Cubic extentsions of the Poincaré algebra

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A systematic study of non-trivial cubic extensions of the four-dimensional Poincaré algebra is undertaken. Explicit examples are given with various techniques (Young tableau, characters etc).

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I. INTRODUCTION

The concept of symmetries has always been an essential tool in the comprehension/description of physical laws. Mathematically symmetries are described by algebras. Of course, not all the mathematical structures are relevant in physics since they have to respect the principles of relativity and quantum mechanics. These consistency conditions lead to a series of no-go theorems \cite{1, 2}, and two wide classes of algebras namely, Lie algebras and Lie superalgebras were intensively used. However, these two structures are not the only ones one is able to consider without contradicting these no-go theorems. It is then appealing to know whether or not new types of symmetry would be relevant in physics. Several approaches beyond Lie (super)algebras have been considered. The main idea of these various approaches is to weaken the hypotheses of the no-go theorems, obtaining new symmetries which go beyond supersymmetry \cite{3, 4, 5, 6, 7}.

In supersymmetric theories, the extensions of the Poincaré algebra are obtained from a “square root” of the translations, “\(QQ \sim P\)” . It is tempting to consider other alternatives where the new algebra is obtained from yet higher order roots. The simplest alternative which we will consider in this paper is “\(QQQ \sim P\)” . It is important to stress that such structures are not Lie (super)algebras (even though they contain a Lie sub-algebra), and as such escape \textit{a priori} the Coleman-Mandula \cite{1} as well as the Haag-Lopuszanski-Sohnius

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no-go theorems [2]. Furthermore, as far as we know, no no-go theorem associated to such
types of extensions has been considered in the literature. In such extensions, the generators
of the Poincaré algebra are obtained as three-fold symmetric products of more fundamental
generators, leading to the “cubic root” of translation. The Lie algebras of order three,
the structures which underlie these extensions, were defined in [8, 9] in full analogy with
supersymmetry and its underlying Lie superalgebra structure. An $F$–Lie algebra admits a
$\mathbb{Z}_F$–gradation ($F = 3$ in this paper), the zero-graded part being a Lie algebra. An $F$–fold
symmetric product (playing the role of the anticommutator in the case $F = 2$) expresses the
zero graded part in terms of the non-zero graded part [8, 9].

Subsequently, a specific $F$–Lie algebra (for $F = 3$) has been studied together with its
implementation in quantum field theory [10, 11]. A general study of the possible non-trivial
extensions of the (1+3)$D$ Poincaré algebra has been undertaken. In this paper we summarize
some of the results established in [12] and we give a systematic way to construct all possible
extensions of the Poincaré algebra when $F = 3$.

The content of this paper is the following. In section 2. we recall the main results
on $F$–Lie algebras. Section 3. is devoted to the general study of Lie algebras of order 3
associated to the Poincaré algebra. Then, several examples are explicitly constructed using
various techniques such as Young tableau, characters etc.

II. LIE ALGEBRAS OF ORDER $F$

The general definition of Lie algebras of order $F$, was given in [8, 9] together with an
inductive way to construct Lie algebras of order $F$ associated with any Lie algebra or Lie
superalgebra. We recall here the main results useful for the sequel. Let $F$ be a positive
integer and define $q = e^{\frac{2\pi i}{F}}$. We consider $\mathfrak{g}$ a complex vector space and $\varepsilon$ an automorphism
of $\mathfrak{g}$ satisfying $\varepsilon^F = 1$. Set $\mathfrak{g}_i \subseteq \mathfrak{g}$ the eigenspace corresponding to the eigenvalue $q^i$ of $\varepsilon$.
Then, we have $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{F-1}$.

**Definition II.1** The vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_F$ is called a (complex) Lie algebra of
order $F$ if it is endowed with the following structure:

1. $\mathfrak{g}$ is a (complex) Lie algebra;

2. $\mathfrak{g}_k, 1 \leq k \leq F - 1$ are representations of $\mathfrak{g}$;
3. there exist multilinear $g$-equivariant maps $\{\cdots\} : S^F(g_k) \to g_0$, where $S^F(D)$ denotes the $F$-fold symmetric product of $D$;

4. for all $Y_1, \cdots, Y_{F+1} \in g_k$ the following “Jacobi identities” hold:

$$
\sum_{i=1}^{F+1} [Y_i, \{Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_{F+1}\}] = 0.
$$

(1)

It should be noted that if $F = 1$, by definition $g = g_0$ and a Lie algebra of order 1 is a Lie algebra; and if $F = 2$, then $g$ is a Lie superalgebra. Therefore, Lie algebras of order $F$ appear as some kind of generalisations of Lie algebras and Lie superalgebras. Moreover, for any $k = 1, \ldots, F - 1$, the $\mathbb{Z}_F$-graded vector spaces $g_0 \oplus g_k$ is a Lie algebra of order $F$. We call these type of algebras elementary Lie algebras of order $F$. From now on we consider only elementary Lie algebras of order 3.

In \cite{9} an inductive process for the construction of Lie algebras of order $F$ starting from a Lie algebra of order $F_1$ with $1 \leq F_1$ was proved.

**Theorem II.2** Let $g_0$ be a Lie algebra and $g_1$ be $g_0$-module such that

(i) $g = g_0 \oplus g_1$ is a Lie algebra of order $F_1 > 1$;

(ii) $g_1$ admits a $g_0$-equivariant symmetric form of order $F_2 > 1$.

