Methods of construction and study of Frankl system self-similar solutions in the hyperbolic case

T Shemyakina\textsuperscript{1} and S Alekseenko\textsuperscript{2}
\textsuperscript{1}Peter the Great Saint-Petersburg Polytechnical University, 29 Politechnicheskaya Street 195251 Saint-Petersburg, Russia, \textsuperscript{2}Nizhny Novgorod State Technical University n a R E Alekseev, 24 Minin Street 603950 Nizhny Novgorod, Russia

E-mail: sh_tat@mail.ru

Abstract. Self-similar solution of the Frankl system in the hyperbolic case was found. The Frankl system is a system of mixed type equations. Under certain conditions, it describes a model of the membrane theory of shells. The Frankl system describes a stationary irrotational motion of an ideal gas in the transition vicinity from subsonic to supersonic speeds. We find a sufficient condition on the initial data that guarantees existence of a global classical solution continued from a local solution. The proof of the nonlocal solvability of the problem in the original variables is based on the additional argument method. It allowed justify and construct a numerical solution. Numerical experiments were carried out for model examples of the Frankl system.

1. Introduction
The main object of research is the Frankl system:
\[
\begin{align*}
\frac{\partial u(x, y)}{\partial y} - P(x, y, u(x, y), v(x, y)) \frac{\partial v(x, y)}{\partial x} &= 0, \\
\frac{\partial u(x, y)}{\partial x} + Q(x, y, u(x, y), v(x, y)) \frac{\partial v(x, y)}{\partial y} &= 0,
\end{align*}
\]
where \(u(x, y), v(x, y)\) are unknown functions; \(P(x, y, u(x, y), v(x, y)) \geq p_0 > 0\), \(p_0, q_0\) are constant, \(Q(x, y, u(x, y), v(x, y)) \geq q_0 > 0\).

The Frankl system is a system of two quasi-linear differential equations of the first order. It is a system of mixed type. Equations with opposite signs represent a system of elliptic type. Equations with the same signs represent a system of hyperbolic type.

For the first time the famous physicist F. I. Frankl \cite{1} has combined the study of the system of equations of mixed type with the study of stationary problems of transonic gas dynamics. Some particular cases of the Frankl system can be found in neutron transport theory, theory of thermoelasticity, in describing of the distribution of electrons in the electric field of the sprite, and so on. Bibliography for mixed type equations can be found in many monographs, in particular J. M. Krikunov \cite{2}, S. A. Chaplygin, A. V. Bitsadze, F. Tricomi, S. V. Falkovich, B. L. Rozhdestvensky and...
N. N. Yanenko are the famous scientists. They studied problems for systems of mixed type equations in particular cases, when the equation system coefficients $P, Q$ possess values of the constants or independent variables. Most often, the study came to the study of differential equations of the second order. Mathematical theory to study a model system of mixed type equations is systematically presented in the monograph by T. V. Chekmarev [3]. He considered the application of this theory to the solution of problems for mixed type systems.

A variety of approaches to the study of mixed systems of differential equations have their advantages and disadvantages. As a rule, the method of research is to convert the nonlinear equations to linear equations in the hodograph plane. Then a linear system of equations is solved. But in the general case, a return to the original variables is a more difficult task than the original task. Application of method of characteristics for quasi-linear equation systems introduces a superposition of unknown functions in the relevant integral equations. In the method of characteristics inverse transformation existence is a condition for original problem solvability. In many cases the problem of inverse function determination is so difficult that it is not solved. The existence of inverse variable transformations is taken as a condition.

To overcome these difficulties we investigate the problem using the method of additional argument [4–6]. We consider the Frankl system in quite a general form. The method we propose does not replace other known methods. It amplifies them and allows to determine more precisely the conditions for the solvability in the original variables. In this work [6] we found sufficient conditions for a nonlocal solvability of the Cauchy problem for a system of two quasi-linear differential equations of first order. At that, the system of equations is represented in characteristic form. However, for the Frankl system with arbitrarily given initial data, conditions for the existence of the global solution established in work [6], are not fulfilled.

In this paper, we prove the global existence of self-similar solutions of the Frankl system in hyperbolic case, subject to the coordination of the initial data. We have done numerical calculations for model problems of the Frankl system. We found numerical Frankl system solution by grid method. We found self-similar solution of the Cauchy problem by the method of additional argument. We have built a numerical solution by the method of successive approximations.

