Domain Growth and Finite-Size-Scaling in the Kinetic Ising Model

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Abstract

This paper describes the application of finite-size scaling concepts to domain growth in systems with a non-conserved order parameter. A finite-size scaling ansatz for the time-dependent order parameter distribution function is proposed, and tested with extensive Monte-Carlo simulations of domain growth in the 2-D spin-flip kinetic Ising model. The scaling properties of the distribution functions serve to elucidate the configurational self-similarity that underlies the dynamic scaling picture. Moreover, it is demonstrated that the application of finite-size-scaling techniques facilitates the accurate determination of the bulk growth exponent even in the presence of strong finite-size effects, the scale and character of which are graphically exposed by the order parameter distribution function. In addition it is found that one commonly used measure of domain size—the scaled second moment of the magnetisation distribution—buries the full extent of these finite-size effects.

PACS numbers: 05.50, 75.10H

1 Introduction

When a system in an initially disordered high temperature state is rapidly quenched to below its transition point, it orders kinetically [1]. Small domains of the new equilibrium phase form and grow self-similarly via the motion of their interfaces. It is now well established that such growth processes involve a characteristic time-dependent length \( R(t) \) – the ‘domain-scale’ – identifiable physically as the average domain size. Moreover, experiments and computer simulations indicate that \( R(t) \) satisfies power law behaviour of the form \( R(t) \sim t^x \), where \( x \) is known as the growth exponent. Interestingly however, it transpires that the value of \( x \) exhibits a remarkably degree of insensitivity to the microscopic description of the system, depending

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only on whether or not the order parameter is a conserved quantity. For systems having a non-conserved order parameter (NCOP) one finds $x = \frac{1}{2}$, while in the conserved case (COP) of phase separation and spinodal decomposition, it now seems quite generally agreed that $x = \frac{1}{3}$.

The existence of whole classes of ostensibly quite disparate physical systems that nevertheless exhibit the same power law growth is, in many respects, reminiscent of the universality classes of critical phenomena. Indeed the similarities between the two fields run rather deep. If one regards the divergence of the domain scale $R(t)$ at late times as analogous to the divergence of the thermal correlation length, then many of the theoretical tools and concepts familiar from critical phenomena also carry over (albeit with some important modifications and differences) to the domain growth context. The concepts of scaling, fixed points and renormalisation group flow processes underpin the contemporary theoretical view of domain growth kinetics, just as they do in critical phenomena.

Notwithstanding these advances however, it is probably fair to say that the present theoretical understanding of domain growth is neither as advanced nor on as firm a footing as critical phenomena. This in part stems from a dearth of exactly solvable models for domain growth, and the consequent lack of an effective test-bed for new theories and conjectures. In view of this situation, it has been common practice to appeal to computer simulation techniques in order to gain physical insight into domain growth processes. Indeed, many such simulations studies have been undertaken, of which perhaps the majority have concentrated on determining the growth law for the system of interest. However other studies have drawn their inspiration from more sophisticated techniques already tried and tested in the context of critical phenomena. One such example is the Monte-Carlo renormalisation group method, which has been successfully adapted to the study of dynamic scaling in both the COP and NCOP cases.

Given the prevalence of computer simulation studies in the field of domain growth and the clear parallels with critical phenomena, it somewhat surprising that hitherto relatively little effort has been directed towards the application of finite-size-scaling (FSS) ideas to dynamic scaling phenomena. Indeed little attention seems to have been given at all to the problems of finite-size effects in simulation studies of domain growth. By contrast, FSS concepts have for years enjoyed considerable success in simulation studies of critical phenomena helping both to mitigate finite size effects, and illuminate the nature of configurational self-similarity. Nevertheless, we are aware of only three papers that consider their utility in the domain growth context. We discuss them in turn.

FSS concepts were first applied to domain growth simulations by Viñals and Jasnow. These authors studied the time-dependent structure factor of the kinetic Ising model following quenches to temperatures well below $T_c$. A FSS expression for the structure factor was proposed, and tested using Monte-Carlo simulations for both COP and NCOP. Good scaling was observed in the NCOP case and a growth exponent consistent with $x = \frac{1}{2}$ was obtained. However the results for the COP case were somewhat less conclusive, possibly reflecting poor statistics and the rather small size of the lattice used.

