lemmas. The first family (which we call uncomplete codes) is the simplest one, and it is studied in depth in Section 3. The analysis of the second family (namely, that of complete codes) is performed in Section 4 and Section 5. Applications to the study of classical Goppa codes are given in Section 6.

1 Preliminaries

Let \( q \) be a prime power and let \( \mathbb{P}^2 \) denote the projective plane of homogeneous coordinates \( (x : y : z) \) over the field \( \mathbb{F}_{q^2} \). Choose a positive divisor of \( q + 1 \), say \( m \), and consider the curve \( Y_m \) defined over \( \mathbb{F}_{q^2} \) by the affine equation \( y^m + y = x^m \). If \( m = q - 1 \), then \( Y_m \) is the well-known Hermitian curve. Here we focus on the more complicated case \( m \neq q + 1 \). In this situation \( Y_m \) has exactly one point at infinity, namely, \( P_\infty = (1 : 0 : 0) \). We notice that \( Y_m \) is a singular plane curve, carrying \( P_\infty \) as a unique singular point. It is well-known that the normalization of \( Y_m \), say \( C_m \), is a
maximal curve. Moreover, $Y_m$ carries $|Y_m(\mathbb{F}_{q^2})| = 1 + q(1 + (q - 1)m)$ points which are $\mathbb{F}_{q^2}$-rational, and its geometric genus (by definition, the geometric genus of $C_m$) is $(q - 1)(m - 1)/2$ (see [18], Example 6.4.2 at page 234). Denote by $\pi : C_m \to Y_m$ the normalization of $Y_m$, and let $i : Y_m \hookrightarrow \mathbb{P}^2$ be the inclusion of $Y_m$ into the projective plane. Since $\pi$ is injective, the composition $u := i \circ \pi : C_m \to \mathbb{P}^2$ is an injective morphism. We define $Q_m \subset C$ by $\pi(Q_m) = P_m$. By [18], Proposition 6.4.1, for any integer $r \geq 0$ the monomials $x^iy^j$ such that

$$i \geq 0, \quad 0 \leq j \leq q - 1, \quad qi + mj \leq r$$

form a basis of the vector space $L(rQ_m)$, the Riemann-Roch space associated to the divisor $rQ_m$ on $C_m$.

**Notation 1** In the sequel, we work with a fixed $q$ and a fixed $m$. Hence, we will always write $Y$ and $C$ instead of $Y_m$ and $C_m$, respectively.

## 2 Definitions and preliminary results

Here we introduce two classes of evaluation codes on a quotient $Y = Y_m$ of the Hermitian curve, and discuss their basic properties.

**Definition 1** Let $E \subseteq Y$ be a zero-dimensional scheme defined over $\mathbb{F}_q$ (the case $E = \emptyset$ may of course be considered). Fix an integer $d > 0$ and set $B := Y(\mathbb{F}_{q^2}) \setminus \langle E_{\text{red}} \cup \{ P_m \} \rangle$. The **uncomplete** code $B(d, -E)$ is the code obtained evaluating the vector space $H^0(\mathbb{P}^2, \mathcal{I}_E(d))$ on the set $B$. The **complete** code $C(d, -E)$ is the code obtained evaluating the vector space $H^0(C, \pi^*\mathcal{O}_Y(d)(\pi^{-1}(-E)))$ on $\pi^{-1}(B)$.

The aim of this paper is to study the minimum distance and the minimum-weight codewords of codes of type $B(d, -E)^\perp$ and $C(d, -E)^\perp$. In Section 6, we will show that many codes on $Y$ of classical interest (such as Goppa one-point and two-point codes) can be easily studied as $C(d, -E)$ codes.

**Proposition 1** Let $B(d, -E)$ and $C(d, -E)$ be as in Definition 1. If $d < q$, then $B(d, -E)$ is a subcode of $C(d, -E)$. Moreover, the minimum distance of $C(d, -E)^\perp$ is at least the minimum distance of $B(d, -E)^\perp$.

**Proof** Since (by assumption) $d < q$, the restriction (and pull-back) map of cohomology groups

$$\rho_{d,E} : H^0(\mathbb{P}^2, \mathcal{I}_E(d)) \to H^0(C, \pi^*\mathcal{O}_Y(d)(\pi^{-1}(-E)))$$

is injective. It follows $B(d, -E) \subseteq C(d, -E)$, and so $B(d, -E)^\perp \supseteq C(d, -E)^\perp$. Since the minimum distance of a code is computed by taking the minimum of the pairwise Hamming distances of the codewords, the second part of the statement easily follows.

