THE WEIGHTED LAPLACIANS ON REAL AND COMPLEX METRIC MEASURE SPACES

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Dedicated to the memory of Professor Shoshichi Kobayashi

Abstract. In this short note we compare the weighted Laplacians on real and complex (Kähler) metric measure spaces. In the compact case Kähler metric measure spaces are considered on Fano manifolds for the study of Kähler-Einstein metrics while real metric measure spaces are considered with Bakry-Émery Ricci tensor. There are twisted Laplacians which are useful in both cases but look alike each other. We see that if we consider noncompact complete manifolds significant differences appear.

1. Introduction

The weighted Laplacians can be considered on smooth metric measure spaces using the exterior derivative $d$ on Riemannian manifolds and the $\partial$-operator on complex Kähler manifolds. We considered in [4] and [5] the weighted $\partial$-Laplacian

$$\Delta_F u = \Delta_F u + \nabla_i \nabla^i u = g^{ij} \nabla_i \nabla_j u + \nabla_i F \nabla^i u$$

on a Fano manifold $M$ where $\Delta_F$ acts on complex-valued smooth functions $u \in C^\infty(M)$. Here, a Fano manifold, by definition, has positive first Chern class, and we have chosen a Kähler form

$$\omega = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j$$

in $2\pi c_1(M)$. The real-valued smooth function $F$ is chosen so that

$$\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} F,$$

and this is possible since the Ricci form

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{ij}).$$

also represents $2\pi c_1(M)$. We say $\lambda$ is an eigenvalue of $\Delta_F$ if $\Delta_F u + \lambda u = 0$ for some nonzero complex-valued function $u$. If we use the weighted volume $d\mu = e^F \omega^m$ we have

$$\int_M g(\partial u, \partial v) d\mu = -\int_M (\Delta_F u) v d\mu = -\int_M u(\Delta_F v) d\mu$$

where

$$g(\partial u, \partial v) = \nabla_i \nabla^i u \bar{v} = g^{ij} \frac{\partial u}{\partial z^j} \frac{\partial v}{\partial \bar{z}^i}.$$
Thus all the eigenvalues are nonnegative real numbers. It is shown that the first non-zero eigenvalue \( \lambda_1(\Delta_F) \) satisfies
\[
(1) \quad \lambda_1(\Delta_F) \geq 1
\]
and the equality holds if and only if the Lie algebra \( \mathfrak{h}(M) \) of all holomorphic vector fields is non-zero. In fact, given a holomorphic vector field \( X \), there corresponds an eigenfunction \( u \) by
\[
X = \text{grad}'u := g^{ij} \frac{\partial u}{\partial z^i} \frac{\partial}{\partial z^j}.
\]
This observation was made before the publication of [3], and was inspired by Shoshichi Kobayashi’s book [15], Chapter III, section 7, “Conformal changes of the Laplacian”. In fact the obstruction to the existence of Kähler-Einstein metrics introduced in [3] was found using the above observation combined with an idea suggested by J.L. Kazdan [13] (see also [14]). See section 2 for more about the application of the above observation to the study of Kähler-Einstein metrics. The proof of (1) follows from the general formula
\[
(2) \quad - \int_M \nabla^i(\Delta_F u) \nabla_i \bar{\nabla} d\mu = \int_M (|\nabla'' u|^2 + |\bar{\partial} u|^2) d\mu,
\]
where
\[
|\nabla'' u|^2 = \nabla \nabla u \nabla_i \nabla_j \bar{\nabla} = g^{ij} g^{\overline{kl}} \nabla_i \nabla_j u \nabla_k \nabla_l.
\]
See also [21], [23], [16] for other applications to the study in Kähler geometry.

