Duality symmetry of Reggeon interactions in multicolour QCD

L.N. Lipatov *
Petersburg Nuclear Physics Institute,
Gatchina, 188350, St.Petersburg, Russia

Abstract
The duality symmetry of the Hamiltonian and integrals of motion for Reggeon interactions in multicolour QCD is formulated as an integral equation for the wave function of compound states of $n$ reggeized gluons. In particular the Odderon problem in QCD is reduced to the solution of the one-dimensional Schrödinger equation. The Odderon Hamiltonian is written in a normal form, which gives a possibility to express it as a function of its integrals of motion.

*Supported by the CRDF, INTAS and INTAS-RFBR grants: RP1-253, 1867-93, 95-0311
1 Introduction

The hadron scattering amplitude at high energies $\sqrt{s}$ in the leading logarithmic approximation (LLA) of the perturbation theory is obtained by calculating and summing all contributions $(g^2 \ln(s))^n$, where $g$ is the coupling constant. In this approximation the gluon is reggeized and the BFKL Pomeron is a compound state of two reggeized gluons [1]. Next-to-leading corrections to the BFKL equation were also calculated [2], which gives a possibility to find its region of applicability. In particular the Möbius invariance of the equation valid in LLA [1] turns out to be violated after taking into account next-to-leading terms.

The asymptotic behaviour $\propto s^{j_0}$ of scattering amplitudes is governed by the $j$-plane singularities of the $t$-channel partial waves $f_j(t)$. The position of these singularities $\omega_0 = j_0 - 1$ for the Feynman diagrams with $n$ reggeized gluons in the $t$-channel is proportional to the eigenvalues of a Schrödinger-like equation [3]. For the multicolour QCD $N_c \to \infty$ the colour structure and the coordinate dependence of the eigenfunctions are factorized [4].

The wave function $f_{m,\tilde{m}}(\vec{p}_1, \vec{p}_2, ..., \vec{p}_n; \vec{p}_0)$ of the colourless compound state $O_{m,\tilde{m}}(\vec{p}_0)$ depends on the two-dimensional impact parameters $\vec{p}_1, \vec{p}_2, ..., \vec{p}_n$ of the reggeized gluons. It belongs to the basic series of unitary representations of the Möbius group transformations

$$\rho_k \to \frac{a \rho_k + b}{c \rho_k + d},$$

where $\rho_k = x_k + iy_k$, $\rho_k^* = x_k - iy_k$ and $a, b, c, d$ are arbitrary complex parameters [1]. For this series the conformal weights

$$m = 1/2 + i\nu + n/2, \quad \tilde{m} = 1/2 + i\nu - n/2$$

are expressed in terms of the anomalous dimension $\gamma = 1 + 2i\nu$ and the integer conformal spin $n$ of the composite operators $O_{m,\tilde{m}}(\vec{p}_0)$. They are related to the eigenvalues

$$M^2 f_{m,\tilde{m}} = m(m-1)f_{m,\tilde{m}}, \quad M^{*2} f_{m,\tilde{m}} = \tilde{m}(\tilde{m}-1)f_{m,\tilde{m}},$$

of the Casimir operators $M^2$ and $M^{*2}$:

$$M^2 = \left( \sum_{k=1}^{n} M_k^a \right)^2 = \sum_{r<s} 2 M_r^a M_s^a = -\sum_{r<s} \rho_{rs}^2 \partial_r \partial_s, \quad M^{*2} = (M^2)^*.$$

Here $M_k^a$ are the Möbius group generators

$$M_k^3 = \rho_k \partial_k, \quad M_k^- = \partial_k, \quad M_k^+ = -\rho_k^2 \partial_k$$

and $\partial_k = \partial/(\partial \rho_k)$.

The wave function $f_{m,\tilde{m}}$ satisfies the Schrödinger equation [4]:

$$E_{m,\tilde{m}} f_{m,\tilde{m}} = H f_{m,\tilde{m}}.$$  

Its eigenvalue $E_{m,\tilde{m}}$ is proportional to the position $\omega_{m,\tilde{m}} = j - 1$ of a $j$-plane singularity of the $t$-channel partial wave:

$$\omega_{m,\tilde{m}} = -\frac{g^2 N_c}{8\pi^2} E_{m,\tilde{m}}.$$
governing the $n$-Reggeon asymptotic contribution to the total cross-section $\sigma_{tot} \sim s^{\omega_{m-m}}$.

In the particular case of the Odderon, being a compound state of three reggeized gluons with the charge parity $C = -1$ and the signature $P_J = -1$, the eigenvalue $\omega_{m-m}^{(3)}$ is related to the high-energy behaviour of the difference of the total cross-sections for interactions of particles $p$ and antiparticles $\bar{p}$ with a target:

$$\sigma_{pp} - \sigma_{p\bar{p}} \sim s^{\omega_{m-m}^{(3)}}. \quad (8)$$

The Hamiltonian $H$ in the multicolour QCD has the property of the holomorphic separability [4]:

$$H = \frac{1}{2}(h + h^*), \quad [h, h^*] = 0, \quad (9)$$

where the holomorphic and anti-holomorphic Hamiltonians

$$h = \sum_{k=1}^{n} h_{k,k+1}, \quad h^* = \sum_{k=1}^{n} h_{k,k+1}^* \quad (10)$$

are expressed in terms of the BFKL operator [4]:

$$h_{k,k+1} = \log(p_k) + \log(p_{k+1}) + \frac{1}{p_k} \log(\rho_{k,k+1}) p_k + \frac{1}{p_{k+1}} \log(\rho_{k,k+1}) p_{k+1} + 2 \gamma. \quad (11)$$

Here $\rho_{k,k+1} = \rho_k - \rho_{k+1}$, $p_k = i \partial/(\partial p_k)$, $p_k^* = i \partial/(\partial \rho_k^*)$, and $\gamma = -\psi(1)$ is the Euler constant.

Owing to the holomorphic separability of $h$, the wave function $f_{m,\bar{m}}(\vec{p}_1, \vec{p}_2, ..., \vec{p}_n; \vec{p}_0)$ has the property of the holomorphic factorization [4]:

$$f_{m,\bar{m}}(\vec{p}_1, \vec{p}_2, ..., \vec{p}_n; \vec{p}_0) = \sum_{r,l} c_{r,l} f_m^r(\rho_1, \rho_2, ..., \rho_n; \rho_0) f_{\bar{m}}^l(\rho_1^*, \rho_2^*, ..., \rho_n^*; \rho_0^*), \quad (12)$$

where $r$ and $l$ enumerate degenerate solutions of the Schrödinger equations in the holomorphic and anti-holomorphic sub-spaces:

$$\epsilon_m f_m = h f_m, \quad \epsilon_{\bar{m}} f_{\bar{m}} = h^* f_{\bar{m}}, \quad E_{m,\bar{m}} = \epsilon_m + \epsilon_{\bar{m}}. \quad (13)$$

Similarly to the case of two-dimensional conformal field theories, the coefficients $c_{r,l}$ are fixed by the single-valuedness condition for the function $f_{m,\bar{m}}(\vec{p}_1, \vec{p}_2, ..., \vec{p}_n; \vec{p}_0)$ in the two-dimensional $\vec{p}$-space.

There are two different normalization conditions for the wave function [4],[5]:

$$\|f\|_1^2 = \int \prod_{r=1}^{n} d^2 \rho_r \left| \prod_{r=1}^{n} \rho_{r,r+1}^{-1} f \right|^2, \quad \|f\|_2^2 = \int \prod_{r=1}^{n} d^2 \rho_r \left| \prod_{r=1}^{n} p_r f \right|^2 \quad (14)$$

compatible with the hermicity properties of $H$. Indeed, the transposed Hamiltonian $h^t$ is related with $h$ by two different similarity transformations [5]:

$$h^t = \prod_{r=1}^{n} p_r h \prod_{r=1}^{n} p_r^{-1} = \prod_{r=1}^{n} \rho_{r,r+1}^{-1} h \prod_{r=1}^{n} \rho_{r,r+1}. \quad (15)$$
Therefore $h$ commutes

$$[h, A] = 0$$ (16)

with the differential operator [4]

$$A = \rho_{12}\rho_{23}\ldots\rho_{n1}\rho_1\rho_2\ldots\rho_n.$$ (17)

Furthermore [5], there is a family $\{q_r\}$ of mutually commuting integrals of motion:

$$[q_r, q_s] = 0, \quad [q_r, h] = 0.$$ (18)

They are given as

$$q_r = \sum_{i_1 < i_2 < \ldots < i_r} \rho_{i_1i_2}\rho_{i_2i_3}\ldots\rho_{i_{r-1}i_r}p_{i_1}p_{i_2}\ldots p_{i_r}.$$ (19)

In particular $q_n$ is equal to $A$ and $q_2$ is proportional to $M^2$.

The generating function for these integrals of motion coincides with the transfer matrix $T$ for the XXX model [5]:

$$T(u) = tr (L_1(u)L_2(u)\ldots L_n(u)) = \sum_{r=0}^{n} u^{n-r} q_r,$$ (20)

where the $L$-operators are

$$L_k(u) = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k \end{pmatrix} = u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left( \begin{array}{cc} 1 \\ -\rho_k \end{array} \right) \left( \begin{array}{c} \rho_k \\ 1 \end{array} \right) p_k.$$ (21)

The transfer matrix is the trace of the monodromy matrix $t(u)$ [6]:

$$T(u) = tr (t(u)), \quad t(u) = L_1(u)L_2(u)\ldots L_n(u).$$ (22)

It can be verified that $t(u)$ satisfies the Yang-Baxter equation [5],[6]:

$$t_{s_1}^{s_2}(u) t_{r_1}^{r_2}(v) t_{s_1}^{s_2}(v-u) = t_{s_1}^{s_2}(v-u) t_{r_1}^{r_2}(v) t_{r_1}^{s_2}(u),$$ (23)

where $l(w)$ is the $L$-operator for the well-known Heisenberg spin model:

$$l_{s_1 s_2}^{s_2}(w) = w \delta_{s_1}^{s_1} \delta_{s_2}^{s_2} + i \delta_{s_1}^{s_1} \delta_{s_2}^{s_2}.$$ (24)

The commutativity of $T(u)$ and $T(v)$

$$[T(u), T(v)] = 0$$ (25)

is a consequence of the Yang-Baxter equation.

If one will parametrize $t(u)$ in the form

$$t(u) = \begin{pmatrix} j_0(u) + j_3(u) \\ j_+(u) \\ j_0(u) - j_3(u) \end{pmatrix},$$ (26)

this equation is reduced to the following Lorentz-covariant relations for the currents $j_\mu(u)$:
\[ [j_\mu(u), j_\nu(v)] = [j_\mu(v), j_\nu(u)] = \frac{i \epsilon_{\mu\nu\rho\sigma}}{2(u-v)} (j^\rho(u) j^\sigma(v) - j^\rho(v) j^\sigma(u)). \]  

(27)

Here \( \epsilon_{\mu\nu\rho\sigma} \) is the antisymmetric tensor (\( \epsilon_{1230} = 1 \)) in the four-dimensional Minkowski space and the metric tensor \( g^{\mu\nu} \) has the signature \((1, -1, -1, -1)\). This form follows from the invariance of the Yang-Baxter equations under the Lorentz transformations.

The generators for the spatial rotations coincide with that of the Möbius transformations

\[ \hat{M} = \sum_{k=1}^{n} \hat{M}_k, \]  

(28)

\[ M_k^3 = \rho_k \partial_k, \quad M_k^1 = \frac{1}{2} (1 - \rho_k^2) \partial_k, \quad M_k^2 = \frac{i}{2} (1 + \rho_k^2) \partial_k. \]  

(29)

The commutation relations for the Lorentz algebra are given below:

\[ [M^s, M^t] = i \epsilon_{stu} M^u, \quad [M^s, N^t] = i \epsilon_{stu} N^u, \quad [N^s, N^t] = i \epsilon_{stu} M^u, \]  

(30)

where \( \hat{N} \) are the Lorentz boost generators.

The commutativity of the transfer matrix \( T(u) \) with the local hamiltonian \( h \) \([5],[7]\)

\[ [T(u), h] = 0 \]  

(31)

is a consequence of the relation:

\[ [L_k(u) L_{k+1}(u), h_{k,k+1}] = -i (L_k(u) - L_{k+1}(u)) \]  

(32)

for the pair Hamiltonian \( h_{k,k+1} \). In turn, this relation follows from the Möbius invariance of \( h_{k,k+1} \) and the identity:

\[ [h_{k,k+1}, \left( \left( \hat{M}_{k,k+1} \right)^2, \hat{N}_{k,k+1} \right)] = 4 \hat{N}_{k,k+1}, \]  

(33)

where

\[ \hat{M}_{k,k+1} = \hat{M}_k + \hat{M}_{k+1}, \quad \hat{N}_{k,k+1} = \hat{M}_k - \hat{M}_{k+1} \]  

(34)

are the Lorentz group generators for the two gluon state.

