On the spectrum of the discrete 1d Schrödinger operator with an arbitrary even potential

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Abstract. The discrete one-dimensional Schrödinger operator is studied in the finite interval of length $N = 2M$ with the Dirichlet boundary conditions and an arbitrary potential even with respect to the spatial reflections. It is shown, that the eigenvalues of such a discrete Schrödinger operator (Hamiltonian), which is represented by the $2M \times 2M$ tridiagonal matrix, satisfy a set of polynomial constrains. The most interesting constrain, which is explicitly obtained, leads to the effective Coulomb interaction between the Hamiltonian eigenvalues. In the limit $M \to \infty$, this constrain induces the requirement, which should satisfy the scattering data in the scattering problem for the discrete Schrödinger operator in the half-line. We obtain such a requirement in the simplest case of the Schrödinger operator, which does not have bound and semi-bound states, and which potential has a compact support.

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1. Introduction

Consider the discrete Schrödinger eigenvalue problem in the one-dimensional chain having even number of sites $N = 2M$, with an arbitrary real even potential $V = \{v_j\}_{j=1}^{N}$:

\[ v_j \psi_l(j) + [2\psi_l(j) - \psi_l(j - 1) - \psi_l(j + 1)] = \lambda_l \psi_l(j), \]
\[ v_{N+1-j} = v_j, \]
\[ j = 1, \ldots, N, \quad \lambda_1 < \lambda_2 < \ldots < \lambda_N. \]

Eigenstates of (1.1) are subjected to the Dirichlet boundary conditions

\[ \psi_l(0) = \psi_l(N + 1) = 0. \]

The discrete Sturm-Liouville problem (1.1)-(1.3) without the parity constrain (1.2) plays an important role in the theory of Anderson localization [1, 2]. The problem (1.1)-(1.3) with an even potential (1.2) naturally arrises in the context of the theory of the thermodynamic Casimir effect [3].
It is proved in this paper, that the eigenvalues of the discrete Sturm-Liouville problem (1.1)-(1.3) satisfy the following equality:

\[
\prod_{m=1}^{M} \prod_{n=1}^{M} (\lambda_{2m-1} - \lambda_{2n}) = 2^M (-1)^{M(M+1)/2}.
\] (1.4)

One can easily check, that (1.4) is satisfied for small \(M = 1, 2, \ldots\) For arbitrary natural \(M\), relation (1.4) is proved in Section 2. Section 3 contains some well-known basic facts about the scattering problem in the half-line for the discrete Schrödinger operator. In the limit \(M \to \infty\), equality (1.4) leads to certain constrains on the scattering data in such a problem, which are derived in Section 4. Concluding remarks are given in Section 5. Proof of (1.4) for the free case \(v_j = 0, j = 1, \ldots, M\) is presented in Appendix A.

2. Discrete Sturm-Liouville problem in the finite interval

In the case of zero potential \(v_j = 0\), the solution of (1.1)-(1.3) reads as

\[
\psi_l(j) = \sin(k_l j),
\]

(2.1)

\[
\lambda_l = \omega(k_l),
\]

(2.2)

\[
k_l = \frac{\pi l}{N + 1},
\]

(2.3)

with

\[
\omega(p) = 4 \sin^2(p/2),
\]

(2.4)

and \(l = 1, \ldots, N\).

For a general real even potential \(v_j\), the eigenstates \(\psi_{2m-1}(j), m = 1, \ldots, M\) are even with respect to the reflection

\[
\psi_{2m-1}(N + 1 - j) = \psi_{2m-1}(j),
\]

(2.5)

whereas eigenstates \(\psi_{2m}(j), m = 1, \ldots, M\) are odd:

\[
\psi_{2m}(N + 1 - j) = -\psi_{2m}(j).
\]

(2.6)

It is useful to consider two associated eigenvalue problems for the even and odd states, which are restricted to the half-chain \(j = 1, \ldots, M\). The eigenvectors \(\psi_{2m-1}(j), j = 1, \ldots, M\), are the eigenstates of the tridiagonal \(M \times M\) matrix \(H^{(ev)}\):

\[
H^{(ev)} = \begin{pmatrix}
  b_1 & -1 & 0 & 0 & 0 & \ldots & 0 \\
  0 & b_2 & -1 & 0 & 0 & \ldots & 0 \\
  -1 & 0 & b_3 & -1 & 0 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 0 & -1 & b_{M-1} & -1 \\
  0 & 0 & \ldots & 0 & 0 & -1 & b_M - 1
\end{pmatrix},
\]

(2.7)

\[
b_j = v_j + 2,
\]
with eigenvalues $\mu_m = \lambda_{2m-1}$, $m = 1, \ldots, M$. Similarly, the eigenvectors $\psi_{2m}(j)$, $j = 1, \ldots, M$, are the eigenstates of the tridiagonal $M \times M$ matrix $H^{(\text{od})}$:

$$H^{(\text{od})} = \begin{pmatrix}
    b_1 & -1 & 0 & 0 & \cdots & 0 \\
    -1 & b_2 & -1 & 0 & \cdots & 0 \\
    0 & -1 & b_3 & -1 & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & \cdots & 0 & -1 & b_{M-1} \\
    0 & 0 & \cdots & 0 & 0 & -1 & b_M + 1
\end{pmatrix}, \quad (2.8)$$

with eigenvalues $\nu_m = \lambda_{2m}$, $m = 1, \ldots, M$. Note, that the matrices $H^{(\text{ev})}$ and $H^{(\text{od})}$ are simply related

$$H^{(\text{od})} - H^{(\text{ev})} = 2P, \quad (2.9)$$

with the projecting matrix $P_{m,m'} = \delta_{m,M} \delta_{m',M}$, $m, m' = 1, \ldots, M$, and rank $P = 1$.

