Cops on the dots in a mathematical model of urban crime and police response

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- Agent-based (discrete) and PDE (continuous) spatial models (focus of this talk)
- Gang rivalries and interactions
- Organized crime
Outline of this talk

- Review prior work in spatio-temporal crime modeling.
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- Discuss extension of prior work to include dynamic police decision-making.
- Introduce a numerical method for the police’s optimization problem.
- Present and discuss numerical results.
Prior work in spatio-temporal crime modeling
Short et al. (*M3AS*, 2008) present an agent-based model of criminal behavior grounded in routine activity theory.
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In the agent-based model, criminals move around in a spatial environment. Each location in this environment has an *attractiveness* value $A$ that increases if a crime has been committed recently nearby.

Criminals are more likely to move to and strike at more attractive locations (where $A$ is greater).
Continuum equations

Though this is a discrete model, it has a continuum limit as a system of coupled PDEs:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} &= \nabla \cdot \left( \nabla \rho - 2\rho \nabla \log A \right) - \rho A + A - A_0, \\
\frac{\partial A}{\partial t} &= \eta \Delta A + \rho A - A + A_0.
\end{align*}
\]
Continuum equations

- Though this is a discrete model, it has a continuum limit as a system of coupled PDEs:

  - Criminal density:
    
    $\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \rho - 2\rho \nabla \log A) - \rho A + \bar{A} - A_0$

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    \]

  - **Attractiveness:**
    \[
    \frac{\partial A}{\partial t} = \eta \nabla A + \rho A - A + A_0
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Continuum equations

\[ \frac{\partial A}{\partial t} = \eta \triangle A + \rho A - A + A_0 \]

\[ \frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \rho - 2\rho \nabla \log A) - \rho A + \bar{A} - A_0 \]

- This is a system of reaction-diffusion equations similar to the Keller-Segel model of chemotaxis.
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- This is a system of reaction-diffusion equations similar to the Keller-Segel model of chemotaxis.
- A linear instability in these equations gives rise to hot spots in solutions, which accords with real crime data.
Adding police

- The original model assumes that police won’t react to the emergence of hot spots.
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\(d\) is a radial function centered at the center of a hot spot that is 1 outside the hot spot and less than 1 inside.
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Dispersion or displacement?

- Depending on the parameters, the addition of deterrence $d$ can disperse or only displace hot spots.
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- Depending on the parameters, the addition of deterrence $d$ can disperse or only displace hot spots.
- Weakly nonlinear stability analysis let us predict which parameter choices lead to one or the other. (Short et al. 2010, *SIADS*.)
Dynamic policing: an optimal control approach

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Cops on the dots in a mathematical model of urban crime and policing
Dynamic policing

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- Rather than prescribing the deterrence factor \( d \) with a specific form constant in time, assume the deterrence arises out of a dynamic deployment of police forces.
- Let \( \kappa(x, y) \) represent the amount of resources the police choose to deploy to point \( (x, y) \). Think of cops walking a beat or squad cars patrolling.

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- Let $\kappa(x, y)$ represent the amount of resources the police choose to deploy to point $(x, y)$. Think of cops walking a beat or squad cars patrolling.
- The effect on the criminals is $d(\kappa(x, y))$, where $d : \mathbb{R}_+ \rightarrow (0, 1]$ is a deterrence function specifying the impact of having a certain amount of police at a given point.
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- Rather than prescribing the deterrence factor $d$ with a specific form constant in time, assume the deterrence arises out of a dynamic deployment of police forces.
- Let $\kappa(x, y)$ represent the amount of resources the police choose to deploy to point $(x, y)$. Think of cops walking a beat or squad cars patrolling.
- The effect on the criminals is $d(\kappa(x, y))$, where $d : \mathbb{R}_+ \rightarrow (0, 1]$ is a deterrence function specifying the impact of having a certain amount of police at a given point.
- $d$ is a smooth, convex, decreasing function; think of $d(k) = e^{-k}$. 
The new system

