Wasserstein Barycenters over Riemannian manifolds

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Abstract

We study barycenters in the space of probability measures on a Riemannian manifold, equipped with the Wasserstein metric. Under reasonable assumptions, we establish absolute continuity of the barycenter of general measures $\Omega \in P(P(M))$ on Wasserstein space, extending on one hand, results in the Euclidean case (for barycenters between finitely many measures) of Agueh and Carlier [1] to the Riemannian setting, and on the other hand, results the Riemannian case of Cordero-Erausquin, McCann, Schmuckenschläger [9] for barycenters between two measures to the multimarginal setting. Our work also extends these results to the case where $\Omega$ is not finitely supported. As applications, we prove versions of Jensen’s inequality on Wasserstein space and a generalized Brunn-Minkowski inequality for a family of (possibly infinitely many) measurable sets on a Riemannian manifold.

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1 Introduction

This paper is devoted to the study of barycenters in Wasserstein space over a Riemannian manifold $M$.

Given a Borel probability measure $\Omega$ on a metric space $(X,d)$, a barycenter of $\Omega$ is defined as a minimizer of $y \mapsto \int_X d^2(x,y)d\Omega(x)$; this definition is chosen in part so that it coincides with the mean, or center of mass, $\int\limits_{\mathbb{R}^n} xd\Omega(x)$ on the Euclidean space $X = \mathbb{R}^n$. Barycenters have been studied extensively by geometers, and have interesting connections to the underlying geometry of $X$; for example, their uniqueness is intimately related to sectional curvature.

The case where the metric space $(X,d) = (P(M), W_2)$ is the space of Borel probability measures on a compact Riemannian manifold $M$, equipped with the $W_2$ distance, holds particular interest, as it gives a nonlinear way to interpolate between a distribution of measures. The Wasserstein, or optimal transport, distance $W_2(\mu, \nu)$ between $\mu, \nu \in P(M)$ is given by
\[ W_2^2(\mu, \nu) = \inf_{\gamma} \int_{M \times M} d^2(x, y) d\gamma(x, y) \]  \hspace{1cm} (1.1)

where \( d \) is the Riemannian distance and the infimum is taken over all probability measures \( \gamma \) on \( M \times M \) whose marginals are \( \mu \) and \( \nu \). It is well known that \( W_2 \) defines a metric on \( P(M) \) (see, e.g., \cite{241}) and therefore it makes sense to talk about barycenters.

**Definition 1.1 (Wasserstein barycenter measure).** Let \( \Omega \) be a probability measure on \( P(M) \). A Wasserstein barycenter of \( \Omega \) is a minimizer among probability measures \( \nu \in P(M) \) of \( \nu \mapsto \int_{P(M)} W_2^2(\mu, \nu) d\Omega(\mu) \).

Existence and uniqueness (under mild conditions) of Wasserstein barycenters are not difficult to establish (see Section 3). When the measure \( \Omega = (1 - t)\delta_{\mu_0} + t\delta_{\mu_1} \) is supported on two points in \( P(M) \), the barycenter measure \( \mu_t \) on \( M \) is equivalent to McCann’s celebrated displacement interpolant \cite{29}. A key property is that \( \mu_t \) is absolutely continuous with respect to volume if \( \mu_0 \) or \( \mu_1 \) is; this fact serves as the foundation for the analysis of convexity type properties of various functionals on the space of absolutely continuous probability measures \( P_{ac}(M) \). The study of functionals which are convex along this interpolation, known as displacement convexity, has been very fruitful; its extensive wealth of applications includes insightful new proofs of geometric and functional inequalities on \( \mathbb{R}^n \), and remarkable generalizations of these inequalities to the Riemannian setting; see, e.g., the books \cite{2,41,42}. In addition, displacement convexity is a fundamental notion in the synthetic treatment of Ricci curvature developed by Sturm \cite{36} \cite{37} and Lott-Villani \cite{26}. An important example of a displacement convex functional is the Boltzman’s \( H \)-functional (or Shannon entropy functional) \( \rho \mapsto \int_M \rho(x) \log \rho(x) d\text{vol}(x) \) on manifolds with nonnegative Ricci curvature.

In the multi-measure setting, with economic applications in mind, Carlier-Ekeland \cite{8} introduced an interpolation between several probability measures, which includes as a special case Wasserstein barycenters of finitely supported measures \( \Omega = \sum_{i=1}^m \lambda_i \delta_{\mu_i} \). In fact, their model is more general, as the distance squared in (1.1) is replaced with a more general cost function. Agueh-Carlier \cite{1} gave a more extensive treatment of Wasserstein barycenters of finitely supported measures \( \Omega \in P(\mathbb{R}^n) \) when the underlying space \( M \) is Euclidean, proving that the barycenter is absolutely continuous with an \( L^\infty \) density if one of the marginals is, as well as convexity type inequalities, which one can interpret as Jensen’s type inequalities for discrete measures and displacement convex functionals, on \( P(\mathbb{R}^n) \). Absolute continuity has also been established for more general interpolations over Euclidean space, as well as barycenters on Hadamard manifolds (simply connected Riemannian manifolds with nonpositive curvature) \cite{33}. Some of these results have been extended by one of us \cite{32} to the...
case where the support of \( \Omega \in P(\mathbb{R}^n) \) is parameterised by a 1-dimensional continuum. Let us note that in addition to economics \([8]\), Wasserstein barycenters in the multi-measure setting have appeared in the literature with applications in image processing \([34]\) and statistics \([4]\).

In this paper, we consider the Wasserstein barycenters of general measures \( \Omega \in P(P(M)) \). In particular, we allow the support of \( \Omega \) to have cardinality greater than 2 (and possibly be infinite) and the underlying domain \( M \) to be general compact Riemannian manifold, without any curvature or topological restrictions. Little is known about Wasserstein barycenters in this case.

Our first main contribution is to establish **absolute continuity with respect to volume measure of the Wasserstein barycenter**, under reasonable conditions on the marginals: see Theorems \(5.1\) and \(6.1\). In previous work \([1]\), \([32]\), and \([33]\), regularity results on barycenters are obtained by exploiting either the special geometry of Euclidean space or the uniform convexity of the distance squared function (in the non-positively curved setting). These tools are not available in the general Riemannian setting, and our argument is based instead on approximations. Indeed, we first consider the case of a finitely supported \( \Omega = \sum_{i=1}^{m} \lambda_i \delta_{\mu_i} \) on \( P(M) \), and adapt an argument of Figalli-Juillet \([13]\) (who studied the two measure case on the Heisenberg group and Alexandrov spaces); this amounts to approximating all but one of the measures \( \mu_i \) by finite sum of Dirac measures, obtaining uniform estimates for the approximating barycenters and passing to the limit. After this, we are able to treat the general case by approximating a general \( \Omega \) on \( P(M) \) by finitely supported measures: see Theorem \(6.1\).

Using the above results, we are then able to establish certain **Wasserstein Jensen’s type inequalities** (see, in particular, Theorems \(7.11\) and \(7.14\)) for a wide variety of displacement convex functionals on \( P(M) \). In fact, we establish two distinct results of this type: one involves \( k \)-displacement convex functionals, and can be interpreted as a generalization of \([42\) Theorem 17.15]. The other involves what we call generalized distortion coefficients; this is closer in spirit to the line of research pioneered by Cordero-Erausquin, McCann and Schmuckenschlager \([9]\), and can be interpreted as a generalization of \([42\ Theorem 17.37].

We note that geometric versions of Jensen’s inequality (that is, versions formulated in terms of barycenters on metric spaces rather than linear averages) are known for measures on finite dimensional smooth manifolds \([12\ Proposition 2]\) and on more general spaces with appropriate sectional curvature bounds (see \([88\) and \([23]\): these sectional curvature bounds are not satisfied by Wasserstein space \( (P(M), W_2) \) \([2]\). Before the present paper, a version of Jensen’s inequality on \( P(M) \), due to Agueh-Carlier \([1]\), was known only when the underlying space \( M \subseteq \mathbb{R}^n \) is Euclidean and the measure \( \Omega \) on \( P(M) \) is finitely supported.

Finally, as an application of the machinery established in this paper, we offer a **random version of the Brunn-Minkowski inequality on a Riemannian manifold**: see Theorem \(8.1\). The classical Brunn-Minkowski inequality involves the interpolation between two sets in Euclidean space; an extension to Riemannian manifolds, with Ricci curvature playing a key role, can be derived.
from the results in [9]. Our result extends this to the interpolation between random sets on $M$. For a finite number of sets, we should note that in Euclidean space this result is easily recoverable using the classical Brunn-Minkowski and induction. For an infinite collection of sets, the Euclidean version is a pre-existing but nontrivial result, known as Vitale’s random Brunn-Minkowski inequality [43]; our work provides a mass transport based proof of it. On the other hand, in the Riemannian case, our result seems to be completely novel, as soon as we interpolate between three or more sets.

Organization of the paper: In the following section, we introduce the notation and terminology we will use throughout the paper. We also recall a few fundamental results from the literature which we will need. In Section 2 we establish a general existence and uniqueness result on the Wasserstein barycenter. In Section 3 we establish some properties of the Wasserstein barycenter, including two balance conditions which will be crucial in subsequent sections. Section 5 is devoted to the proof of the absolute continuity of the Wasserstein barycenter when the measure $\Omega$ has finite support; this result is then exploited to prove absolute continuity of the barycenter of more general $\Omega$ in Section 6. This, in turn, is used in Section 7 where we prove various Wasserstein Jensen’s inequalities. Finally, these results are exploited to establish a random Brunn-Minkowski inequality on curved spaces, in Section 8.

2 Notation, definitions, and preliminary results

In this section, we introduce some notation and terminology which we will use in the rest of the paper, and develop some preliminary results.

2.1 Notation and assumptions

Throughout the rest of the paper, we use the following notation and assumptions:

- $M$ is a complete, compact $n$-dimensional Riemannian manifold
- $d(x, y)$ is the Riemannian distance between two points $x, y$ in $M$.
- $B_r(x)$ is the geodesic ball of radius $r$ in $M$, centred at $x$.
- We will sometimes use the notation $c(x, y) = \frac{1}{2}d^2(x, y)$, where $c$ stands for cost function. A significant property of the function $c$ is the following relation:

$$-D_x c(x, y) = \exp^{-1}_x(y),$$

where $D_x$ denotes the gradient of $c$ with respect to the $x$-variable; although the notation $D_x c$ is often used to denote the differential of $c$ (a
covector) rather than its gradient (a vector), we will often identify vectors and covectors using the Riemannian metric.

