FORMAL LOCAL HOMOLOGY

BY

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Abstract. We introduce a concept of formal local homology modules which is in some sense dual to P. Schenzel’s concept of formal local cohomology modules. The duality theorem and the non-vanishing theorem for formal local homology modules are shown. We also give some conditions for formal local homology modules to be finitely generated or artinian.

1. Introduction. Throughout this paper, \((R, \mathfrak{m})\) will be a local noetherian (commutative) ring with the \(\mathfrak{m}\)-adic topology. Let \(I\) be an ideal of \((R, \mathfrak{m})\) and \(M\) an \(R\)-module. In [S07], P. Schenzel introduced the concept of formal cohomology; the \(i\)th \(I\)-formal cohomology module of \(M\) with respect to \(\mathfrak{m}\) can be defined by

\[
\mathfrak{F}^I_i(M) = \lim_{\leftarrow t} H^i_{\mathfrak{m}}(M/I^tM).
\]

Next, Bijan-Zadeh and Rezaei [BR14], Gu [G14] and Mafi [M13] studied the artinian properties of formal local cohomology modules.

In this paper, we introduce the concept of formal local homology which is in some sense dual to P. Schenzel’s concept of formal local cohomology. The \(i\)th \(I\)-formal local homology module \(\mathfrak{H}^I_{i,J}(M)\) of an \(R\)-module \(M\) with respect to an ideal \(J\) is defined by

\[
\mathfrak{H}^I_{i,J}(M) = \lim_{\leftarrow t} H^i_J(0 : M/I^t).
\]

In the case of \(J = \mathfrak{m}\) we set \(\mathfrak{H}^I_{i,\mathfrak{m}}(M) = \mathfrak{H}^I_{i}(M)\) and speak simply about the \(i\)th \(I\)-formal local homology module.

We also study some basic properties of formal local homology modules \(\mathfrak{H}^I_{i}(M)\) when \(M\) is a linearly compact \(R\)-module, in particular when \(M\) is an artinian \(R\)-module.

The organization of the paper is as follows. In Section 2, we give the definition of the formal local homology modules \(\mathfrak{H}^I_{i,J}(M)\). It is shown that
$H^0_i(\mathfrak{H}_{j,j}(M)) \cong \mathfrak{H}_i(M)$ and $H^1_i(\mathfrak{H}_{j,j}(M)) = 0$ for all $i \neq 0$ (Theorem 2.3). When $M$ is a Hausdorff linearly topologized $R$-module, the Macdonald dual of $M$ is defined by $M^* = \text{Hom}(M, E(R/\mathfrak{m}))$, the set of continuous homomorphisms of $R$-modules $[M62, \S 9]$. The duality theorem (Theorem 2.13) establishes the isomorphisms

$$\mathfrak{H}_i(M^*) \cong \mathfrak{H}_i(M)^*, \quad \mathfrak{H}_i(M)^* \cong \mathfrak{H}_i(M)^*$$

provided $M$ is a linearly compact module over the complete ring $(R, \mathfrak{m})$. In Theorem 2.16 we show that the short exact sequence of artinian modules

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

gives rise to a long exact sequence of $I$-formal local homology modules

$$\cdots \rightarrow \mathfrak{H}_i(M') \rightarrow \mathfrak{H}_i(M) \rightarrow \mathfrak{H}_i(M'') \rightarrow \mathfrak{H}_{i-1}(M') \rightarrow \cdots .$$

In $[R75]$, Robert (and later Kirby $[K90]$) defined the Noetherian dimension $\text{Nd} \dim M$ of a module $M$ as follows: When $M = 0$ we put $\text{Nd} \ dim M = -1$. Then by induction, for any ordinal $\alpha$, we put $\text{Nd} \ dim M = \alpha$ when (i) $\text{Nd} \ dim M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of $M$, there exists a positive integer $m_0$ such that $\text{Nd} \ dim(M_{m+1}/M_m) < \alpha$ for all $m \geq m_0$. Section 2 is closed by the non-vanishing theorem (Theorem 2.18) which says that if $M$ is a non-zero semi-discrete linearly compact $R$-module with $0 \leq \text{Nd} \ dim(0 :_M I) \neq 1$, then

$$\text{Nd} \ dim(0 :_M I) = \max \{ i \mid \mathfrak{H}_i(M) \neq 0 \} .$$

On the other hand, if $M$ is a semi-discrete linearly compact $R$-module such that $\text{Nd} \ dim(0 :_M I) \neq 0$, then

$$\text{Nd} \ dim(0 :_M I) = \max \{ i \mid \mathfrak{H}_i(M) \neq 0 \} .$$

The last section is devoted to finiteness of formal local homology modules. Theorems 3.2 and 3.3 give conditions equivalent to the formal local homology modules $\mathfrak{H}_i(M)$ being finitely generated. In Theorem 3.4 we prove that if $I$ is a principal ideal of $(R, \mathfrak{m})$ and $M$ is an artinian $R$-module, then $\mathfrak{H}_i(M)/I\mathfrak{H}_i(M)$ is a noetherian $\widehat{R}$-module for all $i$. Theorem 3.5 shows that if $M$ is an artinian $R$-module and $s$ a non-negative integer such that $\mathfrak{H}_i(M)$ is a noetherian $\widehat{R}$-module for all $i < s$, then $\mathfrak{H}_s(M)/I\mathfrak{H}_s(M)$ is also a noetherian $\widehat{R}$-module. A question arises: When are the formal local homology modules $\mathfrak{H}_i(M)$ artinian? Theorem 3.6 states that if $M$ is an artinian $R$-module with $\text{Nd} \ dim M = d$, then $\mathfrak{H}_{d-1}(M)$ is an artinian $R$-module. Finally, Theorem 3.7 shows that if $M$ is an artinian $R$-module and $s$ a non-negative integer, then the following statements are equivalent: (i) $\mathfrak{H}_i(M)$ is artinian for all $i > s$, (ii) $\mathfrak{H}_i(M) = 0$ for all $i > s$ and $\text{Ass}(\mathfrak{H}_i(M)) \subseteq \{ \mathfrak{m} \}$ for all $i > s$. 
2. Formal local homology modules. We first recall the concept of linearly compact modules defined by I. G. Macdonald [M62]. A Hausdorff linearly topologized \( R \)-module \( M \) is said to be linearly compact if whenever \( \mathcal{F} \) is a family of closed cosets (i.e., cosets of closed submodules) in \( M \) which has the finite intersection property, then the cosets in \( \mathcal{F} \) have a non-empty intersection. A Hausdorff linearly topologized \( R \)-module \( M \) is called semi-discrete if every submodule of \( M \) is closed. Thus a discrete \( R \)-module is semi-discrete. It is clear that artinian \( R \)-modules are linearly compact with the discrete topology [M73, 3.10]. So the class of semi-discrete linearly compact modules contains all artinian modules. Moreover, when \((R, m)\) is a complete ring, we have \( R = \varprojlim_t R/m^t \) and each \( R/m^t \) is artinian, thus \( R \) is a linearly compact \( R \)-module. If \( M \) is a finitely generated module over the complete ring \((R, m)\), there is a continuous epimorphism \( R^n \to M \), which is an open map, so \( M \) is also linearly compact and semi-discrete (for more details, see [M73 proof of 7.3]).