Because the pair hamiltonian \( h_{k,k+1} \) depends only on the Casimir operator \( \left( \hat{M}_{k,k+1} \right)^2 \), it is diagonal

\[ h_{k,k+1} |m_{k,k+1}\rangle = (\psi(m_{k,k+1}) + \psi(1 - m_{k,k+1}) - 2 \psi(1)) |m_{k,k+1}\rangle \]  

(35)

in the conformal weight representation:

\[ \left( \hat{M}_{k,k+1} \right)^2 |m_{k,k+1}\rangle = m_{k,k+1}(m_{k,k+1} - 1) |m_{k,k+1}\rangle. \]  

(36)

Using the commutation relations between \( \hat{M}_{k,k+1} \) and \( \hat{N}_{k,k+1} \) and taking into account that \( \left( \hat{M}_k \right)^2 = 0 \), one can verify that the operator \( \hat{N}_{k,k+1} \) has non-vanishing matrix elements only
between the states $|m_k,k + 1\rangle$ and $|m_k,k + 1 \pm 1\rangle$. It means that the above identity for $h_{k,k+1}$ is a consequence of the known recurrence relations for the $\psi$-functions:

$$\psi(m) = \psi(m - 1) + 1/(m - 1), \quad \psi(1 - m) = \psi(2 - m) + 1/(m - 1).$$  \tag{37}$$

The pair Hamiltonian $h_{k,k+1}$ can be expressed in terms of the small-$u$ asymptotics

$$\hat{L}_{k,k+1}(u) = P_{k,k+1}(1 + i u h_{k,k+1} + \ldots)$$  \tag{38}$$
of the fundamental $L$-operator $\hat{L}_{k,k+1}(u)$ acting on functions $f(\rho_k, \rho_{k+1})$ \cite{6}. Here $P_{k,k+1}$ is defined by the relation:

$$P_{k,k+1} f(\rho_k, \rho_{k+1}) = f(\rho_{k+1}, \rho_k).$$  \tag{39}$$

The operator $\hat{L}_{k,k+1}$ satisfies the linear equation \cite{6}

$$L_k(u) L_{k+1}(v) \hat{L}_{k,k+1}(u - v) = \hat{L}_{k,k+1}(u - v) L_{k+1}(v) L_k(u).$$  \tag{40}$$

This equation can be solved in a way similar to that for $h_{k,k+1}$

$$\hat{L}_{k,k+1}(u) \sim P_{k,k+1} \sqrt{\Gamma(\hat{m}_{k,k+1} + i u)\Gamma(1 - \hat{m}_{k,k+1} + i u)\Gamma(1 - iu)\Gamma(1 + iu)},$$  \tag{41}$$

where the integral operator $\hat{m}_{k,k+1}$ is defined by the relation

$$\hat{m}_{k,k+1} (\hat{m}_{k,k+1} - 1) = (\hat{M}_k^2 + \hat{M}_{k+1}^2)^2$$  \tag{42}$$

and the proportionality constant, being a periodic function of $\hat{m}_{k,k+1}$ with an unit period, is fixed from the triangle relation

$$\hat{L}_{13}(u) \hat{L}_{23}(v) \hat{L}_{12}(u - v) = \hat{L}_{12}(u - v) \hat{L}_{23}(v) \hat{L}_{13}(u).$$  \tag{43}$$

To find a representation of the Yang-Baxter commutation relations, the algebraic Bethe ansatz is used \cite{6}. To begin with, in the above parametrization of the monodromy matrix $t(u)$ in terms of the currents $j_\mu(u)$, one should construct the pseudovacuum state $|0\rangle$ satisfying the equations

$$j_+(u) |0\rangle = 0.$$  \tag{44}$$

However, these equations have a non-trivial solution only if the above $L$-operators are regularized as

$$L_k^\delta(u) = \left( \begin{array}{c} u + \rho_k p_k - i \delta \\ -p_k^2 p_k + 2 i \rho_k \delta \end{array} \right) \left( \begin{array}{c} p_k \\ u - \rho_k p_k + i \delta \end{array} \right)$$  \tag{45}$$

by introducing an infinitesimally small conformal weight $\delta \to 0$ for reggeized gluons (another possibility is to use the dual space corresponding to $\delta = -1$ \cite{7}). For this regularization the pseudovacuum state is
$$|\delta\rangle = \prod_{k=1}^{n} \rho_{k}^{2i}$$

(46)

It is also an eigenstate of the transfer matrix:

$$T(u)|\delta\rangle = 2 j_{0}(u)|\delta\rangle = ((u - i\delta)^{n} + (u + i\delta)^{n})|\delta\rangle.$$  

(47)

Furthermore, all excited states are obtained by applying the product of the operators $j_{-}(v)$ to the pseudovacuum state:

$$|v_{1}v_{2}...v_{k}\rangle = j_{-}(v_{1})j_{-}(v_{2})...j_{-}(v_{k})|\delta\rangle.$$  

(48)

They are eigenfunctions of the transfer matrix $T(u)$ with the eigenvalues:

$$\tilde{T}(u) = (u + i\delta)^{n} \prod_{r=1}^{k} \frac{u - v_{r} - i}{u - v_{r}} + (u - i\delta)^{n} \prod_{r=1}^{k} \frac{u - v_{r} + i}{u - v_{r}},$$

(49)

providing that the spectral parameters $v_{1}, v_{2}, ..., v_{k}$ are solutions of the set of Bethe equations:

$$\left(\frac{v_{s} - i\delta}{v_{s} + i\delta}\right)^{n} = \prod_{r \neq s} \frac{v_{s} - v_{r} - i}{v_{s} - v_{r} + i}$$

(50)

for $s = 1, 2...k$.

Due to above relations the function

$$Q(u) = \prod_{r=1}^{k} (u - v_{r})$$

(51)

satisfies the Baxter equation [6,7]:

$$\tilde{T}(u)Q(u) = (u - i\delta)^{n} Q(u + i) + (u + i\delta)^{n} Q(u - i),$$

(52)

where $\tilde{T}(u)$ is an eigenvalue of the transfer matrix. Its corresponding eigenfunctions can be expressed in terms of the solution $Q^{(k)}(u)$ of this equation as follows [7]

$$|v_{1}v_{2}...v_{k}\rangle = Q^{(k)}(\hat{u}_{1}) Q^{(k)}(\hat{u}_{2}) ... Q^{(k)}(\hat{u}_{n-1}) |\delta\rangle,$$

(53)

where the integral operators $\hat{u}_{r}$ are zeros of the current $j_{-}(u)$:

$$j_{-}(u) = c \prod_{r=1}^{n-1} (u - \hat{u}_{r}).$$

(54)

Eigenvalues $\epsilon$ of the holomorphic Hamiltonian $h$ also can be expressed in terms of $Q(u)$ [7].

Up to now the Baxter equation was solved only for the case of the BFKL Pomeron ($n = 2$). This is the reason why we use below another approach, based on the diagonalization of the transfer matrix.
2 Duality of Reggeon interactions at large \( N_c \)

The differential operators \( q_r \) and the Hamiltonian \( h \) are invariant under the cyclic permutation of gluon indices \( i \to i + 1 \) (\( i = 1, 2...n \)), corresponding to the Bose symmetry of the Reggeon wave function at \( N_c \to \infty \). It is remarkable that above operators are invariant also under the more general canonical transformation:

\[
\rho_{i-1,i} \to p_i \to \rho_{i,i+1},
\]

combined with reversing the order of the operator multiplication.

This invariance is obvious for the Hamiltonian \( h \) if we write it in the form:

\[
h = h_p + h_\rho,
\]

where

\[
h_p = \sum_{k=1}^{n} \left( \ln(p_k) + \frac{1}{2} \sum_{\lambda = \pm 1} \rho_{k,k+\lambda} \ln(p_k) \rho_{k,k+\lambda}^{-1} + \gamma \right),
\]

\[
h_\rho = \sum_{k=1}^{n} \left( \ln(\rho_{k,k+1}) + \frac{1}{2} \sum_{\lambda = \pm 1} \rho_{k+1(1+\lambda)/2}^{-1} \ln(\rho_{k,k+1}) \rho_{k+1(1+\lambda)/2} + \gamma \right).
\]

Here \( \gamma = -\psi(1) \) is the Euler constant.

The invariance of the transfer matrix can be verified using two equivalent representations for \( q_r \):

\[
q_r = \sum_{i_1 < i_2 < \ldots < i_r} \prod_{k = i_{t+1}}^{i_t} \left( \sum_{k = i_{t+1}}^{i_t} \rho_{k-1,k} p_{i_t} \right) = \sum_{i_1 < i_2 < \ldots < i_r} \prod_{k = i_{t+1}}^{i_t} \left( \rho_{i_t,i_{t+1}} \sum_{k = i_{t+1}}^{i_t} p_k \right).
\]

Note that the supersymmetry corresponds to an analogous generalization of translations to super-translations. Furthermore, the Kramers-Wannier duality in the Ising model and the popular electromagnetic duality \( E \leftrightarrow H \) can be considered as similar canonical transformations [8].

The above duality symmetry is realized as an unitary transformation only for the vanishing total momentum:

\[
\vec{p} = \sum_{r=1}^{n} \vec{p}_r = 0.
\]

In this case one can parametrize gluon momenta in terms of momentum transfers \( k_r \) as follows

\[
p_r = k_r - k_{r+1},
\]

which gives a possibility to present the symmetry transformation in a simpler form:

\[
k_r \to \rho_r \to k_{r+1}, r = 1, 2...n.
\]

Because the operators \( q_r \) compose a complete set of invariants of the transformation, the Hamiltonian \( h \) should be their function

\[
h = h(q_2, q_3, \ldots, q_n),
\]
fixed by the property of its locality. Furthermore, a common eigenfunction of \( q_r \) \((r = 2, \ldots, n)\) is simultaneously a solution of the Schrödinger equation, which means, that the duality symmetry gives an explanation of the integrability of the Reggeon model at \( N_c \to \infty \).

To formulate the duality as an integral equation we work in the two-dimensional impact parameter space \( \vec{p} \), initially without taking into account the property of the holomorphic factorization of the Green functions. The wave function \( \psi_{m, \bar{m}} \) of the composite state with \( \vec{p} = 0 \) can be written in terms of the eigenfunction \( f_{m, \bar{m}} \) of a commuting set of the operators \( q_k \) and \( q_k^* \) for \( k = 1, 2 \ldots n \) as follows

\[
\psi_{m, \bar{m}}(\vec{p}_1^2, \vec{p}_2^2, \ldots, \vec{p}_n^2) = \int \frac{d^2\rho_0}{2\pi} f_{m, \bar{m}}(\vec{p}_1^2, \vec{p}_2^2, \ldots, \vec{p}_n^2, \vec{p}_0^2) .
\] (63)

It is a highest-weight component of the Möbius group representation with the quantum numbers \( m = 1/2 + i\nu + n/2 \) and \( \bar{m} = 1/2 + i\nu - n/2 \) related with eigenvalues of the Casimir operators

\[
\left( \sum_{k=1}^{n} \tilde{M}_k \right)^2 \psi_{m, \bar{m}} = m(m - 1)\psi_{m, \bar{m}} , \quad \left( \sum_{k=1}^{n} \tilde{M}_k^* \right)^2 \psi_{m, \bar{m}} = \bar{m}(\bar{m} - 1)\psi_{m, \bar{m}} .
\] (64)

The other components of this highest-weight representation can be obtained by applying to \( \psi_{m, \bar{m}} \) the Möbius group generators:

\[
\psi_{m, \bar{m}}^{r_1 r_2} = \left( \sum_{k=1}^{n} \rho_k^2 \partial_k \right)^{r_1} \left( \sum_{k=1}^{n} \rho_k^{*2} \partial_k^* \right)^{r_2} \psi_{m, \bar{m}} .
\] (65)

Note that, in accordance with the relations

\[ \tilde{m}^* = 1 - \bar{m} , \quad m^* = 1 - m \] (66)

the conjugate function \( f_{m, \bar{m}}^* \) is transformed as \( f_{1-m, \bar{m}} \):

\[
(f_{m, \bar{m}}(\vec{p}_1^2, \ldots, \vec{p}_n^2, \vec{p}_0^2))^* \sim f_{1-m, \bar{m}}(\vec{p}_1^2, \ldots, \vec{p}_n^2, \vec{p}_0^2) .
\] (67)

Moreover, because of the reality of the Möbius group, the complex-conjugated representations are linearly dependent:

\[
(f_{m, \bar{m}}(\vec{p}_1^2, \ldots, \vec{p}_n^2, \vec{p}_0^2))^* \sim \int d^2\rho_{0\nu} (\rho_{0\nu})^{2m-2}(\rho_{0\nu}^*)^{2\bar{m}-2} f_{m, \bar{m}}(\vec{p}_1^2, \ldots, \vec{p}_n^2, \vec{p}_0^2) .
\] (68)

By considering the limit \( \rho_0 \to \infty \) of this equation, we obtain for \( \psi_{m, \bar{m}} \) the new representation

\[
\psi_{m, \bar{m}}(\vec{p}_1^2, \vec{p}_2^2, \ldots, \vec{p}_n^2) \sim f_{1-m, \bar{m}}(\vec{p}_1^2, \vec{p}_2^2, \ldots, \vec{p}_n^2, \infty) .
\] (69)

Because of the relations

\[
(\psi_{m, \bar{m}}^*(\vec{p}_1^2, \vec{p}_2^2, \ldots, \vec{p}_n^2))^* \sim \psi_{1-m, \bar{m}}^*(\vec{p}_1^2, \vec{p}_2^2, \ldots, \vec{p}_n^2) ,
\] (70)

the functions \( \psi_{m, \bar{m}}^* \) and \( \psi_{m, \bar{m}} \) have the same conformal spin \( n = m - \bar{m} \).

