GRAPHIC BERNSTEIN RESULTS IN CURVED PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract

We generalize a Bernstein-type result due to Albujer and Alías, for maximal surfaces in a curved Lorentzian product 3-manifold of the form $\Sigma_1 \times \mathbb{R}$, to higher dimension and codimension. We consider $M$ a complete spacelike graphic submanifold with parallel mean curvature, defined by a map $f : \Sigma_1 \to \Sigma_2$ between two Riemannian manifolds $(\Sigma_1^m, g_1)$ and $(\Sigma_2^n, g_2)$ of sectional curvatures $K_1$ and $K_2$, respectively. We take on $\Sigma_1 \times \Sigma_2$ the pseudo-Riemannian product metric $g_1 - g_2$. Under the curvature conditions, $\text{Ricci}_1 \geq 0$ and $K_1 \geq K_2$, we prove that, if the second fundamental form of $M$ satisfies an integrability condition, then $M$ is totally geodesic, and it is a slice if $\text{Ricci}_1(p) > 0$ at some point. For bounded $K_1$, $K_2$ and hyperbolic angle $\theta$, we conclude $M$ must be maximal. If $M$ is a maximal surface and $K_1 \geq K_2^+$, we show $M$ is totally geodesic with no need for further assumptions. Furthermore, $M$ is a slice if at some point $p \in \Sigma_1$, $K_1(p) > 0$, and if $\Sigma_1$ is flat and $K_2 < 0$ at some point $f(p)$, then the image of $f$ lies on a geodesic of $\Sigma_2$.

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1 Introduction and statement of main results

The classical Bernstein theorem states that an entire minimal graph in $\mathbb{R}^3$ is a plane. This result has been generalized to graphic hypersurfaces of $\mathbb{R}^{m+1}$ for $m \leq 7$, and for higher dimensions and codimensions under various growth conditions. Calabi in [4] introduced a similar problem in Minkowski space. He considered a maximal (that is, with mean curvature $H = 0$) spacelike hypersurface $M$ in the Lorentz-Minkowski space $\mathbb{R}^{m+1}$ with metric $ds^2 = \sum_{i=1}^{m} (dx_i)^2 - (dx_{m+1})^2$, given by the graph of a function $f$ on $\mathbb{R}^m$ with $|Df| < 1$. In this case the equation for maximality takes the form

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( \frac{\partial f/\partial x_i}{\sqrt{1 - |Df|^2}} \right) = 0.$$ 

Calabi, for $m \leq 4$, and later Cheng and Yau [5] for any $m \geq 2$, proved that any entire solution to the above equation is linear.

Spacelike hypersurfaces of constant mean curvature in Lorentzian spaces have been the subject of investigation in general relativity theory (see for example [3]). Contrarily to the Riemannian case, there are entire spacelike graphs with nonzero constant mean curvature in $\mathbb{R}^{m+1}$ (see [11]), as for example the hyperboloids. On the other hand, under some boundedness assumptions or growth conditions on the Gauss map, Bernstein-type results have been obtained for spacelike submanifolds of the pseudo-Euclidean space $\mathbb{R}^{m+n}$ with parallel mean curvature ([14, 15, 8]).

A natural generalization is to consider maximal spacelike submanifolds in a non-flat ambient space. Albujer and Alías [1] proved a new Calabi-Bernstein-type result for surfaces immersed into a Lorentzian product 3-manifold of the form $\Sigma_1 \times \mathbb{R}$, where $\Sigma_1$ is a Riemannian surface of nonnegative Gauss curvature. In this note, we generalize this result to spacelike graphic submanifolds with parallel mean curvature in a non-flat pseudo-Riemannian product space of any dimension $m+n$, and under less restrictive curvature conditions. Our main tool, as in [11], is the explicit computation of $\Delta \cosh \theta$, where $\theta$ is the hyperbolic angle, a quantity that measures how far is $M$ from a slice, and that was introduced by Chern [7] for the Riemannian version of the Bernstein theorem for surfaces in $\mathbb{R}^3$. This angle plays a similar role to the Gauss map in flat ambient spaces mentioned above.

The main difficulty in higher codimension is the definition of the hyperbolic angle itself. This can be done with the help of a suitable parallel form from the ambient space (see section 2), representing a different approach comparing with the case $m = 2$ and $n = 1$, namely on what concerns the computations involving $\cosh \theta$, that are also considerably more complicated when $n \geq 2$, even for $m = 2$.
Let $N = \Sigma_1 \times \Sigma_2$ be a pseudo-Riemannian product manifold of two Riemannian manifolds $(\Sigma_i, g_i)$ with pseudo-Riemannian metric $\bar{g} = g_1 - g_2$. We denote by $K_i$ and $\text{Ricci}_i$ the sectional curvatures and the Ricci tensor of each $\Sigma_i$, respectively. Assume $\Sigma_1$ is oriented and $M$ is a spacelike graphic submanifold $\Gamma_f = \{(p, f(p)) : p \in \Sigma_1\}$, defined by a map $f : \Sigma_1 \to \Sigma_2$ with $f^* g_2 < g_1$. Let $g$ be the induced metric on $M$. The hyperbolic angle $\theta$ can be defined by

$$\cosh \theta = \frac{1}{\sqrt{\det(g_1 - f^* g_2)}},$$

where the determinant is taken with respect to $g_1$. Note that $\|df\|^2$ is bounded.

