PROFILE OF A TOUCH-DOWN SOLUTION TO A NONLOCAL MEMS MODEL

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Abstract. In this paper, we are interested in the mathematical model of MEMS devices which is presented by the following equation on \((0, T) \times \Omega:\)

\[\frac{\partial u}{\partial t} = \Delta u + \frac{\lambda}{(1 - u)^2 \left(1 + \gamma \int_{\Omega} \frac{1}{1 - u} dx\right)^2}, \quad 0 \leq u < 1,\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) and \(\lambda, \gamma > 0.\) In this work, we have succeeded to construct a solution which quenches in finite time \(T\) only at one interior point \(a \in \Omega.\) In particular, we give a description of the quenching behavior according to the following final profile

\[1 - u(x, T) \sim \theta^* \left[\frac{|x - a|^2}{|\ln |x - a||}\right]^{\frac{1}{3}} \text{ as } x \to a, \theta^* > 0.\]

The construction relies on some connection between the quenching phenomenon and the blowup phenomenon. More precisely, we change our problem to the construction of a blowup solution for a related PDE and describe its behavior. The method is inspired by the work of Merle and Zaag \([13]\) with a suitable modification. In addition to that, the proof relies on two main steps: A reduction to a finite dimensional problem and a topological argument based on Index theory. The highlight of this work is that we handle the nonlocal integral term in the above equation. The interpretation of the finite dimensional parameters in terms of the blowup point and the blowup time allows to derive the stability of the constructed solution with respect to initial data.

1. Introduction.

We are interested in the motion of some elastic membranes which is usually found in Micro-Electro Mechanical System (MEMS) devices, which are available in a variety of electronic devices, such as microphones, transducers, sensors, actuators and so on. Described briefly, MEMS devices contain an elastic membrane which is hanged above a rigid ground plate connected in series with a fixed voltage source and a fixed capacitor. For more details on the physical background and possible applications, we refer the reader to \([3], [7], [9]\) and \([17].\)

For a MEMS device (in \([8]\) and \([6]\)), the distance between the rigid ground plate and the elastic membrane changes with time. It is referred to as the deflection of the membrane. Here, we assume that this distance is very small compared to the device. In fact, we can fully describe the behavior

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of the deflection the following hyperbolic equation

\[
\begin{cases}
\varepsilon^2 \partial_t u + \partial_t u = \Delta u + \frac{\lambda f(x, t)}{(1 - u)^2 \left(1 + \gamma \int_\Omega \frac{1}{1 - u} \, dx\right)^2}, & x \in \Omega, t > 0, \\
u(x, t) = 0, x \in \partial \Omega, t > 0, \\
u(x, 0) = u_0(x), x \in \Omega.
\end{cases}
\]

(1.1)

where \( \Omega \) is considered as the domain of the rigid plate, \( u \) is the deflection of the membrane to the plate, \( \lambda > 0, \gamma > 0 \) and \( f \) is continuous. Here, the distance between the rest position of the membrane and the rigid plate is normalized to 1. When the device is under voltage, \( u \) will vary in the interval \([0,1]\). In addition to that, the parameter \( \lambda \) represents the ratio of the reference electrostatic force to the reference elastic force and \( \varepsilon \) is the ratio of the interaction of the inertial and damping terms in our model. Moreover, the function \( f \) represents the varying dielectric properties of the membrane. See [4] for more details.

In fact, we are interested in a simpler case of (1.1) considered in the following parabolic equation:

\[
\begin{cases}
\partial_t u = \Delta u + \frac{\lambda}{(1 - u)^2 \left(1 + \gamma \int_\Omega \frac{1}{1 - u} \, dx\right)^2}, & x \in \Omega, t > 0, \\
u(x, t) = 0, x \in \Omega, t > 0, \\
u(x, 0) = u_0(x), x \in \Omega.
\end{cases}
\]

(1.2)

Moreover, we are also interested in the following generalization of problem (1.2):

\[
\begin{cases}
\partial_t u = \Delta u + \frac{\lambda}{(1 - u)^p \left(1 + \gamma \int_\Omega \frac{1}{1 - u} \, dx\right)^q}, & x \in \Omega, t > 0, \\
u(x, t) = 0, x \in \Omega, t > 0, \\
u(x, 0) = u_0(x), x \in \Omega,
\end{cases}
\]

(1.3)

where \( p, q > 0 \). Introducing \( Q_T = (0, T) \times \Omega \), where \( T > 0 \), we say that \( u \) is a classical solution of (1.2) if \( u \) is a function in \( C^2(Q_T) \cap C(\bar{Q}_T) \) that satisfies (1.2) at every point in \( Q_T \) as well as the boundary and initial conditions, with

\[
u(x, t) \in [0, 1], \forall x \in \Omega, t \in (0, T).
\]

The local Cauchy problem of (1.2) is solved (see Proposition 1.2.2 page 12 in [10]). Then, either our solution is global in time or there exists \( T > 0 \) such that

\[
\liminf_{t \to T} [1 - u(t, x)] = 0.
\]

(1.5)

We can see that if the above condition occurs, the right-hand side of (1.2) may become singular. This phenomenon is referred to as touch-down in finite time \( T \) in reference to the physical phenomenon, where the membrane ”touches” the rigid ground plate which is placed below. In fact, in our setting, we follow the literature and place the rigid plate at \( u = 1 \), above the membrane which is located at \( u(x, t) \). Note that in case of touch-down, the MEMS device breaks down.

Mathematically, we may refer to the behavior in (1.5) as finite-time quenching. Moreover, \( a \in \Omega \) is a quenching point if and only if there exist sequences \((a_n, t_n) \in \Omega \times (0, T)\) such that

\[
u(a_n, t_n) \to 1, \text{ as } n \to +\infty.
\]
The touch-down phenomenon has been strongly studied in recent decades. Let us for example mention the following result by Guo and Kavallaris in [4]:

Consider \( \Omega \) such that \( |\Omega| \leq \frac{1}{2} \). Then, for all \( \lambda > 0 \) fixed and \( \gamma > 0 \), there exist initial data with a small energy such that problem (1.2) has a solution which quenches in finite time.

In our paper, we are interested in proving a general quenching result with no restriction on any \( \lambda > 0, \gamma > 0 \) and \( C^2 \) bounded domain \( \Omega \). In fact, we do better and give a sharp description of the asymptotics of the solution near the quenching region. The following is the main result:

**Theorem 1.1** (Existence of a touch-down solution). Consider \( \lambda > 0, \gamma > 0 \) and \( \Omega \) a \( C^2 \) bounded domain in \( \mathbb{R}^n \), containing the origin. Then, there exist initial data \( u_0 \in C^\infty(\overline{\Omega}) \) such that the solution of (1.2) quenches in finite time \( T = T(u_0) > 0 \) only at the origin. In particular, the following holds:

(i) The intermediate profile: For all \( t \in [0, T) \)

\[
\frac{(T-t)^{\frac{1}{2}}}{1-u(.,t)} - \theta^* \left( 3 + \frac{9}{8} \frac{|.|^2}{\sqrt{(T-t)}} \frac{1}{(|\ln(T-t)|)} \right)^{-\frac{1}{4}} \leq \frac{C}{\sqrt{|\ln(T-t)|}},
\]

for some \( \theta^* = \theta^*(\lambda, \gamma, \Omega, T) > 0 \).

(ii) The final profile: There exists \( u^* \in C^2(\Omega) \cap C(\overline{\Omega}) \) such that \( u \) uniformly converges to \( u^* \) as \( t \to T \), and

\[
1 - u^*(x) \sim \theta^* \left[ \frac{9}{16} \frac{|x|^2}{|\ln |x||} \right]^{\frac{1}{7}} \text{ as } x \to 0.
\]

**Remark 1.2.** Note that if \( \gamma = 0 \), then our problem coincides with the work of Merle and Zaag [13]. Our paper in then meaningful when \( \gamma \neq 0 \), and the whole issue is how to control the non local term.

**Remark 1.3.** For simplicity, we choose to write our result when the solution quenches at the origin. of course, we can make it quenches at any arbitrary \( a \in \Omega \), simply replace \( x \) by \( x - a \) in the statement.

**Remark 1.4.** In Theorem 1.1, we can describe the evolution of our solution at \( x = 0 \) as follows:

\[
1 - u(0, t) \sim \left( \frac{T-t}{\sqrt{3\theta^*}} \right)^{\frac{5}{3}}, \text{ as } t \to T.
\]

**Remark 1.5.** Note that we can explicitly write the formula of the initial data

\[
u(x, 0) = \frac{\bar{u}(x, 0)}{\bar{u}(x, 0) + 1},
\]

where

\[
\bar{u}(x, 0) = \frac{\bar{\theta}(0)}{\lambda^\frac{1}{2}} U(x, 0),
\]

with

\[
U(x, 0) = T^{-\frac{1}{2}} \left[ \varphi \left( \frac{x}{\sqrt{T}} - \ln T \right) + (d_0 + d_1 \cdot z) \chi_0 \left( \frac{16|x|}{K_0} \right) \right] \chi_1(x, 0) + (1 - \chi_1(x, 0)) H^*(x),
\]

\[
z = \frac{x}{\sqrt{T |\ln T|}},
\]

\[
\chi_1(x, 0) = \chi_0 \left( \frac{|x|}{\sqrt{T |\ln T|}} \right),
\]

and

\[
\varphi(z) = \left( \frac{16|z|}{K_0} \right)^{\frac{1}{2}} \text{ for } |z| \geq 1,
\]

\[
\varphi(z) = 1 - \frac{1}{2} \left( \frac{16|z|}{K_0} \right) \text{ for } |z| \leq 1.
\]
and $\bar{\theta}(0)$ is the unique positive solution of the following equation
\[
\bar{\theta}(0) = \left(1 + \gamma |\Omega| + \frac{\gamma}{\sqrt{\lambda}} \int_{\Omega} U(0) dx\right)^{\frac{2}{3}},
\]
and $\chi_0, \varphi, H^*$ are defined in (2.16), (2.21), (3.12), respectively. Here, $T$ is small enough and the parameters $d_0, d_1$ are fine-tuned in order to get the desired behavior.

**Remark 1.6** (An open question). How big can $\theta^*$ be? This question is related to the work of Merle and Zaag in [13] which corresponds to the case where $\gamma = 0$. For that case, the answer is $\theta^* = \frac{1}{3} \sqrt[3]{\lambda}$. It is very interesting to answer the question in the general case. By a glance to (2.7), we know that $\theta^*$ is strictly greater than $(1 + \gamma |\Omega|)^{\frac{2}{3}}$. Let us define
\[
T_{\text{max}} = \left(\frac{(1 + \gamma |\Omega|)^{\frac{2}{3}}}{\sqrt{\lambda}}, +\infty\right),
\]
and
\[
T = \{\theta^* \in \mathbb{R} \text{ such that (1.6) holds with } u \text{ a positive solution to (1.2), for some } T > 0\}.
\]
Then, by a fine modification in the proof, we can construct a solution such that $\theta^*$ arbitrarily takes large values in $T_{\text{max}}$. In particular, we can prove that $T$ is a dense subset of $T_{\text{max}}$. We would like to make the following conjecture
\[
T = T_{\text{max}}.
\]

Now, we would like to mention that our proof of Theorem 1.1 holds in a more general setting. More precisely, if we consider problem (1.3) in the following regime
\[
n - \frac{2}{p + 1} > 0, \text{ and } q > 0 \text{ and } n \geq 1,
\]
then, Theorem 1.1 changes as follows:

**Theorem 1.7** (Existence of a touch-down solution to (1.3)). Consider $\lambda, \gamma > 0$, and $\Omega$ a $C^2$ bounded domain in $\mathbb{R}^n$ and condition (1.9) holds. Then, there exist initial data $\hat{u}_0$ in $C^\infty(\Omega)$ such that the solution of equation (1.3) touches down in finite time only at the origin. In particular, the following holds:

(i) The intermediate profile, for all $t \in [0, T)$
\[
\left\|\frac{(T - t)^{\frac{4}{p + 1}}}{1 - u(., t)} - \hat{\theta}^* \left(p + 1 + \frac{(p + 1)^2}{4p} \frac{|.|^2}{\sqrt{(T - t)|\ln(T - t)|}}\right)^{-\frac{1}{p + 1}}\right\|_{L^\infty(\Omega)} \leq \frac{C}{\sqrt{|\ln(T - t)|}},
\]
for some $\hat{\theta}^*(\lambda, \gamma, \Omega, p, q) > 0$.

(ii) The exists $\hat{u}^* \in C^2(\Omega) \cap C(\Omega)$ such that $u$ uniformly converges to $\hat{u}^*$ as $t \to T$, and
\[
1 - \hat{u}^*(x) \sim \hat{\theta}^* \left[\frac{(p + 1)^2}{8p} \frac{|x|^2}{|\ln|x||} \right]^{\frac{1}{p + 1}} \text{ as } x \to 0.
\]

**Remark 1.8.** We don’t give the proof of Theorem 1.7 here because the techniques are the same as for Theorem 1.1. In fact, for simplicity, we will only give the proof for the MEMS case $p = q = 2$, considered in equation (1.2) and Theorem 1.1. Of course, all our estimates can be carried on for the general case.
In addition to that, we can apply the techniques of Merle in [11] to create a solution which quenches at arbitrary given points.

**Corollary 1.9.** For any $k$ points $a_1, a_2, ..., a_k$ in $\Omega$, there exist initial data such that (1.3) has a solution which quenches exactly at $a_1, a_2, ..., a_k$. Moreover, the local behavior at each $a_i$ is also given by (1.10), (1.11) by replacing $x$ by $x - a_i$ and $L^\infty(\Omega)$ by $L^\infty(|x - a_i| \leq \omega_0)$, for some $\omega_0 > 0$, small enough.

As a consequence of our techniques, we can derive the stability of the quenching solution which we constructed in Theorem (1.7) under the perturbations of initial data.

**Theorem 1.10 (Stability of the constructed solution).** Let us consider $\hat{u}$, the solution which we constructed in Theorem 1.7 and we also define $\hat{T}$ as the quenching time of the solution and $\hat{\theta}^*$ as the coefficient in front of the profiles (1.10) and (1.11). Then, there exists a open subset $\hat{U}_0$ in $C(\bar{\Omega})$, containing $\hat{u}(0)$ such that for all initial data $u_0 \in \hat{U}_0$, equation (1.3) has a unique solution $u$ quenching in finite time $T(u_0)$ at only one quenching point $a(u_0)$. Moreover, the asymptotics (1.10) and (1.11) hold by replacing $\hat{u}(x,t)$ by $u(x - a(u_0), t)$, and $\hat{\theta}^*$ by some $\theta^*(u_0)$ . Note that, we have

\[(a(u_0), T(u_0), \theta^*(u_0)) \to (0, \hat{T}, \hat{\theta}), \text{ as } ||u_0 - \hat{u}_0||_{C(\bar{\Omega})} \to 0.\]

Let us now comment on the organization of the paper. As we have stated earlier, Theorem 1.1 is a special case of Theorem 1.7. For simplicity in the notations, we only prove Theorem 1.1. The interested reader may derive the general case by inspection. Moreover, we don’t prove Corollary 1.9 and Theorem 1.10, since the former follows from Theorem 1.7 and the techniques of Merle in [11], and the latter follows also from Theorem 1.7 by the method of Merle and Zaag in [14]. In conclusion, we only prove Theorem 1.1 in this paper.

The paper is organized as follows:
- In Section 2, we give a different formulation of the problem, and show how the profile in (1.6) arises naturally.
- In Section 3, we give the proof without technical details.
- In Section 4, we prove the technical details.

Some appendices are added at the end.

### 2. Setting of the problem

#### 2.1. Our main idea

We aim in this subsection at explaining our key idea in this paper. The rigorous proof will be given later. Introducing

\[
\alpha(t) = \frac{\lambda}{\left(1 + \gamma \int_{\Omega} \frac{1}{1 - u(t)} dx\right)^2},
\]

we rewrite (1.2) as the following

\[
\partial_t u = \Delta u + \frac{\alpha(t)}{(1 - u)^2}.
\]

Under this general form, we see our equation (1.2) as a step by step generalization, starting from a much simpler context:

- **Problem 1:** Case where $\alpha(t) \equiv \alpha_0$. This case was considered by Merle and Zaag in [13] where, the authors constructed a solution $u_{\alpha_0}$ satisfying

\[u_{\alpha_0}(x,t) \to 1 \text{ as } (x,t) \to (x_0,T),\]
for some $T > 0$, and $x_0 \in \Omega$. In particular, they gave a sharp description for the quenching profile. Technically, the authors in that work introduced 
\[
\bar{u} = \frac{1}{1-u} - 1 = \frac{u}{1-u},
\]
and constructed a blowup solution for the following equation derived from (2.2):
\[
\partial_t \bar{u} = \Delta \bar{u} - 2\frac{\left| \nabla \bar{u} \right|^2}{\bar{u}} + \alpha_0 \bar{u}^4, \quad \text{with } \alpha(t) \equiv \alpha_0,
\]
(see equation (III), page 1500 in [13] for more details).

- **Problem 2**: Case where $0 < \alpha_1 \leq \alpha(t) \leq \alpha_2$ for all $t > 0$ for some $0 < \alpha_1 < \alpha_2$. This case is indeed a reasonable generalization which follows with no difficulty from the strategy of [13] for Problem 1.

- **Problem 3**: Equation (1.2). Our idea here is to see (1.2) as a coupled system between Problem 2 and (2.1):

\[
\begin{align*}
\frac{\partial_t u}{u} &= \Delta u + \frac{\alpha(t)}{(1-u)^{\gamma}}, \\
\alpha(t) &= \lambda \left(1 + \gamma \int_{\Omega} \frac{1}{u} \right)^{\gamma},
\end{align*}
\]

A simple idea would be to try a kind of fixed-point argument starting from some solution to Problem 1, then defining $\alpha(t)$ according to (2.1) defined with this solution, then solving Problem 2 with this $\alpha(t)$ with the new solution, and so forth.

For this method to work, one has to check whether the iterated $\alpha(t)$ stay away from 0 and $+\infty$, as requested in the context of Problem 2. We checked whether this holds when $u$ solves Problem 1. Fortunately, this was the case, and this gave us a serious hint to treat our equation (1.2) as a perturbation of Problem 1.

In fact, our proof uses no iteration, and we directly apply the strategy of Merle and Zaag in [13] to control the various terms (including the nonlocal term), in order to find a solution which stays near the desired behavior.

2.2. Formulation of the problem

In this section, we aim at setting the mathematical framework of our problem. The rigorous proof will be given later. Our aim is to construct a solution for equation (1.2), defined for all $(x, t) \in \Omega \times [0, T)$, for some $T > 0$ with $0 \leq u(x, t) < 1$, and

\[
u(x, t) \to 1 \text{ as } (x, T) \to (x_0, T),
\]

for some $x_0 \in \Omega$. Without loss of generality, we assume that 
\[
x_0 = 0 \in \Omega.
\]

Introducing,
\[
\bar{u} = \frac{1}{1-u} - 1 = \frac{u}{1-u} \in [0, +\infty),
\]
we derive from (1.2) the following equation on $\bar{u}$

\[
\begin{align*}
\partial_t \bar{u} &= \Delta \bar{u} - 2\frac{\left| \nabla \bar{u} \right|^2}{\bar{u}+1} + \frac{\lambda(\bar{u} + 1)^4}{(1 + \gamma|\Omega| + \gamma \int_{\Omega} \bar{u} dx)^2}, \quad x \in \Omega, t > 0, \\
\bar{u}(x, t) &= 0, \quad x \in \partial \Omega, t > 0, \\
\bar{u}(x, 0) &= \bar{u}_0(x), \quad x \in \bar{\Omega}.
\end{align*}
\]
Our aim becomes then to construct a blowup solution for equation (2.5) such that
\[ \bar{u}(x, 0) \to +\infty \text{ as } t \to T. \]

In order to see our equation as a (not so small) perturbation of the standard case in (2.3), we suggest to make one more scaling by introducing
\[ U(x, t) = \frac{\lambda^2}{\theta(t)} \bar{u}(x, t), \quad U(x, t) \geq 0, \quad \forall (x, t) \in \Omega \times [0, T), \]
with
\[ \theta(t) = \left( 1 + \gamma |\Omega| + \gamma \int_{\Omega} \bar{u}(t) dx \right)^{\frac{2}{3}}. \]

Then, thanks to equation (2.5), we deduce the following equation to be satisfied by \( U \):
\[
\begin{cases}
\partial_t U &= \Delta U - 2 \frac{|\nabla U|^2}{U + \frac{\lambda^2}{\theta(t)}} + \left( U + \frac{\lambda^2}{\theta(t)} \right)^4 \frac{\theta'(t)}{\theta(t)} U, x \in \Omega, t > 0, \\
U(x, t) &= 0, x \in \partial \Omega, t > 0, \\
U(x, 0) &= U_0(x), x \in \bar{\Omega}.
\end{cases}
\]

Note that in the blowup regime, which is our focus, \( U \) is large and equation (2.8) appears indeed as a perturbation of equation (2.3).

Introducing the following notation
\[ \bar{\mu}(t) = \int_{\Omega} U(t) dx, \]
we may rewrite (2.7) as the following equation
\[ \bar{\theta}(t) = \left( 1 + \gamma |\Omega| + \frac{\gamma}{\lambda^2} \bar{\mu}(t) \right)^{\frac{2}{3}}. \]

This way, we may express \( \bar{\theta}(t) \) in terms of \( \bar{u}(t) \), thanks to the following formula
\[
\bar{\theta}(t) = \frac{\sqrt[3]{27A^2 + 3\sqrt[3]{327A^2 + 4A^3B^3 + 18AB^3 + 2B^6}}}{3B^{\frac{1}{3}}} + \frac{B^3}{3} + \frac{\sqrt[3]{2}(6AB + B^4)}{\sqrt[3]{27A^2 + 3\sqrt[3]{327A^2 + 4A^3B^3 + 18AB^3 + 2B^6}}},
\]
where
\[ A = 1 + \gamma |\Omega| \quad \text{and} \quad B = \frac{\gamma}{\lambda^2} \bar{\mu}(t). \]

Particularly, we show here the equivalence between equation (2.5) and (2.8).

**Lemma 2.1 (Equivalence between (2.5) and (2.8)).** Consider \( \lambda > 0, \gamma > 0 \) and \( \Omega \) a bounded domain in \( \mathbb{R}^n \). Then, the following holds:

(i) We consider \( \bar{u} \) a solution of equation (2.5) on \([0, T)\), for some \( T > 0 \) and introduce
\[ U(t) = \frac{\lambda^2}{\theta(t)} \bar{u}(t), \]
where \( \theta(t) = (1 + \gamma |\Omega| + \gamma \int_{\Omega} \bar{u}(t) dx)^{\frac{2}{3}} \). Then, \( U \) is a solution of equation (2.8) on \([0, T)\).

(ii) Otherwise, we consider \( \bar{U} \) a solution of equation (2.8) on \([0, T)\), for some \( T > 0 \) and introduce
\[ \bar{u}(t) = \frac{\theta(t)}{\lambda^2} U(t), \forall t \in [0, T), \]
where $\bar{\theta}(t)$ is defined as in relation (2.10), then $\bar{u}$ is the solution of equation (2.5) on $[0, T)$. In particular, the uniqueness of the solution is preserved.

**Proof.** The proof is easily deduced from the definition in this lemma. We kindly ask the reader to self-check. 

**Remark 2.2.** From settings (2.4) and (2.6) and the local well-posedness of equation (1.2) in the sense of classical solutions (see Proposition 1.2.2 at page 12 in Kavallaris and Suzuki [10]), we can derive the local existence and uniqueness of classical solutions of equations (2.5) and (2.8). Since nonnegativity is preserved for these equation, we will assume that $\bar{u}$ and $U$ are nonnegative.

Thanks to Lemma 2.1, our problem is reduced to constructing a nonnegative solution to (2.8), which blows up in finite time only at the origin. We also aim at describing its asymptotics at the singularity. Since we defined $U$ in (2.6) on purpose so that (2.8) appears as a perturbation of equation (2.3) for $U$ large, it is reasonable to make the following hypotheses:

(i) $1 \leq \bar{\theta}(t) \leq C_0$ for some $C_0 > 0$. Note that from (2.10), we have $\bar{\theta}(t) \geq 1$.

(ii) $|\bar{\theta}'(t)| \ll U^3(t)$ when $U$ large.

It is then reasonable to expect for equation (2.8) the same profile as the one constructed in [13] for equation (2.3). So, it is natural to follow that work by introducing the following Similarity-Variables:

$$W(y, s) = (T - t)^{\frac{1}{2}} U(x, t), \quad s = -\ln(T - t) \quad \text{and} \quad y = \frac{x}{\sqrt{T - t}}. \quad (2.12)$$

Using equation (2.8), we write the equation of $W$ in $(y, s)$ as follows

$$\begin{align*}
\partial_s W &= \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{3} - 2 \frac{\nabla W^2}{W + \frac{1}{\theta(s)}} + \left( W + \frac{W^3}{\theta(s)} \right)^4 - \frac{\theta'(s)}{\theta(s)} W; \\
W(y, s) &= 0, y \in \partial \Omega_s, s > -\ln T, \\
W(y, -\ln T) &= W_0(y), y \in \Omega_s,
\end{align*} \quad (2.13)$$

where

$$\theta(s) = \bar{\theta}(t(s)) = \bar{\theta}(T - e^{-s}), \quad (2.14)$$

and

$$\Omega_s = e^2 \Omega, \quad (2.15)$$

with $\bar{\theta}$ satisfies (2.10) and (2.11).

