Thermoelectric effects in superconductors, and their fundamental importance, were first discussed 70 years ago by Ginzburg [1, 2]. The current rapid revival of interest in this topic is triggered by the advances in fabrication of ferromagnetic/superconductor hybrid structures with broken particle/hole symmetry [3–5]. These effects originate in charge transfer by thermal quasi-particles [6], which is compensated by the countercflow of superconducting current $j_s = -b_n \nabla T$ (where $b_n$ is the normal component thermoelectric coefficient). As it is determined by the dissipative normal current, the direction of the thermally induced supercurrent is time-reversal invariant. We point out here, that multicomponent superconductors which break time-reversal symmetry, have entirely different thermoelectric properties. Namely, there is a generic contribution to the thermally induced supercurrent, whose direction is not invariant under time-reversal operation. Besides, that contribution can be orders of magnitude larger than the ordinary and, in contrast to Ginzburg’s mechanism, be present even at low temperatures. To illustrate this new effect, we consider the $s + is$ state, that breaks time-reversal symmetry, and is widely expected to be realized in pnictide compounds Ba$_{1-x}$K$_x$Fe$_2$As$_2$ and stoichiometric LiFeAs [7, 8].

A fundamentally interesting state that can appear in multicomponent superconducting systems is the so-called $s + is$ state. In addition to the usual gauge symmetry $U(1)$, it is characterized by a broken time-reversal symmetry (BTRS). Not only is it the simplest extension of the most abundant s-wave state that breaks time-reversal symmetry, but also it is expected to arise from various microscopic physics [7–12]. The $s + is$ state can as well be fabricated on demand on the interfaces of superconducting bilayers [13]. Recently it was demonstrated that such physics very likely occurs in strongly hole doped Ba$_{1-x}$K$_x$Fe$_2$As$_2$ [7, 12] as well as in stoichiometric compound LiFeAs [8].

In iron pnictides, the $s + is$ BTRS state originates from the multiband character of superconductivity and several competing pairing channels, as shown schematically on the inset in Fig. 1(a). A typical band structure of Ba$_{1-x}$K$_x$Fe$_2$As$_2$ consists of two hole pockets at the $\Gamma$ point, two electron pockets at $(0, \pi)$ and $(\pi, 0)$, and a third hole pocket at $(\pi, \pi)$. The superconducting coupling here is dominated by the interband repulsion between electron and hole Fermi surfaces, as well as between the two hole pockets at $\Gamma$ and $(\pi, \pi)$. Such a system can be described by a minimal three-band model, assuming that the order parameter components are $\Delta_1$ for electron bands, $\Delta_2$ for the hole bands at $\Gamma$ and $\Delta_3$ for the additional hole pocket [7]. The competition of repulsion forces results in an intrinsically complex order parameter whose components in each band $|\Delta_k|e^{i\theta_k}$ (where $k = 1, 2, 3$ is the band index) possess non-trivial (frustrated) ground-state phase differences $\theta_{kj} = \theta_k - \theta_j$ which are neither 0 nor $\pi$ [7, 9–12, 14]. The ground-state order parameter of the $s + is$ phase is not invariant under the discrete ($\mathbb{Z}_2$) time-reversal transformation associated with complex conjugation $\mathcal{T}(\Delta_k) = \Delta^* _k$. Thus it has a broken $U(1) \times \mathbb{Z}_2$ symmetry. Below, $s \pm is$ will stand for states that are related by time-reversal transformation $\mathcal{T}$. One of the interband phase differences, e.g. $\theta_{12}$ can be used as the order parameter that characterizes the $\mathbb{Z}_2$ phase transition.

The BTRS states with phase frustration $\theta_{kj} \neq 0, \pi$ imply the existence of persistent “intrinsic Josephson-like currents” between the three bands, as shown schematically in Fig. 1(b,c). In analogy with a real-space Josephson effect, these currents have opposite “directions” for the two ground states with opposite values of interband phase differences. Although the underlying microscopic physics is not exotic, nor is any fine tuning is required to form such a state, no experimental observations of such BTRS states have been reported. The reason being the major challenges to distinguish the $s + is$ state from its time-reversal invariant $s_\pm$ and $s_{++}$ cousins. Indeed, the probe of relative phases between components of the order parameter in different bands is a non-trivial task. Few proposals have recently emerged for the indirect observation of BTRS signatures in collective mode spectrum [12, 15–19], unusual topological defects [20, 21] or spontaneous currents near impurities in samples subjected to strain [22].

