POLYNOMIALS WITH LORENTZIAN SIGNATURE AND COMPUTING PERMANENTS

PAPRI DEY

Abstract. We study the class of polynomials whose Hessians evaluated at any interior point of a closed convex cone $K$ have Lorentzian signature. This class is a generalization to the remarkable class of Lorentzian polynomials. We prove that hyperbolic polynomials and conic stable polynomials belong to this class. Finally, we develop a method for computing permanents of nonsingular matrices which belong to a class that includes nonsingular $k$ locally singular matrices via hyperbolic programming.

1. Introduction

We define a homogeneous multivariate polynomial $f(x)$ to be a log-concave polynomial if its Hessian $H_f(x):=(\partial_i \partial_j f)^n_{i,j=1}$ has at most one positive eigenvalue for any $x \in K$, a closed convex cone, and nonsingular for any $x \in \text{int } K$. A log-concave polynomial is called strictly log-concave if its Hessian $H_f(x):=(\partial_i \partial_j f)^n_{i,j=1}$ has Lorentzian signature, i.e., $H_f(x)$ has exactly one positive eigenvalue. Then we define a homogeneous multivariate polynomial $f(x)$ to have Lorentzian signature if $f(x)$ and all of its directional derivatives are log-concave over the interior of some cone $K$. If $f(x)$ and all of its directional derivatives are strictly log-concave over the interior of the cone $K$, we call $f(x)$ has strict Lorentzian signature. Obviously, Lorentzian polynomials studied in [BH20] have Lorentzian signature. The cone $K$ is nonnegative orthant for Lorentzian polynomials. Besides, the coefficients of Lorentzian polynomials are nonnegative and this property of a Lorentzian polynomial induces the support of a Lorentzian polynomial to be an $M$-convex.

We believe that polynomials with Lorentzian signature can be pursued to prove long-standing conjectures, inequalities, bounds, and approximation. Lorentzian polynomials arise in matroid theory, entropy optimization, and negative dependence properties. For an excellent introduction to negative dependence properties see [Pem00]. Stable polynomials, a subclass of Lorentzian polynomials can be found in Gurvits’s work [Gur06a], [Gur06b] that includes solving combinatorial problems like bipartite matching problem with computing permanents, solving Van der Waerden conjecture for doubly stochastic matrices and mixed discriminant, and mixed volume. Properties of stable polynomials and interfacing families are also used to prove Kadison-Singer conjecture [MSS15a] and existence of Ramanujan graphs, and furthermore to construct an infinite families of bipartite Ramanujan graphs of every degree [MSS18]. Completely log-concave polynomials are explored in the context of optimization [AG17] and counting on matroids (for example, strongest Mason’s conjectures on the log-concavity of the number of independent sets of a matroid) [ALGV18], [BH20]. A recent approach to extend the theory of Lorentzian polynomials over cones other than the nonnegative orthant is explored to prove Heron-Rota-Welsh conjecture via Lorentzian polynomials, see [BL21].

In this paper, we provide a few equivalent characterization results in Proposition 4 for a log-concave polynomial. Then, we introduce the notion of polynomials with Lorentzian signature. We show that any hyperbolic polynomial has Lorentzian signature [Theorem 11]. However, the converse need not be true and it has been exemplified. Furthermore, we prove that any conic stable polynomial is a constant multiple of a polynomial with Lorentzian signature [Theorem 15]. Next, we provide a formula for the determinant of a sum of $k$ matrices in terms of mixed discriminant of $k$ tuple of matrices [Proposition 26], which generalizes the matrix determinant lemma. Furthermore, we characterize a class of nonsingular matrices for which permanents can be computed by solving a hyperbolic programming. Nonsingular $k$-locally singular matrices are introduced in the context of $k$ locally psd matrices by Blekherman et. al. in [BDSS20] to solve large scale positive semidefinite programs by applying psd-ness on

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a collection of principal submatrices. We show that the generating polynomial of nonsingular $k$-locally singular matrix is a polynomial with Lorentzian signature, and the permanent of a nonsingular $k$-locally singular matrix can be computed by solving a hyperbolic programming.

The remainder of this paper is organized as follows. In section 2, we discuss preliminary notions and results which are used in the sequel. In section 3, we investigate the properties of polynomials with Lorentzian signature. In section 4, using the notion of mixed discriminant of matrices we generalize the matrix determinant lemma. In section 5, it's shown how hyperbolic programming can be used to compute the permanents of a class of nonsingular matrices. This special class of non-singular matrices includes the class of non-singular $k$-locally singular matrices which has been studied in section 6. In section 7, we discuss the results and conclude with some open questions.

2. Preliminaries

A homogeneous polynomial $f$ of degree $d$ in $n$ variables is hyperbolic with respect to $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and if for all $x \in \mathbb{R}^n$ the univariate polynomial $t \mapsto f(x + te)$ has only real roots. A polynomial $f \in \mathbb{R}[x]$ is called a stable polynomial if it has no roots in the upper half plane, i.e., $f(z) \neq 0 \quad \forall z \in \mathbb{C}^n$ with $\text{im}(z) \in \mathbb{R}^+_0$, the positive orthant. Equivalently, $f \in \mathbb{R}[x]$ is stable if for all $v \in \mathbb{R}^n$ and $u \in \mathbb{R}^n_{>0}$ the univariate polynomial $f(v + tu)$ is real-rooted. A generalization to the notion of stable polynomials is the notion of conic stable polynomials introduced and studied in [JT18a],[JTdW19]. Let $K$ be a closed convex cone in $\mathbb{R}^n$. For a polynomial $f \in \mathbb{R}[z]$, its imaginary projection $\mathcal{I}(f)$ is defined as the projection of the variety of $f$ onto its imaginary part, i.e.,

$$\mathcal{I}(f) = \{\text{im}(z) = (\text{im}(z_1), \ldots, \text{im}(z_n)) : f(z) = 0\},$$

where $\text{im}(\cdot)$ denotes the imaginary part of a complex number.

Definition 1. A polynomial $f \in \mathbb{R}[z]$ is called $K$-stable, if $f(z) \neq 0$ whenever $\text{im}(z) \in \text{int} K$. If $f \in \mathbb{R}[Z]$ on the symmetric matrix variables $Z = (z_{ij})_{n \times n}$ is $S^+_n$-stable, then $f$ is called positive semidefinite-stable (for short, psd-stable) if $S^+_n$ denotes the cone of psd matrices.

Fact 2. A polynomial $f \in \mathbb{R}[x]$ is stable if and only if the (unique) homogenization polynomial w.r.t $x_0$ is hyperbolic w.r.t every vector $e \in \mathbb{R}^{n+1}$ such that $e_0 = 0$ and $e_j > 0$ for all $1 \leq j \leq n$.

Due to Gårding’s foundational work on hyperbolic polynomials [Gär59] it’s known that the hyperbolicity cone corresponding to a hyperbolic polynomial $f$ with respect to $e$, denoted by $\Delta_{++}(f, e)$, defined by $\Delta_{++}(f, e) = \{x \in \mathbb{R}^n : f(x + te) = 0 \Rightarrow t < 0\}$ is an open convex cone. Note that the hyperbolicity cones depend on the spectrum of some matrices.

Log-Concavity. Logarithmic concavity is a property of a sequence of real numbers, and it has a wide range of applications in algebra, geometry, and combinatorics, [Huh18]. A sequence of real numbers $a_0, \ldots, a_d$ is log-concave if

$$a_i^2 \geq a_{i-1}a_{i+1} \quad \text{for all } i$$

When the numbers are positive, log-concavity implies unimodality, i.e., there is an index $i$ such that

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_d$$

for all $i$.

M-convex. A subset $J \subseteq \mathbb{N}^n$ is said to be $M$-convex if it satisfies any one of the following equivalent conditions:

1. **Exchange Property**: For any $\alpha, \beta \in J$ and any index $i$ satisfying $\alpha_i > \beta_i$, there is an index $j$ satisfying $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$.

2. **Symmetric Exchange Property**: For any $\alpha, \beta \in J$ and any index $i$ satisfying $\alpha_i > \beta_i$, there is an index $j$ satisfying $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$ and $\beta - e_j + e_i \in J$.

A proof of the equivalence can be found [Mur98, Chapter 4]. We refer to [Mur98, Section 4.2], [KMT07], [Fuj05] for details on $M$-convex sets. The support of the polynomial $f$ is the subset of $\mathbb{Z}_+^n$ defined by $\text{supp}(f) = \{\alpha \in \mathbb{N}^n : c_\alpha \neq 0\}$.
for $k \in \mathbb{R}^n$. We say that $g$ interlaces $f$ with respect to $e$ if for all $a \in \mathbb{R}^n$ the univariate restriction $g(a + te)$ interlaces $f(a + te)$. For univariate polynomials $f, g \in \mathbb{R}[t]$ with all real roots and $\deg(g) = \deg(f) - 1$, let $\alpha_1 \leq \cdots \leq \alpha_d$ and $\beta_1 \leq \cdots \leq \beta_{d-1}$ be the roots of $f$ and $g$ respectively. We say that $g$ interlaces $f$ if $\alpha_i \leq \beta_i \leq \alpha_{i+1}$ for all $i = 1, \ldots, d - 1$.

