The generalized Erdős-Falconer distance problems in vector spaces over finite fields

Doowon Koh and Chun-Yen Shen

Abstract. In this paper we study the generalized Erdős-Falconer distance problems in the finite field setting. The generalized distances are defined in terms of polynomials, and various formulas for sizes of distance sets are obtained. In particular, we develop a simple formula for estimating the cardinality of distance sets determined by diagonal polynomials. As a result, we generalize the spherical distance problems due to Iosevich and Rudnev [12] and the cubic distance problems due to Iosevich and Koh [11]. Moreover, our results are of higher dimensional version for Vu’s work [24] on two dimension. In addition, we set up and study the generalized pinned distance problems in finite fields. We give a generalization of the work by the authors [2] who studied the pinned distance problems related to spherical distances. Discrete Fourier analysis and exponential sum estimates play an important role in our proof.

Contents

1. Introduction 1
2. Discrete Fourier analysis and exponential sums 4
3. Distance formulas based on the Fourier decays 6
4. Simple formula for generalized Falconer distance problems 9
5. Generalized pinned distance problems 13
References 16

1. Introduction

The Erdős distance problem, in a generalized sense, is a question of how many distances are determined by a set of points. This problem might be the most well-known problem in discrete geometry. One may consider discrete, continuous and finite field formulations of this question. Given finite subsets $E, F$ of $\mathbb{R}^d$, $d \geq 2$, the distance set determined by the sets $E, F$ is defined by $\Delta(E, F) = \{|x - y| : x \in E, y \in F\}$, where $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$. In the case when $E = F$, Erdős [7] asked us to determine the smallest possible size of $\Delta(E, E)$ in terms of the size of $E$. This problem is called the Erdős distance problem and it has been conjectured that $|\Delta(E, E)| \gtrsim |E|^{2/d}$ where $|\cdot|$ denotes the cardinality of the finite set. Taking $E$ as a piece of the integer lattice shows that one can not in general get the better exponent than $2/d$ for the conjecture. For all dimensions $d \geq 2$, this problem has not been solved. In two dimension, the best known result is the work by Katz and Tardos [13], which is based on a previous breakthrough by Solymosi and Tóth [20]. For

1991 Mathematics Subject Classification. 52C10, 11T23.
Key words and phrases: generalized distance sets, Erdős-Falconer distance problems, exponential sums, pinned distances.
the best known results in higher dimensions see [21] and [22]. These results are a culmination of efforts going back to the paper by Erdős [7].

On the other hand, one can also study the continuous analog of the Erdős distance problem, called the Falconer distance problem. This problem is to determine the Hausdorff dimension of compact sets such that the Lebesgue measure of the distance sets is positive. Let \( E \subset \mathbb{R}^d \), \( d \geq 2 \), be a compact set. The Falconer distance conjecture says that if \( \dim(E) > d/2 \), then \( |\Delta(E, E)| > 0 \), where \( \dim(E) \) denotes the Hausdorff dimension of the set \( E \), and \( |\Delta(E, E)| \) denotes one dimensional Lebesgue measure of the distance set \( \Delta(E, E) = \{|x - y| : x, y \in E\} \). Using the Fourier transform method, Falconer [8] proved that if \( \dim(E) > (d + 1)/2 \), then \( |\Delta(E, E)| > 0 \). This result was generalized by Mattila [17] who showed that

\[
\text{if } \dim(E) + \dim(F) > d + 1, \text{ then } |\Delta(E, F)| > 0,
\]

where \( E, F \) are compact subsets of \( \mathbb{R}^d \) and \( \Delta(E, F) = \{|x - y| \in \mathbb{R} : x \in E, y \in F\} \). In particular, he made a remarkable observation that the Falconer distance problem is closely related to estimating the upper bound of the spherical means of Fourier transforms of measures. Using the Mattila’s method, Wolff [26] obtained the best known result on the Falconer distance problem in two dimension. He proved that if \( \dim(E) > 4/3 \), then \( |\Delta(E, E)| > 0 \). The best known results for higher dimensions are due to Erdoǧan [6]. Applying the Mattila’s method and the weighted version of Tao’s bilinear extension theorem [23], he proved that if \( \dim(E) > d/2 + 1/3 \), then \( |\Delta(E, E)| > 0 \), where \( d \geq 2 \) is the dimension. However, the Falconer distance problem is still open for all dimensions \( d \geq 2 \). As a variation of the Falconer distance problem, Peres and Schlag [18] studied the pinned distance problems and showed that the Falconer result can be sharpen. More precisely, they proved that if \( E \subset \mathbb{R}^d \) and \( \dim(E) > (d + 1)/2 \), then \( |\Delta(E, y)| > 0 \) for almost every \( y \in E \), where the pinned distance set \( \Delta(E, y) \) is given by

\[
\Delta(E, y) = \{|x - y| : x \in E\}.
\]

In recent years the Erdős-Falconer distance problem has been also studied in the finite field setting. Let \( \mathbb{F}_q \) be a finite field with \( q \) elements. We denote by \( \mathbb{F}_q^d \), \( d \geq 2 \), the \( d \)-dimensional vector space over the finite field \( \mathbb{F}_q \). Given a polynomial \( P(x) \in \mathbb{F}_q[x_1, \ldots, x_d] \) and \( E, F \subset \mathbb{F}_q^d \), one may define a generalized distance set \( \Delta_P(E, F) \) by the set

\[
\Delta_P(E, F) = \{P(x - y) \in \mathbb{F}_q : x \in E, y \in F\}.
\]

In the case when \( E = F \) and \( P(x) = x_1^2 + x_2^2 \), Bourgain, Katz and Tao [1] first obtained the following nontrivial result on the Erdős distance problem in the finite field setting: if \( q \) is prime with \( q \equiv 3 \pmod{4} \) and \( E \subset \mathbb{F}_q^2 \) with \( |E| = q^\delta \) for some \( 0 < \delta < 2 \), then there exists \( \varepsilon = \varepsilon(\delta) > 0 \) such that

\[
|\Delta_P(E, E)| \gtrsim |E|^{\frac{1}{2} + \varepsilon},
\]

where we recall that if \( A, B \) are positive numbers, then \( A \lesssim B \) means that there exists \( C > 0 \) independent of \( q \), the cardinality of the underlying finite field \( \mathbb{F}_q \) such that \( A \leq CB \). However, if there exists \( i \in \mathbb{F}_q \) with \( i^2 = -1 \), or the field \( \mathbb{F}_q \) is not the prime field, then the inequality (1.2) can not be true in general. For example, if we take \( E = \{(s, is) \in \mathbb{F}_q^2 : s \in \mathbb{F}_q\} \), then \( |E| = q \) but \( |\Delta_P(E, E)| = |\{0\}| = 1 \). Moreover, if \( q = p^2 \) with \( p \) prime, and \( E = \mathbb{F}_p^2 \), then \( |E| = p^2 = q \) but \( |\Delta_P(E, E)| = p = \sqrt{q} \). In view of these examples, Iosevich and Rudnev [12] replaced the question on the Erdős distance problems by the following Falconer distance problem in the finite field setting: how large a set \( E \subset \mathbb{F}_q^d \) is needed to obtain a positive proportion of all distances. They first showed that if \( |E| \geq 2q^{(d+1)/2} \) then one can obtain all distances that is \( |\Delta_P(E, E)| = q \) where
\[ P(x) = x_1^2 + \cdots + x_d^2. \] In addition, they conjectured that \(|E| \gtrsim q^d\) implies that \(|\Delta_P(E,E)| \gtrsim q|E|\). In the case when \(P(x) = x_1^2 + \cdots + x_d^2\), more general conjecture was given by Iosevich and Koh [11]. However, it turned out that in the case \(s = 2\) if one wants to obtain all distances, then arithmetic examples constructed by authors in [9] show that the exponent \((d+1)/2\) is sharp in odd dimensions. The problems in even dimensions are still open. Moreover if one wants to obtain a positive proportion of all distances, then the exponent \((d+1)/2\) was recently improved in two dimension by the authors in [2] who proved that if \(E \subset \mathbb{F}_q^2\) with \(|E| \gtrsim q^{d/3}\), then \(|\Delta_P(E,E)| \gtrsim q|E|\). This result was generalized by Koh and Shen [15] in the sense that if \(E, F \subset \mathbb{F}_q^2\) and \(|E|/|F| \gtrsim q^{d/3}\), then \(|\Delta_P(E,F)| = |\{P(x-y) \in \mathbb{F}_q : x \in E, y \in F\}| \gtrsim q|E|\).