The challenges associated with the observation of the $s + is$ state and its relevance to pnictides motivate our choice of this kind of material for demonstration of the unusual thermoelectric properties of BTRS superconductors. Below we show that in such systems, there is a new mechanism for thermal generation of supercurrent. It is not related to normal thermal currents and leads to thermoelectric properties which strongly differ from that suggested by Ginzburg, in usual superconductors. Al-
Figure 1: (Color online) – (a) Schematic view of the band structure in hole-doped iron pnictide compound Ba$_{1-x}$K$_x$Fe$_2$As$_2$. It consists of two hole pockets at Γ, two electron pockets at (0, π) and (π, 0) and the third hole pocket at (π, π). The superconducting coupling is dominated by the interband repulsion between electron and hole Fermi surfaces, as well as between the two hole pockets at Γ and (π, π). Panels (b,c) schematically show the two degenerate ground-states $s \pm is$. The phases of the order parameter components $\Delta_k = |\Delta_k|e^{i\phi_k}$ are represented by vector diagrams. Circular arrows show the directions of interband Josephson current supported by non-trivial relative phases $\phi_{kj} \neq 0, \pi$. (d) Rod which ends are maintained at different temperatures $T_{1,2}$. The thermophase effect appears due to the temperature-dependent interband phase difference $\phi_{kj} = \phi_{kj}(T)$. The total thermally induced current $j$ will have opposite directions in $s \pm is$ states. Panel (e) sketches a closed circuit in spatially inhomogeneous bulk superconductor with two branches characterized by different values of the thermophase coefficients $\eta^{(1,2)}(T)$. The junctions between branches have different temperatures $T_{1,2}$ and the temperature bias generates magnetic flux through the circuit (7).

though the suggested mechanism does not eliminate the usual thermoelectric effect, it will be shown to exist in a much wider domain of temperatures down to $T \ll T_c$ and be straightforwardly experimentally detectable.

The key idea behind the proposed thermoelectric effect is that, due to the generically temperature-dependent interband phase differences, i.e. $\phi_{kj} = \phi_{kj}(T)$, a temperature bias generates phase gradients of condensate components. Assuming that temperature gradients are small, so that the order parameter is determined by the local temperature, yields the relation

$$\nabla \phi_k = \nabla \phi_0 + \gamma_k(T) \nabla T,$$

where $\phi_0 = \sum_{l=1}^3 \theta_l/3$, and $\gamma_k(T) = \frac{1}{2} \sum_{j \neq k} d\phi_{kj}/dT$ are the thermophase coefficients. Superconducting currents can be generated, as a result of this thermophase effect. In multiband superconductors, the total current has contributions from each band $j = \sum_k j_k$. In units where $h = c = 1$, the partial currents read as $j_k = (\nabla \phi_k - 2\pi A/\Phi_0)/4\pi \lambda_k^2$, where $\lambda_k$ are characteristic constants associated with contributions of a given band to the London penetration depth. $A$ is the vector potential of the magnetic field $B = \nabla \times A$ and $\Phi_0$ the flux quantum. Introducing the notation $Q_0 = \nabla \phi_0 - 2\pi A/\Phi_0$ and the London penetration length $\lambda_L = 1/\sqrt{\Phi_0}$, allows to write the total current as

$$j = \frac{Q_0}{4\pi \lambda_L^2} + \sum_{k \neq l} \frac{\lambda_l^2 - \lambda_k^2}{12\pi \lambda_L^4 \lambda_k^4} \nabla \phi_{kl}.$$  

The first term here is a usual Meissner current while the second part is determined by the gradients of interband phase differences. It describes the charge transfer by counter-currents of several superconducting components.

Observe that according to Eq. (1), the counterflow is excited by the temperature gradient and contributes to the current according to the following generic expression

$$j = \frac{Q_0 + \gamma(T) \nabla T}{4\pi \lambda_L^2(T)} \gamma(T) = \lambda_L^2(T) \sum_k \gamma_k(T) \lambda_k^{-2}(T).$$

The essence of such a thermophase effect is schematically shown in Fig. (1d), for the case of $Q_0 = 0$ so that, according to Eq. (1), the partial currents in each band are given by $j_k = \gamma_k(T) \nabla T/4\pi \lambda_k^2$. Since the thermophase coefficients are opposite for the $s \pm is$ states: $\gamma_k = -\gamma_k$, these thermally induced superconducting currents are sensitive to time-reversal symmetry transformation. That is, for a given temperature bias, the current directions are opposite in $s + is$ and in $s - is$ states.

Deep in the bulk of a superconducting sample the total current (3) should vanish: $j = 0$, which yields the condition for current compensation

$$Q_0 = -\gamma(T) \nabla T.$$  

Integrating Eq. (5) along a closed path and assuming for simplicity that no vortices are trapped in the circuit
The threedimensional Laplacian can be expanded as\(\nabla \cdot \nabla \phi_0 = 0\), we get the thermally induced magnetic flux
\[
\Phi_T = \frac{\Phi_0}{2\pi} \oint \mathbf{dl} \cdot \nabla T \gamma(T). \tag{6}
\]
Since the coefficient \(\gamma(T)\) is in general spatially inhomogeneous along the integration path, the expression (6) yields a finite value of \(\Phi_T\).