It is convenient to allow the potential $\{b_j\}_{j=1}^M$ in the diagonal of the matrices (2.7), (2.8) to take complex values.

**Lemma 2.1** The matrices $H^{(\text{od})}$ and $H^{(\text{ev})}$ defined by (2.7), (2.8) have no common eigenvalues for arbitrary complex potential $\{b_j\}_{j=1}^M$.

**Proof** We will assume that the matrices $H^{(\text{ev})}$ and $H^{(\text{od})}$ have a common eigenvalue $\Lambda$ and come to contradiction.

So, let us suppose that

$$\sum_{j'=1}^M H^{(\text{ev})}_{j,j'} x_{j'} = \Lambda x_j, \quad \sum_{j'=1}^M H^{(\text{od})}_{j,j'} y_{j'} = \Lambda y_j,$$

with nonzero vectors $\{x_j\}_{j=1}^M$, $\{y_j\}_{j=1}^M$. We get

$$\Lambda \sum_{j=1}^M y_j x_j = \sum_{j=1}^M \sum_{j'=1}^M y_{j'} H^{(\text{ev})}_{j,j'} x_{j'} = \sum_{j=1}^M \sum_{j'=1}^M y_{j'} (H^{(\text{od})}_{j,j'} - 2P_{j,j'}) x_{j'} =$$

$$\sum_{j=1}^M \sum_{j'=1}^M y_{j'} H^{(\text{od})}_{j,j'} x_{j'} - 2 \sum_{j=1}^M \sum_{j'=1}^M y_{j'} P_{j,j'} x_{j'} = -2y_M x_M + \Lambda \sum_{j=1}^M y_j x_j.$$

Here we have taken into account, that the matrix $H^{(\text{od})}$ is symmetric. Thus,

$$y_M x_M = 0,$$

which means that at least one of the numbers $y_M$ and $x_M$ is zero. However, if $y_M = 0$, we conclude immediately, that $y_m = 0$ for all $m = 1, \ldots, M$, providing that $\Lambda$ is not an eigenvalue of $H^{(\text{od})}$. Similarly, if $x_M = 0$, we conclude, that $x_m = 0$ for all $m = 1, \ldots, M$, providing that $\Lambda$ is not an eigenvalue of $H^{(\text{ev})}$. This contradiction with the initial assumption proofs the Lemma.

† Really, if $0 = y_M = \psi(j = M)$, then $\psi(j = M + 1) = -\psi(j = M) = 0$, since $\psi(j) = -\psi(2M + 1 - j)$ for all $j = 1, \ldots, 2M$. And since the wave-function $\psi(j)$ takes zero values at two neighbor sites $\psi(M) = \psi(M + 1) = 0$, one can check recursively from (2.1), that $\psi(M - 1) = 0$, $\psi(M - 2) = 0$, $\ldots$, $\psi(1) = 0$, and, therefore, $\psi(j) = 0$ for all $j = 1, \ldots, 2M$. 

$\square$
The sets of eigenvalues \( \{\mu_m\}_{m=1}^M \) and \( \{\nu_m\}_{m=1}^M \) of the matrices \( H^{(ev)} \) and \( H^{(od)} \) are not independent, but are subjected to certain polynomial constrains following from (2.9):

\[
\sum_{m=1}^M \mu_m = \text{Tr} H^{(ev)} = -2 + \text{Tr} H^{(od)} = -2 + \sum_{m=1}^M \nu_m, \tag{2.10}
\]

\[
\sum_{m=1}^M \mu_m^2 = \text{Tr}(H^{(ev)})^2 = \text{Tr}(-2P + H^{(od)})^2 = -4b_M + \sum_{m=1}^M \nu_m^2,
\]

\[
\sum_{m=1}^M \mu_m^3 = \text{Tr}(H^{(ev)})^3 = \text{Tr}(-2P + H^{(od)})^3 = -8 - 6b_M^2 + \sum_{m=1}^M \nu_m^3,
\]

\[
\sum_{m=1}^M \mu_m^4 = \text{Tr}(H^{(ev)})^4 = \text{Tr}(-2P + H^{(od)})^4 = -8b_{M-1} - 24b_M - 8b_M^2 + \sum_{m=1}^M \nu_m^4,
\]

\[
\sum_{m=1}^M \mu_m^5 = \text{Tr}(H^{(ev)})^5 = \text{Tr}(-2P + H^{(od)})^5 = -32 - 10b_{M-1}^2 - 20b_{M-1}b_M - 50b_M^2 - 10b_M^4 + \sum_{m=1}^M \nu_m^5.
\]

**Remark**

1. Equations (2.10) provide a simple way to solve the inverse spectral problem, i.e. to determine one by one the potential \( b_M, b_{M-1}, \ldots, b_1 \), if the both sets of eigenvalues \( \{\mu_m\}_{m=1}^M \) and \( \{\nu_m\}_{m=1}^M \) are known.