In a related analytical study of the time-dependent Ginsburg-Landau model, Guo et al. derived an approximate expression for the FSS properties of the structure factor in the NCOP case. This expression yielded results in qualitative agreement with the earlier simulation studies of Viñals and Jasnow. The authors also considered the cross-over from bulk behaviour to the region where finite-size effects are significant, pointing out that a fuller understanding
of the latter regime requires a clearer elucidation of the role of metastable strip configurations.

In a theoretical analysis of domain growth following a quench into the critical region, Milchev et al. [15] considered the time dependent scaling properties of the order parameter distribution function for NCOP. By generalising well known expressions for the order parameter distribution in the context of static critical phenomena, time-dependent FSS expressions were developed for quenches to $T_c$, as well as to temperatures slightly below and above $T_c$. This work also led to a prediction that critical-slowing-down engenders a weak critical singularity in the proportionality constant (the so-called ‘rate’) of the growth law $R(t) = Bt^{x}$, of the form $B \propto (1 - T/T_c)^{\nu(z-1/x)}$. However no explicit simulation test of either this prediction or the general FSS proposals was carried out.

In the present paper we broaden the application of FSS ideas to domain growth simulations by studying the time-dependent order parameter distribution function of the 2D spin-flip kinetic Ising (SFKI) model for NCOP. The essential features of our study are as follows. We have performed extensive Monte-Carlo simulations of domain growth in the 2D SFKI model following a quench from infinite temperature to points well within the two phase region. We obtain the time-dependent magnetisation distribution function, the structure of which we analyse within a FSS framework. Our analysis bears out the configurational self similarity that lies at the core of the dynamic scaling hypothesis, Moreover, we show that in contrast to the MCRG method, application of FSS techniques facilitates accurate determination of the bulk scaling exponent, even in the presence of strong finite-size effects. The scale and character of these finite-size effects are graphically exposed by our distribution functions.

In addition to the FSS analysis, we have also studied in detail one commonly used measure of domain size, the scaled second moment of the magnetisation distribution $R_M^2(t) \equiv L_d < M^2(t) >$. We show that without a concomitant analysis of configurational structure, this quantity conveys a misleading impression of the extent of finite size effects. We also probe the approach to $t^{1/2}$ scaling in $R_M^2(t)$. It is demonstrated that a substantial transitory period precedes the onset of true $t^{1/2}$ scaling. The duration of this ‘time-lag’ is shown to be rather strongly temperature dependent. A strong temperature dependence is also observed in the proportionality constant of the growth law, in agreement with earlier studies.

2 Background

Lifshitz [2], and later Cahn and Allen [3] proposed a growth equation for NCOP systems on the basis of a phenomenological theory for the Langevin equation associated with a time-dependent Ginzburg-Landau model in the absence of noise. Their principal result, the Lifshitz-Cahn-Allen (LCA) growth law, is expressed as:

$$R(t) = Bt^{1/2}$$

and is valid for late times. The system-specific proportionality constant $B$ is (within the framework of the LCA theory) temperature independent.

The LCA theory has been tested in numerous experimental [16] and computer simulation studies [13, 17, 18, 20, 21]. Many of these simulation studies have concentrated on obtaining the growth law for the system of interest. This is usually achieved by direct measurements of the domain size as a function of time. For this purpose a variety of different measures of domain size can be employed, including the first and second moments of the structure factor, the inverse perimeter length and the scaled second moment of the magnetisation distribution.
A discussion of these quantities and some of their relative merits can be found in Gawlinski et al [17]. Having chosen a measure of domain size, the strategy is then to perform averages over a number of independent quenches, typically numbering up to a few hundred. By plotting the results against \( t^x \), power law scaling can be verified and the growth exponent can be evaluated. Almost without exception, such measurements of domain growth yield values for the growth exponent consistent with the LCA prediction \( x = \frac{1}{2} \).

