Reduction of covers and Hurwitz spaces

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Abstract

In this paper we study the reduction of Galois covers of curves, from characteristic zero to positive characteristic. The starting point is a recent result of Raynaud, which gives a criterion for good reduction for covers of the projective line branched at three points. We use the ideas of Raynaud to study the case of covers of the projective line branched at four points. Under some condition on the Galois group, we generalize the criterion for good reduction of Raynaud. As a new ingredient, we use the Hurwitz space of such covers. Combining our results on reduction of covers with the Hurwitz space approach, we are able to describe the reduction of the Hurwitz space modulo \( p \) and compute the number of covers with good reduction.

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Introduction

Let \( R \) be a complete discrete valuation ring whose residue field \( k \) is algebraically closed of characteristic \( p \) and whose quotient field \( K \) is of characteristic zero. Let \( f_K : Y_K \rightarrow \mathbb{P}^1_K \) be a Galois cover defined over \( K \). We ask ourselves whether \( f_K \) has good reduction. In case \( p \) divides the order of the Galois group, this is a hard question. We cannot expect good reduction, in general.

A recent paper of Raynaud gives a criterion for good reduction in the first case. Let \( f_K : Y_K \rightarrow \mathbb{P}^1_K \) be a Galois cover branched at 0, 1 and \( \infty \). Suppose that \( p \) strictly divides the order of the Galois group \( G \), but not the ramification indices of \( f_K \) and that the center of \( G \) is trivial. Let \( P \cong \mathbb{Z}/p \) be a \( p \)-Sylow of \( G \), and denote by \( n := [N_G(P) : C_G(P)] \) the index of the centralizer of \( P \) in the normalizer of \( P \). Let \( e \) be the absolute ramification index of \( p \) in \( K \). It is shown that the cover has good reduction, provided that \( en < p - 1 \). The idea of the proof is to suppose that \( f_K \) has bad reduction and to study the semistable reduction of \( f_K \). Raynaud proves general structure results on the semistable reduction. Using these techniques, he proves that bad reduction implies \( en \geq p - 1 \).

In this paper, we follow the approach of Raynaud. We consider the reduction of Galois covers \( f_K : Y_K \rightarrow \mathbb{P}^1_K \) branched at four points. We suppose that \( p || G \), but that \( p \) does not divide the ramification indices of \( f_K \). This is the next case to study after the result of Raynaud. However, we put a much stronger condition on the group \( G \). In particular, we assume that \( n = 2 \). The criterion for good reduction given by Raynaud extends to our situation. The stronger condition on the Galois group allows us to get a stronger result. For instance, we are able to describe the semistable model of \( f_K \).

When passing from 3 to 4 branch points, a new aspect arises: the reduction of \( f_K \) might depend on the position of the branch points. It is therefore natural to study the corresponding Hurwitz space, i.e. the moduli space of \( G \)-covers of a certain type, and its reduction to positive characteristic. Hurwitz spaces were first introduced in a purely geometric context, but have since then proved
The statement that group of order 2 \(N\) modular reduction

In this paper, we look at the following situation. Let \(E\) be an elliptic curve with good ordinary reduction. Then there are precisely \(\frac{p^2 - p - 1}{2}\) nonisomorphic \(G\)-covers branched at 4 points, of order \(p\). The pair \((E, P)\) corresponds to a point on \(X_1(p)\). This gives an identification \(H \cong X_1(p)\), for a suitable Hurwitz space \(H\). By the results of Katz and Mazur [17], the reduction of \(X_1(p)\) to characteristic \(p\) is well understood. One can check that the subspace \(\tilde{H}\) of the arithmetic compactification of \(H\) corresponds to the component of \(X_1(p) \otimes \mathbb{F}_p\) parameterizing pairs \((E, P)\) such that \(P = 0\) and hence the isogeny \(\pi: E \to E'/\langle P\rangle\) is inseparable. Even without using the very precise results of [17], the theory of elliptic curves gives the following result on good reduction of Galois covers. Suppose the elliptic curve \(E'_K\) given by the equation \(y^2 = x(x-1)(x-\lambda)\) has good ordinary reduction. Then there are precisely \(p - 1\) nonisomorphic \(G\)-covers branched in 0, 1, \(\infty\) and \(\lambda\), with ramification index 2, which have good reduction. On the other hand, there is no such cover with good reduction if \(E'_K\) has supersingular reduction.

Results

In this paper, we look at the following situation. Let \(G\) be a finite group and \(p\) an odd prime which strictly divides \(|G|\). We assume that the normalizer of a \(p\)-Sylow of \(G\) is a dihedral group. Let \(K\) be as in the beginning, and let \(f_K: Y_K \to \mathbb{P}_K\) be a \(G\)-Galois cover branched at 4 points, of order prime-to-\(p\). We prove that the cover \(f_K\) has either good reduction or a very specific type of bad reduction, which we call modular reduction.

We will briefly explain what this means. Assume that \(f_K\) has bad reduction. Following Raynaud [21, §3.2], we associate to the \(G\)-cover \(f_K\) a \(\Delta\)-cover \(g_K: Z_K \to \mathbb{P}_K\), called the auxiliary cover. Here \(\Delta\) is a subgroup of \(G\), and \(g_K\) is branched in the same points as \(f_K\) and has bad reduction. The statement that \(f_K\) has modular reduction of level \(N\) means essentially that \(\Delta\) is a dihedral group of order \(2N\) (where \(p\nmid N\)) and that \(g_K\) has ramification of order 2. In particular, \(g_K\) gives rise to a \(K\)-point on \(X_1(N)\). This is the link between our results and modular curves.
To explain the construction of $g_K$, we assume for simplicity that the normalizer of a $p$-Sylow of $G$ is of order $2p$ and that the branch points of $f_K$ do not coalesce modulo $p$. Let $f : Y \to X$ be the special fiber of the semistable model of $f_K$. The curve $X$ consists of 5 components: the strict transform of the original component $X_0$, and 4 tails $X_1, \ldots, X_4$ containing the specializations of the branch points $x_i$. The cover $f$ is inseparable over $X_0$ and separable over the tails. Let $E$ be a component of $Y$ above $X_0$. The decomposition group $\Delta := D(E) \subset G$ is dihedral of order $2p$, the inertia group $I(E)$ cyclic of order $p$ (see Fig. 2 in Section 2.4). We obtain $g : Z \to X$ by replacing, for $i = 1, \ldots, 4$, the (disconnected) $G$-cover $f^{-1}(X_i) \to X_i$ by a $\Delta$-cover $Z_i \to X_i$ which is locally, i.e. in an étale neighborhood of $E$, isomorphic to $f^{-1}(X_i) \to X_i$ and tamely ramified above $x_i \in X_i$. This is possible in a unique way, by the Katz–Gabber Lemma, [16]. Using formal patching one can show that $g : Z \to X$ is the reduction of a $\Delta$-cover $g_K : Z_K \to \mathbb{P}^1_K$ which, in some sense, contains all the information about the bad reduction of $f_K$.

Let $H \subset \mathcal{H}_G$ be the subset of the Hurwitz space corresponding to $G$-covers with 4 branch points and prime-to-$p$ ramification. Denote by $\bar{H}$ its arithmetic compactification and by $\bar{H}^{\text{bad}} \subset \bar{H} \otimes \mathbb{F}_p$ the subspace corresponding to bad reduction in characteristic $p$. The map $f : Y \to X$ discussed above corresponds to a $k$-point on $\bar{H}^{\text{bad}}$; the associated map $g : Z \to X$ corresponds essentially to a $k$-point on $X_1(N) \otimes \mathbb{F}_p$. The formal patching argument mentioned above can be used to show that the deformation theory of $f$ and $g$ are equivalent. This gives a strong connection between the subspace $\bar{H}^{\text{bad}} \subset \bar{H}$ and the component of $X_1(N) \otimes \mathbb{F}_p$ corresponding to inseparable isogenies. Using the results of [17] on the reduction of $X_1(N)$, we prove our Reduction Theorem, which describes $\bar{H} \otimes \mathbb{F}_p$. It can be roughly stated as follows (see Fig. 1). The subspaces $\bar{H}^{\text{good}}$ and $\bar{H}^{\text{bad}}$ are smooth curves over $\mathbb{F}_p$; they intersect transversally in the supersingular points, i.e. the points of $\bar{H}^{\text{bad}}$ with supersingular $\lambda$-value. The scheme $\bar{H} \otimes \mathbb{F}_p$ is not reduced, in general; the irreducible components of $\bar{H}^{\text{bad}}$ have multiplicity $p - 1$ or $(p - 1)/2$. Moreover, each irreducible component of $\bar{H}^{\text{bad}}$ is essentially the reduction of a modular curve.

![Figure 1: the Reduction Theorem](image_url)

We call the $\mathbb{Q}$-rational points of $\bar{H}$ lying above 0, 1 and $\infty$ the **cusps** of $\bar{H}$. A cusp corresponds to a degenerate cover $f_{\bar{Q}} : Y_{\bar{Q}} \to X_{\bar{Q}}$, obtained from a smooth $G$-cover by coalescing of branch points. The curve $X_{\bar{Q}}$ is the union of two projective lines intersecting in one point. The cover $f_{\bar{Q}}$ is ramified above the singular point of $X_{\bar{Q}}$, say of order $n$. Our next result, which we call the Cusp Principle, states that this cusp has bad reduction (i.e. its closure in $\bar{H}$ meets $\bar{H}^{\text{bad}}$) if and only if $p|n$. Essentially, this is an application of Raynaud’s result, since the degenerate cover $f_{\bar{Q}}$ is built up from two 3 point covers. The Cusp Principle is the key result in our calculation of the number of $G$-covers with good reduction. The point here is that, via the Hurwitz classification and the braid action, one can explicitly compute the set of cusps of $H$ and decide for each of them whether they have good or bad reduction. In particular, given a finite group $G$ and an odd prime...
verifying all the assumptions made above, we can compute two numbers, \( d \) and \( d_{\text{bad}} \), such that

\[
|\text{Cov}(G, \lambda)^{\text{good}}| = \begin{cases} 
  d - d_{\text{bad}}, & \text{if } \lambda \text{ is ordinary,} \\
  d - \frac{p+1}{p} d_{\text{bad}}, & \text{if } \lambda \text{ is supersingular.}
\end{cases}
\]

Here \( \text{Cov}(G, \lambda)^{\text{good}} \) is the set of isomorphism classes of \( G \)-covers of \( \mathbb{P}^1 \) with good reduction, branched in \( 0, 1, \infty \) and \( \lambda \).

Let \( G = \text{PSL}_2(\ell) \), where \( \ell \) is an odd prime different from \( p \) such that \( p \) exactly divides the order of \( G \), and consider \( G \)-covers branched in 4 points of order \( \ell \). We have computed the cusps of the corresponding Hurwitz spaces, using the computer program \( ho \) [21], for \( \ell \leq 31 \). From this information, we can deduce the complete structure of \( \overline{H}^{\text{bad}} \otimes \mathbb{F}_p \) and the number of covers with good reduction.

This paper owes a lot to Raynaud. It started as an attempt to understand a talk he gave in Oberwolfach, June 1997. In this talk Raynaud presented Example 4.3.2. In the problem session of the same conference, he gave a similar problem as an exercise. In a way, this paper is our solution of this exercise. We would also like to thank Bas Edixhoven for a helpful conversation and for sending his manuscript [7], and Andrew Kresch for comments on an earlier version of Section 1. The second author gratefully acknowledges financial support from the Deutsche Forschungsgemeinschaft.

**Notation**

In this paper we will understand by a \textit{semistable curve} a flat projective morphism \( X \to S \) of schemes whose geometric fibers are reduced connected curves having at most ordinary double points as singularities. We write \( X^{\text{sm}} \) for the subset of smooth points of the morphism \( X \to S \). A \textit{mark} on \( X/S \) is a closed subscheme \( D \subset X^{\text{sm}} \) which is finite and étale over \( S \). The pair \( (X/S, D) \) is called a marked semistable curve. A \textit{stably marked curve} is either a pointed stable curve \( (X/S; x_i) \) in the sense of [19] or a marked semistable curve \( (X/S, D) \) which becomes a pointed stable curve after an étale base change \( S' \to S \). By an \textit{algebraic stack} we mean an algebraic stack in the sense of Deligne–Mumford [5].

**1 Complete Hurwitz spaces**

The goal of this section is to define arithmetic compactifications of Hurwitz spaces for \( G \)-covers. For a given Hurwitz space \( H \), such a compactification \( \overline{H} \) should be a proper model of \( H \) over \( \mathbb{Z} \) whose points in positive characteristic correspond to the reductions of the covers which are parameterized by \( H \). To make this precise, one first has to give the definition of the reduction of a \( G \)-cover. This is done in Section 1.1.

Our definition of the complete Hurwitz space for \( G \)-covers essentially follows the approach of [1]. We let \( \overline{H} \) be the closure of the moduli stack of \( G \)-covers inside a bigger moduli stack parameterizing certain maps between stably marked curves. Then we define \( \overline{H} \) as the coarse moduli space associated to \( \overline{H} \). This is the content of Section 1.2. Section 1.3 discusses a technical problem that arises from our definition. The reader who is not interested in this abstract approach may wish to skip these two sections.

**1.1 Reduction of \( G \)-covers**

In this section we give some terminology and recall some general facts concerning the reduction of \( G \)-covers to positive characteristic. We closely follow [24], §2. However, our definition of the model of a \( G \)-cover is not exactly the same as Raynaud’s. Also, since we allow the bottom curve to degenerate, we have to consider a slightly more general situation than in [24], §2.
**Definition 1.1.1** Let $K$ be a field and $G$ a finite group. A $G$-cover defined over $K$ is a finite separable morphism $f : Y \to X$ of smooth projective and geometrically irreducible $K$-curves together with an isomorphism $G \cong \text{Aut}(Y/X)$ such that $|G| = \deg f$. We say that a $G$-cover $f$ is tame if it is tamely ramified.

Throughout this section, we assume the following situation. Let $R$ be a complete discrete valuation ring with quotient field $K$ of characteristic zero, and residue field $k = \bar{k}$ of characteristic $p > 0$. Let $f_K : Y_K \to X_K$ be a $G$-cover defined over $K$ ($f_K$ is automatically tame). Write $x_{1, K}, \ldots, x_{r, K} \in X_K(K)$ for the branch points and $y_{1, K}, \ldots, y_{n, K} \in Y_K(K)$ for the ramification points of $f_K$. We assume that $2g + r \geq 3$, where $g$ is the genus of $X_K$.

We would like to define a model of the $G$-cover $f_K$ over the ring $R$. After replacing $K$ by a finite extension $K'/K$ and $R$ by its integral closure in $K'$, we may assume that the ramification points $y_{i, K}$ of $f_K$ are $K$-rational. After a further extension of $K$, we may assume that the smooth stably marked curve $(Y_K; y_{i, K})$ extends to a stably marked curve $(Y_R; y_{i, R})$ over $R$. In particular, $Y_R$ is semistable over $R$ and the points $y_{i, K}$ specialize to pairwise distinct, smooth points $y_i$ on the special fiber $Y$ of $Y_R$. Since the stably marked model is unique, the action of $G$ on $Y_K$ extends to $Y_R$. Let $X_R := Y_R/G$ be the quotient scheme and $X$ the special fiber of $X_R$. By [12], Appendix, $X_R$ is again a semistable curve over $R$. Since the ramification points $y_{i, K}$ specialize to pairwise distinct smooth points on $Y$, the branch points $x_{j, K}$ specialize to pairwise distinct smooth points $x_j \in X$. According to [13], the stably marked curve $(X_K; x_{j, K})$ extends to a stably marked curve $(X_R; x_{j, R})$ over $R$, and we have a well defined contraction morphism $X_R \to X_{0, R}$ sending $x_{j, R}$ to $x_{j, R}'$. Let $f_R : Y_R \to X_R$ and $f_{0, R} : Y_R \to X_{0, R}$ be the natural maps and $f : Y \to X$, $f_0 : Y \to X_0$ the induced maps on the special fibers.

**Definition 1.1.2** Let $f_R : Y_R \to X_R$ and $f_{0, R} : Y_R \to X_{0, R}$ be as above. We call $f_R : Y_R \to X_R$ the quotient model and $f_{0, R} : Y_R \to X_{0, R}$ the stable model over $R$ of the $G$-cover $f_K$.

The quotient model and the stable model of $f_K$ exist after a finite extension of $K$. It is clear that these models are stable with respect to any further extension of $K$. In many places it is better to work with the quotient model, because it is a finite map. However, the stable model is easier to study the moduli of. Therefore, our definition of a complete Hurwitz space will be based on the stable model.

**Definition 1.1.3**

(i) The $G$-cover $f_K$ has (potentially) good reduction if (after a finite extension of $K$) the special fiber $f : Y \to X$ of the quotient model of $f_K$ is a tame $G$-cover.

(ii) The $G$-cover $f_K$ has (potentially) admissible reduction if (after a finite extension of $K$) the special fiber $f : Y \to X$ of the quotient model of $f_K$ is a tame admissible cover. By this we mean that $f$ is finite, separable, tamely ramified over the smooth locus $X^m$ and has tame admissible ramification over the ordinary double points of $X$, see [13], §4, or [22], where such a cover is called kummérien.

(iii) The $G$-cover $f_K$ has bad reduction if it does not have potentially admissible reduction.

For the rest of this subsection we will omit the word “potentially” and assume that $K$ is chosen such that the quotient and stable model exist over $R$. Note that $f_K$ has good reduction if and only if it has admissible reduction and $X_R$ is smooth over $R$. If $f_K$ has admissible reduction then $X_R = X_{0, R}$, i.e. quotient and stable model are the same.

Let $W$ be a component of $X$. We call $W$ an original component if it is the strict transform of a component of $X_0$ (otherwise, the map $X \to X_0$ contracts $W$). We will say that $f$ is separable over $W$ if for one (and therefore for all) components $Z$ of $Y$ above $W$ the restriction $f|_Z : Z \to W$ is a separable morphism. Equivalently, the inertia group of $Z$ (the group of elements of $G$ acting
trivially on $Z$) is trivial. Proposition 1.1.4 below extends \[24\], Corollaire 2.4.9, to our situation, which includes admissible reduction. The proof is essentially the same, with \[23\], Théorème 3.2, as additional ingredient.

**Proposition 1.1.4** The $G$-cover $f_K$ has admissible reduction if and only if $f$ is separable over the original components of $X$.

From Proposition 1.1.4 we can deduce the following well known fact.

**Corollary 1.1.5** Assume that the order of $G$ is prime to the characteristic of $k$. Then $f_K$ has potentially admissible reduction. If in addition the branch points $x_{i,K}$ of $f_K$ specialize to pairwise distinct points on the special fiber of a smooth model of $X_K$, then $f_K$ has potentially good reduction.

The following example plays a central role in this paper.

**Example 1.1.6** Let $R$ and $K$ be as before. Choose four $R$-rational points $x_1, \ldots, x_4$ on $\mathbb{P}^1$ such that $x_i \neq x_j$ (m), where $m < R$ is the maximal ideal. Then $(\mathbb{P}^1_R; x_i)$ is a smooth, stably marked curve over $R$. Let $G$ be the dihedral group of order 2$p$, where $p$ is an odd prime, equal to the residue characteristic of $R$. Let $f_K : E_K \to \mathbb{P}^1_K$ be a $G$-cover branched only in the 4 points $x_i$, with ramification of order 2. We have 4$p$ ramification points $y_{i,j}$ on $E_K$, where $1 \leq i \leq 4, 1 \leq j \leq p$ and $f_K(y_{i,j}) = x_i$. The curve $E_K$ has genus 1. After extending $K$ we may assume that the $y_{i,j}$ are $K$-rational. Choosing e.g. $y_{1,1}$ as the origin gives $E_K$ the structure of an elliptic curve. Moreover, $f_K$ can be written as the composition

$$f_K : E_K \xrightarrow{p} E'_K \xrightarrow{2} \mathbb{P}^1_K$$

of a $p$-cyclic isogeny of elliptic curves and a cyclic cover of degree 2. The cover $f_K$ extends to a finite flat morphism $f_R : E_R \to \mathbb{P}^1_R$. Moreover, $E_R$ is an elliptic curve and $f_R$ factors through an isogeny $\pi_R : E_R \to E'_R$.

There are two cases to consider. First, $\pi_R$ might be étale. In this case, the ramification points $y_{i,j}$ extend to disjoint sections $y_{i,j} : \text{Spec} R \to E_R$ and $f : E_R \to \mathbb{P}^1_R$ is tamely ramified along the sections $x_j : \text{Spec} R \to \mathbb{P}^1_R$. In other words, $f_K$ has good reduction. Now assume that $\pi_R$ is not étale. Then its restriction $\pi : E \to E'$ to the special fiber is purely inseparable, and for fixed $i$ the $p$ points $y_{i,j}$, $j = 1, \ldots, p$ specialize to the same point of $E$. We see that $(E_R; y_{i,j})$ is not a stably marked curve. Let $(Z_R; y_{i,j})$ be the extension of $(E_R; y_{i,j})$ to a stably marked curve over $R$ and $q_R : Z_R \to E_R$ the contraction morphism. We can identify $E$ with its strict transform in $Z_R$. The special fiber $Z$ of $Z_R$ has exactly 5 components $E, Z_1, \ldots, Z_4$. For $i = 1, \ldots, 4$, the curve $Z_i$ is smooth and of genus 0, connected to $E$ in one point and contains the specialization of the points $y_{i,j}$ for $j = 1, \ldots, p$. Let $X_R := Z_R/G$ be the quotient. The special fiber $X := X_R \otimes_R k$ has 5 components $X_0, \ldots, X_4$. In fact, $X_0$ is the original component, and for $i = 1, \ldots, 4$, the component $X_i$ is the image of $Z_i$ and contains the specialization of $x_i$. The restriction of $f : Z \to X$ to $X_i$ is a $G$-cover $Z_i \to X_i$ ramified in two points, with ramification of order 2 and 2$p$ (so it is not tame).

### 1.2 Complete Hurwitz stacks

In this section we define the concept of a complete Hurwitz stack, following the idea of \[4\]. Since there are many different versions of Hurwitz stacks, we do this first in detail for one specific kind, namely for $\mathcal{H}_{0,0}^u(G)$, the inner Hurwitz stack for $G$-covers of genus 0 curves with unordered branch points. Then we define several variants of the above. In Section 1.2.4 we look at Hurwitz spaces as coarse moduli spaces.

In this paper we only consider Hurwitz spaces for Galois covers. Moreover, the target curve of the cover will always be of genus 0 and is considered “up to isomorphism”. Thus, we only consider
“reduced” Hurwitz spaces in the terminology used by Fried \[8\]. The genus zero assumption is made only to simplify the notation. It is easy to extend all our definitions to nonreduced Hurwitz spaces. It seems much less trivial to do the same for Hurwitz spaces parameterizing non-Galois covers. For instance, in \[1\] a completion of the classical Hurwitz stack for simple covers is constructed, using the moduli space of stable maps as an ambient space. This construction is more involved than the one we give here. Another problem, discussed in \[1\], Section 4.1, is that “taking quotients by finite groups does not commute with base change”. Therefore, the method proposed in \[2\] of studying complete moduli of non-Galois covers by going to the Galois closure probably does not work very well with our definition of complete Hurwitz stacks. A related problem is discussed in Section \[1.3\].

1.2.1 The ambient stack Let \((X/S, C)\) and \((Y/S, D)\) be stably marked curves, defined over the same scheme \(S\). A morphism of stably marked curves is an \(S\)-morphism \(f : Y \to X\) such that \(f(D) \subset C\). To ease notation, we will usually write \(X\) and \(Y\) instead of \((X/S, C)\) and \((Y/S, D)\). We fix an integer \(r > 0\) and let \(S_{[r]}\) be the following category. Objects of \(S_{[r]}\) are morphisms \(f : Y \to X\) between stably marked curves such that \(X\) has genus 0 and is stably \(r\)-marked. A morphism between an \(S\)-object \(f : (Y, D) \to (X, C)\) and an \(S'\)-object \(f' : (Y', D') \to (X', C')\) of \(S_{[r]}\) consists of a Cartesian diagram

\[
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X
\end{array}
\]

such that \(D' = D \times_S S'\) and \(C' = C \times_S S'\). It is clear that \(S_{[r]}\) is a stack.

Let \(G\) be a finite group and \(S_{[r]}^n(G)\) the following category. Objects of \(S_{[r]}^n(G)\) (over a scheme \(S\)) are pairs \((f, \sigma)\), where \(f : Y \to X\) is an object of \(S_{[r]}\) defined over \(S\) and \(\sigma : G \to \text{Aut}(Y/X)\) is an action of \(G\) on \(Y\) commuting with \(f\) such that the induced action on every geometric fiber of \(f\) is faithful (equivalently, \(\sigma\) induces a closed immersion \(\sigma : G_S \hookrightarrow \text{Aut}(Y/X)\) of group schemes). Mostly we will omit the map \(\sigma\) and simply write \(f : Y \to X\) for an object of \(S_{[r]}^n(G)\). A morphism between two objects \(f' : Y' \to X'\) and \(f : Y \to X\) of \(S_{[r]}^n(G)\) is an \(S_{[r]}\)-morphism \((f, \sigma)\) such that the top arrow \(Y' \to Y\) is \(G\)-equivariant. Again it is clear that \(S_{[r]}^n(G)\) is a stack.