In this paper, we shall study the Erdős-Falconer distance problems for finite fields, associated with the generalized distance set defined as in (1.1). This problem can be considered as a generalization of the spherical distance problems and the cubic distance problems which were studied by Iosevich and Rudnev in [12] and Iosevich and Koh in [11] respectively. The generalized Erdős distance problem was first introduced by Vu [24], mainly studying the size of the distance sets, generated by non-degenerate polynomials \(P(x) \in \mathbb{F}_q[x_1, x_2]\). Using the spectral graph theory, he proved that if \(P(x) \in \mathbb{F}_q[x_1, x_2]\) is a non-degenerate polynomial and \(E \subset \mathbb{F}_q^2\) with \(|E| \gtrsim q|E|\), then we have

\[
|\Delta_P(E,E)| \gtrsim \min\left(q, |E|^{-\frac{1}{2}}\right)
\]

where a polynomial \(P(x) \in \mathbb{F}_q[x_1, x_2]\) is called a non-degenerate polynomial if it is not of the form \(G(L(x_1, x_2))\) where \(G\) is an one-variable polynomial and \(L\) is a linear form in \(x_1, x_2\). In order to obtain the inequality (1.3), the assumption \(|E| \gtrsim q\) is necessary in general setting, which is clear from the following example: if \(P(x) = x_1^2 - x_2^2\) and \(E = \{(t, t) \in \mathbb{F}_q^2 : t \in \mathbb{F}_q\}\) is the line, then we see that \(|E| = q\) and \(|\Delta_P(E,E)| = |\{0\}| = 1\) and so the inequality (1.3) can not be true. Using the Fourier analysis method, Hart, Li, and Shen [10] showed that \(P(x) - b \in \mathbb{F}_q[x_1, x_2]\) does not have any linear factor for all \(b \in \mathbb{F}_q\) if and only if the following inequality holds:

\[
|\Delta_P(E,F)| \gtrsim \min\left(q, \sqrt{|E||F|q^{-\frac{1}{2}}}\right) \quad \text{for all } E, F \subset \mathbb{F}_q^2.
\]

In the finite field setting, results on the Erdős distance problem implies results on the Falconer distance problem. For example, the inequality (1.4) implies that if \(E, F \subset \mathbb{F}_q^2\) with \(|E||F| \gtrsim q^3\), then \(\Delta_P(E,F)\) contains a positive proportion of all possible distances, that is \(|\Delta_P(E,F)| \gtrsim q|E|\).

The purpose of this paper is to develop the two-dimensional work by Vu [24] to higher dimensions. In terms of the Fourier decay on varieties generated by general polynomials, we classify the size of distance sets. In particular, we investigate the size of the generalized Erdős-Falconer distance sets related to diagonal polynomials, that are of the form

\[
P(x) = \sum_{j=1}^d a_j x_j^{c_j} \in \mathbb{F}_q[x_1, \ldots, x_d]
\]

where \(a_j \neq 0\) and \(c_j \geq 2\) for all \(i = 1, \ldots, d\). The polynomial \(P(x) = \sum_{j=1}^d x_j^2\) is related to the spherical distance problem. In this case, the Erdős-Falconer distance problems were well studied by Iosevich and Rudnev [12]. On the other hand, Iosevich and Koh [11] studied the cubic distance problems associated with the polynomial \(P(x) = \sum_{j=1}^d x_j^3\). In addition, Vu’s theorem (1.3) gives us some results on the Erdős-Falconer distance problems in two dimension related to the polynomial \(P(x) = a_1 x_1^{c_1} + a_2 x_2^{c_2}\). As we shall see, our results will recover and extend the aforementioned authors’ work. We also study the generalized pinned distance problems in the finite field setting. As the analogue of the Euclidean pinned distance problem, the authors in [2] considered the following
pinned distance set:
\[ \Delta_P(E, y) = \{ P(x - y) \in \mathbb{F}_q : x \in E \} \]
where \( E \subset \mathbb{F}_q^d, y \in \mathbb{F}_q^d \) and \( P(x) = x_1^2 + \cdots + x_d^2 \). Using the fact that for \( x, x', y \in \mathbb{F}_q^d \),
\[ P(x - y) - P(x' - y) = (P(x) - 2y \cdot x) - (P(x') - 2y \cdot x'), \]
they obtained the following strong result.

**Theorem 1.1.** Let \( E \subset \mathbb{F}_q^d \), \( d \geq 2 \). If \( |E| \geq q^{d+1} \), then there exists \( E' \subset E \) with \( |E'| \sim |E| \) such that
\[ |\Delta_P(E, y)| > \frac{q}{2} \quad \text{for all } y \in E', \]
where \( P(x) = x_1^2 + \cdots + x_d^2 \).

However, if the polynomial \( P(x) \) is replaced by a general polynomial in \( \mathbb{F}_q[x_1, \ldots, x_d] \), then the equality (1.5) cannot be in general obtained. Thus, the main idea for the proof of Theorem 1.1 could not be applied to the generalized pin distance problems. Investigating the Fourier decay on the variety generated by a general polynomial, we shall generalize Theorem 1.1. For instance, our result implies that such fact as above theorem can be obtained if the polynomial \( P \) is a diagonal polynomial with all exponents same.

## 2. Discrete Fourier analysis and exponential sums

In order to prove our main results on the generalized Erdős-Falconer distance problems, the discrete Fourier analysis shall be used as the principle tool. In this section, we review the discrete Fourier analysis machinery for finite fields, and collect well-known facts on classical exponential sums.

### 2.1. Finite Fourier analysis

Let \( \mathbb{F}_q^d, d \geq 2 \), be a \( d \)-dimensional vector space over the finite field \( \mathbb{F}_q \) with \( q \) element. We shall work on the vector space \( \mathbb{F}_q^d \), and throughout the paper, we shall assume that the characteristic of the finite field \( \mathbb{F}_q \) is sufficiently large so that some minor technical problems can be overcome. We denote by \( \chi : \mathbb{F}_q \to \mathbb{S}^1 \) the canonical additive character of \( \mathbb{F}_q \). For example, if \( q \) is prime, then we can take \( \chi(s) = e^{2\pi is/q} \). For the example of the canonical additive character of the general field \( \mathbb{F}_q \), see Chapter 5 in [16].

Let \( f : \mathbb{F}_q^d \to \mathbb{C} \) be a complex valued function on \( \mathbb{F}_q^d \). Then, the Fourier transform of the function \( f \) is defined by
\[ \hat{f}(m) = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(x)\chi(-x \cdot m) \quad \text{for } m \in \mathbb{F}_q^d. \]

We also recall in this setting that the Fourier inversion theorem says that
\[ f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m)\hat{f}(m). \]

Using the orthogonality relation of the canonical additive character \( \chi \), that is \( \sum_{x \in \mathbb{F}_q^d} \chi(x \cdot m) = 0 \) for \( m \neq (0, \ldots, 0) \) and \( \sum_{x \in \mathbb{F}_q} \chi(x \cdot m) = q^d \) for \( m = (0, \ldots, 0) \), we obtain the following Plancherel theorem:
\[ \sum_{m \in \mathbb{F}_q^d} |\hat{f}(m)|^2 = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2. \]

For example, if \( f \) is a characteristic function on the subset \( E \) of \( \mathbb{F}_q^d \), then we see
\[ \sum_{m \in \mathbb{F}_q^d} |\hat{E}(m)|^2 = \frac{|E|}{q^d}, \]
here, and throughout the paper, we identify the set $E \subset \mathbb{F}_q^d$ with the characteristic function on the set $E$, and we denotes by $|E|$ the cardinality of the set $E \subset \mathbb{F}_q^d$.

