Qd(p)-FREE RANK TWO FINITE GROUPS ACT FREELY ON A HOMOTOPY PRODUCT OF TWO SPHERES

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ABSTRACT. A classic result of Swan states that a finite group $G$ acts freely on a finite homotopy sphere if and only if every abelian subgroup of $G$ is cyclic. Following this result, Benson and Carlson conjectured that a finite group $G$ acts freely on a finite complex with the homotopy type of $n$ spheres if the rank of $G$ is less than or equal to $n$. Recently, Adem and Smith have shown that every rank two finite $p$-group acts freely on a finite complex with the homotopy type of two spheres. In this paper we will make further progress, showing that rank two groups that are Qd($p$)-free act freely on a finite homotopy product of two spheres.

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1. Introduction

Recall that the $p$-rank of a finite group $G$, $\text{rk}_p(G)$, is the largest rank of an elementary abelian $p$-subgroup of $G$ and that the rank of a finite group $G$, $\text{rk}(G)$, is the maximum of $\text{rk}_p(G)$ taken over all primes $p$. We can define the homotopy rank of a finite group $G$, $h(G)$, to be the minimal integer $k$ such that $G$ acts freely on a finite CW-complex $Y \simeq S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$. Benson and Carlson [7] state a conjecture that for any finite group $G$, $\text{rk}(G) = h(G)$. The case of this conjecture when $G$ is a rank one group is a direct result of Swan’s theorem [23]. Benson and Carlson’s conjecture has also been verified by Adem and Smith [11, 2] in the case when $G$ is a rank two $p$-group. In addition, recall that Heller [19] has shown that for a finite group $G$, if $h(G) = 2$, then $\text{rk}(G) = 2$. In this paper we will verify this same conjecture for all rank two groups that do not contain a particular type of subquotient.

Before we state our main theorem, we need two definitions.

**Definition 1.** A subquotient of a group $G$ is a factor group $H/K$ where $H, K \subseteq G$ with $K \triangleleft H$. A subgroup $L$ is said to be involved in $G$ if $L$ is isomorphic to a subquotient of $G$. In particular, for a prime $p$, we say that $L$ is $p'$-involved in $G$ if $L$ is isomorphic to a subquotient $H/K$ of $G$ where $K$ has order relatively prime to $p$.

**Definition 2.** Let $p$ be a prime and $E$ be a rank two elementary abelian $p$-group. $E$ is then also a two-dimensional vector space over $\mathbb{F}_p$. We define the quadratic group Qd($p$) to be the semidirect product of $E$ by the special linear group $SL_2(\mathbb{F}_p)$. 
A classic result of Glauberman shows that if for some odd prime $p$ $Qd(p)$ is not involved in a finite group $G$, then the $p$-fusion is controlled by the normalizer of a characteristic $p$-subgroup \[13, 14\]. Although we do not directly use Glauberman’s result here, there are some connections as $p$-fusion plays a integral role in our investigation (see Remark 36). We can now state our main theorem.

**Theorem 3.** Let $G$ be a finite group such that $\text{rk}(G) = 2$. $G$ acts freely on a finite CW-complex $Y \simeq S^{n_1} \times S^{n_2}$ unless for some odd prime $p$, $Qd(p)$ is $p'$-involved in $G$.

The following corollary, which can also be shown using more direct methods, follows immediately from Theorem 3:

**Corollary 4.** If $G$ is a finite group of odd order with $\text{rk}(G) = 2$, then $G$ acts freely on a finite CW-complex $Y \simeq S^{n_1} \times S^{n_2}$.

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## 2. Preliminaries

Recall that Adem and Smith have shown that if $G$ is a rank two $p$-group, then $h(G) = 2$ \[1, 2\]. In their proof they use the following theorem:

**Theorem 5** (Adem and Smith \[2\]). Let $G$ be a finite group and let $X$ be a finitely dominated, simply connected $G$-CW complex such that every isotropy subgroup has rank one. Then for some integer $N > 0$ there exists a finite CW-complex $Y \simeq S^N \times X$ and a free action of $G$ on $Y$ such that the projection $Y \to X$ is $G$-equivariant.

In the present work, we apply Theorem 5 for a rank two group by finding an action of the group on a CW-complex $X \simeq S^n$ such that the isotropy subgroups are of rank one. Adem and Smith in \[2\] also developed a sufficient condition for this type of action to exist.

To explore this sufficient condition, we start by letting $G$ be a finite group. Given a $G$-CW complex $X \simeq S^{N-1}$, there is a fibration $X \to X \times_G EG \to BG$. This fibration gives an Euler class $\beta$, which is a cohomology class in $H^N(G)$. This cohomology class is the transgression in the Serre spectral sequence of the fundamental class of the fiber. Regarding this class, Adem and Smith made the following definition:

**Definition 6** (Adem and Smith \[2\]). Let $G$ be a finite group. The Euler class $\beta$ of a $G$-CW complex $X \simeq S^{N-1}$ is called effective if the Krull dimension of $H^*(X \times_G EG; \mathbb{F}_p)$ is less than $\text{rk}(G)$ for all $p \mid |G|$.

Adem and Smith have proven the following lemma characterizing effective Euler classes:
Lemma 7 (Adem and Smith [2]). Let $\beta \in H^N(G)$ be the Euler class of an action on a finite dimensional $X \simeq S^{N-1}$. $\beta$ is effective if and only if every maximal rank elementary abelian subgroup of $G$ acts without stationary points.

Remark 8. By applying Lemma 7 and Theorem 5, we see that any finite group of rank two, having an effective Euler class of an action on a finite dimensional $X \simeq S^{N-1}$, acts freely on a finite CW-complex $Y \simeq S^m \times S^{N-1}$.

