A convergent boundary-condition conforming adaptive spline-based finite element method for the bi-Laplace operator

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Abstract
We establish the convergence of an adaptive spline-based finite element method of a fourth order elliptic problem.

1 Introduction

Design of optimal meshes for finite element analysis is a topic of extensive research going back to the early seventies. Among the rich variety of strategies explored, the first mathematical framework for automatic optimal mesh generation was laid in the seminal work of Babushka and Rheinboldt [5]. They introduce a class of computable a posteriori error estimates for a general class of variational problems which provides a strategy of extracting localized approximations of the numerical error of the exact solution. The derived computable estimates are shown to form an upper bound with the numerical error, justifying the validity of the estimator, and a lower bounds which ensures efficient refinement; i.e., refinement where only necessary. This established equivalence with the numerical error eluded to promising potential of practicality and robustness. The theoretical results led to a heuristic characterization of optimal meshes through the even distribution a posteriori error quantities over all mesh elements, providing a blue-print for adaptive mesh generation. The first detailed discription and performance analysis for a simple and accessible a posteriori error estimator was conceived for a one-dimensional elliptic and parabolic second-order Poisson-type problems in [4]. The analysis was significantly improved in [28] for two-dimensional scenarios and develops numerous techniques used in the derivation of a posteriori estimates until today.

The first convergence was given in [14] in one-dimensions and later extended to two-dimensions by Dorfler [16]. Combing the advent of a novel marking strategy and finess assumptions of the initial mesh, [16]. Element marking directs the refinement procedure
1 Introduction

to select user-specified ratio of elements with highest error indicators relative to the total estimation. The proposed strategy is later shown to be optimal [15]. The initial mesh assumption was placed to ensure problem datum, such as source function and boundary values, are sufficiently resolved for detection by the solver. The aim is to ensure error reduction of the estimator and thus monotone convergence of numerical error is achieved through contraction of consecutive errors in energy norm at every step, but at the expense of potential over-refinement of the initial mesh. This was achieved using a local counter part of the efficiency estimate described above.

Morin et al [22] came to the realization that the averaging of the data has an unavoidable interference with the estimator error reduction irrespective of quadrature and was due to the averaging of finer features of data brought by finite-dimensional approximations. This averaging was quantified into an oscillation term, a quantity tightly related to the criterion used in the initial mesh assumption used in [16] but it provided a sharper representation of the underlying issue. As a result the initial mesh assumption was removed in [22] and replaced with a Dofler-type marking criterion, separate marking, for the data oscillation. Unfortunately, the relaxation of a fine initial mesh had the unintended consequence of losing the strict monotone behaviour of numerical error decay. It led to the introduction of the interior node property to ensure error reduction with every step so as to ensure two consequence solutions will not be the same unless they are equal to the exact one; but at the expense of introducing over refinement. Each marked element in a two-dimensional triangular mesh undergoes three bisections ensuring an interior node, which furnishes us with a local lower bound and thus recovers strict error reduction with every iteration. The results were extended to saddle-problems in [23] and generalized into abstract Hilbert setting in [24].

For the better part of the 2000’s Morin-Nochetto-Sierbert algorithm (MNS) championed adaptive finite element methods AFEM of linear elliptic problems after which the analysis was refined by Cascon [15] in concrete setting where they did not rely on a local lower bound for convergence which led to the ultimate removal of the costly interior node property and separate marking for oscillation. This was done while achieving quasi-optimal mesh complexity; see [12],[27] for details. It was realized that strict reduction of error in energy norm cannot be guaranteed whenever consecutive numerical solutions coincide but strict monotone decay is be obtain with respect to a suitable quasi-norm. The result was extended to abstract Hilbert by Siebert [26] which is now widely considered state of the art analysis of AFEM among the adaptive community and hinges on the following ingredients: a global upper bound justifying the validity of the a posteriori estimator, a Lipschitz property of the estimator as a function on the discrete finite element trial space indicating suitable sensitivity in variation within the trial space, a Pythagorean-type relation furnished by the variational and discrete forms and any suitable marking strategy akin to that of Dofler; one that aims to equally distributes the elemental error estimates.
2 Problem set up and Adaptive method

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with polygonal boundary \( \Gamma \). For a source function \( f \in L^2(\Omega) \) we consider the following homogenous Dirichlet boundary-valued problem

\[
L u(x) := \Delta^2 u(x) = f(x) \quad \text{in } \Omega
\]
\[
u = \partial u / \partial \nu = 0 \quad \text{on } \Gamma.
\]

The adaptive procedure iterates over the following modules

\[
\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}
\]

The module SOLVE computes a hierarchical polynomial B-spline approximation \( U \) of the solution \( u \) with respect to a hierarchical partition \( P \) of \( \Omega \). A detailed discussion on the nature of such partitions will be carried in Section 2.3. For the module ESTIMATE, we use a residual-based error estimator \( \eta_P \) derived from the a posteriori analysis in Section 3. The module MARK follows the Dölfer marking criterion of [16]. Finally, the module REFINE produces a new refined partition \( P^* \) satisfying certain geometric constraints described in Section 2.3 to ensure sharp local approximation.

2.1 Notation

We begin by laying out the notational conventions and function space definitions used in this presentation. Let \( P \) be a partition of domain \( \Omega \) consisting of square cells \( \tau \) following the structure described in [29],[3]. Denote the collection of all interior edges of cells \( \tau \in P \) by \( \mathcal{E}_P \) and all those along the boundary \( \Gamma \) are to be collected in \( \mathcal{G}_P \). We assume that cells \( \tau \) are open sets in \( \Omega \) and that edges \( \sigma \) do not contain the vertices of its affiliating cell. Let \( \text{diam} (\omega) \) be the longest length within a Euclidian object \( \omega \) and set \( h_\tau := \text{diam} (\tau) \) and \( h_\sigma := \text{diam} (\sigma) \). Then let the mesh-size \( h_P := \max_{\tau \in P} h_\tau \). Define the boundary mesh-size function \( h_\Gamma \in L^\infty(\Gamma) \) by

\[
h_\Gamma(x) = \sum_{\sigma \in \mathcal{G}_P} h_\sigma \mathbb{1}_\sigma(x),
\]

where the \( \mathbb{1}_\sigma \) are the indicator functions on boundary edges. Let \( H^s(\Omega) \), \( s > 0 \), be the fractional order Sobolev space equipped with the usual norm \( \| \cdot \|_{H^s(\Omega)} \); see references [1],[18]. Let \( H^s_0(\Omega) \) be given as the closure of the test functions \( C^\infty_0(\Omega) \) in \( \| \cdot \|_{H^s(\Omega)} \). The semi-norm \( \| \cdot \|_{H^s(\Omega)} \) defines a full norm on \( H^s_0(\Omega) \) by virtue of Poincaré’s inequality.
Moreover, the semi-norm $\|\cdot\|_{L^2(\Omega)}$ defines a norm on $H^2_0(\Omega)$. By $H^{-2}(\Omega) = (H^2_0(\Omega))'$ the dual of $H^2(\Omega)$ with the induced norm

$$
\|F\|_{H^{-2}(\Omega)} = \sup_{v \in H^2_0(\Omega)} \frac{\langle F, v \rangle}{\|v\|_{H^2(\Omega)}}.
$$

(4)

### 2.2 Weak formulation

The natural weak formulation to the PDE (1) reads

Find $u \in H^2_0(\Omega)$ such that $a(u, v) = \ell_f(v)$ for all $v \in H^2_0(\Omega)$,

(5)

where $a : H^2_0(\Omega) \times H^2_0(\Omega) \to \mathbb{R}$ is be the bilinear form $a(u, v) = (\Delta u, \Delta v)_{L^2(\Omega)}$ and $\ell_f(v) = (f, v)_{L^2(\Omega)}$. The energy norm $\|\cdot\| := \sqrt{a(\cdot, \cdot)} \equiv \|\Delta \cdot\|_{L^2(\Omega)}$ is one for which the form $a$ is continuous and coercive on $H^2_0(\Omega)$, with unit proportionality constants. The existence of a unique solution is therefore ensured by Babuska-Lax-Milgram theorem. The variational formulation (5) is consistent with the PDE (1) under sufficient regularity considerations; if $u \in H^4(\Omega) \cap H^2_0(\Omega)$ satisfies (5) then $u$ satisfies (1) in the classical sense by virtue of the Du Bois-Reymond lemma.

### 2.3 Spline spaces and hierarchical partitions

We consider a hierarchical polynomial spline space as the discrete trial and test space. For completeness we describe the construction. Let $S^0 \subset S^1 \subset \cdots \subset S^{L-1}$ be a hierarchy of $L$ tensor-product multivariate spline spaces defined on $\Omega$. For each hierarchy level $\ell$, we obtain $B$-spline polynomial basis $B^\ell$ of degree $r \geq 2$ defined on a tensor-product mesh $G^\ell$ partitioning of $\Omega$ generated by tensorizing translations and dilations of an $r$-th degree cardinal B-spline $b^r$ defined by recursive convolution with the characteristic function:

$$
b^k(x) = \int_0^{k+1} b^{k-1}(x-t) 1_{[0,1]}(t) \, dt, \quad b^0 := 1_{[0,1]}(x).
$$

(6)

Partition $G^{\ell+1}$ is obtained from $G^\ell$ via uniform dyadic subdivisions which will insure the nesting $S^\ell \subset S^{\ell+1}$. That is, if $s \in S^\ell$ then we may express $s$ in terms of $B^{\ell+1}$:

$$
s = \sum_{\beta \in B^{\ell+1}} c^{\ell+1}_\beta(s) \beta,
$$

(7)

where $c^{\ell+1}_\beta(s)$ are the coefficients of $s$ when expressed in $B^{\ell+1}$. From classical spline theory, it is well-known that B-splines are locally linearly independent, they are non-negative, they are supported locally and form a partition of unity. We are now in a position to define hierarchical mesh configuration. A cell $\tau$ of level $\ell$ is said to active if $\tau \in G^\ell$ and $\tau \cap \Omega^{\ell+1} = \emptyset$. A subdomain $\Omega^\ell$ of $\Omega$ is defined as the closure of the union of active Cells $\tau$ in
$G^\ell$. With subdomain hierarchy $\Omega^L = \{\Omega^\ell\}_{\ell=0}^{L-1}$ of closed domains $\Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^{L-1}$, with $\text{int}(\Omega^0) = \Omega$ and $\Omega^L = \emptyset$, we define a hierarchical $P$ partitioning of $\Omega$ as a mesh satisfying the following conditions:

1. Members of $P$ are active cells from $G^\ell$, $0 \leq \ell \leq L - 1$.
2. All cells $\tau$ in $P$ are disjoint.
3. The interior of the closure of the union $\bigcup\{\tau : \tau \in P\}$ is equal to $\Omega$.

A \textit{Hierarchical B-spline} (HB-spline) basis $\mathcal{H}$ with respect to hierarchical partition $P$ is defined as

$$\mathcal{H}_P = \{ \beta \in \mathcal{B}^\ell : \text{supp} \beta \subseteq \bar{\Omega}^\ell \land \text{supp} \beta \nsubseteq \bar{\Omega}^{\ell+1} \}.$$  

A recursive definition is given in [7]. A basis function $\beta$ of level $\ell$ is said to active if $\beta \in \mathcal{B}^\ell \cap \mathcal{H}$, otherwise it is passive. The basis $\mathcal{H}_P$ inherits much of the key properties of tensor-product B-spline bases: they are locally linearly independent, they are non-negative and they have local support [29]. However, the basis does not form a partition of unity which could pose a problem to approximation stability. It is possible to modify $\mathcal{H}_P$ into forming a partition of unity through scaling [29] but instead we use a truncation procedure utilizing the relation (7) to produce a new basis that recovers the partition of unity while preserving all the desirable properties of $\mathcal{H}_P$. We define a truncation operator of a spline function $s \in \mathcal{S}^\ell$:

$$\text{trunc}^{\ell+1}s := \sum_{\beta \in \mathcal{B}^{\ell+1} : \text{supp} \beta \subseteq \bar{\Omega}^{\ell+1}} c^{\ell+1}_\beta(s) \beta.$$  

(8)

In simple terms, the truncation removes contributions coming from active basis functions in $\mathcal{B}^{\ell+1}$ thus reducing the support $s$ from reaching too far into $\Omega^{\ell+1}$. By recursive application of (8) to each spline $\beta \in \mathcal{H}_P$:

$$\text{Trunc}^{\ell+1}\beta := \text{trunc}^{\ell-1}(\text{trunc}^{\ell-2}(\cdots (\text{trunc}^{\ell+1}\beta)))$$  

(9)

we obtain a modified hierarchical B-spline basis, a truncated hierarchical B-spline (THB-spline) basis $\mathcal{T}_P$ with respect to partition $P$:

$$\mathcal{T}_P = \{ \text{Trunc}^{\ell+1}\beta : \beta \in \mathcal{B}^\ell \cap \mathcal{H}_P, \ell = 0 : L - 1 \}.$$  

(10)

The basis $\mathcal{T}_P$ retains all of the aforementioned properties of its hierarchical counterpart $\mathcal{H}_P$ while forming a partition of unity.

