A generalized Jaynes-Cummings model and the relativistic degenerate parametric amplifier

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Abstract

We introduce a generalization of the Jaynes-Cummings model and study some of its properties. We obtain the energy spectrum and eigenfunctions of this model by using the tilting transformation and the squeezed number states of the one-dimensional harmonic oscillator. We show that in the non-relativistic limit, this model can be mapped onto the degenerate parametric amplifier.

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1 Introduction

The Dirac oscillator was first introduced by Ito et al. \cite{1} and Cook \cite{2} and reintroduced in the 80’s by Moshinsky and Szczepaniak \cite{3}. They added the linear term $-imc\omega\beta\alpha \cdot r$ to the relativistic momentum $p$ of the free-particle Dirac equation. This problem reduces to the harmonic oscillator plus a spin-orbit coupling term in the non-relativistic limit.

The Dirac-Moshinsky oscillator has been applied to quark confinement models in quantum chromodynamics, hexagonal lattices and the emulation of graphene in electromagnetic billiards \cite{4}, among others. In particular, the (2+1)-dimensional Dirac-Moshinsky oscillator has been related to quantum optics \cite{5} via the Jaynes-Cummings and Anti-Jaynes-Cummings model.

In quantum optics the Jaynes-Cummings model describes the interaction between a two-level atom with a quantized electromagnetic field \cite{7}. In the rotating wave approximation

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this model have been extensively studied and its exact solution have been founded [8]. These solutions yield quantum collapse and revival of atomic inversion [9], and squeezing of the radiation field [10], among other quantum effects. All these effects have been corroborated experimentally, as can be seen in references [11, 12, 13].

In the 1 + 1-dimensional space, the Dirac-Moshinsky oscillator has been exactly solved by using the theory of the non-relativistic harmonic oscillator [14, 15]. Moreover, Nogami and Toyama constructed the relativistic coherent state for this lower dimensional case [16].

The aim of the present work is to introduce a generalization of the Jaynes-Cummings and we show that our Jaynes-Cummings model can be reduced to the degenerate parametric amplifier in the non-relativistic limit.

This work is organized as it follows. In Section 2, we review the main properties of the Dirac-Moshinsky oscillator in 1+1 dimensions and its mapping to the Anti-Jaynes-Cummings model. In Section 3, we introduce the generalization of the Jaynes-Cummings model in terms of two complex parameters and the creation and annihilation operators of the one-dimensional harmonic oscillator. We decouple the equations for the upper and lower wave functions. We find its energy spectrum and eigenfunctions by using the tilting transformation and the theory of the one-dimensional harmonic oscillator. In Section 4, we show that our model can be reduced to the degenerate parametric amplifier in the non-relativistic limit, for a particular choice of the complex parameters. Finally, we give some concluding remarks.

2 The 1 + 1 Dirac-Moshinsky oscillator and the Anti-Jaynes-Cummings model.

The time-independent Dirac equation for the Dirac-Moshinsky oscillator is given by the Hamiltonian [3]

\[ H_D \Psi = [\alpha \cdot (p - im\omega r) + mc^2 \beta] \Psi = E \Psi, \]

where the Dirac matrices \( \alpha \) and \( \beta \) satisfy the Clifford algebra

\[
\begin{align*}
\alpha_a \alpha_b + \alpha_b \alpha_a &= 2\delta_{ab} 1, \\
\alpha_a \beta + \beta \alpha_a &= 0, \\
\alpha_a^2 &= \beta^2 = 1.
\end{align*}
\]

In 1 + 1 dimensions, the most convenient representations of \( \alpha \) and \( \beta \) are

\[
\alpha = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

With this realization we obtain the coupled equations

\[
\begin{align*}
(E - mc^2) |\Psi_1\rangle &= c(-ip_x + m\omega x)|\Psi_2\rangle, \\
(E + mc^2) |\Psi_2\rangle &= c(ip_x + m\omega x)|\Psi_1\rangle.
\end{align*}
\]