Let $B$ be the second fundamental form of $M$. We state our first main theorem:

**Theorem 1.** Assume $M$ is a complete spacelike graph with parallel mean curvature, and for any $p \in \Sigma_1$, $\text{Ricci}_1(p) \geq 0$ and $K_1(p) \geq K_2(f(p))$. Then we have:

(i) If $\|df\| \|B\|$ is an integrable function on $M$ (and this is the case $M$ compact), then $M$ is totally geodesic. Moreover, if $\text{Ricci}_1(p) > 0$ at some point, then $M$ is a slice, that is, $f$ is constant.

(ii) If $K_1$, $K_2$ and $\cosh \theta$ are bounded, then $M$ is maximal. Furthermore, if $M$ is non-compact, and for $p$ away from a compact set, $K_1(p) - K_2(f(p)) \geq \epsilon$ ($\text{Ricci}_1(p) \geq \epsilon$, respectively), where $\epsilon > 0$ is a constant, then $f$ cannot have rank greater or equal to two (one, respectively) at infinity.

In case $\Sigma_2$ is one-dimensional, in next proposition we replace the boundedness condition on $\cosh \theta$ and $K_1$ given in theorem 1(ii) by weaker conditions, and obtain the same result, by recalling an isoperimetric inequality in [10]:

**Proposition 2.** If $\Sigma_2$ is one dimensional, $\Sigma_1$ is complete, $\text{Ricci}_1 \geq 0$, and $\cosh \theta = o(r)$ when $r \to +\infty$, where $r$ is the distance function to a point of $\Sigma_1$, then $M$ is maximal.

For $M$ a Riemannian surface, using parabolicity arguments we obtain:

**Theorem 3.** If $M$ is a complete maximal spacelike graphic surface, and for each $p \in \Sigma_1$, $K_1(p) \geq \max\{0, K_2(f(p))\}$, then $M$ is totally geodesic. Furthermore:

(i) If $K_1(p) > 0$ at some point $p \in M$, then $M$ is a slice.

(ii) ([4] if $n = 1$, [8]) If $\Sigma_1 = \mathbb{R}^2$ and $\Sigma_2 = \mathbb{R}^n$, then $M$ is a plane.

(iii) If $\Sigma_1$ is flat and $K_2 < 0$ at some point $f(p)$, then either $M$ is a slice or the image of $f$ is a geodesic of $\Sigma_2$. 


Our proof in (ii) gives partially a simpler proof of the same result of Jost and Xin in [8] for the case of surfaces. As a consequence of theorem 3, if \( \Sigma_2 = \mathbb{R} \) we have:

**Corollary 4** ([11]). Let \( M \) be a complete maximal spacelike surface of \( N = \Sigma_1 \times \mathbb{R} \), with pseudo-Riemannian product metric \( g_1 - dt^2 \), and assume \( M \) can be written as the graph of a smooth map \( f : \Sigma_1 \to \mathbb{R} \). If \( K_1 \geq 0 \) then \( M \) is totally geodesic. Moreover, if \( K_1 > 0 \) at some point of \( \Sigma_1 \), then \( M \) is a slice.

This paper is organized as follows. In section 2, we recall some preliminaries of spacelike submanifolds in pseudo-Riemannian manifolds, and compute the Laplacian of \( \cosh \theta \). The proofs of theorem 1 and proposition 2 are given in section 3. In section 4, we discuss the surface case and prove Theorem 3.

## 2 Spacelike submanifolds in pseudo-Riemannian products

Let \( N \) be a \((m+n)\)-dimensional pseudo-Riemannian manifold with non-degenerate metric \( \bar{g} \) of index \( n \), and \( F : M \to N \) a \( m \)-dimensional spacelike submanifold immersed into \( N \). We denote by \( \overline{\nabla}, \nabla \) and \( \nabla^\perp \) the connections on \( N, M \), and the normal bundle \( NM \), respectively, and by \( B \) the second fundamental form of \( M \). We convention the sign of the curvature tensor \( \bar{R} \) of \( N \) is defined by

\[
\bar{R}(X,Y) = [\overline{\nabla}_X,\overline{\nabla}_Y] - \overline{\nabla}_{[X,Y]} \quad \text{and} \quad \bar{R}(X,Y,Z,W) = \bar{g}(\bar{R}(Z,Y)X,X) .
\]

We make use of the indices range, \( i,j,k,\ldots,=1,2,\ldots,m, \alpha,\beta,\ldots,=m+1,\ldots,m+n, \) and \( a,b,c,\ldots,=1,2,\ldots,m+n \), and choose orthonormal frame fields \( \{e_1,\ldots,e_{m+n}\} \) of \( N \), such that restricting to \( M \), \( \{e_1,\ldots,e_m\} \) is a tangent frame (of spacelike vectors), and \( \{e_{m+1},\ldots,e_{m+n}\} \) a normal frame (of timelike vectors). Let \( h_{ij}^\alpha \) be the components of the second fundamental form, \( B(e_i,e_j) = h_{ij}^\alpha e_\alpha \), and \( \bar{R}_{bcd}^i, \bar{R}_{ijkl}^\alpha, \bar{R}_{\beta kli}^\alpha \), the components of the curvature tensors of \( N, M \) and \( NM \), respectively. Using the structure equations of \( N \), we derive (see e.g. [5, 8] for the case \( N = \mathbb{R}^{n+m} \))

