A Note on Extended Euclid’s Algorithm

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Abstract

Starting with the recursive extended Euclid’s algorithm, we apply a systematic approach using matrix notation to transform it into an iterative algorithm. The partial correctness proof derived from the transformation turns out to be very elegant, and easy to follow. The paper provides a connection between recursive and iterative versions of extended Euclid’s algorithm.

1 Introduction

In teaching computer algorithms, many textbooks favor recursive over iterative versions. While the correctness of recursive algorithms can be relatively easy to establish, students may sometimes find it difficult to connect it to the more efficient iterative versions. An example is the extended Euclid’s algorithm from which we compute multiplicative inverses in modular arithmetic. In the textbooks, either the recursive [3, 4, 5, 6] or iterative [1, 2, 8, 9, 10] version is presented, but not both. Moreover, the recursive and iterative versions of extended Euclid’s algorithm look drastically different. Computing multiplicative inverses efficiently is of great importance as it is an indispensable component in the implementation of RSA cryptosystem. As the numbers involved are of hundred of bits in length, it is without doubt that a recursive algorithm may be unacceptably slow.

In this paper, starting with the recursive extended Euclid’s algorithm, we apply a systematic approach to transform it into an iterative algorithm. Using matrix notation, and making use of the associative property of matrix multiplication, one obtains an iterative algorithm. The partial correctness proof turns out to be very clean with the loop invariant expressed using matrices. The paper thus provides a connection between recursive and iterative versions of extended Euclid’s algorithm.
Recursive versions of extended Euclid’s algorithm are provided in textbooks [3, 4, 5, 6]. The textbook [5] by Epp provides a detailed account of the recursive ideas, supplemented with demonstrations by hand calculations. In [1], an iterative extended Euclid’s algorithm is presented while the recursive version is missing. Using mathematical induction, the correctness proof of the iterative algorithm is established in [1] through insightful observation of the properties of the numbers computed during the execution of the algorithm. Then in [2], matrices are used to present the extended Euclid’s algorithm iteratively. However, the development of the iterative codes requires good insights in mathematics, and does not originate from the recursive algorithm, which the book does not give either. Also, the way the iterative codes are given in [2] is not suitable for the construction of a partial correctness proof. Knuth establishes in [8] (Section 1.2.1) the partial correctness proof of the iterative extended Euclid’s algorithm [8, 9] through a flow chart program labeled with assertions. In fact, the non-trivial correctness proof was the only example that Knuth gives in Chapter 1 on basic concepts of his classic book [8] for demonstrating how partial correctness proof can be performed. Even though Knuth’s proof is not difficult to follow, it is still quite mysterious with no indication of how the loop invariant is derived or developed. In contrast, our partial correctness proof is logically developed and step-by-step justified; and the proof’s succinctness and elegance are appealing, not intimidating. In this paper, we do not consider variants [7, 9, 11] of iterative extended Euclid’s algorithms for improved efficiency.

2 Recursive Extended Euclid’s Algorithm

We give in this section a quick review of the design of the recursive extended Euclid’s algorithm [3, 4, 5, 6].

Below is Euclid’s algorithm for computing the greatest common divisor of \( a \) and \( b \), where \( a \geq b \geq 0 \):

\[
\begin{align*}
\text{If } b = 0 & \text{ then } GCD(a, b) = a \\
\text{else } & \gcd(a, b) = \gcd(b, a \mod b)
\end{align*}
\]

\(^1\) In fact, matrices are introduced in [1], not for the presentation of the extended Euclid’s algorithm, only for the development of an algorithm for computing the greatest common divisor for polynomials over a field.
**GCD(a,b): // assume a ≥ b ≥ 0**

if (b == 0) return a
return GCD(b,a%b)

Euclid’s algorithm can be extended to return integers x and y such that GCD(a, b) = ax + by. One can develop the extended Euclid’s algorithm by first taking a leap of faith that such an algorithm EGCD(a, b) can be constructed by extending GCD(a, b) to return (d, x, y) where d = GCD(a, b) and d = ax + by. The algorithm is thus organized as follows:

**EGCD(a,b): // assume a ≥ b ≥ 0**

if (b == 0) return (a,1,0) // a = a 1 + b 0 (base case)
(d,x',y') = EGCD(b,a%b) // d = b x' + (a % b) y' (hypothesis)
/* steps are needed to compute x and y */
return (d,x,y) // d = a x + b y (goal)

We need to determine the steps to compute x and y. From the hypothesis, we deduce that d = bx' + (a % b)y', which can be rewritten as d = bx' + (a - ⌊a/b⌋b)y' = ay' + b(x' - ⌊a/b⌋y'). By letting x = y' and y = x' - ⌊a/b⌋y', the algorithm is finalized as follows:

**EGCD(a,b): // assume a ≥ b ≥ 0**

if (b == 0) return (a,1,0) // a = a 1 + b 0 (base case)
(d,x',y') = EGCD(b,a - ⌊a/b⌋b) // d = b x' + (a % b) y' (hypothesis)
(x,y) = (y',x' - ⌊a/b⌋y')
return (d,x,y) // d = a x + b y (goal)