We now make a topological definition that will allow us to classify certain actions on homotopy spheres.

Definition 9. Consider the set of maps from a space $X$ to a space $Y$. Two such maps $f$ and $g$ are called homotopic if there exists a continuous function $H : X \times [0, 1] \to Y$ such that for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. This induces an equivalence relation on the set of maps from $X$ to $Y$, which makes two maps equivalent if they are homotopic. The resulting equivalence classes are called homotopy classes of maps from $X$ to $Y$; the set of such classes is denoted $[X, Y]$.

Recall that for a group $G$, $BG$ is the classifying space of $G$. Also $[BG, BU(n)]$ represents the homotopy classes of maps from $BG$ to $BU(n)$, which may also be written as $\pi_0\text{Hom}(BG, BU(n))$ (see [12]). In addition, recall that $[BG, BU(n)]$ is isomorphic to the set of isomorphism classes of rank $n$ complex vector bundles over $BG$. In this discussion, complex vector bundles over $BG$ will be thought of as elements of $[BG, BU(n)]$.

Recall that the Euler class of the universal bundle over $BU(n)$ is the top Chern class, which lies in $H^{2n}(BU(n))$. We will call this class $\xi$. A map $\varphi : BG \to BU(n)$ induces a map $\varphi^* : H^{2n}(BU(n)) \to H^{2n}(BG) = H^{2n}(G)$. Thus the image of $\xi$ under $\varphi^*$ is the Euler class of the vector bundle over $BG$ that corresponds to $\varphi$. Let $\beta_\varphi = \varphi^*(\xi)$ be this class, which is the Euler class of a $G$-CW complex that is the unit sphere bundle of the vector bundle. This $G$-CW complex is the fiber as discussed at the outset of Section 2. Recall that if $\varphi$ and $\mu$ are homotopic maps from $BG$ to $BU(n)$, then $\beta_\varphi = \beta_\mu$; therefore, the Euler class of a homotopy class of maps is well defined.

Recall that $G$ is a finite group, $p$ is a prime dividing the order of $G$, and $G_p$ is a Sylow $p$-subgroup of $G$. In addition we will use $\text{Char}_n(G_p)$ to represent the set of degree $n$ complex characters of $G_p$ and will let $\text{Char}_n^G(G_p)$ represent the subset consisting of those degree $n$ complex characters that are the restrictions of class functions of $G$. Also we use the notation $\text{Rep}(G, U(n))$ for the degree $n$ unitary representations of the group $G$, which is given by the equality $\text{Rep}(G, U(n)) = \text{Hom}(G, U(n))/\text{Inn}(U(n))$. Recall that Dwyer and Zabrodsky [12] have shown that for any $p$-subgroup $P$,

$$\text{Char}_n(P) \cong \text{Rep}(P, U(n)) \xrightarrow{\cong} [BP, BU(n)]$$
where the map is $\rho \mapsto B\rho$. So if $\varphi$ is a map $BG \to BU(n)$ and if $P$ is a $p$-subgroup of $G$ for some prime $p$, $\varphi|_P$ is induced by a unitary representation of $P$. The following proposition gives the characterization of effectiveness for the Euler class $\beta_\varphi$.

**Proposition 10** ([1, section 4]). An Euler class $\beta_\varphi$ is effective if and only if for each elementary abelian subgroup $E$ of $G$ with $\text{rk}(E) = \text{rk}(G)$, there is a unitary representation $\lambda : E \to U(n)$ such that both $\varphi_E = B\lambda$ and $\lambda$ does not have the trivial representation as one of its irreducible constituents.

As we proceed we will want to work separately with each prime. Definition 11 will be useful in this endeavor.

**Definition 11.** Let $G$ be a finite group. The Euler class $\beta_\varphi$ is called $p$-effective if for each elementary abelian $p$-subgroup $E$ of $G$ with $\text{rk}(E) = \text{rk}(G)$, there is a unitary representation $\lambda : E \to U(n)$ such that both $\varphi_E = B\lambda$ and $\lambda$ does not have the trivial representation as one of its irreducible constituents.

**Remark 12.** It is clear then that $\beta_\varphi$ is effective if and only if it is $p$-effective for each $p|G|$. Also if $\text{rk}_p(G) < \text{rk}(G)$, then any $\beta_\varphi$ is $p$-effective since it trivially satisfies Definition 11; therefore, we can restate Proposition 10: $\beta_\varphi$ is effective if and only if it is $p$-effective for each prime $p$ such that $\text{rk}_p(G) = \text{rk}(G)$.

Definition 13 concerns the characters of subgroups of $G$.

**Definition 13.** Let $H \subseteq G$ be a subgroup. Let $\chi$ be a character of $H$. We say that $\chi$ respects fusion in $G$ if for each pair of elements $h, k \in H$ that are conjugate in the group $G$, $\chi(h) = \chi(k)$.

**Remark 14.** Let $H \subseteq G$. A sufficient condition for a character of $H$ to respect fusion in $G$ is for the character to be constant on elements of the same order.

Notice that for $\chi \in \text{Char}_n(G_p)$, $\chi$ respects fusion in $G$ if and only if $\chi \in \text{Char}_n^G(G_p)$. In a previous paper [20], the author has shown the following theorem.

**Theorem 15** (Jackson [20, Theorem 1.3]). If $G$ is a finite group that does not contain a rank three elementary abelian subgroup, then the natural mapping

$$\psi_G : [BG, BU(n)] \to \prod_{p|G} \text{Char}_n^G(G_p)$$

is a surjection.

Combining Theorem 15 and Definition 11 we get the following theorem.