- The PDE system becomes

\[
\frac{\partial A}{\partial t} = \eta \Delta A + \rho A - (A - A_0)
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\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \rho - 2\rho \nabla \log A) - \rho A + (\bar{A} - A_0)
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- At each time \( t \), the police minimize the total crime occurring in the domain \( \Omega \) at \( t \), namely \( \int_{\Omega} d(\kappa(x, t))\rho(x, t)A(x, t)dx \).
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\int_{\Omega} d(\kappa(x, t))\rho(x, t)A(x, t)dx.
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- They face a resource constraint \( \int_{\Omega} \kappa dx = K \) for fixed \( K > 0 \) and a positivity constraint \( \kappa \geq 0 \).
The optimization problem

\[ \kappa = \arg \min \left\{ \int_{\Omega} d(k) \rho A \, dx : k \geq 0, \int_{\Omega} k \, dx = K \right\} \]

is convex, so we expect a unique solution to exist.
Dual formulation of the optimization problem

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- **Theorem**: This unique solution is

\[ \kappa = \begin{cases} 
(d')^{-1}(-\lambda/\rho A) & \text{if } \rho A > -\lambda/d'(0), \\
0 & \text{otherwise},
\end{cases} \]

where \( \lambda \) is the Lagrange multiplier associated to the \( L^1 \) constraint \( \int_{\Omega} \kappa \, dx = K \).
Outline of the proof

- Calculus of variations tells us there exists a Lagrange multiplier $\lambda > 0$ such that $d'(\kappa)\rho A + \lambda = 0$ wherever $\kappa > 0$. 

- Thus $\kappa$ is decreasing, so $\kappa$ is an increasing function of $\rho A$.

- Thus there exists some threshold value $C$ of $\rho A$ so that $\kappa > 0$ where $\rho A > C$ but $\kappa = 0$ where $\rho A < C$.

- If $\kappa$ is not continuous across the level set $\{\rho A = C\}$, then it would be advantageous to take some police in $\{C < \rho A < C + \delta\}$ and redeploy them to $\{C - \delta < \rho A < C\}$.

- Thus $\kappa$ must be continuous.

- The choice $C = -\lambda/\kappa'(0)$ allows $\kappa$ to be continuous across $\{\rho A = C\}$.
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- The choice $C = -\lambda/d'(0)$ allows $\kappa$ to be continuous across $\{\rho A = C\}$. 
Free-boundary problem

The system is now

\[ A_t = \eta \Delta A + d(\kappa) \rho A - (A - A_0) \]
\[ \rho_t = \nabla \cdot (\nabla \rho - 2\rho \nabla \log A) - d(\kappa) \rho A + d(\kappa) (\overline{A} - A_0) \]
\[ \kappa = (d')^{-1}(-\lambda/\rho A) \chi\{\rho A > -\lambda/d'(0)\} \]
\[ K = \int_{\{\rho A > -\lambda/d'(0)\}} (d')^{-1}(-\lambda/\rho A) \, dx. \]
Free-boundary problem

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This boundary is the level set \( \{\rho A = -\lambda/d'(0)\} \).
A central finding of Short et al. (2008) was that the homogeneous steady state solution (without police) is linearly unstable in certain parameter regimes. This instability gives rise to hot spots.
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Our model also has a homogeneous steady state solution:

\[ \tilde{A} = d(\tilde{\kappa}) \tilde{B} + A_0, \quad \tilde{\rho} = \frac{\tilde{B}}{\tilde{A}}, \quad \tilde{\kappa} = \frac{K}{|\Omega|}. \]
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Is this linearly stable?
Following Turing stability analysis, let’s do a standard Fourier perturbation of \((A, \rho)\) about \((\tilde{A}, \tilde{\rho})\): 
\[
A = \tilde{A} + \epsilon a, \quad \rho = \tilde{\rho} + \epsilon r
\]
with 
\[
a(x, t) = r(x, t) = e^{\sigma t + in(x_1 + x_2)}.
\]
Choosing the perturbation

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  with \(a(x, t) = r(x, t) = e^{\sigma t + in(x_1 + x_2)}\).

