UNIVERSAL PROPERTIES OF POLYNOMIALS

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Abstract. In this paper we state and prove the universal properties of the bicategory of polynomials, considering both cartesian and general morphisms between these polynomials. The novelty in our approach is that we avoid most of the coherence conditions which would normally arise as a consequence of the complicated nature of polynomial composition; this is done by using the properties of generic bicategories described in our last paper.

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1. Introduction

In this paper we are interested in two constructions on suitable categories \( \mathcal{E} \): the bicategory of spans \( \text{Span}(\mathcal{E}) \) as introduced by Bénabou [1], and the bicategory of polynomials \( \text{Poly}(\mathcal{E}) \) as studied by Gambino, Kock and Weber [6, 14] (all to be reviewed in Section 2). Here we wish to study the universal properties of these constructions; that is, for an arbitrary bicategory \( \mathcal{C} \) we wish to know what it means to give a pseudofunctor \( \text{Span}(\mathcal{E}) \to \mathcal{C} \) or \( \text{Poly}(\mathcal{E}) \to \mathcal{C} \).

In the case of spans, these results have already been established. In particular, given any category \( \mathcal{E} \) with pullbacks, one can form a bicategory denoted \( \text{Span}(\mathcal{E}) \) whose objects are those of \( \mathcal{E} \) and 1-cells are diagrams in \( \mathcal{E} \) of the form below

\[
\begin{array}{ccc}
& s & \\
\downarrow & & \downarrow t \\
\bullet & \downarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & & \bullet
\end{array}
\]
called spans. The universal property of this construction admits a simple description since for every morphism \( f \) in \( \mathcal{E} \) we have adjunctions

\[
\begin{array}{ccc}
& \text{id} & \\
\downarrow & & \downarrow f \\
\bullet & \downarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & & \bullet
\end{array}
\]

in \( \text{Span}(\mathcal{E}) \), and these adjunctions generate all of \( \text{Span}(\mathcal{E}) \).
Indeed, it was noted by Hermida in [7, Theorem A.2] that composing with the canonical embedding $E \hookrightarrow \text{Span}(E)$ describes an equivalence

$$\text{Beck pseudofunctors } E \rightarrow \mathcal{C} \simeq \text{pseudofunctors } \text{Span}(E) \rightarrow \mathcal{C}$$

where a pseudofunctor $F_\Sigma: E \rightarrow \mathcal{C}$ is Beck if for every morphism $f$ in $E$ the 1-cell $F_\Sigma f$ has a right adjoint $F_\Delta f$ in $\mathcal{C}$ (such an $F_\Sigma$ is known as a sinister pseudofunctor), and if the induced pair of pseudofunctors

$$F_\Sigma: E \rightarrow \mathcal{C}, \quad F_\Delta: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$$

satisfy a Beck-Chevalley condition. A natural question to then ask is what these sinister pseudofunctors correspond to when the Beck-Chevalley condition is dropped. This question was solved by Dawson, Paré, and Pronk [4, Theorem 2.15] who showed that composing with the canonical embedding describes an equivalence

$$\text{sinister pseudofunctors } E \rightarrow \mathcal{C} \simeq \text{gregarious functors } \text{Span}(E) \rightarrow \mathcal{C}$$

where gregarious functors are the adjunction preserving op lax normal functors.

An important special case of this is when $\mathcal{C} = \mathbf{Cat}$, where one may consider sinister pseudofunctors $E \rightarrow \mathbf{Cat}$, or equivalently cosinister pseudofunctors $E^{\text{op}} \rightarrow \mathbf{Cat}$. This gives an equivalence between bifibrations over $E$ and gregarious functors $\text{Span}(E) \rightarrow \mathbf{Cat}$. Note also that on $\mathbf{Cat}/E$ there is a KZ pseudomonad $\Gamma_E$ for opfibrations and a coKZ pseudomonad $\Upsilon_E$ for fibrations. This yields (via a pseudo-distributive law) the pseudomonad $\Gamma_E \Upsilon_E$ for bifibrations satisfying the Beck-Chevalley condition [12]. The bifibrations with this property then correspond to pseudofunctors $\text{Span}(E) \rightarrow \mathbf{Cat}$; an archetypal example of this being given by the codomain fibration over $E$.

When considering polynomials it is convenient to assume some extra structure on $E$. In particular, we will take $E$ to be a category with pullbacks, such that for each morphism $f$ in $E$ the “pullback along $f$” functor $\Delta f$ has a right adjoint $\Pi f$. For such a category $E$ (known as a locally cartesian closed category) one can form a bicategory denoted $\text{Poly}(E)$ whose objects are those of $E$ and 1-cells are diagrams in $E$ of the form below

![Diagram](https://example.com/diagram.png)

called polynomials. One can also form a bicategory $\text{Poly}_c(E)$ with the same objects and 1-cells by being more restrictive on the 2-cells (that is, only taking “cartesian” morphisms of polynomials).

The purpose of this paper is to describe the universal properties of these two bicategories of polynomials.

Similar to the case of spans, the universal property of $\text{Poly}(E)$ admits a simple description since for every morphism $f$ in $E$ we have adjunctions

$$\begin{align*}
\text{id} & \quad \dashv \quad f & \text{id} & \quad \dashv \quad f \\
\text{id} & \quad \dashv \quad f & \text{id} & \quad \dashv \quad f
\end{align*}$$

in $\text{Poly}(E)$, and these adjunctions generate all of $\text{Poly}(E)$.

---

1Here “cosinister” means arrows are sent to right adjoint 1-cells instead of left adjoint 1-cells. This is the $F_\Delta$ of such a pair $F_\Sigma:F_\Delta$.

2The bicategory of polynomials can be defined on any category $E$ with pullbacks [14]; however, we will assume local cartesian closure for simplicity.
Using this, we show that in the case of polynomials with general 2-cells, composition with the embedding \( \mathcal{E} \to \text{Poly}(\mathcal{E}) \) describes the equivalence

\[
\text{DistBeck pseudofunctors } \mathcal{E} \to \mathcal{C} \\
\text{pseudofunctors } \text{Poly}(\mathcal{E}) \to \mathcal{C}
\]

where a pseudofunctor \( F_\Sigma : \mathcal{E} \to \mathcal{C} \) is DistBeck if for every morphism \( f \) in \( \mathcal{E} \) the 1-cell \( F_\Sigma f \) has two successive right adjoints \( F_\Delta f \) and \( F_\Pi f \) (such an \( F_\Sigma \) is called a 2-sinister pseudofunctor), and if the induced triple of pseudofunctors

\[
F_\Sigma : \mathcal{E} \to \mathcal{C}, \quad F_\Delta : \mathcal{E}^{op} \to \mathcal{C}, \quad F_\Pi : \mathcal{E} \to \mathcal{C}
\]

satisfy the earlier Beck-Chevalley condition on the pair \( F_\Sigma \) and \( F_\Delta \) in addition to a “distributivity condition” on the pair \( F_\Sigma \) and \( F_\Pi \). Forgetting the distributivity condition yields the notion of a 2-Beck pseudofunctor, so that (1.2) may be seen as a restriction of an equivalence

\[
\text{2-Beck pseudofunctors } \mathcal{E} \to \mathcal{C} \\
\text{gregarious functors } \text{Poly}(\mathcal{E}) \to \mathcal{C}
\]

Similar to earlier, an important special case of this is when \( \mathcal{C} = \text{Cat} \), where one gets an equivalence between fibrations over \( \mathcal{E} \) with sums and products satisfying the Beck–Chevalley condition, and gregarious functors \( \text{Poly}(\mathcal{E}) \to \text{Cat} \).

Note also that on \( \text{Fib}(\mathcal{E}) \) there is a KZ monad \( \Sigma_E \) for fibrations with sums, and a coKZ monad \( \Pi_E \) for fibrations with products. This yields (via a pseudo-distributive law) a pseudomonad \( \Sigma_E \Pi_E \) for fibrations with sums and products, satisfying both the Beck–Chevalley and distributivity conditions. These fibrations then correspond to pseudofunctors \( \text{Poly}(\mathcal{E}) \to \text{Cat} \); the codomain fibration again being an archetypal example.

Another example of this situation is given by taking \( \mathcal{E} \) to be a regular locally cartesian closed category. In this case we have the 2-Beck pseudofunctor \( \text{Sub} : \mathcal{E} \to \text{Cat} \) which sends a morphism \( f : X \to Y \) in \( \mathcal{E} \) to the existential quantifier \( \exists_f : \text{Sub}(X) \to \text{Sub}(Y) \) mapping subobjects of \( X \) to those of \( Y \), which has the two successive right adjoints \( \Delta f \) “pullback along \( f \)” and \( \forall f \) “universal quantification at \( f \)”, thus giving a gregarious functor \( \text{Poly}(\mathcal{E}) \to \text{Cat} \) defined by the assignment

\[
\begin{array}{ccc}
\text{E} & \xrightarrow{p} & \text{B} \\
\text{I} & \xrightarrow{f} & \text{J} \\
\text{Sub} (\text{I}) & \xrightarrow{\Delta f} & \text{Sub} (\text{E}) & \xrightarrow{\forall f} & \text{Sub} (\text{B}) & \xrightarrow{\exists f} & \text{Sub} (\text{J})
\end{array}
\]

The distributivity condition here then amounts to asking that \( \mathcal{E} \) satisfies the internal axiom of choice.

With only cartesian morphisms we do not have the adjunctions on the right in (1.1), thus making the universal property of \( \text{Poly}_c(\mathcal{E}) \) more complicated to state. The universal property of this construction is described as an equivalence

\[
\text{DistBeck triple } \mathcal{E} \to \mathcal{C} \\
\text{pseudofunctors } \text{Poly}_c(\mathcal{E}) \to \mathcal{C}
\]

where a DistBeck triple \( \mathcal{E} \to \mathcal{C} \) is a triple of pseudofunctors

\[
F_\Sigma : \mathcal{E} \to \mathcal{C}, \quad F_\Delta : \mathcal{E}^{op} \to \mathcal{C}, \quad F_\otimes : \mathcal{E} \to \mathcal{C}
\]

such \( F_\Sigma f \vdash F_\Delta f \) for all morphisms \( f \) in \( \mathcal{E} \), with a Beck–Chevalley condition satisfied for the pair \( F_\Sigma \) and \( F_\Delta \), for which \( F_\Delta \) and \( F_\otimes \) are related via invertible Beck–Chevalley coherence data (as we do not have adjunctions \( F_\Delta f \vdash F_\otimes f \) this data does not come for free and must be given instead, subject to suitable coherence.
axioms), such that the pair $F_\Sigma$ and $F_\otimes$ satisfy a distributivity condition as before\(^3\).

There are also weakened versions of the universal property of $\text{Poly}_c(\mathcal{E})$ which arise from dropping these conditions.

An example of this is given by taking $\mathcal{E}$ to be the category of finite sets $\text{FinSet}$ and $\mathcal{C}$ to be the 2-category of small categories $\text{Cat}$. Taking $(\mathcal{A}, \otimes, I)$ to be a symmetric monoidal category such that $\mathcal{A}$ has finite coproducts, we can assign to any finite set $n$ the category $\mathcal{A}^n$ and to any morphism $f : m \to n$ the functors

\[
\begin{align*}
\text{lan}_f : A^m &\to A^n, \quad (a_i : i \in m) \mapsto (\sum_{x \in f^{-1}(j)} a_x : j \in n) \\
(-) \circ f : A^n &\to A^m, \quad (a_j : j \in n) \mapsto (a_{f(i)} : i \in m) \\
\otimes f : A^m &\to A^n, \quad (a_i : i \in m) \mapsto (\otimes_{x \in f^{-1}(j)} a_x : j \in n)
\end{align*}
\]

This gives the data of a Beck triple (that is a DistBeck triple without requiring the distributivity condition). The distributivity condition here holds precisely when the functor $X \otimes (-) : \mathcal{A} \to \mathcal{A}$ preserves finite coproducts for all $X \in \mathcal{A}$.

The reader should note that proving the universal properties concerning the polynomial construction is much more complex than that of the span construction. This is since composition of polynomials is significantly more complicated; this is especially evident in calculations concerning associativity of oplax and pseudofunctors out of polynomials, or calculations involving horizontal composition of general polynomial morphisms.

Fortunately, we are able to avoid these calculations to some extent. This is done by exploiting the fact that both $\text{Span}(\mathcal{E})$ and $\text{Poly}_c(\mathcal{E})$ are “generic bicategories” [13], and hence that oplax functors out of them admit much simpler descriptions. A problem here is that the bicategory $\text{Poly}_c(\mathcal{E})$ does not enjoy this property. However, as $\text{Poly}_c(\mathcal{E})$ embeds into $\text{Poly}(\mathcal{E})$ and both bicategories have the same composition the universal property of the former will assist in proving the latter.

In Section 2 we give the necessary background for this paper. We recall the definitions and basic properties of the bicategories of spans and polynomials, the notions of lax, oplax and gregarious functors, the basic properties of the mates correspondence, and the basic properties of generic bicategories.

In Section 3 we give a proof of the universal properties of spans using the properties of generic bicategories. This is to give a complete and detailed proof of these properties demonstrating our method, before applying it the more complicated setting of polynomials later on.

In Section 4 we give a proof of the universal properties of spans with invertible 2-cells. This is necessary since the universal properties of polynomials with cartesian 2-cells will be described in terms of this property.

In Section 5 we give a proof of the universal properties of polynomials with cartesian 2-cells. It is in this section that our method is of the most use; indeed in our proof we completely avoid coherences involving composition of distributivity pullbacks (the worst coherence conditions which would arise in a direct proof).

In Section 6 we give a proof of the universal properties of polynomials with general 2-cells, by using the corresponding properties for polynomials with cartesian 2-cells and checking some additional coherence conditions concerning naturality with respect to these more general 2-cells.

2. Background

In this section we give the necessary background knowledge for this paper.

\(^3\)The distributivity data need not be given as it may be constructed using the $F_\Sigma, F_\otimes$ Beck coherence data and the adjunctions $F_\Sigma f \dashv F_\otimes f$. 

2.1. The bicategory of spans. Before studying the bicategory of polynomials we will study the simpler and more well known construction of the bicategory of spans, as introduced by Bénabou in 1967 [1].

Definition 1. Suppose we are given a category $\mathcal{E}$ with chosen pullbacks. We may then form the bicategory of spans in $\mathcal{E}$, denoted $\text{Span}(\mathcal{E})$, with objects those of $\mathcal{E}$, 1-cells $A \rightarrow B$ given diagrams in $\mathcal{E}$ of the form

$$
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow^{p} & & \downarrow^{q} \\
A & \rightarrow & B
\end{array}
$$

called spans, composition of 1-cells given by taking the chosen pullback

$$
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow^{\pi_{1}} & & \downarrow^{\pi_{2}} \\
X & \rightarrow & Y \\
\downarrow^{r} & & \downarrow^{s} \\
B & \rightarrow & C
\end{array}
$$

and 2-cells $\nu$ given by those morphisms between the vertices of two spans which yield commuting diagrams of the form

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow^{p} & & \downarrow^{q} \\
X & \rightarrow & Y
\end{array}
$$

The identity 1-cells are given by identity spans $X \xrightarrow{1_{X}} X \xleftarrow{1_{X}} X$ and composition extends to 2-cells by the universal property of pullbacks. The essential uniqueness of the limit of a diagram

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow^{p} & & \downarrow^{q} \\
X & \rightarrow & Y \\
\downarrow^{r} & & \downarrow^{s} \\
B & \rightarrow & C \\
\downarrow^{u} & & \downarrow^{v} \\
D
\end{array}
$$

yields the associators, making $\text{Span}(\mathcal{E})$ into a bicategory.

We denote by $\text{Span}_{\text{iso}}(\mathcal{E})$ the bicategory as defined above, but only taking the invertible 2-cells.

2.2. The bicategory of polynomials. In the earlier defined bicategory of spans the morphisms may be viewed as multivariate linear maps (matrices). In this subsection we recall the bicategory of polynomials, whose morphisms may be viewed as multivariate polynomials.

Before we can define this bicategory we must recall the notion of distributivity pullback as given by Weber [14].

Definition 2. Given two composable morphisms $u: X \rightarrow A$ and $f: A \rightarrow B$ in a category $\mathcal{E}$ with pullbacks, a pullback around $(f, u)$ is a diagram

$$
\begin{array}{ccc}
T & \xrightarrow{p} & X \\
\downarrow^{q} & & \downarrow^{u} \\
Y & \xrightarrow{r} & B
\end{array}
$$

such that the outer rectangle is a pullback. A morphism of pullbacks around $(f, u)$ is a pair of morphisms $s: T \rightarrow T'$ and $t: Y \rightarrow Y'$ such that $p's = p$, $q's = tq$ and $r = r't$. A distributivity pullback around $(f, u)$ is a terminal object in the category of pullbacks around $(f, u)$. If for every $u$ we may form a distributivity pullback around $(f, u)$ we then say that $f$ is exponentiable.
Remark 3. Note that \( f \) is exponentiable if and only if the “pullback along \( f \)” functor \( \Delta_f \) has a right adjoint \([14] \). We will denote this right adjoint by \( \Pi_f \).

The following diagrams are to be the morphisms in the bicategory of polynomials.

**Definition 4.** A polynomial \( P : I \to J \) in a category \( \mathcal{E} \) with pullbacks is a diagram of the form

\[
\begin{array}{ccc}
E & \rightarrow & B \\
\downarrow s & & \downarrow t \\
I & \rightarrow & J \\
\end{array}
\]

where \( p \) is exponentiable.

