Synthesis of Adequate Mathematical Descriptions of Dynamic Systems for the Purposes of Forecasting, Control and Diagnostics

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Abstract. The article proposes to use adequate mathematical descriptions of dynamical systems which in some cases allow making a reasonable forecast of the behaviour of real processes, control of motion and diagnostics in deterministically statement. Two criteria for the adequacy are used (quantitative and qualitative types). The possibilities of adequate mathematical descriptions are investigated if both criteria are met. Several algorithms of synthesis of such descriptions are proposed.

1. Introduction
Methods of mathematical modelling received a powerful impetus for development based on the successes of computer technology [1]. It is difficult to name the area of human activity wherever mathematical modelling methods are used. Scientific works on the application of mathematical modelling methods are millions. However, after obtaining the results of mathematical simulation, problems remain with the effective use of these data. As a rule, a number of important questions are not answered. To what extent the results of mathematical modelling correspond to real physical processes that are being investigated? Is there a basis to use the simulation results to predict the behaviour of real process in future and to control it? In what cases is it possible to diagnose the state of physical processes using mathematical simulation methods?

It is obvious that the efficiency of using the results of mathematical modelling of physical processes depends on the correct choice of the mathematical model of the process and the functions of external influence on this model. The indicator of the correctness of this choice is the good agreement of the simulation results with experiment [3]. At the same time, the question of the correct selection of these data is important [2]. In works [3,4,5] the issues of synthesis of mathematical descriptions of physical processes are considered which ensure the coincidence of the results of mathematical simulation with experiment. Such descriptions are named as adequate mathematical descriptions.

1.1. Some Designations and Definitions
First of all, we introduce into consideration some useful designations and definitions. We will call as the mathematical model of the physical process the set of equations or relations between the characteristics of the mathematical model that reflect the physical laws of the process under investigation. External actions (loads) will be called functions that are present in the mathematical model and which independent from motion of system. Initial and various boundary conditions for the process under study will be named as additional conditions. The totality of the mathematical model of
the physical process, external actions and additional conditions will be called as mathematical
description of the physical process [4,5]. The process of solving the equations of a mathematical
model of a physical process with allowance for external actions and additional conditions will be
called mathematical modelling or simulation. Other definitions can be meeting in the scientific
literature.
We will study problem associated with the synthesis of an adequate mathematical description and its
further use on the example of a dynamical system whose motion is described by a system of ordinary
differential equations.

1.2. Mathematical Descriptions Adequacy of Quantitative Type
Let the physical process be characterized in the general case by a certain number of variables (state
variables) \(\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)\) depending on an infinite number of initial parameters of the process.
The choice of the characteristics of the physical process is determined by the ultimate goals of the
research.
For simplicity of presentation, we will assume that the dynamical system is linear and described by a
system of ordinary differential equations with constant coefficients.

\[
\dot{x} = Ax + Bz, \tag{1}
\]

where \(x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in X\) is a vector function of state variables; \(z(t) =
(z_1(t), z_2(t), \ldots, z_l(t))^T \in Z\) is a vector function of external loads; \(A, B\) are matrices with constant
coefficients of the corresponding dimensions; \(Z, X\) are functional spaces; \((\cdot)^T\) is a sign of
transposition.
Let us denote by \(x^0_1, x^0_2, \ldots, x^0_k\) the initial conditions of the state variables \(x(t) =
(x_1(t), x_2(t), \ldots, x_n(t))^T\) at the initial moment of time \(t_0\).
The possibilities of effective practical use of the results of mathematical simulation are based on the
use of adequate mathematical descriptions in the simulation [3].

**Definition.** An adequate mathematical description of a physical process of quantitative type, with
respect to all variables \(x_1(t), x_2(t), \ldots, x_n(t)\) (ALMD), will be considered the mathematical
description for which the results of the mathematical simulation of the variables \(x_1(t), x_2(t), \ldots, x_n(t)\)
conform with the results of experimental measurements \(\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)\) of the characteristic \(x_1(t), x_2(t), \ldots, x_n(t)\), with the accuracy of the
experiment \(\delta_1, \delta_2, \ldots, \delta_n\) in the selected functional metric on the given segment \(t \in [a, b]\)

\[
\|x_i(t) - \tilde{x}_i(t)\|_{X[a,b]} \leq \delta_i, 1 \leq i \leq n, \tag{2}
\]

where \(\|\|_{X[a,b]}\) is the norm in functional space \(X\). It is assumed that the error of the mathematical
modelling is zero compared with the experimental data error.
If in inequality (2) \(\delta_i\) is replaced on an arbitrary greater value, \(\delta_i', \delta_i' > \delta_i\) then the criterion of
adequacy loses its objectivity. In addition, false adequate mathematical descriptions may appear.
If in the inequality (2) \(\delta_i\) is replaced by an arbitrary smaller value, \(\delta_i', \delta_i' < \delta_i\) then it is possible to get
an erroneous result. In this case, adequate mathematical descriptions are artificially excluded from
consideration.
Experiment and adequate mathematical description are closely related. Different experiments with
different accuracy will correspond to different adequate mathematical descriptions and in each case
there can be several of them.
Let us introduce into consideration one more criterion of adequacy for dynamical systems, which will
be necessary in the future.