Nevertheless, not all aspects of the LCA theory are in accord with observation. Specifically, the neglect of noise in the theory leads to a temperature independent growth law that is at variance with the observations of a number of simulation studies [20, 21, 22, 23]. These demonstrate that temperature does have a significant influence on domain growth, even well away from the critical point. In particular it is found that although \( x = \frac{1}{2} \) appears to hold for all sub-critical temperatures, the magnitude of the proportionality constant \( B \) in the growth law decreases with increasing temperature, the effect being particularly pronounced for temperatures above about 0.6\( T_c \). Current theories suggest that this ‘slowing-down’ of domain growth is largely attributable to the disruption of the smooth interface curvature by thermal roughening. Indeed a reexamination of the LCA theory, this time including noise [24], appears to produce results in qualitative agreement with simulation results. In view of these findings equation (1) should be modified to read

\[
\hat{R}(t, T) = B(T)t^{\frac{1}{2}}
\]

Although direct measurements of the growth exponent from the time dependence of the domain size provide reassuring evidence in support of the dynamic scaling ideas, they represent only a portion of the information potentially available from a simulation. A more stringent and indeed more physically-compelling test of the dynamic scaling hypothesis, is to directly observe the time-dependent statistical self-similarity of the coarse-grained configurational structure. In the NCOP case, the essential character of this structure is uniquely prescribed by the distribution function of the order parameter, the time dependence of which implicitly embodies the growth law. Studies of this quantity therefore represent a potentially exacting test of the dynamic scaling hypothesis, one that transcends measurements of the growth exponent alone. In what follows we develop a FSS approach to the study of the time dependent order parameter distribution.

### 3 Method

Let us denote the time dependent distribution of the magnetisation \( M \) as \( P_L(M, t) \), where \( L \) is the linear extent of the system and \( t \) is the time. Motivated by the work of Milchev et al [14], we make the following finite-size-scaling assumption valid for quenches to temperatures \( T < T_c \):

\[
P_L(M, t) \simeq a(L)\tilde{P}_L(a(L)M, \hat{R}(t)/L, \xi/L)
\]

Here \( \tilde{P}_L \) is a scaling function that is expected to be universal modulo the choice of boundary conditions, \( a(L) \) is a system specific scale factor, \( \hat{R}(t) \) is the domain scale and \( \xi \) is the thermal correlation length.

Now, if the quench temperature lies well below the critical point, \( \xi \) will be negligibly small on the scale of \( L \), and no critical scaling behaviour will be apparent. The only remaining
influence of a finite temperature will be its effect on the growth law prefactor $B(T)$, which is significant even well away from the critical region [24]. It is therefore convenient to simply drop the third argument in equation 3 and absorb all the non-critical temperature dependence into $\overline{R}(t)$ to write:

$$P_L(M, t) \simeq \tilde{P}(M, \overline{R}(t, T)/L)$$

Equation 4 forms the platform upon which our analysis procedure rests. It expresses the basic assertion of the dynamic scaling hypothesis: the spectrum of coarse-grained configurations is expected to be invariant under appropriate rescalings of both length and time. We note that if one accepts a-priori the validity of the dynamic scaling hypothesis, then equation 4 constitutes a definition of the domain scale $\overline{R}(t, T)$. We shall exploit this latter feature to make a detailed evaluation of the growth law within the FSS context. We point out also that insofar as we obtain the scaling behaviour of the whole distribution $P_L(M, t)$, encompassing configurational structure on all length scales, our study goes well beyond that of Viñals and Jasnow [13] who only studied the scaling of the structure factor at three distinct values of wavenumber and did not consider explicitly the nature of finite-size effects or their role within the FSS context.

The essential character of the distribution function $P_L(M)$ is conveyed through the dimensionless fourth order cumulant ratio $G_L$. This quantity is defined in terms of the moments of the magnetisation distribution:

$$G_L \equiv \frac{3}{2} - \frac{<M^4>}{2<M^2>^2}$$

the value of which ranges from zero for a gaussian distribution to unity for a pair of delta functions. The cumulant ratio will prove useful for analysing the scaling properties of the distribution function.

