1/f noise in the Two-Body Random Ensemble

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We show that the spectral fluctuations of the Two-Body Random Ensemble (TBRE) exhibit 1/f noise. This result supports a recent conjecture stating that chaotic quantum systems are characterized by 1/f noise in their energy level fluctuations. After suitable individual averaging, we also study the distribution of the exponent $\alpha$ in the 1/f noise for the individual members of the ensemble. Almost all the exponents lie inside a narrow interval around $\alpha = 1$ suggesting that also individual members exhibit 1/f noise, provided they are individually unfolded.

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Our understanding of quantum chaos has greatly advanced during the last two decades. The pioneering work of Berry and Tabor \cite{1}, and Bohigas, Giannoni and Schmit \cite{2} showed that there exists a close relationship between the energy level fluctuation properties of a quantum system and the large time scale behavior of its classical analogue. In their seminal paper, Bohigas \textit{et al.} conjectured that the fluctuation properties of generic quantum systems which in the classical limit are fully chaotic coincide with those of random matrix theory (RMT). This conjecture is strongly supported by experimental data, many numerical calculations, and analytical work based on semiclassical arguments. Later the interest in these studies was renewed with the discovery that the spectral statistics of quantum disordered systems is also well described by RMT. A review of later developments can be found in references \cite{2,3}.

Thus, RMT plays a fundamental role in quantum chaos studies, though it was originally introduced by Wigner to describe the statistical properties of high-lying energy levels of quantum systems \cite{4}. To describe spectral fluctuations, RMT assumes that physical Hamiltonians can be substituted by “reasonable” random ensembles of Hamiltonian matrices, and introduces convenient statistics of their level spectra. RMT should provide the ensemble average of these statistics. Usually, these statistics are the nearest neighbor spacing distribution \cite{4} introduced to analyze the short range correlations, and the Dyson $\Delta_3$ statistic \cite{5} that allows to study long range correlations.

Recently a new approach, based on traditional methods of time series, has been proposed to analyze spectral fluctuations \cite{6,7}. In this first work we showed that the classical random matrix ensembles (CRME) exhibit 1/f noise in the fluctuations of the excitation energy. We also presented evidences that this is actually a universal property of quantum chaotic systems. The purpose of this brief report is to study whether the 1/f noise present in the spectral statistics of the CRME is also present in the Two-Body Random Ensemble (TBRE). As we shall see below, this question is very pertinent if we want to apply this statistic to the study of the spectral fluctuations of real many-body systems.

The CRME, usually called Gaussian and Circular ensembles, were chosen due to their invariant properties under certain symmetry transformations \cite{8}. For example, the Gaussian Orthogonal Ensemble (GOE) is invariant under orthogonal transformations and is applicable to systems invariant under time-reversal symmetry. However, the GOE represents systems with N-body interactions while normal systems in nature are supposed to be very well described by effective two-body interactions in the mean-field basis. TBRE was introduced to tackle this problem, and in this sense is more appropriated to study atomic nuclei, quantum dots and other mesoscopic systems. This ensemble is constructed from a GOE in the 2-particle Hilbert space and then propagating it to the N-particle Hilbert space by using the direct product structure of this type of spaces. (for that reason this kind of ensembles are also called Embedded GOE (EGOE)) \cite{9,10}.

Given the single particle states $|v_i>, i = 1, 2, \cdots, M$, the two-body Hamiltonian is written as

$$H = \sum_{v_i < v_j, v_k < v_l} <v_k v_l | H | v_i v_j> a_{v_l} a_{v_k} a_{v_i} a_{v_j},$$

where $a_{v_i} (a_{v_i})$ creates (destroys) a fermion in the state $|v_i >$. The two body matrix elements $<v_k v_l | H | v_i v_j>$ are properly antisymmetrized and are taken to be independent Gaussian random variables with

$$<v_k v_l | H | v_i v_j> \sigma^2(1 + \delta_{i(i)}, \delta_{j(j)}).$$

In this equation $\bar{ }$ denotes ensemble average, $\sigma$ is a constant and $\delta$ is the Kronecker delta. Then, the Hamiltonian matrix in the N-particle space is defined in terms of these two-body matrix elements via the direct product structure. The only non-zero N-particle matrix elements are of three types.
where $\Sigma(E)$ is an smooth approximation to the actual step function $\Sigma(E)$ that gives the true number of energy levels from the ground state energy $E_0$ and up to energy $E$. This function is given by

$$\Sigma(E) = \int_{E_0}^{E} \rho(\eta) d\eta.$$  

We have already commented the important analytical result of French and Wong who showed that in the TBRE the mean level density $\overline{\rho}(E)$ goes to Gaussian form in the dilute limit. However, for the dimensions of the matrices used in this work, the corrections to the Gaussian behavior are very important and different for each matrix. Since the use of an accurate unfolding procedure is essential to avoid misleading results for the long range spectral correlations \[14\], we have selected another method. Recently, the problems related to the unfolding procedure in the TBRE have been discussed deeply in Refs. \[15, 16\]. After some tests we have chosen polynomials up to grade 5 to fit the accumulated level density $\Sigma(E)$; higher grades produce spurious long-range correlations. Finally, we have thrown 5% of the eigenvalues in the two spectrum edges.