**Proposition 1.2.1** The stacks \(S_{[r]}\) and \(S_{[r]}^n(G)\) are algebraic, separated and locally of finite type over \(Z\).

**Proof:** The stack \(\mathcal{M}_{g,[n]}\) classifying stably \(n\)-marked curves of fixed genus \(g\) is algebraic, separated and of finite type over \(Z\). A standard Hilbert scheme argument (see e.g. \[2\], Chap. 0.5) shows that \(S_{[r]}\) is algebraic, separated and locally of finite type over \(Z\). Let \(f : Y \to X\) be an object of \(S_{[r]}\) defined over a scheme \(S\). Since \(\text{Aut}(Y/X)\) is a finite \(S\)-group scheme, the functor \(\text{Hom}_S(G_S, \text{Aut}(Y/X))\) is represented by a finite \(S\)-scheme. It is clear that the natural morphism \(S \times_{S_{[r]}} S_{[r]}^n(G) \to \text{Hom}_S(G_S, \text{Aut}(Y/X))\) is a locally closed immersion. Therefore, the forgetful morphism \(S_{[r]}^n(G) \to S_{[r]}\) is relatively representable, separated and of finite type. This completes the proof of the proposition. \(\Box\)

1.2.2 The complete inner Hurwitz stack Let \(r\) and \(G\) be as before. We define the Hurwitz stack \(\mathcal{H}_{[r]}^n(G)\) as follows. Objects of \(\mathcal{H}_{[r]}^n(G)\) over a scheme \(S\) are morphisms \(f : Y \to X\) between smooth \(S\)-curves, together with an action of \(G\) on \(Y\), commuting with \(f\), such that the following holds. The curve \(X/S\) has genus 0 and the geometric fibers of \(f\) are tame \(G\)-covers (see Definition
with exactly \( r \) branch points. Morphisms in \( \mathcal{H}^{\text{in}}_{r}(G) \) are Cartesian diagrams of the form (1) such that the top horizontal arrow is \( G \)-equivariant. We call an object \( f : Y \to X \) of \( \mathcal{H}^{\text{in}}_{r}(G) \) a tame \( G \)-cover, defined over \( S \). It is proved e.g. in [20] that \( \mathcal{H}^{\text{in}}_{r}(G) \) is an algebraic stack, smooth and of finite type over \( \mathbb{Z} \).

Let \( f : Y \to X \) be an \( S \)-object of \( \mathcal{H}^{\text{in}}_{r}(G) \). Then \( f \) is finite and tamely ramified along a divisor \( C \subset X \) which is finite étale of degree \( r \) over \( S \). Hence \((X/S,C)\) is a (smooth) stably \( r \)-marked curve. Moreover, \((Y/S,D)\) is a (smooth) stably marked curve, where \( D := f^{-1}(C) \subset Y \) is the (reduced) inverse image of \( C \). Therefore, we obtain a natural monomorphism

\[
\mathcal{H}^{\text{in}}_{r}(G) \hookrightarrow S^{\text{in}}_{r}(G),
\]

identifying \( \mathcal{H}^{\text{in}}_{r}(G) \) with a full subcategory of \( S^{\text{in}}_{r}(G) \). We will show in Proposition 1.2.4 below that (3) is a locally closed immersion. Note however that we do not need this fact to make the following definition.

**Definition 1.2.2** The complete Hurwitz stack \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) is the closure of \( \mathcal{H}^{\text{in}}_{r}(G) \) inside \( S^{\text{in}}_{r}(G) \) (i.e. the smallest closed substack of \( S^{\text{in}}_{r}(G) \) containing \( \mathcal{H}^{\text{in}}_{r}(G) \) as a full subcategory).

Proposition 1.2.1 shows that \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) is an algebraic stack, separated and locally of finite type over \( \mathbb{Z} \). Let \( k \) be an algebraically closed field and \( f : Y \to X \) an object of \( \mathcal{H}^{\text{in}}_{r}(G) \) defined over \( k \). Choose an étale neighborhood \( U \to \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) of the point \( s : \text{Spec} \ k \to \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) corresponding to \( f \) and let \( \eta : \text{Spec} \ K \to U \) be a generic point of some irreducible component of \( U \). By Exercise II.4.11, \( \eta \) extends to a morphism \( \eta_{R} : \text{Spec} \ R \to U \), where \( R \) is a discrete valuation ring of \( K \) with residue field \( k \), and the restriction of \( \eta_{R} \) to the special point is equal to \( s \). The morphism \( \text{Spec} \ R \to \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) corresponds to an \( R \)-object \( f_{R} : Y_{R} \to X_{R} \) of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) with special fiber \( f : Y \to X \). Since \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) is dense in \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \), the generic fiber \( f_{K} : Y_{K} \to X_{K} \) of \( f_{R} \) is actually an object of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \), i.e. a tame \( G \)-cover. Moreover, \( K \) has characteristic 0. We are essentially (modulo taking the completion of \( R \)) in the situation of Section 1.1. It is clear that \( f_{R} : Y_{R} \to X_{R} \) is the stable model of the \( G \)-cover \( f_{K} \). In particular, \( f_{K} \) has good reduction if and only if \( f \) is an object of \( \mathcal{H}^{\text{in}}_{r}(G) \). We say that \( f \) is a bad cover if \( f_{K} \) has bad reduction (Definition 1.1.3). By Proposition 1.1.4, \( f \) is a bad cover if and only if some irreducible component of \( Y \) has a nontrivial inertia group (with respect to the action of \( G \)).

**Lemma 1.2.3** There is a unique closed reduced substack \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G)_{\text{bad}} \subset \mathcal{H}^{\text{in}}_{r}(G) \) characterized by the following property. For an algebraically closed field \( k \), a \( k \)-object \( f : Y \to X \) of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) is a bad cover if and only if it is an object of the substack \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G)_{\text{bad}} \).

**Proof:** By [4], Lemma II.4.5, it suffices to show that the subset of bad covers is stable under specialization. More precisely, let \( R \) be a discrete valuation ring and \( f_{R} : Y_{R} \to X_{R} \) an object of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) defined over \( R \). Assume that the generic fiber \( f_{K} : Y_{K} \to X_{K} \) of \( f_{R} \) is a bad cover. We have to show that the special fiber \( f : Y \to X \) is a bad cover, too. As remarked above, \( f_{K} \) (resp. \( f \)) is a bad cover if and only if some irreducible component of \( Y_{K} \) (resp. \( Y \)) has a nontrivial inertia group. Clearly, this property is stable under specialization.

By the lemma, \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G)_{\text{adm}} := \tilde{\mathcal{H}}^{\text{in}}_{r}(G) - \tilde{\mathcal{H}}^{\text{in}}_{r}(G)_{\text{bad}} \) is a dense open substack of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \). The discussion before Lemma 1.2.3 shows that the geometric points of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G)_{\text{adm}} \) correspond to tame admissible covers \( f : Y \to X \) over \( k \) which arise as the reduction of tame \( G \)-covers. It follows that an object \( f : Y \to X \) of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G)_{\text{adm}} \) (defined over an arbitrary scheme \( S \)) lies in the full subcategory \( \mathcal{H}^{\text{in}}_{r}(G) \) if and only if \( X/S \) is smooth. Since smoothness of \( X/S \) is an open condition on \( S \), \( \mathcal{H}^{\text{in}}_{r}(G) \) is an open substack of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G)_{\text{adm}} \), and hence an open substack of \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \).

**Proposition 1.2.4** The algebraic stack \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G) \) is reduced, proper and of finite type over \( \mathbb{Z} \) and contains \( \mathcal{H}^{\text{in}}_{r}(G) \) and \( \tilde{\mathcal{H}}^{\text{in}}_{r}(G)_{\text{adm}} \) as dense open substacks.
Proof: It only remains to show that \( \tilde{\mathcal{H}}^m_r(G) \) is reduced and proper. We know that the dense open substack \( \mathcal{H}^m_r(G) \) is reduced, therefore \( \tilde{\mathcal{H}}^m_r(G) \) is reduced as well. To prove properness, we apply [2], Corollaire 7.3.10. We are immediately reduced to the following situation. Let \( R \) be a complete discrete valuation ring with residue field \( k = \bar{k} \) of characteristic \( p \) and quotient field \( K \) of characteristic 0. Let \( f_K : Y_K \to X_K \) be an object of \( \mathcal{H}^m_r(G) \), i.e. a \( G \)-cover. This is the situation of Section 1.1. After a finite extension of \( K \), \( f_K \) has a stable model \( f_R : Y_R \to X_R \) over \( R \) (Definition 1.1.2). Obviously, \( f_R \) is an object of \( S^m_r(G) \). Therefore, \( f_R \) is an object of \( \tilde{\mathcal{H}}^m_r(G) \), by Definition 1.2.2. This proves that \( \tilde{\mathcal{H}}^m_r(G) \) is proper. \( \square \)

The stack \( \mathcal{H}^m_r(G) \) is called the inner Hurwitz stack for \( G \)-covers with \( r \) (unordered) branch points. We will say that the stack \( S^m_r(G) \) is the ambient stack of \( \mathcal{H}^m_r(G) \). We will call the stack \( \tilde{\mathcal{H}}^m_r(G) \) the completion of \( \mathcal{H}^m_r(G) \).

**Remark 1.2.5** It is easy to check that the stack \( \tilde{\mathcal{H}}^m_r(G) \) can be identified with the compactification of \( \mathcal{H}^m_r(G) \) constructed in [3] or [4]. It follows from loc.cit. that \( \tilde{\mathcal{H}}^m_r(G) \) is smooth over \( \mathbb{Z} \) and proper over \( \mathbb{Z}[1/|G|] \).

**Variant 1.2.6** Let \( \mathcal{H}^m_r(G) \) be the stack whose objects are \( S \)-morphisms \( f : Y \to X \) which are locally on \( S \) tame \( G \)-covers. More precisely, after an étale localization \( S' \to S \), there exists an action of \( G \) on \( Y \) such that \( f : Y \to X \) becomes an object of \( \mathcal{H}^m_r(G) \). We call \( \mathcal{H}^m_r(G) \) the absolute Hurwitz stack for \( G \)-covers with \( r \) (unordered) branch points, see also [4]. We embed \( \mathcal{H}^m_r(G) \) into an ambient stack \( S^m_r(G) \). Objects of \( S^m_r(G) \) are pairs \( (f, G) \), where \( f : Y \to X \) is an object of \( S^m_r(S) \) defined over a scheme \( S \) and \( G \subset \text{Aut}(Y/X) \) is an étale subgroup scheme which becomes isomorphic to the constant \( S \)-group scheme \( G_S \) after an étale localization of \( S \). We define the completion \( \tilde{S}^m_r(G) \) as the closure of \( \mathcal{H}^m_r(G) \) inside \( S^m_r(G) \).

**Variant 1.2.7** Let \( \mathcal{H}^m_r(G) \) and \( \mathcal{H}^s_r(G) \) be the inner resp. absolute Hurwitz stack for \( G \)-covers with \( r \) ordered branch points. We can embed \( \mathcal{H}^m_r(G) \) into an ambient stack \( S^m_r(G) \). Objects of \( S^m_r(G) \) are objects \( f : Y \to X \) of \( S^m_r(G) \) together with \( r \) sections \( x_1, \ldots, x_r : S \to X \) such that \( C = \bigcup x_i(S) \) is the mark of the stably marked curve \( X \). We define the completion \( \tilde{\mathcal{H}}^m_r(G) \) as the closure of \( \mathcal{H}^m_r(G) \) inside \( S^m_r(G) \). Similar for \( \tilde{\mathcal{H}}^s_r(G) \).

We obtain a diagram

\[
\begin{array}{ccc}
\mathcal{H}^m_r(G) & \longrightarrow & \mathcal{H}^m_r(G) \\
\downarrow & & \downarrow \\
\tilde{\mathcal{H}}^m_r(G) & \longrightarrow & \tilde{\mathcal{H}}^m_r(G). \\
\end{array}
\]

The vertical arrows in (3) are principal \( \text{Aut}(G) \)-bundles, the horizontal arrows are principal \( \mathfrak{G}_r \)-bundles. All stacks in (3) are algebraic, reduced and proper and of finite type over \( \mathbb{Z} \).

**1.2.3 Complete Hurwitz stacks for a given type** With \( G \) and \( r \) as before, let \( \mathcal{C} = (C_1, \ldots, C_r) \) be an \( r \)-tuple of conjugacy classes of \( G \). We denote by \( \mathbb{Q}(\mathcal{C}) \subset \mathbb{Q}(\zeta_n) \) the field generated by the values \( \chi(\tau) \), where \( \chi \) runs over all irreducible characters of \( G \) and \( \tau \in C_i \), for \( i = 1, \ldots, r \). In other words, \( \mathbb{Q}(\mathcal{C}) \) is the minimal number field over which every class \( C_i \) becomes rational. Let us make a “choice of \( n \)th root of unity over \( \mathbb{Q}(\mathcal{C}) \)”, i.e. we choose an orbit under \( \text{Gal}(\mathbb{Q}(\mathcal{C})) \) of primitive \( n \)th roots of unity, see [27], Section 8.2.1. Let \( \Lambda \subset \mathbb{Q}(\mathcal{C}) \) be a Dedekind domain with fraction field \( \mathbb{Q}(\mathcal{C}) \). We define an open substack

\[
\mathcal{H}^m_r(\mathcal{C})_\Lambda \subset \mathcal{H}^m_r(G) \otimes_{\mathbb{Z}} \Lambda,
\]
corresponding to \(G\)-covers with inertia type \(\mathcal{C}\). To be more precise, let \(f : Y \to (X; x_i)\) be an object of \(\mathcal{H}_r^{\mu}(G) \otimes \Lambda\), defined over an algebraically closed field \(k\). We say that \(f\) has inertia type \(\mathcal{C}\) if \(C_i\) is the conjugacy class associated to the branch point \(x_i\) (with respect to a canonical choice of \(m_i\)th root of unity, induced by the natural map \(\Lambda \to k\)), compare [28], Section 2.2.1. An object of \(\mathcal{H}_r^{\mu}(G) \otimes \Lambda\), defined over an arbitrary scheme \(S\), is said to have inertia type \(\mathcal{C}\) if all its geometric fibers have inertia type \(\mathcal{C}_i\). We define the completion \(\mathcal{H}_r^{\mu}(\mathcal{C})_\Lambda\) as the closure of \(\mathcal{H}_r^{\mu}(\mathcal{C})_\Lambda\) inside \(S_r^{\mu}(G) \otimes \mathbb{Z}\). Clearly, \(\mathcal{H}_r^{\mu}(\mathcal{C})_\Lambda\) is an algebraic stack, proper and of finite type over \(\Lambda\). If the choice of \(\Lambda\) is understood, we will omit it from the notation, and we say that \(\mathcal{H}_r^{\mu}(\mathcal{C})\) is defined over \(\Lambda\), or that \(\Lambda\) is the domain of definition of \(\mathcal{H}_r^{\mu}(\mathcal{C})\).

In a similar way, we can define further variants of complete Hurwitz stacks: \(\mathcal{H}_r^{\mu}(\mathcal{C}), \mathcal{H}_r^{\mu}(\mathcal{C})\) and \(\mathcal{H}_r^{\mu}(\mathcal{C})\). The domains of definition of these stacks are Dedekind domains whose fraction fields are suitable subfields of \(\mathbb{Q}(\mathcal{C})\). For instance, \(\mathcal{H}_r^{\mu}(\mathcal{C})\) can be defined over the ring of integers of the smallest field over which \(\mathcal{C}\) as a tuple is rational, compare [28], Definition 3.15.

### 1.2.4 Complete Hurwitz spaces as coarse moduli spaces

Let \(\mathcal{H} := \mathcal{H}_r^{\mu}(\mathcal{C})\) be the complete Hurwitz stack over \(\Lambda\), as defined in Section 1.2.3. We denote by \(\mathcal{H} := \mathcal{H}_r^{\mu}(\mathcal{C})\) the associated coarse moduli space, and call it the complete Hurwitz space over \(\Lambda\) for \(G\)-covers with inertia type \(\mathcal{C}\). By construction [18], \(\mathcal{H}\) is an algebraic space, proper and of finite type over \(\Lambda\), and contains the usual Hurwitz space \(H = \mathcal{H}_r^{\mu}(\mathcal{C})\) as a dense open subspace. Actually, \(H\) is a scheme and is smooth over \(\Lambda\), see [29]. We define a closed subspace \(H^{\text{bad}}\) as the image of the natural morphism \(H^{\text{bad}} \to H\). Thus, \(H^{\text{adm}} := H - H^{\text{bad}}\) is the coarse moduli space associated to \(H^{\text{adm}} = \mathcal{H} - \mathcal{H}^{\text{bad}}\) and contains \(H\) as a dense open subset. According to [29], \(H^{\text{adm}}\) is a normal scheme.

Let \(\mathcal{H} \to \mathcal{M}_{0,r}\) be the natural forgetful morphism. It is known that \(\mathcal{M}_{0,r}\) is represented by a smooth projective scheme over \(\mathbb{Z}\), see [1]. By the universal property of the coarse moduli space, we obtain a morphism \(\mathcal{H} \to \mathcal{M}_{0,r}\).

**Proposition 1.2.8** Assume that the complete Hurwitz stack \(\mathcal{H}\) is normal and that the forgetful morphism \(\mathcal{H} \to \mathcal{M}_{0,r} \otimes \Lambda\) is relatively representable and finite. Then the complete Hurwitz space \(H\) is a normal scheme, finite over \(\mathcal{M}_{0,r} \otimes \Lambda\).

**Proof:** If \(\mathcal{H}\) is normal and \(\mathcal{H} \to \mathcal{M}_{0,r} \otimes \Lambda\) finite, then the coarse moduli space \(\mathcal{H}\) is the normalization of \(\mathcal{M}_{0,r} \otimes \Lambda\) in \(H\), see [3], Proposition IV.3.10. This proves the proposition. \(\square\)

**Variant 1.2.9** In the same manner, we define complete Hurwitz spaces \(\mathcal{H}_0^{\mu}(G), \mathcal{H}_r^{\mu}(G), \text{etc.}

Proposition 1.2.8 applies as well. There are natural maps between all these variants. For instance, diagram [3] induces an analogous diagram of finite morphisms between algebraic spaces. But unlike the maps in [3], these maps are in general not étale.

### 1.3 Quotient model versus stable model

Let us fix a finite group \(G\) and an integer \(r \geq 0\). Let \(\mathcal{H} := \mathcal{H}_r^{\mu}(G)\) be the complete inner Hurwitz stack defined in Section 1.2. In Section 1.1 we have defined two different models of a \(G\)-cover \(f_K : Y_K \to X_K\), where \(K\) is a complete discrete valued field: the quotient model \(f_R : Y_R \to X_R\) and the stable model \(f_{0,R} : Y_R \to X_{0,R}\). The definition of \(\mathcal{H}\) was made such that the stable model \(f_{0,R} : Y_R \to X_{0,R}\) of the \(G\)-cover \(f_K\) is an object of \(\mathcal{H}\). One drawback of the stable model is that it is in general not a finite map. Over the discrete valuation ring \(R\), one can recover the quotient model from the stable model by taking the quotient scheme \(X_R := Y_R/G\). The problem is that taking quotients does not commute with arbitrary base change. So it is not clear how to define a quotient model of an object \(f_0 : Y \to X_0\) of \(\mathcal{H}\) over an arbitrary scheme \(S\). In this section we propose a definition that works well when \(S\) is either the spectrum of an algebraically closed field or a normal scheme \(S\) with function field of characteristic 0. This will be enough for our purposes.
Definition 1.3.1 Let $S$ be a scheme and $f_0 : Y \to X_0$ an object of $\mathcal{H}$ defined over $S$. Let $f : Y \to X$ be a finite morphism between marked semistable curves commuting with the action of $G$ on $Y$. We say that $f$ is a quotient model of $f_0$ if $f_0$ is the composition of $f$ with an $S$-morphism $X \to X_0$ and if for every geometric fiber $f_s : Y_s \to X_s$ of $f$ the following holds.

(i) The natural morphism $Y_s/G \to X_s$ induces a bijection on geometric points.

(ii) Let $Y_i$ be a component of $Y_s$ and $X_i$ the component of $X_s$ under $Y_i$. Then the degree of $Y_i$ over $X_i$ is equal to the order of the stabilizer $D(Y_i) \subset G$ of $Y_i$.

Proposition 1.3.2 Let $S$ be either the spectrum of an algebraically closed field $k$ or a normal scheme, generically of characteristic 0. Let $f_0 : Y \to X_0$ be an object of $\mathcal{H}$ over $S$. Then there exists a unique quotient model $f : Y \to X$ of $f_0$.

Proof: Let us first assume that $S$ is a normal scheme, generically of characteristic 0. We may assume that $S = \text{Spec } R$ is local, with algebraically closed residue field $k$. The generic fiber $f_K : Y_K \to X_K$ is an admissible cover. In particular, $f_K$ is a quotient model of itself. Let $X := Y/G$. It follows from [16], Proposition 4.2, and [7], Corollary A.7.2.2, that $f : Y \to X$ is a quotient model of $f_0$. To show that it is unique, suppose we have another quotient model $f' : Y \to X'$ of $f_0$. Since $f'$ commutes with the action of $G$ on $Y$, there exists a morphism $\kappa : X \to X'$ of $S$-schemes such that $\kappa \circ f = f'$. We claim that $\kappa$ is an isomorphism of $X_0$-schemes. Since $X$ and $X'$ are flat over $S$ and $\kappa$ is the identity on the generic fiber, it suffices to prove that the restriction of $\kappa$ to the special fiber is an isomorphism of $X_0$-schemes. Hence we have reduced the proposition to the case $S = \text{Spec } k$. Let us now prove this case. We have seen before that every $k$-object $f_0 : Y \to X_0$ of $\mathcal{H}$ is the reduction of a $G$-cover $f_K : Y_K \to X_K$, where $K$ is the quotient field of a discrete valuation ring $R$ with residue field $k$. Therefore, the special fiber $f : Y \to X$ of the quotient model of $f_K$ is a quotient model of $f_0$. It remains to prove its uniqueness. The quotient $X' := Y/G$ is a semistable curve over $k$ and the natural map $X' \to X$ is a bijection on geometric points, by assumption. Let $Y_i$ be a component of $Y$ and $X_i$ resp. $X'_i$ the component of $X$ resp. of $X'$ under $Y_i$. It follows from [14], Proposition IV.2.5, that $X_i$ is the $n$th Frobenius twist $(X'_i)^{F^n}$ of $X'_i$, where $p^n$ is the order of the inertia group $I(Y_i) \subset G$ of $Y_i$. Clearly, this characterizes the map $X' \to X$ and hence the quotient model $f : Y \to X$ uniquely. \qed

2 Semistable reduction

In this section the notations and conventions are as in Section 1.1. Let us recall them briefly. Let $R$ be a complete discrete valuation ring with quotient field $K$ of characteristic zero and residue field $k = k$ of characteristic $p$. Let $f_K : Y_K \to X_K$ be a $G$-cover of smooth curves defined over $K$. Let $g = g(X_K)$ and $r$ the number of branch points of $f_K$. Let $f_R : Y_R \to X_R$ be the quotient model of $f_K$ defined in Definition 1.1.2. Write $f : Y \to X$ for its special fiber. The branch points of $f_K$ specialize to distinct points $x_1, \ldots, x_r$ of the smooth locus of $X$. Let $m_i$ be the ramification index of $x_i$ in $f_K$. Recall that we also defined a different model for $f_K$, called the stable model $f_{0,R} : Y_R \to X_{0,R}$. It is obtained from $f_R$ by composing with a contraction map $X_R \to X_{0,R}$.

We will determine the structure of the reduction $f : Y \to X$ in case the cover $f_K$ has bad reduction, under suitable assumptions (Condition 2.2.2 below). All results of this section rely on the results of [23] and [24]. Results from these papers are recalled very briefly and the reader is referred to these papers for more details.

Condition 2.2.2 plays an essential role in the rest of the paper. It will enable us to compute the structure of $f : Y \to X$. Most importantly, we will assume that the normalizer of a $p$-Sylow group $P$ of $G$ is a dihedral group. We will define an auxiliary cover $g : Z \to X$. The condition on the
normalizer $N_G(P)$ will imply that this auxiliary cover is equivariant under a (dihedral) subgroup of $N_G(P)$. In Section 2.3 we will relate the deformation theory of $f$ to the deformation theory of $g$.

In Section 2.4 we show that $X$ intersect themselves. For an irreducible component $Z$ of the Hurwitz space.

Lemma 2.1.1 A singular point of $Y$ maps to a singular point of $X$, i.e. $G$ acts on $Y$ without inversions.

Proof: The ramification points $y_{i,K}$ specialize to distinct smooth points of $Y$, by construction. It follows from [24], Proposition 2.3.2.b, that there are no inversions. □

Note in particular that the above lemma implies that the irreducible components of $Y$ do not intersect themselves. For an irreducible component $Z$ of $Y$, we denote by $I(Z)$ its inertia group, and $D(Z)$ its decomposition group. Analogously, for a closed point $y$ of $Y$, we will write $I(y)$ for its inertia group $I(y)$. It is equal to the decomposition group of $y$, since $k$ is algebraically closed.

