Calculating the Hawking Temperatures of Conventional Black Holes in the f(R) Gravity Models with the RVB Method

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Abstract
This article aims to calculate the Hawking temperature of black holes in f(R) gravity using the RVB method. Our research shows an integral constant difference between the RVB and general methods in determining the Hawking temperature of black holes. Our study demonstrates the effectiveness of the RVB method in calculating the Hawking temperature of black holes across different f(R) gravity models, indicating its broader potential for further research. Our findings may provide insights for further research and understanding of black hole thermodynamics.

Keywords RVB method · f(R) gravity · Pure geometric model · Hawking temperature

1 Introduction

In the classical view, black holes were believed to be extreme objects from which nothing could escape. However, Hawking and Bekenstein discovered that black holes could possess temperature and entropy, viewed as thermodynamic systems. Various studies [1–9] have shown that the topologically invariant Euler characteristic can characterize the topological properties of black holes.

Calculating the Euler characteristic has proven to be a valuable method for studying essential features of black holes, such as their entropy [9, 10] and the topological properties of the horizon temperature [11, 12]. Recent studies by Robson, Villari and Biancalana [12–14] have demonstrated that the Hawking temperature of a black hole is closely related to its topology. These researchers proposed a topological method based on the Euler characteristic, successfully applied to various black holes, including four-dimensional Schwarzschild black holes, anti-de Sitter black holes, and other Schwarzschild-like or charged black holes. Below, we refer to this method as the RVB method.

Building upon the research conducted by Robson, Villari, and Biancalana [12–14], Liu et al. [15] have successfully computed the Hawking temperature of a charged rotating BTZ black hole under the massive gravity model. They employed the topological method
proposed by the RVB method to achieve this result. Similarly, Xian et al. [16] utilized this method to investigate the Hawking temperature of the global monopole spacetime under the massive gravity model. These studies establish a correlation between the Hawking temperature and the topological properties of black holes.

Previous research has extensively applied the RVB method to different types of black holes under general relativity [12–16]. However, it has remained unclear whether the RVB method can be applied to black holes under f(R) gravity. In this study, we discovered that the Hawking temperature of black holes under f(R) gravity could be readily obtained using the RVB method.

In this study, we investigate the Hawking temperatures of four types of black holes under some gravity models by comparing the RVB method with the regular one. We discovered that an integral constant is required for temperature calculation using the RVB method. This constant can be zero or a parameter term that does not correspond to the Hawking temperature under the conventional method. Our findings indicate that the RVB method can determine the Hawking temperature of black holes under various gravity theories, offering a more straightforward and effective approach to calculating black hole temperatures.

The structure of this paper is organized as follows. The second section introduces the main formula under investigation: the special expression for the Hawking temperature of a two-dimensional black hole system. In the third section, we study the properties of known topological invariants and the RVB method to calculate the Hawking temperature of Schwarzschild-like black holes under f(R) gravity. In the fourth section, we extend the RVB method to calculate the Hawking temperature in RN black hole systems under f(R) gravity. In the fifth section, we use this formula to analyze the characteristics of the Hawking temperature in terms of BTZ black holes under f(R) gravity. In the sixth section, we apply the RVB method to calculate the Hawking temperature of the Kerr-Sen black hole. Finally, in the seventh section, we present the conclusion and discussion of our findings.

2 General Calculation Method and RVB Method for the Hawking Temperature of a Black Hole under the f(R) Gravitational Model

2.1 General Method for Calculating the Temperature of a Black Hole

In the f(R) theory, the metric of a general static spherically symmetric black hole [17–26] is

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2,$$

where $g(r)$ is the general function of the coordinate $r$. We know that surface gravity is

$$\kappa_K = \frac{g'(r_+)}{2}.$$  \hspace{1cm} (2)

Then the black hole temperature is

$$T = \frac{\kappa_K}{2\pi} = \frac{g'(r_+)}{4\pi}.$$  \hspace{1cm} (3)
2.2 RVB Method for Calculating the Temperature of a Black Hole

Black hole systems come in various forms, and although it is relatively easy to determine the temperature of stationary or rotating black holes using simple metrics, many special black holes can be challenging to calculate their temperature when using a complex coordinate system. Therefore, the RVB method relates the Hawking temperature of a black hole to the Euler characteristic $\chi$, which is very useful in calculating the Hawking temperature in any coordinate system. We cannot observe the inner horizon’s temperature since we are located between the cosmological and event horizons. Thus, this paper focuses on the observable Hawking temperature of black holes.

As mentioned in the previous theorem, using a complex coordinate system can pose difficulties in calculating the temperature of certain black holes. In the RVB method, the Euler characteristic $\chi$ is related to the integral of the density $\Pi$ over the boundary of $V^n$, and the value of $\chi$ for a space can be defined as the integral of a certain density function $G$. However, for a manifold with boundary, an important correction to the value of $\chi$ is required, which is given by: [12–14]

$$\chi = \int_{\partial V} \Pi - \int_{\partial M} \Pi.$$ (4)

In previous literature [12–14], the Euler characteristic of an $n$-dimensional compact manifold $M^n$ is defined as the integral of a density form $\Omega$ on the manifold, i.e., $\chi = \int_M \Omega$. In this paper, we aim to study the black hole metric in a special coordinate system, and all our calculations assume that the density form is in the Riemann coordinate system. The boundaries of $V^n$ in $M^{2n-1}$ are an essential concept defined as the fixed point or zero point of the unit vector field in $M^n$. The form $\Omega$ is equal to the exterior derivative of another form $\Pi$ of degree $n-1$, defined by $\Omega = -d\Pi$ in $M^{2n-1}$. The integral of $\Omega$ over $M^n$ is equivalent to the integral over the submanifold $V^n$ of $M^{2n-1}$, and by Stokes’ theorem, it is also equivalent to $\Pi$ on the boundary of $V^n$.