**2. The statement of the problem**

We consider the Frankl system in the hyperbolic case:

\[
\begin{align*}
\frac{\partial u(x, y)}{\partial y} - P(u(x, y), v(x, y)) \frac{\partial v(x, y)}{\partial x} &= 0, \\
\frac{\partial u(x, y)}{\partial x} - Q(u(x, y), v(x, y)) \frac{\partial v(x, y)}{\partial y} &= 0.
\end{align*}
\]

(1)

The Cauchy problem for the system of equations (1) is:

\[
u(0, x) = \varphi(x), \quad v(0, x) = \psi(x), \quad x \in (-\infty, \infty), \quad \varphi(x), \psi(x) \in \overline{C^2}(R^1)
\]

(2)

Formulation of the problem is as follows. Find an unknown functions $u(x, y), v(x, y)$, satisfying the system of equations (1) and the initial conditions (2) in the domain:

\[
\Omega_y = \{(x, y) : x \in (-\infty, \infty), y \in [0, Y], Y > 0\},
\]

**3. Method of problem solving**

The problem (1), (2) is solved by the method of additional argument. The starting system is transformed to the system of symmetric quasi-linear equations with help of Riemann invariants:

\[
\begin{align*}
\frac{\partial u(x, y)}{\partial y} + b(u(x, y), v(x, y)) \frac{\partial v(x, y)}{\partial y} &= 0, \\
\frac{\partial u(x, y)}{\partial y} - b(u(x, y), v(x, y)) \frac{\partial v(x, y)}{\partial y} &= 0.
\end{align*}
\]

(3)
The result is a characteristic shape in the form of a system of differential equations for functions \( z_i(x, y), u(x, y), v(x, y) \):

\[
\begin{align*}
\frac{\partial z_1(x, y)}{\partial y} - e(u(x, y), v(x, y)) \frac{\partial z_1(x, y)}{\partial x} &= z_i(x, y) \cdot F_1(u(x, y), v(x, y), z_i(x, y), z_2(x, y)), \\
\frac{\partial z_2(x, y)}{\partial y} + e(u(x, y), v(x, y)) \frac{\partial z_2(x, y)}{\partial x} &= z_2(x, y) \cdot F_2(u(x, y), v(x, y), z_i(x, y), z_2(x, y)), \\
\frac{\partial u(x, y)}{\partial y} - e(u(x, y), v(x, y)) \frac{\partial u(x, y)}{\partial x} &= \frac{\partial}{\partial y} \left( z_2(x, y) \right), \\
\frac{\partial v(x, y)}{\partial y} - e(u(x, y), v(x, y)) \frac{\partial v(x, y)}{\partial x} &= \frac{\partial}{\partial y} \left( z_2(x, y) \right),
\end{align*}
\]

with the initial conditions:

\[
\begin{align*}
z_i(0, x) &= \frac{\partial}{\partial x} \left( \Phi_1(x) \right) = P(\phi(x), \psi(x)) \psi'(x) + e(\phi(x), \psi(x)) \phi'(x), \\
z_i(0, x) &= \frac{\partial}{\partial x} \left( \Phi_2(x) \right) = P(\phi(x), \psi(x)) \psi'(x) - e(\phi(x), \psi(x)) \phi'(x), \\
u(0, x) &= \phi(x), \\
v(0, x) &= \psi(x),
\end{align*}
\]

where \( e(u(x, y), v(x, y)) = \sqrt{\frac{P(u(x, y), v(x, y))}{Q(u(x, y)v(x, y))}} \).

We define the coordination of initial conditions on demand:

\[
z_i(0, x) = 0, \quad P(\phi(x), \psi(x)) \psi'(x) - e(\phi(x), \psi(x)) \phi'(x) = 0,
\]

then the Cauchy problem can be represented as:

\[
\begin{align*}
\frac{\partial z_2(x, y)}{\partial y} + e(u(x, y), v(x, y)) \frac{\partial z_2(x, y)}{\partial x} &= z_2(x, y) \cdot F_2(u(x, y), v(x, y), z_i(x, y), z_2(x, y)). \\
z_2(0, x) &= 0,
\end{align*}
\]

There exists a trivial solution \( z_j(x, y) = 0 \).