We observe in equation (2.13) that $\Omega_s$ changes as $s \to +\infty$. This is a major difficulty in comparison with the situation where $\Omega = \mathbb{R}^n$. In order to overcome this difficulty, we intend to introduce some cut-off of the solution, so that we reduce to the case $\Omega = \mathbb{R}^n$. Of course, there is a price to pay, in the sense that we will need to handle some cut-off terms. Our model for this will be the work made by Mahmoudi, Nouaili and Zaag in [12] for the construction of a blowup solution to the semilinear heat equation defined on the circle. Let us note that the situation with $\Omega$ bounded was already mentioned in [13]. However, the authors in that work avoided the problem by giving the proof only in the case where $\Omega = \mathbb{R}^n$. In this work, we are happy to handle the case with a bounded $\Omega$, following the ideas of Mahmoudi, Nouaili and Zaag in [12].

More precisely, we introduce the following cut-off function

$$\chi_0 \in C_0^\infty([0, +\infty)), \quad \text{supp}(\chi_0) \subset [0, 2], \quad 0 \leq \chi_0(x) \leq 1, \forall x \quad \text{and} \quad \chi_0(x) = 1, \forall x \in [0, 1]. \quad (2.16)$$

Then, we define the following function

$$\psi_{M_0}(y, s) = \chi_0 \left( M_0 ye^{-\frac{s}{2}} \right), \quad \text{for some} \quad M_0 > 0. \quad (2.17)$$
Let us introduce
\[
 w(y,s) = \begin{cases} 
 W(y,s)\psi_{M_0}(y,s) & \text{if } y \in \Omega, \\
 0 & \text{otherwise}.
 \end{cases} 
\] (2.18)

We remark that \( w \) is defined on \( \mathbb{R}^n \) and \( s \geq -\ln T \) and \( w \equiv 0 \) whenever \( |y| \geq \frac{2}{M_0} e^{\frac{s}{T}} \). Note that \( M_0 \) will be fixed large enough together with others parameters at the end of our proof.

Using equation (2.13), we derive from (2.10) the equation satisfied by \( w \) as follows
\[
 \partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{3} w - 2\frac{|\nabla w|^2}{w + \frac{\lambda e^{-\frac{T}{s}}}{\theta(s)}} + \left( w + \frac{\lambda e^{-\frac{T}{s}}}{\theta(s)} \right)^4 - \left( \theta'(s) \theta(s) \right) w + F(w, W), 
\] (2.19)

where \( F(w, W) \) encapsulates the cut-off terms and is defined as follows
\[
 F(w, W) = \begin{cases} 
 W \left[ \partial_s \psi_{M_0} - \Delta \psi_{M_0} + \frac{1}{2} y \cdot \nabla \psi_{M_0} \right] - 2\nabla \psi_{M_0} \cdot \nabla W \\
 + 2\frac{\nabla W^2 \psi_{M_0}}{W + \frac{\lambda e^{-\frac{T}{s}}}{\theta(s)}} + \psi_{M_0} \left( W + \frac{\lambda e^{-\frac{T}{s}}}{\theta(s)} \right)^4 \left( w + \frac{\lambda e^{-\frac{T}{s}}}{\theta(s)} \right)^4
 & \text{if } y \in \Omega e^{\frac{s}{T}}, \\
 0 & \text{otherwise}.
 \end{cases} 
\] (2.20)

We remark that \( F \equiv 0 \) on the region \( \{ y \in \mathbb{R}^n \mid |y| \leq \frac{1}{M_0} e^{\frac{s}{T}} \text{ or } |y| \geq \frac{2}{M_0} e^{\frac{s}{T}} \} \) and that we have from the conditions (i) and (ii) on \( \dot{\theta}(t) \) on page 8 that
\[
 1 \leq \theta(s) \leq C_0, \text{ and } |\theta'(s)| \ll W^3(y,s).
\]

Making the further assumption that
\[
 \theta'(s) \to 0,
\]
we see that equation (2.23) is almost the same as equation (15) at page 1502 in [13] at least when \( |y| \leq \frac{e^{\frac{s}{T}}}{M_0} \). Hence, it is reasonable to expect for equation (2.19) the same profile as the authors found in [13] for equation (15) in that work, namely
\[
 \varphi(y,s) = \left( 3 + \frac{9}{8} \frac{|y|^2}{s} \right)^{-\frac{4}{9}} + \frac{(3)^{-\frac{4}{9}} n}{4s}, 
\] (2.21)

(note that, this profile was also defined in [13] for a general \( p > 2 \), and that here we need to take \( p = 4 \) hence \( \kappa = (3)^{-\frac{4}{9}} \)). In particular, we would like to construct \( w \) as a perturbation of \( \varphi \). So, we introduce the following function
\[
 q = w - \varphi. \quad (2.22)
\]

Using equation (2.13), we easily write the equation of \( q \)
\[
 \partial_s q = (\mathcal{L} + V)q + T(q) + B(q) + N(q) + R(y,s) + F(w, W), 
\] (2.23)
where

\[ \mathcal{L} = \Delta - \frac{1}{2} y \cdot \nabla + \text{Id}, \quad (2.24) \]

\[ V(y, s) = 4 \left( \varphi^3(y, s) - \frac{1}{3} \right), \quad (2.25) \]

\[ J(q, \theta(s)) = -2 \frac{|\nabla q + \nabla \varphi|^2}{q + \varphi + \frac{\lambda^2 e^{-\frac{4}{\theta(s)}}}{\varphi + \frac{\lambda^2 e^{-\frac{4}{\theta(s)}}}}}, \quad (2.26) \]

\[ B(q) = \left( q + \varphi + \frac{\lambda^2 e^{-\frac{4}{\theta(s)}}}{\theta(s)} \right)^4 - \varphi^4 - 4 \varphi^3 q, \quad (2.27) \]

\[ R(y, s) = -\partial_s \varphi + \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{\varphi}{3} + \varphi^4 - 2 \frac{|\nabla \varphi|^2}{\varphi + \frac{\lambda^2 e^{-\frac{4}{\theta(s)}}}}. \quad (2.28) \]

\[ N(q) = -\frac{\theta'(s)}{\theta(s)} (q + \varphi), \quad (2.29) \]

with \( \theta(s) \) defined in (2.14) and \( F(w, W) \) given in (2.20).

In particular, we assume that \( U \) and \( q \) have good conditions such that Lemmas D.2, D.3, D.4, D.5 and D.6 hold. Then, it is easy to see that all terms in the right-hand side of (2.23) become very small, except for \( (L + V)q \). As a matter of fact, this term plays the most important role in our analysis. Therefore, we show here some main properties on the linear operator \( \mathcal{L} \) and the potential \( V \) (see more details in [1], [2]).

- **Operator \( \mathcal{L} \):** This operator is self-adjoint in \( \mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^n) \), where \( L^2_\rho \) is defined as follows

\[ L^2_\rho(\mathbb{R}^n) = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |f(y)|^2 \rho(y) dy < +\infty \right\}, \]

and

\[ \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{n}{2}}}. \]

This is the spectrum set of operator \( \mathcal{L} \)

\[ \text{Spec}(\mathcal{L}) = \left\{ 1 - \frac{m}{2} \mid m \in \mathbb{N} \right\}. \]

The eigenspace which corresponds to \( \lambda_m = 1 - \frac{m}{2} \) is given

\[ \mathcal{E}_m = \{ h_{m_1}(y_1), h_{m_2}(y_2), \ldots, h_{m_n}(y_n) \mid m_1 + \ldots + m_n = m \}, \]

where \( h_{m_i} \) is the (rescaled) Hermite polynomial in one dimension.

- **Potential \( V \):** It has two important properties:

  (i) The potential \( V(\cdot, s) \to 0 \) in \( L^2_\rho(\mathbb{R}^n) \) as \( s \to +\infty \): In particular, in the region \( |y| \leq K_0 \sqrt{s} \) (the singular domain), \( V \) has some weak perturbations on the effect of operator \( \mathcal{L} \).

  (ii) \( V(y, s) \) is almost a constant on the region \( |y| \geq K_0 \sqrt{s} \): For all \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) and \( s_\epsilon \) such that

\[ \sup_{s \geq s_\epsilon, \frac{|y|}{s} \geq C_\epsilon} \left| V(y, s) - \left( -\frac{4}{3} \right) \right| \leq \epsilon. \]

Note that \(-\frac{4}{3} < -1\) and that the largest eigenvalue of \( \mathcal{L} \) is 1. Hence, roughly speaking, we may assume that \( \mathcal{L} + V \) admits a strictly negative spectrum. Thus, we can easily control our solution in the region \( \{|y| \geq K_0 \sqrt{s} \} \) with \( K_0 \) large enough.
From these properties, it appears that the behavior of $L + V$ is not the same inside and outside of the singular domain $\{ |y| \leq K_0 \sqrt{s} \}$. Therefore, it is natural to decompose every $r \in L^\infty(\mathbb{R}^n)$ as follows:

$$r(y) = r_b(y) + r_e(y) \equiv \chi(y,s)r(y) + (1 - \chi(y,s))r(y),$$

where $\chi(y,s)$ is defined as follows

$$\chi(y,s) = \chi_0 \left( \frac{|y|}{K_0 \sqrt{s}} \right),$$

and $\chi_0$ is given in (2.16). From the above decomposition, we immediately have the following:

$$\text{Supp } (r_b) \subset \{ |y| \leq 2K_0 \sqrt{s} \},$$

$$\text{Supp } (r_e) \subset \{ |y| \geq K_0 \sqrt{s} \}.$$

In addition to that, we are interested in expanding $r_b$ in $L^2_\rho(\mathbb{R}^n)$ according to the basis which is created by the eigenfunctions of operator $L$:

$$r_b(y) = r_0 + r_1 \cdot y + y^T \cdot r_2 \cdot y - 2 \text{ Tr}(r_2) + r_-(y),$$

or

$$r_b(y) = r_0 + r_1 \cdot y + r_\perp(y),$$

where

$$r_i = (P_\beta(r_b))_{\beta \in \mathbb{N}^n, |\beta| = i}, \forall i \geq 0,$$

with $P_\beta(r_b)$ being the projection of $r_b$ on the eigenfunction $h_\beta$ defined as follows:

$$P_\beta(r) = \int_{\mathbb{R}^n} r_b \frac{h_\beta}{\|h_\beta\|_{L^2_\rho}^2} \rho dy, \forall \beta \in \mathbb{N}^n,$$

and

$$r_\perp = P_\perp(r) = \sum_{\beta \in \mathbb{R}^n, |\beta| \geq 2} h_\beta P_\beta(r_b),$$

and

$$r_- = \sum_{\beta \in \mathbb{R}^n, |\beta| \geq 3} h_\beta P_\beta(r_b).$$

In other words, $r_\perp$ is the part of $r_b$ which is orthogonal to the eigenfunctions corresponding to eigenvalues 0 and 1 and $r_-$ is orthogonal to the eigenfunctions corresponding to eigenvalues $1, \frac{1}{\sqrt{s}}$ and 0. We should note that $r_0$ is a scalar, $r_1$ is a vector and $r_2$ is a square matrix of order $n$ and that they are the components of $r_b$ not $r$. Finally, we write $r$ as follows

$$r(y) = r_0 + r_1 \cdot y + y^T \cdot r_2 \cdot y - 2 \text{ Tr}(r_2) + r_-(y) + r_e(y),$$

or

$$r(y) = r_0 + r_1 \cdot y + r_\perp(y) + r_e(y).$$

**A summary of our problem:** Even though we created many extra functions from $U$ to $q$, we always concentrate on solution $U$ to equation (2.8). More precisely, we aim at constructing $U$ blowing up in finite time. Then, we will use equation (2.23) as a crucial formulation in our proof. Indeed, in order to control $U$ blowing up in finite time, it is enough to control the transform function $q$ of $U$ (see definitions (2.12), (2.18) and (2.22)) satisfying

$$\|q(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \to 0, \text{ as } s \to +\infty.$$
3. The proof of the existence result assuming technical details

In this section, we aim at giving a proof without technical details to Theorem 1.1. We would like to summarize the structure of this section as follows:

- **Construction of a shrinking set**: We rely here on the ideas of the Merle and Zaag’s work in [13] to introduce a shrinking set that will guarantee the convergence to zero for $q$ defined in (2.22). This set will constrain our solution as we want. Once our solution is trapped in, we may show the main asymptotics of our solution. In particular, (2.38) holds and our result follows.

- **Preparation of initial data**: We introduce a family of initial data to equation (2.8) depending on some finite set parameters. As a matter of fact, we will choose these parameters such that our solution belongs to the shrinking set for all $t \in [0, T)$.

- **The existence of a trapped solution**: Using a reduction to a finite dimensional problem (corresponding to the finite parameters introduced in our initial data) and a topological argument, we can derive the existence of a blowup solution in finite time, trapped in the shrinking set. More precisely, we show in this part that there exist initial data in that family of initial data such that our solution is completely confined in the shrinking set.

- **The conclusion of Theorem 1.1**: Finally, we rely on the existence of a blowup solution, trapped in the shrinking set to get the conclusion of Theorem 1.1.

### 3.1. Shrinking set

In order to control the solution $U$ blowing up in finite time and satisfying (2.38), we adopt the general ideas given by Merle and Zaag in [13]. For each $K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0$ and $t \in [0, T)$ with $T > 0$, we define

$$
P_1(t) = \left\{ x \in \mathbb{R}^n | |x| \leq K_0 \sqrt{(T-t)|\ln(T-t)|} \right\}, \quad (3.1)$$

$$
P_2(t) = \left\{ x \in \mathbb{R}^n | \frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|} \leq |x| \leq \epsilon_0 \right\}, \quad (3.2)$$

$$
P_3(t) = \left\{ x \in \mathbb{R}^n | |x| \geq \frac{\epsilon_0}{4} \right\}. \quad (3.3)$$

As a matter of fact, we have

$$
\Omega \subset \mathbb{R}^n = P_1(t) \cup P_2(t) \cup P_3(t), \quad \text{for all } t \in [0, T).
$$

We aim at controlling our problem on $P_1(t), P_2(t)$ and $P_3(t)$ as follows:

- **On region $P_1(t)$ (blowup region)**: We control $w$ (see (2.12)) instead of $U$. More precisely, we show that $w$ is a perturbation of the profile $\varphi$ (the blowup profile, introduced in (2.21)). Then, (2.38) will follow from the control of $w$.

- **On region $P_2(t)$ (intermediate region)**: We control a rescaled function $\mathcal{U}$ instead of $U$. More precisely, $\mathcal{U}$ is defined as follows: For all $x \in P_2(t), \xi \in (T-t(x))^{-\frac{1}{3}}(\Omega-x)$ and $\tau \in \left[-\frac{t(x)}{T-t(x)}, 1\right]$, we define

$$
\mathcal{U}(x, \xi, \tau) = (T-t(x))^{\frac{2}{3}} U \left(x + \xi \sqrt{T-t(x)}, (T-t(x)) \tau + t(x)\right), \quad (3.4)
$$

where $t(x)$ is defined as the solution of the following equation

$$
|x| = \frac{K_0}{4} \sqrt{(T-t(x))|\ln(T-t(x))|} \quad \text{and} \quad t(x) < T. \quad (3.5)
$$

We remark that if $\epsilon_0$ is small enough, then $t(x)$ is well defined for all $x \in P_2(t)$. In addition to that, using (3.5), we have the following asymptotic

$$
t(x) \to T, \quad \text{as} \quad x \to 0.
$$

For convenience, we introduce

$$
\varrho(x) = T - t(x). \quad (3.6)
$$
Then, the following holds
\[ \varrho(x) \to 0 \text{ as } x \to 0. \]
As a matter of fact, using (2.8), we write the equation satisfied by \( U \) in \((\xi, \tau) \in \varrho^{-\frac{1}{2}}(\Omega - x) \times \left[-\frac{\hat{t}(x)}{\varrho(x)}, 1\right)\) as follows:
\[
\partial_\tau U = \Delta_\xi U - 2 \frac{\left|\nabla U\right|^2}{U} + \left(U + \frac{\lambda^2 \varrho^2(x)}{\bar{\theta}(\tau)}\right)^4 - \frac{\partial_\tau^2 (t(x))}{\bar{\theta}(\tau)} U, \tag{3.7}
\]
where
\[
\bar{\theta}(\tau) = \bar{\theta}(\tau \varrho(x) + t(x)), \tag{3.8}
\]
with \( \bar{\theta}(t) \) defined in (3.6). We now consider the following domain
\[
|\xi| \leq \alpha_0 \sqrt{\ln(\varrho(x))} \text{ and } \tau \in \left[-\frac{t(x)}{\varrho(x)}, 1\right). \]
When \( \tau = 0 \), we are in region \( P_1(t(x)) \), in fact (note that \( P_1(t(x)) \) and \( P_2(t(x)) \) have some overlapping by definition). From our constraints in \( P_1(t(x)) \), we derive that \( U(x, \xi, 0) \) is flat in the sense that
\[
U(x, \xi, 0) \sim \left(3 + \frac{9 K_0^2}{8 16}\right)^{-\frac{1}{2}}.
\]
Our idea is to show that this flatness will be conserved for all \( \tau \in [0, 1) \) (that is for all \( t \in [t(x), T) \)), in the sense that the solution will not depend that much on space. In one word, \( U \) is regarded as a perturbation of \( \hat{U}(\tau) \), where \( \hat{U}(\tau) \) is defined as follows
\[
\begin{cases}
\partial_\tau \hat{U}(\tau) = \hat{U}^4(\tau), \\
\hat{U}(0) = \left(3 + \frac{9 K_0^2}{8 16}\right)^{-\frac{1}{2}}.
\end{cases} \tag{3.9}
\]
Note that, we can give an explicit formula to the solution of equation (3.9)
\[
\hat{U}(\tau) = \left(3(1 - \tau) + \frac{9 K_0^2}{8 16}\right)^{-\frac{1}{2}}. \tag{3.10}
\]
- On region \( P_3(t)(\text{regular region}) \): Thanks to the well-posedness property of the Cauchy problem for equation (2.23), we control the solution \( U \) as a perturbation of initial data \( U(0) \). Indeed, the blowup time \( T \) will be chosen small in our analysis.

Relying on those ideas, we give in the following the definition of our shrinking set:

**Definition 3.1** (Definition of \( S(t) \)). Let us consider \( T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0, \eta_0 > 0 \) and \( t \in [0, T) \). Then, we introduce the following set
\[
S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, \eta_0, C_0, t) \quad (S(t) \text{ for short}),
\]
as a subset of \( C^{2,1}(\Omega \times (0, t)) \cap C(\bar{\Omega} \times [0, t]) \), containing all functions \( U \) satisfying the following conditions:

(i) **Estimates in \( P_1(t) \)**: We have \( q(s) \in V_{K_0,A}(s) \), where \( q(s) \) is introduced in (2.22), \( s = -\ln(T - t) \) and \( V_{K_0,A}(s) \) is a subset of all function \( r \) in \( L^\infty(\mathbb{R}^n) \), satisfying the following
estimates:
\[
|r_i| \leq \frac{A}{s^2}, \quad (i = 0, 1), \quad \text{and} \quad |r_2| \leq \frac{A^2 \ln s}{s^2},
\]
\[
|r(y)| \leq \frac{A^2}{s^2} (1 + \|y\|^3), \quad \text{and} \quad \|r_x\|_{L^\infty(\mathbb{R}^n)} \leq \frac{A^2}{\sqrt{s}},
\]
\[
|(|\nabla r|\rangle) \leq \frac{A}{s^2} (1 + \|y\|^3), \forall y \in \mathbb{R}^n,
\]
where the definitions of \(r_i, r(y), (\nabla r)\rangle\) are given in (2.32), (2.34) and (2.35), respectively.

(ii) **Estimates in \(P_2(t)\):** For all \(x \in \left[\frac{\kappa}{8} \sqrt{(T-t)} \sqrt{\ln(T-t)} - \epsilon_0\right]\), \(\tau(x, t) = \frac{t-t(x)}{\epsilon_0(x)}\) and \(|\xi| \leq \alpha_0 \sqrt{\ln \rho(x)}\), we have the following
\[
\left| U(x, \xi, \tau(x, t)) - \hat{U}(\tau(x, t)) \right| \leq \delta_0,
\]
\[
|\nabla U(x, \xi, \tau(x, t))| \leq \frac{C_0}{\sqrt{\ln \rho(x)}}
\]
where \(U, \hat{U}\) and \(\rho(x)\) are given (3.4), (3.6) and (3.10), respectively.

(iii) **Estimates in \(P_3(t)\):** For all \(x \in \{|x| \geq \frac{\kappa}{4}\} \cap \Omega\), we have
\[
|U(x, t) - U(x, 0)| \leq \eta_0,
\]
\[
|\nabla U(x, t) - \nabla e^t \Delta U(x, 0)| \leq \eta_0.
\]
In addition to that, we would like to introduce the set \(S^*(K_0, \epsilon_0, A, \delta_0, C_0, \eta_0, T)\) as follows:

**Definition 3.2.** For all \(T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0\), and \(\eta_0 > 0\), we introduce \(S^*(T, K_0, \epsilon_0, A, \delta_0, C_0, \eta_0)\) \((S^*(T)\) for short) as the subset of all functions \(U\) in \(C^{2,1}(\Omega \times (0, T)) \cap C(\Omega \times [0, T])\), satisfying the following: for all \(t \in [0, T]\), we have
\[
U \in S(T, K_0, \epsilon_0, A, \delta_0, C_0, \eta_0, t).
\]

**Remark 3.3.** The shrinking set \(S(t)\) is inspired by the work of Merle and Zaag in [13]. However, we’ve made two major changes:
- A simplification, by removing an unnecessary condition on the second derivative in space in region \(P_2(t)\).
- A smart change in region \(P_3(t)\), by replacing \(\nabla U\) by \(\nabla e^t \Delta U(0)\). This change is crucial since we are working on a bounded domain \(\Omega\).

**Remark 3.4.** The conditions in \(P_2\) and \(P_3\) in Definition 3.1 are designed to make our solution more regular and these conditions help us to control \(U\) in \(P_1\) and \(q(s) \in V_{K_0, A}(s)\). Finally, the main purpose is to satisfy (2.38). In other words, the control \(U\) in \(P_1\) is the main issue.

**Remark 3.5.** In our paper, we use a lot of parameters to control our solution. However, they will be fixed at the end of the proof. In addition to that, we would like to give some conventions on the universal constant in our paper: We use \(C\) for universal constants which depend only \(n, \Omega, \gamma, \lambda\) and we write \(C(K_0, \epsilon_0, ...\)) for constants which depend \(K_0, \epsilon_0, ...\), respectively.

As we mentioned in Remark 3.4, we would like to show here some estimates of the sizes of \(q\) and \(\nabla q\), where \(q\) is the transformed function of \(U\) when \(U \in S(t)\).

**Lemma 3.6** (Sizes of \(q\) and \(\nabla q\)). Let us consider \(K_0 \geq 1\) and \(\epsilon_0 > 0\). Then, there exist \(T_1(K_0, \epsilon_0)\) and \(\eta_1(\epsilon_0)\) such that for all \(\alpha_0 > 0, A > 0, \delta_0 \leq \frac{1}{2} U(0)\) (see (3.10)), \(C_0 > 0, \eta_0 \leq \eta_1, T \leq T_1\) and \(t \in [0, T]\): if \(U \in S(K_0, \epsilon_0, A, \delta_0, C_0, \eta_0, t)\), then, the following holds:
(i) The estimates on \( q \): For all \( y \in \mathbb{R}^n \) and \( s = -\ln(T-t) \), we have

\[
|q(y,s)| \leq \frac{C(K_0)A^2}{\sqrt{s}} \quad \text{and} \quad |q(y,s)| \leq \frac{C(K_0)A^2 \ln s}{s^2} (1 + |y|^3).
\]

(ii) The estimates on \( \nabla q \): For all \( y \in \mathbb{R}^n \), we have

\[
|\nabla q(y,s)| \leq \frac{C(K_0,C_0)A^2}{\sqrt{s}}, \quad |\nabla q(y,s)| \leq \frac{C(K_0,C_0)A^2 \ln s}{s^2} (1 + |y|^3),
\]

and

\[
|(1 - \chi(y,s))\nabla q(y,s)| \leq \frac{C(K_0)}{\sqrt{s}}.
\]

Proof. The conclusion directly follows from the definition of the shrinking set \( S(t) \) and \( V_{K_0,A}(s) \).

In addition to that, these definitions are almost the same as in [13]. Therefore, we kindly refer the reader to see Lemma B.1 at page 1537 in [13]. \( \square \)

### 3.2. Initial data

In this subsection, we will concentrate on introducing our initial data to equation (2.8) so that it is trapped in \( S(0) \). In order to do that, we first introduce the following cut-off function:

\[
\chi_1(x) = \chi_0 \left( \frac{|x|}{\sqrt{T|\ln T|}} \right), \quad (3.11)
\]

where \( \chi_0 \) is given in (2.16). In addition to that, we introduce \( H^* \) as a function in \( C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) satisfying

\[
H^*(x) = \begin{cases} 
\left[ \frac{9}{16} \frac{|x|^2}{|\ln |x||} \right] \frac{1}{3}, & \forall |x| \leq \min \left( \frac{1}{4}d(0,\partial \Omega), \frac{1}{2} \right), x \neq 0, \\
0, & \forall |x| \geq \frac{1}{2}d(0,\partial \Omega), 
\end{cases} \quad (3.12)
\]

and for all \( x \in \mathbb{R}^n, x \neq 0 \), the following condition holds

\[
0 \leq H^*(x) \leq \left[ \frac{9}{16} \frac{|x|^2}{|\ln |x||} \right] \frac{1}{3}.
\]

We now give the definition of our initial data corresponding to equation (2.8): For all \( (d_0,d_1) \in \mathbb{R}^{1+n} \), we define

\[
U_{d_0,d_1}(x,0) = T^{-\frac{1}{3}} \left[ \varphi \left( \frac{x}{\sqrt{T|\ln T|}} - \ln s_0 \right) + (d_0 + d_1 \cdot z)\chi_0 \left( \frac{|z|}{K_0^{\frac{1}{2}}} \right) \chi_1(x) \right. \\
+ \left. H^*(x) \left( 1 - \chi_1(x) \right) \right],
\]

where \( z = \frac{x}{\sqrt{T|\ln T|}} \) and \( \varphi, \chi_0, \chi_1, H^* \) are defined as in (2.21), (2.16), (3.11) and (3.12), respectively.