Then $g = g_0 \oplus g_1$ admits a Lie algebra of order $F_1 + F_2$ structure.

We give now two examples of Lie algebras of order 3 obtained through this construction.

**Example II.3** Let $g_0$ be a semi-simple Lie algebra and $g_1$ its adjoint representation. Let

$\{J_a, a = 1, \cdots, \dim g_0\}$ be a basis of $g_0$ and $\{A_a, a = 1, \cdots, \dim g_0\}$ be the corresponding basis of $g_1$. Let $g_{ab} = Tr(A_aA_b)$ be the Killing form and $f_{abc}$ be the structure constants of $g_0$. Then one can endow $g = g_0 \oplus g_1$ with a Lie algebra of order 3 structure

$$
[J_a, J_b] = f_{ab}^\ c J_c, \quad [J_a, A_b] = f_{ab}^\ c A_c, \quad \{A_a, A_b, A_c\} = g_{ab}J_c + g_{ac}J_b + g_{bc}J_a.
$$

(2)

(This example is a consequence of the Theorem above, modified to include $F_1 = 1$.)
Example II.4 Let \( \mathfrak{g}_0 = \langle L_{\mu\nu}, P_\mu \rangle \) be the \( D \)-dimensional Poincaré algebra and \( \mathfrak{g}_1 = \langle V_\mu \rangle \) be the \( D \)-dimensional vector representation of \( \mathfrak{g}_0 \). The brackets

\[
\begin{align*}
[L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\sigma} L_{\rho\nu} + \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma}, \\
[L_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \\
[L_{\mu\nu}, V_\rho] &= \eta_{\nu\rho} V_\mu - \eta_{\mu\rho} V_\nu, \\
[P_\mu, V_\nu] &= 0, \\
\{V_\mu, V_\nu, V_\rho\} &= \eta_{\mu\nu} P_\rho + \eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu,
\end{align*}
\]

with the metric \( \eta_{\mu\nu} = \text{diag}(1, -1, \cdots, -1) \) endow \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) with an elementary Lie algebra of order 3 structure.

It has been shown that Example II.3 (with \( \mathfrak{g}_0 = \mathfrak{so}(2,3) \)) and Example II.4 (when \( D = 4 \)) are related through an Inönü-Wigner contraction [9]. The algebra of Example II.4, firstly introduced in [9], has been studied in [10] together with its implementation in Quantum Field Theory when \( D = 4 \). Subsequently it has been realised [13] that in arbitrary dimension this algebra acts in a natural geometric way on generalised gauge field or \( p \)-forms.

III. EXTENSION OF THE POINCARÉ ALGEBRA

The \((1 + 3)\)-dimensional Poincaré algebra \( \text{iso}(1,3) \) is given by

\[
[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\sigma} L_{\rho\nu} + \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma}, \quad [L_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu.
\]

(3)

As it is well known the following change of basis (precisely in the complexified of the Lorentz algebra) \( L_{\mu\nu} \rightarrow (J_i = \frac{1}{2} \varepsilon_{ijk} L_{jk}, K_i = L_{0i}) \rightarrow (N_i = \frac{1}{2} (J_i + iK_i), \bar{N}_i = \frac{1}{2} (J_i - iK_i)) \) leads to \( [N_i, N_j] = \varepsilon_{ij}^k N_k, \ [\bar{N}_i, \bar{N}_j] = \varepsilon_{ij}^k \bar{N}_k, \ [N_i, \bar{N}_j] = 0 \) (with \( \varepsilon_{ij}^k = \varepsilon_{ijk} \) the Levi-Civita tensor). Therefore, a finite dimensional irreducible representation of \( \mathfrak{so}(1,3) \) is specified by the eigenvalues of the two Casimir operators \( Q = N_1^2 + N_2^2 + N_3^2 \) and \( \bar{Q} = \bar{N}_1^2 + \bar{N}_2^2 + \bar{N}_3^2 \) which are \( a(a+1) \) and \( b(b+1) \) respectively. We denote \( \mathcal{D}_{a,b}, a, b \in \frac{1}{2}\mathbb{N} \) the corresponding representation of dimension \( (2a+1)(2b+1) \).

Now, from the Poincaré algebra \( \text{iso}(1,3) \) and a given (reducible) representation \( \mathfrak{g}_1 \) of \( \mathfrak{so}(1,3) \) we construct non-trivial extensions of the Poincaré algebra in three steps:

1. we extend the action of \( \mathfrak{so}(1,3) \) on \( \mathfrak{g}_1 \) to the action of \( \text{iso}(1,3) \) on \( \mathfrak{g}_1 \);

2. we study all possible \( \mathfrak{so}(1,3) \)-equivariant mappings from \( S^3(\mathfrak{g}_1) \rightarrow \mathcal{D}_{a,b} \);
A. Finite dimensional representations of the Poincaré algebra

Let \( g_1 = \oplus_i D_{a_i,b_i} \) be an arbitrary reducible finite dimensional representation of \( \mathfrak{so}(1,3) \). We would like to extend the action of \( \mathfrak{so}(1,3) \) on \( g_1 \), to the action of the Poincaré algebra on \( g_1 \). Namely, we would like to calculate \([\mathcal{P}_\mu, D_{a,b}]\). Usually, in field theory, starting from a finite dimensional representation \( g_1 \) of \( \mathfrak{so}(1,3) \), a non-trivial representation of \( \mathfrak{iso}(1,3) \) is realised in terms of an infinite dimensional representation, the field, with \( \mathcal{P}_\mu = \partial_\mu \). Here, we would like to see whether or not \( \mathcal{P}_\mu \) may act non-trivially on \( g_1 \) i.e. can be represented by finite dimensional matrices.