On the other hand a similar idea is commonly used in Riemannian geometry and probability theory with the Bakry-Émery Ricci curvature \( \text{Ric} + \nabla^2 f \) on the weighted Riemannian manifolds \( (M, g, f) \). This means that \( (M, g) \) is a complete Riemannian manifold with the weighted measure \( d\mu = e^{-f} dv \), where \( dv \) denotes the Riemannian volume measure on \( (M, g) \) and \( f \) is a real-valued \( C^2 \)-function. We denote by \( C^\infty(M) \) (resp. \( C^\infty_0(M) \)) the set of real-valued smooth functions (resp. with compact support). For all \( u, v \in C^\infty_0(M) \), the following integration by parts formula holds
\[
\int_M g(\nabla u, \nabla v) d\mu = -\int_M (\Delta f) u v d\mu = -\int_M u (\Delta f) v d\mu,
\]
where \( \Delta_f \) is the called the weighted Laplacian with respect to the volume measure \( \mu \). More precisely, we have
\[
\Delta_f = \Delta - \nabla f \cdot \nabla.
\]
Here we denote by \( \Delta \) the \( d \)-Laplacian: \( \Delta = -d^* d = g^{ij} \nabla_i \nabla_j \) with respect to real coordinates \( (x^1, \ldots, x^n) \). In [1], Bakry and Émery proved that for all \( u \in C_0^\infty(M) \),
\[
(3) \quad \Delta_f |u|^2 - 2(\nabla u, \nabla \Delta_f u) = 2|\nabla^2 u|^2 + 2(\text{Ric} + \nabla^2 f)(\nabla u, \nabla u).
\]
The formula (3) can be viewed as a natural extension of the Bochner-Weitzenböck formula. The equation (2) can also be derived from a similar Weitzenböck type formula. The probabilistic study of the weighted Laplacian was motivated by the hypercontractivity of Markov semigroups. In [8] Gross showed that the hypercontractivity holds if and only if the logarithmic Sobolev inequality holds. Then in [1] Bakry and Émery showed that on a smooth metric measure space the logarithmic Sobolev inequality holds if the Bakry-Émery Ricci tensor is bounded from below by a positive constant, that is if there is a positive constant \( C \) such that
\[
(4) \quad \text{Ric} + \nabla^2 f \geq C g.
\]
Note in this case we have

\[ \lambda_1(\Delta_f) \geq C. \]

See section 3 more about the Bakry-Émery Ricci tensor.

It has been a puzzle (to the author) how the real and complex (Fano) cases are different. One immediate difference is that, while \( \Delta_f \) in the Riemannian case is a real operator, \( \Delta_F \) in the Kähler case is not a real operator unless \( F \) is constant. This means that \( \Delta_F u \) is a complex valued function even if \( u \) is real valued. Therefore the eigenfunctions corresponding to nonzero eigenvalues can not be real valued.

In this paper we see, by comparing with the results of Cheng and Zhou in the real noncompact case, that if we consider noncompact complete manifolds then more significant differences appear between the real and complex weighted Laplacians. For example, when the first nonzero eigenvalue of the twisted Laplacian attains the expected lower bound, it is a discrete spectrum in the real case, but it can be an essential spectrum in the complex case. Moreover if the first nonzero eigenvalue of expected lower bound has multiplicity \( k \) then in the real case the Gaussian soliton of dimension \( k \) splits off, but this is not the case in the complex case. See sections 4 and 5 for more detail.

2. The case of Fano manifolds

Let \( M \) have positive first Chern class, i.e. the first Chern class \( c_1(M) \) contains a positive closed \((1,1)\)-form. This is equivalent to say the anticanonical bundle of \( M \) is ample, and such a manifold is called a Fano manifold. We choose a Kähler from

\[ \omega = \sqrt{-1} g_{\overline{j}} dz^i \wedge d\overline{z}^j \]

in \( 2\pi c_1(M) \). Since the Ricci form

\[ \text{Ric}(\omega) = -\sqrt{-1} \partial \overline{\partial} \log \det(g_{\overline{j}}). \]

also represents \( 2\pi c_1(M) \), there exists a smooth function \( F \) such that

\[ \text{Ric}(\omega) - \omega = \sqrt{-1} \partial \overline{\partial} F. \]

Denote by \( C^\infty_c(M) \) the set of all complex valued functions on \( M \). For \( u \) and \( v \) in \( C^\infty_c(M) \), we consider the \( L^2 \)-inner product with respect to the weighted volume

\[ d\mu = e^F \omega^m \]

and

\[ (\overline{\partial} u, \overline{\partial} v)_F = \int_M g(\overline{\partial} u, \overline{\partial} v) e^F \omega^m. \]

We considered in [4] and [5] the weighted \( \overline{\partial} \)-Laplacian

\[ \Delta_F u = \Delta_{\overline{\partial}} u + \nabla_i F \nabla^i u = g^{\overline{j}i} \nabla_{\overline{j}} \nabla^i u + \nabla_i F \nabla^i u. \]

We then have

\[ (\overline{\partial} u, \overline{\partial} v)_F = -(\Delta_F u, v)_F = -(u, \Delta_F v)_F. \]