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1 Here the subscript $\text{red}$ denotes the reduction of a zero-dimensional scheme.
The next result states technical properties of zero-dimensional subschemes of the projective plane $\mathbb{P}^2$. Lemma 1 is a key-point in our approach, providing a geometric interpretation to certain non-vanishing conditions of cohomology groups. We will use this result many times throughout the paper to get necessary conditions on the supports of some minimum-weight codewords.

**Lemma 1** Fix integers $d > 0$, $z \geq 2$ and a zero-dimensional scheme $Z \subseteq \mathbb{P}^2$ such that $\deg(Z) = z$.

(a) If $z \leq d + 1$, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) = 0$.

(b) If $d + 2 \leq z \leq 2d + 1$, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0$ if and only if there exists a line $L \subseteq \mathbb{P}^2$ such that $\deg(L \cap Z) \geq d + 2$.

**Proof** See [2], Lemma 34.

It is fundamental to our purpose to state Lemma 1 for zero-dimensional schemes $Z \subseteq \mathbb{P}^2$, and not just for finite sets. Moreover, the absence of such a lemma in higher-dimensional projective spaces is the main reason why we are forced to work in the plane, instead of on the normalizations of $Y_m$ curves directly.

### 3 Uncomplete codes

Here we study the geometric properties of uncomplete $B(d, -E)$ codes, find out their dual minimum distance, and characterize the minimum-weight codewords of their dual codes. We notice that the length of a $B(d, -E)$ code is $l + q + q^2 m - q m - |E_{red} \cup \{Q_m\}|$. Indeed, there is no non-zero global section of $\mathcal{I}_E(d)$ vanishing at all the points of $B$ (in the notation of Definition 1).

**Lemma 2** Let $B$ and $B(d, -E)$ be as in Definition 1. Assume $\deg(E) \leq d + 1$. Let $S \subseteq B$ be a subset. There exists a codeword of $B(d, -E)^\perp$ whose support is contained in $S$ if and only if $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$.

**Proof** First of all, we prove that $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$. By contradiction, let us assume $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) > 0$. By Lemma 1 there exists a line $L \subseteq \mathbb{P}^2$ such that $\deg(L \cap E) \geq d + 2$. This contradicts $\deg(E) \leq d + 1$. Recall that the code $B(d, -E)$ is obtained by evaluating on $B$ all the degree $d$ homogeneous forms vanishing on $E$. A subset $S \subseteq B$ contains the support of a minimum-weight codeword of $B(d, -E)^\perp$ if and only if it imposes dependent conditions to $H^0(\mathbb{P}^2, \mathcal{I}_E(d))$. Since $E \cap S = \emptyset$, this is equivalent to say that $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$.

**Proposition 2** Let $B(d, -E)$ be as in Definition 1. Assume $d \leq m - 2$ and $\deg(E) \leq d - 1$. Let $w$ be a codeword of $B(d, -E)^\perp$, and let $S$ denote the support of $w$. Assume $\deg(E \cup S) \leq 2d + 1$. The following facts hold.

1. There exists a line $L \subseteq \mathbb{P}^2$, defined over $\mathbb{F}_q^2$, with $S \subseteq L$ and $\deg((E \cup S) \cap L) \geq d + 2$.
2. Any $S' \subseteq S \cap L$ with $|S'| = d + 2 - \deg(E \cap L)$ is the support of a codeword of $B(d, -E)^\perp$ of weight $d + 2 - \deg(E \cap L)$.
3. If $w$ is a minimum-weight codeword of $\mathcal{B}(d, -E)^\perp$, then $|S| = d + 2 - \deg(E \cap L)$.

Proof. By Lemma 2, we have $h^1(\mathcal{I}_{E,S}(d)) > 0$. Since $\deg(E \cup S) \leq 2d + 1$, by Lemma 1 there exists a line $L \subseteq \mathbb{P}^2$ such that $\deg(L \cap (E \cup S)) \geq d + 2$. Since $\deg(E) \leq d - 1$, we have $|S \cap L| \geq 3$. Since any point of $S$ is defined over $\mathbb{F}_q$, $L$ is defined over $\mathbb{F}_q$ itself.

