Quantum Cosmology via Quantization of Point-Like Lagrangian

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Abstract

The purpose of this paper is to introduce a new way to inquire the quantum cosmology for a certain gravitational theory. Normally, the quantum cosmological model is introduced as the minisuperspace theory which is obtained by reducing the superspace where the Wheeler-DeWitt equation is defined on using the symmetry provided by cosmological principle. Unlike that, the key of our approach is to reinterpret the cosmology in a classical dynamical way using a point-like Lagrangian and then quantize the point-like model. We apply the method into Einstein gravity with and without a cosmological constant and the $f(R)$-gravity, and get their wave equations respectively. By analyzing the exact solution for the quantum cosmology with and without a cosmological constant we demonstrate that the cosmological constant must exist, being a tiny positive number. We also show the possibility of explaining inflation under the quantum version of cosmology.

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I. INTRODUCTION

The motive of studying a quantum cosmological theory mainly emerge from two parts. First of all, classical gravity theory fails in precisely describing topics like the very early universe. To understand those tiny scale objects with huge energy, a successful quantum gravity theory is requisite and then we can apply it into the cosmological case. However since no such theory is available till now, one considerable effort is to view the quantum cosmology as an effective theory which can be approximately obtained by modifying the classical theory.

One of the many attempts following this idea is called the minisuperspace quantization which is first purposed by DeWitt [1]. At that time Wheeler had introduced the idea of superspace, the space of all three-geometries as the arena in which geometrodynamics develops, a particular four-geometry being represented by a trajectory in this space. Misner had just finished applying the Hamiltonian formulation of gravity, developed in the late 1950s and early 1960s, to cosmological models with an eye towards quantization of these cosmologies as model theories of general relativity [2], which is the other motive for studying quantum cosmology. He invented “minisuperspace” and “minisuperspace quantization” or “quantum cosmology” to describe the evolution of cosmological spacetimes as trajectories in the finite-dimensional sector of superspace related to the finite number of parameters that describe $t =$const slices of the models, and the quantum version of such models, respectively.

Cosmological minisuperspaces and their quantum versions were extensively studied in the early 1970s, but interest in them waned after about 1975 and little new work was done until Hawking revived the field in the 1980s [3, 4], emphasizing path-integral approaches. This started a lively resurgence of interest in minisuperspace quantization till now. For a quick review of the minisuperspace quantization see
In this paper, we would like to introduce a new technique in acquiring the quantum system of cosmology via the point-like Lagrangian under a certain gravitational model that is much easier than the normal process but reaches the similar result. The key is to notice that in the minisuperspace theory the degrees of freedom is reduced by applying the cosmological principle to the Wheeler-DeWitt equation and inverse this process by applying the cosmological principle to the Lagrangian before quantizing it. We will apply this technique into a more complicated case, which contains a cosmological constant, and even more general, for the $f(R)$ gravity. We will then give the exact solution to the quantum universe with a cosmological constant and discuss its meaning.

II. POINT-LIKE QUANTUM COSMOLOGY

In the normal sense, we get the cosmological model under a certain kind of gravity by applying the cosmological principle into the gravitational field equation which is obtained from variation of the action. That is to say, setting the metric in the equation to the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 + a^2(t) \left( \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right),$$

one get the equations of motion for the universe which are also called the Friedmann equation.

Surprisingly, this process can be inversed, that is, one can apply the cosmological principle directly into the action before variation. Giving

$$R = 6 \frac{\ddot{a}}{a} + 6 \left( \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right),$$

this modify the Lagrangian to a point-like one with respect to the scale factor. By the variation of the point-like action, it will give exactly the same equations as those
in the normal process. Such method has been extensively used in studying scalar field cosmology [8–10], non-minimally coupled cosmology [11, 12], scalar-tensor theory [13], multiple scalar fields [14], vector field [15], Fermion field [16], f(T) gravity [17], f(R) gravity [18], high order gravity theory [19], Gauss-Bonnet gravity [20] and so on.