Thus, we get the coordination of the initial data for the Frankl system:

\[
\phi'(x) = b(\phi(x), \psi(x)) \psi'(x)
\]

The characteristic form is converted to the following form:

\[
\begin{align*}
\frac{\partial u(x, y)}{\partial y} - e(u(x, y), v(x, y)) \frac{\partial u(x, y)}{\partial x} &= 0, \quad u(x, 0) = \phi(x), \\
\frac{\partial v(x, y)}{\partial y} - e(u(x, y), v(x, y)) \frac{\partial v(x, y)}{\partial x} &= 0, \quad v(x, 0) = \psi(x), \\
\frac{\partial z_i(x, y)}{\partial y} - e(u(x, y), v(x, y)) \frac{\partial z_i(x, y)}{\partial x} &= z_i^2(x, y) \cdot E(u(x, y), v(x, y)), \\
z_i(0, x) &= \Phi_i(x) = 2P(\phi(x), \psi(x)) \psi'(x) = 2e(\phi(x), \psi(x)) \phi'(x),
\end{align*}
\]

where \( E(u(x, y), v(x, y)) = \frac{1}{2e(u, v)} \frac{\partial e(u, v)}{\partial u} + \frac{1}{2P(u, v)} \frac{\partial e(u, v)}{\partial v} > 0 \).

In the domain of \( \Omega_0 = \{ (x, y, s) : x \in (-\infty, \infty), \quad 0 \leq s \leq y \leq Y, \quad Y > 0 \} \) we consider the extended characteristic system, in accordance with the method of additional argument:
We found the solution of the characteristic form as follows: Frankl system (1) with initial conditions (2) has a unique bounded on the set \( x \in \mathbb{R}^d \) solution \( u(x,y) \in C^{k+1}(\Omega_0) \), \( v(x,y) \in C^{k+1}(\Omega_0) \).

The functions \( u(x,y) \), \( v(x,y) \) are determined from the constructed system of equations (4) with \( s = y \). The initial conditions (2) are used for \( 0 \leq y \leq Y_0 \leq Y \) exactly where the constant \( Y_0 \) is determined from the initial data.

We have \( u(x,y) = u_1(x,y,y) \), \( v(x,y) = v_1(x,y,y) \), \( z_1(x,y) = w(x,y,y) \).

Integrating (4) with respect to \( s \), we get the solution of system of ordinary differential equations of the first order:

\[
\begin{aligned}
\frac{d\eta(x,y,s)}{ds} &= -e_1(u_1(x,y,y),v_1(x,y,y)), \\
\frac{du_1(x,y,s)}{ds} &= 0, \\
\frac{dv_1(x,y,s)}{ds} &= 0, \\
\frac{dw(x,y,s)}{ds} &= w^2(x,y,s)E(u_1(x,y,y),v_1(x,y,y)), \\
\end{aligned}
\]

\( \eta(x,y,s), u_1(x,y,y), v_1(x,y,y), w(x,y,y) \) depend not only on the original independent variables \( x,y \), but they depend on an additional argument \( s \).

In the framework of an additional argument in [4–5] we determined the conditions of the local existence of a bounded solution for the Frankl system in the hyperbolic case.

Using \( P(u,v) \in C^{2,2}(V_v), Q(u,v) \in C^{2,2}(V_v), V_v = \{(u,v): u \in [-K,K], v \in [-K,K], K > 0\} \), we get: the Frankl system (1) with initial conditions (2) has a unique bounded on the set \( x \in \mathbb{R}^d \) solution \( u(x,y) \in C^{k+1}(\Omega_0) \), \( v(x,y) \in C^{k+1}(\Omega_0) \).

The functions \( u(x,y) \), \( v(x,y) \) are determined from the constructed system of equations (4) with \( s = y \). The initial conditions (2) are used for \( 0 \leq y \leq Y_0 \leq Y \) exactly where the constant \( Y_0 \) is determined from the initial data.

We have \( u(x,y) = u_1(x,y,y) \), \( v(x,y) = v_1(x,y,y) \), \( z_1(x,y) = w(x,y,y) \).

Integrating (4) with respect to \( s \), we get the solution of system of ordinary differential equations of the first order:

\[
\begin{aligned}
\eta(x,y,s) &= x + (y-s)e_1(u_1(x,y,y),v_1(x,y,y)), \\
u_1(x,y,s) &= \varphi(x + y \cdot e_1(u_1(x,y,y),v_1(x,y,y)), \\
v_1(x,y,s) &= \varphi(x + y \cdot e_1(u_1(x,y,y),v_1(x,y,y)), \\
w(x,y,s) &= \frac{\Phi(x + y \cdot e_1(u_1,v_1))}{1 - s \cdot E(u_1,v_1) \cdot \Phi(x + y \cdot e_1(u_1,v_1))}. \\
\end{aligned}
\]

We found the solution of the characteristic form as follows:

\[
\begin{aligned}
u(x,y) &= \varphi(x + y \cdot e(u(x,y),v(x,y))), \\
v(x,y) &= \varphi(x + y \cdot e(u(x,y),v(x,y))), \\
z_1(x,y) &= \frac{\Phi(x + y \cdot e(u,x,y))}{1 - y \cdot E(u,v) \cdot \Phi(x + y \cdot e(u,v))}. \\
\end{aligned}
\]

Thus, the problem is reduced to the study of global solvability of a system of two algebraic equations:

\[
\begin{aligned}
u(x,y) &= \varphi(x + ye(u(x,y),v(x,y))), \\
v(x,y) &= \varphi(x + ye(u(x,y),v(x,y))). \\
\end{aligned}
\]

The standard method of successive approximations is used to prove the existence of continuous constrained solutions together with derivatives of the first order.