\[
\bar{R}_{jkl}^i = \bar{R}_{jkl}^i - \sum_{\alpha} (h_{ik}^\alpha h_{jl}^{\alpha'} - h_{il}^{\alpha'} h_{jk}^{\alpha}) \quad \text{(Gauss equation)} \quad (2)
\]

\[
\bar{R}_{\beta kli}^\alpha = \bar{R}_{\beta kli}^\alpha - \sum_i (h_{ki}^\alpha h_{li}^{\beta} - h_{li}^{\beta} h_{ki}^{\alpha}) \quad \text{(Ricci equation)}
\]

\[
h_{ij,k}^\alpha - h_{ik,j}^\alpha = -\bar{R}_{ijk}^\alpha = \bar{\alpha}_{ijk} \quad \text{(Codazzi equation)} \quad (3)
\]

The mean curvature of \( F \) is denoted by \( H = \text{trace}_g B = h_{ij}^\alpha e_\alpha \), \( H^\alpha = \Sigma_i h_{ij}^\alpha \). Let \( \Omega \) be a parallel \( m \)-form on \( N \). Similarly to [12], we compute the Laplacian of the
pull back $F^*\Omega$, $\Delta F^*\Omega = \sum_k \nabla_k \nabla_k F^*\Omega - \nabla_{\nabla_{ek} F^*\Omega}$.\\

$$(\nabla_k F^*\Omega)(e_1, \ldots, e_m) = \sum_i \Omega(e_1, \ldots, (\nabla_k e_i - \nabla_k e_i), \ldots, e_m)$$

$$= \sum_i \Omega(e_1, \ldots, B(e_k, e_i), \ldots, e_m). \quad (4)$$

Set $u = u^\top + u^\perp$, with $u^\top \in TM$ and $u^\perp \in NM$. Differentiating (4), we have

$$\Delta F^*\Omega(e_1, \ldots, e_m) = \sum_{ik} \Omega(e_1, \ldots, \nabla_{e_k} B(e_k, e_i) + (\nabla_{e_k} B(e_k, e_i))^\top, \ldots, e_m)$$

$$+ \sum_{k < j} \Omega(e_1, \ldots, B(e_k, e_j), \ldots, B(e_k, e_i), \ldots, e_m)$$

$$+ \sum_{k > j} \Omega(e_1, \ldots, B(e_k, e_j), \ldots, B(e_k, e_i), \ldots, e_m).$$

Using (3), $\sum_k \nabla_{e_k} B(e_k, e_i) = \nabla_{e_i} H + (\bar{\nabla} e_k, e_i) + \bar{\nabla}^\top e_k, e_i)$, and that

$$\sum_{ik} g((\nabla_{e_k} B(e_k, e_i))^\top, e_i) = \sum_{ik} -g(B(e_k, e_i), B(e_k, e_i)) = ||B||^2,$$

we get in components

$$(\Delta F^*\Omega)_{1 \ldots m} = \Omega_{1 \ldots m} ||B||^2 + 2 \sum_{\alpha < \beta, i < j} \Omega_{\alpha \beta ij} \bar{R}_{\alpha i j} + \sum_{\alpha, i} \Omega_{\alpha i} H^\alpha_{ij} - \sum_{\alpha, i, k} \Omega_{\alpha i} \bar{R}_{\alpha i k}, \quad (5)$$

where $\bar{R}_{\alpha i j} = \sum_k h_{ik} h_{jk} - h_{ik} h_{jk}$, and $\Omega_{\alpha \beta ij} = \Omega(e_1, \ldots, e_\alpha, \ldots, e_\beta, \ldots, e_m)$ with $e_\alpha, e_\beta$ occupying the $i$-th and the $j$-th positions. The same meaning is for $\Omega_{\alpha i}$. Here, $|| \cdot ||$ denotes the absolute of the norm of a timelike vector.

Next we consider the pseudo-Riemannian product manifold $N = \Sigma_1 \times \Sigma_2$, with the pseudo-Riemannian metric $\bar{g} = g_1 - g_2$, where $(\Sigma_i, g_i)$ are two Riemannian manifolds of dimension $m$ and $n$, respectively. Let $\pi_i : N \rightarrow \Sigma_i$ denote the corresponding projections.

Suppose $M$ is a spacelike graph of a map $f : \Sigma_1 \rightarrow \Sigma_2$. For any $p \in \Sigma_1$, we consider $\lambda_1^2 \geq \lambda_2^2 \geq \ldots \geq \lambda_m^2 \geq 0$ the eigenvalues of $f^*g_2$. The spacelike condition on $M$ means $\lambda_i^2 < 1$. By the Weyl’s perturbation theorem [13], ordering the eigenvalues in this way, each $\lambda_i^2 : \Sigma_1 \rightarrow [0, 1)$ is a continuous locally Lipschitz function. For each $p$, let $s = s(p) \in \{1, \ldots, m\}$ be the rank of $f$ at $p$, that is, $\lambda_s^2 > 0$ and $\lambda_{s+1}^2 = \ldots = \lambda_m^2 = 0$. Then $s \leq \min\{m, n\}$. We say that $f$ has rank $s$ at infinity, if there is a constant $\varepsilon > 0$ such that $\lambda_s^2 \geq \varepsilon$, away from a compact set $K$. 