2. Let $S_1 \subseteq S \cap L$ be any subset with cardinality at least $d + 2 - \deg(E \cap L)$. We may take, for instance, $S_1 = S \cap L$. Since $\deg(L \cap (S_1 \cup E)) \geq d + 2$, we have $h^1(\mathcal{I}_{S_1 \cup E}(d)) > 0$ (Lemma 1 again). By Lemma 2, there exists a subset $S' \subseteq S_1$ which is the support of some codeword $w'$ of $\mathcal{B}(d, -E)^\perp$.

3. Apply part (2) and the definition of minimum distance.

4 Complete codes: the case $E = \emptyset$

This section and the following one are devoted to the more interesting class of complete codes $\mathcal{C}(d, -E)$. When $E = \emptyset$, we will simply write $\mathcal{C}(d)$ instead of $\mathcal{C}(d, \emptyset)$.

We begin our analysis by studying $\mathcal{C}(d)$ codes, whose geometry is rather simple. In Section 5 we will consider the more general case, and extend our results.

Definition 2 A line $L \subseteq \mathbb{P}^2$ is said to be horizontal if it has an equation of the form $y = a$, for a certain $a \in \mathbb{F}_q$. The line at infinity, of equation $z = 0$, will be denoted by $L_\infty$. Notice that we do not consider this line horizontal.

Lemma 3 Let $\mathcal{C}(d)$ be a complete code with $d \leq m - 2$. Then the minimum distance of $\mathcal{C}(d)^\perp$ is $d + 2$, and $d + 2$ points in the support of a minimum weight codeword of $\mathcal{C}(d)^\perp$ are collinear.

Proof. Since $d \leq m - 2$, there exist $d + 2$ points lying on the intersection of $Y(\mathbb{F}_q) \setminus \{P_a\}$ and a horizontal line of the form $y = a$, with $a \in \mathbb{F}_q \setminus \{0\}$. Such $d + 2$ points contain the support of some codeword of $\mathcal{C}(d)^\perp$, and prove that the minimum distance of $\mathcal{C}(d)^\perp$ is at most $d + 2$. It remains to be shown that the minimum distance of $\mathcal{C}(d)^\perp$ is at least $d + 2$. Consider the restriction (and pull-back) map

$$\rho_d : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \to H^0(C, \pi^*(\mathcal{O}_Y(d))).$$

Since $d < q$, the map $\rho_d$ is injective. Let $S \subseteq C(\mathbb{F}_q) \setminus \{Q_w\}$ be a finite subset of points. Since $\pi^{-1}(P_a) = \{Q_w\}$, we have $|S| = |\pi(S)|$. If the set $S$ imposes dependent conditions to $H^0(C, \pi^*(\mathcal{O}_Y))$ (i.e., if $\pi(S)$ contains the support of a non-zero codeword of $\mathcal{C}(d)^\perp$) then it imposes dependent conditions also to $\text{Im}(\rho_d)$. By the injectivity of $\rho_d$, we have that $\pi(S)$ imposes dependent conditions to $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$. In other words, $h^1(\mathbb{P}^2, \mathcal{I}_S(d)) > 0$. Assume that $S$ is the support of a minimum-weight codeword of $\mathcal{C}(d)^\perp$. We have $|S| \leq d + 2$ (see the first part of the proof). Moreover, the weight of the codeword is exactly $|S|$. Lemma 1 gives $|S| \geq d + 2$ and that $S$ is contained into a line of $\mathbb{P}^2$. This proves that the minimum distance of $\mathcal{C}(d)$ is exactly $d + 2$, and $d + 2$ points of the support of a minimum-weight codeword must be collinear.

If we drop the assumption $d \leq m - 2$, then the proof of Lemma 3 does not work. In fact, the result is not always true when $d > m - 2$ (see Example 2).
Example 2. Set $q := 5$, $m := 2$ and $d := 1$. By writing a simple MAGMA program (the source code is similar to that of Example 4 below) it can be seen that the minimum distance of $C(d)$ is $4 \neq d + 2$. Let $P_{25} = \langle a \rangle$, with $a^2 + 4a + 2 = 0$. Then there exists a minimum-weight codeword of $C(d)$ whose support consists of the following four points.

$$(a^3, a^{11}), \ (a^{21}, a^{22}), \ (a^9, a^{23}), \ (0,0).$$

No three points of them are collinear.