If \( g_1 \) is an irreducible representation then one can show that \( \mathcal{P}_\mu \) acts trivially on \( g_1 \), i.e. \([\mathcal{P}_\mu, g_1] = 0\) \cite{12}. The case where \( g_1 \) is reducible is more involved as can be seen of the following two examples:

1. If \( g_1 = D_{1/2,1/2} \oplus D_{0,0} \) the vector plus the scalar representations of \( \mathfrak{so}(1,3) \), \( \mathcal{P}_\mu \) can be represented by the \( 5 \times 5 \) nilpotent matrices

\[
\mathcal{P}_\mu = \begin{pmatrix}
0 & 0 & 0 & 0 & \delta_\mu^0 \\
0 & 0 & 0 & \delta_\mu^1 & 0 \\
0 & 0 & \delta_\mu^2 & 0 & 0 \\
0 & \delta_\mu^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Indeed, if we denote \( \langle v_\mu, \mu = 0, \ldots, 3 \rangle \) (resp. \( \langle w_0 \rangle \)) a basis of \( D_{1/2,1/2} \) (resp. \( D_{0,0} \)) we have \( \mathcal{P}_\mu(w_0) = v_\mu, \mathcal{P}_\mu(v_\nu) = 0 \).

2. If \( g_1 = D_{1/2,0} \oplus D_{0,1/2} \), since \( D_{1/2,1/2} \otimes D_{1/2,1/2} = D_{1/2,1} \oplus D_{1/2,0} \supset D_{1/2,0} \), \( \mathcal{P}_\mu \) can be represented by the \( 4 \times 4 \) nilpotent matrices \( \mathcal{P}_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ 0 & 0 \end{pmatrix} \)

(with \( \sigma_0 \) the \( 2 \times 2 \) identity matrix and \( \sigma_i, i = 1,2,3 \) the Pauli matrices) such that for \( \psi \in D_{1/2,0}, \chi \in D_{0,1/2} \) we have \( \mathcal{P}_\mu(\psi) = 0, \mathcal{P}_\mu(\chi) = \sigma_\mu \chi \in D_{1/2,0} \).

The general case can be more complicated and his synthesize in Lemma 5.1 of \cite{12}. Here, we just recall the main properties of this technical Lemma. Let \( g_1 \) be a finite-dimensional
(reducible) representation of \( \mathfrak{so}(1, 3) \), such that the space-time translations act non-trivially on \( g_1 \).

Then,

1. \( P_\mu \) are represented by nilpotent matrices;

2. if \( \mathcal{D}_{a,b} \) and \( \mathcal{D}_{c,d} \) are two irreducible representations such that \( P_\mu : \mathcal{D}_{a,b} \rightarrow \mathcal{D}_{c,d} \) non-trivially, then \( \mathcal{D}_{c,d} \subseteq \mathcal{D}_{a,b} \otimes \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \);

3. the representation \( g_1 \) is indecomposable.

B. \( \mathfrak{so}(1, 3) \)–equivariant mappings

1. Construction of \( \mathfrak{so}(1, 3) \)–equivariant mappings

Next, we construct the possible \( \mathfrak{so}(1, 3) \)–equivariant mappings from \( S^3(g_1) \) into \( \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \), with \( g_1 \) an arbitrary representation of \( \mathfrak{so}(1, 3) \). We recall the following isomorphisms of representations of \( GL(A) \times GL(B) \) [14] (p. 80):

\[
S^3(A \oplus B \oplus C) \cong S^3(A) \oplus S^3(B) \oplus S^3(C) \oplus S^2(A) \otimes B \oplus S^2(B) \otimes C \oplus S^2(B) \otimes A \\
\oplus S^2(B) \otimes C \oplus S^2(C) \otimes A \oplus S^2(C) \otimes B \oplus A \otimes B \otimes C,
\]

\[
S^3(A \otimes B) \cong S^3(A) \otimes S^3(B) \oplus S^2(A) \otimes S^2(B) \oplus \Lambda^2(A) \otimes \Lambda^2(B),
\]

\[
S^2(A \otimes B) \cong S^2(A) \otimes S^2(B) \oplus \Lambda^2(A) \otimes \Lambda^2(B),
\]

where \( S^p(D) \) (resp. \( \Lambda^p(D), S^p(D) \)) denotes the irreducible representations \( GL(A) \) symmetric (resp. antisymmetric, corresponding to the Young symmetrizer of the Young diagram \( \boxed{\text{ } } \)).

