Asymptotics and analytic modes for the wave equation in similarity coordinates

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Abstract
We consider the radial wave equation in similarity coordinates within the semigroup formalism. It is known that the generator of the semigroup exhibits a continuum of eigenvalues and embedded in this continuum there exists a discrete set of eigenvalues with analytic eigenfunctions. Our results show that, for sufficiently regular data, the long time behaviour of the solution is governed by the analytic eigenfunctions. The same techniques are applied to the linear stability problem for the fundamental self–similar solution $\chi_T$ of the wave equation with a focusing power nonlinearity. Analogous to the free wave equation, we show that the long time behaviour (in similarity coordinates) of linear perturbations around $\chi_T$ is governed by analytic mode solutions. In particular, this yields a rigorous proof for the linear stability of $\chi_T$ with the sharp decay rate for the perturbations.

1 Introduction
1.1 Motivation
The focusing semilinear wave equation
\begin{equation}
\chi_{tt} - \Delta \chi = \chi^p
\end{equation}
for $\chi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, where $p > 1$ is an odd integer, exhibits radial self–similar solutions, i.e., solutions of the form $\chi(t, x) = (T - t)^{-2/(p-1)} f(|x|/(T - t))$ for a function $f : \mathbb{R} \to \mathbb{R}$ and fixed $T > 0$. In fact, the simplest solution of this type, where $f$ is just a constant, can be obtained by neglecting the Laplacian in Eq. (1) and solving the resulting ordinary differential equation in $t$. We refer to this solution as the fundamental self–similar solution and denote it by $\chi_T$. Although self–similar solutions do not have finite energy, one may use them together with smooth cut–off functions and finite speed of propagation to demonstrate blow up for solutions with smooth compactly supported initial data. This observation immediately raises the question how typical such a self–similar blow up is. Does it happen only for the very special initial data constructed by the procedure described above or can it be observed for a larger set of data? Numerical investigations [1] indicate that the latter is true. Actually, there is a much stronger conjecture, namely that the fundamental self–similar solution describes the blow up behaviour for generic large initial data. This conjecture is based on numerical investigations for the radial equation. In these simulations one observes that the future development of sufficiently large initial data converges to the fundamental self–similar solution near the center $r = 0$ [1]. This indicates that $\chi_T$ has to be stable in some sense. We remark that for $p = 3$ there are also rigorous results in this direction (see [2], [3], [5]). In fact, Merle and Zaag have rigorously proved the full nonlinear stability of a more general family of explicit solutions (which includes $\chi_T$) for the corresponding problem in one space dimension and any
To be precise, this is true only for data that do not have compact support since otherwise Huygens’ principle applies. In order to analyze linear stability of the fundamental self-similar solution it is convenient to introduce similarity coordinates \((τ, ρ)\) defined by \(τ := −\log(T−t)\) and \(ρ := \frac{T−t}{r}\). Since convergence is only expected near \(τ = 0\), one requires \(ρ \in (0, 1)\) which corresponds to the interior of the backward lightcone of the spacetime point \((t, r) = (T, 0)\). Transforming Eq. \((1)\) to similarity coordinates, inserting the ansatz \(χ = χ_τ + ϕ\) and linearizing in \(ϕ\) yields a rather nasty equation of the form

\[
ϕ_{ττ} + ϕ_τ + 2ρϕ_{ρρ} − (1 − ρ^2)ϕ_{ρρ} − \frac{1}{ρ}ϕ_{ρ} − pc_0ϕ = 0
\]

where \(c_0 > 0\) is a constant defined by \(χ_T\). The first step in a heuristic stability analysis is to look for mode solutions, i.e., one inserts the ansatz \(ϕ(τ, ρ) = e^{λτ}u(ρ)\). This yields the generalized eigenvalue problem

\[
−(1 − ρ^2)u'' − \frac{1}{ρ}u' + 2λpu' + [λ(1 + λ) − pc_0]u = 0
\]

which has two singular points at \(ρ = 0\) and \(ρ = 1\). A necessary condition for linear stability of \(χ_T\) is the nonexistence of mode solutions with Re\(λ > 0\). However, it is an entirely nontrivial question what kind of solutions of Eq. \((3)\) one should consider as admissible. In other words, it is not clear what boundary conditions one should impose at the singular point \(ρ = 1\). A basic Frobenius analysis shows that around \(ρ = 1\) there exists an analytic solution and a nonanalytic one where the latter behaves as \(1 − ρ^{1−λ}\) for \(ρ → 1\) (we assume noninteger \(λ\) for simplicity). This shows that the nonanalytic solution becomes more and more regular at the backward lightcone as Re\(λ\) decreases. Hence, if Re\(λ\) is sufficiently small, there is no singular solution which can be excluded a priori. Another difficulty we encounter is the fact that, since this is a highly non self-adjoint problem, the nonexistence of unstable modes does not imply linear stability.

The only way to overcome these obstacles is to look for a well-posed initial value formulation for Eq. \((2)\). It turns out that the machinery provided by semigroup theory can be successfully applied here. Very sketchy, one writes Eq. \((2)\) as a first order system of the form

\[
\frac{d}{dτ}\Phi(τ) = LΦ(τ)
\]

where \(L\) is a spatial differential operator which is realized as an unbounded linear operator acting on a Banach space. The formal solution of this equation is \(Φ(τ) = \exp(τL)Φ(0)\) but this does not make sense mathematically since \(L\) is unbounded. With the help of semigroup theory one is able to construct a well-defined one-parameter family \(S(τ)\) of operators such that the solution of Eq. \((4)\) with initial data \(Φ(0)\) is given by \(Φ(τ) = S(τ)Φ(0)\). Such a formulation solves the two problems described above. First, there exists a well-defined notion of spectrum which implicitly yields the correct boundary condition for Eq. \((4)\), and, secondly, one may use abstract results from semigroup theory to obtain growth bounds for the solutions.