In the approach of Ref. \[18\], the analogy of the energy spectrum with a time series is established in terms of the $\delta_q$ statistic. Using the unfolded energies it is defined as

$$\delta_q = \sum_{i=1}^{q} (s_i - \langle s \rangle) = \epsilon_{q+1} - \epsilon_1 - q.$$  

where $s_i$ are the next-neighbor level spacings, $s_i = \epsilon_{i+1} - \epsilon_i$, with spectral average value $\langle s \rangle = 1$. Note that $\delta_q$ represents the deviation of the excitation energy of the $(q+1)$-th unfolded level from its mean value. Moreover, it is closely related to the level density fluctuations. Indeed, we can write

$$\delta_q = \sum (E_{q+1}) - \Sigma(E_{q+1}) = -\Sigma(E_{q+1}),$$  

if we appropriately shift the ground state energy; thus, it represents the accumulated level density fluctuations at $E = E_{q+1}$.

We will profit of the formal similarity of the $\delta_q$ function with a time series to analyze its properties with numerical techniques, normally used in the domain of complex systems. The most simple procedure is to study the scaling properties of its power spectrum $S(k)$. The latter is defined in terms of the discrete Fourier transform

$$\hat{\delta}_k = \frac{1}{d} \sum_{q=1}^{d-1} \delta_q \exp \left( \frac{2\pi i k q}{d} \right),$$  

in the usual way as

$$S(k) = |\hat{\delta}_k|^2,$$  

where $d \leq D$ is the total number of unfolded levels considered. In the present work $d \approx 0.9D$ We will say that the spectral fluctuations of a Hamiltonian ensemble exhibit $1/f$ noise if the ensemble averaged power spectrum of $\delta_q$ follows a power law of type

$$S(k) \propto \frac{1}{k^\alpha},$$  

or those obtained by permuting the single states. All other matrix elements are zero.

Very few analytic results are known for TBRE, contrary to the classical random matrix ensembles. A very important result is that the level density of the TBRE is Gaussian in the dilute limit, which corresponds to $(N, M) \rightarrow \infty$, $N/M \rightarrow 0$ \[8, 9\], instead of the semi-circular law for the GOE \[10\]. To perform a numerical analysis of the TBRE spectral statistics an important difficulty must be overcome: the TBRE is not ergodic \[10, 11, 12\]. In the present context ergodicity means that the statistical properties of individual ensemble members (and hence those of the physical Hamiltonian) should always coincide with the ensemble average. In order to transform TBRE into an ergodic ensemble the spectrum of each member must be unfolded (see the unfolding description below) individually. In this way, GOE statistics is recovered. For a recent review of TBRE and more generally EGOE see reference \[13\].

In order to establish whether $1/f$ noise is also present in the spectral fluctuations of TBRE, we have studied four ensembles with different matrix sizes. We have treated $N = 6$ “spinless” fermions in $M = 11, 12, 13$ and 14 degenerated states leading to Hilbert space dimensions $D = 462, 924, 1716$ and 3003 respectively. The TBRE matrices were constructed using eqs. (1), (2) and (3) with $\sigma = 1$. There is no relevant energy scale in the model and the only parameter is the dimension of the Hilbert space. We have diagonalized 200 matrices in each case to obtain the ensemble average.

For each Hamiltonian matrix the level density $\rho(E)$ can be separated into a smooth part $\overline{\rho}(E)$, that defines the main trend of the level density, and a fluctuating part $\hat{\rho}(E)$. It is well known that level fluctuations amplitudes are modulated by $\overline{\rho}(E)$; therefore, to compare the statistical properties of different systems or different parts of the same spectrum, the main trend defined by $\overline{\rho}(E)$ must be removed. This procedure, called unfolding, consists in mapping the level energies $E_i$ into new dimensionless levels $\epsilon_i$,

$$E_i \rightarrow \epsilon_i = \Sigma(E_i), \quad i = 1, 2, \cdots, D,$$  

where $\Sigma(E)$ is the ground state energy $E_0$ and up to energy $E$. This function is given by

$$\Sigma(E) = \int_{E_0}^{E} \rho(\eta) d\eta.$$
with $\alpha \approx 1$. For a single Hamiltonian is not clear whether we must impose that the bare power spectrum or some kind of average follows the previous power law. We shall explore three different possibilities below.

