On the Linear AFL: The Non-Basic Case

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1 Introduction

The purpose of this article is to formulate a linear arithmetic fundamental lemma conjecture also for non-basic isogeny classes. Our main result is a reduction of these non-basic cases to the basic one. The context and relevance of these results are as follows.

In general, arithmetic fundamental lemmas are certain identities of intersection numbers on moduli spaces of $p$-divisible groups (RZ spaces) and derivatives of orbital integrals. They are, in particular, local statements, formulated over a $p$-adic local field. Their motivation is global, however, namely they arise from a relative trace formula comparison approach to certain intersection problems of special cycles on Shimura varieties. We refer to the original paper of W. Zhang [26] or his surveys [27, 28] for an explanation of this global-to-local formalism.

There are currently two families of such conjectures. The first comes from the Gan–Gross–Prasad setting of a diagonal embedding of unitary groups $U(V) \to U(V) \times U(V \oplus 1)$. The original conjecture in this context [26] has been proved in work of W. Zhang [29], of W. Zhang and the second author [17, 18] and of Z. Zhang [31]. Several variants have been formulated by Rapoport–Smithling–Zhang [20–22], Y. Liu [15], W. Zhang [30], and Z. Zhang [31]. In all these cases, the global situation is such that intersection takes place in the basic locus. Correspondingly, all the cited works are concerned with basic RZ spaces.

The second family arises from the unitary variant of the Guo–Jacquet restriction problem [5]. That is, the special cycles in question come from an embedding $U_E(V) \to U_F(V)$, where $V$ is a hermitian space for some field extension $F/F_0$ with an additional action of a quadratic extension $E/F$. At inert places of $F/F_0$, this leads to an AFL for basic unitary RZ spaces that is the content of the forthcoming work of Leslie–Xiao–Zhang [10]. At a split place $v$ however, it leads to intersection problems on RZ spaces for all isogeny classes of dimension $1$ and height $2n$ strict $p$-divisible $O_{F_v}$-modules with $O_{E_v}$-action. These are the non-basic linear AFLs from the title of the paper. We mention that the phenomenon of non-basic intersection at split places also occurs for unitary groups over central simple algebras and refer to [8] and [14] for intersection number identities in such situations.

We now describe our results in some detail. They also pertain to the function field setting. We do not exclude the prime 2.
Let $E/F$ be an unramified quadratic extension of a non-archimedean local field $F$ with uniformizer $\pi \in F$. Denote by $\tilde{F}$ the completion of a maximal unramified extension of $F$ and fix an embedding $E \to \tilde{F}$. Let $X$ be a 1-dimensional, height $2n$, not necessarily formal, strict $\pi$-divisible $O_F$-module over the residue field $\mathbb{F}$ of $\tilde{F}$ (cf. Def. [3.1]). Additionally, consider a pair of embeddings

$$\beta = (\beta_1, \beta_2), \quad \beta_i : E \to \text{End}^0(X).$$

The $\pi$-divisible $O_F$-module $X$ decomposes in an essentially unique way as $X = X^0 \times X^1$ with $X^0$ connected and $X^1$ étale. Since $\text{End}^0(X) = \text{End}^0(X^0) \times \text{End}^0(X^1)$, giving a pair $\beta$ is the same as giving analogous pairs $\beta^0, \beta^1$ for the two factors.

Consider now the $\text{RZ}$ space $M$ of strict $\pi$-divisible $O_F$-modules with quasi-isogeny to $X$. It is a formal scheme; abstractly

$$M \cong \coprod_{\mathbb{Z} \times (GL_{2n}(F)/GL_{2n}(O_F))} \text{Spf} O_F[t_1, \ldots, t_{2n-1}], \quad (1.1)$$

where $2n^1$ is the height of the étale factor $X^1$. There are two analogous $\text{RZ}$ spaces $Z_1, Z_2$ that parametrize strict $O_E$-modules together with an $O_F$-linear quasi-isogeny to $(X, \beta_i)$. Forgetting about the $O_E$-action defines closed immersions $Z_1, Z_2 \to M$ that allow to view them as cycles in middle dimension. Assuming that $(\beta_1, \beta_2)$ is regular semi-simple (cf. Def. [2.1]), the centralizer of the image $\beta_1(E) \cup \beta_2(E)$ is an étale $F$-algebra $L \subseteq \text{End}^0(X)$ of degree $n$. Write $L = \prod_{i \in J} L^i$ as product of fields, pick uniformizers $\pi_j \in L^i$ and define $\Gamma = \prod_{j \in J} \pi_j^\infty \subseteq L^\times$. Then $L^\times$ acts compatibly on $Z_1, Z_2$ and $M$, and the quotient $\Gamma \backslash (Z_1 \cap Z_2)$ is artinian. We define

$$\text{Int}(\beta) := \ell_{O_F}(\mathcal{O}_{\Gamma \backslash (Z_1 \cap Z_2)}). \quad (1.2)$$

To $\beta$ one may also associate a matching pair (cf. Def. [2.0])

$$\alpha = (\alpha_0, \alpha_3), \quad \alpha_i : F \times F \to M_{2n}(F),$$

which is unique up to conjugation. Given an element $f \in \mathcal{H} = \mathbb{C}[GL_{2n}(O_F) \backslash GL_{2n}(F)/GL_{2n}(O_F)]$ of the spherical Hecke algebra, one may define an orbital integral for the $GL_n(F \times F) \times GL_n(F \times F)$-action on $GL_{2n}(F)$,

$$O(\alpha, f, s), \quad s \in \mathbb{C}. \quad (1.3)$$

This is an element of $\mathbb{C}[q_F^+, q_F^-]$, where $q_F$ is the residue cardinality of $F$. Under our assumption that $\alpha$ matches $\beta$, the central value $O(\alpha, f, 0)$ vanishes.

**Conjecture 1.1 (Linear AFL).** Assume $\alpha$ and $\beta$ to be as above. Then there is an equality,

$$\frac{1}{\log(q_F)} \frac{d}{ds} \bigg|_{s=0} O(\alpha, 1_{GL_{2n}(O_F)}, s) = \text{Int}(\beta).$$

The basic case is precisely when $X = X^0$ is connected. In this situation, the above conjecture has (up to sign) already been formulated in work of the first author [11] and in his joint work with Howard [7]. In fact, [11] gives a definition of $\text{Int}(\beta, f)$ for every $f \in \mathcal{H}$ and formulates Conj. [1.1] in this generality, while [7] allows the cycles $Z_1, Z_2$ to be defined for two different quadratic extensions $E_1, E_2/F$ (biquadratic setting). We work in the combined generality throughout the paper and refer to Conj. [3.10] below for the general version.

Conj. [1.1] is known when $n = 1$ or 2. The more general version, Conj. [3.10] is known in all cases when $n = 1$, cf. [7][11]. It is currently only known for the unit function $f = 1_{GL_{2n}(O_F)}$ (but possibly $E_1 \neq E_2$) when $n = 2$, cf. [12][13].

The following two are our main results.

**Theorem 1.2.** Write $\beta = (\beta^0, \beta^1)$ for the two components of $\beta$ with respect to $X = X^0 \times X^1$. Then Conj. [1.1] holds for $\beta$ if and only if its hold for the connected component $\beta^0$. 


Corollary 1.3. The AFL (Conj. 1.1) holds for all isogeny classes $X$ whose connected component $X^0$ has height $\leq 4$.

We formulate and prove these two result also in the biquadratic case and for general spherical Hecke functions, cf. Thm. 3.12 and Cor. 3.13. In this generality, however, we have to assume the Guo–Jacquet Fundamental Lemma (resp. its biquadratic variant) for the whole spherical Hecke algebra; it is currently only known for the unit Hecke function and $E_1 = E_2$, see [3].

The proof of Thm. 1.2 relies on the connected-étale sequence $0 \to X^0 \to X \to X^1 \to 0$ of the universal $\pi$-divisible $O_F$-module $M$. It provides a fibration map

$$M \to M^0 \times_{Spf O_F} M^1$$

(1.4)

to the product of the RZ spaces for $X^0$ and $X^1$. If $Y^0$ and $Y^1$ denote their universal $\pi$-divisible $O_F$-modules, then $M$ can be identified with the space of torsion extensions $\text{Ext}^1(Y^1, Y^0)_{\text{tors}}$. This space is in bijection with the Hom-functor $\text{Hom}(T(Y^1), Y^0)$, where $T(Y^1)$ denotes the Tate module. In this way, $M$ is described as the total space of a $\pi$-divisible $O_F$-module over $M^0 \times_{Spf O_F} M^1$. This description allows to relate intersections in the three spaces and we prove the identity

$$\text{Int}(\beta) = |\text{Res}(\text{Inv}(\beta^0), \text{Inv}(\beta^1))|_F O(\beta^1, 1_{GL_{2n}}(O_F)) \text{Int}(\beta^0).$$

(1.5)

Here, $\text{Inv}(\beta^0), \text{Inv}(\beta^1) \in F[T]$ denote the invariants of $\beta^0$ and $\beta^1$ as in Rmk. 2.2. $\text{Res}(\cdot, \cdot)$ denotes the resultant of two polynomials, and $|\cdot|_F$ denotes the normalized absolute value on $F$. A similar identity can be shown for $O(\alpha, f, s)$, leading to the proof of Thm. 1.2. Again, we carry out all these ideas in the biquadratic setting and for arbitrary $f$.

The structure of this article is as follows. In §2 we recall necessary background on matching, orbital integrals and the Guo–Jacquet Fundamental Lemma. This is essentially taken from [7], our own addition being a formulation of the analytic side of Thm. 1.2. In §3 we define our intersection numbers and formulate the AFL. §4 is devoted to the proof of Thm. 1.2, its heart being the fibration argument for §4.1 in §4.2 and a linear algebra computation that produces the resultant factor in §4.3. Finally, our appendices §5 and §6 provide background material on correspondences and the partial Satake transform.

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2 The Fundamental Lemma

This section recalls necessary background on matching, orbital integrals and the fundamental lemma from [5] and [7].

2.1 Invariants

Fix a field $F$ and an integer $n \geq 1$. We consider two quadratic étale extensions $E_1, E_2/F$ and write $\sigma_1, \sigma_2$ for their non-trivial involutions over $F$. Let $B = \prod_{j \in J} B^j$ be the product of a finite family $(B^j)_{j \in J}$ of central simple $F$-algebras. We assume that the degree $[B^j : F] = \dim F(B^j)^{1/2}$ of each factor is even and that the total degree $[B : F] = \sum_{j \in J} [B^j : F]$ is $2n$.

We write $\beta : (E_1, E_2) \to B$ for pairs $(\beta_1 : E_1 \to B, \beta_2 : E_2 \to B)$ of embeddings as $F$-algebras into $B$. Giving a map $\beta_i : E_i \to B$ is the same as giving a tuple of maps $\beta_i^j : E_i \to B^j$. Without
further mentioning, we only consider maps \( \beta \) that make \( B^j \) into a free \( E_i \)-module. This additional specification only matters if \( E_i \cong F \times F \). In particular, pairs \( \beta : (E_1, E_2) \to B \) are the same as tuples of pairs \( \beta^j : (E_1, E_2) \to B^j \).

Let \( E_3 \subset E_1 \otimes_F E_2 \) be the fixed ring under \( \sigma_1 \otimes \sigma_2 \). It is a quadratic étale \( F \)-algebra and we denote by \( \sigma_3 : E_3 \to E_3 \) its non-trivial \( F \)-involution. Given \( \beta \) as above, Howard–Li [7, §2] define a canonical element \( s_\beta \in E_3 \otimes_F B \) that centralizes \( \beta_1(E_1) \) and \( \beta_2(E_2) \). Their definition, which we do not need here, extends verbatim from CSAs to products of CSAs; in fact, simply \( s_\beta = (s_\beta)_j \in J \). They show in [7 Prop. 2.3] that the reduced characteristic polynomial

\[
\text{charred}_{E_3 \otimes_F B/E_3}(s_\beta; T) = \prod_{j \in J} \text{charred}_{E_3 \otimes_F B/E_3}(s_\beta^j; T) \in E_3[T]
\]

is a square and make the following definition.

**Definition 2.1.** (1) The set of invariants of degree \( n \) is the set of monic polynomials \( \delta \in E_3[T] \) of degree \( n \) that satisfy the symmetry

\[
(-1)^n \delta(1 - T) = \sigma_3(\delta(T)).
\]

An invariant \( \delta \) is called regular semi-simple if for every map \( \rho : E_3 \to \overline{F} \), the polynomial \( \rho(\delta) \) has \( n \) distinct non-zero roots.

(2) The invariant \( \text{Inv}(\beta) \in E_3[T] \) of a pair \( \beta \) is the unique monic square root of \( \text{charred}_{E_3 \otimes_F B/E_3}(s_\beta; T) \). It satisfies (2.1) by [7 Prop. 2.3]. A pair \( \beta \) is called regular semi-simple if its invariant is regular semi-simple.

**Remark 2.2.** If \( E_1 = E_2 \), then \( E_3 \cong F \times F \). Fixing such an isomorphism, one may renormalize the invariant by considering only the first component of \( \text{Inv}(\beta; T) \in (F \times F)[T] \). This was done for the formulation of (1.3).

Note that \( \text{Inv}(\beta) = \prod_{j \in J} \text{Inv}(\beta^j) \) and that \( \text{Inv}(\beta) \) only depends on the \( B^\times \)-conjugation orbit of \( \beta \). For regular semi-simple \( \beta \) and simple \( B \), a converse holds:

**Proposition 2.3** ([7 Cor. 2.5.7]). Assume that \( B \) is a central simple \( F \)-algebra. Two regular semi-simple pairs \( \beta, \beta' : (E_1, E_2) \to B \) are \( B^\times \)-conjugate if and only if \( \text{Inv}(\beta) = \text{Inv}(\beta') \).

**Remark 2.4.** The converse direction of Prop. 2.3 does not hold for non-simple \( B \). For example, take \( E_1 = E_2 = F \times F \) and consider two regular semi-simple pairs \( \beta^0, \beta^1 : (E_1, E_2) \to M_2(F) \) with \( \text{Inv}(\beta^0) \neq \text{Inv}(\beta^1) \). Then

\[
(\beta^0, \beta^1), (\beta^1, \beta^0) : (E_1, E_2) \to B := M_2(F) \times M_2(F)
\]

are both regular semi-simple with invariant \( \text{Inv}(\beta^0) \text{Inv}(\beta^1) \). They are not \( B^\times \)-conjugate, however, because \( \beta^0 \) and \( \beta^1 \) are not \( GL_2(F) \)-conjugate.

An immediate question is, given \( E_1, E_2, B \) and a regular semi-simple invariant \( \delta \) as above, whether or not there exists a pair \( \beta : (E_1, E_2) \to B \) with \( \text{Inv}(\beta) = \delta \). This is partially answered by the following universal construction.

**Proposition 2.5** ([7 Prop. 2.5.6]). Given \( E_1, E_2 \) and a regular semi-simple invariant \( \delta \in E_3[T] \) of degree \( n \), there exists an étale \( F \)-algebra \( L_\delta \) of degree \( n \) and a quaternion algebra \( B_\delta/L_\delta \), together with a pair of embeddings \( \beta_\delta : (E_1, E_2) \to B_\delta \), that has the following properties.

1. The image \( \beta_{\delta, 1}(E_1) \cup \beta_{\delta, 2}(E_2) \) generates \( B_\delta \) as \( F \)-algebra.

2. The natural maps \( \beta_{\delta, 1}(E_1) \otimes_F L_\delta \to B_\delta \) are injective and make \( \beta_{\delta, 1}(E_1) \otimes_F L_\delta \) into a maximal commutative \( F \)-subalgebra of \( B \).

4
(3) For every product $B$ of central simple $F$-algebras of total degree $[B : F] = 2n$, and for every pair $\beta : (E_1, E_2) \to B$ of invariant $\delta$, there is a unique embedding of $F$-algebras $\iota : B_\delta \to B$ such that $\beta = \iota \circ \beta_\delta$. Conversely, if $\beta = \iota \circ \beta_\delta$ for an embedding $\iota : B_\delta \to B$ of $F$-algebras, then $\text{Inv}(\beta) = \delta$.

Proof. Starting from $(E_1, E_2, \delta)$, the datum $\beta_\delta : (E_1, E_2) \to B_\delta$ is constructed during the proof of [7 Prop. 2.5.6]. (The algebra $B_\delta$ is denote by $F(\varphi_1, \varphi_2)$ there.) It has properties (1) and (2) by construction. Property (3) is stated in [7 Prop. 2.5.6] for simple $B$. The statement for general $B$ is obtained by the same proof.

Thus pairs $\beta$ of invariant $\delta$ for $B$ are the same as $F$-algebra maps $B_\delta \to B$. Implicit in Prop. 2.5 is moreover the statement that the centralizer of a regular semi-simple pair $\beta$ is an étale $F$-algebra of degree $n$. Namely, it is isomorphic to the center $L_\delta \subset B_\delta$. By the construction of $B_\delta$ in [7], it may be described (up to isomorphism) by

$$L_\delta \cong \langle E_3[T]/(\delta(T)) \rangle_{\sigma_3 = \text{id}}$$

(2.2)

where $\sigma_3$ is extended to $E_3[T]$ as $\sigma_3(T) = 1 - T$.

### 2.2 Matching

From now on, we let $F$ be a non-archimedean local field with normalized absolute value $| \cdot |$ and assume $E_1, E_2$ to be étale field extensions. Put $E_0 = F \times F$. Then the invariant $\text{Inv}(\alpha)$ of a pair $\alpha = (\alpha_0, \alpha_3) : (E_0, E_3) \to M_{2n}(F)$ lies in $E_3[T]$ as well.

**Definition 2.6.** Two regular semi-simple pairs $\alpha : (E_0, E_3) \to M_{2n}(F)$ and $\beta : (E_1, E_2) \to B$ are said to match if $\text{Inv}(\alpha) = \text{Inv}(\beta)$.

Let $\delta \in E_3[T]$ be regular semi-simple and of degree $n$. Let $\alpha_\delta : (E_0, E_3) \to A_\delta$ be the universal datum for $(E_0, E_3, \delta)$ from Prop. 2.5 and let $L_\delta = \text{Cent}(A_\delta)$. By part (2) of Prop. 2.5, the $L_\delta$-algebra $E_0 \otimes_F L_\delta \cong L_\delta \times L_\delta$ embeds into $A_\delta$ which implies $A_\delta \cong M_2(L_\delta)$. In particular, there exists an embedding $\iota : A_\delta \to M_{2n}(F)$. By Prop. 2.5 (3), $\text{Inv}(\iota \circ \alpha_\delta) = \delta$. It follows that for every regular semi-simple invariant $\delta$ of degree $n$, there exists a pair $\alpha : (E_0, E_3) \to M_{2n}(F)$ with $\text{Inv}(\alpha) = \delta$. On the other hand, given $E_1, E_2, B$ and $\delta$, there need not exist a pair $\beta : (E_1, E_2) \to B$ with $\text{Inv}(\beta) = \delta$. Def. 2.6 is asymmetric in this sense.

### 2.3 Orbital integrals for $(E_1, E_2)$

The fundamental lemma compares $\eta$-twisted orbital integrals of pairs $(E_0, E_3) \to M_{2n}(F)$ with orbital integrals of matching pairs $(E_1, E_2) \to M_{2n}(F)$. In this section, we define the second kind. Throughout, $A_{\text{std}} := O_F^{2n}$ denotes the standard lattice and $\mathcal{H} = \mathbb{C}[GL_{2n}(O_F)\backslash GL_{2n}(F)/GL_{2n}(O_F)]$ the spherical Hecke algebra.

**Definition 2.7.** Let $\beta : (E_1, E_2) \to M_{2n}(F)$ be regular semi-simple and write $H_i := \text{Cent}(E_i)^\times$. Choose $g_i \in GL_{2n}(F)$ such that $g_i A_{\text{std}}$ is $O_{E_i}$-stable. For $f \in \mathcal{H}$, define

$$O(\beta, f) := \int_{(H_1 \cap H_2) \backslash (H_1 \times H_2)} f(g^{-1}_1 h^{-1}_2 g_2) dh_1 dh_2.$$  

(2.3)

Here we endow $H_i \cong GL_n(E_i)$ with the Haar measure such that $\text{Vol}(GL_n(O_{E_i})) = 1$. The stabilizer $H_1 \cap H_2$ is the group of units $L^\times$, where $L = \text{Cent}(\beta(E_1) \cup \beta(E_2)) \cong L_\delta$ is an étale $F$-algebra of degree $n$ by Prop. 2.5. We endow it with the measure such that $\text{Vol}(O_L^\times) = 1$. The definition is independent of the chosen $g_i$, once convergence is established:

**Lemma 2.8.** The subset $(H_1 \cap H_2) \backslash (H_1 \times H_2) \subset GL_{2n}(F)$ is closed. In particular, the integrand in (2.3) has compact support.
In particular, we see that in both cases we introduce the following notation: Given two $q$\,(\overline{E})\, normalized absolute value on $E$.

