New solutions to the $sl_q(2)$-invariant Yang-Baxter equations at roots of unity: cyclic representations

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We find the all solutions to the $sl_q(2)$-invariant multi-parametric Yang-Baxter equations (YBE) at $q = i$ defined on the cyclic (semi-cyclic, nilpotent) representations of the algebra. We are deriving the solutions in form of the linear combinations over the $sl_q(2)$-invariant objects - projectors. The direct construction of the projector operators at roots of unity gives us an opportunity to consider all the possible cases, including also degenerated one, when the number of the projectors becomes larger, and various type of solutions are arising, and as well as the inhomogeneous case. We are giving a full classification of the YBE solutions for the considered representations. A specific character of the solutions is the existence of the arbitrary functions.

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1 Introduction

The study of the representation theory of the quantum algebras at roots of unity was begun at the early nineties of the past century [1, 2, 3]. At the same time it stimulated the works on the investigation of the (universal) intertwiner $R$-matrices defined on the non-standard representations (which have no analogies in the case of non-deformed algebra). Such matrices, as well as intertwiner matrices for affine extensions of the quantum algebras were constructed and there were observed the models possessing the quantum algebra symmetry at roots of unity [3, 4, 5, 6, 7, 8, 9, 10, 24]. Especially the simplest case of the quantum algebra $\mathfrak{sl}_q(2)$ at roots of unity ($q^N = 1$) was thoroughly investigated. The connection of the Potts model’s $R$-matrix with the intertwiner $R$-matrices defined on the cyclic representations of the algebra $\hat{\mathfrak{sl}}_q(2)$ ($N \geq 3$) was observed [5, 7]. The $R$-matrices defined on the semi-cyclic (nilpotent) representations of $\mathfrak{sl}_q(2)$ were explored in [9]. As it is known the intertwiner matrices satisfy the Yang-Baxter equations (YBE) [11, 12]. Solutions to the YBE with the non-standard representations are investigated in the series of the papers [6, 8, 9, 24] and some explicit solutions are obtained. However we think that the solutions to the Yang-Baxter equations with the cyclic, semi-cyclic and nilpotent, as well as indecomposable representations of the quantum algebra $\mathfrak{sl}_q(2)$ at roots of unity need a thorough investigation. There is known the decomposition of the intertwiner matrices over the symmetry-invariant objects - projectors [14, 15]. For constructing the projectors explicitly at first one has to determine the fusion rules at roots of unity [3]. Using the detailed rules, formulated in [16] for the highest/lowest weight indecomposable representations, in our previous paper [17] by means of the direct construction of the projection operators, we see that the consideration of the highest/lowest weight indecomposable representations even for the simplest case $q^4 = 1$ gives a large amount of various new solutions. Considering the whole set of the projection operators we ensure the foundation of the all possible solutions for the given representations.

Here we investigate the YBE with the $\mathfrak{sl}_q(2)$-invariant $R$-matrices, defined on the cyclic (semi-cyclic, nilpotent) irreps, again at $q^4 = 1$, which means that we work with $4 \times 4$ matrices. And now also we find rich variety of solutions. As at roots of unity the center of the algebra is enlarged and the cyclic representations are parameterized by means of the continuous parameters (in addition to the eigenvalues of the quadratic Casimir operator), such parameters are involved in the YBE as new parameters, and in general here we deal with the multi-parametric YBE. We would like to
emphasize, that the case of $q^4 = 1$ was investigated in [8], where there were obtained particular solutions, with the matrix elements connected with the Clebsch-Gordan coefficients. The mentioned work contains first hint about a remarkable property of the general solutions defined on the cyclic irreps at $q^4 = 1$, that is the existence of the arbitrary functions. Therein the author noted that the obtained solutions do not exhaust the all list of possible solutions at $q^4 = 1$. In [24] the authors have constructed $R$-matrices defined on the $N$-dimensional irreps of $sl_q(2)$ algebra at roots of unity $q^{2N} = 1$ ($N$-state colored braid matrices), taking the appropriate limit of $q$ from the YBE solutions defined on the infinite dimensional representations at general $q$. The matrices are represented via the Clebsch-Gordan coefficients and are trigonometric functions on the arguments. For the case $N = 2$ this solution corresponds the mentioned solution brought in [8], if to set the arbitrary functions as trigonometric ones. However, as at roots of unity the representation spectra and the fusion rules are changed radically, the use of the limits of the formulas obtained at general $q$ can provide us only with the part of the solutions; the whole set of solutions can be obtained if to construct the states and projectors directly for the exceptional values of $q$ [17], as there can be degenerated situations, when the number of the projection operators becomes larger, compared to the cases at general $q$. We think that the presented technique allows us to pretend the full spectra of the YBE solutions defined on two-dimensional cyclic irreps. The investigation of the solutions by direct constructions with the cyclic (as well as the indecomposable) representations at higher roots of the unity we intend to perform in the further works.

Among the obtained solutions there are entirely new solutions (presented in the subsection 4.3) and also there are such ones, which coincide with the already obtained solutions [8, 10, 20], such as the solution (4.4) [8, 10, 24] or the solutions (4.16, 4.17), which are the particular trigonometric limits of the solutions presented in [20, 22, 23] (see also the citations brought therein), and (4.49) [20]. Thus we unveil the underlying $sl_q(2)$-symmetry of the mentioned solutions (4.16, 4.17, 4.49).

All the obtained solutions have the so-called ”free-fermionic” property [21, 22, 23], which is the peculiarity of the case $q^4 = 1$. The corresponding quantum one-dimensional spin-chain models are the generalizations of the $XY$ model in a transverse magnetic field. This is an expected result, as it is known that the free-fermionic $XX$ model corresponds to the case $q = \pm i$ of the $sl_q(2)$-invariant $XXZ$ model, and also there a correspondence is established between the checkerboard $2d$ Ising model (the $N = 2$ analog of the chiral Potts model) and the free-fermionic $XY$ ($XZ$)
models [5, 13]. In [24] it is stated the correspondence of the obtained \( R \)-matrix at \( N = 2 \) with the trigonometric limit of the tree-parametric (or colored) free-fermionic YBE solutions [22, 23]. The connection of this matrix with the quantum algebras \( gl_q(1|1) \) and \( sl_q(2) \) are shown in [26] and [27].

The paper is organized as follows. In the Section 2 the definition of the quantum algebra \( sl_q(2) \) and it’s representations are brought. The functional representation of the algebra by means of theta functions is constructed for the cyclic (semi-cyclic, nilpotent) irreps. The polynomial representation for the highest/lowest weight irreps can be found e.g. in [16]. In the Section 3 the YB equations for two-dimensional cyclic irreps at \( q = i \) (all the results can be extended for the equivalent case of \( q = -i \)) are formulated, and the general aspects of the investigation by means of the projection operators are explained. In the Section 4 the solutions to the YBE are presented. In the Section 5 the corresponding spin-chain quantum models in general terms are sketched and the summary of the work is given.

2 Algebra and notations

The quantum algebra \( sl_q(2) \) is defined by the generators \( e, f, k_{\pm 1} \) [3, 10]

\[
kek^{-1} = q^2e, \quad kfk^{-1} = q^{-2}f, \quad [e, f] = \lambda^{-1}(k - k^{-1}), \quad \lambda = q - q^{-1}.
\] (2.1)

The quadratic Casimir operator is written as

\[
c = ef + \frac{q^{-1}k + qk^{-1}}{\lambda^2}.
\] (2.2)

At the exceptional values of \( q \) (\( q^N = 1 \)) the center of algebra is enlarged and three new Casimir operators appear: \( k^N, e^N \) and \( f^N \), here \( N = N \) if \( N \) is odd and \( N = N/2 \) if \( N \) is even [1, 3]. One can check this by direct calculations of the corresponding commutators. So the representations are characterized by means of the values of the mentioned operators

\[
e^N = x\mathbb{I}, \quad f^N = y\mathbb{I}, \quad (k_{\pm 1})^N = z^{\pm 1}\mathbb{I} \quad \text{and} \quad c = c\mathbb{I}.
\] (2.3)

The values of the Casimir operators are connected by a relation (2.9) [3], which will be presented further in this section. The representations are grouped into two classes: \( A \)-type representations, having highest and lowest weights, which include usual spin-representations \( V_r \) (typical to the algebra \( sl(2) \)) with the dimensions \( r \leq N \) and the \( 2N \)-dimensional indecomposable representations
$I_A$, arising in the fusions of the spin irreps, and the $B$-type representations, including $N$-dimensional cyclic (semi-cyclic, nilpotent) irreps $V_N$ and the corresponding $2N$-dimensional indecomposable representations $I_B$. For the detailed classification see [3].

Let us present here the general cyclic irrep \{v_1, v_2 \cdots; v_N\}, $v_{i+N} \equiv v_i$ at $q^N = \pm 1$ with the action of the algebra generators:

$$
k \cdot v_i = q^{\varepsilon + 2i}v_i,
$$

$$
ee \cdot v_i = \beta_i v_{i+1},
$$

$$
ef \cdot v_i = \gamma_i v_{i-1},
$$

(2.4)

The algebra relations give

$$
\beta_{i-1}\gamma_i - \gamma_{i+1}\beta_i = [\varepsilon + 2i]q, \prod_{i=1}^N \beta_i = x, \prod_{i=1}^N \gamma_i = y, q^N\varepsilon = z.
$$

(2.5)

The parameters $\beta_i$, $\gamma_i$, connected with the above equations, can be fixed by normalization conditions. Denoting $\alpha_i = \gamma_{i+1}\beta_i$, we find

$$
\alpha_i = \alpha_1 - \sum_{p=2}^i [\varepsilon + 2p]q = \alpha_1 - [i - 1]q[1 + i + \varepsilon]q.
$$

Parameterizing $\alpha_1$ as follows $\alpha_1 = \left[\frac{3\varepsilon + \xi}{2}\right]q\left[\frac{\xi - 3 - \varepsilon}{2}\right]q$, we obtain a compact formula

$$
\alpha_i = \left[i + 1 + \varepsilon + \xi\right]/2\left[\frac{\xi - \varepsilon - 1}{2} - i\right]_q.
$$

(2.6)

The semi-cyclic or nilpotent irreps correspond to the choice $\alpha_N = 0$, which gives the values $\xi = \pm \varepsilon \pm 1 + 2nN$ (modulo $2N$). We can verify that the parameter $\xi$ is connected with the eigenvalue c of the quadratic Casimir operator $c$. Acting by the l.h.s and r.h.s. of the relation (2.2) on the vector state $v_{i+1}$, we find $c = \alpha_i + \frac{q^{\varepsilon + 2i + 1} - q^{-\varepsilon - 2i - 1}}{\lambda^2} = \frac{q^\xi + q^{-\xi}}{\lambda^2}.