2. Excluding one by one the potential \( b_M, b_{M-1}, \ldots, b_1 \) from equations (2.10), one can obtain the infinite set of polynomial constrains of increasing degrees on the eigenvalues \( \{\mu_m\}_{m=1}^M \), \( \{\nu_m\}_{m=1}^M \). No more than \( M \) of constrains in this set can be independent, since the above mentioned eigenvalues are determined by \( M \) parameters \( \{\nu_j\}_{j=1}^M \) as zeroes of the characteristic polynomials of the matrices (2.7) and (2.8).

3. One can easily see from (2.10), that the symmetric polynomials of eigenvalues \( \{\mu_m\}_{m=1}^M \), as well as the symmetric polynomials of eigenvalues \( \{\nu_m\}_{m=1}^M \), can be written as polynomial functions of the potential \( \{b_j\}_{j=1}^M \).

Now we are ready to prove the main

**Theorem 2.2** For arbitrary complex numbers \( \{b_j\}_{j=1}^M \), the eigenvalues \( \{\mu_m\}_{m=1}^M \) and \( \{\nu_m\}_{m=1}^M \) of the matrices \( H^{(od)} \) and \( H^{(ev)} \) defined by (2.7), (2.8) satisfy the equality:

\[
\prod_{m=1}^M \prod_{n=1}^M (\mu_m - \nu_n) = 2^M (-1)^{(M+1)/2}. \tag{2.11}
\]

By restriction of this result to real \( \{b_j\}_{j=1}^M \), we arrive to (1.4).

**Proof** The left-hand side of (2.11) is a symmetric polynomial function of \( \{\mu_m\}_{m=1}^M \). It is also a a symmetric polynomial function of \( \{\nu_m\}_{m=1}^M \). Due to Remark (3), we can
conclude, that the left-hand side of (2.11) can be written as a polynomial function $Q_M(b)$ of the potential \( \{b_j\}_{j=1}^M \):
\[
\prod_{m=1}^M \prod_{n=1}^M (\mu_m - \nu_n) = Q_M(b). \tag{2.12}
\]

It follows from this relation and Lemma 2.1 that the polynomial function $Q_M(b)$ of $M$ complex variables \( \{b_j\}_{j=1}^M \) has no zeros. This means, that this function is a constant, $Q_M(b) = C_M$, which must not depend on the potential \( \{b_j\}_{j=1}^M \).

One can now determine this constant $C_M$ using an appropriate convenient choice of the potential. To this end, we put
\[
b_j = 2, \quad j = 1, \ldots, M, \tag{2.13}
\]
which corresponds to $v_j = 0$, $j = 1, \ldots, M$. Reminding (2.2), we get
\[
\mu_m = 4 \sin^2 \frac{(2m-1)\pi}{2(2M+1)}, \quad \nu_m = 4 \sin^2 \frac{2m\pi}{2(2M+1)}, \tag{2.14}
\]
and equality we need to prove for all natural $M$ takes the form
\[
\prod_{m=1}^M \prod_{n=1}^M \left[ 4 \sin^2 \frac{(2m-1)\pi}{2(2M+1)} - 4 \sin^2 \frac{2n\pi}{2(2M+1)} \right] = 2^M (-1)^{M(M+1)/2}. \tag{2.15}
\]

Proof of this formula is given in Appendix A.

It is interesting to note, that equality (1.4) allows the electrostatic interpretation. Really, let us take the logarithm of the absolute values of the both sides of (1.4), and rewrite the result in the form
\[
- \sum_{m=1}^M \sum_{n=1}^M \ln |x_m^{(A)} - x_n^{(B)}| = -M \ln 2, \tag{2.16}
\]
where $x_m^{(A)} = \lambda_{2m-1}$, and $x_n^{(B)} = \lambda_{2n}$, with $m, n = 1, \ldots, M$ will be treated as space coordinates of two different sets of $M$ particles of types $A$ and $B$, which are distributed along the $x$-axis in the two-dimensional plane. Particles of the $A$ type interlace with particles of the $B$ type, $x_m^{(A)} < x_n^{(B)} < x_{m+1}^{(A)}$. If particles of the same type do not interact with each other, and particles of different types interact via the pair 2$d$ Coulomb potential $u(x^{(A)}, x^{(B)}) = -\ln |x^{(A)} - x^{(B)}|$, then equation (2.16) states simply, that the total Coulomb energy of this system of $2M$ particles should be equal to $-M \ln 2$.

3. Scattering problem for the discrete Schrödinger operator in the half-line

In this Section we briefly summarize some well-known basic results from the scattering theory in the half-line (see, for example [4, 5]) adapted for the the discrete Schrödinger operator [1, 2].

Consider the discrete Schrödinger equation (1.1) in the half-line $j \in \mathbb{N}$
\[
(H\psi)_j = \lambda \psi(j), \tag{3.1}
\]
\[
(H\psi)_j = v_j \psi(j) + [2\psi(j) - \psi(j-1) - \psi(j+1)], \tag{3.2}
\]
\[
j = 1, 2, \ldots, \infty,
\]
supplemented with the Dirichlet boundary condition

\[ \psi(0) = 0. \]  

(3.3)

The potential \( V = \{v_j\}_{j=1}^{\infty} \) in (3.1) is the infinite sequence of real numbers. In the scattering theory, the potential should vanish fast enough at infinity. It is usually required \([4]\), that

\[ \sum_{j=1}^{\infty} j|v_j| < \infty. \]  

(3.4)

For such a potential, the spectrum \( \sigma[H] \) of the operator \( H \) defined by (3.1)-(3.3) consists of the continuous part \( \sigma_{\text{cont}}[H] = (0, 4) \) and a finite number of discrete eigenvalues.