If $\eta$ is only interesting when we only consider spherical Hecke functions in this article. For these, the definition of such integrals $E$ is unramified over $E$. We assume from now on that $E$ aim is to define certain $\eta$- twisted orbital integrals for $E_0$. For general $\eta$, $\lambda \subset V$, we put $[\lambda : \lambda'] = [\lambda : \pi^n \lambda'] - n \dim(V)$ for $n \gg 0$.

**Proof.** Set $G = GL_{2n}(F)$. There is an isomorphism $G/(H_1 \times H_2) \cong H_1/g_1^{-1}g_2$. By Skolem–Noether, via $(g_1, g_2) \mapsto (g_1\beta_1 g_1^{-1}, g_2\beta_2 g_2^{-1})$, the source is in bijection with the $G$-orbits of pairs $(E_1, E_2) \rightarrow M_{2n}(F)$. By Prop. 3.4 the orbit of $\beta$ agrees with the locus where the invariant takes value $\text{inv}(\beta)$. This is, in particular, closed. Since the orbit of $\beta$ maps to $H_1 H_2$ in the target, it follows that $(H_1 \cap H_2) \cap (H_1 \times H_2) \subset GL_{2n}(F)$ is a closed subset as claimed.

The orbital integral $O(\beta, f)$ can also be understood as an $f$-weighted lattice count. Let $\mathcal{L}(\beta_i)$ denote the $O_E$-stable lattices in $F^{2n}$. This set is stable under the $L^\times$-action because $L^\times$ centralizes $\beta_i(E_i)$. Let $g_i \in GL_{2n}(F)$ be fixed as in Def. 2.7 and set $K_i = \text{Stab}(g_i \Lambda_{\text{std}}) \cap H_i$. Then

\[ H_i/K_i \cong \mathcal{L}(\beta_i), \quad h_i \mapsto h_i g_i \Lambda_{\text{std}}. \] (2.4)

As $f$ lies in the spherical Hecke algebra, the restriction $f|_{g_1^{-1}k_1 h_1^{-1}h_2 k_2 g_2}$ is constant for every element $(h_1, h_2) \in H_1 \times H_2$. Moreover, the volume of $L^\times \cap (L^\times \cdot (h_1 K_1 \times h_2 K_2))$ is equal to

\[ \text{Vol}(L^\times \cap h_1 K_1 h_1^{-1} \cap h_2 K_2 h_2^{-1}) = \text{Vol}(\text{Stab}_{L^\times}(h_1 g_1 \Lambda_{\text{std}}, h_2 g_2 \Lambda_{\text{std}})). \] (2.5)

We introduce the following notation: Given two $O_F$-lattices $\Lambda_1, \Lambda_2 \subset F^{2n}$, we write $f(\Lambda_1, \Lambda_2)$ for the value of $f$ at the relative position of $\Lambda_1$ and $\Lambda_2$. That is, $f(\Lambda_1, \Lambda_2) := f(k_1^{-1}k_2)$ where the $k_i \in GL_{2n}(F)$ are chosen with $\Lambda_1 = k_1 \Lambda_{\text{std}}$. Also choose a cocompact, free, discrete subgroup $\Gamma \subseteq L^\times$. Combining (2.4) and (2.5) with Def. 2.7, we obtain

\[ O(\beta, f) = \sum_{(\Lambda_1, \Lambda_2) \in L^\times \cap (\mathcal{L}(\beta_1) \times \mathcal{L}(\beta_2))} \text{Vol}(\text{Stab}_{L^\times}(\Lambda_1, \Lambda_2))^{-1} f(\Lambda_1, \Lambda_2) \] (2.6)

2.4 $\eta$-Twisted orbital integrals for $(E_0, E_3)$

Let $\eta : E_3^\times \rightarrow \{\pm 1\}$ be the character from local class field theory for the extension $E_1 \otimes_F E_2 / E_3$. Our aim is to define certain $\eta$-twisted orbital integrals for regular semi-simple pairs $\alpha : (E_0, E_3) \rightarrow M_{2n}(F)$. We only consider spherical Hecke functions in this article. For these, the definition of such integrals is only interesting when $\eta$ restricts to the trivial character on $O_{E_3}^\times$; cf. [7 Prop. 3.2.7]. Thus we assume from now on that $E_1 \otimes_F E_2 / E_3$ is unramified. Equivalently, we assume that one out of $E_1, E_2$ is unramified over $F$. This leaves us with the following two cases:

(1) If $E_1, E_2$ are both unramified over $F$, then $E_3 \cong F \times F$ and $E_1 \otimes_F E_2 \cong E_1 \times E_1$ is unramified over $E_3$.

(2) If precisely one out of $E_1, E_2$ is ramified over $F$, then $E_1 \otimes_F E_2$ is a biquadratic field extension of $F$ such that $\sigma_1 \otimes \sigma_2$ acts non-trivially on the residue field. It follows that $E_3 / F$ is ramified and $E_1 \otimes_F E_2 / E_3$ unramified.

In particular, we see that in both cases $\eta$ can be defined by $\eta(x) = (-1)^[O_E : x O_E]$, where the index\footnote{If $\Lambda' \subset \Lambda \subset V$ are $O_F$-lattices in a finite dimension $F$-vector space $V$, then we set $[\Lambda : \Lambda'] = \log_q |\Lambda / \Lambda'|$ where $q = q_F$. For general $\Lambda', \Lambda \subset V$, we put $[\Lambda : \Lambda'] = [\Lambda : \pi^n \Lambda'] - n \dim(V)$ for $n \gg 0$.} is meant as $O_F$-lattices in $E_3$. We define $\eta$ to $GL_n(E_3)$ by $\eta(h) := \eta(\det h)$. Denote by $| \cdot |_F$ the normalized absolute value on $F$. Recall that $E_0 = F \times F$. We also define the character

\[ | \cdot | : GL_n(E_0) \rightarrow \mathbb{R}^\times \]

\[ (a, b) \mapsto \frac{\det(a)}{\det(b)}. \] (2.7)
The definition of $O(\beta, f)$ in (2.3) required the choice of auxiliary elements $g_1, g_2 \in GL_{2n}(F)$. The situation for $\eta$-twisted orbital integrals is similar, but we need to be more careful to ensure independence of such an auxiliary choice. Consider a regular semi-simple pair $\alpha : (E_0, E_3) \to M_{2n}(F)$. We write $\mathcal{L}(\alpha_i)$ for the set of $\alpha_i(O_{E_i})$-lattices in $F^{2n}$. Any lattice $\Lambda_0 \in \mathcal{L}(\alpha_0)$ is of the form $\Lambda_0^+ \times \Lambda_0^-$, where $\Lambda_0^+ \subset \alpha_0(1,0)F^{2n}$ is an $\alpha_0(O_F \times O)$-lattice and $\Lambda_0^- \subset \alpha_0(0,1)F^{2n}$ an $\alpha_0(0 \times O_F)$-lattice. Since $\alpha$ is regular semi-simple, one can show (omitted) that the two natural maps obtained from $\alpha$ are bijective. In particular, given $(\Lambda_0, \Lambda_3) \in \mathcal{L}(\alpha_0) \times \mathcal{L}(\alpha_3)$, there are well-defined lattice indices

$$[A_3 : O_{E_3}A_3^+] \oplus [A_3 : O_{E_3}A_3^-] \in \mathbb{Z}. \quad (2.8)$$

**Definition 2.9.** Let $\alpha : (E_0, E_3) \to M_{2n}(F)$ be regular semi-simple. Given $(\Lambda_0, \Lambda_3) \in \mathcal{L}(\alpha_0) \times \mathcal{L}(\alpha_3)$ and $s \in \mathbb{C}$, the transfer factor of $(\Lambda_0, \Lambda_3)$ is defined by

$$\Omega(\Lambda_0, \Lambda_3, s) = (-1)^{[A_3 : O_{E_3}A_3^+] \cdot (q^{[A_3 : O_{E_3}A_3^+] \cdot [A_3 : O_{E_3}A_3^-]} - [A_3 : O_{E_3}A_3^-])^{s/2}. \quad (2.9)$$

This function has the transformation property

$$\Omega(h_0\Lambda_0, h_3\Lambda_3, s) = |h_0|^s \eta(h_3)|\Omega(\Lambda_0, \Lambda_3, s), \quad h_0 \in H_0, \ h_3 \in H_3. \quad (2.10)$$

Choose $g_0, g_3 \in GL_{2n}(F)$ such that $g_i\Lambda_{std} \in \mathcal{L}(\alpha_i)$. For $f \in \mathcal{H}$, define

$$O(\alpha, f, s) := \Omega(g_0\Lambda_{std}, g_3\Lambda_{std}) \cdot \int_{(H_0 \cap H_3)(H_0 \times H_3)} |f(g_0^{-1}h_0^{-1}h_3g_3)| |h_0|^s \eta(h_3) dh_0 dh_3. \quad (2.11)$$

The integrand here is well-defined by (2.1) which states that $\eta$ and $|\cdot|$ are trivial on $H_0 \cap H_3$. Moreover, the normalization of Haar measures in (2.11) is as in Def. (2.7) and the same remarks on convergence apply. Identity (2.10) allows to rewrite (2.11) as

$$O(\alpha, f, s) = \int_{(H_0 \cap H_3)(H_0 \times H_3)} f(g_0^{-1}h_0^{-1}h_3g_3) \Omega(h_0g_0\Lambda_{std}, h_3g_3\Lambda_{std}) dh_0 dh_3. \quad (2.12)$$

As $f$ is spherical, $f(g_0^{-1}h_0^{-1}h_3g_3)$ only depends on the lattices $g_0\Lambda_{std}$ and $\Lambda_3\Lambda_{std}$ and not on the precise choices of $g_0, g_3$. Using that $H_1$ acts transitively on $\mathcal{L}(\alpha_i)$, we see that $O(\alpha, f, s)$ is actually independent of $g_0$ and $g_3$.

There is a lattice count formula for $O(\alpha, f, s)$ that is analogous to (2.6): Let $L$ denote the centralizer of $\alpha$ and pick a cocompact subgroup $\Gamma \subset L^\times$ as before. Then (2.12) gives

$$O(\alpha, f, s) = \sum_{(\Lambda_0, \Lambda_3) \in L^\times \backslash (\mathcal{L}(\alpha_0) \times \mathcal{L}(\alpha_3))} \text{Vol}(\text{Stab}_{L^\times}(\Lambda_0, \Lambda_3))^{-1} f(\Lambda_0, \Lambda_3) \Omega(\Lambda_0, \Lambda_3, s) \quad (2.13)$$

**2.5 Fundamental Lemma**

Recall that we assumed from (2.3) that at least one out of $E_1, E_2$ is unramified over $F$.

**Conjecture 2.10** (Guo–Jacquet Fundamental Lemma). Assume that $\alpha : (E_0, E_3) \to M_{2n}(F)$ and $\beta : (E_1, E_2) \to M_{2n}(F)$ are regular semi-simple and matching. Let $f \in \mathcal{H}$ be a spherical Hecke function. Then

$$O(\alpha, f, 0) = O(\beta, f). \quad (2.14)$$
This has been conjectured by Guo–Jacquet [5] for \( E_1 = E_2 \). The biquadratic case \( (E_1 \neq E_2) \) has been conjectured by Howard–Li [7]. The following cases are known.

1. For \( E_1 = E_2 \) both unramified and the unit Hecke function \( f = 1_{GL_{2n}(O_F)} \), the fundamental lemma is the main result of [5].

2. For general \( E_1, E_2 \) and \( n = 1 \), the conjecture is known for the full spherical Hecke algebra, cf. [7].

3. For general \( E_1, E_2 \) and \( n = 2 \), the conjecture is known for the unit element \( 1_{GL_{2n}(O_F)} \), which is due to the first author [13].

Remark 2.11. FL type identities for the spherical Hecke algebra may sometimes be deduced from the FL for the unit function, using global arguments, which goes back to Clozel [3] and Labesse [9]. This idea is already mentioned in [5] for the above FL conjecture, but does not seem to have been brought to fruition yet.

2.6 Reduction formula

Assume that \( n = n^0 + n^1 \) and that the two regular semi-simple pairs

\[
\alpha : (E_0, E_3) \longrightarrow M_{2n}(F), \quad \beta : (E_1, E_2) \longrightarrow M_{2n}(F)
\]

factor through \( M_{2n^0}(F) \times M_{2n^1}(F) \). Denote their components by \( \alpha^0, \alpha^1 \) resp. \( \beta^0, \beta^1 \). Our aim is to relate the orbital integrals \( O(\alpha, f, s) \) resp. \( O(\beta, f) \) with those for the Levi factors, \( O(\alpha^j, f, s) \) resp. \( O(\beta^j, f), j = 0, 1 \). This relation is due to Guo [5] when \( E_1 = E_2 \).

We work with the following generators (as \( \mathbb{C} \)-vector space) of \( \mathcal{H} \). For \( m \in \mathbb{Z}_{\geq 0} \), write

\[ f(m) = 1_{g \in M_{2n}(O_F), \ det(g) = m} \cdot \tag{2.15} \]

Given a tuple \( m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r \) put

\[ f(m) = f(m_1) \ast f(m_2) \ast \ldots \ast f(m_r). \tag{2.16} \]

Concretely, \( f(m)(g) \) is the number of tuples of \( O_F \)-lattices \( (\Lambda_1, \ldots, \Lambda_r) \) in \( F^{2n} \) such that

\[ gO_F^{2n} = \Lambda_0 \subseteq \Lambda_1 \subseteq \ldots \subseteq \Lambda_r = O_F^{2n}, \quad [\Lambda_i : \Lambda_{i-1}] = m_i \text{ for all } i. \tag{2.17} \]

Also define \( [\pi] = 1_{\pi \in GL_{2n}(O_F)} \) which is invertible with inverse \( [\pi]^{-1} = 1_{\pi^{-1}GL_{2n}(O_F)} \). By Thm. 6.3

\[ \mathcal{H} = \sum_{k \in \mathbb{Z}, r \geq 0, m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r} \mathbb{C} \cdot [\pi]^k \ast f(m). \tag{2.18} \]

We remark that this is not a direct sum, meaning there are non-trivial relations among the generators \([\pi]^k \ast f(m)\). We also mention that

\[ O(\alpha, [\pi]^k \ast f, s) = O(\alpha, f, s), \quad O(\beta, [\pi]^k \ast f) = O(\beta, f) \tag{2.19} \]

for every \( f \in \mathcal{H}, k \in \mathbb{Z} \). These identities are obtained from \( ([\pi]^k \ast f)(g) = f(\pi^{-k}g) \) and a variable substitution in the definitions of \( O(\alpha, f, s) \) and \( O(\beta, f) \). Let \( \mathcal{H}^j = \mathbb{C}[GL_{2n^j}(O_F) \backslash GL_{2n^j}(F)/GL_{2n^j}(O_F)] \), for \( j = 0, 1 \), denote the spherical Hecke algebra of \( GL_{2n^j}(F) \). There is a partial Satake transformation

\[ \mathcal{S} : \mathcal{H} \longrightarrow \mathcal{H}^0 \otimes \mathbb{C} \mathcal{H}^1 \]

which is discussed in detail in (7.2) and Prop. 6.7. It is described on the above generators by

\[ \mathcal{S}([\pi]^k \ast f(m)) = \sum_{m^0, m^1 \in \mathbb{Z}_{\geq 0}^r, m = m^0 + m^1} q^{a^1 |m^0| + a^0 |m^1|} ([\pi]^k \ast f(m^0)) \otimes ([\pi]^k \ast f(m^1)). \tag{2.20} \]

\[ ^2 \text{In the appendix, the notation for } f(m) \text{ is } T_m \text{ or } T_{GL_{2n}, m}, \text{ see 6.3.} \]
Here, \( |m| := \sum_{i=1}^{r} m_i \) denotes the sum of all coefficients. Taking the product of the orbital integrals for \( \alpha_0 \) and \( \alpha_1 \), resp. for \( \beta_0 \) and \( \beta_1 \), defines orbital integrals

\[
O^{\text{Levi}}((\alpha_0, \alpha_1), f, s), \quad O^{\text{Levi}}((\beta_0, \beta_1), f), \quad f \in \mathcal{H}^0 \otimes \mathbb{C} \mathcal{H}^1.
\]

**Theorem 2.12.** Given \( \alpha \) and \( \beta \) as above, there are the following relations of orbital integrals on \( \text{GL}_{2n}(F) \) and its Levi \( \text{GL}_{2n_0}(F) \times \text{GL}_{2n_1}(F) \).

\[
O(\alpha, f, s) = |\text{Disc}_{E_4/F}|^{-\frac{n}{2}} |\text{Res}(\text{Inv}(\alpha^0), \text{Inv}(\alpha^1))|^{-1} \cdot O^{\text{Levi}}((\alpha^0, \alpha^1), S(f), s)
\]

\[
O(\beta, f) = |\text{Disc}_{E_4/F} \cdot \text{Disc}_{E_2/F}|^{-\frac{n}{2}} |\text{Res}(\text{Inv}(\beta^0), \text{Inv}(\beta^1))|^{-1} \cdot O^{\text{Levi}}((\beta^0, \beta^1), S(f)).
\]

(2.21)

Here, we have used the notation \( |x|_F := |\text{Nm}_{E_3/F}(x)|^{1/2} \) for elements \( x \in E_3 \).

**Remark 2.13.** The resultant \( \text{Res}(\text{Inv}(\alpha^0), \text{Inv}(\alpha^1)) \) lies in \( E_3^* \). Namely, \( \alpha \) being regular semi-simple in particular means that \( \text{Inv}(\alpha) = \text{Inv}(\alpha^0)\text{Inv}(\alpha^1) \) is separable, which implies that \( \text{Inv}(\alpha^0) \) and \( \text{Inv}(\alpha^1) \) have no common zeroes. The same applies to all other analogous resultant terms in this article.

**Proof of Thm. 2.12.** These are \[5\] Prop. 2.1 and 2.2] when \( E_1 = E_2 \). We will give a proof for the general case in \[4\].

### 2.7 Vanishing order

We now combine the reduction formula with the following vanishing statement.

**Proposition 2.14 (\[7\] Prop. 3.3.3)).** Assume that \( \alpha : (E_0, E_3) \rightarrow M_{2n}(F) \) is regular semi-simple and matches to a pair \( \beta : (E_1, E_2) \rightarrow B \), where \( B \) is a central division algebra of degree \( 2n \) over \( F \). Then

\[
O(\alpha, f, 0) = 0.
\]

This relies on a functional equation for the orbital integral that relates \( O(\alpha, f, s) \) with \( O(\alpha, f, -s) \), we refer to \[7\].

**Definition 2.15.** Let \( \delta \) be a regular semi-simple invariant and let \( \beta_\delta : (E_1, E_2) \rightarrow B_\delta \) be the universal datum from Prop. 2.3. Denote by \( L_\delta \subseteq B_\delta \) the center, write \( L_\delta = \prod_{j \in J} L^j \) for its factorization into fields and let \( B^j / L^j \) be the corresponding factor of \( B_\delta \). We define the order of \( \delta \) as

\[
\text{ord}(\delta) := |\{ j \in J \mid B^j \text{ is a division algebra} \}|.
\]

(2.22)

**Lemma 2.16.** Let \( L/F \) be a field extension of degree \( d \) and let \( C/L \) be a quaternion algebra. There exists a central division algebra \( D/F \) of degree \( |D : F| = 2d \) together with an embedding \( C \rightarrow D \) if and only if \( C \) is non-split.

**Proof.** Clearly, \( C \) is non-split if there exists an embedding \( C \rightarrow D \) as stated. The converse direction is obtained as follows. We denote the Hasse invariant of a central simple algebra \( D/F \) by \( \text{Ha}(D) \in \mathbb{Q}/\mathbb{Z} \). Recall that \( D \rightarrow \text{Ha}(D) \) defines an isomorphism \( \text{Br}(F) \cong \mathbb{Q}/\mathbb{Z} \). Also recall that for every field extension \( L/F \), the identity \( \text{Ha}(L \otimes_F D) = [L : F] \cdot \text{Ha}(D) \) holds. In particular, \[1\] Thm. 9.12 (c) implies that for every central division algebra \( D/F \) and for every field extension \( L/F \) such that \( [L : F] \mid [D : F] \), there exists an embedding \( L \rightarrow D \) as \( F \)-algebras. In this situation, \[1\] Cor. 9.1 shows that \( \text{Ha}(\text{Cent}_D(L)) = [L : F] \cdot \text{Ha}(D) \). Thus if \( [D : F] = 2[L : F] \), then there exists an embedding \( L \rightarrow D \) and \( \text{Cent}_D(L) \) is a quaternion division algebra over \( L \) which is isomorphic to \( C \) if \( C \) is non-split.