2.2. Exponential sums. Using the discrete Fourier analysis, we shall make an effort to reduce the generalized Erdős-Falconer distance problems to estimating classical exponential sums. Some of our formulas for the distance problems can be directly applied via recent well-known exponential sum estimates. For example, the following lemma is well known and it was obtained by applying cohomological arguments (see Example 4.4.19 in [3]).

**Lemma 2.1.** Let $P(x) = \sum_{j=1}^d a_j x_j^s \in \mathbb{F}_q[x_1, \ldots, x_d]$ with $s \geq 2, a_j \neq 0$ for all $j = 1, \ldots, d$. In addition, assume that the characteristic of $\mathbb{F}_q$ is sufficiently large so that it does not divide $s$. Then,

$$|\hat{V}_t(m)| = \frac{1}{q^d} \left| \sum_{x \in V_t} \chi(-x \cdot m) \right| \lesssim q^{-\frac{d+1}{2}} \quad \text{for all } m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}, \ t \in \mathbb{F}_q \setminus \{0\},$$

and

$$|\hat{V}_0(m)| \lesssim q^{-\frac{d}{2}} \quad \text{for all } m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\},$$

where $V_t = \{x \in \mathbb{F}_q^d : P(x) = t\}$.

However, some theorems obtained by cohomological arguments contain abstract assumptions, and it can be often hard to apply them in practice. In order to overcome this problem, we shall also develop an alternative formula which is closely related to more simple exponential sums. As we shall see, such a simple formula can be obtained by viewing the distance problem in $d$ dimensions as the distance problem for product sets in $(d+1)$-dimensional vector spaces. As a typical application of our simple distance formula, we shall obtain the results on the Falconer distance problems related to arbitrary diagonal polynomials, which take the following forms: $P(x) = \sum_{j=1}^d a_j x_j^{c_j}$ for $c_j \geq 2, a_j \neq 0$ for all $j$. It is shown that such results can be obtained by applying the following well-known Weil’s theorem. For a nice proof of Weil’s theorem, we refer readers to Theorem 5.38 in [16].

**Theorem 2.2.** [Weil’s Theorem] Let $f \in \mathbb{F}_q[s]$ be of degree $c \geq 1$ with $\gcd(c, q) = 1$. Then, we have

$$\left| \sum_{s \in \mathbb{F}_q} \chi(f(s)) \right| \leq (c-1)q^{\frac{1}{2}},$$

where $\chi$ denotes a nontrivial additive character of $\mathbb{F}_q$.

We now collect well-known facts which make a crucial role in the proof of our main results. First, we introduce the cardinality of varieties related to arbitrary diagonal polynomials. The following theorem is due to Weil [25]. See also Theorem 3.35 in [3] or Theorem 6.34 in [16].

**Theorem 2.3.** Let $P(x) = \sum_{j=1}^d a_j x_j^{c_j}$ with $a_j \neq 0, c_j \geq 1$ for all $j = 1, \ldots, d$. For every $t \in \mathbb{F}_q \setminus \{0\}$, we have

$$|\{x \in \mathbb{F}_q^d : P(x) = t\}| \sim q^{d-1}.$$

The following lemma is known as the Schwartz-Zippel Lemma (see [27] and [19]). A nice proof is also given in Theorem 6.13 in [16].

**Lemma 2.4.** [Schwartz-Zippel] Let $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a non zero polynomial with degree $k$. Then, we have

$$|\{x \in \mathbb{F}_q^d : P(x) = 0\}| \leq kq^{d-1}.$$

We also need the following theorem which was implicitly given in [24].
Theorem 2.5. Let $P(x) \in \mathbb{F}_q[x_1, x_2]$ be a non-degenerate polynomial of degree $k \geq 2$. Then there is a set $T \subset \mathbb{F}_q$ with $0 \leq |T| \leq (k-1)$, such that for every $m \in \mathbb{F}_q^2 \setminus \{(0,0)\}$, $t \notin T$,

$$|\hat{V}_t(m)| = \frac{1}{q^2} \sum_{x \in V_t} \chi(-x \cdot m) \lesssim q^{-\frac{2}{d}},$$

where $V_t = \{x \in \mathbb{F}_q^2 : P(x) = t\}$ for $t \in \mathbb{F}_q$.

Remark 2.6. In Theorem 2.5, it is clear that if $t \in T$, then

$$(2.4) \quad |\hat{V}_t(m)| \lesssim q^{-1} \quad \text{for all } m \in \mathbb{F}_q^2.$$ 

This follows immediately from the Schwartz-Zippel lemma and the simple observation that $|\hat{V}_t(m)| \leq q^{-2}|V_t|$.

3. Distance formulas based on the Fourier decays

Following the similar skills due to Iosevich and Rudnev [12], we shall obtain the generalized distance formulas. As an application of the formulas, we will obtain results on the generalized Erdős-Falconer distance problems associated with specific diagonal polynomials $P(x) = \sum_{j=1}^{d} a_j x_j^d$. Let $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a polynomial with degree $\geq 2$. Given sets $E, F \subset \mathbb{F}_q^d$, recall that a generalized pair-wise distance set $\Delta_P(E, F)$ is given by the set

$$\Delta_P(E, F) = \{P(x - y) \in \mathbb{F}_q : x \in E, y \in F\}.$$

For the Erdős distance problems, we aim to find the lower bound of $|\Delta_P(E, F)|$ in terms of $|E|, |F|$. For the Falconer distance problems, our goal is to determine an optimal exponent $s_0 > 0$ such that if $|E||F| \gtrsim q^{s_0}$, then $|\Delta_P(E, F)| \gtrsim q$. In this general setting, the main difficulty on these problems is that we do not know the explicit form of the polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$, generating generalized distances. Thus, we first try to find some conditions on the variety $V_t = \{x \in \mathbb{F}_q^d : P(x) = t\}$ for $t \in \mathbb{F}_q$ such that some results can be obtained for the distance problems. In view of this idea, we have the following distance formula.

Theorem 3.1. Let $E, F \subset \mathbb{F}_q^d$ and $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$. For each $t \in \mathbb{F}_q$, we let

$$(3.1) \quad V_t = \{x \in \mathbb{F}_q^d : P(x) - t = 0\}.$$

Suppose that there is a set $T \subset \mathbb{F}_q$ such that $|V_t| \sim q^{d-1}$ for all $t \in \mathbb{F}_q \setminus T$ and

$$(3.2) \quad \left|\hat{V}_t(m)\right| \lesssim q^{-\frac{d+1}{2}} \quad \text{for all } t \notin T, m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\}.$$

Then, if $|E||F| \geq Cq^{d+1}$ with $C > 0$ sufficiently large, we have

$$|\Delta_P(E, F)| \gtrsim q - |T|.$$

Proof. Consider the counting function $\nu$ on $\mathbb{F}_q$ given by

$$\nu(t) = |\{(x,y) \in E \times F : P(x-y) = t\}|.$$

It suffices to show that $\nu(t) \neq 0$ for every $t \in \mathbb{F}_q \setminus T$. Fix $t \notin T$. Applying the Fourier inversion theorem (2.2) to $V_t(x-y)$ and using the definition of the Fourier transform (2.1), we have

$$\nu(t) = \sum_{x \in E, y \in F} V_t(x-y) = q^{2d} \sum_{m \in \mathbb{F}_q^d} \overline{\hat{E}(m)} \hat{F}(m) \hat{V}_t(m).$$
Write $\nu(t)$ by
\begin{equation}
\nu(t) = q^{2d}\overline{E}(0, \ldots, 0)\widehat{F}(0, \ldots, 0)\widehat{V}_t(0, \ldots, 0) + q^{2d} \sum_{m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} \overline{E}(m)\widehat{F}(m)\widehat{V}_t(m) = I + II.
\end{equation}
From the definition of the Fourier transform, we see
\begin{equation}
0 < I = \frac{1}{q^d} |E||F||V_t|.
\end{equation}
On the other hand, the estimate (3.2) and the Cauchy-Schwarz inequality yield
\begin{equation}
|II| \lesssim q^{2d}q^{-\frac{d+1}{2}} \left( \sum_m |\overline{E}(m)|^2 \right)^{\frac{1}{2}} \left( \sum_m |\widehat{F}(m)|^2 \right)^{\frac{1}{2}}.
\end{equation}
Applying the Plancherel theorem (2.3), we obtain
\begin{equation}
|II| \lesssim q^{\frac{d-1}{2}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}}.
\end{equation}
Since $|V_t| \sim q^{d-1}$ for each $t \in \mathbb{F}_q \setminus T$, comparing (3.4) with (3.5) gives the complete proof. \hfill \square

As a generalized version of spherical distance problems in [12] and cubic distance problems in [11], we have the following corollary.

