Construction of the $K=8$ Fractional Superconformal Algebras

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ABSTRACT

We construct the $K = 8$ fractional superconformal algebras. There are two such extended Virasoro algebras, one of which was constructed earlier, involving a fractional spin (equivalently, conformal dimension) $\frac{6}{5}$ current. The new algebra involves two additional fractional spin currents with spin $\frac{13}{5}$. Both algebras are non-local and satisfy non-abelian braiding relations. The construction of the algebras uses the isomorphism between the $Z_8$ parafermion theory and the tensor product of two tricritical Ising models. For the special value of the central charge $c = \frac{52}{55}$, corresponding to the eighth member of the unitary minimal series, the $\frac{13}{5}$ currents of the new algebra decouple, while two spin $\frac{23}{5}$ currents (level-2 current algebra descendants of the $\frac{13}{5}$ currents) emerge. In addition, it is shown that the $K = 8$ algebra involving the spin $\frac{13}{5}$ currents at central charge $c = \frac{12}{5}$ is the appropriate algebra for the construction of the $K = 8$ (four-dimensional) fractional superstring.

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1. Introduction

The structure of two-dimensional conformal field theories (CFT) is in large part determined by their underlying conformal symmetry algebra [1]. This infinite dimensional algebra organizes all the fields in a CFT into sets of primary fields of definite conformal dimensions, and their associated infinite towers of descendant fields. The fundamental conformal symmetry is the Virasoro algebra,

\[ T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots \]  

where \( T(z) \) is the energy-momentum tensor and the ellipsis refers to further Virasoro descendants of the identity. It turns out that for \( c < 1 \) this symmetry actually determines all unitary models and their complete spectrum [1,2].

It is natural to try to classify \( c \geq 1 \) CFTs by extending the conformal symmetry with new currents [3,4]. The most general extended conformal symmetries can be written as follows. Consider a set of currents \( J_i(z) \), primary with respect to \( T(z) \), with conformal dimensions \( h_i \), where \( i \in \{1, 2, \ldots, N\} \). In other words

\[ T(z)J_i(w) = \frac{h_iJ_i(w)}{(z-w)^2} + \frac{\partial J_i(w)}{(z-w)} + \cdots \]  

The operator product expansions (OPEs) among the currents have the generic form

\[ J_i(z)J_j(w) = q_{ij}(z-w)^{-h_i-h_j}(1 + \cdots) + \sum_k f_{ijk}(z-w)^{-h_i-h_j+h_k} (J_k(w) + \cdots) \]  

where \( q_{ij} \) and \( f_{ijk} \) are structure constants. The ellipses stand for current algebra descendants, whose dimensions differ from those of the identity and the \( J_k \) by positive integers. Since the algebra is chiral, \( i.e. \) independent of \( \bar{z} \), the conformal dimensions are the spins of the fields. The parameters \( c, h_i, q_{ij} \) and \( f_{ijk} \) are not free and must be chosen such that the algebra (1.3) is associative. This condition places
strong constraints on the set of consistent conformal dimensions $h_i$ and restricts the structure constants $q_{ij}$ and $f_{ijk}$ as functions of the central charge $c$. We note that there are known examples where the central charge itself is restricted (e.g. the spin 5/2 current algebra of ref. [4]), and there are also known examples where the number of currents, $N$, is infinite (e.g. the $Z_{t/u}$ parafermion theory of ref. [5]).

Extended conformal algebras naturally fall into three classes:

(1) Local algebras: the simplest type, where all powers of $(z - w)$ appearing in (1.3) are integers. This class includes the most familiar extended algebras, such as the superconformal, Kač-Moody and $W_n$ algebras. These examples are unitary and therefore consist of currents with only integer and half-integer spins. Additionally, there exist non-unitary algebras such as ghost systems where arbitrary spins may be present [6].

When fractional powers of $(z - w)$ appear in (1.3), some of the currents will necessarily have fractional spins. In this case, the algebra is non-local, due to the presence of Riemann cuts in the complex plane. Such algebras are more complicated to construct and analyze than the local ones. Among non-local algebras there is a further division, again along lines of complication.

(2) Abelian non-local algebras: also known as parafermion (PF) or generalized parafermion current algebras, were first constructed by Zamolodchikov and Fateev [7]. They are the simplest type of non-local algebras, involving at most one fractional power of $(z - w)$ in each OPE in (1.3). Any two currents in a PF algebra obey abelian braiding relations, i.e. upon braiding the two currents (analytically continuing one current along a path encircling the other), any correlation function involving these two currents only changes by a phase. The analysis of the associativity conditions for PF theories can be carried out using algebraic methods.

(3) Non-abelian non-local algebras: or non-abelian algebras for short, since they are necessarily non-local. This is the most general class of extended algebras and is the focus of this paper. Their characteristic feature is that their OPEs involve multiple cuts, i.e. there are terms in at least one of the OPEs in (1.3) with
different fractional powers of \((z - w)\). Any two fractional spin currents appearing in one of these OPEs will in general obey non-abelian braiding properties. The analysis of the associativity conditions for non-abelian algebras requires more powerful methods. The first set of non-abelian algebras were constructed in ref. [8].

In general, the holomorphic \(n\)-point correlation functions of the currents,

\[
\langle J_i(z_1)J_j(z_2)\cdots J_k(z_n) \rangle ,
\]

(1.4)
can be expressed as a linear combination of some set of conformal blocks. The relative coefficients of the various conformal blocks are fixed by the closure condition and the associativity condition. The closure condition is simply the requirement that no new currents beyond the currents of the algebra should appear. Associativity is the condition that the particular linear combination of conformal blocks that appears in the \(n\)-point function is invariant under fusion transformations (i.e. duality). For the local algebras, each conformal block involves only integer powers of \((z_i - z_j)\); for the abelian non-local algebras, each correlation function of the parafermion currents has exactly one conformal block, even though it involves fractional powers of \((z_i - z_j)\). Of course, for the most general case we expect each correlation function to have multiple conformal blocks, and the conformal blocks to involve different fractional powers of \((z_i - z_j)\). This general case corresponds to the non-abelian non-local algebras. From this point of view we see that upon braiding the currents, the correlation function \((1.4)\) is, in general, transformed into an independent linear combination of conformal blocks. This reflects the different phases that are picked up upon analytically continuing the different fractional powers of \((z_i - z_j)\).

The different braiding properties of the different types of extended algebras described above are reflected in the moding of their currents. On a given state in any representation of the algebra, we can obtain new states by acting with current modes

\[
J_{-n_1-r_1}^{i_1}J_{-n_2-r_2}^{i_2}\cdots J_{-n_m-r_m}^{i_m}|\Phi\rangle ,
\]

(1.5)
where the \( n_j \) are integers and the \( r_j \) are fractional in general. For local unitary algebras, the half-integer spin currents can have only integer or half-integer modings, and \( r_1 = r_2 = \cdots = r_m = r \) where either \( r = 0 \) or \( r = \frac{1}{2} \), depending on the state \( |\Phi\rangle \). For PF theories, the situation is slightly more complicated [7]. Generically the \( r_i \) are different, determined by the state \( |\Phi\rangle \) and the currents that preceded it. For non-abelian algebras, the moding of a particular current in (1.5) is not unique; it depends both on the state it operates on as well as on the state we want it to create.

Now, the existence of non-abelian algebras and their usefulness in organizing CFTs is illustrated by the \( SU(2) \) WZW coset models. Let us denote the \( SU(2)_K \otimes SU(2)_L/SU(2)_{K+L} \) coset model by \([K,L]\). It is well known that the \([1,L]\) coset series are exactly the unitary models with \( c = 1 - \frac{6}{(L+2)(L+3)} \) [9], so that they are representations of the Virasoro algebra. Next, the \([2,L]\) coset series are representations of the superconformal algebra \( \{T(z), J^{(2)}(z)\} \), where \( J^{(2)}(z) \) is the usual supercurrent. It is also known that the \([4,L]\) coset series are representations of the parafermion current algebra \( \{T(z), \psi_1(z), \psi_2(z)\} \), or \( \{T(z), J^{(4)}(z)\} \) where the spin \( \frac{4}{3} \) current is \( J^{(4)}(z) = \psi_1(z) + \psi_2(z) \) [10]. This pattern strongly suggests the existence of extended algebras for other values of \( K \). This belief was further supported by an explicit construction of the branching functions of the \([K,L]\) coset series based on the assumption that an extended Virasoro symmetry exists [11]. It turns out that the extended Virasoro algebras for \( K \) other than 2 and 4 are not parafermionic, hence they must be non-abelian.

Some of the tools needed for the analysis and construction of special series of non-abelian algebras have already been developed. In particular, in ref. [8] we constructed a series of non-abelian algebras, the fractional superconformal algebras (FSCAs), so-called because they generalize the conventional superconformal algebras. The FSCAs constructed in ref. [8] are minimal in the sense that they contain only one fractional spin current, \( J^{(K)}(z) \), in addition to the energy-momentum tensor \( T(z) \). This \( \{T, J^{(K)}\} \) algebra describes the simplest extended conformal symmetry underlying the \([K,L]\) coset models [11,12]. The explicit form of this
where \( \Delta = (K + 4)/(K + 2) \) is the dimension of the \( J^{(K)} \) current. The structure constant \( \lambda_K(c) \) has the following form [8]

\[
\lambda^2_K(c) = \frac{2K^2(c_{111})^2}{3(K + 4)(K + 2)} \left[ \frac{3(K + 4)^2}{K(K + 2)} \frac{1}{c} - 1 \right],
\]

(1.7)

where

\[
(c_{111})^2 = \frac{\sin^2(\pi \rho) \sin^2(4\pi \rho)}{\sin^3(2\pi \rho) \sin(3\pi \rho) \Gamma(3\rho) \Gamma(4\rho)} \frac{\Gamma^3(\rho) \Gamma^2(4\rho)}{\Gamma^4(2\rho)},
\]

(1.8)

and \( \rho = \frac{1}{K + 2} \). The structure constant \( c_{111} \) is that for the OPE of two spin-one fields to close on another spin-one field in the chiral \( SU(2)_K \) WZW model where all other higher-spin fields are decoupled.

In this paper we shall consider a more complex example of a non-abelian extended algebra which has three fractional spin currents. Specifically, we will show that there are two consistent \( K = 8 \) FSCAs. One is the algebra given in (1.6) for \( K = 8 \). The other is a new algebra which involves, besides the spin \( \frac{6}{5} \) current \( J \), two additional currents \( H_1 \) and \( H_2 \), both with spin \( \frac{13}{5} \). The form for this
\{T(z), J(z), H_1(z), H_2(z)\} algebra will be shown to be

\[
J(z)J(w) = (z-w)^{-\frac{M}{2}}\{1 + \cdots \}
+ s^2 \Lambda(c)(z-w)^{-\frac{G}{2}}\{J(w)+\cdots\}
+ s\Omega(c)(z-w)^{\frac{s}{6}}\{H_1(w)+\cdots+H_2(w)+\cdots\},
\]

\[
J(z)H_1(w) = s\Omega(c)(z-w)^{-\frac{5}{4}}\{J(w)+\cdots\}
+ \frac{13}{14}\Lambda(c)(z-w)^{-\frac{6}{5}}\{H_2(w)+\cdots\},
\]

\[
J(z)H_2(w) = 13\Lambda(c)(z-w)^{-4}\{J(w)+\cdots\},
\]

where the structure constants are given by

\[
s = \sqrt{\frac{2}{3}rx}, \quad r = \sqrt{\frac{5}{2} - 1}, \quad x = \frac{\Gamma^2\left(\frac{G}{2}\right)}{\Gamma\left(\frac{G}{2}\right)\Gamma\left(\frac{M}{2}\right)},
\]

\[
\Lambda(c) = \sqrt{\frac{8(27-5c)}{25c}}, \quad \Omega(c) = \sqrt{\frac{18(55c-52)}{455c}},
\]

and

\[
\Upsilon(c) = \frac{\sqrt{8}(865c-1976)}{5\sqrt{455c(55c-52)}}.
\]

The OPEs involving \(T(z)\) follow easily from (1.1) and (1.2), and the remaining 
\(H_2(z)\) OPEs are found by exchanging \(H_1 \leftrightarrow H_2\) in (1.9).

It is important to point out that the \(\{T, J^{(8)}\}\) FSCA is not a subalgebra of
the \(\{T, J, H_\ell\}\) FSCA, where \(\ell = 1, 2\). They do, however, both have the Virasoro
subalgebra in common, and in addition \(\{T, H_1\}\) and \(\{T, H_2\}\) are subalgebras of
\(\{T, J, H_\ell\}\). For the purposes of classifying CFTs, the existence of two distinct
$K = 8$ FSCAs shows that there are two independent symmetries that can be used in the construction of the $[8, L]$ coset models. The differences between the representation theories of these two FSCAs will be discussed in Sect. 7.

As was mentioned above, the $[K, L]$ coset model has an underlying fractional superconformal symmetry generated by the FSCA. A FSCA current operating on the identity generates a state with precisely its conformal dimension, so that the Virasoro primary field in the appropriate $[K, L]$ coset model with that same conformal dimension can be identified as the FSCA current. In particular, all $[8, L]$ coset models, with $L \geq 2$, have a $\frac{13}{5}$ primary field to associate with the $H_\ell$ FSCA currents. However, the $[1, 8]$ coset model, corresponding to the eighth member of the unitary series, has no such dimension $\frac{13}{5}$ primary field. This apparent inconsistency is resolved when we note that the central charge of this model is $c = \frac{52}{55}$. At precisely this value of $c$, the $H_\ell$ currents become null and decouple from the algebra (1.9). Instead, in the $c = \frac{52}{55}$ unitary model we find a dimension $\frac{23}{5}$ primary field. For $c > \frac{52}{55}$ there also exist $\frac{23}{5}$ currents, denoted by $H'_1$ and $H'_2$, but they can be considered as level-2 current algebra descendants of $H_1$ and $H_2$. For $c = \frac{52}{55}$, however, since the $H_\ell$ are null, the $H'_\ell$ are promoted to be FSCA primary currents. In Sect. 7, calculating directly in the $c = \frac{52}{55}$ unitary model using the methods of Dotsenko and Fateev [13], we compute the structure constants for this $\{T, J, H'_\ell\}$ FSCA. The agreement we find for the $\langle JJJ \rangle$ structure constant between this calculation and the result of eqs. (1.9) and (1.10) provides a non-trivial check on the calculations carried out in this paper.

Although the main results (1.6)-(1.11) are representation independent, in this paper we will construct the $\{T, J^{(8)}\}$ and $\{T, J, H_\ell\}$ FSCAs by using a special representation of the $Z_8$ PF theory. The connection between the $K = 8$ FSCAs and the $Z_8$ PF theory arises as follows. Since the $[8, L]$ coset models form representations of the $K = 8$ FSCAs, so does the $SU(2)_8$ chiral WZW model, since it is the $L \to \infty$ limit of the coset models. In particular, at the special value of the central charge $c = \frac{12}{5}$ (the $SU(2)_8$ central charge), we expect the $K = 8$ FSCA to be embedded in the operator algebra of the $SU(2)_8$ WZW model. Thus, to
construct the $K = 8$ FSCAs for this special value of the central charge, we need to identify $SU(2)_8$ Virasoro primary fields of the appropriate dimensions with the currents $J$, $H_1$ and $H_2$, and then solve the associativity constraints for the chiral $SU(2)_8$ structure constants in order to calculate the $J$, $H_1$ and $H_2$ OPEs. Unfortunately, there are over a hundred potentially non-trivial structure constants in the $SU(2)_8$ WZW model, and among them a great many more associativity constraints (which, though not all independent, must all be checked). Although one could in principle perform a direct computation in the $SU(2)_8$ theory [14], we use instead the following procedure which makes this problem more tractable.

Begin by representing, in the standard way, the $SU(2)_8$ WZW model as the tensor product of a $Z_8$ PF and a free boson $\varphi(z)$. In Sect. 2 we show that the $Z_8$ PF model (which has central charge $c = \frac{7}{4}$) is isomorphic to the tensor product of two tricritical Ising (TCI) models [2,15] (each of which has $c = \frac{7}{10}$). The usefulness of this observation resides in the fact that the TCI model has only five primary fields (besides the identity) and a manageable number of associativity constraints relating the structure constants of their OPEs. In Sect. 3 we solve these associativity conditions for the chiral TCI model and in Sect. 4 we show how all the associative solutions to the TCI$_2$ model can be constructed. The TCI primary fields play the role of a useful book-keeping device for organizing the $Z_8$ PF (or TCI$_2$) fields. In particular we show that there are only four inequivalent solutions of the TCI$_2$ associativity constraints, called $\mathcal{F}1$, $\mathcal{F}2$, $\mathcal{P}1$ and $\mathcal{P}2$, two of which ($\mathcal{F}1$ and $\mathcal{P}1$) form the basis for constructing the $\{T, J, H_1\}$ FSCA, while the other two are related to the $\{T, J^{(8)}\}$ algebra in the same way. (We are not counting as distinct algebras the many associative solutions which can be obtained as subalgebras of these four solutions simply by decoupling various sets of fields.) We also give a set of simple rules for calculating the structure constants of these operator product algebras starting from the structure constants of the chiral TCI model found in Sect. 3.

Given the structure constants of the $Z_8$ PF theory (i.e. the TCI$_2$ model) found in Sect. 4, we can add back in the free boson $\varphi(z)$, implicitly forming associative solutions for the chiral $SU(2)_8$ WZW model. We can then identify the FSCA
currents and calculate their OPEs using our knowledge of the $Z_8$ PF structure constants, and so derive the algebras (1.6)-(1.11) for the special value of the central charge $c = \frac{12}{5}$. In particular, in Sect. 5 we derive the $\{T, J, H_\ell\}$ FSCA for $c = \frac{12}{5}$ in this way from the $\mathcal{F}1$ (or $\mathcal{P}1$) TCI$^2$ model by demanding that the OPEs of those currents close among themselves. To carry this calculation through in terms of the TCI$^2$ fields and the free boson $\varphi(z)$, we find we must calculate the form and normalizations of many Virasoro descendant fields using the conformal Ward identities following from the Virasoro algebra (1.1). This is a second advantage of using the TCI$^2$ isomorphism over calculating directly in the chiral $SU(2)_8$ WZW theory, because the Kač-Moody current algebra Ward identities [16] are substantially more difficult to use.

