DEDEKIND \(\sigma\)-COMPLETE \(\ell\)-GROUPS AND RIESZ SPACES AS VARIETIES

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Abstract. We prove that the category of Dedekind \(\sigma\)-complete Riesz spaces is an infinitary variety, and we provide an explicit equational axiomatization. In fact, we show that finitely many axioms suffice over the usual equational axiomatization of Riesz spaces. Our main result is that \(\mathbb{R}\), regarded as a Dedekind \(\sigma\)-complete Riesz space, generates this category as a quasi-variety, and therefore as a variety. Analogous results are established for the categories of (i) Dedekind \(\sigma\)-complete Riesz spaces with a weak order unit, (ii) Dedekind \(\sigma\)-complete lattice-ordered groups, and (iii) Dedekind \(\sigma\)-complete lattice-ordered groups with a weak order unit.

1. Introduction

Riesz spaces — also known as vector lattices — and the more general lattice-ordered groups are of importance both in analysis (see e.g. [1, 2]) and in algebra (see e.g. [3]). In connections with integration and measure theory the conditionally countably complete Riesz spaces, known as Dedekind \(\sigma\)-complete Riesz spaces, are particularly relevant. We recall that a Riesz space \(G\) is Dedekind \(\sigma\)-complete if for all countable subsets \(S \subseteq G\), if \(S\) admits an upper bound in \(G\), then \(S\) admits a least upper bound. Let us write \(\sigma \mathcal{R}\) for the category whose objects are such Riesz spaces and whose morphisms are the Riesz morphisms (=vector space and lattice homomorphisms) that preserve existing countable suprema. While Riesz spaces and their morphisms form a variety of algebras in the sense of Birkhoff (see e.g. [5]), to our knowledge no equational presentation of the category \(\sigma \mathcal{R}\) is available in the literature. Our first result is an explicit axiomatization of \(\sigma \mathcal{R}\) as an infinitary variety (Theorem 4.3). Specifically, \(\sigma \mathcal{R}\) can be presented as an equationally definable class of algebraic structures with the same primitive operations as Riesz spaces, and one additional operation of countably infinite arity which we write as \(\sup\). Semantically, \(\sup(g, f_1, f_2, \ldots)\) is interpreted in a Dedekind \(\sigma\)-complete Riesz space as \(\sup_{n \geq 1} \{ f_n \wedge g \}\). We prove that, in addition to the usual Riesz space equational axioms, finitely many equations suffice to axiomatize \(\sigma \mathcal{R}\) in this language, and these express very basic properties of \(\sup_{n \geq 1} \{ f_n \wedge g \}\). The set \(\mathbb{R}\) is a fundamental example both of Riesz space and of Dedekind \(\sigma\)-complete Riesz space. It is well known that \(\mathbb{R}\) generates the (finitary) variety of Riesz spaces, see [3, Chapter XV]. Our main result shows that, analogously, \(\mathbb{R}\) generates the infinitary variety of Dedekind \(\sigma\)-complete Riesz spaces (Theorem 6.16). In fact, in Corollary 7.5 we obtain the stronger result that \(\mathbb{R}\) generates this variety as a quasi-variety, too.

Additionally, for each of the results mentioned above we prove an analogous counterpart for the category \(\sigma \mathcal{R}\_u\) whose objects are Dedekind \(\sigma\)-complete Riesz spaces with a designated weak (order) unit, where the morphisms are the Riesz morphisms that preserve both existing countable suprema and the weak unit. Finally,

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we obtain analogous results for corresponding categories of lattice-ordered groups, henceforth shortened to \(\ell\)-groups. For this, we will consider the category \(\sigma\ell\mathcal{G}\) whose objects are Dedekind \(\sigma\)-complete \(\ell\)-groups, and the category \(\sigma\ell\mathcal{G}_u\) whose objects are Dedekind \(\sigma\)-complete \(\ell\)-groups with a designated weak unit.

We assume familiarity with the basic theory of \(\ell\)-groups and Riesz spaces. All needed background can be found, for example, in the standard references [3, 10].

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2. Dedekind \(\sigma\)-complete \(\ell\)-groups are a variety

2.1. Definition of \(\sigma\ell\mathcal{G}\) and \(\mathcal{V}_{\sigma\ell}\).

Definition 2.1. A morphism of \(\ell\)-groups (or \(\ell\)-morphism) \(\varphi: G \to H\) is said to be \(\sigma\)-continuous if \(\varphi\) preserves the existing countable suprema.

Definition 2.2. We denote by \(\sigma\ell\mathcal{G}\) the category whose objects are Dedekind \(\sigma\)-complete \(\ell\)-groups and whose arrows are \(\sigma\)-continuous \(\ell\)-morphisms.

It is well-known that a Dedekind \(\sigma\)-complete \(\ell\)-group is archimedean (if \(a \geq 0\) and \(na \leq b\) for every \(n \geq 1\), then \(a = 0\)) and thus abelian.

Definition 2.3. Let \(\mathcal{V}_{\sigma\ell}\) be the (infinitary) variety described in the following.

Operations of \(\mathcal{V}_{\sigma\ell}\): operations of \(\ell\)-groups and an operation \(\bigvee\) of countably infinite arity. The intended interpretation of \(\bigvee(g, f_1, f_2, \ldots)\) in a Dedekind \(\sigma\)-complete \(\ell\)-group is \(\sup_{n \geq 1}\{f_n \land g\}\), and we adopt the notation

\[
\bigwedge_{n \geq 1} f_n := \bigvee (g, f_1, f_2, \ldots)
\]

Axioms of \(\mathcal{V}_{\sigma\ell}\): The axioms of \(\mathcal{V}_{\sigma\ell}\) are the axioms of \(\ell\)-groups and the following ones (which are seen to be equational, once we rewrite every inequality \(a \leq b\ as \ a \land b = a\))

\[
(A1) \quad \bigvee_{n \geq 1} f_n = \bigvee_{n \geq 1} (f_n \land g);
\]

\[
(A2) \quad \bigvee_{n \geq 1} f_n = (f_1 \land g) \lor \left( \bigvee_{n \geq 2} f_n \right);
\]

\[
(A3) \quad \bigvee_{n \geq 1} (f_n \land h) \leq h.
\]

Lemma 2.4. In \(G \in \mathcal{V}_{\sigma\ell}\), for every \(k \geq 1\), we have

\[
f_k \land g \leq \bigvee_{n \geq 1} f_n.
\]

Proof. By induction on \(k \geq 1\) and applying \((A2)\) we obtain

\[
\bigvee_{n \geq 1} f_n = (f_1 \land g) \lor \cdots \lor (f_k \land g) \lor \left( \bigvee_{n \geq k+1} f_n \right).
\]

Thus \(f_k \land g \leq (f_1 \land g) \lor \cdots \lor (f_k \land g) \lor \left( \bigvee_{n \geq k+1} f_n \right) = \bigvee_{n \geq 1} f_n. \) \(\square\)
2.2. The categories $\sigma\ell G$ and $\mathcal{V}_{\sigma\ell G}$ are isomorphic.