- $P(M)$ is the space of probability measures on $M$ equipped with the weak-* topology, or, equivalently, metrized with the Wasserstein distance $W_2$.
- $\Omega$ is a probability measure on $P(M)$.

## 2.2 Borel measurability of the set $P_{ac}(M)$

Equipped with the distance $W_2$, the space $P(M)$ is a separable metric space. We consider measurable sets with respect to the Borel sigma algebra.

In this subsection we show that the set $P_{ac}(M)$ of absolutely continuous probability measures on $M$, with respect to the $n$-dimensional Hausdorff measure, or equivalently the Riemannian volume, is Borel measurable. We expect that this is already known to experts, but we include it for completeness.

In Section 3, when we show uniqueness of the Wasserstein barycenter of a given probability measure $\Omega$ on $P(M)$, we will need to assume $\Omega(P_{ac}(M)) > 0$.

**Proposition 2.1 (Measurability of $P_{ac}(M)$).** The set $P_{ac}(M) \subset P(M)$, of absolutely continuous probability measures is Borel measurable with respect to the metric topology given by the Wasserstein distance $W_2$, or equivalently with respect to the weak-* topology (the two topologies are equivalent).

**Proof.** Note that a measure $\mu$ being absolutely continuous (with respect to vol) is equivalent to the following statement: for every $\epsilon > 0$, there is $\delta > 0$ such that $\mu(A) \leq \epsilon$ for all Borel sets $A$ with vol$(A) \leq \delta$. This means

$$P_{ac}(M) = \cap_{k \in \mathbb{N}} \cup_{i \in \mathbb{N}} \mathcal{E}_{2^{-k}, 2^{-i}},$$

where the sets $\mathcal{E}_{\epsilon, \delta}$ of probability measures are defined as

$$\mathcal{E}_{\epsilon, \delta} = \{ \mu \in P(M) \mid \mu(A) \leq \epsilon \text{ for all Borel set } A \text{ with vol}(A) \leq \delta \}.$$

To show $P_{ac}(M)$ is a Borel set, we will express it as a countable intersection of countable unions of Borel sets (essentially replacing the set $\mathcal{E}_{2^{-k}, 2^{-i}}$ in (2.1) with a closed set). For this, first define (recall $B_r(x)$ is an open metric ball of radius $r$ at $x$),

$$\mathcal{F} = \{ F \subset M \mid F = \bigcup_{i=1}^m B_{r_i}(x_i) \text{ for some finite sets } \{x_i\}_{i=1}^m \subset M \text{ and } \{r_i\}_{i=1}^m \subset \mathbb{R}_+ \}$$

and consider the subset $\mathcal{B}_{\epsilon, \delta} \subset P(M)$ defined as

$$\mathcal{B}_{\epsilon, \delta} = \{ \mu \in P(M) \mid \mu(F) \leq \epsilon, \forall F \in \mathcal{F} \text{ with } \text{vol}(F) \leq \delta \}.$$

The set $\mathcal{B}_{\epsilon, \delta}$ is a closed subset of $P(M)$, with respect to the weak-* topology: Pick any sequence $\mu_i \in \mathcal{B}_{\epsilon, \delta}$, weakly-* convergent to $\mu_\infty$. Pick a set $F \in \mathcal{F}$, with
vol(F) ≤ δ and F = ∪_{i=1}^{m} B_{r_i}(x_i). Let for each k ∈ N, F_k = ∪_{i=1}^{m} B_{(1-2^{-k})r_i}(x_i) and consider a continuous function f_k : M → ℝ, with support spt f_k ⊂ F and 0 ≤ f_k ≤ 1, and f_k = 1 on F_k. Then, due to the weak-* convergence ε ≥ lim_{i→∞} ∫_M f_k dμ_i = ∫_M f_k dμ_∞. Moreover, μ_∞(F) = lim_{k→∞} ∫_M f_k dμ_∞, since F = ∪_k F_k. This shows that μ_∞(F) ≤ ε, as desired.

Now, clearly E_{ε,δ} ⊂ B_{ε,δ}. We show that B_{ε,2δ} ⊂ E_{ε,δ}: Let μ ∈ B_{ε,2δ}. Let A be an arbitrary Borel set with vol(A) ≤ δ. Pick an arbitrary small number t > 0. One can find an open set U_{A,δ,t}, consisting of finite metric balls U_{A,δ,t} = ∪_{i=1}^{m} B_{r_i}(x_i), with vol( U_{A,δ,t} ) ≤ 2δ and μ(U_{A,δ,t}) ≥ (1-t)μ(A). Then, from the definition of B_{ε,2δ}, μ(U_{A,δ,t}) ≤ ε, therefore, μ(A) ≤ 1/tε. Since t > 0 was arbitrary, this means μ(A) ≤ ε. This shows B_{ε,2δ} ⊂ E_{ε,δ}.

The above paragraph implies P_{ac}(M) = ∩_{k∈N} ∪_{l∈N} E_{2^{-k},2^{-l-1}} = ∩_{k∈N} ∪_{l∈N} B_{2^{-k},2^{-l-1}} which completes the proof, since the latter expression is a countable union of closed (thus Borel) sets.

**Remark 2.2.** Inspection of the previous proof shows that (M, vol) can be replaced with a compact separable metric space (X, ν) equipped with a reference Borel measure ν.

### 2.3 Optimal transport on Riemannian manifolds

Next, we briefly recall some key results in optimal transport on Riemannian manifolds which we will use throughout the paper. We begin with a fundamental result of McCann [28], originally established by Brenier [6] when M = ℝ^n is Euclidean:

**Theorem 2.3 (Optimal transport on M; see Brenier [6], McCann [28]).** Assume μ is absolutely continuous with respect to volume measure. Then the infimum in (1.1) is attained by a unique measure γ. Furthermore, γ = (I×T) #μ is concentrated on the graph of a measurable mapping T over the first marginal, and T takes the form

\[ T(x) = \text{exp}_x(-Du(x)) \]

where u : M → ℝ is a c-convex function; that is

\[ u(x) = \sup_{y ∈ M} -c(x,y) - u^c(y) \]  \hspace{1cm} (2.2)

for some function u^c : M → ℝ, where c(x,y) = 1/2 d^2(x,y).

It is well known that the c-convex function u is a semi-convex function and therefore twice differentiable almost everywhere. At each point where this differentiability holds, the mapping T is differentiable. We now recall a few classical
identities, easily derived from the Brenier-McCann theorem (Theorem 2.3), or, for more general cost functions, from references such as [16] [7] and [27].

Wherever \( u \) is differentiable, we have the first order condition:

\[-D_x c(x, T(x)) = D_x u(x).\]

Differentiating this identity, we obtain

\[ D^2 u(x) + D_x^2 c(x, T(x)) = -D^2_{xz} \bigg|_{z=T(x)} c(x, z) \cdot DT(x). \] (2.3)

It will also be important later to recall the second-order inequality due to (2.2):

\[ D^2 u(x) + D_x^2 c(x, T(x)) \geq 0 \] (2.4)

Taking determinants of (2.3) yields

\[ \det[DT(x)] \] (2.5)

\[ = \det[D^2_{yy} c(x, T(x))]^{-1} \det[D^2_x u(x) + D_x^2 c(x, T(x))]. \]

If both \( \mu \) and \( \nu \) are absolutely continuous with respect to volume, with densities \( f \) and \( g \), respectively, we also have the change of variables formula almost everywhere:

\[ g(T(x)) \det[DT(x)] = f(x), \] (2.6)

which, together with (2.5), implies

\[ f(T(x)) \det[D^2_{yy} c(x, T(x))]^{-1} \det(D^2_x u(x) + D_x^2 c(x, T(x))) \]

\[ = f(x). \] (2.7)

In the present paper we will also make use of a multi-marginal version of the Brenier-McCann theorem (Theorem 2.3), which generalizes from Euclidean space a result of Gangbo and Swiech [17] and is also related to the works of Carlier-Ekeland [8] and Agueh-Carlier [1]. Given probability measures \( \mu_1, \ldots, \mu_m \) on \( M \), the multi-marginal optimal transport problem is to minimize

\[ \int_{\Pi_{i=1}^m M} c(x_1, \ldots, x_m)d\gamma \] (2.8)

over all probability measures \( \gamma \) on the \( m \)-tuple product \( \Pi_{i=1}^m M^m \) whose marginals are the \( \mu_i \)'s. There has recently been substantial interest and progress in understanding this problem in a variety of different settings; see [31] and the references therein. In this paper, we will take the cost function \( c : \Pi_{i=1}^m M \to \mathbb{R} \) to be

\[ c(x_1, \ldots, x_m) = \min_{z \in M} \sum_{i=1}^m \lambda_i d^2(x_i, z) \] (2.9)

where \( \lambda_1, \ldots, \lambda_m > 0 \), with \( \sum_{i=1}^m \lambda_i = 1 \), give weights on the components \( d^2(x_i, z) \) making up the cost function; we will sometimes denote \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_m) \).
In Euclidean space, this coincides with the cost studied by Gangbo and Swiech [17], who proved assertion 1 in Theorem 2.4 below in that setting, extending earlier partial results of Olkin and Rachev [30], Knott and Smith [22] and Ruschendorf and Uckelmann [35].

We have the following theorem:

Theorem 2.4 (Multi-marginal optimal transport on $M$; see [21] Sections 4 and 5). Assume $\mu_1$ is absolutely continuous with respect to $\text{vol}$. 

1. The solution $\gamma$ to (2.8) with cost function (2.9) is concentrated on the graph of a mapping $(F_2, F_3, \ldots, F_m)$ over the first variable and is unique.

2. There exists a unique minimizer $\bar{x}_\lambda(x_1, x_2, \ldots, x_m)$ of $x \mapsto \sum_{i=1}^m \lambda_i d^2(x_i, y)$ for $\gamma$ almost all $(x_1, x_2, \ldots, x_m)$, and moreover this gives a $\gamma$-a.e one-to-one map $\bar{x}_\lambda : \text{spt}\gamma \to M$.

3. Moreover, applying 1 and 2 to a result of Carlier-Ekeland [8, Proof of Proposition 3], we get

\[ \nu := \gamma \#ar{x}_\lambda \]

is the unique Wasserstein barycenter measure of the measures $\mu_1, \ldots, \mu_m$ with weights $\lambda_i$.

In particular, the assertions 2 and 3 will be important for us.

