COUNTABLY COMPACT GROUPS HAVING MINIMAL INFINITE POWERS

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Abstract. We answer the question, raised more than thirty years ago in [10, 12], on whether the power $G^\omega$ of a countably compact minimal Abelian group $G$ is minimal, by showing that the negative answer is equivalent to the existence of measurable cardinals. The proof is carried out in the larger class of sequentially complete groups. We characterize the sequentially complete minimal Abelian groups $G$ such that $G^\omega$ is minimal – these are exactly those $G$ that contain the connected component of their completion. This naturally leads to the next step, namely, a better understanding of the structure of the sequentially complete minimal Abelian groups, and in particular, their connected components which turns out to depend on the existence of Ulam measurable cardinals. More specifically, all connected sequentially complete minimal Abelian groups are compact, if Ulam measurable cardinals do not exist. On the other hand, for every Ulam measurable cardinal $\sigma$ we build a non-compact torsion-free connected minimal $\omega$-bounded Abelian group of weight $\sigma$, thereby showing that the Ulam measurable cardinals are precisely the weights of non-compact sequentially complete connected minimal Abelian groups.

1. Introduction

We denote by $\mathbb{N}$ the set of naturals $\{0, 1, \ldots\}$ (and $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\}$), by $\mathbb{P}$ the set all primes, by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, by $\mathbb{T}$ the unit circle group in $\mathbb{C}$, by $\mathbb{Z}_p$ the $p$-adic integers ($p \in \mathbb{P}$), by $\mathbb{Z}(n)$ the cyclic group of order $n$ ($n \in \mathbb{N}^*$). The cardinality of continuum $2^\omega$ will be denoted also by $\mathfrak{c}$.

All groups considered in this paper (except in §3) will be Abelian, so additive notation will be always used. We denote by $\widehat{G}$ the completion of a topological group $G$ (we do not have to worry about left, right, and two-sided uniformities, since our groups are Abelian), while $c(G)$ denotes the connected component of the neutral element of a group $G$ (for brevity, we call $c(G)$ the connected component of $G$).

1.1. Minimal groups and their products. Following Stephenson [27], call a Hausdorff topological group $(G, \tau)$ minimal if the topology $\tau$ is a minimal element of the partially ordered, with respect to inclusion, set of Hausdorff group topologies.

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on $G$. Compact groups are obviously minimal, the first examples of non-compact minimal groups were given in [27, 17], while the first examples of non-compact countably compact minimal groups were given in [5] (see also [10, 11, 12, 13]).

The problem of preservation of minimality under products (the counterpart of Tychonoff product theorem for compactness) was raised by G. Choquet at the ICM in Nice in 1970. The first counter-example was given in [17]. Since then this problem became a central one in the framework of minimal groups [17, 5, 8, 9, 12, 15, 18, 28, 29]. A general criterion for minimality of arbitrary products of minimal Abelian groups was produced in [5, 9] (see also [9, Theorem 2.4] and [8, Theorem 6.2.3]). The minimality of arbitrary powers of an Abelian group $G$ was characterized earlier by Stoyanov [29], who proved that it is equivalent to the minimality of $G^c$ (see Theorem 2.5 for a more detailed description). Since minimality is preserved by taking direct summands, the minimality of some power $G^\kappa$ implies the minimality of all smaller powers.

It is natural to expect that adding some additional compact-like properties may improve productivity of minimal groups. Indeed, all finite powers of a countably compact minimal group are minimal [12]. On the other hand, there exist pseudo-compact minimal Abelian groups $G$ with $G^{\omega_1}$ non-minimal, but $G^\omega$ minimal (for more detail see [6, 12]).

The choice of compact-like property to match with minimality in this paper is sequential completeness, since it simultaneously generalizes two fundamental properties like countable compactness and completeness. A topological group $G$ is sequentially complete if $G$ is sequentially closed in $\tilde{G}$ (or, equivalently, when every Cauchy sequence in $G$ converges) [13, 14]. We prove that all finite powers of a sequentially complete minimal group are minimal (this is a corollary of a more general fact, see Lemma 2.7). This suggests the following question, set in [10, Question 9] and [12, Question 1.10] in the case of countably compact groups which still remains open since then:

**Question 1.1.** Let $G$ be a sequentially complete minimal Abelian group. Is $G^\omega$ minimal?

We shall see below that this question, as well as the question in [12] regarding countably compact groups, cannot be answered in ZFC.

In order to answer Question 1.1 we characterize first the sequentially complete Abelian groups $G$ such that $G^\omega$ is minimal (i.e., those providing a positive answer to Question 1.1):}

**Theorem A.** For a sequentially complete minimal Abelian group $G$ the following are equivalent:

(a) all powers of $G$ are minimal;

(b) $G^\omega$ is minimal.

(c) $G$ contains $c(\tilde{G})$; hence, $c(G) = c(\tilde{G})$ is compact.

The proof of this theorem is given in §2.2, where we prove it first in the totally disconnected case, when $G^\omega$ is always minimal (Corollary 2.14), i.e., Question 1.1 has positive answer for totally disconnected $G$. The equivalence of (b) and a weaker form of (c) was established in [15, Main Theorem] in the much smaller class of $\omega$-bounded minimal Abelian groups.
On the other hand, we shall see in Corollary 1.6 that the assertion “all powers of a countably compact minimal Abelian groups are minimal” is equivalent to the non-existence of Ulam-measurable cardinals. Let us recall that a cardinal $|X|$ is Ulam-measurable (resp., measurable) if there exists an $\omega_1$-complete (resp., $|X|$-complete) free ultrafilter on $X$ (a filter on a set $X$ is $\kappa$-complete if it is closed under intersections of families of cardinality $< \kappa$). A cardinal is Ulam-measurable if and only if it is greater than or equal to the first measurable cardinal. The assumption that there exist no Ulam-measurable cardinals is known to be consistent with ZFC, while their existence implies consistency of ZFC and hence no proof of that existence is to be expected [22].

Uncountable measurable cardinals are Ulam-measurable, while the least Ulam-measurable cardinal is also measurable.

For the connected case Theorem A gives the following striking consequence characterizing the minimality of $G^\omega$ by compactness of $G$:

**Corollary 1.2.** A connected sequentially complete minimal Abelian group $G$ is compact if and only if $G^\omega$ is minimal.

This corollary shows that in order to positively answer Question 1.1 in the connected case, we have to check whether the connected sequentially complete minimal Abelian group $G$ are compact. This issue turns out to be non-trivial, we shall face it in §1.2 which mainly deals with this aspect of the structure of sequentially complete minimal Abelian groups (see Question 1.4).