Lemma 3 provides the dual minimum distance of $C(d)$ codes, and gives necessary conditions for a set to be the support of a minimum-weight codeword of $C(d)$. It is easily checked that any $d + 2$ points lying on the intersection of $Y$ and a horizontal line are the support of a minimum-weight codeword of $C(d)$. We are going to show that if $(q + 1)/m \geq 3$, then this condition characterize the supports of the minimum-weight codewords of a $C(d)$ code. This gives, in particular, the exact number of the minimum-weight codewords.

Remark 1. Let $i : Y \to \mathbb{P}^2$ be the inclusion of $Y$ in the projective plane, $\pi : C \to Y$ the normalization and $u := C \to \mathbb{P}^2$ the morphism defined by $u := i \circ \pi$. The vector space $V \subseteq L(qQ_m)$ spanned by $\{1, x, y\}$ is the linear system on $C$ (via pull-back through $\pi$) inducing $u$. Set $c := (q + 1)/m$ and observe that the conditions of (1) in Section 1 give

$$\dim L(qQ_m) = h^0(C, \pi^*(\mathcal{O}_Y(1))) = c + 1. \quad (2)$$

Moreover, $\{1, x, y, \ldots, y^{c-1}\}$ form a basis of the vector space $L(qQ_m)$. Assume $c \geq 3$, i.e., $V \subseteq L(qQ_m)$, and consider the linear system $W \subseteq L(qQ_m)$ spanned by $\{1, x, y, y^2\}$. Since $W \supseteq V$ and $V$ has no base points, $W$ has no base points itself. Hence, it induces a morphism $\nu : C \to \mathbb{P}^3$. Since $u$ is injective and $W \supseteq V$, also $\nu$ is injective. Since $u$ has non-zero-differential at each point of $C \backslash \{Q_m\}$, $\nu$ has non-zero differential at each point of $C \backslash \{Q_m\}$. The curve $T := \nu(C) \subseteq \mathbb{P}^3$ is non-degenerate, $\nu : C \to T$ is injective and an isomorphism, except at most at $\nu(Q_m)$. Let $\ell_{\nu(Q_m)} : \mathbb{P}^3 \backslash \{\nu(Q_m)\} \to \mathbb{P}^2$ denote the linear projection from $\nu(Q_m)$. Since $u(Q_m) = (1 : 0 : 0)$, the rational map $\ell_{\nu(Q_m)} : T \dashrightarrow \mathbb{P}^2$ is induced by $(1, y, y^2)$, and the image $\ell_{\nu(Q_m)}(T \backslash \{\nu(Q_m)\})$ is contained into a plane conic $E$. Hence, $T$ is contained in a quadric cone $\Gamma$ with $\nu(Q_m)$ as its vertex. We observe that the fibers of the rational map $\ell_{\nu(Q_m)} : T \dashrightarrow E$ are (outside $Q_m$) the elements of $|mQ_m|$. Take any line $L \subseteq \mathbb{P}^3$ such that $|L \cap C| \geq 3$. Bezout theorem gives $L \subseteq \Gamma$. Hence, $\nu(Q_m) \in L$ and $\nu(C) \cap (L \setminus \nu(Q_m))$ is an element of $|mQ_m|$.

Remark 4 proves in fact the following result.

Lemma 4. Take the set up of Remark 7. Let $P_1, P_2, P_3 \in C(\mathbb{F}_q^3) \backslash \{Q_m\}$ be any three distinct points such that $\nu(P_1), \nu(P_2)$ and $\nu(P_3)$ are collinear. Then $u(P_1), u(P_2), u(P_3)$ lie on an horizontal line of $\mathbb{P}^2$.

Theorem 3. Let $C(d)$ be a complete code. Assume $d \leq m - 2$ and $c = (q + 1)/m \geq 3$. The following facts hold.

1. The minimum distance of $C(d)$ is $d + 2$. 
2. A set $S \subseteq C(\mathbb{F}_{q^2}) \setminus \{Q_{\infty}\}$ is the support of a minimum-weight codeword of $\mathcal{C}(d)^\perp$ if and only if $\pi(S)$ consists of $d + 2$ points lying on a horizontal line.