**Proposition 2.5.** Let $G \in \mathcal{V}_{\sigma\ell G}$. In $G$ we have

\[ \bigvee_{n \geq 1} f_n = \sup \{ f_n \land g \} . \]

**Proof.** By Lemma 2.4, $\bigvee_{n \geq 1} f_n$ is an upper bound of $(f_k \land g)_{k \geq 1}$. Suppose now that $f_n \land g \leq h$ for every $n \geq 1$. Then

\[ \bigvee_{n \geq 1} f_n (A1) = \bigvee_{n \geq 1} (f_k \land g) f_n \land g \leq h \bigvee_{n \geq 1} (f_k \land g \land h) \leq h. \]

\[ \square \]

**Corollary 2.6.** Let $G \in \mathcal{V}_{\sigma\ell G}$. Then $G$ is a Dedekind $\sigma$-complete $\ell$-group.

**Proof.** Let $(f_n)_{n \geq 1} \subseteq G$ and $g \in G$ be such that $f_n \leq g$ for all $n \geq 1$. Then

\[ \bigvee_{n \geq 1} f_n \sup_{n \geq 1} \{ f_n \land g \} f_n \leq g \sup_{n \geq 1} f_n. \]

\[ \square \]

**Lemma 2.7.** Let $\varphi : G \to H$ be a morphism in $\mathcal{V}_{\sigma\ell G}$. Then $\varphi$ is $\sigma$-continuous.

**Proof.** Let $(f_n)_{n \geq 1} \subseteq G$ and $f = \sup_{n \geq 1} f_n$. Then

\[ \varphi \left( \sup_{n \geq 1} f_n \right) f_n \leq f \varphi \left( \sup_{n \geq 1} \{ f_n \land f \} \right) \sup_{n \geq 1} \varphi \left( \bigvee_{n \geq 1} \varphi(f_n) \right) = \sup_{n \geq 1} \varphi(f_n) \sup_{n \geq 1} \varphi(f_n). \]

\[ \square \]

We denote by $U$ the forgetful functor

\[ U : \mathcal{V}_{\sigma\ell G} \to \sigma\ell G \]

that assigns to an object $G \in \mathcal{V}_{\sigma\ell G}$ the set $G$, endowed with the operations of $\ell$-group of $G$ (we forget the operation $\bigvee$). For $\varphi : G \to H$ morphism in $\mathcal{V}_{\sigma\ell G}$, we set $U(\varphi) = \varphi$.

**Proposition 2.8.** The functor $U$ is well-defined.

**Proof.** Every $G \in \mathcal{V}_{\sigma\ell G}$ is a Dedekind $\sigma$-complete $\ell$-group by Corollary 2.6.

Moreover, every $\varphi : G \to H$ morphism in $\mathcal{V}_{\sigma\ell G}$ is an $\ell$-morphism (because $\varphi$ preserves the operations of $\mathcal{V}_{\sigma\ell G}$) which, by Lemma 2.7, is $\sigma$-continuous.

\[ \square \]

We denote by $F$ the functor

\[ F : \sigma\ell G \to \mathcal{V}_{\sigma\ell G} \]

that assigns to an object $G \in \sigma\ell G$ the set $G$, endowed with the operations of $\ell$-group of $G$, and enriched with the operation $\bigvee_{n \geq 1} f_n := \sup_{n \geq 1} \{ f_n \land g \}$. Such supremum exists because $G$ is Dedekind $\sigma$-complete and the countable family $(f_n \land g)_{n \geq 1}$ is bounded from above by $g$. For $\varphi : G \to H$ morphism in $\sigma\ell G$, we set $F(\varphi) = \varphi$.

**Proposition 2.9.** The functor $F$ is well-defined.
Proof. Let $G \in \sigma\ell G$. The axioms of $\ell$-groups are obviously satisfied by $F(G)$. The structure of $\ell$-group (and hence the order) is preserved by the action of $F$ on objects. Moreover, (A1), (A2) and (A3) in Definition 2.3 hold in $F(G)$ because in $G$ we have

1. $\sup_{n \geq 1}(f_n \land g) = \sup_{n \geq 1}((f_n \land g) \land g)$;
2. $\sup_{n \geq 1}(f_n \land g) = (f_1 \land g) \lor \sup_{n \geq 2}(f_n \land g)$;
3. $\sup_{n \geq 1}(f_n \land h \land g) \leq h$.

Let $\varphi: G \to H$ be a morphism in $\sigma\ell G$. $\varphi$ preserves the operations of $\ell$-group. Let $(f_n)_{n \geq 1} \subseteq G$ and $g \in G$. Then

$$\varphi \left( \bigvee_{n \geq 1} f_n \right) \overset{\text{def. of } F}{=} \varphi \left( \sup_{n \geq 1} \{f_n \land g\} \right) \overset{\varphi \text{ preserves countable sups}}{=} \sup_{n \geq 1} \varphi(f_n \land g) =$$

$$\varphi \overset{\text{def. of } F}{=} \sup_{n \geq 1} \{\varphi(f_n) \land \varphi(g)\} \overset{\text{def. of } F}{=} \sup_{n \geq 1} \varphi(f_n).$$

Hence, $\varphi$ preserves the operation $\bigvee$.

$\square$

Theorem 2.10. $U: \mathcal{V}_{\sigma\ell G} \to \sigma\ell G$ and $F: \sigma\ell G \to \mathcal{V}_{\sigma\ell G}$ are inverse functors.

Proof. Let $G \in \sigma\ell G$. Then $UF(G) = G$ because both the functor $F$ and the functor $U$ preserve the operations of $\ell$-groups. Moreover, for $\varphi$ a morphism in $\sigma\ell G$, $UF(\varphi) = U(\varphi) = \varphi$.

Let $G \in \mathcal{V}_{\sigma\ell G}$. Then $FU(G) = G$ because both the functor $F$ and the functor $U$ preserve the operations of $\ell$-groups, and the element $\bigvee_{n \geq 1} f_n$ in $FU(G)$ is, by definition of $F$, the element $\sup_{n \geq 1} \{f_n \land g\}$ in $U(G)$, which, by Proposition 2.8, is the element $\bigvee_{n \geq 1} f_n$ in $G$. Moreover, for $\varphi$ morphism in $\mathcal{V}_{\sigma\ell G}$, $FU(\varphi) = F(\varphi) = \varphi$. $\square$

Corollary 2.11. The category of Dedekind $\sigma$-complete $\ell$-groups is an infinitary variety.

3. DEDEKIND $\sigma$-COMPLETE $\ell$-GROUPS WITH WEAK UNIT ARE A VARIETY

Definition 3.1. An element 1 of an $\ell$-group $G$ is a weak (order) unit iff $1 \geq 0$ and, for all $f \in G$,

$$f \land 1 = 0 \Rightarrow f = 0.$$

3.1. Definition of $\sigma \ell G_u$ and $\mathcal{V}_{\sigma \ell G_u}$.

Definition 3.2. We denote by $\sigma \ell G_u$ the category whose objects are Dedekind $\sigma$-complete $\ell$-groups with a designated weak unit 1 and whose morphisms are $\sigma$-continuous $\ell$-morphisms which preserve such designated weak unit.

