ON THE BILINEAR SQUARE FOURIER MULTIPLIER OPERATORS AND RELATED MULTILINEAR SQUARE FUNCTIONS

ZENGYAN SI, QINGYING XUE, AND KÔZÔ YABUTA

Abstract. Let $n \geq 1$ and $\mathfrak{T}_m$ be the bilinear square Fourier multiplier operator associated with a symbol $m$, which is defined by

$$\mathfrak{T}_m(f_1, f_2)(x) = \left( \int_0^\infty \left| \int_{(\mathbb{R}^n)^2} e^{2\pi ix \cdot (\xi_1 + \xi_2)} m(t\xi_1, t\xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) d\xi_1 d\xi_2 \right|^2 dt \right)^{\frac{1}{2}}.$$ 

Let $s$ be an integer with $s \in [n+1, 2n]$ and $p_0$ be a number satisfying $2n/s \leq p_0 \leq 2$. Suppose that $\nu_2 = \prod_{i=1}^{2} \omega_i^{p/p_i}$ and each $\omega_i$ is a nonnegative function on $\mathbb{R}^n$. In this paper, we show that $\mathfrak{T}_m$ is bounded from $L_{\nu_1}^1(\omega_1) \times L_{\nu_2}^2(\omega_2)$ to $L_p(\nu_2)$ if $p_0 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$. Moreover, if $p_0 > 2n/s$ and $p_1 = p_0$ or $p_2 = p_0$, then $\mathfrak{T}_m$ is bounded from $L_{\nu_1}^1(\omega_1) \times L_{\nu_2}^2(\omega_2)$ to $L_p(\nu_2)$. The weighted end-point $L \log L$ type estimate and strong estimate for the commutators of $\mathfrak{T}_m$ are also given. These were done by considering the boundedness of some related multilinear square functions associated with mild regularity kernels and essentially improving some basic lemmas which have been used before.

1. Introduction

1.1. Background. The multilinear Calderón-Zygmund operators were first introduced and studied by Coifman and Meyer [6, 7], and later on by Grafakos and Torres [11, 18]. Due to the close relationship between the Calderón-Zygmund operators and Littlewood-Paley operators, in the meantime, the multilinear Littlewood-Paley $g$-function and related multilinear Littlewood-Paley type estimates were used in PDE and other fields ([4, 5, 9, 13, 14, 15]). For example, in [15], the authors studied a class of multilinear square functions and applied it to the well-known Kato’s problem. For more works about multilinear Littlewood-Paley type operators, see [3, 29] and the references therein. Recently, in the theory of multilinear operators, efforts have been made to remove or replace the smoothness condition assumed on the kernels, among these achievements are the nice works of Bui and Duong [1], Grafakos, Liu...
and Yang [19], Tomita [30], Grafakos, Miyachi and Tomita [20] and more recent work of Grafakos, He and Honzík [17].

It is also well known that the following $N$-linear ($N \geq 1$) Fourier multiplier operator $T_m$ was introduced by Coifman and Meyer in [8].

$$T_m(f_1, \cdots, f_N)(x) = \frac{1}{(2\pi)^{nN}} \int_{(\mathbb{R}^n)^N} e^{ix \cdot (\xi_1 + \cdots + \xi_N)} m(\xi_1, \cdots, \xi_N) \hat{f_1}(\xi_1) \cdots \hat{f_N}(\xi_N) d\xi.$$  

Suppose that $m$ is a bounded function on $\mathbb{R}^{nN}\setminus\{0\}$ and it satisfies that

$$(1.1) \quad |\partial^{\alpha_1}_{\xi_1} \cdots \partial^{\alpha_N}_{\xi_N} m(\xi_1, \cdots, \xi_N)| \leq C_\alpha (|\xi_1| + \cdots + |\xi_N|)^{-|\langle \alpha_1 \rangle + \cdots + |\alpha_N|)},$$

away from the origin for all sufficiently large multiindices $\alpha_j$. Then, it was shown in [8] that $T_m$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In 2010, by weakening the smoothness condition (1.1), Tomita [30] gave a Hörmander type theorem for $T_m$. Later, Grafakos and Si [21] gave a similar result for the case $p \leq 1$ by using the $L^r$-based Sobolev spaces ($1 < r \leq 2$). Subsequently, Grafakos, Miyachi and Tomita [20] proved that if $m \in L^\infty(\mathbb{R}^{nN})$ satisfies $\sup_{k \in \mathbb{Z}} \|m_k\|_{W^{s_1, \cdots, s_N}(\mathbb{R}^n)} < \infty$ with $s_1, \cdots, s_N > n/2$, then $T_m$ is bounded from $L^2(\mathbb{R}^n) \times \cdots \times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

A weighted version of the results in [30] for $T_m$ was given by Fujita and Tomita [12] under the Hörmander condition with classical $A_p$ weights. Recently, Li and Sun [25] demonstrated a Hörmander type multiplier theorem for $T_m$ with multiple weights. Furthermore, they obtained some weighted estimates for the commutators of $T_m$ with vector version of $\text{BMO}$ functions. Still more recently, Li, Xue and Yabuta [26] considered the estimates about weighted Carleson measure, and consequently they obtained some weighted results of $T_m$ by considering the missing endpoint parts of the results in [12].

1.2. Results on multilinear Fourier multiplier. It is also well known that Lacey [23] studied the following bilinear Littlewood-Paley square function defined by

$$T(f, g)(x) = \left( \sum_{l \in \mathbb{Z}^d} |T_{\phi_l}(f, g)(x)|^2 \right)^{1/2},$$

where the bilinear operator $T_{\phi_l}$ associated with a smooth function $\phi_l$ whose Fourier transform is supported in $\omega_l$ is defined by

$$T_{\phi_l}(f, g)(x) = \int_{(\mathbb{R}^n)^2} \hat{f}(\xi) \hat{g}(\eta) \hat{\phi_l}(\xi - \eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

and $\{\omega_l\}_{l \in \mathbb{Z}^d}$ is a sequence of disjoint cubes. The study on bilinear Littlewood-Paley square function has two motivations: One is Alberto Calderón’s conjectures on bilinear Hilbert transform; Another one is the norm inequalities of Littlewood-Paley type operators.
Our object of investigation is the bilinear square Fourier multiplier operator
\[\mathfrak{T}_m(f_1, f_2)(x) = \left( \int_0^\infty \left| \int_{\mathbb{R}^n} e^{2\pi i x \cdot (\xi_1 + \xi_2)} m(t\xi_1, t\xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) d\xi_1 d\xi_2 \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.\]  

Let \(K_t(x, y_1, y_2) = \frac{1}{t^n m(\frac{x-y_1}{t}, \frac{x-y_2}{t})}\). Then, \(\mathfrak{T}_m\) can be written in the form
\[\mathfrak{T}_m(\vec{f})(x) = \left( \int_0^\infty \left| \int_{\mathbb{R}^n} K_t(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}},\]

The commutator of \(\mathfrak{T}_m\) is defined by
\[\mathfrak{T}_m^\ast (\vec{f})(x) = \sum_{i=1}^2 \left( \int_0^\infty \left| \int_{\mathbb{R}^n} (b_i(x) - b_i(y)) K_t(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}},\]

In this paper, we always assume that \(m \in L^\infty((\mathbb{R}^n)^2)\) and satisfies the conditions
\[|\partial^\alpha m(\xi_1, \xi_2)| \lesssim \frac{(|\xi_1| + |\xi_2|)^{-|\alpha| + \varepsilon_1}}{(1 + |\xi_1| + |\xi_2|)^{\varepsilon_1 + \varepsilon_2}}\]
for some \(\varepsilon_1, \varepsilon_2 > 0\) and \(|\alpha| \leq s\).

The main results of this paper are:

**Theorem 1.1.** Let \(s\) be an integer with \(s \in [n + 1, 2n]\) and \(p_0\) be a number satisfying \(2n/s \leq p_0 \leq 2\). Let \(p_0 \leq p_1, p_2 < \infty, 1/p = 1/p_1 + 1/p_2\), and \(\vec{\omega} \in A_{p_0/p_1}\). Suppose that \(m \in L^\infty((\mathbb{R}^n)^2)\) satisfies (1.3) and that the bilinear square Fourier multiplier operator \(\mathfrak{T}_m\) is bounded from \(L^q_1 \times L^q_2\) into \(L^q_\infty\), for any \(p_0 < q_1, q_2\) and \(1/q = 1/q_1 + 1/q_2\). Then the following weighted estimates hold.

(i) If \(p_1, p_2 > p_0\), then 
\[||\mathfrak{T}_m(\vec{f})||_{L^p(\nu_\vec{\omega})} \leq C ||f_1||_{L^{p_1}(\nu_1)} ||f_2||_{L^{p_2}(\nu_2)}.\]

(ii) If \(p_0 > 2n/s\) and \(p_1 = p_0\) or \(p_2 = p_0\), then
\[||\mathfrak{T}_m(\vec{f})||_{L^p(\nu_\vec{\omega})} \leq C ||f_1||_{L^{p_1}(\nu_1)} ||f_2||_{L^{p_2}(\nu_2)}.\]

**Theorem 1.2.** Let \(s, p_0, p_1, p_2, p, \vec{\omega}, m\) and \(\mathfrak{T}_m\) be the same as in Theorem 1.1. Then the following weighted estimates hold for the commutators of \(\mathfrak{T}_m(\vec{f})\).

(i) If \(p_1, p_2 > p_0\), then for any \(\vec{b} \in BMO^2\), it holds that
\[||\mathfrak{T}_m^\ast (\vec{f})||_{L^p(\nu_\vec{\omega})} \leq C ||\vec{b}||_{BMO} ||f_1||_{L^{p_1}(\nu_1)} ||f_2||_{L^{p_2}(\nu_2)},\]

where \(||\vec{b}||_{BMO} = \max_j ||b_j||_{BMO}\).

(ii) Let \(\vec{\omega} \in A_1\) and \(\vec{b} \in BMO^m\). Then, there exists a constant \(C\) (depending on \(\vec{b}\)) such that
\[\nu_\vec{\omega}\left( \{ x \in \mathbb{R}^n : |\mathfrak{T}_m^\ast (\vec{f})(x)| > t^2 \} \right) \leq C \prod_{j=1}^2 \left( \int_{\mathbb{R}^n} \Phi\left( \frac{|f_j(x)|}{t} \right) \omega_j(x) \right)^{1/2},\]

where \(\Phi(t) = t^{p_0}(1 + \log^+ t)^{p_0}\). 

---

ON THE BILINEAR SQUARE FOURIER MULTIPLIER OPERATORS ... 3
The above results still hold for $m$-linear square Fourier multiplier operators. An example will be given in section 2, which shows that the assumption that $\mathcal{T}_m$ is bounded from $L^{q_1} \times L^{q_2}$ into $L^{q_\infty}$ in Theorems 1.1 and 1.2 is reasonable. The proofs of Theorems 1.1 and 1.2 will be based on the results of multilinear square functions obtained in the next subsection.

1.3. Results on multilinear square functions. In order to state more known results, we need to introduce some definitions.

**Definition 1.1 (Multilinear operator and multilinear square function).** Let $K$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ and $T_t = t^{-m}K(.\cdot t)$. Then, the multilinear operator $T$ and multilinear square function $T$ are defined by

\[
\mathcal{T}\tilde{f}(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m
\]

and

\[
T(f) = \left( \int_0^\infty \left( \int_{(\mathbb{R}^n)^m} K_t(x, y_1, \cdots, y_m) \prod_{j=1}^m f_j(y_j) dy_1 \cdots dy_m \right)^2 dt \right)^{\frac{1}{2}},
\]

where $\tilde{f} = (f_1, \ldots, f_m) \in S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n)$ and all $x \notin \bigcap_{j=1}^m \text{supp} f_j$.

**Definition 1.2 (Bui and Duong’s condition, [2]).** Let $S_j(Q) = 2^j Q \setminus 2^{j-1} Q$ if $j \geq 1$, and $S_0(Q) = Q$. Then, assume that the following two conditions hold

(h1) For all $1 \leq p_0 \leq q_1, q_2, \ldots, q_m < \infty$ and $0 < q < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$, $T$ maps $L^{q_1} \times \cdots \times L^{q_m}$ into $L^{q_\infty}$.

(h2) There exists $\delta > n/p_0$ so that for the conjugate exponent $p'_0$ of $p_0$, one has

\[
\left( \int_{S_{j_1}(Q)} \cdots \int_{S_{j_m}(Q)} |K(z, \vec{y}) - K(x, \vec{y})|^{p'_0} d\vec{y} \right)^{1/p'_0} \leq C \frac{|x - z|^{m(\delta - n/p_0)}}{|Q|^{m\delta/n}} 2^{-m\delta j_0}
\]

for all ball $Q$, all $x, z \in 1/2 Q$ and $(j_1, \ldots, j_m) \neq (0, \ldots, 0)$, where $j_0 = \max_{k=1,\ldots,m} \{j_k\}$.