**Theorem 16.** Let $G$ be a finite group with $\text{rk}(G) = 2$, $p$ a prime dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. If there is a character $\chi$ of $G_p$ that respects fusion in $G$ and has the property that $[\chi|_E, 1_E] = 0$ for each $E \subseteq G_p$ elementary abelian with $\text{rk}(E) = 2$, then there is a complex vector bundle over $BG$ whose Euler class is $p$-effective.
**Remark 17.** Let $p$ be a prime, $G_p$ a $p$-group of rank $n$, and $\chi$ a character of $G_p$. A sufficient condition to guarantee that $[\chi|_E, 1_E] = 0$ for each $E \subseteq G_p$ elementary abelian with $\text{rk}(E) = \text{rk}(G_p)$ is that $\chi(1) = c(p^n - 1)$ and $\chi(g) = -c$ for each element of $G_p$ of order $p$. In this case, $\chi|_E = c(\rho - 1)$ where $\rho$ is the regular character of the group $E$.

**Theorem 16** leads to definition **18**:

**Definition 18.** Let $G$ be a finite group, $p$ a prime dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. A character $\chi$ of $G_p$ is called a $p$-effective character of $G$ under two conditions: $\chi$ respects fusion in $G$; and for each $E \subseteq G_p$ elementary abelian with $\text{rk}(E) = \text{rk}(G)$, $[\chi|_E, 1_E] = 0$.

**Remark 19.** Let $G$ be a finite group, $p$ a prime dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. A character $\chi$ of $G_p$ is $p$-effective if $\chi$ is both constant on elements of the same order and $[\chi|_E, 1_E] = 0$ for each $E \subseteq G_p$ elementary abelian with $\text{rk}(E) = \text{rk}(G)$. This follows from Remark 14.

Applying Definition 18, **Theorem 16** can be restated as follows: If $G$ has a $p$-effective character, then there is a complex vector bundle over $BG$ whose Euler class is $p$-effective.

We have now essentially proved **Theorem 20**, which is a modified form of a theorem by Adem and Smith [2, Theorem 7.2].

**Theorem 20** (See [2, Theorem 7.2]). Let $G$ be a finite group with $\text{rk}(G) = 2$. If for each prime $p$ dividing the order of $G$ there exists a $p$-effective character of $G$, then $G$ acts freely on a finite CW-complex $Y \simeq S^{N_1} \times S^{N_2}$.

**Proof.** For each prime $p$ dividing the order of $G$, let $\chi_p$ be a $p$-effective character of $G$. There exists an integer $n$ such that $\chi_p(1)$ divides $n$ for each $p$ dividing the order of $G$ giving

$$\left(\frac{n}{\chi_{p_1}(1)}, \frac{n}{\chi_{p_2}(1)}, \ldots, \frac{n}{\chi_{p_k}(1)}\right) \in \prod_{p|\text{rk}(G)} \text{Char}_G^n(G_p).$$

By Theorem 15 there exists an element $\varphi \in [BG, BU(n)]$ such that

$$\psi_G(\varphi) = \left(\frac{n}{\chi_{p_1}(1)}, \frac{n}{\chi_{p_2}(1)}, \ldots, \frac{n}{\chi_{p_k}(1)}\right).$$

The Euler class of the homotopy class $\varphi$ is then an effective Euler class of $G$. **Theorem 20** then follows from Remark 8.

**Theorem 20** reduces the problem of showing that a rank two finite group $G$ acts freely on a finite CW-complex $Y \simeq S^{N_1} \times S^{N_2}$ to the problem of demonstrating that for each prime $p$ there is a $p$-effective character of $G$. It is important to state that not all rank two finite groups have a $p$-effective character for each prime $p$. An example of a finite group that does not contain a $p$-effective character for a particular prime is given in the following remark, which will be proven in Section 6.
Remark 21. Let $p$ be an odd prime. The group $\text{PSL}_3(\mathbb{F}_p)$ does not have a $p$-effective character.

The groups $\text{PSL}_3(\mathbb{F}_p)$, for odd primes $p$, are the only finite simple groups of rank 2 that were not shown by Adem and Smith to act freely on a finite CW-complex $Y \simeq S^{N_1} \times S^{N_2}$ \cite{1} \cite{2}.

3. A sufficient condition for $p$-effective characters

In Section 3, we establish a sufficient condition for the existence of a $p$-effective character by first recalling the following definitions.

Definition 22. Let $G$ be a finite group and $H$ and $K$ subgroups such that $H \subset K$. We say that $H$ is strongly closed in $K$ with respect to $G$ if for each $g \in G$, $gHg^{-1} \cap K \subseteq H$.

Remark 23. Let $G$ be a finite group and $H$ and $K$ subgroups such that $H \subset K$. We say that $H$ is weakly closed in $K$ with respect to $G$ if for each $g \in G$ with $gHg^{-1} \subseteq K$, $gHg^{-1} = H$. For a subgroup $H$ of prime order, $H$ is strongly closed in $K$ with respect to $G$ if and only if $H$ is weakly closed in $K$ with respect to $G$.

Definition 24. Let $P$ be a $p$-group. We use $\Omega_1(P)$ to denote a $p$-subgroup of $P$ generated by all of the order $p$ elements of $P$. In particular, if $P$ is abelian, $\Omega_1(P)$ is elementary abelian, and if $P$ is cyclic, then $\Omega_1(P)$ has order $p$.

Lemma 25. If $E$ is an elementary abelian subgroup of a $p$-group $P$, then $\langle E, \Omega_1(Z(P)) \rangle$ is an elementary abelian subgroup of $P$. In particular, if $E$ is a maximal elementary abelian subgroup of $P$ under inclusion, then $\Omega_1(Z(P)) \subseteq E$.