**Corollary 2.17.** Let \( \alpha : (E_0, E_3) \rightarrow M_{2n}(F) \) be a regular semi-simple pair. Then

\[
\text{ord}_{s=0} O(\alpha, f, s) \geq \text{ord}(\text{Inv}(\alpha)).
\]

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Proof. Let $\delta = \text{Inv}(\alpha)$ and let $\beta_\delta: (E_1, E_2) \to B_\delta$ be the universal datum for $(E_1, E_2, \delta)$. Write $L_\delta = \prod_{j \in J} L^j$ and $B_\delta = \prod_{j \in J} B^j$ as before. Let $\delta = \prod_{j \in J} \delta^j$ be the corresponding factorization of $\delta$, see (2.23), and note that $B^j = B_{\delta^j}$. By Lem. 2.10 $B^j$ embeds into a central division algebra of degree $2[L^j : F]$ if it is a division algebra. Then Prop. 2.14 states that $O(\alpha^j, f^j) = 0$ for every spherical Hecke function $f^j$ on $GL_2(L^j; F)$ and every pair $\alpha^j : (E_0, E_3) \to GL_2(L^j; F)$ of invariant $\delta^j$. Combining this observation with the product formula Thm. 2.12 and performing induction on $|J|$, we obtain the corollary.

Let $q_F$ be the residue cardinality of $F$. We next consider the normalized first derivative

$$
\partial \text{Orb}(\alpha, f) := \frac{1}{\log(q_F)} \frac{d}{ds}|_{s=0} O(\alpha, f, s).
$$

(2.23)

Corollary 2.18. Assume that $\alpha : (E_0, E_3) \to M_{2n}(F)$ is regular semi-simple and that $f \in \mathcal{H}$ is such that

$$
O(\alpha, f, 0) = 0, \quad \partial \text{Orb}(\alpha, f) \neq 0.
$$

(2.24)

Then $\text{ord}(\text{Inv}(\alpha)) = 1$. In particular, there are uniquely determined $n^0, n^1$ with $n = n^0 + n^1$ such that $\alpha$ matches a pair $\beta : (E_1, E_2) \to D_{1/2n^0} \times M_{2n^1}(F)$. Here, $D_\lambda$ denotes the central division algebra of Hasse invariant $\lambda$ over $F$.

Proof. Set $\text{ord}(\alpha) = \text{ord}(\text{Inv}(\alpha))$. Cor. 2.17 provides $\text{ord}(\alpha) \leq 1$ and we are required to exclude the case $\text{ord}(\alpha) = 0$. This relies on the functional equation [7, Prop. 3.2.8], saying

$$
O(\alpha, f, s) = \varepsilon(\alpha, s)O(\alpha, f, -s)
$$

(2.25)

for a factor $\varepsilon(\alpha, s) \in \pm F^*|_F$ that only depends on $\alpha$. The proof of [7, Prop. 3.3.3], combined with Thm. 2.12 and Lem. 2.10 shows that the sign of $\varepsilon(\alpha, s)$ is $(-1)^{\text{ord}(\alpha)}$. Assume it were positive, say $\varepsilon(\alpha, s) = q_F^s$. Taking the derivative $(d/ds)|_{s=0}$ of (2.23), we obtain that

$$
\log(q_F) \partial \text{Orb}(\alpha, f) = r \log(q_F)O(\alpha, f) - \log(q_F) \partial \text{Orb}(\alpha, f).
$$

This is impossible under our assumptions (2.24). Thus necessarily $\text{ord}(\alpha) = 1$. The existence of $(n^0, n^1)$ and of the pair $\beta$ is now evident from Lem. 2.10. Take $n^0 = [L^{j_0} : F]$ and $n^1 = n - n^0$, where $j_0 \in J$ is the unique index in (2.22) such that $B^{j_0}$ is division.

3 The Arithmetic Fundamental Lemma

3.1 Quadratic cycles

Let $F$, $E_1$ and $E_2$ as well as $n \geq 1$ be as before. In particular, $E_1$ and $E_2$ are quadratic étale field extensions of $F$ of which at least one is unramified. Denote by $\bar{F}/F$ the completion of a maximal unramified extension and by $\bar{F}$ its residue field. Let $\bar{E}$ be a composite of $\bar{F}$, $E_1$ and $E_2$, and fix embeddings $E_1, E_2 \to \bar{E}$.

Definition 3.1. Let $S$ be an $O_F$-scheme such that $\pi \in O_S$ is locally nilpotent.

1. Assume $F$ is $p$-adic. A strict $\pi$-divisible $O_F$-module over $S$ is a pair $(X, \iota)$ consisting of a $p$-divisible group $X$ over $S$ and a strict action $\iota : O_F \to \text{End}(X)$. The latter means $\iota(a) = a$ on $\text{Lie}(X)$.

2. Assume $F \cong F_q((\pi))$ is a local function field. A strict $\pi$-divisible $O_F$-module over $S$ is a $\pi$-divisible group $(X, \iota)$ over $S$ (for $F$) in the sense of [6] Def. 7.1 that is strict, meaning that $\iota(a) = a$ on $\text{Lie}(X)$.

3These are called $\pi$-divisible local Anderson modules in [6]. Our terminology is chosen for uniformity with the $p$-adic case.
The theories of these two types of objects are completely parallel, at least to the extent required by this article. We also remark that the objects in (2) are equivalent to minuscule local $O_F$-shtuka, cf. [6 Thm. 8.3].

Let $X/F$ be a strict $\pi$-divisible $O_F$-module of height $2n$ and dimension 1 and consider a pair of embeddings

$$\beta : (E_1, E_2) \to \text{End}^0(X).$$

Considering its identity component and maximal étale quotient, $X$ decomposes in a unique way as a product $X = X^0 \times X^1$ with $X^0$ connected and $X^1$ étale. The heights of $X^0$ and $X^1$ are even because of the existence of $\beta$ and we write $\text{ht}(X^0) = 2n^0$ and $\text{ht}(X^1) = 2n^1$. Then

$$\text{End}(X) \cong O_D \times M_{2n^1}(O_F)$$

where $D/F$ denotes a central division algebra of Hasse invariant $1/2n^0$. The datum of $\beta$ is thus equivalent to that of two pairs

$$\beta^0 : (E_1, E_2) \to \text{End}^0(X^0), \quad \beta^1 : (E_1, E_2) \to \text{End}^0(X^1).$$

**Lemma 3.2.** Assume that $\beta$ is regular semi-simple. Then $\text{ord}(\text{Inv}(\beta)) = 1$. Conversely, for every regular semi-simple invariant $\delta$ with $\text{ord}(\delta) = 1$, there is a unique pair $(X, \beta)$ as above up to isogeny such that $\text{Inv}(\beta) = \delta$.

**Proof.** Write $L_\delta = \prod L^j$ and $B_\delta = \prod B^j$ as in Def. 2.14. As seen during the proof of Cor. 2.18 there exists an $F$-algebra embedding $B_\delta \to D_{1/2n^0} \times M_{2n^1}(F)$ if and only if there is a unique index $j_0$ with $B^{j_0}$ a division algebra and $[L_{j_0} : F] = n^0$. The pairs $(2n^0, 2n^1)$ with $n^0 + n^1 = n$ on the other hand are in bijection with the possible isogeny classes of $X$ (Dieudonné–Manin classification). Finally, any embedding of $B_\delta$ into $D_{1/2n^0} \times M_{2n^1}(F)$ has to be a product of two embeddings

$$B^{j_0} \to D_{1/2n^0}, \quad \prod_{j \neq j_0} B^j \to M_{2n^1}(F)$$

and hence all such embeddings are conjugate by Skolem–Noether. This proves the uniqueness up to isogeny. \[\square\]

**Remark 3.3.** An analogous uniqueness statement holds for local Shimura data of EL type for inner forms of $GL_{2n}(F)$, see [14 Prop. 4.5]. It would be interesting to know if there is a more group-theoretic explanation for this phenomenon.

Let $M \to \text{Spf} \, O_E$ denote the RZ space of $X$, that is

$$M(S) = \left\{ (X, \rho) \bigg| \begin{array}{l} X/S \text{ a strict } \pi\text{-divisible } O_F\text{-module} \\ \rho : \mathcal{S} \times_S X \to \mathcal{S} \times \text{Spec}_E \mathbb{X} \text{ a quasi-isogeny} \end{array} \right\}.$$ (3.1)

Here, $\mathcal{S} := F \otimes_{O_E} S$ denotes the special fiber of $S$. Analogously define, for $i = 1, 2$,

$$Z(\beta_i)(S) = \left\{ (X, \rho) \bigg| \begin{array}{l} X/S \text{ a strict } \pi\text{-divisible } O_{E_i}\text{-module} \\ \rho : \mathcal{S} \times_S X \to \mathcal{S} \times \text{Spec}_{E_i} \mathbb{X} \beta_i \text{ an } O_{E_i}\text{-linear quasi-isogeny} \end{array} \right\}.$$ (3.2)

The above spaces are well-known to be formally of finite type and formally smooth over $\text{Spf} \, O_E$. The relative dimension of $M$ is $2n - 1$, that of the $Z(\beta_i)$ is $n - 1$. Moreover, the reduced subschemes $M_{\text{red}}$ and $Z(\beta_i)_{\text{red}}$ are 0-dimensional. The forgetful maps $Z(\beta_1), Z(\beta_2) \to M$ are closed immersions and allow to view the $Z(\beta_i)$ as cycles on $M$. 

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3.2 Hecke correspondences

Throughout this article, we work with a non-standard definition of integral models of Hecke operators that is derived from our non-standard presentation \([2.13]\) of the Hecke algebra. Therefore, we begin with some general remarks on Hecke correspondences that justify this approach and provide the comparison with the definitions in \([11]\) and \([23]\).

Write \(Z^c(M)\) for the group of cycles on \(M\) of codimension \(c\), similarly for \(M \times \text{Spf} O_E\). By correspondence on \(M\) we mean an element of

\[
\text{Corr}(M) := \{ z \in Z^{2n-1}(M \times \text{Spf} O_E) \mid p_1, p_2 : \text{Supp}(z) \to M \text{ both finite} \}. \tag{3.3}
\]

Correspondences form a ring through convolution and act on \(Z^*(M)\); we refer to Appendix \((3)\) for these definitions and their properties.

Recall that \(\mathcal{H}\) denotes the spherical Hecke algebra of \(GL_{2n}(F)\) with respect to \(GL_{2n}(O_F)\). Given an indicator function \(1_\mu \in \mathcal{H}\), there is a definition of Hecke correspondence \([R(1_\mu)] \in \text{Corr}(M)\). It is uniquely characterized by the requirement that its generic fiber parametrizes pairs \(((X_1, \rho_1), (X_2, \rho_2))\) such that \(\rho_2^{-1} \rho_1 : X_1 \to X_2\) is of type \(\mu\). The general definition of \([R(f)]\), \(f \in \mathcal{H}\) is obtained by linear extension. Given cycles \(z_1, z_2\) on \(M\) of complementary dimension and such that \(\text{Supp}[R(f)] \cap (\text{Supp} z_1 \times \text{Spf} O_E \times \text{Supp} z_2)\) is artinian, there is an intersection number \([z_1, [R(f)] \ast z_2]\). In this way, the Hecke action on cycles on \(M\) and the resulting intersection numbers are unambiguously defined.

An explicit construction of \([R(f)]\) in terms of formal scheme models \(R(f) : M \times M \to M \times M\) is given in \([11] \S 6.2\) or \([23] \S 4.3\), it relies on the notion of Drinfeld level structure. By Prop. \((5.8)\) the Hecke action \([R(f)] : Z^*(M) \to Z^*(M)\) is then expressed as

\[
[R(f)] \ast z = p_{1,*}[R(f) \times_M M \rho_2^{-1}(z)].
\]

(The bracket notation indicates passage to the cycle.) Assume furthermore that \(Z_1, Z_2 \subset M\) are closed formal subschemes of pure complementary dimension, both defined by regular sequences, and such that \(R(f) \cap (Z_1 \times \text{Spf} O_E \times Z_2)\) is artinian. The mentioned models \(R(f)\) are even regular, so Cor. \((3.10)\) implies

\[
([Z_1], [R(f)] \ast [Z_2]) = \ell_{O_E}(\text{Supp} R(f) \times_M M (Z_1 \times Z_2)). \tag{3.4}
\]

This provides the promised link of the intersection numbers from \([11] \& 13\) (given by the RHS) with the canonical ones on the LHS. We now come to our definition of Hecke correspondences, which, by the above arguments, lead to the same intersection numbers.

**Definition 3.4.** (1) Given \(m \geq 0\), denote by \((p_1, p_2) : R(m) \to M \times \text{Spf} O_E\) the functor

\[
R(m)(S) = \left\{ (X_0 \xrightarrow{\varphi} X_1, \rho) \mid (X_1, \rho) \in M(S), \varphi \text{ an } O_F\text{-linear isogeny of height } m \right\}. \tag{3.5}
\]

The map to \(M \times \text{Spf} O_E\) is given by

\[
(X_0 \xrightarrow{\varphi} X_1, \rho) \mapsto ((X_0, \rho \circ \varphi), (X_1, \rho)).
\]

It is a closed immersion that identifies \(R(m)\) with the locus of those pairs \(((X_0, \rho_0), (X_1, \rho_1))\) such that the quasi-isogeny \(\rho_1^{-1} \rho_0 : X_0 \to X_1\) is an isogeny of degree \(m\).

(2) For a tuple \(m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}\), we define

\[
R(m) := R(m_1) \times_{p_2, M, p_1} R(m_2) \times_{p_2, M, p_1} \cdots \times_{p_2, M, p_1} R(m_r)
\]

by composition. In other words, \(R(m)\) represents the functor

\[
R(m)(S) = \left\{ (X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_r} X_r, \rho) \mid \text{Each } X_i \text{ a } \pi\text{-divisible strict } O_F\text{-module, } \right. \ \ 
\left. \ (X_r, \rho) \in M(S), \varphi_i \text{ an } O_F\text{-linear isogeny of height } m_i \right\}. \tag{3.6}
\]
We often write \((X\bullet, \rho)\) for such chains of isogenies. The map to \(M \times_{\text{Spf} \mathcal{O}_k} M\) takes the chain \((X\bullet, \rho)\) to \(((X_0, \rho \circ \varphi_r \circ \ldots \circ \varphi_1), (X_r, \rho))\).

(3) For \(k \in \mathbb{Z}\), multiplication in the framing defines an automorphism
\[
[\pi]^k : M \to M, \quad (X, \rho) \mapsto (X, \pi^k \rho).
\]
Taking its graph, we view \([\pi]^k : M \to M \times_{\text{Spf} \mathcal{O}_k} M\) as correspondence.

**Proposition 3.5.** For any \(m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}\), the formal scheme \(R(m)\) is Cohen–Macaulay and the two projection maps \(p_1, p_2 : R(m) \to M\) finite flat. In particular, Def. 3.4 provides a model for the Hecke correspondence,
\[
\left[\pi^k \times_{p_2, M, p_1} R(m)\right] = \left[R([\pi]^k * f(m))\right] \text{ in Corr}(M).
\]

**Proof.** First consider the case \(r = 1\) and the first projection \(p_1 : R(m) \to M\). Let \((X, \rho) \in M(S)\). The fiber of \(p_1\) over \((X, \rho)\) is the set of \(O_F\)-linear isogenies \(X \to Y\) of height \(m\) to strict \(\pi\)-divisible \(O_F\)-modules \(Y\), up to isomorphism in \(Y\). Considering the occurring kernels, this set is in bijection with certain finite locally free subgroups \(H \subset X\) that we now describe.

In the \(p\)-adic setting, \(H\) has to be \(O_F\)-stable, of order \(q_F^m\), and such that \(O_F\) acts on \(X/H\) strictly, meaning \(\iota(a) = a\) in \(\text{End}(\operatorname{Lie}(X/H))\). Varying \(S\) and \((X, \rho)\), this description shows that \(p_1\) is relatively representable in projective schemes.

In the function field setting, we still consider \(O_F\)-stable \(H\) of order \(q_F^m\). We also need to make sure that \(X/H\) is again a \(\pi\)-divisible \(O_F\)-module however. This is the case if and only if each truncation \((X/H)[\pi^r], r \geq 0\), is balanced as \(\mathbb{F}_q\)-module scheme. (Being balanced is a condition on the primitive elements in the Hopf algebra of the group scheme, cf. [19 Def. 5.13]). This property is equivalent to being a strict \(\mathbb{F}_q\)-module scheme, cf. [6 Rmk. 5.3]. It is required for the truncations of a \(\pi\)-divisible \(O_F\)-module by definition, cf. [6 Def. 7.1]. We are given that all \(X[\pi^r]\) are balanced. Then all \((X/H)[\pi^r]\) are balanced if and only if \(H\) is balanced, because this property is well-behaved in exact sequences. The condition for a finite locally free group scheme to be balanced is open and closed on the base, cf. [19 Rmk. 5.17]. Finally, we impose that the \(O_F\)-action on \(X/H\) is strict. In this way, we have again shown that \(p_1\) is relatively representable in projective schemes.

Now consider the special fiber \(\text{Spec} \mathbb{F} \times_M R(m) \to \text{Spec} \mathbb{F}\). It is of dimension 0 because a 1-dimensional \(\pi\)-divisible \(O_F\)-module over \(\mathbb{F}\) has at most finitely many subgroup of order \(q_F^m\). Thus \(p_1\) is finite and we obtain \(\dim R(m) \leq \dim M\).

The map \(R(m) \to M \times_{\text{Spf} \mathcal{O}_k} M\) is a closed immersion and \(M \times_{\text{Spf} \mathcal{O}_k} M\) is regular. We claim that \(R(m)\) is locally defined by at most \(2n - 1\) equations. Let \(X_i = p_i^*(X)\) be the pullback of the universal \(\pi\)-divisible \(O_F\)-module from the \(i\)-th component, \(i = 1, 2\). Write \(R(m) = V(I)\) for an ideal sheaf \(I\) and consider the Hodge filtrations of \(X_1|_{V(I^2)}\) and \(X_2|_{V(I^2)}\) as well as the map on \((O_F\text{-linear})\)
\[
\begin{align*}
0 & \to F_1 \to D_{X_1}(V(I^2)) \to \operatorname{Lie}(X_1|_{V(I^2)}) \to 0 \\
0 & \to F_2 \to D_{X_2}(V(I^2)) \to \operatorname{Lie}(X_2|_{V(I^2)}) \to 0.
\end{align*}
\]
(3.7)

Note that \(I/I^2 \subset \mathcal{O}_{V(I^2)}\) is defined by the condition that it is the minimal ideal \(J \subset \mathcal{O}_{V(I^2)}\) such that the universal quasi-isogeny \(p_2^{-1} \rho_1 : X_1|_{V(I^2)} \to X_2|_{V(I^2)}\) is an isogeny mod \(J\). (Here, \((X_1, \rho_1, X_2, \rho_2)\) denotes the universal point over \(M \times_{\text{Spf} \mathcal{O}_k} M\).) By Grothendieck–Messing theory, \(I/I^2\) equals the ideal defined by the condition that the composite map \(F_1 \to \operatorname{Lie}(X_2|_{V(I^2)})\) in (3.7) vanishes. The

\[\text{We refer to [6 Thm. 9.8] for the fact that the deformation theory of } \pi\text{-divisible } O_F\text{-modules in equal characteristic is the same as that in the } p\text{-adic setting.}\]
source is a vector bundle of rank $2n - 1$, the target a line bundle. Thus $R(m)$ is locally defined by $2n - 1$ equations as claimed. In particular, also $\dim R(m) \geq \dim M$ everywhere, so $R(m)$ is pure of dimension $2n$. It follows from the smoothness of $M$ that $R(m)$ is Cohen–Macaulay.

The flatness of both $p_1$ and $p_2$ is now deduced from “Miracle Flatness”: A local homomorphism $R \rightarrow S$ of noetherian local rings with $R$ regular and $S$ Cohen–Macaulay is flat if and only if its fiber dimension is constant, see [25, 00R4].

The obtained properties now extend to general $r$: The composition of finite flat correspondences is finite flat, so both $p_1, p_2 : R(m_1, \ldots, m_r) \rightarrow M$ are finite flat. Being finite flat over a regular local ring implies Cohen–Macaulay.

Finally, the identity $[[\pi]^k \times_{p_2, M, p_1} R(m)] = [R ([\pi]^k * f(m))]$ is clear for generic fibers. Since $R(m)$ is flat over $M$, it is also flat over $O_E$, so the identity extends to cycles on $M$. □

### 3.3 Intersection numbers

**Definition 3.6.** For $m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}$ and $\beta$ as before, we define $I(\beta, m)$ by the Cartesian diagram

$$
\begin{array}{ccc}
I(\beta, m) & \longrightarrow & Z(\beta_2) \times_{\text{Spf } O_E} Z(\beta_1) \\
\downarrow & & \downarrow \\
R(m) & \longrightarrow & M \times_{\text{Spf } O_E} M.
\end{array}
$$

**Lemma 3.7.** Assume that $\beta$ is regular semi-simple. Then $I(\beta, m)$ is a (possibly infinite) union of artinian schemes.