We will also need the following universal property of polynomial composition.

**Proposition 5.** \([14, \text{Prop. 3.1.6}] \) Suppose we are given two polynomials \( P : I \to J \) and \( Q : J \to K \). Consider a category with objects given by commuting diagrams of the form

\[
\begin{array}{ccc}
A_1 & \rightarrow & A_2 & \rightarrow & A_3 \\
\downarrow I & & \downarrow J & & \downarrow K \\
E & \rightarrow & B & \rightarrow & M \\
\downarrow s & & \downarrow t & & \downarrow u \\
I & \rightarrow & J & \rightarrow & M \\
\end{array}
\]

for which the left and right squares are pullbacks (but not necessarily the middle), and morphisms given by triples \( (A_i \rightarrow B_i : i = 1, 2, 3) \) rendering commutative the diagram

\[
\begin{array}{ccc}
A_1 & \rightarrow & A_2 & \rightarrow & A_3 \\
\downarrow I & & \downarrow J & & \downarrow K \\
E & \rightarrow & B & \rightarrow & M \\
\downarrow s & & \downarrow t & & \downarrow u \\
I & \rightarrow & J & \rightarrow & M \\
\end{array}
\]

Then in this category, the outside composite in the diagram formed below (which is a polynomial \( I \to K \))

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow pb & & \downarrow dpb \\
E & \rightarrow & B \\
\downarrow s & & \downarrow t \\
I & \rightarrow & J \\
\end{array}
\]

is a terminal object.

**Definition 6.** Suppose we are given a category \( \mathcal{E} \) with chosen pullbacks and distributivity pullbacks. We may then form the bicategory of polynomials with cartesian 2-cells in \( \mathcal{E} \), denoted \( \text{Poly}_c(\mathcal{E}) \), with objects those of \( \mathcal{E} \), 1-cells \( A \to B \) given by polynomials, composition of 1-cells given by forming the diagram \((2.1)\) just above, and cartesian 2-cells given by pairs of morphisms \((\sigma, \nu)\) rendering commutative the diagram

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow pb & & \downarrow dpb \\
E & \rightarrow & B \\
\downarrow s & & \downarrow t \\
I & \rightarrow & J \\
\end{array}
\]
such that the middle square is a pullback. The identity 1-cells are given by identity polynomials \( X \xrightarrow{1_X} X \xrightarrow{1_X} X \xrightarrow{1_X} X \). Composition of 2-cells and the associators may be recovered from Proposition 5 above.

**Definition 7.** Suppose we are given a category \( \mathcal{E} \) with chosen pullbacks and distributivity pullbacks. We may then form the **bicategory of polynomials with general 2-cells**, denoted \( \text{Poly} (\mathcal{E}) \), with objects and 1-cells as in \( \text{Poly}_e (\mathcal{E}) \), and 2-cells given by diagrams as below on the left below

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {X};
  \node (B) at (2,0) {Y};
  \node (C) at (0,-2) {M};
  \node (D) at (2,-2) {N};
  \node (E) at (1,-1) {};\node (F) at (1,-1.5) {};\draw[->](A) to node[swap] {s} (C);
  \node (G) at (1.5,-1) {};\draw[->](B) to node[sloped,above] {t} (D);
  \node (H) at (0.5,-1) {};\draw[->](C) to node[swap] {u} (G);
  \node (I) at (1,-1) {};\draw[->](D) to node[swap] {v} (I);
  \node (J) at (1,-1) {};\draw[->](A) to node[swap] {\varepsilon_1} (B);
  \node (K) at (1,-1) {};\draw[->](C) to node[swap] {f_1} (D);
  \node (L) at (1,-1) {};\draw[->](B) to node[swap] {p} (E);
  \node (M) at (1,-1) {};\draw[->](D) to node[swap] {g} (F);
  \node (N) at (1,-1) {};\draw[->](E) to node[swap] {q} (C);
\end{tikzpicture}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {X};
  \node (B) at (2,0) {Y};
  \node (C) at (0,-2) {M};
  \node (D) at (2,-2) {N};
  \node (E) at (1,-1) {};\node (F) at (1,-1.5) {};\draw[->](A) to node[swap] {s} (C);
  \node (G) at (1.5,-1) {};\draw[->](B) to node[sloped,above] {t} (D);
  \node (H) at (0.5,-1) {};\draw[->](C) to node[swap] {u} (G);
  \node (I) at (1,-1) {};\draw[->](D) to node[swap] {v} (I);
  \node (J) at (1,-1) {};\draw[->](A) to node[swap] {\varepsilon_2} (B);
  \node (K) at (1,-1) {};\draw[->](C) to node[swap] {f_2} (D);
  \node (L) at (1,-1) {};\draw[->](B) to node[swap] {p} (E);
  \node (M) at (1,-1) {};\draw[->](D) to node[swap] {g} (F);
  \node (N) at (1,-1) {};\draw[->](E) to node[swap] {q} (C);
\end{tikzpicture}
\end{array}
\]

regarded equivalent to the diagram on the right provided both indicated regions are pullbacks.

For the other operations of this bicategory such as the composition operation on 2-cells we refer the reader to the equivalence \( \text{Poly} (\mathcal{E}) \simeq \text{PolyFun} (\mathcal{E}) \) [6] where \( \text{PolyFun} (\mathcal{E}) \) is the bicategory of polynomial functors, described later in Example 11.

**Remark 8.** Note that it suffices to give local equivalences \( \text{PolyFun} (\mathcal{E})_{X,Y} \simeq \text{Poly} (\mathcal{E})_{X,Y} \) since from this it follows that the bicategorical structure on \( \text{PolyFun} (\mathcal{E}) \) endows the family of hom-categories \( \text{Poly} (\mathcal{E})_{X,Y} \) with the structure of a bicategory via doctrinal adjunction [8]. This describes the bicategory structure on \( \text{Poly} (\mathcal{E}) \).

2.3. **Morphisms of bicategories.** There are a few types of morphisms between bicategories we are interested in for this paper. These include oplax functors, lax functors, pseudofunctors, gregarious functors and sinister pseudofunctors. After the following trivial definition we will recall these notions.

**Definition 9.** Given two bicategories \( \mathcal{A} \) and \( \mathcal{B} \), a **locally defined functor** \( F : \mathcal{A} \to \mathcal{B} \) consists of

- for each object \( X \in \mathcal{A} \) an object \( FX \in \mathcal{B} \);
- for each pair of objects \( X,Y \in \mathcal{A} \), a functor \( F_{XY} : \mathcal{A}_{XY} \to \mathcal{B}_{FX,FY} \),

subject to no additional conditions.

It is one of the main goals of this paper to show that many of the coherence conditions arising from the following associativity diagram may be largely avoided, at least when \( \mathcal{A} \) is \( \text{Span} (\mathcal{E}) \) or \( \text{Poly}_e (\mathcal{E}) \) for a suitable category \( \mathcal{E} \).

**Definition 10.** Given two bicategories \( \mathcal{A} \) and \( \mathcal{B} \), a **lax functor** \( F : \mathcal{A} \to \mathcal{B} \) is a locally defined functor \( F : \mathcal{A} \to \mathcal{B} \) equipped with

- for each object \( X \in \mathcal{A} \), a 2-cell \( \lambda_X : 1_{FX} \to F1_X \);
- for each triple of objects \( X,Y,Z \in \mathcal{A} \) and pair of morphisms \( f : X \to Y \) and \( g : Y \to Z \), a 2-cell \( \varphi_{g,f} : Fg \cdot Ff \Rightarrow Fgf \) natural in \( g \) and \( f \),

such that the constraints render commutative the associativity diagram

\[
\begin{align*}
(Fh \cdot (Fg \cdot Ff)) & \xrightarrow{\varphi_{Fg,Ff}} Fh \cdot F(gf) \\
& \xrightarrow{\varphi_{Fh,F(gf)}} F(h(gf)) \\
& \xrightarrow{\alpha_{Fh,F(gf),Ff}} F(h(gf))
\end{align*}
\]
for composable morphisms \( h, g \) and \( f \). In addition, the nullary constraint cells must render commutative the diagrams

\[
\begin{align*}
Ff \cdot 1_X & \xrightarrow{\varphi_{f \cdot 1_X}} Ff \cdot F(1_X) \\
Ff & \xrightarrow{\varphi_{f \cdot 1_X}} Ff \cdot 1_X
\end{align*}
\]

\[
\begin{align*}
1_{FY} \cdot Ff & \xrightarrow{\lambda_{Ff}} F(1_Y) \cdot Ff \\
1_{FY} & \xrightarrow{\lambda_{Ff}} F(1_Y)
\end{align*}
\]

for all morphisms \( f : X \to Y \). If the constraint cells \( \varphi \) and \( \lambda \) are required invertible, then this is the definition of a pseudofunctor. If the direction of the constraints is reversed, this is the definition of an oplax functor. If the nullary constraints \( \lambda \) are invertible (in either the lax or oplax case) we then say our functor is normal.

**Example 11.** It is well known that given a category \( \mathcal{E} \) with pullbacks there is a pseudofunctor \( \text{Span} (\mathcal{E}) \to \text{Cat} \) which assigns an object \( X \in \mathcal{E} \) to the slice category \( \mathcal{E}/X \) and on spans is defined the assignment

\[
\begin{array}{c}
\xymatrix{ 
I & E 
\ar[r]^p & B \\
I & J 
\ar[ur]^t}
\end{array}
\]

\[
\xymatrix{ 
\mathcal{E}/I 
\ar[r]^\Delta & \mathcal{E}/B 
\ar[r]^-\Sigma & \mathcal{E}/J
}
\]

where \( \Sigma_t \) is the “composition with \( t \)” functor, and \( \Delta_s \) is the “pullback along \( s \)” functor (the right adjoint of \( \Sigma_s \)).

If \( \mathcal{E} \) is locally cartesian closed, meaning that for each morphism \( p \) the functor \( \Delta_p \) has a further right adjoint denoted \( \Pi_p \), then there is also such a canonical functor out of \( \text{Poly} (\mathcal{E}) \) \([6]\) and \( \text{Poly}_c (\mathcal{E}) \) \([14]\), which assigns an object \( X \in \mathcal{E} \) to the slice category \( \mathcal{E}/X \) and on polynomials is defined the assignment

\[
\begin{array}{c}
\xymatrix{ 
I & E_p 
\ar[r]^p & B \\
I & J 
\ar[ur]^t}
\end{array}
\]

\[
\xymatrix{ 
\mathcal{E}/I 
\ar[r]^\Delta & \mathcal{E}/E 
\ar[r]^-\Pi_p & \mathcal{E}/B 
\ar[r]^-\Sigma & \mathcal{E}/J
}
\]

A functor isomorphic to one as on the right above is known as a polynomial functor. The objects of \( \mathcal{E} \), polynomial functors, and strong natural transformations form a 2-category \( \text{PolyFun} (\mathcal{E}) \) \([6]\).

**Remark 12.** In the subsequent sections we are interested in pseudofunctors mapping into a general bicategory \( \mathcal{C} \), not just \( \text{Cat} \), however we will still use the above example to motivate our notation.

The following is a special type of oplax functor which turns up when studying the universal properties of the span construction \([4, 5]\). This notion will also be useful for studying the universal properties of the polynomial construction.

**Definition 13.** \([4, \text{Definition 2.4}]\) We say an oplax normal functor of bicategories \( F : \mathscr{A} \to \mathscr{B} \) is gregarious (also known as jointed) if for any pair of 1-cells \( f : A \to B \) and \( g : B \to C \) in \( \mathscr{A} \) for which \( g \) has a right adjoint, the constraint cell \( \varphi_{g \cdot f} : F(gf) \to Fg \cdot Ff \) is invertible.

There is also an alternative characterization of gregarious functors worth mentioning, which establishes gregarious functors as a natural concept.

**Proposition 14.** \([4, \text{Propositions 2.8 and 2.9}]\) An oplax normal functor of bicategories \( F : \mathscr{A} \to \mathscr{B} \) is gregarious if and only if it preserves adjunctions; that is, if for every adjunction \( (f \dashv u : A' \to A, \eta, \varepsilon) \) in \( \mathscr{A} \) there exists 2-cells \( \varphi : 1_{FA} \to Fu \cdot Ff \).
and \( \pi : Ff \cdot Fu \to 1_{FA} \) which exhibit \( Ff \) as left adjoint to \( Fu \) and render commutative the squares

\[
\begin{array}{c}
F(1_A) \xrightarrow{F\eta} F(u f) \\
\lambda_A \downarrow \quad \varphi_{u,f} \downarrow \quad \lambda_{A'} \downarrow \quad F\varepsilon \downarrow \\
1_{FA} \xrightarrow{\pi} Fu \cdot Ff \\
\end{array}
\]

We also need a notion of morphism between lax, oplax, gregarious or pseudo-functors. It will be convenient here to use Lack’s icons [10], defined as follows.

**Definition 15.** Given two lax functors \( F, G : \mathcal{A} \to \mathcal{B} \) which agree on objects, an icon \( \alpha : F \to G \) consists of a family of natural transformations

\[
\begin{array}{c}
Fg \cdot Ff \xrightarrow{\varphi_{g,f}} F(gf) \\
\alpha_g \ast \alpha_f \downarrow \quad \alpha_g \downarrow \\
Gg \cdot Gf \xrightarrow{\psi_{g,f}} G(gf) \\
\end{array}
\]

with components rendering commutative the diagrams

\[
\begin{array}{c}
FG \cdot GF \xrightarrow{\varphi_{g,f} \cdot \psi_{g,f}} F(gf) \\
\alpha_g \ast \alpha_f \downarrow \quad \alpha_g \downarrow \\
\end{array}
\]

for composable morphisms \( f \) and \( g \) in \( \mathcal{A} \). Similarly, one may define icons between oplax functors.

An important point about icons is that there is a 2-category of bicategories, oplax (lax) functors, and icons. For convenience, we make the following definition.

**Definition 16.** We denote by \( \text{Icon} \) (resp. \( \text{Greg} \)) the 2-category of bicategories, pseudo (resp. gregarious) functors and icons.

### 2.4. Mates under adjunctions.

We now recall the basic properties of mates [9]. Given two pairs of adjoint morphisms

\[
\eta_1, \varepsilon_1 : f_1 \dashv u_1 : B_1 \to A_1, \quad \eta_2, \varepsilon_2 : f_2 \dashv u_2 : B_2 \to A_2
\]

in a bicategory \( \mathcal{A} \), we say that two 2-cells

\[
\begin{array}{c}
A_1 \xrightarrow{\eta_1} B_1 \\
f_1 \downarrow \quad \psi_{\alpha} \downarrow \quad \eta_1 \downarrow \\
B_1 \xrightarrow{u_1} A_2
\end{array}
\]

\[
\begin{array}{c}
A_1 \xrightarrow{\eta_2} B_1 \\
f_2 \downarrow \quad \psi_{\beta} \downarrow \quad \eta_2 \downarrow \\
B_1 \xrightarrow{u_2} A_2
\end{array}
\]

are mates under the adjunctions \( f_1 \dashv u_1 \) and \( f_2 \dashv u_2 \) if \( \beta \) is given by the pasting

\[
\begin{array}{c}
A_1 \xrightarrow{g} A_2 \\
\psi_{\eta_1} \downarrow \quad \psi_{\eta_2} \downarrow \\
B_1 \xrightarrow{\psi_{\alpha}} B_2
\end{array}
\]
or equivalently, \( \alpha \) is given by the pasting

\[
\begin{array}{ccc}
A_1 \xrightarrow{id} A_1 \xrightarrow{g} A_2 \\
\downarrow f_1 \downarrow \psi_\eta_1 \downarrow u_1 \xrightarrow{\psi \beta} u_2 \xrightarrow{\psi \varepsilon_2} f_2 \\
B_1 \xrightarrow{h} B_2 \xrightarrow{id} B_2
\end{array}
\]

It follows from the triangle identities that taking mates in this fashion defines a bijection between 2-cells \( f_2 g \rightarrow hf_1 \) and 2-cells \( gu_1 \rightarrow u_2 h \).

Moreover, it is well known that this correspondence is functorial. Given another adjunction \( \eta_3, \varepsilon_3 : f_3 \dashv u_3 : B_3 \rightarrow A_3 \) and 2-cells as below

\[
\begin{array}{ccc}
A_1 \xrightarrow{g} A_2 \xrightarrow{m} A_3 \\
\downarrow f_1 \downarrow \psi \alpha_t \downarrow f_3 \\
B_1 \xrightarrow{h} B_2 \xrightarrow{n} B_3
\end{array}
\quad
\begin{array}{ccc}
A_1 \xrightarrow{g} A_2 \xrightarrow{m} A_3 \\
\downarrow u_1 \downarrow \psi \alpha_r \downarrow u_2 \\
B_1 \xrightarrow{h} B_2 \xrightarrow{n} B_3
\end{array}
\]

where \( \alpha_t \) and \( \alpha_r \) respectively correspond to \( \beta_t \) and \( \beta_r \) under the mates correspondence, it follows that the pasting of \( \alpha_t \) and \( \alpha_r \) corresponds to the pasting of \( \beta_t \) and \( \beta_r \) under the mates correspondence. Moreover, the analogous property holds for pasting vertically. These vertical and horizontal pasting properties\(^4\) are often referred to as functoriality of mates.