Let \( I \) be an ideal of \((R, m)\) and \( M \) an \( R \)-module. It is well-known that the \( i \)th local cohomology module \( H^i_I(M) \) of \( M \) with respect to \( I \) can be defined by

\[
H^i_I(M) = \lim_{\leftarrow t} \text{Ext}^i_R(R/I^t, M).
\]

When \( i = 0 \), \( H^0_I(M) \cong \bigcup_{t>0} (0 :_M I^t) = \Gamma_I(M) \).

In [S07], P. Schenzel introduced the concept of formal cohomology, defining the \( i \)th \( I \)-formal cohomology module of \( M \) with respect to \( m \) by

\[
\mathfrak{F}^i_I(M) = \lim_{\leftarrow t} H^i_m(M/I^tM).
\]

Note that the \( i \)th local homology module \( H^i_I(M) \) of an \( R \)-module \( M \) with respect to \( I \) can be defined by

\[
H^i_I(M) = \lim_{\leftarrow t} \text{Tor}^R_i(R/I^t, M) \quad (\text{CN01}).
\]

When \( i = 0 \), \( H^0_I(M) \cong \varprojlim_t M/I^tM = \Lambda_I(M) \), the \( I \)-adic completion of \( M \).

From the concept of formal cohomology and the concept of local homology we suggest the following definition.

**Definition 2.1.** Let \( I, J \) be ideals of \( R \). The \( i \)th \( I \)-formal local homology module \( \mathfrak{F}^i_{I,J}(M) \) of an \( R \)-module \( M \) with respect to \( J \) is defined by

\[
\mathfrak{F}^i_{I,J}(M) = \lim_{\leftarrow t} H^i_J(0 :_M I^t).
\]

In the case of \( J = m \) we set \( \mathfrak{F}^i_{I,m}(M) = \mathfrak{F}^i_I(M) \) and speak simply about the \( i \)th \( I \)-formal local homology module.
Remark 2.2. (i) From [CN08, 3.1(i)] we know that $H^J_i(0 :_M I^t)$ has a natural structure of a module over the ring $A_J(R)$, and hence $\mathfrak{z}^I_{i,J}(M)$ also has a natural structure of a module over $A_J(R)$. In particular, $\mathfrak{z}^I_{i,J}(M)$ has a natural structure of a module over $\hat{R}$.

(ii) If $M$ is finitely generated, then $H^J_i(0 :_M I^t) = 0$ for all $i > 0$, all ideals $I, J$ and all integers $t$ by [CN01, 3.2(ii)], and so $\mathfrak{z}^I_{i,J}(M) = 0$ for all $i > 0$.

In the following theorem we compute the local cohomology modules of an $I$-formal local homology module $\mathfrak{z}^I_{i,J}(M)$.

Theorem 2.3. Let $M$ be an $R$-module. Then

$$H^I_i(\mathfrak{z}^I_{j,J}(M)) \cong \begin{cases} 0, & i \neq 0, \\ \mathfrak{z}^I_{j,J}(M), & i = 0, \end{cases}$$

for any integer $j$.

Proof. We have

$$H^I_i(\mathfrak{z}^I_{j,J}(M)) = H^I_i\left(\lim_{t \to} H^J_i(0 :_M I^t)\right) \cong \lim_{t \to} H^I_i(H^J_i(0 :_M I^t)).$$

Assume that the ideal $I$ is generated by $r$ elements $x_1, \ldots, x_r$. Set $\underline{x}(s) = (x^s_1, \ldots, x^s_r)$ and let $H^i(\underline{x}(s), N)$ be the $i$th Koszul cohomology module of an $R$-module $N$ with respect to $\underline{x}(s)$. We have

$$H^I_i(\mathfrak{z}^I_{j,J}(M)) \cong \lim_{t \to} \lim_{s \to} H^I(\underline{x}(s), H^J_j(0 :_M I^t)).$$

Note that

$$\underline{x}(s)H^J_j(0 :_M I^t) = 0 \quad \text{for all } s \geq t.$$

Then

$$\lim_{s \to} H^I(\underline{x}(s), H^J_j(0 :_M I^t)) \cong \begin{cases} 0, & i \neq 0, \\ H^J_j(0 :_M I^t), & i = 0. \end{cases}$$

By passing to direct limits $\lim_{t \to}$ we obtain the required conclusion.

Corollary 2.4. Let $M$ be an $R$-module and $i$ an integer such that $0 : \mathfrak{z}^I_{i,J}(M) I = 0$. Then $\mathfrak{z}^I_{i,J}(M) = 0$.

Proof. It follows from 2.3 that

$$\mathfrak{z}^I_{i,J}(M) = \Gamma_I(\mathfrak{z}^I_{i,J}(M)) = \bigcup_{t>0} (0 : \mathfrak{z}^I_{i,J}(M) I^t).$$

As $0 : \mathfrak{z}^I_{i,J}(M) I = 0$, we conclude that $\mathfrak{z}^I_{i,J}(M) = 0$.

If $M$ is a linearly compact $R$-module, then $M$ has a natural structure of a linearly compact module over $\hat{R}$ by [CN08, 7.1]. We have the following lemma.
Lemma 2.5. Let $M$ be a linearly compact $R$-module. Then
\[ \mathcal{F}_{i,J}^I(M) \cong \mathcal{F}_{i,J}^\hat{R}(M) \quad \text{for all } i \geq 0. \]

Proof. The natural homomorphism $R \to \hat{R}$ gives by [CN01, 3.7] isomorphisms
\[ H^I_i(0 :_M I^t) \cong H^I_\hat{R}_i(0 :_M I^t \hat{R}) \]
for all $i \geq 0$. By passing to direct limits, we have the isomorphisms
\[ \mathcal{F}_{i,J}^I(M) \cong \mathcal{F}_{i,J}^\hat{R}(M) \]
as required. ■

Lemma 2.6. Let $I, J$ be ideals of $R$ and $M$ a linearly compact $R$-module. If $M$ is $J$-separated (meaning that $\bigcap_{t>0} J^t M = 0$), then
\[ \mathcal{F}_{i,J}^I(M) \cong \begin{cases} 0, & i \neq 0, \\ \Gamma_I(M), & i = 0. \end{cases} \]

Proof. As $M$ is $J$-separated, $0 :_M I^t$ is also $J$-separated for all $t > 0$. It follows from [CN08, 3.8] that
\[ H^J_i(0 :_M I^t) \cong \begin{cases} 0, & i \neq 0, \\ 0 :_M I^t, & i = 0. \end{cases} \]
By passing to direct limits we obtain the conclusion. ■

By [CN01, 3.3(i)] and [CN08, 3.3], the local homology modules $H^I_i(M)$ are linearly compact and $J$-separated for all $i$. Hence we have the following immediate consequence.

Corollary 2.7. Let $M$ be a linearly compact $R$-module. Then
\[ \mathcal{F}_{i,J}^I(H^J_j(M)) \cong \begin{cases} 0, & i \neq 0, \\ \Gamma_I(H^J_j(M)), & i = 0, \end{cases} \]
for all $j$.

