THE GRADED DIFFERENTIAL GEOMETRY OF MIXED SYMMETRY TENSORS

ANDREW JAMES BRUCE & EDUARDO IBARGUENGOTIA

Abstract. We show how the theory of \( \mathbb{Z}^2 \)-manifolds - which are a non-trivial generalisation of supermanifolds - may be useful in a geometrical approach to mixed symmetry tensors such as the dual graviton. The geometric aspects of such tensor fields on both flat and curved space-times are discussed.

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1. Introduction

Recall that differential forms are covariant tensor fields that are completely antisymmetric in their indices. Furthermore, it is well-known that supermanifolds offer a convenient set-up in which to deal with differential forms. In particular, differential forms can be understood as functions on the supermanifold \( ITM \) known as the antitangent bundle. This supermanifold is constructed by taking the tangent bundle of a manifold and then declaring the fibre coordinates to be Grassmann odd. Moreover, the antitangent bundle canonically comes equipped with an odd vector field which ‘squares to zero’, this vector field is identified with the de Rham differential. Mixed symmetry tensor fields are covariant tensors fields with more than one set of antisymmetrised indices. Mixed symmetry tensor fields represent a natural generalisation of differential forms in which the tensors are neither fully symmetric nor antisymmetric. From a representation theory point of view, they correspond to Young diagrams with more than one column. In physics, such tensor fields appear in the context of higher spin fields, dual gravitons, double dual gravitons etc. as found in various formulations of supergravity and string theory. In particular, the particle spectrum of string theory contains beyond the massless particles of the effective supergravity theory, an infinite tower of massive particles of ever higher spin. In the tensionless limit, these higher spin excitations become massless. Thus, if one wants to consider the effective theory beyond the effective supergravity theory, one is forced to contend with mixed symmetry tensors. Moreover, it is known that in string theory certain mixed symmetry tensors couple to exotic branes [4]. To our knowledge, the first study of mixed symmetry tensors field from a physics perspective was Curtright [9] who studied a generalised version of gauge theory. For a review of mixed symmetry tensors, including some historical remarks, the reader may consult Campoleoni [3]. Recently, Chatzistavrakidis et al [5] showed how to reformulate Galileon action functionals in an index-free framework using a generalised notion of a supermanifold. The reader should also note that these results are part of Khoo’s PhD dissertation [15]. In particular, the theory involves two sets of Grassmann variables that mutually commute. Seemingly unknown to the authors is that they are really employing particular examples of \( \mathbb{Z}^2 \)-manifolds with \( n = 2 \). Moreover, the formalism of bi-forms (and multi-forms) as developed by Dubois-Violette & Henneaux [11], de Medeiros & Hull [10], and Bekaert & Boulanger [2], is naturally accommodated within this setting.

The aim of this note is to show how the theory of \( \mathbb{Z}^2 \)-manifolds offers a further geometric perspective on mixed symmetry tensor fields. The locally ringed space approach to \( \mathbb{Z}^2 \)-manifolds is currently work in progress initially started by Covolo et al. [6, 7, 8]. However, with the basic tenets in place, the time is ripe to seek applications and links with known constructions. Very loosely, \( \mathbb{Z}^2 \)-manifolds are ‘manifolds’ in which we have \( \mathbb{Z}^2 \)-graded, \( \mathbb{Z}^2 \)-commutative coordinates. The sign rules are controlled by the standard scalar product on \( \mathbb{Z}^2 \). Hence, in general, we have sets of coordinates that anti-commute amongst themselves while commuting across the sets. This is exactly what we require in order to describe mixed symmetry tensors. The one complication is that, in general, there are also formal coordinates that are not nilpotent. This means that we must consider formal power series and not just polynomials in the formal coordinates. However, with the applications to mixed symmetry tensors in mind, we will not need to dwell on this subtlety. We will concentrate on mixed tensors with two ‘blocks’ of antisymmetric indices and so we will only employ very particular \( \mathbb{Z}^2 \)-manifolds with no non-nilpotent formal coordinates.