Proof. Let $Z = \Omega_1(Z(P))$. Since $Z$ is central, the subgroup $\langle E, \Omega_1(Z(P)) \rangle$ is abelian. The product is generated by elements of order $p$ and so must be elementary abelian. \hfill \qed

Corollary 26. Let $P$ be a $p$-group with $\text{rk}(P) = \text{rk}(Z(P))$. $\Omega_1(Z(P))$ is the unique elementary abelian subgroup of $P$ that is maximal under inclusion. In addition, $\Omega_1(P) = \Omega_1(Z(P))$, and $\Omega_1(Z(P))$ is strongly closed in $P$ with respect to $G$.

Proof. If $E$ is a maximal elementary abelian subgroup of $P$ under inclusion, then $\Omega_1(Z(P)) \subseteq E$ by Lemma 25. Equality holds since $\text{rk}(E) \leq \text{rk}(P) = \text{rk}(\Omega_1(Z(P)))$. \hfill \qed

Again we let $p$ be a prime dividing the order of a finite group $G$ and let $G_p \in \text{Syl}_p(G)$. We show that if there is a subgroup of $Z(G_p)$ that is strongly closed in $G_p$ with respect to $G$, then $G$ has a $p$-effective character in the following proposition.

Proposition 27. Let $G$ be a finite group, $n = \text{rk}(G)$, $p$ a prime divisor of $|G|$ with $\text{rk}_p(G) = n$, and $G_p$ a Sylow $p$-subgroup of $G$. If there exists $H \subseteq Z(G_p)$ such that $H$ is non-trivial and strongly closed in $G_p$ with respect to $G$, then $G$ has a $p$-effective character.
Definition 30 (see [17]). Let \( P \subseteq G \) be a \( p \)-centric subgroup. \( P \) is said to be principal \( p \)-radical if \( O_p(N_G(P)/PC_G(P)) = \{1\} \).

Lemma 31 (Diaz, Ruiz, Viruel [10, Section 3]). Let \( G \) be a finite group and \( p > 2 \) a prime with \( \mathrm{rk}_p(G) = 2 \). Let \( G_p \in \text{Syl}_p(G) \) and \( P \) be a principal \( p \)-radical subgroup of \( G \). If \( P \neq G_p \) and if \( P \) is metacyclic, then \( P \cong \mathbb{Z}/p^n \times \mathbb{Z}/p^m \) with either \( p = 3 \) or \( n = 1 \).
Lemma 32 ([10], Lemma 3.11]). Let $G$ be a subgroup of $GL_2(p)$ for an odd prime $p$. If $O_p(G) = \{1\}$ and if $p$ divides the order of $G$, then $SL_2(p) \subseteq G$.

Lemma 33. Let $H$ be a finite group and $p$ an odd prime such that $O_p(H) = \{1\}$ and $C_H(O_p(H)) \subseteq O_p(H)$. If $O_p(H)$ is homocyclic abelian of rank two and is not a Sylow $p$-subgroup of $H$, then $H$ has a subgroup isomorphic to $Qd(p)$.

Proof. Let $P = O_p(H) = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$. We may consider the group of automorphisms $Aut(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n})$ as two by two matrices with coefficients in $\mathbb{Z}_{p^n}$ and with determinants not divisible by $p$. Reduction modulo $p$ gives the following short exact sequence of groups:

$$1 \to Q \to Aut(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}) \xrightarrow{\rho} GL_2(\mathbb{F}_p) \to 1$$

with $Q$ a $p$-group. Notice that $H/P \subseteq Aut(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n})$. $Q \cap H/P = \{1\}$ because $O_p(H) = P$; therefore, $\rho$ is an injection when restricted to $H/P$. Notice that $\rho(H/P)$ has order divisible by $p$, due to $P$ not being a Sylow $p$-subgroup of $H$; thus, by Lemma 32, $SL_2(p) \subseteq \rho(H/P)$. A restriction of $\rho$ is now an isomorphism between a subgroup of $\mathbb{F}_p$ and $SL_2(\mathbb{F}_p)$. Let $\alpha \in K$ be preimage under $\rho$ of the scalar matrix with $-1$ diagonal entries in $SL_2(\mathbb{F}_p)$. Let $\pi$ be the quotient map $H \to H/P$. We see that $\alpha$ has a preimage under $\pi$ that has order 2. Call this preimage $\beta$. $P$ can be viewed as two-dimensional vector space over $\mathbb{F}_{p^n}$ where $p^n$ is the exponent of $P$. Notice that $\beta$ acts on elements of $P$ by scalar multiplication by $-1$ so it has no nontrivial fixed points in $P$. Let $C$ be the centralizer of $\beta$ in $H$. $\pi|_C$ is an isomorphism since $C \cap P = \{1\}$. Also $K \subseteq \pi(C)$ because $K$ is contained in the centralizer of $\alpha = \pi(\beta)$ in $H/P$. It follows that $C$ has a subgroup $L$ isomorphic to $K$, hence, isomorphic to $SL_2(\mathbb{F}_p)$. $\langle L, \Omega_1(\hat{P}) \rangle$ is a subgroup of $H$, which is isomorphic to $Qd(p)$. \qed

Remark 34. Notice that the hypothesis of Lemma 33 which states that $O_p(H) = \{1\}$ and $C_H(O_p(H)) \subseteq O_p(H)$ implies that $H$ is $p$-constrained.

Theorem 35. Let $G$ be a finite group and $p > 2$ a prime with $\text{rk}_p(G) = 2$. Let $G_p \in \text{Syl}_p(G)$. If $\Omega_1(Z(G_p))$ is not strongly closed in $G_p$ with respect to $G$, then $Qd(p)$ is $p'$-involved in $G$.

