Effective Diffusions with Intertwined Structures

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Abstract

Let $p : N \to M$ be a surjective map of smooth manifolds. We are concerned with singular perturbation problems associated to a pair of second order positive definite differential operators with no zero order terms, that are intertwined by $p$. We discuss the associated random perturbations of stochastic differential equations and present a number of examples including perturbation to geodesic flows and construction of a Brownian motion on $S^2$ through homogenisation of SDE’s on the Hopf fibration.

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1 Introduction

The principal idea of singular perturbation is to deduce long term trends of a complex system from that of a relatively simple one for which some observables are known. We are mainly concerned with random perturbation of stochastic differential equations with intertwined diffusion structures. A diffusion operator $B$ on a smooth finite dimensional manifold $N$ is a second order differential operator with positive definite symbol and vanishing zero order term. Its symbol $\sigma^B$ is a real valued bilinear map on the cotangent bundle, determined by

$$\sigma^B(dg_1, dg_2) = \frac{1}{2} (B(g_1 g_2) - (Bg_1)g_2 - g_1(Bg_2)),$$
where \( g_1, g_2 \) are real-valued \( C^2 \) functions on \( N \). Let \( n = \dim(N) \). In a local coordinate system, let

\[
B g = \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 g}{\partial y_i \partial y_j} + \sum_{k=1}^{n} b_k \frac{\partial g}{\partial y_k}
\]

where \( a_{i,j} \) and \( b_k \) are smooth functions with the \( n \times n \) matrix valued function \( (a_{i,j}) \) positive symmetric. Then for \( y \in N \),

\[
\sigma_y^B (dg_1(y), dg_2(y)) = \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j}(y) \frac{\partial g_1}{\partial y_i}(y) \frac{\partial g_2}{\partial y_j}(y).
\]

The operator \( B \) is said to be elliptic if the symbol is strictly positive and to have constant rank if the rank of its symbol \( \sigma_u^B \) is constant in \( u \).

Let \( p : N \to M \) be a smooth onto map between smooth manifolds \( N \) and \( M \). Denote by \( Tp : TN \to TM \) the differential of \( p \). If \( B \) is an operator on \( N \) and \( A \) an operator on \( M \) such that

\[
B(f \circ p) = (Af) \circ p
\]

for all real valued \( C^2 \) functions \( f \) on \( M \), we say that \( B \) and \( A \) are intertwined or \( B \) is over \( A \). The simplest intertwining is given by projection of a product space to one of the factor spaces.

Let \( T_u p : T_u N \to T_{p(u)} M \) be the differential of the map \( p \) at \( u \), a linear map between tangent spaces. Its kernel is said to be a vertical tangent space, a subspace of the tangent space \( T_u N \) and is denoted by \( VT_u N \). The vector bundle \( VTN = \bigcup_u VT_u N \) is called the vertical tangent bundle. We say that a diffusion operator \( B^0 \) on \( N \) is vertical if \( B^0(f \circ p) = 0 \) for any \( C^2 \) function \( f \) on \( M \).

Suppose that \( B \) is over \( A \) and that \( \{ \sigma^A_x, x \in M \} \) has constant rank then there is a unique smooth lifting map

\[
h_u : \text{Image} [\sigma^A_{p(u)}] \subset T_{p(u)} M \to \text{Image} (\sigma^B_u) \subset T_u N
\]

such that \( T_u p \circ h_u \) is the identity map (Proposition 2.1.2 in Elworthy-LeJan-Li[11]). The image \( h_u \) induces a smooth distribution, called the horizontal distribution associated to \( A \). In the case of \( A \) elliptic let \( HT_u N \) be the image of \( h_u \), then

\[
T_u N = HT_u N \oplus VT_u N.
\]

In the case where \( p : N \to M \) is a principal bundle this induces an Ehresmann connection on \( N \) with a corresponding connection 1-form.
Let $A = \frac{1}{2} \sum_{i=1}^{m} L X_i, L X_i + L X_0$, where $X_i$ are vector fields and $L X_i$ denotes Lie differentiation in the direction of $X_i$. For short we also write $A = \frac{1}{2} \sum_{i=1}^{m} X_i^2 + X_0$. Suppose that $X_0 \in \text{Image } \sigma^A$ we say $A$ is cohesive. An elliptic operator is cohesive. The horizontal lift of $A$ is:

$$A^H = \frac{1}{2} \sum \tilde{X}_i^2 + \tilde{X}_0$$

where $\tilde{X}_i(u)$ is the horizontal lift of the tangent vector $X_i(p(u))$. By Theorem 2.2.5 in [11], there is a unique vertical diffusion $B^v$ such that $B = A^H + B^v$. If $u$ is a regular point of the map $p$, the $B^v$ diffusion starting from $u$ stays in the sub-manifold $p^{-1}(p(u))$. The vertical diffusion operator is ‘elliptic’ if $\sigma^B: VT_u N \times VT_u N \rightarrow \mathbb{R}$ is strictly positive definite.

Let $A$ be a cohesive diffusion on $M$ and $B^0$ a vertical diffusion. Let $L^\epsilon = \frac{1}{2} A^H + B^0$, or $L^\epsilon = A^H + \frac{1}{2} B^0$ where $\epsilon$ is a real number which we take to zero. We would like to understand the asymptotic behaviour of the solutions of $\frac{\partial}{\partial \epsilon} = L^\epsilon$ as $\epsilon$ goes to zero.

For simplicity write $L^\epsilon = \frac{1}{\epsilon} L_0 + L_1$. Let us expand $f^\epsilon(t, y)$, a solution of the parabolic differential equation $\frac{\partial}{\partial \epsilon} f^\epsilon(t, y) = L^\epsilon f^\epsilon(t, y)$ in $\epsilon$: $f^\epsilon = \frac{1}{\epsilon} f_0 + f_1 + \epsilon f_2 + o(\epsilon)$. What can we say about $f_i$? Is there a diffusion operator $\tilde{L}$ such that $f_1$ solves $\frac{\partial}{\partial \epsilon} = \tilde{L}$? See e.g. the book of Arnold [3] in the context of perturbation to Hamiltonian systems and a recent book of Stuart-Pavliotis [26] on multi-scale methods.

We now introduce notations concerning Markov processes associated to diffusion operators. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space. Let $\phi_t(y, \omega)$ be a family of strong Markov stochastic process with values in a manifold $N$ with $\phi_0(y) = y$. The $\omega$ variable will be suppressed for simplicity. Define a linear operator $P_t$ on the space of bounded functions by $P_t f(y) = \mathbb{E}_t f(\phi_t(y, \omega))$, where $\mathbb{E}$ denotes integration with respect to $P$. Then $(P_t, t \geq 0)$ is a semigroup of operators with $P_0$ the identity map. Its infinitesimal generator $\mathcal{L}$ is defined by $\mathcal{L} f = \lim_{t \rightarrow 0} \frac{P_t f - f}{t}$ with domain the set of functions such that the limit exists. For such $f$, $P_t f$ solves the parabolic equation $\frac{\partial}{\partial t} = \mathcal{L}$ with initial function $f$. On the other hand given any diffusion operator $\mathcal{L}$, there is a strong Markov process whose infinitesimal generator is $\mathcal{L}$. This can be seen by introducing a Hörmander form representation of $\mathcal{L}$ and a stochastic differential equation. When $\mathcal{L}$ is reasonably smooth the Markov process is continuous in $t$ for almost surely all $\omega \in \Omega$ and the Markov process is said to be a diffusion (process) with generator $\mathcal{L}$. If $\mathcal{L} = \frac{1}{2} \Delta$, where $\Delta$ is the Laplacian operator for a Riemannian metric, we say the diffusion process is a Brownian motion for that metric.

The dynamic picture of the perturbation problem is as following. Let $y^\epsilon_t$ be a diffusion operator with initial value $y^\epsilon_0$ associated to $\mathcal{L}_0 + \epsilon \mathcal{L}_1$. The
\[ y_\epsilon \] process follows roughly the orbit determined by \( \mathcal{L}_0 \) with negligible differences. However on a large time scale of order \( 1/\epsilon \), the deviation of the perturbed orbit from the unperturbed one becomes visible. In general we ask whether there is a function of \( y_\epsilon \) that does not change with time when \( \epsilon = 0 \). If so, denote this function by \( F \). The variable \( F(y_\epsilon) \) should vary slowly when \( \epsilon \rightarrow 0 \) and in the limit \( \mathcal{F}(y_\epsilon) \) may converge to a Markov process whose probability distribution is determined by a diffusion operator \( \bar{L} \). It is desirable to find all conserved quantities and their explicit probability distribution, i.e. the limiting diffusion operator \( \bar{L} \), which is said to be the effective motion. There is extensive literature on this and we refer to the following books and the references therein: Bensoussan-Lions-Papanicolaou [5] and Freidlin-Wentzell [14].

We now consider the problem at the level of stochastic differential equations. Every diffusion operator, if sufficiently smooth, can be represented as sum of squares of vector fields, the so called Hörmander form representation. We write the two diffusion operators \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) in Hörmander form: \( \mathcal{L}_0 = \frac{1}{2} \sum_{i=1}^{m} X_i X_i + X_0 \), and \( \mathcal{L}_1 = \frac{1}{2} \sum_{i=1}^{m} Y_i Y_i + Y_0 \). Let \( (b^i_t, w^i_t) \) be independent one dimensional Brownian motions on a given filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, P) \) with the usual assumptions. Let \( \phi_\epsilon(y) \) denote the solution to the stochastic differential equations (SDE) driven by the vector fields \( \frac{1}{\sqrt{\epsilon}} X_i, \frac{1}{\epsilon} X_0, Y_i, Y_0, i = 1, \ldots, m \), with initial point \( y \):

\[
\text{d}y_\epsilon^i = \frac{1}{\sqrt{\epsilon}} \sum_i X_i(y_\epsilon) \circ \text{d}b^i_t + \frac{1}{\epsilon} X_0(y_\epsilon) \text{d}t + \sum_j Y_j(y_\epsilon) \circ \text{d}w^j_t + Y_0(y_\epsilon) \text{d}t.
\]

The solutions are continuous Markov processes (diffusion processes) with generator \( \mathcal{L}' \).

**Main Results.** The structure of the paper is as following. In section 2, we present several intertwining examples to illustrate the terminologies. In section 3 SDE’s on the Hopf vibration are investigated. We construct stochastic process with generator a hypoelliptic horizontal diffusion on \( S^3 \) from two vector fields one of which the vertical vector field induced by the circle action and the other a horizontal vector field induced from an element of the Lie algebra of norm 1. In particular we construct a Brownian motion on \( S^2 \) through homogenisation.

In section 4 the state space is the frame bundle of a Riemannian manifold. The frame bundle is closely related to the tangent bundle of the manifold, notably suitable first order differential equations on the frame bundle correspond to second order differential equations on the manifold. It is also the natural space to record the position and orientation of a particle. For intertwined diffusion models on the frame bundle of a complete
Riemannian manifold $M$ there are two basic slow motions: the horizontal diffusions and the vertical diffusions. The horizontal diffusions appear naturally in Malliavin calculus. The solution flow of a vertical diffusion can be considered as a random evolution of a linear frame and is an interesting object in geometry, e.g. it was examined in Brendle-Schoen [8] in connection with the question as to whether positive isotropic curvature condition is preserved by R. Hamilton’s ODE.

In section 4.1 Perturbations to vertical diffusions on the orthonormal frame bundle, whose projection to the manifold is a fixed point are studied. The effective motion on the manifold is a diffusion which is of the same ‘type’ as the perturbation (Theorem 4.1). Special care has to be taken in this case to avoid assumptions on the injectivity radius of the orthonormal frame bundle. In section 4.2 a suitable perturbation model to the geodesic flow, a first-order ODE on the frame bundle, is developed. The perturbation is of Ornstein-Uhlenbeck type and in the limit we see a diffusion with generator $\frac{1}{n(n-1)} \Delta_H$, $\Delta_H$ being the horizontal Laplacian, and a rescaled Brownian motion to the manifold. This study relates to approximation of Brownian motions and by extension that of stochastic differential equations. We would also like to compare this to the philosophy in Bismut [7], that $\ddot{x} = \frac{1}{T}(-\dot{x} + \dot{w})$ interpolates between classical Brownian motion ($T \to 0$) and the geodesic flow ($T \to \infty$). In section 4.4 perturbation to the the semi-elliptic horizontal flow is considered and we obtain an effective motion that is transversal to the holonomy bundle.

In terms of ellipticity our SDEs have the following features. The unperturbed system can be elliptic or hypoelliptic. The perturbations could be elliptic, hypoelliptic or degenerate. We used two types of scalings: the standard scaling and the scaling of ‘Ornstein-Uhlenbeck type’. A related problem on commutation of linearisation with averaging is discussed in a paper in preparation [21].

2 Some Basic Examples

Intertwined structures occur naturally. One such standard example is projection of a product space. The other example is a principal bundle $p : N \to M$ with a group action $G$ which acts freely. The latter introduces a twist in the product structure. These will include projections of groups to their quotient groups and that of manifolds to their moduli spaces. See [111] and Liao [22] for a discussion of diffusion processes on symmetric spaces. We give some examples which illustrate the procedure of averaging with intertwined structures and the local structure of frame bundles. Let
us begin with the trivial example of a cylinder $\mathbb{R} \times S^1$. Let $z$ denote the $S^1$ direction, $p(x, z) = z$, $A = \frac{\partial}{\partial x}$, and $B = \sin z \frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2}$. Then $A^H = \frac{\partial^2}{\partial z^2}$. The effective motion associated to $\text{hypoelliptic flows}$ $\sin z \frac{\partial}{\partial x} + \frac{1}{\epsilon} \frac{\partial^2}{\partial z^2}$ converges to a simple completely degenerate motion.