Lemma 2.1.2 (a) Let $Z$ be an irreducible component of $Y$. Then $I(Z)$ is a $p$-group and a normal subgroup of the decomposition group $D(Z)$.

(b) Let $y_R: \text{Spec}(R) \to Y_R$ be a section whose image is contained in the smooth locus of $Y_R$. Let $Z$ be the irreducible component of $Y$ on which $y := y_R \otimes k$ lies. Write $m = p^n a$, with $\gcd(n,p) = 1$, for the ramification index of $y_K$ in $f$. Then $I(Z)$ is normal in $I(y)$ and $I(y)/I(Z)$ is cyclic of order $n$.

(c) Any branch point $x_{i,K}$ of $f_K$ whose ramification index $m_i$ is prime-to-$p$, specializes to a component of $X$ over which $f$ is separable.

Proof: Part (a) and (b) follow from [23], Lemme 6.3.3. Let $x_{i,R}$ be a branch point such that the ramification index $m_i$ of $x_{i,K}$ is prime-to-$p$. Suppose it specializes to a component $W$ of $X$ over which $f$ is inseparable. Then the inertia group of any component $Z$ of $Y$ mapping to $W$ is nontrivial. Hence $x_i := x_{i,R} \otimes k$ will be branched of order $p^a m_i$ with $a > 0$, by Part (b). This is in contradiction with the assumption that the ramification points specialize to distinct points on $Y$. This proves (c). □

Lemma 2.1.3 Let $y$ be a singular point of $Y$. Let $Z_1, Z_2$ be the two irreducible components of $Y$ passing through $y$.

(a) The inertia group $I(y)$ is an extension of a cyclic group of order prime-to-$p$ by a $p$-group.

(b) The groups $I(Z_1)$ and $I(Z_2)$ are normal subgroups of $I(y)$ and $<I(Z_1), I(Z_2)>$ is the $p$-Sylow group of $I(y)$.

Proof: Part (a) follows from [24], Proposition 2.3.2.a, since $G$ acts without inversions. Part (b) is proved analogous to [21], Lemme 6.3.6.iii. The assumptions in that lemma differ from the assumptions in the present case, but the proof goes through. □
2.2 First properties

We will suppose now that \( f_K : Y_K \to X_K \) is a \( G \)-cover branched at four points \( x_1, \ldots, x_4 \), where \( X_K \) has genus zero. Let \( m_i \) be the ramification index of a point above \( x_i \). We suppose that \( f_K \) has bad reduction, and denote by \( f : Y \to X \) the special fiber of the quotient model of \( f_K \).

**Notation 2.2.1** Let \( U \) be the union of the components \( W \) of \( X \) such that \( f \) is inseparable over \( W \). Let \( P \) be a \( p \)-Sylow group of \( G \). We denote by \( N_G(P) \) the normalizer of \( P \) in \( G \) and by \( C_G(P) \) the centralizer of \( P \) in \( G \).

**Condition 2.2.2** In the rest of the paper we will assume the following conditions to hold:

(a) \( p \neq 2 \),
(b) \( m_1, \ldots, m_4 \) prime-to-\( p \),
(c) \( p | |G| \),
(d) \( N_G(P) \) is a dihedral group.

**Example 2.2.3** Here are some examples of groups for which Condition 2.2.2(d) is satisfied. Let \( G = PSL_2(\ell^n) \), where \( \ell > 2 \) is a prime. Suppose that \( p \neq \ell \) is a prime exactly dividing \( |G| = (\ell^n - 1)\ell^n/2 \). Then \( N_G(P) \) is a dihedral group of order \( \ell^n - 1 \) or order \( \ell^n + 1 \), [15], Abschnitt II.8. Special cases are \( G = PSL_2(5) = A_5 \) and \( G = PSL_2(9) = A_6 \).

Recall that there is a map \( X \to X_0 \) which contracts some components; the strict transforms of the components of \( X_0 \) in \( X \) are called the \emph{original} components. Since we assumed that \( r = 4 \), there are two possibilities for \( X_0 \): it can be smooth or not. In case \( X_0 \) is singular, it will consist of two genus zero components which meet in a unique point. We will call these components \( W_1 \) and \( W_2 \) and will also write \( W_1, W_2 \) for their strict transform in \( X \). Similarly, we will write \( X_0 \) for the strict transform of \( X_0 \) in \( X \), in case \( X_0 \) is smooth.

Suppose \( X_0 \) is not smooth. Since \( X_0 \) is stably marked, there will be exactly two branch points specializing to each of the components \( W_i \). In \( X \) the two components \( W_1 \) and \( W_2 \) are connected by a chain of \( \mathbb{P}^1 \)'s, since the dual graph of \( X \) is a tree. Let \( \Lambda \) the union of \( W_1, W_2 \) and the components connecting the two. We will say that a component \( W \) of \( X \) is a \emph{tail} if it is not contained in \( \Lambda \) and meets the rest of \( X \) in a unique point.

In case \( X_0 \) is smooth we just put \( \Lambda = \{X_0\} \). The definition of \emph{tail} then becomes the usual definition of tail, i.e. one views the dual graph of \( X \) to be oriented from \( X_0 \).

**Lemma 2.2.4** Suppose that \( f_K \) has bad reduction. Let \( W \) be an irreducible component of \( X \). Then \( f|_W \) is separable if and only if \( W \) is a tail.

In case \( X_0 \) is smooth, the statement of this lemma is the same as [24], Lemme 3.1.2, except that the model we are looking at is slightly different from the one considered in that paper. The definition of tail is made in such a way that \( f \) will be separable exactly over the tails of \( X \). The proof of Lemma 2.2.4 relies on Condition 2.2.2. If one does not assume the condition, it will not be true in general that \( f \) will be exactly separable over the tails. For counter-examples see for example [1] for \( p = 2 \) or [20] for general \( p \).

**Proof:** We split the proof up in two parts. First we consider the statement of the lemma for components \( W \) which are not contained in \( \Lambda \). Then we show that \( f \) is inseparable for all components contained in \( \Lambda \).

Let \( W \) be a component of \( X \) such that \( f|_W \) is separable and \( W \) is not contained in \( \Lambda \). The proof of [24], Proposition 2.4.8, carries over to this situation and shows that \( W \) is connected to the rest of \( X \) in a single point, i.e. \( W \) is a tail.

Now suppose that \( W \) is a tail of \( X \). Let \( Z \) be a component of \( Y \) which maps to \( W \). In case there is a branch point specializing to \( W \), the cover is separable over \( W \) by Lemma 2.1.2(c) and Condition
2.2.2 (b). Suppose there are no branch points specializing to $W$ and $Z \to W$ is inseparable. Then $I(Z)$ is a nontrivial $p$-group which is a normal subgroup of $D(Z)$. Let $Z' = Z/I(Z)$. Since $p$ exactly divides the order of $G$, it follows that $D(Z)/I(Z)$ is of order prime-to-$p$. Then $Z' \to W$ is Galois of prime-to-$p$ order and branched at at most one point, hence trivial. Since $Z \to Z' = W$ is purely inseparable, $Z$ is a component of $Y$ of genus zero which meets the rest of $Y$ in a single point. By assumption, there is no ramification point specializing to $Z$. This contradicts the minimal character of $Y$. Hence $f|_W$ is separable.

We are now going to prove the lemma for components contained in $\Lambda$, i.e. we have to show that $f$ is inseparable over the components contained in $\Lambda$.

Suppose that $X_0$ is smooth. The cover $f$ is inseparable over $X_0$, by Proposition 1.1.4. This finishes the proof of the lemma for $X_0$ smooth.

Suppose that $X_0$ is singular and suppose that there exists a component of $\Lambda$ over which $f$ is separable. Let $U$ be the union of the components of $X$ over which the cover is inseparable. If $f$ is separable over both $W_1$ and $W_2$, then the reduction is admissible by Proposition 1.1.4. It is no restriction to suppose that $f$ is inseparable over $W_1$. Let $U'$ be the connected component of $U$ containing $W_1$. The assumption on $\Lambda$ implies that $U'$ does not contain $W_2$. The number of branch points specializing to $U'$ is at most two. Let $\bar{U}'$ be the union of $U'$ and the components of $X$ which are adjacent to $U'$.

We orient the dual graph of $\bar{U}'$ starting from $W_1$. Let $B'$ be the set of tails of $\bar{U}'$. Let $b_0$ be the unique component of $\bar{U}'$ on the geodesic connecting $W_1$ and $W_2$ over which $f$ is separable. By assumption, such a component exist. For $b \in B' - \{b_0\}$, we say that the corresponding tail $X_b$ is primitive if one of the branch points specializes to this component. Otherwise, we will call the tail new. Denote the set of primitive (resp. new) tails by $B'_\text{prim}$ (resp. $B'_\text{new}$). The tail $X_b$ of $\bar{U}'$ meets the rest of $u'$ in a unique point $x_b$. Let $y_b$ be a point of $Y$ mapping to $x_b$. By Lemma 2.1.2, $I(y_b)$ is an extension of a cyclic group of order $n_b$ prime-to-$p$, by a cyclic group of order $p$. Denote the conductor by $h_b$, and put $\sigma_b = h_b/n_b$. Recall that the ramification at $y_b$ is wild. This means that $I(y_b)$ is a subgroup of $N_G(P)$. We have assumed (Condition 2.2.2) that $N_G(P)$ is a dihedral group. It follows from [24], Lemme 1.1.2, that $\sigma_b \equiv 1/2$ (mod $Z$). In particular, for $b \in B'$, we have $\sigma_b \geq 1/2$.

Analogous to [24], Section 3.4, one proves the following vanishing cycle formula:

$$\sum_{b \in B'_\text{new}} (\sigma_b - 1) = -2 + \sum_{b \in B'_\text{prim}} (1 - \sigma_b) + (1 - \sigma_{b_0}).$$

(One proves this formula by constructing an “auxiliary cover” $Z' \to \bar{U}'$ and showing that it can be lifted to characteristic zero. The vanishing cycle formula follows then from the Riemann–Hurwitz formula applied to the generic fiber of the lift.) Furthermore, for $b \in B'_\text{new}$ we have that $\sigma_b - 1 \geq 1/2$. This follows from a genus consideration, [24], Proposition 3.3.5. The inequalities for $\sigma_b$ together with the fact that $|B'_{\text{prim}}| \leq 2$, implies that

$$\frac{|B'_{\text{new}}|}{2} \leq -2 + \frac{|B'_{\text{prim}}| + 1}{2} \leq -\frac{1}{2}.$$

Which is impossible. This concludes the proof. □

Remark 2.2.5 The notation and results used in the above proof will be introduced and explained in more detail in the next section. The reason for introducing the notation twice is that now that we proved Lemma 2.2.4, the notation can be simplified considerably.

2.3 The auxiliary cover

In this subsection we will suppose that Condition 2.2.3 is satisfied. Furthermore, we suppose that the cover $f_K$ has bad reduction. We start by introducing some more notation. Similar notation is
used in the proof of Lemma \[2.2.4\].

Let \( \mathcal{B} \) be the set of tails of \( X \). Every tail \( X_b \) of \( X \) contains a unique singular point \( x_b \) of \( X \). Choose a singular point \( y_b \) of \( Y \) mapping to \( x_b \) and let \( Y_b \) be the component of \( Y \) through \( y_b \), which is mapping to \( X_b \). Denote by \( h_b \) the conductor of \( Y_b \rightarrow X_b \) at \( y_b \), and by \( n_b \) the order of the prime-to-\( p \) ramification. Put \( \sigma_b = h_b/n_b \); this is the jump in the higher ramification groups of \( D(y_b) \), in the upper numbering.

Let \( P \) be a \( p \)-Sylow group of \( G \). Let \( U \) be the union of all components of \( X \) over which \( f \) is inseparable. Let \( V \) be any connected component of \( f^{-1}(U) \). Let \( y \) be a singular point of \( V \) and \( Z_1 \) and \( Z_2 \) be the components passing through \( y \). By Lemma \[2.2.4\], the inertia groups \( I(Z_1), I(Z_2) \) are cyclic of order \( p \), since \( Z_1 \) and \( Z_2 \) do not map to tails of \( X \). Therefore, by Lemma \[2.1.3\], we have that \( I(Z_1) = I(Z_2) \) and both are equal to the \( p \)-part of \( I(y) \). Here we use that \( p \) divides the order of \( G \). It follows that all irreducible components of \( V \) have the same \( p \)-Sylow subgroup of \( G \) as inertia group. Therefore, there is a connected component \( V \) of \( f^{-1}(U) \) such that \( I(V) = P \).

We will always assume \( V \) to be chosen like this.

As a consequence of Lemma \[2.2.4\], the construction of auxiliary covers as in \[24\], Section 3.2, goes through in our slightly different context. Since our notation differs from the notation of \[24\], we will recall the result.

**Proposition 2.3.1** There exists a cover \( g_K : Z_K \rightarrow X_K \) which is Galois with group \( D(V) \) and has a quotient model \( g_R : Z_R \rightarrow X_R \). It is uniquely characterized by the following properties:

(a) There exists a suitable open \( \Omega \) of \( X_R \), which contains \( X_b -\{x_b\} \) for \( b \in \mathcal{B} \), such that \( Z_R \rightarrow X_R \) is tamely ramified over \( \Omega \) and unramified outside the sections \( x_i \) (\( i = 1 \ldots 4 \)).

(b) There exists an étale neighborhood \( X'_R \rightarrow X_R \) of \( U \subset X \subset X_R \) such that \( Y'_R \rightarrow X'_R \simeq \text{Ind}^{D(V)}_{D(V)}(Z'_R \rightarrow X'_R) \), where \( Z'_R = Z_R \times_{X_R} X'_R \) and \( Y'_R = Y_R \times_{X_R} X'_R \).

We call \( g_K \) the auxiliary cover associated to \( f_K \).

**Proof:** \[24\], Proposition 3.2.6.

The (special fiber of the) auxiliary cover looks as follows. As above, we let \( V \) be a connected component of \( f^{-1}(U) \) with inertia group \( P \). Restricted to \( U \), the auxiliary cover is just \( V \rightarrow U \).

Let \( X_b \) be a tail of \( X \) and \( x_b \) the unique point of \( X_b \) which is singular in \( X \); we may suppose \( x_b = \infty \). Let \( \Delta_b \) be the inertia group of a point \( y_b \) above \( x_b \) which lies on \( V \). Then, by the Katz–Gabber Lemma \[4\], there exist a cover \( Z_b \rightarrow X_b \) unbranched outside \( 0, \infty \) and at most tamely ramified at \( 0 \) which locally around \( \infty \), if we induce it up to a \( G \)-cover, agrees with \( Y \rightarrow X \).

Now \( Z \rightarrow X|_{X_b} = \text{Ind}^{D(V)}_{D(V)}(Z_b \rightarrow X_b) \). Note that by construction, the \( \sigma_b \) for \( b \in \mathcal{B} \) for the cover \( g : Z \rightarrow X \), are the same as for the original cover.

**Lemma 2.3.2** For \( b \in \mathcal{B}_{\text{new}} \), we have \( \sigma_b - 1 \geq 1/2 \).

**Proof:** This follows from \[24\], Proposition 3.3.5 and Lemme 1.1.2, and the assumption that \( N_G(P) \) is a dihedral group.

The following formula reflects the condition that the genus of \( Y \) has to be equal to the genus of the generic fiber \( Y_K \). It is proved in \[24\], Section 3, using the auxiliary cover \( g_R : Z_R \rightarrow X_R \).

**Proposition 2.3.3** (Vanishing cycle formula)

\[ \sum_{b \in \mathcal{B}_{\text{new}}} (\sigma_b - 1) = -2 + \sum_{b \in \mathcal{B}_{\text{prim}}} (1 - \sigma_b). \]
2.4 Modular reduction

Let \( f_K : Y_K \to X_K \) be a \( G \)-cover for which Condition 2.2.2 holds. In this subsection we will show that \( f_K \) has either good reduction or modular reduction. Essentially, the property of having modular reduction means that the auxiliary cover introduced in the previous section is a cover \( Z_K \to \mathbb{P}^1_K \) with Galois group a dihedral group and ramification of order 2 as discussed in Example 1.1.6.

**Definition 2.4.1** Suppose \( f_K \) has bad reduction. We will say that \( f_K \) has modular reduction if the following conditions are satisfied.

(a) The curve \( X \) has four primitive tails and no new tails. Every irreducible component of \( X \) is either an original component or a tail.

(b) Let \( E \) be a connected component of \( f^{-1}(X_0) \). Then \( D(E) \) is a dihedral group of order \( 2N \) for some \( N \) divisible by \( p \).

(c) Let \( X_b \) be a tail of \( X \). Let \( x_b \) be the unique singular point of \( X_b \) in \( X \) and let \( y_b \) be a point of \( Y \) mapping to \( x_b \). Then the inertia group \( I(y_b) \) is a dihedral group of order \( 2p \) and \( \sigma_b = 1/2 \).

If \( f_K : Y_K \to X_K \) has modular reduction, then we will say that the special fiber \( f : Y \to X \) of the quotient model of \( f_K \) is of modular type.

The integer \( N \) defined above will be called the level of \( f \). This terminology reflects the relation between \( f \) of modular type and modular curves.

**Remark 2.4.2** Suppose that \( f_K : Y_K \to X_K \) has modular reduction of level \( N \). The auxiliary cover \( g_K : Z_K \to X_K \) corresponding to \( f_K \) is a Galois cover with Galois group the dihedral group \( \Delta \) of order \( 2N \), branched at \( x_1, \ldots, x_4 \) of order 2. The reduction \( g : Z \to X \) is inseparable over \( X_0 \); it is separable over the tails. Note that \( g^{-1}(X_0) \) may be identified with \( E \) in the definition; it is, after choice of a base point, a generalized elliptic curve.

**Figure 2:** modular reduction of level \( N = p \)

**Proposition 2.4.3** Let \( f_K : Y_K \to X_K \) be as before, in particular we suppose that Condition 2.2.2 is satisfied and that \( f_K \) has bad reduction. Then \( f_K \) has modular reduction.
Proof: Suppose that $f_K$ has bad reduction. Proposition 2.3.3 implies that
\[ \sum_{b \in \mathbb{B}_{\text{new}}} \sigma_b - 1 = -2 + \sum_{b \in \mathbb{B}_{\text{prim}}} 1 - \sigma_b. \]
Recall that $\sigma_b \geq 3/2$ for $b \in \mathbb{B}_{\text{new}}$ and $\sigma \geq 1/2$ for $b \in \mathbb{B}_{\text{prim}}$, Lemma 2.3.2. Hence $|\mathbb{B}_{\text{new}}|/2 \leq 0$. This implies that $|\mathbb{B}_{\text{new}}| = 0$ and $\sigma_b = 1/2$ for $b \in \mathbb{B}_{\text{prim}}$. Moreover, all $x_i$ specialize to components over which $f$ is separable, so $|\mathbb{B}_{\text{prim}}| = 4$.

The decomposition group of a singular point $y_b$ of $Y$ contains a dihedral group of order $2p$, since $\sigma_b = 1/2$. (It cannot be Abelian, since then $\sigma_b$ would be an integer by the Hasse–Arf Theorem.) Since $D(y_b) = I(y_b)$ is a cyclic-by-$p$ group, $D(y_b)$ is isomorphic to a dihedral group of order $2p$. This proves Part (c) of Definition 2.4.1.

The decomposition group $D(V)$ is a subgroup of $N_G(P)$ which is a dihedral group, by assumption. Note that $D(V)$ is not cyclic, since it contains (a conjugate of) the inertia group of some point $y_b$, which is dihedral. This proves Part (b).

The only thing left to show is the second part of Part (a). Let $W$ be a component of $X$ which is neither a tail nor an original component and let $Z$ be a component of $Y$ which maps to $W$. Above we have shown that there are exactly four tails. This implies that $W$ meets the rest of $X$ in two points. There is no branch point specializing to $W$, so the maximal separable subcover $Z' \to W$ of $Z \to W$ is branched at at most two points. Moreover, $I_Z$ is a cyclic group of order $p$, since $W$ is not a tail. This implies that the degree of $Z' \to W$ is prime-to-$p$, so it is a cyclic cover of $\mathbb{P}^1$ branched at two points. But then $Z$ has genus zero and meets the rest of $Y$ in exactly two points. This contradicts the minimality of $Y$.

This shows that the cover has modular reduction. \qed

2.5 Reduction of degenerate covers

In this section we will study the reduction behavior of the degenerate covers corresponding to the cusps of the Hurwitz space. Note that the theory of semistable reduction we developed so far does not apply here, since we always assumed that the generic fibers of our curves were smooth.

The situation in this section is somewhat different from that in the previous sections. Let $R$ be a complete discrete valuation ring whose quotient field $k$ is of characteristic zero and whose residue field $k = \bar{k}$ has characteristic $p$. Let $(X_K; x_1, \ldots, x_4)$ be a stably marked $K$-curve of genus zero, which we suppose to be singular. Let $X_{1,K}$ and $X_{2,K}$ be the two irreducible components of $X_K$. Let $\tau$ be the singular point of $X_K$. Let $f_K : Y_K \to X_K$ be an admissible $G$-Galois cover ramified at $x_1, \ldots, x_4$. Let $\rho$ be a point of $Y_K$ mapping to $\tau$ and let $Y_{1,K}$ and $Y_{2,K}$ be the components of $Y_K$ passing through $\rho$, where we suppose that $Y_{i,K}$ maps to $X_{i,K}$. Let $G_i$ be the decomposition group $Y_{i,K}$. Let $X_{i,R}$ and $Y_{i,R}$ be the closure of $X_{i,K}$ and $Y_{i,K}$ in $X_R$. Since $\mathcal{H}^0_{\mathcal{F}}(G)$ is proper, $f_K$ extends uniquely to a map $f_{0,R} : Y_R \to X_{0,R}$ between stably marked covers over $\text{Spec} R$. Let $f_i : Y_{i,R} \to X_{i,R}$ be the corresponding morphism. We will denote the special fibers of $Y_{i,R}, X_{i,R}, Y_{i,R}, X_{i,R}$ by $Y, X, Y_i, X_i$, respectively. The mark $C$ on $X_R$ can be written as a union $C = C'_1 \cup C'_2$, where $C'_i$ is a mark on $X_{i,R}$. Let $\tau$ be the unique singular point of $X_K$, we denote its (unique) extension to $X_R$ also by $\tau$. Let $C_i = C'_i \cup \{\tau\}$. Let $D'_i$ be the restriction of the mark of $Y_{i,R}$ to $Y_{i,R}$ and write $D_i$ for the union of $D'_i$ with the points of $Y_{i,R}$ which are singular in $Y_R$. The next lemma follows immediately.

Lemma 2.5.1 The morphism $f_{i,R} : (Y_{i,R}, D_i) \to (X_{i,R}, C_i)$ constructed above is a morphism of stably marked curves.

The covers $f_{i,K}$ constructed above are covers of a projective line branched at three points. So for these covers we can apply the criterion for good reduction proved by Raynaud [24]. Actually, since here we put a stronger condition on $G$ than in [24], we can show that the covers $f_{i,K}$ have
good reduction iff $p$ does not divide the ramification indices, Lemma 2.5.2. In other words, the condition on the field of definition in the Theorem of Raynaud is not needed in this case.

Proposition 2.5.3 is the key result in this section. It describes the reduction of the degenerate covers to characteristic $p$. The idea is that we can understand the reduction of such a degenerate cover, because we understand the reduction of the two three point covers of which it is made. Proposition 2.5.3 will be used in Section 5 to explicitly describe the bad part of the Hurwitz space.

**Lemma 2.5.2** Let $f_K : Y_K \to X_K \cong \mathbb{P}_k^1$ be a $G$-cover branched at three points $0, 1, \infty$. Suppose that $p$ exactly divides the order of $G$ and the normalizer of a $p$-Sylow group of $G$ is dihedral. Denote the ramification indices of $f_K$ by $m_1, m_2, m_3$. Let $f_R : Y_R \to X_R$ be the corresponding inertia groups are nontrivial. Moreover, they contain a dihedral group of order $2$. This proves Part (c).

Proof: This follows immediately from the vanishing cycle formula [24], Section 3.4, combined with the estimates for $\sigma_t$ from Lemma 2.3.3.

**Proposition 2.5.3** Let $f_K : Y_K \to X_K$ be as above and let $f : Y \to X$ be its reduction. Let $n$ be the order of the ramification of $f_K$ above the singular point $\tau$. We denote by $\tau_k$ the image of $\tau$ on the special fiber. Let $\rho_k$ be a point on $Y$ above $\tau_k$.

(a) The cover $f_K$ has admissible reduction iff $p \not| n$.

(b) The inertia group $I(\rho_k)$ has order $n$.

(c) If $f_K$ has bad reduction, then $f$ is of modular type.

(d) Suppose $f_K$ has bad reduction, then $n$ divides the level $N$ of $f$. Let $Z_1$ and $Z_2$ be the irreducible components of $Y$ passing through $\rho_k$. Then $D(Z_1)$ and $D(Z_2)$ are dihedral groups of order $2n$.

Proof: The cover $f : Y \to X$ is the reduction of a degenerate cover, by assumption. However, there will be covers representing points in the interior of the Hurwitz stack in characteristic zero which specialize to $f$. Therefore, in case $f_K$ has bad reduction, $f$ will be of modular type by Proposition 2.4.3. This proves Part (c).