1.3. Mathematical Descriptions Adequacy of Qualitative Type
Let the mathematical model of the physical process (1) depend continuously on the parameter vector \(p\)
(for example, the mass of the elements, the rigidity of the elastic elements, etc.). In this case, there is a
one-to-one correspondence between the components of the parameter vector \(p\) and real physical
elements. This important correspondence will be called the main correspondence (MC). In addition,
we will assume that the connections between the elements of the mathematical model (1) correspond to the known physical laws (Hooke's law, Newton's laws, etc.). The implementation of such a correspondence will be considered the implementation of the criterion of mathematical description adequacy of the qualitative type [3, 4].

Next, we will consider several algorithms for synthesizing mathematical descriptions with the adequacy of the quantitative type.

2. Synthesis of Mathematical Descriptions with Adequacy of Quantitative Type

Two basic approaches to the problem of synthesizing mathematical descriptions of physical processes are known in the literature:

1) mathematical model is given a priori with inexact parameters and then model of external actions is being determined for which the condition of mathematical description adequacy system (1) is valid [5];

2) some model of external actions is given a priori, and then mathematical model is being chosen for which the condition of adequacy for system (1) is valid [6, 7].

We will assume that all components of the vector function of the external load
\[ \mathbf{z}(t) = (z_1(t), z_2(t), \ldots, z_m(t))^T \in Z \]
are unknown and some of the state variables
\[ \mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \]
are obtained experimentally.

The solution to the problem of synthesizing an adequate mathematical description (AMD) within the framework of the first approach is described in [4, 5]. This solution is based on reducing the original linear dynamical system to a number of linear subsystems of system (1) with one of the unknown external loads \( z_j(t) \) and one known state variable \( x_i(t) \) by the method of "sections". Such transformation is not always possible.

Further, we obtain m Volterra equations of the first kind (ill-posed problem) for all unknown functions of the external loads \( z_j(t) \), \( 1 \leq j \leq m \) from obtained subsystems by using the method of Green's functions for linear equations [3, 5]:
\[
\int_{t_0}^t N_k(t - \tau)z_k(\tau)d\tau = P_k(t),
\]
where \( N_k(t - \tau), P_k(t) \) are known functions.

Rewrite the equation (3) in the form (the indices were removed) [4, 6]:
\[
A_pz = u_\delta,
\]
where \( A_p \) is a completely continuous operator (\( p \) is vector parameters of mathematical model), \( A_p: Z \to U, z \in Z, u_\delta \in U, u_\delta \) is initial experimental data (graphic), \( z \) is unknown function, \( Z, U \) are functional spaces.

Further, we shall suppose that the element \( u_\delta \) in the equation (4) is given with a known error:
\[
\|u_\delta - u_{\text{ex}}\|_U \leq \delta,
\]
where \( \delta \) is constant, \( \delta > 0 \), where \( u_{\text{ex}} \) is exact initial data.

Let's denote by \( Q_{\delta,p} \) the set of possible solutions of equation (4):
\[
Q_{\delta,p} = \{ z : \|A_pz - u_\delta\|_U \leq \delta \}.
\]
The set \( Q_{\delta,p} \) is unbounded at any \( \delta \) [8]. Thus, the initial mathematical model of physical process (1) and any function from the set \( Q_{\delta,p} \) provide an adequacy of results of mathematical simulation [3, 8].

Next, the approach proposed by Phillips and Tikhonov is used [8]. Consider now the following extreme problem:
\[
\Omega[z_{\delta,p}] = \inf_{z \in Q_{\delta,p}} \Omega[z],
\]
where functional \( \Omega[z] \) has been defined on set \( Z \).

Under certain conditions the solution to the extreme problem (5) exists, is unique and belongs to set of possible solutions \( Q_{\delta,p} \) [3].

We obtain all components of the vector function \( z(t) \) after solving equations of type (4). This vector function provides an adequate linear mathematical description of the physical process (ALMD) together with the dynamic system (1).
The examples of adequate mathematical descriptions are given in [3, 5].