**Corollary 3.2.** Let $P(x) = \sum_{j=1}^d a_j x_j^s \in \mathbb{F}_q[x_1, \ldots, x_d]$ for $s \geq 2$ integer and $a_j \neq 0$. Suppose that the characteristic of $\mathbb{F}_q$ is sufficiently large. If $|E||F| \geq Cq^{d+1}$ for $E, F \subset \mathbb{F}_q^*$, then $|\Delta_P(E, F)| = q - 1$, where $C > 0$ is a sufficiently large constant.

**Proof.** The statement in Corollary 3.2 follows immediately from Theorem 3.1 along with Lemma 2.1 and Theorem 2.3. \hfill \square

Under the assumptions in Corollary 3.2, we do not know whether the distance set $\Delta_P(E, F)$ contains zero or not. However, if $E = F$, then $0 \in \Delta_P(E, F)$. In this case, the distance set contains all possible distances.

Theorem 3.1 may provide us of an exact size of distance set $\Delta_P(E, F)$ and it may be a useful theorem for the Falconer distance problems for finite fields. However, if $|E||F|$ is much smaller than $q^{d+1}$, then Theorem 3.1 does not give any information about the size of the distance set $\Delta_P(E, F)$. Now, we introduce another generalized distance formula which is useful for the Erdős distance problems in the finite field setting.

**Theorem 3.3.** Let $E, F \subset \mathbb{F}_q^d$ and $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$. For each $t \in \mathbb{F}_q$, the variety $V_t$ is defined as in (3.1). Suppose that there exists a set $A \subset \mathbb{F}_q$ with $|A| \sim 1$ such that
\begin{equation}
|\widehat{V}_t(m)| \lesssim q^{-\frac{d+1}{2}} \quad \text{for all } t \notin A, m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}
\end{equation}
and
\begin{equation}
|\widehat{V}_t(m)| \lesssim q^{-\frac{d}{2}} \quad \text{for all } t \in A, m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}.
\end{equation}
If $|E||F| \geq Cq^d$ for some $C > 0$ sufficiently large, then we have
\begin{equation}
|\Delta_P(E, F)| \gtrsim \min \left( q, q^{-\frac{(d-1)}{2}} \sqrt{|E||F|} \right).
\end{equation}
PROOF. From (3.3) and (3.4), we see that for every \( t \in \mathbb{F}_q \),

\[
\nu(t) = \left| \{(x, y) \in E \times F : P(x - y) = t\} \right|
= \frac{1}{q^d} |E||F||V_t| + q^{2d} \sum_{m \in \mathbb{F}_q \setminus \{0, \ldots, 0\}} \overline{E}(m) \overline{F}(m) \overline{V_t}(m).
\]

where we also used the Schwartz-Zippel lemma (Theorem 2.4). From the Cauchy-Schwarz inequality and the Plancherel theorem (2.3), we therefore see that for every \( t \in \mathbb{F}_q \),

\[
\nu(t) \lesssim \frac{|E||F|}{q} + q^d \sqrt{|E||F|} \left( \max_{m \neq (0, \ldots, 0)} |\overline{V_t}(m)| \right).
\]

From our hypotheses (3.6), (3.7), it follows that

\[
\nu(t) \lesssim \frac{|E||F|}{q} + q^{d+1} \sqrt{|E||F|} \quad \text{if} \quad t \notin A
\]

and

\[
\nu(t) \lesssim \frac{|E||F|}{q} + q^d \sqrt{|E||F|} \quad \text{if} \quad t \in A.
\]

By these inequalities and the definition of the counting function \( \nu(t) \), we see that

\[
|E||F| = \sum_{t \in \Delta_P(E, F)} \nu(t) = \sum_{t \in A \setminus \Delta_P(E, F)} \nu(t) + \sum_{t \in (\mathbb{F}_q \setminus A) \setminus \Delta_P(E, F)} \nu(t)
\]

\[
\lesssim \frac{|E||F|}{q} + q^d \sqrt{|E||F|} + \left( \frac{|E||F|}{q} + q^{d+1} \sqrt{|E||F|} \right) |\Delta_P(E, F)|,
\]

where we used the fact that |\( A \)| \( \sim 1 \). Note that if |\( E||F| \geq Cq^d \) for some \( C > 0 \) sufficiently large, then |\( E||F| \sim |E||F| + \frac{|E||F|}{q} + q^d \sqrt{|E||F|} \). From this fact and above estimate, we conclude that if |\( E||F| \geq Cq^d \) for some \( C > 0 \) sufficiently large, then

\[
|\Delta_P(E, F)| \gtrsim \frac{|E||F|}{q} + q^{d+1} \sqrt{|E||F|}
\]

which completes the proof. \( \square \)

**Remark 3.4.** From the proof of Theorem 3.3, it is clear that if \( A \) is an empty set, then we can drop the assumption that |\( E||F| \geq Cq^d \) for some \( C > 0 \) sufficiently large. As an example showing that \( A \) can be an empty set, Koh [14] showed that if the dimension \( d \geq 3 \) is odd and \( P(x) = \sum_{j=1}^d a_j x_j^2 \) with \( a_j \neq 0 \), then |\( \overline{V_t}(m) \)| \( \lesssim q^{-(d+1)/2} \) for all \( m \neq (0, \ldots, 0), t \in \mathbb{F}_q \).

Combining Theorem 3.3 with Lemma 2.1, the following corollary immediately follows.

**Corollary 3.5.** Let \( P(x) = \sum_{j=1}^d a_j x_j^2 \in \mathbb{F}_q[x_1, \ldots, x_d] \) for \( s \geq 2 \) integer and \( a_j \neq 0 \). Assume that the characteristic of \( \mathbb{F}_q \) is sufficiently large. If \( E, F \subset \mathbb{F}_q^d \) with |\( E||F| \geq Cq^d \) for some \( C > 0 \) sufficiently large, then we have

\[
|\Delta_P(E, F)| \gtrsim \min \left( q, q^{-(d+1)/2} \sqrt{|E||F|} \right).
\]

As pointed out in Remark 3.4, if \( s = 2 \) and \( d \) is odd, then the conclusion in Corollary 3.5 holds without the assumption that |\( E||F| \gtrsim q^d \).
4. Simple formula for generalized Falconer distance problems

In previous section, we have seen that the distance problems are closely related to decays of the Fourier transforms on varieties. In order to apply Theorem 3.1 or Theorem 3.3, we must estimate the Fourier decay of the variety \( V_t = \{ x \in \mathbb{F}_q^d : P(x) = t \} \). In general, it is not easy to estimate the Fourier transform of \( V_t \). To do this, we need to show the following exponential sum estimate holds: for every \( m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\} \),

\[
\hat{V}_t(m) = q^{-d} \sum_{x \in V_t} \chi(-x \cdot m) = q^{-d-1} \sum_{(x, s) \in \mathbb{F}_q^{d+1}} \chi(sP(x) - m \cdot x - st) \lesssim q^{-d+1},
\]

where the second equality follows from the orthogonality relation of the canonical additive character \( \chi \). In other words, we must show that for \( m \neq (0, \ldots, 0) \),

\[
\sum_{(x, s) \in \mathbb{F}_q^{d+1}} \chi(sP(x) - m \cdot x - st) \lesssim q^{-d+1}.
\]

Can we find a more useful, easier formula for distance problems than the formulas given in Theorem 3.1 or Theorem 3.3? If we are just interested in getting the positive proportion of all distances, then the answer is yes. We do not need to estimate the size of \( V_t \) and we just need to estimate more simple exponential sums. We have the following simple formula.