2.4 Admissible partitions

For local and stable approximation we need to control the influence of each basis function. With additional restrictions on the structure of partitions $P$ we can guarantee that the
number of basis functions acting on any point is bounded and that the diameter of the support of a basis function is comparable to any cell in its support. A partition \( P \) is said to be admissible if the truncated basis functions in \( T \) which have support on \( \tau \in P \) belong to at most two levels successive levels. The support extension of a cell \( \tau \in G^\ell \) with respect to level \( k \leq \ell \) is defined as

\[
S(\tau, k) := \left\{ \tau' \in G^k : \exists \beta \in B^k \text{ s.t supp } \beta \cap \tau' \neq \emptyset \land \text{ supp } \beta \cap \tau \neq \emptyset \right\}
\]

Note that the support extension consists of cells from the tensor-product mesh \( G^k \). To assess the locality of the basis; i.e., the influence of basis functions have on active cells, it is useful to consider a support extension consisting of all active cells belonging to its support regardless of level. For \( \tau \in P \) define

\[
\omega_{\tau} = \bigcup_{\ell=0}^{L-1} S(\tau, \ell) \cap P \equiv \{ \tau' \in P : \text{ supp } \beta \cap \tau' \neq \emptyset \implies \text{ supp } \beta \cap \tau \neq \emptyset \},
\]

indicating the collection of all supports for basis function \( \beta \)'s whose supports intersect \( \tau \).

Analogously, we denote the support extension for an edge \( \sigma \in \mathcal{E} \cup \mathcal{G} \) by

\[
\omega_{\sigma} = \{ \tau \in P : \text{ supp } \beta \cap \tau \neq \emptyset \implies \text{ supp } \beta \cap \tau \neq \emptyset, \ \sigma \subset \partial \tau \}.
\]

The following auxiliary subdomain provides a way to ensure mesh admissibility

\[
U^\ell := \bigcup \left\{ \tau : \tau \in G^\ell \land S(\tau, \ell) \subseteq \Omega^\ell \right\}
\]

Lemma 2.1. Let \( \Omega^L \) be a subdomain hierarchy with respect to partition \( P \) of domain \( \Omega \). If

\[
\Omega^\ell \subseteq U^{\ell-1}
\]

for \( \ell = 2 : L - 1 \), then \( P \) is an admissible partition.

Proof. See [10].

In other words, \( U^\ell \) represents the biggest subset of \( \Omega^\ell \) so that the set of B-splines in \( B^\ell \) whose support is contained in \( \Omega^\ell \) spans the restriction of \( S^\ell \) to \( U^\ell \).

2.5 The adaptive method

We now discuss the modules \textbf{SOLVE}, \textbf{ESTIMATE}, \textbf{MARK} and \textbf{REFINE} in detail.
The module SOLVE

The space of piecewise polynomials of degree \( r \geq 2 \) defined on a partition \( P \) will be given by

\[
P^r_P(\Omega) = \prod_{\tau \in P} P_r(\tau).
\] (16)

Assuming we have at our disposal a polynomial B-spline space \( X_P \subset P^r_P(\Omega) \cap H^2_0(\Omega) \) then the discrete problem reads

\[
U = \text{SOLVE}[P,f] : \text{Find } U \in X_P \text{ such that } a(U,V) = \ell_f(V) \text{ for all } V \in X_P.
\] (17)

The linear system is numerically stable and consistent with (5) in the sense that \( a(u,U) = \ell_f(V) \) for every \( V \in X_P \) and therefore we are provided with Galerkin orthogonality:

\[
a(u - U,V) = 0 \quad \forall V \in X_P.
\] (18)

Moreover, the spline solution will serve as an optimal approximation to \( u \) in \( X_P \) with respect to \( \| \cdot \| \). Indeed, we have for any \( V \in X_P \),

\[
\| u - U \|^2 \leq \frac{1}{C_{\text{coer}}} a_P(u - U,u - V) \leq \frac{C_{\text{cont}}}{C_{\text{coer}}} \| u - U \| \| u - V \|.
\] (19)

The module ESTIMATE

For a continuous function \( v \) we define the jump operator across interface \( \sigma \).

\[
\mathbb{J}_\sigma(v) = \lim_{t \to 0} [v(x + t\sigma) - v(x - t\sigma)], \quad x \in \sigma.
\] (20)

The adaptive refinement procedure of method (2) will aim to reduce the error estimations instructed by the cell-wise error indicators:

\[
\eta_P^2(V,\tau) = h^4_r \| f - \mathcal{L}V \|^2_{L^2(\tau)} + \sum_{\sigma \subset \partial \tau} h^3_\sigma \left( \| \mathbb{J}_\sigma \left( \frac{\partial \mathcal{L}V}{\partial n} \right) \|^2_{L^2(\tau)} + h_\sigma \| \mathbb{J}_\sigma(\Delta V) \|^2_{L^2(\tau)} \right).
\] (21)

We can define the indicators on subsets of \( \Omega \) via:

\[
\eta_P^2(V,\omega) = \sum_{\tau \in P: \tau \subset \omega} \eta_P^2(V,\tau), \quad \omega \subseteq \Omega
\] (22)

To each cell \( \tau \) in mesh \( P \) the error indicators (21) will assign error estimations:

\[
\{ \eta_\tau : \tau \in P \} = \text{ESTIMATE}[U,P] : \eta_\tau := \eta_P(U,\tau)
\] (23)

The module SOLVE

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We can define the indicators on subsets of \( \Omega \) via:

\[
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\] (22)

To each cell \( \tau \) in mesh \( P \) the error indicators (21) will assign error estimations:

\[
\{ \eta_\tau : \tau \in P \} = \text{ESTIMATE}[U,P] : \eta_\tau := \eta_P(U,\tau)
\] (23)
2 Problem set up and Adaptive method

The module MARK

We follow the Dorfler marking strategy \[ \theta \leq 1, \]

\[
\sum_{\tau \in M} \eta^2_P(U, \tau) \geq \theta \sum_{\tau \in P} \eta^2_P(U, \tau).
\]

(24)

To ensure minimal cardinality of \( M \) in the marking strategy one typically undergoes QuickSort which has an average complexity of \( \mathcal{O}(n \log n) \) to produce the indexing set \( J \).

The module REFINE

The refinement framework is designed to preserve the structure described in the previous section hinges on extending the marked cells obtained from module MARK to a set \( \omega_{P \rightarrow P_*} \) for which the new mesh \( P_* \) is admissible. We define the neighbourhood of \( \tau \in P \cap \mathcal{G}_\ell \) as

\[
N(P, \tau) := \left\{ \tau' \in P \cap \mathcal{G}_\ell^{-1} : \exists \tau'' \in S(\tau, \ell), \ 0 \leq \tau'' \right\}
\]

when \( \ell - 1 > 0 \), and \( N(P, \tau) = \emptyset \) otherwise. To put in concrete terms, the neighbourhood \( N(P, \tau) \) of an active cell in \( \mathcal{G}^\ell \) consist of active cells \( \tau' \) of level \( \ell - 1 \) overlapping the support extension of \( \tau \) with respect to level \( \ell \). Procedure \texttt{REFINE} will ensure that for a constant

**Algorithm 1** Recursive refinement \( P_* \leftarrow \texttt{recursive_refine} [P, \tau] \)

1: for all \( \tau' \in N(P, \tau) \) do
2: \( P \leftarrow \texttt{recursive_refine} [P, \tau'] \)
3: end for
4: if \( \tau \in P \) then
5: \( \{\tau_j\}_{j=1}^4 \leftarrow \texttt{dyadic-refine} \tau \)
6: \( P_* \leftarrow (P \setminus \tau) \cup \{\tau_j\}_{j=1}^4 \)
7: end if

**Algorithm 2** Carry admissible mesh refinement \( P_* \leftarrow \texttt{mesh_refine} [P, M, m] \)

1: for \( \tau \in M \) do
2: \( P \leftarrow \texttt{recursive_refine} [P, \tau] \)
3: end for
4: \( P_* \leftarrow P \)

c_{\text{shape}} > 0, depending only on the polynomial degree of the spline space, all considered partitions therefore will satisfy the shape-regularity constraints:

\[
\sup_{P \in \mathcal{P}} \max_{\tau \in P} \# \{ \tau \in P : \tau \in \omega_\tau \} \leq c_{\text{shape}} \quad \text{(finite-intersection property),}
\]

\[
\sup_{P \in \mathcal{P}} \max_{\tau \in P} \frac{\text{diam}(\omega_\tau)}{h_\tau} \leq c_{\text{shape}} \quad \text{(graded).}
\]

(25)
For any two partitions $P_1, P_2 \in \mathcal{P}$ there exists a common admissible partition in $\mathcal{P}$, called the overlay and denoted by $P_1 \oplus P_2$, such that
\[
\#(P_1 \oplus P_2) \leq \#P_1 + \#P_2 - \#P_0.
\]
(26)
Moreover, shown in [11], if the sequence $\{P_\ell\}_{\ell \geq 1}$ is obtained by repeating the step $P_{\ell+1} := \text{REFINE} [P_\ell, \mathcal{M}_\ell]$ with $\mathcal{M}_\ell$ any subset of $P_\ell$, then for $k \geq 1$ we have that
\[
\#P_k - \#P_\ell \leq \Lambda \sum_{\ell=1}^{k} \#\mathcal{M}_\ell.
\]
(27)
where $\Lambda > 0$ which will depend on the polynomial degree $r$.

3 A posteriori estimates

We define the residual quantity $\mathcal{R} \in H^{-2}(\Omega)$ by
\[
\langle \mathcal{R}, v \rangle = a(u - U, v), \quad v \in H^2_0(\Omega).
\]
(28)
In view of continuity and coercivity of the bilinear form we readily have sharp a posteriori estimates for $u - U$
\[
C_{\text{cont}}^{-1} \|\mathcal{R}\|_{H^{-2}(\Omega)} \leq \|u - U\| \leq C_{\text{coer}}^{-1} \|\mathcal{R}\|_{H^{-2}(\Omega)}.
\]
(29)
The quantity $\|\mathcal{R}\|_{H^{-2}(\Omega)}$ is computable since it only depends on available discrete approximation of solution $u$. We follow the techniques devised in [28],[2] to approximate $\|\mathcal{R}\|_{H^{-2}(\Omega)}$.

