Homogenization of the time-dependent heat equation on planar one-dimensional periodic structures

Matko Ljulj a, Kersten Schmidt b, Adrien Semin b and Josip Tambača a

aDepartment of Mathematics, Faculty of Science, University of Zagreb, Zagreb, Croatia; bFachbereich Mathematik, AG Numerik und Wissenschaftliches Rechnen, Technische Universität Darmstadt, Darmstadt, Germany

ABSTRACT
In this paper we consider the homogenization of a time-dependent heat conduction problem on a planar one-dimensional periodic structure. On the edges of a graph the one-dimensional heat equation is posed, while the Kirchhoff junction condition is applied at all (inner) vertices. Using the two-scale convergence adapted to homogenization of lower-dimensional problems we obtain the limit homogenized problem defined on a two-dimensional domain that is occupied by the mesh when the mesh period $\delta$ tends to 0. The homogenized model is given by the classical heat equation with the conductivity tensor depending on the unit cell graph only through the topology of the graph and lengths of its edges. We show the well-posedness of the limit problem and give a purely algebraic formula for the computation of the homogenized conductivity tensor. The analysis is completed by numerical experiments showing a convergence to the limit problem where the convergence order in $\delta$ depends on the unit cell pattern.

1. Introduction
In this paper we deal with homogenization of the time-dependent heat conduction on a periodic graph $\Gamma^\delta$ of period $\delta$ in a rectangular subdomain of $\mathbb{R}^2$ (see Figure 1 for examples of such domains $\Gamma^\delta$ and Figure 2 for examples of unit cell pattern). On edges of the graph the heat conduction is modeled by the one-dimensional heat equation

$$\rho c_p \partial_1 u^\delta - \partial_1 (a \partial_1 u^\delta) = f^\delta \quad \text{on } \Gamma^\delta$$

(1)

where $\partial_1$ is the first order derivative along the edges of $\Gamma^\delta$. The Kirchhoff junction law [1,2] is applied at the (inner) vertices and the problem is completed by initial and boundary condition and will be detailed in Section 2. Such a model on a lower-dimensional domain can be justified as limit model of the heat conduction on a graph-like domain with thin edges – also called fat graph – with the technique of [1] or [3] (see also [4] for a higher order model for the wave equation).

For elliptic problems on lower-dimensional domains Bouchitté et.al. [5–8] and Zhikov [9] introduced and used an extension of the two-scale convergence on measures to obtain limit problems. It is based on an extension of the two-scale convergence on surfaces that was introduced by Neuss–Radu in [10] and in [11] and by Allaire, Damlamian and Hornung in [12] to incorporate boundary conditions with oscillating coefficients. The two-scale convergence on measures was applied in [6,7] in a variational approach using the $\Gamma$–convergence for the energy functional and scalar problems. In [13] the $\Gamma$–convergence is used to obtain a macroscopic behavior of a graphene sheet modeled by a
Figure 1. Example of one-dimensional periodic domain $\Gamma^\delta$ for values of $\delta$ in the planar rectangle $\Omega$ with side lengths $L_1 = 4$, $L_2 = 3$ for the pattern configuration shown in Figure 2(b). (a) $\delta = 0.5$; (b) $\delta = 0.25$.

Figure 2. Two different examples of valid patterns $\Gamma_\gamma$.

hexagonal lattice of elastic bars. In [8] the technique of fattening of lower-dimensional problems is also applied for the vector unknown variational problem. In [14] the measure generalization is used to analyze the elasticity problem on singular structures. A stationary problem on a curvilinear graph similar to the one considered in this paper was considered in [15,16]. It seems that [16] is the first paper on the subject. In this research we also use, in essence, the same notion of the two-scale convergence. However, since we are dealing with the time dependent problem, we have to include the time variable in its definition. We do it in the setting from [10] and [11] and not generalized to singular measures. The results shown in this work for an evolution equation have similarities with results in [6] and [9] for an equilibrium problem that was considered in [6] in a minimization formulation. However, our arguments rely more on graph theory and differential geometry.

The homogenization of the time-dependent heat equation on planar one-dimensional periodic structures has not been studied so far and shall be addressed in this work. Using the *mesh two-scale*
convergence we rigorously derive the limit homogenized model which is given by the heat conduction problem on the two dimensional domain occupied by the mesh. We focus on the homogenization of the domain so in order to simplify the procedure we assume that the conductivity is scalar and non-oscillatory. Furthermore we restrict to non-oscillatory initial temperature. As in the homogenization of the heat equation the time can be kept as a parameter in the problem, see e.g. [17]. Thus it is not surprising that the homogenized conductivity tensor is the same as for the stationary diffusion equation; compare with [6], see also [18] for a formal asymptotic expansion. In difference to the previous works we will introduce a purely algebraic formula for the computation of the homogenized conductivity tensor. The homogenized tensor depends on the unit cell graph only through the topology of the unit cell graph (incidence matrix of the graph) and the lengths of the edges of the graph. Depending on the unit cell geometries it can be a multiple of the unit matrix or a diagonal or non-diagonal matrix valued function. Moreover, the homogenized conductivity tensor turns out to be positive definite which then leads to the existence and uniqueness result for the limit problem.

The thermal conduction problem, however on the ‘fattened’ rectangular graph, was first studied in the paper [19]. Further, in [20] the homogenization of the heat equation on a 3D fattened graph for a simple geometry like in Figure 2(a) is done first and the thickness is taken to zero last. In our approach we start with the one-dimensional model (i.e. the thickness is first taken to zero) and then we do the homogenization. As already noted in [6] the limit model for the stationary diffusion problem on fattened graphs in 3D that is obtained when thickness of the fat edges and the period tend both to zero is the same independently of the way the limiting is performed. Thus it is not a surprise that the models in [20] coincide with the two-scale limit of the one-dimensional limit model. However, it is important to consider homogenization on the mesh objects directly to develop techniques where thickening is not possible, either because three-dimensional equations are too complicated or do not exist or existing techniques using an approach based on measures can not be applied.

The article is organized as follows. In Section 2 we describe the problem in detail and present the main results: a priori stability estimates, the homogenized model and weak convergence as well as convergence in norm. In order to do asymptotics of the problem we need compactness results. We use the compactness result for $L^2(\Gamma^\delta)$ from [11] and derive the compactness result for $H^1(\Gamma^\delta)$ and for $\partial_t u^\delta$ in $L^2(\Gamma^\delta)$. Some results from graph theory give us technical conditions that allow us to prove the compactness theorem. Then in Section 3.4 we prove the a priori estimates for $u^\delta$, the solution of the problem on the one-dimensional mesh $\Gamma^\delta$. In Section 3.1 we prove some technical results and in Section 3.2 the main properties of the homogenized model. Using two-scale convergences that follow from the compactness theorems we are able to take the limit in the $\delta$-problem and obtain the equation for the corrector and the limit problem. The corrector equation leads to the formulation of the canonical problem which is then used to build the homogenized tensor (Section 3.3). In Section 3.4 we prove the corresponding convergence in norm. In Section 4 we formulate the algebraic method to compute the homogenized tensor and compute it for five different patterns. In Section 5 we compare by numerical experiments the solution of the heat equation on the $\delta$-periodic graph with the one of the homogenized problem.

2. Description of the problem and the main result

In this section, we formulate the model including the geometrical setting we consider in this paper. We start with the description of the geometry.

Let us consider a one-dimensional manifold $\Gamma Y \subset [0,1]^2$ – the unit mesh – that can be described by a connected and oriented graph structure $(V, E)$ (please see [21,22] for the usual terminology in graph theory), where $V$ and $E$ denote respectively the set of vertices and the set of edges. We assume that each edge $e \in E$ admits a $W^{1,\infty}$ and bijective arc length parametrization $\gamma$ from an open interval, that is directed in accordance with the orientation of the graph. Then, we denote for $y \in e$ the unitary tangent vector $t(y) := \gamma'(\gamma^{-1}(y))$, see Figure 2 for two different examples.
We will consider trails and circuits that follow or may not follow the orientation of the graph. A trail is a walk (a finite sequence of neighboring vertices and edges) in which all edges are distinct and a circuit is a closed trail. If $\Gamma_Y$ is transformed to an undirected graph $\Gamma'_Y$ where each edge keeps its parametrization, then we denote by $T(\Gamma'_Y)$ the set of all trails in $\Gamma'_Y$. Moreover, we define for each trail $T$ of the undirected graph $\Gamma'_Y$ of $\Gamma_Y$ the orientation function $\chi_T : \Gamma_Y \rightarrow \{-1, 0, +1\}$ that takes the value $\pm 1$ on all points $y$ in an interior of an edge $e$ of $T$, the sign indicating if the edge is passed according to the orientation of $e$ in the oriented graph $\Gamma_Y$ ($+1$) or not ($-1$), and 0 otherwise.

As the unit mesh $\Gamma_Y$ shall be connected in all directions to repetitions of itself translated by $e_1 = (1, 0)$ or $e_2 = (0, 1)$ we assume that it touches each side of the unit square in such a way that

$$\exists y_1, y_2 \in \Gamma_Y : y_1 + e_1, y_2 + e_2 \in \Gamma_Y.$$  

Note here that $y_1$ and $y_1 + e_1$ as well as $y_2$ and $y_2 + e_2$ are connected by a trail of $\Gamma'_Y$ since $\Gamma_Y$ is connected. This trail becomes a circuit in the graph $\Gamma'_{Y,\delta}$ obtained by identifying all such points on the sides of the square $[0, 1]^2$, i.e. $y$ with $y + e_1$ and $y$ with $y + e_2$ for $y \in \delta[0, 1]^2$ – and call them opposite points – meaning that they correspond to the same vertex of $\mathcal{V}$. Moreover, all corner points $y \in \{(0, 0), (1, 0), (0, 1), (1, 1)\} \cap \Gamma_Y$, if they exist, are identified and represented by one vertex in $\mathcal{V}$.

The set of all circuits in $\Gamma'_{Y,\delta}$, the undirected version of the periodic graph $\Gamma_{Y,\delta}$, we denote by $C(\Gamma'_{Y,\delta})$.

Let $\Omega$ be a rectangular domain of lengths $L_1$ and $L_2$, such that the ratio $L_1/L_2$ belongs to $\mathbb{Q}$. Then, we consider only periods $\delta > 0$ such that $N_1^2 = L_1/\delta$ and $N_2^2 = L_2/\delta$ are integers and refer as for any value of $\delta^2$ if a statement holds for any such $\delta$.

Finally, we consider the plane mesh $\Gamma^\delta$ (see Figure 1 for an illustration) defined by

$$\Gamma^\delta := \bigcup_{n_1=0}^{N_1^2-1} \bigcup_{n_2=0}^{N_2^2-1} \Gamma_{n_1, n_2}^\delta, \quad \Gamma_{n_1, n_2}^\delta := \delta (\Gamma_Y + n_1 e_1 + n_2 e_2)$$

that can be described by a connected graph, where each vertex of $\Gamma^\delta$ corresponds to exactly one point in $\Gamma_Y$, i.e. there is no identification of points and so no periodic structure of $\Gamma^\delta$. Note that opposite points in neighboring repetitions of $\Gamma_{n_1, n_2}^\delta$ coincide, making them the same vertex of $\Gamma^\delta$.

We introduce now the notion of derivatives and integrals on the meshes $\Gamma^\delta$ and $\Gamma_Y$.

**Definition 2.1 (Derivative):** Let $e$ be an oriented edge with a $W^{1,\infty}$ and bijective arc length parametrization $y$ that is in accordance with the orientation of $e$ and let $u$ be a continuous function on $e$ such that $u \circ y$ is differentiable. We define its derivative, for all points $x \in e$, and $s = y^{-1}(x)$ by

$$\partial_t u(x) := \frac{d}{ds} (u \circ y)(s).$$

Note that the derivative is orientation dependent, and since the technical results for the first-order derivatives will be used we have to use oriented graphs. However, due to the second-order derivative in (1) the solution $u^\delta$ and the variational problems will not be orientation-dependent, so any fixed orientation will suit the setting. Edge by edge we can extend this derivative on functions defined on graphs $\Gamma_Y$ and $\Gamma^\delta$.

**Definition 2.2 (Slow and fast derivative):** Given a function $u$ on $\Omega \times \Gamma_Y$. If $u(\cdot, y)$ is differentiable on $\Omega$ for all $y \in \Gamma_Y$ we define its slow derivative by

$$\nabla_x u(x, y) \cdot t(y).$$

Moreover, we define its fast derivative $\partial_{\Gamma^\delta, \gamma}$ for all $x \in \Omega$ on which $u(x, \cdot)$ is differentiable on $\Gamma_Y$ as the derivative $\partial_{\Gamma^\delta}$ with respect to $y$. 
For a function $u$ differentiable on $\Omega \times \Gamma_Y$, the function $x \mapsto u(x, \frac{x}{\delta})$ is differentiable on $\Gamma^\delta$ and by the chain rule,

$$\partial^\Gamma \left( u \left( x, \frac{x}{\delta} \right) \right) = \nabla_x u \left( x, \frac{x}{\delta} \right) \cdot t \left( \frac{x}{\delta} \right) \frac{1}{\delta} \partial^\gamma u \left( x, \frac{x}{\delta} \right).$$

(5)

Note that even for functions $u$ that are constant in $y$ (independent of $y$), its slow derivative $\nabla_x u(x) \cdot t(y)$ depends on the fast variable $y$ through the tangent field on $\Gamma_Y$. This means especially that for functions that are differentiable on $\Omega$ it holds for any $\delta > 0$

$$\partial^\Gamma u(x) = \nabla u(x) \cdot t \left( \frac{x}{\delta} \right) \quad \text{on } \Gamma^\delta \subset \Omega.$$

(6)

For simplicity, also if $u$ is a function of $(t, x) \in (0, t_1) \times \Omega$ we use $\nabla u$ for the gradient of $u$ with respect to $x$.

**Definition 2.3 (Integral):** Given a function $u$ defined on $\Gamma_Y$, we say that $u$ is integrable on $\Gamma_Y$ if $u$ is integrable on each edge $e \in \mathcal{E}$. In that case, the integral of $u$ over $\Gamma_Y$ is given by

$$\int_{\Gamma_Y} u(y) \, ds(y) := \sum_{e \in \mathcal{E}} \int_e u(y) \, ds(y),$$

where the integral of $u$ over the edge $e$ is defined as the curve integral of the first kind, that is $\int_0^\ell u(y(s)) \, ds$ for an arc length parametrization $y : [0, \ell] \to e$.