Proof. Notice that $\text{rk}(Z(G_p)) = 1$ by Corollary 26. Let $Z = \Omega_1(Z(G_p))$. By hypothesis, $Z$ is not strongly closed in $G_p$ with respect to $G$. Recall that Alperin’s Fusion Theory 3 states that given a weak conjugation family $\mathcal{F}$, there exists an $(S, T) \in \mathcal{F}$ with $Z \subseteq S \subseteq G_p$ and with $Z$ not strongly closed in $S$ with respect to $T$. The collection of all pairs $(P, N_G(P))$, where $P \subseteq G_p$ is a principal $p$-radical subgroup of $G$, is the weak conjugation family of Goldschmidt [15] Theorem 3.4] (see also [20]). There exists $P \subseteq G_p$ that is a principal $p$-radical subgroup of $G$ with $Z \subseteq P$ and with $Z$ not strongly closed in $P$ with respect to $N_G(P)$. Notice that
and $y$. Notice that this result does not depend on the rank of $G$.

Definition 37:

**Remark 36.** The results this section are connected to the work of Glauberman on characteristic subgroups and $p$-stability (see [13, 14]). To describe the connection, we first need to define the Thompson subgroup of a $p$-group $P$: the Thompson subgroup $J(P)$ is then the subgroup of $P$ generated by the abelian subgroups of $P$ that have maximal order among all abelian subgroups of $P$. Glauberman in [13] shows that if $G$ is a finite group and $p$ an odd prime with $G_p \in \text{Syl}_p(G)$ such that $Qd(p)$ is not involved in $G$, then $Z(J(G_p))$ is strongly closed in $G_p$ with respect to $G$. If, in addition, $Z(J(G_p))$ is contained in every elementary abelian $p$-subgroup of $G_p$ with the same rank as $G$, then $G$ has a $p$-effective character; in particular, $G$ has a $p$-effective character for an odd prime $p$ if $G$ is $Qd(p)$-free and $J(G_p) = G_p$. Notice that this result does not depend on the rank of $G$.

5. Prime two

The argument in this section will be done in two parts. First we will show, regarding the prime two, that if the sufficiency condition of Proposition 27 does not hold for $G$ and for prime 2, then $G_2 \in \text{Syl}_2(G)$ is either dihedral, semi-dihedral, or wreathed. In the second part we will deal with these exceptional cases. We start with two definitions.

**Definition 37 (p. 191)).** Recall that a 2-group is semi-dihedral (sometimes called quasi-dihedral) if it is generated by two elements $x$ and $y$ subject to the relations that $y^2 = x^{2n} = 1$ and $yxy^{-1} = x^{-1+2^{n-1}}$ for some $n \geq 3$. 

The argument in this section will be done in two parts. First we will show, regarding the prime two, that if the sufficiency condition of Proposition 27 does not hold for $G$ and for prime 2, then $G_2 \in \text{Syl}_2(G)$ is either dihedral, semi-dihedral, or wreathed. In the second part we will deal with these exceptional cases. We start with two definitions.
Definition 38 ([16 p. 486]). A 2-group is called wreathed if it is generated by three elements $x$, $y$, and $z$ subject to the relations that $x^{2^n} = y^{2^n} = z^2 = 1$, $xy = yx$, and $zxz^{-1} = y$ with $n \geq 2$.

Notice that the dihedral group of order 8 is wreathed. Now that we have these definitions, we will state Proposition 39, which is crucial in our discussion of 2-subgroups of rank two. The proof of Proposition 39 follows the proof of a theorem by Alperin, Brauer, and Gorenstein [5, Proposition 7.1].

Proposition 39 (See Proposition 7.1 of [5] and its proof.). Let $G$ be a finite group and $G_2 \in \text{Syl}_2(G)$. Suppose that $\text{rk}(G_2) = 2$. If $\Omega_1(Z(G_2))$ is not strongly closed in $G_2$ with respect to $G$, then $G_2$ is either dihedral, semi-dihedral, or wreathed.

Below we briefly review the proof of Proposition 7.1 of [5], including the changes needed to prove Proposition 39.

Again notice that $\text{rk}(Z(G_2)) = 1$ by Corollary 26. If $G_2$ does not contain a normal rank 2 elementary abelian subgroup, then $G_2$ must be either dihedral or semi-dihedral (see [10 Theorem 5.4.10]). We may assume that $G_2$ has a normal rank 2 elementary abelian subgroup, which we will call $V$. Letting $T = C_{G_2}(V)$, we notice that $[G_2 : T] \leq 2$. Also $G_2 \neq T$ since $\text{rk}(Z(G_2)) = 1$; therefore, $[G_2 : T] = 2$.

Letting $A_G(V) = N_G(V)/C_G(V)$, we notice that $A_G(V)$ is isomorphic to a subgroup of $\text{Aut}(V) \cong \Sigma_3$ and that $2 || A_G(V) ||$. Thus either $A_G(V) \cong \Sigma_3$ or $2 = |A_G(V)|$.

From the argument in [5], we get the following results: if $A_G(V) \cong \Sigma_n$, then $G_2$ is wreathed, including the case of $D_8$; and if $|A_G(V)| = 2$, then $G_2 \cong D_8$. This finishes the discussion of Proposition 39.

We have seen that if $G$ is a rank two finite group, it has a 2-effective character unless a Sylow 2-subgroup is dihedral, semi-dihedral, or wreathed. In the rest of this section, we will show that in each of these three cases $G$ also has a 2-effective character, which will complete the proof that a rank two finite group has a 2-effective character. The author proved the following three lemmas in his doctoral thesis [21]; the proofs are included here for completeness. We will start with a lemma that will be used in the proof of Proposition 40.