Definition 3.3. Let $\mathcal{V}_{\sigma \ell G_u}$ be the (infinitary) variety described in the following.

Operations of $\mathcal{V}_{\sigma \ell G_u}$: operations of $\mathcal{V}_{\sigma \ell G}$ (see Definition 2.3) and the constant symbol 1.

Axioms of $\mathcal{V}_{\sigma \ell G_u}$: The axioms of $\mathcal{V}_{\sigma \ell G}$ are the axioms of $\mathcal{V}_{\sigma \ell G}$ (see Definition 2.3) and

$$\bigvee_{n \geq 1} ([f] \land n1) = [f].$$
3.2. Every $G \in \mathcal{V}_{\sigma\ell G_u}$ is a Dedekind $\sigma$-complete $\ell$-group with weak unit.

Every $G \in \mathcal{V}_{\sigma\ell G_u}$ is a Dedekind $\sigma$-complete $\ell$-group in an obvious way, as shown in Section 2.4. This section shows that 1 is a weak unit for $G$.

**Lemma 3.4.** Let $G$ be an abelian $\ell$-group. Let $a, b, c \in G$. If $a \land c = 0$ and $b \land c = 0$, then $(a + b) \land c = 0$.

**Proof.** Let us suppose $a \land c = 0$ and $b \land c = 0$. As a first consequence, $a, b, c \geq 0$. $0 = 0 + 0 = a \land c + b \land c + (a + b) \land c \geq (a + b) \land c \land c = (a + b) \land c \geq 0$. Thus $(a + b) \land c = 0$. $\square$

**Lemma 3.5.** Let $G$ be an abelian $\ell$-group. Let $a \in G$. Then $(na)^- = na^-$. $\square$

**Proposition 3.6.** Let $G \in \mathcal{V}_{\sigma\ell G_u}$. The element 1 of $G$ is a weak unit.

**Proof.**

(1) Claim: $1 \geq 0$.

By Item (1) in Definition 3.3, taking $f = 0$, we have $|0| = |0|$.

Thus $0 = \sup_{n \geq 1} \{|0| \land n1\} = \sup_{n \geq 1} - (n1)^- \leq \sup_{n \geq 1} - (n1)^- = -1^-$. Thus $1^- = 0$, and $1 = 1^+ + 1^- = 1^+ \geq 0$.

(2) Claim: $f \land 1 = 0 \Rightarrow f = 0$.

Suppose $f \land 1 = 0$. As a first consequence, $f \geq 0$. By induction on $n$, Lemma 3.4 proves $f \land n1 = 0$ for every $n \geq 1$. Thus,

$f \leq 0 \Rightarrow f \land 1 = 0 \land n1 = 0. \square$

3.3. The categories $\sigma\ell G_u$ and $\mathcal{V}_{\sigma\ell G_u}$ are isomorphic. We denote by $U_u$ the forgetful functor

$U_u : \mathcal{V}_{\sigma\ell G_u} \to \sigma\ell G_u$

that assigns to an object $G \in \mathcal{V}_{\sigma\ell G_u}$ the set $G$, endowed with the operations of $\ell$-group of $G$, and with the interpretation in $G$ of the constant symbol 1 as the designated weak unit (we forget the operation $\overline{\lor}$). For $\varphi : G \to H$ morphism in $\mathcal{V}_{\sigma\ell G_u}$, we set $U_u(\varphi) := \varphi$.

**Proposition 3.7.** The functor $U_u$ is well-defined.

**Proof.** Let $G \in \mathcal{V}_{\sigma\ell G_u}$. Then $G$ is a Dedekind-$\sigma$-complete $\ell$-group by Corollary 2.9 and the element 1 of $G$ is a weak unit by Proposition 3.6.

Let $\varphi : G \to H$ be a morphism in $\mathcal{V}_{\sigma\ell G_u}$. $\varphi$ is a $\sigma$-continuous $\ell$-morphism, as shown in Proposition 2.3. Moreover, $\varphi$ preserves the element 1, by definition of morphism in $\mathcal{V}_{\sigma\ell G_u}$. $\square$

**Lemma 3.8.** Let $G$ be an $\ell$-group, $I \neq \emptyset$ a set and $(x_i)_{i \in I} \subseteq G$. If $\sup_{i \in I} x_i$ exists, then, for every $a \in G$, $\sup_{i \in I} \{a \land x_i\}$ exists and

$$a \land \left(\sup_{i \in I} x_i\right) = \sup_{i \in I} \{a \land x_i\}.$$ 

**Proof.** See Proposition 6.1.2 in [3]. $\square$
Proposition 3.9. Let $G \in \sigma \ell G_w$, $f \in G^+$. Then
\[ f = \sup_{n \geq 1} \{ f \wedge n1 \}. \]

Proof. Set $f_n := f \wedge n1$ and $g := \sup_{n \geq 1} f_n$.
\[
\begin{align*}
    f_{n+1} &= f \wedge (n+1)1 = f \wedge (n1 + 1) \wedge f = \\
    &= (f \wedge n1 + 1) \wedge f = (f_n + 1) \wedge f.
\end{align*}
\]
\[
\sup_{n \geq 1} f_n \underset{n \geq 2}{\sup} f_n = \sup_{n \geq 1} (f_{n+1} + 1) \wedge f = (g + 1) \wedge f.
\]

From $g = (g + 1) \wedge f$ we obtain $0 = ((g + 1) \wedge f) - g$. By distributivity of $+$ over $\wedge$, we obtain $0 = ((g + 1) - g) \wedge (f - g) = 1 \wedge (f - g)$. Being 1 a weak unit, we obtain $f - g = 0$, hence the thesis. □

We denote by $F_u$ the functor
\[ F_u : \sigma \ell G_u \to V_{\sigma \ell G_u} \]
that assigns to an object $G \in \sigma \ell G_u$ the set $G$, endowed with the operations of $\ell$-group of $G$, and enriched with the operation $\bigvee f_n := \sup_{n \geq 1} \{ f_n \wedge g \}$ and with the designated weak unit of $G$ as the interpretation of the constant symbol 1. For $\varphi : G \to H$ be a morphism in $\sigma \ell G_u$, we set $U_u(\varphi) := \varphi$.

Proposition 3.10. The functor $F_u$ is well-defined.

Proof. Let $G \in \sigma \ell G_u$. Then $F_u(G)$ satisfies the axioms of $V_{\sigma \ell G}$, by Proposition 2.9. Moreover, the axiom $\bigvee (|f| \wedge n1) = |f|$ holds in $F_u(G)$, by Proposition 3.9.

Let $\varphi : G \to H$ be a morphism in $\sigma \ell G_u$. $\varphi$ preserves the operations of $V_{\sigma \ell G}$, as shown in Proposition 2.9. Moreover, $\varphi$ preserves the weak unit 1, by definition of morphism in $\sigma \ell G_u$. □

Theorem 3.11. $U_u : V_{\sigma \ell G} \to \sigma \ell G_u$ and $F_u : \sigma \ell G_u \to V_{\sigma \ell G_u}$ are inverse functors.