3.1 Approximation in $X_P$

To quantify the approximation power of $X_P$, we use a quasi-interpolant $I_P : L^2(\Omega) \to X_P$; see [8],[7],[6] for the detailed construction of $I_P$. The following theorem summarizes the local approximation properties of $I_P$. Various spline-based quasi-interpolants have been studied extensively and amounts to choosing dual-functionals $\lambda$. A suitable choice for B-spline basis is [6]. To each level $\ell$ we assume we have in hand
\[
I^\ell(v) = \sum_{\beta \in B^\ell} \lambda_\beta(v) \beta, \quad v \in L^2(\Omega)
\]
(30)
such that $I^\ell(s) = s$ for every $s \in S^\ell$. In [7] it is shown that it is sufficient to define $I_P$ with
\[
I_P(v) = \sum_{\ell=0}^{L-1} \sum_{\beta \in B^\ell \cap H_P} \lambda_\beta(v) \text{Trunc}^{\ell+1} \beta, \quad v \in L^2(\Omega)
\]
(31)
with each $\lambda_\beta$ being that of the one in the level-wise interpolant (30).
Theorem 3.1 (Quasi-interpolation). There exists a quasi-interpolation projection operator $I_P : L^2(\Omega) \to X_P$ such that, for a constant $c_{\text{shape}} > 0$, independent of the refinement, and $0 \leq t \leq 2$,
\[
\|I_P v\|_{L^2(\tau)} \leq c_{\text{shape}} \|v\|_{L^2(\omega_\tau)} \quad \text{for } \tau \in P \text{ and } v \in L^2(\omega_\tau)
\] (32)
and the approximation properties
\[
\forall v \in H^2(\omega_\tau), \quad |v - I_P v|_{H^t(\tau)} \leq c_{\text{shape}} h_\tau^{2-t} |v|_{H^2(\omega_\tau)} \quad \forall v \in H^2(\omega_\tau)
\] (33)
and
\[
\forall \sigma \in E_P \cup G_P, \quad |v - I_P v|_{H^t(\sigma)} \leq c_{\text{shape}} h_\sigma^{3/2-t} |v|_{H^2(\omega_\sigma)} \quad \forall v \in H^2(\omega_\sigma)
\] (34)

Recall the general trace theorem \cite{1},\cite{18} for cells $\tau \in P$ and edges $\sigma \in G_P$ with $\sigma \subset \partial \tau$. For a constant $d_0 > 0$
\[
\|v\|_{L^2(\sigma)}^2 \leq d_0 \left( h_\sigma^{-1} \|v\|_{L^2(\tau)}^2 + h_\sigma \|\nabla v\|_{L^2(\tau)}^2 \right) \quad \forall v \in H^1(\Omega). \tag{35}
\]

Lemma 3.2 (Auxiliary discrete estimate). Let $\tau \in P$. Then for $d_1 > 0$, depending only on polynomial degree $r$, for $0 \leq s \leq t \leq r + 1$ we have
\[
|V|_{H^t(\tau)} \leq d_1 h_\tau^{s-t} |V|_{H^s(\tau)} \quad \forall V \in P_r(\tau), \tag{36}
\]
and if $\sigma \subset \partial \tau$, for a constant $d_2 > 0$ we have
\[
\|V\|_{L^2(\sigma)} \leq d_2 h_\sigma^{-1/2} \|V\|_{L^2(\tau)} \quad \forall V \in P_r(\tau), \tag{37}
\]
where $d_2 := d_0 \max\{1,d_1\}$.

Remark 3.3. The constants $d_1$, $d_0$, $d_2$ all depend on the polynomial degree and the reference cell or edge; $\hat{\tau} = [0,1]^2$ or $\hat{\sigma} = [0,1]$. From now, for a simpler presentation of the analysis, we combined all these constants, and their powers into a unifying constant $c_*$.

3.2 The global upper bound

We prove that the proposed error estimator is reliable.

Lemma 3.4 (Estimator reliability). Let $P$ be a partition of $\Omega$ satisfying Conditions (25). The module \texttt{ESTIMATE} produces a posteriori error estimate $\eta_P$ for the discrete error such that for a constants $C_{\text{rel}} > 0$,
\[
\|u - U\|^2 \leq C_{\text{rel}} \eta_P^2(U, \Omega) \tag{38}
\]
with constants depending only on $c_{\text{shape}}$. 
Proof. In this proof we will derive a localized quantification for the residual $\mathcal{R}$. In view of definition (28) we follow standard procedure and integrate by parts to obtain

$$\langle \mathcal{R}, v \rangle = a(u - U, v) \quad \forall v \in H^2_0(\Omega)$$

We will derive a localized quantification for the residual $\mathcal{R}_P$ which will provide a sharp upper-bound estimate for residual.

$$\langle \mathcal{R}, v \rangle = \sum_{\tau \in P} \left( \int_{\tau} (f - \mathcal{L}U) v + \int_{\partial \tau} \Delta U \frac{\partial v}{\partial n_{\tau}} - \int_{\partial \tau} \frac{\partial \Delta U}{\partial n_{\tau}} v \right)$$

(39)

Expressing all the integrals over cell boundaries as integrals over edges,

$$\sum_{\tau \in P} \left( \int_{\partial \tau} \Delta U \frac{\partial v}{\partial n_{\tau}} - \int_{\partial \tau} \frac{\partial \Delta U}{\partial n_{\tau}} v \right) = \sum_{\sigma \in \mathcal{E}_P} \left( \int_{\sigma} J_{\sigma} \left( \frac{\partial \Delta U}{\partial n_{\sigma}} \right) v - \int_{\sigma} J_{\sigma} (\Delta U) \frac{\partial v}{\partial n_{\sigma}} \right).$$

(40)

We have

$$\|\langle \mathcal{R}, v \rangle\| \leq \sum_{\tau \in P} \|f - \mathcal{L}U\|_{L^2(\tau)} \|v|_{L^2(\tau)} + \sum_{\sigma \in \mathcal{E}_P} \left\| J_{\sigma} \left( \frac{\partial \Delta U}{\partial n_{\sigma}} \right) \right\|_{L^2(\sigma)} \|(v - \mathcal{I}_P v)\|_{L^2(\sigma)} \leq \sum_{\tau \in P} \|R\|_{L^2(\tau)} \|v|_{L^2(\tau)} + \sum_{\sigma \in \mathcal{E}_P} \|J_{\sigma} (\Delta U)\|_{L^2(\sigma)} \left\| \frac{\partial \Delta U}{\partial n_{\sigma}} \right\|_{L^2(\sigma)}.$$

(41)

We define the interior residual $R = f - \mathcal{L}U$ and jump terms $J_1 = J_{\sigma} \left( \frac{\partial \Delta U}{\partial n_{\sigma}} \right)$ and $J_2 = J_{\sigma} (\Delta U)$ across each interior edge $\sigma$. Starting with the first three terms in (41), we use the approximation results from Lemma 3.2 to estimate interior residual terms

$$\sum_{\tau \in P} \|R\|_{L^2(\tau)} \|v - \mathcal{I}_P v\|_{L^2(\tau)} \leq \sum_{\tau \in P} \|R\|_{L^2(\tau)} c_1 h_{\tau}^2 \|v|_{L^2(\tau)} \leq c_1 \left( \sum_{\tau \in P} h_{\tau}^4 \|R\|_{L^2(\tau)}^4 \right)^{1/2} \left( \sum_{\tau \in P} \|v|_{H^2(\omega_{\tau})}^2 \right)^{1/2}.$$ 

(42)

As for the interior edge jump terms,

$$\sum_{\sigma \in \mathcal{E}_P} \|J_1\|_{L^2(\sigma)} \|(v - \mathcal{I}_P v)\|_{L^2(\sigma)} \leq \sum_{\sigma \in \mathcal{E}_P} \|J_1\|_{L^2(\sigma)} c_1 h_{\sigma}^{3/2} \|v|_{L^2(\omega_{\sigma})} \leq c_1 \left( \sum_{\sigma \in \mathcal{E}_P} h_{\sigma}^3 \|J_1\|_{L^2(\sigma)}^2 \right)^{1/2} \left( \sum_{\sigma \in \mathcal{E}_P} \|v|_{H^2(\omega_{\sigma})}^2 \right)^{1/2}.$$ 

(43)
and
\[
\sum_{\sigma \in E_P} \|J_2\|_{L^2(\sigma)} \left\| \frac{\partial}{\partial n_\sigma} (v - I_P v) \right\|_{L^2(\sigma)} \leq \sum_{\sigma \in E_P} \|J_2\|_{L^2(\sigma)} c_1 h_\sigma^{1/2} \|v\|_{L^2(\omega_\sigma)},
\]
\[
\leq c_1 \left( \sum_{\sigma \in E_P} h_\sigma \|J_2\|_{L^2(\sigma)}^2 \right)^{1/2} \left( \sum_{\sigma \in E} \|v\|_{H^2(\omega_\sigma)}^2 \right)^{1/2} .
\]
(44)

From the finite-intersection property (25) we have \( \sum_{\sigma \in \mathcal{E}} \|v\|_{H^2(\omega_\sigma)}^2 \leq c_{\text{shape}} \|v\|_{H^2(\Omega)}^2 \) and using (25) we have \( \sum_{\sigma \in \mathcal{E}} \|v\|_{H^2(\tau(\sigma))}^2 \leq c_{\text{shape}} \|v\|_{H^2(\Omega)}^2 \). Summing up we arrive at
\[
\| \langle \mathcal{R}, v \rangle \| \leq c_1 c_{\text{shape}} \left\{ \left( \sum_{\tau \in P} h_\tau^4 \|R\|_{L^2(\tau)}^2 \right)^{1/2} + \left( \sum_{\sigma \in E_P} h_\sigma^3 \|J_1\|_{L^2(\sigma)}^2 \right)^{1/2} \right\} \|v\|_{H^2(\Omega)}
\]
(45)

3.3 Global lower bound

We will define extension operators \( E_\sigma : C(\sigma) \to C(\tau) \) for all edges \( \sigma \subset \partial \tau \). Let \( \hat{\tau} = [0, 1] \times [0, 1] \) and \( \hat{\sigma} = [0, 1] \times \{0\} \). Let \( F_\tau : \mathbb{R}^2 \to \mathbb{R}^2 \) be the affine transformation comprising of translation and scaling mapping \( \hat{\tau} \) onto \( \tau \) and \( \hat{\sigma} \) onto \( \sigma \). Define \( \hat{E} : C(\hat{\sigma}) \to C(\hat{\tau}) \) via
\[
\hat{E} v(x, y) = v(x) \quad \forall x \in \hat{\sigma}, \quad (x, y) \in \hat{\tau}, \quad v \in C(\hat{\sigma}).
\]
(46)

To this end, let \( \sigma \) be an edge of a cell \( \tau \in P \), then define \( E_\sigma : C(\sigma) \to C(\tau) \) via
\[
E_\sigma v := \hat{E}(v \circ F_\tau) \circ F_\tau^{-1}.
\]
(47)

In other words extending the values of \( v \) from \( \sigma \) into \( \tau \) along inward \( n_\sigma \). Let \( \psi_\tau \) be any smooth cut-off function with the following properties:
\[
\text{supp} \psi_\tau \subseteq \tau, \quad \psi_\tau \geq 0, \quad \max_{x \in \tau} \psi_\tau(x) \leq 1.
\]
(48)

Furthermore, we will define for \( \sigma = \partial \tau_1 \cap \partial \tau_2 =: D_\sigma \) two \( C^1 \) cut-off functions \( \psi_\sigma \) and \( \chi_\sigma \) via the following
\[
\text{supp} \chi_\sigma \subseteq D_\sigma, \quad \text{supp} \psi_\sigma \subseteq D_\sigma, \quad \psi_\tau \geq 0, \quad \max_{x \in \tau} \psi_\tau(x) \leq 1,
\]
(49)

such that
\[
\frac{\partial \psi_\sigma}{\partial n_\sigma} \equiv 0, \quad \chi_\sigma \equiv 0 \quad \text{and} \quad d_1 h_\sigma^{-1} \psi_\sigma \leq \frac{\partial \psi_\sigma}{\partial n_\sigma} \leq d_2 h_\sigma^{-1} \psi_\sigma \quad \text{along} \ \sigma.
\]
(50)
Let $\hat{\sigma}$ and $\hat{\tau}$ retain the same meanings as before. Let $\hat{\tau}_1 = \tau$ and let $\hat{\tau}_2 = \{(x,y) \in \mathbb{R}^2 : (x,-y) \in \hat{\tau}\}$ and let $\textbf{n} = (0,-1)$. Now let $F_\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the affine map that maps $\hat{D}$ onto $D_\sigma$ and define
\[
\psi_\sigma = \hat{\psi} \circ F_\sigma^{-1}, \quad \chi_\sigma = \hat{\chi} \circ F_\sigma^{-1}.
\] (51)

unit normal vector on $\hat{\sigma}$. Let $g(y)$ be the cubic polynomial satisfying
\[
g^{(4)}(y) = 0 \text{ for } y \in (0,1) \text{ such that } g'(0) = g(1) = g'(1) = 0 \text{ and } g(0) = 1. \tag{52}
\]