Lemma 2.8. Let $I, J$ be ideals of $R$ and $M$ an artinian $R$-module. If $M$ is $I$-separated (meaning that $\bigcap_{t>0} I^t M = 0$), then
\[ \mathcal{F}_{i,J}^I(M) \cong H^I_i(M) \quad \text{for all } i. \]

Proof. As $M$ is an $I$-separated artinian $R$-module, there is a positive integer $n$ such that $I^n M = 0$. Then $0 :_M I^n = M$. Therefore
\[ \mathcal{F}_{i,J}^I(M) = \lim_{\longleftarrow t} H^I_i(0 :_M I^t) \cong H^J_i(M) \quad \text{for all } i. \quad \Box \]

If $M$ is an artinian $R$-module, then $M$ is linearly compact and discrete. Moreover, in the case $J = \mathfrak{m}$, it follows from [CN01, 4.6] that $H^\mathfrak{m}_i(M)$ is a
noetherian $\hat{R}$-module. Therefore, from [2.5] [2.6] and [2.8] we have the following immediate consequences.

**Corollary 2.9.** Let $I, J$ be ideals of $R$ and $M$ an artinian $R$-module. The following statements are true:

(i) $\mathfrak{F}_i^I(M) \cong \mathfrak{F}_i^{\hat{R}}(M)$ for all $i \geq 0$.

(ii) If $M$ is $J$-separated, then

$$\mathfrak{F}_{i,J}^I(M) \cong \begin{cases} 0, & i \neq 0, \\ M, & i = 0. \end{cases}$$

(iii) If $M$ is $I$-separated, then

$$\mathfrak{F}_i^I(M) \cong H^m_i(M)$$

and so $\mathfrak{F}_i^I(M)$ is a noetherian $\hat{R}$-module for all $i$.

In order to state the duality theorem we recall the concepts of Matlis dual and Macdonald dual. Let $M$ be an $R$-module and $E(R/m)$ the injective envelope of $R/m$. The module $D(M) = \text{Hom}(M, E(R/m))$ is called the Matlis dual of $M$. If $M$ is a Hausdorff linearly topologized $R$-module, then the Macdonald dual of $M$ is defined by $M^* = \text{Hom}(M, E(R/m))$ (with “Hom” italicized), which is the set of continuous homomorphisms of $R$-modules (see [M62, §9]). Note that $E(R/m)$ is an artinian $R$-module, so $E(R/m)$ is linearly compact with the discrete topology. The topology on $M^*$ is defined as in [M62, 8.1]. Moreover, if $M$ is semi-discrete, then the topology of $M^*$ coincides with that induced on it as a submodule of $E(R/m)^M$, where $E(R/m)^M = \prod_{x \in M} (E(R/m))^x$, $(E(R/m))^x = E(R/m)$ for all $x \in M$ (see [M62, 8.6]). A Hausdorff linearly topologized $R$-module $M$ is called linearly discrete if every $m$-primary quotient of $M$ is discrete. It is clear that $M^* \subseteq D(M)$, and equality holds if and only if $M$ is semi-discrete, by the following lemma.

**Lemma 2.10 ([M62, 5.8]).** Let $M$ be a Hausdorff linearly topologized $R$-module. Then $M$ is semi-discrete if and only if $M^* = D(M)$.

**Lemma 2.11 ([M62, 9.3, 9.12, 9.13]).** Let $(R, \mathfrak{m})$ be a complete local noetherian ring.

(i) If $M$ is linearly compact, then $M^*$ is linearly discrete (hence semi-discrete). If $M$ is semi-discrete, then $M^*$ is linearly compact.

(ii) If $M$ is linearly compact or linearly discrete, then we have a topological isomorphism $\omega : M \xrightarrow{\cong} M^{**}$.

**Lemma 2.12.** Let $(R, \mathfrak{m})$ be a complete local noetherian ring.

(i) If $M$ is finitely generated, then $M^*$ is artinian.

(ii) If $M$ is artinian, then $M^*$ is finitely generated.
Proof. (i) As $M$ is a finitely generated module over the complete local noetherian ring $(R, \mathfrak{m})$, it follows from [M62, 7.3] that $M$ is linearly compact and semi-discrete. Then $M^* = D(M)$ by 2.10. Now, the conclusion follows from [S90, 3.4.11].

(ii) An artinian $R$-module is linearly compact with the discrete topology. Then $M^* = D(M)$ by 2.10. Finally, the conclusion follows from [S90, 3.4.12].

We have the following duality theorem.

**Theorem 2.13.** Let $(R, \mathfrak{m})$ be a complete ring and $M$ a linearly compact $R$-module. Then

$$\mathfrak{F}_i^l(M^*) \cong \mathfrak{F}_i^l(M) \quad \text{for all } i.$$  

**Proof.** By [CN08, 6.7],

$$M^*/I^t M^* \cong (0 :_M I^t)^*, \quad (M/I^t M)^* \cong 0 :_M I^t \quad \text{for all } t > 0.$$  

Therefore

$$\mathfrak{F}_i^l(M^*) = \varinjlim_{t} H_i^m(0 :_M I^t) \cong \varinjlim_{t} H_i^m((M/I^t M)^*)$$  

$$\cong \varinjlim_{t} (H_i^m(M/I^t M))^* \quad \text{(see [CN08, 6.4(ii)])}$$  

$$= \mathfrak{F}_i^l(M)^* \quad \text{(see [M62, 9.14])},$$  

$$\mathfrak{F}_i^l(M) = \varprojlim_{t} H_i^m(M^*/I^t M^*) \cong \varprojlim_{t} H_i^m((0 :_M I^t)^*)$$  

$$\cong \varprojlim_{t} (H_i^m(0 :_M I^t))^* \quad \text{(see [CN08, 6.4(ii)])}$$  

$$= \mathfrak{F}_i^l(M)^* \quad \text{(see [M62, 2.6])}.$$  

The proof is complete. ■

**Corollary 2.14.** Let $(R, \mathfrak{m})$ be a complete ring and $M$ a linearly compact $R$-module. Then

$$\mathfrak{F}_i^l(M) \cong \mathfrak{F}_i^l(M^*), \quad \mathfrak{F}_i^l(M) \cong \mathfrak{F}_i^l(M^*)^* \quad \text{for all } i.$$  

**Proof.** Combining 2.11(ii) with 2.13 yields

$$\mathfrak{F}_i^l(M) \cong \mathfrak{F}_i^l(M^{**}) \cong \mathfrak{F}_i^l(M^*)^*, \quad \mathfrak{F}_i^l(M) \cong \mathfrak{F}_i^l(M^{**}) \cong \mathfrak{F}_i^l(M^*)^*.$$  

In order to state Theorem 2.16 about the long exact sequence of formal local homology modules we need the following lemma.

**Lemma 2.15.**

(i) A continuous homomorphism $f : M \to N$ of linearly topologized $R$-modules induces continuous homomorphisms $\varphi_i : \mathfrak{F}_i^l(M) \to \mathfrak{F}_i^l(N)$ of formal local cohomology modules for all $i$. 

(ii) If $M$ is a semi-discrete linearly compact $R$-module, then $\mathcal{F}^i_f(M)$ is linearly compact for all $i$.