Let \( g_1 \) be a representation of \( \mathfrak{so}(1, 3) \) and let \( \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \) be the vector representation of \( \mathfrak{so}(1, 3) \). Using the first equation given in [14], since \( g_1 \) is a reducible representation of \( \mathfrak{so}(1, 3) \), \( S^3(g_1) \) reduces to three types of terms (i) \( S^3(\mathcal{D}) \), (ii) \( S^2(\mathcal{D}) \otimes \mathcal{D}' \) and (iii) \( \mathcal{D} \otimes \mathcal{D}' \otimes \mathcal{D}'' \) with \( \mathcal{D}, \mathcal{D}', \mathcal{D}'' \) three irreducible representations. Thus all possible \( \mathfrak{so}(1, 3) \)–equivariant mappings are of the type (i) \( S^3(\mathcal{D}) \rightarrow \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \), (ii) \( S^2(\mathcal{D}) \otimes \mathcal{D}' \rightarrow \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \) and (iii) \( \mathcal{D} \otimes \mathcal{D}' \otimes \mathcal{D}'' \rightarrow \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \). We now
characterise more precisely these mappings. (From now one, \( \mathcal{D}_{a,b} \) is written \( \mathcal{D}_{a,b} = \mathcal{D}_{a,0} \otimes \mathcal{D}_{0,b} \) with \([Q, \mathcal{D}_{a,0}] = a(a + 1)\mathcal{D}_{a,0}, \ [Q, \mathcal{D}_{0,b}] = 0, \ [\bar{Q}, \mathcal{D}_{a,0}] = 0, \ [\bar{Q}, \mathcal{D}_{0,b}] = b(b + 1)\mathcal{D}_{0,b} \) where \( Q, \bar{Q} \) are the two Casimir operators of \( \mathfrak{so}(1,3) \).)

(i) Type I. \( \mathfrak{so}(1,3) \)– equivariant mappings: \( S^3(\mathcal{D}) \longrightarrow D_{1,\frac{1}{2}}^{3,1} \)

Let \( \mathcal{D} = \mathcal{D}_{a,b} = \mathcal{D}_{a,0} \otimes \mathcal{D}_{0,b} \) with \( a, b \in \frac{1}{2} \mathbb{N} \), \( D_{1,\frac{1}{2}}^{3,1} \subseteq D_{a,b} \otimes D_{a,b} \otimes D_{a,b} \) if \( a \) and \( b \) are half-integer. From the second equation of (I) \( D_{\frac{1}{2},\frac{1}{2}}^{3,1} \subseteq S^3(\mathcal{D}_{a,b}) \) if either

\[
I_S : D_{\frac{1}{2},0} \subseteq S^3(\mathcal{D}_{a,0}) \text{ and } D_{0,\frac{1}{2}} \subseteq S^3(\mathcal{D}_{0,b});
\]

\[
I_A : D_{\frac{1}{2},0} \subseteq \Lambda^3(\mathcal{D}_{a,0}) \text{ and } D_{0,\frac{1}{2}} \subseteq \Lambda^3(\mathcal{D}_{0,b});
\]

\[
I_M : D_{\frac{1}{2},0} \subseteq S(\mathcal{D}_{a,0}) \text{ and } D_{0,\frac{1}{2}} \subseteq S(\mathcal{D}_{0,b}).
\]

In particular, when \( \mathcal{D} = \mathcal{D}_{a,a} \) and \( a \) a half-integer, the mapping \( S^3(\mathcal{D}_{a,a}) \longrightarrow D_{\frac{1}{2},\frac{1}{2}} \) is always \( \mathfrak{so}(1,3) \)– equivariant and is called type \( I_0S, I_0A, I_0M \) respectively. The extension of the Poincaré algebra given in Example IIA is of type \( I_0M \) with \( a = b = \frac{1}{2} \).

(ii) Type II. \( \mathfrak{so}(1,3) \)– equivariant mappings: \( S^2(\mathcal{D}) \otimes \mathcal{D}' \longrightarrow D_{1,\frac{1}{2}}^{3,1} \)

Let \( \mathcal{D} = \mathcal{D}_{a,b} \) and \( \mathcal{D}' = \mathcal{D}_{c,d} \), \( D_{\frac{1}{2},\frac{1}{2}}^{3,1} \subseteq \mathcal{D}_{a,b} \otimes \mathcal{D}_{a,b} \otimes \mathcal{D}_{a,b} \otimes \mathcal{D}_{c,d} \) if \( c, d \) are half-integer and there exists an \( n = 0, \cdots, 2a \) such that \( 2a - n - b = \frac{1}{2} \) or \( b - 2a + n = \frac{1}{2} \). From the third equation of (I) \( D_{\frac{1}{2},\frac{1}{2}}^{3,1} \subseteq S^2(\mathcal{D}_{a,b}) \otimes \mathcal{D}_{c,d} \) if either

\[
II_S : D_{\frac{1}{2},0} \subseteq S^2(\mathcal{D}_{a,0}) \otimes \mathcal{D}_{c,0} \text{ and } D_{0,\frac{1}{2}} \subseteq S^2(\mathcal{D}_{0,b}) \otimes \mathcal{D}_{0,d};
\]

\[
II_A : D_{\frac{1}{2},0} \subseteq \Lambda^2(\mathcal{D}_{a,0}) \otimes \mathcal{D}_{c,0} \text{ and } D_{0,\frac{1}{2}} \subseteq \Lambda^2(\mathcal{D}_{0,b}) \otimes \mathcal{D}_{0,d}.
\]

We call these \( \mathfrak{so}(1,3) \)– equivariant mappings, mappings of type \( II_S \) and \( II_A \) respectively.