We say \( \lambda \) is an eigenvalue of \( \Delta_F \) if \( \Delta_F u + \lambda u = 0 \) for some nonzero complex-valued function \( u \). Since \( \Delta_F \) is self-adjoint by (10) all the eigenvalues are nonnegative real numbers.
Theorem 2.1 (c.f. [4], [5]). The first non-zero eigenvalue \( \lambda_1(\Delta_F) \) of \( \Delta_F \) satisfies
\[
\lambda_1(\Delta_F) \geq 1
\]
and the equality holds if and only if the Lie algebra \( \mathfrak{h}(M) \) of all holomorphic vector fields is non-zero. In fact, for a holomorphic vector field \( X \), there corresponds an eigenfunction \( u \) by
\[
X = \text{grad}'u := g^{ij} \frac{\partial u}{\partial z^j} \frac{\partial}{\partial z^i}.
\]

If we pick another Kähler form \( \tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi \) with \( \varphi \in C^\infty(M) \), then \( \tilde{u} := u + u^i\varphi_i \) is the first eigenfunction corresponding to the holomorphic vector field \( X \) in the previous theorem. Using this I proved the following theorem in the first version of [3].

Theorem 2.2 ([3]). On a Fano manifold if we define \( f : \mathfrak{h}(M) \to \mathbb{C} \) by
\[
f(X) = \int_M X F \omega^m
\]
then \( f \) is indecent of the choice of \( \omega \in 2\pi c_1(M) \). In particular if \( f \neq 0 \) then \( M \) does not admit a Kähler-Einstein metric.

If we use the above correspondence in Theorem 2.1 we have a complex valued smooth function \( u \) such that \( X = \text{grad}'u \) and that \( \Delta_F u + u = 0 \). It follows that
\[
X F = -\Delta_F u - u
\]
and that
\[
f(X) = -\int_M u \omega^m.
\]
This implies the following result which is known as Mabuchi’s theorem [17] for toric Fano manifolds. But this is generally true even if \( M \) is not toric.

Theorem 2.3. The character \( f \) vanishes if and only if, for the action of the maximal torus of the reductive part of the automorphism group, the barycenter of the moment map lies at 0.

Proof. We have the Chevalley decomposition \( \mathfrak{h}(M) = \mathfrak{h}_r \oplus \mathfrak{h}_u \) where \( \mathfrak{h}_r \) is the maximal reductive sub algebra and \( \mathfrak{h}_u \) is the unipotent radical. But by [18] \( f \) vanishes on \( \mathfrak{h}_u \). It follows that \( f \) vanishes if and only if \( f \) vanishes on \( \mathfrak{h}_r \). But any element of the automorphism in the reductive group is contained in a maximal torus. Therefore \( f \) vanishes if and only if it vanishes on the maximal abelian subalgebra. Thus the theorem follows from (12). \( \square \)

Note also in passing that the theorem of Matsushima [19], saying that the Lie algebra \( \mathfrak{h}(M) \) is reductive when \( M \) is a Kähler-Einstein manifold, follows from the Theorem 2.3 since if \( (M, g) \) is Kähler-Einstein then we can take \( F = 0 \) and \( \Delta_F = \Delta_g \) is a real operator so that \( \mathfrak{h}(M) \) is a complexification of the purely imaginary eigenfunctions, the gradient vectors of which are Killing vector fields.

3. Hypercontractivity, logarithmic Sobolev inequality and Bakry-Émery-Ricci tensor

In this section we review the historical background of Bakry-Émery Ricci tensor in probability theory. We refer the reader to the lecture notes [11] for more details. A Markov semigroup \( P_t = e^{t\mathcal{L}} \), \( t > 0 \), is said to have hypercontractivity if
\[
\|P_t u\|_{L^p(\mu)} \leq \|u\|_{L^q(\mu)}
\]
where \( p = p(t) = 1 + (q - 1)e^{\frac{q}{2}} > 1, \ c \in [1, \infty) \) is a constant and \( \mu \) is a probability measure.

In [3], L.Gross showed that \( P_t = e^{t\mathcal{L}} \) satisfies hypercontractivity if and only if the following logarithmic Sobolev inequality is satisfied:

\[
\mu(u^2 \log u^2) \leq c \mu(u(-\mathcal{L}u)) + \mu(u^2) \log \mu(u^2)
\]

where \( c \in (0, \infty) \) is a constant independent of \( u \).