Proof. Let $G \in \sigma \ell G_u$. Then $U_u F_u(G) = G$ since both the functor $F_u$ and the functor $U_u$ preserve the operations of $\ell$-groups and the designated unit. Moreover, for $\varphi$ morphism in $\sigma \ell G_u$, $U_u F_u(\varphi) = U_u(\varphi) = \varphi$.

Let $G \in V_{\sigma \ell G_u}$. Then $F_u U_u(G) = G$ because the operations of $V_{\sigma \ell G}$ over $G$ are preserved by $F_u U_u$, as shown in Theorem 2.10 and $1_{F_u U_u(G)} = 1_{U_u(G)} = 1_G$. Moreover, for $\varphi$ morphism in $V_{\sigma \ell G_u}$, $F_u U_u(\varphi) = F_u(\varphi) = \varphi$. □

Corollary 3.12. The category of Dedekind $\sigma$-complete $\ell$-groups with weak unit is an infinitary variety.

4. Dedekind $\sigma$-complete Riesz spaces are a variety

In this section we will omit the proofs, since the assertions can be reduced to (or done in analogy with) the results in Section 2.
4.1. Definition of $\sigma RS$ and $V_{\sigma RS}$.

**Definition 4.1.** We denote by $\sigma RS$ the category whose objects are Dedekind $\sigma$-complete Riesz spaces and whose morphisms are morphisms of Riesz spaces (or Riesz morphisms) which are $\sigma$-continuous.

**Definition 4.2.** Let $V_{\sigma RS}$ be the (infinitary) variety described in the following.

*Operations of $V_{\sigma RS}$:* operations of Riesz spaces and an operation of countably infinite arity $\bigvee$.

*Axioms of $V_{\sigma RS}$:* The axioms of $V_{\sigma RS}$ are the axioms of Riesz spaces and the following ones (which are seen to be equational, once we rewrite every inequality $a \leq b$ as $a \wedge b = a$)

(A1) $\left( \bigvee_{n \geq 1} f_n \right)^g = \left( \bigvee_{n \geq 1} (f_n \wedge g) \right)^g$;  

(A2) $\left( \bigvee_{n \geq 1} f_n \right)^g = \left( f_1 \wedge g \right) \lor \left( \bigvee_{n \geq 2} f_n \right)^g$;  

(A3) $\left( \bigvee_{n \geq 1} (f_n \wedge h) \right) \leq h$.

4.2. The categories $\sigma RS$ and $V_{\sigma RS}$ are isomorphic. We denote by $T$ the forgetful functor

$T: V_{\sigma RS} \to \sigma RS$

that assigns to an object $G \in V_{\sigma RS}$ the set $G$, endowed with the operations of Riesz space of $G$ (we forget the operation $\bigvee$). For $\varphi: G \to H$ morphism in $V_{\sigma RS}$, we set $T(\varphi) := \varphi$.

We denote by $E$ the functor

$E: \sigma RS \to V_{\sigma RS}$

that assigns to an object $G \in \sigma RS$ the set $G$, endowed with the operations of Riesz space of $G$, and enriched with the operation $\left( \bigvee_{n \geq 1} f_n \right)^g := \sup_{n \geq 1} \{f_n \wedge g\}$. For $\varphi: G \to H$ morphism in $\sigma RS$, we set $E(\varphi) := \varphi$.

**Theorem 4.3.** $T: V_{\sigma RS} \to \sigma RS$ and $E: \sigma RS \to V_{\sigma RS}$ are inverse functors.

**Corollary 4.4.** The category of Dedekind $\sigma$-complete Riesz spaces is an infinitary variety.

5. Dedekind $\sigma$-complete Riesz spaces with weak unit are a variety

In this section we will omit the proofs, since the assertions can be reduced to (or done in analogy with) the results in Section 3.

5.1. Definition of $\sigma RS_u$ and $V_{\sigma RS_u}$.

**Definition 5.1.** We denote by $\sigma RS_u$ the category whose objects are Dedekind $\sigma$-complete Riesz spaces with a designated weak unit $1$, and whose morphisms are $\sigma$-continuous Riesz morphisms which preserve such designated weak unit.

**Definition 5.2.** Let $V_{\sigma RS_u}$ be the (infinitary) variety described in the following.

*Operations of $V_{\sigma RS_u}$:* operations of $V_{\sigma RS}$ (see Definition 4.2) and the constant symbol $1$.

*Axioms of $V_{\sigma RS_u}$:* The axioms of $V_{\sigma RS_u}$ are the axioms of $V_{\sigma RS}$ (see Definition 4.2) and

$\left( \bigvee_{n \geq 1} (|f| \wedge 1) \right) = |f|$.
5.2. The categories $\sigma\mathbb{RS}_u$ and $\mathcal{V}_{\sigma\mathbb{RS}_u}$ are isomorphic. We denote by $T_u$ the forgetful functor
\[ T_u : \mathcal{V}_{\sigma\mathbb{RS}_u} \to \sigma\mathbb{RS}_u \]
that assigns to an object $G \in \mathcal{V}_{\sigma\mathbb{RS}_u}$ the set $G$, endowed with the operations of Riesz space of $G$ and with the interpretation in $G$ of the constant symbol $1$ as the designated weak unit (we forget the operation $\vee$). For $\varphi : G \to H$ morphism in $\mathcal{V}_{\sigma\mathbb{RS}_u}$, we set $T_u(\varphi) := \varphi$.

We denote by $E_u$ the functor
\[ E_u : \sigma\mathbb{RS}_u \to \mathcal{V}_{\sigma\mathbb{RS}_u} \]
that assigns to an object $G \in \sigma\mathbb{RS}_u$ the set $G$, endowed with the operations of Riesz space of $G$, and it is enriched with the operation $\vee$ $f_n := \sup_{n \geq 1} \{f_n \wedge g\}$ and with the designated weak unit of $G$ as the interpretation of the constant symbol $1$. For $\varphi : G \to H$ morphism in $\sigma\mathbb{RS}_u$, we set $E_u(\varphi) := \varphi$.

Theorem 5.3. $T_u : \mathcal{V}_{\sigma\mathbb{RS}_u} \to \sigma\mathbb{RS}_u$ and $E_u : \sigma\mathbb{RS}_u \to \mathcal{V}_{\sigma\mathbb{RS}_u}$ are inverse functors.

Corollary 5.4. The category of Dedekind $\sigma$-complete Riesz spaces with weak unit is an infinitary variety.

6. The varieties $\mathcal{V}_{\sigma\mathbb{EG}}$, $\mathcal{V}_{\sigma\mathbb{EG}_u}$, $\mathcal{V}_{\sigma\mathbb{RS}}$, $\mathcal{V}_{\sigma\mathbb{RS}_u}$ are generated by $\mathbb{R}$

The set $\mathbb{R}$ has a canonical structure of $\ell$-group (operations: $\{0, +, -, \vee, \wedge\}$) and of Riesz space (operations: $\{0, +, -, \vee, \wedge\} \cup \{\lambda \cdot - | \lambda \in \mathbb{R}\}$). Additionally, we may consider the operation $\vee f_n = \sup_{n \geq 1} \{f_n \wedge g\}$ and the element $1$ (thought as operation of arity $0$).