In the course of the simulation work, we shall also consider one commonly used measure of domain size, the scaled second moment of the density distribution [19]:

$$R^2_M(t) \equiv L^d <M^2(t)>$$

with $d = 2$ in the simulations to be described below. This quantity is formally equivalent to the zero wavevector value of the structure factor. Its behaviour will be compared with the results from the FSS analysis of the magnetisation distribution.

4 Monte-Carlo Studies

4.1 Computational details

The Monte-Carlo simulations [25] reported here were all performed using a Glauber algorithm. Four system sizes were studied having linear extent $L = 32, 64, 128$ and 256. Following convention, periodic boundary conditions were employed throughout. In order to avoid spurious ‘marching’ effects associated with sequential ‘typewriter’ updating, update sites were visited randomly [17]. In this paper, time is expressed in units of Monte-Carlo steps (MCS) and the update of a single spin is taken as $1/L^2$ of one MCS.

The simulations were implemented on the Parsytec ‘Supercluster’ Transputer array of the Interdisziplinäres Zentrum für Wissenschaftliches Rechnen der Universität Heidelberg. Copies
of the program were run on each of 64 Transputers. This simple but effective form of parallelism considerably enhances the rate of data acquisition, allowing the accumulation of very high statistics.

The time-dependent magnetisation distribution \( P_L(M, t) \) was obtained initially in the form of a histogram recording the magnetisation \( M \) as a function of time. This histogram was built up over many independent quenches or ‘runs’. As a consistency check, both positive and negative values of the magnetisation were accumulated separately. Owing to limitations on Transputer on-chip memory, it was not possible to histogram separately all the \( L^2 + 1 \) magnetisation states. A binning procedure was therefore implemented, with the magnetisation states being distributed over 513 bins. In order to preserve the symmetry of the binning process with respect to zero magnetisation, the \( M = 0 \) state was allocated its own bin. The other \( L^2 \) magnetisation states were distributed uniformly over the remaining 512 bins. The weight in the zero magnetisation bin was subsequently rescaled to take account of the smaller number of contributing magnetisation states.

The bulk of the simulation work reported here was performed at a reduced coupling of \( J/\kappa_B T = 1 \), corresponding to a quench temperature \( T \approx 0.44 T_c \). For the \( L = 32, 64, 128 \) and 256 systems, respective observation periods of 500, 1000, 1300 and 2000 MCS were utilised. The number of runs performed in each case was \( 4 \times 10^5, 1.3 \times 10^5, 6.4 \times 10^4 \) and \( 5.5 \times 10^4 \) respectively. We note that these statistics far exceed those of previously reported studies by at least an order of magnitude.

4.2 Finite-size-scaling results

In figure 1 we present the time evolution of the magnetisation distribution function \( P_L(M, t) \) for the \( L = 32 \) and \( L = 128 \) system sizes. The functional form of these distributions serves to elucidate the nature of the ordering processes in the system. The salient features are best illustrated by considering the form of the distribution function at a number of discrete timeslices. For convenience we shall employ the \( L = 32 \) data for this purpose, but our remarks apply equally to all system sizes.

Figure 2 shows the \( L = 32 \) magnetisation distribution at times \( t = 20, 50, 75, 100, 175, 256, 512 \) MCS. Clearly at very early times, the distribution is sharply peaked around \( M = 0 \), reflecting the fact that the system is still highly disordered and only small domains have begun to form. As time progresses however, and the domains grow ever larger on the scale of the system, the distribution spreads out and weight is transferred from the central portion towards the periphery. Interestingly though, this broadening of the distribution is not uniform as one might expect. Instead the distribution starts to develop quite early on into a three peaked structure. We discuss the origin of each of these peaks in turn.

The first peak is centred on \( M = 0 \) and represents metastable strip configurations. These strip configurations comprise a single domain which spans the system along one of the lattice directions. At the temperature studied, strip configurations start to appear for \( t^* \approx 0.2L \). Since the strip interface is essentially flat, there is no driving force for further growth, and decay can only proceed by thermal fluctuations \( [22] \). Consequently strip domains are extremely long-lived, being essentially stable on the timescale of the other growth processes we consider.

