Classical Electrodynamics with Dual Potentials

M. Baker
University of Washington, Seattle, WA 98105

James S. Ball
University of Utah, Salt Lake City, UT 84112

and

F. Zachariasen
California Institute of Technology, Pasadena, CA 91125

Abstract

We present Dirac’s method for using dual potentials to solve classical electrodynamics for an oppositely charged pair of particles, with a view to extending these techniques to non-Abelian gauge theories.

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I. Solving Maxwell’s Equations with Dual Potentials

Dirac[1] showed that Maxwell’s equations could be extended to include both electrically and magnetically charged particles by connecting the magnetically charged particles to strings. In the absence of magnetically charged particles one can apply Dirac’s method to ordinary electrodynamics by connecting electrically charged particles to strings. In this formulation Maxwell’s equations become equations for dual potentials \( C_\mu \) whose sources are the polarization currents produced by the Dirac strings. The potentials themselves depend upon the location of the strings but they yield the same string independent electromagnetic fields as the usual procedure.

If in addition the dual potentials \( C_\mu \) are minimally coupled to Higgs fields, these fields necessarily carry magnetic charge. Such a theory describes the motion of electrically charged particles connected by Dirac strings in a dual superconductor. If extended to non-Abelian gauge theory\(^2\) it becomes a concrete realization of the Mandelstam ’tHooft\[^3,\,^4\] picture of color confinement as a manifestation of dual superconductivity. We have found that in order to understand this mechanism for confinement it is very helpful to first have a clear picture of how dual potentials work in ordinary electrodynamics. Therefore, in this paper we present an elementary discussion of Dirac’s method applied to electrodynamics and work out some simple examples. Our discussion is entirely pedagogical and contains nothing new although some specific results obtained here may not be readily accessible elsewhere. Our goal here is to make the following paper as comprehensible as possible.

Consider first a sourceless linear dielectric medium. Then Maxwell’s equations

\[
\begin{align*}
\nabla \cdot \vec{D} &= 0 \\
\nabla \cdot \vec{B} &= 0 \\
\n\nabla \times \vec{H} - \partial_0 \vec{D} &= 0 \\
\n\nabla \times \vec{E} + \partial_0 \vec{B} &= 0 \\
\vec{D} &= \varepsilon \vec{E} \\
\vec{B} &= \mu \vec{H},
\end{align*}
\]

(1.1)

can be solved by introducing vector potentials in either of two ways. The conventional choice is to write

\[
\vec{B} \equiv \nabla \times \vec{A}, \quad \vec{E} \equiv -\partial_0 \vec{A} - \nabla A_0,
\]

(1.2a)
in which case eqs. (1.1b) become kinematical identities and the dynamics is contained in eqs. (1.1a). The vector $A^\mu = (A_0, \vec{A})$ is called the vector potential. The alternate (dual) choice is to write

$$\vec{D} = -\nabla \times \vec{C}, \quad \vec{H} = -\partial_0 \vec{C} - \nabla C_0,$$

(1.2b)

in which case eqs. (1.1a) are kinematical identities and eqs. (1.1b) contain the dynamics. The vector $C^\mu = (C_0, \vec{C})$ is called the dual vector potential.

Let us first use $C_\mu$ to solve the source free Maxwell eqs. (1.1) in order to get accustomed to using the dual potential. We first write eqs. (1.2b) in covariant form by defining:

$$G_{0k} = H_k, \quad G_{ij} = \epsilon_{ijk} D_k,$$

(1.3)

so that eqs. (1.2b) take the form

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu.$$

(1.4)

In a relativistic medium $\epsilon = \frac{1}{\mu}$. Then using eq. (1.3) we can write the constitutive equations as

$$E_i = \frac{\mu}{2} \epsilon_{ijk} G_{jk}, \quad B_i = \mu G_{0i},$$

(1.5)

and Maxwell’s eqs. (1.1b) as

$$\partial^\alpha \mu G_{\alpha\beta} = 0.$$

(1.6)

Eqs. (1.6) for $C_\mu$ have the same form as the usual Maxwell equations for $A_\mu$, obtained from (1.1a), with the replacement $\mu \rightarrow \epsilon$, and they are solved in the same way. Eq. (1.5) then gives the electromagnetic fields $\vec{E}$ and $\vec{B}$ in terms of $C_\mu$.

Electric current sources $j_\mu = (\rho, \vec{j})$ appear only in eqs. (1.1a) and not in eqs. (1.1b). Hence in the presence of electric currents eqs. (1.1b) remain valid and are still kinematic identities in terms of $A^\mu$. Eqs. (1.1a), in contrast, are no longer
identities in terms of $C^\mu$. However, Dirac has shown how to generalize eqs. (1.2b) in order to satisfy eqs. (1.1a) with dual potentials $C^\mu$ even in the presence of electric currents.