Proof. (i) The continuous homomorphism $f : M \to N$ induces continuous homomorphisms $M/I^tM \to N/I^tN$ for all $t > 0$. By an argument analogous to that used in [CN08, proof of 2.5] we get continuous homomorphisms $\text{Ext}^i_R(R/m^s, M/I^tM) \to \text{Ext}^i_R(R/m^s, N/I^tN)$ for all $s, t > 0$ and $i$. By passing to direct limits $\lim_{\to t}$ we have continuous homomorphisms of local cohomology modules $H^i_m(M/I^tM) \to H^i_m(N/I^tN)$. Now, by passing into inverse limits $\lim_{\leftarrow t}$ we get continuous homomorphisms of formal local cohomology modules $\varphi_i : \mathcal{F}^i_f(M) \to \mathcal{F}^i_f(N)$ for all $i$.

(ii) By [CN08, 7.9] the local cohomology modules $H^i_m(M/I^tM)$ are artinian for all $t$ and $i$. Note that $\mathcal{F}^i_f(M) = \lim_{\leftarrow t} H^i_m(M/I^tM)$. Therefore, the conclusion follows from [M62, 3.7, 3.10].

Theorem 2.16. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of artinian modules. Then there is a long exact sequence of $I$-formal local homology modules

$$\cdots \to \mathcal{F}^i_1(M') \to \mathcal{F}^i_1(M) \to \mathcal{F}^i_1(M'') \to \mathcal{F}^i_{-1}(M') \to \cdots.$$ 

Proof. By [S89, 1.11], an artinian module over a local noetherian ring $(R, m)$ has a natural structure of artinian module over $\hat{R}$, and a subset of $M$ is an $R$-submodule if and only if it is an $\hat{R}$-submodule. Thus, from 2.9 we may assume that $(R, m)$ is a complete ring. We now consider the short exact sequence of artinian $R$-modules

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$ 

Note that the artinian $R$-modules are linearly compact and discrete, so the homomorphisms $f, g$ are continuous. Combining [CN08, 6.5] with 2.12 we have the short exact sequence of finitely generated $R$-modules

$$0 \to M'' \xrightarrow{f^*} M^* \xrightarrow{g^*} M'^* \to 0$$

where the induced homomorphisms $f^*, g^*$ are continuous. It gives rise by [S07, 3.11] to a long exact sequence of $I$-formal local cohomology modules

$$\cdots \to \mathcal{F}^{i-1}_1(M'^*) \to \mathcal{F}^i_1(M'') \xrightarrow{g_i} \mathcal{F}^i_1(M^*) \xrightarrow{f_i} \mathcal{F}^i_1(M'^*) \to \cdots.$$ 

From 2.15(ii), the $I$-formal local cohomology modules $\mathcal{F}^i_1(M^*), \mathcal{F}^i_1(M''), \mathcal{F}^i_1(M'^*)$ are linearly compact and the homomorphisms of the long exact sequence are inverse limits of the homomorphisms of artinian modules. Note that homomorphisms of artinian modules are continuous, because the topologies on the artinian modules are discrete. Moreover, the inverse limits of
continuous homomorphisms are also continuous. Thus the homomorphisms of the long exact sequence are continuous. Therefore by [CN08, 6.5] the long exact sequence induces an exact sequence

\[ \cdots \to (\tilde{\delta}^i_1(M^*)^*) \to (\tilde{\delta}^i_2(M^*)^*) \to (\tilde{\delta}^i_3(M^*)^*) \to (\tilde{\delta}^{i-1}_1(M^*)^*) \to \cdots. \]

The conclusion now follows from 2.14. \[ \Box \]

We now recall the concept of Noetherian dimension of an \( R \)-module \( M \), denoted by \( \text{Ndim} \ M \). This notion was originally introduced by R. N. Roberts [R75] under the name of Krull dimension. Later, D. Kirby [K90] changed the terminology to Noetherian dimension to avoid confusion with well-known Krull dimension of finitely generated modules.

Let \( M \) be an \( R \)-module. When \( M = 0 \) we put \( \text{Ndim} \ M = -1 \). Then by induction, for any ordinal \( \alpha \), we put \( \text{Ndim} \ M = \alpha \) when (i) \( \text{Ndim} \ M < \alpha \) is false, and (ii) for every ascending chain \( M_0 \subseteq M_1 \subseteq \cdots \) of submodules of \( M \), there exists a positive integer \( m_0 \) such that \( \text{Ndim}(M_{m+1}/M_m) < \alpha \) for all \( m \geq m_0 \). Thus \( M \) is non-zero and finitely generated if and only if \( \text{Ndim} \ M = 0 \). If \( 0 \to M'' \to M' \to 0 \) is a short exact sequence of \( R \)-modules, then \( \text{Ndim} \ M = \max\{\text{Ndim} M'', \text{Ndim} M'\} \).

**Proposition 2.17.** Let \( I, J \) be ideals of \( R \) and \( M \) an artinian \( R \)-module. If \( \text{Ndim}(0 :_M I) = 0 \), then

\[ \tilde{\delta}^I_{i,J}(M) \cong \begin{cases} 0, & i \neq 0, \\ M, & i = 0. \end{cases} \]

**Proof.** Since \( \text{Ndim}(0 :_M I) = 0 \), \( 0 :_M I \) has finite length by [R75, p. 269], and so \( 0 :_M I \) is \( J \)-separated. It follows from [CN08, 3.8] that

\[ H^J_i(0 :_M I^t) \cong \begin{cases} 0, & i \neq 0, \\ 0 :_M I^t, & i = 0, \end{cases} \]

for all \( t > 0 \). By passing to direct limits we have

\[ \tilde{\delta}^I_{i,J}(M) \cong \begin{cases} 0, & i \neq 0, \\ \Gamma^I_i(M), & i = 0. \end{cases} \]

Now the conclusion follows from [S89, 1.4], as \( M \) is an artinian module over the local ring \((R, m)\). \[ \Box \]

Recall that the (Krull) dimension \( \dim R M \) of a non-zero \( R \)-module \( M \) is the supremum of the lengths of chains of primes in the support of \( M \) if this supremum exists, and \( \infty \) otherwise. For convenience, we set \( \dim M = -1 \) if \( M = 0 \). Note that if \( M \) is non-zero and artinian, then \( \dim M = 0 \). If \( M \) is finitely generated, then \( \dim M = \max\{\dim R/p \mid p \in \text{Ass} M\} \).
By [CN08, 4.10], if $M$ is a non-zero semi-discrete linearly compact $R$-module, then

\[
\text{Ndim } \Gamma_m(M) = \max \{ i \mid H^m_i(M) \neq 0 \} \quad \text{if } \Gamma_m(M) \neq 0,
\]
\[
\text{Ndim } M = \max \{ i \mid H^m_i(M) \neq 0 \} \quad \text{if } \text{Ndim } M \neq 1.
\]

We have the following non-vanishing theorem for formal local homology modules.

**Theorem 2.18.** Let $M$ be a non-zero semi-discrete linearly compact $R$-module. Then

(i) $\text{Ndim}(0 :_M I^t) = \text{Ndim}(0 :_M I)$ if $0 \leq \text{Ndim}(0 :_M I) \neq 1$; 
(ii) $\text{Ndim}(0 :_\Gamma_m(M) I^t) = \text{Ndim}(0 :_\Gamma_m(M) I)$ if $\text{Ndim}(0 :_\Gamma_m(M) I) \neq 0$.