This implies that in case $f_K$ has bad reduction, $f_0 : Y \to X_0$ does not contract any components to $\tau$. This is clearly also the case if $f_K$ does not have bad reduction. Since $f_i : Y_i \to X_i$ for $i = 1, 2$ are flat, we have $g(Y_i) = g(Y_i, k)$. Moreover, $g(Y) = g(Y_K)$. Now

$$g(Y) = (|G|/|G_1|)g(Y_1) + (|G|/|G_2|)g(Y_2) + 1 - |G|/|G_1| - |G|/|G_2| + |G|/|I(\rho_k)|$$

and

$$g(Y_K) = (|G|/|G_1|)g(Y_{1,K}) + (|G|/|G_2|)g(Y_{2,K}) + 1 - |G|/|G_1| - |G|/|G_2| + |G|/n.$$ 

This shows that $|I(\rho_k)| = n$.

Suppose $p/n$. Then Part (b) of Lemma 2.1.2 implies that the covers $f_i, K$ have bad reduction for $i = 1, 2$. It follows that $f_K$ has bad reduction. Conversely, suppose that $f_K$ has bad reduction. Part (c) implies that $f_K$ has modular reduction. In particular it follows that $I(Z_i)$ is $p$-cyclic. Part (b) of Lemma 2.1.2 implies that $p/n$.

Suppose that $f_K$ has bad reduction. The decomposition groups $D(Z_i)$ are subgroups of $N_G(P)$, since the corresponding inertia groups are nontrivial. Moreover, they contain a dihedral group of order $2p$, by the definition of modular reduction (Definition 2.4.1). It follows that the $D(Z_i)$ are dihedral groups. The maximal separable subcover of $Z_i \to X_i$ is $f_i' : Z_i' := Z_i/I(Z_i) \to X_i$. Note that $f_i'$ is branched at three points of order $2, 2, n/p$ hence $g(Z_i') = 0$. It follows that the degree of $f_i'$ is $2n/p$, hence $D(Z_i)$ is a dihedral group of order $2n$. By definition of the level $N$ of $f$, the subgroup of $N_G(P)$ generated by $D(Z_1)$ and $D(Z_2)$ is a dihedral group of order $2N$; this implies that $n|N$. 

\[\square\]
3 A Reduction Theorem

In this section we prove a theorem about the structure of \( \bar{H} \otimes \mathbb{F}_p \), where \( \bar{H} \) is a complete Hurwitz space for \( G \)-covers satisfying Condition 2.2.2. As explained in the introduction, this theorem relies on, and in some sense extends, the results of Katz and Mazur on the reduction of the modular curve \( X_1(p) \). This is somewhat surprising. Like modular curves, the Hurwitz spaces we look at are curves, defined over small number fields, and arise as quotients of the upper half plane by discrete subgroups of \( \text{GL}_2(\mathbb{Z}) \), see \( \text{[3]} \). However, these groups are non-congruence subgroups, in general.

3.1 Statement of the main results

3.1.1 Let \( G \) be a finite group, \( \mathcal{C} = (C_1, C_2, C_3, C_4) \) a class vector in \( G \) of length 4 and \( p \) an odd prime. We denote by \( m_i \) the order of the elements of \( C_i \). We assume that Condition 2.2.2 holds, with respect to \( G, p \) and \( m_i \).

Let \( K := K(C) \subset \mathbb{Q}(\zeta_m) \) be the minimal field over which the classes \( C_i \) are rational, see Section 1.2.3. We choose a prime ideal \( \mathfrak{p} \) of \( K \) dividing \( p \), and denote by \( \Lambda := \mathcal{O}_{\mathfrak{p}} \) its local ring. We let \( H := \bar{H}^{\mathfrak{p}}(C) \) be the complete Hurwitz space over \( \Lambda \), as defined in Section 1.2.4. By construction, \( H \) is an algebraic space, proper and of finite type over \( \Lambda \). It contains the Hurwitz space \( H := H^{\mathfrak{p}}(C) \) as a dense open subscheme. The scheme \( H \) is smooth over \( \Lambda \); its generic fiber \( H \otimes K \) is the reduced Hurwitz curve studied e.g. in \( \text{[3]} \).

Let \( H^{\text{bad}} \subset H \) be the closed subspace corresponding to bad covers. Since bad covers occur only in positive characteristic, \( H^{\text{bad}} \) is a closed subspace of \( H \otimes \mathbb{F}_q \), where \( \mathbb{F}_q \) is the residue field of \( \Lambda \). The complement \( H^{\text{adm}} := H - H^{\text{bad}} \) is a dense open subscheme and corresponds to admissible covers. Let \( H^{\text{good}} \) be the closure of \( H^{\text{adm}} \otimes \mathbb{F}_q \) in \( H \otimes \mathbb{F}_q \). Note that this is an abuse of notation, since \( H^{\text{good}} \) has nontrivial intersection with \( H^{\text{bad}} \), in general. There is a natural map \( H \to \mathbb{P}^1_{\Lambda} \otimes \Lambda \).

We say that an \( \mathbb{F}_p \)-rational point \( s \) of \( H^{\text{bad}} \) is supersingular if the corresponding value \( \lambda(s) \in \mathbb{F}_p \) is supersingular.

Theorem 3.1.1

(i) The complete Hurwitz space \( H \) is a normal scheme of dimension 2. The natural map \( H \to \mathbb{P}^1_{\Lambda} \otimes \Lambda \) is finite and flat.

(ii) The subspaces \( H^{\text{bad}} \) and \( H^{\text{good}} \) are smooth projective curves over \( \mathbb{F}_q \). They intersect transversally in the supersingular points.

In the rest of Section 3.1 we state a number of results on the local structure of \( H \) and explain how Theorem 3.1.1 can be deduced from them. Let us give a brief outline. On the open subset \( H^{\text{adm}} \subset H \), Theorem 3.1.1 is known to hold, see \( \text{[29]} \). Therefore, it suffices to look at \( H \) in a neighborhood of a point corresponding to a bad cover. Theorem 3.1.1 below describes the universal deformation ring of such a bad cover. Essentially, this theorem implies that the complete Hurwitz stack \( \mathcal{H} \) associated to \( H \) is regular and that a version of Theorem 3.1.1 holds for \( \mathcal{H} \). To finish the proof of Theorem 3.1.1, we have to study the monodromy action, i.e. the action of the group of automorphisms of a bad cover on its universal deformation ring. Propositions 3.1.4 describes this action, and Theorem 3.1.7 follows. Under some extra hypotheses (Condition 3.1.6), we can improve our results on the monodromy action, and we can actually show that \( H \) is regular. The relevant statement is made in Proposition 3.1.7.

The proofs of Theorem 3.1.1, Proposition 3.1.4 and Proposition 3.1.7 are postponed to Section 3.3.

3.1.2 The universal deformation ring Let \( \mathcal{H} := \bar{H}^{\mathfrak{p}}(C) \) be the complete Hurwitz stack over \( \Lambda \), associated to \( H \), and let \( \mathcal{H}^{\text{bad}}, \mathcal{H}^{\text{good}} \subset \mathcal{H} \) be the closed substacks corresponding to the closed
subspaces $\tilde{H}^{\text{bad}}, \tilde{H}^{\text{good}} \subset \tilde{H}$. We look at the following situation. Let $k$ be an algebraically closed field of characteristic $p$ and

$$f_0 : Y \longrightarrow (X_0; x_i)$$

an object of $\tilde{H}^{\text{bad}}$, defined over $k$. In the rest of this section we will mostly write “$Y$” instead of “$f_0 : Y \rightarrow (X_0; x_i)$” for this object; we understand that the curve $Y$ is equipped with an action of the group $G$, a mark $D \subset Y$ and a map $f_0$ to the stably marked curve $(X_0; x_i)$.

Let $R_Y$ be the strict complete local ring of $\tilde{H}$ at the $k$-point corresponding to $Y$. Let $Y_{\text{univ}}$ be the object of $\tilde{H}_k(C)$ corresponding to the tautological morphism $\text{Spec} R_Y \rightarrow \tilde{H}$. By a general property of algebraic stacks, $Y_{\text{univ}}$ is the universal deformation of $Y$ as object of $\tilde{H}$. This means the following. Let $W(k)$ denote the ring of Witt vectors over $k$ and $\tilde{C}_k$ the category of complete local Noetherian $W(k)$-algebras with residue field $k$. We let $\text{Def}(Y)$ be the functor which assigns to $R \in \tilde{C}_k$ the set of isomorphism classes of deformations of $Y$ over $R$. Then $Y_{\text{univ}}$ defines an equivalence

$$\text{Hom}_{\tilde{C}_k}(R_Y, \cdot ) \sim \text{Def}(Y).$$

The closed substack $\tilde{H}^{\text{bad}} \subset \tilde{H}$ (resp. $\tilde{H}^{\text{good}} \subset \tilde{H}$) corresponds to a subfunctor $\text{Def}(Y)^{\text{bad}} \subset \text{Def}(Y)$ (resp. $\text{Def}(Y)^{\text{good}} \subset \text{Def}(Y)$), which is represented by a quotient ring $R_Y^{\text{bad}}$ (resp. $R_Y^{\text{good}}$) of $R_Y$.

We denote by $\text{Def}(X_0; x_i)$ the deformation functor for $(X_0; x_i)$ as object of $\mathcal{M}_{0,4}$ and by $R_{X_0}$ the universal deformation ring. The morphism $\tilde{H} \rightarrow \mathcal{M}_{0,4}$ induces a transformation $\text{Def}(Y) \rightarrow \text{Def}(X; x_i)$, hence a morphism $R_{X_0} \rightarrow R_Y$. The ring $R_{X_0}$ is of the form $R_{X_0} = W(k)[[u]]$. For instance, if $X_0$ is smooth we may assume that $X_0 = \mathbb{P}^1$, $x_1 = 0$, $x_2 = 1$, $x_3 = \infty$, $x_4 = \lambda_0 \in k - \{0,1\}$ and $w = \lambda - \lambda_0$. We say that $Y$ is supersingular (resp. ordinary) if $\lambda_0$ is supersingular (resp. ordinary). If $X_0$ is singular, we will also say that $Y$ is ordinary.

**Theorem 3.1.2** The ring $R_Y$ is regular of dimension 2 and a finite flat extension of $R_{X_0}$. Moreover, there exists a regular sequence $(t, \pi)$ for $R_Y$ such that

1. $R_Y^{\text{bad}} = R_Y/(\pi) \cong k[[t]]$,
2. The induced (finite) morphism $R_{X_0} \otimes_{W(k)} k \rightarrow R_Y^{\text{bad}}$ is inseparable of degree $p$. Its separable part $R_{X_0} \otimes_{W(k)} k \rightarrow (R_Y^{\text{bad}})^p \cong k[[t^p]]$ is tamely ramified (of degree $d$); if $X_0$ is smooth then $d = 1$.
3. If $Y$ is ordinary, then $p = u \pi^{p-1}$ for a unit $u \in R_Y$ and $R_Y^{\text{good}} = 0$.
4. If $Y$ is supersingular, then $p = u \pi^{p-1}$, where $u \in R_Y$ is a local equation for $\tilde{H}_k(C)^{\text{good}}$. More precisely, $R_Y^{\text{good}} = R_Y/(u) \cong k[[\pi]]$, and $(u, \pi)$ is a regular sequence for $R_Y$.

In Section 3.3 we will prove this theorem in the case that the group $G$ is dihedral and the conjugacy classes $C_i$ represent reflections. In this case, the $G$-cover $f : Y \rightarrow X_0$ corresponds essentially to a point on the modular curve $X_1(N)$, and Theorem 3.1.2 follows from the results of [7]. We will prove the general case in Section 3.3 by reduction to the dihedral case.

**Corollary 3.1.3** The complete Hurwitz stack $\tilde{H}$ is regular of dimension 2. The natural morphism $\tilde{H} \rightarrow \mathcal{M}_{0,4}$ is finite and flat. The closed substacks $\tilde{H}^{\text{bad}}$ and $\tilde{H}^{\text{good}}$ are smooth over $\mathbb{F}_q$, and intersect transversally in the supersingular point.
3.1.3 The monodromy action  Corollary [3.1.3] together with Proposition [3.1.4] implies Part (i) of Theorem [3.1.1]. To prove Part (ii) of Theorem [3.1.1] we use the general fact that the coarse moduli scheme is locally the quotient of (an étale cover of) the corresponding algebraic stack by a finite group action. Actually, it suffices to look at the strict complete local rings. Hence we are led to study the monodromy action of the automorphisms of $Y$ on the universal deformation ring $R_Y$.

Let $Y$ be as in Section [3.1.2]. We write $\text{Aut}_k(Y)$ for the group of $k$-linear automorphisms of $Y$, considered as object of $\mathcal{H}$. Since $(X_0;x_i)$ has no nontrivial automorphism, an element $\sigma \in \text{Aut}_k(Y)$ is a $k$-automorphism of $Y$ such that $f_0 \circ \sigma = f_0$ and $g \circ \sigma = \sigma \circ g$ for all $g \in G$. By the universal property of $Y_{\text{univ}}$, for each $\sigma$ there exist a unique automorphism $\gamma : Y \to Y$ such that $\sigma$ lifts to a unique $\gamma$-semilinear automorphism $\sigma_{\text{univ}} : Y_{\text{univ}} \to Y_{\text{univ}}$. We call $\gamma \in \text{Aut}_k(R_Y)$ the monodromy action of $\sigma$. We define the monodromy group $\Gamma$ of $Y$ as the image of the homomorphism $\text{Aut}_k(Y) \to \text{Aut}_{\text{univ}}(Y_{\text{univ}})$. Let $Y_\eta$ be the geometric generic fiber of $Y_{\text{univ}}$ (note that $R_Y$ is a domain, by Theorem [3.1.1]). Identifying the group of $\eta$-automorphisms of $Y_\eta$ with $C_G$, the center of $G$, we obtain a natural exact sequence

\[(5) \quad 1 \to C_G \to \text{Aut}_k(Y) \to \Gamma \to 1.\]

Let $s : \text{Spec} \, k \to \bar{H}$ be the geometric point of the Hurwitz space corresponding to $Y$. The strict complete local ring of $\bar{H}$ is the ring of $\Gamma$-invariants of $R_Y$:

\[(6) \quad \hat{O}_{\bar{H},s} = R_Y^\Gamma.\]

3.1.4 The absolute monodromy action  To study the action of $\Gamma$ on $R_Y$, it will turn out to be useful to enlarge $\Gamma$ and look at the absolute monodromy group $\Gamma^{ab}$.

Let $\hat{\mathcal{H}}^{ab}_k(C)$ be the absolute version of the complete Hurwitz stack $\mathcal{H}$. Let us for the moment consider $Y$ and $Y_{\text{univ}}$ as objects of $\hat{\mathcal{H}}^{ab}_k(C)$. In other words, we forget the embedding $G \to \text{Aut}(Y/X)$, and only retain its image, see Variant [3.1.4]. It is still true that $Y_{\text{univ}}$ is the universal deformation of $Y$. But we obtain a bigger automorphism group, in general. We denote by $\text{Aut}^{ab}_k(Y)$ the group of $k$-automorphisms of $Y$ as object of $\hat{\mathcal{H}}^{ab}_k(C)$. An element $\sigma \in \text{Aut}^{ab}_k(Y)$ is a $k$-automorphism of $Y$ such that $f_0 \circ \sigma = f_0$ and $g \circ \sigma = \sigma \circ g$ for all $g \in G$. We find that $G$ is a normal subgroup of $\text{Aut}^{ab}_k(Y)$, and $\text{Aut}^{ab}_k(Y)$ is the centralizer of $G$ in $\text{Aut}^{ab}_k(Y)$. The group $\text{Aut}^{ab}_k(Y)$ acts on $R_Y$; this action extends the action of $\text{Aut}_k(Y)$. We write $\Gamma^{ab}$ for the image of $\text{Aut}^{ab}_k(Y)$ in $\text{Aut}_{\text{univ}}(Y_{\text{univ}})$. Clearly, $\Gamma \subseteq \Gamma^{ab}$, and the exact sequence (3) becomes

\[(7) \quad 1 \to G \to \text{Aut}^{ab}_k(Y) \to \Gamma^{ab} \to 1.\]

Proposition 3.1.4  There exist two characters $\chi^{\text{bad}} : \Gamma^{ab} \to \mathbb{F}_p^\times$ and $\chi^{\text{adm}} : \Gamma^{ab} \to \mu_d$ (here $\mu_d \subseteq R_Y$ denotes the set of $d$th roots of unity, for $d$ as in Theorem [3.1.2] (ii)) with the following properties. The parameters $t, \pi \in R_Y$ in Theorem [3.1.2] can be chosen such that for all $\gamma \in \Gamma^{ab}$

$$\gamma(t) = \chi^{\text{adm}}(\gamma) \cdot t, \quad \gamma(\pi) = \chi^{\text{bad}}(\gamma) \cdot \pi \pmod{\pi^2}.$$  

Moreover, the homomorphism $(\chi^{\text{bad}}, \chi^{\text{adm}}) : \Gamma^{ab} \to \mathbb{F}_p^\times \times \mu_d$ is injective.

We will prove this proposition in Section [3.3.3]. In the rest of this subsection we will show that Theorem [3.1.2] and Proposition [3.1.4] together imply Theorem [3.1.1]. Since $\Gamma \subseteq \Gamma^{ab}$, Proposition [3.1.4] holds also for $\Gamma$.

The closed subscheme $\hat{\mathcal{H}}^{\text{bad}} \subseteq \hat{\mathcal{H}}$ is the image of the natural morphism $\hat{\mathcal{H}}^{\text{bad}} \to \hat{\mathcal{H}}$, by definition. It follows that the strict complete local ring $\hat{O}_{\hat{\mathcal{H}}^{\text{bad}},s}$ is the image of the natural morphism $R_Y^\Gamma \to R_Y^{\text{bad}}$. By Proposition [3.1.4] the order of $\Gamma$ is prime-to-$p$. Therefore, the natural map $R_Y^\Gamma \to (R_Y^{\text{bad}})^\Gamma$
is surjective, hence $\hat{O}_{\bar{H}, s}^{\text{bad}} = (R_{\bar{Y}_{s}}^{\text{bad}})^{\Gamma}$, see [17], A.7. Now Theorem 3.1.2 (i) and Proposition 3.1.4 imply

\[ (8) \quad \hat{O}_{\bar{H}, s}^{\text{bad}} = (R_{\bar{Y}_{s}}^{\text{bad}})^{\Gamma} \cong k[[t^{d}]], \]

where $d \equiv d \pmod{d}$ is the order of $\chi^{\text{adm}}(\Gamma)$. We have shown that $\bar{H}$ is a smooth curve. The same argument shows that

\[ (9) \quad \hat{O}_{\bar{H}, s}^{\text{good}} = (R_{\bar{Y}_{s}}^{\text{good}})^{\Gamma} \cong \begin{cases} 0 & \text{if } Y \text{ is ordinary}, \\ k[[\bar{\pi}^{\mu}]] & \text{if } Y \text{ is supersingular}, \end{cases} \]

where $\mu(p-1)$ is the order of $\chi^{\text{adm}}(\Gamma)$. We conclude that $\bar{H}^{\text{good}}$ is a smooth curve and that $\bar{H}^{\text{good}} \to \mathbb{P}_{\lambda}^{1}$ is ramified of order $(p-1)/\mu$ in the supersingular points. Let us assume that $Y$ is supersingular and denote by $\bar{H}_{p}^{\text{red}}$ (resp. by $\bar{H}_{p}^{\text{red}}$) the closed subscheme $(\bar{H} \otimes \mathbb{F}_{p})^{\text{red}} \subset \overline{H}$ (resp. the closed substack $(\bar{H} \otimes \mathbb{F}_{p})^{\text{red}} \subset \bar{H}$). By Theorem 3.1.3 (iv), $\bar{H}_{p}^{\text{red}} \times_{\bar{H}} \Spec R_{Y} = R_{Y}/(\pi u) \cong k[[\bar{\pi}, \bar{u}]]$. Taking invariants and arguing as before, we get

\[ (10) \quad \hat{O}_{\bar{H}, s}^{\text{red}, \text{p}} = (R_{Y}/(\pi u))^{\Gamma} \cong k[[\bar{\pi}^{\mu}, \bar{u} \mid \bar{\pi}^{\mu} \bar{u} = 0]] \]

(note that $\chi^{\text{adm}} = 1$, since $X_{0}$ is smooth). This completes the proof of Theorem 3.1.2, modulo the proofs of Theorem 3.1.3 and Proposition 3.1.4.

Translating the statements of Theorem 3.1.2 into geometric properties of the map $\bar{H} \to \mathbb{P}_{\lambda}^{1}$, we obtain the following corollary.

**Corollary 3.1.5** Let $s$ be the point on $\bar{H}^{\text{bad}}$ corresponding to $Y$.

(i) The natural map $\bar{H}^{\text{bad}} \to \mathbb{P}_{\lambda}^{1} \otimes \mathbb{F}_{q}$ is finite, with inseparability degree $p$. Its separable part $(\bar{H}^{\text{bad}})^{(p)} \to \mathbb{P}_{\lambda}^{1} \otimes \mathbb{F}_{q}$ is tamely ramified at $\lambda = 0, 1, \infty$ and étale everywhere else. More precisely, its ramification index in $s^{(p)}$ is equal to $[\text{Im}(\chi^{\text{adm}}) : \mu_{d}]$.

(ii) The natural map $\bar{H}^{\text{good}} \to \mathbb{P}_{\lambda}^{1} \otimes \mathbb{F}_{q}$ is tamely ramified at $\lambda = 0, 1, \infty$ and the supersingular values of $\lambda$, and is étale everywhere else. Its ramification index in $s \in \bar{H}^{\text{good}} \cap \bar{H}^{\text{bad}}$ is $[\chi^{\text{bad}}(\Gamma) : \mathbb{F}_{p}^{\times}]$ (this happens if and only if $Y$ is supersingular).

(iii) Let $m_{s}$ be the multiplicity of $\bar{H}$ in a neighborhood of $s$ in $\bar{H} \otimes \mathbb{F}_{q}$, i.e. the length of the Artinian local ring $\hat{O}_{\bar{H}, s}$, where $\eta : \Spec k((t^{d})) \to \bar{H}$ comes from equation (3). Then $m_{s} = [\chi^{\text{bad}}(\Gamma) : \mathbb{F}_{p}^{\times}]$.

3.1.5 **Regularity** We have a better control of the monodromy group $\Gamma$, if we assume that — in addition to Condition 2.2.2 — the following holds.

**Condition 3.1.6**

(e) The center $C_{G}$ of $G$ is trivial.

(f) Let $G'$ be a subgroup of $G$ which contains an element of order $p$ and an element of one of the classes $C_{i}$, $i = 1, \ldots, 4$. Then $G = G'$.

**Proposition 3.1.7** Assume that Condition 3.1.6 holds. Let $N$ be the level of $Y$. Then $\chi^{\text{adm}}|_{\Gamma} = 1$ and

\[ \Gamma \cong \begin{cases} \mathbb{Z}/2 & \text{if } N = p, \\ \mathbb{Z} / 2 & \text{if } N > p. \end{cases} \]
We will prove Proposition 3.1.7 in Section 3.3.4.

Corollary 3.1.8 If Condition 3.1.6 holds, then $H$ is regular.

Proof: It follows from Theorem 3.1.2 and Propositions 3.1.4 and 3.1.7 that $R^p_Y$ has a regular sequence $(t, \pi')$, where $\pi' \sim \pi^\mu$ and $\mu = 1, 2$. □

Remark 3.1.9 We do not know of any example in which either $\bar{H}$ or $\bar{H}_{4}^{\text{ab}}(C)$ is not regular. On the other hand, the Hurwitz spaces $\bar{H}_{4}^{\text{un}}(C)$ and $\bar{H}_{4}^{\text{ab}}(C)$ behave like modular curves. We have to replace the $\lambda$-line with the $j$-line. For the special values $j = 0$ and $j = 1728$, the monodromy action may become more complicated then in Proposition 3.1.4. Therefore, $\bar{H}_{4}^{\text{un}}(C)$ and $\bar{H}_{4}^{\text{ab}}(C)$ are very often not regular. However, Theorem 3.1.1 remains true.

3.2 Dihedral covers and generalized elliptic curves

In this section we show that Theorem 3.1.2 is true in the case the group $G$ is a dihedral group of order $2N$, where $p \mid N$, and $C$ consists of 4 times the conjugacy class of reflections. In order to prove this, we relate the deformation theory of a bad cover, which arises as the reduction of a $G$-cover of type $C$, to the deformation theory of a generalized elliptic curve endowed with a certain level structure. Once this is achieved, Theorem 3.1.2 follows from the results of [17] on the reduction of the modular curve $X_1(p)$.

3.2.1 Generalized elliptic curves We start by recalling some definitions from [3], [17] and [1]. A generalized elliptic curve over a scheme $S$ is a semistable curve $E/S$ of genus 1 together with a section $0 : S \to E^{\text{un}}$ and an $S$-morphism $+: E^{\text{un}} \times_S E \to E$, verifying the following properties. The geometric fibers of $E/S$ are either smooth or “$n$-gons”. The restriction of $+$ to $E^{\text{un}}$ gives $E^{\text{un}}$ the structure of a commutative group scheme with identity 0. Moreover, $E^{\text{un}}$ acts on $E$ by “rotation”, see [3], Definition II.1.12. If $E/S$ is smooth of genus 1 and $0 : S \to E$ a section, then there exists one and only one such group law $+: E \times E \to E$, and we call $(E, 0)$ an elliptic curve over $S$.

