Kurtoses and high order cumulants: Insights from resummed perturbation theory

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Abstract. Cumulants of conserved charges provide important information about the physics of the quark-gluon plasma around the phase transition region, as they are by construction sensitive to changes in the degrees of freedom of the system. In this brief proceedings contribution, I report on recent results for two such quantities from two different improved perturbative frameworks, as well as discuss their relevance for heavy ion experiments.

1. Introduction
The weak coupling expansion of the QCD pressure shows few signs of convergence, and only at extremely high temperatures (see e.g. [1]). It is therefore crucial to try to improve the situation by attempting to continue the perturbative results as close to the phase transition region as possible. This is highlighted by the fact that the quark-gluon plasma (QGP) is argued [2] to reach a temperature of only a few times the pseudo-critical temperature of the deconfinement transition, $T_c = 154 \pm 9\text{MeV}$ [3], where the plasma is clearly not extremely weakly coupled.

When the quark chemical potentials $\mu_f$ are turned on to study the phase diagram of the theory (see [4, 5] for present and [6, 7] for future collider experiments), lattice Monte Carlo simulations become inapplicable due to the so-called sign problem of QCD [8]. There have been many attempts to find a solution to this problem, but so far the most successful approach amounts to merely Taylor expanding the pressure in powers of $\mu_f/T$, giving access to only moderate densities. The corresponding coefficients are related to cumulants of conserved quantum numbers in the system, and give information about their correlations and fluctuations, being practical probes of changes occurring in the degrees of freedom of the system across the transition region.

The limitation of lattice methods on the $T$-$\mu$ plane is one of the main reasons to attack the problem of fluctuations and correlations of conserved charges — and ultimately the equation of state — from a resummed perturbation theory point of view, where there is no sign problem. These quantities can indeed be extended to very large values of the ratio $\mu_f/T$, hence giving information about how the QGP behaves, as it becomes more and more dense.

2. Statistical QCD and cumulants
Given the Hamiltonian $H_{QCD}$ of Quantum Chromodynamics, the corresponding partition function $Z_{QCD}(T, \mu_f; V)$ is written in the form $Z_{QCD} \equiv \text{Tr} \exp[-(H_{QCD} - \sum_f \mu_f Q_f)/T]$. Here, $Q_f$ and $\mu_f$ denote respectively conserved charges and the corresponding chemical potentials.

In the present work, we will mainly consider the up, down and strange quark conserved charges (with their chemical potentials $\mu_u$, $\mu_d$, $\mu_s$), but one can equivalently express the partition function in term of the baryon, electric charge and strangeness conserved numbers (with their chemical potentials $\mu_B$, $\mu_Q$, $\mu_S$). In addition, $Z_{QCD}$ gives access to a host of...
equilibrium thermodynamic quantities such as the pressure and entropy, following relations such as
\[ p_{\text{QCD}} = \frac{\partial(T \log Z_{\text{QCD}})}{\partial V \rightarrow \infty} T/V \log Z_{\text{QCD}} \]

or \[ S_{\text{QCD}} = \frac{\partial(T \log Z_{\text{QCD}})}{\partial T}. \]

From the partition function, it is easy to see that the mean and (co)variance of two conserved charges, i.e. \( \langle Q_f \rangle \) and \( \langle (Q_f - \langle Q_f \rangle) \cdot (Q_g - \langle Q_g \rangle) \rangle \), can be expressed in terms of derivatives with respect to the corresponding chemical potentials. Thus, we can write
\[ \langle Q_f \rangle = T \frac{\partial}{\partial \mu_f} \log Z_{\text{QCD}}, \quad \langle (Q_f - \langle Q_f \rangle) \cdot (Q_g - \langle Q_g \rangle) \rangle = T^2 \frac{\partial^2}{\partial \mu_f \partial \mu_g} \log Z_{\text{QCD}}, \]

defining quantities called the first and second order cumulants of the related conserved charges.

Based on the above, it is clear that when considering the covariance of two different charges, such as \( Q_a \) and \( Q_b \), one is probing the correlation between the two related flavors, while when considering the variance of the same charge, one is probing its fluctuations. The former quantity gives information about possible bound-state survival [9], while the latter is related to how the system reacts to small increases of density. It also follows that equation (1) relates the cumulants to the equation of state of the system via derivatives with respect to chemical potentials. These quantities, typically referred to as susceptibilities, are denoted by
\[ \chi_{u_i, d_j, s_k ... \, (T, \{ \mu_f \})} \equiv \frac{\partial^n p_{\text{QCD}}(T, \{ \mu_f \})}{\partial \mu_{u_i} \partial \mu_{d_j} \partial \mu_{s_k} ...} \quad \text{with} \quad n = i + j + k + ... , \]

where instead of quark numbers one may also consider any other conserved charge.