### 1.2 The problem of analytic modes

For simplicity one may first develop a semigroup formulation for the free wave equation, i.e., Eq. \((2)\) with \(c_0 = 0\). This problem has recently been considered \([2]\) and we have shown that there exists a semigroup \(S_0(τ)\) that yields the time evolution in energy space, i.e., for very rough data. It should be remarked that this is an interesting result per se, at least from the mathematical point of view, since the semigroup generator is highly non self-adjoint. In fact, it is not even normal and its spectrum has a remarkable structure: It consists (essentially) of a continuum of eigenvalues filling a left half-plane in the set of complex numbers. We review the corresponding results in Sec. \([2]\). A special subset \(\{0, −1, −2, \ldots\}\) of the point spectrum consists of eigenvalues with analytic eigenfunctions. From the point of view of semigroup theory there is no reason to consider these “analytic eigenvalues” as distinguished. However, in numerical evolutions one observes that the asymptotic behaviour (for \(τ → ∞\)) of solutions is exactly described by the analytic eigenvalues and eigenfunctions \([3]\). Therefore, the question is how to explain this behaviour. Note that this is not a mere effect of preservation of regularity. In the abstract approach, preservation

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of regularity is expressed by the fact that domains of powers of the generator \( L_0 \) are invariant under the time evolution, i.e., if \( \Phi(0) \in \mathcal{D}(L_0^k) \) for \( k \in \mathbb{N} \) then \( S_0(\tau)\Phi(0) \in \mathcal{D}(L_0^k) \). But one cannot get rid of "nonanalytic eigenvalues" by prescribing data in \( \mathcal{D}(L_0^k) \) since any eigenvector of \( L_0 \) is by definition also an eigenvector of \( L_0^k \). However, in Sec. 3.1 we show that another class of higher Sobolev spaces, denoted by \( \mathcal{H}^{2k} \), remains invariant under \( S_0 \). A key observation in this respect is a certain commutator property exhibited by the generator \( L_0 \), see Lemma 3.1 below. The spaces \( \mathcal{H}^{2k} \) are suitable to get rid of the continuum eigenvalues and only analytic ones remain. More precise, we show that initial data in \( \mathcal{H}^{2k} \) can be expanded in a sum of the first \( 2k \) analytic eigenfunctions of \( L_0 \) plus a remainder whose time evolution decays faster than the rest. This result shows in particular that the long time behaviour of solutions with smooth initial data is described by the analytic modes as is observed numerically.

1.3 Application to the semilinear wave equation

Numerical studies of Eq. (2) exhibit a very similar behaviour as described above for the free wave equation: The large \( \tau \) behaviour of linear perturbations around \( \psi \) is precisely described by analytic modes, i.e., analytic solutions of Eq. (3). The techniques explained above for the free wave equation carry over to this problem. We obtain the analogous result (see Theorem 4.1 below) which shows that the long time behaviour is indeed given by the analytic modes. In particular, this result yields a rigorous proof for the linear stability of the fundamental self–similar solution of Eq. (1) with the sharp decay rate for the residual spectra by \( \sigma \) continuous embedded in \( Y \).

To improve readability we write vectors as boldface letters and the components are numbered by lower indices, e.g., \( \mathbf{u} = (u_1, u_2)^T \). The notation \( X \hookrightarrow Y \) for two normed vector spaces \( X, Y \) means that \( X \) is continuously embedded in \( Y \). When given an inner product \( (\cdot, \cdot)_X \) on a vector space \( X \) we denote the induced norm by \( \| \cdot \|_X \), i.e., \( \| \cdot \|_X := \sqrt{(\cdot, \cdot)_X} \). The Cartesian product \( X \times Y \) of two vector spaces \( X \) and \( Y \) with inner products \( (\cdot, \cdot)_X \) and \( (\cdot, \cdot)_Y \) is implicitly assumed to be equipped with the inner product \( (u|v)_{X \times Y} := (u_1|v_1)_X + (u_2|v_2)_Y \). For a Banach space \( X \) we denote by \( B(X) \) the space of bounded linear operators on \( X \). For a closed operator \( L : \mathcal{D}(L) \subset X \to X \) we set \( \mathcal{R}_L(\lambda) := (\lambda - L)^{-1} \) whenever the right–hand side exists. The resolvent set of \( L \) is denoted by \( \rho(L) \) and the point, continuous and residual spectra by \( \sigma_p(L), \sigma_c(L) \) and \( \sigma_r(L) \), respectively (see [2] for the precise definitions). Finally, the expression \( A \lesssim B \) means that there exists a \( C > 0 \) such that \( A \leq CB \).

1.4 Notations

To improve readability we write vectors as boldface letters and the components are numbered by lower indices, e.g., \( \mathbf{u} = (u_1, u_2)^T \). The notation \( X \hookrightarrow Y \) for two normed vector spaces \( X, Y \) means that \( X \) is continuously embedded in \( Y \). When given an inner product \( (\cdot, \cdot)_X \) on a vector space \( X \) we denote the induced norm by \( \| \cdot \|_X \), i.e., \( \| \cdot \|_X := \sqrt{(\cdot, \cdot)_X} \). The Cartesian product \( X \times Y \) of two vector spaces \( X \) and \( Y \) with inner products \( (\cdot, \cdot)_X \) and \( (\cdot, \cdot)_Y \) is implicitly assumed to be equipped with the inner product \( (u|v)_{X \times Y} := (u_1|v_1)_X + (u_2|v_2)_Y \). For a Banach space \( X \) we denote by \( B(X) \) the space of bounded linear operators on \( X \). For a closed operator \( L : \mathcal{D}(L) \subset X \to X \) we set \( \mathcal{R}_L(\lambda) := (\lambda - L)^{-1} \) whenever the right–hand side exists. The resolvent set of \( L \) is denoted by \( \rho(L) \) and the point, continuous and residual spectra by \( \sigma_p(L), \sigma_c(L) \) and \( \sigma_r(L) \), respectively (see [2] for the precise definitions). Finally, the expression \( A \lesssim B \) means that there exists a \( C > 0 \) such that \( A \leq CB \).