When charged particles are present eqs. (1.1a) become

\[
\begin{align*}
\nabla \cdot \vec{D} &= \rho, \\
\nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t},
\end{align*}
\tag{1.7}
\]

Suppose that the total charge $Q = \int \rho d\vec{x} = 0$. (If $Q \neq 0$, then there will be Dirac strings extending to infinity, but nothing essential will be changed.) Then we can always find a polarization vector $\vec{P}$ and a magnetization vector $\vec{M}$ so that

\[
\rho = -\nabla \cdot \vec{P}, \quad \vec{J} = \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t},
\tag{1.8}
\]

where $\vec{P}$ is the dipole moment per unit volume and $\vec{M}$ is the magnetic moment per unit volume. Inserting eq. (1.8) into (1.7), we obtain

\[
\nabla \cdot (\vec{D} + \vec{P}) = 0, \quad \nabla \times (\vec{H} - \vec{M}) - \frac{\partial (\vec{D} + \vec{P})}{\partial t} = 0.
\]

Hence,

\[
\vec{D} = -\nabla \times \vec{C} - \vec{P}, \quad \vec{H} = -\nabla \vec{C}_0 - \frac{\partial \vec{C}}{\partial t} + \vec{M}.
\tag{1.9}
\]

Eqs. (1.7) then become kinematical identities and eqs. (1.1b) contain the dynamics as before. Using the definitions (1.3) of $G_{\mu\nu}$, we can write eqs. (1.9) in the covariant form

\[
G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + G_{s_{\mu\nu}},
\tag{1.10}
\]

where the tensor $G_{s_{\mu\nu}}$ has components

\[
G_{s_{0k}} = M_k, \quad G_{s_{ij}} = -\epsilon_{ijk} P_k,
\tag{1.11}
\]

and we have now specialized to the case where $\vec{P}$ and $\vec{M}$ arise from Dirac strings connecting the charged particles; hence the superscript “s” on $G_{s_{\mu\nu}}$. Eq. (1.10) is just
the generalization of eq. (1.4) to account for the presence of charged particles. Eq. (1.11) shows that $G^{s}_{\mu\nu}$ is the dual of the polarization tensor. Eqs. (1.1b) and (1.1c) are unchanged so that eqs. (1.5) and (1.6) remain the same as does the definition (1.3) of $G_{\mu\nu}$. The effect of the charged particles is to change the relation (1.4) between $G_{\mu\nu}$ and $C_{\mu}$ to (1.10) where $G^{s}_{\mu\nu}$ is determined in terms of $\rho$ and $\vec{j}$ by solving eqs. (1.8) for $\vec{F}$ and $\vec{M}$. This is done in Section II for a pair of oppositely charged particles.

Substituting eq. (1.10) into eq. (1.6) we obtain

$$\partial^\alpha \mu (\partial_\alpha C_\beta - \partial_\beta C_\alpha) = -\partial^\alpha \mu G^{s}_{\alpha\beta}, \quad (1.12)$$

which determines the dual potentials $C_{\mu}$ in terms of $G^{s}_{\mu\nu}$. Eqs. (1.12) provide an alternate form of Maxwell’s equations which are completely equivalent to the usual form expressed in terms of the vector potential $A_{\mu}$, namely

$$\partial^\alpha \epsilon (\partial_\alpha A_\beta - \partial_\beta A_\alpha) = j_\beta. \quad (1.13)$$

All text books on electricity and magnetism could be rewritten using only dual potentials $C_{\mu}$ satisfying eq. (1.12) and the same electromagnetic forces between charged particles would be obtained. The potentials themselves, however, could be completely different. For example in a dielectric medium having a wave number dependent dielectric constant $\epsilon(q) \to 0$ as $q^2 \to 0$ (corresponding to antiscreening at large distances), the potentials $A_{\mu}$ determined from eq. (1.13) would be singular at large distances, while the dual potentials $C_{\mu}$ satisfying eq. (1.12) with $\mu = 1/\epsilon \to \infty$ as $q^2 \to 0$ would be screened at large distances. Use of the potentials $A_{\mu}$ to describe this system would introduce singularities which do not appear in the dual potentials $C_{\mu}$. Hence the dual potentials are the natural choice to describe a medium with long range antiscreening.

Note that for $\mu = \epsilon = 1$, $\vec{B} = \vec{H}$, $\vec{D} = \vec{E}$ and substituting eqs. (1.9) in (1.1b) gives the equation for the dual potentials in three-dimensional notation:

$$\tilde{\nabla} \cdot (-\nabla C_0 - \partial_0 \vec{C}) = -\tilde{\nabla} \cdot \vec{M}, \quad (1.14)$$
\[ \nabla \times (-\nabla \times \vec{C}) + \partial_0 (\nabla \times \vec{C} - \nabla \vec{C}_0) = \nabla \times \vec{P} - \frac{\partial \vec{M}}{\partial t}. \] (1.15)

These equations are identical to eq. (1.12) with \( \mu = 1 \). They have the same form as the equations for \( A_\mu \), the ordinary vector potentials in a polarizable medium with \( \vec{P} \) and \( \vec{M} \) interchanged. For example \(-\nabla \cdot \vec{M}\) is the source of \( \vec{C}_0 \). However eqs. (1.14) and (1.15) describe the electrodynamics of electrically charged particles moving in the vacuum and \( \vec{P} \) and \( \vec{M} \) are the polarization and magnetization respectively of the Dirac strings attached to these particles, as we shall now see.