2.4 Geometric barycenters on Riemannian manifolds: volume distortion

In the remaining part of the present section, we discuss geometric barycenters on a Riemannian manifold and introduce the volume distortion constants associated to them. Given a probability measure $\lambda$ on $M$, we define its set of barycenters as

\[ BC(\lambda) = \text{argmin} \left( y \mapsto \int_M c(y, x) d\lambda(x) \right) \]

We introduce the notation:

\[ bc_\lambda(x_1, \ldots, x_m) = BC \left( \sum_{i=1}^m \lambda_i \delta_{x_i} \right), \]

for the barycenter of the discrete measure with weights $\lambda_1, \ldots, \lambda_m > 0$.

We will also require the following notion:

Definition 2.5 (Volume distortion). Let $\lambda$ be a Borel probability measure on $M$ with a unique barycenter $\bar{x}$ (that is, such that $BC(\lambda)$ is a singleton). We define the barycentric volume distortion coefficients at $y \notin \text{cut}(\bar{x})$

\[ \alpha_\lambda(y) := \frac{\det[-D^2_{xz}]_{z=z}c(y, z)}{\det[\int_M D^2_{xz}]_{z=z}c(x, z)d\lambda(x)]} \quad (2.10) \]

where $D^2_{xz}c(x, z)$ denotes the Hessian of the function $z \mapsto c(x, z)$, and the determinants are computed in exponential local coordinates at $\bar{x}$ and $y$. 

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Remark 2.6 (Justification of the name volume distortion for $\alpha_\lambda$). Volume distortion coefficients were introduced in \cite{9}, to compare the volume of small ball and the set of points which are a fixed distance along geodesics starting from points in that ball and ending at some fixed point.

The quantity $\alpha_\lambda(x)$ captures in the distortion of volume between a point $x \in \text{spt} (\lambda)$ and the barycenter of a measure $\lambda$, in a sense which we make precise below.

Suppose

1. $\lambda = \sum_{i=1}^m \lambda_i \delta_{x_i}$ has finite support and assume that, for $y$ near $x_j$,

   $$BC(\sum_{i \neq j}^m \lambda_i \delta_{x_i} + \lambda_j \delta_y)$$

   is a singleton;

2. the function $y \mapsto BC(\sum_{i \neq j}^m \lambda_i \delta_{x_i} + \lambda_j \delta_y)$ is differentiable at $x_j$.

We claim that, for a fixed index $j$,

$$\alpha_\lambda(x_j) = \lim_{r \to 0} \frac{\text{vol}(BC(\lambda, B_r(x_j)))}{\text{vol}(B_{\lambda_r}(x_j))}$$

where $BC(\lambda, B_r(x_j)) = \bigcup_{y \in B_r(x_j)} BC(\sum_{i \neq j}^m \lambda_i \delta_{x_i} + \lambda_j \delta_y)$.

In particular, when $\lambda = t\delta_x + (1-t)\delta_y$, we have $\alpha_\lambda(x) = v_{1-t}(y, x)$, where $v_{1-t}(y, x)$ is the volume distortion coefficient of \cite{9}.

Proof. From assumption 1, we can define $\bar{x}(x_1, ..., x_m) = BC(\sum_{i=1}^m \lambda_i \delta_{x_i})$.

Now, the function

$$z \mapsto \sum_{i=1}^m \lambda_i c(x_i, z)$$

is differentiable near $z = \bar{x}(x_1, ..., x_m)$ (for a proof, see e.g. \cite{21} Lemma 3.1); moreover, we have

$$\sum_{i=1}^m \lambda_i D_z c(x_i, z) = 0.$$

From assumption 2, we can differentiate the last equation with respect to $x_j$, which yields

$$\sum_{i=1}^m \lambda_i D^2_{zz} c(x_i, \bar{x}) \cdot D_{x_j} \bar{x} + \lambda_j D^2_{zz} c(x_j, \bar{x}) = 0,$$

hence, after taking determinants and rearranging, we have,

$$\det(D_{x_j} \bar{x}) = \frac{\lambda_j^n \det[-D^2_{zz} c(x_j, \bar{x})]}{\det[\sum_{i=1}^m \lambda_i D^2_{zz} c(x_i, \bar{x})]}$$

(2.11)
Notice that the absolute value of the left-hand side of (2.11) is the volume distortion
\[
\lim_{r \to 0} \frac{\text{vol}(BC(\lambda, B_r(x_j)))}{\text{vol}(B_r(x_j))},
\]
since all the terms on the right-hand side are nonnegative, dividing (2.11) by \(\lambda^n_j\) yields the desired result.

Before concluding this section, we prove a couple of lemmas, relating the \(\alpha_\lambda\) to the Ricci curvature of \(M\). Let us fix the notation:

\[
S_K(d) = \begin{cases} 
\frac{\sin(\sqrt{K}d)}{\sqrt{K}d} & \text{if } K > 0 \\
1 & \text{if } K = 0 \\
\frac{\sinh(\sqrt{-K}d)}{\sqrt{-K}d} & \text{if } K < 0
\end{cases}
\] (2.12)

We will need the following lemma, whose proof is based on an argument in [9].

**Lemma 2.7.** Suppose \(-K \leq 0\) is a lower bound for the Ricci curvature on \(M\). Then

\[
\text{tr}[(D^2_{xy}c(x,y))] \leq n \frac{\sqrt{K}d(x,y)}{\tanh(\sqrt{K}d(x,y))}.
\]

The proof is exactly as in [9, Lemma 3.12], but we take the trace over an orthonormal basis to get from sectional to Ricci curvature.

**Lemma 2.8.** Suppose \(K\) is a lower bound for the Ricci curvature on \(M\). Then

\[
\det(-D^2_{xy}c(x,y)) \geq [S_K(d(x,y))]^{-n+1}
\]
where \(S_K\) is given in (2.12).

**Proof.** Note that \(-D^2_{xy}c(x,y)\) is the inverse of \(d \exp_x(\cdot)\) evaluated at \(-D_xc(x,y)\). Therefore, by the Bishop-Gromov volume comparison theorem (see, e.g., [9], section 11.10, Theorem 15),

\[
t \mapsto \det(-D^2_{xy}c(x,y)) \cdot [S_K(d(x,y))]^{-n-1}
\]
is nondecreasing along a geodesic \(y_t\) starting at \(y_0 = x\). As this function is 1 when \(t = 0\), the result follows. \(\square\)

**Proposition 2.9 (Distortion under \(\text{Ric} \geq 0\)).** Suppose the Ricci curvature of \(M\) is everywhere nonnegative, i.e., \(\text{Ric} \geq 0\). Then, for any \(x \in M\) and \(\lambda \in P(M)\), we have

\[\alpha_\lambda(x) \geq 1.\]

**Proof.** Minimality of \(z \mapsto \int_M c(x,z)d\lambda(x)\) at the barycenter \(\bar{x}\), combined with semi-concavity of \(z \mapsto c(x,z)\) and Fatou’s lemma yields

\[
\int_M D^2_{zz}|_{z=\bar{x}}c(x,z)d\lambda(x) \geq 0
\]

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as a matrix (notice that until this moment we do not need any assumption on the curvature). Now, as $\lambda$ is a probability measure, Lemma 2.7 with $K = 0$ implies
\[
\text{tr}\left[ \int_M D_{zz}^2 \bigg|_{z = \bar{x}} c(x, z) d\lambda(x) \right] \leq n;
\]
applying the geometric-arithmetic mean inequality to the nonnegative matrix
\[
\int_M D_{zz}^2 \bigg|_{z = \bar{x}} c(x, z) d\lambda(x)
\]
yields
\[
\det\left[ \int_M D_{zz}^2 \bigg|_{z = \bar{x}} c(x, z) d\lambda(x) \right] \leq 1.
\]
Combining this with the inequality $\det[-D_{yz}^2 \bigg|_{z = \bar{x}} c(y, z)] \geq 1$ (from Lemma 2.8 with $K = 0$), yields the desired result.

More generally, if $\text{Ric} \geq -K$ (for $K \geq 0$), we have
\[
\alpha_\lambda(x) \geq C(diam(M), K, n) \quad (2.13)
\]
where $\text{diam}(M)$ is the diameter of the manifold and
\[
C(diam(M), K, n) := \begin{cases} 
1 & \text{for } K = 0, \\
(S_{-K}(\text{diam}(M))^{n+1} \cdot \frac{\sqrt{K} \text{diam}(M)}{\text{tanh}(\sqrt{K} \text{diam}(M))})^{-n} & \text{for } K > 0.
\end{cases}
\]

3 The Wasserstein barycenter: existence and uniqueness

Let us recall the notion of a Wasserstein barycenter of a probability measure $\Omega$ on $P(M)$ (Definition 1.1 in the introduction). Wasserstein barycenters were considered previously by Agueh-Carlier, who established existence and uniqueness results for finitely supported measures $\Omega \in P(P(\mathbb{R}^n))$ when the underlying space is Euclidean [1]. Other variants of these results can be found in [8] [32] and [33].

We present below a general existence and uniqueness result, which encompasses the earlier results found in [1] [8] [32] and [33]. The proof is essentially the same as the argument found in [32], but is included in the interest of completeness.

**Theorem 3.1 (Existence and uniqueness of the Wasserstein barycenter).** Recall the assumptions and notation in Section 2.7. If $\Omega(P_{ac}(M)) > 0$, then there exists a unique Wasserstein barycenter of $\Omega$. 
Proof. Due to compactness of $M$, the set $P(M)$ of probability measures on $M$ is weak-* compact, or, equivalently, the Wasserstein space $(P(M), W_2)$ is compact. Now, for any $\mu$, the mapping $\nu \mapsto W^2_2(\nu, \mu)$ is uniformly Lipschitz on Wasserstein space, and therefore so too is $\nu \mapsto \int_{P(M)} W^2_2(\mu, \nu) d\Omega(\mu)$. Therefore, existence of a minimizer follows immediately.

The uniqueness will follow from the fact that, with respect to linear interpolation of measures, the functional $\nu \mapsto \int_{P(M)} W^2_2(\mu, \nu) d\Omega(\mu)$ is convex and the convexity is strict if $\Omega(P_{ac}(M)) > 0$. We prove this below.