With this we can define in particular the length $|\Gamma_Y| := \int_{\Gamma_Y} \, ds(y)$ of $\Gamma_Y$, and the length $|\Gamma^\delta| = |\Omega||\Gamma_Y|/\delta$ of $\Gamma^\delta$ is defined for any $\delta > 0$ where $|\Omega|$ is the area of $\Omega = [0, L_1] \times [0, L_2]$.

For an oriented graph $\mathcal{G}$ we denote by $L^2(\mathcal{G})$ the Banach space of functions $u$ such that $|u|^2$ is integrable on $\mathcal{G}$ and by $H^1(\mathcal{G})$ the Hilbert space

$$H^1(\mathcal{G}) = \{ u \in L^2(\mathcal{G}) : \partial^\Gamma u \in L^2(\mathcal{G}), u \text{ continuous on } \mathcal{G} \},$$

the latter equipped with the norm

$$\|v\|^2_{H^1(\mathcal{G})} = \|v\|^2_{L^2(\mathcal{G})} + \|\partial^\Gamma v\|^2_{L^2(\mathcal{G})}.$$

In particular we will need to consider the spaces $L^2(\Gamma_Y)$, $H^1(\Gamma_Y)$ and $H^1(T)$ for any trail $T$ of $\Gamma_Y$. Note that orientation on a graph has to be chosen in order to define derivatives but there is no difference between $L^2(\mathcal{G})$ and $L^2(\mathcal{G}')$ and $H^1(\mathcal{G})$ and $H^1(\mathcal{G}')$, when $\mathcal{G}'$ is the undirected graph of $\mathcal{G}$.

As $\Gamma_{Y,\#}$ is a periodic manifold of $[0, 1]^2$ the continuity of $u \in H^1_0(\Gamma_Y) := H^1(\Gamma_{Y,\#})$ means that $u(y_1) = u(y_2)$ whenever $y_1$ and $y_2$ correspond to the same vertex of $\Gamma_Y$ in particular in the opposite points. Note, that $H^1(T)$ for trails $T$ of $\Gamma_Y$ allows for functions that are discontinuous at the opposite points.

Let $\Gamma_D \subset \partial \Omega$ be a part of the boundary of $\Omega$ that corresponds to a finite union of line segments of positive length. For any $\delta$, let us define $\Gamma_D^\delta := \partial \Gamma^\delta \setminus \Gamma_D$. By assumption on $\Gamma_D$, $\Gamma_D^\delta$ is different from the empty set for sufficiently small $\delta$. We then define the space $\mathcal{H}^\delta$ as the subspace of functions $v \in H^1(\Gamma^\delta)$ whose trace vanishes on $\Gamma_D^\delta$, i.e.

$$\mathcal{H}^\delta := \{ v \in H^1(\Gamma^\delta) : v = 0 \text{ on } \Gamma_D^\delta \}.$$ 

We consider now the following problem given in the weak formulation for a time $t_1 > 0$, a source term $f^\delta \in L^2(0, t_1; L^2(\Gamma^\delta))$ and an initial data $u_{\text{init}} \in \mathcal{H}^\delta$: find $u^\delta$ in $L^2(0, t_1; \mathcal{H}^\delta)$ with $\partial_t u^\delta$ in $L^2(0, t_1; L^2(\mathcal{G}^\delta))$ such that for all $v^\delta \in \mathcal{H}^\delta$ and a.e. $t \in (0, t_1)$

$$\int_{\Gamma^\delta} \rho c_p \partial_t u^\delta(t, x) v^\delta(x) \, ds(x) + \int_{\Gamma^\delta} a(x) \partial^\Gamma u^\delta(t, x) \partial^\Gamma v^\delta(x) \, ds(x) = \int_{\Gamma^\delta} f^\delta(t, x) v^\delta(x) \, ds(x), \quad (7a)$$
Here, $\rho$ is the mass density of the material and $c_p$ is the specific heat capacity that are considered as constants, $u_{\text{init}} : \Gamma^\delta \to \mathbb{R}$ is a given initial temperature and the thermal conductivity $a : \overline{\Omega} \to \mathbb{R}$ is a positive continuous function. Thus there exist
\[
a_{\text{min}} = \min_{x \in \Omega} a(x) > 0.
\] (8)

It implies
\[
a_{\text{min}} \leq a(x) \leq \|a\|_{L^\infty(\Omega)}, \quad x \in \Gamma^\delta.
\]

The problem (7a) is the weak formulation of the heat conduction problem (1) with heat source $f^\delta$, initial condition given by $u_{\text{init}}$, homogeneous Dirichlet conditions on $\Gamma^\delta_D$ and homogeneous Neumann conditions on the remaining part of the boundary. Also note that the solution $u^\delta$ of the variational formulation (7a) is independent of the orientation on the graph even so the definition of the derivative $\partial_t$ is orientation-dependent (with difference only in sign) since the curve integrals of the first kind are orientation independent and in the second term in the left-hand side of (7a) the opposite signs cancel out.

Following the notations of [6], we introduce $\mu(y) := \frac{1}{|\Gamma^\delta|} ds(y)$ as the normalized periodic one-dimensional Hausdorff measure on $\Gamma^\delta$, therefore the measure associated to $\Gamma^\delta$ is just given as $\mu_\delta(x) = \delta \mu(x/\delta)$. As $\delta \to 0$, the domain $\Gamma^\delta$ tends to a dense subdomain of $\Omega$, so that the natural idea is to consider a limit problem of (7a) on $\Omega$. Following the two-scale homogenization theory [6,11,12], the measure that has to be considered for the limit case is not the usual Lebesgue measure $L^2(\Omega)$ on $\Omega$ but the tensorial product $L^2(\Omega) \otimes \mu_\delta \Gamma^\delta$.

To stress the passage from the 1D mesh to the 2D domain we call the associated two-scale convergence in this paper the mesh two-scale convergence. The convergence is given for functions defined on a mesh and was already introduced in the literature, see e.g. [10,12]. However, since we have the time $t$ as an additional parameter we consider the mesh two-scale convergence with a parameter where we follow the work of Neuss–Radu [10,11].

**Definition 2.4 (Mesh two-scale convergence):** We say that sequence of functions $(v^\delta) \subset L^2(0, t_f; L^2(\Gamma^\delta))$ with $\delta > 0$ for $\delta \to 0$ is ‘mesh two-scale convergent’ with limit $v^0 \in L^2(0, t_f; L^2(\Omega; L^2(\Gamma^\delta)))$ if for each $\psi \in C^\infty([0, t_f]; C^\infty(\Omega; H^1(\Gamma^\delta)))$ we have
\[
\lim_{\delta \to 0} \delta \int_0^{t_\delta} \int_{\Gamma^\delta} v^\delta(t, x) \psi \left( t, x, \frac{x}{\delta} \right) ds(x) dt = \int_0^t \int_{\Omega} \int_{\Gamma^\delta} v^0(t, x, y) \psi(t, x, y) ds(y) dx dt. \tag{9}
\]

In this case we use the abbreviation $v^\delta \xrightarrow{m2s} v^0$.

Note that with a simple scaling argument we have that for any $\delta > 0$
\[
\delta \|\mathbb{I}_{\Gamma^\delta}\|^2_{L^2(\Gamma^\delta)} = \|\mathbb{I}_{\Omega \times \Gamma^\delta}\|^2_{L^2(\Omega \times \Gamma^\delta)},
\]

where $\mathbb{1}_X$ is the characteristic function of a set $X$. This explains the factor $\delta$ in the left-hand side of the mesh two-scale convergence and the factor $\sqrt{\delta}$ in the various $L^2$-norm estimates that will follow.

Having in mind the mesh two-scale convergence we consider $f^\delta$ in (7a) to depend on the fast variable as well, i.e. there exists a function $f$ on $(0, t_f) \times \Omega \times \Gamma^\delta$ such that $f^\delta(t, x) = f(t, x, x/\delta)$ on $(0, t_f) \times \Gamma^\delta$ for all $\delta > 0$. In the following lemma we give uniform bounds on the data of the $\delta$-dependent problem. They are then used to show its well-posedness and a priori stability estimates on its solution $u^\delta$ as stated in Theorem 2.8. The a priori estimates will be needed in the proof of

\[
u^\delta(0, x) = u_{\text{init}}(x), \quad x \in \Gamma^\delta. \tag{7b}
\]
the mesh two-scale convergence the homogenized model (Theorem 2.9) in Section 3.3 and of the convergence in norm (Theorem 2.11) in Section 3.4.

**Lemma 2.5 (Uniform bound of the data):** (a) Let \( f \in L^2(0, t_f; L^2(\Gamma_Y; C(\overline{\Omega}))) \) and \( f^\delta(t, x) = f(t, x, \tfrac{x}{\delta}) \) on \((0, t_f) \times \Gamma^\delta\) for \( \delta > 0\). Then, for any \( \delta > 0 \) it holds that \( f^\delta \in L^2(0, t_f; L^2(\Gamma^\delta)) \) and

\[
\delta \| f^\delta \|_{L^2(0, t_f; L^2(\Gamma^\delta))}^2 \leq |\Omega| \| f \|_{L^2(0, t_f; L^2(\Gamma_Y; L^\infty(\Omega)))}^2. \tag{10}
\]

(b) If \( u_{\text{init}} \in C^1(\overline{\Omega}) \) such that \( u_{\text{init}} = 0 \) on \( \Gamma_D \). Then, for any \( \delta > 0 \), \( u_{\text{init}} \in \mathcal{H}^\delta \) and

\[
\delta \| u_{\text{init}} \|_{H^1(\Gamma^\delta)}^2 \leq |\Omega| (1 + 2|\Gamma_Y|) \| u_{\text{init}} \|_{W^{1, \infty}(\Omega)}^2. \tag{11}
\]

**Proof:** From the definition of norm we get

\[
\delta \| f^\delta \|_{L^2(0, t_f; L^2(\Gamma^\delta))}^2 = \delta \sum_{n_1, n_2} \int_0^{t_f} \delta \int_{\Gamma_Y} |f(t, \delta(y + n_1 e_1 + n_2 e_2), y)|^2 \, ds(y) \, dt \\
\leq \delta^2 \sum_{n_1, n_2} \int_0^{t_f} \int_{\Gamma_Y} \|f(t, \cdot, y)\|_{L^2(\Omega)}^2 \, ds(y) \, dt \leq |\Omega| \| f \|_{L^2(0, t_f; L^2(\Gamma_Y; L^\infty(\Omega)))}^2,
\]

and, hence, \( f^\delta \in L^2(0, t_f; L^2(\Gamma^\delta)) \) for any \( \delta > 0 \). Just in the same way we obtain

\[
\delta \| u_{\text{init}} \|_{L^2(\Gamma^\delta)}^2 \leq |\Omega| \| u_{\text{init}} \|_{L^\infty(\Omega)}^2,
\]

and similarly

\[
\delta \| \partial_n u_{\text{init}} \|_{L^2(\Gamma^\delta)}^2 = \delta \sum_{n_1, n_2} \int_{\Gamma_Y} \| \nabla_x u_{\text{init}}(t, \delta(y + n_1 e_1 + n_2 e_2)) \cdot f(y) \|^2 \, ds(y) \\
\leq \delta^2 \sum_{n_1, n_2} \int_{\Gamma_Y} 2 \| \nabla_x u_{\text{init}}(t, \cdot) \|^2_{L^\infty(\Omega)} \, ds(y) \leq 2 |\Omega| |\Gamma_Y| \| \nabla_x u_{\text{init}} \|_{L^\infty(\Omega)}^2.
\]

Hence, \( u_{\text{init}} \in \mathcal{H}^\delta \) and the proof is complete. \( \square \)

**Lemma 2.6:** Let \( \delta > 0, f^\delta \in L^2(0, t_f; L^2(\Gamma^\delta)) \) and \( u_{\text{init}} \in \mathcal{H}^\delta \). Then the problem (7a) has a unique solution that satisfies

\[
u^\delta \in L^2(0, t_f; \mathcal{H}^\delta \cap H^2(\Gamma^\delta)) \quad \partial_t u^\delta \in L^2(0, t_f; L^2(\Gamma^\delta)),
\]

where \( \Gamma^\delta := \Gamma^\delta \setminus \mathcal{Y} \) the graph without its vertices and \( H^2(\Gamma^\delta) \) the Sobolev space of functions that are in \( H^2(e) \) on each edge \( e \) of \( \Gamma^\delta \).

**Proof:** The assumption on \( f \) implies that for any \( \delta > 0 \) that \( f^\delta \in L^2(0, t_f; L^2(\Gamma^\delta)) \subset L^2(0, t_f; (\mathcal{H}^\delta)'\) and \( u_{\text{init}} \in \mathcal{H}^\delta \subset L^2(\Gamma^\delta)\). Hence, the classical solution theory for parabolic equations gives the existence and uniqueness of the solution \( u^\delta \) of (7a) that satisfies \( u^\delta \in L^2(0, t_f; \mathcal{H}^\delta) \), \( \partial_t u^\delta \in L^2(0, t_f; (\mathcal{H}^\delta)') \), cf. the abstract results in [23, Theorem 1 and 2 on pages 512 and 513].