**Proof.** This follows a posteriori from the length computations in [7][11]. We give an alternative a priori argument. Let $\text{Spf } A \subset I(\beta, f)$ be an affine closed formal subscheme, given by some chain $(X_\bullet, \rho) \in R(m)(\text{Spf } A)$. Then $X := X_r$ algebraizes to a strict $\pi$-divisible $O_F$-module $X$ over $\text{Spec } A$ that comes with a pair of embeddings

$$(\rho^{-1}\beta_1\rho, (\rho\varphi_r \cdots \varphi_1)^{-1}\beta_2(\rho\varphi_r \cdots \varphi_1)) : (O_{E_1}, O_{E_2}) \longrightarrow \text{End}^0(X).$$

The intersection of its image with $\text{End}(X)$ generates an order in a quaternion algebra $B$ over an étale $F$-algebra $L$ of degree $n$ because $\beta$ is regular semi-simple, see Prop. [25]. Moreover $B \neq M_2(L)$ because $B$ embeds into $\text{End}^0(X)$. Our task is to show that $\rho$ algebraizes, i.e. that $(X_\bullet, \rho) \in R(m)(\text{Spec } A)$.

It is well-known that $p$-divisible groups in characteristic 0 are étale. The analogous statement holds in the function field setting. Namely let $X/S$ be a strict $\pi$-divisible $O_F$-module. The three properties $X$ being $\pi^\infty$-torsion, $\pi \in O_S^*$ and $I(\pi) = \pi$ in $\text{End}(\text{Lie}(X))$ imply $\text{Lie}(X) = 0$. The only strict $\pi$-divisible $O_F$-module of height $2n$ over a geometric point with $\pi \neq 0$ is thus $(F/O_F)^{2n}$ (up to isomorphism). Its endomorphism ring is $M_{2n}(O_F)$ and there is no possibility for a pair of invariant $\text{Inv}(\beta)$ as in [10]. Thus $A$ has to be $\pi$-torsion, $k \gg 0$.

Concerning $O_F$-fields with $\pi = 0$, the geometric isogeny class of $X$ is the only one of height $2n$ and dimension 1 that admits a pair of embeddings of invariant $\text{Inv}(\beta)$, see Lem. [12]. Thus all fibers of $X$ are of the same geometric isogeny class. In particular, the connected component of identity $X^0$ is again a strict $\pi$-divisible $O_F$-module, cf. [16] Prop. II.4.9 ($p$-adic case) and [6] Prop. 10.16I (function field case).

Let $A \rightarrow R$ be a local homomorphism to the ring of integers in a complete, algebraically closed, non-archimedean field $K$. Denote its residue field by $k$. We use index notation $X_R$, $X_K$, $X_k$ etc. to denote base changes to $R$, $K$, $k$ etc. The map

$$\text{End}(X_R) \longrightarrow \text{End}(X_K) = O_{D_{1/2n}} \times M_{2n}(O_F)$$

is injective, so the reduction map

$$\text{End}(X_R) \longrightarrow \text{End}(X_k)$$

is also injective.
is a bijection. If we manage to find an isogeny \( \tau : X_R \to X_R \), then the composition \((\rho \tau^{-1})\tau : X_R \to X_R\) is a \((\text{Spec } R)\)-valued point of \( M \) that agrees with the given \( \text{Spf } R \)-valued one. But any map \( \text{Spec } R \to M \) factors through one of the closed points of \( M \) because \( M \) is a disjoint union of formal spectra of complete local rings. Since \( R \) was arbitrary, this implies the statement of the lemma.

Since \( R \) is perfect, we may write \( X_R = X_R^0 \times X_R^1 \) and construct \( \tau \) component-wise

\[
\tau^0 : X_R^0 \longrightarrow X_R^0, \quad \tau^1 : X_R^1 \longrightarrow X_R^1.
\]

Both \( X_R^0 \) and \( X_R^1 \) are isomorphic to \((F/OF)^{2n^0}\) and we take \( \tau \) as any isomorphism. For \( \tau^0 \), we consider \( X_R^0 \cong \text{Spf } R[t] \) as a formal group. (See [6, Cor. 10.12] for the equivalence of formal \( O_F \)-modules and connected strict \( \pi \)-divisible \( O_F \)-modules.) Since special and generic fiber of \( X_R^0 \) are both of height \( 2n^0 \), and since \( K \) is algebraically closed, we may normalize the coordinate such that multiplication by \( \pi \in O_F \) is expressed by

\[
[\pi](t) = t^{2n^0}.
\]

The power series describing addition and the \( O_F \)-action all commute with \([\pi](t)\) and now have coefficients in the finite field \( F_{q^{2n_0}} \). In other words, \( X_R^0 \) comes via base change from \( F \) and is thus isomorphic to \( X_R^0 \).

Let \( L := L(\beta) \subset \text{End}^0(\mathfrak{X}) \) denote the centralizer of \( \beta_1(E_1) \cup \beta_2(E_2) \). Its units \( L^\times \) act on all objects in \( \mathfrak{X} \) by transport of structure along \( \rho \).

**Lemma 3.8.** Assume that \( \beta \) is regular semi-simple. Then the quotient \( L^\times \backslash \pi_0(\mathcal{I}(\beta, m)) \) is finite.

**Proof.** Since \( I(\beta, m)(\mathbb{F}) = \pi_0(I(\beta, m)) \), this is purely a statement about \( \mathbb{F} \)-points. Given a \( \pi \)-divisible group \( X \) over \( \mathbb{F} \), we write \( X = X^0 \times X^1 \) for its decomposition into connected and étale part. Then every \( \mathbb{F} \)-point \((X, \rho) \in M(\mathbb{F})\) is (in a unique way) a product \((X^0 \times X^1, \rho^0 \times \rho^1)\), where \( \rho^0 : X^0 \to \mathbb{X}^0 \) and \( \rho^1 : X^1 \to \mathbb{X}^1 \) are two quasi-isogenies. Moreover, giving an isogeny \( \varphi_i : X_{i-1} \to X_i \) of height \( m_i \) is the same as giving a pair of isogenies

\[
\varphi^0_i : X^0_{i-1} \longrightarrow X^0_i, \quad \varphi^1_i : X^1_{i-1} \longrightarrow X^1_i
\]

with \( \text{ht}(\varphi^0_i) + \text{ht}(\varphi^1_i) = m_i \). In this way, we obtain

\[
\pi_0(I(\beta, m)) = \prod_{m^0 + m^1 = m} \pi_0(I(\beta^0, m^0)) \times \pi_0(I(\beta^1, m^1)) \tag{3.10}
\]

where \( m^0, m^1 \in \mathbb{Z}_{\geq 0} \). Moreover, the centralizer is a product \( L = L^0 \times L^1 \) and its units act diagonally on the right of \( \mathfrak{X} \). This reduces us to the two cases of \( \mathfrak{X} \) being connected or étale.

If \( \mathfrak{X} \) is connected, then \( \pi_0(M) \cong \mathbb{Z} \) via \((X, \rho) \mapsto \text{ht}(\rho)\). The element \( \pi \in F^\times \) acts through translation by \( 2n \), proving the statement in this case. If \( \mathfrak{X} \) is étale, then \( \pi_0(M) \cong GL_{2n}(F)/GL_{2n}(O_F) \) and the argument is the same as for the convergence of (2.7).

In fact, \([R(f)]\) is invariant under the diagonal action of \( \text{Aut}^0(\mathfrak{X}) \) on \( M \times_{\text{Spec } O_E} M \) for every \( f \). Lem. 3.7 and Lem. 3.8 imply that \( \text{Supp}[R(f)] \times_M M \) \((Z(\beta_2) \times Z(\beta_1))\) is artinian and finite modulo \( L^\times \) whenever \( \beta \) is regular semi-simple.

**Definition 3.9.** Let \( \beta \) be regular semi-simple. Endow \( L = L(\beta)^\times \) with the measure such that \( \mu(O_L^\times) = 1 \). Let \( \Gamma \subseteq L^\times \) be a cocompact subgroup that acts freely on \( \pi_0(M) \). For \( f \in \mathcal{H} \), define the intersection number

\[
\text{Int}(\beta, f) := \mu(\Gamma \backslash L^\times)^{-1}(\Gamma \backslash [R(f)], \Gamma \backslash (Z(\beta_2) \times Z(\beta_1)))
\]

\[
= \sum_{x \in L^\times \backslash \pi_0(\text{Supp}[R(f)] \times Z(\beta_2) \times Z(\beta_1))} \mu(\text{Stab}(x))^{-1}([R(f)]_x, (Z(\beta_2) \times Z(\beta_1))_x). \tag{3.11}
\]
The intersection of the first expression is taken on $\Gamma \backslash (M \times M)$. The intersections of the second happen on the connected component $x$. More concretely, write $f = \sum_{i \in I} \lambda_i \cdot [\pi]^{k_i} \ast f(m^{(i)})$ for some finite index set $I$. (Recall that the $f(m^{(i)})$ were defined in (2.10).) Then, by Prop. 3.5 and Cor. 5.10

$$\text{Int}(\beta, f) = \sum_{i \in I} \lambda_i \cdot \text{Int}(\beta, m^{(i)})$$

where the summands on the right hand side are defined by

$$\text{Int}(\beta, m) := \mu(\Gamma \backslash L^\times)^{-1} \ell_{O_E}(\mathcal{O}_{\Gamma \backslash I(\beta, m)})$$

$$= \sum_{x \in L^\times \backslash \pi_0(I(\beta, m))} \mu(\text{Stab}(x))^{-1} \ell_{O_E}(\mathcal{O}_{I(\beta, m), x}).$$  \hspace{1cm} (3.12)

Here, we have also used the following analog of (2.11):

$$\text{Int}(\beta, f) = \text{Int}(\beta, [\pi]^{k} \ast f), \quad f \in \mathcal{H}, \; k \in \mathbb{Z}.$$  \hspace{1cm} (3.13)

Namely, $[\pi]^{k} \ast R(f)$ is defined by composing $R(f)$ with the graph of the automorphism $\pi^{-k} : M \rightarrow M$, $(X, \rho) \mapsto (X, \pi^{-k} \rho)$, and this automorphism maps $Z(\beta_1)$ to itself.

### 3.4 The AFL

Recall that $E_0 = F \times F$, and that $E_3 \subset E_1 \otimes_F E_2$ is the fixed ring under $\sigma_1 \otimes \sigma_2$. Recall further that

$$\partial \text{Orb}(\alpha, f) = \frac{1}{\log(q_f)} \left. \frac{d}{ds} \right|_{s=0} O(\alpha, f, s)$$  \hspace{1cm} (14.04)

denotes the normalized first derivative. By Cor. 2.18 and Lem. 3.2, the following conjecture concerns precisely those invariants $\delta$ with $\text{ord}(\delta) = 1$.

**Conjecture 3.10.** Let $\alpha : (E_0, E_3) \rightarrow M_{2n}(F)$ be a regular semi-simple pair of embeddings that matches to a pair $\beta : (E_1, E_2) \rightarrow \text{End}^n(\mathcal{X})$. Then, for any $f \in \mathcal{H}$, there is an equality

$$\partial \text{Orb}(\alpha, f) = \text{Int}(\beta, f).$$

**Remark 3.11.** (1) In case $\mathcal{X} = \mathcal{X}^0$ connected, this is essentially the conjecture of Howard and the first author [7] (resp. the first author [11] if $E_1 = E_2$). The conjectures stated there are slightly different, however, because they do not take into account connected components. This is remedied with the definition in (3.12). The phenomenon does not yet occur for $n = 1$ or $n = 2$ because in these cases $L^\times$ acts transitively on $\pi_0(I(\beta, f))$. It is, unfortunately, quite challenging to verify an explicit example for $n = 3$ that shows that $\text{Int}(\beta, f)$ is the correct definition. We offer a slightly different identity in Ex. 3.14 below that provides some evidence however. We also remark that our definition of $\text{Int}(\beta, f)$ is the one that comes naturally from a global intersection problem.

(2) The case $n = 1$ of Conj. 3.10 is proved in [7,11]. The case of $n = 2$ and $f = 1_{GL_4(O_F)}$ is shown in [12] (for $E_1 = E_2$) and the forthcoming work [13] (for $E_1 \neq E_2$).

(3) Assume $E_1 = E_2 = E$ is unramified over $F$. Given any pair $\beta$, one may write $\beta_2 = \mu \beta_1 \mu^{-1}$ for some $\mu \in D^\times$. Assume that $|m| \not\equiv v_D(\gamma)$ modulo 2. Then $R(m) \cap (Z(\beta_2) \times Z(\beta_1)) = \emptyset$. One can also show (omitted) that the orbital integral then vanishes identically, $O(\alpha, f, s) = 0$. In particular, Conj. 3.10 holds in that case.

It is enough to know Conj. 3.10 for a $C$-basis of $\mathcal{H}$. Our main result to this end is the following analogue of the analytic reduction formula Thm. 2.12.
Theorem 3.12. For every \( m \in \mathbb{Z}_{\geq 0} \), the following identity holds:

\[
\text{Int}(\beta, m) = |\text{Disc}_{E_1/F} \cdot \text{Disc}_{E_2/F}|^{-\frac{a_0 + a_1}{2}} F |\text{Res} \left( \text{Inv}(\beta^0), \text{Inv}(\beta^1) \right)|^{-1} \sum_{m = m^0 + m^1} q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|} |\text{Int}(\beta^0, m^0) \text{O}(\beta^1, f(m^1))|.
\]

(3.15)

Here, we have used the notation \( |x|_F := |Nm_{E_3/F}(x)|^{1/2} \) for elements \( x \in E_3 \).

Comparing this with (2.21), we obtain the following corollary.

Corollary 3.13. Assume that the FL (Conj. 2.11) holds. Then the AFL (Conj. 3.11) holds in all cases if it holds in all basic cases, meaning in all cases where \( X = X^0 \) is connected.

More precisely, the AFL then holds for some \( X \), some \( f(m) \) and some \( \beta \) if it holds for \( \beta^0 \) and every \( f(m^0) \) with \( 0 \leq m^0 \leq m \).

In particular, the AFL holds whenever \( \text{ht}(X^0) = 2 \). It holds for the unit function \( f = 1_{GL_{2n}(O_F)} \) if \( \text{ht}(X^0) = 4 \).

Proof. We show the more precise, second statement. Consider \( X = X^0 \times X^1 \) of height \( 2n = 2n^0 + 2n^1 \), a pair \( \beta = (\beta^0, \beta^1) : (E_1, E_2) \to D_{2n^0} \times M_{2n^1}(F) \), and some \( m \in \mathbb{Z}_{\geq 0}^2 \). Let \( \alpha^0 : (E_0, E_3) \to M_{2n^0}(F) \) match \( \beta^0 \) and let \( \alpha^1 : (E_0, E_3) \to M_{2n^1}(F) \) match \( \beta^1 \). In particular, their direct sum matches \( \beta \). Set \( \delta^j := \text{Inv}(\alpha^j) = \text{Inv}(\beta^j) \) in the following. Also define

\[
\Theta = |\text{Disc}_{E_1/F} \cdot \text{Disc}_{E_2/F}|^{-\frac{a_0 + a_1}{2}} F |\text{Res} \left( \delta^0, \delta^1 \right)|^{-1}
\]

and note that this expression equals

\[
|\text{Disc}_{E_1/F}|^{-\frac{a_0 + a_1}{2}} F |\text{Res} \left( \delta^0, \delta^1 \right)|^{-1}.
\]

Namely, taking \( E_1 \) to be unramified, \( \text{Disc}_{E_1/F} \) is trivial and

\[
\text{Disc}_{E_2/F} = \text{Disc}_{E_1 \otimes_F E_2/E_1} = \text{Disc}_{E_1 \otimes_F E_3/E_1} = \text{Disc}_{E_3/F}
\]

where the outer identities follow from the unramifiedness of \( E_1/F \) and where the middle one follows from \( E_1 \otimes_F E_3 \cong E_1 \otimes_F E_2 \). Now we apply Thm. 2.12. Substituting the concrete expression for the partial Satake transformation \( S(f(m)) \) from (2.20), taking the derivative at \( s = 0 \), and applying the vanishing of \( O(\alpha^0, f(m^0)) \) for every \( m^0 \in \mathbb{Z}_{\geq 0}^2 \) (see Prop. 2.14), this theorem states that

\[
\partial \text{Orb}(\alpha, f(m)) = \Theta \sum_{m = m^0 + m^1} q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|} \partial \text{Orb}(\alpha^0, f(m^0)) O(\alpha^1, f(m^1)).
\]

(3.16)

On the other hand, Thm. 3.12 states that

\[
\text{Int}(\beta, m) = \Theta \sum_{m = m^0 + m^1} q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|} \text{Int}(\beta^0, m^0) O(\beta^1, f(m^1)).
\]

(3.17)

The fundamental lemma (Conj. 2.10), which we assumed to hold, states that \( O(\alpha^1, f(m^1)) \) equals \( O(\beta^1, f(m^1)) \) for all \( m^1 \). Furthermore, our assumption also is that \( \partial \text{Orb}(\alpha, f(m^0)) = \text{Int}(\beta^0, m^0) \) for all occurring \( m^0 \). Comparing (3.16) and (3.17) then shows that \( \partial \text{Orb}(\alpha, f(m)) = \text{Int}(\beta, m) \) as was to be shown. The last statement of the corollary follows from the known cases of the AFL in Rmk. 3.11. \[ \Box \]
The proof of Thm. 3.12 will be given in the next section. Here, we conclude with the promised example on connected component counts.

**Example 3.14.** Assume that $X = X^0$ is connected and that $E = E_1 = E_2$ is unramified over $F$. Let $L/F$ be an unramified field extension of degree $n$ and choose an embedding $L \to D = \text{End}^0(X)$. Write $B = \text{Cent}_D(L)$ for its centralizer and let $\beta : (E, E) \to B$ be a pair of embeddings of $F$-algebras whose image generates $B$ over $F$. This situation can only exist if $n$ is odd because $E \otimes_F L$ then embeds into $B$ by Prop. 2.5 (2) and hence has to be a field. Denote by $Z = Z(O_L, \beta) \subseteq M$ the closed formal subscheme of $(X, \rho)$ such that

$$\rho^{-1}O_L \rho, \rho^{-1}\beta_1(O_E) \rho, \rho^{-1}\beta_2(O_E) \rho \text{ all } \subseteq \text{End}(X).$$

Let $\Gamma = \pi^Z$. (Note that $\pi$ is also a uniformizer for $L$.) Then $\Gamma \backslash Z(O_L, \beta)$ has $2n$ connected components that correspond to the $2n$ possible characters of $O_{\beta_1(E)L}$, acting on the Lie algebra. Imposing compatible Kottwitz conditions for the two $O_E$-actions leaves us with only $n$ connected components. Each is isomorphic to the linear AFL of $GL\_2 \_L$. So

$$\ell_{O_L}(\Gamma \backslash Z) = n \text{Int}(\beta \otimes \text{id}_L, 1_{GL_2(O_L)}).$$

Let $\alpha : (E_0, E_3) \to M_{2n}(F)$ match $\beta$. The centralizer of $\alpha$ is isomorphic to $L$. In (2.13), we may sum only over the sets

$L(O_L[\alpha]) \subseteq L(\alpha_0), \ L(O_L[\alpha_3]) \subseteq L(\alpha_3)$

of lattices $\Lambda$ that are also $O_L$-stable to obtain a rational function in $q^s$,

$$O(O_L[\alpha], 1_{GL_{2n}(O_F)}, s) = \sum_{(\alpha_0, \Lambda_3) \in L \times (L(O_L[\alpha_0]) \times L(O_L[\alpha_3]))} \text{Vol}_L(\alpha_0, \Lambda_3) \Omega(\Lambda_0, \Lambda_3, s),$$

with $\delta(\Lambda_0, \Lambda_3) = 1$ if $\Lambda_0 = \Lambda_3$ and 0 otherwise. Then

$$O(O_L[\alpha], 1_{GL_{2n}(O_F)}, s) = O(\alpha \otimes \text{id}_L, 1_{GL_2(O_L)}, s),$$

(3.18)

and

$$\partial \text{Orb}(O_L[\alpha], 1_{GL_{2n}(O_F)}) = n \partial \text{Orb}(\alpha \otimes \text{id}_L, 1_{GL_2(O_L)}),$$

(3.19)

where the factor $n$ stems from the difference in normalization by $q_F$ or $q_L = q_F^n$. In this way, we have recovered the AFL conjecture over $L$ for $n = 2$ and the unit Hecke function, but only with the right convention for connected component counting.

## 4 Reduction to elliptic invariant

The purpose of this section is to prove the reduction formulas Thm. 2.12 and Thm. 3.12. The next three sections reduce this to a degree computation that will be accomplished in 4.4

### 4.1 Fibration of lattice sets

Let $V = V^0 \oplus V^1$ be a direct sum decomposition of $V = F^{2n}$ with $V^0 \cong F^{2n_0}$ and $V^1 \cong F^{2n_1}$. Denote by $p^V : V \to V^j$ the two projection maps. The fundamental observation is that there is a bijection of $O_F$-lattices $X \subseteq V$ and triples

$$\left\{ (X^0, X^1, s) \mid X^0 \subseteq V^0, X^1 \subseteq V^1 \text{ both } O_F \text{-lattices, } \begin{array}{l} s : X^1 \to V^0/X^0 \text{ any } O_F \text{-linear map} \end{array} \right\}.$$

It is given by

$$X \mapsto \left( X^0 := X \cap V^0, \ X^1 := p^1(X), \ s_X := [X^1 \to X^0 \to V^0/X^0] \right)$$

(4.1)
where $t : X^1 \to X$ is any choice of splitting for $p^1$. The inverse is given by
\[
(X^0, X^1, s) \mapsto X^0 + \{ (x_1), x_1 \in X^1 \}
\]
where $\bar{s} : X^1 \to V^0$ is any lift of $s$. The bijection moreover satisfies the following two functoriality properties.

