THE HADWIGER-NELSON PROBLEM WITH TWO FORBIDDEN DISTANCES

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Abstract. In 1950 Edward Nelson asked the following simple-sounding question:

How many colors are needed to color the Euclidean plane $\mathbb{E}^2$ such that no two points distance 1 apart are identically colored?

We say that 1 is a forbidden distance. For many years, we only knew that the answer was 4, 5, 6, or 7. In a recent breakthrough, de Grey [2] proved that at least five colors are necessary.

In this paper we consider a related problem in which we require two forbidden distances, 1 and $d$. In other words, for a given positive number $d \neq 1$, how many colors are needed to color the plane such that no two points distance 1 or $d$ apart are assigned the same color? We find several values of $d$, for which the answer to the previous question is at least 5. These results and graphs may be useful in constructing simpler 5-chromatic unit distance graphs.

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1. Introduction

Let \( (\mathbb{E}^2, \| \cdot \|) \) be the 2-dimensional space with the usual Euclidean norm, and let \( \mathcal{D} \) be a subset of \((0, \infty)\). The distance graph defined by \( \mathcal{D} \) is the graph \( G(\mathbb{E}^2, \mathcal{D}) \) whose vertices are points in \( \mathbb{E}^2 \) and whose edges are the pairs of points \( \{x, y\} \) such that \( \|x - y\| \in \mathcal{D} \).

The graph \( G \) has chromatic number \( \chi(\mathbb{E}^2, \mathcal{D}) = k \in \mathbb{N} \) if there exists a \( k \)-coloring of \( G \), i.e. a function from \( \mathbb{E}^2 \) to \( \{1, 2, \ldots, k\} \) that maps adjacent vertices to different values, and \( k \) is minimal with this property.

A long-standing open problem is to determine the chromatic number of \( \mathbb{E}^2 \) when the distance set \( \mathcal{D} \) reduces to only one element, say \( \mathcal{D} = \{1\} \).

**Open Problem 1.1. The Hadwiger-Nelson Question.** How many colors are needed to color the points of the plane such that no two points distance 1 apart are assigned the same color?

According to Soifer [10], the question was raised by Edward Nelson in 1950. It is easy to show that

\[
4 \leq \chi(\mathbb{E}^2, \{1\}) \leq 7.
\]

The upper bound follows from the existence of a tiling of the plane by regular hexagons, of diameter strictly between \( 2/\sqrt{7} \) and 1, which can be assigned seven colors in a periodic manner to form a 7-coloring of the plane with no two identically colored points distance 1 apart. This construction first appeared in a paper by Hadwiger [7].

The lower bound is due to Leo and William Moser [9]. Rotate a unit rhombus around one of the vertices of degree 2 until the opposite vertex is at distance 1 from its original position. The resulting 7-vertex graph, called the Mosers’s spindle, has chromatic number 4 - see figure 1.

![Figure 1. Mosers’ spindle construction](image-url)
Very recently, de Grey [2] managed to improve the lower bound to \( \chi(\mathbb{E}^2) \geq 5 \). In an attempt to better understand the structure and properties of 5-chromatic unit distance graphs, it is of interest to consider that case when the set of distances \( \mathcal{D} \) contains more than one element.

Exoo [5] studied the case when the set is an entire interval, \( \mathcal{D} = [1, d] \), and proved that 
\[
\chi(\mathbb{E}^2, [1, 1.00853 \ldots]) \geq 5 \quad \text{and} \quad \chi(\mathbb{E}^2, [1, d]) = 7 \quad \text{for} \quad 1.13475 \ldots 5 < d < 1.38998 \ldots.
\]

The case when the distance set consists of all odd positive integers was investigated by Ardal, Maňuch, Rosenfeld, Shelah, and Stacho [1]. They proved that 
\[
\chi(\mathbb{E}^2, \{1, 3, 5, 7, 9, 11, \ldots\}) \geq 5.
\]

An interesting case occurs when \( |\mathcal{D}| = 2 \), that is, when there are exactly two forbidden distances. We will be studying the following:

**Problem 1.2.** Find values \( d \neq 1 \) such that \( \chi(\mathbb{E}^2, \{1, d\}) \geq 5 \).

Despite being a natural question, it appears that until very recently [11] the problem had not been considered in this explicit form. The only relevant result we could find is due to Katz, Krebs and Shaheen [8], who proved 
\[
\chi(\mathbb{E}^2, \{1, \sqrt{11}/3\}) \geq 5.
\]

However, their initial motivation was different, as we will explain later on, when we will also provide a new proof of this result.

There are two good reasons we believe that Problem 1.2 is worth looking into. On one hand, it should be easier to prove that \( \chi(\mathbb{E}^2, \{1, d\}) \geq 5 \) than to prove \( \chi(\mathbb{E}^2, \{1\}) \geq 5 \). On the other hand, as recently shown in [6], one can prove results of the following type

**Theorem 1.3.** There exist values of \( d \neq 1 \) such that
\[
(2) \quad \text{If} \quad \chi(\mathbb{E}^2, \{1\}) = 4 \quad \text{then} \quad \chi(\mathbb{E}^2, \{1, d\}) = 4.
\]

The authors of [6] used \( d = \sqrt{11}/3 \) but most likely such an implication can be proved for other values of \( d \). They then proved that \( \chi(\mathbb{E}^2, \{1, \sqrt{11}/3\}) \geq 5 \), by constructing a 5-chromatic \( \{1, \sqrt{11}/3\} \)-graph, and therefore obtained that \( \chi(\mathbb{E}^2, \{1\}) \geq 5 \).