With the regularity of \( f^\delta \) and \( u_{\text{init}} \) we can apply classical regularity results edge by edge to get that \( u^\delta \in L^2(0, t_f; \mathcal{H}^\delta \cap H^2(\Gamma^\delta)) \), and \( \partial_t u^\delta \in L^2(0, t_f; L^2(\Gamma^\delta)) \), see e.g. [24, Theorem 5 in Chapter 7]. \( \square \)
Lemma 2.7 (A priori stability estimates): Let for $\delta > 0$ \( \sqrt{\delta} \| f^\delta \|_{L^2([0,t]\times L^2(\Gamma^\delta))} \), \( \sqrt{\delta} \| u_{\text{init}} \|_{L^2(\Gamma^\delta)} \) and \( \sqrt{\delta} \| \partial_{\Gamma} u_{\text{init}} \|_{L^2(\Gamma^\delta)} \) be uniformly bounded with respect to $\delta$, and let $u^\delta$ be the unique solution of (7a). Then there is $C > 0$ independent of $\delta$ such that

\[
\sqrt{\delta} \| u^\delta \|_{L^\infty([0,t]\times L^2(\Gamma^\delta))} \leq C,
\]

\[
\sqrt{\delta} \| \partial_{\Gamma} u^\delta \|_{L^2([0,t]\times L^2(\Gamma^\delta))} \leq C,
\]

\[
\sqrt{\delta} \| \partial_t u^\delta \|_{L^2([0,t]\times L^2(\Gamma^\delta))} \leq C.
\]

Proof: First testing for any $\tau \in (0,t)$ the variational formulation (7a) with $u^\delta(t,x)$ we find for any $t \in (0,\tau)$

\[
\frac{1}{2} \frac{d}{dt} \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} + a_{\min} \left\| \partial_{\Gamma} u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \leq \int_\Omega f^\delta(t,x) u^\delta(t,x) \, d\mu_\delta(x).
\]

Then, applying the Cauchy-Schwartz inequality and the Young inequality we get

\[
\frac{1}{2} \frac{d}{dt} \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} + a_{\min} \left\| \partial_{\Gamma} u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \leq \frac{1}{4\eta} \left\| f^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} + \eta \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)}.
\]

Integrating this relation over $t \in (0,\tau)$ we obtain

\[
\frac{1}{2} \left\| u^\delta(\tau,\cdot) \right\|^2_{L^2(\Gamma^\delta)} + a_{\min} \int_0^\tau \left\| \partial_{\Gamma} u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \, dt \leq \frac{1}{4\eta} \int_0^\tau \left\| f^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \, dt + \eta \int_0^\tau \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \, dt + \frac{1}{2} \left\| u_{\text{init}} \right\|^2_{L^2(\Gamma^\delta)}.
\]

We take then $\eta = 1/(4\delta t)$ such that

\[
\eta \int_0^\tau \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \leq \frac{1}{4} \sup_{t \in (0,t)} \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)},
\]

so that relation (12) gives

\[
\frac{1}{2} \left\| u^\delta(\tau,\cdot) \right\|^2_{L^2(\Gamma^\delta)} + a_{\min} \int_0^\tau \left\| \partial_{\Gamma} u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \, dt \leq \delta \int_0^\tau \left\| f^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \, dt + \frac{1}{4} \sup_{t \in (0,t)} \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} + \frac{1}{2} \left\| u_{\text{init}} \right\|^2_{L^2(\Gamma^\delta)}.
\]

Now, by assumption there exists a constant $D$ such that, for any $\delta$,

\[
\delta \int_0^\tau \left\| f^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \, dt \leq D
\]

and

\[
\delta \left\| u_{\text{init}} \right\|^2_{L^2(\Gamma^\delta)} \leq D.
\]

(1) Relation (14) implies

\[
\delta \left\| u^\delta(\tau,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \leq (2\delta t + 1)D + \delta \frac{1}{2} \sup_{t \in (0,t)} \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)},
\]

that gives

\[
\delta \sup_{t \in (0,t)} \left\| u^\delta(t,\cdot) \right\|^2_{L^2(\Gamma^\delta)} \leq (4\delta t + 2)D.
\]
Theorem 2.8 (Well-posedness of the $\delta$-dependent problem (7a)).

Proof: The assumptions on $u$ under additional regularity assumption on the data the convergence in norm in Theorem 2.11 holds for all $\delta > 0$ with $f \in L^2(0, t_1; L^2(\Gamma^\delta \cap H^2(\Gamma^\delta)))$ and $u_{\text{init}} \in H^1(\Gamma^\delta)$ with $u_{\text{init}} = 0$ on $\Gamma_D$. Then, problem (7a) has a unique solution $u^\delta \in L^2(0, t_1; H^\delta \cap H^2(\Gamma^\delta))$ with $\partial_t u^\delta \in L^2(0, t_1; L^2(\Gamma^\delta))$ and there is $C > 0$ such that for all $\delta > 0$

$$\sqrt{\delta} \left( \| u^\delta \|_{L^2(0, t_1; L^2(\Gamma^\delta)))} + \| \partial_t u^\delta \|_{L^2(0, t_1; L^2(\Gamma^\delta)))} + \| \partial_t u^\delta \|_{L^2(0, t_1; L^2(\Gamma^\delta)))} \right) \leq C. \quad (16)$$

\textbf{Proof:} The assumptions on $f$ and $\text{u}_{\text{init}}$ imply by Lemma 2.5 for any $\delta > 0$ that $f^\delta \in L^2(0, t_1; L^2(\Gamma^\delta))$. Hence by Lemma 2.6 the solution of (7a) exists and satisfies $u^\delta \in L^2(0, t_1; H^\delta \cap H^2(\Gamma^\delta))$ and $\partial_t u^\delta \in L^2(0, t_1; L^2(\Gamma^\delta))$. As with Lemma 2.5 the assumptions of Lemma 2.7 are fulfilled we conclude the \textit{a priori} estimate (16). \hfill \blacksquare

In the following two theorems the main results of the paper are stated, namely

1. the mesh two-scale convergence $u^\delta$ to the solution $u^0$ of the homogenized problem in Theorem 2.9 that will be proved in Section 3.3, where the properties of the homogenized model will be shown in Section 3.2, and

2. under additional regularity assumption on the data the convergence in norm in Theorem 2.11 that will be proved in Section 3.4.

Theorem 2.9 (Mesh two-scale convergence to the homogenized problem): Let $\Gamma_Y$ be connected and satisfies (2) and let the assumptions of Theorem 2.8 be fulfilled. Then the sequence $(u^\delta)_{\delta > 0}$ of solutions of (7a) for $\delta \to 0$ mesh two-scale converges to the function $u^0$ which is the unique solution of the problem: find

$$u^0 \in L^2(0, t_1; H) \quad \text{such that} \quad \partial_t u^0 \in L^2(0, t_1; L^2(\Omega)),$$

where $H = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}$, such that for all $v \in H$

$$\int_{\Omega} \rho c_p \partial_t u^0(t, x) v(x) \, dx + \int_{\Omega} a(x) A_{\text{hom}} \nabla u^0(t, x) \cdot \nabla v(x) \, dx = \int_{\Omega} f_{\text{hom}}(t, x) v(x) \, dx, \quad (17a)$$

$$u^0(0, x) = u_{\text{init}}(x), \quad x \in \Omega, \quad (17b)$$
where
\[ f_{\text{hom}} := \frac{1}{|Y|} \int_Y f(\cdot, \cdot, y) \, ds(y) \] (18)
and where the constant 2-by-2 symmetric and positive definite matrix \( A_{\text{hom}} \) is given by
\[ A_{\text{hom}} := \frac{1}{|Y|} \int_Y (t(y) + \partial_1 \phi(y)) \left( t(y) + \partial_1 \phi(y) \right)^T \, ds(y), \] (19)
where \( \phi \in H^1_\#(\Gamma_Y)^2 \) is the unique solution (up to an additive constant) of the canonical problem
\[ \int_{\Gamma_Y} \partial_1 \phi(y) \partial_1 \psi(y) \, ds(y) = -\int_{\Gamma_Y} t(y) \partial_1 \psi(y) \, ds(y), \quad \forall \psi \in H^1_\#(\Gamma_Y). \] (20)
Moreover, \( u^\delta|_{t=0} \) mesh two-scale converges to the initial data \( u_{\text{init}} \) of (17a).

Solvability of the problem (20) will be stated in Lemma 3.7, the symmetry and positive definiteness of the homogenized tensor \( A_{\text{hom}} \) in Lemma 3.8 and the existence and uniqueness of the problem (17a) in Lemma 3.9.

Remark 2.10: We have already noted that \( u^\delta \) is orientation independent. Furthermore since \( A_{\text{hom}} \) is with its definition (19) orientation independent the same holds for \( u^0 \).

Theorem 2.11 (Convergence to the homogenized problem in norm): Let the assumption of Theorem 2.8 be fulfilled and let in addition \( u_{\text{init}} \in H^3(\Omega) \) and
\[ f \in L^2(0, t_f; L^2(\Gamma_Y; H^2(\Omega))) \quad \text{s.t.} \quad \partial_t f \in L^2(0, t_f; L^2(\Gamma_Y; L^2(\Omega))). \]
Then it holds in the limit \( \delta \to 0 \)
\[ \sqrt{\delta} \| u^\delta - u^0 \|_{C([0,t_f];L^2(\Gamma^\delta))} \to 0, \quad \sqrt{\delta} \| \partial_1 (u^\delta - u^0 - \delta u^1) \|_{L^2(0, t_f; L^2(\Gamma^\delta))} \to 0, \] (21)
where \( u^0 \) is the solution of (17a), \( u^1(t,x,y) = \nabla u^0(t,x) \cdot \phi(y) \) and \( \phi \) is an arbitrary solution of the canonical problem (20).

The convergence to the limit solution in \( L^2(\Gamma^\delta) \), where for a convergence of the gradients in \( L^2(\Gamma^\delta) \) the first order corrector has to be added, is in accordance with the usual homogenization of elliptic and parabolic equations with periodic pattern [17].

3. The homogenized model

3.1. Technical preliminaries

Lemma 3.1: Let \( G \) be a oriented graph that may be periodic or not where each of its edges admits a \( W^{1,\infty} \) and bijective arc length parametrization. Let, moreover, \( T \) be a trail of \( G' \) from \( x_0 \) to \( x_1 \), and let \( \varphi \in H^1(T) \). Then,
\[ \int_T \partial_1 \varphi(x) \chi_T(x) \, ds(x) = \varphi(x_1) - \varphi(x_0). \] (22)

Proof: As \( T \) represents a continuous and piecewise differentiable curve the lemma is a direct consequence of the definition of the derivative (4) and the fundamental theorem of calculus. ■
As simple consequence of the orientation dependence of the derivative and Lemma 3.1 it holds for any circuit \( C \in C(\Gamma'_{Y,e}) \) and any function \( \varphi \in H^1_\#(\Gamma_Y) \) that
\[
\int_{\Gamma_Y} \partial_\gamma \varphi(x) \chi_C(x) \, ds(x) = 0.
\] (23)

Note that the formula (22) also holds for any walk in \( G' \) and consequently formula (23) holds for closed walks.

A form of reciprocal of (23) is true as well: if the integral of a function \( \psi \) along any circuit \( C \in C(\Gamma'_{Y,e}) \) multiplied with the orientation function \( \chi_C \) is equal to zero, then \( \psi \) possesses a potential \( \varphi \in H^1_\#(\Gamma_Y) \).

**Lemma 3.2:** Let \( \psi \in L^2(\Gamma_Y) \) and all circuits \( C \in C(\Gamma'_{Y,e}) \)
\[
\int_{\Gamma_Y} \psi(y) \chi_C(y) \, ds(y) = 0.
\]

Then there exists \( \varphi \in H^1_\#(\Gamma_Y) \) such that \( \partial_\gamma \varphi = \psi \) almost everywhere on \( \Gamma_Y \).

**Proof:** We start defining \( \varphi(y_0) = 0 \) on an arbitrary vertex \( y_0 \in \Gamma_Y \).

Then, for any other vertex \( y_1 \in \Gamma_Y \) we fix a trail \( T \) of \( \Gamma'_Y \) from \( y_0 \) to \( y_1 \) and define
\[
\varphi(y_1) = \int_T \psi(y_\gamma) \chi_T(y) \, ds(y).
\]

As for any other trail \( T' \) of \( \Gamma'_Y \) from \( y_0 \) to \( y_1 \) it holds by assumption of the lemma that for the closed walk \( C \) concatenating \( T \) and \( T'_- \), the inverse trail to \( T' \),
\[
0 = \int_T \psi(y_\gamma) \chi_T(y) \, ds(y) + \int_{T'_-} \psi(y_\gamma) \chi_{T'_-}(y) \, ds(y).
\]

As \( \int_{T'_-} \psi(y_\gamma) \chi_{T'_-}(y) \, ds(y) = \int_{T'_-} \psi(y_\gamma)(-\chi_{T'}(y)) \, ds(y) \) the definition of \( \varphi(y_1) \) is independent of the choice of the trail.

Finally, for all points \( y \) in the interior of an edge \( e \in E \) with arc length parametrization \( y \) let \( \ell = y^{-1}(y) \) and \( \varphi(y) = \varphi(y(0)) + \int_0^\ell \psi(y(s)) \, ds \), which gives a continuous definition on its starting and end point of \( e \) and, hence, due to the trail independence on all vertices of \( V \). With the definition of the derivative \( (4) \) and the fundamental theorem of calculus it holds \( \partial_\gamma \varphi = \psi \) on each edge and so \( \varphi \in H^1_\#(\Gamma_Y) \). This completes the proof.

**Lemma 3.3:** Let \( G \) be strongly connected graph and for \( v \in L^2(G) \), all \( \psi \in H^1(G) \) and all circuits \( C \in C(G') \) it holds
\[
\int_G v(y) \partial_\gamma \psi(y) \chi_C(y) \, ds(y) = 0.
\] (24)

Then, \( v \) is constant on \( G \).

**Proof:** For a strongly connected graph \( G \) any edge \( e \in E(G) \) belongs to at least one circuit \( C \in C(G') \).
Restricting to test functions with support only on \( e \) we find that
\[
\int_e v(y) \partial_\gamma \psi(y) \, ds(y) = 0 \quad \forall \psi \in C^\infty_0(e).
\]

This implies that \( v \) takes constant values on all edges \( e \in E(G) \).
Now, let $e_1$ and $e_2$ be two edges of $G$ with a common vertex $x \in \mathcal{E}(e_1) \cap \mathcal{E}(e_2)$. By strong connectivity of $G$ these two edges belong to some circuit $C$. Then, restricting to test functions supported on $\tilde{\Gamma} := e_1 \cup e_2 \cup \{x\}$ the equality (24) implies
\[
0 = \int_{e_1 \cup e_2} v(y) \partial_\Gamma \psi(y) \chi_C(y) \, ds(y) \quad \forall \psi \in H^1_0(\tilde{\Gamma}).
\]
Then, integrating by parts and using that $\partial_\Gamma v = 0$ on $e_1$ and $e_2$ we find
\[
0 = (v|_{e_1} - v|_{e_2}) \psi(x) \quad \forall \psi(x) \in \mathbb{R}.
\]
As $v$ takes the same constant value on two arbitrary neighboring edges it is constant on the whole $\tilde{\Gamma}$.