II. The Fields of a Pair of Oppositely Charged Particles

We now apply the results of the previous section to the case of two particles of charge \( e(-e) \) moving along trajectories \( \vec{x}_1(t)(\vec{x}_2(t)) \) in free space with \( \mu = \epsilon = 1 \). Then

\[ \rho(\vec{x}, t) = e[\delta^3(\vec{x} - \vec{x}_1(t)) - \delta^3(\vec{x} - \vec{x}_2(t))], \] (2.1)

and

\[ \vec{j}(\vec{x}, t) = e[\vec{v}_1 \delta^3(\vec{x} - \vec{x}_1(t)) - \vec{v}_2 \delta^3(\vec{x} - \vec{x}_2(t))], \] (2.2)

where \( \vec{v}_i = \frac{d\vec{x}_i}{dt}, i = 1,2 \). We must find a polarization \( \vec{P} \) and magnetization \( \vec{M} \) satisfying eq. (1.8) with \( \rho \) and \( \vec{j} \) given by eqs. (2.1) and (2.2). The solution of this problem was given by Dirac.\[1\] Let \( \vec{y}(\sigma, t) \) be any line \( L(t) \) connecting \( \vec{x}_2(t) \) and \( \vec{x}_1(t) \) i.e., \( \vec{y}(\sigma_1, t) = \vec{x}_1(t), \vec{y}(\sigma_2, t) = \vec{x}_2(t), \sigma_2 \leq \sigma \leq \sigma_1 \). (See Figure 1.) On each element \( d\vec{y} \) of \( L \) place a dipole moment \( d\vec{p} = e d\vec{y} \). It is evident from Fig. 1 that the charge and current density produced by the sum of these dipoles is that due to the pair of moving oppositely charged particles, namely eqs. (2.1) and (2.2). To obtain (2.1) formally we note that the dipole moment per unit volume \( \vec{P} \) is

\[ \vec{P}(x) = e \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} d\vec{y} \delta(\vec{x} - \vec{y}) = e \int_{\sigma_2}^{\sigma_1} d\sigma \frac{\partial \vec{y}(\sigma, t)}{\partial \sigma} \delta(\vec{x} - \vec{y}(\sigma, t)). \] (2.3)
Then

\[ -\nabla \cdot \vec{P} = -e \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} \, d\vec{y} \cdot \nabla_x \delta(\vec{x} - \vec{y}) = \rho(\vec{x}). \tag{2.4} \]

Furthermore since the line element \( d\vec{y} \) is moving with velocity \( \vec{v} = \frac{\partial}{\partial t} \vec{y}(\sigma, t) \), the string \( L \) in Fig. 1 has a magnetization

\[ \vec{M} = e \int_{\vec{x}_2}^{\vec{x}_1} d\vec{y} \times \frac{\partial \vec{y}}{\partial t} \delta(\vec{x} - \vec{y}) = e \int_{\sigma_2}^{\sigma_1} d\sigma \frac{\partial \vec{y}}{\partial \sigma} \times \frac{\partial \vec{y}}{\partial t} \delta(\vec{x} - \vec{y}(\sigma, t)). \tag{2.5} \]

Next we show explicitly that eq. (2.3) for \( \vec{P} \) and (2.5) for \( \vec{M} \) give via eq. (1.8) the current density. From eq. (2.3) we have

\[
\frac{\partial \vec{P}}{\partial t} = e \frac{\partial}{\partial t} \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} d\vec{y}(\sigma, t) \delta(\vec{x} - \vec{y}(\sigma, t)) \]

\[ = e \left[ \frac{d\vec{x}_1}{dt}(t) \delta(\vec{x} - \vec{x}_1(t)) - \frac{d\vec{x}_2}{dt}(t) \delta(\vec{x} - \vec{x}_2(t)) \right] \]

\[ + \frac{e}{\delta t} \left[ \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} d\vec{y}(\sigma, t + \delta t) \delta(\vec{x} - \vec{y}(\sigma, t + \delta t)) - \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} d\vec{y}(\sigma, t) \delta(\vec{x} - \vec{y}(\sigma, t)) \right] \]

\[ = \vec{j}(\vec{x}, t) + \frac{e}{\delta t} \oint d\vec{y} \delta(\vec{x} - \vec{y}). \tag{2.6} \]

The first term on the right-hand side of eq. (2.6) arises from differentiating with respect to the end points with the path fixed. The line integral in eq. (2.6) is over a closed contour running from \( \vec{x}_2(t) \) to \( \vec{x}_1(t) \) along the path \( \vec{y}(\sigma, t + \delta t) \) and returning
to \( \tilde{x}_2(t) \) along \( \tilde{y}(\sigma, t) \). (See Fig. 2.) We denote \( \tilde{y}(\sigma, t + dt) - \tilde{y}(\sigma, t) = \delta \tilde{y} \) and the element of area \( d\tilde{y} \times \delta \tilde{y} \equiv d\tilde{S} \). Then by Stokes’ theorem
\[
\frac{e}{\dot{\delta}t} \oint d\tilde{y} \delta (\tilde{x} - \tilde{y}) = -\frac{e}{\dot{\delta}t} \int d\tilde{S} \times \nabla_{\tilde{y}} \delta (\tilde{x} - \tilde{y})
\]
\[
= e \int d\tilde{y} \times \frac{\delta \tilde{y}}{\dot{\delta}t} \times \nabla_{\tilde{x}} \delta (\tilde{x} - \tilde{y}) = -\nabla \times \tilde{M}. \tag{2.7}
\]
Eqs. (2.6) and (2.7) yield eq. (1.8) with \( \tilde{j} \) given by eq. (2.2) as asserted.