(iii) Type III. \( \mathfrak{so}(1,3) \)– equivariant mappings: \( \mathcal{D} \otimes \mathcal{D}' \otimes \mathcal{D}'' \longrightarrow D_{1,\frac{1}{2}}^{3,1} \).

Let \( \mathcal{D} = \mathcal{D}_{a,b}, \mathcal{D}' = \mathcal{D}_{c,d} \) and \( \mathcal{D}'' = \mathcal{D}_{e,f} \) (\( a \geq c \geq e \)). If \( a + c + e \) is half-integer and if there exists an \( n = 0, \cdots, 2c \) such that \( a + c - n - e = \frac{1}{2} \) or \( e - a - c + n = \frac{1}{2} \) (plus similar relations for \( b, d, f \)) then \( D_{\frac{1}{2},\frac{1}{2}}^{3,1} \subseteq \mathcal{D}_{a,b} \otimes \mathcal{D}_{c,d} \otimes \mathcal{D}_{e,f} \). Thus, there are many \( \mathfrak{so}(1,3) \)–equivariant
mappings of these types.

2. Explicit construction using Young projectors

In this subsection, we give an explicit $\mathfrak{so}(1,3)$-equivariant mapping by mean of Young projectors. Let $\mathcal{D} = \mathcal{D}_{1,0} \otimes \mathcal{D}_{0,1}$. Using conventional notations for spinors, let $\mathcal{D}_{1,0} = \langle \psi_\alpha, \alpha = 1, 2 \rangle$ and $\mathcal{D}_{0,1} = \langle \bar{\chi}^\dot{\alpha}, \dot{\alpha} = 1, 2 \rangle$ be the spinor representations of $\mathfrak{so}(1,3)$. We introduce the Dirac $\Gamma-$matrices

$$\Gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix},$$

with $\sigma_\mu = (\sigma_0, \sigma_i)$ and $\bar{\sigma}_\mu = (\sigma_0, -\sigma_i)$. The index structure of the $\sigma-$matrices is as follow $\sigma_\mu \rightarrow \sigma_{\mu \alpha \dot{\alpha}}, \bar{\sigma}_\mu \rightarrow \bar{\sigma}_{\mu \dot{\alpha} \alpha}$. We also define $\psi_\alpha = \varepsilon_{\alpha \beta} \psi_\beta, \bar{\psi}^\alpha = \varepsilon^{\alpha \beta} \bar{\psi}_\beta, \bar{\chi}^{\dot{\alpha}} = \bar{\varepsilon}_{\dot{\alpha} \dot{\beta}} \bar{\chi}^\dot{\beta}, \chi^{\dot{\alpha}} = \varepsilon^{\dot{\alpha} \dot{\beta}} \bar{\chi}_\dot{\beta}$ with the antisymmetric matrices $\varepsilon, \bar{\varepsilon}$ given by $\varepsilon_{12} = \bar{\varepsilon}_{12} = -1, \epsilon^{12} = \bar{\epsilon}^{12} = 1$.

Now, we consider the representation

$$\mathcal{D}'_{1,0} \cong \mathfrak{S}_3 \left( \mathcal{D}_{1,0} \right).$$

We introduce the projector (Young symmetriser)$^1$

$$P = \frac{1}{3}(1 - (12))(1 + (13)) = \frac{1}{3}(1 - (12) + (13) - (123))$$

with

$$(a \ b) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, (a \ b \ c) = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

two cycles of $\mathfrak{S}_3$, the group of permutation with three elements. A direct calculation gives

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$^1$ Usually the other convention is taken for the Young symmetriser : $P = \frac{1}{3}(1 + (13))(1 - (12))$. But here it is more convenient to choose $\frac{1}{3}(1 - (12))(1 + (13))$ for the calculus.
\[ P(\psi_\alpha \otimes \psi_\beta \otimes \psi_\gamma) = \varepsilon_{\alpha\beta\lambda} \lambda_\gamma \] (6)

with \( D'_{\frac{1}{2}, 0} = \langle \lambda_\alpha, \alpha = 1, 2 \rangle \), \( \lambda_\alpha = (\psi_1 \otimes \psi_2 - \psi_2 \otimes \psi_1) \otimes \psi_\alpha - \psi_\alpha \otimes (\psi_1 \otimes \psi_2 - \psi_2 \otimes \psi_1) \) (the same result can be obtained using the usual calculus of the Clebsch-Gordan coefficients).