Let $A$ be a finite Abelian group and $E/S$ a generalized elliptic curve. A weak $A$-structure on $E$ is a group homomorphism $\phi : A \to E^{\text{un}}(S)$ such that the Cartier divisor $\phi(A) := \sum_{a \in A} \phi(a)$ is a subgroup-scheme of $E^{\text{un}}$. A weak $A$-structure $\phi$ is called an $A$-structure if $\phi(A)$ meets every irreducible component of every geometric fiber of $E/S$. The following two examples are classical. For $A = \mathbb{Z}/n$, an $A$-structure is called a $\Gamma_1(n)$-structure. For $A = \mathbb{Z}/n \times \mathbb{Z}/n$, an $A$-structure is called a $\Gamma(n)$-structure.

3.2.2 $\Gamma_2(N)$-structures We fix an integer $N > 0$ and an odd prime number $p$ such that $p \mid N$. We define the Abelian group

$$A := \mathbb{Z}/2N \times \mathbb{Z}/2.$$  

Similarly, we let $A' := \mathbb{Z}/2N' \times \mathbb{Z}/2$, where $N = pN'$, and let $\tau : A \to A'$ be the natural projection.

Definition 3.2.1 A $\Gamma_2(N)$-structure on a generalized elliptic curve $E/S$ is an $A$-structure $\phi : A \to E(S)$, with $A$ as above. We say that $\phi$ is étale if the subscheme $\phi(A) \subset E$ is étale over $S$ (equivalently, the induced map $\phi_* : A \to E_2(K)$ is injective for all geometric points $s : \text{Spec} k \to S$). We say that $\phi$ is $p$-local if $\ker \phi \subset A$ has order $p$.

The following is obvious from [17]:

Remark 3.2.2
(i) If $2N$ is invertible on $S$ then $\phi$ is étale.

(ii) If $\phi$ is $p$-local then $S$ is an $\mathbb{F}_p$-scheme. Moreover, $\phi = \phi' \circ \tau$, where $\phi' : A' \to E(S)$ is an étale $\Gamma_2(N')$-structure.

**Proposition 3.2.3** Let $G$ be a dihedral group of order $2N$ and $f : E \to \mathbb{P}^1_K$ a $G$-cover, branched in $4$ points with ramification index $2$. After a finite extension of $K$, there exists a map $\phi : A \to E(K)$, whose image $\phi(A) \subset Y$ is the set of ramification points of $f$, such that $(E, \phi)$ is an elliptic curve with an étale $\Gamma_2(N')$-structure.

**Proof:** The cover $f$ factors through an étale $N$-cyclic cover $\pi : E \to E'$, and $E$ and $E'$ are smooth projective curves of genus 1. We may assume that all ramification points of $f$ are $K$-rational. Let us choose one ramification point $0 \in E(K)$, and set $0' := \pi(0)$. Now we can regard $\pi$ is an $N$-cyclic isogeny between elliptic curves. Moreover, the branch points of $f$ are precisely the points of $E$ lying above the 2-torsion points of $E'$. Therefore, the set of branch points is a subgroup of $E[2N](K)$, of order $4N$, and contains a point of order $2N$. It is clear that this subgroup is isomorphic to $A$.

Consider the exact sequence

$$
0 \to \mathbb{Z}/N \to A \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 0
$$

of Abelian groups, where $a \in \mathbb{Z}/N$ is send to $(2a, 0) \in A$ and $(a, b) \in A$ is send to $(\bar{a}, \bar{b})$. Let $(E, \phi)$ be a generalized elliptic curve with $\Gamma_2(N)$-structure. The image $\phi(\mathbb{Z}/N) \subset E$ is a subgroup scheme of $E^\text{sm}$, finite and flat over $S$. Since $E^\text{sm}$ acts on $E$, we can form the quotient scheme $E' := E/\phi(\mathbb{Z}/N)$. One checks fiber by fiber that $E'/S$ is an étale, branched $\Gamma_2(N)$-structure. Via the exact sequence (11), $E'$ is endowed with a $\Gamma(2)$-structure $\bar{\phi} : \mathbb{Z}/2 \times \mathbb{Z}/2 \to E'(S)$. Let $[-1] : E' \to E'$ be the canonical involution (see [1], Chapitre II) and $X_0 := E'/<[-1]>$ the quotient. We write $f_\phi : E \to X_0$ for the natural map. We choose a bijection $\alpha : \{1, 2, 3, 4\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and let $x_j \in X_0(S)$ be the image of $\phi(\alpha(j))$, for $j = 1, \ldots, 4$.

**Proposition 3.2.4** Suppose $2$ is invertible on $S$.

(i) The curve $(X_0; x_j)$ is stably marked, of genus 0.

(ii) If $\phi$ is étale, then $f_\phi : E \to X_0$ is an admissible cover, ramified of order 2 along the sections $x_j$. There is a natural $G$-action on $E$, where $G$ is dihedral of order $2N$, such that $X_0 = E/G$. In addition $E/S$ is smooth then $f_\phi$ is a tame $G$-cover.

**Proof:** It suffices to prove the proposition in the case $S = \text{Spec} \ k$, where $k$ is an algebraically closed field. Carrying a $\Gamma(2)$-structure, $E'$ is either smooth or a 2-gon. Since $[-1]$ is the identity on $E'[2] \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the points $x_1, \ldots, x_4 \in X_0$ are distinct and smooth. If $E$ is smooth then $X_0 \cong \mathbb{P}_k^1$. Otherwise, $[-1]$ restricts to an involution on each component of $E'$ and interchanges the two singular point. In this case, $X_0$ is the union of two projective lines meeting transversally in one point, and each component of $X_0$ contains two of the points $x_1, \ldots, x_4$. This proves (i).

We have seen that $E' \to X_0$ is ramified at $x_1, \ldots, x_4$ of order 2 and étale everywhere else. Assume that $E/S$ is smooth and $\phi(A) \subset E$ is étale. Then $\pi : E \to E'$ is an étale $N$-cyclic isogeny. It follows that $f_\phi$ is a dihedral Galois cover, ramified at $x_1, \ldots, x_4$ of order 2. In case $E$ is singular, the map $f_\phi$ may be ramified in the singular points. However, using the description of the group law on a Néron polygon given in [1], Chapitre II, one checks that $f_\phi$ is admissible. \qed
3.2.3 **Deformation** Let $k$ be an algebraically closed field of characteristic $p$. We fix a
generalized elliptic curve $E$ over $k$ and a $\Gamma_2(N)$-structure $\phi : A \to E(k)$. We assume that $\phi$ is
$p$-local. Let $\text{Def}(E, \phi)$ denote the deformation functor classifying isomorphism classes of
deformations $(E_R, \phi_R)$ of $(E, \phi)$ over complete local $W(k)$-algebras $R$ with residue field $k$. Let $\text{Def}(E, \phi)_\text{loc}$
be the subfunctor corresponding to deformations $(E_R, \phi_R)$ where $\phi_R$ is $p$-local.

**Proposition 3.2.5** The functor $\text{Def}(E, \phi)$ has a universal deformation ring $R_\phi$. The ring $R_\phi$
regular of dimension $2$; there exists a regular sequence $(t, \pi)$ with the following properties.

(i) The ring $R^\text{loc}_\phi := R_\phi/(\pi) \cong k[[t]]$ is the universal deformation ring for $\text{Def}(E, \phi)_\text{loc}$.

(ii) If $E$ is ordinary, then $p = u\pi^{p-1}$, where $u \in R^\phi_\phi$.

(iii) If $E$ is supersingular, then $p = u\pi^{p-1}$, and $(u, \pi)$ is another regular sequence for $R_\phi$.

**Proof:** Let us first assume that $E$ is smooth. We write $\text{Def}(E, \phi|_{\mathbb{Z}/p})$ for the functor classifying
deformations of $E$ together with the $\Gamma_1(p)$-structure $\phi|_{\mathbb{Z}/p}$. Since $A \cong A' \times \mathbb{Z}/p$ and the order of
$A'$ is prime-to-$p$, the morphism $\text{Def}(E, \phi) \to \text{Def}(E, \phi|_{\mathbb{Z}/p})$ that sends $(E_R, \phi_R)$ to $(E_R, \phi_R|_{\mathbb{Z}/p})$
is an equivalence. Therefore, if $E$ is smooth, the proposition is a direct consequence of [7], Section
13.5. Namely, the universal deformation ring $R_\phi$ can be identified with the strict complete local
ring of the moduli stack $\mathcal{M}(\Gamma_1(p))$ at the point corresponding to $(E, \phi|_{\mathbb{Z}/p})$.

It is clear from [7] how to extend this to the general case. Actually, since $p \parallel N$, the situation here
is somewhat easier than in [7]. Since $\phi$ is $p$-local, the number of components of $E$ is prime-to-$p$ (more
precisely, if $E$ is singular, the number of components is $2n$, where $n|N'$). By [7], Théorème III.1.2,
the generalized elliptic curve $E$ admits a universal deformation $E_{R_0}$ over $R_0 = W(k)[[t]]$. As in the
smooth case, the morphism $\text{Def}(E, \phi) \to \text{Def}(E, \phi|_{\mathbb{Z}/p})$ is an equivalence, if we regard $\phi|_{\mathbb{Z}/p}$ as a
weak $\mathbb{Z}/p$-structure. It follows from [7] that $\text{Def}(E, \phi)$ admits a universal deformation $(E_{\text{univ}}, \phi_{\text{univ}})$
over a ring $R_\phi$. In fact, $\text{Spec } R_\phi$ is a closed subscheme of $E_{R_0}[p]^{\times}$ and $E_{\text{univ}} = E_{R_0} \otimes_{R_0} R_\phi$. We
regard $t$ as an element of $R_\phi$ via the natural morphism $R_0 = W(k)[[t]] \to R_\phi$. Let us choose a formal
parameter $T$ of $E_{\text{univ}}$ along the 0-section (see [7], Section 2.2.3). The point $P_{\text{univ}} := \phi_{\text{univ}}(2N', 0)$ is
a point of exact order $p$ on $E_{\text{univ}}$. Since $\phi$ is $p$-local, $P_{\text{univ}}|E = \phi(2N', 0) = 0$. Hence we may regard
$\pi := T(P_{\text{univ}})$ as an element of $R_\phi$. We claim that $(t, \pi)$ is a regular sequence for $R_\phi$ such that (i),
(ii) and (iii) hold. We have already mentioned that, if $E$ is smooth, this is proved in [7]. If $E$ is singular,
situation is essentially the same as for $E$ smooth and ordinary, see [7]. Namely, we may choose $T$
such that $E_{R_0}[p]^{\times} = \text{Spec } R_0[T | \Phi_p(1+T) = 0]$, where $\Phi_p(X) = (X^p - 1)/(X - 1)$. Therefore,
$R_\phi = W(k)[\zeta_p][[t]]$ and $\pi = \zeta_p - 1$. This completes the proof of the proposition. \(\square\)

Let $(E_R, \phi_R)$ be a deformation of $(E, \phi)$. Following Section 3.2.2 we associate to every deformation
$(E_R, \phi_R)$ of $(E, \phi)$ a finite map $f_{\phi_R} : E_R \to X_{0,R}$ and sections $x_{1,R}, \ldots, x_{n,R} : \text{Spec } R \to X_{0,R}$
such that $(X_{0,R}; x_{j,R})$ is a stably marked curve of genus $0$. This gives rise to a morphism

\[
(12) \quad \text{Def}(E, \phi) \longrightarrow \text{Def}(X_{0}; x_j)
\]

def of deformation functors and hence to a $W(k)$-algebra morphism $R_{X_0} \to R_{\phi}$.

**Proposition 3.2.6** The ring $R_\phi$ is a finite and flat extension of $R_{X_0} \cong W(k)[[w]]$. Modulo $\pi$, we
obtain a finite extension

\[
R_{X_0} \otimes_{W(k)} k \cong k[[w]] \longrightarrow R^\text{loc}_\phi \cong k[[t]]
\]

with inseparability degree $p$. Its separable part $k[[w]] \to (R^\text{loc}_\phi)^p = k[[t^p]]$ is tamely ramified of
degree $d$. Here

\[
d = \begin{cases} 
1 & \text{if } E \text{ is smooth,} \\
N'/n & \text{if } E \text{ is a } 2n\text{-gon.}
\end{cases}
\]
Proof: Let $E'' := E/\phi(Z/p)$ be the quotient by the subgroup scheme $\phi(Z/p) \subset E^{\text{sm}}$ and $\phi' : A' \to E''(k)$ the induced $\Gamma_2(N')$-structure. Clearly, the map $f_{\phi'} : E \to X_0$ induced by $\phi$ factors through the projection $E \to E''$. The resulting map $f_{\phi''} : E'' \to X_0$ is the map induced by $\phi''$. We see that the morphism (13) can be written as the composition of two morphisms, as follows:

$$\text{Def}(E, \phi) \to \text{Def}(E'', \phi'') \to \text{Def}(X_0; x_j).$$

Let $R_{\phi''}$ be the universal deformation ring of $\text{Def}(E'', \phi'')$. By Proposition 3.2.4 (ii), the map $f_{\phi''} : E'' \to X_0$ is admissible, and is “Galois” with dihedral Galois group of order $2N'$. If $E$ is singular, then $f_{\phi''}$ is ramified of order $d$ over the unique singular point of $X_0$, where $d$ is as in the statement of the proposition. It is not hard to see that any deformation of the admissible cover $f_{\phi''}$ together with the group action corresponds to a unique deformation of $(E'', \phi'')$. Therefore, it follows from 29 that $R_{\phi''} = R_{X_0} [z \mid z^d = w] \cong W(k)[[z]]$. Here we identify $R_{X_0}$ with $W(k)[[w]]$, such that, if $X_0$ is singular, $w$ is the deformation parameter of the singular point. We are reduced to showing that $R_{\phi''} \to \tilde{R}_\phi$ is finite and flat, and purely inseparable of degree $p$ modulo $\pi$.

Let $\tilde{B} := R_{\phi''} \otimes_{W(k)} k \cong k[[z]]$ and $\tilde{B} := B^{1/p} \cong k[[z^{1/p}]]$. Let $(E''_B, \phi''_B)$ be the deformation of $(E'', \phi'')$ corresponding to the universal morphism $R_{\phi''} \to \tilde{B}$. We can define a generalized elliptic curve $E_B'$ over $\tilde{B}$ such that $E''_B = E_B^{(p)}$ is the $p$th power Frobenius twist of $E_B'$. The relative Frobenius $F : E_B' \to E_B''$ is a $p$-cyclic, purely inseparable “isogeny” whose kernel is generated by the 0-section of $E_B$ (which is a point of exact order $p$). Moreover, there exists a unique $p$-local $\Gamma_2(N)$-structure $\phi_B : A \to E_B(\tilde{B})$ such that $\phi''_B = F \circ \phi_B$. Conversely, let $(E, \phi, R) = (E_{\phi''}, \phi''_B)$ be any $p$-local deformation of $(E, \phi)$. By 27, we can canonically identify $E_R'' := E_R/\phi_R(Z/p)$ with $E_{\phi''_B}$ and the quotient map $E_R \to E''_R$ with the Frobenius $F : E_R \to E_{\phi''_B}^{(p)}$. It follows that $\tilde{B} = R^*_\phi$. Hence $R_{\phi''} \to \tilde{R}_\phi$ is purely inseparable of degree $p$ modulo $\pi$. Nakayama’s Lemma shows that $R_{\phi}$ is finite over $R_{\phi''}$. Since a finite morphism between two regular local rings of the same dimension is automatically flat (see 3, V.3.8), the proposition is proved. \hfill \Box

3.2.4 Stabilization Let $(E_R, \phi_R)$ be a deformation of $(E, \phi)$. A stabilization of $(E_R, \phi_R)$ is a morphism $q_R : Z_R \to E_R$ between semistable $R$-curves together with a map $\psi_R : A \to Z_R(R)$ such that (i) $(Z_R, \psi_R(A))$ is a stably marked curve, (ii) $\phi_R = q_R \circ \psi_R$, and (iii) $q_R$ is the contraction of the marked semistable curve $(Z_R, \psi_R(A'))$ (see 19).

Lemma 3.2.7 For every deformation $(E_R, \phi_R)$, there exists a stabilization $(Z_R, \psi_R)$. Assume that there exists a dense open subset $U \subset \text{Spec} R$ such that $\phi_U$ is étale. Then $(Z_R, \psi_R)$ is unique up to unique isomorphism.

Proof: If $\phi_U$ is étale, then $(E_U, \phi_U(A))$ is stably marked, so necessarily $Z_U = E_U$. Therefore, the uniqueness follows from the fact that the moduli stack of stably marked curves is separated, see 19.

The notion of stabilization is certainly compatible with base change $R \to R'$. Hence it suffices to prove the existence of stabilization for the universal deformation $(E^{\text{univ}}, \phi^{\text{univ}})$. Recall that $\pi = T(\phi^{\text{univ}}(2N')) \in R_{\phi}$, where $T$ is a formal parameter of $E^{\text{univ}}$ along the 0-section. Relative to $T$, the formal group of $E^{\text{univ}}_{\phi^{\text{univ}}}$ is given by a power series $\Phi(T_1, T_2) = T_1 + T_2 + \ldots \in R[[T_1, T_2]]$. Since $\phi_{\text{univ}}$ is a group homomorphism, we have

$$T(\phi^{\text{univ}}(2N'm, 0)) \equiv m \pi \pmod{\pi^2}, \quad m \in Z/p.$$  

Let $q_{\text{univ}} : Z_{\text{univ}} \to E_{\text{univ}}$ be the blowup of $E_{\text{univ}}$ along the closed subscheme $\phi^{\text{univ}}(A') \cap (\pi) \subset E_{\text{univ}}$. Since $\phi^{\text{univ}}(A')$ consists of $|A'| = 4N'$ pairwise disjoint sections $\text{Spec} R_{\phi} \to E_{\text{univ}}$, the blowup $Z_{\text{univ}}$ is a semistable curve over $R_{\phi}$. Denote its special fiber by $Z$. Then

$$Z = E \cup \bigcup_{b \in A'} Z_b.$$
where \( Z_b \cong \mathbb{P}^1_k \) is connected to \( E \) in the point \( z_b := \phi(b) \). For \( b \in A' \), the translate \( T_b := T - \phi_{univ}(b) \) is a formal parameter of \( E \) along the section \( \phi_{univ}(b) \). By the definition of \( Z_{univ} \), \( T_b := T_b/\pi \) is a regular function in a neighborhood of \( Z_b - \{ z_b \} \subset Z_{univ} \) and defines an isomorphism \( Z_b \cong \mathbb{P}^1_k \) mapping \( z_b \) to \( \infty \). For \( a \in A \), let \( \psi_{univ}(a) \) be the closure of \( \phi_{univ}(a) \) in \( E_{univ}(K) = Z_{univ}(K) \) in \( Z_{univ} \). This defines a map \( \psi_{univ} : A \to Z_{univ}(R) \) such that \( \phi_{univ} = \psi_{univ} \circ \psi_{univ} \). Write \( a = b + c \), with \( b \in A' \) and \( c \in \mathbb{Z}/p \). Using [6], Théorème II.3.2, we can apply \( \psi_{univ}(a) \) to conclude that there exists at most one morphism \( + \) satisfying the claimed properties.

As in the proof of the lemma, let \( Z_{univ} \) be the stabilization of the universal deformation \((E_{univ}, \phi_{univ})\). Let \( f_{0,univ} : Z_{univ} \to X_{0,univ} \) be the composition of \( q_{univ} \) with the map \( f_{\phi_{univ}} : E_{univ} \to X_{0,univ} \) induced by \( \phi_{univ} \). Let \( K \) be the fraction field of \( R_{\phi} \). Since \( \phi_K \) is étale, \( Z_K = E_K \). By Proposition 3.2.3, \( f_{0,K} : Z_K \to X_{0,K} \) is a \( G \)-cover, branched at 4 points of order 2, where \( G \) is dihedral of order \( 2N \). In other words, \( Z_K \) is a \( K \)-object of the Hurwitz stack \( \mathcal{H}_4^G(G) \). By the uniqueness of stabilization, the action of \( G \) extends to \( Z_{univ} \). Therefore, \( Z_{univ} \), together with the mark \( \psi_{univ}(A) \), the action of \( G \) and the map \( f_{0,univ} \), is an \( R_{\phi} \)-object of the complete Hurwitz stack \( \mathcal{H}_4^G(G) \). In particular, the special fiber \( Z \) of \( Z_{univ} \) is a \( k \)-object of \( \mathcal{H}_4^G(G)_{\text{bad}} \). Via pullback, we obtain a morphism

\[
(15) \quad \text{Def}(E, \phi) \to \text{Def}(Z)
\]
of deformation functors, compatible with the morphisms \( \text{Def}(E, \phi) \to \text{Def}(X_0; x_j) \) and \( \text{Def}(Z) \to \text{Def}(X_0; x_j) \).

**Proposition 3.2.8** The morphism \( [\mathbb{F}] \) is an isomorphism; it induces an isomorphism between \( \text{Def}(E, \phi)_{\text{loc}} \) and \( \text{Def}(Z)_{\text{bad}} \).

**Proof:** The first statement is equivalent to the assertion that \( Z_{univ} \) is the universal deformation of \( Z \). We construct, by construction, the maximal subset \( U \subset \text{Spec} R_b \) such that \( \psi_U \) is étale is precisely the maximal subset such that \( Z_U \to X_{0,U} \) is an admissible cover. Therefore, the first statement of the proposition implies the second.

Let \( Z_b \) be a deformation of \( Z \). We denote by \( \psi : A \to Z(k) \) the restriction of \( \psi_{univ} \) to \( Z \). Clearly, \( \psi \) lifts uniquely to a map \( \psi_R : A \to Z_R(R) \) such that \( \psi_R(A) \) is the mark of the stably marked curve \( Z_R \). Let \( q_R : Z_R \to E_R \) be the contraction of the marked semistable curve \( (Z_R, \psi_R(A')) \), and let \( \phi_R := q_R \circ \psi_R \). We claim that the assignment \( Z_R \to (E_R, \phi_R) \) defines a morphism \( \text{Def}(Z) \to \text{Def}(E, \phi) \), which is the inverse of \( [\mathbb{F}] \). This is clear from the construction and the following lemma.

**Lemma 3.2.9** There exists a unique morphism \( +_R : E_0^R \times E_R \to E_R \) such that \( (E_R, +_R, \phi_R(0)) \) is a generalized elliptic curve and \( \phi_R \) is a \( \Gamma_2(N) \)-structure.

**Proof:** By construction, \( E_R \) is a semistable \( R \)-curve of genus 1. We claim that every geometric fiber \( E_s \) of \( E_R \) is either smooth or a Néron polygon, and that \( \phi_R(A) \) meets every irreducible component of \( E_s \). In fact, this is an open condition on \( \text{Spec} R \), and it is true for the special fiber \( E \).

Let us first prove the uniqueness of \( +_R \). Suppose a morphism \( +_R \) satisfying the conditions of the lemma exists. Since \( \phi_R : A \to E_R^R(R) \) is a group homomorphism (with respect to the group law induced by \( +_R \)), it induces an action of \( A \) on \( E_R \) via translation. Since \( \phi_R \) is a \( \Gamma_2(N) \)-structure, \( A \) acts transitively on the set of irreducible components of every geometric fiber. By definition of this action, \( \phi_R \) is equivariant (here \( A \) acts on itself by translation). But \( (E_R, \psi_R(A')) \) is stably marked, so there can exists at most one \( A \)-action on \( E_R \) with this property. Therefore, we can apply Théorème II.3.2, to conclude that there exists at most one morphism \( +_R \) with the claimed properties.
The assignment $Z_R \mapsto (E_R, \phi_R)$ is clearly compatible with base change $R \to R'$. Hence it suffices to prove the existence of $+R$ in the case $R = R_Z$, i.e., when $Z_R$ is the universal deformation of $Z$. Let $U \subset \text{Spec } R$ be the maximal subset such that $Z_U$ is a $G$-cover. By the construction of the complete Hurwitz stack $\tilde{H}^1_G(U)$, $U$ is open and dense. Moreover, $Z_U = E_U$ is a smooth curve of genus 1. By \cite{3}, Proposition 2.7, there exists a unique structure of elliptic curve on $(E_U, \phi_U(0))$. As in the proof of Proposition 3.2.3 one shows that $\phi_U$ is a $G_2(N)$-structure. In particular, $A$ acts on $E_U$ such that $\phi_U$ is equivariant. Since $(E_R, \phi_R(A'))$ is stably marked and $U \subset \text{Spec } R$ is dense, this action extends uniquely to $E_R$, and $\phi_R$ is equivariant. By \cite{3}, Proposition 2.7, the induced action of $A$ on $\text{Pic}^0 E_R/R$ is trivial. Therefore, we can apply \cite{3}, Théorème II.3.2, to show that there exists a structure of generalized elliptic curve on $E_R$ such that the action of $a \in A$ on $E_R$ is given by translation with $\phi_R(a)$. It remains to show that $\phi_R$ is a weak $A$-structure. But this is a closed condition on $\text{Spec } R$ (see \cite{17}) and it is true on $U \subset \text{Spec } R$. Hence it is true on $\text{Spec } R$. This concludes the proof of Lemma 3.2.9 and Proposition 3.2.8.