Taking into account the hermicity properties of the total Hamiltonian:
\[ H^+ = \prod_{k=1}^{n} | \rho_{k,k+1} |^{-2} H \prod_{k=1}^{n} | \rho_{k,k+1} |^2 = \prod_{k=1}^{n} | \rho_k |^2 H \prod_{k=1}^{n} | \rho_k |^{-2}, \]  

(71)

the solution \( \psi_{m,m}^{\pm} \) of the complex-conjugated Schrödinger equation for \( \overrightarrow{p} = 0 \) can be expressed in terms of \( \psi_{m,m}^{-} \) as follows:

\[ \psi_{m,m}^{\pm}(\overrightarrow{p}_1, \overrightarrow{p}_2, \ldots, \overrightarrow{p}_n) = \prod_{k=1}^{n} | \rho_{k,k+1} |^{-2} \left( \psi_{m,m}^{-}(\overrightarrow{p}_1, \overrightarrow{p}_2, \ldots, \overrightarrow{p}_n) \right)^* . \]  

(72)

If one performs the Fourier transformation of \( \psi_{m,m}^{\pm} \) to the momentum space

\[ \Psi_{m,m}^{-}(\overrightarrow{p}_1, \overrightarrow{p}_2, \ldots, \overrightarrow{p}_n) = \int \prod_{k=1}^{n-1} \frac{d^2 \rho_{k-1,k}}{2\pi} \prod_{k=1}^{n} e^{i \overrightarrow{p}_k \cdot \overrightarrow{p}} \rho_k \psi_{m,m}^{\pm}(\overrightarrow{p}_1, \overrightarrow{p}_2, \ldots, \overrightarrow{p}_n) \]  

(73)

with substituting the arguments

\[ \overrightarrow{p}_k \rightarrow \overrightarrow{\rho}_{k,k+1}, \]  

(74)

the new expression \( \Psi_{m,m}^{-}(\overrightarrow{p}_1, \overrightarrow{p}_2, \ldots, \overrightarrow{p}_n) \) will have the same properties as the initial function \( \psi_{m,m}^{-}(\overrightarrow{p}_1, \overrightarrow{p}_2, \ldots, \overrightarrow{p}_n) \) under rotations and dilatations.

Moreover, in accordance with the above duality symmetry it satisfies the same set of equations as \( \psi_{m,m}^{-} \) and therefore these two functions can be chosen to be proportional:

\[ \psi_{m,m}^{-}(\overrightarrow{p}_1, \overrightarrow{p}_2, \ldots, \overrightarrow{p}_n) = c_{m,m} \Psi_{m,m}^{-}(\overrightarrow{p}_1, \overrightarrow{p}_2, \ldots, \overrightarrow{p}_n) . \]  

(75)

The proportionality constant \( c_{m,m} \) is determined from the condition that the norm of the function \( \psi_{m,m}^{-} \):

\[ \| \psi_{m,m}^{-} \|^2 = \int \prod_{k=1}^{n-1} \frac{d^2 \rho_{k-1,k}}{2\pi} \psi_{m,m}^{\pm} \psi_{m,m}^{-} \]  

(76)

is conserved after this transformation.

Because \( \psi_{m,m}^{-} \) is also an eigenfunction of the integrals of motion \( A \) and \( A^* \) with their eigenvalues \( \lambda_m \) and \( \lambda_m^* = \lambda_m \):

\[ A \psi_{m,m}^{-} = \lambda_m \psi_{m,m}^{-}, \quad A^* \psi_{m,m}^{-} = \lambda_m^* \psi_{m,m}^{-}, \]  

(77)

one can verify that, for the unitarity of the duality transformation, the constant \( c_{m,m} \) should be chosen as

\[ c_{m,m} = | \lambda_m | 2^n , \]  

(78)

for an appropriate phase of \( \psi_{m,m}^{-} \). Here the factor \( 2^n \) appears due to the relation \( \partial_\mu^2 = 4 \partial_\mu \partial^\mu \).

This value of \( c_{m,m} \) is compatible also with the requirement that two subsequent duality transformations are equivalent to the cyclic permutation \( i \rightarrow i + 1 \) of gluon indices.

Thus, the duality symmetry takes the form of the following integral equation for \( \psi_{m,m}^{-} \):

\[ \psi_{m,m}^{-}(\overrightarrow{p}_1, \ldots, \overrightarrow{p}_n) = | \lambda_m | 2^n \int \prod_{k=1}^{n-1} \frac{d^2 \rho_{k-1,k}}{2\pi} \prod_{k=1}^{n} e^{i \overrightarrow{\rho}_{k,k+1} \cdot \overrightarrow{p}_k} \rho_k \psi_{m,m}^{\pm}(\overrightarrow{p}_1, \ldots, \overrightarrow{p}_n) . \]  

(79)
Note that the validity of this equation in the Pomeron case \( n = 2 \) can be verified from the relations

\[
f_{m,\tilde{m}}(\bar{\rho}_1^\lambda; \ldots; \bar{\rho}_n^\lambda) \sim \left( \frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^m \left( \frac{\rho_{12}^*}{\rho_{10}\rho_{20}^*} \right)^{\tilde{m}}, \quad \psi_{m,\tilde{m}}(\bar{\rho}_1^\lambda) \sim (\rho_{12})^{1-m}(\rho_{12}^*)^{1-\tilde{m}},
\]

\[
|\lambda_m| \, 2^2 \int \frac{d^2\rho_{12}}{2\pi} \, e^{i\bar{\rho}_1^\lambda \rho_{12}^*} (\rho_{12}^*)^m = e^{i\delta(m,\tilde{m})} \rho_{12}^{1-m} \rho_{12}^* \rho_{10}^{2-m} \, |\lambda_m| = |m(1-m)|, \quad (80)
\]

Let us use for \( f_{m,\tilde{m}} \) the conformally covariant anzatz

\[
f_{m,\tilde{m}}(\bar{\rho}_1^\lambda, \ldots; \bar{\rho}_n^\lambda) = \prod_{i=1}^{n} \frac{\rho_{i,i+1}^*}{\rho_{i0}^{*2}} \prod_{i=1}^{n} \frac{\rho_{i,i+1}}{\rho_{i0}^{2}}, \quad f_{m,\tilde{m}}(\bar{x}_1^\lambda, \ldots, \bar{x}_n^\lambda), \quad (81)
\]

where the anharmonic ratios \( x_r \) (\( r = 1, 2, \ldots n \)) of the gluon coordinates are chosen as follows

\[
x_r = \frac{\rho_{r-1,r} \rho_{r+1,r}}{\rho_{r-1,0} \rho_{r+1,r}}; \quad \prod_{r=1}^{n} x_r = (-1)^n; \quad \sum_{r=1}^{n} (-1)^r \prod_{k=r+1}^{n} x_k = 0. \quad (82)
\]

The function \( f_{m,\tilde{m}}(\bar{x}_1^\lambda, \ldots, \bar{x}_n^\lambda) \) is invariant under certain modular transformations as a consequence of the Bose symmetry.

For the physically interesting case \( m, \tilde{m} \to 1/2 \) we can calculate \( \psi_{m,\tilde{m}} \), taking into account the logarithmic divergence of the integral at \( \bar{\rho}_0^\lambda \to \infty \):

\[
\lim_{m,\tilde{m} \to 1/2} \psi_{m,\tilde{m}}(\bar{\rho}_1^\lambda, \ldots, \bar{\rho}_n^\lambda) = \frac{f(\bar{z}_1^\lambda, \ldots, \bar{z}_n^\lambda)}{1-m-\tilde{m}} \prod_{k=1}^{n} |\rho_{k,k+1}|^{1/n}, \quad \tilde{z}_r = \frac{\rho_{r-1,r}}{\rho_{r+1,r}}. \quad (83)
\]

Let us change the variables \( z_r \) to new ones \( y_k \) as follows:

\[
y_k = \rho_{k+1,k}/\rho_{n,n-1} = (-1)^{n-k-1} \prod_{r=k+1}^{n-1} z_r; \quad y_{n-1} = 1, \quad \sum_{k=1}^{n} y_k = 0. \quad (84)
\]

The duality equation for the wave function \( f(\bar{y}_1^\lambda, \ldots, \bar{y}_n^\lambda) \) can be written in the form:

\[
f(\bar{y}_1^\lambda, \ldots, \bar{y}_n^\lambda) = |\lambda| \int \prod_{k=1}^{n-2} \frac{d^2y_k'}{2\pi} K(\bar{y}; \bar{y}') \prod_{k=1}^{n} \frac{|y_k|^{1/n}}{|y_k'|^{2-1/n}} f(y_1, \ldots, y_n). \quad (85)
\]

The integral kernel \( K(\bar{y}; \bar{y}') \) is given below:

\[
K(\bar{y}; \bar{y}') = 2^n \int \frac{d^2\rho_{n,n-1}}{2\pi |\rho_{n,n-1}|^3} \exp \left( -i \sum_{k=1}^{n} \left( \bar{\rho}_k, \frac{y_k' - \rho_{k+1,k}' - \rho_{k,n-1}'\bar{y}_{n-2}}{} \right) \right) \quad (86)
\]

and is calculated analytically:

\[
K(\bar{y}; \bar{y}') = -2^n \sum_{k=1}^{n} \left( \frac{y_k' - \rho_{k,n-1}'\bar{y}_{n-2}}{} \right) \quad (87)
\]

To simplify the duality equation, we consider below the compound state of three reggeized gluons.
3 Duality equation for the Odderon wave function

In the case of the Odderon the conformal invariance fixes the solution of the Schrödinger equation [5]

\[
 f_{m,\tilde{m}}(\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3; \vec{\rho}_0) = \left( \frac{\rho_{12} \rho_{23} \rho_{31}}{\rho_{10}^2 \rho_{20}^2 \rho_{30}^2} \right)^{m/3} \left( \frac{\rho_{12}^* \rho_{23}^* \rho_{31}^*}{\rho_{10}^* \rho_{20}^* \rho_{30}^*} \right)^{\tilde{m}/3} f_{m,\tilde{m}}(\vec{\tau}) \tag{88}
\]

up to an arbitrary function \( f_{m,\tilde{m}}(\vec{\tau}) \) of one complex variable \( x \) being the anharmonic ratio of four coordinates

\[
x = \frac{\rho_{12} \rho_{30}}{\rho_{10} \rho_{32}}. \tag{89}
\]

Note that, owing to the Bose symmetry of the Odderon wave function, \( f_{m,\tilde{m}}(\vec{\tau}) \) has the following modular properties:

\[
f_{m,\tilde{m}}(\vec{\tau}) = (-1)^{(\tilde{m}-m)/3} f_{m,\tilde{m}}(\vec{\tau}/|x|^2) = (-1)^{(\tilde{m}-m)/3} f_{m,\tilde{m}}(\vec{1} - \vec{\tau}) \tag{90}
\]

and satisfies the normalization condition

\[
\|f_{m,\tilde{m}}\|^2 = \int \frac{d^2x}{|x(1-x)|^{4/3}} |f_{m,\tilde{m}}(\vec{\tau})|^2, \tag{91}
\]

compatible with the modular symmetry.

After changing the integration variable from \( \rho_0 \) to \( x \) in accordance with the relations

\[
\rho_0 = \frac{\rho_{31} \rho_{12}}{\rho_{12} - x \rho_{32}} + \rho_1, \quad d\rho_0 = \frac{\rho_{31} \rho_{12} \rho_{32}}{(\rho_{12} - x \rho_{32})^2} dx \tag{92}
\]

the Odderon wave function \( \psi_{m,\tilde{m}}(\vec{\rho}_{ij}) \) at \( \vec{q} = 0 \) can be written as

\[
\psi_{m,\tilde{m}}(\vec{\rho}_{ij}) = \left( \frac{\rho_{23}}{\rho_{12} \rho_{31}} \right)^{m-1} \left( \frac{\rho_{23}^*}{\rho_{12}^* \rho_{31}^*} \right)^{\tilde{m}-1} \chi_{m,\tilde{m}}(\vec{\tau}), \quad z = \frac{\rho_{12}}{\rho_{32}}, \tag{93}
\]

where

\[
\chi_{m,\tilde{m}}(\vec{\tau}) = \int \frac{d^2x}{2\pi |x - z|^{4/3}} \left( \frac{(x - z)^3}{x(1-x)} \right)^{2m/3} \left( \frac{(x^* - z^*)^3}{x^*(1-x^*)} \right)^{2\tilde{m}/3}. \tag{94}
\]

In fact this function is proportional to \( f_{1-m,1-\tilde{m}}(\vec{\tau}) \):

\[
\chi_{m,\tilde{m}}(\vec{\tau}) \sim (x(1-x))^{2(m-1)/3} (x^*(1-x^*))^{2(\tilde{m}-1)/3} f_{1-m,1-\tilde{m}}(\vec{\tau}), \tag{95}
\]

which is a realization of the discussed linear dependence between two representations \((m, \tilde{m})\) and \((1-m, 1-\tilde{m})\).