We construct the $\{T, J, H_\ell\}$ FSCA for arbitrary central charge in Sect. 6 by turning on a background charge for $\varphi$ and demanding closure of the operator product algebra. In addition, using the $\mathcal{F}1$ TCI$^2$ structure constants, we construct appropriate screening charges and show that they commute with all the currents in the $\{T, J, H_\ell\}$ FSCA with background charge. These can then be used to solve for the spectrum and correlation functions of the FSCA using Feigin-Fuchs techniques [13,17,12]. In Sect. 7 we carry out the same steps starting with the $\mathcal{F}2$ (or $\mathcal{P}2$) TCI$^2$ model and a boson with background charge to construct the $\{T, J^{(8)}\}$ algebra, recovering the results (1.6)-(1.8) of ref. [8]. Using the $\mathcal{F}2$ operator algebra we can also construct the relevant screening charges for the $\{T, J^{(8)}\}$ FSCA. At this point we will make a few comments on the differences between the representation theories of the two $K = 8$ FSCAs we have constructed. At the central charge $c = \frac{52}{55}$, the $H_\ell$ decouple and the FSCA changes from $\{T, J, H_\ell\}$ to $\{T, J, H'_\ell\}$. We use the $c = \frac{52}{55}$ unitary model to construct this latter FSCA and then we compare it with the $\{T, J, H_\ell\}$ and $\{T, J^{(8)}\}$ FSCAs.

The motivation for studying the $K = 8$ FSCAs in particular (out of all possible $K$) is the observation that this algebra, at central charge $c = \frac{12}{5}$ (i.e. zero background charge), is the world-sheet basis of the $K = 8$ fractional superstring theory with critical space-time dimension four [18]. In Sect. 8 we show that the
single-current algebra \( \{T, J^{(8)}\} \) does not allow the coupling of space-time fermions in the fractional superstring, whereas the \( \{T, J, H_\ell\} \) algebra does. Indeed, we show explicitly how to derive the space-time Dirac equation satisfied by the massless fermion states of the fractional superstring using the \( \{T, J, H_\ell\} \) algebra.

Finally, we have collected technical discussions in three appendices. In Appendix A we derive the associativity conditions for the chiral TCI model following Dotsenko and Fateev [13]. Appendix B is a compilation of a special subset of the associative structure constants for the \( \mathcal{F}1 \) TCI\(^2\) model found in Sect. 4. In Appendix C we compute the OPEs between various Virasoro descendant fields in terms of the structure constants for the primary field OPEs using the conformal Ward identities.

2. The \( Z_8 \)-parafermion and the tricritical Ising model

After a brief introduction to the \( Z_8 \) PF theory we show how it corresponds to the tensor product of two TCI models.

The operator content of the chiral \( Z_8 \) PF theory can be realized by the \( SU(2)_8/U(1) \) coset model [7]. The chiral \( SU(2)_8 \) WZW theory [3] has central charge \( c_{WZW} = \frac{12}{5} \) and consists of holomorphic primary fields \( \Phi_m^j(z) \) of conformal dimension \( j(j+1)/10 \). The indices \( j, m \in \mathbb{Z}/2 \) label \( SU(2) \) representations where \( 0 \leq j \leq 4 \) and \( |m| \leq j \) with \( j - m \in \mathbb{Z} \). When we factor a \( U(1) \) subgroup out of \( SU(2)_8 \), we correspondingly factor the primary fields as

\[
\Phi_m^j(z) = \phi_m^j(z) \exp \left\{ \frac{m}{2} \varphi(z) \right\}. \tag{2.1}
\]

Here \( \varphi \) is the free \( U(1) \) boson normalized so that \( \langle \varphi(z)\varphi(w) \rangle = +\ln(z - w) \). The \( \phi_m^j(z) \) are Virasoro primary fields in the \( Z_8 \) PF theory with conformal dimensions:

\[
h_m^j = \frac{j(j+1)}{10} - \frac{m^2}{8} \quad \text{for } |m| \leq j. \tag{2.2}
\]

The central charge of the \( Z_8 \) PF theory is then \( c = c_{WZW} - c_\varphi = \frac{7}{5} \). The definition of the \( \phi_m^j \) fields can be consistently extended to the case where \( |m| > j \) by the
rules
\[
\phi^j_m = \phi^j_{m+8} = \phi^{4-j}_{m-4}.
\] (2.3)

The fusion rules of the parafermion fields follow from those of the \( SU(2)_8 \) theory:
\[
[\phi^j_{m_1}] \times [\phi^j_{m_2}] \sim \sum_{j=|j_1-j_2|}^r [\phi^j_{m_1+m_2}] \tag{2.4}
\]
where \( r = \min\{j_1 + j_2, 8 - j_1 - j_2\} \). The sectors \([\phi^j_m]\) include the primary fields \( \phi^j_m \) and a tower of higher-dimension fields (with dimensions differing by integers) defined as in (2.1) from the Kač-Moody current algebra descendants of the \( \Phi^j_m \).

It follows from the fusion rules (2.4) that a special set of fields, called PF currents, form a closed algebra. The PF currents, denoted \( \psi_\ell \), are defined by
\[
\psi_\ell \equiv \phi^0_\ell = \phi^0_{\ell-8}.
\] (2.5)

By (2.3) and (2.2), the \( \psi_\ell \) have conformal dimensions \( \ell(8 - \ell)/8 \). The operator product algebra they satisfy is called the \( \mathbb{Z}_8 \) PF current algebra:
\[
\psi_\ell(z)\psi_m(w) = \frac{c_{\ell,m}}{(z-w)^{s(\ell,m)}} \psi_{\ell+m}(w) + \cdots,
\] (2.6)
where, if \(-4 \leq \ell, m \leq 4\), then \( s(\ell, m) = \frac{\ell m}{4} + |\ell| + |m| - |\ell + m| \). The currents are normalized so that \( c_{\ell,-\ell} = 1 \). Notice, in particular, that the \( \psi_4 \) PF current has conformal dimension 2 and satisfies the OPE
\[
\psi_4(z)\psi_4(w) = \frac{1}{(z-w)^4} + \frac{(4/c)T_{\mathbb{Z}_8}(w)}{(z-w)^2} + \frac{(2/c)\partial T_{\mathbb{Z}_8}(w)}{(z-w)} + \cdots
\] (2.7)
where, for example, the factor \( 4/c \) in front of the \( \mathbb{Z}_8 \) PF energy-momentum tensor \( T_{\mathbb{Z}_8}(w) \) follows from conformal invariance (since \( T_{\mathbb{Z}_8} \) is itself a descendant of the identity) and the ellipsis denotes further descendants of the identity. Here \( c = \frac{7}{5} \) is the central charge of the \( \mathbb{Z}_8 \) PF theory.
Now we will examine in some depth an interesting and useful representation of the $Z_8$ PF theory. This representation does not realize the full set of $Z_8$ PF fields, but does realize the subset of fields (forming a closed operator product algebra) necessary for the construction of the $K = 8$ FSCAs.

Zamolodchikov observed [4] that an operator product algebra with central charge $c$ and a dimension 2 operator which, together with the energy-momentum tensor, forms a closed operator subalgebra can be written in a new basis to be the direct product of two algebras each with central charge $\frac{c}{2}$. We noted above that the $Z_8$ PF has an additional dimension 2 operator besides the energy-momentum tensor $T_{Z_8}(z)$, namely the current $\psi_4(z)$. Specifically, from the Virasoro algebra (1.1), the $\psi_4\psi_4$ OPE (2.7) and the fact that $\psi_4$ is a primary dimension 2 field

$$T_{Z_8}(z)\psi_4(w) = \frac{2\psi_4(w)}{(z - w)^2} + \frac{\partial \psi_4(w)}{(z - w)} + \ldots ,$$

(2.8)

it follows that the two combinations

$$T_1 = \frac{1}{2} \left( T_{Z_8} + \sqrt{\frac{c}{2}} \psi_4 \right) ,$$

$$T_2 = \frac{1}{2} \left( T_{Z_8} - \sqrt{\frac{c}{2}} \psi_4 \right) ,$$

(2.9)

satisfy separate Virasoro algebras with central charge $c/2$, and that the $T_1T_2$ OPE is regular. Since the $Z_8$ PF theory has central charge $c = \frac{7}{5}$, it follows that $T_1$ and $T_2$ are the energy-momentum tensors for two (distinct) $c = \frac{7}{10}$ CFTs. Furthermore, since the $Z_8$ PF theory is unitary and the only unitary $c = \frac{7}{10}$ conformal field theory is the tricritical Ising (TCI) model [2], it is natural to investigate writing the $Z_8$ PF theory as the tensor product of two TCI models (which we will denote by TCI$^2$).

The first thing to check is whether or not the fields in the $Z_8$ PF model have corresponding fields in the TCI$^2$ model. The dimensions $\Delta_{r,s}$ of the primary fields
\( \Phi_{r,s} \) in the TCI model are \([1]\)

\[
\Delta_{r,s} = \frac{(5r - 4s)^2 - 1}{80}, \quad 1 \leq r \leq 3, \quad 1 \leq s \leq 4.
\]

This gives six distinct fields with dimensions \( \{0, \frac{1}{10}, \frac{3}{5}, \frac{3}{2}, \frac{7}{10}, \frac{3}{80}\} \). The simplest primary fields in a TCI\(^2\) model are the products of primary fields in each TCI factor. The following table summarizes their dimensions:

| \( T_1 \) | \( T_2 \) | \( \Phi_{1,1}^{(1)} \) | \( \Phi_{1,2}^{(1)} \) | \( \Phi_{1,3}^{(1)} \) | \( \Phi_{1,4}^{(1)} \) | \( \Phi_{2,1}^{(1)} \) | \( \Phi_{2,2}^{(1)} \) |
|---|---|---|---|---|---|---|---|
| \( + \) | \( 0 \) | \( 0 \) | \( \frac{1}{10} \) | \( \frac{3}{5} \) | \( \frac{3}{2} \) | \( \frac{7}{10} \) | \( \frac{3}{80} \) |
| \( \Phi_{1,1}^{(2)} \) | \( 0 \) | \( \frac{1}{10} \) | \( \frac{3}{5} \) | \( \frac{3}{2} \) | \( - \) | \( - \) |
| \( \Phi_{1,2}^{(2)} \) | \( \frac{1}{10} \) | \( \frac{1}{10} \) | \( \frac{1}{5} \) | \( \frac{7}{10} \) | \( \frac{8}{5} \) | \( - \) | \( - \) |
| \( \Phi_{1,3}^{(2)} \) | \( \frac{3}{5} \) | \( \frac{3}{5} \) | \( \frac{7}{10} \) | \( \frac{6}{5} \) | \( \frac{21}{10} \) | \( - \) | \( - \) |
| \( \Phi_{1,4}^{(2)} \) | \( \frac{3}{2} \) | \( \frac{3}{2} \) | \( \frac{8}{5} \) | \( \frac{21}{10} \) | \( 3 \) | \( - \) | \( - \) |
| \( \Phi_{2,1}^{(2)} \) | \( \frac{7}{10} \) | \( - \) | \( - \) | \( - \) | \( - \) | \( \frac{7}{5} \) | \( \frac{19}{10} \) |
| \( \Phi_{2,2}^{(2)} \) | \( \frac{3}{80} \) | \( - \) | \( - \) | \( - \) | \( - \) | \( \frac{19}{10} \) | \( \frac{3}{10} \) |

The dashes in the table represent dimensions of fields that do not appear in the \( Z_8 \) PF. Their decoupling must be explicitly demonstrated, which we will do later on, but for now we will simply ignore them. There exist infinitely many other primary fields formed from appropriate combinations of descendant fields from each TCI factor. For example, \( \Phi_{1,2}^{(1)} \partial \Phi_{1,2}^{(2)} - [\partial \Phi_{1,2}^{(1)}] \Phi_{1,2}^{(2)} \) is a primary field of dimension \( \frac{6}{5} \). In general, these more complicated primary fields will have dimensions differing from those given in (2.11) by the addition of a positive integer.

Now we turn to the dimensions of the fields in the \( Z_8 \) PF model. The dimensions \( h^j_m \) of the basic set of Virasoro primary fields are given by eq. (2.2). As noted earlier, the full set of primaries of the PF theory have dimensions of the form \( h^j_m + \text{positive integers} \). Examining (2.2) one finds that the half-integral spin fields, \( j \in \mathbb{Z} + \frac{1}{2} \), have no counterpart in the TCI\(^2\) model, so that we will only be
able to find a representation for the integral spin fields. As these form a closed algebra among themselves, the absence of the half-integral spins is self-consistent. In fact we will not need these half-integral spin fields to construct the $K = 8$ supercurrents. Furthermore, only integral spin fields enter in the $K = 8$ fractional superstring partition function [18]. The dimensions of the integral spin fields are summarized in the following table.

\[
\begin{array}{c|cccc}
  j & 4 & 3 & 2 & 1 & 0 \\
  m & & & & \\
  \hline \\
  4 & 2 & \frac{15}{8} & \frac{3}{2} & \frac{7}{8} & 0 \\
  3 & \frac{6}{5} & \frac{43}{40} & \frac{7}{10} & \frac{3}{40} \\
  2 & \frac{3}{5} & \frac{19}{40} & \frac{1}{10} & \frac{3}{40} \\
  1 & \frac{1}{5} & \frac{3}{40} & & & \\
  0 & 0 & & & & \\
\end{array}
\]  

(2.12)

Notice the pleasing fact that every field in the TCI$^2$ table has at least one partner in the $Z_8$ PF table (mod 1) and vice-versa.

Next we will check the equivalence of the characters of the TCI$^2$ model and the $Z_8$ PF theory. The characters $\chi_{r,s}(q)$ corresponding to the primary fields $\Phi_{r,s}$ of the TCI model are [19]

\[
\eta(q)\chi_{r,s}(q) = q^{\frac{j}{2} R} \sum_{n \in \mathbb{Z}} (q^{\alpha_n} - q^{-\alpha_n}),
\]

(2.13)

where $\eta(q)$ is the Dedekind $\eta$-function, and $q = e^{2\pi i \tau}$ where $\tau$ is the modular parameter of the torus. The characters $Z_m^j(q)$ for the $Z_8$ PF sectors $[\phi_m]$ are related to the known string functions $c^{2j}_{2m}$ by $Z_m^j(q) = \eta(q)c^{2j}_{2m}(q)$ [20]. This gives the expression for the $Z_8$ PF characters

\[
\eta^2(q)Z_m^j(q) = q^{h_m+\frac{j}{40}} \sum_{u,v=0}^{\infty} (-1)^{u+v} q^{\frac{1}{2}h_u + \frac{1}{2}h_v} q^{u(u+1)+v(v+1)+9uv} \\
\times \left[q^{u(j+m)+v(j-m)} - q^{9-2j+u(9-j-m)+v(9-j+m)}\right].
\]

(2.14)
The PF characters satisfy the identities $Z^j_m = Z^j_{m+8} = Z^{4-j}_{m-4}$ by virtue of the PF field identifications (2.3). Comparing these character formulas, we find the following nine identities between the $Z_8$ PF and TCI$^2$ characters:

\[
\begin{align*}
Z_0^0 + Z_{10}^4 &= [\chi_0^2 + \chi_{10}^2] q^{-\frac{c}{24}} \\
Z_0^1 + Z_0^3 &= [\chi_{1}^2 + \chi_3^2] q^{-\frac{c}{24}} \\
Z_0^2 &= [\chi_0 \chi_3 + \chi_{10} \chi_2] q^{-\frac{c}{24}} \\
Z_0^0 + Z_1^4 &= [\chi_{2}^2] q^{-\frac{c}{24}} \\
Z_1^1 + Z_1^3 &= [\chi_{3}^2] q^{-\frac{c}{24}} \\
Z_2^2 &= [\chi_0 \chi_3 + \chi_{10} \chi_2] q^{-\frac{c}{24}} \\
Z_1^0 + Z_0^4 &= [\chi_{1}^2 + \chi_3^2] q^{-\frac{c}{24}} \\
Z_1^1 + Z_3^3 &= [\chi_{5}^2] q^{-\frac{c}{24}} \\
Z_2^1 &= [\chi_{15} \chi_5] q^{-\frac{c}{24}} \\
Z_0^0 + Z_2^4 &= [\chi_{15} \chi_5] q^{-\frac{c}{24}} \\
Z_2^2 &= [\chi_0 \chi_3 + \chi_{10} \chi_2] q^{-\frac{c}{24}} \\
Z_2^0 &= [\chi_0 \chi_3 + \chi_{10} \chi_2] q^{-\frac{c}{24}}
\end{align*}
\]

for central charge $c = \frac{7}{5}$. This makes the identification of the $Z_8$ PF fields with the TCI$^2$ fields more explicit.

The final check on the equivalence of these two theories would be to show that the $Z_8$ PF fusion rules (2.4) are the same as those for the TCI$^2$ model. That this is indeed the case will be made clear in the subsequent discussion [see eqs. (4.3)-(4.5)].

Notice that in the above identification of $Z_8$ PF fields as TCI$^2$ fields, the PF characters only appear in the particular combinations $Z^j_m + Z^{4-j}_{m}$. These are precisely the combinations that appear in the modular invariant partition function of the $K = 8$ fractional superstring [18].
3. Solving the TCI associativity constraints

We present the associativity constraints, following ref. [13], of the chiral TCI model four-point functions and use them to construct the TCI operator product algebra. In the next section we expand this discussion to classify the TCI\(^2\) operator product algebras.

What is needed to construct the TCI chiral algebra is a complete list of the associative transformation properties of all non-vanishing four-point correlation functions. That is, for a given four-point function

\[
\langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\phi_l(z_l) \rangle,
\]

we need to know the transformation matrix between the conformal blocks as \(z_i \rightarrow z_j \quad (z_k \rightarrow z_l)\) and as \(z_j \rightarrow z_k \quad (z_l \rightarrow z_i)\). These matrices are called fusion matrices and are denoted \(\alpha\). In general, determining \(\alpha\) for (3.1) is a difficult and as yet unsolved problem, but for the minimal models in general and the TCI model in particular it can be solved using the Feigin-Fuchs technique. Dotsenko and Fateev [13] show how to construct the fusion matrices in these cases. In Appendix A we review and summarize the Feigin-Fuchs technique and use it to construct the \(\alpha\) matrices for the TCI model.