We begin by studying $\nu \mapsto W^2_2(\nu, \mu)$. Let $\nu_0, \nu_1 \in P(M)$. Let $\gamma_i$ be optimal couplings between $\nu_i$ and $\mu$, for $i = 0, 1$, respectively. We set $\nu_s = s\nu_1 + (1-s)\nu_0$ and $\gamma_s = s\gamma_1 + (1-s)\gamma_0$. Noting that $\gamma_s$ has $\nu_s$ and $\mu$ as its marginals, we have

$$W^2_2(\mu, \nu_s) \leq \int_{M \times M} d(x, y)^2 d\gamma_s = s \int_{M \times M} d(x, y)^2 d\gamma_1 + (1-s) \int_{M \times M} d(x, y)^2 d\gamma_0 = sW^2_2(\mu, \nu_1) + (1-s)W^2_2(\mu, \nu_0)$$

(3.1)

This yields convexity of the function $\nu \mapsto W^2_2(\nu, \mu)$.

Next, we will show this convexity is strict if $\mu \in P_{ac}(M)$. By the Brenier-McCann theorem (Theorem 2.3), there exists a unique optimal map $F_s : spt(\mu) \to spt(\nu_s)$ for each $s$, such that the unique optimal measure $\gamma_s \in \Gamma(\mu, \nu_s)$ is concentrated on the graph $\{(x, F_s(x))\}$.

We need to show that, assuming $\nu_0 \neq \nu_1$ and $0 < s < 1$, the inequality (3.1) is strict. Note first that the inequality is strict unless $\gamma_s$ is an optimal coupling between $\mu$ and $\nu_s$; by the uniqueness result, this means we must have $\gamma_s = \gamma_s = (Id, F_s)\#\mu$. That is, $\gamma_s$ is concentrated on the graph of $F_s$.

On the other hand, $\gamma_s$ is concentrated on the union of two graphs, $F_0(x)$ and $F_1(x)$:

$$\gamma_s = s(Id, F_1)\#\mu + (1-s)(Id, F_0)\#\mu.$$ 

This is possible only if $F_0 = F_1 = F_s \mu$ almost everywhere, which, in turn, implies $\nu_0 = (F_0)\#\mu = (F_1)\#\mu = \nu_1$. This yields strict convexity of $\nu \mapsto W^2_2(\nu, \mu)$ whenever $\mu$ is absolutely continuous with respect to volume.

Finally, integrating $\nu \mapsto W^2_2(\nu, \mu)$ with respect to $\Omega$ yields convexity of the functional $\nu \mapsto \int_{P(M)} W^2_2(\mu, \nu) d\Omega(\mu)$, and the convexity is strict under the assumption $\Omega(P_{ac}(M)) > 0$. This implies uniqueness of its minimizer, the Wasserstein barycenter of $\Omega$.

Remark 3.2. By inspecting the above proof, it is clear that Theorem 3.1 holds for more general spaces on which the optimal maps, $T\#\mu = \nu$, exist uniquely for any arbitrary absolutely continuous source measure $\mu$. This includes for example, Alexandrov spaces [3].
Example 3.3. As an illuminating example, consider the round sphere. If the \( \Omega = \frac{1}{2} [\delta_{\text{north}} + \delta_{\text{south}}] \) be the sum of two Dirac measures supported on the north and south pole, then its Wasserstein barycenter is not unique: any probability measure supported on the equator is a Wasserstein barycenter. However, if we smear out one of the Dirac measures making it absolutely continuous, then the resulting Wasserstein barycenter will be a unique (in fact absolutely continuous) measure supported near the equator.

4 Properties of the Wasserstein barycenter: first and second order balance

We develop here several properties of the Wasserstein barycenter which we will use later on. For some of these, we will need to assume that the Wasserstein barycenter is absolutely continuous with respect to volume. Conditions on \( \Omega \) ensuring this absolutely continuity will be presented later on. The main results of this section are Theorems 4.4 and 4.6 which are crucial for later sections.

4.1 Differentiability of family of dual potentials

The key results of this subsection are (4.1) and (4.2) for derivatives of the integral of a measurable family of dual potentials for optimal transport problems. We first establish the almost everywhere second differentiability of a certain measurable family of dual potentials. More precisely,

**Lemma 4.1 (a.e. x and \( \Omega \text{-a.e. } \mu \)).** Let \( \bar{\mu} \in P(M) \) and for each \( \mu \in P(M) \), let \( u_\mu \) be the dual potential (determined modulo an additive constant) for the optimal transport problem \((\Omega)\) between \( \bar{\mu} \) and \( \mu \). Let \( \Omega \) be a Borel probability on \( P(M) \). For volume almost all \( x \), \( x \mapsto u_\mu(x) \) is twice differentiable for \( \Omega \)-almost all \( \mu \in P(M) \).

**Proof.** The proof is a simple application of Fubini’s theorem. Let \( A \subset P(M) \times M \) be the set of points where the twice differentiability fails. We are to show that its projection onto \( P(M) \), namely, \( A_x = \{ \mu : (\mu, x) \in A \} \) has \( \Omega \)-measure zero, for almost all \( x \) (notice that the set \( A \) is measurable). Assume by contradiction that \( A_x \) has positive \( \Omega \)-measure for some non-measure zero set of \( x \). Then there exists \( \epsilon \geq 0 \) and a set \( B \subseteq M \) with \( |B| > 0 \) such that \( \Omega(A_x) \geq \epsilon \) for all \( x \in B \). Therefore, using Fubini’s theorem, we have

\[
(\Omega \times \text{vol})(A) = \int_M \int_{P(M)} \chi_A d\Omega d\text{vol} \geq \int_B \int_{P(M)} \chi_A d\Omega d\text{vol} = \int_B \int_{P(M)} \chi_{A_x} d\Omega d\text{vol} = \int_B \Omega(A_x) d\text{vol}(x) \geq \text{vol}(B) \epsilon > 0
\]

On the other hand, for each \( \mu \), the dual potential \( u_\mu \) is a semi-convex function \([9]\) (recall \( M \) is compact), so due to Alexandrov’s second differentiability theorem,
$u_\mu$ is twice differentiable at Lebesgue (vol) a.e. points; i.e. $A_\mu = \{x : (\mu, x) \in A\}$ has zero volume. So we have

$$(\Omega \times \text{vol})(A) = \int_{P(M)} \int_M \chi_A d\text{vol}\Omega = \int_{P(M)} \int_M \chi_{A_\mu} d\text{vol}\Omega = \int_{P(M)} 0 d\Omega = 0.$$ 

The contradiction implies the desired result.

From Lemma 4.1, we see that for Lebesgue almost every $x$, the maps $\mu \mapsto \nabla_x u_\mu(x)$ and $\mu \mapsto \nabla_2^2 x u_\mu(x)$ are well defined $\Omega(\mu)$ a.e. Now, an essential ingredient in our work is the function $x \mapsto \int_{P(M)} u_\mu(x) d\Omega(\mu)$, where $u_\mu$ is the dual potential function given in Lemma 4.1. Note that this function is Lipschitz and semi-convex since each $u_\mu$ is uniformly Lipschitz and semi-convex (recalling that $M$ is compact). By Rademacher’s theorem and Alexandrov’s second differentiability theorem, this function is twice differentiable for Lebesgue almost everywhere $x$. Moreover, applying Lemma 4.1 for almost every $x$, we immediately have the following:

**Proposition 4.2** (Derivatives inside the integral $\int_{P(M)} d\Omega$).

\[
\nabla_x \int_{P(M)} u_\mu(x) d\Omega(\mu) = \int_{P(M)} \nabla_x u_\mu(x) d\Omega(\mu), \tag{4.1}
\]

\[
\nabla_2^2_x \int_{P(M)} u_\mu(x) d\Omega(\mu) \geq \int_{P(M)} \nabla_2^2 u_\mu(x) d\Omega(\mu). \tag{4.2}
\]

**Proof.** This can be seen by applying the dominated convergence theorem for (4.1) due to uniform Lipschitzness of $u_\mu$ and Fatou’s lemma for (4.2) due to the semi-convexity of $u_\mu$. \qed

### 4.2 First and second order balance at the Wasserstein barycenter

We now consider the Wasserstein barycenter measure $\bar{\mu}$ of $\Omega \in P(P(M))$, and the dual potentials $u_\mu$ for optimal transport problems (4.1) from $\bar{\mu}$ to $\mu$. Using the equations (4.1) and (4.2), we will establish the main results of this section, namely, the first and second order balance between the $u_\mu$’s with respect to $\Omega$:

**Theorem 4.4** We begin with a lemma relating barycenters on the manifold $M$ to Wasserstein barycenters on $P(M)$:

**Lemma 4.3** (Riemannian barycenter from Wasserstein barycenter).

Let $\bar{\mu}$ be a Wasserstein barycenter of the measure $\Omega$ on $P(M)$ and assume $\bar{\mu}$ is absolutely continuous with respect to volume; let $T_{\mu}$ be an optimal map from $\bar{\mu}$ to $\mu$. Let $\lambda_z = (\mu \mapsto T_{\mu}(z))_{\#} \Omega$. Then, for $\bar{\mu}$ almost every $z$, $z$ is a barycenter of $\lambda_z$.

If, in addition, $\Omega(P_{ac}(M)) > 0$, then for $\bar{\mu}$ almost every $z$, $z$ is the unique barycenter of $\lambda_z$. 

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Proof. We first show that $\bar{\mu}$-a.e. $z$ is a barycenter $\lambda_z$. The proof is by contradiction; suppose not. Then there exists a set $A \subset M$ with $\bar{\mu}(A) > 0$ and for all $z \in A$, $z$ is not a barycenter of $\lambda_z$. We define $g : \text{spt}(\bar{\mu}) \to M$ by letting $g(z) \in BC(\lambda_z)$ be a measurable selection of the barycenters. Then, for all $z \in \text{spt}(\bar{\mu})$, we have

$$
\int_{P(M)} d^2(z, T_\mu(z))d\Omega(\mu) \geq \int_{P(M)} d^2(g(z), T_\mu(z))d\Omega(\mu)
$$

and the inequality is strict on the set $A$ of positive $\bar{\mu}$ measure. We define the probability measure

$$
\nu := g \# \bar{\mu}.
$$

For each $\mu$, the measure $\gamma_\mu := (g, T_\mu) \# \bar{\mu}$ is then a coupling of $\mu$ and $\nu$, and we have

$$
W_2^2(\mu, \nu) \leq \int_{M \times M} d^2(z, x)d\gamma_\mu(z, x) = \int_{\text{spt}(\bar{\mu})} d^2(g(z), T_\mu(z))d\bar{\mu}(z).
$$

Therefore,

$$
\int_{P(M)} W_2^2(\mu, \nu)d\Omega(\mu) \leq \int_{P(M)} \int_{\text{spt}(\bar{\mu})} d^2(g(z), T_\mu(z))d\bar{\mu}(z)d\Omega(\mu)
$$

where the second and fourth lines follow from Fubini’s theorem and the strict inequality in the third line follows from (4.3) (and the fact that (4.3) is strict on a set of $\bar{\mu}$ positive measure). This contradicts the fact that $\bar{\mu}$ is a barycenter of $\Omega$, completing the proof that $z$ is a barycenter of $\lambda_z$ for $\bar{\mu}$-a.e. $z$.