Now consider a closed circuit with two branches made up of different superconductors, for example the \(\text{Ba}_{1-x}\text{K}_x\text{Fe}_2\text{As}_2\) compound with different doping levels. The thermophysical coefficient \(\gamma(T)\) in this case has a stepwise discontinuity along the circuit, determined by the values \(\gamma^{(1,2)}(T)\) at the different branches. This is shown schematically in Fig. 1(c). Assuming that the junctions between branches have different temperatures \(T_{1,2}\), Eq. (6) yields an induced magnetic flux through the circuit given by
\[
\Phi_T = \frac{\Phi_0}{2\pi} \int_{T_1}^{T_2} dT [\gamma^{(1)}(T) - \gamma^{(2)}(T)]. \tag{7}
\]

The maximal possible magnitude of \(\Phi_T\) can be estimated by considering one of the branches to be in the time-reversal invariant state, so that e.g. \(\gamma^{(2)}(T) = 0\) and \(\lambda_1 \ll \lambda_2\). Then, from Eq. (6), we obtain \(\Phi_T/\Phi_0 \approx \delta\theta_{12}/4\pi\) where \(\delta\theta_{12} = \theta_{12}(T_2) - \theta_{12}(T_1)\).

The resulting thermally induced flux can have a giant magnitude compared to that produced by a usual thermoelectric effect. Below, we will introduce a realistic microscopic model for the \(s + is\) state, to demonstrate that the interband phase difference can have a significant variation as a function of temperature \(\max(\delta\theta_{12}) \sim \pi\). Therefore the resulting flux \(\Phi_T \sim \Phi_0\) is much larger than the typical value of \(\Phi_T \sim 10^{-3}\Phi_0\) in usual superconducting thermoelectric circuits [1, 2]. Note another principal difference from the ordinary thermophysical effect: in usual superconductors, the thermoelectric coefficient is proportional to the normal phase density and therefore is significant only in the vicinity of \(T_c\) [23, 24]. But by contrast the thermophysical effect in BTRS state does not involve normal current and therefore exists down to the very low temperatures \(T < T_c\).

To give a microscopic basis for this physics, we now proceed to calculating thermophysical coefficients for the \(s + is\) superconducting state in \(\text{Ba}_{1-x}\text{K}_x\text{Fe}_2\text{As}_2\). Within the minimal three-band model, it is parameterized by two pairing constants characterizing the strength of interband hole-hole \(u_{hh}\) and electron-hole \(u_{he}\) repulsions (as shown schematically in Fig. 1). For certain parameters \(u_{hh}, u_{he}\), the competition between different interband repulsion channels yields phase frustration in superconducting components residing in different bands (i.e. the tendency for ground states with non-trivial phase differences between \(\Delta_{1,2,3}\)). For the model case of identical hole bands in the \(\Gamma\) and \((\pi, \pi)\) points, the multicomponent order parameter has a symmetric form
\[
\Delta = e^{i\vartheta_0} [\Delta_1 e^{i\vartheta_1/2}, \Delta_2 e^{-i\vartheta_2/2}, -|\Delta_3|], \tag{2}\]
where \(\vartheta_0\) is the common phase. Hence within this model the thermophysical coefficients in Eq. (3) are given by \(\gamma_1(T) = -\gamma_2(T) = -\gamma_3 = 0\). The phase diagram, shown in Fig. 2, is obtained by solving a non-linear self-consistency problem (see Appendix for details). It demonstrates the line of \(T_{\text{Z2}}\) BTRS phase transition that occurs at the temperature \(T_{\text{Z2}} \lesssim T_c\). In the vicinity of \(T_{\text{Z2}}\), the thermophysical coefficients diverge as \(\gamma(T) \sim (T_{\text{Z2}} - T)^{-1/2}\). This is consistent with the fact that the effective potential for the phase difference gets soft near the symmetry-restoring phase transition. In Fig. 2(b) the overall profile of the function \((T_{\text{Z2}} - T)\gamma(T)\) is shown, where the prefactor \((T_{\text{Z2}} - T)\) is added to remove the divergence. The typical temperature dependencies of \(\theta_{12}(T)\) and \(\gamma(T)\) are shown in Fig. 2(c,d) for particular coupling parameters. The magnitude of the total variation of the relative phase \(\theta_{1h}\) as a function of temperature can be estimated from the typical dependence in Fig. 2(c). There \(\delta\theta_{12} \approx 0.4\pi\), so that the thermally generated flux is \(\Phi_T \sim 0.2\Phi_0\).