For simplicity, let us confine our discussion to the flat FRW universe (κ = 0). Recall the point-like Lagrangian for a flat FRW universe under Einstein gravity, written as

\[ L = \sqrt{-g} R = -6a \dot{a}^2 \]  \hspace{1cm} (3)

Its canonical momentum is

\[ \pi_a = \frac{\partial L}{\partial \dot{a}} = -12a \ddot{a} \]  \hspace{1cm} (4)

from which we can get its Hamiltonian

\[ H = \pi_a \dot{a} - L = -6a \ddot{a}^2 = -\frac{\pi_a^2}{24a} \]  \hspace{1cm} (5)

Following the process of standard canonical quantization, we should replace the canonical momentum \( \pi_a \) by the operator of momentum \(-i \partial_a\) and get the wave equation of the scale factor \( a \)

\[ i \frac{\partial}{\partial t} \phi(t, a) = \frac{1}{24a} \left( \frac{\partial}{\partial a} \right)^2 \phi(t, a) \]  \hspace{1cm} (6)

Since the scale factor \( a \) is a non-negative real number in cosmology, this wavefunction should be a function defined only on the right half of the real line with its value being a complex number at any given time.

Assuming \( \phi(t, a) \) have the form of \( e^{-i \epsilon t} \phi(\epsilon; a) \) and plugging it into eq.(6), we get the eigen equation of energy

\[ \frac{1}{24a} \phi''(a) = \epsilon \phi(a) \hspace{1cm} a \geq 0 \]  \hspace{1cm} (7)
The solution of this equation depends on whether the eigenvalue is positive or not.

When \( \epsilon > 0 \) we can first rescale the variable \( \xi = (24\epsilon)^{1/3}a \) to drop the parameters in eq.(7) and modify it to

\[
\frac{d^2\phi}{d\xi^2} - \xi \phi = 0 , \quad \xi \geq 0 .
\] (8)

Through a variable substitution of \( z = 2\xi^{3/2}/3 \) and introducing a new function \( u(z) \) with the relationship \( \phi(z) = \sqrt{\xi}u(z) \), we get

\[
u'' + \frac{1}{z}u' - \left(1 + \frac{1}{(3z)^2}\right)u = 0 ,
\] (9)

which is a modified Bessel equation whose solutions can be described by a linear combination of the first modified Bessel function \( I_{1/3}(z) \) and the second modified Bessel function \( K_{1/3}(z) \).

However \( I_{1/3}(z) \) increases in exponential form when \( z \) goes to infinity, which shows great divergent trend and can not be accepted as a reasonable wavefunction. Therefore, recovering all the transformations we made, we get the eigen function with a given eigenvalue described by eq.(7) of

\[
\phi^+ (\epsilon; a) = (24\epsilon)^{1/4}\sqrt{a}K_{1/3} \left( \frac{4\sqrt{6}\epsilon}{3a^{3/2}} \right) .
\] (10)

The asymptotic expansion of the second modified Bessel function with huge arguments can be described as

\[
K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z}\right) ,
\] (11)

which indicates our wavefunction \( \phi^+ \) behaves like

\[
\phi^+(\epsilon; a) \sim a^{4/3} e^{-4\sqrt{6\epsilon}a^{3/2}/3} ,
\] (12)

and will decay to zero at an extremely quick rate with \( a \) getting huge enough. This means \( \phi^+ \) is normalizable and is suitable for being a wavefunction.
For tiny arguments, the second modified Bessel function has the following asymptotic form,
\[ K_\nu(z) \sim \frac{\Gamma(\nu)}{2} \left( \frac{2}{z} \right)^\nu. \] (13)

Applying this into the wavefunction eq.(10), one can find that when \( a \to 0 \), the wavefunction
\[ \phi^\pm(\epsilon ; a) \sim \frac{3^{1/3}}{2} \Gamma \left( \frac{1}{3} \right) \approx 1.93 \] (14)
hits a nonzero constant therefore cannot avoid the possibility of a cosmological singularity.