**Example 1.** Take $N = \mathbb{R}^3$ with the Heisenberg group structure. For $(x, y, z) \in \mathbb{R}^3$ define $p(x, y, z) = (x, y)$. Let $Y_1 = \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial z}$ and $Y_2 = \frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial z}$ be the left invariant vector fields on $N$ associated to $(1,0,0)$ and $(0,1,0)$. Let $A^H = \frac{1}{2}(Y_1^2 + Y_2^2)$. Consider $B^0 = \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial z}$. Then $\mathcal{L}^\epsilon = \frac{1}{2}(Y_1^2 + Y_2^2) + \frac{1}{\epsilon} B^0$ is over the cohesive operator $A = \frac{1}{2}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ on $\mathbb{R}^2$. The associated horizontal lift is $(u, v) \mapsto (u, v, \frac{1}{2} x v - \frac{1}{2} y u)$. The vector fields $Y_1, Y_2$ are respectively the horizontal lifts of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ and $A^H$ is the horizontal lift of $A$ which is seen to have constant rank 2. The ‘slow part’ of the fully elliptic $\mathcal{L}^\epsilon$ diffusion converges weakly to the $A^H$ diffusion, a hypo-elliptic diffusion, as expected. We illustrate what does it mean by the ‘slow part’ by the following trivial, but explicit, SDE:

$$
\begin{align*}

dx^\epsilon_t &= \cos(z^\epsilon_t) \circ db^1_t - \sin(z^\epsilon_t) \circ db^2_t, \\
\quad dy^\epsilon_t &= \sin(z^\epsilon_t) \circ db^1_t + \cos(z^\epsilon_t) \circ db^2_t, \\
\quad dz^\epsilon_t &= \frac{1}{2}(x^\epsilon_t \sin(z^\epsilon_t) - y^\epsilon_t \cos(z^\epsilon_t)) \circ db^1_t + \frac{1}{2}(x^\epsilon_t \cos(z^\epsilon_t) + y^\epsilon_t \sin(z^\epsilon_t)) \circ db^2_t \\
&+ \frac{1}{\sqrt{\epsilon}} dw_t - \frac{1}{\epsilon} z^\epsilon_t dt.
\end{align*}
$$

Two obvious slow variables are $(x^\epsilon_t, y^\epsilon_t)$. In our terminology the slow part is the solution to the following SDE parametrized by $z^\epsilon_t$:

$$
\begin{align*}

\tilde{dx}^\epsilon_t &= \cos(z^\epsilon_t) \circ db^1_t - \sin(z^\epsilon_t) \circ db^2_t, \\
\quad \tilde{dy}^\epsilon_t &= \sin(z^\epsilon_t) \circ db^1_t + \cos(z^\epsilon_t) \circ db^2_t, \\
\quad \tilde{dz}^\epsilon_t &= \frac{1}{2}(\tilde{x}^\epsilon_t \sin(z^\epsilon_t) - \tilde{y}^\epsilon_t \cos(z^\epsilon_t)) \circ db^1_t + \frac{1}{2}(\tilde{x}^\epsilon_t \cos(z^\epsilon_t) + \tilde{y}^\epsilon_t \sin(z^\epsilon_t)) \circ db^2_t.
\end{align*}
$$

The law of the process in independent of $\epsilon$ and is $\frac{1}{2}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - y \frac{\partial^2}{\partial x \partial z} + \frac{1}{2}(x^2 + y^2) \frac{\partial^2}{\partial x \partial y}$, c.f. Example 4.1 for an example on the frame bundle that is in the same spirit. The third component of the Markov process associated to $\mathcal{L}^\epsilon$ converges to the stochastic area of the limits of the first two components: $\frac{1}{2} \int_0^t \tilde{x}^\epsilon_s d\tilde{y}^\epsilon_s - \frac{1}{2} \int_0^t \tilde{y}^\epsilon_s d\tilde{x}^\epsilon_s$ converges. This means taking stochastic area and taking $\epsilon \to 0$ commute, as expected.

This example can extend to the case of a general connection given by $\mathcal{h}(x,y,z)(u,v) = (u,v,r_1 u + r_2 v)$ allowing the functions $r_i$ to depend on $z$. Assuming that $(\frac{\partial r_1}{\partial y} - r_1 \frac{\partial r_2}{\partial y})^2$ is strictly positive, $\mathcal{h}$ is determined by the operator $A^H := \frac{1}{2} \left( \frac{\partial}{\partial x} + r_1 \frac{\partial}{\partial z} \right)^2 + \frac{1}{2} \left( \frac{\partial}{\partial y} + r_2 \frac{\partial}{\partial z} \right)^2$ and $A$, page 21 Elworthy-LeJan-Li[11]. Let $X_1 = \frac{\partial}{\partial x} + \frac{1}{2} r_1 \frac{\partial}{\partial z}$ and $X_2 = \frac{\partial}{\partial y} + r_2 \frac{\partial}{\partial z}$. The first prolongation of
span\\{X_1, X_2\}, i.e. span\\{X_1, X_2, [X_1, X_2]\} has full rank at each point. Consider a function \(\alpha\) such that the invariant measure \(\mu\) of \((\gamma - r_1^2 - r_2^2)\frac{\partial^2}{\partial z^2} + \alpha \frac{\partial}{\partial z}\) is the standard Gaussian measure. Let \(\gamma\) be such that \(\gamma - r_1^2 - r_2^2 > c > 0\) some \(c\). The \(A^H + \frac{1}{c^2}(\gamma - r_1^2 - r_2^2)\frac{\partial^2}{\partial z^2} + \alpha \frac{\partial}{\partial z}\) diffusion converges to an elliptic diffusion on the Heisenberg group if \(r_i\) are not constants a.s. with respect to \(\mu\).

**Example 3.** A non-relativistic quantum mechanical diffusion lives naturally in \(\mathbb{R}^3 \times SO(3)\), the orthonormal frame bundle of \(\mathbb{R}^3\). Its spatial projection lives in \(\mathbb{R}^3\). Studies associated to quantum mechanical equations, mainly the continuity equation describing the probability density of the quantum equation, have intertwined structures on \(p : \mathbb{R}^3 \times SO(3) \to \mathbb{R}^3\). I am grateful to D. Elworthy to bring my attention to the paper of Wallstrom [28] where limits of stochastic processes in \(\mathbb{R}^3 \times SO(3)\) are discussed. The Bopp-Haag-equations have one free parameter \(I\) and its solutions converge to that of an equation with Pauli Hamiltonian as \(I \to 0\). The Bopp-Haag-Dankel stochastic mechanical diffusions \(\mathbb{R}^3 \times SO(3)\) were introduced by Dankel, describing a diffusion particle with definite position and orientation. The Bopp-Haag-Dankel diffusions on \(\mathbb{R}^3 \times SO(3)\) are given by a simple SDE with drift given by a Pauli spinor (solution of quantum equation associated with Pauli Hamiltonian with parameter \(I\)). In [28] it was shown that for spin \(\frac{1}{2}\) wave functions and regular potentials the process parametrized by \(I\) converge to a Markovian process onto \(\mathbb{R}^3\), due to the averaging out of the orientational motion. The spatial projection describes the spatial motion of the particle without its orientation.

**Example 4.** Let \(G = SO(n)\) and \(\pi : \mathbb{R}^n \times G \to \mathbb{R}^n\) the projection to its first component. Let \(g = so(n)\) be the Lie algebra of the Lie group \(G\). For each \(x \in \mathbb{R}^n\), let \(h_x : T_x\mathbb{R}^n \sim \mathbb{R}^n \to g\) be a linear map varying smoothly in \(x\). The map \((x, v) \mapsto (x, h_x(v))\) can be considered as the horizontal lifting map through \((x, I)\) where \(I\) is the identity matrix. This induces on \(\mathbb{R}^n\) a non-trivial covariant differentiation \(\nabla\). Let \(e \in \mathbb{R}^n\), consider the SDE
\[dx_t = \epsilon_1 g_t \circ db_t + \epsilon g_t e dt\]
\[dg_t = \epsilon_1 h_x^*(g_t \circ db_t) g_t + e h_x(g_t e) g_t dt + \sqrt{\delta} \sum_{k=1}^{p} Z_k(x_t, g_t) \circ dw^k_t + \delta Z_0(x_t, g_t)dt.\]

where \((b^i_t, w^k_t)\) are independent 1-dimensional Brownian motions, \(b_t = (b^1_t, \ldots, b^n_t)\), \(w_t = (w^1_t, \ldots, w^n_t)\), and \(Z_k : \mathbb{R}^n \times G \to TG\) with \(Z_k(x, g) \in T_g G\). We denote by \(\circ\) Stratonovich integration, which must be used in the manifold setting and the correction term should be computed. When \(h = 0\) this corresponds to the flat connection. We consider three types of scalings: 1) \(\delta = 1, \epsilon_1 = \sqrt{\epsilon}\)
and $\epsilon \to 0$; 2) $\epsilon_1 = \epsilon = 1$ and $\delta \to 0$. For the third type take $\epsilon_1 = 0$, $\epsilon = 1$ and $\delta \to \infty$. In case 1) it turns out that the solution $x_i$ is a slow variable, despite the involvement of $g_i$.

Let us now describe the model using the language of orthonormal frame bundles. Let $v \mapsto h_{(x,g)}(v)$ be the horizontal lifting map. Then $h_{(x,g)}(v) = h_{(x,I)}(v)g$ where $I$ is the identity matrix. Denote by $h_x(v)$ the lifting map at $(x, I)$. Define $\theta_{(x,g)}(v, w) = g^{-1}(v)$. For $(v, A) \in \mathbb{R}^n \times \mathfrak{g}$, define $\varpi_{(x,I)}(v, A) = A - h_x(v)$ and

$$\varpi_{(x,g)}(v, A g) = g^{-1}\varpi_{(x,I)}(v, w)g = g^{-1}Ag - g^{-1}h_x(v)g.$$ 

The above example is the orthonormal frame bundle, $\mathbb{R}^n \times SO(n) \to \mathbb{R}^n$, of $\mathbb{R}^n$ with the group $G = SO(n)$ acting on the right of $\mathbb{R}^n \times SO(n)$. This describes the local structure of the orthonormal frame bundle of a Riemannian manifold. The frame bundle of $\mathbb{R}^n$ can also be represented as the Euclidean group. See section 4 for limiting theorems concerning the mentioned SDE in the context of a general manifold.

### 3 Homogenisation on the Hopf Fibration

Hopf fibration occurs in multiple situations in physics: in quantum systems and in mechanics, see e.g. Urbanke [27] for an account. Hopf fibration is the principle bundle $\pi : S^3 \to S^2$ with $S^1$ acting on the right. Here $S^n$ denotes the $n$-sphere. As pointed out by M. Berger in 1962, this is a non-trivial collapsing manifold. The sphere $S^3$ is equipped with a metric inherited from $\mathbb{R}^4$. The collapsing was achieved by shrinking the length by a scale of $\epsilon$ along the Hopf fibration direction and leaving the orthogonal directions unchanged. In a paper in preparation we study the dynamics associated to collapsing manifold [20].

It is convenient to consider the representation by unitary groups: $S^3$ is identified with $SU(2)$, $S^1$ with $U(1)$ and $S^2$ with $SU(2)/U(1)$. A typical element of $SU(2)$ may be expressed as $(z, w)$, where $z, w \in \mathbb{C}$ are such that $|z|^2 + |w|^2 = 1$, or as a matrix $\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$. The right action by $e^{i\theta} \in U(1)$ is $(z, w) \mapsto (e^{i\theta}z, e^{i\theta}w)$, which can be considered as right multiplication in the group $SU(2)$ by elements of the form $e^{i\theta} \sim (e^{i\theta}, 0)$:

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$ 

The Hopf map $\pi : SU(2) \to S^2$ is a submersion,

$$\pi(z, w) = (Re(2z\bar{w}), Im(2z\bar{w}), |z|^2 - |w|^2).$$
The map $T_u\pi$ can be better visualised if $S^3$ is treated as a subset of $\mathbb{R}^4$, writing $z = y_1 + iy_2, \ w = y_3 + iy_4, \ y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, 

$$T_y\pi = 2 \begin{pmatrix} y_3 & y_4 & y_1 & y_2 \\ -y_4 & y_3 & y_2 & -y_1 \\ y_1 & y_2 & -y_3 & -y_4 \end{pmatrix}.$$ 

The vertical tangent spaces are the kernels of $T\pi$. It is easy to check that the vector field $V(y_1, y_2, y_3, y_4) = -y_2\partial_1 + y_1\partial_2 - y_4\partial_3 + y_3\partial_4$ is vertical. Back to the principal bundle picture, $V((z, w)) := (iz, iw)$ is the fundamental vertical vector field, associated to $i$ in the Lie algebra of $U(1)$.