Here is another application of Theorem A. The general criterion for minimality of arbitrary products of Abelian groups from [8, Theorem 6.2.3], simplifies in the case of sequentially complete group (due to Lemma 2.7), so that applying the criterion from [8, Corollary 6.2.8] one gets the following corollary (for a proof see §2.2).

**Corollary 1.3.** Let $\{G_i\}_{i \in I}$ be a family of sequentially complete Abelian groups. Then $G = \prod_{i \in I} G_i$ is minimal if and only if every countable subproduct is minimal. In such a case there exists a co-finite subset $J \subseteq I$ such that all groups $G_i^\omega$, $i \in J$, are minimal.

Corollary 1.2 implies that if the groups $G_i$ in the above theorem are also connected, then $G$ is minimal if and only if all but finitely many of the groups $G_i$ are compact.

According to [22, Theorem 1.5] the last part of Corollary 1.3 cannot be inverted: there exists a family $\{G_n : n \in \mathbb{N}\}$ of $\omega$-bounded Abelian groups such that all powers $G_n^\lambda$ are minimal, but $\prod_{n=0}^{\infty} G_n$ is not minimal.

1.2. **The structure of sequentially complete minimal groups.** Ulam-measurable cardinals already appeared in the context of comparison between compactness and countable compactness [1], [4], [32]. Relaxing countable compactness to sequential completeness and adding minimality to the mix we consider the following

**Question 1.4.** Is every connected sequentially complete minimal Abelian group $G$ compact?

Connectedness cannot be omitted here, since there exist minimal $\omega$-bounded proper dense subgroups of $\mathbb{Z}_{\omega}^\omega$ [11] (they are obviously totally disconnected).

According to Corollary 1.2 a positive answer to Question 1.1 is equivalent to a positive answer to Question 1.4.
It turns out that the answer to Question 1.4 depends on the existence of measurable cardinals, hence Question 1.4 and Question 1.1 cannot be answered in ZFC.

**Theorem B.** Let $\alpha \geq \omega$ be a cardinal. Then the following conditions are equivalent:

(a) $\alpha$ is Ulam-measurable;

(b) there exists a non-compact connected, sequentially complete, minimal Abelian group $G$ of weight $\alpha$;

(c) there exists a non-compact connected, $\omega$-bounded, minimal torsion-free Abelian group $G$ of weight $\alpha$.

Moreover, if $\alpha$ is the least Ulam-measurable cardinal, then there exists a connected non-compact minimal torsion-free Abelian group $G$ of weight $\alpha$ that is $\beta$-bounded for every $\beta < \alpha$.

The proof of this theorem is given in §2.2. Since the least Ulam-measurable cardinal is a measurable cardinal, we prove a slightly more general version of the last assertion of the theorem. Namely, for every measurable cardinal $\alpha$ there exists a connected non-compact minimal torsion-free Abelian group $G$ of weight $\alpha$ that is $\beta$-bounded for every $\beta < \alpha$. It should be noted here that the last property is very close to compactness, since a space $X$ that is $w(X)$-bounded must be compact.

On the other hand, the implication (b) $\to$ (a) of the above theorem follows from the fact that a connected sequentially complete minimal Abelian group of non-measurable size is compact due to item (b) of the following:

**Theorem C.** Let $G$ be a sequentially complete minimal Abelian group.

(a) If $c(G)$ is compact, then $c(G) = c(\tilde{G})$ is compact and $G/c(G)$ is sequentially complete and minimal.

(b) If $w(c(G))$ is not Ulam-measurable, then $c(G)$ is compact.

In particular, a sequentially complete Abelian group $G$ with non Ulam-measurable $w(c(G))$ is minimal if and only if $c(G)$ is compact and $G/c(G)$ is minimal (the sufficiency uses the three space property, see Proposition 2.8 or [18]). This reduces the study of countably compact minimal Abelian groups with small connected component to that of totally disconnected groups.

Now we come back to Question 1.1. First we give a sufficient condition for a positive answer:

**Corollary 1.5.** Let $G$ be a sequentially complete minimal Abelian group such that $w(c(G))$ is not Ulam measurable. Then $G^\omega$ is minimal.

Indeed, the hypotheses of the corollary imply that $c(G) = c(\tilde{G})$ is compact (in view of Theorem C(b)), so $G^\omega$ is minimal, by Theorem A. This proves item (a) of the next corollary, in view of Theorem B. Actually, the next corollary shows that Question 1.1 cannot be answered neither for countably compact groups, nor for sequentially complete groups.

**Corollary 1.6.** (a) Under the assumption that there exist no Ulam-measurable cardinals, all powers of a sequentially complete minimal Abelian group are minimal, i.e., Question 1.1 has a positive answer.

(b) Under the assumption that there exist Ulam-measurable cardinals, there exists an $\omega$-bounded minimal Abelian group $G$ such that $G^\omega$ is not minimal, Question 1.1 has a negative answer.
Item (b) follows from item (b) of Theorem B, which ensures the existence of a non-compact connected, sequentially complete, minimal Abelian group $G$ of weight $\alpha$, whenever $\alpha$ is an Ulam-measurable cardinal. Then $G^{\omega}$ is not minimal, according to Corollary 1.2.

Finally, we offer a description of the sequentially complete Abelian group $G$ such that $G^{\omega}$ is minimal, i.e., those satisfying the equivalent conditions of Theorem A. They turn out to be extensions of a compact group by a minimal Abelian group that is a direct product of bounded minimal $p$-groups.

**Theorem D.** Let $G$ be a sequentially complete Abelian group such that $G^{\omega}$ is minimal. Then there exists a compact subgroup $N$ of $G$ such that $G/N \cong \prod_{p \in P} B_p$, where $B_p$ is a bounded $p$-torsion minimal group for every prime $p$.

This theorem is proved in §2.2. The subgroup $N$ is “large”, it contains the compact subgroup $c(G)$ and much more (for a stronger and more precise form of the theorem see Theorem 2.17).

The paper is organized as follows. In §2.1 we provide background on minimality and sequential completeness. In particular, we recall two criteria for minimality of dense subgroups of compact Abelian groups and we give a factorization theorem for totally disconnected sequentially complete minimal Abelian group (Theorem 2.10). Section 2.2 contains the proofs of Theorems A, B, C, D and Corollary 1.3. A key point of §2.2 is Lemma 2.15 which is the clue to the proofs of Theorems A, C and D.

**Notation and terminology.** Let $G$ be a group and $A$ be a subset of $G$. We denote by $\langle A \rangle$ the subgroup of $G$ generated by $A$. The group $G$ is **divisible** if for every $g \in G$ and $n \in \mathbb{N}^*$ the equation $nx = g$ has a solution in $G$. For an Abelian group $G$ we set $G[n] = \{x \in G : nx = 0\}$, for $n \in \mathbb{N}^*$, and $\text{Soc}(G) = \bigoplus_{p \in P} G[p]$.