From (3.13), we would like to give the definition of initial data corresponding to equation (2.23), \( q_{d_0,d_1}(s_0) \) with \( s_0 = -\ln T \):

\[
q_{d_0,d_1}(y,s_0) = e^{-\frac{d_0}{4}} U_{d_0,d_1} \left( ye^{-\frac{d_0}{4}},0 \right) - \varphi(y,s_0), \quad (3.14)
\]

where and \( \psi_M, \varphi \) and \( U_{d_0,d_1} \) are introduced in (2.17), (2.21) and (3.13), respectively.
Remark 3.7. We would like to explain in brief how our initial data $U_{d_0,d_1}$ has naturally the form shown in (3.13). As we mentioned at the beginning of this section, our purpose is to control initial data in $S(0)$. More precisely, our initial data have to satisfy items (i) and (ii) in Definition 3.1. As a matter of fact, when $T$ is small enough, the second term in the right hand side of (3.13) is zero on $P_1(0)$. Then, our initial data has only the first term and we adopt the idea given in [14] (see also [13], [5]), we use $d_0, d_1$ in order to control $q(s_0)$ in $V_{K_0,A}(s_0)$. In addition to that, we would like to mention that Proposition 3.13 below states that when $q$ is trapped in $V_{K_0,A}(s)$, it has only two components $(q_0,q_1)(s)$ which may attain their upper bound, the others being strictly less than their upper bound specified in the definition of $V_{K_0,A}(s)$. This is indeed the reason to use $(d_0,d_1)$ in our initial data. More precisely, these $1+n$ parameters allows us to a reduction to a finite dimensional problem. We now mention the control in $P_2$. In that region, $|x|$ is small enough and we may consider that $U$ is near the final profile

$$\left(\frac{9}{16}\frac{|x|^2}{\ln|y|}\right)^{-\frac{1}{8}}.$$ 

As a matter of fact, it is reasonable to introduce $H^*$ as the main asymptotic of our initial data in $P_2(0)$. Using some priori estimates, we can derive good estimates in $P_2(0)$. More precisely, the following proposition is our statement:

Proposition 3.8 (Preparation of initial data). There exists $K_2 > 0$ such that for all $K_0 \geq K_2$ and $\delta_2 > 0$, there exist $\alpha_2(K_0,\delta_2) > 0$ and $C_2(K_0) > 0$ such that for every $\alpha_0 \in (0, \alpha_2]$, there exists $\epsilon_2(K_0,\delta_2,\alpha_0) > 0$ such that for every $\epsilon_0 \in (0, \epsilon_2]$ and $A \geq 1$, there exists $T_2(K_0,\delta_2,\epsilon_0, A,C_2) > 0$ such that for all $T \leq T_2$ and $s_0 = -\ln T$. The following holds:

(I) We can find a set $D_A \subset [-2,2] \times [-2,2]^n$ such that if we define the following mapping

$$\Gamma: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^n$$

$$(d_0,d_1) \mapsto (q_0,q_1)(s_0),$$

then, $\Gamma$ is affine, one to one from $D_A$ to $\hat{V}_A(s_0)$, where $\hat{V}_A(s)$ is defined as follows

$$\hat{V}_A(s) = \left[ -\frac{A}{s^2}, \frac{A}{s^2} \right]^{1+n}.$$ (3.15)

Moreover, we have

$$\Gamma|_{\partial D_A} \subset \partial \hat{V}_A(s_0),$$

and

$$\deg \left( \Gamma|_{\partial D_A} \right) \neq 0,$$ (3.16)

where $q_0, q_1$ are defined as in (2.36), considered as the components of $q_{d_1,d_1}(s_0)$, which is a transform function of $U_{d_0,d_1}(0)$, given in (2.22).

(II) We now consider $(d_0,d_1) \in D_A$. Then, initial data $U_{d_0,d_1}(0)$ belongs to

$$S(K_0,\epsilon_0, \alpha_0, A, \delta_2, C_2, 0,0) = S(0),$$

where $S(0)$ is defined in Definition 3.1. Moreover, the following estimates hold

(i) Estimates in $P_1(0)$: We have $q_{d_0,d_1}(s_0) \in V_{K_0,A}(s_0)$ and

$$|q_0(s_0)| \leq \frac{A}{s_0^2}, \quad |q_{1,j}(s_0)| \leq \frac{A}{s_0^2}, \quad |q_{2,i,j}(s_0)| \leq \frac{\ln s_0}{s_0^2}, \forall i,j \in \{1,\ldots,n\},$$

$$|q_-(y,s_0)| \leq \frac{1}{s_0^2}(|y|^3 + 1), \quad |\nabla q_-(y,s_0)| \leq \frac{1}{s_0^2}(|y|^3 + 1), \forall y \in \mathbb{R}^n,$$

and

$$q_0 \equiv 0,$$

where the components of $q_{d_0,d_1}(s_0)$ are defined in (2.34).
(ii) Estimates in $P_2(0)$: For every $|x| \in \left[ \frac{K_0}{4}\sqrt{T} \ln T, \epsilon_0 \right]$, $\tau_0(x) = -\frac{t(x)}{\varrho(x)}$ and $|\xi| \leq \alpha_0 \sqrt{\ln \varrho(x)}$, we have

$$|\mathcal{U}(x, \xi, \tau_0(x)) - \hat{\mathcal{U}}(\tau_0(x))| \leq \delta_2 \text{ and } |\nabla_\xi \mathcal{U}(x, \xi, \tau_0(x))| \leq \frac{C_2}{\sqrt{\ln \varrho(x)}},$$

where $\mathcal{U}, \hat{\mathcal{U}}$, and $\varrho(x)$ are defined as in (3.4), (3.10) and (3.6) respectively.

**Proof.** The proof of Proposition 3.8 will be given in Appendix A. We now assume that this proposition holds and continue to get to the conclusion of Theorem 1.1.

### 3.3. Existence of a solution trapped in $S^*(T)$

In this subsection, we would like to derive the existence of a blowup solution $U$ to equation (2.8), trapped in $S^*(T)$. As we said earlier, our proof will be a (non-trivial) adaptation of the proof designed by Merle and Zaag in [13] for the more standard case (2.3). However, in comparison with (2.3), we observe in equation (2.8) a new feature, the nonlocal term involving $\bar{\theta}(t)$. As a matter of fact, it is important to study this term and its derivative. In particular, in the works which we used to make the main idea for our work (such as [13], [14], [5]), the authors only studied for constant coefficients parabolic equations. Hence, it makes a main highlight in our work. For that reason, we show here some estimates on $\bar{\theta}(t)$ (also on $\bar{\mu}(t)$). The following is our statement:

**Proposition 3.9** (Some estimates of $\bar{\theta}(t)$ and $\bar{\mu}(t)$). Let us consider $\lambda > 0, \gamma > 0$ and $\Omega$ a $C^2$ bounded domain. Then, there exists $K_3 > 0$ such that for all $K_0 \geq K_3, \delta_0 > 0$, there exist $\alpha_3(K_0, \delta_0) > 0$ such that for all $\alpha_0 \leq \alpha_3$, there exists $\varepsilon_3(K_0, \delta_0, \alpha_0) > 0$ such that for all $\varepsilon_0 \leq \varepsilon_3$ and $A \geq 1, C_0 > 0, \eta_0 > 0$, there exists $T_3 > 0$ such that for all $T \leq T_3$ the following holds: Assuming $U$ is a non-negative solution of equation (2.8) on $[0, t_1]$, for some $t_1 < T$, $U \in S(T, K_0, \varepsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t) = S(t)$ for all $t \in [0, t_1]$ and $U(0) = U_{d_0,d_1}(0)$, given in (3.13) with some $(d_0, d_1) \in \mathbb{R}^{1+n}, |d_0|, |d_1| \leq 2$, the following statements holds:

(i) For all $t \in [0, t_1]$, $\bar{\mu}(t)$ and $\bar{\theta}(t)$ are positive and these estimates hold

$$0 \leq \bar{\mu}(t) \leq C, \quad 1 \leq \bar{\theta}(t) \leq C.$$  \hfill (3.17)

Moreover, for all $t \in (0, t_1)$, we have

$$|\bar{\mu}'(t)| \leq C(T - t)^{\frac{4n-8}{6}} |\ln(T - t)|, \quad |\bar{\theta}'(t)| \leq C(T - t)^{\frac{2n-6}{6}} |\ln(T - t)|.$$ \hfill (3.19)

(ii) In particular, if $U \in S(t)$ for all $t \in [0, T)$, then $\bar{\mu}(t)$ and $\bar{\theta}(t)$ converge respectively to $\bar{\mu}_T$ and $\bar{\theta}_T \in \mathbb{R}^*_+$ as $t \to T$.

**Remark 3.10.** Although we know from item (ii) that $\bar{\theta}(t)$ converges to $\bar{\theta}_T$, we don’t know how big is $\bar{\theta}_T$. In particular, the dependence of these constants on $\gamma, \lambda, \Omega$ and $T, t_0, \ldots$, is not clear yet.

**Proof.** We can see that item (ii) is a direct consequence of (i). So, we give only the proof of item (i). Using (2.9) and the fact that $U(t) \geq 0$ for all $t$, we derive the following

$$\bar{\mu}(t) = \int_{\Omega} U(t) dx \geq 0.$$

In addition to that, we write

$$\bar{\mu}(t) \leq \int_{\Omega} U(t) dx \leq \int_{P_1(t)} U(t) dx + \int_{P_2(t)} U(t) dx + \int_{P_3(t)} U(t) dx,$$ \hfill (3.21)
where $P_1(t), P_2(t), P_3(t)$ are given in (3.1), (3.2) and (3.3), respectively. Remembering $\varrho(x)$, defined in (3.6), we see that the following holds

$$\varrho(x) \sim \frac{8}{K_0^2 |\ln |x||} \text{ as } x \to 0.$$  

In particular, using Definition 3.1, we get the following estimates: for all $t \in [0, t_1]$

On $P_1(t), |U(x, t)| \leq (T - t)^{-\frac{3}{8}} \left[ \frac{CA^2}{\sqrt{|\ln (T - t)|}} + |\varphi(0, -\ln (T - t))| \right] \leq C(T - t)^{-\frac{3}{8}},$

On $P_2(t), |U(x, t)| \leq \varrho^{-\frac{4}{7}}(x) \left[ \tilde{U}(\tau(x, t)) + \delta_0 \right] \leq C \left[ \frac{|x|^2}{|\ln |x||} \right]^{-\frac{3}{8}},$

On $P_3(t), |U(x, t)| \leq |U(x, 0)| + \eta_0 \leq |U(x, 0)| + 1,$

provided that $K_0 \geq K_{3.1} \delta_0 \leq 1$ and $\eta_0 \leq 1$. Integrating $U$ on each $P_i(t), i = 1, 2, 3$, we obtain the following

$$\int_{P_1(t)} U(t) dx \leq C(K_0)(T - t)^{\frac{\theta'}{\theta(t)} - \frac{3}{8}} |\ln (T - t)|^{\frac{\theta'}{\theta(t)}},$$

$$\int_{P_2(t)} U(t) dx \leq C \int_{|x| \leq \epsilon_0} \left[ \frac{|x|^2}{|\ln |x||} \right]^{-\frac{3}{8}} dx \leq C \epsilon_0^{\frac{n-4}{2}} |\ln \epsilon_0|^{\frac{3}{8}},$$

$$\int_{P_3(t)} U(t) \leq \left[ \int_{\frac{\theta}{4} \leq |x|, x \in \Omega} [H^n + 1] dx \right],$$

where $H^n$ is defined in (3.12). Using (3.21) and the above estimates, it is easy to obtain the following estimate

$$\mu(t) \leq C, \text{ for all } t \in [0, t_1],$$

provided that $K_0 \geq K_{3.2}(\lambda, \gamma), \epsilon_0 \leq \epsilon_{3.1}(\lambda, \gamma), \eta_0 \leq \eta_{3.1}(\lambda, \gamma)$ and $T \leq T_{3.1}(K_0, \lambda, \gamma)$. This yields (3.17) and (3.18) also follows by using (2.10) and (3.7). We now give a proof to (3.19). Integrating two sides of equation (2.8), we get the following ODE

$$\bar{\theta}'(t) + \frac{\partial'(t)}{\partial(t)} \bar{\mu}(t) = \int_\Omega \Delta U(t) dx + \int_\Omega \left( \left( U(t) + \frac{\lambda^\frac{3}{8}}{\theta(t)} \right)^4 - 2 \frac{|\nabla U(t)|^2}{U(t) + \frac{\lambda^\frac{3}{8}}{\theta(t)}} \right) dx. \quad (3.22)$$

We aim at proving the following estimate

$$\left| \int_\Omega \left( \left( U(t) + \frac{\lambda^\frac{3}{8}}{\theta(t)} \right)^4 - 2 \frac{|\nabla U(t)|^2}{U(t) + \frac{\lambda^\frac{3}{8}}{\theta(t)}} \right) dx \right| \leq C(T - t)^{\frac{3n-8}{6}} |\ln (T - t)|^n. \quad (3.23)$$

In order to do so, we first prove that

$$\int_\Omega U^4(t) dx \leq C(T - t)^{\frac{3n-8}{6}} |\ln (T - t)|^n, \quad (3.24)$$

$$\int_\Omega \frac{|\nabla U(t)|^2}{U(t) + \frac{\lambda^\frac{3}{8}}{\theta(t)}} dx \leq C(T - t)^{\frac{3n-8}{6}} |\ln (T - t)|^n. \quad (3.25)$$
The techniques of proofs (3.24) and (3.25) are the same. Therefore, we only give here the proof of (3.25). Let us consider

$$I(x,t) = \frac{|\nabla U(x,t)|^2}{U(x,t) + \frac{\lambda^*}{\theta(t)}}.$$ 

Then,

$$\int_{\Omega} I(x,t)dx \leq \int_{P_1(t)} I(x,t)dx + \int_{P_2(t)} I(x,t)dx + \int_{P_3(t)} I(x,t)dx.$$ 

Now we claim the following lemma:

**Lemma 3.11.** Under the hypothesis in Proposition 3.9, for all $t \in (0, t_1]$, the following estimates hold:

- **On $P_1(t)$:** $I(x,t) \leq C(K_0)(T-t)^{-\frac{4}{3}}$, (3.26)
- **On $P_2(t)$:** $I(x,t) \leq C(K_0)g^{-\frac{1}{4}}(x) \leq C(K_0)\left[\frac{|x|^2}{\ln|x|}\right]^{-\frac{4}{3}}$, (3.27)
- **On $P_3(t)$:** $I(x,t) \leq C\left(|\nabla U(x,0)|^2 + \eta_0^2\right) = C(|\nabla H^*(x)| + \eta_0^2)$. (3.28)

**Proof.** From the definition of $S(t)$, we easily derive (3.28). So, we only give here the proofs of (3.26) and (3.27). We now start with (3.26). Let us consider $x \in P_1(t)$ and we use the condition of $U$ in $P_1(t)$ to get the following

$$\frac{1}{C(K_0)}(T-t)^{-\frac{4}{3}} \leq U(x,t) \leq C(K_0)(T-t)^{-\frac{1}{3}}.$$ (3.29)

In addition to that, thanks to item $(ii)$ in Lemma 3.6, we get

$$|\nabla y W\left(\frac{x}{\sqrt{t-t}}, -\ln(T-t)\right)| \leq \frac{C(K_0)A^2}{\sqrt{\ln(T-t)}},$$

which yields

$$|\nabla U(x,t)| \leq C(K_0)(T-t)^{-\frac{2}{3}}.$$ (3.30)

Then, (3.26) follows by (3.29) and (3.30).

We now consider $x \in P_2(t)$. It is easy to derive from item $(ii)$ in Definition 3.1 that

$$\frac{1}{C(K_0)}g^{\frac{1}{4}}(x) \leq U(x,t) \leq C(K_0)g^{\frac{1}{4}}(x),$$

$$|\nabla U(x,t)| \leq Cg^{-\frac{1}{4}}(x),$$

provided that $\delta_0 \leq \delta_{3,1}$ and $\epsilon_0 \leq \epsilon_{3,2}$. This gives (3.27) and concludes the proof of Lemma 3.11. □

We now continue the proof of Proposition 3.9. Considering $t \in (0, t_1)$ and taking the integral on two sides of (3.26), we write

$$\int_{P_1(t)} |I(x,t)|dx \leq C(K_0) \int_{|x| \leq K_0\sqrt{(T-t)|\ln(T-t)|}}(T-t)^{-\frac{4}{3}}dx$$

$$\leq C(K_0)(T-t)^{-\frac{2}{3}} |\ln(T-t)|^{\frac{a}{2}}.$$ 

Integrating the two sides of (3.27) and using the following fact

$$g(x) \sim \frac{8}{K_0^2 |\ln|x||}$$ as $x \to 0$, we get
we obtain the following
\[
\int_{P_2(t)} |I(x,t)| \leq C(K_0) \left[ \frac{n-\delta}{\epsilon_0} \ln \epsilon_0 \left( \frac{4}{3} - ((T-t) |\ln(T-t)|) \right) \right. \\
\left. \int_{\partial \Omega} \left( \frac{2\gamma}{3\lambda^3} \frac{\bar{\mu}(t)}{\theta(t)} \right)^{\frac{2}{3}} \right]
\]

In addition to that, from (3.28), we have
\[
\int_{P_3(t)} |I(x,t)| dx \leq C.
\]

Hence, (3.25) holds.
In addition to that, using (F.3), we can derive that
\[
\int_{\Omega} \Delta U(t) dx < \infty, \forall t \in (0,t_1).
\]

Therefore, we have
\[
\lim_{v \to 0} \int_{\{x, d(x, \partial \Omega) > v\}} \Delta U dx = \int_{\Omega} \Delta U(t) dx.
\]

Moreover, for all \( v > 0 \) small enough and from item (iii) of Definition (3.1), we have
\[
\left| \int_{\{x, d(x, \partial \Omega) > v\}} \Delta U dx \right| = \left| \int_{\partial \{x, d(x, \partial \Omega) > v\}} \nu(x) \cdot \nabla U(x,t) dS \right| \leq C. \tag{3.31}
\]

This implies that
\[
\int_{\Omega} \Delta U(t) dx \leq C. \tag{3.32}
\]

Hence, from (3.22), (3.23) and (3.32), we derive the following
\[
\left| \bar{\mu}'(t) + \frac{\bar{\theta}'(t)}{\bar{\theta}(t)} \bar{\mu}(t) \right| \leq C(T-t)^{\frac{3n-8}{6}} |\ln(T-t)|^n. \tag{3.33}
\]

In addition to that, from the relation between \( \bar{\mu} \) and \( \bar{\theta} \) in (2.10), we write
\[
\frac{\bar{\theta}'(t)}{\bar{\theta}(t)} = \frac{2\gamma}{3\lambda^3} \left( 1 - \frac{2\gamma}{3\lambda^3} \frac{\bar{\mu}(t)}{\theta(t)} \right)^{-1} \left( 1 + \gamma |\Omega| + \frac{2\gamma}{3\lambda^3} \frac{\bar{\theta}(t) \bar{\mu}(t)}{\theta(t)} \right)^{-\frac{1}{3}} \bar{\mu}'(t).
\]

We also have the fact that
\[
\sqrt{\bar{\theta}(t)} \geq \frac{\gamma}{\lambda^3} \bar{\mu}(t),
\]

which yields that
\[
1 \leq \left( 1 - \frac{2\gamma}{3\lambda^3} \frac{\bar{\mu}(t)}{\theta(t)} \right)^{-1} \left( 1 + \gamma |\Omega| + \frac{2\gamma}{3\lambda^3} \frac{\bar{\theta}(t) \bar{\mu}(t)}{\theta(t)} \right)^{\frac{1}{3}} \leq 3.
\]

Hence, \( \bar{\theta}'(t) \) and \( \bar{\mu}'(t) \) have the same sign and we can use (3.33) to conclude that
\[
|\bar{\mu}'(t)| \leq C(T-t)^{\frac{3n-8}{6}} |\ln(T-t)|^n. \tag{3.34}
\]

This yields (3.19) and (3.20). Thus, we get the conclusion of the proof of Proposition 3.9. \(\square\)
Proposition 3.12 (Existence of a solution to equation (2.8), confined in $S^*$). We can find parameters $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0, \eta_0 > 0$ such that there exist $(d_0, d_1) \in \mathbb{R} \times \mathbb{R}^n$ such that with initial data $U_{d_0,d_1}(0)$ (given in (3.13)), the solution $U$ of equation (2.8) exists on $\Omega \times [0, T)$ and

$$U \in S^*(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, T),$$

where $S^*(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, T)$ given in (3.2).

Proof. As a matter of fact, this Proposition plays a central role in our problem. In other words, it will imply Theorem 1.1 (see subsection 3.4 below). The proof of this Proposition will be presented in two steps:

- First step: We use a reduction of our problem to a finite dimensional one. More precisely, we prove that the controlling $U$ in $S(t)$ for all $t \in [0, T)$ is reduced to the control of $(q_0, q_1)(s)$ in $\hat{V}_A(s)$ (see Proposition 3.13 below).

- Second step: In this step, we aim at proving that there exist $(d_0, d_1) \in \mathbb{R}^{1+n}$ such that $U \in S^*(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, T)$ with suitable parameters. Then, the conclusion follows from a topological argument based on Index theory.

We now give two main steps with more technical details:

a) Reduction to a finite dimensional problem: In this step, we derive that the control of $U \in S(t)$ with $t \in [0, T)$ is reduced to the control of the transform function $q(s)$ such that two first components $(q_0, q_1)(s)$ are trapped in $\hat{V}_A(s)$ (see (3.15)), where $s = -\ln(T - t)$. More precisely, the following proposition is our statement:

Proposition 3.13 (Reduction to a finite dimensional problem). There exist $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0, \eta_0 > 0$ such that the following holds: We consider $U$ a solution of equation (2.8) that exists on $[0, t_1]$, for some $t_1 < T$, with initial data $U_{d_0,d_1}(0)$ given in (3.13), for some $(d_0, d_1) \in \mathcal{D}_A$. We also assume that we have $U \in S(t)$ for all $t \in [0, t_1]$ and $U \in \partial S(t_1)$ (see the definition of $S(t) = S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ in Definition 3.1 and the set $\mathcal{D}_A$ given in Proposition 3.8). Then, the following statements hold:

(i) We have $(q_0, q_1)(s_1) \in \partial \hat{V}_A(s_1)$, where $(q_0, q_1)(s)$ are components of $q(s)$ given in (2.36) and $q(s)$ is the transform function of $U$ defined in (2.22) and $s_1 = \ln(T - t_1)$.

(ii) There exists $\nu_0 > 0$ such that for all $\nu \in (0, \nu_0)$, we have

$$(q_0, q_1)(s_1 + \nu) \notin \hat{V}_A(s_1 + \nu).$$

Consequently, there exists $\nu_1 > 0$ such that

$$U \notin S(t_1 + \nu), \forall \nu \in (0, \nu_1).$$

The idea of the proof is inspired (in a non trivial way) by the ideas given by Merle and Zaag in [13]. Since the proof is long and technical, we leave it to Section 4. Therefore, we assume here that Proposition 3.13 holds and go forward to the conclusion of Proposition 3.12.

b) Topological argument and the conclusion of Proposition 3.12: In this step, by using Proposition 3.13 and a topological argument based on Index theory, we conclude Proposition 3.12. More precisely, we prove that there exist $T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0$ and $(d_0, d_1) \in \mathcal{D}_A$ such that with initial data $U_{d_0,d_1}(0)$ (defined in (3.13)), the solution of equation (2.8) exists on $[0, T)$ and belongs to $S^*(T)$ where $S^*(T)$ is defined in Definiton 3.2. Indeed, let us consider parameters $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0, \eta_0 > 0$ such that Propositions 3.8 and 3.13 hold. Using Proposition 3.8, we have the following

$$\forall (d_0, d_1) \in \mathcal{D}_A, \quad U_{d_0,d_1}(0) \in S(0).$$

In particular, it follows from Proposition 1.2.2 page 12 in Kavallaris and Suzuki [10] together with Lemma 2.1 that equation (2.8) is locally in time well-posed in $C^{2,1}(\Omega \times (0,T_0)) \subset C(\Omega \times [0,T_0])$. 


for some $T_0 > 0$. Therefore, for every $(d_0, d_1) \in \mathcal{D}_A$, we define $t^*(d_0, d_1) \in [0, T)$ as the maximum time, satisfying

$$U_{d_0, d_1} \in S(t), \forall t \in [0, t^*(d_0, d_1)),$$

where $U_{d_0, d_1}$ is the solution of (2.8) corresponding to initial data $U_{d_0, d_1}(0)$, introduced in (3.13). Then, we have two possible cases:

a) Either $t^*(d_0, d_1) = T$ for some $(d_0, d_1) \in \mathcal{D}_A$, then, we get the conclusion of the proof.

b) Or $t^*(d_0, d_1) < T$, for all $(d_0, d_1) \in \mathcal{D}_A$. This case in fact never occurs, as we will show in the following.

