Group Theoretical Interpretation of the $CPT$-theorem

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Abstract

An algebraic description of basic discrete symmetries (space inversion $P$, time reversal $T$, charge conjugation $C$ and their combinations $PT$, $CP$, $CT$, $CPT$) is studied. Discrete subgroups $\{1, P, T, PT\}$ of orthogonal groups of multidimensional spaces over the fields of real and complex numbers are considered in terms of fundamental automorphisms of Clifford algebras. The fundamental automorphisms form a finite group of order 4. The charge conjugation is represented by a pseudoautomorphism of the Clifford algebra. Such a description allows one to extend the automorphism group. It is shown that an extended automorphism group ($CPT$-group) forms a finite group of order 8. The group structure and isomorphisms between the extended automorphism groups and finite groups are studied in detail. It is proved that there exist 64 different realizations of $CPT$-group. An extension of universal coverings (Clifford-Lipschitz groups) of the orthogonal groups is given in terms of $CPT$-structures which include well-known Shirokov-Dąbrowski $PT$-structures as a particular case.

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1 Introduction

Importance of discrete transformations is well-known, many textbooks on quantum theory began with description of the discrete symmetries, and famous Lüders–Pauli $CPT$–Theorem is a keystone of quantum field theory. A fundamental notion of antiparticle immediately relates with the charge conjugation $C$. The requirement of invariance concerning the each of the discrete transformations gives rise to certain consequences which can be verified in experience. So, from the invariance concerning the $CPT$-transformation it follows an identity of the masses and full life times of particles and antiparticles, from the invariance concerning the time reversal $T$ we have certain relations between the forward and reverse reaction cross-sections. In turn, from the invariance concerning the charge conjugation $C$ it follows an absence of the reactions forbidden by the conservation law of charge parity, and from the invariance concerning the space inversion $P$ and time reversal $T$ it follows an absence of the electric dipol moment of particles. As follows from experience, for the processes defined by strong and electromagnetic interactions there exists an invariance with respect to the all discrete transformations. In contrast to strong and electromagnetic interactions, for the weak interaction, as shown by experiment, there is no invariance concerning space inversion $P$, but there is invariance with respect to $CP$-transformation. Moreover, there
are experimental evidences confirming $CP$–violation (a decay of the neutral $K$–mesons). It is clear that the analysis of the discrete symmetries allows us to reveal the most profound structural characteristics of the matter.

However, usual practice of definition of the discrete symmetries from the analysis of relativistic wave equations does not give a full and consistent theory of the discrete transformations. In the standard approach, except a well studied case of the spin $j = 1/2$ (Dirac equation), a situation with the discrete symmetries remains unclear for the fields of higher spin $j > 1/2$. It is obvious that a main reason of this is an absence of a fully adequate formalism for description of higher–spin fields (all widely accepted higher–spin formalisms such as Rarita–Schwinger approach \cite{20}, Bargmann–Wigner \cite{1} and Gel’fand–Yaglom \cite{10} multispinor theories, and also Joos–Weinberg $2(2j + 1)$–component formalism \cite{14} \cite{31} have many intrinsic contradictions and difficulties). The first attempt of going out from this situation was initiated by Gel’fand, Minlos and Shapiro in 1958 \cite{11}. In the Gel’fand–Minlos–Shapiro approach, the discrete symmetries are represented by outer involutory automorphisms of the Lorentz group (there are also other realizations of the discrete symmetries via the outer automorphisms, see \cite{17} \cite{15} \cite{27}). At present, the Gel’fand–Minlos–Shapiro ideas have been found further development in the works of Buchbinder, Gitman and Shelepin \cite{5} \cite{12}, where the discrete symmetries are represented by both outer and inner automorphisms of the Poincaré group.

Discrete symmetries $P$ and $T$ transform (reflect) space and time (two the most fundamental notions in physics), but in the Minkowski 4–dimensional space–time continuum \cite{18} space and time are not separate and independent. By this reason a transformation of one (space or time) induces a transformation of another. Therefore, discrete symmetries should be expressed by such transformations of the continuum, that transform all its structure totally with a full preservation of discrete nature. The only possible candidates on the role of such transformations are automorphisms. In such a way, the idea of representation of the discrete symmetries via the automorphisms of the Lorentz group (‘rotation’ group of the 4–dimensional continuum) is appeared in the Gel’fand–Minlos–Shapiro approach, or via the automorphisms of the Poincaré group (motion group of the 4–dimensional continuum) in the Buchbinder–Gitman–Shelepin approach.

In 1909, Minkowski showed \cite{18} that a causal structure of the world is described by a 4–dimensional pseudo–Euclidean geometry. In accordance with \cite{18} the quadratic form \(x^2 + y^2 + z^2 - c^2t^2\) remains invariant under the action of linear transformations of the four variables $x, y, z$ and $t$, which form a general Lorentz group $G$. As is known, the general Lorentz group $G$ consists of a proper orthochronous Lorentz group $G_0$ and three reflections (discrete transformations) $P, T, PT$, where $P$ and $T$ are space and time reversal, and $PT$ is a so–called full reflection. The discrete transformations $P, T$ and $PT$ added to an identical transformation form a finite group. Thus, the general Lorentz group may be represented by a semidirect product $G_0 \odot \{1, P, T, PT\}$. In 1958, Shirokov pointed out \cite{25} \cite{26} that an universal covering of the inhomogeneous Lorentz group has eight inequivalent realizations. Later on, in the eighties this idea was applied to a general orthogonal group $O(p, q)$ by Dąbrowski \cite{9}. As is known, the orthogonal group $O(p, q)$ of the real space $\mathbb{R}^{p,q}$ is represented by the semidirect product of a connected component $O_0(p, q)$ and a discrete subgroup $\{1, P, T, PT\}$. Further, a double covering of the orthogonal group $\tilde{O}(p, q)$ is a Clifford–Lipschitz group $\text{Pin}(p, q)$ which is completely constructed within a Clifford algebra $\mathcal{C}_{p,q}$. In accordance with squares of elements of the discrete subgroup ($a = P^2$, $b = T^2$, $c = (PT)^2$) there exist eight double coverings (Dąbrowski
groups [9] of the orthogonal group defining by the signatures \((a, b, c)\), where \(a, b, c \in \{-, +\}\). Such in brief is a standard description scheme of the discrete transformations. However, in this scheme there is one essential flaw. Namely, the Clifford–Lipschitz group is an intrinsic notion of the algebra \(\mathcal{O}_{p,q}\) (a set of the all invertible elements of \(\mathcal{O}_{p,q}\)), whereas the discrete subgroup is introduced into the standard scheme in an external way, and the choice of the signature \((a, b, c)\) of the discrete subgroup is not determined by the signature of the space \(\mathbb{R}^{p,q}\). Moreover, it is suggest by default that for any signature \((p, q)\) of the vector space there exist the all eight kinds of the discrete subgroups. It is obvious that a consistent description of the double coverings of \(O(p, q)\) in terms of the Clifford–Lipschitz groups \(\text{Pin}(p, q) \subset \mathcal{O}_{p,q}\) can be obtained only in the case when the discrete subgroup \(\{1, P, T, PT\}\) is also defined within the algebra \(\mathcal{O}_{p,q}\). Such a description has been given in the works [28, 29, 30], where the discrete symmetries are represented by fundamental automorphisms of the Clifford algebras.

So, the space inversion \(P\), time reversal \(T\) and their combination \(PT\) correspond to an automorphism \(\star\) (involution), an antiautomorphism \(\sim\) (reversion) and an antiautomorphism \(\overline{\star}\) (conjugation), respectively. Group theoretical structure of the discrete transformations is a central point in this work. The fundamental automorphisms of the Clifford algebras are compared to elements of the finite group formed by the discrete transformations. In its turn, a set of the fundamental automorphisms, added by an identical automorphism, forms a finite group \(\text{Aut}(\mathcal{O})\), for which in virtue of the Wedderburn–Artin Theorem there exists a matrix representation. In such a way, an isomorphism \(\{1, P, T, PT\} \simeq \text{Aut}(\mathcal{O})\) plays a central role and allows us to use methods of the Clifford algebra theory at the study of a group theoretical structure of the discrete transformations. First of all, it allows us to classify the discrete groups into Abelian \(\mathbb{Z}_2 \otimes \mathbb{Z}_2\), \(\mathbb{Z}_4\) and non–Abelian \(D_4\), \(Q_4\) finite groups, and also we establish a dependence between the finite groups and signature of the spaces in case of real numbers. It is shown that the division ring structure of \(\mathcal{O}_{p,q}\) imposes hard restrictions on existence and choice of the discrete subgroup, and the signature \((a, b, c)\) depends upon the signature of the underlying space \(\mathbb{R}^{p,q}\). Moreover, this description allows us to incorporate the Gel’fand–Minlos–Shapiro automorphism theory into Shirokov–Dąbrowski scheme and further to unite them on the basis of the Clifford algebra theory.

Other important discrete symmetry is the charge conjugation \(C\). In contrast with the transformations \(P, T, PT\), the operation \(C\) is not space–time discrete symmetry. This transformation is first appeared on the representation spaces of the Lorentz group and its nature is strongly different from other discrete symmetries. By this reason in the section 3 the charge conjugation \(C\) is represented by a pseudoautomorphism \(A \rightarrow \overline{A}\) which is not fundamental automorphism of the Clifford algebra. All spinor representations of the pseudoautomorphism \(A \rightarrow \overline{A}\) are given in Theorem [5]. An introduction of the transformation \(A \rightarrow \overline{A}\) allows us to extend the automorphism group \(\text{Aut}(\mathcal{O})\) of the Clifford algebra. It is shown that automorphisms \(A \rightarrow A^*, A \rightarrow \overline{A}, A \rightarrow A^*, A \rightarrow \overline{A}, A \rightarrow \overline{A}, A \rightarrow \overline{A}\) and \(A \rightarrow \overline{A}^*\) form a finite group of order 8 (an extended automorphism group \(\text{Ext}(\mathcal{O}) = \{\text{Id}, \star, \sim, \overline{\star}, \overline{\star}, \overline{\star}, \overline{\star}\}\)). The group \(\text{Ext}(\mathcal{O})\) is isomorphic to a \(CPT\)-group \(\{1, P, T, C, CP, CT, CPT\}\). There exist isomorphisms between \(\text{Ext}(\mathcal{O})\) and finite groups of order 8. A full number of different realizations of \(\text{Ext}(\mathcal{O})\) is equal to 64. This result allows us to define extended universal coverings \((CPT\text{-structures})\) of the orthogonal groups. It is shown that the eight Shirokov-Dąbrowski \(PT\)-structures present a particular case of general \(CPT\)-structures.
2 Clifford–Lipschitz groups

Theory of spinors and spinor representations (universal coverings of the groups) is closely related with the foundations of quantum mechanics. As is known, any quantized system $S$ corresponds to a complex separable Hilbert space $\mathcal{H}$. At this point, a physical state is represented by a vector $|x\rangle \in \mathcal{H}$ and $\langle x | x \rangle = 1$. In turn, any physical observable $a$ (for example, energy, electric charge and so on) corresponds to a self-conjugated operator $A$ in the space $\mathcal{H}$. The spectrum of $A$ coincides with a set of all possible values of $a$. Quantum mechanics does not give a definite value of the magnitude $a$ in the state $|x\rangle$. We have here only a mathematical expectation of $a$:

$$\langle x | A | x \rangle = \text{Sp} P_x,$$

where $P_x$ is a Hermitean projection operator ($P_x^+ = P_x$) on the state $|x\rangle$. The unit eigenvectors of $P_x$ are distinguished by the scalar phase factor. All these vectors lead to identical physical predictions and, therefore, they describe one and the same state. In turn, the operators $P_x$ in itself are observables. Indeed, the magnitude

$$\text{Sp} P_x P_y = |\langle x | y \rangle|^2$$

(1)

is a location probability of the physical system $S$ in the state $|x\rangle$ (or $|y\rangle$).

For the each quantized system $S$ there exists an invariance group $G$. It means that $G$ acts on the system $S$ and there exists an isomorphism between $S$ and $g(S)$, $g \in G$. Let $P_{gx}$ be the state obtained from $P_x$ via the transformation $g \in G$. Then, the group $G$ is the invariance group if all probabilities in (1) are invariant:

$$\forall |x\rangle \in \mathcal{H}, \quad \forall g \in G, \quad \text{Sp} P_{gx_1} P_{gx_2} = \text{Sp} P_x P_x,$$

or

$$|\langle gx_1 | gx_2 \rangle|^2 = |\langle x_1 | x_2 \rangle|^2.$$

It means that $G$ acts on $\mathcal{H}$ isometrically. As it shown in [32, 33, 2], the mapping $|x\rangle \rightarrow gx$ is either unitary operator $U(g)$ or antiunitary operator $V(g)$ on the Hilbert space $\mathcal{H}$. Let $\mathcal{V}(\mathcal{H})$ be a group of both unitary and antiunitary operators on $\mathcal{H}$, and let $\mathcal{U}(\mathcal{H})$ be a subgroup of unitary operators. The group $\mathcal{U}(\mathcal{H})$ is a subgroup of index 2 of the group $\mathcal{V}(\mathcal{H})$, and by this reason $\mathcal{U}(\mathcal{H})$ is an invariant subgroup of $\mathcal{V}(\mathcal{H})$. The transformations $\mathcal{U}(g)$ for $g \in G$ generate a subgroup $\mathcal{E}(G)$ of the group $\mathcal{V}(\mathcal{H})$, which is an extension of $G$ by means of the group $U$ (multiplication of the vectors of $\mathcal{H}$ by the phase factor $e^{i\alpha}$, where $\alpha$ is a real number; at this point the state remains unaltered), that is,

$$G \xrightarrow{f} \text{Aut} U,$$

where a kernel of the mapping Ker $f$ is an invariant subgroup $G_+ \subset G$ of index 2, which acts as the group of unitary transformations, and a nontrivial element of the mapping image Im $f$ is the complex conjugation. Therefore, one can say that $G_+$ acts as a linear unitary projective representation. For example, if $G_+$ is a rotation group $SO(3)$, then its projective representations are coincide with linear irreducible unitary representations of the group $SU(2)$ (the group $SU(2)$ is an universal covering of $SO(3)$). In turn, universal coverings of the orthogonal groups are completely defined within Clifford–Lipschitz groups.
The Lipschitz group $\Gamma_{p,q}$, also called the Clifford group, introduced by Lipschitz in 1886 \[10\], may be defined as the subgroup of invertible elements $s$ of the algebra $\mathcal{C}_{p,q}$:

$$\Gamma_{p,q} = \left\{ s \in \mathcal{C}_{p,q}^+ \cup \mathcal{C}_{p,q}^- \mid \forall x \in \mathbb{R}^{p,q}, \ sx s^{-1} \in \mathbb{R}^{p,q} \right\}.$$  

The set $\Gamma_{p,q}^+ = \Gamma_{p,q} \cap \mathcal{C}_{p,q}^+$ is called special Lipschitz group \[7\].

Let $N : \mathcal{C}_{p,q} \to \mathcal{C}_{p,q}$, $N(x) = x^x$. If $x \in \mathbb{R}^{p,q}$, then $N(x) = x(-x) = -x^2 = -Q(x)$. Further, the group $\Gamma_{p,q}$ has a subgroup

$$\text{Pin}(p, q) = \left\{ s \in \Gamma_{p,q} \mid N(s) = \pm 1 \right\}. \quad (2)$$  

Analogously, a spinor group $\text{Spin}(p, q)$ is defined by the set

$$\text{Spin}(p, q) = \left\{ s \in \Gamma_{p,q}^+ \mid N(s) = \pm 1 \right\}. \quad (3)$$  

It is obvious that $\text{Spin}(p, q) = \text{Pin}(p, q) \cap \mathcal{C}_{p,q}^+$. The group $\text{Spin}(p, q)$ contains a subgroup

$$\text{Spin}_+(p, q) = \left\{ s \in \text{Spin}(p, q) \mid N(s) = 1 \right\}. \quad (4)$$  

It is easy to see that the groups $O(p, q)$, $SO(p, q)$ and $SO_+(p, q)$ are isomorphic, respectively, to the following quotient groups

$$O(p, q) \simeq \text{Pin}(p, q)/\mathbb{Z}_2, \; \text{SO}(p, q) \simeq \text{Spin}(p, q)/\mathbb{Z}_2, \; \text{SO}_+(p, q) \simeq \text{Spin}_+(p, q)/\mathbb{Z}_2,$$

where the kernel $\mathbb{Z}_2 = \{ 1, -1 \}$. Thus, the groups $\text{Pin}(p, q)$, $\text{Spin}(p, q)$ and $\text{Spin}_+(p, q)$ are the double coverings of the groups $O(p, q)$, $SO(p, q)$ and $SO_+(p, q)$, respectively.

Further, since $\mathcal{C}_{p,q}^+ \simeq \mathcal{C}_{q,p}^+$, then

$$\text{Spin}(p, q) \simeq \text{Spin}(q, p).$$

In contrast with this, the groups $\text{Pin}(p, q)$ and $\text{Pin}(q, p)$ are non–isomorphic. Denote $\text{Spin}(n) = \text{Spin}(n, 0) \simeq \text{Spin}(0, n)$.

**Theorem 1** \[3\]. *The spinor groups*

\[
\text{Spin}(2), \; \text{Spin}(3), \; \text{Spin}(4), \; \text{Spin}(5), \; \text{Spin}(6)
\]

*are isomorphic to the unitary groups*

$$U(1), \; Sp(1) \sim SU(2), \; SU(2) \times SU(2), \; Sp(2), \; SU(4).$$

Over the field $\mathbb{F} = \mathbb{R}$ in the case of $p - q \equiv 1, 5 \pmod 8$ the algebra $\mathcal{C}_{p,q}$ is isomorphic to a direct sum of two mutually annihilating simple ideals $\frac{1}{2}(1 \pm \omega)\mathcal{C}_{p,q}$: $\mathcal{C}_{p,q} \simeq \frac{1}{2}(1 + \omega)\mathcal{C}_{p,q} \oplus \frac{1}{2}(1 - \omega)\mathcal{C}_{p,q}$, where $\omega = e_{12...p+q}$, $p - q \equiv 1, 5 \pmod 8$. At this point, the each ideal is isomorphic to $\mathcal{C}_{q,p-1}$ or $\mathcal{C}_{q,p-1}$. Therefore, for the Clifford–Lipschitz groups we have the following isomorphisms

$$\text{Pin}(p, q) \simeq \text{Pin}(p, q - 1) \cup \text{Pin}(p, q - 1)$$

$$\simeq \text{Pin}(q, p - 1) \cup \text{Pin}(q, p - 1). \quad (5)$$
Or, since $\mathcal{O}_{p,q-1} \simeq C_{p,q}^+ \subset C_{p,q}$, then according to [4]

$$\text{Pin}(p,q) \simeq \text{Spin}(p,q) \bigcup \text{Spin}(p,q)$$

if $p - q \equiv 1, 5 \pmod{8}$.

Further, when $p - q \equiv 3, 7 \pmod{8}$, the algebra $\mathcal{O}_{p,q}$ is isomorphic to a complex algebra $\mathbb{C}_{p+q-1}$. Therefore, for the Pin groups we obtain

$$\text{Pin}(p,q) \simeq \text{Pin}(p,q - 1) \bigcup e_{12...p+q} \text{Pin}(p,q - 1)$$

$$\simeq \text{Pin}(q,p - 1) \bigcup e_{12...p+q} \text{Pin}(q,p - 1)$$

if $p - q \equiv 1, 5 \pmod{8}$ and correspondingly

$$\text{Pin}(p,q) \simeq \text{Spin}(p,q) \cup e_{12...p+q} \text{Spin}(p,q).$$

In case of $p - q \equiv 3, 7 \pmod{8}$ we have isomorphisms which are analogous to [6]-[7], since $\omega \mathcal{O}_{p,q} \sim C_{p,q}$. Generalizing these results, we obtain the following

**Theorem 2.** Let $\text{Pin}(p,q)$ and $\text{Spin}(p,q)$ be the Clifford-Lipschitz groups of the invertible elements of the algebras $\mathcal{O}_{p,q}$ with odd dimensionality, $p - q \equiv 1, 3, 5, 7 \pmod{8}$. Then

$$\text{Pin}(p,q) \simeq \text{Pin}(p,q - 1) \bigcup \omega \text{Pin}(p,q - 1)$$

$$\simeq \text{Pin}(q,p - 1) \bigcup \omega \text{Pin}(q,p - 1)$$

and

$$\text{Pin}(p,q) \simeq \text{Spin}(p,q) \bigcup \omega \text{Spin}(p,q),$$

where $\omega = e_{12...p+q}$ is a volume element of $\mathcal{O}_{p,q}$.

In case of low dimensionalities from Theorem 1 and Theorem 2 it immediately follows

**Theorem 3.** For $p + q \leq 5$ and $p - q \equiv 3, 5 \pmod{8}$,

$$\text{Pin}(3,0) \simeq SU(2) \cup iSU(2),$$

$$\text{Pin}(0,3) \simeq SU(2) \cup eSU(2),$$

$$\text{Pin}(5,0) \simeq Sp(2) \cup eSp(2),$$

$$\text{Pin}(0,5) \simeq Sp(2) \cup iSp(2).$$

**Proof.** Indeed, in accordance with Theorem 2 $\text{Pin}(3,0) \simeq \text{Spin}(3) \cup e_{123} \text{Spin}(3)$. Further, from Theorem 1 we have $\text{Spin}(3) \simeq SU(2)$, and a square of the element $\omega = e_{123}$ is equal to $-1$, therefore, $\omega \sim i$. Thus, $\text{Pin}(3,0) \simeq SU(2) \cup iSU(2)$. For the group $\text{Pin}(0,3)$ a square of $\omega$ is equal to $+1$, therefore, $\text{Pin}(0,3) \simeq SU(2) \cup eSU(2)$, $e$ is a double unit. As expected, $\text{Pin}(3,0) \not\simeq \text{Pin}(0,3)$. The isomorphisms for the groups $\text{Pin}(5,0)$ and $\text{Pin}(0,5)$ are analogously proved. □
In turn, over the field $\mathbb{F} = \mathbb{C}$ there exists a complex Clifford–Lipschitz group
$$\Gamma_n = \{ s \in \mathbb{C}_n^+ \cup \mathbb{C}_n^- | \forall x \in \mathbb{C}^n, sxs^{-1} \in \mathbb{C}^n \}.$$ 

The group $\Gamma_n$ has a subgroup
$$\text{Pin}(n, \mathbb{C}) = \{ s \in \Gamma_n | N(s) = \pm 1 \}.$$ 

(8)

$\text{Pin}(n, \mathbb{C})$ is an universal covering of the complex orthogonal group $O(n, \mathbb{C})$. When $n \equiv 1 \pmod{2}$ we have
$$\text{Pin}(n, \mathbb{C}) \cong \text{Pin}(n-1, \mathbb{C}) \bigcup \mathbb{E}_{12...n} \text{Pin}(n-1, \mathbb{C}).$$ 

(9)

On the other hand, there exists a more detailed version of the $\text{Pin}$–group (2) proposed by Dąbrowski in 1988 [9]. In general, there are eight double coverings of the orthogonal group $O(p,q)$ [9, 4]:
$$\rho^{a,b,c} : \text{Pin}^{a,b,c}(p,q) \longrightarrow O(p,q),$$

where $a, b, c \in \{+,-\}$. As is known, the group $O(p,q)$ consists of four connected components: identity connected component $O_0(p,q)$, and three components corresponding to space inversion $P$, time reversal $T$, and a combination of these two $PT$, i.e., $O(p,q) = (O_0(p,q)) \cup P(Q_0(p,q)) \cup T(O_0(p,q)) \cup PT(O_0(p,q))$. Further, since the four–element group (reflection group) $\{1, P, T, PT\}$ is isomorphic to the finite group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ (Gauss–Klein viergruppe [23, 24]), then $O(p,q)$ may be represented by a semidirect product $O(p,q) \cong O_0(p,q) \odot (\mathbb{Z}_2 \otimes \mathbb{Z}_2)$. The signs of $a, b, c$ correspond to the signs of the squares of the elements in $\text{Pin}^{a,b,c}(p,q)$ which cover space inversion $P$, time reversal $T$ and a combination of these two $PT$ ($a = -P^2$, $b = T^2$, $c = -(PT)^2$ in Dąbrowski’s notation [9] and $a = P^2$, $b = T^2$, $c = (PT)^2$ in Chamblin’s notation [6] which we will use below). An explicit form of the group $\text{Pin}^{a,b,c}(p,q)$ is given by the following semidirect product
$$\text{Pin}^{a,b,c}(p,q) \cong \frac{(\text{Spin}_+(p,q) \odot C^{a,b,c})}{\mathbb{Z}_2},$$

(10)

where $C^{a,b,c}$ are the four double coverings of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. All the eight double coverings of the orthogonal group $O(p,q)$ are given in the following table:

| $a$ $b$ $c$ | $C^{a,b,c}$ | Remark |
|-------------|-------------|---------|
| ++ +        | $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ | $PT = TP$ |
| + + -        | $\mathbb{Z}_2 \otimes \mathbb{Z}_4$ | $PT = TP$ |
| + - +        | $\mathbb{Z}_2 \otimes \mathbb{Z}_4$ | $PT = TP$ |
| ++ -        | $\mathbb{Z}_2 \otimes \mathbb{Z}_4$ | $PT = TP$ |
| -- -        | $\mathbb{Q}_4$ | $PT = -TP$ |
| -- +        | $\mathbb{D}_4$ | $PT = -TP$ |
| + + -        | $\mathbb{D}_4$ | $PT = -TP$ |
| + + +        | $\mathbb{D}_4$ | $PT = -TP$ |

Here $\mathbb{Z}_4$, $\mathbb{Q}_4$, and $\mathbb{D}_4$ are complex, quaternion, and dihedral groups, respectively. According to [9] the group $\text{Pin}^{a,b,c}(p,q)$ satisfying the condition $PT = -TP$ is called Cliffordian, and respectively non–Cliffordian when $PT = TP$. 