**Definition 1.3 (Xue and Yan’s condition, [32]).**

For any $t \in (0, \infty)$, we assume that $K_t(x, y_1, \ldots, y_m)$ satisfies the following conditions: there is a positive constant $A > 0$, such that

\[
\left( \int_0^\infty |K_t(z, y_1, \cdots, y_m) - K_t(x, y_1, \cdots, y_m)|^{2} dt \right)^{\frac{1}{2}} \leq \frac{A |z - x|^\gamma}{(\sum_{j=1}^m |x - y_j|)^{m\gamma + \gamma}},
\]

whenever $|z - x| \leq \frac{1}{2} \max_{j=1}^m |x - y_j|$; and

\[
\left( \int_0^\infty |\tilde{K}_t(x, \vec{y}) - \tilde{K}_t(x, y_1, \ldots, y'_j, \ldots, y_m)|^{2} dt \right)^{\frac{1}{2}} \leq \frac{A |y_j - y'_j|^\gamma}{(\sum_{j=1}^m |x - y_j|)^{m\gamma + \gamma}},
\]
whenever $|y_j - y'_j| \leq \frac{1}{2} \max_{j=1}^{m} |x - y_j|$

\begin{equation}
\left( \int_{0}^{\infty} |K_t(x, y_1, \ldots, y_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq \frac{A}{(\sum_{j=1}^{m} |x - y_j|)^{mn}}.
\end{equation}

In 2013, Bui and Duong [2] studied the boundedness of $\mathcal{T}$ on product of weighted Lebesgue spaces with the kernel satisfies the more weaker regularity conditions (h1) and (h2). It should be pointed out that, under the assumptions (h1) and (h2), the multilinear operator $\mathcal{T}$ defined in (1.4) may not fall under the scope of the theorem of multilinear Calderón-Zygmund singular integral operators. In 2015, Xue and Yan [32] established the multiple-weighted norm inequalities for multilinear square function $T$

Motivated by the above two works, we introduce the following new condition to study the boundedness of multilinear square function and the associated commutators.

**Definition 1.4 (New condition).** Let $1 \leq p_0 < \infty$. Let $S_j(Q) = 2^j Q \setminus 2^{j-1}Q$ if $j \geq 1$, and $S_0(Q) = Q$. Then, assume that

- (H1) For all $p_0 \leq q_1, q_2, \ldots, q_m < \infty$ and $0 < q < \infty$ with $1/q_1 + \ldots + 1/q_m = 1/q$, $T$ maps $L^{q_1} \times \cdots \times L^{q_m}$ into $L^{q,\infty}$.

- (H2) There exists $\delta > n/p_0$ so that for the conjugate exponent $p'_0$ of $p_0$, one has

\begin{equation}
\left( \int_{S_{j_0}(Q)} \cdots \int_{S_{j_1}(Q)} \left( \int_{0}^{\infty} |K_t(z, y) - K_t(x, y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \frac{dy}{y} \right)^{1/p'_0} \leq C \frac{|x - z|^{m(\delta-n/p_0)}}{|Q|^{m\delta/n}} 2^{-m\delta j_0}
\end{equation}

for all balls $Q$, all $x, z \in 1/2Q$ and $(j_1, \ldots, j_m) \neq (0, \ldots, 0)$, where $j_0 = \max_{k=1,\ldots,m} \{j_k\}$.

- (H3) There exists some positive constant $C > 0$ such that

\begin{equation}
\left( \int_{S_{j_0}(Q)} \cdots \int_{S_{j_1}(Q)} \left( \int_{0}^{\infty} |K_t(x, y_1, \ldots, y_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \frac{dy}{y} \right)^{1/p_0} \leq C \frac{2^{-mnj_0/p_0}}{|Q|^{m/p_0}}
\end{equation}

for all balls $Q$ with center at $x$ and $(j_1, \ldots, j_m) \neq (0, \ldots, 0)$, where $j_0 = \max_{k=1,\ldots,m} \{j_k\}$.

**Definition 1.5 (Commutators of multilinear square operator).** The commutators of multilinear square operator $T$ with BMO functions $\vec{b} = (b_1, b_2, \ldots, b_m)$ are defined by

\begin{equation}
T_{\vec{b}}(\vec{f})(x) = \sum_{i=1}^{m} \left( \int_{0}^{\infty} \int_{(\mathbb{R}^n)^m} (b_i(x) - b_i(y_i))K_t(x, y) \prod_{j=1}^{m} f_j(y_j) dy_1 \ldots dy_m \left| \frac{dt}{t} \right|^{\frac{1}{2}} \right)
\end{equation}

for any $\vec{f} = (f_1, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$ and all $x \notin \bigcap_{j=1}^{m} \text{supp} f_j$. 
We obtain the following weighted estimates.

**Theorem 1.3.** Let $T$ be the multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some $1 \leq p_0 < \infty$. Then, for any $p_0 \leq p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $\vec{\omega} \in A_{F/p_0}$, the following weighted estimates hold.

1. If there is no $p_i = p_0$, then $\|T(\vec{f})\|_{L^p(\vec{\nu}_w)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\nu_{i,w_i})}$.

2. If there is a $p_i = p_0$, then $\|T(\vec{f})\|_{L^p(\vec{\nu}_w)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\nu_{i,w_i})}$.

As for the commutators of $T$, we obtain the following weighted estimates.

**Theorem 1.4.** Let $T$ be the multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some $1 \leq p_0 < \infty$. Let $\vec{b} \in BMO^m$. Then, for any $p_0 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $\vec{\omega} \in A_{\vec{F}/p_0}$, we have

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(\vec{\nu}_w)} \leq C\|\vec{b}\|_{BMO} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\nu_{i,w_i})},$$

where $\|\vec{b}\|_{BMO} = \max_j \|b_j\|_{BMO}$.

**Theorem 1.5.** Let $T$ be the multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some $1 \leq p_0 < \infty$. Let $\vec{b} \in BMO^m$. Let $\vec{\omega} \in A_{\vec{F}}$ and $\vec{b} \in BMO^m$. Then, there exists a constant $C$ (depending on $\vec{b}$) such that

$$\nu_2(\{x \in \mathbb{R}^n : |T_{\vec{b}}\vec{f}(x)| > t^m\}) \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{t}\right) \omega_j(x)\right)^{1/m},$$

where $\Phi(t) = t^{p_0}(1 + \log^+ t)^{p_0}$.

We organize this paper as follows: Section 2 contains one example concerning with the new assumption on $\Sigma_m$. Section 3 will be devoted to establish two key propositions related to multilinear square Fourier multiplier operator, which can be used to prove Theorem 1.1-1.2. In section 4, we will give the proofs of Theorem 1.3 and Theorem 1.4. Section 5 will be devoted to give the proof of Theorem 1.5.

Throughout this paper, the notation $A \lesssim B$ stands for $A \leq CB$ for some positive constant $C$ independent of $A$ and $B$.

2. An example

In this section, an example will be given to show that there are some multilinear square Fourier multiplier operators which are bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Thus, the assumption that $\Sigma_m$ is bounded from $L^{q_1} \times L^{q_2}$ into $L^{q,\infty}$ in Theorem 1.1-1.2 is reasonable.

Let

$$\tilde{T}_m(\vec{f})(x) = \int_0^\infty \int_{(\mathbb{R}^n)^4} e^{2\pi ix \cdot (\xi_1 + \xi_2 + \xi_3 + \xi_4)} m(t\xi_1, t\xi_2)m(t\xi_3, t\xi_4) \prod_{i=1}^4 \tilde{f}_i(\xi_i) d\xi_i \frac{dt}{t}.$$
Example 2.1. Suppose that \( m(0, 0) = 0 \) and there exists some \( \varepsilon > 0 \) such that
\[
(2.1) \quad |\partial^\alpha m(\xi_1, \xi_2)| \leq (1 + |\xi_1| + |\xi_2|)^{-s-\varepsilon}, \quad \text{for all } |\alpha| \leq 2n + 1.
\]
Then, there exists a constant \( \delta \), with \( 0 < \delta \leq 1 \), such that

(i) \( \tilde{T}_m \) is bounded from \( L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \times L^{q_3}(\mathbb{R}^n) \times L^{q_4}(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for \( 2 - \delta < q_1, q_2, q_3, q_4 < \infty \) with \( 1/q = 1/q_1 + 1/q_2 + 1/q_3 + 1/q_4 \).

(ii) \( \tilde{\Sigma}_m \) is bounded from \( L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for \( 2 - \delta < q_1, q_2 < \infty \) with \( 1/q = 1/q_1 + 1/q_2 \).

Proof. (i) Let \( \tilde{m}(\xi_1, \xi_2, \xi_3, \xi_4) = \int_0^\infty m(t\xi_1, t\xi_2) m(t\xi_3, t\xi_4) dt \). Then \( \tilde{T}_m \) can be written as a Fourier multiplier operator in the following form:
\[
\tilde{T}_m(f)(x) = \int_{(\mathbb{R}^n)^4} e^{2\pi i x \cdot (\xi_1 + \xi_2 + \xi_3 + \xi_4)} \tilde{m}(\xi_1, \xi_2, \xi_3, \xi_4) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) \hat{f}_4(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4,
\]

Next, we will show that \( \tilde{m} \) is a multiplier by considering two cases.

Case (a): \( 1 \leq |\alpha| \leq 2n + 1 \). We have
\[
|\partial^\alpha \tilde{m}(\xi_1, \xi_2, \xi_3, \xi_4)| = \left| \int_0^\infty |\partial^\alpha (m(t\xi_1, t\xi_2) m(t\xi_3, t\xi_4))| \frac{dt}{t} \right|
\leq \int_0^\infty \frac{|\partial^\alpha (1 + |t\xi_1| + |t\xi_2|)^{s+\varepsilon}(1 + |t\xi_3| + |t\xi_4|)^{s+\varepsilon}|}{t} dt
\leq \int_0^\infty \frac{|1 + t(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)|^{s+\varepsilon}}{(1 + t)^{s+\varepsilon}} dt
= \frac{1}{(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|)^{s+\varepsilon}} \int_0^\infty \frac{s^{s+\varepsilon}}{(1 + s)^{s+\varepsilon}} ds.
\]

Case (b): \( |\alpha| = 0 \).

By using the mean-value theorem and the assumption \( m(0, 0) = 0 \), we may obtain that \( |m(\xi_1, \xi_2)| \leq |\xi_1| + |\xi_2| \). Thus, together with the boundedness of \( m \), it yields that \( |m(\xi_1, \xi_2)| \leq (|\xi_1| + |\xi_2|)^{1/2} \) for \( \xi_1, \xi_2 \in \mathbb{R}^n \). Therefore, we have
\[
|\tilde{m}(\xi_1, \xi_2, \xi_3, \xi_4)| = \left| \int_0^\infty (m(t\xi_1, t\xi_2) m(t\xi_3, t\xi_4)) \frac{dt}{t} \right|
\leq \int_0^\infty \frac{(|t\xi_1| + |t\xi_2|)^{1/4} (|t\xi_3| + |t\xi_4|)^{1/4}}{(1 + |t\xi_1| + |t\xi_2|)^{1/2} (1 + |t\xi_3| + |t\xi_4|)^{1/2}} dt
\leq \int_0^\infty \frac{(t(|\xi_1| + |\xi_2|))(t(|\xi_3| + |\xi_4|))^{1/4}}{(1 + t(|\xi_1| + |\xi_2| + |\xi_3| + |\xi_4|))^{1/2}} dt
\leq \int_0^\infty \frac{s^{1/2}}{(1 + s)^{1/2}} ds < \infty.
\]

Note that \( 2n + 1 > 4n/2 \), then by Theorem 1 in [21], one obtains that there exists \( 0 < \delta \leq 1 \) such that \( \tilde{T}_m \) is bounded from \( L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \times L^{q_3}(\mathbb{R}^n) \times L^{q_4}(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for \( 2 - \delta < q_1, q_2, q_3, q_4 \) with \( 1/q = 1/q_1 + 1/q_2 + 1/q_3 + 1/q_4 \).
(ii) Note that
\[
\mathcal{T}_m(f)(x) = \int_0^\infty \int_{(\mathbb{R}^n)^4} e^{2\pi i x \cdot (\xi_1 + \xi_2 - \xi_3 - \xi_4)} m(t\xi_1, t\xi_2) \bar{m}(t\xi_3, t\xi_4) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) dt \times \hat{f}_1(\xi_3) \hat{f}_2(\xi_4) d\xi_1 d\xi_2 d\xi_3 d\xi_4 \frac{dt}{t}.
\]
Then, as a consequence of (i), we obtain that \(\mathcal{T}_m\) is bounded from \(L^q(\mathbb{R}^n)\) to \(L^q(\mathbb{R}^n)\) for \(2 - \delta < q_1, q_2 < \infty\) with \(1/q = 1/q_1 + 1/q_2\). \(\square\)

3. Proofs of Theorems 1.1 and 1.2

This section will be devoted to prove Theorems 1.1 and 1.2 by showing that the associated kernel of \(\mathcal{T}_m\) satisfies the conditions (H2) and (H3) in Definition 1.4. The following two propositions provide a foundation for our analysis.

Proposition 3.1. Let \(s \in \mathbb{N}\) satisfy \(n + 1 \leq s \leq 2n\). Suppose \(m \in L^\infty((\mathbb{R}^n)^2)\) satisfies
\[
|\partial^\alpha m(\xi_1, \xi_2)| \lesssim (|\xi_1| + |\xi_2|)^{-|\alpha| + \varepsilon_1} (1 + |\xi_1| + |\xi_2|)^{\varepsilon_1 + \varepsilon_2},
\]
for some \(\varepsilon_1, \varepsilon_2 > 0\) and \(|\alpha| \leq s\). Then, for any \(2n/s < p \leq 2\), there exist \(C > 0\) and \(\delta > n/p\), such that
\[
\left( \int_{S_j(Q)} \int_{S_k(Q)} \left( \int_0^\infty \left| \hat{m} \left( \frac{x - y_1}{t}, \frac{x - y_2}{t} \right) - \hat{m} \left( \frac{\bar{x} - y_1}{t}, \frac{\bar{x} - y_2}{t} \right) \right|^2 dt \frac{t^\delta}{t^{4n+1}} \right)^{\frac{p}{2}} dy_1 dy_2 \right)^{\frac{1}{p}} \leq C \frac{|x - \bar{x}|^{2(\delta - n/p)}}{|Q|^{2\delta/n}} 2^{-2\delta \max(j, k)}
\]
for all balls \(Q\), all \(x, \bar{x} \in 1/2Q\) and \((j, k) \neq (0, 0)\).