Indeed, assuming by contradiction that case b) hold and using the continuity of the solution in time and the definition of the maximum time $t^*(d_0, d_1)$, we have

$$U_{d_0, d_1}(t^*(d_0, d_1)) \notin \partial S(t^*(d_0, d_1)).$$

Thanks to the finite dimensional reduction property given in item (i) of Proposition 3.13, we derive the following

$$(q_0, q_1)(s_s(d_1, d_2)) \in \partial \hat{\mathcal{V}}_A(s_s(d_0, d_1)), $$

where $q_0, q_1$ are defined in (2.36) as the components of $q_{d_0,d_1}$, which is a transformed function of $U_{d_0,d_1}$ (see (2.22)) and $s_s(d_0, d_1) = -\ln(T - t^*(d_0, d_1))$. Then, we may define the following mapping

$$A: \mathcal{D}_A \rightarrow ([-1, 1] \times [-1, 1]^n)$$

$$(d_0, d_1) \mapsto \frac{s_s^2(d_0, d_1)}{A}(q_0, q_1)(s_s(d_0, d_1)).$$

From the definition of $t^*(d_1, d_2)$, the components $(q_0, q_1)$ and the transversal crossing property given in item (ii) in Proposition 3.13, we see that $A$ is continuous on $\mathcal{D}_A$. In addition to that, from item (i) of Proposition 3.8, we can derive that for all $(d_0, d_1) \in \partial \mathcal{D}_A$

$$(q_0, q_1)(s_0) \in \partial \hat{\mathcal{V}}_A(s_0), \quad s_0 = -\ln T.$$ 

However, using item (ii) of Proposition 3.13 again and the definition of $t^*(d_0, d_1)$ we deduce that

$$t^*(d_0, d_1) = 0,$$

which yields

$$s_s(d_0, d_1) = s_0 \text{ and } \Lambda(d_0, d_1) = \frac{s_s^2}{A}(d_0, d_1),$$

where $\Lambda$ is defined in item (1) of Proposition 3.8. Hence, thanks to (3.16), we conclude

$$\deg \left(\Lambda |_{\mathcal{D}_A}\right) \neq 0.$$ 

In fact, such a mapping $\Lambda$ can not exist by using Index theory. Hence, case b) doesn’t occur only case a) occurs. Thus, the conclusion of Proposition 3.12 follows. 

3.4. The conclusion of Theorem 1.1

In this subsection, we would like to give a complete proof of Theorem 1.1. We now consider the solution $U$ which has been constructed in Proposition 3.12. Then, $U$ exists on $[0, T)$ and

$$U(t) \in S(t), \forall t \in [0, T).$$

Using item (i) in Definition 3.1, we have the following

$q$ exists on $[-\ln T, +\infty)$ and $\|q(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{\sqrt{s}}, \forall s \in [-\ln T, +\infty), \quad (3.35)$
for some constant $C > 0$. Thanks to (2.4), (2.6), (2.12) and (2.18), we have
\[
\left\| \frac{(T - t)^{\frac{2}{3}} \Lambda^\frac{1}{3}}{\theta(t)(1 - u(t,.))} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\sqrt{|\ln(T - t)|}} \tag{3.36}
\]
Using (3.18) and (3.20), we can derive that $\dot{\theta}(t)$ converges to $\theta_T > 0$ with
\[
|\dot{\theta}(t) - \theta_T| \leq C(T - t)^{\frac{1}{T}}, \forall t \in [0, T).
\]
This implies (1.6) with $\theta^* = \frac{\theta_T}{\Lambda^\frac{1}{3}}$. Then, item (i) of Theorem 1.1 follows. We now prove that $u$ quenches only at 0. Indeed, from the above estimate, we can derive that 0 is a quenching point of $u$. Now, we aim at proving that $x \in \Omega \setminus \{0\}$ are not quenching points of $u$. In fact, relying on relations (2.4) and (2.6), it is enough to prove the following Lemma:

**Proposition 3.14.** The solution $U$ satisfies the following statements:

(i) For all $x \in \Omega \setminus \{0\}$, there exists $\nu(x) > 0$ such that
\[
\limsup_{t \to T} \sup_{|x'| = \nu(x)} U(x', t) < +\infty. \tag{3.37}
\]

(ii) For all $x \in \Omega \setminus \{0\}$, $\lim_{t \to T} U(x, t)$ exists. In particular, if we define $U^*(x) = \lim_{t \to T} U(x, t)$, for all $x \in \Omega \setminus \{0\}$, then $u^* \in C(\Omega \setminus \{0\})$, and $U(t)$ uniformly converges to $u^*$ on every compact subset of $\Omega \setminus \{0\}$. In particular, we have the following asymptotic behavior
\[
U^*(x) \sim \left[\frac{9}{32} \ln |x| \right]^{-\frac{1}{4}}, \text{ as } x \to 0. \tag{3.38}
\]

**Proof.** We consider $U$ the solution constructed in Proposition 3.12. The proof will be given in two parts:

- **The proof of item (i):** The proof follows from the definition of shrinking set $S(t)$. Let us consider two cases: $|x| > \frac{T}{4}, x \in \Omega$ and $|x| \leq \frac{T}{4}, x \in \Omega$.
  
  + The case where $|x| > \frac{T}{4}, x \in \Omega$: Using item (iii) of Definition 3.1, we conclude that for all $t \in [0, T)$,
    \[
    U(x, t) \leq U(x, 0) + \eta_0 < +\infty.
    \]

Then, (3.37) follows.
  
  + The case where $|x| \leq \frac{T}{4}, x \in \Omega$: For every $x$ in that region, we can find $t_x$ close to $T$ such that $|x| \in \left[\frac{\kappa_0}{4} \sqrt{(T - t_x)|\ln(T - t_x)|}, \epsilon_0 \right]$. Moreover, we derive that $|x| \in \left[\frac{\kappa_0}{4} \sqrt{(T - t)|\ln(T - t)|}, \epsilon_0 \right]$ for all $t \in [t_x, T)$. Considering $t \in [t_x, T)$ and using item (ii) in Definition 3.1, we derive the following
    \[
    U(x + \xi \sqrt{\rho(x)}, t) \leq \rho^{-\frac{1}{2}}(x) \left[U(\tau(x, t)) + \delta_0 \right], \forall |\xi| \leq \alpha_0 \sqrt{\ln \rho(x)}.
    \]

This estimate directly implies (3.37).

- **The proof of item (ii):** By using parabolic regularity and the technique given by Merle in [11], item (i) and Lemma F.1, we may derive that there exists a function $U^* \in C(\Omega \setminus \{0\})$ such that $U(x, t) \to U^*(x)$, as $t \to T$, for all $x \in \Omega, x \neq 0$. Moreover, one can prove that the convergence is uniform on every compact subset of $\Omega \setminus \{0\}$. It remains to give the asymptotic behavior (3.38). We consider $x_0 \in \Omega$ such that $|x_0|$ is small enough. We first introduce the following functions: $U(x_0, \xi, \tau)$ is defined in (3.4) and
\[
\mathcal{V}(x_0, \xi, \tau) = \nabla \xi U(x_0, \xi, \tau), \tag{3.39}
\]
where $\xi \in e^{-\frac{1}{4}}(x_0)(\Omega - x_0) \subset \mathbb{R}^n$ and $\tau \in \left[-\frac{t(x_0)}{\varrho(x_0)}, 1\right)$, where $t(x_0)$ and $\varrho(x_0)$ are defined as in (3.5) and (3.6), respectively. We aim at proving the following estimates:

$$\sup_{\tau \in [0,1], |\xi| \leq |\ln(\varrho(x_0))|^{\frac{1}{4}}} |U(x_0, \xi, \tau) - \hat{U}(\tau)| \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}},$$

(3.40) and

$$\sup_{\tau \in [0,1], |\xi| \leq 2|\ln(\varrho(x_0))|^{\frac{1}{4}}} |V(x_0, \xi, \tau)| \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}},$$

(3.41) and

$$\sup_{\tau \in [\tau_0,1], |\xi| \leq \frac{1}{2}|\ln(\varrho(x_0))|^{\frac{1}{4}}} |\partial_\tau U(x_0, \xi, \tau)| \leq C(x_0),$$

(3.42)

for some $\tau_0 \in (0,1)$, fixed, and we also recall that $\hat{U}(\tau)$ is introduced in (3.10).

We see that (3.41) follows from the fact that $U \in S(t), \forall t \in [0,T)$ and item (ii) of Definition 3.1. Thus, we only need to give the proofs of (3.40) and (3.42).

- The proof of (3.40): We write here the equation of $U$ from (3.9)

$$\partial_\tau U = \Delta_\xi U - 2\frac{|\nabla_\xi U|^2}{U} + \left(U + \frac{\lambda^4_2 \varrho^{\frac{1}{4}}(x_0)}{\hat{\varrho}(\tau)}\right)^4 - \frac{\hat{\varrho}'(\tau)}{\hat{\varrho}(\tau)} U,$$

(3.43)

where $\hat{\varrho}(\tau) = \hat{\varrho}(\varrho(x_0) + t(x_0))$ is given in (3.8). From (3.36) with $t = t(x_0)$, we derive that

$$\sup_{|\xi| \leq 6|\ln(\varrho(x_0))|^{\frac{1}{4}}} |U(x_0, \xi, 0) - \hat{U}(0)| \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}}. $$

(3.44)

In addition to that, from item (ii) of Definition 3.1, we have for all $|\xi| \leq 6|\ln(\varrho(x_0))|^{\frac{1}{4}}$ and $\tau \in [0,1)$:

$$U(x_0, \xi, \tau) \geq \frac{1}{2}\hat{U}(0),$$

(3.45)

$$U(x_0, \xi, \tau) \leq \frac{3}{2}\hat{U}(1),$$

(3.46)

provided that $\delta_0 \leq \frac{1}{2}\hat{U}(0)$. We now consider $U(\xi, \tau)$ as follows

$$U(\xi, \tau) = U(x_0, \xi, \tau) - \hat{U}(\tau),$$

where $\xi \in e^{-\frac{1}{4}}(x_0)(\Omega - x_0)$ and $\tau \in [0,1)$.

We then derive an equation satisfied by $U$

$$\partial_\tau U = \Delta_\xi U + G_1 + G_2,$$

(3.47)

where $G_1, G_2$ are defined as follows

$$G_1(\xi, \tau) = -2\frac{|\nabla U|^2}{U} + \frac{\hat{\varrho}'(\tau)}{\hat{\varrho}(\tau)} U,$$

$$G_2(\xi, \tau) = \left(U + \frac{\lambda^4_2 \varrho^{\frac{1}{4}}(x_0)}{\hat{\varrho}(\tau)}\right)^4 - \hat{U}(\tau).$$

Next, we derive from the definition of $\hat{\varrho}(\tau)$, Proposition 3.9 and the fact that for all $\tau \in (0,1)$,

$$|\hat{\varrho}'(\tau)| \leq C \varrho^{\frac{1}{12}}(x_0)(1 - \tau)^{-\frac{11}{12}},$$

and

$$1 \leq \hat{\varrho}(\tau) \leq C.$$
Hence, from (3.41), (3.45) and (3.46), we deduce that for all $\tau \in [0, 1], |\xi| \leq 2 |\ln \varrho(x_0)|^{\frac{1}{4}}$

$$|G_1(\xi, \tau)| = \left| -2 \frac{\nabla U(x_0, \xi, \tau)^2}{U(x_0, \xi, \tau)} + \frac{\varrho_+ \varrho(x_0)}{\varrho(\tau)} \left( U(x_0, \xi, \tau) \right) \right|$$

$$\leq \frac{C}{|\ln \varrho(x_0)|^{\frac{1}{4}}} \left( (1 - \tau)^{-\frac{13}{12}} + 1 \right).$$

In addition to that, we derive from (3.46) that

$$|G_2(\xi, \tau)| \leq C|U(x_0, \xi, \tau)| + \frac{C}{|\ln \varrho(x_0)|^{\frac{1}{4}}}.$$  

We now recall the cut-off function $\chi_0$, defined as in (2.16), then, we introduce

$$\phi_1(\xi) = \chi_0 \left( \frac{|\xi|}{|\ln(\varrho(x_0))|^{\frac{1}{4}}} \right).$$

As a matter of fact, we have some rough estimates on $\phi_1$

$$\|\nabla \phi_1\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}} \text{ and } \|\Delta \phi_1\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}}. \quad (3.48)$$

Let us define $U_1(\xi, \tau) = \phi_1(\xi) U(\xi, \tau)$, for all $\xi \in \mathbb{R}^n$ and $\tau \in [0, 1)$. Then, $U_1$ satisfies the following equation

$$\partial_\tau U_1 = \Delta U_1 - 2 \nabla \phi_1 \cdot \nabla U - \Delta \phi_1 U + \phi_1 G_1(\xi, \tau) + \phi_1 G_2(\xi, \tau).$$

Using Duhamel’s principal, we write an integral equation satisfied by $U_1$

$$U_1(\tau) = e^{\tau \Delta} U_1(0) + \int_0^\tau e^{(\tau - \sigma) \Delta} \left[ -2 \nabla \phi_1 \cdot \nabla U - \Delta \phi_1 U + \phi_1 G_1 + \phi_1 G_2 \right](\sigma) d\sigma.$$

This implies that for all $\tau \in [0, 1)$, we have

$$\|U_1(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}} + C \int_0^\tau \|U_1(\cdot, \sigma)\|_{L^\infty(\mathbb{R}^n)} d\sigma.$$  

Thanks to Granwall’s inequality, we get the following

$$\|U_1(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}}, \forall \tau \in [0, 1),$$

which yields (3.40).

Using (3.42), we can derive that the limit $\lim_{\tau \to 1} U(x_0, 0, \tau)$ exists. In addition to that, we derive from (3.40) that

$$U^*(x_0) = \lim_{\tau \to 1} U(x_0, 0, \tau) = \left( \frac{9}{8} \left( \frac{K_0^2}{16} \varrho(x_0) \right)^{-\frac{1}{2}} \right) \sim \left( \frac{9}{16} \frac{|x_0|^2}{|\ln |x_0||} \right)^{-\frac{1}{2}} \text{ as } x_0 \to 0.$$

This is the conclusion of (3.38). So, we get the proof in Proposition 3.14 and we also get the complete conclusion of Theorem 1.1.
4. Reduction to a finite dimensional problem

This section plays a central role in our analysis. In fact, it is devoted to the proof of Proposition 3.13. More precisely, this section has two parts:

- In the first subsection, we prove priori estimates on $U$ in $P_1(t), P_2(t)$ and $P_3(t)$ when $U$ is trapped in $S(t)$.

- The second subsection is devoted to the conclusion of Proposition 3.13. In fact, we use the first subsection to derive that $U$ satisfies almost all the conditions in $S(t)$ with strict bounds, except for the bounds on $q_0(s)$ and $q_1(s)$, with $s = -\ln(T - t)$. This means that in order to control $U$ in $S(t)$, we need to control only $(q_0, q_1)(s)$ in $\hat{V}_A(s)$, defined in (3.15). In addition to that, we also prove the outgoing transversal crossing property. It means that if the solution $U$ touches the boundary of $S(t_1)$ for some $t_1 \in (0, T)$, then, $U$ will be outside $S(t)$ for all $t \in (t_1, t_1 + \nu)$ with $\nu$ small enough. In one word, this is the reduction to a finite dimensional problem: the control of two components $(q_0, q_1)(s)$ in $\hat{V}_A(s)$.

4.1. A priori estimates

We proceed in 3 steps: $a, b$ and $c$, respectively devoted to parts $P_1(t), P_2(t)$ and $P_3(t)$.

a) We aim in the following Proposition at proving a priori estimates for $U$ in $P_1(t)$:

**Lemma 4.1.** There exists $K_4 > 0, A_4 > 0$ such that for all $K_0 \geq K_4, A \geq A_4$ and $l^* > 0$ there exists $T_4(K_0, A, l^*)$ such that for all $\varepsilon_0 > 0, \alpha_0 > 0, \delta_0 > 0, \eta_0 > 0, C_0 > 0, T \leq T_4$ and for all $l \in [0, l^*]$, the following holds: Assume that we have the following conditions:

- We consider initial data $U(0) = U_{d_0, d_1}(0)$, given in (3.13) and $(d_0, d_1) \in D_A$, given in Proposition 3.8 such that $(q_0, q_1)(s_0)$ belongs to $\hat{V}_A(s_0)$, where $s_0 = -\ln T$, $\hat{V}_A(s)$ is defined in (3.15) and $q_0, q_1$ are components of $q_{d_0, d_1}(s_0)$, a transform function of $U$, defined in (2.22).

- We have $U \in S(T, K_0, \varepsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [T - e^{-\sigma}, T - e^{-(s + l)}]$, for some $\sigma \geq s_0$ and $l \in [0, l^*]$.

Then, the following estimates hold:

(i) For all $s \in [\sigma, \sigma + l]$, we have

$$|q_0(s) - q_0(s)| + \left| q_{1,i}'(s) - \frac{1}{2} q_{1,i}(s) \right| \leq \frac{C}{s^2}, \forall i \in \{1, \ldots, n\},$$

and

$$\left| q_{2,i,j}'(s) + \frac{2}{s} q_{2,i,j}(s) \right| \leq \frac{CA}{s^2}, \forall i, j \in \{1, \ldots, n\},$$

where $q_1 = (q_{1,j})_{1 \leq j \leq n}$, $q_2 = (q_{2,i,j})_{1 \leq i \leq n, 1 \leq j \leq n}$ and $q_1, q_2$ are defined in (2.32).

(ii) Control of $q_-(y, s)$: For all $s \in [\sigma, \sigma + l], y \in \mathbb{R}^n$ we have the two following cases:

- The case where $\sigma \geq s_0$:

$$|q_-(y, s)| \leq C \left( e^{\frac{\sigma^2}{2}} + A^2 e^{-(s-\sigma)^2} + (s - \sigma) \right) \frac{(1 + |y|^3)}{s^2},$$

- The case where $\sigma = s_0$

$$|q_-(y, s)| \leq C(1 + (s - \sigma)) \frac{(1 + |y|^3)}{s^2}.$$

(iii) Control of the gradient term of $q$: For all $s \in [\sigma, \sigma + l], y \in \mathbb{R}^n$, we have the two following cases:

- The case where $\sigma \geq s_0$:

$$|(\nabla q)_\perp(y, s)| \leq C \left( e^{\frac{\sigma^2}{2}} + e^{-(s-\sigma)^2} + (s - \sigma) + \sqrt{s - \sigma} \right) \frac{(1 + |y|^3)}{s^2},$$

- The case where $\sigma = s_0$.
- The case where $\sigma = s_0$

$$\|(\nabla q)_{\perp}(y, s)\| \leq C \left(1 + (s - \sigma) + \sqrt{s - \sigma}\right) \frac{(1 + |y|^3)}{s^2}. \quad (4.6)$$

(iii) Control of the outside part $q_\epsilon$: For all $s \in [\sigma, \sigma + \lambda]$, we have the two following cases:

- The case where $\sigma \geq s_0$:

$$\|q_\epsilon(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \leq C \left(A^2 e^{-\frac{s - \sigma}{2}} + A e^{(s - \sigma)} + 1 + (s - \sigma)\right) \frac{1}{\sqrt{s}}, \quad (4.7)$$

- The case where $\sigma = s_0$

$$\|q_\epsilon(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \leq C \left(1 + (s - \sigma)\right) \frac{1}{\sqrt{s}}. \quad (4.8)$$

**Proof.** The proof of this proposition relies completely on techniques given by Merle and Zaag in [13]. As a matter of fact, the equation (2.23) is quite the same as in that paper if we ignore some perturbations which will be very small in our analysis. More precisely, thanks to Lemmas D.1, D.2, D.3, D.4, D.5 and D.6, we assert that the techniques in [13] hold in our case. Hence, we kindly refer the reader to Lemma 3.2 at page 1523 in [13] for more details. 

This implies an a priori estimates in $P_1(t)$ as follows:

**Proposition 4.2** (A priori estimates in $P_1(t)$). There exist $K_5 \geq 1$ and $A_5 \geq 1$ such that for all $K_0 \geq K_5$, $A \geq A_5$, $\epsilon_0 > 0$, $\alpha_0 > 0$, $\delta_0 \leq \frac{1}{2\lambda} U(0)$, $C_0 > 0$, $\eta_0 > 0$, there exists $T_5(K_0, \epsilon_0, A, \alpha_0, \delta_0, C_0, \eta_0)$ such that for all $T \leq T_5$, the following holds: If $U$ is a nonnegative solution of equation (2.8) satisfying $U \in S(T, K_0, \epsilon_0, A, \alpha_0, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, t_5]$ for some $t_5 \in [0, T]$, and initial data $U(0) = U_{d_0, d_1}$ given in (3.13) for some $d_0, d_1 \in \mathcal{P}_A$ given in Proposition 3.8, then, for all $s \in [-\ln T, -\ln(T - t_5)]$, we have the following:

$$\forall i, j \in \{1, \cdots, n\}, \quad |q_{2, i, j}(s)| \leq \frac{A^2 \ln s}{2s^2}, \quad \left\|q_{\cdot, (\cdot, s)}\right\|_{L^\infty} \leq \frac{A}{2s^2}, \quad \left\|\frac{\nabla q_{(\cdot, s)}}{1 + |y|^3}\right\|_{L^\infty} \leq \frac{A}{2s^2} \quad \text{and} \quad \|q_\epsilon(s)\|_{L^\infty} \leq \frac{A^2}{2\sqrt{s}},$$

where $q$ is a transformed function of $U$ given in (2.22).

**Proof.** The proof is a consequence of Lemma 4.1. In particular, the proof is the same as in the work of Merle and Zaag in [14]. Hence, we refer the reader to Proposition 3.7, page 157 in that work. 

b) We now show a priori estimate on $U$ in $P_2(t)$. We start with the following lemma:

**Lemma 4.3** (A priori estimates in the intermediate region). There exists $K_6$ and $A_6 > 0$, such that for all $K_0 \geq K_6$, $A \geq A_6$, $\delta_6 > 0$, there exists $\alpha_6(K_6, \delta_6) > 0$, $C_6(K_0, A) > 0$ such that for all $\alpha_0 \leq \alpha_6$, $C_0 > 0$, there exists $\epsilon_6(\alpha_0, A, \delta_6, C_0)$ such that for all $\epsilon_0 \leq \epsilon_6$, there exists $T_6(\epsilon_0, A, \delta_6, C_0)$ and $\eta_6(\epsilon_0, A, \delta_6, C_0) > 0$ such that for all $T \leq T_6$, $\eta_0 \leq \eta_6$, $\delta_0 \leq \frac{1}{2} \left(3 + \frac{9K_4^2}{16}\right)^{-\frac{1}{2}}$, the following holds: if $U \in S(T, K_0, \epsilon_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, t_4]$, for some $t_4 \in [0, T]$, then, for all $|x| \in \left[\frac{K_0^2}{T - t_4} \ln(T - t_4), \epsilon_0\right]$, we have:
Proof. We leave the proof to Appendix B.

Using the above lemma, we now give a priori estimates in $P_2(t)$. The following is our statement:

**Proposition 4.4** (A priori estimates in $P_2(t)$). There exists $K_7 > 0$ and $A_7 > 0$ such that for all $K_0 \geq K_7, A \geq A_7$, there exists $\delta_7 \leq \frac{1}{4}U(0)$ and $C_7(K_0, A)$ such that for all $\delta_0 \leq \delta_7, C_0 \geq C_7$ there exists $\alpha_7(K_0, \delta_0)$ such that for all $\alpha_0 \leq \alpha_7$, there exist $\epsilon_0 \leq \epsilon_7$, there exists $T_7(\epsilon_0, A, \delta_0, C_0) > 0$ such that for all $T \leq T_7$ the following holds: We assume that we have $U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, t)$ for all $t \in [0, T_7]$ for some $T_7 \in [0, T)$, then, for all $|x| \in \left[\frac{K_0}{4} \sqrt{(T-t_0)} \ln(T-t_0), \epsilon_0\right], |\xi| \leq a_0 \sqrt{\ln \varrho(x)}$ and $\tau \in \left[\max\left(0, -\frac{t_0(x)}{\varrho(x)}, 0\right), \frac{t_0(x) - t(x)}{\varrho(x)}\right]$, we have

$$|U(x, \xi, \tau_0) - \hat{U}(\tau_0)| \leq \delta_0$$

and

$$|\nabla U(x, \xi, \tau)| \leq \frac{C_0}{2 \sqrt{\ln \varrho(x)}}$$

where $\varrho(x) = T - t(x)$.