Eqs. (1.14) and (1.15) with \( \tilde{P} \) and \( \tilde{M} \) given by eqs. (2.3) and (2.5) respectively determine \( C_{\mu} \). To obtain the explicit form of the covariant version of these equations we note that

\[
G_{\mu \nu}^s = -\epsilon_{\mu \nu \alpha \beta} \int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \frac{\partial y^\alpha}{\partial \tau} \frac{\partial y^\beta}{\partial \sigma} \delta^4(x - y), \tag{2.8}
\]
where
\[
x^{\mu} = (t, \tilde{x}), \quad y^{\mu} = (y^0, \tilde{y}),
\]
and
\[
d\tau = \sqrt{(dy^0)^2 - (d\tilde{y})^2}.
\]
Eq. (2.8) is the standard covariant form for the Dirac string field \( G_{\mu \nu}^s \).\cite{1} To show that the expressions (1.11) and (2.8) for \( G_{\mu \nu}^s \) are the same first set \( \mu = 0 \) and \( \nu = k \) in eq. (2.8):
\[
G_{0k}^s = -\epsilon_{k \mu \nu \alpha \beta} \int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \frac{\partial y^\mu}{\partial \tau} \frac{\partial y^n}{\partial \sigma} \delta^4(\tilde{x} - \tilde{y}) \delta(y_0 - t)
\]
\[
= e \int_{\sigma_2}^{\sigma_1} \left( \frac{\partial \tilde{y}}{\partial \sigma} \times \frac{\partial \tilde{y}}{\partial t} \right)_k \delta^3(\tilde{x} - \tilde{y}) = M_k. \tag{2.9a}
\]
Next set $\mu = i$ and $\nu = j$ in eq. (2.8):

$$G^s_{ij} = -\varepsilon_{ijk} \int_{\tau_1}^{\tau_2} d\tau_1 \int_{\sigma_2}^{\sigma_1} d\sigma \frac{\partial y^0}{\partial \tau} \frac{\partial y^k}{\partial \sigma} \delta^3(\vec{x} - \vec{y}) \delta(\tau_0 - t)$$

$$= -\varepsilon_{ijk} \int_{\sigma_2}^{\sigma_1} d\sigma \frac{\partial y^k}{\partial \sigma} \delta^3(\vec{x} - \vec{y}) = -\varepsilon_{ijk} P_k.$$  \hspace{1cm} (2.9b)

Thus eq. (2.8) is just the covariant version of eqs. (2.3) and (2.5).

Eq. (1.12) with $\mu = 1$, and $G^s_{\mu\nu}$ given by eq. (2.8) is the covariant form of eqs. (1.14) and (1.15) determining the dual potential produced by a pair of oppositely charged particles moving in the vacuum. The resulting $C_\mu$ will depend upon the location of the string, but this dependence will drop out in the expression for the electromagnetic field tensor $G_{\mu\nu}$. We will show in the next section how eqs. (1.14) and (1.15) with $\vec{P}$ and $\vec{M}$ given by eqs. (2.3) and (2.5) produce the usual expressions for the electric and magnetic fields of slowly moving particles.

To conclude this section we note that the equation of motion (1.12) with $\mu = 1$, namely

$$\partial^\mu G_{\mu\nu} = 0,$$  \hspace{1cm} (2.10)

can be obtained from a Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} = \frac{1}{2} (\vec{H}^2 - \vec{D}^2).$$  \hspace{1cm} (2.11)

We will see in Section V that this Lagrangian gives not only the field equations but also the particle equations of motion.

III. Coulomb’s Law and the Biot Savart Law

To understand better how dual potentials work we will solve eqs. (1.14) and (1.15) for slowly moving particles. First consider charges at rest. Then $\vec{M} = 0$ and $\vec{P}$...
is time independent, and eq. (1.14) becomes \(\nabla^2 C_0 = 0\), (i.e., \(C_0 = 0\)), and eq. (1.15) reduces to

\[
\nabla \times (-\nabla \times \vec{C}_D) = \nabla \times \vec{P},
\]

where we have denoted the static solution \(\vec{C} = \vec{C}_D\) (for Dirac). Eq. (3.1) has the form of the equation for the vector potential due to a polarization current produced by the superposition (2.3) of dipoles. Thus, \(\vec{C}_D\) is just the vector potential produced by a superposition of point dipoles of strength \(-ed\vec{y}\) distributed uniformly along the string (see eq. 2.3), i.e., \(\vec{C}_D\) is given by[5]

\[
\vec{C}_D(\vec{x}) = -\frac{e}{4\pi} \int_{\vec{x}_2}^{\vec{x}_1} d\vec{y} \times \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3},
\]