Before we proceed to our results, we state the following generalization of Mosers’ spindle idea.

**Theorem 1.4.** The Spindle Method. Let \( G \) be a finite graph with vertex set \( V = \{1, 2, \ldots, n\} \) and edge set \( E \). Assume that the chromatic number of \( G \), \( \chi(G) = k \), and that in every \( k \)-coloring of \( G \), vertices 1 and 2 are colored identically.

Let \( G' \) be a copy of \( G \) such that \( 1 = 1' \) and \( 2 \neq 2' \). Then the chromatic number of the graph \( H \) whose edge set is \( E \cup E' \cup \{\{2, 2'\}\} \) is \( \geq k + 1 \).

**Proof.** Assume that \( H \) is \( k \)-colorable and let \( c \) be such a \( k \)-coloring. Then \( c(1) = c(2) \) by our assumption about \( G \). At the same time, since \( G' \) is a copy of \( G \), it follows that \( c(1') = c(1) = c(2') \). But then
vertices 2 and 2' have the same color, which violates the condition that the endpoints of edge \{2, 2'\} must be colored differently.

We will also use the following definition very frequently.

**Definition 1.5.** For a given positive real number \(d \neq 1\), a \(\{1, d\}\)-graph is a finite graph whose vertices are points in the Euclidean plane \(E^2\), and whose edges are obtained by connecting two points whenever the distance between them is either 1 or \(d\).

It is clear that given \(d \neq 1\), if one can find a \(\{1, d\}\)-graph whose chromatic number is 5, then it would immediately follow that \(\chi(E^2, \{1, d\}) \geq 5\). Most of our paper is dedicated to constructing such graphs, for various values of \(d\).

Also note that any \(\{1, d\}\)-graph can be transformed into a \(\{1, 1/d\}\)-graph by an appropriate scaling. For this particular reason, we will always consider only the case \(d > 1\).

Hence, we want to construct \(\{1, d\}\)-graphs that have chromatic number 5. Ideally, these graphs would have not too many vertices, as computing the chromatic number of large graphs is computationally expensive.

The natural idea is to require such a graph to contain many small subgraphs that are 4-chromatic. The smallest 4-chromatic graph is of course \(K_4\), the complete graph on 4 vertices. The question now becomes, for what values of \(d \neq 1\) can \(K_4\) be represented as a \(\{1, d\}\)-graph? Luckily for us, this problem was already solved by Erdős and Kelly [4], and later by Einhorn and Schoenberg [3].

**Theorem 1.6.** [4, 3] The only values \(d > 1\) for which the complete graph \(K_4\) can be embedded as a \(\{1, d\}\)-graph are: \(d = (\sqrt{5} + 1)/2\), \(d = \sqrt{3}\), \(d = (\sqrt{6} + \sqrt{2})/2\), and \(d = \sqrt{2}\).

![Figure 2. Complete graphs realized as \(\{1, d\}\)-graphs](image)

Here and later we make the convention of representing the edges of length 1 in red and the edges of length \(d\) in blue - see figure 2. Note that if \(d = (\sqrt{5} + 1)/2\), then \(K_5\), (the complete graph on 5 vertices) can be expressed as a \(\{1, d\}\)-graph. Since \(\chi(K_5) = 5\), we immediately have that

**Theorem 1.7.**

\[
\chi \left( E^2, \left\{ 1, \frac{\sqrt{5} + 1}{2} \right\} \right) = \chi \left( E^2, \left\{ 1, \frac{\sqrt{5} - 1}{2} \right\} \right) \geq 5.
\]
2. Two simple cases: \( d = \sqrt{3} \) and \( d = (\sqrt{6} + \sqrt{2})/2 \).

If \( d \in \{ \sqrt{3}, (\sqrt{6} + \sqrt{2})/2 \} \), then there are two different ways to realize \( K_4 \) as a \( \{1, d\} \)-graph, as shown in figure 2. As we shall see shortly, this is very useful as in both cases we can represent \( K_5 \setminus e \), the complete graph on 5 vertices with a missing edge, as a \( \{1, d\} \)-graph. Since in every 4-coloring of \( K_5 \setminus e \) the endpoints of the missing edge must be colored the same, we can then employ the spindle technique to construct a 5-chromatic \( \{1, d\} \)-graph. Details are presented below.

**Theorem 2.1.**

\[
\chi \left( \mathbb{E}^2, \{1, \sqrt{3}\} \right) = \chi \left( \mathbb{E}^2, \{1, 1/\sqrt{3}\} \right) \geq 5.
\]

**Proof.** Consider the following five points:

1. \((0, 0)\), 2. \((2, 0)\), 3. \((1/2, -\sqrt{3}/2)\), 4. \((1, 0)\), 5. \((1/2, \sqrt{3}/2)\).

It is easy to see that with the exception of the pair \{1, 2\}, every other two points are either 1 or \( \sqrt{3} \) apart. In other words, these points define a \( \{1, \sqrt{3}\} \)-graph which is a complete \( K_5 \) with a missing edge, as shown in figure 3.