Let us now consider several characteristic examples which should arise in concrete experimental set-ups based on circuits containing TRSB \(s + is\) superconductors. To that end, we use the three-component Ginzburg-Landau (GL) theory (8), although the effects will be similar in the microscopic formalism. The GL expansion, derived from the microscopic three-band model (see Appendix),

\[\int dt \mathbf{dl} \cdot \nabla \varphi_0 = 0\), we get the thermally induced magnetic flux
\[\Phi_T = \frac{\Phi_0}{2\pi} \oint \mathbf{dl} \cdot \nabla T \gamma(T). \tag{6}\]
Since the coefficient \(\gamma(T)\) is in general spatially inhomogeneous along the integration path, the expression (6) yields a finite value of \(\Phi_T\).

Now consider a closed circuit with two branches made up of different superconductors, for example the \(\text{Ba}_{1-x}\text{K}_x\text{Fe}_2\text{As}_2\) compound with different doping levels. The thermophysical coefficient \(\gamma(T)\) in this case has a stepwise discontinuity along the circuit, determined by the values \(\gamma^{(1,2)}(T)\) at the different branches. This is shown schematically in Fig. 1(c). Assuming that the junctions between branches have different temperatures \(T_{1,2}\), Eq. (6) yields an induced magnetic flux through the circuit given by
\[\Phi_T = \frac{\Phi_0}{2\pi} \int_{T_1}^{T_2} dT [\gamma^{(1)}(T) - \gamma^{(2)}(T)]. \tag{7}\]

The maximal possible magnitude of \(\Phi_T\) can be estimated by considering one of the branches to be in the time-reversal invariant state, so that e.g. \(\gamma^{(2)}(T) = 0\) and \(\lambda_1 \ll \lambda_2\). Then, from Eq. (6), we obtain \(\Phi_T/\Phi_0 \approx \delta\theta_{12}/4\pi\) where \(\delta\theta_{12} = \theta_{12}(T_2) - \theta_{12}(T_1)\).

The resulting thermally induced flux can have a giant magnitude compared to that produced by a usual thermoelectric effect. Below, we will introduce a realistic microscopic model for the \(s + is\) state, to demonstrate that the interband phase difference can have a significant variation as a function of temperature \(\max(\delta\theta_{12}) \sim \pi\). Therefore the resulting flux \(\Phi_T \sim \Phi_0\) is much larger than the typical value of \(\Phi_T \sim 10^{-3}\Phi_0\) in usual superconducting thermoelectric circuits [1, 2]. Note another principal difference from the ordinary thermophysical effect: in usual superconductors, the thermoelectric coefficient is proportional to the normal phase density and therefore is significant only in the vicinity of \(T_c\) [23, 24]. But by contrast the thermophysical effect in BTRS state does not involve normal current and therefore exists down to the very low temperatures \(T < T_c\).

To give a microscopic basis for this physics, we now proceed to calculating thermophysical coefficients for the \(s + is\) superconducting state in \(\text{Ba}_{1-x}\text{K}_x\text{Fe}_2\text{As}_2\). Within the minimal three-band model, it is parameterized by two pairing constants characterizing the strength of interband hole-hole \(u_{hh}\) and electron-hole \(u_{he}\) repulsions (as shown schematically in Fig. 1). For certain parameters \(u_{hh}, u_{he}\), the competition between different interband repulsion channels yields phase frustration in superconducting components residing in different bands (i.e. the tendency for ground states with non-trivial phase differences between \(\Delta_{1,2,3}\)). For the model case of identical hole bands in the \(\Gamma\) and \((\pi, \pi)\) points, the multicomponent order parameter has a symmetric form
\[\Delta = e^{i\vartheta_0} [\Delta_1 e^{i\vartheta_1/2}, \Delta_2 e^{-i\vartheta_2/2}, -|\Delta_3|], \tag{2}\]
where \(\vartheta_0\) is the common phase. Hence within this model the thermophysical coefficients in Eq. (3) are given by \(\gamma_1(T) = -\gamma_2(T) = -\gamma_3 = 0\). The phase diagram, shown in Fig. 2, is obtained by solving a non-linear self-consistency problem (see Appendix for details). It demonstrates the line of \(T_{\text{Z2}}\) BTRS phase transition that occurs at the temperature \(T_{\text{Z2}} \lesssim T_c\). In the vicinity of \(T_{\text{Z2}}\), the thermophysical coefficients diverge as \(\gamma(T) \sim (T_{\text{Z2}} - T)^{-1/2}\). This is consistent with the fact that the effective potential for the phase difference gets soft near the symmetry-restoring phase transition. In Fig. 2(b) the overall profile of the function \((T_{\text{Z2}} - T)\gamma(T)\) is shown, where the prefactor \((T_{\text{Z2}} - T)\) is added to remove the divergence. The typical temperature dependencies of \(\theta_{12}(T)\) and \(\gamma(T)\) are shown in Fig. 2(c,d) for particular coupling parameters. The magnitude of the total variation of the relative phase \(\theta_{1h}\) as a function of temperature can be estimated from the typical dependence in Fig. 2(c). There \(\delta\theta_{12} \approx 0.4\pi\), so that the thermally generated flux is \(\Phi_T \sim 0.2\Phi_0\).