The fusion rules of the TCI model are [1]

\[
\begin{align*}
\Phi_{1,2} \times \Phi_{1,2} & \sim \Phi_{1,1} + \Phi_{1,3} & \Phi_{1,2} \times \Phi_{2,1} & \sim \Phi_{2,2} \\
\Phi_{1,2} \times \Phi_{1,3} & \sim \Phi_{1,2} + \Phi_{1,4} & \Phi_{1,3} \times \Phi_{2,1} & \sim \Phi_{2,2} \\
\Phi_{1,2} \times \Phi_{1,4} & \sim \Phi_{1,3} & \Phi_{1,4} \times \Phi_{2,1} & \sim \Phi_{2,1} \\
\Phi_{1,3} \times \Phi_{1,3} & \sim \Phi_{1,1} + \Phi_{1,3} & \Phi_{1,2} \times \Phi_{2,2} & \sim \Phi_{2,1} + \Phi_{2,2} \\
\Phi_{1,3} \times \Phi_{1,4} & \sim \Phi_{1,2} & \Phi_{1,3} \times \Phi_{2,2} & \sim \Phi_{2,1} + \Phi_{2,2} \\
\Phi_{1,4} \times \Phi_{1,4} & \sim \Phi_{1,1} & \Phi_{1,4} \times \Phi_{2,2} & \sim \Phi_{2,2} \\
\Phi_{2,1} \times \Phi_{2,1} & \sim \Phi_{1,1} + \Phi_{1,4} \\
\Phi_{2,1} \times \Phi_{2,2} & \sim \Phi_{1,2} + \Phi_{1,3} \\
\Phi_{2,2} \times \Phi_{2,2} & \sim \Phi_{1,1} + \Phi_{1,2} + \Phi_{1,3} + \Phi_{1,4}
\end{align*}
\]

(3.2)
These fusion rules are maximal in the sense that no additional fields may appear on the right-hand side of a given OPE; on the other hand, some fields may decouple or have vanishing structure constants and thereby *not* appear. The task of finding an associative solution to the chiral TCI model is a straightforward if laborious one: apply the associativity constraints of Appendix A to all non-vanishing four-point functions to determine a consistent set of structure constants to insert into the fusion rules above. Additionally, the multiplicity of the fields must be left free a priori and determined by the fusion rules and associativity constraints. For example, in the $Z_{K \geq 3}$ PF theory there are two distinct dimension $1 - \frac{1}{K}$ operators, namely $\psi_1$ and $\psi_{-1}$.

We consider only diagonalizable algebras so that we are free to choose the normalization

$$q_{ij} = \delta_{ij}$$

in (1.3). In other words, the OPE of any field with itself closes on the identity with coefficient unity. This normalization implies some symmetries among the structure constants which we will use. First of all, associativity of three-point functions implies that structure constants are cyclically symmetric. Specifically, the following two ways of expanding the same three-point function must give the same result,

$$\langle \phi_i \phi_j \phi_k \rangle = c_{ijk} \langle [\phi_k] \phi_k \rangle = c_{ijk},$$

$$\langle \phi_i \phi_j \phi_k \rangle = c_{jki} \langle \phi_i [\phi_i] \rangle = c_{jki}.$$  \hspace{1cm} (3.4)

Second of all, because the models we are working with are unitary the interchange of two indices effects the action of complex conjugation so that we have

$$c_{ijk} = c_{jki} = c_{kij} = c_{ijk}^* = c_{ikj}^* = c_{kji}^*. \hspace{1cm} (3.5)$$

Using these symmetries we can proceed to calculate the structure constants for the chiral TCI algebra. (Note that the only normalization freedom left after imposing (3.3) is in the sign of the fields. Thus the structure constants derived below will only be fixed up to possible signs.)
We will not explicitly show the entire construction, but as an illustrative example we will show that the $\Phi_{2,1}$ field decouples from the chiral TCI algebra, and then we will simply state the final result. Therefore, consider the OPE of the $\Phi_{2,1}$ field with itself

$$\Phi_{2,1}\Phi_{2,1} = \Phi_{1,1} + c_{(2,1)(2,1)(1,4)} \Phi_{1,4}, \quad (3.6)$$

and the four-point function

$$\langle \Phi_{1,4}\Phi_{2,1}\Phi_{1,4}\Phi_{2,1} \rangle. \quad (3.7)$$

This four-point function has only one conformal block and its (one by one) fusion matrix is $\alpha = -1$ (see Appendix A). Thus the associativity constraint derived from (3.7) is

$$-c_{(1,4)(2,1)(2,1)}^2 = c_{(2,1)(1,4)(2,1)}^2. \quad (3.8)$$

As the structure constants are cyclically symmetric this implies that $c_{(2,1)(2,1)(1,4)} = 0$. Checking to see if this result is consistent we look at the four-point function consisting of all $\Phi_{2,1}$ fields, i.e.

$$\langle \Phi_{2,1}\Phi_{2,1}\Phi_{2,1}\Phi_{2,1} \rangle. \quad (3.9)$$

The $\alpha$ matrix for this four-point function [from table (A8)] gives the following associativity constraint

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 8 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_{(2,1)(2,1)(1,4)}^2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ c_{(2,1)(2,1)(1,4)}^2 \end{pmatrix}. \quad (3.10)$$

However, plugging the value $c_{(2,1)(2,1)(1,4)} = 0$ into this matrix equation gives a contradiction which is resolved only by decoupling the $\Phi_{2,1}$ field from the chiral TCI algebra. Similar reasoning shows that the $\Phi_{2,2}$ field must also decouple.
In the preceding argument we implicitly took the multiplicity of the $\Phi_{2,1}$ field to be one. The argument, however, can be repeated assuming $n$ copies of $\Phi_{2,1}$ and it is not difficult to see that the result is the same: all $n$ $\Phi_{2,1}$ fields must decouple.

A solution does exist though for the remaining four fields, and a little work yields the operator product algebra (here $z$ and $w$ dependences as well as Virasoro descendants are suppressed)

\[
\begin{align*}
\Phi_{1,2}\Phi_{1,2} &= \Phi_{1,1} - s\Phi_{1,3}, \\
\Phi_{1,2}\Phi_{1,3} &= -s\Phi_{1,2} + u\Phi_{1,4}, \\
\Phi_{1,2}\Phi_{1,4} &= u\Phi_{1,3}, \\
\Phi_{1,3}\Phi_{1,3} &= \Phi_{1,1} + s\Phi_{1,3}, \\
\Phi_{1,3}\Phi_{1,4} &= u\Phi_{1,2}, \\
\Phi_{1,4}\Phi_{1,4} &= \Phi_{1,1},
\end{align*}
\]

(3.11)

where the structure constants $u$ and $s$ are given by

\[
\begin{align*}
u &= \sqrt{\frac{3}{7}}, \\
s &= \sqrt{\frac{2}{3} r x},
\end{align*}
\]

(3.12)

where

\[
\begin{align*}
r &= \frac{\sqrt{5} - 1}{2}, \\
x &= \frac{\Gamma^2 \left( \frac{2}{5} \right)}{\Gamma \left( \frac{4}{5} \right) \Gamma \left( \frac{2}{5} \right)}.
\end{align*}
\]

(3.13)

Again, we find that increasing the multiplicity of the $\Phi_{1,i}$ fields does not yield any new associative solutions, so that (3.11) is the only chiral associative TCI algebra.

The new $K = 8$ algebra, i.e. the \{T, J, H_\ell\} FSCA, can be built essentially from the direct tensoring of two TCI algebras given above plus a free boson. We construct this \{T, J, H_\ell\} FSCA in Sects. 5 and 6. However, solving a TCl^2 associative algebra introduces additional solutions. The technical details and subtleties of this procedure, which will be needed at the end of Sect. 6 to construct screening charges, is presented in the next section.
4. Solving the TCI\(^2\) associativity constraints

In this section we construct the associative solutions to the TCI\(^2\) operator product algebra. This involves an extended technical discussion. However, for the purpose of constructing the \(K = 8\) FSCAs, only the two TCI\(^2\) solutions, denoted \(\mathcal{F}1\) and \(\mathcal{F}2\), are relevant. We use \(\mathcal{F}1\) to construct the \(\{T, J, H\ell\}\) algebra and screening charges and \(\mathcal{F}2\) to do the same for the \(\{T, J^{(8)}\}\) FSCA. In particular, the results for the \(\mathcal{F}1\) algebra OPEs needed to construct the new \(\{T, J, H\ell\}\) FSCA are given below in eqs. (4.13)-(4.15).

We now consider solving the associativity constraints on the TCI\(^2\) operator product algebra. The obvious solution is the tensor product algebra constructed by simply multiplying the associative TCI algebra found in the last section with another copy of itself. We call this algebra \(\mathcal{P}1\). However, solving the associativity constraints for a tensor product algebra also yields new solutions which cannot be expressed as the product of the structure constants of two simple algebras. This must be the case for the TCI\(^2\) model, because at the level of solving a consistent associative operator product algebra, the mathematics of constructing a chiral tensor product of two TCI models \((c = \frac{7}{10} + \frac{7}{10} = \frac{7}{5}, \bar{c} = 0)\) is the same as that of constructing a non-chiral TCI model \((c = \frac{7}{10}, \bar{c} = \frac{7}{10})\). In the latter case, we know there exists a left-right symmetric model (see for example [15]) in which the magnetic spin operator, with \((h, \bar{h}) = \left(\frac{4}{80}, \frac{3}{80}\right) \sim \Phi_{2,2}(\!z\!\!z)\Phi_{2,2}(\!\bar{z}\!\bar{z})\), enters. Such a field does not appear in the simple product of two chiral TCI models.

We are, of course, interested in the former problem—constructing a chiral tensor product of two TCI models (TCI\(^2\))—because our objective is to find a representation of the \(c = \frac{7}{5}, Z_8\) PF theory. To this end consider two TCI models with energy-momentum tensors \(T_1\) and \(T_2\), (where \(T_{Z8} = T_1 + T_2\)), whose primary fields are \(\Phi_{i,j}^{(1)}\) and \(\Phi_{i,j}^{(2)}\) respectively. It is convenient to define the following fields
in the tensor product model

\[ \beta_{ij} \sim \Phi_{1,i}^{(1)} \Phi_{1,j}^{(2)}, \]
\[ \lambda_{ij}, \bar{\lambda}_{ij} \sim \Phi_{2,i}^{(1)} \Phi_{2,j}^{(2)}. \]  
(4.1)

These identifications are only symbolic since the structure constants for the \( \beta_{ij}, \lambda_{ij} \) and \( \bar{\lambda}_{ij} \) fields are generically not the product of structure constants for the \( \Phi_{i,j} \) fields, as we have already discussed. Note especially that the definition for the \( \lambda_{ij} \) fields is identical to that for the \( \bar{\lambda}_{ij} \) fields. This is because, as could be guessed from the presence of order and disorder fields with the same conformal dimensions in the non-chiral, left-right diagonal, TCI model, the \( \lambda \) fields naturally split. In other words, the \( \lambda \) fields have multiplicity 2. There are in fact solutions where the \( \lambda \) fields have multiplicity 1, but these turn out to be simple subalgebras of the larger operator algebras that we will construct with both \( \lambda \) and \( \bar{\lambda} \) fields. Further increasing the multiplicities of the \( \beta \) and \( \lambda, \bar{\lambda} \) fields does not yield any new solutions.

A priori, the “crossed” fields \( \Phi_{1,i}^{(1)} \Phi_{2,j}^{(2)} \) and \( \Phi_{2,i}^{(1)} \Phi_{1,j}^{(2)} \) must also be considered, but arguments similar to those that ruled out the existence of the \( \Phi_{2,1} \) and \( \Phi_{2,2} \) fields in a chiral TCI model are applicable here also. The natural, and mathematically forced, exclusion of the “crossed” fields is actually reassuring since they have no counterpart in the \( Z_8 \) PF theory that we are trying to represent.

The TCI\(^2\) operator algebra for the 24 simple primary fields \( \beta_{ij}, \lambda_{ij} \) and \( \bar{\lambda}_{ij} \) contains several hundred distinct structure constants allowed by the fusion rules, although many of them will turn out to be zero. As mentioned before, the TCI\(^2\) model contains an infinite number of additional fields which are composed of descendants of the simple primaries with respect to \( T_1 \) or \( T_2 \), but are primary with respect to the full energy-momentum tensor \( T_1 + T_2 \). Thus, using the conformal Ward identities for each TCI factor separately (\( T_1 \) or \( T_2 \)), we can derive the structure constants of any TCI\(^2\) primary from those of the simple primaries (4.1). This is a crucial simplifying feature, for it allows us to solve the associativity constraints for the TCI\(^2\) theory by looking only at a finite operator algebra.
A useful way of organizing the 24 simple primary fields is summarized in the following table.

| 1 | $F_4$ | $\beta_{33}$ | $\beta_{22}$ | $\beta_{32}$ | $\beta_{23}$ | $\lambda_{22}$ | $\bar{\lambda}_{22}$ |
|---|---|---|---|---|---|---|---|
| 2 | $F_3$ | $\beta_{31}$ | $\beta_{24}$ | $\beta_{34}$ | $\beta_{21}$ | $\lambda_{21}$ | $\bar{\lambda}_{21}$ |
| 2 | $F_2$ | $\beta_{13}$ | $\beta_{42}$ | $\beta_{12}$ | $\beta_{43}$ | $\lambda_{12}$ | $\bar{\lambda}_{12}$ |
| 0 | $F_1$ | $\beta_{11}$ | $\beta_{44}$ | $\beta_{14}$ | $\beta_{41}$ | $\lambda_{11}$ | $\bar{\lambda}_{11}$ |

| $j$ | $\beta^{(1)}$ | $\beta^{(2)}$ | $\beta^{(3)}$ | $\beta^{(4)}$ | $\lambda$ | $\bar{\lambda}$ |
|---|---|---|---|---|---|---|
| $|m|$ | 0 | 0 | 2 | 2 | 1 | 1 |

In this table $F_i$ refers to all the fields in the corresponding row and $\beta^{(i)}$, $\lambda$ and $\bar{\lambda}$ to all fields the corresponding columns. The potential usefulness of this grouping is made apparent by the fact that the fields in these rows and columns correspond, by the identifications made in Sect. 2, to $Z_8$ PF fields with $j$ and $m$ quantum numbers given above. Note that we have used the PF field identifications (2.3) to restrict $j$ and $|m|$ (though not necessarily simultaneously) to the set \{0, 1, 2\}. Indeed, from the fusion rules (3.2) we can summarize the fusion rules for the TCI$^2$ fields neatly in terms of fusion rules for the row and column labels. Specifically, we have the $F_i \times F_j$ fusion rules

\[
\begin{align*}
F_1 \times F_i & \sim F_i & F_3 \times F_3 & \sim F_1 + F_3 \\
F_2 \times F_2 & \sim F_1 + F_2 & F_3 \times F_4 & \sim F_2 + F_4 \\
F_2 \times F_3 & \sim F_4 & F_4 \times F_4 & \sim F_1 + F_2 + F_3 + F_4 \\
F_2 \times F_4 & \sim F_3 + F_4
\end{align*}
\]

Similarly, the $\beta^{(i)} \times \beta^{(j)}$ fusion rules can be summarized by all permutations of the five fusion rules

\[
\begin{align*}
\beta^{(1)} \times \beta^{(i)} & \sim \beta^{(i)}, \\
\beta^{(2)} \times \beta^{(3)} & \sim \beta^{(4)}.
\end{align*}
\]

Now in general, from (3.2) we have $\lambda \times \lambda$, $\lambda \times \bar{\lambda}$, $\bar{\lambda} \times \bar{\lambda}$, $\bar{\lambda} \times \lambda \sim \beta^{(1)} + \beta^{(2)} + \beta^{(3)} + \beta^{(4)}$. It turns out, however, that in all general solutions we can define new $\lambda$ and $\bar{\lambda}$ fields
(by taking linear combinations of the old fields) so that they have the fusion rules

\begin{align*}
\lambda \times \lambda & \sim \bar{\lambda} \times \bar{\lambda} \sim \beta^{(1)} + \beta^{(2)}, \\
\lambda \times \bar{\lambda} & \sim \beta^{(3)} + \beta^{(4)}.
\end{align*}

(4.5)

The fusion rules of the \( \beta^{(i)} \) with \( \lambda \) or \( \bar{\lambda} \) follow from (4.5). To calculate the fusion of any two specific simple primary fields in (4.2), we can use the intersection of the fusion rules (4.3) with (4.4) and (4.5).

The main result of this section is that the TCI\(^2\) algebra has exactly four distinct (maximal) associative solutions which we call \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{F}_1 \) and \( \mathcal{F}_2 \). We will begin by describing these four algebras and their interrelation. In particular, we will present a set of simple rules for calculating the structure constants of any of the algebras given those of the \( \mathcal{F}_1 \) algebra. In eq. (4.6) below and Appendix B we list all the \( \mathcal{F}_1 \) structure constants. At the end of this section we will briefly outline the arguments that lead to this classification and construction of the TCI\(^2\) associative operator algebras.

The \( \mathcal{F}_1 \) operator algebra is the largest one and includes all 24 simple primary fields in (4.2). The \( \mathcal{F}_2 \) algebra contains the 12 fields in the \( F_1 \) and \( F_4 \) rows, \textit{i.e.} the \( F_2 \) and \( F_3 \) fields decouple, and is \textit{not} a subalgebra of \( \mathcal{F}_1 \). Now the \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) algebras each have subalgebras in which only the fields in the \( \beta^{(i)} \) columns appear, \textit{i.e.} the \( \lambda \) and \( \bar{\lambda} \) fields decouple. These two subalgebras will be denoted \( \tilde{\mathcal{P}}_1 \) and \( \tilde{\mathcal{P}}_2 \), respectively. They are related in a simple way to the two remaining algebras \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), which, however, are not subalgebras of any other associative operator product algebras. In particular, the operator content of \( \mathcal{P}_i \) and \( \tilde{\mathcal{P}}_i \) are the same. \( \mathcal{P}_1 \) contains only the 16 \( \beta_{ij} \) fields while \( \mathcal{P}_2 \) contains only the 8 \( \beta_{ij} \) fields in the rows \( F_1 \) and \( F_4 \). As we mentioned before, the \( \mathcal{P}_1 \) algebra is the simple tensor product of two associative chiral TCI operator algebras, found in the last section.
The interrelationships between these algebras are summarized in the diagram

\[
P_1 \xleftarrow{\mathcal{C}} \overset{\mathcal{R}}{\sim} P_1 \subset F_1
\]

\[
P_2 \xleftarrow{\mathcal{C}} \overset{\mathcal{R}}{\sim} P_2 \subset F_2
\]

(4.6)

The row and column maps, \( \mathcal{R} \) and \( \mathcal{C} \) respectively, are given by simple rules, and allow us to construct all the operator algebras starting from \( F_1 \). In particular, the column map \( \mathcal{C} \) is given by the rule that we multiply the structure constants coupling fields in the \( \beta^{(2)} \), \( \beta^{(3)} \) and \( \beta^{(4)} \) columns by a factor of \(-i\), while leaving all other structure constants the same:

\[
\mathcal{C} : \langle \beta^{(2)} \beta^{(3)} \beta^{(4)} \rangle \overset{-i}{\longleftrightarrow} \langle \beta^{(2)} \beta^{(3)} \beta^{(4)} \rangle.
\]

(4.7)

The map for these \( \beta^{(i)} \) fields in other orderings is determined by the structure constant symmetries (3.5). Note that the sign of \( i \) in (4.7) is actually of no consequence, since we have not fixed the signs of the normalizations of the fields; however, the phase \( i \) itself cannot be defined away by any change in the normalization of the fields.

