The index of Lie poset algebras II - types B, C, and D

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Abstract

After the fashion of type-A Lie poset algebras studied by Coll and Gerstenhaber, we define posets of types B, C, and D. These posets encode the matrix forms of certain Lie algebras which lie between the algebras of upper-triangular and diagonal matrices. We also investigate the index and spectral theories of these algebras. For an important restricted class, we develop combinatorial index formulas and, in particular, characterize posets corresponding to Frobenius Lie algebras. In this latter case we show that the spectrum is binary; that is, consists of an equal number of 0’s and 1’s.

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1 Introduction

The study of “Lie poset algebras” was initiated by Coll and Gerstenhaber in [3], where the deformation theory of such algebras was investigated. The authors define Lie poset algebras as subalgebras of $A_n -1 = \mathfrak{sl}(n)$ which lie between the subalgebras of upper-triangular and diagonal matrices; we will refer to such Lie subalgebras of $\mathfrak{sl}(n)$ as type-A Lie poset algebras. In [7], the authors extend the notion of Lie poset algebra to the other classical families of Lie algebras. Here, we define posets which encode the matrix forms of such Lie poset algebras and initiate an investigation into their index and spectral theories.

Formally, the index of a Lie algebra $\mathfrak{g}$ is defined as

$$\text{ind } \mathfrak{g} = \min_{F \in \mathfrak{g}^*} \dim(\ker(B_F)),$$

where $B_F$ is the skew-symmetric Kirillov form defined by $B_F(x, y) = F([x, y])$, for all $x, y \in \mathfrak{g}$. Of particular interest are those Lie algebras which have index zero, and are called Frobenius. A functional $F \in \mathfrak{g}^*$ for which $\dim \ker(B_F) = \text{ind } \mathfrak{g} = 0$ is likewise called Frobenius. Given a Frobenius Lie algebra $\mathfrak{g}$ and a Frobenius functional $F \in \mathfrak{g}^*$, the map $\mathfrak{g} \to \mathfrak{g}^*$ defined by $x \mapsto B_F(x, -)$ is an isomorphism. The inverse image of $F$ under this isomorphism, denoted $\hat{F}$, is called a principal element of $\mathfrak{g}$. In [18], Ooms shows that the eigenvalues (and multiplicities) of $\text{ad}(\hat{F}) = [\hat{F}, -] : \mathfrak{g} \to \mathfrak{g}$ do not depend on the choice of principal element $\hat{F}$ (see also [15]). It follows that the spectrum of $\text{ad}(\hat{F})$ is an invariant of $\mathfrak{g}$, which we call the spectrum of $\mathfrak{g}$.

Recently, there has been a motivated push to determine combinatorial index formulas for certain families of Lie algebras. Families for which such formulas have been found include seaweed algebras and type-A Lie poset algebras [5, 6, 7, 9, 12, 16, 19, 20, 21]. In this article, we consider the analogues of type-A Lie poset algebras in the other classical types: $B_k = \mathfrak{so}(2k + 1)$, $C_k = \mathfrak{sp}(2k)$, and $D_k = \mathfrak{so}(2k)$; such

1Frobenius algebras are of special interest in deformation and quantum group theory stemming from their connection with the classical Yang-Baxter equation (see [13, 14]).
algebras are called type-B, C, and D Lie poset algebras, respectively. We find that these Lie poset algebras are encoded by posets whose underlying sets are of the form \{-n, \ldots, -1, 0, 1, \ldots, n\} in type B and of the form \{-n, \ldots, -1, 1, \ldots, n\} in types C and D. Furthermore, we fully characterize the index and spectral theories of type-B, C, and D Lie poset algebras whose underlying posets have the property that there are no relations between pairs of positive (resp. negative) integers.

The organization of this paper is as follows. Section 2 sets the combinatorial definitions and notation needed from the theory of posets. In Sections 3 and 4 we formally introduce type-B, C, and D Lie poset algebras and the posets which encode them. Finally, Sections 5 and 6 deal with the index and spectral theories of types-B, C, and D Lie poset algebras.

2 Posets

A finite poset \((\mathcal{P}, \preceq_{\mathcal{P}})\) consists of a finite set \(\mathcal{P}\) together with a binary relation \(\preceq_{\mathcal{P}}\) which is reflexive, antisymmetric, and transitive. When no confusion will arise, we simply denote a poset \((\mathcal{P}, \preceq_{\mathcal{P}})\) by \(\mathcal{P}\), and \(\preceq_{\mathcal{P}}\) by \(\preceq\). Throughout, we let \(\leq\) denote the natural ordering on \(\mathbb{Z}\). Two posets \(\mathcal{P}\) and \(\mathcal{Q}\) are isomorphic if there exists an order-preserving bijection \(\mathcal{P} \to \mathcal{Q}\).

**Remark 1.** If \(|\mathcal{P}| = n\), then there exists a poset \((\{1, \ldots, n\}, \preceq')\) and \(f: (\mathcal{P}, \preceq_{\mathcal{P}}) \to (\{1, \ldots, n\}, \preceq')\) such that \(\preceq' \subset \preceq_{\mathcal{P}}\subset\leq\).

If \(x \preceq y\) and there exists no \(z \in \mathcal{P}\) satisfying \(x, y \neq z\) and \(x \preceq z \preceq y\), then \(x \preceq y\) is a covering relation. Covering relations are used to define a visual representation of \(\mathcal{P}\) called the Hasse diagram—a graph whose vertices correspond to elements of \(\mathcal{P}\) and whose edges correspond to covering relations (see, for example, Figure 1 (a)). A poset \(\mathcal{P}\) is connected if the Hasse diagram of \(\mathcal{P}\) is connected as a graph.

Given a subset \(S \subset \mathcal{P}\), the induced subposet generated by \(S\) is the poset \(\mathcal{P}_S\) on \(S\), where \(i \preceq_{\mathcal{P}_S} j\) if and only if \(i \preceq_{\mathcal{P}} j\). A totally ordered subset \(S \subset \mathcal{P}\) is called a chain. One can define a simplicial complex \(\Sigma(\mathcal{P})\) by having chains of cardinality \(n+1\) in \(\mathcal{P}\) define the \(n\)-dimensional faces of \(\Sigma(\mathcal{P})\) (see, for example, Figure 1 (b)).

**Example 1.** Let \(\mathcal{P} = \{1, 2, 3, 4\}\) with \(1 \preceq 2 \preceq 3, 4\). In Figure 1 we illustrate (a) the Hasse diagram of \(\mathcal{P}\) and (b) \(\Sigma(\mathcal{P})\).

![Figure 1: (a) Hasse Diagram \(\mathcal{P}\) and (b) \(\Sigma(\mathcal{P})\)](image)

Two operations for combining posets which will be useful later are as follows:

**Definition 1.** Given two posets \(\mathcal{P}\) and \(\mathcal{Q}\) which are disjoint as sets, the disjoint union of \(\mathcal{P}\) and \(\mathcal{Q}\) is the poset \(\mathcal{P} + \mathcal{Q}\) on the union \(\mathcal{P} \cup \mathcal{Q}\) such that \(s \preceq t\) in \(\mathcal{P} + \mathcal{Q}\) if either

- \(s, t \in \mathcal{P}\) and \(s \preceq_{\mathcal{P}} t\), or
- \(s, t \in \mathcal{Q}\) and \(s \preceq_{\mathcal{Q}} t\).
Remark 2. If there are more than two posets $P_i$, for $1 \leq i \leq n$, then we denote their disjoint union as $\sum_{i=1}^{n} P_i$.