**Proof.** We leave the proof to Appendix C.

**Remark 4.5.** Unlike what Merle and Zaag did in [13], we don’t require any condition in $\nabla^2 U$ in $P_2(t)$ (see Definition 3.1), as we have already stated in Remark 3.3. Accordingly, our a priori estimates in $P_2(t)$ will be simpler than those of [13], as one may see from the proof given in Appendix C.

c) We now give a priori estimates on $U$ in $P_3(t)$:

**Proposition 4.6** (A priori estimates in $P_3$). Let us consider $K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 \in [0, \frac{1}{2}U(0)], C_0 > 0, \eta_0 > 0$. Then, there exists $T_8(\eta_0) > 0$ such that for all $T \leq T_8$, the following holds: We assume that $U$ is a nonnegative solution of (2.8) on $[0, t_8]$ for some $t_8 < T$, and $U \in S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, t_8]$ and initial data $U(0) = U_{d_0, d_1}$ given in (3.13) with $|d_0|, |d_1| \leq 2$. Then, for all $|x| \geq \frac{\epsilon_0}{4}$ and $t \in (0, t_8]$,

$$|U(x, t) - U(x, 0)| \leq \frac{\eta_0}{2},$$

$$|\nabla U(x, t) - \nabla u^i \Delta U(x, 0)| \leq \frac{\eta_0}{2}.$$  

**Remark 4.7.** As we have mentioned in Remark 3.3, we draw the attention of the reader to the change we have made with respect to the work of Merle and Zaag in [13]: We compare $\nabla U(t)$ to $\nabla u^i \Delta U(0)$ and not to $\nabla U(0)$ in [13] and this is crucial, since we are working on a bounded domain.

Following the remark, we have just stated, we give in the following a crucial parabolic estimate for the free Dirichlet heat semi-group in $\Omega$:  

(i) For all $|\xi| \leq \frac{7}{4}a_0 \sqrt{\ln \varrho(x)}$ and $\tau \in \left[\max\left(0, -\frac{t(x)}{\varrho(x)}, 0\right), \frac{t(x) - t(x)}{\varrho(x)}\right]$, the transformed function $U(x, \xi, \tau)$ defined in (3.4) satisfies the following:

$$|\nabla U(x, \xi, \tau)| \leq \frac{2C_0}{\sqrt{\ln \varrho(x)}},$$

$$\nabla U(x, \xi, \tau) \geq \frac{1}{4} \left(3 + \frac{9K_0^2}{8}\right)^{-\frac{1}{3}},$$

$$|\nabla U(x, \xi, \tau)| \leq 4.$$  

(ii) For all $|\xi| \leq 2a_0 \sqrt{\ln \varrho(x)}$ and $\tau_0 = \max\left(0, -\frac{t(x)}{\varrho(x)}\right)$, we have

$$|U(x, \xi, \tau_0) - \hat{U}(\tau_0)| \leq \delta_0$$

and

$$|\nabla U(x, \xi, \tau)| \leq \frac{C_0}{\sqrt{\ln \varrho(x)}}.$$  

Proof. We leave the proof to Appendix B.
Lemma 4.8 (A parabolic regularity on the linear problem). Let us consider initial data $U_{d_0,d_1}$, given in (3.13), for some $|d_0|,|d_1| \leq 2$. If we define

$$L(t) = e^{t\Delta} U_{d_0,d_1}, t \in (0,T].$$

Then, $L(t) \in C(\bar{\Omega} \times [0,T]) \cap C^\infty(\Omega \times (0,T])$. Moreover, the following holds

$$\|\nabla_x L(t)\|_{L^\infty(|x| \geq \frac{\epsilon_0}{8}, x \in \Omega)} \leq C(\epsilon_0), \forall [0,T],$$

(4.14)

where $\epsilon_0$ introduced in the definition of $U_{d_0,d_1}$.

Proof. See Appendix E

The proof of Proposition 4.6. We rewrite the equation satisfied by $U$ as follows

$$\partial_t U = \Delta U + G(U),$$

where

$$G(U) = -2 \frac{|\nabla U|^2}{U} + \left( U + \frac{\lambda^+}{\theta(t)} \right)^4 - \frac{\theta'(t)}{\theta(t)} U.$$ 

We remark that in order to get the conclusion, it is enough to prove that for all $x \in \Omega, |x| \geq \frac{\epsilon_0}{8}$ and $t \in (0,t_8]$, we have the following estimates

$$|U_1(x,t) - U_1(x,0)| \leq \frac{\eta_0}{2},$$

(4.15)

$$|\nabla U_1(x,t) - \nabla e^{t\Delta} U_1(x,0)| \leq \frac{\eta_0}{2},$$

(4.16)

where $U_1(x,t) = \exp \left( \int_0^t \frac{\theta'(s)}{\theta(s)} ds \right) U(x,t)$. Using the equation satisfied by $U$, we may derive an equation satisfied by $U_1$ as follows:

$$\partial_t U_1 = \Delta U_1 + G_1,$$

(4.17)

where $G_1(t) = \exp \left( \int_0^t \frac{\theta'(s)}{\theta(s)} ds \right) \left[ -2 \frac{|\nabla U|^2}{U + \frac{\lambda^+}{\theta(t)}} + \left( U + \frac{\lambda^+}{\theta(t)} \right)^4 \right]$. In particular, from the fact that $U \in S(t)$ and Proposition 3.9, we can derive the following

$$\left| \exp \left( \pm \int_0^t \frac{\theta'(s)}{\theta(s)} ds \right) \right| \leq 2.$$

Moreover, from item (iii) of Definition of 3.1 and Lemma 4.8, we derive the following:

$$|G_1(x,t)| \leq C(K_0, \epsilon_0, \eta_0), \forall |x| \geq \frac{\epsilon_0}{8} \text{ and } \forall t \in (0,t_8].$$

In the following, we first prove (4.15) then (4.16).

+ The proof of (4.15): We consider a cut-off function $\chi_2 \in C_0^\infty(\bar{\Omega})$ such that $\chi_2 = 1$ for all $|x| \geq \frac{\epsilon_0}{8}, x \in \Omega$ and $\chi_2 = 0$ for all $|x| \leq \frac{\epsilon_0}{8}$ and $|\nabla \chi_2| + |\Delta \chi_2| \leq C(\epsilon_0)$. If we define $U_2 = U_1 \chi_2$, then $U_2$ satisfies the following

$$\partial_t U_2 = \Delta U_2 + G_2,$$

where

$$G_2(U) = -2 \nabla U_1 \cdot \nabla \chi_2 - \Delta \chi_2 U_1 - \chi_2 G_1.$$

Using the estimate of $G_1$ and the following fact

$$|\nabla U_1(x,t)| + |U_1(x,t)| \leq C(K_0, \epsilon_0, \eta_0), \forall |x| \geq \frac{\epsilon_0}{8} \text{ and } t \in [0,t_8],$$

we have

$$|U_2(x,t)| \leq C(K_0, \epsilon_0, \eta_0), \forall |x| \geq \frac{\epsilon_0}{8} \text{ and } t \in [0,t_8].$$

Hence,

$$|U_1(x,t) - U_1(x,0)| \leq \frac{\eta_0}{2},$$

(4.15)

and

$$|\nabla U_1(x,t) - \nabla e^{t\Delta} U_1(x,0)| \leq \frac{\eta_0}{2},$$

(4.16)
which is a consequence of the fact that $U \in S(t)$ (particularly items (i) and (iii) in Definition 3.1), we conclude the following

$$\|G_2(x, t)\|_{L^\infty(\Omega)} \leq C(K_0, \epsilon_0, C_0, \eta_0), \forall|x| \geq \frac{\epsilon_0}{8} \text{ and } \forall t \in [0, t_8].$$

We now use a Duhamel formula to write $U_2$ as follows

$$U_2(t) = e^{t\Delta}U_2(0) + \int_0^t e^{(t-s)\Delta} (G_2(U(\tau))) d\tau, \quad (4.18)$$

where $e^{t\Delta}$ stands for the Dirichlet heat semi-group on $\Omega$ (see more in Appendix E). In particular, we have for all $U_0 \in L^\infty(\Omega),

$$\|e^{t\Delta}U_0\|_{L^\infty(\Omega)} \leq \|U_0\|_{L^\infty(\Omega)}.$$ 

Therefore,

$$|U_2(t) - U_2(0)| \leq |U_2(t) - e^{t\Delta}U_2(0)| + |e^{t\Delta}U_2(0) - U_2(0)| \leq \left| \int_0^t e^{(t-s)\Delta}G_2(s)ds \right| + |e^{t\Delta}U_2(0) - U_2(0)| \leq C(K_0, \epsilon_0, C_0, \eta_0)T + \|e^{t\Delta}(U_2(0)) - U_2(0)\|_{L^\infty(\Omega)}.$$ 

In addition to that, because $U_2(0)$ is smooth and has a compact support in $\Omega$, we can prove that

$$\|e^{t\Delta}(U_2(0)) - U_2(0)\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to 0,$$

which yields the fact that

$$\|U_2(t) - U_2(0)\|_{L^\infty(\Omega)} \leq \frac{\eta_0}{2},$$

provided that $T \leq T_{8,1}(K_0, \epsilon_0, C_0, \eta_0)$. This concludes the proof of (4.15).

+ The proof of (4.16): We derive from (4.18) the following fact:

$$\nabla U_2(t) = \nabla e^{t\Delta}U_2(0) + \int_0^t \nabla e^{(t-s)\Delta}G_2(\tau)d\tau.$$ 

This implies that

$$|\nabla U_2(t) - \nabla e^{t\Delta}U_1(0)| \leq |\nabla e^{t\Delta}U_2(0) - \nabla e^{t\Delta}U_1(0)| + \left| \int_0^t \nabla e^{(t-s)\Delta}G_2(\tau)d\tau \right|.$$ 

Using (E.3) and Lemma (E.1) below, we derive that

$$\left| \int_0^t \nabla e^{(t-s)\Delta}G_2(\tau)d\tau \right| \leq C(K_0, \epsilon_0, C_0, \eta_0) \int_0^t \frac{1}{\sqrt{t-s}}d\tau \leq C(K_0, \epsilon_0, C_0, \eta_0)\sqrt{T}.$$ 

In order to finish the proof, it is enough to prove that for all $|x| \geq \frac{\epsilon_0}{4}$, we have

$$|\nabla e^{t\Delta}U_2(0) - \nabla e^{t\Delta}U_1(0)| \leq \frac{\eta_0}{4}, \quad (4.19)$$
provided that \( T \leq T_{8.2} \). Indeed, using the definition of the Dirichlet heat semi-group and Lemma E.1 below, we may write the following:

\[
|\nabla e^{t\Delta}U_2(0) - \nabla e^{t\Delta}U_1(0)| = \left| \int_{\Omega} \nabla_x G(x, y, t, 0)(1 - \chi_2(y))U_{d_0,d_1}(y)dy \right|
\]

\[
\leq \frac{C}{t^{n+1}} \left( \frac{|x-y|^2}{t} \right)^n |U_{d_0,d_1}(y)|dy
\]

\[
\leq C(\epsilon) \sqrt{t} \int_{|y| \leq \frac{\epsilon}{6}} |U_{d_0,d_1}(y)|dy
\]

\[
\leq C(\epsilon_0) \sqrt{T} \|U_{d_0,d_1}\|_{L^1(\Omega)} \leq C(\epsilon_0) \sqrt{T}.
\]

This yields (4.19), provided that \( T \leq T_{8.3}(\epsilon_0) \). In particular, from the definitions of \( U_2 \) and \( U_1 \), we can derive (4.16). Finally, we get the conclusion of Proposition 4.6. \( \Box \)

### 4.2. The conclusion of the proof of Proposition 3.13

It this part, we aim at giving a complete proof to Proposition 3.13:

The proof of Proposition 3.13. We first choose the parameters \( K_0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, \delta_1 > 0, C_0 > 0, \eta_0 > 0 \) and \( T > 0 \) such that Propositions 3.8, 4.2, 4.4 and 4.6 hold. In particular, the constant \( T \) will be fixed small later. Then, the conclusion of the proof follows as we will show in the following. We now consider \( U \), a solution of equation (2.8), with initial data \( U_{d_0,d_1}(0) \), defined in Definition 3.13 and satisfying the following:

\[
U \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, C_0, \eta_0, t) = S(t),
\]

for all \( t \in [0, t_*] \) for some \( t_* \in (0, T) \) and \( u \in \partial S(t_*) \).

(i) Using Propositions 4.2, 4.4 and 4.6, we can derive that

\[
(q_1, q_2)(s_*) \in \partial \hat{V}_A(s_*),\tag{4.20}
\]

where \( s_* = \ln(T - t_*) \).

(ii) Using item (i), we derive that either

\[
|q_0(s_*)| = \frac{A}{s_*^2},
\]

or there exists \( j_0 \in \{1, \ldots, n\} \) such that

\[
|q_{1,j_0}(s_*)| = \frac{A}{s_*^2}.
\]

Then, without loss of generality, we can suppose that the first case occurs, because the argument is the same in other cases. Hence, using (4.1) in Lemma 4.1, we see that

\[
|q'_0(s) - q_0(s)| \leq \frac{C}{s_*^2}.
\]

Therefore, we obtain that the sign of \( q'_0(s_*) \) is opposite to the sign of

\[
\frac{d}{ds} \left( \frac{\epsilon_0 A}{s_*^2} \right)(s_*),
\]
provided that $A \geq 2C$, where $\epsilon_0 = \pm 1$ and $q_0(s_*) = \epsilon_0 \frac{A}{s^2}$. This means that the flow of $q_0$ is transverse outgoing on the bounds of the shrinking set

$$-\frac{A}{s^2} \leq q_0(s) \leq \frac{A}{s^2}.$$

It follows then that $(q_0,q_1)(s)$ leaves $\hat{\mathcal{V}}(s)$ at $s_*$. Thus, we conclude item (ii). Finally, we get the conclusion of Proposition 3.13

\[ \square \]

A. Preparation of initial data

In this section, we give the proof of Proposition 3.8. More precisely, we aim at proving the following lemma which directly implies Proposition 3.8:

**Lemma A.1.** There exists $K_2 > 0$ such that for all $K_0 \geq K_2, \delta_2 > 0$, there exist $\alpha_2(K_0, \delta_2) > 0, C_2 > 0$ such that for all $\alpha_0 \in (0, \alpha_2]$ there exists $\alpha_2(K_0, \delta_2, \alpha_0) > 0$ such that for all $\epsilon_0 \in (0, \epsilon_2]$ and $A \geq 1$, there exists $T_2(K_0, \delta_2, \epsilon_0, A, C_2) > 0$ such that for all $T \in (0, T_2)$, there exists a subset $\mathcal{D}_A \subset [-2, 2]^{1+n}$ such that the following properties hold: Assume that initial data $U_{d_0,d_1}(0)$ is given as in (3.13), then:

A) For all $(d_0, d_1) \in \mathcal{D}_A$, we have initial data $U(0) = U_{d_0,d_1}(0) \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_2, 0, C_2, 0, 0)$. In particular, we have the following:

(i) Estimates in $P_1(0)$: we have the transformed function $q(s_0)$ of $U_{d_0,d_1}(0)$, trapped in $V_{K_0,A}(s_0)$, where $s_0 = -\ln T$ and we have also the following estimates:

$$|q_0(s_0) - \frac{Ad_0}{s_0^2}| + |q_{1,j}(s_0) - \frac{Ad_{1,j}}{s_0^2}| \leq Ce^{-s_0}, \text{ for all } j \in \{1, \ldots, n\},$$

$$|q_{2,i,j}(s_0)| \leq \frac{\ln s_0}{s_0^2}, \text{ for all } i, j \in \{1, \ldots, n\},$$

$$|q_-(y, s_0)| \leq \frac{1}{s_0^2}(1 + |y|^3), \quad |(\nabla_y q)_\bot(y, s_0)| \leq \frac{1}{s_0^2}(1 + |y|^3), \text{ for all } y \in \mathbb{R}^n,$$

and

$$q_+(s_0) \equiv 0,$$

where the components of $q$ are defined in (2.37).

(ii) Estimates in $P_2(0)$: For all $|x| \in \left[\frac{K_0}{4} \sqrt{T} \ln T, \epsilon_0\right]$ and and $|\xi| \leq \alpha_0 \sqrt{\ln g(x)}$, we have

$$|\mathcal{U}(x, \xi, \tau_0(x)) - \hat{\mathcal{U}}(\tau_0(x))| \leq \delta_2, \text{ and } |\nabla_\xi \mathcal{U}(x, \xi, \tau_0(x))| \leq \frac{C_2}{\sqrt{\ln g(x)}},$$

where $\tau_0(x) = -\frac{t(x)}{\varrho(x)}$ and $\mathcal{U}, \hat{\mathcal{U}}, t(x), g(x)$ are given in (3.4), (3.5), (3.6) and (3.10).

B) We have the following

$$(d_0, d_1) \in \mathcal{D}_A \text{ if and only if } (q_0, q_1)(s_0) \in \hat{\mathcal{V}}_A(s_0)$$

and

$$(d_0, d_1) \in \partial \mathcal{D}_A \text{ if and only if } (q_0, q_1)(s_0) \in \partial \hat{\mathcal{V}}_A(s_0),$$

where $\hat{\mathcal{V}}_A(s)$ given in (3.15).

**Proof.** We see that part B) directly follows from item (i) of part A). In addition to that, our definition is almost the same as in [19] (see also the work of Ghoul, Nguyen and Zaag [5], the works of Merle and Zaag [13, 14]). So, we kindly refer the reader to see the proofs of the existence of the set $\mathcal{D}_A$, item i in A) and part B) in Proposition 4.5 in [19]. Here we only give the proof of item (ii) in part A). We now consider $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, \delta_2 > 0, C_2 > 0, \eta_0 > 0$. We
aim at proving that if these constants are suitably chosen, then for all \( x \in \left[ \frac{2K_0}{T} \ln |T|, \epsilon_0 \right] \) and \(|\xi| \leq 2\alpha_0 \sqrt{|\ln g(x)|}\), where \( g(x) \) given in (3.5), we have the following

\[
\left| \mathcal{U}(x, \xi, \tau_0(x)) - \hat{\mathcal{U}}(\tau_0(x)) \right| \leq \delta_2, \quad |\nabla \xi \mathcal{U}(x, \xi, \tau_0(x))| \leq \frac{C_2}{\sqrt{|\ln g(x)|}}.
\]

We observe from the definition of \( t(x) \) given in (3.5) that if \( \alpha_0 \leq \alpha_{2,1} \) and \( \epsilon_0 \leq \epsilon_{2,1} \), then, for all \( x \in \left[ \frac{2K_0}{T} \ln |T|, \epsilon_0 \right] \) and \(|\xi| \leq 2\alpha_0 \sqrt{|\ln g(x)|}\), we have

\[
|\xi \sqrt{g(x)}| \leq \frac{|x|}{2},
\]

which yields

\[
\frac{r_0}{2} \leq \frac{|x|}{2} \leq |x + \xi \sqrt{T(x)}| \leq \frac{3}{2}|x|, \quad \text{with} \quad r_0 = \frac{K_0}{4} \sqrt{T} |\ln T|.
\]

Hence, for all \( x \in \left[ \frac{2K_0}{T} \ln |T|, \epsilon_0 \right] \), we have

\[
\chi \left( 16(x + \xi \sqrt{g(x)}) \sqrt{T}, -\ln T \right) \chi_1(x + \xi \sqrt{g(x)}) = 0,
\]

where \( \chi \) and \( \chi_1 \) are defined in (2.31) and (3.11), respectively. Therefore, from (3.13) and the definition of \( \mathcal{U} \) in (3.4), we may derive that for all \( x \in \left[ \frac{2K_0}{T} \ln |T|, \epsilon_0 \right] \) and \(|\xi| \leq 2\alpha_0 \sqrt{|\ln g(x)|}\),

\[
\mathcal{U}(x, \xi, \tau_0) = (I) \chi_1 \left( x + \xi \sqrt{g(x)} \right) + (II) \left( 1 - \chi_1(x + \xi \sqrt{g(x)}) \right),
\]

where

\[
(I) = \left( \frac{g(x)}{T} \right)^{\frac{1}{2}} \left( 3 + \frac{9}{8} \frac{|x + \xi \sqrt{g(x)}|^2}{T \ln |T|} \right)^{-\frac{1}{2}},
\]

and

\[
(II) = g^{\frac{1}{2}}(x) H^*(x + \xi \sqrt{g(x)}),
\]

with \( H^*(x) \) given in (3.12). In addition to that, from the definition of \( g(x) \), given in (3.6), we obtain the following asymptotics

\[
\ln g(x) \sim 2 \ln |x| \quad \text{and} \quad g(x) \sim \frac{8}{R_0^2} \frac{|x|^2}{\ln |x|} \quad \text{as} \quad |x| \to 0.
\]

Besides that, we introduce \( r_0 = \frac{K_0}{4} \sqrt{T} |\ln T| \) and \( R_0 = \sqrt{T} |\ln T| \). Then, the following holds

\[
g(r_0) \sim T, \quad g(R_0) \sim \frac{16}{K_0^2} T |\ln T| \quad \text{and} \quad g(2R_0) \sim \frac{64}{R_0^2} T |\ln T| \quad \text{as} \quad T \to 0.
\]

We aim in the following at giving some estimates on \( \mathcal{U}(x, \xi, \tau_0(x)) \) and \( \nabla \xi \mathcal{U}(x, \xi, \tau_0(x)) \).

- Estimate on \( \mathcal{U} \): From the definition of the cut-off function \( \chi_1 \) given in (3.11), it is enough to prove that for all \(|x| \in \left[ r_0, 2 + \frac{1}{100} R_0 \right] \) and \(|\xi| \leq 2\alpha_0 \sqrt{|\ln g(x)|}\), we have

\[
\left| \frac{1}{T} - \hat{\mathcal{U}}(\tau_0) \right| \leq \frac{\delta_2}{2}, \quad \text{(A.4)}
\]

on one hand and also that for all \(|x| \in \left[ \frac{99}{100} R_0, \epsilon_0 \right] \) and \(|\xi| \leq 2\alpha_0 \sqrt{|\ln g(x)|}\), we have

\[
\left| \frac{1}{T} - \hat{\mathcal{U}}(\tau_0) \right| \leq \frac{\delta_2}{2}, \quad \text{(A.5)}
\]
on the other hand. Indeed, let us start with the proof of (A.4): We consider \( |x| \in \left[ r_0, (2 + \frac{1}{100}) R_0 \right] \) and \( |\xi| \leq 2\alpha_0 \sqrt{\ln \rho(x)} \). Then, we write the following:

\[
|I_1 - \tilde{U}(\tau_0(x))| = \left| \left( \frac{3}{\rho(x)} + \frac{|x + \xi \sqrt{\rho(x)}|^2}{8 |\rho(x)| \ln T} \right)^{-\frac{1}{2}} - \left( \frac{3}{\rho(x)} + \frac{9 K_0^2}{8 16} \right)^{-\frac{1}{2}} \right|
\]

\[
= \left| \left( \frac{3}{\rho(x)} + \frac{9 K_0^2}{8 16} + \frac{9}{8} \left[ \frac{|x + \xi \sqrt{\rho(x)}|^2}{|\rho(x)| \ln T} - \frac{K_0^2}{16} \right] \right)^{-\frac{1}{2}} - \left( \frac{3}{\rho(x)} + \frac{9 K_0^2}{8 16} \right)^{-\frac{1}{2}} \right|.
\]

In addition to that, we have

\[
\frac{|x + \xi \sqrt{\rho(x)}|^2}{\rho(x)|\ln T|} - \frac{K_0^2}{16} = \frac{|x|^2}{\rho(x)|\ln T|} \left( 1 + 2 \frac{x \cdot \xi}{|\rho(x)|} \frac{\sqrt{\rho(x)}}{|x|^2} + \frac{|\xi|^2 \rho(x)}{|x|^2} \right) - \frac{K_0^2}{16} = \frac{K_0^2}{16} \ln \rho(x) \left( 1 + 2 \frac{x \cdot \xi}{|\rho(x)|} \frac{\sqrt{\rho(x)}}{|x|^2} + \frac{|\xi|^2 \rho(x)}{|x|^2} \right) - \frac{K_0^2}{16}.
\]

Besides that, we also have the following:

\[
\frac{|x \cdot \xi|}{|x|^2} \sqrt{\rho(x)} \leq 4\alpha_0,
\]

\[
\frac{|\xi|^2}{|x|^2} \rho(x) \leq 4\alpha_0^2.
\]

Moreover, for all \( |x| \in \left[ r_0, (2 + \frac{1}{100}) R_0 \right] \), we derive from (A.3) that

\[
\frac{\ln \rho(x)}{|\ln T|} \sim 1, \text{ as } T \to 0.
\]

So, the following holds

\[
\left| \frac{|x + \xi \sqrt{\rho(x)}|^2}{\rho(x)|\ln T|} - \frac{K_0^2}{16} \right| \to 0,
\]

as \((\alpha_0, T) \to (0, 0)\). From this fact, we can derive that if \( T \leq T_{2,1}(K_0, \delta_2), \alpha_0 \leq \alpha_{2,2}(K_0, \delta_2), \) we have

\[
|I_1 - \tilde{U}(\tau_0(x))| \leq C(K_0) \left| \frac{|x + \xi \sqrt{T(x)}|^2}{T(x)|\ln T|} - \frac{K_0^2}{16} \right| \leq \frac{\delta_1}{2}.
\]

This concludes the proof of (A.4).

