Functional Pearl: The Distributive $\lambda$-Calculus

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Abstract

We introduce a simple extension of the $\lambda$-calculus with pairs—called the distributive $\lambda$-calculus—obtained by adding a computational interpretation of the valid distributivity isomorphism $A \Rightarrow (B \land C) \equiv (A \Rightarrow B) \land (A \Rightarrow C)$ of simple types. We study the calculus both as an untyped and as a simply typed setting. Key features of the untyped calculus are confluence, the absence of clashes of constructs, that is, evaluation never gets stuck, and a leftmost-outermost normalization theorem, obtained with straightforward proofs. With respect to simple types, we show that the new rules satisfy subject reduction if types are considered up to the distributivity isomorphism. The main result is strong normalization for simple types up to distributivity. The proof is a smooth variation over the one for the $\lambda$-calculus with pairs and simple types.

Keywords  $\lambda$-calculus type isomorphisms rewriting normalization

1 Introduction

The topic of this paper is an extension of the $\lambda$-calculus with pairs, deemed the distributive $\lambda$-calculus, obtained by adding a natural computational interpretation of the distributivity isomorphism of simple types:

$$A \Rightarrow (B \land C) \equiv (A \Rightarrow B) \land (A \Rightarrow C) \quad (1)$$

Namely, one extends the calculus with the following commutation rules:

$$\langle t, s \rangle u \rightarrow \langle tu, su \rangle \quad \pi_i(\lambda x.t) \rightarrow \lambda x.\pi_i t \quad i = 1, 2$$

The aim of this paper is showing that the distributive $\lambda$-calculus is a natural system, and contributions are in both the typed and untyped settings.

We study the untyped setting to show that our calculus makes perfect sense also without types. This is to contrast with System I, another calculus providing computational interpretations of type isomorphisms recently introduced by Díaz-Caro and Dowek [5], that does not admit an untyped version—the relationship between the two is discussed below.
**Typing Up to Distributivity and Subject Reduction.** At the typed level, the key point is that simple types are here considered *up to distributivity*. In this way, the apparently ad-hoc new rules do satisfy the subject reduction property.

Consider for instance $\pi_1(\lambda x.t)$: working up to the distributivity isomorphism—so that isomorphic types type the same terms—the subterm $\lambda x.t$ may now have both the arrow type $A \Rightarrow (B \land C)$ and the conjunctive type $(A \Rightarrow B) \land (A \Rightarrow C)$, so that $\pi_1(\lambda x.t)$ can be typed with $A \Rightarrow B$. Distributivity also allows for the type to be preserved—that is, subject reduction holds. According to the arrow type, indeed, the body $t$ of the abstraction has type $B \land C$ and thus the reduct of the commutation rule $\pi_1(\lambda x.t) \rightarrow \lambda x.\pi_1 t$ can also be typed with $A \Rightarrow B$. The other commutation rule can be typed similarly.

**Overview of the Paper.** For the untyped setting, we show that the distributive $\lambda$-calculus is confluent, its closed normal forms are values, and it has a leftmost-outermost normalization theorem, exactly as for the $\lambda$-calculus (without pairs).

With respect to types, we show subject reduction and strong normalization of the distributive $\lambda$-calculus with simple types up to distributivity.

**The Pearl.** The proofs in the paper are remarkably smooth. The properties for the untyped calculus are immediate. Confluence follows by the fact that the calculus is an orthogonal higher-order rewriting system [1,9,10]. The leftmost-outermost normalization theorem, similarly, follows by an abstract result by van Ramsdonk [12], because the calculus verifies two additional properties of orthogonal higher-order rewriting system from which leftmost-outermost normalization follows. Finally, the fact that closed normal forms are values—what we call *progress*—is obtained via a straightforward induction.

For the typed setting, the given argument for subject reduction goes smoothly through. The main result of the paper is that the simply typed distributive $\lambda$-calculus is strongly normalizing. The proof follows Tait’s reducibility method. In particular, the interpretation of types is the same at work for the $\lambda$-calculus with pairs and projections (that is, without distributive rules). The key point is to prove that the two sides of the distributivity isomorphism have the same interpretation. This can be proved with two easy lemmas. Everything else is as in the case without distributive rules.

**Type Isomorphisms and System I.** As shown by Bruce, Di Cosmo and Longo [4] the isomorphisms of simple types can be completely characterized by distributivity (that is, equation (1)) plus the following three (for more about type isomorphisms see Di Cosmo’s short survey [6] or book [5]):

- **Commutativity**: $A \land B \equiv B \land A$
- **Associativity**: $(A \land B) \land C \equiv A \land (B \land C)$
- **Currying**: $(A \land B) \Rightarrow C \equiv A \Rightarrow (B \Rightarrow C)$

At the inception of Díaz-Caro and Dowek’s System I [8], there is the idea of turning all these type isomorphisms into computational principles.Precisely, these isomorphisms give rise to some equations $t \sim s$ between terms, such as $(t,s) \sim (s,t)$ for the commutativity of conjunctions, for instance. The result of
Díaz-Caro and Dowek is that the $\lambda$-calculus with pairs extended with 5 such equations (distributivity induces 2 equations) is strongly normalizing modulo.

**System I Rests on Types.** The equations of System I, while well behaved with respect to termination, come with two drawbacks. First, the calculus is not confluent. Second, the definitions of the rewriting rules and of the equations depend on types, so that it is not possible to consider an untyped version. Both issues are easily seen considering the commutativity equation. Consider $t = \pi_1(s, u)$. If pairs are commutative, $t$ can rewrite to both $s$ and $u$:

$$s \leftarrow \pi_1(s, u) \sim \pi_1(u, s) \rightarrow u$$

which breaks both confluence and subject reduction (if $s$ has type $A$ and $u$ has type $B$). To recover subject reduction, one uses a projection $\pi_A$ indexed by a type rather than a coordinate so that (if $s$ has type $A$ and $u$ has type $B$):

$$s \leftarrow \pi_A(s, u) \sim \pi_A(u, s) \rightarrow s$$

note that in order to apply the rule we need to know the type of $s$. Moreover, confluence is not recovered—if both $s$ and $u$ have type $A$ then the result may non-deterministically be $s$ or $u$, according to System I. Díaz-Caro and Dowek in [8] indeed adopt a sort of proof-irrelevant point of view, for which subject reduction is more important than confluence for normalization: types guarantee the existence of a result (strong normalization), and this guarantee is stable by evaluation (subject reduction), while uniqueness of the result is abandoned (no confluence).

**System I and the Distributive $\lambda$-Calculus.** The two issues of System I are not due only to the commutativity isomorphism, as the currying and associativity isomorphisms also contributes to them. The distributive $\lambda$-calculus essentially restricts System I by keeping only the distributive isomorphism, which is the only one not hindering confluence and the possibility of defining the calculus independently from the type system.