5
We take an orthonormal basis \( \{a_i\}_{i=1,\ldots,m} \) of \( T_p\Sigma_1 \) of eigenvectors of \( f^*g_2 \) with corresponding eigenvalues \( \lambda_i^2 \). Set \( a_{i+m} = df(a_i)/|df(a_i)| \) for \( i \leq s \). This constitutes an orthonormal system in \( T_{f(p)}\Sigma_2 \), that we complete to give an orthonormal basis \( \{a_\alpha\}_{\alpha=m+1,\ldots,m+n} \) for \( T_{f(p)}\Sigma_2 \). Moreover, changing signs of the \( \lambda_i \) if necessary, we can write \( df(a_i) = -\lambda_i a_\alpha \), where \( \lambda_i a_\alpha = \delta_{\alpha,m+i} \lambda_i \) meaning \( = 0 \) if \( i > s \) or \( \alpha > m+s \). Therefore

\[
e_i = \frac{1}{\sqrt{1-\sum_{\beta} \lambda_i^2 \delta_{\beta,i}}} (a_i + \sum_{\beta} \lambda_i \beta a_\beta), \quad i = 1, \ldots, m \tag{6}
\]

\[
e_\alpha = \frac{1}{\sqrt{1-\sum_{\alpha} \lambda_j^2 \delta_{\alpha,j}}} (a_\alpha + \sum_{j} \lambda_j a_j), \quad \alpha = m+1, \ldots, m+n \tag{7}
\]

form orthonormal basis of \( T_pM \) and \( N_pM \), respectively. We may assume \( e_i \) to be positively oriented. We also identify \( M = \Gamma_f \) with \( \Sigma_1 \) with the graph metric \( g = g_1 - f^*g_2 \), and consider \( \lambda_i \) as functions on the variable \( x = (p,f(p)) \in M \) identified with the variable \( p \in \Sigma_1 \), through the diffeomorphism \( \pi_1|_{\Sigma_1} : M \to \Sigma_1 \). Let \( \Omega \) be the volume form of \( (\Sigma_1,g_1) \), which is a parallel \( m \)-form on \( N \), and \( \Omega' \) the one of \( (\Sigma_1,g) \). The ratio \( \Omega/\Omega' \) is given by

\[
\Omega_{1\ldots m} = \star(\pi_1|_{\Sigma_1})^*\Omega = \pi_1^*\Omega(e_1,\ldots,e_m) = \frac{1}{\sqrt{\prod_{i=1}^m (1-\lambda_i^2)}} = \frac{1}{\sqrt{\det(g_1-f^*g_2)}},
\]

where \( \star \) is the star operator on \( M \). This quantity is \( \geq 1 \) and is \( \cosh \theta \) defined in \( \Omega \). Then, \( \cosh \theta \) is identically equal to 1 if and only if \( f \) is a constant map, that is, \( M \) is a slice. Furthermore, if \( \cosh \theta \) is bounded, what means \( \lambda_i^2 \leq 1 - \delta \), where \( \delta > 0 \) is a constant, then \( g_1 \geq g \geq \delta g_1 \). In this case, \( \Sigma_1 \) is complete (compact) if and only if \( M \) is so. The singular values \( \lambda_i \) of \( f \) are constant maps if \( F \) is a totally geodesic immersion. To see this, we first note that \( F \) is totally geodesic if and only if \( f : \Sigma_1 \to \Sigma_2 \) is a totally geodesic map (a proof of this is similar to the Riemannian case [9], remark 2). Parallel transport of \( a_i \) along geodesics of \( \Sigma_1 \) starting from \( p \), shows that \( f^*g_2(a_i,a_j) = \delta_{ij} \lambda_i^2 \) is constant.

With respect to the frames \( \{e_i\} \) and \( \{e_\alpha\} \) given in (6), (7), we have

\[
2 \sum_{\alpha < \beta, i < j} \Omega_{\alpha\beta i j} \bar{R}^\alpha_{\beta ij} = 2 \sum_{\alpha, \beta, k, i < j} \lambda_i \alpha \lambda_j \beta (h_ik^\alpha h_j^\beta - h_ik^\beta h_j^\alpha) \cosh \theta. \tag{8}
\]

We denote by \( R_1 \) and \( R_2 \) the curvature tensors of \( \Sigma_1 \) and \( \Sigma_2 \). Since \( \bar{R}_{\kappa k} = \sum_\beta \bar{R}_{\kappa k} g^{\alpha\beta} = \)
\[ \bar{R}_{ik} \]

Consider for \( i \neq j \) the two-planes \( P_{ij} = \text{span}\{a_i, a_j\} \), \( P'_{ij} = \text{span}\{a_{m+i}, a_{m+j}\} \). Since \( \lambda_i \) is diagonal, we have

\[
\sum_{\alpha, \beta} \Omega_{\alpha i} \bar{R}_{\alpha k} = \sum_{i,j} \frac{\cosh \theta \lambda_i^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} \left( R_1(a_i, a_j, a_i, a_j) - \lambda_i^2 R_2(a_{m+i}, a_{m+j}, a_{m+i}, a_{m+j}) \right).
\]