**Proof.** (i) We begin by proving that $\text{Ndim}(0 :_M I^t) = \text{Ndim}(0 :_M I)$ for all $t > 0$.

As $M$ is a semi-discrete linearly compact $R$-module, [CN08, 7.1,7.2] shows that $M$ has a natural structure of a semi-discrete linearly compact module over $\hat{R}$ and $\text{Ndim}_R M = \text{Ndim}_{\hat{R}} M$. Thus, we may assume that $(R, \mathfrak{m})$ is a complete ring. First, we consider the special case when $M$ is artinian. Then $D(M)$ is a finitely generated $R$-module by Matlis dual. We have

\[
\text{dim } D(M)/I^t D(M) = \text{dim } D(M)/ID(M).
\]

Combining 2.10 with [CN08 7.4] yields

\[
\text{Ndim}(0 :_M I^t) = \text{dim } D(0 :_M I^t) = \text{dim } D(M)/I^t D(M) = \text{dim } D(M)/ID(M) = \text{Ndim}(0 :_M I).
\]

We now assume that $M$ is a semi-discrete linearly compact $R$-module. By [Z83 Theorem] there is a short exact sequence

\[
0 \to N \to M \to A \to 0,
\]

where $N$ is finitely generated and $A$ is artinian. It induces an exact sequence

\[
0 \to 0 :_N I^t \xrightarrow{f} 0 :_M I^t \xrightarrow{g} 0 :_A I^t \xrightarrow{\delta} \text{Ext}_R^1(R/I^t, N).
\]

Then we have two short exact sequences

\[
0 \to 0 :_N I^t \xrightarrow{f} 0 :_M I^t \to \text{Im } g \to 0,
\]
\[
0 \to \text{Im } g \to 0 :_A I^t \rightarrow \text{Im } \delta \to 0.
\]

Since $0 :_N I^t$ and $\text{Im } \delta$ are finitely generated $R$-modules, we get $\text{Ndim}(0 :_N I^t) = \text{Ndim} \text{Im } \delta = 0$. It follows that

\[
\text{Ndim}(0 :_M I^t) = \text{Ndim } \text{Im } g = \text{Ndim}(0 :_A I^t) = \text{Ndim}(0 :_M I).
\]
The rest of the proof is analogous to that of (i). 

\[ d = \text{Ndim}(0 :_M I) = \text{Ndim}(0 :_M I^t) = \max\{ i \mid H_i^m(0 :_M I^t) \neq 0 \} \]

d for all \( t > 0 \). Thus \( H_d^m(0 :_M I^t) \neq 0 \) and \( H_i^m(0 :_M I^t) = 0 \) for all \( i > d \) and \( t > 0 \). The short exact sequences for all \( t > 0 \),

\[ 0 \to 0 :_M I^t \to 0 :_M I^{t+1} \to 0 :_M I^{t+1}/0 :_M I^t \to 0, \]

induce exact sequences

\[ \cdots \to H_{d+1}^m(0 :_M I^{t+1}/0 :_M I^t) \to H_d^m(0 :_M I^t) \to H_d^m(0 :_M I^{t+1}) \to \cdots \]

by \[ \text{CN08} \ 3.7 \]. As \( \text{Ndim}(0 :_M I^{t+1}/0 :_M I^t) \leq d \), \[ \text{CN08} \ 4.8 \] shows that \( H_{d+1}^m(0 :_M I^{t+1}/0 :_M I^t) = 0 \). Thus the homomorphisms

\[ H_d^m(0 :_M I^t) \to H_d^m(0 :_M I^{t+1}) \]

are injective for all \( t > 0 \). Therefore \( \lim_{\rightarrow \to} H_d^m(0 :_M I^t) \neq 0 \) and \( \lim_{\rightarrow \to} H_i^m(0 :_M I^t) = 0 \) for all \( i > d \). Hence (i) is proved.

(ii) It is clear that \( 0 :_{\Gamma_m(M)} I = \Gamma_m(0 :_M I) \). Set \( d = \text{Ndim}(0 :_{\Gamma_m(M)} I) = \text{Ndim}(0 :_{\Gamma_m(M)} I^t) \). From \[ \text{CN08} \ 4.10(i) \] we have

\[ d = \text{Ndim}(0 :_{\Gamma_m(M)} I^t) = \text{Ndim} \Gamma_m(0 :_M I^t) = \max\{ i \mid H_i^m(0 :_M I^t) \neq 0 \}. \]

The rest of the proof is analogous to that of (i).

**Corollary 2.19.** Let \( M \) be an artinian \( R \)-module such that \( 0 :_M I \neq 0 \). Then

\[ \text{Ndim}(0 :_M I) = \max\{ i \mid \mathfrak{z}_i^I(M) \neq 0 \}. \]

**Proof.** Since an artinian \( R \)-module is semi-discrete linearly compact and \( \Gamma_m(M) = M \), the conclusion follows from \[ \text{2.18(ii)} \].

**3. The finiteness of formal local homology modules.** Recall that the co-support \( \text{Cosupp}_R(M) \) of an \( R \)-module \( M \) is the set of primes \( \mathfrak{p} \) such that there exists a cocyclic homomorphic image \( L \) of \( M \) with \( \text{Ann}(L) \subseteq \mathfrak{p} \) \[ \text{Y95} \ 2.1 \]. Recall also that a module is cocyclic if it is a submodule of \( E(R/\mathfrak{m}) \). If \( 0 \to N \to M \to K \to 0 \) is an exact sequence of \( R \)-modules, then \( \text{Cosupp}_R(M) = \text{Cosupp}_R(N) \cup \text{Cosupp}_R(K) \) \[ \text{Y95} \ 2.7 \].

**Lemma 3.1.** Let \( M \) be an artinian \( R \)-module. Then

\[ \text{Cosupp}_R(\mathfrak{z}_0^I(M)) \cap V(1R) \subseteq V(mR). \]

**Proof.** By \[ \text{2.5} \] we may assume that \( (R, \mathfrak{m}) \) is a complete ring. We note that \( M^* \) is a finitely generated \( R \)-module. Combining \[ \text{2.14} \] with \[ \text{Y95} \ 2.9 \] yields

\[ \text{Cosupp}_R(\mathfrak{z}_0^I(M)) = \text{Cosupp}_R(\mathfrak{z}_0^I(M^*)) \]

\[ \subseteq \text{Cosupp}_RD((\mathfrak{z}_0^I(M^*))) = \text{Supp}_R(\mathfrak{z}_0^I(M^*)). \]
By the proof of [S07, 4.3] we have
\[ \text{Cosupp}_R(\bar{\mathfrak{g}}_0^I(M)) \cap V(I) \subseteq \text{Supp}_R(\bar{\mathfrak{g}}_0^I(M^*) \cap V(I) \subseteq V(m). \]

The following gives a property equivalent to the formal local homology module \( \bar{\mathfrak{g}}_i^I(M) \) being finitely generated for all \( i < s \).

**Theorem 3.2.** Let \( M \) be an \( R \)-artinian module and \( s \) a positive integer. Then the following statements are equivalent:

(i) \( \bar{\mathfrak{g}}_i^I(M) \) is a finitely generated \( \hat{R} \)-module for all \( i < s \);

(ii) \( I \subseteq \sqrt{0 :_R \bar{\mathfrak{g}}_i^I(M)} \) for all \( i < s \).