To describe the row map \( \mathcal{R} \), we must first mention that we will always write the structure constants of the TCI\(^2\) operator algebras as functions of the formal “variables” \( r \) and \( x \). The numerical value of any such structure constant is found, however, by letting these “variables” take the values

\[
r = \frac{\sqrt{5} - 1}{2}, \quad x = \frac{\Gamma^2 \left( \frac{2}{3} \right)}{\Gamma \left( \frac{1}{5} \right) \Gamma \left( \frac{3}{5} \right)},
\]

(4.8)

that we are derived from the chiral TCI associativity conditions in Sect. 3. Consider the \( r \) and \( x \) dependence of the OPEs of fields in the rows \( F_i \). For the \( F_1 \) and \( P_1 \) algebras, these OPEs are (symbolically)

\[
F_i \cdot F_j \sim \sum_k (r x)^{a_k} F_k,
\]

(4.9)

where the exponents \( a_k \) can take values in the set \( a_k \in \{0, \frac{1}{2}, 1\} \). In other words, the \( r \) and \( x \) dependence of these structure constants is always proportional to \( r x \)
raised to one of the above-mentioned powers; furthermore, these powers are the same for all fields in a given row $F_k$. The rule for the row map $\mathcal{R}$ is now easy to state: first decouple the $F_2$ and $F_3$ fields and then, treating the different powers $(r x)^{ak}$ as independent parameters, make the substitutions

$$\mathcal{R} : \begin{cases} (r x)^0 & \rightarrow 1 , \\ (r x)^{1/2} & \rightarrow 0 , \\ (r x)^1 & \rightarrow \frac{x}{\sqrt{r}} . \end{cases} \quad (4.10)$$

For example, in the $\mathcal{F}1$ algebra the $F_4 \cdot F_4$ OPEs (4.3) can be written symbolically as,

$$F_4 \cdot F_4 \sim F_1 + \sqrt{r x} F_2 + \sqrt{r x} F_3 + r x F_4 , \quad (4.11)$$

where the factors $\sqrt{r x}$ and $r x$ indicate the dependence on $r$ and $x$ in their respective structure constants. The row map (4.10) then leads to the $\mathcal{F}2$ OPEs

$$F_4 \cdot F_4 \sim F_1 + \frac{x}{\sqrt{r}} F_4 , \quad (4.12)$$

where all other numerical factors in the structure constants remain the same.

We will now write down the structure constants of the $\mathcal{F}1$ operator algebra. To start with, the OPEs of the $\beta^{(i)}$ fields (i.e. the $\tilde{\mathcal{P}}1$ subalgebra) can be succinctly written as

$$\mathcal{F}1 : \quad \beta_{ij}^{(r)} \beta_{kl}^{(s)} = \sum_{t=1}^{4} d_{rst} \sum_{m,n=1}^{4} c_{ikm} c_{jln} \beta_{mn}^{(t)} , \quad (4.13)$$

where $c_{ijk}$ and $d_{rst}$ obey the structure constant symmetries (3.5). Their non-zero components have the values,

$$c_{1ii} = 1 ,$$

$$c_{333} = -c_{223} = \sqrt{\frac{2}{3}} r x ,$$

$$c_{234} = \sqrt{\frac{3}{7}} , \quad (4.14)$$
and
\[
d_{1rr} = 1, \\
d_{234} = i.
\] (4.15)

Note that the \( c_{ijk} \) are just the chiral TCI structure constants derived in Sect. 3. Indeed, if we operate with the column map (4.7) on these structure constants, we obtain the \( \mathcal{P}_1 \) operator algebra structure constants. This has the effect of letting \( d_{rst} \rightarrow d'_{rst} \) in (4.13), where
\[
d'_{1rr} = d'_{234} = 1.
\] (4.16)

The inclusion of the column label superscripts and \( \sum_{t=1}^{4} d'_{rst} \) in (4.13) becomes redundant in this case, so that we obtain
\[
\mathcal{P}_1 : \quad \beta_{ij} \beta_{kl} = \sum_{m,n=1}^{4} c_{ikm} c_{jln} \beta_{mn},
\] (4.17)

which is just the direct product of two chiral TCI associative solutions, as advertised.

It remains to include the fields in the \( \lambda \) and \( \bar{\lambda} \) columns in the \( \mathcal{F}_1 \) algebra. The couplings of these fields are found by solving the associativity constraints, as discussed below. A unique maximal solution is found; a compilation of the resulting structure constants is given in Appendix B. The other maximal TCI\(^2\) operator algebras can be found by acting with the \( \mathcal{R} \) and \( \mathcal{C} \) maps as outlined above (4.6). The \( \mathcal{F}_2 \) operator algebra found by this procedure has structure constants that are equivalent (up to normalizations) to those found by Qiu in ref. [15] for the non-chiral left-right symmetric TCI model.

So far we have simply described the four inequivalent associative TCI\(^2\) operator algebras. We will now briefly describe what is involved in solving the TCI\(^2\) associativity constraints. The basic observation is that, since the simple primary fields of the TCI\(^2\) model are just products of the primaries of each TCI factor, the fusion matrices for the four-point correlators of the TCI\(^2\) simple primaries are simply the
tensor product of the two fusion matrices for the associated four-point function for each TCI factor separately. Let us give an example which illustrates this point. Consider the $\langle \beta_{33} \beta_{33} \beta_{33} \beta_{33} \rangle$ four-point function. Now, from (4.1) $\beta_{33} = \Phi^{(1)}_{1,3} \Phi^{(2)}_{1,3}$, where the superscripts denote the two TCI factors. The general $\Phi_{1,3} \Phi_{1,3}$ OPE in the chiral TCI model is, (3.2),

$$\Phi_{1,3} \Phi_{1,3} = \Phi_{1,1} + c_{333} \Phi_{1,3} .$$  \hspace{1cm} (4.18)

The associativity condition from the $\langle \Phi_{1,3} \Phi_{1,3} \Phi_{1,3} \Phi_{1,3} \rangle$ four-point function is [from eq. (A8) in Appendix A] the eigenvalue equation

$$\begin{pmatrix} r & \frac{3r}{2x} \\ \frac{2x}{x} & -r \end{pmatrix} \begin{pmatrix} 1 \\ c_{333}^2 \end{pmatrix} = \begin{pmatrix} 1 \\ c_{333}^2 \end{pmatrix}, \hspace{1cm} (4.19)$$

which has the unique solution

$$c_{333}^2 = \frac{2}{3} rx . \hspace{1cm} (4.20)$$

The fact that the matrix in (4.19) has exactly one $+1$ eigenvalue is at the heart of the uniqueness of the single chiral TCI model. However, when this matrix is tensored with itself to find the associativity constraint for the $\langle \beta_{33} \beta_{33} \beta_{33} \beta_{33} \rangle$ four-point function, we find that it now has two $+1$ eigenvalues which implies that it has a line of solutions. One of these solutions, of course, is just the direct product of the chiral TCI solution, giving the $\beta_{33} \beta_{33}$ OPE

$$\beta_{33} \beta_{33} = \beta_{11} + c_{333} \beta_{13} + c_{333} \beta_{31} + c_{333}^2 \beta_{33} , \hspace{1cm} (4.21)$$

for $c_{333}$ given by (4.20). When the associativity constraints of additional four-point functions are considered*, without decoupling the $\beta_{13}$ and $\beta_{31}$ fields (i.e., for solutions in which these two fields have multiplicity one, not zero), we find that

---

* A set of four-point functions sufficient to show that (4.21) is the only consistent solution is, for example: $\langle \beta_{13} \beta_{33} \beta_{33} \beta_{33} \rangle$, $\langle \beta_{13} \beta_{33} \beta_{13} \beta_{33} \rangle$ and $\langle \beta_{13} \beta_{33} \beta_{31} \beta_{33} \rangle$. 

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only the solution (4.21), out of the line of possible solutions, is consistent. (This solution corresponds to either the \( P_1 \) or \( F_1 \) operator algebras, since the \( \beta_{33}\beta_{33} \) OPE happens to be the same for both algebras.) Decoupling \( \beta_{13} \) and \( \beta_{31} \) leads to another solution on the line—the only other consistent solution—which is a result of the eigenvalue equation

\[
\begin{pmatrix}
  r^2 & \frac{3r^2}{2x} & \frac{3r^2}{2x} & \frac{9r^2}{4x^2} \\
  \frac{2}{3}xr & -r^2 & r & -\frac{3r^2}{2x} \\
  \frac{2}{3}xr & r & -r^2 & -\frac{3r^2}{2x} \\
  \frac{4}{3}x^2 & -\frac{2}{3}xr & -\frac{2}{3}xr & r^2
\end{pmatrix}
\begin{pmatrix}
  1 \\
  0 \\
  0 \\
  \frac{4x^2}{9r}
\end{pmatrix}
= \begin{pmatrix}
  1 \\
  0 \\
  0 \\
  \frac{4x^2}{9r}
\end{pmatrix}.
\]

(4.22)

This leads to the OPE

\[
\beta_{33}\beta_{33} = \beta_{11} + \frac{2}{3}\sqrt{r}\beta_{33}.
\]

(4.23)

This OPE is part of the \( F_2 \) and \( P_2 \) operator algebras; it is easy to check that (4.21) and (4.23) are related by the \( R \) map.

This example illustrates how, even though the chiral TCI model has a single solution to its associativity constraints, the tensor product of two such models may have multiple solutions. Indeed, the pattern illustrated in the above example for the \( \beta^{(1)} \) fields extends also to the fields in the other columns \( \beta^{(i)}, \lambda \) and \( \bar{\lambda} \). To show that the complete set of solutions to the associativity conditions for the TCI\(^2\) operator product algebra is \( F_1, F_2, P_1, P_2 \) and their subalgebras, requires the systematic and unilluminating examination of all the four-point associativity constraints.

This completes the construction of the four maximal TCI\(^2\) operator algebras. We use \( F_1 \) (or \( P_1 \)) to construct the \( \{ T, J, H_\ell \} \) FSCA and its associated screening charges in Sects. 5 and 6. In Sect. 7 we show how the \( F_2 \) (or \( P_2 \)) algebra leads to the \( \{ T, J^{(8)} \} \) FSCA that was constructed in ref. [8].
5. Construction of the \( \{T, J, H_\ell\} \) FSCA for \( c = \frac{12}{5} \)

In this section we will construct the set of \( K = 8 \) FSCA currents \( \{T, J, H_\ell\} \) and derive their structure constants for the special value of the central charge \( c = \frac{12}{5} \). As explained in the introduction, at this value of the central charge the \( K = 8 \) FSCA is embedded in the tensor product of the TCI\(^2\) model with a free boson.

We set the normalization of the boson \( \varphi(z) \) by \( \langle \varphi(z) \varphi(w) \rangle = + \ln(z - w) \) so that its energy-momentum tensor is

\[
T_\varphi(z) = + \frac{1}{2} : \partial \varphi(z) \partial \varphi(z) : .
\]  

The dimension 1 primary field \( \partial \varphi(z) \) will play a special role in what follows. Its OPEs are

\[
\begin{align*}
T_\varphi(z)T_\varphi(0) &= \frac{1}{2} z^{-4} + 2 z^{-2} T_\varphi(0) + \cdots \\
T_\varphi(z) \partial \varphi(0) &= z^{-2} \partial \varphi(0) + z^{-1} \partial^2 \varphi(0) + \cdots \\
\partial \varphi(z) T_\varphi(0) &= z^{-2} \partial \varphi(0) + 0 + \cdots \\
\partial \varphi(z) \partial \varphi(0) &= z^{-2} + 2 T_\varphi(0) + \cdots .
\end{align*}
\]  

The full energy-momentum tensor for the TCI\(^2\) plus boson theory is then \( T = T_{Z8} + T_\varphi \). Here the TCI\(^2\) (or \( Z_8 \) PF) energy-momentum tensor \( T_{Z8} = T_1 + T_2 \) where the \( T_i \) are the energy-momentum tensors for each TCI factor separately.

To construct our fractional supersymmetry current \( J \) we need to find all dimension \( \frac{6}{5} \) fields primary with respect to \( T \). Since we want to generalize the idea of the superconformal current to one which transforms bosons to parafermion fields, we impose, as an anzatz, the condition that only the derivatives of \( \varphi \) enter into the expression for the currents. This means, in particular, that the vertex operators \( \exp\{\alpha \varphi(z)\} \) need not be considered. With this restriction the complete list of dimension \( \frac{6}{5} \) primary fields in the TCI\(^2\) model tensored with a free boson consists of the three fields \( \beta_{33}, \beta_{22} \partial \varphi, \) and \( \beta'_{22} \). Here \( \beta'_{22} \) is a TCI\(^2\) primary field made from
descendant fields from each TCI factor:

\[ \beta'_{22} = \partial \Phi^{(1)}_{1,2}\Phi^{(2)}_{1,2} - \Phi^{(1)}_{1,2}\partial \Phi^{(2)}_{1,2}. \]  

(5.3)

Therefore, \( J \) is a linear combination of these three fields.

The conditions we impose on this linear combination are that the \( JJ \) OPE close back on itself (\( i.e. \) no other linearly independent dimension \( \frac{6}{5} \) field enters) and that it close on the identity with coefficient unity. The calculation of this OPE, however, uses the structure constants of the \( \text{TCI}^2 \) theory. Therefore we must first choose which of the four inequivalent \( \text{TCI}^2 \) operator product algebras, found in Sect. 4, to use to calculate the \( K = 8 \) FSCA currents. Now, it so happens that the OPEs of \( \text{TCI}^2 \) fields listed above that contribute to the \( J \) current are left unchanged by the “column map” \( C \) introduced in the last section. This means that the OPEs for these fields in the \( P1 \) and \( F1 \) \( \text{TCI}^2 \) operator algebras are the same; and similarly for the OPEs in the \( P2 \) and \( F2 \) algebras. Thus, as far as the calculation of the currents is concerned, there are really only two independent choices of \( \text{TCI}^2 \) algebra to make. However, we will later find that only for the \( Fi \) algebras can we construct appropriate screening charges for the FSCAs when the boson field \( \varphi \) has background charge. Thus, one should view the construction of the FSCAs based on the \( Pi \) algebras as reflecting the existence of “special” unitary representations of the FSCAs for central charge \( c = \frac{12}{5} \).

In this and the next section, we will construct the \( K = 8 \) FSCAs based on the \( F1 \) (or \( P1 \)) \( \text{TCI}^2 \) operator algebras. We will find that the \( \{T, J, H_\ell\} \) algebra, described in the introduction, emerges in this case. In Sect. 7 we will construct the \( \{T, J^{(8)}\} \) FSCA which is based on \( F2 \) (or \( P2 \)).

We can now proceed with the determination of the \( J \) current. Using the \( F1 \) (or \( P1 \)) OPEs described in the last section, we can compute the relevant \( \text{TCI}^2 \) OPEs. Suppressing all primary fields but the identity, \( \beta_{22}, \beta'_{22} \) and \( \beta_{33} \) on the right hand
sides of these OPEs, we find

\begin{align}
\beta_{22}\beta_{22} &= 1 + s^2\beta_{33} \\
\beta_{22}'\beta_{22}' &= -\frac{2}{5} - \frac{4}{5}s^2\beta_{33} \\
\beta_{33}\beta_{33} &= 1 + s^2\beta_{33} \\
\beta_{22}'\beta_{22}' &= 0
\end{align}

(5.4)

where \(s = \sqrt{\frac{2}{5}r_x}\) was defined in Sect. 3, and where we have also suppressed the \(z\) and \(w\)-dependence of the fields and their coefficients in the OPEs. The only technical difficulty encountered in computing (5.4) occurs in the \(\beta_{22}'\beta_{33}\) OPE. In this case the TCI Virasoro Ward identities must be used to deduce the coefficient of the \(\Phi_{1,2}\) descendant that contributes to \(\beta_{22}'\) (5.3). In particular, the relevant coefficient is that of the second term of eq. (C12) in Appendix C, where the Virasoro Ward identities are systematically solved for the first few descendants. Since \(J\) is a linear combination of the dimension \(\frac{6}{5}\) fields mentioned before, and requiring that the \(JJ\) OPE close back on \(J\), and on the identity with coefficient unity, it is easy to see that \(J\) is uniquely fixed to be

\[J(z) = \frac{1}{\sqrt{2}}\beta_{33}(z) + \frac{1}{\sqrt{2}}\beta_{22}(z)\partial\varphi(z).\]  

(5.5)

(Note that \(\beta_{22}'\) does not, in the end, contribute to \(J\).)

By keeping track of fields with dimensions other than zero and \(\frac{6}{5}\) (which correspond to the identity and \(J\), respectively) in the \(JJ\) OPE, we can calculate any other currents that couple in the FSCA. We find that that the \(JJ\) OPE closes on additional fields with dimensions \(\frac{3}{5}\) (mod 1). Since the \(\beta_{33}\beta_{33}\) and \(\beta_{22}\beta_{22}\) OPEs both close on the fields \(\beta_{13}\) and \(\beta_{31}\), with dimensions \(\frac{3}{5}\), we might expect that there are additional currents in the FSCA of dimension \(\frac{3}{5}\). In fact, the \(\frac{3}{5}\) fields all cancel among themselves as do the \(\frac{8}{5}\) fields so that the new FSCA currents have dimensions \(\frac{13}{5}\). It turns out that there are two of them (they could be combined and written as one, but they naturally split) which we call \(H_1\) and \(H_2\). Performing the calculation using the techniques for computing Virasoro descendants outlined
in Appendix C we find the following form for $H_1$:

$$H_1(z) = \sqrt{\frac{91}{120}} \left[ -\frac{75}{142} \partial^2 \beta_{13}(z) + \frac{55}{71}(L_{-2}\beta_{13})(z) + \frac{6}{71}\beta'_1(z) \
- \beta_{13}(z)T_\varphi(z) - \sqrt{\frac{3}{7}}\beta_{42}(z)\partial_\varphi(z) \right],$$

(5.6)

where the field $\beta'_1$, primary with respect to $T_{Z8}$, is defined by

$$\beta'_1(z) = -\frac{15}{26}\partial^2 \Phi_{1,3}^{(2)}(z) + \frac{11}{13}(L_{-2}\Phi_{1,3}^{(2)})(z) - \frac{5}{7}T_1(z)\Phi_{1,3}^{(2)}(z).$$

(5.7)

The notation for descendant fields is context dependent. In particular, the $L_{-2}$ in (5.6) refers to the mode of the TCI$^2$ energy-momentum tensor $T_{Z8} = T_1 + T_2$, while the $L_{-2}$ in (5.7) refers to the mode of the energy-momentum tensor $T_2$ of the second TCI factor. Thus, for example, $(L_{-2}\beta_{13}) = T_1\Phi_{1,3}^{(2)} + (L_{-2}\Phi_{1,3}^{(2)}).$ The form of $H_2$ is found by making the substitutions $\beta_{13} \rightarrow \beta_{31}$ and $\beta_{42} \rightarrow \beta_{24}$ in (5.6) and (5.7). The normalization factor $\sqrt{\frac{91}{120}}$ is included to make the $H_1H_1$ OPE close on the identity with coefficient unity.