2 Semigroup formulation in energy space

In this section we review results recently obtained by the author [2] on a semigroup formulation of the free wave equation in similarity coordinates. We define similarity coordinates \( (\tau, \rho) \) as explained in the introduction by \( \tau := -\log(T-t), \rho := \frac{t}{r^2} \) and consider the radial wave equation on \((3+1)\) Minkowski space,

\[ \ddot{\psi} - \frac{\dot{\psi}}{r} - \frac{2}{\rho} \psi = 0. \]

Substituting \( \psi(t, r) := r\tilde{\psi}(t, r) \) yields

\[ \tilde{\psi}_{tt} - \tilde{\psi}_{rr} = 0 \]

with the boundary condition \( \tilde{\psi}(t, 0) = 0 \) for all \( t \). We write this equation as a first order system

\[ \partial_t \Psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_r \Psi \]
where $\Psi := (\psi_1, \psi_2)^T$. Changing to similarity coordinates we obtain

$$\partial_r \Phi = \begin{pmatrix} -\rho & 1 \\ 1 & -\rho \end{pmatrix} \partial_r \Psi$$

(5)

where $\Phi(\tau, \rho) := \Psi(T - e^{-\tau}, \rho e^{-\tau})$.

Let $\mathcal{H} := L^2(0, 1) \times L^2(0, 1)$, $\mathcal{D}(\tilde{L}_0) := \{u \in C^1[0, 1] \times C^1[0, 1] : u_1(0) = 0\}$ and

$$\tilde{L}_0 u(\rho) := \begin{pmatrix} -\rho u_1'(\rho) + u_2'(\rho) \\ u_1'(\rho) - \rho u_2'(\rho) \end{pmatrix}.$$ 

$\tilde{L}_0 : \mathcal{D}(\tilde{L}_0) \subset \mathcal{H} \rightarrow \mathcal{H}$ is a densely defined linear operator on the Hilbert space $\mathcal{H}$. An operator formulation of Eq. (3) is given by

$$\frac{d}{d\tau} \Phi(\tau) = \tilde{L}_0 \Phi(\tau)$$

for a strongly differentiable function $\Phi : [0, \infty) \rightarrow \mathcal{H}$. We have the following result [2].

**Theorem 2.1.** The operator $\tilde{L}_0$ is closable and its closure $\bar{L}_0$ generates a strongly continuous one-parameter semigroup $S_0 : [0, \infty) \rightarrow \mathcal{B}(\mathcal{H})$ satisfying $\|S_0(\tau)\|_{\mathcal{B}(\mathcal{H})} \leq e^{\frac{\rho}{\tau}}$ for all $\tau > 0$.

The spectrum of $\bar{L}_0$ is given by $\sigma_\mu(\bar{L}_0) = \{\lambda \in \mathbb{C} : \mathrm{Re} \lambda < \frac{1}{2}\}$, $\sigma_\sigma(\bar{L}_0) = \{\lambda \in \mathbb{C} : \mathrm{Re} \lambda = 1/2\}$, $\sigma_\sigma(\bar{L}_0) = \emptyset$.

### 3 Semigroup formulation for more regular data

#### 3.1 Invariance of higher Sobolev spaces

We show that a certain class of higher Sobolev spaces is invariant under the semigroup $S_0$. For $k \in \mathbb{N}_0$ we set

$$\mathcal{H}^{2k} := \{u \in H^{2k}(0, 1) \times H^{2k}(0, 1) : u^{(2j)}(0) = u^{(2j+1)}(0) = 0, j \in \mathbb{N}_0, j < k\}$$

and define an operator $D^2 : \mathcal{H} \rightarrow \mathcal{H}$ by $D^2 u := u''$. We have $\mathcal{H} = \mathcal{H}^0$ and equip $\mathcal{H}^{2k}$ with the inner product $(u|v)_{\mathcal{H}^{2k}} := (u|v)_{\mathcal{H}} + (D^{2k} u|D^{2k} v)_{\mathcal{H}}$. The following lemma summarizes elementary properties.

**Lemma 3.1.**

1. $\mathcal{H}^{2k}$ is a Hilbert space.
2. $\mathcal{H}^{2(k+1)}$ is a dense subspace of $\mathcal{H}^{2k}$ and the inclusion $\mathcal{H}^{2(k+1)} \subset \mathcal{H}^{2k}$ is continuous.
3. The operator $D^2$ satisfies $D^2 \mathcal{H}^{2(k+1)} \subset \mathcal{H}^{2k}$.
4. We have $\mathcal{H}^{2(k+1)} \subset \mathcal{D}(\bar{L}_0)$ and $\bar{L}_0 \mathcal{H}^{2(k+1)} \subset \mathcal{H}^{2k}$.
5. $D^2$ and $\bar{L}_0$ satisfy the commutator relation $D^2 \bar{L}_0 u = \bar{L}_0 D^2 u - 2D^2 u$ for all $u \in \mathcal{H}^4$.

**Proof.** The proof is straightforward by inserting the definitions and using well known properties of Sobolev spaces.

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As usual we define the part $L_{0,k}$ of $\bar{L}_0$ in $\mathcal{H}^{2k}$ by $\mathcal{D}(L_{0,k}) := \{u \in \mathcal{D}(\bar{L}_0) \cap \mathcal{H}^{2k} : L_0 u \in \mathcal{H}^{2k}\}$ and $L_{0,k} u := L_0 u$. We show that $L_{0,k}$ generates a semigroup on $\mathcal{H}^{2k}$.

**Proposition 3.1.** The operator $L_{0,k}$ generates a strongly continuous one-parameter semigroup $S_{0,k} : [0, \infty) \rightarrow \mathcal{B}(\mathcal{H}^{2k})$ satisfying $\|S_{0,k}\|_{\mathcal{B}(\mathcal{H}^{2k})} \leq e^{\frac{\rho}{k}}$.

**Proof.** By Lemma 3.1 we immediately observe that $L_{0,k}$ is densely defined since $\mathcal{H}^{2(k+1)} \subset \mathcal{D}(L_{0,k})$.