To prove the second assertion, we must show that we must have $g(z) = z$ $\bar{\mu}$ almost everywhere, under the additional assumption $\Omega(P_{ac}(M)) > 0$. By Theorem 3.1 the barycenter $\bar{\mu}$ is unique, and therefore, we must have $\nu = g \# \bar{\mu} = \bar{\mu}$, and so we have equality throughout the preceding string of inequalities. In particular the first line in (4.3) becomes

$$
\int_{P(M)} W_2^2(\mu, \nu)d\Omega(\mu) = \int_{P(M)} \int_{\text{spt}(\bar{\mu})} d^2(g(z), T_\mu(z))d\bar{\mu}(z)d\Omega(\mu).
$$

This implies that for $\Omega$ almost every $\mu$, the plan $(g, T_\mu)\bar{\mu}$ is an optimal plan between $\nu = \bar{\mu}$ and $\mu$; as $(Id, T_\mu)\bar{\mu}$ is the unique such optimal plan, by the
Brenier-McCann theorem (Theorem 2.3), this implies
\[(g, T_\mu) \# \bar{\mu} = (\text{Id}, T_\mu) \# \bar{\mu},\]
for \(\Omega\) almost all \(\mu\). In particular, as \(\Omega(P_{ac}(M)) > 0\), the preceding holds for some \(\mu \in P_{ac}(M)\). For such a \(\mu\), this means that for \(\bar{\mu}\) almost all \(z\), there exists some \(y\) such that
\[(z, T_\mu(z)) = (g(y), T_\mu(y)).\]
Now, as \(\mu\) is absolutely continuous, \(T_\mu\) is injective \(\bar{\mu}\) almost everywhere; hence, \(T_\mu(z) = T_\mu(y)\) implies \(z = y\). The preceding equation then means that, \(\bar{\mu}\) almost everywhere, we have \(z = y\), and therefore,
\[z = g(z),\]
as desired.

We now prove the following key property that gives a balance condition for the first and second order derivatives of the functions \(u_\mu\). It will play an important role in obtaining estimates on the density of the Wasserstein barycentre as well as the proof of our Wasserstein Jensen’s inequalities in the later part of the paper: Theorem 4.6 and Sections 5.2, 6 and 7.

**Theorem 4.4 (The first and second order balance at the Wasserstein barycenter).** Let \(\bar{\mu}\) be the Wasserstein barycenter of a measure \(\Omega\) on \(P(M)\). Assume \(\bar{\mu}\) is absolutely continuous and let \(T_\mu(x) = \exp_x(\nabla u_\mu(x))\) be the optimal map pushing \(\bar{\mu}\) forward to \(\mu\), where \(u_\mu\) is the associated dual potential. Then for \(\bar{\mu}\) almost all \(x\) we have

1st order balance: \[\int_{P(M)} \nabla u_\mu(x) d\Omega(\mu) = 0, \tag{4.5}\]

2nd order balance: \[\int_{P(M)} \nabla^2 u_\mu(x) d\Omega(\mu) \leq 0. \tag{4.6}\]

**Proof of the 1st order balance (4.5).** Fix an arbitrary \(x\) where the map \(T_\mu\) is well defined (which holds \(\bar{\mu}\) a.e.). Since each \(u_\mu\) is a \(c\)-convex function, for its \(c\)-dual \(u_\mu^c\), we have
\[u_\mu(y) \geq -\frac{d^2(y, T_\mu(x))}{2} - u_\mu^c(T_\mu(x))\]
for any \(y \in M\) and \(\mu \in P(M)\), with equality when \(y = x\). Integrating against \(\Omega\), we have
\[\int_{P(M)} u_\mu(y) d\Omega(\mu) \geq \int_{P(M)} -\frac{d^2(y, T_\mu(x))}{2} d\Omega(\mu) - \int_{P(M)} u_\mu^c(T_\mu(x)) d\Omega(\mu), \tag{4.7}\]
with equality when \(y = x\).
On the other hand, by Lemma 4.3, for \( \bar{\mu} \) almost every \( x \), we have that \( x \) is the barycenter of \( \lambda_z = (\mu \mapsto T_\mu(z))_\# \Omega \); that is, a minimizer of

\[
f_x : y \mapsto \int_{P(M)} d^2(y, T_\mu(x)) d\Omega(\mu).
\]

Therefore, the latter function \( f_x \), which is semi-concave is differentiable at \( x \): due to semi-concavity, there is \( C > 0 \) such that the function \( f_x(y) - C \text{dist}^2(x, y) \) is locally geodesically concave near \( x \). Minimality at \( x \) implies \( f_x(y) - C \text{dist}^2(x, y) \geq f_x(x) - C \text{dist}^2(x, y) \). Since \( y \mapsto f_x(x) - C \text{dist}^2(x, y) \) has vanishing derivative at \( x \), concavity of \( f_x(y) - C \text{dist}^2(x, y) \) implies that locally the function \( y \mapsto f_x(y) - C \text{dist}^2(x, y) \) is also locally bounded from above by the constant \( f_x(x) \). This implies the differentiability of \( f_x \) at \( x \) as well as

\[
\nabla_y \bigg|_{y=x} f_x(y) = \nabla_y \bigg|_{y=x} \int_{P(M)} d^2(T_\mu(x), y) d\Omega(\mu) = 0. \tag{4.8}
\]

By assumption, \( \bar{\mu} \) is absolutely continuous, so the Lebesgue a.e. first and second order differentiability of the function \( y \mapsto \int_{P(M)} u_\mu(y) d\Omega(\mu) \) implies \( \bar{\mu} \)-a.e. first and second order differentiability. Since equality holds in (4.7) at \( y = x \), for \( \bar{\mu} \)-a.e. \( x \), we have

\[
\nabla_y \bigg|_{y=x} \int_{P(M)} u_\mu(y) d\Omega(\mu) = \nabla_y \bigg|_{y=x} \int_{P(M)} d^2(T_\mu(x), y) d\Omega(\mu) = 0, \tag{4.9}
\]

where the last equality follows from (4.8). Equation (4.5) for \( \bar{\mu} \)-a.e. \( x \) now follows from (4.1).

**Proof of the 2nd order balance (4.6).** This follows immediately by applying Lemma 4.5 below to

\[
\psi(x) = \int_{P(M)} u_\mu(x) d\Omega(\mu),
\]

noting (4.9), and then using (4.2).

**Lemma 4.5 (Vanishing derivatives imply vanishing Hessian).** Let \( \bar{\mu} \) be an absolutely continuous measure on \( M \). Let \( \psi : M \to \mathbb{R} \) be a Lipschitz and geodesically semi-convex function. Suppose for \( \bar{\mu} \)-a.e. \( x \),

\[
\nabla_x \psi(x) = 0
\]

Then, for \( \bar{\mu} \)-a.e. \( x \),

\[
\nabla^2_x \psi(x) = 0.
\]

**Proof.** We first find a relevant set of full \( \bar{\mu} \) measure. Since \( \bar{\mu} \) is absolutely continuous and \( \nabla_x \psi(x) = 0 \) for \( \bar{\mu} \)-a.e. \( x \), there exists a full \( \bar{\mu} \) measure set \( S \) (i.e. \( \bar{\mu}(M \setminus S) = 0 \)) with the following properties at each \( x \in S \):
1. $\psi$ is second order differentiable (in the Alexandrov sense) at $x$;
2. $\nabla \psi(x) = 0$;

Moreover, due to absolute continuity of $\bar{\mu}$ and the Lebesgue density theorem, the set
$$S' = \{x \in S \mid \lim_{r \to 0} \frac{\text{vol}(S \cap B_r(x))}{\text{vol}(B_r(x))} = 1\}$$
has full $\bar{\mu}$ measure. It suffices to show $\nabla^2 \psi(x) = 0$ on $S'$.

Now, fix an arbitrary $x \in S'$. We recall that the Hessian $\nabla^2 \psi$ satisfies for each $v \in T_xM$, (see, e.g. [42, Theorem 14.25]),
$$\nabla^2 \psi(x)v = \partial \psi(\exp_x v) - \nabla \psi(x) + o(|v|) \quad \text{as } v \to 0,$$
$$= \partial \psi(\exp_x v) + o(|v|) \quad \text{(since } x \in S' \subset S),$$
where $\partial \psi$ denotes the subdifferential, which coincides with $\nabla \psi$ at differentiable points. Therefore, whenever $\exp_x v \in S$, we see
$$\nabla^2 \psi(x)v = o(|v|). \quad (4.10)$$

Now, notice that since $x$ is a density point of $S$ (by definition of $S'$), for each unit vector $w \in T_xM$, $|w| = 1$, there is a sequence of $v_k$ such that $\exp_x v_k \in S$, and $\frac{v_k}{|v_k|} \to w$ and $|v_k| \to 0$ as $k \to \infty$. Therefore,
$$\nabla^2 \psi(x)w = \lim_{k \to \infty} \nabla^2 \psi(x) \frac{v_k}{|v_k|} = \text{use (4.10)} \lim_{k \to \infty} \frac{o(|v_k|)}{|v_k|} = 0.$$

This shows that $\nabla^2 \psi(x) = 0$, completing the proof. \hfill \Box

### 4.3 Jacobian determinants of optimal maps from a Wasserstein barycenter

Theorem 4.6 (Jacobian determinant inequality for the Wasserstein barycenter). Assume that the Wasserstein barycenter $\bar{\mu}$ of the measure $\Omega$ on
$P(M)$ is absolutely continuous. Letting $T_\mu$ denote the optimal map from $\bar{\mu}(x)$ to $\mu$, consider the measure on $M$ given by

$$\lambda_x := \int_{P(M)} \delta_{T_\mu(x)} d\Omega(\mu), \quad (4.11)$$

which is defined with respect to a.e. $x$ (due to Lemma 4.1). Then, for $\bar{\mu}$-a.e. $x$,

$$1 \geq \int_{P(M)} \alpha_{\lambda_x}^{1/n}(T_\mu(x)) \det^{1/n} DT_\mu(x) d\Omega(\mu).$$