**Lemma 3.4:** For $v \in H^1_0(\Gamma^\delta)$ and any circuit $C \in \mathcal{C}(\Gamma^\delta_Y)$ it holds
\[
\int_{\Gamma^\delta} \partial_\Gamma v(x) \chi_C \left( \frac{x}{\delta} \right) \, ds(x) = 0.
\]

**Proof:** We decompose the support of $\chi_C \left( \frac{x}{\delta} \right)$ into the union of circuits $(C_i) \in \mathcal{C}(\Gamma^\delta_Y)$ and trails $(T_j) \in \mathcal{T}(\Gamma^\delta_Y)$, where each trail $T_j$ is of length $\ell_j$, respectively, and admits a parametrization $\gamma_j$ such that $\gamma_j(0)$ and $\gamma_j(\ell_j)$ belongs to $\partial \Omega$ and so $v(\gamma_j(0)) = v(\gamma_j(\ell_j)) = 0$. Using then Lemma 3.1 on each circuit $C_i$ and on each trail $T_j$ leads to the desired result.

The following theorem is the key theorem in the classical homogenization theory transferred to the considered setting of periodic meshes. Its proof follows exactly the one in the usual theory (see [10, Theorem 1.5.5]).

**Theorem 3.5:** Let the sequence $(v^\delta) \subseteq L^2(0, t_\delta; L^2(\Gamma^\delta_Y))$ with $\delta > 0$ and $\delta \to 0$ be such that there is $C > 0$ and it holds for all $\delta > 0$ that
\[
\sqrt{\delta} \| v^\delta \|_{L^2(0, t_\delta; L^2(\Gamma^\delta_Y)))} \leq C.
\]
Then there is a subsequence that we again denote by $(v^\delta)$ and a function $v^0 \in L^2(0, t_\delta; L^2(\Omega; L^2(\Gamma_Y)))$ such that $v^\delta$ mesh two-scale converges to $v^0$.

**Remark 3.6:** The time-dependent mesh two-scale convergence implies a posteriori the usual mesh two-scale convergence, since Theorem 3.5 is also valid when the family $(v^\delta)$ is independent of time.

Note, that Definition 2.4 of the mesh two-scale convergence uses highly smooth test functions, but it can be generalized to functions with less regularity. Especially, test functions in $L^2(0, t_\delta; L^2(\Gamma_Y; C(\Omega)))$, which we call admissible functions, can be considered.

### 3.2. Properties of the homogenized model

**Lemma 3.7:** The problem (20) has a unique solution up to an additive constant in $\mathbb{R}^2$.

**Proof:** First note that if $\phi$ is a solution of (20), then for any constant $C \in \mathbb{R}^2$, $\phi + C$ is also a solution of (20). Therefore, we consider the two equations of (20) separately and restrict the problem to the periodic space with vanishing average $H^1_0(\Gamma_Y) := \{ \theta \in H^1_0(\Gamma_Y) \text{suchthat} \int_{\Gamma_Y} \theta(y) \, dy = 0 \}$, on which the semi-norm $\| \partial_\Gamma \theta \|_{L^2(\Gamma_Y)}$ is a norm. Problem (20) becomes then a classical elliptic problem and its
well-posedness follows from the Lax–Milgram theorem. Note that no compatibility condition, that is sometimes called necessary condition, is needed due to the special right hand side.

Lemma 3.8: The matrix $A_{\text{hom}}$ defined in (19) is symmetric and positive definite.

Proof: From the definition we see that $A_{\text{hom}}$ is symmetric and positive semidefinite. Therefore to show that it is positive definite we only have to see that it is injective. Let $\xi \in \mathbb{R}^2$ be in the kernel of $A_{\text{hom}}$. Thus $A_{\text{hom}} \xi = 0$ and thus

$$0 = \xi^T A_{\text{hom}} \xi = \frac{1}{|\Gamma_Y|} \int_{\Gamma_Y} \xi^T (t + \partial_t \phi) \left( t + \partial_t \phi \right)^T \xi \, ds(y) = \frac{1}{|\Gamma_Y|} \int_{\Gamma_Y} \left( t + \partial_t \phi \right) \cdot \xi \, ds(y).$$

This implies that

$$(t(y) + \partial_t \phi(y)) \cdot \xi = 0, \quad \text{for a.e. } y \in \Gamma_Y.$$  

Let $T \in \mathcal{T}(\Gamma'_Y)$ be an arbitrary trail of $\Gamma'_Y$ parametrized by $y_T$, which connects opposite points $y_1$ and $y_1 + e_1$ that exist by (2). By summing up results of Lemma 3.1 for the functions $y \mapsto \phi(y)$ and $y \mapsto y_j$ for $j \in \{1, 2\}$ we find

$$0 = \int_{\Gamma_Y} (\partial_t \phi(y) + t(y)) \chi_T(y) \, ds(y) \cdot \xi = 0 \cdot \xi + e_1 \cdot \xi = e_1 \cdot \xi.$$  

Similarly, we take an arbitrary trail $T \in \mathcal{T}(\Gamma'_Y)$ connecting opposite points $y_2$ and $y_2 + e_2$ that again exists by (2) to obtain $e_2 \cdot \xi = 0$. Therefore $e_1 \cdot \xi = e_2 \cdot \xi = 0$ which implies $\xi = 0$ and so $A_{\text{hom}}$ is injective.

The following lemma states the existence and uniqueness of the limit problem (17a).

Lemma 3.9 (Well-posedness of the limit problem): The problem (17a) has a unique solution that satisfies

$$u^0 \in L^2(0, t; \mathcal{H}) \cap L^2(0, t; H^2(\Omega)), \quad \partial_t u^0 \in L^2(0, t; L^2(\Omega)).$$

Proof: The statement is a direct consequence of the symmetry and the positive-definiteness of the matrix $A_{\text{hom}}$ by Lemma 3.8, the Poincaré inequality [23, Theorem 1 on page 558] and usual regularity theory for linear parabolic equations (see [24]).

3.3. Proof of theorem 2.9: mesh two-scale convergence to homogenized model

Proof of Theorem 2.9: With the uniform stability estimates (16) the assumptions of Theorem 3.5 are fulfilled, and there exist $u^\delta, z^0, w^0 \in L^2(0, t; L^2(\Omega), L^2(\Gamma_Y)))$ and a sub-sequence of $(u^\delta)$ that is again denoted by $(u^\delta)$ such that

$$u^\delta \overset{m2s}{\rightharpoonup} u^0, \quad \partial_t u^\delta \overset{m2s}{\rightharpoonup} z^0, \quad \partial_t u^\delta \overset{m2s}{\rightharpoonup} w^0.$$  

The remainder of the proof is in seven steps. In step 1 and 2 we prove that $u^0$ is independent of $y$, first for strongly connected graphs and then for more general graphs. Then, in step 3 we show that $w^0 = \partial_t u^0$ and that there exists $u^1 \in L^2(0, t; L^2(\Omega))$ with $\partial_t w^0 \in L^2(0, t; L^2(\Omega))$ such that $z^0 = \nabla u^0 \cdot t + \partial_t y u^1$. We shall also prove that boundary (step 4) and initial conditions (step 5) are respected when passing to the limit. Finally, in step 6 and 7 we show that the limit solution $u^0$ satisfies the variational equation (17a) with the homogenized source $f_{\text{hom}}$ and homogenized tensor $A_{\text{hom}}$ defined by (18) and (19), respectively.
1. Independence of $u^0$ from $y$ for strongly connected graphs. Let us suppose that the graph is strongly connected. We take an arbitrary $w \in C_c^\infty(\Omega)$ and $\theta \in H^1_{\Gamma Y}$. Now, we let $\delta \to 0$ and apply the convergences in (25) on all three addends in the integral on the right-hand side. Due to the factor $\delta^2$, the first two addends vanish while the last remains leading to

$$0 = \int_0^{t_1} \phi(t) \int_{\Omega_Y} \int_{\Gamma Y} u^0(t, x, y) \partial_\Gamma \theta(y) \, \gamma_Y(y) \, ds(y) \, dx \, dt.$$ 

Then, the arbitrariness of $w$ and $\phi$ implies

$$0 = \int_{\Gamma Y} u^0(t, x, y) \partial_\Gamma \theta(y) \, \gamma_Y(y) \, ds(y)$$

and in view of Lemma 3.3 we find that $u^0$ is constant on $\Gamma_Y$.

2. Independence of $u^0$ from $y$ for general graphs Let $\Gamma_Y$ be an arbitrary graph in the unit cell that satisfies the assumptions of Theorem 2.9. Let $\Gamma_{Y,-}$ be the graph obtained from $\Gamma_Y$ by reversing orientations of all edges, and let $\Gamma_{Y,+} := \Gamma_Y \cup \Gamma_{Y,-}$. This graph has the same number of vertices as $\Gamma_Y$ and the double number of edges which topologically coincide, but we include both possible orientations. Thus, for each edge $e_0$ in $\Gamma_Y$ there are two edges $e_1$ and $e_2$ in $\Gamma_{Y,+}$ which coincide with $e_0$, and such that $e_0$ and $e_1$ have the same orientation, opposite from the orientation of $e_2$. It is clear that $G_Y$ is strongly connected. We analogously define $\Gamma_{Y,-}$ and $\Gamma_{Y,+}$.

For all $\delta > 0$ and $u^\delta$ on $[0, t_1] \times \Gamma^\delta$ we define $\tilde{u}^\delta$ on $[0, t_1] \times \tilde{\Gamma}^\delta$ such that the value on both doubled edges in $\Gamma^\delta$ coincide with the values of $u^\delta$ on the original edge. In other words, for original edge $e_0$ in $\Gamma^\delta$ and its copies $e_1, e_2$ in $\tilde{\Gamma}^\delta$ (one oriented equally, one oriented oppositely) we have

$$\tilde{u}^\delta|_{x \in e_1} = \tilde{u}^\delta|_{x \in e_2} = u^\delta|_{x \in e_0}.$$ 

We easily see that we also have

$$\partial t \tilde{u}^\delta|_{x \in e_1} = \partial t \tilde{u}^\delta|_{x \in e_2} = \partial t u^\delta|_{x \in e_0} \quad \text{and} \quad \partial_\Gamma \tilde{u}^\delta|_{x \in e_1} = -\partial_\Gamma \tilde{u}^\delta|_{x \in e_2} = \partial_\Gamma u^\delta|_{x \in e_0},$$

so we see that families

$$\sqrt{\delta} \|u^\delta\|_{L^2(0,t_1;L^2(\Gamma^\delta))}, \quad \sqrt{\delta} \|\partial_\Gamma u^\delta\|_{L^2(0,t_1;L^2(\Gamma^\delta))}, \quad \sqrt{\delta} \|\partial_\Gamma v^\delta\|_{L^2(0,t_1;L^2(\tilde{\Gamma}^\delta))}$$

are bounded (by $2C$, where $C$ is the constant in (16)), so all families have convergent subsequences (still denoted by $\tilde{u}^\delta$). Furthermore $\tilde{\Gamma}_Y$ is strongly connected, so we can apply the result obtained in the step 1. Thus the two-scale limit $\tilde{u}^0$ of the sequence $(\tilde{u}^\delta)_\delta$ is independent of the fast variable. This means especially that $\tilde{u}^0$ takes the same value for both edges in $\tilde{\Gamma}_Y$ of one edge in $\Gamma_Y$. 


Now, we define \( u^0 : (0, t_t) \times \Omega \times \Gamma_Y \rightarrow \mathbb{R} \), \((t, x, y) \mapsto \tilde{\nu}^0(t, x)\) and show that it is the two-scale limit of \( u^\delta \). For this we consider for each test function \( \psi \in L^2(0, t_t; L^2(\Gamma_Y; C(\Omega))) \) a corresponding function \( \tilde{\psi} \in L^2(0, t_t; L^2(\Gamma_Y; C(\Omega))) \) with \( \tilde{\psi}(\cdot, \cdot, y) := \psi(\cdot, \cdot, y) \) if \( y \in \Gamma_Y \) and \( \tilde{\psi}(\cdot, \cdot, y) := 0 \) otherwise. This yields

\[
\delta \int_0^{t_t} \int_{\Gamma^\delta} u^\delta(t, x) \psi(t, x, \frac{x}{\delta}) \, ds(x) \, dt = \delta \int_0^{t_t} \int_{\Gamma^\delta} \tilde{\nu}^0(t, x) \tilde{\psi}(t, x, \frac{x}{\delta}) \, ds(x) \, dt \\
\rightarrow \int_0^{t_t} \int_{\Omega} \int_{\Gamma_Y} \tilde{\nu}^0(t, x) \tilde{\psi}(t, x, y) \, ds(y) \, dx \, dt = \int_0^{t_t} \int_{\Omega} \int_{\Gamma_Y} u^0(t, x) \psi(t, x, y) \, ds(y) \, dx \, dt.
\]

This proves that \( u^0 \), which does not depend on \( y \) by its definition, is the two-scale limit of the sequence \((u^\delta)_\delta\).

3. Form of limits. First we shall prove that \( w^0 = \partial_t u^0 \). To do so, we take an arbitrary admissible function \( \psi \) such that \( \psi(0, \cdot, \cdot) = \psi(t_t, \cdot, \cdot) = 0 \), so we have

\[
0 = \delta \int_0^{t_t} \frac{d}{dt} \int_{\Gamma^\delta} u^\delta(t, x) \psi(t, x, \frac{x}{\delta}) \, ds(x) \, dt,
\]

so that

\[
\delta \int_0^{t_t} \int_{\Gamma^\delta} \partial_t u^\delta(t, x) \psi(t, x, \frac{x}{\delta}) \, ds(x) \, dt = -\delta \int_0^{t_t} \int_{\Gamma^\delta} u^\delta(t, x) \partial_t \psi(t, x, \frac{x}{\delta}) \, ds(x) \, dt.
\]

Using then the two-scale convergence of \( u^\delta \) to \( u^0 \) and \( \partial_t u^\delta \) to \( w^0 \) leads to

\[
\int_0^{t_t} \int_{\Omega} \int_{\Gamma_Y} w^0(t, x, y) \psi(t, x, y) \, ds(y) \, dx \, dt = -\int_0^{t_t} \int_{\Omega} \int_{\Gamma_Y} u^0(t, x) \partial_t \psi(t, x, y) \, ds(y) \, dx \, dt.
\]

Due to the arbitrariness of the function \( \psi \), it turns out that \( \partial_t u^0 \) exists, that \( \partial_t u^0 = w^0 \) and thus \( \partial_t u^0 \in L^2(0, t_t; L^2(\Omega)) \).