The corresponding reality property for the Möbius group representations can be presented in the form of the integral relation

\[
\chi_{m,\tilde{m}}(\vec{\tau}) = \int \frac{d^2x}{2\pi} (x - z)^{2m-2} (x^* - z^*)^{2\tilde{m}-2} \chi_{1-m,1-\tilde{m}}(\vec{\tau}), \tag{96}
\]
for an appropriate choice of phases of the functions \( \chi_{m, \tilde{m}} \) and \( \chi_{1-m, 1-\tilde{m}} \). The function \( \chi_{m, \tilde{m}}(\vec{z}) \) satisfies the modular relations

\[
\chi_{m, \tilde{m}}(\vec{z}) = (-1)^{m-\tilde{m}} z^{2m-2\tilde{m}} z^{2\tilde{m}-2} \chi_{m, \tilde{m}}(\vec{z}/|z|^2) = (-1)^{m-\tilde{m}} \chi_{m, \tilde{m}}(\vec{1} - \vec{z}). \tag{97}
\]

The duality property of the wave function \( \psi_{m, \tilde{m}}(\vec{\rho}_{ij}) \) can be written in a form of the integral equation:

\[
\psi_{m, \tilde{m}}(\vec{\rho}_{ij}) = |\lambda_m| \left( \frac{d^2 \rho_{12} d^2 \rho_{23}}{2\pi} \frac{d^2 \rho_{13} \exp(i(\vec{\rho}_{21} \vec{\rho}_{23} + \vec{\rho}_{31} \vec{\rho}_{31}))}{|\rho_{12} \rho_{23} \rho_{31}|^2} \right) \psi^*_{m, \tilde{m}}(\vec{\rho}_{ij}), \tag{98}
\]

where \( |\lambda_m|^2 \) is the corresponding eigenvalue of the differential operator

\[
|A|^2 = |\rho_{12} \rho_{23} \rho_{31} p_1 p_2 p_3|^2. \tag{99}
\]

The function \( \psi^*_{m, \tilde{m}}(\vec{\rho}_{ij}) \) is transformed as \( \psi_{1-m, 1-\tilde{m}}(\vec{\rho}_{ij}) \):

\[
\psi^*_{m, \tilde{m}}(\vec{\rho}_{ij}) \sim \psi_{1-m, 1-\tilde{m}}(\vec{\rho}_{ij}). \tag{100}
\]

In terms of \( \chi_{m, \tilde{m}}(\vec{z}) \) the above duality equation looks as follows

\[
\frac{\chi_{m, \tilde{m}}(\vec{z})}{|\lambda_m|} = \int \frac{d^2 z'}{2\pi} \exp(-i(\vec{\rho}_{32} \vec{\rho}_{31})^m \left( \chi^*_{1-m, 1-\tilde{m}}(\vec{z}) \right)^m \frac{R(\vec{z}, \vec{z}')}{|\rho_{32} \rho_{31}|^2} \chi_{m, \tilde{m}}(\vec{z})), \tag{101}
\]

where

\[
R(\vec{z}, \vec{z'}) = \int \frac{d^2 \rho_{32} / (2\pi)}{|\rho_{32}|^4 \rho_{32}} \exp(i(\vec{\rho}_{32} \vec{\rho}_{13})^m \chi_{m, \tilde{m}}(\vec{z})). \tag{102}
\]

This integral is calculated to be

\[
R(\vec{z}, \vec{z'}) = C_{m, \tilde{m}} Z^{1-m} (Z^*)^{1-\tilde{m}} \tag{103}
\]

with

\[
C_{m, \tilde{m}} = \frac{2 e^{i\delta(m, \tilde{m})}}{|m| (2 - m) \Gamma(2 - m) \Gamma(2 - \tilde{m})}, \tag{104}
\]

where the phase \( \delta(m, \tilde{m}) \) was defined above and

\[
Z = z z^* - \frac{1}{z} = z(2 - m) \tag{105}
\]

Further, we introduce the new integration variable \( z' \rightarrow z^* \), which effectively leads to the substitution \( \chi_{m, \tilde{m}}(\vec{z}^*) \rightarrow \chi_{m, \tilde{m}}(\vec{z}) \). By performing the modular transformation \( z = \frac{1}{z^*} \rightarrow z \) and taking into account the modular relation

\[
\chi_{m, \tilde{m}}(\vec{z}) = (1 - z)^{2m-2}(1 - z^*)^{2\tilde{m}-2} \chi_{m, \tilde{m}}((\vec{1} - \vec{z})/|1 - z|^2), \tag{106}
\]
one can rewrite the above duality equation for $\chi_{m,\tilde{m}}(\tilde{z}^\dagger)$ as

$$ \frac{\chi_{m,\tilde{m}}(\tilde{z}^\dagger)}{|\lambda_m| C_{m,\tilde{m}}} = \int \frac{d^2 z'}{2\pi} \left( \frac{z'(1-z')z(1-z)}{z'-z} \right)^{m-1} \left( \frac{z'(1-z')z(1-z)}{z'-z} \right)^{\tilde{m}-1} \chi^*_{m,\tilde{m}}(z'). \quad (107) $$

This equation for $\chi_{m,\tilde{m}}(\tilde{z}^\dagger)$ corresponds to the symmetry of the Odderon wave function under the involution $p_k \leftrightarrow \frac{1}{2} \varepsilon_{klm} p_l m$.

It can be written in the pseudo-differential form:

$$ z(1-z) (i\partial)^{2-m} z^* (1-z^*) (i\partial^*)^{2-\tilde{m}} \varphi_{1-m,1-\tilde{m}}(\tilde{z}^\dagger) = |\lambda_m,\tilde{m}| \left( \varphi_{1-m,1-\tilde{m}}(\tilde{z}^\dagger) \right)^*, \quad (108) $$

where

$$ \varphi_{1-m,1-\tilde{m}}(\tilde{z}^\dagger) = 2^{1-m-\tilde{m}} (z(1-z))^{1-m} (z^*(1-z^*))^{1-\tilde{m}} \chi_{m,\tilde{m}}(\tilde{z}^\dagger). \quad (109) $$

Note, that for a self-consistency of the above equation its right-hand side should be orthogonal to the zero modes of the operator $(i\partial)^{2-m} (i\partial^*)^{2-\tilde{m}}$. Due to the Bose symmetry of the wave function it is enough to impose on it only one integral constraint:

$$ \int \frac{d^2 z}{|z(1-z)|^2} \left( \varphi_{1-m,1-\tilde{m}}(\tilde{z}^\dagger) \right)^* = 0. \quad (110) $$

The normalization condition for the wave function

$$ \|\varphi_{m,\tilde{m}}\|^2 = \int \frac{d^2 x}{|x(1-x)|^2} |\varphi_{m,\tilde{m}}(\tilde{x})|^2 $$

is compatible with the duality symmetry.

The holomorphic and anti-holomorphic factors of $f_{m,\tilde{m}}(\rho_1^\dagger, \rho_2^\dagger, \rho_3^\dagger; \rho^\dagger)$ are eigenfunctions of the integrals of motion $A$ and $A^*$:

$$ A f_m = \lambda_m f_m, \quad A^* f_{\tilde{m}} = \lambda_{\tilde{m}} f_{\tilde{m}}, \quad \lambda_m = (\lambda_m)^*. \quad (111) $$

In the limit $m \to 1/2$, $\tilde{m} \to 1/2$, corresponding to the ground state, the integral over $x$ for the wave function $\psi_{m,\tilde{m}}$ at $q = 0$ is calculated, since the main contribution appears from the singularity at $x = z$:

$$ \psi_{m,\tilde{m}}(\rho_1^\dagger) \simeq \frac{1}{(m + \tilde{m} - 1)} |\rho_{12}\rho_{23}\rho_{31}|^{1/3} f(z, z^*), \quad z = \rho_{12}/\rho_{32}, \quad (112) $$

where $f(x, x^*) = f_{1/2,1/2}(x, x^*)$. This means that one can obtain for $f(x, x^*)$ the following equation in the $x$-representation:

$$ |x(1-x)|^{1/3} f(x, x^*) = -\frac{4|\lambda|}{\pi} \int \frac{d^2 y}{|y(1-y)|^{3/3}} \frac{|y-x|}{f(y, y^*)} \quad (113) $$

for a definite choice of the phase of the ground-state function $f$.

The common factor in front of the integral is in agreement with the condition of conservation of the norm of the wave function after this canonical transformation (only its sign can be changed). The above integral relation is reduced to the following pseudo-differential equation
where we introduced the functions $\phi(x, x^*) = |\lambda| \varphi(x, x^*)$, 
(114)

and use the relation

$$|p|^3 |x| = - |p|^3 \int_{-\pi/2}^{\pi/2} \frac{d\phi}{2\pi} \int_{-\infty}^{\infty} \frac{d|\vec{p}|}{|\vec{p}|} \exp(-i \vec{p} \cdot \vec{x}) = -\frac{\pi}{4} \delta^2(\vec{x}) .$$
(116)

For self-consistency of the pseudo-differential equation one should impose on the wave function the following constraint:

$$\int \frac{d^2 x}{|x(1-x)|^2} \varphi(x, x^*) = 0 .$$
(117)

Let us derive the duality equation for $\varphi_{m, \tilde{m}}(\vec{x}^*)$ for general values of $m$ and $\tilde{m}$ by using different arguments. We start with the conformally covariant anzatz for the holomorphic factor $f_m(\rho_1, \rho_2, \rho_3; \rho_0)$:

$$f_m(\rho_1, \rho_2, \rho_3; \rho_0) = \left( \frac{\rho_{12} \rho_{23} \rho_{31}}{\rho_{10} \rho_{20} \rho_{30}} \right)^{m/3} f_m(x) , \quad x = \frac{\rho_{12} \rho_{30}}{\rho_{10} \rho_{32}} .$$
(118)

In the $x$ representation the integral of motion $A = \rho_{12} \rho_{23} \rho_{31} p_1 p_2 p_3$ is an ordinary differential operator. When acting on $f_m(x)$, it can be presented in the following form

$$\frac{i^3}{x} \left( \frac{m}{3} (x - 2) + x(1 - x) \partial \right) \frac{x}{1 - x} \left( \frac{m}{3} (1 + x) + x(1 - x) \partial \right) \frac{1}{x} \left( \frac{m}{3} (1 - 2x) + x(1 - x) \partial \right)$$

$$= X^{-m/3} A_{m} X^{m/3} , \quad X = x(1 - x) , \quad A_{m} = a_{1-m} a_{m} ,$$
(119)

where

$$a_{m} = x (1 - x) p^{m+1} , \quad a_{1-m} = x (1 - x) p^{2-m} , \quad p = i \partial .$$
(120)

Therefore the differential equation $A f_m = \lambda_m f_m$ for eigenfunctions $f_m$ and eigenvalues $\lambda_m$ is equivalent to the system of two pseudo-differential equations

$$a_{m} \varphi_m = l_m \varphi_{1-m} , \quad a_{1-m} \varphi_{1-m} = l_{1-m} \varphi_{m} ,$$
(121)

where we introduced the functions $\varphi_m$ and $\varphi_{1-m}$ in accordance with the definitions

$$\varphi_m \equiv X^{m/3} f_m , \quad \varphi_{1-m} \equiv X^{(1-m)/3} f_{1-m}$$
(122)

and the eigenvalues $l_m$ and $l_{1-m}$ are related with the eigenvalue $\lambda_m$ of $A_{m}$ as follows

$$l_{1-m} l_{m} = \lambda_{m} .$$
(123)

The function $\varphi_{1-m}(x)$ has the conformal weight equal to $1 - m$. It is important that there is another relation between $\varphi_{m}$ and $\varphi_{1-m}$

$$\varphi_{1-m} = R_{m}(x, p) \varphi_{m} ,$$
(124)
where
\[ R_m(x, p) = X^{-m+1} p^{1-2m} X^{-m}, \quad R_{1-m}(x, p) = (R_m(x, p))^{-1}. \]

This follows from the fact that, for the Möbius group, the complex-conjugated representations \( O_{m,\tilde{m}}(\mathbb{P}^0) \) and \( O_{1-m,1-\tilde{m}}(\mathbb{P}^0) \) are linearly dependent. To obtain the above expression for \( R_m(x, p) \) one should use for the Odderon wave function \( f_{m,\tilde{m}}(\rho^1, \rho^2, \rho^3; \rho^0) \) the conformal anzatz and the property of holomorphic factorization. The transformation

\[ \varphi'(\vec{x}) = U \varphi(\vec{x}), \quad U = R_m(x, p) R_{\tilde{m}}(x^*, p^*) \]

is unitary for our choice of the norm:

\[ \|\varphi'\|^2 = \|\varphi\|^2. \]

Because

\[ \int \frac{d^2x}{|x(1-x)|^2} (\varphi(\vec{x}))^* A_m(x) A_{\tilde{m}}(x^*) \varphi(\vec{x}) = \|A_m \varphi\|^2, \]

the eigenvalue of the operator \( A_{\tilde{m}}(x^*) \) is complex conjugated to \( \lambda_m \):

\[ A_m(x) \varphi(\vec{x}) = \lambda_m \varphi(\vec{x}), \quad A_{\tilde{m}}(x^*) \varphi(\vec{x}) = \lambda_m^* \varphi(\vec{x}). \]

Note that due to its Möbius covariance \( \varphi(\vec{x}) \) is the sum of products of the eigenfunctions having opposite signs of their eigenvalues \( \lambda \):

\[ \varphi(\vec{x}) = \sum C_{ik} \left( \varphi_{m,\lambda}^i(x) \varphi_{m,-\lambda}^k(x^*) + \varphi_{m,-\lambda}^i(x) \varphi_{m,\lambda}^k(x^*) \right). \]

Since under the simultaneous transformations \( x \leftrightarrow x^* \) and \( m \leftrightarrow \tilde{m} \) this function should be symmetric for fixed \( \lambda \), we conclude, that eigenvalues satisfy one of two relations

\[ \lambda_m = \pm(\lambda_m)^* \]

and therefore \( \lambda_m \) can be purely real or imaginary. It turns out, that \( \lambda_m \) is imaginary as a consequence of the modular invariance [9].