**Remark 17.** Given an adjunction \( \eta, \varepsilon : f \dashv u : B \rightarrow A \) the left square below

\[
\begin{array}{ccc}
A \xrightarrow{f} A \\
\downarrow f \downarrow \psi \id \downarrow \id \\
B \xrightarrow{u} B
\end{array}
\quad
\begin{array}{ccc}
A \xrightarrow{f} A \\
\downarrow u \downarrow \psi \varepsilon \downarrow \id \\
B \xrightarrow{u} B
\end{array}
\]

corresponds to the right above via the mates correspondence, allowing one to see the counit of an adjunction as an instance of the mates correspondence. A similar calculation may be done for the units. This will allow us to see calculations involving units and counits as functoriality of mates calculations.

One consequence of the mates correspondence which will be of interest to us is the following lemma; a special case of \([4, \text{Lemma } 2.13]\), showing that the component of an icon between gregarious functors at a left adjoint 1-cell is invertible.

**Lemma 18.** Suppose \( F, G : \mathcal{A} \rightarrow \mathcal{B} \) are gregarious functors between bicategories which agree on objects. Suppose that \( \alpha : F \rightarrow G \) is an icon. Suppose that a given 1-cell \( f : X \rightarrow Y \) has a right adjoint \( u \) in \( \mathcal{A} \) with unit \( \varepsilon \) and counit \( \eta \). Then the 2-cell \( \alpha_f : Ff \rightarrow Gf \) has an inverse given by the mate of \( \alpha_u : Fu \rightarrow Gu \).

**Proof.** As \( f \dashv u \) we have \( Ff \dashv Fu \) via counit

\[
\begin{array}{c}
Ff \cdot Fu \xrightarrow{\varphi_{f,u}} F(fu) \xrightarrow{F\varepsilon} F1_Y \xrightarrow{\lambda_Y} 1_{FY}
\end{array}
\]

and unit

\[
1_{FX} \xrightarrow{\lambda_{X}^{-1}} F1_X \xrightarrow{F\eta} F(uf) \xrightarrow{\varphi_{u,f}} Fu \cdot Ff
\]

\(^4\)There are also nullary pasting properties which we will omit.
and similarly $Gf \dashv Gu$. That the mate of $\alpha_u$ constructed as the pasting

\[
\begin{array}{ccccccccc}
F X & \xext{1_{FX}} & F X & \xext{1_{FX}} & F X & \xext{F f} & FY & \xext{1_{FY}} & FY & \xext{1_{FY}} & FY \\
\uparrow \lambda^{-1}_X & \quad & \uparrow \lambda^{-1}_X & \quad & \uparrow \lambda^{-1}_X & \quad & \uparrow \lambda^{-1}_X & \quad & \uparrow \lambda^{-1}_X & \quad & \uparrow \lambda^{-1}_X \\
F X & \xext{1_{FX}} & F X & \xext{1_{FX}} & F X & \xext{1_{FX}} & FX & \xext{1_{FX}} & G f & \xext{G f} & FY \\
\end{array}
\]

is the inverse of $\alpha_f$ is a simple calculation which we will omit (as the details are in [4, Lemma 2.13]).

**Remark 19.** Under the conditions of the above lemma we have corresponding functors $F^\co, G^\co : \mathcal{A}^\co \to \mathcal{B}^\co$ which are adjunction preserving (gregarious), and an icon $\alpha^\co : G^\co \to F^\co$. Thus noting $u \dashv f$ in $\mathcal{A}^\co$ we see that in $\mathcal{B}^\co$, $\alpha^\co_u : Gf \to Ff$ has an inverse given as the mate of $\alpha^\co_f$. It follows that $\alpha_u$ has an inverse given as the mate of $\alpha_f$ in $\mathcal{B}$.

2.5. **Adjunctions of spans and polynomials.** Later on we will need to discuss gregarious functors out of bicategories of spans and bicategories of polynomials, and so an understanding of the adjunctions in these bicategories will be essential.

We first recall the classification of adjunctions in the bicategory of spans. A proof of this classification is given in [3, Proposition 2], but this proof does not readily generalize to the setting of polynomials. We therefore give a simpler proof using the properties of the mates correspondence.

**Proposition 20.** Up to isomorphism, all adjunctions in $\text{Span}(\mathcal{E})$ are of the form

\[(2.2)
\begin{array}{ccccccc}
X & \xleftarrow{1_X} & f & \xrightarrow{\alpha} & Y \\
\downarrow s & & & & \downarrow t \\
X & \xleftarrow{\pi_1} & Y & \xleftarrow{\pi_2} & Y
\end{array}
\]

with unit and counit

\[(2.3)
\begin{array}{cccccccc}
\bullet & \xleftarrow{s} & \bullet & \xrightarrow{t} & \bullet \\
\downarrow u & & & & \downarrow v \\
\bullet & \xleftarrow{id} & \bullet & \xrightarrow{id} & \bullet
\end{array}
\]

where $(X \times_Y X, \pi_1, \pi_2)$ is the pullback of $f$ with itself.

**Proof.** It is simple to check the above defines an adjunction. We now check that all adjunctions have this form, up to isomorphism. To do this, suppose we are given an adjunction of spans

\[
\begin{array}{ccccccc}
s & \xleftarrow{1_X} & \bullet & \xrightarrow{\alpha} & t \\
\downarrow \pi_1 & & & & \downarrow \pi_2 \\
X & \xleftarrow{\pi_1} & \bullet & \xrightarrow{\pi_2} & Y
\end{array}
\]

and denote the unit of this adjunction (actually a representation of the unit using the universal property of pullback) by

\[
\begin{array}{cccccccc}
\bullet & \xleftarrow{\alpha_u} & \bullet & \xrightarrow{\alpha_v} & \bullet \\
\downarrow \alpha & & & & \downarrow \alpha \\
\bullet & \xleftarrow{id} & \bullet & \xrightarrow{id} & \bullet
\end{array}
\]
noting that \( v \beta \) is the identity. We then factor this unit as

\[
1 \rightarrow (s, t); (h, 1) \xrightarrow{id, \beta} (s, t); (u, v)
\]

where the first morphism is represented by

\[ \xymatrix{ & 1 \ar[r] & (s, t) \ar[r] & (u, v) \ar[r] & 1 \ar[r] & (h, 1) & \ar[r] & (u, v) } \]

and \( \beta : (h, 1) \rightarrow (u, v) \) is pictured on the right in (2.3). Under the mates correspondence this yields two morphisms

\[
(u, v) \rightarrow (h, 1) \xrightarrow{\beta} (u, v)
\]

which must compose to the identity. As the first morphism of spans is necessarily \( v \) we have also established \( \beta v \) as the identity, and hence \( v \) as an isomorphism. This allows us to construct an isomorphism of right adjoints \( (u, v) \rightarrow (f, 1) \) for an \( f \) as in Figure 2.2, corresponding to an isomorphism of left adjoints \( (1, f) \rightarrow (s, t) \) and hence showing \( s \) is invertible also. \( \Box \)

Remark 21. If we restrict ourselves to the bicategory \( \text{Span}_\text{iso}(\mathcal{E}) \) then we only have adjunctions as above when \( f \) is invertible (necessary to construct the counit).

In the case of polynomials there are more adjunctions to consider.

**Proposition 22.** Up to isomorphism, every adjunction in \( \text{Poly}(\mathcal{E}) \) is a composite of adjunctions of the form

\[ \xymatrix{ X \ar[r]^{1_X} & X \ar[r]^{f} & Y \ar[r]_{1_Y} & Y } \]

with unit and counit

\[ \xymatrix{ X \ar[r]^{1_X} & X \ar[r]^{f} & Y \ar[r]_{1_Y} & Y \ar[r]^{1_Y} & X \ar[r]_{1_X} & X } \]

and

\[ \xymatrix{ X \ar[r]^{1_X} & X \ar[r]^{f} & Y \ar[r]_{1_Y} & Y \ar[r]^{1_Y} & X \ar[r]_{1_X} & X } \]

with unit and counit

\[ \xymatrix{ Y \ar[r]^{1_Y} & Y \ar[r]^{f} & Y \ar[r]_{1_Y} & Y \ar[r]^{1_Y} & X \ar[r]_{1_X} & X } \]
Proof. It is simple to check that the above define adjunctions of polynomials, indeed this is almost the same calculation as in the case of spans. We now check that all adjunctions have this form, up to isomorphism. To do this, suppose we are given an adjunction of polynomials

\[
\begin{array}{ccc}
\bullet & \xrightarrow{s} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{u} & \bullet
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\bullet & \xleftarrow{t} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xleftarrow{v} & \bullet
\end{array}
\]

and denote the unit of this adjunction by

\[
\begin{array}{ccc}
\bullet & \xrightarrow{id} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{\alpha_1} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{\beta_1} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{\beta_2} & \bullet
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\bullet & \xrightarrow{\alpha_2} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{\beta_2} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{\beta_1} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{id} & \bullet
\end{array}
\]

noting that \(v \beta_2\) is the identity. We then factor this unit as, where \((\beta_1, \beta_2) : (h, q', 1) \rightarrow (u, q, v)\) is the cartesian morphism of polynomials pictured on the right above,

\[
1 \xrightarrow{} (s, p, t) ; (h, q', 1) \xrightarrow{id ; (\beta_1, \beta_2)} (s, p, t) ; (u, q, v)
\]

which under the mates correspondence yields two morphisms

\[
(u, q, v) \xrightarrow{} (h, q', 1) \xrightarrow{(\beta_1, \beta_2)} (u, q, v)
\]

which must compose to the identity; that is, a diagram below

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\beta_2} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{\beta_1} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{id} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{\alpha_2} & \bullet \\
\downarrow &       & \downarrow \\
\bullet & \xrightarrow{\alpha_1} & \bullet
\end{array}
\]

composing to the identity, showing \(\beta_2 v\) is the identity, and hence that \(v\) is invertible. This allows us to construct an isomorphism of right adjoints \((u, q, v) \rightarrow (f, g, 1)\) for some \(f\) and \(g\), corresponding to an isomorphism of left adjoints \((g, 1, f) \rightarrow (s, p, t)\) and hence showing \(p\) is invertible also. □

Remark 23. If we restrict ourselves to the bicategory \(\text{Poly}_c(\mathcal{E})\) then to have the second adjunction of Proposition 22 we require \(f\) to be invertible.

2.6. Basic properties of generic bicategories. The bicategories of spans \(\text{Span}(\mathcal{E})\) and bicategories of polynomials with cartesian 2-cells \(\text{Poly}_c(\mathcal{E})\) defined above both satisfy a special property: they contain a special class of 2-cells (which one may think of as the “diagonal” 2-cells\(^6\)) such that any 2-cell into a composite of 1-cells \(\alpha : c \rightarrow a; b\) factors uniquely as some diagonal 2-cell \(\delta : c \rightarrow l; r\) pasted with 2-cells \(\alpha_1 : l \rightarrow a\) and \(\alpha_2 : r \rightarrow b\). A bicategory \(\mathcal{A}\) with this property is called generic.

One of the main properties of generic bicategories \(\mathcal{A}\) is that oplax functors out of them admit an alternative description, similar to the description of a comonad.

\(^5\)Here the cartesian part of the morphism of polynomials is represented using Proposition 5.

\(^6\)Formally, these diagonals are defined as the generic morphisms against the composition functor. See [13] for details.
In particular, for a locally defined functor $L: \mathcal{A} \to \mathcal{C}$ one may define a bijection between coherent binary and nullary oplax constraint cells

$$\varphi_{a,b}: L(a; b) \to La; Lb, \quad \lambda_X: L1_X \to 1_X$$

and “coherent” comultiplication and counit maps

$$\Phi_S: Lc \to Ll; Lr, \quad \Lambda_\varepsilon: Ln \to 1_X$$

indexed over diagonal maps $\delta: c \to l; r$ and augmentations (2-cells into identity 1-cells) $\varepsilon: n \to 1_X$. Indeed, given the data $(\varphi, \lambda)$ the comultiplication maps $\Phi_\delta$ and counit maps $\Lambda_\varepsilon$ are given by the composites

$$Le \to L(l; r) \xrightarrow{\varphi_{l,r}} Ll; Lr$$

$$Ln \xrightarrow{\lambda_X} L1_X \xrightarrow{\lambda_X} 1_X$$

and conversely given the data $(\Phi, \Lambda)$ the oplax constraints $\varphi_{a,b}$ and $\lambda_X$ are recovered by factoring the identity 2-cell through a diagonal as on the left below and defining the right diagram to commute.

Trivially, we recover each unit $\lambda_X: L(1_X) \to 1_X$ as the component of $\Lambda$ at $\id_{1_X}$.

For the full statement concerning the bicategories of spans and cartesian polynomials, see Proposition 28 and Proposition 44 respectively.

### 3. Universal properties of spans

In this section, we give a complete proof of the universal properties of spans [4] using the properties of generic bicategories [13]. This is to demonstrate our method in the simpler case of spans before applying it to polynomials in Section 5.

#### 3.1. Stating the universal property

Before stating the universal property we recall that we have two canonical embeddings into the bicategory of spans given by the pseudofunctors denoted

$$(\_)_\Sigma: \mathcal{E} \to \text{Span}(\mathcal{E}), \quad (\_)_\Delta: \mathcal{E}^{\text{op}} \to \text{Span}(\mathcal{E})$$

These are defined on objects by sending an object of $\mathcal{E}$ to itself, and are defined on each morphism in $\mathcal{E}$ by the assignments

$$(\_)_\Sigma: \quad X \xrightarrow{f} Y \quad \mapsto \quad X \xleftarrow{1_Y} X \xrightarrow{f} Y$$

$$(\_)_\Delta: \quad X \xrightarrow{f} Y \quad \mapsto \quad Y \xrightarrow{f} X \xleftarrow{1_X} X$$

The embedding $(\_)_\Sigma$ has the property that it is both sinister and satisfies the Beck condition; two properties which we now define.

**Definition 24.** Let $\mathcal{E}$ be a category seen as a locally discrete 2-category, and let $\mathcal{C}$ be a bicategory. We say a pseudofunctor $F: \mathcal{E} \to \mathcal{C}$ of bicategories is **sinister** if for every morphism $f$ in $\mathcal{E}$ the 1-cell $Ff$ has a right adjoint in $\mathcal{C}$.

Supposing further that $\mathcal{E}$ has pullbacks, for any pullback square in $\mathcal{E}$ as on the left below, we may apply $F$ and compose with pseudofunctionality constraints giving
an invertible 2-cell as in the middle square below, and then take mates to get a 2-cell as on the right below

\[
\begin{array}{c}
\begin{tikzcd}
\bullet & Ff' \\
\downarrow & \downarrow \\
\bullet & Fg'
\end{tikzcd}
\end{array}
\quad \sim
\begin{array}{c}
\begin{tikzcd}
\bullet & Fg' \\
\downarrow & \downarrow \\
\bullet & Fg
\end{tikzcd}
\end{array}
\]

We say the sinister pseudofunctor \( F: \mathcal{E} \to \mathcal{C} \) satisfies the Beck condition if every such \( b_{f',g'}^{f,g} \) as on the right above is invertible.

**Remark 25.** Note that \( b_{f',g'}^{f,g} \) as above may be defined for any commuting square, not just a pullback. We call such a \( b_{f',g'}^{f,g} \) the Beck 2-cell corresponding to the commuting square, but should not expect it to be invertible if the square is not a pullback (even if the Beck condition holds).

We denote by \( \text{Sin}(\mathcal{E}, \mathcal{C}) \) the category of sinister pseudofunctors \( \mathcal{E} \to \mathcal{C} \) and invertible icons, and \( \text{Beck}(\mathcal{E}, \mathcal{C}) \) the subcategory of sinister pseudofunctors satisfying the Beck condition. The universal property of spans is then the following result, as given by Hermida \([7]\) and Dawson, Paré, and Pronk \([4]\, \text{Theorem 2.15}\).

**Theorem 26** (Universal Properties of Spans). Given a category \( \mathcal{E} \) with chosen pullbacks, composition with the canonical embedding \( (\text{---})_{\Sigma}: \mathcal{E} \to \text{Span}(\mathcal{E}) \) defines the two equivalences of categories

\[
\begin{align*}
\text{Greg}(\text{Span}(\mathcal{E}), \mathcal{C}) & \simeq \text{Sin}(\mathcal{E}, \mathcal{C}) \\
\text{Icon}(\text{Span}(\mathcal{E}), \mathcal{C}) & \simeq \text{Beck}(\mathcal{E}, \mathcal{C})
\end{align*}
\]

for any bicategory \( \mathcal{C} \).

### 3.2. Proving the universal property

Before proving Theorem 26 we will need to show that given a sinister pseudofunctor \( \mathcal{E} \to \mathcal{C} \) one may reconstruct an oplax functor \( \text{Span}(\mathcal{E}) \to \mathcal{C} \). The following lemma and subsequent propositions describe this construction.