At a given \( \lambda \), equations (3.1), (3.2) with omitted boundary condition (3.3) have two linearly independent solutions, and the general solution of (3.1), (3.2) can be written as their linear combination. For two sequences \( \{\psi_1(j)\}_{j=0}^{\infty} \), and \( \{\psi_2(j)\}_{j=0}^{\infty} \), one can define the Wronskian

\[ W[\psi_1, \psi_2]_j = \psi_1(j)\psi_2(j+1) - \psi_1(j+1)\psi_2(j), \quad j = 0, 1, 2, \ldots \]  

(3.5)

It is straightforward to check, that the Wronskian of two solutions of equations (3.1), (3.2) does not depend on \( j \).

Let us turn now to the scattering problem associated with equations (3.1)-(3.3), which corresponds to the case \( 0 < \lambda < 4 \). Instead of parameter \( \lambda \in \sigma_{\text{cont}}[H] \), it is also convenient to use the momentum \( p \) and the related complex parameter \( z = e^{ip} \):

\[ \lambda = 2 - 2 \cos p = 2 - z - z^{-1}. \]

Three solutions of (3.1), (3.3) are important for the scattering problem.

- The regular solution \( \varphi(j, p) \), which is fixed by the boundary condition

\[ \varphi(0, p) = 0, \quad \varphi(1, p) = 1. \]  

(3.6)

- Two Jost solutions \( f(j, p) \), and \( f(j, -p) \), which are determined by their behavior at large \( j \to \infty \), and describe the out- and in-waves, respectively,

\[ f(j, \pm p) \to \exp(\pm ipj) = z^{\pm j}, \quad \text{at} \quad j \to \infty. \]  

(3.7)

The regular solution \( \varphi(j, p) \) can be represented as a linear combination of two Jost solutions,

\[ \varphi(j, p) = \frac{i}{2 \sin p} [F(p)f(j, -p) - F(-p)f(j, p)], \quad 0 < p < \pi. \]  

(3.8)

The complex coefficient \( F(p) \) in the above equation is known as the Jost function. It is determined by (3.8) for real momenta \( p \) in the interval \( p \in (-\pi, \pi) \), where it satisfies the relation

\[ F(-p) = [F(p)]^*, \]  

(3.9)

and can be written as

\[ F(p) = \exp[\sigma(p) - i\eta(p)]. \]  

(3.10)
At large $j \to \infty$, the regular solution behaves as
\[
\varphi(j, p) \to \frac{A(p)}{\sin p} \sin[pj + \eta(p)], \quad j \to +\infty,
\]
where $A(p) = \exp[\sigma(p)]$ is the scattering amplitude, and $\eta(p)$ is the scattering phase. The latter can be defined in such a way, that $\eta(-p) = -\eta(p)$ for $-\pi < p < \pi$.

The following exact representation holds for the Jost function $F(p)$ in terms of the regular solution $\varphi(j, p)$:
\[
F(p) = 1 + \sum_{j=1}^{\infty} e^{ipj} v_j \varphi(j, p), \quad (3.11)
\]

(cf. equation (1.4.4) in [4] in the continuous case).

For $|z| = 1$, denote by $\hat{F}(z)$ the Jost function $F(p)$ expressed in the complex parameter $z$: $F(p) = \hat{F}(z = e^{ip})$. The function $\hat{F}(z)$ can be analytically continued into the circle $|z| < 1$, where it has finite number of zeros $\{a_n\}_{n=1}^{\infty}$. These zeroes determine the discrete spectrum $\{\lambda_n\}_{n=1}^{\infty}$ of the problem (3.1)-(3.3):
\[
\lambda_n = 2 - a_n - a_n^{-1}, \quad \text{for} \quad n = 1, \ldots, \infty. \quad (3.12)
\]

Of course, these eigenvalues are real in the boundary problem with a real potential.

To simplify further analysis, we shall consider in the sequel the potentials which satisfy the following requirements:

(i) The potential $V$ should have a compact support, i.e.
\[
v_j = 0, \quad \text{for all} \quad j > J, \quad (3.13)
\]
with some natural $J$.

(ii) The corresponding Jost function $\hat{F}(z)$ should not have zeroes inside the circle $|z| < 1$, i.e. $\infty = 0$. In other words, the spectrum $\sigma[H]$ should be purely continuous.

(iii) The Jost function $\hat{F}(z)$ should take non-zero values at $z = \pm 1$: $\hat{F}(1) \neq 0$, and $\hat{F}(-1) \neq 0$.

Conditions (ii) and (iii) imply, that the operator $H$ does not have bound and semi-bound states [4], respectively.

For the potential satisfying (3.13), only $J$ initial terms survive in the sum in the right-hand side of (3.11). Since $\varphi(j, p)$ is a polynomial of the spectral parameter $\lambda = 2 - z - z^{-1}$ of the order $j - 1$, the Jost function (3.11) expressed in the parameter $z$ is a polynomial of the degree $2J - 1$:
\[
\hat{F}(z) = 1 + \sum_{j=1}^{2J-1} c_j(V) z^j = \prod_{n=1}^{2J-1} \left[ 1 - \frac{z}{a_n(V)} \right], \quad (3.14)
\]

where the coefficients $c_j(V)$ polynomially depend on the potential $v_j$, $j = 1, \ldots, J$. Due to the constrains (iii), (iii), we get
\[
|a_n(V)| > 1, \quad \text{for all} \quad n = 1, \ldots, J. \quad (3.15)
\]
Spectrum of discrete Schrödinger operator with even potential

Let us periodically continue the scattering phase \( \eta(p) \) from the interval \((-\pi, \pi)\) to the whole real axis \( p \in \mathbb{R} \). It follows from (3.15), that for a potential satisfying (i)-(iii), the scattering phase is an analytical odd \( 2\pi \)-periodical function in the whole real axis: \( \eta(p) \in C^\infty (\mathbb{R}/2\pi \mathbb{Z}) \).