7
3 Discrete symmetries and Clifford algebras

In Clifford algebra $\mathcal{C}$ there exist four fundamental automorphisms.

1) **Identity**: An automorphism $A \to A$ and $e_i \to e_i$.

   This automorphism is an identical automorphism of the algebra $\mathcal{C}$. $A$ is an arbitrary element of $\mathcal{C}$.

2) **Involution**: An automorphism $A \to A^*$ and $e_i \to -e_i$.

   In more details, for an arbitrary element $A \in \mathcal{C}$ there exists a decomposition $A = A' + A''$, where $A'$ is an element consisting of homogeneous odd elements, and $A''$ is an element consisting of homogeneous even elements, respectively. Then the automorphism $A \to A^*$ is such that the element $A''$ is not changed, and the element $A'$ changes sign: $A^* = -A' + A''$.

   If $A$ is a homogeneous element, then

   $A^* = (-1)^k A$,  \hspace{1cm} (11)

   where $k$ is a degree of the element. It is easy to see that the automorphism $A \to A^*$ may be expressed via the volume element $\omega = e_{12...p+q}$:

   $A^* = \omega A \omega^{-1}$,  \hspace{1cm} (12)

   where $\omega^{-1} = (-1)^{(p+q)(p+q-1)/2} \omega$. When $k$ is odd, the basis elements $e_{i_1i_2...i_k}$ the sign changes, and when $k$ is even, the sign is not changed.

3) **Reversion**: An antiautomorphism $A \to \tilde{A}$ and $e_i \to e_i$.

   The antiautomorphism $A \to \tilde{A}$ is a reversion of the element $A$, that is the substitution of each basis element $e_{i_1i_2...i_k} \in A$ by the element $e_{i_ki_{k-1}...i_1}$:

   $e_{i_ki_{k-1}...i_1} = (-1)^{\frac{k(k-1)}{2}} e_{1i_2...i_k}$.

   Therefore, for any $A \in \mathcal{C}_{p,q}$ we have

   $\tilde{A} = (-1)^{\frac{k(k-1)}{2}} A$.  \hspace{1cm} (13)

4) **Conjugation**: An antiautomorphism $A \to \tilde{A}^*$ and $e_i \to -e_i$.

   This antiautomorphism is a composition of the antiautomorphism $A \to \tilde{A}$ with the automorphism $A \to A^*$. In the case of a homogeneous element from the formulae (11) and (13), it follows

   $\tilde{A}^* = (-1)^{\frac{k(k+1)}{2}} A$.  \hspace{1cm} (14)

   One of the most fundamental theorems in the theory of associative algebras is

**Theorem 4** (Wedderburn–Artin). Any finite–dimensional associative simple algebra $\mathfrak{A}$ over the field $\mathbb{F}$ is isomorphic to a full matrix algebra $M_n(\mathbb{K})$, where a natural number $n$ defined unambiguously, and a division ring $\mathbb{K}$ defined with an accuracy of isomorphism.

In accordance with this theorem all properties of the initial algebra $\mathfrak{A}$ are isomorphically transferred to the matrix algebra $M_n(\mathbb{K})$. Later on we will widely use this theorem. In its turn, for the Clifford algebra $\mathcal{C}_{p,q}$ over the field $\mathbb{F} = \mathbb{R}$ we have an isomorphism $\mathcal{C}_{p,q} \simeq \text{End}_\mathbb{K}(I_{p,q}) \simeq M_{2m}(\mathbb{K})$, where $m = \frac{p+q}{2}$, $I_{p,q} = \mathcal{C}_{p,q}f$ is a minimal left ideal of $\mathcal{C}_{p,q}$, and
$\mathbb{K} = f\mathcal{O}_{p,q}f$ is a division ring of $\mathcal{O}_{p,q}$. The primitive idempotent of the algebra $\mathcal{O}_{p,q}$ has a form

$$f = \frac{1}{2}(1 \pm e_{a_1})\frac{1}{2}(1 \pm e_{a_2}) \cdots \frac{1}{2}(1 \pm e_{a_k}),$$

where $e_{a_1}, e_{a_2}, \ldots, e_{a_k}$ are commuting elements with square 1 of the canonical basis of $\mathcal{O}_{p,q}$ generating a group of order $2^k$. The values of $k$ are defined by a formula $k = q - r_{q-p}$, where $r_i$ are the Radon–Hurwitz numbers \cite{19, 13}, values of which form a cycle of the period 8: $r_{i+8} = r_i + 4$. The values of all $r_i$ are

$$
\begin{array}{cccccccc}
  i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
  r_i & 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3
\end{array}
$$

The all Clifford algebras $\mathcal{O}_{p,q}$ over the field $\mathbb{F} = \mathbb{R}$ are divided into eight different types with a following division ring structure:

I. Central simple algebras.

1) Two types $p - q \equiv 0, 2 \pmod{8}$ with a division ring $\mathbb{K} \simeq \mathbb{R}$.

2) Two types $p - q \equiv 3, 7 \pmod{8}$ with a division ring $\mathbb{K} \simeq \mathbb{C}$.

3) Two types $p - q \equiv 4, 6 \pmod{8}$ with a division ring $\mathbb{K} \simeq \mathbb{H}$.

II. Semi-simple algebras.

4) The type $p - q \equiv 1 \pmod{8}$ with a double division ring $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$.

5) The type $p - q \equiv 5 \pmod{8}$ with a double quaternionic division ring $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$.

Over the field $\mathbb{F} = \mathbb{C}$ there is an isomorphism $\mathbb{C}_n \simeq M_{2n/2}(\mathbb{C})$ and there are two different types of complex Clifford algebras $\mathbb{C}_n$: $n \equiv 0 \pmod{2}$ and $n \equiv 1 \pmod{2}$.

In virtue of the Wedderburn–Artin theorem the all fundamental automorphisms of $\mathcal{O}$ are transferred to the matrix algebra. Matrix representations of the fundamental automorphisms of $\mathcal{C}_n$ were first obtained by Rashevskii in 1955 \cite{21}: 1) Involution: $A^* = WAW^{-1}$, where $W$ is a matrix of the automorphism $*$ (matrix representation of the volume element $\omega$); 2) Reversion: $\tilde{A} = EA^T E^{-1}$, where $E$ is a matrix of the antiautomorphism $\sim$ satisfying the conditions $E_i E - E_i^T = 0$ and $E^T = (-1)^{\frac{m(m-1)}{2}}E_i$ here $E_i = \gamma(e_i)$ are matrix representations of the units of the algebra $\mathcal{O}$; 3) Conjugation: $\tilde{A}^* = C A^T C^{-1}$, where $C = E W^T$ is a matrix of the antiautomorphism $\bar{\gamma}$ satisfying the conditions $C E^T + E_i C = 0$ and $C^T = (-1)^{\frac{m(m+1)}{2}}C$.

In the recent paper \cite{23} it has been shown that space reversal $P$, time reversal $T$ and combination $PT$ are correspond respectively to the fundamental automorphisms $A \rightarrow A^*$, $A \rightarrow \tilde{A}$ and $A \rightarrow \tilde{A}^*$.

**Proposition 1.** Let $\mathcal{O}_{p,q}$ ($p + q = 2m$) be a Clifford algebra over the field $\mathbb{F} = \mathbb{R}$ and let $\text{Pin}(p,q)$ be a double covering of the orthogonal group $O(p,q) = O_0(p,q) \oplus \{1, P, T, PT\} \simeq O_0(p,q) \circ (\mathbb{Z}_2 \otimes \mathbb{Z}_2)$ of transformations of the space $\mathbb{R}^{p,q}$, where $\{1, P, T, PT\} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ is a group of discrete transformations of $\mathbb{R}^{p,q}$, $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ is the Gauss–Klein group. Then there is an isomorphism between the group $\{1, P, T, PT\}$ and an automorphism group $\{\text{Id}, *, \sim, \bar{\gamma}\}$ of the algebra $\mathcal{O}_{p,q}$. In this case, space inversion $P$, time reversal $T$ and combination $PT$ are correspond to the fundamental automorphisms $A \rightarrow A^*$, $A \rightarrow \tilde{A}$ and $A \rightarrow \tilde{A}^*$.
An equivalence of the multiplication tables of the groups \( \{1, P, T, PT\} \) and \( \text{Aut}(\mathcal{O}) = \{\text{Id}, *, \sim, \tilde{*}\} \) proves this isomorphism (in virtue of the commutativity \((\tilde{A}^*) = (\tilde{A})^*\) and the involution conditions \((*)^2 = (\sim)^2 = \text{Id}\):

|    | \text{Id} | \star | \sim | \tilde{*} |
|----|----------|------|------|--------|
| \text{Id} | \text{Id} | \star | \sim | \tilde{*} |
| \star | \text{Id} | \star | \sim | \tilde{*} |
| \sim | \sim | \text{Id} | \star | \tilde{*} |
| \tilde{*} | \tilde{*} | \sim | \text{Id} | \star |

Further, in the case \(P^2 = T^2 = (PT)^2 = \pm 1\) and \(PT = -TP\) there is an isomorphism between the group \(\{1, P, T, PT\}\) and an automorphism group \(\text{Aut}(\mathcal{O}) = \{I, W, E, C\}\). So, for the Dirac algebra \(\mathbb{C}_4\) in the canonical \(\gamma\)-basis there exists a standard (Wigner) representation \(P = \gamma_0\) and \(T = \gamma_1\gamma_3\) \([3]\), therefore, \(\{1, P, T, PT\} = \{1, \gamma_0, \gamma_1\gamma_3, \gamma_0\gamma_1\gamma_3\}\). On the other hand, in the \(\gamma\)-basis an automorphism group of \(\mathbb{C}_4\) has a form \(\text{Aut}(\mathbb{C}_4) = \{I, W, E, C\} = \{1, \gamma_0\gamma_1\gamma_2\gamma_3, \gamma_1\gamma_3, \gamma_0\gamma_2\}\). It has been shown \([28]\) that \(\{1, P, T, PT\} = \{1, \gamma_0, \gamma_1\gamma_3, \gamma_0\gamma_1\gamma_3\} \simeq \text{Aut}(\mathbb{C}_4) \simeq \mathbb{Z}_4\), where \(\mathbb{Z}_4\) is a complex group with the signature \((+, -, -)\). Generalizations of these results onto the algebras \(\mathbb{C}_n\) are contained in the following two theorems:

**Theorem 5** \([28]\). Let \(\text{Aut} = \{I, W, E, C\}\) be the automorphism group of the algebra \(\mathbb{C}_{p+q}\) \((p + q = 2m)\), where \(W = \varepsilon_1\varepsilon_2\cdots\varepsilon_m\varepsilon_{m+1}\varepsilon_{m+2}\cdots\varepsilon_{p+q}\), and \(E = \varepsilon_1\varepsilon_2\cdots\varepsilon_m\), \(C = \varepsilon_{m+1}\varepsilon_{m+2}\cdots\varepsilon_{p+q}\), \(\varepsilon_m = 1\) (mod 2), and \(\varepsilon_{m+1}\varepsilon_{m+2}\cdots\varepsilon_{p+q}\). Let \(\text{Aut}_-\) and \(\text{Aut}_+\) be the automorphism groups, in which all elements correspondingly commute \((m \equiv 0\) (mod 2)) and anticommute \((m \equiv 1\) (mod 2))

Then over the field \(\mathbb{F} = \mathbb{C}\) there are only two non–isomorphic groups: \(\text{Aut}_- \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2\) for the signature \((+, +, +)\) if \(n \equiv 0\) (mod 4) and \(\text{Aut}_+ \simeq \mathbb{Q}_4 / \mathbb{Z}_2\) for the signature \((-,-,-)\) if \(n \equiv 2\) (mod 4).

**Theorem 6** \([28]\). Let \(\text{Pin}_{a,b,c}(p,q)\) be a double covering of the complex orthogonal group \(O(n,\mathbb{C})\) of the space \(\mathbb{C}^n\) associated with the complex algebra \(\mathbb{C}_n\). Squares of the symbols \(a, b, c \in \{-, +\}\) correspond to squares of the elements of the finite group \(\mathbb{A} = \{I, W, E, C\} : a = W^2, b = E^2, c = C^2\), where \(W, E\) and \(C\) are correspondingly the matrices of the fundamental automorphisms \(\mathbb{A} \rightarrow \mathbb{A}^*, \mathbb{A} \rightarrow \tilde{\mathbb{A}}\) and \(\mathbb{A} \rightarrow \tilde{\mathbb{A}}^*\) of \(\mathbb{C}_n\). Then over the field \(\mathbb{F} = \mathbb{C}\) for the algebra \(\mathbb{C}_n\) there exist two non–isomorphic double coverings of the group \(O(n,\mathbb{C})\):

1) Non–Cliffordian groups

\[
\text{Pin}^{+,-,+}(n,\mathbb{C}) \simeq (\text{Spin}_+(n,\mathbb{C}) \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2),
\]

if \(n \equiv 0\) (mod 4) and

\[
\text{Pin}^{+,-,+}(n,\mathbb{C}) \simeq \text{Pin}^{+,-,+}(n-1,\mathbb{C}) \cup e_{12} \cdot \text{Pin}^{+,-,+}(n-1,\mathbb{C}),
\]

if \(n \equiv 1\) (mod 4).

2) Cliffordian groups

\[
\text{Pin}^{-,-,-}(n,\mathbb{C}) \simeq (\text{Spin}_-(n,\mathbb{C}) \otimes \mathbb{Q}_4) / \mathbb{Z}_2,
\]

10
if $n \equiv 2 \pmod{4}$ and
\[
\text{Pin}^{\sim \sim}(n, \mathbb{C}) \simeq \text{Pin}^{\sim \sim}(n - 1, \mathbb{C}) \cup e_{12 \ldots n} \text{Pin}^{\sim \sim}(n - 1, \mathbb{C}),
\]
if $n \equiv 3 \pmod{4}$.

A consideration of the discrete symmetries over the field of real numbers is a much more complicated problem. First of all, in contrast to the field of complex numbers over the field $\mathbb{F} = \mathbb{R}$ there exist eight different types of the Clifford algebras and five division rings, which, in virtue of the Wedderburn–Artin Theorem, impose hard restrictions on existence and choice of the matrix representations for the fundamental automorphisms.

**Theorem 7.** Let $\mathcal{O}_{p,q}$ be a Clifford algebra over a field $\mathbb{F} = \mathbb{R}$ and let $\text{Aut}(\mathcal{O}_{p,q}) = \{1, W, E, C\}$ be a group of fundamental automorphisms of the algebra $\mathcal{O}_{p,q}$. Then for eight types of the algebras $\mathcal{O}_{p,q}$ there exist, depending upon a division ring structure of $\mathcal{O}_{p,q}$, following isomorphisms between finite groups and groups $\text{Aut}(\mathcal{O}_{p,q})$ with different signatures $(a, b, c)$, where $a, b, c \in \{-, +\}$:

1) $\mathbb{K} \simeq \mathbb{R}$, types $p - q \equiv 0, 2 \pmod{8}$.

If $E = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$ and $C = \mathcal{E}_{1}\mathcal{E}_{2}\cdots\mathcal{E}_{p}$, then Abelian groups $\text{Aut}_{-}(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with the signature $(+, +, +)$ and $\text{Aut}_{-}(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(+, -, -)$ exist at $p, q \equiv 0 \pmod{4}$ and $p, q \equiv 2 \pmod{4}$, respectively, for the type $p - q \equiv 0 \pmod{8}$, and also Abelian groups $\text{Aut}_{-}(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(-, -, +)$ and $\text{Aut}_{-}(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(-, +, -)$ exist at $p \equiv 0 \pmod{4}$, $q \equiv 2 \pmod{4}$ and $p \equiv 2 \pmod{4}$, $q \equiv 0 \pmod{4}$ for the type $p - q \equiv 2 \pmod{8}$, respectively.

If $E = \mathcal{E}_{1}\mathcal{E}_{2}\cdots\mathcal{E}_{p}$ and $C = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$, then non–Abelian groups $\text{Aut}_{+}(\mathcal{O}_{p,q}) \simeq D_4/\mathbb{Z}_2$ with the signature $(+, -, +)$ and $\text{Aut}_{+}(\mathcal{O}_{p,q}) \simeq D_4/\mathbb{Z}_2$ with the signature $(+, +, -)$ exist at $p, q \equiv 3 \pmod{4}$ and $p, q \equiv 1 \pmod{4}$, respectively, for the type $p - q \equiv 0 \pmod{8}$, and also non–Abelian groups $\text{Aut}_{+}(\mathcal{O}_{p,q}) \simeq Q_4/\mathbb{Z}_2$ with $(-, -, +)$ and $\text{Aut}_{+}(\mathcal{O}_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(-, +, +)$ exist at $p \equiv 3 \pmod{4}$, $q \equiv 1 \pmod{4}$ and $p \equiv 1 \pmod{4}$, $q \equiv 3 \pmod{4}$ for the type $p - q \equiv 2 \pmod{8}$, respectively.

2) $\mathbb{K} \simeq \mathbb{H}$, types $p - q \equiv 4, 6 \pmod{8}$.

If $E = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ is a product of $k$ skewsymmetric matrices (among which $l$ matrices have a square $+1$ and $t$ matrices have a square $-1$) and $C = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ is a product of $p + q - k$ symmetric matrices (among which $h$ matrices have a square $+1$ and $g$ have a square $-1$), then at $k \equiv 0 \pmod{2}$ for the type $p - q \equiv 4 \pmod{8}$ there exist Abelian groups $\text{Aut}_{-}(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with $(+, +, +)$ and $\text{Aut}_{-}(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4$ with $(+, -, -)$ if $l - t, h - g \equiv 0, 1, 4, 5 \pmod{8}$ and $l - t, h - g \equiv 2, 3, 6, 7 \pmod{8}$, respectively. And also at $k \equiv 0 \pmod{2}$ for the type $p - q \equiv 6 \pmod{8}$ there exist $\text{Aut}_{-}(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4$ with $(-, -, +)$ and $\text{Aut}_{-}(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4$ with $(-, +, -)$ if $l - t \equiv 0, 1, 4, 5 \pmod{8}$, $h - g \equiv 2, 3, 6, 7 \pmod{8}$ and $l - t \equiv 2, 3, 6, 7 \pmod{8}$, $h - g \equiv 0, 1, 4, 5 \pmod{8}$, respectively.

Inversely, if $E = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ is a product of $p + q - k$ symmetric matrices and $C = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ is a product of $k$ skewsymmetric matrices, then at $k \equiv 1 \pmod{2}$ for the type $p - q \equiv 4 \pmod{8}$ there exist non–Abelian groups $\text{Aut}_{+}(\mathcal{O}_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(+, +, +)$ and $\text{Aut}_{+}(\mathcal{O}_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(+, -, -)$ if $h - g \equiv 2, 3, 6, 7 \pmod{8}$, $l - t \equiv 0, 1, 4, 5 \pmod{8}$ and $h - g \equiv 0, 1, 4, 5 \pmod{8}$, $l - t \equiv 2, 3, 6, 7 \pmod{8}$, respectively. And also at $k \equiv 1 \pmod{2}$ for the type $p - q \equiv 6 \pmod{8}$ there exist $\text{Aut}_{+}(\mathcal{O}_{p,q}) \simeq Q_4/\mathbb{Z}_2$ with $(-, -, -)$ and $\text{Aut}_{+}(\mathcal{O}_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(-, +, +)$ if $h - g, l - t \equiv 2, 3, 6, 7 \pmod{8}$ and
h - g, l - t \equiv 0, 1, 4, 5 \pmod{8}, \text{ respectively.}

3) \mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}, \mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}, \text{ types } p - q \equiv 1, 5 \pmod{8}.

For the algebras \( \mathcal{A}_{p,0} \) of the types \( p - q \equiv 1, 5 \pmod{8} \) there exist Abelian automorphism groups with the signatures \((-,-,+), (-,+,-)\) and non-Abelian automorphism groups with the signatures \((-,-,-), (+,-,-)\). Correspondingly, for the algebras \( \mathcal{A}_{p,0} \) of the types \( p - q \equiv 1, 5 \pmod{8} \) there exist Abelian groups with \((+,+,+), (+,-,-)\) and non-Abelian groups with \((+,-,+), (+,+,-)\). In general case for \( \mathcal{A}_{p,q} \), the types \( p - q \equiv 1, 5 \pmod{8} \) admit all eight automorphism groups.

4) \mathbb{K} = \mathbb{C}, \text{ types } p - q \equiv 3, 7 \pmod{8}.

The types \( p - q \equiv 3, 7 \pmod{8} \) admit the Abelian group \( \text{Aut}_- (\mathcal{A}_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) with the signature \((+,+,+)\) if \( p \equiv 0 \pmod{2} \) and \( q \equiv 1 \pmod{2} \), and also non-Abelian group \( \text{Aut}_+ (\mathcal{A}_{p,q}) \simeq \mathbb{Q}_4/\mathbb{Z}_2 \) with the signature \((-,-,-)\) if \( p \equiv 1 \pmod{2} \) and \( q \equiv 0 \pmod{2} \).