Then

\[
-\nabla \times \vec{C}_D = \frac{e}{4\pi} \left\{ \int_{\vec{x}_2}^{\vec{x}_1} d\vec{y} \cdot \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3} - \int_{\vec{x}_2}^{\vec{x}_1} d\vec{y} \cdot \frac{\nabla \times (\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3} \right\}
\]

\[
= \frac{e}{4\pi} \left\{ \int_{\vec{x}_2}^{\vec{x}_1} d\vec{y} 4\pi \delta(\vec{x} - \vec{y}) + \frac{(\vec{x} - \vec{x}_1)}{|\vec{x} - \vec{x}_1|^3} - \frac{(\vec{x} - \vec{x}_2)}{|\vec{x} - \vec{x}_2|^3} \right\}
\]

\[
= \vec{P} + \vec{D}_C,
\]

where

\[
\vec{D}_C \equiv \vec{D}_{\text{Coulomb}} = \frac{e}{4\pi} \left( \frac{(\vec{x} - \vec{x}_1)}{|\vec{x} - \vec{x}_1|^3} - \frac{(\vec{x} - \vec{x}_2)}{|\vec{x} - \vec{x}_2|^3} \right),
\]

so that

\[
\vec{D} = -\nabla \times \vec{C}_D - \vec{P} = \vec{D}_C.
\]

The above elementary derivation of Coulomb’s law indicates that it really isn’t too much harder to work with dual potentials and strings than to work with ordinary
potentials and localized charges. The string cancellation mechanism in eq. (3.5) is depicted in Fig. 3 in which we have taken the string to be a straight line connecting $\vec{x}_2$ and $\vec{x}_1$. We see that $-\vec{\nabla} \times \vec{C}_{D}$ gives a divergence free field distribution. The singular field passing through the line $L$ is cancelled by the singular polarization $\vec{P}$, leaving a Coulomb field with a source at $\vec{x}_1$ and a sink at $\vec{x}_2$.

Next, let us solve eqs. (1.14) and (1.15) to first order in $\vec{v}_1$ and $\vec{v}_2$ and to zero order in the accelerations $\vec{\dot{v}}_1$ and $\vec{\dot{v}}_2$. First look at eq. (1.15). We choose the gauge $\vec{\nabla} \cdot \vec{C} = 0$. (Note $\vec{\nabla} \cdot \vec{C}_{D} = 0$). Then eq. (1.14) becomes

$$-\nabla^2 C_{0D} = -\vec{\nabla} \cdot \vec{M},$$ (3.6)

where we have denoted the solution $C_0 = C_{0D}$. Eq. (3.6) has the form of the equation for the scalar potential due to a polarization charge produced by the superposition (2.5) of dipoles. Hence $C_{0D}$ is just the scalar potential produced by a superposition of point dipoles of strength $ed\vec{y} \times \dot{\vec{y}}$ distributed uniformly along the string, i.e., $C_{0D}$ is

$$C_{0D} = \frac{e}{4\pi} \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} (d\vec{y} \times \dot{\vec{y}}) \cdot \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3}. \quad (3.7)$$

From eqs. (2.5) and (3.7) we see that the time derivative of $\vec{M}$ and $\vec{C}_{0D}$ do not contain terms linear in the velocities. The same is true for $\partial^2_{\vec{y}}\vec{C}_{D}$ calculated from eq. (3.2) with $\vec{x}_1 \rightarrow \vec{x}_1(t), \vec{x}_2 \rightarrow \vec{x}_2(t)$. Hence to first order in the velocities eq. (1.15) reduces to eq. (3.1), and so $\vec{C} = \vec{C}_{D}$ and $\vec{D} = \vec{D}_{C}$.

To calculate $\vec{H}$ we use eq. (1.9) with $\vec{C} = \vec{C}_{D}$ and $C_0 = C_{0D}$. We first calculate

$$-\frac{\partial}{\partial t} \vec{C}_{D} = \frac{e}{4\pi} \int_{\vec{x}_2(t)}^{\vec{x}_1(t)} d\vec{y} \times \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3}. \quad (3.8)$$

The evaluation of the right-hand side of eq. (3.8) parallels that of eq. (2.6) and we
obtain

\[-\frac{\partial}{\partial t} \vec{C}_D = \vec{H}_{BS} + \frac{e}{\delta t} \oint d\vec{y} \times \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3}, \quad (3.9)\]

where the term

\[\vec{H}_{BS} \equiv \vec{H}_{\text{Biot Savart}} \equiv \frac{e}{4\pi} \left\{ \vec{v}_1 \times \frac{(\vec{x} - \vec{x}_1(t))}{|\vec{x} - \vec{x}_1(t)|^3} - \vec{v}_2 \times \frac{(\vec{x} - \vec{x}_2(t))}{|\vec{x} - \vec{x}_2(t)|^3} \right\}, \quad (3.10)\]

arises from time differentiation of \(\vec{x}_1(t)\) and \(\vec{x}_2(t)\) in eq. (3.8) leaving the path fixed. The line integral in (3.9), over the same contour occurring in eq. (2.6), arises from moving the string keeping the endpoints fixed. Paralleling eq. (2.7) we then apply Stokes’ theorem to obtain

\[\frac{e}{\delta t} \oint d\vec{y} \times \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3} = \frac{e}{\delta t} \int \left( \frac{d\vec{s}}{4\pi} \times \vec{\nabla}_y \right) \times \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3} \]

\[\begin{aligned}
= e \int_{x_2(t)}^{x_1(t)} \left( d\vec{y} \times \frac{\delta \vec{y}}{\delta t} \right) \left( \vec{\nabla}_y \cdot \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3} \right) \\
+ \vec{\nabla}_x \frac{e}{4\pi} \int d\vec{y} \times \frac{\delta \vec{y}}{\delta t} \cdot \frac{(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^3} \\
= -M + \vec{\nabla} C_{0D}. \quad (3.11)
\end{aligned}\]