When \( \epsilon < 0 \), in order to keep \( \xi \) being positive, we should rescale the variable in form of \( \xi = (-24\epsilon)^{1/3}a \). Now equation eq.(7) becomes
\[ \frac{d^2\phi}{d\xi^2} + \xi \phi = 0 \quad , \quad \xi \geq 0. \] (15)

Again we apply the substitution \( z = 2^{3/2}/3 \) and let \( \phi = \sqrt{\xi}u \), then we end up with a Bessel equation
\[ u'' + \frac{1}{z} u' + \left( 1 - \frac{1}{(3z)^2} \right) u = 0 \quad . \] (16)

Normally we choose the Bessel function of first kind \( J_{1/3}(z) \) as one of the basis for its solution space. The other base can have different choice among which the most convenient should be the Bessel function of second kind \( Y_{1/3}(z) \) also known as the Weber function or the Neumann function.

So for every distinct energy level, the quantum system has the degenerate degree of two with its independence basis chosen as
\[ \phi_1^-(\epsilon ; a) = (-24\epsilon)^{\frac{1}{3}} \sqrt{a} J_{\frac{1}{3}} \left( \frac{4\sqrt{-6\epsilon}}{3}a^{\frac{3}{2}} \right), \] (17)
\[ \phi_2^- (\epsilon ; a) = (-24\epsilon)^{\frac{1}{3}} \sqrt{a} Y_{\frac{1}{3}} \left( \frac{4\sqrt{-6\epsilon}}{3}a^{\frac{3}{2}} \right), \] (18)
For great arguments, the Bessel functions behave like the following

\[
J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \left( \cos \left( z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + O(z^{-1}) \right),
\]

\[
Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \left( \sin \left( z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + O(z^{-1}) \right).
\]

Therefore, when \( a \) goes to infinity, the wavefunctions have asymptotic expansions

\[
\phi^-_1(\epsilon; a) \sim a^{-\frac{3}{4}} \cos \left( \frac{4\sqrt{-6\epsilon}}{3} a^{\frac{3}{2}} - \frac{5\pi}{12} \right) + O(a^{-\frac{7}{4}}),
\]

\[
\phi^-_2(\epsilon; a) \sim a^{-\frac{3}{4}} \sin \left( \frac{4\sqrt{-6\epsilon}}{3} a^{\frac{3}{2}} - \frac{5\pi}{12} \right) + O(a^{-\frac{7}{4}}),
\]

that somehow behaves like sine-cosine with a decay to the power of \(-1/4\). However this rate of descend is too slow to allow the wavefunction to be normalizable, which means they will be more like free particles.

For small arguments, the Bessel functions are approximated by

\[
J_\nu(z) \sim \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^\nu
\]

\[
Y_\nu(z) \sim -\frac{\Gamma(\nu)}{\pi} \left( \frac{2}{z} \right)^\nu \cot(\nu\pi) \left( \frac{z}{2} \right)^\nu.
\]

Apply these into eq.(17) and eq.(18), and it is easy to see

\[
\phi^-_1(\epsilon; a) \sim O(a),
\]

\[
\phi^-_2(\epsilon; a) \sim -\frac{3^{1/3}}{\pi} \Gamma \left( \frac{1}{3} \right) + O(a).
\]

Both two behave linearly for tiny \( a \), and \( \phi^-_1 \) vanishes at \( a = 0 \) meanwhile \( \phi^-_2 \) hits a constant at around \(-1.23\).

As shown from our discussion, only \( \phi^-_1 \) is capable of avoiding the cosmological singularity. However, such a function that cannot be normalized will give the average value of \( a \) being infinitely big, which describing a particle can exist at any place in the classical situation but cannot be reasonably explained under a cosmological model. Therefore, we would better turn to consider a quantum cosmological model under some more generalized gravitational models like gravity with a cosmological constant.
III. QUANTUM COSMOLOGY WITH A COSMOLOGICAL CONSTANT

The point-like Lagrangian for a flat FRW universe under the gravitational model with a cosmological constant has the form

\[ \mathcal{L} = -6aa^2 + \Lambda a^3 \quad . \]  