The Euler characteristic [1–5] can be described:

$$\chi = \int_{r_0} \Pi - \int_{r_H} \Pi - \int_{r_0} \Pi = -\int_{r_H} \Pi.$$ (5)

In a word, in the calculation of the Euler characteristic, the outer boundary is permanently canceled out. Therefore, the integral should only be related to the Killing horizon.

According to references [12–16, 19], the topological formula can obtain the Hawking temperature of a two-dimensional black hole:

$$T_H = \frac{\hbar c}{4\pi k_B \chi} \sum_{j \leq \chi} \int_{r_{H_j}} \sqrt{|g|} |Rdr|,$$ (6)

among them, $\hbar$ is Planck’s constant, $c$ is the speed of light, $k_B$ is the Boltzmann constant, $g$ is the determinant of the metric, $R$ is the Ricci scalar, and $r_{H_j}$ is the location of the Killing horizon. This paper employs the natural unit system, where $\hbar = c = k_B = 1$. The Euler characteristic $\chi$, which depends on the spatial coordinate $r$, represents the Killing level in Euclidean geometry. Through transformation, in this paper, $|g|=1$.

From $|g|=1$, the Hawking temperature can be rewritten by using Eq. (6)
where $r_{c(-)}$ represents the radius of the additional horizon (cosmological horizon or inner horizon), and $r_+$ is the radius of the event horizon. In the Euclidean coordinate system, the period $\tau$ is fixed on the event horizon, but for the cosmological horizon, a conic singularity is present, and a boundary must be introduced to remove it.

\[ T_H = -\frac{1}{2} \left( \frac{1}{4\pi} \int_{r_{c(-)}} Rdr - \frac{1}{4\pi} \int_{r_+} Rdr \right), \]  

(7)

3 Hawking Temperature of Schwarzschild-Like Black Holes Under f(R) Gravity Obtained by RVB Method

In this section, we briefly review the f(R) static black hole solution and its thermodynamics when the constant Ricci curvature is not equal to 0. The general form of its action is given by:

\[ I = \frac{1}{2} \int d^4x \sqrt{-g}f(R) + S_{\text{mat}}. \]  

(8)

3.1 In the Case of Constant Ricci Curvature (Initial Conditions)

3.1.1 Schwarzschild-de Sitter-f(R) Black Holes

A spherically symmetric solution with constant scalar curvature $R_0$ is considered a simple but important example. By comparison with [27–36], the Schwarzschild solution ($R_0 = 0$) or the Schwarzschild-de Sitter solution is

\[ ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2, \]  

(9)

where

\[ g(r) = 1 - \frac{2M}{r} - \frac{R_0 r^2}{12}. \]  

(10)

Calculating a Schwarzschild spherical symmetric solution with constant scalar curvature $R_0$ under the f(R) model is not difficult. The surface gravity on the event horizon and the cosmological horizon can be expressed as:

\[ \kappa_h = \frac{R_0}{48} \frac{r_+^{-1}(r_c - r_+)(r_+ - r_-)}, \]

\[ \kappa_c = \frac{R_0}{48} \frac{r_c^{-1}(r_c - r_+)(r_c - r_-)}, \]  

(11)

where $r_c$ is the radius of the cosmological horizon, $r_+$ is the radius of the event horizon, and $r_-$ is the radius of the inner horizon.

It is worth noting that the Schwarzschild-de Sitter black hole in the f(R) model has two horizons, the event horizon and the cosmological horizon, both of which emit Hawking radiation. In the Euclidean coordinate system, the two-dimensional line element is [10, 17]
\[ ds^2 = g(r) d\tau^2 + \frac{dr^2}{g(r)}. \]  

(12)

Thus the Ricci scalar is

\[ R = \frac{d^2}{dr^2} g(r) = -\frac{4M}{r^3} - 1/6R_0. \]  

(13)

From \(|g|=1\), the Hawking temperature is gotten by Eqs. (6) and (7):

\[ T_H = -\frac{1}{4\pi} \left( \frac{1}{4\pi} \int_{r_c} Rdr - \frac{1}{4\pi} \int_{r_+} Rdr \right). \]  

(14)

There were two Killing horizons at that time, the Euler characteristic is 2, and we get

\[ T_H = \kappa_h/(2\pi) + \kappa_c/(2\pi) + C. \]  

(15)

\(r_c\) is the radius of the cosmological horizon, \(r_+\) is the radius of the event horizon, both are the radius of the Killing horizon, and \(C\) is the integral constant. In the Euclidean coordinate system, the period \(\tau\) is already fixed on the event horizon. However, the cosmological horizon has a conic singularity, which needs to be removed by introducing a boundary.

When comparing Eq. (15) with the Hawking temperature calculated using the conventional method, we observe a difference in the integral constant used in the calculation.

\[ T_H = \kappa_h/(2\pi) + \kappa_c/(2\pi), \]  

(16)

we noticed \(C\) is 0.

### 3.1.2 Black Hole in the Form of \(f(R)\) Theory:

\[ f(R) = R - qR^{\beta+1} \frac{\alpha \beta + \alpha + e}{\beta + 1} + q eR^{\beta+1} \ln \left( \frac{a^\beta R^\beta}{c} \right) \]

One of the forms of \(f(R)\) gravity is [17–26]

\[ f(R) = R - qR^{\beta+1} \frac{\alpha \beta + \alpha + e}{\beta + 1} + q eR^{\beta+1} \ln \left( \frac{a^\beta R^\beta}{c} \right). \]  

(17)

where \(0 \leq e \leq \frac{\beta}{4} \left( 1 + \frac{2}{\beta} \alpha \right), q = 4a^\beta_0/c(\beta + 1), \alpha \geq 0, \beta \geq 0\) and \(a_0 = \ell_p^2, \alpha\) and \(c\) are constants. Since \(R \neq 0\), this \(f(R)\) theory has no Schwarzschild solution. Its metric form is similar to Eq. (9) and

\[ g(r) = 1 - \frac{2m}{r} + \beta_1 r, \]  

(18)

where \(m\) is related to the mass of the black hole, and \(\beta_1\) is a model parameter.