Now, we shall prove that for the limit \( z^0 = \nabla u^0 \cdot t + \partial_{\Gamma_Y} u^t \) holds. To do so, let us consider an arbitrary circuit \( C \in C(\Gamma^\delta) \) and an arbitrary \( w \in D(\Omega) \). Since the function \( x \mapsto \nu^\delta(t, x)w(x) \) is in \( H^1(\Gamma^\delta) \) applying Lemma 3.4 and multiplying the equality by \( \delta \phi \) with a function \( \phi \in C^\infty([0, t_t]) \) and integrating over \([0, t_t]\) leads to

\[
0 = \delta \int_0^{t_t} \phi(t) \int_{\Gamma^\delta} \partial_{\Gamma'_Y} \left( u^\delta(t, x)w(x) \right) \chi_C \left( \frac{x}{\delta} \right) \, ds(x).
\]

Then (6) implies

\[
\delta \int_0^{t_t} \phi(t) \int_{\Gamma^\delta} \partial_{\Gamma'_Y} u^\delta(t, x)w(x) \chi_C \left( \frac{x}{\delta} \right) \, ds(x) \, dt = -\delta \int_0^{t_t} \phi(t) \int_{\Gamma^\delta} u^\delta(t, x) \nabla w(x) \cdot t \left( \frac{x}{\delta} \right) \chi_C \left( \frac{x}{\delta} \right) \, ds(x) \, dt.
\]

Now, taking the limit \( \delta \rightarrow 0 \) and applying the mesh two scale limits (25) on each side of the equation we obtain that

\[
\int_0^{t_t} \phi(t) \int_{\Gamma_Y} \int_{\Omega} z^0(t, x, y)w(x) \chi_C(y) \, ds(y) \, dx \, dt = -\int_0^{t_t} \phi(t) \int_{\Gamma_Y} \int_{\Omega} u^0(t, x) \nabla w(x) \cdot t(y) \chi_C(y) \, ds(y) \, dx \, dt.
\]
for all \( w \in C_c^\infty(\Omega) \) and all \( \phi \in C^\infty([0, T]) \). Arbitrariness of \( \phi \) then implies

\[
\int_\Omega \int_{\Gamma_y} z^0(t, x, y) w(x) \chi_C(y) \, ds(y) \, dx = -\int_\Omega u^0(t, x) \nabla_x w(x) \cdot \left( \int_{\Gamma_y} t(y) \chi_C(y) \, ds(y) \right) \, dx.
\]

Let us denote

\[
t_C := \int_{\Gamma_y} t(y) \chi_C(y) \, ds(y).
\]

Thus we obtain

\[
\int_\Omega \left( \int_{\Gamma_y} z^0(t, x, y) \chi_C(y) \, ds(y) \right) w(x) \, dx = -\int_\Omega u^0(t, x) \nabla_x w(x) \cdot t_C \, dx. \tag{27}
\]

Then, by assumption on \( \Gamma_y \), if we consider trails \( T_i \) that connect the opposite points \( y_i \) and \( y_i + e_i \) (i.e. the first and the last vertices are \( y_i \) and \( y_i + e_i \) \( i = 1, 2 \)) it is a circuit \( C_i \) on \( \Gamma'_y \). Now, using Lemma 3.1 with the function \( \phi_j(y) = y \cdot e_j \) (then (6) implies \( \partial_{\Gamma} \phi_j(y) = t(y) \cdot e_j \) for \( j \in \{1, 2\} \) leads to

\[
t_{C_i} \cdot e_j := \int_{\Gamma_y} t(y) \cdot e_j \chi_C(y) \, ds(y) = \int_{\Gamma_y} t(y) \cdot e_j \chi_{T_i}(y) \, ds(y) = \int_{\Gamma_y} \partial_{\Gamma} \phi_j(y) \chi_{T_i}(y) \, ds(y)
\]

and, hence, \( t_{C_i} = e_i, \, i \in \{1, 2\} \). That means that there are at least two linear independent vectors \( t_C \) and the equality (27) defines the weak derivative \( \nabla_x u^0 \). Since the function

\[(t, x) \mapsto \int_{\Gamma_y} z^0(t, x, y) \chi_C(y) \, ds(y)\]

belongs to \( L^2(0, t; L^2(\Omega)) \) the weak derivative \( \nabla_x u^0 \) belongs to \( L^2(0, t; L^2(\Omega)) \) as well. Hence, \( u^0 \in L^2(0, t; L^2(\Omega)) \). Now, with the equality \( \nabla w \cdot t_C = \frac{1}{2}(w t_C) \) as \( t_C \) is a constant vector integrating by parts on the right-hand side of (27) we obtain

\[
\int_\Omega \int_{\Gamma_y} w(x) \left( z^0(t, x, y) - \nabla_x u^0(t, x) \cdot t(y) \right) \chi_C(y) \, ds(y) \, dx = 0.
\]

Then, arbitrariness of \( w \) implies

\[
\int_{\Gamma_y} \left( z^0(t, x, y) - \nabla_x u^0(t, x) \cdot t(y) \right) \chi_C(y) \, ds(y) = 0.
\]

As we assumed arbitrariness of \( C \in C(\Gamma_{y, \delta}) \) Lemma 3.2 now implies that there exists a function \( u^1(t, x, \cdot) \in H^1_0(\Gamma_y) \) such that

\[
\partial_{\Gamma,y} u^1(t, x, y) = z^0(t, x, y) - \nabla_x u^0(t, x) \cdot t(y). \tag{28}
\]

Since the right hand side is in \( L^2(0, t; L^2(\Omega; L^2(\Gamma_y))) \) we conclude that \( u^1 \in L^2(0, t; H^1(\Gamma_y)) \).

4. Boundary conditions. Let us take a function \( w \in C^1(\Omega) \) such that \( w|_{\partial\Omega \setminus \Gamma_D} = 0 \), i.e. it vanishes on the complement of the boundary of the Dirichlet boundary. As the function \( x \mapsto \delta u^0(t, x) w(x) \) is in \( H^1(\Gamma^\delta) \) it follows from Lemma 3.4 for circuits introduced in the previous step that

\[
\int_{\Gamma^\delta} \partial_{\Gamma} u^0(t, x) w(x) \chi_C(c_i) \left( \frac{x}{\delta} \right) \, ds(x) = -\delta \int_{\Gamma^\delta} u^0(t, x) \partial_{\Gamma} w(x) \chi_C(c_i) \left( \frac{x}{\delta} \right) \, ds(x), \quad i = 1, 2.
\]

Multiplying the equation by \( \phi \in C^\infty([0, t]) \) and integrating over \( [0, t] \) we obtain in the limit \( \delta \to 0 \) using (28) and (6)

\[
\int_0^t \phi(t) \int_\Omega \int_{\Gamma_y} (\nabla_x u^0(t, x) \cdot t(y) + \partial_{\Gamma,y} u^1(t, x, y)) w(x) \chi_C(y) \, ds(y) \, dx \, dt
\]
\[=-\int_0^t \phi(t) \int_{\Omega} \int_{\Gamma_Y} u^0(t,x) \nabla w(x) \cdot t(y) \chi_{C_i}(y) \, ds(y) \, dx \, dt. \quad (29)\]

Therefore, using \( t_{C_i} = e_i, i = 1, 2 \), (29) becomes

\[\int_0^t \phi(t) \int_{\Omega} \left( \nabla u^0(t,x) \cdot e_i + \int_{\Gamma_Y} \partial_{\Gamma_Y} u^1(t,x,y) \chi_{C_i}(y) \, ds(y) \right) \, w(x) \, dx \, dt \]

\[=-\int_0^t \phi(t) \int_{\Omega} u^0(t,x) \nabla w(x) \cdot e_i \, dx \, dt, \quad i = 1, 2.\]

Since \( u^1 \) is \( \Gamma_Y \) periodic, using Lemma 3.1 with \( \mathcal{G} = \Gamma_Y \), it holds

\[\int_{\Gamma_Y} \partial_{\Gamma_Y} u^1(x,y) \chi_{C_i}(y) \, ds(y) = 0, \quad i = 1, 2.\]

Thus, using arbitrariness of \( \phi \in C^\infty([0, T]) \) we obtain that

\[e_i \cdot \int_{\Omega} \nabla u^0(t,x) w(x) \, dx = -e_i \cdot \int_{\Omega} u^0(t,x) \nabla w(x) \, dx.\]

After partial integration in the right hand side it follows that

\[e_i \cdot \int_{\Gamma_D} u^0(t,x) w(x) n(x) \, ds(x) = 0,\]

where \( n \) is the unit outer normal on \( \partial \Omega \). Due to the arbitrariness of the function \( w \), we deduce that the function \((t,x) \mapsto e_i \cdot n(x) u^0(t,x)\) vanishes on \((0,t_1) \times \Gamma_D\). This is only possible for \( i = 1, 2 \) if \( u^0 \) vanishes on \( \Gamma_D \).

5. Initial condition. Let us take that \( u^\delta|_{t=0} \) mesh two-scale converges to the limit \( u_{init} \in L^2(0,t_1; L^2(\Omega; L^2(\Gamma_Y))) \). Let us take any \( \psi \in C_0^\infty(\Omega, H_1^1(\Gamma_Y)) \) and \( \phi \in C^\infty(0,t_1) \) such that \( \phi(0) = 1, \phi(t_1) = 0 \). Then partial integration in \( t \) gives us

\[\delta \int_0^t \int_{\Gamma^\delta} \partial_t u^\delta(t,x) \psi \left( x, \frac{x}{\delta} \right) \phi(t) \, ds(x) \, dt \]

\[=-\delta \int_0^t \int_{\Gamma^\delta} u^\delta(t,x) \psi \left( x, \frac{x}{\delta} \right) \phi'(t) \, ds(x) \, dt - \delta \int_{\Gamma^\delta} u^\delta(0,x) \psi \left( x, \frac{x}{\delta} \right) \, ds(x).\]

Now, in the limit \( \delta \to 0 \) we obtain

\[\int_0^t \int_{\Omega} \int_{\Gamma_Y} \partial_t u^0(t,x) \psi(x,y) \phi(t) \, ds(y) \, dx \, dt\]

\[=-\int_0^t \int_{\Omega} \int_{\Gamma_Y} u^0(t,x) \psi(x,y) \phi'(t) \, ds(y) \, dx \, dt - \int_{\Omega} \int_{\Gamma_Y} u_{init}(x) \psi(x,y) \, ds(y) \, dx.\]

Integrating the left hand side by parts we find

\[\int_0^t \int_{\Omega} \int_{\Gamma_Y} \partial_t u^0(t,x) \psi(x,y) \phi(t) \, ds(y) \, dx \, dt\]

\[=-\int_0^t \int_{\Omega} \int_{\Gamma_Y} u^0(t,x) \psi(x,y) \phi'(t) \, ds(y) \, dx \, dt - \int_{\Omega} \int_{\Gamma_Y} u^0(0,x) \psi(x,y) \, ds(y) \, dx.\]
By subtracting last two results we obtain

\[
\int_{\Omega} \int_{\Gamma_y} (u_{\text{init}}(x) - u^0(0,x)) \psi(x,y) \, ds(y) \, dx = 0.
\]

Arbitrariness of \( \psi \) implies \( u_{\text{init}} = u^0|_{t=0} \).

6. Form of the corrector term \( u^1 \). We consider a test function \( v^\delta(x) = v(x) \theta \left( \frac{x}{\delta} \right) \) as product of a slow varying function \( v \in \mathcal{H} \) and a fast varying function \( \theta \in H^1_\delta(\Gamma_y) \). Multiplying (7a) by \( \delta^2 \phi \) for \( \phi \in C^\infty([0,t_1]) \), integrating over \([0,t_1]\) and using this particular test function leads to

\[
\delta^2 \int_0^t \phi(t) \int_{\Gamma_y} \partial_t \delta^\theta(t,x) \theta \left( \frac{x}{\delta} \right) v(x) \, ds(x) \, dt
\]

\[
+ \delta \int_0^t \phi(t) \int_{\Gamma_y} a(x) \partial_t \delta^\theta(t,x) \left( \partial_{\Gamma_y} \theta \left( \frac{x}{\delta} \right) \right) v(x) \, ds(x) \, dt + \delta \theta \left( \frac{x}{\delta} \right) v(x) \cdot t \left( \frac{x}{\delta} \right) \, ds(x) \, dt
\]

\[
\delta^2 \int_0^t \phi(t) \int_{\Gamma_y} f \left( t, x, \frac{x}{\delta} \right) \theta \left( \frac{x}{\delta} \right) v(x) \, ds(x) \, dt.
\]  

Under the hypothesis of Lemma 2.5 and using the Cauchy-Schwartz inequality, the right-hand side of (30) tends to 0 as \( \delta \) tends to 0, independently of the choice of \( v, \theta \) and \( \phi \). Furthermore the first term on the left hand side also tends to zero by the a priori estimates from Theorem 2.8. The second term in the left hand side of (30) is split in two. We use the Cauchy-Schwarz inequality and the a priori estimate from Theorem 2.8 for \( \partial_{\Gamma} \delta^\theta \) to conclude that the term with additional \( \delta \) tends to zero as \( \delta \to 0 \). We use then the mesh two-scale convergence of \( \partial_{\Gamma} \delta^\theta \) to \( \partial_{\Gamma} u^0 + \partial_{\Gamma_y} u^1 \) for the test function

\[
\psi : (x,y) \mapsto a(x) v(x) \partial_{\Gamma} \theta(y) \in L^2(\Omega; L^2(\Gamma_y)),
\]

to take the limit in the remaining term on the left hand side of (30). What remains in the limit is

\[
\int_0^t \phi(t) \int_{\Omega} \int_{\Gamma_y} a(x) \left( \partial_{\Gamma} u^0(t,x) + \partial_{\Gamma_y} u^1(t,x,y) \right) v(x) \partial_{\Gamma} \theta(y) \, ds(y) \, dx \, dt = 0.
\]  

Since \( v \) is an arbitrary function of \( \mathcal{H} \) and \( \phi \) arbitrary in \( C^\infty([0,t_1]) \), we obtain that

\[
\int_{\Gamma_y} \left( \partial_{\Gamma} u^0(t,x) + \partial_{\Gamma_y} u^1(t,x,y) \right) \partial_{\Gamma} \theta(y) \, ds(y) = 0.
\]