We recall some notions of compactness-like conditions in topological groups and spaces. A Tychonoff space $X$ is **pseudocompact** if every continuous real-valued function on $G$ is bounded, and **countably compact** if every countable open cover has a finite subcover (equivalently, every sequence in $X$ has a cluster point). For an infinite cardinal $\alpha$, a group $G$ is **$\alpha$-bounded** if every subset of cardinality $\leq \alpha$ of $G$ is contained in a compact subgroup of $G$ (some authors use the term $\omega$-bounded for a different property). Obviously, $\omega$-boundedness implies countable compactness, and countable compactness implies pseudocompactness.

For undefined symbols or notions see [8, 19, 20].

2. **Proofs of the main results**

2.1. **Background on minimality and sequential completeness.** A topological group $G$ is **precompact** if its completion $\tilde{G}$ is compact (or, equivalently, if for any open $U \neq \emptyset$ in $G$ there is a finite subset $F \subseteq G$ such that $F + U = G$). Pseudo-compact groups are precompact. The following fundamental theorem of Prodanov and Stoyanov [25] (see also [8, Theorem 2.7.7] for an alternative proof) says that the minimal Abelian groups are precisely the (dense) subgroups of the compact Abelian groups:

**Theorem 2.1.** [25] Every minimal Abelian group is precompact.
A subgroup $H$ of a topological Abelian group $G$ is \textit{essential} if every non-trivial closed subgroup $N$ of $G$ meets $H$ non-trivially. The following criterion for minimality of dense subgroups, given by Stephenson \cite{27} and Prodanov \cite{24}, describes the minimal Abelian groups as the dense essential subgroups of the compact Abelian groups:

\textbf{Theorem 2.2.} Let $G$ be a compact Abelian group and $H$ be a dense subgroup of $G$. Then $H$ is minimal if and only if $H$ is essential in $G$.

Actually, it is possible to carry out the test of essentiality with only closed subgroups $N$ of $G$ that are either cyclic $p$-group or copies of $\mathbb{Z}$, with $p \in \mathbb{P}$:

\textbf{Lemma 2.3.} \cite{8} Theorem 4.3.7 \ Let $G$ be a compact Abelian group. A dense subgroup $H$ of $G$ is essential if and only if $\text{Soc}(G) \leq H$ and for every prime $p$ the subgroup $H$ non-trivially meets every subgroup $N \leq G$ with $N \cong \mathbb{Z}_p$.

The reduction in the above lemma is based on a localization technique invented by Stoyanov \cite{28} (see also \cite{8} Chapter 4) that we recall now. For a prime $p$ an element $x$ of a topological Abelian group $G$ is called \textit{quasi-$p$-torsion} if either $x$ is $p$-torsion or $\langle x \rangle$ is isomorphic to $\mathbb{Z}$ equipped with the $p$-adic topology. The set $\text{td}_p(G)$ of all quasi-$p$-torsion elements of $G$ is a subgroup of $G$. In these terms, Lemma 2.3 ensures that $H$ is essential in $G$ if and only if $\text{td}_p(H)$ is essential in $\text{td}_p(G)$ for every $p$ (see \cite{8} Theorem 4.3.7).

Here we collect some properties of the subgroup $\text{td}_p(-)$ that will be used in the sequel, for further properties see \cite{8} Chapter 4],

\textbf{Fact 2.4.} \cite{8} \begin{enumerate}
\item Let $f : H \to G$ be a continuous homomorphisms of topological Abelian groups. Then for every prime $p$:
\begin{enumerate}
\item $f(\text{td}_p(H)) \leq \text{td}_p(G)$, if $H$ is compact and $f$ is surjective, then $f(\text{td}_p(H)) = \text{td}_p(G)$;
\item if $f$ is an embedding, then $f(\text{td}_p(H)) = \text{td}_p(G) \cap f(H)$.
\end{enumerate}
\item If $\{G_i : i \in I\}$ is a family of topological Abelian groups, then $\text{td}_p(\prod_{i \in I} G_i) = \prod_{i \in I} \text{td}_p(G_i)$.
\item If $G$ is a totally disconnected compact Abelian group, then for every $p \in \mathbb{P}$ the subgroup $\text{td}_p(G)$ of $G$ is closed and $G \cong \prod_{p \in \mathbb{P}} \text{td}_p(G)$ topologically.
\end{enumerate}

Following \cite{28}, call a topological Abelian group $G$ \textit{strongly $p$-dense} if there exists $k \in \mathbb{N}^*$ such that $p^k \text{td}_p(G) \subseteq G$. This technical property turns out to be the key towards complete understanding of minimality of powers:

\textbf{Theorem 2.5.} \cite{28} Theorem 3.5 \ For a minimal Abelian group $G$ the following are equivalent:

\begin{enumerate}
\item all powers of $G$ are minimal;
\item $G^c$ is minimal;
\item $G$ is strongly $p$-dense for every prime $p$.
\end{enumerate}

For the proof of Theorem A we need the following lemma:

\textbf{Lemma 2.6.} If a sequentially complete Abelian group $G$ is strongly $p$-dense for some prime $p$, then $G \supseteq c(G)$.
Proof. In fact $c(\tilde{G})$ is connected hence divisible, therefore $td_p(c(\tilde{G}))$ is divisible as well. Now strong $p$-density implies $td_p(c(\tilde{G})) \subseteq G$. Since $td_p(c(\tilde{G}))$ is sequentially dense in $c(\tilde{G})$ by \[13\] Theorem 2.9(a)], it follows also that $G \cap c(\tilde{G})$ is sequentially dense in $c(\tilde{G})$. Now the sequential completeness of $G$ yields $G \supseteq c(\tilde{G})$. □

The next lemma resolves the problem of minimality for finite products of sequentially complete minimal groups:

Lemma 2.7. Let $G$ be a sequentially complete minimal group. Then $G \times H$ is minimal for every minimal group $H$. In particular, any finite product of sequentially complete minimal groups is minimal.

Proof. Fix an arbitrary prime $p$ and pick $x \in td_p(G)$. It suffices to check that $x$ is contained in a compact subgroup of $G$, then \[8\] Proposition 6.1.13 (see also \[9\] Proposition 4.1) applies. The case when $C = \langle x \rangle$ is finite is trivial. Assume that $C$ is infinite. Then $C$ is metizable, being isomorphic to $\mathbb{Z}$ equipped with the $p$-adic topology. Hence its compact closure $\overline{C}$ in $\tilde{G}$ is contained in $G$, since $G$ is sequentially complete. □

The minimality is not preserved by taking quotients (even quotients of minimal group with respect to finite subgroups may fail to be minimal). In item (b) of the next lemma one can find an instance when minimality is preserved by taking quotients. In item (a) we recall a “three space property” from \[18\].