Definition 2. If $P$ is a poset, define its dual poset $P^*$ by the following rules:

- $i, j \in P^*$ if $i, j \in P$ and
- $j \preceq_{P^*} i$ if $i \preceq_P j$.

3 Lie poset algebras

Let $P$ be a finite poset and $k$ be an algebraically closed field of characteristic zero, which we may take to be the complex numbers. The associative poset algebra $A(P) = A(P, k)$ is the span over $k$ of elements $e_{i,j}$, for $i, j \in P$ satisfying $i \preceq j$, with multiplication given by setting $e_{i,j}e_{k,l} = e_{i,l}$ if $j = k$ and 0 otherwise. The trace of an element $\sum e_{i,j}$ is $\sum c_{i,i}$.

We can equip $A(P)$ with the commutator product $[a, b] = ab - ba$, where juxtaposition denotes the product in $A(P)$, to produce the Lie poset algebra $g(P) = g(P, k)$. If $|P| = n$, then both $A(P)$ and $g(P)$ may be regarded as subalgebras of the algebra of $n \times n$ upper-triangular matrices over $k$. Such a matrix representation is realized by replacing each basis element $e_{i,j}$ by the $n \times n$ matrix $E_{i,j}$ containing 1 in the $i,j$-entry and 0’s elsewhere. The product between elements $e_{i,j}$ is then replaced by matrix multiplication between the $E_{i,j}$.

Example 2. Let $P$ be the poset of Example 1. The matrix form of elements in $g(P)$ is illustrated in Figure 2, where the *’s denote potential non-zero entries.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & * & * & * & * \\
2 & 0 & * & * & * \\
3 & 0 & 0 & * & 0 \\
4 & 0 & 0 & 0 & * \\
\end{array}
\]

Figure 2: Matrix form of $g(P)$, for $P = \{1, 2, 3, 4\}$ with $1 \leq 2 \leq 3, 4$

Remark 3. Let $b$ be the Borel subalgebra consisting of all $n \times n$ upper-triangular matrices and $h$ be its Cartan subalgebra of diagonal matrices. Any subalgebra $g$ lying between $h$ and $b$ is then a Lie poset algebra; for $g$ is then the span over $k$ of $h$ and those $E_{i,j}$ which it contains, and there is a partial order on $P = \{1, \ldots, n\}$ compatible with the linear order by setting $i \preceq j$ whenever $E_{i,j} \in g$.

Remark 4. Restricting to trace-zero matrices results in a subalgebra of $A_{n-1} = sl(n)$, referred to as a type-A Lie poset algebra \cite{3, 7, 8}. As stated in the introduction, Coll and Mayers \cite{7} initiated an investigation into the index and spectral theories of Type-A Lie poset algebras.

Considering Remarks 3 and 4, we make the following definition.

Definition 3. Let $g$ be a simple Lie algebra, realized as a linear Lie algebra. Let $b \subset g$ be the Borel subalgebra consisting of all upper-triangular matrices and $h$ be its Cartan subalgebra of diagonal matrices. A Lie subalgebra $p \subset g$ satisfying $h \subset p \subset b$ is called a Lie poset subalgebra of $g$. If $g$ is $sl(n)$, $so(2n+1)$, $sp(2n)$, or $so(2n)$, for $n \in \mathbb{N}$, then $p$ is called a type-A, type-B, type-C, or type-D Lie poset algebra, respectively.

4 Posets of types B, C, and D

In this section, we provide definitions for posets of types B, C, and D, which encode matrix forms that define Lie poset algebras of types B, C, and D.
Remark 5. Recall that the subalgebra of upper-triangular matrices of
• $\mathfrak{sp}(2n)$, i.e., type C, consists of all $2n \times 2n$ upper-triangular matrices of the form given in Figure 3 with $B = B^T$.
• $\mathfrak{so}(n)$, for $n$ even, i.e., type D, consist of all $n \times n$ upper-triangular matrices of the form given in Figure 3 with $B = -B^T$.
• $\mathfrak{so}(n)$, for $n$ odd, i.e., type B, consist of all $n \times n$ upper-triangular matrices of the form given in Figure 3 with $B = -B^T$, and a 0 on the diagonal separating $A$ and $-A^T$.

Figure 3: Matrix form for other classical families

Remark 6. Throughout the remainder of this article, unless stated otherwise, we assume that the rows and columns of a $2n \times 2n$ (resp. $2n+1 \times 2n+1$) matrix are labeled by $\{-n, \ldots, -1, 1, \ldots, n\}$ (resp. $\{-n, \ldots, -1, 0, 1, \ldots, n\}$). Given this assumption, a basis for a
• type-C Lie poset algebra can be taken which consists of elements of the form $E_{-i,-i} - E_{i,i}$, for all $i \in [n]$, $E_{i,-j} - E_{j,i}$, for $i, j \in [n]$, $E_{-i,-j} + E_{j,-i}$, for $i, j \in [n]$, and $E_{-i,i}$, for $i \in [n]$.
• type-B or D Lie poset algebra can be taken which consists of elements of the form $E_{-i,-i} - E_{i,i}$, for all $i \in [n]$, $E_{i,-j} - E_{j,i}$, for $i, j \in [n]$, and $E_{-i,j} - E_{j,i}$, for $i, j \in [n]$.

Definition 4. A type-C poset is a poset $\mathcal{P} = \{-n, \ldots, -1, 1, \ldots, n\}$ such that
1. if $i \preceq \mathcal{P} j$, then $i \leq j$; and
2. if $i \neq -j$, then $i \preceq \mathcal{P} j$ if and only if $-j \preceq \mathcal{P} -i$.
A type-D poset is a poset $\mathcal{P} = \{-n, \ldots, -1, 1, \ldots, n\}$ satisfying 1 and 2 above as well as
3. $-i \npreceq \mathcal{P} i$.
A type-B poset is a poset $\mathcal{P} = \{-n, \ldots, -1, 0, 1, \ldots, n\}$ satisfying 1 through 3 above.

Example 3. In Figure 4, we illustrate the Hasse diagram of the type-C (and D) poset $\mathcal{P} = \{-3, -2, -1, 1, 2, 3\}$ with $-2 \preceq 1, 3$; $-3 \preceq 2$; and $-1 \preceq 2$. Note that adding 0 to $\mathcal{P}$ and a vertex labeled 0 to the Hasse diagram of Figure 4 results in a type-B poset and its corresponding Hasse diagram.

Figure 4: Hasse Diagram

Theorem 1. Type-C (resp. B or D) posets $\mathcal{P}$ are in bijective correspondence with type-C (resp. B or D) Lie poset algebras $\mathfrak{p}$ as follows:
• $-i, i \in \mathcal{P}$ if and only if $E_{-i,-i} - E_{i,i} \in \mathfrak{p}$;
\begin{itemize}
  \item $-i \preceq_P -j$ and $j \preceq_P i$ if and only if $E_{-i,-j} - E_{i,i} \in \mathfrak{p}$;
  \item $-i \preceq_P j$ and $-j \preceq_P i$ if and only if $E_{-i,j} + E_{-j,i} \in \mathfrak{p}$ (resp. $E_{-i,j} - E_{-j,i} \in \mathfrak{p}$);
  \item $-i \preceq_P i$ if and only if $E_{-i,i} \in \mathfrak{p}$.
\end{itemize}

**Proof.** Closure of $\mathfrak{p}$ under the Lie bracket is equivalent to transitivity of $\preceq_P$. Thus, considering Remark 6, $\mathfrak{p}$ forms a type-C (resp. B or D) Lie poset algebra. Conversely, such a basis encodes the relation restrictions required in a type-C (resp. B or D) poset. \hfill \Box

**Remark 7.** Note that as in the type-A case, type-C (resp. B or D) posets $\mathcal{P}$ determine the matrix form of the corresponding type-C (resp. B or D) Lie poset algebra by identifying which entries of a $|\mathcal{P}| \times |\mathcal{P}|$ matrix can be non-zero. In particular, the $i,j$-entry can be non-zero if and only if $i \preceq_P j$.