To be precise, we do not simply restrict to distributivity, but we also change its computational interpretation. First, we do not consider equations, but rewriting rules, and also we consider the rule $\pi_i(\lambda x.t) \rightarrow \lambda x.\pi_i t$ that was not part of System I, while we remove both equations:

$$\lambda x.(t, s) \sim (\lambda x.t, \lambda x.s) \quad \pi_i(ts) \sim \lambda x.(\pi_i t)s \quad i = 1, 2$$

The main reason is that they would make much harder to establish confluence of the calculus, because they introduce various critical pairs—the distributive $\lambda$-calculus is instead trivially confluent, because it is an orthogonal higher-order rewriting system, and all such systems are confluent.

To sum up, System I aims at being a maximal enrichment of the $\lambda$-calculus with computation principles induced by type isomorphisms, while the distributive $\lambda$-calculus rather is a minimal extension aiming at being as conservative as possible with respect to the $\lambda$-calculus, and in particular at being definable without types.

\footnote{Such a rule was however present in an early version of System I, see [7].}
Clashes. Let us point out a pleasant by-product of the distributive rewriting rules that we adopt. A nice property of the $\lambda$-calculus is that there can never be \textit{clashes} of constructors. In logical terms, there is only one introduction rule (corresponding to the abstraction constructor) and only one elimination rule (application) and they are duals, that is, they interact via $\beta$-reduction. Extensions of the $\lambda$-calculus usually lack this property. Typically, extending the $\lambda$-calculus with pairs $(t, s)$ (and of course projections $\pi_1 t$ and $\pi_2 t$) introduces the following two clashes:

$$(t, s) u \quad \pi_i (\lambda x.t) \quad i = 1, 2$$

where an elimination constructor (application or projection) is applied to the wrong introduction rule (pair or abstraction). These clashes are stuck, as there are no rules to remove them, and it is not clear whether it makes any sense to consider such an unrestricted $\lambda$-calculus with pairs.

Our distributive rules deal exactly with these clashes, removing them by commuting constructors. Concretely, the absence of clashes materializes as a \textit{progress} property: all closed normal forms are values, that is, their outermost constructor corresponds to an introduction rule.

Related work. Beyond Díaz-Caro and Dowek’s System I, the only work we are aware of bearing some analogies to ours is Arbiser, Miquel, and Ríos’ $\lambda$-calculus with constructors \cite{arbiser2009pattern}, where the $\lambda$-calculus is extended with constructors and a pattern matching construct that commutes with applications. They show it to be confluent and even having a separation theorem akin to Bohm’s. The calculus has been further studied in a typed setting by Petit \cite{petit2008lambda}, but type isomorphisms play no role in this case.

Another related work is Aït-Kaci and Garrigue’s label-selective $\lambda$-calculus \cite{aikaci1997lambda}, which considers the $\lambda$-calculus plus the only type isomorphism for the implication: $$A \Rightarrow B \Rightarrow C \equiv B \Rightarrow A \Rightarrow C$$ In order to avoid losing confluence and subject reduction, they introduce a labeling system to the arguments, so that the application order becomes irrelevant.

2 The Untyped Distributive $\lambda$-Calculus

The language of the distributive $\lambda$-calculus $\lambda_{\text{dist}}$ is given by the following grammar:

$$\text{Terms} \quad t, s, u \quad ::= \quad x \mid \lambda x.t \mid ts \mid (t, s) \mid \pi_1 t \mid \pi_2 t$$

The rewriting rules are first given at top level:

\textbf{Rules at top level}

\textbf{Standard rules}\n
\begin{align*}
(\lambda x.t)s & \Rightarrow_\beta t\{x\leftarrow s\} \\
\pi_i (t_1, t_2) & \Rightarrow \pi_i t_i \quad i = 1, 2
\end{align*}

\textbf{Distributive rules}\n
\begin{align*}
(t, s) u & \Rightarrow \pi_\lambda (tu, su) \\
\pi_i (\lambda x.t) & \Rightarrow \pi_\lambda \lambda x.\pi_1 t \quad i = 1, 2
\end{align*}

\footnote{With conjunction, this isomorphism is a consequence of currying and commutativity.}
Then, we extend them to be applied wherever in a term. We formulate such an extension using contexts, that are terms where exactly one subterm has been replaced with a hole ⟨⟩:

\[
\text{Contexts} \quad C, D, E \ :::= \ ⟨⟩ | \ λx.C | Ct | tC | ⟨C, t⟩ | ⟨t, C⟩ | π_1C | π_2C
\]

The operation of replacing the hole ⟨⟩ of a context C with a given term t is called plugging and it is noted \( C⟨t⟩ \). As usual, plugging can capture variables.

Now we can define the contextual closure of the top level rules.

\[
\text{Contextual closure} \quad \frac{t \mapsto a \, s}{C⟨t⟩ \mapsto a \, C⟨s⟩} \quad a \in \{β, π_1, π_2, @x, πλ\}
\]

The contextual closure is given with contexts as a compact way of expressing the closure of all rules by all constructors, in the proofs sometimes we consider the closure by a single constructor. We use \( \rightarrow_{\text{dist}} \) for the union of all the rewriting rules defined above.

**Values and Neutral Terms.** Two subsets of terms play a special role in the following, terms whose outermost constructor corresponds to a logical introduction rule (values) and elimination rule (neutral terms), plus—in both cases—variables.

**Definition 2.1** (Values and neutral terms).

- **Values:** a term is value if it is either a variable \( x \), an abstraction \( λx.t \), or a pair \( ⟨t, s⟩ \).
- **Neutral terms:** a term is neutral if it is either a variable \( x \), an application \( ts \), or a projection \( π_i t \).

Sometimes, neutral terms are also required to be normal. Here they are not.

**Progress.** The first property that we show is that all closed normal forms are values. Please note that evaluation is not call-by-value, here the aim is simply to stress that in the distributive λ-calculus there are no clashes, i.e. closed-normal neutral terms.

**Proposition 2.2** (Progress). If \( t \) is a closed normal form then it is a value.

**Proof.** By induction on \( t \). Cases:

- **Variable:** impossible, since \( t \) is closed.
- **Abstraction or pair:** then the statement holds.
- **Application, i.e. \( t = su \).** Since \( t \) is normal and closed, so is \( s \). Then, by i.h. \( s \) is a value, that is, either an abstraction or a pair. In the first case, rule \( β \) applies and in the second case rule \( @x \) applies. Hence, in any case \( t \) is not in normal form, absurd. Therefore, \( t \) cannot be an application in normal form.
- **Projection, i.e. \( t = π_is \).** Since \( t \) is normal and closed, so is \( s \). Then, by i.h. \( s \) is a value, that is, either an abstraction or a pair. In the first case, rule \( πλ \) applies and in the second case rule \( π_i \) applies. Therefore, \( t \) cannot be a projection in normal form. □
Substitution. For the proof of strong normalization we shall need a basic property of substitution with respect to rewriting steps.