Put $\phi(y) = g(y)1_{[0,1]} + g(-y)1_{[-1,0]}$. Now define $G$ to be the quartic polynomial satisfying
\[
G^{(5)}(y) = 0 \text{ for } y \in (0,1) \text{ such that } G(0) = G(1) = G'(1) = 0 \text{ and } G'(0) = \phi(0). \tag{53}
\]

Put $\eta(y) = G(y)1_{[0,1]} - G(-y)1_{[-1,0]}$. Finally, let $H = x^2(1-x)^2(1-y)^2$ and let $\Phi(x,y) = H(x,y)1_{y \geq 0} + H(x,-y)1_{y \leq 0}$. Finally set
\[
\hat{\psi}(x,y) = \Phi(x,y)\phi(y), \quad \hat{\chi}(x,y) = \Phi(x,y)\eta(y), \quad (x,y) \in \hat{D}. \tag{54}
\]

**Lemma 3.5** (Localizing estimates). Let $P$ be a partition of $\Omega$ satisfying Conditions (25). Let $\tau$ be a cell in partition $P$. For a constant $c_m > 0$ depending only on polynomial degree $m$,
\[
\|q\|^2_{L^2(\tau)} \leq d_4 \int_\tau \psi_\tau q^2 \quad \forall q \in \mathbb{P}_m(\tau), \tag{55}
\]

Let $\sigma$ be an edge in $E_P$ and let $\tau_1$ and $\tau_2$ be cells from $P$ for which $\sigma \subseteq \overline{\tau_1} \cap \overline{\tau_2}$. We also have
\[
\|q\|^2_{L^2(\sigma)} \leq d_4 \int_\sigma \psi_\sigma q^2 \tag{56}
\]

and
\[
\|\psi_\sigma^{1/2}E_\sigma q\|_{L^2(\tau)} \leq d_4 h_\sigma^{1/2}\|q\|_{L^2(\sigma)} \quad \forall \tau \in D_\sigma \tag{57}
\]

holding for every $q \in \mathbb{P}_m(\sigma)$.

**Proof.** Relations (55) and (56) are proven in the same fashion as in [2], [28]. We focus on (57). We prove that $q \mapsto \|\psi_\sigma^{1/2}E_\sigma q\|_{L^2(\tau)}$ is a norm on $\mathbb{P}_m(\sigma)$.
\[
\|E_\sigma q\|_{L^2(\tau)} = | \tau |^{1/2}\|\hat{E}(q \circ F_\tau)\|_{L^2(\hat{\tau})}. \tag{58}
\]

It clear that $\hat{q} \in \mathbb{P}_m(\hat{\sigma})$ is identically zero if and only if its extension $\hat{E}\hat{q}$ is identically zero on $\hat{\tau}$. So $\hat{q} \mapsto \|\hat{E}\hat{q}\|_{L^2(\hat{\tau})}$ is an equivalent norm on $\mathbb{P}_m(\hat{\sigma})$ we have
\[
\|\hat{E}(q \circ F_\tau)\|_{L^2(\hat{\tau})} \leq c\|q \circ F_\tau\|_{L^2(\hat{\sigma})} = c h_\sigma^{-1/2}\|q\|_{L^2(\sigma)} \tag{59}
\]

so with $c|\tau|^{1/2}h_\sigma^{1/2} \leq \overline{\tau}h_\sigma^{1/2}$
\[
\|\psi_\sigma^{1/2}E_\sigma q\|_{L^2(\tau)} \leq \overline{\tau}h_\sigma^{1/2}\|q\|_{L^2(\sigma)}. \tag{60}
\]
Lemma 3.6 (Estimator efficiency). Let $P$ be a partition of $\Omega$ satisfying conditions (25). The module ESTIMATE provides a posteriori error estimate of the discrete solution error such that

$$C_{\text{eff}} \eta_P^2(U, \Omega) \leq \|u - U\|^2 + \text{osc}_P^2(\Omega).$$

with constant $C_{\text{eff}}$ depending only on $c_{\text{shape}}$.

Proof. The proof is carried by localizing the error contributions coming from the cells residuals $R$ and edge jumps $J_1$ and $J_2$. For $\tau \in P$ let $\psi_\tau \in H^1_s(\tau)$ be as in (48) and let $\mathbf{R}$ be a polynomial approximation of $\mathbf{R}|_\tau$ by means of an $L^2$-orthogonal projection. Using the norm-equivalence relation (55) of Lemma 3.5

$$\|\mathbf{R}\|_{L^2(\tau)}^2 \leq d_4 \int_\tau \mathbf{R}(\mathbf{R}_{\psi_\tau}) \leq d_4 \|\mathbf{R}\|_{H^{-2}(\tau)} \|\mathbf{R}_{\psi_\tau}\|_{H^2(\tau)}.$$  \hfill (62)

From (36) and (48), $\|\mathbf{R}_{\psi_\tau}\|_{H^2(\tau)} \leq d_4 h_{\tau}^2 \|\mathbf{R}\|_{L^2(\tau)}$ with a constant $d_4 > 0$ which depends on the polynomial degree $r$ and (62) now reads $h_{\tau}^2 \|\mathbf{R}\|_{L^2(\tau)} \leq d_4 \|\mathbf{R}\|_{H^{-2}(\tau)}$. Then

$$h_{\tau}^2 \|R\|_{L^2(\tau)} \leq h_{\tau}^2 \|\mathbf{R}\|_{L^2(\tau)} + h_{\tau}^2 \|R - \mathbf{R}\|_{L^2(\tau)},
\leq d_4 \|R\|_{H^{-2}(\tau)} + h_{\tau}^2 \|R - \mathbf{R}\|_{L^2(\tau)},
\leq d_4 \|\mathbf{R}\|_{H^{-2}(\tau)} + d_4 \|R - \mathbf{R}\|_{H^{-2}(\tau)} + h_{\tau}^2 \|R - \mathbf{R}\|_{L^2(\tau)},
\leq d_4 \|\mathbf{R}\|_{H^{-2}(\tau)} + h_{\tau}^2 \|R - \mathbf{R}\|_{L^2(\tau)},$$

Recognizing that $R - \mathbf{R} = f - \mathbf{f}$, define $\text{osc}_P(\tau) := h_{\tau}^2 \|f - \mathbf{f}\|_{L^2(\tau)}$. We turn our attention to the jump terms across the interior edges. We begin with the edge residual $J_1$. Let an edge $\sigma \in \mathcal{E}_P$ and cells $\tau_1, \tau_2 \in P$ be such that $\sigma \subset \partial \tau_1 \cap \partial \tau_2$ and denote $D_{\sigma} = \overline{\tau_1} \cup \overline{\tau_2}$. If $v \in H^1_0(D_{\sigma})$ then

$$\langle \mathcal{R}, v \rangle = \int_{D_{\sigma}} \mathbf{R} v + \int_{\sigma} J_1 v - \int_{\sigma} J_2 \frac{\partial v}{\partial \mathbf{n}_{\sigma}}.$$  \hfill (64)

Let $\psi_{\sigma}$ be the bubble function (51) and constantly extend the values of $J_1$ in directions $\pm \mathbf{n}_{\sigma}$; i.e., into each of $\tau_i$, and set $v_{\sigma} = \psi_{\sigma} J_1$. Then (64) reads

$$d_4 \|J_1\|_{L^2(\sigma)}^2 \leq \int_{\sigma} \psi_{\sigma} J_1^2 = \langle \mathcal{R}, v_{\sigma} \rangle - \int_{D_{\sigma}} R v_{\sigma}.$$  \hfill (65)

From (37) and (51) we have the estimates $\|\psi_{\sigma} E_{\sigma} J_1\|_{H^2(D_{\sigma})} \leq d_1 h_{\sigma}^{-2} \|E_{\sigma} J_1\|_{L^2(D_{\sigma})}$ and $\|\psi_{\sigma} E_{\sigma} J_1\|_{L^2(D_{\sigma})} \leq d_4 h_{\sigma}^{1/2} \|J_1\|_{L^2(\sigma)}$ which we apply to (65) to obtain

$$d_4 \|J_1\|_{L^2(\sigma)}^2 \leq (d_1 h_{\sigma}^{-2} \|\mathcal{R}\|_{H^{-2}(D_{\sigma})} + \|R\|_{L^2(\sigma)}) \|J_1\|_{L^2(\sigma)},
\leq d_2 h_{\sigma}^{1/2} (d_1 h_{\sigma}^{-2} \|\mathcal{R}\|_{H^{-2}(D_{\sigma})} + \|R\|_{L^2(\sigma)}) \|J_1\|_{L^2(\sigma)},$$

(66)
where the last line follows from (57). Now let \( \chi_\sigma \) be the function (56), extend the values of \( J_2 \) into \( D_\sigma \) and set \( w_\sigma = \chi_\sigma J_2 \). We then have
\[
\langle \mathcal{R}, w_\sigma \rangle = \int_{D_\sigma} R w_\sigma - h_\sigma^{-1} \int_\sigma \psi_\sigma J_2^2.
\] (67)

Similarly, we obtain
\[
d_4 h_\sigma^{-1} \| J_2 \|_{L^2(\sigma)}^2 \leq d_2 h_\sigma^{1/2} \left( d_1 h_\sigma^{-2} \| \mathcal{R} \|_{H^{-2}(D_\sigma)} + \| R \|_{L^2(D_\sigma)} \right) \| J_2 \|_{L^2(\sigma)}.
\] (68)

We have from (66) and (68)
\[
h_\sigma^3 \| J_1 \|_{L^2(\sigma)}^2 + h_\sigma \| J_2 \|_{L^2(\sigma)}^2 \leq \frac{d d_4}{d_2} \| \mathcal{R} \|_{H^{-2}(D_\sigma)} + \frac{d_4}{d_2} h_\sigma^4 \| R \|_{L^2(D_\sigma)}^2.
\] (69)

Summing up we have
\[
\eta_P^2(V, \tau) = h_\sigma^4 \| R \|_{L^2(\tau)}^2 + \sum_{\sigma \in \partial \tau} \left( h_\sigma^3 \| J_1 \|_{L^2(\sigma)}^2 + h_\sigma \| J_2 \|_{L^2(\sigma)}^2 \right),
\]
\[
\leq h_\sigma^4 \| R \|_{L^2(\tau)}^2 + \frac{d_4}{d_2} \sum_{\sigma \in \partial \tau} \left( d_1 \| \mathcal{R} \|_{H^{-2}(D_\sigma)} + h_\sigma \| R \|_{L^2(D_\sigma)}^2 \right),
\]
\[
\leq (1 + c_{\text{shape}}) h_\sigma^4 \| R \|_{L^2(\omega_\tau)}^2 + \frac{d d_4}{d_2} \sum_{\sigma \in \partial \tau} \| \mathcal{R} \|_{H^{-2}(D_\sigma)}^2.
\] (70)

we arrive at
\[
\eta_P^2(V, \tau) \leq (1 + c_{\text{shape}}) \left( d_1 \| \mathcal{R} \|_{H^{-2}(\tau)} + \text{osc}_P^2(\omega_\tau) \right) + \frac{d d_4}{d_2} \sum_{\sigma \in \partial \tau} \| \mathcal{R} \|_{H^{-2}(D_\sigma)}^2.
\] (71)

Note that
\[
\sum_{\sigma \in \partial \tau} \| \mathcal{R} \|_{H^{-2}(D_\sigma)}^2 \leq C_{\text{cont}} \sum_{\sigma \in \partial \tau} \| u - U \|_{H^2(D_\sigma)}^2 \leq c_{\text{shape}} C_{\text{cont}} \| u - U \|_{H^2(\Omega)}^2
\] (72)

### 3.4 Discrete upper bound

Here we show that the estimator is capable of local quantification of the difference between two consecutive discrete spline solutions.