**Theorem 8.** Let \( \text{Pin}^{a,b,c}(p,q) \) be a double covering of the orthogonal group \( O(p,q) \) of the real space \( \mathbb{R}^{p,q} \) associated with the algebra \( \mathcal{A}_{p,q} \). The squares of symbols \( a, b, c \in \{-,+\} \) correspond to the squares of the elements of a finite group \( \text{Aut}(\mathcal{A}_{p,q}) = \{1, W, E, C\} : a = W^2, b = E^2, c = C^2 \), where \( W, E \) and \( C \) are the matrices of the fundamental automorphisms \( A \to A^*, A \to \bar{A} \) and \( A \to A^* \) of the algebra \( \mathcal{A}_{p,q} \), respectively. Then over the field \( \mathbb{F} = \mathbb{R} \) in dependence on a division ring structure of the algebra \( \mathcal{A}_{p,q} \), there exist eight double coverings of the orthogonal group \( O(p,q) \):

1) A non-Cliffordian group

\[
\text{Pin}^{+,+,+}(p,q) \simeq \frac{(\text{Spin}_0(p,q) \circ \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)}{\mathbb{Z}_2},
\]

exists if \( \mathbb{K} \simeq \mathbb{R} \) and the numbers \( p \) and \( q \) form the type \( p - q \equiv 0 \pmod{8} \) and \( p,q \equiv 0 \pmod{4} \), and also if \( p - q \equiv 4 \pmod{8} \) and \( \mathbb{K} \simeq \mathbb{H} \). The algebras \( \mathcal{A}_{p,q} \) with the rings \( \mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}, \mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H} \) \((p - q \equiv 1, 5 \pmod{8})\) admit the group \( \text{Pin}^{+,+,+}(p,q) \) if in the direct sums there are addendums of the type \( p - q \equiv 0 \pmod{8} \) or \( p - q \equiv 4 \pmod{8} \). The types \( p - q \equiv 3, 7 \pmod{8} \), \( \mathbb{K} \simeq \mathbb{C} \) admit a non-Cliffordian group \( \text{Pin}^{+,+,+}(p+q-1,\mathbb{C}) \) if \( p \equiv 0 \pmod{2} \) and \( q \equiv 1 \pmod{2} \). Further, non-Cliffordian groups

\[
\text{Pin}^{a,b,c}(p,q) \simeq \frac{(\text{Spin}_0(p,q) \circ (\mathbb{Z}_2 \otimes \mathbb{Z}_4))}{\mathbb{Z}_2},
\]

with \((a,b,c) = (+,-,-)\) exist if \( p - q \equiv 0 \pmod{8} \), \( p,q \equiv 2 \pmod{4} \) and \( \mathbb{K} \simeq \mathbb{R} \), and also if \( p - q \equiv 4 \pmod{8} \) and \( \mathbb{K} \simeq \mathbb{H} \). Non-Cliffordian groups with the signatures \((a,b,c) = (-,+,+)\) and \((a,b,c) = (-,-,+))\) exist over the ring \( \mathbb{K} \simeq \mathbb{R} \) \((p - q \equiv 2 \pmod{8})\) if \( p \equiv 2 \pmod{4} \), \( q \equiv 0 \pmod{4} \) and \( p \equiv 0 \pmod{4} \) and \( q \equiv 2 \pmod{4} \), respectively, and also these groups exist over the ring \( \mathbb{K} \simeq \mathbb{H} \) if \( p - q \equiv 6 \pmod{8} \). The algebras \( \mathcal{A}_{p,q} \) with the rings \( \mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}, \mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H} \) \((p - q \equiv 1, 5 \pmod{8})\) admit the group \( \text{Pin}^{+,-,+}(p,q) \) if in the direct sums there are addendums of the type \( p - q \equiv 0 \pmod{8} \) or \( p - q \equiv 4 \pmod{8} \), and also admit the groups \( \text{Pin}^{-+,+}(p,q) \) and \( \text{Pin}^{-,+,+}(p,q) \) if in the direct sums there are addendums of the type \( p - q \equiv 2 \pmod{8} \) or \( p - q \equiv 6 \pmod{8} \).

2) A Cliffordian group

\[
\text{Pin}^{-,+}(p,q) \simeq \frac{(\text{Spin}_0(p,q) \circ \mathbb{Q}_4)}{\mathbb{Z}_2},
\]
exists if \( K \simeq \mathbb{R} \ (p-q \equiv 2 \pmod{8}) \) and \( p \equiv 3 \pmod{4}, q \equiv 1 \pmod{4} \), and also if \( p-q \equiv 6 \pmod{8} \) and \( K \simeq \mathbb{H} \). The algebras \( \mathcal{A}_{p,q} \) with the rings \( K \simeq \mathbb{R} \oplus \mathbb{R}, K \simeq \mathbb{H} \oplus \mathbb{H} \) (\( p-q \equiv 1, 5 \pmod{8} \)) admit the group \( \text{Pin}^{-,-}(p,q) \) if in the direct sums there are addends of the type \( p-q \equiv 2 \pmod{8} \) or \( p-q \equiv 6 \pmod{8} \). The types \( p-q \equiv 3, 7 \pmod{8} \), \( K \simeq \mathbb{C} \) admit a Cliffordian group \( \text{Pin}^{-,-}(p+q-1, \mathbb{C}) \), if \( p \equiv 1 \pmod{2} \) and \( q \equiv 0 \pmod{2} \). Further, Cliffordian groups

\[
\text{Pin}^{a,b,c}(p,q) \simeq \frac{(\text{Spin}_0(p,q) \otimes D_4)}{\mathbb{Z}_2},
\]

with \( (a,b,c) = (-,+,+), (+,-,+), (-,-,+), (-,+,+) \) exist if \( K \simeq \mathbb{R} \ (p-q \equiv 2 \pmod{8}) \) and \( p \equiv 1 \pmod{4}, q \equiv 3 \pmod{4} \), and also if \( p-q \equiv 6 \pmod{8} \) and \( K \simeq \mathbb{H} \). Cliffordian groups with the signatures \( (a,b,c) = (+,-,+), (-,-,+), (-,+,+) \) exist over the ring \( K \simeq \mathbb{R} \ (p-q \equiv 0 \pmod{8}) \) if \( p,q \equiv 3 \pmod{4} \) and \( p,q \equiv 1 \pmod{4} \), respectively, and also these groups exist over the ring \( K \simeq \mathbb{H} \) if \( p-q \equiv 4 \pmod{8} \). The algebras \( \mathcal{A}_{p,q} \) with the rings \( K \simeq \mathbb{R} \oplus \mathbb{R}, K \simeq \mathbb{H} \oplus \mathbb{H} \) (\( p-q \equiv 1, 5 \pmod{8} \)) admit the group \( \text{Pin}^{-,+}(p,q) \) if in the direct sums there are addends of the type \( p-q \equiv 2 \pmod{8} \) or \( p-q \equiv 6 \pmod{8} \), and also admit the groups \( \text{Pin}^{+,+}(p,q) \) and \( \text{Pin}^{+,+}(p,q) \) if in the direct sums there are addends of the type \( p-q \equiv 0 \pmod{8} \) or \( p-q \equiv 4 \pmod{8} \).

4 Pseudoautomorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \) and charge conjugation

As is known, the algebra \( \mathbb{C}_n \) is associated with a complex vector space \( \mathbb{C}^n \). Let \( n = p+q \), then an extraction operation of the real subspace \( \mathbb{R}^{p,q} \) in \( \mathbb{C}^n \) forms the foundation of definition of the discrete transformation known in physics as a charge conjugation \( C \). Indeed, let \( \{e_1, \ldots, e_n\} \) be an orthobasis in the space \( \mathbb{C}^n \), \( e_i^2 = 1 \). Let us remain the first \( p \) vectors of this basis unchanged, and other \( q \) vectors multiply by the factor \( i \). Then the basis

\[
\{e_1, \ldots, e_p, ie_{p+1}, \ldots, ie_{p+q}\}
\]

allows us to extract the subspace \( \mathbb{R}^{p,q} \) in \( \mathbb{C}^n \). Namely, for the vectors \( \mathbb{R}^{p,q} \) we take the vectors of \( \mathbb{C}^n \) which decompose on the basis (15) with real coefficients. In this way, we obtain a real vector space \( \mathbb{R}^{p,q} \) endowed (in general case) with a non–degenerate quadratic form

\[
Q(x) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \ldots - x_{p+q}^2,
\]

where \( x_1, \ldots, x_{p+q} \) are coordinates of the vector \( x \) in the basis (15). It is easy to see that the extraction of \( \mathbb{R}^{p,q} \) in \( \mathbb{C}^n \) induces an extraction of a real subalgebra \( \mathcal{A}_{p,q} \) in \( \mathbb{C}_n \). Therefore, any element \( \mathcal{A} \in \mathbb{C}_n \) can be unambiguously represented in the form

\[
\mathcal{A} = A_1 + iA_2,
\]

where \( A_1, A_2 \in \mathcal{A}_{p,q} \). The one-to-one mapping

\[
\mathcal{A} \rightarrow \overline{\mathcal{A}} = A_1 - iA_2
\]

transforms the algebra \( \mathbb{C}_n \) into itself with preservation of addition and multiplication operations for the elements \( \mathcal{A} \); the operation of multiplication of the element \( \mathcal{A} \) by the number
transforms to an operation of multiplication by the complex conjugate number. Any mapping of \( \mathbb{C}_n \) satisfying these conditions is called a pseudoautomorphism. Thus, the extraction of the subspace \( \mathbb{R}^{p,q} \) in the space \( \mathbb{C}^n \) induces in the algebra \( \mathbb{C}_n \) a pseudoautomorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \) \cite{21,22}.

Let us consider a spinor representation of the pseudoautomorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \) of the algebra \( \mathbb{C}_n \) when \( n \equiv 0 \pmod{2} \). In the spinor representation the every element \( \mathcal{A} \in \mathbb{C}_n \) should be represented by some matrix \( \mathcal{A} \), and the pseudoautomorphism \( \Pi \) takes a form of the pseudoautomorphism of the full matrix algebra \( M_{2n/2} \):

\[
\mathcal{A} \rightarrow \overline{\mathcal{A}}.
\]

On the other hand, a transformation replacing the matrix \( \mathcal{A} \) by the complex conjugate matrix, \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \), is also some pseudoautomorphism of the algebra \( M_{2n/2} \). The composition of the two pseudoautomorphisms \( \mathcal{A} \rightarrow \mathcal{A} \) and \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \), \( \mathcal{A} \rightarrow \mathcal{A} \rightarrow \overline{\mathcal{A}} \), is an internal automorphism \( \mathcal{A} \rightarrow \mathcal{A} \) of the full matrix algebra \( M_{2n/2} \):

\[
\overline{\mathcal{A}} = \Pi \mathcal{A} \Pi^{-1},
\]

where \( \Pi \) is a matrix of the pseudoautomorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \) in the spinor representation. The sufficient condition for definition of the pseudoautomorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \) is a choice of the matrix \( \Pi \) in such a way that the transformation \( \mathcal{A} \rightarrow \Pi \mathcal{A} \Pi^{-1} \) transfers into itself the matrices \( E_1, \ldots, E_p, iE_{p+1}, \ldots, iE_{p+q} \) (the matrices of the spinbasis of \( \mathcal{C}_{p,q} \)), that is,

\[
E_i \rightarrow E_i = \Pi E_i \Pi^{-1} \quad (i = 1, \ldots, p+q).
\]

**Theorem 9.** Let \( \mathbb{C}_n \) be a complex Clifford algebra for \( n \equiv 0 \pmod{2} \) and let \( \mathcal{C}_{p,q} \subset \mathbb{C}_n \) be its subalgebra with a real division ring \( \mathbb{K} \cong \mathbb{R} \) when \( p - q \equiv 0,2 \pmod{8} \) and quaternionic division ring \( \mathbb{K} \cong \mathbb{H} \) when \( p - q \equiv 4,6 \pmod{8} \), \( n = p + q \). Then in dependence on the division ring structure of the real subalgebra \( \mathcal{C}_{p,q} \) the matrix \( \Pi \) of the pseudoautomorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \) has the following form:

1) \( \mathbb{K} \cong \mathbb{R} \), \( p - q \equiv 0,2 \pmod{8} \).

The matrix \( \Pi \) for any spinor representation over the ring \( \mathbb{K} \cong \mathbb{R} \) is proportional to the unit matrix.

2) \( \mathbb{K} \cong \mathbb{H} \), \( p - q \equiv 4,6 \pmod{8} \).

\[
\Pi = E_{a_1} E_{a_2} \cdots E_{a_n} \quad \text{when} \quad a \equiv 0 \pmod{2} \quad \text{and} \quad \Pi = E_{b_1} E_{b_2} \cdots E_{b_s} \quad \text{when} \quad b \equiv 1 \pmod{2},
\]

where a complex matrices \( E_{a_1} \) and real matrices \( E_{b_1} \) form a basis of the spinor representation of the algebra \( \mathcal{C}_{p,q} \) over the ring \( \mathbb{K} \cong \mathbb{H} \), \( a + b = p + q \), \( 0 < t \leq a \), \( 0 < s \leq b \). At this point,

\[
\Pi \Pi = \begin{cases} 
1 & \text{if} \ a, b \equiv 0,1 \pmod{4}, \\
-1 & \text{if} \ a, b \equiv 2,3 \pmod{4},
\end{cases}
\]

where \( l \) is the unit matrix.

**Proof.** The algebra \( \mathbb{C}_n \) \( (n \equiv 0 \pmod{2}, n = p + q) \) in virtue of \( \mathbb{C}_n = \mathbb{C} \otimes \mathcal{C}_{p,q} \) and definition of the division ring \( \mathbb{K} \cong f \mathcal{C}_{p,q} f \) \((f \text{ is a primitive idempotent of the algebra } \mathcal{C}_{p,q})\) has four different real subalgebras: \( p - q \equiv 0,2 \pmod{8} \) for the real division ring \( \mathbb{K} \cong \mathbb{R} \) and \( p - q \equiv 4,6 \pmod{8} \) for the quaternionic division ring \( \mathbb{K} \cong \mathbb{H} \).

1) \( \mathbb{K} \cong \mathbb{R} \).
Since for the types \( p - q \equiv 0,2 \pmod{8} \) there is an isomorphism \( \mathcal{A}_{p,q} \simeq M_{\frac{p+q}{2}}(\mathbb{R}) \) (Wedderburn–Artin Theorem), then all the matrices \( \mathcal{E}_i \) of the spinbasis of \( \mathcal{A}_{p,q} \) are real and \( \tilde{\mathcal{E}}_i = \mathcal{E}_i \). Therefore, in this case the condition (18) can be written as follows

\[
\mathcal{E}_i \rightarrow \mathcal{E}_i = \Pi \mathcal{E}_i \Pi^{-1},
\]

whence \( \mathcal{E}_i \Pi = \Pi \mathcal{E}_i \). Thus, for the algebras \( \mathcal{A}_{p,q} \) of the types \( p - q \equiv 0,2 \pmod{8} \) the matrix \( \Pi \) of the pseudoautomorphism \( \mathcal{A} \rightarrow \mathcal{A} \) commutes with all the matrices \( \mathcal{E}_i \). It is easy to see that \( \Pi \sim I \).

2) \( K \simeq \mathbb{H} \).

In turn, for the quaternionic types \( p - q \equiv 4,6 \pmod{8} \) there is an isomorphism \( \mathcal{A}_{p,q} \simeq M_{\frac{p+q}{2}}(\mathbb{H}) \). Therefore, among the matrices of the spinbasis of the algebra \( \mathcal{A}_{p,q} \) there are matrices \( \mathcal{E}_a \) satisfying the condition \( \tilde{\mathcal{E}}_a = -\mathcal{E}_a \). Let \( a \) be a quantity of the complex matrices, then the spinbasis of \( \mathcal{A}_{p,q} \) is divided into two subsets. The first subset \( \{ \mathcal{E}_a = -\mathcal{E}_a \} \) contains complex matrices, \( 0 < t \leq a \), and the second subset \( \{ \mathcal{E}_s = \mathcal{E}_s \} \) contains real matrices, \( 0 < s \leq p + q - a \). In accordance with a spinbasis structure of the algebra \( \mathcal{A}_{p,q} \simeq M_{\frac{p+q}{2}}(\mathbb{H}) \), the condition (18) can be written as follows

\[
\mathcal{E}_{\alpha t} \rightarrow -\mathcal{E}_{\alpha t} = \Pi \mathcal{E}_{\alpha t} \Pi^{-1}, \quad \mathcal{E}_{\beta s} \rightarrow \mathcal{E}_{\beta s} = \Pi \mathcal{E}_{\beta s} \Pi^{-1}.
\]

Whence

\[
\mathcal{E}_{\alpha t} \Pi = -\Pi \mathcal{E}_{\alpha t}, \quad \mathcal{E}_{\beta s} \Pi = \Pi \mathcal{E}_{\beta s}.
\]

Thus, for the quaternionic types \( p - q \equiv 4,6 \pmod{8} \) the matrix \( \Pi \) of the pseudoautomorphism \( \mathcal{A} \rightarrow \mathcal{A} \) anticommutes with a complex part of the spinbasis of \( \mathcal{A}_{p,q} \) and commutes with a real part of the same spinbasis. From (19) it follows that a structure of the matrix \( \Pi \) is analogous to the structure of the matrices \( \mathbb{E} \) and \( \mathbb{C} \) of the antiautomorphisms \( \mathcal{A} \rightarrow \mathcal{A}^1 \) and \( \mathcal{A} \rightarrow \mathcal{A}^* \), correspondingly (see Theorem 7), that is, the matrix \( \Pi \) of the pseudoautomorphism \( \mathcal{A} \rightarrow \mathcal{A} \) of the algebra \( \mathbb{C}_n \) is a product of only complex matrices, or only real matrices of the spinbasis of the subalgebra \( \mathcal{A}_{p,q} \).

So, let \( 0 < a < p + q \) and let \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \) be a matrix of \( \mathcal{A} \rightarrow \mathcal{A} \), then permutation conditions of the matrix \( \Pi \) with the matrices \( \mathcal{E}_{\beta_s} \) of the real part \( (0 < s \leq p + q - a) \) and with the matrices \( \mathcal{E}_{\alpha_t} \) of the complex part \( (0 < t \leq a) \) have the form

\[
\Pi \mathcal{E}_{\beta_s} = (-1)^a \mathcal{E}_{\beta_s} \Pi,
\]

\[
\Pi \mathcal{E}_{\alpha_t} = (-1)^{a-t} \sigma(\alpha_t) \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_{a-t}} \mathcal{E}_{\alpha_{a-t+1}} \cdots \mathcal{E}_{\alpha_a},
\]

\[
\mathcal{E}_{\alpha_t} \Pi = (-1)^{t-1} \sigma(\alpha_t) \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_{t-1}} \mathcal{E}_{\alpha_{t+1}} \cdots \mathcal{E}_{\alpha_a},
\]

that is, when \( a \equiv 0 \pmod{2} \) the matrix \( \Pi \) commutes with the real part and anticommutes with the complex part of the spinbasis of \( \mathcal{A}_{p,q} \). Correspondingly, when \( a \equiv 1 \pmod{2} \) the matrix \( \Pi \) anticommutes with the real part and commutes with the complex part. Further, let \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_{p+q-a}} \) be a product of the real matrices, then

\[
\Pi \mathcal{E}_{\beta_s} = (-1)^{p+q-a-s} \sigma(\beta_s) \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_{s-1}} \mathcal{E}_{\beta_{s+1}} \cdots \mathcal{E}_{\beta_{p+q-a}},
\]

\[
\mathcal{E}_{\beta_s} \Pi = (-1)^{s-1} \sigma(\beta_s) \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_{s-1}} \mathcal{E}_{\beta_{s+1}} \cdots \mathcal{E}_{\beta_{p+q-a}}.
\]
\[ \Pi \varepsilon_{\alpha t} = (-1)^{p+q-a} \varepsilon_{\alpha t} \Pi, \]  

(23)

that is, when \( p + q - a \equiv 0 \mod(2) \) the matrix \( \Pi \) anticommutes with the real part and commutes with the complex part of the spinbasis of \( \mathcal{C}_{p,q} \). Correspondingly, when \( p+q-a \equiv 1 \mod(2) \) the matrix \( \Pi \) commutes with the real part and anticommutes with the complex part.

The comparison of the conditions (20)–(21) with the condition (19) shows that the matrix \( \Pi = \varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \cdots \varepsilon_{\alpha_a} \) exists only at \( a \equiv 0 \mod(2) \), that is, \( \Pi \) is a product of the complex matrices \( \varepsilon_{\alpha_t} \) of the even number. In turn, a comparison of (22)–(23) with (19) shows that \( \Pi = \varepsilon_{\beta_1} \varepsilon_{\beta_2} \cdots \varepsilon_{\beta_{p+q-a}} \) exists only at \( p + q - a \equiv 1 \mod(2) \), that is, \( \Pi \) is a product of the real matrices \( \varepsilon_{\beta_s} \) of the odd number.

Let us calculate now the product \( \Pi \hat{\Pi} \). Let \( \Pi = \varepsilon_{\beta_1} \varepsilon_{\beta_2} \cdots \varepsilon_{\beta_{p+q-a}} \) be a product of the real matrices. Since \( \hat{\varepsilon}_{\beta_s} = \varepsilon_{\beta_s} \), then \( \hat{\Pi} = \Pi \) and \( \Pi \hat{\Pi} = \Pi^2 \). Therefore,$$
\Pi \hat{\Pi} = (\varepsilon_{\beta_1} \varepsilon_{\beta_2} \cdots \varepsilon_{\beta_{p+q-a}})^2 = (-1)^{(p+q-a)(p+q-a-1)} \cdot 1. \tag{24}
$$

Further, let \( \Pi = \varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \cdots \varepsilon_{\alpha_a} \) be a product of the real matrices. Then \( \hat{\varepsilon}_{\alpha_t} = -\varepsilon_{\alpha_t} \) and \( \hat{\Pi} = (-1)^a \Pi = \Pi \), since \( a \equiv 0 \mod(2) \). Therefore,$$
\Pi \hat{\Pi} = (\varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \cdots \varepsilon_{\alpha_a})^2 = (-1)^{a(a-1)/2} \cdot 1. \tag{25}
$$

Let \( p + q - a = b \) be a quantity of the real matrices \( \varepsilon_{\beta_s} \) of the spinbasis of \( \mathcal{C}_{p,q} \), then \( p + q = a + b \). Since \( p + q \) is always even number for the quaternionic types \( p-q \equiv 4, 6 \mod(2) \), then \( a \) and \( b \) are simultaneously even or odd numbers. Thus, from (24) and (25) it follows

\[ \Pi \hat{\Pi} = \begin{cases} 
1, & \text{if } a, b \equiv 0, 1 \mod(4), \\
-1, & \text{if } a, b \equiv 2, 3 \mod(4), 
\end{cases} \]

which required to be proved.

In the present form of quantum field theory complex fields correspond to charged particles. Thus, the extraction of the subalgebra \( \mathcal{C}_{p,q} \) with the real ring \( \mathbb{K} \simeq \mathbb{R} \) in \( \mathbb{C}_n \), \( p-q \equiv 0, 2 \mod(8) \), corresponds to physical fields describing truly neutral particles such as photon and neutral mesons (\( \pi^0, \eta^0, \rho^0, \omega^0, \varphi^0, K^0 \)). In turn, the subalgebras \( \mathcal{C}_{p,q} \) with the ring \( \mathbb{K} \simeq \mathbb{H} \), \( p-q \equiv 4, 6 \mod(8) \), correspond to charged or neutral fields.

## 5 Extended automorphism groups

An introduction of the pseudoautomorphism \( \mathcal{A} \to \overline{\mathcal{A}} \) allows us to extend the automorphism set of the complex Clifford algebra \( \mathbb{C}_n \). Namely, we add to the four fundamental automorphisms \( \mathcal{A} \to \mathcal{A}, \mathcal{A} \to \mathcal{A}^*, \mathcal{A} \to \tilde{\mathcal{A}}, \mathcal{A} \to \tilde{\mathcal{A}}^* \) the pseudoautomorphism \( \mathcal{A} \to \overline{\mathcal{A}} \) and following three combinations:

1) A pseudoautomorphism \( \mathcal{A} \to \overline{\mathcal{A}}^* \). This transformation is a composition of the pseudoautomorphism \( \mathcal{A} \to \overline{\mathcal{A}} \) with the automorphism \( \mathcal{A} \to \mathcal{A}^* \).

2) A pseudoantiautomorphism \( \mathcal{A} \to \overline{\mathcal{A}} \). This transformation is a composition of \( \mathcal{A} \to \overline{\mathcal{A}} \) with the antiautomorphism \( \mathcal{A} \to \overline{\mathcal{A}}^* \).

3) A pseudoantiautomorphism \( \mathcal{A} \to \overline{\mathcal{A}}^* \) (a composition of \( \mathcal{A} \to \overline{\mathcal{A}} \) with the antiautomorphism \( \mathcal{A} \to \overline{\mathcal{A}}^* \)).
Thus, we obtain an automorphism set of \( C_n \) consisting of the eight transformations. Let us show that the set \( \{ \text{Id}, \star, \sim, \tilde{x}, \overline{x}, \overline{\sim}, \overline{\star}, \overline{x} \} \) forms a finite group of order 8 and let us give a physical interpretation of this group.

**Proposition 2.** Let \( C_n \) be a Clifford algebra over the field \( \mathbb{F} = \mathbb{C} \) and let \( \text{Ext}(C_n) = \{ \text{Id}, \star, \sim, \tilde{x}, \overline{x}, \overline{\sim}, \overline{\star}, \overline{x} \} \) be an extended automorphism group of the algebra \( C_n \). Then there is an isomorphism between \( \text{Ext}(C_n) \) and the full CPT-group of the discrete transformations, \( \text{Ext}(C_n) \simeq \{ 1, P, T, PT, C, CP, CT, CPT \} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \). In this case, space inversion \( P \), time reversal \( T \), full reflection \( PT \), charge conjugation \( C \), transformations \( CP \), \( CT \) and the full CPT-transformation correspond to the automorphism \( A \rightarrow A^\star \), antiautomorphisms \( A \rightarrow \overline{A} \), \( A \rightarrow \overline{\overline{A}} \), pseudoautomorphisms \( A \rightarrow \overline{A} \), \( A \rightarrow \overline{\overline{A}} \), pseudoantiautomorphisms \( A \rightarrow \overline{A} \) and \( A \rightarrow \overline{\overline{A}} \), respectively.