One can veryfy the validity of the following relation

\[ R_m(x, p) a_{1-m} a_m = a_m a_{1-m} R_m(x, p), \]

if the following identity is used:

\[ X^{-m} a_{1-m} a_m X^m = x^2(1-x)^2 p^3 + 2(1+m) x(1-x) (i p (1-2x) p - p) + m(1+m) \left( x(1-x) p - (1-2x)^2 p + i m (1-2x) \right). \]

In particular this relation means that two eigenvalues of \( A_m \) coincide

\[ \lambda_m = \lambda_{1-m} \]

for the eigenfunctions \( \varphi_m \) and \( \varphi_{1-m} \) linearly related by \( R_m(x, p) \). Let us introduce the operators

\[ S_m(x, p) = R_{1-m}(x, p) a_m, \quad S_{1-m}^t(x, p) = a_{1-m} R_m(x, p), \]
leaving the function \( \varphi_m \) in the same space. Due to the above formulas these operators commute with one another:

\[
A_m = S^t_{1-m}(x, p) S_m(x, p) = S_m(x, p) S^t_{1-m}(x, p). 
\] (134)

Therefore we obtain two duality equations for the wave function whose conformal weight is equal to \( m \):

\[
S_m(x, p) \varphi_m = L^{(m)}_1 \varphi_m, \quad S^t_{1-m}(x, p) \varphi_m = L^{(m)}_2 \varphi_m. 
\] (135)

As a consequence of the relation

\[
L^{(m)}_1 L^{(m)}_2 = \lambda 
\] (136)

between the eigenvalues of \( S_m \) and \( S^t_{1-m} \), the second equation follows from the first one.

In the particular case \( m = 1/2 \), the equation for the holomorphic factors \( \varphi(x) \) looks especially simple:

\[
x(1 - x)p^{3/2} \varphi_{\pm \sqrt{\lambda}}(x) = \pm \sqrt{\lambda} \varphi_{\pm \sqrt{\lambda}}(x) 
\] (137)

and can be reduced in the \( p \)-representation to the Schrödinger equation with the potential \( V(p) = p^{-3/2} \). For \( m = \tilde{m} = 1/2 \) the total odderon wave function \( \varphi(x, x^*) \), symmetric under the above canonical transformation, is a solution of the equation:

\[
|x(1 - x)|^2 |p|^{3} \varphi(x, x^*) = |\lambda| \varphi(x, x^*), 
\] (138)

where in accordance with its hermicity properties the eigenvalue of the operator \( \sqrt{\mathcal{A}} \) for the anti-holomorphic factor \( \varphi(x^*) \) is taken to be equal to \( \lambda^* \).

4 Single-valuedness condition

There are three independent solutions \( \varphi_i^{(m)}(x, \lambda) \) of the third-order ordinary differential equation

\[
a_{1-m} a_m \varphi = -ix(1 - x) \left( x(1 - x) \partial^2 + (2 - m)(1 - 2x) \partial - 1 + m \right) \partial \varphi = \lambda \varphi 
\] (139)

for each eigenvalue \( \lambda \). In the region \( x \to 0 \) they can be chosen as follows

\[
\varphi_r^{(m)}(x, \lambda) = \sum_{k=1}^{\infty} d^{(m)}_k(\lambda) x^k, \quad d^{(m)}_1(\lambda) = 1. 
\] (140)

\[
\varphi_s^{(m)}(x, \lambda) = \sum_{k=0}^{\infty} a^{(m)}_k(\lambda) x^k + \varphi_r^{(m)}(x, \lambda) \ln x, \quad a^{(m)}_1 = 0, 
\] (141)

\[
\varphi_f^{(m)}(x, \lambda) = \sum_{k=0}^{\infty} c^{(m)}_{k+m}(\lambda) x^{k+m}, \quad c^{(m)}_m(\lambda) = 1. 
\] (142)

The appearance of \( \ln x \) in \( \varphi_s^{(m)}(x, \lambda) \) is related with the degeneracy of the differential equation in the small-\( x \) region. There is an ambiguity in the definition of \( \varphi_s^{(m)}(x, \lambda) \) because one can
add to it the function \( \varphi^{(m)}_r(x, \lambda) \) with an arbitrary coefficient. We have chosen \( a^m_0 = 0 \) to remove this uncertainty.

Taking into account that for \( \rho_1 \to \rho_2 \) the operator product expansion is applicable, the functions \( \varphi^{(m)}_i(x, \lambda) \) can be considered as contributions of the holomorphic composite operators \( O^{(M)}_i(\rho_1) \) with the conformal weights \( M = 0, m \) and 1 for \( i = s, f \) and \( r \) correspondingly. In this interpretation the above degeneracy is related with the existence of the conserved vector current (for \( m = 1/2 \) there is also a conserved fermion current).

Due to the above differential equation the coefficients \( a, c \) and \( d \) satisfy the following recurrence relations

\[
i\lambda a^{(m)}_k = \left( a^{(m)}_{k+1} + d^{(m)}_{k+1} \frac{d}{dk} \right) k(k+1)(k+1-m) - \left( a^{(m)}_k + d^{(m)}_k \frac{d}{dk} \right) k(k-m)(2k-m) \\
+ \left( a^{(m)}_{k-1} + d^{(m)}_{k-1} \frac{d}{dk} \right) (k-1)(k-m)(k-1-m),
\]

\[
i\lambda c^{(m)}_{k+m} = (k+m)(k+m+1)(k+1)c^{(m)}_{k+m+1} - (k+m)k(2k+m)c^{(m)}_{k+m} \\
+ (k+m-1)k(k-1)c^{(m)}_{k+m-1},
\]

\[
i\lambda d^{(m)}_k = k(k+1)(k+1-m)d^{(m)}_{k+1} - k(k-m)(2k-m)d^{(m)}_k \\
+ (k-1)(k-1-m)d^{(m)}_{k-1}.
\]

(143) (144)

In particular, from the equation for \( a^{(m)}_k \) at \( k = 0 \), since \( d^{(m)}_1 = 1 \), we obtain

\[
a^{(m)}_0 = \frac{i}{\lambda} (m-1).
\]

(145)

The introduced functions have simple analytic properties in the vicinity of the point \( x = 0 \). In particular, \( \varphi^{(m)}_r(x, \lambda) \) is regular here and is transformed under the modular transformation

\[
x \to x' = -x/(1-x)
\]

(146)

as

\[
\varphi^{(m)}_r(x', \lambda) = -(1-x)^m \varphi^{(m)}_r(x, -\lambda).
\]

(147)

The functions \( \varphi^{(m)}_s(x, \lambda) \) and \( \varphi^{(m)}_f(x, \lambda) \) have singularities at \( x = 0 \), which leads to different results for their analytic continuations to negative values of \( x \):

\[
\varphi^{(m)}_s(x', \lambda) = -(1-x)^m \left( \varphi^{(m)}_s(x, -\lambda) \pm i\pi \varphi^{(m)}_r(x, -\lambda) \right),
\]

\[
\varphi^{(m)}_f(x', \lambda) = \exp(\pm i\pi m) (1-x)^m \varphi^{(m)}_f(x, -\lambda).
\]

(148)

Therefore, from the Bose symmetry of the Odderon wave function

\[
f_{m, \tilde{m}}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3; \tilde{p}_0^*) = \left( \frac{\rho_{23}}{\rho_{20} \rho_{30}} \right)^m \left( \frac{\rho_{23}^*}{\rho_{20}^* \rho_{30}^*} \right)^{\tilde{m}} \varphi_{m, \tilde{m}}(x, x^*),
\]

(149)
combined with the single-valuedness condition near \( \vec{x} = 0 \), we obtain for the total wave function the following representation:

\[
\varphi_{m,\tilde{m}}(x, x^{*}) = \varphi_{f}(x, \lambda) \varphi_{f}^{*}(x^{*}, \lambda^{*}) + c_{1} \left( \varphi_{s}(x, \lambda) \varphi_{s}^{*}(x^{*}, \lambda^{*}) + \varphi_{r}(x, \lambda) \varphi_{r}^{*}(x^{*}, \lambda^{*}) \right) \\
+ c_{2} \varphi_{r}(x, \lambda) \varphi_{r}^{*}(x^{*}, \lambda^{*}) + (\lambda \to -\lambda) \tag{150}
\]

The complex coefficients \( c_{1}, c_{2} \) and the eigenvalues \( \lambda \) are fixed from the conditions of the single-valuedness of \( f_{m,\tilde{m}}(\vec{p}_{1}, \vec{p}_{2}, \vec{p}_{3}; \vec{p}_{0}) \) at \( \vec{p}_{3} = \vec{p}_{i}^{*} (i = 1, 2) \) and the Bose symmetry [9]. It is sufficient to require its invariance under the transformation \( \vec{p}_{2}^{*} \leftrightarrow \vec{p}_{3}^{*} \) corresponding to the symmetry of \( \varphi_{m,\tilde{m}}(x, x^{*}) \)

\[
\varphi_{m,\tilde{m}}(x, x^{*}) = \varphi_{m,\tilde{m}}(1 - x, 1 - x^{*}). \tag{151}
\]

For this purpose, one should analytically continue the functions \( \varphi_{i}(x) \) in the region near the point \( x = 1 \) and calculate from the differential equation the monodromy matrix \( C_{rk}^{(m)} \) defined by the relations

\[
\varphi_{r}^{(m)}(x, \lambda) = \sum_{k} C_{rk}^{(m)} \varphi_{k}^{(m)}(1 - x, -\lambda), \tag{152}
\]

\[
\varphi_{r}^{*}(x^{*}, \lambda^{*}) = \sum_{k} C_{rk}^{(m)} \varphi_{k}^{*}(1 - x^{*}, -\lambda^{*}). \tag{153}
\]

Owing to the single-valuedness condition and the Bose symmetry of \( f_{m,\tilde{m}} \) we obtain a set of linear equations for parameters \( c_{1} \) and \( c_{2} \) with coefficients expressed in terms of \( C_{rk}^{(m)} \) and \( C_{rk}^{(\tilde{m})} \). The spectrum of \( \lambda \) is fixed by the condition of selfconsistency of these equations [9].

To derive the relations among parameters \( c_{1}, c_{2} \) and \( \lambda \) following from the duality symmetry, it is convenient to introduce the operators

\[
S_{m,\tilde{m}} = S_{m}(x, p) S_{\tilde{m}}(x^{*}, p^{*}), \quad S_{m,\tilde{m}}^{+} = S_{1-m}^{t}(x, p) S_{1-\tilde{m}}^{t}(x^{*}, p^{*}) \tag{154}
\]

where \( S_{m}(x, p) \) and \( S_{1-m}^{t}(x, p) \) were defined in the previous section. These operators are hermitially conjugated each to another and have the property:

\[
|A_{m}(x)|^{2} = S_{m,\tilde{m}}^{+} S_{m,\tilde{m}} = S_{m,\tilde{m}} S_{m,\tilde{m}}^{+}. \tag{155}
\]

In particular it means, that they have common eigenfunctions

\[
S_{m,\tilde{m}} \varphi_{m,\tilde{m}}(x, x^{*}) = \frac{|\lambda|^{2}}{c_{1}} e^{i\theta} \varphi_{m,\tilde{m}}(x, x^{*}), \quad S_{m,\tilde{m}}^{+} \varphi_{m,\tilde{m}}(x, x^{*}) = c_{1} e^{-i\theta} \varphi_{m,\tilde{m}}(x, x^{*}) \tag{156}
\]

where

\[
e^{i\theta} = (-i)^{m-\tilde{m}} \frac{\Gamma(m) \Gamma(1 + m) \Gamma(1 + \tilde{m})}{\Gamma(1 - m) \Gamma(2 - m) \Gamma(2 - \tilde{m})}. \tag{157}
\]

The eigenvalues are obtained from the small-\( x \) asymptotics of these equations.

Because the operators \( S_{m,\tilde{m}} \) and \( S_{m,\tilde{m}}^{+} \) are hermitially conjugated, we have

\[
|c_{1}| = |\lambda|. \tag{158}
\]

18
In the particular case \( m = \tilde{m} = 1/2 \), where \( S_{1/2,1/2} = |S_{1/2}|^2 \), the coefficient \( c_1 \) is positive:

\[
    c_1 = |\lambda| .
\]

Another relation can be derived if we shall take into account, that the complex conjugated representations \( \varphi_{m,\tilde{m}} \) and \( \varphi_{1-b,1-b} \) of the Möbius group are related by the unitary operator \( U = R_m(x,p)R_{\tilde{m}}(x^*,p^*) \), defined in the previous section:

\[
    e^{i\gamma}(\varphi_{m,\tilde{m}})^* = U \varphi_{m,\tilde{m}},
\]

where \( e^{i\gamma} \) is an eigenvalue of this operator. By calculating the right hand side of this equation at \( x \to 0 \), we obtain

\[
    e^{i\gamma} = (-1)^{m-\tilde{m}} \frac{\Gamma(2-\tilde{m})}{\Gamma(1+\tilde{m})} \frac{\Gamma(2-m)}{\Gamma(1+m)} \frac{c_1}{c_1}, \quad \text{Im } \frac{c_2}{c_1} = \text{Im } (m^{-1} + \tilde{m}^{-1}).
\]  

One can verify from the numerical results of ref. [9] that both relations for \( c_1 \) and \( c_2 \) are fulfilled. For example, we have for the ground-state eigenfunction with \( m = \tilde{m} = 1/2 \):

\[
    i \lambda = 0.205257506, \quad c_1 = 0.205257506
\]

and for one of the excited states with \( m = \tilde{m} = 1/2 + i 3/10 \):

\[
    i \lambda = 0.247227544, \quad c_1 = 0.247186043 - i 0.004529717, \quad c_2 = -1.156524786 - i 0.415163678, \quad |c_1| = |\lambda|, \quad \text{Im } \frac{c_2}{c_1} = 2 \text{Im } (1/2 + i 3/10)^{-1}.
\]

After the Fourier transformation of \( \varphi_{m,\tilde{m}}(\mp^*) \) to the momentum space \( \mp^* \) the regular terms near the points \( \mp^* = 0 \) and \( \mp^* = 1 \) do not give any contribution to its asymptotic behaviour at \( \mp^* \to \infty \). The requirement of the holomorphic factorization and single-valuedness of the wave function in the momentum space leads to the quantization of \( \lambda \).