**Theorem 4.1.** Let \( P(x) \in \mathbb{F}_q[x_1, \ldots, x_d] \) be a polynomial. Given \( E, F \subset \mathbb{F}_q^d \), define the distance set

\[
\Delta_P(E, F) = \left\{ P(x - y) \in \mathbb{F}_q : x \in E, y \in F \right\}.
\]

Suppose that the following estimate holds: for every \( m \in \mathbb{F}_q^d \) and \( s \neq 0 \),

\[
\left| \sum_{x \in \mathbb{F}_q^d} \chi(sP(x) + m \cdot x) \right| \lesssim q^d.
\]

Then, if \( |E||F| \geq Cq^{d+1} \) with \( C > 0 \) sufficiently large, then \( |\Delta_P(E, F)| \gtrsim q \).

Notice that the estimate (4.2) is easier than the estimate (4.1). We shall see that Theorem 4.1 can be obtained by studying the distance problem related to the generalized paraboloid in \( \mathbb{F}_q^{d+1} \). The details and the proof of Theorem 4.1 will be given in the next subsections. Using Theorem 4.1, we have the following corollary.

**Corollary 4.2.** Let \( P(x) = \sum_{j=1}^d a_j x_j^{c_j} \) for \( c_j \geq 2 \) integers, \( a_j \neq 0 \), and \( \gcd(c_j, q) = 1 \) for all \( j \). Let \( E, F \subset \mathbb{F}_q^d \). Define \( \Delta_P(E, F) = \{ P(x - y) \in \mathbb{F}_q : x \in E, y \in F \} \). If \( |E||F| \geq Cq^{d+1} \) with \( C > 0 \) sufficiently large, then \( |\Delta_P(E, F)| \gtrsim q \).

**Proof.** From Theorem 4.1, it suffices to show that the estimate (4.2) holds. However, this is an immediate result from Weil’s theorem (Theorem 2.2) and the proof is complete.

**Remark 4.3.** We stress that Corollary 3.2 does not imply Corollary 4.2 above. Considering the diagonal polynomial \( P(x) = \sum_{j=1}^d a_j x_j^{c_j} \), if the exponents \( c_j \) are distinct, then Corollary 3.2 does not give any information. Authors in this paper have not found any reference which shows that for \( m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\} \), and \( t \neq 0 \),

\[
|\hat{V}_t(m)| \lesssim q^{-d+1},
\]

where \( V_t = \{ x \in \mathbb{F}_q^d : \sum_{j=1}^d a_j x_j^{c_j} = t \} \) and all \( c_j \) are not same. Thus, we can not apply Theorem 3.1 to obtain such result as in Corollary 4.2. In conclusion, Theorem 4.1 can be very powerful to study the generalized Falconer distance problems. We remark that using some powerful results from algebraic geometry we can find more concrete examples of polynomials satisfying (4.2) or (4.1). For example, see Theorem 8.4 in [4] or Theorem 9.2 in [5].
4.1. Distance problems related to generalized paraboloids. In this subsection, we shall find a useful theorem which yields the simple distance formula in Theorem 4.1. Let $E, F \subset \mathbb{F}_q^d$ are product sets. In the case when $E = F$ and $P(x) = x_1^2 + \cdots + x_d^2$, it is well known in [2] that if $|E| |F| \geq q^{2d^2/(2d-1)}$, then $|\Delta_P(E, F)| \geq q$. This improves the Falconer exponent $(d+1)/2$. Here, we also study the generalized Falconer distance problems for product sets, related to the generalized paraboloid distances which are different from the usual spherical distance. If a distance set is related to usual spheres or paraboloids, then we can take advantage of the explicit forms in the varieties. In these settings, if $E$ and $F$ are product sets in $\mathbb{F}_q^d$, we may easily get the improved Falconer distance result, $|E| |F| \geq q^{2d^2/(2d-1)}$. However, the polynomial generating a distance set is not given in an explicit form, then the generalized distance problem can be hard. We are interested in getting the improved Falconer result on the generalized distance problems for product sets, associated with generalized paraboloids defined as in below. Moreover, we aim to apply the result to proving Theorem 4.1. To achieve our aim, we shall work on $\mathbb{F}_q^{d+1}$ in stead of $\mathbb{F}_q^d, d \geq 1$. We now introduce the generalized paraboloid in $\mathbb{F}_q^{d+1}$. Given a polynomial $P(x) = \mathbb{F}_q[x_1, \ldots, x_d]$ and $t \in \mathbb{F}_q$, we define the generalized paraboloid $V_t \subset \mathbb{F}_q^{d+1}$ as the set
\[
V_t = \{(x, x_{d+1}) \in \mathbb{F}_q^d \times \mathbb{F}_q : P(x) - x_{d+1} = t\},
\]
It is clear that $|V_t| = q^d$ for all $t \in \mathbb{F}_q$, because if we fix $x \in \mathbb{F}_q^d$, then $x_{d+1}$ is uniquely determined. If the polynomial is given by $P(x) = x_1^2 + \cdots + x_d^2$, then $V_0$ is exactly the usual paraboloid in $\mathbb{F}_q^{d+1}$. Let $H(x, x_{d+1}) = P(x) - x_{d+1}$, where $H$ is a polynomial in $\mathbb{F}_q[x_1, \ldots, x_d, x_{d+1}]$. Given $E^*, F^* \subset \mathbb{F}_q^{d+1}$ and $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$, consider the generalized distance set
\[
\Delta_H(E^*, F^*) = \{H(x - y, x_{d+1} - y_{d+1}) \in \mathbb{F}_q : (x, x_{d+1}) \in E^*, (y, y_{d+1}) \in F^*\},
\]
where $H(x, x_{d+1}) = P(x) - x_{d+1}$. One may have the following question. What kinds of conditions on the polynomial $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ do we need to get the improved Falconer exponent for the distance problems associated with the product sets $E^*$ and $F^*$ in $\mathbb{F}_q^{d+1}$? The following theorem may answer for above question.

**Theorem 4.4.** Let $P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a polynomial with degree $\geq 2$ satisfying the following condition: for each $s \neq 0$ and $m \in \mathbb{F}_q$,
\[
(4.3) \quad \left| \sum_{x \in \mathbb{F}_q^d} \chi(sP(x) + m \cdot x) \right| \lesssim q^d.
\]
If $E^* = E \times E_{d+1}$ and $F^* = F \times F_{d+1}$ are product sets in $\mathbb{F}_q^d \times \mathbb{F}_q$, and $\frac{|E^*| |F^*|}{|E_d+1|} \geq C q^{d+1}$ with $C > 0$ sufficiently large, then we have
\[
|\Delta_H(E^*, F^*)| = \{H(x - y, x_{d+1} - y_{d+1}) \in \mathbb{F}_q : (x, x_{d+1}) \in E^*, (y, y_{d+1}) \in F^*\} \gtrsim q,
\]
where $H(x, x_{d+1}) = P(x) - x_{d+1}$.