**Proof.** The group \( \{ 1, P, T, PT, C, CP, CT, CPT \} \) at the conditions \( P^2 = T^2 = (PT)^2 = C^2 = (CP)^2 = (CT)^2 = (CPT)^2 = 1 \) and commutativity of all the elements forms an Abelian group of order 8, which is isomorphic to a cyclic group \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \). The Cayley tableau of this group has a form

|   | 1 | P | T | PT | C | CP | CT | CPT |
|---|---|---|---|----|---|----|----|-----|
| 1 | 1 | P | T | PT | C | CP | CT | CPT |
| P | P | 1 | PT | T | CP | C | CT | CT |
| T | T | PT | 1 | P | CT | CPT | C | CP |
| PT | PT | T | P | 1 | CPT | CT | CP | C |
| C | C | CP | CT | CPT | 1 | P | T | PT |
| CP | CP | C | CPT | CT | P | 1 | PT | T |
| CT | CT | CPT | C | CP | T | PT | 1 | P |
| CPT | CPT | CT | CP | C | PT | T | P | 1 |

In turn, for the extended automorphism group \( \{ \text{Id}, \star, \sim, \tilde{x}, \overline{x}, \overline{\sim}, \overline{\star}, \overline{x} \} \) in virtue of commutativity \( (\overline{\overline{A}}) = (\overline{\overline{A}}) \), \( (\overline{A}^\star) = (\overline{A})^\star \), \( (\overline{\overline{A}}) = (\overline{A}) \), \( (\overline{A}^\star) = (\overline{A})^\star \) and an involution property \( \star \star = \sim \sim = \overline{\overline{\sim}} = \text{Id} \) we have a following Cayley tableau

|   | Id | * | ~ | * | ~ | * | ~ |
|---|----|---|---|---|---|---|---|
| Id | Id | * | ~ | * | ~ | * | ~ |
| * | * | Id | ~ | * | ~ | * | ~ |
| ~ | ~ | * | Id | ~ | * | ~ | * |
| * | ~ | * | Id | ~ | * | ~ | * |
| ~ | ~ | * | Id | ~ | * | ~ | * |
| * | * | ~ | * | ~ | * | * | Id |
| ~ | ~ | * | ~ | * | ~ | * | Id |
| * | * | ~ | * | ~ | * | * | Id |
The identity of multiplication tables proves the group isomorphism

\[ \{1, P, T, PT, C, CP, CT, CPT\} \simeq \{\text{Id}, \ast, \bar{\ast}, \bar{\ast}, \bar{\ast}, \bar{\ast}, \bar{\ast}\} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2. \]

Further, in the case of \( P^2 = T^2 = \ldots = (CPT)^2 = \pm 1 \) and anticommutativity of the elements we have an isomorphism between the \( CPT \)-group and a group \( \text{Ext}(\mathbb{C}_n) \). The elements of \( \text{Ext}(\mathbb{C}_n) \) are spinor representations of the automorphisms of the algebra \( \mathbb{C}_n \). The Wedderburn–Artin Theorem allows us to define any spinor representaions for the automorphisms of \( \mathbb{C}_n \).

### 5.1 Pseudoautomorphism \( A \rightarrow \overline{A}^* \)

Let us find a spinor representation of the pseudoautomorphism \( A \rightarrow \overline{A}^* \). The transformation \( A \rightarrow \overline{A}^* \) is a composition of the pseudoautomorphism \( A \rightarrow \overline{A} \) and the automorphism \( A \rightarrow A^* \). Under action of \( A \rightarrow A^* \) we have \( e_i \rightarrow -e_i \), where \( e_i \) are the units of \( C_{p,q} \). In turn, under action of \( A \rightarrow \overline{A} \) the units \( e_i \) remain unaltered, \( e_i \rightarrow e_i \). Therefore, under action of the pseudoautomorphism \( A \rightarrow \overline{A}^* \) we obtain \( e_i \rightarrow -e_i \).

As it shown previously, the transformations \( A \rightarrow A^* \) and \( A \rightarrow \overline{A} \) in the spinor representation are defined by the expressions \( A^* = WAW^{-1} \) and \( \overline{A} = \Pi\hat{A}\Pi^{-1} \). The order of the composition of these transformations is not important (\( A^* = (\overline{A})^* = (A^*)^* \)). Indeed, if \( W \) is a real matrix, then

\[ \overline{A}^* = W\Pi\hat{A}\Pi^{-1}W^{-1} = \Pi(WAW^{-1})\Pi^{-1}, \]

or

\[ \overline{A}^* = (W\Pi\hat{A}W\Pi)^{-1} = (\Pi\hat{W})\hat{A}(\Pi\hat{W})^{-1}. \] (26)

Otherwise, we have \( (\overline{A}^*) = \Pi\hat{W}AW^{-1}\Pi^{-1} \). Let us assume that \( W \) is a complex matrix, then \( \hat{W} = -W \) and, therefore, \( (\overline{A}^*) = \Pi(-W)\hat{A}(-W^{-1})\Pi^{-1} = (\Pi\hat{W})\hat{A}(\Pi\hat{W})^{-1}. \) Thus, the relation (26) is always fulfilled.

Let \( K = \Pi\hat{W} \) be a matrix of the pseudoautomorphism \( A \rightarrow \overline{A}^* \). Then (26) can be written as follows

\[ \overline{A}^* = K\hat{A}K^{-1}. \] (27)

Since under action of the pseudoautomorphism \( A \rightarrow \overline{A}^* \) we have \( e_i \rightarrow -e_i \), in the spinor representation we must demand \( \varepsilon_i \rightarrow -\varepsilon_i \) also, or

\[ \varepsilon_i \rightarrow -\varepsilon_i = K\hat{e}_iK^{-1}. \] (28)

In the case of real subalgebras \( C_{p,q} \) with the ring \( \mathbb{K} \simeq \mathbb{R} \) we have \( \hat{e}_i = e_i \) and the relation (28) takes a form

\[ \varepsilon_i \rightarrow -\varepsilon_i = K\varepsilon_iK^{-1}, \]

whence

\[ \varepsilon_iK = -K\varepsilon_i, \]

that is, the matrix \( K \) is always anticommutates with the matrices of the spinbasis. However, for the ring \( \mathbb{K} \simeq \mathbb{R} \) the matrix \( \Pi \) of \( A \rightarrow \overline{A} \) is proportional to the unit matrix, \( \Pi \sim I \) (Theorem 9). Therefore, in this case we have \( K \sim W \).
In the case of real subalgebras $\mathcal{O}_{p,q}$ with the quaternionic ring $\mathbb{K} \simeq \mathbb{H}$ the spinbasis is divided into two parts: a complex part $\{ \hat{E}_{ \alpha t} = -\hat{E}_{\alpha t} \}$, $(0 < t \leq a)$, where $a$ is a number of the complex matrices of the spinbasis, and a real part $\{ \hat{E}_{ \beta s} = \hat{E}_{\beta s} \}$, where $p + q - a$ is a number of the real matrices, $(0 < s \leq p + q - a)$. Then, in accordance with the spinbasis structure of the algebra $\mathcal{O}_{p,q} \simeq M_{2^{p+q}}(\mathbb{H})$, the relation (28) can be written as follows

$$\hat{E}_{ \alpha t} \rightarrow \hat{E}_{\alpha t} = K\hat{E}_{\alpha t}K^{-1}, \quad \hat{E}_{\beta s} \rightarrow -\hat{E}_{\beta s} = K\hat{E}_{\beta s}K^{-1}.$$  

Whence

$$E_{\alpha, t} = \pm E_{\alpha, t}, \quad E_{\beta, s} = \pm E_{\beta, s}.$$  

Thus, for the quaternionic types $p - q \equiv 4, 6 \mod 8$ the matrix $K$ of the pseudoautomorphism $\mathcal{A} \rightarrow \mathcal{A}^\ast$ commutes with the complex part and anticommutes with the real part of the spinbasis of $\mathcal{O}_{p,q}$. Hence it follows that a structure of the matrix $K$ is analogous to the structure of the matrix $\Pi$ of the pseudoautomorphism $\mathcal{A} \rightarrow \mathcal{A}$ (see Theorem 2), that is, the matrix $K$ of $\mathcal{A} \rightarrow \mathcal{A}^\ast$ is a product of only complex or only real matrices.

So, let $0 < a \leq p + q$ and let $K = \hat{E}_{\beta_1}\hat{E}_{\beta_2}\cdots\hat{E}_{\beta_a}$ be the matrix of the pseudoautomorphism $\mathcal{A} \rightarrow \mathcal{A}^\ast$, then permutation conditions of the matrix $K$ with the matrices $E_{\beta_s}$ of the real part $(0 < s \leq p + q - a)$ and the matrices $E_{\alpha_t}$ of the complex part $(0 < t \leq a)$ have the form

$$KE_{\beta_s} = (-1)^{s}\hat{E}_{\beta_s},$$

$$KE_{\alpha_t} = (-1)^{a-t}\sigma(\alpha_t)E_{\alpha_1}\hat{E}_{\alpha_2}\cdots\hat{E}_{\alpha_{a-1}}E_{\alpha_{a+1}}\cdots E_{\alpha_a};$$

$$E_{\alpha_t}K = (-1)^{t-1}\sigma(\alpha_t)E_{\alpha_1}\hat{E}_{\alpha_2}\cdots\hat{E}_{\alpha_{a-1}}E_{\alpha_{a+1}}\cdots E_{\alpha_a};$$

(30)

(31)

that is, at $a \equiv 0 \mod 2$ $K$ commutes with the real part and anticommutes with the complex part of the spinbasis. Correspondingly, at $a \equiv 1 \mod 2$ $K$ anticommutes with the real and commutes with the complex part. Further, let $K = \hat{E}_{\beta_1}\hat{E}_{\beta_2}\cdots\hat{E}_{\beta_{p+q-a}}$ be a product of the real matrices of the spinbasis, then

$$KE_{\beta_s} = (-1)^{p+q-a-s}\sigma(\beta_s)E_{\beta_1}\hat{E}_{\beta_2}\cdots\hat{E}_{\beta_{s-1}}E_{\beta_{s+1}}\cdots E_{\beta_{p+q-a}};$$

$$E_{\beta_s}K = (-1)^{s-1}\sigma(\beta_s)E_{\beta_1}\hat{E}_{\beta_2}\cdots\hat{E}_{\beta_{s-1}}E_{\beta_{s+1}}\cdots E_{\beta_{p+q-a}};$$

(32)

(33)

that is, at $p + q - a \equiv 0 \mod 2$ the matrix $K$ anticommutes with the real part and commutes with the complex part of the spinbasis. Correspondingly, at $p + q - a \equiv 1 \mod 2$ $K$ commutes with the real and anticommutes with the complex part.

A comparison of the conditions (30)–(31) with (29) shows that the matrix $K = E_{\alpha_1}\hat{E}_{\alpha_2}\cdots\hat{E}_{\alpha_a}$ exists only if $a \equiv 1 \mod 2$. In turn, a comparison of the conditions (32)–(33) with (29) shows that the matrix $K = \hat{E}_{\beta_1}\hat{E}_{\beta_2}\cdots\hat{E}_{\beta_{p+q-a}}$ exists only if $p + q - a \equiv 0 \mod 2$.

Let us find now squares of the matrix $K$. In accordance with obtained conditions there exist two possibilities:

1) $K = E_{\alpha_1}\hat{E}_{\alpha_2}\cdots\hat{E}_{\alpha_a}$, $a \equiv 1 \mod 2$.

$$K^2 = \begin{cases} +1, & \text{if } a_+ - a_- \equiv 1, 5 \mod 8, \\ -1, & \text{if } a_+ - a_- \equiv 3, 7 \mod 8. \end{cases}$$
where $a_+$ and $a_-$ are numbers of matrices with $'+'$- and $'-'$-squares in the product $\mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_n}$.

2) \( K = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_{p+q-a}} \), \( p + q - a \equiv 0 \) (mod 2).

\[
K^2 = \begin{cases} 
+1, & \text{if } b_+ - b_- \equiv 0, 4 \text{ (mod 8)}, \\
-1, & \text{if } b_+ - b_- \equiv 2, 6 \text{ (mod 8)},
\end{cases}
\]

where $b_+$ and $b_-$ are numbers of matrices with $'+'$- and $'-'$-squares in the product $\mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_{p+q-a}}$, respectively.

### 5.2 Pseudoantiautomorphism $\mathcal{A} \to \bar{\mathcal{A}}$

The pseudoantiautomorphism $\mathcal{A} \to \bar{\mathcal{A}}$ is the composition of the pseudoautomorphism $\mathcal{A} \to \bar{\mathcal{A}}$ with the antiautomorphism $\mathcal{A} \to \bar{\mathcal{A}}$. Under action of $\mathcal{A} \to \bar{\mathcal{A}}$ the units $e_i$ remain unaltered, $e_i \to e_i$. Analogously, under action of $\mathcal{A} \to \bar{\mathcal{A}}$ we have $e_i \to e_i$. Therefore, under action of the pseudoantiautomorphism $\mathcal{A} \to \bar{\mathcal{A}}$ the units $e_i$ remain unaltered also, $e_i \to e_i$.

The spinor representations of the transformations $\mathcal{A} \to \bar{\mathcal{A}}$ and $\mathcal{A} \to \bar{\mathcal{A}}$ are defined by the expressions $\bar{\mathcal{A}} = \mathcal{E}A^T\mathcal{E}^{-1}$ and $\bar{\mathcal{A}} = \Pi A\Pi^{-1}$, respectively. Let us find a spinor representation of the transformation $\mathcal{A} \to \bar{\mathcal{A}}$. The order of the composition of these transformations is not important, $\bar{\mathcal{A}} = (\bar{\mathcal{A}}) = (\bar{\mathcal{A}})$. Indeed,

\[
\bar{\mathcal{A}} = \mathcal{E}(\Pi A\Pi^{-1})^T = \Pi(\mathcal{E}A^T\mathcal{E}^{-1})\Pi^{-1}
\]

or,

\[
\bar{\mathcal{A}} = (\Pi A)\Pi^{-1} = (\Pi E)(\mathcal{A})^T(\Pi E)^{-1},
\]

since $\Pi^{-1} = \Pi^T$ and $\bar{\mathcal{A}} = \Pi E(A^T)^T\Pi^{-1} = \Pi E(A^T)^T\Pi^{-1}$ in the case when $E = E$ is a real matrix and $\bar{\mathcal{A}} = \Pi E(A^T)^T\Pi^{-1} = \Pi(-E)(-A^T)^T\Pi^{-1} = \Pi E(A^T)^T(\Pi E)^{-1}$ in the case when $E = -E$ is a complex matrix. Let $S = \Pi E$ be a matrix of the pseudoanti-automorphism $\mathcal{A} \to \bar{\mathcal{A}}$ in the spinor representation. Then (33) can be rewritten as follows

\[
\bar{\mathcal{A}} = S(A^T)S^{-1}.
\]

Since under action of the transformation $\mathcal{A} \to \bar{\mathcal{A}}$ we have $e_i \to e_i$, in the spinor representation we must demand $\mathcal{E}_i \to \bar{\mathcal{E}}_i$ also, or

\[
\mathcal{E}_i \to \bar{\mathcal{E}}_i = S\bar{\mathcal{E}}_i S^{-1}.
\]

In the case of real subalgebras $\mathcal{A}_{p,q}$ with the ring $\mathbb{K} \simeq \mathbb{R}$ we have $\bar{\mathcal{E}}_i = \mathcal{E}_i$ and, therefore, the relation (36) takes a form

\[
\mathcal{E}_i \to \bar{\mathcal{E}}_i = S\bar{\mathcal{E}}_i S^{-1}.
\]

Let $\{\mathcal{E}_{\gamma_i}\}$ be a set of symmetric matrices ($\mathcal{E}_{\gamma_i}^T = \mathcal{E}_{\gamma_i}$) and let $\{\mathcal{E}_{\delta_j}\}$ be a set of skewsymmetric matrices ($\mathcal{E}_{\delta_j}^T = -\mathcal{E}_{\delta_j}$) of the spinbasis of the algebra $\mathcal{A}_{p,q}$. Then from the relation (37) it follows

\[
\mathcal{E}_{\gamma_i} \to \bar{\mathcal{E}}_{\gamma_i} = S\mathcal{E}_{\gamma_i} S^{-1}, \quad \mathcal{E}_{\delta_j} \to \bar{\mathcal{E}}_{\delta_j} = -S\mathcal{E}_{\delta_j} S^{-1}.
\]
Whence 
\[ \mathcal{E}_{\gamma} S = S \mathcal{E}_{\gamma}, \quad \mathcal{E}_{\delta} S = -S \mathcal{E}_{\delta}, \]
that is, the matrix $S$ of the pseudoantiautomorphism $A \rightarrow \overline{A}$ in the case of $\mathbb{K} \simeq \mathbb{R}$ commutes with the symmetric part and anticommutes with the skewsymmetric part of the spinbasis of $\mathcal{O}_{p,q}$. In virtue of Theorem 9 over the ring $\mathbb{K} \simeq \mathbb{R}$ the matrix $\Pi$ of the pseudoautomorphism $A \rightarrow \overline{A}$ is proportional to the unit matrix, $\Pi \sim I$. Therefore, in this case we have $S \sim E$ and an explicit form of $S$ coincides with $E$.

Further, in case of the quaternionic ring $\mathbb{K} \simeq \mathbb{H}$, $p - q \equiv 4, 6 \mod 8$, a spinbasis of $\mathcal{O}_{p,q}$ contains both complex matrices $\mathcal{E}_{\alpha}$ and real matrices $\mathcal{E}_{\beta}$, among which there are symmetric and skewsymmetric matrices. It is obvious that the sets of complex and real matrices do not coincide with the sets of symmetric and skewsymmetric matrices. Let $\{\mathcal{E}_{\alpha}\}$ be a complex part of the spinbasis, then the relation (38) takes a form

\[ \mathcal{E}_{\alpha} \rightarrow \mathcal{E}_{\alpha} = -S \mathcal{E}^T_{\alpha} S^{-1}. \] (38)

Correspondingly, let $\{\mathcal{E}_{\gamma}\}$ and $\{\mathcal{E}_{\delta}\}$ be the sets of symmetric and skewsymmetric matrices of the complex part. Then the relation (38) can be written as follows

\[ \mathcal{E}_{\gamma} \rightarrow \mathcal{E}_{\gamma} = -S \mathcal{E}_{\gamma} S^{-1}, \quad \mathcal{E}_{\delta} \rightarrow \mathcal{E}_{\delta} = S \mathcal{E}_{\delta} S^{-1}. \]

Whence 
\[ \mathcal{E}_{\gamma} S = -S \mathcal{E}_{\gamma}, \quad \mathcal{E}_{\delta} S = S \mathcal{E}_{\delta}. \] (39)

Therefore, the matrix $S$ of the pseudoantiautomorphism $A \rightarrow \overline{A}$ anticommutes with the complex symmetric matrices and commutes with the complex skewsymmetric matrices of the spinbasis of $\mathcal{O}_{p,q}$.

Let us consider now the real part $\{\mathcal{E}_{\beta}\}$ of the spinbasis of $\mathcal{O}_{p,q}$, $p - q \equiv 4, 6 \mod 8$. In this case the relation (38) takes a form

\[ \mathcal{E}_{\beta} \rightarrow \mathcal{E}_{\beta} = S \mathcal{E}^T_{\beta} S^{-1}. \] (40)

Let $\{\mathcal{E}_{\beta}\}$ and $\{\mathcal{E}_{\beta}\}$ be the sets of real symmetric and real skewsymmetric matrices, respectively. Then the relation (40) can be written as follows

\[ \mathcal{E}_{\gamma} \rightarrow \mathcal{E}_{\gamma} = S \mathcal{E}_{\gamma} S^{-1}, \quad \mathcal{E}_{\beta} \rightarrow \mathcal{E}_{\beta} = -S \mathcal{E}_{\beta} S^{-1}. \]

Whence 
\[ \mathcal{E}_{\gamma} S = S \mathcal{E}_{\gamma}, \quad \mathcal{E}_{\beta} S = -S \mathcal{E}_{\beta}. \] (41)

Thus, the matrix $S$ of the transformation $A \rightarrow \overline{A}$ commutes with the real symmetric matrices and anticommutes with the real skewsymmetric matrices of the spinbasis of $\mathcal{O}_{p,q}$.

Let us find now an explicit form of the matrix $S = \Pi E$. In accordance with Theorem 2 for the quaternionic types $p - q \equiv 4, 6 \mod 8$ the matrix $\Pi$ takes the two different forms: 1) $\Pi = E_{\alpha_1} E_{\alpha_2} \cdots E_{\alpha_a}$ is the product of complex matrices at $a \equiv 0 \mod 2$; 2) $\Pi = E_{\beta_1} E_{\beta_2} \cdots E_{\beta_b}$ is the product of real matrices at $b \equiv 1 \mod 2$. In turn, for the matrix $E$ of the antiautomorphism $A \rightarrow \overline{A}$ over the ring $\mathbb{K} \simeq \mathbb{H}$ (see Theorem 7) we have the following two forms: 1) $E = E_{j_1} E_{j_2} \cdots E_{j_k}$ is the product of skewsymmetric matrices at $k \equiv 0 \mod 2$; 2) $E = E_{i_1} E_{i_2} \cdots E_{i_{p+q-k}}$ is the product of symmetric matrices at $k \equiv 1 \mod 2$. Thus, in
accordance with definition \( S = \Pi \mathbf{E} \) we have four different products: \( S = \varepsilon_{\alpha_1\alpha_2\ldots\alpha_n} \varepsilon_{j_1j_2\ldots j_k}, \) \( S = \varepsilon_{\alpha_1\alpha_2\ldots\alpha_n} \varepsilon_{i_1i_2\ldots i_{p+q-k}}, \) \( S = \varepsilon_{\beta_1\beta_2\ldots\beta_n} \varepsilon_{j_1j_2\ldots j_k}, \) \( S = \varepsilon_{\beta_1\beta_2\ldots\beta_n} \varepsilon_{i_1i_2\ldots i_{p+q-k}}. \) It is obvious that in the given products there are identical matrices.

Let us examine the first product \( S = \varepsilon_{\alpha_1\alpha_2\ldots\alpha_n} \varepsilon_{j_1j_2\ldots j_k}. \) Since in this case \( \Pi \) contains all the complex matrices of the spinbasis, among which there are symmetric and skewsymmetric matrices, and \( \mathbf{E} \) contains all the skewsymmetric matrices of the spinbasis, then \( \Pi \) and \( \mathbf{E} \) contain a quantity of identical matrices (complex skewsymmetric matrices). Let \( m \) be a number of the complex skewsymmetric matrices of the spinbasis of the algebra \( \mathcal{O}_{p,q}, p - q \equiv 4, 6 \mod 8. \) Then the product \( S = \Pi \mathbf{E} \) takes a form

\[
\varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \cdots \varepsilon_{j_1} \varepsilon_{j_2} \cdots \varepsilon_{j_k} = (-1)^N \sigma(i_1) \sigma(i_2) \cdots \sigma(i_m) \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_s},
\]

where the indices \( c_1, \ldots, c_s \) present itself a totality of the indices \( \alpha_1, \ldots, \alpha_n, \beta_1, \beta_2, \ldots j_k \) obtained after removal of the indices occurred twice; \( N \) is a number of inversions.

First of all, let us remark that \( \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_s} \) is an even product, since the original product \( \Pi \mathbf{E} \) is the even product also. Besides, the product \( \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_s} \) contains all the complex symmetric matrices and all the real skewsymmetric matrices of the spinbasis.