By Proposition 3.2.8, we can identify the universal deformation rings $R_\phi$ and $R_Z$, and regard $Z_{\text{unv}}$ as the universal deformation of $Z$. In view of Proposition 3.2.5 and Proposition 3.2.6, we obtain:

**Corollary 3.2.10** Theorem 3.1.2 is true for $Y = Z$.

**Remark 3.2.11** Let $X_0(N)_{Z(p)}$ and $X_1(N)_{Z(p)}$ be the arithmetic models over $\mathbb{Z}(p)$ of the modular curves $X_0(N)$ and $X_1(N)$, as defined in \cite{3} and \cite{17}. The results of this section imply that $X_0(N)_{Z(p)} \cong \tilde{H}^n_1(C)$ and $X_1(N)_{Z(p)} \cong \tilde{H}^n_1(C)$, where

$$C := \begin{cases} (2A, 2A, 2A) & \text{if } N \text{ is odd}, \\ (2A, 2A, 2B, 2B) & \text{if } N \text{ is even}, \end{cases}$$

and $2A, 2B$ denote the conjugacy classes of the “reflections” in a dihedral group of order $2N$. Let $X_2(N)$ be the coarse moduli space for generalized elliptic curves with $\Gamma_2(N)$-structure. One can show that $X_2(N)_{Z(p)} \cong \tilde{H}^n_1(C)$, where $C = (2A, 2A, 2B, 2B)$ is the tuple of conjugacy classes in a dihedral group of order $4N$, as in (16).

### 3.3 Proof of the Reduction Theorem

We are now ready to complete the proof of the Reduction Theorem. The main argument is given in Section 3.3.3, where we compare the deformation theory of $Y$ to the deformation theory of the special fiber $Z$ of the associated auxiliary cover. Since we have modular reduction, Theorem 3.1.2 follows from the results of Section 3.3. In Section 3.3.3 and Section 3.3.4 we prove Proposition 3.1.4 and Proposition 3.1.7, using the same method: we first reduce the dihedral case and then use the results of Section 3.3.

**3.3.1 The auxiliary cover** Let $f_0 : Y \to X_0$ be as in Section 3.1.2. As we have seen in the paragraph following Definition 1.2.2 there exists a complete discrete valuation ring $R$ with residue field $k$ and quotient field $K$ of characteristic 0 such that $f_0 : Y \to X_0$ is the reduction of a $G$-cover $f_K : Y_K \to \mathbb{P}^1_K$. More precisely, the $G$-cover $f_K$ has a stable model $f_{0,R} : Y_{R} \to X_{0,R}$ over $R$, with special fiber $f_0$. We denote by $f_R : Y_R \to X_R$ the quotient model of $f_K$ and by $f : Y \to X$ its special fiber (see Section 1.1). Note that $f_0$ factors through $f$ and that $f$ does not depend on the choice of the lift $f_K$, by Proposition 1.3.2.

We are assuming that Condition 2.2.2 holds. Therefore, it follows from Proposition 2.4.3 that the $G$-cover $f_K$ has modular reduction of level $N$, where $N$ is some integer diving $|G|$ and divisible by $p$ (see Definition 2.4.1). Let $g_K : Z_K \to \mathbb{P}^1_K$ be the auxiliary cover associated to $f_K$. The cover $g_K : Z_K \to \mathbb{P}^1_K$ is a $\Delta$-cover, where $\Delta \subset G$ is a dihedral group of order $2N$; it has the same
branch locus as \( f_K \), but with ramification of order 2. Let \( C_{\text{aux}} \) be the inertia type of \( g_K \), and let \( \mathcal{H}_{\text{aux}} := \mathcal{H}^1(\mathcal{C}_{\text{aux}}, \mathcal{O}_E) \) be the complete Hurwitz stack over \( Z_{\text{D}} \) for \( \Delta \)-covers with inertia type \( C_{\text{aux}} \).

By definition, \( g_K \) is a \( K \)-object of \( \mathcal{H}_{\text{aux}} \). Let \( g_R : Z_R \to X_R \) and \( g_{0,R} : Z_R \to X_{0,R} \) be the quotient and the stable model of \( g_K \). By definition, \( g_{0,R} \) is an \( R \)-object of \( \mathcal{H}_{\text{aux}} \) extending \( g_K \); its special fiber \( g_0 : Z \to X_0 \) is a \( K \)-object of \( \mathcal{H}^1_{\text{aux}} \). In the sequel, we will denote this object simply by \( Z \).

According to Section 3.2.2, there exists a map \( \phi_K : A \to Z_K(K) \) such that \( (Z_K, \phi_K) \) is an elliptic curve with \( \Gamma_2(N) \)-structure, and the cover \( g_K : Z_K \to X_K \) is induced by \( \phi_K \). We extend \( \phi_K \) to a map \( \psi_R : A \to Z_R(R) \); let \( q_R : Z_R \to \mathcal{E}_R \) be the contraction of \( (Z_R, \psi_R(A')) \) and let \( \phi_R := q_R \circ \psi_R \). By Lemma 3.2.4, \((E_R, \phi_R)\) is a generalized elliptic curve with \( \Gamma_2(N) \)-structure.

Let \((E, \phi)\) be the special fiber of \((E_R, \phi_R)\). Clearly, we are in the situation of Section 3.2.4. In particular, Theorem 3.1.2 is true for \( Y = Z \), by Corollary 3.2.10.

### 3.3.2 Proof of Theorem 3.1.2

By the construction of the auxiliary cover, there exists an étale map \( U^{(1)} \to X \), covering \( X_0 \), and a \( G \)-equivariant isomorphism

\[
Y \times_X U^{(1)} \cong \text{Ind}_{\Delta}^G(Z \times_X U^{(1)}),
\]

of \( X \)-schemes. Over the open subset \( U^{(2)} \), the map \( g \) is tamely ramified along the mark \( C \subset X \).

Let \( X_{\text{aux}} := Z_{\text{aux}}/\Delta \). Since \( Z_R \) is regular, the natural morphism \( g_{\text{aux}} : Z_{\text{aux}} \to X_{\text{aux}} \) is a quotient model of \( Z_{\text{aux}} \), by Proposition 3.3.2. In particular, \( X_{\text{aux}} \) is a semistable curve and carries a natural mark \( C_{\text{aux}} \subset X_{\text{aux}} \).

**Construction 3.3.1** Let \( R \in \mathcal{C}_k \) be Artinian, and let \( Z_R \) be a deformation of \( Z \) over \( R \). There exist a unique morphism \( Z_R \to R \) such that \( Z_R = Z_{\text{aux}} \otimes_R R \). Let \( X_R := X_{\text{aux}} \otimes_R R \). Then \( Z_R \to X_R \) is a quotient model of \( Z_R \). Note that the pair \((U^{(1)}, U^{(2)})\), where \( U^{(1)} \to X \) and \( U^{(2)} \to X \) are as above, is an étale covering of \( X \). For \( i = 1, 2 \), let \( Y^{(i)} := Y \times_X U^{(i)} \) and \( Z^{(i)} := Z \times_X U^{(i)} \). For \( i, j = 1, 2 \), let \( U^{(i,j)} := U^{(i)} \times_X U^{(j)} \), \( Y^{(i,j)} := Y \times_X U^{(i,j)} \) and \( Z^{(i,j)} := Z \times_X U^{(i,j)} \). Since \( R \) is Artinian, the étale covering \((U^{(1)}, U^{(2)})\) of \( X \) extends uniquely to an étale covering \((U^{(1)}_R, U^{(2)}_R)\) of \( X_R \). Actually, \( U^{(2)}_R = X_R \times X_0 \). Over \( U^{(2)} \), the finite map \( f : Y \to X \) is tamely ramified along \( C \).

By a theorem of Grothendieck and Murre, there exists a unique extension of \( Y^{(2)} \to U^{(2)} \) to a finite morphism \( Y^{(2)}_R \to U^{(2)}_R \) which is tamely ramified along \( C_R \subset X_R \).

Define

\[
Y^{(1)}_R := \text{Ind}_{\Delta}^G(Z_R \times_{X_R} U^{(1)}_R).
\]

By (17), \( Y^{(1)}_R \otimes_R k = Y^{(1)} \). We claim that there exist \( G \)-equivariant isomorphisms

\[
\alpha^{(i,j)}_R : Y^{(i)}_R \times_{U^{(i)}_R} U^{(i,j)}_R \cong Y^{(j)}_R \times_{U^{(j)}_R} U^{(i,j)}_R, \quad i,j = 1, 2
\]

of \( U^{(i,j)} \)-schemes which extend the identity on \( Y^{(i,j)} \). For \((i, j) \neq (1, 1)\), this follows again from Grothendieck–Murre. For \((i, j) = (1, 1)\), we obtain (19) via the canonical identification

\[
Y^{(1)}_R \times_{U^{(1)}_R} U^{(1,1)}_R \cong \text{Ind}_{\Delta}^G(Z_R \times_{X_R} U^{(1,1)}_R) \cong Y^{(1)}_R \times_{U^{(1)}_R} U^{(1,1)}_R
\]

(on the left hand side we use the first projection \( U^{(1,1)}_R \to U^{(1)}_R \), on the right hand side we use the second projection). It is clear that the isomorphisms (19) verify the obvious cocycle condition. Therefore, there exist a finite morphism \( f_R : Y_R \to X_R \) such that \( Y^{(i)}_R = Y_R \times_{X_R} U^{(i)}_R \), for \( i = 1, 2 \).

Construction 3.3.1 associates to any deformation \( Z_R \in \text{Def}(Z)(R) \) over an Artinian ring \( R \in \mathcal{C}_k \) a finite map \( f_R : Y_R \to X_R \) which extends \( f : Y \to X \). Moreover, the \( G \)-action on \( Y \) and the mark \( D \subset Y \) extend to \( Y_R \). Since \( X_R \) is projective over \( R \), we can apply Grothendieck’s Existence Theorem and extend this construction to the case of an arbitrary ring \( R \in \mathcal{C}_k \). We claim that
constructed curve $Y_R$, together with the $G$-action, the natural morphism $f_{0,R} : Y_R \to X_{0,R}$ and the mark $D_R \subset Y_R$, is an object of $\mathcal{H}$. It suffices to prove this in the case $R = R_Z$, $Z_R = Z_{\text{univ}}$. So let $Y_{RZ} \to X_{\text{univ}}$ be the map associated to $Z_{\text{univ}}$ by Construction 3.3.1. Since the generic fiber of $g_{\text{univ}} : Z_{\text{univ}} \to X_{\text{univ}}$ is a $\Delta$-cover, the generic fiber of the map $Y_{RZ} \to X_{\text{univ}}$ is a $G$-cover, i.e. an object of $\mathcal{H}$. By construction, $Y_{RZ} \to X_{\text{univ}}$ is an object of $S^n_G(G)$. Therefore, $Y_{RZ} \to X_{\text{univ}}$ is an object of $\mathcal{H}$. We have shown that Construction 3.3.1 defines a morphism of functors

$$\text{Def}(Z) \longrightarrow \text{Def}(Y).$$

To complete the proof of Theorem 3.1.2, we have to show that (21) is an isomorphism. We need two lemmas.

**Lemma 3.3.2** Let $R \in \mathcal{C}_k$ be an Artinian $k$-algebra. Let $Z_R$ be a deformation of $Z$ over $R$ and $Y_R$ its image under (21). If $Y_R \cong Y \otimes_k R$ is a trivial deformation, then $Z_R \cong Z \otimes_k R$ is trivial, too.

**Proof:** It suffices to show that the deformation $(E_R, \phi_R)$ of $(E, \phi)$ corresponding to $Z_R$ is trivial. Since $Y_R \cong Y \otimes_k R$, the closed embedding $E \hookrightarrow Y$ extends to a closed embedding $E \otimes_k R \hookrightarrow Y_R$. By Construction 3.3.1 and descent, we obtain a closed embedding $E \otimes_k R \hookrightarrow Z_R$, extending the closed embedding $E \hookrightarrow Z$. Composition with the contraction morphism $g_R : Z_R \to E_R$ yields a morphism $E \otimes_k R \to E_R$ which restricts to the identity on the special fiber. Since both $E \otimes_k R$ and $E_R$ are flat and of finite type over $R$, it is an isomorphism. By construction, this isomorphism identifies $\phi_R$ with $\phi \otimes_k R$. This proves the lemma.

**Lemma 3.3.3** Let $R \in \mathcal{C}_k$ be a normal domain, with fraction field of characteristic 0. Let $Y_R$ be a deformation of $Y$ over $R$. Then there exists a deformation $Z_R$ of $Z$ over $R$ whose image under (21) is isomorphic to $Y_R$.

**Proof:** Let $X'_R := Y_R/G$. By Proposition 1.3.2, the natural map $f_R : Y_R \to X'_R$ is a quotient model of $Y_R$. In particular, $X'_R$ is a semistable curve with special fiber $X$. We claim that there exists a maximal open subset $U'_R \subset X'_R$, containing $C'R$, over which $f_R$ is tamely ramified along $C'R$. Clearly, $U'_R \supset Z_R$ contains the generic fiber and $U'_R \otimes_k k = X - X_0$.

We claim that there exists a finite morphism $g_R : Z_R \to X'_R$, extending $g : Z \to X$, with $\Delta$ acting on $Z_R$, characterized by the following two properties: (i) over $U'_R$, the map $g_R$ is tamely ramified along $C'R$, (ii) over an étale neighborhood of $X'_R \setminus U'_R$, $Y_R$ is isomorphic to $\text{Ind}_Z^X(Z_R)$. In fact, one can construct $g_R$ using the same method as in Construction 3.3.1. It follows that the generic fiber of $g_R$ is a $\Delta$-cover, hence an object of $\mathcal{H}_{\text{univ}}$. Therefore, $Z_R \in \text{Def}(Z)(R)$. By construction, $g_R : Z_R \to X'_R$ is a quotient model of $Z_R$. The uniqueness of the quotient model implies $X'_R = X_{\text{univ}} \otimes_k Z_R$. A formal verification shows that $Y_R$ is the image of $Z_R$ under (21).

We are now going to complete the proof of Theorem 3.1.2. The morphism (21) induces a homomorphism $R_Y \to R_Z$ of local rings. We have to show that it is an isomorphism. Let $m_Y < R_Y$, $m_Z < R_Z$ be the maximal ideals. Lemma 3.3.2 implies that for every $N > 0$ the $k$-module $R_Z/m_Y R_Z$ is generated by the images of 1 and $(m_Z)^N$. It follows that $R_Z/m_Y R_Z = k$. So $R_Y \to R_Z$ is surjective, by Nakayama’s Lemma.

Let $p \in \text{Spec} R_Y$ be a generic point. The quotient $A := R_Y/p$ is a complete local domain with residue field $k$ and fraction field $K$ of characteristic 0. The integral closure $\hat{A}$ of $A$ in $K$ is again a complete local domain with residue field $k$. So by Lemma 3.3.3, the morphism $R_Y \to \hat{A}$ factors via $R_Y \to R_Z$. Therefore, $I := \text{Ker}(R_Y \to R_Z) \subset p$. It follows that $I$ is contained in the nilradical of $R_Y$. But $R_Y$ is reduced, so $I = 0$ and (21) is an isomorphism. This completes the proof of Theorem 3.1.2. □
3.3.3 Proof of Proposition 3.1.4 We fix an element \( \gamma \in \Gamma_{ab} \) and choose an automorphism \( \sigma : Y \to Y \) in \( \text{Aut}_{k}(Y) \) which induces \( \gamma \). Such an automorphism \( \sigma \) is unique up to composition with an element of \( G \) (later in the proof we will give a “canonical” choice). Recall that the generalized elliptic curve \( \Gamma \) is a closed subscheme of \( Y \). Let \( E_0 \) be the identity component of \( E \). Since \( E_0 \) is an irreducible component of \( Y \) above \( X_0 \), we may assume \( \sigma|_{E_0} = \text{Id}_{E_0} \), after composing \( \sigma \) with an appropriate element of \( G \). Since \( E \) is either irreducible or a Néron polygon, it follows that \( \sigma \) induces an automorphism \( \sigma|_{E} : E \to E \) of the \( k \)-curve \( E \) which normalizes the action of \( \Delta = D(E) \) and commutes with \( f_{0}|_{E} : E \to X_0 \). By the construction of \( Z \), \( \sigma|_{E} \) extends uniquely to an automorphism \( \sigma_{Z} : Z \to Z \) such that \( \sigma_{Z} \) and \( \sigma \) agree in an étale neighborhood of \( E \). Note that \( \sigma_{Z} \) normalizes the action of \( \Delta \) and commutes with \( g_{0} : Z \to X_0 \). In other words, \( \sigma_{Z} \in \text{Aut}_{k}^{Z}(Z) \). Therefore, \( \sigma_{Z} \) lifts to a \( \gamma \)-semilinear automorphism \( \sigma_{Z, \text{univ}} : Z_{\text{univ}} \to Z_{\text{univ}} \), for some \( W(k) \)-algebra automorphism \( \gamma \) of \( R_{Y} = R_{Z} \). Let \( \hat{E} \) be the formal completion of \( Z_{\text{univ}} \) along \( E \subset Z_{\text{univ}} \). It follows from Construction 3.3.1 that we can identify \( \hat{E} \) with the formal completion of \( Y_{\text{univ}} \) along \( E \subset Y_{\text{univ}} \), and that \( \sigma_{Z, \text{univ}}|_{\hat{E}} = \sigma_{\text{univ}}|_{\hat{E}} \). In particular, \( \gamma \) is \( \gamma \). Therefore, to prove Proposition 3.1.4, we may assume that \( Y = Z \) and \( G = \Delta \).

Recall that there exists a map \( \psi_{\text{univ}} : A \to Z_{\text{univ}}(R_{Z}) \) such that the following holds: (i) \( \psi_{\text{univ}}(A) \) is the mark of the stably marked curve \( Z_{\text{univ}} \); (ii) \( (Z_{\text{univ}}, \psi_{\text{univ}}) \) is the stabilization of \( (E_{\text{univ}}, \phi_{\text{univ}}) \) and (iii) \( (E_{\text{univ}}, \phi_{\text{univ}}) \) is the universal deformation of its special fiber \( (E, \phi) \). Let \( \psi : A \to Z(k) \) be the restriction of \( \psi_{\text{univ}} \) to \( Z \) and let \( Z_{0} \) be the component of \( Z \) which meets \( E \) in \( 0 \). By the proof of Lemma 3.2.7, we may identify \( Z_{0} \) with \( \mathbb{P}^{1}_{k} \) such that \( \infty \in Z_{0} \) is the point where \( Z_{0} \) meets \( E \) and such that \( \psi(2N'a, 0) = a \in \mathbb{F}_{p} \subset Z(k) \). It is clear that \( \sigma : Z \to Z \) restricts to an automorphism of \( Z_{0} \) which fixes \( 0 \) and permutes the points \( \psi(2N'a, 0) \). Therefore, \( \sigma \) acts on \( Z_{0} \) as \( z \to cz + b \), with \( b, c \in \mathbb{F}_{p}, c \neq 0 \). Composing \( \sigma \) by an element of the decomposition group \( D(Z_{0}) \subset G \), we may assume that \( b = 0 \). This determines \( \sigma \) uniquely. Moreover,

\[
\chi^{\text{bad}}(\gamma) := c^{-1}
\]

defines a homomorphism \( \chi^{\text{bad}} : \Gamma_{ab} \to \mathbb{P}_{\mathbb{Z}}^{2} \). Since \( \sigma_{\text{univ}} \) is an automorphism of \( Z_{\text{univ}} \) as stably marked curve, it descents to an automorphism \( \hat{\sigma}_{\text{univ}} : E_{\text{univ}} \to E_{\text{univ}} \) such that \( q_{\text{univ}} \circ \sigma_{\text{univ}} = \hat{\sigma}_{\text{univ}} \circ q_{\text{univ}} \), where \( q_{\text{univ}} : Z_{\text{univ}} \to E_{\text{univ}} \) is the projection morphism. Let \( P_{\text{univ}} := \phi_{\text{univ}}(2N'a, 0) \subset E_{\text{univ}} \). Since \( \phi_{\text{univ}} \) is a group homomorphism, we have \( a \cdot P_{\text{univ}} = \phi_{\text{univ}}(2N'a, 0), \) for \( a \in \mathbb{Z}/p \). Using the definition of \( \chi^{\text{bad}} \), we obtain

\[
(22) \quad \hat{\sigma}_{\text{univ}}(a \cdot P_{\text{univ}}) = \chi^{\text{bad}}(\gamma)^{-1} \cdot a \cdot P_{\text{univ}}, \quad a \in \mathbb{Z}/p.
\]

In particular, \( \hat{\sigma}_{\text{univ}} \) fixes the 0-section of \( E_{\text{univ}} \). Similar to the proof of Lemma 3.2.7, one shows that \( \hat{\sigma} \) is a \( \gamma \)-semilinear automorphism of the generalized elliptic curve \( E_{\text{univ}} \), i.e. is compatible with the “group law” on \( E_{\text{univ}} \). In particular, \( \sigma|_{E} : E \to E \) is a \( k \)-linear automorphism of the generalized elliptic curve \( E \).

Recall that \( E_{\text{univ}} = E_{R_{0}} \otimes_{R_{0}} R_{Z} \), where \( E_{R_{0}} \) is the universal deformation of \( E \), defined over \( R_{0} = W(k)[[t]] \subset R_{Z} \). It follows that \( \hat{\sigma}_{\text{univ}} \) is induced by a \( \gamma_{0} \)-linear automorphism \( \sigma_{R_{0}} : E_{R_{0}} \to E_{R_{0}} \), where \( \gamma_{0} := \gamma|_{R_{0}} \) is the monodromy action of \( \sigma|_{E} \) on \( R_{0} \). If \( E \) is smooth then \( \sigma|_{E} = \text{Id}_{E} \). So in this case we have \( \gamma(t) = t \), as claimed in Proposition 3.1.4. If \( E \) is not smooth, it is isomorphic to the standard 2\( n \)-gon, for some \( n \), see [4], Section 2.11. So we identify the group of components \( E_{m}^{\text{univ}}/E_{0}^{\text{univ}} \) with \( \mathbb{Z}/2n \) and the individual components \( E_{i}, \quad i \in \mathbb{Z}/2n \), with \( \mathbb{P}_{k}^{1} \). According to [4], Proposition II.1.10, the automorphism \( \sigma|_{E} \) is given by the formula

\[
\sigma|_{E}(x, i) = (\zeta^{i}x, i), \quad x \in \mathbb{P}^{1}, \quad i \in \mathbb{Z}/2n,
\]

for some \( 2n \)th root of unity \( \zeta \). But \( \sigma|_{E_{i}} : E_{i} \to E_{i} \) lifts to an element of the decomposition group \( D(E_{i}) \subset G \), which is dihedral of order \( 2N'/n \), by Proposition II.5.3(d). We conclude that \( \zeta \) is the \( d \)th root of unity, where \( d \) is the greatest common divisor of \( 2n \) and \( N'/n \). We set \( \chi^{\text{adm}}(\gamma) := \zeta \). It
is obvious that this defines a group homomorphism \( \chi^{adm} : \Gamma \to \mu_d \). If we identify \( E_{univ} \) with the Tate elliptic curve \( \mathbb{G}_m/q^2 \), where \( q = f^{2n} \), then we find \( \gamma(t) = \zeta \cdot t \), see [3], Chapitre VII.

It remains to prove the formula \( \gamma(\pi) \equiv \chi^{bad}(\gamma) \cdot \pi \pmod{\pi^2} \). Recall that \( \pi = T(P_{univ}) \), where \( T \) is a formal parameter of \( E_{univ} \), along the 0-section. Actually, \( T \) was chosen as a formal parameter of \( E_{\mathbf{R}_0} \), see the proof of Proposition 3.2.3. It follows from the preceding discussion that we may assume \( \tau_{univ}(T) = T \). Using (22) and the formal group law on \( E_{univ} \), we get

\[
\pi' := T(\tau_{univ}(P_{univ})) \equiv \chi^{bad}(\gamma)^{-1} \cdot \pi \pmod{\pi^2}.
\]

By definition, we have

\[
T - \gamma(\pi') = \tau_{univ}(T - \pi') = u(T - \pi), \quad u \in R_Z[[T]]^\times.
\]

Comparing coefficients, we find \( u \equiv 1 \pmod{\pi} \), and therefore \( \gamma(\pi') \equiv \pi \pmod{\pi^2} \). With (23), we conclude that \( \gamma(\pi) \equiv \chi^{bad}(\gamma) \cdot \pi \pmod{\pi^2} \). \( \square \)

### 3.3.4 Proof of Proposition 3.1.7

We assume that Condition 3.1.6 holds. Recall that the curve \( X \) consists of five components \( X_0, \ldots, X_4 \), where \( X_i \) contains the specialization of the branch point \( x_i \), for \( i = 1, \ldots, 4 \). Fix \( i \in \{1, \ldots, 4\} \) and let \( W_i \) be a component of \( f^{-1}(X_i) \). The restriction of \( f : Y \to X \) to \( W_i \) is a \( D(W_i) \)-Galois cover \( W_i \to X_i \), branched at 2 points. Over \( x_i \), the cover \( W_i \to X_i \) is tamely ramified, with inertia type \( C_i \). Over the point where \( X_i \) meets \( X_0 \), we have wild ramification, of order \( 2p \). It follows from Condition 3.1.6 (f) that \( D(W_i) = G_i \), i.e. \( W_i = f^{-1}(X_i) \to X_i \) is a \( G \)-cover. Let \( w \in W_i \) be a point where \( W_i \) meets \( E \subset Y \). The inertia group \( I(w) \) is dihedral of order \( 2p \).