Let $(u_j) \subset \mathcal{D}(L_{0,k})$ with $u_j \rightarrow u$ and $L_{0,k} u_j \rightarrow f$ both in $\mathcal{H}^{2k}$. Since $\mathcal{H}^{2k} \subset \mathcal{H}$ (Lemma 3.1) this implies $u_j \rightarrow u$, $L_0 u_j \rightarrow f$ in $\mathcal{H}$ and by the closedness of $L_0$ we conclude $u \in \mathcal{D}(\bar{L}_0) \cap \mathcal{H}^{2k}$ and $L_0 u = f \in \mathcal{H}^{2k}$ which shows $u \in \mathcal{D}(L_{0,k})$ and we have proved that $L_{0,k}$ is closed.

By using the commutator relation from Lemma 3.1 and integration by parts (cf. [2]) we obtain

$$\mathrm{Re}(L_{0,k} u | u)_{\mathcal{H}^{2k}} = \mathrm{Re}\left( (L_0 u | u)_{\mathcal{H}} + (L_0 D^{2k} u | D^{2k} u)_{\mathcal{H}} - 2k \|D^{2k} u\|_{\mathcal{H}}^2 \right) \leq \frac{1}{2} \|u\|_{\mathcal{H}^{2k}}^2$$

for all $u \in \mathcal{H}^{2(k+1)}$ and by a density argument this estimate holds in fact for all $u \in \mathcal{D}(L_{0,k})$. 

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Let \( f \in \mathcal{H}^{2k} \cap C^{\infty}(0,1)^2 \) and define \( F(\rho) := f_1(\rho) + \rho f_2(\rho) + \int_0^\rho f_3(\xi)d\xi \), \( u_2(\rho) := \frac{1}{\rho^k} \int_0^1 F(\xi)d\xi \) and \( u_1(\rho) := \rho u_2(\rho) - \int_0^\rho f_2(\xi)d\xi \). Then the Taylor series expansion for \( u_1 \) around \( \rho = 0 \) up to order \( 2k - 1 \) contains only odd powers of \( \rho \) whereas the analogous series for \( u_2 \) up to order \( 2k \) contains only even powers of \( \rho \). This shows that \( u \) satisfies the appropriate boundary conditions at \( \rho = 0 \) and we conclude that \( u \in \mathcal{H}^{2k} \cap D(L_0) \). Furthermore, a direct computation yields \( (1 - L_0)u = f \) which shows that \( u \in D(L_{0.0}) \) and \( 1 - L_{0.0} \) has dense range.

Invoking the Lumer–Phillips Theorem (see e.g. [9], p. 56, Theorem 4.2.6) finishes the proof.

Based on this result we are able to conclude the invariance of \( \mathcal{H}^{2k} \) under the semigroup \( S_0 \).

**Lemma 3.2.** The space \( \mathcal{H}^{2k} \) is \( L_0 \)-admissible, i.e., it is an invariant subspace of \( S_0(\tau) \), \( \tau > 0 \), and the restriction of \( S_0(\tau) \) to \( \mathcal{H}^{2k} \) is a strongly continuous semigroup on \( \mathcal{H}^{2k} \) satisfying \( \|S_0(\tau)u\|_{\mathcal{H}^{2k}} \leq e^{\frac{\tau}{2}}\|u\|_{\mathcal{H}^{2k}} \) for all \( u \in \mathcal{H}^{2k} \) and \( \tau > 0 \).

**Proof.** Let \( f \in \mathcal{H}^{2k} \) and \( \lambda \in \rho(L_0) \). Proposition 3.3 implies that there exists a \( u \in D(L_{0.0}) \) such that \( (\lambda - L_{0.0})u = f \). However, since \( L_{0.0} \subset L_0 \), we have \( (\lambda - L_0)u = f \) and thus, \( R_{L_0}(\lambda)f = u \in \mathcal{H}^{2k} \). This shows that \( R_{L_0}(\lambda)\mathcal{H}^{2k} \subset \mathcal{H}^{2k} \). By Lemma 3.1, the embedding \( \mathcal{H}^{2k} \subset \mathcal{H} \) is continuous and therefore, the claim follows from Proposition 3.1 and the theorem on admissible spaces (see e.g. [11], p. 123, Theorem 5.5).

### 3.2 Decoosition

We improve the growth estimate \( \|S_0(\tau)\|_{\mathcal{H}^{2k} \mathcal{B}(\mathcal{H}^{2k})} \leq e^{\frac{\tau}{2}} \) by a decomposition of the initial data space \( \mathcal{H}^{2k} \). Let \( N \) denote the set of all \( u \in \mathcal{H}^{2k} \) such that \( D^{2k}u = 0 \). \( N \) is a finite dimensional (and hence closed) subspace of \( \mathcal{H}^{2k} \). Thus, there exists a bounded orthogonal projection \( P \in \mathcal{B}(\mathcal{H}^{2k}) \) associated to it. We immediately obtain \( \|S_0(\tau)u\|_{\mathcal{H}^{2k}} \leq \|S_0(\tau)Pu\|_{\mathcal{H}^{2k}} + \|S_0(\tau)(I - P)u\|_{\mathcal{H}^{2k}} \) for any \( u \in \mathcal{H}^{2k} \). In order to keep things simple, we analyse the two parts \( \|S_0(\tau)Pu\|_{\mathcal{H}^{2k}} \) and \( \|S_0(\tau)(I - P)u\|_{\mathcal{H}^{2k}} \) separately.

**Lemma 3.3.** The subspace \( N \) is spanned by \( 2k \) analytic functions \( u(\cdot, \lambda_j) \), \( j = 1, 2, \ldots, 2k \) where each \( u(\cdot, \lambda_j) \) is an eigenfunction of \( L_0 \) with eigenvalue \( \lambda_j = -j + 1 \).