**Figure 3.** A 4-colorable \( K_5 \setminus e \) as a \( \{1, \sqrt{3}\} \)-graph converted to a 5-chromatic \( \{1, \sqrt{3}\} \)-graph via the spindle method

It follows that for any 4-coloring of this graph, vertices 1 and 2 must be assigned the same color. One can then apply the spindle technique by rotating the graph around vertex 1 until the image of vertex 2 is one unit apart from its original position. It is a simple exercise to find that the rotation angle can be chosen as \( \arccos(7/8) \). The coordinates of the four new vertices can be readily computed:

- \( 2' (7, \sqrt{15})/4 \), \( 3' (7 + 3\sqrt{5}, -7\sqrt{3} + \sqrt{15})/16 \), \( 4' (7, \sqrt{15})/8 \), \( 5' (7 - 3\sqrt{5}, 7\sqrt{3} + \sqrt{15})/16 \).

One obtains a 9-vertex, 19 edges, \( \{1, \sqrt{3}\} \)-graph, which requires 5 colors. \( \square \)
One can use exactly the same approach to prove the following:

**Theorem 2.2.**

\[ \chi \left( \mathbb{E}^2, \left\{ 1, \left( \sqrt{6} + \sqrt{2} \right)/2 \right\} \right) = \chi \left( \mathbb{E}^2, \left\{ 1, \left( \sqrt{6} - \sqrt{2} \right)/2 \right\} \right) \geq 5. \]

**Proof.** Consider the following five points:

1. \((0, 0)\), 2. \((\sqrt{6}, -\sqrt{2})/2\), 3. \((-\sqrt{2}, -\sqrt{2})/2\), 4. \((-\sqrt{2} + \sqrt{6}, -\sqrt{2} - \sqrt{6})/4\), 5. \((-\sqrt{2} + \sqrt{6}, \sqrt{2} + \sqrt{6})/4\).

It is easy to see that with the exception of the pair \((1, 2)\), every other pair of points is either 1 or \((\sqrt{6} + \sqrt{2})/2\) apart. In other words, these points define a \(\{1, (\sqrt{6} + \sqrt{2})/2\}\)-graph which is a complete \(K_5\) with a missing edge, as illustrated in figure 4.

It follows that for any 4-coloring of this graph, vertices 1 and 2 must be assigned the same color. One can then apply the spindle technique by rotating the graph around vertex 1 until the image of vertex 2 is one unit apart from its original position. It is a simple exercise to find that the rotation angle can be chosen as \(\arccos(3/4)\). The coordinates of the new four vertices can be readily computed:

\[ 2' \left( 3\sqrt{6} + \sqrt{14}, -3\sqrt{2} + \sqrt{42} \right)/8, \]
\[ 3' \left( -3\sqrt{2} + \sqrt{14}, -3\sqrt{2} - \sqrt{14} \right)/8, \]
\[ 4' \left( -3\sqrt{2} + 3\sqrt{6} + \sqrt{14} + \sqrt{42}, -3\sqrt{2} - 3\sqrt{6} - \sqrt{14} + \sqrt{42} \right)/16, \]
\[ 5' \left( -3\sqrt{2} + 3\sqrt{6} - \sqrt{14} - \sqrt{42}, 3\sqrt{2} + 3\sqrt{6} - \sqrt{14} + \sqrt{42} \right)/16. \]

One obtains a 9-vertex, 19 edges, \(\{1, (\sqrt{6} + \sqrt{2})/2\}\)-graph, which requires 5 colors. \(\square\)
3. The case \( d = \sqrt{2} \).

Theorem 3.1.

\[ \chi \left( \mathbb{R}^2, \{1, \sqrt{2}\} \right) \geq 5. \]

As mentioned in the introduction, this result was proved by Katz, Krebs and Shaheen [8]. They first proved the following:

**Theorem 3.2.** [8] Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function such that \( f(A) + f(B) + f(C) + f(D) = 0 \) holds whenever \( ABCD \) is a unit square. Then \( f(P) = 0 \) for all \( P \in \mathbb{R}^2 \).

The proof is far from being easy. However, once established, it easily implies Theorem 3.1 as follows.

**Proof.** [8] Assume that it is possible to color the plane with four colors (red, purple, green and blue), such that no two points distance 1 or \( \sqrt{2} \) apart are colored the same. Observe that all four colors are needed for the vertices of a unit square. Define a function \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f(P) = 3 \) if \( P \) is colored purple, and \( f(P) = -1 \) otherwise. Then \( f(A) + f(B) + f(C) + f(D) = 0 \) whenever \( ABCD \) is a unit square. By Theorem 3.2 \( f \) must be the zero function, which contradicts the definition of \( f \). \( \square \)

The above proof is a beautiful example of an indirect argument since a 5-chromatic \( \{1, \sqrt{2}\} \)-graph is not constructed explicitly. It is easy however to build such a graph, which we present below.

**Proof.** Consider the \( \{1, \sqrt{2}\} \)-graph whose vertices have the following coordinates:

- \( 1 (1/2,1/2) \), \( 2 (0,0) \), \( 3 (1,0) \), \( 4 (1,1) \), \( 5 (0,1) \), \( 6 (1/4 - \sqrt{7}/4, 3/4 - \sqrt{7}/4) \),
- \( 7 (3/4 + \sqrt{7}/4, 1/4 + \sqrt{7}/4) \), \( 8 (1/2, \sqrt{7}/2) \), \( 9 (1/4 - \sqrt{7}/4, 1/4 + \sqrt{7}/4) \), \( 10 (-\sqrt{7}/2, 1/2) \),
- \( 11 (1 + \sqrt{7}/2, 1/2) \), \( 12 (3/4 + \sqrt{7}/4, 3/4 - \sqrt{7}/4) \), \( 13 (1/2, 1 - \sqrt{7}/2) \).