The Lie algebra $su(2)$ is the set of matrices such that $A + \overline{A}^T = 0$ and with zero trace:

$$\begin{pmatrix} ia & \beta \\ -\beta & -ia \end{pmatrix}, \quad a \in \mathbb{R}, \beta \in \mathbb{C}. $$

We take a real valued inner product on $su(2)$, $\langle A, B \rangle := \frac{1}{2} \text{trace } AB^*$, and the following orthonormal basis:

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. $$

Note that $X_1$ is adjoint invariant under the circle action and so is the linear span of $\{X_2, X_3\}$. Denote by $X^*_i$ the left invariant vector fields associated to $X_i$. Let us define a distribution $D = \text{span}\{X^*_2, X^*_3\}$, which is obviously left invariant with respect to the group action on $S^3$. The span of the left invariant vector fields is also right invariant under the circle action. This is due to the fact that $ue^{i\theta}X_i \in D_{ue^{i\theta}} = u(X_i e^{-2i\theta})e^{i\theta} \in D_{ue^{i\theta}}$ for $i = 2, 3$. Then $T_uS^3 = [\ker T_u\pi] \oplus D_u$ defines an Ehresmann connection on the principal bundle and a horizontal lifting map.

Let $\nabla^L$ be the left invariant linear connection and $\nabla$ the Levi-Civita connection for the bi-invariant Riemannian metric on the Lie group $SU(2)$. Denote by $\Delta$ the Laplacian on $S^3$. Let $\Delta_H = \sum_{i=2}^{3} \nabla^L df(X^*_i, X^*_i) = \sum_{i=2}^{3} \mathcal{L}_{X_i} L_{X_i}$ be the hypoelliptic Laplacian corresponding to the Horizontal distribution generated by the left invariant vector fields $\{X^*_2, X^*_3\}$.

3.1 Construction of Brownian Motion on $S^2$ by Homogenisation

Let $Y_0$ to be vector in $\text{span}\{X_2, X_3\}$. Since $Y_0e^{i\theta}$ remains in $\text{span}\{X_2, X_3\}$ the SDE $du^*_t = u_t^*Y_0 g^*_t dt + \frac{1}{\sqrt{t}} u^*_tX_1 \circ \sigma dB_t$ on $S^3$ makes sense for any stochastic process $g^*_t \in U(1)$. Note that $\{X_1, X_2, X_3\}$ is a Milnor frame [23] with structural constants $(-2, -2, -2)$, 

$$[X_1, X_2] = -2X_3, \quad [X_2, X_3] = -2X_1, \quad [X_3, X_1] = -2X_2.$$
If \( Y_0 = c_2X_2 + c_3X_3 \neq 0 \), span\{\( X_1, Y_0, [Y_0, X_1] \} = \) span\{\( X_1, X_2, X_3 \). By the structural equations \([Y_0, X_1] = 2c_2X_3 - 2c_3X_2 \) and \{\( X_1, Y_0, [Y_0, X_1] \)\} is linearly independent following from the non-degeneracy of the matrix

\[
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & c_2 & -2c_3 \\
  0 & c_3 & 2c_2 \\
\end{pmatrix}.
\]

It follows that the SDE under discussion is hypoelliptic.

The Hopf map \( \pi \) projects a curve \( u_t \) in \( SU(2) \) to one in \( S^2 \). A curve \( x_t \) in \( S^2 \) lifts to a horizontal curve \( \tilde{x}_t \) in \( SU(2) \) through the horizontal lifting map induced by the Ehresmann connection.

**Theorem 3.1** Let \((b_t)\) be a one dimensional Brownian motion and take \( u_0 \in SU(2) \). Let \((u^\epsilon_t, g^\epsilon_t)\) be the solution to the following SDE on \( SU(2) \times U(1) \), with \( u^\epsilon_0 = u_0 \) and \( g^\epsilon_0 = 1 \),

\[
du^\epsilon_t = u^\epsilon_t Y_0 g^\epsilon_t dt + \frac{1}{\sqrt{\epsilon}} u^\epsilon_t X_1 \circ db_t, \quad dg^\epsilon_t = \frac{1}{\sqrt{\epsilon}} g^\epsilon_t X_1 \circ db_t.
\]

Let \( x^\epsilon_t = \pi(u^\epsilon_t) \) and \( \tilde{x}_t \) its horizontal lift. Then \( \tilde{x}_t \) converges in probability to the hypoelliptic diffusion with generator \( \tilde{L}F = \frac{1}{2}|Y_0|^2 \Delta_H \). If \( Y_0 \) is a unit vector, \( x^\epsilon_t \) converges in law to the Brownian motion on \( S^2 \).

**Proof** Let \( a_t^\epsilon \in S^1 \) be such that \( u^\epsilon_t = \tilde{x}_t a_t^\epsilon \) where \( \tilde{x}_t \) is the horizontal lift of \( x^\epsilon_t \) through \( u_0 \) using the connection determined by \{\( X^*_2, X^*_3 \)\}. Then \( a_0^\epsilon = 1 \) and

\[
d\tilde{x}_t = TR_{(a_t^\epsilon)^{-1}} \circ du^\epsilon_t + ((a_t^\epsilon)^{-1} \circ da_t^\epsilon)^\ast(u^\epsilon_t).
\]

All stochastic integration involved in the above equation are Stratonovich integrals. Thus

\[
d\tilde{x}_t = TR_{(a_t^\epsilon)^{-1}} \left(u^\epsilon_t Y_0 g^\epsilon_t dt + \frac{1}{\sqrt{\epsilon}} u^\epsilon_t X_1 \circ db_t\right) + ((a_t^\epsilon)^{-1} \circ da_t^\epsilon)^\ast(\tilde{x}_t).
\]

Since \( \tilde{x}_t \) is horizontal, \( \omega(d\tilde{x}_t) = 0 \), we obtain

\[
(a_t^\epsilon) \circ d(\tilde{a}_t^\epsilon)^{-1} = -\omega_{\tilde{x}_t} \left(\frac{1}{\sqrt{\epsilon}} u^\epsilon_t X_1 (a_t^\epsilon)^{-1} \circ db_t\right) = \frac{1}{\sqrt{\epsilon}} a_t^\epsilon X_1 (a_t^\epsilon)^{-1} \circ db_t.
\]

It follows that \( d(\tilde{a}_t^\epsilon)^{-1} = -\frac{1}{\sqrt{\epsilon}} a_t^\epsilon X_1 (a_t^\epsilon)^{-1} \circ db_t \), \( (a_t^\epsilon) = (g^\epsilon_t) \), and

\[
d\tilde{x}_t = u^\epsilon_t Y_0 dt + \frac{1}{\sqrt{\epsilon}} u^\epsilon_t X_1 (g^\epsilon_t)^{-1} \circ db_t - \frac{1}{\sqrt{\epsilon}} \tilde{x}_t a_t^\epsilon X_1 (a_t^\epsilon)^{-1} \circ db_t = \tilde{x}_t g^\epsilon_t Y_0 dt.
\]
Since there is no Stratonovich correction term for \( dg_t = g_t X_1 \circ d\beta_t \), the corresponding infinitesimal generator is \( \frac{1}{2} \Delta_{S^1} \) where \( \Delta_{S^1} \) is the Laplacian on \( S^1 \). Let \( F : S^3 \to \mathbb{R} \) be any smooth function. Since \( Y_0 \in \text{span}\{X_2, X_3\} \),

\[
F(\tilde{x}_t^r) = F(u_0) + \int_0^t dF(\tilde{x}_s^r g_s Y_0) ds = F(u_0) + \sum_{j=2}^3 \int_0^t dF(\tilde{x}_s^r X_j) (\tilde{x}_s^r X_j, \tilde{x}_s^r g_s Y_0) ds
\]

\[
= F(u_0) + \sum_{j=2}^3 \int_0^t dF(\tilde{x}_s^r X_j)(X_j, g_s Y_0) ds.
\]

The two real valued functions on \( S^1 \), \( g \mapsto \langle X_2, g Y_0 \rangle \) and \( g \mapsto \langle X_3, g Y_0 \rangle \), are eigenfunctions of \( \Delta_{S^1} \). Then for \( j = 2, 3 \),

\[
dF(\tilde{x}_t^r X_j)(X_j, g_t Y_0) - dF(u_0 X_j)(X_j, Y_0)
\]

\[
= \sum_{k=2}^3 \int_0^t \nabla^L dF(\tilde{x}_s^r X_k, \tilde{x}_s^r X_j)(X_j, g_s Y_0)(X_k, g_s Y_0) ds + \frac{1}{\epsilon} \int_0^t dF(\tilde{x}_s^r X_j)(X_j, g_s X_1 Y_0) db_s
\]

\[
+ \frac{1}{\epsilon} \int_0^t dF(\tilde{x}_s^r X_j)(X_j, g_s^2 X_0 Y_0) ds.
\]

Applying the identities \( X_1^2 = -I \) we obtain

\[
F(\tilde{x}_t^r) = F(u_0) - \epsilon \sum_{j=2}^3 (dF(\tilde{x}_t^r X_j)(X_j, g_t Y_0) - dF(u_0 X_j)(X_j, Y_0))
\]

\[
+ \epsilon \sum_{j,k=2}^3 \int_0^t \nabla^L dF(\tilde{x}_s^r X_k, \tilde{x}_s^r X_j)(X_j, g_s Y_0)(X_k, g_s Y_0) ds (3.1)
\]

\[
+ \epsilon \sum_{j=2}^3 \int_0^t dF(\tilde{x}_s^r X_j)(X_j, g_s^2 X_1 Y_0) db_s.
\]

Since \( F \) is a smooth function on compact manifolds, the probability distribution of \( \{\tilde{x}_t^r, \epsilon > 0\} \) is tight, see Lemma [3.2] below. We now move to the canonical probability space with the standard filtration \( \mathcal{F}_t \). By Lemma [3.3] below, conditioning on the filtration \( \mathcal{F}_s \), the canonical filtration on the canonical probability space,

\[
\epsilon \sum_{j,k=2}^3 \int_0^t \nabla^L dF(\tilde{x}_s^r X_k, \tilde{x}_s^r X_j)(X_j, g_s Y_0)(X_k, g_s Y_0) ds
\]

converges to

\[
\sum_{j,k=2}^3 \int_0^t \nabla^L dF(u X_k, u X_j) \int_{S^1} \langle X_j, g Y_0 \rangle \langle X_k, g Y_0 \rangle dgd ds.
\]
Here $dg$ is the Haar measure on $S^1$. It is easy to check that
\[
\int_{S^1} \langle X_2, gY_0 \rangle \langle X_3, gY_0 \rangle dg = 0,
\]
either by direct computation or note that there is $g' \in U(1)$ such that $g'X_2 = -X_2$ and $g'X_3 = X_3$ and using the translation invariance of the Haar measure. Since there is an element of $S^1$ that maps $X_2$ to $X_3$,
\[
\int_{S^1} \langle X_2, gY_0 \rangle^2 dg = \int_{S^1} \langle X_3, gY_0 \rangle^2 dg.
\]
Note that
\[
\sum_{j=1}^2 \int_{S^1} \langle X_j, gY_0 \rangle \langle X_j, gY_0 \rangle dg = |gY_0|^2 = |Y_0|^2.
\]
We conclude that $\tilde{x}_t^\epsilon$ converges in distribution and its law is determined by the generator $\mathcal{L}F(u) = \frac{1}{2}|Y_0|^2 \nabla L dF(uX_2, uX_2) + \frac{1}{2}|Y_0|^2 \nabla L dF(uX_3, uX_3)$. Since $\nabla L X_i^\epsilon = 0$ for $i = 2, 3$, $\sum_{i=2}^3 \nabla L d(f \circ \pi)(Y_i^\epsilon, Y_i^\epsilon) = \operatorname{trace} \nabla d f$. Note also the Riemannian metric on $S^2$ is that induced from $S^3$, the process $x_i^\epsilon$ has generator $\frac{1}{2}|Y_0|^2 \Delta_{S^2}$ and is a Brownian motion when $Y_0$ is a unit vector.

**Lemma 3.2** Let $\mu^\epsilon$ be the probability distributions of the stochastic processes $(\tilde{x}_t^\epsilon, t \geq 0)$ from the theorem. Then $\{\mu^\epsilon, \epsilon > 0\}$ is relatively compact.