The uncoupled equations for the two components |\( \Psi_1 \rangle \) and |\( \Psi_2 \rangle \) can be solved by using the non-relativistic theory of the quantum harmonic oscillator. The eigenvalues are [14, 15]

\[
E_n = \pm mc^2 \sqrt{1 + \frac{2|n|\hbar \omega}{mc^2}}, \quad n = 0, \pm 1, \pm 2, ...
\]
where the upper sign should be chosen for \( n \geq 0 \) and the lower one for \( n < 0 \). The normalized eigenfunction \(|\Psi\rangle\) is

\[
|\Psi\rangle = \begin{pmatrix}
\sqrt{\frac{\lambda(E+mc^2)}{2^m} H_{|n|}} e^{-\lambda x^2/2} \\
\sqrt{\frac{\lambda(E-mc^2)}{2^m} H_{|n|-1}} e^{-\lambda x^2/2}
\end{pmatrix},
\]

where \( H_n(x) \) is the Hermite polynomial and \( \lambda = \sqrt{\frac{m\omega}{\hbar}} \). If we introduce the usual creation and annihilation operators

\[
a = \sqrt{\frac{m\omega}{2\hbar}} x + \frac{i}{\sqrt{2m\omega\hbar}} p_x, \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} x - \frac{i}{\sqrt{2m\omega\hbar}} p_x,
\]

the coupled equations (4) and (5) can be written in the simplified form \[16\]

\[
(E - mc^2) |\Psi_1\rangle = c\sqrt{2m\omega}\hbar a^\dagger |\Psi_2\rangle,
\]

\[
(E + mc^2) |\Psi_2\rangle = c\sqrt{2m\omega}\hbar a |\Psi_1\rangle.
\]

From these equations we can rewrite the Hamiltonian of the 1 + 1-dimensional Dirac-Moshinsky oscillator as

\[
H = \delta (\sigma_- a + \sigma_+ a^\dagger) + mc^2 \sigma_3,
\]

where \( \delta = c\sqrt{2m\omega}\hbar \), \( \sigma_+ \) and \( \sigma_- \) are the spin raising and lowering operators, and \( \sigma_3 \) is the Pauli matrix. The equation \[11\] formally is the Hamiltonian of the Anti-Jaynes-Cummings model of the quantum optics. Similarly, the 2 + 1-dimensional Dirac-Moshinsky oscillator has been mapped onto the Anti-Jaynes-Cummings oscillator by using the chiral creation and annihilation operators \[5\]. The same conclusion was obtained for the 2 + 1-dimensional Dirac-Moshinsky oscillator coupled to an external magnetic field, by using the complex coordinate \( z \) and its conjugate momentum \( p_z \) \[17\].

### 3 The generalized Jaynes-Cummings model.

From equation \[11\], we introduce the generalized Jaynes-Cummings model as

\[
H = \hbar \left[ \sigma_- (g^* a + f a^\dagger) + \sigma_+ (ga^\dagger + f^* a) \right] + mc^2 \sigma_3,
\]

where \( g \) and \( f \) are two general complex parameters. The coupled equations for the spinor components \(|\Psi_1\rangle\) and \(|\Psi_2\rangle\) are

\[
\hbar (g a^\dagger + f^* a) |\Psi_2\rangle = (E - mc^2) |\Psi_1\rangle,
\]

\[
\hbar (g^* a + fa^\dagger) |\Psi_1\rangle = (E + mc^2) |\Psi_2\rangle.
\]

Notice that these equations are the generalization of equations \[9\] and \[10\]. The uncoupled equations for \(|\Psi_1\rangle\) and \(|\Psi_2\rangle\) are easily obtained from above expressions, which result to be

\[
\hbar^2 (|g|^2 a^\dagger a + |f|^2 a a^\dagger + gfa^\dagger + f^* g^* a^2) |\Psi_1\rangle = (E^2 - m^2 c^4) |\Psi_1\rangle,
\]

\[15\]
\[ \hbar^2((|g|^2a a^\dagger + |f|^2a^\dagger a + gfa^\dagger a + f^*g^*a^2)|\Psi_2) = (E^2 - m^2c^4)|\Psi_2\rangle. \] (16)