**Derivative Relaxation.** It’s a well-known fact that if $f$ has degree $d$ and is hyperbolic with respect to $e$, then for $k = 0, 1, \ldots, d$, the $k$-th directional derivative in the direction $e$, i.e.,

$$D_e^{(k)}f(x) = \frac{d^k}{dt^k}f(x + te)|_{t=0}$$

is also hyperbolic with respect to $e$. This is one familiar way to produce new hyperbolic polynomials by taking directional derivatives of hyperbolic polynomials in directions of hyperbolicity [AG70, Section 3.10]. A beautiful construction can be found by Renegar [Ren04] in the context of optimization. Moreover, the hyperbolicity cones of the directional derivatives form a sequence of **relaxations** of the original hyperbolicity cone as follows.

$$\lambda_+(D_e^{(k)}f, e) \supseteq \lambda_+(D_e^{(k-1)}f, e) \supseteq \cdots \supseteq \lambda_+(f, e).$$

Someone interested in this topic can see [Ren04], [SP15], [Sau18] for the details.

### 3. Polynomials with Lorentzian Signature

Gårding in his pioneering work [Går59] has shown that if $f(x)$ is a hyperbolic of degree $d$ w.r.t $e$, $\log(f(x))$ is concave, by showing that $f(x)^{1/d}$ is concave and homogeneous of degree 1 in $\Lambda_+(f, e)$, and vanishes on the boundary of $\Lambda_+(f, e)$, see [Gül97] for a compact proof.

#### 3.1. The space of Log-Concave Polynomials

Let $n$ and $d$ be nonnegative integers, and $\mathbb{R}_d^n$ denote the set of degree $d$ homogeneous polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. The **Hessian** of $f \in \mathbb{R}[x_1, \ldots, x_n]$ is the symmetric matrix

$$H_f(x) = (\partial_i \partial_j f)^n_{i,j=1},$$

where $\partial_i$ denotes the partial derivative $\frac{\partial}{\partial x_i}$. Motivated by Gårding’s result and inspired by the ideas to define the notion of the log-concave polynomials with non-negative coefficients in [AG18], we define that $f \in \mathbb{R}[x_1, \ldots, x_n]$ is **log-concave** over the interior of a closed convex cone $K$ if $\log(f)$ is a concave function over the interior of the cone $K$. Note that the definition makes sense if $f(e) > 0$ for all $e \in \text{int} K$, equivalently, $\log(f)$ is defined at all points of $\text{int} K$. Besides, the entries of the Hessian of $\log(f)$ at a point in $K$ are continuous functions in the coefficients of $f$. Thus, a homogeneous polynomial $f$ is log-concave if and only if the Hessian of $\log(f)$ is negative semidefinite at all points of $a \in K$ where it is defined if and only if the Hessian evaluated at $x \in K$ has at most one positive eigenvalue. For simplicity, we assume that the zero polynomial, constant polynomials and linear polynomials are log-concave. Then, equivalently, $f(x) \in \mathbb{R}[x]$ is log-concave if for any two points $a, b \in K$, and $\lambda \in [0, 1]$ we have

$$f(\lambda a + (1 - \lambda)b) \geq f(a)^\lambda \cdot f(b)^{1-\lambda}.$$ 

In order to show an analogous equivalent characterization result for the notion of log-concavity over $K \subset \mathbb{R}^n$ we use the same line of thoughts mentioned in [AG18]. Note that if a symmetric matrix $Q \in \mathbb{R}^{n \times n}$ has exactly one positive eigenvalue, $QP^T$ has exactly one positive eigenvalue where $P \in \mathbb{R}^{m \times n}$. We also use Cauchy’s interlacing Theorem, [HJ12, Theorem 4.3.17] which states that if $R \in \mathbb{R}^{(n-1) \times (n-1)}$ is a principal submatrix of $Q \in \mathbb{R}^{n \times n}$, then the eigenvalues of $R$ interlaces the eigenvalues of $Q$. We show the following result which will be used in the sequel.

**Lemma 3.** Let $Q \in \mathbb{R}^{n \times n}$ be a symmetric matrix with at most one positive eigenvalue. Then for any $a \in \mathbb{R}^n$ such that $a^TQA > 0$, the $n \times n$ matrix $(a^TQA) \cdot Q - t(Qa)(Qa)^T$ is negative semidefinite for all $t \geq 1$. More precisely, $Q$ has exactly one positive eigenvalue if and only if there exists $a \in \mathbb{R}^n$ such that $a^TQA > 0$ with $Q$ having at most one positive eigenvalue.
Proof. Note that if $a^T Q a \leq 0$ for all $a \in \mathbb{R}^n$, $Q$ is negative semidefinite. If there exists $a \in \mathbb{R}^n$ such that $a^T Q a > 0$, it’s sufficient to prove the result for $t = 1$. Let $b \in \mathbb{R}^n$ and consider the $2 \times n$ matrix $P$ with rows $a^T$ and $b^T$. Then

$$PQP^T = \begin{bmatrix} a^T Q a & a^T Q b \\ b^T Q a & b^T Q b \end{bmatrix}$$

Since $Q$ has exactly one eigenvalue, by using the Cauchy’s interlacing theorem mentioned above, we can conclude that $PQP^T$ has exactly one positive eigenvalue. That implies $\det(PQP^T) = a^T Q a \cdot b^T Q b - a^T Q b \cdot b^T Q a \leq 0$. Therefore, $b^T ((a^T Q a) \cdot Q - (Q a)(Q a)^T) b \leq 0$ for all $b \in \mathbb{R}^n$.

**Proposition 4.** Let $f \in \mathbb{R}[x]$ be homogeneous of degree $d \geq 2$. Fix a point $a \in K$ with $f(a) > 0$, and let $H_f(a) := Q$. The following are equivalent.

1. $f$ is strictly log-concave at $x = a$.
2. $Q$ has the Lorentzian signature $(+, - , \ldots, -)$, equivalently, exactly one positive eigenvalue.
3. $x \mapsto x^T Q x$ is negative definite on $n - 1$ dimensional space
4. the matrix $(a^T Q a) Q - (Q a)(Q a)^T$ is negative definite.

Proof. Euler’s Identity states that for a homogeneous polynomial $f$ of degree $d$, $(\nabla f \cdot x) = \sum_{i=1}^{n} x_i \partial_i f = d f(x)$. Using this on $f$ and $\partial_i f$ one can easily get the following identities.

$$H_f x = (d - 1) \cdot \nabla f(x), \ x^T H_f x = d(d - 1) \cdot f(x)$$

Besides, the Hessian of $\log(f)$ at $x = a$ equals

$$H_{\log f}(a) = \left( \frac{f \nabla f - \nabla f \nabla f^T}{f^2} \right)_{x=a} = d(d - 1) \frac{a^T Q a \cdot Q - \frac{d}{a^T Q a}}{(a^T Q a)^2}$$

$(1 \Leftrightarrow 2)$ Thus, by eq. (4) negative definiteness of the Hessian of $\log(f)$ at $a \in K$ is equivalent to the fact that the Hessian $H_f(a) =: Q$ of $f$ evaluated at $a \in K$ has the Lorentzian signature.

$(1 \Rightarrow 3)$ If $f$ is strictly log-concave at $x = a$, then the Hessian of $\log(f(x))$ at $x = a$ is negative definite. Then by eq. (4) $(a^T Q a) \cdot Q - \frac{d}{a^T Q a} (Q a)(Q a)^T$ is negative definite. Therefore, since $a^T Q a > 0$, so its restriction to the linear space $(Q a)^\perp = \{x \in \mathbb{R}^n | x^T Q a = 0 \}$ implies that $H_{\log f}(a) = \frac{d(d - 1)}{a^T Q a} Q$ is negative definite on this linear space. That means the quadric $x \mapsto x^T Q x$ is negative definite on $(Q a)^\perp$. Since $(Q a)^\perp$ is $n - 1$ dimensional for $Q a \neq 0$, therefore, $x \mapsto x^T Q x$ is negative definite on $n - 1$ dimensional space.

$(3 \Rightarrow 4)$ Since $f(a) > 0$ implies $a^T Q a > 0$ and $Q$ is nonsingular, so by Equation (3) in Lemma 3 $\det PQP^T < 0$. Hence the result follows from Lemma 3.

$(4 \Rightarrow 1)$ $(a^T Q a) \cdot Q - (Q a)(Q a)^T$ is negative definite implies $(a^T Q a) \cdot Q - \frac{d}{a^T Q a} (Q a)(Q a)^T$ is negative definite. Then it follows from eq. (4).

$(2 \Rightarrow 5)$ Consider a $b \in K \cap \mathbb{R}^n$ such that $Q b \neq 0$. Then construct a $2 \times n$ matrix $P$ with rows $a^T$ and $b^T$. Since $a^T Q a > 0$ and $Q$ has Lorentzian signature, so by Equation (3) in Lemma 3 $\det PQP^T < 0$, i.e., $(a^T Q b)^2 > (a^T Q a)(b^T Q b)$. This implies $Q$ is negative definite on the hyperplane $\{b \in \mathbb{R}^n : a^T Q b = 0 \}$ for $Q b \neq 0$.

$(5 \Rightarrow 3)$ Since $Q b \neq 0$ and $x \mapsto x^T Q x$ is negative definite on $(Q b)^\perp$ for every $b \in K$, so $x \mapsto x^T Q x$ is negative definite on $n - 1$ dimensional linear space.

**Corollary 5.** Polynomial $f(x)$ is log-concave at $x = a$ in $K$ if and only if $H_f(a)$ has at most one positive eigenvalue if and only if $x \mapsto x^T Q x$ is negative semidefinite on $n - 1$ dimensional space. think

Proof. Since $K$ is a closed convex cone, so it follows from Proposition 4.

Next we show log-concavity is preserved by taking partial derivatives. A similar statement can be found in [ALGV18, Lemma 2.1].

**Proposition 6.** For a homogeneous polynomial $f \in \mathbb{R}_n^d$ of degree $\geq 3$, $f$ is log-concave at $a$ if and only if $D_a(f)$ is log-concave at $a$.