**Example 4.** Let $\mathcal{P}$ be the poset of Example 3. The matrix form encoded by $\mathcal{P}$ and defining the corresponding type-C (resp. D) Lie poset algebra is illustrated in Figure 5, where $*$'s denote potential non-zero entries.

\begin{center}
\[
\begin{pmatrix}
-3 & * & 0 & 0 & * & 0 \\
* & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\end{center}

Figure 5: Matrix form for $\mathcal{P} = \{-3, -2, -1, 1, 2, 3\}$ with $-2 \leq 1, 3; -3 \leq 2; \text{ and } -1 \leq 2$

**Remark 8.** Given a type-C poset $\mathcal{P}$, we denote the corresponding type-C Lie poset algebra by $\mathfrak{g}_C(\mathcal{P})$; furthermore, we define the following basis for $\mathfrak{g}_C(\mathcal{P})$:

$$\mathcal{B}_C(\mathcal{P}) = \{E_{-i,-i} - E_{i,i} \mid -i, i \in \mathcal{P}\} \cup \{E_{-i,j} + E_{-j,i} \mid -i, -j, i, j \in \mathcal{P}, -j \preceq i, -i \preceq j\} \cup \{E_{-i,i} \mid -i, i \in \mathcal{P}, -i \preceq i\}.$$  

Similarly, given a type-D (resp. B) poset $\mathcal{P}$ we denote the corresponding type-D (resp. B) Lie poset algebra by $\mathfrak{g}_D(\mathcal{P})$ (resp. $\mathfrak{g}_B(\mathcal{P})$) and define the basis $\mathcal{B}_D(\mathcal{P})$ (resp. $\mathcal{B}_B(\mathcal{P})$) as follows:

$$\{E_{-i,-i} - E_{i,i} \mid -i, i \in \mathcal{P}\} \cup \{E_{-i,j} - E_{-j,i} \mid -i, -j, i, j \in \mathcal{P}, -j \preceq i, -i \preceq j\}.$$  

We continue to set the combinatorial notation for posets of types B, C, and D.

**Remark 9.** All previous notions defined for posets $\mathcal{P}$, e.g., Hasse diagram of $\mathcal{P}$, connected, $\Sigma(\mathcal{P})$, etc, still apply.

Given a type-B, C, or D poset $\mathcal{P}$, let $\mathcal{P}^+ = \mathcal{P}_{\mathbb{Z}_{\geq 0}}$ and $\mathcal{P}^- = \mathcal{P}_{\mathbb{Z}_{< 0}}$; that is, $\mathcal{P}^+$ (resp. $\mathcal{P}^-$) is the poset induced by the positive (resp. negative) elements of $\mathcal{P}$.

**Remark 10.** By property 2 of Definition 4, we have that $\mathcal{P}^+$ is isomorphic to $(\mathcal{P}^-)^*$.

Let $\text{Rel}_\pm(\mathcal{P})$ denote the set of relations $x \preceq_P y$ such that $x \in \mathcal{P}^-$ and $y \in \mathcal{P}^+$. We call $\mathcal{P}$ separable if $\text{Rel}_\pm(\mathcal{P}) = \emptyset$, and non-separable otherwise

**Remark 11.** If $\mathcal{P}$ is a separable, then $\mathcal{P}$ is the disjoint union of $\mathcal{P}^+$ and $\mathcal{P}^-$.  

We say that $\mathcal{P}$ is of height $(i,j)$ if $i$ (resp. $j$) is one less than the largest cardinality of a chain in $\mathcal{P}^+$ (resp. $\mathcal{P}^-$).

To end this section, we introduce a condensed graphical representation for height-$(0,1)$, type-B, C, or D posets which will be used in the following section.
Definition 5. Given a height-(0, 1), type-B, C, or D poset \( P \), we define the relation graph \( RG(P) \) as follows:

- each pair of elements \(-i, i \in P \) are represented by a single vertex in \( RG(P) \) labeled by \( i \in P^+ \) (omitting the vertex representing 0 in type B);
- if \(-i \preceq j \) in \( P \), then there is an edge connecting vertex \( i \) and vertex \( j \) in \( RG(P) \).

Remark 12. If \( P \) is a height-(0, 1), type-B poset, then 0 is not related to any other element of \( P \).

Remark 13. If \(-i \preceq i \) in \( P \), then vertex \( i \) defines a self-loop in \( RG(P) \). Note that \( RG(P) \) can only contain self-loops if \( P \) is a type-C poset.

Example 5. In Figure 6, we illustrate the (a) Hasse diagram and (b) relation graph corresponding to the height-(0, 1), type-C poset \( P = \{-3, -2, -1, 1, 2, 3\} \) with \(-2 \preceq 1, 2, 3; -3 \preceq 2; \) and \(-1 \preceq 2\).

\[
\begin{array}{c}
3 & 2 & 1 \\
-3 & -2 & -1
\end{array}
\quad
\begin{array}{c}
3 & 2 & 1 \\
0
\end{array}
\]

Figure 6: (a) Hasse Diagram and (b) Relation Graph

5 Index

In this section, we develop combinatorial formulas for the index of type-B, C, and D Lie poset algebras corresponding to connected, height-(0, 1), type-B, C, and D posets, respectively.

It will be convenient to use an alternative characterization of the index. Let \( g \) be an arbitrary Lie algebra with basis \( \{x_1, \ldots, x_n\} \). The index of \( g \) can be expressed using the commutator matrix, \(( [x_i, x_j] )_{1 \leq i \leq j \leq n}\), over the quotient field \( R(g) \) of the symmetric algebra \( Sym(g) \) as follows (see [11]).

Theorem 2. The index of \( g \) is given by

\[
\text{ind } g = n - \text{rank}_{R(g)}( [x_i, x_j] )_{1 \leq i \leq j \leq n}
\]

Example 6. Consider \( g_A(P(1, 1)) \); that is, the collection of upper-triangular matrices in \( \mathfrak{sl}(2) \). A Chevalley basis for \( g_A(P(1, 1)) \) is given by \( \{x_1, x_2\} \), where \([x_1, x_2] = 2x_2\). The commutator matrix \(( [x_i, x_j] )_{1 \leq i \leq j \leq 2}\) is illustrated in Figure 7. Since the rank of this matrix is two, it follows from Theorem 2 that \( g_A(P(1, 1)) \) is Frobenius.

\[
\begin{bmatrix}
0 & 2x_2 \\
-2x_2 & 0
\end{bmatrix}
\]

Figure 7: Commutator matrix

Remark 14. To ease notation, row and column labels of commutator matrices will be bolded and matrix entries (elements of \( R(g(P)) \)) will be unbolded. Furthermore, we will refer to the row corresponding to \( x \) in a commutator matrix – and by a slight abuse of notation, in any equivalent matrix – as row \( x \).