Using (6) this gives the unit pattern problem for \( u^1 \):

\[
\int_{\Gamma_y} \partial_{\Gamma_y} u^1(t,x,y) \partial_{\Gamma} \psi(y) \, ds(y) = -\nabla_x u^0(t,x) \cdot \int_{\Gamma_y} t(x) \partial_{\Gamma} \psi(y) \, ds(y).
\]

which has by Lemma 3.7 a unique solution up to an additive constant in \( y \) (note that \( t \) and \( x \) are parameters here). With this, we can express the solution of (32) using the solution of the canonical problems (20) as \( u^1(t,x,y) = \nabla_x u^0(t,x) \cdot \phi(y) + \tilde{u}^1(t,x) \) with a function \( \tilde{u}^1 \) independent of the fast variable \( y \) and

\[
\partial_{\Gamma_y} u^1(t,x,y) = \nabla_x u^0(t,x) \cdot \nabla_{\Gamma} \phi(y).
\]  

7. Variational formulation for the unique limit solution \( u^0 \). Next we take a test function \( v^\delta(t,x) = \delta v(x) \phi(t) \), where \( v \in H^1(\Omega) \) is independent of the fast variable \( y \) and \( \phi \in C^\infty([0,t_1]) \), in (7a). We
obtain
\[
\delta \int_0^t \phi(t) \int_{\Gamma} \partial_t u^\delta(t,x)v(x) \, ds(x) \, dt + \delta \int_0^t \phi(t) \int_{\Gamma} a(x) \partial_\Gamma u^\delta(x) \partial_\Gamma v(x) \, ds(x) \, dt
\]
\[
= \delta \int_0^t \phi(t) \int_{\Gamma} f(t,x)v(x) \, ds(x) \, dt.
\]

Then using the mesh two-scale convergence of \( \partial_\Gamma u^\delta \) (respectively \( \partial_t u^\delta \)) to \( \partial_\Gamma u^0 + \partial_\Gamma y u^1 \) (resp. \( \partial_t u^0 \)), we take the limit when \( \delta \) tends to zero and obtain
\[
\int_0^t \phi(t) \int_{\Omega} \partial_t u^0(t,x)v(x) \, ds(x) \, dt
\]
\[
+ \int_0^t \phi(t) \int_{\Omega} a(x) \left( \nabla u^0(x) \cdot t(y) + \partial_\Gamma y u^1(x,y) \right) \nabla v(x) \cdot t(y) \, ds(y) \, dx \, dt
\]
\[
= |\Gamma| \int_0^t \phi(t) \int_{\Omega} f_{\text{hom}}(t,x)v(x) \, dx \, dt,
\]
where \( f_{\text{hom}} \) was defined in (18). Using arbitrariness of \( \phi \) in \( C^\infty([0,t]) \) we obtain that for almost every \( t \in [0,t] \) we have
\[
\int_{\Omega} \partial_t u^0(t,x)v(x) \, ds(x) + \frac{1}{|\Gamma|} \int_{\Omega} \int_{\Gamma} a(x) \left( \nabla u^0(t,x) \cdot t(y) + \partial_\Gamma y u^1(t,x,y) \right) \nabla v(x) \cdot t(y) \, ds(y) \, dx
\]
\[
= \int_{\Omega} f_{\text{hom}}(t,x)v(x) \, dx,
\]
Using the solution representation for \( \partial_\Gamma y u^1 \) from (33), \( \partial_\Gamma y u^1(t,x,y) = \nabla u^0(t,x) \cdot \nabla_\Gamma \phi(y) \), we obtain
\[
\int_{\Omega} \partial_t u^0(t,x)v(x) \, ds(x) + \frac{1}{|\Gamma|} \int_{\Omega} \int_{\Gamma} a(x) \left( \int_{\Gamma} t(y) \left( t(y) + \partial_\Gamma \phi(y) \right)^T \, ds(y) \right) \nabla u^0(t,x) \cdot \nabla v(x) \, dx
\]
\[
= \int_{\Omega} f_{\text{hom}}(t,x)v(x) \, dx.
\]

Testing the canonical problem (20) by the two functions \( \psi = \phi_i, i = 1, 2 \) we obtain that
\[
\int_{\Gamma} \partial_\Gamma \phi(y)(t(y) + \partial_\Gamma \phi(y))^T \, ds(y) = 0.
\]
Thus the matrix \( A_{\text{hom}} \), defined in (19), can be also written as
\[
A_{\text{hom}} = \int_{\Gamma} t(y) \left( t(y) + \partial_\Gamma \phi(y) \right)^T \, ds(y)
\]
and, hence, in view of (34) the two-scale limit \( u^0 \) satisfies (17a). Since \( A_{\text{hom}} \) is positive definite by Lemma 3.8 similarly to Lemma 2.6 existence and uniqueness of the limit problem (17a) follow.

As usual, uniqueness of the solution of the limit problem then also implies the mesh two-scale convergences for the whole family \( (u^\delta)_\delta \) and their derivatives in (25).

In Theorem 2.9 we assumed the stability estimate of the solutions \( u^\delta \) of (1) stated in Theorem 2.8. Repeating the first five steps in the proof of Theorem 2.9 for any sequence of function \( (v^\delta)_\delta \) satisfying the same stability estimate we can state similarly the following statement.
Corollary 3.10: Let $\Gamma_Y$ be connected and satisfies (2). Let the sequence $(\psi_\delta)_\delta \in L^2(0, t_1; H^1(\Gamma^s))$ with $\delta \to 0$ be such that $\partial_t \psi_\delta \in L^2(0, t_1; L^2(\Gamma^s))$ for any $\delta > 0$ be such that there is $C > 0$ such that for all $\delta > 0$ one has

$$\sqrt{\delta} \| \psi_\delta \|_{L^2(0, t_1; L^2(\Gamma^s))} \leq C, \quad \sqrt{\delta} \| \partial_t \psi_\delta \|_{L^2(0, t_1; L^2(\Gamma^s))} \leq C, \quad \sqrt{\delta} \| \partial_t \psi_\delta \|_{L^2(0, t_1; L^2(\Gamma^s))} \leq C.$$ 

Then there is a sub-sequence of $(\psi_\delta)_\delta$ (still denoted by $\psi_\delta$) and functions $\psi_0 \in L^2(0, t_1; H^1(\Omega))$ and $\psi_1 \in L^2(0, t_1; H^1(\Omega))$ with $\partial_t \psi_0 = L^2(0, t_1; L^2(\Omega))$ such that

$$\psi_\delta \rightharpoonup \psi_0, \quad \partial_t \psi_\delta \rightharpoonup \partial_t \psi_0 \quad \text{in} \quad L^2(0, t_1; L^2(\Omega))$$

Additionally, if $(\psi_\delta)_\delta \subseteq L^2(0, t_1; H^1(\Omega))$, then the limit $\psi_0$ belongs to $L^2(0, t_1; \mathcal{H})$. Also, if $\psi_0 \vert_{t=0}$ mesh two-scale converges, its limit is $\psi_0 \vert_{t=0}$.

3.4. Proof of theorem 2.11: convergence in norm to homogenized model

Proof of Theorem 2.11: By assumption on $f$ we find that $f_{\text{hom}} \in L^2(0, t_1; H^2(\Omega))$ and $\partial_t f_{\text{hom}} \in L^2(0, t_1; L^2(\Omega))$ and by assumption on $u_{\text{init}}$ using [24, Chap. 7, Theorem 6] we conclude

$$u_0 \in L^2(0, t_1; H^3(\Omega)), \quad \partial_t u_0 \in L^2(0, t_1; H^2(\Omega)).$$

Therefore $\nabla u_0 \in L^2(0, t_1; H^2(\Omega))$ and thus

$$(t, x, y) \mapsto \nabla_x u_0(t, x, y) \cdot f(y) \in L^2(0, t_1; H^2(\Omega); L^2(\Gamma_0)) \quad \text{for} \quad x \in \Omega,$$

Moreover $f \in L^2(0, t_1; H^2(\Gamma_0; C(\Omega)))$, so $f$ is an admissible test function.

Therefore for any $\delta$ the integrals in the following $\delta$ family and all the terms in the following computation are well defined

$$\Lambda^\delta(t) := \frac{1}{2} \delta \int_{\Gamma_0} \left( u_\delta^\delta(t, x) - u_0^\delta(t, x) \right)^2 \text{d}s(x)$$

$$\quad + \delta \int_0^t \int_{\Gamma_0} a(x) \partial_t^\delta \left( u_\delta^\delta(t, x) - u_0^\delta(t, x) - \delta u_1^\delta \left( \tau, x, \frac{x}{\delta} \right) \right)^2 \text{d}s(x) \text{d}\tau. \quad (35)$$

Applying the Newton–Leibniz theorem in the first term of $\Lambda^\delta$ leads to

$$\Lambda^\delta(t) = \delta \int_0^t \int_{\Gamma_0} \left( \partial_t u_\delta^\delta(t, x) - \partial_t u_0^\delta(t, x) \right) \left( u_\delta^\delta(t, x) - u_0^\delta(t, x) \right) \text{d}s(x) \text{d}\tau$$

$$\quad + \delta \int_0^t \int_{\Gamma_0} a(x) \partial_t^\delta \left( u_\delta^\delta(t, x) - u_0^\delta(t, x) - \delta u_1^\delta \left( \tau, x, \frac{x}{\delta} \right) \right)^2 \text{d}s(x) \text{d}\tau. \quad (36)$$

Now we use (7a) for the particular test function $v(x) := u_\delta^\delta(t, x) - u_0^\delta(t, x) \in H^1(\Gamma^s)$ to resolve the quadratic terms in $u_\delta^\delta$, so that $\Lambda^\delta(t)$ becomes

$$\Lambda^\delta(t) = -\delta \int_0^t \int_{\Gamma_0} \partial_t u_0^\delta(t, x) \left( u_\delta^\delta(t, x) - u_0^\delta(t, x) \right) \text{d}s(x)$$

$$\quad + \delta \int_0^t \int_{\Gamma_0} a(x) \partial_t^\delta \left( -u_0^\delta(t, x) - \delta u_1^\delta \left( \tau, x, \frac{x}{\delta} \right) \right)$$

$$\times \partial_t^\delta \left( u_\delta^\delta(t, x) - u_0^\delta(t, x) - \delta u_1^\delta \left( \tau, x, \frac{x}{\delta} \right) \right) \text{d}s(x) \text{d}\tau$$

$$\quad + \delta \int_0^t \int_{\Gamma_0} a(x) \partial_t^\delta \partial_t^\delta \left( u_\delta^\delta(t, x) \partial_t^\delta \left( -u_0^\delta(t, x) - \delta u_1^\delta \left( \tau, x, \frac{x}{\delta} \right) \right) \right) \text{d}s(x) \text{d}\tau$$

$$\quad - \delta \int_0^t \int_{\Gamma_0} f \left( \tau, x, \frac{x}{\delta} \right) \left( u_\delta^\delta(t, x) - u_0^\delta(t, x) \right) \text{d}s(x) \text{d}\tau.$$
We use then the weak convergence stated by (25) and the fact that \( f \) is admissible. Therefore \( \Lambda^\delta(t) \) converges to the limit functional \( \Lambda(t) \) defined by

\[
\Lambda(t) := -\int_0^t \int_\Omega \int_{\Gamma_Y} a(x)(\nabla u^0(\tau, x) \cdot f(y) + \partial_{\Gamma_Y} u^1(\tau, x, y)) \partial_{\Gamma_Y} u^1(\tau, x, y) \, dy \, dx \, d\tau
\]

\[
= -\int_0^t \int_\Omega \left( \int_{\Gamma_Y} \partial_{\Gamma_Y} \phi(y) \left( f(y) + \partial_{\Gamma_Y} \phi(y) \right)^T \, ds(y) \right) \nabla u^0(\tau, x) \cdot \nabla u^0(\tau, x) \, dx \, d\tau,
\]

\[
= -|\Gamma_Y| \int_0^t \int_\Omega A_{\text{hom}} \nabla u^0(\tau, x) \cdot \nabla u^0(\tau, x) \, dx \, d\tau.
\]

The homogenized tensor \( A_{\text{hom}} \) is positive definite by Lemma 3.8 and thus \( \Lambda(t) \leq 0 \). However, \( \Lambda^\delta(t) \geq 0 \) which is both possible only if \( \Lambda(t) = 0 \). Taking \( t = t_t \) leads to

\[
\sqrt{\delta} \| \partial\Gamma(u^\delta - u^0 - \delta u^1) \|_{L^2(0, t_t; L^2(\Gamma^s))} \to 0.
\]

We also deduce pointwise convergence

\[
\sqrt{\delta} \| u^\delta(t, \cdot) - u^0(t, \cdot) \|_{L^2(\Gamma^s)} \to 0,
\]

for any \( t \in (0, t_t) \). Let us denote by \( \Lambda^1_2(t) \) and \( \Lambda^2_2(t) \) the first and the second term, respectively, in the right hand side of (36). Thus \( \Lambda^\delta(t) = \Lambda^1_2(t) + \Lambda^2_2(t) \). For the second term we have \( \sup_{t \in [0, t_t]} \Lambda^2_2 = \Lambda^2_2(t_t) \) since the function under integral sign is positive. Thus pointwise convergence of \( \Lambda^\delta \) implies convergence in \( L^\infty \). For \( \Lambda^\delta_1 \) we show that it is equicontinuous.

\[
|\Lambda^\delta_1(t + \Delta t) - \Lambda^\delta_1(t)| \leq \delta \int_t^{t+\Delta t} \int_{\Gamma^s} |\partial_{\tau}(u^\delta(\tau, x) - u^0(\tau, x))(u^\delta(\tau, x) - u^0(\tau, x))| \, ds(x) \, d\tau
\]

\[
\leq \left( \delta \int_t^{t+\Delta t} \int_{\Gamma^s} (\partial_{\tau}(u^\delta(\tau, x) - u^0(\tau, x)))^2 \, ds(x) \, d\tau \right)^{\frac{1}{2}}
\]

\[
+ \left( \delta \int_t^{t+\Delta t} \int_{\Gamma^s} (u^\delta(\tau, x) - u^0(\tau, x))^2 \, ds(x) \, d\tau \right)^{\frac{1}{2}}.
\]

From the \textit{a priori} estimates in Theorem 2.8 we have that the first term is uniformly bounded with respect to \( \delta \). For the second term we use the Newton–Leibnitz formula and obtain

\[
|\Lambda^\delta_1(t + \Delta t) - \Lambda^\delta_1(t)| \leq C \left( \delta \int_t^{t+\Delta t} \int_0^\tau \int_{\Gamma^s} (u^\delta(s, x) - u^0(s, x))^2 \, ds(x) \, ds \, d\tau \right)^{\frac{1}{2}}
\]

\[
\leq C \left( 2\delta \int_t^{t+\Delta t} \int_0^\tau \int_{\Gamma^s} \partial_{\tau}(u^\delta(\zeta, x) - u^0(\zeta, x))(u^\delta(s, x) - u^0(\zeta, x)) \, ds(x) \, ds \, d\tau \right)^{\frac{1}{2}}
\]

\[
\leq C \left( 2 \int_t^{t+\Delta t} \Lambda^\delta_1(\tau) \, d\tau \right)^{\frac{1}{2}}.
\]

Since \( \Lambda^\delta_1 \) is uniformly bounded we have that

\[
|\Lambda^\delta_1(t + \Delta t) - \Lambda^\delta_1(t)| \leq C(\Delta t)^{\frac{1}{2}}.
\]

Thus \( \Lambda^\delta_1 \) is equicontinuous and since it pointwisely converges to 0 we get

\[
\|\Lambda^\delta\|_{L^\infty(0, \Delta t)} \to 0,
\]

as \( \delta \) tends to 0. Thus (21) is proved. \( \blacksquare \)
4. Computation of the homogenized tensor $A_{\text{hom}}$

In this section, we give a more practical way to compute the matrix $A_{\text{hom}}$ than solving (20). Even though the geometries in Figure 3 are only with straight edges the following analysis also refers to curved edges.