The notation \( \delta(\lambda) \) will be used for the scattering phase \( \eta(p) \) expressed in terms of the spectral parameter \( \lambda: \eta(p) = \delta(\lambda = 2 - 2 \cos p) \), for \( 0 \leq \lambda \leq 4 \), and \( 0 \leq p \leq \pi \). Conditions (i), (ii) guarantee, that

\[
\delta(0) = \delta(4) = 0. \tag{3.16}
\]

4. Constrains on the scattering data in the discrete Schrödinger scattering problem in the half-line

It is shown in this Section, that the scattering phase \( \delta(\lambda) \) in the boundary problem (3.1)-(3.3) for the discrete Schrödinger operator in the half-line with an arbitrary potential \( V \) obeying (i)-(iii) should satisfy the constrain

\[
\int_0^4 d\lambda \delta(\lambda) \frac{\lambda - 2}{\lambda(\lambda - 4)} + \frac{1}{\pi} \int_0^4 d\lambda_1 \delta(\lambda_1) \mathcal{P} \int_0^4 d\lambda_2 \frac{\delta'(\lambda_2)}{\lambda_2 - \lambda_1} = 0, \tag{4.1}
\]

where \( \mathcal{P} \) indicates the principal value integral. It is straightforward to rewrite the above constrain in the equivalent form in terms of the Jost function \( \hat{F}(z) \):

\[
\ln \hat{F}(z = 1) + \ln \hat{F}(z = -1) + \oint_{|z|=1} \frac{dz}{2\pi i} \ln[\hat{F}(1/z)] \frac{d\ln[\hat{F}(z)]}{dz} = 0, \tag{4.2}
\]

where the integration path in the right-hand side is gone in the counter-clockwise direction.

To prove (4.1), let us consider the discrete Schrödinger eigenvalue problem (1.1)-(1.3) in the finite interval \( 1 \leq j \leq N = 2M, \) with \( M > J \), and with the even potential \( V^{(M)} = \{v_j^{(M)}\}_{j=1}^{2M} \), which restriction to the interval \([1, M]\) coincides with that of the potential \( V = \{v_j\}_{j=1}^{\infty} \):

\[
v_j^{(M)} = \begin{cases} v_j, & \text{if } j \leq J, \\ 0, & \text{if } J < j \leq 2M - J, \\ v_{2M+1-j}, & \text{if } 2M - J < j \leq 2M. \end{cases} \tag{4.3}
\]

It is easy to see, that the spectrum \( \{\lambda_l\}_{l=1}^{2M} \) of the problem (1.1)-(1.3) with such a potential can be expressed in terms of the scattering phase \( \eta(p) \) of the corresponding semi-infinite problem (3.1)-(3.3) by the relations

\[
\lambda_l = \omega(p_l), \quad l = 1, \ldots, 2M, \quad (4.4)
\]

\[
(2M + 1) p_l + 2 \eta(p_l) = (2M + 1) k_l, \quad (4.5)
\]

where \( \omega(p) = 2 - 2 \cos p \), and \( k_l = \pi l/(2M + 1) \). Solving equation (4.5) with respect to \( p_l \) one obtains at large \( M \):