By using the \( SU(1,1) \) Lie algebra realization in terms of the Bose operators \( a, a^\dagger \) (see Appendix, equation (37)), we can write the uncoupled equation for \( |\Psi_1\rangle \) as

\[ \left[ 2K_0\hbar^2(|g|^2 + |f|^2) + 2gf\hbar^2K_+ + 2g^*f^*\hbar^2K_- + \frac{\hbar^2}{2}(|f|^2 - |g|^2) \right]|\Psi_1\rangle = (E^2 - m^2c^4)|\Psi_1\rangle, \] (17)

where we have used the property \( aa^\dagger = a^\dagger a + 1 \). In order to remove the operators \( K_\pm \), we apply the tilting transformation to the above Klein-Gordon-type Hamiltonian \( H_{KG}|\Psi_1\rangle = E|\Psi_1\rangle \). To do this we proceed as in references [18, 19]. Since \( D(\xi)D^\dagger(\xi) = 1 \), equation (17) can be written as

\[ D^\dagger(\xi)H_{KG}D(\xi)D^\dagger(\xi)|\Psi_1\rangle = (E^2 - m^2c^4)D^\dagger(\xi)|\Psi_1\rangle, \] (18)

where \( D(\xi) \) is the \( SU(1,1) \) displacement or squeezing operator and \( \xi = -\frac{i}{2}\tau e^{-i\phi} \) (see Appendix). If we define the tilted Hamiltonian \( H'_{KG} = D^\dagger(\xi)H_{KG}D(\xi) \) and the wave function \( |\Psi'_1\rangle = D^\dagger(\xi)|\Psi_1\rangle \), this equation can be written as \( H'_{KG}|\Psi'_1\rangle = (E^2 - m^2c^4)|\Psi'_1\rangle \).

By choosing the coherent state parameter as \( \tau = \text{tanh}^{-1}(4gf/(|g|^2 + |f|^2)) \), we obtain that the tilted Hamiltonian \( H'_{KG} \) is

\[ H'_{KG} = 2K_0\hbar^2(|g|^2 - |f|^2) + \frac{\hbar^2}{2}(|f|^2 - |g|^2), \] (19)

where \( 2K_0 \) is the Hamiltonian of the one-dimensional harmonic oscillator. Thus, by using that the energy spectrum of the one-dimensional harmonic oscillator is \( n + \frac{1}{2} \), we obtain that the energy spectrum of the generalized Jaynes-Cummings is

\[ E = \pm \sqrt{\hbar^2(|g|^2 - |f|^2)n + m^2c^4}. \] (20)

If we apply the same procedure to the uncoupled equation for the other spinor component \( |\Psi_2\rangle \) we obtain the energy spectrum

\[ E = \pm \sqrt{\hbar^2(|g|^2 - |f|^2)(n' + 1) + m^2c^4}. \] (21)

Thus, since both components \( |\Psi_1\rangle \) and \( |\Psi_2\rangle \) belong the the same energy, we obtain that the relationship between the quantum numbers is \( n = n' + 1 \). Moreover, if we set \( f = 0 \) in the generalized Jaynes-Cummings model of equation (12), the standard Dirac-Moshinsky oscillator is recovered. Also, under this setting, if we identify \( \delta = gh \), the spectrum (20) is simplified to that of equation (6).

The eigenfunctions of the tilted Hamiltonian \( H'_{KG} \) are those of the one-dimensional harmonic oscillator

\[ \Psi'_1(x) = \sqrt{\frac{1}{\pi^{1/4}(2n!)(\hbar)^{1/2}}} e^{-\frac{x^2}{2\hbar}} H_n(x), \] (22)

where \( H_n(x) \) are the Hermite polynomials. Therefore, the eigenfunctions of the degenerate parametric amplifier are obtained from \( |\Psi'_1\rangle = D(\xi)|\Psi'_1\rangle \). Similar results hold for \( |\Psi_1\rangle \). These states are known as the squeezed number states of the one-dimensional harmonic oscillator [20]. The action of the squeezing operator \( D(\xi) \) on \( |\Psi'_1\rangle \) is a long calculation. However it has
been calculated by Nieto in reference [20] (see Appendix). By using these results we are able to construct the spinor $|\Psi\rangle$ in the form

$$|\Psi\rangle = \frac{1}{\pi^{1/4}(F_1)^{1/2}}e\left(-\frac{1}{2}x^2F_2\right)\begin{pmatrix} A_n \left(\frac{F_3}{2^n n!}\right)^{n/2}H_n \left(\frac{x}{F_4}\right) \\ B_n \left(\frac{F_3}{2^n n!}\right)^{(n-1)/2}H_{n-1} \left(\frac{x}{F_4}\right) \end{pmatrix}, \quad n = 1, 2, ... \quad (23)$$

where $F_1, F_2, F_3$ and $F_4$ are functions depending on the coherent state parameters, given in the Appendix. Since the squeezed number states are already normalized, after normalization we obtain that the spinor $|\Psi\rangle$ is