![Figure 5. Unit edges (in red) and \( \sqrt{2} \)-edges (in blue) of the \( \{1, \sqrt{2}\} \)-graph in Theorem 3.1](image-url)
It is a simple matter to verify that the following 20 pairs are unit-distance apart

\{1, 6\}, \{1, 7\}, \{1, 9\}, \{1, 12\}, \{2, 3\}, \{2, 5\}, \{2, 9\}, \{3, 4\}, \{3, 7\}, \{4, 5\}, \{4, 12\}, \{5, 6\}, \{6, 10\},
\{6, 13\}, \{7, 8\}, \{7, 11\}, \{8, 9\}, \{9, 10\}, \{11, 12\}, \{12, 13\},

while the following 14 pairs are distance \(\sqrt{2}\) apart:

\{2, 4\}, \{2, 8\}, \{2, 10\}, \{2, 12\}, \{3, 5\}, \{3, 6\}, \{3, 8\}, \{3, 11\}, \{4, 9\}, \{4, 11\}, \{4, 13\}, \{5, 7\}, \{5, 10\}, \{5, 13\}.

We will show that this graph requires five colors. Let us assume that four colors suffice: purple, red, green and blue. Since every two of the vertices 2, 3, 4, and 5 are adjacent, one can assume that \(c(2) = \text{purple}, c(3) = \text{red}, c(4) = \text{green}, \text{and} c(5) = \text{blue}\). By the symmetry of the graph one can assume that \(c(1) = \text{purple}\), as well. But then \(c(6) = c(7) = \text{green}\) since both vertices 6 and 7 are adjacent to 1, 3 and 5. Since vertex 8 is adjacent to 2, 3, and 7; then \(c(8) = \text{blue}\). Similarly, vertex 9 is adjacent to vertices 1, 4, and 8; hence \(c(9) = \text{red}\). Vertex 10 is adjacent to vertices 2, 5, 6, and 9, which all have different colors. Hence a fifth color is needed. We reached the desired contradiction.  

\[\Box\]

4. An exotic looking distance

At this point we exhausted the possibilities that \(K_4\), the complete graph on four vertices, can be represented as a \(\{1, d\}\)-graph. (see Theorem 1.6). Recall that we would like our graph to contain many small subgraphs which are hard to 4-color. We already used \(K_5 \setminus e\), the complete graph on five vertices with a missing edge, in dealing with the cases \(d = \sqrt{3}\) and \(d = (\sqrt{6} + \sqrt{2})/2\).

There are no other useful subgraphs of order 5 left to consider since \(K_5 \setminus \{e, e'\}\), the \(K_5\) with two missing edges can be 3-colored if it is \(K_4\)-free. Naturally, we look at subgraphs of order 6. The question becomes: Are there any 4-chromatic graphs of order 6 which do not contain \(K_4\) as a subgraph?

The answer to the above question is affirmative. In fact, it is not difficult to show that there is exactly one graph with the desired properties: the wheel graph \(W_6\).

There are several values of \(d\) for which \(W_6\) can be realized as a \(\{1, d\}\)-graph, including

\[d = \sqrt{2 + \sqrt{2}}, \quad d = \frac{1}{2}\sqrt{4 + 2\sqrt{2}}, \quad d = \frac{1}{2}\sqrt{8 + 2\sqrt{6} - 2\sqrt{2}}.\]

We will, however, consider a value which is somewhat more interesting. For the remainder of this section we will take

\[d = \frac{1}{2}\sqrt{3^{1/4} \cdot 2\sqrt{2} + 2\sqrt{3} + 2}.\]

Note the \(3^{1/4}\) which appears in the expression of \(d\) above. The surprising fact is that despite the relatively complicated value of \(d\), there are 16 different ways to embed \(W_6\) as a \(\{1, d\}\)-graph - see figure 11 in the appendix. This does not happen for any of the other distances listed in (3). It is therefore to be expected that it should be not too difficult to construct a 5-chromatic \(\{1, d\}\)-graph. We show the details below.
Lemma 4.1. Consider the $\{1,d\}$-graph whose vertices have the following coordinates:

1. $(3^{1/4}\sqrt{2} - \sqrt{3} + 1, 3^{3/4}\sqrt{2} + \sqrt{3} + 1)/4$, 
2. $(-3^{1/4}\sqrt{2} + \sqrt{3} - 1, 3^{3/4}\sqrt{2} + \sqrt{3} - 1)/4$, 
3. $(3^{1/4}\sqrt{2} + \sqrt{3} + 1, 3^{3/4}\sqrt{2} + \sqrt{3} - 1)/4$, 
4. $(-3^{1/4}\sqrt{2} - \sqrt{3} - 1, 3^{3/4}\sqrt{2} + \sqrt{3} - 1)/4$, 
5. $(3^{1/4}\sqrt{2} + \sqrt{3} + 1, 3^{3/4}\sqrt{2} + \sqrt{3} + 3)/4$, 
6. $(-3^{1/4}\sqrt{2} - \sqrt{3} - 1, 3^{3/4}\sqrt{2} + \sqrt{3} + 3)/4$, 
7. $(3^{3/4}\sqrt{2} - 3^{1/4}\sqrt{2}, 3^{3/4}\sqrt{2} - 3^{1/4}\sqrt{2} + 2\sqrt{3} - 2)/4$, 
8. $(-3^{3/4}\sqrt{2} + 3^{1/4}\sqrt{2}, 3^{3/4}\sqrt{2} - 3^{1/4}\sqrt{2} + 2\sqrt{3} - 2)/4$, 
9. $(1,0)$, 
10. $(-1,0)$, 
11. $(1, \sqrt{3})/2$, 
12. $(-1, \sqrt{3})/2$, 
13. $(0,0)$.