We liken the current situation to the early days of supersymmetry and in particular the initial works on superspace methods. In particular, physicists worked rather formally with commuting and anticommuting coordinates largely unaware of that the mathematical theory of supermanifolds was concurrently being developed in the Soviet Union by Berezin and collaborators. We speculate that \( \mathbb{Z}^2 \)-manifolds will shed light on various aspects of theoretical physics and here we suggest just one potentially useful facet.

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2. Basics of $\mathbb{Z}_2^n$-geometry

The first reference to $\mathbb{Z}_2^n$-manifolds (coloured manifolds) is Molotkov [16] who developed a functor of points approach. The locally ringed space approach to $\mathbb{Z}_2^n$-manifolds is presented in [6]. We will draw upon this heavily and not present proofs of any formal statements. We work over the field $\mathbb{R}$ and in our notation $\mathbb{Z}_2^2 := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (n-times). A $\mathbb{Z}_2^n$-graded algebra is an $\mathbb{R}$-algebra with a decomposition into vector spaces $\mathcal{A} := \oplus_{\gamma \in \mathbb{Z}_2^n} \mathcal{A}_\gamma$, such that the multiplication respect the $\mathbb{Z}_2^n$-grading, i.e., $\mathcal{A}_\alpha \cdot \mathcal{A}_\beta \subset \mathcal{A}_{\alpha + \beta}$. Furthermore, we will always assume the algebras to be associative and unital. If for any pair of homogeneous elements $a \in \mathcal{A}_\alpha$ and $b \in \mathcal{A}_\beta$ we have that

$$a \cdot b = (-1)^{(\alpha, \beta)} b \cdot a,$$

where $\langle - , - \rangle$ is the standard scalar product on $\mathbb{Z}_2^n$, then we have a $\mathbb{Z}_2^n$-commutative algebra.

The basic objects we will employ are smooth $\mathbb{Z}_2^n$-manifolds. Essentially, such objects are ‘manifolds’ equipped with both standard commuting coordinates and formal coordinates of non-zero supermanifolds - we have formal coordinates that are $\mathbb{Z}_2^n$-nilpotent.

In order to keep track of the various formal coordinates, we need to introduce a notation on how we fix the order of elements in $\mathbb{Z}_2^n$, we do this lexicographically. For example, with this choice of ordering $\mathbb{Z}_2^n = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Note that other choices of ordering have appeared in the literature. A tuple $q = (q_1, q_2, \ldots, q_N)$, where $N = 2^n - 1$ provides all the information about the formal coordinates. We can now recall the definition of a $\mathbb{Z}_2^n$-manifold.

**Definition 2.1.** A (smooth) $\mathbb{Z}_2^n$-manifold of dimension $p|q$ is a locally $\mathbb{Z}_2^n$-ringed space $M := (M, \mathcal{O}_M)$, which is locally isomorphic to the $\mathbb{Z}_2^n$-ringed space $\mathbb{R}^{p|q} := (\mathbb{R}^p, \mathcal{O}_{\mathbb{R}^p}[[\xi]])$. Local sections of $M$ are formal power series in the $\mathbb{Z}_2^n$-graded variables $\xi$ with smooth coefficients,

$$\mathcal{O}_M(U) \simeq C^\infty(U)[[\xi]] := \left\{ \sum_{\alpha \in \mathbb{N}^N} \xi^\alpha f_\alpha \mid f_\alpha \in C^\infty(U) \right\},$$

for ‘small enough’ open domains $U \subset M$. Morphisms between $\mathbb{Z}_2^n$-manifolds are morphisms of $\mathbb{Z}_2^n$-ringed spaces, that is, pairs $\Phi = (\phi, \phi^*) : (M, \mathcal{O}_M) \rightarrow (N, \mathcal{O}_N)$ consisting of a continuous map $\phi : M \rightarrow N$ and sheaf morphism $\phi^* : \mathcal{O}_N \rightarrow \mathcal{O}_M$, i.e., a family of $\mathbb{Z}_2^n$-algebra morphisms $\phi^*_\gamma : \mathcal{O}_N(V) \rightarrow \mathcal{O}_M(\phi^{-1}(V))$, where $V \subset N$ is open. We will refer to the global sections of the structure sheaf $\mathcal{O}_M$ as functions on $M$ and denote them as $C^\infty(M) := \mathcal{O}_M(M)$.