The additional term of \( \Lambda a^3 \) does not contain the derivative of \( a \), thus will not change the form of canonical moment we get in eq.(4). So the Hamiltonian in this case is

\[ \mathcal{H} = -\frac{\pi a^2}{24a} - \Lambda a^3 \quad , \]

and quantum system should be described by the following wave equation

\[ i\frac{\partial}{\partial t}\psi(t, a) = \frac{1}{24a} \left( \frac{\partial}{\partial a} \right)^2 \psi(t, a) - \Lambda a^3 \psi(t, a) \quad . \]

Its eigen equation is given by

\[ \psi''(a) - 24\Lambda a^4 \psi(a) = 24\epsilon a \psi(a) \quad . \]

If we assume the solution has the form of some \( \exp(\varphi(a)) \), we can get a new equation

\[ \varphi''(a) + \varphi^2(a) = 24\epsilon a + 24\Lambda a^4 \quad . \]

Notice that if \( \varphi(a) \sim a^3 \), then

\[ \varphi'(a) \sim a^2, \quad \varphi''(a) \sim a, \quad \varphi^2(a) \sim a^4 \quad , \]

which implies that if we carefully choose a special \( \epsilon_0 \) as eigenvalue, it might allow \( \varphi''(a) \) and \( \varphi^2(a) \) equal to \( 24\epsilon_0 a \) and \( 24\Lambda a^4 \) separately. Letting \( \varphi \) be the form of \( ka^3 \) where \( k \) is an undetermined coefficient, we hope that

\[ 6ka = 24\epsilon_0 a \quad , \]

\[ 9k^2a^4 = 24\Lambda a^4 \quad . \]
It is obvious that if and only if when $6\varepsilon_0^2 = \Lambda$, a solution exists with $k = 4\varepsilon_0$ and

$$
\psi_{\varepsilon_0}(a) = e^{\pm 4\varepsilon_0 a^3}, \quad \varepsilon_0 = \sqrt{\frac{\Lambda}{6}}.
$$

(35)

However this solution does not exist when $\Lambda$ is negative since energy should always be a real number.

The special solution gives us a hint to get exact ones. Divide $a^4\psi(a)$ at both sides of eq.(30) and obtain

$$
\frac{1}{a^4} \frac{\psi''}{\psi} = \frac{24\varepsilon}{a^3} + 24\Lambda.
$$

(36)

When $a$ goes to infinity, we can see that $\psi$ approximately obeys the asymptotic equation

$$
\frac{1}{a^4} \frac{\psi''}{\psi} = 24\Lambda.
$$

(37)

Assuming the general solution of $\psi(a)$ has the form

$$
\psi(a) = u(a) \cdot e^{\nu a^3}
$$

(38)

similar to eq.(31) and $u(a)$ descends to a constant $u_\infty$ when $a$ is large, it is easy to calculate out that

$$
\frac{1}{a^4} \frac{\psi''}{\psi} = \frac{1}{a^4} \frac{6u_\infty \nu a e^{\nu a^3} + 9u_\infty \nu^2 a^4 e^{\nu a^3}}{u_\infty e^{\nu a^3}} \sim 9\nu^2 + O(a^{-3}).
$$

(39)

Clearly, set $9\nu^2 = 24\Lambda$ and eq.(37) is ensured.

Applying eq.(38) into the eigen equation eq.(30), we get the equation for $u(a)$ as

$$
u''(a) + 6\nu a^2 \nu'(a) + (6\nu - 24\varepsilon)au(a) = 0.
$$

(40)

With substitution $z = -2\nu a^3$ and $u = aw$, rearranging eq.(40) we get

$$zw''(z) + \left(\frac{4}{3} - z\right)w'(z) - \left(\frac{2}{3} - \frac{4\varepsilon}{3\nu}\right)w(z) = 0.
$$