**Proof.** (i)\( \Rightarrow \) (ii). For \( i < s \), as \( \bar{\mathfrak{g}}_i^I(M) \) is a finitely generated \( \hat{R} \)-module, the increasing chain of submodules of \( \bar{\mathfrak{g}}_i^I(M) \)
\[ 0 :_R \bar{\mathfrak{g}}_i^I(M) I \subseteq 0 :_R \bar{\mathfrak{g}}_i^I(M) I^2 \subseteq \cdots \subseteq 0 :_R \bar{\mathfrak{g}}_i^I(M) I^t \subseteq \cdots \]

is stationary: there is a positive integer \( r \) such that \( 0 :_R \bar{\mathfrak{g}}_i^I(M) I^t = 0 :_R \bar{\mathfrak{g}}_i^I(M) I^r \) for all \( t \geq r \). It follows from 2.3 that
\[ \bar{\mathfrak{g}}_i^I(M) = \bigcup_{t > 0} \left( 0 :_R \bar{\mathfrak{g}}_i^I(M) I^t \right) = 0 :_R \bar{\mathfrak{g}}_i^I(M) I^r. \]

Therefore \( I^r \bar{\mathfrak{g}}_i^I(M) = 0 \) and so \( I \subseteq \sqrt{0 :_R \bar{\mathfrak{g}}_i^I(M)} \).

(ii)\( \Rightarrow \) (i). We use induction on \( s \). When \( s = 1 \), we have
\[ \bar{\mathfrak{g}}_0^I(M) = \lim_{\to t} \Lambda_m(0 :_M I^t) = \lim_{\to t} \lim_{\to k} (0 :_M I^t)/m^k(0 :_M I^t). \]

As \( 0 :_M I^t \) is artinian, there is \( k_t \) such that \( m^k(0 :_M I^t) = m^{k_t}(0 :_M I^t) \) for all \( k \geq k_t \). Then
\[ \lim_{\to k} (0 :_M I^t)/m^k(0 :_M I^t) = (0 :_M I^t)/m^{k_t}(0 :_M I^t) \]

and
\[ \bar{\mathfrak{g}}_0^I(M) = \lim_{\to t} (0 :_M I^t)/m^{k_t}(0 :_M I^t) \]
\[ \cong \lim_{\to t} (0 :_M I^t)/\lim_{\to t} m^{k_t}(0 :_M I^t) = M/\lim_{\to t} m^{k_t}(0 :_M I^t). \]

Thus, \( \bar{\mathfrak{g}}_0^I(M) \) is artinian and hence \( \text{Cosupp}_R(\bar{\mathfrak{g}}_0^I(M)) = V(0 :_R \bar{\mathfrak{g}}_0^I(M)) \) by [Y95, 2.3]. Moreover, from the hypothesis we have \( V(0 :_R \bar{\mathfrak{g}}_0^I(M)) \subseteq V(I) \), so \( V(0 :_R \bar{\mathfrak{g}}_0^I(M)) \subseteq V(I \hat{R}) \). It follows from 3.1 that
\[ \text{Cosupp}_R(\bar{\mathfrak{g}}_0^I(M)) = V(0 :_R \bar{\mathfrak{g}}_0^I(M)) \subseteq V(m \hat{R}) \]

and so the \( \hat{R} \)-module \( \bar{\mathfrak{g}}_0^I(M) \) has finite length by [NOS, 3.9].

Let \( s > 1 \). As \( M \) is artinian, there is a positive integer \( m \) such that \( I^t M = I^m M \) for all \( t \geq m \). Set \( K = I^m M \). Then by 2.16 the short exact
sequence of artinian $R$-modules

$$0 \to K \to M \to M/K \to 0$$
gives rise to a long exact sequence

$$\cdots \to \hat{\mathcal{F}}_{i+1}(M/K) \to \hat{\mathcal{F}}_{i}(K) \to \hat{\mathcal{F}}_{i}(M) \to \hat{\mathcal{F}}_{i}(M/K) \to \cdots.$$  

It is clear that $M/K$ is $I$-separated, so $\hat{\mathcal{F}}_{i}(M/K)$ is a finitely generated $\hat{R}$-module for all $i$ by 2.9. Thus, the proof will be complete if we show that $\hat{\mathcal{F}}_{i}(K)$ is a finitely generated $\hat{R}$-module for all $i < s$. We know that $I \subseteq \sqrt{0 : \hat{\mathcal{F}}_{i}(M/K)}$, as $\hat{\mathcal{F}}_{i}(M/K)$ is a finitely generated $\hat{R}$-module for all $i$. By the hypothesis, we have $I \subseteq \sqrt{0 : \hat{\mathcal{F}}_{i}(K)}$ for all $i < s$. Since $IK = K$, there is an $x \in I$ such that $xK = K$ by [M73, 2.8]. Then there is a positive integer $r$ such that $x^r \hat{\mathcal{F}}_{i}(K) = 0$ for all $i < s$. Now the short exact sequence

$$0 \to 0 :_K x^r \to K \xrightarrow{x^r} K \to 0$$
induces a short exact sequence

$$0 \to \hat{\mathcal{F}}_{i}(K) \to \hat{\mathcal{F}}_{i-1}(0 :_K x^r) \to \hat{\mathcal{F}}_{i-1}(K) \to 0$$

for all $i < s$. It follows that $I \subseteq \sqrt{0 : \hat{\mathcal{F}}_{i-1}(0 :_K x^r)}$ for all $i < s$. By the inductive hypothesis, $\hat{\mathcal{F}}_{i-1}(0 :_K x^r)$ is a finitely generated $\hat{R}$-module for all $i < s$. Therefore $\hat{\mathcal{F}}_{i}(K)$ is a finitely generated $\hat{R}$-module for all $i < s$ and the proof is complete. ■

The following theorem gives a property equivalent to $\hat{\mathcal{F}}_{i}(M)$ being finitely generated for all $i > s$.

**Theorem 3.3.** Let $M$ be an artinian $R$-module and $s$ a non-negative integer. Then the following statements are equivalent:

(i) $\hat{\mathcal{F}}_{i}(M)$ is a finitely generated $\hat{R}$-module for all $i > s$;

(ii) $I \subseteq \sqrt{0 : \hat{\mathcal{F}}_{i}(M)}$ for all $i > s$.

**Proof.** (i)⇒(ii). The argument is similar to that used in the proof of 3.2

(ii)⇒(i). We proceed by induction on $d = \text{Ndim } M$. When $d = 0$, we have $\text{Ndim}(0 :_M I) = 0$ and so $\hat{\mathcal{F}}_{i}(M) = 0$ for all $i > 0$.