**Proof** Write $y_t^\epsilon = \tilde{x}_t^\epsilon$ for simplicity. Let $\mu_n$ be a subsequence from $\{\mu^\epsilon\}$ corresponding to a sequence of numbers $\epsilon_n$. We wish to prove that it has a weakly convergent subsequence. It is sufficient to prove that the family of measures $\mu_n$ is tight, i.e. for every $\delta > 0$ there exists a compact set $K_\delta \subset M$ such that $\mu_n(K_\delta) > 1 - \delta$ for all $n$. As probability measures on the space of continuous paths on $M$, $\mu_n(\sigma : \sigma(0) = y_0) = 1$ where $\sigma : [0, 1] \to M$ is a continuous path on $M$. For any $y_1, y_2 \in M$, let $\phi : M \times M \to \mathbb{R}$ be a smooth function that agrees with the Riemannian distance function when $d(y_1, y_2) < a/2$ where $a$ is the injectivity radius of $M$ and $\phi(y_1, y_2) = 1$ when $d(y_1, y_2) > 2a$. This is possible by taking $\phi = \alpha \circ d$ where $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is a suitable bump function with $\alpha$ the identity function on $[0, a/2]$. Then $\phi$ is a distance function on $M$ that generates the same topology as $d$. The family of measures $\{\mu_n\}$ is tight if for any $a, \eta > 0$ there exists $0 < \delta < 1$ such that there is an $\epsilon_0 > 0$, with
\[
P\left( \omega : \sup_{|s-t| < \delta} \phi(y_s^\epsilon, y_t^\epsilon) > a \right) < \eta, \quad \text{when } \epsilon < \epsilon_0.
\]
In the proof of the Theorem, take \( F(y) = \phi(y, u) \). Then by formula (3.1),
\[
\mathbb{E} \sup_{s \leq \delta} \phi^2(y_s, u) \leq \phi^2(y_0, u) + C \epsilon + \epsilon \delta
\]
for some constant \( C \). Let \( \phi'_s(y, u) \) denote \( y'_s(u) \) with \( y'_0(u) = y \). Let \( \theta_s \)
denotes the shift operator in the Wiener space. By the Cocycle property, for \( s < t \),
\[
\mathbb{E} \sup_{|s-t| \leq \delta} \phi^2(y_s^n, y'_t^n) = \mathbb{E} \mathbb{E} \left\{ \sup_{|s-t| \leq \delta} \phi^2(z, \phi'_{t-s}(z, \theta_{t-s}(u))) \middle| y_s^n = z \right\} \leq C \epsilon + \epsilon \delta
\]
and the required tightness holds. \( \square \)

Let \( y^n_t \) be a family of Markov processes on a Riemannian manifold \( M \) that is relatively compact. Represent this as the coordinate process on path space with measure \( \mu^n \), the distribution of \( y' \). Suppose that \( \mu_n \) converges weakly to \( \tilde{\mu} \). Let \( F : M \to \mathbb{R} \) be a smooth function with compact support. Let \( \mathcal{A} \) be a diffusion operator. Suppose that
\[
\int f \left( F(X_t) - F(X_0) - \int^t_s \mathcal{A}F(X_r)dr \right) d\mu_n \to 0
\]
for any function \( f \) that is measurable with respect to \( \mathcal{F}_s \) where \( (\mathcal{F}_s, s \geq 0) \) is the canonical filtration. Then \( \tilde{\mu} \) is the probability distribution of a \( \mathcal{A} \)-diffusion. In fact letting \( M^F_t = F(X_t) - F(X_0) - \int^t_s \mathcal{A}F(X_r)dr \), then \( M^F_t \) is a \( \mu \) martingale and \( \mu \) is the solution to the martingale problem associated to \( \mathcal{A} \). The following lemma reflects this philosophy. Let \( \epsilon_n \) be a sequence converges to zero. We are interested in the term
\[
\epsilon \sum_{j,k=2}^3 \int \nabla^l dF(x_s^\epsilon X_k, x_s^\epsilon X_j) \langle X_j, g^\epsilon_s Y_0 \rangle \langle X_k, g^\epsilon_s Y_0 \rangle ds.
\]

**Lemma 3.3** Let \((y^n_t, h^n_t)\) be a family of \( SU(2) \times U(1) \) valued stochastic processes on a probability space such that the law of \((y^n_t, h^n_t)\) agrees with that of \((\tilde{x}^\epsilon, \tilde{g}^\epsilon_t)\) in Theorem 3.1. Let \((\tilde{x}^\epsilon, \tilde{g}^\epsilon)\) be a weakly convergent sequence. Let \( y^n_t := y^n_t \).

We may assume that \( y^n_t \) converges to \( \tilde{y} \) almost surely. Let \( F : SU(2) \to \mathbb{R} \) be a smooth function. Define
\[
\mathcal{L}F(u) = \sum_{j,k=2}^3 \nabla^l dF(uX_k, uX_j) \int_{S^3} \langle X_j, gY_0 \rangle \langle X_k, gY_0 \rangle dg,
\]
\[
\mathcal{L}F(y^n_s, g^n_s) = \sum_{j,k=2}^3 \nabla^l dF(y^n_s X_k, y^n_s X_j) \langle X_j, g^n_s Y_0 \rangle \langle X_k, g^n_s Y_0 \rangle.
\]
Then the following convergence holds in $L^1$,

$$
\epsilon \int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}} \mathcal{A} F(y^\epsilon_r, g^\epsilon_r) dr \to \int_{s}^{t} \tilde{L} F(y^\epsilon_s) dr,
$$

and for any real valued bounded function $\phi$ on the path space,

$$
\mathbb{E}\phi(y^n_r, r \leq s) \left( F(y^n_t) - F(y^n_s) - \int_{s}^{t} \tilde{L} F(y^n_r) dr \right) \to 0.
$$

**Proof** By formula (3.1),

$$
F(y^n_t) - F(y^n_s) - \int_{s}^{t} \tilde{L} F(y^n_r) dr = \epsilon \sum_{j=2}^{3} dF(y^\epsilon_{t,j} X_j)(X_j, g Y_0) - \epsilon \sum_{j=2}^{3} dF(y^\epsilon_{s,j} X_j)(X_j, g Y_0)
$$

$$
+ \epsilon \int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}} \mathcal{A}^n F(y^\epsilon_r, g^\epsilon_r) dr - \int_{s}^{t} \tilde{L} F(y^n_r) dr.
$$

It is sufficient to prove that

$$
\epsilon \int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}} \mathcal{A}^n F(y^\epsilon_r, g^\epsilon_r) dr \to \int_{s}^{t} \tilde{L} F(y^n_r) dr.
$$

Let $t_0 = s < t_1 < \cdots < t_n = t$ be a division of $[s, t]$ with appropriate scale. Let $\Delta t_i = t_{i+1} - t_i$. Assume that $\Delta t_i = \sqrt{\epsilon}$ so $\frac{\Delta t_i}{\epsilon}$ is large. On each interval $[\frac{s}{\epsilon}, \frac{s+1}{\epsilon}],$.

$$
\left| \nabla^L dF(y^\epsilon_{t_i} X_k, y^\epsilon_{t_i} X_j) - \nabla^L dF(y^\epsilon_{t_{i+1}} X_k, y^\epsilon_{t_{i+1}} X_j) \right| \leq C |y^\epsilon_{t_i} - y^\epsilon_{t_{i+1}}| \sim o(\sqrt{\epsilon} \Delta t_i),
$$

where $\sim$ means in the order of, after taking expectations. Let $dg$ be the Haar measure on $S^1$. It is the invariant measure for $g^\epsilon_t$ where $g^\epsilon_t$ is solution to $dg^\epsilon_t = \frac{1}{\epsilon} g^\epsilon_t X_t dt$. By Birkhoff’s ergodic theorem on $S^1$ we obtain,

$$
\epsilon \int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}} \nabla^L dF(y^\epsilon_{t_i} X_k, y^\epsilon_{t_i} X_j)(X_j, g Y_0) \langle X_k, g Y_0 \rangle ds
$$

$$
\sim \sum_{i} \Delta t_i \nabla^L dF(y^\epsilon_{t_i} X_k, y^\epsilon_{t_i} X_j) \int_{\frac{s}{\epsilon}}^{\frac{t_{i+1}}{\epsilon}} \langle X_j, g Y_0 \rangle \langle X_k, g Y_0 \rangle ds
$$

$$
\sim \sum_{i} \Delta t_i \nabla^L dF(y^\epsilon_{t_i} X_k, y^\epsilon_{t_i} X_j) \int_{S^1} \langle X_j, g Y_0 \rangle \langle X_k, g Y_0 \rangle dg
$$

$$
\sim \int_{\frac{s}{\epsilon}}^{\frac{t}{\epsilon}} \nabla^L dF(g^\epsilon_t X_k, g^\epsilon_t X_j) dr \int_{S^1} \langle X_j, g Y_0 \rangle \langle X_k, g Y_0 \rangle dg
$$

$$
\sim \int_{s}^{t} \nabla^L dF(g^\epsilon_t X_k, g^\epsilon_t X_j) dr \int_{S^1} \langle X_j, g Y_0 \rangle \langle X_k, g Y_0 \rangle dg.
$$

\[\square\]
4 Perturbed Systems On Principal Bundles

Let $M$ be a smooth finite dimensional manifold. The frame bundle $\pi : FM \to M$ is a principal bundle with group action $GL(n, \mathbb{R})$. Its total space is the collection of all linear isomorphisms $u : \mathbb{R}^n \to T_{\pi(u)}M$. Given a Riemannian metric on $M$, the orthonormal frame bundle $\pi : OM \to M$ is a reduced bundle with group action $O(n)$ and the fibre at $u : \mathbb{R}^n \to T_{\pi(u)}M$ consisting of isometric linear maps. The total space of the frame bundle or the orthonormal frame bundle is a manifold in its own right. If $M$ is orientated the orthonormal frame bundle consists of two components.

The group will be denoted by $G$ and the right action by the component $\tilde{u}$ so $\tilde{u}$ is the collection of all linear isomorphisms $u : \mathbb{R}^n \to \mathbb{R}^n$. Let $\pi$ be a smooth finite dimensional manifold. The frame bundle $\pi : FM \to M$ consisting of isometric linear maps. The total space of the frame bundle $\pi : OM \to M$ is a principal bundle with group action $SO(n)$.

Denote by $TOM$ the tangent space of $OM$ and by $VT_uOM$ the naturally defined vertical sub-bundle, $VT_uOM = \ker[T_u\pi]$. If $A$ belongs to the Lie algebra $\mathfrak{so}(n)$, denote by $A^*$ the fundamental vertical vector field on $OM$ induced by right multiplication,

$$A^*(u) = \frac{d}{dt}|_{t=0} \exp(tA).$$

Then an o.n.b of $\mathfrak{so}(n)$ induces a family of vertical vector fields that spans $VTOM$ and $VTOM$ is an integrable sub-bundle of the tangent bundle $OM$.

A connection $\nabla$ on the tangent space of $M$ induces a splitting of the tangent spaces of $T_uOM$,

$$T_uOM = HT_uOM \oplus VT_uOM.$$

Let $HTOM = \sqcup_u HT_uOM$ and $VTOM = \sqcup_u VT_uOM$. Then $HTOM$ is a right invariant distribution and the splitting is an Ehresmann connection of $TOM$. Tangent vectors or vector fields are called horizontal (respectively vertical) if they take values in $TOM$ (respectively in $VTOM$). This determines a linear connection $\nabla$ on $M$, a connection 1-form $\omega_u : T_uOM \to \mathfrak{so}(n)$, and a horizontal lifting map $h_u : T_{\pi(u)}M \to T_uOM$.

To each $e \in \mathbb{R}^n$, there is associated a standard (or basic) horizontal vector field $H(e)$ given by $u \mapsto H(u)(e) \equiv h_u(ue)$. For $A \in \mathfrak{so}(n)$, $[H(e), A^*]$ is a horizontal vector field. For $e, \tilde{e} \in \mathbb{R}^n$ the vertical part of $[H(e), H(\tilde{e})]$ is given by the curvature form $\Omega$, $[H(e), H(\tilde{e})] = 2\Omega(H(e), H(\tilde{e}))$. Let $\{e_i\}$ be an orthonormal basis of $\mathbb{R}^n$ and define $H_i = H(u)(e_i)$.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space with the usual assumptions and let $\{w_i^j, h_i^j, 1 \leq j \leq p, 1 \leq l \leq m\}$ be independent one dimensional
Brownian motions. Let \( w_t = (w_t^1, \ldots, w_t^m) \) and \( b_t = (b_t^1, \ldots, b_t^p) \). Let \( \{X_l, l = 0, 1, 2, \ldots, m\} \) be a family of horizontal vector fields and \( \{Z_j, j = 0, 1, \ldots, p\} \) a family of vertical vector fields. Consider the SDE

\[
\begin{align*}
\left\{ \begin{array}{l}
du_t^i = \sqrt{\epsilon_1} \sum_{l=1}^m X_l(u_t^i) \circ db_t^l + \epsilon X_0(u_t^i) dt + \sqrt{\delta} \sum_{j=1}^p Z_j(u_t^i) \circ dw_t^j + \delta Z_0(u_t^i) dt, \\
u_0^i = u_0.
\end{array} \right.
\end{align*}
\]

(4.1)

here \( \epsilon_1, \epsilon \) and \( \delta \) are parameters. The infinitesimal generator of the SDE is

\[
\frac{1}{2} \epsilon_1 \sum_{l=1}^m X_l^2 + \epsilon X_0^2 + \delta \sum_{j=1}^m Z_j^2 + \delta Z_0.
\]

4.1 Perturbation to Vertical Flows

A Riemannian connection \( \nabla \) on \( M \) is a connection that is compatible with the Riemannian metric, with possibly a non-vanishing torsion \( T \). We take the horizontal bundle on the principle bundle induced by this connection. We will assume that the connection is complete, i.e. every geodesic extends to all finite time parameter. This is so if every standard horizontal vector field is complete. Let \( \varpi : T_uOM \rightarrow so(n) \) be the connection 1-form, corresponding to the given Riemannian connection \( \nabla \), which is determined by adjoint invariance and its values on fundamental vertical vector fields: \((R_g)^* \varpi = \text{ad}(g^{-1}) \varpi \) and \( \varpi(A) = A \). Let \( \theta_u : T_uOM \rightarrow \mathbb{R}^n \) be the canonical 1-form such that \( \theta_u(h_u(ue)) = e \). Let \( \nabla \) be the direct sum connection on \( TOM \). For any vector \( v \in T_uOM \) and vector field \( U \) on \( OM \),

\[
\nabla_v U = \varpi^{-1} d(\varpi(U))(v) + \theta^{-1} d(\theta(U))(v).
\]

This connection \( \nabla \) has zero curvature and a non-vanishing torsion in general. A formula for the torsion of \( \nabla \) is given in Li [21].