To relate the values of the Casimir operators [3, 10] one can start from the relation (2.2) in form:

$$
eyf = c - \frac{q^{-1}k + qk^{-1}}{\lambda^2},
$$

acting the l.h.s and r.h.s of it on the states of an $N$-dimensional cyclic irrep and multiplying the results, which in fact will form the determinants (invariant quantity) of the corresponding
where the parametrization $c = q$. So we arrive at:

$$\prod_{k=1}^{N} [a + k]_q = \lambda^{-\mathcal{N}} (q^{N\alpha + \mathcal{N}(\mathcal{N}+1)/2} + (-1)^\mathcal{N} q^{-N\alpha - \mathcal{N}(\mathcal{N}+1)/2}) \equiv \Phi(\alpha). \quad (2.7)$$

So we arrive at:

$$\prod_{s=1}^{N} \alpha_s = \prod_{s=1}^{N} \left( c - \frac{q^{2s-1} + q^{-2s-\varepsilon + 1}}{\lambda^2} \right) \equiv \prod_{s=1}^{N} \left( \frac{q^\varepsilon + q^{-\varepsilon}}{\lambda^2} - \frac{q^{2s-1} + q^{-2s-\varepsilon + 1}}{\lambda^2} \right) = \quad (2.8)$$

$$= \prod_{s=1}^{N} \left[ \frac{\xi}{2} + \frac{1}{2}(\varepsilon - 1) + s \right]_q \left[ \frac{\xi}{2} - \frac{1}{2}(\varepsilon - 1) - s \right]_q = \lambda^{-2\mathcal{N}} \left( q^{N\varepsilon} + q^{-N\varepsilon} + (-q)^\mathcal{N} (z + z^{-1}) \right),$$

where the parametrization $c = \frac{\xi + \varepsilon - \xi}{2}$ is used. Thus,

$$xy = \lambda^{-2\mathcal{N}} (q^{N\varepsilon} + q^{-N\varepsilon} + (\mp 1)^\mathcal{N} (z + z^{-1})). \quad (2.9)$$

Taking into account the relation (2.9) the cyclic irreps have three independent characteristics. Besides of the parameters $\varepsilon, \xi$ in the presented representation space (2.4) we can introduce the third independent parameter $\omega$ by fixing the parameters $\beta_i, \gamma_i$ in the following general way: $\beta_i = \sqrt{\alpha_i} f(\varepsilon, \xi, \omega, i), \quad \gamma_i = \sqrt{\alpha_i - 1}/f(\varepsilon, \xi, \omega, i - 1)$, with a function $f(\varepsilon, \xi, \omega, i)$. Particularly we can take

$$\beta_i = \left[ i + \frac{1 + \varepsilon + \xi}{2} \right]_q \left[ \omega + i \right]_q, \quad \gamma_i = \left[ \frac{\xi - \varepsilon + 1}{2} \right]_q / \left[ \omega + i - 1 \right]_q. \quad (2.10)$$

Here the parameters $\varepsilon, \xi, \omega$ are related by the constraints (2.9), $x = \Phi(\frac{1 + \varepsilon + \xi}{2})/\Phi[\omega]$ and $y = \Phi(\frac{\xi - \varepsilon + 1}{2})/\Phi[\omega]$. In respect to $q^\varepsilon$ and $q^\omega$ these constraints are the equations of the $\mathcal{N}$-th degree and have different solutions of number $\mathcal{N}$. The solutions with different $\xi$ ($\xi_i = \xi_0 + i, \quad i = 1, ..., \mathcal{N}$) are connected with different values of the quadratic Casimir operator, while the solutions with different $\omega$ ($\omega_n = \omega_0 + 2n, \quad n = 1, ..., \mathcal{N}$) are entirely equivalent.

Any cyclic representation with the given Casimir values $\{x, y, z, c\}$ can be characterized by the quantities $\{x, y, \varepsilon, \xi\}$. The semi-cyclic irreps with the condition $\alpha_N = 0$ can be defined as follows: $\beta_i = \alpha_i$ and $\gamma_i = 1 + (y - 1)\delta_{i,1}$, when $x = 0$ and there is a highest weight ($\nu_\mathcal{N}$); or $\beta_i = 1 + (x - 1)\delta_{i,N}$ and $\gamma_i = \alpha_i$, when $y = 0$ and there exists a lowest weight ($\nu_1$).

The quantum algebra is characterized by co-product, definition of which has some ambiguity, when we check the consistency of the co-product with the algebra relations. In the case of the
general values of $q$ the generators on the tensor product of two representations can be chosen in the following general form:

$$\Delta[k] = k \otimes k, \quad \Delta[e] = k^a \otimes e + e \otimes k^b, \quad \Delta[f] = k^c \otimes f + f \otimes k^d,$$

which is obviously consistent with the scale part of the symmetry $(2.1)$. Then unwanted terms in the algebra relations cancel at $d = -a, c = -b$ and $a - b = \pm 1$. This provides one-parameter families of the co-products $\Delta$ and $\tilde{\Delta} \equiv P \Delta P$ ($P$ is a permutation map):

$$\Delta[k^\pm] = k^\pm \otimes k^\pm, \quad \Delta[e] = k^a \otimes e + e \otimes k^{a+1}, \quad \Delta[f] = k^{-a-1} \otimes f + f \otimes k^{-a},$$

$$\Delta[k^\pm] = k^\pm \otimes k^\pm, \quad \Delta[e] = k^a \otimes e + e \otimes k^{a-1}, \quad \Delta[f] = k^{-a+1} \otimes f + f \otimes k^{-a}.$$  \hspace{1cm} (2.11, 2.12)

However, when $q$ takes exceptional values ($q^N = \pm 1$) only integer (integer and half-integer) values of $a$ are acceptable. One can check this statement straightforward in the following way. If we suppose that the operator $k^a$ satisfies the algebra relation $k^a e = q^{2a} e k^a$, then we come to $k^a e^N = q^{2aN} e^N k^a$. As the operator $e^N$ belongs to the center, it follows that $q^{2aN} = 1$, i.e. the number $a$ (2a) must be integer if $q^N = -1$ ($q^N = 1$).

In the further discussion we use the formula $(2.12)$ with the value $a = 1$. Then the operation $\tilde{\Delta}$ corresponds to $(2.11)$ with $a = 0$. These two operations are connected with the intertwiner matrix $R$ defined on the space $V \otimes V$:

$$R \Delta = \tilde{\Delta} R.$$ \hspace{1cm} (2.13)

It occurs that the irreps (representations), on which the intertwiner is defined, must have correlated parameters: the values of the extended center are mutually connected due to the relations $(2.13)$. For general $N$ the elements of the center $e^N, f^N, k^{\pm N}$ have the same co-products as the generators $e, f, k^{\pm 1}$:

$$\Delta[e^N] = k^N \otimes e^N + e^N \otimes 1, \quad \Delta[f^N] = 1 \otimes f^N + f^N \otimes k^{-N}, \quad \Delta[k^{\pm N}] = k^{\pm N} \otimes k^{\pm N}.$$ \hspace{1cm} (2.14)

Implying the relation $(2.13)$ for the elements of the center $e^N, f^N$ on the tensor product of two cyclic representations with the characteristics $\{x_i, y_i, z_i\}$ and $\{x_j, y_j, z_j\}$, we arrive at $[3]

$$z_i x_j + x_i z_j = x_i z_j + x_j, \quad y_j + y_i z_j^{-1} = y_i + z_i^{-1} y_j.$$ \hspace{1cm} (2.15)
2.1 Functional representation of the algebra

The algebra (2.1) can be realized in terms of finite-difference operators acting on the space of complex valued functions as follows:

\[ e = q^{\gamma/2} e^t [\partial - \alpha] q^{\omega}, \quad f = q^{-\gamma/2 - \epsilon \partial} e^{-t} [\beta - \partial] q^{\omega}, \quad k = q^{2\delta - \alpha - \beta}. \]  

(2.16)

The parameter \( \gamma \) is related to the rescaling of the generators \( e \) and \( f \), while the parameter \( \epsilon \) is related to an automorphism \( e \to e^{k^{\ell/2}}, f \to k^{-\ell/2} f \). The parameters \( \alpha \) and \( \beta \) are also defined up to common shift. For the spin-irreps the representation space is isomorphic to the space of polynomials of \( e^t \). Then the half-sum \( (\alpha + \beta)/2 = \ell \) has sense of the spin of the representation.

The functional realization for cyclic representations, containing three independent parameters can be obtained from (2.16) by a transformation:

\[ e' = e \sum_{n=0}^{N-1} e_n q^{\epsilon n - n + 2n\partial}, \quad f' = f \sum_{n=0}^{N-1} f_n q^{\epsilon n - n + 2n\partial}, \quad k' = q^{\chi k}, \]

with some \( e_n, f_n, \chi \) which can be defined from the algebra relations. From the another hand we can simply apply the realization (2.10) to the appropriate chosen functional space. The role of monomials for the cyclic representations can play the following theta-functions with characteristics:

\[ \theta_r(t) = \sum_{n=-\infty}^{\infty} e^{i\pi \tau (n+\frac{r}{N})^2} e^{i(n\omega + t)}, \quad r = 1, \ldots N. \]  

(2.17)

The parameter \( \tau \) is specified by one requirement:

\[ \text{Im} \tau > 0, \]

ensuring the convergence of theta-series. The functions (2.17) form basis in the space of entire functions of order \( N \) [28].

The cyclic property is implied in this realization by the fact that shifts induced by derivative \( \partial_t \) on \( r \) units are defined by modulo \( N \) due to the periodicity of theta-functions (2.17). In order to find an operator realization of generators corresponding to (2.10) acting on basis (2.17) one should just replace the parameter \( i \) in the expressions of the matrix elements of generators by derivative \( \partial \), and use (2.16) with fixed \( \epsilon \). The resulting expressions are:

\[ e = e^t \left[ \partial - \frac{1 + \epsilon + \xi}{2} \right] q^{\omega \partial} e^{\frac{\xi}{N\tau} (2\partial + 1)}, \quad f = e^{-\frac{\xi}{N\tau} (2\partial + 1)} e^{-t} \left[ \frac{\xi - \epsilon + 1}{2} \right] q^{\omega \partial - 1}, \quad k = q^{\epsilon + 2\partial}. \]  

(2.18)
One can verify by straight construction that at $q^4 = 1$ and $x \neq 0$, $y \neq 0$, there are two possible types of non-reducible representations, which are 2-dimensional cyclic irreps and 4-dimensional indecomposable representations (of $A$ or $B$ class) [3]. The tensor product of two general cyclic irreps usually decomposes into a sum of two another cyclic irreps with definite values of the Casimir operators. It follows from (2.14), that the parameters $x$, $y$, $z$ are the same for two cyclic irreps arisen in the fusion. Indecomposable representation can appear for some special cases, under the necessary (but not sufficient) condition that the values of the quadratic Casimir operator are coinciding.