**Lemma 2.3 (Substitutivity of \( \to_{\text{dist}} \)).**

1. **Left substitutivity:** if \( t \to_{\text{dist}} t' \) then \( t\{x\leftarrow s\} \to_{\text{dist}} t'\{x\leftarrow s\} \).

2. **Right substitutivity:** if \( s \to_{\text{dist}} s' \) then \( t\{x\leftarrow s\} \to_{\text{~}}^* t\{x\leftarrow s'\} \).

**Proof.** The first point is an easy induction on the relation \( \to_{\text{dist}} \), the second one on \( t \). Details in the Appendix.

Confluence. The distributive \( \lambda \)-calculus is an example of orthogonal higher-order rewriting system [10], that is a class of rewriting systems for which confluence always holds, because of the good shape of its rewriting rules.

**Theorem 2.4 (Confluence).** The distributive \( \lambda \)-calculus is confluent, that is, if \( s_1 \to_{\text{dist}}^* t \to_{\text{dist}}^* s_2 \) then there exists \( u \) such that \( s_1 \to_{\text{dist}}^* u \to_{\text{dist}}^* s_2 \).

Leftmost-Outermost Normalization. A classic property of the ordinary \( \lambda \)-calculus is the (untyped) normalization theorem for leftmost-outermost (shortened to LO) reduction. The theorem states that LO reduction \( \to_{\text{LO}} \) is normalizing, that is, \( \to_{\text{LO}} \) reaches a normal form from \( t \) whenever \( t \) has a \( \beta \) reduction sequence to a normal form. The definition of LO reduction \( \to_{\text{LO}} \) on ordinary \( \lambda \)-terms is given by:

\[
\begin{align*}
(\lambda x.t)s & \to_{\text{LO}} t\{x\leftarrow s\} \\
\lambda x.t & \to_{\text{LO}} \lambda x.s \\
t & \to_{\text{LO}} s & \text{is neutral} \\
tu & \to_{\text{LO}} su & u \text{ is neutral and normal} \\
t & \to_{\text{LO}} s & u \to_{\text{LO}} us \\
\pi_i \langle t_1, t_2 \rangle & \to_{\text{LO}} t_i \\
\langle t, s \rangle u & \to_{\text{LO}} \langle tu, su \rangle \\
\pi_i (\lambda x.t) & \to_{\text{LO}} \lambda x.\pi_i t \\
\pi_i t & \to_{\text{LO}} \pi_i s \\
\langle t, u \rangle & \to_{\text{LO}} \langle s, u \rangle & u \text{ is normal} \\
\langle u, t \rangle & \to_{\text{LO}} \langle u, s \rangle & t \to_{\text{LO}} s
\end{align*}
\]

By exploiting an abstract result by van Ramsdonk, we obtain a LO normalization theorem for \( \lambda_{\text{dist}} \) for free. Leftmost-outermost reduction \( \to_{\text{LO}} \) can indeed be defined uniformly for every orthogonal rewriting system. For the distributive \( \lambda \)-calculus we simply consider the previous rules with respect to terms in \( \lambda_{\text{dist}} \), and add the following clauses:

\[
\begin{align*}
\pi_i \langle t_1, t_2 \rangle & \to_{\text{LO}} t_i \\
\langle t, s \rangle u & \to_{\text{LO}} \langle tu, su \rangle \\
\pi_i (\lambda x.t) & \to_{\text{LO}} \lambda x.\pi_i t \\
\pi_i t & \to_{\text{LO}} \pi_i s \\
\langle t, u \rangle & \to_{\text{LO}} \langle s, u \rangle & u \text{ is normal} \\
\langle u, t \rangle & \to_{\text{LO}} \langle u, s \rangle & t \to_{\text{LO}} s
\end{align*}
\]

In [12], van Ramsdonk shows that every orthogonal higher-order rewriting system that is fully extended and left normal has a LO normalization theorem. These requirements, similarly to orthogonality, concern the shape of the rewriting rules—see [12] for exact definitions. Verifying that the distributive \( \lambda \)-calculus is fully extended and left normal is a routine check, omitted here to avoid defining formally higher-order rewriting systems. The theorem then follows.

3Precisely, on the one hand van Ramsdonk in [12] shows that full extendedness implies that outermost-fair strategies are normalizing. On the other hand, left-normality implies that leftmost-fair rewriting is normalizing. Then, the LO strategy is normalizing.
Theorem 2.5 (Leftmost-outermost normalization). If \( t \rightarrow_{\text{dist}}^* s \) and \( s \) is \( \rightarrow_{\text{dist}} \)-normal then \( t \rightarrow_{\text{LO}}^* s \).

3 Simple Types Up To Distributivity

In this section we define the simply typed distributive \( \lambda \)-calculus and prove subject reduction.

The type system. The grammar of types is given by

\[
A ::= \tau | A \Rightarrow A | A \land A
\]

where \( \tau \) is a given atomic type.

The relation \( \equiv \) denoting type isomorphism is defined by

\[
\begin{align*}
A &\equiv A \\
B &\equiv B \\
A &\equiv C \\
A \Rightarrow B &\equiv B \Rightarrow A \\
A \Rightarrow B &\equiv C \\
A &\equiv B \\
A &\equiv A \land B \\
A &\equiv A \land C \\
A &\equiv (A \Rightarrow B) \land (A \Rightarrow C)
\end{align*}
\]

The typing rules are:

\[
\begin{align*}
&\Gamma, x : A \vdash x : A \quad (ax) \\
&\Gamma \vdash t : A \Rightarrow A \vdash B \quad (\Rightarrow) \\
&\Gamma \vdash \lambda x.t : A \Rightarrow B \\
&\Gamma \vdash \pi_i t : A \land B \quad (\pi_i \Leftarrow) \\
&\Gamma \vdash t : A \land B \vdash s : B \\
&\Gamma \vdash \pi_1 t : A \land B \\
&\Gamma \vdash \pi_2 t : B \\
&\Gamma \vdash t : A \land B \vdash \pi_1 t : A \\
&\Gamma \vdash t : A \land B \vdash \pi_2 t : B \\
&\Gamma \vdash t : A \land B \vdash \pi_1 t : A \\
&\Gamma \vdash t : A \land B \vdash \pi_2 t : B \\
&\Gamma \vdash s : B \Rightarrow A \\
&\Gamma \vdash u : A \\
&\Gamma \vdash \pi_i s : B_1 \land B_2
\end{align*}
\]

Note rule \( \equiv \): it states that if \( t \) is typable with \( A \) then it is also typable with \( B \) for any type \( B \equiv A \). It is the key rule for having subject reduction for the distributive \( \lambda \)-calculus.