3. Possible Algorithm of Forecast Estimation with Guaranty
A possible algorithm for predicting the behavior of only one variable \( x_k(t) \), \( 1 \leq k \leq n \) of a dynamic system (1) in a deterministic setting is considered for simplicity. Let us admission, something each vector component system (1) in a deterministic setting is considered for simplicity. The simple algorithm for the synthesis of an adequate mathematical description for the purposes of adequacy of the qualitative type (criterion MS).

The adequate mathematical description with a vector of parameters of dynamical system (1) be carried out. It will also assume that system (1) satisfies the criterion of the possible or desired spread of the parameters of the mathematical model of the dynamic process. The union of physical parameters in some integral parameters of a mathematical model is inadmissible, since this violates the MC.

The maximum deviation \( \hat{x}_{k,p_i}(t) \) of the state variable \( x_k(t) \) from zero for any possible vector \( p \in D \) is obtained with guarantee. The union of physical parameters in some integral parameters of a mathematical model is inadmissible, since this violates the MC.

Another algorithm for synthesizing an adequate mathematical description for the purpose of motion forecasting of the variable \( x_k(t) \) of the dynamic system (1) has the form:

- Two sets of possible parameters \( \{\hat{p}_i\} = D_m \in D \) (\( i = \overline{1,m} \)) and \( \{\hat{p}_j\} = D_{n_1} \in D \) (\( i = \overline{1,m_1} \)) are selected with a certain steps in the area of possible changes parameters of the mathematical model \( D \); external loads will not change in the future.
- For each parameter of the mathematical model \( p_i \) and a given measurement of the variable state \( \hat{x}_k(t) \), a vector functions of external actions \( z_{p_i} = (z_1_{p_i}, z_2_{p_i}, \ldots, z_{l_{p_i}})^T \) are determined which together with the mathematical model (1) provides an adequate mathematical description of the physical process.
- The state variables \( \hat{x}_{p_j,k}^{\hat{p}_i}(t), (i = \overline{1,m}) \) are determined by mathematical simulation of system (1) with parameters \( \hat{p}_j \in D, (j = \overline{1,m_1}) \) and are desired characteristics \( \hat{x}_{p_j,k}^{\hat{p}_i} \).

Then we define of mathematical model (1) vector parameters \( \hat{p}_1, (\hat{p}_1 \in D_{m_1} \in D) \) and the vector functions of external loads \( z_{p_i} (\hat{p}_1 \in D_m \in D) \) that satisfies the condition

\[
\hat{x}_{p_1,k}^{\hat{p}_1} = \max_j \max_i \hat{x}_{p_j,k}^{\hat{p}_i}. \tag{6}
\]

The adequate mathematical description with a vector of parameters \( \hat{p}_1 \) and function of external load...
makes possible to obtain the maximum deviation $\tilde{x}_{p_k^l}$ of the variable state $x_k(t)$ for any possible vector $p \in D$ with guarantee. Physical parameters of external action are assumed to be unchanged.

It is possible to forecast the behaviour of a dynamic system to estimate the minimum of the maximum deviation of a state variable $x_k(t)$. To this end it is necessary to change the condition (6):

$$\tilde{x}_{p_0^l} = \min_j \min_i \tilde{x}_{p_j^k}.$$  (7)

To predict other characteristics of the physical process, it is necessary to change the definition of these characteristics and change the conditions of the type (6), (7).

If the parameters of the dynamic system do not change in the future, but significant changes in the external loads are possible, then forecasting algorithms are possible according to the following scheme:

- Several problems of the synthesis of ALMD are solved within the framework of the first approach for various experiments with parameter $p_0 \in D$ and we obtain models of external loads $z_{1,p_0}^1, z_{2,p_0}^2, \ldots, z_{d,p_0}^d$.
- By analyzing the results, general patterns in the behavior of these models can be found and a "general" model of external influence $z_{\alpha,p_0}^\alpha$ formed.
- We can predict the motion of variable $x_k(t)$ of system (1) with the model of external load $z_{\alpha,p_0}^\alpha$ and parameter $p_0 \in D$.

An example of such an algorithm is the problem of assessing the coefficient of dynamism of mechanical vibrations in the main mechanical line of a rolling mill [5].

4. On the Possibilities of Using Adequate Mathematical Descriptions for the Purposes of Control and Diagnostics

The problem of controlling the motion of dynamical systems belongs to one of the problems of systems theory that have important practical applications. The most significant works in this area have already a sufficient history [9]. However, most of these works are devoted to the problem of moving a physical object from one point in space to another. The problem of moving a physical object along a given trajectory remains poorly understood. In addition, the algorithms for constructing control for the problem of motion along a given trajectory, the problem of realizability of the constructed algorithms for practical problems and the issues of the adequacy of the control problem for real physical problems remain without attention [10].