Let us now consider several characteristic examples which should arise in concrete experimental set-ups based on circuits containing TRSB \(s + is\) superconductors. To that end, we use the three-component Ginzburg-Landau (GL) theory (8), although the effects will be similar in the microscopic formalism. The GL expansion, derived from the microscopic three-band model (see Appendix),
Figure 3: (Color online) – A bimetallic ring consisting of two branches in the $s \pm is$ state, but having different chemical composition (see Appendix). Both junctions are maintained at different temperatures $T_{1,2}$, so that there is a thermal gradient that varies linearly with the polar angle. This is sketched in panel (a). The resulting thermally induced magnetic field is shown on panel (c). Its overall sign depends on which of $Z_2$ BTRS ground states (i.e. $s + is$ or $s - is$) is realized. This configuration supports nonzero flux $\Phi_T$ in the inner hole, that is more or less independent of the diameter of the inner hole. The total flux, on the other hand depends on the diameter of the sample (b).

reads as

$$
\mathcal{F} = \sum_{k=1}^{3} \frac{D_k}{2} \left[ \left( \nabla + i \frac{2\pi}{\Phi_0} A \right) \Delta_k \right]^2 + \alpha_k |\Delta_k|^2 + \beta_k \frac{B^2}{2}.
$$

where $j, k$ are band indices, and $D_k$ are diffusion constants. We model temperature dependence of the coefficients as $\alpha_k \simeq \alpha_k^0 (T/T_k - 1)$ ($\alpha_k^0$ and $T_k$ being characteristic constants, further details on the derivation and choice of parameters are given in Appendix). The numerical studies of the GL equations are performed within a finite element formulation [25] and using a non-linear conjugate gradient algorithm, with the standard condition that no current flows through the sample’s boundaries.

First we consider the case of a bimetallic superconducting ring with both branches made of $s + is$ superconductor, but having different doping. Thus the GL coefficients in (8) should have different values in the upper and lower branches. The junctions between branches are maintained at different temperatures $T_{1,2}$ as sketched in Fig. 3(a). Provided the annulus is thick enough, the total flux through the area circumvented by the outer boundary is given by Eq. (7). Note that the distribution of magnetic field inside the ring is largely concentrated near the junction between two branches as shown in the Fig. 3. The dependence of the total magnetic flux through the sample area is shown in Fig. 3(b). It stems from the contribution to the total current (2), produced by the gradients of relative interband phases $\theta_{kj}$, in the presence of imbalanced relative densities [28].

Above, we assumed that the sample is mono-domain. In realistic systems, in general there will be topological defects: domain walls between that separate $s \pm is$ phases. Consider now a multi-domain sample. It is known that such domain walls do not carry magnetic field at constant temperature [21]. Strikingly, in the presence of a temperature gradient along the domain wall, it generates a thermally induced magnetic field, as shown in Fig. 4. There, the thermophase coefficients $\gamma(T)$ have opposite signs in $s + is$/$s - is$ domains. Therefore, in the vicinity of the interface between them, there should be net superconducting current and the thermally induced magnetic field $B_T$. Similarly the total compensation of current is impossible near other topological defects where the thermophase coefficients are spatially inhomogeneous. Note that there is no phase winding in this flux-carrying configuration and the thermally induced flux $B_T$ is not quantized [29].

This non-trivial thermoelectric effect in $s + is$ phases can be employed to obtain an advanced functionality in the practical applications of certain iron-based superconductors. For this purpose one can manipulate the symmetry of superconducting states by simultaneously applying temperature bias and an external mag-
netic field $H_{ext}$ which in general removes the degeneracy of $s + is/s - is$ states. As a result it is possible to prepare a particular state by cooling the system through a BTRS transition. As an example we suggest a scheme of a memory cell based on the non-trivial thermoelectric properties of bimetallic rings shown in Fig. 3. There, the $s \pm is$ states have the opposite values of $B_T$. In such a system the value of the $Z_2$ index, which distinguishes $s \pm is$ states, can be considered as a bit of information. Let us now describe the writing/reading protocols. First the external field $H_{ext}$ interacts with thermally induced currents and removes the $s \pm is$ degeneracy. Hence the particular $s + is$ or $s - is$ state can be selected when such a system is cooled through the BTRS phase transition, in the presence of both external field and temperature bias $\delta T = T_2 - T_1$. After cooling, both the external field and temperature bias can be removed, leaving the system in the ground state characterized by the $Z_2$ index $\text{sgn}(\delta T H_{ext})$. The read-out protocol, i.e. the measurement of $Z_2$ index, can be implemented by applying the temperature bias (in the absence of external field) and measuring the direction of induced magnetic flux.

