Signed Enumeration of Upper-Right Corners in Path Shuffles

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Abstract

We resolve a conjecture of Albert and Bousquet-Mélou enumerating quarter-planar walks with fixed horizontal and vertical projections according to their upper-right-corner count modulo 2. In doing this, we introduce a signed upper-right-corner count statistic. We find its distribution over planar walks with any choice of fixed horizontal and vertical projections. Additionally, we prove that the polynomial counting loops with a fixed horizontal and vertical projection according to the absolute value of their signed upper-right-corner count is \((x + 1)^{\text{positive}}\). Finally, we conjecture an equivalence between \((x + 1)^{\text{positive}}\) of the generating function for upper-right-corner count and signed upper-right-corner count, leading to a reformulation of a conjecture of Albert and Bousquet-Mélou on which their asymptotic analysis of permutations sortable by two stacks in parallel relies.

1 Introduction

When studying walks on \(\mathbb{Z}^2\) with unit steps, it is natural to restrict oneself to walks which project vertically on a fixed vertical path \(V\) and horizontally on a fixed horizontal path \(H\), where \(V\) (resp. \(H\)) comprises North and South steps (resp. East and West steps) \([1]\). Such paths correspond with shuffles, or interleavings, of \(V\) and \(H\). If \(|V| = v\) and \(|H| = h\), there are \(\binom{v + h}{h}\) such shuffles. It will be convenient to denote East, West, North, and South steps by \(\rightarrow\), \(\leftarrow\), \(\uparrow\), and \(\downarrow\).

Given a path, several natural statistics arise. In this paper, we focus on variants of peak-count, which counts the number of NW (i.e., \(\Uparrow\)) and ES (i.e., \(\Uparrow\)) corners. If the path lies within the first quadrant, then the peak-count is visually the number of peaks pointing away from the origin. Equivalently, peak-count counts occurrences of \(\rightarrow\downarrow\) and \(\uparrow\leftarrow\) in a shuffle of \(V\) and \(H\). Albert and Bousquet-Mélou introduced peak-count and showed it to have applications in the study of permutations sortable by two stacks in parallel \([1]\). They also proved that peak-count has the same distribution over shuffles as do several other statistics (Proposition 14 of \([1]\), which was originally observed by Julien Courtiel and Olivier Bernardi independently).
Albert and Bousquet-Mélou studied several generating functions involving peak-count of planar walks in depth. Although very little is known in the case where the path’s horizontal and vertical projections are fixed, Albert and Bousquet-Mélou posed the following conjecture, which they attribute to Julien Courtiel [1].

**Conjecture 1.1** (Albert and Bousquet-Mélou, Conjecture (P1) on pp. 32 [1]).

Let \( H \) (resp. \( V \)) be a path beginning and ending at the origin of half-length \( i \) (resp. \( j \)) on the alphabet \( \{\rightarrow, \leftarrow\} \) (resp. \( \{\uparrow, \downarrow\} \)). Then the polynomial that counts walks of the shuffle class of \( HV \) according to the number of \( \uparrow \) and \( \downarrow \) corners takes the value \((i+j)i\) when evaluated at \(-1\). Equivalently, the shuffle class of \( HV \) contains \((i+j)i\) more shuffles with even peak-count than with odd peak-count.

Albert and Bousquet-Mélou note that, among other things, proving the conjecture would eliminate the need for the somewhat lengthy proof of their Proposition 15 [1].

We prove Conjecture 1.1 and extend it to the case where \( H \) and \( V \) are arbitrary words on the alphabets \( \{\rightarrow, \leftarrow\} \) and \( \{\uparrow, \downarrow\} \) respectively. This corresponds with considering arbitrary walks on the plane, rather than only planar loops.

In order to study peak-count modulo 2, it is sufficient to study what we call **signed peak-count**, which is the number of \( \uparrow \) corners minus the number of \( \downarrow \) corners. While signed peak-count and peak-count are guaranteed to share the same parity, it turns out that signed peak-count exhibits nice behavior allowing for it to be completely enumerated.

**Theorem 1.2.** Let \( V \) be a word comprising \( u \uparrow \)’s and \( d \downarrow \)’s. Let \( H \) be a word comprising \( r \rightarrow \)’s and \( l \leftarrow \)’s. Then the number of shuffles of \( V \) and \( H \) with signed peak-count \( k \) is

\[
\binom{r+u}{u-k} \binom{l+d}{d+k}.
\]

Note that Theorem 1.2 immediately leads to a formula for the difference between the number of even and odd peak-count shuffles of \( V \) and \( H \).

\[
\sum_k (-1)^k \binom{r+u}{u-k} \binom{l+d}{d+k}.
\]

(1)

In certain key cases, Formula (1) simplifies. When \( r = l \) and \( u = d \) (as is the case in Conjecture 1.1), Formula (1) becomes

\[
\sum_k (-1)^k \binom{r+u}{u-k} \binom{r+u}{u+k} = \binom{r+u}{u}.
\]

by equation (30) of [6].

Additionally, when \( r = u \) and \( l = d \), Formula (1) becomes

\[
\sum_k (-1)^k \binom{2r}{r-k} \binom{2l}{l+k},
\]

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which is known [6, Equation 29] to be the super Catalan number,

\[ S(r, k) = \frac{(2r)! (2k)!}{r! k! (r + k)!}. \]

Finding a combinatorial interpretation for the super Catalan number is an open problem; when \( r \leq 3 \) or \( |r - k| \leq 4 \), combinatorial interpretations have been found (\([2], [3]\)). Note that our result does not yield a combinatorial interpretation because it examines the difference between the cardinalities of two sets, rather than the cardinality of a single set. As far as we know, however, it is the first non-contrived example of a combinatorial problem in which the super Catalan numbers appear.

Peak-count is intimately connected to the study of permutations sortable by two stacks in parallel. Albert and Bousquet-Mélou use the statistic to both characterize the generating function enumerating such permutations and to analyze the asymptotic number of them \([1]\). The asymptotic analysis, in particular, brings peak-count to the spotlight, as the analysis relies on two conjectures about peak-count which remain open. In addition to the results already discussed, this paper makes progress on the first of these conjectures (Conjecture 10 of \([1]\)), which concerns the \((x + 1)\)-positivity of the generating function for peak-count of quarter-planar loops. In particular, our results reduce the conjecture to a new conjecture of a different flavor concerning the relationship between peak-count and signed peak-count.