**Lemma 3.7 (Estimator discrete reliability).** Let \( P \) be a partition of \( \Omega \) satisfying conditions (25) and let \( P_* = \text{REFINE}[P, R] \) for some refined set \( R \subseteq P \). If \( U \) and \( U_* \) are the respective solutions to (17) on \( P \) and \( P_* \), then for a constants \( C_{\text{Rel,1}}, C_{\text{Rel,2}} > 0 \), depending only on \( c_{\text{shape}} \),
\[
\| U_* - U \|_2^2 \leq C_{\text{Rel,1}} \eta_P^2(U, \omega_{R \rightarrow P_*})
\] (73)
where \( \omega_{R \rightarrow P_*} \) is understood as the union of support extensions of refined cells from \( P \) to obtain \( P_* \).
Proof. Let \( e_* = U_* - U \). First note that if \( V \in X_P \) then in view of the nesting \( X_P \subset X_{P_*} \),
\[
a(U_* - U, e_*) = a(U_* - U, e_* - V).
\] (74)

To localize, we form disconnected subdomains \( \Omega_i \subseteq \Omega \), \( i \in J \), each formed from the interiors of connected components of \( \Omega_* = \bigcup_{\tau \in R_{P \rightarrow P_*}} \tau \). Then to each subdomain \( \Omega_i \) we form a partition \( P_i = \{ \tau \in P : \tau \subset \Omega_i \} \), interior edges \( E_i = \{ \sigma \in E_P : \sigma \subset \partial \tau, \tau \in P_i \} \), and a corresponding finite-element space \( X_i \). Let \( I_i : H^2(\Omega_i) \rightarrow X_i \). Let \( V \in X_P \) be an approximation of \( e_* \) be given by
\[
V = e_* 1_{\Omega \setminus \Omega_*} + \sum_{i \in J} (I_i e_*) \cdot 1_{\Omega_i}. \] (75)

Then \( e_* - V \equiv 0 \) on \( \Omega \setminus \Omega_* \) and performing integration by parts will yield
\[
a(U_* - U, e_* - V) = \sum_{i \in J} \left[ \sum_{\tau \in P_i} \langle R, e_* - I_i e_* \rangle_\tau + \sum_{\sigma \in E_i} \{ \langle J_1, e_* - I_i e_* \rangle_\sigma + \langle J_2, e_* - I_i e_* \rangle_\sigma \} \right],
\] (76)

Following the same procedure carried in Lemma 3.4 we have
\[
\sum_{\tau \in P_i} \langle R, e_* - I_i e_* \rangle_\tau + \sum_{\sigma \in E_i} \{ \langle J_1, e_* - I_i e_* \rangle_\sigma + \langle J_2, e_* - I_i e_* \rangle_\sigma \} \leq c_1 \left( \sum_{\tau \in P_i} \eta_P^2(U, \tau) \right)^{1/2} \left( \sum_{\tau \in P_i} \| e_* \|^2_{H^2(\omega_\tau)} \right)^{1/2} \] (77)

Set \( \omega_{R_{P \rightarrow P_*}} = \bigcup \{ \omega_\tau : \tau \in R_{P \rightarrow P_*} \} \). We therefore have
\[
\| U_* - U \|^2 \leq C_{\text{dRel}} \eta_P(U, \omega_{R_{P \rightarrow P_*}}) \| e_* \|_{H^2(\Omega)} \] (78)

4 Convergence

In section we show that the derived computable estimator (22) when used to direct refinement will result in decreased error. This will hinge on the estimator Lipschitz property of Lemma 4.1. To show that procedure (2) exhibits convergence we must be able to relate the errors of consecutive discrete solutions. The symmetry of the bilinear form, consistency of the formulation and finite-element spline space nesting will readily provide that via Galerkin Pythagoras in Lemma 4.3.
4.1 Error reduction

**Lemma 4.1** (Estimator Lipschitz property). Let \( P \) be a partition of \( \Omega \) satisfying conditions (25). There exists a constant \( C_{\text{lip}} > 0 \), depending only on \( c_{\text{shape}} \), such that for any cell \( \tau \in P \) we have

\[
|\eta_P(V, \tau) - \eta_P(W, \tau)| \leq C_{\text{lip}}|V - W|_{H^2(\omega_\tau)},
\]

holding for every pair of finite-element splines \( V \) and \( W \) in \( \mathcal{X}_P \).

**Proof.** Let \( V \) and \( W \) be finite-element splines in \( \mathcal{X}_P \) and let \( \tau \) be a cell in partition \( P \).

\[
\eta_P(V, \tau) - \eta_P(W, \tau) = h_\tau^2 (\|f - \mathcal{L}V\|_{L^2(\tau)} - \|f - \mathcal{L}W\|_{L^2(\tau)}) + \sum_{\sigma \subset \partial \tau} h_\sigma^{1/2} (\|\mathcal{J}_\sigma (\Delta V)\|_{L^2(\sigma)} - \|\mathcal{J}_\sigma (\Delta W)\|_{L^2(\sigma)}) + \sum_{\sigma \subset \partial \tau} h_\sigma^{3/2} (\|\mathcal{J}_\sigma (\partial_\sigma \Delta V)\|_{L^2(\sigma)} - \|\mathcal{J}_\sigma (\partial_\sigma \Delta W)\|_{L^2(\sigma)}).
\]

Treating the interior term,

\[
\|f - \mathcal{L}V\|_{L^2(\tau)} - \|f - \mathcal{L}W\|_{L^2(\tau)} \leq |V - W|_{H^4(\tau)} \leq d_1 h_\tau^{-2} |V - W|_{H^2(\tau)},
\]

Treating the edge terms we have

\[
\|\mathcal{J}_\sigma (\Delta V)\|_{L^2(\sigma)} - \|\mathcal{J}_\sigma (\Delta W)\|_{L^2(\sigma)} \leq \|\mathcal{J}_\sigma (\Delta V - \Delta W)\|_{L^2(\sigma)}.
\]

Let \( \tau' \) from \( P \) be a cell that shares the edge \( \sigma \), i.e \( \tau' \) is an adjacent cell to \( \tau \). For any finite-element spline \( V \in \mathcal{X}_P \) we have

\[
\|\mathcal{J}_\sigma (V)\|_{\sigma} \leq d_2 \left( h_\sigma^{-1/2} \|V\|_{\tau} + h_\sigma^{-1/2} \|V\|_{\tau'} \right) \leq d_2 h_\sigma^{-1/2} \|V\|_{\omega_\tau}.
\]

Replacing \( V \) with \( \Delta V - \Delta W \) gives

\[
h_\sigma^{1/2} \|\mathcal{J}_\sigma (\Delta V - \Delta W)\|_{\sigma} \leq d_1 d_2 |V - W|_{H^2(\omega_\tau)}.
\]

Similarly, we have

\[
h_\sigma^{3/2} (\|\mathcal{J}_\sigma (\partial_\sigma \Delta V)\|_{L^2(\sigma)} - \|\mathcal{J}_\sigma (\partial_\sigma \Delta W)\|_{L^2(\sigma)}) \leq d_1 d_2 |V - W|_{H^2(\omega_\tau)}.
\]

It then follows from (80)

\[
|\eta_P(V, \tau) - \eta_P(W, \tau)| \leq d_1 (|V - W|_{H^2(\tau)} + 2 d_2 |V - W|_{H^2(\omega_\tau)}) \leq d_1 (1 + 2 d_2) |V - W|_{H^2(\omega_\tau)}.
\]
Lemma 4.2 (Estimator error reduction). Let \( P \) be a partition of \( \Omega \) satisfying conditions (25), let \( \mathcal{M} \subseteq P \) and let \( P_* = \text{REFINE}[P,\mathcal{M}] \). There exists constants \( \lambda \in (0, 1) \) and \( C_{\text{est}} > 0 \), depending only on \( c_{\text{shape}} \), such that for any \( \delta > 0 \) it holds that for any pair of finite-element splines \( V \in \mathbb{X}_P \) and \( V_* \in \mathbb{X}_{P_*} \) we have

\[
\eta^2_{P_*}(V, \Omega) \leq (1 + \delta) \left\{ \eta^2_P(V, \Omega) - \frac{1}{2} \eta^2_P(V, \mathcal{M}) \right\} + c_{\text{shape}}(1 + \frac{1}{\delta})\|V - V_*\|^2. \tag{87}
\]

Proof. Let \( \mathcal{M} \subseteq P \) be a set of marked elements from partition \( P \) and let \( P_* = \text{REFINE}[P,\mathcal{M}] \).

For notational simplicity we denote \( \mathbb{X}_{P_*} \) and \( \eta_{P_*} \) by \( \mathbb{X}_s \) and \( \eta_s \), respectively. Let \( V \) and \( V_* \) be the respective finite-element splines from \( \mathbb{X}_P \) and \( \mathbb{X}_s \). Let \( \tau \) be a cell from partition \( P_* \). In view of the Lipschitz property of the estimator (Lemma 4.1) and the nesting \( \mathbb{X}_P \subseteq \mathbb{X}_s \),

\[
\eta^2_s(V, \tau) \leq \eta^2_s(V, \tau) + |V - V_*|^2_{H^2(\omega_\tau)} + 2\eta_s(V, \tau)|V - V_*|^2_{H^2(\omega_\tau)}. \tag{88}
\]

Given any \( \delta > 0 \), an application of Young’s inequality on the last term gives

\[
2\eta_s(V, \tau)|V - V_*|^2_{H^2(\omega_\tau)} \leq \delta \eta^2_s(V, \tau) + \frac{1}{\delta}\|V - V_*\|^2_{H^2(\omega_\tau)}. \tag{89}
\]

We now have

\[
\eta^2_s(V, \tau) \leq (1 + \delta)\eta^2_s(V, \tau) + (1 + \frac{1}{\delta})\|V - V_*\|^2_{H^2(\omega_\tau)}. \tag{90}
\]

Recalling that the partition cell are disjoint with uniformly bounded support extensions, we may sum over all the cells \( \tau \in P_* \) to obtain

\[
\eta^2_s(V, P_*) \leq (1 + \delta)\eta^2_s(V, P_*). \tag{91}
\]

It remains to estimate \( \eta^2_s(V, P_*) \). Let \( |\mathcal{M}| \) be the sum areas of all cells in \( \mathcal{M} \). For every marked element \( \tau \in \mathcal{M} \) define \( P_{*,\mathcal{M}} = \{ \text{child}(\tau) : \tau \in \mathcal{M} \} \). Let \( b > 0 \) denote the number of bisections required to obtain the conforming partition \( P_* \) from \( P \). Let \( \tau_b \) be a child of a cell \( \tau \in \mathcal{M} \). Then \( h_{\tau_b} \leq 2^{-1}h_{\tau} \). Noting that \( V \in \mathbb{X}_P \) we have no jumps within \( \tau \)

\[
\eta^2_s(V, \tau_b) = h^4_{\tau_b}\|f - \mathcal{L}V\|^2_{s_{\tau_b}} \leq (2^{-1}h_{\tau})^4\|f - \mathcal{L}V\|^2_{s_{\tau}}, \tag{92}
\]

summing over all children

\[
\sum_{\tau_b \in \text{children}(\tau)} \eta^2_s(V, \tau_b) \leq 2^{-1}\eta^2_s(V, \tau), \tag{93}
\]

and we obtain by disjointness of partitions an estimate on the error reduction

\[
\sum_{\tau_b \in P_{*,\mathcal{M}}} \eta^2_s(V, \tau_b) \leq 2^{-1}\eta^2_s(V, \mathcal{M}). \tag{94}
\]