Proposition 2.8. Let $G$ be a topological Abelian group and let $K$ be a compact subgroup of $G$. Then

(a) \[18\] if $G/K$ is minimal, then $G$ is minimal;
(b) if $G$ is minimal and $K$ is connected, then $G/K$ is minimal.

Proof. Let us only mention that (a) was proved in \[18\] without the assumption that $G$ is Abelian and compactness of $K$ is relaxed to the conjunction of minimality and completeness.

(b) First we note that $G/K$ is precompact, by Theorem 2.1 Since $K$ is compact, this implies that $G$ is precompact too, so $G$ is compact and $G/K \cong \hat{G}/K$. In order to apply Lemma 2.3 and Theorem 2.1 we need to check first that $\text{Soc}(\hat{G}/K) \leq G/K$. Pick $a' = a + K \in \text{Soc}(\hat{G}/K)$. Then $pa \in K$. Since $K$ is a connected compact Abelian group, $K$ is divisible. Hence, $pa = pb$, for some $b \in K$. Then $t = a - b \in \text{Soc}(\hat{G}) \leq G$, by Lemma 2.3 and Theorem 2.1 since $G$ is minimal. Therefore, $a' = t + b + K = t + K \in G/K$. In order to conclude with the application of Lemma 2.3, consider a subgroup $\mathbb{Z}_p \cong \mathbb{N}^* \leq \hat{G}/K$ and put $q(N_1) = N$, where $q : \hat{G} \rightarrow \hat{G}/K$ is the canonical homomorphism. In order to see that the short exact sequence $0 \rightarrow K \rightarrow L \rightarrow N \rightarrow 0$ splits, consider its dual short exact sequence $0 \rightarrow \hat{N} \rightarrow \hat{N}_1 \rightarrow \hat{K} \rightarrow 0$ which splits, due to the divisibility of $\hat{N} \cong \mathbb{Z}(p^\infty)$. Hence, $L \cong K \times N_1$, where $N_1 \cong \mathbb{Z}_p$ is a closed subgroup of $L$ with $q(N_1) = N$, so clearly the restriction $q|_{N_1} : N_1 \rightarrow N$ is an isomorphism. Again by Lemma 2.3 and Theorem 2.1 applied to the minimal group $G$, $N_1 \cap G \neq \{0\}$. Since, $q|_{N_1}$ is an isomorphism, this implies $N \cap (G/K) \neq \{0\}$. □

Corollary 2.9. Let $G$ be a topological Abelian group such that $c(G)$ is compact. Then $G$ is minimal if and only if $G/c(G)$ is minimal.
Now we prove that the property (3) from Fact 2.1 can be extended to sequentially complete minimal groups:

**Theorem 2.10.** Let $G$ be a sequentially complete minimal totally disconnected Abelian group. Then for every $p \in \mathbb{P}$ the subgroup $td_p(G)$ of $G$ is closed and $G \cong \prod_{p \in \mathbb{P}} td_p(G)$ topologically.

**Proof.** The completion $\tilde{G}$ of $G$ is compact by Prodanov-Stoyanov’s Theorem 2.1. Moreover, $\tilde{G}$ is totally disconnected by Theorem C (its proof does not depend on the proof of the current theorem). Hence, $\tilde{G} = \prod_{p} K_p$, where $K_p = td_p(\tilde{G})$ for every $p \in \mathbb{P}$, by Fact 2.4(3). Hence, for each $p \in \mathbb{P}$ the subgroup $td_p(G) = G \cap K_p$ of $G$ is closed, by items (1b) and (3) of Fact 2.4. Since $G$ is sequentially complete, to prove that $P := \prod_{p \in \mathbb{P}} td_p(G) \leq G$ it suffices to check that the subgroup $S = \bigoplus_{p \in \mathbb{P}} td_p(G)$ of $G$ is sequentially dense in $P$. In fact, let $x = (x_p) \in P$. Then obviously $x \in \prod_{p \in \mathbb{P}} \langle x_p \rangle$. As this subgroup of $P$ is metrizable, $x \in \overline{\bigoplus_{p \in \mathbb{P}} \langle x_p \rangle}$ and $\bigoplus_{p \in \mathbb{P}} \langle x_p \rangle \subseteq S$, this proves the sequential density of $S$ in $P$.

In order to prove the desired inclusion $G \leq P$ it suffices to see that the canonical projection $\pi_p : \prod_{q \in \mathbb{P}} K_q \to K_p$ satisfies $\pi_p(G) \leq td_p(G)$, where we consider $\pi_p(G)$, as well as $K_p = \pi_p(K)$, as subgroups of the product $K$. Since, $td_p(G) = G \cap K_p$ and obviously $\pi_p(G) \leq K_p$, it suffices to check that $\pi_p(G) \leq G$.

To prove this assertion take $x = (x_q)_{q \in \mathbb{P}} \in G$, fix a prime $p$ and let $p_1, \ldots, p_n$ be an enumeration of the set $\mathbb{P} \setminus \{p\}$. Then we can consider $y = \pi_p(x)$ as the element of the product, such that $\pi_p(y) = x_p$, while $\pi_{p_q}(y) = 0$ for each $q \in \mathbb{N}^*$. We have to prove that $y \in G$. By the Chinese theorem of remainders that there exists a sequence $\{k_n\}$ in $\mathbb{Z}$ such that

$$k_n \equiv 1 \pmod{p^n} \quad \text{and} \quad k_n \equiv 0 \pmod{(p_1 \cdots p_n)^n}$$

for each $n \in \mathbb{N}^*$. Then $k_n \to 1$ in the $p$-adic topology of $\mathbb{Z}$ and $k_n \to 0$ in the $p_m$-adic topology of $\mathbb{Z}$ for each $m \in \mathbb{N}^*$. This yields

$$\lim_n k_n x_p = x_p \quad \text{in} \quad A_p \quad \text{and} \quad \lim_n k_n x_{p_m} = 0 \quad \text{in} \quad A_{p_m} \quad \text{for each} \quad m \in \mathbb{N}^*$$

since the topology of $A_q$ is coarser than the $q$-adic topology of $A_q$ for every $q \in \mathbb{P}$. Since $G$ is sequentially complete, $\lim k_n x \in G$. On the other hand, (2.1) entails $y = \lim k_n x \in G$.