The study of permutations sortable by two stacks in parallel traces back to Section 2.2.1 of *The Art of Computer Programming* \([7]\), in which Knuth characterized the permutations sortable by a single stack, as well as those sortable by a double-ended queue. This spawned the works of Even and Itai \([5]\), Pratt \([8]\), and Tarjan \([10]\) which extended Knuth’s investigations to more general networks of stacks and queues, including the permutations sortable by two stacks in parallel. The 1971 paper of Even and Itai \([5]\) characterized the permutations sortable by two stacks in parallel, allowing for the detection of such a permutation in polynomial time. The 1973 paper of Pratt \([8]\) then characterized the list of minimal permutations which cannot be sorted by two stacks in parallel, connecting the problem to the rapidly growing field of permutation pattern avoidance.\(^1\) Despite this, however, the enumeration of the permutations sortable by two stacks in parallel remained open for more than forty years, until finally being solved by Albert and Bousquet-Mélou in 2014 along with the introduction of peak-count.

As mentioned previously, Albert and Bousquet-Mélou’s asymptotic analysis of the number of sortable permutations relies on two conjectures concerning peak-count \([1]\). We make progress on the first:

\(^1\)Interestingly, the origin of the study of permutation pattern avoidance is often also traced back to Section 2.2.1 of *The Art of Computer Programming* \([7]\). The first serious study of the combinatorial structures of pattern-avoiding permutations, however, was conducted by Simion and Schmidt in 1985 \([9]\).

\(^2\)Several notions of an enumeration exist. Here we mean a counting technique which can be performed in polynomial time. This excludes, for example, the 2012 algorithm of Denton \([4]\) which counts the permutations in time \(O(n^5 2^n)\).
Conjecture 1.3 (Albert, Bousquet-Mélou, Conjecture 10 on pp. 12 [1]). *The generating function as a function of a variable $x$ for peak-count of quarter-planar loops of a given length can be expressed as a positive polynomial in $\mathbb{Z}[x+1]$ (i.e., is $(x+1)$-positive).*

In Section 3, we prove a variant of Conjecture 1.3 applying to signed peak-count rather than to peak-count. In particular, we prove that the polynomial counting loops according to the absolute value of their signed peak-count is $(x+1)$-positive, even when we restrict ourselves to loops with fixed horizontal and vertical projections. In Section 4, we go on to pose Conjecture 4.2 which reduces the study of $(x+1)$-positivity of peak-count to the study of $(x+1)$-positivity of the absolute value of signed peak-count. Due to our results on $(x+1)$-positivity of absolute signed peak-count, it follows that Conjecture 4.2 implies Conjecture 1.3.

The outline of this paper is as follows. In Section 2, we enumerate signed peak-count of shuffles; additionally, we enumerate peak-count modulo 2. In Section 3, we prove our results on $(x+1)$-positivity. In Section 4, we conclude with conjectures and directions of future work.

## 2 Enumerating Signed peak-count

In this section, we derive a formula for the number of shuffles with a given signed peak-count. To accomplish this, we introduce a subtly related statistic called shifted In-Vert and, under certain conditions, we provide a bijection establishing the equidistribution of shifted In-Vert and signed peak-count. We begin by introducing some conventions.

For the rest of this section, let $r, l, u, d$ be non-negative integers. Moreover, fix $\mathcal{V}$ (resp. $\mathcal{H}$) to be a word comprising $r \rightarrow$ steps (resp. $u \uparrow$ steps) and $l \leftarrow$ steps (resp. $d \downarrow$ steps). In particular, $\rightarrow, \leftarrow, \uparrow$, and $\downarrow$ correspond to East, West, North, and South respectively.

**Definition 2.1.** A shuffle of $\mathcal{V}$ and $\mathcal{H}$ is an intertwining of the steps in $\mathcal{V}$ and $\mathcal{H}$. In addition, a shuffle has the $\uparrow$ in its zero-th position.

**Remark 2.2.** The convention of $\uparrow$ preceding the first step is non-standard. But it will facilitate the definition of a statistic called shifted In-Vert.

We will use $\Sigma$ to denote the set of shuffles of $\mathcal{V}$ and $\mathcal{H}$.

**Example 2.3.** If $\mathcal{V} = \uparrow \downarrow$ and $\mathcal{H} = \rightarrow \rightarrow$, then $\Sigma$ is the set:

$$\{ \uparrow \rightarrow \rightarrow \uparrow \downarrow, \uparrow \rightarrow \uparrow \rightarrow \downarrow, \uparrow \rightarrow \uparrow \rightarrow \downarrow, \uparrow \rightarrow \uparrow \rightarrow \rightarrow, \uparrow \rightarrow \rightarrow \rightarrow \rightarrow \}$$
Definition 2.4. For $\sigma \in \Sigma$, we use $\#(A, B)_\sigma$ to denote the number of occurrences in $\sigma$ of an element of set $A$ followed immediately by an element of set $B$. If $A$ or $B$ contains only a single element, we omit set braces around that element. For example, $A = \{\leftarrow\}$ is written as $A = \leftarrow$.

Definition 2.5. We use the following short-hands for certain sets: Horiz = $\{\rightarrow, \leftarrow\}$ is the set of horizontal steps, Vert = $\{\uparrow, \downarrow\}$ is the set of vertical steps, In = $\{\leftarrow, \downarrow, \uparrow\}$ is the set of inward steps, and Out = $\{\rightarrow, \uparrow\}$ is the set of outward steps.

The names inward and outward take the perspective of the first quadrant of the plane, in which $\leftarrow$, $\downarrow$ and $\uparrow$ all point inward towards the origin, and both $\rightarrow$ and $\uparrow$ point outward away from the origin.

Example 2.6. The value $\#(\text{In}, \uparrow)_\sigma$ counts instances of a horizontal step followed by an $\uparrow$. For example, if $\sigma = \overline{\downarrow}, \uparrow, \leftarrow, \downarrow, \rightarrow, \overline{\downarrow}, \uparrow, \uparrow$, then $\#(\text{In}, \uparrow)_\sigma = 2$ and the instances are over-lined.