$$|\Psi\rangle = \frac{1}{\pi^{1/4}(F_1)^{1/2}}e\left(-\frac{1}{2}x^2F_2\right)\begin{pmatrix} \sqrt{\frac{E\pm mc^2}{2E}} \left(\frac{F_3}{2^n n!}\right)^{n/2}H_n \left(\frac{x}{F_4}\right) \\ \mp i \sqrt{\frac{E\pm mc^2}{2E}} \left(\frac{F_3}{2^n n!}\right)^{(n-1)/2}H_{n-1} \left(\frac{x}{F_4}\right) \end{pmatrix}. \quad (24)$$

4 The relativistic degenerate parametric amplifier.

A particular case of the generalized Jaynes-Cummings model proposed in equation (12), is obtained if we set the complex parameters $g$ and $f$ as

$$g = \frac{imc^2\sqrt{2\mu}}{\hbar}e^{-i\phi}, \quad f = \frac{\chi}{\sqrt{2\mu}}e^{-i\phi}, \quad \mu = \frac{\hbar\omega}{mc^2}. \quad (25)$$

By written $E = mc^2 + \epsilon$, in the non-relativistic limit ($\epsilon \ll mc^2$) we obtain that the uncoupled equation for $|\Psi_1\rangle$ becomes

$$\left[ \left(\hbar\omega + \frac{\hbar^2\chi^2}{4\omega^2}\right)a^\dagger a + \frac{\hbar\chi^2}{4\omega^2} - i\hbar \frac{\chi}{2} \left( a^2 e^{2i\phi} - a^2 e^{-2i\phi} \right) \right] |\Psi_1\rangle = \epsilon |\Psi_1\rangle. \quad (26)$$

Formally, the operator on the left hand is the Hamiltonian of the time-independent degenerate parametric amplifier [13, 21]. Here, $\chi$ is the coupling constant (proportional to the second-order susceptibility of the medium and to the amplitude of the pump), $\phi$ is the phase of the pump field and $\omega$ is its frequency. Because of this particular case, the generalized Jaynes-Cummings model (equation (12)), is a relativistic version of the degenerate parametric amplifier.

The exact energy spectrum for $g$ and $f$ given by (25) is obtained from equation (20)

$$E = \pm mc^2 \sqrt{1 + \left(\frac{2\hbar\omega}{mc^2} - \frac{\hbar^2\chi^2}{2\omega mc^2}\right)n}. \quad (27)$$

This energy spectrum is the relativistic version to that of the degenerate parametric amplifier previously calculated in reference [18]. Moreover, if $\chi = 0$, this energy spectrum is reduced to that of the 1 + 1-dimensional Dirac-Moshinsky oscillator of equation (13).
5 Concluding remarks

We introduced a generalization of the Jaynes-Cummings model in terms of the creation and annihilation operators of the one-dimensional harmonic oscillator. We obtained the exact solution of this problem by using the tilting transformation and the squeezed number states of the one-dimensional harmonic oscillator. For a particular choice of the generalized model parameters, we constructed the relativistic version of the degenerate parametric amplifier.

The generalization of the Jaynes-Cummings model introduced in this work was constructed with only one oscillator and it is a linear combination of the Jaynes-Cummings and Anti-Jaynes-Cummings model. This formulation can be extended to a model that includes two oscillators. This new generalization shall permit us to have a wider range of applications and will be reported in a future work.

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6 Appendix

Three operators $K_\pm, K_0$ close the $su(1,1)$ Lie algebra if they satisfy the commutation relations \[ [K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \] (28)

The action of these operators on the Fock space states $\{|k, n\rangle, n = 0, 1, 2, \ldots\}$ is

$K_+|k, n\rangle = \sqrt{(n+1)(2k+n)}|k, n+1\rangle,$ \hspace{1cm} (29)

$K_-|k, n\rangle = \sqrt{n(2k+n-1)}|k, n-1\rangle,$ \hspace{1cm} (30)

$K_0|k, n\rangle = (k+n)|k, n\rangle,$ \hspace{1cm} (31)

where $|k, 0\rangle$ is the lowest normalized state. The Casimir operator $K^2$ for any irreducible representation of this group is given by

$K^2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+)$ \hspace{1cm} (32)

and satisfies the relationship $K^2 = k(k-1)$. Thus, a representation of $su(1,1)$ algebra is determined by the number $k$, called the Bargmann index. The discrete series are those for which $k > 0$.