Subject reduction. The proof of subject reduction is built in a standard way, from a generation and a substitution lemma, plus a straightforward lemma on the shape of isomorphic types.

Lemma 3.1 (Generation). Let \( \Gamma \vdash t : A \). Then,

1. If \( t = x \), then \( \Gamma = \Gamma', x : B \) and \( B \equiv A \).
2. If \( t = \lambda x.s \), then \( \Gamma, x : B \vdash s : C \) and \( B \Rightarrow C \equiv A \).
3. If \( t = \langle s_1, s_2 \rangle \), then \( \Gamma \vdash s_i : B_i \), for \( i = 1, 2 \), and \( B_1 \land B_2 \equiv A \).
4. If \( t = su \), then \( \Gamma \vdash s : B \Rightarrow A \), \( \Gamma \vdash u : A \).
5. If \( t = \pi_is \), then \( \Gamma \vdash s : B_1 \land B_2 \) and \( B_i = A \).
Proof. Formally, the proof is by induction on $\Gamma \vdash t : A$, but we rather give an informal explanation. If $t$ is a value ($x, \lambda x.s,$ or $\langle s_1, s_2 \rangle$) then the last rule may be either the corresponding introduction rule or $\equiv$, and the statement follows. If $t$ is not a value there are two similar cases. If $t = su$ what said for values still holds, but we can say something more. Note indeed that if $A \equiv C$ and $\Gamma \vdash s : B \Rightarrow C$ then since $C$ is a sub-formula of $B \Rightarrow C$ we can permute the $\equiv$ rule upwards and obtain $\Gamma \vdash s : B \Rightarrow A$. Similarly if $t = \pi_i s$, which is also an elimination rule.

Lemma 3.2 (Substitution). If $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$, then $\Gamma \vdash t\{x\leftarrow s\} : B$.

Proof. Easy induction on the derivation of $\Gamma, x : A \vdash t : B$. Details in the Appendix.

Lemma 3.3 (Equivalence of types).

1. If $A \land B \equiv C \land D$ then $A \equiv C$ and $B \equiv D$.

2. If $A \Rightarrow B \equiv C \Rightarrow D$ then $A \equiv C$ and $B \equiv D$.

3. If $A \land B \equiv C \Rightarrow D$ then $D \equiv D_1 \land D_2$, $A \equiv D_1$ and $B \equiv C \Rightarrow D_2$.

Proof. By induction on the definition of $\equiv$.

Theorem 3.4 (Subject reduction). If $\Gamma \vdash t : A$ and $t \rightarrow_{\text{dist}} s$, then $\Gamma \vdash s : A$.

Proof. By induction on $t \rightarrow_{\text{dist}} s$ using the generation lemma (Lemma 3.1). We first deal with the cases of the rules applied at top level:

- **\beta-rule**: $(\lambda x.t)s \rightarrow_{\beta} t\{x\leftarrow s\}$. By generation, $\Gamma \vdash \lambda x.t : B \Rightarrow A$, $\Gamma \vdash s : B$. Again by generation, $\Gamma, x : C \vdash t : D$, with $C \Rightarrow D \equiv B \Rightarrow A$, so by Lemma 3.3, $C \equiv B$ and $D \equiv A$. Then, by rule ($\equiv$) we have $\Gamma \vdash s : C$, and so, by the substitution lemma (Lemma 3.2) we have $\Gamma \vdash t\{x\leftarrow s\} : D$, therefore, by rule ($\equiv$), $\Gamma \vdash t\{x\leftarrow s\} : A$.

- **Projection**: $\pi_i(t_1, t_2) \rightarrow_{\pi_i} t_i$. By generation, $\Gamma \vdash \langle t_1, t_2 \rangle : B_1 \land B_2$ with $B_i = A$. By generation again, $\Gamma \vdash t_i : C_i$ with $C_1 \land C_2 \equiv B_1 \land B_2$. Therefore, by rule ($\equiv$), $\Gamma \vdash t_i : A$.

- **Pair-application**: $(t, s)u \rightarrow_{\text{dist}} \langle tu, su \rangle$. By generation, $\Gamma \vdash \langle t, s \rangle : B \Rightarrow A$ and $\Gamma \vdash u : B$. By generation again, $\Gamma \vdash t : C$ and $\Gamma \vdash s : D$ with $C \land D \equiv B \Rightarrow A$. By Lemma 3.3, $A \equiv A_1 \land A_2$, $C \equiv B \Rightarrow A_1$ and $D \equiv B \Rightarrow A_2$. Then,

$$
\begin{align*}
\Gamma \vdash t : C & \\
\Gamma \vdash t : B \Rightarrow A_1 & (\equiv) \\
\Gamma \vdash u : B & (\Rightarrow_t) \\
\Gamma \vdash tu : A_1 & \\
\Gamma \vdash s : D & (\equiv) \\
\Gamma \vdash s : B \Rightarrow A_2 & (\equiv) \\
\Gamma \vdash su : A_2 & (\Rightarrow_s) \\
\Gamma \vdash \langle tu, su \rangle : A_1 \land A_2 & (\land_i) \\
\Gamma \vdash \langle tu, su \rangle : A & (\equiv)
\end{align*}
$$

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• **Projection-abstraction:** \( \pi_i(\lambda x.t) \mapsto \lambda x.\pi_i t \). By generation, \( \Gamma \vdash \lambda x.t : B_1 \land B_2 \) with \( B_i = A \). By generation again, \( \Gamma, x : C \vdash t : D_i \), with \( C \Rightarrow D \equiv B_1 \land B_2 \). Then, by Lemma \[4.3\], \( D \equiv D_1 \land D_2 \), \( B_1 \equiv C \Rightarrow D_1 \), and \( B_2 \equiv C \Rightarrow D_2 \). Then, \( A = C \Rightarrow D_i \) and so,

\[
\begin{align*}
\Gamma, x : C \vdash t : D &
\quad (\equiv) \\
\Gamma, x : C \vdash \pi_i t : D_i &
\quad (\lambda_e) \\
\Gamma \vdash \lambda x.\pi_i t : C \Rightarrow D_i &
\quad (\Rightarrow_i)
\end{align*}
\]

The inductive cases are all straightforward. We give one of them, the others are along the same lines. Let \( \lambda x.t \to_{\text{dist}} \lambda x.s \) because \( t \to_{\text{dist}} s \). By generation, \( \Gamma, x : B \vdash t : C \), with \( B \Rightarrow C \equiv A \). By i.h., \( \Gamma, x : B \vdash s : C \), so, by rules \((\Rightarrow_i)\) and \((\equiv)\), \( \Gamma \vdash \lambda x.s : A \). \( \square \)

### 4 Strong normalisation

Here we prove strong normalization using Tait’s reducibility technique. The key point shall be proving that the interpretation of types is stable by distributivity.