Proof. It follows from the fact that the Hessian $H_D_a f(a) = (d - 2)H_f(a)$ using Euler’s identity $d \cdot f = \sum_{i=1}^{n} x_i \partial_i f$ on the polynomial $\partial_i \partial_j f$. 


Remark 7. Note that the sign of the coefficients of \( f(x) \) remains unchanged by taking derivatives until the coefficients vanish since the coefficient of monomial \( x_1^{k_1} \) in any \( f \) is a positive multiple of \( \partial_1^{k_1} \cdots \partial_n^{k_n} f(x) \) at \( x = 0 \).

**Proposition 8.** If \( f(x) \in \mathbb{R}_n^d \) is log-concave at \( x = a \), then \( f(Ax) \) is log-concave at some point (need not be \( a \)) for any nonsingular \( n \times n \) matrix \( A \).

**Proof.** Since \( f \) is log-concave at \( x = a \), so \( H_f(a) \) has at most one positive eigenvalue by Corollary 5. Using Euler’s Identity we have \( x^T H_f x = d(d - 1) \cdot f(x) \). Therefore, \( x^T A^T H_f A x = d(d - 1) \cdot f(Ax) \). Say \( H_f(a) =: Q \). By Sylvester’s law of inertia, the signatures of \( Q \) and \( A^TQA \) are the same as they are congruent to each other. Hence, the result.

### 3.2. Polynomials with Lorentzian signature

In this subsection we introduce the notion of a polynomial with Lorentzian signature. It’s shown in [DP18, Proposition 3.1 ] that a homogeneous quadratic polynomial \( f(x) = x^T Q x \) is hyperbolic w.r.t \( x \in \mathbb{R}^n \) such that \( f(x) > 0 \) if and only if its Hessian \( H_f = Q \) can have at most one positive eigenvalue. Moreover, it’s shown in [DP18, Theorem 4.5] that the hyperbolicity cone associated with a hyperbolic quadratic polynomial is a full dimensional cone if and only if the matrix representation \( Q \) is not negative semidefinite. Thus, the interesting case is when a quadratic hyperbolic polynomial has a full dimensional hyperbolicity cone, i.e., a quadratic polynomial such that its Hessian has the Lorentzian signature.

Up to normalization (since every quadratic in \( \mathbb{R}^n \) is affinely equivalent to a quadric given by one of the three normal forms) it’s of the form \( x_1^2 - (x_2^2 + \cdots + x_n^2) \), cf. [DGT21] for details. In this case, there are two full dimensional hyperbolicity cones

\[
\{ x \in R_+ \times \mathbb{R}^n : x_1^2 > \sum_{j=2}^{n} x_j^2 \} \quad \text{and} \quad \{ x \in R_- \times \mathbb{R}^n : x_1^2 > \sum_{j=2}^{n} x_j^2 \}
\]

containing \((1,0,\ldots,0)\) and \((-1,0,\ldots,0)\) respectively.

Consider the homogeneous bivariate cubic polynomial \( f = x_1^3 - x_2^3x_2 + x_2^3 \). In this case, \( \partial_1(f) \) and \( \partial_2(f) \) are log-concave at \( x \in \mathbb{R}^n \), but for example, neither \( \sum_{i=1}^2 \partial_i f \) nor \( f \) is log-concave at \( a = (1,1) \). Also note that \( f \) is not a hyperbolic polynomial. In this example, we see that though \( \partial_1 f \) and \( \partial_2 f \) are log-concave at any \( x \in \mathbb{R}^2 \) and there exist \( b = (0,1) \), \( c = (1,0) \in \mathbb{R}^2 \) such that \( D_b \partial_1 f = D_c \partial_2 f \not= 0 \), but \( \partial_1 f + \partial_2 f \) is not log-concave at \((1,1)\). Thus, the observation about quadratic polynomials; besides, the notion of Lorentzian polynomials motivates us to define the notion of polynomials with Lorentzian signature which is preserved under all directional derivatives. In [Gur09], Gurvits defines \( f \) to be strongly log-concave if for all \( \alpha \in \mathbb{N}^n \), \( \partial^\alpha f \) is identically zero or \( \partial^\alpha f \) is log-concave on \( \mathbb{R}^n_{>0} \), the positive orthant. In [AGV18], Anari et al. define \( f \) to be completely log-concave if for all \( m \in \mathbb{N} \) and any \( m \times m \) matrix \( A = (a_{ij}) \) with nonnegative entries,

\[
\left( \prod_{i=1}^m D_i \right) f \text{ is identically zero or } \left( \prod_{i=1}^m D_i \right) f \text{ is log-concave on } \mathbb{R}^n_{>0}
\]

where \( D_i \) is the differential operator \( \sum_{j=1}^n a_{ij} \partial_j \).

On the other hand, Brändén et al. [BH20] define the notion of Lorentzian polynomials whose supports are \( M \)-convex, and show that the notion of Lorentzian polynomials is equivalent [cf. [BH20, Theorem 2.30]] to other two notions of strongly log-concave introduced in [Gur09] and completely log-concave introduced in [AGV18]. The proof is based on showing that the supports of completely log-concave and strongly log-concave are \( M \)-convex. Note that the support of hyperbolic polynomials need not be \( M \)-convex. Also note that if \( f \) is hyperbolic w.r.t \( e \), it is hyperbolic w.r.t \(-e\). If \( f \) is irreducible, there are only two hyperbolicity cones, one is \( \Lambda_{++}(f,e) \) containing \( e \) and other one is the negative of \( \Lambda_{++}(f,e) \) containing \(-e\), see [LPR05, Prop. 2]. Let \( H_n^+ \) denote the space of hyperbolic polynomials \( f \) of degree \( d \) such that \( f(a) > 0 \) for all \( a \in \Lambda_{++}(f,e) \). The space \( H_n^+ \) can be associated with the set of \( n \times n \) symmetric matrices that have at most one positive eigenvalue. More precisely, a homogeneous quadratic polynomial \( f \) is a hyperbolic polynomial w.r.t a point \( e \) such that \( f(e) > 0 \) if and only if its Hessian has at most one positive eigenvalue. A homogeneous quadratic polynomial \( f \) is a strictly hyperbolic polynomial w.r.t a point \( e \) such that \( f(e) > 0 \) if and only if it has Lorentzian signature or in other words, it’s strongly log-concave. Besides, if we assume that quadratic \( f \) has only positive coefficients and its Hessian has Lorentzian signature (i.e., exactly one positive eigenvalue), then \( f \) is a univariate quadratic polynomial. Thus, quadratic Lorentzian polynomials are univariate quadratic polynomials.
So, by exploiting the nature of quadratic hyperbolic polynomials we define the notion of polynomials with Lorentzian signature as follows. First, we define a topology on the space \( \mathbb{R}^d_n \) of degree \( d \) homogeneous polynomials using the Euclidean norm for the coefficients, cf. [Nui68].

**Definition 9.** A homogeneous polynomial \( f \in \mathbb{R}^d_n \) is said to be polynomial with Lorentzian signature if for all \( a_1, \ldots, a_d \in \text{int } K \), (where \( K \) is a closed convex cone) the symmetric bilinear form

\[
(x, y) \mapsto D_x D_y d_{a_1} \cdots d_{a_d} f
\]

has at most one positive eigenvalue. By convention, we say that zero polynomial is a polynomial with Lorentzian signature.

**Theorem 11.** For a homogeneous polynomial \( f \in \mathbb{R}^d_n \), its hyperbolicity cone \( \Lambda \) w.r.t all \( a \) with Lorentzian signature where the cone \( \Lambda \) is the hyperbolicity cone \( \Lambda \) of \( f \) with Lorentzian signature.

**Definition 10.** For a homogeneous polynomial \( f \in \mathbb{R}^d_n \) of degree \( d \) and with Lorentzian signature.

**Proof.** Let \( f \in \mathcal{H}^d_n \). That means \( f \in \mathbb{R}^d_n \) is a hyperbolic polynomial w.r.t \( e \) such that \( f(e) > 0 \). Then \( f \) is hyperbolic w.r.t all \( a \in \Lambda_{++}(f, e) \) and \( f(a) > 0 \). Gårding [Gar59] has shown that hyperbolic polynomial \( f \) is log-concave on its hyperbolicity cone \( \Lambda_{++}(f, e) \). So, the Hessian \( H_f(a) \) for all \( a \in \Lambda_{++}(f, e) \) has at most one positive eigenvalue by Corollary 5. Since the hyperbolicity is preserved by directional derivatives in the directions of hyperbolicity, so using eq. (2) \( D_a(f) \) is hyperbolic w.r.t all \( a \in \Lambda_{++}(f, e) \). Thus, we conclude that hyperbolic \( f \) is a polynomial with Lorentzian signature where the cone \( K \) is the hyperbolicity cone \( \Lambda_{++}(f, e) \).}

**Theorem 12.** Any hyperbolic polynomial \( f \in \mathcal{H}^d_n \) is a polynomial with Lorentzian signature.

**Proof.** Let \( f \in \mathcal{H}^d_n \). That means \( f \in \mathbb{R}^d_n \) is a hyperbolic polynomial w.r.t \( e \) such that \( f(e) > 0 \). Then \( f \) is hyperbolic w.r.t all \( a \in \Lambda_{++}(f, e) \) and \( f(a) > 0 \). Gårding [Gar59] has shown that hyperbolic polynomial \( f \) is log-concave on its hyperbolicity cone \( \Lambda_{++}(f, e) \). So, the Hessian \( H_f(a) \) for all \( a \in \Lambda_{++}(f, e) \) has at most one positive eigenvalue by Corollary 5. Since the hyperbolicity is preserved by directional derivatives in the directions of hyperbolicity, so using eq. (2) \( D_a(f) \) is hyperbolic w.r.t all \( a \in \Lambda_{++}(f, e) \). Thus, we conclude that hyperbolic \( f \) is a polynomial with Lorentzian signature where the cone \( K \) is the hyperbolicity cone \( \Lambda_{++}(f, e) \). □

**Theorem 12.** For a homogeneous polynomial \( f \in \mathbb{C}[z] \), the following are equivalent.
(1) \( f \) is \( K \)-stable.
(2) \( \mathcal{I}(f) \cap \text{int } K = \emptyset. \)
(3) \( f \) is hyperbolic w.r.t. every point in \( \text{int } K \).