\[
p_l = k_l - \frac{2 \eta(k_l)}{2M + 1} + \frac{4 \eta(k_l) \eta'(k_l)}{(2M + 1)^2} + O(M^{-3}). \tag{4.6}
\]
For an arbitrary \( M > J \), we get from (4.4):
\[
\sum_{m=1}^{M} S_m = 0, \tag{4.7}
\]
where
\[
S_m = \sum_{n=1}^{M} [\ln |\omega(p_{2n-1}) - \omega(p_{2m})| - \ln |\omega(k_{2n-1}) - \omega(k_{2m})|]. \tag{4.8}
\]
Proceeding to the large-\( M \) limit, one finds after substitution of (4.6) into (4.8) and expansion the result in 1/(2\( M + 1 \)):
\[
S_m = S_m^{(0)} + S_m^{(1)} + O(M^{-2}), \tag{4.9}
\]
where
\[
S_m^{(0)} = \frac{2}{2M + 1} \sum_{n=1}^{M} \frac{\omega'(k_{2n}) \eta(k_{2m}) - \omega'(k_{2n-1}) \eta(k_{2m})}{\omega(k_{2n-1}) - \omega(k_{2m})}, \tag{4.10}
\]
\[
S_m^{(1)} = \frac{2}{(2M + 1)^2} \sum_{n=1}^{M} \left\{ \frac{\omega'(k_{2n}) \eta^2(k_{2m}) - \omega'(k_{2n-1}) \eta^2(k_{2m})}{\omega(k_{2n-1}) - \omega(k_{2m})} \right\} . \tag{4.11}
\]
In the right-hand side of the second equation we can safely [up to the terms of order \( O(M^{-2}) \)] replace the sum in \( n \) by the integral over the momentum \( q \):
\[
S_m^{(1)} = \frac{1}{\pi(2M + 1)} \int_{0}^{\pi} dq \left\{ \frac{\omega'(q) \eta^2(q) - \omega'(k_{2m}) \eta^2(q)}{\omega(q) - \omega(k_{2m})} \right\} \right) + O(M^{-2}). \tag{4.12}
\]
Calculation of the large-\( M \) asymptotics of \( S_m^{(0)} \) is more delicate. First, we extend summation in (4.10) in the index \( n \) from 1 till 2\( M + 1 \)
\[
S_m^{(0)} = \frac{2}{2M + 1} \sum_{n=1}^{M} R_m(k_{2n-1}) =
\frac{2}{2M + 1} \left[ \frac{R_m(k_{2M+1})}{2} + \frac{1}{2} \sum_{n=1}^{2M+1} R_m(k_{2n-1}) \right], \tag{4.13}
\]
where
\[
R_m(q) = \frac{\omega'(k_{2m}) \eta(k_{2m}) - \omega'(q) \eta(q)}{\omega(q) - \omega(k_{2m})}. \tag{4.14}
\]
In (4.13) we have taken into account the reflection symmetry \( R_m(q) = R_m(2\pi - q) \) of the function (4.14), providing \( R_m(k_{2n-1}) = R_m(k_{2(2M+1-n)+1}) \).
Since \( k_{2M+1} = \pi, \) and \( \eta(\pi) = 0, \omega(\pi) = 4, \) we get from (4.14)
\[
R_m(k_{2M+1}) = \frac{\omega'(k_{2m}) \eta(k_{2m})}{4 - \omega(k_{2m})}. \tag{4.15}
\]
The sum in the second line of (4.13) reads as
\[
\sum_{n=1}^{2M+1} R_m(k_{2n-1}) = \sum_{n=1}^{2M+1} R_m \left( 2\pi \frac{n-1/2}{2M+1} \right). \tag{4.16}
\]
Since \(R_m(q) \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})\), this sum can be replaced with exponential accuracy by the integral at large \(M \to \infty\):
\[
\sum_{n=1}^{2M+1} R_m \left( 2\pi \frac{n-1/2}{2M+1} \right) = \frac{2M+1}{2\pi} \int_0^{2\pi} dq R_m(q) + o(M^{-\mu}), \tag{4.17}
\]
where \(\mu\) is an arbitrary positive number, see formula 25.4.3 in [6]. Taking into account (4.14), the integral in the right-hand side can be written as
\[
\int_0^{2\pi} dq R_m(q) = \omega'(k_{2m}) \eta(k_{2m}) \mathcal{P} \int_0^{2\pi} \frac{dq}{\omega(q) - \omega(k_{2m})} - \mathcal{P} \int_0^{2\pi} dq \frac{\omega'(q) \eta(q)}{\omega(q) - \omega(k_{2m})} = 2 \omega'(k_{2m}) \eta(k_{2m}) \mathcal{P} \int_0^{\pi} \frac{dq}{\omega(q) - \omega(k_{2m})} = -2 I[\omega(k_{2m})] = -2 I[\omega(k_{2m})], \tag{4.18}
\]
where
\[
I(\Lambda) = \mathcal{P} \int_0^{\lambda} d\lambda \frac{\delta(\lambda)}{\Lambda - \lambda}, \quad \text{with } 0 < \Lambda < 4. \tag{4.19}
\]
In the second line of (4.18) we have taken into account the equality
\[
\mathcal{P} \int_0^{\pi} \frac{dq}{[\omega(q) - \omega(k)]^\nu} \equiv \frac{1}{2} \lim_{\epsilon \to +0} \left\{ \int_0^{\pi} \frac{dq}{[\omega(q+i\epsilon) - \omega(k)]^\nu} + \int_0^{\pi} \frac{dq}{[\omega(q-i\epsilon) - \omega(k)]^\nu} \right\} = 0, \tag{4.20}
\]
with \(0 < \kappa < \pi\), and \(\nu = 1\).

Collecting (4.13)-(4.19), we get
\[
S_m^{(0)} = -\frac{I[\omega(k_{2m})]}{\pi} - \frac{1}{2M+1} \frac{\omega'(k_{2m}) \eta(k_{2m})}{4 - \omega(k_{2m})} + O(M^{-2}). \tag{4.21}
\]

Thus, we obtain from (4.21) the equality
\[
\lim_{M \to \infty} \sum_{m=1}^{M} S_m^{(0)} + \lim_{M \to \infty} \sum_{m=1}^{M} S_m^{(1)} = 0, \tag{4.22}
\]
where \(S_m^{(0)}\) and \(S_m^{(1)}\) are given by equations (4.21), and (4.11), respectively.