We can obtain from the duality equation and reality condition also the representations for the coefficients \( c_{1,2} \) in terms of integrals from \( \varphi_{m,\tilde{m}}(x,x^*) \) over the fundamental region of the modular group, where the expansion in \( x \) is convergent. These relations allow one to calculate the coefficients \( c_{1,2} \) without using the single-valuedness condition.

## 5 Hamiltonian and integrals of motion

The holomorphic Hamiltonian \( \hat{h} \) for the compound state of \( n \) Reggeons for \( N_c \to \infty \) commutes with the transfer matrix \( \hat{T}(u) \) owing to the following relation for \( \hat{h}_{k,k+1} \):

\[
    [h_{k,k+1}, T(u)] = -i \text{tr} (L_1(u) \ldots L_{k-1}(u) (L_k(u) - L_{k+1}(u)) L_{k+2}(u) \ldots L_n(u)).
\]

It can be considered as a linear equation for \( h_{k,k+1} \). The formal solution of this equation can be written as

\[
    h_{k,k+1} = \lim_{t \to \infty} \left( i \int_0^t dt' \exp(i T(u) t') [h_{k,k+1}, T(u)] \exp(-i T(u) t') + h_{k,k+1}(t) \right),
\]
where \( h_{k,k+1}(t) \) is the time-dependent operator

\[
h_{k,k+1}(t) = \exp(i T(u) t) \ h_{k,k+1} \ \exp(-i T(u) t) .
\]  

(166)

Since the integral term is cancelled in the sum of \( h_{k,k+1} \), we can substitute

\[
h_{k,k+1} \rightarrow h_{k,k+1}(t).
\]  

(167)

At \( t \rightarrow \infty \) as a result of rapid oscillations of off-diagonal matrix elements, each pair Hamiltonian is diagonalized in the representation, where the transfer matrix is diagonal, and therefore it is a function of the integrals of motion \( \hat{q}_k \):

\[
h_{k,k+1}(\infty) = f_{k,k+1}(\hat{q}_2, \hat{q}_3, ... \hat{q}_n) .
\]  

(168)

Its dependence from the spectral parameter \( u \) disappears in this limit and the total Hamiltonian is

\[
h = h(\hat{q}_2, \hat{q}_3, ... \hat{q}_n) = \sum_{k=1}^{n} f_{k,k+1}(\hat{q}_2, \hat{q}_3, ... \hat{q}_n).
\]  

(169)

All operators \( O(t) \) satisfy the Heisenberg equations

\[
- i \ \frac{d}{dt} O(t) = [T(u) , O(t)]
\]  

(170)

with certain initial conditions. In the case of the pair Hamiltonian the initial conditions are

\[
h_{k,k+1}(0) = \psi(\hat{m}_{k,k+1}) + \psi(1 - \hat{m}_{k,k+1}) - 2\psi(1) ,
\]  

(171)

where the quantities \( \hat{m}_{k,k+1} \) are related to the pair Casimir operators as

\[
\hat{m}_{k,k+1}(\hat{m}_{k,k+1} - 1) = M_{k,k+1}^2 = -\rho_{k,k+1}^2 \partial_k \partial_{k+1} .
\]  

(172)

In the case of the Odderon, \( h \) does not depend on time if \( h_{k,k+1}(t) \) is determined as

\[
h_{k,k+1}(t) = e^{itA} h_{k,k+1} e^{-itA} ,
\]  

(173)

and \( h_{k,k+1}(\infty) \) is a function of the total conformal momentum \( \hat{M}^2 = \hat{m}(\hat{m} - 1) \) and of the integral of motion \( q_3 = A \), which can be written as follows:

\[
A = \frac{i^3}{2} \left[ M_{12}^2 , M_{13}^2 \right] = \frac{i^3}{2} \left[ M_{23}^2 , M_{12}^2 \right] = \frac{i^3}{2} \left[ M_{13}^2 , M_{23}^2 \right] .
\]  

(174)

Using these formulas and the following relations among the Möbius group generators \( \hat{M}_r \)

\[
M_{ir}^2 - M_{kr}^2 = 2 \left( \hat{M}_i - \hat{M}_k , \hat{M}_r \right) ,
\]  

(175)

\[
[h_{ik}, [M_{ik}^2 , \hat{M}_i - \hat{M}_k]] = 4 \left( \hat{M}_i - \hat{M}_k \right) ,
\]  

(176)

we can verify the commutation relations
\[ i[h_{12}, A] = M_{13}^2 - M_{23}^2, \quad i[h_{13}, A] = M_{23}^2 - M_{12}^2, \quad i[h_{23}, A] = M_{12}^2 - M_{23}^2, \quad (177) \]
from which it is obvious that \( A \) commutes with \( h \).

In a general case of \( n \) reggeized gluons, one can use the Clebsch-Gordan approach, based on the construction of common eigenfunctions of the total momentum \( \hat{M} \) and a set \( \{ \hat{M}_k \} \) of the commuting sub-momenta, to find all operators \( M_{k,k+1}^2 \) in the corresponding representation. However to calculate \( h \) we should perform an unitary transformation to the representation, where \( T(u) \) is diagonal, because in this case for \( t \to \infty \) the off-diagonal matrix elements of \( M_{k,k+1}^2 \) disappear and their diagonal elements depend only on \( q_r \):

\[
\begin{align*}
  f_{k,k+1}(q_2, q_3, \ldots q_n) &= \langle q_2, \ldots q_n | h_{k,k+1} | q_2, \ldots q_n \rangle, \\
  \hat{q}_k | q_2, \ldots q_n \rangle &= q_k | q_2, \ldots q_n \rangle. \quad (178)
\end{align*}
\]

Let us consider, for example, the interaction between particles 1 and 2. The transfer matrix, which should be diagonalized, can be written as follows

\[
T(u) = \left( u^2 - \frac{1}{2} \vec{L}^2 \right) d_{3\ldots n}(u) + \left( i u \vec{L} - \frac{1}{4} \left[ \vec{L}^2, \vec{N} \right] \right) \hat{d}_{3\ldots n}(u), \quad (179)
\]

where the differential operators \( d_{3\ldots n}(u) \) and \( \hat{d}_{3\ldots n}(u) \) are independent of \( \vec{p}_1^0 \) and \( \vec{p}_2^n \). They are related to the monodromy matrix \( t_{3\ldots n}(u) \) for particles 3, 4, ..., \( n \) as follows

\[
d_{3\ldots n}(u) = tr t_{3\ldots n}(u), \quad \hat{d}_{3\ldots n}(u) = tr(\hat{\sigma} t_{3\ldots n}(u)), \quad t_{3\ldots n}(u) = L_3(u)\ldots L_n(u) \quad (180)
\]
and the matrix \( t_{3\ldots n}(u) \) satisfies the Yang-Baxter equations with a hidden Lorentz symmetry.

The operators \( \vec{L} \) and \( \vec{N} \) are constructed in terms of the Möbius group generators of particles 1 and 2:

\[
\vec{L} = \vec{M}_1 + \vec{M}_2, \quad \vec{N} = \vec{M}_1 - \vec{M}_2, \quad M_k^+ = \rho_k \partial_k, \quad M_k^- = -\rho_k^2 \partial_k, \quad M_k^0 = \partial_k. \quad (181)
\]
They have the commutation relations, corresponding to the Lorentz algebra:

\[
\begin{align*}
  \left[ L^z, L^\pm \right] &= \pm L^\pm, \quad \left[ L^+, L^- \right] = 2L^\pm, \quad \left[ L^z, N^\pm \right] = \pm N^\pm, \\
  \left[ L^+, N^- \right] &= 2N^z, \quad \left[ N^z, N^\pm \right] = \pm L^\pm, \quad \left[ N^+, N^- \right] = 2L^z. \quad (182)
\end{align*}
\]

Let us introduce the Polyakov basis for the wave function of the composite state of two gluons with the conformal weight \( M \):

\[
| \rho_{0\gamma}, M \rangle = \left( \frac{\rho_{12}}{\rho_{10} \rho_{20}} \right)^M. \quad (183)
\]
Here \( \rho_{0\gamma} \) enumerates the components of the infinite-dimensional irreducible representation of the conformal group.
One can verify that the representation of the generators \( \hat{L} \) and \( \hat{N} \) in this basis is given by

\[
M^{\pm} | \rho_0, M \rangle = \left( \rho_1 \partial_1 + \rho_2 \partial_2 \right) | \rho_0, M \rangle = -(\rho_0 \partial_0 + M) | \rho_0, M \rangle, \\
M^{-} | \rho_0, M \rangle = (\partial_1 + \partial_2) | \rho_0, M \rangle = -\partial_0 | \rho_0, M \rangle
\]

and

\[
\left( N^{\pm} - \rho_0 N^{-} \right) | \rho_0, M \rangle = \frac{M}{M - 1} \partial_0 | \rho_0, M - 1 \rangle, \\
\left( N^{+} + 2\rho_0 N^{\pm} - \rho_0^2 N^{-} \right) | \rho_0, M \rangle = -2M | \rho_0, M - 1 \rangle, \\
\frac{2M - 1}{M(M - 1)} N^{-} | \rho_0, M \rangle = | \rho_0, M + 1 \rangle + \frac{1}{(M - 1)^2} \partial_0^2 | \rho_0, M - 1 \rangle.
\]

Note that there is a simple relation among the generators, provided that they act on the state \( | \rho_0, M \rangle \):

\[
[N^z - \rho_0 N^-, N^+ + 2\rho_0 N^z - \rho_0^2 N^-] = M^+ + 2\rho_0 M^z - \rho_0^2 M^- = 0.
\]

The eigenfunction of the transfer matrix \( T(u) \) can be written as a superposition of the states \( | \rho_0, M \rangle \) with various values of \( \rho_0 \) and \( M \):

\[
f_m(\rho_1, \rho_2, ..., \rho_n; \rho_0) = \sum_M \int d\rho_0 | \rho_0, M \rangle \ f_{m,M}(\rho_0, \rho_3, ..., \rho_n; \rho_0), \quad (187)
\]

where \( m \) is the conformal weight of the composite state. The function \( f_{m,M}(\rho_0, \rho_3, ..., \rho_n; \rho_0) \) in accordance with the Möbius symmetry, has the form

\[
f_{m,M}(\rho_0, \rho_3, ..., \rho_n; \rho_0) = (\rho_0^{-1})^{m+M-1} \prod_{r=3}^{n} \left( \frac{\rho_r}{\rho_0} \right)^{-\frac{m+M}{2}} \psi(x_1, x_2, ..., x_{n-3}), \quad (188)
\]

where \( x_r \) are independent anharmonic ratios constructed from the coordinates \( \rho_0, \rho_3, ..., \rho_n \).

Because of its Möbius invariance, the transfer matrix \( T(u) \) after acting on \( f_m(\rho_1, ..., \rho_n; \rho_0) \) gives again a superposition of the states \( | \rho_0, M \rangle \), but with the coefficients which are linear combinations of \( f_{m,M} \) and \( f_{m,M \pm 1} \). Therefore for its eigen function the coefficients satisfy some recurrence relations, and the problem of the diagonalization of the transfer matrix \( T(u) \) is reduced to the solution of these recurrence relations. For \( n \geq 3 \) in the sub-channel \( \rho_{1,2} \) the recurrence relations depend on matrix elements of the operators \( \overrightarrow{d}_{3...n}(u) \) and \( d_{3...n}(u) \) between the wave functions \( f_{m,M}(\rho_0, \rho_3, ..., \rho_n; \rho_0) \) which should be chosen in such a way, to provide the property of \( f_m(\rho_1, \rho_2, ..., \rho_n; \rho_0) \) to be a representation of the cyclic group of transformations \( i \rightarrow i + 1 \).

In the appendix we consider these relations in the first non-trivial case \( n = 3 \). In the next section the relation between the Odderon Hamiltonian and its integral of motion \( A \) is discussed from another point of view.
6 Odderon Hamiltonian in the normal order

Let us write down the pair Hamiltonian as follows \[4\]

\[ h_{12} = \log(\rho_1^2 \partial_1) + \log(\rho_2^2 \partial_2) - 2 \log(\rho_{12}) - 2 \psi(1). \] (189)

This representation allows us to present the total Hamiltonian for \( n \) reggeized gluons in the form invariant under the M"obius transformations

\[ h = \sum_{k=1}^{n} \left( \log \left( \frac{\rho_{k+2,0} \rho_{k,k+1}^2}{\rho_{k+1,0} \rho_{k+1,k+2}} \partial_k \right) + \log \left( \frac{\rho_{k-2,0} \rho_{k,k-1}^2}{\rho_{k-1,0} \rho_{k-1,k-2}} \partial_k \right) - 2 \psi(1) \right), \] (190)

where \( \rho_0 \) is the coordinate of the composite state.