Proof. Denote the left-side of (3.2) by \(A_{j,k}(m, Q)(x, \bar{x})\), and let \(Q = B(x_0, R)\). Let \(u = ax\) \((a > 0)\) and \(s = at\), one obtains that
\[
A_{j,k}(m, Q)(x, \bar{x}) = a^{-2n/p'} \left( \int_{S_j(Q^a)} \int_{S_k(Q^a)} \left( \int_0^\infty \left| \hat{m} \left( \frac{x^a - u_1}{at}, \frac{x^a - u_2}{at} \right) - \hat{m} \left( \frac{\bar{x}^a - u_1}{at}, \frac{\bar{x}^a - u_2}{at} \right) \right|^2 dt \frac{t^{\delta}}{t^{4n+1}} \right)^{\frac{p}{2}} du_1 du_2 \right)^{\frac{1}{p'}}
\]
\[
= a^{2n/p} \left( \int_{S_j(Q^a)} \int_{S_k(Q^a)} \left( \int_0^\infty \left| \hat{m} \left( \frac{x^a - u_1}{s}, \frac{x^a - u_2}{s} \right) \right|^2 dt \frac{t^\delta}{t^{4n+1}} \right)^{\frac{p}{2}} du_1 du_2 \right)^{\frac{1}{p'}}
\]
\[-m\left(\frac{\bar{x}^a - u_1}{s}, \frac{\bar{x}^a - u_2}{s}\right)\left|\frac{ds}{s^{4n+1}}\right|^\frac{q'}{2} du_1 du_2\right]^\frac{1}{n'}

= a^{2n/p}A_{j,k}(m, Q^a)(x^a, \bar{x}^a),

where \(Q^a = B(ax_0, aR)\), \(a^a = ax\) and \(\bar{x}^a = a\bar{x}\). Therefore, taking \(a = 1/(2^{\max(j,k)}R)\), the desired estimate (3.2) follows from the following fact:

\[A_{j,k}(m, Q^a)(x^a, \bar{x}^a) \lesssim |x^a - \bar{x}^a|^{2(\delta - n/p)}\]

Thus, we only need to show (3.2) in the case \(R = 1/2^{\max(j,k)}\). In addition, we may assume \(|h| = |x - \bar{x}| < 1/2\) and \(k \geq j\) (hence \(k \geq 1\)). Hence, for \(Q = B(x_0, 2^{-k})\) and \(\delta > n/p\), we need to show that

\[(3.3) \quad A_{j,k}(m, Q)(x, \bar{x}) \lesssim |x - \bar{x}|^{2(\delta - n/p)}.

Let \(\Psi \in \mathcal{S}(\mathbb{R}^{2n})\) with \(\text{supp} \Psi \subseteq \{(\xi, \eta) : 1/2 \leq |\xi| + |\eta| \leq 2\}\) and

\[
\sum_{j \in \mathbb{Z}} \Psi(2^{-j}\xi, 2^{-j}\eta) = 1, \quad \text{for all } (\xi, \eta) \in (\mathbb{R}^{2n}) \setminus \{0\}.
\]

Now, we can write

\[m(\xi, \eta) = \sum_{j \in \mathbb{Z}} m_j(\xi, \eta) := \sum_{j \in \mathbb{Z}} \Psi(2^{-j}\xi, 2^{-j}\eta)m(\xi, \eta)
\]

and hence \(\text{supp } m_j \subseteq \{(\xi, \eta) : 2^{j-1} \leq |\xi| + |\eta| \leq 2^{j+1}\}\).

By changing variables, to prove (3.3), it is sufficient to show that for \(Q = B(x_0, 2^{-k})\), the following inequality holds:

\[
\left(\int_{S_j(Q_x)} \int_{S_h(Q_x)} \left(\int_0^\infty \hat{m}(y + h, z + h) - \hat{m}(y, z) dt \right)^{\frac{q'}{2}} \, dy dz \right)^{\frac{1}{n'}} \leq C|h|^{2(\delta - n/p)},
\]

where \(h = x - \bar{x}\) and \(Q_x = Q - \bar{x}\). We prove this in the following three cases.

(a) The case \(2n/p < s < 2n/p + 1\). First, we note that (3.1) remains valid for any smaller positive number than \(\varepsilon_1\). Thus, one may take \(\varepsilon_1\) sufficiently close to \(s - 2n/p\) so that \(0 < \varepsilon_1 < s - 2n/p\).

For any interval \(I\) in \(\mathbb{R}_+\), we introduce the notion \(A_{\varepsilon}\) and \(A_{\varepsilon}(I)\) as follows.

\[A_{\varepsilon} := \left(\int_{S_j(Q_x)} \int_{S_h(Q_x)} \left(\int_0^\infty \hat{m}(y + h, z + h) - \hat{m}(y, z) dt \right)^{\frac{q'}{2}} \, dy dz \right)^{\frac{1}{n'}};\]

\[A_{\varepsilon}(I) := \left(\int_{S_j(Q_x)} \int_{S_h(Q_x)} \left(\int_I \hat{m}(y + h, z + h) - \hat{m}(y, z) \right)^{\frac{q'}{2}} \, dy dz \right)^{\frac{1}{n'}}.\]

Since \(Q_x = B(x_0 - \bar{x}, 1/2^k)\), we have \(2^{-2} \leq |y + h| \leq 2\) and \(|z + h| \leq 2^{j-k+1}\) for all \(y \in S_k(Q_x)\) and \(z \in S_j(Q_x)\). Therefore, it yields that

\[A_{\varepsilon}(I) \lesssim \left(\int_{|z| \leq 2^{j-k+1}} \int_{2^{-2} \leq |y| \leq 2} \left(\int_I \hat{m}(y, z) \left|\frac{dt}{t^{4n+1}}\right|^{\frac{q'}{2}} \, dy dz \right)^{\frac{1}{n'}}.\]
Note that $|y| \sim 1$ in the above integration domain, by the Minkowski inequality and the Haussdorf-Young inequality, for $|\alpha| = s$, we have

$$A_\ell(I) \lesssim \left( \int_{|z| \leq 2^{j-k+1}} \int_{2^{-2} \leq |y| \leq 2} \left( \int_I |y^\alpha| \left| \tilde{m}_\ell \left( \frac{y}{t^\alpha}, \frac{z}{t} \right) \right|^2 \frac{dt}{t^{4n+1}} \right)^{\frac{p'}{2}} \frac{dydz}{t} \right)^{\frac{1}{p'}}$$

$$\leq \left( \int_I \left( \int_{|z| \leq 2^{j-k+1}} \int_{2^{-2} \leq |y| \leq 2} \left| y^\alpha \tilde{m}_\ell \left( \frac{y}{t^\alpha}, \frac{z}{t} \right) \right|^p \frac{dydz}{t} \right)^{\frac{2}{p'}} \frac{dt}{t^{4n+1}} \right)^{\frac{1}{2}}$$

$$= \left( \int_I \left( \int_{R^n} \int_{R^n} |\partial_\xi^2 m_\ell(\xi, \eta)|^p d\xi d\eta \right)^{\frac{2}{p}} \frac{dt}{t^{2|\alpha|-4n/p-1}} \right)^{\frac{1}{2}}$$

$$\lesssim \left( \int_{R^n} \int_{R^n} |\partial_\xi^2 m_\ell(\xi, \eta)|^p d\xi d\eta \right)^{\frac{1}{p'}}$$

Hence, we obtain

$$(3.4) \quad A_\ell(I) \lesssim \frac{(2^J)^{c_1 - |\alpha| + 2n/p}}{(1 + 2^J)^{c_1 + c_2}} \left( \int_I t^{2|\alpha|-4n/p-1} dt \right)^{\frac{1}{2}}$$.

Now, setting $\varphi_\ell(\xi, \eta) = m_\ell(\xi, \eta)(e^{2\pi i \eta t \cdot (\xi + \eta)} - 1)$, we have

$$\tilde{m}_\ell \left( \frac{y + h}{t}, \frac{z + h}{t} \right) - \tilde{m}_\ell \left( \frac{y}{t}, \frac{z}{t} \right) = \varphi_\ell \left( \frac{y}{t}, \frac{z}{t} \right).$$

Proceeding the same argument as before, we have

$$A_\ell(I) \lesssim \left( \int_{S_j(Q_s)} \int_{S_k(Q_s)} \left( \int_I |y^\alpha| \left| \tilde{m}_\ell \left( \frac{y + h}{t}, \frac{z + h}{t} \right) - \tilde{m}_\ell \left( \frac{y}{t}, \frac{z}{t} \right) \right|^2 \frac{dt}{t^{4n+1}} \right)^{\frac{p'}{2}} \frac{dydz}{t} \right)^{\frac{1}{p'}}$$

$$= \left( \int_{S_j(Q_s)} \int_{S_k(Q_s)} \left| y^\alpha \varphi_\ell \left( \frac{y}{t}, \frac{z}{t} \right) \right|^p \frac{dydz}{t} \right)^{\frac{2}{p'}} \frac{dt}{t^{4n+1}}$$

$$= \left( \int_{S_j(Q_s)} \int_{S_k(Q_s)} \left| y^\alpha \varphi_\ell(y, z) \right|^p \frac{dydz}{t} \right)^{\frac{2}{p'}} \frac{dt}{t^{2|\alpha|-4n/p-1}}$$

$$\leq \left( \int_{R^n} \int_{R^n} |\partial_\xi^2 \varphi_\ell(\xi, \eta)|^p d\xi d\eta \right)^{\frac{2}{p'}} \frac{dt}{t^{2|\alpha|-4n/p-1}}$$

$$= \left( \int_{R^n} \int_{R^n} |\partial_\xi^2 [m_\ell(\xi, \eta)(e^{-2\pi i \eta t \cdot (\xi + \eta)} - 1)]|^p d\xi d\eta \right)^{\frac{2}{p'}} \frac{dt}{t^{2|\alpha|-4n/p-1}}$$

By the following fact

$$|\partial_\xi^2 [m_\ell(\xi, \eta)(e^{-2\pi i \eta t \cdot (\xi + \eta)} - 1)]| \lesssim \frac{2^J |h|}{t} \left( \frac{2^J}{1 + 2^J} \right)^{c_1 + c_2} + \sum_{\beta=1}^{n} \left( \frac{|h|}{t} \right)^{\beta} \left( \frac{2^J}{1 + 2^J} \right)^{c_1 + c_2},$$
it yields that

\[(3.5)\]
\[
A_\ell(I) \lesssim \left( \int_1^\ell \frac{2^\ell |h|}{t} \frac{(2^\ell)^{\varepsilon_1 - |\alpha|}}{1 + 2^\ell \varepsilon_1 + \varepsilon_2} + \sum_{\beta = 1}^{\max(\beta, 1)} \left( \frac{|h|}{t} \right)^\beta \frac{(2^\ell)^{\varepsilon_1 - |\alpha| + \beta}}{1 + 2^\ell \varepsilon_1 + \varepsilon_2} \right)^\frac{1}{2} 2^{4n\ell/p} t^{2|\alpha| - 4n/p - 1} dt \right)^{\frac{1}{2}}.
\]

Now, we fix sufficiently small \(\varepsilon > 0\) so that \(\varepsilon(s - 2n/p) < \min\{\varepsilon_1, \varepsilon_2\}\). Then, if \(2^\ell|h| \geq 1\), noting \(2n/p < s < 2n/p + 1\) and using \((3.4)\) for \(I = (0, (2^\ell|h|)^{1+\varepsilon})\), we have

\[A_\ell((0, (2^\ell|h|)^{1+\varepsilon})) \lesssim 2^{-\ell(\varepsilon_1 + 2n/p + \max(\beta, 1))} \beta (2^\ell|h|)^{(1+\varepsilon)(s-2n/p)} = |h|^{(1+\varepsilon)(s-2n/p)} 2^{\ell(\varepsilon(s-2n/p) - \varepsilon_2)}\]

By \((3.5)\) for \(I = [(2^\ell|h|)^{1+\varepsilon}, \infty)\), we have

\[A_\ell([(2^\ell|h|)^{1+\varepsilon}, \infty)) \lesssim \sum_{\beta = 0}^{|\alpha|} |h|^{\max(\beta, 1)} 2^{\ell(-|\alpha| + 2n/p + \max(\beta, 1))} (2^\ell|h|)^{(1+\varepsilon)(s-2n/p - \max(\beta, 1))}
\]

Thus, noting \(\varepsilon(s - 2n/p) - \varepsilon_2 < 0\) and \(|h| < 1\), we obtain

\[(3.6)\]
\[
\sum_{2^\ell|h| \geq 1} A_\ell \lesssim \sum_{2^\ell|h| \geq 1} |h|^{(1+\varepsilon)(s-2n/p)} 2^{\ell(\varepsilon(s-2n/p) - \varepsilon_2)} + \sum_{2^\ell|h| \geq 1} |h|^{-\max(\beta, 1) + (1+\varepsilon)(s-2n/p)} 2^{\ell(\varepsilon(s-2n/p)-\max(\beta, 1))}
\]

\[\leq |h|^{s-2n/p + \varepsilon_2} + \sum_{\beta = 0}^{|\alpha|} |h|^{s-2n/p} \lesssim |h|^{s-2n/p}.
\]