To clarify, we reduce the angular degrees of freedom of spacetime because the definition of the Killing horizon does not involve them. In the Euclidean coordinate system, the two-dimensional metric is given as follows [10, 17], and the Euler characteristic is equal to 1:
Thus the Ricci scalar is
\[ R = \frac{d^2 g(r)}{dr^2} = -\frac{4m}{r^3}. \] (20)

From \(|g|=1\), the Hawking temperature writes
\[ T_H = \frac{2m}{r^+_H} / (4\pi) + C. \] (21)

C is the integral constant, \( r_+ \) is the radius of the event horizon, which is also the only radius of the Killing horizon,
\[ r_+ = \frac{\sqrt{8\beta_1 m + 1} - 1}{2\beta_1}. \] (22)

The Hawking temperature under the conventional method is
\[ T_H = \left( \frac{2m}{r^+_H} + \beta_1 \right) / (4\pi), \] (23)
we noticed C is \( \beta_1 / (4\pi) \).

### 3.2 In the Case of Non-Constant Ricci Curvature (Initial Conditions)

#### 3.2.1 Black Hole in the Form of f(R) Theory: \( f(R) = R + 2\alpha \sqrt{R} \)

The format of the metric is similar to Eq. (9) [30], where
\[ g(r) = \frac{1}{2} + \frac{1}{3\alpha r}. \] (24)

The Hawking temperature is obtained using the RVB method. In this case, the Euler characteristic equals 1, and the Killing horizon coincides with the event horizon. The twodimensional line element in the Euclidean coordinate system is given by: [10, 17]
\[ ds^2 = g(r)d\tau^2 + \frac{dr^2}{g(r)}. \] (25)

Thus the Ricci scalar is
\[ R = 2 / (3\alpha (r^3)). \] (26)

The Hawking temperature writes
\[ T_H = -1 / ((12\pi\alpha) r_+^2) + C. \] (27)

C is the integral constant. At the event horizon, \( g(r_+) = 0 \), resulting in...
The Hawking temperature under the conventional method is
\[ T_H = -1/((12\pi)\alpha r_+^2). \]  
(29)
In contrast, C is 0.

### 3.2.2 Black Hole in the Form of f(R) Theory: \( f(R) = R + 2\alpha \sqrt{R - 4\Lambda - 2\Lambda} \)

The second f(R) model considered with non-constant curvature is given by literature \([17–28]\)
\[ f(R) = R + 2\alpha \sqrt{R - 4\Lambda - 2\Lambda}. \]  
(30)
The constant \( \alpha < 0 \) is the integration constant, which can be interpreted as the cosmological constant. The metric form is similar to Eq. (9).

In the Euclidean coordinate system, the two-dimensional line element is \([10, 17]\)
\[ ds^2 = g(r)d\tau^2 + \frac{dr^2}{g(r)}, \]  
(31)
where
\[ g(r) = \frac{1}{2} + \frac{1}{3\alpha r} - \frac{\Lambda}{3} r^2. \]  
(32)
Since there is only one positive root of \( g(r) = 0 \), the Ricci scalar is given by:
\[ R = 2/(3\alpha(r^3)) - 2\Lambda/3. \]  
(33)
Using the RVB method, we see that
\[ T_H = -1/((12\pi)\alpha r_+^2) - 2\Lambda r_+/(12\pi) + C. \]  
(34)
C is the integral constant.

The Hawking temperature under the conventional method is
\[ T_H = -1/((12\pi)\alpha r_+^2) - 2\Lambda r_+/(12\pi). \]  
(35)
In contrast, C is 0.

If the cosmological constant is positive, the resulting Euler characteristic becomes 2, and the metric becomes isomorphic to the example discussed in Section 3.1.1. This conclusion is consistent with the findings presented in that section.

### 3.2.3 Black Hole in the Form of f(R) Theory: \( f(R) = R - \frac{\mu^4}{R} \) and \( f(R) = R - \lambda \exp(-\xi R) \)

The solution to this model in four dimensions is given by \([30–36]\), with a metric form similar to Eq. (9). In the case of a negative cosmological constant, the solution is given by:
where we should set \( \Lambda = \pm \frac{\mu^2}{2d} \sqrt{d^2 - 4} \) or \( \lambda = \frac{2d\Lambda e^{\varphi}}{d+2\varphi} k_c \), is a constant; it can take 1, -1, 0.

To ensure consistency with the negative terminal of \( \Lambda \) for \( d \) dimensions, we should set \( \Lambda \) as negative. In the Euclidean coordinate system, the line element for a two-dimensional space can be expressed as [10, 17]:

\[
 ds^2 = g(r) d\tau^2 + \frac{dr^2}{g(r)}. \tag{37}
\]

Thus the Ricci scalar is

\[
 R = \frac{d^2}{dr^2} g(r) = -(d-3)(d-2) \frac{M}{r^{d-1}} - 4/((d-1)(d-2))\Lambda. \tag{38}
\]

The Hawking temperature writes

\[
 T_H = -1/((d-1)(d-2))\Lambda r_+ + (d-3)M/(4\pi r_+^{d-2}) + C. \tag{39}
\]

\( C \) is the integral constant.