We now aim at proving (A.5). We consider \( |x| \in \left[ \frac{90}{100} R_0, \epsilon_0 \right] \) and \( |\xi| \leq 2\alpha_0 \sqrt{\ln \rho(x)} \). Using the definition of (II), we write as follows

\[
|\text{II} - \tilde{U}(\tau_0(x))| = \left| \left( \frac{9}{16 \rho(x)|\ln |x + \xi \sqrt{\rho(x)}||} \right)^{-\frac{1}{3}} - \left( \frac{3}{\rho(x)} + \frac{9 K_0^2}{8 16} \right)^{-\frac{1}{3}} \right|
\]

\[
= \left| \left( \frac{9 K_0^2}{8 16} + \frac{9}{16} \left( \frac{|x + \xi \sqrt{\rho(x)}|^2}{\rho(x)|\ln |x + \xi \sqrt{\rho(x)}||} - \frac{K_0^2}{8} \right) \right)^{-\frac{1}{3}} - \left( \frac{9 K_0^2}{8 16} + \frac{3}{\rho(x)} \right)^{-\frac{1}{3}} \right|.
\]
Besides that, the function $\varrho(x)$ is radial in $x$, and increasing in $|x|$ when $|x|$ is small enough. Then, for all $\epsilon_0 \leq \epsilon_{2,1}$ and $|x| \in \left[ \frac{99}{100} R_0, \epsilon_0 \right]$, we have

$$\left| \frac{T}{\varrho(x)} \right| \leq \frac{T}{\varrho \left( \frac{99}{100} R_0 \right)} \leq C(K_0) |\ln T|^{-1} \rightarrow 0 \text{ as } T \rightarrow 0. \quad (A.6)$$

In addition to that, we have

$$\left| \frac{x + \xi \sqrt{\varrho(x)}}{\varrho(x) \ln |x + \xi \sqrt{\varrho(x)}|} \right|^2 - \frac{K^2_0}{8} = \frac{1}{\varrho(x) \ln |x + \xi \sqrt{\varrho(x)}|} \left[ |x|^2 + 2x \cdot \xi \sqrt{\varrho(x)} + |\xi|^2 \varrho(x) \right] - \frac{K^2_0}{8}$$

$$= \frac{K^2_0}{16} \left[ \frac{|\ln \varrho(x)|}{\ln |x + \xi \sqrt{\varrho(x)}|} - 2 + 4\alpha_0 \frac{|\ln \varrho(x)|}{|\ln |x + \xi \sqrt{\varrho(x)}|} \right] + 4\alpha_0^2 \frac{|\ln \varrho(x)|}{|\ln |x + \xi \sqrt{\varrho(x)}|} \right].$$

In particular, we have the following fact

$$\ln \varrho(x) \sim 2 \ln |x|, \text{ as } |x| \sim 0,$$

$$\frac{1}{|\ln |x + \xi \sqrt{\varrho(x)}||} \sim \frac{1}{|\ln |x||}, \text{ as } \alpha_0 \rightarrow 0.$$ This yields

$$\left| \frac{x + \xi \sqrt{\varrho(x)}}{\varrho(x) \ln |x + \xi \sqrt{\varrho(x)}|} \right|^2 - \frac{K^2_0}{8} \rightarrow 0 \text{ as } (\epsilon_0, \alpha_0) \rightarrow (0, 0). \quad (A.7)$$

From (A.6) and (A.7), we derive that

$$\left| (II) - \hat{U}(\tau_0(x)) \right| \leq C(K_0) \left[ \left| \frac{x + \xi \sqrt{\varrho(x)}}{\varrho(x) \ln |x + \xi \sqrt{\varrho(x)}|} - \frac{K^2_0}{8} \right| + \frac{T}{\varrho(x)} \right] \leq \frac{\delta_2}{2},$$

provided that $\alpha \leq \alpha_{2,3}(K_0, \delta_2), \epsilon_0 \leq \alpha_{2,2}(K_0, \delta_2, \alpha_0)$ and $T \leq T_{2,3}$. Thus, (A.5) holds. Finally, we get the conclusion that for all $|x| \in \left[ \frac{K_0}{4} \sqrt{T |\ln T|}, \epsilon_0 \right]$ and $|\xi| \leq 2\alpha_0 \sqrt{|\ln \varrho(x)|}$, we have

$$\left| U(x, \xi, \tau_0(x)) - \hat{U}(\tau_0(x)) \right| \leq \delta_2.$$

*Estimate on $\partial_2 U$:* From the definition of $U(x, \xi, \tau_0(x)) = U \left( x, \xi, -\frac{t(x)}{\varrho(x)} \right)$ given in (3.4) and the expression (3.13) of initial data, we decompose $\nabla_2 U$ as follows

$$\partial_2 U(x, \xi, \tau_0(x)) = B_1 + B_2 + B_3,$$
Then, we have

\[
B_1 = \left[ -\frac{3}{4} \frac{\varrho(x)}{T^{\frac{3}{4}} \ln T} \left( x + \xi \sqrt{\varrho(x)} \right) \left( 3 + \frac{9}{8} \frac{|x + \xi \sqrt{\varrho(x)}|^2}{T \ln T} \right)^{-\frac{3}{4}} \right] \chi_1(x + \xi \sqrt{\varrho(x)}), \quad (A.8)
\]

\[
B_2 = \frac{\varrho(x)}{T} \nabla H^*(x + \xi \sqrt{\varrho(x)}) \left( 1 - \chi_1(x + \xi \sqrt{\varrho(x)}) \right), \quad (A.9)
\]

\[
B_3 = \left[ \left( \frac{\varrho(x)}{T} \right)^{\frac{1}{4}} \left( 3 + \frac{9}{8} \frac{|x + \xi \sqrt{\varrho(x)}|^2}{T \ln T} \right)^{-\frac{1}{4}} + 3 \frac{\ln \varrho(x)}{T^\frac{5}{4}} - \varrho(x) H^*(x + \xi \sqrt{\varrho(x)}) \right] \times \sqrt{\varrho(x)} \nabla \chi_1(x + \xi \sqrt{\varrho(x)}). \quad (A.10)
\]

It is enough to prove the following estimates:

- Estimate of \( B_1 \): For all \(|x| \in \left[ 0; (2 + \frac{1}{100}) R_0 \right]\) and \(|\xi| \leq 2\alpha_0 \sqrt{|\ln \varrho(x)|}\) we have

\[
|B_1| \leq \frac{C(K_0)}{\sqrt{|\ln \varrho(x)|}}. \quad (A.11)
\]

- Estimate of \( B_2 \): For all \(|x| \in \left[ \frac{99}{100} R_0, \epsilon_0 \right]\) and \(|\xi| \leq 2\alpha_0 \sqrt{|\ln \varrho(x)|}\), we have

\[
|B_2| \leq \frac{C(K_0)}{\sqrt{|\ln \varrho(x)|}}. \quad (A.12)
\]

- Estimate of \( B_3 \): For all \(|x| \in \left[ \frac{99}{100} R_0, (2 + \frac{1}{100}) R_0 \right]\) and \(|\xi| \leq 2\alpha_0 \sqrt{|\ln \varrho(x)|}\), we have

\[
|B_3| \leq \frac{C(K_0)}{\sqrt{|\ln \varrho(x)|}}. \quad (A.13)
\]

We now start the proof:

- Estimate of \( B_1 \): We have the fact that for all \(|z| \geq 1\)

\[
\left( 3 + \frac{9}{8} |z|^2 \right)^{-\frac{5}{4}} \leq C |z|^{-\frac{8}{3}}.
\]

Then,

\[
|B_1| \leq C \frac{\varrho(x)}{T^{\frac{3}{4}} \ln T} \frac{T^{\frac{5}{4}} \ln T^{\frac{3}{4}}}{|x + \xi \sqrt{\varrho(x)}|^{\frac{5}{4}}}
\]

\[
\leq C \frac{\varrho(x)}{|x + \xi \sqrt{\varrho(x)}|^{\frac{5}{4}}}.
\]

Using (A.1), we obtain the following:

\[
|B_1| \leq C \frac{\varrho(x)}{|x|^{\frac{3}{4}}} \ln T^{\frac{1}{4}}.
\]

In addition to that, for all \(|x| \in \left[ r_0, (2 + \frac{1}{100}) R_0 \right]\), we have

\[
|\ln \varrho(x)| \sim |\ln T|, \text{ as } T \to 0.
\]

Then, we have

\[
|B_1| \leq C \frac{\varrho(x)}{K_0 \varrho(x)} \ln \varrho(x) \frac{|\ln T|^{\frac{1}{4}}}{|x|^{\frac{3}{4}}} \leq \frac{C}{\sqrt{|\ln \varrho(x)|}},
\]

provided that \( K_0 \geq K_{2,3}, T \leq T_{2,4} \). This yields (A.11).
- **Estimate of $B_2$:** From the definition of $H^*(x)$, when $|x| \leq \epsilon_0$, $\epsilon_0$ small enough, we have

$$H^*(x) = \left[ \frac{9}{16} \frac{|x|^2}{\ln |x|} \right]^{-\frac{1}{4}}.$$  

This implies

$$|\nabla H^*(x)| \leq C \frac{\ln |x|^{\frac{1}{4}}}{|x|^{\frac{7}{4}}}.$$  

Hence,

$$|B_2| \leq C \frac{\rho_0^{\frac{1}{2}}(x) \ln |x|^{\frac{1}{4}}}{|x|^{\frac{7}{4}}} \leq C \frac{\ln |x|^{\frac{1}{4}}}{|\ln \varrho(x)|^{\frac{1}{4}}} \frac{1}{\sqrt{|\ln \varrho(x)|}},$$  

on one hand. On the other hand, we have the following

$$|\ln \varrho(x)| \sim 2|\ln |x||, \quad \text{as } x \to 0.$$  

Thus, (A.12) holds provided that $\epsilon_0 \leq \epsilon_{2.4}(K_0)$.

- **Estimate of $B_3$:** We first use the definition of $\chi_1$ in (3.11) to write

$$|\nabla \chi_1(x)| \leq \frac{C}{\sqrt{T} \ln T}.$$  

We now consider $|x| \in \left[ \frac{99}{100} R_0, (2 + \frac{1}{100}) R_0 \right]$ and $|\xi| \leq 2\alpha_0 \sqrt{|\ln \varrho(x)|}$. We define

$$B_4 = \left( 3T + \frac{9}{8} \frac{|x + \xi \sqrt{\varrho(x)}|^2}{|\ln T|} \right)^{-\frac{3}{4}} + \frac{3 - 4n}{T^{\frac{1}{4}} |\ln T|} - H^*(x + \xi \sqrt{\varrho(x)}).$$  

Then,

$$B_3 = B_4 \rho_0^{\frac{1}{2}}(x) \nabla \chi_1 \left( x + \xi \sqrt{\varrho(x)} \right).$$  

We now aim at giving some estimates on $B_4$ as follows: Using the fact that $|x| \in \left[ \frac{99}{100} R_0, (2 + \frac{1}{100}) R_0 \right]$, $|\xi| \leq 2\alpha_0 \sqrt{|\ln \varrho(x)|}$ and (A.1), we can derive that

$$\frac{1}{C} T |\ln T| \leq \frac{|x + \xi \sqrt{\varrho(x)}|^2}{|\ln T|} \leq C T |\ln T|,$$  

$$\frac{1}{C} T |\ln T| \leq \frac{|x + \xi \sqrt{\varrho(x)}|^2}{|\ln |x + \xi \sqrt{\varrho(x)}||} \leq C T |\ln T|.$$  

This implies that

$$|B_4| \leq C (T |\ln T|)^{-\frac{3}{4}}.$$  

Hence, we estimate $B_3$:

$$|B_3| \leq C (T |\ln T|)^{-\frac{3}{4}} \rho_0^{\frac{1}{2}}(x) \nabla \chi_1 \left( x + \xi \sqrt{\varrho(x)} \right)$$  

$$\leq C \rho_0^{\frac{1}{2}}(x) \frac{1}{T^{\frac{3}{4}}} \frac{1}{|\ln T|^{\frac{3}{4}}}.$$  

In addition to that, for all $|x| \in \left[ \frac{99}{100} R_0, (2 + \frac{1}{100}) R_0 \right]$, we use (A.3) to deduce that

$$|\varrho(x)| \leq C T |\ln T|,$$  

and we also have the following fact

$$|\ln \varrho(x)| \sim |\ln T|, \quad \text{as } T \to 0.$$
So, we conclude that

$$|B_3| \leq \frac{C}{\sqrt{|\ln g(x)|}}$$

provided that $K_0 \geq K_{2.4}, \epsilon_0 \leq \epsilon_{2.5}(K_0, \alpha_0)$ and $T \leq T_{2.5}(K_0)$. Thus, we get the conclusion of (A.13). Finally, the conclusion of Lemma A.1 follows. \(\square\)

**B. A priori estimates in the intermediate region**

In this section, we aim at giving the proof of Lemma 4.3. Because our definitions are the same as in [13], estimates in this Proposition follow in the same way as in that work. Hence, we kindly refer the reader to Lemma 2.6 in page 1515 in that work for the proof of (4.9) and item (ii). It happens that, although the authors in [13] gave a statement which is similar to (4.10), they did not give the proof. For that reason, we give here the proof of (4.10) and (4.11).

- The proof of (4.10): We consider $|x| \in \left[ \frac{K_0}{T} \sqrt{(T - t_s)} ln(T - t_s), \epsilon_0 \right], |\xi| \leq \frac{7}{4} \alpha_0 \sqrt{|\ln g(x)|}$ and $\tau \in \left[ \max \left(0, -\frac{t(x)}{g(x)} \right), \frac{t_s - t(x)}{g(x)} \right]$. As a matter of fact, there exists $t \in [0, t_s]$ such that

$$\tau = \frac{t - t(x)}{g(x)}.$$

Let us define

$$X = x + \xi \sqrt{g(x)}.$$

We aim at considering the three following cases:

+ The case where $|X| \leq \frac{K_0}{T} \sqrt{(T - t_s)} ln(T - t_s)$.

We write

$$U(x, \xi, \tau) = g^{\frac{3}{2}}(x)U(X, t).$$

We have the fact that $X \in P_1(t)$. Then, using item (i) in Definition 3.1 together with item (i) in Lemma 3.6, we get

$$(T - t)^{-\frac{3}{2}} U(X, t) = W(Y, s), \quad \text{where} \quad Y = \frac{X}{\sqrt{T - t}}, s = -\ln(T - t)$$

$$\geq \left( 3 + \frac{|X|^2}{(T - t) \ln(T - t)} \right)^{-\frac{1}{2}} \frac{CA^2}{\sqrt{s}}$$

$$\geq \frac{1}{2} \left( 3 + \frac{9 K_0^2}{8} \right)^{-\frac{1}{2}},$$

provided that $T \leq T_{6.1}(K_0, A)$. This yields

$$U(x, \xi, \tau) \geq \left( \frac{g(x)}{T - t} \right)^{-\frac{3}{2}} \frac{1}{2} \left( 3 + \frac{9 K_0^2}{8} \right)^{-\frac{1}{2}}.$$

In addition to that, the function $|x| \mapsto g(x)$ is increasing when $|x|$ is small enough. This implies that

$$g(x) \leq g \left( \frac{K_0}{4(1 - \frac{7}{4} \alpha_0)} \sqrt{(T - t_s) \ln(T - t_s)} \right).$$

From (3.5), (3.6) and (A.2), we derive that

$$g \left( \frac{K_0}{4(1 - \frac{7}{4} \alpha_0)} \sqrt{(T - t_s) \ln(T - t_s)} \right) \sim \frac{8}{K_0^2} \frac{K_0^2}{16(1 - \frac{7}{4} \alpha_0)^2} \frac{2(T - t_s) \ln(T - t_s)}{|\ln(T - t_s)|} = \frac{(T - t)}{(1 - \frac{7}{4} \alpha_0)^2},$$

as $T \to 0$. Hence, we have

$$\frac{g(x)}{T - t} \leq 4,$$
provided that \( \alpha_0 \leq \frac{2}{7}, T \leq T_{6,2} \). Finally, we get
\[
U(x, \xi, \tau) \geq \frac{1}{4} \left( 3 + \frac{8 K_0^2}{9 \cdot 16} \right)^{-\frac{1}{8}}.
\]

+ The case where \( |X| \in \left[ \frac{K_0}{4} \sqrt{(T-t)} |\ln(T-t)|, \epsilon_0 \right] \). In other words, we have \( X \in P_2(t) \). We write as follows
\[
U(x, \xi, \tau) = \varrho(x) U(X, t).
\]

In addition to that, using item (ii) in the definition of \( S(t) \) (see Definition 3.1), we get the following:
\[
U(X, t) = \varrho^{-\frac{1}{3}}(X) U(X, 0, \frac{t-t(X)}{\varrho(X)}) \geq \varrho^{-\frac{1}{3}}(X) \frac{1}{2} \left( 3 + \frac{9 K_0^2}{8 \cdot 16} \right)^{-\frac{1}{8}},
\]
provided that \( \delta_0 \leq \frac{1}{2} \left( 3 + \frac{9 K_0^2}{8 \cdot 16} \right)^{-\frac{1}{8}} \). In particular, using the fact that
\[
(1 - \frac{7}{4} \alpha_0)|x| \leq |X| \leq (1 + \frac{7}{4} \alpha_0)|x|.
\]

Then, we get
\[
\frac{1}{2} \left( \frac{\varrho(x)}{\varrho(X)} \right)^{\frac{1}{3}} \geq \frac{1}{2},
\]
provided that \( \alpha_0 \leq \alpha_{7,2}(K_0) \) and \( |x| \leq \epsilon_{7,2}(K_0, \alpha_0) \). This yields that
\[
U(x, \xi, \tau) \geq \frac{1}{4} \left( 3 + \frac{9 K_0^2}{8 \cdot 16} \right)^{-\frac{1}{8}}.
\]

+ The case where \( |X| \geq \epsilon_0 \). This means \( X \in P_3(t) \). We first have the following fact
\[
U(x, \xi, \tau) = \varrho^\frac{1}{3}(x) U(X, t) \geq \frac{1}{2} \varrho^\frac{1}{3}(x) U(X, 0),
\]
provided that \( \eta_0 \leq \frac{1}{2} \) and \( \epsilon_0 \leq \epsilon_{6,3} \). We remark also that \( |X| \leq (1 + \frac{7}{4} \alpha_0)|x| \leq (1 + \frac{7}{4} \alpha_0) \epsilon_0 \leq \frac{3}{2} \epsilon_0 \). Then,
\[
U(X, 0) = \left[ \frac{9}{16} \frac{|X|^2}{|X||} \right]^{-\frac{1}{8}}.
\]
Moreover, using (A.2) and (B.1), we get
\[
\varrho^\frac{1}{3}(x) U(X, 0) \geq \frac{1}{\sqrt{2}} \left[ 9 \frac{K_0^2}{8 \cdot 16} \right]^{-\frac{1}{3}} \geq \frac{1}{2} \left( 3 + \frac{9 K_0^2}{8 \cdot 16} \right)^{-\frac{1}{3}},
\]
provided that \( \alpha_0 \leq \alpha_{6,4}, \epsilon_0 \leq \epsilon_{6,3} \).

As a matter of fact, we obtain the following
\[
U(x, \xi, \tau) \geq \frac{1}{4} \left( 3 + \frac{9 K_0^2}{8 \cdot 16} \right)^{-\frac{1}{8}}.
\]

This completely concludes the proof of (4.10).

- The proof of (4.11): The idea of the proof is similar to the first one. We also consider three cases
  + The case where \( |X| \leq \frac{K_0}{4} \sqrt{(T-t)} |\ln(T-t)| \). This implies that \( X \in P_1(t) \). We write here
    \[
    U(x, \xi, \tau) = \varrho^\frac{1}{3}(x) U(X, t).
    \]
Using item (i) in the definition of $S(t)$ (see Definition 3.1), together with item (i) in Lemma 3.6, we derive that

$$|U(X, t)| \leq (T - t)^{-\frac{1}{8}} \left[ \left( 3 + \frac{9}{8(T-t)} |X|^2 \right)^{-\frac{1}{4}} + \frac{CA^2}{\sqrt{|\ln(T-t)|}} \right] \leq 2(T - t)^{-\frac{1}{4}},$$

provided that $T \leq T_{6.5}$. In addition to that, from the following fact

$$\frac{K_0}{4} \sqrt{\varrho(X) \ln \varrho(X)} = |X| \leq \frac{K_0}{4} \sqrt{(T-t) \ln(T-t)},$$

this yields that

$$\varrho(X) \leq T - t.$$

Then,

$$U(x, \xi, \tau) \leq 2 \left( \frac{\varrho(x)}{\varrho(X)} \right)^\frac{1}{3}.$$

On the other hand, using (B.1), we can derive

$$\frac{\varrho(x)}{\varrho(X)} \leq 2,$$

provided that $\alpha_0 \leq \alpha_{6.4}$. This also yields that

$$U(x, \xi, \tau) \leq 4.$$  \hspace{1cm} \text{(B.2)}

+ The case where $|X| \in \left[ \frac{K_0}{4} \sqrt{(T-t) \ln(T-t)}, \epsilon_0 \right]$. This means $X \in P_2(t)$. We write

$$U(x, \xi, \tau) = \varrho^\frac{1}{3}(x) \varrho^{-\frac{1}{3}}(X) U(X, 0, \frac{t-t(X)}{\varrho(X)}).$$

Hence, we derive from item (ii) of Definition 3.1, the fact that $U \in S(t)$ and (B.1) that

$$U(x, \xi, \tau) \leq \left( \frac{\varrho(x)}{\varrho(X)} \right)^\frac{1}{3} U(X, 0, \frac{t-t(X)}{\varrho(X)}) \leq 4,$$

provided that $K_0 \geq K_{6.2}, \alpha_0 \leq \alpha_{6.4}(K_0), \delta_0 \leq \delta_{6.1}$.  \hspace{1cm}

+ The case where $|X| \geq \epsilon_0$. The result follows from item (iii) of Definition 3.1.  \hspace{1cm}

Hence, (4.11) follows. Finally, we get the conclusion of Lemma 4.3.

C. A priori estimate on $P_2(t)$

In this section, we aim at giving the proof of Proposition 4.4

The proof of Proposition 4.4. We first choose parameters $K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, \delta_6$ such that Lemma 4.3 holds. Then, items (i) and (ii) in that Lemma hold. We would like to prove that: for all

$$|x| \in \left[ \frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}, \epsilon_0 \right], |\xi| \leq \alpha_0 \sqrt{\ln \varrho(x)}, \tau \in \left[ \max \left( 0, \frac{t-t(x)}{\varrho(x)} \right), \frac{t_7 - t(x)}{\varrho(x)} \right] = [\tau_0, \tau_1],$$

the following holds

$$|U(x, \xi, \tau) - \bar{U}(\tau)| \leq \frac{\delta_0}{2},$$

$$|\nabla_{\xi} U(x, \xi, \tau)| \leq \frac{C_0}{2 \sqrt{|\ln \varrho(x)|}}.$$  \hspace{1cm} \text{(C.1)}

$$|\nabla_{\xi} U(x, \xi, \tau)| \leq \frac{C_0}{2 \sqrt{|\ln \varrho(x)|}}.$$  \hspace{1cm} \text{(C.2)}
We first recall equation (3.7)
\[ \partial_{\tau}U = \Delta_{\tau}U - 2\frac{|\nabla U|^2}{U + \frac{\lambda^4 \theta^\frac{1}{2}(x)}{\theta(\tau)}} + \left( U + \frac{\lambda^4 \theta^\frac{1}{2}(x)}{\theta(\tau)} \right)^4 - \frac{\tilde{\theta}'(\tau)}{\theta(\tau)}U. \]

- The proof of (C.1): We first introduce the following function
\[ Z(\xi, \tau) = U(x, \xi, \tau) - \bar{U}(\tau). \]
Using (3.7), we write the following equation
\[ \partial_{\tau}Z = \Delta Z + \left( U + \frac{\tilde{\theta}(\tau)\theta^\frac{1}{2}(x)}{\lambda^4} \right)^4 - \bar{U}(\tau)^4 + G(\xi, \tau), \]
where
\[ G(\xi, \tau) = -2\frac{|\nabla U|^2}{U + \frac{\lambda^4 \theta^\frac{1}{2}(x)}{\theta(\tau)}} - \frac{\tilde{\theta}'(\tau)}{\theta(\tau)}U. \]
Using Proposition 3.9 and the definition of \( \tilde{\theta}(\tau) \) in (3.8), we derive that
\[ \left| \tilde{\theta}'(\tau) \right| \leq C \frac{|\theta^\frac{1}{2}|}{\theta^\frac{1}{2}}(1 - \tau)^{-\frac{11}{24}}. \tag{C.3} \]
Hence, from Lemma 4.4, we derive the following: for all \( |\xi| \leq \frac{7}{4} \alpha_0 \sqrt{|\ln \rho(x)|} \) and \( \tau \in [\tau_0, \tau_f] \),
\[ |G(\xi, \tau)| \leq \frac{C}{|\ln \rho(x)|}\left((1 - \tau)^{-\frac{11}{24}} + 1\right), \]
provided that \( |x| \leq \epsilon_{\tau,2}(K_0, \delta_0) \). In particular,
\[ \left| \left( U + \frac{\tilde{\theta}(\tau)\theta^\frac{1}{2}(x)}{\lambda^4} \right)^4 - \bar{U}(\tau)^4 \right| \leq C \left( |Z| + \theta^\frac{1}{2}(x) \right). \]

We here define \( \chi_1(\xi) = \chi_0 \left( \frac{|\xi|}{\sqrt{\ln \rho(x)}} \right) \), where \( \chi_0 \in C^\infty_0(\mathbb{R}) \), \( \chi_0(x) = 1, \forall|x| \leq \frac{\xi}{4}, \chi_0(x) = 0, \forall|x| \geq \frac{7}{4} \), and \( 0 \leq \chi_0 \leq 1 \). As a matter of fact, we have the following estimates
\[ |\nabla \chi_1| \leq \frac{C}{\sqrt{|\ln \rho(x)|}} \quad \text{and} \quad |\nabla^2 \chi_1| \leq \frac{C}{|\ln \rho(x)|}. \tag{C.4} \]
Introducing
\[ Z_1(\xi, \tau) = \chi_2(\xi)Z(\xi, \tau), \]
we then write an equation satisfied by \( Z_1 \)
\[ \partial_{\tau}Z_1 = \Delta Z_1 + G_1(\xi, \tau), \]
where \( G_1 \) satisfies the following: for all \( |\xi| \leq \frac{7}{4} \alpha_0 \sqrt{|\ln \rho(x)|} \)
\[ |G_1(x, \xi, \tau)| \leq C(|Z_1| + \frac{1}{|\ln \rho(x)|}\left((1 - \tau)^{-\frac{11}{24}} + 1\right), \]
Using Duhamel’s principal, we derive the following
\[ ||Z_1(\tau)||_{L^\infty} \leq (\delta_6 + \frac{C}{|\ln \rho(x)|^{\frac{1}{2}}} + C \int_{\tau_0}^{\tau} ||Z_1(s)||_{L^\infty} ds \]
\[ \leq 2\delta_6 + C \int_{0}^{\tau} ||Z_1(s)||_{L^\infty} ds. \]
Using Gronwall’s inequality, we get the following

$$\|Z_1\|_{L^\infty(\mathbb{R}^n)} \leq 2C\delta_0.$$ 

In particular, if we choose $C_0 \geq 4C\delta_0$, then (C.1) follows.