The $SU(1,1)$ Perelomov coherent states are defined as the action of the displacement operator $D(\xi)$ onto the lowest normalized state $|k, 0\rangle$ as \[ |\zeta\rangle = D(\xi)|k, 0\rangle = (1 - |\zeta|^2)k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}} \zeta^n|k, n\rangle, \] (33)
The displacement operator $D(\xi)$ is defined in terms of the creation and annihilation operators $K_+, K_-$ as

$$D(\xi) = \exp(\xi K_+ - \xi^* K_-),$$

where $\xi = -\frac{1}{2}\tau e^{-i\varphi}$, $-\infty < \tau < \infty$ and $0 \leq \varphi \leq 2\pi$. The so-called normal form of the squeezing operator is given by

$$D(\xi) = \exp(\zeta K_+) \exp(\eta K_0) \exp(-\zeta^* K_-),$$

where $\zeta = -\tanh(\frac{1}{2}\tau) e^{-i\varphi}$ and $\eta = -2 \ln \cosh |\xi| = \ln(1 - |\xi|^2)$ \cite{24}.

The $SU(1, 1)$ Perelomov number coherent state $|\zeta, k, n\rangle$ is defined as the action of the displacement operator $D(\xi)$ onto an arbitrary excited state $|k, n\rangle$ \cite{19}

$$|\zeta, k, n\rangle = \sum_{s=0}^{\infty} \frac{\zeta^s}{s!} \sum_{j=0}^{n} \frac{(-\zeta^*)^j}{j!} e^{\eta(k+j-n)} \frac{\sqrt{\Gamma(2k+n)\Gamma(2k+n-j+s)}}{\Gamma(2k+n-j)} \times \frac{\sqrt{\Gamma(n+1)\Gamma(n-j+s+1)}}{\Gamma(n-j+1)} |k, n-j+s\rangle.$$

(36)

It can be seen that the operators

$$K_0 = \frac{1}{2} \left( a^+ a + \frac{1}{2} \right), \quad K_+ = \frac{1}{2} a^+, \quad K_- = \frac{1}{2} a^2.$$

(37)

satisfy the $su(1, 1)$ Lie algebra commutation relations \cite{28}. In this expressions $a$ and $a^+$ are the creation and annihilation operators of the one-dimensional harmonic oscillator. The number states of the harmonic oscillator decompose into two invariant subspaces, spanned by even number states $|2n\rangle$ and odd numbers states $|2n+1\rangle$, $n = 0, 1, \ldots$. For this case, the displacement operator $D(\xi)$ is called squeeze operator $S(\xi)$ and is given by

$$D(\xi) = S(z) = e^{\left( \frac{1}{2} z a^2 - \frac{1}{2} z^* a^2 \right) + (F_3)^{n/2} \left( \frac{F_3}{2^n n!} \right)^{1/2} H_n \left( \frac{x}{F_4} \right)},
$$

(38)

If we apply the squeeze operator to a number state $|n\rangle$ (for $n$ even) of the harmonic oscillator we obtain

$$S(z)|n\rangle = \frac{1}{\pi^{1/4} (F_1)^{1/2}} e^{(-\frac{1}{4} x^2 F_2)} \left( \frac{F_3}{2^n n!} \right)^{1/2} H_n \left( \frac{x}{F_4} \right),$$

(39)

where $H_n$ are the Hermite polynomials and $F_1, F_2, F_3$ and $F_4$ are functions of the coherent state parameters. These functions and are given explicitly by \cite{20}

$$F_1 = \cosh r + e^{i\varphi} \sinh r,$$

(40)

$$F_2 = \frac{1 - i \sinh \varphi \sinh r (\cosh r + e^{i\varphi} \sinh r)}{(\cosh r + \cos \varphi \sinh r)(\cosh r + e^{i\varphi} \sinh r)},$$

(41)

$$F_3 = \frac{\cosh r + e^{-i\varphi} \sin \varphi \sinh r}{\cosh r + e^{i\varphi} \sin \varphi \sinh r},$$

(42)

$$F_4 = (\cosh^2 r + \sinh^2 r + 2 \cos \varphi \cosh r \sinh r)^{1/2}.$$

(43)

A similar expression is valid when $n$ is odd.
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