The existence and location of a possible critical point on the phase diagram of QCD is often investigated through the behavior of cumulants. Indeed, the variance of the baryon number \( \chi_{B_2}(T, \mu_B) \) is expected to display a critical-like behavior with a sharp peak at the critical point, as it is highly sensitive to changes of density in the medium. This quantity can be estimated via a Taylor expansion containing higher order cumulants at vanishing chemical potential (see e.g. [10] for discussion and results in the two flavor case). For some recent reviews on heavy ion collisions and in particular the use of cumulants in statistical QCD, the reader is referred to [2].

3. Resummed perturbative QCD

3.1. Dimensional Reduction inspired resummation

It is well known [11] that dimensional reduction allows the dynamics of the energy scales of order \( \lambda T \) and smaller to be properly accounted for through an effective field theory, which for QCD is called Electrostatic QCD (EQCD) [12, 13]. This effective theory turns out to account for all of the infrared divergences typically encountered in naive weak coupling expansions [14]. Thus, EQCD not only provides a way out of the so-called infrared catastrophe, but in addition allows for weak coupling calculations to be carried out to high orders in perturbation theory upon a careful matching procedure with the full theory.

From the above fact, one can write the QCD pressure \( p_{\text{QCD}} \) as the sum of \( p_{\text{hard}}(g) \) and \( T p_{\text{EQCD}}(m_{E, g}, \lambda_E, \zeta) \), where the function \( p_{\text{hard}} \) gives the contribution of the hard scale (\( \sim T \)) and is available through a strict loop expansion in QCD, while \( p_{\text{EQCD}} \) gives the contribution of the soft (\( \sim gT \)) and the ultrasoft (\( \sim g^2 T \)) scales, being accessible from the EQCD partition function. The EQCD parameters \( m_{E}(g), g_E(g), \lambda_E(g) \) and \( \zeta(g) \) are functions of the temperature and quark chemical potentials, and admit expansions in powers of the QCD gauge coupling \( g \).

Typically, when computing a physical quantity, one expands the EQCD parameters in powers of \( g \). However, as first suggested in [1] and later successfully applied to the case of the pressure at zero chemical potential in [15], one can simply consider both \( p_{\text{hard}} \) and \( p_{\text{EQCD}} \) functions of the EQCD parameters, and not expand them in \( g \). This resums certain higher order contributions to the pressure, containing all correct contributions up to and including order \( g^6 \log g \) [16, 17] when expanded in powers of \( g \), and leads to a considerable decrease in the theoretical uncertainties through a substantial reduction of the renormalization scale dependence. Thus it also improves the convergence properties of the series. We will denote these results as the ‘DR ones’.

More details about the implementation of the resummation at finite chemical potentials can be found in [18], while general details about dimensional reduction and EQCD can be found in the original references [12, 13] as well as in [17, 19, 20] for a generalization to finite density.
3.2. Hard-thermal-loop perturbation theory

Another way of resumming important higher order contributions to thermodynamic quantities is to make use of a variationally improved perturbation theory framework (see [21, 22]), for which the basic idea is to introduce a physically relevant (usually mass) term that is added and subtracted from the Lagrangian density of the original theory. Treating the added piece with the non-interacting part of the action while adding the subtracted one to the interaction terms of it, one ends up interpolating between the original theory and a theory having dressed propagators and vertices. The procedure is of course such that working consistently to a given order in a perturbative expansion, the final results correspond to the correct original theory.

In a gauge field theory such as QCD the situation is somewhat more complicated, as one cannot simply add and subtract a local mass term for the gauge fields without breaking the gauge invariance. Instead, one can make use of the well-known hard-thermal-loop effective action [23, 24], which plays the role of a non-local mass term. This approach was first applied in [25, 26, 27], and is known as Hard-Thermal-Loop perturbation theory (HTLpt).

HTLpt amounts to reorganizing the Lagrangian density of QCD such that \( \mathcal{L}_{\text{HTLpt}} = [\mathcal{L}_{\text{QCD}} + (1 - \delta) \mathcal{L}_{\text{HTL}}]|_{g \to \sqrt{g}} + \Delta \mathcal{L}_{\text{HTL}} \). \( \mathcal{L}_{\text{HTL}} \) is a gauge invariant HTL improvement term given by the HTL effective action, \( \delta \) is a formal expansion parameter set to one after having Taylor expanded the path integral up to some order in it, and \( \Delta \mathcal{L}_{\text{HTL}} \) is a counter term necessary to cancel further ultraviolet divergences introduced by the resulting reorganization of the perturbative series. Essentially, this reorganization amounts to shifting the ground state of the expansion from an ideal gas of massless particles to an ideal gas of massive quasiparticles [28, 29].

More details about the implementation of the HTLpt procedure at finite chemical potentials can be found in [30, 18] as well as in [28, 31]. For related developments, see also [32].