**Example 2.2** (The local model). The locally $\mathbb{Z}_2^n$-ringed space $\mathcal{U}^{p|q} := (\mathcal{U}^p, \mathcal{O}_{\mathcal{U}^p}[[\xi]])$, where $\mathcal{U}^p \subset \mathbb{R}^p$ is naturally a $\mathbb{Z}_2^n$-manifold – we refer to such $\mathbb{Z}_2^n$-manifolds as $\mathbb{Z}_2^n$-superdomains of dimension $p|q$. We can employ (natural) coordinates $(x^a, \xi^\alpha)$ on any $\mathbb{Z}_2^n$-superdomain, where $x^a$ form a coordinate system on $\mathcal{U}^p$ and the $\xi^\alpha$ are formal coordinates.

Many of the standard results from the theory of supermanifolds pass over to $\mathbb{Z}_2^n$-manifolds. For example, the topological space $M$ comes with the structure of a smooth manifold of dimension $p$, hence our suggestive notation. Moreover, there exists a canonical projection $\epsilon : \mathcal{O}(M) \rightarrow C^\infty(M)$. What makes $\mathbb{Z}_2^n$-manifolds a very workable form of noncommutative geometry is the fact that we have well-defined local models. Much like the theory of manifolds, one can construct global geometric concepts via the glueing of local geometric concepts. That is, we can consider a $\mathbb{Z}_2^n$-manifold as being cover by $\mathbb{Z}_2^n$-superdomains together with specified glueing information given by coordinate transformations, composed by homomorphisms

$$\Psi_{\beta\alpha} := \Psi_{\beta}^{-1}\Psi_{\alpha} : \Psi_{\alpha}^{-1}(\Psi_{\alpha}(U_\alpha) \cap \Psi_{\beta}(U_\beta)) \rightarrow \Psi_{\beta}^{-1}(\Psi_{\alpha}(U_\alpha) \cap \Psi_{\beta}(U_\beta)),$$

which are labelled by the different local models $(U_\alpha, C^\infty(U_\alpha)[[\xi]])$, $\{\Psi_{\alpha} : U_\alpha \rightarrow \Psi_{\alpha}(U_\alpha) \subset M\}$, whenever $U_\alpha \cap U_\beta \neq \emptyset$; and a graded unital $\mathbb{R}$-algebra morphism $\Psi_{\beta\alpha}^* : C^\infty(U_\alpha)[[\xi]] \rightarrow C^\infty(U_\alpha)[[\xi]].$

We have the chart theorem ([6, Theorem 7.10]) that basically says that morphisms between $\mathbb{Z}_2^n$-superdomains can be completely described by local coordinates and that these local morphisms can then be extended uniquely to morphisms of locally $\mathbb{Z}_2^n$-ringed spaces. This allows one to proceed to describe the theory much as one would on a standard smooth manifold in terms of local coordinates. Indeed, we will employ the standard abuses of notation when dealing with coordinate transformations and morphisms. In particular, the explicit way of computing change of coordinates concerning any geometrical object are well understood and work identically as in classical differential geometry. In essence, one need only take into account that $\mathbb{Z}_2^n$-degree needs to be preserved under any permissible changes of coordinates. For example, vector fields are defined as $\mathbb{Z}_2^n$-graded derivations of the global sections, $X \in \text{Der}(C^\infty(M) \subset \text{End}(C^\infty(M)))$, that are compatible with restrictions. That is, given some open subset $U \subset M$, we can always ‘localise’ the vector field, i.e., $X|_U = X_U \in \text{Der}(\mathcal{O}_M(U))$. 