Let $R_{ij}$ is an intertwiner matrix of the quantum algebra $\mathfrak{sl}_q(2)$ defined on the space $V_i \otimes V_j$, when $V_i$ and $V_j$ are cyclic irreps. Hereafter we shall denote by index $i$ in $V_i$ the characteristic index of the representation space (and not the dimension, as it was in the previous discussion), now fixing the dimension of the irreps as $r = 2$. As it is known the intertwiner matrices $R_{ij}$, $\tilde{R}_{ij} = \sum_{kl} R_{ij}$ (defined on $V_i \otimes V_j$) which have the commutativity properties

$$\Delta R_{ij} = R_{ij} \Delta, \quad \Delta \tilde{R}_{ij} = \tilde{R}_{ij} \Delta,$$

satisfy to the Yang-Baxter equations [10, 14, 15]

$$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij} \quad \text{or} \quad \tilde{R}_{ij}\tilde{R}_{jk}\tilde{R}_{ij} = \tilde{R}_{jk}\tilde{R}_{ij}\tilde{R}_{jk}.$$

The spectral parameter dependent YBE, when $R_{ij}$ depends on $\mathbb{C}$-valued spectral parameters $u_i$, $u_j$, can be achieved by the affine extension of the quantum algebra (or by any so-called "baxterization" procedure). In the present situation the parameters arise naturally connected with the characteristic parameters of the representations $V_i$, $V_j$. When the operators in the l.h.s. and r.h.s. of the YBE act on the tensor product $V_1 \otimes V_2 \otimes V_3$, where $V_i$ ($i = 1, 2, 3$) are characterized by the parameters $\{x_i, y_i, z_i\}$, then we can take the YB equations in the following form (the YBE here are inhomogeneous in the sense that the representation spaces $V_i$, $V_j$ on which $\tilde{R}_{ij}$-matrix acts in general have different characteristics)

$$R_{12}(x_1, y_1, z_1)R_{13}(x_1, y_1, z_1)R_{23}(x_2, y_2, z_2) = R_{23}(x_2, y_2, z_2)R_{13}(x_1, y_1, z_1)R_{12}(x_1, y_1, z_1) \quad \text{or} \quad \tilde{R}_{12}(x_1, y_1, z_1)\tilde{R}_{13}(x_1, y_1, z_1)\tilde{R}_{23}(x_2, y_2, z_2) = \tilde{R}_{23}(x_2, y_2, z_2)\tilde{R}_{13}(x_1, y_1, z_1)\tilde{R}_{12}(x_1, y_1, z_1).$$

When $N = 2$, we define two dimensional irreps $V_i$ so, that the algebra generators have the
following general matrix representations on it:

\[ e_i = \begin{pmatrix} 0 & x_i^a \\ x_i \ y_i^a & 0 \end{pmatrix}, \quad f_i = \begin{pmatrix} 0 & \bar{y}_i \\ y_i \ y_i^a & 0 \end{pmatrix}, \quad k_i = e^{\varepsilon_i} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \]  

(3.4)

Here the algebra relations imply \( 2y_i^a x_i^a = \cosh \varepsilon_i + \sqrt{4x_i y_i + (\cosh \varepsilon_i)^2} \). So \( e_i^2 = x_i \mathbb{I}, f_i^2 = y_i \mathbb{I}, k_i^2 = -e^{2\varepsilon_i} \mathbb{I} \) (we set \( z_i = -e^{2\varepsilon_i} \)) and \( c_i = \mp \sqrt{x_i y_i + (\cosh \varepsilon_i)^2} / 4 \mathbb{I} \), where \( \mathbb{I} \) is the unit operator. In the further discussion instead of the parameters \( x_i, y_i, z_i \) we are fixing the parameters \( x_i, c_i, \varepsilon_i \). The value of \( y_i^a \), using the above relation, can be written as \( y_i^a = \frac{1}{y_i} \frac{\cosh \varepsilon_i + c_i}{\cosh \varepsilon_i - c_i} \). Then the parameter \( x_i^a \) is just a parameter connected with the automorphism of the algebra: it can be cancelled by the automorphism: \( g_i \to U_i g_i U_i^{-1}, g = e, f, k, \) and \( U = \begin{pmatrix} \sqrt{x_i y_i} & 0 \\ 0 & \frac{1}{\sqrt{x_i y_i}} \end{pmatrix} \). However for more generality we take the matrices dependent over the parameters \( x_i^{a,ij} \).

The generators from the center of the algebra \( \check{c} = e^2, f^2, k^2, c \) are proportional to the identity operator on the irreps, and the relation (3.1) means, that an intertwiner can exist only on the such vector spaces’ products \( V_i \otimes V_j \), on which, particularly, \( \Delta_{ij} [\check{c}] = \Delta_{ji} [\check{c}] \). This means, as it was stated in the Section 2 for general values of \( \mathcal{N} \) and as we can verify by straight derivation, that the following relations must be fulfilled:

\[ x_j (1 + e^{2\varepsilon_i}) = x_i (1 + e^{2\varepsilon_j}), \quad c_j \cosh \varepsilon_i = \pm c_i \cosh \varepsilon_j, \]  

(3.5)

where instead of the parameter \( y \) in (2.15) we use the eigenvalues of the quadratic Casimir operator.

Summarizing, we see that the intertwiner matrices \( R_{ij} \) for the general cyclic irreps depend on the representation characteristics \( \varepsilon_i, x_i^a \) and \( \varepsilon_j, x_j^a \), as the remaining parameters \( x_i, x_j, c_i, c_j \) can be obtained from the relations (3.5), introducing appropriate constants \( x_i / (1 + e^{2\varepsilon_i}) = x_0, c_i / \cosh \varepsilon_i = c_0 \). The parameters \( x_0 \) and \( c_0 \) are the same for the all three \( \check{R} \)-matrices, so these are constant parameters and can not be considered as spectral parameters. Let \( c_j \cosh \varepsilon_i = c_i \cosh \varepsilon_j \), then the YB equations can be presented as:

\[
R_{12}(u_1, u_2; x_1^a, x_2^a)R_{13}(u_1, u_3; x_1^a, x_3^a)R_{23}(u_2, u_3; x_2^a, x_3^a) = \]
\[
R_{23}(u_2, u_3; x_2^a, x_3^a)R_{13}(u_1, u_3; x_1^a, x_3^a)R_{12}(u_1, u_2; x_1^a, x_2^a), \]  

(3.6)

Here for more generality we introduced additional spectral parameters \( u_i \). However we shall see that it is not necessary to separate these parameters, they appear naturally.
As in our previous works [17, 18], here we shall look for the YBE solutions in the form of linear composition of the invariant operators - projectors. We consider as projector operators a definite basis (linearly independent and complete set) in the space of the algebra invariant operators which are commutative with the algebra generators in the given representation space. Let the last consists of the irreps \( V_i \), which have different characteristics. Then the projectors \( P_i \) are defined as the matrices which act on the irreps \( V_i \) as unity matrices and vanish on the another irreps: 
\[ P_i \cdot V_j = \delta_{ij} V_j. \]
When there are irreps \( V_i, i = 1, \ldots, p \), with the same characteristics then there are also the projectors 
\[ P_{ij} \cdot V_k = P_{jk} V_i. \]
The projectors satisfy the following relations:
\[ \sum_i P_i = I, \quad P_i P_j = \delta_{ij} P_i, \quad P_k P_{ij} = \delta_{ki} P_{ij}, \quad P_{ij} P_{kr} = \delta_{jk} P_{ir}. \]  
(3.7)

In general the tensor product \( V_i \otimes V_j \) decomposes into two cyclic irreps, on which the Casimir operators \( e^2, f^2 \) and \( k^2 \) have the same values (on the tensor product they act as the operators proportional to unity matrix), and the Casimir operator \( c \) has two different values \( c_{ij}, \bar{c}_{ij} \), differing by a sign 
\[ c_{ij} = -\bar{c}_{ij} = -ic_i \sinh [\varepsilon_i + \varepsilon_j]/\cosh \varepsilon_i. \]
Taking into account this, we can denote the spaces in the tensor expansion as \( V_{ij}^{\pm} \):
\[ V_i \otimes V_j = V_{ij}^{+} \oplus V_{ij}^{-}. \]

As we intend to investigate the \( B \)-type representations step by step, here we do not consider the indecomposable representations. It is worthy to mention however that for the cases described by \( c_i \cosh \varepsilon_j = \pm c_j \cosh \varepsilon_i \), the tensor product \( V_2 \otimes V_2 = V_2 \oplus V_2 \) under the condition \( e^{2(\varepsilon_i + \varepsilon_j)} = 1 \) deforms into \( V_2 \otimes V_2 = F_{3,1}^{(4)} \) [16], which is an \( A \)-type indecomposable representation. Now the \( \hat{R} \)-matrix, (as well as any invariant matrix) decomposes into the sum of the projectors \( P_{\hat{Z}} \) and \( P_{\hat{Z}} \) (see for the description the work [17]). As the number of projection operators does not increase, the new projectors can be found as the limit cases of the linear combinations of the non-deformed projectors \( P_{\pm} \), and as a result no new solutions to YBE arise [17], all the solutions can be obtained from the presented solutions taking a proper limit \( \varepsilon_j \to -\varepsilon_i \). When \( c_i = c_j = 0 \) the tensor product remains the same.

4 Solutions to YBE

We analyze in this section the solutions to YBE defined on the tensor product of three two-dimensional cyclic irreps. Semi-cyclic and nilpotent cases can be obtained taking the particular limits.
Below we consider separately three different cases corresponding to the relations (3.5): \( c_j \cosh \varepsilon_i = c_i \cosh \varepsilon_j \), \( c_j \cosh \varepsilon_i = -c_i \cosh \varepsilon_j \) and \( c_j = c_i = 0 \). The case \( \cosh \varepsilon_i = \cosh \varepsilon_j = 0 \), which occurs to be degenerated, also will be considered.

### 4.1 \( c_j \cosh \varepsilon_i = c_i \cosh \varepsilon_j \)

At first let us explore the case \( c_j \cosh \varepsilon_i = c_i \cosh \varepsilon_j \). There are two projectors here \( P_+ = -(c - c_i)\mathbb{1})/(2c_{ij}) \) and \( P_- = (c + c_i)\mathbb{1})/(2c_{ij}) \): \( P_\pm \cdot V_{ij}^\pm = V_{ij}^\pm \). The commutativity relation (3.1) means that \( \hat{R}_{ij} \) is a sum over the ”projectors” \( \hat{P}_\pm = P_{ij}P_{ij} \), where \( P_{ij} \) is an identical transformation map \( V\{x_i, y_i, z_i\} \otimes V\{x_j, y_j, z_j\} \rightarrow V\{x_i, y_i, z_i\} \otimes V\{x_j, y_j, z_j\} \).

The operator \( P_{ij} \) depends for the discussed case on the parameters \( \varepsilon_i, \varepsilon_j, (x_i^0, x_j^0) \),

\[
P_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & x_i^0 \frac{1 + e^{2\varepsilon_j}}{1 + e^{2\varepsilon_i}} & \frac{i(e_i - e_j)}{1 + e^{2\varepsilon_i}} & 0 \\
0 & \frac{i(e_i - e_j)}{1 + e^{2\varepsilon_i}} & x_j^0 \frac{1 + e^{2\varepsilon_j}}{1 + e^{2\varepsilon_i}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (4.1)

This projector operator has the following properties, \( P_{ij}P_{ji} = I \) and \( P_{ii} = I \).