In the case \(2^\ell|h| < 1\), using \((3.4)\) for \(I = (0, (2^\ell|h|)^{1-\varepsilon})\), we have

\[A_\ell((0, (2^\ell|h|)^{1-\varepsilon})) \lesssim 2^{\ell(-s+2n/p+\varepsilon_1)} (2^\ell|h|)^{(1-\varepsilon)(s-2n/p)} = |h|^{(1-\varepsilon)(s-2n/p)} 2^{\ell(-\varepsilon(s-2n/p) + \varepsilon_1)}\]

Further more, by using \((3.5)\) for \(I = [(2^\ell|h|)^{1-\varepsilon}, \infty)\), we have

\[A_\ell([(2^\ell|h|)^{1-\varepsilon}, \infty)) \lesssim \sum_{\beta = 0}^{|\alpha|} |h|^{\max(\beta, 1)} 2^{\ell(-s+2n/p+\max(\beta, 1))} (2^\ell|h|)^{(1-\varepsilon)(s-2n/p - \max(\beta, 1))}
\]

\[= \sum_{\beta = 0}^{|\alpha|} |h|^{\max(\beta, 1) + (1-\varepsilon)(s-2n/p)} 2^{-\ell(s-2n/p-\max(\beta, 1))}.
\]

By the fact that \(\varepsilon(s - 2n/p) - \varepsilon_1 < 0\) and \(|h| < 1\), we obtain

\[(3.7)\]
\[
\sum_{2^\ell|h| < 1} A_\ell \lesssim \sum_{2^\ell|h| < 1} |h|^{(1-\varepsilon)(s-2n/p)} 2^{\ell(\varepsilon(s-2n/p) + \varepsilon_1)}
\]
By the Hölder inequality, it yields that

\[ \left| h \right|^{s-2n/p-\varepsilon} + \sum_{\beta=0}^{2\varepsilon} \left| h \right|^{s-2n/p} \lesssim \left| h \right|^{s-2n/p-\varepsilon} + \left| h \right|^{s-2n/p}. \]

Noting that \( 0 < \varepsilon < s - 2n/p \) and taking \( \delta = (s - \varepsilon)/2 \), by (3.6) and (3.7), it holds that

\[
\left( \int_{S_j(Q)} \int_{S_k(Q)} \left( \int_0^\infty \left| \bar{m} \left( \frac{y + h}{t}, \frac{z + h}{t} \right) - \bar{m} \left( \frac{y}{t}, \frac{z}{t} \right) \right|^{2} \frac{dt}{t^{4n+1}} \right)^{\frac{p}{2}} dydz \right)^{\frac{1}{p}} \leq \sum_{\ell \in \mathbb{Z}} A_{\ell} \lesssim \left| h \right|^{2(\delta-n/p)},
\]

This leads to the conclusion of Proposition 3.1 in the case \( 2n/p < s < 2n/p + 1 \).

(b) The case \( 2n/p < s = 2n/p + 1 \). First, we Choose \( 1 < p_0 < p \) such that \( 2n/p_0 < s \). Then \( p_0 \) satisfies \( 2n/p_0 < s = 2n/p + 1 < 2n/p + 1 \). Hence, for all balls \( Q \), all \( x, \bar{x} \in \frac{1}{2}Q \) and \( (j, k) \neq (0, 0) \), by the step (a), we have

\[
\left( \int_{S_j(Q)} \int_{S_k(Q)} \left( \int_0^\infty \left| \bar{m} \left( \frac{y + h}{t}, \frac{z + h}{t} \right) - \bar{m} \left( \frac{y}{t}, \frac{z}{t} \right) \right|^{2} \frac{dt}{t^{4n+1}} \right)^{\frac{p}{2}} dydz \right)^{\frac{1}{p}} \leq C \frac{|h|^{2\delta-2n/p_0}}{|Q|^{2\delta/n}} 2^{-2\delta \max(j,k)}.
\]

By the Hölder inequality, it yields that

\[
\left( \int_{S_j(Q)} \int_{S_k(Q)} \left( \int_0^\infty \left| \bar{m} \left( \frac{y + h}{t}, \frac{z + h}{t} \right) - \bar{m} \left( \frac{y}{t}, \frac{z}{t} \right) \right|^{2} \frac{dt}{t^{4n+1}} \right)^{\frac{p}{2}} dydz \right)^{\frac{1}{p}} \leq (2^{n(j+k)}|Q|^2)^{\frac{1}{p_0}} \left( \int_{S_j(Q)} \int_{S_k(Q)} \left( \int_0^\infty \left| \bar{m} \left( \frac{y + h}{t}, \frac{z + h}{t} \right) - \bar{m} \left( \frac{y}{t}, \frac{z}{t} \right) \right|^{2} \frac{dt}{t^{4n+1}} \right)^{\frac{p}{2}} dydz \right)^{\frac{1}{p}} \times \left( \frac{dt}{t^{4n+1}} \right)^{\frac{p}{2}} \frac{1}{p_0} |Q|^{\frac{2\delta}{p_0}} \frac{1}{|Q|^{2\delta/n}} \frac{1}{2^{2\delta \max(j,k)}} \right. \]

\[
\lesssim (2^{n \max(j,k)}|Q|^2)^{\frac{1}{p_0}} \left( \left| h \right|^{2\delta-2n/p_0} |Q|^{\frac{2\delta}{p_0}} \frac{1}{|Q|^{2\delta/n}} \frac{1}{2^{2\delta \max(j,k)}} \right. \]

\[
= \left| h \right|^{(2\delta-2n/p_0+2n/p)\cdot n} |Q|^{\frac{2\delta-2n/p_0+2n/p}{n} \cdot 2^{-2(2\delta-2n/p_0+2n/p)} \max(j,k)}. \]

Therefore, taking \( \delta - n/p_0 + n/p > n/p \) as \( \delta \) newly, we obtain the desired estimate.

(c) The case \( 2n/p + 1 < s \leq 2n \). In this case there is an integer \( l \) such that \( 2n/p + l < s \leq 2n/p + 1 + l \). Then it follows that \( 2n/p < s - l \leq 2n/p + 1 \). Thus, regarding \( s - l \) as \( s \), we may deduce this case to the previous case (a) or case (b). This completes the proof of Proposition 3.1.

□
Proposition 3.2. Let \( s \in \mathbb{N} \) with \( n + 1 \leq s \leq 2n \). Let \( m \in L^\infty((\mathbb{R}^n)^2) \) and satisfy
\[
|\partial^\alpha m(\xi_1, \xi_2)| \lesssim (|\xi_1| + |\xi_2|)^{-|\alpha|} \quad \text{for } |\alpha| \leq s,
\]
and
\[
|m(\xi_1, \xi_2)| \lesssim \frac{(|\xi_1| + |\xi_2|)^{\varepsilon_1}}{(1 + |\xi_1| + |\xi_2|)^{\varepsilon_1 + \varepsilon_2}}, \quad \text{for some } \varepsilon_1, \varepsilon_2 > 0.
\]
Then, for \( 2n/s < p \leq 2 \), there exists a constant \( C > 0 \), such that the following inequality holds for all balls \( Q \) with center at \( x \) and \( (j, k) \neq (0, 0) \).
\[
\left( \int_{S_j(Q)} \int_{S_k(Q)} \left( \int_0^\infty \left| \tilde{m}\left( \frac{x - y_1}{t}, \frac{x - y_2}{t} \right) \right|^2 \frac{dt}{t^{4n+1}} \right)^{\frac{p'}{2}} dy_1 dy_2 \right)^{\frac{1}{p'}} \leq C \frac{1}{|Q|^{2/p}} 2^{-2n\max(j,k)/p}.
\]

Proof. Let \( Q = B(x, R) \), \( u = ax \) \((a > 0)\) and \( s = at \), we have
\[
B_{j,k}(m, Q)(x) := \left( \int_{S_j(Q)} \int_{S_k(Q)} \left( \int_0^\infty \left| \tilde{m}\left( \frac{x - y_1}{t}, \frac{x - y_2}{t} \right) \right|^2 \frac{dt}{t^{4n+1}} \right)^{\frac{p'}{2}} dy_1 dy_2 \right)^{\frac{1}{p'}}
\]
\[
= a^{2n/p} \left( \int_{S_j(Q^a)} \int_{S_k(Q^a)} \left( \int_0^\infty \left| \tilde{m}\left( \frac{x^a - u_1}{t}, \frac{x^a - u_2}{t} \right) \right|^2 \frac{ds}{s^{4n+1}} \right)^{\frac{p'}{2}} du_1 du_2 \right)^{\frac{1}{p'}}
\]
where \( Q^a = B(ax, aR) \), \( x^a = ax \). So, taking \( a = 1/(2^{\max(j,k)}R) \), the estimate \( B_{j,k}(m, Q^a)(x^a) \lesssim 1 \) implies the desired estimate. Thus, we only need to show \( (3.10) \) in the case \( R = 1/(2^{\max(j,k)}) \). We may also assume \( k \geq j \) and hence \( k \geq 1 \). Then, for \( Q = B(x, 2^{-k}) \), it sufficient to show that
\[
B_{j,k}(m, Q)(x) \lesssim 1.
\]
By changing variables, it is enough to show that
\[
\left( \int_{S_j(Q_x)} \int_{S_k(Q_x)} \left( \int_0^\infty \left| \tilde{m}\left( \frac{y}{t}, \frac{z}{t} \right) \right|^2 \frac{dt}{t^{4n+1}} \right)^{\frac{p'}{2}} dydz \right)^{\frac{1}{p'}} \leq C \frac{1}{|Q_x|^{2/p}} 2^{-2n\max(j,k)/p},
\]
where \( Q_x = Q - x \).
For every interval \( I \) in \( \mathbb{R}_+ \), let
\[
B_{j,k}(m, Q, I)(x) := \left( \int_{S_j(Q_x)} \int_{S_k(Q_x)} \left( \int_I \left| \tilde{m}\left( \frac{y}{t}, \frac{z}{t} \right) \right|^2 \frac{dt}{t^{4n+1}} \right)^{\frac{p'}{2}} dydz \right)^{\frac{1}{p'}}.
\]
The Minkowski inequality, together with Haussdorff-Young inequality implies that
\[
B_{j,k}(m, Q, I)(x)
\]
\[
\lesssim (2^k R)^{-|\alpha|} \left( \int_{S_j(Q_x)} \int_{S_k(Q_x)} \left( \int_I y^\alpha \left| \tilde{m}\left( \frac{y}{t}, \frac{z}{t} \right) \right|^2 \frac{dt}{t^{4n+1}} \right)^{\frac{p'}{2}} dydz \right)^{\frac{1}{p'}}
\]
\[
\lesssim (2^k R)^{-|\alpha|} \left( \int_I \left( \int_{S_j(Q_x)} \int_{S_k(Q_x)} \left| y^\alpha \left| \tilde{m}\left( \frac{y}{t}, \frac{z}{t} \right) \right| dydz \right)^{\frac{p'}{2}} \frac{dt}{t^{4n+1}} \right)^{\frac{1}{2}}
\]
This completes the proof of Proposition 3.2.

(a) The case true, whose proofs will be postponed to the next sections.

In virtue of $2^k R = 1$, taking $|\alpha| = s$ and $I = [2^{(1+\epsilon)}, \infty)$, the estimate in (3.8) implies that

$$B_{j,k}(m_{\ell}, Q, [2^{(1+\epsilon)}, \infty)) \lesssim 2^{\ell(1+\epsilon)(-2n/p) + 2\ell \varepsilon_1} 2^{2\ell(2n/p)} = 2^{\ell(\varepsilon_1 - 2\varepsilon_2/n/p)}.$$ 

Next, we consider two cases according to the value of $\ell$.

Case (a). $\ell < 0$. In this case, taking $|\alpha| = 0$ and $I = [2^{(1+\epsilon)}, \infty)$, the estimate in (3.9) implies that

$$B_{j,k}(m_{\ell}, Q, [2^{(1+\epsilon)}, \infty)) \lesssim 2^{\ell(1+\epsilon)(-2n/p) + 2\ell \varepsilon_1} 2^{2\ell(2n/p)} = 2^{\ell(\varepsilon_1 - 2\varepsilon_2/n/p)}.$$ 

In virtue of $2^k R = 1$, taking $|\alpha| = s$ and $I = [0, 2^{(1+\epsilon)}]$, the estimate in (3.8) implies that

$$B_{j,k}(m_{\ell}, Q, [0, 2^{(1+\epsilon)}]) \lesssim 2^{\ell(1+\epsilon)(s-2n/p) + 2\ell \varepsilon(s-2n/p)} = 2^{\ell \varepsilon(s-2n/p)}.$$ 

Hence,

$$B_{j,k}(m_{\ell}, Q, [0, \infty)) \lesssim 2^{\ell(\varepsilon_1 - 2\varepsilon_2/n/p) + 2\ell \varepsilon(s-2n/p)}.$$

Case (b). $\ell \geq 0$. By repeating the same arguments as in case (a), we get

$$B_{j,k}(m_{\ell}, Q, [2^{(1-\epsilon)}, \infty)) \lesssim 2^{\ell(1-\epsilon)(-2n/p) + 2\ell \varepsilon_2} 2^{2\ell(2n/p)} = 2^{\ell(2n/p - \varepsilon_2)}$$

and

$$B_{j,k}(m_{\ell}, Q, [0, 2^{(1-\epsilon)}]) \lesssim 2^{\ell(1-\epsilon)(s-2n/p) + 2\ell \varepsilon(s-2n/p)} = 2^{-\ell \varepsilon(s-2n/p)}.$$ 

Therefore,

$$B_{j,k}(m_{\ell}, Q, [0, \infty)) \lesssim 2^{\ell(2n/p - \varepsilon_2)} + 2^{-\ell \varepsilon(s-2n/p)}.$$ 

Choosing $\varepsilon > 0$ so that $2n\varepsilon/p < \min(\varepsilon_1, \varepsilon_2)$, we obtain from case (a) and case (b)

$$B_{j,k}(m, Q)(x) \leq \sum_{\ell < 0} B_{j,k}(m_{\ell}, Q, [0, \infty)) + \sum_{\ell \geq 0} B_{j,k}(m_{\ell}, Q, [0, \infty))$$ 

$$\lesssim \sum_{\ell < 0} [2^{\ell(\varepsilon_1 - 2\varepsilon_2/n/p) + 2\ell \varepsilon(s-2n/p)} + 2^{\ell(2n/p - \varepsilon_2)} + 2^{-\ell \varepsilon(s-2n/p)}]$$

$$\lesssim 1.$$ 

This completes the proof of Proposition 3.2. □

Proofs of Theorems 1.1 and 1.2 First, we assume that Theorems 1.3 and 1.5 are true, whose proofs will be postponed to the next sections.