Proceeding along the same lines with \( D_{0, \frac{1}{2}} \) and introducing \( D'_{0, \frac{1}{2}} \)

\[ D'_{0, \frac{1}{2}} \cong \{ D_{0, \frac{1}{2}} \} = \langle \bar{\rho}^\alpha, \alpha = 1, 2 \rangle \]

we obtain

\[ P(\psi_\alpha \otimes \psi_\beta \otimes \psi_\gamma) \otimes P(\bar{\chi}_\dot{\alpha} \otimes \bar{\chi}_\dot{\beta} \otimes \bar{\chi}_\dot{\gamma}) = \varepsilon_{\alpha\beta\dot{\gamma}} \bar{\varepsilon}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \lambda_\gamma \otimes \bar{\rho}_\gamma. \]

Symmetrising the R.H.S. we then get

\[ S(\psi_\alpha \otimes \bar{\chi}_\dot{\alpha}) \otimes (\psi_\beta \otimes \bar{\chi}_\dot{\beta}) \otimes (\psi_\gamma \otimes \bar{\chi}_\dot{\gamma}) = \varepsilon_{\alpha\beta\dot{\gamma}} \bar{\varepsilon}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \lambda_\gamma \otimes \bar{\rho}_\gamma + \varepsilon_{\gamma\alpha\dot{\gamma}} \bar{\varepsilon}_{\dot{\gamma}\dot{\alpha}\dot{\gamma}} \lambda_\beta \otimes \bar{\rho}_\beta + \varepsilon_{\beta\gamma\dot{\gamma}} \bar{\varepsilon}_{\dot{\beta}\dot{\gamma}\dot{\gamma}} \lambda_\alpha \otimes \bar{\rho}_\alpha. \] (7)

Now, from the isomorphism of \( D_{\frac{3}{2}, 0} \otimes D_{0, \frac{1}{2}} \) with the vector representation, we have the correspondence

\[ V_\mu = \bar{\sigma}_\mu \dot{\alpha} \psi_\alpha \otimes \bar{\chi}_\dot{\alpha}, \quad \psi_\alpha \otimes \bar{\chi}_\dot{\alpha} = \frac{1}{2} \sigma_\mu \alpha \lambda_\alpha V_\mu; \]
\[ P_\mu = \bar{\sigma}_\mu \dot{\alpha} \lambda_\alpha \otimes \bar{\rho}_\dot{\alpha}, \quad \lambda_\alpha \otimes \bar{\rho}_\dot{\alpha} = \frac{1}{2} \sigma_\mu \alpha \lambda_\alpha P_\mu, \] (8)

(thus \( \langle P_\mu, \mu = 0, \cdots, 3 \rangle \sim D_{\frac{3}{2}, \frac{1}{2}}, \langle V_\mu, \mu = 0, \cdots, 3 \rangle \sim D'_{\frac{3}{2}, \frac{1}{2}} \)) and equations (7) reduce to

\[ S(V_\mu \otimes V_\nu \otimes V_\rho) = \eta_{\mu\nu} P_\rho + \eta_{\nu\rho} P_\mu + \eta_{\rho\mu} P_\nu, \] (9)

which is the trilinear bracket of the algebra given in Example II.4.
3. Decomposition of $\mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D}$ with respect to symmetry of Young diagrams

The $\mathfrak{so}(1,3)$—equivariant mappings of type $I$ lead immediately to an interesting question. Let $\mathcal{D}_a$ be an irreducible representation of $\mathfrak{sl}(2)^2$. It is well known that $\mathcal{D}_a \otimes \mathcal{D}_a = S^2(\mathcal{D}_a) \oplus \Lambda^2(\mathcal{D}_a)$ with $S^2(\mathcal{D}_a) = \mathcal{D}_{2a} \oplus \mathcal{D}_{2a-2} \oplus \cdots$, $\Lambda^2(\mathcal{D}_a) = \mathcal{D}_{2a-1} \oplus \mathcal{D}_{2a-3} \oplus \cdots$, that is one can identified each irreducible summand of $\mathcal{D}_a \otimes \mathcal{D}_a$ in the symmetric or antisymmetric part of $\mathcal{D}_a \otimes \mathcal{D}_a$. If we now consider $\mathcal{D}_a \otimes \mathcal{D}_a \otimes \mathcal{D}_a$ similarly we have the following decomposition

$$\mathcal{D}_a \otimes \mathcal{D}_a \otimes \mathcal{D}_a = S^3(\mathcal{D}_a) \oplus S^3(\mathcal{D}_a) \oplus \Lambda^3(\mathcal{D}_a)$$

and we would like to identify in which symmetry of the Young tableau is a given irreducible summand. To answer to this question, we firstly have to recall some known results on the character.

Let $\mathfrak{g}$ be a semisimple Lie algebra and denote $\Lambda$ the weight lattice. The integral group ring $\mathbb{Z}[\Lambda]$ on the abelian group $\Lambda$ is the free $\mathbb{Z}$—module $\mathbb{Z}^{(\Lambda)}$. If we write $e^\lambda$ the basis element of $\mathbb{Z}[\Lambda]$ corresponding to the weight $\lambda$ we have the multiplication law $e^\lambda e^\mu = e^{\lambda+\mu}$. If now, we introduce the representation ring $R(\mathfrak{g})$ the character homomorphism is defined by

$$\text{ch} : R(\mathfrak{g}) \rightarrow \mathbb{Z}[\Lambda]$$

$$V \mapsto \sum \dim V_\lambda e^\lambda$$

where $V_\lambda$ is the weight space corresponding to the weight $\lambda$ of the representation space $V$ [14] (p. 375). The representation ring also has the structure of $\lambda$—ring, this allows to express the character of the symmetric and exterior power through the formulæ

$$\sum_{n \in \mathbb{N}} \text{ch} (S^n(V)) T^n = \exp \left( \sum_{m \geq 1} \frac{1}{m} \Psi^m(\text{ch}(V)) T^m \right),$$

$$\sum_{n \in \mathbb{N}} \text{ch} (\Lambda^n(V)) T^n = \exp \left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \Psi^m(\text{ch}(V)) T^m \right),$$