Proof. We have for $\Omega$-a.e. $\mu$ and a.e. $x$ (so for $\bar{\mu}$-a.e. $x$ for absolutely continuous $\bar{\mu}$),

$$D^2 u_\mu(x) + D^2_{xx} c(x, T_\mu(x)) = -D^2_{xy} c(x, T_\mu(x)) DT_\mu(x).$$

Rearranging, integrating against $\Omega$ and using (4.6), yields

$$\int_{P(M)} -D^2_{xy} c(x, T_\mu(x)) DT_\mu(x) d\Omega(\mu)$$

$$= \int_{P(M)} \left[ D^2 u_\mu(x) + D^2_{xx} c(x, T_\mu(x)) \right] d\Omega(\mu)$$

$$\leq \int_{P(M)} D^2_{xx} c(x, T_\mu(x)) d\Omega(\mu) \quad \text{(by (4.6))}$$

Note that each $-D^2_{xy} c(x, T_\mu(x)) DT_\mu(x)$ is positive semi-definite by the $c$-convexity of $u_\mu$, and so is $\int_{P(M)} D^2_{xx} c(x, T_\mu(x)) d\Omega(\mu)$ because $x$ is the barycenter of $\lambda_x$ by Lemma 4.3. Therefore,

$$\det \left[ \int_{P(M)} -D^2_{xy} c(x, T_\mu(x)) DT_\mu(x) d\Omega(\mu) \right]$$

$$\leq \det \left[ \int_{P(M)} D^2_{xx} c(x, T_\mu(x)) d\Omega(\mu) \right].$$

Combining Minkowski’s determinant inequality with Jensen’s inequality then yields:

$$\int_{P(M)} \det^{1/n} [-D^2_{xy} c(x, T_\mu(x))] \det^{1/n} [DT_\mu(x)] d\Omega(\mu)$$

$$\leq \det^{1/n} \left[ \int_{P(M)} D^2_{xx} c(x, T_\mu(x)) d\Omega(\mu) \right].$$

This yields the desired inequality, by the definitions of $\lambda_x$ (see (4.11)) and $\alpha_{\lambda_x}$ (see (2.10)). \qed
5 Absolute continuity of the Wasserstein barycenter of finitely many measures

In this section, we first establish absolute continuity of the Wasserstein barycenter \( \bar{\mu} \) (which is itself a measure on \( M \)) of finitely many probability measures \( \mu_i, i = 1, ..., m \) with weights \( \lambda_i \geq 0, i = 1, ..., m \); see Theorem 5.1. This result is interesting in its own right, but will also prove to be crucial to obtain derivative estimates on optimal maps from \( \bar{\mu} \) to the measures \( \mu_i \): see Theorem 5.6. We will use it in Section 6 to treat general probability measures \( \Omega \) on \( P(M) \), by an approximation argument.

We regard the Wasserstein barycenter as the metric barycenter of the probability measure \( \Omega = \sum_{i=1}^{m} \lambda_i \delta_{\mu_i} \) on the space of probability measures \( P(M) \) equipped with the Wasserstein metric \( W_2 \), where \( \delta_{\mu_i} \) denotes the Dirac measure on \( P(M) \) concentrated at \( \mu_i \in P(M) \), i.e. for each Borel set (with respect to the weak-* topology) \( \Gamma \subset P(M) \), it satisfies

\[
\delta_{\mu_i}(\Gamma) = \begin{cases} 
1 & \text{if } \mu_i \in \Gamma, \\
0 & \text{otherwise.}
\end{cases}
\]

In our main results, we also assume that \( \Omega(P_{ac}(M)) > 0 \) (that is, at least one of the \( \mu_i \) is absolutely continuous with respect to volume and \( \lambda_i \neq 0 \)).

We now state the main result of this section:

**Theorem 5.1 (Absolute continuity of the Wasserstein barycenter for finitely many measures).** Assume \( \mu_1 \) is absolutely continuous with respect to volume on \( M \) and let \( \lambda_1 > 0 \). Then the Wasserstein barycenter \( \bar{\mu} \in P(M) \) of the measure \( \Omega = \sum_{i=1}^{m} \lambda_i \delta_{\mu_i} \in P(P(M)) \) is absolutely continuous with respect to volume on \( M \).

The above result was shown in the Euclidean case \( M = \mathbb{R}^n \) by Agueh and Carlier [1]. However, their method exploits the underlying Euclidean geometry and the special algebraic structure of the multi-marginal transport in a pivotal way: in particular, in the Euclidean setting, the barycenter of the points \( x_1, ..., x_m \in \mathbb{R}^n \) with respective weights \( \lambda_1, ..., \lambda_m \) is nothing but the algebraic average \( \sum_{i=1}^{m} \lambda_i x_i \). To demonstrate this special nature of the Euclidean space, we provide an alternative but simpler proof to [1] of the absolute continuity of the Wasserstein barycenter in that case:

**Proof of Theorem 5.1 when \( M = \mathbb{R}^n \) is the Euclidean space.** (See [1] for a different proof.) By Theorem 2.4, assertion 3, the Wasserstein barycenter measure \( \bar{\mu} \) is given by \( (\bar{x}_\lambda)_{\#} \gamma \), where \( \gamma \in P(\Pi_{i=1}^{m} M) \) is the Kantrovich solution of the multi-marginal problem (2.8), and \( \bar{x}_\lambda \) is the barycenter map with weights \( \lambda_i \)'s: in this Euclidean setting, \( \bar{x}_\lambda(x_1, ..., x_m) = \sum_{i=1}^{m} \lambda_i x_i \).

Let \( (x_1, ..., x_m), (x'_1, ..., x'_m) \in \text{spt } \gamma \), and denote \( z = \bar{x}_\lambda(x_1, ..., x_m) = \sum_{i=1}^{m} \lambda_i x_i \), \( z' = \bar{x}_\lambda(x'_1, ..., x'_m) = \sum_{i=1}^{m} \lambda_i x'_i \). Using the monotonicity of the support of \( \gamma \),
namely,
\[ \sum_{i=1}^{m} \lambda_i (|z - x_i|^2 + |z' - x_i'|^2) \leq \sum_{i=1}^{m} (\lambda_i |z' - x_i|^2 + |z - x_i|^2), \]
we get the following very special inequality:
\[ |z - z'|^2 \geq \sum_{i=1}^{m} \lambda_i^2 |x_i - x_i'|^2 \]  
(5.1)

From this we see that the inverse map \((\bar{x}_\lambda)^{-1} : \text{spt}\bar{\mu} \rightarrow \text{spt}\gamma\) is Lipschitz with constant \(\frac{1}{\lambda_1}\), and therefore the composition of the inverse and the projection \(\pi_1 : \text{spt}\gamma \rightarrow \text{spt}\mu_1\) is Lipschitz as well, also with Lipschitz constant \(\frac{1}{\lambda_1}\). Since \(\mu_1\) is absolutely continuous, and this composition pushes \(\bar{\mu}\) forward to \(\mu_1\), this immediately implies \(\bar{\mu}\) is absolutely continuous. Moreover, if \(\mu_1 \in L^\infty(M)\), then,
\[ \left\| \frac{d\bar{\mu}}{dx} \right\|_\infty \leq \frac{1}{\lambda_1} \left\| \frac{d\mu_1}{dx} \right\|_\infty \]
where \(\frac{d\mu}{dx}\) denotes the Radon-Nikodym derivative. This completes the proof in the Euclidean case.

Neither this proof, nor a related argument on very special Riemannian manifolds (simply connected manifolds with nonpositive curvature) seem suited to handle the general Riemannian case. In particular, an analogue of the inequality (5.1) is not known in that context.

Instead, our proof is inspired by the method of Figalli and Juillet, who established absolute continuity of Wasserstein geodesics (also known as McCann’s displacement interpolants) over the Heisenberg group and Alexandrov spaces, using a very nice approximation argument.

We devote the following subsection to the proof of Theorem 5.1.

### 5.1 Proof of the absolute continuity of the Wasserstein barycenter for finitely many measures

The proof requires a few lemmata.

Recall that \(M\) is an \(n\)-dimensional Riemannian manifold. To fix notation, for each Borel set \(E \subseteq M\), and \(m - 1\) points \(x_2, \ldots, x_m\) and the weights \(\lambda_i \geq 0\), \(i = 1, \ldots, m\) (and \(\sum_i \lambda_i = 1\)), let
\[ b_{C\lambda}(E, x_2, \ldots, x_m) := \bigcup_{x \in E} b_{C\lambda}(x, x_2, \ldots, x_m). \]  
(5.2)

A crucial geometric property of this set is given in the following lemma, which roughly speaking, provides a Lipschitz inverse map of \(y \mapsto b_{C\lambda}(y, x_2, \ldots, x_m)\), implying bounded volume distortion. The underlying principle behind the proof
is similar to the idea behind the proof of Theorem 4.6 if the gradients of uniformly semi-concave functions (e.g. \(\text{dist}^2\)) are balanced in the sense of (4.5), then the second derivatives of the functions are bounded uniformly from both above and below. For the uniform estimates in what follows, it is important that the points \(x_2, ..., x_m\) are fixed.

**Lemma 5.2 (A Lipschitz inverse to the barycentre map).** Assume \(\lambda_1 > 0\) in (5.2). Then, there exists a map

\[
G_{\lambda;x_2,\ldots,x_m} : bc_{\lambda}(M, x_2, \ldots, x_m) \to M
\]

such that for each Borel set \(E\),

1. \(E = G_{\lambda;x_2,\ldots,x_m}(bc_{\lambda}(E, x_1, \ldots, x_m))\);
2. \(G_{\lambda;x_2,\ldots,x_m}\) is uniformly locally Lipschitz with a Lipschitz constant \(C = C(\lambda, M)\) depending only on \(\lambda = (\lambda_1, \ldots, \lambda_m)\) and \(M\) (that is, not on \(x_2, \ldots, x_m\)).

In particular, this implies

\[
\text{vol}(G_{\lambda;x_2,\ldots,x_m}(E \cap bc_{\lambda}(M, x_2, \ldots, x_m))) \leq C^n \text{vol}(E \cap bc_{\lambda}(M, x_2, \ldots, x_m))
\]

for any Borel set \(E \subset M\).