3. The number of the minimum-weight codewords of $\mathcal{C}(d)^\perp$ is

$$(q - 1)(q^2 - 1) \binom{m}{d + 2}.$$ 

Proof Part (1) is a particular case of Lemma 3. Now we prove part (2). By Lemma 3, the minimum distance of $\mathcal{C}(d)^\perp$ is $d + 2$, and $d + 2$ points in the support of a minimum-weight codeword must be collinear. Assume that $S := \{P_1, P_2, P_3, \ldots, P_{d + 2}\}$ is a set of $d + 2$ points contained into a line which is not horizontal. If $S$ is the support of a codeword of $\mathcal{C}(d)^\perp$, then it is the support of a minimum-weight codeword. By Lemma 1, the points $P_4, \ldots, P_{d + 2}$ impose independent conditions to the vector space $H^0(\mathbb{F}_2, O_{\mathbb{P}^2}(d - 1))$. Moreover, there exists a degree $d - 1$ plane curve, say $X$, which contains $P_4, \ldots, P_{d + 2}$, but contains neither $P_1$, nor $P_2$, nor $P_3$. Hence, it is enough to show that $P_1, P_2$ and $P_3$ impose independent conditions to $H^0(C, \pi^*(\mathcal{O}_Y(1)))$. This follows from Remark 1 and Lemma 4. To get the third part, observe that the minimum-weight codewords of any code having a fixed support form a line of the code (by definition of minimum distance).

If we drop the assumption $c \geq 3$, then the proof of Lemma 4 does not work. In fact, Theorem 3 is not true in general when $c = 2$ (see Example 4).

Example 4 In our usual notation, choose $q := 5$, $m := 3$ and $d := 1$. It follows $c = (q + 1)/m = 2$. Let $\mathcal{C}(1)$ be the code obtained evaluating $H^0(\mathbb{F}_2, O_{\mathbb{P}^2}(d - 1))$ on the set $\pi^{-1}(B)$. A basis of this vector space is $\{1, x, y\}$. The following MAGMA program constructs the code $\mathcal{C}(1)^\perp$, prints a random minimum-weight codeword and its support.

```magma
q:=5; m:=3; F<a>:=GF(q^2); A<x,y>:=AffineSpace(F,2); f:=y^q+y-x^m; X:=Curve(A,f); pts:=Points(X); npts:=#pts; P<u,v>:=PolynomialRing(F,2); B:=\[
\{0,0\}, \{0,1\}, \{1,0\}\]; nf:=3; rows:=\[
\]; for i in [1..nf] do;
    for j in [1..npts] do;
        Append(~rows[i], Evaluate(u^B[j][1]*v^B[j][2], [pts[j][1],pts[j][2]]));
    end for;
end for;
for i in [1..nf] do;
    mww:=MinimumWord(D);
supp:=\[
\]; if mww[j] ne 0 then Append(~supp, [pts[j][1],pts[j][2]]);
end if;
end for;
```

On our Linux machine (AMD processor, 32 bits) the output is the following.
In this section we discuss the general case of \( C(d, -E) \) codes, extending our previous results.

**Notation 5** Let \( L_0 \) be the line in \( \mathbb{P}^2 \) of equation \( y = 0 \). This line intersects \( Y \) only at \((0 : 0 : 1)\) with multiplicity \( m \). We denote by \( \Lambda \) the set of all the plane horizontal lines different from \( L_0 \). Moreover, we denote by \( \Theta \) the set of the other lines of \( \mathbb{P}^2 \) defined over \( \mathbb{F}_q \) which do not pass through \( P_\infty \). Finally, set \( \alpha_1 := \max_{L \in \Lambda} \deg(L \cap E) \) and \( \alpha_2 := \max_{L \in \Theta} \deg(L \cap E) \). We recall that the line of equation \( z = 0 \) is denoted by \( L_\infty \).

**Remark 2** We notice that \( \alpha_1 \) is in fact easy to compute. Indeed, the geometry of \( Y \) implies \( \alpha_1 = \max_{L \in \Lambda} |L \cap E_{\text{red}}| \). Note also that \(|\Lambda| = q - 1\).

**Theorem 6** Let \( C(d, -E) \) be a complete code (see Definition 1). Assume \( \deg(E) \leq d - 1 \) and \( d \leq m - 2 \).