**Proof.** Let $E^*, F^* \subset \mathbb{F}_q^{d+1}$ be product sets given by the forms: $E^* = E \times E_{d+1}$ and $F^* = F \times F_{d+1}$ in $\mathbb{F}_q^d \times \mathbb{F}_q$. In addition, assume that $\frac{|E^*| |F^*|}{|E_d+1|} \gtrsim q^{d+1}$. Let $x^*, y^* \in \mathbb{F}_q^{d+1}$. As before, consider the counting function $\nu$ on $\mathbb{F}_q$ given by
\[
\nu(t) = |\{(x^*, y^*) \in E^* \times F^* : H(x^* - y^*) = t\}|,
\]
For each $t \in \mathbb{F}_q$, let
\[
V_t = \{x^* \in \mathbb{F}_q^{d+1} : H(x^*) - t = 0\}.
\]
We are interested in measuring the lower bound of the distance set $\Delta_H(E^*, F^*)$ defined by
\[
\Delta_H(E^*, F^*) = \{H(x^* - y^*) \in \mathbb{F}_q : x^* \in E^*, y^* \in F^*\}.
\]
In \((d+1)\) dimension, applying the Fourier inversion theorem \((2.2)\) to the function \(V_t(x^*-y^*)\) and using the definition of the Fourier transforms \((2.1)\), we have
\[
\nu(t) = \sum_{x^*, y^* \in E^*} V_t(x^*-y^*)
= q^{2(d+1)} \sum_{m^* \in \mathbb{F}^{d+1}_q} \overline{E^*(m^*)}\hat{F}^*(m^*)\hat{V}_t(m^*)
= q^{2(d+1)} \sum_{m^* \in \mathbb{F}^{d+1}_q} \overline{E^*(m^*)}\hat{F}^*(m^*)\hat{V}_t(m^*)
\]
\[
= \frac{|E^*||F^*|}{q} + q^{2(d+1)} \sum_{m^* \in \mathbb{F}^{d+1}_q} \overline{E^*(m^*)}\hat{F}^*(m^*)\hat{V}_t(m^*)
\]

Squaring the \(\nu(t)\) and summing it over \(t \in \mathbb{F}_q\) yield that
\[
\sum_{t \in \mathbb{F}_q} \nu^2(t) = \frac{|E^*|^2|F^*|^2}{q} + 2q^{2d+1}|E^*||F^*| \sum_{m^* \in \mathbb{F}^{d+1}_q} \overline{E^*(m^*)}\hat{F}^*(m^*) \sum_{t \in \mathbb{F}_q} \hat{V}_t(m^*)
+ q^{4(d+1)} \sum_{m^*, \xi^* \in \mathbb{F}^{d+1}_q} \overline{E^*(m^*)}\hat{F}^*(m^*) \overline{E^*(\xi^*)}\hat{F}^*(\xi^*) \sum_{t \in \mathbb{F}_q} \hat{V}_t(m^*)\hat{V}_t(\xi^*)
= I + II + III.
\]

Observe that I and II are given by
\[
(4.4) \quad I = \frac{|E^*|^2|F^*|^2}{q} \quad \text{and} \quad II = 0,
\]
where II = 0 follows immediately from the fact that \(\sum_{t \in \mathbb{F}_q} \hat{V}_t(m^*) = 0\) for \(m^* \neq (0, \ldots, 0)\). In order to estimate III, first observe that for \(m^* = (m, m_{d+1}) \in \mathbb{F}^{d+1}_q\),
\[
\hat{V}_t(m^*) = \frac{1}{q^{d+1}} \sum_{x \in \mathbb{F}_q^d} \chi(-m_{d+1}P(x) - m \cdot x)\chi(tm_{d+1}).
\]

It therefore follows that for \(m^* = (m, m_{d+1}), \xi^* = (\xi, \xi_{d+1}) \in \mathbb{F}^{d+1}_q\),
\[
\hat{V}_t(m^*)\hat{V}_t(\xi^*) = \frac{1}{q^{2(d+1)}} \sum_{x, y \in \mathbb{F}_q^d} \chi(t(m_{d+1} + \xi_{d+1}))\chi(-m_{d+1}P(x) - m \cdot x)\chi(-\xi_{d+1}P(y) - \xi \cdot y).
\]

Notice that if \(m^* \neq (0, \ldots, 0)\) and \(m_{d+1} = 0\), then \(\hat{V}_t(m^*)\) vanishes. In addition, observe that if \(m_{d+1} + \xi_{d+1} \neq 0\), then \(\sum_{t \in \mathbb{F}_q} \hat{V}_t(m^*)\hat{V}_t(\xi^*)\) also vanishes and if \(m_{d+1} + \xi_{d+1} = 0\), then \(\sum_{t \in \mathbb{F}_q} \chi(t(m_{d+1} + \xi_{d+1})) = q\). From these observations together with a change of a variable, \(m_{d+1} \to s\), we obtain that
\[
III = q^{2d+3} \sum_{m, \xi \in \mathbb{F}^{d+1}_q} \sum_{s \in \mathbb{F}_q^d} \overline{E^*(m, s)}\hat{F}^*(m, s)\overline{E^*(\xi, -s)}\hat{F}^*(\xi, -s)W(m, \xi, s, P),
\]
where \(W(m, \xi, s, P) = \sum_{x, y \in \mathbb{F}_q^d} \chi(-sP(x) - m \cdot x)\chi(sP(y) - \xi \cdot y)\). Our assumption \((4.3)\) implies that for each \(s \neq 0\) and \(m, \xi \in \mathbb{F}_q^d\),
\[
(4.5) \quad |W(m, \xi, s, P)| \leq \sum_{x, y \in \mathbb{F}_q^d} \chi(-sP(x) - m \cdot x)\chi(sP(y) - \xi \cdot y) \lesssim q^d.
\]
Since \( E^* = E \times E_{d+1} \) and \( F^* = F \times F_{d+1} \), it is clear that
\[
\widehat{E}^*(m,s) = \widehat{E}(m)\widehat{E}_{d+1}(s) \quad \text{and} \quad \widehat{F}^*(m,s) = \widehat{F}(m)\widehat{F}_{d+1}(s).
\]

Using this fact along with the inequality (4.5), we see that
\[
|\text{III}| \lesssim q^{3(d+1)} \left( \sum_{m \in \mathbb{F}_q^d} \left| \widehat{E}(m) \widehat{F}(m) \right|^2 \right)^{2} \left( \sum_{s \in \mathbb{F}_q \setminus \{0\}} \left| \widehat{E}_{d+1}(s)\widehat{F}_{d+1}(s) \right|^2 \right). 
\]

Using the Cauchy-Schwarz inequality and the trivial bound \(|\widehat{F}_{d+1}(s)| \leq |\widehat{F}_{d+1}(0)| = \frac{|F_{d+1}|}{q}\), we obtain that
\[
|\text{III}| \lesssim q^{3d+1}|F_{d+1}|^2 \left( \sum_{m \in \mathbb{F}_q^d} \left| \widehat{E}(m) \right|^2 \right)^{2} \left( \sum_{s \in \mathbb{F}_q} \left| \widehat{F}(m) \right|^2 \right) \left( \sum_{s \in \mathbb{F}_q} \left| \widehat{E}_{d+1}(s) \right|^2 \right).
\]

Using the Plancherel theorem (2.3) yields the following:

\[
|\text{III}| \lesssim q^d |E||E_{d+1}||F||F_{d+1}|^2 = q^d |E^*||F^*| |F_{d+1}|. 
\]

Putting estimates (4.4), (4.6) together, we conclude that
\[
\sum_{t \in \mathbb{F}_q} \nu^2(t) \lesssim \frac{|E^*|^2 |F^*|^2}{q} + q^d |E^*||F^*| |F_{d+1}|.
\]

By the Cauchy-Schwarz inequality, we see that
\[
|E^*|^2 |F^*|^2 = \left( \sum_{t \in \Delta_H(E^*,F^*)} \nu(t) \right)^2 \leq |\Delta_H(E^*,F^*)| \left( \sum_{t \in \mathbb{F}_q} \nu^2(t) \right).
\]

Thus, we have proved the following:
\[
|\Delta_H(E^*,F^*)| \gtrsim \min \left( q, q^{-d} |E^*||F^*| |F_{d+1}|^{-1} \right).
\]

This implies that if \( \frac{|E^*||F^*|}{|F_{d+1}|} \gtrsim q^{d+1} \), then
\[
|\Delta_H(E^*,F^*)| \gtrsim q,
\]

which completes the proof.

\[\Box\]