(1) Assume $X \subseteq Y$. Then clearly $X^0 \subseteq Y^0$ and $X^1 \subseteq Y^1$. We claim that also the following diagram commutes
\[
\begin{array}{c}
X^1 \\
\downarrow s_X \\
V^0/X^0 \\
\downarrow s_V \\
Y^1 \\
\end{array}
\]
Namely $(\bar{s}_X(x^1), x^1) \in X$ lying in $Y$ means that there is $y^0 \in Y^0$ such that $\bar{s}_X(x^1) = \bar{s}_Y(x^1) + y_0$. Conversely, if for two lattices $X$ and $Y$ one has $X^j \subseteq Y^j$ and also that (4.2) commutes, then $X \subseteq Y$.

(2) Let $\zeta = (\zeta^0, \zeta^1) \in \text{End}(V^0) \times \text{End}(V^1) \subseteq \text{End}(V)$ be an endomorphism. Then $X$ is $\zeta$-stable if and only if each $X^j$ is $\zeta^j$-stable and $\zeta^0 \circ s = s \circ \zeta^1$.

### 4.2 Application to orbital integrals

Consider a regular semi-simple pair $\alpha : (E_0, E_3) \to M_{2n}(F)$. Let $B$ be the quaternion algebra generated by the image of $\alpha$, denote by $L = \text{Cent}(B)$ its center. Assume that $L = L^0 \times L^1$ is a product with eigenspace decomposition $V = V^0 \oplus V^1$. Given $m = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r$, define
\[
L(\alpha, m) = \left\{ (X_0, X_1, \ldots, X_r) \left| \begin{array}{l}
X_0 \subset V \text{ an } O_{E_0}\text{-lattice}, X_i \subset V \text{ an } O_{E_i}\text{-lattice}, \\
X_i \subset V \text{ an } O_F\text{-lattice for } i = 1, \ldots, r-1, \\
X_{i-1} \subseteq X_i \text{ of index } m_i
\end{array} \right. \right\}. \quad (4.3)
\]
Note that this definition matches (2.17). The above discussion provides a map
\[
L(\alpha, m) \rightarrow \prod_{m^0, m^1 \in \mathbb{Z}_{\geq 0}^r, m = m^0 + m^1} L(\alpha^0, m^0) \times L(\alpha^1, m^1)
\]
\[
(X_0, \ldots, X_r) \mapsto \left( (X_0^0, \ldots, X_r^0), (X_0^1, \ldots, X_r^1) \right). \quad (4.4)
\]
We use the shorter notation $X^*, X_0^*$ and $X_1^*$ in the following. Let $\Gamma^j \subset L^{j, \infty}$ be a subgroup with $\Gamma^j \times O^{\times}_{L^j} \cong L^{j, \infty}$ and put $\Gamma = \Gamma^0 \times \Gamma^1 \subseteq L^\times$. Then (4.3) is $\Gamma$-equivariant. Using (2.13) and the description of $f(m)$ in (2.17), we obtain
\[
O(\alpha, [\pi]^k \ast f(m), s) = \sum_{X^* \in \Gamma \setminus L(\alpha, m)} \Omega(X^*, X_0, s). \quad (4.5)
\]
The same results give
\[
O(\alpha^0, [\pi]^k \ast f(m^0), s) O(\alpha^1, [\pi]^k \ast f(m^1), s)
\]
\[
= \sum_{(X_0^*, X_1^*) \in \Gamma^0 \times L(\alpha^0, m^0) \times \Gamma^1 \setminus L(\alpha^1, m^1)} \Omega(X^*_0, X_0^*, s) \Omega(X^*_1, X_1^*, s). \quad (4.6)
\]
Observe now that the transfer factor from (2.9) is multiplicative: If $X^* \mapsto (X_0^*, X_1^*)$ in (4.4), then
\[
\Omega(X^*, X_0, s) = \Omega(X^*_0, X_0^*, s) \Omega(X^*_1, X_1^*, s). \quad (4.7)
\]
By the explicit description of the partial Satake transform (2.20), the reduction formula Thm. 2.12 (for $E_0$ and $E_3$) is now reduced to the following proposition.
Proposition 4.1. Each fiber of \( \mathcal{L}(\alpha^0, m^0) \times \mathcal{L}(\alpha^1, m^1) \) has

\[
\left| \text{Disc}_{E_1/F} \right|^{-\alpha^0/m^0} \left| \text{Res} \left( \text{Inv}(\alpha^0), \text{Inv}(\alpha^1) \right) \right|^{-1} \cdot q^{n^0|m^0+n^0|m^1}
\]

many elements. In other words, given any pair \((X_0^0, X_1^0) \in \mathcal{L}(\alpha^0, m^0) \times \mathcal{L}(\alpha^1, m^1)\), there are many tuples \((\gamma_0, \ldots, \gamma_r)\) of \(O_F\)-linear maps \(\gamma_i : X_i^0 \to V^0/X_i^0\) that fit into a commutative diagram

\[
\begin{array}{ccccccc}
X_0^0 & \rightarrow & X_1^0 & \rightarrow & \cdots & \rightarrow & X_r^0 \\
\gamma_0 & \downarrow & \gamma_1 & \downarrow & \cdots & \downarrow & \gamma_r \\
V^0/X_0^0 & \rightarrow & V^0/X_1^0 & \rightarrow & \cdots & \rightarrow & V^0/X_r^0
\end{array}
\]

and such that \(\gamma_0\) is \(O_{E_2}\)-linear, and such that \(\gamma_r\) is \(O_{E_0}\)-linear.

Given a regular semi-simple pair \(\beta : (E_1, E_2) \to M_{2n}(F)\) there is a completely analogous definition of a lattice set \(\mathcal{L}(\beta, m)\). If \(\beta\) factors through \(M_{2n}(F) \times M_{2n}(F)\), all the above considerations apply to provide a fibration

\[
\mathcal{L}(\beta, m) \rightarrow \prod_{m=m_0}^{m^1} \mathcal{L}(\beta^0, m^0) \times \mathcal{L}(\beta^1, m^1).
\]

In this way, Thm. 2.12 (for \(E_1\) and \(E_2\)) reduces to the following statement.

Proposition 4.2. Each fiber of \(\mathcal{L}(\beta^0, m^0) \times \mathcal{L}(\beta^1, m^1)\) has

\[
\left| \text{Disc}_{E_1/F} \cdot \text{Disc}_{E_2/F} \right|^{-\beta^0/m^0} \left| \text{Res} \left( \text{Inv}(\beta^0), \text{Inv}(\beta^1) \right) \right|^{-1} \cdot q^{n^0|m^0+n^0|m^1}
\]

many elements. In other words, given any pair \((X_0^0, X_1^0) \in \mathcal{L}(\beta^0, m^0) \times \mathcal{L}(\beta^1, m^1)\), there are many tuples \((\gamma_0, \ldots, \gamma_r)\) of \(O_F\)-linear maps \(\gamma_i : X_i^0 \to V^0/X_i^0\) that fit into a commutative diagram

\[
\begin{array}{ccccccc}
X_0^0 & \rightarrow & X_1^0 & \rightarrow & \cdots & \rightarrow & X_r^0 \\
\gamma_0 & \downarrow & \gamma_1 & \downarrow & \cdots & \downarrow & \gamma_r \\
V^0/X_0^0 & \rightarrow & V^0/X_1^0 & \rightarrow & \cdots & \rightarrow & V^0/X_r^0
\end{array}
\]

and such that \(\gamma_0\) is \(O_{E_2}\)-linear, and such that \(\gamma_r\) is \(O_{E_1}\)-linear.

4.3 Fibration of RZ spaces

Our aim is to adapt the previous arguments to strict \(\pi\)-divisible \(O_F\)-modules. We begin with some general observations about the connected-étale sequence. Let \(S\) be a scheme with \(\pi \in O_S\) locally nilpotent. Assume that \(X/S\) has fiberwise constant étale rank. Then, by [16 Prop. II.4.9] resp. [7 Prop. 10.16], it is an extension

\[
0 \rightarrow X^0 \rightarrow X \rightarrow X^1 \rightarrow 0
\]

of its maximal étale quotient by its identity component. We view \(\pi\)-divisible \(O_F\)-modules as sheaves for the fpqc-topology in the following. Then the maximal étale quotient \(X^1\) has the canonical presentation

\[
0 \rightarrow T \rightarrow V \rightarrow X^1 \rightarrow 0
\]

where

\[
T := TX^1 := \lim_{n \geq 0} X^1[\pi^n] \quad \text{and} \quad V := VX^1 := TX^1[\pi^{-1}]
\]

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denote the Tate module resp. rational Tate module of $X^1$. The natural map $V \to X^1$ in $\mathbb{F}_\ell$ is defined by

$$\pi^{-n}((\ldots,x_2,x_1,x_0)) \mapsto x_n,$$

where $(\ldots, x_2, x_1, x_0) \in TX^1$.

Applying $\text{Hom}(-, X^0)$ and taking into account that multiplication by $\pi$ is an automorphism of $V$ while $\text{Hom}(T, X^0)$ is torsion, we obtain an exact sequence

$$0 \to \text{Hom}(T, X^0) \to \text{Ext}^1(X^1, X^0) \to \text{Ext}^1(V, X^0). \tag{4.14}$$

The leftmost term here is torsion, the rightmost term torsion-free, so

$$\text{Hom}(T, X^0) = \text{Ext}^1(X^1, X^0)_{\text{torsion}}.$$

By definition, an extension

$$0 \to X^0 \to X \to X^1 \to 0$$

here is torsion if and only if there exists an integer $N \geq 1$ and a commutative diagram

$$\begin{array}{c}
0 \to X^0 \to X^0 \oplus X^1 \to X^1 \to 0 \\
\downarrow \downarrow \downarrow \downarrow \\
0 \to X^0 \to X \to X^1 \to 0.
\end{array} \tag{4.15}$$

**Proposition 4.3.** (1) An extension $0 \to X^0 \to X \to X^1 \to 0$ is torsion if and only if there exists an isogeny $\varphi : X \to X^0 \oplus X^1$.

(2) Assume $\varphi^0 : X^0 \to Y^0$ and $\varphi^1 : X^1 \to Y^1$ are two homomorphisms with $Y^1$ étale. Assume furthermore that we are given two torsion extensions

$$0 \to X^0 \to X \to X^1 \to 0 \quad \text{and} \quad 0 \to Y^0 \to Y \to Y^1 \to 0.$$ Then there exists a (necessarily unique) map $\varphi : X \to Y$ that restricts to $\varphi_0$ and extends $\varphi_1$ if and only if the following diagram commutes

$$\begin{array}{c}
TX^1 \xrightarrow{T\varphi^1} TY^1 \\
\downarrow \downarrow \\
X^0 \xrightarrow{\varphi^0} Y^0.
\end{array}$$

**Proof.** (1) The two $\pi$-divisible $O_F$-modules in the middle column of (4.15) have the same height and the kernel of the middle vertical arrow of (4.15) identifies with $X^1[\pi^N]$, is hence finite locally free. This shows the “only if” assertion.

For the converse, we observe that any homomorphism $\varphi : X \to X^0 \oplus X^1$ provides two homomorphisms $\varphi^0 = \varphi|_{X^0} : X^0 \to X^0$ and $\varphi^1 = (\varphi \mod X^0) : X^1 \to X^1$. If $\ker(\varphi)$ is finite locally free, then also the intersection $\ker(\varphi^0) = \ker(\varphi) \cap X^0$ has this property because it is open and closed in $\ker(\varphi)$. Thus $\varphi^0$ is an isogeny. Then $\ker(\varphi^1) = \ker(\varphi)/\ker(\varphi^0)$ is also finite locally free, so $\varphi^1$ is an isogeny. Let $\psi^0$ and $\psi^1$ be such that $\varphi^1 \psi^1 = \pi^N$ and $\psi^0 \varphi^0 = \pi^N$ for some $N \geq 0$. Then the class of $X$ in $\text{Ext}^1$ is annihilated by $\pi^2N$.

(2) The connecting homomorphism in (4.14) is such that $X$ is constructed from a map $s_X : TX^1 \to X^0$ as $X = (X^0 \oplus VX^1)/TX^1$. The claim is then obvious.

Apply these considerations now to $(X, \rho)$, the universal pair over $M$. Let

$$0 \to X^0 \to X \to X^1 \to 0 \tag{4.16}$$
be its connected-étale sequence and let $S \to M$ be a point. Both $X^0$ and $X^1$ inherit a strict $O_F$-action and $\rho$ provides a pair of quasi-isogenies
\[
\rho^0 : \overline{S} \times_S X^0 \to \overline{S} \times_{\text{Spec} \mathbb{F}} X^0, \quad \rho^1 : \overline{S} \times_S X^1 \to \overline{S} \times_{\text{Spec} \mathbb{F}} X^1
\] (4.17)
over the special fiber $\overline{S} = \mathbb{F} \otimes_{O_F} S$. It follows that $\overline{S} \times_S X$ is isogeneous to $\overline{S} \times_S (X^0 \times X^1)$ and hence $X$ is isogeneous to $X^0 \times X^1$. The extension class $S \times_M X$ is thus torsion by Prop. [4.3](1) and comes from a map $TX^1 \to X^0$.

Let $M_{X^0}$ and $M_{X^1}$ denote the RZ spaces for the two indicated strict $\pi$-divisible $O_F$-modules. Denote by $(Y^0, \rho^0)$ and $(Y^1, \rho^1)$ the universal points of $M_{X^0}$ resp. $M_{X^1}$. By abuse of notation, we continue to write $(Y^j, \rho^j)$ in place of $p_j^*(Y^j, \rho^j)$ for their pullbacks to $M_{X^0} \times M_{X^1}$. The Hom-functor $\text{Hom}(TY^1, Y^0)$ is then itself a strict $\pi$-divisible $O_F$-module. (In fact, since $M_{X^0} \times M_{X^1}$ is a union of formal spectra of strict complete local rings, it is isomorphic to $(Y^0)^{2n^1}$. Thus we obtain a map
\[
M_X \to \text{Hom}(TY^1, Y^0)
\]
\[
(X, \rho) \mapsto ((X^0, \rho^0), (X^1, \rho^1), \text{extension class of } X).
\] (4.18)

Here, we view $\text{Hom}(TY^1, Y^0)$ as a sheaf on $M_{X^0} \times M_{X^1}$ and $(X^j, \rho^j)$ denotes the image point in $M_{X^j}$. Prop. [4.3](1) proves the following proposition.

**Proposition 4.4.** The map (4.18) is an isomorphism. It moreover has the following functoriality property. Given $\zeta = (\zeta^0, \zeta^1) \in \text{End}^0(X^0) \times \text{End}^0(X^1)$, the closed formal subscheme $Z(\zeta) \subseteq M_X$ where $\rho^{-1}\zeta \rho$ is a homomorphism agrees with the subfunctor of $\zeta$-linear homomorphisms in
\[
(Z(\zeta^0) \times Z(\zeta^1)) \times (M_{X^0} \times M_{X^1}) \text{Hom}(TY^1, Y^0).
\] (4.19)

Here, an $S$-valued point $(X^0, \rho^0, X^1, \rho^1, h : TX^0 \to X^1)$ is called $\zeta$-linear if
\[
(\rho^1)^{-1}\zeta^1 \rho^1 \circ h = h \circ T((\rho^0)^{-1}\zeta^0 \rho^0).
\]

Consider now a regular semi-simple pair $\beta : (E_1, E_2) \to \text{End}^0(\mathbb{X})$, write $\beta^j : (E_1, E_2) \to \text{End}^0(\mathbb{X}^j)$, $j = 0, 1$ for its components. Then we have the three intersections $I(\beta, m)$, $I(\beta^0, m^0)$ and $I(\beta^1, m^1)$ that result from (4.18) applied to $X, X^0$ and $X^1$, respectively. The above considerations provide a map
\[
\Pi : I(\beta, m) \to \prod_{m=m^0+m^1} I(\beta^0, m^0) \times I(\beta^1, m^1)
\]
\[
(X_0 \xrightarrow{\zeta^0} X_1 \xrightarrow{\zeta^3} \ldots \xrightarrow{\zeta^0} X_r, \rho) \mapsto \left( (X_0 \xrightarrow{\zeta^0} \ldots \xrightarrow{\zeta^0} X^0_r, \rho^0) \right) \right).
\] (4.20)

**Proposition 4.5.** The map (4.20) is finite locally free. Its degree over $I(\beta^0, m^0) \times I(\beta^1, m^1)$ equals
\[
|\text{Disc}_{E_1/F} \cdot \text{Disc}_{E_2/F}| F^{n^0} \cdot n^0 |m^0| + n^0 |m^1|. \]

(4.21)

The proof will be given in the next section. Here we note the following description of $I(\beta, m)$, which is provided by Prop. [4.3](1) and Prop. [4.4](1). Namely
\[
I(\beta, m) \subseteq \prod_{i=0}^r \text{Hom}_{O_F}(TY_i^1, Y_i^0) |\prod_{m=m^0+m^1} I(\beta^0, m^0) \times I(\beta^1, m^1)
\]
is the subfunctor of those tuples $(\gamma_i : TY_i^1 \to Y_i^0)_{i=0}^r$ such that the diagram
\[
\begin{array}{ccc}
TY_0^1 & \xrightarrow{T \psi_0^1} & TY_1^1 & \xrightarrow{T \psi_1^1} & \cdots & \xrightarrow{T \psi_r^1} & TY_r^1 \\
\gamma_0 & \downarrow & \gamma_1 & \downarrow & \cdots & \downarrow & \gamma_r \\
Y_0^0 & \xrightarrow{\psi_0^0} & Y_1^0 & \xrightarrow{\psi_2^0} & \cdots & \xrightarrow{\psi_r^0} & Y_r^0
\end{array}
\] (4.22)

commutes, such that $\gamma_0$ is $O_{E_2}$-linear, and such that $\gamma_r$ is $O_{E_1}$-linear.
4.4 Degree computation

This section proves Prop. 4.1, Prop. 4.2 and Prop. 4.5 in parallel, the arguments being the same for all three cases. Our main interest lies in the RZ space intersection, so we use the terminology of $\pi$-divisible $O_F$-modules. Fix a pair $\beta : (E_1, E_2) \to M_{2n}(\mathbb{F})$ (resp. $\beta : (E_1, E_2) \to M_{2n}(F)$) that preserve a decomposition $V = V^0 \oplus V^1$. Fix lattice chains $\Lambda^0 \in \mathcal{L}(\alpha^0, m^0)$ and $\Lambda^1 \in \mathcal{L}(\alpha^1, m^1)$ (resp. $\Lambda^0 \in \mathcal{L}(\beta^0, m^0)$ and $\Lambda^1 \in \mathcal{L}(\beta^1, m^1)$). Taking $S = \text{Spec} \mathbb{F}$, the datum $\Lambda^0$ is equivalent to that of a chain $\Lambda^0 := \Lambda^0 \otimes_{O_F} (F/O_F)$ of étale $\pi$-divisible $O_F$-modules of type $(\alpha^0, m^0)$, (resp. $(\beta^0, m^0)$). Similarly, $T^0$ is the chain of Tate modules of $T^0 \otimes_{O_F} (F/O_F)$. In this way, all arguments about $\pi$-divisible $O_F$-modules in the following apply literally and provide the proofs of Prop. 4.4 and Prop. 4.5 We only formulate the case of $(E_1, E_2)$ however, leaving it to the reader to substitute for $(E_0, E_3)$.