Throughout this section, given a type-C poset \( P \), we set

\[
C(g_C(P)) = ([x_i, x_j])_{1 \leq i \leq j \leq n}, \text{ where } \{x_1, \ldots, x_n\} = B_C(P).
\]
Theorem 3. If \( P \) is a type-D poset, then \( \text{ind} g_C(P) = \text{ind} g_D(P) \).

**Proof.** Going from \( g_C(P) \) to \( g_D(P) \), the only changes that occur are basis vectors of the form \( E_{-i,j} + E_{-j,i} \) become \( E_{i,j} - E_{-j,i} \). Such changes lead to purely aesthetic changes in the corresponding commutator matrices. \( \square \)

Remark 15. For ease of discourse, all remaining results of this section will be stated in terms of type-C Lie poset algebras; but, with only minor modifications to proofs, it can be seen that all results hold in the type-B and D cases as well.

Theorem 4. If \( \{P_i\}_{i=1}^n \) is a collection of type-C posets on disjoint sets, then \( \text{ind} g_C(\bigcup_{i=1}^n P_i) = \sum_{i=1}^n \text{ind} g_C(P_i) \).

**Proof.** Note that \( B_C(\bigcup_{i=1}^n P_i) = \bigcup_{i=1}^n B_C(P_i) \) where the \( B_C(P_i) \), for \( i = 1, \ldots, n \), form mutually disjoint sets. Order the basis elements so that those corresponding to \( B_C(P_i) \) occur first, following by those corresponding to \( B_C(P_2) \), etc. With this ordering on the elements of \( B_C(\bigcup_{i=1}^n P_i) \), the commutator matrix \( C(g_C(\bigcup_{i=1}^n P_i)) \) is block diagonal, with each block corresponding to \( C(g_C(P_i)) \), for \( i = 1, \ldots, n \). \( \square \)

Remark 16. Considering Theorem 4, to determine index formulas for type-C Lie poset algebras, it suffices to consider type-C Lie poset algebras corresponding to connected, type-C posets.

Theorem 5. If \( P \) is a separable, type-C poset, then \( \text{ind} g_C(P) = \text{ind} g(P^+) = \text{ind} g_A(P^+) + 1 \).

**Proof.** Note that

\[
\{x_1, \ldots, x_n\} = \{E_{i,i} \mid i \in P^+\} \cup \{E_{i,j} \mid i \preceq_\text{p+} j\}
\]

and

\[
\{y_1, \ldots, y_n\} = \left\{\sum_{i=1}^{\text{ind} P^+} E_{i,i}\right\} \cup \{E_{i,i} - E_{i+1,i+1} \mid 1 \leq i \leq |P^+| - 1\} \cup \{E_{i,j} \mid i \preceq_\text{p+} j\}
\]

both form bases for \( g(P^+) \), while

\[
\{z_1, \ldots, z_{n-1}\} = \{E_{i,i} - E_{i+1,i+1} \mid 1 \leq i \leq |P^+| - 1\} \cup \{E_{i,j} \mid i \preceq_\text{p+} j\}
\]

forms a basis for \( g_A(P^+) \). Replacing \( E_{i-i,i} \) by \( E_{i,i} \) and \( E_{-j-i,j} \) by \( E_{i,j} \) in \( C(g_C(P)) \) results in \( (x_i, x_j)_{1 \leq i \leq j \leq n} \), establishing the first equality. The second equality follows by comparing the commutator matrices \( (y_i, y_j)_{1 \leq i \leq j \leq n} \) and \( (z_i, z_j)_{1 \leq i \leq j \leq n-1} \), for \( g(P^+) \) and \( g_A(P^+) \), respectively. \( \square \)

Corollary 1. If \( P \) is a type-C poset such that \( g_C(P) \) is Frobenius, then \( P \) is non-separable.

Corollary 2. If \( P \) is a height-(0, 0), type-C poset, then \( \text{ind} g_C(P) = |P^+| \).

**Proof.** This follows since any commutator matrix corresponding to \( g(P^+) \) is the \( |P^+| \times |P^+| \) zero-matrix. \( \square \)

### 5.1 Matrix reduction

In this section, we describe an algorithm for reducing \( C(g_C(P)) \), for \( P \) a connected, height-(0, 1), type-C poset.

As a first step in our matrix reduction, we order the row and column labels of \( C(g_C(P)) \), i.e., the elements of \( B_C(P) \), as follows:

1. the elements \( E_{-i-i} - E_{i,i} \) in increasing order of \( i \) in \( \mathbb{Z} \);
2. the elements \( E_{-i,j} + E_{-j,i} \) in increasing lexicographic order of \( (i,j) \), for \( i < j \), in \( \mathbb{Z} \times \mathbb{Z} \).
With this ordering, since height-(0, 1), type-C posets have no non-trivial transitivity relations, \( C(\mathcal{g}_C(\mathcal{P})) \) has the following form:

\[
\begin{pmatrix}
0 & -B(\mathcal{P})^T \\
B(\mathcal{P}) & 0
\end{pmatrix}
\]

where \( B(\mathcal{P}) \) has rows labeled by basis elements of the form \( E_{-1j} + E_{-j1} \) and columns labeled by basis elements of the form \( E_{-1-1} - E_{1i} \), and \( -B(\mathcal{P})^T \) has these labels reversed. Thus, since \( \text{rank}(B(\mathcal{P})) = \text{rank}(B(\mathcal{P})^T) \), to calculate the index, it suffices to determine the rank of \( B(\mathcal{P}) \).

Now, in order to define our matrix reduction, we catalogue the form of collections of rows in \( B(\mathcal{P}) \) which correspond to certain substructures of \( RG(\mathcal{P}) \) as well as row operations to reduce such collections of rows. To condense illustrations, columns of zeros will be omitted.

**Paths:** If \( RG(\mathcal{P}) \) has a path consisting of the sequence of vertices \( i_1, \ldots, i_n \), then the corresponding rows and columns of \( B(\mathcal{P}) \) have the form illustrated in Figure 8.

\[
\begin{pmatrix}
E_{-11,i_2} + E_{-i_2,i_1} \\
E_{-12,i_3} + E_{-i_3,i_2} \\
\vdots \\
E_{-i_n-1,i_n} + E_{-i_n,i_n-1}
\end{pmatrix}
\begin{pmatrix}
E_{i_1,i_2} - E_{i_2,i_1} \\
E_{i_2,i_3} - E_{i_3,i_2} \\
\vdots \\
E_{i_{n-1},i_n} - E_{i_n,i_{n-1}}
\end{pmatrix}
\begin{pmatrix}
E_{i_n,i_{n-1}} - E_{i_{n-1},i_n} \\
\vdots \\
E_{i_{n-1},i_n} - E_{i_n,i_{n-1}}
\end{pmatrix}
\]

Figure 8: Path

**Definition 6.** If \( RG(\mathcal{P}) \) contains a path consisting of the sequence of vertices \( i_1, \ldots, i_n \), then define the row operation \( \text{Path}(i_1, \ldots, i_n) \) on \( C(\mathcal{g}_C(\mathcal{P})) \) to be

\[
(E_{-i_n,i_{n-1}} + E_{-i_{n-1},i_n}) + \sum_{j=1}^{n-1} (-1)^j \frac{E_{-i_n,i_{n-1}} + E_{-i_{n-1},i_n}}{E_{-i_n-j,i_{n-1}-j} + E_{-i_{n-1}-j,i_n-j}} (E_{-i_{n-j},i_{n-j-1}} + E_{-i_{n-1-j},i_n})
\]

performed at row \( E_{-11,i_n} + E_{-1n,i_1} \).