We denote by $\mathbb{K}$ the (compact) Pontryagin dual of the discrete group $\mathbb{Q}$, i.e., $\mathbb{K}$ is the group of all homomorphisms $\mathbb{Q} \to \mathbb{T}$ equipped with the topology of pointwise convergence. Since the Pontryagin dual of a compact torsion-free Abelian group is a discrete divisible Abelian group, every compact torsion-free Abelian group is a product of copies of $\mathbb{K}$ and the groups $\mathbb{Z}_p$. Hence, the next proposition reduces various problems related to minimal Abelian groups $G$ to the case of dense minimal subgroups of the powers $\mathbb{K}^\alpha$ or $\mathbb{Z}_p^\alpha$, where $\alpha = w(G)$.

**Proposition 2.11.** Let $G$ be a minimal Abelian group. Then there exists a compact torsion-free Abelian group $K$ with $w(K) = w(G)$, a dense minimal subgroup $G_1$ of $K$ and a compact totally disconnected subgroup $N$ of $G_1$ such that:

(a) the quotient group $G_1/N$ is isomorphic to $G$ and consequently $K/N \cong \tilde{G}$;
(b) for every $p \in \mathbb{P}$ $G$ is strongly $p$-dense if and only if $G_1$ is strongly $p$-dense;
(c) $G_1$ is countably compact (resp., sequentially complete, $\omega$-bounded, totally disconnected) if $G$ is countably compact (resp., sequentially complete, $\omega$-bounded, totally disconnected).

In case $G$ is a minimal quasi $p$-torsion group for some prime $p$, then $K = \mathbb{Z}_p^w(G)$.

**Proof.** (a) and the part of (c) about sequential completeness are proved in [13, Proposition 2.7]. The remaining part of (c) concerning total disconnectedness, countable compactness and $\omega$-boundedness is clear.

To check (b) denote by $q : K \rightarrow \tilde{G}$ the quotient map. Since $q(td_p(K) = td_p(\tilde{G})$ ([8]), strong $p$-density of $G_1$ provides $k \in \mathbb{N}$ with $p^k td_p(K) \leq G_1$, which obviously implies strong $p$-density of $G$. If $G$ is strongly $p$-dense, say $p^k td_p(\tilde{G}) \leq G$, then for $x \in td_p(G_1)$ one has $q(x) \in td_p(\tilde{G})$, so $p^k q(x) = q(p^k x) \in G$. Therefore, $p^k x \in G_1$. □

**Corollary 2.12.** If $G$ is a sequentially complete minimal Abelian group with $c(G) = c(\tilde{G})$, then $G/c(G)$ is sequentially complete and minimal.

**Proof.** By Proposition 2.11 there exists a compact torsion-free Abelian group $K$ with $w(K) = w(G)$, a dense sequentially complete minimal subgroup $G_1$ of $K$ and a compact totally disconnected subgroup $N$ of $G_1$ such that $G_1/N \cong G$. Let $f : K \rightarrow K/N \cong \tilde{G}$ be the canonical projection. Since $f$ is surjective, we have $f(c(K)) = c(\tilde{G})$. Since $c(G) = c(\tilde{G})$, $G_1$ contains $c(K)$. Moreover, as $K$ is torsion-free, one has $K = c(K) \times D$, where $D$ is a compact totally disconnected subgroup of $K$. Then $G_1 = c(K) \times G_2$, where $G_2 = D \cap G_1$ is a closed subgroup of $G_1$. Hence $G_2$ is sequentially complete and minimal. As $G/c(G) \cong G_1/c(K) = (c(K) \times G_2)/c(K) \cong G_2$, we deduce that $G/c(G)$ is sequentially complete and minimal. □

Let us note that as far as only minimality of $G/c(G)$ is concerned, Corollary 2.9 applies, so that only compactness of $c(G)$ suffices to this end (so the equality $c(G) = c(\tilde{G})$ is not needed).

2.2. **Proof of Theorems A, B, C, D and Corollary 1.3.** The major step towards the proof of Theorem A concerns the totally disconnected case. The following proposition and its immediate corollary go in this direction:

**Proposition 2.13.** A totally disconnected sequentially complete minimal Abelian group is strongly $p$-dense for every prime $p$.

Applying Theorem 2.5 one obtains:

**Corollary 2.14.** All powers of a totally disconnected sequentially complete minimal Abelian group are minimal.

We split the proof of Proposition 2.13 in several steps.

The next lemma is a local version of Proposition 2.13 and provides an essential tool for the proof of Theorem C(a) as well. In the case of subgroups of $\mathbb{Z}_p^w$ (which is the key to the proof in the general case of the lemma), it was proved in [15, Main Lemma] for $\omega$-bounded groups $G$ by means of a specific functorial correspondence, based on Pontryagin duality, between precompact groups covered by their compact subgroups and linearly topologized groups introduced in [30]. This duality technique essentially used the assumption of $\omega$-boundedness and cannot work...
even under the assumption of countable compactness. The proof given here in the sequentially complete case is completely different, it has purely topological nature.

**Lemma 2.15.** Let \( p \) be a prime number. If \( G \) is a dense sequentially complete minimal subgroup \( G \) of a pro-\( p \)-group \( A \), then there exists \( k \in \mathbb{N} \) such that \( p^k A \subseteq G \).

**Proof.** We first show that the proof can be reduced to the case when \( A = \mathbb{Z}_p^\omega \). By Proposition 2.11 A is a quotient of \( B = \mathbb{Z}_p^\omega \), with \( \alpha = w(A) \) and \( G_1 = f^{-1}(G) \) is a sequentially complete minimal subgroup of \( B \). If there exists \( k \in \mathbb{N}^* \) such that \( p^k B \subseteq G_1 \), then clearly, \( p^k A \subseteq G \).

Obviously, \( A = \mathbb{Z}_p^\omega \) is a \( \mathbb{Z}_p \)-module. Moreover, since for \( a \in G \) the cyclic \( \mathbb{Z}_p \)-submodule \( \mathbb{Z}_p a \cong \mathbb{Z}_p \) is compact metrizable, so it is contained in the sequentially complete subgroup \( G \). Hence, \( G \) is a submodule of \( A \). By the minimality of \( G \) and Theorem 2.2 for \( 0 \neq x \in \mathbb{Z}_p \) the cyclic \( \mathbb{Z}_p \)-submodule \( \mathbb{Z}_p x = (x) \) non-trivially meets \( G \), hence \( A/G \) is torsion \( \mathbb{Z}_p \)-module, so a \( p \)-group. We need to prove that it is a bounded \( p \)-group. Clearly, it is enough to prove that for every sequence \( \{x_k : k \in \mathbb{N}^* \} \subseteq A \) there exists \( n \in \mathbb{N}^* \) (independent on \( k \)) such that \( p^n x_k \in G \) for all \( k \).