**Lemma 4.1:** The function $\phi = (\phi_1, \phi_2)$ is a solution to the canonical problem (20) if and only if $s \mapsto \phi \circ \gamma_i(s) + \gamma_i(s)$ is affine, for all $i = 1, \ldots, N_E$ and the Kirchhoff junction condition holds at all vertices, i.e.

$$\sum_{i=1}^{N_E} (\phi \circ \gamma_i + \gamma_i)'(\ell_i) - \sum_{i=1}^{N_E} (\phi \circ \gamma_i + \gamma_i)'(0) = 0$$

for all vertices $v_j \in V, j = 1, \ldots, N_V$.

**Proof:** We first prove that any solution of (20) is affine and then prove that it satisfies the Kirchhoff junction conditions. If $\phi$ solves (20), then we have

$$0 = \int_{\Gamma Y} (\partial_{\Gamma \gamma} \phi + t) \cdot \partial_{\Gamma \gamma} \psi \, ds(y) = \sum_{i=1}^{N_E} \int_0^{\ell_i} [(\partial_{\Gamma \gamma} \phi + t) \cdot \partial_{\Gamma \gamma} \psi] \circ \gamma_i(s) \, ds$$

$$= \sum_{i=1}^{N_E} \int_0^{\ell_i} ((\phi \circ \gamma_i)'(s) + \gamma_i'(s)) \cdot (\psi \circ \gamma_i)'(s) \, ds.$$  \hspace{1cm} (38)

In the last equation we see that if we test the expression with function $\psi \in C^\infty_c(\gamma_i((0, \ell_i)), \mathbb{R}^2)$ (for a particular $i$), after integration by parts, we obtain that the distributional derivative of $(\phi \circ \gamma_i)'(s) + \gamma_i'(s)$ is zero, so we get that $s \mapsto \phi \circ \gamma_i(s) + \gamma_i(s)$ is affine for all $i = 1, \ldots, N_E$. Performing partial integration in (38) now gives

$$\sum_{i=1}^{N_E} \left[ (\phi \circ \gamma_i)'(s) + \gamma_i'(s) \right] \cdot (\psi \circ \gamma_i)(s) \bigg|_0^{\ell_i} = 0.$$

Inserting continuous and affine test function $\psi$ having value $e_1$ (respectively $e_2$) at a vertex $v_j \in V$ and 0 at all other vertices we obtain the Kirchhoff junction conditions.

To prove the converse follow exactly the opposite statements. $\blacksquare$
In view of this lemma, the function

\[
x \mapsto q(x) := \phi(x) + x
\]

defined on \( \Gamma_Y \) is affine on each edge, continuous on \( \Gamma_Y \), but not periodic.

Before we proceed let us introduce some more definitions. For vertex \( v \in V \) we define \( \eta(v) \) as the vector equal to \( e_1 \) or \( e_2 \) if \( v \) lies on the right or top side of the unit cell (excluding the corners), respectively, \( e_1 + e_2 \) if \( v \) lies on the right top corner and \( 0 \) otherwise. Assuming that vertices in \( V \) are indexed, with \( v_{\pi(i)} \) we denote vertex equal to \( v_i \) if \( v_i \) is not located on the upper or right boundary of the unit cell, and otherwise the vertex \( v_i^T \) such that \( v_i \) and \( v_i^T \) are opposite vertices. Thus we have identities \( v_{\pi(i)} = v_i - \eta(v_i) \) and

\[
q(v_i) = q(v_{\pi(i)}) + \eta(v_i), \quad i \in \{1, \ldots, N_V\}.
\]

Since from this identity we see that values of function \( q \) are uniquely defined in vertices on the right and upper boundary once they are defined in all other vertices, we define

\[
b_j = \frac{d}{ds}(q \circ \gamma_j)(\cdot), \quad Q_i = q(v_i),
\]

for \( i = 1, \ldots, N_{V'} \), \( j = 1, \ldots, N_E \), where \( V' \) denotes the set of all vertices not located on the right and upper boundary of the unit cell. Without loss of generality, those vertices are indexed as first \( N_{V'} \) vertices in the set \( V \). We will show that the matrix \( A_{\text{hom}} \) can be found without solving the canonical problem (20), but by solving linear system of equations in terms of \( 2N_E + 2N_{V'} \) unknowns defined in (40).

We introduce incidence matrix \( A_\mathcal{T} \in M_{2N_E,2N_E}(\mathbb{R}) \) of the oriented periodic graph \( \Gamma_Y \), such that \( 2 \times 2 \) submatrix on intersection of rows \( 2i-1, 2i \) and columns \( 2j-1, 2j \) is equal to \( I \) if the \( j \)th edge enters the \( i \)th vertex (i.e. \( \gamma_j(\ell_j) = v_i \)), equal to \( -I \) if the \( j \)th edge leaves the \( i \)th vertex (i.e. \( \gamma_j(0) = v_i \)), and \( 0 \) otherwise. For a similar argument see [25]. Thus, equations in (37) can be written, introducing the notation \( b = (b_1^T, \ldots, b_{N_E}^T)^T \), by

\[
A_\mathcal{T} b = 0.
\]

Secondly, for any edge \( e_j, j = 1, \ldots, N_E \), which connects vertices \( v_{i_1} = \gamma_j(\ell_j) \) and \( v_{i_2} = \gamma_j(0) \) we use the definition of \( b_j \) and \( Q_i \) in (40) for a function \( q \) and the Newton–Leibnitz theorem to obtain

\[
Q_{\pi(i_1)} - Q_{\pi(i_2)} = q(v_{\pi(i_1)}) - q(v_{\pi(i_2)})
= (q(v_{i_1}) - q(v_{i_2})) - (\eta(v_{i_1}) - \eta(v_{i_2}))
= \int_0^{\ell_j} \frac{d}{ds}(q \circ \gamma_j)(s) ds - (\eta(\gamma_j(\ell_j)) - \eta(\gamma_j(0)))
= \ell_j b_j - (\eta(\gamma_j(\ell_j)) - \eta(\gamma_j(0))).
\]

Hence, introducing the diagonal matrix \( L \in M_{2N_E,2N_E}(\mathbb{R}) \) whose \( (2j-1) \)th and \( 2j \)th diagonal entries are equal to \( \ell_j \), the vector \( f \in \mathbb{R}^{2N_E} \) whose \( (2j-1) \)th and \( 2j \)th component are the first and the second component of \( \eta(\gamma_j(\ell_j)) - \eta(\gamma_j(0)) \), respectively, and \( Q = (Q_1^T, \ldots, Q_{N_{V'}}^T)^T \) the equations in (42) can be written as

\[
A_\mathcal{T}^T Q + L b = f.
\]

Equations (41) and (43) together form a linear system for \( (b, Q) \):

\[
\begin{bmatrix}
L & A_\mathcal{T}^T \\
A_\mathcal{T} & 0
\end{bmatrix}
\begin{bmatrix}
b \\
Q
\end{bmatrix}
= \begin{bmatrix}
f \\
0
\end{bmatrix}.
\]
Lemma 4.2: System (44) uniquely defines $b$, and consequently it uniquely defines $A_{\text{hom}}$.

Proof: First, (43) is equivalent to

$$b = L^{-1}(f - A_T^T Q).$$

(45)

Inserting this expression in the second equation in (44) – which is (41) – we get

$$A_T L^{-1} A_T^T Q = A_T L^{-1} f.$$

(46)

To show the solvability of this system, i.e. the existence of $Q$, we need to prove the condition from the Kronecker–Capelli theorem [26, p. 56]:

$$A_T L^{-1} f \in \text{Im}(A_T L^{-1} A_T^T) = \text{Ker}(A_T L^{-1} A_T^T)^\perp = \text{Ker}(A_T^T)^\perp = \text{Im} A_T,$$

which is clear. Thus, there is a solution to (46). All solutions are described with $Q \in Q_0 + \text{Ker} A_T^T$ where $Q_0$ is a fixed solution. Plugging it back to (45) we get

$$b = L^{-1}(f - A_T^T Q) = L^{-1}(f - A_T^T Q_0),$$

which is unique. Thus the matrix $A_{\text{hom}}$ is uniquely defined as well since

$$A_{\text{hom}} = \frac{1}{|\Gamma_Y|} \int_{\Gamma_Y} (\partial_T \phi(y) + t(y))(\partial_T \phi(y) + t(y))^T \, ds(y) = \frac{1}{|\Gamma_Y|} \sum_{j=1}^{N_E} \ell_j b_j b_j^T,$$

(47)

and $|\Gamma_Y| = \sum_{j=1}^{N_E} \ell_j$.

As a contrary to $b$, vector $Q$ is not uniquely defined by the system (44), since due to Theorem 4.2.4 in [27], $\text{Ker} A_T^T$ is two-dimensional. This is in accordance with Lemma 3.7, since $\phi$ is defined uniquely up to an additive constant.

Remark 4.3: From the system (44), from Lemma 4.2 and from the expression for $A_{\text{hom}}$ we see that all properties of the homogenized problem come solely from the connectivity properties of the vertices in the oriented graph $\Gamma_Y$ and lengths of its edges. Thus, positions of vertices in the unit cell or differential geometry properties of edges forming the graph do not play a role in the definition of the matrix $A_{\text{hom}}$ and thus in the homogenized problem.

Example 4.1: Let us find the operator $A_{\text{hom}}$ for the geometry shown in Figure 3(a). The graph consists of 4 edges and 5 vertices with two of them being located on the right or upper boundary of the unit cell. That is the reason why the matrix $A_T$ is $6 \times 8$ matrix and it is equal to

$$A_T = \begin{bmatrix} I_2 & 0 & -I_2 & 0 \\ 0 & I_2 & 0 & -I_2 \\ -I_2 & -I_2 & I_2 & I_2 \end{bmatrix},$$

where $I_n$ denotes the $n \times n$ identity matrix. Since all edges have equal length $\frac{1}{2}$, the diagonal matrix $L$ is simply given by $L = \frac{1}{2} I_{2N_E}$, and, hence, $L^{-1} = 2 I_{2N_E}$. As the first and second edge are ending on the
right or upper boundary and, hence, \( \eta(\gamma_1(\ell_1)) - \eta(\gamma_1(0)) = (1,0)^T \) and \( \eta(\gamma_2(\ell_2)) - \eta(\gamma_2(0)) = (0,1)^T \) we find 

\[
f = (1,0,0,1,0,0,0,0)^T.
\]

Now, we have everything explicitly defined to compute the homogenized tensor \( A_{\text{hom}} \). We proceed as in the proof of Lemma 4.2. First, we seek one solution \( Q^0 \) of (46) which is in our case

\[
\begin{bmatrix}
4I_2 & 0 & -4I_2 \\
0 & 4I_2 & -4I_2 \\
-4I_2 & -4I_2 & 8I_2
\end{bmatrix}
= 2
\begin{bmatrix}
e_1 \\
e_2 \\
-e_1 - e_2
\end{bmatrix}.
\]

Such a solution is \( Q^0 = \begin{bmatrix} \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0 \end{bmatrix}^T \). Now, inserting \( Q^0 \) into (45) we get \( b_1 = b_3 = e_1, \ b_2 = b_4 = e_2 \), and finally inserting it into (47) leads to \( A_{\text{hom}} = \frac{1}{2}I_2 \).

So, in the most simple unit-cell pattern, that was discussed in Example 4.1 the homogenized tensor is just a multiple of the identity matrix and in the homogenized model we have just the two-dimensional Laplacian scaled by a factor.

In the following two examples, we consider a unit-cell pattern with a high level of symmetry for which the homogenized tensor \( A_{\text{hom}} \) is a diagonal matrix but not just a multiple of the identity matrix \( I_2 \) or a non-diagonal matrix, respectively.

**Example 4.2:** We consider the unit-cell graph \( \Gamma_Y \) in Figure 3(b) that is composed of one vertical line and two lines that go through the center of the unit cell where the latter two intersect in an angle of \( 2\varphi \), where \( \varphi \in (0, \pi/4) \). The corresponding graph is composed of two edges of length \( \frac{1}{2} \) and four edges of length \( 1/(2\cos \varphi) \).