Inserting (8) and (9) into (5), and using the fact that the star operator is parallel, we get

\[
\Delta \cosh \theta = \cosh \theta \left\{ ||B||^2 + 2 \sum_{k,i<j} \lambda_i \lambda_j h^{m+i}_{ik} h^{m+j}_{jk} - 2 \sum_{k,i<j} \lambda_i \lambda_j h^{m+j}_{ik} h^{m+i}_{jk} 
+ \sum_i \left( \frac{\lambda_i^2}{1 - \lambda_i^2} \text{Ricci}_1(a_i, a_i) + \sum_{j \neq i} \frac{\lambda_i^2 \lambda_j^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} [K_1(P_{ij}) - K_2(P'_{ij})] \right) \right\} 
+ \sum_{\alpha, i} \Omega_{\alpha i} H_{\alpha i},
\]

(10)

### 3 Proof of main results

It is convenient to recall lemma 3.1 in [1], which is also valid in our setting.

**Lemma 5** ([1], for \( n = 2, m = 3 \)). Let \( M \) be a spacelike \( m \)-dimensional submanifold immersed into \( N = \Sigma_1 \times \Sigma_2 \). Then \( \Sigma_1 \) is necessarily complete if \( M \) is complete. In this case \( \pi_{1|\Sigma} : M \to \Sigma_1 \) is a covering map.

We also recall the well-known Omori-Cheng-Yau maximum principle:
Proof of Theorem 1. By (4) we have

\[ d \cosh \theta(e_k) = \sum_{\alpha} \Omega(\pi(e_1), \ldots, \pi(e_{\alpha}), \ldots, \pi(e_m)) h_{ik}^\alpha = \cosh \theta \sum_i \lambda_i h_{ik}^{m+i} \]  

(11)

which implies

\[ \frac{|\nabla \cosh \theta|^2}{\cosh^2 \theta} = \sum_k (\sum_i \lambda_i h_{ik}^{m+i})^2 = \sum_{i,k} (\lambda_i h_{ik}^{m+i})^2 + 2 \sum_{i,j} \lambda_i \lambda_j h_{ik}^{m+i} h_{jk}^{m+j}. \]  

(12)

We shall calculate \( \Delta \ln(\cosh \theta) = (\cosh \theta \Delta(\cosh \theta) - |\nabla \cosh \theta|^2) \cosh^{-2} \theta \). From (10) and (12), and the assumption \( H \) parallel, that is \( H_{ij} = 0 \), we have,

\[ \Delta \ln(\cosh \theta) = ||B||^2 - \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2 - 2 \sum_{i,k,j} \lambda_i \lambda_j h_{ik}^{m+i} h_{jk}^{m+j} \]

(13)

\[ + \sum_i \left( \frac{\lambda_i^2}{1-\lambda_i^2} \text{Ricci}_1(a_i, a_i) + \sum_{j \neq i} \frac{\lambda_i \lambda_j^2}{(1-\lambda_i^2)(1-\lambda_j^2)} [K(P_{ij}) - K(P'_{ij})] \right) \]

(14)

First we need to compute the terms on the right hand side of (13). Since \( F \) is spacelike, at each point \( p \in M \) there exists a positive constant \( \delta = \delta(p) \leq 1 \) such that \( \lambda_i^2 \leq 1 - \delta \) for any \( 1 \leq i \leq m \). Thus, \( |\lambda_i \lambda_j| \leq 1 - \delta \) for any \( i \) and \( j \). We note that \( \lambda_i = 0 \) for \( i > \min(m,n) \). Therefore, we have

\[ ||B||^2 \geq \sum_{i,k,j} (h_{ik}^{m+i})^2 = \sum_{i,k,j} [(h_{ik}^{m+j})^2 + (h_{jk}^{m+i})^2] + \sum_{i,k} (h_{ik}^{m+i})^2, \]

where we keep in mind that \( h_{ik}^{m+j} = 0 \) when \( m + j > m + n \) (because it is the only possible meaning). So the terms in (13) satisfy at \( p \)

\[ ||B||^2 - \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2 - 2 \sum_{i,k,j} \lambda_i \lambda_j h_{ik}^{m+i} h_{jk}^{m+j} \]

\[ \geq \delta ||B||^2 + (1 - \delta) \left\{ \sum_{i,k,j} [(h_{ik}^{m+j})^2 + (h_{jk}^{m+i})^2] + \sum_{i,k} (h_{ik}^{m+i})^2 \right\} \]

\[ - (1 - \delta) \sum_{i,k,j} (h_{ik}^{m+i})^2 - 2 (1 - \delta) \sum_{i,k,j} |h_{ik}^{m+j}||h_{jk}^{m+i}| \]

\[ \geq \delta ||B||^2. \]  

(15)
From the curvature assumptions, (14) \( \geq 0 \). Consequently, we have at each \( p \) a differential inequality,

$$
\Delta \ln(\cosh \theta) \geq \delta(p)||B||^2 \geq \frac{\delta(p)}{m}||H||^2.
$$

(16)