Definition 2.7. The signed peak-count of $\sigma \in \Sigma$ is $\#(\uparrow, \leftarrow)_\sigma - \#(\rightarrow, \downarrow)_\sigma$.

We begin by reformulating signed peak-count:

Lemma 2.8. The signed peak-count of $\sigma \in \Sigma$ can be expressed as $\#(\uparrow, \text{In})_\sigma - \#(\text{Out}, \downarrow)_\sigma$.

Proof. By definition, signed peak-count is $\#(\uparrow, \leftarrow)_\sigma - \#(\rightarrow, \downarrow)_\sigma$. This is equal to $\#(\uparrow, \leftarrow)_\sigma + \#(\uparrow, \downarrow)_\sigma - \#(\rightarrow, \downarrow)_\sigma - \#(\uparrow, \downarrow)_\sigma$, which combines to

$\#(\uparrow, \text{In})_\sigma - \#(\text{Out}, \downarrow)_\sigma$.

Next we introduce shifted In-Vert, which is easier to study than peak-count. Interestingly, Proposition 2.14 will show that the two statistics are equidistributed in certain key cases.

Definition 2.9. The In-Vert of a shuffle $\sigma \in \Sigma$ is $\#(\text{In}, \text{Vert})_\sigma$. The shifted In-Vert is $\#(\text{In}, \text{Vert})_\sigma - d$. (Recall that $d$ is the number of $\downarrow$’s in $\sigma$.)

It will be useful to have the following reformulation of shifted In-Vert.

Lemma 2.10. The shifted In-Vert of $\sigma \in \Sigma$ can be expressed as $\#(\text{In}, \uparrow)_\sigma - \#(\text{Out}, \downarrow)_\sigma$.

Proof. Because shuffles cannot begin with $\downarrow$ (due to the $\swarrow$), every $\downarrow$ is preceded by either an inward or an outward step. Consequently, we can rewrite $d$ as $(\#(\text{Out}, \downarrow)_\sigma + \#(\text{In}, \downarrow)_\sigma)$. Thus the shifted In-Vert $\#(\text{In}, \text{Vert})_\sigma - d$ expands to $(\#(\uparrow)_\sigma + \#(\text{In}, \downarrow)_\sigma) - (\#(\text{Out}, \downarrow)_\sigma + \#(\text{In}, \downarrow)_\sigma)$.
which simplifies to
\[ \#(\text{In}, \uparrow)_\sigma - \#(\text{Out}, \downarrow)_\sigma. \]

Next we construct an involution \( f : \Sigma \to \Sigma \) such that, under certain conditions, the shifted In-Vert of \( f(\sigma) \) is the same as the signed peak-count of \( \sigma \).

**Definition 2.11.** A out-run in \( \sigma \in \Sigma \) is a (non-empty) maximal contiguous subsequence of outward steps.

**Definition 2.12.** The flip \( f(\sigma) \) of a shuffle \( \sigma \in \Sigma \), is obtained by replacing every out-run in \( \sigma \) with the same out-run written backwards.

**Example 2.13.** For example, if \( \sigma = \uparrow, \rightarrow, \rightarrow, \uparrow, \leftarrow, \downarrow, \uparrow, \rightarrow, \leftarrow, \leftarrow, \uparrow \), then the out-runs are \( \rightarrow, \rightarrow, \uparrow \), in addition to \( \uparrow, \rightarrow, \) and \( \uparrow \). Thus the flip of \( \sigma \) is \( f(\sigma) = \uparrow, \uparrow, \rightarrow, \rightarrow, \leftarrow, \downarrow, \rightarrow, \uparrow, \leftarrow, \leftarrow, \uparrow \).

**Proposition 2.14.** Suppose the final step of \( V \) is \( \downarrow \). Then for all \( \sigma \in \Sigma \) the shifted In-Vert of \( f(\sigma) \) is the same as the signed peak-count of \( \sigma \).

**Proof.** Since the final step of \( V \) is \( \downarrow \), if \( \sigma \) ends with a out-run, then that out-run must consist only of \( \rightarrow \)'s. Thus every out-run in \( \sigma \) containing at least one \( \uparrow \) is both preceded and followed by an inward step. Consequently, \( \#(\text{In}, \uparrow)_{f(\sigma)} = \#(\uparrow, \text{In})_{\sigma} \), since flipping a shuffle reverses the order of every out-run. Additionally, it is easy to see that \( \#(\text{Out}, \downarrow)_{f(\sigma)} = \#(\text{Out}, \downarrow)_{\sigma} \).

By Lemma 2.10, the shifted In-Vert of \( f(\sigma) \) is \( \#(\text{In}, \uparrow)_{f(\sigma)} - \#(\text{Out}, \downarrow)_{f(\sigma)} \), which thus equals \( \#(\uparrow, \text{In})_{\sigma} - \#(\text{Out}, \downarrow)_{\sigma} \). By Lemma 2.8 this is exactly the signed peak-count of \( \sigma \). \qed

Next, we enumerate In-Vert in the case where the final letter in \( V \) is \( \downarrow \).

**Proposition 2.15.** Let \( V \) and \( H \) be vertical and horizontal paths with a total of \( u \uparrow \)'s, \( d \downarrow \)'s, \( r \rightarrow \)'s, and \( l \leftarrow \)'s. Suppose the final step of \( V \) is \( \downarrow \). Then the number of shuffles \( \sigma \in \Sigma \) with In-Vert \( k \) is
\[
\binom{r + u}{u + d - k} \binom{l + d}{k}.
\]

**Proof.** Pick \( u + d - k \) of the \( \rightarrow \)'s and \( \uparrow \)'s in \( H \) and \( V \). Pick \( k \) of the \( \leftarrow \)'s and \( \downarrow \)'s in \( H \) and \( V \). If the final \( \downarrow \) in \( V \) is picked, then replace it by picking \( \uparrow \) instead. We call the steps which have just been picked blue, and the remaining steps red. We call such a coloring a blue-red coloring. By definition, there are \( \binom{r + u}{u + d - k} \binom{l + d}{k} \) blue-red colorings.