![Diagram](image)

**FIG. 1:** Ensemble averaged power spectra of four TBRE with different dimensions. The best $1/f^\alpha$ fit is also shown. The curves have been displaced vertically to avoid overlapping between them.

The results obtained for the ensemble averaged power spectra are shown in Fig. 1 using a log-log scale. It clearly seen that the calculated points spread along straight lines. The line slopes, i.e., the power spectrum exponents are obtained by means of a least-squares fit and their values are $\alpha = 1.09 \pm 0.04$, $\alpha = 1.08 \pm 0.01$, $\alpha = 1.07 \pm 0.01$ and $\alpha = 1.07 \pm 0.01$ for $(N,M) =$ (6,11), (6,12), (6,13) and (6,14), respectively. The exponents are very close to one, confirming that there is $1/f$ noise in the TBRE. Thus, we obtain a new and powerful check of the conjecture that links the spectral statistics of the TBRE with that of the GOE.

Moreover, in order to study to what extent the $\delta_\eta$ statistic is also meaningful for individual spectra, we have randomly selected a member pertaining to the TBRE ensemble with $(N,M) =$ (6,14). The upper plot of Fig. 2 shows the power spectrum of the $\delta_\eta$ function for this member. Although this result suggests the existence of a power law, the calculated points are widely spread around the mean behavior, and therefore other different curves can be used to fit the data points. Performing a least square fit to a straight line we obtain a value $\alpha = 1.10 \pm 0.07$, but the error seems unreliable. Following B. Mandelbrot, the problem arises because of the double logarithmic plot: spectral components must never be plotted raw, only after suitable averaging [17]. One of the best procedures to perform this average consists in dividing the high frequency portion of the logarithmic frequency axis into equal bins and averaging the power spectrum components in each bin. The result of this procedure is shown in the second plot of fig. 2: the averaged data points are no more widely spread, but all of them fall near the mean behavior. If we perform a least square fit to this new set of data points we obtain $\alpha = 1.10 \pm 0.06$; therefore, the fuzzy behavior is confirmed to be related just to the double logarithmic plot. An alternative averaging procedure is to calculate a running or spectral average, $\langle S(k) \rangle$. Since in this case the dimension is large enough, we can divide the whole spectrum in 10 different sets of 256 consecutive levels. Then, the $\delta_\eta$ power spectrum is calculated for each level set and in order to reduce fluctuations and clarify the main trend a running average is performed using these sets. The bottom plot of Fig. 2 displays the result of this calculation using open circles. A least squares fit leads to the following exponent $\alpha = 1.02 \pm 0.05$, which is very similar to that obtained by means of an ensemble average. This result is compatible with the ergodic properties of TBRE once the spectra are individually unfolded. Note that the different lengths of the three power spectra shown in this figure are due to the fact that in the first and the second cases we are actually using the whole sequence, but in the third one we use sequences of 256 consecutive levels.

To make this discussion more quantitative, we have calculated the exponent $\alpha$ for the 200 matrices of the ensemble with $(N,M) =$ (6,14) using the binning method previously described. The average value is $\langle \alpha \rangle = 1.06$ and the width of the distribution is $\sigma_\alpha = 0.08$. Figure 3 shows an histogram of the distribution together with a
FIG. 3: Histogram of the distribution of $\alpha$ for the $(N, M) = (6, 14)$ TBRE.

Gaussian defined by the previous parameters that seems to fit the data very well. Although this result has been obtained for a particular sample of a particular ensemble $((N, M) = (6, 14))$, it suggests that individual members also are characterized by $1/f$ noise.

We have confirmed that the spectral fluctuations of the TBRE exhibit $1/f$ noise. This behavior supports the previously stated conjecture that chaotic quantum systems are characterized by $1/f$ noise in their energy level fluctuations. We have also shown that individual members have $1/f$ noise in their excitation energy fluctuations provided they are individually unfolded and the power spectrum of the $\delta_q$ function is appropriately averaged. Actually, the distribution of the $\alpha$ exponent in the $1/f^\alpha$ law is a Gaussian centered near $\alpha = 1$ with a quite small width. Therefore, the spectral fluctuations of atomic nuclei, quantum dots and mesoscopic systems can be studied by means of the scaling properties of the power spectrum of the $\delta_q$ function. The advantages of this new statistic are perfectly used in some recent works about the nuclear masses [13, 12], where the $1/f$ noise of different series of fluctuations in the nuclear masses along the nuclear chart was explored. Depending of the definition of the fluctuations, different exponents in the power law were found. $1/f$ noise was shown to be a powerful tool to investigate spectral correlations in this kind of experimental data.

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