The following lemma in [JT18a] establish a reduction of multivariate \( K \)-stability to univariate stable polynomials.

**Lemma 13.** A multivariate polynomial \( f \in \mathbb{C}[z] \setminus \{0\} \) is \( K \)-stable if and only if for all \( x, y \in \mathbb{R}^n \) with \( y \in \text{int } K \) the univariate polynomial \( t \mapsto f(x + ty) \) is stable.

Note that a polynomial need not be stable in order to be a \( K \)-stable polynomial. For example \( (x_1 + x_3)^2 - x_2^2 \) is not a stable polynomial but psd-stable where \( K \) is the cone of positive semidefinite matrices. See [DGT21] for a comparison among stable, psd-stable and determinantal polynomials. By [JT18b], the hyperbolicity cones of a homogeneous polynomial \( f \) coincide with the components of \( \mathcal{I}(f)^c \), where \( \mathcal{I}(f)^c \) denotes the complement of \( \mathcal{I}(f) \). This implies:

**Corollary 14.** A hyperbolic polynomial \( f \in \mathbb{C}[z] \) is \( K \)-stable if and only if \( \text{int } K \subseteq \Lambda_+(f, e) \) for some hyperbolicity direction \( e \) of \( f \).

The proof is based on the observation that a hyperbolic polynomial \( f \in \mathbb{C}[z] \) is \( K \)-stable if and only if \( \text{int } K \subseteq \mathcal{I}(f)^c \). See [DGT21] for the details.

Note that the nonzero coefficients of a homogeneous stable polynomial have the same sign [COSW04, Theorem 6.1]. Using this result any homogeneous stable polynomial with nonnegative coefficients is Lorentzian polynomial, cf. [BH20]. Here we show an analogous result to the fact any homogeneous stable polynomial is a constant multiple of a Lorentzian polynomial. Let \( K^d_n \) denote the space of degree \( d \), \( K \) stable \( n \)-variate polynomials such that \( f(a) > 0 \) for all \( a \in \text{int } K \).

**Theorem 15.** Any \( K \)-stable polynomial \( f \in \mathbb{R}^d_n \) is a constant multiple of a polynomial with Lorentzian signature.

*Proof.* Let \( f \in \mathbb{R}^d_n \) be a \( K \)-stable polynomial. Then by Theorem 12 and Corollary 14 \( f \) is a hyperbolic polynomial w.r.t all \( a \in \text{int } K \subseteq \Lambda_+(f, e) \) such that \( f(e) > 0 \) (w.l.o.g one can assume it, otherwise, need to consider the point \(-e\)). Say \( H_f(e) := Q \). Gårding [Går59] has shown that hyperbolic polynomial \( f \) is log-concave on its hyperbolicity cone \( \Lambda_+(f, e) \). So, the Hessian \( H_f(a) \) for all \( a \in \text{int } K \) has at most one positive eigenvalue by Corollary 5. Since the hyperbolicity is preserved by directional derivatives in the directions of hyperbolicity, so \( K \)-stability is preserved by directional derivatives \( a \in \text{int } K \). Thus, using Corollary 14 and eq. (2), \( D_a \) is an open map sending \( K^d_n \) to \( K^{d-1}_n \). Thus, \( K^d_n \) is contained in the space of polynomials with Lorentzian signature. \( \square \)

**Remark.** Not all polynomials with Lorentzian signature are hyperbolic polynomials. For example, \( f(x) = -2x_1^2 + 12x_1x_2 + 18x_1x_2^2 - 8x_3^2 \) is not hyperbolic w.r.t \( (1, \ldots, 1) \) although it’s log-concave at \( e = (1, \ldots, 1) \) and moreover, since all of its directional derivatives \( D_a f \) are hyperbolic quadratic polynomials with Lorentzian signature for any \( a \in \mathbb{R}^*_{\geq 0} \), so it’s a polynomial with Lorentzian signature over a cone \( K \subset \mathbb{R}^*_{\geq 0} \) containing \( e \). In fact, this example also shows that hyperbolicity of \( D_e f \) does not imply that hyperbolicity of \( f(x) \) as opposed to log-concavity shown in Proposition 6.

Let \( c \) be a fixed positive real number, \( K \) be a cone, and let \( f(x) \in \mathbb{R}[x] \) be the polynomial of degree \( d \), not necessarily homogeneous.

**Definition 17.** We say that \( f \) is \( c \)-Rayleigh if it satisfies the following inequality.

\[
\partial^\alpha f(x) \partial^{\alpha-\epsilon_i-e_j} f(x) \leq c \partial^{\alpha+e_i} f(x) \partial^{\alpha+e_j} f(x) \quad \forall i, j \in [n], \alpha \in \mathbb{Z}^n_+, x \in K
\]

For an multi-affine (means degree of each variable at most one) polynomial \( f \), \( c \)-Rayleigh condition is equivalent to

\[
f(a) \partial_i \partial_j f(a) \leq c \partial_i f(a) \partial_j f(a) \quad \text{for all } a \in K, \quad \text{and } i, j \in [n]
\]

It’s shown in [BH20] that if \( f \) is a Lorentzian polynomial, then \( f \) is \( 2(1 - \frac{1}{d}) \)-Rayleigh. This can be seen as an analog of the Hodge-Riemann relations for homogeneous stable polynomials, cf. [Huh18]. Here we show that the \( 2(1 - 1/d) \) Rayleigh condition is not true in general for polynomials with Lorentzian signature.
Example 18. Consider the polynomial \( f = -\sum_{i=1}^{4} x_i^4 + 2(x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_2^2 x_3 + x_2^2 x_4 + x_3^2 x_4) + 8x_1 x_2 x_3 x_4. \) This is a polynomial with Lorentzian signature. In fact, it’s a hyperbolic polynomial w.r.t \( e \). Thus, \( H_f(a) = Q \) has exactly one positive eigenvalue for all \( a \in \Lambda_{++}(f,e) \), and the hyperbolicity cone contains \( e = (1,\ldots,1) \). Note that the hyperbolicity cone does not contain \( e_i, i = 1,\ldots,4 \). By Theorem 11 \( f \) is a polynomial with Lorentzian signature over its hyperbolicity cone. Although, \( f \) is not a \( 2(1-\frac{1}{3}) \)-Rayleigh, but it’s \( 2 \)-Rayleigh since \( 16 \cdot 8 \leq 2(1-1/4) \cdot 64 \) but \( 16 \cdot 8 \leq 2 \cdot 64 \).

3.3. Generating Polynomials with Lorentzian Signature. In this subsection, we provide a means to create polynomials with Lorentzian signature which need not be Lorentzian.

**Proposition 19.** The polynomial \( f_A(x) = \det(\sum_{i=1}^{n} x_i A_i) \) where \( A_i = \sum_{j=1}^{n} a_{ij} E_j, E_j = e_j e_j^T, e_j \) are the standard canonical basis of \( \mathbb{R}^n \), is a hyperbolic polynomial w.r.t some direction \( e \in \mathbb{R}^n \) if \( A = (a_{ij}) \) is nonsingular.

**Proof.** Observe that \( \det(\sum_{i=1}^{n} x_i A_i) = \det(I \text{ Diag}(\sum_{i=1}^{n} x_i a_{ij}) I^T) \) where \( I \) is the identity of matrix of size \( n \). Thus, \( \det(\sum_{i=1}^{n} x_i A_i) = \det(\text{ Diag}(\sum_{i=1}^{n} x_i a_{ij})) \). A determinantal polynomial is a hyperbolic polynomial if there exists a direction in which the linear span of its coefficient matrices is positive definite as it is shown that any non-empty semidefinite slice is a hyperbolicity cone, see [LPR05]. So, it is sufficient to find such a direction. In particular, we find a direction \( e \in \mathbb{R}^n \) such that

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} e_i a_{ij} E_j = I
\]

Then it can be translated into a linear system of equations such that \( \sum_{j=1}^{n} e_i a_{ji} = 1 \) for all \( 1 \leq i \leq n \) \( \Rightarrow A^T e = 1 \) where \( A^T = (a_{ji}), e = (e_1,\ldots,e_n) \) and \( 1 \) is the all ones vector. Therefore, if \( A \) is an invertible matrix, the linear system is consistent and one can find the direction \( e \) in \( \mathbb{R}^n \). Thus, \( f_A \) is a hyperbolic polynomial w.r.t some direction \( e \in \mathbb{R}^n \).

**Special Cases**

(1) If \( A \) is a nonsingular matrix with nonnegative entries, the generating polynomial \( f_A \) is a stable polynomial. A doubly stochastic matrix is an example of a nonnegative matrix.

(2) If \( A \) is an \( n \times n \) (symmetric) positive definite matrix, the generating polynomial \( f_A \) is a hyperbolic polynomial where the monomial \( x_1 \ldots x_n \) appears with positive coefficient.