We introduce the notation: 
\( C_\varphi = \max\{ \sup_{(x,y) \in \mathbb{R}^d} \Phi(x), \sup_{(x,y) \in \mathbb{R}^d} \Phi'(x) \} \), 
\( C_v = \max\{ \sup_{(x,y) \in \mathbb{R}^d} \varphi(x), \sup_{(x,y) \in \mathbb{R}^d} \varphi'(x) \} \).
\[ N_u = \max \left\{ \sup_{\alpha_i} \left| \frac{\partial u}{\partial x} \right|, \sup_{\beta_j} \left| \frac{\partial u}{\partial y} \right| \right\}, \quad N_v = \max \left\{ \sup_{\alpha_i} \left| \frac{\partial v}{\partial x} \right|, \sup_{\beta_j} \left| \frac{\partial v}{\partial y} \right| \right\}. \]

Solutions \( u(x, y), v(x, y) \) have an estimations:
\[ |u(x, y)| \leq C_{\varphi}, \quad |v(x, y)| \leq C_{\psi}. \] (6)
We demand the implementation of inequalities:
\[ \Phi_1(x \cdot y \cdot e(u, v)) = 2P(\varphi, \psi)\psi'(x + y \cdot e(u, v)) \leq 0, \]
\[ \Phi_2(x + y \cdot e(u, v)) = 2e(\varphi, \psi)\varphi'(x + y \cdot e(u, v)) \leq 0, \]
not to a function \( z(x, y) \) become infinite.
Then for the derivatives of initial functions, we obtain the following conditions:
\[ \varphi'(x) \leq 0, \quad \psi'(x) \leq 0. \] (7)
Derivatives of functions \( u(x, y), v(x, y) \) have the following form:
\[ \frac{\partial u}{\partial x} = \varphi' \left( 1 - y \left( \frac{\varphi' e}{\partial u} + \psi' e \frac{\partial e}{\partial v} \right) \right), \quad \frac{\partial u}{\partial y} = \varphi' \left( 1 - y \left( \frac{\varphi' e}{\partial u} + \psi' e \frac{\partial e}{\partial v} \right) \right), \]
\[ \frac{\partial v}{\partial x} = \psi' \left( 1 - y \left( \frac{\varphi' e}{\partial u} + \psi' e \frac{\partial e}{\partial v} \right) \right), \quad \frac{\partial v}{\partial y} = \psi' \left( 1 - y \left( \frac{\varphi' e}{\partial u} + \psi' e \frac{\partial e}{\partial v} \right) \right). \]

We obtain the estimates of derivatives of functions \( u(x, y), v(x, y) \):
\[ \left| \frac{\partial u(x, y)}{\partial x} \right| \leq N_u, \quad \left| \frac{\partial u(x, y)}{\partial y} \right| \leq N_u, \quad \left| \frac{\partial v(x, y)}{\partial x} \right| \leq N_v, \quad \left| \frac{\partial v(x, y)}{\partial y} \right| \leq N_v, \] (8)
then we obtain the main condition:
\[ \varphi'(x) \frac{\partial e}{\partial u} + \psi'(x) \frac{\partial e}{\partial v} \leq 0. \] (9)

Global assessments received (6), (8) for the functions \( u(x, y), v(x, y) \) provide an opportunity to extend the solution on any given interval \([0, Y]\). We take as initial values \( u(x, Y_0), v(x, Y_0) \) and we will extend the solution on some interval \([Y_0, Y]\). Then, we take as initial values \( u(x, Y), v(x, Y) \) and we will extend the solution to the interval \([Y_0, Y]\). The range of solvability interval will not decrease, as it is defined by global assessments (6), (8). As a result, the solution \( u(x, y), v(x, y) \) for finite number of steps can be extended to the entire specified period \([0, Y]\).

The uniqueness of the solution \( u(x, y), v(x, y) \) is proved by using similar global assessments, which allowed establishing the convergence of successive approximations in the works of authors [4 - 6].

Eventually we get the following results.