Let us find now permutation conditions between the matrix \( S = \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_s} \) and the units of the spinbasis of \( \mathcal{O}_{p,q}, p - q \equiv 4, 6 \mod 8. \) For the complex symmetric matrices \( \varepsilon_{\alpha_\gamma} \) and complex skewsymmetric matrices \( \varepsilon_{\alpha_s} \) we have

\[
S \varepsilon_{\alpha_\gamma} = (-1)^{s-\gamma} \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_{s-2}},
\]

\[
\varepsilon_{\alpha_\gamma} S = (-1)^{\gamma-1} \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_{s-2}},
\]

(42)

\[
S \varepsilon_{\alpha_s} = (-1)^{s} \varepsilon_{\alpha_s} S,
\]

(43)

that is, the matrix \( S \) always anticommutes with the complex symmetric matrices and always commutes with the complex skewsymmetric matrices, since \( s \equiv 0 \mod 2. \) Further, for the real symmetric matrices \( \varepsilon_{\beta_\gamma} \) and real skewsymmetric matrices \( \varepsilon_{\beta_s} \) we have

\[
S \varepsilon_{\beta_\gamma} = (-1)^{s-\gamma} \varepsilon_{\beta_\gamma} S,
\]

(44)

\[
S \varepsilon_{\beta_s} = (-1)^{s-\delta} \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_{s-2}},
\]

\[
\varepsilon_{\beta_s} S = (-1)^{\delta-1} \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_{s-2}}.
\]

(45)

Therefore, \( S \) always commutes with the real symmetric matrices and always anticommutes with the real skewsymmetric matrices.

A comparison of the obtained conditions (42)–(45) with (39) and (41) shows that \( S = \varepsilon_{c_1} \varepsilon_{c_2} \cdots \varepsilon_{c_s} \) automatically satisfies the conditions, which define the matrix of the pseudoanti-automorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}}. \)

Let examine the second product \( S = \varepsilon_{\alpha_1\alpha_2\ldots\alpha_n} \varepsilon_{i_1i_2\ldots i_{p+q-k}}. \) In this case \( \Pi \) contains all the complex matrices of the spinbasis, among which there are symmetric and skewsymmetric matrices, and \( \mathbf{E} \) contains all the symmetric matrices of the spinbasis. Thus, \( \Pi \) and \( \mathbf{E} \) contain a quantity of identical complex symmetric matrices. Let \( l \) be a number of the complex symmetric matrices, then for the product \( S = \Pi \mathbf{E} \) we obtain

\[
\varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \cdots \varepsilon_{\alpha_s} \varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_{p+q-k}} = (-1)^N \sigma(i_1) \sigma(i_2) \cdots \sigma(i_l) \varepsilon_{d_1} \varepsilon_{d_2} \cdots \varepsilon_{d_g},
\]

where \( d_1, \ldots, d_g \) present itself a totality of the indices \( \alpha_1, \ldots, \alpha_n, i_1, \ldots, i_{p+q-k} \) obtained after removal of the indices occurred twice. The product \( S = \varepsilon_{d_1} \varepsilon_{d_2} \cdots \varepsilon_{d_g} \) is odd, since the original
product \( \Pi E \) is odd also. Besides, \( \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_p} \) contains all the complex skewsymmetric matrices and all the real symmetric matrices of the spinbasis.

Let us find permutation conditions of the matrix \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_p} \) with the units of the spinbasis of \( \mathcal{O}_{p,q} \). For the complex part of the spinbasis we have

\[
S \mathcal{E}_{\alpha_i} = \begin{pmatrix} -1 \end{pmatrix}^g \mathcal{E}_{\alpha_i} S, \\
S \mathcal{E}_{\alpha_i} = \begin{pmatrix} -1 \end{pmatrix}^{g-\delta} \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_{g-2}}, \\
\mathcal{E}_{\alpha_i} S = \begin{pmatrix} -1 \end{pmatrix}^{\delta-1} \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_{g-2}},
\]

that is, the matrix \( S \) anticommutes with the complex symmetric matrices and commutes with the complex skewsymmetric matrices, since \( g \equiv 1 \text{ (mod 2)} \). For the real part of the spinbasis of \( \mathcal{O}_{p,q} \) we obtain

\[
S \mathcal{E}_{\beta_i} = \begin{pmatrix} -1 \end{pmatrix}^{g-\gamma} \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_{g-2}}, \\
\mathcal{E}_{\beta_i} S = \begin{pmatrix} -1 \end{pmatrix}^{\gamma-1} \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_{g-2}}, \\
S \mathcal{E}_{\beta_i} = \begin{pmatrix} -1 \end{pmatrix}^{\delta-1} \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_{g-2}}.
\]

Therefore, \( S \) commutes with the real symmetric matrices and anticommutes with the real skewsymmetric matrices.

Comparing the obtained conditions (46)–(49) with the conditions (39) and (41) we see that \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_p} \) automatically satisfies the conditions which define the matrix of the transformation \( A \to A \).

Let us consider now the third product \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{f_k} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k} \). In this product the matrix \( \Pi \) of the pseudoautomorphism \( A \to A \) contains all the real matrices of the spinbasis, among which there are symmetric and skewsymmetric matrices, and the matrix \( E \) of \( A \to A \) contains all the skewsymmetric matrices of the spinbasis, among which there are both the real and complex matrices. Therefore, \( S \) contains a quantity of identical real skewsymmetric matrices. Let \( u \) be a number of the real skewsymmetric matrices of the spinbasis of the algebra \( \mathcal{O}_{p,q} \), \( p - q \equiv 4, 6 \text{ (mod 8)} \), then for the product \( S \) we obtain

\[
\mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k} = (-1)^{N} \sigma(i_1) \sigma(i_2) \cdots \sigma(i_u) \mathcal{E}_{e_1} \mathcal{E}_{e_2} \cdots \mathcal{E}_{e_h},
\]

where the indices \( e_1, e_2, \ldots, e_n \) present itself a totality of the indices \( \beta_1, \ldots, \beta_b, j_1, \ldots, j_k \) obtained after removal of the indices occurred twice. The product \( S = \mathcal{E}_{e_1} \mathcal{E}_{e_2} \cdots \mathcal{E}_{e_h} \) is odd, since the original product \( \Pi E \) is odd also. It is easy to see that \( \mathcal{E}_{e_1} \mathcal{E}_{e_2} \cdots \mathcal{E}_{e_h} \) contains all the real symmetric matrices and all the complex skewsymmetric matrices of the spinbasis. Therefore, the matrix \( S = \mathcal{E}_{e_1} \mathcal{E}_{e_2} \cdots \mathcal{E}_{e_h} \) is similar to the matrix \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_p} \), and its permutation conditions with the units of the spinbasis of \( \mathcal{O}_{p,q} \) are equivalent to the relations (46)–(49).

Finally, let us examine the fourth product \( S = \mathcal{E}_{b_1} \mathcal{E}_{b_2} \cdots \mathcal{E}_{b_h} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \). In turn, this product contains a quantity of identical real symmetric matrices. Let \( v \) be a number of the real symmetric matrices of the spinbasis of \( \mathcal{O}_{p,q} \), then

\[
\mathcal{E}_{b_1} \mathcal{E}_{b_2} \cdots \mathcal{E}_{b_h} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} = (-1)^{N} \sigma(i_1) \sigma(i_2) \cdots \sigma(i_v) \mathcal{E}_{f_1} \mathcal{E}_{f_2} \cdots \mathcal{E}_{f_w},
\]

where \( f_1, \ldots, f_w \) present itself a totality of the indices \( \beta_1, \ldots, \beta_b, i_1, \ldots, i_{p+q-k} \) obtained after removal of the indices occurred twice. The product \( S = \mathcal{E}_{f_1} \mathcal{E}_{f_2} \cdots \mathcal{E}_{f_w} \) is even, since the
original product $\Pi E$ is even also. It is easy to see that $\mathcal{E}_{f_1}\mathcal{E}_{f_2}\cdots\mathcal{E}_{f_w}$ contains all the real skew-symmetric matrices and all the complex symmetric matrices of the spin basis. Therefore, the matrix $S = \mathcal{E}_{f_1}\mathcal{E}_{f_2}\cdots\mathcal{E}_{f_w}$ is similar to the matrix $S = \mathcal{E}_{c_1}\mathcal{E}_{c_2}\cdots\mathcal{E}_{c_s}$, and its permutation conditions with the units of the spin basis are equivalent to the relations (42), (43).

Thus, from the four products we have only two non-equivalent products. The squares of the non-equivalent matrices $S = \mathcal{E}_{c_1}\mathcal{E}_{c_2}\cdots\mathcal{E}_{c_s}$ ($s \equiv 0 \pmod{2}$) and $S = \mathcal{E}_{d_1}\mathcal{E}_{d_2}\cdots\mathcal{E}_{d_g}$ ($g \equiv 1 \pmod{2}$) are

$$S^2 = (\mathcal{E}_{c_1}\mathcal{E}_{c_2}\cdots\mathcal{E}_{c_s})^2 = \begin{cases} +1, & \text{if } u + l \equiv 0, 4 \pmod{8}, \\
-1, & \text{if } u + l \equiv 2, 6 \pmod{8}; \end{cases} \quad (50)$$

$$S^2 = (\mathcal{E}_{d_1}\mathcal{E}_{d_2}\cdots\mathcal{E}_{d_g})^2 = \begin{cases} +1, & \text{if } m + v \equiv 1, 5 \pmod{8}, \\
-1, & \text{if } m + v \equiv 3, 7 \pmod{8}. \end{cases} \quad (51)$$

### 5.3 Pseudoantiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$

The pseudoantiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$, that defines the $CPT$-transformation, is a composition of the pseudoautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}$, antiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ and automorphism $\mathcal{A} \rightarrow \mathcal{A}^*$. Under action of the automorphism $\mathcal{A} \rightarrow \mathcal{A}^*$ the units of $\mathcal{O}_{p,q}$ change the sign, $e_i \rightarrow -e_i$. In turn, under action of the transformations $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ and $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ the units remain unaltered, $e_i \rightarrow e_i$. Therefore, under action of the pseudoantiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$ the units change the sign, $e_i \rightarrow -e_i$.

The spinor representations of the transformations $\mathcal{A} \rightarrow \mathcal{A}^*$, $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ and $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ have the form: $\mathcal{A}^* = WAW^{-1}$, $\bar{\mathcal{A}} = E\mathcal{A}^T\bar{E}^{-1}$ and $\bar{\mathcal{A}} = \Pi \bar{\mathcal{A}} \Pi^{-1}$. Let us find a matrix of the transformation $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$. We will consider the pseudoantiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$ as a composition of the pseudoautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ with the antiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}$. The spinor representation of $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$ is $\bar{\mathcal{A}}^* = \mathcal{C}^T\mathcal{C}^{-1}$, where $\mathcal{C} = EW$. Since $\bar{\mathcal{A}}^* = (\bar{\mathcal{A}})^* = (\mathcal{A}^*)$, then

$$\bar{\mathcal{A}}^* = \mathcal{C} \left( \Pi \bar{\mathcal{A}} \Pi^{-1} \right)^T \mathcal{C}^{-1} = \Pi \left( \mathcal{C}^T \mathcal{C}^{-1} \right)^{\Pi^{-1}},$$

or

$$\bar{\mathcal{A}}^* = (\mathcal{C} \Pi \bar{\mathcal{A}} \Pi^{-1}) \mathcal{C}^{-1} = (\mathcal{C} \Pi \mathcal{C}^{-1} \Pi^{-1}), \quad (52)$$

since $\Pi^{-1} = \Pi^T$ and $\bar{\mathcal{A}}^* = \Pi \hat{\mathcal{C}} \left( \hat{\mathcal{C}}^T \right)^{-1} \Pi^{-1}$ in the case when $\hat{\mathcal{C}} = \mathcal{C}$ is a real matrix, and also $\bar{\mathcal{A}}^* = \Pi \hat{\mathcal{C}} \left( \hat{\mathcal{C}}^T \right)^{-1} \Pi^{-1} = \Pi \bar{\mathcal{C}} \bar{\mathcal{C}}^{-1} \Pi^{-1}$ in the case when $\hat{\mathcal{C}} = -\mathcal{C}$ is a complex matrix.

Let $\mathcal{F} = \Pi \mathcal{C}$ (or $\mathcal{F} = \Pi EW$) be a matrix of the pseudoantiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$. Then the relation (52) can be written as follows

$$\bar{\mathcal{A}}^* = \mathcal{F} \mathcal{A}^T \mathcal{F}^{-1}.$$  \hspace{1cm} (53)

Since under action of the transformation $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$ we have $e_i \rightarrow -e_i$, in the spinor representation we must demand $\mathcal{E}_i \rightarrow -\mathcal{E}_i$ also, or

$$\mathcal{E}_i \rightarrow -\mathcal{E}_i = \mathcal{F} \mathcal{E}_i \mathcal{F}^{-1}. \quad (54)$$
In case of the real subalgebras \( \mathcal{O}_{p,q} \) with the ring \( \mathbb{K} \cong \mathbb{R} \), \( p - q \equiv 0, 2 \pmod{8} \), we have \( \dot{E}_i = E_i \) for all matrices of the spinbasis and, therefore, the relation \( (54) \) takes a form

\[
E_i \longrightarrow -E_i = FE_i F^{-1}.
\]

Let \( \{ E_{\gamma_i} \} \cup \{ E_{\delta_j} \} \) be a spinbasis of the algebra \( \mathcal{O}_{p,q} \) over the ring \( \mathbb{K} \cong \mathbb{R} \), \( (E_{\gamma_i}^T = E_{\gamma_i}, E_{\delta_j}^T = -E_{\delta_j}) \). Then the relation \( (55) \) can be written in the form

\[
E_{\gamma_i} \longrightarrow -E_{\gamma_i} = FE_{\gamma_i} F^{-1}, \quad E_{\delta_j} \longrightarrow E_{\delta_j} = FE_{\delta_j} F^{-1}.
\]

Whence

\[
E_{\gamma_i} F = -FE_{\gamma_i}, \quad E_{\delta_j} F = FE_{\delta_j},
\]

that is, the matrix \( F \) of the pseudoantiautomorphism \( A \rightarrow \overline{A} \) in case of the ring \( \mathbb{K} \cong \mathbb{R} \) anticommutes with the symmetric part of the spinbasis of \( \mathcal{O}_{p,q} \) and commutes with the skewsymmetric part. In virtue of Theorem \( \Box \) the matrix \( \Pi \) of the pseudoautomorphism \( A \rightarrow \overline{A} \) over the ring \( \mathbb{K} \cong \mathbb{R} \) is proportional to the unit matrix, \( \Pi \cong I \). Therefore, in this case \( F \cong C \) (\( F \cong EW \)) and an explicit form of the matrix \( F \) coincides with \( C \) (see Theorem \( \Box \)).

In case of the quaternionic ring \( \mathbb{K} \cong \mathbb{H} \), \( p - q \equiv 4, 6 \pmod{8} \), the spinbasis of \( \mathcal{O}_{p,q} \) contains both complex matrices \( E_{\alpha_t} \) and real matrices \( E_{\beta_r} \). For the complex part the relation \( (54) \) takes a form

\[
E_{\alpha_t} \longrightarrow E_{\alpha_t} = FE_{\alpha_t} F^{-1},
\]

or, taking into account complex symmetric and complex skewsymmetric components of the spinbasis, we obtain

\[
E_{\alpha_t} \longrightarrow E_{\alpha_t} = FE_{\alpha_t} F^{-1}, \quad E_{\alpha_s} \longrightarrow E_{\alpha_s} = -FE_{\alpha_s} F^{-1}.
\]

Whence

\[
E_{\alpha_t} F = FE_{\alpha_t}, \quad E_{\alpha_s} F = -FE_{\alpha_s}.
\]

(56)

For the real part of the spinbasis of \( \mathcal{O}_{p,q} \) from \( (54) \) we obtain

\[
E_{\beta_r} \longrightarrow -E_{\beta_r} = FE_{\beta_r} F^{-1},
\]

or, taking into account symmetric and skewsymmetric components of the real part of the spinbasis, we find

\[
E_{\beta_r} \longrightarrow -E_{\beta_r} = FE_{\beta_r} F^{-1}, \quad E_{\beta_s} \longrightarrow E_{\beta_s} = FE_{\beta_s} F^{-1}.
\]

Whence

\[
E_{\beta_r} F = -FE_{\beta_r}, \quad E_{\beta_s} F = FE_{\beta_s}.
\]

(57)

Therefore, the matrix \( F \) of the pseudoantiautomorphism \( A \rightarrow \overline{A} \) commutes with the complex symmetric and real skewsymmetric matrices, and also \( F \) anticommutes with the complex skewsymmetric and real symmetric matrices of the spinbasis of \( \mathcal{O}_{p,q} \), \( p - q \equiv 4, 6 \pmod{8} \).

Let us find an explicit form of the matrix \( F = \Pi C \). It is easy to see that in virtue of \( F = \Pi C = (\Pi E) W = SW \) the matrix \( F \) is a dual with respect to the matrix \( S \) of the pseudoantiautomorphism \( A \rightarrow \overline{A} \). In accordance with Theorem \( \Box \) for the quaternionic types \( p - q \equiv 4, 6 \pmod{8} \) the matrix \( \Pi \) has two different forms: \( \Pi = E_{\alpha_1} E_{\alpha_2} \cdots E_{\alpha_n} (\dot{E}_{\alpha_i} = -E_{\alpha_i}) \).
\[ a \equiv 0 \pmod{2}; \quad \Pi = \epsilon_{\beta_1} \epsilon_{\beta_2} \cdots \epsilon_{\beta_b} (\hat{\epsilon}_{\beta_s} = \epsilon_{\beta_s}), \quad b \equiv 1 \pmod{2}. \] In turn, for the quaternionic types the matrix \( C \) of the antiautomorphism \( A \to \widetilde{A}^\star \) has the two forms (see Theorem 7): 1) \( C = \epsilon_{i_1} \epsilon_{i_2} \cdots \epsilon_{i_{p+q-k}} \) is the product of all symmetric matrices of the spinbasis of \( \mathcal{O}_{p,q} \) at \( p + q - k \equiv 0 \pmod{2} \); 2) \( C = \epsilon_{j_1} \epsilon_{j_2} \cdots \epsilon_{j_k} \) is the product of all skewsymmetric matrices of the spinbasis at \( k \equiv 1 \pmod{2} \). Thus, in accordance with definition \( F = \Pi C \) we have four products: \( F = \epsilon_{\alpha_1} \alpha_2 \cdots \alpha_a \epsilon_{i_1} \epsilon_{i_2} \cdots i_{p+q-k}, \quad F = \epsilon_{\alpha_1} \alpha_2 \cdots \alpha_a \epsilon_{j_1} \epsilon_{j_2} \cdots j_k, \quad F = \epsilon_{\beta_1} \beta_2 \cdots \beta_b \epsilon_{i_1} \epsilon_{i_2} \cdots i_{p+q-k}, \quad F = \epsilon_{\beta_1} \beta_2 \cdots \beta_b \epsilon_{j_1} \epsilon_{j_2} \cdots j_k. \)

Let us examine the first product \( F = \epsilon_{\alpha_1} \alpha_2 \cdots \alpha_a \epsilon_{i_1} \epsilon_{i_2} \cdots i_{p+q-k} \). In this case \( \Pi \) contains all the complex matrices of the spinbasis, among which there are symmetric and skewsymmetric matrices. In turn, \( C \) contains all the symmetric matrices, among which there are both complex and real matrices. It is obvious that in this case \( \Pi C \) contains a quantity of identical complex symmetric matrices. Therefore, the product \( F \) consists of all the complex skewsymmetric matrices and all the real symmetric matrices of the spinbasis. The product \( F \) is even, since the original product \( \Pi C \) is even also. It is easy to see that \( F \) coincides with the product \( \epsilon_{d_1} \epsilon_{d_2} \cdots \epsilon_{d_g} \) at \( g \equiv 0 \pmod{2} \).

Let us find permutation conditions of the matrix \( F = \epsilon_{d_1} \epsilon_{d_2} \cdots \epsilon_{d_g} \) with the units of the spinbasis of \( \mathcal{O}_{p,q}, \quad p - q \equiv 4, 6 \pmod{8} \). For the complex and real parts we obtain

\[
\begin{align*}
F \epsilon_{\alpha_1} &= (-1)^g \epsilon_{\alpha_1} F, \\
F \epsilon_{\alpha_2} &= (-1)^{g-\delta} \epsilon_{d_1} \epsilon_{d_2} \cdots \epsilon_{d_{g-2}}, \\
\epsilon_{\alpha_3} F &= (-1)^{\delta-1} \epsilon_{d_1} \epsilon_{d_2} \cdots \epsilon_{d_{g-2}}, \\
F \epsilon_{\beta_1} &= (-1)^{g-\gamma} \epsilon_{d_1} \epsilon_{d_2} \cdots \epsilon_{d_{g-2}}, \\
\epsilon_{\beta_2} F &= (-1)^{\gamma-1} \epsilon_{d_1} \epsilon_{d_2} \cdots \epsilon_{d_{g-2}}, \\
F \epsilon_{\beta_3} &= (-1)^{g-\delta} \epsilon_{\beta_3} F.
\end{align*}
\]

Therefore, since \( g \equiv 0 \pmod{2} \) the matrix \( F \) always commutes with the complex symmetric and real skewsymmetric matrices and always anticommutes with the complex skewsymmetric and real symmetric matrices of the spinbasis. A comparison of the permutation conditions (58)–(61) with the conditions (56)–(57) shows that \( F = \epsilon_{d_1} \epsilon_{d_2} \cdots \epsilon_{d_g} \) at \( g \equiv 0 \pmod{2} \) automatically satisfies the conditions which define the matrix of the pseudoantiautomorphism \( A \to \widetilde{A}^\star \).

Let examine the second product \( F = \epsilon_{\alpha_1} \alpha_2 \cdots \alpha_a \epsilon_{j_1} \epsilon_{j_2} \cdots j_k \). This product contains all the complex part of the spinbasis and all the skewsymmetric matrices. Therefore, in the product \( \Pi C \) there is a quantity of identical complex skewsymmetric matrices. The product \( F \) is odd and consists of all the complex symmetric and real skewsymmetric matrices of the spinbasis. It is easy to see that in this case \( F \) coincides with the product \( \epsilon_{c_1} \epsilon_{c_2} \cdots \epsilon_{c_s} \) at \( s \equiv 1 \pmod{2} \). Permutation conditions of the matrix \( F = \epsilon_{c_1} \epsilon_{c_2} \cdots \epsilon_{c_s} \) with the units of the spinbasis are

\[
\begin{align*}
F \epsilon_{\alpha_1} &= (-1)^{s-\gamma} \epsilon_{c_1} \epsilon_{c_2} \cdots \epsilon_{c_{s-2}}, \\
\epsilon_{\alpha_2} F &= (-1)^{\gamma-1} \epsilon_{c_1} \epsilon_{c_2} \cdots \epsilon_{c_{s-2}}, \\
F \epsilon_{\alpha_3} &= (-1)^{s-\delta} \epsilon_{c_1} \epsilon_{c_2} \cdots \epsilon_{c_{s-2}}, \\
F \epsilon_{\beta_1} &= (-1)^{s-\gamma} \epsilon_{\beta_1} F, \\
F \epsilon_{\beta_2} &= (-1)^{s-\delta} \epsilon_{\beta_1} F, \\
F \epsilon_{\beta_3} &= (-1)^{s-\delta} \epsilon_{\beta_1} F, \\
\epsilon_{\beta_3} F &= (-1)^{s-\delta} \epsilon_{c_1} \epsilon_{c_2} \cdots \epsilon_{c_{s-2}}.
\end{align*}
\]

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Therefore, since \( s \equiv 1 \) (mod 2) the matrix \( F \) always commutes with the complex symmetric and real skew-symmetric matrices and always anticommutes with the complex skew-symmetric and real symmetric matrices of the spinbasis. Comparing the conditions (52)–(55) with the conditions (56) and (57) we see that \( F = E_{c_1} E_{c_2} \cdots E_{c_s} \) at \( s \equiv 1 \) (mod 2) identically satisfies the conditions which define the matrix of the transformation \( A \to \bar{A}^* \).

The third product \( F = \prod C = E_{\beta_1 \beta_2 \cdots \beta_k} E_{j_1 j_2 \cdots j_k} \) contains all the real part and all the symmetric matrices of the spinbasis. Therefore, in the product \( \Pi C \) there is a quantity of identical real symmetric matrices. Thus, the product \( F \) is odd and consists of all the real skew-symmetric and complex symmetric matrices of the spinbasis. It is easy to see that we came again to the matrix \( F = E_{c_1} E_{c_2} \cdots E_{c_s} \) (mod 2)) with the permutation conditions (52)–(55).