Furthermore, if this open is ‘small enough’, we can employ local coordinates \((x^\alpha, \xi^\alpha)\) and write
\[
X_U = X^a(x, \xi) \frac{\partial}{\partial x^a} + X^\alpha(x, \xi) \frac{\partial}{\partial \xi^\alpha}.
\]
Under changes of local coordinates
\[
x^\nu = x^\nu(x, \xi), \quad \xi^\alpha = \xi^\alpha(x, \xi),
\]
remembering the abuses of notation and that \(\mathbb{Z}^2\)-degree is preserved, the induced transformation law on the components of the vector field follow from the chain rule and are given by
\[
X^\nu = X^b \frac{\partial x^\nu}{\partial x^b} + X^\alpha \frac{\partial x^\nu}{\partial \xi^\alpha}, \quad X^\alpha = X^b \frac{\partial x^\nu}{\partial x^b} + X^\beta \frac{\partial \xi^\nu}{\partial \xi^\beta}.
\]
See Covolo et al. [8, Lemma 2.2] for details. The reader can easily verify that the \(\mathbb{Z}^2\)-graded commutator of two vector fields is again a vector field and that the obvious \(\mathbb{Z}^2\)-graded version of the Jacobi identity holds.

As is customary in classical differential geometry, we will not write out the restrictions of geometric objects explicitly and simply write objects in terms of there components in some chosen local coordinate system. In other words, one can work locally on \(\mathbb{Z}^2\)-manifolds in more-or-less the same way as one works on classical manifolds and indeed, supermanifolds. The glaring exception here is the theory of integration on \(\mathbb{Z}^2\)-manifolds which is expected to be quite involved (see Poncin [17] for work in this direction).

3. Mixed symmetry tensors over Minkowski space-time

Consider \(D\)-dimensional Minkowski space-time \(M = (\mathbb{R}^D, \eta)\). The Poincaré transformations we write as
\[
x^\mu \mapsto x'^\mu = x^\nu \Lambda^\nu_\mu + a^\mu,
\]
where \(a^\mu\) are the local gauge potentials. We now wish to construct a \(\mathbb{Z}^2\)-manifold built from \(M\) in a canonical way. In particular, consider
\[
\mathcal{M} := TM[[0, 1]] \times_M TM[[1, 0]],
\]
where we have indicated the assignment of the \(\mathbb{Z}^2\)-grading to the fibre coordinates on each tangent bundle. It is straightforward to see that we do indeed obtain a \(\mathbb{Z}^2\)-manifold in this way by using coordinates (see [6, Proposition 6.1]). Specifically, we can always employ (global) coordinates of the form
\[
\left( \frac{x^\mu}{(0, 0)}, \frac{\xi^\nu}{(0, 1)}, \frac{\theta^\rho}{(1, 0)} \right),
\]
where we have signalled the assignment of \(\mathbb{Z}^2\)-grading. Note that we have the non-trivial \(\mathbb{Z}^2\)-commutation rules
\[
\xi^\mu \xi^\nu = -\xi^\nu \xi^\mu, \quad \theta^\rho \theta^\nu = -\theta^\nu \theta^\rho, \quad \xi^\mu \theta^\nu = \pm \theta^\nu \xi^\mu.
\]
Thus, while each ‘species’ of non-zero degree coordinate are themselves nilpotent, across ‘species’ they commute. This is, of course, very different to the case of standard supermanifolds. The Poincaré transformations induce the obvious linear coordinate transformations on the formal coordinates
\[
\xi'^\nu = \xi^\mu \Lambda^\nu_\mu, \quad \theta'^\nu = \theta^\rho \Lambda^\nu_\rho.
\]
Clearly, these transformation laws respect the assignment of \(\mathbb{Z}^2\)-grading and satisfy (rather trivially) the cocycle condition. Thus, we do indeed obtain a \(\mathbb{Z}^2\)-manifold in this way. As the coordinate transformations respect the obvious bundle structure and do not ‘mix’ the non-zero degree coordinates we have an example of a so-called split \(\mathbb{Z}^2\)-manifold [7]. The fact that we do not, in this case, have non-zero degree coordinates that are not nilpotent means that we only deal with polynomials in the formal coordinates.