(11)

---

2 Since $\mathfrak{so}(1,3,\mathbb{C}) \sim \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

3 This formal character can be identified with the usual character of the corresponding representation of the Lie group $G$ (associated to the Lie algebra $\mathfrak{g}$), restricted to the Cartan subgroup [14] (p. 381).
where $\Psi^m$ is the Adams linear operator $\Psi^m(e^\lambda) = e^{m\lambda}$ (exercise 11, p. 231). Developing both side of (11) up to $T^3$ we get

$$
\begin{align*}
\text{ch}(S^2(V)) &= \frac{1}{2} \Psi^2(\text{ch}V) + \frac{1}{2}(\text{ch}V)^2 \\
\text{ch}(S^3(V)) &= \frac{1}{3} \Psi^3(\text{ch}V) + \frac{1}{2} \text{ch}V \Psi^2(\text{ch}V) + \frac{1}{6}(\text{ch}V)^3
\end{align*}
$$

(12)

and

$$
\begin{align*}
\text{ch}(\Lambda^2(V)) &= -\frac{1}{2} \Psi^2(\text{ch}V) + \frac{1}{2}(\text{ch}V)^2 \\
\text{ch}(\Lambda^3(V)) &= \frac{1}{3} \Psi^3(\text{ch}V) - \frac{1}{2} \text{ch}V \Psi^2(\text{ch}V) + \frac{1}{6}(\text{ch}V)^3.
\end{align*}
$$

(13)

Finally, since

$$(\text{ch}V)^3 = \text{ch}(S^3(V)) + \text{ch}(S^2(V) + \text{ch}(\Lambda^3(V))),$$

we get

$$
\text{ch}(S^2(V)) = -\frac{2}{3} \Psi^3(\text{ch}V) + \frac{2}{3}(\text{ch}V)^3.
$$

(14)

For $\mathfrak{sl}(2)$ the weight lattice in one dimensional, and if we denote $(e^n)_{n \in \mathbb{Z}}$ a basis of the integral ring group $\mathbb{Z}[\Lambda]$, for the representation $\mathcal{D}_a$ the character (10) reduces to

$$
\text{ch}(\mathcal{D}_a) = e^{-a} + e^{-a+1} + \cdots + e^{a-1} + e^a.
$$

(15)

Let us mention that (15) also comes from the Weyl character formula. Indeed, if we denote $\mu$ the positive weight of $\mathfrak{sl}(2)$, $\rho = \sum_{\text{positive weight } \lambda} \frac{\lambda}{2} = \frac{\mu}{2}$. For the representation $\mathcal{D}_a$ the highest weight is $\lambda = a\mu$ and the Weyl character formula simplifies to

$$
\text{ch}(\mathcal{D}_a) = \frac{e^{a\mu} - e^{-(a\mu)}}{e^\rho - e^{-\rho}}
$$

and gives (15) (with $e^\ell \to e^{\mu\ell}$). Finally, Eqs (12)-(13)-(14) enable us to decompose each irreducible summand of $\mathcal{D}_a \otimes \mathcal{D}_a \otimes \mathcal{D}_a$ with respect to the symmetry of Young tableau. For instance, if $\mathcal{D}_a = \mathcal{D}_{\frac{1}{2}}$, one obtains
\[ S^3(D_{\frac{1}{2}}) = D_{\frac{1}{2}}, \quad S^3(D_{\frac{3}{2}}) = D_{\frac{1}{2}} \oplus D_{\frac{3}{2}}. \] (16)

and for \( D_{\frac{3}{2}} \) one gets

\[ S^3(D_{\frac{3}{2}}) = D_{\frac{3}{2}} \oplus D_{\frac{1}{2}} \oplus D_{\frac{3}{2}}, \quad S^3(D_{\frac{3}{2}}) = 2D_{\frac{3}{2}} \oplus 2D_{\frac{1}{2}} \oplus 2D_{\frac{3}{2}}, \quad A^3(D_{\frac{3}{2}}) = D_{\frac{3}{2}}. \] (17)

Thus only a type \( I_M \) mapping can be obtained with \( D_{\frac{1}{2}} \) or \( D_{\frac{3}{2}} \), and the decomposition (16) corresponds to (0). In fact, more generally, it can be proven when \( a \in \frac{1}{2}\mathbb{N} \) only types \( I_M \) mappings are allowed.

C. Lie algebras of order three associated with the Poincaré algebra

Combining the two results established in this section (study of the finite dimensional representations of \( \text{iso}(1,3) \)) and study of the \( \text{so}(1,3) \)–equivariant mappings from \( S^3(g_1) \rightarrow D_{\frac{1}{2},\frac{1}{2}} \) non-trivial extensions of the Poincaré algebra associated to a Lie algebra of order 3, \( g = \text{iso}(1,3) \oplus g_1 \) are obtained as follow.

1. We consider \( g_1 \) a given finite-dimensional (reducible) representation of \( \text{so}(1,3) \) such that there exists an \( \text{so}(1,3) \)–equivariant mapping from \( S(g_1) \) into \( D_{\frac{1}{2},\frac{1}{2}} \) (of type I and/or type II and/or type III above).