In (4.1) take \( \epsilon_1 = \epsilon \) and \( \delta = 1 \). Let \( \mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1 \) where

\[
\mathcal{L}_1 = \frac{1}{2} \sum_{l=1}^m L_{X_l} L_{X_l} + L_{X_0}, \quad \mathcal{L}_0 = \frac{1}{2} \sum_{j=1}^p L_{Z_j} L_{Z_j} + L_{Z_0}.
\]

**Theorem 4.1** Assume that \( M \) has positive injectivity radius, \( \{\varpi_u[Z_j(u)]\}_{j=1}^m \) spans \( g \), and the vector fields \( \{X_l, l \geq 0\} \) and \( \{\nabla_{X_l} X_l, l \geq 1\} \) have linear growth. Let \( u_t^i \) be a solution with initial value \( u_0 \in OM \), to the SDE

\[
du_t^i = \sqrt{\epsilon} \sum_{l=1}^m X_l(u_t^i) \circ db_t^l + \epsilon X_0(u_t^i) dt + \sum_{j=1}^p Z_j(u_t^i) \circ dw_t^j Z_0(u_t^i) dt.
\]

(4.2)

Let \( x_t^e = \pi(u_t^i) \) and \( \tilde{x}_t^e \) its horizontal lift. Let \( \mu_u \) be the invariant measure of the following SDE on \( G \):

\[
dg_t = \sum_{j=1}^m TL_{g_t} \varpi[Z_j(u g_t)] \circ dw_t^j + TL_{g_t} \varpi[Z_0(u g_t)] dt.
\]
Define
\[
\begin{align*}
  b(u) &= \int_G \left( \frac{1}{2} \sum_{l=1}^p \tilde{\nabla}_{\xi_l} \tilde{\xi}_l(ug) + \chi_0(ug) \right) d\mu_u(g) \\
  a_{i,j}(u) &= \int_G \sum_{l=1}^p \langle TR_{g^{-1}} \tilde{\xi}_l(ug), H_i(u) \rangle \langle TR_{g^{-1}} \tilde{\xi}_l(ug), H_j(u) \rangle d\mu_u(g),
\end{align*}
\]

Then \( \tilde{x}^\epsilon_t \) converges weakly with limiting generator \( \mathcal{L} \). For \( F : OM \to \mathbb{R} \) smooth with compact support,
\[
\mathcal{L}F(u) = dF(b(u)) + \frac{1}{2} \sum_{i,j=1}^p a_{i,j}(u)\nabla dF(H_i(u), H_j(u)).
\]

**Proof** Since \( \tilde{x}^\epsilon_t \) and \( u^\epsilon_t \) belong to the same fibre we may define \( g^\epsilon_t \in G \) by \( u^\epsilon_t = \tilde{x}^\epsilon_t g^\epsilon_t \). If \( a_t \) is a \( C^1 \) curve in \( G \)
\[
\frac{d}{dt} \Big|_{t=0} u_{a_t} = \frac{d}{dr} \big|_{r=0} u_{a_t} a_t^{-1} a_{r+t} = (a^{-1}_t a_t)^*(u_{a_t}).
\]
It follows that
\[
du^\epsilon_t = TR_{g^\epsilon_t}^* \tilde{x}^\epsilon_t + (TL_{g^\epsilon_t}^{-1})^* d\nu(u^\epsilon_t).
\]
Since right translation of horizontal vectors are horizontal, \( \varpi(u^\epsilon_t) = TL_{g^\epsilon_t}^{-1} d\nu(u^\epsilon_t) \)
and
\[
d\nu(u^\epsilon_t) = \sum_{j=1}^m TL_{g^\epsilon_t} \varpi[Z_j(\tilde{x}_t^\epsilon g^\epsilon_t)] d\nu(u^\epsilon_t) + TL_{g^\epsilon_t} \varpi[Z_0(\tilde{x}_t^\epsilon g^\epsilon_t)] dt.
\] (4.3)

By Itô’s formula, \( d\tilde{x}^\epsilon_t = \sqrt{\epsilon} \sum_{l=1}^p T\pi(\tilde{\xi}_l(u^\epsilon_t)) d\nu_l + \epsilon T\pi(\tilde{\xi}_0(u^\epsilon_t)) dt \)
so
\[
d\tilde{x}^\epsilon_t = \mathbf{h}_{\tilde{x}_t^\epsilon}(\circ d\tilde{x}^\epsilon_t) = \sqrt{\epsilon} \sum_{l=1}^p \mathbf{h}_{\tilde{x}_t^\epsilon} [T\pi(\tilde{\xi}_l(u^\epsilon_t))] d\nu_l + \epsilon \mathbf{h}_{\tilde{x}_t^\epsilon} [T\pi(\tilde{\xi}_0(u^\epsilon_t))] dt.
\]
By assumption on the vector fields \( \tilde{\xi}_l \), the above SDE does not explode and \( \pi(u^\epsilon_t) \) exists for all time. In terms of the group action, we have
\[
d\tilde{x}^\epsilon_t = \sqrt{\epsilon} \sum_{l=1}^p \mathbf{h}_{\tilde{x}_t^\epsilon} [T\pi(\tilde{\xi}_l(u^\epsilon_t))] d\nu_l + \epsilon \mathbf{h}_{\tilde{x}_t^\epsilon} [T\pi(\tilde{\xi}_0(u^\epsilon_t))] dt.
\] (4.4)
Let \( \mu^\epsilon \) be the laws of the \( \tilde{x}^\epsilon_t \). We first show that \( \{\mu^\epsilon\} \) is tight. By Prohorov’s theorem a family of probability measures is tight if it is relatively compact. Since \( \tilde{x}_0^\epsilon = u_0 \) it suffices to estimate the modulus of continuity and show...
that for all positive numbers \( a, \eta \), there exists \( \delta > 0 \) such that for all \( \epsilon \) reasonably small, see Billingsley [6] Ethier-Kurtz [13],

\[
P(\omega : \sup_{|s-t|<\delta} d(\tilde{x}_t^\epsilon, \tilde{x}_s^\epsilon) > a) < \eta.
\]

Here \( d \) denotes a distance function on \( OM \). The Riemannian distance function is not smooth on the cut locus. The cut locus of \( OM \) is in general not predictable by that of \( M \). To avoid any assumption on the cut locus of \( OM \) we construct a new distance function that preserves the topology of \( OM \).

Let \( x \in M \) and \( 2a \) the minimum of 1 and the injectivity radius of \( M \). Let \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a smooth concave function such that \( \phi(r) = r \) when \( r < a \) and \( \phi(r) = 1 \) when \( r \geq 2a \), e.g. \( \phi \) is the convolution of \( \min(1, r) \) with a standard mollifier supported in the set \( \{ r : |r - \frac{3a}{2}| < a/2 \} \). Let \( \rho \) and \( \tilde{\rho} \) be respectively the Riemannian distance on \( M \) and \( OM \). Then \( \phi \circ \rho \) and \( d := \phi \circ \tilde{\rho} \) are distance functions. For \( u \in \pi^{-1}(x) \),

\[
\phi \circ \tilde{\rho}(u, \tilde{x}_t^\epsilon) = (\phi \circ \tilde{\rho})(u, \tilde{x}_0^\epsilon) + \int_0^t d(\phi \circ \tilde{\rho}) \left( \sqrt{\epsilon} \sum_{l=1}^p \eta_{\tilde{x}_l^\epsilon} [T\pi(X_l(u_s))] \circ dB_s^l \right)
\]

\[
+ \int_0^t \epsilon d(\phi \circ \tilde{\rho}) \eta_{\tilde{x}_l^\epsilon} [T\pi(X_l_0(u_s^\epsilon))] \, ds
\]

\[
= (\phi \circ \tilde{\rho})(u, \tilde{x}_0^\epsilon) + \int_0^t d(\phi \circ \rho) \left( \sqrt{\epsilon} \sum_{l=1}^p [T\pi(X_l(u_s))] dB_s^l \right)
\]

\[
+ \epsilon \sum_{l=1}^p \int_0^t \nabla d(\phi \circ \rho) (T\pi(X_l(u_s)), T\pi(X_l(u_s^\epsilon))) \, ds
\]

\[
+ \epsilon \int_0^t d(\phi \circ \rho) \left( \frac{1}{2} \sum_{l=1}^p \nabla_{T\pi(X_l)}(T\pi \circ X_l)(u_s) + T\pi(X_l_0(u_s^\epsilon)) \right) ds.
\]

Since \( \phi \circ \rho \) has compact support and the vector fields concerned have linear growth, \( |T\pi(X_l(u_s^\epsilon))| \leq C(1 + \rho(u_s^\epsilon, u)) \leq [C + C\tilde{\rho}(\tilde{x}_s^\epsilon, \tilde{u}_s^\epsilon)] + C\tilde{\rho}(u, \tilde{x}_s^\epsilon) \) some \( u \in OM \). The quantity \( C + C\tilde{\rho}(\tilde{x}_s^\epsilon, \tilde{u}_s^\epsilon) \) is bounded from the compactness of \( G \) and it follows that \( \mathbb{E}[\phi \circ \tilde{\rho}(u, \tilde{x}_t^\epsilon)^2] \leq C_1(t)((\phi \circ \tilde{\rho})^2(u, \tilde{x}_0^\epsilon) + \epsilon t) \) for some constant \( C \). By the Markov property and the estimates below the required tightness follows,

\[
\mathbb{E}[\phi \circ \tilde{\rho}(\tilde{x}_t^\epsilon, \tilde{x}_s^\epsilon)^2] \leq C_1|t - s|.
\]

By the right invariance of the horizontal lift,

\[
\eta_{\tilde{x}_l^\epsilon} [T\pi_{\tilde{\pi}_{G_2}}(X_l(\tilde{x}_s^\epsilon, \tilde{g}_s))] = TR_{\tilde{g}_s}(X_l(u_s^\epsilon)).
\]
Let $F : OM \to \mathbb{R}$ be a smooth function. For $\nabla$, the canonical direct sum connection on $OM$ associated to $\nabla$,

$$
F(\tilde{x}_t) = F(u_0) + \sqrt{\epsilon} \sum_{l=1}^{p} \int_0^t \left( \nabla_{\nabla} X_l(u_s) + X_l(u_s) \right) dB_s^l
$$

$$
+ \frac{1}{2} \epsilon \sum_{l=1}^{p} \int_0^t \nabla dF \left( TR_{g_\epsilon}^{-1} X_l(u_s) + TR_{g_\epsilon}^{-1} X_l(u_s) \right) ds
$$

$$
+ \frac{1}{2} \epsilon \sum_{l=1}^{p} \int_0^t dF \left( TR_{g_\epsilon}^{-1} X_l(u_s) + TR_{g_\epsilon}^{-1} X_l(u_s) \right) ds.
$$

By tightness and Prohorov’s theorem we may take a sequence $\epsilon_n \to 0$ with the property that $\tilde{x}_{\epsilon_n}^n_t$ converges in law to a probability measure $\mu$. Let $X$ be the coordinate process on the path space and $G_t = \sigma \{ X_s : 0 \leq s \leq t \}$. Since $G$ is compact the following term has at most quadratic growth,

$$
\int_G \langle TR_{g_\epsilon^{-1}} X_l(u_g), H_i(u) \rangle \langle TR_{g_\epsilon^{-1}} X_l(u_g), H_j(u) \rangle \mu(u) dg,
$$

and by the same argument $\int_G \sum_{l=1}^{p} \nabla X_l(u_g) \mu(u) dg$ has linear growth. To identify the limiting process it suffices to show that for all real-valued smooth functions $F$ on $OM$ with compact support,

$$
\int \left( F(X_t) - F(X_s) - \int_s^t \tilde{L} F(X_r) dr \right) g \, d\mu_t
$$

converges to zero where $g$ is any real-valued bounded $G_s$-measurable function on the Wiener space and $X_t$ the canonical process.

Let $z^n_t$ be a sequence of random variables whose law agrees with that of $\tilde{x}_{\epsilon_n}^n_t$ for some sequence $\epsilon_n$ and $z^n_t$ converges almost surely. Let $g$ be a $\{ z^n_s, s \leq t \}$-adapted bounded function. For $t \geq s$,

$$
\mathbb{E} g \left( F(z^n_t) - F(z^n_s) - \int_s^t \tilde{L} F(z^n_r) dr \right) = \mathbb{E} \left[ g \int_s^t (A^n F - \tilde{L} F)(z^n_r) dr \right] \to 0,
$$

where $A^n F$ is given by the bounded variation part in the formula for $F(\tilde{x}_t)$. The convergence holds since $G$ is compact and also the invariant measure $\mu_G$ for the elliptic SDE (4.3) is ergodic. The proof is standard and follows from the Lemma below. See e.g. Hasminskii [15], Papanicolaou-Stroock-Varadhan [25].
Lemma 4.2 Let $f$ be a bounded function with bounded derivative then

$$\int_s^t A^\epsilon f(\tilde{x}_r^\epsilon)dr = \int_s^t \dot{L} f(\tilde{x}_r^\epsilon)dr + R(f, \epsilon, s, t)$$

where $(E \sup_{s \leq t} |R(f, \epsilon, s, t)|^\beta)^{\frac{1}{\beta}} \leq C(t) \epsilon^{\frac{1}{\beta}}$ for any $\beta > 1$.

The proof is completely analogous of that of Lemma 3.2 in [19]. In sub-intervals whose length is very small compared to $1/\epsilon$ we consider $\tilde{x}_s^\epsilon$ as constants, and apply the ergodic theorem on each interval. With the size of the sub-intervals chosen correctly, the sum over all sub-intervals of the limits forms a Riemann sum. The convergence follows from the Cocycle property of the flows, estimates for the rate of convergence in the ergodic theorem and the regularity of the function $A^\epsilon f$.

In the theorem above the assumption on the injectivity radius can be removed in the case of the projection being a Brownian motion with bounded drift. See e.g. the estimates in [18]. We look into two special cases, when the horizontal vector fields are either right invariant (lifts of vector fields on the manifold $M$) or the standard horizontal vector fields.