For the remaining cells \( T \in P \setminus \mathcal{M} \), the estimator monotonicity implies \( \eta_{P_*}(V, T) \leq \eta_{P}(V, T) \). Decompose the partition \( P \) as union of marked cells in \( \mathcal{M} \) and their complement \( P \setminus \mathcal{M} \).
to conclude the total error reduction obtained by \textbf{REFINE} and the choice of Dorfler parameter $\theta$

$$
\eta^2_P(V, \Omega) \leq \eta^2_P(V, \Omega \setminus \mathcal{M}) + 2^{-1} \eta^2_P(V, \mathcal{M}) = \eta^2_P(V, \Omega) - \frac{1}{2} \eta^2_P(V, \mathcal{M}).
$$

(95)

\begin{proof}
At first we express

$$
a(u - U_s, u - U_s) = a(u - U, u - U) - a(U_s - U, U_s - U) + a(U - U_s, u - U_s) + a(u - U_s, U - U_s).
$$

(97)

Recognizing that $a(u - U_s, U - U_s) = a(U - U_s, u - U) = 0$, we arrive at

$$
a(u - U_s, u - U_s) = a(u - U, u - U) - a(U_s - U, U_s - U).
$$

(98)

\end{proof}

\begin{lemma}[Galerkin Pythagoras]
Let $P$ and $P_s$ be partitions of $\Omega$ satisfying conditions (25) with $P_s \geq P$ and let $U \in \mathcal{X}_P$ and $U_s \in \mathcal{X}_P$ be the spline solutions to (17). Then

$$
\|u - U_s\|^2 = \|u - U\|^2 - \|U_s - U\|^2.
$$

(96)

\end{lemma}

\begin{proof}
At first we express

$$
a(u - U_s, u - U_s) = a(u - U, u - U) - a(U_s - U, U_s - U) + a(U - U_s, u - U_s) + a(u - U_s, U - U_s).
$$

(97)

Recognizing that $a(u - U_s, U - U_s) = a(U - U_s, u - U) = 0$, we arrive at

$$
a(u - U_s, u - U_s) = a(u - U, u - U) - a(U_s - U, U_s - U).
$$

(98)

\end{proof}

\begin{theorem}[Convergence of conforming AFEM]
For a contractive factor $\alpha \in (0, 1)$ and a constant $C_{\text{est}} > 0$, given any successive mesh partitions $P$ and $P_s$ satisfying conditions (25), $f \in L^2(\Omega)$ and Dorfler parameter $\theta \in (0, 1]$, the adaptive procedure $\textbf{AFEM}[P, f, \theta]$ with produce two successive solutions $U \in \mathcal{X}_P$ and $U_s \in \mathcal{X}_{P_s}$ to problem (17) for which

$$
\|u - U_s\|^2 + C_{\text{est}} \eta^2_P(U_s, \Omega) \leq \alpha(\|u - U\|^2 + C_{\text{est}} \eta^2_P(U, \Omega)).
$$

(99)

\end{theorem}

\begin{proof}
Adopt the following abbreviations:

$$
e_P = \|u - U\|, \quad E_s = \|U_s - U\|,
\eta_P = \eta_P(U, \Omega), \quad \eta_P(\mathcal{M}) = \eta_P(U, \mathcal{M}).
$$

(100)

Define constants $q_{\text{est}}(\theta, \delta) := (1 + \delta)(1 - \frac{\theta^2}{2})$ and $C_{\text{est}}^{-1} := e_{\text{shape}}(1 + \frac{1}{\theta})$ so that in view of Dorfler $-\eta^2_P(\mathcal{M}) \leq -\theta^2 \eta^2_P$,

$$
(1 + \delta) \left\{ \eta^2_P(\Omega) - \frac{1}{2} \eta^2_P(\mathcal{M}) \right\} \leq q_{\text{est}} \eta^2_P,
$$

(102)

and (87) reads $\eta^2_{P_s} \leq q_{\text{est}} \eta^2_P + C_{\text{est}}^{-1} E^2_s$. Together with Galerkin orthogonality,

$$
e^2_P + C_{\text{est}} \eta^2_P \leq e^2_P - E^2_s + C_{\text{est}} \left( q_{\text{est}} \eta^2_P + C_{\text{est}}^{-1} E^2_s \right) = e^2_P + q_{\text{est}} C_{\text{est}} \eta^2_P
$$

(103)
Let \( \alpha \) be a positive parameter and express \( e_P^2 = \alpha e_P^2 + (1 - \alpha) e_P^2 \). Invoking on the reliability estimate \( e_P^2 \leq C_{\text{rel}} \eta_P^2 \) on one of the decomposed terms gives

\[
e_P^2 + C_{\text{est}} \eta_P^2 \leq \alpha e_P^2 + [(1 - \alpha) C_{\text{rel}} + q_{\text{est}} C_{\text{est}}] \eta_P^2.
\]

Choose \( \delta > 0 \) with \( \delta < \frac{\theta^2}{2 - \theta^2} \) so that \( q_{\text{est}} \in (0, 1) \) and we may choose a contractive \( \alpha < 1 \) for which \((1 - \alpha) C_{\text{rel}} + q_{\text{est}} C_{\text{est}} = \alpha C_{\text{est}}\). Indeed,

\[
\alpha = q_{\text{est}} C_{\text{est}} + C_{\text{rel}} C_{\text{est}} < 1.
\]

**Remark 4.5.** Observe that if \( C(\delta) := C_{\text{rel}} / C_{\text{est}}(\delta) \),

\[
\alpha = \frac{(1 + \delta)(1 - \frac{\theta^2}{2}) + C}{1 + C} = 1 - \frac{1 + \delta}{1 + C} \left( \frac{\theta^2}{2} - \frac{\delta}{1 + \delta} \right),
\]

and \((1 + \delta)/(1 + C) \to 0\) as \( \delta \to 0 \). Then it is clear that contractive factor \( \alpha \) deteriorates as \( \theta \to 0 \). We have \( \alpha \approx 1 - c \theta^2 \) for some constant \( c < 1 \) depending on \( \delta \).

5 Quasi-optimality of AFEM

The selection of marked cells within every loop of the AFEM dictated by procedure \textbf{MARK} is determined by the error indicators (21). Hence the decay rate of the adaptive method in terms of the DOFs is heavily dependant on the estimator (22) which, in view of the Global Upper Bound (3.4), may exhibit slower decay than the energy norm whenever over-resolution occurs. In view of the Global Lower Bound (3.6), the quality of the estimator strongly depends on the resolution of the right-hand-side source function \( f \) on the mesh resulting from the averaging process the finite element approximation yields manifesting in the oscillation term \( \text{osc}_P(f, \Omega) \). The fact that the estimator is equivalent to the error in the energy norm up-to the oscillation term

\[
C_{\text{eff}} \eta^2(U, \Omega) - \text{osc}_P^2(f, \Omega) \leq \| u - U \|^2 \leq C_{\text{rel}} \eta^2_P(U, \Omega)
\]

motivates measuring the decay rate of the total-error

\[
\rho_P(v, V, g) := \left( \| v - V \|^2 + \text{osc}_P^2(g, \Omega) \right)^{1/2},
\]

which in the asymptotic regime, due to estimator dominance over oscillation, can be made equivalent to the quasi-error

\[
\rho_P^2(u, U, f) \approx \eta^2_P(U, P) \approx \| u - U \|^2 + C_{\text{est}} \eta^2_P(U, P)
\]

In what follows we show that Cea’s lemma holds for the total-error norm, that is, that the finite element solution \( U \) is an optimal choice from \( \mathcal{X}_P \) in total-error norm.
Lemma 5.1 (Optimality of total error). Let $u$ be the solution of (5) and for all $P \in \mathcal{P}$ let $U \in X_P$ be the discrete solution to (17). Then,

$$\rho_P^2(u, U, f) \leq \inf_{V \in X_P} \rho_P^2(u, V, f).$$

(110)

Proof. In view of Galerkin orthogonality and the symmetry of the bilinear form $a(u - U, U - V) = a(U - V, u - U) = 0$ we have

$$a(u - V, u - V) = a(u - U, u - U) + a(U - V, U - V)$$

(111)

and we have $\|u - V\|^2 = \|u - U\|^2 + \|U - V\|^2$. Therefore,

$$\rho_P^2(u, U, f) \leq \|u - U\|^2 + \text{osc}_P^2(f, \Omega) + \|U - V\|^2$$

(112)

$$= \|u - V\|^2 + \text{osc}_P^2(f, \Omega)$$

\[
\rho_P^2(u, V, f)
\]

5.1 AFEM approximation class

In order to assess the performance of AFEM (2), the rate of decay of error in terms of DOFs, we consider the following nonlinear approximation classes that govern the adaptive finite element problem:

$$A_s = \left\{ v \in H_0^2(\Omega) : \sup_{N > 0} N^s \inf_{P \in \mathcal{P}_N} \inf_{V \in X_P} \|v - V\| < \infty \right\}$$

(113)

$$O_s = \left\{ g \in L^2(\Omega) : \sup_{N > 0} N^s \inf_{P \in \mathcal{P}_N} \|h_P^2(g - \Pi g)\|_{L^2(\Omega)} < \infty \right\}$$

(114)

If $(u, f) \in A_s \times O_s$ then nonlinear-approximation theory dictates there exists an admissible partition $P \in \mathcal{P}_N$ for which $u$ can be approximated in $X_P$ with an error proportional to $N^{-s}$. We hope that the proposed AFEM (2) will generate a sequence of partitions $P_\ell$ for which $\rho^2_{P_\ell}(u, U_\ell, f)$ decays with order $(\# P_\ell)^{-s}$. We define the approximation class described by the total-error norm (108)

$$A_s = \left\{ v \in H_0^2(\Omega) : |v|_{A_s} := \sup_{N > 0} N^s \inf_{P \in \mathcal{P}_N} E_P(v) < \infty \right\}$$

(115)

where

$$E_P(v) = \inf_{V \in X_P} \rho_P(v, V, \mathcal{L}v), \quad v \in H_0^2(\Omega)$$

(116)
Remark 5.2. We will restrict values $s \in (0, r/2]$. For values $s > r/2$ resulting approximation spaces will consist of polynomials only. Valid values for rate $s$ will be refined and made more precise when we characterize the aforementioned approximation classes in terms of smoothness.

Lemma 5.3 (Equivalence of classes). Let $u$ be the weak solution to (5). If $u \in A_s$ and $f \in O_s$ then $u \in A_s$.

Proof. By assumption we have two admissible partitions $P_1, P_2 \in \mathcal{P}_N$ and a finite-element spline $V \in X_{P_1}$ such that $\|u - V\| \leq N^{-s}$ and $\text{osc}_{P_2}(f) \leq N^{-s}$. Invoking Mesh Overlay (26) we obtain an admissible partition $P := P_1 \oplus P_2$ for which $\#P \leq 2N$ and because of spline space nesting we have

$$\|u - V\| + \text{osc}_P(f) \leq N^{-s}.$$  

5.2 Quasi-optimality

The contraction achieved in the convergence proof is ensured by the Dorfler marking strategy. However the relationship between the Dorfler strategy and error reduction in the total-error norm goes deeper than asserted in Theorem 4.4. In the following lemma we show that if $R_{P \rightarrow P_*}$ is a set of refined elements resulting in a reduction of error in contractive sense, then necessarily the Dorfler property holds for the set $\omega_{R_{P \rightarrow P_*}}$. The fact will be instrumental in proving that the cardinality of marked cells will keep the partition cardinality at each refinement step proportional to the optimal quantity dictated by nonlinear approximation.