Eqs. (3.9) and (3.11) then yield

\[\vec{H} = -\frac{\partial \vec{C}_D}{\partial t} - \vec{\nabla} C_{0D} + M = \vec{H}_{BS}. \quad (3.12)\]

Thus we see that the Biot Savart magnetic field \(\vec{H}_{BS}\) comes from the time derivative of the limits \(\vec{x}_2(t)\) and \(\vec{x}_1(t)\) in the integral for \(\vec{C}_D\), eq. (3.8). The remaining string dependent part of \(\frac{\partial \vec{C}_D}{\partial t}\) cancels the contribution to \(\vec{H}\) coming from \(C_{0D}\) and \(M\).
We now use the solutions for $\vec{C}, \vec{C}_0, \vec{D}$ and $\vec{H}_{BS}$ to eliminate the fields in the Lagrangian $L$ given by

$$L = \int d\vec{x}L = \frac{1}{2} \int d\vec{x}(\vec{H}^2 - \vec{D}^2),$$

(3.13)

to second order in the velocities of the charged particles. The Lagrangian $L$, defined as the integral over the Lagrangian density (2.11) then becomes a function $L = L(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2)$ only of the positions and velocities of the charged particles. To higher order in the velocities one cannot eliminate the field degrees of freedom in $L$ because of the presence of radiation.

For particles at rest we have $\vec{C} = \vec{C}_0, C_0 = 0, \vec{H} = 0, \vec{D} = \vec{D}_C$ and

$$L(\vec{v}_1 = \vec{v}_2 = 0) = -\int d\vec{x} \frac{1}{2} \vec{D}_C^2 = \frac{-e^2}{4\pi|\vec{x}_1 - \vec{x}_2|},$$

(3.14)

where the self-energy has been subtracted. To first order in the velocities, $\vec{C} = \vec{C}_D$ and $\vec{D} = \vec{D}_C$ given by eq. (3.4) with $\vec{x}_1 \to \vec{x}_1(t), \vec{x}_2 \to \vec{x}_2(t)$. In other words the static field configuration follows adiabatically the motion of the charged particles. Furthermore since the Lagrangian $L$ is stationary about static solutions of the field equations we have

$$-\int d\vec{x} \frac{1}{2} \vec{D}^2 = -\frac{e^2}{4\pi|\vec{x}_1(t) - \vec{x}_2(t)|},$$

valid to second order in the velocities $\vec{v}_1$ and $\vec{v}_2$.

All the velocity dependence in $L$ then comes from $\int \vec{H}^2$ which to second order in the velocities is

$$\frac{1}{2} \int d\vec{x} \vec{H}^2 = \frac{1}{2} \int d\vec{x} (\vec{H}_{BS})^2 = -\frac{1}{2} \frac{e^2}{4\pi \bar{R}} \left[ \vec{v}_1 \cdot \vec{v}_2 + \frac{\vec{v}_1 \cdot \vec{R} \vec{v}_2 \cdot \vec{R}}{\bar{R}^2} \right],$$

(3.15)

where the self-energies have again been subtracted out and where $\bar{R} = \vec{x}_1(t) - \vec{x}_2(t)$. 

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Hence, we obtain for the second order Lagrangian $L$ (first obtained by Darwin\cite{7})

$$L(\vec{x}_1, \vec{x}_2; \vec{v}_1, \vec{v}_2) = \frac{1}{2} \int d\vec{x}[\vec{H}_{BS}^2 - \vec{D}_C^2]$$

$$= -\frac{e^2}{4\pi R} \frac{1}{2} \frac{e^2}{4\pi R} \left[ \vec{v}_1 \cdot \vec{v}_2 + \frac{\vec{v}_1 \cdot \vec{R} \vec{v}_2 \cdot \vec{R}}{R^2} \right].$$ (3.16)

As a final remark we connect the notation of this paper to that used in our previous work\cite{2} on QCD where we have introduced string fields $\vec{D}_s$ and $\vec{H}_s$ defined as

$$\vec{D}_s \equiv -\vec{P}, \quad \vec{H}_s \equiv \vec{M},$$ (3.17)

so that eqs. (1.9) take the form

$$\vec{D} = -\vec{\nabla} \times \vec{C} + \vec{D}_s, \quad \vec{H} = -\vec{\nabla} C_0 - \frac{\partial \vec{C}}{\partial t} + \vec{H}_s.$$ (3.18)

The fields $\vec{D}_s$ and $\vec{H}_s$ then cancel the string contributions to $-\vec{\nabla} \times \vec{C}$ and $-\vec{\nabla} C_0 - \frac{\partial \vec{C}}{\partial t}$ yielding fields $\vec{D}$ and $\vec{H}$ free of string singularities. For slowly moving particles this mechanism is explicitly exhibited by eqs. (3.3) and (3.11).