Then in every 4-coloring of this graph, vertices 1 and 2 must be assigned the same color.

**Figure 6.** Unit edges (in red) and $d$-edges (in blue) for the graph in Lemma 4.1

Proof. It is easy to verify that the thirteen points listed above determine 19 unit edges:

$\{1,3\}, \{1,5\}, \{1,8\}, \{1,12\}, \{2,4\}, \{2,6\}, \{2,11\}, \{3,5\}, \{3,9\}, \{4,6\}, \{4,10\}, \{9,11\}, \{9,13\}, \{10,12\}, \{10,13\}, \{11,12\}, \{11,13\}, \{12,13\}$,

and 14 edges of length $d$:

$\{1,4\}, \{1,6\}, \{1,13\}, \{2,3\}, \{2,5\}, \{2,13\}, \{3,8\}, \{3,13\}, \{4,7\}, \{4,13\}, \{5,7\}, \{6,8\}, \{7,10\}, \{8,9\}$.
Assume by contradiction that there exists a 4-coloring \( c : \{1, 2, \ldots, 12, 13\} \to \{\text{red, purple, green, blue}\} \) such that \( c(1) \neq c(2) \). Without loss of generality, assume that \( c(1) = \text{purple} \) and \( c(2) = \text{red} \). Since vertices 3 and 13 are both adjacent to vertices 1 and 2 and adjacent to each other, we can assume that \( c(3) = \text{green} \) and \( c(13) = \text{blue} \). With this setup, we can now show that the colors of the remaining vertices are uniquely determined.

Vertex 4 is adjacent to 1, 2, and 13, hence \( c(4) = \text{green} \). Vertex 5 is adjacent to 1, 2, and 3, hence \( c(5) = \text{blue} \). Vertex 6 is adjacent to 1, 2, and 4, hence \( c(6) = \text{blue} \). Vertex 7 is adjacent to 2, 4, and 5, hence \( c(7) = \text{purple} \). Vertex 8 is adjacent to 1, 3, and 6, hence \( c(8) = \text{red} \). Vertex 9 is adjacent to 3, 8, and 13, hence \( c(9) = \text{purple} \). Vertex 10 is adjacent to 4, 7, and 13, hence \( c(10) = \text{red} \). Vertex 11 is adjacent to 2, 9, and 13, hence \( c(11) = \text{green} \). Vertex 12 is adjacent to 1, 10, and 13, hence \( c(12) = \text{green} \).

However, \( c(11) = c(12) = \text{green} \) is impossible since vertices 11 and 12 are adjacent. We obtained the desired contradiction. \( \square \)

We are now in position to prove the following

**Theorem 4.2.**

\[
\chi \left( \mathbb{E}^2, \left\{1, \frac{1}{2}\sqrt{3^{1/4} \cdot 2\sqrt{2} + 2\sqrt{3} + 2}\right\}\right) \geq 5.
\]

**Proof.** Consider the graph defined in Lemma 4.1. Any 4-coloring must assign vertices 1 and 2 the same color. Moreover, the distance between these two vertices is \((3^{1/4}\sqrt{2} - \sqrt{3} + 1)/2 = 0.564\ldots > 1/2\). Hence, we can apply the spindle technique to this graph by rotating about vertex 1 until the image of vertex 2 is at distance 1 from its original position. The resulting \( \{1, d\} \)-graph has 25 vertices, 67 edges, and it is 5-chromatic. \( \square \)

5. **A special subset of \( \mathbb{E}^2 \)**

For the rest of the paper we focus on graphs whose vertices belong to the following subset of \( \mathbb{E}^2 \):

\[
\Lambda := \left\{ \left( \frac{a\sqrt{3}}{12} + \frac{b\sqrt{11}}{12}, \frac{c}{12} + \frac{d\sqrt{33}}{12} \right) : a, b, c, d, \text{integers} \right\}.
\]

To simplify the presentation, in the sequel we will use the following notation:

\[
[a, b, c, d] := \left( \frac{a\sqrt{3}}{12} + \frac{b\sqrt{11}}{12}, \frac{c}{12} + \frac{d\sqrt{33}}{12} \right).
\]

One reason for this choice is that Mosers’ 7-vertex 4-chromatic unit distance graph, shown in figure 1, has such an embedding. With the convention in (6), the coordinates of the vertices can be written

\[
[0, 0, 0, 0], [0, 0, 12, 0], [6, 0, 6, 0], [6, 0, 18, 0], [0, 2, 10, 0], [5, 1, 5, -1], [5, 3, 15, -1].
\]
6. Two more distances, and a slightly different approach

The next two results are easy to prove; however, we believe the method of proof is significant.