**Example 7.** The result of applying \( \text{Path}(i_1, \ldots, i_n) \) to the matrix of Figure 8 is illustrated in Figure 9.

\[
\begin{pmatrix}
E_{-11,i_2} + E_{-i_2,i_1} \\
E_{-12,i_3} + E_{-i_3,i_2} \\
\vdots \\
E_{-i_n-1,i_n} + E_{-i_n,i_n-1}
\end{pmatrix}
\begin{pmatrix}
E_{i_1,i_2} - E_{i_2,i_1} \\
E_{i_2,i_3} - E_{i_3,i_2} \\
\vdots \\
E_{i_{n-1},i_n} - E_{i_n,i_{n-1}}
\end{pmatrix}
\begin{pmatrix}
E_{i_n,i_{n-1}} - E_{i_{n-1},i_n} \\
\vdots \\
E_{i_{n-1},i_n} - E_{i_n,i_{n-1}}
\end{pmatrix}
\]

Figure 9: Path

**Self-loop:** If \( RG(\mathcal{P}) \) has a self-loop at vertex \( i_1 \), then the corresponding rows and columns of \( B(\mathcal{P}) \) have the form illustrated in Figure 10.

\[
E_{-11,i_1} \begin{pmatrix}
E_{i_1,i_1} - E_{i_2,i_1} \\
-2E_{-11,i_1}
\end{pmatrix}
\]

Figure 10: Self-loop
Cycles: If $RG(P)$ has a cycle consisting of $n > 1$ vertices $i_1, \ldots, i_n$, then the corresponding rows and columns of $B(P)$ have the form illustrated in Figure 11.

\[
\begin{pmatrix}
E_{-i_1,i_n} \oplus E_{-i_n,i_1} & E_{-i_2,i_1} - E_{i_1,i_2} & E_{-i_3,i_2} - E_{i_2,i_3} & \cdots & E_{-i_n,i_{n-1}} - E_{i_{n-1},i_n} & E_{-i_n,i_n} \\
0 & -E_{-i_1,i_2} - E_{i_2,i_1} & -E_{-i_1,i_2} - E_{i_2,i_1} & \cdots & 0 & 0 \\
0 & 0 & -E_{-i_2,i_3} - E_{i_3,i_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -E_{-i_n,i_1} - E_{i_1,i_n} & -E_{-i_n,i_1} - E_{i_1,i_n} \\
\end{pmatrix}
\]

Figure 11: Cycle

**Definition 7.** If $RG(P)$ contains a cycle consisting of $n > 1$ vertices $i_1, \ldots, i_n$, then define the row operation $Row_c(i_1, \ldots, i_n)$ on $C(qC(P))$ to be

\[
(E_{-i_1,i_n} + E_{-i_n,i_1}) + \sum_{j=1}^{n-1} (-1)^j \frac{E_{-i_1,i_n} + E_{-i_n,i_1}}{E_{-i_j,i_{j+1}} + E_{-i_{j+1},i_j}} (E_{-i_j,i_{j+1}} + E_{-i_{j+1},i_j})
\]

performed at row $E_{-i_1,i_n} + E_{-i_n,i_1}$.

**Example 8.** The result of applying $Row_c(i_1, \ldots, i_n)$ to the matrix of Figure 11, for $n$ even, is illustrated in Figure 12.

\[
\begin{pmatrix}
E_{-i_1,i_n} \oplus E_{-i_n,i_1} & E_{-i_2,i_1} - E_{i_1,i_2} & E_{-i_3,i_2} - E_{i_2,i_3} & \cdots & E_{-i_n,i_{n-1}} - E_{i_{n-1},i_n} & E_{-i_n,i_n} \\
0 & -E_{-i_1,i_2} - E_{i_2,i_1} & -E_{-i_1,i_2} - E_{i_2,i_1} & \cdots & 0 & 0 \\
0 & 0 & -E_{-i_2,i_3} - E_{i_3,i_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -E_{-i_n,i_1} - E_{i_1,i_n} & -E_{-i_n,i_1} - E_{i_1,i_n} \\
\end{pmatrix}
\]

Figure 12: Reduced Cycle 1

**Remark 17.** Note that if we disregard the newly added zero row in Figure 12, then the configuration of rows is the same as that corresponding to the cycle defined by $i_1, \ldots, i_n$ with the edge between $i_1$ and $i_n$ removed.

**Definition 8.** If $RG(P)$ contains an odd cycle consisting of $n > 1$ vertices $i_1, \ldots, i_n$, then define the row operation $Row_o(i_1, \ldots, i_n)$ on $C(qC(P))$ to be $Row_o(i_1, \ldots, i_n)$ followed by multiplying row $(E_{-i_1,i_n} + E_{-i_n,i_1})$ by $-E_{-i_1,i_n} + E_{-i_n,i_1}$.

**Example 9.** The result of applying $Row_o(i_1, \ldots, i_n)$ to the matrix of Figure 11, for $n$ odd, is illustrated in Figure 13.

\[
\begin{pmatrix}
E_{-i_1,i_n} \oplus E_{-i_n,i_1} & E_{-i_2,i_1} - E_{i_1,i_2} & E_{-i_3,i_2} - E_{i_2,i_3} & \cdots & E_{-i_n,i_{n-1}} - E_{i_{n-1},i_n} & E_{-i_n,i_n} \\
0 & -E_{-i_1,i_2} - E_{i_2,i_1} & -E_{-i_1,i_2} - E_{i_2,i_1} & \cdots & 0 & 0 \\
0 & 0 & -E_{-i_2,i_3} - E_{i_3,i_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -E_{-i_n,i_1} - E_{i_1,i_n} & -E_{-i_n,i_1} - E_{i_1,i_n} \\
\end{pmatrix}
\]

Figure 13: Reduced Cycle 2
Remark 18. Note that the configuration of rows in Figure 13 is the same as that corresponding to the cycle defined by $i_1, \ldots, i_n$ with the edge between $i_1$ and $i_n$ removed and with $i_2$ defining a self-loop. If node $i_2$ already defined a self-loop, then row $E_{-i_1,i_n} + E_{-i_n,i_1}$ can be reduced to a zero row in the obvious way.

Now, we come to the matrix reduction algorithm.

Remark 19. In the algorithm outlined below, the relation graph $RG(P)$ of a connected, height-$(0, 1)$, type-$C$ poset $P$ is used as a bookkeeping device to guide the reduction. In this way, not only are the rows of $B(P)$ in $C(g_C(P))$ altered, but also row and column labels as well as $RG(P)$.

Matrix Reduction Algorithm: Let $P$ be a connected, height-$(0, 1)$, type-$C$ poset

**Step 1:** Set $G_1 = RG(P)$, $M_1 = B(P)$, $\Gamma_1 = (G_1, M_1)$, and $k = 1$.

**Step 2:** Check $G_k$ for self-loops.

- If $G_k$ has a self-loop at vertex $i$ and vertex $i$ is adjacent to a vertex $j$, go to **Step 3**.
- If $G_k$ contains self-loops and no vertex defining a self-loop is adjacent to any other vertex, halt.
- If $G_k$ has no self-loops, go to **Step 4**.