Given a blue-red coloring \( C \) and a shuffle \( \sigma \in \Sigma \), we say that \( C \) fits \( \sigma \) if the blue steps in \( C \) are exactly the steps in \( \sigma \) which precede a vertical step. We will
show that for every $\sigma$ with In-Vert $k$ there is a $C$ fitting $\sigma$, and that for every $C$ there is exactly one $\sigma$ with In-Vert $k$ which $C$ fits. Consequently, the number of blue-red colorings $C$ is equal to the number of $\sigma \in \Sigma$ with In-Vert $k$.

First we show that for every $\sigma \in \Sigma$ with In-Vert $k$ there is a blue-red coloring $C$ which fits $k$. Select $C$ to color a step blue if it precedes a vertical step in $\sigma$ and red otherwise. Because $\sigma$ has In-Vert $k$, exactly $k$ inward steps will be blue, while $u + d - k$ outward steps will be blue. Additionally, because the final step of $V$ is $\downarrow$, it cannot precede a vertical step in $\sigma$, and will not be blue. Thus $C$ is a valid blue-red coloring which fits $\sigma$.

Next, in order to complete the proof, we show that every blue-red coloring fits exactly one $\sigma \in \Sigma$ with In-Vert $k$. We may uniquely construct $\sigma$ as follows. By definition, the first step is $\uparrow$. Given the first $j$ steps, the $(j + 1)$-th step must be vertical if the $j$-th step is blue, and horizontal otherwise. Therefore, the $(j + 1)$-th step is either forced to be the next unused step of $V$ or forced to be the next unused step of $H$. Consequently, all of $\sigma$ is uniquely determined by the blue-red coloring. Observe that the construction uses exactly $u + d$ steps from $V$ since exactly $u + d$ steps are blue, and thus also uses exactly $r + l$ steps from $H$. Consequently, the construction is well-defined, yielding a valid shuffle $\sigma$ with In-Vert $k$.

Using Proposition 2.14 and Proposition 2.15 it is easy to enumerate signed peak-count when the final step of $V$ is $\downarrow$. In fact, with a little more work, one can remove the restriction that $V$ ends with $\downarrow$. This brings us to our main result of the section.

**Theorem 2.16.** Let $V$ and $H$ be vertical and horizontal paths with a total of $u \uparrow$’s, $d \downarrow$’s, $r \rightarrow$’s, and $l \leftarrow$’s. The number of $\sigma \in \Sigma$ with signed peak-count $k$ is

$$\binom{r + u}{u - k} \cdot \binom{l + d}{d + k}.$$

**Proof.** There are two cases which Proposition 2.14 gives us for free.

- **Case 1:** The final step of $V$ is $\downarrow$. By Proposition 2.14 the number of shuffles with signed peak-count $k$ is the number with shifted In-Vert $d + k$, and thus In-Vert $d + k$. By Proposition 2.15 this is

$$\binom{r + u}{u + d - (d + k)} \cdot \binom{l + d}{(d + k)} = \binom{r + u}{u - k} \cdot \binom{l + d}{d + k}.$$

- **Case 2:** The final step of $H$ is $\leftarrow$. The signed peak-count of a shuffle is the negative of the signed peak-count of the complement shuffle in which all $\rightarrow$’s become $\uparrow$’s, all $\leftarrow$’s become $\downarrow$’s, and vice-versa. By case (1), the number of such complement shuffles with signed peak-count $-k$ is

$$\binom{u + r}{r + k} \cdot \binom{d + l}{l - k} = \binom{r + u}{u - k} \cdot \binom{l + d}{d + k}.$$
Note that if one of $\mathcal{H}$ or $\mathcal{V}$ is empty, then the theorem is trivially true. If neither Case 1 nor Case 2 holds, and neither $\mathcal{H}$ nor $\mathcal{V}$ are empty paths, then the final steps of $\mathcal{V}$ and of $\mathcal{H}$ must both be outward. We will prove this final case by induction on $r + u$, using the previous cases as base cases.

- **Case 3: The final steps of $\mathcal{V}$ and of $\mathcal{H}$ are both outward.** First consider shuffles ending with $\rightarrow$. Ignoring the final step, we see by induction that the number with signed peak-count $k$ is

$$\binom{(r-1)+u}{u-k} \binom{l+d}{d+k}. $$

Next consider shuffles ending with $\uparrow$. Ignoring the final step, we see by induction that the number with signed peak-count $k$ is

$$\binom{r+(u-1)}{(u-1)-k} \binom{l+d}{d+k}. $$

Summing over the two sub-cases yields

$$\binom{r+u}{u-k} \binom{l+d}{d+k}. $$

Recall that peak-count and signed peak-count share parity. Consequently, Theorem 2.16 gives a formula for peak-count modulo 2:

**Corollary 2.17.** The number of $\sigma \in \Sigma$ with even peak-count minus the number with odd peak-count is

$$\sum_k (-1)^k \binom{r+u}{u-k} \binom{l+d}{d+k}. $$

When $r = l$ and $u = d$, this resolves the Conjecture 1.1 since

$$\sum_k (-1)^k \binom{r+u}{u-k} \binom{r+u}{u+k} = \binom{r+u}{u}. $$

(See Equation (30) of [6].) When $a = u$ and $l = d$, the sum yields the super Catalan numbers. (See Equation (29) of [6].)

### 3 (x+1)-Positivity of Signed Peak-Count of Loops

A polynomial in $\mathbb{Z}[x]$ is *(x+1)-positive* if it is in $\mathbb{N}_0[x+1]$, where $\mathbb{N}_0$ denotes the non-negative integers.

Recall that Conjecture 1.3, which is Conjecture 10 of [1], states that the polynomial counting quarter-planar loops of a given length by peak-count is
(x + 1)-positive. In particular, this result is then used in an analysis of the asymptotic number of permutations sortable by two stacks in parallel.

In this section, we prove that a similar but stronger statement can be made about the absolute value of signed peak-count of loops. This is of particular interest due to conjectures we pose in Section 4 connecting (x + 1)-positivity of peak-count and signed peak-count. In fact, our conjectures combined with Theorem 3.13 imply Conjecture 1.3.