The second peak, which we term the ‘dynamic peak’, corresponds to configurations that are still growing, i.e. those that have not evolved into strip configurations. An example of a dynamic configuration is a circular domain embedded in a ‘sea’ of the opposite phase. At short times, the dynamic peak is to be found near \( M = 0 \) where it is superimposed on the central peak
of metastable configurations. As time progresses however and the domains grow, the dynamic peak ‘moves’ to successively larger $M$ values where it can be resolved from the metastable peak. Since metastable states are manufactured from dynamic ones, strip configurations necessarily have magnetisation values that lie under the dynamic peak at their moment of formation. We find that in approximately 35% of runs, the system evolves into a metastable strip state.

The third peak, which we term the ‘saturation peak’, represents equilibrium configurations which start to appear at approximately the same time as strip configurations. This peak is very narrow (its width varies like $L^{-d}$) and for low temperatures is situated very close to $M = \pm 1$. The saturation peak draws its weight from the large-$M$ tail of the dynamic peak. All those surviving dynamic configurations that escape entrapment as metastable strips, eventually become saturated. Thus for late times (though still short compared to the characteristic lifetime of metastable strips), the system enters a quasi-asymptotic regime where all growth has effectively ceased and the distribution function comprises only the central metastable peak and the saturation peak. It is to be expected that the quasi-asymptotic form of the distribution function is $L$-independent.

We turn now to an examination of the time-dependent configurational self-similarity that underlies the dynamic scaling picture. Specifically we consider the transformations (rescalings of time) required to perform the data collapse of distribution functions for different $L$. In figure 3, we show example timeslices of the distribution function $P_L(M, t)$ for various choices of $L$ and $t$. The data correspond to three distinct values of $z \equiv t^{\frac{d}{2}} / L$ (cf. equations 2 and 4), namely $z \approx 0.17, 0.28$ and 0.31. In each instance the timeslices shown were chosen specifically to optimise the quality of the data collapse as quantified by the matching of the cumulant ratio. Figure 3(a) shows the case $z \approx 0.17$ for the four system sizes $L = 32, 64, 128$ and 256, at respective times $t = 25, 110, 480, 1952$. The case $z \approx 0.28$ is shown in figure 3(b) for the $L = 64$ and $L = 128$ systems at respective times $t = 313$ and 1300. Finally, figure 3(c) shows the data for $z \approx 0.31$ for system sizes $L = 32$ and $L = 64$, at times $t = 100$ and $t = 420$ respectively.

In each of the examples shown in figure 3(a)–(c), the quality of the data collapse is high, although the extent of finite-size effects differs markedly between the three cases. In case (a) there is nothing in the structure of the distribution functions to suggest that finite-size effects are yet significant. On the other hand, for cases (b) and (c) there is clearly a high proportion of static (metastable and saturated) configurations, as evidenced by the triple-peaked structure of the distribution function. We term this latter regime the strongly finite-size scaling region. At first sight, the observation of time-dependent scaling even in the presence of static configurations, is somewhat counter-intuitive. We wish to argue however, that this is actually a feature of our finite-size-scaling approach.

To this end it is instructive to consider the nature of typical configurations in the strongly finite-size region ($z > 0.2$). In this regime one finds both dynamic configurations and static configurations, the former of which grow like $t^{\frac{d}{2}}$ while the latter, of course, don’t grow at all. The crucial point underpinning the observation of scaling behaviour, is that metastable or saturated configurations are only formed when a dynamic configuration spans the system. Thus their formation rate at a given time $t$ (and hence their magnetisation distribution) is entirely mediated by the distribution of dynamic clusters which barely span the system at time $t$. Now, since the distribution of these barely-spanning dynamic clusters is controlled by the scaling variable $R(t, T)/L$, it follows that the whole distribution (comprising both dynamic and static contributions) also scales with this variable. Accordingly $t^{\frac{d}{2}}$ FSS behaviour is observed.
even at times when a significant proportion of configurations have stopped growing. Moreover, measurements of the growth exponent $x$ extracted from the scaling behaviour of the distribution functions are expected to be reliable at times long after the simple picture of bulk growth (with an associated physically well-defined domain size) has broken down. This is, we feel, analogous to the situation in critical phenomena, where application of FSS techniques permit accurate measurements of critical exponents even when the correlation length exceeds the system size.