It takes some easy verifications to see that
\[
\begin{align*}
(1) \ & \mathbb{R}, \left\{0, +, -, \vee, \wedge, \overline{\vee} \right\} \in \mathcal{V}_{\sigma\mathbb{EG}}; \\
(2) \ & \mathbb{R}, \left\{0, +, -, \vee, \wedge, \overline{\vee}, 1 \right\} \in \mathcal{V}_{\sigma\mathbb{EG}_u}; \\
(3) \ & \mathbb{R}, \left\{0, +, -, \vee, \wedge, \overline{\vee}, \lambda \right\} \cup \{\lambda \cdot - | \lambda \in \mathbb{R}\} \in \mathcal{V}_{\sigma\mathbb{RS}}; \\
(4) \ & \mathbb{R}, \left\{0, +, -, \vee, \wedge, \overline{\vee}, 1 \right\} \cup \{\lambda \cdot - | \lambda \in \mathbb{R}\} \in \mathcal{V}_{\sigma\mathbb{RS}_u}.
\end{align*}
\]

We will show that
\[
\begin{align*}
(1) \ & \text{the variety } \mathcal{V}_{\sigma\mathbb{EG}} \text{ is generated by } \mathbb{R}, \left\{0, +, -, \vee, \wedge, \overline{\vee} \right\}; \\
(2) \ & \text{the variety } \mathcal{V}_{\sigma\mathbb{EG}_u} \text{ is generated by } \mathbb{R}, \left\{0, +, -, \vee, \wedge, \overline{\vee}, 1 \right\}; \\
(3) \ & \text{the variety } \mathcal{V}_{\sigma\mathbb{RS}} \text{ is generated by } \mathbb{R}, \left\{0, +, -, \vee, \wedge, \overline{\vee}, \lambda \right\} \cup \{\lambda \cdot - | \lambda \in \mathbb{R}\}; \\
(4) \ & \text{the variety } \mathcal{V}_{\sigma\mathbb{RS}_u} \text{ is generated by } \mathbb{R}, \left\{0, +, -, \vee, \wedge, \overline{\vee}, 1 \right\} \cup \{\lambda \cdot - | \lambda \in \mathbb{R}\}.
\end{align*}
\]

The proof of these results depends on Theorem 6.5 below. A proof can be found in [8], and can also be recovered from the combination of [6] and [7]. The theorem and its variants have a long history: for a fuller bibliographic account please see [6].

Definition 6.1. Given a set $X$, a $\sigma$-ideal of subsets of $X$ is a family $\mathcal{F}$ of subsets of $X$ such that
Remark 6.3. Let \( \sigma \) be a \( \sigma \)-ideal of subsets of \( X \). For \( f, g \in R^X \), we write \( f \sim_I g \) if \( \{ x \in X \mid f(x) \neq g(x) \} \in I \). \( \sim_I \) is an equivalence relation on \( R^X \) and we denote by \( \overline{R^X}_I \) the quotient of \( R^X \) by \( \sim_I \). The operations of \( \ell \)-groups, the operation \( \overline{\vee} \), and the constant 1 are well-defined in \( \overline{R^X}_I \) in the standard way (for example \( \overline{\vee}_{n \geq 1} [f_n] = \left[ \overline{\vee}_{n \geq 1} f_n \right] \)) with the map \( [-] : R^X \to \overline{R^X}_I \) preserving the designated operation. This works because the operations at issue are finitary or countably infinitary.

Remark 6.2. Let \((A, \leq)\) and \((B, \leq)\) be partially ordered sets, and \( \varphi : A \to B \) a surjective order-preserving map. Then \( \varphi \) preserves every existing supremum.

Remark 6.3. For \( X \) a set and \( I \) a \( \sigma \)-ideal of subsets of \( X \), the \( \ell \)-morphism \( [-] : R^X \to \overline{R^X}_I \) is \( \sigma \)-continuous. Indeed, \( [-] \) is a surjective order-preserving map and therefore, by Remark 6.2, it preserves every existing supremum.

Proposition 6.4. For \( X \) a set and \( I \) a \( \sigma \)-ideal of subsets of \( X \), the \( \ell \)-group \( \overline{R^X}_I \) is Dedekind \( \sigma \)-complete.

Proof. Let \( ([f_n]_{n \geq 1} \in \overline{R^X}_I \) be a countable collection in \( \overline{R^X}_I \) bounded from above by \([g] \). The collection \( (f_n \land g)_{n \geq 1} \) is bounded from above by \( g \) and therefore it admits supremum. By Remark 6.3, also the collection \( ([f_n] \land [g])_{n \geq 1} = ([f_n] \land [g])_{n \geq 1} = ([f_n]_{n \geq 1} \) admits supremum.

Theorem 6.5 (Loonis-Sikorski Theorem for Riesz spaces). Let \( G \) be a Dedekind \( \sigma \)-complete Riesz space. Then there exist a set \( X \), a boolean \( \sigma \)-ideal \( I \) of subsets of \( X \) and an injective \( \sigma \)-continuous Riesz morphism \( \varphi : G \hookrightarrow \overline{R^X}_I \).

Proof. See [8].

6.1. The variety \( V_{\sigma \ell G} \) is generated by \( \mathbb{R} \).

Theorem 6.6 (Loonis-Sikorski Theorem for \( \ell \)-groups). Let \( G \) be a Dedekind \( \sigma \)-complete \( \ell \)-group. Then there exist a set \( X \), a boolean \( \sigma \)-ideal \( I \) of subsets of \( X \) and an injective \( \sigma \)-continuous \( \ell \)-morphism \( \varphi : G \hookrightarrow \overline{R^X}_I \).

Proof. There exists a Dedekind \( \sigma \)-complete Riesz space \( H \) and an injective \( \sigma \)-continuous \( \ell \)-morphism \( \iota : G \hookrightarrow H \); see, e.g., [9]. Applying Theorem 6.3 to the Dedekind \( \sigma \)-complete Riesz space \( H \), we obtain an injective \( \sigma \)-continuous Riesz morphism \( \varphi' : H \hookrightarrow \overline{R^X}_I \). The composition \( \varphi = \varphi' \circ \iota : G \hookrightarrow \overline{R^X}_I \) is an injective \( \sigma \)-continuous \( \ell \)-morphism, since both \( \iota \) and \( \varphi' \) are injective \( \sigma \)-continuous \( \ell \)-morphisms.

Remark 6.7. \( \overline{R^X}_I \) may be thought of as an object of \( V_{\sigma \ell G} \) in two ways:

1. \( \overline{R^X}_I \) inherits from \( R^X \) the structure of \( \ell \)-group. By Proposition 6.4, with such structure \( \overline{R^X}_I \) is Dedekind \( \sigma \)-complete. Hence \( \overline{R^X}_I \in \sigma \ell G \), and \( F(\overline{R^X}_I) \in V_{\sigma \ell G} \).