**Proof.** For \( j = 1, 2, \ldots, 2k \) set \( u(\rho, \lambda_j) := (1 - \rho)^{1-\lambda_j} - (1 + \rho)^{1-\lambda_j} \), \( u_1(\rho, \lambda_j) := \rho u'(\rho, \lambda_j) + (\lambda_j - 1)u(\rho, \lambda_j) \) and \( u_2(\rho, \lambda_j) := u'(\rho, \lambda_j) \). Then \( u(\cdot, \lambda_j) \) is an eigenfunction of \( L_0 \) with eigenvalue \( \lambda_j \) as a straightforward computation shows. Observe that \( u(\cdot, \lambda_j) \) is an odd function (binomial theorem, all even powers cancel) and therefore, \( u_1(\cdot, \lambda_j) \) is odd and \( u_2(\cdot, \lambda_j) \) is even. This shows that \( u(\cdot, \lambda_j) \in \mathcal{H}^{2k} \) for each \( j \). Note further that \( u_1(\cdot, \lambda_j) \) and \( u_2(\cdot, \lambda_j) \) are polynomials of degree strictly smaller than \( 2k \). Thus, \( D^{2k}u(\cdot, \lambda_j) = 0 \) for each \( j \) and we conclude \( u(\cdot, \lambda_j) \in N \). A function in \( \mathcal{H}^{2k} \) satisfies \( 2k \) boundary conditions and thus, the space \( N \), which consists of \( u \in \mathcal{H}^{2k} \) with \( D^{2k}u = 0 \), is \( 2k \)-dimensional. However, the \( 2k \) eigenfunctions \( u(\cdot, \lambda_j) \), \( j = 1, 2, \ldots, 2k \) belong to \( N \) and they are linearly independent.

**Lemma 3.2** shows that we can calculate the time evolution \( S_0(\tau)Pu \) explicitly: Since \( Pu \in \mathcal{N} \) we can expand it in terms of analytic eigenfunctions \( u(\cdot, \lambda_j) \) of \( L_0 \),

\[
P{u} = \sum_{j=1}^{2k} c_j u(\cdot, \lambda_j),
\]

where \( \lambda_j = -j + 1 \) and \( c_1, c_2, \ldots, c_{2k} \in \mathbb{C} \) are the expansion coefficients. Hence, we obtain

\[
S_0(\tau)Pu = \sum_{j=1}^{2k} c_j e^{\lambda_j \tau} u(\cdot, \lambda_j).
\]

In order to estimate \( S_0(\tau)(I - P)u \) we need a few preparations. First, we state the following general observation.

**Lemma 3.4.** Let \( X,Y \) be Hilbert spaces with \( X \subset Y \) and \( A : X \to Y \) a linear bounded surjective mapping. Then there exists a constant \( c > 0 \) such that \( \|Au\|_Y \geq c\|u\|_X \) for all \( u \in (\ker A)^\perp \).
Proof. By the boundedness of $A$ we conclude that $\ker A$ is closed and thus, we have the decomposition $X = \ker A \oplus (\ker A)^\perp$. Hence, the mapping $A|_{(\ker A)^\perp}$ is bijective. The closed graph theorem implies that its inverse is bounded as well and thus, there exists a $c > 0$ such that $\|Au\|_Y \geq c\|u\|_X$ for all $u \in (\ker A)^\perp$.

Now we are able to prove the desired estimate for $S_0(\tau)(I - P)u$.

\begin{proposition}
1. The subspace $\mathcal{N}^\perp \subset \mathcal{H}^{2k}$ is invariant under the semigroup $S_0$.

2. The mapping $u \mapsto \|D^{2k}u\|_H$ defines a norm on $\mathcal{N}^\perp$ which is equivalent to $\|\cdot\|_{\mathcal{N}^{2k}}$.

3. For $u \in \mathcal{N}^\perp$ we have the estimate $\|S_0(\tau)u\|_{\mathcal{N}^{2k}} \lesssim e^{(\frac{1}{2} - 2k)\tau} \|u\|_{\mathcal{N}^{2k}}$ for all $\tau > 0$.
\end{proposition}

\begin{proof}
1. Let $f \in \mathcal{N}^\perp$ (orthogonal complement in $\mathcal{H}^{2k}$), $f \neq 0$, $\lambda \in \rho(L_0)$ and set $u := R_{L_0}(\lambda)f$. Then we have $u \in \mathcal{H}^{2k}$ (cf. proof of Lemma 3.2). Suppose $u \notin \mathcal{N}^\perp$, i.e., $u \notin \mathcal{N}$. Then the commutator relation from Lemma 5.1 yields $D^{2k}f = D^{2k}(\lambda - L_0)u = L_0D^{2k}u = 0$ which implies $f \in \mathcal{N}$ and we infer $f = 0$, a contradiction. Thus, we have $R_{L_0}(\lambda)\mathcal{N}^\perp \subset \mathcal{N}^\perp$. Since $\mathcal{N}^\perp$ is closed in $\mathcal{H}^{2k}$ and $S_0(\tau)|_{\mathcal{N}^{2k}}$ defines a semigroup on $\mathcal{H}^{2k}$ by Lemma 3.2 the claim follows from [11], p. 121, Theorem 5.1.

2. The mapping $\mathcal{D}^{2k} : \mathcal{H}^{2k} \to \mathcal{H}$ is linear, bounded and surjective. According to Lemma 5.1 we have
\[
\|u\|_{\mathcal{N}^{2k}} \lesssim \|D^{2k}u\|_H
\]
for all $u \in \mathcal{N}^\perp = (\ker D^{2k})^\perp$. Trivially, we have $\|D^{2k}u\|_H \leq \|u\|_{\mathcal{N}^{2k}}$.