Since \( C_G = 1 \), by Condition 3.1.6 (e), every element \( \gamma \in \Gamma \) is induced by a unique element \( \sigma \in \text{Aut}_k(Y) \). It is clear that \( \sigma \) fixes the component \( W_i \) and that \( \sigma(w) \) is again a singular point of \( Y \). Moreover, there exists an element \( \tau \in G \) such that \( \sigma' := \tau^{-1} \circ \sigma \in \text{Aut}_k(Y) \) fixes \( w \).

The element \( \tau \) is unique up to composition with an element of \( I(w) \) and normalizes \( I(w) \). For \( h \in I(w) \), we have \( \sigma' h(\sigma')^{-1} = \tau^{-1} h \). By Condition 2.2.2 (d), we may assume that \( \sigma' \) centralizes \( I(w) \).

According to the Katz–Gabber Lemma, there exists a unique \( I(w) \)-cover \( Z_i \to X_i \) which is isomorphic to \( W_i \to X_i \) in an étale neighborhood of \( w \) and tamely ramified above \( x_i \in X_i \). In fact, \( Z_i \) is a component of \( g^{-1}(X_i) \), where \( g : Z \to X \) is the auxiliary cover associated to \( f \).

The automorphism \( \sigma'_{|W_i} \) induces an isomorphism \( \sigma'_{|Z_i} \) of \( Z_i \) centralizing \( I(w) \). Recall that there exists an isomorphism \( Z_i \cong \mathbb{P}_k^1 \) such that the action of \( I(w) \) on \( Z_i \) is generated by the translation \( z \mapsto z + 1 \) and the reflection \( z \mapsto -z \). Using this identification it is easy to see that \( \sigma'_{|Z_i} = \tau_{|W_i} \).

It follows that \( \tau \in G \) centralizes the action of \( D(W) = G \) on \( W \). But \( G \) has trivial center by Condition 3.1.6 (e), so \( \sigma_{|W} = \text{Id}_{W_i} \).

We have shown that \( \sigma \) is the identity on all components of \( Y \) except possibly on those that lie above \( X_0 \). Therefore, \( \sigma \) restricts to an automorphism \( \sigma_{|E} : E \to E \) which fixes all points where \( E \) meets another component of \( Y \). Recall that the set of points where \( E \) meets another component is the image of a \( \Gamma_2(N') \)-structure \( \phi' : \mathbb{A}' = \mathbb{Z}/2N' \times \mathbb{Z}/2 \to E(k) \), with \( N' = N/p \).

In particular, \( \sigma \) fixes \( 0 \in E \) and is thus an isomorphism of \( E \) as generalized elliptic curve (see the proof of Proposition 3.1.4 above).

We can now finish the proof of Proposition 3.1.7 by applying the classification of automorphism groups of generalized elliptic curves, see [3]. Assume that \( N' > 1 \). Using that \( \phi'(\mathbb{A}') \) meets every component of \( E \) and contains points of order \( > 2 \) we conclude that \( \sigma_{|E} = \text{Id}_{E} \). Therefore, \( \chi^{bad}(\gamma) = \chi^{adm}(\gamma) = 1 \), so \( \gamma = 1 \).

Suppose \( N' = 1 \). Then \( \sigma_{|E} \in \langle -1 \rangle \), where \( \langle -1 \rangle : E \to E \) is the canonical involution of \( E \). Therefore, \( \chi^{bad}(\gamma) = \pm 1 \) and \( \chi^{adm}(\gamma) = 1 \). It remains to be shown that \( \langle -1 \rangle : E \to E \) lifts to an element \( \sigma \in \text{Aut}_k(Y) \) inducing \( \gamma \in \Gamma \) with \( \chi^{bad}(\gamma) = -1 \). We can set \( \sigma_{|W_i} := \text{Id}_{W_i} \) for \( W_i := f^{-1}(X_i) \), \( i = 1, \ldots, 4 \), and define \( \sigma_{|\tau(E)} \) as the canonical involution on the generalized elliptic curve \( \tau(E) \), for all \( \tau \in G \). This completes the proof. \( \square \)
4 Applications to good reduction

In this section we apply the results obtained in the previous sections to questions of good reduction of Galois covers. We extend Raynaud’s criterion for good reduction to our situation (Theorem 4.1.2). Since we are in a very special situation, we get a somewhat sharper bound. For covers with 4 branch points, this result is not useful to produce covers with good reduction, in practice. The rigidity method, which can be used to construct covers over fields with low ramification, hardly ever works for 4 branch points.

However, our results on the reduction of the Hurwitz space allow us to compute the number of covers with good reduction, for given type and position of the branch points (Theorem 4.2.3). The rough idea is this. In characteristic 0, the structure of the Hurwitz space $\mathcal{H}$ as a cover of the $\lambda$-line is known, and has a nice description in terms of the braid action. Our Reduction Theorem describes the structure of $\mathcal{H} \otimes \mathbb{F}_p$. The Cusp Principle links these two results. It states that a cusp of $\mathcal{H}$ has good reduction if and only if it corresponds to an admissible cover with prime-to-$p$ ramification over the singular point.

4.1 Good and bad reduction

4.1.1 Let $G$ be a finite group and $\mathcal{C} = (C_1, C_2, C_3, C_4)$ be a 4-tuple of conjugacy classes of $G$. Let $K_0$ be a field of characteristic 0 such that the individual conjugacy classes $C_i$ are rational over $K_0$, i.e. $\mathbb{Q}(\mathcal{C}) \subset K_0$. We fix an algebraic closure $\bar{K}_0$ of $K_0$. We choose an element $\lambda \in K_0 - \{0, 1\}$ and define $\text{Cov}(\mathcal{C}, \lambda)$ as the set of isomorphism classes of $G$-covers $f : Y \to \mathbb{P}^1$, defined over $\bar{K}_0$, of type $(\mathcal{C}; 0, 1, \infty, \lambda)$. In other words, $\text{Cov}(\mathcal{C}, \lambda) = \pi^{-1}(\lambda)$, where

$$\pi : H^\text{an}_4(\mathcal{C}) \to \mathbb{P}^1_\lambda - \{0, 1, \infty\}$$

is the natural map from the inner Hurwitz space of $G$-covers of type $\mathcal{C}$ to the $\lambda$-line, and we see $\lambda$ as a $\bar{K}_0$-rational point on $\mathbb{P}^1_\lambda$.

Since the domain of definition of $H^\text{an}_4(\mathcal{C})$ is contained in $K_0$ (by assumption), we obtain a natural action of $\text{Gal}(\bar{K}_0/K_0)$ on $\text{Cov}(\mathcal{C}, \lambda)$. In more concrete terms, this action is given as follows. For $\sigma \in \text{Gal}(\bar{K}_0/K_0)$ and $f \in \text{Cov}(\mathcal{C}, \lambda)$, we can form the twisted cover $\sigma f : \sigma Y \to \mathbb{P}^1$ by applying $\sigma$ to the coefficients of the equations defining $f$ and the action of $G$ on $Y$. To each $f \in \text{Cov}(\mathcal{C}, \lambda)$ we can associate the field of moduli of $f$ (relative to $K_0$), i.e. the fixed field of all $\sigma \in \text{Gal}(\bar{K}_0/K_0)$ such that $\sigma f \cong f$. Equivalently, $K$ is the field of rationality of the point on $H^\text{an}_4(\mathcal{C})$ corresponding to $f$. If the center of $G$ is trivial then $f$ has a unique model $f_K : Y_K \to \mathbb{P}^1_K$ over $K$. See e.g. [10].

4.1.2 In the situation of 4.1.1, we will now make the following assumptions. The field $K_0$ is complete with respect to a discrete valuation $v$. We denote by $R_0$ the corresponding valuation ring. The residue field $k_0$ of $v$ is assumed to be algebraically closed of characteristic $p$, where $p$ is an odd prime number. We assume that Conditions 2.2.2 and 3.1.6 hold, with respect to the class vector $\mathcal{C}$ and the prime $p$. Finally, we assume that

$$\lambda \not\equiv 0, 1, \infty \pmod{v}.$$  

For a $G$-cover $f : Y \to \mathbb{P}^1$ in $\text{Cov}(\mathcal{C}, \lambda)$ we can ask whether it has good or bad reduction at $v$, in the sense of Section 1.1 (under Condition 26, good reduction is equivalent to admissible reduction). If a cover $f \in \text{Cov}(\mathcal{C}, \lambda)$ has good reduction, its reduction $f_k : Y_k \to \mathbb{P}^1_k$ is a $G$-cover over the field $k$ of positive characteristic, of type $(\mathcal{C}; 0, 1, \infty, \lambda)$ ($\lambda \in k$ denotes the residue of $\lambda \in K_0$). By a theorem of Grothendieck, all $G$-covers over $k$ of type $(\mathcal{C}; 0, 1, \infty, \lambda)$ arise as the reduction of a unique $G$-cover $f \in \text{Cov}(\mathcal{C}, \lambda)$ with good reduction. This motivates the following question.

Question 4.1.1 How many covers $f \in \text{Cov}(\mathcal{C}, \lambda)$ have good reduction at $v$?
Theorem 4.2.3 below answers this question in an explicit way. Its proof relies on the Reduction Theorem 3.1.1 and the Cusp Principle, Proposition 1.2.1.

4.1.3 Let $\bar{H} = \bar{H}^w(\mathbb{C})$ be the completion of the Hurwitz space $H = H^w(\mathbb{C})$, see Section 4. We understand that $\bar{H}$ is defined over $\Lambda := \mathcal{O}_{\mathbb{C}}[p]$. Here $\mathbb{Q}(\mathbb{C}) \subset K_0$ is as in Section 1.2.3 and $p$ is the prime ideal corresponding to the restriction of $\nu$ to $\mathbb{Q}(\mathbb{C})$. By Theorem 3.1.1 (i), $\bar{H}$ is a normal scheme of dimension 2, proper and flat over $\Lambda$. Let $\pi \in \text{Cov}(\mathbb{C}, \lambda)$ and be $K$ its field of moduli (relative to $K_0$). Since $K/K_0$ is finite, $\nu$ extends uniquely to $K$. We denote by $R$ the corresponding valuation ring. Since $\bar{H}$ is proper over $\Lambda$, the morphism $\text{Spec} K \to \bar{H}$ corresponding to $f$ extends uniquely to a morphism $\phi : \text{Spec} R \to \bar{H}$, giving rise to a $k$-rational point $s : \text{Spec} k \to \bar{H}$. By the definition of $\bar{H}_{\text{bad}}$, $f$ has good (resp. bad) reduction if and only if $s \not\in \bar{H}_{\text{bad}}$ (resp. $s \in \bar{H}_{\text{bad}}$).

We can be more precise. Since we assume the center of $G$ to be trivial, $f$ descents to a unique $G$-cover $f_K : Y_K \to \mathbb{P}^1_k$ defined over $K$. Let $K'/K$ be the minimal extension of $K$ over which $f_K \otimes K'$ has a stable model $f_{0,R'} : Y_{R'} \to \mathbb{P}^1_{R'}$ over the valuation ring $R' \subset K'$ (see Section 1.1). We denote by $f_{0,k} : Y_k \to \mathbb{P}^1_k$ the special fiber of $f_{0,R'}$. The morphism $f_{0,k}$ (together with the induced marks on $Y_k$ and $\mathbb{P}^1_k$ and the $G$-action on $Y_k$) is an object of the Hurwitz stack $\bar{H}$ associated to $H$ and corresponds to the $k$-point $s : \text{Spec} k \to \bar{H}$. The stable model $f_{0,R'}$ is a deformation of $f_{0,k}$. Hence it corresponds to a unique morphism

$$(27) \quad R_{Y_k} \to R'$$

of $W(k)$-algebras, where $R_{Y_k}$ is the universal deformation ring of $f_{0,k}$, see Section 3.1.2. The extension $K'/K$ is Galois, and its Galois group is a subgroup of $\Gamma$, the universal monodromy group of $f_{0,k}$, see Section 3.1.3. The morphism $(27)$ is equivariant with respect to the injection $\text{Gal}(K'/K) \hookrightarrow \Gamma$. The morphism

$$(28) \quad \hat{O}_{\bar{R},s} = R_{Y_k}^p \to R$$

obtained from $(27)$ by taking invariants corresponds to the morphism $\phi : \text{Spec} R \to \bar{H}$. If $f$ has good reduction, then $\Gamma = 1$ and $K' = K$. On the other hand, if $f$ has bad reduction, it has modular reduction of level $N$, where $N$ is an integer strictly divisible by $p$, see Proposition 2.4.3.

We say that $\lambda \in K_0$ is ordinary (resp. supersingular) if the elliptic curve $y^2 = x(x-1)(x-\lambda)$ has ordinary (resp. supersingular) reduction modulo $v$. By [14, Corollary IV.4.22], $\lambda$ is supersingular if and only if $h_p(\lambda) \equiv 0 \pmod{v}$, where $h_p(X) \in \mathbb{Z}[X]$ is an explicit polynomial of degree $(p-1)/2$. By a theorem of Igusa, $h_p(X)$ is separable modulo $p$, i.e. there are exactly $(p-1)/2$ supersingular values $\lambda$.

**Theorem 4.1.2** Assume that $f \in \text{Cov}(\mathbb{C}, \lambda)$ has bad reduction of level $N$. Let $K$ be the field of moduli of $f$, relative to $K_0$, and denote by $e$ the ramification index of $p$ in $K$. Then

$$(29) \quad e \geq \begin{cases} \frac{(p-1)}{2} & \text{if } N = p \\ p - 1 & \text{if } N > p. \end{cases}$$

Moreover, if $\lambda$ is supersingular then the inequality $(29)$ is strict.

**Proof:** As explained above, the cover $f \in \text{Cov}(\mathbb{C}, \lambda)$ gives rise to a local ring homomorphism $R_{Y_k} \to R'$, where $R_{Y_k}$ is the universal deformation ring of the reduction of $f$ and $R'/R$ is the minimal extension over which $f$ has a stable model. We denote by $\pi \in R'$ the image of the element of $R_{Y_k}$ with the same name, given by Theorem 3.1.2. Clearly, $v(\pi) > 0$. Since $\pi^{p-1} | p$ (Theorem 3.1.2 (iii) and (iv)), the ramification index of $p$ in $R'$ is at least $p - 1$. But $\text{Gal}(K'/K) \hookrightarrow \Gamma$, so the inequality $(29)$ follows from Proposition 3.1.7. If $\lambda$ is supersingular then $p = \pi^{p-1} u$, where $v(u) > 0$, showing that the inequality $(29)$ is strict. \hfill $\square$

**Remark 4.1.3** The inequality $e \geq (p-1)/2$ can also be deduced from [24], Corollaire 4.2.5 and Théorème 5.1.1.
4.2 The Hurwitz space as cover of the λ-line

4.2.1 The Hurwitz classification and braid action Let us for the moment consider the morphism \( \pi \) in \((25)\) as an unramified cover of Riemann surfaces. We choose a point \( \lambda_0 \in ( - \infty, 0 ) \) on the negative real line and a point \( x_0 \in \{ x \in \mathbb{C} \mid \text{Im} \, x > 0 \} \) on the upper half plane. Let \( \gamma_i \) be the unique element of \( \pi_1( \mathbb{P}^1 - \{ 0, 1, \infty, \lambda_0 \}, x_0 ) \) represented by a simple loop which crosses the real line exactly twice, turning around the \( i \)-th point in the list \( (0, 1, \infty, \lambda_0) \), in counterclockwise orientation. The group \( \pi_1( \mathbb{P}^1 - \{ 0, 1, \infty, \lambda_0 \}, x_0 ) \) is generated by the \( \gamma_i, i = 1, \ldots, 4 \), with the only relation \( \prod_i \gamma_i = 1 \). With these choices made, there is a canonical bijection

\[
\pi^{-1}(\lambda_0) \cong \text{Ni}_4^\ast(\mathbb{C}) := \{ g = (g_1, \ldots, g_4) \mid G = <g_i>, g_i \in C_i, \prod_i g_i = 1 \} / G,
\]

(here \( G \) acts on the set of tuples \( g \) by diagonal conjugation).

The cover \( \pi \) induces an action of \( \Pi := \pi_1( \mathbb{P}^1 - \{ 0, 1, \infty \}, \lambda_0 ) \) on \( \pi^{-1}(\lambda_0) \), hence on \( \text{Ni}_4^\ast(\mathbb{C}) \) via \((30)\). To describe this action explicitly, we denote by \( \tilde{\mathcal{H}}_4 \) the Hurwitz braid group on 4 strings, with generators \( Q_1, Q_2, Q_3 \) and relations \((3)\), \((1.a-c)\). We identify \( \tilde{\mathcal{H}}_4 \) with the fundamental group of \( \mathcal{U}_4 := \{ x \in \mathbb{P}^1(\mathbb{C}) \mid |x| = 4 \} \), with base point \( x_0 = \{ 0, 1, \infty, \lambda_0 \} \). Thus we obtain an embedding

\[
\Pi := \pi_1( \mathbb{P}^1 - \{ 0, 1, \infty \}, \lambda_0 ) \hookrightarrow \tilde{\mathcal{H}}_4.
\]

The elements

\[
a_0 := Q_3Q_2Q_1^{-1}Q_3^{-1}, \quad a_1 := Q_3Q_2Q_3^{-1}, \quad a_\infty := Q_3^2,
\]

lie in the image of \((33)\) and define standard generators of \( \Pi \). In particular, \( a_0 a_1 a_\infty = 1 \), and \( a_w \) is represented by a simple loop around \( w \), for \( w \in \{ 0, 1, \infty \} \). One can check using the formula \((3)\), \((1.d)\), that the induced action of \( \Pi \) on \( \text{Ni}_4^\ast(\mathbb{C}) \) is given by

\[
[g_1, g_2, g_3, g_4] a_w = \begin{cases} 
[g_1^{-1}g_2^{-1}g_3^{-1}g_4^{-1}], & \text{if } w = 0, \\
[g_1, g_2^{-1}g_3^{-1}g_4^{-1}, g_4^{-1}], & \text{if } w = 1, \\
[g_1, g_2^{-1}, g_3^{-1}, g_4^{-1}], & \text{if } w = \infty
\end{cases}
\]

(\( w \) use the notation \( g^\gamma = \gamma^{-1} g \gamma \)).

4.2.2 The cusps Let \( \tilde{\pi} : \tilde{H} \to \mathbb{P}^1_\mathbb{Q} \) denote the canonical map, which extends \( \pi \). We say that a \( \mathbb{Q} \)-point \( c \) on \( \tilde{H} \) is a cusp if \( \tilde{\pi}(c) \in \{ 0, 1, \infty \} \). We write \( \text{Cusps}(\mathbb{C}, w) \) for the set of cusps above \( w \in \{ 0, 1, \infty \} \). The cusps are the ramification points of the finite, tamely ramified morphism \( \tilde{\pi}_\mathbb{Q} \). By Section \([2,3]\) we can identify cusps with certain braid orbits:

\[
\text{Cusps}(\mathbb{C}, w) \cong \text{Ni}_4^\ast(\mathbb{C}) / \langle a_w \rangle, \quad w \in \{ 0, 1, \infty \}.
\]

Let us fix a cusp \( c \) above \( w \in \{ 0, 1, \infty \} \), represented by a class \( [g] \in \text{Ni}_4^\ast(\mathbb{C}) \). The conjugacy class of the element \( \gamma \in G \) associated to \( [g] \) in \((33)\) does only depend on the orbit of \( [g] \) under the action of \( a_w \), and is thus canonically associated to the cusp \( c \). We call \( n := \text{ord}(\gamma) \) the order of the cusp \( c \). Note that the length of the \( a_w \)-orbit of \( [g] \) divides \( n \).

As a point on \( H \), the cusp \( c \) corresponds to an admissible cover

\[
f_K : Y_K \to X_K
\]

between stably marked curves over a number field \( K \), together with an action of \( G \) on \( Y_K \). The bottom curve \( X_K \) is singular, consisting of two components \( X_{1,K}, X_{2,K} \cong \mathbb{P}^1_\mathbb{K} \), intersecting in one
point. It is shown e.g. in [29], Section 4.3.3, that the order \( n \) is the ramification index of \( f \) above the singular point of \( X_K \).

We choose once and for all a valuation \( \bar{v} \) of \( \bar{Q} \) extending the valuation of \( Q(C) \) corresponding to the prime ideal \( p \). Since \( \bar{H} \) is proper over \( \Lambda \), the \( \bar{Q} \)-valued point \( c \) reduces (with respect to the valuation \( \bar{v} \)) to an \( \bar{F}_p \)-valued point \( \bar{c} \) on \( \bar{H} \). We say that the cusp \( c \) has good (resp. bad) reduction if \( \bar{c} \notin \bar{H}^{bad} \) (resp. \( \bar{c} \in \bar{H}^{bad} \)). If \( c \) has bad reduction then the point \( \bar{c} \in \bar{H}^{bad} \) corresponds to a bad cover \( f : Y \to X \) of modular type of level \( N \), for some integer \( N \) strictly divisible by \( p \). We say that \( c \) has bad reduction of level \( N \).

**Proposition 4.2.1 (The Cusp Principle)** A cusp \( c \) of order \( n \) has bad reduction if and only if \( p|n \). In this case, \( n \) divides the level \( N \).

**Proof:** After a finite extension of \( K \), the admissible cover \( f_K \) corresponding to \( c \) extends to an \( R \)-object \( f_R : Y_R \to X_R \) of the Hurwitz stack \( \mathcal{H} \), where \( R \) is the valuation ring of \( K \) corresponding to \( \bar{v}|_K \). The special fiber \( f : Y \to X \) of \( f_R \) is the object of \( \mathcal{H} \) which induces the \( \bar{F}_p \)-point \( \bar{c} \) on \( \bar{H} \). We are exactly in the situation of Section 2.2. The Cusp Principle follows from Proposition 2.5.3. \( \square \)

### 4.2.3 The bad components

According to Theorem 3.1.1, \( \bar{H}^{bad} \otimes \bar{F}_p \) is a smooth projective curve over \( \bar{F}_p \). We pick a connected component \( W \subset \bar{H}^{bad} \otimes \bar{F}_p \). We call \( W \) a bad component. The geometric points of \( W \) correspond to bad covers of modular type. It is clear that the level \( N \) of these covers is constant on \( W \). Therefore, we may call \( N \) the level of \( W \). We let \( \eta \) be the generic point of \( W \), and we denote by \( m \) the multiplicity of \( W \) inside \( \bar{H}^{bad} \). By definition, \( m \) is the length of the local ring of \( \eta \) on \( \bar{H} \otimes \bar{F}_p \).

**Proposition 4.2.2** If \( N = p \) then \( m = (p - 1)/2 \). Otherwise, \( m = p - 1 \).

**Proof:** This follows directly from Corollary 3.1.3 (iii) and Proposition 3.1.7. \( \square \)

By Corollary 3.1.3 (i), the natural map \( W \to \mathbb{P}_1^1 \otimes \bar{F}_p \) is inseparable, with inseparability degree \( p \). Moreover, the induced map \( W^p \to \mathbb{P}_1^1 \otimes \bar{F}_p \) is tamely ramified in \( 0, 1, \infty \), and étale everywhere else. We are interested in describing it in more detail. We write \( N = pN' \), and let \( X_2(N') \) be the coarse moduli space for generalized elliptic curves with \( \Gamma_2(N') \)-structure, see Section 3.3. We fix a bijection \( \alpha : \{1, 2, 3, 4\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \); this determines a finite map \( X_2(N') \to \mathbb{P}_1^1 \) by Proposition 3.2.4 (i). Since \( N' \) is prime-to-\( p \), \( X_2(N') \otimes \bar{F}_p \) is a smooth projective curve over \( \bar{F}_p \), and the map \( X_2(N') \otimes \bar{F}_p \to \mathbb{P}_1^1 \otimes \bar{F}_p \) is finite, tamely ramified in \( 0, 1, \infty \) and étale everywhere else.

**Proposition 4.2.3** There exists a finite map \( h : W \to X_2(N') \otimes \bar{F}_p \), compatible with the maps to \( \mathbb{P}_1^1 \otimes \bar{F}_p \). This map is the composition of a purely inseparable map of degree \( p \) and an étale map.