The matrix \( \hat{R}_{ij}^+(u) = \hat{P}_+ + f_{ij} \hat{P}_- \),

\[
\hat{R}_{ij}^+ = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & x_i^0 (e^{\varepsilon_i - e_j} f_{ij}) \cosh [\varepsilon_i] & \frac{i(f_{ij} \cosh [\varepsilon_i] - \cosh [\varepsilon_j])}{\sinh [\varepsilon_i + \varepsilon_j]} & 0 \\
0 & \frac{i(f_{ij} \cosh [\varepsilon_i] - \cosh [\varepsilon_j])}{\sinh [\varepsilon_i + \varepsilon_j]} & x_j^0 (e^{\varepsilon_i - e_j} f_{ij}) \cosh [\varepsilon_j] & 0 \\
0 & 0 & 0 & f_{ij}
\end{pmatrix},
\] (4.2)

admits a general solution with \( f_{ij} = (f_i + e^{\varepsilon_i + \varepsilon_j} f_j)/(e^{\varepsilon_i + \varepsilon_j} f_i + f_j) \). Here the coefficients \( f_i, f_j \) are arbitrary, and enter into the solution as \( f_i/f_j \), so we can denote that proportion as \( \frac{f_i}{f_j} \equiv f(\varepsilon_i, x_i^0, \{u_i\}) \) with arbitrary function \( f(\varepsilon_i, x_i^0, \{u_i\}) \) and a set of the spectral parameters \( \{u_i, u_j\} \).

The corresponding matrix is

\[
\hat{R}_{ij}^+(u, e^{\varepsilon_i}, e^{\varepsilon_j}, x_i^0, x_j^0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & x_i^0 \frac{(1 + e^{2\varepsilon_i}) f_i}{1 + e^{2\varepsilon_j}} & \frac{i e_i - e_j}{1 + e^{2\varepsilon_i}} & 0 \\
0 & \frac{i e_i - e_j}{1 + e^{2\varepsilon_i}} & x_j^0 \frac{(1 + e^{2\varepsilon_j}) f_j}{1 + e^{2\varepsilon_i}} & 0 \\
0 & 0 & 0 & \frac{f_i + e^{\varepsilon_i + \varepsilon_j}}{1 + e^{2\varepsilon_j}}
\end{pmatrix}.
\] (4.3)
Note, that the matrix (4.3) for the particular homogeneous case \( \varepsilon_i = \varepsilon_j, \quad x_i^a = x_j^a \), is a \( \tilde{R} \)-matrix, describing the \( XX \)-model in the transverse magnetic field (\( \cos \varepsilon \)). Setting \( f_i/f_j = e^{u_i-u_j} \equiv e^u \), and after consecutive replacements \( e^{2\varepsilon} = e^{2u_0}, \quad u \rightarrow iu \) and \( u_0 \rightarrow iu_0 + i\pi/2 \) and multiplying the matrix by an overall function \( \sin(u + u_0) \), we shall come to the \( \tilde{R} \)-matrix, which describes the \( XX \)-model in the transverse field \( \cos u_0 \) [19, 10, 24].

\[
\tilde{R}_{ij}(u) = \begin{pmatrix}
sin(u + u_0) & 0 & 0 & 0 \\
0 & e^{iu} \sin u_0 & \sin u & 0 \\
0 & \sin u & e^{-iu} \sin u_0 & 0 \\
0 & 0 & 0 & \sin(u_0 - u)
\end{pmatrix}.
\] (4.4)

This matrix satisfies to the simple YBE

\[
R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v).
\] (4.5)

As there are classified the YBE solutions with general non-homogeneous \( 4 \times 4 \) \( R \)-matrices with eight non-zero matrix elements (like eight-vertex model’s \( R \)-matrix [13]) depending on the one spectral parameter (”difference property”: \( R(u, v) = R(u - v) \)) [13, 20], we know that there are limited kind of such solutions and all the interesting cases are restricted with the cases of the \( XYZ \)-model’s matrix and the ”free-fermionic” non-homogeneous extensions, one of which is just the matrix brought above. At \( q = i \) the \( sl_q(2) \) invariant matrices defined on the irreps all have free-fermionic property: \( \tilde{R}^{00}_{00} \tilde{R}^{11}_{11} = \tilde{R}^{01}_{01} \tilde{R}^{10}_{10} - \tilde{R}^{10}_{01} \tilde{R}^{01}_{10} \) (see the Summary).

At the end of this subsection we want to mention the relation of the solution (4.3) to the one obtained in the paper [8]. These two solutions can be related by an automorphism of the matrix \( R_{ij} \), written as

\[
R_{p_ip_j}^{n_in_j} \Rightarrow R_{n_in_j}^{p_ip_j} \frac{f_{n_in_j}}{f_{p_ip_j}},
\] (4.6)

induced from the transformations \( e_{n,,j} \rightarrow f_{n,,j} e_{n,,j} \) of the vector basis \( e_{n,,j} \) \( (n, j = 0, 1, p, j = 0, 1) \) of the space \( V_{i,j} \) with a function \( f_{n,,j} = 1 + i e^{\varepsilon\delta_{n,,j}} \). So, only the matrix elements \( \tilde{R}_{01}^{01} \) and \( \tilde{R}_{10}^{10} \) transform correspondingly into the functions

\[
\frac{x^a}{x_i} \frac{(1-i e^{\varepsilon})}{1+e^{\varepsilon}e^{e_i}} \quad \text{and} \quad \frac{x^a}{x_j} \frac{(1-i e^{\varepsilon})}{1+e^{\varepsilon}e^{e_j}}.
\]

Now let us represent the next solutions to the YBE with the cyclic representations.
4.2 \( c_j \cosh \varepsilon_i = -c_i \cosh \varepsilon_j \)

In the case \( c_j \cosh \varepsilon_i = -c_i \cosh \varepsilon_j \) the transformation operator \( P_{ij} \) has the following matrix representation

\[
P_{ij} = \begin{pmatrix}
\frac{i(e^{\varepsilon_i}+e^{\varepsilon_j})}{1-e^{\varepsilon_i+\varepsilon_j}} & 0 & 0 & \frac{x_i(1+e^{2\varepsilon_j})}{(1-e^{\varepsilon_i+\varepsilon_j})}x_j^2 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
x_i^2x_j^2(1+e^{2\varepsilon_i}) & 0 & 0 & \frac{i(e^{\varepsilon_i}+e^{\varepsilon_j})}{-1+e^{\varepsilon_i+\varepsilon_j}}
\end{pmatrix}.
\]

(4.7)

As we see there is a dependence from the parameter \( x_i \). Recalling, that \( x_i/(1+e^{2\varepsilon_i}) = x_j/(1+e^{2\varepsilon_j}) \), we can use an independent parameter \( x_0 = x_i/(1+e^{2\varepsilon_i}) \) instead of \( x_i \). And then the matrix \( \tilde{R}_{ij} = P_{ij}(P_+ + g_{ij}P_-) \) is

\[
\tilde{R}_{ij} = \begin{pmatrix}
\frac{-i(g_{ij}\cosh [\varepsilon_i]+\cosh [\varepsilon_j])}{\sinh [\varepsilon_i+\varepsilon_j]} & 0 & 0 & \frac{-2x_0(g_{ij}+e^{\varepsilon_i+\varepsilon_j})\cosh [\varepsilon_i]\cosh [\varepsilon_j]}{\sinh [\varepsilon_i+\varepsilon_j]}x_j^2 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
\frac{-x_i^2x_j^2(g_{ij}+e^{-\varepsilon_i-\varepsilon_j})}{2x_0\sinh [\varepsilon_i+\varepsilon_j]} & 0 & 0 & \frac{i(\cosh [\varepsilon_i]+g_{ij}\cosh [\varepsilon_j])}{\sinh [\varepsilon_i+\varepsilon_j]}
\end{pmatrix}.
\]

(4.8)

The matrix of this kind have to be considered in the YBE together with the matrix \( \tilde{R}_{ij}^+ = P_{ij}(P_+ + f_{ij}P_-) \) in the following non-homogeneous YBE,

\[
R_{13}^+(u_1, u_2; e^{\varepsilon_1,c_1,x_1^2}, e^{\varepsilon_2,c_2,x_2^2})R_{13}^-(u_1, u_3; e^{\varepsilon_1,c_1,x_1^2}, e^{\varepsilon_3,c_3,x_3^2})R_{23}^-(u_2, u_3; e^{\varepsilon_2,c_2,x_2^2}, e^{\varepsilon_3,c_3,x_3^2})R_{23}^+(u_2, u_3; e^{\varepsilon_2,c_2,x_2^2}, e^{\varepsilon_3,c_3,x_3^2}) = \]

(4.9)

where the conditions \( c_1 \cosh \varepsilon_2 = c_2 \cosh \varepsilon_1, c_1 \cosh \varepsilon_3 = -c_3 \cosh \varepsilon_1 \) and \( c_2 \cosh \varepsilon_3 = -c_3 \cosh \varepsilon_2 \) work.

The solutions to the presented YBE are of this graceful form

\[
f_{ij} = \frac{f[u_i, \varepsilon_i, x_i^2] + e^{\varepsilon_i+\varepsilon_j}f[u_j, \varepsilon_j, x_j^2]}{f[u_i, \varepsilon_i, x_i^2]e^{\varepsilon_i+\varepsilon_j} + f[u_j, \varepsilon_j, x_j^2]}, \quad g_{ij} = \frac{f[u_i, \varepsilon_i, x_i^2] - e^{\varepsilon_i+\varepsilon_j}g[u_j, \varepsilon_j, x_j^2]}{f[u_i, \varepsilon_i, x_i^2]e^{\varepsilon_i+\varepsilon_j} + g[u_j, \varepsilon_j, x_j^2]},
\]

(4.10)

where the functions \( f[u, \varepsilon, x^2], g[u, \varepsilon, x^2] \) are arbitrary. Note, that the solution \( \tilde{R}_{ij}^+ \) coincides with the general solution obtained in the previous subsection.

Note, that the resemblance of the functions \( f_{ij} \) and \( g_{ij} \) is not casual, as the constraint \( c_j \cosh \varepsilon_i = -c_i \cosh \varepsilon_j \) can be transformed into \( c_j \cosh \varepsilon_i = c_i \cosh (\varepsilon_j + i\pi) \) (corresponding to the case discussed in the previous subsection), which means that we can consider the space \( V_j \) having parameter
\((e_j + i\pi)\) instead of \(e_j\), which does not change the values of \(z_j, x_j, y_j\), but interchanges the vector states: \(\{v_1, v_2\}_j \rightarrow \{v_2, v_1\}_j\), explaining thus the difference between the matrix forms of (4.7) and (4.1). And moreover, we can extend this observation for the case with general \(N\). Then the relations between the characteristics of two cyclic irreps \(V_i, V_j\), on which an intertwiner is defined can be presented as follows from the general constraints (2.9) and (2.15) \((q^N = \pm 1)\):

\[
\frac{x_i}{(z_i^{1/2} - z_i^{-1/2})^2} = \frac{x_j}{(z_j^{1/2} - z_j^{-1/2})^2} = \frac{q^{N\xi_i/2} + (\mp 1)^N q^{-N\xi_i/2}}{z_i^{1/2} - z_i^{-1/2}} = \frac{q^{N\xi_j/2} + (\mp 1)^N q^{-N\xi_j/2}}{z_j^{1/2} - z_j^{-1/2}}.
\]

(4.11)

The second equations connected with the quadratic Casimir operators with two signs can be relate one to another by the change \(z_j^{1/2} \rightarrow -z_j^{1/2}\).