Lemma 5.4 (Optimal Marking). Let $U = \text{SOLVE}[P, f]$, let $P_\ast$ be any refinement of $P$ and let $U_\ast = \text{SOLVE}[P_\ast, f]$. If for some positive $\mu < 1$

$$\|u - U_\ast\|^2 + \text{osc}_{P_\ast}^2(f, P_\ast) \leq \mu (\|u - U\|^2 + \text{osc}_P^2(f, P)),$$  

and $R_{P \rightarrow P_*}$ denotes collection of all elements in $P$ requiring refinement to obtain $P_\ast$ from $P$, then for $\theta \in (0, \theta_\ast)$ we have

$$\eta_P(U, \omega_{R_{P \rightarrow P_*}}) \geq \theta \eta_P(U, P)$$  

Proof. Let $\theta < \theta_\ast$, the parameter $\theta_\ast$ to be specified later, such that the linear contraction of the total error holds for $\mu := 1 - \frac{\theta^2}{\theta_\ast^2} > 0$. The Efficiency Estimate (61) together with the assumption (117)

$$(1 - \mu)C_{\text{eff}} \eta_P^2(U, P) \leq (1 - \mu) \rho_P^2(u, U, f)$$

$$= \rho_P^2(u, U, f) - \rho_{P_\ast}^2(u_\ast, U_\ast, f)$$

$$= \|u - U\|^2 - \|u - U_\ast\|^2 + \text{osc}_P^2(f, \Omega) - \text{osc}_{P_\ast}^2(f, \Omega)$$  

(118)
In view of Galerkin pythagorus gives \(\|u - U\|^2 - \|u - U_\ast\|^2 = \|U - U_\ast\|^2\). \(R_{P \rightarrow P_\ast} \subset P\) so \(\text{osc}_P^2(f, \Omega) - \text{osc}_{P_\ast}^2(f, \Omega) \leq \text{osc}_P^2(f, \omega_{R_{P \rightarrow P_\ast}})\). Estimator asymptotic dominance over oscillation \(\text{osc}_P^2(U, \tau) \leq \eta_P^2(U, \tau)\) and Discrete Upper Bound (73)

\[
(1 - \mu)C_{\text{est}}\eta_P^2(U, P) \leq (1 + C_{\text{dRel}})\eta_P^2(U, \omega_{R_{P \rightarrow P_\ast}})
\]

By definition \(\theta^2 = (1 - \mu)\theta_\ast^2 < \theta_\ast^2\) we arrive at \(\theta^2\eta_P^2(U, P) \leq \eta_P^2(U, \omega_{R_{P \rightarrow P_\ast}})\) for \(\theta^2 < \frac{C_{\text{est}}}{1 + C_{\text{dRel}}} =: \theta_\ast^2\).

**Lemma 5.5** (Cardinality of Marked Cells). Let \(\{(P_\ell, X_\ell, U_\ell)\}_{\ell \geq 0}\) be sequence generated by AFEM \((P_0, f; \varepsilon, \theta)\) for admissible \(P_0\) and the pair \(u \in A_s\) for some \(s > 0\) then

\[
\#\mathcal{M}_\ell \leq \left(1 - \frac{\theta_\ast^2}{\theta_\ell^2}\right) \frac{1}{\theta_\ell^2} |u|_{A_s}^\frac{1}{2} \left\{\|u - U_\ell\|^2 + \text{osc}_\ell^2(f, P_\ell)\right\}^{-\frac{1}{2}}
\]

**Proof.** Assume that the marking parameter satisfies the hypothesis of Theorem 5.4 and suppose that \(u \in A_s\) for some \(s > 0\). Set \(\mu = 1 - \frac{\theta_\ast^2}{\theta_\ell^2}\) and let \(\varepsilon := \mu \rho_\ell(u, U_\ell, f)^2\). Then by definition of \(A_s\) there exists an admissible partition \(P_\ell\) and a spline \(V_\varepsilon \in X_\varepsilon\) for which

\[
\rho_\varepsilon^2(u, V_\varepsilon, f) \leq \varepsilon^2 \quad \text{with} \quad \#P_\varepsilon - \#P_0 \leq |u|_{A_s}^{1/s} \varepsilon^{-1/s}
\]

Let \(P_\ast := P_\ell \oplus P_\ell\) be the overlay partition of \(P_\varepsilon\) and \(P_\ell, \ell \geq 0\), and let \(U_\ast \in X_\ast\) be the corresponding spline solution. In view of Optimality of Total Error in Lemma 5.1 and the fact \(P_\ast \geq P_\varepsilon\) makes \(X_\ast \supseteq X_\varepsilon\) and

\[
\rho_\ast^2(u, U_\ast, f) = \rho_\varepsilon^2(u, V_\varepsilon, f) \leq \varepsilon^2 = \mu \rho_\ell^2(u, U_\ell, f)
\]

From Optimal Marking of Lemma 5.4 we have \(R_{P_\ast \rightarrow P_\ell} \subset P_\ell\) satisfying Dorfler property for \(\theta < \theta_\ast\).

\[
\#\mathcal{M}_\ell \leq \#R_{P_\ast \rightarrow P_\ell} \leq \#P_\ast - \#P_\ell
\]

In view of overlay property \(\#P_\ast \leq \#P_\varepsilon + \#P_\ell - \#P_0\) in (26) and definition of \(\varepsilon\) we arrive at

\[
\#\mathcal{M}_\ell \leq \#P_\varepsilon - \#P_0 \leq \mu^{-1/2s} |u|_{A_s}^{1/s} \rho_\ell(u, U_\ell, f)^{-1/2}
\]

**Theorem 5.6** (Quasi-optimality). If \(u \in A_s\) and \(P_0\) is admissible, then the call \(\text{AFEM}[P_0, f, \varepsilon, \theta]\) generates a sequence \(\{(P_\ell, X_\ell, U_\ell)\}_{\ell \geq 0}\) of strictly admissible partitions \(P_\ell\), conforming finite-element spline spaces \(X_\ell\) and discrete solutions \(U_\ell\) satisfying

\[
\rho_\ell(u, U_\ell, f) \leq \Phi(\theta)(|u, f|_{A_s}(\#P - \#P_0)^{-s})
\]

with \(\Phi(\theta) = (1 - \theta^2/\theta_\ast^2)^{-1/2}\)
Proof. Let $\theta < \theta_*$ be given and assume that $u \in \mathcal{K}^s(\rho)$. We will show that the adaptive procedure AFEM will produce a sequence $\{(P_\ell, X_\ell, U_\ell)\}_{\ell \geq 0}$ such that $\rho_\ell \leq (\#P_\ell - \#P_0)^{-s}$. Let $A(\theta, s) := (1 - \theta^2/\theta_*^2)^{-1/2s}|u|_{\mathcal{K}^s}^{1/s}$ Cardinality of Marked Cells \((121)\) and \((27)\) yields

$$\#P_\ell - \#P_0 \leq A(\theta, s) \sum_{j=0}^{\ell-1} \rho_j^{-1/s}.$$

In view of Convergence Theorem \(4.4\), we have for a factor $C_{\text{est}} > 0$ and a contractive factor $\alpha \in (0, 1)$

$$e_\ell^2 + C_{\text{est}} \eta_\ell^2 \leq \alpha^{2(\ell-j)} (e_j^2 + C_{\text{est}} \eta_j^2), \quad j = 1, \ldots, \ell - 1,$$

holding for any iteration $\ell \geq 0$. At each intermediate step, the Efficiency Estimate \((61)\) makes $e_j^2 + \gamma \eta_j^2 \leq (1 + C_{\text{est}}/C_{\text{eff}}) \rho_j^2$ so we may write

$$\rho_j^{-1/s} \leq \alpha^{j/s} \left(1 + \frac{C_{\text{est}}}{C_{\text{eff}}}\right)^{1/2s} (e_\ell^2 + C_{\text{est}} \eta_\ell^2)^{-1/s}.$$

We sum \((127)\) over $j = 0: \ell - 1$ and we recover the total-error from the quasi-error using estimator domination over oscillation,

$$\sum_{j=0}^{\ell-1} \rho_j^{-1/s} \leq \sum_{j=0}^{\ell-1} \alpha^{j/s} \left(1 + \frac{C_{\text{est}}}{C_{\text{eff}}}\right)^{1/2s} (e_\ell^2 + C_{\text{est}} \eta_\ell^2)^{-1/2s}.$$

We obtain

$$\#P_\ell - \#P_0 \leq M(\theta, s) \left(e_\ell^2 + C_{\text{est}} \eta_\ell^2\right)^{-1/2s} \sum_{j=1}^\ell \alpha^{j/s},$$

where $M(\theta, s) = A(\theta, s) \left(1 + \frac{C_{\text{est}}}{C_{\text{eff}}}\right)^{1/2s}$ and $\sum_{j=1}^\ell \alpha^{j/s} \leq \alpha^{1/s}(1 - \alpha^{1/s})^{-1} =: S(\theta, s)$ for any $\ell \geq 1$.

$$\#P_\ell - \#P_0 \leq S(\theta, s) M(\theta, s) \rho_\ell(u, U_\ell, f)^{-1/2}$$

From Remark 4.5

$$\frac{\alpha^{1/s}}{1 - \alpha^{1/s}} \leq$$

\(\square\)

6 Characterization of approximation classes

In this section we characterize the approximation classes of the previous section. Namely, we will express $\mathcal{A}^s$, $\mathcal{O}^s$ and $\mathcal{K}^s$ in terms of Besov smoothness spaces. Let $m \geq 1$ be an integer and $h > 0$, we define the $m$-th order forward difference operator $\Delta_h^m$ recursively via

$$\Delta_h^m := \Delta_h[\Delta_h^{m-1}], \quad \Delta_h = T_h - I, \quad T_h f(t) = f(t + h).$$
For $G \subset \mathbb{R}^d$ convex with $\text{diam} G = 1$, we define the Besov space via modulus the of smoothness

$$
\omega_m(f, t)_p := \sup_{h \leq t} \| \Delta_h^m f \|_{L^p(G_m h)}, \quad G_{m h} = \left\{ x \in G : [x + m h]^d \subset G \right\}, \quad (130)
$$

and $\omega_m(f, t)_p := 0$ for values $t$ such that $x + m h \notin G$. Note that if $f \in \mathbb{P}_{m-1}(G)$ then $\omega_m(f, t)_p = 0$, moreover if $\omega_m(f, t)_p = o(t^m)$ then necessarily we have $f \in \mathbb{P}_{m-1}(G)$. For values $\alpha > 0$, $0 < q, p \leq \infty$ we characterize Besov spaces $\mathcal{B}_q^\alpha := \mathcal{B}_q^\alpha(L^p(G))$ in terms of $\omega_m(f, t)_p$:

$$
\mathcal{B}_q^\alpha(G) = \{ f \in L^p(G) : |v|_{\mathcal{B}_q^\alpha(G)} < \infty \} \quad (131)
$$

where the semi-norm reads

$$
|f|_{\mathcal{B}_q^\alpha(G)} = \| t \mapsto t^{-\alpha - 1/q} \omega_m(f, t)_p \|_{L^q(0, \infty)} \quad (132)
$$

Note that if $m \geq \alpha - \max\{0, 1/q - 1\}$ then different choices of $m$ with result in quasi-norms $| \cdot |_{\mathcal{B}_{q, p, m}}$ that are equivalent to each other. On the other hand if $m < \alpha - \max\{0, 1/q - 1\}$ then the Besov space $\mathcal{B}_{q, p, m}$ is a polynomial space of degree $m - 1$. We will need some tools for the following analysis. Let $G \subset \mathbb{R}^2$ We will make use of the Whitney-type estimate: $0 < p \leq \infty$

$$
\inf_{\pi \in \mathcal{P}_r} \| f - \pi \|_{L^p(G)} \leq \omega_r(f, \text{diam} G)_p \quad \forall f \in L^p(G) \quad (133)
$$

We have the $\omega_{r+1}(f, \text{diam} G)_p \leq |f|_{\mathcal{B}_{p, p}}$ and whenever $\alpha < r$ we also have the continuous embedding $\mathcal{B}_r \hookrightarrow \mathcal{B}_{p, p}$. We have

$$
\inf_{\pi \in \mathcal{P}_r} \| f - \pi \|_{L^p(G)} \leq |f|_{\mathcal{B}_r} \quad \forall f \in \mathcal{B}_r \quad (134)
$$

Let $H^0 := L^2$ and for $\beta > 0$ let $H^\beta := W^\beta_2$. We have the embedding for $\alpha > 0$ and $0 < p \leq \infty$

$$
\mathcal{B}_p^\alpha(G) \hookrightarrow H^\beta(G) \quad \text{if} \quad \alpha - \beta > 2 \left( \frac{1}{p} - \frac{1}{2} \right). \quad (135)
$$

Following result is due to Binev [13] which we include for completeness.