**Step 3:** Set $k = l$. Form $\Gamma_{l+1} = (G_{l+1}, M_{l+1})$ as follows

1. perform

$$
(E_{-i,j} + E_{-j,i}) - \frac{E_{-i,j} + E_{-j,i}}{2E_{-i,i}}E_{-i,i}
$$

at row $E_{-i,j} + E_{-j,i}$ in $M_l$.

2. If vertex $j$ does not define a self-loop, then
   3. multiply row $E_{-i,j} + E_{-j,i}$ by $\frac{E_{-i,j} + E_{-j,i}}{2E_{-i,i}}$ in $M_l$,
   4. replace the row label $E_{-i,j} + E_{-j,i}$ by $E_{-j,j}$ in $M_l$, and
   5. remove the edge between nodes $i$ and $j$ in $G_k$ and add a self-loop at vertex $j$.
   6. Set $k = l + 1$ and go to **Step 2**.

- If vertex $j$ defines a self-loop, then
   3. remove the edge between vertices $i$ and $j$ in $G_l$,
   4. Note that row $E_{-i,j} + E_{-j,i}$ is now a zero row.
   5. Set $k = l + 1$ and go to **Step 2**.

**Step 4:** Check $G_k$ for odd cycles containing $n > 1$ vertices

- If $G_k$ has such an odd cycle: Let $i_1, \ldots, i_n$ be the vertices of a largest odd cycle in $G_k$; if there are more than one, assuming $i_1 < i_j$ for $j = 2, \ldots, n$, take $(i_1, \ldots, i_n)$ to be the lexicographically least in $\mathbb{Z}^n$. Go to **Step 5**.
- If $G_k$ has no such odd cycles go to **Step 6**.

**Step 5:** Set $l = k$. Form $\Gamma_{l+1} = (G_{l+1}, M_{l+1})$ as follows

1. perform Row$_o(i_1, \ldots, i_n)$ in $M_l$,

2. replace the labels of row and column $E_{-i_1,i_n} + E_{-i_n,i_1}$ by $E_{-i_2,i_2}$ in $M_l$,

3. remove the edge between vertices $i_1$ and $i_n$, and add a self-loop at vertex $i_2$ in $G_l$, and

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4. Set \( k = l + 1 \) and go to Step 2

**Step 6:** Check \( G_k \) for even cycles:

- If \( G_k \) has an even cycle: Let \( i_1, \ldots, i_n \) be the vertices of a largest even cycle in \( G_k \); if there are more than one, assuming \( i_1 < i_j \) for \( j = 2, \ldots, n \), take \( (i_1, \ldots, i_n) \) to be the lexicographically least in \( \mathbb{Z}^n \). Go to **Step 7**.

- If \( G_k \) has no even cycles, go to **Step 8**.

**Step 7:** Set \( l = k \). Form \( \Gamma_{l+1} = (G_{l+1}, M_{l+1}) \) as follows
  1. perform \( Row_c(i_1, \ldots, i_n) \) in \( M_l \),
  2. remove the edge between vertices \( i_1 \) and \( i_n \) in \( G_l \),
  3. note that row \( \mathbf{E}_{-i_1,i_n} + \mathbf{E}_{-i_n,i_1} \) is now a zero row,
  4. Set \( k = l + 1 \) and go to **Step 2**.

**Step 8:** Set \( l = k \). Form \( \Gamma_{l+1} = (G_{l+1}, M_{l+1}) \) as follows
  1. Take \( i_1 \) minimal in \( \mathbb{Z} \) such that \( i_1 \) is a degree one vertex of \( G_l \).
  2. For every vertex \( i_n \neq i_1 \), if \( i_1, i_2, \ldots, i_{n-1}, i_n \) is the sequence of nodes in the unique path in \( G_l \) from \( i_1 \) to \( i_n \), then perform \( \text{Path}(i_1, \ldots, i_n) \) at row \( \mathbf{E}_{-i_n,i_n-1} + \mathbf{E}_{-i_n-1,i_n} \) in \( M_l \).
  3. Halt.

**Remark 20.** Since we only consider finite posets \( \mathcal{P} \), the above algorithm must halt in a finite number of steps.

### 5.2 Index formula

In this section, we determine an index formula for connected, height-(0,1), type-C Lie poset algebras. Throughout, we let \( V(\mathcal{P}) \) and \( E(\mathcal{P}) \) denote, respectively, the set of vertices and edges of \( RG(\mathcal{P}) \).

**Lemma 1.** Let \( \mathcal{P} \) be a connected, height-(0,1), type-C poset.

- If \( RG(\mathcal{P}) \) contains an odd cycle, then the rank of \( B(\mathcal{P}) \) in \( C(g_C(\mathcal{P})) \) is \( |V(\mathcal{P})| \);
- if \( RG(\mathcal{P}) \) contains no odd cycles, then the rank of \( B(\mathcal{P}) \) in \( C(g_C(\mathcal{P})) \) is \( |V(\mathcal{P})| - 1 \);

**Proof.** The proof breaks into 4 cases.

**Case 1:** If \( RG(\mathcal{P}) \) contains a self-loop, then the algorithm proceeds by removing adjacent edges to vertices defining self-loops and making sure, post edge removal, that such adjacent vertices define self-loops. Thus, as \( RG(\mathcal{P}) \) is assumed to be connected, this implies that the algorithm above halts with \( \Gamma_n = (G_n, M_n) \), where

- \( G_n = RG(\mathcal{P}') \) for the poset \( \mathcal{P}' \) satisfying \( \mathcal{P}' = \mathcal{P} \) as sets and \( -i \preceq_{\mathcal{P}'} i \) for all \( i \in \mathcal{P} \), and
- \( M_n \) is the \( B(\mathcal{P}') \) block in \( C(g_C(\mathcal{P}')) \) with (potentially) additional zero rows;

that is, each of the \( |\mathcal{P}^+| = |V(\mathcal{P})| \) columns of \( M_n \) has a corresponding unique row with nonzero entry in that column. Thus, the result follows.
Case 2: If $RG(P)$ contains an odd cycle consisting of $n > 1$ vertices and no self-loops, then the algorithm starts by removing an edge from an odd cycle and adding a self-loop. From here, the algorithm proceeds, and the result follows, as in Case 1.

Case 3: If $RG(P)$ is a tree, then the algorithm halts at $\Gamma_n = (G_n, M_n)$, where if $i_1$ is the specified degree one vertex of $\text{Step 8}$, then given $i_j$ and $i_k$ adjacent in $RG(P)$ with $i_j$ contained in the unique path from $i_1$ to $i_k$ we have that row $E_{-i_1,i_k} - E_{-i_k,i_j}$ is the unique row with nonzero entry in column $E_{-i_k,-i_k} - E_{i_k,i_k}$; that is, all $|P^*| - 1 = |V(P)| - 1$ rows of $M_n$ are linearly independent and the result follows.

Case 4: If $RG(P)$ contains an even cycle and no odd cycles, then the algorithm removes edges from even cycles (introducing zero rows), until the resulting graph is a tree. From here, the algorithm proceeds, and the result follows, as in Case 3.

As a result of Lemma 1, we get the following.

**Theorem 6.** If $P$ is a connected, height-(0,1), type-C poset, then

$$\text{ind} \, g_C(P) = |E(P)| - |V(P)| + 2\delta_o,$$

where $\delta_o$ is the indicator function for $RG(P)$ containing no odd cycles.