(41)
which is a modified confluent hypergeometric equation that is also known as Kum-
mer’s equation. It has a solution described by Kummer’s function defined as

\[ M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{\alpha^{(n)} z^n}{\gamma^{(n)} n!} , \]  

where in our case \( \alpha = \frac{2}{3} - \frac{4}{3} \epsilon / \nu \) and \( \gamma = \frac{4}{3} \). Here the symbol \( x^{(n)} \) stands for a rising factorial defined as

\[ x^{(0)} = 1 , \]
\[ x^{(n)} = x(x+1) \cdots (x+n-1), \quad n \geq 1 . \]

Since \( \gamma \) is not an integer, another solution independent to the first one can be simply introduced as

\[ w(z) = z^{1-\gamma} M(\alpha - \gamma + 1, 2 - \gamma, z) . \]

Recovering from the substitution we made and considering \( \nu \) can be either positive or negative, it seems that we will get four independent eigen functions for any given eigenvalue \( \epsilon \)

\[ \psi_{1,2}^{\pm} (\epsilon; a) = ae^{\pm \nu a^3} M \left( \frac{2}{3} \mp \frac{4\epsilon}{3\nu} \frac{4}{3}, \mp 2\nu a^3 \right) , \]

where \( \nu = 2\sqrt{6\Lambda}/3 \). However Kummer’s function obeys the property of Kummer’s transformation

\[ M(\alpha, \gamma, z) = e^z M(\gamma - \alpha, \gamma, -z) . \]

So in fact one can check that \( \psi_{1,2}^{+} = \psi_{1,2}^{-} \), and there are only two independent solutions for each \( \epsilon \).

For great arguments, Kummer’s function can be approximately expanded as

\[ M(\alpha, \gamma, z) \sim \Gamma(\gamma) \left( \frac{e^z z^{\alpha-\gamma} + (-z)^{-\alpha}}{\Gamma(\alpha) + \Gamma(\gamma - \alpha)} \right) . \]
Applying it into eq.(59) and eq.(60), we get the asymptotic expansions of these wavefunctions when $a$ goes to infinity,

$$\psi_1 \sim \Theta(2, -\nu) + \Theta(2, \nu) \quad ,$$

$$\psi_2 \sim \Theta(1, -\nu) + \Theta(1, \nu) \quad ,$$

where

$$\Theta(n, \nu) = \Gamma\left(\frac{2n}{3}\right) \frac{e^{\nu a^3} a^{-1+4\epsilon/\nu}}{(2\nu)^{(n/3-4\epsilon/3\nu)} \Gamma(n/3 + 4\epsilon/3\nu)} .$$

Since $\nu = 2\sqrt{6\Lambda}/3$, the sign of $\Lambda$ will decide whether $\nu$ is real, and therefore decide how the asymptotic expansions behave.

When $\Lambda < 0$, $\nu = i(2\sqrt{-6\Lambda}/3)$ is an imaginary number and makes the modulus of both $\exp(\pm \nu a^3)$ and $a^{\pm 4\epsilon/\nu}$ being unit for any real $a$. So the asymptotic expansions can be simplified to

$$\psi_{1,2} \sim O(a^{-1}) \quad .$$

This descending with the order of minus one is too slow, thus neither of the two eigenfunctions is normalizable.

When $\Lambda > 0$, with $\nu$ being a real number, the behavior of $\Theta$ is completely determined by the exponential term $\exp(\pm \nu a^3)$. Fortunately, we know that $\Gamma(z)$ explodes at non-positive integer points, thus a carefully selected eigenvalue $\epsilon$ can make the exploded term $\Theta(n, \nu)$ in the expansions vanish.

For $\psi_1$ it requires

$$\epsilon_n^{(1)} = -(3n + 2) \sqrt{\frac{\Lambda}{6}} \quad , \quad n = 0, 1, \ldots$$

Even so, we have the fact that for $M(\alpha, \gamma + 1, z)$ whose $\alpha$ is a non-positive integer, it can be described by Laguerre function

$$L_n^{(\gamma)}(z) := \binom{n + \gamma}{n} M(-n, \gamma + 1, z) \quad .$$
So the eigen states $|n^{(1)}\rangle$ for eigenvalue $\epsilon^{(1)}_n$ after normalization is written as

$$
\langle a|n^{(1)}\rangle = c_n a e^{-\nu a^3} L_n^{(\frac{1}{2})}(2\nu a^3),
$$

where $c_n$ is the normalization factor.