IV. The Dual Lagrangian and the Equations of Motion of the String\cite{8}

The action $S$ describing the electromagnetic interactions of a particle of charge $e$ and mass $m_1$ with a particle of charge $-e$ and mass $m_2$ is

$$S = -m_1 \int_{t_1}^{t_2} \sqrt{1 - \vec{v}_1^2} dt - m_2 \int_{t_2}^{t_1} \sqrt{1 - \vec{v}_2^2} dt + \int d^4 x \mathcal{L},$$ (4.1)

where $\mathcal{L}$ is given by eqs. (2.11) and (1.10). Varying $C_\mu$ in the action $S$ gives the field equation (2.10). To obtain the equations of motion for the particles and for
the string which connects them we vary the string coordinates: \( y^\lambda \to y^\lambda + \delta y^\lambda \) and correspondingly vary the particle positions \( x^\lambda_1 \to x^\lambda_1 + \delta x^\lambda_1, x^\lambda_2 \to x^\lambda_2 + \delta x^\lambda_2 \) such that

\[
\delta x^\lambda_1(\tau) = \delta y^\lambda(\sigma_1, \tau)
\]

\[
\delta x^\lambda_2(\tau) = \delta y^\lambda(\sigma_2, \tau).
\]

(4.2)

Denote

\[
\frac{\partial y^\lambda}{\partial \sigma} = y'^\lambda, \frac{\partial y^\lambda}{\partial \tau} = \dot{y}^\lambda.
\]

(4.3)

Then

\[
\delta \left( -\frac{1}{4} \int d^4 x G^{\mu\nu} G_{\mu\nu} \right) = -\frac{1}{2} \int d^4 x G^{\mu\nu}(x) \delta G^s_{\mu\nu}(x),
\]

(4.4)

where

\[
\delta G^s_{\mu\nu} = -\epsilon_{\mu\nu\lambda\alpha} \int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \left\{ \left[ \delta y'^\lambda y'^\alpha + \dot{y}^\lambda \delta y'^\alpha \right] \delta(x - y) + \dot{y}^\lambda y'^\alpha \partial_\beta \delta(x - y) \delta y^\beta \right\}.
\]

(4.5)

Hence,

\[
-\frac{1}{2} \int d^4 x G^{\mu\nu} \delta G^s_{\mu\nu} = \epsilon_{\mu\nu\lambda\alpha} \int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \left\{ -\delta y^\lambda \frac{d}{d\tau}(G^{\mu\nu} y'^\alpha) - \delta y'^\lambda \frac{d}{d\sigma}(G^{\mu\nu} y) + \dot{y}^\lambda y'^\alpha \partial_\beta G^{\mu\nu} + \dot{y}'^\lambda y^\alpha \partial_\beta G^{\mu\nu} \right\}
\]

\[
= \epsilon_{\mu\nu\lambda\alpha} \int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \left\{ -\delta y^\lambda \frac{d}{d\tau}(G^{\mu\nu} y'^\alpha) - \delta y'^\lambda \frac{d}{d\sigma}(G^{\mu\nu} y) \right\}
\]

\[
+ \dot{y}^\beta \delta y'^\lambda y^\alpha \partial_\beta G^{\mu\nu} + \frac{d}{d\tau}(\delta y^\lambda G^{\mu\nu} y'^\alpha) + \frac{d}{d\sigma}(\delta y'^\lambda G^{\mu\nu} y)
\]

\[
= \epsilon_{\mu\nu\lambda\alpha} \int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \left\{ -\delta y^\lambda y'^\alpha \dot{y}^\beta \partial_\beta G^{\mu\nu} - \delta y'^\lambda y^\alpha \dot{y}^\beta \partial_\beta G^{\mu\nu} \right\}
\]
\[ + \delta y^\beta \dot{y}^\lambda y'^\alpha \partial_\beta G^{\mu\nu} + \frac{d}{d\sigma} (\delta y'^\alpha G^{\mu\nu} \dot{y}^\lambda) \} \]

\[ = \frac{e}{2} \int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} \left\{ (\delta y^\lambda y'^\alpha \dot{y}^\beta) (-\epsilon_{\mu\nu\lambda\alpha} \partial_\beta G^{\mu\nu} - \epsilon_{\mu\nu\beta\lambda} \partial_\alpha G^{\mu\nu} \right. \]

\[ + \epsilon_{\mu\nu\beta\alpha} \partial_\lambda G^{\mu\nu} \right) + \epsilon_{\mu\nu\lambda\alpha} \frac{d}{d\sigma} (\delta y'^\alpha G^{\mu\nu} \dot{y}^\lambda) \} \right\}, \quad (4.6) \]

where we have used the fact that the variations of \( \delta y^\lambda \) vanish at \( \tau_2 \) and \( \tau_1 \).