2. Since \( \mathbb{R} \in V_{\sigma \ell G} \), and \( \sim_I \) is a congruence in the language of \( V_{\sigma \ell G} \), \( \overline{R^X}_I \) inherits a structure of object of \( V_{\sigma \ell G} \) as a quotient of a power of \( \mathbb{R} \). We will denote this object by \( F(\overline{R^X}_I) \).
Lemma 6.8. \( F \left( \frac{R^X}{I} \right) = F(\frac{R^X}{I}) \), i.e. \( F \left( \frac{R^X}{I} \right) \) and \( F(\frac{R^X}{I}) \) identify the same object of \( \mathcal{V}_{\sigma\ell G} \).

Proof. The operations of \( \ell \)-groups are the same in \( F \left( \frac{R^X}{I} \right) \) and \( F(\frac{R^X}{I}) \) because in both cases they are defined in the standard way (for example \([f] \lor [g] := [f \lor g] \)).

Let us see how the operation \( \tilde{V} \) is defined in \( F \left( \frac{R^X}{I} \right) \). Let \( ([f_n])_{n \geq 1} \subseteq F \left( \frac{R^X}{I} \right) \), \( [g] \in F \left( \frac{R^X}{I} \right) \).

\[
\bigg[ \bigg[ \bigg[ \bigg[ \bigg[ [f_n] \bigg] \bigg] \bigg] \bigg] \bigg] = \sup \{ [f_n] \land [g] \} = \sup \{ [f_n] \land g \} \]

We further know \( \sup \{ [f_n] \land g \} (x) = \sup_{n \geq 1} \{ f_n(x) \land g(x) \} \), since the order in the \( \ell \)-group \( R^X \) is defined by \( f \leq g \iff f \lor g = g \), and \( (f \lor g)(x) := f(x) \lor g(x) \).

In \( F(\frac{R^X}{I}) \), \( \tilde{V} \) is defined as follows.

\[
\tilde{V} \left( [g] \right) = \bigg[ \bigg[ \bigg[ \bigg[ \bigg[ [f_n] \bigg] \bigg] \bigg] \bigg] \bigg] = \sup \{ [f_n] \land g \} = \sup \{ [f_n] \land g \} \]

We know \( \left( \sup_{n \geq 1} [f_n] \right) (x) = \sup_{n \geq 1} f_n(x) = \sup_{n \geq 1} \{ f_n(x) \land g(x) \} \).

We have shown that the operation \( \tilde{V} \) is the same in \( F \left( \frac{R^X}{I} \right) \) and \( F(\frac{R^X}{I}) \). \( \square \)

Theorem 6.9. The variety \( \mathcal{V}_{\sigma\ell G} \) is generated by \( R \).

Proof. Let \( G \in \mathcal{V}_{\sigma\ell G} \). \( U(G) \in \sigma\ell G \). By Theorem 6.6 we have an injective \( \sigma \)-continuous \( \ell \)-morphism \( \varphi: U(G) \hookrightarrow \frac{R^X}{I}, \) \( U(G) \in \sigma\ell G, \frac{R^X}{I} \in \sigma\ell G \) (as shown in Item (1) in Remark 6.7), and \( \varphi \) is a morphism in \( \sigma\ell G \). We can therefore apply the functor \( F \) to \( \varphi \), obtaining an injective morphism in \( \mathcal{V}_{\sigma\ell G} \)

\[
F(\varphi): F(U(G)) \hookrightarrow F \left( \frac{R^X}{I} \right)
\]

We see that \( F(\varphi) \) injects \( G \) in \( F \left( \frac{R^X}{I} \right) \), which, by Lemma 6.8, is a quotient of a power of \( R \) in \( \mathcal{V}_{\sigma\ell G} \). \( \square \)

6.2. The variety \( \mathcal{V}_{\sigma\ell G_u} \) is generated by \( R \).

Lemma 6.10. Let \( \varphi: G \twoheadrightarrow H \) be a surjective \( \ell \)-morphism. For \( h \in H^+ \), there exists an element \( g \in G^+ \) such that \( h = \varphi(g) \).

Proof. Let \( f \) be such that \( \varphi(f) = h \). Then \( \varphi(f^+) = \varphi(f)^+ = h^+ = h \). Thus \( g := f^+ \) satisfies the desired properties. \( \square \)

Theorem 6.11 (Loomis-Sikorski Theorem for \( \ell \)-groups with weak unit). Let \( G \) be a Dedekind \( \sigma \)-complete \( \ell \)-group with weak unit. Then there exist a set \( X \), a boolean \( \sigma \)-ideal \( I \) of subsets of \( X \) and an injective \( \sigma \)-continuous \( \ell \)-morphism \( \varphi: G \hookrightarrow \frac{R^X}{I} \) such that \( \varphi(1_G) = [1_{R^X}] \).

Proof. By Theorem 6.6, we have an injective \( \sigma \)-continuous \( \ell \)-morphism \( \psi: G \hookrightarrow \frac{R^Y}{I} \).

Since \( 1_G \geq 0 \), we have \( \psi(1_G) \geq 0 \). By Lemma 6.10, we can choose an element \( u \in (R^Y)^+ \) such that \( [u]_I \psi(1_G) \). Define \( X := \{ x \in X \mid u(x) > 0 \} \). Define \( I \) := \( \{ J \cap X \mid J \in \mathcal{F} \} \). Define \( I \) := \( \{ J \subseteq X \mid J \in \mathcal{F} \} \). Note that \( I \) is a \( \sigma \)-ideal of subsets
We have \( \text{ker}([-\|\cdot\| \circ r]) \subseteq \text{ker}([-\|\cdot\| \circ m]) \). Indeed, let \( f \in \text{ker}([-\|\cdot\|]) \). Then \( \{y \in Y \mid f(y) \neq 0\} \subseteq J \). Hence \( f_{|X} \circ \mu \neq 0 \). i.e. \( f \in \text{ker}([-\|\cdot\| \circ -\|\cdot\|]) \).

Since \( \ker([-\|\cdot\|]) \subseteq \ker([-\|\cdot\| \circ -\|\cdot\|]) \), by the universal property of the quotient there exists a unique \( \ell \)-morphism \( \rho: \mathbb{R}^Y \rightarrow \mathbb{R}^X \) such that the following diagram commutes.

Being \( -\|\cdot\| \) and \( -\|\cdot\| \) surjective, \( \rho \) must be surjective. By Remark 6.2, \( \rho \) preserves the existing countable suprema.

Define the map \( m: \mathbb{R}^X \xrightarrow{\sim} \mathbb{R}^X \) by \( (m(f))(x) = \frac{1}{\mu(x)} f(x) \). Note that, by definition of \( X \), \( u(x) > 0 \) for every \( x \in X \), and therefore the map is well defined. For every \( \lambda \in \mathbb{R}, \lambda > 0 \), the map \( m\lambda: \mathbb{R} \xrightarrow{\sim} \mathbb{R} \) defined by \( g \mapsto \lambda g \) is an \( \ell \)-isomorphism. This suggests that \( m \) is an \( \ell \)-isomorphism. In fact, its inverse is the \( \ell \)-morphism \( m^{-1}: \mathbb{R}^X \xrightarrow{\sim} \mathbb{R}^X \) defined by \( (m^{-1}(g))(x) = u(x)g(x) \).