In the second term, we can replace with sufficient accuracy the summation in \(m\) by integration in the momentum \(k\):
\[
\sum_{m=1}^{M} S_m^{(1)} = \frac{1}{2\pi^2} \int_0^{\pi} \frac{dk}{\Omega} \int_0^{\pi} dq \frac{[\omega(q) \eta^2(q)]' - [\omega'(k) \eta^2(k)]'}{\omega(q) - \omega(k)}
- \frac{1}{2\pi^2} \int_0^{\pi} \frac{dk}{\Omega} \int_0^{\pi} dq \left[ \frac{\omega'(k) \eta(k) - \omega'(q) \eta(q)}{\omega(q) - \omega(k)} \right]^2 + O(M^{-1}). \tag{4.23}
\]
The first integral in the right-hand side vanishes due to equality (4.20) with \(0 < k < \pi\), and \(\nu = 1\). The second line in (4.23) can be transformed as follows

\[
- \frac{1}{2\pi^2} \int_0^\pi dk \int_0^\pi dq \left[ \frac{\omega'(k) \eta(k) - \omega'(q) \eta(q)}{\omega(q) - \omega(k)} \right]^2 = \quad (4.24)
\]

\[
- \frac{1}{\pi^2} \int_0^\pi dk [\omega'(k) \eta(k)]^2 \mathcal{P} \int_0^\pi dq \frac{dq}{[\omega(q) - \omega(k)]^2} + \frac{1}{\pi^2} \int_0^\pi dk \mathcal{P} \int_0^\pi dq \frac{\omega'(k) \eta(k) \omega'(q) \eta(q)}{[\omega(q) - \omega(k)]^2}.
\]

The first term in the right-hand side vanishes due to equality (4.20) with \(\nu = 2\). Then, after a simple algebra we obtain from the second term in the right-hand side of (4.24)

\[
\lim_{M \to \infty} \sum_{m=1}^M S_m^{(1)} = \frac{1}{\pi^2} \int_0^4 d\lambda_1 \delta(\lambda_1) \mathcal{P} \int_0^4 d\lambda_2 \frac{\delta'(\lambda_2)}{\lambda_2 - \lambda_1}. \quad (4.25)
\]

Let us turn now to calculation of the first term in the left-hand side of equality (4.22). At large \(M\), one obtains

\[
\sum_{m=1}^M S_m^{(0)} = - \frac{1}{2\pi} \int_0^\pi dk \frac{\omega'(k) \eta(k)}{4 - \omega(k_2m)} - \frac{1}{\pi} \sum_{m=1}^M I[\omega(k_2m)] + O(M^{-1}). \quad (4.26)
\]

The first term in the right-hand side equals to \(I(4)/(2\pi)\). The large-\(M\) asymptotics of the sum in the right-hand side can be found as follows

\[
\sum_{m=1}^M I[\omega(k_2m)] = - \frac{I(0)}{2} + \frac{1}{2} \sum_{m=1}^{2M+1} I[\omega(k_2m)] = - \frac{I(0)}{2} + \frac{2M + 1}{4\pi} \int_0^{2\pi} dk I[\omega(k)] + O(M^{-\mu}), \quad (4.27)
\]

where \(\mu\) is an arbitrary positive number. In deriving (4.27) we have taken into account, that \(I[(\omega(k)]\) is the 2\(\pi\)-periodical function of \(p\) in \(\mathbb{R}\), which is continuous with all its derivatives [i.e., \(I[(\omega(k)] \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})\)], and formula 25.4.3 in [6]. The integral in the right-hand side vanishes due to equality (4.20) with \(\nu = 1\):

\[
\int_0^{2\pi} dk I[\omega(k)] = \int_0^{2\pi} dk \mathcal{P} \int_0^4 d\lambda \frac{\delta(\lambda)}{\lambda - \omega(k)} = \int_0^4 d\lambda \delta(\lambda) \mathcal{P} \int_0^{2\pi} \frac{dk}{\lambda - \omega(k)} = 0. \quad (4.28)
\]

Collecting (4.26)- (4.28), we come to the simple formula

\[
\lim_{M \to \infty} \sum_{m=1}^M S^{(0)}_m = \frac{I(0) + I(4)}{2\pi}. \quad (4.29)
\]

From (4.19), (4.22), (4.25), (4.29), we arrive to the final result (4.1).

Similarly to (1.4), equation (4.1) also admits the electrostatic interpretation. Let us treat the function \(\rho(\lambda) = 2\delta'(\lambda)/\pi\) as the electric charge density, which is distributed in the linear interval \(0 < \lambda < 4\). Requirement (3.16) implies, that the total electric charge of this distribution is zero,

\[
\int_0^4 d\lambda \rho(\lambda) = 0.
\]
It is straightforward to rewrite (4.1) in terms of the function $\rho(\lambda)$:

$$\frac{1}{2} \int_0^4 d\lambda_1 \rho(\lambda_1) \int_0^4 d\lambda_2 \rho(\lambda_2) \ln |\lambda_1 - \lambda_2| - \int_0^4 d\lambda \rho(\lambda) \left[ q_1 \ln \lambda + q_2 \ln(4 - \lambda) \right] - q_1 q_2 \ln 4 = -\frac{\ln 2}{2},$$

(4.30)

where $q_1 = q_2 = -1/2$. The left-hand side of the above equality represents the Coulomb energy of the continuous charge distribution $\rho(\lambda)$ located in the interval $(0, \lambda)$ in the two-dimensional plane, and two point charges $q_1 = q_2 = -1/2$ located at the points $\lambda_1 = 0$ and $\lambda_2 = 4$.

5. Conclusions

We have studied some general spectral properties of the one-dimensional Sturm-Liouville problem for the discrete Schrödinger equation with the Dirichlet boundary conditions. Both cases of the finite-interval and semi-infinite problems were considered.