The data collapse shown in figure 3 gives particular examples of dynamic scaling. However in order to demonstrate the full extent of the scaling behaviour that resides in our distributions, it is expedient to consider the time-dependence of the cumulant ratio. Figure 4(a) shows the measured cumulant ratio (as prescribed in equation 5) expressed as a function of time for all four of the system sizes studied. Before examining the cumulant scaling behaviour however, we should point out two general features of the data. Firstly at short times the data for $L = 128$ and $L = 256$ appears somewhat noisy. This effect is however simply an artifact of our binning procedure and reflects the fact that for early times (and especially for larger system sizes), the distribution is very narrow and contributes significant weight to only a small number of bins. Measurements of the cumulant ratio from data distributed over so few bins exhibit an enhanced degree of sensitivity to small statistical deviations from symmetry. Secondly, at late times the data for the $L = 32$ system shows the cumulant ratio appearing to saturate at a value $G_L \approx 0.8$, well short of its true asymptotic value of unity. This leveling-off of $G_L$ reflects the cessation of growth that occurs in the quasi-asymptotic regime. The discrepancy in the cumulant value signals the presence of metastable strip configurations.

In order to expose the time-dependent scaling behaviour of the cumulant ratio, we have reexpressed the data of figure 4(a) in terms of the variable $z \equiv t^{1/2}/L$. The rescaled results are shown in figure 4(b). Except at short times, where as already noted, measurements of the cumulant are affected by noise, the curves are to a good approximation parallel to one another (there is some deviation for the $L = 32$ system, and these are addressed in the discussion section). We note that this parallelism extends well beyond $z = 0.2$, thus corroborating our assertion that $P_L(M, t)$ manifests $t^{1/2}$ scaling even in the presence of strong finite-size effects. Interestingly, however although the curves for $L = 128$ and $L = 256$ coincide for late times, the data for $L = 32$ and $L = 64$ are displaced from the limiting (large $L$) case by an amount that decreases with increasing $L$. To effect coincidence and achieve scaling for the smaller system sizes, it is therefore necessary to make an empirical redefinition of the domain scale $R(t, T)$ to include an additive $L$-dependent term viz:

$$R(t, T, L) = A(L) + B(T)t^{1/2}$$

where $A(L)$ is a system specific quantity. Use of this modified form of the domain scale in the scaling variable $z \equiv R/L$ allows the scaled cumulant data to be brought into coincidence for all system sizes. Specifically, we find that the $L = 32$ and $L = 64$ data can be brought into correspondence with the $L = 128$ and $L = 256$ data by choices $A(32) \approx 0.7, A(64) \approx 0.8$.

### 4.3 The scaled second moment: $R_M^2(t) \equiv L^d < M^2(t) >$

One commonly employed measure of the domain size (actually the square of the domain size) in simulation studies of NCOP growth, is the scaled second moment of the magnetisation, $R_M^2(t) \equiv L^d < M^2(t) > \cite{19}$. Although a number of simulation studies of this quantity have been carried out, it is in practice rather difficult to obtain highly accurate measurements due to
its lack of self averaging [14]. This implies that the relative statistical error cannot be reduced by increasing the system size, but only by performing a very large number of runs—a task that is computationally intensive. Perhaps as a consequence, none of the previous studies of $R^2_M(t)$ have provided very detailed information on its behaviour.

In the present study, $R^2_M(t)$ is obtained as a by-product of our measurements of $P_L(M,t)$. Moreover owing to the large number of runs needed to accumulate our magnetisation distribution functions, we have been able to obtain very precise estimates of its behaviour. These estimates were extracted directly from the measured second moment of the magnetisation distributions and are shown in figure 5 expressed as a function of time. In contrast to the scaled cumulant ratios, the $R^2_M(t)$ data for all systems sizes coincide to a very high degree of accuracy. Furthermore the data also appears linear in time, as indeed one would expect on the basis of the LCA growth law. The data remains linear until $t^{1/2} \approx 0.4L$ at which point the curves tail off.