Since \( -\|\cdot\| \) and \( -\|\cdot\| \circ m \) are surjective \( \ell \)-morphism with same kernel, using the universal property of quotients we obtain that there exists an \( \ell \)-isomorphism \( \eta: \mathbb{R}^X \xrightarrow{\sim} \mathbb{R}^X \) which makes the following diagram commute.
We have the following diagram.

\[
\begin{array}{cccc}
\mathbb{R}^Y & \xrightarrow{-|X} & \mathbb{R}^X & \xrightarrow{m} & \mathbb{R}^X \\
\downarrow & & \downarrow & & \downarrow \\
I & \xrightarrow{\psi} & \mathbb{R}^Y & \xrightarrow{\rho} & \mathbb{R}^X \\
\end{array}
\]

\[G \xleftarrow{\psi} \mathbb{R}^Y \xrightarrow{\rho} \mathbb{R}^X \]

Define \( \varphi := \eta \circ \rho \circ \psi : G \to \mathbb{R}^X \). Then \( \varphi \) has the desired properties. Indeed,

1. \( \varphi \) is a \( \sigma \)-continuous \( \ell \)-morphism since \( \eta, \rho \) and \( \psi \) are such.
2. We show that \( \varphi \) is injective. In fact, being \( \eta \) an \( \ell \)-isomorphism, it is enough to show that \( \rho \circ \psi \) is injective. Let \( g \in G \) be such that \( (\rho \circ \psi)(g) = 0 \). Let \( f \in \mathbb{R}^Y \) such that \( [f]_\mathcal{I} = \psi(g) \). Then \( [f]_\mathcal{I} \mathcal{I} = \rho([f]_\mathcal{I}) = \rho(\psi(g)) = 0 \). This means \( I := \{ x \in X \mid f(x) \neq 0 \} \neq \emptyset \). Let \( \{ y \in Y \mid (u \land [f])(y) \neq 0 \} \neq \emptyset \). Hence \( 0 = [u \land [f]]_\mathcal{I} = [u]_\mathcal{I} \land ([f]_\mathcal{I} = \psi(1_G) \land \psi(g)) = \psi(1_G \land g) \). By injectivity of \( \psi \), it follows that \( 1_G \land g = 0 \), hence \( g = 0 \), hence \( 0 = g \).
3. \( \varphi(1_G) = [1]_\mathbb{R}^X \). Indeed

\[
\varphi(1_G) = \eta \rho \psi(1_G) = \eta \rho([u]_\mathcal{I}) = [m(r(u))]_\mathcal{I} = [m(u)_\mathcal{I}]_\mathcal{I}
\]

Thus \( m(u)_\mathcal{I} = 1 \).

Lemma 6.12. \([1]_\mathbb{R}^X \) is a weak unit for \( \mathbb{R}^X \).

**Proof.** First, \([1] \geq [0] \) since \([\cdot]\) preserves the order. Second, let \([1] \land [f] = 0 \). Then \([1] \land [f] = 0 \). Thus \( \{ x \in X \mid f(x) \neq 0 \} \neq \emptyset \). Hence \( [f] = 0 \).

Remark 6.13. \( \mathbb{R}^X \) may be thought of as an object of \( \mathcal{V}_{\sigma \ell G_u} \) in two ways:

1. \( \mathbb{R}^X \) inherits from \( \mathbb{R}^X \) the structure of \( \ell \)-group. By Proposition 6.3, with such structure \( \mathbb{R}^X \) is Dedekind \( \sigma \)-complete. By Lemma 6.12, \([1]_\mathbb{R}^X \) is a weak unit for \( \mathbb{R}^X \). Thus \( \mathbb{R}^X \in \sigma \ell G_u \), and \( F_u \left( \mathbb{R}^X \right) \in \mathcal{V}_{\sigma \ell G_u} \).
2. Since \( \mathbb{R} \in \mathcal{V}_{\sigma \ell G_u} \), \( \sim_\mathcal{I} \) is a congruence in the language of \( \mathcal{V}_{\sigma \ell G_u} \), \( \mathbb{R}^X \) inherits a structure of object of \( \mathcal{V}_{\sigma \ell G_u} \) as quotient of a power of \( \mathbb{R} \). We will denote this object as \( F_u \left( \mathbb{R} \right)^X \).

Lemma 6.14. \( F_u \left( \mathbb{R}^X \right) = F_u \left( \mathbb{R} \right)^X \), i.e. \( F_u \left( \mathbb{R}^X \right) \) and \( F_u \left( \mathbb{R} \right)^X \) identify the same object of \( \mathcal{V}_{\sigma \ell G_u} \).

**Proof.** As far as it concerns the operations of \( \ell \)-groups and the operation \( \sim_\mathcal{I} \), they are defined in the same ways in the two structures, as already shown in Lemma 6.8. Moreover, in \( F_u \left( \mathbb{R}^X \right) \), the constant 1 is defined as \( 1_{\mathbb{R}^X} := [1]_\mathbb{R}^X \), which coincides with the interpretation of the constant symbol 1 in \( F_u \left( \mathbb{R} \right)^X \).

Theorem 6.15. The variety \( \mathcal{V}_{\sigma \ell G_u} \) is generated by \( \mathbb{R} \).
Proof. Let $G \in \mathcal{V}_{\sigma\ell G_u}$. $U_u(G) \in \sigma\ell G_u$. By Theorem 6.11 we have an injective $\sigma$-continuous $\ell$-morphism $\varphi: U_u(G) \hookrightarrow \mathbb{R}^X$ such that $\varphi(1_G) = [1_{\mathbb{R}^X}]$. $U_u(G) \in \sigma\ell G_u$, $\mathbb{R}^X \in \sigma\ell G_u$ (as shown in Item 1 in Remark 6.13) and $\varphi$ is a morphism in $\sigma\ell G_u$. We can therefore apply the functor $F_u$ to $\varphi$, and we obtain an injective morphism in $\mathcal{V}_{\sigma\ell G_u}$

$$F_u(\varphi): F_u(U_u(G)) \hookrightarrow F_u \left( \mathbb{R}^X \right)$$

$F_u(\varphi)$ injects $G$ in $F_u \left( \mathbb{R}^X \right)$, which, by Lemma 6.14 is a quotient of a power of $\mathbb{R}$. □

6.3. The varieties $\mathcal{V}_{\sigma \mathbb{R}_\Sigma}$ and $\mathcal{V}_{\sigma \mathbb{R}_u}$ are generated by $\mathbb{R}$. The proofs of the following two theorems are analogous to the proofs of Theorem 6.9 and Theorem 6.15.