### 4.2. Proof of Theorem 4.1

We prove that the general paraboloid distance problem for product sets in \( \mathbb{F}_q^{d+1} \) implies the generalized distance problem in \( \mathbb{F}_q^d \). Namely, Theorem 4.1 can be obtained as a corollary of Theorem 4.4. In order to prove Theorem 4.1, first fix \( E,F \subset \mathbb{F}_q^d \) with \( |E||F| \geq Cq^{d+1} \) with \( C > 0 \) large. Let \( E^* = E \times \{0\} \subset \mathbb{F}_q^{d+1} \) and \( F^* = F \times \{0\} \subset \mathbb{F}_q^{d+1} \). Observe that \( |E| = |E^*|, |F| = |F^*|, \) and
\[
|\Delta_P(E,F)| = |\{ P(x-y) \in \mathbb{F}_q : x \in E, y \in F \}|
\]
\[
= |\Delta_H(E^*,F^*)| = |\{ H(x-y,x_{d+1}-y_{d+1}) \in \mathbb{F}_q : (x,x_{d+1}) \in E^*, (y,y_{d+1}) \in F^* \}|
\]

where \( H(x,x_{d+1}) = P(x) - x_{d+1} \). The assumption (4.2) in Theorem 4.1 implies that the conclusion of Theorem 4.4 holds: if \( \frac{|E^*||F^*|}{|F_{d+1}|} \gtrsim q^{d+1} \), then \( |\Delta_H(E^*,F^*)| \gtrsim q \). Since \( |\{0\}| = 1, |E^*| = \)
\(|E|, |F^*| = |F|, \) and \(|\Delta_H(E^*, F^*)| = |\Delta_P(E, F)|\), we therefore conclude that if \(|E||F| \gtrsim q^{d+1}\), then \(|\Delta_P(E, F)| \gtrsim q\). Thus, the proof of Theorem 4.1 is complete.

5. Generalized pinned distance problems

We find the conditions on the polynomial \(P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]\) such that the desirable results for generalized pinned distance problems hold. First, let us introduce some notation associated with the pinned distance problems. Let \(P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]\) be a polynomial. For each \(t \in \mathbb{F}_q\), we define a variety \(V_t\) by

\[
V_t = \{ x \in \mathbb{F}_q^d : P(x) = t \}.
\]

The Schwartz-Zippel Lemma (Lemma 2.4) says that \(|V_t| \lesssim q^{d-1}\) for all \(t \in \mathbb{F}_q\). Let \(E \subset \mathbb{F}_q^d\). Given \(y \in \mathbb{F}_q^d\), we denote by \(\Delta_P(E, y)\) a pinned distance set defined as

\[
\Delta_P(E, y) = \{ P(x - y) \in \mathbb{F}_q : x \in E \}.
\]

We are interested in finding the element \(y \in \mathbb{F}_q^d\) and the size of \(E \subset \mathbb{F}_q^d\) such that \(|\Delta_P(E, y)| \gtrsim q\). We have the following theorem.

**Theorem 5.1.** Let \(T \subset \mathbb{F}_q^d\), with \(|T| \sim 1\). Suppose that the varieties \(V_t\), generated by a polynomial \(P(x) \in \mathbb{F}_q[x_1, \ldots, x_d]\), satisfy the following: for all \(m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\},\)

\[
\begin{align*}
|\widehat{V}_t(m)| & \lesssim q^{\frac{d+1}{2}} \quad \text{if} \ t \notin T \\
|\widehat{V}_t(m)| & \lesssim q^{\frac{d}{2}} \quad \text{if} \ t \in T.
\end{align*}
\]

Let \(E, F \subset \mathbb{F}_q^d\). If \(|E||F| \geq Cq^{d+1}\) with \(C > 0\) large enough, then there exists \(F_0 \subset F\) with \(|F_0| \sim |F|\) such that

\[
|\Delta_P(E, y)| \gtrsim q \quad \text{for all} \ y \in F_0.
\]

**Proof.** Using the pigeonhole principle, it suffices to prove that if \(|E||F| \gtrsim q^{d+1}\), then

\[
\frac{1}{|F|} \sum_{y \in F} |\Delta_P(E, y)| \gtrsim q.
\]

For each \(t \in \mathbb{F}_q^d\) and \(y \in F\), consider the counting function \(\nu_y(t)\) given by

\[
\nu_y(t) = |\{ x \in E : P(x - y) = t \}| = |\{ x \in E : x - y \in V_t \}|.
\]

Applying the Fourier inversion transform to the function \(V_t(x - y)\) and using the definition of the Fourier transform, we see that

\[
\begin{align*}
\nu_y(t) &= \sum_{x \in \mathbb{F}_q^d} E(x)V_t(x - y) = q^d \sum_{m \in \mathbb{F}_q^d} \overline{E}(m)\widehat{V}_t(m)\chi(-m \cdot y) \\
&= q^d \overline{E}(0, \ldots, 0)\widehat{V}_t(0, \ldots, 0)\chi(0) + q^d \sum_{m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}} \overline{E}(m)\widehat{V}_t(m)\chi(-m \cdot y) \\
&= \frac{|E||V_t|}{q^d} + q^d \sum_{m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}} \overline{E}(m)\widehat{V}_t(m)\chi(-m \cdot y).
\end{align*}
\]
Squaring the $\nu_y(t)$ and summing it over $y \in F$ and $t \in \mathbb{F}_q$, we see that

$$
\sum_{y \in F} \sum_{t \in \mathbb{F}_q} \nu_y^2(t) = \sum_{y \in F} \sum_{t \in \mathbb{F}_q} \frac{|E|^2|V_t|^2}{q^{2d}} + \sum_{y \in F} \sum_{t \in \mathbb{F}_q} 2|E||V_t| \sum_{m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} E(m)\hat{V}_t(m)\chi(-m \cdot y) + \sum_{y \in F} \sum_{t \in \mathbb{F}_q} q^{2d} \sum_{m, \xi \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} \overline{E}(m)\hat{V}_t(m)\chi(-m \cdot y)\overline{E}(\xi)\hat{V}_t(\xi)\chi(-\xi \cdot y)
$$

$$
= A + B + C.
$$

Since $|V_t| \lesssim q^{d-1}$ for all $t \in \mathbb{F}_q$, it is clear that

$$
|A| \lesssim \frac{|E|^2|F|}{q}.
$$

To estimate $|B|$, first use the definition of the Fourier transform and find the maximum value of the sum in $t \in \mathbb{F}_q$ with respect to $m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}$. Namely, we have

$$
|B| \leq 2q^d|E| \left( \max_{m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} \sum_{t \in \mathbb{F}_q} |V_t||\hat{V}_t(m)| \right) \sum_{m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} |\overline{E}(m)||\hat{F}(m)|.
$$

From the assumptions, (5.1), (5.2), $|T| \sim 1$, and the fact that $|V_t| \lesssim q^{d-1}$ for all $t \in \mathbb{F}_q$, we see that the maximum value term is $\lesssim q^{(d-1)/2}$. If we use the Cauchy-Schwarz inequality and the Plancherel theorem, then we also see that

$$
\sum_{m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} |\overline{E}(m)||\hat{F}(m)| \leq \frac{|E|^\frac{d}{2}|F|^\frac{1}{2}}{q^d}.
$$

Therefore, the value $B$ can be estimated by

$$
|B| \lesssim q^{\frac{d-1}{2}}|E|^\frac{d}{2}|F|^\frac{1}{2}.
$$

Now we estimate the value $C$. Using a change of the variable, $\xi \rightarrow -\xi$, we see

$$
\sum_{m, \xi \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} \overline{E}(m)\hat{V}_t(m)\chi(-m \cdot y)\overline{E}(\xi)\hat{V}_t(\xi)\chi(-\xi \cdot y)
$$

$$
= \left| \sum_{m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} \overline{E}(m)\hat{V}_t(m)\chi(-m \cdot y) \right|^2
$$

which is always a nonnegative real number. In order to obtain an upper bound of the term $C$, we therefore expand the sum over $y \in F$ to the sum over $y \in \mathbb{F}_q^d$ and we compute the sum in $y$ by using the orthogonality relation of the canonical additive character $\chi$. It therefore follows that

$$
|C| \leq q^{3d} \sum_{m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} \sum_{t \in \mathbb{F}_q} |\hat{V}_t(m)|^2|\overline{E}(m)|^2
$$