- We do not perturb \(\kappa\). Instead, let it be the solution to the optimal control problem given this \(A\) and \(\rho\). Calculus of variations lets us separate \(O(\epsilon)\) and \(o(\epsilon)\) terms.
The linearized system

The linearized system is

\[
\begin{pmatrix}
-\eta n^2 - 1 + D(\tilde{\kappa})\tilde{\rho} & D(\tilde{\kappa})\tilde{A} \\
\frac{2\tilde{\rho}}{\tilde{A}} n^2 - d(\tilde{\kappa})\tilde{\rho} & -n^2 - d(\tilde{\kappa})\tilde{A}
\end{pmatrix}
\begin{pmatrix}
a \\
r
\end{pmatrix}
= \sigma
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\]

where \( D(\tilde{\kappa}) = d(\tilde{\kappa}) - d'(\tilde{\kappa})^2 / d''(\tilde{\kappa}) \).
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The system is stable when both eigenvalues \( \sigma \) are negative.
Exponential deterrence: unconditional linear stability

- For some choices of $d$, the system is unconditionally linearly stable.

- [Additional points or details can be added here if available.]
For some choices of $d$, the system is unconditionally linearly stable.

For example, if $d(k) = e^{-k}$, then $D(k) = 0$ identically, so the system becomes

\[
\begin{pmatrix}
-\eta n^2 - 1 & 0 \\
\frac{2\tilde{\rho}}{\tilde{A}} n^2 - d(\tilde{\kappa})\tilde{\rho} & -n^2 - d(\tilde{\kappa})\tilde{A}
\end{pmatrix}
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\end{pmatrix} \begin{pmatrix} a \\ r \end{pmatrix} = \sigma \begin{pmatrix} a \\ r \end{pmatrix}.$$  

- The matrix is triangular, so the eigenvalues are the diagonal entries.

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- Contrast with the original Short model.
Cops on the dots in a mathematical model of urban crime and policing.
We discretize space into a square grid of length $N$ with periodic boundary conditions. We discretize time into equally spaced steps.
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We use standard spectral methods to get $A^{n+1}$ and $\rho^{n+1}$ from $(A^n, \rho^n, \kappa^n)$.
We discretize space into a square grid of length $N$ with periodic boundary conditions. We discretize time into equally spaced steps.

We use standard spectral methods to get $A^{n+1}$ and $\rho^{n+1}$ from $(A^n, \rho^n, \kappa^n)$.

Solving for $\kappa^{n+1}$ from $(A^{n+1}, \rho^{n+1})$ can be efficient with a novel algorithm.
A dual method for $\kappa$

- As before in the continuous theory, $\kappa_{i,j}^{n+1}$ is positive if $\rho_{i,j}^{n+1} A_{i,j}^{n+1}$ exceeds some threshold value and 0 otherwise.
A dual method for $\kappa$

- As before in the continuous theory, $\kappa_{i,j}^{n+1}$ is positive if $\rho_{i,j}^{n+1} A_{i,j}^{n+1}$ exceeds some threshold value and 0 otherwise.
- The problem then reduces to finding the threshold value $-\lambda^{n+1} / d'(0)$ and all the indices $(i, j)$ for which $\rho_{i,j}^{n+1} A_{i,j}^{n+1} > -\lambda^{n+1} / d'(0)$. 

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Easy case: Police deploy everywhere

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Easy case: Police deploy everywhere

- If $-\lambda^{n+1}/d'(0)$ is less than all of the values of $\rho_{i,j}^{n+1} A_{i,j}^{n+1}$, then the police deploy everywhere.
- In this case we need to find $\lambda^{n+1}$ solving

$$H(\lambda) := \sum_{j=1}^{N^2} (d')^{-1}(\lambda/f_j) = K.$$
Easy case: Police deploy everywhere

- If \(-\lambda^{n+1}/d''(0)\) is less than all of the values of \(\rho_{i,j}^{n+1}A_{i,j}^{n+1}\), then the police deploy everywhere.
- In this case we need to find \(\lambda^{n+1}\) solving