Let \( g(r)=0 \), we get

\[
 r_+ = -((2^{1/3} \times (2k - 3dk + d^2k))/(216MA^2 - 324dMA^2 + 108d^2MA^2 + \sqrt{-864(2k - 3dk + d^2k)^3}\Lambda^3 + (216MA^2 - 324dMA^2 + 108d^2MA^2)^2))^{1/3}) \nonumber
\]

\[
 - \frac{1}{6 \times 2^{1/3} \Lambda} (216MA^2 - 324dMA^2 + 108d^2MA^2 \nonumber
\]

\[
 + \sqrt{-864(2k - 3dk + d^2k)^3}\Lambda^3 + (216MA^2 - 324dMA^2 + 108d^2MA^2)^2)^{1/3}. \tag{40}
\]

The Hawking temperature under the conventional method is

\[
 T_H = -1/((d-1)(d-2))\Lambda r_+ + (d-3)M/(4\pi r_+^{d-2}). \tag{41}
\]

By comparison, we get \( C \) is 0.

If the cosmological constant is positive, the Euler characteristic equals 2, and the metric is isomorphic to the one discussed in Section 3.1.1. Thus, the conclusion reached in this case is in line with the findings of Section 3.1.1.

### 3.2.4 Black Hole in the Form of f(R) Theory: \( df(R)/dR = 1 + \alpha r \)

The metric form is similar to Eq. (9) [36] where

\[
 g(r) = C_2 r^2 + \frac{1}{2} + \frac{1}{3\alpha r} + \frac{C_1}{r} \left[ 3\alpha r - 2 - 6\alpha^2 r^2 + 6\alpha^3 r^3 \ln \left( 1 + \frac{1}{\alpha r} \right) \right]. \tag{42}
\]

\( C_1 \) and \( C_2 \) are constants.

Using the RVB method, we determined that the event horizon coincides with the Killing horizon. Specifically, for a two-dimensional space in the Euclidean coordinate system, the metric can be written as follows: [10, 17]:

\[ \square \] Springer
So the Ricci scalar is

\[ R = 2C_2 + \frac{2}{3ar^3} - \frac{4C_1}{r^3} - C_1 \frac{12a^2}{(1/ar + 1)r^2} - \frac{6aC_1}{(1/ar + 1)^2 r^2} + 12a^3C_1 \ln \left( \frac{1}{ar + 1} \right). \]  \hspace{1cm} (44)

The Hawking temperature is

\[ T_H = \left( 2C_2 r_+ - \frac{1}{3ar_+^2} + \frac{2C_1}{r_+^2} + C_1(12a^3 \ln \left( \frac{1}{ar_+ + 1} \right) r_+ - \frac{6a^2}{(1/ar_+ + 1)^2}) \right) /(4\pi) + C. \]  \hspace{1cm} (45)

C is the integral constant. The Hawking temperature under the conventional method is

\[ T_H = \left( 2C_2 r_+ - \frac{1}{3ar_+^2} + \frac{2C_1}{r_+^2} + C_1(12a^3 \ln \left( \frac{1}{ar_+ + 1} \right) r_+ - \frac{6a^2}{(1/ar_+ + 1)^2}) \right) /(4\pi). \]  \hspace{1cm} (46)

In contrast, C is \(-6a^2C_1/(4\pi)\).

If the function \( g(r) \) has two additional positive roots, the resulting metric is equivalent to the example presented in Section 3.1.1, and the conclusion agrees with that section’s findings.

### 3.2.5 Black Hole in the Form of \( f(R) \) Theory: \( f(R) = R + \Lambda + \frac{\frac{R + \Lambda}{\frac{R}{R_0} + 2/\alpha}}{\frac{R + \Lambda}{R_0}} \ln \frac{R + \Lambda}{R_0} \)

The metric form of the space as mentioned above is similar to Eq. (9), [37]:

\[ g(r) = 1 - \frac{2M}{r} + \beta r - \frac{\Lambda r^2}{3}. \]  \hspace{1cm} (47)

\( \beta > 0. \)

The RVB method can calculate the Hawking temperature, where the Euler characteristic equals 1, and the event horizon is the unique Killing horizon. In the Euclidean coordinate system, the metric for a two-dimensional space can be expressed as [10, 17]:

\[ ds^2 = g(r) d\tau^2 + \frac{dr^2}{g(r)}. \]  \hspace{1cm} (48)

So the Ricci scalar is

\[ R = -\frac{4M}{r^3} - \frac{2\Lambda}{3}. \]  \hspace{1cm} (49)

When \( g(r) = 0 \) has only one positive root, the Euler characteristic is 1,

\[ T_H = \left( \frac{2M}{r_+^2} - \frac{2\Lambda r_+}{3} \right) /(4\pi) + C. \]  \hspace{1cm} (50)
C is the integral constant.

Let \( g(r) = 0 \), we get

\[
\frac{r_+}{\Lambda} + \frac{2^{1/3} \times (-9\beta^2 - 9\Lambda)}{3\Lambda \left[ -54\beta^3 - 81\beta\Lambda + 162\Lambda^2 + \sqrt{4(-9\beta^2 - 9\Lambda)^3 + (-54\beta^3 - 81\beta\Lambda + 162\Lambda^2)^2} \right]^{1/3}} - \frac{(54\beta^3 - 81\beta\Lambda + 162\Lambda^2 + \sqrt{4(-9\beta^2 - 9\Lambda)^3 + (-54\beta^3 - 81\beta\Lambda + 162\Lambda^2)^2})^{1/3}}{3 \times 2^{1/3}\Lambda}.
\] (51)

The Hawking temperature under the conventional method is

\[
T_H = \frac{2M}{r_+^2} - \frac{2\Lambda r_+}{3} + \beta)/(4\pi).
\] (52)

By contrast, \( C = \beta/(4\pi) \).

If the cosmological constant is positive, the resulting metric is isomorphic to the one presented in Section 3.1.1, and the conclusion drawn from this case is consistent with Section 3.1.1. Additionally, the Euler characteristic in this situation is equal to 2.