Consider the following $\pi$-divisible $O_F$-modules over $S$,

\[
H_i = \text{Hom}_{O_F}(T_i, Y_i), \quad i = 0, \ldots, r
\]

\[
C_i = \text{Hom}_{O_F}(T_{i-1}, Y_i), \quad i = 1, \ldots, r
\]

as well as

\[
H^+_0 = \text{Hom}_{O_{E_2}}(T_0, Y_0), \quad H^-_0 = \text{Hom}_{O_{E_2\text{-conj}}}(T_0, Y_0),
\]

\[
H^+_i = \text{Hom}_{O_{E_1}}(T_i, Y_i), \quad H^-_i = \text{Hom}_{O_{E_1\text{-conj}}}(T_i, Y_i).
\]

Here, we used the notation

\[
\text{Hom}_{O_{E_2\text{-conj}}}(T_0, Y_0) = \{ f : T_0 \to Y_0 \mid f(at) = a^{a_2} f(t) \forall a \in O_{E_2}\}
\]

in the first line and an analogous notation in the second. We will now define a homomorphism

\[
\Phi = \begin{pmatrix}
-p_0 \\
-R_1 & L_1 \\
\vdots & \ddots & \ddots \\
-R_r & L_r \\
-p_r
\end{pmatrix} : \bigoplus_{i=0}^r H_i \longrightarrow \bigoplus_{i=0}^r C_i \oplus H^-_i
\]

such that $\ker(\Phi)$ precisely describes the tuples $(\gamma_0, \ldots, \gamma_r)$ in [4.20], [4.12] and [4.22]. The components $R_i$ and $L_i$ are the composition to the right and left in each square,

\[
R_i : H_{i-1} \longrightarrow C_i, \quad \gamma \mapsto [Y_{i-1} \to Y_i] \circ \gamma, \quad i = 1, \ldots, r
\]

\[
L_i : H_i \longrightarrow C_i, \quad \gamma \mapsto \gamma \circ [T_{i-1} \to T_i], \quad i = 1, \ldots, r
\]

The condition $L_i(\gamma_i) - R_i(\gamma_{i-1}) = 0$ precisely expresses commutativity of the square

$$
\begin{array}{ccc}
T_{i-1} & \longrightarrow & T_i \\
\gamma_{i-1} \downarrow & & \downarrow \gamma_i \\
Y_{i-1} & \longrightarrow & Y_i
\end{array}
$$

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The kernels of $p_2^- : H_0 \to H_0^-$ and $p_1^- : H_r \to H_r^-$ on the other hand should precisely be the $O_{E_2}$-linear (resp. $O_{E_1}$-linear) elements. Pick elements $\zeta_i$ such that $O_{E_i} = O_F[\zeta_i]$. Define
\[
\begin{align*}
\tilde{p}_2^- : & \quad H_0 \to H_0^-, \; f \mapsto [t \mapsto f(\zeta_2 t) - \zeta_1 f(t)] \\
\tilde{p}_1^- : & \quad H_r \to H_r^-, \; f \mapsto [t \mapsto f(\zeta_1 t) - \zeta_1 f(t)].
\end{align*}
\]
(4.28)

It is easily checked that $p_2^-$ and $p_1^-$ have image in $H_0^-$ resp. $H_r^-$ as claimed. For example,
\[
\begin{align*}
p_1^-(f)(\zeta_1 t) & = f(\zeta_1^2 t) - \zeta_1 f(\zeta_1 t) \\
& = (\zeta_1 + \zeta_1^2) f(\zeta_1 t) - \zeta_1 \zeta_1^2 f(t) - \zeta_1 f(\zeta_1 t) \\
& = \zeta_1^2 f(\zeta_1 t) - \zeta_1 f(t) \\
& = \zeta_1^2 p_1^-(f)(t).
\end{align*}
\]

This concludes the definition of (4.20).

**Lemma 4.6.** Let $E/F$ be an étale quadratic extension and $T$ an $O_E$-lattice. Let $Y$ be a $\pi$-divisible group with $O_E$-action, write $O_E = O_F[\zeta]$. Then $\text{Hom}_{O_E}(T, Y)$ is again a $\pi$-divisible $O_F$-module and equals the kernel of
\[
p : \text{Hom}_{O_E}(T, Y) \to \text{Hom}_{O_E\text{-conj}}(T, Y), \; f \mapsto [t \mapsto f(\zeta t) - \zeta f(t)].
\]

In particular, the following sequence is exact,
\[
0 \to \text{Hom}_{O_E}(T, Y) \xrightarrow{\iota} \text{Hom}_{O_E}(T, Y) \xrightarrow{p} \text{Hom}_{O_E\text{-conj}}(T, Y) \to 0.
\]
(4.29)

**Proof.** Choosing a basis of $T$, it suffices to consider $T = O_E$. Then both $\text{Hom}_{O_E}(O_E, Y)$ and $\text{Hom}_{O_E\text{-conj}}(O_E, Y)$ are isomorphic to $Y$ via $(y_1, y_2) \mapsto y_1$ and hence a $\pi$-divisible $O_F$-module as claimed. Then the cokernel of $\iota$ is a $\pi$-divisible $O_F$-module and has the same height as $\text{Hom}_{O_E\text{-conj}}(T, Y)$, so maps isomorphically onto it.

In particular, $\ker(\Phi)$ precisely describes the fiber of (4.23). The maps $R_i$ and $L_i$ are isogenies, so we have quasi-isogenies $L_r^{-1}R_i : H_{i-1} \to H_i$ that allow to define a quasi-homomorphism
\[
\tilde{p}_1^- := p_2^- R_r^{-1} R_r^{-1} \cdots L_r^{-1} R_1 : H_0 \to H_r^-.
\]

**Lemma 4.7.** The homomorphism $\Phi$ is an isogeny if and only if the quasi-homomorphism
\[
\begin{pmatrix} p_2^- \\ p_1^- \end{pmatrix} : H_0 \to H_0^- \oplus H_r^-
\]
is a quasi-isogeny. If both are quasi-isogenies, then their degrees are related by
\[
\deg(\Phi) = \deg \left( \begin{pmatrix} p_2^- \\ p_1^- \end{pmatrix} \right) \prod_{i=1}^r \deg(L_i).
\]
(4.30)

**Proof.** We may multiply $\Phi$ from the left by a quasi-isogeny of degree 1 in the following way,
\[
\begin{pmatrix} I_2^- \\ \vdots \\ -p_1^- L_r^{-1} \\ I_n \end{pmatrix} \begin{pmatrix} p_2^- \\ -R_1^- L_1 \\ \vdots \\ -R_r^- \end{pmatrix} = \begin{pmatrix} p_2^- \\ -R_1^- L_1 \\ \vdots \\ -R_r^- \end{pmatrix} \begin{pmatrix} p_2^- \\ -R_1^- L_1 \\ \vdots \\ -R_r^- \end{pmatrix}.
\]

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Just like in (4.29), the projections from (4.32) restrict to surjections \( f \) quasi-homomorphisms. We call a quasi-homomorphism following notion: Given two \( O \)-modules. These are \( \tilde{F} \)-vector spaces of dimensions \( 2n, 2n^0 \) and \( 2n^1 \), respectively. Also put

\[ P = N(\text{Hom}_{O_F}(T\mathcal{X},\mathcal{X}^0)) = \text{Hom}(N^1, N^0) \]

which has \( \tilde{F} \)-dimension \( 4n^0 n^1 \). Define the following subspaces of \( P \), each of half that dimension,

\[ P_1^+ = \text{Hom}_{E_1}(N^1, N^0), \quad P_1^- = \text{Hom}_{E_1,\text{conj}}(N^1, N^0) \]

\[ P_2^+ = \text{Hom}_{E_2}(N^1, N^0), \quad P_2^- = \text{Hom}_{E_2,\text{conj}}(N^1, N^0). \]

We also consider the projection maps to \( P_i^\pm \), for \( i = 1, 2 \),

\[ q_i^+: P \rightarrow P_i^+, \quad f \mapsto [t \mapsto f(\zeta_i t) - \zeta_i^q f(t)] \]

\[ q_i^-: P \rightarrow P_i^-, \quad f \mapsto [t \mapsto f(\zeta_i t) - \zeta_i^q f(t)]. \]

The given points \( (Y^1_j, \rho^1_j, \varphi^1_j) \in I(\beta^j, m^1)(S) \) define lattices in the above \( \tilde{F} \)-vector spaces. Let \( M(-) \) denotes the integral Dieudonné module functor over the special point of \( S \). Set

\[ M_1^0 = N(\rho_0^0)(M(Y^0_r)), \quad M_2^0 = N(\rho_0^0)(M(Y^0_0)) \]

\[ M_1^1 = N(\rho_1^0)(M(Y^1_r)), \quad M_2^1 = N(\rho_1^0)(M(Y^1_0)). \]  

(4.33)

Then \( M_1^0 \subset N^0 \) and \( M_1^1 \subset N^1 \) are \( O_{E_1} \)-stable, while \( M_2^0 \) and \( M_2^1 \) are \( O_{E_2} \)-stable. We note for later use that there is an inclusion \( M_2^0 \subseteq M_1^0 \) of index \( |m^0| \) because of the chain of isogenies

\[ Y^0_0 \xrightarrow{\varphi^0_1} Y^1_0 \xrightarrow{\varphi^0_2} \cdots \xrightarrow{\varphi^0_r} Y^0_r \]

that satisfies \( \rho_0^0 = \rho_0^1 \circ \varphi^0_r \circ \cdots \circ \varphi^0_1 \). In the same way, \( M_2^1 \subseteq M_1^1 \) with index \( |m^1| \).

Passing to Hom-spaces, these lattices provide Dieudonné lattices in \( P \) and the \( P_i^\pm \),

\[ \Lambda_i = \text{Hom}_{O_F}(M_1^i, M_1^0) \]

\[ \Lambda_i^+ = \text{Hom}_{O_{E_1}}(M_1^i, M_1^0) \]

\[ \Lambda_i^- = \text{Hom}_{O_{E_1,\text{conj}}}(M_1^i, M_1^0). \]

(4.34)

Just like in (4.29), the projections from (4.32) restrict to surjections \( q_i^\pm: \Lambda_i \rightarrow \Lambda_i^\pm \). Let us adopt the following notion: Given two \( O_F \)-lattices or \( O_F \)-lattices \( \Lambda \) and \( \Lambda' \), we write \( f: \Lambda \rightarrow \Lambda' \) for homomorphisms \( f: \Lambda[\pi^{-1}] \rightarrow \Lambda'[\pi^{-1}] \) together with the datum of the two lattices. We call such morphisms quasi-homomorphisms. We call a quasi-homomorphism \( f \) a quasi-isogeny if \( f: \Lambda[\pi^{-1}] \rightarrow \Lambda'[\pi^{-1}] \) is bijective. In this case, the degree of \( f \) is defined as the valuation of the ratio \( \det(f(\Lambda'))/\det(\Lambda) \).
Proposition 4.8. Assume that any quasi-isogeny \( z \) if one of the six maps appearing below is a quasi-isogeny, then all of them are. In this case, the degrees of \( \Lambda \) is a quasi-isogeny and its degree is
\[
|\text{Disc}_{E/\mathbb{F}} \cdot \text{Disc}_{E_2/\mathbb{F}}|^{-\frac{n^0}{2}} \cdot |\text{Res}(\text{Inv}(\beta^0), \text{Inv}(\beta^1))|^{-1} \cdot q^{n^1|m^n| - n^0|m^1}|.
\]

Here, we have used the notation \(|x|_F := |\text{Nm}_{E/F}(x)|^{1/2}\) for elements \( x \in E_3 \).

We first prove two lemmas.

**Lemma 4.9.** If one of the six maps appearing below is a quasi-isogeny, then all of them are. In this case, the degrees of
\[
(q_1^+, q_2^-) : \Lambda_1 \rightarrow \Lambda_1^- \times \Lambda_2^-
\]
all agree. Similarly, the degrees of
\[
(q_1^-, q_2^+) : \Lambda_2 \rightarrow \Lambda_1^- \times \Lambda_2^-
\]
all agree.

**Proof.** We prove the claim for the first triple of maps. Consider the exact sequence from (4.29) to obtain the following commutative diagram
\[
0 \rightarrow \Lambda_1^+ \rightarrow \Lambda_1 \rightarrow \Lambda_1^- \rightarrow 0
\]
(Note that \( q_1^- : \Lambda_1 \rightarrow \Lambda_1^- \) is surjective even if \( E_1/F \) is ramified; the argument is as for the map \( p \) in (4.29).) The claim follows from this diagram for the two maps \( (q_1^+, q_2^-) \) and \( q_2^-|_{\Lambda_1^+} \) in (4.37). Consider now any quasi-isogeny \( z \in \text{End}_{E/F}^0(\mathcal{X}_0) \) that Galois commutes with both \( E_1 \) and \( E_2 \) in the sense that \( z\mathcal{G}_i = \mathcal{G}_i^0 \cdot z \) for both \( i = 1, 2 \). (Such elements always exist; a specific choice is given by the element \( z \) from [7 (2.4.2)]. It Galois commutes with both \( E_i \) by [7 Prop. 2.4.2] and is invertible because \( \beta^0 \) is regular semi-simple. The latter implies \( t \) in loc. cit. to be invertible by definition, then [7 Prop. 2.4.1] implies \( z \) invertible.) The element \( z \) intertwines \( q_2^-|_{\Lambda_1^+} \) and \( q_2^+|_{\Lambda_1^-} \) in the sense that the following diagram commutes.

\[
\begin{array}{ccc}
\Lambda_1^+ & \xrightarrow{f} & \Lambda_1^- \\
q_2^- \downarrow & & q_2^+ \downarrow \\
\Lambda_2^- & \xrightarrow{f} & \Lambda_2^+. \\
\end{array}
\]
Since $z$ is a quasi-isogeny and since the horizontal maps have the same degree, this proves the claimed shared properties of the first triple. The case of the second triple is the same by symmetry. Moreover, $Λ_1$ and $Λ_2$ are lattices in the same isocrystal, so the first three maps are quasi-isogenies if and only if the second triple is.

**Lemma 4.10.** The endomorphism $q_1^+ q_2^- |_{Λ_1^+} : Λ_1^+ → Λ_1^+$ is a quasi-isogeny of degree

$|\text{Disc}_{E_1/F} \cdot \text{Disc}_{E_2/F} |_{E_1}^{-n_0 n_1} |\text{Res}(\text{Inv}(β^0), \text{Inv}(β^1))|_{E_1}^{-2}$.

**Proof.** For any $f ∈ Λ_1^+$, we have $β_0^1(x) ⋅ f = f ⋅ β_1^1(x)$ for every $x ∈ E_1$. From definition \(132\), we obtain that

\[
q_1^+ (q_2^- (f)) = \left( f ⋅ β_1^1(ζ_1) - β_0^1(ζ_2) ⋅ f \right) ⋅ β_1^1(ζ_1)
\]

definition of $q_2^-$ in \(132\)
\[
- β_1^0(ζ_1^{v_1}) + \left( β_2^0(ζ_2^{v_2}) ⋅ f - f ⋅ β_2^1(ζ_2^{v_2}) \right)
\]

equals $q_2^- (f)$, see \(4.31\) below
\[
= f ⋅ w(β^1) - w(β^0) ⋅ f
\]

with the two elements

\[
w(β^0) = (β_2^0(ζ_2) ⋅ β_1^0(ζ_1) + β_2^0(ζ_1) ⋅ β_1^0(ζ_2))
\]
\[
w(β^1) = (β_2^1(ζ_2) ⋅ β_1^1(ζ_1) + β_1^1(ζ_1) ⋅ β_2^1(ζ_2)).
\]

We have also used the simple observation

\[
f ⋅ ζ_i - ζ_i ⋅ f = ζ_i^{v_1} ⋅ f - f ⋅ ζ_i^{v_1}
\]

for all maps $f$, which results from $f ⋅ \text{tr}(ζ_i) = \text{tr}(ζ_i) ⋅ f$. The elements $w(β^0)$ and $w(β^1)$ are precisely the elements $w$ for the two pairs $β^0$ and $β^1$ from \(7\) (2.4.1)]. They have the property that each of them is both $O_{E_1}$- and $O_{E_2}$-linear, i.e. $w(β)$ centralizes $β^i$, see \(7\) Prop. 2.4.2. Note that $\text{End}_{E_1}(X^1) ≅ M_{n_1}(E_1)$, also write $D = \text{End}_{E_1}(X^0)$. Thus we may view

\[
w(β^0) ∈ D, \quad w(β^1) ∈ M_{n_1}(E_1)
\]

and $q_1^+ q_2^- |_{Λ_1^+} = w(β^1) ⊗ 1 - 1 ⊗ w(β^0) ∈ M_{n_1}(E_1) ⊗_{E_1} D$. We recall a basic fact about the resultant. Assume we are given $A ∈ M_n(k)$ and $B ∈ M_m(k)$ where $k$ is any field. Then

\[
\det \left[ A ⊗ 1 - 1 ⊗ B : M_{n×m}(k) → M_{n×m}(k), \quad f → Af - fB \right] = \text{Res}(P_A, P_B),
\]

where $P_A$ and $P_B$ denote the characteristic polynomials of $A$ and $B$. (This identity is clear if $A$ and $B$ are diagonalizable. The general case follows from a Zariski density argument and by passing to the algebraic closure of $k$.) Put differently, the determinant of $A ⊗ 1 - 1 ⊗ B ∈ M_{n,k} ⊗_k M_{m,k}$, viewed as $(nm × nm)$-matrix is $\text{Res}(P_A, P_B)$. Since determinants may be computed after scalar extension, these considerations also apply to central simple algebras and we obtain

\[
\text{deg}(q_1^+ q_2^- |_{Λ_1^+}) = \left|\text{Nrd}_{M_{n^2}(D)/E_1} (w(β^1) ⊗ 1 - 1 ⊗ w(β^0))\right|_{E_1}^{-1}
\]
\[
= \left|\text{Res}(\text{charred}(w(β^0)), \text{charred}(w(β^1)))\right|_{E_1}^{-1}
\]
\[
= \left|\text{Res}(\text{charred}(w(β^0)), \text{charred}(w(β^1)))\right|_{F}^{-1}. \quad (4.42)
\]

It is shown in \(7\) Prop. 2.4.1] that the two characteristic polynomials here are related to the invariants of $β^0$ and $β^1$ by a linear transformation. More precisely, one has $w(β^i) = d s_β^i + c$ in $E_3 ⊗_{E_2} M_{2n^2}(F)$ resp. $E_3 ⊗_{E_2} D_{1/2n^2}$ where $c, d ∈ E_3$ are the following two elements,

\[
c = - (ζ_1 ⊗ ζ_2^{v_2} + ζ_1^{v_1} ⊗ ζ_2), \quad d = (ζ_1 - ζ_1^{v_1}) ⊗ (ζ_2 - ζ_2^{v_2})
\]

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Taking the square root proves Prop. 4.8. Substituting (4.35) and Lem. 4.10, this means that (4.46) is a quasi-isogeny and that its degree is given by (4.36). Then Lem. 4.7 states that \( \rho \) is an isogeny. Composing with the quasi-isogenies from Lem. 4.9 and Lem. 4.10 already show that (4.35) is a quasi-isogeny. From Lem. 4.7 we obtain

\[
\text{deg}((q_1^-, q_2^-) : \Lambda_2 \rightarrow \Lambda_1^- \times \Lambda_2^-)^2 = \text{deg}(\Lambda_2 \rightarrow \Lambda_1^-) \cdot \text{deg}((q_1^-, q_2^-) : \Lambda_1^- \times \Lambda_2^-) \cdot \text{deg}((q_1^-, q_2^-) : \Lambda_2^- \rightarrow \Lambda_1^-) = \text{deg}(\Lambda_1 \rightarrow \Lambda_2^-) \cdot \text{deg}((q_1^+, q_2^-) : \Lambda_1^+ \rightarrow \Lambda_2^+).
\]

Substituting (4.36) and Lem. 4.10 this means that \( \text{deg}((q_1^-, q_2^-) : \Lambda_2 \rightarrow \Lambda_1^- \times \Lambda_2^-)^2 \) equals

\[
|\text{Disc}_{E_1/F} \cdot \text{Disc}_{E_2/F}|_{F}^{-n^2-n^1} \cdot |\text{Res}(\text{Inv}(\beta^0), \text{Inv}(\beta^1))|_{F}^{-1} \cdot q^{2n^1 \cdot |m^0| - 2n^0 \cdot |m^1|}.
\]

Taking the square root proves Prop. 4.8.

**Proof of Propositions 4.1, 4.2, 4.5 and Theorems 2.12, 3.12** We take up the setting of Lem. 4.7. Recall that this lemma concerned the quasi-homomorphism

\[
\binom{p_2}{p_1} : \text{Hom}(T_0, Y_0) \rightarrow \text{Hom}_{O_{E_2}}^\text{conj}(T_0, Y_0) \times \text{Hom}_{O_{E_1}}^\text{conj}(T_r, Y_r).
\]

Composing with the quasi-isogenies \( \rho_0^0, \rho_0^1, \rho_1^0, \rho_1^1 \), and applying the Dieudonné module functor \( M(\_\_\_) \) over the special fiber, (4.40) is sent to (4.40) is sent to (4.40) is sent to (4.40) is sent to (4.40) is sent to (4.40) is sent to (4.40) is sent to (4.40) is sent to (4.40) is sent to (4.40). Thus Prop. 4.8 shows that (4.40) is a quasi-isogeny and that its degree is given by (4.36). Then Lem. 4.7 states that \( \Phi \) is an isogeny. Clearly,

\[
\prod_{i=1}^{r} \text{deg}(L_i) = q^{2n^0 \cdot |m^1|}.
\]

Combining (4.36) with (4.36), we obtain

\[
\text{deg}(\Phi) = |\text{Disc}_{E_1/F} \cdot \text{Disc}_{E_2/F}|_{F}^{-n^2-n^1} \cdot |\text{Res}(\text{Inv}(\beta^0), \text{Inv}(\beta^1))|_{F}^{-1} \cdot q^{n^1 \cdot |m^0| + n^0 \cdot |m^1|}.
\]

This finishes the proof of all our results.

**5 Appendix I: Local intersection theory**

This section provides some background on intersection theory on regular local formal schemes. Its first aim is to give a precise definition of cycles, correspondences and the action of the latter on the former. Its second aim is to isolate some cases in which intersection numbers may be computed as lengths. These allow us to work with our non-standard definition of Hecke correspondences in §3 and §4 of the paper.