To conclude, we demonstrated that multicomponent superconductors with broken time-reversal symmetry, and in particular the $s + is$ state, feature a giant thermoelectric effect of principally different nature than that in single-component superconductors. It originates in thermally induced intercomponent counter-flows, in contrast to the counter-flows of normal and superconducting currents in the mechanism originally discussed by Ginzburg. Although related effects should be present in various multicomponent superconductors, along with multicomponent superfluids, we focused on the $s + is$ states where the effect is generic (irrespective of the model).

The work was supported by the Knut and Alice Wallenberg Foundation through a Royal Swedish Academy of Sciences Fellowship, by the Swedish Research Council, by the Swedish National Infrastructure for Computing (SNIC) at National Supercomputer Center at Linköping, Sweden.

[1] V. L. Ginzburg, Zhurnal Eksperimental’noi i Teoreticheskoi Fiziki 14, 177 (1944).
[2] V. L. Ginzburg, Rev. Mod. Phys. 76, 981 (2004).
[3] F. Giazotto, T. T. Heikkilä, and F. S. Bergeret, Phys. Rev. Lett. 114, 067001 (2015).
[4] A. Ozaeta, P. Virtanen, F. S. Bergeret, and T. T. Heikkilä, Phys. Rev. Lett. 112, 057001 (2014).
[5] P. Machon, M. Eschrig, and W. Belzig, Phys. Rev. Lett. 110, 047002 (2013).
[6] D. J. Van Harlingen, D. F. Heidel, and J. C. Garland, Phys. Rev. B 21, 1842 (1980).
[7] S. Maiti, M. M. Korshunov, and A. V. Chubukov, Phys. Rev. B 85, 014511 (2012).
[8] F. Ahn, I. Eremin, J. Knolle, V. B. Zabolotnyy, S. V. Borisenko, B. Büchner, and A. V. Chubukov, Phys. Rev. B 89, 144513 (2014).
[9] V. Stanev and Z. Tesanović, Phys. Rev. B 81, 134522 (2010).
[10] R. Thomale, C. Platt, W. Hanke, J. Hu, and B. A. Bernevig, Phys. Rev. Lett. 107, 117001 (2011).
[11] K. Suzuki, H. Usui, and K. Kuroki, Phys. Rev. B 84, 144514 (2011).
[12] S. Maiti and A. V. Chubukov, Phys. Rev. B 87, 144511 (2013).
[13] T. K. Ng and N. Nagaosa, Europhysics Letters 87, 17003 (2009).
[14] J. Carlström, J. Garaud, and E. Babaev, Phys. Rev. B 84, 134518 (2011).
[15] G. Blumberg, A. Mialitsin, B. S. Dennis, M. V. Klein, N. D. Zhigadlo, and J. Karpinski, Phys. Rev. Lett. 99, 227002 (2007).
[16] S.-Z. Lin and X. Hu, Phys. Rev. Lett. 108, 177005 (2012).
[17] V. Stanev, Phys. Rev. B 85, 174520 (2012).
[18] M. Marciani, L. Fanfarillo, C. Castellani, and L. Benfatto, Phys. Rev. B 88, 214508 (2013).
[19] M. Khodas, A. V. Chubukov, and G. Blumberg, Phys. Rev. B 89, 245134 (2014).
[20] J. Garaud, J. Carlström, and E. Babaev, Phys. Rev. Lett. 107, 197001 (2011).
[21] J. Garaud and E. Babaev, Phys. Rev. Lett. 112, 017003 (2014).
[22] S. Maiti, M. Sigrist, and A. Chubukov, ArXiv e-prints (2014), arXiv:1412.7439 [cond-mat.supr-con].
[23] Y. M. Gal’Perin, V. L. Gurevich, and V. I. Kozub, ZhETF Pisma Redaktsii 17, 867 (1973).
[24] Y. M. Gal’perin, V. L. Gurevich, and V. I. Kozub, Zhurnal Eksperimental’noi i Teoreticheskoi Fiziki 66, 1387 (1974).
[25] F. Hecht, J. Numer. Math. 20, 251 (2012).
[26] E. Babaev, L. D. Faddeev, and A. J. Niemi, Phys. Rev. B65, 100512 (2002).
[27] E. Babaev, Phys. Rev. B79, 104506 (2009).
[28] The unconventional pattern of magnetic field induced by counter-flows is related to the general property of non-Meissner contribution to electrodynamics, in multicomponent superconductors. The essence of the physics leading to the Meissner and, in a broader sense, to the Anderson-Higgs effect is that the magnetic field is described by a massive vector field theory. As shown in Refs. [26, 27], multicomponent systems cannot generally be described by a massive vector field theory. Rather, it can be mapped onto a massive vector field theory coupled to another field associated with the charge-carrying counter-flows. The later terms lead to contributions which are typically negligible or unobservable in ordinary multiband superconductors due for example to interband couplings. By contrast as discussed in this paper, the thermoelectric effect in the $s + is$ state, makes the interband phase fluctuations propagate into the bulk. This produces the unconventional magnetic signatures.
[29] Magnetic flux penetration into a superconductor without phase winding/vortices is not restricted to three-band superconductors. This can be seen from the general relation for magnetic flux obtained from the expression supercurrent $\Phi = \oint A dt = \oint J/\rho (r) + \sum_{i=1,...,N} (\Delta r(r))i \nabla \delta i(r)/\rho (r)| (\rho (r)$ being the total density). Even if there is no phase winding and if we
select the integration contour where \( J \to 0 \), the integral will in general be nonzero, in the presence of gradients of relative phases \( \theta_i - \theta_j \) and densities.