One approach to proving a polynomial is (x + 1)-positive is to evaluate the polynomial at (x − 1) and then to prove that the coefficients of the new polynomial are non-negative. For absolute value of signed peak-count, however, this approach does not easily yield results. Instead, we introduce (x + 1)-positive building-block polynomials out of which we construct the polynomial counting shuffles by absolute value of signed peak-count.

We begin by establishing notation.

**Definition 3.1.** Let \( V \) (resp. \( H \)) be any path consisting of \( u \uparrow \)'s (resp. \( r \rightarrow \)'s) and \( d \downarrow \)'s (resp. \( l \leftarrow \)'s). Let \( F(r, l, u, d) \) be the generating function in the variable \( x \) counting shuffles of \( V \) and \( H \) according to the absolute value of the signed peak-count.

By Theorem 2.16, \( F(r, l, u, d) \) depends only on \( r, l, u, d \) and not \( V \) or \( H \), making it well defined. The main result of this section is Theorem 3.13 which proves that \( F(m, m, n, n) \) is (x + 1)-positive and results in a formula for \( F(m, m, n, n) \) expanded in powers of (x + 1). To accomplish this, we define a set of words \( W_m \) and a simple statistic called absolute even count such that \( F(m, m, n, n) \) counts words in \( W_m \) according to the statistic. We then partition \( W_m \) so that the generating function counting a given part according to absolute even-count is in a family of (x + 1)-positive building-block polynomials. We begin by constructing these building blocks.

**Definition 3.2.** The polynomial \( \text{Bin}_k(n) \in \mathbb{Z}[x] \) is defined by

\[
\text{Bin}_k(n) = \sum_{i \geq 0} x^i \binom{n}{i + k}.
\]

It is not hard to show that \( \text{Bin}_k(n) \) is (x + 1)-positive.

**Lemma 3.3.** Let \( n \) and \( k \) be positive integers. Then,

\[
\text{Bin}_k(n) = \sum_{i \geq 0} \binom{n - i - 1}{k - 1} (x + 1)^i.
\]

**Proof.** Observe that \( \text{Bin}_k(n) \) is the generating function counting subsets \( S \subseteq \{1, \ldots, n\} \) of size at least \( k \) according to \( |S| - k \).

Consider only sets \( S \subseteq \{1, \ldots, n\} \) such that the \( k \)-th largest element of \( S \) is \( j \). Then there are \( \binom{n - j}{k - 1} \) options for the \( k \) smallest elements of \( S \). Moreover, since
we can choose whether each of \{j+1, j+2, \ldots, n\} is in \(S\), the total contribution of such paths to \(\text{Bin}_k(n)\) is 
\[
\binom{j-1}{k-1}(x+1)^{n-j}.
\]

Summing over values for \(j\), we get 
\[
\text{Bin}_k(n) = \sum_{j=1}^{n} \binom{j-1}{k-1}(x+1)^{n-j} = \sum_{i=0}^{n} \binom{n-i-1}{k-1}(x+1)^i.
\]

Note that Lemma 3.3 could also have easily been proven by routinely using Zeilberger’s creative telescoping [11]. However, we prefer our combinatorial proof as it highlights one of the types of arguments that may prove useful to the reader when studying \((x+1)\)-positivity.

We will now use \(\text{Bin}_k(n)\) to build a more interesting building block:

**Definition 3.4.** Let \(\text{Bin}_k(n)\) be the generating function counting subsets \(S \subseteq [n]\) according to \(|S| - k\).

The next result, Proposition 3.5, establishes that \(\text{Bin}_k(n)\) is \((x+1)\)-positive. Interestingly, the expansion of \(\text{Bin}_k(n)\) in powers of \((x+1)\) has constant coefficient 0. Like Lemma 3.3 Proposition 3.5 could also be shown with Zeilberger’s creative telescoping. We find it more straightforward, however, to simply interpret \(\text{Bin}_k(n)\) in terms of its cousin \(\text{Bin}_{n-k}(n')\).

**Proposition 3.5.** Let \(n\) and \(k\) be positive integers and \(k \leq n\). Then,
\[
\text{Bin}_k(n) = \sum_{i \geq 0} \left( \binom{n-i-1}{k-1} + \binom{n-i-1}{n-k-1} \right) (x+1)^i.
\]

**Proof.** Recall that \(\text{Bin}_k(n)\) counts subsets \(S \subseteq [n]\) of size at least \(k\) according to \(|S| - k\). On the other hand, \(\text{Bin}_{n-k}(n)\) counts subsets \(S \subseteq [n]\) of size at least \(n-k\) according to \(|S| - n + k\). Equivalently, considering the complement sets, \(\text{Bin}_{n-k}(n)\) counts subsets of size at most \(k\) according to \((n-|S|)-n+k=k-|S|\). Thus
\[
\text{Bin}_k(n) + \text{Bin}_{n-k}(n) = \text{Bin}^k(n) + \binom{n}{k},
\]
since we are double-counting subsets of size \(k\).

It follows from Lemma 3.3 that for \(k > 0\),
\[
\text{Bin}^k(n) = \sum_{i \geq 0} \left( \binom{n-i-1}{k-1} (x+1)^i + \sum_{i \geq 0} \binom{n-i-1}{n-k-1} (x+1)^i - \binom{n}{k} \right)
\]
\[
= \sum_{i > 0} \left( \binom{n-i-1}{k-1} + \binom{n-i-1}{n-k-1} \right) (x+1)^i + \binom{n-1}{k-1} + \binom{n-1}{n-k-1} - \binom{n}{k}.
\]
Reducing the final three terms to zero, we get the desired formula. \(\square\)
Having designed \((x+1)\)-positive building-block polynomials into which we will partition \(F(m, m, n, n)\), we next introduce the idea of absolute even-count.

**Definition 3.6.** Let \(W^m_n\) be the set of binary words with \(2m\) zeroes and \(2n\) ones.

**Definition 3.7.** The even-count of a binary word \(w\) is the number of ones in even-indexed positions. The shifted even-count is

\[
(\text{# ones in even positions}) - \frac{1}{2}(\text{# ones}).
\]

The absolute even-count is the absolute value of the shifted even-count.