The hyperbolicity cone associated with \( f_A \) is given by

\[
\Lambda_{++}(f_A,e) = \{ x \in \mathbb{R}^n : t \mapsto f_A(x + t e) \text{ has negative roots} \} = \{ x \in \mathbb{R}^n : f_A(x + t e) \neq 0 \Rightarrow t \geq 0 \}
\]

The curve defined by \( f_A \) and the hyperbolicity cone \( \Lambda_{++}(f_A,1) \).
Corollary 20. $f_A$ is a polynomial with strict Lorentzian signature, and the $\mathcal{H}_{f_A}(e)$ and for any $v \in \Lambda_{++}(f, e)$, $\mathcal{H}_{D_v f}(e)$ have exactly one positive eigenvalue, or the Lorentz signature.

4. Mixed Discriminant

In this section we generalize the matrix determinant lemma using the notion of mixed discriminant of matrices. At first, we report a result of [Dey21] which uniquely express each coefficient of a determinantal multivariate polynomial $\det(\sum_i^n (x_i A_i))$ in terms of the mixed discriminant of coefficient matrices $A_i, i = 1, \ldots, n$. There are various instructive ways to define the notion of mixed discriminant of $n$-tuple of $n \times n$ matrices, see [Gur06b], [BR97] for details. A clever use of mixed discriminant of $k(\leq n)$-tuple of $n \times n$ matrices can be found in [MSS15b] to establish a connection between the mixed characteristic polynomials with diagonal matrices and matching polynomials introduced by Heilmann and Leib [HL72] of a bipartite graph. We follow the constructive definition of the notion of mixed discriminant of $k(\leq n)$-tuple of $n \times n$ matrices (need not be distinct) introduced in [Dey20] to avoid the scalar factors appearing in other definitions. Notice that the mixed discriminant of matrices is called the generalized mixed discriminant in [Dey20].

Definition 21. Consider the $n \times n$ matrices $A^{(l)} = (a^{(l)}_{ij})$ for $l = 1, \ldots, n$. The mixed discriminant (MD) of a tuple of matrices $(A^{(1)}, \ldots, A^{(n)})$ is defined as

$$D(A^{(1)}, \ldots, A^{(1)}, A^{(2)}, \ldots, A^{(2)}, \ldots, A^{(n)}, \ldots, A^{(n)}) = \sum_{\alpha \in S[k]} \sum_{\sigma \in \widehat{S}} a^{(1)}_{\alpha_1 \sigma_1} \ldots a^{(1)}_{\alpha_k \sigma_k}$$

where $k_j \in \{0, 1, \ldots, n\}$ with $k = \sum_{j=1}^n k_j$ and $S[k]$ is the order preserving $k$-cycles in the symmetric group $S_n$ i.e.,

$$\alpha = (\alpha_1, \ldots, \alpha_k) \in S[k] \Rightarrow \alpha_1 < \alpha_2 < \cdots < \alpha_k,$$

and $\widehat{S}$ is the set of all distinct permutations of $\{1, \ldots, 1, \ldots, n, \ldots, n\}$.

It’s proved in [Dey20]

Theorem 22. (Mixed Discriminant Theorem) The coefficients of a multivariate determinantal polynomial $\det(I + \sum_i^n x_i A_i) \in \mathbb{R}[x]$ of degree $d$ are uniquely determined by the generalized mixed discriminants of the coefficient matrices $A_i$ as follows. If the degree of a monomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ is $k_1 + k_2 + \cdots + k_n = k \leq d$, then the coefficient $f_{k_1 \ldots k_n}$ of $(x_1^{k_1} \cdots x_n^{k_n})$ is given by

$$D(A^{(1)}_{k_1}, A^{(2)}_{k_2}, \ldots, A^{(n)}_{k_n}) = D(I^{(d-k)}_{d-k}, A^{(1)}_{k_1}, A^{(2)}_{k_2}, \ldots, A^{(n)}_{k_n})$$

A couple of well-known facts about mixed discriminant of a tuple of $d \times d$ matrices.

1. $\text{tr } A_i = D(I, \ldots, I, A_i)$ the coefficient of $x_i$ for all $i = 1, \ldots, n$ in $\det(I + \sum_{i=1}^n x_i A_i)$.

2. $\det A_i = D(A^{d-1}_{i, \ldots, i})$ the coefficient of $x_i^d$ for all $i = 1, \ldots, n$ where $d$ is the degree of the polynomial $\det(I + \sum_{i=1}^n x_i A_i)$, equal to the size $d$ of the coefficient matrix.

Another basic fact about mixed discriminant which will be used later, see [Gur06b].

Fact 23. If $X, Y; A_i, 1 \leq i \leq n$ are $n \times n$ complex matrices and $\alpha_i, i \leq n$ are complex numbers then the following identity holds:

$$D(X\alpha_1 A_1 Y, \ldots, X\alpha_i A_i Y, \ldots, X\alpha_n A_n Y) = \det(X) \cdot \det(Y) \prod_{i=1}^n \alpha_i D(A_1, \ldots, A_n)$$
In a homogeneous setting, as a consequence of Theorem 22 we have the following results.

**Corollary 24.** Consider any \( k \leq n \)-tuple of \( n \times n \) matrices \( A_i \) for \( i = 1, \ldots, n \).

\[
\det(\sum_{i=1}^{k} x_i A_i) = \sum_{i=1}^{k} \det(A_i) x_i^n + \sum_{k=\mathbb{N}_0^2, k_j \in \{0, \ldots, d-1\}} D(A_1, \ldots, A_1, A_2, \ldots, A_2, \ldots, A_n, \ldots, A_n) x_1^{k_1} \ldots x_n^{k_n}.
\]

where \( k \in \mathbb{N}_0^k := \{(k_1, \ldots, k_n) \in \mathbb{N}_0^n | \sum_{j=1}^{n} k_j = d \} \) and \( d \leq n \) is the degree of the polynomial \( \det(\sum_{i=1}^{k} x_i A_i) \). As a special case for a 2-tuple of \( n \times n \) matrices we have

\[
\det(x_1 A_1 + x_2 A_2) = \det(A_1) x_1^n + \det(A_2) x_2^n + \sum_{k=\mathbb{N}_0^2, k_j \in \{0, \ldots, d-1\}} D(A_1, \ldots, A_1, A_2, \ldots, A_2) x_1^{k_1} x_2^{k_2}
\]

**Remark 25.** If none of the terms in the r.h.s of Equation (6) is nonzero, the degree of the polynomial must not be equal to the size of the matrices. If any of \( A_i \)'s is invertible, the degree of the polynomial \( d = n \).

Moreover, we generalize the matrix determinant lemma and provide another proof for the matrix determinant lemma by substituting \( x_i = 1 \) for all \( 1 \leq i \leq k \leq n \) in Equation (6).

**Proposition 26.**

\[
\det(\sum_{i=1}^{k} A_i) = \sum_{i=1}^{k} \det(A_i) + \sum_{k=\mathbb{N}_0^2, k_j \in \{0, \ldots, d-1\}} D(A_1, \ldots, A_1, A_2, \ldots, A_2, \ldots, A_n, \ldots, A_n)
\]

\[
\det(A_1 + A_2) = \det(A_1) + \det(A_2) + \sum_{k=\mathbb{N}_0^2, k_j \in \{0, \ldots, d-1\}} D(A_1, A_1, A_2, A_2, \ldots, A_2)
\]

**Another Proof of Matrix Determinant Lemma:**

\[
\det(A + uv^T) = (1 + v^T A^{-1} u) \det(A)
\]

for a nonsingular matrix \( A \) and column vectors \( u, v \).

**Proof.**

\[
det(A + uv^T) = det(A) + D(A, \ldots, A, uv^T) \text{ by Proposition 26}
\]

\[
= det(A) + det(A) D(I, \ldots, I, A^{-1} uv^T) \text{ since } A \text{ is nonsingular,}
\]

\[
= det(A) + det(A) \text{tr}(A^{-1} uv^T)
\]

\[
= det(A)(1 + v^T A^{-1} u)
\]

**Remark 27.** This result provides an explicit formula for the determinant of the sum of any finite (in particular, two) matrices which is not equal to the sum of the determinant of the matrices.

5. **Permanent**

In this section, we propose a novel technique to express the permanent of a nonsingular matrix (need not be a nonnegative matrix) via some hyperbolic polynomial. This point of view enables us to identify the VdW stable family for which the coefficient of \( x_1 \ldots x_n \) is the permanent of some nonnegative matrix. Moreover, we characterize the class \( M \) of nonsingular matrices for which computing the permanent of a matrix \( A \in M \) is equivalent to solving a hyperbolic programming (a special type of convex programming for which self-concordant barrier is known and the interior point method can be applied). The permanent of a square \( n \times n \) matrix \( A = (a_{ij}) \) is

\[
\text{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i, \sigma(i)}
\]

where \( S_n \) is the symmetric group of degree \( n \). This is almost the same as the definition of the determinant of a matrix except the signatures of the permutations are not taken into account in the definition of the permanent of a matrix.

The problem of finding the upper and lower bounds of the permanents of various types of matrices including \((1, 0), (1, 0, -1), \) doubly stochastic, and nonnegative matrices have been studied by a large group of people from
various areas of mathematics and theoretical computer science since 20th century. Here is a list of a few interesting articles on this fascinating topic [Sch78], [MM65], [Gly10].

**Theorem 28.** Minc’s conjecture: Let $A = (a_{ij}) \in \{0, 1\}^{n \times n}$. Then $\text{Per}(A) \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{n}}$ where $r_i := \sum_{j=1}^{n} a_{ij}$.

It was proved by Brégman, 1973 [Brè73]. A concise proof of this result is due to Schrijver [Sch78] out of several known proofs.

**Theorem 29.** The Van der Waerden conjecture: If $A = (a_{ij})$ is an $n \times n$ doubly stochastic matrix, $\text{Per}(A) \geq \frac{n!}{n^n}$.