Same discussion also holds for solution $\psi_2$, giving its eigenvalue

$$
\epsilon^{(2)}_n = -(3n + 1) \sqrt{\frac{\Lambda}{6}}, \quad n = 0, 1, \ldots
$$

and the normalized eigen state

$$
\langle a|n^{(2)}\rangle = d_n e^{-\nu a^3} L_n^{(-\frac{1}{2})}(2\nu a^3),
$$

where $d_n$ is its normalization factor.

From the definition of Kummer’s function eq.(42) we can see that for tiny arguments,

$$
\psi_1(\epsilon; a) \sim a(1 - \nu a^3)(1 + (2\epsilon + \nu) a^3),
$$

$$
\psi_2(\epsilon; a) \sim (1 - \nu a^3)(1 + (4\epsilon + \nu) a^3).
$$

Therefore when $a$ goes to zero, $\psi_1$ vanishes and $\psi_2$ hits the unit, hence the cosmological singularity is naturally avoided only in the first case. Thus we will only choose the first set of eigen states to be the basis of the quantum system: $\epsilon_n = \epsilon^{(1)}_n$ and $|n\rangle = |n^{(1)}\rangle$

For any real number $\alpha$, the first two Laguerre polynomials are

$$
L_0^{(\gamma)} = 1,
$$

$$
L_1^{(\gamma)} = 1 + \gamma - x.
$$

So the eigen states with eigenvalue of the highest two are

$$
\langle a|0\rangle = c_0^{(1)} a e^{-\nu a^3},
$$

$$
\langle a|1\rangle = c_1^{(1)} a \left(\frac{4}{3} - a\right) e^{-\nu a^3}.
$$
If a certain state is a combination of only these two states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ at initial ($|\alpha|^2 + |\beta|^2 = 1$), it will evolve with respect to the cosmological time $t$ as

$$|\psi,t\rangle = \alpha|0\rangle e^{-i\varepsilon_0 t} + \beta|1\rangle e^{-i\varepsilon_1 t}.$$  \hspace{1cm} (65)

The evolution of the average measurement $\bar{a}^2$ can be calculated out:

$$\bar{a}^2(t) = \langle \psi, t | a^2 | \psi, t \rangle$$

$$= |\alpha|^2 \langle 0 | a^2 | 0 \rangle + \alpha^* \beta \langle 0 | a^2 | 1 \rangle e^{i(\varepsilon_0 - \varepsilon_1) t} + |\beta|^2 \langle 1 | a^2 | 1 \rangle + \alpha \beta^* \langle 1 | a^2 | 0 \rangle e^{-i(\varepsilon_0 - \varepsilon_1) t}$$

$$= 2 \Re(\alpha^* \beta e^{i(\varepsilon_0 - \varepsilon_1) t}) \langle 0 | a^2 | 1 \rangle + (|\alpha|^2 \langle 0 | a^2 | 0 \rangle + |\beta|^2 \langle 1 | a^2 | 1 \rangle)$$

$$= 2 c_{0,1} |\alpha||\beta| \cos \left( \frac{\sqrt{6} \Lambda}{2} t - \theta \right) + (c_{0,0} |\alpha|^2 + c_{1,1} |\beta|^2)$$

where $c_{n,m} = \langle n | a^2 | m \rangle$ and $\theta = \text{Arg}(\alpha) - \text{Arg}(\beta)$.