### 4 Hawking Temperature of RN Black Holes Under \( f(R) \) Gravity by RVB Method

This section will briefly overview four-dimensional charged black holes in an \( f(R) \) gravitational background with a constant Ricci scalar curvature. Relevant literature on this topic includes references [13, 31, 37–39]. The corresponding action for this system can be expressed as follows:

\[
S = \int_{\mathcal{M}} d^4x \sqrt{-g} [f(R) - F_{\mu\nu}F^{\mu\nu}].
\] (53)

#### 4.1 Solutions of the Black Hole with Constant Ricci Curvature

##### 4.1.1 \( f(R) – RN – deSitter \) Black Hole

We compare the Reissner-Nordstrom black hole in de Sitter spacetime with the spherically symmetric solution of \( f(R) \) gravity with a constant curvature scalar \( R_0 \), or initial Ricci curvature, under the same model. For instance, we can consider the RN black hole solution, where \( R_0 = 0 \). In this solution, \( g(r) \) takes the form of \( g(r) = 1 - \frac{2M}{r} - \frac{R_0 r^2}{12} + \frac{Q^2}{r^2} \), and the corresponding metric is given by [33]:

\[
ds^2 = -\left(1 - \frac{2M}{r} - \frac{R_0 r^2}{12} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} - \frac{R_0 r^2}{12} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\] (54)

where \( R_0(>0) \) is the cosmological constant.

Let \( g(r) = 0 \). Three solutions can be obtained,
1 - \frac{2M}{r} - \frac{R_0 r^2}{12} + \frac{Q^2}{r^2} = 0. \tag{55}

r_1 is a negative root, which has no physical meaning; \( r_i \) is the smallest positive root, corresponding to the inner horizon of the black hole; \( r_e \) is a smaller positive root, corresponding to the outer horizon of the black hole; \( r_c \) is the largest positive root. The three surface gravities are

\[ \kappa^1 = \frac{R_0}{48} r_i^{-2}(r_i - r_1)(r_c - r_i)(r_e - r_i), \]
\[ \kappa^2 = \frac{R_0}{48} r_e^{-2}(r_c - r_1)(r_c - r_i)(r_e - r_c), \tag{56} \]
\[ \kappa^3 = \frac{R_0}{48} r_c^{-2}(r_c - r_1)(r_c - r_i)(r_c - r_e). \]

We can employ the RVB method to determine the Hawking temperature. The resulting Euler characteristic for this solution is equal to 2. In the Euclidean coordinate system, the corresponding two-dimensional metric is given by [10, 17]:

\[ ds^2 = g(r) d\tau^2 + \frac{dr^2}{g(r)}, \tag{57} \]

so the Ricci scalar is

\[ R = \frac{d^2}{dr^2} g(r) = -\frac{4M}{r^3} - 1/6R_0 + 6Q^2/r^4. \tag{58} \]

By using Eq. (7), we obtain the following:

\[ T_H = -\frac{1}{2} \left( \frac{1}{4\pi} \int_{r_i} R dr - \frac{1}{4\pi} \int_{r_e} R dr \right), \tag{59} \]

and

\[ T_H = \kappa_2/(2\pi) + \kappa_3/(2\pi) + C. \tag{60} \]

\( r_c \) and \( r_e \) refer to the radii of the cosmological and event horizons, respectively, which are both Killing horizons. \( C \) is an integral constant. Since we are situated between the cosmological and event horizons, the temperature of the inner horizon cannot be observed. In the Euclidean signature, the period \( \tau \) is fixed at the event horizon. Therefore, it is necessary to introduce a boundary to remove the conic singularity exhibited by the cosmological horizon.

The Hawking temperature under the conventional method is

\[ T_H = \kappa_2/(2\pi) + \kappa_3/(2\pi). \tag{61} \]

By contrast, \( C \) is 0.

4.1.2 Black Hole in the Form of \( f(R) \) Theory: \( f(R) = R - \alpha R^n \)

The example we focus on is [16, 30]
The metric form is similar to Eq. (54), where

\[ f(R) = R - \alpha R^n. \]  

The metric is similar to Eq. (54), where

\[ g(r) = 1 - \frac{2m}{r} + \frac{q^2}{br^2}. \]  

\[ b = f'(R_0) \] and the two parameters \( m \) and \( q \) are proportional to the black hole mass and charge, respectively [14]

\[ M = mb, \quad Q = \frac{q}{\sqrt{b}}. \]  

In the Euclidean coordinate system, the line element for a two-dimensional space can be expressed as [10, 17]:

\[ ds^2 = g(r) d\tau^2 + \frac{dr^2}{g(r)}, \]  

so the Ricci scalar is

\[ R = \frac{d^2}{dr^2} g(r) = -\frac{4m}{r^3} + \frac{6q^2}{br^4}. \]  

We got

\[ T_H = \left( \frac{m}{r_+^2} - \frac{q^2}{br_+^3} + \frac{m}{r_-^2} - \frac{q^2}{br_-^3} \right)/(4\pi) + C. \]  

\( r_+ \) and \( r_- \) represent the radii of the inner horizon and event horizon, respectively. Both of these radii correspond to the radius of the Killing horizon. \( C \) is the integral constant.

\[ r_+ = \sqrt{m^2 - Q^2} + m, \quad r_- = m - \sqrt{m^2 - Q^2}. \]  

The Hawking temperature under the conventional method is

\[ T_H = \left( \frac{m}{r_+^2} - \frac{q^2}{br_+^3} + \frac{m}{r_-^2} - \frac{q^2}{br_-^3} \right)/(4\pi). \]  

By contrast, \( C \) is 0.