The space of \((p,q)\)-forms on \(M\) we define as
\[
\Omega^{(p,q)}(M) := C^\infty(M)_{(p,q)},
\]
where we naturally have the \(\mathbb{N} \times \mathbb{N}\)-grading given by the polynomial order in each coordinate. By considering all possible degrees we obtain a unital \(\mathbb{Z}^2\)-commutative algebra
\[
\Omega(M) := C^\infty(M) = \bigoplus_{(p,q) \in \mathbb{N} \times \mathbb{N}} \Omega^{(p,q)}(M),
\]
which we refer to as the algebra of bi-forms. Note that we naturally, have a \(C^\infty(M) = \Omega^{(0,0)}(M)\) module structure on the space of all bi-forms.

In coordinates, any \((p,q)\)-form can be written as
\[
\omega^{(p,q)}(x, \xi, \theta) = \frac{1}{p!q!} \theta^{\rho_1} \cdots \theta^{\rho_p} \xi^{\mu_1} \cdots \xi^{\mu_q} \omega_{\mu_{q-\cdots-q_1}[\nu_{q-\cdots-q_1}]}(x).
\]
Due to the \(\mathbb{Z}^2\)-commutation rules, we have the relation that \(\omega_{[\nu_{q-\cdots-q_1}][\nu_{q-\cdots-q_1}]} = \omega_{[\nu_{q-\cdots-q_1}][\nu_{q-\cdots-q_1}]}\) and \(\omega_{[\nu_{q-\cdots-q_1}][\nu_{q-\cdots-q_1}]} = \omega_{[\nu_{q-\cdots-q_1}][\nu_{q-\cdots-q_1}]}\) Note that we will not insist on any further relations in general.
Example 3.1. The dual graviton in $D$-dimensions is a $(1, D - 3)$-form and so is given in coordinates as

$$C(x, \xi, \theta) = \frac{1}{(D - 3)!} \theta^\mu \xi^\nu \cdots \xi^{D - 3} C_{\mu \nu \cdots \rho \sigma \tau}(x).$$

Similarly, the double dual graviton in $D$-dimensions of a $(D - 3, D - 3)$-form and so is given in coordinates as

$$D(x, \xi, \theta) = \frac{1}{(D - 3)!(D - 3)!} \theta^\mu \theta^\nu \cdots \theta^{D - 3} \xi^\sigma \xi^{D - 3} D_{\mu \nu \cdots \rho \sigma \tau \cdots \theta \theta \cdots \theta}(x).$$

See Hull [13, 14] for details of the rôle of dual gravitons and double dual gravitons in electromagnetic duality in gravitational theories.

Canonically, the algebra of bi-forms on $D$-dimensional Minkowski space-time comes equipped with a pair of de Rham differentials. These differentials we consider as homological vector fields on the $Z_2^D$-manifold $\mathcal{M}$. That is, they ‘square to zero’, i.e., $2d^2 = [d, d] = 0$. In coordinate we have

$$d_{(1, 1)} = \xi^\mu \frac{\partial}{\partial x^\mu}, \quad d_{(1, 0)} = \theta^\mu \frac{\partial}{\partial \xi^\mu}.$$

It is important to note that do indeed have a pair of vector fields in this way. In particular, the partial derivatives change under Poincaré transformations as

$$\frac{\partial}{\partial x^\mu} = \Lambda^\rho_\mu \frac{\partial}{\partial x^\rho}, \quad \frac{\partial}{\partial \xi^\nu} = \Lambda^\rho_\nu \frac{\partial}{\partial \xi^\rho}, \quad \frac{\partial}{\partial \theta^\rho} = \Lambda^\rho_\rho \frac{\partial}{\partial \theta^\rho}.$$

Thus, the pair of de Rham differentials are well-defined. It is also clear that they $Z_2^D$-commute, i.e,

$$[d_{(1, 0)}, d_{(0, 1)}] := d_{(1, 0)} \circ d_{(0, 1)} - d_{(0, 1)} \circ d_{(1, 0)} = 0.$$

In this way, we obtain a de Rham bi-complex. Also, note that the interior product and Lie derivative can also be directly ‘doubled’.