Consider the group \( \mathbb{Z}_p^\omega \) equipped with the \( p \)-adic topology \( \tau_p \). A base of the neighbourhoods of \( 0 \) in the topology \( \tau_p \) is given by the open subgroups \( V_n = p^n \mathbb{Z}_p^\omega \), \( n \in \mathbb{N}^* \). Then \( \mathbb{Z}_p^\omega \) is a complete metrizable topological group. Let \( E \subseteq \mathbb{Z}_p^\omega \) be the set of all sequences \( \{a_k : k \in \mathbb{N}^* \} \) with \( \lim a_k = 0 \).

**Claim 2.16.** \( E \) is a closed pure submodule of \( \mathbb{Z}_p^\omega \), i.e., \( E \cap p^m \mathbb{Z}_p^\omega = p^m E \) for all \( m \in \mathbb{N}^* \).

**Proof.** Obviously, \( E \) is a submodule of \( \mathbb{Z}_p^\omega \). To check that \( E \) is closed, pick \( c = (c_k) \in \mathbb{Z}_p^\omega \setminus E \). Then \( c_k \not\to 0 \), so there exists \( m \) such that for every \( k \in \mathbb{N}^* \) there exists \( l \geq k \) such that \( c_l \notin p^m \mathbb{Z}_p \). Then \( (c + V_m) \cap E = \emptyset \). Indeed, if \( a = (a_k) \in E \), then there exists \( t > k \) such that \( a_s \in p^m \mathbb{Z}_p \) for all \( s \geq t \). Taking now \( s \geq t \) such that \( c_s \notin V_m \) we have \( a_s - c_s \notin p^m \mathbb{Z}_p \), as \( a_s \in p^m \mathbb{Z}_p \). Hence \( a - c \notin V_m \) and \( a \notin c + V_m \). This proves that \( (c + V_m) \cap E = \emptyset \).

It remains to check that \( E \cap p^m \mathbb{Z}_p^\omega = p^m E \) for all \( m \in \mathbb{N}^* \). To check the non-trivial inclusion \( E \cap p^m \mathbb{Z}_p^\omega \subseteq p^m E \) for \( m \in \mathbb{N}^* \) pick \( a = (a_k) \in E \cap p^m \mathbb{Z}_p^\omega \). Then \( a_k = p^m b_k \) for some \( b_k \in \mathbb{Z}_p^\omega \) and for all \( k \). Since obviously \( b_k \to 0 \) in \( \mathbb{Z}_p^\omega \), \( b = (b_k) \in E \), so \( a = p^m b \in p^m E \).

Now we define a map \( \varphi : E \to A \) putting \( \varphi(e_n) = x_n \), where \( \{e_n : n \in \mathbb{N}^* \} \) is the canonical base of the dense submodule \( E' = \mathbb{Z}_p^\omega \) of \( E \). This assignment can be uniquely extended to a \( \mathbb{Z}_p \)-homomorphism \( \varphi : E' \to A \) which is obviously continuous, so can be further (uniquely) extended to a continuous \( \mathbb{Z}_p \)-homomorphism \( \varphi : E \to A \) satisfying \( \{a_k \} \xrightarrow{\varphi} \sum a_k x_k \).

The \( \mathbb{Z}_p \)-submodule \( F = \varphi^{-1}(G) \) of \( E \) is sequentially closed (and hence closed) and \( E/F \), being isomorphic to a submodule of \( A/G \), is a \( p \)-group. Therefore, for every \( s \in E \) there exists \( n \in \mathbb{N}^* \) such that \( p^ns \in F \), so \( E/F = \bigcup_{n \in \mathbb{N}^*} (E/F)[p^n] \) and \( (E/F)[p^n] \) is closed in \( E/F \) equipped with the quotient topology. Let \( q : E \to E/F \) be the canonical homomorphism. Then \( E_n = q^{-1}((E/F)[p^n]) \) is closed in \( E \) and \( E = \bigcup_{n \in \mathbb{N}^*} E_n \). By the Baire category theorem implies \( E_k \) has a non-empty interior for some \( k \in \mathbb{N}^* \). Since \( E_k \) is a subgroup, this yields that \( F_k \) is open. Hence there
exists an open subgroup $E \cap V_m = p^m E$ in $E$ such that $p^k(p^m E) = p^{m+k} E \subseteq F$ for some $k \in \mathbb{N}^*$. This means, that for every sequence $\{a_i\} \subseteq p^{m+k} E$ one has $\varphi(\{a_i\}) \in G$. In particular, for every $i \in \mathbb{N}^*$ we have $p^{m+k} x_i = \varphi(p^{m+k} e_i) \in G$. □

Now we prove first Theorem C that will be used below in the proof of Proposition 2.13 and Theorem B.

**Proof of Theorem C.** Both (a) and (b) are proved in [13, Theorem 3.6] in full detail although the proof is quite long and heavy. Furthermore, (a) and (b) are not articulated, with the blanket assumption that $w(c(G))$ is not Ulam measurable. We preferred to split (a) and (b) so that it becomes clear where the relevant assumption that $w(c(G))$ is not Ulam measurable is used (only in item(b)), as well as the crucial role of Lemma 2.15 in the proof of (a).

(a) The proof of the fact that $c(G) = c(\tilde{G})$ whenever $c(G)$ is compact can be found in Steps 1 and 3 of the proof of [13, Theorem 3.6]. Both steps essentially use Lemma 2.15 in the particular (but essential) case when $A$ is a power of $\mathbb{Z}_p$.

We prefer to point out a self-contained argument for deducing the remaining part of (a) from the equality $c(G) = c(\tilde{G})$, rather than refer to [13]. Indeed, Corollary 2.12 implies that this equality yields minimality and sequential completeness of $G/c(G)$.

(b) The proof of this item is Step 2 in the proof of [13, Theorem 3.6]. □.

**Proof of Proposition 2.13.** We have to prove that a totally disconnected sequentially complete minimal Abelian group $G$ is strongly $p$-dense for every prime $p$.

By Theorems 2.4 and C, the completion $K$ of $G$ is compact and totally disconnected, hence $K = \prod_p K_p$, where $K_p = \text{td}_p(K)$ is a pro-$p$-group, according to Fact 2.4(3). By Theorem 2.10 $G = \prod_p G_p$, where $G_p = \text{td}_p(G) = G \cap K_p$ is a dense essential subgroup of $K_p$. Moreover, $G_p$ is sequentially complete, being a closed subgroup of $G$. By Lemma 2.15 there exists $k_p \in \mathbb{N}^*$ with $p^{k_p} K_p \subseteq G_p \subseteq G$. □

**Proof of Theorem A.** We have to prove that for a minimal sequentially complete Abelian group $G$ the following are equivalent:

(a) all powers of $G$ are minimal;
(b) $G^{\omega}$ is minimal.
(c) $G$ contains $c(G)$; hence, $c(G) = c(\tilde{G})$ is compact.