Finally, the fourth product \( F = E_{\beta_1 \beta_2 \cdots \beta_k} E_{j_1 j_2 \cdots j_k} \) contains all the real part and all the skew-symmetric matrices of the spinbasis. Therefore, in the product \( \Pi C \) there is a quantity of identical real skew-symmetric matrices. This product is equivalent to the matrix \( F = E_{d_1} E_{d_2} \cdots E_{d_g} \) (mod 2)) with the permutation conditions (55)–(58).

As with the pseudoantiautomorphism \( A \to \bar{A} \), from the four products we have only two non-equivalent products. Let us find squares of the non-equivalent matrices \( F = E_{d_1} E_{d_2} \cdots E_{d_g} \) (mod 2)) and \( F = E_{c_1} E_{c_2} \cdots E_{c_s} \) (mod 2)):

\[
F^2 = (E_{d_1} E_{d_2} \cdots E_{d_g})^2 = \begin{cases} +1, & \text{if } m + v \equiv 0, 4 \pmod{8}, \\ -1, & \text{if } m + v \equiv 2, 6 \pmod{8}; \end{cases} \tag{66}
\]

\[
F^2 = (E_{c_1} E_{c_2} \cdots E_{c_s})^2 = \begin{cases} +1, & \text{if } u + l \equiv 3, 7 \pmod{8}, \\ -1, & \text{if } u + l \equiv 1, 5 \pmod{8}. \end{cases} \tag{67}
\]

6 The structure of \( \text{Ext}(C_n) \)

As noted previously, the group \( \text{Ext}(C_n) \) is a finite group of order eight. This group contains as a subgroup the automorphism group \( \text{Aut}_\pm(C_n) \) (reflection group). Moreover, in the case of \( \Pi \sim I \) (the subalgebra \( \mathcal{O}_{p,q} \) has the ring \( \mathbb{K} \simeq \mathbb{R}, p - q \equiv 0, 2 \) (mod 8)) the group \( \text{Ext}(C_n) \) is reduced to its subgroup \( \text{Aut}_\pm(C_n) \). The structure of the groups \( \text{Aut}_\pm(C_n), \text{Aut}_\pm(\mathcal{O}_{p,q}) \) is studied in detail (see Theorems 5 and 7).

There are six finite groups of order eight (see Table 1). One is cyclic and two are direct group products of cyclic groups, hence these three are Abelian. The remaining three groups are the quaternion group \( Q_4 \) with elements \( \{ \pm 1, \pm i, \pm j, \pm k \} \), the dihedral group \( D_4 \), and the group \( \mathbb{Z}_4 \otimes \mathbb{Z}_2 \). All these groups are non–Abelian. As is known, an important property of each finite group is its order structure. The order of a particular element \( \alpha \) in the group is the smallest integer \( p \) for which \( \alpha^p = 1 \). The following table lists the number of distinct elements in each group which have order 2, 4, or 8 (the identity 1 is the only element of order 1).
Table 1. Finite groups of order 8.

| Type    | Order structure |
|---------|-----------------|
| $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ | Abelian 7 |
| $\mathbb{Z}_4 \otimes \mathbb{Z}_2$ | 3 4 |
| $\mathbb{Z}_8$ | 1 2 4 |
| $D_4$ | Non–Abelian 5 2 |
| $Q_4$ | 1 6 |
| $\ast \mathbb{Z}_4 \otimes \mathbb{Z}_2$ | 3 4 |

Of course, $\mathbb{Z}_8$ does not occur as a $G(p, q)$ (Salingaros group), since every element of $G(p, q)$ has order 1, 2, or 4. The groups $\mathbb{Z}_4 \otimes \mathbb{Z}_2$ and $\ast \mathbb{Z}_4 \otimes \mathbb{Z}_2$ have the same order structure, but their signatures $(a, b, c, d, e, f, g)$ are different. Moreover, the group $\ast \mathbb{Z}_4 \otimes \mathbb{Z}_2$ presents a first example of the finite group of order 8 which has an important physical meaning.

**Example 1.** Let us consider a Dirac algebra $\mathbb{C}_4$. In the algebra $\mathbb{C}_4$ we can evolve four different real subalgebras $\mathcal{A}_{1,3}$, $\mathcal{A}_{3,1}$, $\mathcal{A}_{4,0}$, $\mathcal{A}_{0,4}$. Let us evolve the spacetime algebra $\mathcal{A}_{1,3}$. The algebra $\mathcal{A}_{1,3}$ has the quaternionic division ring $\mathbb{K} \simeq \mathbb{H}$ ($p - q \equiv 6 \pmod{8}$) and, therefore, admits the following spinor representation (the well known $\gamma$-basis):

$$
\begin{align*}
\gamma_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\
\gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \\
\gamma_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\end{align*}
$$

The famous Dirac equation in the $\gamma$–basis looks like

$$
\left( i\gamma_0 \frac{\partial}{\partial x_0} - i\gamma_1 \frac{\partial}{\partial x} - m \right) \psi(x_0, \mathbf{x}) = 0.
$$

The invariance of the Dirac equation with respect to $P$–, $T$–, and $C$–transformations leads to the following representation (see, for example, [3] and also many other textbooks on quantum field theory):

$$
P \sim \gamma_0, \quad T \sim \gamma_1 \gamma_3, \quad C \sim \gamma_2 \gamma_0.
$$

Thus, we can form a finite group of order 8 ($CPT$–group)

$$
\{1, P, T, PT, C, CP, CT, CPT\} \sim \{1, \gamma_0, \gamma_1 \gamma_3, \gamma_0 \gamma_1 \gamma_3, \gamma_2 \gamma_0, \gamma_2, \gamma_2 \gamma_0 \gamma_1 \gamma_3, \gamma_2 \gamma_1 \gamma_3\}.
$$

It is easy to verify that a Cayley tableau of this group has a form.
Hence it follows that the $CPT$–group $\{70\}$ is a non–Abelian finite group of the order structure $(3, 4)$. In more details, it is the group $\tilde{\mathbb{Z}}_4 \otimes \mathbb{Z}_2$ with the signature $(+, -, -, +, -, -)$.

**Theorem 10.** Let $\mathbb{C}_n$ be the Clifford algebra over the field $\mathbb{F} = \mathbb{C}$ and let $\text{Ext}(\mathbb{C}_n) = \{I, W, E, C, \Pi, K, S, F\}$ be an extended automorphism group of the algebra $\mathbb{C}_n$, where $W$, $E$, $C$, $\Pi$, $K$, $S$, $F$ are spinor representations of the transformations $A \to A^*$, $A \to \bar{A}$, $A \to \bar{A}^*$, $A \to \bar{A}$, $A \to \bar{A}^*$. Then over the field $\mathbb{F} = \mathbb{C}$ in dependence on a division ring structure of the real subalgebras $\mathbb{A}_{p,q} \subset \mathbb{C}_n$ there exist following isomorphisms between finite groups and groups $\text{Ext}(\mathbb{C}_n)$:

1) $\mathbb{K} \simeq \mathbb{R}$, types $p - q \equiv 0, 2 \pmod{8}$.
In this case the matrix $\Pi$ of the pseudoautomorphism $A \to \bar{A}$ is proportional to the unit matrix (identical transformation) and the extended automorphism group $\text{Ext}(\mathbb{C}_n)$ is reduced to the group of fundamental automorphisms $\text{Aut}_+(\mathbb{C}_n)$.

2) $\mathbb{K} \simeq \mathbb{H}$, types $p - q \equiv 4, 6 \pmod{8}$.
In dependence on a spinbasis structure of the subalgebra $\mathbb{A}_{p,q}$ there exist the following group isomorphisms: $\text{Ext}_-(\mathbb{C}_n) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with the signature $(+, +, +, +, +, +)$ and $\text{Ext}(\mathbb{C}_n) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with $(+, +, +, -, -, -, -)$ for the type $p - q \equiv 4 \pmod{8}$ if $m, n, l, u \equiv 0 \pmod{2}$, where $m$ and $l$ are quantities of complex skewsymmetric and symmetric matrices,
and \( u \) and \( v \) are quantities of real skewsymmetric and symmetric matrices of the spinbasis of \( \alpha_{p,q} \). Correspondingly, at \( m,v,l,u \equiv 0 \pmod{2} \) there exist Abelian groups \( \text{Ext}_- (\mathbb{C}_n) \cong \mathbb{Z}_4 \otimes \mathbb{Z}_2 \) with \( (+,−,−,d,e,f,g) \) for the type \( p−q \equiv 4 \pmod{8} \) and \( \text{Ext}_- (\mathbb{C}_n) \cong \mathbb{Z}_4 \otimes \mathbb{Z}_2 \) with \( (−,+,−,d,e,f,g) \) for the type \( p−q \equiv 6 \pmod{8} \), where among the symbols \( d,e,f,g \) there are two pluses and two minuses.

If \( m,v,l,u \equiv 1 \pmod{2} \) or if among \( m,v,l,u \) there are both even and odd numbers, then there exists the non–Abelian group \( \text{Ext}_+ (\mathbb{C}_n) \cong Q_4 \) with the signatures \( (−,−,−,d,e,f,g) \) for the type \( p−q \equiv 6 \pmod{8} \) and \( \text{Ext}_+ (\mathbb{C}_n) \cong D_4 \) with \( (−,+−,d,e,f,g) \) for the type \( p−q \equiv 4 \pmod{8} \). And also there exist \( \text{Ext}_+ (\mathbb{C}_n) \cong D_4 \) with \( (+,+,−,d,e,f,g) \) for the type \( p−q \equiv 6 \pmod{8} \), where among the symbols \( d,e,f,g \) there are three pluses and one minus. Besides, there exist the groups \( \text{Ext}_+ (\mathbb{C}_n) \cong Q_4 \) with the signatures \( (+,−,−,−,−,−) \) for \( p−q \equiv 4 \pmod{8} \) and \( (−,+,−,−,−,−) \) for \( p−q \equiv 6 \pmod{8} \). And also there exist the groups \( \text{Ext}_+ (\mathbb{C}_n) \cong D_4 \) with \( (+,+,−,d,e,f,g) \) for the type \( p−q \equiv 6 \pmod{8} \) and \( (a,b,c,+,+,++) \) for the type \( p−q \equiv 6 \pmod{8} \) (among the symbols \( a,b,c \) there are one plus and two minuses, and among \( d,e,f,g \) there are two pluses and two minuses). There exist the non–Abelian group \( \text{Ext}_+ (\mathbb{C}_n) \cong Q_4 \) with the signatures \( (+,−,−,−,−,−) \) for \( p−q \equiv 4 \pmod{8} \) and \( (−,+,−,−,−,−) \) for \( p−q \equiv 6 \pmod{8} \), where among the symbols \( d,e,f,g \) there are two pluses and two minuses. And also there exist \( \text{Ext}_+ (\mathbb{C}_n) \cong Q_4 \) with \( (+,−,−,d,e,f,g) \) for the type \( p−q \equiv 4 \pmod{8} \) and \( (−,+,−,d,e,f,g) \) for the type \( p−q \equiv 6 \pmod{8} \), where among the symbols \( d,e,f,g \) there are three pluses and one minus. The full number of the different signatures \( (a,b,c,d,e,f,g) \) is equal to 64.

**Proof.** First of all, it is necessary to define permutation relations between the elements of the group \( \text{Ext} \). We start with the matrix of the pseudoautomorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \) (permutation relations between the elements \( \mathcal{W}, \mathcal{E} \) and \( \mathcal{C} \) are found in Theorem 7). As is known, for the types \( p−q \equiv 4,6 \pmod{8} \) the matrix \( \Pi \) exists in the two different forms: 1) \( \Pi = \mathcal{E}_{a_1} \mathcal{E}_{a_2} \cdots \mathcal{E}_{a_n} \) is the product of all complex matrices of the spinbasis at \( a \equiv 0 \pmod{2} \); 2) \( \Pi = \mathcal{E}_{b_1} \mathcal{E}_{b_2} \cdots \mathcal{E}_{b_n} \) is the product of all real matrices of the spinbasis at \( b \equiv 1 \pmod{2} \).

Let us consider permutation relations of \( \Pi \) with the matrix \( \mathcal{K} \) of the pseudoautomorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \). The matrix \( \mathcal{K} \) also exists in the two different forms: \( \mathcal{K} = \mathcal{E}_{a_1} \mathcal{E}_{a_2} \cdots \mathcal{E}_{a_n} \) at \( a \equiv 1 \pmod{2} \) and \( \mathcal{K} = \mathcal{E}_{b_1} \mathcal{E}_{b_2} \cdots \mathcal{E}_{b_n} \) at \( b \equiv 0 \pmod{2} \). In virtue of the definition \( \mathcal{K} = \Pi \mathcal{W} \), where \( \mathcal{W} = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_{p+q} \) is the spinor representation of the automorphism \( \mathcal{A} \rightarrow \mathcal{A}^* \), the matrix \( \Pi = \mathcal{E}_{a_1} \mathcal{E}_{a_2} \cdots \mathcal{E}_{a_n} \) corresponds to a matrix \( \mathcal{K} = \mathcal{E}_{b_1} \mathcal{E}_{b_2} \cdots \mathcal{E}_{b_n} \), since \( n = p+q \) is always even for the types \( p−q \equiv 4,6 \pmod{8} \). Correspondingly, for the matrix \( \Pi = \mathcal{E}_{b_1} \mathcal{E}_{b_2} \cdots \mathcal{E}_{b_n} \) we obtain \( \mathcal{K} = \mathcal{E}_{a_1} \mathcal{E}_{a_2} \cdots \mathcal{E}_{a_n} \), where \( a,b \equiv 1 \pmod{2} \). It is easy to see that in both cases we have a relation

\[
\Pi \mathcal{K} = (−1)^{ab} \mathcal{K} \Pi, \tag{71}
\]

that is, at \( a,b \equiv 0 \pmod{2} \) the matrices \( \Pi \) and \( \mathcal{K} \) always commute and at \( a,b \equiv 1 \pmod{2} \) always anticommute.

Let us find now permutation relations of \( \Pi \) with the matrix \( \mathcal{S} \) of the pseudoanti-automorphism \( \mathcal{A} \rightarrow \overline{\mathcal{A}} \). As is known, the matrix \( \mathcal{S} \) exists in the two non-equivalent forms: 1) \( \mathcal{S} = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s} \) is the product of all complex symmetric and real skewsymmetric matrices

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at \( s \equiv 0 \) (mod 2); 2) \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_g} \) is the product of all complex skewsymmetric and real symmetric matrices at \( g \equiv 1 \) (mod 2). From \( S = \Pi E \) it follows that \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \) corresponds to \( S = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s} \) if \( E = \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k} \) and to \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_g} \) if \( E = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \). In turn, the matrix \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \) corresponds to \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_g} \) if \( E = \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k} \) and to \( S = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s} \) if \( E = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \). Thus, taking into account that \( S = \Pi E \), we obtain

\[
\Pi S = (-1)^{\frac{a(a-1)}{2} + \tau} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k},
\]
\[
\Pi I = (-1)^{\frac{a(a-1)}{2} + \tau - m + a k} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \quad (72)
\]

for the matrices \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \) and \( S = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s} \), where \( m \) is the number of complex skewsymmetric matrices of the spinbasis of \( \mathcal{O}_{p,q}^r \), \( p - q \equiv 4, 6 \) (mod 8). Since a comparison \( a k \equiv 0 \) (mod 2) holds always, then the matrices \( \Pi \) and \( S \) commute at \( m \equiv 0 \) (mod 2) and anticommute at \( m \equiv 1 \) (mod 2). Correspondingly,

\[
\Pi S = (-1)^{\frac{a(a-1)}{2} + \tau} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}},
\]
\[
\Pi I = (-1)^{\frac{a(a-1)}{2} + \tau - l + a (p + q - k)} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \quad (73)
\]

for the matrices \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \) and \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_g} \), where \( l \) is the number of complex symmetric matrices of the spinbasis. Since \( a (p + q - k) \equiv 0 \) (mod 2) \( a \equiv 0 \) (mod 2), \( p + q - k \equiv 1 \) (mod 2)), then in this case the matrices \( \Pi \) and \( S \) commute at \( l \equiv 0 \) (mod 2) and anticommute at \( l \equiv 1 \) (mod 2). Further, we have

\[
\Pi S = (-1)^{\frac{b(b-1)}{2} + \tau} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k},
\]
\[
\Pi I = (-1)^{\frac{b(b-1)}{2} + \tau - u + b k} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \quad (74)
\]

for the matrices \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \) and \( S = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_g} \), where \( u \) is the number of real skewsymmetric matrices of the spinbasis. Since \( b k \equiv 0 \) (mod 2) \( b \equiv 1 \) (mod 2), \( k \equiv 0 \) (mod 2)), then \( \Pi \) and \( S \) commute at \( u \equiv 0 \) (mod 2) and anticommute at \( u \equiv 1 \) (mod 2). Finally,

\[
\Pi S = (-1)^{\frac{b(b-1)}{2} + \tau} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}},
\]
\[
\Pi I = (-1)^{\frac{b(b-1)}{2} + \tau - v + b (p + q - k)} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \quad (75)
\]

for the matrices \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \) and \( S = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s} \), where \( v \) is the number of real symmetric matrices. Therefore, permutation conditions of the matrices \( \Pi \) and \( S \) in this case have the form \( b (p + q - k) \equiv v \) (mod 2), that is, \( \Pi \) and \( S \) commute at \( v \equiv 0 \) (mod 2) and anticommute at \( v \equiv 1 \) (mod 2).

Now we find permutation conditions of \( \Pi \) with the matrix \( F \) of the pseudoantiautomorphism \( \mathcal{A} \rightarrow A^* \). In turn, the matrix \( F \) exists also in the two non-equivalent forms: \( F = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_g} \) at \( g \equiv 0 \) (mod 2) and \( F = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s} \) at \( s \equiv 1 \) (mod 2). From the definition \( F = \Pi C \) it follows that \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \) corresponds to \( F = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_g} \) if \( C = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \) and to \( F = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s} \) if \( C = \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k} \). The matrix \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \) corresponds to \( F = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s} \) if \( C = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \) and to \( F = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_g} \) if \( C = \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k} \). Thus, taking into account that \( F = \Pi C \), we obtain

\[
\Pi F = (-1)^{\frac{a(a-1)}{2} + \tau} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}},
\]
\[
F \Pi = (-1)^{\frac{a(a-1)}{2} + \tau - l + a (p + q - k)} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \quad (76)
\]
for the matrices $\Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a}$ and $F = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_q}$. It is easy to see that $\Pi$ and $F$ commute at $l \equiv 0 \pmod{2}$ and anticommute at $l \equiv 1 \pmod{2}$, since $a, p + q - k \equiv 0 \pmod{2}$. Analogously,

$$\Pi F = (-1)^{\frac{a(a-1)}{2} + \tau} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k},$$

$$FI = (-1)^{\frac{a(a-1)}{2} + \tau - m + ak} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k}$$ (77)

for $\Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a}$ and $F = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_q}$. Therefore, $\Pi$ and $F$ commute at $m \equiv 0 \pmod{2}$ and anticommute at $m \equiv 1 \pmod{2}$, since $a \equiv 0 \pmod{2}$, $k \equiv 1 \pmod{2}$. Further, we have

$$\Pi F = (-1)^{\frac{b(k-1)}{2} + \tau} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}},$$

$$FI = (-1)^{\frac{b(k-1)}{2} + \tau - v + b(p+q-k)} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}}$$ (78)

for the matrices $\Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b}$ and $F = \mathcal{E}_{c_1} \mathcal{E}_{c_2} \cdots \mathcal{E}_{c_s}$. In this case, $\Pi$ and $F$ commute at $v \equiv 0 \pmod{2}$ and anticommute at $v \equiv 1 \pmod{2}$, since $b \equiv 1 \pmod{2}$, $p + q - k \equiv 0 \pmod{2}$. Finally,

$$\Pi F = (-1)^{\frac{b(k-1)}{2} + \tau} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k},$$

$$FI = (-1)^{\frac{b(k-1)}{2} + \tau - u + bk} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k}$$ (79)

for $\Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b}$ and $F = \mathcal{E}_{d_1} \mathcal{E}_{d_2} \cdots \mathcal{E}_{d_q}$. Therefore, in this case permutation conditions of the matrices $\Pi$ and $F$ have the form $bk \equiv u \pmod{2}$, that is, $\Pi$ and $F$ commute at $u \equiv 1 \pmod{2}$ and anticommute at $u \equiv 0 \pmod{2}$, since $b, k \equiv 1 \pmod{2}$.

Let us define now permutation conditions of $\Pi$ with the matrices $W$, $E$ and $C$ of the transformations $\mathcal{A} \to \mathcal{A}^*$, $\mathcal{A} \to \tilde{\mathcal{A}}$ and $\mathcal{A} \to \tilde{\mathcal{A}}^*$, respectively. First of all, according to Theorem 3 in the case of subalgebras $\mathcal{A}_{p,q}$ with the real ring $\mathbb{K} \simeq \mathbb{R}$ (types $p - q \equiv 0, 2 \pmod{8}$) the matrix $\Pi$ is proportional to the unit matrix and, therefore, $\Pi$ commutes with $W$, $E$ and $C$. In case of the ring $\mathbb{K} \simeq \mathbb{H}$ (types $p - q \equiv 4, 6 \pmod{8}$) the matrix $\Pi$ exists in the two forms: $\Pi_{a}$ at $a \equiv 0 \pmod{2}$ and $\Pi_{b}$ at $b \equiv 1 \pmod{2}$. Since $a + b = p + q$, then the matrix $W$ can be represented by a product $\mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b}$ and for $\Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a}$ we have

$$\Pi W = (-1)^{\frac{a(a-1)}{2}} \sigma(\alpha_1) \sigma(\alpha_2) \cdots \sigma(\alpha_a) \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b},$$

$$W \Pi = (-1)^{\frac{a(a-1)}{2} + ba} \sigma(\alpha_1) \sigma(\alpha_2) \cdots \sigma(\alpha_a) \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\alpha_a}. \quad (80)$$

Hence it follows that $\Pi$ and $W$ commute at $ab \equiv 0 \pmod{2}$ and anticommute at $ab \equiv 1 \pmod{2}$, but $a \equiv 0 \pmod{2}$ and, therefore, the matrices $\Pi$ and $W$ always commute in this case. Taking $\Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b}$, we find the following conditions

$$\Pi W = (-1)^{\frac{b(k-1)}{2} + ab} \sigma(\beta_1) \sigma(\beta_2) \cdots \sigma(\beta_b) \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a},$$

$$W \Pi = (-1)^{\frac{b(k-1)}{2}} \sigma(\beta_1) \sigma(\beta_2) \cdots \sigma(\beta_b) \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a}. \quad (81)$$

Hence it follows that $ab \equiv 1 \pmod{2}$, since in this case $a, b \equiv 1 \pmod{2}$ ($p + q = a + b$ is an even number). Therefore, at $b \equiv 1 \pmod{2}$ the matrices $\Pi$ and $W$ always anticommute.

Let us find now permutation conditions between $\Pi$ and the matrix $E$ of the antiautomorphism $\mathcal{A} \to \tilde{\mathcal{A}}$. As is known (see Theorem 7), the matrix $E$ exists in the two non-equivalent
forms. First of all, if \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_m} \) and \( \mathcal{E} = \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_u} \), then it is obvious that \( \Pi \) and \( \mathcal{E} \) contain \( m \) identical complex skewsymmetric matrices. We can represent the matrices \( \Pi \) and \( \mathcal{E} \) in the form of the following products: 

\[
\Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_l} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_m} \quad \text{and} \quad \mathcal{E} = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_m} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_u},
\]

where \( l \) and \( u \) are the numbers of complex symmetric and real skewsymmetric matrices, respectively. Therefore,

\[
\Pi \mathcal{E} = (-1)^{\frac{m(m-1)}{2}} \sigma(i_1)\sigma(i_2)\cdots\sigma(i_m)\mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_l} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_u},
\]

\[
\mathcal{E} \Pi = (-1)^{\frac{m(m-1)}{2}+m(u+l)} \sigma(i_1)\sigma(i_2)\cdots\sigma(i_m)\mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_l} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_u},
\]

that is, \( \Pi \) and \( \mathcal{E} \) commute at \( m(u+l) \equiv 0 \) (mod 2) and anticommute at \( m(u+l) \equiv 1 \) (mod 2).