Let \((M, g, e^{-f}dV_g)\) be a smooth metric measure space. This means that \((M, g)\) is a Riemannian manifold with the Riemannian volume element \( dV_g \), that \( f \) is a smooth function on \( M \), and that we consider the twisted volume element \( e^{-f}dV_g \).

In [1], Bakry and Émery showed that if a metric measure space \((M, g, e^{-f}dV_g)\) has finite measure \( \int_M e^{-f}dV_g < \infty \), and \( \text{Ric}_f := \text{Ric} + \nabla^2 f \geq \lambda g \) with a positive constant \( \lambda > 0 \), then the logarithmic Sobolev inequality holds with respect to the measure \( d\mu = e^{-f}dV_g \) and \( \mathcal{L}u = \Delta_f u = \Delta u - \nabla f \cdot \nabla u \). In particular, \( P_t = e^{t\mathcal{L}} \) satisfies hypercontractivity. Note that Morgan [20] later proved that if \( \text{Ric}_f \geq \lambda g \) for some positive constant \( \lambda > 0 \) then the weighted measure is finite: \( \int_M e^{-f}dV_g < \infty \).

The key observation of Bakry and Émery is the following Weitzenböck type formula.

\[
\Delta_f |\nabla u|^2 - 2(\nabla u, \nabla \Delta_f u) = 2|\nabla^2 u|^2 + 2(\text{Ric} + \nabla^2 f)(\nabla u, \nabla u).
\]

As an immediate consequence of (13) we obtain the following spectral gap.

**Proposition 3.1** ([1], [12], [20]; see also [2]). *In the situation as above, if a non constant function \( u \in L^2(e^{-f}dV_g) \) satisfies \( \Delta_f u + \mu u = 0 \) then \( \mu \geq \lambda \). That is, the first nonzero eigenvalue \( \lambda_1(\Delta_f) \) satisfies \( \lambda_1(\Delta_f) \geq \lambda \).*

4. **Spectrum of complete smooth metric measure spaces**

Given a smooth metric measure space \((M, g, e^{-f}dV_g)\) we set \( L^2_0(M) \) to be the closure of the set \( C_0^\infty(M) \) of real-valued smooth functions on \( M \) with compact support with respect to \( L^2(d\mu) \)-norm, and \( H^1_0(M) \) to be the closure of the set \( C_0^\infty(M) \) of real-valued smooth functions \( u \) on \( M \) with compact support with respect to the norm

\[
||u||_{H^1_0} = \left( \int_M (u^2 + |\nabla u|^2) d\mu \right)^{\frac{1}{2}}.
\]

By the result of Bakry and Émery, if \( \text{Ric}_f \geq \lambda g \) for some positive constant \( \lambda > 0 \) then we have the logarithmic Sobolev inequality

\[
\int_M u^2 \log u^2 d\mu \leq C \int_M |\nabla u|^2 d\mu
\]

for \( u \in H^1_0(M) \) with \( \int_M u^2 d\mu = 1 \).

It is known (c.f. [12], [2]) that if the logarithmic Sobolev inequality holds then we have the compact embedding \( H^1_0(M) \hookrightarrow L^2_0(M) \), and the spectrum of the twisted Laplacian \( \Delta_f \) is discrete. We denote by \( \lambda_1(\Delta_f) \) the first nonzero eigenvalue of \( \Delta_f \) with eigenfunction in \( H^1_0(M) \).

Now we recall a recent result of Cheng and Zhou [2]. First let us see two examples.

**Example 4.1.** Let \((\mathbb{R}^n, g_{\text{can}}, \frac{|x|^2}{2})\) be the Gaussian soliton, that is \( g_{\text{can}} \) is the flat metric \( \frac{1}{2}|\nabla|^{2}|x|^2 \) with \( f = \frac{|x|^2}{2} \) so that \( \text{Ric}_f = 0 + \frac{1}{2}\nabla x \cdot \nabla x \). Then we have \( \lambda_1(\Delta_f) = \)
\[ \frac{1}{2}. \] This is because Proposition 3.4 shows \( \lambda_1(\Delta_f) \geq \frac{1}{2} \). But the equality is attained by \( u = x^1 \) since \( \Delta u + \nabla u \cdot \nabla f = dx^1 \cdot \left( \frac{1}{2} \sum x^i dx^i \right) = \frac{x^1}{2} = \frac{u}{2} \).