Lemma 40. Let $G$ be a finite group, let $n = \text{rk}(G)$, let $p$ a prime divisor of $|G|$ with $\text{rk}_p(G) = n$, and let $G_p \in \text{Syl}_p(G)$. If $G_p$ is abelian, $G_p \triangleleft G$, or $G$ is $p$-nilpotent, then $G$ has a $p$-effective character.

Proof. In each case we will show that $G$ has a $p$-effective character by showing that $Z(G_p)$ is strongly closed in $G_p$ with respect to $G$ using Proposition 27. If $G_p$ is abelian, $Z(G_p) = G_p$ is obviously strongly closed in $G_p$ with respect to $G$ (see Corollary 26). In the case where $G_p \triangleleft G$, $Z(G_p) \triangleleft G$ because $Z(G_p)$ is a characteristic subgroup of $G_p$; therefore, $Z(G_p)$ is
strongly closed in $G_p$ with respect to $G$. If $G$ is $p$-nilpotent, then two elements of $G_p$ are conjugate in $G$ if and only if they are conjugate in $G_p$. Thus, in this case as well, $Z(G_p)$ is strongly closed in $G_p$ with respect to $G$. □

**Lemma 41.** If $P$ is a dihedral or semi-dihedral 2-group such that $|P| = 2^n$ with $n \geq 3$, then there is a character $\chi$ of $P$ such that

$$
\chi(g) = \begin{cases}
3 \cdot 2^{n-3} & \text{if } g = 1 \\
-2^{n-3} & \text{if } g \text{ is an involution} \\
2^{n-3} & \text{otherwise.}
\end{cases}
$$

**Proof.** Let $N$ be the commutator subgroup $[P, P]$ and notice that $P/N \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Since $Z(P) \cong \mathbb{Z}/2$, $Z(P) \subseteq N$; therefore, there is a non-trivial irreducible character $\lambda$ of $P/N$ with $\lambda(gN) = 1$ for $g \in Z(P)$ such that if $g \in P \setminus Z(P)$ is an involution, then $\lambda(gN) = -1$. Let $\varphi(g) = \lambda(gN)$, which is a character of $P$.

We will define another character of $P$ by first inductively defining a character of each $D_{2m}$.

We start by letting $\psi_3$ be the character of $D_8$ such that

$$
\psi_3(g) = \begin{cases}
2 & \text{if } g = 1 \\
-2 & \text{if } g \in Z(D_8) \setminus \{1\} \\
0 & \text{otherwise.}
\end{cases}
$$

For the induction, let $\psi_k = \text{Ind}_{D_{2k-1}}^{D_{2k}} \psi_{k-1}$. $\psi_k$ is a character of $D_{2k}$ for each $k \geq 3$. We see that if $P$ is dihedral, then $\chi = 2^{n-3} \varphi + \psi_n$; therefore, Lemma 41 holds for dihedral 2-groups. On the other hand, if $P$ is semidihedral, $\chi = 2^{n-3} \varphi + \text{Ind}_{D_{2n-1}}^{P} \psi_{n-1}$. □

**Lemma 42.** Let $G$ be a finite group with $S \in \text{Syl}_2(G)$ a wreathed 2-group. If $\text{rk}(G) = 2$ and $G$ has no normal subgroup of index 2, then $G$ has a 2-effective character.

**Proof.** Let $S$ be a Sylow 2-subgroup of $G$. We will say that $S$ is generated by $x$, $y$, and $z$ as in Definition 38. Let $\alpha$ be any primitive $(2^n)^{th}$ root of unity. We will define a degree 3 complex representation of $S$ as follows. $\kappa : S \to GL_3(\mathbb{C})$ is given by

$$
\kappa(x) = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha^{-2}
\end{pmatrix},
\kappa(y) = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \alpha^{-2} & 0 \\
0 & 0 & \alpha
\end{pmatrix},
\kappa(z) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
$$
Let $\nu$ be the complex character of $S$ associated to $\kappa$: for each $g \in S$, $\nu(g) = \text{tr}(\kappa(g))$. We will now show that $\nu$ is a 2-effective character for $G$.

First notice the character values on involutions. The elements of $S$ that are involutions are $x^{2n-1}$, $y^{2n-1}$, $(xy)^{2n-1}$, and those conjugate in $S$ to $z$. Clearly $\nu$ takes the value $-1$ for each of these elements. If $H$ is a rank 2 elementary abelian subgroup of $S$, then $[\nu|_H, 1_H] = 0$.

Let $S_0$ be the subgroup of $S$ generated by $x$ and $y$. $S_0$ is an abelian normal subgroup of index 2. We emphasize that $z$ is conjugate to an element of $S_0$, in particular to $x^{2n-1}$, and $[N_G(S_0) : C_G(S_0)] = 6$ (see [9, Proposition 2G*]). Now we look at fusion of elements of $S_0$. Any two elements of $S_0$ that are conjugate in $G$ are conjugate in $N_G(S_0)$ (see [9, Proposition 2D]). $N_G(S_0)$ is generated by $C_G(S_0)$, the element $z$, and an element $\eta \in G$ such that $\eta x^{-1} = y$, $\eta y^{-1} = x^{-1} y^{-1}$, and $\eta (x^{-1} y^{-1})\eta^{-1} = x$ [9, Proposition 2B].

To show that $\nu$ respects fusion of elements in $S_0$, it is enough to show that if $g \in S_0$, then $\nu(\mu) = \nu(\mu g^{-1}) = \nu(\eta g \eta^{-1})$, assuming that such an $\eta$ exists. Notice that for any $g \in S_0$, there are integers $k$ and $l$ with $g = x^ky^l$. We see then that $zg^{-1} = x^l y^k$ and $\nu(x^k y^l) = \alpha^{k+l} + \alpha^{-2} \alpha^{-2k+l} = \nu(\alpha x^l y^k)$. Also $\eta g \eta^{-1} = x^{-k} y^l \eta^{-1}$ and $\nu(x^k y^l) = \alpha^{k+l} + \alpha^{-k} \alpha^{-2k+l} = \nu(\alpha x^l y^k)$.