This solution suggests a pulsing universe with a characteristic time of $4\pi \sqrt{1/6\Lambda}$. Actually although we do not know what state the universe is at a certain time, we can prove that it always rebounds with the same characteristic time no matter how the initial coefficient of each eigen state is given. It is clear that the evolution of $\bar{a}^2$ is always described by

$$\bar{a}^2(t) = \sum_{n,m \geq 0} \eta_{nm} \cos((\varepsilon_n - \varepsilon_m) t + \theta_{nm}).$$ \hspace{1cm} (66)

Recalling eq.(54), we can see $\varepsilon_n - \varepsilon_m$ is always some integer times of $\sqrt{6\Lambda}/2$ which ensures that $\bar{a}^2(t)$ has a period of $4\pi \sqrt{1/6\Lambda}$.

Considering the universe as we observed is experiencing an accelerating expansion through its whole life till now, it is reasonable to assume it is still in the first quarter of the period, implying the cosmological constant $\Lambda$ should be no bigger than $\pi^2/6T_0^2$ where $T_0$ stands for the cosmological time till now.

Another interesting thing is that, observing the combination of solutions can provide a square wave, this quantum system must have some special states that may
let $a^2$ rise as fast as possible at some certain time $t_0$. For the simplest case, considering a state composed by eigen states with real coefficients $|\psi, t\rangle = \sum_n \tau_n e^{-i\varepsilon_n t} |n\rangle$, if its scale factor evolves like

$$a^2(t) = A \sum_{k=0}^{N} \frac{1}{2k-1} \cos \left( \frac{\sqrt{6}A}{2} (2k-1) t \right) + C \quad , \quad (67)$$

then choose only the coefficients of the first $2N - 1$ states to be non-zero and real, we know they will satisfy the polynomial system

$$\sum_{n=0}^{2N-l-1} \tau_n \tau_{n+l} c_{n,n+l} = \frac{A}{4} \left( \frac{1 - (-1)^l}{l} \right) , \quad 1 \leq l \leq 2N - 1 \quad , \quad (68)$$

There are in total $2N$ coefficients needed to be fixed. The normalization condition together with eq.(68) exactly give the same number of equations from which the coefficients can be solved. Therefore we can satisfy eq.(67) for any $N$ as large as we wish. That gives the possibility of an expansion of $a$ at any velocity, which may generate an inflation with the speed even faster than exponential level as normal understanding.

**IV. QUANTUM COSMOLOGY OF $f(R)$ GRAVITY**

For more general cases, we consider the quantum model of a flat FRW universe under $f(R)$ gravity. More detailed discussion on the point-like model of $f(R)$ universe can be found in the works of Capozziello [18]. The point-like Lagrangian with no term of matter will be like

$$\mathcal{L} = (f - f_R R)a^3 - 6f_{RR} \dot{R}a^2 \dot{a} - 6f_R a \ddot{a}^2 \quad , \quad (69)$$

and the canonical momentums for $a$ and $R$ are, respectively

$$\pi_a = -6f_{RR} \dot{R}a^2 - 12f_R \dot{a} \quad , \quad (70)$$

$$\pi_R = -6f_{RR} a^2 \dot{a} \quad . \quad (71)$$
So the canonical energy is

\[ E_L = -(f - f_R) a^3 - 6 f_{RR} \dot{R} a^2 \dot{a}^2 - 6 f_R a \dot{a}^2 \]  
(72)

From eq.(71) we can directly read that

\[ \dot{a} = -\frac{\pi R}{6 f_{RR} a^2} \]  
(73)

Plug it into eq.(70) and get

\[ 6 f_{RR} \dot{R} a^2 = -\pi a + 2 \frac{f_R}{a f_{RR}} \pi R \]  
(74)

Applying them to eq.(72), the Hamiltonian for the system becomes

\[ \mathcal{H} = -(f - f_R) a^3 + \frac{\pi R}{6 f_{RR} a^2} \left( -\pi a + \frac{2 f_R}{a f_{RR}} \pi R \right) - 6 f_R a \left( -\frac{\pi R}{6 f_{RR} a^2} \right)^2 \]

\[ = -(f - f_R) a^3 - \frac{1}{6 a^2} \left( \frac{\pi R}{f_{RR}} \right) a^2 + \frac{1}{6 a^3} \left( \frac{f_R}{f_{RR}^2} \right) a^2 \]  