**Theorem 6.16.** The variety $\mathcal{V}_{\sigma \mathbb{R}_\Sigma}$ is generated by $\mathbb{R}$.

**Theorem 6.17.** The variety $\mathcal{V}_{\sigma \mathbb{R}_u}$ is generated by $\mathbb{R}$.

6.4. Conclusion. Summing Section 6 up, we have the following results.

**Theorem 6.18.** (1) The variety $\mathcal{V}_{\sigma \ell G}$ is generated by $\left( \mathbb{R}, \left\{ 0, +, -, \lor, \land, \neg \right\} \right)$;

(2) The variety $\mathcal{V}_{\sigma \ell G_u}$ is generated by $\left( \mathbb{R}, \left\{ 0, +, -, \lor, \land, \neg, 1 \right\} \right)$;

(3) The variety $\mathcal{V}_{\sigma \mathbb{R}_\Sigma}$ is generated by $\left( \mathbb{R}, \left\{ 0, +, -, \lor, \land, \neg \right\} \cup \left\{ \lambda \cdot - | \lambda \in \mathbb{R} \right\} \right)$;

(4) The variety $\mathcal{V}_{\sigma \mathbb{R}_u}$ is generated by $\left( \mathbb{R}, \left\{ 0, +, -, \lor, \land, \neg, 1 \right\} \cup \left\{ \lambda \cdot - | \lambda \in \mathbb{R} \right\} \right)$.

7. The quasi-varieties $\mathcal{V}_{\sigma \ell G}, \mathcal{V}_{\sigma \ell G_u}, \mathcal{V}_{\sigma \mathbb{R}_\Sigma}, \mathcal{V}_{\sigma \mathbb{R}_u}$ are generated by $\mathbb{R}$

**Definition 7.1.** Given a similarity type $\mathcal{L}$, a quasi-equation with countably many premises (in $\mathcal{L}$) is an expression of the following form.

$$\tau_1 = \rho_1 \quad \tau_2 = \rho_2 \quad \Rightarrow \tau = \rho$$

where $\tau$ is a set and $\tau, \rho, \tau_1, \rho_1, \tau_2, \rho_2, \ldots$ are terms in $\mathcal{L}$. We say that such a quasi-equation holds in an $\mathcal{L}$-algebra $A$ if the conclusion holds in $A$ whenever each premise hold in $A$.

**Example 7.2.** One example of quasi-equation with countably many premises is the archimedean property.

If $a \geq 0$ and $na \leq b$ for every $n \geq 1$, then $a = 0$.

**Lemma 7.3.** For $a, b, a_1, b_1, a_2, b_2, \ldots \in \mathbb{R}$, if

$$[a_1 = b_1, a_2 = b_2, \ldots] \Rightarrow a = b$$

then

$$\bigvee_{n,k \geq 1} k|a_n - b_n| = |a - b|.$$
Proof. Let us suppose \([a_1 = b_1, a_2 = b_2, \ldots] \Rightarrow a = b\). This means \(a = b\) or there exists \(m \geq 1\) such that \(a_m \neq b_m\). If \(a = b\), then \(\bigvee_{n,k \geq 1} k|a_n - b_n| = 0 = |a - b|\). If, instead, \(a_m \neq b_m\) for some \(m \geq 1\), then \(|a - b| \geq \bigvee_{n,k \geq 1} k|a_n - b_n| \geq |a_m - b_m| \geq 0\).

\[0 = |a - b|\]

\[\bigvee_{n,k \geq 1} k|a_n - b_n| \geq |a - b|\].

\[\square\]

**Theorem 7.4.** Let \(\mathbb{Z}\) be the variety generated by \((\mathbb{R}, \mathcal{L})\), where \(0, +, -, \vee, \land, \overline{\vee} \in \mathcal{L}\). If a quasi-equation with countably many premises in the language \(\mathcal{L}\) holds in \(\mathbb{R}\), then it holds in every \(G \in \mathbb{Z}\).

Proof. Let us suppose that the quasi-equation

\[\tau_1 = \rho_1, \tau_2 = \rho_2, \ldots \Rightarrow \tau = \rho\]

holds in \(\mathbb{R}\). By Lemma 7.3, the equation

\[\bigvee_{n,k \geq 1} k|\tau_n - \rho_n| = |\tau - \rho|\]

holds in \(\mathbb{R}\), and therefore in every \(G \in \mathbb{Z}\). We now see that in such a \(G\), if \(\tau_n = \rho_n\) for all \(n\), then

\[|\tau - \rho| = \bigvee_{n,k \geq 1} k|\tau_n - \rho_n| = \bigvee_{n,k \geq 1} 0 = 0\]

In conclusion, \(\tau = \rho\).

\[\square\]

**Corollary 7.5.** If a quasi-equation with countably many premises in the language of \(\mathcal{V}_{\sigma G}\) (resp. \(\mathcal{V}_{\sigma RS}, \mathcal{V}_{\sigma RS_u}\)) holds in \(\mathbb{R}\), then it holds in every object of \(\mathcal{V}_{\sigma G}\) (resp. \(\mathcal{V}_{\sigma RS}, \mathcal{V}_{\sigma RS_u}\)).

Proof. By Theorem 6.18, the hypotheses of Theorem 7.4 are satisfied by \(\mathcal{V}_{\sigma G}, \mathcal{V}_{\sigma RS}\) and \(\mathcal{V}_{\sigma RS_u}\), and the thesis follows.

\[\square\]

**Example 7.6.** We illustrate Corollary 7.5 with an example. Since the archimedean property is a quasi-equation with countably many premises, and since in \(\mathbb{R}\) such property is easily seen to hold, by Corollary 7.5 the well-known fact that every Dedekind \(\sigma\)-complete \(\ell\)-group is archimedean immediately follows.

**Example 7.7.** We illustrate Corollary 7.5 with another example. Corollary 7.5 ensures that the following statement, which is a particular case of the general distributivity law in Lemma 3.3, can be proved just by checking its validity for \(G = \mathbb{R}\).

Let \(G\) be a Dedekind \(\sigma\)-complete \(\ell\)-group, and let \((x_n)_{n \geq 1} \subseteq G\). If \(\sup_{n \geq 1} x_n\) exists, then, for every \(a \in G\), \(\sup_{n \geq 1} \{a \land x_n\}\) exists and

\[a \land \left(\sup_{n \geq 1} x_n\right) = \sup_{n \geq 1} \{a \land x_n\}\]

Indeed, the statement is equivalent to

\[b = \sup_{n \geq 1} x_n \Rightarrow a \land b = \sup_{n \geq 1} \{a \land x_n\}\]

i.e. to a conjunction of quasi-equations with countably many premises, as the next proposition shows.
Proposition 7.8. Let $G \in \mathcal{V}_{\sigma \ell G}$, $(f_n)_{n \geq 1} \subseteq G$ and $g \in G$. Then

\[ g = \sup_{n \geq 1} f_n \iff \begin{cases} 
  g = \bigvee_{n \geq 1} f_n \\
  f_1 \wedge g = f_1 \\
  f_2 \wedge g = f_2 \\
  \vdots 
\end{cases} \]

Proof. If $g = \sup_{n \geq 1} f_n$, then $f_n \wedge g = f_n$ for every $n \geq 1$. Moreover, by Proposition 2.5:

\[ \bigvee_{n \geq 1} f_n = \sup_{n \geq 1} \{ f_n \wedge g \} = \sup_{n \geq 1} f_n = g. \]

Suppose now $f_n \wedge g = f_n$ for every $n \geq 1$ and $g = \bigvee_{n \geq 1} f_n$. Then, by Proposition 2.5:

\[ g = \bigvee_{n \geq 1} f_n = \sup_{n \geq 1} \{ f_n \wedge g \} = \sup_{n \geq 1} f_n. \]

□

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