1. The minimum distance of \( C(d, -E) \) is greater or equal than \( d + 2 - \max\{\alpha_1, \alpha_2\} \).
2. If \( \alpha_1 \leq \alpha_2 \), then the minimum distance of \( C(d, -E) \) is exactly \( d + 2 - \alpha_1 \). Moreover, the number of the minimum-weight codewords of \( C(d, -E) \) is at least \((q - 1)(q^2 - 1)\binom{\frac{m}{d + 2 - \alpha_1}}{\frac{m}{d + 2 - \alpha_1}}\).
3. Assume \( d \leq m - 4 \), \( \alpha_1 = \alpha_2 \) and \( d \geq \alpha_1 + 1 \). Then the support of any minimum-weight codeword of \( C(d, -E) \) is contained into a horizontal line \( L \in \Lambda \). Moreover, the number of the minimum-weight codewords of \( C(d, -E) \) is exactly \((q - 1)(q^2 - 1)\binom{\frac{m}{d + 2 - \alpha_1}}{\frac{m}{d + 2 - \alpha_1}}\).

**Proof** Since \( d < q \), the restriction (and pull-back) map

\[ \rho_{d,E} : H^0(\mathbb{P}^2, \mathcal{I}_E(d)) \to H^0(C, \pi^*(\mathcal{I}_Y(d))(\pi^{-1}(-E))) \]

is injective. Let \( S \) be the support of a minimum-weight codeword of \( C(d, -E) \). In this situation the weight of the codeword is exactly \(|S|\) (and not only smaller or equal than \(|S|\)). By definition, \( S \) imposes dependent conditions to the vector spaces \( H^0(C, \pi^*(\mathcal{I}_Y(d))(\pi^{-1}(-E))) \) and \( H^0(\mathbb{P}^2, \mathcal{I}_E(d)) \) (here we used the injectivity of \( \rho_{d,E} \)). As in the proof of Lemma 2, we have \( h^1(\mathbb{P}^2, \mathcal{I}_{E,S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0 \).
By Lemma[1] there exists a line $L \subseteq \mathbb{P}^2$ such that $\deg (L \cap (E \cup S)) \geq d + 2$. Since $S \cap E = \emptyset$ and $L$ cannot be $L_0$ nor $L_{\infty}$, we have

$$d + 2 \leq \deg (L \cap (E \cup S)) \leq \max \{ \alpha_1, \alpha_2 \} + |S|.$$ 

It follows $|S| \geq d + 2 - \max \{ \alpha_1, \alpha_2 \}$. This proves the first statement. If $\alpha_1 \geq \alpha_2$ then, by 1., we have that the minimum distance of $\mathcal{C}(d, -E)^{\perp}$ is at least $d - 2 - \alpha_1$. Since, for any $L \in A, d + 2 - \alpha_1$ points in $Y(\mathbb{F}^2) \cap (L \setminus (L \cap E_{\text{red}}))$ contain the support of a codeword of $\mathcal{C}(d, -E)^{\perp}$, we have that the minimum distance of $\mathcal{C}(d, -E)^{\perp}$ is exactly $d + 2 - \alpha_1$. The lower bound on the number of the minimum-weight codewords trivially follows. Now we prove the last statement. If $\alpha_1 = \alpha_2 = 0$, then $E = \emptyset$ and the thesis is just Lemma[3]. Hence, we may assume $\alpha_1 > 0$. Let $S \subseteq \mathbb{P}^2$ be the image in $\mathbb{P}^2$ of the support of a codeword with minimum-weight $d + 2 - \alpha_1$. By Lemma[1] there exists a line $L \in \Lambda \cup \Theta$ such that $S \subseteq L$. Write $S = \{ P_1, \ldots, P_{d+2-\alpha_1} \}$. As in the proof of Theorem[3] it is enough to find a degree $d - 1$ plane curve $X \supseteq E \cup \{ P_1, \ldots, P_{d+2-\alpha_1} \}$ such that $P_i \notin X$ for $i = 1, 2, 3$. Set $A := \{ P_1, \ldots, P_{d+2-\alpha_1} \}$. Let $\text{Res}_L(A \cup E)$ denote the residual scheme of $A \cup E$ with respect to $L$, i.e., the closed subscheme of $\mathbb{P}^2$ with $\mathcal{I}_{A \cup E} : \mathcal{I}_L$ as its ideal sheaf[2]. We have an exact sequence of sheaves

$$0 \to \mathcal{I}_{\text{Res}_L(E)}(d - 2) \to \mathcal{I}_{A \cup E}(d - 1) \to \mathcal{I}_{A \cup (E \cap L)}(d - 1) \to 0. \tag{3}$$