It transpires on closer inspection, however, that the data is not as linear as a cursory examination would suggest. In figure 6 we show the data of figure 5 with the $t$ dependence divided out. This clearly shows that $t^{1/2}$ scaling does not in fact set in until after an initial transitory period—approximately 300 MCS are required for $R^2_M(t)/t$ to attain 95% of its late-time value. This ‘time-lag’ manifests the fact that scaling behaviour only becomes apparent once the domains have grown large on the scale of the largest microscopic length (in this case the lattice spacing). We note additionally that for the $L = 32$ there is no real ‘window’ of $t^{1/2}$ scaling at all since all growth ceases well before 300 MCS.

An investigation of the influence of temperature on domain growth has also been performed. Measurements of $R^2_M(t)$ at short times ($t < 300$ MCS) were made for the $L = 128$ system at a variety of temperatures between $T = 0$ and $T_c$. For each temperature studied, $2.5 \times 10^4$ runs were performed. The results, presented in figure 7(a), clearly show that domain growth slows as the temperature is increased, in accord with earlier findings [20, 21, 22, 23]. In order to quantify the degree of this slowing-down, we have obtained estimates for the square of the proportionality constant $B(T)$ of the growth law. These estimates, representing the gradient of the $R^2_M(t)$ curve at $t = 300$ MCS, are plotted as a function of temperature in figure 7(b). Also shown is the theoretical expression of Grant and Gunton [24], the arbitrary vertical scale of which has been chosen so as to match the simulation data at $T = 0$. The observed behaviour is in qualitative agreement with the theoretical prediction, although the theory substantially underestimates the temperature at which the growth rate vanishes.

Finally, the effect of temperature on the approach to scaling has also been examined, by dividing out the time dependence. The results, figure 7(c), demonstrate that the approach to scaling is more protracted at higher temperatures, presumably reflecting the slower growth rate. Nevertheless, all indications are that $t^{1/2}$ scaling is still obeyed for late times. We note however, that the strength of the roughening effect in the near-critical region will almost certainly mask the proposed weak critical singularity of the growth law proportionality constant [14], complicating any verification of its existence by computer simulation methods.

### 5 Discussion

In this paper we have explored the application of finite-size-scaling techniques to the domain growth problem. Our results for the matching of the distribution function $P_L(M,t)$, as expressed through our scaled cumulant data, bear out the time-dependent configurational
self-similarity that underlies the dynamic scaling hypothesis. Moreover, we find that when analysed within a FSS framework, the distributions yield accurate estimates for the bulk scaling exponent $n = 1/2$, even in the regime where the system is strongly afflicted by finite size effects. Clearly therefore the present method should prove useful in circumventing the limitations imposed by finite computer resources in studies of domain growth.

We have also examined in detail the behaviour of the scaled second moment of the magnetisation $R_M^2(t) \equiv L^d < M^2(t) >$, used as a measure of the domain size. Although this quantity exhibits $t^{1/2}$ scaling for late times, our results shows that appreciable deviations from this behaviour persist until at least 300 MCS. Indeed for the $L = 32$ case studied here, the system never attains the asymptotic scaling regime because all growth stops well before 300 MCS. This effect is presumably responsible for the observation (noted in section 4.2), that the scaled cumulant value for the $L = 32$ system is not quite parallel to those of the larger system sizes (cf. figure 4(b)). Evidently therefore, system sizes of at least $L = 64$ must be employed if anything approaching ‘true’ $t^{1/2}$ scaling is to be observed at all.

In this context it is also interesting to note that $R_M^2(t)$ exhibits few of the finite-size effects manifested by the distributions themselves. Thus for instance, the data of figure 5 coincide for all system sizes, in contrast to the behaviour of the scaled cumulant ratio of figure 4(b). Also, $R_M^2(t)$ is apparently insensitive to the presence of metastable strip and saturated configurations, remaining stubbornly linear (at the temperature studied) up to $t^{1/2} \approx 0.4L$, corresponding to $R_M = 0.65L$. In contrast, the evidence of our distributions shows that metastable and saturated configurations actually appear at substantially earlier times than this, being already evident for $t^{1/2} = 0.2L$, corresponding to $R_M = 0.32L$.