(a) The case $p_0 > 2n/s$. By Proposition 3.1 and Proposition 3.2, it is easy to see that the associated kernel of $\mathcal{K}_m$ satisfies the conditions (H2) and (H3). Since we have supposed (H1) from the beginning, applying Theorems 1.3 and 1.5, we obtain the desired conclusions in Theorem 1.1 and Theorem 1.2.

(b) The case $p_0 = 2n/s$. By the property of $A_p$ weights, there exists a real number $\tilde{p}_0$.
satisfying $p_0 = 2n/s < \tilde{p}_0 < \min(p_1, p_2, 2)$ and $\tilde{\omega} \in A_{\tilde{p}/\tilde{p}_0}$ (see [1] or [24]). Therefore, by step (a), we finish the proof of Theorem 1.1 and Theorem 1.2.

4. Proofs of Theorems 1.3 and 1.4

Let us recall the definition of $A_\beta$ weights introduced by Lerner et al. [24].

**Definition 4.1.** Let $\vec{P} = (p_1, \ldots, p_m)$ and $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 \leq p_1, \ldots, p_m < \infty$. Given $\vec{\omega} = (\omega_1, \ldots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i^{p/p_i}$. We say that $\vec{\omega}$ satisfies the $A_\beta$ condition if

$$\sup_Q \left( \frac{1}{|Q|} \int_Q \prod_{i=1}^{m} \omega_i^{\frac{p_i}{p}} \right) \prod_{i=1}^{m} \left( \frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{\frac{1}{p_i'}} < \infty,$$

when $p_i = 1$, $\left( \frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{\frac{1}{p_i'}}$ is understood as $(\inf_Q \omega_i)^{-1}$.

The new maximal function $\mathcal{M}_p$ can be defined by

$$\mathcal{M}_p(f)(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \left( \frac{1}{|Q|} \int_Q |f_j(y_j)|^p dy_j \right)^{1/p}.$$

When $p = 1$, we get $\mathcal{M}_1 = \mathcal{M}$, which was introduced in [24].

In order to prove our results, we need the following lemmas.

**Lemma 4.1.** For any $1 < p_0 \leq p_1, \ldots, p_m < \infty$ and $p$ so that $1/p = 1/p_1 + \cdots + 1/p_m$ and $\vec{\omega} \in A_{\vec{p}/p_0}$, where $\vec{p}/p_0 = (p_1/p_0, \ldots, p_m/p_0)$, the following weighted estimates hold.

1. If there is no $p_i = p_0$, then $\|\mathcal{M}_{p_0}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}$.
2. If there is a $p_i = p_0$, then $\|\mathcal{M}_{p_0}(\vec{f})\|_{L^{p, \infty}(\nu_{\vec{\omega}})} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}$.

**Proof.** The proof of (1) was given in [2]. The proof of (2) is similar to (1), we omit the proof. \qed

**Lemma 4.2.** ([10]) Let $\omega$ be an $A_\infty$ weight. Then there exist constant $C$ and $\rho > 0$ depending upon the $A_\infty$ condition of $\omega$ such that, for all $\lambda, \varepsilon > 0$,

$$\omega(\{y \in \mathbb{R}^n : Mf(y) > \lambda, M^\#f(y) \leq \lambda + \varepsilon\}) \leq C \varepsilon^\rho \omega(\{y \in \mathbb{R}^n : Mf(y) > \frac{1}{2} \lambda\}).$$

As consequences, we have the following estimates for $\delta > 0$.

1. Let $\varphi : (0, \infty) \to (0, \infty)$ be a doubling, that is, $\varphi(2a) \leq C \varphi(a)$ for $a > 0$. Then, there exists a constant $C$ depending upon the $A_\infty$ condition of $\omega$ and doubling condition of $\varphi$ such that

$$\sup_{\lambda > 0} \varphi(\lambda) \omega(\{y \in \mathbb{R}^n : Mf(y) > \lambda\}) \leq C \sup_{\lambda > 0} \varphi(\lambda) \omega(\{y \in \mathbb{R}^n : M^\#f(y) > \lambda\}).$$

for every function such that the left-hand side is finite.
(2) Let \( 0 < p < \infty \). There exists a positive constant \( C \) depending upon the \( A_\infty \) condition and \( p \) such that

\[
\left( \int_{\mathbb{R}^n} (M_\delta f(x))^p \omega(x) dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} (M^2_\delta f(x))^p \omega(x) dx \right)^{1/p}
\]

for every function such that the left-hand side is finite.

Lemma 4.3. Let \( T \) be a multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some \( 1 \leq p_0 < \infty \). For any \( 0 < \delta < \min\{1, \frac{p_0}{m}\} \), there is a constant \( C < \infty \) such that for any bounded and compactly supported \( f_j, (j = 1, \ldots, m) \).

\[
M^*_\delta T(\tilde{f})(x) \leq CM_{p_0}(\tilde{f})(x).
\]

Proof. Fix a point \( x \in \mathbb{R}^n \) and a ball \( Q \) containing \( x \). For \( 0 < \delta < \min\{1, \frac{p_0}{m}\} \), we only need to show that there exists a constant \( c_Q \) such that

\[
\left( \frac{1}{|Q|} \int_Q |T(\tilde{f})(z) - c_Q|^{\delta} \, dz \right)^{1/\delta} \leq C M_{p_0, j}(\tilde{f})(x).
\]

For each \( j = 1, \ldots, m \), we decompose \( f_j = f_j^0 + f_j^\infty \), where \( f_j^0 = f_j \chi_{Q^*} \), \( Q^* \) is the ball with center at \( x \) and having eight times bigger radius than \( Q \).

First, we claim that

\[
\left( \int_0^\infty \left| \int_{\mathbb{R}^m} K_t(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j^{\alpha_j}(y_j) \, dy_j \right|^2 \frac{dt}{t} \right)^{1/2} < \infty, \quad \text{for } \vec{\alpha} \neq \vec{0}.
\]

In fact, set

\[
c_{Q, \vec{\alpha}} = \left( \int_0^\infty \left| \int_{\mathbb{R}^m} K_t(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j^{\alpha_j}(y_j) \, dy_j \right|^2 \frac{dt}{t} \right)^{1/2}.
\]

By the Minkowski inequality, we have

\[
c_{Q, \vec{\alpha}} \leq \int_{\mathbb{R}^m} \left( \int_0^\infty \left| K_t(x, y_1, \ldots, y_m) \right|^2 \frac{dt}{t} \right)^{1/2} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)| \, dy.
\]

To estimate \( c_{Q, \vec{\alpha}} \), we may assume \( \alpha_1 = \cdots = \alpha_l = 0 \) and \( \alpha_{l+1} = \cdots = \alpha_m = \infty \). Since \( \tilde{f}_j \in L_{p_0}^\infty(\mathbb{R}^n) \), there exists the smallest \( j_0 \in \mathbb{N} \) satisfying \( \text{supp} \tilde{f} \subset 2^{j_0}Q^* \). Then, by using the Hölder inequality and condition (H3), one may obtain that

\[
c_{Q, \vec{\alpha}} \leq \left( \int_{(2^{j_0}Q^*)^{m-l} \times (S_0(Q^*))^l} \left( \int_0^\infty \left| K_t(x, y) \right|^2 \frac{dt}{t} \right)^{\nu_0/2} \, dy \right)^{1/\nu_0}
\times \left( \int_{(2^{j_0}Q^*)^{m-l} \times (S_0(Q^*))^l} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)|^{p_0} \, dy \right)^{1/p_0}
\leq \sum_{k=1}^{j_0} \left( \int_{(S_k(Q^*))^{m-l} \times (S_0(Q^*))^l} \left( \int_0^\infty \left| K_t(x, y) \right|^2 \frac{dt}{t} \right)^{\nu_0} \, dy \right)^{1/\nu_0} \int_{\mathbb{R}^n} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)|^{p_0} \, dy \right)^{1/p_0}
\]
\begin{align*}
&\leq C \sum_{k=1}^{j_0} \frac{2^{-nk/p_0}}{|Q|^{m/p_0}} \left( \int_{\mathbb{R}^{nm}} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)|^{p_0} d\bar{y} \right)^{1/p_0} < \infty.
\end{align*}

Let $c_{Q,t} = \sum_{\alpha, \alpha \neq \bar{\alpha}} \int_{\mathbb{R}^{nm}} K_t(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j^{\alpha_j}(y_j) d\bar{y}$ and $c_Q = \left( \int_0^\infty |c_{Q,t}|^{2 \frac{dt}{t}} \right)^{1/2}$, where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_i = 0$ or $\infty$. Then we have
\begin{align*}
&\left( \frac{1}{|Q|} \int_Q \left| T(f)(z) - c_Q \right|^\delta (dz) \right)^{1/\delta} \\
&\leq C \left( \frac{1}{|Q|} \int_Q \left( \int_0^\infty \int_{\mathbb{R}^{nm}} K_t(z, \bar{y}) \prod_{j=1}^m f_j^{0}(y_j) d\bar{y} \right)^{2 \frac{dt}{t}} \right)^{\delta/2} (dz) \right)^{1/\delta} \\
&\quad + C \sum_{\alpha, \alpha \neq \bar{\alpha}} \left( \frac{1}{|Q|} \int_Q \left( \int_0^\infty \left| K_t(z, \bar{y}) - K_t(x, \bar{y}) \right| \prod_{j=1}^m f_j^{\alpha_j}(y_j) d\bar{y} \right)^{2 \frac{dt}{t}} \right)^{\delta/2} (dz) \right)^{1/\delta} \\
&= I_0 + C \sum_{\alpha \neq \bar{\alpha}} I_{\alpha}.
\end{align*}

By condition (H1), $T$ maps $L^{p_0} \times \cdots \times L^{p_0}$ into $L^{p_0/m, \infty}$. This together with the Kolmogorov inequality tells us that
\begin{align*}
I_0 \leq C \left\| T(f^0) \right\|_{L^{p_0/m, \infty}(Q^*_0)} \leq C \prod_{j=1}^m \left( \frac{1}{|Q^*_j|} \int_{Q^*_j} \left| f_j(z) \right|^{p_0} dz \right)^{p_0} \leq C \mathcal{M}_{p_0}(f^0)(x).
\end{align*}

To estimate $I_{\alpha}$ for $\alpha \neq \bar{\alpha}$, we may assume $\alpha_1 = \cdots = \alpha_l = \infty$ and $\alpha_{l+1} = \cdots = \alpha_m = 0$. By condition (H2), it yields that
\begin{align*}
&\int_{\mathbb{R}^{nm}} \left( \int_0^\infty \left| K_t(z, \bar{y}) - K_t(x, \bar{y}) \right|^{2 \frac{dt}{t}} \right)^{1/2} \prod_{j=1}^m |f_j^{\alpha_j}(y_j)| d\bar{y} \\
&\leq C \int_{(Q^*)_0^{c} \times (Q^*)^{m-l}} \left( \int_0^\infty \left| K_t(z, \bar{y}) - K_t(x, \bar{y}) \right|^{2 \frac{dt}{t}} \right)^{1/2} \prod_{j=1}^l |f_j^{\infty}(y_j)| dy_j \prod_{j=l+1}^m |f_j^{0}(y_j)| dy_j \\
&\quad \times \prod_{j=1}^l \left| f_j^{\infty}(y_j) \right| dy_j \prod_{j=l+1}^m \left| f_j^{0}(y_j) \right| dy_j \\
&\leq C \sum_{j_1, \ldots, j_l \geq 1} \left( \int_{(Q^*)^{m-l}} \prod_{j=1}^l S_{j_l}(Q^*) \cdots \prod_{j=1}^l \left( \int_0^\infty \left| K_t(z, \bar{y}) - K_t(x, \bar{y}) \right|^{2 \frac{dt}{t}} \right)^{1/p_0} \right)^{1/p_0} \\
&\quad \times \prod_{j=1}^l \left( \int_{2^{j_l}Q^*} |f_j(y_j)|^{p_0} dy_j \right)^{1/p_0} \prod_{j=l+1}^m \left( \int_{Q^*} |f_j(y_j)|^{p_0} dy_j \right)^{1/p_0} \\
&\leq C \sum_{j_1, \ldots, j_l \geq 1} \frac{|x - z|^{m(\delta - n/p_0)}}{|Q^*|^{m\delta/n} 2^{-m\delta_0}}.
\end{align*}
Thus, we finish the proof of Lemma 4.3. □

**Lemma 4.4.** Suppose \( K_t \) satisfies (H3) for some \( 1 \leq p_0 < \infty \). Suppose \( f_i \in C_c^\infty(\mathbb{R}^n) \) and supp \( f_i \subset B(0, R) \) for any \( i = 1, \cdots, m \). Then there is a constant \( C < \infty \) such that for \( |x| > 3R \), the following estimate holds uniformly.