**Example 3.8.** The word 1010110000 has absolute even-count \(|1 - 2| = 1|.

Absolute even-count’s usefulness stems from its connection to signed peak-count, which is described in the following lemma.

**Lemma 3.9.** If \(H\) is a horizontal loop with \(m\) steps right and left, and \(V\) is a vertical loop with \(n\) steps up and down, then number of shuffles of \(V\) and \(H\) with signed peak-count of absolute value \(k\) is the same as the number of elements of \(W^m_n\) with absolute even-count \(k\).

Consequently, \(F(m, m, n, n)\) counts elements of \(W^m_n\) by absolute even-count.

**Proof.** Theorem 2.16 tells us that the number of shuffles of \(V\) and \(H\) with signed peak-count of absolute value \(k\) is

\[
2 \binom{m+n}{n-k} \binom{m+n}{n+k}.
\]

On the other hand, in order for an element of \(W^m_n\) to have \(n+k\) ones in even positions, we get to choose \(n+k\) of the \(m+n\) even positions to take value one, and then \(n-k\) of the \(m+n\) odd positions to take value one. Similarly, in order for an element of \(W^m_n\) to have \(n-k\) ones in even positions, we get to choose \(n-k\) of the even positions to be ones, and \(n+k\) of the odd positions to be ones. It follows that the number of elements in \(W^m_n\) with absolute even-count \(k\) is also

\[
2 \binom{m+n}{n-k} \binom{m+n}{n+k}.
\]

It will be useful for us to think about absolute even-count using as simple reformulation involving the notion of an odd-indexed pair.

**Definition 3.10.** An odd-indexed pair in a word \(w\) is any pair of letters \((w(2i+1), w(2i+2))\).

**Example 3.11.** The binary word 10011100 partitions into odd-indexed pairs as 10|01|11|00.
Lemma 3.12. For a given \( w \in W_n^m \), the absolute even-count of \( w \) is
\[
\left| \left( \# \text{ (0, 1) odd-indexed pairs in } w \right) - \frac{1}{2} \left( \# \text{ (a, b) odd-indexed pairs in } w \text{ with } a \neq b \right) \right|.
\]

Proof. For a given \( w \in W_n^m \), the contribution of a given odd-indexed pair \( (a, b) \) to the shifted even-count is 0 if \( a = b \); \(-\frac{1}{2}\) if \( a = 1 \) and \( b = 0 \); and \( \frac{1}{2} \) if \( b = 1 \) and \( a = 0 \). Therefore, the shifted even-count of \( w \) is
\[
\frac{1}{2} \left( \# \text{ (0, 1) odd-indexed pairs in } w \right) - \frac{1}{2} \left( \# \text{ (1, 0) odd-indexed pairs in } w \right),
\]
which can be rewritten as
\[
\left( \# \text{ (0, 1) odd-indexed pairs in } w \right) - \frac{1}{2} \left( \# \text{ (a, b) odd-indexed pairs in } w \text{ with } a \neq b \right).
\]

We are now prepared to present the main result of the section. Theorem 3.13 describes \( F(m, m, n, n) \) as a polynomial in \((x + 1)\).

Theorem 3.13. The polynomial \( F(m, m, n, n) \) is \((x + 1)\)-positive. More precisely, \( F(m, m, n, n) \) equals
\[
\binom{m + n}{n} + 2 \sum_{i>0} (x + 1)^i \sum_{0 \leq k < n} \binom{m + n - k}{k} \binom{m + n - k}{2n - 2k - i - 1} \binom{2n - 2k - i - 1}{n - k - 1}.
\]

Proof. Lemma 3.9 allows us to interpret \( F(m, m, n, n) \) as counting elements of \( W_n^m \) according to their absolute even-count. We will partition the elements of \( W_n^m \) so that each part has absolute even-count enumerated by \( \text{Bin}^k(2k) \) for some \( k \), allowing us to apply Proposition 3.5 in order to prove the \((x + 1)\)-positivity of \( F(m, m, n, n) \).

For \( 0 \leq i < m + n \), define \( i \)-toggling of a word \( w \in W_n^m \) as swapping the values in the \((i + 1)\)-th odd-indexed pair. Say that two words are \( \text{toggle-equivalent} \) if they can be reached from each other by repeated toggling (for various \( i \)). Denote the toggle-equivalence class of \( w \in W_n^m \) by \( T(w) \). For example, \( T(110110) \) is
\[
\{11|01|10, 11|10|10, 11|01|01, 11|10|01\},
\]
where we use \(|\) to separate odd-indexed pairs.

Call the set of \( i \) such that \( w(2i + 1) = 1 \) and \( w(2i + 2) = 1 \) the anchor \( A(w) \), and the set of \( i \) such that \( w(2i) \neq w(2i + 1) \) the base \( B(w) \). Then we can characterize \( T(w) \) in terms of \( w \)'s anchor and base, observing that
\[
T(w) = \{ w' : w' \in W_n^m, A(w') = A(w), B(w') = B(w) \}.
\]
Equation (2) tells us that we can think of each element $w'$ of $T(w)$ as being represented by the set $S_{w'} = \{ i \mid w'(2i + 1) = 0, w'(2i + 2) = 1 \}$, with each of the subsets $S \subseteq B(w)$ representing exactly one element in $T(w)$. Moreover, by Lemma 3.12, the absolute even-count of each element $w'$ is then given by $|S_{w'}| - |B(w)|/2$. It follows that the elements of $T(w)$ are counted by $\text{Bin}^{B(w)}/2(|B(w)|)$ with respect to absolute even-count.

At this point, we have shown that $F(m, m, n, n)$ can be expressed as the sum of polynomials of the form $\text{Bin}_r^j$, each of which we know to be $(x+1)$-positive due to Proposition 3.5. Having established the $(x+1)$-positivity of $F(m, m, n, n)$, it remains to derive a formula for it as a polynomial in $(x+1)$.