Rewriting a % b as a - ⌊a/b⌋b, the algorithm can be given as follows:

**EGCD(a,b): // assume a ≥ b ≥ 0**

if (b == 0) return (a,1,0) // a = a 1 + b 0 (base case)
(d,x',y') = EGCD(b,a - ⌊a/b⌋b) // d = b x' + (a % b) y' (hypothesis)
(x,y) = (y',x' - ⌊a/b⌋y')
return (d,x,y) // d = a x + b y (goal)

### 3 From Recursion to Iteration

We notice the similarity between the computing of (y', x' - ⌊a/b⌋y') from (x', y') and the computing of (b, a - ⌊a/b⌋b) from (a, b). We can capture the computation using matrix multiplication. Let \( A = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \). Then
\[ \begin{array}{c}
\alpha \\
\beta
\end{array} = \begin{array}{c}
\beta \\
\alpha
\end{array} A \quad \text{and} \quad \begin{array}{c}
\alpha \\
\beta
\end{array} - \lfloor \alpha / \beta \rfloor \begin{array}{c}
\beta \\
\alpha
\end{array} = \begin{array}{c}
\alpha \\
\beta
\end{array} A.
\]

Our goal is to derive an iterative extended Euclid’s algorithm EGCD. But, for technical convenience, for the moment we reduce the functionality of EGCD to eGCD, which performs the same as before except that it no longer returns \( d \), the greatest common divisor of \( a \) and \( b \). Furthermore, eGCD takes vectors for both input and output. Using matrix notation, we derive the codes for eGCD([\( a \) \( b \)]) which returns \([x \ y]\) such that gcd(a, b) = ax + by as follows:

\[
eGCD([a \ b]):\quad \text{// assume } a \geq b \geq 0
\]

\[
\text{if } (b == 0) \text{ return } [1 \ 0]
\]

\[
\text{return } eGCD \left( [a \ b] \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \right) \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix}
\]

As the recursion progresses, the recursion stack remembers the sequence of \( 2 \times 2 \) matrices generated. It is only when the recursion halts that we start multiplying the result \([1 \ 0]\) with the sequence of matrices in the reverse order of their generations. However, since matrix multiplication is associative, we can indeed multiply the \( 2 \times 2 \) matrices as they are generated. That is, we no longer need to remember all the matrices generated until the program halts. This insight allows us to derive an iterative algorithm.

Each time the program advances to the next round, the matrix \( \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \) is multiplied to the left side of the product of \( 2 \times 2 \) matrices generated so far. In addition, \( \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \) is multiplied to the right side of \([a \ b]\) to give the new \([a \ b]\). By maintaining the product of \( 2 \times 2 \) matrices generated as \( \begin{bmatrix} c & e \\ d & f \end{bmatrix} \), the codes for eGCD([a, b]) becomes:
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\begin{align*}
e\text{GCD}([a \ b]): \quad & \begin{bmatrix} c & e \\ d & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\quad & \text{while } (b \neq 0) \\
\quad & \begin{bmatrix} c & e \\ d & f \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \begin{bmatrix} c & e \\ d & f \end{bmatrix} \\
\quad & [a \ b] = [a \ b] \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix}
\end{align*}

\text{return } \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c & e \\ d & f \end{bmatrix}

As the matrix \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} is the transpose of itself, we have

\begin{align*}
\begin{bmatrix} c & d \\ e & f \end{bmatrix} &= \begin{bmatrix} c & d' \\ e & f \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix}
\end{align*}

Next, we give the codes of eGCD([a, b]) in a compact way as follows:

\begin{align*}
e\text{GCD}([a, b]): \quad & \begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\quad & \text{while } (b \neq 0) \\
\quad & \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix}
\end{align*}

\text{return } [c, e]

By the logic of the original Euclid’s algorithm, the greatest common divisor is given by the value of \(a\) when the program \text{GCD} halts. It turns out that the same is true for the program eGCD. We therefore obtain iterative codes for \text{EGCD} by augmenting eGCD to return \((a, c, e)\). Furthermore, the current codes of eGCD are treating the input parameters \(a\) and \(b\) as working variables, which values change over the execution of the program. By separating the input parameters from the working variables, we obtain the iterative EGCD algorithm as follows:
EGCD(α, β):

\[
\begin{bmatrix}
a & b \\
c & d \\
e & f \\
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

while (b != 0)

\[
\begin{bmatrix}
a & b \\
c & d \\
e & f \\
\end{bmatrix} = \begin{bmatrix}
a & b \\
c & d \\
e & f \\
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & -\lfloor a/b \rfloor \\
\end{bmatrix}
\]

return (a, c, e)

Note that the codes above are the same as that given in Knuth’s Algorithm X in page 342 of [9] except that we are using matrix notations whereas Knuth uses vectors.