3. We equip $\mathcal{N}^\perp$ with the inner product $(u|v)_{\mathcal{N}^\perp} := (D^{2k}u|D^{2k}v)_H$. Then $\mathcal{N}^\perp$ is a Hilbert space and $S_0(\tau)|_{\mathcal{N}^\perp}$ defines a semigroup on $\mathcal{N}^\perp$ by assertions 1 and 2 from above. The generator of this semigroup is $L_0|_{\mathcal{N}^\perp}$, the part of $L_0$ in $\mathcal{N}^\perp$, and it satisfies
\[
\Re(L_0|_{\mathcal{N}^\perp} u|v)_{\mathcal{N}^\perp} = \Re(L_0D^{2k}u|D^{2k}v)_H - 2k\|u\|^2_{\mathcal{N}^\perp} \leq \left(\frac{1}{2} - 2k\right) \|u\|^2_{\mathcal{N}^\perp}
\]
for $u \in \mathcal{D}(L_0|_{\mathcal{N}^\perp})$ where we have used the commutator relation from Lemma 5.1 iteratively. This estimate implies $\|S_0(\tau)u\|_{\mathcal{N}^\perp} \lesssim e^{(\frac{1}{2} - 2k)\tau} \|u\|_{\mathcal{N}^\perp}$. However, by assertion 2 above, the norm $\|\cdot\|_{\mathcal{N}^\perp}$ is equivalent to $\|\cdot\|_{\mathcal{N}^{2k}}$ and we arrive at the claim.
\end{proof}

Proposition 3.2 implies that $\|S_0(\tau)(I - P)u\|_{\mathcal{N}^{2k}} \lesssim e^{(\frac{1}{2} - 2k)\tau} \|u\|_{\mathcal{N}^{2k}}$ since $(I - P)u \in \mathcal{N}^\perp$ and this completes the investigation of the free wave equation in similarity coordinates. We end up with the result that the time evolution can be estimated as
\[
\|S_0(\tau)u\|_{\mathcal{N}^{2k}} \lesssim \sum_{j=1}^{2k} c_j e^{\lambda_j \tau} + e^{(\frac{1}{2} - 2k)\tau} \|u\|_{\mathcal{N}^{2k}}
\]
for initial data $u \in \mathcal{H}^{2k}$ and $\lambda_j = -j + 1$ if we assume the eigenfunctions $u(\cdot, \lambda_j)$ to be normalized. This completely answers the question of the role of the analytic modes. If the initial data are sufficiently regular then the long time behaviour of the solution is dominated by the first analytic eigenmodes. This is exactly what is observed numerically. We also emphasize that this result implies a certain completeness property of the analytic modes. Sufficiently regular initial data can be expanded in a sum of analytic modes plus a remainder which decays faster. We summarize the results in a theorem.

\begin{theorem}
Let $u \in \mathcal{H}^{2k}$. Then there exist constants $c_1, \ldots, c_{2k} \in \mathbb{C}$ and a function $g \in \mathcal{H}^{2k}$ such that
\[
u = \sum_{j=1}^{2k} c_j u(\cdot, \lambda_j) + g
\]
and $\|S_0(\tau)g\|_{\mathcal{N}^{2k}} \leq e^{(\frac{1}{2} - 2k)\tau} \|g\|_{\mathcal{N}^{2k}}$ where $u(\cdot, \lambda_j)$ are normalized analytic eigenfunctions of $L_0$ with eigenvalues $\lambda_j = -j + 1$. In particular, we have
\[
\|S_0(\tau)u\|_{\mathcal{N}^{2k}} \lesssim \sum_{j=1}^{2k} c_j e^{\lambda_j \tau} + e^{(\frac{1}{2} - 2k)\tau} \|u\|_{\mathcal{N}^{2k}}
\]
for all $\tau > 0$.
\end{theorem}
4 Application to the semilinear wave equation

We apply the previously obtained results to the linear stability problem for the fundamental self–similar solution of Eq. 1. To this end we construct a semigroup $S$ acting on $\mathcal{H}^{2k}$ that describes the time evolution of linear perturbations of the fundamental self–similar solution. Throughout this section we restrict ourselves to $k \in \mathbb{N}$ since the case $k = 0$ has already been investigated in 2.

4.1 Operator formulation

Let $L' \in \mathcal{B}(\mathcal{H})$ be defined by

$$L'\mathbf{u}(\rho) := \left( \begin{array}{c} p_{c0} \int_0^\rho u_2(\xi)d\xi \\ 0 \end{array} \right)$$

where the constant $c_0$ is given by the fundamental self–similar solution $\chi_T$. Note that $L'$ leaves $\mathcal{H}^{2k}$ invariant as an odd–even argument easily shows. Furthermore, $D^{2k}$ and $L'$ commute, i.e., $D^{2k}L'u = L'D^{2k}\mathbf{u}$ for all $\mathbf{u} \in \mathcal{H}^{2k}$.

An operator formulation for the linear stability problem Eq. (2) is given by

$$\frac{d}{d\tau} \Phi(\tau) = L\Phi(\tau)$$

where $L := L_0 + L'$ and the Bounded Perturbation Theorem (see e.g. [3]) immediately yields the existence of a semigroup $S : [0, \infty) \to \mathcal{B}(\mathcal{H})$ satisfying $\|S(\tau)\mathbf{u}\|_{\mathcal{H}^{2k}} \leq e^{(\frac{1}{2} + p_{c0})\tau}\|\mathbf{u}\|_{\mathcal{H}^{2k}}$ for all $\mathbf{u} \in \mathcal{H}^{2k}$ and $\tau > 0$.

4.2 Invariant subspaces

Lemma 4.1. The space $\mathcal{H}^{2k}$ is $L$–admissible, i.e., it is an invariant subspace of $S(\tau)$, $\tau > 0$, and the restriction of $S(\tau)$ to $\mathcal{H}^{2k}$ is a strongly continuous semigroup on $\mathcal{H}^{2k}$ satisfying $\|S(\tau)\mathbf{u}\|_{\mathcal{H}^{2k}} \leq e^{(\frac{1}{2} + p_{c0})\tau}\|\mathbf{u}\|_{\mathcal{H}^{2k}}$ for all $\mathbf{u} \in \mathcal{H}^{2k}$ and $\tau > 0$.

Proof. The part of $L$ in $\mathcal{H}^{2k}$ is given by $L_{0,k} + L'|_{\mathcal{H}^{2k}}$ since $L'\mathcal{H}^{2k} \subset \mathcal{H}^{2k}$. Thus, Proposition 3.1 and the Bounded Perturbation Theorem show that the part of $L$ in $\mathcal{H}^{2k}$ generates a strongly continuous one–parameter semigroup on $\mathcal{H}^{2k}$ with the same growth bound as $S$. Applying the same argument as in the proof of Lemma 3.2 yields the claim.