Lemma 6.1. If \( \chi \left( \mathbb{E}^2, \left\{ 1, \sqrt{3/2 + \sqrt{33}/6} \right\} \right) = 4 \) then \( \chi \left( \mathbb{E}^2, \left\{ 1, \sqrt{3/2 + \sqrt{33}/6}, 1/\sqrt{3} \right\} \right) = 4. \)

Proof. Suppose that \( \chi \left( \mathbb{E}^2, \left\{ 1, \sqrt{3/2 + \sqrt{33}/6} \right\} \right) = 4 \); that is, there exists a 4-coloring of the plane, \( c: \mathbb{E}^2 \to \{\text{red, purple, green, blue}\} \) such that no two points which are either distance 1 or distance \( \sqrt{3/2 + \sqrt{33}/6} \) apart are assigned the same color. We will show that for this particular coloring, no two points distance \( 1/\sqrt{3} \) apart can be colored identically either. Indeed, for the sake of reaching a contradiction, assume that two such points exist, and denote them by \( A \) and \( B \). Choose a system of coordinates such that \( A = (0,0) = [0,0,0,0] \) and \( B = (1/\sqrt{3},0) = [4,0,0,0] \). Here we used notation (6) for the coordinates. Consider next the \( \{1, \sqrt{3/2 + \sqrt{33}/6}\} \)-graph induced by \( A, B \), and the following seven additional vertices:

\[
1 \ [1,3,3, -1], 2 \ [0,0,12,0], 3 \ [-1, -3, 9, 1], 4 \ [-1, 3, 3, 1], 5 \ [-2, 0, 6, 0], 6 \ [0,0,6,2], 7 \ [5,3,9,1].
\]

Assume that \( c(A) = c(B) = \text{blue} \), as shown in figure 7. Note that vertex \( A \) is adjacent to vertices 1, 2, 3, and 4, while vertex \( B \) is adjacent to vertices 5, 6 and 7. Hence, none of vertices 1 through 7 can be colored blue; that is, we can only use the remaining three colors for these vertices. However, it is easy to see that the subgraph induced by these seven vertices cannot be 3-colored.

![Figure 7](image-url)

**Figure 7.** A \( \{1, \sqrt{3/2 + \sqrt{33}/6}\} \)-graph. Unit edges are in red, \( \sqrt{3/2 + \sqrt{33}/6} \) edges are in blue. The distance between points \( A \) and \( B \) is \( 1/\sqrt{3} \).
Indeed, since vertices 3, 5, and 6 form a triangle, one can assume that \( c(3) = \text{red}, \ c(5) = \text{green}, \) and \( c(6) = \text{purple} \). But this choice implies in order that \( c(4) = \text{red}, \ c(7) = \text{green}, \) and \( c(2) = \text{purple} \). Finally, vertex 1 is adjacent to 2, 4, and 7, and these vertices are already colored with different colors. Hence, the subgraph induced by vertices 1 through 7 cannot be 3-colored. We have obtained the desired contradiction. \( \square \)

The above result combined with Theorem 2.1 immediately implies the following:

**Theorem 6.2.** \( \chi \left( \mathbb{E}^2, \left\{ 1, \sqrt{3/2 + \sqrt{33}/6} \right\} \right) \geq 5. \)

**Proof.** Suppose the opposite is true.

Then by Lemma 6.1 it follows that \( \chi \left( \mathbb{E}^2, \left\{ 1, \sqrt{3/2 + \sqrt{33}/6}, 1/\sqrt{3} \right\} \right) = 4, \) with eventually gives that \( \chi \left( \mathbb{E}^2, \left\{ 1, 1/\sqrt{3} \right\} \right) = 4. \) But this contradicts the result in Theorem 2.1. \( \square \)

**Observation.** One can construct an explicit 5-chromatic \( \left\{ 1, \sqrt{3/2 + \sqrt{33}/6} \right\} \)-graph as follows:

Start with the 9-vertex graph from Theorem 2.1 scaled down by a factor of \( \sqrt{3} \). This graph has 13 edges of length \( 1/\sqrt{3} \) and 6 edges of length 1, and it is a 5-chromatic \( \left\{ 1, 1/\sqrt{3} \right\} \)-graph.

Next, for each edge \( AB \) of length \( 1/\sqrt{3} \), add to this graph the seven vertices of the graph constructed in Lemma 6.1. The resulting graph has \( 9 + 13 \cdot 7 = 100 \) vertices, and it is a 5-chromatic \( \left\{ 1, \sqrt{3/2 + \sqrt{33}/6} \right\} \)-graph. Most likely, much smaller graphs with this property do exist. The advantage of our proof is that we do not have to deal with the 100-vertex graph directly; instead a lower bound for the chromatic number of this graph follows by considering two much smaller graphs, both of order 9.

Using the same technique as in Lemma 6.1 we can prove the following

**Lemma 6.3.** If \( \chi \left( \mathbb{E}^2, \left\{ 1, \sqrt{5/3} \right\} \right) = 4 \) then \( \chi \left( \mathbb{E}^2, \left\{ 1, \sqrt{5/3}, 1/\sqrt{3} \right\} \right) = 4. \)

**Proof.** Suppose that \( \chi \left( \mathbb{E}^2, \left\{ 1, \sqrt{5/3} \right\} \right) = 4; \) that is, there exists a 4-coloring of the plane \( c : \mathbb{E}^2 \to \{ \text{red}, \text{purple}, \text{green}, \text{blue} \} \) such that no two points which are either distance 1 or distance \( \sqrt{5/3} \) apart are assigned the same color. We will show that for this particular coloring, no two points distance \( 1/\sqrt{3} \) apart can be colored identically either.