**Proof:** Let \( s : \text{Spec } k \to W \) be a geometric point, corresponding to a cover \( f_0 : Y \to X_0 \) over \( k \). Following Section 3, we associate to \( f_0 \) a generalized elliptic curve \( E \) over \( k \), together with a \( \Gamma_2(N) \)-structure \( \phi \). We may assume that the ordering of the branch points \( x_1, \ldots, x_4 \in X_0 \) is compatible with the bijection \( \alpha : \{1, 2, 3, 4\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \) we have chosen above. As in the proof of Proposition 3.2.6, we let \( E'' := E/\phi(\mathbb{Z}/p) \) and \( \phi'' \) be the induced \( \Gamma_2(N') \)-structure. The pair \( (E'', \phi'') \) gives rise to a geometric point \( s' : \text{Spec } k \to X_2(N') \otimes \bar{F}_p \). One easily checks that \( (E'', \phi'') \) is unique up to isomorphism. In other words, \( s \mapsto h(s) := s' \) is a well defined map on geometric points. We claim that this map is induced by a finite morphism \( h : W \to X_2(N') \otimes \bar{F}_p \), as in the statement of the proposition. In order to prove this claim, it suffices to show the following. Let \( R_s \) (resp. \( R_{s'} \)) be the complete local ring of \( s \) on \( W \) (resp. of \( s' \) on \( X_2(N') \otimes \bar{F}_p \)). Then there exists a finite morphism \( h^*: R_{s'} \to R_s \) of local \( \bar{F}_p \)-algebras, purely inseparable of degree \( p \), with the following property. Let \( \bar{K} \) be an algebraic closure of the fraction field of \( R_s \), \( \bar{s} : \text{Spec } \bar{K} \to W \) the tautological point and \( \bar{s}' : \text{Spec } \bar{K} \to X_2(N') \otimes \bar{F}_p \) the point induced by \( h^* \). Then \( h(\bar{s}) = \bar{s}' \).
Following Section 3.1.4, we identify $R_s = \hat{\Omega}_{W,s}$ with $(R^W_{\Gamma})^{-1} = k[[t]]^\Gamma$. By Proposition 3.1.4, $\Gamma$ acts trivially on $k[[t]]$, so $R_s = k[[t]]$. In particular, the point $\tilde{s}$ corresponds to the generic fiber of $Y_{\text{red}} \otimes_{R_s} k[[t]]$. In the same way, we can identify $R_s$ with $R_{\phi''}$, where $R_{\phi''}$ is the universal deformation ring of $(E'', \phi'')$ and $\Gamma'$ the corresponding monodromy group. By the proof of Proposition 3.2.6 we have $R_{\phi''} = k[[t^p]]$. Moreover, $\Gamma'$ is trivial, because

$$\text{Aut}(E'', \phi'') = \begin{cases} \mathbb{Z}/2 & \text{if } N' = 1, \\ 1 & \text{otherwise.} \end{cases}$$

If we define $h_s^*$ as the natural injection $k[[t^p]] \hookrightarrow k[[t]]$, the proposition follows. \hfill

**Remark 4.2.4** One can show that the étale part $h^{(p)} : W^{(p)} \to X_2(N') \otimes \overline{\mathbb{F}}_p$ of $h$ is in fact an isomorphism. In this sense, bad components are “modular”.

### 4.2.4 The number of covers with good reduction

Let us denote by $\text{Cov}(\mathbb{C}, \lambda)^{\text{good}}$ the subset of $\text{Cov}(\mathbb{C}, \lambda)$ containing the covers with good reduction. Define

$$d := | \text{Ni}^\lambda_1(\mathbb{C})|, \quad d^{\text{bad}} := | \{ [g] \in \text{Ni}^\lambda_1(\mathbb{C}) | p | \text{ord}(g_3g_4) \} |.$$

We know from (33) that $d = \deg \bar{\pi} = |\text{Cov}(\mathbb{C}, \lambda)|$. Using the Cusp Principle and the Reduction Theorem, we can show:

**Theorem 4.2.5** We have

$$|\text{Cov}(\mathbb{C}, \lambda)^{\text{good}}| = \begin{cases} d - d^{\text{bad}}, & \text{if } \lambda \text{ is ordinary,} \\ d - \frac{p+1}{p}d^{\text{bad}}, & \text{if } \lambda \text{ is supersingular.} \end{cases}$$

In particular, if $\lambda$ is ordinary and $\text{Ni}^\lambda_1(\mathbb{C}) \neq \emptyset$ then $\text{Cov}(\mathbb{C}, \lambda)^{\text{good}} \neq \emptyset$.

**Proof:** According to Theorem 3.1.1, $H^{\text{good}}$ is a smooth curve over $\mathbb{F}_q$ and the natural map $\bar{\pi}^{\text{good}} : H^{\text{good}} \to \mathbb{P}^1_\lambda \otimes \mathbb{F}_q$ is finite. Let $S := H^{\text{good}} \cap H^{\text{bad}}$. By Theorem 3.1.1 (iii), $S$ contains exactly the points on $H^{\text{bad}}$ with a supersingular $\lambda$-value. By definition, we have $H^{\text{good}} - S = H^{\text{adm}} \otimes \mathbb{F}_q$. It follows that $H^{\text{good}} - S \to \mathbb{P}^1_\lambda \otimes \mathbb{F}_q$ is tamely ramified in $0$, $1$ and $\infty$ and étale everywhere else. Comparing the degrees of $\bar{\pi}^{\text{good}}$ and of $H^{\text{adm}} \to \mathbb{P}^1_\lambda$ above $\infty$, we get

$$\deg \bar{\pi}^{\text{good}} = \sum c e_c = d - d^{\text{bad}}.$$  

Here $c$ runs over the set of cusps above $\infty$ with good reduction, and $e_c$ denotes the ramification index of $\bar{\pi}$ in $c$ (which is equal to the length of the $a_\infty$-orbit of $\text{Ni}^\lambda_1(\mathbb{C})$). The second equality in (35) is a consequence of the Cusp Principle, Proposition 4.2.1.

In case $\lambda$ is ordinary, the statement of the theorem follows directly from (33). Assume that $\lambda$ is supersingular, and let $S_\lambda$ be the set of points in $S$ above $\lambda$. For $s \in S_\lambda$, we denote by $m_s$ the ramification index of $\pi^{\text{good}}$ in $s$. We obtain

$$|\text{Cov}(\mathbb{C}, \lambda)^{\text{good}}| = \deg \bar{\pi}^{\text{good}} - \sum_{s \in S_\lambda} m_s.$$  

According to Corollary 3.1.3 (ii) and (iii), $m_s$ equals the multiplicity of the bad component $W_s$ meeting $H^{\text{good}}$ in $s$. By Corollary 3.1.5 (i), the natural map $W_s \to \mathbb{P}^1_\lambda \otimes \mathbb{F}_p$ is the composition of a purely inseparable map of degree $p$ and a map which is étale away from $0$, $1$ and $\infty$. Therefore,

$$d = \deg \bar{\pi}^{\text{good}} + p \cdot \sum_{s \in S_\lambda} m_s.$$  

The equations (35), (36) and (37) together imply Theorem 1.2.3 in the supersingular case. \hfill\-box

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4.3 Examples

In this section, we explain how the results we obtained can be used to compute the reduction of the Hurwitz space, in an explicit example. We take \( G = PSL_2(\ell) \), where \( \ell \neq p \) is a prime such that \( p \) exactly divides \( |G| = (\ell^2 - 1)/2 \). Recall that \( G \) has two conjugacy classes of order \( \ell \), which we denote by \( \ell A \) and \( \ell B \). We take \( C = (\ell A, \ell A, \ell A) \) or \( C = (\ell A, \ell A, \ell B, \ell B) \). The normalizer of a \( p \)-Sylow group of \( G \) for \( p \neq \ell \) is a dihedral group of order \( \ell + 1 \) or \( \ell - 1 \), depending on whether \( p|\ell + 1 \) or \( p|\ell - 1 \), \([4]\). Note that Condition 3.1.7 is also satisfied. We are interested in computing the reduction to characteristic \( p \) of \( \tilde{H} := H^\text{red}(\mathbb{C}) \).

We will explain the algorithm for computing the reduction of \( \tilde{H} \) in the special case \( \ell = 11 \) and \( C = (\ell A, \ell A, \ell B, \ell B) \). After that, we give a table with the reduction of \( \tilde{H} \) for \( \ell \leq 31 \), to all odd primes \( p \neq \ell \) exactly dividing the order of \( G \).

Take \( \ell = 11 \) and \( C = (\ell A, \ell A, \ell B, \ell B) \). Since \( |G| = 660 \), we know that \( \tilde{H} \) has good reduction to characteristic \( p \neq 2, 3, 5, 11 \). The reduction to characteristic 2 and 11 we cannot compute using our methods. Let us first take \( p = 3 \). The normalizer of a 3-Sylow group is a dihedral group of order 12, so the possible levels associated to the bad components are 3 and 6. To decide which ones occur, we want to apply the Cusp Principle \([1.2.1]\). For this we need to know the order of the cusps in characteristic zero.

Using the program \( ho, [21] \), we compute the irreducible components \( Z \) of \( \tilde{H} \otimes \mathbb{Q} \), and the ramification indices of \( Z \to \mathbb{P}_\lambda^3 \), for each irreducible component. Let us relate these ramification indices to the order of the cusps. Let \( s \) be a cusp of \( \tilde{H} \otimes \mathbb{Q} \) defined over \( K \), and let \( f_K : Y_K \to X_K \) be the corresponding admissible cover. Let \( \tau \) be the unique singular point of \( X_K \) and \( \rho \) a singular point of \( Y_K \). Let \( n \) be the ramification index of \( \rho \) and \( G_1 \) and \( G_2 \) the decomposition groups of the two components of \( Y_K \) passing through \( \rho \). Using the description of the normalizers of elements in \( G = PSL_2(\ell) \) given in \([12]\), Abschnitt II.8, one easily checks that the ramification index \( e \) of \( s \) in \( \tilde{H} \otimes \mathbb{C} \to \mathbb{P}_\lambda^3 \) is equal to \( n \), unless \( n = \ell \) and \( G_i \simeq \mathbb{Z}/\ell \) for some \( i \), see. Section 4.2.1. In particular, \( p|\ell \iff p|e \).

\[
\begin{array}{|c|c|c|}
\hline
\ell & C = (\ell A, \ell A, \ell B, \ell B) & \text{ramification} \\
\hline
11 & \text{deg} & g & \text{num} \\
\hline
2^1; -; 1^2 & 2 & 0 & 1 \\
2^2 6^2; -; 1^4 3^4 & 16 & 1 & 1 \\
2^1 1^2 6^2 5^2; -; 5^2 11^4 3^4 & 33 & 2 & 1 \\
\hline
\end{array}
\]

The notation is as follows. Each row corresponds to an irreducible component; the last entry of each row gives the number of isomorphic components. The first entry gives the ramification of \( \tilde{H} \otimes \mathbb{C} \to \mathbb{P}_\lambda^3 \) over \( 0, 1, \infty \). Here \( 2^1 \) means one ramification point of order 2, and \( 1^2 \) means two ramification points of order 1. A “−” indicated that over this point, the ramification indices are the same as over the previous point. The next entries give the genus of the components and its degree over the \( \lambda \)-line.

The Cusp Principle implies that the first component \( W_1 \) has good reduction to characteristic 3, since 3 does not divide the order of any of the cusps. The second and third component, which we will denote by \( W_2 \) and \( W_3 \), have bad reduction to characteristic 3. Since both these components have a cusp of order \( n = 6 \) and the order of the cusp divides the level of the bad component it reduces to, we see that in both cases there will be a bad component of level \( N = 6 \). A bad component of level 6 is a cover of \( X_3(\mathbb{N}') \otimes \mathbb{F}_3 \), purely inseparable of degree 3, with \( \mathbb{N}' = \mathbb{N}/p = 2 \), see Remark \([1.2.4]\). The curve \( X_2(2) \otimes \mathbb{F}_3 \) is a cover of degree 2 of the \( \lambda \)-line, branched at 0 and 1 and unbranched at \( \infty \). In Proposition 1.2.2 we computed the multiplicity of the bad components. A bad component of level 6 has multiplicity \( p - 1 = 2 \). We conclude that the reduction of the irreducible components \( W_2 \) and \( W_3 \) each have one bad component, and it is of level 6.

Now let us have a look at the good components. The good and the bad components intersect over the supersingular \( \lambda \)'s. In characteristic 3, there is only one supersingular \( \lambda \), namely \( \lambda = -1 \). In characteristic 2, we have four supersingular \( \lambda \)'s, namely \( \lambda = 1, -1, \ell - 1, \ell + 1 \).

The decomposition groups of the \( \lambda \)-lines are dihedral groups of order 2, corresponding to the fixed point sets of the \( \lambda \)-actions, where \( \lambda \) is a generator of a cyclic subgroup of order 2 in the Galois group. The reduction of the \( \lambda \)-lines to characteristic 2 is represented by the following covers of degree 2 of the \( \lambda \)-line:

\[
\begin{array}{|c|c|}
\hline
\ell & \text{ramification} \\
\hline
11 & 2^1; -; 1^2 \\
2^2 6^2; -; 1^4 3^4 & 16 \\
2^1 1^2 6^2 5^2; -; 5^2 11^4 3^4 & 33 \\
\hline
\end{array}
\]

This table represents the reduction of the \( \lambda \)-lines to characteristic 2, where \( 2^1 \) means one ramification point of order 2, and \( 1^2 \) means two ramification points of order 1. A “−” indicated that over this point, the ramification indices are the same as over the previous point. The next entries give the genus of the components and its degree over the \( \lambda \)-line.

In characteristic 3, there is only one supersingular \( \lambda \), namely \( \lambda = -1 \).
(mod 3). This means that the good and bad components meet in two points. Since the multiplicity of the bad component is two, the good components will be ramified of order two in these intersection points. From this we can compute the number of covers with good reduction, for each value of $\lambda$.

$$|\text{Cov}(W_2, \lambda)_{\text{good}}| = \begin{cases} 4 & \text{if } \lambda \not\equiv -1 \pmod{3}, \\ 0 & \text{if } \lambda \equiv -1 \pmod{3}. \end{cases}$$

$$|\text{Cov}(W_3, \lambda)_{\text{good}}| = \begin{cases} 21 & \text{if } \lambda \not\equiv -1 \pmod{3}, \\ 17 & \text{if } \lambda \equiv -1 \pmod{3}. \end{cases}$$

In general we are not able to calculate the number of good components. However, since the degree of $W_2$ is sufficiently small, we can describe what its good part looks like. As remarked before, the degree of $W_2^{\text{good}}$ over $\mathbb{P}_\lambda^1$ is 4 and it is ramified over $-1$ at two points of order two. Outside the supersingular $\lambda$’s, the ramification is as in characteristic zero. So over 0 and 1, there are two ramification points of order two, and over $\infty$ it is unramified. From this it follows that $W_2^{\text{good}}$ is connected.

Now let us have a look at $p = 5$. In this case, the components $W_1$ and $W_2$ both have good reduction, since 5 does not divide the order of any of the cusps. The component $W_3$ has bad reduction. Note that the normalizer of a 5-Sylow group of $G$ is a dihedral group of order 10, so the only possibility for the level is 5. A bad component of level 5 is a cover of $X_2(1) \otimes \overline{\mathbb{F}}_5$, purely inseparable of degree 5. The curve $X_2(1) \otimes \overline{\mathbb{F}}_5$ is isomorphic to the $\lambda$-line. It has multiplicity $(p - 1)/2 = 2$. In characteristic 5, there are two supersingular $\lambda$-values: the primitive sixth roots of unity. In these points the good part will have an “extra” ramification of order two. So as before we are able to compute the number of covers with good reduction.

$$|\text{Cov}(W_3, \lambda)_{\text{good}}| = \begin{cases} 23 & \text{if } \lambda \not\equiv \zeta_6, \zeta_6^5 \pmod{5}, \\ 21 & \text{if } \lambda \equiv \zeta_6, \zeta_6^5 \pmod{5}. \end{cases}$$

The results are summarized in the following lemma.

**Lemma 4.3.1** Let $W_1, W_2, W_3$ be the three irreducible components of $H^n_1(PSL_2(11)) \otimes \overline{\mathbb{Q}}$, as described above.

(a) Then $W_1$ has good reduction to characteristic $p \neq 2, 11$.

(b) The component $W_2$ has good reduction to characteristic $p \neq 2, 3, 11$. In characteristic 3, it as two irreducible components: a bad component of level 6 and a good component. The degree of the good component over the $\lambda$-line is 4.

(c) The component $W_3$ has good reduction to characteristic $p \neq 2, 3, 5, 11$. In characteristic 3, there is one bad component, of level 6, and the degree of the good part over the $\lambda$-line is 21. In characteristic 5, there is one bad component, of level 5, and the degree of the good part over the $\lambda$-line is 23.

The Hurwitz space might have irreducible components of large degree having good reduction at many primes. For example, take $\ell = 31$ and $\underline{C} = (\ell A, \ell A, \ell B, \ell B)$. The Hurwitz space in characteristic zero has an irreducible component of genus 37, whose degree over the $\lambda$-line is 128. The cusps of this component all have ramification index a power of 2. We conclude from the Cups Principle that this component has good reduction to all primes $p \neq 2, 31$. For details, see the table below.
Example 4.3.2 (Raynaud) Let $H$ be the inner Hurwitz space parameterizing Galois covers of $\mathbb{P}^1$ with Galois group $A_5$ which are branched at four points of order 3. Let $\bar{H}$ be its completion, over $\mathbb{Z}_{(5)}$. (Raynaud considered the absolute Hurwitz space; it is easy to make the adaption to that case.) In characteristic zero, $\bar{H} \otimes \bar{\mathbb{Q}}$ is connected and has degree 18 over the $\lambda$-line. Over $0, 1, \infty$ this cover has ramification $3^25^21^2$. We conclude as above that the reduction of the Hurwitz space to characteristic 5 has one bad component of level 5. So the good degree is 8. If $\lambda$ reduces to a supersingular value, there are 6 covers with good reduction.

The above example was presented by Raynaud in his talk in Oberwolfach, June 1997. In the problem session of the same conference he proposed the exercise of computing the number of covers with good reduction to characteristic 3 for $G = A_5$ and ramification of order 5. The answer to this exercise appears in the first rows of the table below.

The following table describes the reduction of the Hurwitz space $\bar{H} := \bar{H}^{\text{in}}(C)$, where $G = \text{PSL}_2(\ell)$ and $C$ is either $(\ell A, \ell A, \ell B, \ell B)$ or $(\ell A, \ell A, \ell A, \ell A)$. Every row corresponds to an isomorphism class of irreducible components in $\bar{H}$; the entry “num” gives the number of isomorphic components. The first column gives the $\ell$, the second column gives the class vector. The third column gives the ramification of $\bar{H} \to \mathbb{P}^1$ in characteristic zero over $0, 1, \infty$. Here $a^b$ means: $b$ ramification points of order $a$ and $-$ means: the same as the previous point. The entries “deg” and $g$ give the degree over the $\lambda$-line and the genus of the component (in characteristic zero). The last three entries describe the reduction for odd primes different from $\ell$ which exactly divide the order of $G$. No statement is made for other primes. A dash means: the component has good reduction to all such primes. Each prime $p$ such that the component has bad reduction to characteristic $p$, is listed on a separate row. Under “bad components”, for each prime, all the bad components are listed. The last entry gives the degree of the good part over the $\lambda$-line. This is the number of covers with good reduction, for $\lambda$ ordinary. The number of covers with good reduction for supersingular $\lambda$ can be computed by Theorem 4.2.5. A component has good reduction to characteristic $p$ for odd primes $p \neq \ell$ which are not listed and which strictly divide the order of $G$. The prime $\ell = 13$ is missing from the table because our computer refused to run $ho$ for $\text{PSL}_2(13)$. The prime $\ell = 17$ is missing because there are no primes $p \neq 17$ which exactly divide the order of $\text{PSL}_2(17)$.
| $\ell$ | Ni | ramification | deg | $g$ | num | $p$ | bad comp | gdeg |
|---|---|---|---|---|---|---|---|---|
| 5 | AABB | $2^4; -; 1^2$ | 2 | 0 | 1 | -- | -- | -- |
| 5 | AABB | $3^4; 3^2; -$ | 5 | 0 | 1 | 3 | $1 \times N = 3$ | 2 |
| 5 | AAAA | $5^3; 1^2; -; -$ | 10 | 0 | 1 | 3 | $1 \times N = 3$ | 7 |
| 7 | AABB | $4^2; -; 1^4; 2^2$ | 8 | 0 | 1 | -- | -- | -- |
| 7 | AABB | $1^4; 3^4; -; 7^1; 3^1; 2^2$ | 14 | 0 | 1 | 3 | $1 \times N = 3$ | 11 |
| 7 | AAAA | $1^2; 2^2; -$ | 2 | 0 | 0 | 3 | -- | -- |
| 7 | AAAA | $2^3; -; -$ | 7 | 0 | 1 | 3 | $1 \times N = 3$ | 4 |
| 11 | AABB | $2^4; -; 1^2$ | 2 | 0 | 1 | -- | -- | -- |
| 11 | AABB | $2^6; -; 1^4; 3^4$ | 16 | 1 | 1 | 3 | $1 \times N = 6$ | 4 |
| 11 | AABB | $2^3; 6^2; 5^2; -; 5^2; 11^1; 3^4$ | 33 | 2 | 1 | 3 | $1 \times N = 6$ | 21 |
| 11 | AAAA | $1^3; -; -$ | 4 | 0 | 0 | 0 | 1 | 3 | $1 \times N = 3$ | 1 |
| 11 | AAAA | $3^5; -; -$ | 22 | 3 | 1 | 3 | $4 \times N = 3$ | 10 |
| 19 | AABB | $2^4; -; 1^4; 2^2$ | 8 | 0 | 1 | -- | -- | -- |
| 19 | AABB | $2^4; -; 1^6; 5^8$ | 48 | 9 | 1 | 5 | $1 \times N = 10$ | 16 |
| 19 | AABB | $2^1; 3^3; 9^1; 10^1; -; 5^8; 3^3; 19^1$ | 95 | 17 | 1 | 5 | $1 \times N = 10$ | 55 |
| 19 | AAAA | $1^2; 5^1; -$ | 12 | 1 | 2 | 3 | $1 \times N = 5$ | 2 |
| 19 | AAAA | $3^5; 9^1; -$ | 76 | 18 | 1 | 5 | $4 \times N = 5$ | 36 |
| 29 | AABB | $4^2; -; 1^4; 2^2$ | 8 | 0 | 1 | -- | -- | -- |
| 23 | AABB | $2^4; 4^1; 12^4; -; 1^1; 6^3; 8^2; 2^4$ | 64 | 13 | 1 | 3 | $1 \times N = 12$ | 16 |
| 23 | AABB | $1^1; 4^1; 11^1; 12^4; -; 6^1; 11^3; 9^1; 2^2; 3^1$ | 138 | 32 | 1 | 3 | $1 \times N = 12$ | 90 |
| 23 | AAAA | $1^2; 2^1; -$ | 2 | 0 | 0 | 0 | 1 | 3 | $1 \times N = 3$ | 1 |
| 23 | AAAA | $3^1; -; -$ | 4 | 0 | 0 | 3 | $1 \times N = 3$ | 1 |
| 23 | AAAA | $2^6; -; 3^1; 4^4$ | 16 | 1 | 2 | 3 | $1 \times N = 6$ | 4 |
| 23 | AAAA | $2^3; 3^1; 11^6; -$ | 115 | 24 | 1 | 3 | $3 \times N = 6, 2 \times N = 3$ | 67 |
| 29 | AABB | $2^4; -; 1^2$ | 2 | 0 | 1 | -- | -- | -- |
| 29 | AABB | $2^4; 1^4; -; 1^4; 7^2; 1^2$ | 96 | 25 | 1 | 7 | $1 \times N = 14$ | 12 |
| 29 | AABB | $5^3; 2^2; 14^3; 15^1; -; 1^1; 15^1; 7^2; 2^5$ | 203 | 54 | 1 | 3 | $1 \times N = 15, 1 \times N = 3$ | 128 |
| 29 | AABB | $5^3; 15^1; 7^2; 2^5$ | 203 | 54 | 1 | 3 | $1 \times N = 15, 1 \times N = 3$ | 128 |
| 29 | AAAA | $1^2; 7^2; -$ | 24 | 4 | 4 | 7 | $1 \times N = 7$ | 3 |
| 29 | AAAA | $1^1; 4^3; 7^2; 5^1; 15^1; 29^1; -; -$ | 232 | 54 | 1 | 3 | $1 \times N = 15, 1 \times N = 3$ | 157 |
| 31 | AABB | $1^6; -; 1^6; 2^8; 4^8; 8^8$ | 128 | 37 | 1 | -- | -- | -- |
| 31 | AABB | $1^9; 3^5; 16^5; 15^1; -; 5^9; 15^4; 31^1; 2^8; 4^3; 8^2$ | 248 | 67 | 1 | 3 | $1 \times N = 15, 1 \times N = 3$ | 173 |
| 31 | AAAA | $2^2; -; 1^2$ | 2 | 0 | 0 | 3 | -- | -- |
| 31 | AAAA | $1^2; 2^2; 4^2; -$ | 8 | 0 | 0 | 3 | -- | -- |
| 31 | AAAA | $1^2; 2^2; 4^2; -$ | 32 | 0 | 0 | 3 | -- | -- |
| 31 | AAAA | $3^2; 15^2; 5^2; 4^8; 8^8; -; -$ | 217 | 51 | 1 | 3 | $1 \times N = 15, 1 \times N = 3$ | 142 |
| 31 | AAAA | $2^2; 4^2; -$ | 32 | 0 | 0 | 3 | -- | -- |
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