**Definition 4.1 (Basic definitions and notations).**

- **SN terms:** we write \( \text{SN} \) for the set of strongly normalising terms.
- **One-step reducts:** the set \( \{ s \mid t \to_{\text{dist}} s \} \) of all the one-step reducts of a term \( t \) is noted \( \text{Red}(t) \).
- **Evaluation length:** \( \text{eval}(t) \) is the length of the longest path starting from \( t \) to arrive to a normal form
- **Size:** \( \text{size}(t) \) is the size of the term \( t \) defined in the usual way.

**The interpretation of types.** The starting point of the reducibility technique is the definition of the interpretation of types, which is the standard one.

**Definition 4.2 (Interpretation of types).**

\[
\begin{align*}
[\tau] & := \text{SN} \\
[A \Rightarrow B] & := \{ t \mid \forall s \in [A], ts \in [B] \} \\
[A \land B] & := \{ t \mid \pi_1 t \in [A] \text{ and } \pi_2 t \in [B] \}
\end{align*}
\]

**The reducibility properties.** The next step is to prove the standard three properties of reducibility. The proof is standard, that is, the distributive rules do not play a role here.

**Lemma 4.3 (Properties of the interpretation).** For any type \( A \) the following properties of its interpretation are valid.

- **CR1** \([A] \subseteq \text{SN}\).
- **CR2** If \( t \in [A] \) and \( t \to_{\text{dist}} s \), then \( s \in [A] \).
- **CR3** If \( t \) is neutral and \( \text{Red}(t) \subseteq [A] \), then \( t \in [A] \).
Proof.

CR1 By induction on $A$. Cases:

- $[\tau] = \text{SN}$.
- Let $t \in [A \Rightarrow B]$. Then, for all $s \in [A]$, we have $ts \in [B]$. By i.h., $[B] \subseteq \text{SN}$, so $ts \in \text{SN}$, and hence, $t \in \text{SN}$.
- Let $t \in [A \land B]$. Then, in particular, $\pi_1 t \in [A]$. By i.h., $[A] \subseteq \text{SN}$, so $\pi_1 t \in \text{SN}$, and hence, $t \in \text{SN}$.

CR2 By induction on $A$. Cases:

- Let $t \in [\tau] = \text{SN}$. Then if $t \rightarrow \text{dist} s$, we have $s \in \text{SN} = [\tau]$.
- Let $t \in [A \Rightarrow B]$. Then, for all $u \in [A]$, we have $tu \in [B]$. By i.h. on $B$, since $tu \rightarrow \text{dist} su$, we have $su \in [B]$ and so $s \in [A \Rightarrow B]$.
- Let $t \in [A_1 \land A_2]$. Then, $\pi_1 t \in [A_i]$, for $i = 1, 2$. By i.h. on $A_i$, since $\pi_1 t \rightarrow \text{dist} \pi_i s$, we have $\pi_i s \in [A_i]$ and so $s \in [A_1 \land A_2]$.

CR3 By induction on $A$. Let $t$ be neutral. Cases:

- Let $\text{Red}(t) \subseteq [\tau] = \text{SN}$. Then $t \in \text{SN} = [\tau]$.
- Let $\text{Red}(t) \subseteq [A \Rightarrow B]$. Then for each $t' \in \text{Red}(t)$, we have that for all $s \in [A]$, $t's \in [B]$. Since $ts$ is neutral, if we show that $\text{Red}(ts) \subseteq [B]$ then the i.h. on $B$ gives $ts \in [B]$ and so $t \in [A \Rightarrow B]$.

Since, by CR1 on $[A]$, we have $s \in \text{SN}$, we show that $\text{Red}(ts) \subseteq [B]$ by a second induction on $\text{size}(s)$. The possible reducts of $ts$ are:

- $t's$, with $t \rightarrow \text{dist} t'$, which is in $[B]$ by hypothesis,
- $ts'$, with $s \rightarrow \text{dist} s'$, then by the second induction hypothesis $\text{Red}(ts') \subseteq [B]$ and by i.h. $ts' \in [B]$.

Note that since $t$ is neutral there are no other reductions from $ts$.

- Let $\text{Red}(t) \subseteq [A_1 \land A_2]$. Then for each $t' \in \text{Red}(t)$, we have that $\pi_i t' \in [A_i]$, for $i = 1, 2$. We show that $\text{Red}(\pi_i t) \subseteq [A_i]$, which—since $\pi_i$ is neutral—by i.h. implies $\pi_i t \in [A_i]$, and so $t \in [A_1 \land A_2]$.

Since $t$ is neutral, its only possible reducts have the form $\pi_i t'$, with $t \rightarrow \text{dist} t'$, which are in $[A_i]$ by hypothesis. \qed

Stability of the interpretation by isomorphism. Finally, we come to the point where distributivity plays a role. Here we prove that the interpretation of types is stable by $\equiv$, that is, if $A \equiv B$ then $[A] = [B]$. We need an auxiliary lemma stating a sort of stability by anti-reduction of $[A]$ with respect to the standard rewriting rules of $\beta$ and projection.

Lemma 4.4.

1. If $t, s \in \text{SN}$ and $t \{x \leftarrow s\} \in [A]$ then $(\lambda x. t)s \in [A]$.
2. If $t_i \in [A_i]$ then $\pi_i(t_1, t_2) \in [A_i]$, for $i = 1, 2$.

Proof.
Lemma 4.5
(Stability by isomorphism)

Proof. By induction on \( \text{eval}(t) + \text{eval}(s) \). We show that \( \text{Red}((\lambda x. t)s) \subseteq [A] \), and obtain the statement by CR3. Cases:

- \( (\lambda x. t)s \rightarrow_{\text{dist}} (\lambda x. t')s \) with \( t \rightarrow_{\text{dist}} t' \). We can apply the \( i.h. \) because if \( t \rightarrow_{\text{dist}} t' \) then \( t\{x\leftarrow s\} \rightarrow_{\text{dist}} t'\{x\leftarrow s\} \) by left substitutivity of \( \rightarrow_{\text{dist}} \) (Lemma 2.31), and \( t\{x\leftarrow s\} \in [A] \) by CR2. By \( i.h. \), \( (\lambda x. t')s \in [A] \).

- \( (\lambda x. t)s \rightarrow_{\text{dist}} (\lambda x. t)s' \) with \( s \rightarrow_{\text{dist}} s' \). We can apply the \( i.h. \) because if \( s \rightarrow_{\text{dist}} s' \) then \( t\{x\leftarrow s\} \rightarrow_{\text{dist}} t\{x\leftarrow s'\} \) by right substitutivity of \( \rightarrow_{\text{dist}} \) (Lemma 2.31), and \( t\{x\leftarrow s'\} \in [A] \) by CR2. By \( i.h. \), \( (\lambda x. t)s' \in [A] \).