- The proof of (C.2): We rely on the idea as for the proof of (C.1). We consider $Z_2(\xi, \tau) = \chi_1 U(x, \xi, \tau) \exp \left( \int_{\tau_0}^\tau \frac{\partial'(s)}{\theta(s)} ds \right)$, where $\chi_1$ given in the proof of (C.1). Then, we can derive an equation satisfied by $Z_2$ as follows

$$\partial_\tau Z_2 = \Delta Z_2 + \chi_1 U^4 \exp \left( \int_{\tau_0}^\tau \frac{\partial'(s)}{\theta(s)} ds \right) + G_2(\xi, \tau),$$

where $G_2$ defined by

$$G_2(\xi, \tau) = \exp \left( \int_{\tau_0}^\tau \frac{\partial'(s)}{\theta(s)} ds \right) \left[ -2\nabla \chi_1 \cdot \nabla U - \Delta \chi_1 U - \frac{\chi_1 |\nabla U|^2}{U + \chi_1 \frac{\theta'_\tau}{\theta(\tau)}} + \chi_1 \left( U + \frac{\chi_1}{\theta(\tau)} \right)^4 - \chi_1 U^4 \right].$$

In particular, from (C.3), we can get the following fact

$$\left| \exp \left( \pm \int_{\tau_0}^\tau \frac{\partial'(s)}{\theta(s)} ds \right) \right| \leq 2, \forall \tau \in [\tau_0, \tau],$$

as $|x| \leq \epsilon_{8,1}$. Then, using the results in Lemma 4.3, we can deduce the following

$$\|G_2(\cdot, \tau)\|_{L^\infty} \leq \frac{C}{|\ln \theta(x)|}, \forall \tau \in [\tau_0, \tau],$$

provided that $|x| \leq \epsilon_{8,2}(K_0)$. We write $Z_2$ in the following integral equation

$$Z_2(\tau) = e^{(\tau-\tau_0)\Delta} Z_2(\tau_0) + \int_{\tau_0}^\tau e^{(\tau-\sigma)\Delta} \left[ \chi_1 U^4(\sigma) \exp \left( \int_{\tau_0}^\sigma \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) + G_2(\sigma) \right] d\sigma.$$ (C.7)

We now aim at proving the following estimates:

$$\left\| \nabla e^{(\tau-\tau_0)\Delta} Z_2(\tau_0) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_6 + C}{\sqrt{|\ln \theta(x)|}},$$

(C.8)

$$\left\| \nabla e^{(\tau-\sigma)\Delta} \left( \chi_1 U^4(\sigma) \exp \left( \int_0^\sigma \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq C \|\nabla Z_2\|_{L^\infty(\mathbb{R}^n)} + \frac{C}{\sqrt{|\ln \theta(x)|}}.$$ (C.9)

- The proof of (C.8): We write $e^{(\tau-\tau_0)\Delta} Z_2(\tau_0)$ as follows

$$e^{(\tau-\tau_0)\Delta} Z_2(\tau_0)(\xi, \tau) = \int_{\mathbb{R}^n} \frac{e^{-\frac{|\xi-\xi'|^2}{4\pi(\tau-s)}}}{(4\pi(\tau-s))^\frac{n}{2}} \chi_1(\xi') U(x, \xi', \tau_0(x)) \exp \left( \int_0^\sigma \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi'.$$
This yields
\[
\left| \nabla e^{(\tau - \tau_0)} \Delta Z_2(\tau_0)(\xi, \tau) \right| = \left\| \nabla e^{(\tau - s)} \Delta \left( \chi_1 \mathcal{U}^4(s) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) \right)(\xi, s) \right\|
\]
\[
= \int_{\mathbb{R}^n} \nabla e^{(\tau - s)} \Delta \chi_1(\xi') \mathcal{U}(x, \xi', \tau) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi' |
\]
\[
= \int_{\mathbb{R}^n} \nabla e^{(\tau - s)} \Delta \chi_1(\xi') \mathcal{U}(x, \xi', \tau) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi' |
\]
\[
\leq \int_{\mathbb{R}^n} e^{(\tau - s)} \Delta \chi_1(\xi') \mathcal{U}(x, \xi', \tau) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi' |
\]
\[
+ \int_{\mathbb{R}^n} e^{(\tau - s)} \Delta \chi_1(\xi') \mathcal{U}(x, \xi', \tau) \nabla \mathcal{U}(x, \xi', s) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi' .
\]
Thus, using the above estimate, the result of item (ii) in Lemma 4.3 and (C.4), we can conclude (C.8).

+ The proof of (C.9):
\[
\left| \nabla e^{(\tau - s)} \Delta \left( \chi_1 \mathcal{U}^4(s) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) \right)(\xi, s) \right|
\]
\[
= \int_{\mathbb{R}^n} \nabla e^{(\tau - s)} \Delta \chi_1(\xi') \mathcal{U}(x, \xi', \tau) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi' |
\]
\[
= \int_{\mathbb{R}^n} \nabla e^{(\tau - s)} \Delta \chi_1(\xi') \mathcal{U}(x, \xi', \tau) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi' |
\]
\[
\leq \int_{\mathbb{R}^n} e^{(\tau - s)} \Delta \chi_1(\xi') \mathcal{U}(x, \xi', \tau) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi' |
\]
\[
+ \int_{\mathbb{R}^n} e^{(\tau - s)} \Delta \chi_1(\xi') \mathcal{U}(x, \xi', \tau) \nabla \mathcal{U}(x, \xi', s) \exp \left( \int_{\tau_0}^{s} \frac{\partial'(\sigma)}{\theta(\sigma)} d\sigma \right) d\xi' .
\]
Then, using (4.11), (C.4) and the definition of \( Z_2(s) \), we get the following
\[
\left| \nabla e^{(r-s)\Delta} \left( \chi \mu^4(s) \exp \left( \int_{\tau_0}^{s} \frac{\ddot{\theta}(\sigma)}{\dot{\theta}(\sigma)} d\sigma \right) \right) (\xi, s) \right| \leq C\| \nabla Z_2(s) \|_{L^\infty(\mathbb{R}^n)} + \frac{C}{\sqrt{\ln \rho(x)}}
\]
which yields (C.9).

We now come back to the proof of (C.2). We use (C.7), (C.8) and (C.9) to obtain the following
\[
\| \nabla Z_2(\tau) \|_{L^\infty(\mathbb{R}^n)} \leq C_0 + C\sqrt{\ln \rho(x)} + C \int_{\tau_0}^{\tau} \| \nabla Z_2(s) \|_{L^\infty(\mathbb{R}^n)}.
\]
Thanks to Gronwall’s inequality, we derive the following
\[
\| \nabla Z_2(\tau) \|_{L^\infty(\mathbb{R}^n)} \leq \frac{C(C_0)}{\sqrt{\ln \rho(x)}}
\]
In addition to that, from the definition of \( Z_2 \), we deduce that for all \(|\xi| \leq \alpha_0 \sqrt{\ln \rho(x)}\),
\[
Z_2(\xi, \tau) = \mu(x, \xi, \tau) \exp \left( \int_{\tau_0}^{\tau} \frac{\ddot{\theta}(\sigma)}{\dot{\theta}(\sigma)} d\sigma \right).
\]
This implies that
\[
|\nabla \xi \mu(x, \xi, \tau)| \leq \frac{2C(C_0)}{\sqrt{\ln \rho(x)}}
\]
Finally, if we take \( C_0 \geq 4C(C_0) \), then
\[
|\nabla \xi \mu(x, \xi, \tau)| \leq \frac{C_0}{2\sqrt{\ln \rho(x)}},
\]
which implies (C.2).

\[\square\]

D. Some bounds on terms in equation (2.23)

In this section, we give essential ingredients for the proof of Lemma 4.1. More precisely, we will estimate some functions involved in equation (2.23): \( V, J, B, R, N \) and \( F \). In fact, as we explained in the proof Section right after Lemma 4.1, we choose not to prove Lemma 4.1, in order to avoid lengthy estimates already mentioned by Merle and Zaag in [13]. The interested reader may use our estimates in this section and follow the proof of Lemma 3.2 on page 1523 in [13] in order to check the argument.

Let us first give some estimates on \( V(y, s) \):

**Lemma D.1** (Expansion and bounds on the potential \( V \)). We consider \( V \) defined in (2.25). Then, the following holds: \( V \) is bounded on \( \mathbb{R}^n \times [1, +\infty) \) and for all \( s \geq 1 \)
\[
|V(y, s)| \leq C \frac{(1 + |y|^2)}{s}, \forall y \in \mathbb{R}^n,
\]
and
\[
V(y, s) = -\left( \frac{|y|^2 - 2n}{4s} \right) + \tilde{V}(y, s),
\]
where \( \tilde{V} \) satisfies the following
\[
|\tilde{V}(y, s)| \leq C(K_0) \frac{(1 + |y|^4)}{s^2}, \forall |y| \leq K_0 \sqrt{s}.
\]

**Proof.** The proof is easily derived from the explicit formula of \( V \). We kindly refer the readers to self-check or see Lemma B.1, page 1270 in [15] with \( p = 4 \). \[\square\]
We now give a bound on the quadratic term $B(q)$.

**Lemma D.2** (A bound on $B(q)$). Let us consider $B(q)$ defined in (2.27). If $\theta(s) \geq 1$, for all $s$ and $|q| \leq 1$, then, the following holds

$$|B(q)| \leq C(K_0) \left( |q|^2 + e^{-\frac{t}{s}} \right).$$

*Proof.* By using Newton binomial formula, the conclusion directly follows. \qed

Next, we aim at giving some bounds on $J(q, \theta(s))$. The following is our statement:

**Lemma D.3** (Bound on $J(q, \theta(s))$). For all $K_0 > 0$, $A \geq 1$ and $\epsilon_0 > 0$, there exist $\eta_0(\epsilon_0)$ and $T_0(K_0, \epsilon_0, A)$ such that for all $c_0 > 0, C_0 > 0$ and $T \leq T_0, \delta_0 \leq \frac{1}{2}\hat{U}(0)$ and $\eta_0 \leq \eta_0$, the following holds: If $U \in S(T, K_0, \epsilon_0, c_0, A, \delta_0, C_0, \eta_0, t)$, and for all $s \in [0, T)$, then, for all $y \leq 2K_0 \sqrt{s}$, $s = -\ln(T - t)$, we have the following estimates:

\[
\left| \left( J(q, \theta(s)) + 4 \frac{\nabla \varphi \cdot \nabla q}{\varphi + \frac{\hat{\theta}(s)}{\theta(s)}} \right) \right| \leq C(K_0, A) \left( \frac{|y|^2}{s^2} |q| + s^{-1}|q|^2 + |\nabla q|^2 \right),
\]

\[
|J(q, \theta(s))| \leq C(K_0, A) \left( \frac{|q|}{s} + \frac{|\nabla q|}{\sqrt{s}} \right),
\]

where $q$ is a transformed function of $U$ given in (2.22) and $J(q, \theta(s))$ is defined in (2.26). In particular, for all $y \in \mathbb{R}^n$, we have

\[
|\left( 1 - \chi(y, s) \right)T(q, \theta(s))| \leq C(K_0, C_0) \min \left( \frac{1}{s}, \frac{|y|^3}{s^{3/2}} \right).
\]

*Proof.* The techniques of the proof of estimates (D.1), (D.2) and (D.3) are the same. Although, function $J(q, \theta)$ is our work has some differences from the work of Merle and Zaag in [13], we assert that the proof still holds with our model. In order to show this argument, we kindly ask to refer the reader to check Lemma B.4 in that work. For that reason, we only give the proof of (D.1) and (D.2) here, and we leave the proof of (D.3) for the reader to be done similarly as for comparison Lemma B.4 in [13]. We now consider $|y| \leq 2K_0 \sqrt{s}$, and introduce $G(h) = \frac{-2 |\nabla \varphi + h\nabla q|^2}{\varphi + \frac{\hat{\theta}(s)}{\theta(s)} + hq} + \frac{2 |\nabla \varphi|^2}{\varphi + \frac{\hat{\theta}(s)}{\theta(s)} + hq}, h \in [0, 1]$. Then, we have the following:

\[
G'_h(h) = \frac{2q |\nabla \varphi + h\nabla q|^2}{\left( \varphi + \frac{\hat{\theta}(s)}{\theta(s)} + hq \right)^2} - 4 \frac{\nabla q(\nabla \varphi + h\nabla q)}{\varphi + \frac{\hat{\theta}(s)}{\theta(s)} + hq},
\]

\[
G''_h(h) = -4q^2 \frac{|\nabla \varphi + h\nabla q|^2}{\left( \varphi + \frac{\hat{\theta}(s)}{\theta(s)} + hq \right)^3} + 8q \frac{\nabla q(\nabla \varphi + h\nabla q)}{\varphi + \frac{\hat{\theta}(s)}{\theta(s)} + hq} - 4 \frac{|\nabla q|^2}{\varphi + \frac{\hat{\theta}(s)}{\theta(s)} + hq}.
\]

Using a Taylor expansion of $G(h)$ on $[0, 1]$, at $h = 0$, we get the following:

\[
G(1) = G(0) + G'(0) + \int_0^1 (1 - h)G''(h)dh.
\]

Using the following facts

\[
G(1) = J(q, \theta(s)), G(0) = 0,
\]
we write the following
\[ T(q, \theta(s)) = \left( \frac{2q|\nabla \varphi|^2}{\varphi + \frac{q}{s} e^{-\frac{y}{\theta(s)}}} - \frac{4\nabla \varphi \cdot \nabla q}{\varphi + \frac{q}{s} e^{-\frac{y}{\theta(s)}}} \right) + \int_0^1 (1 - h)G''(h)dh \]

From the definition of \( \varphi \) given in (2.21), we can derive that for all \( s \geq 1 \) and \( y \in \mathbb{R}^n \), we have
\[ \frac{|\nabla \varphi(y, s)|^2}{\varphi^2(y, s)} \leq C \frac{|y|^2}{s} \quad \text{and} \quad |\nabla \varphi(y, s)| \leq Cs^{-\frac{1}{2}}. \]

In addition to that, using Lemma 3.6, we can prove that there exists \( s_9(A, K_0) \) such that for all \( s \geq s_9 \), \( h \in [0, 1] \) and \( |y| \leq 2K_0\sqrt{s} \), we have the following
\[ |F''(h)(y, s)| \leq C(A, K_0) \left( \frac{|q|^2}{s} + |\nabla q|^2 \right) \leq C(A, K_0) \left( \frac{|q|}{s} + \frac{|\nabla q|}{\sqrt{s}} \right). \]

Thus, (D.1) and (D.2) follow. \( \square \)

We now aim at giving some estimates on \( R \). The following is our statement:

**Lemma D.4 (Bounds on \( R \)).** Let us consider \( R \) defined in (2.28). We assume that \( \theta(s) \geq 1 \), for all \( s \geq 1 \). Then, for all \( s \geq 1 \) and \( y \in \mathbb{R}^n \), the following holds:
\[ \left| R(y, s) - \frac{c_1}{s^2} \right| \leq C \frac{(1 + |y|^3)}{s^3}, \]
and
\[ |\nabla R(y, s)| \leq C \frac{(1 + |y|^3)}{s^3}. \]

In particular,
\[ \|R(., s)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{s}. \]

**Proof.** The function \( R \), in our work is different from the definition in [13] (up to a very small difference). Hence, the proof of [13] holds in our case with minor adaptation. Accordingly, we kindly refer the reader to check Lemma B.5 page 1541 in that work. \( \square \)

We now give some estimates on \( N \). The control of this term is a new contribution of our study. In addition to that, it is a direct consequence of Proposition 3.9 on the control \( \theta(t) \). The following is our statement:

**Lemma D.5 (Bound on \( N(q, \theta(s)) \)).** There exists \( K_{10} > 0 \) such that for all \( K_0 \geq K_{10}, A > 0 \) and \( \delta_0 \leq \frac{1}{2} \left( 3 + \frac{q}{s} \right)^{-\frac{1}{2}} \), there exist \( \alpha_{10}(K_0, \delta_0) > 0 \) and \( C_{10}(K_0) > 0 \) such that for every \( \alpha_0 \in (0, \alpha_{10}] \) we can find \( \epsilon_{10}(K_0, \delta_0, \alpha_0) > 0 \) such that for every \( \alpha_0 \in (0, \epsilon_{10}], \eta_0 \leq 1 \), there exists \( T_{10}(K_0) > 0 \) such that for all \( T \leq T_{10} \), the following holds: Assume that \( U \) is a nonnegative solution of equation (2.8) on \([0, t_{10}]\) for some \( t_{10} \leq T_{10} \), and initial data \( U(0) = U_{d_0, d_1} \) given in (3.13) for some \( (d_0, d_1) \in \mathbb{R} \times \mathbb{R}^n \) satisfying \( |d_0|, |d_1| \leq 2 \), and \( U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t) \) for all \( t \in [0, t_{10}] \). Then, for all \( s = -\ln(T - t) \) with \( t \in [0, t_{10}] \), the following estimate holds:
\[ \|N(q, \theta(s))\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{s^{2019}}, \]

where \( N(q, \theta(s)) \) is defined in (2.29).
Proof. Using the fact that $U$ is in $S(t)$, item (i) in Definition 3.1 and item (i) of Lemma 3.6, we derive that
\[ \|(q + \varphi)(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \leq C. \]
Hence, it is enough to find a bound on the following quantity
\[ \frac{\theta'(s)}{\theta(s)}. \]
(see in definition (2.29)). As a matter of fact, using Proposition 3.9, it is clear to have the following
\[ \left| \frac{\theta'(s)}{\theta(s)} \right| = \left| \frac{\theta'(t)}{\theta(t)} \right| \leq C e^{\frac{s_0 - s}{\theta_0}} |s^n|. \]
Hence, there exists $s_{10}$ large enough such that for all $s \geq s_0 \geq s_{10}$, we can write
\[ \|N(q, \theta(s))\|_{L^\infty(\mathbb{R}^n)} \leq C e^{-\frac{s}{s_{10}}} \leq \frac{1}{s^{2019}}, \]
which yields the conclusion of the proof.

Finally, we give a bound on $F(w, W)$. As a matter of fact, this is an important bridge that connects the problems in $\mathbb{R}^n$ and in a bounded domain. In other words, it is created by the localization around blowup region. Fortunately, this term is controled as a small perturbation in our analysis. More precisely, the following is our statement:

**Lemma D.6 (Bound on $F(w, W)$).** Let us consider $F(w, W)$, defined in (2.20). Then, there exists $\epsilon_{11} > 0$ such that $K_0 > 0$, $\epsilon_0 \leq \epsilon_{11}$, $\alpha_0 > 0$, $A \geq 0$, $\delta_0 > 0$, $C_0 > 0$, $\eta_0 > 0$, there exists $T_{11} > 0$ such that for all $T \leq T_{11}$, the following holds: Assuming that $U \in S(T, K_0, \epsilon_0, A, \delta_0, C_0, \eta_0, t)$, for all $t \in [0, t_{11}]$, for some $t_{11} \in [0, T)$, then, we have
\[ \|F(w, W)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{s^{2019}}, \]
where $s = -\ln(T - t)$.

**Proof.** From the definition of $F$, it is enough to consider $|y| \in \left[ \frac{\epsilon - \epsilon_0}{2\epsilon_0}, \frac{2\epsilon + \epsilon_0}{2\epsilon_0} \right]$. We now take $\epsilon_0 \leq \frac{1}{2\epsilon_0}$, then, this domain corresponds to the region $P_3(t)$ where our solution $U$ is regarded as a perturbation of initial data. Using the fact that $U$ is in $S(t)$, then, we can derive from item (iii) in Definition 3.1 that
\[ |W(y, s)| \leq C(K_0) e^{-\frac{s}{3}}, \]
\[ |\nabla_y W(y, s)| \leq C(K_0) e^{-\frac{s}{3}}. \]
In addition to that, from definition (2.18), we deduce that
\[ |w(y, s)| \leq C(K_0, M_0) e^{-\frac{s}{3}}, \]
\[ |\nabla w(y, s)| \leq C(K_0, M_0) e^{-\frac{s}{3}}. \]
On the other hand, using the definition of $\psi_{M_0}$ given in (2.17), we get the following
\[ \left| \partial_s \psi_{M_0} - \Delta \psi_{M_0} + \frac{1}{2} y \cdot \nabla \psi_{M_0} \right| \leq C(M_0). \]
In fact, using the above estimate, we can get the conclusion if $s \geq s_0(K_0)$.

\[ \square \]
E. The Dirichlet heat semi-group on $\Omega$

In this section, we aim at giving some main properties of the Dirichlet heat semi-group $\left(e^{t\Delta}\right)_{t>0}$ (see more details in [18] or chapter 16 in [16]). In particular, we prove the parabolic regularity estimate of Lemma 4.8. We consider the following equation

$$
\begin{cases}
\partial_t U - \Delta U &= 0 \text{ in } \Omega \times (0, T), \\
U &= 0 \text{ in } \partial\Omega \times (0, T), \\
U(x, 0) &= U_0(x) \text{ in } \Omega.
\end{cases}
$$

(E.1)

In particular, one can prove that there exists $G(x, y, t, \tau), t \geq \tau$ nonnegative, symmetric in $x, y$, i.e $G(x, y, t, \tau) = G(y, x, t, \tau)$ and defined in $\Omega \times \Omega \times (0, T) \times (0, T)$ with the following condition

$$
\begin{cases}
(\partial_t - \Delta)G(x, y, t, \tau) &= \delta(x - y)\delta(t - \tau), \\
G(x, y, \tau, \tau) &= 0 \text{ and } G(x, y, t, \tau) = 0 \text{ if } x \in \partial\Omega.
\end{cases}
$$

(E.2)

Moreover, for all $f \in L^\infty(\Omega)$, we have

$$(e^{t\Delta} f)(x) = \int_\Omega G(x, y, t, 0) f(y) dy.$$  

(E.3)

Hence, we can write the solution of equation (E.1) as follows

$$U(t) = e^{t\Delta}(U_0).$$

We now consider furthermore the following non-homogeneous equation

$$
\begin{cases}
\partial_t U - \Delta U &= F \text{ in } \Omega \times (0, T), \\
U &= 0 \text{ in } \partial\Omega \times (0, T), \\
U(x, 0) &= U_0(x) \text{ in } \Omega.
\end{cases}
$$

(E.4)

If $F \in C(\Omega \times (0, T))$, $u_0 \in C(\Omega)$ and $\Omega$ is $C^2$, bounded domain in $\mathbb{R}^n$. Then, we can prove that there locally exists a classical solution of problem (E.4). Then, by using Duhamel principal, the solution satisfies the following integral equation

$$U(t) = e^{t\Delta}(U_0) + \int_0^t e^{(t-s)\Delta} F(s) ds.$$

Sometimes, we also call $G(x, y, t, \tau)$ the Green function. Let us give in the following the main properties of the Green function:

**Lemma E.1.** Let us consider the Green function called $G(x, y, t, \tau)$ above. Then, the following holds: for all $(x, y, t, \tau) \in \Omega \times \Omega \times (0, T) \times (0, T)$ and integer numbers $r, s$, we have

$$
\left| \partial_t^r \partial_{x_1}^s \cdots \partial_{x_n}^s G(x, y, t, \tau) \right| \leq C(t - \tau)^{-\frac{n+2r+s}{2}} \exp \left( -c(\Omega) \frac{|x - y|^2}{t - \tau} \right).
$$

**Proof.** We kindly refer the reader to see Theorem 16.3, page 413 in [16].

We now prove in the following Lemma 4.8

**The proof of Lemma 4.8.** From the definition of the semigroup $e^{t\Delta}$, it is easy to derive that $L(t) \in C(\Omega \times [0, T]) \cap C^\infty(\Omega \times (0, T)]$. Hence, it is enough to give the proof of (4.14). Indeed, we first derive the support of $U_{d_0, d_1} = \{ |x| \leq \frac{d_0}{2} d(0, \partial\Omega) \}$. We now consider two following regions:

$$
\begin{align*}
\Omega_1 &= \left\{ \frac{6}{8} \leq |x| \leq \frac{7}{8} d(0, \partial\Omega) \right\}, \\
\Omega_2 &= \left\{ |x| > \frac{3}{4} d(0, \partial\Omega) \right\} \cap \Omega.
\end{align*}
$$
In addition to that, we can write $L_1(t)$ as follows

$$L(x, t) = \int_{\Omega} G(x, y, t, 0) U_{d_0, d_1}(y) dy = \int_{\{y \leq \frac{1}{2}d(0, \partial \Omega)\}} G_{\Omega}(x, y, t, 0) U_{d_0, d_1}(y) dy,$$

which yields

$$\nabla L(x, t) = \int_{\{y \leq \frac{1}{2}d(0, \partial \Omega)\}} \nabla_x G(x, y, t, 0) U_{d_0, d_1}(y) dy.$$  (E.6)