Example 4.1 (The Right Invariant Case) Let $X_l, l = 0, 1, 2, \ldots, m$, be vector fields on $M$. Define $X_l(u) = h_u(X_l(\pi(u)))$ and we have

$$du_l^\epsilon = \sqrt{\epsilon} \sum_{i=1}^p X_i(u_l^\epsilon) \circ db_l^i + \epsilon X_0(u_l^\epsilon) dt + \sum_{j=1}^m Z_j(u_l^\epsilon) \circ dw_l^j + Z_0(u_l^\epsilon) dt.$$ 

The projection $\pi(u_l^\epsilon)$ satisfies $dx_l^\epsilon = \sqrt{\epsilon} \sum_{i=1}^p X_l(x_l^\epsilon) \circ db_l^i + \epsilon X_0(x_l^\epsilon) dt$. For all $\epsilon$, $x_l^\epsilon$ are $\frac{1}{2} \sum L_{X_l} L_{X_i} + L_{X_0}$-diffusions. The horizontal lift $\tilde{x}_l_t$ of $x_l_t$ are $\frac{1}{2} \sum L_{X_l} L_{X_i} + L_{X_0}$-diffusions.

Example 4.2 (The Rotational Invariant Case) Let $\{e_l\}_{l=1}^n$ be an o.n.b. of $\mathbb{R}^n$, $e_0 \in \mathbb{R}^n$. Let $H_l(u) \equiv H(u)(e_l)$, and $H_0(u) \equiv H(u)(e_0)$ be horizontal vector fields. We have

$$du_l^\epsilon = \sqrt{\epsilon} \sum_{i=1}^n H_l(u_l^\epsilon) \circ db_l^i + \epsilon H_0(u_l^\epsilon) dt + \sum_{j=1}^m Z_j(u_l^\epsilon) \circ dw_l^j + Z_0(u_l^\epsilon) dt.$$ 

Write $\tilde{x}_l^\epsilon = u_l^\epsilon(g_l^\epsilon)^{-1}$. Then

$$d\tilde{x}_l^\epsilon = \sqrt{\epsilon} H(\tilde{x}_l^\epsilon)(g_l^\epsilon \circ db_l^i) + \epsilon H(\tilde{x}_l^\epsilon)(g_l^\epsilon e_0) dt. \quad (4.5)$$
Its ‘formal’ Stratonovitch correction term vanishes. If $\hat{x}_0^\epsilon = u_0$ then $g_0^\epsilon$ is the identity matrix. Write $du_t^\epsilon = dt$ and let $Z_j = \sigma_k^j A_j^\epsilon$ where $\{A_j\}$ is an o.n.b of $so(n)$. Then
\[ dg_t^\epsilon = \sum_{j,k} \sigma_k^j(u_t^\epsilon)g_t^\epsilon A_j \circ dw_t^k. \]

If $\sigma_k^j$ are constants the process $g_t^\epsilon$ is independent of $\epsilon$ and $\hat{x}_t^\epsilon$ is a Markov process on $OM$. If furthermore $e_0 = 0$, the law of $\hat{x}_t^\epsilon$, and hence that of $x_t^\epsilon$, is independent of $\epsilon$. This follows from the independence of $g_t^\epsilon$ and $\{b_t\}$. Finally $\hat{x}_t^\epsilon$ is a horizontal Brownian motion with projection $x_t$ a Markov process and a Brownian motion on $M$.

Remark: More generally if $\{\Phi_t(u)\}$ is a family of Markov processes on $OM$ with the property that $\Phi_t(u) \overset{law}{=} \Phi_t(u)\psi_t(g)$ for some $\psi_t(g) \in G$ and $\sigma\{\pi(\Phi_t(u)) | r \leq s\} = \sigma\{\Phi_r(u) : r \leq s\}$, then $\pi(\Phi_t(u))$ is a Markov process. Denote by $Q_t(u_0, du)$ the law of $\Phi_t(u_0)$ and let $f : M \rightarrow \mathbb{R}$ be a Borel measurable function, $x_t = \pi(\Phi_t(u_0))$,
\[ \mathbb{E}\{f(x_t) | \sigma\{x_r, r \leq s\}\} = \int (f \circ \pi)(u)Q_{t-s}(\hat{x}_s, du). \]

It follows that $\int (f \circ \pi)(u)Q_{t-s}(\hat{x}_s, du) = \int (f \circ \pi)(u\psi_t(g))Q_{t-s}(\hat{x}_s g, du) = \int (f \circ \pi)(u)Q_{t-s}(\hat{x}_s, du)$. So $\mathbb{E}\{f(x_t) | \sigma\{x_r, r \leq s\}\}$ depends only on $x_s = \pi(\hat{x}_s)$. When $e_0 = 0$, the flow of $\mathbf{4.5}$ satisfies the rotational invariance condition and the horizontal lift of $x_t$ is a function of the path $(x_r, r \leq t)$.

Example 4.3 Let $\alpha : M \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth map so that $\alpha(x) \in \mathbb{L}(\mathbb{R}^n; \mathbb{R}^n)$. Let $\{e_i\}_{i=1}^n$ be an o.n.b of $\mathbb{R}^n$, $e_0 \in \mathbb{R}^n$. Consider
\[ du_t^\epsilon = \sqrt{\epsilon} \sum_{i=1}^n h_u[\alpha(\pi(u))e_i] \circ db_t + \epsilon h_u[\alpha(\pi(u))e_0](u_t^\epsilon) dt + \sum_{j=1}^m Z_j(u_t^\epsilon) \circ dw_t^j + Z_0(u_t^\epsilon) dt. \]

The projection $x_t^\epsilon = \pi(u_t^\epsilon)$ satisfies:
\[ dx_t^\epsilon = \sqrt{\epsilon} \sum_{i=1}^n u_t^\epsilon \alpha(x_t^\epsilon)(e_i) \circ db_t + \epsilon u_t^\epsilon \alpha(x_t^\epsilon)(e_0) dt = \sqrt{\epsilon} u_t^\epsilon \alpha(x_t^\epsilon) \circ db_t + \epsilon u_t^\epsilon \alpha(x_t^\epsilon)(e_0) dt. \]

Let $\hat{x}_t^\epsilon$ be the horizontal lifting map of $x_t^\epsilon$ and $g_t^\epsilon$ be an element of $G$ determined by $u_t^\epsilon = \hat{x}_t^\epsilon g_t^\epsilon$. Then $d\hat{x}_t^\epsilon = \sqrt{\epsilon} H(\hat{x}_t^\epsilon)g_t^\epsilon \alpha(x_t^\epsilon) \circ db_t + \epsilon H(\hat{x}_t^\epsilon)g_t^\epsilon \alpha(x_t^\epsilon)(e_0) dt$.

When $\alpha(x)$ is not trivial the bounded variation term for $f(x_t)$, where $f : M \rightarrow \mathbb{R}$ is a smooth function, will involve $\sum_t \nabla df(u_t^\epsilon \alpha(x_t^\epsilon) e_i, u_t^\epsilon \alpha(x_t^\epsilon) e_i)$ which is no longer a trace. It will also involve the derivative of $X_t$. In this case it is useful to consider the system as perturbation of the vertical SDE about which we know a lot more.
4.2 Perturbation of Ornstein-Uhlenbeck Type

We now describe the relation between horizontal equations on frame bundles and geodesic flows. Let \( P = GL(M) \) be the linear frame bundle over \( M \). A vector field on a frame bundle can be considered as a second order differential equation on the underlying manifold as below. Fix \( e_0 \in \mathbb{R}^n \) and \( H \) the isotropy group at \( e_0 \) of the action \( G = GL(n, \mathbb{R}) \) on \( \mathbb{R}^n \). The tangent bundle \( TM \) can be considered as a fibre bundle associated with the principal fibre bundle \( P \) with fibre \( \mathbb{R}^n \). The total space \( E \) is \( P \times \mathbb{R}^n / \sim \) where the equivalent class is determined by \([u, e] \sim [ug^{-1}, ge]\), any \( g \in G \). Elements of the form \( ug \) where \( g \in H \) belong to the same equivalence class. It can be identified with the quotient bundle \( P/H \), whose element containing \( u \) is the equivalence class of the form \( \{ug, g \in H\} \). Denote by \( \xi_0 \) the coset \( H \).

Let \( \alpha \) be the associated map:

\[
\alpha_{e_0} : u \in P \rightarrow ue_0 \in TM.
\]

This induces a map \( w \in T_uP \rightarrow T_u\alpha_{e_0}(w) \in T_{ue_0}TM \). Each element \( v \in TM \) has a representation \( v = ue_0 \), where \( u \) is unique up to right translation by elements of \( H \). Furthermore a right invariant vector field \( W \) on \( P \) induces a vector field on \( TM \). In fact if \( v = ue_0' = ue_0 \) there is \( g \in G \) with \( u' = ug \) and \( e_0' = g^{-1}e_0 \). Since \( \alpha_{e_0}(u) = \alpha_{e_0}(R_gu) \),

\[
T_u\alpha_{e_0}(W(u)) = T_{u'}\alpha_{e_0'}(TR_gW(u)) = T_{u'}\alpha_{e_0'}W(u').
\]

This map \( W \in \Gamma TP \mapsto X_W \in \Gamma TT M \) is independent of the choices of \( e_0 \). Fix \( e_0 \). Any vector field \( W \) that is invariant by right translations of elements of \( H \) induces a vector field on \( TM \). Consider a horizontal distribution determined by a connection on \( TM \) and let \( W(u) = H_u(e_0) \) be the fundamental horizontal vector field associated to \( e_0 \), the induced vector field is a geodesic spray \( X \), i.e. in local co-ordinates \( X(x, v) = (x, v, Z(x, v)) \) and \( Z(x, sv) = s^2Z(s, v) \), which corresponds to the geodesic flow equation on \( TM \):

\[
dv_t = -\Gamma_{\sigma_t}(vt)(vt), \quad \dot{\sigma}_t = vt, \quad \sigma(0) = \pi(u), \quad v(0) = ue_0.
\]

Here \( \Gamma \) denotes the Christoffel symbol. The corresponding horizontal flow on \( P \) is given by \( \dot{u}_t = H(u_t)(e_0) \).

Based on E. Nelson’s Ornstein-Uhlenbeck theory of Brownian motions [24] we ask the following question. What happens if we replace the driving Brownian motion \( dw_t \) by \( vt dt \) where \( v_t \) is an Ornstein-Uhlenbeck process? Consider the position process \( z_t \) in \( \mathbb{R}^n \) with velocity process satisfy
the Langevin equation:

\[ dv_t^\epsilon = -\frac{1}{\epsilon} v_t^\epsilon dt + \frac{1}{\epsilon} dw_t, \quad z_t^\epsilon = v_t^\epsilon. \]

where \( w_t \) is a Brownian motion with values in \( \mathbb{R}^n \) and \( z_0 = 0 \). The \( z_t^\epsilon \) process converges to \( w_t \) as \( \epsilon \to 0 \). The convergence holds almost surely and in fact the result holds if \( w_t \) is replaced by any continuous function. We now interpret the convergence in terms of homogenisation. First we rescale the variables in space and time and setting \( \tilde{v}_t = \sqrt{\epsilon} v_t, \tilde{z}_t = \sqrt{\epsilon} z_t \). It is easy to see that \( \tilde{z}_t^\epsilon \) is the slow variable and \( \sqrt{\epsilon} z_t^\epsilon \) converges to a Brownian motion. In fact

\[ d\tilde{v}_t^\epsilon = -\frac{1}{\epsilon} \tilde{v}_t^\epsilon dt + \frac{1}{\sqrt{\epsilon}} dw_t, \quad \tilde{z}_t^\epsilon = \tilde{v}_t^\epsilon. \]

We must take care to take this model to the orthonormal bundle. First we are not allowed to rescale variables in non-linear spaces. We should not rescale the frame variable, in the orthonormal frame bundle, in space either.

We shall consider the orthonormal frame bundle of an \( n \)-dimensional manifold with group action by \( G = O(n) \) or \( G = SO(n) \) if the manifold is oriented so that \( OM \) is a connected manifold of its own right. In the latter case we assume that \( n > 1 \). Let \( e_0 \in \mathbb{R}^n \) be a unit vector and \{\( A_k, k = 1, 2, \ldots, N = n(n - 1)/2 \)\} be elements of \( g \), and \( A_0 \in g \). Let \( A_k^* \) be the corresponding fundamental vertical vector field corresponding to \( A_k \). Consider

\[ du_t^\epsilon = H(u_t^\epsilon)(e_0)dt + \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{N} A_k^*(u_t^\epsilon) \circ dw_t^k + \frac{1}{\epsilon} A_0^*(u_t^\epsilon)dt. \]

For ‘\( \epsilon = \infty \)’, the equation can be considered as the ‘geodesic flow’ equation, as explained earlier. If \( x_t^\epsilon = \pi(u_t^\epsilon) \) then \( \dot{x}_t^\epsilon = u_t^\epsilon e_0 \). Note that the change of the velocity of the motion on \( M \) is always unitary. Due to the fast rotation, the geodesic has rapid changing directions and we expect to see a jittering motion and indeed we obtain a scaled Brownian motion in the limit if the rotational motion is elliptic.