Now we prove (i). From (16), \( \ln(\cosh \theta) \) is a subharmonic function on a complete Riemannian manifold \( M \), and by (11), \( ||\nabla \ln(\cosh \theta)|| \leq C||df||||B|| \), where \( C > 0 \) is a constant. Under the integrability condition of \( ||df||||B|| \), we have integrability of \( ||\nabla \ln(\cosh \theta)|| \), and applying Yau’s special Stokes’ theorem (corollary of \$1 \) (16) we conclude that \( \Delta \ln(\cosh \theta) = 0 \). From (16) we obtain \( ||B||^2 = 0 \), and therefore \( M \) is totally geodesic. Then all singular values \( \lambda_i \) are constant functions. Moreover if there exists at least one point \( p_0 \in \Sigma_1 \) such that Ricci \( \lambda(p_0) > 0 \), then we easily obtain from (14) that \( \lambda_i = 0 \) for any \( i = 1, \cdots, m \), and thus \( f \) is a constant map, that is, \( M \) is a slice.

(ii) Using an o.n. basis \( E_i \) that at a given point diagonalizes the Ricci tensor \( \text{Ricci}^M \) of \( M \), and applying Gauss equation (2), we have for each \( s \)

$$
\text{Ricci}^M(E_s, E_s) = \sum_{j \neq s} \left\{ \bar{R}(E_s, E_j, E_s, E_j) - \sum_{\alpha} (h_{s}^{\alpha}h_{jj}^{\alpha} - h_{sj}^{\alpha}h_{sj}^{\alpha}) \right\}
$$

$$
= \sum_{j \neq s} \bar{R}(E_s, E_j, E_s, E_j) + \sum_{\alpha} \left( h_{ss}^{\alpha} - \frac{1}{2}H^{\alpha} \right)^2 - \frac{1}{4}||H||^2 + \sum_{\alpha, j \neq s} (h_{sj}^{\alpha})^2,
$$

(17)

where all components appearing in this expression are with respect to this frame. Since \( M \) has parallel mean curvature, \( ||H|| \) is constant, and so \( \text{Ricci}^M \) is bounded from below whenever \( \sum_j \bar{R}(E_s, E_j, E_s, E_j) \) is so. Using the other frame, we have

$$
\sum_j \bar{R}(E_s, E_j, E_s, E_j) = \sum_{i,j,k} A_{si}A_{sj} \bar{R}(e_i, e_j, e_k, e_j),
$$

where \( A_{si} = g(E_s, e_i) \), defines an orthogonal matrix. As in section 2, we have

$$
\sum_{j \neq i,k} \bar{R}(e_i, e_j, e_k, e_j) = \sum_{j \neq i,k} \left( R_1(a_i, a_j, a_k, a_j) - \lambda_i \lambda_k \lambda_j^2 R_2(a_{m+i}, a_{m+j}, a_{m+k}, a_{m+j}) \right).
$$

(18)

Since \( K_1 \) and \( K_2 \) are bounded, the same holds for \( R_1(a_i, a_j, a_k, a_j) \) and \( R_2(a_{m+i}, a_{m+j}, a_{m+k}, a_{m+j}) \). Boundedness of \( \cosh \theta \) means there is a positive lower bound \( \delta \) for all \( \delta(p) \), and (18) is bounded. Thus, from Proposition 6, there exists a sequence \( \{ p_k \} \subset M \) such that \( \lim_{k \to \infty} \Delta \ln(\cosh \theta)(p_k) \leq 0 \). Then by (16) we conclude \( H = 0 \), because \( ||H|| \) is constant. Now we assume for \( p \) away from a compact
set $K$ of $\Sigma_1$, $K_1(p) - K_2(f(p)) \geq \varepsilon$, where $\varepsilon > 0$ is constant. Again by (15) and (14), we have for $p \notin K$,

$$\Delta \ln(\cosh \theta) \geq \delta \|B\|^2 + \sum_i \left( \frac{\lambda_i^2}{(1 - \lambda_i^2)} \text{Ricci}_1(a_i, a_i) + \sum_{j \neq i} \frac{\lambda_i^2 \lambda_j^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} \varepsilon \right) \geq 0. \quad (19)$$

First we assume $p_k \notin K$ for $k$ large. From (19) $\lim_{k \to \infty} \lambda_i(p_k) \lambda_j(p_k) = 0$, for $i \neq j$, and so $\lim_{k \to \infty} \lambda_i(p_k) = 0$ for $i \geq 2$. In particular, $f$ cannot have rank greater or equal to two at infinity. If $p_k \in K$ for a subsequence, then $\ln(\cosh \theta)$ attains a maximum at some limit point $p_\infty$ of $p_k$, and being a subharmonic map (see (16)), by the strong maximum principle $\cosh \theta$ is constant, and so, by (16), $B = 0$, because $\delta(p_\infty) \neq 0$. Consequently all $\lambda_i$ are constant, and $0 = (19)$ holds for $p \notin K$, what implies $\lambda_i(p) = 0$ for all $i \geq 2$. The case $\text{Ricci}_1 \geq \varepsilon$ is similar. \qed

We observe the boundedness condition on $\theta$ is equivalent to the boundedness condition on the Gauss map of Jost and Xin [8] in case $K_1 = K_2 = 0$. Hence, by their Bernstein theorem we have:

**Proposition 7** ([8]). If $\Sigma_1 = \mathbb{R}^m$, $\Sigma_2 = \mathbb{R}^n$, and $M$ is a graphic parallel submanifold with bounded hyperbolic angle, then $M$ is a plane.