Analogously, if \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_m} \) and \( \mathcal{E} = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \), then it is easy to see that in this case \( \Pi \) and \( \mathcal{E} \) contain \( l \) identical complex symmetric matrices. Then \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_m} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \) and \( \mathcal{E} = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \) (\( v \) is the number of all real symmetric matrices of the spinbasis) and

\[
\Pi \mathcal{E} = (-1)^{\frac{l(l-1)}{2}} \sigma(i_1)\sigma(i_2)\cdots\sigma(i_l)\mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_m} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_v},
\]

\[
\mathcal{E} \Pi = (-1)^{\frac{l(l-1)}{2}+l(m+v)} \sigma(i_1)\sigma(i_2)\cdots\sigma(i_l)\mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_m} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_v}.
\]

Therefore, in this case \( \Pi \) and \( \mathcal{E} \) commute at \( l(m+v) \equiv 0 \) (mod 2) and anticommute at \( l(m+v) \equiv 1 \) (mod 2).

In turn, the matrices \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \) and \( \mathcal{E} = \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_m} \) contain \( u \) identical real skewsymmetric matrices. Therefore, \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_v} \) and

\[
\Pi \mathcal{E} = (-1)^{\frac{u(u-1)}{2}} \sigma(i_1)\sigma(i_2)\cdots\sigma(i_u)\mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_m},
\]

\[
\mathcal{E} \Pi = (-1)^{\frac{u(u-1)}{2}+u(m+v)} \sigma(i_1)\sigma(i_2)\cdots\sigma(i_u)\mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_m}.
\]

Hence it follows that \( \Pi \) and \( \mathcal{E} \) commute at \( u(m+v) \equiv 0 \) (mod 2) and anticommute at \( u(m+v) \equiv 1 \) (mod 2).

Finally, the matrices \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \) and \( \mathcal{E} = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \) contain \( v \) identical real symmetric matrices. Therefore, in this case \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_v} \) and

\[
\Pi \mathcal{E} = (-1)^{\frac{v(v-1)}{2}} \sigma(i_1)\sigma(i_2)\cdots\sigma(i_v)\mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_v},
\]

\[
\mathcal{E} \Pi = (-1)^{\frac{v(v-1)}{2}+v(u+l)} \sigma(i_1)\sigma(i_2)\cdots\sigma(i_v)\mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_v},
\]

that is, in this case \( \Pi \) and \( \mathcal{E} \) commute at \( v(u+l) \equiv 0 \) (mod 2) and anticommute at \( v(u+l) \equiv 1 \) (mod 2).

It is easy to see that permutation conditions of \( \Pi \) with the matrix \( \mathcal{C} \) of the antiautomorphism \( \mathcal{A} \to \mathcal{A}^* \) are analogous to the conditions \( \text{[82]} \) and \( \text{[83]} \). Indeed, at \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_m} \) and \( \mathcal{C} = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \) these conditions are analogous to \( \text{[83]} \), that is, \( l(m+v) \equiv 0,1 \) (mod 2). For the matrices \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_m} \) and \( \mathcal{C} = \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k} \) we obtain the condition \( \text{[82]} \), that is, \( m(u+l) \equiv 0,1 \) (mod 2). In turn, for \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \) and \( \mathcal{C} = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{p+q-k}} \) we have the condition \( \text{[83]} \), that is, \( v(u+l) \equiv 0,1 \) (mod 2). Finally, the matrices \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_u} \) and \( \mathcal{C} = \mathcal{E}_{j_1} \mathcal{E}_{j_2} \cdots \mathcal{E}_{j_k} \) correspond to \( \text{[84]} \) with \( m(u+l) \equiv 0,1 \) (mod 2).
Let us consider now permutation conditions of the matrix $\mathbf{K}$ of $\mathcal{A} \to \mathcal{A}^\ast$ with other elements of the group $\text{Ext}(\mathbb{C}_a)$. As it has been shown previously, the structure of $\mathbf{K}$ is analogous to the structure of $\Pi$, that is, $\mathbf{K} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_n}$ at $a \equiv 1 \pmod{2}$ and $\mathbf{K} = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_b}$ at $b \equiv 0 \pmod{2}$. Therefore, permutation conditions for $\mathbf{K}$ are similar to the conditions for $\Pi$. Permutation conditions between $\mathbf{K}$ and $\Pi$ are defined by the relation (71).

Coming to the matrix $\mathbf{S}$ of the pseudoantiautomorphism $\mathcal{A} \to \mathcal{A}$, we see that permutation conditions between $\mathbf{K}$ and $\mathbf{S}$ are analogous to (72)–(75). Namely, if $\mathbf{K} = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_b}$ and $\mathbf{S} = \mathcal{E}_{c_1}\mathcal{E}_{c_2}\cdots\mathcal{E}_{c_c}$, then from (73) it follows that $\mathbf{K}$ and $\mathbf{S}$ commute at $v \equiv 0 \pmod{2}$ and anticommute at $v \equiv 1 \pmod{2}$, since in this case $b \equiv 0 \pmod{2}$ and $p + q - k \equiv 0 \pmod{2}$. Analogously, if $\mathbf{K} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ and $\mathbf{S} = \mathcal{E}_{d_1}\mathcal{E}_{d_2}\cdots\mathcal{E}_{d_d}$, then from (73) we find that $\mathbf{K}$ and $\mathbf{S}$ commute at $l \equiv 0 \pmod{2}$ and anticommute at $l \equiv 1 \pmod{2}$, since $a \equiv 1 \pmod{2}$, $p + q - k \equiv 0 \pmod{2}$ for this case. Further, if $\mathbf{K} = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_b}$ and $\mathbf{S} = \mathcal{E}_{d_1}\mathcal{E}_{d_2}\cdots\mathcal{E}_{d_d}$, then from (74) we find that in this case $\mathbf{K}$ and $\mathbf{S}$ commute at $u \equiv 0 \pmod{2}$ and anticommute at $u \equiv 1 \pmod{2}$, since $b \equiv 0 \pmod{2}$, $k \equiv 1 \pmod{2}$. Finally, if $\mathbf{K} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ and $\mathbf{S} = \mathcal{E}_{c_1}\mathcal{E}_{c_2}\cdots\mathcal{E}_{c_c}$, then from (72) it follows that $\mathbf{K}$ and $\mathbf{S}$ commute at $m \equiv 1 \pmod{2}$ and anticommute at $m \equiv 0 \pmod{2}$, since in this case $a, k \equiv 1 \pmod{2}$.

In like manner we can find permutation conditions of $\mathbf{K}$ with the matrix $\mathbf{F}$ of the pseudoantiautomorphism $\mathcal{A} \to \mathcal{A}^\ast$. Indeed, for $\mathbf{K} = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_b}$ and $\mathbf{F} = \mathcal{E}_{d_1}\mathcal{E}_{d_2}\cdots\mathcal{E}_{d_d}$ from (74) it follows that $\mathbf{K}$ and $\mathbf{F}$ commute at $u \equiv 0 \pmod{2}$ and anticommute at $u \equiv 1 \pmod{2}$, since in this case $b, k \equiv 0 \pmod{2}$. Further, if $\mathbf{K} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ and $\mathbf{F} = \mathcal{E}_{c_1}\mathcal{E}_{c_2}\cdots\mathcal{E}_{c_c}$, then from (74) we find that $\mathbf{K}$ and $\mathbf{F}$ commute at $m \equiv 0 \pmod{2}$ and anticommute at $m \equiv 1 \pmod{2}$, since $b \equiv 0 \pmod{2}$, $k \equiv 1 \pmod{2}$. In turn, for the matrices $\mathbf{K} = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_b}$ and $\mathbf{F} = \mathcal{E}_{c_1}\mathcal{E}_{c_2}\cdots\mathcal{E}_{c_c}$ from (73) we obtain that $\mathbf{K}$ and $\mathbf{F}$ commute at $v \equiv 0 \pmod{2}$ and anticommute at $v \equiv 1 \pmod{2}$, since $b \equiv 0 \pmod{2}$, $p + q - k \equiv 1 \pmod{2}$. Finally, if $\mathbf{K} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ and $\mathbf{F} = \mathcal{E}_{d_1}\mathcal{E}_{d_2}\cdots\mathcal{E}_{d_d}$, then from (72) it follows that $\mathbf{K}$ and $\mathbf{F}$ commute at $l \equiv 1 \pmod{2}$ and anticommute at $l \equiv 0 \pmod{2}$, since in this case $a, p + q - k \equiv 1 \pmod{2}$.

It is easy to see that in virtue of similarity of the matrices $\mathbf{K}$ and $\Pi$, permutation conditions of $\mathbf{K}$ with the elements of $\text{Aut}_1(\mathcal{O}_{p,q})$ are analogous to the conditions for $\Pi$. Indeed, if $\mathbf{K} = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_b}$, then from (81) it follows that $\mathbf{K}$ and $\mathbf{W}$ always commute, since in this case $a, b \equiv 0 \pmod{2}$. In turn, if $\mathbf{K} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$, then from (80) we see that $\mathbf{K}$ and $\mathbf{W}$ always anticommute, since $a, b \equiv 1 \pmod{2}$. Further, if $\mathbf{K} = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_b}$ and $\mathbf{E} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$, then from (81) it follows that $\mathbf{K}$ and $\mathbf{E}$ commute at $u(m + v) \equiv 0 \pmod{2}$ and anticommute at $u(m + v) \equiv 1 \pmod{2}$. For the matrices $\mathbf{K} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ and $\mathbf{E} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ from (82) it follows $m(u + l) \equiv 0, 1 \pmod{2}$. Correspondingly, for the matrices $\mathbf{K} = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_b}$ and $\mathbf{E} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ from (83) we obtain the condition $v(u + l) \equiv 1 \pmod{2}$. For $\mathbf{K} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ and $\mathbf{E} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ from (83) we have $l(m + v) \equiv 0, 1 \pmod{2}$.

Let consider permutation conditions of the matrix $\mathbf{S}$ of the transformation $\mathcal{A} \to \mathcal{A}$ with other elements of the group $\text{Ext}(\mathbb{C}_a)$. Permutation conditions of $\mathbf{S}$ with the matrices $\Pi$ and $\mathbf{K}$ have been defined previously (see (72)–(75)). Now we define permutation conditions of $\mathbf{S}$ with the matrix $\mathbf{F}$ of the pseudoantiautomorphism $\mathcal{A} \to \mathcal{A}^\ast$ and elements of the subgroup.
contain identical matrices. Then

$$S \equiv F \equiv 0 \pmod{2},$$

that is, in this case S and F always commute, since $s, g \equiv 0 \pmod{2}$. If $S = E_{d_1}E_{d_2}\cdots E_{d_g}$ and $F = E_{c_1}E_{c_2}\cdots E_{c_s}$, where $s, g \equiv 1 \pmod{2}$, then from (86) it follows that S and F always anticommute.

Further, if $S = E_{c_1}E_{c_2}\cdots E_{c_s}$, then permutation conditions of S with the matrix W of the automorphism $A \to A^*$ have the form:

$$SW = (-1)^{\frac{s(s-1)}{2}} \sigma(c_1)\sigma(c_2)\cdots \sigma(c_s)E_{d_1}E_{d_2}\cdots E_{d_g},$$

$$WS = (-1)^{\frac{s(s-1)}{2} + sg} \sigma(c_1)\sigma(c_2)\cdots \sigma(c_s)E_{d_1}E_{d_2}\cdots E_{d_g},$$

(87)

that is, in this case S always commutes with W, since $s, g \equiv 0 \pmod{2}$. In turn, the matrix $S = E_{d_1}E_{d_2}\cdots E_{d_g}$ always anticommutes with W, since $s, g \equiv 1 \pmod{2}$.

Let us define now permutation conditions between S and E. If $S = E_{c_1}E_{c_2}\cdots E_{c_s}$ and $E = E_{j_1}E_{j_2}\cdots E_{j_k}$, then the product SE contains u identical real skewsymmetric matrices. Hence it follows that $S = E_{c_1}E_{c_2}\cdots E_{c_i}E_{i_1}E_{i_2}\cdots E_{i_u}$ and $E = E_{i_1}E_{i_2}\cdots E_{i_u}E_{j_1}E_{j_2}\cdots E_{j_m}$, then

$$SE = (-1)^{\frac{s(s-1)}{2}} \sigma(i_1)\sigma(i_2)\cdots \sigma(i_u)E_{c_1}E_{c_2}\cdots E_{c_i}E_{i_1}E_{i_2}\cdots E_{i_u}E_{j_1}E_{j_2}\cdots E_{j_m},$$

$$ES = (-1)^{\frac{s(s-1)}{2} + u(l+m)} \sigma(i_1)\sigma(i_2)\cdots \sigma(i_u)E_{c_1}E_{c_2}\cdots E_{c_i}E_{i_1}E_{i_2}\cdots E_{i_u}E_{j_1}E_{j_2}\cdots E_{j_m},$$

(88)

that is, S and E commute at $u(l+m) \equiv 0 \pmod{2}$ and anticommute at $u(l+m) \equiv 1 \pmod{2}$.

In turn, the products $S = E_{d_1}E_{d_2}\cdots E_{d_g}$ and $E = E_{j_1}E_{j_2}\cdots E_{j_k}$ contain m identical complex skewsymmetric matrices. Therefore, $S = E_{d_1}E_{d_2}\cdots E_{d_m}E_{i_1}E_{i_2}\cdots E_{i_u}$, $E = E_{i_1}E_{i_2}\cdots E_{i_m}E_{j_1}E_{j_2}\cdots E_{j_u}$ and

$$SE = (-1)^{\frac{m(m-1)}{2}} \sigma(i_1)\sigma(i_2)\cdots \sigma(i_u)E_{d_1}E_{d_2}\cdots E_{d_m}E_{i_1}E_{i_2}\cdots E_{i_u},$$

$$ES = (-1)^{\frac{m(m-1)}{2} + m(v+u)} \sigma(i_1)\sigma(i_2)\cdots \sigma(i_u)E_{d_1}E_{d_2}\cdots E_{d_m}E_{i_1}E_{i_2}\cdots E_{i_u},$$

(89)

that is, permutation conditions between S and E in this case have the form $m(v+u) \equiv 0, 1 \pmod{2}$.

Further, the products $S = E_{d_1}E_{d_2}\cdots E_{d_g}$ and $E = E_{i_1}E_{i_2}\cdots E_{i_{p+q-k}}$ contain v identical real symmetric matrices. Then $S = E_{d_1}E_{d_2}\cdots E_{d_m}E_{i_1}E_{i_2}\cdots E_{i_v}$, $E = E_{i_1}E_{i_2}\cdots E_{i_v}E_{i_1}E_{i_2}\cdots E_{i_v}$ and

$$SE = (-1)^{\frac{(v-1)}{2} \sigma(i_1)\sigma(i_2)\cdots \sigma(i_v)E_{d_1}E_{d_2}\cdots E_{d_m}E_{i_1}E_{i_2}\cdots E_{i_v}},$$

$$ES = (-1)^{\frac{(v-1)}{2} + v(l+m)} \sigma(i_1)\sigma(i_2)\cdots \sigma(i_v)E_{d_1}E_{d_2}\cdots E_{d_m}E_{i_1}E_{i_2}\cdots E_{i_v},$$

(90)

that is, permutation conditions between S and E are $v(m+l) \equiv 0, 1 \pmod{2}$.

Finally, the products $S = E_{c_1}E_{c_2}\cdots E_{c_s}$ and $E = E_{d_1}E_{d_2}\cdots E_{d_{p+q-k}}$ contain l identical complex symmetric matrices and, therefore, $S = E_{c_1}E_{c_2}\cdots E_{c_m}E_{i_1}E_{i_2}\cdots E_{i_l}$, $E = E_{i_1}E_{i_2}\cdots E_{i_l}E_{i_1}E_{i_2}\cdots E_{i_l}$. Then
\[ SE = (-1)^{\frac{u+v}{2}} \sigma(i_1)\sigma(i_2) \cdots \sigma(i_l)E_{c_1}E_{c_2} \cdots E_{c_s}E_{i_1}E_{i_2} \cdots E_{i_v}, \]

\[ ES = (-1)^{\frac{u+v}{2} + (u+v)} \sigma(i_1)\sigma(i_2) \cdots \sigma(i_l)E_{c_1}E_{c_2} \cdots E_{c_s}E_{i_1}E_{i_2} \cdots E_{i_v} \]  

(91)

and permutation conditions for S and E in this case have the form \( l(u + v) \equiv 0, 1 \pmod{2} \).

It is easy to see that permutation conditions between S and C are analogous to (88)–(91). Indeed, if \( S = E_{c_1}E_{c_2} \cdots E_{c_s} \) and \( C = E_{i_1}E_{i_2} \cdots E_{i_{p+q-k}} \), then from (91) it follows the comparison \( l(u + v) \equiv 0, 1 \pmod{2} \). In turn, if \( S = E_{d_1}E_{d_2} \cdots E_{d_y} \) and \( C = E_{i_1}E_{i_2} \cdots E_{i_{p+q-k}} \), then from (91) we obtain \( v(m + l) \equiv 0, 1 \pmod{2} \). Further, for \( S = E_{d_1}E_{d_2} \cdots E_{d_y} \) and \( C = E_{j_1}E_{j_2} \cdots E_{j_k} \) from (88) it follows that \( m(v + u) \equiv 0, 1 \pmod{2} \). Analogously, for \( S = E_{c_1}E_{c_2} \cdots E_{c_s} \) and \( C = E_{j_1}E_{j_2} \cdots E_{j_k} \) from (88) we have \( u(l + m) \equiv 0, 1 \pmod{2} \).

Finally, let us consider permutation conditions of the matrix \( F \) of \( A \rightarrow A^* \) with other elements of the group \( \text{Ext}(\mathbb{C}_n) \). Permutation conditions of \( F \) with the matrices \( \Pi, K \) and \( S \) have been found previously (see (70)–(73), (88)). Now we define permutation conditions between \( F \) and the elements of the subgroups \( \text{Aut}_\pm(C_{p,q}) \). It is easy to see that permutation conditions between \( F \) and \( W \) are equivalent to (77), that is, \( F = E_{d_1}E_{d_2} \cdots E_{d_y} \) always commute with \( W \), since \( s, g \equiv 0 \pmod{2} \), and \( F = E_{c_1}E_{c_2} \cdots E_{c_s} \) always anticommute with \( W \), since in this case \( s, g \equiv 1 \pmod{2} \). In turn, permutation conditions between \( F \) and \( E \) are equivalent to (88)–(91). Indeed, if \( F = E_{d_1}E_{d_2} \cdots E_{d_y} \) and \( E = E_{j_1}E_{j_2} \cdots E_{j_k} \), then from (88) it follows the comparison \( m(v + u) \equiv 0, 1 \pmod{2} \). For \( F = E_{c_1}E_{c_2} \cdots E_{c_s} \) and \( E = E_{j_1}E_{j_2} \cdots E_{j_k} \) from (88) we have \( u(l + m) \equiv 0, 1 \pmod{2} \). Analogously, for \( F = E_{c_1}E_{c_2} \cdots E_{c_s} \) and \( E = E_{i_1}E_{i_2} \cdots E_{i_{p+q-k}} \) from (91) we obtain \( l(u + v) \equiv 0, 1 \pmod{2} \), and for \( F = E_{d_1}E_{d_2} \cdots E_{d_y} \) and \( E = E_{i_1}E_{i_2} \cdots E_{i_{p+q-k}} \) from (91) it follows \( v(m + l) \equiv 0, 1 \pmod{2} \). It is easy to see that permutation conditions between \( F \) and \( C \) are equivalent to (88)–(91). Namely, for \( F = E_{d_1}E_{d_2} \cdots E_{d_y} \), \( C = E_{i_1}E_{i_2} \cdots E_{i_{p+q-k}} \) from (91) we obtain \( v(m + l) \equiv 0, 1 \pmod{2} \), for \( F = E_{c_1}E_{c_2} \cdots E_{c_s} \), \( C = E_{j_1}E_{j_2} \cdots E_{j_k} \), and \( E = E_{i_1}E_{i_2} \cdots E_{i_{p+q-k}} \) from (91) it follows \( l(u + v) \equiv 0, 1 \pmod{2} \). Correspondingly, for \( F = E_{c_1}E_{c_2} \cdots E_{c_s} \), \( C = E_{j_1}E_{j_2} \cdots E_{j_k} \), \( E = E_{i_1}E_{i_2} \cdots E_{i_{p+q-k}} \) from (88) we find \( u(l + m) \equiv 0, 1 \pmod{2} \), and for \( F = E_{d_1}E_{d_2} \cdots E_{d_y} \), \( C = E_{j_1}E_{j_2} \cdots E_{j_k} \), \( E = E_{i_1}E_{i_2} \cdots E_{i_{p+q-k}} \) from (88) we see that \( m(v + u) \equiv 0, 1 \pmod{2} \).

Now, we are in a position to define a detailed classification for extended automorphism groups \( \text{Ext}(\mathbb{C}_n) \). First of all, since for the subalgebras \( C_{p,q} \) over the ring \( \mathbb{K} \simeq \mathbb{R} \) the group \( \text{Ext}(\mathbb{C}_n) \) is reduced to \( \text{Aut}_\pm(\mathbb{C}_n) \), then all essentially different groups \( \text{Ext}(\mathbb{C}_n) \) correspond to the subalgebra \( C_{p,q} \) with the quaternionic ring \( \mathbb{K} \simeq \mathbb{H} \), \( p - q \equiv 4, 6 \) (mod 8). Let us classify the groups \( \text{Ext}(\mathbb{C}_n) \) with respect to their subgroups \( \text{Aut}_\pm(\mathbb{C}_{p,q}) \). Taking into account the structure of \( \text{Aut}_\pm(\mathbb{C}_{p,q}) \) at \( p - q \equiv 4, 6 \) (mod 8) (see Theorem 7) we obtain for the group \( \text{Ext}(\mathbb{C}_n) = \{ E, W, E, \Pi, K, S, F \} \) the following possible realizations:

\[
\begin{align*}
\text{Ext}^1(\mathbb{C}_n) &= \{ 1, E_{12-p+q}, E_{j_1j_2} \cdots E_{j_k}, E_{i_1i_2} \cdots E_{i_{p+q-k}}, E_{a_1a_2} \cdots E_{a_s}, E_{b_1b_2} \cdots E_{b_t}, E_{d_1d_2} \cdots E_{d_y} \}, \\
\text{Ext}^2(\mathbb{C}_n) &= \{ 1, E_{12-p+q}, E_{j_1j_2} \cdots E_{j_k}, E_{i_1i_2} \cdots E_{i_{p+q-k}}, E_{b_1b_2} \cdots E_{b_t}, E_{a_1a_2} \cdots E_{a_s}, E_{d_1d_2} \cdots E_{d_y} \}, \\
\text{Ext}^3(\mathbb{C}_n) &= \{ 1, E_{12-p+q}, E_{i_1i_2} \cdots E_{i_{p+q-k}}, E_{j_1j_2} \cdots E_{j_k}, E_{a_1a_2} \cdots E_{a_s}, E_{b_1b_2} \cdots E_{b_t}, E_{d_1d_2} \cdots E_{d_y} \}, \\
\text{Ext}^4(\mathbb{C}_n) &= \{ 1, E_{12-p+q}, E_{i_1i_2} \cdots E_{i_{p+q-k}}, E_{j_1j_2} \cdots E_{j_k}, E_{b_1b_2} \cdots E_{b_t}, E_{a_1a_2} \cdots E_{a_s}, E_{c_1c_2} \cdots E_{c_y}, E_{d_1d_2} \cdots E_{d_y} \}.
\end{align*}
\]

The groups \( \text{Ext}^1(\mathbb{C}_n) \) and \( \text{Ext}^2(\mathbb{C}_n) \) have Abelian subgroups \( \text{Aut}_\pm(\mathbb{C}_{p,q}) \) (\( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) or \( \mathbb{Z}_4 \)). In turn, the groups \( \text{Ext}^3(\mathbb{C}_n) \) and \( \text{Ext}^4(\mathbb{C}_n) \) have only non-Abelian subgroups \( \text{Aut}_+(\mathbb{C}_{p,q}) \) (\( \mathbb{Q}_4/\mathbb{Z}_2 \) or \( \mathbb{D}_4/\mathbb{Z}_2 \)).