- \( (\lambda x. t)s \rightarrow_{\beta} t\{x\leftarrow s\} \), which is in \( [A] \) by hypothesis.

2. By CR1 we have \( t_i \in SN \). By induction on \( \text{eval}(t_1) + \text{eval}(t_2) \). The possible reducts of \( \pi_i(t_1, t_2) \) are:

- \( t_i \), because of a \( \rightarrow_{\pi_i} \) step. Then \( t_i \in [A_1] \) by hypothesis.

- \( \pi_i(t'_1, t_2) \), with \( t_1 \rightarrow_{\text{dist}} t'_1 \). We can apply the \( i.h. \) because \( [A_1] \ni t_1 \rightarrow_{\text{dist}} t'_1 \) which is in \( [A_1] \) by CR2. Then \( \pi_i(t'_1, t_2) \in [A] \) by \( i.h. \).

- \( \pi_i(t_1, t'_2) \), with \( t_2 \rightarrow_{\text{dist}} t'_2 \). As the previous case, just switching coordinate of the pair.

\[ \square \]

Lemma 4.5 (Stability by isomorphism). If \( A \equiv B \), then \( [A] = [B] \).

Proof. By induction on \( A \equiv B \). The only interesting case is the base case \( A \Rightarrow B_1 \land B_2 \equiv (A \Rightarrow B_1) \land (A \Rightarrow B_2) \). The inductive cases follow immediately from the \( i.h. \).

We prove \( [A \Rightarrow B_1 \land B_2] = [(A \Rightarrow B_1) \land (A \Rightarrow B_2)] \) by proving the double inclusion.

- Let \( t \in [A \Rightarrow B_1 \land B_2] \). Then for all \( s \in [A] \) we have \( ts \in [B_1 \land B_2] \), so

\[ \pi_i(ts) \in [B_i] \quad (2) \]

We need to prove that \( (\pi_i(ts)) \in [B_i] \). Since this term is neutral, we prove that \( \text{Red}((\pi_i(ts)) \subseteq [B_i] \) and conclude by CR3. By CR1 and \( (2) \), \( t \) and \( s \) are in \( SN \), so we proceed by induction on \( \text{eval}(t) + \text{eval}(s) \). The possible one-step reducts fired from \( (\pi_i(ts)) \) are:

- \( (\pi_i(t'))s \), with \( t \rightarrow_{\text{dist}} t' \), then \( i.h. \) applies.

- \( (\pi_i(t)s') \), with \( s \rightarrow_{\text{dist}} s' \), then \( i.h. \) applies.

- \( t_is \), if \( t = (t_1, t_2) \). Since \( \pi_i(ts) = \pi_i((t_1, t_2)s) \rightarrow_{\text{dist}} \pi_i(t_1s, t_2s) \rightarrow_{\text{dist}} t_is \), by \( (2) \) and CR2 we have \( t_is \in [B_i] \).

- \( (\lambda x. \pi_iu)s \) if \( t = \lambda x. u \). Then we can apply Lemma 4.5\( \Box \) since we know that \( u \) and \( s \) are \( SN \) and that \( \pi_i(ts) = \pi_i((\lambda x. u)s) \rightarrow_{\beta} \pi_iu\{x\leftarrow s\} \) which by \( (2) \) and CR2 is in \( [B_i] \). We obtain \( (\lambda x. \pi_iu)s \in [B_i] \)

- Let \( t \in [(A \Rightarrow B_1) \land (A \Rightarrow B_2)] \). Then \( \pi_i(t) \in [A \Rightarrow B_1] \), and so for all \( s \in [A] \), we have \( (\pi_i(ts)) \in [B_i] \). By CR1 we have \( t, s \in SN \), so we proceed by induction on \( \text{eval}(t) + \text{eval}(s) \) to show that \( \text{Red}(\pi_i(ts)) \subseteq [B_i] \), which implies \( \pi_i(ts) \in [B_i] \) and so \( ts \in [B_1 \land B_2] \), and then \( t \in [A \Rightarrow B_1 \land B_2] \). The possible reducts of \( \pi_i(ts) \) are:
Proof. By induction on the derivation of $\Gamma \vdash \theta$.

Lemma 4.7

The last step is to prove what is usually called **Adequacy**, that is, that typability of $t$ with $A$ implies that $t \in \llbracket A \rrbracket$, up to a substitution $\theta$ playing the role of the typing context $\Gamma$. The proof is standard, the distributive rules do not play any role.

**Definition 4.6** (Valid substitution). We say that a substitution $\theta$ is valid with respect to a context $\Gamma$ (notation $\theta \models \Gamma$) if for all $x : A \in \Gamma$, we have $\theta x \in \llbracket A \rrbracket$.

**Lemma 4.7** (Adequacy). If $\Gamma \vdash t : A$ and $\theta \models \Gamma$, then $\theta t \in \llbracket A \rrbracket$.

Proof. By induction on the derivation of $\Gamma \vdash t : A$.

- $\Gamma, x : A \vdash x : A$ (ax) Since $\theta \models \Gamma, x : A$, we have $\theta x \in \llbracket A \rrbracket$.
- $\Gamma, x : A \vdash t : B \quad \Gamma \vdash \lambda x.t : A \Rightarrow B$ ($\Rightarrow_i$)

By i.h., if $\theta' \models \Gamma, x : A$, then $\theta' t \in \llbracket B \rrbracket$. Let $s \in \llbracket A \rrbracket$, we have to prove that $\theta(\lambda x.t)s = (\lambda x.\theta t)s \in \llbracket B \rrbracket$. By CR1, $s, \theta t \in \mathsf{SN}$, so we proceed by a second induction on $\text{size}(s) + \text{size}(\theta t)$ to show that $\text{Red}(\lambda x.\theta t)s \subseteq \llbracket B \rrbracket$, which implies $(\lambda x.\theta t)s \in \llbracket B \rrbracket$. The possible reducts of $(\lambda x.\theta t)s$ are:

- $(\lambda x.t')s$, with $\theta t \rightarrow \text{dist} t'$, then the second i.h. applies.
- $(\lambda x.t)s'$, with $s \rightarrow \text{dist} s'$, then the second i.h. applies.
- $\theta t(x\leftarrow s)$, then take $\theta' = \theta, x \mapsto s$ and notice that $\theta' \models \Gamma, x : A$, so $\theta' t \in \llbracket B \rrbracket$.

- $\Gamma \vdash t : A \Rightarrow B \quad \Gamma \vdash s : B$ ($\Rightarrow_e$)

By i.h., $\theta t \in \llbracket A \Rightarrow B \rrbracket$ and $\theta s \in \llbracket B \rrbracket$, so, by definition, $\theta t \theta s = \theta(t s) \in \llbracket B \rrbracket$.