We consider below in more detail the Odderon. Using for its wave function the conformal anzatz

\[ f_m(\rho_1, \rho_2, \rho_3; \rho_0) = \left( \frac{\rho_{23}}{\rho_{20}\rho_{30}} \right)^m \varphi_m(x), \quad x = \frac{\rho_{12}\rho_{30}}{\rho_{10}\rho_{32}}, \] (191)

one can obtain the following Hamiltonian for the function \( \varphi_m(x) \) in the space of the anharmonic ratio \( x \) \[4\]

\[ h = 6\gamma + \log(x^2\partial) + \log((1-x)^2\partial) + \log \left( \frac{x^2}{1-x}((1-x)\partial + m) \right) + \]

\[ \log \left( \frac{1}{1-x}((1-x)\partial + m) \right) + \log \left( \frac{(1-x)^2}{x}(x\partial - m) \right) + \log \left( \frac{1}{x}(x\partial - m) \right), \] (192)

It is convenient to introduce the logarithmic derivative \( P \equiv x\partial \) as a new momentum. Using the relations \[4\]:

\[ \log(x^2\partial) = \log(x) + \psi(1-P), \quad \log(\partial) = -\log(x) + \psi(-P), \]

\[ \log(x^2\partial) = \log(\partial) + 2\log(x) - \frac{1}{P}, \]

\[ ((1-x)\partial + m) = (1-x)^{1+m} \partial (1-x)^{-m}, \quad x\partial - m = x^{1+m} \partial x^{-m}, \] (193)

one can transform this Hamiltonian to the normal form:

\[ \frac{h}{2} = -\log(x) + \psi(1 - P) + \psi(-P) + \psi(m - P) - 3\psi(1) + \sum_{k=1}^{\infty} x^k f_k(P), \] (194)

where

\[ f_k(P) = -\frac{2}{k} + \frac{1}{2} \left( \frac{1}{P + k - m} + \frac{1}{P + k} \right) + \sum_{t=0}^{k} \frac{c_t(k)}{P + t}. \] (195)
Here
\[ c_t(k) = \frac{(-1)^{k-t} \Gamma(m+t) ((t-k)(m+t)+mk/2)}{k \Gamma(m-k+t+1) \Gamma(t+1) \Gamma(k-t+1)}. \] (196)

On the other hand the differential operators \( a_m \) and \( a_{1-m} \) can be written in terms of the quantities \( P \) and \( x \) as follows
\[
\begin{align*}
a_m & = i^{-1-m}x^{-m}(1-x) \frac{\Gamma(m-P+1)}{\Gamma(-P)}, \\
a_{1-m} & = i^{-2+m}x^{-1+m}(1-x) \frac{\Gamma(-P-m+2)}{\Gamma(-P)}.
\end{align*}
\] (197) (198)

Using the above representation for \( h \) and the following expression for the integral of motion:
\[
B = i a_{1-m} a_m = \frac{(1-x)}{x} ((1-x)P - 1 - x + xm) P (P - m),
\] (199)
one can verify their commutativity
\[
[h,B] = 0.
\] (200)

Therefore \( h \) is a function of \( B \).

In particular for large \( B \) this function should have the form:
\[
\frac{h}{2} = \log(B) + 3\gamma + \sum_{r=1}^{\infty} \frac{c_r}{B^{2r}}.
\] (201)

The first two terms of this asymptotic expansion were calculated in ref. \[4\]. The series is constructed in inverse powers of \( B^2 \), because \( h \) should be invariant under all modular transformations, including the inversion \( x \rightarrow 1/x \) under which \( B \) changes its sign. The same functional relation should be valid for the eigenvalues \( \varepsilon/2 \) and \( \mu = i\lambda \) of these operators:
\[
\frac{\varepsilon}{2} = \log(\mu) + 3\gamma + \sum_{r=1}^{\infty} \frac{c_r}{\mu^{2r}}.
\] (202)

For large \( \mu \) it is convenient to consider the corresponding eigenvalue equations in the \( P \) representation, where \( x \) is the shift operator
\[ x = \exp(-\frac{d}{dP}), \] (203)

after extracting from eigenfunctions of \( B \) and \( h \) the common factor
\[
\varphi_m(P) = \Gamma(-P) \Gamma(1-P) \Gamma(m-P) \exp(i\pi P) \Phi_m(P).
\] (204)

The function \( \Phi_m(P) \) can be expanded in series over \( 1/\mu \)
\[
\Phi_m(P) = \sum_{n=0}^{\infty} \mu^{-n} \Phi_{m}^{n}(P), \Phi_{m}^{0}(P) = 1,
\] (205)
where the coefficients $\Phi^n_m(P)$ turn out to be the polynomials of order $4n$ satisfying the recurrence relation:

$$
\Phi^n_m(P) = \sum_{k=1}^{P} (k-1)(k-1-m) \left((k-m)\Phi^{n-1}_m(k-1) + (k-2)\Phi^{n-1}_{1-m}(k-1-m)\right)
- \frac{1}{2} \sum_{k=1}^{m} (k-1)(k-1-m) \left((k-m)\Phi^{n-1}_m(k-1) + (k-2)\Phi^{n-1}_{1-m}(k-1-m)\right),
$$

valid due to the duality equation written below for a definite choice of the phase of $\Phi_m(P)$

$$
x^{-m}(1-x)(P-m)(P-m+1) \Phi_m(P) = \mu^{m} \Phi_{1-m}(P)
$$

with the use of the substitution $x \mu \to x$.

Note that the summation constants $\Phi^0_m(0)$ in the above recurrence relation have the anti-symmetry property

$$
\Phi^0_n(0) = -\Phi^0_{1-m}(0),
$$

which guarantees the fulfilment of the relation

$$
\Phi^0_n(m) = \Phi^0_{1-m}(0)
$$

being a consequence of the duality relation

$$
\Phi^0_n(P) = \Phi^0_{1-m}(P-m) + (P-1)(P-m)(P-m-1)\Phi^0_{1-m}(P-1).
$$

The symmetric part of $\Phi^0_m(0)$ under the substitution $m \leftrightarrow 1-m$ would simply modify the normalization constant for $\Phi_m(P)$.

The energy can be expressed in terms of $\Phi_m(P)$ as follows:

$$
\frac{\varepsilon}{2} = \log(\mu) + 3\gamma + \frac{\partial}{\partial P} \log \Phi_m(P)
$$

$$
+ (\Phi_m(P))^{-1} \sum_{k=1}^{\infty} \mu^{-k} f_k(P-k) \Phi_m(P-k) \prod_{r=1}^{k} (P-r)(P-r+1)(P-r-m+1)
$$

and it should not depend on $P$ due to the commutativity of $h$ and $B$.

By solving the recurrence relations for $\Phi^0_n(P)$ and putting the result in the above expression, we obtain the following asymptotic expansion for $\varepsilon/2$:

$$
\frac{\varepsilon}{2} = \log(\mu) + 3\gamma + \left(\frac{3}{448} + \frac{13}{120}(m-1/2)^2 - \frac{1}{12}(m-1/2)^4\right) \frac{1}{\mu^2} + 
$$

$$
\left(-\frac{4185}{2050048} - \frac{2151}{49280}(m-1/2)^2 + \ldots\right) \frac{1}{\mu^4} + \left(\frac{965925}{37044224} + \ldots\right) \frac{1}{\mu^6} + \ldots.
$$

This expansion can be used with a certain accuracy even for the smallest eigenvalue $\mu = 0.20526$, corresponding to the ground-state energy $\varepsilon = 0.49434$ [9]. For the first excited state with the same conformal weight $m = 1/2$, where $\varepsilon = 5.16930$ and $\mu = 2.34392$ [9],

\[25\]
the energy can be calculated from the above asymptotic series with a good precision. The analytic approach, developed in this section, should be compared with the method based on the Baxter equation [10].

In the conclusion, we note that the remarkable properties of the Reggeon dynamics are presumably related with supersymmetry. In the continuum limit $n \to \infty$ the above duality transformation coincides with the supersymmetric translation, which is presumably connected with the observation [11], that in this limit the underlying model is a twisted $N = 2$ supersymmetric topological field theory. Additional arguments supporting the supersymmetric nature of the integrability of the reggeon dynamics were given in ref. [12]. Namely, the eigenvalues of the integral kernels in the evolution equations for quasi-partonic operators in the $N = 4$ supersymmetric Yang-Mills theory are proportional to $\psi(j - 1)$, which means that these evolution equations in the multicolour limit are equivalent to the Schrödinger equation for the integrable Heisenberg spin model similar to the one found in the Regge limit [7]. Note that at large $N_c$ the $N = 4$ Yang-Mills theory is guessed to be related with the low-energy asymptotics of a superstring model [13].

Acknowledgements

I want to thank the Universität Hamburg for its hospitality during my stay in Germany, where the basic part of this work was done. I thank G. Altarelli, J. Ellis and other participants of the CERN theory seminar for their interest in my talk. Fruitful discussions with L. Faddeev, A. Neveu, V. Fateev, A. Zamolodchikov, J. Bartels, A. Martin, B. Nicolescu, P. Gauron, E. Antonov, M. Braun, A. Bukhvostov, S. Derkachev, A. Manashov, G. Volkov, R. Kirschner, L. Szymanowski and J. Wosiek were helpful.

Appendix

Here we consider consequences of the conformal weight representation for the Odderon. In this case the total Hamiltonian is

$$h(m, \lambda) = h_{12} + h_{23} + h_{31}.$$

The eigenvalue of $h$ is expressed in terms of its matrix elements:

$$h(m, \lambda) = \sum_{k=1}^{3} \langle m, \lambda | h_{k,k+1} | m, \lambda \rangle,$$

where $| m, \lambda \rangle$ is a normalized eigenfunction of two commuting operators

$$\left(\sum_{k=1}^{3} M_k^{2}\right) \ | m, \lambda \rangle = m(m - 1) \ | m, \lambda \rangle, \ A \ | m, \lambda \rangle = \lambda \ | m, \lambda \rangle.$$
Let us consider for definiteness the interaction in the channel 12, where \( M \) is the pair conformal weight. If one will construct the matrix \( V^\lambda_M(m) \) performing the unitary transformation between the \( M- \) and \( \lambda - \) representations

\[
| m, \lambda \rangle = \sum_M V^\lambda_M(m) | m, M \rangle, \quad M^2_{12} | m, M \rangle = M(M - 1) | m, M \rangle, \quad \sum_M V^\lambda_M V^M_\lambda = \delta^\lambda_M,
\]

then the diagonal matrix elements of the pair Hamiltonian \( h_{12} \) can be calculated as

\[
\langle m, \lambda | h_{12} | m, \lambda \rangle = \sum_M h(M) V^\lambda_M(m) V^M_\lambda(m), \quad h(M) = \psi(M) + \psi(1 - M) + 2\gamma.
\]

We shall derive below the recurrence relations for \( V^\lambda_M \). To begin with, we note that, according to the commutation relations for the Lorentz algebra generators, there are non-vanishing matrix elements of the boost operator \( \vec{N} \) only between the states \( | M \rangle \) and \( | M \pm 1 \rangle \). This is valid also for the matrix elements of the operators \( A \) and \( M^2_{13} - M^2_{23} \) according to the relations

\[
\langle M' | A | M \rangle = \frac{i^3}{4} (M' - M)(M' + M - 1) \langle M' | M^2_{13} - M^2_{23} | M \rangle,
\]

\[
\langle M \pm 1 | M^2_{13} - M^2_{23} | M \rangle = 2M^2_3 \langle M \pm 1 | \vec{N} | M \rangle.
\]

Thus, for the common eigenfunctions \( | m, M \rangle \) of two commuting Casimir operators

\[
\left( \sum_{k=1}^3 M_k^2 \right)^2 | m, M \rangle = m(m - 1) | m, M \rangle
\]

and

\[
M^2_{12} | m, M \rangle = M(M - 1) | m, M \rangle
\]

we have

\[
A | m, M \rangle = \frac{i^3}{2} \left( MC^{+}_{m,M} | m, M + 1 \rangle - (M - 1)C^{-}_{m,M} | m, M - 1 \rangle \right),
\]

where the coefficients \( C^{\pm}_{m,M} \) are defined by the relations

\[
\left( M^2_{13} - M^2_{23} \right) | m, M \rangle = C^{+}_{m,M} | m, M + 1 \rangle + C^{-}_{m,M} | m, M - 1 \rangle.
\]

One can obtain from the above equations the following recurrence relation for the unitary matrix \( V^\lambda_M(m) \):

\[
\lambda V^\lambda_M(m) = \frac{i^3}{2} \left( (M - 1)C^{+}_{m,M-1}V^\lambda_{M-1}(m) - MC^{-}_{m,M+1}V^\lambda_{M+1}(m) \right).
\]

To calculate the matrix elements \( C^{\pm}_{m,M} \) of the operator \( M^2_{13} - M^2_{23} \), we use the above representation of the generators \( \vec{L} \) and \( \vec{N} \) in the Polyakov basis and obtain

\[
\left( M^2_{13} - M^2_{23} \right) | \rho^\nu, M \rangle = \left( 2N^z M^x_3 + N^+ M^x_3 + N^- M^x_3 \right) | \rho^\nu, M \rangle
\]
\[
2M \left( \frac{1}{M - 1} \rho_{30'} \partial_{\rho'} - 1 \right) | \rho_{0'} M - 1 \rangle \partial_3 - \rho_{30'}^2 N^+ | \rho_{0'} M \rangle \partial_3.
\]

Owing to the Möbius invariance, the three-gluon state \(| m, M \rangle\), with the conformal weights \(m\) and \(M\), can be written as a superposition of the Polyakov functions

\[
f_{m,M}(\rho_1, \rho_2, \rho_3; \rho_0) = \int_L d\rho' | \rho_{0'} M \rangle (\rho_{0'}^M)^{-m+M-1} \left( \frac{\rho_{30}}{\rho_{30'}} \right)^{-M-m+1}
\]

with various integration contours \(L\). By integrating the terms in \(M_3^2\) with derivatives of \(| \rho_{0'} M \rangle\) by parts and using the relations

\[
-\rho_{30'}^2 \partial_3 (\rho_{0'}^M)^{-m+M-1} \left( \frac{\rho_{30}}{\rho_{30'}} \right)^{-M-m+1} = -(M + m - 1) (\rho_{0'}^M)^{-m+M} \left( \frac{\rho_{30}}{\rho_{30'}} \right)^{-M-m},
\]

\[
\left( \frac{1}{1 - 2M} \partial_{\rho'}^2 \rho_{30'}^2 - 2 \partial_{\rho'} \rho_{30'} - 2(M - 1) \right) \partial_3 (\rho_{0'}^M)^{-m+M-1} \left( \frac{\rho_{30}}{\rho_{30'}} \right)^{-M-m+1} =
\]

\[-(M + m - 1) \frac{(m - M)(m - M + 1)}{2M - 1} (\rho_{0'}^M)^{-m+M-2} \left( \frac{\rho_{30}}{\rho_{30'}} \right)^{-M-m+2},
\]

one can obtain the recurrence relation for the function \(f_{m,M} = f_{m,M}(\rho_1, \rho_2, \rho_3; \rho_0)\):

\[
\left( M_{13}^2 - M_{23}^2 \right) f_{m,M} = \frac{M(M + m - 1)}{1 - 2M} \left((M - 1) f_{m,M+1} + \frac{(m - M)(m - M + 1)}{M - 1} f_{m,M-1} \right).
\]