There are $\binom{m+n}{k}$ $(m+n-k)$ distinct toggle-equivalence classes containing words with anchors of size $k$. In particular, there are $\binom{m+n}{k}$ options for $A(w)$, for each of which there are $\binom{m+n-k}{2n-2k}$ options for $B(w)$. Therefore the distribution of absolute even-count over $W_n^m$ is given by the polynomial

$$
\sum_{0 \leq k \leq n} \binom{m+n}{k} \binom{m+n-k}{2n-2k} \text{Bin}^{n-k}(2n-2k).
$$

By Proposition 3.5, when $k \neq n$, $\text{Bin}^{n-k}(2n-2k)$ equals

$$
\sum_{i>0} \left( \binom{2n-2k-i-1}{n-k-1} + \binom{(2n-2k)-i-1}{(2n-2k)-(n-k)-1} \right) (x+1)^i
$$

$$
= 2 \sum_{i>0} \binom{2n-2k-i-1}{n-k-1} (x+1)^i.
$$

When $k = n$, $\text{Bin}^{n-k}(2n-2k) = 1$. Plugging this along with Equation (4) into Equation (3), the distribution of absolute even-count over $W_n^m$ is

$$
\binom{m+n}{n} \binom{m+n-k}{0} + 2 \sum_{0 \leq k < n} \binom{m+n}{k} \binom{m+n-k}{2n-2k} \sum_{i>0} \binom{2n-2k-i-1}{n-k-1} (x+1)^i
$$

$$
= \binom{m+n}{n} + 2 \sum_{i>0} (x+1)^i \sum_{0 \leq k < n} \binom{m+n}{k} \binom{m+n-k}{2n-2k} \binom{2n-2k-i-1}{n-k-1}.
$$

It turns out that Theorem 2.16 extends to proving the $(x+1)$-positivity of $F(m, n, m, n)$ at no additional cost. Indeed, Proposition 3.14 shows that the $(x+1)$-positivity of $F(r, l, u, d)$ is always symmetric in $l$ and $u$. In addition, since $F(r, l, u, d) = F(u, d, r, l)$, it follows that $(x+1)$-positivity is also symmetric in $d$ and $r$.

\[\text{Proposition 3.5} \text{ technically doesn’t cover } \text{Bin}^0(0) = 1, \text{ which is obviously also } (x+1)-\text{positive.}\]
Proposition 3.14. The polynomial $F(r, l, u, d)$ is $(x+1)$-positive if and only if $F(r, u, l, d)$ is $(x+1)$-positive. More precisely,

$$F(r, l, u, d) \cdot (r + l)!(u + d)! = F(r, u, l, d) \cdot (r + u)!(l + d)!.$$ 

Proof. Observe that

$$F(r, l, u, d) = \sum_i x^{|i|} \binom{r + u}{u + i} \binom{l + d}{d - i},$$

by Theorem 2.16. Pulling out a factor of $\binom{r + l}{r + l}(u + d)!$, we obtain

$$F(r, l, u, d) = \frac{(r + u)!(l + d)!}{(r + l)!(u + d)!} \sum_i x^{|i|} \binom{r + l}{l + i} \binom{u + d}{d - i}$$

$$= \frac{(r + u)!(l + d)!}{(r + l)!(u + d)!} F(r, u, l, d),$$

completing the proof. \qed

4 Conjectures and Open Questions

In this section, we state conjectures relating peak-count and signed peak-count, and we discuss directions for future work. Recall that Albert and Bousquet-Mélou’s recent asymptotic analysis of permutations sortable by two stacks in parallel relies on two conjectures, the most interesting of which is Conjecture 1.3. Our main conjecture, Conjecture 4.2, implies Conjecture 1.3, and may prove easier for future researchers to tackle.

We call a path vertical (resp. horizontal) if it comprises $\uparrow$ and $\downarrow$ (resp. $\rightarrow$ and $\leftarrow$) steps. A path is quarter-planar if it resides in the quarter plane $\{(x, y) | x \geq 0, y \geq 0\}$. A quarter-planar vertical or horizontal path is referred to as positive.

Fix $V$ to be a vertical positive path with $u$ up-steps and $d$ down-steps. Let $Q$ be the set of quarter-planar paths with $r$ right-steps, $l$ left-steps, and vertical projection $V$. Let $G_1$ be the generating function for peak-count over $Q$ and $G_2$ be the generating function for signed peak-count over $Q$. We will state our conjectures in terms of $G_1$ and $G_2$. Additionally, computations indicate that all of our conjectures are still true when we modify $Q$ to consider planar paths rather than quarter-planar paths and $V$ to be any vertical path rather than necessarily a positive one.

We begin by re-stating a conjecture of [1], originally posed as Conjecture (P2). Conjecture 4.1 implies several other conjectures of [1], including Conjecture 1.3, the primary conjecture on which their asymptotic analysis of two-stack sortable permutations relies.

\textsuperscript{4}Note that the modified versions of the conjectures neither imply or are implied by the un-modified versions. The modified versions may be easier though. For example, the modified version of Conjecture 4.1 is solved in [1], and the modified version of Conjecture 1.3 is straightforward.
Conjecture 4.1 (Albert and Bousquet-Mélou, Conjecture (P2) on pp. 32 [1]).

If \( a = b \) and \( c = d \), then \( G_1 \) is \((x+1)\)-positive.

Our main conjecture, Conjecture 4.2, states a surprising equivalence between \((x+1)\)-positivity of \( G_1 \) and \( G_2 \) (tested for \( u, d, r, l \leq 5 \)). In particular, given Conjecture 4.2, our analysis of \((x+1)\)-positivity of signed peak-count in Theorem 3.13 implies Conjecture 4.1.

Conjecture 4.2. The polynomial \( G_1 \) is \((x+1)\)-positive if and only if \( G_2 \) is \((x+1)\)-positive.

Since peak-count and signed peak-count are of equal parity, the constant coefficients of \( G_1 \) and \( G_2 \) as polynomials in \((x+1)\) are equal. Computations show, however, that \((x+1)\)-positivity of \( G_1 \) and \( G_2 \) does not depend only on this coefficient.

By Theorem 2.16, \( G_2 \) depends only on \( r, l, u, d \). Thus Conjecture 4.2 implies the following.

Conjecture 4.3. The \((x+1)\)-positivity of \( G_1 \) depends only on \( r, l, u, \) and \( d \).