\[
T(\vec{f})(x) \leq C M_{p_0}(\vec{f})(x).
\]

**Proof.** By the Minkowski inequality, the Hölder inequality, the support property of \( f_j \) and condition (H3), we obtain

\[
T(\vec{f})(x) \leq \int_{B(0,R)^m} \left( \int_0^\infty \left| K_t(x,y_1,\cdots,y_m) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \prod_{j=1}^m |f_j(y_j)| dy_j
\]

\[
\leq \int_{\left( B(x,\frac{2}{3}|x|) \right) \setminus \left( B(x,\frac{2}{3}|x|) \right)^m} \left( \int_0^\infty \left| K_t(x,y_1,\cdots,y_m) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \prod_{j=1}^m |f_j(y_j)| dy_j
\]

\[
\leq \left( \int_{\left( B(x,\frac{2}{3}|x|) \right) \setminus \left( B(x,\frac{2}{3}|x|) \right)^m} \left( \int_0^\infty \left| K_t(x,y_1,\cdots,y_m) \right|^2 \frac{dt}{t} \right)^{\frac{2}{p_0}} dy_j \right)^{1/p_0}
\]

\[
\times \left( \int_{\left( B(x,\frac{2}{3}|x|) \right) \setminus \left( B(x,\frac{2}{3}|x|) \right)^m} \prod_{j=1}^m |f_j(y_j)|^{p_0} dy_j \right)^{1/p_0}
\]

\[
\leq CM_{p_0}(\vec{f})(x),
\]

which concludes the proof of Lemma 4.4. □

**Proof of Theorem** 4.3 First, we will show that (1) in Theorem 4.3 is true. By Lemma 4.1, we may assume that \( ||M_{p_0,\vec{f}}||_{L^p(\nu_2)} \) is finite. Without loss of generality, we further assume that each \( f_i > 0 \), \( f_i \in C_c^\infty(\mathbb{R}^n) \) and \( \nu_2 \) are bounded functions. Now,
we claim that \(\int_{\mathbb{R}^n} (T(\tilde{f}))^p \nu_\omega dx < \infty\). In fact,

\[
\left(\int_{\mathbb{R}^n} (T(\tilde{f}))^p \nu_\omega dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} (M_\delta(T(\tilde{f})))^p \nu_\omega dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} (M_{p\delta}(T(\tilde{f})))^p \nu_\omega dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} |f_i|^{p_i} \nu_i dx \right)^{1/p_i}.
\]

By the H"older inequality, it yields that

\[
\int_{3B} (T(\tilde{f}))^p \nu_\omega dx \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}} < \infty.
\]

On the one hand, by using Lemma [4.3], it holds that

\[
\int_{(3B)^c} (T(\tilde{f}))^p \nu_\omega dx \leq \int_{(3B)^c} (M_{p\delta}(\tilde{f}))^p \nu_\omega dx < \infty.
\]

Now, we are in a position to prove \(\int_{\mathbb{R}^n} (M_\delta T(\tilde{f}))^p \nu_\omega dx < \infty\). Since \(\omega \in A_\infty\), then there exists \(q_0 > 1\), such that \(\omega \in A_{q_0}\). We may take \(\delta > 0\), small enough and \(p/\delta > q_0\) such that \(\omega \in A_{p/\delta}\). Then, the boundedness of \(M\) yields that

\[
\int_{\mathbb{R}^n} (M_\delta T(\tilde{f}))^p \nu_\omega dx < \int_{\mathbb{R}^n} (T(\tilde{f}))^p \nu_\omega dx < \infty.
\]

Thus, the desired estimates follows by using Fefferman-Stein's inequality,

\[
\left(\int_{\mathbb{R}^n} (T(\tilde{f}))^p \nu_\omega dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} (M_\delta(T(\tilde{f})))^p \nu_\omega dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} (M_{p\delta}(T(\tilde{f})))^p \nu_\omega dx \right)^{1/p} \leq C \left(\int_{\mathbb{R}^n} |f_i|^{p_i} \nu_i dx \right)^{1/p_i}.
\]

The proof of Theorem [1.3] (2) can be treated similarly as that in Theorem [1.3] (1), with only a slight modifications. Thus, we omit the proof of it.

Thus we complete the proof of Theorem [1.3].

Next, we turn to the proof of Theorem [1.4]. We prepare several lemmas.

**Lemma 4.5.** Suppose \(K_i\) satisfies (H3) for some \(1 \leq p_0 < \infty\). Suppose \(f_i \in C^\infty_c(\mathbb{R}^n)\) and \(\text{supp } f_i \subset B(0, R)\) for any \(i = 1, \ldots, m\). Then there is a constant \(C < \infty\) such that for \(|x| > 3R\) and bounded function \(b_j(x)\), \(j = 1, \ldots, m\), the following estimate holds uniformly.

\[
T_{\tilde{b}}(\tilde{f})(x) \leq C\|\tilde{b}\|_{\infty} M_{p_0}(\tilde{f})(x).
\]

**Proof.** We can use the same arguments as in Lemma [4.4] to finish the proof.

**Lemma 4.6.** Let \(T\) be a multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some \(1 \leq p_0 < \infty\). Then, for any \(0 < \delta < \varepsilon < \min\{1, \frac{p_0}{m}\}\) and \(q_0 > p_0\), there is a constant \(C < \infty\) such that for any bounded and compactly supported \(f_j(j = 1, \ldots, m)\), the following inequality holds

\[
M_{\delta}^\varepsilon(T_{\tilde{b}}(\tilde{f}))(x) \leq C\|\tilde{b}\|_{BMO} (M_{q_0}(\tilde{f})(x) + M_{\varepsilon}(T(\tilde{f}))(x)).
\]
Proof. We may assume \( \vec{b} = (b, 0, \ldots, 0) \). Fix a point \( x \in \mathbb{R}^n \) and a ball \( Q \) containing \( x \). For \( 0 < \delta < \varepsilon < \min\{1, \frac{\ln m}{m}\} \), we need to show that there exists a constant \( c_Q \) such that

\[
\left( \frac{1}{|Q|} \int_Q |T_{\vec{b}}(\vec{f})(z) - c_Q|^\delta dz \right)^{1/\delta} \leq C||\vec{b}||_{\text{BMO}}(\mathcal{M}_{q_0}(\vec{f})(x) + M_\varepsilon(T(\vec{f}))(x)).
\]

For any constant \( c_Q \), we have

\[
\left( \frac{1}{|Q|} \int_Q |T_{\vec{b}}(\vec{f})(z) - c_Q|^\delta dz \right)^{1/\delta} \leq C\left( \frac{1}{|Q|} \int_Q |b(z) - b_Q|^\delta \left( \int_0^\infty \int_{\mathbb{R}^n} K_t(z, \vec{y})( \prod_{j=1}^m f_j(y_j) d\vec{y} \right)^2 \frac{dt}{t} dz \right)^{1/\delta} + C\left( \frac{1}{|Q|} \int_Q \left( \int_0^\infty \int_{\mathbb{R}^n} K_t(z, \vec{y})(b(y_1) - b_{Q^*}) \prod_{j=1}^m f_j(y_j) d\vec{y} \right)^2 \frac{dt}{t} \right)^{1/2} - c_Q \left( \frac{1}{|Q|} \int_Q \right)^{1/\delta} : = I + II.
\]

The Hölder inequality gives that

\[
I \leq C\left( \frac{1}{|Q^*|} \int_{Q^*} \delta_p d\vec{y} \right)^{1/2} \left( \frac{1}{|Q^*|} \int_{Q^*} |T(\vec{f})(z)|^{p\delta} dz \right)^{1/p\delta} \leq C||\vec{b}||_{\text{BMO}}M_\varepsilon(T(\vec{f}))(x),
\]

where we have chosen \( p > 1 \) so that \( \delta p < \varepsilon < p_0/m \) and \( \delta p' > 1 \).

Now for each \( j \) we decompose \( f_j = f_j^0 + f_j^\infty \), where \( f_j^0 = f_j \chi_{Q^*} \), \( j = 1, \ldots, m \), and \( Q^* = 8Q \). Then

\[
\prod_{j=1}^m f_j(y_j) = \sum_{\vec{a}} f_{1}^{\alpha_1}(y_1) \cdots f_{m}^{\alpha_m}(y_m),
\]

where \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \) with \( \alpha_i = 0 \) or \( \infty \).

Now, we introduce the notion, \( c_{Q,t} = \left( \int_0^\infty |c_{Q,t}|^{2 \frac{dt}{t}} \right)^{1/2} \), where

\[
c_{Q,t} = \sum_{\vec{a} \neq \vec{0}} \int_{\mathbb{R}^nm} (b(y_1) - b_{Q^*}) K_t(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j^{\alpha_j}(y_j) d\vec{y}.
\]

Similarly as before, the finiteness of \( c_Q \) follows from the condition \( \text{(H3)} \). Moreover,

\[
II \leq C\left( \frac{1}{|Q|} \int_Q \left( \int_0^\infty \int_{\mathbb{R}^n} K_t(z, \vec{y})(b(y_1) - b_Q) \prod_{j=1}^m f_j(y_j) d\vec{y} - c_{Q,t} \right)^2 \frac{dt}{t} \delta^2/2 dz \right)^{1/\delta} \leq C\left( \frac{1}{|Q|} \int_Q \left( \int_0^\infty \int_{\mathbb{R}^n} K_t(z, \vec{y})(b(y_1) - b_{Q^*}) \prod_{j=1}^m f_j^0(y_j) d\vec{y} \right)^2 \frac{dt}{t} \right)^{1/2} + \sum_{\vec{a} \neq \vec{0}} \frac{1}{|Q|} \int_Q \left( \int_0^\infty \int_{\mathbb{R}^n} (K_t(z, \vec{y}) - K_t(x, \vec{y}))(b(y_1) - b_{Q^*}) \right).
\]
\[ \times \prod_{j=1}^{m} |f_{2j}^{n_j}(y_j)|dy_j^{2\frac{dt}{t}} |z|^{\delta/2}dz \bigg)^{1/\delta} = II_0 + II_{\vec{a}, \vec{a} \neq \vec{0}}. \]

The condition (H1), together with the Kolmogorov inequality \((p_0 < q_0)\) gives that
\[ II_0 \leq C \left( \frac{1}{|Q|} \int_{Q} |\mathbb{T}|((b - b_{Q^*})f_0^1, \ldots, f_0^m)(z)|^\delta dz \right)^{1/\delta} \]
\[ \leq C \mathbb{L}((b - b_{Q^*})f_0^1, \ldots, f_0^m)_{L^{p_0/m, \infty}(Q, \frac{d\nu}{|Q|})} \]
\[ \leq C \left( \frac{1}{|Q|} \int_{Q} |(b(z) - b_{Q^*})f_0^0(z)|^{p_0} dz \right)^{1/p_0} \prod_{j=2}^{\infty} \left( \frac{1}{|Q|} \int_{Q} |f_{j}^0(z)|^{p_0} dz \right)^{1/p_0} \]
\[ \leq C \|\vec{b}\|_{BMO} M_{q_0}(\vec{f})(x). \]

A similar argument as in the proof of Lemma 4.3 will lead to that
\[ \int_{\mathbb{R}^m} \left( \int_{0}^{\infty} |K_i(z, \vec{y}) - K_i(x, \vec{y})|^{2\frac{dt}{t}} \bigg)^{1/2} |(b(y_1) - b_{Q^*})| \prod_{j=1}^{m} |f_{2j}^{n_j}(y_j)|dy_j \]
\[ \leq C \sum_{j_0 \geq 1} \frac{|x - z|^{m(\delta - n/p_0)}}{|Q^*|^{m\delta/m}} m_2^{-m\delta_{j_0}2\delta_{j_0}n/p_0} |Q^*|^{m/p_0} \]
\[ \times \left( \frac{1}{|2^{j_0}Q^*|} \int_{2^{j_0}Q^*} |(b(y_1) - b_{Q^*})f_1(y_1)|^{p_0} dy_1 \right)^{1/p_0} \prod_{j=2}^{m} \left( \frac{1}{|2^{j_0}Q^*|} \int_{2^{j_0}Q^*} |f_{j}(y_j)|^{p_0} dy_j \right)^{1/p_0} \]
\[ \leq C \|\vec{b}\|_{BMO} M_{q_0}(\vec{f})(x). \]

Here, \(\delta > n/p_0\) and \(x, z \in Q\). Then, by Minkowski’s inequality, we get
\[ II_{\vec{a}, \vec{a} \neq \vec{0}} \leq C \left( \frac{1}{|Q|} \int_{Q} \left( \int_{\mathbb{R}^m} \left( \int_{0}^{\infty} |K_i(z, \vec{y}) - K_i(x, \vec{y})|^{2\frac{dt}{t}} \bigg)^{1/2} |(b(y_1) - b_{Q^*})| \right) \right)^{1/\delta} \prod_{j=1}^{m} |f_{2j}^{n_j}(y_j)|dy_j^{\delta} \bigg)^{1/\delta} \]
\[ \leq C \mathcal{M}_{q_0}(\vec{f})(x). \]

Then, the proof of Lemma 4.6 is finished. \(\square\)

**Proof of Theorem 1.4.** We may assume that \(\|\vec{b}\|_{BMO} = 1\). By repeating the same arguments as in the proof of Theorem 1.3 (1), we get \(\int_{\mathbb{R}^n}(T(f))(\vec{y}) \nu_{\vec{z}}dx \) and \(\int_{\mathbb{R}^n}(M_{\vec{b}}(f))(\vec{y}) \nu_{\vec{z}}dx \) are finite. Since \(\nu_{\vec{z}} \in A_{p_0/m_0} \), Theorem 1.3 (1) gives that
\[ \left( \int_{\mathbb{R}^n}(M_{\vec{b}}(T(f))(\vec{y}) \nu_{\vec{z}}dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n}(T(f)) \nu_{\vec{z}}dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n}|f_i|^{p_{0}(\vec{z})}dx \right)^{1/p}. \]
It is known from [24] that if \(\vec{\omega} \in A_{p_{0}/p_0} \), then there exists \(q_0 > p_0\) such that \(\vec{\omega} \in A_{p_{0}/q_0} \). Then
\[ \|\mathcal{M}_{q_0}(\vec{f})\|_{L^p(\nu_{\vec{z}})} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_{i}(\vec{z}_{ii})}}. \]
Thus, we have
\[
\left( \int_{\mathbb{R}^n} (T_b(f))^{p} \nu_\omega dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} (M_{\delta}(T_b(f)))^{p} \nu_\omega dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} (M_{\delta}^k(T_b(f)))^{p} \nu_\omega dx \right)^{1/p} \\
\leq C \left( \int_{\mathbb{R}^n} (M_{\kappa_0}(f))^{p} \nu_\omega dx \right)^{1/p} + C \left( \int_{\mathbb{R}^n} (M_{\lambda}(T(f)))^{p} \nu_\omega dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |f_i|^{p_i} \omega_i dx \right)^{1/p_i}.
\]

Hence, we complete the proof of Theorem 1.4.