We can now write down the $\{T, J, H_\ell\}$ algebra for $c = \frac{12}{5}$. Performing some lengthy calculations relying heavily on Appendix C, we find the operator product algebra

$$JJ = 1 + s^2\sqrt{2}J + s\sqrt{\frac{120}{91}}(H_1 + H_2),$$

$$JH_1 = s\sqrt{\frac{120}{91}}J + \frac{13}{7\sqrt{2}}H_2,$$

$$JH_2 = s\sqrt{\frac{120}{91}}J + \frac{13}{7\sqrt{2}}H_1,$$

$$H_1H_1 = 1 - s\sqrt{\frac{10}{273}}H_1,$$

$$H_2H_2 = 1 - s\sqrt{\frac{10}{273}}H_2,$$

$$H_1H_2 = \frac{13}{7\sqrt{2}}J.$$

(5.8)
where recall that $s = \sqrt{\frac{2}{3}} r x$. In (5.8) we have suppressed the $z$ and $w$ dependences (which can easily be restored) as well as the Virasoro descendant fields.

There is an important point to make about descendant fields on the right hand side of the OPEs (5.8). In addition to all the Virasoro descendants, there will be, in general, an infinite number of Virasoro primaries. Every new Virasoro primary entering on the right hand side of (5.8) will have dimensions $1, \frac{6}{5}$ or $\frac{13}{5} (\text{mod 1})$. Since they add no new cuts to the FSCA OPEs, they can all be considered to be current algebra descendant fields. We will refer to such a field whose dimension differs from that of the current $J$ by an integer $n$, as a level-$n$ current algebra descendant of $J$. Identical definitions apply to the currents $1$ and $H_{\ell}$.

For instance, consider the $J J \sim 1 + \cdots$ OPE. We find explicitly that

$$J(z) J(0) = z^{-\frac{12}{5}} \times$$

$$\left\{ 1 + z^2 T(0) + \frac{z^3}{2} \partial T(0) + \frac{z^4}{34} \{7L_{-2} T + 3 \partial^2 T\} (0) + \cdots \right\} + \frac{3z^4}{119N} \left[ G(0) + \frac{z}{2} \partial G(0) + \cdots \right] + \cdots ,$$

(5.9)

where $G$ is a dimension 4 Virasoro primary of the following form,

$$G = N \left[ 17 \beta_{44} \partial \phi - \frac{5}{2} (L_{-2} T_1 + L_{-2} T_2 + 2T_1 T_\varphi + 2T_2 T_\varphi) + \frac{35}{54} L_{-2} T_\varphi - \frac{7}{36} \partial^2 T_\varphi + \frac{305}{7} T_1 T_2 + \frac{3}{4} (\partial^2 T_1 + \partial^2 T_2) \right].$$

(5.10)

Thus $G$ is a level-4 current algebra descendant of the identity. This field $G$ also enters the $H_1 H_1$ OPE:

$$H_1 H_1 = 1 + \cdots + \frac{39}{2380N} G + \cdots ,$$

(5.11)

and similarly for $H_2 H_2$. The point is that any given current algebra descendant, such as $G$, may or may not appear in a given representation of the $\{T, J, H_\ell\}$ algebra. One could decide to include $G$ among the defining currents of the FSCA,
with the effect of reducing the number of representations of the algebra. On the other hand, the currents $H_1$ and $H_2$ must be included in the definition of the chiral algebra because they are associated with new cuts in the FSCA OPEs.

6. The \{\(T, J, H_\ell\)\} algebra at arbitrary central charge

In this section we extend the construction of the \(\{T, J, H_\ell\}\) algebra to general central charge by adding a background charge to the \(\varphi\) boson. The boson energy-momentum tensor is then

\[
T_\varphi(z) = \frac{1}{2} : \partial \varphi(z) \partial \varphi(z) : - \alpha_0 \partial^2 \varphi(z). \tag{6.1}
\]

With this normalization, the central charge for the boson then becomes,

\[
c_\varphi = 1 - 12\alpha_0^2, \tag{6.2}
\]

and the total central charge for the boson plus TCI\(^2\) theory is

\[
c = \frac{12}{5} - 12\alpha_0^2. \tag{6.3}
\]

\(\partial \varphi\) is no longer a primary field so its OPEs with itself and \(T_\varphi\) are more complicated than in (5.2). The OPEs with the background charge \(\alpha_0\) turned on are

\[
T_\varphi(z)T_\varphi(0) = \frac{1 - 12\alpha_0^2}{2} z^{-4} + 2z^{-2}T_\varphi(0) + \cdots
\]
\[
T_\varphi(z)\partial \varphi(0) = 2\alpha_0 z^{-3} + z^{-2}\partial \varphi(0) + z^{-1}\partial^2 \varphi(0) + \cdots \tag{6.4}
\]
\[
\partial \varphi(z)T_\varphi(0) = -2\alpha_0 z^{-3} + z^{-2}\partial \varphi(0) + 0 + \cdots
\]
\[
\partial \varphi(z)\partial \varphi(0) = z^{-2} + 2T_\varphi(0) + 2\alpha_0 \partial^2 \varphi(0) + \cdots .
\]

Now we are in a position to construct the \(J, H_1\) and \(H_2\) currents for a general central charge \(c\).
We start with $J$ and notice right away that as written in (5.5) it is no longer primary because $\beta_{22} \partial \varphi$ is not. However, using the OPEs (6.4), it is found that the following combination is primary

$$\beta_{22}(z) \partial \varphi(z) - 5\alpha_0 \partial \beta_{22}(z).$$

(6.5)

Demanding the same conditions that we did when constructing $J$ for $c = \frac{12}{5}$, namely, that the $JJ$ OPE close on the identity with coefficient unity and that it close on itself without any new dimension $\frac{6}{5}$ fields entering, the form of $J$ for general $c$ is then fixed to be

$$J(z) = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1 - 5 \alpha_0^2}} \left[ \beta_{22}(z) \partial \varphi(z) - 5\alpha_0 \partial \beta_{22}(z) \right] + \frac{1}{\sqrt{2}} \sqrt{\frac{1 + 4 \alpha_0^2}{1 - 5 \alpha_0^2}} \beta_{33}(z).$$

(6.6)

To discover the form for $H_1$ and $H_2$ we calculate the OPE of $J$ with itself and pick out all dimension $\frac{13}{5}$ pieces. The fact that the $\frac{3}{5}$ and $\frac{8}{5}$ pieces cancel among themselves and drop out is certainly not obvious, but explicit calculation demonstrates that this is indeed the case. We find

$$H_1 = N_H \left[ -\frac{75}{142} \left( 1 + \frac{49}{3} \alpha_0^2 \right) \partial^2 \beta_{13} + \frac{6}{11} \left( 1 - 31 \alpha_0^2 \right) \beta'_{13} \\
- \beta_{13} T_\varphi - \frac{3}{2} \alpha_0 \beta_{13} \partial^2 \varphi + \frac{5}{2} \alpha_0 \partial \beta_{13} \partial \varphi \\
+ \frac{55}{71} \left( 1 + \frac{156}{11} \alpha_0^2 \right) L_{-2} \beta_{13} \\
+ \sqrt{\frac{3(1 + 4 \alpha_0^2)}{7}} \left( -\beta_{42} \partial \varphi + \frac{5}{8} \alpha_0 \partial \beta_{42} + \frac{7}{8} \alpha_0 \beta'_{42} \right) \right],$$

(6.7)

where $\beta'_{42}$ is the Virasoro primary field

$$\beta'_{42}(z) = \partial \Phi^{(1)}_{1,4}(z) \Phi^{(2)}_{1,2}(z) - 15 \Phi^{(1)}_{1,4}(z) \partial \Phi^{(2)}_{1,2}(z).$$

(6.8)

As before, $H_2$ is obtained by substituting $\beta_{13} \to \beta_{31}$ and $\beta_{42} \to \beta_{24}$ in the above formulas. The normalization constant $N_H$ is determined by calculating the $H_1 H_1$.
OPE and normalizing it to close on the identity with coefficient unity. Doing this gives,

\[ N_H = \sqrt{\frac{91}{120 (1 - 5\alpha_0^2)(1 - \frac{33}{4}\alpha_0^2)}}. \]  

(6.9)

The result for the complete OPEs for \( J, H_1 \) and \( H_2 \) can now be presented (again suppressing the \( z \) and \( w \) dependences)

\[ JJ = 1 + s \sqrt{\frac{120}{91}} \sqrt{\frac{1 - \frac{33}{4}\alpha_0^2}{1 - 5\alpha_0^2}} (H_1 + H_2) + s^2 \sqrt{2} \sqrt{\frac{1 + 4\alpha_0^2}{1 - 5\alpha_0^2}} J, \]

\[ JH_1 = s \sqrt{\frac{120}{91}} \sqrt{\frac{1 - \frac{33}{4}\alpha_0^2}{1 - 5\alpha_0^2}} J + \frac{13}{7\sqrt{2}} \sqrt{\frac{1 + 4\alpha_0^2}{1 - 5\alpha_0^2}} H_2, \]

\[ JH_2 = s \sqrt{\frac{120}{91}} \sqrt{\frac{1 - \frac{33}{4}\alpha_0^2}{1 - 5\alpha_0^2}} J + \frac{13}{7\sqrt{2}} \sqrt{\frac{1 + 4\alpha_0^2}{1 - 5\alpha_0^2}} H_1, \]

\[ (6.10) \]

\[ H_1H_1 = 1 - s \sqrt{\frac{10}{273}} \frac{1 - \frac{510}{7}\alpha_0^2}{\sqrt{(1 - 5\alpha_0^2)(1 - \frac{33}{4}\alpha_0^2)}} H_1, \]

\[ H_2H_2 = 1 - s \sqrt{\frac{10}{273}} \frac{1 - \frac{510}{7}\alpha_0^2}{\sqrt{(1 - 5\alpha_0^2)(1 - \frac{33}{4}\alpha_0^2)}} H_2, \]

\[ H_1H_2 = \frac{13}{7\sqrt{2}} \sqrt{\frac{1 + 4\alpha_0^2}{1 - 5\alpha_0^2}} J. \]

This is the \( \{T, J, H_\ell\} \) FSCA algebra presented in the introduction, where (6.3) was used to write the structure constants in terms of the central charge instead of the background charge.

By constructing the FSCA OPEs with arbitrary background charge we have derived the dependence of the FSCA structure constants on the central charge.
However, we have not shown that the \( \{T, J, H_\ell\} \) operator algebra possesses an interesting representation theory. In particular, the representation in which we constructed it, namely \( \text{TCI}^2 \otimes \{\varphi, \alpha_0\} \) (the second factor representing the boson with background charge), is generically a nonunitary theory. To show that the \( \{T, J, H_\ell\} \) FSCA actually has unitary representations for \( c \neq \frac{12}{5} \), we follow the Feigin-Fuchs approach \([13,17]\) of constructing a projection of the \( \text{TCI}^2 \otimes \{\varphi, \alpha_0\} \) Fock space to a unitary subspace. The crucial step in this procedure is the construction of “screening charges,” dimension zero operators which commute with all the currents of the algebra. The screening charges \( Q_\pm \) can be written as line integrals of dimension 1 currents \( V_\pm(z) \)

\[
Q_\pm = \oint \frac{dz}{2\pi i} V_\pm(z) .
\]  

(6.11)

The condition that the screening charges commute with the FSCA currents is equivalent to the condition that the screening currents \( V_\pm(z) \) have zero (or a total derivative) as the residue of the single pole term in their OPEs with the currents.

Using the \( \mathcal{F}1 \) \( \text{TCI}^2 \) OPEs, we can construct such screening currents for the \( \{T, J, H_\ell\} \) FSCA. They are found to be of the form

\[
V_+(z) = \lambda_{11}(z)S_+(z), \quad V_-(z) = \bar{\lambda}_{11}(z)S_-(z),
\]

(6.12)

where \( S_\pm(z) \) are the dimension \( \frac{1}{8} \) fields, primary with respect to \( T_\varphi(z) \) (6.1),

\[
S_\pm(z) = e^{\alpha_\pm \varphi(z)}, \quad \alpha_\pm = -\alpha_0 \pm \sqrt{\alpha_0^2 + \frac{1}{4}},
\]

(6.13)

so that \( V_\pm(z) \) are dimension 1 primary fields with respect to \( T(z) \). Note that these screening currents involve \( \lambda \) and \( \bar{\lambda} \) fields, which do not appear in the \( \mathcal{P}1 \) \( \text{TCI}^2 \) operator algebra, defined in Sect. 4. Thus, even though the \( \{T, J, H_\ell\} \) algebra at arbitrary background charge (6.10) could be constructed in the \( \mathcal{P}1 \) model, we find that its screening charges cannot.
To show that the associated screening charges do indeed commute with the FSCA currents, we have to evaluate the OPEs of $V_\pm(z)$ with those currents. Because $V_\pm(z)$ were constructed to be Virasoro primary fields of dimension one, they satisfy the OPEs

$$V_\pm(z)T(w) = \frac{V_\pm(w)}{(z-w)^2} + \text{regular terms},$$  \hspace{1cm} (6.14)

and thus $Q_\pm$ automatically commute with $T(z)$.

The $V_\pm$ OPEs with the other currents are conveniently written in terms of the following fields:

\begin{align*}
\frac{1}{3} : & \quad U_+(z) = \lambda_{22}(z)S_+(z), \\
\frac{2}{3} : & \quad W^{(1)}_+(z) = \lambda_{12}(z)S_+(z), \\
\frac{8}{5} : & \quad X^{(1)}_+(z) = \lambda'_{12}(z)S_+(z), \\
\frac{8}{5} : & \quad Y^{(1)}_+(z) = 5\partial\lambda_{12}(z)S_+(z) - 19\lambda_{12}(z)\partial S_+(z),
\end{align*}

(6.15)

where the fraction preceding each charge is its conformal dimension and we have defined the following field, primary with respect to $T_{Z8}(z)$,

$$\lambda'_{12}(z) = 3\partial\Phi^{(1)}_{2,1}(z)\Phi^{(2)}_{2,2}(z) - 35\Phi^{(1)}_{2,1}(z)\partial\Phi^{(2)}_{2,2}(z).$$  \hspace{1cm} (6.16)

We define more fields $W^{(2)}_+$, $X^{(2)}_+$ and $Y^{(2)}_+$ by making the substitution $\lambda_{12} \to \lambda_{21}$ in (6.15); we also define the fields with minus subscripts letting $S_+ \to S_-$ and $\lambda \to \bar{\lambda}$. Now, using Appendices B and C, we calculate the relevant OPEs to find

$$V_\pm(z)J(w) = \pm\sqrt{\frac{2}{1 - 5\alpha_0^2}} \frac{U_\pm(w)}{(z-w)^2} + \text{reg.}$$

(6.17)

$$V_\pm(z)H_\ell(w) = -i\sqrt{\frac{3}{4}} N_H \left[ -4\alpha_0\alpha_\pm \frac{W^{(\ell)}_\pm(w)}{(z-w)^3} + \frac{10}{3} \alpha_0\alpha_\pm \frac{\partial W^{(\ell)}_\pm(w)}{(z-w)^2} \right.$$

$$+ \frac{16}{399}(2 - 15\alpha_0\alpha_\pm) \frac{X^{(\ell)}_\pm(w)}{(z-w)^2} - \frac{2}{5i}(3 - 32\alpha_0\alpha_\pm) \frac{Y^{(\ell)}_\pm(w)}{(z-w)^2} \right] + \text{reg.}$$

Since the single pole term vanishes in these OPEs, we have confirmed that the screening charges $Q_\pm$ commute with $J$ and the $H_\ell$. In fact, from Appendix C, it is
easy to check that the level 2 Virasoro descendants of $W_\pm^{(\ell)}(z)$ and level 1 Virasoro descendants of $U_\pm(z)$, $X_\pm^{(\ell)}(z)$ and $Y_\pm^{(\ell)}(z)$ enter with coefficient zero in the above OPEs. The non-trivial thing to check is that there are no new Virasoro primaries of dimension $\frac{6}{5}$ (for the $V_\pm J$ OPEs) or $\frac{13}{5}$ (for the $V_\pm H_\ell$ OPEs) contributing single pole terms in (6.17).

Following from the properties of the screening charge, it is not hard to see that a BRST cohomology analysis of the type carried out in ref. [12] can be performed starting from the $\{T, J, H_\ell\}$ FSCA. In this way we can, in principle, construct a sequence of unitary representations of the $\{T, J, H_\ell\}$ FSCA at special values of the background charge. We will return to this point in the next section.

7. Relation between the various $K = 8$ FSCAs

We will now construct the FSCA based on the $\mathcal{F}_2$ (or $\mathcal{P}_2$) TCI$^2$ operator algebras. In other words, we simply repeat the steps carried out in Sects. 5 and 6 using the $\mathcal{F}_2$ OPEs instead of those of the $\mathcal{F}_1$ operator product algebra.

It turns out that $J^{(8)}$ has the identical form as $J$ defined in (5.5). However, because $J^{(8)}$ is constructed using the $\mathcal{F}_2$ model OPEs, we find that the $H_1$ and $H_2$ currents do not appear in the $J^{(8)} J^{(8)}$ OPE. In fact, by the row map $\mathcal{R}$ (4.10) which relates the $\mathcal{F}_1$ and $\mathcal{F}_2$ OPEs, the TCI$^2$ fields which entered into the definition of the $H_\ell$ currents (5.6) decouple from the $\mathcal{F}_2$ operator product algebra. In other words, $J^{(8)} J^{(8)}$ closes only on the identity and $J^{(8)}$. The background charge is turned on in the $\{T, J^{(8)}\}$ theory as in Sect. 6; the result for $J^{(8)}$ is the same expression (6.6). From (6.10), decoupling $H_1$ and $H_2$ and letting $s^2 \to \frac{2}{3} \frac{x^2}{\sqrt{r}}$ (which is the action of the $\mathcal{R}$ map), we find the $\langle J^{(8)} J^{(8)} J^{(8)} \rangle$ structure constant to be

$$
\lambda_8^2(c) = \frac{32}{45} \left( \frac{27}{5c} - 1 \right) \frac{x^2}{r},
$$

(7.1)
in agreement with (1.7) for $K = 8$, the result found in ref. [8]. The screening current for the $\{T, J^{(8)}\}$ algebra are the same as those found for the $\{T, J, H_\ell\}$
algebra (6.11)-(6.13), and the screening charge commutes with $J^{(8)}$ by virtue of a $V_\pm J$ OPE identical to that in (6.17).