- $\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2 \quad \Gamma \vdash \langle t_1, t_2 \rangle : A_1 \land A_2$ ($\land_i$)

By i.h., $\theta t_i \in \llbracket A_i \rrbracket$, for $i = 1, 2$. By CR1 we have $\theta t_i \in \mathsf{SN}$, hence we proceed by a second induction on $\text{size}(\theta t_1) + \text{size}(\theta t_2)$ to show that $\text{Red}(\pi_i(\theta t_1, \theta t_2)) \subseteq \llbracket A_i \rrbracket$, which, by CR3 implies $\pi_i(\theta t_1, \theta t_2) \in \llbracket A_i \rrbracket$ and so $\theta(\pi_i(\theta t_1, \theta t_2)) \in \llbracket A_i \rightarrow \llbracket A_i \land A_2 \rrbracket$.

The possible one-step reducts of $\pi_i(\theta t_1, \theta t_2)$ are:

- $\pi_i(t', \theta t_2)$, with $\theta t_1 \rightarrow \text{dist} t'$, then the second i.h. applies.
\[ \pi_i(\theta t_1, t'), \text{ with } \theta t_2 \rightarrow_{\text{dist}} t', \text{ then the second i.h. applies.} \]

\[ \theta t \in \llbracket A \rrbracket. \]

\[ \Gamma \vdash t : A_1 \land A_2 \]

\[ \begin{array}{ll}
\Gamma \vdash i t : \pi_i = \theta t \in \llbracket A \rrbracket.
\end{array} \]

\[ \Gamma \vdash t : A \equiv B \]

\[ \begin{array}{ll}
\Gamma \vdash t : B \equiv \theta t \in \llbracket A \rrbracket, \text{ so, by Lemma 4.5, } \\
\theta t \in \llbracket B \rrbracket.
\end{array} \]

Theorem 4.8 (Strong normalisation). If \( \Gamma \vdash t : A \), then \( t \in \text{SN} \).

\[ \begin{array}{ll}
\text{Proof.} \text{ By Lemma 4.7, if } \theta \models \Gamma, \text{ then } \\
\theta t \in \llbracket A \rrbracket, \text{ so, by CR3, variables—which are neutral } \\
terms—are in all the interpretations, and so the identity substitution is valid in any context, in particular, in } \Gamma. \text{ Hence, } t \in \llbracket A \rrbracket. \text{ By CR1, } \llbracket A \rrbracket \subseteq \text{SN}. \text{ Hence, } \\
t \in \text{SN}. \end{array} \]

5 Discussion and conclusions

The Unit Type. The point of the paper is the fact that the distributive rewriting rules and typing up to distributivity perfectly marry together. The elimination of clashes, on the other hand, is a nice consequence of our approach that should not be taken too seriously, because it does not scale up, as we now show.

Let’s consider the extension of the distributive \( \lambda \)-calculus with the unit type \( \top \) and a construct * of type \( \top \). In this extended setting it is still possible to interpret distributivity as in the previous sections, and all our results still holds. There are however two new clashes, namely \( * u \) and \( \pi_i * \). If one makes the further step of eliminating them via new rules and type them up to new isomorphisms, then unfortunately normalization breaks, as we now show.

Consider their natural commutation rules:

\[ * u \rightarrow * \quad \pi_i * \rightarrow * \quad i = 1, 2 \]

To have subject reduction along the same lines of what we did, one needs to work up to the following two isomorphisms:

\[ A \Rightarrow \top \equiv \top \quad \top \land \top \equiv \top \]

Note that \( A \Rightarrow \top \equiv \top \) has to be valid for any type \( A \), in particular it is true for \( \top \), giving \( \top \Rightarrow \top \equiv \top \). Now, unfortunately, one can type the diverging term \( \Omega := (\lambda x.xx)(\lambda x.xx) \), as the following derivation shows, and in fact all the terms of the ordinary \( \lambda \)-calculus—said differently strong normalization breaks.

This example also reinforces the fact, already stressed in the introduction, that interpretations of type isomorphisms tend to break key properties. Distributivity, instead, is somewhat special, as it admits an interpretation that is conservative with respect to the properties of the underlying calculus.
**Additional Distributivity Rules.** It is possible to add the two following distributive rewriting rules:

\[ \lambda x. (t, s) \rightarrow \langle \lambda x.t, \lambda x.s \rangle \]
\[ \pi_i(ts) \rightarrow (\pi_i t)s \quad i = 1, 2 \]

Subject reduction and strong normalization still hold. The problem is that the rewriting system is no longer orthogonal, since the following critical pairs are now possible:

\[
\begin{align*}
\pi_i(\lambda x.(t_1, t_2)) & \rightarrow \pi_i(\lambda x.t_1, \lambda x.t_2) \\
\lambda x.(t_1, t_2) & \rightarrow \lambda x.t_i
\end{align*}
\]
\[
\begin{align*}
(\lambda x.(t, s))u & \rightarrow (t\{x\leftarrow u\}, s\{x\leftarrow u\}) \\
(\lambda x.t, \lambda x.s)u & \rightarrow ((\lambda x.t)u, (\lambda x.s)u)
\end{align*}
\]

\[
\begin{align*}
\pi_i((t_1, t_2)s) & \rightarrow (\pi_i(t_1, t_2))s \\
\pi_i((\lambda x.t)s) & \rightarrow \pi_i(t\{x\leftarrow s\})
\end{align*}
\]
\[
\begin{align*}
\pi_i(t_1 s, t_2 s) & \rightarrow t_is \\
(\pi_i(\lambda x.t))s & \rightarrow (\lambda x.\pi_i t)s
\end{align*}
\]

While the pairs on the left side are easy to deal with, those on the right side have an unpleasant closing diagram and make the rewriting system much harder to study.

**Conclusions.** We have extended the $\lambda$-calculus with pairs with two additional commutation rules inspired by the distributivity isomorphism of simple types, and showed that it is a well behaved setting. In the untyped case, confluence, progress, and leftmost-outermost normalization are obtained essentially for free. In the typed case, subject reduction up to distributivity holds, as well as strong normalization. The proof of strong normalization, in particular, is a smooth adaptation of Tait’s standard reducibility proof for the $\lambda$-calculus with pairs.

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A Proofs Appendix

Lemma 2.3 (Substitutivity of $\rightarrow_{\text{dist}}$).

1. **Left substitutivity**: if $t \rightarrow_{\text{dist}} t'$ then $t[x\leftarrow s] \rightarrow_{\text{dist}} t'[x\leftarrow s]$.

2. **Right substitutivity**: if $s \rightarrow_{\text{dist}} s'$ then $t[x\leftarrow s] \rightarrow_{\text{dist}}^* t[x\leftarrow s']$.