4.1.3 Black Hole in the Form of \( f(R) \) Theory: \( f(R) = R - \lambda \exp(-\xi R) + \kappa R^2 \)

To simplify our calculations, we define \( \lambda = \frac{R_0^d}{2\xi^d} \) and \( \kappa = -\frac{1+2\xi R_0}{R(2+\xi R_0)} \), where \( \xi \) is a free parameter. For this study, we set \( d \) to be 4. The metric for a four-dimensional space can be expressed in a form similar to Eq. (54) [35], where

\[ g(r) = k - \frac{2\lambda}{(d-1)(d-2)} r^2 - \frac{M}{r^{d-3}} + \frac{Q^2}{r^{d-2}}. \]  

\( k \) is a constant; it can take 1, -1, 0. In the Euclidean coordinate system, the two-dimensional metric is [10, 17]
so the Ricci scalar is

\[ R = \frac{d^2}{dr^2} g(r) = -(d - 3)(d - 2) \frac{M}{r^{d-1}} - 4/((d - 1)(d - 2)) \Lambda + (d - 1)(d - 2) Q^2 / r^d. \]  

(72)

To determine their values, we must fix specific parameters such as \( \lambda \) and \( \kappa \). The metric form suggests that the Hawking temperature calculation for this static spherically symmetric black hole is consistent with the result obtained through the conventional method (where the integral constant is 0). When the cosmological constant is negative, the resulting Euler characteristic is equal to 1, and the Hawking temperature can be expressed as follows:

\[ T_H = -1/((d - 1)(d - 2)\pi) \Delta r_+ + (d - 3)M/(4\pi(r_+)^{d-2}) - (d - 2)Q^2/(4\pi(r_+)^{d-1}) + C. \]  

(73)

C is the integral constant, and \( r_+ \) is the radius of the event horizon.

The Hawking temperature under the conventional method is

\[ T_H = -1/((d - 1)(d - 2)\pi) \Delta r_+ + (d - 3)M/(4\pi(r_+)^{d-2}) - (d - 2)Q^2/(4\pi(r_+)^{d-1}). \]  

(74)

C is 0 when contrast is made.

If the cosmological constant is positive, the results agree with Section 4.1.1, and the conclusion remains the same.

4.2 The Solution of RN Black Hole has a Non-Constant Ricci Curvature

4.2.1 Black Hole in the Form of \( f(R) \) Theory: \( f(R) = 2a \sqrt{R - \alpha} \)

One of the versions of the \( f(R) \) model is described in reference [30]:

\[ f(R) = 2a \sqrt{R - \alpha}. \]  

(75)

In this model, the parameter \( \alpha \), which is associated with an effective cosmological constant, is expressed in units of \([\text{distance}]^{-1}\), and it satisfies the condition \( \alpha > 0 \). The expression for the static spherically symmetric black hole in this model is as follows:

\[ ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2, \]  

(76)

where

\[ g(r) = \frac{1}{2} \left( 1 - \frac{\alpha r^2}{6} + \frac{2Q}{r^2} \right). \]  

(77)

And \( Q \) is the integration constant. The event horizon of the black hole is located at: (a) when \( \alpha > 0 \) and \( Q > 0 \), \( r_s = \sqrt{3\alpha + a\sqrt{9 + 12aQ}/\alpha} \); (b) when \( \alpha > 0 \), \( Q < 0 \) and \( \alpha Q > -3/4 \), \( r_s = \sqrt{3a - a\sqrt{9 + 12aQ}/\alpha} \); (c) when \( \alpha < 0 \) and \( Q < 0 \), \( r_s = \sqrt{3/\alpha - a\sqrt{9 + 12aQ}/\alpha} \); (d) when \( \alpha > 0 \) and \( Q = 0 \), \( r_+ = \sqrt{6/\alpha} \).
When $\alpha > 0$ and $Q > 0$, or $\alpha < 0$ and $Q < 0$, or $\alpha > 0$ and $Q = 0$, there is only one positive root of $g(r) = 0$ that has physical significance. At this point, the Euler characteristic is equal to 1, and $|g|$ equals 1.

In the Euclidean coordinate system, the two-dimensional metric is given by [10, 17]:

$$ds^2 = g(r)d\tau^2 + \frac{dr^2}{g(r)}.$$  \hspace{1cm} (78)

The Ricci scalar is as follows

$$R = \frac{d^2}{dr^2} g(r) = 6Q/r^4 - \alpha/6.$$  \hspace{1cm} (79)

Using the RVB method, we get

$$T_H = (-\frac{2Q}{r_c^4} - \frac{r_c^2 \alpha}{6})/(4\pi) + C.$$  \hspace{1cm} (80)

$C$ is the integral constant.

The Hawking temperature under the traditional method is

$$T_H = (-\frac{2Q}{r_c^4} - \frac{r_c^2 \alpha}{6})/(4\pi).$$  \hspace{1cm} (81)

In contrast, $C$ is 0.

Upon varying the values of $\alpha$ and $Q$, we observe that the equation $g(r) = 0$ has two positive roots that correspond to the event horizon and the cosmological horizon, respectively. Both of these horizons are Killing horizons, and their structure is consistent with the description presented in Section 4.1.1. If $\alpha > 0$ and $Q < 0$, the resulting Euler characteristic is 2. In this case, we can use the RVB method to calculate the Hawking temperature, which is given by:

$$T_H = \frac{1}{2} \left( \frac{1}{4\pi} \int_{r_c} Rdr - \frac{1}{4\pi} \int_{r_e} Rdr \right).$$  \hspace{1cm} (82)

$$T_H = \kappa_2/(2\pi) + \kappa_3/(2\pi) + C.$$  \hspace{1cm} (83)

$r_c$ and $r_e$ represent the radii of the cosmological horizon and event horizon, respectively. Both of these radii correspond to the radius of the Killing horizon. $C$ is an integral constant.