**Lemma 6.1.** Let $v \in \mathcal{B}_p^\alpha(\Omega)$ for $\alpha \geq 0$, $0 < p < \infty$ and let $\delta > 0$.

$$
e(\tau, P) = |\tau|^{\delta} |v|_{\mathcal{B}_p^\alpha(\omega)}, \quad \omega = \tau \text{ or } \omega_r. \quad (136)
$$

**Given any $\varepsilon > 0$, the adaptive procedure**

1. $\mathcal{M}_t \leftarrow \{ \tau \in P_t : e(\tau, P_t) > \varepsilon \}$
2. while $\mathcal{M}_t \neq \emptyset$ do
   1. $P_{t+1} \leftarrow \text{REFINE}(P_t, \mathcal{M}_t)$
   2. $\mathcal{M}_{t+1} \leftarrow \{ \tau \in P_{t+1} : e(\tau, P_{t+1}) > \varepsilon \}$


where we invoked \( \sum \) finite number of steps \( L \). If reduce by a factor 1/4, we therefore obtain integer for which \( \tau \) error quantities \( \varepsilon \) for values \( \alpha < r - 1 + \max\{0, 1/p - 1\} \) and \( \frac{\alpha}{2} \geq \frac{1}{p} - \frac{1}{2} \) and 0 < \( p < \infty \).
Proof. Let \( \pi \in \mathbb{P}_r(\omega_r) \).

\[
|v - \Pi P v|_{H^2(\tau)} \leq |v - \pi|_{H^2(\tau)} + |\Pi P (\pi - v)|_{H^2(\tau)} \\
\leq |v - \pi|_{H^2(\tau)} + c_{\text{shape}} |\pi - v|_{H^2(\omega_r)} \lesssim C_{\text{shape}} |v - \pi|_{H^2(\omega_r)}. \quad (138)
\]

Let \( \omega_r = T(G) \) and \( \hat{v} = v \circ T \). For \( \alpha < r - 1 + \max\{0, 1/p - 1\} \) we have nontrivial Besov spaces \( \mathcal{B}_{p,p}^{2+\alpha}(G) \) and if \( \frac{1}{p} \leq \frac{\alpha + 1}{2} \) we have the continuous embedding \( \mathcal{B}_{p,p}^{2+\alpha}(G) \hookrightarrow H^2(G) \). Together with the facts \( |\hat{v}|_{\mathcal{B}_{p,p}^{2+\alpha}(G)} \approx h^2 + \alpha - 2/p |v|_{\mathcal{B}_{p,p}^{2+\alpha}(\omega_r)} \) and \( |\hat{\pi}|_{\mathcal{B}_{p,p}^{2+\alpha}(G)} = 0 \) we arrive at

\[
h_r |v - \pi|_{H^2(\omega_r)} \approx |\hat{v} - \hat{\pi}|_{H^2(G)} \lesssim \|\hat{v} - \hat{\pi}\|_{L^2(G)} + |\hat{v}|_{\mathcal{B}_{p,p}^{2+\alpha}(G)} \quad (139)
\]

Invoking (134),

\[
\inf_{\pi \in \mathbb{P}_r(\omega_r)} h_r |v - \pi|_{H^2(\omega_r)} \leq |\hat{v}|_{\mathcal{B}_{p,p}^{2+\alpha}(G)} \approx h_r^{2+\alpha - 2/p} |v|_{\mathcal{B}_{p,p}^{2+\alpha}(\omega_r)} \quad (140)
\]

we obtain

\[
\inf_{\pi \in \mathbb{P}_r(\omega_r)} |v - \pi|_{H^2(\omega_r)} \leq h_r^{\alpha + 1 - 2/p} |v|_{\mathcal{B}_{p,p}^{2+\alpha}(\omega_r)} \approx \tau^\delta |v|_{\mathcal{B}_{p,p}^{2+\alpha}(\omega_r)} \quad (141)
\]

with \( \delta := \frac{\alpha + 1}{2} - \frac{1}{p} > 0 \). We have the local estimate

\[
\forall \tau \in \mathcal{P}, \quad |v - \Pi P v|_{H^2(\tau)} \lesssim C_{\text{shape}} |\tau|^{\delta} |v|_{\mathcal{B}_{p,p}^{2+\alpha}(\omega_r)} \quad (142)
\]

and

\[
\|v - \Pi P v\|^2 = \sum_{\tau \in \mathcal{P}} |v - \Pi P v|_{H^2(\tau)}^2 \lesssim \sum_{\tau \in \mathcal{P}} \epsilon(\tau, P)^2 \quad (143)
\]

we have in view of Lemma 6.1 with \( \omega = \omega_r \), there exists an admissible mesh \( P \in \mathcal{P} \) such that

\[
\|v - \Pi P v\|^2 \lesssim \#P \varepsilon^2 \quad \text{with} \quad \#P - \#P_0 \lesssim \|P|_{\mathcal{B}_{p,p}^{2+\alpha}(\Omega)}^{(1+\delta p)} \varepsilon^{-p/(1+\delta p)} \quad (144)
\]

Noting that \( p\delta = p(\alpha + 1)/2 - 1 \) so \( p/(1 + \delta p) = 2/(\alpha + 1) \) let \( N = \#P \) and let \( \varepsilon = N^{-(\alpha+1)/2} |v|_{\mathcal{B}_{p,p}^{2+\alpha}(\Omega)} \) then

\[
\|v - \Pi P v\| \lesssim |v|_{\mathcal{B}_{p,p}^{2+\alpha}(\Omega)} N^{-\alpha/2} \quad \text{and} \quad \#P - \#P_0 \leq N.
\]

Let \( s = \frac{\alpha}{2} \) then

\[
|v|_{A^s} = \sup_{N > 0} N^s \inf_{P \in \mathcal{P}_N} \inf_{V \in \mathcal{K}_P} \|u - V\| \lesssim \sup_{N > 0} N^s \|v - \Pi P v\| \lesssim |v|_{\mathcal{B}_{p,p}^{2+\alpha}(\Omega)} < \infty.
\]

\( \square \)
Theorem 6.3. We have $\mathcal{B}^{\alpha}_{p,p}(\Omega) := \mathcal{B}^{\alpha}_{p,p;r-3}(\Omega) \subset \mathcal{O}^s$ with $s = \frac{\alpha + 1}{2}$ for values $\alpha < r - 3 + \max\{0,1/p - 1\}$ and $\frac{\alpha}{2} \geq \frac{1}{p} - \frac{1}{r}$, $0 < p < \infty$.

Proof. Let $\pi \in \mathbb{P}_{r-4}^s(\omega_r)$.

$$\|f - \Pi f\|_{L^2(\tau)} \leq \|f - \pi\|_{L^2(\tau)} + \|\Pi(\pi - f)\|_{L^2(\tau)} \leq \|f - \pi\|_{L^2(\tau)} + \|\pi - f\|_{H^2(\tau)} \leq \|f - \pi\|_{L^2(\tau)}. \leqno(145)$$

Let $\tau = T(G)$ and $\hat{f} = f \circ T$. For $\alpha < r - 3 + \max\{0,1/p - 1\}$ we have nontrivial Besov spaces $\mathcal{B}^{\alpha}_{p,p}(G)$ and if $\frac{1}{p} \leq \frac{\alpha + 1}{2}$ we have the continuous embedding $\mathcal{B}^{\alpha}_{p,p}(G) \hookrightarrow L^2(G)$. Together with the facts $|\hat{f}|_{\mathcal{B}^{\alpha}_{p,p}(\tau)} = h^{-2/p}_{\tau} |f|_{\mathcal{B}^{\alpha}_{p,p}(\tau)}$ and $|\hat{\pi}|_{\mathcal{B}^{\alpha}_{p,p}(\tau)} = 0$ we arrive at

$$h_{\tau}^{-1} \|f - \pi\|_{L^2(\tau)} = |\hat{f} - \hat{\pi}|_{L^2(\tau)} \leq |\hat{f} - \hat{\pi}|_{L^{\infty}(G)} + |\hat{f}|_{\mathcal{B}^{\alpha}_{p,p}(G)} \leqno(146)$$

Invoking (134),

$$\inf_{\pi \in \mathbb{P}_{r-4}^s(\tau)} h_{\tau}^{-1} \|f - \pi\|_{L^2(\tau)} \leq |\hat{f}|_{\mathcal{B}^{\alpha}_{p,p}(\tau)} = h_{\tau}^{-1} |f|_{\mathcal{B}^{\alpha}_{p,p}(\tau)} \leqno(147)$$

we obtain

$$\inf_{\pi \in \mathbb{P}_{r}(\tau)} \|f - \pi\|_{L^2(\tau)} \leq h_{\tau}^{\alpha + 1/2} |f|_{\mathcal{B}^{\alpha}_{p,p}(\tau)} \leqno(148)$$

We have

$$\text{osc}_P(f) \approx \sum_{\tau \in P} h_{\tau}^{2} \|f - \Pi f\|_{L^2(\tau)} \leq \sum_{\tau \in P} h_{\tau}^{\alpha + 3/2} |f|_{\mathcal{B}^{\alpha}_{p,p}(\tau)} \leq \sum_{\tau \in P} |\tau|^\delta |f|_{\mathcal{B}^{\alpha}_{p,p}(\tau)} \leqno(149)$$

with $\delta := \frac{\alpha + 3}{2} - \frac{1}{p}$; let $e(\tau,P) = |\tau|^\delta |f|_{\mathcal{B}^{\alpha}_{p,p}(\tau)}$. We have in view of Lemma 6.1 with $\omega = \tau$, there exists an admissible mesh $P \in \mathcal{P}$ such that

$$\text{osc}_P^2(f) \leq \# P \varepsilon^2 \text{ with } \# P - \# P_0 \leq |v|_{\mathcal{B}^{\alpha}_{p,p}(\Omega)}^{p/(1 + \delta p)} \varepsilon^{-p/(1 + \delta p)} \leqno(150)$$

Noting that $p \delta = p(\alpha + 3)/2 - 1$ so $p/(1 + \delta p) = 2/(\alpha + 3)$, let $N = \# P$ and let $\varepsilon = N^{-(\alpha + 3)/2} |v|_{\mathcal{B}^{\alpha}_{p,p}(\Omega)}$ then

$$\text{osc}_P(f) \leq |f|_{\mathcal{B}^{\alpha}_{p,p}(\Omega)} N^{-(\alpha + 1)/2} \text{ and } \# P - \# P_0 \leq N. \leqno(150)$$

Let $s = \frac{\alpha + 1}{2}$ then $|f|_{\mathcal{O}^s} < \infty$. \hfill \square

The previous two results in combination with Lemma 5.3 yields a one-sided characterization of the AFEM approximation class $\mathbb{H}^s$:

Corollary 6.4 (One-sided characterization for $\mathbb{H}^s$). Let $u$ be the weak solution to (5). If $u \in (\mathcal{B}^{2s+2}_{p,p};(\Omega) \cap H^2_0(\Omega))$ with $1/p - 1/2 \leq 2s < r - 1 + \max\{0,1/p - 1\}$ for some $0 < p < \infty$ and $\mathcal{L} u \in (\mathcal{B}^{2s-1}_{q,q};(\Omega) \cap L^2(\Omega))$ with $1/q - 1/2 \leq 2s < r - 3 + \max\{0,1/q - 1\}$ for some $0 < q < \infty$, then $u \in \mathbb{H}^s$. 

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