Canonically we also have a pair of vector fields of $Z_2^D$-degree $(1, 1)$, given by

$$\Delta_{(1, 1)} = \xi^\mu \frac{\partial}{\partial \theta^\mu}, \quad \Delta_{(1, 0)} = \theta^\mu \frac{\partial}{\partial \xi^\mu}.$$

A direct calculation shows that the non-trivial $Z_2^D$-commutators are

$$[\Delta_{(0, 1)}, d_{(1, 0)}] = d_{(1, 1)}; \quad [\Delta_{(1, 0)}, d_{(0, 1)}] = d_{(0, 1)}.$$

Rather conveniently, we can understand the metric as a $(1, 1)$-form and the inverse of the metric as a second-order differential operator given by

$$\eta := \theta^\mu \xi^\nu \eta_{\mu \nu}, \quad \eta^{-1} := \eta^{\mu \nu} \frac{\partial^2}{\partial \xi^\mu \partial \theta^\nu},$$

respectively.

Example 3.2. Consider the Curtright field on $D = 5$ Minkowski space-time [9]. Such a field is understood to be the electromagnetic dual of the graviton field. In our language, the Curtright field is an example of a $(1, 2)$-form and as such can be written in coordinates as

$$C(x, \xi, \theta, \eta) = \frac{1}{2!} \theta^\rho \xi^\sigma \xi^\lambda C_{\mu \rho \sigma \xi \xi \lambda}(x).$$

There is a further symmetry condition on the Curtright field, i.e., $C_{\mu \nu \rho} + C_{\rho \mu \nu} + C_{\nu \rho \mu} = 0$. This condition can be expressed as

$$\Delta_{(1, 1)} C = \frac{1}{2!} \xi^\rho \xi^\sigma \xi^\lambda \left( \frac{\partial C_{\mu \nu \rho \sigma}}{\partial x^\lambda} + \frac{\partial C_{\nu \rho \mu \sigma}}{\partial x^\lambda} + \frac{\partial C_{\rho \mu \nu \sigma}}{\partial x^\lambda} \right) = \frac{1}{2!} \theta^\rho \xi^\sigma \xi^\lambda F_{\mu \nu \sigma \rho \lambda}(x),$$

which we recognise (up to possible conventions) to be the 

**Curtright field strength.** Applying $d_{(1, 0)}$ to the Curtright field strength yields

$$E := d_{(1, 0)} (d_{(0, 1)} C) = \frac{1}{2!} \theta^\rho \theta^\sigma \theta^\lambda \xi^\mu \xi^\nu \left( \frac{\partial F_{\mu \nu \sigma \rho \lambda}}{\partial x^\rho} - \frac{\partial F_{\mu \nu \rho \sigma \lambda}}{\partial x^\rho} \right) = \frac{1}{2} \theta^\rho \theta^\sigma \theta^\lambda \xi^\mu \xi^\nu \xi^\lambda E_{\mu \nu \sigma \rho \lambda}(x),$$

which we recognise (up to possible conventions) to be the **Curtright curvature tensor**, which is fully gauge invariant, see Bekaert, Boulanger & Henneaux [1] for details. Similarly the **Curtright–Ricci tensor** and its trace (again, up to conventions) can be constructed by applying the inverse metric, i.e.,