**Lemma 27.** Let \( \mathcal{E} \) be a category with pullbacks seen as a locally discrete 2-category, and let \( \mathcal{C} \) be a bicategory. Suppose \( F: \mathcal{E} \to \mathcal{C} \) is a given sinister pseudofunctor, and for each morphism \( f \in \mathcal{E} \) define \( F_{\Delta f} := Ff \) and take \( F_{\Delta} \) to be a chosen right adjoint of \( Ff \) (choosing \( F_{\Delta} \) to strictly preserve identities). We may then define local functors

\[
L_{X,Y}: \text{Span}(\mathcal{E})_{X,Y} \to \mathcal{C}_{LX,LY}, \quad X,Y \in \mathcal{E}
\]

by the assignment

\[
\begin{array}{c}
\begin{tikzcd}
X & T & Y \\
\downarrow & \downarrow & \downarrow \\
S & \bullet & \bullet
\end{tikzcd}
\end{array}
\quad \mapsto
\begin{array}{c}
\begin{tikzcd}
FX & FT & FY \\
\downarrow & \downarrow & \downarrow \\
FS & \bullet & \bullet
\end{tikzcd}
\end{array}
\]

where \( \alpha \) is the mate of the isomorphism on the left below

\[
\begin{array}{c}
\begin{tikzcd}
FX & FT \\
\downarrow & \downarrow \\
FX & FX
\end{tikzcd}
\end{array}
\quad \approx
\begin{array}{c}
\begin{tikzcd}
FX & FX \\
\downarrow & \downarrow \\
FX & FX
\end{tikzcd}
\end{array}
\]
under the adjunctions $F\Sigma s \dashv F\Delta s$ and $F\Sigma u \cdot F\Sigma f \dashv F\Delta f \cdot F\Delta u$, and $\gamma$ is the mate of the isomorphism on the right above under the adjunctions $F\Sigma f \dashv F\Delta f$ and $1_{FY} \dashv 1_{FY}$.

Proof. Functoriality is clear from functoriality of mates and the associativity condition and unitary conditions on $F$. $\square$

To show that these local functors can be endowed with the structure of an oplax functor it will be useful to recall the following reduced description of such an oplax structure, obtained via the theory of Subsection 2.6.

**Proposition 28.** [13] Let $\mathcal{E}$ be a category with pullbacks and denote by $\text{Span}(\mathcal{E})$ the bicategory of spans in $\mathcal{E}$. Let $\mathcal{C}$ be a bicategory. Then to give an oplax functor

$$L : \text{Span}(\mathcal{E}) \to \mathcal{C}$$

is to give a locally defined functor

$$L_{X,Y} : \text{Span}(\mathcal{E})_{X,Y} \to \mathcal{C}_{LX,LY}, \quad X,Y \in \mathcal{E}$$

with comultiplication and counit maps

$$\Phi_{s,h,t} : L(s,t) \to L(s,h) ; L(h,t), \quad \Lambda_{h} : L(h,h) \to 1_{LX}$$

for every respective diagram in $\mathcal{E}$ such that:

1. for any triple of morphisms of spans as below

$$\begin{array}{ccc}
X & \xleftarrow{u} & Z \\
\downarrow{s} & & \downarrow{t} \\
T & \rightarrow & Y
\end{array} \hspace{1cm} \begin{array}{ccc}
X & \xleftarrow{u} & Z \\
\downarrow{s} & & \downarrow{t} \\
T & \rightarrow & Y
\end{array}
$$

we have the commuting diagram

$$L(u,v) \xrightarrow{\Phi_{u,k,v}} L(u,k) ; L(k,v)$$

$$L(s,t) \xrightarrow{\Phi_{s,h,t}} L(s,h) ; L(h,t)$$

2. for any morphism of spans as on the left below

$$\begin{array}{ccc}
X & \xleftarrow{p} & N \\
\downarrow{q} & & \downarrow{q} \\
X & \rightarrow & X
\end{array} \hspace{1cm} \begin{array}{ccc}
X & \xleftarrow{p} & N \\
\downarrow{q} & & \downarrow{q} \\
X & \rightarrow & X
\end{array}
$$

the diagram on the right above commutes;

3. for all diagrams of the form

$$\begin{array}{ccc}
W & \xleftarrow{s} & X \\
\downarrow{h} & & \downarrow{h} \\
X & \rightarrow & Y
\end{array} \hspace{1cm} \begin{array}{ccc}
W & \xleftarrow{s} & X \\
\downarrow{h} & & \downarrow{h} \\
X & \rightarrow & Y
\end{array} \hspace{1cm} \begin{array}{ccc}
X & \xleftarrow{h} & Y \\
\downarrow{h} & & \downarrow{h} \\
X & \rightarrow & Y
\end{array}
$$
in \( \mathcal{E} \), we have the commuting diagram

\[
\begin{array}{ccc}
L(s,t) & \xrightarrow{\Phi_{s,h,t}} & L(s,h) ; L(h,t) \\
L(s,k) ; L(k,t) & \xrightarrow{\Phi_{s,k,t}} & L(s,k) ; L(k,t) \\
\end{array}
\]

(4) for all spans \((s,t)\) we have the commuting diagrams

\[
\begin{array}{ccc}
L(s,s) ; L(s,t) & \xrightarrow{\Phi_{s,s,t}} & L(s,s) ; L(s,t) \\
L(s,t) ; L(t,t) & \xrightarrow{\Phi_{s,t,t}} & L(s,t) ; L(t,t) \\
\end{array}
\]

We now prove that the locally defined functor \( L \) above may be endowed with an oplax structure.

**Proposition 29.** Let \( \mathcal{E} \) be a category with pullbacks seen as a locally discrete 2-category, and let \( \mathcal{C} \) be a bicategory. Suppose \( F: \mathcal{E} \to \mathcal{C} \) is a given sinister pseudofunctor. Then the locally defined functor \( L_{X,Y}: \text{Span}(\mathcal{E})_{X,Y} \to \mathcal{C}_{LX,LY} \), \( X,Y \in \mathcal{E} \) as in Lemma 27 canonically admits the structure of an oplax functor.

**Proof.** By Proposition 28, to equip the locally defined functor \( L \) with an oplax structure is to give comultiplication maps \( \Phi_{s,h,t}: L(s,t) \to L(s,h) ; L(h,t) \) and counit maps \( \Lambda_h: L(h,h) \to 1_{LX} \) for diagrams of the respective forms

satisfying naturality, associativity, and unitary conditions. To do this, we take each \( \Phi_{s,h,t} \) and \( \Lambda_h \) to be the respective pastings

\[
(3.1)
\]

Associativity of comultiplication is trivial; indeed, given a diagram of the form

\[
\begin{array}{ccc}
W & \xleftarrow{s} & X \\
& h \downarrow & \downarrow t \\
Y & \xrightarrow{k} & Z \\
\end{array}
\]

the pasting

\[
\begin{array}{c}
FW \xrightarrow{F \Delta s} FT \xrightarrow{F \Delta h} FY \xrightarrow{F \Delta k} FT \xrightarrow{F \Delta t} FZ
\end{array}
\]

is the same regardless of which order we compose in, a direct consequence of middle four interchange. The unitary axioms are also trivial, an immediate consequence of the triangle identities for an adjunction.
For the naturality condition, suppose we are given a triple of morphisms of spans

and note that we have the commuting diagram

\[
\begin{array}{ccc}
L(u,v) & \Phi_{u,k,v} & L(u,k); L(k,v) \\
\downarrow Lf & & \downarrow Lf;Lf \\
L(s,t) & \Phi_{s,h,t} & L(s,h); L(h,t)
\end{array}
\]

since the top composite is

and the bottom composite is

where the unlabeled 2-cells are as in Lemma 27. That these pastings agree is a standard functoriality of mates calculation. We omit the naturality of counits calculation, as it is a simpler functoriality of mates calculation. □

Remark 30. Is is trivial that each \( \lambda_X \) given by \( \Lambda \) at \( 1_X \) is invertible above.

We now check that the structure given above has its oplax constraints given by Beck 2-cells.

Lemma 31. Let the oplax functor \( L : \text{Span}(\mathcal{E}) \rightarrow \mathcal{C} \) be constructed as in Proposition 29. Then the binary oplax constraint cell on \( L \), at a composite of spans constructed as below

\[
\begin{array}{ccc}
& M & \\
& b & c \\
X & Y & Z
\end{array}
\]

(3.2)
is given by the Beck 2-cell for the pullback appropriately whiskered by \( F_\Delta a \) and \( F_\Sigma d \).

**Proof.** Given composable spans \((a, b)\) and \((c, d)\) the composite is given by the diagram (3.2). We then have an induced diagonal

\[
\delta_{ac', h, db'} : (ac', db') \to (ac', h); (h, db')
\]

and morphisms \(c' : (ac', h) \to (a, b)\) and \(b' : (h, db') \to (c, d)\) for which

\[
(a, b); (c, d) \overset{\delta_{\alpha', h, db'}}\longrightarrow (ac', h); (h, db') \overset{c', b'}\longrightarrow (a, b); (c, d)
\]

is the identity on \((a, b); (c, d)\). Hence the oplax constraint cell corresponding to the comultiplication maps \( \Phi \), namely

\[
\varphi_{(a, b), (c, d)} : L ((a, b); (c, d)) \to L (a, b) ; L (c, d)
\]

is given by the pasting

\[
\begin{array}{ccc}
FM & \overset{\varphi_{(a, b), (c, d)}}\longrightarrow & FM \\
\downarrow_{F a} & & \downarrow_{F a} \\
FX & \overset{\varphi_{(a, b), (c, d)}}\longrightarrow & FY \\
\downarrow_{F \Delta a} & & \downarrow_{F \Delta a} \\
FT & \overset{\varphi_{(a, b), (c, d)}}\longrightarrow & FS \\
\end{array}
\]

\[
\begin{array}{ccc}
FM & \overset{\varphi_{(a, b), (c, d)}}\longrightarrow & FM \\
\downarrow_{F b} & & \downarrow_{F b} \\
FY & \overset{\varphi_{(a, b), (c, d)}}\longrightarrow & FZ \\
\downarrow_{F \Delta b} & & \downarrow_{F \Delta b} \\
FT & \overset{\varphi_{(a, b), (c, d)}}\longrightarrow & FS \\
\end{array}
\]

where \( h = bc' = cb' \). It is an easy consequence of functoriality of mates that this pasting is the usual Beck 2-cell for the pullback with the appropriate whiskerings. \( \square \)

Finally, we will need the following lemma, a consequence of Lemma 18, in order to complete the proof.

**Lemma 32.** Suppose \( L, K : \text{Span}(\mathcal{E}) \to \mathcal{C} \) are given gregarious functors. Then any icon \( \alpha : L \to K \) is necessarily invertible.

**Proof.** We take identities to be pullback stable for simplicity, so that we have \((s, t) = (s, 1) ; (1, t)\). Let us consider the component of such an icon \( \alpha \) at a general span \( s, t \). Since \( \alpha \) is an icon, the diagram

\[
\begin{array}{ccc}
L (s, t) & \overset{\varphi}{\longrightarrow} & L (s, 1) ; L (1, t) \\
\downarrow_{\alpha_{(s,t)}} & & \downarrow_{\alpha_{(s,1)} ; \alpha_{(1,t)}} \\
K (s, t) & \overset{\psi}{\longrightarrow} & K (s, 1) ; K (1, t)
\end{array}
\]

commutes. By Lemma 18 we know \( \alpha_{(1,t)} \) is invertible, and by its dual we know \( \alpha_{(s,1)} \) is invertible. As \( F \) and \( G \) are gregarious \( \varphi \) and \( \psi \) are invertible above. Hence \( \alpha_{(s,t)} \) is invertible. \( \square \)

We now know enough for a complete proof of the universal properties of the span construction as given by Dawson, Paré, Pronk and Hermida.

**Proof of Theorem 26.** We consider the assignment of Theorem 26, i.e. composition with the embedding \((-)_\Sigma : \mathcal{E} \to \text{Span}(\mathcal{E})\) written as the assignment

\[
\begin{array}{ccc}
\text{Span}(\mathcal{E}) & \overset{F}{\longrightarrow} & \mathcal{C} \\
\downarrow_{G} & & \downarrow_{G_\Sigma} \\
\mathcal{E} & \overset{F_\Sigma}{\longrightarrow} & \mathcal{C}
\end{array}
\]

We start by proving the first universal property.
Well defined. This is clear by Corollary 32.

Fully faithful. That the assignment \( \alpha \mapsto \alpha \Sigma \) is bijective follows from the condition \( (\alpha_{ij}^{-1})^* = \alpha_{ij} \) forced by Lemma 18, and the commutativity of Figure 3.4. One need only check that any collection \( \alpha_{s,t} : F(s,t) \to G(s,t) \)
satisfying these two properties necessarily defines an icon. Indeed, that such an \( \alpha \) is locally natural is a simple consequence of functoriality of mates and \( \alpha \Sigma \) being an icon. To see that such an \( \alpha \) then defines an icon, note that each \( \Phi_{s,h,t} \) may be decomposed as the commuting diagram

\[
\begin{array}{ccc}
F(s) & \xrightarrow{\Phi_{s,h,t}} & F(h) \times F(t) \\
\Phi_{s,t} \downarrow & & \downarrow \Phi_{h,t}^{-1} \Phi_{h,t}^{-1} \\
F(s,1) \times F(1,t) & \xrightarrow{F(1,1) \Phi_{1,h,1}: F(1,1)} & F(1,1) \times F(h,1) \times F(1,1)
\end{array}
\]

and so the commutativity of the diagram\(^7\)

\[
\begin{array}{ccc}
F(s,t) & \xrightarrow{\Phi_{s,h,t}} & F(s,h) \times F(h,t) \\
\alpha_{s,t} \downarrow & & \downarrow \alpha_{s,h}: \alpha_{h,t} \\
G(s,t) & \xrightarrow{\Psi_{s,h,t}} & G(s,h) \times G(h,t)
\end{array}
\]

amounts to asking that the pastings

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\eta_{Gh}} & \bullet \\
\downarrow F_{s} & & \downarrow \eta_{Gh} \Phi_{s,t} \\
G_{s} & \xrightarrow{G_{s}} & G_{s} \\
\downarrow F_{s} \eta_{Gh} & & \downarrow \Phi_{s,t} \eta_{Gh} \\
G_{s} & \xrightarrow{G_{s}} & G_{s} \\
\downarrow F_{s} & & \downarrow \Phi_{s,t} \\
\bullet & \xrightarrow{id} & \bullet
\end{array}
\]

and

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\eta_{Gh}} & \bullet \\
\downarrow F_{s} & & \downarrow \eta_{Gh} \Phi_{s,t} \\
G_{s} & \xrightarrow{G_{s}} & G_{s} \\
\downarrow F_{s} \eta_{Gh} & & \downarrow \Phi_{s,t} \eta_{Gh} \\
G_{s} & \xrightarrow{G_{s}} & G_{s} \\
\downarrow F_{s} & & \downarrow \Phi_{s,t} \\
\bullet & \xrightarrow{id} & \bullet
\end{array}
\]

agree; which is easily seen by expanding \( \alpha_{\Delta i} \) in terms of \( \alpha_{\Sigma i}^{-1} \) and using the triangle identities. The nullary icon condition is trivial. This shows that \( \alpha \) indeed admits the structure of an icon.

Essentially surjective. Given any sinister pseudofunctor \( F : \mathcal{E} \to \mathcal{C} \) we take the gregarious functor \( L : \text{Span}(\mathcal{E}) \to \mathcal{C} \) from Proposition 29 and note that \( L_\Sigma = F \).

We now verify the second universal property.

Restrictions. The second property is a restriction of the first. Indeed, given a pseudofunctor \( L : \text{Span}(\mathcal{E}) \to \mathcal{C} \) the corresponding pseudofunctor \( L_\Sigma : \mathcal{E} \to \mathcal{C} \) satisfies the Beck condition, since the embedding \( (-)_\Sigma : \mathcal{E} \to \text{Span}(\mathcal{E}) \) satisfies the Beck condition. Moreover, given a sinister pseudofunctor \( F : \mathcal{E} \to \mathcal{C} \) which satisfies the Beck condition, the corresponding map \( \text{Span}(\mathcal{E}) \to \mathcal{C} \) is pseudo since the oplax constraint cells of this functor are Beck 2-cells by Lemma 31.

\[\text{□}\]

\(^7\)This diagram is equivalent to the binary coherence condition on such an icon.
4. Universal properties of spans with invertible 2-cells

In this section we derive the universal property of the bicategory of spans with invertible 2-cells, denoted $\text{Span}_{\text{iso}}(E)$. Indeed, an understanding of this universal property will be required for stating the universal property of polynomials with cartesian 2-cells $\text{Poly}_c(E)$ described in the next section.

4.1. Stating the universal property. The embeddings $(-)_\Sigma$ and $(-)_\Delta$ into $\text{Span}_{\text{iso}}(E)$ are defined the same as in the case of spans with the usual 2-cells. The difference here is that we no longer have adjunctions $f_\Sigma \dashv f_\Delta$ in general, a fact which we will emphasize by replacing the symbol $\Sigma$ with $\otimes$. Consequently the universal property is more complicated to state, and so we will need some definitions.