In order to a contradiction, assume that two such points exist and denote them by \( A \) and \( B \). Choose a system of coordinates such that \( A = (0, 0) = [0, 0, 0, 0] \) and \( B = (1/\sqrt{3}, 0) = [4, 0, 0, 0] \). Again, we are using notation \( [x] \) for the coordinates.

Consider next the \( \left\{ 1, \sqrt{5/3} \right\} \)-graph induced by \( A, B, \) and the following 31 additional vertices:

\[ [2, 0, 0, 2], [-2, 0, 0, 2], [1, 3, -3, 1], [1, -3, 3, 1], [-1, 3, -3, -1], [-1, -3, 3, -1], [0, 0, -12, 0], [-1, 3, 3, 1], [-1, -3, -3, 1], [6, 0, 6, 0], [6, 0, -6, 0], [5, 1, 5, -1], [5, -1, -5, -1], [1, 3, 3, -1], [1, -3, -3, -1], [5, 3, -3, 1], [5, -3, 3, 1], [10, 0, 0, -2], [3, -3, 3, -1], [3, 3, 3, 1], [3, -3, -3, 1], [7, -3, -9, 1], [1, 3, 9, 1], [1, -3, -9, 1], [5, 3, -3, 1], [5, -3, -3, -1], [7, 3, -9, 1], [7, -3, -9, -1], [-2, 0, 6, 0], [-2, 0, -6, 0], [1, 3, -9, -1]. \]
Assume that $c(A) = c(B) = \text{blue}$, as shown in Figure 8. It can be verified that vertex $A$ is adjacent to vertices 1 through 15, while vertex $B$ is adjacent to vertices 16 through 31 (and to vertices 1 and 2 as well). Hence, none of vertices 1 through 31 can be colored blue. So we can only use the remaining three colors for these vertices. However, it can be checked that the subgraph induced by these 31 vertices, regarded as a $\{1, \sqrt{5/3}\}$-graph, cannot be 3-colored. We used both Maple and Sage to verify this assertion.

Again, we combine the above Lemma with Theorem 2.1 to obtain the following

**Theorem 6.4.** $\chi\left(\mathbb{E}^2, \left\{1, \sqrt{5/3}\right\}\right) \geq 5$.

**Proof.** Suppose the opposite is true.

Then by Lemma 6.3 it follows that $\chi\left(\mathbb{E}^2, \left\{1, \sqrt{5/3}, 1/\sqrt{3}\right\}\right) = 4$, which eventually gives that $\chi\left(\mathbb{E}^2, \{1, 1/\sqrt{3}\}\right) = 4$. This contradicts the result in Theorem 2.1.

□
7. Another two distances

One may argue that quite a few of the values of \( d \) we studied so far are rather complicated. In this section we are trying to rectify this situation by considering two simple values: \( d = 2 \) and \( d = 2/\sqrt{3} \).

Theorem 7.1.

\[
\chi(\mathbb{E}^2, \{1, 2\}) \geq 5.
\]

*Proof.* Consider the 26-vertex \( \{1, 2\} \)-graph whose vertices are given by the following coordinates:

\[
[-2, 0, 0, 2], [2, 0, 0, 2], [0, 0, 0, 0], [0, 0, 0, 4], [0, 0, -6, 2], [0, 0, 6, 2], [-1, -3, 3, 1], [1, 3, 3, 3], [-3, -3, 3, 1],
[3, 3, 3, 1], [-1, -3, -3, 1], [1, 3, -3, 1], [-4, 0, 0, 0], [4, 0, 0, 0], [3, -3, -3, 1], [-3, 3, -3, 1], [1, -3, -3, 3],
[-1, 3, -3, 3], [1, -3, 3, 1], [-1, 3, 3, 1], [-2, 0, 6, 0], [2, 0, 6, 0], [-2, 0, -6, 0], [2, 0, -6, 0], [0, -6, 0, 2], [0, 6, 0, 2].
\]

![Figure 9. A 26-vertex \( \{1, 2\} \)-graph with chromatic number 5](image)

It can be verified that this graph has 75 unit edges, and 10 edges of length 2. The graph is shown in figure 9 and it can be checked that it is 5-chromatic. Again, we used both Maple and Sage to verify this. Given the relatively small order of this graph, a computer-free proof is certainly possible. Since the midpoint of any edge of length 2 is also a vertex of the graph, these long edges are shown with curved line segments.

The surprising fact is that a relatively small number of long edges (only 10 of them), is sufficient to raise the chromatic number from 4 to 5.