**Proof.** To start, by Theorem 2, we know that

$$\text{ind} \, g_C(P) = \text{dim}(C(g_C(P))) - \text{rank}(C(g_C(P))),$$

where $\text{dim}(C(g_C(P))) = |E(P)| + |V(P)|$. Furthermore, $\text{rank}(C(g_C(P))) = 2 \cdot \text{rank}(B(P))$. By Lemma 1, we know that if $RG(P)$ contains an odd cycle, then $\text{rank}(B(P)) = |V(P)|$; that is,

$$\text{ind} \, g_C(P) = |E(P)| + |V(P)| - 2|V(P)| = |E(P)| - |V(P)|.$$

Otherwise, $\text{rank}(B(P)) = |V(P)| - 1$ so that

$$\text{ind} \, g_C(P) = |E(P)| + |V(P)| - 2(|V(P)| - 1) = |E(P)| - |V(P)| + 2.$$

**Theorem 7.** If $P$ is a height-(0,1), type-C poset, then $g_C(P)$ is Frobenius if and only if each connected component of $RG(P)$ contains a single cycle which consists of an odd number of vertices.

**Proof.** Combining Theorem 4 and Corollary 2, we find that $P$ is a disjoint union of Frobenius, height-(0,1), type-C posets. Assuming $P$ is connected, note that $|E(P)| - |V(P)| \geq -1$ with equality when $RG(P)$ is a tree. Thus, by Theorem 6, if $P$ is Frobenius, then $RG(P)$ must contain an odd cycle. If $RG(P)$ contains an odd cycle, then $|E(P)| - |V(P)| \geq 0$ with equality if and only if $RG(P)$ contains a single odd cycle. Therefore, the result follows.

**Remark 21.** To ease discourse in the following section, type-C (resp. $B$, $D$) posets corresponding to Frobenius, type-C (resp. $B$, $D$) Lie poset algebras are referred to as Frobenius, type-C (resp. $B$, $D$) posets.

### 6 Spectrum

In this section, given a Frobenius type-B, C, or D Lie poset algebra generated by a height-(0,1) poset $P$ we determine the form of a particular Frobenius functional (see Theorem 8) as well as its corresponding principal element (see Theorem 9). With a principal element in hand, we are then able to determine the form of the spectrum for such Lie algebras (see Theorem 10).
Remark 22. As in previous sections, all results and proofs will be in terms of type-C Lie poset algebras. With only minor modifications in the proofs, though, it is easily seen that all results also apply to type-B and D Lie poset algebras.

Remark 23. Throughout this section, we let $E_{i,j}$ denote the functional which returns the $i,j$-entry of a matrix.

Theorem 8. If $\mathcal{P}$ is a Frobenius, height-$(0,1)$, type-C poset and

$$F_{\mathcal{P}} = \sum_{i \leq j} E_{-i,j}^* + \sum_{i \leq j} \delta_{i \leq j} \cdot E_{-i,i}^*,$$

where $\delta_{-}$ is the Kronecker delta function, then $F_{\mathcal{P}}$ is a Frobenius functional on $g_C(\mathcal{P})$.

Remark 24. Throughout this section, we will assume that if $\mathcal{P}$ is a Frobenius, height-$(0,1)$, type-C poset such that $RG(\mathcal{P})$ contains an odd cycle consisting of the vertices $\{i_1, \ldots, i_n\}$, then $i_1 < i_2 < \ldots < i_n$. Such an assumption does not limit the results of this section since there exists a type-C poset $\mathcal{P}'$ isomorphic to $\mathcal{P}$ for which $RG(\mathcal{P}')$ has the given property.

Lemma 2. If $\mathcal{P}$ is a Frobenius, height-$(0,1)$, type-C poset, then $B \in g_C(\mathcal{P}) \cap \ker(F_{\mathcal{P}})$ must satisfy $E_{i,i}^*(B) = 0$, for all $i \in \mathcal{P}$.

Proof. Given $B \in \ker(F) \cap g_d(\mathcal{P})$, we have

1. $F([E_{-i,j} + E_{-j,i}, B]) = -E_{-i,j}^*(B) + E_{j,i}^*(B)$, for all $-j, -i, i, j \in \mathcal{P}$ satisfying $-i \leq j$ and $i < j$; and
2. $F([E_{-i,i}, B]) = -E_{-i,i}^*(B) + E_{i,i}^*(B) = 0$, for $-i, i \in \mathcal{P}$ satisfying $-i \leq i$.

As a result of condition 1,

$$E_{-i,j}^*(B) = E_{j,i}^*(B),$$

for all $-j, -i, i, j \in \mathcal{P}$ contained in a connected component of $RG(\mathcal{P})$ satisfying $-i \leq j$ and $i < j$. Considering each connected component $C$ of $RG(\mathcal{P})$ separately, the proof breaks into two cases.

Case 1: $C$ contains a self-loop, say at vertex $i_1$. Condition 2 and the fact that $B \in \mathfrak{sp}(\mathcal{P})$ combine to imply that $E_{i_1,i_1}^*(B) = E_{-i_1,-i_1}^*(B) = 0$. Thus, considering (1) and the fact that $C$ is connected, we may conclude that $E_{i,i}^*(B) = 0$ for all $i \in \mathcal{P}$ contained in $C$.

Case 2: $C$ contains an odd cycle consisting of $n > 1$ elements $\{i_1, \ldots, i_n\}$ satisfying $i_1$ is adjacent to $i_n$ and $i_2, i_j$ is adjacent to $i_{j-1}$ and $i_{j+1}$, for $2 \leq j \leq n - 1$, and $i_1 < \ldots < i_n$. Restricting condition 1 to $\{i_1, \ldots, i_n\}$ we find that

$$E_{i_1,i_2}^*(B) = -E_{i_2,i_1}^*(B)$$
$$E_{i_2,i_3}^*(B) = -E_{i_3,i_2}^*(B)$$
$$\ldots$$
$$E_{i_{n-1},i_n}^*(B) = -E_{i_n,i_{n-1}}^*(B)$$
$$E_{i_n,i_1}^*(B) = -E_{i_1,i_n}^*(B);$$

but this implies that $E_{-i_1,-i_1}^*(B) = E_{i_1,i_1}^*(B)$. Thus, the result follows as in Case 1. \square

Lemma 3. If $\mathcal{P}$ is a Frobenius, height-$(0,1)$, type-C poset, then $B \in g_C(\mathcal{P}) \cap \ker(F_{\mathcal{P}})$ must satisfy $E_{i,j}^*(B) = 0$, for all $-i, j \in \mathcal{P}$ satisfying $-i \leq j$. 

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Proof. Note that $B \in \ker(F_{P}) \cap g_{C}(P)$ must satisfy
\[
F([E_{-i,-i} - E_{i,i}, B]) = \sum_{i \leq j} E_{-i,j}^{*}(B) + \sum_{k < i} E_{-k,i}^{*}(B) + 2 \cdot \delta_{-i \leq i} \cdot E_{-i,i}^{*}(B) = 0,
\]
for all $-i, i \in P$. First, we show that $E_{-i,j}^{*}(B) = 0$, for all $-i, j \in P$ satisfying $-i \leq j$, $\{j, i\}$ does not define
an edge of an odd cycle in $RG(P)$, and $j \neq i$. Let $\Gamma_{1} = RG(P)$

**Step 1**: Consider all $-i, i \in P$ for which $i$ is a vertex of degree one in $\Gamma_{1}$, say $i$ is adjacent to $j$, then
\[
F([E_{-i,-i} - E_{i,i}, B]) = E_{-i,j}^{*}(B) = 0 \text{ (or } E_{-j,i}^{*}(B) = 0). \]
Since $B \in \sp(|P|)$, this further implies that
\[
E_{-j,i}^{*}(B) = 0 \text{ (or } E_{-i,j}^{*}(B) = 0). \]
Removing each such vertex $i$ and edge $\{i, j\}$ of $\Gamma_{1}$ results in $\Gamma_{2}$.