Observe that $\deg (\text{Res}_L(E)) - \alpha_1 \leq d - 2$, and so we use [2]. Lemma 34, in order to compute $h^1(\mathcal{I}_{\text{Res}_L(E)}(d - 2)) = 0$. Hence, the restriction map

$$\rho : H^0(\mathbb{P}^2, \mathcal{I}_{A \cup E}(d - 1)) \to H^0(L, \mathcal{I}_{A \cup (E \cap L)}(d - 1))$$

turns out to be surjective. Observe that $E$ and each $P_i$ are defined over $\mathbb{F}_q$. Take a degree $d - 1$ effective divisor, $F$, defined over $\mathbb{F}_q$; and containing $A \cup (E \cap L)$, but not containing any $P_i$ with $i \leq 3$. Let $f$ be an equation of $F$ defined over $\mathbb{F}_q$. Take $f_1 \in H^0(\mathbb{P}^2, \mathcal{I}_{E \cup A}(d - 1))$ defined over $\mathbb{F}_q$ and such that $\rho(f_1) = f$. Finally, choose $X := \{ f_1 = 0 \}$.

6 Goppa codes of classical interest

In this section we apply our analysis to classical problems in geometric Coding Theory. More precisely, we are going to study Goppa codes arising from quotients of the Hermitian curve. Since $Y = Y_n$ curves are not smooth, when writing “ Goppa code on $Y$ ” we always mean “ Goppa code on $C$ ” (the normalization of $Y$). The points of $Y$ will be identified with those of $C$ through the injectivity of the normalization $\pi : C \to Y$ (see Section[1]).

**Definition 3** Let $q$ be a prime power and $n \geq 2$ be an integer. We say that codes $\mathcal{C}, \mathcal{D} \subseteq \mathbb{F}_q^n$ are strongly isometric if there exists a vector $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ of non-zero components such that

$$\mathcal{C} = x \mathcal{D} := \{ (x_1 v_1, \ldots, x_n v_n) \in \mathbb{F}_q^n \ | \ (v_1, \ldots, v_n) \in \mathcal{D} \}.$$
The notation will be $C \sim D$, and this clearly defines an equivalence relation on the set of codes in $\mathbb{F}_q^n$.

**Remark 3** Take the setup of Definition 3. Then $C \sim D$ if and only if $C^\perp \sim D^\perp$. Indeed, if $C = xD$ then $C^\perp = x^{-1}D^\perp$, where $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$. A strongly isometry of codes preserves the minimum distance of a code, its weight distribution and the supports of its codewords.

**Remark 4** Let $X$ be a smooth projective curve defined over $\mathbb{F}_q$, and let $D$ and $D'$ be divisors on $X$. Take points $P_1, \ldots, P_n \in X(\mathbb{F}_q)$ which do not appear neither in the support of $D$, nor in the support of $D'$. Set $D := \sum_{i=0}^n P_i$. It is known (see [14], Remark 2.16) that if $D \sim D'$ (as divisors) then $C(D, D) \sim C(D', D')$. By Remark 3, we have also $C(D, D)^\perp \sim C(D', D')^\perp$. Studying Goppa codes up to strong isometries is a well-established praxis (see [14] again).

Here we show that many Goppa codes on $Y$ (more precisely, on $C$) can be studied, up to strong isometries, by using the results of the previous sections.

**Remark 5** Let $Y = Y_n$ be a quotient of the Hermitian curve over the finite field $\mathbb{F}_{q^2}$, and denote by $\pi: C \to Y$ its normalization. The curve $C$ carries the following identity of vector spaces (see [18], Proposition 6.4.1):

$$L(qQ_m) = H^0(C, \pi^*(\mathcal{O}_Y(1))).$$

Moreover, since $C$ is maximal, for any pair of rational points $P, Q \in C(\mathbb{F}_{q^2})$ we get a linear equivalence $(q + 1)P \sim (q + 1)Q$ (see for instance [16], Lemma 1).

**Corollary 1** (one-point codes) Let $0 \leq r \leq (m - 2)q$ be an integer. Denote by $C_r$ the (Goppa) one-point code on $Y$ obtained evaluating the vector space $L_r$ on the set $B := Y(\mathbb{F}_{q^2}) \setminus \{P_m\}$. Write $r = dq + e$ with $0 \leq e \leq q - 1$, and assume $0 \leq e \leq d - 1$.