As before, we denote by $\mathcal{N}$ the space of all $\mathbf{u} \in \mathcal{H}^{2k}$ such that $D^{2k}\mathbf{u} = 0$ and by $\mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}^{2k}$.

Proposition 4.1. The subspace $\mathcal{N}^\perp$ is invariant under the semigroup $S$. Furthermore, we have the estimate

$$\|S(\tau)\mathbf{u}\|_{\mathcal{H}^{2k}} \lesssim e^{(\frac{1}{2} + p_{c0} - 2k)\tau}\|\mathbf{u}\|_{\mathcal{H}^{2k}}$$

for all $\mathbf{u} \in \mathcal{N}^\perp$ and $\tau > 0$.

Proof. Let $\lambda > \frac{1}{2} + p_{c0}$. Then $R_L(\lambda)\mathcal{H}^{2k} \subset \mathcal{H}^{2k}$ since $\mathcal{H}^{2k}$ is $L$–admissible by Lemma 1.1 (cf. [11], p. 123, Theorem 5.5). Set $\mathbf{u} := R_L(\lambda)f$ for an $f \in \mathcal{N}^\perp$, $f \neq 0$, and suppose $\mathbf{u} \notin \mathcal{N}^\perp$. Then $\mathbf{u} \in \mathcal{N}$ and the commutator relations show $D^{2k}f = D^{2k}(\lambda - L)\mathbf{u} = 0$ which implies $f \in \mathcal{N}$ and we infer $f = 0$, a contradiction. Hence, $R_L(\lambda)\mathcal{N}^\perp \subset \mathcal{N}^\perp$ and, since $\mathcal{N}^\perp$ is closed in $\mathcal{H}^{2k}$, the claim follows from [11], p. 121, Theorem 5.1.

We define an inner product on $\mathcal{N}^\perp$ by $(\mathbf{u} | \mathbf{v})_{\mathcal{N}^\perp} := (D^{2k}\mathbf{u} | D^{2k}\mathbf{v})_{\mathcal{H}}$. The induced norm $\| \cdot \|_{\mathcal{N}^\perp}$ is equivalent to $\| \cdot \|_{\mathcal{H}^{2k}}$ on $\mathcal{N}^\perp$ by Proposition 3.2 and thus, $\mathcal{N}^\perp$ equipped with $(\cdot | \cdot)_{\mathcal{N}^\perp}$ is a Hilbert space since $\mathcal{N}^\perp$ is closed in $\mathcal{H}^{2k}$. The restriction $S(\tau)_{\mathcal{N}^\perp}$ is a semigroup on $\mathcal{N}^\perp$ and its generator is the part of $L$ in $\mathcal{N}^\perp$, denoted by $L_{\mathcal{N}^\perp}$. The generator satisfies

$$\text{Re}(L_{\mathcal{N}^\perp}\mathbf{u} | \mathbf{u})_{\mathcal{N}^\perp} =$$

$$\text{Re} \left( (L_0D^{2k}\mathbf{u} | D^{2k}\mathbf{u})_{\mathcal{H}} + (L'D^{2k}\mathbf{u} | D^{2k}\mathbf{u})_{\mathcal{H}} \right) - 2k(D^{2k}\mathbf{u} | D^{2k}\mathbf{u})_{\mathcal{H}}$$

$$\leq \left( \frac{1}{2} + p_{c0} - 2k \right) \|\mathbf{u}\|_{\mathcal{N}^\perp}^2$$
for all \( u \in \mathcal{D}(L_{N^2}) \) where we have used the commutator relation from Lemma 3.1 and the fact that \( L' \) and \( D^{2k} \) commute, as already remarked. This, however, implies the growth estimate \( ||S(\tau)u||_{N^2} \leq e^{(\frac{1}{2} + p_c - 2k)\tau} ||u||_{N^2} \) for all \( u \in N^2 \) and \( \tau > 0 \) and by the equivalence of the norms \( || \cdot ||_{N^2} \) and \( || \cdot ||_{N^2} \) (Proposition 4.1) we arrive at the claim.

Note that the growth estimate given in Proposition 4.1 is certainly not optimal, however, we do not make any attempts to improve it since we can make \( k \) arbitrarily large.

### 4.3 Analytic modes

The point spectrum of \( L \) can be calculated by solving a second order ODE. To this end we recall the definition of the operator \( T(\rho) \) from \( 2 \). The domain of \( T(\lambda) \) is \( \mathcal{D}(T(\lambda)) := \{u \in H^1(0,1) : u \in H^2_{0,\text{loc}}(0,1), u(0) = 0\} \) and \( T(\lambda)u := t(\lambda)u \) where

\[
 t(\lambda)u(\rho) := -(1 - \rho^2)u''(\rho) + 2\lambda pu'(\rho) + [\lambda(\lambda - 1) - p\mathfrak{c}]u(\rho).
\]

Then \( \lambda \in \sigma_p(L) \) if and only if \( \dim \ker T(\lambda) = 1 \) (see \( 2 \), Proposition 2).

Observe that \( t(\lambda)u = 0 \) corresponds exactly to Eq. 3 in view of the substitution \( u(\rho) \mapsto pu(\rho) \).

Thus, as explained in the introduction, the semigroup approach implicitly yields the correct boundary condition for the generalized eigenvalue problem Eq. 3 that defines mode solutions.

The general solution of \( t(\lambda)u = 0 \) can be given in terms of Legendre functions and hence, the point spectrum can be calculated explicitly. In \( 1 \) and \( 3 \) it has been shown \(^2\) that \( t(\lambda)u = 0 \) has a nontrivial analytic solution if and only if \( \lambda = \lambda_j^\pm \) where \( \lambda_j^+ := 1 + \frac{2}{p-1} - 2j \) and \( \lambda_j^- := -\frac{2\mathfrak{c}}{p-1} - 2j \) for a \( j \in \mathbb{N}_0 \). We will refer to \( \lambda_j^\pm \) as analytic eigenvalues. Moreover, it follows from \( 1 \) that the analytic functions \( u(\cdot, \lambda_j^\pm) \) satisfying \( t(\lambda_j^\pm)u(\cdot, \lambda_j^\pm) = 0 \) are in fact odd \(^3\) polynomials of degree \( 2j + 1 \).