$$
\leq q^{3d} \left( \max_{m \in \mathbb{F}_q^d \setminus \{0, \ldots, 0\}} \sum_{t \in \mathbb{F}_q} |\hat{V}_t(m)|^2 \right) \sum_{m \in \mathbb{F}_q^d} |\overline{E}(m)|^2.
$$
Using the Plancherel theorem and the assumption of the Fourier decay of $V_t$, we see that
\[
\sum_{m \in \mathbb{F}_q^d} |\hat{E}(m)|^2 = \frac{|E|}{q^d} \quad \text{and} \quad \max_{m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\}} \sum_{t \in \mathbb{F}_q} |\hat{V}_t(m)|^2 \lesssim q^{-d}.
\]
Putting these facts together yields the upper bound of the value $|C|:
\begin{equation}
(5.6) \quad |C| \lesssim q^d |E|.
\end{equation}
From (5.4), (5.5), and (5.6), we obtain the following estimate:
\[
\sum_{t \in \mathbb{F}_q} \sum_{t' \in \mathbb{F}_q} \nu^2_y(t) \lesssim \frac{|E|^2 |F|}{q} + q \frac{d^2}{q^2} |E|^3 |F|^\frac{1}{2} + q^d |E|.
\]
Observe that if $|E||F| \geq Cq^{d+1}$ for $C > 0$ sufficiently large, then
\begin{equation}
(5.7) \quad \sum_{y \in \mathbb{F}_q} \sum_{t \in \mathbb{F}_q} \nu^2_y(t) \lesssim \frac{|E|^2 |F|}{q}.
\end{equation}
We are ready to finish the proof. For each $y \in \mathbb{F}_q$, if we note that $\sum_{t \in \Delta_P(E,y)} \nu_y(t) = |E|$ and then apply the Cauchy-Schwarz inequality, then we see
\[
|E|^2 |F|^2 = \left( \sum_{y \in \mathbb{F}_q} \sum_{t \in \Delta_P(E,y)} \nu_y(t) \right)^2 \leq \left( \sum_{y \in \mathbb{F}_q} |\Delta_P(E,y)| \right) \left( \sum_{y \in \mathbb{F}_q} \sum_{t \in \mathbb{F}_q} \nu^2_y(t) \right) \lesssim \left( \sum_{y \in \mathbb{F}_q} |\Delta_P(E,y)| \right) \frac{|E|^2 |F|}{q},
\]
where the last line follows from the estimate (5.7). Thus, the estimate (5.3) holds and we complete the proof of Theorem 5.1. \[\square\]

**Remark 5.2.** Let $E, F \subset \mathbb{F}_q^d \setminus \{0\}$. We note that if $P(x_1,\ldots,x_d) = a_1x_1^s + \cdots + a_dx_d^s$ satisfies the assumptions in Corollary 3.2, then there exists a subset $F_0$ of $F$ with $|F_0| \sim |F|$ such that
\[
|\Delta_P(E,y)| \gtrsim q \quad \text{for all } y \in F_0.
\]
This is an immediate result from Theorem 5.1 and Lemma 2.1. In terms of the generalized Falconer distance problem, this result sharpens the statement of Corollary 3.2. On the other hand, Corollary 3.2 gives us the exact number of the elements in the distance set.

We close this paper by introducing a corollary of Theorem 5.1, which sharpens and generalizes the Vu’s result (1.3).

**Corollary 5.3.** Let $P(x) \in \mathbb{F}_q[x_1,x_2]$ be a non-degenerate polynomial. If $|E||F| \geq Cq^3$ for $E, F \subset \mathbb{F}_q^2$ and $C > 0$ sufficiently large, then there exists a subset $F_0$ of $F$ with $|F_0| \sim |F|$ such that
\[
|\Delta_P(E,y)| \gtrsim q \quad \text{for all } y \in F_0.
\]
**Proof.** The proof follows immediately by applying Theorem 5.1 along with Theorem 2.5 and (2.4) in Remark 2.6. \[\square\]
References

[1] J. Bourgain, N. Katz, and T. Tao, *A sum-product estimate in finite fields, and applications*, Geom. Funct. Anal. 14 (2004), 27–57.

[2] J. Chapman, M. Erdoğan, D. Hart, A. Iosevich, and D. Koh, *Pinned distance sets, Wolff’s exponent in finite fields and sum-product estimates*, (2009), arxiv.org.

[3] T. Cochrane, *Exponential sums and the distribution of solutions of congruences*, Inst. of Math., Academia Sinica, Taipei, (1994).

[4] P. Deligne, *La conjecture de Weil I*, Publ. Math. I.H.E.S. 43 (1973), 273–308.

[5] J. Denef and F. Loeser, *Weights of exponential sums, intersection cohomology and Newton polyhedra*, Invent. Math. 106 (1991), 275–294.

[6] M. Erdoğan, *A bilinear Fourier extension theorem and applications to the distance set problem*, Internat. Math. Res. Notices 23 (2005), 1411-1425.

[7] P. Erdős, *On sets of distances of n points*, Amer. Math. Monthly 53, (1946), 248–250.

[8] K. Falconer, *On the Hausdorff dimension of distance sets*, Mathematika, 32 (1985), 206–212.

[9] D. Hart, A. Iosevich, D. Koh and M. Rudnev, *Averages over hyperplanes, sum-product theory in vector spaces over finite fields and the Erdős-Falconer distance conjecture*, Trans. Amer. Math. Soc. (2010) To appear.

[10] D. Hart, L. Li and C. Shen, *Fourier analysis and expanding phenomena in finite fields*, preprint (2010), arxiv.org.

[11] A. Iosevich and D. Koh, *The Erdős-Falconer distance problem, exponential sums, and Fourier analytic approach to incidence theorems in vector spaces over finite fields*, SIAM Journal of Discrete Mathematics, Vol 23, no 1 (2008), 123–135.

[12] A. Iosevich and M. Rudnev, *Erdős distance problem in vector spaces over finite fields*, Trans. Amer. Math. Soc. 359 (2007), 6127-6142.

[13] N. Katz and G. Tardos, *A new entropy inequality for the Erdős distance problem*, Contemp. Math. 342, Towards a theory of geometric graphs, 119-126, Amer. Math. Soc., Providence, RI (2004).

[14] D. Koh, *Extension and averaging operators for finite fields*, preprint (2009), arxiv.org.

[15] D. Koh and C. Shen, *Sharp extension theorems and Falconer distance problems for algebraic curves in two dimensional vector spaces over finite fields*, preprint (2010), arxiv.org.

[16] R. Lidl and H. Niederreiter, *Finite fields*, Cambridge University Press, (1993).

[17] P. Mattila, *Spherical averages of Fourier transforms of measures with finite energy: dimension of intersections and distance sets*, Mathematika 34(1987), 207–228.

[18] Y. Peres and W. Schlag, *Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions*, Duke Math. J. 102 (2000), no. 2, 193–251.

[19] J. Schwartz, *Fast probabilistic algorithms for verification of polynomial identities*, J. ACM, 27(4): (1980), 701–717.

[20] J. Solymosi and C. Tóth, *Distinct distances in the plane*, Discrete Comput. Geom. 25 (2001), no. 4, 629–634.

[21] J. Solymosi and V. Vu, *Distinct distances in high dimensional homogeneous sets in: Towards a Theory of Geometric Graphs (J. Pach, ed.), Contemporary Mathematics, vol. 342, Amer. Math. Soc. (2004).*

[22] J. Solymosi and V. Vu, *Near optimal bounds for the number of distinct distances in high dimensions*, Combinatorica, Vol 28, no 1 (2008), 113–125.

[23] T. Tao, *A sharp bilinear restriction estimate for paraboloids*, Geom. Funct. Anal. 13 (2003), 1359–1384.

[24] V. Vu, *Sum-product estimates via directed expanders*, Math. Res. Lett. 15 (2008), 375–388.

[25] A. Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. 55 (1949), 497–508.

[26] T. Wolff, *Decay of circular means of Fourier transforms of measures*, Internat. Math. Res. Notices 1999, 547–567.

[27] R. Zippel, *Probabilistic algorithms for sparse polynomials*, In Proceedings of the International Symposium on Symbolic and Algebraic Computation, 216–226, Springer-Verlag, 1979.