In summary, we have constructed the currents, their structure constants and the screening charges for both $K = 8$ FSCAs. This was done for the general \{T, J^{(K)}\} FSCAs in ref. [8].

It is well-known that the existence of screening charges is very important. For example, the construction of the characters of the minimal series in the Feigin-Fuchs approach relies on the screening current of the Virasoro algebra [17,19]. The screening charges of the \{T, J^{(K)}\} algebra were used in a similar fashion to construct the characters of its representations [11,12]. The physical picture behind this construction can be described as follows. In general, the (true) Fock space of our model is generated by the repeated operations of the negative modes of the currents (the creation operators) on the primary states. In the absence of background charge, this is equivalent to the Fock space generated by the repeated operations of the negative modes of the PF currents and the boson field on the primary states. We will refer to this latter Fock space as the PF$\otimes\{\varphi, \alpha_0\}$ Fock space. As we turn on the background charge $\alpha_0$ of the boson, the PF$\otimes\{\varphi, \alpha_0\}$ Fock space contains states that are absent from the true Fock space; furthermore, some of the states in the true Fock space become null (i.e. descendant and primary) and must be removed. Thus, if we wish to start from the PF$\otimes\{\varphi, \alpha_0\}$ Fock space, both these sets of spurious states must be removed to obtain the correct Fock space. Because they commute with the currents of the algebra (by construction), the screening charges are the appropriate tools to perform this surgery.

To be more precise, we can construct a BRST operator $Q$ from the screening currents of the \{T, J^{(K)}\} FSCAs [12]. For a set of discrete values of the background charge $\alpha_0$ of the boson, we can construct the characters of the representations of the FSCA via the BRST cohomology. These characters turn out to be precisely the branching functions of the $SU(2)_K \otimes SU(2)_L/SU(2)_{K+L}$, or simply $[K,L]$. 
coset theory, where the central charge is given by

$$c = \frac{2(K - 1)}{K + 2} + 1 - 12\alpha_0^2 = \frac{3K}{K + 2} + \frac{3L}{L + 2} - \frac{3(K + L)}{K + L + 2}.$$  \hspace{1cm} (7.2)

Now let us turn our attention to the other algebra we have constructed for the $K = 8$ case, namely the $\{T, J, H_\ell\}$ FSCA. Since this algebra has the same screening currents as the $\{T, J^{(8)}\}$ FSCA, we expect to obtain the same $[8, L]$ coset theories as representations, i.e. their branching functions are also representations of the $\{T, J, H_\ell\}$ FSCA for the central charges given in (7.2). However, there are a couple of important differences between the representation theories of these two $K = 8$ FSCAs.

First, the size of the branching function representations (reflecting the field content of the representations) of these two FSCAs are different. In general, the (holomorphic part of the) primary fields of the extended conformal algebra have the form

$$V_{j,p}(z) = \phi_j^p(z)e^{ip\varphi(z)},$$  \hspace{1cm} (7.3)

where $\phi_j^p$ is a PF field, and the momentum $p$ of the boson belongs to a well-defined set of discrete momenta [11,12], whose precise values do not concern us here. As we have seen in earlier sections, some of the PF fields decouple from the current $J^{(8)}(z)$. This means that some of the branching functions that are present in the coset theory are actually missing from the representations of the $\{T, J^{(8)}\}$ FSCA. To be specific, the $\phi_2^2$ PF field (which corresponds to $\beta_{12}$ and $\beta_{21}$ in the TCI$^2$ notation) decouples from $J^{(8)}$. This means that the character (and the associated branching functions) corresponding to the primary fields $V_{2,p}$ are not representations of the $\{T, J^{(8)}\}$ FSCA. To generate them we must use the $\{T, J, H_\ell\}$ FSCA. This is true generically for other values of $K$ besides $K = 8$. In particular, for $K > 4$, we expect the existence of FSCAs other than the simplest $\{T, J^{(K)}\}$ FSCAs constructed in ref. [8].
The second point concerns the range of central charges for which we can find unitary coset representations of two FSCAs corresponding to the same $K$. Since $J^{(K)}$, when written in terms of PF fields, contains the $\phi^1_0$ field which satisfies the fusion rule [see (2.4)]
\begin{equation}
\phi^1_0 \times \phi^1_0 \sim 1 + \phi^1_0 + \phi^2_0 ,
\end{equation}
we expect generically that the OPE of $J^{(K)}$ with itself will generate a new current $H$ involving the dimension $6/(K + 2) \phi^2_0$ field. In fact, the current $H$ can be shown [8] to involve level 2 PF descendants of $\phi^2_0$, and so in general has conformal dimension $2 + 6/(K + 2)$. Note that for $K = 8$, this is just the value, $\frac{13}{5}$, that we found above for the dimension of the $H_\ell$ currents of the $\{T, J, H_\ell\}$ algebra. On the other hand, the $[K, L]$ coset models, which should form representations of the FSCA algebras, correspond for $L = 1$ to the minimal unitary series, with central charge $c_{L=1} = 1 - 6/(K + 2)(K + 3)$. Now, the current $J^{(K)}(z)$ operating on the identity generates a state with conformal dimension $(K + 4)/(K + 2)$. Since the unitary minimal model always contains the primary field $\Phi_{3,1}(z)$ with conformal dimension $(K + 4)/(K + 2)$, this field is identified with the current $J^{(K)}$ in the $[K, 1]$ coset models. The $\Phi_{3,1}\Phi_{3,1}$ OPE is (for $K \geq 4$) [1]
\begin{equation}
\Phi_{3,1}\Phi_{3,1} \sim 1 + \lambda \Phi_{3,1} + \mu \Phi_{5,1} ,
\end{equation}
where $\Phi_{5,1}$ is the dimension $4 + 6/(K + 2)$ primary field, and $\lambda$ and $\mu$ are structure constants that have to be determined by associativity. Since the dimension of $\Phi_{5,1}$ is larger than that of the $H_\ell$ currents by two units and there is no other field with the dimension of the $H_\ell$ in this minimal model, an algebra involving the $H_\ell$ cannot be represented by this minimal model.

The $\{T, J^{(K)}\}$ algebras avoid this problem, of course, by decoupling the $H$ currents, as we have seen explicitly above in the $K = 8$ case. This corresponds to an associativity solution in which $\mu = 0$ in (7.5). This problem appears to be present, though, if we try to apply the $\{T, J, H_\ell\}$ algebra to the $[8, 1]$ coset model,
which is identified as the eighth member of the unitary minimal series with central charge \(c_{L=1} = \frac{52}{55}\) and which does not have a primary field of dimension \(\frac{13}{5}\). It turns out that in this case the problem is resolved again by the decoupling of the \(H_\ell\) currents from the \(\{T, J, H_\ell\}\) FSCA. Indeed, precisely at \(c = c_{L=1}\) the \(\{T, J, H_\ell\}\) algebra structure constant \(\Omega(c)\) vanishes (1.10), so that the \(H_\ell\) decouple in the \(J(z)J(w)\) OPE. More to the point, the \(\Upsilon(c)\) structure constant (1.11) diverges at this special value of the central charge, showing that the \(H_\ell\) currents must decouple from the algebra as a whole. That is, to avoid the divergence we must renormalize \(H_\ell\) resulting in their having zero norm and hence their being null states in this representation.

Note that the resulting reduced \(\{T, J\}\) algebra at \(c = c_{L=1}\) is still different from the \(\{T, J^{(8)}\}\) FSCA; in particular they have different structure constants for coupling three \(J\) currents [corresponding to the \(\lambda\) structure constant in (7.5)]. Since \(\lambda\) for the reduced \(\{T, J\}\) algebra is different from the value it takes in the \(\{T, J^{(8)}\}\) algebra, by associativity of the \(\Phi_{3,1}\) four-point function, \(\mu\) in eq. (7.5) must be different for the two algebras also. In particular, \(\mu\) will be non-zero for the reduced \(\{T, J\}\) algebra, and thus the dimension \(\frac{23}{5}\) \(\Phi_{5,1}\) field will couple to these currents. This new current actually enters with multiplicity two, and will be denoted \(H'_1(z)\) and \(H'_2(z)\). For \(c > \frac{52}{55}\) the \(H'_\ell\) are level-2 current algebra descendants of the \(H_\ell\), but at \(c = \frac{52}{55}\) the \(H_\ell\) are null so that the \(H'_\ell\) are the FSCA primary currents.

Using the TCI\(^2\otimes\{\varphi, \alpha_0\}\) representation for the \(K = 8\) FSCA developed in this paper, the form for \(H'_\ell\) could be constructed and the structure constants analogous to (1.10)-(1.11) for this \(\{T, J, H'_\ell\}\) FSCA could, in principle, be calculated. In practice, however, this would require a significant expansion of the calculation of Virasoro descendants given in Appendix C. Fortunately, we can use the methods of ref. [13] to calculate directly in the \(c = \frac{52}{55}\) unitary model. An additional advantage is that we can compare the \(\langle JJJ\rangle\) structure constant found from either method and check for consistency of our whole picture.

To start with, the fusion rules for the unitary model force us to consider, in
addition to the fields $\Phi_{1,1} \equiv 1$, $\Phi_{3,1}$ and $\Phi_{5,1}$, the fields $\Phi_{7,1}$ and $\Phi_{9,1}$. These additional fields do not destroy the basic $K = 8$ FSCA, because modulo 1 the $\Phi_{7,1}$ and $\Phi_{9,1}$ fields have the same dimension as $\Phi_{3,1}$ and $\Phi_{1,1}$, respectively. We make the following definitions

\[
\begin{align*}
1 & \sim \Phi_{1,1} \quad \Delta = 0, \\
J & \sim \Phi_{3,1} \quad \Delta = \frac{6}{5}, \\
H'_1, \quad H'_2 & \sim \Phi_{5,1} \quad \Delta = \frac{23}{5}, \\
J' & \sim \Phi_{7,1} \quad \Delta = \frac{51}{5}, \\
I' & \sim \Phi_{9,1} \quad \Delta = 18,
\end{align*}
\]

(7.6)

where the conformal dimensions, $\Delta$, of these fields appear in the right hand column. Therefore, we see that we can interpret $J'$ as a level-9 current algebra descendant of $J$, and $I'$ as a level-18 current algebra descendant of the identity.

Calculating directly in the $c = \frac{52}{55}$ unitary model we can construct the $\{T, J, H'_\ell\}$ FSCA for this central charge, analogous to (1.9)-(1.11), and we find explicitly that

\[
\begin{align*}
J \ J &= 1 + r x f_{333} J + \sqrt{r x} f_{335} H'_1 + \sqrt{r x} f_{335} H'_2 \\
J \ H'_1 &= \sqrt{r x} f_{335} \left[ J + \frac{f_{357}}{f_{335}} J' \right] + f_{355} H'_2 \\
J \ H'_2 &= \sqrt{r x} f_{335} \left[ J - \frac{f_{357}}{f_{335}} J' \right] + f_{355} H'_1 \\
H'_1 \ H'_1 &= [1 + f_{559} J'] + \sqrt{r x} f_{555} H'_1 \\
H'_2 \ H'_2 &= [1 - f_{559} J'] + \sqrt{r x} f_{555} H'_2 \\
H'_1 \ H'_2 &= f_{355} \left[ J + \frac{f_{557}}{f_{355}} J' \right]
\end{align*}
\]

(7.7)

where the $f_{ijk}$ obey the structure constant symmetry properties (3.3) and (3.5),
and are given by

\[
\begin{align*}
    f_{333} &= \frac{2 \cdot 7}{3} \sqrt{\frac{2}{13}}, \\
    f_{335} &= \frac{2^2 \cdot 5 \cdot 7}{13} \sqrt{\frac{2}{13}}, \\
    f_{355} &= \frac{3 \cdot 23}{7 \cdot 17} \sqrt{2 \cdot 13}, \\
    f_{555} &= -\frac{3^4 \cdot 19 \cdot 37 \cdot 47}{5^2 \cdot 7 \cdot 13} \sqrt{\frac{2}{5 \cdot 7 \cdot 17 \cdot 23}}.
\end{align*}
\]

(7.8)

Now we can make the non-trivial check that from eq. (7.8) and eq. (1.10) we have

\[
s^2 \Lambda(c = \frac{52}{55}) = rxf_{333},
\]

(7.9)

so that the \(\{T, J, H'\}\) FSCA is indeed consistent with the \(\{T, J, H\}\) one.

In the associativity constraints of the \(c = \frac{52}{55}\) unitary model, the fields \(J'\) and \(I'\) are not automatically included in the current blocks of \(J\) and \(I\), respectively, as they are in the \(\text{TCI}^2 \otimes \{\varphi, \alpha_0\}\) representation. Therefore, they must be included as separate fields and their associativity constraints checked also. We find the unique associative solution

\[
\begin{align*}
    I' I' &= 1, \\
    I' J &= +f_{379}J', \\
    I' J' &= -f_{379}J, \\
    I' H'_1 &= +f_{559}H'_1, \\
    I' H'_2 &= -f_{559}H'_2, \\
    J J' &= f_{379}I' + rxf_{377}J' - \sqrt{rxf_{3757}}H'_1 + \sqrt{rxf_{3757}}H'_2, \\
    J' J' &= 1 + rxf_{377}J + \sqrt{rxf_{577}}H'_1 + \sqrt{rxf_{577}}H'_2, \\
    J' H'_1 &= -\sqrt{rxf_{3757}}J + \sqrt{rxf_{577}}J' + f_{557}H'_2, \\
    J' H'_2 &= +\sqrt{rxf_{3757}}J + \sqrt{rxf_{577}}J' - f_{557}H'_1.
\end{align*}
\]

(7.10)
where the new structure constants are given by

\[
\begin{align*}
    f_{357} &= -i \frac{1}{53} \sqrt{\frac{3 \cdot 17 \cdot 19 \cdot 37 \cdot 47}{2 \cdot 7 \cdot 13}} \\
    f_{377} &= -\frac{17 \cdot 19 \cdot 29}{3 \cdot 13 \cdot 23} \sqrt{\frac{2}{13}} \\
    f_{379} &= i \sqrt{\frac{2 \cdot 7 \cdot 23 \cdot 29 \cdot 59 \cdot 79}{3 \cdot 13 \cdot 17 \cdot 37 \cdot 47 \cdot 67}} \\
    f_{557} &= -i \frac{3 \cdot 19 \cdot 29}{5^5 \cdot 7^2} \sqrt{\frac{2 \cdot 3 \cdot 19 \cdot 23 \cdot 37 \cdot 47}{5}} \\
    f_{559} &= \frac{23}{5^7} \sqrt{\frac{17 \cdot 19 \cdot 29 \cdot 59 \cdot 79}{2 \cdot 5 \cdot 7 \cdot 67}} \\
    f_{577} &= \frac{19^2 \cdot 29^2 \cdot 59 \cdot 79}{13 \cdot 5^8} \sqrt{\frac{17}{2 \cdot 5 \cdot 7 \cdot 23}}.
\end{align*}
\]

(7.11)

Note that the \( f_{557} \) structure constant has one of its indices barred to remind the reader that this structure constant actually couples the distinct fields \( H'_1 \) and \( H'_2 \). Thus, by the structure constant symmetry (3.5), \( f_{557} = -f_{557} \); this distinction between \( H'_1 \) and \( H'_2 \) does not have to be made in any other structure constant since the relevant ones are all real.

Since the \( \Omega \) and \( \Upsilon \) structure constants of the \( \{T, J, H_\ell\} \) FSCA are both finite at other values of the central charge, we should not encounter a similar decoupling of the \( H_1 \) and \( H_2 \) in any other representation of the \( K = 8 \) \( \{T, J, H_\ell\} \) FSCA. Indeed, it is easy to check that the \( K = 8, L \geq 2 \) coset models all have primary fields with conformal dimension \( \frac{13}{5} \).

The Feigin-Fuchs framework gives a plausible explanation of this special behavior of the FSCA representations for low-\( L \) cosets. The BRST cohomology argument is based on turning on the background charge of the boson while leaving the PF part of the theory untouched. As the background charge increases, the effective central charge of the boson decreases. The BRST cohomology can be viewed as a reduction of the size of the Fock space to suit the reduced central charge. However, conservatively, we may want to avoid having to reduce the central charge of the
boson to less than zero, since in that case we will have to reduce the size of the Fock space of the PF model itself. For the $K = 8$ case, i.e. for the $[8, L]$ coset models, we see that the boson has zero central charge precisely when $L = 2$, and it is easy to check that the primary field with conformal dimension $\frac{13}{5}$ is still present in the $L = 2$ coset model. However, the $L = 1$ model (the minimal model), whose boson has effective central charge $c_\varphi = -\frac{5}{11}$, no longer has a dimension $\frac{13}{5}$ field.

8. Application to the $K = 8$ Fractional Superstring

In this section, we turn to the application of the $K = 8$ FSCAs to fractional superstrings. Briefly, the $K = 8$ fractional superstring [18] propagates in four-dimensional Minkowski space-time, and has a supersymmetric particle spectrum. On the (two-dimensional) string world-sheet, the fractional superstring is built from four copies of the $Z_8$ PF theories and four coordinate bosons (denoted $X^\mu$). Thus, the CFT underlying this string is a four-fold tensor product of the $c = \frac{12}{5}$ theory discussed in Sect. 5, with a Minkowski metric. In particular, the free boson fields (with no background charge) of each of the four $c = \frac{12}{5}$ theories, which we called $\varphi$, are to be identified with the coordinate bosons $X^\mu$ ($\mu = 0, 1, 2, 3$) of the string theory, whose radii are infinite. The connection between the world-sheet and space-time field content of the $K = 8$ fractional superstring is made by way of the $K = 8$ FSCA. Basically, the FSCA currents generate the physical state conditions on the string Fock space. This means that the non-negative modes of the currents annihilate the physical states of the fractional superstring. More detailed discussions of fractional superstring theories can be found in refs. [18,21].

We have constructed in this paper two inequivalent FSCAs at $K = 8$. It is natural to ask which is the correct one to describe the $K = 8$ fractional superstring. We will argue below, by an examination of the physical state conditions for the massless fermion states of the fractional superstring, that the $\{T, J, H_\ell\}$ FSCA is the relevant algebra.
In the $K = 8$ fractional superstring the space-time fermion states come from the $\phi_{\pm 2}^2$ PF fields on the world-sheet [18], which are the $\beta_{12}$ and $\beta_{21}$ fields in our TCI$^2$ notation. To be more precise, the massless fermion state in four space-time dimensions is given by

$$|\Psi\rangle = \left(\prod_{\mu=0}^{3} \sigma^\mu(z)\right) e^{ip^\mu X(z)}|0\rangle,$$ (8.1)

where $\sigma^\mu(z)$ stands for either $\beta_{12}^\mu(z)$ or $\beta_{21}^\mu(z)$. The $\mu$ index reflects the fact that we are tensoring together four copies of the $Z_8$ PF theory as well as the coordinate bosons $X^\mu(z)$ to obtain a Minkowski space-time interpretation. Thus $p^\mu$ is the Minkowski space-time momentum. The only non-trivial physical state conditions for the state (8.1) (i.e., the only non-negative modes of the FSCA currents that do not identically annihilate that state) are $L_0$ and $J_0$. Here $L_0$ is the zero mode of the total energy-momentum tensor; the effect of its physical state condition is simply to show that (8.1) is massless. $J_0$ refers to either the zero mode of the (total) $J^{(8)}$ current of the $\{T, J^{(8)}\}$ FSCA, or the zero mode of the (total) $J$ current of the $\{T, J, H_\ell\}$ FSCA.