To prove the implication (b) $\rightarrow$ (a) assume that $G^{\omega}$ is minimal. Since for every prime $p$ every closed monothetic subgroup of $\text{td}_p(G)$ is compact, it follows from [5] (see also [8, Corollary 6.2.8]) that all powers of $G$ are minimal, i.e., (a) holds.

To prove the the implication (a) $\rightarrow$ (c) assume that all powers $G^\alpha$ are minimal. Then by Theorem 2.5 $G$ is strongly $p$-dense for some prime $p$. By Lemma 2.6 $G$ contains $K = c(\tilde{G})$. Since $K$ is divisible, this yields $\text{td}_p(K)$ is divisible too. Hence $G$ contains $\text{td}_p(K)$. According to [13, Theorem 2.9(a)] $\text{td}_p(K)$ is sequentially dense in $K$. Therefore, the sequential completeness of $G$ implies $K \subseteq G$. In particular, $c(G) = K$ is compact.

For the proof of the implication (c) $\rightarrow$ (b) assume that $c(G) = c(\tilde{G})$ holds true. By Corollary 2.12 $G/c(G)$ is minimal and sequentially complete. Since $G/c(G)$ is totally disconnected, Corollary 2.14 yields that $(G/c(G))^\omega$ is minimal. Since
\((G/c(G))^\omega \cong G^\omega /(c(G))^\omega\) and \(c(G)^\omega\) is compact, we deduce that \(G^\omega\) is minimal, by Proposition 2.8. □

**Proof of Corollary 1.3.** We have to prove that if \(\{G_i\}_{i \in I}\) is a family of sequentially complete Abelian groups, then \(G = \prod_{i \in I} G_i\) is minimal if and only if every countable subproduct is minimal and in such a case there exists a co-finite subset \(J \subseteq I\) such that all groups \(G_i, i \in J\), are minimal.

The first part follows from [8, Theorem 6.2.7]. According to the same theorem, there exists a co-finite subset \(J \subseteq I\) such that all groups \(G_i, i \in J\), are strongly \(p\)-dense. According to Theorem 2.5 this yields minimality of \(G^\omega_i\). □

**Proof of Theorem B.** The implication \((c) \rightarrow (b)\) is trivial. The implication \((b) \rightarrow (a)\) follows from Theorem C.

(a) \rightarrow (c)\ Let \(\alpha\) be Ulam measurable and let \(I\) be a set of cardinality \(\alpha\). Since the cardinal \(\alpha\) is Ulam measurable, there exists a countably complete ultrafilter \(\mathcal{F}\) on \(I\). Following [8, Proposition 3.6.1], in the sequel we use the following properties of \(\mathbb{K}\) and related notation. Taking the dual of the short exact sequence \(0 \rightarrow Z \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/Z \rightarrow 0\) one obtains a short exact sequence \(0 \rightarrow \mathbb{Q}/Z \rightarrow \mathbb{K} = \mathbb{Q} \rightarrow \mathbb{H} = T \rightarrow 0\). Here we denote by \(\mathbb{H}\) the closed subgroup \(\mathbb{Q}/Z \cong \prod_p \mathbb{Z}_p\) so \(\mathbb{K}\), that \(\mathbb{H}_p := td_p(\mathbb{H}) \cong \mathbb{Z}_p\), up to isomorphism.

The subgroup \(B_F = \bigcup_{S \in \mathcal{F}} \mathbb{K}^{(S)} \subseteq \mathbb{K}^I\) is dense, since \(\mathbb{K}^{(I)} \leq B_F\) and the former subgroup is dense. Since the ultrafilter \(\mathcal{F}\) is countably complete, the family of subgroups \(\{\mathbb{K}^{(S)}\}_{S \in \mathcal{F}}\) is \(\omega\)-directed, hence the subgroup \(B_F\) is \(\omega\)-bounded. Then the group \(G_F := \mathbb{H}^I + B_F\) is \(\omega\)-bounded as well, as \(\mathbb{H}^I\) is compact. So far we proved that \(G_F\) is a dense \(\omega\)-bounded subgroup of \(\mathbb{K}^I\). To see that it is not compact, it suffices to note that it is a proper subgroup of \(\mathbb{K}^I\) (indeed, if \(a \in \mathbb{K} \setminus \mathbb{H}\), then the point \(a\) of \(\mathbb{K}\) with all coordinates equal to \(a\) does not belong to \(G_F\)).

It only remains to check that \(G_F\) is minimal. According to Theorem 2.2, we have to prove that \(G_F\) is essential in \(\hat{G}_F = \mathbb{K}^I\). By Lemma 2.3 it suffices to check that for every prime \(p\) and for every \(x = (x_i) \in td_p(\mathbb{K}) = (td_p(\mathbb{K}))^I\) there exists \(n \in \mathbb{N}^*\) such that \(p^n x \in G_F\). Let \(supp(x) = \{i \in I : x_i \neq 0\}\). For \(n \in \mathbb{N}^*\) set \(T_n = \{i \in supp(x) : p^n x_i \notin \mathbb{H}_p\}\). Obviously \(T_1 \supseteq T_2 \supseteq \ldots\) is a chain in \(X\) and \(\bigcap T_n = \emptyset\) as \(i \in supp(x)\) yields there exists \(k\), so that \(i \notin T_k\), as \(x_i \in td_p(\mathbb{K}) = \bigcup_{k=0}^{\infty} p^{-k} \mathbb{H}_p\). Hence

\[
I = \bigcup_{n=1}^{\infty} I \setminus T_n.
\]

Since \(\mathcal{F}\) is a countably complete ultrafilter, it follows from [3] that \(I \setminus T_n_0 \in \mathcal{F}\) for some \(n_0 \in \mathbb{N}^*\). Hence \(\mathbb{K}^{T_n_0} \subset B_F \subseteq G_F\). Split \(x = x' + x''\), such that \(supp(x') \cap supp(x'') = \emptyset\) and \(supp(x') \subseteq T_n_0\). Then \(supp(x'') \subseteq I \setminus T_n_0\) so that \(p^n x'' \in \mathbb{H}^I \subseteq G_F\), while \(x' \in \mathbb{K}^{T_n_0} \subseteq G_F\). Consequently, \(p^n x \in G_F\).