**Ginzburg-Landau expansion**

We use the microscopic self-consistency equation (9) to derive the GL expansion.

\[
\Delta = 2\pi T \Lambda \sum_{n=0}^{N_d} F(\omega_n),
\]

(9)

where \( F = (F_1, F_2, F_3) \) and \( F_k(\omega_n) = \Delta_k / \sqrt{\omega_n^2 + |\Delta_k|^2} \), \( \omega_n = (2n + 1)\pi T \), with \( n \in \mathbb{Z} \) is the fermionic Matsubara frequency, \( T \) the temperature and \( N_d = \Omega_d / (2\pi T) \) is a cut-off at Debye frequency. In the diffusive case the anomalous functions in each band can be expanded as follows

\[
F_j = \frac{\Delta_j}{\omega_n} - \frac{\Delta_j |\Delta_j|^2}{2\omega_n^2} + D_j (\nabla - i A / \phi_0)^2 \Delta_j,
\]

(10)

where \( D_j \) is the diffusion coefficient. Then we get for the summation over Matsubara frequencies

\[
2\pi T \sum_{n=0}^{\infty} 1/\omega_n = G_0 + \tau ,
\]

(11a)

\[
2\pi T \sum_{n=0}^{\infty} 1/(2\omega_n^3) = 0.1/T^2 ,
\]

(11b)

\[
2\pi T \sum_{n=0}^{\infty} 1/(2\omega_n^5) = 0.4/T ,
\]

(11c)

where \( \tau = (1 - T/T_c) \). We normalize \( \Delta \) by \( T_c \) and \( \xi_0 = \sqrt{\pi D_0/8T_c} \), where \( D_0 \) is some arbitrary diffusion constant.

\[
(G_0 + \tau - \hat{\Lambda}^{-1}) \Delta - b |\Delta|^2 \cdot \Delta = \hat{D} (\Pi^2 \Delta)
\]

(12)

where \( b = 0.1, \Pi = \nabla - i e A, \Delta = (\Delta_1, \Delta_2, \Delta_3), |\Delta|^2 = (|\Delta_1|^2, |\Delta_2|^2, |\Delta_3|^2) \), and \( \hat{D} = (\hat{D}_1, \hat{D}_2, \hat{D}_3) \) where \( \hat{D}_j = D_j / D_0 \). The diffusion coefficients can be different. In the diffusive case the mixed gradient terms are absent.

Let us consider the case of the intraband dominated pairing which can be described by a three-component GL theory in the vicinity of \( T_c \). This regime is described by the following coupling matrix

\[
\hat{\Lambda} = \begin{pmatrix}
\lambda & -\eta_b & -\eta_e \\
-\eta_b & \lambda & -\eta_e \\
-\eta_e & -\eta_e & \lambda
\end{pmatrix}
\]