2. We extend the action of \( \text{so}(1,3) \) onto \( g_1 \) to an action of the Poincaré algebra \( \text{iso}(1,3) \) on \( g_1 \) as in Lemma 5.1 of [12].

3. Assuming that \( g = \text{iso}(1,3) \oplus g_1 \) is a Lie algebra of order 3, means that some identities have to be satisfied. These identities come from the point 1.-3. in the Definition II.1 and from the identity (1):

\[
\begin{align*}
0 &= [[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2], \quad J_1 \\
0 &= [[X_1, X_2], Y_3] + [[X_2, Y_3], X_1] + [[Y_3, X_1], X_2], \quad J_2 \\
0 &= [X_1, \{Y_1, Y_2, Y_3\}] - \{[X_1, Y_1], Y_2, Y_3\} - \{Y_1, [X_1, Y_2], Y_3\} - \{Y_1, Y_2, [X_1, Y_3]\}, \quad J_3 \\
0 &= [Y_1, \{Y_2, Y_3, Y_4\}] + [Y_2, \{Y_3, Y_4, Y_1\}] + [Y_3, \{Y_4, Y_1, Y_2\}] + [Y_4, \{Y_1, Y_2, Y_3\}], \quad J_4
\end{align*}
\]
(for all \(X_1, X_2, X_3 \in \mathfrak{so}(1, 3), Y_1, Y_2, Y_3, Y_4 \in \mathfrak{g}_1\)). The identities \(J_1-J_2\) are satisfied by definition and the identity \(J_3\) by construction. However, the identity \(J_4\) put severe constraints. Indeed, imposing \(J_4\) automatically leads to a trivial action of \(P_\mu\) on \(\mathfrak{g}_1\) i. e. \([P_\mu, \mathfrak{g}_1] = 0\) \[12\]. (It should be noticed that if from the very beginning we have assumed that the generators of space-time translations commute with the generators of \(\mathfrak{g}_1\), the identity \(J_4\) would have been trivially satisfied.)

The point 1-3 above means that any representation \(\mathfrak{g}_1\) of \(\mathfrak{so}(1, 3)\) such that \(D_{\frac{1}{2}, \frac{1}{2}} \subset S^3(\mathfrak{g}_1)\) leads to a possible non-trivial extension of the Poincaré algebra. At that point among the three types of brackets of a Lie algebra of order 3, (1) \([\mathfrak{so}(1, 3), \mathfrak{so}(1, 3)] \subset \mathfrak{so}(1, 3)\), (2) \([\mathfrak{so}(1, 3), \mathfrak{g}_1] \subset \mathfrak{g}_1\) and \(\{\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1\} \subset D_{\frac{1}{2}, \frac{1}{2}}\), only the last ones are still unspecified. Indeed, we just know that there exists an \(\mathfrak{so}(1, 3)\)–equivariant mapping from \(S^3(\mathfrak{g}_1)\) into \(D_{\frac{1}{2}, \frac{1}{2}}\). The precise form of these brackets, not yet known, is obtained with identity \(J_3\). The point 1. above ensures that the identity \(J_3\) will give non-trivial brackets.

For instance, if one start with \(\mathfrak{g}_1 = D_{\frac{1}{2}, \frac{1}{2}} = \langle V_\mu \rangle\), we know that there exists an \(\mathfrak{so}(1, 3)\)–equivariant mapping \(S^3(D_{\frac{1}{2}, \frac{1}{2}}) \rightarrow D_{\frac{1}{2}, \frac{1}{2}}\). To give the precise form of the trilinear brackets, we introduce the eigenvectors of the Cartan subalgebra of \(\mathfrak{so}(1, 3)\):

\[
V_{++}, V_{+-}, V_{--} = \gamma_1 P_{--}, \quad V_{-+}, V_{-}, V_{+} = \gamma_2 P_{--},
\]

\[
V_{++}, V_{-+}, V_{-} = \gamma_1 P_{--}, \quad V_{-+}, V_{-}, V_{+} = \gamma_2 P_{--},
\]

\[
V_{++}, V_{-}, V_{+} = \gamma_1 P_{--}, \quad V_{++}, V_{--}, V_{+} = \gamma_2 P_{--}.
\]

Imposing the Jacoby identity \(J_3\) we obtain \(\alpha_2 = -\frac{1}{2} \alpha_1, \beta_1 = \alpha_1, \beta_2 = -\frac{1}{2} \alpha_1, \gamma_1 = \frac{1}{2} \alpha_1, \gamma_2 = -\alpha_1, \delta_1 = \frac{1}{2} \alpha_1, \delta_2 = -\alpha_1\), and the brackets \[18\] reproduce the trilinear brackets of Example \[11\]: \(\{V_\mu, V_\nu, V_\rho\} = \eta_{\mu\rho} P_\rho + \eta_{\nu\rho} P_\rho + \eta_{\mu\nu} P_\nu\) \[12\]. Thus, the only Lie algebra of order 3 associated to \(\mathfrak{g} = \mathfrak{so}(1, 3) \oplus D_{\frac{1}{2}, \frac{1}{2}}\) is the Lie algebra of order 3 given in Example \[11\].

This general study shows that many Lie algebras of order 3 extending non-trivially the \((1 + 3)\)–dimensional Poincaré algebra can be defined.
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