**Proof.** We claim that for each \(z \in bc_{\lambda}(M, x_2, \ldots, x_m)\), there exists a unique point, which we will define to be \(G_{\lambda;x_2,\ldots,x_m}(z)\) satisfying

\[
z \in bc_{\lambda}(G_{\lambda;x_2,\ldots,x_m}(z), x_2, \ldots, x_m).
\]

Existence of \(G_{\lambda;x_2,\ldots,x_m}(z)\) follows from the definition of \(bc_{\lambda}(M, x_2, \ldots, x_m)\). To see uniqueness, we define

\[
g(z) = \frac{1}{\lambda_1} \sum_{i=2}^{m} \lambda_i d^2(x_i, z),
\]

and recall that for each \(x\), every \(z \in bc_{\lambda}(x, x_2, \ldots, x_m)\) is not in the cutlocus of any \(x_i\) [21]. Therefore, \(g(z)\) is twice differentiable at each \(z \in bc_{\lambda}(M, x_2, \ldots, x_m)\).

Now, for any point \(y\) such that

\[
z \in bc_{\lambda}(y, x_2, \ldots, x_m),
\]

we have \(\nabla_w \big|_{w = z} d^2(y, w) = -\nabla_z g(z)\) or, equivalently, \(y = \exp_z \nabla g(z)\). Therefore, \(\exp_z \nabla g(z)\) is the only point with the desired property, establishing uniqueness, as well as the formula

\[
G_{\lambda;x_2,\ldots,x_m}(z) = \exp_z \nabla g(z).
\]

It follows by definition that

\[
E = G_{\lambda;x_2,\ldots,x_m}(bc_{\lambda}(E, x_1, \ldots, x_m)),
\]

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proving assertion 1.

To prove the second assertion, first observe that due to minimality of \( w \mapsto d^2(G_{\lambda;x_2,...,x_m}(z), w) + g(w) \) at \( w = z \), we have

\[
\nabla^2_w d^2(G_{\lambda;x_2,...,x_m}(z), w) + \nabla^2 g(z) \geq 0.
\]

It is well known that the function \( z \mapsto d^2(y, z) \) is semi-concave and satisfies the estimate \( \nabla^2 d^2(y, z) \leq C' \) for some \( C' \) depending only on \( M \) (see, e.g. [9]); it follows from the definition of \( g \) that we have \( \nabla^2 g(z) \leq \frac{C'}{\lambda} \). The inequality above then yields \( \nabla^2_d g(z) \geq -C' \) for each \( z \in bc_{\lambda}(x, x_2, ..., x_m) \).

Now, by the same reasoning, it follows that for each \( z \in bc_{\lambda}(x, x_2, ..., x_m) \), and each \( i \), we have \( \nabla^2 d^2(x_i, z) \geq -\lambda_i C' \). Set \( K_\lambda = \max_i \frac{(1-\lambda_i) C'}{\lambda_i} \).

By continuity of \( \nabla^2 d^2(y, z) \) away from the cutlocus, and compactness of \( M \), there exists an \( r > 0 \) such that for each \( y \) and each \( \tilde{z} \) with \( \nabla^2 d^2(y, \tilde{z}) \geq -K_\lambda \), we have \( \nabla^2 d^2(y, z) \geq -K_\lambda - 1 \) for \( z \in B_r(\tilde{z}) \).

Therefore, for each \( \tilde{z} \in bc_{\lambda}(x, x_2, ..., x_m) \), we obtain \( |\nabla^2 g(\tilde{z})| \leq K \), for \( K = \max\{ \frac{C'}{\lambda}, K_\lambda + 1 \} \) on \( B_r(\tilde{z}) \); as the exponential map is Lipschitz, we obtain that \( G_{\lambda;x_2,...,x_m}(z) = \exp_{y,} \nabla g(\tilde{z}) \) is Lipschitz on \( B_r(\tilde{z}) \), and therefore on \( B_r(\tilde{z}) \cap bc_{\lambda}(x, x_2, ..., x_m) \), with a Lipschitz constant \( C \) depending only on \( \lambda \) and \( M \).

The property

\[
\text{vol}(G_{\lambda;x_2,...,x_m}(E \cap bc_{\lambda}(M, x_2, ..., x_m))) \leq C^n \text{vol}(E \cap bc_{\lambda}(M, x_2, ..., x_m))
\]

now follows from standard arguments in geometric measure theory; see, for example, [9] Propositions 12.6 and 12.12. \( \square \)

**Lemma 5.3** (Absolute continuity of the Wasserstein barycenter when all but one marginal is discrete). Assume \( \mu_1 \) is a probability measure, absolutely continuous with respect to volume on \( M \) and let \( \lambda_1 > 0 \). Moreover, assume that \( \mu_i \), for \( i = 2, ..., m \) are discrete measures on \( M \), i.e., each \( \mu_i \), \( i \geq 2 \) is of the form \( \mu_i = \sum_{j=1}^{N_i} \delta_{x^i_j} \) with \( x^i_j \in M \). Let \( \bar{\mu} \) be the unique Wasserstein barycenter measure of the measures \( \mu_i \) with weights \( \lambda_i \), \( i = 1, ..., m \).

Then there is a finite collection of points \( \{(x^1_j, ..., x^m_j)\}_{j=1}^{L} \subseteq \prod_{i=2}^{m} M \) such that, for any Borel \( E \subset M \), we have

\[
\bar{\mu}(E) = \sum_{j=1}^{L} \mu_1(G_j(bc_j \cap E))
\]

where, using the notation from the last lemma,

\[
\begin{align*}
G_j &:= G_{\lambda;x^2_j,...,x^m_j} \\
bc_j &:= \text{spt} \bar{\mu} \cap bc_{\lambda}(M, x^2_j, ..., x^m_j)
\end{align*}
\]

**Proof.** From Theorem 2.4 there exists a unique multi-marginal optimal plan, say \( \gamma \), for the \( m \) measures \( \mu_1, \mu_2, ..., \mu_m \); moreover, the Wasserstein barycenter measure \( \bar{\mu} \) is given by \( \bar{\mu} = \tilde{x}_{\lambda} \# \gamma \).

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Since the measures $\mu_i, i = 2, \ldots, m$ are discrete measures on $M$, the support $\text{spt}\gamma$ is an almost everywhere disjoint union of sets $W_j, j = 1, \ldots, L$ satisfying
\[
\pi_2 \times \ldots \times \pi_m(W_j) = \{(x^1_j, \ldots, x^m_j)\},
\]
for some $x^l_j \in M$, in the support of the discrete measure $\mu_i$ for $i = 2, \ldots, m$.

From Theorem 2.4, the maps $\pi_1 : \text{spt}\gamma \to \text{spt}\mu_1$ and $\bar{x}_\lambda : \text{spt}\gamma \to \text{spt}\bar{\mu}$ are one-to-one and onto ($\gamma$-a.e.); thus, the correspondence $G_j$, restricted to $\text{spt}\bar{\mu}$ is one-to-one and onto $\pi_1(W_j)$ ($\mu_1$ a.e.) as well. Note that this is the only place where we use optimality of $\gamma$ (or the optimality of $\bar{\mu}$). The above bijections now give a partition of $\text{spt}\mu_1 = \bigcup_{j=1}^L \pi_1(W_j)$, and a partition $\text{spt} \bar{\mu} = \bigcup_{j=1}^L bc_j$, up to sets of $\mu_1$ measure 0 and sets of $\bar{\mu}$ measure 0, respectively.

Therefore, from $\bar{\mu} = \bar{x}_\lambda \gamma$ and $\pi_1 \gamma = \mu_1$, we see, for any Borel set $E \subset M$,
\[
\bar{\mu}(E) = \mu_1(\bigcup_{j=1}^L G_j(bc_j \cap E)) \quad \text{(5.3)}
\]
\[
= \sum_{j=1}^L \mu_1(G_j(bc_j \cap E)) \quad \text{(5.4)}
\]
(since the $G_j(bc_j)$’s are disjoint modulo a set of $\mu_1$ measure 0.)

establishing the claim. \hfill \square

We also need the following continuity of the Wasserstein barycenter under weak-* convergence.

**Lemma 5.4 (Continuity of Wasserstein barycenter in the weak-* topology).** Suppose $\mu^N_i$ converges in the weak-* topology to $\mu_i$ for each $i$, as $N \to \infty$. Then any weakly-* convergent subsequence of Wasserstein barycenter measure $\bar{\mu}^N$ of the measures $\mu^N_1, \ldots, \mu^N_m$ with weights $\lambda_1, \ldots, \lambda_m$ converges in the weak-* topology to a Wasserstein barycenter measure $\bar{\mu}$ of the measures $\mu_1, \ldots, \mu_m$, with the same weights.

**Proof.** For each measure $\nu$, we have by definition
\[
\sum_{i=1}^m \lambda_i W^2_2(\mu_i^N, \bar{\mu}^N) \leq \sum_{i=1}^m \lambda_i W^2_2(\mu_i^N, \nu)
\]
Now, passing to any weak-* convergent subsequence $\bar{\mu}^N \to \bar{\mu}$ and taking the limit in the preceding inequality and using continuity of $W_2$ with respect to the weak* topology, we obtain
\[
\sum_{i=1}^m \lambda_i W^2_2(\mu_i, \bar{\mu}) \leq \sum_{i=1}^m \lambda_i W^2_2(\mu_i, \nu).
\]
As $\nu$ was arbitrary, this implies that $\bar{\mu}$ is by definition a barycenter of the $\mu_i$’s, as desired. \hfill \square
We now prove the main theorem of this section:

**Proof of Theorem 5.1** We approximate each \(\mu_2, \ldots, \mu_m\) in the weak-* topology by linear combinations \(\mu_{N2}, \ldots, \mu_{Nm}\) of \(N\) Dirac masses. Note that by Lemma 5.4, the Wasserstein barycenters \(\tilde{\mu}^N\) converges in the weak-* topology to \(\tilde{\mu}\), the Wasserstein barycenter measure of the original measures \(\mu_1, \ldots, \mu_m\), which is unique due to Theorem 3.1 and the assumption that \(\mu_1\) is absolutely continuous.