Next we use the identity

\[ - (\epsilon_{\mu\nu\lambda\alpha} \partial_\beta + \epsilon_{\mu\nu\beta\lambda} \partial_\alpha + \epsilon_{\mu\nu\alpha\beta} \partial_\lambda) G^{\mu\nu} = 2\epsilon_{\mu\lambda\alpha\beta} \partial_\nu G^{\nu\mu} \quad (4.7) \]

and obtain

\[ - \frac{1}{2} \int dx G^{\mu\nu} \delta G^s_{\mu\nu} = e \int_{\tau_2}^{\tau_1} d\tau \int_{\sigma_2}^{\sigma_1} d\sigma \delta y^\lambda y'^\alpha \dot{y}^\beta \epsilon_{\mu\lambda\alpha\beta} \partial_\nu G^{\nu\mu} \]

\[ + \frac{e}{2} \epsilon_{\mu\nu\lambda\alpha} \int_{\tau_2}^{\tau_1} d\tau (\delta x^\alpha_1 G^{\mu\nu}(x_1) \dot{x}_1^\lambda - \delta x^\alpha_2 G^{\mu\nu}(x_2) \dot{x}_2^\lambda). \quad (4.8) \]

We must add to the above variation that of the particle action \( S_P \):

\[ \delta S_P = \delta \int_{\tau_2}^{\tau_1} d\tau \left[ -m_1 \sqrt{\dot{x}_1^2} - m_2 \sqrt{\dot{x}_2^2} \right] \]

\[ = \int_{\tau_2}^{\tau_1} d\tau \left( -m_1 \dot{x}_1^\alpha \delta \dot{x}_1^\alpha - m_2 \dot{x}_2^\alpha \delta \dot{x}_2^\alpha \right) \]

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The total change in the action \( \delta S \) due to a change in particle coordinates is

\[
\delta S = \delta S_P + \delta \left( -\frac{1}{4} \int dx G_{\mu\nu} G^{\mu\nu} \right) =
\]

\[
\int_{\tau_2}^{\tau_1} d\tau \left[ \delta x_1^\alpha \left( m_1 \ddot{x}_{1\alpha} + \frac{e}{2} \epsilon_{\mu\nu\lambda\alpha} G^{\mu\nu}(x_1) \dot{x}_1^\lambda \right) 
+ \delta x_2^\alpha \left( m_2 \ddot{x}_{2\alpha} - \frac{e}{2} \epsilon_{\mu\nu\lambda\alpha} G^{\mu\nu}(x_2) \dot{x}_2^\lambda \right) \right].
\]

In proceeding from (4.8) to (4.10) we used the field equation (2.10) which eliminates the string contribution to the variation of the action in eq. (4.8).

If we had introduced further interactions\(^{(2)}\) of the \( C_\mu \) so that \( \partial_\mu G^{\mu\nu} \neq 0 \), then there would have been additional variations of the action arising from the first term in eq. (4.8). In that case Hamilton’s principle \( \delta S = 0 \) gives

\[
\frac{e}{2} \epsilon_{\mu\nu\lambda\alpha} G^{\mu\nu}(y) = 0,
\]

in addition to the Lorentz force equations

\[
m_1 \ddot{x}_{1\alpha} = \frac{e}{2} \epsilon_{\alpha\mu\nu\lambda} G^{\mu\nu}(x_1) \dot{x}_1^\lambda,
\]

and

\[
m_2 \ddot{x}_{2\alpha} = \frac{e}{2} \epsilon_{\alpha\mu\nu\lambda} G^{\mu\nu}(x_2) \dot{x}_2^\lambda,
\]

following from eq. (4.10). Thus eq. (4.11) provides a boundary condition along the strings upon the current \( \partial_\nu G^{\nu\mu} \).
V. Conclusions

Our purpose in writing this paper is pedagogical, though of course, motivated by our interest in using dual potentials in QCD. We have seen how normal classical electrodynamics can be handled completely in terms of dual potentials, and that the use of these potentials gives the solution to the conventional Maxwell equations for the electric and magnetic fields and leads to the usual Lorentz force law for the motion of charged particles. While dual potentials provide a somewhat awkward way to solve electrodynamics when charges are present, they are nevertheless the natural variables to describe a dielectric medium with long distance anti-screening. This is the underlying reason for their utility in describing long range QCD.
References

1. P.A.M. Dirac, Phys. Rev., 74 (1948) 817.

2. M. Baker, J. S. Ball and F. Zachariasen, Phys. Rev., D47 (1993) 3021 and references given there.

3. S. Mandelstam, Phys. Rep., 23C (1976) 245.

4. G. ’t Hooft, in Proc. Europ. Phys. Soc. Conf. on High Energy Physics (1975) edited by A. Zichichi (Editrice Compositori) Bologna, (1976) p.1225.

5. D.J. Griffiths, Introduction to Electrodynamics, Prentice Hall, (1989) p.149.

6. Reference 5 Page 239.

7. L. Landau and E. Lifshitz, Classical Theory of Fields, Pergamon Press (1975) p.168.

8. This section just repeats in somewhat more detail the corresponding calculation in Reference 1 and is included here for completeness.

Figure Captions

Fig.1 Dirac String $L(t)$ connecting oppositely charged particles.

Fig.2 Closed contour describing path of line integral on right-hand side of eq. (2.6).

Fig.3 Diagram representing string cancellation mechanism of eq. (3.5).
This figure "fig1-1.png" is available in "png" format from:

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