3.3. Technical details

Working at four- and one-loop levels in the DR and HTLpt schemes, respectively, we must assign values to parameters. Following [33], we choose to apply the Fastest Apparent Convergence (FAC) scheme to the NLO expression of \( g_E \). We obtain \( \Lambda_{\text{central}} = 1.445 \times 2 \pi T \) and \( 1.291 \times 2 \pi T \) for the central values with two and three flavors [30, 18], and then vary these values by a factor of two in order to estimate the sensitivity of the results with respect to this choice. Regarding the running of the coupling, we use a two-loop expression in the DR case and one-loop running in the HTLpt result. To fix the value of \( \Lambda_{\text{QCD}} \) we use the recent lattice result \( \alpha_s(1.5 \text{ GeV}) = 0.326 \) from [34], demanding that our one- and two-loop couplings agree with it for \( \Lambda = 1.5 \text{ GeV} \). This respectively gives \( \Lambda_{\text{QCD}} = 176 \text{ MeV} \) and \( 283 \text{ MeV} \) for \( N_f = 3 \), as well as \( \Lambda_{\text{QCD}} = 204 \text{ MeV} \) and \( 324 \text{ MeV} \) for \( N_f = 2 \). These values are then varied by \( \pm 30 \text{ MeV} \).

In the plots appearing in the following section, the bands corresponding to our perturbative results are composed by varying all of the parameters mentioned here in their respective ranges.

4. Results

4.1. Kurtoses

Roughly speaking, the kurtosis is a measure of how strongly peaked a given quantity is, and is defined as the ratio of the corresponding fourth and second order cumulants, the latter one squared. In figure 1, we display this quantity for the (light) quark and baryon numbers, further multiplying it by the second order cumulants for the sake of comparison with lattice data. Such a quantity can be used as a measure of how fast a phase transition is (see e.g. [35]).

In figure 1 (left), we plot our DR and HTLpt results together with lattice data from [36, 37, 38], which at temperatures around \( T = 300 - 400 \text{ MeV} \) seem to agree nicely with the one-loop HTLpt band, but soon start approaching the DR prediction. It should be noted that the latter reproduces the overall trend of the lattice data better. On the right figure, we see similar trends in both predictions, with the difference that the DR one seems to converge to the Stefan Boltzmann limit much faster than in the case of the light quark kurtosis. This trend is seen to be in accordance with the displayed lattice data from [37, 39], and agrees with the expectation that the medium is much less sensitive to the hadronic degrees of freedom in this range of temperatures. Moreover, in figure 1 (right), we display the corresponding three-loop HTLpt
prediction obtained using the cumulants computed in [28]. Although the band is relatively large at low temperature, this result is also seen to reproduce the qualitative trend of the lattice data, furthermore agreeing quite well with the DR one. From the one-loop to the three-loop HTLpt prediction, we notice a reasonably good convergence of the series for this quantity.

4.2. Sixth order cumulants

In figure 2, we next display sixth order cumulants for the light quark number in the cases of three (left) and two (right) quark flavors. Such higher order cumulants are expected to be very sensitive probes of the hadronic freeze-out [40]. For these quantities, the two first orders of perturbation theory vanish, and the weak coupling expansions only start at order $g^3$, rendering the properties of the weak coupling series much poorer than for the lower order cumulants. Indeed, we see from both the left and right figures that our DR predictions are positive for all displayed temperatures, while the HTLpt results are consistently negative. The latter apparently agree better with the (unfortunately not continuum extrapolated) lattice data of [10, 41]. However, it should be recalled our HTLpt predictions are of one-loop order, and based on the very good agreement between our DR results and the recent three-loop HTLpt calculations of [28] (see [18] for a detailed comparison), it is probable that the forthcoming three-loop HTLpt prediction of [42] for the sixth order cumulant will also turn out to be positive.

It is interesting to note that the leading plasmon contribution to the weak coupling expansion of the sixth order cumulant is negative, while it can be seen by truncating our DR results at various orders that the sign of the result is turned positive by the higher order contributions. Noting that lattice studies seem to favor negative values for sixth order cumulants [10, 40, 41] above the transition region, we suspect that including further perturbative orders to the current DR result, it will eventually change in the temperature range considered. This will, however, likely not be verified in the near future, as the technical challenges involved in e.g. a full five-loop determination of the quantity are formidable.

5. Conclusion

We first considered kurtoses regarding the light quark and baryon number operators for three flavor QCD. We saw how resummed perturbation theory provides a very successful description of lattice data down to $2 - 3 \ T_c$, which is clearly an indication that a weakly interacting quasiparticles picture is well suited for the fermionic sector of the theory, even at that range of temperature. Also, information on the relevant degrees of freedom was highlighted by the qualitative difference of fluctuations between light quark and baryon conserved charges. Next, we looked at sixth order cumulants in the cases of two and three flavor QCD, encountering quantities less successfully described within resummed perturbation theory. This could be understood from the fact that the weak coupling expansions only start at a relatively high order.
Figure 2. The sixth order cumulant for the light quark number, evaluated for $N_f = 3$ (left) and $N_f = 2$ (right), with lattice data from [41] (left) and [10] (right). The dashed curves inside the bands correspond to the central values of the renormalization and QCD scales.

Such quantities giving crucial information about the QGP degrees of freedom, I hope these results will be of practical use in the present and future analysis of heavy ion collision data.

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