**Theorem.** Suppose the functions \( P(u(x, y), v(x, y)) \geq p_0 > 0, \quad Q(u(x, y), v(x, y)) \geq q_0 > 0, \)
\( p_0, q_0 \) are constant , \( P(u, v) \in C^{2, 2}(\Omega_v), \quad Q(u, v) \in C^{2, 2}(\Omega_v). \)
Let the functions \( \phi(x), \psi(x) \in C^2(R^1) \) satisfy the conditions:
1. \( \phi'(x) = b(\phi(x), \psi(x))\psi'(x) \left( b(u, v) = \sqrt{P(u, v) \cdot Q(u, v)} \right), \)
2. \( \phi'(x) \leq 0, \quad \psi'(x) \leq 0, \)
3. \( \phi'(x) \frac{\partial e}{\partial u} + \psi'(x) \frac{\partial e}{\partial v} \leq 0. \)

Then for any \( Y \) the Cauchy problem (1), (2) has a unique bounded on the set \( x \in R^1 \) solution \( u(x, y) \in C^{2, 1}(\Omega_x), \quad v(x, y) \in C^{2, 1}(\Omega_x). \)
Points to note. Functions \( u(x, y), v(x, y) \) become infinite, if both of the inequalities:

\[
\phi'(x) > 0, \quad \psi'(x) > 0, \quad \frac{\partial \phi}{\partial u} + \frac{\partial \psi}{\partial v} > 0.
\]

4. Calculations

Numerical calculations were carried out for many modeling tasks of the Frankl system, in particular [7 – 9]. In works [8 – 9] we studied the problem of modeling processes in the environment.

In this article, numerical calculations are carried out for model problem Frankl system (1), (2) with the following data:

\[
P(u, v) = v^2, \quad Q(u, v) = 1.0, \quad e(u, v) = v, \quad b(u, v) = v,
\]

\[
\phi(x) = 1.0 - 0.5x + 0.125x^2, \quad \psi(x) = 1.0 - 0.5x, \quad x \in [0.0, 1.0], \quad y \in [0.0, 1.0],
\]

The number of dividing points portioning the interval of variables set \( x: \quad m = 6, \quad y: \quad n = 11 \).

Analytical solution of the problem has the following form:

\[
u(x, y) = \frac{(x - 2)^2 + (y + 2)^2}{2(y + 2)^2}, \quad v(x, y) = \frac{(2 - x)}{(y + 2)}
\]

The first experiment. The problem was solved by grid method (Godunov scheme).

Boundary conditions have the form:

\[
u(x_0, y) = \phi_1(y) = 0.5 + \frac{2}{(y + 2)^2}, \quad u(x_m, y) = \phi_2(y) = 0.5 + \frac{2}{(y + 2)^2},
\]

\[
u(x_0, y) = \psi_1(y) = \frac{2}{(y + 2)}, \quad v(x_m, y) = \psi_2(y) = \frac{1}{(y + 2)}.
\]

We have error of approximation of the numerical solution to the exact solution:

\[
\delta u = 0.3750, \quad \delta v = 0.2955.
\]

The second experiment. The problem was solved by the method of successive approximations according to the formulas:

\[
u(x, y) = \phi(x + ye(u(x, y), v(x, y))),
\]

\[
v(x, y) = \psi(x + ye(u(x, y), v(x, y))).
\]

We have error of approximation of the numerical solution to the exact solution:

\[
itr = 1, \quad \delta u = 0.2778, \quad \delta v = 0.3333;
\]

\[
itr = 2, \quad \delta u = 0.0972, \quad \delta v = 0.1667;
\]

\[
itr = 3, \quad \delta u = 0.0590, \quad \delta v = 0.0833.
\]

The third experiment. The problem is solved by the hodograph method.

\[
x = \frac{1}{2} c_1 u^2 + \frac{1}{4} c_2 v^4 + \frac{1}{3} c_3 v^3 + c_4 u + c_5,
\]

\[
y = c_1 u v + c_2 u + c_3 v + c_5.
\]

where the coefficients \( c_i \) are const, \( i = 1, 2, 3, 4, 5 \).

In this example we found constants, but the formulas of the functions \( u(x, y), v(x, y) \) are very tedious. We do not show them.

5. Conclusion

In the present work we proposed one more method to study the Frankl system of hyperbolic case. This is the method of additional argument. This method made it possible to prove the global existence of self-similar solution, subject to the approval of the initial data. Requirements for the initial data are quite simple. In addition, we found the algorithm for finding the numerical solution of the problem. Numerical calculations were performed on model problems of the Frankl system. Application of
additional argument method has simplified numerical calculations and increased accuracy of estimation evaluation. In further studies we suggest the use of hybrid methods, based on methods of grids and additional argument.

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