### 4.3 \(c_i = c_j = 0\)

The next case corresponds to the situation, when \(c_i = c_j = 0\). Now two eigenvalues of the Casimir operator \(c_{ij}\) coincide one with another and equal to 0. It means that there are four linear independent projection operators, which compose the \(\hat{P}\)-matrix. We denote them as \(P_{ij} \cdot \{P_{++}, P_{--}, P_{+-}, P_{-+}\}\). The first two operators act on the each of two cyclic representations as identity operator and vanish on the other irrep \(\{P_{\pm} \cdot V^\pm_{ij} = V^\pm_{ij}, P_{\pm} \cdot V^\mp_{ij} = 0\}\), meanwhile two other projectors transpose one irrep with the other \(\{P_{\mp} \cdot V^\pm_{ij} = V^\mp_{ij}, P_{\mp} \cdot V^\mp_{ij} = 0\}\). The transformation operator \(P_{ij}\) now can be written as

\[
P_{ij} = \begin{pmatrix}
    (x_1^2 e^{e_j} \cosh [x_j] - e^{-e_j} (x_2^2) e^{e_j} \cosh [x_j]) & 0 & 0 & 2x_0(e^{e_j} \cosh [x_j] / x_1^2 - e^{-e_j} (\cosh [x_j] / x_1^2)^2) \\
    0 & 0 & 1 & 0 \\
    0 & 1 & 0 & 0 \\
    2x_0 \sinh [x_j] & 0 & 0 & (x_1^2 e^{e_j} \cosh [x_j] - e^{-e_j} (x_2^2) e^{e_j} \cosh [x_j]) \\
\end{pmatrix}.
\]

(4.12)

Then the matrix \(\hat{R}_{ij} = P_{ij} (P_{++} + f_{ij} P_{--} + g_{ij} P_{+-} + h_{ij} P_{-+})\) has the following form

\[
\hat{R}_{ij} = \begin{pmatrix}
    (x_1^2 e^{e_j} \cosh [x_j]) & 0 & 0 & 2x_0(e^{e_j} \cosh [x_j])^2 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -x_1^2 e^{-e_j} \cosh [x_j] & 0 & 0 & (x_1^2 e^{e_j} \cosh [x_j])^2 \\
\end{pmatrix} + f_{ij} \begin{pmatrix}
    (x_1^2 e^{e_j} \cosh [x_j]) & 0 & 0 & 2x_0 e^{e_j} \cosh [x_j] \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    i(x_1^2) e^{-e_j} & 0 & 0 & x_1^2 e^{e_j} \cosh [x_j] \\
\end{pmatrix}.
\]

(4.13)
Among the YB equations there is simple relation on the coefficient function \( f_{ij} \)

\[
    f_{ik} = f_{ij} f_{jk}, \quad (4.14)
\]

which expresses the *factorizable* property of \( f_{ij} \). It means that we can take \( f_{ik} = f_{i}/f_{k} \), with the functions \( f_{a} (a = i, k) \) depending only of the parameters with the index \( a \).

At first let us explore two simple cases.

When \( g_{ij} = 0 \) and \( h_{ij} = 0 \), then there is one solution to YBE with the following value of the *factorizable* function \( f_{ij} \)

\[
    f_{ij} = \frac{e^{\varepsilon_{j} \cosh [\varepsilon_{j}] x_{0}^{a}}}{e^{\varepsilon_{j} \cosh [\varepsilon_{j}] x_{0}^{a}}} \left( 1 \pm \sqrt{1 + f_{0}(\cosh [\varepsilon_{j}]^{2})} \right),
\]

where \( f_{0} \) is a constant.

When the expression for the \( \hat{R} \)-matrix includes only the projectors \( P_{+-} \) and \( P_{-+} \), then there is no solution to the YBE.

For obtaining the general solutions let us consider at first the homogeneous solutions which satisfy the conditions \( x_{a}^{i} = x_{a}^{j} \), \( \varepsilon_{i} = \varepsilon_{j} \), and explore the one-parametric YBE equations (4.5).

As we have stated, the function \( f_{ij} \) can be presented as \( f_{ik} = f_{i}/f_{k} \). If \( f_{i} \) depends only on the state parameters \( x_{a}^{i} \) and \( \varepsilon_{i} \), then in the homogeneous case \( f_{ij} = 1 \). There are two such spectral-parameter dependent solutions. One is written as \( f_{ij} = 1 \) and \( g_{ij} = -h_{ij} = \tanh [u] \) \( (u \) is an additive spectral parameter)

\[
    \hat{R}^{*}(u) = \left( \begin{array}{cccc}
        1 & 0 & 0 & -e^{\alpha} \tanh [u] \\
        0 & 1 & \tanh [u] & 0 \\
        0 & -\tanh [u] & 1 & 0 \\
        e^{-\alpha} \tanh [u] & 0 & 0 & 1 \\
    \end{array} \right), \quad e^{\alpha} = \frac{2e^{\varepsilon_{j} \cosh [\varepsilon_{j}] x_{0}^{a}}}{(x_{a}^{i})^{2}}. \quad (4.16)
\]
The function \( f \) here is not casual, as the eigenvalues \( \varepsilon \) inhomogeneous case, taking that the function \( f \) of \( \varepsilon \). This is just a trigonometric limit of the \( \tilde{R} \)-matrix of the 2d Ising model [13, 19, 20].

If \( f_i \) has also an extra argument \( u_i \) (spectral parameter), then in the homogeneous case we take \( f_i = f[u_i] \), and \( f_{ij} = f[u_i]/f[u_j] \). As we are exploring now one parametric YBE \((4.5)\), we require that the function \( f_{ij} \) depends on the difference of the spectral parameters, which dictates the choice of \( f[u_i] \) as an exponential function, and \( f_{ij} = e^{u_i - u_j} \equiv e^{a_i} \). Then we shall come to the solution \((4.4)\) obtained in the subsection 4.1. The generalization of this solution to the inhomogeneous case is

\[
R^{ss}(u) = \begin{pmatrix}
1 - \frac{\tanh [u]}{\cosh \varepsilon_i} & 0 & 0 & -e^{-\alpha} \tanh [u] \\
0 & 1 & \tanh [\varepsilon_i] \tanh [u] & 0 \\
0 & \tanh [\varepsilon_i] \tanh [u] & 1 & 0 \\
-e^{-\alpha} \tanh [u] & 0 & 0 & 1 + \frac{\tanh [u]}{\cosh \varepsilon_i}
\end{pmatrix}.
\]

(4.17)

This is just a trigonometric limit of the \( \tilde{R} \)-matrix of the 2d Ising model [13, 19, 20].

The second solution corresponds to \( f_{ij} = 1 \) and \( g_{ij} = h_{ij} = \tanh [u] \tanh [\varepsilon_i] \).

\[
f_{ij} = \frac{(x_i^a)^2 f[\varepsilon_i, x_i^a, \{u_i\}]}{(x_j^a)^2 f[\varepsilon_j, x_j^a, \{u_j\}]} (1 + e^{2\varepsilon_i}) (1 + e^{2\varepsilon_j})
\]

(4.18)

\[
g_{ij} = \frac{1}{x_i^a x_j^a} e^{\varepsilon_i} (1 + e^{2\varepsilon_i}) \frac{f[\varepsilon_i, x_i^a, \{u_i\}]}{f[\varepsilon_i, x_i^a, \{u_i\}]} - e^{\varepsilon_j} (1 + e^{2\varepsilon_j})
\]

(4.19)

\[
h_{ij} = \frac{1}{x_i^a x_j^a} e^{\varepsilon_j} (1 + e^{2\varepsilon_j}) \frac{f[\varepsilon_i, x_i^a, \{u_i\}]}{f[\varepsilon_i, x_i^a, \{u_i\}]} - e^{\varepsilon_i} (1 + e^{2\varepsilon_i})
\]

(4.20)

The function \( f[\varepsilon_i, x_i^a, \{u_i\}] \) is an arbitrary function. In the particular homogeneous case when \( \varepsilon_i = \varepsilon_j \), \( x_i^a = x_j^a \) and \( f_{ij} = e^{2(u_i - u_j)} \equiv e^{2u} \), we have \( f_{ij} = e^{2u} \), \( g_{ij} = h_{ij} = ie^{a_i} \sinh [u] / \cosh [\varepsilon_i] \), the corresponding \( R \)-matrix coincides with the solution \((4.4)\). And one can observe, that in the inhomogeneous case, taking \( f[\varepsilon_i, x_i^a, \{u_i\}] / f[\varepsilon_j, x_j^a, \{u_j\}] = (1 + e^{2\varepsilon_i}) f_i / (1 + e^{2\varepsilon_j}) f_i \), after some normalization calculations this is the solution \( \tilde{R}'_{ij} \) \((4.3)\) which we have in the subsection 4.1. The appearance of the solution \( \tilde{R}'_{ij} \) here is not casual, as the eigenvalues \( c_{i,j} \) are not presented in the projectors evidently, so the values \( c_{i,j} = 0 \) are also permissible in the case discussed in the subsection 4.1. This solution, with the choice \( \frac{1}{f_j} = e^{2u} \) is also equivalent to the trigonometric limit of the free-fermionic elliptic solutions [22, 23], after fixing the elliptic module as \( k = 0 \).