(13)

where \( \eta_e, \eta_b \ll \lambda \). The critical temperature is determined by the equation \( G_0 = \min(\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}) \), where \( \lambda_1, \lambda_2 = (2\lambda - \eta_b \pm \sqrt{8\eta_b^2 + \eta_e^2})/[2(\lambda^2 - \eta_b \eta_e - 2\eta_e^2)] \) and \( \lambda_3^{-1} = 1/(\lambda + \eta_b) \) are the positive eigenvalues of the matrix

\[
\hat{\Lambda}^{-1} = X \begin{pmatrix}
\lambda_2 - \eta_e^2 & \eta_e + \lambda \eta_e & \eta_e (\lambda + \eta_b) \\
\eta_e + \lambda \eta_e & \lambda_2 - \eta_e^2 & \eta_e (\lambda + \eta_b) \\
\eta_e (\lambda + \eta_b) & \eta_e (\lambda + \eta_b) & \lambda_2 - \eta_e^2
\end{pmatrix}
\]

(14)

where \( X = 1/[(\lambda_2 - \lambda \eta_b - 2\eta_e^2)(\lambda + \eta_b)] \). Since we assume that \( \eta_e, \eta_b > 0 \) and \( \eta_e, \eta_b \ll \lambda \) the critical temperature is given by the smallest eigenvalue \( G_0 = 1/(\lambda + \eta_b) \) so that

\[
G_0 \hat{\Lambda} - \hat{\Lambda}^{-1} = - \begin{pmatrix}
a_1 & a_1 & a_2 \\
a_1 & a_1 & a_2 \\
a_2 & a_2 & a_3
\end{pmatrix}
\]

(15)

where

\[
\begin{align*}
a_1 &= (\eta_e^2 + \lambda \eta_b)/X \\
a_2 &= \eta_e (\lambda + \eta_b)/X \\
a_3 &= (2\eta_e^2 - \eta_e^2 + \lambda \eta_b)/X.
\end{align*}
\]

(16a, 16b, 16c)

E.g. for \( \lambda = 1 \) and \( \eta_b = 0.1, \eta_e = 0.2 \) we get \( a_1 = 0.2244, a_2 = 0.1282 \) and \( a_3 = 0.1923 \). For \( \lambda = 1 \) and \( \eta_e = \eta_b = 0.1 \) we get \( a_1 = a_2 = a_3 = 0.1136 \).

Then we get the system of GL equations

\[
\begin{align*}
(\tau - a_1) & \Delta_1 - a_1 \Delta_2 - a_2 \Delta_3 - b \Delta_1 |\Delta_1|^2 + \hat{D}_1 (\Pi^2 \Delta_1) = 0 \\
(\tau - a_2) & \Delta_2 - a_1 \Delta_1 - a_2 \Delta_3 - b \Delta_2 |\Delta_2|^2 + \hat{D}_2 (\Pi^2 \Delta_2) = 0 \\
(\tau - a_3) & \Delta_3 - a_1 \Delta_1 - a_2 \Delta_2 - b \Delta_3 |\Delta_3|^2 + \hat{D}_3 (\Pi^2 \Delta_3) = 0
\end{align*}
\]

where \( b = 0.1, \Pi = \nabla - i e A \) and \( \hat{D}_j = D_j / D_0 \). The diffusion coefficients can be different. In the diffusive case the mixed gradient terms are absent. The free energy functional whose variations gives the Ginzburg-Landau equations reads as

\[
F = \frac{B^2}{2} + \sum_{k=1}^{3} \left\{ \hat{D}_k |\Pi \Delta_k|^2 + \alpha_k |\Delta_k|^2 + \frac{\beta_k}{2} |\Delta_k|^4 \right\}
\]

(17)

where \( \beta_k = b, \eta_{12} = a_1, \eta_{13} = \eta_{23} = a_2, \alpha_k = \alpha_k^0(T/T_k - 1) \), and \( \alpha_k^0 = 1 - a_k, T_k = T_c (1 - a_k) \).

**Details of simulations**

In order to investigate the response to the thermal gradients, we minimize numerically the free energy (17), in zero external field, with the standard condition that no current flows through the sample’s boundaries. In simulations we use the microscopically derived coefficients (16). The theory is discretized within a finite element formulation and the minimization is performed with a non-linear gradient algorithm.
Details for the bimetallic ring

In Fig. 3 of the main text we consider the case of a bimetallic that consists of two branches in the $s + is$ state, but with different chemical composition. The difference in the chemical composition is modelled by modulating the Ginzburg-Landau coefficients (16) in the lower branch to be 85% of their value in the upper branch:

$$a_i^{\text{lower}} = 0.85 a_i^{\text{upper}}.$$ 

Thus the $a_i$ vary stepwise while passing from one branch to the other. The interfaces between both branches are maintained at different temperatures $T_1$ and $T_2$. Here this is the whole interface that is maintained at a given temperature, and thus the temperature is a linear function of the polar angle. This set-up determines the spatially inhomogeneous coefficients $\alpha_k$ and $\eta_{kj}$. 