For the finite-interval problem with even number of sites $2M$ and an arbitrary even potential, it was shown, that its eigenvalues should satisfy the infinite set of polynomial constrains of increasing degrees. Though the number of these constrains is infinite, no more than $M$ of them are independent. It is simple to find few initial small-degree polynomials in this set, but explicit calculation of subsequent polynomials of higher degrees becomes more and more difficult. Nevertheless, we have obtained in the explicit form one polynomial constrain from this set, which has the degree $M$, see equation (1.4). It leads to the effective Coulomb interaction between the eigenvalues, which correspond to even and odd eigenstates.

The scattering problem for the discrete one-dimensional Schrödinger equation in the half-line has been analysed as the $M \to \infty$ limit of the described above $2M$-site discrete Sturm-Liouville problem in the finite-interval. It was proved, that the scattering phase of the discrete scattering problem (3.1)-(3.3) should satisfy condition (4.1), if: (i) the potential has a compact support, (ii) the spectrum of the Hamiltonian is purely continuous, $\sigma[H] = (0, 4)$, and (iii) the Jost function takes nonzero values on its end points $\lambda = 0$ and $\lambda = 4$. Constrain (4.1) admits the electrostatic interpretation (4.30), as its finite-interval counterpart (1.4).

We did not try to prove (4.1) for the most general case of the discrete semi-infinite scattering problem. It is natural to expect, however, that it should hold for some more general class of potentials, which vanish fast enough at infinity, though do not have a compact support. On the other hand, in the case of the potentials with discrete spectrum and/or semi-bound states (the latter appear if the Jost function has zeroes at the end points of the continuous spectrum, see [4]), some modified forms of (4.1) should also exist.

We believe, that obtained results could be useful for the theory of Anderson localisation and for the theory of random matrices.
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Appendix A. Proof of equality (2.15)

Let us start from the following auxiliary

\textbf{Lemma Appendix A.1} The following equality holds for all natural \( M \) and integer \( n \):

\[
\prod_{m=1}^{2M+1} \left\{ 4 \sin^2 \left[ \frac{(2m-1)\pi}{2(2M+1)} - \alpha \right] - 4 \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right\} = 4 \cos^2[\alpha(2M+1)].
\] (A.1)

\textbf{Proof} Denote

\[
g(\alpha) = \prod_{m=1}^{2M+1} \left\{ 4 \sin^2 \left[ \frac{(2m-1)\pi}{2(2M+1)} - \alpha \right] - 4 \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right\}.
\] (A.2)

The symmetry properties of the function \( g(\alpha) \)

\[
g(-\alpha) = g(\alpha),
\]

\[
g\left( \alpha + \frac{\pi}{2M+1} \right) = g(\alpha)
\]

follow from (A.2).

Function \( g(\alpha) \) is analytical in the complex \( \alpha \)-plane and has the second order zeroes at the points

\[
\alpha_l = \frac{\pi}{2(2M+1)} + \frac{\pi l}{2M+1}, \quad l = 0, \pm 1, \pm 2, \ldots
\] (A.3)

It follows from (A.3) that the function

\[
R(\alpha) = \frac{g(\alpha)}{4 \cos^2[\alpha(2M+1)]}
\] (A.4)

is analytical and has no zeroes in the complex \( \alpha \)-plane. Furthermore, this function is rational in the variable \( z = e^{i\alpha} \) and approaches to 1 at \( z \to \infty \) and at \( z \to 0 \). Therefore, \( R(\alpha) = 1 \).

Putting \( \alpha = 0 \) in (A.1) we find

\[
\prod_{m=1}^{2M+1} \left\{ 4 \sin^2 \left[ \frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right\} = 4.
\] (A.5)

Since

\[
\prod_{j=1}^{2M+1} \left\{ 4 \sin^2 \left[ \frac{(2j-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right\} = 4 \left( 1 - \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right)^2,
\]

\[
\prod_{m=1}^{M} \left\{ 4 \sin^2 \left[ \frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right\}^2 4 \left( 1 - \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right),
\]
Spectrum of discrete Schrödinger operator with even potential

we get

\[ \prod_{m=1}^{M} \left\{ 4 \sin^2 \left[ \frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right\}^2 = \left[ \cos \frac{\pi n}{2M+1} \right]^{-2}, \]

or

\[ \prod_{n=1}^{M} \left\{ 4 \sin^2 \left[ \frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right\} = (-1)^n \left[ \cos \frac{\pi n}{2M+1} \right]^{-1}. \quad (A.6) \]

To fix the sign of the right-hand side of (A.6), we have taken into account that just the first \( n \) factors in the product in the left-hand side are negative at \( n = 1, \ldots, M \).

Thus,

\[ \prod_{n=1}^{M} \prod_{m=1}^{M} \left\{ 4 \sin^2 \left[ \frac{(2m-1)\pi}{2(2M+1)} \right] - 4 \sin^2 \left[ \frac{2n\pi}{2(2M+1)} \right] \right\} = \prod_{n=1}^{M} \left( -1 \right)^n \cos \frac{\pi n}{2M+1}. \quad (A.7) \]

To determine the product in the right-hand side we use formula 1.392.1 in Ref. [7]:

\[ \sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin \left( x + \frac{\pi k}{n} \right). \quad (A.8) \]

For \( n = 2M + 1, x = \pi/2 \), we get from (A.8)

\[ \prod_{k=0}^{2M} \cos \left( \frac{\pi k}{2M+1} \right) = 2^{-2M} (-1)^M, \]

providing

\[ \prod_{k=1}^{M} \cos \left( \frac{\pi k}{2M+1} \right) = 2^{-M}. \quad (A.9) \]

Substitution of (A.9) into (A.7) leads finally to (2.15).

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