$$\eta^{-1}(E) = \frac{1}{2} \theta^\rho \xi^\sigma \xi^\lambda \eta_{\mu \nu \rho \sigma \lambda}(x) = \frac{1}{2!} \theta^\rho \xi^\sigma \xi^\lambda E_{\mu \nu \rho \sigma \lambda}(x),$$

$$\eta^{-1}(\eta^{-1}(E)) = \xi^\lambda \eta_{\mu \nu \rho \sigma \lambda}(x) = \xi^\lambda E_{\lambda \mu \nu \rho \sigma \lambda}(x).$$
Remark 3.3. The procedure to describe mixed symmetry tensors with more antisymmetric ‘blocks’ is clear. In particular, if we have $n$ such blocks, then we should consider the $\mathbb{Z}_2^n$-manifold

$$\mathcal{M} := TM[[0, \ldots, 0, 1]] \times_M TM[[0, \ldots, 0, 1, 0]] \times_M \cdots \times_M TM[[1, \ldots, 0, 0]],$$

where we have signalled the $\mathbb{Z}_2^n$-degree of the fibre coordinates. Note that we have a canonical de Rham differential in each sector. Thus, the previous statements of this section can be generalised verbatim.

4. Mixed symmetry tensors over curved space-times

Directly extending the constructions to curved space-times $(M, g)$ is not possible. This was already noticed in [5], albeit with no reference to $\mathbb{Z}_2^n$-manifolds. The two de Rham differentials cannot be naively be considered as vector fields on $\mathcal{M} = TM[[0, 1]] \times_M TM[[1, 0]]$. The resolution to this problem is the standard one: we use the Levi-Civita connection to lift the vector fields. The $\mathbb{Z}_2^n$-manifold $\mathcal{M}$ comes equipped with natural coordinates

$$\left( \frac{dx^\nu}{(0, 0)}, \frac{\zeta^\nu}{(0, 1)}, \frac{\theta^\nu}{(1, 0)} \right),$$

where again we have signalled the assignment of $\mathbb{Z}_2^n$-grading. The permissible changes of local coordinates are

$$x^\mu = x^\mu(x), \quad \zeta^\nu = \xi^\nu \frac{\partial x^\nu}{\partial x^\nu}, \quad \theta^\nu = \theta^\nu \frac{\partial \tilde{x}^\nu}{\partial x^\nu}.$$

As standard, we define a covariant derivative

$$\nabla_\mu := \frac{\partial}{\partial x^\mu} - \xi^\nu \Gamma^\rho_{\mu \nu} \frac{\partial}{\partial \theta^\rho} - \theta^\nu \Gamma^\rho_{\nu \mu} \frac{\partial}{\partial \theta^\rho},$$

where $\Gamma^\rho_{\mu \nu}$ are the Christoffel symbols of the Levi-Civita connection. We then define the covariant de Rham derivatives as

$$\nabla_{(0, 1)} := \xi^\mu \nabla_\mu = \xi^\mu \frac{\partial}{\partial x^\mu}, \quad \nabla_{(1, 0)} := \theta^\mu \nabla_\mu = \theta^\mu \frac{\partial}{\partial \theta^\mu},$$

remembering that the Christoffel symbols are symmetric in the lower indices, i.e., the Levi-Civita connection is torsion free. Due to the transformation rules for the Christoffel symbols both these covariant de Rham derivatives are well-defined vector fields on $\mathcal{M}$. However, in general, we lose the fact that these vector fields are homological and that they commute. This is in stark contrast to the case of standard differential forms where the covariant derivative (with respect to any torsionless connection) reduces to the de Rham differential. Direct calculation shows that

$$[\nabla_{(0, 1)}, \nabla_{(0, 1)}] = R_{(0, 1)} = \theta^\mu \xi^\lambda \xi^\nu R_{\mu \nu \lambda}(x) \frac{\partial}{\partial \theta^\rho};$$

$$[\nabla_{(1, 0)}, \nabla_{(1, 0)}] = R_{(1, 0)} = \xi^\nu \theta^\lambda \theta^\sigma R_{\nu \lambda \sigma}(x) \frac{\partial}{\partial \xi^\rho};$$