The Hawking temperature under normal conditions is

$$T_H = \kappa_2/(2\pi) + \kappa_3/(2\pi).$$  \hspace{1cm} (84)

By contrast, $C$ is 0. This conclusion is in line with what was stated in Section 4.1.1.

$$\kappa_2 = \frac{R_0}{48} r_c^{-2} (r_c - r_1) (r_c - r_i) (r_c - r_e),$$  \hspace{1cm} (85)

$$\kappa_3 = \frac{R_0}{48} r_c^{-2} (r_c - r_1) (r_c - r_i) (r_c - r_e).$$

$R_0$ is the initial Ricci curvature.
4.2.2 Black Hole in the Form of $f(R)$ Theory: $f(R) = R - \lambda \exp(-\xi R) + \kappa R^n + \eta \ln(R)$

The metric form is similar to Eq. (54) [35], where

$$g(r) = k_1 - \frac{2\Lambda}{(d-1)(d-2)} r^2 - \frac{M}{r^{d-3}} + \frac{Q^2}{r^{d-2}}. \quad (86)$$

We take $d=4$,

$$\lambda = \frac{R + \kappa R^n - (R + n\kappa R^n) \ln R}{(1 + \xi R \ln R) e^{-\xi R}},$$

$$\eta = -\frac{(1 + \xi R) R + (n + \xi R) \kappa R^n}{1 + \xi R \ln R}, \quad (87)$$

where $\xi$ is a free parameter, and $k_1$ is a constant; it can take 1, -1, 0.

When the cosmological constant is positive, the structure of the black hole solution is consistent with that of the case where the cosmological constant is negative, as described in Section 4.1.1. We find that the equation $g(r) = 0$ has two positive roots, corresponding to the event horizon and the cosmological horizon, which are Killing horizons. Therefore, the conclusion is the same for the negative cosmological constant case.

We apply the RVB method to calculate the Hawking temperature,

$$T_H = -\frac{1}{2} \left( \frac{1}{4\pi} \int_{r_e}^{r_i} R dr - \frac{1}{4\pi} \int_{r_e}^{r_i} R dr \right). \quad (88)$$

$$T_H = \kappa_2/(2\pi) + \kappa_3/(2\pi) + C. \quad (89)$$

In the Euclidean signature, the period $\tau$ is already fixed on the event horizon, so the cosmological horizon has a conic singularity that needs to be removed by introducing a suitable boundary.

The Hawking temperature under normal conditions is

$$T_H = \kappa_2/(2\pi) + \kappa_3/(2\pi). \quad (90)$$

In contrast, $C$ is 0. This conclusion is in line with what was stated in Section 4.1.1.

$$\kappa_2 = \frac{R_0}{48} r_c^{-2} (r_e - r_1) (r_e - r_i) (r_c - r_e),$$

$$\kappa_3 = \frac{R_0}{48} r_c^{-2} (r_e - r_1) (r_e - r_i) (r_c - r_c). \quad (91)$$

$R_0$ is the initial Ricci curvature.

When the cosmological constant is negative, the two-dimensional metric is [10, 17]

$$ds^2 = g(r) d\tau^2 + \frac{dr^2}{g(r)}, \quad (92)$$

so the Ricci scalar is

$$R = \frac{d^2}{dr^2} g(r) = -(d - 3)(d - 2) \frac{M}{r^{d-1}} - 4/((d - 1)(d - 2)) \Lambda + (d - 1)(d - 2) Q^2 / r^d. \quad (93)$$
Only one positive root has physical significance when \( g(r) = 0 \). At this point, the Euler characteristic equals 1, and the event horizon is the only Killing horizon. By utilizing the RVB method, we can obtain the following expression:

\[
T_H = -1/((d - 1)(d - 2)\pi)\Lambda r_+ + (d - 3)M/(4\pi(r_+)^{d-2}) - (d - 2)Q^2/(4\pi(r_+)^{d-1}) + C. 
\]  

(94)

\( C \) is the integral constant, and \( r_+ \) is the radius of the event horizon.

Hawking temperature under the traditional method is

\[
T_H = -1/((d - 1)(d - 2)\pi)\Lambda r_+ + (d - 3)M/(4\pi(r_+)^{d-2}) - (d - 2)Q^2/(4\pi(r_+)^{d-1}). 
\]  

(95)

By contract, \( C \) is 0.

### 5 Hawking Temperature of the BTZ Black Hole Under \( f(R) \) Gravity Obtained by RVB Method

In the following two models, the cosmological constant is negative.

#### 5.1 Black Hole in the Form of \( f(R) \) Theory: \( f(R) = -4\eta^2 M \ln(-6\Lambda - R) + \xi R + R_0 \)

We bring the following solution: \([27, 28]\)

\[
ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\phi^2 \]  

(96)

where

\[
g(r) = -\Lambda r^2 - M(2\eta r + \xi). \]  

(97)

The curvature scalar is

\[
R = -2\Lambda - \frac{4M\eta}{r}. \]  

(98)

In the Euclidean coordinate system, the metric for a two-dimensional space can be expressed as \([10, 17]\):

\[
ds^2 = g(r)d\tau^2 + \frac{dr^2}{g(r)}. \]  

(99)

Thus the Ricci scalar is

\[
R = -2\Lambda. \]  

(100)

Using the RVB method, we can calculate the Hawking temperature,

\[
T_H = \frac{-\Lambda r_+}{2\pi} + C. \]  

(101)
C is the integral constant.

Let \( g(r) = 0 \), we get

\[
r_+ = -\frac{\sqrt{M(M\eta^2 - \Lambda^2)} + M\eta}{\Lambda}, \tag{102}
\]

\( r_+ \) is the event horizon radius and the unique Killing horizon radius.

The Hawking temperature calculation under the conventional method is

\[
T = \frac{-\Lambda r_+ - M\eta}{2\pi}. \tag{103}
\]

By contrast, \( C = \frac{-M\eta}{2\pi} \).