**Example 4.2.** Consider \( S^{n-k}(\sqrt{2(n-k-1)}) \times \mathbb{R}^k \) for \( n-k \geq 2 \) and \( k \geq 1 \). Take \( f = \frac{|u|^2}{4} \) with \( t \in \mathbb{R}^k \). Then we have \( \text{Ric}_f = \frac{1}{2} g \) and \( \lambda_1(\Delta_f) = \frac{1}{2} \), and the corresponding eigenfunctions are linear functions in \( \mathbb{R}^k \).

**Theorem 4.3** ([2]). Let \( (M^n, g, e^{-f} dV_g) \) be a complete smooth metric measure space with Ric \( \geq \lambda g \) for a positive constant \( \lambda \). If \( \lambda_1(\Delta_f) = \lambda \) with multiplicity \( k \) then \( M \) is isometric to \( S^{n-k} \times \mathbb{R}^k \) with \( \lambda_1(\Delta^V_f) > \lambda \) and \( f \) is written in the form \( f(p, t) = f(p, 0) + \frac{1}{2} |t|^2 \) where \( p \in \Sigma \) and \( t \in \mathbb{R}^k \).

In the case of shrinking Ricci soliton

\[ (14) \quad \text{Ric}_f = \lambda g, \]

it is shown in [2] that we have \( \lambda \leq \lambda_1(\Delta_f) \leq 2 \lambda \). This is because we always have

\[ (15) \quad \Delta_f f + 2 \lambda f = 0, \]

and \( f \) belongs to \( H^1_f(M) \). In the case of a compact shrinking soliton, a lower bound of \( \lambda_1(\Delta_f) \) can be given in terms of the diameter, and combined with (15) we can obtain a universal lower bound of the diameter (7), (9).

5. **Complete Kähler metric measure spaces**

We say that \( (M, g, e^F dV_g) \) is a complete Kähler metric measure space if \( (M, g) \) is a complete Kähler manifold with

\[ (16) \quad \text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} F, \]

where

\[ (17) \quad \omega = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j \]

is the Kähler form and \( F \) is a smooth function. If \( M \) is compact \( M \) is naturally a Fano manifold. Note also \((M, g)\) is a gradient Kähler-Ricci soliton if in addition

\[ \nabla'' \nabla'' F = 0. \]

By the same arguments of Morgan [20] we can show that the weighted volume \( \int_M e^F dV_g \) is finite. We consider the same weighted Laplacian \( \Delta_F \) as in the Fano case in section 2. Namely, \( \Delta_F \) is given by the same formula (10) and acts on the complex-valued functions \( C^k_c(M) \).

**Theorem 5.1.** Let \( (M, g, e^F dV_g) \) be a complete Kähler metric measure space. Then \( \lambda_1(\Delta_F) \geq 1 \) and there is an imbedding of the 1-eigenspace \( \Lambda_1 \) to the Lie algebra \( \mathfrak{h}(M) \) of all holomorphic vector fields on \( M \).

Unlike the real case in section 3, \( \Lambda_1 \) can be infinite dimensional so that 1 is an essential spectrum. For example, consider \((\mathbb{C}^n, g_{\text{can}}, e^{-|z|^2} dV_{g_{\text{can}}})\) with \( n \geq 2 \) where \( g_{\text{can}} = \nabla z \cdot \nabla \bar{z} \). With \( F = -|z|^2 \), \((\mathbb{C}^n, g_{\text{can}}, e^{-|z|^2} dV_{g_{\text{can}}})\) is a complete Kähler metric measure space, or even gradient shrinking Kähler-Ricci soliton. Let \( v(z_2, \cdots, v_n) \) be polynomials in \( z_2, \cdots, z_n \), and put \( u = v z_1 \). Then we see

\[ \Delta_F u = -u. \]
The space of all polynomials is of course infinite dimensional. Note that the imbedding in Theorem 5.1 is not surjective since there are non-integrable holomorphic functions in \( z_2, \ldots, z_n \).

Note also a result of the same type as Theorem 4.3 also does not hold in Kähler situation as the following example shows. Let \( \Sigma \) be an \( n \)-dimensional Fano manifold with \( \dim h(\Sigma) = k \). Then there is no splitting of Euclidean factor. Moreover, as seen above, the first eigenspace of the Euclidean factor \( \mathbb{C}^k \) is infinite dimensional.

Note that in the case of Kähler-Ricci solitons the logarithmic Sobolev inequality holds for real-valued functions (with respect to weighted \( d \)-Laplacian). Similar arguments were given in Gross [9] and Gross and Qian [10].

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