Since, by the constructions of Sects. 5 and 7, both $J$ currents have the same form, namely,

$$J(z) = \frac{1}{\sqrt{2}} \sum_{\mu=0}^{3} \left( \frac{\beta_{22}^\mu(z)}{2} \partial X_\mu(z) + (\beta_{33}^\mu)^\mu(z) \right)$$ (8.2)

(note that the total current for the tensor product theory is the sum of the currents for each factor), it might seem that there will be no difference in their action on the state (8.1). However, in the $\{T, J^{(8)}\}$ FSCA we learned that the $\beta_{12}$ and $\beta_{21}$ fields actually decouple from the fractional supercurrent $J^{(8)}$. This means that there is no $J_0$ physical state condition on the state (8.1). Since the Dirac equation for the massless fermion state $|\Psi\rangle$ should come from the $J_0$ physical state condition (and there is no other condition it could come from), it is clear that the $\{T, J^{(8)}\}$

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FSCA cannot be the appropriate worldsheet symmetry for the $K = 8$ fractional superstring.

Let us see the consequences of the $J_0$ physical state condition

$$J_0|\Psi\rangle = 0 ,$$

(8.3)

when $J$ is the $\{T, J, H_\ell\}$ FSCA current. In this case, by the $F1$ TCI\(^2\) operator product algebra, the $\beta_{12}$ and $\beta_{21}$ fields couple to the $\beta_{22}$ field. Following from (4.13) we have the OPEs

$$\beta_{22}(z)\beta_{12}(w) = +i(z-w)^{-\frac{1}{5}}\beta_{21}(w) - is(z-w)^{\frac{2}{5}}\beta_{23}(w),$$

$$\beta_{22}(z)\beta_{21}(w) = -i(z-w)^{-\frac{1}{5}}\beta_{12}(w) + is(z-w)^{\frac{2}{5}}\beta_{32}(w).$$

(8.4)

Let us consider the moding of $\beta_{22}$. Define

$$|\beta_{ij}\rangle = \lim_{z \to 0} \beta_{ij}(z)|0\rangle,$$

(8.5)

then, following from the OPEs (8.4) and the dimensions of the $\beta_{ij}$ fields, we find

$$\langle \beta_{22}\rangle_0|\beta_{12}\rangle = +i|\beta_{21}\rangle,$$

$$\langle \beta_{22}\rangle_0|\beta_{21}\rangle = -i|\beta_{12}\rangle,$$

$$\langle \beta_{22}\rangle_{-\frac{3}{5}}|\beta_{12}\rangle = -is|\beta_{23}\rangle,$$

$$\langle \beta_{22}\rangle_{-\frac{3}{5}}|\beta_{21}\rangle = +is|\beta_{32}\rangle.$$  \hspace{1cm} (8.6)

Thus $\langle \beta_{22}\rangle_0(\beta_{22})_0|\sigma\rangle = |\sigma\rangle$ for $|\sigma\rangle = |\beta_{12}\rangle$ or $|\beta_{21}\rangle$. If we now add back in the space-time index $\mu$, and demand that the $\beta_{22}$ fields corresponding to different space-time dimensions anticommute (which we can always do by the inclusion of appropriate Klein factors), we obtain the following anticommutation relations

$$\{\langle \beta_{22}\rangle_0^\mu, (\beta_{22})_0^\nu\}|\Psi\rangle = g^{\mu\nu}|\Psi\rangle .$$

(8.7)

This is just the Clifford algebra acting on $|\Psi\rangle$. Thus $|\Psi\rangle$ lies in a spinor represen-
ation of the four-dimensional Lorentz algebra, so that we can write

$$|\Psi \rangle = |\alpha, p \rangle u_\alpha(p) .$$  \hspace{1cm} (8.8)

Here $u_\alpha(p)$ is the Dirac spinor wave-function of the massless state.

Using the explicit form for $J(z)$ given in eq. (8.2) we can easily derive the equation of motion satisfied by the $u_\alpha(p)$ spinor wave-function from the $J_0$ physical state condition. Specifically, by virtue of (8.7), the $\beta_{22}$ zero modes can be identified with Dirac gamma matrices when acting on $|\Psi \rangle$:

$$(\beta_{22})^\mu_0 = \frac{i\gamma^\mu}{\sqrt{2}} .$$  \hspace{1cm} (8.9)

Since the $\beta_{33}$ term in $J_0$ automatically annihilates $|\Psi \rangle$ (from dimensional considerations), it follows from the physical state condition (8.3) that

$$J_0|\Psi \rangle = \frac{1}{\sqrt{2}} \sum_{\mu=0}^{3} (\beta_{22})^\mu_0 p_\mu |\Psi \rangle = \frac{i}{2} \slashed{p} |\Psi \rangle = 0 ,$$  \hspace{1cm} (8.10)

and thus $\slashed{p} u = 0$. This is the Dirac equation for a massless (space-time) fermion. This leads us to conclude that the $\{T, J, H_\ell\}$ FSCA is the correct worldsheet symmetry algebra for the $K = 8$ fractional superstring theory. Of course, the demonstration of the consistency of the $K = 8$ fractional superstring as a whole involves a highly non-trivial analysis that remains to be carried out.

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APPENDIX A

In this appendix we give a very brief review of the Feigin-Fuchs technique as applied by Dotsenko and Fateev \[13\] to the minimal models. Actually we will focus only on the unitary series. We then tabulate all the fusion matrices for the TCI model.

The basic idea is to represent a unitary model with $c = 1 - \frac{6}{p(p+1)}$, by a single free boson with background charge. The most important information necessary to construct associative algebras is the transformation properties of the conformal blocks for four-point functions. Within the bounds of the unitary series this can be done by solving the differential equations that the correlation functions satisfy. As these differential equations are constructed by using the null states in the models, for large $p$ the differential equations will become practically intractable. It turns out that since we are interested in only the second member of the unitary series, with $p = 4$, the differential equations technique \textit{can} be applied—in fact that is the method used by Qiu in ref. \[15\]. But even at this low a level the methods of Dotsenko and Fateev \[13\] are clearly easier. They map the minimal model primary fields onto exponentials of a free boson with background charge, thereby expressing correlation functions as multiple integrals. The normalizations for these integrals and some of their transformation properties under fusions were first computed by Dotsenko and Fateev; a general formula for the transformation matrices under fusion is derived in ref. \[8\].

Consider a general four-point function of primary fields in some level $p$ unitary model,

$$G = \langle \phi_i \phi_j \phi_k \phi_l \rangle.$$  \hfill (A.1)

(Usually, minimal model primaries are written $\Phi_{n,m}$, but we are merely being symbolic as we do not want to hide the ideas behind too much notational baggage.) Taking the normalization $\phi_i \phi_j = \delta_{i,j}$ the function $G$ can be expanded in the fol-
owing two ways,

\[
G_{i \to j, k \to l} = \sum_m c_{ijm} c_{klm} F_m, \\
G_{j \to k, l \to i} = \sum_n c_{jkn} c_{lin} F'_n,
\]  

(A.2)

where \( F_m \) and \( F'_n \) are the conformal blocks, properly normalized, and the \( c_{ijk} \) are the structure constants appearing in the primary field OPEs:

\[
\phi_i \phi_j = \sum_k c_{ijk} \phi_k.
\]  

(A.3)

Now using the normalization integrals of ref. [13] and the transformation matrices of ref. [8], it is straightforward to construct the “fusion matrix” relating the two different sets of conformal blocks in (A.2). We will always denote the fusion matrices by \( \alpha \). They satisfy the following matrix equation involving the structure constants:

\[
\sum_m \alpha_{n,m} c_{ijm} c_{klm} = c_{jkn} c_{lin}.
\]  

(A.4)

We refer to this equation as the associativity condition for the \( \langle \phi_i \phi_j \phi_k \phi_l \rangle \) four-point function. For any algebra (not just the minimal models), given the complete set of \( \alpha \) matrices, all associative algebras can be constructed.

We now present an exhaustive list of the \( \alpha \) matrices for the TCI model. For ease of notation we abbreviate the fields in the correlation functions by \( i \) and \( \hat{j} \), where

\[
i = \Phi_{1,i}, \quad \hat{j} = \Phi_{2,j}.
\]  

(A.5)

One appealing feature of this notation is that the field 1 is actually the identity. Now the following table is a complete list of the \( \alpha \) matrices for all four-point
functions with only one conformal block.

| 4-point | α | 4-point | α | 4-point | α | 4-point | α |
|---------|---|---------|---|---------|---|---------|---|
| ⟨2224⟩ | 1 | ⟨2211⟩ | 1/2 | ⟨2212⟩ | 1 | ⟨2422⟩ | 1/2 |
| ⟨2244⟩ | 3/7 | ⟨2121⟩ | −1 | ⟨2122⟩ | 1 | ⟨2242⟩ | −1 |
| ⟨2424⟩ | 1 | ⟨2311⟩ | 1 | ⟨2312⟩ | 3/7 | ⟨3422⟩ | 1/2 |
| ⟨2334⟩ | −1 | ⟨2131⟩ | 1 | ⟨3212⟩ | −1/2 | ⟨3242⟩ | 1 |
| ⟨2343⟩ | 1 | ⟨3311⟩ | 3/4 | ⟨2132⟩ | 3 | ⟨4422⟩ | 1/56 |
| ⟨3344⟩ | 3/7 | ⟨3131⟩ | 1 | ⟨2412⟩ | 7/6 | ⟨4242⟩ | −1 |
| ⟨3434⟩ | 1 | ⟨4411⟩ | 7/8 | ⟨4212⟩ | −1/6 |

Permuting the order of the fields in the four-point functions in (A.6) does not lead to new associativity constraints, by virtue of the symmetries of the structure constants (3.3) and (3.5), and by the form of the associativity constraint (A.4).

As an example of how to use the above table, consider the four-point function ⟨3412⟩. Since α = 7/4 for this correlation function we can write,

\[ \frac{7}{4} c_{342} c_{\hat{1}\hat{2}} = c_{41\hat{1}} c_{2\hat{3}\hat{1}}. \]  

(A.7)

Of course, we know that a single chiral TCI model has no consistent solution with the \( \hat{1} \) and \( \hat{2} \) fields present (see Sect. 3), but tensoring this four-point function with itself and then solving gives the following constraint

\[ \frac{49}{16} C^{(33)(44)(22)} c_{(ii)(\hat{2}\hat{2})} = c_{(44)(\hat{1}\hat{1})(ii)} c_{(\hat{2}\hat{2})(33)(ii)}. \]  

(A.8)

Identifying the \( ii \) indices with \( \beta \) fields and the \( \hat{j}\hat{j} \) indices with either \( \lambda \) or \( \bar{\lambda} \) fields (see Sect. 4) gives an associativity constraint on the structure constants of the TCI2
model. In Appendix B, where a complete solution to the associativity constraints of the TCI\(^2\) model involving the \(\lambda\) and \(\bar{\lambda}\) fields is given, one can check that (A.8) is indeed satisfied.

The following table summarizes all of the \(\alpha\) matrices for those four-point functions with exactly two conformal blocks.

| 4−point | \(\alpha\) | 4−point | \(\alpha\) | 4−point | \(\alpha\) |
|---------|---------|---------|---------|---------|---------|
| \(\langle 2222 \rangle\) | \(\begin{pmatrix} r & 3r/2x \\ 3x & -r \end{pmatrix}\) | \(\langle 2222 \rangle\) | \(\begin{pmatrix} 1/2r & -3r/2x \\ x & 3r \end{pmatrix}\) | \(\langle 111\bar{1} \rangle\) | \(\frac{1}{\sqrt{2}}\) \(\begin{pmatrix} 1 & 8/7 \\ 7/8 & -1 \end{pmatrix}\) |
| \(\langle 3333 \rangle\) | \(\begin{pmatrix} r & 3r/2x \\ 3x & -r \end{pmatrix}\) | \(\langle 2222 \rangle\) | \(\begin{pmatrix} -r & 1r/2x \\ 2x & r \end{pmatrix}\) | \(\langle 1\bar{1}2\bar{2} \rangle\) | \(\frac{1}{\sqrt{2}}\) \(\begin{pmatrix} 1/2 & 4 \\ 3/4 & -6 \end{pmatrix}\) |
| \(\langle 2233 \rangle\) | \(\begin{pmatrix} 2x & r \\ 3r & -9r/14x \end{pmatrix}\) | \(\langle 2322 \rangle\) | \(\begin{pmatrix} 3r/4x & -7r \\ 1/2r & 14/3x \end{pmatrix}\) | \(\langle 1\bar{2}\bar{1}2 \rangle\) | \(\frac{1}{\sqrt{2}}\) \(\begin{pmatrix} -1 & 2/3 \\ 3/2 & 1 \end{pmatrix}\) |
| \(\langle 2323 \rangle\) | \(\begin{pmatrix} -r & 14/9x \\ 9x/14 & r \end{pmatrix}\) | \(\langle 2\bar{2}22 \rangle\) | \(\begin{pmatrix} r & 3r/2x \\ 2/3x & -r \end{pmatrix}\) | \(\langle 1\bar{2}\bar{2}2 \rangle\) | \(\frac{1}{\sqrt{2}}\) \(\begin{pmatrix} 1 & 2 \\ 1/2 & -1 \end{pmatrix}\) |
| \(\langle 3\bar{3}2\bar{2} \rangle\) | | \(\langle 3\bar{2}3\bar{2} \rangle\) | \(\begin{pmatrix} 3r/4x & 9r/14x \\ 1/6x & -1/2r \end{pmatrix}\) | |
| \(\langle 3\bar{2}\bar{3}2 \rangle\) | | \(\langle 3\bar{2}\bar{2}2 \rangle\) | \(\begin{pmatrix} r & -9r/2x \\ -2/9x & -r \end{pmatrix}\) | |
| \(\langle 3\bar{2}\bar{2}\bar{2} \rangle\) | | | | |

(A.9)

A few words of explanation about this table are in order. We will explain the conventions used in (A.9) by way of an example. For instance, the \(\langle 23\bar{2}\bar{2} \rangle\) four-point function gives rise to the associativity condition

\[
\begin{pmatrix} \frac{3}{4}r & -7r \\ \frac{1}{2}r & \frac{14}{3}x \end{pmatrix} \begin{pmatrix} c_{23\bar{2}\bar{2}} \\ c_{2\bar{3}\bar{2}2} \end{pmatrix} = \begin{pmatrix} c_{3\bar{2}\bar{1}\bar{2}} \\ c_{3\bar{2}\bar{2}\bar{2}} \end{pmatrix}.
\]

(A.10)

The pattern in eq. (A.10) persists for all entries in the table. That is, the fields that are fused to, \emph{i.e.} the third index of the \(c_{ijks}\), are placed in increasing numerical order down the column matrix of structure constants.

The final four-point function to consider is the only one with more than two conformal blocks, that is \(\langle 2\bar{2}\bar{2}\bar{2} \rangle\). It has, in fact, four conformal blocks and implies
We see that the convention used for the table of two conformal block four-point functions persists here also. This completes the associative information about the TCI model needed to construct consistent algebras.
APPENDIX B

In this appendix we present a compilation of the $F1$ TCI$^2$ OPEs involving the $\lambda$ and $\bar{\lambda}$ fields. The $\lambda\lambda$ OPEs are

\[
\begin{align*}
\lambda_{11}\lambda_{11} &= \beta_{11} - \frac{7}{8} \beta_{44}, \\
\lambda_{11}\lambda_{12} &= -\lambda_{12}\lambda_{11} = i\sqrt{\frac{3}{4}} \beta_{13} - i\sqrt{\frac{7}{16}} \beta_{42}, \\
\lambda_{11}\lambda_{21} &= -\lambda_{21}\lambda_{11} = i\sqrt{\frac{3}{4}} \beta_{31} - i\sqrt{\frac{7}{16}} \beta_{24}, \\
\lambda_{11}\lambda_{22} &= +\lambda_{22}\lambda_{11} = -\frac{3}{4} \beta_{33} + \frac{1}{2} \beta_{22}, \\
\lambda_{12}\lambda_{21} &= +\lambda_{21}\lambda_{12} = +\frac{3}{4} \beta_{33} - \frac{1}{2} \beta_{22}, \\
\lambda_{12}\lambda_{12} &= \beta_{11} + \sqrt{\frac{rx}{6}} \beta_{13} + \sqrt{\frac{7rx}{8}} \beta_{42} + \frac{1}{8} \beta_{44}, \\
\lambda_{21}\lambda_{21} &= \beta_{11} + \sqrt{\frac{rx}{6}} \beta_{31} + \sqrt{\frac{7rx}{8}} \beta_{24} + \frac{1}{8} \beta_{44}, \\
\lambda_{21}\lambda_{22} &= -\lambda_{22}\lambda_{21} = i\sqrt{\frac{3}{4}} \beta_{13} + \sqrt{\frac{rx}{8}} \beta_{33} + i\sqrt{\frac{7}{16}} \beta_{42} + i\sqrt{\frac{rx}{2}} \beta_{22}, \\
\lambda_{12}\lambda_{22} &= -\lambda_{22}\lambda_{12} = i\sqrt{\frac{3}{4}} \beta_{31} + \sqrt{\frac{rx}{8}} \beta_{33} + i\sqrt{\frac{7}{16}} \beta_{24} + i\sqrt{\frac{rx}{2}} \beta_{22}, \\
\lambda_{22}\lambda_{22} &= \beta_{11} + \sqrt{\frac{rx}{6}} (\beta_{13} + \beta_{31}) + \frac{rx}{6} \beta_{33} - rx\beta_{22} - \sqrt{\frac{rx}{56}} (\beta_{42} + \beta_{24}) - \frac{1}{56} \beta_{44},
\end{align*}
\]

(B.1)

According to our table of TCI$^2$ field groupings (4.2), these OPEs can be symbolically written

\[
\lambda \cdot \lambda \sim \beta^{(1)} + \beta^{(2)}.
\]

(B.2)

Now the $\bar{\lambda}\bar{\lambda}$ OPEs are easily constructed from the $\lambda\lambda$ ones by writing

\[
\bar{\lambda} \cdot \bar{\lambda} \sim \beta^{(1)} - \beta^{(2)}.
\]