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H(\lambda) := \sum_{j=1}^{N^2} (d')^{-1}(\lambda/f_j) = K.
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- Standard iterative methods suffice for this.
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- If $-\lambda^{n+1}/d''(0)$ is less than all of the values of $\rho^n_{i,j} A^{n+1}_{i,j}$, then the police deploy everywhere.
- In this case we need to find $\lambda^{n+1}$ solving

$$H(\lambda) := \sum_{j=1}^{N^2} (d')^{-1}(\lambda/f_j) = K.$$ 

- Standard iterative methods suffice for this.
- Unfortunately you must compute $H$ at every iteration.
Less easy case: Police do not deploy everywhere

- Sort the values of $\rho^{n+1} A^{n+1}$ in descending order. Where we had a 2D $N \times N$ vector we now have $f$, a 1D vector of length $N^2$. 

- Finding the cutoff crime level $-\lambda/d'(0)$ is equivalent to finding the cutoff index $J$ so that

$$G(J) := J \sum_{j=1}^{J} \left( d'(0) f_{J} / f_{j} \right) = K.$$ 

$G$ is increasing, so this has a unique solution.

We use a discrete false position (linear interpolation) method to find $J$ iteratively.
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$$G(J) := \sum_{j=1}^{J} (d')^{-1}(d'(0)f_J/f_j) = K.$$
Less easy case: Police do not deploy everywhere

- Sort the values of $\rho^{n+1} A^{n+1}$ in descending order. Where we had a 2D $N \times N$ vector we now have $f$, a 1D vector of length $N^2$.

- Finding the cutoff crime level $-\lambda/d'(0)$ is equivalent to finding the cutoff index $J$ so that

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- We use a discrete false position (linear interpolation) method to find $J$ iteratively.
Speed

- Two issues might make us worried about speed:
  - Sorting an $N^2$-long vector every time step.
  - Evaluating $(d')^{-1}$ up to $N^2$ times every iteration of the inner loop.

Remember the sort between time steps. It's expensive the first time, but you get to start close to the correct sorting every time step.

Remember $J$ or $\lambda$ from the last time step to reduce the number of iterations this time.

In practice solving for $(A_{n+1}, \rho_{n+1})$ takes longer than solving for $\kappa_{n+1}$. 
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Cops on the dots in a mathematical model of urban crime and police response.
Results
Few police resources: $K = 300$
Many police resources: $K = 700$
Intermediate police resources: $K = 500$
Long-term intermediate, sped up 50x
For different values of $K$, the maximum value of $\kappa$ at time $t$ is plotted for two classes of solutions. The solid line comes from solutions seeded with an initial condition of stable hot spots. The dashed line comes from solutions seeded with the homogeneous equilibrium values. Inset pictures are snapshots of $A$. 

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Interpretation

- Introducing only a few police into an environment with hot spots of crime can lower the maximum level of crime but displace the crime to other areas.

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Cops on the dots in a mathematical model of urban crime and policing
Interpretation

- Introducing only a few police into an environment with hot spots of crime can lower the maximum level of crime but displace the crime to other areas.
- If you have enough police and have them pursue the crime as it’s displaced, you have the potential to remove hot spots entirely.

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Cops on the dots in a mathematical model of urban crime and police
Introducing only a few police into an environment with hot spots of crime can lower the maximum level of crime but displace the crime to other areas.

If you have enough police and have them pursue the crime as it’s displaced, you have the potential to remove hot spots entirely.

But if you don’t have enough police you can end up with “warm worms” where some areas still have higher crime than others.
Future work
Open mathematical questions

- Linear stability for general deterrence
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- Linear stability for general deterrence
- Bifurcation theory of the two stable states
Open mathematical questions

- Linear stability for general deterrence
- Bifurcation theory of the two stable states
- Status of the "warm worms"
Extensions of the model

- Delayed, incomplete, or noisy police information

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Extensions of the model

- Delayed, incomplete, or noisy police information
- More strategic criminal decisions: game theory?
Questions?