4 Partial Correctness Proof

In this section, we prove the partial correctness of the iterative EGCD algorithm. First, we need to determine the loop invariant. During the execution of the iterative EGCD algorithm, \( \begin{bmatrix} c & d \\ e & f \end{bmatrix} \) is the product of the 2×2 matrices of the form \( \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \) generated so far in the while loop. On the other hand, \( \begin{bmatrix} a & b \end{bmatrix} \) is \( \begin{bmatrix} \alpha & \beta \end{bmatrix} \) multiplied with the same sequence of 2×2 matrices. Thus, \( \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} \). Next, since \( \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} = \begin{bmatrix} b & a \% b \end{bmatrix} \) and recall that GCD(a, b) = GCD(b, a%b) by the logic of Euclid’s algorithm, the greatest common divisor of the current values of a and b must equal the greatest common divisor of the previous values of a and b, which in turn must equal the greatest common divisor of the initial values of a and b, which are α and β respectively. Therefore, GCD(a, b) = GCD(α, β).

We propose the loop invariant to consist of the two relationship

\[
\begin{bmatrix}
a & b \\
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
\end{bmatrix} \begin{bmatrix}
c & d \\
e & f \\
\end{bmatrix} \]

and GCD(a, b) = GCD(α, β).

Next, we will complete the partial correctness proof step by step.

We need to verify that the the loop invariant holds when the while loop is first entered. After the initial assignment statement before the while loop
is executed, we have \([a\ b] = [\alpha\ \beta]\) and \(\begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\), for which the loop invariant can be easily verified to be valid.

We now turn the attention to the assignment statement within the while loop. The precondition of the assignment statement is the loop invariant augmented with \(b \neq 0\). Even though the assignment statement involves division by \(b\), it is a safe operation as the precondition guarantees that \(b\) is non-zero.

From \([a\ b] = [\alpha\ \beta]\left[ \begin{array}{cc} c & d \\ e & f \end{array} \right]\) of the loop invariant, we can preserve equality by multiplying to each side of the equation the matrix \(\begin{bmatrix} 0 & 1 \\ 1 & -\lfloor \frac{a}{b} \rfloor \end{bmatrix}\) as \(b \neq 0\). Together with the facts that \(\text{GCD}(b, a\%b) = \text{GCD}(a, b)\) when \(b \neq 0\) (by the logic of Euclid’s algorithm) and \(\text{GCD}(a, b) = \text{GCD}(\alpha, \beta)\) (by the loop invariant), we deduce that the following holds:

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor \frac{a}{b} \rfloor \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor \frac{a}{b} \rfloor \end{bmatrix}, \text{ and}
\]

\[\text{GCD}(b, a\%b) = \text{GCD}(a, b) = \text{GCD}(\alpha, \beta), \text{ and } b \neq 0\]

After performing the assignment statement, the value \([a\ b]\left[ \begin{array}{cc} 0 & 1 \\ 1 & -\lfloor \frac{a}{b} \rfloor \end{array} \right]\) is re-written as the new \([a\ b]\); \(\begin{bmatrix} c & d \\ e & f \end{bmatrix} \left[ \begin{array}{cc} 0 & 1 \\ 1 & -\lfloor \frac{a}{b} \rfloor \end{array} \right]\) is re-written as the new \(\begin{bmatrix} c & d \\ e & f \end{bmatrix}\); \(\text{GCD}(b, a\%b)\) is rewritten as \(\text{GCD}(a, b)\) since the new \([a\ b]\) has taken on the value \([a\ b]\left[ \begin{array}{cc} 0 & 1 \\ 1 & -\lfloor \frac{a}{b} \rfloor \end{array} \right]\ = [b\ a\%b]\); however, the value of \(a\%b\), which becomes the new \(b\) value, can no longer be guaranteed to be non-zero. Putting the observations together, the loop invariant again holds after the assignment statement.

Finally, we consider the correctness when the loop is exited. Upon exiting the while loop when \(b = 0\), the loop invariant still holds. Instantiating \(b\) to 0 in the condition \(\text{GCD}(a, b) = \text{GCD}(\alpha, \beta)\) of the loop invariant, we have \(a = \text{GCD}(a, 0) = \text{GCD}(a, b) = \text{GCD}(\alpha, \beta)\). From the condition \([a\ b] = [\alpha\ \beta]\left[ \begin{array}{cc} c & d \\ e & f \end{array} \right]\) of the loop invariant, we deduce that \(a = \alpha c + \beta e\). Therefore, the return of the triple \((a, c, e)\) by \(\text{EGCD}\) is correct as \(a = \text{GCD}(\alpha, \beta)\).


5 Conclusion

In this paper, using the leap of faith technique, the recursive extended Euclid’s algorithm is first constructed. Taking the original Euclid’s algorithm for granted, we do not need any new mathematics insights in extending Euclid’s algorithm. Next, applying only simple program transformation technique, and using the associative property of matrix multiplication, we derive the iterative algorithm. The clean process easily pinpoints the loop invariant from which a solid and easy-to-follow partial correctness proof of the iterative algorithm is established.

The treatment given in the paper demonstrates the maturity of computer science as a discipline. Relying only on computer science principles, we successfully derive the iterative extended Euclid’s algorithm. Our approach complements other developments of the iterative algorithm [1, 2, 8, 9, 10] that require more mathematics insights in their presentations.

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