Let $d > 0$. As $M$ is artinian, there is a positive integer $m$ such that $I^t M = I^m M$ for all $t \geq m$. Set $K = I^m M$. By an argument similar to that in the proof of 3.2, we only need to prove that $\hat{\mathcal{F}}_{i}(K)$ is a finitely generated $\hat{R}$-module for all $i > s$. From the hypothesis, we have $I \subseteq \sqrt{0 : \hat{\mathcal{F}}_{i}(K)}$ for all $i > s$. Since $IK = K$, there is an $x \in I$ such that $xK = K$ by [M73, 2.8]. Then there is a positive integer $r$ such that $x^r \hat{\mathcal{F}}_{i}(K) = 0$ for all $i > s$. Set $y = x^r$. The short exact sequence

$$0 \to 0 :_K y \to K \xrightarrow{y} K \to 0$$
induces a short exact sequence

$$0 \to \hat{\mathcal{F}}_{i}(K) \to \hat{\mathcal{F}}_{i-1}(0 :_K y) \to \hat{\mathcal{F}}_{i-1}(K) \to 0$$
for all \( i > s \). It follows that \( I \subseteq \sqrt{0 : \mathfrak{F}_{i-1}^l(0:K)} \) for all \( i > s \). By \cite[3.7]{CN08}, \( \text{Ndim}(0 :_K y) \leq d - 1 \). From the inductive hypothesis, \( \mathfrak{F}_{i-1}^l(0 :_K y) \) is a finitely generated \( \hat{R} \)-module for all \( i > s \). Therefore \( \mathfrak{F}_{i}^l(K) \) is a finitely generated \( \hat{R} \)-module for all \( i > s \) and the proof is complete. \( \blacksquare \)

We can ask what can be said about the module \( \mathfrak{F}_{i}^l(M)/I\mathfrak{F}_{i}^l(M) \). When \( I \) is a principal ideal we have the following answer.

**Theorem 3.4.** Let \( I \) be a principal ideal of \((R, m)\) and \( M \) an artinian \( R \)-module. Then \( \mathfrak{F}_{i}^l(M)/I\mathfrak{F}_{i}^l(M) \) is a noetherian \( \hat{R} \)-module for all \( i \).

**Proof.** Assume that \( I \) is generated by an element \( x \). As \( M \) is artinian, there is a positive integer \( m \) such that \( x^tM = x^mM \) for all \( t \geq m \). Set \( K = x^mM \). The short exact sequence of artinian \( R \)-modules

\[
0 \to K \xrightarrow{f} M \xrightarrow{g} M/K \to 0
\]
gives rise to a long exact sequence of formal local homology modules

\[
\cdots \to \mathfrak{F}_{i+1}^l(M/K) \xrightarrow{\delta_{i+1}} \mathfrak{F}_{i}^l(K) \xrightarrow{f_i} \mathfrak{F}_{i}^l(M) \xrightarrow{g_i} \mathfrak{F}_{i}^l(M/K) \to \cdots.
\]

Hence we have short exact sequences

\[
0 \to \text{Im} f_i \to \mathfrak{F}_{i}^l(M) \to \text{Im} g_i \to 0,
\]

\[
0 \to \text{Im} \delta_{i+1} \to \mathfrak{F}_{i}^l(K) \to \text{Im} f_i \to 0.
\]

These induce the exact sequences

\[
\text{Im} f_i/I \text{Im} f_i \to \mathfrak{F}_{i}^l(M)/I\mathfrak{F}_{i}^l(M) \to \text{Im} g_i/I \text{Im} g_i \to 0,
\]

\[
\text{Im} \delta_{i+1}/I \text{Im} \delta_{i+1} \to \mathfrak{F}_{i}^l(K)/I\mathfrak{F}_{i}^l(K) \to \text{Im} f_i/I \text{Im} f_i \to 0.
\]

Note that \( M/K \) is \( I \)-separated. By \cite[2.9]{2} \( \mathfrak{F}_{i}^l(M/K) \) is a noetherian \( \hat{R} \)-module and so \( \text{Im} g_i/I \text{Im} g_i \) is a noetherian \( \hat{R} \)-module for all \( i \). Thus, the proof will be complete once we show that \( \mathfrak{F}_{i}^l(K)/I\mathfrak{F}_{i}^l(K) \) is a noetherian \( \hat{R} \)-module for all \( i \).

As \( xK = K \), there is a short exact sequence

\[
0 \to 0 :_K x \to K \xrightarrow{x} K \to 0.
\]

It gives rise to a long exact sequence

\[
\cdots \to \mathfrak{F}_{i}^l(0 :_K x) \to \mathfrak{F}_{i}^l(K) \xrightarrow{x} \mathfrak{F}_{i}^l(K) \to \mathfrak{F}_{i-1}^l(0 :_K x) \to \cdots.
\]

Since \( 0 :_K x \) is \( I \)-separated, \( \mathfrak{F}_{i}^l(0 :_K x) \) is a noetherian \( \hat{R} \)-module for all \( i \) by \cite[2.9]{2}. It follows from the long exact sequence that \( \mathfrak{F}_{i}^l(K)/x\mathfrak{F}_{i}^l(K) \) is a noetherian \( \hat{R} \)-module for all \( i \). Therefore so is \( \mathfrak{F}_{i}^l(K)/I\mathfrak{F}_{i}^l(K) \) for all \( i \), and the proof is complete. \( \blacksquare \)

The following theorem shows a sufficient condition for the \( \hat{R} \)-module \( \mathfrak{F}_{i}^l(M)/I\mathfrak{F}_{i}^l(M) \) to be noetherian.
THEOREM 3.5. Let $M$ be an artinian $R$-module and $s$ a non-negative integer. If $\mathcal{F}_i^I(M)$ is a noetherian $\hat{R}$-module for all $i < s$, then $\mathcal{F}_s^I(M)/I\mathcal{F}_s^I(M)$ is a noetherian $\hat{R}$-module.

Proof. We use induction on $s$. When $s = 0$, it follows from the proof of 3.2 that $\mathcal{F}_0^I(M)$ is a quotient module of $M$, so it is artinian. By [Y95 2.3],

$$\text{Cosupp}_{\hat{R}}(\mathcal{F}_0^I(M)/I\mathcal{F}_0^I(M)) = V(0 :_{\hat{R}} \mathcal{F}_0^I(M)/I\mathcal{F}_0^I(M)) = V(I\hat{R} + (0 :_{\hat{R}} \mathcal{F}_0^I(M))) = V(I\hat{R}) \cap V(0 :_{\hat{R}} \mathcal{F}_0^I(M)) = V(I\hat{R}) \cap \text{Cosupp}_{\hat{R}}(\mathcal{F}_0^I(M)).$$

It follows from 3.1 that $\text{Cosupp}_{\hat{R}}(\mathcal{F}_0^I(M)/I\mathcal{F}_0^I(M)) \subseteq V(m\hat{R})$ and hence the $\hat{R}$-module $\mathcal{F}_0^I(M)/I\mathcal{F}_0^I(M)$ has finite length.