The extension for the first matrix \((4.16)\) with the parameters \( x_i^a \neq x_j^a \), \( \varepsilon_i \neq \varepsilon_j \) can be written as

\[
f_{ij} = \frac{(x_i^a)^2}{(x_j^a)^2} \frac{1 + e^{2\varepsilon_j}}{1 + e^{2\varepsilon_i}}
\]

(4.21)

\[
g_{ij} = -h_{ij} = (1 + e^{2\varepsilon_j}) \frac{x_i^a}{x_j^a} \frac{\pm(h[\varepsilon_i, x_i^a, \{u_i\}] - h[\varepsilon_j, x_j^a, \{u_j\}])}{h[\varepsilon_i, x_i^a, \{u_i\}] + h[\varepsilon_j, x_j^a, \{u_j\}] + \pm(i e^{\varepsilon_i} - i e^{\varepsilon_j})}
\]

(4.22)
The function \( h[\varepsilon, x^a, \{u\}] \), here and below too, is an arbitrary function. Two solutions with different signs can be mapped one to another by the shift of the variables \( \varepsilon_{i,j} \rightarrow \varepsilon_{i,j} + i\pi \) and transformation \( h_{ij} \rightarrow -h_{ij} \), \( g_{ij} \rightarrow -g_{ij} \). The corresponding \( \tilde{R} \) matrix, after normalization, with multiplication by a function, has the form (we choose the case with upper sign in (4.22) and use the notations

\[
\tilde{h}_i = h[\varepsilon_i, x^a_i, \{u_i\}], \quad h_j = h[\varepsilon_j, x^a_j, \{u_j\}] \quad \text{and} \quad \tilde{h}_i = h[\varepsilon_i, x^a_i, \{u_i\}], \quad \tilde{h}_j = h[\varepsilon_j, x^a_j, \{u_j\}]
\]

\[
\tilde{R}^*_{ij} = \begin{pmatrix}
\tilde{h}_i + h_j & 0 & 0 & -x_0(\varepsilon^i - i)(\varepsilon^j - i)(\tilde{h}_i - h_j) \\
0 & \frac{x^a_i(x^a_j - i)(\tilde{h}_i + \tilde{h}_j)}{x^a_i(x^a_j + i)} & h_j - h_i & 0 \\
0 & 0 & \frac{x^a_i(x^a_j - i)}{x^a_i(x^a_j + i)}(\tilde{h}_i + h_j) & 0 \\
x_0(x^a_i + i)(x^a_j + i) & 0 & 0 & h_i + h_j
\end{pmatrix}
\] (4.23)

This matrix, after an appropriate re-parametrization can be brought to the form of the two-parametric solution of YBE [20], see also (4.49).

The extension of the second solution (4.17) for the inhomogeneous case is

\[
f_{ij} = \frac{(x^a_i)^2 1 + e^{2\varepsilon_{ij}}}{(x^a_j)^2 1 + e^{2\varepsilon_{ij}}} \quad (4.24)
\]

\[
g_{ij} = h_{ij} + 2x^a_i e^{\varepsilon_i} - e^{\varepsilon_j} \quad (4.25)
\]

\[
h_{ij} = \frac{x^a_i h[\varepsilon_i, x^a_i, \{u_i\}](1 \pm i(e^{\varepsilon_i - e^{\varepsilon_j}}) - e^{\varepsilon_i + e_j} - h[\varepsilon_j, x^a_j, \{u_j\}](1 \pm i(e^{\varepsilon_i - e^{\varepsilon_j}}) - e^{\varepsilon_i + e_j})}{(1 + e^{2\varepsilon_{ij}})(h[\varepsilon_i, x^a_i, \{u_i\}] + h[\varepsilon_j, x^a_j, \{u_j\}])]} \quad (4.26)
\]

By redefinition of the arbitrary functions \( h[\varepsilon_i, x^a_i, \{u_i\}] \), it is possible to change the appearance of the functions \( h_{ij} \), \( g_{ij} \). Particularly, one can bring the parametrization in (4.22) to the form

\[
h_{ij} = \frac{x^a_i h[\varepsilon_i, x^a_i, \{u_i\}](1 \pm (e^{\varepsilon_j} - e^{\varepsilon_i}) + e^{\varepsilon_i + e_j}) - h[\varepsilon_j, x^a_j, \{u_j\}](1 \pm (e^{\varepsilon_i} - e^{\varepsilon_j}) + e^{\varepsilon_i + e_j})}{(1 + e^{2\varepsilon_{ij}})(h[\varepsilon_i, x^a_i, \{u_i\}] + h[\varepsilon_j, x^a_j, \{u_j\}])} \quad (4.26)
\]

The particular homogeneous cases (4.16, 4.17) correspond to the choice \( h[\varepsilon_i, x^a_i, \{u_i\}] = e^{2(\varepsilon_i - u_i)} = e^{2u} \). Note, that this solution with the same choice of the function \( h[\varepsilon_i, x^a_i, \{u_i\}] \), but in inhomogeneous case \( \varepsilon_i = \varepsilon_j \) is equivalent to the trigonometric limit of the elliptic solutions [22, 23], with the elliptic module \( k = 1 \) (for the parameterizations presented in [23], one must perform some transformations, such as \( \varepsilon_i = \varphi_i + \pi/2 \) and then the automorphism (4.6), with appropriate chosen functions \( f_{ij} \)).

The matrix representation of the solutions (4.24-4.26) is the following (the case with upper sign), where we have used the notations

\[
\tilde{R}^*_{ij} = \frac{x^a_i}{x^a_j(1 + e^{2\varepsilon_{ij}})} \times \quad (4.27)
\]

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Further we represent all the constant primary conditions (constant solutions to YBE), i.e. when the already obtained case (4.15) is one of the such solutions, which has not included in the three families of the solutions (4.21-4.22), (4.24-4.26) and (4.18-4.20), presented in the previous paragraph.

Hereafter we omit the variables $x^a$ and $u$ in the arguments of the functions, as the variables $u_i$ are not involved immediately in the YBE, and the variables $x^a_i$ can be eliminated by the appropriate normalization of the functions. However, when we obtain a dependence from an arbitrary function on the parameter $\varepsilon_i$, then we can involve in the argument the remaining variables as well.

We take for the function $f_{ij}$ a general parametrization (4.18)

$$f_{ij} = \frac{(x^a_i)^2}{(x^a_j)^2} \frac{f[\varepsilon_i]}{f[\varepsilon_j]}.$$  

The obtained solutions $\tilde{R}(\varepsilon_i, \varepsilon_j, x^a_i, x^a_j; u_i, u_j)$ contain arbitrary functions on the variables $\varepsilon_i, x^a_i, u_i$. This dependence from the arbitrary functions has a remarkable property of "factorization", in the sense, that the functions appear in the matrix elements only in the form of the ratio $\frac{f[h|\varepsilon_i, x^a_i; \{u_i\}]}{f[h|\varepsilon_j, x^a_j; \{u_j\}]}$.

In this way it gives us an opportunity to choose the dependence from the extra arguments (spectral parameters $u_i$, $u_j$) in difference form via the exponential functions, $\frac{f[h|\varepsilon_i, x^a_i; \{u_i\}]}{f[h|\varepsilon_j, x^a_j; \{u_j\}]} \approx e^{u_i-u_j}$, and for the argument $u_{ij} = u_i - u_j$ the YB equations have ordinary form (4.5). The mentioned property comes from the fact, that we have obtained the above inhomogeneous solutions as generalizations to the solutions of the YBE (4.5).

But, as we can see, there is possible to obtain more general inhomogeneous solutions, where the dependence from the arbitrary functions has not the discussed "factorization" property. The solutions of the functions $f_{ij}$, $g_{ij}$, $h_{ij}$ to the YBE for the homogeneous cases, i.e. at the values $\varepsilon_i = \varepsilon_j$, $x^a_i = x^a_j$, can be viewed as primary conditions for the general inhomogeneous solutions. Further we represent all the constant primary conditions (constant solutions to YBE), i.e. when also $u_i = u_j$ (spectral parameter dependent ones $u_i \neq u_j$ with YBE (4.5) are presented above), and their extensions.

\* The most fruitful case corresponds to the primary conditions $f_{ii} = 1$, $g_{ii} = h_{ii} = 0$. Note that the already obtained case (4.15) is one of the such solutions, which has not included in the three families of the solutions (4.21-4.22), (4.24-4.26) and (4.18-4.20), presented in the previous paragraph.
For presenting the general solutions with the mentioned primary conditions \( f_{ii} = 1, \ g_{ii} = h_{ii} = 0 \) we denote

\[
\tilde{g}_{ij} = ie^{\varepsilon_i}(1 + e^{2\varepsilon_j}) + \frac{x_j^a}{x_i^a}(1 + e^{2\varepsilon_i})(e^{\varepsilon_i+\varepsilon_j}g_{ij} + h_{ij}),
\]

\[
\tilde{h}_{ij} = \frac{f[\varepsilon_j]}{f[\varepsilon_i]} \left( e^{\varepsilon_i} \frac{f[\varepsilon_j]}{f[\varepsilon_i]} - e^{\varepsilon_j} + i \frac{x_j^a}{x_i^a}(e^{\varepsilon_i+\varepsilon_j}h_{ij} + g_{ij}) \right).
\] (4.29)

From the YBE we obtain the following consistency conditions for the solutions \((g_0)\) is a constant\)

\[
\tilde{h}_{ij} = 0 \quad \text{or} \quad e^{\varepsilon_j}(1 + e^{2\varepsilon_i})^2 \frac{f[\varepsilon_i]}{f[\varepsilon_j]} + i(1 + e^{2\varepsilon_j}) \tilde{g}_{ij} = \frac{g_0}{(f[\varepsilon_j])^2}.
\] (4.30)

One can unveil the meaning of the above conditions, representing the \(\tilde{R}\)-matrix (4.13) in terms of the functions \(\tilde{h}_{ij}, \ \tilde{g}_{ij}\). It appears that \(\tilde{R}^{00}_{11} \approx \tilde{h}_{ij}\), and \(\tilde{R}^{11}_{00} \approx (e^{\varepsilon_j}(1 + e^{2\varepsilon_i})^2 \frac{f[\varepsilon_i]}{f[\varepsilon_j]} + i(1 + e^{2\varepsilon_j}) \tilde{g}_{ij})\).

The consistency conditions simply imply \(\tilde{R}^{00}_{11} = 0\) or \(\tilde{R}^{11}_{00} = 0\) (when \(g_0 = 0\)), or \(\tilde{R}^{11}_{11} = \frac{\tilde{R}^{00}_{11}}{(f[\varepsilon_i]f[\varepsilon_j])}\).

At first let us consider the case \(\tilde{h}_{ij} = 0\). The solutions now have the forms (the function \(g_{ij}\) can be obtained from the equation (4.29))

\[
f_{ij} = \frac{(x_i^a)^2 f[\varepsilon_i]}{(x_j^a)^2 f[\varepsilon_j]}, \quad h_{ij} = \frac{\frac{x_j^a}{x_i^a}[e^{\varepsilon_j} - \tilde{h}[\varepsilon_j]](1 + e^{2\varepsilon_j}) \frac{f[\varepsilon_i]}{f[\varepsilon_j]} - [e^{\varepsilon_i} - \tilde{h}[\varepsilon_i]](1 + e^{2\varepsilon_j})}{(1 + e^{2\varepsilon_i})(1 + e^{2\varepsilon_j})},
\] (4.31)

where the functions \(\tilde{h}[\varepsilon]\) and \(f[\varepsilon]\) are interrelated/interdependent. Let \(\tilde{h}[\varepsilon]\) is an arbitrary function, then the general solutions contain a constant number \(f_0\) and

\[
f[\varepsilon] = \frac{(1 + f_0)\tilde{h}[\varepsilon] \pm \sqrt{(1 + f_0^2)\tilde{h}[\varepsilon]^2 - 2f_0}}{1 + e^{2\varepsilon}}.
\] (4.32)

Of course, one can reverse the dependence in the relation (4.32) and write the function \(\tilde{h}[\varepsilon]\) in terms of the arbitrary function \(f[\varepsilon]\), then we shall come to the formula

\[
h_{ij} = \frac{x_j^a \cosh \varepsilon_i f[\varepsilon_i](1 \pm i \sqrt{e^{-2\varepsilon_j} + f_0(\cosh \varepsilon_j f[\varepsilon_j])^2}) - \cosh \varepsilon_j f[\varepsilon_j](1 \pm i \sqrt{e^{-2\varepsilon_i} + f_0(\cosh \varepsilon_i f[\varepsilon_i])^2})}{2f[\varepsilon_j] \cosh \varepsilon_i \cosh \varepsilon_j}.
\] (4.33)

When in (4.31) the function \(\tilde{h}[\varepsilon]\) = 0, then the condition (4.32) is not required, the function \(f[\varepsilon]\) is arbitrary, and we come to the solution (4.18, 4.19, 4.20).