A related theorem is given in Dowell [9] stating that an Ornstein-Uhlenbeck position process on 2-uniformly smooth Banach manifolds converges. Those are manifolds modelled on 2-uniformly smooth Banach spaces. By a 2-uniformly smooth Banach space \( B \) we mean one with the property that there is a constant \( C > 0 \) such that \[ ||x + y||^2 + ||x - y||^2 \leq 2||x||^2 + C||y||^2 \]
holds for all \( x, y \in B \). The iterated Ornstein-Uhlenbeck processes in [9] and the settings of manifolds with Lorenzo metrics, see e.g. Bailleul [4], are also worth exploring. We expect interesting results arise for processes with infinite-dimensional noise. For a related work, central limit theorem for geodesic flows, we refer to Enriquez-Franchi-LeJan [12].

Let \( A_k^* \) denote also the left invariant vector field on \( G \) induced by \( A_k \in g \): \( A_k^*(g) = g A_k \). Let \( A = \frac{1}{2} \sum_k (A_k^*)^2 + A_0^* \). Denote by \( \Delta^L \) the left invariant Laplacian on \( G \). For all \( \epsilon \), let \( u_0^\epsilon = u_0 \) and \( x_0 = \pi(u_0) \). Let \( \{e_i\} \) be an orthonormal basis of \( \mathbb{R}^n \) and we may take \( e_1 = e_0 \).

**Theorem 4.3** Let \( M \) be a compact Riemannian manifold, \( u_0 \in OM \). Let \( u_t^\epsilon \) be the solution to the SDE on \( OM \):

\[
du_t^\epsilon = H(u_t^\epsilon)(e_0)dt + \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^N A_k^*(u_t^\epsilon) \circ dw_t^k + \frac{1}{\epsilon} A_0^*(u_t^\epsilon)dt, \quad u_0^\epsilon = u_0.
\]

(4.6)

If \( A \) is elliptic then \( \pi(u_t^\epsilon) \) and its horizontal lift converges in law. If furthermore \( A = \frac{1}{2} \Delta^L \) then \( \pi(u_t^\epsilon) \) converges in law to a rescaled Brownian motion with generator \( \frac{4}{n(n-1)}H \). Its horizontal lift converges in law to the diffusion process on \( OM \) with generator \( \frac{4}{n(n-1)}\Delta_H \).

**Proof** Define the Lie group valued process \( g_t^\epsilon \) by \( u_t^\epsilon = \tilde{x}_t^\epsilon g_t^\epsilon \) and \( g_0^\epsilon = I \), the unit matrix. Following earlier computations and using the fact that \( A^* \) is right invariant and \( (R_g)^* \overline{\nu} = ad(g^{-1})\overline{\nu} \),

\[
dx_t^\epsilon = \tilde{x}_t^\epsilon g_t^\epsilon e_0 dt, \quad d\tilde{x}_t = H(\tilde{x}_t^\epsilon)(g_t^\epsilon e_0)dt, \quad dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^N g_t^\epsilon A_k \circ dw_t^k + \frac{1}{\epsilon} g_t^\epsilon A_0 dt.
\]

For any smooth function \( F : OM \to \mathbb{R} \) with compact support,

\[
F(\tilde{x}_t^\epsilon) = F(u_0) + \int_0^t dF(\tilde{x}_s^\epsilon)H(\tilde{x}_s^\epsilon)g_s^\epsilon e_0 ds.
\]

As in the proof of the previous theorem, we see that the family \( \{\tilde{x}_t^\epsilon\} \) is tight and that it converges in law as \( \epsilon \to 0 \). For each \( u \in OM \) there is a solution \( h_t : G \to \mathbb{R} \) to the equation

\[
A h_t(u, g) = dF(u)(H(u)e_i) \langle ge_0, e_i \rangle.
\]

For \( n > 1 \), \( \int gOdg = \int gdg \) where \( dg \) is the Haar measure normalised to be a probability measure and \( O \) any matrix in \( G \). The integral of \( ge_0 \) with
respect to the Haar measure on $G$ vanishes. For $G = SO(n)$ it follows also from $\int_{SO(n)} g e_0 dg = \int_{S^{n-1}} s ds$, see Proposition 3.2.1 in Krantz-Parks [17]. Denoting by $D_1 h_i$ and $D_2 h_i$ the differential of $h_i$ with respect to the first and the second variable respectively,

$$h_i(\tilde{x}_s^t, g_s^0) = h(u_0, I) + \frac{1}{\epsilon} \sum_k \int_0^t (D_2 h_i)(x_{s}^t, g_{s}^0) (g_{s}^k A_k)dw_s^k + \frac{1}{\epsilon} \int_0^t A h_i(\tilde{x}_s^t, g_s^0)ds$$

$$+ \int_0^t (D_1 h_i)(\tilde{x}_s^t, g_{s}^0) (H(u_s^t)g_s^0 e_0)ds.$$

Plug this back to $F(\tilde{x}_s^t)$ to see that

$$F(\tilde{x}_s^t) = F(u_0) + \epsilon \sum_i \left( h_i(\tilde{x}_s^t, g_s^0) - h(u_0, I) \right) - \sqrt{\epsilon} \sum_{k,j} \int_0^t (D_2 h_i)(x_{s}^t, g_{s}^0) (A_k g_s^j)dw_s^k$$

$$- \epsilon \sum_i \int_0^t (D_1 h_i)(\tilde{x}_s^t, g_{s}^0) (H(u_s^t)g_s^0 e_0)ds.$$

The tightness of the law of $\{\tilde{x}_s^t\}$ can be proved similar to that of the previous theorems, c.f. Lemma 3.2. Furthermore for $s < t$ the following convergence holds in $L^1$,

$$- \epsilon \int_s^t (D_1 h_i)(\tilde{x}_s^t, g_{s}^0) (H(u_s^t)g_s^0 e_0)ds$$

$$\rightarrow - \int_s^t \nabla dF (H(\tilde{x}_s^t)e_j, H(\tilde{x}_s^t)e_i)ds \int_G \langle ge_0^i, e_j \rangle A^{-1} (ge_0, e_i) d\mu(g),$$

where $\mu$ is the unique invariant measure for the $A$ diffusion on $G$. The proof is similar to that of Lemma 3.3, taking into account of the following computation

$$-(D_1 h_i)(u, g) (H(u)(ge_0)) = -\nabla dF (H(u)ge_0, H(u)e_i) A^{-1} (ge_0, e_i)$$

$$= -\sum_j \nabla dF (H(u)e_j, H(u)e_i) \langle ge_0^i, e_j \rangle A^{-1} (ge_0, e_i).$$

Now we assume that $A = \frac{1}{2} \Delta$. We may assume that $\{A_k\}$ is an orthonormal basis of $g$ and $A_0 = 0$ and let

$$a_{i,j} = -\int_G \langle ge_0^i, e_j \rangle (\frac{1}{2} \Delta)^{-1} (ge_0, e_i) dg,$$

where $g$ is the Haar measure on $G$. For $i \neq j$ the cross term $a_{i,j}$ vanishes. There is an element $O \in G$ such that $O e_i = -e_i$ and $O e_j = e_j$. 
Furthermore \( \sum_l A_l^2 = -\frac{n-1}{2} I \) and \( \sum_l g A_l^2 = -\frac{n-1}{2} g I \). It is easy to see that \( (\frac{1}{2}\Delta^L)^{-1}\langle ge_0, e_i \rangle = -\frac{1}{n-1}\langle ge_0, e_i \rangle \). The integral \( \int \langle ge_0, e_i \rangle^2 dg \) is independent of \( i \). In fact \( \int \langle ge_0, e_i \rangle^2 dg = \int \langle ge_0, Oe_i \rangle^2 dg \), for any \( O \in G \). Since \( \int \sum_i \langle ge_0, e_i \rangle^2 dg = 1 \) it follows that
\[
a_{i,i} = \frac{4}{n-1} \int_G \langle ge_0, e_i \rangle^2 \, dg = \frac{4}{(n-1)n}.\]

Finally we see that
\[
\sum_{i,j} \nabla dF(H(u)e_j, H(u)e_i) \langle ge_0, e_j \rangle A^{-1}(ge_0, e_i) = \frac{4}{(n-1)n} \Delta_H.\]

The two operators \( \Delta_H \) and \( \Delta \) are intertwined by \( \pi \), and \( x_t^* \) converges to a Brownian motion. \( \square \)

### 4.3 Another Intertwined Pair

At this point we discuss a question asked to me by J. Norris. Since the process on the orthonormal frame bundle encodes the Riemannian metric we expect to see the Riemannian metric manifesting itself in some form, e.g. in the form of the corresponding Laplacian operator. Does the system below have a non-degenerate limit which is not necessarily associated to the given Riemannian metric on \( M \)? In general the intertwined system would look like the following,

\[
du_t^i = CH(u_t^i) \circ db_t^i \ + \ \frac{1}{\sqrt{\epsilon}} H(u_t^i) V(x_t^i, g_t^i) dt \ + \ \frac{1}{\epsilon} A_{k_t^i}(u_t^i) \circ dw_t^k \ + \ \frac{1}{\epsilon} A_{0_t^i}(u_t^i) dt.
\]

\[
dx_t^i = Cu_t^i \circ db_t^i \ + \ \frac{1}{\sqrt{\epsilon}} u_t^i V(x_t^i, g_t^i) dt.
\]

Below we compute a simple case. The argument, with suitable adjustments, remains valid for the general case.

**Example 4.4** For simplicity consider \( \mathbb{R}^n \times SO(n) \) with the standard connection, and the SDE

\[
dg_t^i = \frac{1}{\sqrt{\epsilon}} g_t^i A_k \circ dw_t^k
\]

\[
dx_t^i = \delta g_t^i \circ db_t + \frac{1}{\sqrt{\epsilon}} g_t^i V(x_t^i, g_t^i) dt. \tag{4.7}
\]
Here $V$ is a $\mathbb{R}^n$ valued function such that $\int_G gV(x, g)dg = 0$ where $dg$ is the Haar measure. For example take $V(g)$ to be a function of even powers of $g$. We assume that $V$ is suitably bounded with its partial derivatives in $x$ suitably bounded. The parameter $\delta$ is to be chosen.

Letting $A_k^*(g) = gA_k$. $\mathcal{L}_0 = \frac{1}{2} \sum_k (A_k^*)^2$, assume that it is $\frac{1}{2} \Delta^L$. Taking $\delta = \sqrt{\epsilon}$, formal computation by multi scale analysis shows that:

Claim. The limiting law for $x^\epsilon_t$ is governed by the partial differential equation on $\mathbb{R}^n$:

$$\frac{\partial \rho}{\partial t} = - \int L_{gV(x, g)\partial x} \mathcal{L}_0^{-1}(gV(x, g)\rho) \, dg, \quad (4.8)$$

where the integral is with respect to the Haar measure on $SO(n)$.

If $\delta = 1$ it ought to have, in addition, a $\Delta_M$ term on the right hand side:

$$\frac{\partial \rho}{\partial t} = \Delta_M f_t - \int L_{uV(x, u)\partial x} \mathcal{L}_0^{-1}(L_{uV(x, u)\rho})dv(u),$$

which we do not discuss rigorously. A drift term in the $g$ equation can also be added. Another interesting regime to consider is $\sum \delta_i g_i^\epsilon \circ db_i$ instead of $g_i^\epsilon \circ db$ with $\delta_i$ takes values from $\{1, \sqrt{\epsilon}\}$. In this case, a non-Laplacian like equation would follow. In the case that $\delta_i$ are all equal and $V(x, g)$ is independent of $x$, the system can be interpreted as an intertwined pair through time scaling.

Equation (4.8) can be deduced by the methodology below. Let $f : M \rightarrow \mathbb{R}$ be a smooth compactly supported function and $\Delta_M$ the Laplacian on $M$. Then

$$f(x_t^\epsilon) = f(x_0) + \delta \int_0^t df(g_s^\epsilon db_s) + \frac{1}{2} \mathbb{E} \int_0^t \Delta_M f(x_s^\epsilon)ds + \frac{1}{\sqrt{\epsilon}} \int_0^t df(g_s^\epsilon V(x_s^\epsilon, g_s^\epsilon))ds.$$

If $h$ is solution to $\mathcal{L}_0h(x, g) = df(xV(x, g))$, then

$$\frac{1}{\sqrt{\epsilon}} \int_0^t df(g_s^\epsilon V(x_s^\epsilon, g_s^\epsilon))ds = \sqrt{\epsilon} h(x_t^\epsilon, u_t^\epsilon) - \sqrt{\epsilon} h(x_0, u_0) - \sqrt{\epsilon} \delta \int_0^t \partial_x h(x_s^\epsilon, g_s^\epsilon)g_s^\epsilon \, db_s$$

$$- \int_0^t \partial_y h(g_s^\epsilon A_kdw^\epsilon_s) - \sqrt{\epsilon \delta^2} \int_0^t \Delta_M h(x_s^\epsilon, g_s^\epsilon)ds$$

$$- \int_0^t L_{g_s^\epsilon V} \partial_x h(x_s^\epsilon, g_s^\epsilon)ds.$$

Since $\delta = \sqrt{\epsilon}$, it is now easy to observe that \{x_t^\epsilon\} is a tight family. Since $g_t^\epsilon$ is a fast ergodic motion and $x_t^\epsilon$ does not move much as $t \rightarrow 0$, under suitable conditions,

$$\lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow 0} \frac{\mathbb{E} f(x_t^\epsilon) - f(x_0)}{t} = \lim_{t \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{\mathbb{E} f(x_t^\epsilon) - f(x_0)}{t} = \mathcal{L}_{gV\partial x} h(x_0).$$

has the required limit.
4.4 Perturbation to Horizontal Diffusions

Let $M$ be a compact connected $n$-dimensional smooth Riemannian manifold with connection $\nabla$. Let $\omega$ be the corresponding connection 1-form on the orthonormal frame bundle $\mathcal{O}M$ with Lie group $G$, where $G$ is taken to be $O(n)$ or $SO(n)$ depending whether $M$ is oriented. The horizontal bundle is integrable when and only when the curvature tensor of $\nabla$ vanishes. The Lie brackets of two fundamental horizontal vector fields will in general contribute to a vertical motion. However perturbation to horizontal flows can still be discussed and in this case we should consider not its projection to the manifold $M$ unless the connection $\nabla$ is flat, but its motion transversal to the holonomy bundle.