- We consider the case where $x \in \Omega_2$ : Thanks to Lemma E.1 and (E.6), we have

$$|\nabla L(x, t)| \leq \int_{\{y \leq \frac{1}{2}d(0, \partial \Omega)\}} |\nabla_x G(x, y, t, 0)||U_{d_0, d_1}(y)| dy
\leq \int_{\{y \leq \frac{1}{2}d(0, \partial \Omega)\}} C \exp \left( -c_\Omega \frac{|x-y|^2}{t} \right) \frac{t^{\frac{n+1}{2}}}{|x-y|^{n+1}} |U_{d_0, d_1}(y)| dy
\leq C \int_{\{y \leq \frac{1}{2}d(0, \partial \Omega)\}} \frac{|U_{d_0, d_1}(y)|}{|x-y|^{n+1}} dy$$

Because $x \in \Omega_2$, we have the following fact

$$\frac{1}{|x-y|^{n+1}} \leq C.$$

This yields the following

$$|\nabla L(x, t)| \leq C \int_{\{y \leq \frac{1}{2}d(0, \partial \Omega)\}} |U_{d_0, d_1}(y)| dy.$$

In addition to that, using (3.13), we have the following

$$\int_{\{y \leq \frac{1}{2}d(0, \partial \Omega)\}} |U_{d_0, d_1}(y)| dy = \int_{\{y \leq 2\sqrt{T} \ln T\}} |U_{d_0, d_1}(y)| dy + \int_{2\sqrt{T} \ln T \leq |y| \leq \frac{1}{2}d(0, \partial \Omega)} |U_{d_0, d_1}(y)| dy
\leq \int_{\{y \leq 2\sqrt{T} \ln T\}} T^{-\frac{n}{2}} \left( \frac{|y|}{\sqrt{T} \ln T} \right)^{\frac{n+1}{2}} \chi_0 \left( \frac{|y|}{\sqrt{T} \ln T} \right) \chi_0 \left( \frac{|y|}{\sqrt{T} \ln T} \right) dy
+ \int_{2\sqrt{T} \ln T \leq |y| \leq \frac{1}{2}d(0, \partial \Omega)} (1 - \chi_1(y)) H^*(y) dy \leq C,$$

which yields

$$|\nabla L(x, t)| \leq C,$$  (E.7)

It is similar to prove the following estimate

$$|\nabla^2 L(x, t)| \leq C,$$  (E.8)

- We consider the case where $x \in \Omega_1$ : Let us define $\phi(x)$ as a function in $C_0^\infty (\mathbb{R}^n)$ and satisfying the following conditions

$$\phi(x) = 0 \text{ if } |x| \geq \frac{11}{12} d(0, \partial \Omega),$$
$$\phi(x) = 1 \text{ if } |x| \leq \frac{7}{8} d(0, \partial \Omega).$$

Then, we also introduce the following function

$$L_1(x, t) = \phi(x) \nabla L(x, t).$$
Using Duhamel’s formula, we get

\[ L_1(t) = e^{t}L_1(0) + \int_0^t e^{(t-s)} \left[ -2\nabla \phi \cdot \nabla^2 L - \Delta \phi \nabla L \right](s) ds. \]  

(E.10)

We now aim at proving the following fact:

\[ \parallel e^{(t-s)}(\Delta \phi \nabla L)(s) \parallel_{L^\infty(\Omega)} \leq C \parallel L_1(s) \parallel_{L^\infty(\Omega)} + C, \]  

(E.11)

\[ \parallel e^{(t-s)}(\nabla \phi \cdot \nabla^2 L)(s) \parallel_{L^\infty(\Omega)} \leq C \parallel L_1(s) \parallel_{L^\infty(\Omega)} \frac{1}{\sqrt{t-s}} + C \left( 1 + \frac{1}{ \sqrt{t-s} } \right), \]  

(E.12)

- The proof of (E.11): We have the following fact:

\[ \parallel (\Delta \phi \nabla L) \parallel_{L^\infty(\Omega)} \leq | I_{\{ |x| \leq \frac{1}{2}d(0,\partial\Omega) \}} \Delta \phi \nabla L | + | I_{\{ |x| > \frac{1}{2}d(0,\partial\Omega) \}} \Delta \phi \nabla L |
\leq C |\phi \nabla L | + C = C |L_1| + C. \]

Then, by using the monotonicity of the operator \( e^{(t-s)} \Delta \), we derive directly (E.11).

- The proof of (E.12): From the definition of operator \( e^{(t-s)} \Delta \), we can write the following

\[ e^{(t-s)} \Delta (\nabla \phi \cdot \nabla^2 L(s)) = \int_{\Omega} G(x, y, t, s) \nabla \phi(y) \cdot \nabla^2 L(y, s) dy. \]

We consider \( j \in \{1, \ldots, n\} \), and integrate by part, we get the following

\[
\int_{\Omega} \sum_{i=1}^{n} G(x, y, t, s) \partial_{y_i} \phi(y) \partial^2_{y_j, y_j} L(y) dy = - \int_{\Omega} \left( \nabla_y G(x, y, t, s) \cdot \nabla \phi + G(x, y, t, s) \Delta(y) \right) \partial_{y_j} L(y, s) dy
- \int_{\Omega} \nabla_y G(x, y, t, s) \cdot \nabla \phi \partial_{y_j} L(y, s) dy
- \int_{\Omega} G(x, y, t, s) \Delta(y) \partial_{y_j} L(y, s) dy.
\]

Using the definition of \( \phi \) in the above and (E.7), we have the following fact:

\[
|\nabla L| = | I_{\{ |x| \leq \frac{1}{2}d(0,\partial\Omega) \}} \nabla L | + | I_{\{ |x| > \frac{1}{2}d(0,\partial\Omega) \}} \nabla L |
= | I_{\{ |x| \leq \frac{1}{2}d(0,\partial\Omega) \}} \phi \nabla L | + | I_{\{ |x| > \frac{1}{2}d(0,\partial\Omega) \}} \nabla L |
\leq |L_1| + C.
\]

Then,

\[
\left| \int_{\Omega} \sum_{i=1}^{n} G(x, y, t, s) \partial_{y_i} \phi(y) \partial^2_{y_j, y_j} L(y) dy \right| \leq (\parallel L_1(s) \parallel_{L^\infty(\Omega)} + C) \parallel \nabla_y G(x, y, t, s) \cdot \nabla \phi dy \parallel
+ (\parallel L_1(s) \parallel_{L^\infty(\Omega)} + C) \parallel G(x, y, t, s) \Delta \phi dy \parallel
\leq (\parallel L_1(s) \parallel_{L^\infty(\Omega)} + C) \left[ \frac{C}{\sqrt{t-s}} + C \right],
\]

which implies (E.12). We now use (E.10), (E.11) and (E.12) to deduce the following

\[
\parallel L_1(t) \parallel_{L^\infty} \leq C \parallel \nabla U_{d_0,d_1} \parallel_{L^1(\Omega)} + \int_0^t \left[ C \left( 1 + \frac{1}{\sqrt{t-s}} \right) \parallel L_1(s) \parallel_{L^\infty} + C \left( 1 + \frac{1}{\sqrt{t-s}} \right) \right] ds. \]  

(E.13)
Indeed, we write
\[ \| L_1(t) \|_{L^\infty} \leq C \| \nabla U_{d_0,d_1} \|_{L^1(\Omega)}. \]
We admit the following fact which we will be proved at the end:
\[ \| \nabla U_{d_0,d_1} \|_{L^1(\Omega)} \leq CT^{-\frac{1}{2}} + C(\epsilon_0). \] (E.14)
This estimate gives a rough estimation on \( L_1 \) as follows
\[ \| L_1(t) \|_{L^\infty(\Omega)} \leq CT^{-\frac{1}{2}} + C(\epsilon_0). \] (E.15)
Let us improve this estimate. We come back to identity (E.10) and consider the set of all \( x \in \Omega \) such that \( |x| \geq \frac{\epsilon_0}{8} \). By using the definition of \( U_{d_0,d_1} \) in (3.13), we first prove the following fact
\[ \| e^{t\Delta} (\nabla U_{d_0,d_1}) \|_{L^\infty(|x| \geq \frac{\epsilon_0}{8}, x \in \Omega)} \leq C(\epsilon_0). \] (E.16)
Indeed, we write
\[ e^{t\Delta} (\nabla U_{d_0,d_1}) = \int_{\Omega} G(x,y,t,0) \nabla y U_{d_0,d_1}(y) dy = \int_{|y| \leq \frac{\epsilon_0}{16}} G(x,y,t,0) \nabla y U_{d_0,d_1}(y) dy \]
\[ + \text{Bound on } I_1: \text{ Using integration by parts, we get the following:} \]
\[ I_1 = - \int_{|y| \leq \frac{\epsilon_0}{16}} \nabla y G(x,y,t,0) U_{d_0,d_1}(y) dy + \int_{|y| = \frac{\epsilon_0}{16}} G(x,y,t,0) U_{d_0,d_1}(y) \eta(y) dS. \]
From Lemma E.1, we derive that
\[ |I_1(x,t)| \leq \int_{|y| \leq \frac{\epsilon_0}{16}} \frac{\exp \left( -c \Omega \frac{|x-y|^2}{t} \right)}{t^{n+1}} |U_{d_0,d_1}(y)| dy + C(\epsilon_0) \]
\[ \leq \int_{|y| \leq \frac{\epsilon_0}{16}} \exp \left( -c \Omega \frac{|x-y|^2}{t} \right) \frac{\exp \left( -c \Omega \frac{|x-y|^2}{t} \right)}{t^{n+1}} \frac{1}{|x-y|^{n+1}} |U_{d_0,d_1}(y)| dy + C(\epsilon_0) \]
\[ \leq C(\epsilon_0) \| U_{d_0,d_1} \|_{L^1(\Omega)} + C(\epsilon_0) \leq C_1(\epsilon_0) \]
\[ + \text{Bound on } I_2: \text{ It is easy to prove that} \]
\[ \| \nabla U_{d_0,d_1}(\cdot) \|_{L^\infty(|y| \leq \frac{\epsilon_0}{16}, d(0,\partial \Omega))} \leq C(\epsilon_0). \]
This yields directly that
\[ |I_2(x,t)| \leq C(\epsilon_0) \int_{\frac{\epsilon_0}{16}}^{\epsilon_0} |G(x,y,t,0) dy| \leq C(\epsilon_0). \]
Hence, we get the conclusion the proof of (E.16). Using (E.13), (E.15) and (E.16), we get the following: for all \( |x| \geq \frac{\epsilon_0}{8}, x \in \Omega \)
\[ |L_1(x,t)| \leq C(\epsilon_0) + C \int_0^t \left( 1 + \frac{1}{\sqrt{t-s}} \right) T^{-\frac{1}{2}} ds \leq C(\epsilon_0), \]
provided that \( T < 1 \). This yields that for all \( x \in \Omega_1 \)
\[ |\nabla L(x,t)| \leq C(\epsilon_0). \] (E.17)
Finally, (4.14) follows from (E.7) and (E.17), which will conclude the proof of Lemma 4.8. However, in order to finish the proof we need to prove (E.14): Indeed, from the definition of \( U_{d_0,d_1} \) given in (3.13), we write

\[
\nabla_x U_{d_0,d_1}(x) = I_1(x) + I_2(x) + I_3(x) + I_4(x),
\]

where

\[
I_1 = T^{-\frac{1}{2}} \left[ -\frac{3}{4} \left( 3 + \frac{9}{8} \frac{|x|^2}{T|\ln T|} \right) |x|^{-\frac{7}{3}} \frac{x}{T|\ln T|} + \frac{d_1}{\sqrt{T|\ln T|}} \chi_0 \left( \frac{x}{T|\ln T|} \right) \right] + \frac{d_1 \cdot x}{\sqrt{T|\ln T|}} \chi_0' \left( \frac{x}{T|\ln T|} \right) \frac{x}{|x|T|\ln T|} \chi_0 \left( \frac{|x|}{T|\ln T|} \right),
\]

\[
I_2 = T^{-\frac{1}{2}} \left[ \left( 3 + \frac{9}{8} \frac{|x|^2}{T|\ln T|} \right) |x|^{-\frac{7}{3}} \frac{x}{T|\ln T|} + \left( d_0 + \frac{d_1 \cdot x}{\sqrt{T|\ln T|}} \right) \chi_0 \left( \frac{|x|}{T|\ln T|} \right) \right]
\]

\times \chi_0' \left( \frac{|x|}{T|\ln T|} \right) \frac{x}{|x|T|\ln T|},
\]

\[
I_3 = \left( 1 - \chi_0 \left( \frac{x}{T|\ln T|} \right) \right) \nabla H^*(x),
\]

\[
I_4 = -\chi_0' \left( \frac{x}{T|\ln T|} \right) \frac{x}{|x|T|\ln T|} H^*(x).
\]

As a matter of fact, we have the following

\[
\| \nabla U_{d_0,d_1} \|_{L_1} \leq \int_{\Omega} |I_1(x)|dx + \int_{\Omega} |I_2(x)|dx + \int_{\Omega} |I_3(x)|dx + \int_{\Omega} |I_4(x)|dx.
\]

In particular, we have

\[
\text{Supp}(I_1) \subset \{|x| \leq 2T|\ln T|\},
\]

\[
\text{Supp}(I_2) \subset \{T|\ln T| \leq |x| \leq 2T|\ln T|\},
\]

\[
\text{Supp}(I_3) \subset \{T|\ln T| \leq |x| \leq \frac{1}{2} d(0, \partial \Omega)\},
\]

\[
\text{Supp}(I_4) \subset \{T|\ln T| \leq |x| \leq 2T|\ln T|\}.
\]

By some simple upper bounds on \( I_1 \) and \( I_2 \), we can derive that

\[
\int_{\Omega} |I_1(x)|dx \leq CT^{-\frac{3}{2}} + C \quad \text{and} \quad \int_{\Omega} |I_2(x)|dx \leq CT^{-\frac{3}{2}} + C.
\]

We now aim at estimating \( I_3 \) and \( I_4 \).

+ Estimate on \( I_3 \): We write as follows

\[
\int_{\Omega} |I_3(x)|dx = \int_{\sqrt{T|\ln T| \leq |x| \leq \min(\frac{1}{2} d(0, \partial \Omega))} |I_3(x)|dx + \int_{\min(\frac{1}{2} d(0, \partial \Omega)) \leq |x| \leq \frac{1}{2} d(0, \partial \Omega)} |I_3(x)|dx
\]

\leq \int_{\sqrt{T|\ln T| \leq |x| \leq \min(\frac{1}{2} d(0, \partial \Omega))} |I_3(x)|dx + C.
\]

In addition to that,

\[
\int_{\sqrt{T|\ln T| \leq |x| \leq \min(\frac{1}{2} d(0, \partial \Omega))} |I_3(x)|dx \leq C \int_{\sqrt{T|\ln T| \leq |x| \leq \min(\frac{1}{2} d(0, \partial \Omega))} |x|^{-\frac{3}{2}} \ln |x|^{\frac{3}{2}}dx
\]

\leq CT^{-\frac{3}{2}} + C.
This implies that
\[ \int_{\Omega} |I_3(x)| dx \leq C T^{-\frac{3}{2}} + C. \]

+ Estimate on \( I_4 \): We have
\[ \int_{\sqrt{T} \ln T \leq |x| \leq 2\sqrt{T} \ln T} |I_4(x)| dx \leq \frac{C}{\sqrt{T} \ln T} \int_{\sqrt{T} \ln T \leq |x| \leq 2\sqrt{T} \ln T} |\ln |x||^{\frac{5}{2}} |x|^{-\frac{7}{2}} dx \leq C T^{-\frac{3}{2}}. \]

From the above estimates, we can conclude (E.14). We also finish the proof of Lemma 4.8. \( \square \)

F. Some Parabolic estimates

In this section, we aim at giving some estimates on \( U, \nabla U, \nabla^2 U \). More precisely, the following is our statement:

**Lemma F.1** (Parabolic estimates on \( U \)). We consider \( U \) a solution to equation (2.8) and \( U \in S(T, K_0, \epsilon_0, \omega_0, A, \delta_0, C_0, \eta_0, t) \), for all \( t \in [0, t_1] \) for some \( t_1 \leq T \). Then, the following estimates follows: for all \( t \in [0, T] \)
\[
\|U(\cdot, t)\|_{L^\infty(\Omega)} \leq C(K_0, A)(T - t)^{-\frac{3}{2}}, \tag{F.1}
\]
\[
\|\nabla U(\cdot, t)\|_{L^\infty(\Omega)} \leq C(K_0, A) \frac{(T - t)^{-\frac{5}{2}}}{|\ln(T - t)|^{\frac{3}{2}}}, \tag{F.2}
\]
\[
\|\nabla^2 U(\cdot, t)\|_{L^\infty(\Omega)} \leq C(K_0, A)(T - t)^{-c}, \tag{F.3}
\]
for some constant \( c = c(K_0, A) > 0 \).

In particular, we have the following local convergence: We assume furthermore that \( U \in S(t) \), for all \( t < T \). Then, for all \( x \in \Omega \) there exist \( R_x > 0, t_x \in [0, T) \) such that the following holds
\[
\|\partial_t U(\cdot, t)\|_{L^\infty(B(x, R_x))} \leq C(K_0, A, T, x), \forall t \in [t_x, T). \tag{F.4}
\]

**Remark F.2.** We would like to remark that from (F.4) and the definition of the shrinking set \( S(t) \) (see Definition 3.1), we ensure for all \( x_0 \in \Omega \setminus \{0\}, U(x_0, t) \) is convergent as \( t \to T \).

**Proof.** We see that estimates (F.1) and (F.2) directly follow from the definition of the shrinking set and Lemma 3.6. For that reason, we only give here the proofs of (F.3) and (F.4).

- The proof of (F.3): From (2.4), (2.6), we consider \( u \) defined as follows:
\[
u(x, t) = 1 - \frac{1}{1 + \lambda \frac{|U(x, t)|}{\theta(t)}}. \tag{F.5}\]

Then, \( u \) satisfies (1.2) and \( u(0) \) is in \( C^\infty_0(\Omega) \). We now derive an equation satisfied by \( \nabla^2 u \) as follows:
\[
\partial_t \nabla^2 u = \Delta(\nabla^2 u) + H_1 \nabla^2 u + H_2, \tag{F.6}
\]
where \( H_1 = \left(\frac{2\lambda}{\theta^2(t) (1 - u)}\right) \right) \) and \( H_2 = (H_{2,i,j})_{i,j \leq n} \) is a square matrix with
\[
H_{2,i,j} = 6 \frac{\partial_i u \partial_j u}{(1 - u)^4}.
\]

Using the definition of \( u \), (3.18) and two estimates (F.1) and (F.2), we can derive the following fact: for all \( t \in [0, T) \),
\[
\|H_1(t)\|_{L^\infty(\Omega)} \leq C(K_0, A)(T - t)^{-1},
\]
\[
\|H_2(t)\|_{L^\infty(\Omega)} \leq C(K_0, A)(T - t)^{-\frac{3}{2}}.
\]
We write $\nabla^2 u$ under the integral equation following

$$\nabla^2 u(t) = e^{t\Delta} (\nabla^2 u(0)) + \int_0^t e^{(t-s)\Delta} \left[ H_1(s)\nabla^2 u + H_2(s) \right] (s) ds.$$ 

This implies that

$$\| \nabla^2 u(t) \|_{L^\infty(\Omega)} \leq \| e^{t\Delta} (\nabla^2 u(0)) \|_{L^\infty(\Omega)} + C(K_0, A) \int_0^t \left( \frac{1}{T-s} \| \nabla u(s) \|_{L^\infty(\Omega)} + (T-s)^{-\frac{2}{3}} \right) ds.$$ 

Besides that, we can prove that there exists $c_1 > 0$ such that

$$\| e^{t\Delta} (\nabla^2 u(0)) \|_{L^\infty(\Omega)} \leq C(T-t)^{-c_1}.$$ 

Thanks to Growall’s lemma, we get the following

$$\| \nabla^2 u \|_{L^\infty(\Omega)} \leq C(K_0, A)(T-t)^{-c_2},$$ 

with some constant $c_2 > 0$.

Finally, from the relation between $u$ and $U$, we can get the conclusion of (F.3).

- The proof of (F.4): By using the definitions (3.2) and (3.2) of $P_2(t)$ and $P_3(t)$, respectively, if we consider an arbitrary $x \in \Omega \setminus \{0\}$, then, there exist $t_x, r_x$ such that

$$\text{the ball of radius } r_x, \text{ centred } B(x, r_x) \in P_2(t) \cup P_3(t), \forall t \in [t_x, T).$$

Then, using the definition of the shrinking set $S(t)$, given in Definition 3.1 and the fact that $u \in S(t)$ for all $t \in [t_x, T)$, we derive that there exists $C(K_0, x)$ such that for all $t \in [t_x, T)$, we have

$$\| U(\cdot, t) \|_{L^\infty(B(x, r_x))} \leq C(K_0, x). \quad (F.7)$$

In addition to that, we derive from Proposition 3.9, we have

$$1 \leq \bar{\theta}(t) \leq C, \text{ and } |\bar{\theta}'(t)| \leq C(T-t)^{\frac{n-2}{n}} |\ln(T-t)|^n \leq (T-t)^{-\frac{2n}{n+2}}, \quad (F.8)$$

for all $t \in [t_x, T)$.

We recall $u$, defined in (F.5). We now derive an equation satisfied by $\partial_t u$

$$\partial_t (\partial_t u) = \Delta \partial_t u + H_1 \partial_t u + H_3(t), \quad (F.9)$$

where

$$H_1(t) = \frac{2\lambda}{\bar{\theta}^2(t)(1-u)^2}, \quad H_3(t) = -\frac{3\lambda}{(1-u)^2 \bar{\theta}^4(t)}.$$ 

We then introduce the following cut-off function: $\phi \in C_0^\infty(\mathbb{R}^n)$ which satisfying

$$\phi(z) = 1 \text{ if } |z-x| \leq \frac{r_x}{2}, \text{ and } \phi(z) = 0 \text{ if } |z-x| \geq \frac{3}{4}r_x \text{ and } 0 \leq \phi(z) \leq 1, \forall z \in \mathbb{R}^n.$$ 

Particularly, we also define

$$v(z, t) = \phi(z) \partial_t u(z, t) \text{ for all } z \in \mathbb{R}^n.$$ 

Using (F.9), we can derive an equation satisfied by $v(t)$ as follows

$$\partial_t v = \Delta v - 2\text{div}(\nabla \phi \partial_t u) + \Delta \phi \partial_t u + H_1 v(t), \quad (F.10)$$

Using (F.3), (F.5) (F.7) and the fact that $U$ is nonnegative solution, we can deduce that

$$\| \nabla \phi \partial_t u(t) \|_{L^\infty(\mathbb{R}^n)} \leq C(K_0, A, x)(T-t)^{-c},$$

and

$$\| \Delta \phi \partial_t u(t) \|_{L^\infty(\mathbb{R}^n)} \leq C(K_0, A, x)(T-t)^{-c}.$$ 

Moreover, we can derive from (F.7) and (F.8) that

$$\| I_{|z-x| \leq r_x} H_1(t) \|_{L^\infty(\mathbb{R}^n)} \leq C(K_0, A, x),$$
where
\[ \|H_3(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(K_0, x)(T - t)^{-\frac{11}{12}}. \]
We now deduce from (F.10) that \( v \) satisfies the following integral equation
\[
v(t) = e^{(t-t_x)\Delta} v(t_x) + \int_{t_x}^t e^{(t-s)\Delta} \left[ -2\text{div}(\nabla \phi \partial_t u) + \Delta \phi \partial_t u + H_1(s) \right] ds,
\]
where \( e^{t\Delta} \) stands for the heat semigroup on \( \mathbb{R}^n \). Then, we get the following
\[
|v(t)| \leq C(K_0, A, x)(1 + (T - t)^{-c+1}) + 2 \left| \int_{t_x}^t e^{(t-s)\Delta} \text{div}(\nabla \phi \partial_t u) ds \right|.
\]
In particular, we have
\[
\left| e^{(t-s)\Delta} \text{div}(\nabla \phi \partial_t u) \right| \leq \frac{C}{\sqrt{t-s}} \| \nabla \phi \partial_t u \|_{L^\infty(\mathbb{R}^n)} \leq C(K_0, x) \frac{(T - t)^{-c}}{\sqrt{t-s}}.
\]
This implies that
\[
|v(t)| \leq C(K_0, A, x)(1 + (T - t)^{-c+1}) + C(K_0, A, x) \int_{t_x}^t \frac{(T - s)^{-c}}{(t - s)^{\frac{11}{12}}} ds.
\]
+ If \(-c + \frac{1}{4} \geq 0\). This gives us that
\[
|v(t)| \leq C(K_0, A, x),
\]
which yields the conclusion of our proof.
+ Otherwise, we use the above estimate to derive that
\[
|v(t)| \leq C(K_0, A, T, x)(T - t_x)^{-c+\frac{1}{4}}.
\]
We can see that by using a parabolic estimate as we have done. We can improve our estimate on \( |v(t)| \) from \( C(K_0, A, x)(T - t)^{-c} \) to \( C(K_0, A, x)(T - t)^{-c+\frac{1}{4}} \). Hence, we can repeat with a finite steps to get the conclusion of the proof. We kindly refer the reader to check this argument.

\[ \square \]

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