Now let us consider the case \(\tilde{h}_{ij} \neq 0\) in (4.30). When \(g_0 = 0\), then we have the solutions with arbitrary functions \(f[\varepsilon]\) and constant \(f_0\):

\[
h_{ij} = \frac{x_j^a i(\cosh \varepsilon_i f[\varepsilon_i] - \cosh \varepsilon_j f[\varepsilon_j]) \pm \sqrt{f_0 + e^{2\varepsilon_i}(\cosh \varepsilon_i f[\varepsilon_i])^2} \mp \sqrt{f_0 + e^{2\varepsilon_j}(\cosh \varepsilon_j f[\varepsilon_j])^2}}{2f[\varepsilon_j] \cosh \varepsilon_i \cosh \varepsilon_j}.
\] (4.34)
In the case \(g_0 \neq 0\) the general solutions are of the following form with arbitrary \(f[\varepsilon]\) and \(g_0\)

\[
\bar{h}_{ij} = \frac{x_i^a (1 - e^{2(\varepsilon_i + \varepsilon_j)}) f[\varepsilon_i] \left(\bar{f}[\varepsilon_i] e^{\varepsilon_i} \left(\bar{f}[\varepsilon_i] \bar{h}[\varepsilon_i] - \bar{f}[\varepsilon_i] \bar{h}[\varepsilon_i]\right) + g_0 \left(\bar{f}[\varepsilon_i] \bar{h}[\varepsilon_i] - \bar{f}[\varepsilon_i] \bar{h}[\varepsilon_i]\right)\right)}{g_0 (1 + \bar{f}[\varepsilon_i] \bar{f}[\varepsilon_i] \bar{h}[\varepsilon_i] \bar{h}[\varepsilon_i]) + e^{\varepsilon_i} \bar{f}[\varepsilon_i] \left(\bar{f}[\varepsilon_i] \bar{h}[\varepsilon_i] + g_0^2 \bar{h}[\varepsilon_i] \bar{h}[\varepsilon_i]\right)}
\]

(4.35)

where \(\bar{f}[\varepsilon] = [1 + e^{2\varepsilon}] f[\varepsilon]\) and (below \(h_0\) is an arbitrary number)

\[
\bar{h}[\varepsilon] = \frac{(\bar{f}[\varepsilon])^2 - 1}{\bar{f}[\varepsilon] h_0 \pm \sqrt{(\bar{f}[\varepsilon] h_0)^2 + (\bar{f}[\varepsilon])^2 - g_0^2 ((\bar{f}[\varepsilon])^2 - 1)}}.
\]

(4.36)

The functions \(h_{ij}, g_{ij}\) can be obtained then using the relations (4.29) and the second equation in (4.30). Let us remind once again that the arbitrary function \(f[\varepsilon]\) can have also an extra argument \(u\), and in this case taking the homogeneous limit \(\varepsilon_i = \varepsilon_j, x_i^a = x_j^a\) (two identical irreps), but keeping \(u_i \neq u_j\), we shall have two-parametric solution \(R(u_i, u_j)\) to YBE. The solutions (4.21-4.22), (4.24-4.26) obtained previously in this subsection and containing arbitrary functions \(h[\varepsilon_i, x_i^a, \{u_i\}]\) correspond to the exceptional cases of (4.35, 4.36), with the property \(\bar{f}[\varepsilon] = 1, g_0\) \((\bar{f}[\varepsilon] = constant)\).

If we impose additional requirements \(g_{ij} = 0, h_{ij} = 0\), it will fix the function \(f_{ij}\), as in the case (4.15). For completeness, let us present all the particular cases. When \(h_{ij} = 0\) and/or \(g_{ij} = 0\), then under the conditions \(h_{ij} \neq 0, g_{ij} \neq 0\) (the second relation in (4.30)) we shall come to the solution (4.15). For the mentioned conditions, there are another particular solutions also: when \(h_{ij} = 0\) they are \(f[\varepsilon] = \frac{1}{\cosh[\varepsilon]}\) and \(g_{ij} = i \frac{\sinh[\varepsilon_i - \varepsilon_j]}{\cosh[\varepsilon_i]}, \) when \(g_{ij} = 0\), the solutions are \(f[\varepsilon] = \frac{e^{-2\varepsilon}}{\cosh[\varepsilon]}\) and \(h_{ij} = -i \frac{e^{\varepsilon_i - \varepsilon_j} \sinh[\varepsilon_i - \varepsilon_j]}{\cosh[\varepsilon_i]}\). The condition \(h_{ij} = 0\) brings to the specific solutions

\[
h_{ij} = 0, \quad f[\varepsilon] = \frac{1 \pm e^{-\varepsilon} \sqrt{f_0 e^{\varepsilon} \cosh[\varepsilon] - 1}}{\cosh[\varepsilon]} - e^{\varepsilon_i}, \quad g_{ij} = \frac{e^{\varepsilon_i} f[\varepsilon_i]}{f[\varepsilon_j]} - e^{\varepsilon_j}
\]

and

\[
g_{ij} = 0, \quad f[\varepsilon] = \frac{e^{-2\varepsilon} (1 \pm \sqrt{1 + f_0 e^{\varepsilon} \cosh[\varepsilon]})}{\cosh[\varepsilon]}, \quad h_{ij} = \frac{e^{-\varepsilon_i} f[\varepsilon_i]}{f[\varepsilon_j]} - e^{-\varepsilon_j}
\]

The conditions \(h_{ij} \neq 0, g_{ij} = 0\) (see the second relation in (4.30)) imply

\[
h_{ij} = 0, \quad f[\varepsilon] = \frac{e^{-\varepsilon} (1 \pm \sqrt{1 + f_0 e^{\varepsilon} \cosh[\varepsilon]})}{e^{2\varepsilon} \cosh[\varepsilon] - 1}, \quad g_{ij} = \frac{e^{-\varepsilon_i} f[\varepsilon_i] \cosh[\varepsilon_i]}{f[\varepsilon_j] \cosh[\varepsilon_j]} - e^{-\varepsilon_j} \cosh[\varepsilon_j]
\]

and

\[
g_{ij} = 0, \quad f[\varepsilon] = \frac{e^{\varepsilon} \pm \sqrt{f_0 e^{\varepsilon} \cosh[\varepsilon] - 1}}{e^{2\varepsilon} \cosh[\varepsilon] - 1}, \quad h_{ij} = \frac{e^{\varepsilon_i} f[\varepsilon_i] \cosh[\varepsilon_i]}{f[\varepsilon_j] \cosh[\varepsilon_j]} - e^{\varepsilon_j} \cosh[\varepsilon_j]
\]

As it was stated the variables \(x^a\) can be eliminated from the YBE by the appropriate normalization of the functions \(f_{ij}, g_{ij}\) and \(h_{ij}\): \(f_{ij} \rightarrow \frac{x_i^a}{x_j^a} f_{ij}, g_{ij} \rightarrow \frac{\bar{x}_i^a}{\bar{x}_j^a} g_{ij}\) and \(h_{ij} \rightarrow \frac{\bar{x}_i^a}{\bar{x}_j^a} h_{ij}\). For expelling the variables \(x^a\) from the \(\hat{R}\)-matrix (4.13), there is need also an additional vector space renormalization. Hereafter in the formulas we omit the variables \(x^a\).
Next group of the solutions is equipped with the primary conditions
\[ f_{ii} = 1, \quad g_{ii} = h_{ii} = \pm 1, \]  
\[ f_{ii} = 1, \quad g_{ii} = h_{ii} = \pm \tanh[\varepsilon_i]. \]  
(4.37)

At the first we investigate the case \( g_{ii} = h_{ii} = \pm \tanh[\varepsilon_i] \). Here we have two solutions
\[ f_{ij} = \frac{1 + e^{2\varepsilon_j}}{1 + e^{2\varepsilon_i}}, \quad g_{ij} = \frac{\pm(1 - e^{\varepsilon_i + \varepsilon_j}) + i(e^{\varepsilon_i} - e^{\varepsilon_j})}{1 + e^{2\varepsilon_i}}, \quad h_{ij} = \frac{\pm(1 - e^{\varepsilon_i + \varepsilon_j}) - i(e^{\varepsilon_i} - e^{\varepsilon_j})}{1 + e^{2\varepsilon_i}}. \]  
(4.38)

Then for the case \( g_{ii} = -h_{ii} = 1 \) we obtain
\[ f_{ij} = \frac{1 + e^{2\varepsilon_j}}{1 + e^{2\varepsilon_i}}, \quad g_{ij} = i - e^{\varepsilon_j}, \quad h_{ij} = i - e^{\varepsilon_i}. \]  
(4.39)

When \( g_{ii} = -h_{ii} = -1 \), then we have
\[ f_{ij} = \frac{1 + e^{2\varepsilon_j}}{1 + e^{2\varepsilon_i}}, \quad g_{ij} = i + e^{\varepsilon_j}, \quad h_{ij} = i + e^{\varepsilon_i}. \]  
(4.40)

As we can see, the previous group of the solutions exclude the normalization condition, i.e. \( \tilde{R}_{ii}(\varepsilon, \varepsilon, x^a, x^a, 0) \neq \mathbb{I} \). Another group of such solutions is
\[ f_{ij} = 0, \quad g_{ij} = \frac{-ie^{\varepsilon_j - \varepsilon_i}}{2\cosh\varepsilon_i}, \quad h_{ij} = \frac{-i}{2\cosh\varepsilon_i}, \]  
(4.41)

The remaining case can be presented by the following matrix \( \tilde{R}_{ij} = f_{ij}\tilde{P}_{-} + g_{ij}\tilde{P}_{+} + h_{ij}\tilde{P}_{++} \).

Here the existing solutions to YBE are (we can set \( f_{ij} = 1 \), as there is a normalization freedom)
\[ g_{ij} = \frac{ie^{\varepsilon_j - \varepsilon_i}}{2\cosh\varepsilon_j}, \quad h_{ij} = \frac{i f_{ij}}{2\cosh\varepsilon_j}, \]  
\[ g_{ij} = \frac{-e^{\varepsilon_i} f_{ij}}{i \pm e^{\varepsilon_j}}, \quad h_{ij} = \frac{i f_{ij}}{\pm i + e^{\varepsilon_j}}, \]  
\[ g_{ij} = \frac{(ie^{\varepsilon_i} \pm 1)f_{ij}}{1 + e^{2\varepsilon_j}}, \quad h_{ij} = \frac{(i \mp e^{\varepsilon_i})f_{ij}}{1 + e^{2\varepsilon_j}}. \]  
(4.46)

So, we exhausted all the possible solutions with the condition \( c_i = c_j = 0 \).