This conjecture is particularly surprising given that \( G_1 \) itself does not depend only on \( r, l, u, \) and \( d \). Can one characterize for which \( r, l, u, d \) \( G_1 \) and \( G_2 \) are \((x+1)\)-positive? Proposition 3.14 shows that \((x+1)\)-positivity of \( G_2 \) is unchanged by swapping \( d \) and \( r \) or \( u \) and \( l \).

In the proof of Theorem 3.13, we essentially partitioned shuffles of vertical and horizontal loops \( V \) and \( H \) so that each part has a signed peak-count distribution counted by the generating function \( \text{Bin}^j(2j) \) for some \( j \in \mathbb{N}_0 \). It is natural to ask whether this proof technique extends to analyzing \( G_2 \) for arbitrary \( r, l, u, d \). Observe that \( B = \{1\} \cup \left\{ \frac{\text{Bin}^j(2j)}{2} : j \in \mathbb{N}\right\} = \{1\} \cup \{\text{Bin}^j(2j-1) : j \in \mathbb{N}\} \) contains a single monic polynomial of each degree, implying that any polynomial in \( \mathbb{Z}[x] \) can be expressed uniquely as a linear combination of elements in \( B \). If a polynomial can be expressed as linear combination of elements of \( B \) with non-negative coefficients, we call it toggle-buildable. The following conjecture has been tested for \( r, l, u, d \leq 5 \).

Conjecture 4.4. The generating function \( G_2 \) is toggle-buildable exactly when \( G_2 \) is \((x+1)\)-positive.

However, the same cannot be said for peak-count (i.e., \( G_1 \)), even when \( r = l \) and \( u = d \) (so that we are considering quarter-planar loops with a fixed vertical projection). For example, when \( r = l = 4 \) and \( u = d = 2 \), the polynomial \( G_1 \), which equals \( x^2 + 12x + 15 = (1 + x)^2 + 10(1 + x) + 4 \), is \((x+1)\)-positive but not toggle-buildable.

Our proof of Theorem 3.13 suggests several additional questions. In the proof of Theorem 3.13, we discuss \( F(m, m, n, n) \) in terms of the absolute even-count of binary words. Assuming the final step of \( V \) is \( \downarrow \), however, then the \footnote{Note that \( 2\text{Bin}^j(2j-1) = \text{Bin}^j(2j) \) for \( j \in \mathbb{N} \).} 
\footnote{We do not require the coefficient of each \( \text{Bin}^j(2j-1) \) to be even, as is the case when \( a = b \) and \( c = d \). In fact, such a requirement would make Conjecture 1.1 untrue.}
proof of Proposition 2.15 provides a straightforward bijection between binary words in $W^m_n$ with absolute even-count $k$ and shuffles of $V$ and $H$ with shifted $\text{In-Vert} k$. Furthermore, if we apply Proposition 2.14 then we obtain a bijection between binary words in $W^m_n$ with absolute even count $k$ and shuffles of $V$ and $H$ with signed peak-count $k$. This leads us to pose several open problems:

1. When $i$-toggling a binary word changes its even-count by one, it also changes the signed peak-count of the corresponding shuffle by one. Computer computations indicate that, in fact, when $i$-toggling changes signed peak-count by one, it often also only changes peak-count by one. If one could characterize when this is the case, then toggling could be a useful tool for proving Conjectures 4.1 and 4.2.

2. Conjecture 4.4 suggests that the the toggling argument in the proof of Theorem 3.13 may have the potential to be extended to $F(m, n, m, n)$ (without using the trick from the proof of Proposition in which we multiply the expression by a non-integer amount before analyzing it). In particular, such an extension might yield a combinatorial interpretation for the super Catalan number. Indeed, the shuffles in toggle-equivalence classes of size one would be exactly those counted by the constant term of $F(m, n, m, n)$ expanded in powers of $(x + 1)$, of which there are a super Catalan number by Corollary 2.17.

3. Suppose $H$ is of the form

$$→, →, →, \ldots, →, ←, \ldots, ←, ←,$$

and $V$ is of the form

$$↑, ↑, ↑, \ldots, ↑, ↓, \ldots, ↓, ↓,$$

each with the same number of outward steps as inward steps. Then any shuffle of $V$ and $H$ cannot contain both $(↑, ←)$ and $(→, ↓)$ pairs. Thus the absolute value of the signed peak-count is equal to the peak-count of such shuffles. Consequently, Theorem 3.13 proves the $(x+1)$-positivity of the generating function counting these shuffles by their peak-count.

However, we have been unable to find a simple combinatorial description for the action of $i$-toggling on these shuffles (i.e., the result of $i$-toggling the corresponding binary word). Does one exist? And is there an analogue of toggling for peak-count rather than signed peak-count?

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References

[1] Michael Albert and Mireille Bousquet-Mélou. Permutations sortable by
two stacks in parallel and quarter plane walks. European Journal of Com-
binatorics, 43:131–164, 2015.

[2] Emily Allen and Irina Gheorghiciuc. A weighted interpretation for the
super Catalan numbers. Journal of Integer Sequences, 17(2):3, 2014.

[3] Xin Chen and Jane Wang. The super Catalan numbers $s(m, m + s)$ for
$s \leq 4$. arXiv preprint arXiv:1208.4196, 2012.

[4] Daniel Denton. Methods of computing deque sortable permutations given
complete and incomplete information. arXiv preprint arXiv:1208.1532,
2012.

[5] Shimon Even and Alon Itai. Queues, stacks and graphs. Theory of Machines
and Computations, pages 71–86, 1971.

[6] Ira M Gessel. Super ballot numbers. Journal of symbolic computation,
14(2):179–194, 1992.

[7] Donald Ervin Knuth. The art of computer programming: sorting and
searching, volume 3. Pearson Education, 1998.

[8] Vaughan R Pratt. Computing permutations with double-ended queues,
parallel stacks and parallel queues. In Proceedings of the fifth annual ACM
symposium on Theory of computing, pages 268–277. ACM, 1973.

[9] Rodica Simion and Frank W Schmidt. Restricted permutations. European
Journal of Combinatorics, 6(4):383–406, 1985.

[10] Robert Tarjan. Sorting using networks of queues and stacks. Journal of
the ACM (JACM), 19(2):341–346, 1972.

[11] Doron Zeilberger. The method of creative telescoping. Journal of symbolic
computation, 11(3):195–204, 1991.