**Proof.**

1. By induction on the relation $\rightarrow_{\text{dist}}$. Base cases:

   - Let $t = (\lambda y.u)r \mapsto_{\beta} u[y\leftarrow r] = t'$. Then,
     
     
     $t[x\leftarrow s] = ((\lambda y.u)r[x\leftarrow s] = (\lambda y.u[x\leftarrow s])r[x\leftarrow s] \mapsto_{\beta} (u[x\leftarrow s])y\leftarrow r(x\leftarrow s) = (u[y\leftarrow r])[x\leftarrow s] = t'[x\leftarrow s]$

   - Let $t = \pi_i(u_1, u_2) \mapsto_{\pi_i} u_i = t'$. Then,
     
     $t[x\leftarrow s] = (\pi_i(u_1, u_2))[x\leftarrow s] = \pi_i(u_1[x\leftarrow s], u_2[x\leftarrow s]) \mapsto_{\pi_i} u_i[x\leftarrow s] = t'[x\leftarrow s]$

   - Let $t = (u, r)p \mapsto_{\alpha_x} (up, rp) = t'$. Then,
     
     $t[x\leftarrow s] = ((u, r)p[x\leftarrow s] = (u[x\leftarrow s], r[x\leftarrow s])(p[x\leftarrow s]) \mapsto_{\alpha_x} (u[x\leftarrow s]p[x\leftarrow s], r[x\leftarrow s]p[x\leftarrow s]) = (up, rp)[x\leftarrow s] = t'[x\leftarrow s]$

   - Let $t = \pi_i(\lambda y.u) \mapsto_{\pi_i} \lambda y.\pi_i u = t'$. Then,
     
     $t[x\leftarrow s] = \pi(\lambda y.u)[x\leftarrow s] = \pi(\lambda y.u[x\leftarrow s]) \mapsto_{\pi_i} \lambda y.\pi_i(u[x\leftarrow s]) = (\lambda y.\pi_i u)[x\leftarrow s] = t'[x\leftarrow s]$

We treat the inductive cases compactly via contexts. First note that a straightforward induction on $C$ shows that $C(t[x\leftarrow s]) = C[x\leftarrow s](t[x\leftarrow s])$, where the substitution $C[x\leftarrow s]$ on contexts is defined as expected. Now, consider $t = C(u) \mapsto_{\alpha} C(r) = t'$ with $u \mapsto_{\alpha} r$, for some $a \in \{\beta, \oplus, \pi_1, \pi_2, \pi_\lambda\}$. By i.h., $u[x\leftarrow s] \mapsto_{\alpha} r[x\leftarrow s]$. Hence,

$t[x\leftarrow s] = C(u)[x\leftarrow s] = C[x\leftarrow s](u[x\leftarrow s])
\mapsto_{\alpha} C[x\leftarrow s](r[x\leftarrow s]) = C[r][x\leftarrow s] = t'[x\leftarrow s]$

2. By induction on $t$.

   - Let $t = x$. Then,
     
     $t[x\leftarrow s] = s \rightarrow_{\text{dist}} s' = t[x\leftarrow s']$

   - Let $t = y$. Then,
     
     $t[x\leftarrow s] = y \rightarrow_{\text{dist}}^* y = t[x\leftarrow s']$

   - Let $t = \lambda y.u$. By i.h., $u[x\leftarrow s] \rightarrow_{\text{dist}}^* u[x\leftarrow s']$. Then,
     
     $t[x\leftarrow s] = \lambda y.u[x\leftarrow s] \rightarrow_{\text{dist}}^* \lambda y.u[x\leftarrow s'] = t[x\leftarrow s']$
Lemma 3.2 (Substitution). If $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$, then $\Gamma \vdash t[x-s] : B$.

Proof. By induction on the derivation of $\Gamma, x : A \vdash t : B$.

- Let $\Gamma, x : A \vdash x : A$ as a consequence of rule $(ax)$. Then, $x[x-s] = s$, and we have $\Gamma \vdash s : A$.
- Let $\Gamma, y : B, x : A \vdash y : B$ as a consequence of rule $(ax)$. Then, $y[x-s] = y$, and by rule $(ax)$, $\Gamma, y : B \vdash y : B$.
- Let $\Gamma, x : A \vdash t : B$ as a consequence of $\Gamma, x : A \vdash t : C, C \equiv B$ and rule $(\equiv)$. Then, by $i.h.$, $\Gamma \vdash t[x-s] : C$, so, by rule $(=\equiv)$, $\Gamma \vdash t[x-s] : B$.
- Let $\Gamma, x : A \vdash \lambda y.t : B \Rightarrow C$ as a consequence of $\Gamma, x : A, y : B \vdash t : C$ and rule $(\Rightarrow_i)$. Then, by $i.h.$, $\Gamma, y : B \vdash t[x-s] : C$, so, by rule $(\Rightarrow_i)$, $\Gamma \vdash \lambda y.t[x-s] : B \Rightarrow C$. Notice that $\lambda y.t[x-s] = (\lambda y.t)x-s$.
- Let $\Gamma, x : A \vdash tr : B$ as a consequence of $\Gamma, x : A \vdash t : C \Rightarrow B, \Gamma, x : A \vdash r : C$, and rule $(\Rightarrow_e)$. Then, by $i.h.$, $\Gamma \vdash t[x-s] : C \Rightarrow B$ and $\Gamma \vdash r[x-s] : C$, so, by rule $(\Rightarrow_e)$, $\Gamma \vdash t[x-s]r[x-s] : B$. Notice that $t[x-s]r[x-s] = (tr)x-s$.
- Let $\Gamma, x : A \vdash \langle t_1, t_2 \rangle : B_1 \land B_2$ as a consequence of $\Gamma, x : A \vdash t_i : B_i, i = 1, 2$, and rule $(\land_i)$. Then, by $i.h.$, $\Gamma \vdash t_i[x-s] : B_i$, so, by rule $(\land_i)$, $\Gamma \vdash \langle t_1[x-s], t_2[x-s] \rangle : B_1 \land B_2$. Notice that $\langle t_1[x-s], t_2[x-s] \rangle = \langle t_1, t_2 \rangle x-s$.
- Let $\Gamma, x : A \vdash \pi_1 t : B$ as a consequence of $\Gamma, x : A \vdash t : B \land C$ and rule $(\land_{e_1})$. Then, by $i.h.$, $\Gamma \vdash t[x-s] : B \land C$, so, by rule $(\land_{e_1})$, $\Gamma \vdash \pi_1(t[x-s]) : B$. Notice that $\pi_1(t[x-s]) = \pi_1 t[x-s]$.
- Let $\Gamma, x : A \vdash \pi_2 t : B$ as a consequence of $\Gamma, x : A \vdash t : B \land B$ and rule $(\land_{e_1})$. Analogous to previous case.