Next we will check fusion in elements of order at most $2^n$. If $g \in S \setminus S_0$ is of order at most $2^n$, then there is an integer $h$ with $g$ conjugate to $x^{-h} y^{-h} z$ in $S$ (see [9, Lemma 4]). In particular, $\nu(g) = \nu(x^{-h} y^{-h} z)$. Recall that $x^{-h} y^{-h} z$ is conjugate to $x^{2n-1-h} y^{-h}$ in $G$ (see [9, Proposition 2F]). We see that $\nu(x^{-h} y^{-h} z) = -\alpha^{-2h} = \alpha^{2n-1-h} \alpha^{-2h} = \nu(x^{2n-1-h} y^{-h})$.

At this point we have finished the fusion of elements of order at most $2^n$. Notice that all elements of order larger than $2^n$ must have order $2^{m+1}$ and must be in $S \setminus S_0$. Suppose that $g \in S$ is of order $2^{m+1}$ and is conjugate to another element $g'$ of $S$. It immediately follows that $g, g' \in S \setminus S_0$ and that $g$ and $g'$ are conjugate in $S$ (see [9, Proposition 2E]). So any character of $S$, in particular $\nu$, must agree on $g$ and $g'$. This concludes the proof of Lemma 42.

Proposition 43 (Jackson [21]). If $G$ is a finite group with a dihedral, semi-dihedral, or wreathed Sylow 2-subgroup such that rk$(G) = 2$, then $G$ has a 2-effective character.

Proof. Let $G_2 \in \text{Syl}_2(G)$ and let $n$ be the integer with $|G_2| = 2^n$. If $G_2$ is either dihedral or semi-dihedral, then there is a character of $G_2$ given in Lemma 11 such that

$$\chi(g) = \begin{cases} 3 \cdot 2^{n-3} & \text{if } g = 1 \\ -2^{n-3} & \text{if } g \text{ is an involution} \\ 2^{n-3} & \text{otherwise} \end{cases}$$
Since $\chi$ is constant on elements of the same order, it must respect fusion in $G$ as in Remark 19. It is also clear that if $E \subseteq G_2$ is a rank 2 elementary abelian subgroup, $[\chi|_E, 1_E] = 0$ (see Remark 17). $\chi$ is, thus, a 2-effective character of $G$.

We may now assume that $G_2$ is wreathed and is generated by $x$, $y$, and $z$ as in Definition 38. In addition, let $S_0$ be the abelian subgroup of $G_2$ generated by $x$ and $y$. Since $G_2$ is wreathed, one of the following holds (see [4, Proposition 2, p. 11]):

1. $G$ has no normal subgroups of index 2,
2. $Z(G_2)$ is strongly closed in $G_2$ with respect to $G$,
3. $G$ has a normal subgroup $K$ with Sylow 2-subgroup $S_0$ generated by $x$ and $y$, or
4. $G$ is 2-nilpotent.

Case 1 was treated in Lemma 42. Case 2 has been treated in Proposition 27.

We now treat case 3. Suppose $G$ has a normal subgroup $K$ with Sylow 2-subgroup $S_0$ generated by $x$ and $y$. Notice that $G_2 \cap K = S_0$. We see that $S_0$ is weakly closed in $G_2$ with respect to $G$. By a standard result of Burnside (see [6, 37.6]), $N_G(S_0)$ controls fusion in $C_{G_2}(S_0) = S_0$. Also notice that any element of $G_2 \setminus S_0$ is conjugate in $G$ to only those elements to which it is conjugate in $G_2$. Using the Frattini factor subgroup of $S_0$, we notice that $N_G(S_0)/C_G(S_0)$ is isomorphic to a subgroup of $\Sigma_3$ (see [9, Page 264] or [4, Page 12]).

We may assume that $Z(G_2)$ is not strongly closed in $G_2$ with respect to $G$, otherwise the group is treated in case 2. In particular, this implies that $\Omega_1(Z(G_2)) = \langle (xy)^{2^{n-1}} \rangle$ is not strongly closed in $G_2$ with respect to $G$. The involution $(xy)^{2^{n-1}}$ must then be conjugate in $N_G(S_0)$ to another involution in $S_0$. Thus, $N_G(S_0)/C_G(S_0)$ must not be a two group and must also have order divisible by two since $S_0 < G_2$. $N_G(S_0)/C_G(S_0) \cong \Sigma_3$, and we notice this symmetric group permutes the elements $x$, $y$, and $xy$; furthermore, the character $\chi$ of $G_2$ given in Lemma 42 is again a 2-effective character in this case.

Case 4 was treated in Lemma 40.

To conclude Section 5 we state the following theorem whose proof has now been completed.

**Theorem 44.** Every rank two finite group $G$ has a 2-effective character.

6. **Counterexample**

Looking at Section 5 one may wonder if a similar argument can be made for odd primes. However, this is not the case as we will show in the following proposition.