Next we prove proposition 2. We are assuming $\Sigma_1$ complete and oriented. Let $\mathcal{O}$ be an open set of $\Sigma_1$. The Cheeger constant of $\mathcal{O}$ is defined by

$$h(\mathcal{O}) = \inf_D \frac{A_1(\partial D)}{V_1(D)},$$

where $D$ ranges over all open submanifolds of $\mathcal{O}$ with compact closure in $\mathcal{O}$ and smooth boundary, and $A_1(\partial D)$ and $V_1(D)$ are respectively, the induced volumes of $\partial D$ and $D$ for the metric $g_1$. If $\Sigma_1$ is closed, we adopt the same definition for $h(\Sigma_1)$, that is zero in this case. We fix a point $p$ of $\Sigma_1$, and $B_s$ denotes the open geodesic ball at $p$ of radius $s$.

**Lemma 8** ([2]). If $\text{Ricci}_1 \geq 0$, then for any $r > 0$, $h(B_r) \leq \frac{C}{r}$, where $C > 0$ is a constant that does not depend on $s$. In particular $h(\Sigma_1) = 0$.

**Proposition 9** ([10]). If $\Sigma_2$ is one-dimensional and $M$ is a graphic spacelike hypersurface $\Gamma_f$, then on a open bounded set $D$ of $\Sigma_1$, with smooth boundary

$$\inf_D \|H\| \leq \frac{1}{m} \frac{b_D}{\sqrt{1 - b^2_D}} \frac{A_1(\partial D)}{V_1(D)},$$

where $b_D = \sup_D \|\nabla f\|_1$.\[10]
In particular, if $M$ has constant mean curvature, the Cheeger constant of $(\Sigma_1, g_1)$ vanish, and the hyperbolic angle is bounded, then $M$ is maximal.

Now, proposition 2 follows directly as an application of previous lemma and the inequality in proposition 9, by taking $D \subset B_r$, and that for $n = 1$, $\cosh \theta = (1 - \| \nabla f \|^2)^{-1/2}$.

We should note that a certain nonnegativeness condition on the curvature of $\Sigma_1$ plays a fundamental role in this type of results. If $\Sigma_1$ is the $m$-hyperbolic space $\mathbb{H}^m$ there are examples of complete entire graphic hypersurfaces with constant mean curvature $c$, for any $c$, and with bounded hyperbolic angle, as can be shown by the following proposition. The function $r(x) = \ln \left(\frac{1+|x|}{1-|x|}\right)$ is the distance function in $\mathbb{H}^m$ to 0, for the Poincaré model:

**Proposition 10 ([10]).** Let $c$ be any constant and $f_c : \mathbb{H}^m \to \mathbb{R}$ defined by:

$$f_c(x) = \int_0^{r(x)} \frac{c}{(\sinh r)^{m-1}} \int_0^r (\sinh t)^{m-1} dt \frac{dr}{\sqrt{1 + \left(\frac{c}{(\sinh r)^{m-1}} \int_0^r (\sinh t)^{m-1} dt\right)^2}}.$$

Then $f_c$ is smooth on all $\mathbb{H}^m$, and for each $c, d \in \mathbb{R}$, $\Gamma(f_c + d) \subset \mathbb{H}^m \times \mathbb{R}$ is a complete spacelike graph of bounded hyperbolic angle, with $|\nabla f_c|^2 \leq \frac{c^2}{(m-1)^2} / \left(1 + \frac{c^2}{(m-1)^2}\right) < 1$ and constant mean curvature given by $\langle H, \nu \rangle = \frac{c}{m}$, where $\nu = (-\nabla f_c, 1) / \sqrt{1 + |\nabla f_c|^2}$ is the unit timelike normal to the graph. Furthermore, \{\$\Gamma(f_c + d)(x) : x \in \mathbb{H}^m, d \in \mathbb{R}\$\} (with $c$ fixed) and \{\$\Gamma(f_c + d + c)(x) : x \in \mathbb{H}^m, c \in \mathbb{R}\$\} (with $d$ fixed) define foliations of $\mathbb{H}^m \times \mathbb{R}$ by hypersurfaces, with the same constant mean curvature $c$, and with constant mean curvature parameterized by the leaf, respectively.

The constant mean curvature parameter $c$ (or $1/c$) in the second example of the proposition, can be interpreted as a natural “time function” of geometric nature. The existence of such foliations have been considered in general relativity. For $c = 0$ the above examples are slices. In [11] there are several existence theorems and explicit examples of complete maximal graphic surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that are not slices.
4 Surface case \((m = 2)\)

Proof of Theorem 3. We calculate

\[
\Delta \left( \frac{1}{\cosh \theta} \right) = -\frac{\Delta \cosh \theta}{(\cosh \theta)^2} + \frac{2|\nabla \cosh \theta|^2}{(\cosh \theta)^3}.
\]