Let $s > 0$. As $M$ is artinian, there is a positive integer $m$ such that $I^tM = I^mM$ for all $t \geq m$. Set $K = I^mM$, the short exact sequence of artinian $R$-modules

$$0 \to K \xrightarrow{f} M \xrightarrow{g} M/K \to 0$$

gives rise to a long exact sequence

$$\cdots \to \mathcal{F}_{i+1}^I(M/K) \xrightarrow{\delta_{i+1}} \mathcal{F}_i^I(K) \xrightarrow{f_i} \mathcal{F}_i^I(M) \xrightarrow{g_i} \mathcal{F}_i^I(M/K) \to \cdots.$$ 

By an argument similar to that in the proof of 3.4 we only need to prove that $\mathcal{F}_i^I(K)/I\mathcal{F}_i^I(K)$ is a noetherian $\hat{R}$-module. Since $IK = K$, there is an $x \in I$ such that $xK = K$ by [M73 2.8]. Now the short exact sequence

$$0 \to 0 :_{K} x \to K \xrightarrow{x} K \to 0$$

gives rise to a long exact sequence

$$\cdots \to \mathcal{F}_i^I(K) \xrightarrow{x} \mathcal{F}_i^I(K) \to \mathcal{F}_{i-1}^I(0 :_{K} x) \to \mathcal{F}_{i-1}^I(K) \xrightarrow{x} \mathcal{F}_{i-1}^I(K) \to \cdots.$$ 

By 2.9 $\mathcal{F}_i^I(M/K)$ is a noetherian $\hat{R}$-module for all $i$. From the first long exact sequence, we deduce that $\mathcal{F}_i^I(K)$ is a noetherian $\hat{R}$-module for all $i < s$ by the hypothesis. Therefore $\mathcal{F}_i^I(0 :_{K} x)$ is a noetherian $\hat{R}$-module for all $i < s - 1$. It follows from the inductive hypothesis that $\mathcal{F}_{s-1}^I(0 :_{K} x)/I\mathcal{F}_{s-1}^I(0 :_{K} x)$ is a noetherian $\hat{R}$-module. From the last exact sequence we have the short exact sequence

$$0 \to \mathcal{F}_s^I(K)/x\mathcal{F}_s^I(K) \to \mathcal{F}_{s-1}^I(0 :_{K} x) \to 0 :_{\mathcal{F}_{s-1}^I(K)} x \to 0.$$

It induces an exact sequence

$$\text{Tor}_1^{\hat{R}}(R/I, 0 :_{\mathcal{F}_{s-1}^I(K)} x) \to \mathcal{F}_s^I(K)/I\mathcal{F}_s^I(K) \to \mathcal{F}_{s-1}^I(0 :_{K} x)/I\mathcal{F}_{s-1}^I(0 :_{K} x).$$

Since $0 :_{\mathcal{F}_{s-1}^I(K)} x$ is a noetherian $\hat{R}$-module, so is $\text{Tor}_1^{\hat{R}}(R/I, 0 :_{\mathcal{F}_{s-1}^I(K)} x)$. 

It follows that $\tilde{F}_s^I(K)^I\tilde{F}_s^I(K)$ is a noetherian $\tilde{R}$-module and the proof is complete. ■

Let $M$ be an artinian $R$-module. In general, the formal local homology modules $\tilde{F}_i^I(M)$ are not noetherian $\tilde{R}$-modules. But when $i = \text{Ndim } M$, we have the following theorem.

**THEOREM 3.6.** Let $M$ be an artinian $R$-module with $\text{Ndim } M = d$. Then $\tilde{F}_d^I(M)$ is a noetherian $\tilde{R}$-module.

**Proof.** We use induction on $d = \text{Ndim } M$. When $d = 0$, $M$ has finite length. It follows from 2.17 that $\tilde{F}_0^I(M) \cong M$, so $\tilde{F}_0^I(M)$ is a noetherian $\tilde{R}$-module.

Let $d > 0$. As $M$ is artinian, there is a positive integer $m$ such that $I^tM = I^mM$ for all $t \geq m$. Set $K = I^mM$. By 2.16 the short exact sequence of artinian $R$-modules

$$0 \to K \to M \to M/K \to 0$$

gives rise to a long exact sequence

$$\cdots \to \tilde{F}_{i+1}(M/K) \to \tilde{F}_i^I(K) \to \tilde{F}_i^I(M) \to \tilde{F}_i^I(M/K) \to \cdots .$$

Since $IK = K$, there is an $x \in I$ such that $xK = K$ by [M73, 2.8]. By [CN08, 3.7], $\text{Ndim}(0 :_K I) \leq \text{Ndim}(0 :_K x) \leq d - 1$. Hence $\tilde{F}_d^I(K) = 0$ by 2.19. Thus, the long exact sequence yields the exact sequence

$$0 \to \tilde{F}_d^I(M) \to \tilde{F}_d^I(M/K).$$

It is clear that $M/K$ is $I$-separated, so $\tilde{F}_i^I(M/K)$ is a finitely generated $\tilde{R}$-module for all $i$ by 2.9. Therefore, $\tilde{F}_d^I(M)$ is a noetherian $\tilde{R}$-module. The proof is complete. ■

The following theorem gives properties equivalent to $\tilde{F}_i^I(M)$ being artinian for all $i > s$.

**THEOREM 3.7.** Let $M$ be an artinian $R$-module and $s$ a non-negative integer. Then the following statements are equivalent:

(i) $\tilde{F}_i^I(M)$ is artinian for all $i > s$;
(ii) $\tilde{F}_i^I(M) = 0$ for all $i > s$;
(iii) $\text{Ass}(\tilde{F}_i^I(M)) \subseteq \{m\}$ for all $i > s$.

**Proof.** (i)$\Rightarrow$(ii). We use induction on $d = \text{Ndim } M$. If $d = 0$, then $\text{Ndim}(0 :_M I) = 0$. By 2.19, $\tilde{F}_0^I(M) = 0$ for all $i > 0$.

Let $d > 0$. As $M$ is artinian, there is a positive integer $m$ such that $m^tM = m^mM$ for all $t \geq m$. Set $K = m^mM$. By 2.16 the short exact sequence of artinian $R$-modules

$$0 \to K \to M \to M/K \to 0$$

...
gives rise to a long exact sequence
\[ \cdots \to F^I_{i+1}(M/K) \to F^I_i(K) \to F^I_i(M) \to F^I_i(M/K) \to \cdots. \]

It is clear that $M/K$ is $I$-separated, so $F^I_i(M/K) \cong H^m_i(M/K)$ by 2.9. Moreover, $M/K$ is also $m$-separated, so $H^m_i(M/K) = 0$ for all $i > 0$ by [CN08 3.8]. Hence

\[ F^I_i(K) \cong F^I_i(M) \quad \text{for all } i > 0. \]

Thus, the proof will be complete if we show that $F^I_i(K) = 0$ for all $i > s$.

By the hypothesis, $F^I_i(K)$ is artinian for all $i > s$. As $mK = K$, there is an $x \in m$ such that $xK = K$ by [M73 2.8]. Now the short exact sequence
\[ 0 \to 0 :_K x \to K \xrightarrow{x} K \to 0 \]
gives rise to an exact sequence
\[ \cdots \to F^I_{i+1}(K) \to F^I_i(0 :_K x) \to F^I_i(K) \xrightarrow{x} F^I_i(K) \to \cdots. \]

By [CN08 4.7], $\text{Ndim}(0 :_K x) \leq d - 1$. Then the inductive hypothesis gives $F^I_i(0 :_K x) = 0$ for all $i > s$ and we have an exact sequence
\[ 0 \to F^I_i(K) \xrightarrow{x} F^I_i(K) \quad \text{for all } i > s. \]

It follows that $0 :\cong F^I_i(K) = 0$ for all $i > s$. Since $F^I_i(K)$ is artinian for all $i > s$, we conclude that $F^I_i(K) = 0$ for all $i > s$.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i). The argument is similar to that in the proof of (i)$\Rightarrow$(ii) and we use the artinian criterion of [M90 1.3].

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