The possibility of using adequate mathematical models for the purposes of diagnostics of dynamic systems follows from the properties of such mathematical models. Let's consider some physical cyclic process. Let us for one cycle of functioning of a dynamic system of type (1) in a normal situation, the state variables $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in X, t \in [0,T]$ are obtained experimentally and an adequate mathematical description (AMD) is determined. Let such a model is $A_{\alpha,p_0}z_{p_0^l}, p_0 \in D$.

The nature of the change in the components of the external load function $z_0(t) = (z_1(t), z_2(t), \ldots, z_l(t))^T \in Z$ will correspond to the normal functioning of the physical process.

If the normal functioning of the dynamic system (1) is disrupted, the state variables $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ of the dynamic system will inevitably change and, accordingly, the components of the vector of the external load $z_0(t) = (z_1(t), z_2(t), \ldots, z_l(t))^T$ will change. In addition, the system usually becomes nonlinear. The change $z_0(t)$ can be used to judge the nature of violations in the functioning. Periodic control indirect measurements of the vector $z_0(t)$ provide information about the real current state of the physical process.

It should be noted that the use of AMD for such purposes requires special approaches when constructing a system (1). In addition, in this case, additional theoretical difficulties arise, taking into account the inaccuracy of constructing a dynamic system (1). In other words, it is necessary in this case to take into account the operator error in equation (4).
The articles devoted to this approach to diagnostic problems are still unknown to the author. However, some preliminary theoretical substantiation was made in the works [5, 11].

5. Conclusion

The paper proposes an approach for improving the efficiency of using the methods of mathematical simulation for dynamic systems for the purposes of forecasting, control and diagnostics. It is based on the use of adequate mathematical descriptions. Two criteria for the adequacy of mathematical descriptions to real physical processes are used. Some algorithms for the synthesis of adequate models for the purposes of forecasting control and diagnostics are proposed. The analysis of the possibilities of such descriptions is carried out. If the qualitative criterion of adequacy is not met, then the results of forecasting the motion of the dynamic system are satisfactory only in a small neighbourhood of the data for which an adequate mathematical description is constructed. In this case, the output is possible using the appropriate neural network. A significant problem remains when solving control and diagnostics problems with an inaccurately specified operator in equation (4). There are still open questions about when this error should be taken into account and when not.

References

[1] Samarskii A and Mikhailov A 2018 *Principles of Mathematical Modeling Ideas, Methods, Examples* (CRC Press Taylor & Francis Group Boca Raton London New York) p 376
[2] Shang Yilun 2019 Subgraph Robustness of Complex Networks Under Attacks IEEE Transactions on Systems, Man, and Cybernetics: Systems. Vol.49, is. 4, pp.821-832
[3] Menshikov Yu 2020 *Synthesis of Adequate Mathematical Descriptions of Physical Processes, Monograph* (Cambridge Scholars Publishing) p 173
[4] Menshikov Yu 2013 *Synthesis of Adequate Mathematical Description as Solution of Special Inverse Problems European Journal of Mathematical Sciences vol. 2, n.3* pp 256-271
[5] Menshikov Yu and Poljakov N 2009 *Identification of Models of External Actions* (Dnepropetrovsk, Ukraine Publishing House Science and Education) p 188
[6] Gubarev V 2008 Method of iterative identification of many-dimensional systems with inexact data. Part 1. Theoretical basics J. Problems of Control and Information 2 (Kiev Ukraine) pp.8-26
[7] Stepashko V 2008 Method of critical dispersions as analytical apparatus of theory of inductive modeling Problems of Control and Information, 2 (Kiev Ukraine) pp.8-26.
[8] Tikhonov A and Arsenin V 1979 *Methods of incorrect problems solution* (Moscow, Science) p 278
[9] Krasovsky N 1968 *Theory of motion control* (Science Moscow) p 478
[10] Menshikov Yu 2019 Controlling the motion of a dynamic system along a given trajectory using stable adequate mathematical descriptions Abstracts of the Scientific-Practical Conference on Modern Computational and Experimental Methods for Determining the Characteristics of Rocket and Space Technology (December 10-12, Dnipro Ukraine) p 16
[11] Menshikov Yu 2013 Identification of Rotor Unbalance as Inverse Problem of Measurement. *Journal Advances of Pure Mathematics*, vol.3, no.9A, pp 20-25