**Definition 33.** Given a category $E$ with chosen pullbacks, we may define the category of *Lax Beck pairs* on $E$, denoted $\text{LaxBeckPair}(E, C)$. This category has objects given by pairs of pseudofunctors $F_\otimes : E \to C$, $F_\Delta : E^{\text{op}} \to C$ which agree on objects, equipped with, for each pullback square

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 f'' \\
 g'' \\
 f \\
 \end{array}
\end{array}
\end{array} & \xymatrix{ & F_\otimes f'' \ar[dr] & \\
 & F_\Delta g'' \ar[dr] & \\
 & F_\otimes f \ar[dr] & \\
 & \end{array}
\end{array}
\]

in $E$ as on the left, a 2-cell as on the right (which we call a Beck 2-cell). The collection of these Beck 2-cells comprise the “Beck data” denoted $F b$ (or just $b$), and are required to be coherent in that given any double pullback of the respective forms

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 f''_1 \\
 g''_1 \\
 f_1 \\
 \end{array}
\end{array}
\end{array} & \xymatrix{ & F_\otimes f''_1 \ar[dr] & \\
 & F_\Delta g''_1 \ar[dr] & \\
 & F_\otimes f_1 \ar[dr] & \\
 & \end{array}
\end{array}
\]

we have

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 f'' \\
 g'' \\
 f \\
 \end{array}
\end{array}
\end{array} & \xymatrix{ & F_\otimes f'' \ar[dr] & \\
 & F_\Delta g'' \ar[dr] & \\
 & F_\otimes f \ar[dr] & \\
 & \end{array}
\end{array}
\]

and

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 f''_1 \\
 g''_1 \\
 f_1 \\
 \end{array}
\end{array}
\end{array} & \xymatrix{ & F_\otimes f''_1 \ar[dr] & \\
 & F_\Delta g''_1 \ar[dr] & \\
 & F_\otimes f_1 \ar[dr] & \\
 & \end{array}
\end{array}
\]

and

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 f'' \\
 g'' \\
 f \\
 \end{array}
\end{array}
\end{array} & \xymatrix{ & F_\otimes f'' \ar[dr] & \\
 & F_\Delta g'' \ar[dr] & \\
 & F_\otimes f \ar[dr] & \\
 & \end{array}
\end{array}
\]

and

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 f''_1 \\
 g''_1 \\
 f_1 \\
 \end{array}
\end{array}
\end{array} & \xymatrix{ & F_\otimes f''_1 \ar[dr] & \\
 & F_\Delta g''_1 \ar[dr] & \\
 & F_\otimes f_1 \ar[dr] & \\
 & \end{array}
\end{array}
\]
and

\[ F_\Delta(g_1 g_2) \cong F_\Delta(g'_1 g'_2) \]

In addition, the Beck 2-cells corresponding to the pullbacks

\[
\begin{array}{c}
\bullet \\
\downarrow \text{id} \\
\bullet
\end{array}
\quad \begin{array}{c}
\bullet \\
\downarrow \text{id} \\
\bullet
\end{array}
\quad \begin{array}{c}
\bullet \\
\downarrow \text{id} \\
\bullet
\end{array}
\quad \begin{array}{c}
\bullet \\
\downarrow \text{id} \\
\bullet
\end{array}
\]

when pasted with the appropriate nullary constraints as in

\[
\begin{array}{c}
\bullet \\
\downarrow \text{id} \\
\bullet
\end{array}
\quad \begin{array}{c}
\bullet \\
\downarrow \text{id} \\
\bullet
\end{array}
\quad \begin{array}{c}
\bullet \\
\downarrow \text{id} \\
\bullet
\end{array}
\quad \begin{array}{c}
\bullet \\
\downarrow \text{id} \\
\bullet
\end{array}
\]

are required to be identities. We refer to these conditions as the Beck–Chevalley coherence conditions. A morphism in this category \((F_\otimes, F_\Delta, F b) \to (G_\otimes, G_\Delta, G b)\) is a pair of icons \(\alpha : F_\otimes \to G_\otimes\) and \(\beta : F_\Delta \to G_\Delta\) such that for each pullback square as on the left below

\[ (4.2) \]

the right diagram commutes. The category \textbf{BeckPair} \((\mathcal{E}, \mathcal{C})\) is the subcategory of \textbf{LaxBeckPair} \((\mathcal{E}, \mathcal{C})\) containing objects \((F_\otimes, F_\Delta, F b)\) such that every Beck 2-cell in \(F b\) is invertible.

Before we can state the universal property, we will need to describe how lax Beck pairs arise from suitable functors out of \textbf{Span}_\text{iso} \((\mathcal{E})\).

**Definition 34.** Let \(\mathcal{E}\) be a category with pullbacks (chosen such that identities pullback to identities) and let \(\mathcal{C}\) be a bicategory. Then the category

\[ \textbf{Greg}_{\otimes, \Delta}(\textbf{Span}_\text{iso} \,(\mathcal{E}) \, , \, \mathcal{C}) \]

has objects given by those gregarious functors of bicategories \textbf{Span}_\text{iso} \((\mathcal{E}) \to \mathcal{C}\) which restrict to pseudofunctors when composed with the canonical embeddings \((-)_\otimes : \mathcal{E} \to \textbf{Span}_\text{iso} \,(\mathcal{E})\) and \((-)_\Delta : \mathcal{E} \to \textbf{Span}_\text{iso} \,(\mathcal{E})\). Moreover, we require that
each oplax constraint

\[ F\left(\begin{array}{c}
\bullet
\downarrow^s
\downarrow t
\end{array}\right) \rightarrow F\left(\begin{array}{c}
\bullet
\downarrow^\text{id}
\downarrow q
\end{array}\right) ; F\left(\begin{array}{c}
\bullet
\downarrow^\text{id}
\downarrow t
\end{array}\right) \]

be invertible. The morphisms of this category are icons.

**Proposition 35.** Let \( \mathcal{E} \) be a category with pullbacks (chosen such that identities pullback to identities) and let \( \mathcal{C} \) be a bicategory. We then have a functor

\[ (-)_\otimes\Delta : \text{Greg}_\otimes\Delta (\text{Span}_{\text{iso}} (\mathcal{E}), \mathcal{C}) \rightarrow \text{LaxBeckPair} (\mathcal{E}, \mathcal{C}) \]

defined by the assignment taking such a gregarious functor \( F : \text{Span}_{\text{iso}} (\mathcal{E}) \rightarrow \mathcal{C} \) to the pair of pseudofunctors

\[ F_\otimes : \mathcal{E} \rightarrow \mathcal{C} , \quad F_\Delta : \mathcal{E}^{\text{op}} \rightarrow \mathcal{C} \]
equipped with Beck 2-cells given by, for each pullback square as on the left below (with the chosen pullback on the right below)

\[
\begin{array}{ccc}
\bullet
\downarrow^g
\downarrow f
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
\bullet
\downarrow^\sim g
\downarrow \sim f
\end{array}
\]

the composite of:

1. the inverse of an oplax constraint cell

\[ F\left(\begin{array}{c}
\bullet
\downarrow^g
\downarrow f
\end{array}\right) ; F\left(\begin{array}{c}
\bullet
\downarrow^\sim g
\downarrow \sim f
\end{array}\right) \rightarrow F\left(\begin{array}{c}
\bullet
\downarrow^g
\downarrow f
\end{array}\right) \]

2. the application of \( F \) to the induced isomorphism of pullbacks

\[ F\left(\begin{array}{c}
\bullet
\downarrow^g
\downarrow f
\end{array}\right) \rightarrow F\left(\begin{array}{c}
\bullet
\downarrow^\sim g
\downarrow \sim f
\end{array}\right) \]

3. the oplax constraint cell

\[ F\left(\begin{array}{c}
\bullet
\downarrow^g
\downarrow f
\end{array}\right) \rightarrow F\left(\begin{array}{c}
\bullet
\downarrow^\sim g
\downarrow \sim f
\end{array}\right) ; F\left(\begin{array}{c}
\bullet
\downarrow^g
\downarrow f
\end{array}\right) \]

**Proof.** We must check the Beck 2-cells defined as above satisfy the required coherence conditions. The nullary condition on the Beck 2-cells is trivially equivalent to the nullary condition on the constraints of \( F \). To see the “horizontal double pullback condition” holds, we note that since \( F : \text{Span}_{\text{iso}} (\mathcal{E}) \rightarrow \mathcal{C} \) is oplax normal, we have a resulting natural transformation

\[ N (F) : N (\text{Span}_{\text{iso}} (\mathcal{E})) \rightarrow N (\mathcal{C}) \]

where the functor \( N : \text{Bicat} \rightarrow [\Delta^{op}, \text{Set}] \) is given by the geometric nerve [11]. In particular (as in [2]), on 2-simplices the assignment

\[
\begin{array}{ccc}
\bullet
\downarrow^a
\downarrow b
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
\bullet
\downarrow^F a
\downarrow F b
\end{array}
\]

\[
\begin{array}{ccc}
\bullet
\downarrow^c
\end{array}
\]

\[
\begin{array}{ccc}
\bullet
\downarrow^F c
\end{array}
\]
where $\overline{F\alpha}$ is $F\alpha$ composed with the appropriate oplax constraint cell, satisfies the condition that

\[
\begin{array}{c}
\bullet \\
\downarrow^\beta \\
\downarrow^\alpha \\
\end{array}
\quad = 
\begin{array}{c}
\bullet \\
\downarrow^\gamma \\
\downarrow^\delta \\
\end{array}
\]

implies that

\[
\begin{array}{c}
\bullet \\
\downarrow^\gamma \\
\downarrow^\delta \\
\end{array}
\quad = 
\begin{array}{c}
\bullet \\
\downarrow^\gamma \\
\downarrow^\delta \\
\end{array}
\]

Now consider the three spans

\[
\begin{array}{c}
\bullet \\
\downarrow^\gamma \\
\downarrow^\delta \\
\end{array}
\quad = 
\begin{array}{c}
\bullet \\
\downarrow^\gamma \\
\downarrow^\delta \\
\end{array}
\]

which we denote by shorthand as $(1, f_2), (1, f_1)$ and $(g, 1)$ respectively (where $f_1$, $f_2$ and $g$ are as in the left diagram in Figure 4.1). Applying this fact to the equality below, where each of the four regions contains a canonical isomorphism or equality of spans

\[
\begin{array}{c}
\bullet \\
\downarrow^\gamma \\
\downarrow^\delta \\
\end{array}
\quad = 
\begin{array}{c}
\bullet \\
\downarrow^\gamma \\
\downarrow^\delta \\
\end{array}
\]

then gives the horizontal double pullback condition (after composing with the appropriate pseudofunctoriality constraints of $F\Sigma$ and constraints of the form (4.3)). The proof of the vertical condition is similar. Finally, it is clear the canonical assignment on morphisms is well defined, and the assignment given by composing with the canonical embeddings is trivially functorial. □

We can now state the universal property of $\text{Span}_{\text{iso}}(\mathcal{E})$.

**Theorem 36.** Given a category $\mathcal{E}$ with chosen pullbacks (chosen such that identities pullback to identities), the functor $(-)\otimes\Delta$ of Proposition 35 defines the two equivalences of categories

\[
\text{Greg} \otimes \Delta (\text{Span}_{\text{iso}}(\mathcal{E}), \mathcal{C}) \simeq \text{LaxBeckPair}(\mathcal{E}, \mathcal{C})
\]

\[
\text{Icon} (\text{Span}_{\text{iso}}(\mathcal{E}), \mathcal{C}) \simeq \text{BeckPair}(\mathcal{E}, \mathcal{C})
\]

for any bicategory $\mathcal{C}$.

**4.2. Proving the universal property.** We prove Theorem 36 directly, as the properties of generic bicategories cannot be used here. Also, for simplicity we assume without loss of generality that $\mathcal{C}$ is a 2-category and that the gregarious functors in question strictly preserve identities. This is justified since every bicategory is equivalent to a 2-category and every oplax normal functor is isomorphic to one which preserves identity 1-cells strictly.

**Proof of Theorem 36.** We start by proving the first universal property. We must prove that the functor

\[
(-)\otimes\Delta : \text{Greg} \otimes \Delta (\text{Span}_{\text{iso}}(\mathcal{E}), \mathcal{C}) \rightarrow \text{LaxBeckPair}(\mathcal{E}, \mathcal{C})
\]

defines an equivalence of categories.
Essentially Surjective. Given such a pair $F_\Sigma$ and $F_\Delta$ with Beck data $b$ we may define local functors

$$L_{X,Y} : \text{Span}_{\text{iso}}(E)_{X,Y} \rightarrow \mathcal{C}_{X,Y}, \quad X,Y \in E$$

by the assignment (suppressing pseudofunctoriality of $F_\Sigma$ and $F_\Delta$)

$$X \xrightarrow{F} Y \quad \mapsto \quad FX \xrightarrow{F_\Delta u} FM \xrightarrow{\psi_{1,1}} FM \xrightarrow{F_\varphi} FY$$

which is functorial by the Beck coherence conditions. An oplax constraint cell

$$L\left( \begin{array}{ccc} \bullet & \bullet & \bullet \\ u & v & w \end{array} \right) \rightarrow L\left( \begin{array}{ccc} \bullet & \bullet & \bullet \\ u' & v' & w' \end{array} \right) ; L\left( \begin{array}{ccc} \bullet & \bullet & \bullet \\ p & q & r \end{array} \right)$$

is given by (suppressing pseudofunctoriality of $F_\otimes$ and $F_\Delta$)

That these constraints satisfy the identity conditions trivially follows from the unit condition on the Beck 2-cells. For the associativity condition, suppose we are given diagrams of chosen pullbacks as below, with $p$ the induced isomorphism of generalized pullbacks, that is the associator for the triple $(a, b), (c, d), (e, f))$,

$$\text{then we must check that}$$

$$L(a g i, f j) \xrightarrow{\varphi} L(a g, d h) ; L(e, f) \xrightarrow{\varphi} (L(a, b) ; L(c, d) ; L(e, f))$$

$$L(a m, f n) \xrightarrow{\varphi} L(a, b) ; L(c k, f l) \xrightarrow{\varphi} (L(a, b) ; (L(c, d) ; L(e, f)))$$

commutes. The top path is a pasting of Beck 2-cells corresponding to the left diagram below, and the bottom path is the pasting of Beck 2-cells corresponding to the right diagram below (suppressing pseudofunctoriality constraints of $F_\Sigma$ and
For checking the oplax constraint cells are natural, consider a pair of morphisms of spans

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Fully Faithful. Suppose we are given two gregarious functors $F, G : \text{Span}_{\text{iso}}(\mathcal{E}) \to \mathcal{C}$ along with their restrictions $F_\otimes, G_\otimes$ and $F_\Delta, G_\Delta$ and families of Beck 2-cells $^b \varphi$ and $^b \psi$.

We first check the assignment of icons is surjective. Suppose we are given icons $\alpha : F_\otimes \to G_\otimes$ and $\beta : F_\Delta \to G_\Delta$ such that (4.2) holds. Then we may define an icon $\gamma : F \to G$ on each span $(s, t)$ by

$$F(s, t) \xrightarrow{\gamma_{s, t}} G(s, t)$$

where $\varphi$ and $\psi$ are the appropriate oplax constraint cells (necessarily invertible above). Now (4.2) forces $\gamma$ to be locally natural, as it suffices to check naturality on generating 2-cells, that is diagrams such as

$$\begin{array}{ccc}
\bullet & \xrightarrow{f} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{g} & \bullet
\end{array}$$

with $f$ invertible (this only needs trivial pullbacks corresponding to $b_{1,1}^{f,f}$). For checking $\gamma$ is an icon, the identity condition on $\gamma$ is from that of $\alpha$ and $\beta$. The composition condition is precisely (4.2).

We now check that the assignment of icons is injective. Suppose two given icons $\sigma, \delta$ both restrict to icons $\alpha$ and $\beta$. Then since the icons $\sigma$ and $\delta$ respect the composite of the spans

$$\begin{array}{ccc}
\bullet & \xleftarrow{f} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xleftarrow{g} & \bullet
\end{array}$$

both $\sigma$ and $\delta$ must satisfy the diagram 4.4 (in place of $\gamma$) and so are equal.

Restrictions. It is clear from the above that the oplax constraints are invertible precisely when the Beck data is invertible. □

5. Universal Properties of Polynomials with Cartesian 2-Cells

In this section we prove the universal property of the bicategory of polynomials with cartesian 2-cells, denoted $\text{Poly}_{c}(\mathcal{E})$. We will keep the proof as analogous to the case of spans as possible, though it still becomes somewhat more complicated.

5.1. Stating the universal property. This universal property of $\text{Poly}_{c}(\mathcal{E})$ turns out to be an amalgamation of that of $\text{Span}(\mathcal{E})$ and $\text{Span}_{\text{iso}}(\mathcal{E})$; in particular to give a pseudofunctor $\text{Poly}_{c}(\mathcal{E}) \to \mathcal{C}$ is to give a pair of pseudofunctors

$$\text{Span}(\mathcal{E}) \to \mathcal{C}, \quad \text{Span}_{\text{iso}}(\mathcal{E}) \to \mathcal{C}$$

which “$\Delta$-agree”, that is coincide on objects and on spans of the form

$$Y \xleftarrow{f} X \xrightarrow{1_X} X$$
with an additional condition asking that certain “distributivity morphisms” be invertible. For the purposes of the proof we will give a slightly different but equivalent description, for which we will need the following definitions.

**Definition 37.** Given a category $\mathcal{E}$ with chosen pullbacks, we may define the category of *Lax Beck triples* from $\mathcal{E}$ to a bicategory $\mathcal{C}$, denoted $\text{LaxBeckTriple}(\mathcal{E}, \mathcal{C})$.

An object consists of a triple of pseudofunctors which agree on objects

$$F_\Sigma : \mathcal{E} \to \mathcal{C}, \quad F_\Delta : \mathcal{E}^{\text{op}} \to \mathcal{C}, \quad F_\otimes : \mathcal{E} \to \mathcal{C}$$

such that $F_\Sigma f \dashv F_\Delta f$ for all morphisms $f$ in $\mathcal{E}$, along with “Beck data” denoted by $F b$ and consisting of for each pullback square

![Diagram](5.1)

in $\mathcal{E}$ as on the left, a 2-cell as on the right subject to the binary and nullary Beck coherence conditions as in Definition 33.