5. PROOF OF THEOREM 1.5

We begin with some basic lemmas.

**Lemma 5.1.** For \(0 < \alpha < \infty\), let
\[
\Phi(t) = t(1 + \log^+ t)^\alpha, \quad 0 < t < \infty.
\]
Then it is a Young function and its complementary Young function is equivalent to
\[
\Phi_1(t) = \int_0^t \varphi_1(s) ds, \quad \text{where } \varphi_1(t) = \begin{cases} t^{1/\alpha}, & 0 < t < 1 \\ e^{t^{1/\alpha}-1}, & t \geq 1. \end{cases}
\]

**Proof.** Let
\[
\Phi_0(t) = \begin{cases} t^{1+\alpha}, & 0 < t < 1 \\ t(1 + \log t)^\alpha, & 1 \leq t < \infty. \end{cases}
\]

Then \(\Phi_0(t) \sim \Phi(t)\) and
\[
\phi_0(t) = \Phi_0'(t) = \begin{cases} (1 + \alpha)t^\alpha, & 0 < t < 1 \\ (1 + \log t)^\alpha + \alpha(1 + \log t)^{\alpha-1}, & 1 < t < \infty. \end{cases}
\]

Futhermore,
\[
\phi_0'(t) = \begin{cases} \alpha(1 + \alpha)t^{\alpha-1}, & 0 < t < 1 \\ \frac{\alpha(1 + \log t)^{\alpha-1}}{t} + \frac{\alpha(\alpha-1)(1 + \log t)^{\alpha-2}}{t}, & 1 < t < \infty. \end{cases}
\]

So, \(\Phi_0(t)\) is also a Young function. Let
\[
\phi_1(t) = \begin{cases} t^\alpha, & 0 < t < 1 \\ (1 + \log t)^\alpha, & 1 \leq t < \infty. \end{cases}
\]

Then we see that \(\phi_1(t) < \phi_0(t) \leq (1 + \alpha)\phi_1(t)\) \((0 < t < \infty)\) and
\[
\phi_1'(t) = \begin{cases} \frac{\alpha t^{\alpha-1}}{t}, & 0 < t < 1 \\ \frac{\alpha(1 + \log t)^{\alpha-1}}{t}, & 1 < t < \infty. \end{cases}
\]

Hence \(\Phi_1(t) = \int_0^t \phi_1(s) ds\) is a Young function and is equivalent to \(\Phi_0(t)\) and so to \(\Phi(t)\).

The inverse function of \(\phi_1(t)\) is given by
\[
\bar{\phi}_1(t) = \begin{cases} t^{1/\alpha}, & 0 < t < 1 \\ e^{t^{1/\alpha}-1}, & t \geq 1. \end{cases}
\]
ON THE BILINEAR SQUARE FOURIER MULTIPLIER OPERATORS ... 23

which completes the proof of Lemma 5.1.

□

Lemma 5.2. Let $b \in \text{BMO}(\mathbb{R}^n)$ and $f \in L^p \log^p L(\mathbb{R}^n)$ for some $1 \leq p < \infty$. Then, for any ball $Q$, $x \in Q$ and $j \in \mathbb{N}_0$, there exists a constant $C > 0$ such that

$$\left( \frac{1}{|2^j Q|} \int_{2^j Q} |(b(y) - b_Q) f(y)|^p dy \right)^{1/p} \leq C(j + 1) \| b \|_{\text{BMO}} M_{L^p \log^p L} f(x).$$

Proof. (a) The case $j = 0$. Let $Q$ be a ball in $\mathbb{R}^n$. Let $\Phi(t)$ and $\Phi_1(t)$ be in Lemma 5.1 as $\alpha = p$. Then by the Hölder inequality in Orlicz spaces, it holds that

$$\frac{1}{|Q|} \int_Q |(b(y) - b_Q) f(y)|^p dy \leq C \| b \|_{\Phi} \| ((b(y) - b_Q)/\| b \|_\Phi) \|^p \| f \|^p_{\Phi,Q}.$$

Note that

$$\Phi_1(t) \leq \int_0^t e^{s^{1/\alpha}} ds =: \Psi(t).$$

Thus, for any $c > 0$, we have

$$\frac{1}{|Q|} \int_Q \Phi_1 \left( \frac{|b(y) - b_Q|^p}{\| b \|_\Phi^p} \right) dy \leq \frac{1}{|Q|} \int_Q \Psi \left( \frac{|b(y) - b_Q|^p}{\| b \|_\Phi^p} \right) dy = \frac{1}{|Q|} \int_Q \int_0^\infty e^{s^{1/\alpha}} ds \left( \frac{|b(y) - b_Q|^p}{\| b \|_\Phi^p} \right) dy ds = \int_0^\infty e^{s^{1/\alpha}} \left( \frac{1}{|Q|} \int_Q \chi_{\{ |b(y) - b_Q|^p / \| b \|_\Phi^p > s \}} \right) dy ds.$$

On the other hand, by the John-Nirenberg inequality, there exist positive constants $c_1$ and $c_2$ such that

$$|\{ x \in Q : |b(x) - b_Q| > \lambda \}| \leq c_2 |Q| e^{-c_1 \lambda / \| b \|_\Phi}, \quad \lambda > 0.$$

Hence, choosing $c$ big enough such that $c_1 c > 1$, we get

$$\frac{1}{|Q|} \int_Q \Phi_1 \left( \frac{|b(y) - b_Q|^p}{\| b \|_\Phi^p} \right) \leq \int_0^\infty e^{s^{1/\alpha}} c_2 c e^{-c_1 c s^{1/\alpha}} ds = c_2 \int_0^\infty e^{-(c_1 c - 1) s^{1/\alpha}} ds < \infty,$$

which shows that the norm $\|(b(y) - b_Q)/\| b \|_\Phi\|^p\|_{\Phi,Q}$ is bounded by a constant depending on $p, c_1, c_2$. Combining this with (5.1) gives

$$\left( \frac{1}{|Q|} \int_Q |(b(y) - b_Q) f(y)|^p dy \right)^{1/p} \leq C \| b \|_{\text{BMO}} M_{L^p \log^p L} f(x),$$

for any $x \in Q$.

(b) The case $j \in \mathbb{N}$. By the Minkowski inequality and step (a), one obtains

$$\left( \frac{1}{|2^j Q|} \int_{2^j Q} |(b(y) - b_Q) f(y)|^p dy \right)^{1/p} \leq C(j + 1) \| b \|_{\text{BMO}} M_{L^p \log^p L} f(x).$$
\[
\begin{align*}
&\leq \sum_{i=1}^{j} \left( \frac{1}{|2^{i}Q|} \int_{2^{i}Q} |(b_{2iQ} - b_{2^{i-1}Q})f(y)|dy \right)^{1/p} + \left( \frac{1}{|2^{i}Q|} \int_{2^{i}Q} |(b(y) - b_{2iQ})f(y)|dy \right)^{1/p} \\
&\leq \sum_{i=1}^{j} \frac{2^{n}}{|2^{i}Q|} \int_{2^{i}Q} |(b(y) - b_{2iQ})|dy \left( \frac{1}{|2^{i}Q|} \int_{2^{i}Q} |f(y)|dy \right)^{1/p} + C \|b\|_{BMO} \|f\|_{pQ}\Phi \leq C(j + 1) \|b\|_{BMO} M_{L \log^p L}(f)(x).
\end{align*}
\]

This completes the proof of Lemma 5.2.

Using Lemma 5.2, we can improve Lemma 4.6 as follows.

**Lemma 5.3.** Let \( T \) be a multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some \( 1 \leq p_0 < \infty \). Then, for any \( 0 < \delta < \varepsilon < \min\{1, \frac{m}{m}\} \), there is a constant \( C < \infty \) such that for any bounded and compactly supported \( f_j, j = 1, \ldots, m \),

\[
M_{\Phi}^x(T_{\tilde{b}}(\tilde{f}))(x) \leq C \|\tilde{b}\|_{BMO} \left( \sum_{i=1}^{m} \mathcal{M}_{\Phi}^{(i)}(\tilde{f})(x) + M_{\varepsilon}(T(\tilde{f}))(x) \right),
\]

where \( \Phi(t) = t^{p_0}(1 + \log^+ t)^{p_0} \) and

\[
\mathcal{M}_{\Phi}^{(i)}(\tilde{f})(x) = \sup_{Q \ni x} \|f_i\|_{\Phi, Q} \prod_{j \neq i} \left( \frac{1}{|Q|} \int_{Q} |f_j(y)|dy \right)^{1/p_0}.
\]

The proof of Theorem 1.3 will be based on the following lemmas.

**Lemma 5.4.** Let \( T \) be a multilinear square function with a kernel satisfying condition condition (H1), (H2) and (H3) for some \( 1 \leq p_0 < \infty \). Let \( \omega \) be an \( A_\infty \) weight and let \( \Phi(t) = t^{p_0}(1 + \log^+ t)^{p_0} \). Suppose that \( \bar{b} \in BMO^m \). Then, there exists a constant \( C \) (independent of \( \bar{b} \)) such that the following inequality holds

\[
\sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbb{R}^n : |T_{\bar{b}} \tilde{f}(y)| > t^m \}) \leq C \sum_{i=1}^{m} \sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbb{R}^n : |\mathcal{M}_{\Phi}^{(i)} \tilde{f}(x)| > t^m \}),
\]

for all bounded vector function \( \tilde{f} = (f_1, \ldots, f_m) \) with compact support.

**Proof.** We borrow some ideas from Theorem 3.19 in [24]. We may assume \( \|b_j\|_{BMO} = 1 \), for \( j = 1, \ldots, m \). It is enough to prove the result for

\[(5.2) \quad T_{b}(\bar{f})(x) = \left( \int_{0}^{\infty} \left| \int_{(\mathbb{R}^n)^m} (b_1(x) - b_1(y_1)K_t(x, \tilde{y}) \prod_{j=1}^{m} f_j(y_j)dy_1 \ldots dy_m \right| \frac{dt}{t} \right)^{1/2}.
\]
Let $0 < \delta < \varepsilon < 1/m$. By the Lebesgue differentiation theorem, it suffices to prove
\[
\sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : |M_\delta(T_{\vec{b}}\vec{f})(y)| > t^m \right\}
\leq C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : \mathcal{M}_\phi^{(1)}(\vec{f})(y) > t^m \right\}.
\]
(5.3)

However, Lemma 4.2 yields that
\[
\sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : |M_\delta(T_{\vec{b}}\vec{f})(y)| > t^m \right\}
\leq C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : |M_\delta^2(T_{\vec{b}}\vec{f})(y)| > t^m \right\},
\]
(5.4)

whenever the left-hand side is finite. Therefore, (5.3) follows from
\[
\sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : |M_\delta^2(T_{\vec{b}}\vec{f})(y)| > t^m \right\}
\leq C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : \mathcal{M}_\phi^{(1)}(\vec{f})(y) > t^m \right\}.
\]
(5.5)

In order to use Fefferman-Stein inequality, we claim the following inequalities hold:
\[
\sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : |M_\delta(T_{\vec{b}}\vec{f})(y)| > t^m \right\} < \infty.
\]
(5.6)

and
\[
\sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : |M_\varepsilon(T\vec{f})(y)| > t^m \right\} < \infty.
\]
(5.7)