**Step k**: Consider all $-i, i \in P$ for which $i$ is a vertex of degree one in $\Gamma_{k}$, say $i$ is adjacent to $j$, then taking
into account the results **Step 1** through **Step k − 1**, we must have
\[
F([E_{-i,-i} - E_{i,i}, B]) = E_{-i,j}^{*}(B) = 0 \text{ (or } E_{-j,i}^{*}(B) = 0). \]
Once again, since $B \in \sp(|P|)$, this further implies that
\[
E_{-j,i}^{*}(B) = 0 \text{ (or } E_{-i,j}^{*}(B) = 0). \]
Removing each such vertex $i$ and edge $\{i, j\}$ of $\Gamma_{k}$ results in $\Gamma_{k+1}$.

Since $RG(P)$ is finite, there must exist $m$ for which the connected components of $\Gamma_{m}$ are odd cycles. Thus,
$E_{-i,j}^{*}(B) = 0$ for all $-i, j \in P$ satisfying $-i \leq j$, $\{j, i\}$ does not define an edge of an odd cycle in $RG(P)$,
and $j \neq i$. It remains to consider $E_{i,j}^{*}(B)$ corresponding to components of $\Gamma_{m}$, which break into two cases:

**Case 1**: Components consisting of a self-loop at vertex $i$. In this case, utilizing the results of **Step 1** through **Step m** above, we must have
\[
F([E_{-i,-i} - E_{i,i}, B]) = 2 \cdot E_{-i,i}^{*}(B) = 0. \]

**Case 2**: Components consisting of an odd cycle with $n > 1$ elements $\{i_{1}, \ldots, i_{n}\}$, where $i_{1}$ is adjacent to $i_{n}$ and $i_{2}$ is adjacent to $i_{1}$ and $i_{3}$, for $2 \leq j \leq n - 1$, and $i_{1} < \ldots < i_{n}$. Restricting equation (2) to
$\{i_{1}, \ldots, i_{n}\}$ and utilizing the results of **Step 1** through **Step m** above, we find
\[
F([E_{-i_{1},-i_{1}} - E_{i_{1},i_{1}}, B]) = E_{-i_{1},i_{2}}^{*}(B) + E_{-i_{1},i_{n}}^{*}(B) = 0
\]
\[
F([E_{-i_{2},-i_{2}} - E_{i_{2},i_{2}}, B]) = E_{-i_{1},i_{2}}^{*}(B) + E_{-i_{2},i_{3}}^{*}(B) = 0
\]
\[
\vdots
\]
\[
F([E_{-i_{n-1},-i_{n-1}} - E_{i_{n-1},i_{n-1}}, B]) = E_{-i_{n-1},i_{n-1}}^{*}(B) = 0
\]
\[
F([E_{-i_{n},-i_{n}} - E_{i_{n},i_{n}}, B]) = E_{-i_{1},i_{n}}^{*}(B) + E_{-i_{n-1},i_{n}}^{*}(B) = 0.
\]
Thus,
\[
E_{-i_{1},i_{n}}^{*}(B) = -E_{i_{1},i_{2}}^{*}(B) = -E_{i_{2},i_{3}}^{*}(B) = \ldots = -E_{i_{n-2},i_{n-1}}^{*}(B) = -E_{i_{n-1},i_{n}}^{*}(B);
\]
that is, $E_{i_{1},i_{n}}^{*}(B) = -E_{i_{1},i_{n}}^{*}(B) = 0$ and thus $E_{i_{j},i_{j+1}}^{*}(B) = 0$, for $j = 1, \ldots, n - 1$. Since $B \in \sp(|P|)$,
we also get that $E_{i_{j},i_{j+1}}^{*}(B) = E_{i_{j+1},i_{j}}^{*}(B) = 0$, for $j = 1, \ldots, n - 1$. The result follows. \(\square\)
Proof of Theorem 8. By Lemma 2 and Lemma 3, if $B \in \ker(F_P) \cap \mathfrak{g}_C(P)$, then $B = 0$. \hfill \Box

Remark 25. Given a poset $P$ and the functional $F_P$ as in Theorem 8, to determine the form of the principal element $\hat{F}_P = \sum_{i \in P} c_i E_{i,i}$ note that, since $\hat{F}_P \in sp(\mathbb{F}[P])$, it must be the case that ($*$) $c_i = -c_{-i}$, for all $i \in P$. Furthermore, since $F_P = ad \hat{F}_P$, it must be the case that (**$c_{-i} < c_j = 1$, for $-i, j \in P$ with $E_{i,j}^* = c_{-i}$ a summand of $F_P$. It should be clear that ($*$) and (**$ combine to completely characterize $\hat{F}_P$.

Theorem 9. If $P$ is a Frobenius, height-(0,1), type-C poset, then $\hat{F}_P = \sum_{i \in P} c_i E_{i,i}$ satisfies

$$c_i = \begin{cases} \frac{1}{2}, & i \in P^+; \\ -\frac{1}{2}, & i \in P^-.
\end{cases}$$

Proof. For each edge of $RG(P)$, between, say, vertices $i$ and $j$ with $i < j$, we get the conditions $c_{-i} = 1 + c_j$ and $c_{-j} = 1 + c_i$. Thus, for each connected component $C$ of $RG(P)$ we have that $c_{-i} = c_{-j}$ and $c_i = c_j$, for all $i, j$ representing vertices of $C$ with $-i, -j \in P^-$ and $i, j \in P^+$. For $i$ representing a vertex in $C$ this implies that $c_{-i} = 1 + c_i = 1 - c_{-i}$; that is, $c_{-i} = \frac{1}{2}$ and $c_i = -\frac{1}{2}$. The result follows. \hfill \Box

Theorem 10. If $P$ is a Frobenius, height-(0,1), type-C poset, then $\mathfrak{g}_C(P)$ has a spectrum consisting of an equal number of 0’s and 1’s.

Proof. Consider the basis $\mathcal{B}_C(P)$ for $\mathfrak{g}_C(P)$. Given the form of $\hat{F}_P$ found in Theorem 9, we see that basis elements contained in the set

$$\{E_{-i,-i} - E_{i,i} \mid -i, i \in P\}$$

are eigenvectors of $ad(\hat{F}_P)$ with eigenvalue 0, and basis elements contained in the set

$$\{E_{-i,j} + E_{-j,i} \mid -i, -j, i, j \in P, -j \leq i, -i \leq j\} \cup \{E_{-i,i} \mid -i, i \in P, -i \leq i\}$$

are eigenvectors of $ad(\hat{F}_P)$ with eigenvalue 1. By Theorem 7, we must have that

$$|\{E_{-i,-i} - E_{i,i} \mid -i, i \in P\}| = |P^+| = |\{E_{-i,j} + E_{-j,i} \mid -i, -j, i, j \in P, -j \leq i, -i \leq j\} \cup \{E_{-i,i} \mid -i, i \in P, -i \leq i\}|.$$

Therefore, since $\mathcal{B}_C(P)$ is a basis for $\mathfrak{g}_C(P)$, the result follows. \hfill \Box

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