(B.3)
For instance
\[ \lambda_{11} \lambda_{22} = -\frac{3}{4} \beta_{33} + \frac{1}{2} \beta_{22} \]
\[ \Rightarrow \bar{\lambda}_{11} \bar{\lambda}_{22} = -\frac{3}{4} \beta_{33} - \frac{1}{2} \beta_{22}. \]  

(B.4)

Another set of OPEs to construct are the \( \lambda \bar{\lambda} \) ones which we tackle next. The structure constants of the \( \lambda \bar{\lambda} \) OPEs are determined by the following rule. Consider the following ‘algebra’ which is not associative:

\[ \Phi_{2,1} \Phi_{2,1} \sim \Phi_{1,1} + \sqrt{\frac{7}{8}} \Phi_{1,4}, \]
\[ \Phi_{2,1} \Phi_{2,2} \sim \sqrt{\frac{1}{2}} \Phi_{1,2} + \sqrt{\frac{3}{4}} \Phi_{1,3}, \]
\[ \Phi_{2,2} \Phi_{2,2} \sim \Phi_{1,1} + \sqrt{r} \Phi_{1,2} + \sqrt{\frac{r x}{6}} \Phi_{1,3} + \sqrt{\frac{1}{56}} \Phi_{1,4}. \]  

(B.5)

The \( \sim \) relation is to emphasize that (B.5) is not a true associative algebra, but merely a building block for one. Taking a ‘direct product’ (i.e. simply multiplying the structure constants) of the (B.5) ‘algebra’ with itself yields the correct structure constants up to phases for the \( \lambda \bar{\lambda} \) OPEs [as well as for those in the list (B.1)]. However, the determination of the proper phases can only be gotten by explicit calculation. They are summarized in the following table:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\lambda \times \bar{\lambda} & \bar{\lambda}_{11} & \bar{\lambda}_{12} & \bar{\lambda}_{21} & \bar{\lambda}_{22} \\
\hline
\lambda_{11} & +\omega_{+} \beta_{14} + \omega_{-} \beta_{41} & -\omega_{-} \beta_{12} + \omega_{+} \beta_{43} & -\omega_{-} \beta_{34} + \omega_{+} \beta_{21} & -\omega_{+} \beta_{32} - \omega_{-} \beta_{23} \\
\hline
\lambda_{12} & +\omega_{-} \beta_{12} - \omega_{+} \beta_{43} & -\omega_{+} \beta_{12} + \omega_{-} \beta_{43} & +\omega_{+} \beta_{32} + \omega_{-} \beta_{23} & +\omega_{-} \beta_{32} + \omega_{+} \beta_{23} \\
\hline
\lambda_{21} & +\omega_{-} \beta_{34} - \omega_{+} \beta_{21} & +\omega_{+} \beta_{32} + \omega_{-} \beta_{23} & +\omega_{+} \beta_{34} - \omega_{-} \beta_{21} & +\omega_{+} \beta_{34} + \omega_{-} \beta_{23} \\
\hline
\lambda_{22} & -\omega_{+} \beta_{32} - \omega_{-} \beta_{23} & +\omega_{-} \beta_{32} + \omega_{+} \beta_{23} & -\omega_{-} \beta_{12} - \omega_{+} \beta_{43} & +\omega_{+} \beta_{14} + \omega_{-} \beta_{41} \\
\hline
\end{array}
\]

(B.6)
where $\omega_{\pm}$ are the eighth roots of unity

$$\omega_{\pm} = e^{\pm i \frac{\pi}{4}}. \quad (B.7)$$

For example, from (B.5) and (B.6), the following OPEs can be constructed,

$$\lambda_{22}\lambda_{12} = \sqrt{\frac{3}{224}}\omega_{-}\beta_{34} + \sqrt{\frac{3rx}{4}}\omega_{-}\beta_{32} + \sqrt{\frac{1}{2}}\omega_{+}\beta_{21} + \sqrt{\frac{rx}{12}}\omega_{+}\beta_{23},$$

$$\bar{\lambda}_{12}\lambda_{22} = \sqrt{\frac{3}{224}}\omega_{+}\beta_{34} + \sqrt{\frac{3rx}{4}}\omega_{+}\beta_{32} + \sqrt{\frac{1}{2}}\omega_{-}\beta_{21} + \sqrt{\frac{rx}{12}}\omega_{-}\beta_{23}. \quad (B.8)$$

The $\bar{\lambda}\lambda$ OPEs as well as the $\beta\lambda$ and $\beta\bar{\lambda}$ OPEs, can be deduced from the above OPEs by the symmetries of the structure constants (3.5). This completes the construction of the $F1\ TCI^2$ operator algebra.

**APPENDIX C**

In this appendix we use the conformal Ward identities to compute the coefficients of the first few descendant fields appearing on the right hand side of OPEs of fields which are themselves Virasoro descendants. The basic principles for this derivation were first outlined in ref. [1], and the result (C12)-(C13) below was explicitly calculated when $i = j$. The method outlined here is the logical basis for any such calculation, and can be reformulated in various ways.

Let us first discuss how to compute the OPEs between descendants of primary fields in general, given the structure constants for the primaries themselves. That is, we want to show how to calculate the $\beta_{ijk}^{\{\vec{m}\}\{\vec{n}\}\{\vec{p}\}}$ coefficients in the following OPE

$$\phi_{i}^{\{\vec{m}\}}(z)\phi_{j}^{\{\vec{n}\}}(0) = \sum_{k} \sum_{\{\vec{p}\}} c_{ijk} z^{-\Delta_{k}-\Delta_{i}-\Delta_{j}+p-m-n} \beta_{ijk}^{\{\vec{m}\}\{\vec{n}\}\{\vec{p}\}} \phi_{k}^{\{\vec{p}\}}(0), \quad (C.1)$$
where
\[ \phi^{\{\vec{p}\}}_k(0) = (L_{-\{\vec{p}\}}\phi_k)(0), \]
\[ L_{-\{\vec{p}\}} = \hat{L}_{-p_1} \hat{L}_{-p_2} \cdots \hat{L}_{-p_t}, \]
\[ \hat{L}_{-p_i} = \begin{cases} L_{-p_i}, & \text{if } p_i \neq 0, \\ 1, & \text{if } p_i = 0, \end{cases} \]
\[ p = p_1 + p_2 + \cdots + p_t, \]

and
\[ \beta^{\{0\}\{0\}\{0\}}_{ijk} \equiv 1. \]

Note that we have defined \( \hat{L}_{-p} \) so that \( \phi^{\{0\}}(z) = \phi(z) \).

We can determine the \( \beta^{\{\vec{m}\}\{\vec{n}\}\{\vec{p}\}}_{ijk} \) by evaluating the three-point function of descendants,
\[ \langle \phi^{\{\vec{m}\}}_i(z_1)\phi^{\{\vec{n}\}}_j(z_2)\phi^{\{\vec{p}\}}_k(z_3) \rangle, \]
in two different ways. First, take the limit as \( z_1 \to z_2 \) by using the OPE (C.1) to reduce the three-point function (C.4) to the sum over two-point functions
\[ c_{ijk} \sum_{\{\vec{q}\}} z_{12}^{\Delta_i-\Delta_j-q-m-n} \beta^{\{\vec{m}\}\{\vec{n}\}\{\vec{q}\}}_{ijk} \langle \phi^{\{\vec{q}\}}_k(z_2)\phi^{\{\vec{p}\}}_k(z_3) \rangle, \]
where \( z_{ij} = z_i - z_j \). The two-point functions in eq. (C.5) can be calculated using the conformal Ward identities. One finds that their coordinate dependence is \( z_{23}^{-2\Delta_k-q-p} \) and their normalization depends only on \( \Delta_k \) and the central charge \( c \).

On the other hand, the three-point function (C.4) can be calculated directly in terms of the structure constant \( c_{ijk} \), again using the conformal Ward identities. To be more explicit, the general descendant field \( \phi^{\{\vec{m}\}}_i(z) \) can be written as the following integral
\[ \phi^{\{\vec{m}\}}_i(z) = \oint \frac{d\zeta_1 T(\zeta_1)}{(\zeta_1 - z)^{m_1+1}} \cdots \oint \frac{d\zeta_t T(\zeta_t)}{(\zeta_t - z)^{m_t+1}} \phi_i(z), \]
where the contours of integration are nested circles enclosing \( z \). Thus, the two-point functions in eq. (C.5) and the three-point function (C.4) can be expressed as the
integrals of multi-point functions involving only insertions of the energy momentum
tensor along with the primary fields. Now, conformal invariance implies the Ward
identity [1]

\[
\langle T(\zeta_1) \cdots T(\zeta_M) \phi_{k_1}(z_1) \cdots \phi_{k_N}(z_N) \rangle \\
= \left\{ \sum_{i=1}^{N} \left[ \frac{\Delta_{k_i}}{(\zeta_1 - z_i)^2} + \frac{1}{\zeta_1 - z_i} \frac{\partial}{\partial z_i} \right] + \sum_{j=2}^{M} \left[ \frac{2}{\zeta_{1j}^2} + \frac{1}{\zeta_{1j}} \frac{\partial}{\partial \zeta_{1j}} \right] \right\} \\
\times \langle T(\zeta_2) \cdots T(\zeta_{M-1}) T(\zeta_{M}) \phi_{k_1}(z_1) \cdots \phi_{k_N}(z_N) \rangle \\
+ \sum_{j=2}^{M} \frac{c_{1j}}{\zeta_{1j}^2} \langle T(\zeta_2) \cdots T(\zeta_{j-1}) T(\zeta_{j+1}) \cdots T(\zeta_{M}) \phi_{k_1}(z_1) \cdots \phi_{k_N}(z_N) \rangle,
\]

(C.7)

where \( \zeta_{ij} = \zeta_i - \zeta_j \). Using this Ward identity repeatedly, an \( n \)-point function of
descendant fields can be written in terms of the \( n \)-point function of the primary
fields. Again by conformal invariance, the two and three-point functions of primary
fields are

\[
\langle \phi_i(z_1) \phi_j(z_2) \rangle = \delta_{ij} z_{12}^{-2\Delta_i},
\]

\[
\langle \phi_i(z_1) \phi_j(z_2) \phi_k(z_3) \rangle = c_{ijk} z_{12}^{-\delta_i} z_{13}^{-\delta_j} z_{23}^{-\delta_k},
\]

where we have introduced the useful combinations of dimensions

\[
\delta_i = -\Delta_i + \Delta_j + \Delta_k,
\]

\[
\delta_j = +\Delta_i - \Delta_j + \Delta_k,
\]

\[
\delta_k = +\Delta_i + \Delta_j - \Delta_k.
\]

(C.8)

(C.9)

Now if we systematically compute the three-point function (C.4) for all fields
\( \phi_k^{\{\vec{p}\}}(z_3) \) using the two different methods outlined above, then we can compare
the expansion (C.5) with the direct calculation of eq. (C.4) and thereby deduce
the \( \beta^{\{\vec{m}\}\{\vec{n}\}\{\vec{p}\}}_{ijk} \) coefficients.

For the purposes of this paper we only need to know the \( \beta^{\{\vec{m}\}\{\vec{n}\}\{\vec{p}\}}_{ijk} \) for

\[
\{\vec{m}\}, \{\vec{n}\}, \{\vec{p}\} \subseteq \{\{0\}, \{1\}, \{1,1\}, \{2\}\}.
\]

(C.10)

It is sufficient to calculate those with \( \{\vec{m}\}, \{\vec{n}\} \subseteq \{\{0\}, \{2\}\} \), since those with \( \{1\} \)
or \( \{1,1\} \) can be simply reached via differentiation (recall that \( L_{-1} = \partial \) when acting
on a primary field). Denoting by $\mathcal{M}$ the matrix of inner products of the level-2 Virasoro descendants of $\phi_k$:

$$\mathcal{M} = \begin{pmatrix} 4\Delta_k (2\Delta_k + 1) & 6\Delta_k \\ 6\Delta_k & 4\Delta_k + \frac{c}{2} \end{pmatrix},$$

we present the results of these calculations in the form of the four OPEs between the fields $\phi_i, \phi_i^{(2)}$ and $\phi_j, \phi_j^{(2)}$:

$$\phi_i(z)\phi_j(0) = c_{ijk}z^{-\delta_k} \left[ \phi_k(0) + \frac{\delta_j}{2\Delta_k} \phi_k^{(1)}(0) \\
+ z^2(a\phi_k^{(1,1)} + b\phi_k^{(2)}) + \cdots \right],$$

where

$$\mathcal{M}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \delta_j(\delta_j + 1) \\ \delta_j + \Delta_i \end{pmatrix}.$$  

$$\phi_i^{(2)}(z)\phi_j(0) = c_{ijk}z^{-\delta_k-2} \left[ (\delta_k + \Delta_j)\phi_k(0) + z \frac{f}{2\Delta_k} \phi_k^{(1)}(0) \\
+ z^2(a\phi_k^{(1,1)} + b\phi_k^{(2)}) + \cdots \right],$$

$$f = (\delta_k + \Delta_i)(\delta_j + 2) - 3\delta_k,$$

$$\mathcal{M}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f(\delta_j + 3) + 3\delta_j(-\delta_k + 1) \\ (\delta_k + \Delta_j)(\delta_j + \Delta_i + 4) - 9\delta_k + 4\Delta_i + \frac{c}{2} \end{pmatrix}.$$  

$$\phi_i(z)\phi_j^{(2)}(0) = c_{ijk}z^{-\delta_k-2} \left[ (\delta_k + \Delta_i)\phi_k(0) + z \frac{f}{2\Delta_k} \phi_k^{(1)}(0) \\
+ z^2(a\phi_k^{(1,1)} + b\phi_k^{(2)}) + \cdots \right],$$

$$f = (\delta_k + \Delta_i)(\delta_j - 2) + 3\delta_k,$$
\[ \mathcal{M} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f(\delta_j - 1) + 3(\delta_j \delta_k + \delta_i) \\ (\delta_k + \Delta_i)(\delta_j + \Delta_i - 2) + 4\Delta_j + \frac{c}{2} \end{pmatrix}. \] (C.19)

\[ \phi_i^{(2)}(z)\phi_j^{(2)}(0) = c_{ijk}z^{-4}\left[ g\phi_k(0) + z\frac{f}{2\Delta_k}\phi_k^{(1)}(0) + z^2(a\phi_k^{(1,1)} + b\phi_k^{(2)}) + \cdots \right]. \] (C.20)

\[ g = \frac{c}{2} + 11(\Delta_i + \Delta_j) - 7\Delta_k + 2\Delta_i^2 + 5\Delta_i\Delta_j + 2\Delta_j^2 - 3(\Delta_i + \Delta_j)\Delta_k + \Delta_k^2, \] (C.21)

\[ f = g\delta_j + 3(\delta_k + 2)(\Delta_j - \Delta_i), \] (C.22)

\[ \mathcal{M} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f(\delta_j + 1) + 6\Delta_k(\delta_k + \Delta_i) + 3(\delta_k + 1)[2\Delta_k + \delta_j(\Delta_j - \Delta_i)] \\ g(\delta_j + \Delta_i + 2) - 9(\delta_k + \Delta_i)(\delta_k + 2) + (4\Delta_i + \frac{c}{2})(\delta_k + \Delta_i) + (4\Delta_j + \frac{c}{2})(\delta_k + \Delta_j) \end{pmatrix}. \] (C.23)

REFERENCES

1. A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* B241 (1984) 333.
2. D. Friedan, Z. Qiu and S. Shenker, *Phys. Rev. Lett.* 52 (1984) 1575.
3. V.G. Knizhnik and A.B. Zamolodchikov, *Nucl. Phys.* B247 (1984) 83.
4. A.B. Zamolodchikov, *Theor. Math. Phys.* 63 (1985) 1205.
5. C. Ahn, S.-w. Chung and S.-H.H. Tye, *Nucl. Phys.* B365 (1991) 191.
6. See *e.g.* D. Freidan in *Unified String Theories*, ed. by M. Green and D. Gross, World Scientific (1986).
7. A.B. Zamolodchikov and V.A. Fateev, *Sov. Phys. J.E.T.P.* 62 (1985) 215.
8. P.C. Argyres, J. Grochocinski and S.-H.H. Tye, *Nucl. Phys.* **B367** (1991) 217.

9. P. Goddard, A. Kent and D. Olive, *Comm. Math. Phys.* **103** (1986) 105.

10. V.A. Fateev and A.B. Zamolodchikov, *Sov. Phys. J.E.T.P.* **71** (1988) 451.

11. D. Kastor, E. Martinec and Z. Qiu, *Phys. Lett.* **200B** (1988) 434; J. Bagger, D. Nemeschansky and S. Yankielowicz, *Phys. Rev. Lett.* **60** (1988) 389; F. Ravanini, *Mod. Phys. Lett.* **A3** (1988) 397.

12. S.-w. Chung, E. Lyman and S.-H.H. Tye, *Int. J. Mod. Phys.* **A7** #14 (1992).

13. Vl.S. Dotsenko and V.A. Fateev, *Nucl. Phys.* **B240** (1984) 312; *Nucl. Phys.* **B251** (1985) 691.

14. Vl.S. Dotsenko, *Nucl. Phys.* **B338** (1990) 747; *Nucl. Phys.* **B358** (1991) 547.

15. Z. Qiu, *Nucl. Phys.* **B270** (1986) 205; D. Friedan, Z. Qiu and S. Shenker, *Phys. Lett.* **151B** (1985) 37.

16. A.B. Zamolodchikov and V.A. Fateev, *Sov. J. Nucl. Phys.* **43** (1986) 657.

17. G. Felder, *Nucl. Phys.* **B317** (1989) 215.

18. P.C. Argyres and S.-H.H. Tye, *Phys. Rev. Lett.* **67** (1991) 3339.

19. A. Rocha-Caridi in *Vertex Operators in Mathematics and Physics*, MSRI Publication **3** (Springer, Heidelberg 1984) 451.

20. V.G. Kac and D. Peterson, *Bull. AMS* **3** (1980) 1057; *Adv. Math.* **53** (1984) 125; J. Distler and Z. Qiu, *Nucl. Phys.* **B336** (1990) 533.

21. K.R. Dienes and S.-H.H. Tye, *Model-Building for Fractional Superstrings*, Cornell preprint CLNS 91/1100, McGill preprint McGill/91-29 (November 1991); P.C. Argyres, K.R. Dienes and S.-H.H. Tye, *New Jacobi-like Identities for ZK Parafermion Characters*, CLNS 91/1113, McGill/91-37 (January 1992); P.C. Argyres, E. Lyman and S.-H.H. Tye, *Low-Lying States of the Six-Dimensional Fractional Superstring*, CLNS 91/1121 (February 1992).