As we see the solutions with the normalization condition \( \tilde{R}_{ii} = \mathbb{I} \) contain arbitrary constants and arbitrary functions, forming so families of the solutions.
4.4 $\cosh \varepsilon = 0$

The solutions of the equation $c_i \cosh \varepsilon_j = \pm c_j \cosh \varepsilon_i$ include also the case with $\cosh \varepsilon_{i,j} = 0$, which we did not consider above as it corresponds to the value $z_{i,j} = -1$. But here there is an interesting property of the algebra. Two dimensional representation of the algebra now has two linearly independent generators, as here $f_i = (c_i/x_i)e_i$, $f_j = (c_j/x_j)e_j$ and $k_{i,j} = \text{diag}\{-1,1\}$. But the co-product defined above give good defined four dimensional representations for all three generators. The fusion here corresponds to the case $V_2 \otimes V_2 = V_2 \oplus V_2$, where in the summand there are two dimensional cyclic irreps with the values $e^2 = x = x_i + x_j$, $f^2 = y = y_i + y_j$, $k^2 = 1$, the quadratic Casimir $c$ has two different values on the irreps, differing by the signs, $\pm c_{i,j}$, $c_{i,j} = \sqrt{(x_i + x_j)(c_i^2x_j + c_j^2x_i)/(x_ix_j)}$. The projection operators $P^{-}_{ij}$, $P^{+}_{ij}$ can now be constructed as well:

$$
\tilde P^{\pm}_{ij} = \begin{pmatrix}
\frac{c_i+c_j \pm c_{i,j}}{\mp 2c_{i,j}} & 0 & 0 & \frac{c_{i,j} - c_i \mp c_j}{\pm 2c_{i,j}} \\
0 & \frac{c_{i,j}c_i + c_{i,j}x_i}{c_{i,j} + 2c_{i,j}x_i} & \frac{c_i - c_{i,j}}{\mp 2c_{i,j}} & 0 \\
0 & c_{i,j}c_j + c_{i,j}x_j & \frac{c_{i,j} - c_i}{\pm 2c_{i,j}} & 0 \\
\frac{c_{i,j}c_i - c_{i,j}x_i}{\mp 2c_{i,j}x_i} & 0 & 0 & \frac{c_i + c_j \mp c_{i,j}}{\pm 2c_{i,j}}
\end{pmatrix},
$$

(4.48)

The simplest matrix $\tilde R_{ij} = \tilde P^{+}_{ij} + f_{ij}\tilde P^{-}_{ij}$ satisfying to the YBE is a constant $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and corresponds to the value $f_{ij} = 1$. The next solution has the value $f_{ij} = -1$ and the corresponding matrix, after multiplication by $-c_{i,j}/(2\sqrt{c_{i,j}})$ and redefining the parameters, $c_{i,j} = e^{2u_{i,j}}$, $x_{i,j} = e^{2w_{i,j}}$, is the following (the notations $u_{ij} = u_i - u_j$ and $w_{ij} = w_i - w_j$ are used)

$$
\tilde R(u_{ij}, w_{ij}) = \begin{pmatrix}
\cosh[u_{ij}] & 0 & 0 & e^{w_{1}+w_{2}} \sinh[w_{ij} - u_{ij}] \\
0 & e^{w_{ij}} \cosh[u_{ij} - w_{ij}] & \sinh[u_{ij}] & 0 \\
0 & \sinh[-u_{ij}] & e^{-w_{ij}} \cosh[u_{ij} - w_{ij}] & 0 \\
e^{-w_{1}-w_{2}} \sinh[u_{ij} - w_{ij}] & 0 & 0 & \cosh[u_{ij}]
\end{pmatrix},
$$

(4.49)

This is simply the two-parametric solution [20].

Note, that for the nilpotent or semi-cyclic irreps this case equivalent to $c_i = c_j = 0$. 

[20]
5 Summary

In this article we have obtained the all spectra of the solutions to the $sl_q(2)$-invariant YBE at $q^4 = 1$ defined on the cyclic irreps. We would like to discuss here some peculiarities of the obtained solutions and the corresponding integrable models.

We want at first to pay the attention into the following interesting point regarding to the appearance of the arbitrary functions in the solutions of the investigated YBE with inhomogeneous behavior. The homogeneous choice of the irrep parameters $\{x_i, z_i, c_i\} = \{x_j, z_j, c_j\}$ then give us ”baxterised” YBE solutions. It is because of the arbitrary functions. Indeed, as we have noted the arbitrary functions can be parameterized besides of the irrep characteristics also by some external parameters: $\{u_i\}, \{u_j\}$. Then taking the homogeneous case in the solutions, we can keep $\{u_i\} \neq \{u_j\}$, and as result we shall have spectral-parameter dependent solutions to the homogeneous spectral-parameter dependent YBE. In this case of the homogeneous limit the arbitrariness of the functions has not meaning that there is a family of the solutions, as now the function $f(u_i)$ is just a transformation (reparametrization) of the spectral parameter $u_i$. We can separate the obtained solutions as really ”baxterised” ones, which include arbitrary functions, and ”just” inhomogeneous solutions, where all the functions are fixed (the arbitrariness in this case can be presented only by arising of some constants), and their homogeneous limits are the solutions to the constant YBE.

As it was observed before the existed $4 \times 4$ solutions to $sl_q(2)$-invariant YBE at $q = \pm i$ all have the ”free-fermionic” property [13, 5, 8, 10, 24]. We can show here that this is valid for all the solutions on two dimensional cyclic irreps and moreover: all the matrices in the form \( \tilde{R}_{ij}(u) = \sum f_a(u) \tilde{P}_{ij}^a \), with the obtained projection matrices (for all three cases discussed in the Section 4), independent from the functions $f_a(u)$ (with arbitrary $f_a(u)$), possess the following relation on the matrix elements:

\[
\tilde{R}^0_{00}(u)\tilde{R}^{11}_{11}(u) + \tilde{R}^0_{01}(u)\tilde{R}^{10}_{10}(u) = \tilde{R}^0_{01}(u)\tilde{R}^{10}_{10}(u) + \tilde{R}^0_{00}(u)\tilde{R}^{11}_{11}(u).
\]

(5.1)

The chain models corresponding to the obtained solutions all have the form of the $XY$ models in the transverse field. Let us present a general expression for the corresponding quantum Hamiltonian operators. The transfer matrix approach in the theory of the integrable models implies, that the first logarithmic derivative (at the normalization point) of the transfer matrix defined on a one-dimensional chain as $\tau(u) = tr_j \prod_i R_{ij}(u)$, coincides with the Hamiltonian operator of the integrable
quantum spin-chain model, $H = i\hbar \tau(u)/(\tau(u) du)|_{u=0}$. It means that the expansion of the $R$-matrix near the point $u = 0$, where $R(0) = I$, gives interaction terms $h_{i,i+1}$ in the elementary cell of the nearest-neighbor Hamiltonian operator $H = \sum_i h_{i,i+1}$. At $u = 0$ we have

$$R^{00}_{00}(0) = R^{11}_{11}(0) = R^{01}_{01}(0) = R^{10}_{10}(0) = 1$$

and the remaining elements are vanishing. Expanding near that point the relation (5.1), we can see, that the derivatives at that point satisfy to the following relation

$$R^{00}_{00}(0) + R^{11}_{11}(0) = R^{01}_{01}(0) + R^{10}_{10}(0).$$

The expansion of the $R(u)$-matrix for the $h_{i,i+1}$ (we omit the overall coupling constant) gives the following relation, where we use the Pauli matrices $\sigma^+ = (0 1 0\ 0 0 1)\), $\sigma^- = (0 0 1\ 0 1 0)$, $\sigma^z = (1 0 0\ 0 -1 0)$,

$$h_{i,i+1} = \frac{1}{4} \left( R^{00}_{00}(0) + R^{11}_{11}(0) + R^{01}_{01}(0) + R^{10}_{10}(0) \right) + \frac{1}{2} \left( R^{00\prime}_{00}(0) - R^{11\prime}_{11}(0) + R^{01\prime}_{01}(0) - R^{10\prime}_{10}(0) \right) \sigma^z_i + \frac{1}{2} \left( R^{00\prime\prime}_{00}(0) - R^{11\prime\prime}_{11}(0) + R^{01\prime\prime}_{01}(0) - R^{10\prime\prime}_{10}(0) \right) \sigma^z_{i+1} + \frac{1}{4} \left( R^{00\prime\prime\prime}_{00}(0) - R^{11\prime\prime\prime}_{11}(0) + R^{01\prime\prime\prime}_{01}(0) - R^{10\prime\prime\prime}_{10}(0) \right) \sigma^z_i \sigma^z_{i+1}.$$

We see that the coupling before the interaction term $\sigma^z_i \sigma^z_{i+1}$ vanishes for the matrices with the "free-fermionic" property. In terms of the scalar fermions this summand corresponds to the four fermions’ interaction. The fermionic representation can be performed by the Jordan-Wigner transformation [10], or by a simple method brought in the work [19] for a general $4 \times 4$ $R$-matrix. The chain models with the local terms (5.4) in the Hamiltonian describe some inhomogeneous $XY$ models in a transverse field, and the Hamiltonian operators in the representation of the fermionic creation and annihilation operators have only quadratic nearest-neighbored hopping terms.

As example, the solution (4.4) just describes the $XX$-model in the transverse magnetic field

$$H = J \sum_i \left( \sigma^z_i \sigma^z_{i+1} + \sigma^z_{i+1} \sigma^z_i + \hbox{cosh } u_0 \sigma^z_i \right).$$

Another Hamiltonian operator corresponding to the general solution (4.3), when we normalize the matrix so that $\alpha = 0$, and $f_i/f_j = 1$, and the transfer matrix is expanded near the point $\varepsilon_i = \varepsilon_j = \varepsilon$, is written as

$$H = J \sum_i \left( i(\sigma^+_i \sigma^-_{i+1} - \sigma^+_i \sigma^-_{i}) + e^\varepsilon(\sigma^z_{i+1} - \sigma^z_i) \right).$$

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Note that this relatively simple free-fermion description arises in four-dimensional matrixes case $q = \pm i$. Higher roots of unity, starting from $9 \times 9$-dimensional case, corresponding to $q^3 = 1$, lead to much more rich variety of solutions and contain also higher interaction terms in the corresponding one-dimensional quantum chain Hamiltonian operators. These cases will be considered elsewhere.

The more interesting results we expect to find are connected with the cyclic indecomposable representations, as in the case for the highest/lowest weight indecomposable representations, considered in [17]. There we have found solutions, which correspond to the one chain Hamiltonian operators with the interactions.

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