The conjecture was proved by D.I. Falikman [Fal81] and the uniqueness part of the conjecture about the bound which claims that the bound is attained uniquely at $A = J_n$ where each entry of $J_n$ is 1/n was proved by G.P. Egorychev [Ego81].

The permanent of a matrix has attracted a lot of attention due to its connections with combinatorial objects which are in general difficult to study. The permanent of an $n \times n$ matrix $A = (a_{ij})$ is equal to the number of weighted directed cycle covers of a digraph $G$ on $n$ vertices labelled by $\{1, \ldots, n\}$ and $(a_{ij})$ be the weighted adjacency matrix for $G$. The permanent of a $(0,1)$ matrix counts the number of perfect matchings in a bipartite graph. Gurvits’s pioneering work in [Gur06a] has first expressed $\text{Per}(A)$ via stable polynomials as follows.

**Theorem 30.** Let $A = (a_{ij}), a_{ij} \geq 0$ be the adjacency matrix of some bipartite graph $G$ and consider the following polynomial $f_A(x_1, \ldots, x_n) = \prod_{i=1}^{n} (\sum_{j=1}^{n} a_{ij} x_j)$. Then

$$\text{Per}(A) = \prod_{k=1}^{n} \partial_{x_0} |_{x_k=0} f_A = \#\text{perfect matchings in } G.$$  

Moreover, using Stirling’s formula $\frac{n!}{\sqrt{2\pi n} n^{-n}}$ tends to unity for $n$ tends to $\infty$, cf. [Fel68], see also [Fri79] for the lower bound result. Linial et.al. in [LSW98] proposed a deterministic strongly polynomial algorithm to approximate the permanent of a nonnegative $n \times m$ matrix to within a multiplicative factor of $e^n$. Gurvits’s way of looking at the permanent of a matrix as the coefficient of certain repetitive derivatives of a stable polynomial enables him to approximate the permanent of a nonnegative matrix within a factor $\frac{e^n}{n^m}$ (for any fixed integer $m$) by a deterministic polynomial time oracle algorithm, see [Gur06a].

**Fact 31.** Van-der-Waerden conjecture: For a non-$n \times n$ doubly stochastic matrix, $\text{Per}(A) \geq \frac{n!}{n^n} \geq (1/e)^n$

Indeed, this implies Schrijver’s perfect matching inequality which was conjectured by Schrijver in [Sch98] and proved by Schrijver and Valliant in [SV80]. In fact, Gurvits introduced the notion of VdW-family in [Gur06a] to unify Van der Waerden / Schrijver-Valliant/ Bapat conjectures in terms of homogeneous polynomials with nonnegative coefficients.

Gurvits introduced the notion of polynomial capacity. Consider an $n$-variate homogeneous polynomial $f \in \mathbb{R}_+^n [x]$ of degree $n$ with nonnegative real coefficients. Define the capacity of $f$ as

$$\text{Cap}_\alpha(f(x)) := \inf_{x > 0} \frac{f(x)}{x^\alpha}, \text{ for } \alpha \in \mathbb{R}_+^n.$$  

Here are some known facts about polynomial capacity for a nonnegative polynomial which will be used in the sequel. For $\mu \in \text{supp}(f)$, we have $f(1, \ldots, 1) \geq \text{Cap}_\mu(f(x)) = \inf_{x > 0} \frac{f(x)}{x^\mu}$ $\geq f_\mu$ and thus, $\text{Cap}_1(f)$ could be a good approximation to $f_1$ (all-ones coefficient). Based on the following crucial observation

$$\log \text{Cap}_\alpha(f) = \inf_{y \in \mathbb{R}_+} [\langle y, \alpha \rangle + \log \sum_{\mu} f_\mu e^{\langle y, \mu \rangle}] \text{ via } x \to e^y.$$  

It turns out that computing polynomial capacity is a convex programming (or geometric programming) problem since $f_\mu > 0$. In particular, $\log(\text{Cap}_1(f)) = \inf_{\sum_{1 \leq i \leq n} \mu_i = 0, y_i = 0} \log(f(e^y))$, see [Gur06a] for details.

**Definition 32.** VdW Stable Family is a stratified set of homogeneous polynomials, i.e., a class $F = \bigcup_{1 \leq n < \infty} F_d$, where $F_d \in \mathbb{R}_+^n [x]$ such that

- if $f \in F_d, d > 1$, then for all $1 \leq i \leq n$ the polynomials $\partial_i f \in F_{d-1}$ and
- $\text{Cap}_1(\partial_{x_k} |_{x_k=0} f) \geq \left(\frac{n_k-1}{n_k}\right)^{n_k-1} \text{Cap}_1(f)$ where $n_k$ is the maximum degree of $x_k$ in $f$. 

Questions: Are there any other stable polynomials whose coefficients of $x_1 \ldots x_n$ are the permanents of some nonnegative matrices?

**Proposition 33.** Consider the determinantal polynomial $\det(\sum_{i=1}^n x_i A_i)$ where each $A_i$ is in the nonnegative linear span of $n$ rank one matrices, i.e., $A_i = \sum_{j=1}^n a_{ij} v_j v_j^T$ for all $i, j = 1, \ldots, n$ and $a_{ij} \geq 0$. The coefficient $x_1 \ldots x_n$ is $D(A_1, \ldots, A_n) = \det(V) \text{Per}(A) \det(V^T)$ where $A = (a_{ij})$, nonnegative matrix.

**Proof.** Note that $A_i = V D_i V^T$ where $V = [v_1 \ v_2 \ \ldots \ v_n]$ and $D_i = \text{Diag}(a_{1i}, \ldots, a_{ni})$ with $a_{ij} \geq 0$. Thus each coefficient matrix $A_i$ is positive semidefinite. Then by using Branden’s result, $\sum_{i=1}^n a_{ij} v_j v_j^T$ is stable polynomial for any choice of $V \in GL_n$. On the other hand, polynomial $\det(\sum_{i=1}^n x_i A_i) = \det(\sum_{i=1}^n V(x_i D_i) V^T) = \det(V) \det(\sum_{i=1}^n x_i D_i) \det(V^T)$. Then the rest follows from Theorem 22 and Fact 23.

**Remark 34.** When $A_i$’s are diagonal matrices, $D(A_1, \ldots, A_n) = \text{Per}(A)$ and when $A_i$’s are simultaneously diagonalizable matrices, $D(A_1, \ldots, A_n) = \text{Per}(A)$.

Thus, we characterize the class of VdW family of stable polynomials in $n$ variables such that the coefficient of $x_1 \ldots x_n$ which is the mixed discriminant $D(A_1, \ldots, A_n)$ of $n$-tuple symmetric matrices is the permanent of some $n \times n$ nonnegative matrix $A$ up to scaling.

However, Gurvits’s clever way to view the permanent of a nonnegative matrix as the repetitive derivatives of a stable polynomial evaluated at zero leads to a way to visualize the permanent of any nonsingular matrix (the entries could be negative and complex numbers). In this paper, following the similar concepts we propose a technique to compute the permanent of any nonsingular matrix by using a special type of hyperbolic polynomials.

**Proposition 35.** Consider the polynomial $f_A(x) = \det(\sum_{i=1}^n x_i A_i)$ where $A_i = \sum_{j=1}^n a_{ij} E_j, E_j = e_j e_j^T, e_j$ are the standard canonical basis of $\mathbb{R}^n$ and matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $\det A \neq 0$. Then

$$\text{Per}(A) = \prod_{k=1}^n \partial_{x_k}|_{x_k=0} f_A$$

**Proof.** The permanent of $A = (a_{ij}), a_{ij} \in \mathbb{R}$ is the coefficient of $x_1 \ldots x_n$ of the determinantal polynomial $\det(\sum_{i=1}^n x_i A_i)$ where $A_i = \sum_{j=1}^n a_{ij} E_j, E_j = e_j e_j^T, e_j$ are the standard canonical basis of $\mathbb{R}^n$. The coefficient of $x_1 \ldots x_n$ is the mixed discriminant $D(A_1, \ldots, A_n)$ by Theorem 22. Observe that $\det(\sum_{i=1}^n x_i A_i) = \det(\text{Diag}(\sum_{i=1}^n x_i a_{ij}, 1 \leq j \leq n))$. Since the coefficient matrices $A_i$ are diagonal, $D(D_1, \ldots, D_n) = \text{Per}(A^T) = \text{Per}(A)$. The mixed discriminant of diagonal matrices is the permanent of matrix $A$ follows from the definitions of the permanent and mixed discriminant.

**Question:** Can we approximate the permanent of a nonsingular matrix by a deterministic poly-time oracle algorithm? Computing the permanent of a matrix is #P-hard (Valiant 1979) even if the matrix entries are all either 0 or 1. Note that we cannot use polynomial capacity because $f_A$ in Proposition 35 need not have nonnegative coefficients. Thus, we cannot directly use Gurvits’s machinery. By Proposition 19 the polynomial $f_A(x) = \det(\sum_{i=1}^n x_i A_i)$ where $A_i = \sum_{j=1}^n a_{ij} E_j, E_j = e_j e_j^T, e_j$ are the standard canonical basis of $\mathbb{R}^n$, is a hyperbolic polynomial w.r.t some $e \in \mathbb{R}^n$ if $A$ is nonsingular. Thus, $f_A$ in Proposition 35 is a hyperbolic polynomial but not a stable polynomial. The directional derivatives of hyperbolic polynomials in the directions of hyperbolicity are hyperbolic, cf. [ABG70], [Ren04] and Güler has shown in [Gü1997] that long step interior point method [cf. [NN94],[BGLS01],[Nes03, Chapter 4],[NT08] for excellent survey on this topic] can be applied to hyperbolic barrier function $-\log f(x)$ while solving hyperbolic programming. Now we are ready to characterize the class $\mathcal{M}$ of nonsingular matrices.