**Proposition 45.** Let $p$ be an odd prime and $G$ be a rank two finite group. If $Qd(p)$ is involved in $G$ and if $G$ and $Qd(p)$ have isomorphic Sylow $p$-subgroups, then $G$ does not have a $p$-effective character. In particular, $Qd(p)$ does not have a $p$-effective character for an odd prime $p$. 
Proof. Let \( G_p \in \text{Syl}_p(G) \). Notice that \( G_p \), which is isomorphic to a Sylow \( p \)-subgroup of \( \text{Qd}(p) \), is an extra-special \( p \)-group of size \( p^3 \) and exponent \( p \). Let \( \chi_1, \ldots, \chi_n \) be the irreducible characters of \( G_p \) and assume that \( \chi_1 \) is the trivial character. Notice from the structure of \( G_p \) that if \( \chi_i(1) \neq 1 \), then for \( g \in G_p \), \( \chi_i(g) \neq 0 \) if and only if \( g \in Z(G_p) \). Also notice that \( Z(G_p) \) is cyclic of order \( p \) and \( Z(G_p) \) is not strongly closed in \( G_p \) with respect to \( G \). Let \( x \in Z(G_p) \) and \( y \in G_p \setminus Z(G_p) \) such that there is a \( g \in G \) with \( y^g = x \). Suppose \( \chi \) is a character of \( G_p \) that is a \( p \)-effective character of \( G \). There exists \( a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0} \) such that 
\[
\chi = \sum_{i=1}^{n} a_i \chi_i. \quad \text{Since} \quad \chi \quad \text{is a} \quad p \text{-effective character of} \quad G, \quad \text{it respects fusion in} \quad G, \quad \text{which implies} \quad \chi(x) = \chi(y) \quad \text{and} \quad a_1 = 0. \quad \text{Suppose that} \quad a_i > 0 \quad \text{for some} \quad i > 1 \quad \text{with} \quad \chi_i(1) = 1. \quad \text{Fixing this} \quad i, \quad \text{let} \quad E_i = \{g \in G_p | \chi_i(g) = 1\}. \quad \text{Notice that for such an} \quad i, \quad E_i \quad \text{is a rank two elementary abelian subgroup of} \quad G_p. \quad \text{This implies that} \quad [\chi_i|_{E_i}, 1_{E_i}] = 1, \quad \text{so} \quad [\chi|_{E_i}, 1_{E_i}] > 0, \quad \text{which contradicts the assumption that} \quad \chi \quad \text{is a} \quad p \text{-effective character of} \quad G; \quad \text{therefore, for each} \quad i \quad \text{such that} \quad \chi_i(1) = 1, \quad a_i = 0. \quad \text{Now} 
\[
\chi(x) = \sum_{i \text{ such that } \chi_i(1) \neq 1} a_i \chi_i(x) + \sum_{i \text{ such that } \chi_i(1) = 1} a_i = \sum_{i \text{ such that } \chi_i(1) = 1} a_i \chi_i(y) = \chi(y). 
\]
\( \chi(x) = 0; \) therefore, \( \chi(x) = 0. \quad \chi(z) = 0 \quad \text{for all} \quad z \in Z(G_p) \setminus \{1\} \quad \text{because} \quad Z(G_p) \quad \text{is a cyclic group of order} \quad p. \quad \text{Notice that in showing} \quad \chi(z) = 0 \quad \text{for each} \quad z \in Z(G_p) \setminus \{1\}, \quad \text{we see that} \quad \chi(g) = 0 \quad \text{for all} \quad g \in G_p \setminus Z(G_p) \quad \text{by the structure of} \quad G; \quad \text{thus,} \quad \chi(g) = 0 \quad \text{for all} \quad g \in G_p \setminus \{1\}. \quad \chi \quad \text{must be identically zero, which contradicts the definition of} \quad p \text{-effective character.} \quad \square \n
Lemma 46. Let \( G \) be a finite group with \( p \) a prime dividing \( |G| \) and \( H \subseteq G \). Suppose that \( p \) divides \( |H| \) and that \( \text{rk}_p(G) = \text{rk}_p(H) \). If \( G \) has a \( p \)-effective character, so does \( H \).

Proof. We may assume that \( \text{rk}_p(H) = \text{rk}(G) \). Let \( G_p \in \text{Syl}_p(G) \) such that \( H_p = G_p \cap H \in \text{Syl}_p(H) \). Let \( \chi \) be a character of \( G_p \) that is a \( p \)-effective character of \( G \). \( \chi|_{H_p} \) is a character of \( H_p \), which is not identically zero. \( \chi|_{H_p} \) respects fusion in \( H \) because \( \chi \) respects fusion in \( G \). Any maximal rank elementary abelian subgroup of \( H_p \) is also a maximal rank elementary abelian subgroup of \( G_p \); therefore, \( \chi|_{H_p} \) is a \( p \)-effective character of \( H \). \( \square \)

Combining Lemma 46 with Theorems 35 and 44, we get the following theorem from which Remark 2.1 follows:

Theorem 47. Let \( G \) be a finite group of rank two and let \( p \) be a prime dividing \( |G| \). \( G \) has \( p \)-effective character if and only if either \( p = 2 \) or both \( p > 2 \) and \( G \) does not \( p' \)-involve \( \text{Qd}(p) \).

In showing that a finite group \( G \) of rank two acts freely on a finite complex \( Y \simeq S^n \times S^m \), we show that \( G \) acts on a finite complex \( X \simeq S^m \) with isotropy groups of rank one and then we apply Theorem 5. Ozgun Unlu 24 and Grodal 18 have shown separately that for each odd prime \( p \), \( \text{Qd}(p) \) cannot act on any finite complex homotopy equivalent to a sphere with rank one isotropy groups. From both of their proofs it is clear that for any odd prime \( p \) and rank
two group $G$ that $p'$-involves $Qd(p), G$ cannot act on any finite complex homotopy equivalent to a sphere with rank one isotropy groups.

Combining this discussion with Theorem 47 and the discussion in Section 2, we conclude with the following proposition:

**Proposition 48.** Let $G$ be a finite group of rank two. $G$ acts on some finite complex homotopy equivalent to a sphere with rank one isotropy groups if and only if for each odd prime $p, G$ does not $p'$-involve $Qd(p)$.

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