\( \square \)
Theorem 7.2.

\[ \chi(\mathbb{E}^2, \{1, 2/\sqrt{3}\}) \geq 5. \]

Proof. Consider the 103-vertex \( \{1, 2/\sqrt{3}\} \)-graph whose vertices are given by the following coordinates:

\[
\begin{align*}
&\{0, 0, 0\}, [6, 0, 6, 0], [0, 0, 12, 0], [-6, 0, 6, 0], [-6, 0, -6, 0], [0, 0, -12, 0], [6, 0, -6, 0], [0, -2, -10, 0], [0, 2, -10, 0], [0, -2, 10, 0], [0, 2, 10, 0], \\
&[-2, 0, 0, -2], [2, 0, 0, 2], [-2, 0, 0, 2], [2, 0, 0, 2], [-5, 1, -5, -1], [5, -1, -5, -1], [-5, -1, -5, 1], [5, 1, -5, 1], [-1, 3, -3, -1], [1, -3, -3, -1], \\
&[-1, -3, -3, 1], [1, 3, -3, 1], [-1, -3, 3, -1], [1, 3, 3, -1], [-1, -3, 3, 1], [1, -3, 3, 1], [-5, -1, -5, -1], [5, 1, -5, -1], [-5, 1, 5, 1], [5, -1, 5, 1], [5, -1, 5, 1], \\
&[0, -6, -6, 0], [0, 6, 6, 0], [-3, 3, -3, -3], [3, -3, -3, -3], [3, -3, -3, 3], [3, 3, -3, -3], [-3, -3, -3, -3], [3, 3, 3, -3], \\
&[-3, 3, 3, 3], [3, -3, 3, 3], [-4, 0, 0, 0], [4, 0, 0, 0], [-2, 0, -6, 0], [2, 0, -6, 0], [-2, 0, 6, 0], [2, 0, 6, 0], [0, -2, -2, 0], [0, 2, -2, 0], [0, -2, 2, 0], \\
&[0, 2, 2, 0], [1, -1, -1, 1], [-1, 1, -1, 1], [1, 1, -1, 1], [-1, -1, 1, 1], [-1, -1, 1, 1], [1, -1, 1, 1], [1, -1, 1, 1], [8, 0, 0, 0], \\
&[4, 0, 12, 0], [-4, 0, 12, 0], [-4, 0, 12, 0], [4, 0, -12, 0], [0, -4, -4, 0], [4, 0, -12, 0], [0, -4, -4, 0], [0, -4, -4, 0], [-2, 2, -2, -2], \\
&[2, 2, -2, -2], [-2, -2, -2, -2], [-2, -2, -2, -2], [2, 2, 2, 2], [-4, 2, 2, 2], [2, 2, 2, 2], [-2, -2, 2, 2], [-4, 2, -2, 2], [4, -2, 2, 2], [-4, 2, 2, 2], \\
&[4, -2, 2, 2], [-1, 1, -7, 1], [1, 1, -7, 1], [-3, 1, -5, 1], [-5, 1, 1, 1], [3, -5, 1, 1], [-1, 7, 1, -1], [1, 1, 7, -1], [1, 1, 7, -1], [-1, 1, 7, -1], [-1, 1, 7, -1], \\
&[4, 2, -2, 0], [-4, -2, -2, 0], [4, 2, 2, 0], [-1, -1, -7, 1], [1, 1, -7, 1], [-1, 1, -7, 1], [-3, -1, -5, 1], [3, -1, -5, 1], [-3, -1, -5, 1], [3, -1, 5, 1], [3, 1, 5, 1], [-1, 1, 7, 1], [1, 1, 7, 1].
\end{align*}
\]

Of course, this graph is much too large to be handled directly. It has 312 unit edges, 177 edges of length \(2/\sqrt{3}\), and chromatic number 5. Despite its size, Sage takes only a couple of minutes to verify this. A list of vertices is available at the url [12].
8. Conclusions and directions of future research

Suppose one wants to color the plane such that no two points distance 1 or \(d\) apart can be colored identically. We proved that at least five colors are needed for \(d\) taking any of the following values:

\[
\frac{\sqrt{5} + 1}{2}, \sqrt{3}, \frac{\sqrt{6} + \sqrt{2}}{2}, \frac{1}{2} \sqrt{3^{1/4} \cdot 2\sqrt{2} + 2\sqrt{3} + 2}, \sqrt{3/2 + \sqrt{33}/6}, \frac{\sqrt{5}}{\sqrt{3}}, \frac{2}{\sqrt{3}}.
\]

In a couple of instances we proved results of the following form:

If \(\chi(\mathbb{E}^2, \{1, d\}) = 4\) then \(\chi(\mathbb{E}^2, \{1, d, d'\}) = 4\).

This allowed us to reduce showing that \(\chi(\mathbb{E}^2, \{1, d\}) \geq 5\) to proving that \(\chi(\mathbb{E}^2, \{1, d, d'\}) \geq 5\) instead. As already mentioned, if \(d = \sqrt{11/3}\) one can prove an implication of the form:

If \(\chi(\mathbb{E}^2, \{1\}) = 4\) then \(\chi(\mathbb{E}^2, \{1, d\}) = 4\).

It is tempting to propose the following

**Conjecture 8.1.** There exist values \(d \neq 1\) such that

If \(\chi(\mathbb{E}^2, \{1\}) = 5\) then \(\chi(\mathbb{E}^2, \{1, d\}) = 5\).

If one can then find a 6-chromatic \(\{1, d\}\)-graph the above conjecture would immediately imply that \(\chi(\mathbb{E}^2) \geq 6\). At this time however, no such graphs are known.

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Figure 11. Sixteen different embeddings of the wheel graph $W_6$ as a $\{1, d\}$-graph. Here $d = \frac{1}{2} \sqrt{3^{1/4} \cdot 2\sqrt{2} + 2\sqrt{3} + 2}$. 