A morphism $(F_\Sigma, F_\Delta, F_\otimes, F b) \to (G_\Sigma, G_\Delta, G_\otimes, G b)$ in this category consists of an invertible icon $\beta : F_\Delta \to G_\Delta$ and (usual) icon $\gamma : F_\otimes \to G_\otimes$ such that for each pullback square in $\mathcal{E}$ as above, the diagram

![Diagram](5.2)

commutes.

There are a number of conditions which may be imposed on a lax Beck triple; these are defined as follows.

**Definition 38.** Given a lax Beck triple $(F_\Sigma, F_\Delta, F_\otimes, F b)$ from $\mathcal{E}$ to a bicategory $\mathcal{C}$, we say that:

1. the $\Delta \otimes$ condition holds if each component of the Beck data $F_\otimes b_{f,g}'$ is invertible;
2. the $\Sigma \Delta$ condition holds if each component of the $F_\Sigma - F_\Delta$ Beck data is invertible\(^8\);
3. the $\Sigma \otimes$ condition (distributivity condition) holds if for any distributivity pullback in $\mathcal{E}$ as on the left below

![Diagram](5.3)

the corresponding “distributivity morphism” defined as on the right above is invertible.

---

\(^8\)This is equivalent to asking the gregarious functor $\text{Span}(\mathcal{E}) \to \mathcal{C}$ resulting from $F_\Sigma$ be a pseudofunctor.
In particular, we define a *Beck triple* to be a lax Beck triple such that both conditions (1) and (2) hold, and a *DistBeck triple* to be a Beck triple also satisfying (3). We denote the corresponding subcategories of $\text{LaxBeckTriple}(\mathcal{E};\mathcal{C})$ as $\text{BeckTriple}(\mathcal{E};\mathcal{C})$ and $\text{DistBeckTriple}(\mathcal{E};\mathcal{C})$ respectively.

There are a number of canonical embeddings into $\text{Poly}_{c}(\mathcal{E})$ to mention; the most obvious being the embeddings

$(-)_{\Sigma} : \mathcal{E} \to \text{Poly}_{c}(\mathcal{E})$, $(-)_{\Delta} : \mathcal{E}^{\text{op}} \to \text{Poly}_{c}(\mathcal{E})$, $(-)_{\otimes} : \mathcal{E} \to \text{Poly}_{c}(\mathcal{E})$

which are defined on objects by sending an object of $\mathcal{E}$ to itself, and are defined on each morphism in $\mathcal{E}$ by the assignments

$(-)_{\Sigma} : X \xrightarrow{f} Y \mapsto X \xleftarrow{1_{X}} X \xrightarrow{1_{X}} X \xrightarrow{f} Y$

$(-)_{\Delta} : X \xrightarrow{f} Y \mapsto Y \xleftarrow{1_{Y}} X \xrightarrow{1_{X}} X \xrightarrow{f} Y$

$(-)_{\otimes} : X \xrightarrow{f} Y \mapsto Y \xleftarrow{1_{Y}} X \xrightarrow{f} Y \xleftarrow{1_{Y}} Y$

We also have the inclusion $(-)_{\Sigma \Delta} : \text{Span}(\mathcal{E}) \to \text{Poly}_{c}(\mathcal{E})$ of spans into polynomials given by the assignment

The less obvious embedding $(-)_{\Delta \otimes} : \text{Span}_{\text{iso}}(\mathcal{E}) \to \text{Poly}_{c}(\mathcal{E})$ is the canonical embedding of spans with invertible 2-cells into polynomials, given by the assignment

where one must note the appropriate square is a pullback since $f$ is invertible.

We will need to consider gregarious functors which restrict to pseudofunctors on the embeddings we have just defined, and so we make the following definition.

**Definition 39.** Let $\mathcal{E}$ be a locally cartesian closed category, let $\mathcal{C}$ be a bicategory and form the category $\text{Greg}(\text{Poly}_{c}(\mathcal{E});\mathcal{C})$. We define $\text{Greg}_{\otimes}(\text{Poly}_{c}(\mathcal{E});\mathcal{C})$ as the subcategory of gregarious functors $F : \text{Poly}_{c}(\mathcal{E}) \to \mathcal{C}$ such that the restriction $F_{\otimes} : \mathcal{E} \to \mathcal{C}$ is pseudo. Define $\text{Greg}_{\Sigma \Delta \otimes}(\text{Poly}_{c}(\mathcal{E});\mathcal{C})$ as the subcategory of gregarious functors for which both restrictions $F_{\Sigma \Delta} : \text{Span}(\mathcal{E}) \to \mathcal{C}$ and $F_{\Delta \otimes} : \text{Span}_{\text{iso}}(\mathcal{E}) \to \mathcal{C}$ are pseudo.

**Remark 40.** Note that a gregarious functor $F : \text{Poly}_{c}(\mathcal{E}) \to \mathcal{C}$ automatically restricts to pseudofunctors $F_{\Sigma}$ and $F_{\Delta}$. This is why we have omitted these conditions.
Also note that oplax constraints of the form
\[ F \left( \begin{array}{ccc} s & t & \text{id} \\ \downarrow & & \downarrow \\ \bullet & \bullet & \bullet \end{array} \right) \rightarrow F \left( \begin{array}{ccc} s & \text{id} & \text{id} \\ \downarrow & & \downarrow \\ \bullet & \bullet & \bullet \end{array} \right); F \left( \begin{array}{ccc} \text{id} & t & \text{id} \\ \downarrow & & \downarrow \\ \bullet & \bullet & \bullet \end{array} \right) \]
are automatically invertible by gregariousness.

We now have enough to state the universal property of polynomials.

**Theorem 41 (Universal Properties of Polynomials: Cartesian Setting).** Given a category \( E \) with chosen pullbacks and distributivity pullbacks, denote by \( T \) the composite operation

\[
\begin{array}{l}
\text{Greg}_{\otimes} (\text{Poly}_{\Sigma}(E), \mathcal{C}) \\
\downarrow \\
\text{Greg} (\text{Span}(E), \mathcal{C}) \times \text{Greg}_{\Sigma \otimes \Delta} (\text{Span}_{\text{iso}}(E), \mathcal{C}) \\
\downarrow \\
\text{LaxBeckTriple}(E, \mathcal{C})
\end{array}
\]

where the first operation is composition with the embeddings \((-)_{\Sigma \otimes} \) and \((-)_{\Delta \otimes} \), and the second is composition with the embedding \((-)_{\Sigma} \) and translation under the universal property of \( \text{Span}_{\text{iso}}(E) \). Then \( T \) defines the equivalences of categories

\[
\begin{array}{l}
\text{Greg}_{\otimes} (\text{Poly}_{\Sigma}(E), \mathcal{C}) \simeq \text{LaxBeckTriple}(E, \mathcal{C}) \\
\text{Greg}_{\Sigma \otimes \Delta \otimes} (\text{Poly}_{\Sigma}(E), \mathcal{C}) \simeq \text{BeckTriple}(E, \mathcal{C}) \\
\text{Icon}(\text{Poly}_{\Sigma}(E), \mathcal{C}) \simeq \text{DistBeckTriple}(E, \mathcal{C})
\end{array}
\]

for any bicategory \( \mathcal{C} \).

**Remark 42.** There are five other equivalences of categories since each of the three independent conditions \( \Sigma \Delta, \Delta \otimes \) and \( \Sigma \otimes \) of Definition 38 may or may not be enforced (giving a total of eight conditions). However, as the three above appear to be the most useful, we will not mention the others.

5.2. **Proving the universal property.** Before proving Theorem 41 we will need to show that given a lax Beck triple \( E \rightarrow \mathcal{C} \) one may reconstruct an oplax functor \( \text{Poly}_{\Sigma}(E) \rightarrow \mathcal{C} \). The following lemma and subsequent propositions describe this construction.

**Lemma 43.** Let \( E \) be a locally cartesian closed category seen as a locally discrete 2-category, and let \( \mathcal{C} \) be a bicategory. Suppose we are given a lax Beck triple consisting of pseudofunctors

\[ F_{\Sigma}: E \rightarrow \mathcal{C}, \quad F_{\Delta}: \mathcal{C}_{\text{op}} \rightarrow \mathcal{C}, \quad F_{\otimes}: E \rightarrow \mathcal{C} \]

and Beck 2-cells \( b \). We may then define local functors

\[ L_{X,Y}: \text{Poly}_{\Sigma}(E)_{X,Y} \rightarrow \mathcal{C}_{LX,LY}, \quad X,Y \in E \]

by the assignment...
where α is the mate of the isomorphism on the left below

\[
\begin{array}{ccc}
FE & \xrightarrow{1_{FE}} & FE \\
F_{\Sigma}s & \cong & F_{\Sigma}u \cdot F_{\Sigma}f \\
FX & \xrightarrow{1_{FX}} & FX \\
\end{array}
\quad
\begin{array}{ccc}
FB & \xrightarrow{F_{\Sigma}t} & FY \\
F_{\Sigma}g & \cong & 1_{FY} \\
FN & \xrightarrow{F_{\Sigma}v} & FY \\
\end{array}
\]

under the adjunctions \(F_{\Sigma}s \dashv F_{\Delta}s\) and \(F_{\Sigma}u \cdot F_{\Sigma}f \dashv F_{\Delta}f \cdot F_{\Delta}u\), \(\gamma\) is the mate of the isomorphism on the right above under the adjunctions \(F_{\Sigma}g \dashv F_{\Delta}g\) and \(1_{FY} \dashv 1_{FY}\), and \(b_{\rho}^{f}_{U}f\) (simply denoted \(b\) for convenience) is the component of the Beck data at the given pullback.

Proof. The local functor \(L_{X,Y}\) sends the components of the composite

\[
\begin{array}{ccc}
E & \xrightarrow{p} & B \\
X & \xrightarrow{f} & Y \\
& \xrightarrow{m} & \\
M & \xrightarrow{h} & N \\
& \xrightarrow{k} & \\
T & \xrightarrow{v} & S \\
\end{array}
\]

to the pasting (where we have inserted squares below which paste to identities)

\[
\begin{array}{ccc}
FE & \xrightarrow{id} & FE \\
F_{\Delta}s & \cong & F_{\Delta}h \cdot F_{\Delta}f \\
FX & \xrightarrow{\alpha_{2}} & FM \\
F_{\Delta}m & \cong & F_{\Delta}h \cdot F_{\Delta}f \\
FT & \xrightarrow{id} & FS \\
\end{array}
\quad
\begin{array}{ccc}
FB & \xrightarrow{id} & FB \\
F_{\Sigma}g & \cong & F_{\Sigma}g \cdot F_{\Sigma}f \\
FY & \xrightarrow{\alpha_{1}} & FN \\
F_{\Sigma}v & \cong & F_{\Sigma}g \cdot F_{\Sigma}f \\
FS & \xrightarrow{id} & FS \\
\end{array}
\]

By the condition on vertical composition of Beck 2-cells \(b\), and functoriality of mates, this is equal to applying \(L_{X,Y}\) to the composite. That the identity maps are preserved is similar to the case of spans, but using the vertical nullary condition on Beck 2-cells \(b\).

As in the case of spans, it will be helpful to recall the reduced description of an oplax structure on local functors out of the bicategory of polynomials.

**Proposition 44.** [13] Let \(\mathcal{E}\) be a locally cartesian closed category and denote by \(\textbf{Poly}_{c}(\mathcal{E})\) the bicategory of polynomials in \(\mathcal{E}\) with cartesian 2-cells. Let \(\mathcal{C}\) be a bicategory. Then to give an oplax functor

\[
L : \textbf{Poly}_{c}(\mathcal{E}) \rightarrow \mathcal{C}
\]

is to give a locally defined functor

\[
L_{X,Y} : \textbf{Poly}_{c}(\mathcal{E})_{X,Y} \rightarrow \mathcal{C}_{LX,LY}, \quad X,Y \in \mathcal{E}
\]

with comultiplication and counit maps

\[
\Phi_{s,p_{1},h,p_{2},t} : L(s, p, t) \rightarrow L(s, p_{1}, h) \cdot L(h, p_{2}, t), \quad \Lambda_{h} : L(h, 1, h) \rightarrow 1_{LX}
\]

for every respective diagram in \(\mathcal{E}\),

\[
\begin{array}{ccc}
E & \xrightarrow{P_{1}} & T \xrightarrow{P_{2}} & B \\
X & \xrightarrow{h} & Y & \xrightarrow{t} & Z \\
X & \xrightarrow{h} & X \xrightarrow{t} & X
\end{array}
\]
where we assert $p = p_1; p_2$ on the left, such that:

(1) for any morphisms of polynomials as below

we have the commuting diagram

(2) for any morphism of polynomials as on the left below

the diagram on the right above commutes;

(3) for all diagrams of the form

in $\mathcal{E}$, we have the commuting diagram

(4) for all polynomials $(s, p, t)$ the diagrams
We now prove that the locally defined functor \( L \) above may be endowed with an oplax structure.

**Lemma 45.** Let \( \mathcal{E} \) be a locally cartesian closed category seen as a locally discrete 2-category, and let \( \mathcal{C} \) be a bicategory. Suppose we are given a lax Beck triple

\[
F_\Sigma : \mathcal{E} \to \mathcal{C}, \quad F_\Delta : \mathcal{E}^{\text{op}} \to \mathcal{C}, \quad F_\otimes : \mathcal{E} \to \mathcal{C}
\]

with Beck 2-cells \( \beta \). Then the locally defined functor

\[
L_{X,Y} : \text{Poly}._c(\mathcal{E})_{X,Y} \to \mathcal{C}_{LX,LY}, \quad X,Y \in \mathcal{E}
\]
as in Lemma 43 canonically admits the structure of an oplax functor.

**Proof.** By Proposition 44, to equip the locally defined functor \( L \) with an oplax structure is to give comultiplication maps \( \Phi_{s,p,1} \), \( \Phi_{h,p,2},t \) and counit maps \( \Lambda_h \) for all diagrams of the respective forms,

\[
\begin{array}{ccc}
E & \overset{p_1}{\longrightarrow} & T \\
\downarrow p_2 & & \downarrow t \\
X & \longrightarrow & B
\end{array}
\]

satisfying naturality, associativity, and unitary conditions. To do this, we take each \( \Phi_{s,p,1,h,p,2},t \) to be the pasting

\[
\begin{array}{ccc}
FX & \overset{F_{\otimes}}{\longrightarrow} & FE \\
\downarrow F_{\otimes} & \quad & \downarrow F_{\otimes} \\
FT & \overset{F_{\otimes}F_{\Delta}}{\longrightarrow} & FY \\
\downarrow F_{\otimes} & \quad & \downarrow F_{\otimes} \\
FB & \overset{F_{\otimes}F_{\Delta}}{\longrightarrow} & FZ \\
\end{array}
\]

and each \( \Lambda_h \) to be the pasting

\[
\begin{array}{ccc}
FX & \overset{F_{\otimes}1_x}{\longrightarrow} & FX \\
\downarrow F_{\otimes} & \quad & \downarrow F_{\otimes} \\
FT & \overset{F_{\otimes}F_{\Delta}}{\longrightarrow} & FT \\
\downarrow F_{\otimes} & \quad & \downarrow F_{\otimes} \\
FX & \overset{F_{\Delta}1_x}{\longrightarrow} & FX \\
\end{array}
\]

Associativity of comultiplication is almost trivial; indeed, given a diagram of the form

\[
\begin{array}{ccc}
O & \overset{a}{\longrightarrow} & G \\
\downarrow s & \quad & \downarrow b \\
W & \longrightarrow & H \\
\downarrow h & \quad & \downarrow c \\
X & \longrightarrow & K \\
\downarrow t & \quad & \downarrow k \\
Y & \longrightarrow & Z
\end{array}
\]

both paths in the associativity of comultiplication condition compose to

\[
\begin{array}{ccc}
FW & \overset{F_{\otimes}a}{\longrightarrow} & FO \\
\downarrow F_{\otimes} & \quad & \downarrow F_{\otimes} \\
FG & \overset{F_{\otimes}b}{\longrightarrow} & FG \\
\downarrow F_{\otimes} & \quad & \downarrow F_{\otimes} \\
FX & \overset{F_{\otimes}c}{\longrightarrow} & FY \\
\downarrow F_{\otimes} & \quad & \downarrow F_{\otimes} \\
FH & \overset{F_{\otimes}k}{\longrightarrow} & FH \\
\downarrow F_{\otimes} & \quad & \downarrow F_{\otimes} \\
FK & \overset{F_{\otimes}t}{\longrightarrow} & FK \\
\end{array}
\]

by associativity of the constraints of \( F_\otimes \). The unitary axioms are also almost trivial, a consequence of the triangle identities for an adjunction and the unitary axioms on \( F_\otimes \).
For the naturality condition, suppose we are given a triple of cartesian morphisms of polynomials

and consider the diagram

\[
\begin{array}{ccc}
L(u, q, v) & \Phi_{u,q_1,v,q_2,v} & L(u, k, q_2, v) \\
L(f, g) & & L(f, k, c, g) \\
L(s, p, t) & \Phi_{s,p_1,h,p_2,t} & L(s, h, p, t)
\end{array}
\]

Now the top composite is

\[
\begin{array}{ccc}
FR & \cong & FI \\
FS & \Phi_{u,q_1,v,q_2,v} & FS \\
FY & \Phi_{s,p_1,h,p_2,t} & FY \\
FE & \Phi_{u,q_1,v,q_2,v} & FE
\end{array}
\]

where the unlabeled 2-cells are as in Lemma 43, and one may rewrite the pasting of the three middle triangles above as an “identity square” and pasting with \(\eta_{\mathbb{F}h}\).