To prove the last assertion of the theorem assume now that \(\alpha\) is measurable. Then we can choose the filter \(\mathcal{F}\) complete, i.e., stable under intersections of families of cardinality < \(\alpha\). As above, the family of subgroup \(\{\mathbb{K}^{(S)}\}_{S \in \mathcal{F}}\) is \(\beta\)-directed for every \(\beta < \alpha\), hence the subgroup \(B_F = \bigcup_{S \in \mathcal{F}} \mathbb{K}^{(S)}\) of \(\mathbb{K}^I\), defined as above, is \(\beta\)-bounded for every \(\beta < \alpha\). Then also the group \(G_F = \mathbb{H}^I + B_F\) is \(\beta\)-bounded for every \(\beta < \alpha\). The rest of the proof works in the same way. □
The following theorem provides a proof of a stronger version of Theorem D:

**Theorem 2.17.** Let $G$ be a sequentially complete Abelian group such that $G^\sim$ is minimal. Then for every $p \in \mathbb{P}$ there exists a $k_p \in \mathbb{N}^*$ such that $G$ contains the compact subgroup $N = \{p^{k_p+1}td_p(G) : p \in \mathbb{P}\}$ (the closure taken in $\tilde{G}$) and $G/N \cong \prod_{p \in \mathbb{P}} B_p,$ where $B_p$ is a bounded $p$-torsion minimal group for every prime $p.$

**Proof.** By Theorem A, $c(G) = c(\tilde{G})$ is compact. By Corollary 2.12, $G_1 := G/c(G)$ is minimal and sequentially complete. Put $A = \tilde{G}/c(G)$ and let $q : \tilde{G} \to \tilde{G}/c(G)$ be the canonical homomorphism. Clearly $A$ is the completion of $G_1.$ Put for brevity $A_p = td_p(A).$ By Fact 2.4, $A = \prod_p A_p,$ since $A$ is totally disconnected. Proposition 2.13 implies that for every prime $p$ there exists $k_p \in \mathbb{N}^*$ such that

$$N_p := p^{k_p}A_p \leq G_p$$

Hence, $S_1 = \bigoplus_{p \in \mathbb{P}} pN_p \leq G_1.$ Since $S_1$ is sequentially dense in the closed subgroup $N_1 = \prod_{p \in \mathbb{P}} pN_p$ of $A$ and $G_1$ is sequentially complete, we deduce that $N_1 \leq G_1.$ Hence, $N = q^{-1}(N_1) \leq G.$ By Fact 2.4, $q(td_p(\tilde{G})) = A_p$ for every prime $p,$ hence the subgroup $S = \langle p^{k_p+1}td_p(\tilde{G}) : p \in \mathbb{P}\rangle$ of $\tilde{G}$ satisfies $q(S) = S_1 \leq N_1,$ so $S \leq N \leq G.$ Hence, $S \leq L := \overline{S} \leq N$ and $S_1 = q(S) \leq q(L) \leq q(N) = N_1 = \overline{S_1}.$ Since $q(L)$ is compact, we deduce that $q(L) = q(N).$ As $\ker q = c(\tilde{G}) \leq L \leq N$ by (the proof of) Lemma 2.6 we conclude that $L = N.$ This proves the first assertion of the theorem.

To prove the second one we put $B_p = A_p/pN_p = A_p/p^{k_p+1}A_p$ and note that $A/N_1 \cong \prod_{p \in \mathbb{P}} B_p.$ Since $G/N \cong G_1/N_1,$ it is enough to prove that $G_1/N_1$ is minimal. According to Theorem 2.22, this amounts to check that $G_1/N_1$ is an essential subgroup of $A/N_1.$ Since for every prime $p$ the group $B_p$ is a bounded $p$-group, it suffices to check that $\text{Soc}(B_p) \leq G_1/N_1,$ according to Lemma 2.3.

The minimality of $G_1$ and Lemma 2.3 yield that $\text{Soc}(A) = \bigoplus_{p \in \mathbb{P}} A[p] \leq G_1.$ By virtue of (2.3), we obtain

$$A[p] + p^{k_p}A_p \leq G_1.$$  

To check that $\text{Soc}(B_p) \leq G_1/N_1,$ pick $\overline{a} = a + pN_p \in \text{Soc}(B_p),$ then $p\overline{a} = 0$ in $B_p,$ so $pa \in pN_p.$ Consequently, $pa = p^{k_p+1}b$ for some $b \in A_p.$ Hence, $t := a - p^{k_p}b \in A[p].$ Now (2.4) gives $a = t + p^{k_p}b \in G.$ Therefore, $\overline{a} \in G_1/N_1.$  

\[\square\]

3. Final comments and questions

The groups considered in this section are not necessarily Abelian.

Sequential completeness is not preserved under taking continuous homomorphic images. Following 14, call a group $G$ $h$-sequentially complete if all continuous homomorphic images of $G$ are sequentially complete. Countably compact groups are obviously $h$-sequentially complete. Nilpotent sequentially $h$-complete groups are precompact 14, Theorem 3.6, while any precompact sequentially $h$-complete group is pseudocompact 14, Theorem 3.9.

**Question 3.1.** (1) Find an example of a precompact $h$-sequentially complete group that is not countably compact.

(2) Find an example of a pseudocompact sequentially complete group that is not $h$-sequentially complete.
(3) Find an example of a (precompact) sequentially complete group that has non-sequentially-complete quotients.

(4) Are precompact sequentially complete minimal groups pseudocompact?

(5) Are sequentially complete minimal totally (hereditarily) disconnected groups zero-dimensional?

In some cases (4) has a positive answer – when the group has large hereditarily pseudocompact (in particular, countably compact) subgroups. In view of [6, Theorem 1.2] (5) is true for \( h \)-sequentially complete groups. On the other hand, it was shown in [13] that (5) is true for sequentially complete minimal Abelian groups.

**Question 3.2.** Which of the following properties of an infinite topological group guarantee the existence of non-trivial converging sequences: i) countably compact and minimal; ii) countably compact and totally minimal; iii) totally minimal?

The following question was left open in [13]:

**Question 3.3.** Let \( G \) be a sequentially complete minimal Abelian group. Is then \( G/c(G) \) sequentially complete?

Finally, we recall an old open problem raised by the second named author closely related to the topic of this paper. We now cancel our assumption that all groups under consideration are Abelian.

**Question 3.4.** (Uspenskij [31]) Is it true that arbitrary products of complete minimal groups are minimal? What about infinite powers?

Note that the answer is positive in the Abelian case, since minimal complete Abelian groups are compact, by Theorem 2.1. “Complete” in Question 3.4 means complete with respect to the two-sided uniformity. The question also makes sense for the case of Weil-complete groups, that is, groups complete with respect to the left (or right) uniformity. A particular instance of such groups are the locally compact ones. The above question seems to be open even in the case of locally compact minimal groups (many instances of positive answer in this case can be found in [26], yet no general result seems to be available to the best of our knowledge).

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