Its matrices \( A_I \) and \( L \) and vector \( f \) are

\[
A_I = \begin{bmatrix}
-I_2 & 0 & 0 & I_2 & 0 & 0 \\
0 & -I_2 & 0 & 0 & I_2 & 0 \\
0 & 0 & -I_2 & 0 & 0 & I_2
\end{bmatrix},
\]

\[
L = \text{diag}
\begin{bmatrix}
\frac{1}{2\cos \varphi}, \frac{1}{2\cos \varphi}, \frac{1}{2\cos \varphi}, \frac{1}{2\cos \varphi}, \frac{1}{2\cos \varphi}, \frac{1}{2\cos \varphi}
\end{bmatrix}
\otimes I_2,
\]

and \( f = (-1,0,-1,0,1,0,0,0,0,0,0,0)^T \). Again by solving first (45) and then plugging it in (45) and (47) we get one particular solution \( Q^0 \), the vector \( b \) and the homogenized tensor \( A_{\text{hom}} \):

\[
Q^0 = \frac{1}{2}
\begin{bmatrix}
e_1 \\
e_1 \\
-e_2 \\
0
\end{bmatrix}, \ b =
\begin{bmatrix}
-\cos \varphi e_1 \\
-\cos \varphi e_1 \\
\cos \varphi e_1 \\
\cos \varphi e_1
\end{bmatrix},\ A_{\text{hom}} = \frac{\cos \varphi}{\cos \varphi + 2}
\begin{bmatrix}
2 \cos \varphi & 0 \\
0 & 1
\end{bmatrix}.
\]

For all angles \( \varphi \in (0, \pi/4) \) the homogenized tensor \( A_{\text{hom}} \) is not a multiple of the identity matrix.

**Example 4.3:** We consider now the unit cell graph in Figure 3(c) for which the homogenized tensor \( A_{\text{hom}} \) is even not diagonal. It consists of 3 edges and 4 vertices with 2 of them being on the right or
upper side of unit cell. From that (and other properties of the graph) we see that

$$A_T = \begin{bmatrix} -I_2 & 0 & I_2 \\ I_2 & 0 & -I_2 \end{bmatrix}.$$  

We notice that the second edge of the graph is actually a loop in the corresponding periodic graph and consequently, it does not appear in the matrix $A_T$ since two identity matrices cancel out in matrix, on for entering and one for leaving the same vertex.

Moreover, the matrix $L$ and right-hand side $f$ are equal to

$$L = \begin{bmatrix} \sqrt{2} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} I_2 \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ e_2 \\ e_1 - e_2 \end{bmatrix}.$$  

In the same way as in the previous example, we find one particular solution $Q^0$, the vector $b$ and the homogenized tensor $A_{\text{hom}}$:

$$Q^0 = \begin{bmatrix} \frac{1}{4} (e_1 - e_2) \\ \frac{1}{4} (e_1 - e_2) \\ -
\frac{1}{4} (e_1 - e_2) \end{bmatrix}, \quad b = \begin{bmatrix} \frac{\sqrt{2}}{2} (e_1 - e_2) \\ e_2 \\ \frac{\sqrt{2}}{2} (e_1 - e_2) \end{bmatrix}, \quad A_{\text{hom}} = \frac{1}{1 + \sqrt{2}} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} + 1 \end{bmatrix},$$

where the latter is not diagonal.

5. Numerical experiments

In this section we compare the solutions of the derived homogenized model (17a) and corresponding $\delta$-problem (7a) for the three pattern introduced in Example 4.1–4.3 by numerical simulations using the numerical C++ library Concepts [28].

For this we apply first a semi-discretization in space using the finite element method with continuous and piecewise polynomial functions for both, the $\delta$-dependent problem (7a) on the mesh $\Gamma^\delta$ as well as for the homogenized problem (17a) on the domain $\Omega$. For this each edge of the mesh $\Gamma^\delta$ is subdivided into smaller edges to obtain a one-dimensional finite element mesh and the domain $\Omega$ is partitioned into non-overlapping quadrilateral cells.

Denoting the stiffness matrix of the semi-discretized $\delta$-dependent problem (7a) by $K^\delta$, the mass matrix by $M^\delta$, the right hand side vector by $F^\delta(t)$, the time-dependent solution vector (associated to $u^\delta(t, \cdot)$) by $U^\delta(t)$ and the initial vector (associated to $u_{\text{init}}$) by $U_{\text{init}}^\delta$ the semi-discrete problem reads

$$M^\delta \frac{\partial}{\partial t} U^\delta(t) + K^\delta U^\delta(t) = F^\delta(t), \quad t > 0, \quad U^\delta(0) = U_{\text{init}}^\delta.$$  

To impose the Dirichlet boundary conditions a penalization is used.

For the time-discretization of (48) we are using the Crank–Nicholson scheme that is of second order in the (uniform) time step $\Delta T$. Then, with $F_n^\delta = F^\delta(n\Delta T)$ the solution vectors $U_n^\delta = U^\delta(n\Delta T)$ at time $t_n = n\Delta T$ fulfill the linear systems

$$\left( M^\delta + \Delta T \frac{\Delta}{2} K^\delta \right) U_{n+1}^\delta = \left( M^\delta - \Delta T \frac{\Delta}{2} K^\delta \right) U_n^\delta + \Delta T \frac{\Delta}{2} M^\delta \left( F_{n+1}^\delta + F_n^\delta \right), \quad n \in \mathbb{N},$$

with the initial condition $U_0^\delta = U_{\text{init}}^\delta$.  

Discretizing the homogenized problem (17a) in the same way we obtain solution vectors $U_n^0$ at time $t_n = n\Delta T$. We project this discrete solution at each time onto the finite element mesh of $\Gamma^\delta$, where $P^\delta(U_n^0)$ is the projection of the vector $U_n^0$, to compute an approximation of the relative $L^2(\Gamma^\delta)$ error

$$E(t_n) := \sqrt{\left(\left((U_n^\delta - P^\delta(U_n^0))^T M^\delta (U_n^\delta - P^\delta(U_n^0))\right) / \left((U_n^\delta)^T M^\delta U_n^\delta\right)\right)}.$$ 

For all simulations we consider the domain $\Omega = (0,1)^2$, the coefficients $\rho c_p = a = 1$, the source term

$$f^\delta(t,x) = f(t,x,\frac{x}{\delta}) = 4 \exp(-196| x - (0.5,0.5) |^2) \exp(-3t)$$

that depends only on the macroscopic variable and, hence, $f_{\text{hom}} = f^\delta$, and the initial data $u_{\text{init}} = 0$. We found the discretization error small in comparison to the modeling error when splitting each edge of $\Gamma^\delta(n_1,n_2)$ uniformly into three (smaller) edges on which polynomials of degree 2 are used. In this way the discretization error decreases with decreasing period $\delta$. For the homogenized solution, we use a uniform mesh of 16 square cells and polynomials of degree 6. For the time discretization of the Crank-Nicholson scheme we used as time step $\Delta T = 0.002$.

In Figure 4 the discretized solution of the heat equation (1) on the mesh $\Gamma^\delta$ with $\delta = 1/16$ – for illustrative purposes on a thickened graph of thickness $1/160$ – and the homogenized solution, i.e. the solution of (17a), is illustrated for Examples 4.1–4.3 at time $t_f = 2$. For Example 4.2 the angle is $\phi = \tan^{-1}(0.5)$. Note, that in all figures in this section the same colorbar is used that scales from 0 (blue) to $1.4 \cdot 10^{-4}$ (red). For Example 4.1, for which the homogenized tensor is a multiple of the identity matrix $I_2$, we observe the temperature decays from the mid-point approximately the same.

![Figure 4](image-url)
Figure 5. Plot of the relative $L^2(\Gamma^d)$ error $E(t_f)$ with respect to the period $\delta$ for $t_f = 2$ for the three unit cell patterns in Figure 3. For Example 4.2 the angle is $\phi = \tan^{-1}(0.5)$.

Figure 6. Two pattern $\Gamma^r$ with the same homogenized tensor $A_{\text{hom}} = \frac{1}{2}I_2$ as the one of Example 4.1 (top row) and temperature distribution on the mesh $\Gamma^h$ for $\delta = 1/16$ at time $t_f = 2$ (bottom row).
way in all directions. For Example 4.2, for which the homogenized tensor is diagonal but not a multiple of $I_2$, we observe a faster decay in one axis direction. For Example 4.3, for which the homogenized tensor is not even diagonal, we observe a faster decay in another direction. In all three examples the homogenized solution is in very good agreement with the solution of the heat equation on the mesh $\Gamma^\delta$.

In Figure 5 the (approximative) relative $L^2(\Gamma^\delta)$-error $\mathcal{E}(t_\delta)$ is shown as a function of the period $\delta$. We observe a linear convergence to 0 for Example 4.2 and Example 4.3. Moreover for Example 4.1 the limit solution shows a quadratic convergence in $\delta$. This goes along with the formal asymptotic expansion which for the geometry in this example gives that the first order corrector is zero.

For the curiosity of the reader we show two patterns in Figure 6 that extended in all direction represent the same mesh as the mesh of pattern in Example 4.1, cf. Figure 3(a). One can easily verify that for the two patterns the homogenized tensor equals $\frac{1}{2}I_2$ as for the one of Example 4.1, i.e. the solution on the mesh $\Gamma^\delta$ has macroscopically at leading order the same behavior. This we observe in the numerical experiments (cf. Figure 4(a) and Figure 6).

**Acknowledgments**

The authors would like to thank Luka Grubišić (University of Zagreb), Herbert Egger (TU Darmstadt) and Andro Mikelić (University of Lyon 1) for fruitful discussions.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

The authors gratefully acknowledge the financial support of their research cooperation ‘Asymptotic and algebraic analysis of nonlinear eigenvalue problems for biomechanical and photonic devices’ through the bilateral program ‘Procope’ between the German Academic Exchange Service (DAAD) based on funding of the German Federal Ministry of Education and Research [project ID 57334847] and the Croatian Ministry of Science (contract number 910-08/16-01/00209). The research was partly conducted during the stay of the first author at the Graduate School Computational Engineering (CE) at the Technical University (TU) Darmstadt and he is grateful to the support and hospitality. Work of the first and the last author has been supported by the [grant number HRZZ 2735] of the Croatian Science Foundation and the COST Action: CA18232 - Mathematical models for interacting dynamics on networks.

**ORCID**

Matko Ljulj http://orcid.org/0000-0003-1831-7228
Kersten Schmidt http://orcid.org/0000-0001-7729-6960
Josip Tambača http://orcid.org/0000-0003-1792-6056

**References**

[1] Kuchment P. Graph models for waves in thin structures. Waves Random Media. 2002;12(4):R1–R24.
[2] Rubinstein J, Schatzman M. Variational problems on multiply connected thin strips. I. Basic estimates and convergence of the Laplace spectrum. Arch Ration Mech Anal. 2001;160(4):271–308.
[3] Marušić S, Marušić-Paloka E. Two-scale convergence for thin domains and its applications to some lower-dimensional models in fluid mechanics. Asymptot Anal. 2000;23(1):23–57.
[4] Joly P, Semin A. Construction and analysis of improved Kirchoff conditions for acoustic wave propagation in a junction of thin slots. ESAIM Proc. 2008;25:44–67.
[5] Bouchitté G, Butazzo G, Seppecher P. Energies with respect to a measure and applications to low dimensional structures. Calc Var Partial Differ Equ. 1997;5:37–54.
[6] Bouchitté G, Fragalà I. Homogenization of thin structures by two-scale method with respect to measures. SIAM J Math Anal. 2001;32(6):1198–1226.
[7] Bouchitté G, Fragalà I. Homogenization of elastic thin structures: a measure-fattening approach. Special issue on optimization (Montpellier, 2000). J Convex Anal. 2002;9(2):339–362.
[8] Bouchitté G, Fragalà I, Rajesh M. Homogenization of second order energies on periodic thin structures. Calc Var Partial Differ Equ. 2004;20(2):175–211.

[9] Zhikov VV. On an extension of the method of two-scale convergence and its applications. Math Sb. 2000;191:973–1014.

[10] Neuss-Radu M. Some extensions of two-scale convergence. C R Acad Sci Paris Sér I Math. 1996;322(9):899–904.

[11] Radu M. Homogenization techniques [Diploma thesis], Cluj-Napoca/Romania: University Heidelberg/Germany; 1992.

[12] Allaire G, Damlamian A, Hornung U. Two-scale convergence on periodic surfaces and applications. Proceedings of the International Conference on Mathematical Modelling of Flow through Porous Media; Singapore: World Scientific Publication; 1996. p. 15–25.

[13] Le Dret H, Raoult A. Homogenization of hexagonal lattices. Netw Heterog Media. 2013;8(2):541–572.

[14] Zhikov VV. Averaging of problems in the theory of elasticity on singular structures. (Russian). Izv Ross Akad Nauk Ser Mat. 2002;66(2):81–148. translation in Izv Math 2002;66(2):299–365.

[15] Maz'ya VG, Slutskii AS. Averaging of a differential operator on a fine periodic curvilinear net. Math Nachr. 1987;133:107–133. (in Russian).

[16] Maz'ya VG, Slutskij AS. Averaging of differential equations on a fine mesh. Sov Math Dokl. 1987;35:371–375.

[17] Cioranescu D, Donato P. An introduction to homogenization. New York: The Clarendon Press, Oxford University Press; 1999. (Oxford Lecture Series in Mathematics and its Applications; 17).

[18] Panasenko G. Multi-scale modelling for structures and composites. Dordrecht: Springer; 2005.

[19] Panasenko GP. Averaging processes in framework structures. Math USSR-Sb. 1983;122(2):220–231. (in Russian). English translation in Math.USSR Sbornik, 1985;50(1):213–225.

[20] Cioranescu D, Saint-Jean-Paulin J. Homogenization of reticulated structures. New-York: Springer-Verlag; 1999.

[21] Bondy JA, Murty USR. Graph theory with applications. New York: American Elsevier Publishing Co. Inc.; 1976.

[22] Cioabă SM, Ram Murty M. A first course in graph theory and combinatorics. New Delhi: Hindustan Book Agency; 2009. (Texts and Readings in Mathematics; vol. 55).

[23] Dautray R, Lions J-L. Mathematical analysis and numerical methods for science and technology. Berlin: Springer-Verlag; 1992. (Evolution problems. I; vol. 5).

[24] Evans LC. Partial differential equations. Providence: American Mathematical Society; 1998.

[25] Tambaca J, Žugec B. A biodegradable elastic stent model. Math Mech Solids. 2019;24(8):2591–2618.

[26] Shafarevich IR, Remizov AO. Linear algebra and geometry. Berlin: Springer Science & Business Media; 2012.

[27] Jungnickel D. Graphs, networks and algorithms. New York (NY): Springer; 2011. (Algorithms and Computation in Mathematics).

[28] Fraunfelder P., Lage C. Concepts – an object-oriented software package for partial differential equations. ESAIM: Math Model Numer Anal. 2002;36(5):937–951.