So the wave equation that describes this quantum system is

\[ i \frac{\partial}{\partial t} \Psi = -a^3 (f - f_R) \Psi + \frac{1}{6 a^2} \frac{1}{f_{RR} \partial a} \frac{\partial}{\partial a} \Psi - \frac{1}{a^3} \frac{f_R}{f_{RR}^2} \left( \frac{\partial}{\partial \dot{R}} \right)^2 \Psi \]  
(75)

\( \Psi(t, a, R) \) is a function of cosmological time \( t \), scale factor \( a \) and Ricci scalar \( R \).

The equation relies on the form of \( f(R) \) to be exactly solved. However as a linear partial differential equation, its coefficients of all the second order terms satisfy the fact that \( \Delta = 1/(36 a^4 f_{RR}^2) \) is positive on the whole \( a - R \) plane. Therefore the eigen equation of the operator \( \mathcal{H} \) is a hyperbolic equation and can be transformed into a wave equation.

We need to point out that \( a \) and \( R \) have been separated via Palatini formalism. In this case, their relation is linked by one of the equations of motion rather than a given definition. So after the quantization, this relation must have degenerated to be statistically satisfied. That means even a flat universe of small scale or a huge scale universe with large curvature which are not normally allowed in the classical case will also have contribution to the possibility.
V. CONCLUSION AND DISCUSSIONS

The purpose of this paper has been to introduce a new approach to inquire the minisuperspace model without seeking the Wheeler-DeWitt equation for a certain gravitational theory.

The trick is to apply the cosmological principle directly to the action of a gravitational system before variation, and reform the Lagrangian of geometry to a classical point-like one. It is obvious that such a process of taking the metric of a cosmological model which is truncated by an enormous degree of imposed symmetry and simply plugging it into a quantization procedure may not give an answer that is in any way an exact solution. However, we have seen that the variation of this point-like Lagrangian gives the right equation of motion (the Friedmann equation) to describe the universe.

By quantizing this semi-classical system described by the point-like Lagrangian, we represent a quantum system that is very similar to the minisuperspace through reducing the superspace where the Wheeler-DeWitt equation is defined on. The only difference is, for solving our quantum equation we need the concept of the eigenvalue $\epsilon$ of the Hamiltonian of the system which does not exist in the classical minisuperspace theory.

It is very natural to apply our technique beyond the Einstein gravity to the gravitational model with cosmological constant and more general $f(R)$ gravity with the help of Palatini formalism and respectively get their quantum cosmological model. This especially opens the gate for considering quantum cosmology of $f(R)$ gravity.

As the second aspect of our work shown in this paper, we give the exact solutions of the quantum systems we get under Einstein gravity with and without cosmological constant. By analyzing the solutions, we firstly exhibit that to make the quantum system describing our universe physically legal, the cosmological constant is indis-
pensable otherwise the statistical scale of the universe will be infinite. Secondly, we prove that all the possible states in such a legal quantum cosmological model predict pulsing universe with the same period of a cosmological characteristic time that is inversely proportional to the square root of the cosmological constant. Considering the enormous time the universe has existed, the cosmological constant must be extremely tiny. Thirdly, we show that this quantum system contains states that allow an expansion at any speed as fast as possible, which could probably provide a motivation for the inflation.

At last, we would like to give some comments on our theory. The method used in our theory is directly quantizing the point-like Lagrangian which should not be equivalent exactly to the original Lagrangian of the corresponding gravity theories but is only equivalent to the Friedmann equation. Hence our approach has the drawback of not being an exact quantum theory of cosmology, which also presents in the minisuperspace approach. But our method still gives some interesting results and more importantly raises a very interesting question as following. Now we know that there are two ways of reducing the enormous degrees of freedom in quantum cosmology. One way is applying the cosmological principle to the Wheeler-DeWitt equation, and the other is, as shown in this paper, applying the cosmological principle to the Lagrangian. Then, what is the difference between these two approaches? This query should surely cause our attention for the further research.
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