First, we need to make sure the decision problem works, i.e., the coefficient of $x_1 \ldots x_n$ is positive, equivalently, $D(A_1, \ldots, A_n) > 0$. Thus, if the hyperbolic polynomial $f(x) > 0$ for some $x \in A_{+\perp}(f, e)$ with $x_i > 0$, then we can define the polynomial capacity of $f(x)$ as

$$\text{Cap}_\alpha(f(x)) := \inf_{x > 0} \mathbb{E}(\alpha x) \frac{f(x)}{x^\alpha}, \text{ for } \alpha \in \mathbb{R}^n_+$$
Note that this is a conic geometric programming if the coefficients of \( f(x) \) are nonnegative, cf [CS14], [BLNW20]. However, in the context of \( f(x) \) being hyperbolic, the coefficients of \( f(x) \) need not be nonnegative. In particular, for \( \alpha = 1 \) we have

\[
\text{Cap}_1(f(x)) := \inf_{x_i > 0 \land x \in A_{++}(f,e)} \frac{f(x)}{x_1 \cdots x_n} \\
\Rightarrow \log \text{Cap}_1(f) = \inf_{x_i > 0 \land x \in A_{++}(f,e)} \left[ -\sum_{i=1}^{n} \log x_i + \log f(x) \right] \\
\Rightarrow \log \text{Cap}_1(f) = \inf_{x_i > 0} \sum_{i=1}^{n} \log x_i = 0 \land x \in A_{++}(f,e) \left[ \log f(x) \right]
\]

Recall that Gårding’s result shows that the function \( \log f(x) \) is concave for a hyperbolic polynomial \( f(x) \), cf. [Går59], [Gü97]. Consider the class \( \mathcal{M} \) of nonsingular matrices \( A \) such that \( D(A_1,\ldots,A_n) > 0 \) and the generating polynomials \( f_A(x) \) are hyperbolic w.r.t some \( x \) with \( x_i > 0, 1 \leq i \leq n \). Then the problem of approximating the permanent of such a nonsingular matrix \( A \in \mathcal{M} \) can be translated into computing the capacity of the generating polynomial \( f_A \) which is equivalent to solving a hyperbolic programming problem. Therefore, one can use interior point method and Ellipsoid method, and furthermore, it would be interesting to find a deterministic polynomial time algorithm for approximating the permanent of a nonsingular matrix \( A \in \mathcal{M} \) such that \( D(A_1,\ldots,A_n) > 0 \) which can be a topic for future work.

6. \( k \)-locally PSD matrices

In this section we show that the class \( \mathcal{M} \) is nonempty. In order to identify the class \( \mathcal{M} \) we need to characterize the set of matrices whose entries are negative but the permanents are positive. Interestingly, the set of \( k \)-locally psd matrices, introduced and studied by Blekherman et.al. [BDSS20] is a prominent candidate for the class \( \mathcal{M} \).

\[
S(n,k) = \{ X \in S^n : \text{all } k \times k \text{ principal submatrices of } X \text{ are PSD} \}
\]

be the set of \( k \)-locally PSD matrices. However, a matrix \( M \) is \((n,k)\)-locally singular if it lies in \( S(n,k) \) and all of the \( k \times k \) minors of \( M \) are singular. An interesting class of locally singular matrices in \( S(n,k) \) are nonsingular locally singular matrices, abbreviated as NLS. We show that \( \text{Per}(M) > 0 \) when \( M \) is a NLS using a beautiful structure theorem proved in [BDSS20]. Another important fact is \( DMD \in S(n,k) \) where \( D \) is a diagonal matrix with non-zero diagonal entries and \( DMD \) is called diagonally congruent to \( M \), see [BDSS20] for details.

Theorem 36. Let \( n - 1 > k > 2 \) with \( k \) being close to \( n \) or \( (n,k) = (4,2) \). Suppose that \( M \) is a NLS in \( S(n,k) \). Then \( M \) must be diagonally congruent to \( G(n,k) \) where \( G(n,k) = \frac{k}{k-1} I - \frac{1}{\sqrt{n}} \mathbf{1} \mathbf{1}^T \).

We prove the following result to show the permanent of NLS is positive.

Lemma 37. The permanent \( \text{Per}(D'AD') \) is \( \det(D')^2 \text{Per}(A) \) for any complex matrix \( A \) and diagonal matrix \( D' \) with nonzero entries.

Proof. Say \( A = (a_{ij}) \) is a \( n \times n \) matrix, then \( \text{Per}(A) \) is the coefficient of \( x_1 \ldots x_n \) in the determinantal polynomial \( \det(\sum_{i=1}^{n} x_i D_i) \) where \( D_i = \text{Diag}(a_{ij}) \). Thus, by Proposition 35 \( \text{Per}(A) = D(D_1,\ldots,D_n) \). Then it follows from Fact 23 as \( \text{Per}(D'AD') = D(D'D_1D',\ldots,D'D_nD') = \det(D')^2 \text{Per}(A) \).

Proposition 38. When \( n \) is an even integer, \( \text{Per}(M) \) is positive for any NLS \( M \in S(n,k) \) where \( n - 1 > k > 2 \) or \( (n,k) = (4,2) \). When \( n \) is an odd integer, \( \text{Per}(M) \) is positive for any NLS \( M \in S(n,k) \) where \( n - 1 > k > \sqrt{2(n - 1)} \).

Proof. Using Theorem 36 we know that \( M \) is diagonally congruent to \( G(n,k) \). On the other hand, we derive that

\[
\text{Per}(G(n,k)) = \left( \frac{k}{k-1} \right)^n \sum_{i=0}^{n} \left( -\frac{1}{k} \right)^i \frac{n!}{(n-i)!} = \left\{ \begin{array}{ll}
\frac{1}{(k-1)^n} \sum_{i=0}^{n} (-1)^{n-i} i^{k^{-i}} \left( \frac{(n-i)!}{(n-1)!} \right) & \text{when } n \text{ is even} \\
\frac{1}{(k-1)^n} \sum_{i=0}^{n} (-1)^i i^{k^{-i}} \left( \frac{(n-i)!}{(n-1)!} \right) & \text{when } n \text{ is odd}
\end{array} \right.
\]

When \( n \) is an even integer, \( \text{Per}(G(n,k)) = \frac{n!}{(k-1)^n} \sum_{i=0}^{n} (-1)^{n-i} i^{k^{-i}} \left( \frac{(n-i)!}{(n-1)!} \right) = \frac{n!}{(k-1)^n} \sum_{i=0}^{n} (-1)^i \frac{i^{k^{-i}}}{\sqrt{n}} \). The r.h.s summation is always a finite positive number. In fact, when \( n \to \infty \), it converges to \( e^{-k} \) for any \( k \) since it is the Maclaurin
series of $e^{-k}$ as an infinite sum. When $n$ is odd, 
\[
\text{Per}(G(n, k)) > 0
\]

\[
\Leftrightarrow k[k^{n-1} + n(n-1)k^{n-3} + \cdots + n!] > n[k^{n-1} + (n-1)(n-2)k^{n-3} + \cdots + (n-1)!]
\]

\[
\Leftrightarrow \frac{k^{n-1} + \sum_{j=1}^{n-1} k^{-(2j+1)} \prod_{i=1}^{j}(n-2i+2)(n-2i+1)}{\sqrt{n}} > \frac{n}{k}
\]

\[
\Leftrightarrow \frac{1 + \sum_{j=1}^{n-1} k^{-j/2}[n^2+3n+2+4(j)^2-(4n+6)]^j!}{1 + \sum_{j=1}^{n-1} k^{-j/2}[n^2+n+4(j)^2-(4n+2)]} > \frac{n-k}{k}
\]

\[
\Leftrightarrow 2(n-1)\left[k^{-3} + \sum_{j=1}^{n-1} k^{-2j-n}(2j)\right] > \frac{n-k}{k}
\]

\[
\Leftrightarrow k^2 \left[k^{-3} + \sum_{j=1}^{n-1} k^{-2j-n}(2j)\right] > \frac{n-k}{k}
\]

The above inequality is valid if the l.h.s is a finite number when $n$ is large. Note that the coefficients of the likewise terms of the expression inside the parentheses of the numerator are greater equal to the coefficients of likewise terms of the expression inside the parentheses of the denominator. Thus, $\text{Per}(G(n, k)) > 0$ if $k > \sqrt{2(n-1)}$ when $n$ is odd. Then irrespective of $n$ being even or odd the rest of the theorem follows from Lemma 37 that $\text{Per}(M) = \text{Per}(D(G(n, k))D) = |\text{det}(D')|^2 \text{Per}(G(n, k))$ where the diagonal entries of a diagonal matrix $D$ are nonzero. Hence the proof.

**Remark 39.** When $n$ is even, $\text{Per}(-G(n, k)) = (-1)^n \text{Per}(G(n, k)) = \text{Per}(G(n, k)) > 0.$

**Remark 40.** It’s noticed while doing experiments in Mathematica that $\text{Per}(G(n, k)) > 0$ even for some $k < \sqrt{2(n-1)}$. For example, $(n, k) = (9, 3), \text{Per}(G(n, k)) < 0$, but $\text{Per}(G(n, k)) > 0$ for $k \geq 4 = \sqrt{2(9-1)}; (n, k) = (7, 3), \text{Per}(G(n, k)) > 0, k \geq 3 < \sqrt{2(7-1)}$; and $(n, k) = (19, 6), \text{Per}(G(n, k)) < 0$, but $\text{Per}(G(n, k)) > 0$ for $k \geq 7 > \sqrt{2(19-1)}$.