So, by (10) and (12) we have,

\[
\Delta \left( \frac{1}{\cosh \theta} \right) = -\frac{\Delta \cosh \theta}{(\cosh \theta)^2} - 2 \sum_{k,i<j} \lambda_i \lambda_j h_{ik} h_{jk}^{m+i} h_{jk}^{m+j} - 2 \sum_{k,i<j} \lambda_i \lambda_j h_{ik} h_{jk}^{m+i} h_{jk}^{m+j} - 2 \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2
\]

\[
+ \sum_{i=1}^2 \lambda_i^2 \left( (1-\lambda_i^2) K_1 - K_2 (a_3, a_4) \right).
\]

Since \(M\) is maximal and \(m = 2\), we have

\[
||B||^2 - 2 \sum_{k,i<j} \lambda_i \lambda_j h_{ik} h_{jk}^{m+i} h_{jk}^{m+j} - 2 \sum_{k,i<j} \lambda_i \lambda_j h_{ik} h_{jk}^{m+i} h_{jk}^{m+j} - 2 \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2
\]

\[
= ||B||^2 - 2 \lambda_1 \lambda_2 \left[ h_1^{m+1} h_2^{m+2} + h_2^{m+1} h_1^{m+2} + h_2^{m+1} h_2^{m+2} + h_1^{m+2} h_2^{m+1} \right]
\]

\[
- 2 \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2
\]

\[
\geq \sum_{i<j,k} [(h_{ik}^{m+i})^2 + (h_{jk}^{m+j})^2] + \sum_{i,k} (h_{ik}^{m+i})^2 - 2 \sum_{i,k} \lambda_i^2 (h_{ik}^{m+i})^2
\]

\[
\geq \sum_k (h_{1k}^{m+2})^2 + (h_{2k}^{m+1})^2 - (h_{1k}^{m+1})^2 - (h_{2k}^{m+2})^2
\]

\[
= 0.
\]

Therefore by assumption on the curvatures, (20) becomes

\[
\Delta \left( \frac{1}{\cosh \theta} \right) \leq -\frac{1}{\cosh \theta} \left( \sum_{i=1,2} \lambda_i^2 \left[ (1-\lambda_i^2) K_1 - K_2 (a_3, a_4) \right] \right) \leq 0.
\]

By Gauss equation the Gauss curvature of \(M\) is given by

\[
K_M = R_{1212} = \bar{R}_{1212} - \sum_\alpha (h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2) = \bar{R}_{1212} + \sum_\alpha [(h_{11}^\alpha)^2 + (h_{12}^\alpha)^2],
\]
where similarly to (18),

\[
\bar{R}_{1212} = \frac{1}{(1 - \lambda_1^2)(1 - \lambda_2^2)} \left[ K_1 - \lambda_1^2 \lambda_2^2 K_2(a_3, a_4) \right] \geq 0.
\]

Consequently, the Gauss curvature of \( M \) is nonnegative, and so \( M \) is parabolic, in the sense that any nonnegative superharmonic function on the surface is constant. By (22), \( \cosh \theta \) is constant, and the inequalities in (21) are identities. From these identities, we immediately have

\[ h_{ij}^{\alpha} = 0 \quad \text{for} \quad \alpha \geq 5 \]

and

\[
\sum_{i,k=1}^{2} (h_{ik}^{2+i})^2 = \sum_{i,k=1}^{2} \lambda_i^2 (h_{ik}^{2+i})^2.
\]

Since \( \lambda_i < 1 \) for \( i = 1, 2 \), the last equality implies \( h_{ij}^3 = h_{ij}^4 = 0 \). Therefore, \( M \) is totally geodesic and \( \lambda \) and \( \cosh \theta \) are constant. From (22) we conclude that, if at some point \( K_1(p) > 0 \), then \( f \) is constant. Now assume \( K_1 \) is identically zero, and \( K_2 < 0 \) at some point \( f(p) \). By (22), \( \lambda_1 \lambda_2 = 0 \). Hence, the rank of \( f \) is zero or one, and since \( f : \Sigma_1 \to \Sigma_2 \) is a totally geodesic map, the image of \( f \) lies on a geodesic of \( \Sigma_2 \). In case \( \Sigma_i \) are Euclidean spaces, a totally geodesic surface of \( \mathbb{R}^{2+n} \) is a plane.

So we have the following corollary of Theorem 3 for \( K_2 = -1 \):

**Corollary 11.** If \( M \) is a complete maximal spacelike graph of the pseudo–Riemannian product \( \mathbb{R}^2 \times \mathbb{H}^n \), defined by a map \( f : \mathbb{R}^2 \to \mathbb{H}^n \), then either \( f \) is constant or its image lies on a geodesic of \( \mathbb{H}^n \).

If \( \Sigma_2 \) is complete, there are trivial examples of complete totally geodesic spacelike graphs in \( \mathbb{R}^m \times \Sigma_2 \), with image of \( f \) a non-constant geodesic. Let \( \gamma : \mathbb{R} \to \Sigma_2 \) be an entire geodesic with \( \| \gamma'(0) \|^2 < 1 \). Then \( f : \mathbb{R}^m \to \Sigma_2 \), given by \( f(x_1, \ldots, x_m) = \gamma(x_1) \), is a totally geodesic map with image \( \gamma \), and the graph of \( f \) is a complete totally geodesic spacelike immersion. Note that we are using two facts: geodesics of \( \Sigma_2 \) are just the totally geodesic maps from \( \mathbb{R} \) into \( \Sigma_2 \), and totally geodesic graphs are just the graphs of totally geodesic maps (see section 2).
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