Let us start with the group \( \text{Ext}^1(\mathbb{C}_n) \). All the elements of \( \text{Ext}^1(\mathbb{C}_n) \) are even products, that is, \( p + q \equiv 0 \pmod{2} \), \( k \equiv 0 \pmod{2} \), \( a \equiv 0 \pmod{2} \), \( b \equiv 0 \pmod{2} \), \( s \equiv 0 \pmod{2} \).
and $g \equiv 0 \pmod{2}$. At this point, the elements $l, W, E, C$ form Abelian subgroups $Z_2 \otimes Z_2$ or $Z_4$ (see Theorem 7). In virtue of (71) the element $\Pi$ commutes with $K$, and from (72) it follows that $\Pi$ commutes with $S$ at $m \equiv 0 \pmod{2}$. From (76) we see that $\Pi$ commutes with $F$ at $l \equiv 0 \pmod{2}$, and from (80) it follows that $\Pi$ commutes with $W$. The conditions (82) show that $\Pi$ commutes with $E$ at $m(u + l) \equiv 0 \pmod{2}$ and commutes with $C$ at $l(m + v) \equiv 0 \pmod{2}$. Further, the element $K$ commutes with $S$ at $v \equiv 0 \pmod{2}$ and commutes with $F$ at $u \equiv 0 \pmod{2}$. And also $K \in \text{Ext}^1$ always commutes with $W$ and commutes correspondingly with $E$ and $C$ at $u(m + v) \equiv 0 \pmod{2}$ and $v(u + l) \equiv 0 \pmod{2}$. From (80) it follows that the element $S$ always commutes with $F$ and $W$. The conditions (83) and (91) show that $S$ commutes with $E$ and $C$ correspondingly at $u(l + m) \equiv 0 \pmod{2}$ and $l(u + v) \equiv 0 \pmod{2}$. From virtue of (87) the element $F$ always commutes with $W$. Finally, from (80) and (90) it follows that $F$ commutes with $E$ and $C$ correspondingly at $m(v + u) \equiv 0 \pmod{2}$ and $v(m + l) \equiv 0 \pmod{2}$. Thus, the group $\text{Ext}^1(C_n)$ is Abelian at $m, l, u, v \equiv 0 \pmod{2}$. In case of the subgroup $\text{Aut}_r(O_{p,q}) \simeq Z_2 \otimes Z_2 \otimes Z_2$ with the signature $(+, +, +, +, +, +, +)$ for the elements $\Pi, K, S$ and $F$ with positive squares $(m - l \equiv 0, 4 \pmod{2}, v - u \equiv 0, 4 \pmod{2}, u + l \equiv 0, 4 \pmod{2})$ and $(m + v \equiv 0, 4 \pmod{2})$. It is easy to see that for the type $p - q \equiv 4 \pmod{8}$ there exists also $\text{Ext}^1(C_n) \simeq Z_4 \otimes Z_2$ with the signature $(+, +, +, -,-,-,-)$ and the subgroup $Z_2 \otimes Z_2$, where $m - l \equiv 2, 6 \pmod{8}, v - u \equiv 2, 6 \pmod{8}$, and $m + v \equiv 0, 2, 6 \pmod{8}$. Further, for the type $p - q \equiv 4 \pmod{8}$ there exist Abelian groups $\text{Ext}^1(C_n) \simeq Z_4 \otimes Z_2$ with the signatures $(+, +, +, d, e, f, g)$ and subgroups $Z_4$, where among the symbols $d, e, f, g$ there are two pluses and two minuses. Correspondingly, at $m, v, l, u \equiv 0 \pmod{2}$ for the type $p - q \equiv 6 \pmod{8}$ there exist Abelian groups $\text{Ext}^1(C_n) \simeq Z_4 \otimes Z_2$ with the signatures $(-, +, +, -,-,-,-)$ and $(-, +, +, d, e, f, g)$ if $m - u \equiv 0, 1, 4, 5 \pmod{8}$, and $m - l \equiv 2, 3, 6, 7 \pmod{8}$ and $m - u \equiv 2, 3, 6, 7 \pmod{8}$, and $v - u \equiv 0, 1, 4, 5 \pmod{8}$.

It is easy to see that from the comparison $k, a, b, s, g \equiv 0 \pmod{2}$ it follows that the numbers $m, v, l, u$ are simultaneously even or odd. The case $m, v, l, u \equiv 0 \pmod{2}$, considered previously, gives rise to the Abelian groups $\text{Ext}^1$. In contrast to this, the case $m, v, l, u \equiv 1 \pmod{2}$ gives rise to non-Abelian groups $\text{Ext}^1$ with Abelian subgroups $Z_4$ and $Z_2 \otimes Z_2$ (it follows from (72)). Namely, in this case we have the group $\text{Ext}^1 \simeq \tilde{Z}_4 \otimes Z_2$ with the subgroup $Z_4$ for $p - q \equiv 4, 6 \pmod{8}$ (it should be noted that signatures of the groups $Z_4 \otimes Z_2$ and $\tilde{Z}_4 \otimes Z_2$ do not coincide), the group $\text{Ext}^1 \simeq Q_4$ with the subgroup $Z_4$ for $p - q \equiv 4, 6 \pmod{8}$ and $\text{Ext}^1 \simeq D_4$ with $Z_2 \otimes Z_2$ for the type $p - q \equiv 4 \pmod{8}$.

Let us consider now the group $\text{Ext}^2(C_n)$. In this case among the elements of $\text{Ext}^2$ there are both even and odd elements: $k \equiv 0 \pmod{2}$, $b \equiv 1 \pmod{2}$, $a \equiv 1 \pmod{2}$, $s \equiv 1 \pmod{2}$. At this point, the elements $l, W, E, C$ form Abelian subgroups $Z_2 \otimes Z_2$ and $Z_4$. In virtue of (73) the element $\Pi$ always anticommutes with $K$, and from (80) it follows that the elements $S$ and $F$ always anticommute. Therefore, all the groups $\text{Ext}^2$ are non-Abelian. Among these groups there are the following isomorphisms: $\text{Ext}^2_+ \simeq \tilde{Z}_4 \otimes Z_2$ with the signatures $(+, +, +, d, e, f, g)$ for the type $p - q \equiv 4 \pmod{8}$ and $(+, +, +, d, e, f, g)$, $(-, +, +, -,-,-,-)$ for $p - q \equiv 6 \pmod{8}$, and among the symbols $d, e, f, g$ there are two pluses and two minuses; $\text{Ext}^2_+ \simeq Q_4$ with $(+, +, +, d, e, f, g)$ for $p - q \equiv 4 \pmod{8}$ and $(+, +, +, -,-,-,-)$ for $p - q \equiv 6 \pmod{8}$; $\text{Ext}^2_+ \simeq D_4$ with $(a, b, c, +, +, +, +)$ and $(+, +, +, d, e, f, g)$, where among $a, b, c$ there are two minuses and one plus, and among $d, e, f, g$ there are two pluses and two minuses. For all the groups $\text{Ext}^2$ among the numbers $m, v, l$ there are both even and odd numbers.
Let consider the group $\text{Ext}^3(\mathbb{C}_n)$. First of all, the groups $\text{Ext}^3$ contain non-Abelian subgroups $\text{Aut}_+(\mathcal{A}_{p,q})$ (the elements $E$ and $C$ are odd). Therefore, all the groups $\text{Ext}^3$ are non-Abelian. Among these groups there are the following isomorphisms: $\text{Ext}^3_+ \simeq D_4$ with $(+, -, +, d, e, f, g)$ and $(+, +, -, d, e, f, g)$ for the type $p - q \equiv 4$ (mod 8), where among $d$, $e$, $f$, $g$ there are three pluses and one minus; $\text{Ext}^3_+ \simeq Q_4$ with $(-, -, -, d, e, f, g)$, where among $d$, $e$, $f$, $g$ there are one plus and three minuses; $\text{Ext}^3_+ \simeq D_4$ with $(+, +, +, d, e, f, g)$, where among $d$, $e$, $f$, $g$ there are three pluses and one minus (the type $p - q \equiv 6$ (mod 8)).

Besides, there exist the groups $\text{Ext}^3_+ \simeq \mathbb{Z}_4 \otimes \mathbb{Z}_2$ with the signatures $(+, -, +, d, e, f, g)$, $(+, +, -, d, e, f, g)$ for the type $p - q \equiv 4$ (mod 8) and $(-, -, -, d, e, f, g)$, $(-, +, +, d, e, f, g)$ for $p - q \equiv 6$ (mod 8), where among $d$, $e$, $f$, $g$ there are one plus and three minuses.

Finally, let us consider the group $\text{Ext}^4(\mathbb{C}_n)$. These groups contain non–Abelian subgroups $\text{Aut}_+(\mathcal{A}_{p,q})$ and, therefore, all $\text{Ext}^4$ are non–Abelian. The isomorphism structure of $\text{Ext}^4$ is similar to $\text{Ext}^3$.

It is easy to see that a full number of all possible signatures $(a, b, c, d, e, f, g)$ is equal to $2^7 = 128$. At this point, we have eight signature types: (seven ‘+’), (one ‘−’, six ‘+’), (two ‘−’, five ‘+’), (three ‘−’, four ‘+’), (four ‘−’, three ‘+’), (five ‘−’, two ‘+’), (six ‘−’, one ‘+’), (seven ‘−’). However, only four types from enumerated above correspond to finite groups of order 8: (seven ‘+’) $\rightarrow \mathbb{Z}_2 \otimes \mathbb{Z}_4 \otimes \mathbb{Z}_2$, (two ‘−’), (five ‘+’) $\rightarrow D_4$, (four ‘−’, three ‘+’) $\rightarrow \mathbb{Z}_4 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ and (six ‘−’, one ‘+’) $\rightarrow Q_4$. Therefore, for the group $\text{Ext}$ there exist 64 different realizations.

Example 2. Let us study an extended automorphism group of the Dirac algebra $\mathbb{C}_4$. We evolve in $\mathbb{C}_4$ the real subalgebra with the quaternionic ring. Let it be the spacetime algebra $\mathcal{A}_{p,q}$ with a spinbasis defined by the matrices (68). We define now elements of the group $\text{Ext}(\mathbb{C}_4)$. First of all, the matrix of the automorphism $\mathcal{A} \rightarrow \mathcal{A}^*$ has a form: $W = \gamma_0 \gamma_1 \gamma_2 \gamma_3$. Further, since
\[
\gamma_0^T = \gamma_0, \quad \gamma_1^T = -\gamma_1, \quad \gamma_2^T = -\gamma_2, \quad \gamma_3^T = -\gamma_3,
\]
then in accordance with Theorem 7 the matrix $E$ of the antiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ is an even product of skewsymmetric matrices of the spinbasis (68), that is, $E = \gamma_1 \gamma_3$. From the definition $C = EW$ we find that the matrix of the antiautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$ has a form $C = \gamma_0 \gamma_2$. $\gamma$–basis contains three real matrices $\gamma_0$, $\gamma_1$ and $\gamma_3$, therefore, for the matrix of the pseudoautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ we obtain $\Pi = \gamma_0 \gamma_1 \gamma_3$ (see Theorem 9). Further, in accordance with $K = \Pi W$ for the matrix of the pseudoautomorphism $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$ we have $K = \gamma_2$. Finally, for the pseudoantiautomorphisms $\mathcal{A} \rightarrow \bar{\mathcal{A}}$, $\mathcal{A} \rightarrow \bar{\mathcal{A}}^*$ from the definitions $S = \Pi E$, $F = \Pi C$ it follows $S = \gamma_0$, $F = \gamma_1 \gamma_2 \gamma_3$. Thus, we come to the following extended automorphism group:

\[
\text{Ext}(\mathbb{C}_4) \simeq \{I, W, E, C, \Pi, K, S, F\} \simeq \\
\{I, \gamma_0 \gamma_1 \gamma_2 \gamma_3, \gamma_1 \gamma_3, \gamma_0 \gamma_2, \gamma_0 \gamma_1 \gamma_3, \gamma_2, \gamma_0, \gamma_1 \gamma_2 \gamma_3\}. \quad (92)
\]

The Cayley tableau of this group has a form
As follows from this tableau, the group $\text{Ext}(\mathbb{C}_4)$ is non-Abelian. $\text{Ext}_+(\mathbb{C}_4)$ contains Abelian group of spacetime reflections, $\text{Aut}^{-}_+(\mathbb{C}^{1,3}) \simeq \mathbb{Z}_4$, as a subgroup. It is easy to see that the group (92) is a group of the form $\text{Ext}^2_+$ with order structure $(3, 4)$. More precisely, the group (92) is a finite group $\ast \mathbb{Z}_4 \otimes \mathbb{Z}_2$ with the signature $(-, -, +, -, -, +, +)$.

Coming back to example 1, we see that the groups (70) and (92) are isomorphic,

$$\{1, P, T, PT, C, CP, CT, CPT\} \simeq \{I, W, E, C, \Pi, K, S, F\} \simeq \ast \mathbb{Z}_4 \otimes \mathbb{Z}_2.$$

Moreover, the subgroups of spacetime reflections of these groups are also isomorphic:

$$\{1, P, T, PT\} \simeq \{I, W, E, C\} \simeq \mathbb{Z}_4.$$

Thus, we come to the following result: the finite group (70), derived from the analysis of invariance properties of the Dirac equation with respect to discrete transformations $C$, $P$ and $T$, is isomorphic to an extended automorphism group of the Dirac algebra $\mathbb{C}_4$. This result allows us to study discrete symmetries and their group structure for physical fields of any spin (without handling to analysis of relativistic wave equations).
7 CPT–structures

As it has been shown previously, there exist 64 different signatures \((a,b,c,d,e,f,g)\) for the extended automorphism group \(\text{Ext}(\mathbb{C}_n)\) of the complex Clifford algebra \(\mathbb{C}_n\). At this point, the group of fundamental automorphisms, \(\text{Aut}_\pm(\mathbb{C}_p,q)\), which has 8 different signatures \((a,b,c)\), is defined as a subgroup of \(\text{Ext}(\mathbb{C}_n)\). As is known, the Clifford–Lipschitz group \(\text{Pin}(n,\mathbb{C})\) (an universal covering of the complex orthogonal group \(O(n,\mathbb{C})\)) is completely constructed within the algebra \(\mathbb{C}_n\) (see definition (8)). If we take into account spacetime reflections, then according to [25, 26, 9], there exist 8 types of universal covering \((\text{PT–structures})\) described by the group \(\text{Pin}^{a,b,c}(n,\mathbb{C})\). As it shown in [28, 29], the group \(\text{Pin}^{a,b,c}(n,\mathbb{C})\) (correspondingly \(\text{Pin}^{a,b,c}(p,q)\) over the field \(\mathbb{F} = \mathbb{R}\)) is completely defined within the algebra \(\mathbb{C}_n\) (correspondingly \(\mathbb{C}_p,q\)) by means of identification of the reflection subgroup \(\{1, P, T, PT\}\) with the automorphism group \(\{\text{Id}, \star, \sim, \bar{x}\}\) of \(\mathbb{C}_n\) (correspondingly \(\mathcal{O}_{p,q}\)).

In turn, the pseudoautomorphism \(A \rightarrow \bar{A}\) of \(\mathbb{C}_n\) (correspondingly \(\mathcal{O}_{p,q}\)) allows us to give a further generalization of the Clifford–Lipschitz group (8) (correspondingly (2)). We claim that there exist 64 types of the universal covering \((\text{CPT–structures})\) for the complex orthogonal group \(O(n,\mathbb{C})\):

\[
\text{Pin}^{a,b,c,d,e,f,g}(n,\mathbb{C}) \simeq \frac{(\text{Spin}_+(n,\mathbb{C}) \circ C^{a,b,c,d,e,f,g})}{\mathbb{Z}_2},
\]

(93)

where \(C^{a,b,c,d,e,f,g}\) are five double coverings of the group \(\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2\). All the possible double coverings \(C^{a,b,c,d,e,f,g}\) are given in the following table:

| a b c d e f g | \(C^{a,b,c,d,e,f,g}\) | Type     |
|--------------|-----------------|----------|
| + + + + + + + | \(\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2\) | Abelian  |
| three ‘+’ and four ‘−’ | \(\mathbb{Z}_4 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2\) |          |
| one ‘+’ and six ‘−’ | \(Q_4 \otimes \mathbb{Z}_2\) | Non–Abelian |
| five ‘+’ and two ‘−’ | \(D_4 \otimes \mathbb{Z}_2\) |          |
| three ‘+’ and four ‘−’ | \(\mathbb{Z}_4 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2\) |          |

The group \(\mathbb{Z}_4\) with non–Abelian \(C^{a,b,c,d,e,f,g}\) we will call Cliffordian and respectively non–Cliffordan when \(C^{a,b,c,d,e,f,g}\) is Abelian.

Analogously, over the field \(\mathbb{F} = \mathbb{R}\) there exist 64 universal coverings of the real orthogonal group \(O(p,q)\):

\[
\rho^{a,b,c,d,e,f,g} : \text{Pin}^{a,b,c,d,e,f,g} \rightarrow O(p,q),
\]

where

\[
\text{Pin}^{a,b,c,d,e,f,g}(p,q) \simeq \frac{(\text{Spin}_+(p,q) \circ C^{a,b,c,d,e,f,g})}{\mathbb{Z}_2}.
\]

(94)

It is easy to see that in case of the algebra \(\mathcal{O}_{p,q}\) (or subalgebra \(\mathcal{O}_{p,q} \subset \mathbb{C}_n\)) with the real division ring \(\mathbb{K} \simeq \mathbb{R}\), \(p − q \equiv 0, 2 \pmod{8}\), CPT–structures, defined by the groups (93) and (94), are reduced to the eight Shirokov–Dąbrowski \(PT–structures\).

Further, using the well-known isomorphism (9), we obtain for the group \(O(n,\mathbb{C})\) with odd dimensionality the following universal covering:

\[
\text{Pin}^{a,b,c,d,e,f,g}(n,\mathbb{C}) \simeq \text{Pin}^{a,b,c,d,e,f,g}(n−1,\mathbb{C}) \bigcup e_{12..n} \text{Pin}^{a,b,c,d,e,f,g}(n−1,\mathbb{C}).
\]
Correspondingly, in virtue of $O_{p,q} \simeq \mathbb{C}_{n-1}$ ($p - q \equiv 3, 7 \pmod{8}$, $n = p + q$) and \[\mathbb{C}\] for the group $O(p,q)$ with odd dimensionality we have
\[\text{Pin}^{a,b,c,d,e,f,g}(p,q) \simeq \text{Pin}^{a,b,c,d,e,f,g}(n-1, \mathbb{C})\]
for $p - q \equiv 3, 7 \pmod{8}$ and
\[\text{Pin}^{a,b,c,d,e,f,g}(p,q) \simeq \text{Pin}^{a,b,c,d,e,f,g}(p-1) \cup e_{12...n} \text{Pin}^{a,b,c,d,e,f,g}(p,q-1),\]
\[\text{Pin}^{a,b,c,d,e,f,g}(p,q) \simeq \text{Pin}^{a,b,c,d,e,f,g}(q-1) \cup e_{12...n} \text{Pin}^{a,b,c,d,e,f,g}(q,p-1)\]
for the types $p - q \equiv 1, 5 \pmod{8}$.

**Theorem 11.** Let $\text{Pin}^{a,b,c,d,e,f,g}(n, \mathbb{C})$ be an universal covering of the complex orthogonal group $O(n, \mathbb{C})$ of the space $\mathbb{C}^n$ associated with the complex algebra $\mathbb{C}_n$. Squares of the symbols $a, b, c, d, e, f, g \in \{-, +\}$ correspond to squares of the elements of the finite group $\text{Ext} = \{1, W, E, C, \Pi, K, S, F\}$: $a = W^2$, $b = E^2$, $c = C^2$, $d = \Pi^2$, $e = K^2$, $f = S^2$, $g = F^2$, where $W$, $E$, $C$, $\Pi$, $K$, $S$, $F$ are matrices of the automorphisms $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$. Then over the field $\mathbb{F} = \mathbb{C}$ in dependence on a division ring structure $\mathbb{K} = f\mathbb{C}_p q f$ of the real subalgebras $\mathbb{C}_p q \subset \mathbb{C}_n$, there exist the following universal coverings ($\text{CPT}$-structures) of the group $O(n, \mathbb{C})$:

1) $\mathbb{K} \simeq \mathbb{R}$, $p - q \equiv 0, 2 \pmod{8}$.
In this case $\text{CPT}$-structures are reduced to the eight Shirokov–Dąbrowski $\text{PT}$-structures
\[\text{Pin}^{a,b,c}(n, \mathbb{C}) \simeq \frac{(\text{Spin}_+(n, \mathbb{C}) \circ C^{a,b,c})}{\mathbb{Z}_2},\]
where $C^{a,b,c}$ are double coverings of the group $\{1, P, T, PT\} \simeq \{1, W, E, C\} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

2) $\mathbb{K} \simeq \mathbb{H}$, $p - q \equiv 4, 6 \pmod{8}$.
In this case we have 64 universal coverings:

a) Non–Cliffordian group
\[\text{Pin}^{+;++;++;++;++}(n, \mathbb{C}) \simeq \frac{(\text{Spin}_+(n, \mathbb{C}) \circ \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)}{\mathbb{Z}_2}\]
exists if the subalgebra $\mathbb{C}_p q$ admits the type $p - q \equiv 4 \pmod{8}$. Non–Cliffordian groups
\[\text{Pin}^{a,b,c,d,e,f,g}(n, \mathbb{C}) \simeq \frac{(\text{Spin}_+(n, \mathbb{C}) \circ \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)}{\mathbb{Z}_2}\]
exist with the signature $(+, +, +, -, -, -)$ when $\mathbb{C}_p q$ has the type $p - q \equiv 4 \pmod{8}$, and also these groups with the signatures $(a, b, c, d, e, f, g)$ exist when $p - q \equiv 6 \pmod{8}$, where among the symbols $a$, $b$, $c$ there are two minuses and one plus, and among $d$, $e$, $f$, $g$ – two pluses and two minuses.

b) Cliffordian groups
\[\text{Pin}^{a,b,c,d,e,f,g}(n, \mathbb{C}) \simeq \frac{(\text{Spin}_+(n, \mathbb{C}) \circ Q_4 \otimes \mathbb{Z}_2)}{\mathbb{Z}_2}\]
exist with the signature $(+, -, -, -, -, -)$ when $p - q \equiv 4 \pmod{8}$ and with the signatures $(-, +, -, -, -, -)$, $(-, +, -, -, -, -)$ when $p - q \equiv 6 \pmod{8}$. And also these groups
exist with the signature \((-,-,-,d,e,f,g)\) if \(p-q \equiv 6 \pmod{8}\), where among the symbols \(d, e, f, g\) there are one plus and three minuses. Cliffordian groups

\[
\text{Pin}^{a,b,c,d,e,f,g}(n, \mathbb{C}) \cong \frac{(\text{Spin}_+(n, \mathbb{C}) \circ D_4 \otimes \mathbb{Z}_2)}{\mathbb{Z}_2}
\]

exist with the signatures \((+,-,-,+,+,+)+\) and \((+,+,+,d,e,f,g)\) when \(\mathcal{O}_{p,q}\) has the type \(p-q \equiv 4 \pmod{8}\) and among \(d, e, f, g\) there are two minuses and two pluses, and also these groups exist with \((+,-,+,d,e,f,g)\) and \((+,+,-,d,e,f,g)\), where among \(d, e, f, g\) there are three pluses and one minus. Cliffordian groups of this type exist also with the signatures \((a,b,c,+,+,+,+)\) and \((-,+,+,-,d,e,f,g)\) when \(p-q \equiv 6 \pmod{8}\), where among \(a, b, c\) there are two minuses and one plus, and among \(d, e, f, g\) there are three pluses and one minus. Cliffordian groups

\[
\text{Pin}^{a,b,c,d,e,f,g}(n, \mathbb{C}) \cong \frac{(\text{Spin}_+(n, \mathbb{C}) \circ Z_4 \otimes Z_2 \otimes Z_2)}{\mathbb{Z}_2}
\]

exist with the signatures \((+,-,-,d,e,f,g)\) when \(p-q \equiv 4 \pmod{8}\) and among the symbols \(d, e, f, g\) there are two pluses and two minuses, and also these groups exist with \((+,-,+,d,e,f,g)\) and \((+,+,-,d,e,f,g)\), where \(p-q \equiv 4 \pmod{8}\) and among \(d, e, f, g\) there are one plus and three minuses. Cliffordian groups of this type exist also with \((-,+,-,d,e,f,g)\) and \((-,-,+,d,e,f,g)\) when \(p-q \equiv 6 \pmod{8}\), where among \(d, e, f, g\) there are two pluses and two minuses. And also these groups exist with \((-,-,-,d,e,f,g)\) if \(p-q \equiv 6 \pmod{8}\), where among \(d, e, f, g\) there are three pluses and one minus.

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