1. If $e = 0$, then the minimum distance of $C_{r,e}^\perp$ is $d + 2$ and the minimum-weight codewords of $C_{r,e}^\perp$ are at least $(q - 1)/d - 1$. Moreover, if $(q + 1)/m \geq 3$, then minimum-weight codewords of $C_{r,e}^\perp$ are exactly $(q - 1)(q - 1)\binom{m}{d-1}$.

2. If $e > 0$, then the minimum distance of $C_{r,e}^\perp$ is $d + 1$ and the minimum-weight codewords of $C_{r,e}^\perp$ are at least $(q - 1)(q - 1)\binom{m}{d-1}$.

**Proof** By Remark 5 and Remark 4 we have $C_r \sim C(d, -eP_m)$. Now apply Lemma 3 Theorem 3 and Theorem 4.

**Corollary 2** (two-point codes) Let $a, b$ be integers such that $a + b > 0$, and let $P \in Y(\mathbb{F}_{q^2}) \setminus \{P_m\}$ be a rational point different from $P_m$. Denote by $C(a, b, P)$ the (Goppa) two-point code on $Y$ obtained evaluating the vector space $L(aP_m + bP)$ on the set $B := Y(\mathbb{F}_{q^2}) \setminus \{P_m, P\}$. There always exist integer $d, d', b'$ such that

1. $d > 0$, $d', b' \geq 0$;
2. $aP_m + bP \sim dqP_m - dP_m - b'P$.

Assume that these integers $d, d', b'$ can be chosen with the properties...
(3) \( d \leq m - 2; \)
(4) \( 0 \leq a' + b' \leq d - 1; \)
(5) \( b' > 0. \)

Denote by \( P_0 \in Y(\mathbb{F}_{q^2}) \) the point of coordinates \((0:0:1)\). The following facts hold.

(A) If \( a = 0 \) and \( P = P_0 \), then the minimum distance of \( C(a, b, P)^\perp \) is greater or equal than \( d + 1 \).

(B) If either \( a = 0 \) and \( P \neq P_0 \), or \( a > 0 \) and \( P = P_0 \), then the minimum distance of \( C(a, b, P)^\perp \) is exactly \( d \) and the minimum-weight codewords of \( C(a, b, P)^\perp \) are at least \((q - 1)(q^2 - 1){m \choose d + 1}\). Moreover, if \( d \leq m - 4 \) then the equality holds.

(C) If \( a > 0 \) and \( P \neq P_0 \), then the minimum distance of \( C(a, b, P)^\perp \) is exactly \( d \), and the minimum-weight codewords of \( C(a, b, P)^\perp \) are at least \((q - 1)(q^2 - 1){m \choose d}\).

Proof The first part of the proof trivially follows from Remark[5]. Set \( E := a'P_\infty + b'P \), and observe that \( C(a', b', P)^\perp \sim C(d, -E) \). By Remark[4] we can apply Theorem[6] in the notation of the cited theorem, in case (A) we have \( \alpha_1 = 0 \) and \( \alpha_2 = 1 \), in case (B) we have \( \alpha_1 = \alpha_2 = 1 \). Finally, in case (C) we have \( \alpha_1 = 2, \alpha_2 = 1 \).

Remark 6 Let \( C(\mathcal{T}, D) \) be any non-trivial Goppa code on the curve \( Y \). Let \( \{ P_1, \ldots, P_r \} \) be the support of \( D \). There always exist integers \( d > 0 \) and \( (a_i)_{i=1}^r \) such that \( a_i \geq 0 \) for each \( i \) and

\[
D \sim dqP_\infty - \sum_{i=1}^r a_iP_i.
\]

By setting \( E := \sum_{i=1}^r a_iP_i \), we have \( C(D, E) \sim C(d, -E) \). Moreover, if \( \sum_{i=1}^r a_i \leq d - 1 \) then the code \( C(D, E)^\perp \) is described in any case by Theorem[6].

7 Conclusions

In this paper we study the dual minimum distance and minimum weight of codes arising from quotients of the Hermitian curve through geometric construction. We describe the parameters of such codes from a cohomological point of view, and geometrically characterize the supports of their minimum-weight codewords, deriving explicit formulas for their number. Our analysis is then applied to study the dual codes of one-point and two-point Goppa codes on a quotient of the Hermitian curve.

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