It follows that this is equal to the bottom composite given by the pasting

\[
\begin{array}{ccc}
FR & \cong & FI \\
FS & \Phi_{u,q_1,v,q_2,v} & FS \\
FY & \Phi_{s,p_1,h,p_2,t} & FY \\
FE & \Phi_{u,q_1,v,q_2,v} & FE
\end{array}
\]

using the horizontal binary axiom on elements of \(\mathbb{F}\); thus showing naturality of comultiplication. Naturality of counits is similar to the case of spans (except that one must use the horizontal nullary axiom on elements of \(\mathbb{F}\)) and so will be omitted.

\[\square\]

It will be useful to have a description of the oplax constraint cells \(\varphi\) corresponding to our comultiplication maps \(\Phi\). This is described by the following lemma.
Lemma 46. Let the oplax functor \( L: \text{Poly}_e(\mathcal{E}) \rightarrow \mathcal{C} \) be constructed as in Proposition 45. Then the binary oplax constraint cell on \( L \) at a composite of polynomials constructed as below

\[
\begin{array}{c}
\xymatrix{ & H \ar[r]^{p_1} & M \ar[r]^{p_2} & K \\
A \ar[r]_m & B \ar[r]_z \ar[ur]^{pb} & C \ar[r]_d & D \ar[ur]_{pb} \\
& X \ar[ur]_a & & \ar[ul]_b & Y \ar[ur]_c & & \ar[ul]_d & Z }
\end{array}
\]

is given by the pasting

\[
\begin{array}{c}
\xymatrix{ & F_{\underline{p_1}w} \ar[r]^{\Phi_{\underline{p_1}w}} & F_{\underline{p_2}w} \ar[r]^{\Phi_{\underline{p_2}w}} & F_{\underline{p_2}w} \ar[r]^{\Phi_{\underline{p_2}w}} & F_{\underline{p_2}w} \\
FX \ar[r]_{\Phi_{\underline{p_1}w}} \ar[ur]_{\Phi_{\underline{p_1}w}} & F \ar[r]_{\Phi_{\underline{p_2}w}} \ar[r]_{\Phi_{\underline{p_2}w}} & F \ar[r]_{\Phi_{\underline{p_2}w}} \ar[r]_{\Phi_{\underline{p_2}w}} & F \ar[ur]_{\Phi_{\underline{p_2}w}} \\
& FA \ar[r]_{\Phi_{\underline{p_1}w}} \ar[ur]_{\Phi_{\underline{p_1}w}} & FB \ar[r]_{\Phi_{\underline{p_2}w}} \ar[r]_{\Phi_{\underline{p_2}w}} & FC \ar[r]_{\Phi_{\underline{p_2}w}} \ar[r]_{\Phi_{\underline{p_2}w}} & FD \ar[ur]_{\Phi_{\underline{p_2}w}} \\
& F_{\underline{p_1}w} \ar[r]_{\Phi_{\underline{p_1}w}} & F_{\underline{p_1}w} \ar[r]_{\Phi_{\underline{p_1}w}} & F_{\underline{p_1}w} \ar[r]_{\Phi_{\underline{p_1}w}} & F_{\underline{p_1}w} \ar[ur]_{\Phi_{\underline{p_1}w}} \\
& F_{\underline{p_2}w} \ar[r]_{\Phi_{\underline{p_2}w}} & F_{\underline{p_2}w} \ar[r]_{\Phi_{\underline{p_2}w}} & F_{\underline{p_2}w} \ar[r]_{\Phi_{\underline{p_2}w}} & F_{\underline{p_2}w} \ar[ur]_{\Phi_{\underline{p_2}w}} \\
& F_{\underline{p_2}w} \ar[r]_{\Phi_{\underline{p_2}w}} & F_{\underline{p_2}w} \ar[r]_{\Phi_{\underline{p_2}w}} & F_{\underline{p_2}w} \ar[r]_{\Phi_{\underline{p_2}w}} & F_{\underline{p_2}w} \ar[ur]_{\Phi_{\underline{p_2}w}} }
\end{array}
\]

where \( p = p_2p_1 \) and \( h = bx = cy \).

Proof. Given composable polynomials \((a, m, b)\) and \((c, n, d)\) the composite is given by the terminal diagram as in (5.4). We then have an induced diagonal

\[
\delta_{aw, p, dz}: (aw, p, dz) \rightarrow (aw, p_1, h); (h, p_2, dz)
\]

and morphisms \((w, x): (aw, p_1, h) \rightarrow (a, m, b)\) and \((y, z): (h, p_2, dz) \rightarrow (c, n, d)\) for which

\[
(a, m, b); (c, n, d) \xrightarrow{(w, x);(y, z)} (aw, p_1, h); (h, p_2, dz) \xrightarrow{(a, m, b);(c, n, d)}
\]

is the identity on \((a, m, b); (c, n, d)\). It follows that the binary oplax constraint cell corresponding to the comultiplication maps \( \Phi \), namely

\[
\varphi_{(a, m, b);(c, n, d)}: L ((a, m, b); (c, n, d)) \rightarrow L (a, m, b); L (c, n, d)
\]

is given by (5.5). \(\square\)

We now know enough for a complete proof of the universal properties of the polynomials with cartesian 2-cells.

Proof of Theorem 41. We consider the assignment

\[
\Upsilon: \text{Greg}_\otimes (\text{Poly}_e(\mathcal{E}), \mathcal{C}) \rightarrow \text{LaxBeckTriple}(\mathcal{E}, \mathcal{C})
\]

of Theorem 41, i.e. given a gregarious functor \( F: \text{Poly}_e(\mathcal{E}) \rightarrow \mathcal{C} \) which restricts to a pseudofunctor when composed with \((-)_\otimes \), we extract, via Theorem 36, the pseudofunctors \( F_\Sigma, F_\Delta \) and \( F_\otimes \) equipped with the Beck data \( b \). This data defines a lax Beck triple \( \mathcal{E} \rightarrow \mathcal{C} \).

Well defined. Given an icon \( \alpha: F \Rightarrow G: \text{Poly}_e(\mathcal{E}) \rightarrow \mathcal{C} \) we know \( \alpha_\Delta: F_\Delta \rightarrow G_\Delta \) is invertible, as it is a restriction of an icon \( \alpha_{\Sigma \Delta}: F_{\Sigma \Delta} \rightarrow G_{\Sigma \Delta} : \text{Span}(\mathcal{E}) \rightarrow \mathcal{C} \) which is necessarily invertible by Lemma 32.

We start by proving the first universal property.
Fully faithful. That the assignment \( \alpha \mapsto (\alpha_\Delta, \alpha_\otimes) \) is bijective follows from the fact the assignment \( \sigma_\Sigma \Delta \mapsto \alpha_\Delta \) is bijective, and the necessary commutativity of

\[
\begin{align*}
F(s,p,t) \xrightarrow{\varphi} F(s,1,1) & ; F(1,p,1) ; F(1,1,t) \\
G(s,p,t) \xrightarrow{\psi} G(s,1,1) & ; G(1,p,1) ; G(1,1,t)
\end{align*}
\]

where \( \varphi \) and \( \psi \) must be invertible constraints since \( F \) and \( G \) are gregarious.

Again, that \( \left( \alpha_{\Sigma \Delta}^{-1} \right)_{\varphi} = \alpha_\Delta \) is forced by Lemma 18 and one need only check that any collection

\[
\alpha_{s,p,t} : F(s,p,t) \to G(s,p,t)
\]

satisfying this property and (5.6) necessarily defines an icon.

We omit the calculation showing \( \alpha \) is locally natural. Indeed this calculation is almost the same as in the case of spans, except we must interchange a Beck 2-cell with the components \( \alpha_\Delta \) and \( \alpha_\otimes \) using the condition (5.2).

To see that such an \( \alpha \) then defines an icon, note that each \( \Phi_{s,p_1,h,p_2,t} \) may be decomposed as the commuting diagram

\[
\begin{align*}
F(s,p,t) & \xrightarrow{\Phi_{s,p_1,h,p_2,t}} F(s,p_1,h) ; F(h,p_2,t) \\
F(s,p_1,1) & ; F(1,p_2,t) \xrightarrow{F(s,p_1,1) ; \Phi_{1,h,1,1} ; F(1,p_2,t)} F(s,p_1,1) ; F(1,1,h) ; F(h,1,1) ; F(1,1,p_2,t)
\end{align*}
\]

and so the commutativity of the diagram

\[
\begin{align*}
F(s,p,t) & \xrightarrow{\Phi_{s,p_1,h,p_2,t}} F(s,p_1,h) ; F(h,p_2,t) \\
G(s,p,t) & \xrightarrow{\Phi_{s,p_1,h,p_2,t}} G(s,p_1,h) ; G(h,p_2,t)
\end{align*}
\]

amounts to checking that the pastings

and

agree. This is almost the same calculation as in spans except here we must use that \( \alpha_\otimes \) is an icon.
ESSENTIALLY SURJECTIVE. Suppose we are given a lax Beck triple \((F_\Sigma, F_\Delta, F_\Pi, b)\).
Then by Proposition 45, we get an oplax normal functor \(F: \text{Poly}_c(\mathcal{E}) \to \mathcal{C}\) (which
is gregarious as a consequence Proposition 22, and clearly restricts to a pseudo-
functor on \(\otimes\)), and this constructed \(F\) clearly restricts to the same lax Beck triple
when \(T\) is applied.

We now prove the remaining two universal properties, seen as restrictions of the first.

REstrictions. It is clear that for any Greg\(\otimes\) functor \(F: \text{Poly}_c(\mathcal{E}) \to \mathcal{C}\) we may
write \(F \cong \tilde{F}\) where \(\tilde{F}\) is given by sending \(F\) to its lax Beck triple and recovering a
map \(\tilde{F}: \text{Poly}_c(\mathcal{E}) \to \mathcal{C}\) under the above equivalence.

Also it is clear that \(F\) (or equivalently \(\tilde{F}\)) restricts to pseudofunctors on \(\Sigma \Delta\) and
\(\Delta \otimes\) precisely when this lax Beck triple is a Beck triple. This is seen by using the
general expression for an oplax constraint cell (5.5) on composites of polynomials
\((s,1,t);(u,1,v)\) and \((s,t,1);(u,v,1)\).

Now as each oplax constraint cell may be constructed from “Beck composites”
as above and “distributivity composites” of the form \((1,1,u);(1,f,1)\) (by the proof
of [6, Prop. 1.12]), it follows that asking \(F\) be pseudo corresponds to asking that,
in addition, the oplax constraint cells for composites \((1,1,u);(1,f,1)\) be invertible.
But this is precisely the \(\Sigma \otimes\) distributivity condition.

\[\square\]

6. Universal properties of polynomials

In this section we prove the universal property of the bicategory of polynomi-
als with general 2-cells, denoted \(\text{Poly}(\mathcal{E})\). As this bicategory is not generic, the
methods of the previous section do not directly apply. However, as composition in
\(\text{Poly}_c(\mathcal{E})\) and \(\text{Poly}(\mathcal{E})\) is the same we can still apply some results of the previous
section to help prove this universal property.

6.1. Stating the universal property. The universal property of \(\text{Poly}(\mathcal{E})\) ends
up being simpler to state then that of \(\text{Poly}_c(\mathcal{E})\) due to the existence of more
adjunctions. To state this property we will first require a strengthening of the
notions “sinister” and “Beck” pseudofunctor given in Section 3.

Definition 47. Let \(\mathcal{E}\) be a category with pullbacks, and let \(\mathcal{C}\) be a bicategory. We
denote by \(\text{2Sin}(\mathcal{E},\mathcal{C})\) the subcategory of \(\text{Sin}(\mathcal{E},\mathcal{C})\) consisting of pseudofunctors
\(F: \mathcal{E} \to \mathcal{C}\) for which \(Ff\) has two successive right adjoints for every morphism
\(f \in \mathcal{E}\). We denote by \(\text{2Beck}(\mathcal{E},\mathcal{C})\) the subcategory of \(\text{2Sin}(\mathcal{E},\mathcal{C})\) consisting of
those pseudofunctors which in addition satisfy the Beck condition.

Remark 48. The above Beck condition is on the pair \(F_\Sigma-F_\Delta\), but one could also ask
a Beck condition on the pair \(F_\Delta-F_\Pi\). The reason for not using the latter is that the
Beck 2-cells (arising from adjunctions \(F_\Delta f \dashv F_\Pi f\)) are not in the direction required
for constructing a lax Beck triple, and are invertible if and only if the former Beck
2-cells are invertible.

The following lemma will be needed to describe a distributivity condition which
may be imposed on such pseudofunctors.

Lemma 49. Let \(\mathcal{E}\) be a locally cartesian closed category seen as a locally discrete
2-category, and let \(\mathcal{C}\) be a bicategory. Suppose \(F: \mathcal{E} \to \mathcal{C}\) is a given 2-Beck pseudo-
functor, and for each morphism \(f \in \mathcal{E}\) define \(F_\Sigma f := Ff\), take \(F_\Delta f\) to be a chosen
right adjoint of \(Ff\) (choosing \(F_\Sigma\) to strictly preserve identities), and take \(F_\Pi f\) to
be a chosen right adjoint of \(F_\Delta f\) (again choosing \(F_\Pi\) to strictly preserve identities).
We may then define a Beck triple with underlying pseudofunctors
\[F_\Sigma: \mathcal{E} \to \mathcal{C}, \quad F_\Delta: \mathcal{E}^{\text{op}} \to \mathcal{C}, \quad F_\Pi: \mathcal{E} \to \mathcal{C}\]
and for each pullback as on the left below, the mate of the middle isomorphism below whose existence is asserted by the Beck condition

(6.1)

\[
\begin{array}{ccc}
f' & \xrightarrow{g'} & g \\
\downarrow & \quad & \downarrow \\
\bullet & \xrightarrow{f} & \bullet \\
\end{array}
\quad \Rightarrow 
\begin{array}{ccc}
F \circ f' & \xrightarrow{F \circ g'} & F \circ g \\
\downarrow & \quad & \downarrow \\
F \circ f & \xrightarrow{F \circ g} & F \circ g \\
\end{array}
\quad \Rightarrow 
\begin{array}{ccc}
F \circ f' & \xrightarrow{F \circ g'} & F \circ g \\
\downarrow & \quad & \downarrow \\
F \circ f & \xrightarrow{F \circ g} & F \circ g \\
\end{array}
\]

defining the Beck data as on the right above.

**Proof.** One needs to check that the defined Beck data satisfies the necessary coherence conditions, but this trivially follows from functoriality of mates. Also, every component of the Beck data $b_{f,g}^{f',g'}$ defined as in the above lemma must be invertible. This is since an isomorphism of left adjoints must correspond to an isomorphism of right adjoints under the mates correspondence.

It will be useful to give the Beck triples arising this way a name, and so we make the following definition.

**Definition 50.** We call a Beck triple $\mathcal{E} \to \mathcal{C}$ cartesian if for every morphism $f \in \mathcal{E}$ there exists adjunctions $F \circ f \dashv F \Pi f$ and the $\Delta \Pi$ Beck data corresponds to the $\Sigma \Delta$ data via the mates correspondence as in (6.1).

We may also ask that a cartesian Beck triple (or the corresponding 2-Beck functor) satisfies a distributivity condition.

**Definition 51.** Given the assumptions and data of Lemma 49, we say a 2-Beck pseudofunctor $F: \mathcal{E} \to \mathcal{C}$ satisfies the distributivity condition if the cartesian Beck triple recovered from this lemma satisfies the distributivity condition of Definition 38; meaning this cartesian Beck triple is a DistBeck triple.

Similar to the case of $\text{Poly}_\epsilon (\mathcal{E})$, we again have embeddings

$(-)_{\Sigma} : \mathcal{E} \to \text{Poly}(\mathcal{E})$, $(-)_{\Delta} : \mathcal{E}^{\text{op}} \to \text{Poly}(\mathcal{E})$, $(-)_{\Pi} : \mathcal{E} \to \text{Poly}(\mathcal{E})$

The main difference here is that with these embeddings we have triples of adjunctions $f \Sigma \dashv f \Pi \dashv f \Delta$ for every morphism $f \in \mathcal{E}$.

Trivially we have the inclusion $(-)_{\Sigma \Delta} : \text{Span}(\mathcal{E}) \to \text{Poly}(\mathcal{E})$ of spans into polynomials given by the assignment

The less obvious embedding $(-)_{\Delta \Pi} : \text{Span}(\mathcal{E})^{\text{co}} \to \text{Poly}(\mathcal{E})$ is the canonical embedding of spans with reversed 2-cells into polynomials, given by the assignment

We now have enough to state the universal property of polynomials.
Theorem 52 (Universal Properties of Polynomials: General Setting). Given a category $E$ with chosen pullbacks and distributivity pullbacks, composition with the canonical embedding $(-)_E : E \to \text{Poly}(E)$ defines the two equivalences of categories

- $\text{Greg}(\text{Poly}(E), C) \simeq \text{2Beck}(E, C)$
- $\text{Icon}(\text{Poly}(E), C) \simeq \text{DistBeck}(E, C)$

for any bicategory $C$.

Remark 53. One might ask if there is a universal property without the Beck condition being required. The problem is that if the restrictions to $\text{Span}(E)$ and $\text{Span}(E)^{co}$ are only required gregarious, but not pseudo, we do not have a canonical way to construct the necessary $\Delta \Pi$ Beck data, and so such a universal property would be unnatural.

6.2. Proving the universal property. Before proving Theorem 52 we will need to show how to reconstruct a gregarious functor $\text{Poly}(E) \to C$ from a 2-Beck pseudofunctor $E \to C$. The following proposition describes this construction.