Clearly therefore caution should be exercised when attempting to gauge the extent of finite-size effects without an accompanying analysis of configurational structure. Particular care is called for when performing Monte Carlo renormalisation group (MCRG) studies of domain growth. In such studies one typically applies a local blocking transformation, the action of which is intended to produce configurations whose statistical properties are appropriate to an earlier time. Evidently however such a procedure must fail when applied to metastable or saturated configurations because repeated applications of a blocking transformation cannot produce a non-spanning domain. The MCRG method is therefore strictly only applicable in the effectively bulk regime where no metastable or saturated configuration have yet evolved. In this respect, the present FSS method possesses a major advantage over the MCRG approach. The size of our coarse-graining length is simply that of the system itself, and the effect of changing this length is studied not by a blocking transformation, but by comparison with independent simulations of different sizes. Consequently the difficulties associated with blocking transformations are circumvented.

Finally with regard to extensions of the present work, it has been suggested that the appearance of metastable strip configurations may merely be an artifact of the use of periodic boundary conditions [17, 27]. It would now be of interest to explore in detail the degree to which the boundary conditions affect the nature and extent of finite-size effects, particularly with regard to the stability of strip configurations. We intend to report on such extensions in future work.
Acknowledgments

NBW and Ch.M acknowledge financial support from the Graduiertenkolleg für Modellierung und wissenschaftliches Rechnen in Mathematik und Naturwissenschaften, IWR Universität Heidelberg.

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Figure 1: The time evolution of (a) the $L = 32$, and (b) the $L = 128$ magnetisation distribution functions $P(L, M, t)$ for times in the range $0 – 512$ MCS and $0 – 1300$ MCS respectively. The distribution at each instant is normalised to unit integrated weight. In (a) the saturation peaks at $M \simeq \pm 1$ have been truncated in order to accentuate the central structure.

Figure 2: Timeslices of the $L = 32$ magnetisation distribution $P(M, t)$ at times $t = 20, 50, 75, 100, 175, 256, 512$ MCS. All distributions are normalised to unit integrated weight.

Figure 3: The data collapse of the normalised magnetisation distribution $P(M)$ for various system sizes and times. (a) $L = 32, 64, 128$ and 256 at times $t = 25, 110, 480$ and 1952 MCS respectively. (b) $L = 64$ and $L = 128$ at times $t = 313$ and $t = 1300$ MCS respectively. (c) $L = 32$ and $L = 64$ at times $t = 100$ and $t = 420$ MCS respectively.
Figure 4: (a) The cumulant ratio $G_L$ of the magnetisation distribution, prescribed in equation 3 and expressed as a function of time. The data shown corresponds to the four system sizes $L = 32, 64, 128$ and $256$. (b) The data of (a) reexpressed as a function of the variable $z = t^4/L$. In the interests of clarity, only data for $0 < z < 0.4$ is shown and the data points have been suppressed. The full data set is shown in the inset.

Figure 5: (a) The quantity $R^2_M(t)$, defined in the text, and expressed as a function of time. Data is shown for four system sizes $L = 32, 64, 128$ and $256$. Errors are less than 0.5%. The data points have been suppressed for clarity. (b) Log-log plot of the data of (a).

Figure 6: The quantity $R^2_M(t)/t$ expressed as a function of time for the four system sizes $L = 32, 64, 128$ and $256$.

Figure 7: (a) $R^2_M(t)$ for the $L = 128$ system at a selection of temperatures in the range 0–$T_c$. Errors are less than 1%. (b) The gradient of $R^2_M(t)$ at $t = 300$ MCS for various values of $T/T_c$. The dashed line through the data points serves merely to guide the eye. Errors are comparable with symbol sizes. Also shown (solid line) is the theoretical prediction of Grant and Gunton [24], scaled in the manner described in the text. (c) The data of (a) with the time dependence divided out.