Admitting the claim first, we will prove (5.5). Lemma 4.3, Lemma 4.6 and Fefferman-Stein inequality yield that
\[
\sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : |M_\delta(T_{\vec{b}}\vec{f})(y)| > t^m \right\}
\leq C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : \mathcal{M}_\phi^{(1)}(\vec{f})(y) + M_\varepsilon(T\vec{f})(y) > t^m \right\}
\leq C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : \mathcal{M}_\phi^{(1)}(\vec{f})(y) > t^m \right\}
\]
\[
+ C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : M_\varepsilon(T\vec{f})(y) > t^m \right\}
\leq C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : \mathcal{M}_\phi^{(1)}(\vec{f})(y) > t^m \right\}
\]
\[
+ C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : \mathcal{M}_p(\vec{f})(y) > t^m \right\}
\leq C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\left\{ y \in \mathbb{R}^n : \mathcal{M}_\phi^{(1)}(\vec{f})(y) > t^m \right\}.
\]
Now, we only need to show that (5.6) holds, by the reason that the proof of (5.7) is very similar but much easier. We assume that the $b_j$ and $\omega$ are bounded. Suppose that $\text{supp } f \subset B(0,R)$. Hence, since $\Phi(t) \geq t^{p_0}$ and $0 < \delta < 1/m$, it follows that
\[
\sup_{t>0} \frac{1}{\Phi(1/t)} \omega(\{y \in \mathbb{R}^n : M_\delta(|T_{\vec{b}}\tilde{f}|)(y) > t^m\}) \leq C||\omega||_{L^\infty} \sup_{t>0} \frac{1}{\Phi(1/t)} \{y \in \mathbb{R}^n : M_{m\delta}(|T_{\vec{b}}\tilde{f}|^{1/m})(y) > t\} \leq C \sup_{t>0} t^{p_0} \{y \in \mathbb{R}^n : |T_{\vec{b}}\tilde{f}(y)|^{1/m} > t\} \leq C \sup_{t>0} t^{p_0} \{y \in B_{3R} : |T_{\vec{b}}\tilde{f}(y)|^{1/m} > t\} + \sup_{t>0} t^{p_0} \{y \in B_{3R}^c : |T_{\vec{b}}\tilde{f}(y)|^{1/m} > t\} = I + II.
\]
We first consider the contribution of $I$. Taking $r > 1$, by the Assumption (H1) and the Hölder inequality, we have
\[
I \leq C \int_{B(0,3R)} |T_{\vec{b}}\tilde{f}(y)|^{p_0/m} dy \leq CR^{(1-1/r)n} \left( \int_{\mathbb{R}^n} |T_{\vec{f}}(y)|^{p_0/r} dy \right)^{1/r} < \infty.
\]
For the contribution of $II$, note that we may control $|T_{\vec{b}}\tilde{f}(x)|$ by $M_{p_0}\tilde{f}(x)$ if we assume that $b$ is bounded. Then
\[
II^m \leq C t^{mp_0} \{y \in \mathbb{R}^n : M_{p_0}\tilde{f}(y)^{1/m} > t\}^m \leq (||M_{p_0}\tilde{f}||_{L^{p_0/m,\infty}})^{p_0} \leq C \left( \prod_{i=1}^m \|f_i(x)\|_{p_0} \right)^{p_0} < \infty.
\]
Thus, the claim (5.5) is proved. Hence, we finish the proof of Lemma 5.4. \hfill \Box

**Lemma 5.5.** Let $1 \leq p_0 < \infty$ and $\vec{\omega} \in A_\vec{f}$. Then, there exists a constant $C$ such that for $1 \leq i \leq m$
\[
\nu_{\vec{\omega}}(\{x \in \mathbb{R}^n : |M^{(i)}_{\phi}(\tilde{f}(x)) > t^m\}) \leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \Phi\left( \frac{|f_j(x)|}{t} \right) \omega_j(x) dx \right)^{1/m},
\]
where $\Phi(t) = t^{p_0} (1 + \log^+ t)^{p_0}$.

**Proof.** Some ideas will be taken from the proof of Theorem 3.17 in [24]. By homogeneity, we may assume that $t = 1$ and $\tilde{f} \geq 0$. Set
\[
\Omega = \{x \in \mathbb{R}^n : M^{(i)}_{\phi}(\tilde{f}(x)) > 1\}.
\]
It is easy to see that $\Omega$ is open and we may assume that it is not empty. To estimate the size of $\Omega$, it is enough to estimate the size of every compact set $F$ contained in $\Omega$. We note that we may use cubes in place of balls in the definition of maximal functions. Now, we can cover any such $F$ by a finite family of cubes $Q_j$ for which
\[
1 < \|f_i\|_{\phi, Q_j} \prod_{j=2}^m (f_j)_{Q_j}.
\]
Let $\Phi_0$.

Using the fact that $\Phi$ is submultiplicative, it yields that

$$F \subset \bigcup_i 3Q_i.$$ 

By homogeneity,

$$1 < \left\| f_1 \prod_{j=2}^m ((f_j)_{Q_i})^{1/p_0} \right\|_{\Phi, Q_i}$$

and by the properties of the norm $\| \cdot \|_{\Phi, Q_i}$, this is the same as

$$1 < \frac{1}{|Q_i|} \int_{Q_i} \Phi \left( f_1 \prod_{j=2}^m ((f_j)_{Q_i})^{1/p_0} \right) dy.$$ 

Using the fact that $\Phi$ is submultiplicative, it yields that

$$1 < \frac{1}{|Q_i|} \int_{Q_i} \Phi (f_1) dy \prod_{j=2}^m \Phi ((f_j)_{Q_i})^{1/p_0}).$$

Let $\Phi_0(t) = t(1 + \log^+ t)^{p_0}$. By the Jensen inequality, we have

$$\Phi \left( (f_j)_{Q_i} \right)^{1/p_0)} = \frac{1}{|Q_i|} \int_{Q_i} |f_j(y)|^{p_0} dy \left( 1 + \frac{1}{p_0} \log^+ \frac{1}{|Q_i|} \int_{Q_i} |f_j(y)|^{p_0} dy \right)^{p_0}$$

$$\leq \frac{1}{|Q_i|} \int_{Q_i} |f_j(y)|^{p_0} dy \left( 1 + \log^+ \frac{1}{|Q_i|} \int_{Q_i} |f_j(y)|^{p_0} dy \right)^{p_0}$$

$$= \Phi_0 \left( \frac{1}{|Q_i|} \int_{Q_i} |f_j(y)|^{p_0} dy \right)$$

$$\leq \int_{Q_i} \Phi_0(|f_j|^{p_0}) (y) dy$$

$$= \frac{1}{|Q_i|} \int_{Q_i} |f_j(y)|^{p_0} (1 + \log^+ |f_j(y)|^{p_0} dy)^{p_0} dy$$

Finally, by the condition assumed on the weights and the Hölder inequality at, one obtains discrete level,

$$\nu_\omega(F)^m \approx \left( \sum_i \nu_\omega(Q_i) \right)^m \leq p_0^{p_0} \left( \sum_i \prod_{j=1}^m \inf_Q \omega_j^{1/m} |Q_i|^{1/m} \left( \frac{1}{|Q_i|} \int_{Q_i} \Phi (f_j) dy \right)^{1/m} \right)^m$$

$$\leq p_0^{p_0} \left( \sum_{j=1}^m \left( \int_{Q_i} \Phi (f_j(y)) \omega_j(y) dy \right)^{1/m} \right)^m \leq p_0^{p_0} \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \Phi (|f_j(x)| \omega_j(x) dx \right).$$

which concludes the proof of Lemma 5.5. \qed
Using the above lemmas, now, we can show Theorem 1.5.

**Proof of Theorem 1.5.** It is enough to prove the result for the operator $T_b$ defined in (5.2). By homogeneity we may assume $t = 1$. Since $\Phi$ is submultiplicative, Lemma 5.4 and Lemma 5.5 yield that

\[
\begin{align*}
\nu_\omega \left( \left\{ x \in \mathbb{R}^n : T_b \vec{f}(x) > 1 \right\} \right)^m & \leq C \sup_{t > 0} \frac{1}{\Phi(1/t)^m} \nu_\omega \left( \left\{ x \in \mathbb{R}^n : T_b \vec{f}(x) > t^m \right\} \right)^m \\
& \leq C \sup_{t > 0} \frac{1}{\Phi(1/t)^m} \nu_\omega \left( \left\{ x \in \mathbb{R}^n : \mathcal{M}_\Phi^{(1)} \vec{f}(x) > t^m \right\} \right)^m \\
& \leq C \sup_{t > 0} \frac{1}{\Phi(1/t)^m} \prod_{j=1}^m \int_{\mathbb{R}^n} \Phi\left(\frac{|f_j(x)|}{t}\right) \omega_j(x) \, dx \\
& \leq C \sup_{t > 0} \frac{1}{\Phi(1/t)^m} \prod_{j=1}^m \int_{\mathbb{R}^n} \Phi(|f_j(x)|) \Phi(1/t) \omega_j(x) \, dx \\
& \leq C \prod_{j=1}^m \int_{\mathbb{R}^n} \Phi(|f_j(x)|) \omega_j(x) \, dx.
\end{align*}
\]

(5.8)

This complete the proof of Theorem 1.5.

**References**

[1] T. A. Bui and X. T. Duong, *On commutators of vector BMO functions and multilinear singular integrals with non-smooth kernels*, J. Math. Anal. Appl. 371 (2010) 80-84.

[2] T. A. Bui and X. T. Duong, *Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers*, Bull. Sci. Math. 137 (1) (2013), 63-75.

[3] X. Chen, Q. Xue, K. Yabuta, *On multilinear Littlewood-Paley operators*, Nonlinear Anal. 115 (2015) 25-40.

[4] R. R. Coifman, D. Deng, Y. Meyer, *Domains de la racine carré de certains opérateurs différentiels accrétifs*, Ann. Inst. Fourier (Grenoble) 33 (1983), 123-134.

[5] R. R. Coifman, A. McIntosh, Y. Meyer, *L’intégrale de Cauchy définit un opérateur borné sur $L^2$ pour les courbes lipschitziennes*, Ann. Sci. Math. 116 (1982), 361-387.

[6] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. 212 (1975), 315-331.

[7] R. R. Coifman and Y. Meyer, *Commutateurs d’intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier, Grenoble 28 (1978), 177-202.

[8] R. R. Coifman and Y. Meyer, *Au-delà des opérateurs pseudo-différentiels*, Asterisque 57, 1978.

[9] G. David, J. L. Journé, *Une caractérisation des opérateurs intgraux singuliers bornes sur $L^2(\mathbb{R}^n)$*, C. R. Math. Acad. Sci. Paris 296 (1983), 761-764.

[10] C. Fefferman and E. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), 107-115.

[11] L. Grafakos and R. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. 165 (2002), 124-164.

[12] M. Fujita and N. Tomita, *Weighted norm inequalities for multilinear Fourier multipliers*, Trans. Amer. Math. Soc. 364 (2012), 6335–6353.

[13] E. B. Fabes, D. Jerison, C. Kenig, *Multilinear Littlewood-Paley estimates with applications to partial differential equations*, Proc. Natl. Acad. Sci. 79 (1982), 5746-5750.
[14] E. B. Fabes, D. Jerison, C. Kenig, Necessary and sufficient conditions for absolute continuity of elliptic harmonic measure, Ann. of Math. 119 (1984), 121-141.
[15] E. B. Fabes, D. Jerison, C. Kenig, Multilinear square functions and partial differential equations, Amer. J. Math. 107 (1985), 1325-1368.
[16] J. García-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.
[17] L. Grafakos, D. He and P. Honzik, Rough bilinear singular integrals, available at: http://arxiv.org/abs/1509.06099.
[18] L. Grafakos, R. H. Torres, Maximal operator and weighted norm inequalities for multilinear singular integrals, Indiana Univ. Math. J. 51(5) (2002), 1261-1276.
[19] L. Grafakos, L. Liu and D. Yang, Multiple weighted norm inequalities for maximal multilinear singular integrals with non-smooth kernels, Proc. Roy. Soc. of Edinburgh Sect. A 141 (2011), 755-775.
[20] L. Grafakos, A. Miyachi, N. Tomita, On multilinear Fourier multipliers of limited smoothness. Canad. J. Math. 65 (2) (2013), 299-330.
[21] L. Grafakos, Z. Si, The Hörmander multiplier theorem for multilinear operators, J. Reine Angew. Math. 668 (2012), 133-147.
[22] G. Hu and C.-C. Lin, Weighted norm inequalities for multilinear singular integral operators and applications, Anal. Appl. (Singap.) 12 (3)(2014), 269-291.
[23] M. Lacey, On bilinear Littlewood-Paley square functions, Publ. Mat. 40(2) (1996), 387-396.
[24] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, Adv. Math. 220(4) (2009), 1222-1264.
[25] K. Li and W. Sun, Weighted estimates for multilinear Fourier multipliers, Forum Math. 27(2)(2015), 1101-1116
[26] W. Li, Q. Xue and K. Yabuta, Weighted version of Carleson measure and multilinear Fourier multiplier, Forum Math. 27 (2) (2015), 787-805.
[27] G. Lu, P. Zhang, Multilinear Calderón-Zygmund operators with kernels of Dini’s type and applications, Nonlinear Anal. 107 (2014), 92-117.
[28] D. Maldonado, V. Naibo, Weighted norm inequalities for paraproducts and bilinear pseudodifferential operators with mild regularity, J. Fourier Anal. Appl. 15 (2009), 218-261.
[29] S. Shi, Q. Xue, K. Yabuta, On the boundedness of multilinear Littlewood-Paley g∗λ function, J. Math. Pures Appl. 101(3) (2014), 394-413.
[30] N. Tomita, A Hörmander type multiplier theorem for multilinear operators, J. Funct. Anal. 259(2010), 2028-2044.
[31] Q. Xue, X. Peng, K. Yabuta, On the theory of multilinear Littlewood-Paley g function, J. Math. Soc. Japan 67 (2) (2015), 535-559.
[32] Q. Xue and J. Yan, On multilinear square function and its applications to multilinear Littlewood-Paley operators with non-convolution type kernels, J. Math. Anal. Appl. 422 (2015), 1342-1362.

Zengyan Si, School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, People’s Republic of China
E-mail address: zengyan@hpu.edu.cn

Qingying Xue, School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People’s Republic of China
E-mail address: qyxue@bnu.edu.cn

Kôzô Yabuta, Research Center for Mathematical Sciences, Kwansei Gakuin University, Gakuen 2-1, Sanda 669-1337, Japan