Proposition 54. Let $E$ be a locally cartesian closed category seen as a locally discrete 2-category, and let $C$ be a bicategory. Suppose $F : E \to C$ is a given 2-Beck pseudofunctor, and for each morphism $f \in E$ define $F_\Sigma f := Ff$, take $F_\Delta f$ to be a chosen right adjoint of $Ff$ (choosing $F_\Delta$ to strictly preserve identities), and take $F_\Pi f$ to be a chosen right adjoint of $F_\Delta f$ (again choosing $F_\Pi$ to strictly preserve identities). We may then:

1. define a lax Beck triple as in Lemma 49;
2. define a gregarious functor $L : \text{Poly}(E) \to C$ satisfying the $\Sigma \Delta$ and $\Delta \Pi$ Beck conditions;
3. define local functors

$$L : \text{Poly}(E)_{X,Y} \to \mathcal{C}^{LX,LY}, \quad X, Y \in E$$

assigning each general morphism of polynomials

$$\begin{array}{ccc}
E & \xleftarrow{p} & B \\
\downarrow{e} & & \downarrow{t} \\
X & \xleftarrow{\alpha} & Y \\
\downarrow{f} & & \downarrow{g} \\
S & \xleftarrow{w} & B \\
\downarrow{u} & & \downarrow{v} \\
M & \xleftarrow{\beta} & N
\end{array}$$

assigning each general morphism of polynomials

$$\begin{array}{ccc}
FE & \xleftarrow{F_\Pi p} & FB \\
\downarrow{F_\Delta s} & & \downarrow{F_\Delta t} \\
FX & \xleftarrow{F_\alpha} & FY \\
\downarrow{F_\Delta u} & & \downarrow{F_\Delta v} \\
FM & \xleftarrow{F_\Pi q} & FN
\end{array}$$

$$\begin{array}{ccc}
& \xleftarrow{\psi} & \\
& \downarrow{\phi} & \downarrow{\gamma} \\
& \xi & \\
& \downarrow{\delta} & \downarrow{\epsilon} \\
& \phi & \\
& \downarrow{\theta} & \downarrow{\omega} \\
& \xi & \\
& \downarrow{\eta} & \downarrow{\zeta} \\
& \phi & \\
$$

$$\begin{array}{ccc}
& \xleftarrow{\psi} & \\
& \downarrow{\phi} & \downarrow{\gamma} \\
& \xi & \\
& \downarrow{\delta} & \downarrow{\epsilon} \\
& \phi & \\
& \downarrow{\theta} & \downarrow{\omega} \\
& \xi & \\
& \downarrow{\eta} & \downarrow{\zeta} \\
& \phi & \\
$$


where of the diagrams

\[
\begin{array}{cccc}
F_1 S & F S & F E & F Y \\
F_S \cdot F e & F_S \cdot F e & F_u \cdot F s & F_y \cdot F s \\
F_1 X & F X & F Y & F Y \\
\end{array}
\]

(a) \( \alpha \) is constructed as the mate of the left diagram under the adjunctions \( F_S \cdot F e \dashv F\Delta e \cdot F S \) and \( F_S \cdot F e \dashv F_S \cdot F e \);
(b) \( \mu \) is constructed as the mate of the middle diagram under the adjunctions \( F\Delta e \cdot F S \dashv 1_{FB} \cdot 1_{FB} \);
(c) \( \beta \) is the component of the Beck data at the given pullback;
(d) \( \gamma \) is the mate of the isomorphism on the right above under the adjunctions \( F_S \cdot F e \dashv F\Delta e \cdot F S \) and \( 1_{FY} \cdot 1_{FY} \).

(4) define a gregarious functor \( L : \text{Poly}(E) \to \mathcal{C} \).

Proof. We divide the proof into a number of parts.

**Part 1.** See Lemma 49.

**Part 2.** It then follows from Theorem 41 that this cartesian Beck triple gives rise to a gregarious functor \( \mathcal{L} : \text{Poly}(E) \to \mathcal{C} \). The \( \Sigma\Delta \) invertibility condition translates to an \( \Delta\Pi \) invertibility condition via the mates correspondence; an isomorphism of left adjoints must correspond to an isomorphism of right adjoints. Therefore each component of the Beck data \( \beta \) is invertible.

**Part 3.** The goal here is to show that we have local functors

\[ L : \text{Poly}(E)_{X,Y} \to \mathcal{C}L_{X,Y}, \quad X, Y \in E \]

We first note, for well definedness, that given two general morphisms of polynomials as below

\[
\begin{array}{ccc}
X & S_1 \xrightarrow{p_{1,2}} B & Y \\
\downarrow u & \downarrow f_1 & \downarrow g \\
M & N & Q \\
\end{array}
\]

\[
\begin{array}{ccc}
X & S_2 \xrightarrow{p_{2,3}} B & Y \\
\downarrow u & \downarrow f_2 & \downarrow g \\
M & N & Q \\
\end{array}
\]

equivalent in that there exists an isomorphism \( \nu : S_1 \to S_2 \) such that \( f_2\nu = f_1 \) and \( e_2\nu = e_1 \), it follows from a straightforward functoriality of mates calculation that \( L_{X,Y} \) assigns both morphisms of polynomials to equal pastings.

As local functoriality with respect to cartesian morphisms was shown in Lemma 43, local functoriality with respect to “triangle morphisms” is a straightforward functoriality of mates calculation, and the case of a triangle morphism followed by a cartesian morphism is almost by definition, it suffices to consider the case of a cartesian morphism followed by a triangle morphism (the only non trivial case to consider).
Suppose we are given a composite of polynomial morphisms as on the left below

\[ \begin{array}{c}
\begin{array}{ccc}
E & \overset{p}{\to} & B \\
\downarrow^{s} & \downarrow^{pb} & \downarrow^{t} \\
X & \overset{f}{\to} & Y
\end{array}
& \quad & \begin{array}{ccc}
\begin{array}{ccc}
E & \overset{p}{\to} & B \\
\downarrow^{s} & \downarrow^{pb} & \downarrow^{t} \\
X & \overset{f'}{\to} & Y
\end{array}
\end{array}
\end{array} \]

evaluated as the diagram on the right above. We must check that

\[ (6.2) \]

\[ \begin{array}{c}
\begin{array}{ccc}
FX & \overset{\Delta s}{\to} & FM \\
\downarrow^{F\Delta s} & \downarrow^{F\Delta f} & \downarrow^{F\Delta e} \\
FJ & \overset{\Delta e}{\to} & FN
\end{array}
\end{array} \]

To see this, we paste both sides with the inverse of the b appearing on the right above, and check that have an equality. Starting with the observation that the left side pasted with this inverse is the left diagram below, we see

\[ \begin{array}{c}
\begin{array}{ccc}
FX & \overset{\Delta s}{\to} & FM \\
\downarrow^{F\Delta s} & \downarrow^{F\Delta f} & \downarrow^{F\Delta e} \\
FJ & \overset{\Delta e}{\to} & FN
\end{array}
\end{array} \]

upon realizing m as a whiskering of a unit and canceling the b. Transferring the unit along the mates correspondence gives the left diagram below

\[ \begin{array}{c}
\begin{array}{ccc}
FX & \overset{\Delta s}{\to} & FM \\
\downarrow^{F\Delta s} & \downarrow^{F\Delta f} & \downarrow^{F\Delta e} \\
FJ & \overset{\Delta e}{\to} & FN
\end{array}
\end{array} \]

which is seen as the right diagram after writing the whiskering of the unit back in terms of m. This is clearly the right side of Figure 6.2 with the pasting of the 2-cell b having been undone.

PART 4. The goal here is to show that this now defines a gregarious functor

\[ L: \text{Poly}(\mathcal{E}) \to \mathcal{C} \]

Now, as we already have a gregarious functor \( \overline{L}: \text{Poly}_c(\mathcal{E}) \to \mathcal{C} \), given by the restriction of \( L \) to the cartesian setting, and composition of 1-cells \( \text{Poly}_c(\mathcal{E}) \) and \( \text{Poly}(\mathcal{E}) \) is defined the same way, it suffices to check that that the oplax constraint
data $\varphi, \lambda$ of $\mathcal{T}$ defines oplax constraint data on $L$. Indeed, $\varphi$ and $\lambda$ are already known to satisfy the nullary and associativity axioms and so we need only check naturality of the constraint data with respect to our larger class of 2-cells.

Taking $\theta : P \to P''$ and $\phi : Q \to Q''$ to be general morphisms of polynomials, canonically decomposed into triangle parts $\theta_i, \phi_i$ and cartesian parts $\theta_c, \phi_c$, we note that to to see that the left diagram commutes below

$$L(P; Q) \xrightarrow{\varphi_{P,Q}} LP; LQ$$

it suffices to check that the top square in the right diagram commutes. This is since the bottom square on the right commutes by naturality of the constraint data with respect to our larger class of 2-cells.

The naturality condition then amounts to checking that

$$\text{we first check the naturality condition for right whiskerings of triangle morphisms. The whiskering of such an } x \text{ is constructed as the induced map } x' \text{ into the pullback as in the diagram below}$$

$$\text{and so we need only prove naturality for whiskerings of triangle morphisms.}$$

We now check left whiskerings of triangle morphisms, which is significantly more complicated than the above situation. To simplify this calculation, we consider only

$$\text{which is similar to the calculation in Figure 6.2.}$$
simpler triangle morphisms of the form \( x: (u_1, 1, 1) \to (u_2, x, 1) \). It will turn out that it suffices to consider only these simpler triangle morphisms.

To construct the left whiskering of the triangle morphism \( x \) by a polynomial we first construct the two relevant composites of polynomials as below:

\[
\begin{array}{c}
\text{Construct left whiskering of } x \\
\end{array}
\]

Now since we have a factorization of pullbacks as below:

\[
\begin{array}{c}
\text{Factorization of pullbacks}
\end{array}
\]

it follows that we have an induced cartesian morphism of polynomials:

\[
\begin{array}{c}
\text{Induced cartesian morphism}
\end{array}
\]

where \( h \) and \( j \) are the pullback of \( p \) with \( u_2' \). We then give the factorization of pullbacks:

\[
\begin{array}{c}
\text{Factorization of pullbacks}
\end{array}
\]

and see the morphism of polynomials resulting from this whiskering is given by:

\[
\begin{array}{c}
\text{Whiskering morphism}
\end{array}
\]
The naturality condition then follows from seeing that, where \( z = u_1 t'_1 = t u'_1 \),

is equal to the pasting, using an analogue of Figure 6.2,

which is equal to, where \( d = u_2 t'_2 e = t u'_2 e \), noting \( dm = zy \) and \( u'_2 = u'_1 y \),
Finally resulting in

Now, we wish to prove the naturality condition for any left whiskering of a general triangle morphism, written \( P; \theta \). This can be written as \( P; (\theta_2; R) \) for a simpler triangle morphism \( x \) as above, and so we are trying to show region (1) commutes (suppressing associators in \( \mathcal{C} \))

\[
L(P; ((u_1, 1, 1); R)) \xrightarrow{\varphi} LP; L((u_1, 1, 1); R) \xrightarrow{\varphi} LP; L(u_1, 1, 1); LR
\]

We now note that for the commutativity of (1) it suffices to prove the outside diagram above commutes, as both constraints \( \varphi \) are invertible in region (2) by gregariousness, and region (2) is known to commute as naturality for right whiskerings has been shown.

As associativity of the constraints has been verified, this is the same as showing that the outside of

\[
L((P; (u_1, 1, 1)); R) \xrightarrow{\varphi} L(P; (u_1, 1, 1)); LR \xrightarrow{\varphi} LP; L(u_1, 1, 1); LR
\]

commutes. But the left square commutes as naturality with respect to right whiskerings is known for both triangle and cartesian morphisms, and the right square above commutes by naturality of left whiskerings of \( \theta_2 \). This gives the result. \( \square \)

We now have enough to complete the proof of Theorem 52.

**Proof of Theorem 52.** We consider the assignment of Theorem 52, i.e. composition with the embedding \( (-)_\Sigma : \mathcal{E} \to \text{Poly}(\mathcal{E}) \) written as the assignment

\[
\text{Poly}(\mathcal{E}) \xrightarrow{\varphi} \mathcal{E} \xleftarrow{\alpha_\Sigma} \mathcal{C}
\]

We start by proving the first universal property.

**Well defined.** That each icon is invertible is seen by restricting to spans and applying Corollary 32.
FULLY FAITHFUL. That the assignment \( \alpha \mapsto \alpha_\Sigma \) is injective follows from the necessary commutativity of

\[
\begin{array}{ccc}
F(s, p, t) & \xrightarrow{\varphi} & F(s, 1, 1); F(1, p, 1); F(1, 1, t) \\
\downarrow^{\alpha_{(s, p, t)}} & & \downarrow^{\alpha_{(s, 1, 1)}; \alpha_{(1, p, 1)}; \alpha_{(1, 1, t)}} \\
G(s, p, t) & \xrightarrow{\psi} & G(s, 1, 1); G(1, p, 1); G(1, 1, t)
\end{array}
\]

where \( \varphi \) and \( \psi \) are invertible by gregariousness, and the identities \( \alpha_{\Sigma j} = \left( \alpha_{\Sigma j}^{-1} \right)^* \) and \( \alpha_{\Sigma j}^* = \left( \alpha_{\Sigma j}^{-1} \right)^* \) forced by Lemma 18. For surjectivity, one need only check any collection \( \alpha \) consisting of 2-cells

\[
\alpha_{s,p,t} : F(s, p, t) \to G(s, p, t)
\]

satisfying these properties defines an icon. As composition is the same in \( \text{Poly}_c(\mathcal{E}) \), the compatibility of the collection \( \alpha \) with the oplax constraint cells is the same calculation as in Section 5. Thus one need only check local naturality of \( \alpha \). As local naturality with respect to the cartesian morphisms is already known, one need only consider triangle morphisms. But local naturality with respect to triangle morphism is almost the same calculation as in the case of spans; this is expected as the triangle morphisms arise from the canonical embedding \((-)_{\Delta^I} : \text{Span}^{\circ}(\mathcal{E}) \to \text{Poly}(\mathcal{E})\).

ESSENTIALLY SURJECTIVE. Given any 2-Beck pseudofunctor \( F : \mathcal{E} \to \mathcal{C} \) we take the gregarious functor \( L : \text{Poly}(\mathcal{E}) \to \mathcal{C} \) from Proposition 54 and note that \( L_{\Sigma} = F \).

We now deduce the second universal property.

RESTRICTIONS. The second property is a restriction of the first. Indeed, given a pseudofunctor \( L : \text{Poly}(\mathcal{E}) \to \mathcal{C} \) the corresponding pseudofunctor \( L_{\Sigma} : \mathcal{E} \to \mathcal{C} \) satisfies the distributivity condition since \( L_{\Sigma} \) is also the restriction of the pseudofunctor \( \overline{T} : \text{Poly}_c(\mathcal{E}) \to \mathcal{C} \). Moreover, given a 2-Beck pseudofunctor \( F : \mathcal{E} \to \mathcal{C} \) which satisfies the distributivity condition, the corresponding map \( \text{Poly}(\mathcal{E}) \to \mathcal{C} \) is pseudo since the map \( \text{Poly}_c(\mathcal{E}) \to \mathcal{C} \) (with the same constraint data) arising from the cartesian Beck triple is pseudo.

\[\square\]

References

[1] J. Bénabou, Introduction to bicategories, in Reports of the Midwest Category Seminar, Springer, Berlin, 1967, pp. 1–77.
[2] M. Bullejos, E. Faro, and V. Blanco, A full and faithful nerve for 2-categories, Appl. Categ. Structures, 13 (2005), pp. 223–233.
[3] A. Carboni, S. Kasangian, and R. Street, Bicategories of spans and relations, J. Pure Appl. Algebra, 33 (1984), pp. 259–267.
[4] R. J. M. Dawson, R. Pare, and D. A. Pronk, Universal properties of Span, Theory Appl. Categ., 13 (2004), pp. No. 4, 61–85.
[5] The span construction, Theory Appl. Categ., 24 (2010), pp. No. 13, 302–377.
[6] N. Gambino and J. Kock, Polynomial functors and polynomial monads, Math. Proc. Cambridge Philos. Soc., 154 (2013), pp. 153–192.
[7] C. Hermida, Representable multicategories, Adv. Math., 151 (2000), pp. 164–225.
[8] G. M. Kelly, Doctrinal adjunction, in Category Seminar (Proc. Sem., Sydney, 1972/1973), Springer, Berlin, 1974, pp. 257–280. Lecture Notes in Math., Vol. 420.
[9] G. M. Kelly and R. Street, Review of the elements of 2-categories, in Category Seminar (Proc. Sem., Sydney, 1972/1973), Springer, Berlin, 1974, pp. 75–103. Lecture Notes in Math., Vol. 420.
[10] S. Lack, Icons, Appl. Categ. Structures, 18 (2010), pp. 289–307.
[11] R. Street, The algebra of oriented simplexes, J. Pure Appl. Algebra, 49 (1987), pp. 283–335.
[12] T. von Glehn, Polynomials and models of type theory, PhD thesis, University of Cambridge, 2015.
[13] C. Walker, *Generic bicategories*, arXiv eprint, (2018). under review; available at http://arxiv.org/abs/1805.01703.

[14] M. Weber, *Polynomials in categories with pullbacks*, Theory Appl. Categ., 30 (2015), pp. No. 16, 533–598.

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