Very similar to the free wave equation, the subspace \( N \) is again spanned by analytic eigenfunctions of \( L \).

**Lemma 4.2.** The subspace \( N \) is spanned by \( 2k \) analytic functions \( u(\cdot, \lambda_j^\pm) \) for \( j = 0, 1, 2, \ldots, k - 1 \) where each \( u(\cdot, \lambda_j^\pm) \) is an eigenfunction of \( L \) with eigenvalue \( \lambda_j^\pm \), \( \lambda_j^+ = 1 + \frac{2}{p-1} - 2j \) and \( \lambda_j^- = -\frac{2\mathfrak{c}}{p-1} - 2j \).

**Proof.** The space \( N \) is \( 2k \)-dimensional as already remarked in the proof of Lemma 3.3. Let \( u(\cdot, \lambda_j^\pm) \neq 0 \) satisfy \( t(\lambda_j^\pm)u(\cdot, \lambda_j^\pm) = 0 \) for \( j = 0, 1, \ldots, k - 1 \) and define \( u(\cdot, \lambda_j^\pm) \) by \( u_1(\rho, \lambda_j^\pm) := pu(\rho, \lambda_j^\pm) + (\lambda_j^\pm - 1)u(\rho, \lambda_j^\pm) \) and \( u_2(\rho, \lambda_j^\pm) := u'(\rho, \lambda_j^\pm) \). Then \( u(\cdot, \lambda_j^\pm) \in \mathcal{H}^{2k} \cap \mathcal{D}(L) \) since \( u_1(\cdot, \lambda_j^\pm) \) is an odd polynomial and \( u_2(\cdot, \lambda_j^\pm) \) is an even polynomial. A straightforward computation shows \( Lu(\cdot, \lambda_j^\pm) = \lambda_j^\pm u(\cdot, \lambda_j^\pm) \) and thus, \( u(\cdot, \lambda_j^\pm) \) is an eigenfunction of \( L \) with eigenvalue \( \lambda_j^\pm \). The functions \( u_1(\cdot, \lambda_j^\pm) \) and \( u_2(\cdot, \lambda_j^\pm) \) are polynomials of degree strictly smaller than \( 2k \) and therefore, \( \mathcal{D}^{2k} u(\cdot, \lambda_j^\pm) = 0 \) which shows that \( u(\cdot, \lambda_j^\pm) \in N \) for all \( j = 0, 1, 2, \ldots, k - 1 \). However, since the \( u(\cdot, \lambda_j^\pm) \) are eigenfunctions with different eigenvalues they are linearly independent and they form a set of \( 2k \) linearly independent functions in the \( 2k \)-dimensional space \( N \).

### 4.4 The time evolution of linear perturbations

We have collected all the necessary preliminaries to conclude the analogous result to Theorem 4.1.

**Theorem 4.1.** Let \( u \in \mathcal{H}^{2k} \). Then there exist \( 2k \) constants \( c^*_0, \ldots, c^*_{k-1} \in \mathbb{C} \) and a function \( g \in \mathcal{H}^{2k} \) such that

\[
 u = \sum_{j=0}^{k-1} \left( c^*_j u(\cdot, \lambda_j^+) + c^*_j u(\cdot, \lambda_j^-) \right) + g
\]

and \( ||S(\tau)g||_{\mathcal{H}^{2k}} \leq e^{(\frac{1}{2} + p_c - 2k)\tau} ||g||_{\mathcal{H}^{2k}} \) where \( u(\cdot, \lambda_j^\pm) \) are normalized analytic eigenfunctions of \( L \) with eigenvalues \( \lambda_j^\pm = 1 + \frac{2}{p-1} - 2j \) and \( \lambda_j^- = -\frac{2\mathfrak{c}}{p-1} - 2j \). In particular, we have

\[
 ||S(\tau)u||_{\mathcal{H}^{2k}} \leq \sum_{j=0}^{k-1} \left( c^*_j e^{\lambda_j^\pm \tau} + c^*_j e^{\lambda_j^- \tau} \right) + e^{(\frac{1}{2} + p_c - 2k)\tau} ||u||_{\mathcal{H}^{2k}}
\]

\(^2\)Note that in our convention the whole spectrum is shifted to the right by \( \frac{2}{p-1} \) compared to \( 1 \).

\(^3\)In \( 1 \) the corresponding analytic "eigenfunctions" are even polynomials of degree \( 2j \) but, according to our convention, one has to multiply them by \( \rho \).
for all \( \tau > 0 \).

By making \( k \) sufficiently large we infer that the long time behaviour of smooth perturbations is governed by the analytic modes and this is exactly what is observed numerically. Furthermore, the largest eigenvalue \( \lambda^+_{0} = 1 + \frac{p-1}{2} \) is known to emerge from the time translation symmetry of the original problem (cf. [2]). This apparent instability is merely an effect of the similarity coordinates and it is therefore called the gauge instability. Thus, for studying the question of linear stability we only allow perturbations with \( c^+_0 = 0 \) (notation as in Theorem 4.1), i.e., perturbations such that the gauge instability is not present. Theorem 4.1 shows that the time evolution of sufficiently regular perturbations \( u \in \mathcal{H}^{2k} \) with \( c^+_0 = 0 \) decays as \( \| S(\tau)u \|_{\mathcal{H}^{2k}} \lesssim e^{-\frac{p-1}{2}\tau} \| u \|_{\mathcal{H}^{2k}} \) for \( \tau \to \infty \) and this estimate is clearly sharp. Hence, the decay is exactly described by the largest analytic eigenvalue apart from the gauge instability. We conclude that the fundamental self–similar solution for the wave equation with a focusing power nonlinearity is linearly stable.

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