The most important input in the following is Serre’s Vanishing Conjecture Thm. 5.4 which is due (independently) to Roberts [24] and Gillet–Soulé [3]. In fact, we present the theory of cycles (in our context) as a special case of [41], which thus becomes our main reference. None of the statements that follow is new.
5.1 Cycle intersection

Let $M = \text{Spf} A$ be the formal spectrum of a regular complete local ring of dimension $n$. We write $M^{(c)}$ for the set of integral closed formal subschemes $Z \subseteq M$ of codimension $c$. These are the same as integral closed subschemes of $\text{Spec} A$ and we write $\eta_Z \in \text{Spec} A$ for the generic point of $Z$. Denote by

$$Z^c(M) := \bigoplus_{Z \in M^{(c)}} Z \cdot [Z] \tag{5.1}$$

the group of cycles of codimension $c$. The next definition is the standard way of defining cycles from coherent sheaves: Take the maximal dimensional irreducible components of its support with appropriate multiplicities.

**Definition 5.1.** (1) For a finite type $A$-module $E$ with $\text{codim} \text{Supp}(E) \geq c$, we define

$$[E]^c := \sum_{Z \in M^{(c)}} \ell_{A_{\eta_Z}} (E_{\eta_Z}) \cdot [Z] \in Z^c(M). \tag{5.2}$$

Here, $\ell$ is our notation for the length of an artinian module, while $A_{\eta_Z}$ and $E_{\eta_Z}$ denote the localizations at $\eta_Z$.

(2) Given a bounded complex of finite type $A$-modules $E^\bullet$, we write $\text{Supp}(E^\bullet) := \bigcup_{i \in I} \text{Supp}(H^i(E^\bullet))$. If $\text{codim} \text{Supp}(E^\bullet) \geq c$, we define

$$[E^\bullet]^c := \sum_{i \in \mathbb{Z}} (-1)^i [H^i(E^\bullet)]^c \in Z^c(M). \tag{5.3}$$

In fact, every such complex $E^\bullet$ is quasi-isomorphic to a perfect complex because $A$ is regular. (A complex is perfect if it is bounded and if all its terms are finite free.) For $Y \subseteq \text{Spec} A$ closed, let $K^Y_0(M)$ denote the $K$-group of perfect complexes acyclic outside $Y$, cf. [4, §1]. Let $K_0^Y(Y)$ denote the $K$-group of coherent sheaves with support on $Y$. We write $(E^\bullet)^c$ (resp. $(E)$) for the class of a perfect complex (resp. coherent sheaf) in these $K$-groups. Then, by [4, Lem. 1.9],

$$K^Y_0(M) \xrightarrow{\cong} K^Y_0(Y) \xrightarrow{\cup} \sum_{i \in \mathbb{Z}} (-1)^i [H^i(E^\bullet)]. \tag{5.4}$$

It is clear that $(5.3)$ descends to a map $K^Y_0(Y) \to Z^c(M)$ whenever $\text{codim} Y \geq c$ and that $(5.2)$ then comes from composition with $(5.4)$. There is an associative, commutative cup product

$$\cup: K^Y_0(M) \times K^Z_0(M) \to K^{Y \cap Z}(M) \xrightarrow{\cup} \bigoplus_{i \in \mathbb{Z}} \bigoplus_{i \in \mathbb{Z}} (E^\bullet_i) \cdot (E^\bullet'_i) \tag{5.5}$$

Every cycle $z \in Z^c(M)$ may be viewed as an element of $K^Y_0(\text{Supp} z)$ by linear extension from the map $M^{(c)} \ni Z \to \mathcal{O}_Z$. Applying the cup product and composing with the cycle map of Def. 5.1 provides an associative, commutative intersection product of cycles whenever intersections are non-degenerate.

**Definition 5.2.** Given $Z_1 \in M^{(c_1)}$ and $Z_2 \in M^{(c_2)}$ such that $\text{codim}(Z_1 \cap Z_2) \geq c_1 + c_2$, put

$$[Z_1] \cdot [Z_2] := \Big[ \mathcal{O}_{Z_1} \otimes_{\mathcal{O}_M} \mathcal{O}_{Z_2} \Big]^{c_1+ c_2} \in Z^{c_1+ c_2}(M). \tag{5.6}$$

This is feasible because, as mentioned before, the $\mathcal{O}_Z$, are quasi-isomorphic to perfect complexes. Write $(Z^{c_1}(M) \times Z^{c_2}(M))_{\text{proper}}$ for the set of those pairs $(z_1, z_2)$ with $\text{codim}(\text{Supp}(z_1) \cap \text{Supp}(z_2)) \geq c_1 + c_2$. Then $(5.6)$ extends linearly to a map

$$(Z^{c_1}(M) \times Z^{c_2}(M))_{\text{proper}} \to Z^{c_1+ c_2}(M).$$
Remark 5.3. Formula (5.6) may be written more concretely as
\[
[Z_1] \cdot [Z_2] := \sum_{Z \in M^{(c_1 + c_2)}} \sum_{i \geq 0} (-1)^i \ell_{A_{n_Z}} \left( \text{Tor}_i^{A_{n_Z}}(\mathcal{O}_{Z_1, n_Z}, \mathcal{O}_{Z_2, n_Z}) \right) \cdot [Z] \in Z^{c_1 + c_2}(M). \tag{5.7}
\]

Theorem 5.4 (Serre’s Vanishing Conjecture, [4 Thm. 5.4]). Let \( E_1^1 \) and \( E_2^1 \) be two perfect complexes of \( \mathcal{A} \)-modules with codim Supp\( (E_1^1) \) + codim Supp\( (E_2^1) \) > \( n = \dim \mathcal{A} \). Then
\[
[E_1^1]^{l_{M^1}} [E_2^1]^{c_1} = [E_1^1 \otimes_{\mathcal{O}_M} E_2^1]^{c_1 + c_2}. \tag{5.8}
\]

Proof. The left hand side of (5.8) is defined by viewing \([E_1^1]^{c_1}\) and \([E_2^1]^{c_2}\) as elements of \( K_0(\text{Supp}(E_1^1)) \) by linear extension of the map \( M^{(c)} \ni Z \mapsto \mathcal{O}_{Z} \). In \( K_0(\text{Supp}(E_1^1)) \) we may write
\[
[E_1^1]^{c_1} = (E_1^1)^{\varepsilon_1},
\]
for some coherent sheaf \( \varepsilon_1 \) with codim Supp\( \varepsilon_1 > c_1 \). Applying Thm. 5.4 to the localizations \( A_{n_Z} \) for \( Z \in M^{(c_1 + c_2)} \) we obtain
\[
\left[ \varepsilon_1 \otimes_{\mathcal{O}_M} E_2^1 \right]^{c_1 + c_2} = \left[ E_1^1 \otimes_{\mathcal{O}_M} \varepsilon_2 \right]^{c_1 + c_2} = \left[ \varepsilon_1 \otimes_{\mathcal{O}_M} \varepsilon_2 \right]^{c_1 + c_2} = 0.
\]

\[ \square \]

5.2 Correspondences

Let \( W \) denote a complete DVR. Assume from now on that \( M \) is local, formally of finite type, and formally smooth of relative dimension \( n \) over \( \text{Spf} W \). We define an intersection number as follows.

Definition 5.6. The degree \( \deg : Z^n(M) \xrightarrow{\sim} \mathbb{Z} \) is defined as \( \deg(\mathcal{O}_Z) = \ell_W(\mathcal{O}_Z) \). Composing with the intersection product (5.6) defines the intersection number of properly intersecting cycles,
\[
(\ , \ , ) : (Z^c(M) \times Z^{n-c}(M))_{\text{proper}} \longrightarrow \mathbb{Z}, \quad (z_1, z_2) = \deg(z_1 \cdot z_2). \tag{5.9}
\]

The formal smoothness provides that \( M \times_{\text{Spf} W} M \) is again regular and of dimension \( 2n - 1 \). Consider the diagram
\[
\begin{array}{ccc}
M \times_{\text{Spf} W} M & \xrightarrow{p_2} & M \\
p_1 \downarrow & & \downarrow \\
M. & & \\
\end{array}
\]

There is a pullback of cycles
\[
p_2^* : Z^c(M) \longrightarrow Z^c(M \times_{\text{Spf} W} M), \quad [Z] \longmapsto [p_2^{-1}(Z)].
\]

There is also a pushforward of cycle whose support is finite (via \( p_1 \)) over \( M \),
\[
p_1_* : Z^c(M \times_{\text{Spf} W} M)_{\text{finite}} \longrightarrow Z^{c-n+1}(M), \quad [Z] \longmapsto \deg(Z \to p_1(Z)) [p_1(Z)].
\]

These considerations similarly apply to the various projection maps
\[
p_{ij} : M \times_{\text{Spf} W} M \times_{\text{Spf} W} M \longrightarrow M \times_{\text{Spf} W} M.
\]
**Definition 5.7.** (1) Denote by $\text{Corr}(M) \subseteq Z^{n-1}(M \times_{\text{Spf} W} M)$ all those elements whose support is finite over $M$ via both $p_1$ and $p_2$. Define the bilinear map

$$
\ast : \text{Corr}(M) \times \text{Corr}(M) \longrightarrow \text{Corr}(M)
$$

$$(C_1, C_2) \mapsto C_1 \ast C_2 := p_{13,\ast} (p_{12}^{\ast}(C_1) \cdot p_{23}^{\ast}(C_2)).
$$

(5.10)

Note that the intersection $\text{Supp} p_{12}^{\ast}(C_1) \cap \text{Supp} p_{23}^{\ast}(C_2)$ is finite over $M$ (via both $p_1$ and $p_3$) and hence has the expected codimension.

(2) Define the bilinear map

$$
\text{Corr}(M) \times Z^{c}(M) \longrightarrow Z^{c}(M)
$$

$$(C, z) \mapsto C \ast z := p_{1,\ast} (C \cdot p_{2}^{\ast}(z)).
$$

(5.11)

Again, the occurring intersection is of the expected codimension and finite over $M$ (via both $p_1$ and $p_2$).

Let $R$ be a formal scheme, let $p_1, p_2 : R \rightarrow M$ be two finite maps and assume that $R$ is pure of dimension $n$. Then also $(p_1, p_2) : R \rightarrow M \times_{\text{Spf} W} M$ is finite and $R$ defines the correspondence $[R] := [\mathcal{O}_R]^{\ast n-1}$.

**Proposition 5.8.** (1) Definitions (5.10) and (5.11) make $\text{Corr}(M)$ into a ring and $Z^{c}(M)$ into a left $\text{Corr}(M)$-module.

(2) Moreover, if $C_1 = [R_1]$ and $C_2 = [R_2]$ for finite maps $R_i \rightarrow M \times_{\text{Spf} W} M$ from purely $n$-dimensional formal schemes $R_i$, then simply

$$
C_1 \ast C_2 = [R_1 \times_{p_2, M, p_1} R_2].
$$

(5.12)

(3) Similarly, if $C \in \text{Corr}(M)$ has the form $C = [R]$ and $z = [Z] \in Z^{c}(M)$, then

$$
C \ast z = p_{1,\ast} [R \times_{p_2, M} Z].
$$

(5.13)

**Proof.** Statement (1) claims associativity of (5.10) and (5.11). Using the projection formula [3 (1.7.2)], this reduces to associativity of the cycle intersection for

$$
p_{12}^{\ast}C_1 \cdot p_{23}^{\ast}C_2 \cdot p_{34}^{\ast}C_3, \quad \text{resp.} \quad p_{12}^{\ast}C_1 \cdot p_{23}^{\ast}C_2 \cdot p_{34}^{\ast}z
$$

on $M^4$ resp. $M^3$, where $C_i \in \text{Corr}(M)$ and $z \in Z^{c}(M)$. For example, the first expression may be rewritten as

$$
p_{14,\ast} (p_{12}^{\ast}C_1 \cdot p_{23}^{\ast}C_2 \cdot p_{34}^{\ast}C_3) = p_{13,\ast} (p_{12}^{\ast} (p_{12}^{\ast}C_1) \cdot p_{23}^{\ast}C_2 \cdot p_{34}^{\ast}C_3)
$$

$$
= p_{13,\ast} (p_{12}^{\ast}C_1 \cdot p_{12}^{\ast} (p_{23}^{\ast}C_2 \cdot p_{34}^{\ast}C_3))
$$

$$
= p_{13,\ast} (p_{12}^{\ast}C_1 \cdot p_{23}^{\ast}C_2 \cdot p_{34}^{\ast}C_3)
$$

$$
= C_1 \ast (C_2 \ast C_3).
$$

We omit further details.

Let now $R_1, R_2 \rightarrow M \times_{\text{Spf} W} M$ be as in Statement (2). By definitions of the two sides in (5.12), we need to see the equality

$$
p_{13,\ast} [\mathcal{O}_{p_{12}}^{-1}(R_1) \otimes_{\mathcal{O}_{M \times_{\text{Spf} W} M}} \mathcal{O}_{p_{23}}^{-1}(R_2)]^{2n-2} = p_{13,\ast} [\mathcal{O}_{R_1} \otimes_{p_2} \mathcal{O}_{M \times_{\text{Spf} W} M} \mathcal{O}_{p_2}^{-1} \mathcal{O}_{R_2}]^{2n-2}.
$$

(5.14)

In fact, we claim this before applying $p_{13,\ast}$ and we show the slightly stronger statement that all higher Tor-terms on the left hand side of (5.14) have support in codimension $\geq 2n - 1$. To see this, replace both $\mathcal{O}_{R_i}$ by perfect complexes $E_i^{\ast}$ on $M \times_{\text{Spf} W} M$. Then there is an isomorphism of the following kind, which may be checked term by term,

$$
(E_1^{\ast} \otimes_{W} A) \otimes_{A \otimes_{W} A} (A \otimes_{W} E_2^{\ast}) \cong (E_1^{\ast} \otimes_{p_2} A, p_1^{\ast} \mathcal{O}_{R_2}^{\ast}).
$$

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The maps $p_1^*, p_2^*: A \to A \otimes_W A$ are flat, so the $E_i^*$ provide flat resolutions of $\mathcal{O}_{R_i}$ as $A$-modules. Thus we find that

$$\mathcal{O}_{p_1^{-1}R_1} \otimes_{\mathcal{O}_{M \times M}} \mathcal{O}_{p_2^{-1}R_2} \cong \mathcal{O}_{R_1} \otimes_{p_1^* \mathcal{O}_{M \times M}} p_1^* \mathcal{O}_{R_2}. \quad (5.15)$$

But $A$ is an integral domain, so the various projection maps $R_1, R_2 \to M$ are generically flat. Thus the higher Tor-terms on the right hand side of (5.15) have support in dimension $< \dim M$ as claimed, proving (2). The proof of (3) is identical and omitted.

We finally recall that intersection numbers may often be computed as lengths. This relies on the following fact about Cohen-Macaulay rings.

Lemma 5.9 (Tag 02JN). Let $R$ be a Cohen–Macaulay noetherian local ring. A sequence of elements $x_1, \ldots, x_r \in R$ is regular if and only if $\dim(R/(x_1, \ldots, x_r)) = \dim(R) - r$.

Corollary 5.10. (1) Let $Z_1, Z_2 \subseteq M$ be closed formal subschemes. Assume that $Z_1$ is Cohen-Macaulay and that $Z_2$ is defined by a regular sequence. Assume further that $\text{codim} Z_1 + \text{codim} Z_2 = n$ and that $Z_1 \cap Z_2$ is artinian. Then

$$([Z_1], [Z_2]) = \ell_W(\mathcal{O}_{Z_1 \cap Z_2}).$$

(2) Assume that $R$ is Cohen–Macaulay of pure dimension $n$ and that $p_1, p_2 : R \to M$ are two finite maps. Let $Z_1, Z_2 \subseteq M$ be closed formal subschemes that are defined by regular sequences and such that $\text{codim} Z_1 + \text{codim} Z_2 = n$. Assume further that $R \times M \times \text{Spf} W M (Z_1 \times \text{Spf} W Z_2)$ is artinian. Then

$$([Z_1], [R] \cdot [Z_2]) = \ell_W (R \times M \times \text{Spf} W M (Z_1 \times \text{Spf} W Z_2)).$$

Proof. (1) This is standard. Assume $Z_2 = V(x_1, \ldots, x_c)$ for a regular sequence $x_1, \ldots, x_c$. By Lem. 5.10 and our dimension assumptions, $x_1|_{Z_1}, \ldots, x_c|_{Z_1}$ define a regular sequence on $Z_1$. Resolving $\mathcal{O}_{Z_2}$ by the Koszul complex of $(x_1, \ldots, x_c)$ shows that $\text{Tor}_i^\mathcal{O}_M(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = 0$ for $i > 0$. Thus $\mathcal{O}_{Z_1 \cap Z_2}$ represents $\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_{M \times M}} \mathcal{O}_{Z_2}$. By Cor. 5.5 $\ell_W(\mathcal{O}_{Z_1 \cap Z_2})$ then agrees with $([Z_1], [Z_2])$.

(2) By the projection formula [H (1.7.2)] or by Prop. 5.8

$$([Z_1], [R] \cdot [Z_2]) = \text{deg}([R] \cdot p_1^*[Z_1] \cdot p_2^*[Z_2]).$$

Cor. 5.5 provides the first equality in

$$p_1^*[Z_1] \cdot p_2^*[Z_2] = [\mathcal{O}_{p_1^{-1}(Z_1)} \otimes_{\mathcal{O}_{M \times M}} \mathcal{O}_{p_2^{-1}(Z_2)}]^n = [Z_1 \times \text{Spf} W Z_2]^n.$$
The \( \mathbb{C} \)-coefficient (partial) Satake transformation is the following map of spherical Hecke algebras
\[
S : \mathbb{C}[G(O_F) \backslash G(F)/G(O_F)] \longrightarrow \mathbb{C}[L(O_F) \backslash L(F)/L(O_F)]
\]
\[
f \mapsto Sf; \quad Sf(l) := \delta_l(l)^{\frac{1}{2}} \int_{U(F)} f(lu) \, du
\]  \hspace{1cm} (6.2)
where \( du \) is the Haar measure normalized by \( U(O_F) \).

**Lemma 6.1.** The partial Satake transformation \((6.2)\) is a map of \( \mathbb{C} \)-algebras.

**Proof.** The integral
\[
S[f \ast g](l) = \delta_l(l)^{\frac{1}{2}} \int_{U(F)} \int_{G(F)} f(y^{-1}lu)g(y)dy \, du
\]
is invariant under \( y \mapsto yk \) for \( k \in G(O_F) \). The Iwasawa decomposition provides a \( P \)-equivariant isomorphism
\[
P(F)/P(O_F) \cong G(F)/G(O_F).
\]
The measure on the two spaces here is given by the left invariant Haar measure \( dl \, du \) on \( P = LU \). Therefore
\[
S[f \ast g](l) = \delta_l(l_2^{-1}l_1^{\frac{1}{2}}) \delta_l(l_2^{\frac{1}{2}}) \int_{U(F)} \int_{L(F)} f(l_2^{-1}l_1u)g(l_2u)du \, dv.
\]
Applying the change of variables \( u \mapsto l^{-1}l_2u_2l_2^{-1}l_1u \), we conclude that
\[
S[f \ast g](l) = \int_{L(F)} \left( \delta_l(l_2^{-1}l_1^{\frac{1}{2}}) \int_{U(F)} f(l_2^{-1}l_1u)du \cdot \delta_l(l_2^{\frac{1}{2}}) \int_{U(F)} g(l_2u)du \right) \, dv = (S[f] \ast S[g])(l)
\]
as desired. \( \square \)

We next specialize to the usual Satake transformation, which is the case of the partition \( n = 1+1+\cdots+1 \). We write \( T \) for the diagonal torus and \( U \) for the unipotent radical, now consisting of upper triangular unipotent matrices. Since \( T \) is abelian, we have \( T(O_F) \backslash T(F)/T(O_F) = T(F)/T(O_F) \).

Denote by \( x_i : T \longrightarrow \{0,1\} \) the characteristic function of the subset
\[
O_F^x \times \cdots O_F^x \times \bigotimes_{\text{i-th component}} \pi O_F^x \times \cdots \times \pi O_F^x \subset T.
\]
Then each \( x_i \) is invertible in \( \mathbb{C}[T(F)/T(O_F)] \) and there is the explicit description
\[
\mathbb{C}[T(F)/T(O_F)] = \mathbb{C}[x_1^\pm, \cdots, x_n^\pm].
\]  \hspace{1cm} (6.3)
The monomial \( x_1^{m_1} \cdots x_n^{m_n} \) on the right hand side is simply the characteristic function of the subset
\[
\pi^{m_1}O_F^x \times \pi^{m_2}O_F^x \times \cdots \times \pi^{m_n}O_F^x.
\]
Put \( \text{Mat}_{k \times n}^{\text{det} = 0}(O_F) := \text{Mat}_{n \times n}(O_F) \cap \text{GL}_n(F) \). Let \( W = N(T)/T \) be the Weyl group. For any two \( O_F \)-modules \( \Lambda_1 \) and \( \Lambda_2 \), we write \( \Lambda_1 \subset \Lambda_2 \) to denote a (not necessarily strict) inclusion \( \Lambda_1 \subset \Lambda_2 \) with \( \Lambda_2/\Lambda_1 = q^k \), where \( q = |O_F/\pi O_F| \).