### 5.2 Black Hole in the Form of \( f(R) \) Theory: \( f(R) = -2\eta M \ln(6\Lambda + R) + R_0 \)

In the case where \( \Phi(r) = 0 \), the charged \((2 + 1)\)-dimensional solution under pure \( f(R) \)-gravity can be expressed using the metric given in Eq. (96), as shown in [27, 28], where

\[
g(r) = -\Lambda r^2 - Mr - \frac{2Q^2}{3\eta r}. \tag{104}
\]

The two-dimensional line element is [10, 17]

\[
ds^2 = g(r)d\tau^2 + \frac{dr^2}{g(r)}, \tag{105}
\]

at this point \(|g| = 1\), the Ricci scalar is

\[
R = -\frac{4Q^2}{3\eta r^3} - 2\Lambda. \tag{106}
\]

Applying the RVB method, we obtain the following:

\[
T_H = -\frac{\Lambda r_+}{2\pi} + \frac{Q^2}{6\pi \eta r_+^2} + C. \tag{107}
\]

C is the integral constant.

Let \( g(r) = 0 \), we get

\[
r_+ = \frac{1}{3} \left\{ \frac{M}{\Lambda} - \frac{M^2\eta}{\Lambda \left(M^3\eta^3 + 9Q^2\eta^2\Lambda^2 + 3\sqrt{2M^3Q^2\eta^5\Lambda^2} + 9Q^4\eta^4\Lambda^4 \right)^{1/3}} - \right.
\]

\[
\left. \frac{M^3\eta^3 + 9Q^2\eta^2\Lambda^2 + 3\sqrt{2M^3Q^2\eta^5\Lambda^2} + 9Q^4\eta^4\Lambda^4 \right)^{1/3}}{\eta\Lambda} \right\}. \tag{108}
\]
\( r_+ \) is the event horizon radius and the unique Killing horizon radius.

The calculation of the Hawking temperature using the conventional method is as follows:

\[
T = -\frac{\Lambda r_+}{2\pi} - \frac{M}{4\pi} + \frac{Q^2}{6\pi \eta r_+^2}.
\]  \hspace{1cm} (109)

By contrast, \( C \) is \( -\frac{M}{4\pi} \).

### 6 Hawking Temperature of Spherically Symmetric KERR-SEN Black Holes Obtained by the RVB Method

A specific solution exists for spherically symmetric black holes in certain modified gravity theories, such as the Kerr-Sen solution. The spacetime line element for the Kerr-Sen black hole is given by: [29]:

\[
ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\Omega^2,
\]  \hspace{1cm} (110)

where

\[
g(r) = \left( r^2 - 2M'r + a^2 \right)/\left( r_+^2 + a^2 \right).
\]  \hspace{1cm} (111)

The two-dimensional line element is [10, 17]

\[
ds^2 = g(r)d\tau^2 + \frac{dr^2}{g(r)},
\]  \hspace{1cm} (112)

the Ricci scalar is

\[R = 2/\left( r_+^2 + a^2 \right).\]  \hspace{1cm} (113)

The two horizons are

\[r_\pm = M' \pm \sqrt{M'^2 - a^2}.\]  \hspace{1cm} (114)

The effective mass is

\[M' = M - b1 = M - \frac{Q^2}{2M}.\]  \hspace{1cm} (115)

Note, \( r_\pm \) is the radius of the inner and outer horizons of the black hole, and \( b1 \) is the parameter related to the coordinate extension.

We find the Hawking temperature in this case [12–14]

\[
T_H = -\frac{1}{2} \left( \frac{1}{4\pi} \int_{r_-} Rdr - \frac{1}{4\pi} \int_{r_+} Rdr \right),
\]  \hspace{1cm} (116)

where \( r_- \) is the radius of the inner horizon, \( r_+ \) is the radius of the event horizon.


\[ T_H = \frac{\sqrt{(M')^2 - a^2}}{4\pi M'(M' + \sqrt{(M')^2 - a^2})} + C, \]  \tag{117}

C is the integral constant. Compared to the temperature obtained from the conventional method, the Hawking temperature \( T_H \), which is an essential thermodynamic quantity corresponding to a black hole, can be calculated at the poles as:

\[ T_H = \frac{1}{2\pi} \lim_{r \to r_+} \sqrt{g^{rr}} \partial_r \sqrt{-g_H(\theta = 0)}, \]  \tag{118}

we get

\[ T = \frac{\sqrt{(M')^2 - a^2}}{4\pi M'(M' + \sqrt{(M')^2 - a^2})}. \]  \tag{119}

By contrast, C is 0.

### 7 Conclusion and Discussion

This study uses the Euler characteristic in topological formulations to investigate the Hawking temperature of four significant black holes in various f(R) gravity theories. Our findings reveal a difference in the integral constant between the temperature results obtained by the RVB method and the conventional method. However, the integral constant can be determined accurately by using the standard definition to calculate the Hawking temperature. Therefore, the topological method can also be applied to calculating the temperature of black holes in conventional f(R) gravity. Additionally, we observed that the integral constants for different f(R) gravity theories are distinct, as they depend on the gravitational theory corresponding to each f(R) form.

The interpretation of the constant C in black hole physics depends on the specific properties of the black hole studied in a particular gravitational model. The metric and curvature scalar is often closely related to physical quantities such as the black hole’s mass, spin, and charge. In some gravity models, C may be related to the mass of the black hole or the graviton. By measuring it, we can gain information about the mass of the black hole or the graviton, enabling us to analyze the details of the gravity theory related to it. The significance of the constant C is that it represents the characteristic of modified gravity theories. The constant C also reflects the behavior of black holes in modified gravity theories, and each modified gravity theory shows different values of C, even if we set C=0.

Our interest lies in exploring the issue of the appearance of the constant of integration C, which can be confusing for many. Meanwhile, the RVB method is a valuable tool for studying the Hawking temperature of black holes as it employs a geometric approach. However, our research has revealed that there is still room for discussion regarding the RVB method. This discovery may have significant implications for further exploration of the thermodynamics of black holes and deeper physics. In our upcoming research, we plan to connect the temperature Green’s function method with the RVB...
method to address the issue of the undetermined constant \( C \). This research could shed new light on the behavior of black holes and further our understanding of the universe.

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**Declarations**

**Declarations** All partial information is available.

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