**Corollary 41.** For any integer $k \in \{1 + \sqrt{2(n-1)}, \ldots, n-1\}$, $\text{Per}(G(4, 2)) > \cdots > \text{Per}(G(n, k)) > \text{Per}(G(n, k+1)) > \cdots > \text{Per}(G(n, n-1))$

Following [BKS+21], consider $\mathcal{T}$ be the class of functions $T : S^n \to \mathbb{R}$ so that $T$ is a unitarily invariant matrix norm, that means the norm depends entirely on the eigenvalues or in particular, the trace function. The Schatten $p$-norms, $||M||_p = \left(\sum_i |\lambda_i(M)|^p\right)^{1/p}$ for $p \geq 1$, including the Frobenius norm as a special case of the Schatten $p$-norm when $p = 2$ are examples of unitarily invariant matrix norms.

**Theorem 42.** Let $k \in \{2, \ldots, n\}$. For any $M \in S(n, k)$ such that $\text{tr}(M) = 1$,

\[
\text{Per}(M) \leq \text{Per}(\tilde{G}(n, k)),
\]

The upper bound of the permanent of this class of matrices is 1/32. The upper bound is attained at $\tilde{G}(4, 2)$.

**Proof.** We obtain that $\text{Per}(\tilde{G}(n, k)) = \frac{n!}{n^n} \left(\frac{k}{k-1}\right)^n \sum_{i=0}^{n} \left(\frac{-1}{k}\right)^i \frac{1}{(n-i)!}$. Thus, for any integer $n$, and $k > \sqrt{2(n-1)}$, $\text{Per}(\tilde{G}(n, k)) \geq \text{Per}(\tilde{G}(n+1, k))$. On the other hand, using Lemma 37 $\text{Per}(D'AD') = |\text{det}(D')|^2 \text{Per}(A)$ for any complex matrix $A$ and diagonal matrix $D'$ with nonzero entries. Thus, when we normalize the matrix to get $\text{tr}(M) = 1$, $\text{Per}(M) \leq \text{Per}(\tilde{G}(n, k))$. Then using Corollary 41 we have the following result.

$\text{Per}(\tilde{G}(4, 2)) > \text{Per}(\tilde{G}(n, k)) > \text{Per}(M)$

**Remark 43.** $\text{Per}(\tilde{G}(4, 2)) = \frac{\text{Per}(G(4, 2))}{4^4} = \frac{8}{3^4} = \frac{1}{32} > \frac{1}{4}$.

**Conjecture 44.** Let $k \in \{2, \ldots, n\}$. Let $T \in \mathcal{T}$ and $\tilde{G}(n, k) = \frac{G(n, k)}{T(G(n, k))}$. For any $M \in S(n, k)$ with $T(M) = 1$, the permanent of $M$ is at most as large the permanent of $\tilde{G}(n, k)$, i.e.,

\[
\text{Per}(M) \leq \text{Per}(\tilde{G}(n, k)), \text{ for all } M \in S(n, k) \text{ s.t } T(M) = 1
\]
The bound on $\text{Per}(M)$ is tight since $G(n, k) \in S(n, k)$ achieves the bound.

However, all diagonal entries of $G(n, k)$ are identically 1, and all off-diagonal entries are identically $\frac{-1}{k-1}$. In particular, when $k = 2$, all diagonal entries of $G(n, k)$ are identically $-1$, and all off-diagonal entries are identically $-1$. So, it’s a $(1, -1)$ matrix. It’s interesting to compare the matrix $C(n, n)$ studied in [Sei84] with $G(n, 2)$. In fact, $C(n, n) = -G(n, 2)$. It’s shown in [Sei84] that $\text{Per}(C(n, m)) > 0$ for all $n \geq 4$ and $0 \leq m \leq n$ where $n \times m$ matrix $C(n, m) = c_{ij}, 0 \leq m \leq n$ is given by

$$C(n, m) = \begin{cases} c_{ij} = -1, & \text{for } n - m < i = j \\ 1 & \text{otherwise} \end{cases}$$

Following the above result we have

**Proposition 45.** $\text{Per}(\overline{-G(n, 2)})$ is positive for $n \geq 4$ where $\overline{-G(n, 2)}$ is the polar cone of $G(n, 2)$.

**Remark 46.** Note that locally singularity conditions may not be necessary to show that the permanent of any nonsingular $k$ locally psd matrix is positive. But as of now it’s not clear how to classify any nonsingular matrix $M \in S(n, k)$ without the structure theorem 36.

### 6.1 Why NLS matrices.

**Proposition 47.** Consider a NLS $A = (a_{ij})$ in $S(n, k)$, where $n - 1 > k > 2$ or $(n, k) = (4, 2)$ and the generating polynomial $f_A = \det(\sum_{i=1}^{n} x_i A_i)$ where $A_i = \text{Diag}(a_{ij})$ for all $1 \leq j \leq n$. Then $f_A$ is a hyperbolic polynomial, and the decision problem of membership of the monomial $x_1 \ldots x_n$ in $f_A$, i.e., $(1, \ldots, 1) \in \text{supp}(f_A)$ is true. Furthermore, $D(A_1, \ldots, A_n) > 0$ if $n - 1 > k > \sqrt{2(n-1)}$ when $n$ is odd or if $n - 1 > k > 2$ when $n$ is even or $(n, k) = (4, 2)$.

**Proof.** By the structure theorem 36 $A = (a_{ij}) = D'G(n, k)/D'$ is $S(n, k)$ for $n - 1 > k > 2$ or $(n, k) = (4, 2)$ where $D'$ is a diagonal matrix with nonzero diagonal entries. Note that $f_A = \det(\sum_{i=1}^{n} x_i A_i) = \det \left( \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} E_j \right)$ where $E_j = e_j e_j^T$ and $e_j$ form the standard canonical basis of $\mathbb{R}^n$. Then by Proposition 19 the generating polynomial $f_A$ is a hyperbolic polynomial w.r.t $e = (1, \ldots, 1)$, all-ones vector since $A$ is nonsingular. By Proposition 38 Per($G(n, k)$) is non-zero. So, $(1, \ldots, 1) \in \text{supp}(f_A)$. Then $D(A_1, \ldots, A_n) > 0$ follows from Proposition 35 and Proposition 38.

**Lemma 48.** Consider a NLS $A = (a_{ij})$ such that $-A \in S(n, 2)$, $n \geq 4$ and the generating polynomial $f_A = \det(\sum_{i=1}^{n} x_i A_i)$ where $A_i = \text{Diag}(a_{ij})$ for all $1 \leq j \leq n$. Then $f_A$ is a hyperbolic polynomial, and the decision problem of membership of the monomial $x_1 \ldots x_n$ in $f_A$, i.e., $(1, \ldots, 1) \in \text{supp}(f_A)$ is true. Furthermore, $D(A_1, \ldots, A_n) > 0$ if $n \geq 4$.

Recall that Güler [Gü97] showed that the associated hyperbolicity cone is open, convex and may contain the entire lines. Following the terminology mentioned in [Gü97] a polynomial is said to be a complete polynomial if the lineality space $L(f) = L(\Lambda_{++}(f, e)) = \{0\}$ where

$$L(f) = \{x \in \mathbb{R}^n : f(y + tx) = f(y), t \in \mathbb{R}, y \in \mathbb{R}^n\}$$

$$L(\Lambda_{++}(f, e)) = \{x \in \mathbb{R}^n : \Lambda_{++}(f, e) + x = \Lambda_{++}(f, e)\}$$

Then, he showed that $f$ is a complete polynomial if and only if the associated hyperbolicity cone is regular.

**Remark 49.** Consider the hyperbolicity cone $\Lambda_{++}(f_A, e)$ of the generating polynomial $f_A$ where $A \in S(n, k)$ is nonsingular; $n$ is even and $n - 1 > k > 2$, or $(n, k) = (4, 2)$. Then $\Lambda_{++}(f_A, e)$ is not a regular cone, equivalently, $f_A$ is not a complete polynomial.

**Example 50.** Consider $A = (a_{ij}) = G(4, 2)$. Then the generating polynomial is

$$f_A = - \sum_{i=1}^{4} x_i^4 + 2 \sum_{i,j \in \{1,2,3,4\}, i < j} x_i^2 x_j^2 + 8x_1 x_2 x_3 x_4.$$
The Hessian of \( f_A \) has one positive eigenvalue.

\[
H_{f_A}(1, \ldots, 1) = 16 \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix} =: 16B
\]

where \( B \) is the adjacency matrix of the complete graph \( K_4 \).

7. Conclusions and Open Questions

The crux of the article is to propose and study homogeneous polynomials with Lorentzian signature that generalizes Lorentzian polynomials. We show that hyperbolic polynomials and \( K \)-(conic) stable polynomials are members of the proposed class of polynomials. As an immediate consequence of generating polynomials with Lorentzian signature, we establish a connection with mixed discriminant of matrices and permanents of nonsingular (need not be nonnegative) matrices via hyperbolic polynomials. Another insightful result that we obtain in this article is the characterization of nonsingular matrices for which permanents can be computed by solving hyperbolic programming using long-step interior point methods, which include nonsingular \( k \)-locally singular matrices.

We conjecture that every polynomial with Lorentzian signature is the limit of polynomials with strict Lorentzian signature, which would imply that the space of polynomials with Lorentzian signature is closed. We contemplate that the notion of polynomials with Lorentzian signature will unlock many possibilities for extending several results of Lorentzian polynomials into a general framework. Another natural question arises in this context is to propose a deterministic polynomial time algorithm to approximate permanents of the special class of nonsingular matrices.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211