**Definition 6.2.** For any \( 0 \leq k \leq n \), resp. for \( 0 \leq k \), define the following functions on \( G(F) \):
\[
S_k(g) = \begin{cases} 1 & \text{if } \pi O_F^n \subset_{k-n} g O_F^n \subset O_F^n; \\ 0 & \text{otherwise,} \end{cases} \quad T_k(g) = \begin{cases} 1 & \text{if } g O_F^n \subset_{k} O_F^n; \\ 0 & \text{otherwise.} \end{cases}
\]  \hspace{1cm} (6.4)
For any $-n \leq k < 0$, resp. for $k < 0$, put $S_k(g) := S_{-k}(g^{-1})$ and $T_k(g) := T_{-k}(g^{-1})$. In particular $S_n$, $S_0$ and $S_{-n}$ are the characteristic functions of $\pi G(O_F)$, $G(O_F)$ and $\pi^{-1}G(O_F)$, respectively. For example, $(S_{-n} \ast f)(g) = f(\pi g)$ for every Hecke function $f$.

Our first result in this appendix is the following theorem.

Theorem 6.3. Let $G^{(k)}$ be the preimage of $\pi^k O_F^k$ under the map $\det : \text{Mat}_{n \times n}(O_F) \rightarrow O_F$ and let

$$\mathcal{H}^{(k)} := \mathbb{C}\left[G(O_F)\backslash G^{(k)}/G(O_F)\right], \quad \mathcal{H} := \bigoplus_{k=0}^{\infty} \mathcal{H}^{(k)}. \quad (6.5)$$

The following statements hold.

1. The Satake transformation defines an isomorphism from $\mathcal{H}^{(k)}$ to the vector space of degree $k$ homogeneous symmetric polynomials, and consequently from $\mathcal{H}$ to the algebra of symmetric polynomials

$$S : \mathcal{H}^{(k)} \overset{\cong}{\rightarrow} \mathbb{C}[x_1, \ldots, x_n]^{W=\text{id}, \deg=k}, \quad S : \mathcal{H} \overset{\cong}{\rightarrow} \mathbb{C}[x_1, \ldots, x_n]^{W=\text{id}}. \quad (6.6)$$

2. Either set $\{S_i\}_{0 \leq i \leq n}$ or $\{T_i\}_{0 \leq i \leq n}$ generates $\mathcal{H}$ as a $\mathbb{C}$-algebra.

3. Either set $\{S_{-n}\} \cup \{S_i\}_{0 \leq i \leq n}$ or $\{S_{-n}\} \cup \{T_i\}_{0 \leq i \leq n}$ generates $\mathbb{C}[G(O_F)\backslash G(F)/G(O_F)]$ as a $\mathbb{C}$-algebra.

Note that (3) follows immediately from (2). The proof of (1) and (2) will be given in the next few sections and will be complete after this. Part (1) is actually well-known and may be found in more general form in [2], but we include a proof for convenience. We begin here with the observation that it is sufficient to prove surjectivity in (1):

Lemma 6.4. Assume $S$ maps $\mathcal{H}$ surjectively onto $\mathbb{C}[x_1, \ldots, x_n]^{W=\text{id}}$. Then $S : \mathcal{H} \cong \mathbb{C}[x_1, \ldots, x_n]^{W=\text{id}}$ is bijective.

Proof. Since $\det(g \cdot U) = \det(g)$, we have $S(\mathcal{H}^{(k)}) \subset \mathbb{C}[x_1, \ldots, x_n]^{\deg=k}$ for any integer $k$. By counting double cosets via the Cartan decomposition,

$$\# \left( G(O_F)\backslash G^{(k)}/G(O_F) \right) = \# \{k_1 \leq k_2 \leq \cdots \leq k_n : k_1 + k_2 + \cdots + k_n = k \}$$

$$= \dim \mathbb{C}[x_1, \ldots, x_n]^{W=\text{id}, \deg=k},$$

we conclude that the $\mathbb{C}$-vector spaces $\mathcal{H}^{(k)}$ and $\mathbb{C}[x_1, \ldots, x_n]^{W=\text{id}, \deg=k}$ are of the same dimension and the lemma follows.

6.2 Satake transformation in terms of orbital integrals

In this section we prove the inclusion

$$S(\mathcal{H}) \subset \mathbb{C}[x_1, \ldots, x_n]^{W=\text{id}}. \quad (6.7)$$

Let $U_{n-1} := U$ and inductively define $U_{i-1} := [U_i, U_i]$ as the derived subgroup, meaning the minimal normal subgroup such that $U_i/U_{i-1}$ is abelian. Explicitly,

$$U_i = \{(a_{ij})_{n \times n} : a_{k,k} = 1 \text{ and } a_{k,k+j} = 0 \text{ for } j < 0 \text{ and for } 0 < j < n-i \}.$$

Then

$$U_i/U_{i-1} \cong \{(a_{1,1+n-i}, a_{2,2+n-i}, \ldots, a_{i,n})\} \cong F^{\oplus i}$$

and for any $t = \text{diag}(t_1, \ldots, t_n) \in T$, the map

$$U \rightarrow U; \quad u \mapsto t^{-1}ut^{-1}$$

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induces linear transformations on $U_i/U_{i-1}$ with eigenvalues $\{t_k^{-1}t_{k+n-i} - 1\}_{1 \leq i \leq n}$. Assume that the $t_i$ are pairwise different. Then, denoting by $du$ the Haar measure on $U$, this implies that

$$d(t^{-1}utu^{-1}) = \prod_{i=1}^{n-1} \prod_{k=1}^{i} |t_k^{-1}t_{k+n-i} - 1|_F \cdot du = |\det t|_F^{-n+1} \cdot \delta_T(t)^{-\frac{1}{2}} \prod_{1 \leq i < j \leq n} |t_i - t_j|_F \cdot du.$$  

Note that the product of $|t_i - t_j|_F$ for $1 \leq i < j \leq n$ equals to $|\text{Disc}(\text{char}_i)|_F^2$, where $\text{char}_i(\lambda) = \det(\lambda \cdot \text{id} - t)$ is the characteristic polynomial of $t$, and where $\text{Disc}$ denotes the discriminant. We have

$$S[f](t) = \delta_T(t)^{\frac{1}{2}} \int_{U(F)} f(t(t^{-1}utu^{-1})) d(t^{-1}utu^{-1})$$

$$= |\det t|_F^{-n+1} |\text{Disc}(\text{char}_i)|_F^{\frac{1}{2}} \int_{U(F)} f(utu^{-1}) du.$$  

Since $\text{GL}_n(F) = \text{GL}_n(O_F)U(F)T(F)$ by the Iwasawa decomposition and since $f$ is $\text{GL}_n(O_F)$-invariant, the occurring orbital integral is

$$\int_{U(F)} f(utu^{-1}) du = \int_{\text{GL}_n(F)/T(F)} f(hth^{-1}) dh.$$  

This expression only depends on $t$ up to conjugation. Therefore, the Satake transformation is Weyl-invariant as claimed in [16.7].

### 6.3 Satake transformation of miniscule Hecke functions

We next determine the images $S[S_k]$. Write $V_n = F^n$ and let $V_n \supset V_{n-1} \supset \cdots \supset V_0 = \{0\}$ be the unique $U$-invariant filtration such that $V_i/V_{i-1} \cong F$. For any $t = \text{diag}(t_1, t_2, \ldots, t_n)$ and any $u \in U$, the linear transformation $tu : V_n \rightarrow V_n$ induces scaling by $t_i$ on $V_i/V_{i-1}$. Therefore, $O_F^n \supset tuO_F^n \supset \pi \cdot O_F^n$ implies that $t_i \in O_F^n \cup \pi \cdot O_F^n$. This shows that

$$S[S_k] = \sum_{I \subset \{1, \ldots, n\} \atop \#I = k} C_I \cdot \prod_{i \in I} x_i$$

for some constants $C_I$. The polynomial on the right hand side is $W$-invariant by [6.7], so the constants $C_I$ are actually independent of $I$.

Now we determine $C_{\{n-k+1, \ldots, n\}}$. Consider the element $t = \text{diag}(1, 1, \cdots, 1, \pi, \cdots, \pi)$ and note that $S_k(tu) = 1$ if and only if $u \in U(O_F)$. Therefore $S[S_k](t) = \delta_T(t)^{\frac{1}{2}} = q^{-\frac{k(n-k)}{2}}$ which implies that

$$S[S_k] = q^{-\frac{k(n-k)}{2}} \sum_{I \subset \{1, \ldots, n\} \atop \#I = k} \prod_{i \in I} x_i$$

is (a non-zero scalar multiple of) an elementary symmetric polynomial. It is well-known that these generate $\mathbb{C}[x_1, \cdots, x_n]^{W = \text{id}}$, so

$$S(H) = \mathbb{C}[x_1, \cdots, x_n]^{W = \text{id}}.$$  

At this point, we have shown (2) of Thm. 6.3 for the set $\{S_k\}_{0 \leq k \leq n}$.

### 6.4 Generating symmetric polynomials

The next result provides an alternative generating set for the algebra of symmetric polynomials. Define

$$s_k(x_1, \cdots, x_n) := \sum_{I \subset \{1,2,\cdots, n\} \atop \#I = k} \prod_{i \in I} x_i \quad \text{and} \quad b_m(x_1, \cdots, x_n) := \sum_{m_1 + \cdots + m_n = m} \prod_{i=1}^{n} x_i^{m_i}. \quad (6.8)$$
It was already said and used that \( \{ s_k \}_{0 \leq k \leq n} \) is a set of generators of \( \mathbb{C}[x_1, \ldots, x_n]^W = \text{id} \).

**Proposition 6.5.** The set \( \{ b_0(x_1, \ldots, x_n) \}_{0 \leq k \leq n} \) generates \( \mathbb{C}[x_1, \ldots, x_n]^W = \text{id} \) as a \( \mathbb{C} \)-algebra.

**Proof.** Note that \( b_0 = s_0 = 1 \). We claim that, for all \( k \geq 1 \),

\[
\sum_{i=0}^{k} (-1)^i b_is_{k-i} = 0. \tag{6.9}
\]

To prove this, consider the polynomial expansion

\[
\prod_{i=1}^{n} (X - x_i) = \sum_{i=0}^{n} (-1)^i s_i X^{n-i}.
\]

and that of its reciprocal

\[
\prod_{i=1}^{n} (X - x_i)^{-1} = \sum_{i=0}^{\infty} b_i X^{-i-n}.
\]

The claim follows by multiplying these two equations together. The proposition now follows by induction from the identities \( s_k = -\sum_{i=1}^{k} (-1)^i b_is_{k-i}, \, k \geq 1 \). \(\square\)

### 6.5 Satake transformation of \( T_m \)

**Proposition 6.6.** We have

\[
S[\mathbf{T}_m] = q^{\frac{m(n-1)}{2}} b_m(x_1, \ldots, x_n).
\]

**Proof.** Consider a tuple \( m_\bullet = (m_1, \ldots, m_n) \) of integers \( m_i \geq 0 \) and put \( |m_\bullet| = m_1 + \cdots + m_n \). Let

\[
t = \text{diag}(\pi^{m_1}, \pi^{m_2}, \ldots, \pi^{m_n}) \in T, \quad u = (a_{ij})_{1 \leq i, j \leq n} \in U.
\]

We have

\[
\det(tu) = \det(t) = \pi^{\left\lfloor \frac{|m_\bullet|}{2} \right\rfloor}.
\]

Note that \( u \in U \) implies that \( a_{ij} = 0 \) if \( j < i \) and \( a_{ii} = 1 \) for \( 1 \leq i \leq n \). We have

\[
\int_{U(F)} \mathbf{T}_m(tu) \, du = 1_{x = O_{\mathbb{F}}} \left( \pi^{\left\lfloor \frac{|m_\bullet|}{2} \right\rfloor} \right) \prod_{i=1}^{n} \prod_{j=i+1}^{n} \int_{F} 1_{O_{\mathbb{F}}} \left( \pi^{m_i} a_{ij} \right) da_{ij}
\]

\[
= 1_{x = O_{\mathbb{F}}} \left( \pi^{\left\lfloor \frac{|m_\bullet|}{2} \right\rfloor} \right) \prod_{i=1}^{n} |\pi^{-m_i}|^{-\left\lfloor \frac{(n-i)}{2} \right\rfloor}.
\]

Therefore, if \( |m_\bullet| = m \), then

\[
S[\mathbf{T}_m](t) = \prod_{i=1}^{n} |\pi^{m_i}|^{-\left\lfloor \frac{(n-i)}{2} \right\rfloor} \prod_{i=1}^{n} |\pi^{-m_i}|^{-\left\lfloor \frac{(n-i)}{2} \right\rfloor} = \prod_{i=1}^{n} |\pi^{m_i}|^{-\left\lfloor \frac{(n-i)}{2} \right\rfloor} = \left| \pi^m \right|^{-\left\lfloor \frac{n}{2} \right\rfloor},
\]

which implies

\[
S[\mathbf{T}_m] = \left| \pi^m \right|^{-\left\lfloor \frac{n}{2} \right\rfloor} \sum_{m=m_1+\cdots+m_n} \prod_{i=1}^{n} x_i^{m_i} = q^{\frac{m(n-1)}{2}} b_m(x_1, \ldots, x_n),
\]

where \( b_m \) is the polynomial from (6.8). \(\square\)

Combining Prop. 6.6 with Prop. 6.5 and the bijectivity of the Satake transformation proves that \( \{ \mathbf{T}_m \}_{0 \leq m \leq n} \) is a set of generators for \( \mathcal{H} \). The proof of Thm. 6.3 is now complete.
6.6 Partial Satake transformation of $T_m$

We lastly come to the second result of this appendix, namely an explicit formula for the partial Satake transformation of $T_m$. Suppose the partition in question is $n = n_1 + n_2$ and put $g = (g_1, g_2) \in L = \text{GL}_{n_1}(F) \times \text{GL}_{n_2}(F)$. To avoid confusion, we write $T_{\text{GL}_{n,m}}$ for the function $T_m$ on $GL_n(F)$.

**Proposition 6.7.** For any integer $m \geq 0$, we have

$$S[T_{\text{GL}_{n,m}}] = \sum_{m_1 + m_2 = m} q^{\frac{m_1^2 + m_2^2 + m_1 + m_2}{2}} T_{\text{GL}_{n_1,m_1}} \otimes T_{\text{GL}_{n_2,m_2}}.$$

**Proof.** For any

$$u := \begin{pmatrix} I_{n_1} & u \\ I_{n_2} & \end{pmatrix} \in U_{(n_1,n_2)}(F),$$

we have

$$T_{\text{GL}_{n,m}}(g \cdot u) = 1 \iff \begin{cases} (g_1, g_2) \in \text{Mat}_{n_1 \times n_1}(O_F) \times \text{Mat}_{n_2 \times n_2}(O_F), \\ g_1 u \in \text{Mat}_{n_1 \times n_2}(O_F), \text{ and} \\ \det g_1 \cdot \det g_2 \in \pi^m O_F^\times. \end{cases}$$

Therefore, for any $g = (g_1, g_2) \in \text{Mat}_{n_1 \times n_1}(O_F) \times \text{Mat}_{n_2 \times n_2}(O_F)$ with $\det g_1 \cdot \det g_2 \in \pi^m O_F^\times$, we find

$$S(T_m)(g) = \frac{|\det(g_1)|_{F}^{\frac{n_1}{2}}}{|\det(g_2)|_{F}^{\frac{n_2}{2}}} \int_{\text{Mat}_{n_1 \times n_2}(O_F)} \mathbf{1}_{\text{Mat}_{n_1 \times n_2}(O_F)}(g_1 \cdot u) du = |\det(g_1)|_{F}^{\frac{n_1}{2}} \cdot |\det(g_2)|_{F}^{\frac{n_2}{2}},$$

as desired. \[\square\]

**References**

[1] P. K. Draxl, *Skew fields*, London Mathematical Society Lecture Note Series 81, Cambridge University Press, Cambridge, 1983.

[2] B. H. Gross, *On the Satake isomorphism*, in *Galois representations in arithmetic algebraic geometry*, LMS Lecture Notes 254, 1998, 223–238.

[3] L. Clozel, *The fundamental lemma for stable base change*, Duke Math. J. 61 (1990), no. 1, 255–302.

[4] H. Gillet, C. Soulé, *K-théorie et nullité des multiplicités d’intersection*, C. R. Acad. Sci. Paris Sér. I Math. 300 (1985), no. 3, 71–74.

[5] J. Guo, *On a generalization of a result of Waldspurger*, Canad. J. Math. 48 (1996), no. 1, 105–142.

[6] U. Hartl, R. K. Singh, *Local shtukas and divisible local Anderson modules*, Canad. J. Math. 71 (2019), no. 5, 1163–1207.

[7] B. Howard, Q. Li, *Intersections in Lubin–Tate space and biquadratic Fundamental Lemmas*, Preprint (2020), arXiv:2010.07365

[8] N. Hultberg, A. Mihatsch, *A linear AFL for quaternion algebras*, Preprint (2023), arXiv:2308.02458

[9] J.-P. Labesse, * Fonctions élémentaires et lemme fondamental pour le changement de base stable*, Duke Math. J. 61 (1990), no. 2, 519–530.

[10] S. Leslie, J. Xiao, W. Zhang, *Unitary Friedberg–Jacquet periods and theirs twists: RTF, FL and AFL*, in preparation.

[11] Q. Li, *An intersection formula for CM cycles in Lubin–Tate spaces*, Duke Math. J. 171 (2022), no. 9, 1923–2011.

[12] Q. Li, *A proof of the linear arithmetic fundamental lemma for GL_4*, Canad. J. Math. 74 (2022), no. 2, 381–427.

[13] Q. Li, *A computational proof for the biquadratic Linear AFL for GL_4*, in preparation.

[14] Q. Li, A. Mihatsch, *Arithmetic transfer for inner forms of GL_{2n}*, Preprint (2023), arXiv:2307.11716

[15] Y. Liu, *Fourier–Jacobi cycles and arithmetic relative trace formula (with an appendix by Chao Li and Yihang Zhu)*, Camb. J. Math. 9 (2021), no. 1, 1–147.

[16] W. Messing, *The crystals associated to Barsotti-Tate groups: With applications to abelian schemes*, Lecture Notes in Mathematics 264, Springer-Verlag, Berlin-New York, 1972.
[17] A. Mihatsch, Local Constancy of Intersection Numbers, Algebra Number Theory, 16 (2022), no. 2, 505–519.
[18] A. Mihatsch, W. Zhang, On the Arithmetic Fundamental Lemma Conjecture over a general p-adic field, J. Eur. Math. Soc. (2023), [published online first].
[19] T. Poguntke, Group schemes with $\mathbb{F}_q$-action., Bull. Soc. Math. France 145 (2017), no. 2, 345–380.
[20] M. Rapoport, B. Smithling, W. Zhang, Arithmetic diagonal cycles on unitary Shimura varieties, Compos. Math. 156 (2020), no. 9, 1745–1824.
[21] M. Rapoport, B. Smithling, W. Zhang, Regular formal moduli spaces and arithmetic transfer conjectures, Math. Ann. 370 (2018), no. 3-4, 1079–1175.
[22] M. Rapoport, B. Smithling, W. Zhang, On the arithmetic transfer conjecture for exotic smooth formal moduli spaces, Duke Math. J. 166 (2017), no. 12, 2183–2336.
[23] M. Rapoport, B. Smithling, W. Zhang, On Shimura varieties for unitary groups, Pure Appl. Math. Q. 17 (2021), no. 2, 773–837.
[24] P. Roberts, The vanishing of intersection multiplicities of perfect complexes, Bull. Amer. Math. Soc. 13 (1985), no. 2, 127–130.
[25] The Stacks Project Authors, Stacks Project (2018), https://stacks.math.columbia.edu.
[26] W. Zhang, On arithmetic fundamental lemmas, Invent. Math. 188 (2012), no. 1, 197–252.
[27] W. Zhang, Gross–Zagier formula and arithmetic fundamental lemma, Fifth International Congress of Chinese Mathematicians, pt. 1, 2, 447–459, Amer. Math. Soc., Providence, RI, 2012.
[28] W. Zhang, Periods, cycles, and $L$-functions: a relative trace formula approach, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, 487–521, World Sci. Publ., Hackensack, NJ, 2018.
[29] W. Zhang, Weil representation and arithmetic fundamental lemma, Ann. of Math. (2) 193 (2021), no. 3, 863–978.
[30] W. Zhang, More Arithmetic Fundamental Lemma conjectures: the case of Bessel subgroups, Preprint (2021), arXiv:2108.02086.
[31] Z. Zhang, Maximal parahoric arithmetic transfers, resolutions and modularity, Preprint (2021), arXiv:2112.11994.