$$[\nabla_{(1, 0)}, \nabla_{(0, 1)}] = R_{(1, 1)} = \xi^\mu \theta^\lambda \theta^\sigma R_{\mu \nu \lambda}(x) \frac{\partial}{\partial \theta^\rho} - \theta^\mu \xi^\lambda \xi^\nu R_{\mu \nu \lambda}(x) \frac{\partial}{\partial \xi^\rho},$$

where $R_{\mu \nu \lambda}$ is the Riemann curvature of the Levi-Civita connection (similar expressions can be found in [12]). The vector fields $\Delta_{(0, 1)}$ and $\Delta_{(1, 0)}$ have exactly the same local form as on Minkowski space-time. A direct calculation shows that

$$[\Delta_{(0, 1)}, \nabla_{(0, 1)}] = \nabla_{(0, 1)}, \quad [\Delta_{(1, 0)}, \nabla_{(1, 0)}] = \nabla_{(1, 0)};$$

where one has to take care with the signs due to the $\mathbb{Z}_2^n$-grading.

Example 4.1. The covariant Riemann tensor is an example of a $(2, 2)$-form on $(M, g)$:

$$R(x, \xi, \theta) = \frac{1}{2!} \partial^\rho \theta^\mu \theta^\nu \xi^\sigma \xi^\tau R_{\rho \sigma \mu \nu}(x),$$

here $R_{\rho \sigma \mu \nu} := g_{\rho \lambda} R_{\sigma \mu \nu}^\lambda$ and $R_{\lambda \sigma \mu \nu}$ is the Riemann curvature of the Levi-Civita connection. A direct computation shows that the first Bianchi identity can be written as

$$\Delta_{(0, 1)} R = \frac{1}{3!} \theta^\rho \xi^\nu \xi^\mu \xi^\sigma \left( R_{\nu \lambda \sigma} + R_{\nu \lambda \rho} + R_{\nu \rho \lambda} \right) = 0.$$

Similarly, a direct computation shows that the second Bianchi identity can be written as

$$\nabla_{(0, 1)} R = \frac{1}{2!} \partial^\rho \theta^\mu \theta^\nu \xi^\sigma \xi^\tau \left( \partial R_{\rho \lambda \sigma \mu \nu} - \Gamma^\omega_{\nu \lambda} R_{\rho \sigma \mu \nu} - \Gamma^\omega_{\rho \lambda} R_{\nu \sigma \mu \nu} + \partial R_{\rho \lambda \sigma} \right) \xi^\lambda \xi^\nu \xi^\mu \xi^\sigma + \left( \partial R_{\lambda \sigma \mu \nu} - \Gamma^\omega_{\nu \lambda} R_{\mu \sigma \rho \omega} - \Gamma^\omega_{\rho \lambda} R_{\mu \sigma \rho \omega} \right) = 0.$$
5. Concluding Remarks

As remarked in the introduction, differential forms on a manifold $M$ are naturally understood as functions of the antitangent bundle $\Pi T M$, which itself canonically comes equipped with the de Rham differential, here understood as a homological vector field. Similarly, bi-forms on a (pseudo-)Riemannian manifold $(M, g)$, are naturally understood as functions on the $\mathbb{Z}_2$-manifold $T M[(0, 1)] \times_M T M[(1, 0)]$, which canonically comes equipped with the odd vector fields (generally, non-homological) $\nabla_{(0,1)}$ and $\nabla_{(1,0)}$. Similar statements can be made for more general multi-forms.

While the goals of this note have been modest, we hope that the observations here will prove useful in further studies of mixed symmetry tensors. In particular, it is well-known that constructing consistent theories of interacting mixed symmetry tensors is problematic. We hope that further geometric insight can be gained via $\mathbb{Z}_2$-manifolds and that this will lead to a better understanding of how to build actions involving mixed symmetry tensors.

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