Due to its Möbius covariance \(f_{m,M}(\rho_1, \rho_2, \rho_3; \rho_0)\) can be presented in the form

\[
f_{m,M}(\rho_1, \rho_2, \rho_3; \rho_0) = \left( \frac{\rho_{12}^2 \rho_{23}^2 \rho_{31}^2}{\rho_{10}^2 \rho_{20}^2 \rho_{30}^2} \right)^{m/3} f_{m,M}(x),
\]

where \(x\) is the anharmonic ratio

\[
x = \frac{\rho_{12} \rho_{30}}{\rho_{10} \rho_{32}}.
\]

By introducing the new integration variable

\[
x' = \frac{\rho_{12} \rho_{30}'}{\rho_{10} \rho_{32}}
\]

we obtain the following expression for \(f_{m,M}(x)\):

\[
f_{m,M}(x) = (x(1 - x))^{2m/3} \int dx' (1 - x')^{-M} (x' - x)^{-m+M-1} \left( \frac{x'}{x} \right)^{-M-m+1}.
\]

This function satisfies the following differential equation

\[
M_{12}^2(x) f_{m,M}(x) = M(M - 1) f_{m,M}(x)
\]

and the recurrence relation
The pair Casimir operators in the $x$-representation are given below:

$$
M^2_{12}(x) = \frac{x}{1-x} \left( \frac{m}{3} (1 - 2x) + x(1-x)\partial \right) \frac{1}{x} \left( \frac{m}{3} (1 + x) + x(1-x)\partial \right),
$$

$$
M^2_{13}(x) = \frac{1-x}{x} \left( \frac{m}{3} (1 - 2x) + x(1-x)\partial \right) \frac{1}{1-x} \left( \frac{m}{3} (x - 2) + x(1-x)\partial \right),
$$

$$
M^2_{23}(x) = -\frac{1}{x(1-x)} \left( \frac{m}{3} (x - 2) + x(1-x)\partial \right) \left( \frac{m}{3} (1 + x) + x(1-x)\partial \right)
$$

and satisfy the relation:

$$
M^2_{12}(x) + M^2_{13}(x) + M^2_{23}(x) = m(m - 1).
$$

The function $f_{m,M}(x)$ can be expressed for two different choices of the integration contour $L$ through the hypergeometric functions:

$$
f^1_{m,M}(x) = \frac{\Gamma(1-M)}{\Gamma(M)\Gamma(2-2M)} (x(1-x))^{2m/3} x^{1-M-m} F(m + 1 - M, 1 - M; 2 - 2M; x),
$$

$$
f^2_{m,M}(x) = \frac{\Gamma(m+M)}{\Gamma(2M)\Gamma(1+m-M)} (x(1-x))^{2m/3} x^{M-m} F(m + M, M; 2M; x),
$$

where

$$
F(a, b; c; x) = 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \ldots .
$$

Moreover, the above eigenvalue equation for $f_{m,M}(x)$ is equivalent to the hypergeometric equation for $F(a, b; c; x)$:

$$
x(1-x) \frac{d^2}{dx^2} F + (c - (a + b + 1)x) \frac{d}{dx} F - abF = 0,
$$

because

$$
(x(1-x))^{-2m/3} x^{-M+m} M^2_{12}(x) (x(1-x))^{2m/3} x^{M-m} = M(M-1) + x \left( x(1-x) \frac{d^2}{dx^2} + (2M - (m+2M+1)x) \frac{d}{dx} - M(M+m) \right).
$$

In an analogous way, using the relation

$$
(x(1-x))^{-2m/3} x^{-M+m} \left( \frac{M^2_{13}(x) - M^2_{23}(x)}{x(1-x)} \right)^{2m/3} x^{M-m} =
$$
\[
\frac{1}{x}(M(2 - x) + (1 - m)x + (2 - x)x\partial) (M(1 - x) - m + x(1 - x)\partial)
\]

and extracting from it the hypergeometric differential operator with the coefficient chosen to cancel the term with the second derivative, we obtain that the above recurrence relation for \( f_{m,M}(x) \) is equivalent to the following relation for the hypergeometric function \( F_M(x) = F(M, M + m; 2M; x) \):

\[
\left( m(m - 1) + M(2m - 1 - M) + 2(1 - m)(1 - x)\partial + \frac{2M(M - m)}{x} \right) F_M(x)
\]

\[
= (M - m) \left( \frac{(M - 1)(M + m)(M + m - 1)}{2(2M - 1)(2M + 1)} x F_{M+1}(x) + \frac{2M}{x} F_{M-1}(x) \right).
\]

One can verify its validity in the \( k \)-th order of the expansion in \( x \) by taking into account the algebraic identity:

\[
(m + 2k)(m - 1) + M(2m - 1 - M) + 2 \left( 1 - m + \frac{M(M - m)}{k + 1} \right) \left( \frac{M + k)(M + m + k)}{2M + k} \right)
\]

\[
= \frac{(M - m)(M + m - 1)}{2M - 1} \left( \frac{(M - 1)k}{2M + k} + \frac{M(2M + k - 1)}{k + 1} \right).
\]

Let us now return to the problem of finding the recurrence relations for the matrix \( V^\lambda_M(m) \) performing the unitary transformation between the \( M \)- and \( \lambda \)-representations. As earlier, it is convenient to work with the function \( \phi_m(x) \) defined by the relation

\[
f_m(x) = (x(1 - x))^{-m/3} \phi_m(x).
\]

It satisfies the equation

\[
a_{1-m} a_m \phi_m(x) = \lambda \phi_m(x),
\]

where

\[
a_m = x(1 - x)(i\partial)^{m+1}.
\]

One can search the solution of the above eigenvalue equation as the linear combination

\[
\phi_m(x) = \sum_M C_M(m) F^m_M(x)
\]

of the eigenfunctions of the operator \( M^2_{12} \). The coefficients \( C_M(m) \) would coincide with the matrix \( V^\lambda_M(m) \), if the functions \( F^m_M(x) \) would be normalized. However, in accordance with above formulas we define \( F^m_M(x) \) by the following expression:

\[
F^m_M(x) = \frac{\Gamma(2M)\Gamma(1 + m - M)}{\Gamma(m + M)} (x(1 - x))^{m/3} f_{m,M}(x) = x^m(1 - x)^m F(m + M, M; 2M; x).
\]

The transition to the renormalized functions can be easily done.

According to the above presentation the quantities \( F^m_M(x) \) satisfy the following equality:

\[
a_{1-m} a_m F^m_M(x) = \frac{3}{2} M(M - 1)(M - m) \left( \frac{(M + m)(M + m - 1)}{2(2M - 1)(2M + 1)} F^m_{M+1}(x) - 2 F^m_{M-1}(x) \right).
\]
Because the right-hand side of this equality is zero for $M = 0, 1$ and $m$, one can restrict the summation over $M$ in the eigenfunction $\varphi_m(x)$ to two series: $M = r$ and $M = m + r$ with $r = 0, 1, 2, \ldots$. However, in the first case the coefficient in front of $F^m_0(x)$ can not be calculated through the coefficient in front of $F^m_0(x)$. Due to this degeneracy of the equation one should introduce for $r \geq 1$ the more complicated function $\Phi^m_r(x)$:

$$
\Phi^m_r(x) = \lim_{M \to r} \frac{d}{dM} F^m_M(x).
$$

The recurrence relation for these functions can be obtained by differentiating the relation for $F^m_M(x)$. In particular, for $r = 1$ we obtain

$$
a_{1-m} a_m \Phi^m_1(x) = \frac{i^3}{2} (1 - m) \left( \frac{(1 + m) m}{6} F^m_2(x) - 2F^m_0(x) \right).
$$

Thus, in accordance with the small-$x$ behaviour of the eigenfunctions $\varphi(x)$, discussed above, we write the linearly independent solutions in the form:

$$
\varphi^{(m)}_r(x, \lambda) = \sum_{k=1}^{\infty} \Delta^m_k(\lambda) F^m_k(x).
$$

$$
\varphi^{(m)}_s(x, \lambda) = \sum_{k=0}^{\infty} (\alpha^m_k(\lambda) F^m_k(x) + \Delta^m_k(\lambda) \Phi^m_k(x)),
$$

$$
\varphi^{(m)}_f(x, \lambda) = \sum_{k=0}^{\infty} \gamma^m_k(\lambda) F^m_{k+m}(x),
$$

The coefficients $\alpha^m_k$, $\gamma^m_k$ and $\Delta^m_k$ satisfy the recurrence relations:

$$
i \lambda \alpha^m_k(\lambda) = \left( \alpha^m_{k+1}(\lambda) + \beta^m_{k+1}(\lambda) \frac{d}{dk} \right) k(k+1)(k-m+1)
$$

$$
- \frac{1}{4} \left( \alpha^m_{k-1}(\lambda) + \beta^m_{k-1}(\lambda) \frac{d}{dk} \right) (k-1)(k-2)(k-m-1) \frac{(k+m-1)(k+m-2)}{(2k-3)(2k-1)},
$$

$$
i \lambda \Delta^m_k(\lambda) = - \frac{1}{4} (k-1)(k-2)(k-m-1) \frac{(k+m-1)(k+m-2)}{(2k-3)(2k-1)} \Delta^m_{k-1}(\lambda)
$$

$$
+ k(k+1)(k-m+1) \Delta^m_{k+1}(\lambda), \quad \Delta^m_1(\lambda) = 1;
$$

$$
i \lambda \gamma^m_k(\lambda) = - \frac{1}{4} (k+m-1)(k+m-2)(k-1) \frac{(k+2m-1)(k+2m-2)}{(2k+2m-3)(2k+2m-1)} \gamma^m_{k-1}(\lambda)
$$

$$
+ (k+m)(k+m+1)(k+1) \gamma^m_{k+1}(\lambda), \quad \gamma^m_0(\lambda) = 1.
$$

If we compare these relations with the derived above analogous recurrence relations for the coefficients of the expansion of $\varphi^{(m)}_r(x)$ in the series over $x$, it is obvious that the factors in front of the corresponding quantities with the index $k+1$ coincide. Furthermore, the quantities $\alpha_k$, $\gamma_k$ and $\Delta_k$ are absent in the right-hand side of these relations contrary to the previous case, where the similar factors in front of $a_k$, $c_k$ and $d_k$ are non-zero.
References

1. L.N. Lipatov, Sov. J. Nucl. Phys. 23 (1976) 642;
V.S. Fadin, E.A. Kuraev and L.N. Lipatov, Phys. Lett. B60 (1975) 50;
Ya.Ya. Balitsky and L.N. Lipatov, Sov. J. Nucl. Phys. 28 (1978) 822;
L.N. Lipatov, Sov. Phys. JETP 63 (1986) 904;
2. V.S. Fadin, L.N. Lipatov, Phys. Lett. B429 (1998) 127;
3. J.Bartels, Nucl. Phys. B175 (1980) 365;
J. Kwiecinski and M. Prascalowicz, Phys. Lett. B94 (1980) 413;
4. L.N. Lipatov, Phys. Lett. B251 (1990) 284; B309 (1993) 394
5. L.N. Lipatov, hep-th/9311037, Padua preprint DFPD/93/TH/70, unpublished;
6. R.J. Baxter, Exactly Solved Models in Statistical Mechanics, (Academic Press, New York, 1982);
V.O. Tarasov, L.A. Takhtajan and L.D. Faddeev, Theor. Math. Phys. 57 (1983) 163;
7. L.N. Lipatov, Sov. Phys. JETP Lett. 59 (1994) 571;
L.D. Faddeev and G.P. Korchemsky, Phys. Lett. B342 (1995) 311;
8. C. Montonen and D. Olive, Phys. Lett. 72 (1977) 117;
N. Seiberg and E. Witten, Nucl. Phys. 426 (1994) 19;
9. L.N. Lipatov, Recent Advances in Hadronic Physics, Proceedings of the Blois conference (World Scientific, Singapore, 1997);
R.Janik and J. Wosiek, hep-ph/9802100, Crakow preprint TPJU-2/98;
M.A. Braun, hep-ph/9801352, St.Petersburg University preprints;
M.A. Braun, P. Gauron and B. Nicolescu, preprint LPTPE/UP6/10/July 98;
M. Praszalowicz and A. Rostworowski, hep-ph/9805243, Crakow preprint TPJU-8/98;
10. R. Janik and J. Wosiek, Phys. Rev. Lett. 79 (1997) 2935;
11. J. Ellis and N.E. Mavromatos, preprint OUTP-98-51P, hep-ph/9807451;
12. L.N. Lipatov, Perspectives in Hadronic Physics, Proceedings of the ICTP conference (World Scientific, Singapore, 1997).
13. J. Maldacena, Adv. Theor. Math. Phys. 2 : 231 (1998), hep-th/9711200.