Let $u_0 \in \mathcal{O}M$ and $\tau : [0,1] \to M$ be a $C^1$ curve with $\tau(0) = \pi(u_0)$. Let $\tilde{\tau}$ be the horizontal lift of $\tau$ through $u_0$. The parallel displacement $\tilde{\tau}_1$ of $u_0$ can be written as $u_0a$ some $a \in G$. The set of such $a$ that represents parallel displacements of $u_0$ forms a subgroup of $G$ and is called the holonomy group with reference point $u_0 \in \mathcal{O}M$ which we denote by $\Phi(u_0)$. In another word $a \in \Phi(u_0)$ if $u_0$ and $u_0a$ are connected by a horizontal curve. Denote by $\Phi_0(u_0)$ the restricted holonomy group which contains only loops that are homotopic to the identity loop. By Theorem 4.2, in Kobayashi-Nomizu [16], $\Phi(u_0)$ is a closed subgroup and a sub-manifold of $G$ with $\Phi_0(u_0)$ its identity component. Since $M$ is connected all holonomy groups are isomorphic.

Two points of $\mathcal{O}M$ are equivalent if they are connected by a $C^1$ horizontal curve. For each $u$ in $\mathcal{O}M$ let $P(u_0)$ be the holonomy bundle through $u_0$, it consists of all $u \in \mathcal{O}M$ such that $u \sim u_0$, i.e. $u$ and $u_0$ are connected by a horizontal curve. We may consider $\mathcal{O}M$ as disjoint union of sets of the form $P(u)$. Let $H = \Phi(u_0)$, which acts on $P$ on the right, and $P/H$ be the modulus space of $P$ with respect to the equivalent relation. Then $P/H$ is a smooth manifold and it can be identified with the associated bundle with fibre $G/H$ and the equivalent relation: $(uh^{-1}, h\xi)$ where $\xi$ denotes the coset corresponding to the identity. We identify $(u, a\xi)$ with the orbit in $P/H$ that contains $ua$. Denote by $\Pi_1 : P \to P/H$ the natural projection. The main task in the proof of the theorem below is to make sense of freezing the conserved ‘variable’ and averaging out the ‘fast ‘variable’.

**Theorem 4.4** Let $M$ be a connected and compact Riemannian manifold with a Riemannian connection $\nabla$. Consider

$$
\begin{align*}
du^e_t &= H(u^e_t) \circ dt + H_0(u^e_t)dt + \sqrt{\epsilon} \sum_{k=1}^{m} Z_k(u^e_t) \circ dw^k_t + \epsilon Z_0(u^e_t)dt, \\
\end{align*}
$$

(4.9)

$$
\begin{align*}
u^0_e &= u_0,
\end{align*}
$$
where $H_0$ is a horizontal vector field and $Z_k$ are vertical vector fields. Then
\[ \Pi_1(u_0') \] converges in law, which is identified in (4.10) below. Furthermore define $g'_t$ by $u'_t = \tilde{x}'_t g'_t$, where $x'_t = \pi(u'_t)$ and $\tilde{x}'_t$ its horizontal lift. The projection of $g'_t$ to the space of cosets, $G/\Phi(u_0)$, converges weakly.

**Proof** By the holonomy theorem of Ambrose-Singer [1] the Lie algebra of $\Phi(u_0)$ is a subspace of $g$ and is spanned by matrices of the form $\Omega_{w_1, w_2}$ where $w_1, w_2$ are horizontal vectors at $T_p$ and $v \in P(u_0)$. If $u \sim v$ then $\Phi(u) = \Phi(v)$. Let $H = \Phi(u_0)$, a manifold whose dimension is denoted $n_0$.

We define a distribution $S$ on $OM$: $S = \{ T_u(P(u)) : u \in OM \}$. It is of constant rank, $n + n_0$. This distribution is differentiable and involutive and $P(u)$ is the maximal integral manifold of $S$ through $u$. Note that the holonomy bundles are translations of each other: $P(u_0a) = P(u_0), a \in G$. If $u$ is equivalent to $v$, the maximal integral manifolds through them are identical.

Let $u_t$ be the solution starting from $u_0$ of the equation
\[
du_t = H(u_t) \circ db_t + H_0(u_t) dt.
\]
Then $u_t$ is constant in $t$. To see this let $f$ be a $BC^\infty$ function on $P/\Phi(u_0)$ and denote by $\Pi_1: P \rightarrow P/\Phi(u_0)$ the projection. Then
\[
f([u_t]) = f([u_0]) + \int_0^t df(T\Pi_1(H(u_s)) \circ db_s + \int_0^t df(T\Pi_1(H_0(u_s))) ds.
\]
By the Reduction Theorem, page 83 of Kobayashi-Nomizu[16], each holonomy bundle $P(u)$ is a reduced bundle with structure group $\Phi(u)$ and the connection in $OM$ is reducible to a connection in $P(u)$. Hence
\[ T_u(P(u)) = HT_uOM \oplus VT_u(p(u)). \]
In particular we have $T\Pi_1(HTOM) = 0$ and $f([u_t]) = f([u_0])$.

We have shown that the solution to the horizontal SDE stays in $P(u_0)$ for all time. The horizontal SDE, restricted to the maximal integrable manifold $P(u_0)$, satisfies the Hörmander conditions and is ergodic with a unique invariant measure $\mu_{P(u_0)}$. Fix a point $u_0 \in P$ with $x_0 = \pi(u_0)$. Let $\nu$ be the Haar measure on $H = \Phi(u_0)$. Denote by $\nu_a$ the Haar measure on $\Phi(u_0a)$. Note that if $v = u_0a$ some $a \in G$, let $u \in \Phi(u_0)$ and a horizontal curve $\alpha$ with $\alpha_0 = u_0, \alpha_1 = u_0g, g \in \Phi(u_0)$. Then $\beta = \alpha_0a$ is horizontal with $\beta_0 = v$ and $\beta_1 = u_0ga = va^{-1}ga$. Consequently $\Phi(u_0a) = \text{ad}(a^{-1})\Phi(u_0)$. If $a \in H$, $\Phi(u_0a) = \Phi(u_0)$ and $\nu_a = \nu$.

Denote by $N_a$ the following fibre of the holonomy bundle $P(u_0)$:
\[ N_a = \pi^{-1}(x) \cap P(u_0), \quad x = \pi(u). \]
Locally $N_{u_0} = M \times \{ \text{ad}(a^{-1}\Phi(u_0)) \}$ and $\mu_{P(u_0)} = dx \times dv_a$ where $dx$ is the volume measure of the manifold $M$.

Let $F : OM \to \mathbb{R}$ be a $BC^1$ function, we the integral

$$
\tilde{F}[P(u)] := \int_{P(u)} Fd\mu_{P(u)}
$$

is defined to be a number depending on a transversal of $P(u)$. On each fibre of the holonomy bundle $P(u_0)$ we choose a reference element $v(x)$, which determines reference elements on holonomy bundles $P(u)$, due to that $v(x)u$ is an element of $P(u_0)$ where $u \in \pi^{-1}(x)$. For any $u \in P(u_0)$ there is $g \in H$ such that $u = v(\pi(u))g$. We define

$$
\int_{P(u_0)} Fd\mu_{P(u_0)} := \int_{M} \int_{P(u_0) \cap \pi^{-1}(x)} F(v(x)g)d\nu(g)dx.
$$

The resulting number is independent of the choice of $v$. To see this let $v'$ be another choice then $v' = vh$ some $h \in H$ and $u = va = v'h^{-1}a$. Since $G$ is a compact group, the Haar measure is bi-invariant,

$$
\int_{P(u_0) \cap \pi^{-1}(x)} F(v(x)g)d\nu_a(g) = \int_{P(u_0) \cap \pi^{-1}(x)} F(v'(x)h^{-1}a)d\nu_a(g)
$$

$$
= \int_{P(u_0) \cap \pi^{-1}(x)} F(v'(x)a')d\nu_x(g').
$$

Similarly if $u = v(x)ag \in P(u_0a)$ the following integral is well defined:

$$
\int_{P(u)} Fd\mu_{P(u)} := \int_{M} \int_{P(u) \cap \pi^{-1}(x)} F(v(x)ag)d\nu_a(g)dx.
$$

Evaluate $f : P/\Phi(u_0) \to \mathbb{R}$ at $u'_i$, denoting $\Pi_i(u)$ by $[u]$, to see that

$$
f([u'_i]) = f([u'_0]) + \sqrt{\epsilon} \int_{0}^{t} df(T\Pi_{1}(Z_{k}(u'_s))) \circ dw^k_s + \epsilon \int_{0}^{t} df(T\Pi_{1}(Z_{0}(u'_s))) ds.
$$

Let $\mathfrak{m}$ be the Lie algebra of $H$ and let $A_i, i = 1, \ldots, n_0$ be an o.n.b. of $\mathfrak{m}$. Let $B_j, j = n_0 + 1, \ldots, N$ be an o.n.b. of the vertical part of the distribution $S$ at $u_0$. Define $A_j = \varpi_{u_0}(B_j) \in \mathfrak{g}$. Consider the family of fundamental vertical vector fields $\{A'_j(u), j > n_0\}$, restricted to $P(u_0)$. Then $T\Pi_{1}(A'_j) = 0$ for $i \leq n_0$ and for $j > n_0, T\Pi_{1}(A'_j) = A'_i([u])$.

Writing $Z_k$ in terms of the basis $\{A_k\}$, $Z_k = \sum_j \sigma_k^j A'_j$, we have

$$
f([u'_i]) = f([u'_0]) + \sqrt{\epsilon} \sum_{k = n_0+1}^{N} \sum_{j = n_0+1}^{N} \int_{0}^{t} \sigma_k^j(u'_s) df(A'_j([u'_s])) \circ dw^k_s
$$

$$
+ \epsilon \sum_{j = n_0+1}^{N} \int_{0}^{t} \sigma_0^j(u'_s) df(A'_j([u'_s])) ds.
$$
The process $[u^t]\epsilon$ is in general not Markov. It is however clear, following the standard method as used earlier, that the probability distributions \( \{u^t, \epsilon > 0 \} \) is tight and any sequence of $[u^t]$ has a convergent subsequence with the same limit. The limit can be identified below. Define \( a_{i,j}(u) = \sum_{k\geq 1} \sigma_{i}^{k} \sigma_{j}^{k} d\mu_{P(u)} \), and \( \bar{Z} = \sum_{j=n_{0}+1}^{N} \sigma_{j}^{0} A_{j}^{*} \). For \( i, j \geq n_{0} \) define

\[
\delta_{0}^{j}([u]) = \int_{P(u)} \left( \sigma_{0}^{j} + \frac{1}{2} \sum_{k \geq 1} d\sigma_{k}^{j}(Z_{k}) \right) d\mu_{P(u)}.
\]

Then

\[
\mathcal{L}f([u]) = \sum_{i,j=n_{0}+1}^{N} a_{i,j}([u]) \nabla d f (A_{i}^{*}, A_{j}^{*}) + df(Z([u])).
\]

The rest of the proof for the convergence is similar to that of Theorem 4.1.

For the second statement consider the process $g^t$ defined by \( u^t = \pi(u^t)g^t \). Let \( p : G \rightarrow G/H \) be the canonical homomorphism. For \( a \in G \) denote by \( [a] \) the left coset of \( H \) that contains \( a \). Finally note that, if \( [u] \) denotes an element of of \( P/H \) that contains \( u \),

\[
[u^t] = [\pi(u^t)]g^t = [u_{0}]g^t,
\]

and the motion in \( [u^t] \) is essentially the motion of \( g^t \), while the latter is considered to be a representative of \( G/\Phi(u_{0}) \). Specifically,

\[
dg^t = \sqrt{\epsilon} \sum_{k=1}^{m} \sum_{j=1}^{N} \sigma_{k}^{j}(u^t)g_{i}^{k} A_{j} \circ dw_{i}^{k} + \epsilon \sum_{j=1}^{N} \sigma_{0}^{j}(u^t)g_{i}^{j} A_{j},
\]

\[
d[g^t] = \sqrt{\epsilon} \sum_{k=1}^{m} \sum_{j=n_{0}+1}^{N} \sigma_{k}^{j}(u^t)[g_{i}^{j}] A_{j} \circ dw_{i}^{k} + \epsilon \sum_{j=n_{0}+1}^{N} \sigma_{0}^{j}(u^t)[g_{i}^{j}] A_{j}.
\]

The second statement thus follows.

According to Theorem 8.2 in Kobayashi-Nomizu [16] if \( OM \) is connected there is a connection on \( OM \) such that \( P(u) = OM \). On the other extreme if the curvature vanishes the orthonormal frame bundle \( OM \) foliates. This is so for a Lie group with the Left or right invariant connection. If \( M \) is simply connected the curvature zero case corresponds to the product bundle with the trivial connection. See also section 6.2 in Elworthy-LeJan-Li [11] for a discussion on the equivalence of the stochastic holonomy and holonomy and Arnaudon-Thalmaier [2] for work on Yang-Mills Fields and random holonomy.
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