We establish absolute continuity of \(\tilde{\mu}\) by contradiction. If \(\tilde{\mu}\) is not absolute continuous, then there is a Lebesgue measure zero set, say \(S\), such that \(\tilde{\mu}(S) \geq \delta\), for some positive \(\delta > 0\). We can choose small (open) neighbourhoods of \(S\), say, \(U_k\) with \(\text{vol}(U_k) \leq 2^{-k}\); as \(S \subseteq U_k\), we have \(\tilde{\mu}(U_k) \geq \delta\). Now, due to the weak-* convergence, for a large \(N_k\), we have \(\tilde{\mu}^{N_k}(U_k) \geq \delta/2\). Then, by Lemma 5.3 we have

\[
\tilde{\mu}^{N_k}(U_k) = \mu_1(\bigcup_{j=1}^{L_{N_k}} G_j(bc_j \cap U_k)).
\]

But, note that from Lemma 5.2 assertion 2,

\[
\text{vol}(\bigcup_{j=1}^{L_{N_k}} G_j(bc_j \cap U_k)) \leq C^n \text{vol}(U_k).
\]

This implies there is a sequence of Borel sets \(V_k\) (i.e. \(V_k = (\bigcup_{j=1}^{L_{N_k}} G_j(bc_j \cap U_k))\)) with \(\text{vol}(V_k) \lesssim 2^{-k}\), and

\[
\mu_1(V_k) \geq \delta/2.
\]

This is a contradiction and completes the proof, since the absolute continuity of \(\mu_1\) (with respect to vol) is equivalent to the following property: for every \(\epsilon > 0\), there is \(\eta > 0\) such that \(\mu(A) \leq \epsilon\) for all Borel sets \(A\) with \(\text{vol}(A) \leq \eta\).

\[\square\]

### 5.2 Upper bound of the density of the Wasserstein barycenter

In this subsection, we prove that the density of the barycenter can be controlled by the densities of the marginals (see Theorem 5.6 below). Aside from being interesting in its own right, this result will be used in the proof of Theorem 6.1 on absolute continuity of the Wasserstein barycenter of a general measure on \(P(M)\).

Before presenting the main theorem of the section, we introduce the following notation:

**Definition 5.5 (The set \(A_L\)).** For \(0 < L < \infty\), let \(A_L\) be the set of Borel probability measures on \(M\), absolutely continuous with respect to volume, whose densities have \(L^\infty\) norm less than or equal to \(L\).

Note that, since the bound on the \(L^\infty\) norm is preserved under weak-* convergence, \(A_L\) is a weakly-* closed, and thus Borel measurable, subset of \(P(M)\). Here is the main theorem of the section:

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Theorem 5.6 (Upper bound on the density of the Wasserstein barycenter). Fix $L > 0$. Let $\bar{\mu}^N = f^N d\text{vol}$ be the barycenter of the measures $\mu_i$, $i = 1, 2, ..., N$, with weights $\lambda_i$ and assume that at least some of the $\mu_i = g_i d\text{vol}$ belong to $A_L$ (note that this condition ensures uniqueness and absolute continuity of the barycenter $\bar{\mu}^N$, by Theorems 3.1 and 5.1, respectively). Then we have

$$C \| \bar{f}^N \|_\infty \leq \left[ \sum_{\mu_i \in A_L} \lambda_i \right]^{-n} \sup_{\mu_i \in A_L} \| g_i \|_\infty,$$

where $C$ is the constant from (2.13).

Proof. Let $T_i^N$ be the optimal maps from $\bar{\mu}^N$ to $\mu_i$. Note that, in this setting, Theorem 4.6 reduces to

$$1 \geq \sum_{i=1}^N \lambda_i \lambda_i \left( T_i^N(x) \right)^{1/n} \det^{1/n} DT_i^N(x)$$

(5.5)

for almost every $x$.

The remainder of the proof follows an argument in [22]. For a.e. $x \in \text{spt} \bar{\mu}^N$, we have the following Jacobian determinant equations

$$\det DT_i^N(x) = \frac{\bar{f}^N(x)}{g_i(T_i^N(x))}.$$  

Using this and (2.13), we rearrange (5.5) to get

$$C \bar{f}^N(x) \leq \left[ \sum_{\mu_i \in A_L} \lambda_i \frac{1}{g_i(T_i^N(x))} \right]^{-n}$$

for $C = C(\text{diam}(M), K, n)$ in (2.13). Applying convexity of $0 < t \mapsto t^{-n}$ to this, we see

$$C \bar{f}^N(x) \leq \left[ \sum_{\mu_i \in A_L} \lambda_i \right]^{-n-1} \sum_{\mu_i \in A_L} \lambda_i g_i(T_i^N(x)).$$

In particular,

$$C \| \bar{f}^N \|_\infty \leq \left[ \sum_{\mu_i \in A_L} \lambda_i \right]^{-n} \sup_{\mu_i \in A_L} \| g_i \|_\infty.$$

\hfill \Box

6 Absolute continuity of the Wasserstein barycenters of general distributions

In this section, we establish the absolute continuity of the Wasserstein barycenter of a general measure $\Omega$ on $P(M)$ under a reasonable assumption:
Theorem 6.1 (Absolute continuity of barycenters of general measures on $P(M)$). Let $\Omega$ be a probability measure on Wasserstein space $P(M)$ over an $n$-dimensional compact Riemannian manifold $M$. Assume that $\text{Ric}_M \geq K$ for $K \in \mathbb{R}$. Assume $\Omega(\mathcal{A}_L) > 0$. Then, the Wasserstein barycenter measure $\bar{\mu}$ of $\Omega$ is absolutely continuous on $M$ with density $\bar{f}$ satisfying

$$\|\bar{f}\|_\infty \leq \frac{L}{C \Omega(\mathcal{A}_L)^n}$$

where $C = C(M)$ is the constant given in (2.13).

The proof is by approximation; as $P(M)$ is itself a complete separable metric space, we can approximate the measure $\Omega$ by finitely supported measures $\Omega^N$, which have absolutely continuous Wasserstein barycenters as shown in Theorem 5.1 (see Lemma 6.2 below). To pass to the limit, we require the uniform estimates from Theorem 5.6, which then require the technical but reasonable hypothesis $\Omega(\mathcal{A}_L) > 0$.

For details, we start with the following topological lemma (without proof) for an approximation argument.

Lemma 6.2 (Approximation by Dirac deltas; see, e.g. [42, Theorem 6.18]). For any complete separable metric space $X$, its Wasserstein space $P(X)$ is also a complete separable metric space. Moreover, for each Borel probability measure $\nu$ on $X$, there exists a sequence $\nu^N = \sum_{i=1}^N \lambda_i \delta_{x_i}$ of finitely supported probability measures on $X$ converging in the weak-* topology to $\nu$.

Proof of Theorem 6.1 Decompose

$$\Omega = \Omega(\mathcal{A}_L) \Omega_{\mathcal{A}_L} + (1 - \Omega(\mathcal{A}_L)) \Omega_{P(M)\setminus\mathcal{A}_L},$$

where $\Omega_{\mathcal{A}_L}$ and $\Omega_{P(M)\setminus\mathcal{A}_L}$ denote the probability measures obtained by restricting $\Omega$ to $\mathcal{A}_L$ and $P(M) \setminus \mathcal{A}_L$, respectively, then normalizing.

It is clear that $\mathcal{A}_L$ is closed in the weak-* topology of $P(M)$, and therefore is itself a complete separable metric space, and so applying Lemma 6.2 to $X = \mathcal{A}_L$ yields a sequence $\Omega^N$ of finitely supported probability measures on $\mathcal{A}_L$ converging in the weak-* topology (on $P(\mathcal{A}_L)$) to $\Omega_{\mathcal{A}_L}$. Similarly, applying Lemma 6.2 to $X = P(M)$ we can find a sequence $\Omega^N_{P(M)\setminus\mathcal{A}_L}$ of finitely supported probability measures on $P(M)$ converging in the weak-* topology on $P(P(M))$ to $\Omega_{P(M)\setminus\mathcal{A}_L}$ (note however, that the $\Omega^N_{P(M)\setminus\mathcal{A}_L}$ need not be supported on the open set $\Omega_{P(M)\setminus\mathcal{A}_L}$).

We then have that the finitely supported measures

$$\Omega^N := \Omega(\mathcal{A}_L) \Omega^N_{\mathcal{A}_L} + (1 - \Omega(\mathcal{A}_L)) \Omega^N_{P(M)\setminus\mathcal{A}_L}$$

converge weakly-* to $\Omega$, and by construction, for each $N$ we have $\Omega^N(\mathcal{A}_L) \geq \Omega(\mathcal{A}_L) > 0$. We will denote $\Omega^N = \sum_{i=1}^N \lambda_i \delta_{\mu_i}$. Denote by $\bar{\mu}^N \in P(M)$ the Wasserstein barycenter of $\Omega^N$; $\bar{\mu}^N$ is unique and absolutely continuous as a result of Theorems 5.1 and 5.6 and the fact that $\Omega^N(\mathcal{A}_L) > 0$. 

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Let $T^N_i$ be the optimal maps from $\bar{\mu}^N$ to $\mu_i$. Let $\bar{f}^N$, $g^N_i$ be the density functions for the absolutely continuous measures $\bar{\mu}^N$ and $\mu_i \in \mathcal{A}_L$, respectively. From Theorem 5.6

$$C\|\bar{f}^N\|_\infty \leq \left[ \sum_{\mu_i \in \mathcal{A}_L} \lambda_i \right]^{-n} \sup_{\mu_i \in \mathcal{A}_L} \|g_i\|_\infty \leq \frac{L}{\Omega(\mathcal{A}_L)^n}.$$ 

Now, for any open $A \subseteq M$, we can pass to the weak-* limit, as $\bar{\mu}(A) \leq \liminf_{N \to \infty} \bar{\mu}^N(A)$, to obtain,

$$\bar{\mu}(A) \leq \frac{1}{C} \liminf_{N \to \infty} \int_A \bar{f}^N(x)dx \leq \frac{L}{C\Omega(\mathcal{A}_L)^n \text{vol}(A)}.$$

For a non open set $A \subseteq M$, we get the same inequality using an approximation. This inequality is equivalent to the desired bound on $\|\bar{f}\|_\infty$, and so the proof is complete.

Once the absolute continuity of the barycenter is established, one can use Theorem 4.6 to obtain refined estimates on its density in terms of the generalized volume distortion coefficients.

**Corollary 6.3 (Density estimates with volume distortion).** Assume the conditions in Theorem 6.1, and that $\Omega$-a.e $\mu$ is absolutely continuous. Denote the density of the barycentre $\bar{\mu}$ by $\bar{f}$, the density of the measure $\mu$ by $f_\mu$, and the optimal map between $\bar{\mu}$ and $\mu$ by $T_\mu$. Then, for almost all $x$, we have

$$\bar{f}(x) \leq \left[ \int_{P(M)} \frac{\alpha^{1/n}_{\lambda_x}(T_\mu(x))}{(f_\mu(T_\mu(x)))^{1/n}} d\Omega(\mu) \right]^{-n}.$$

**Proof.** From Lemma 4.1 for a.e. $x$, $T_\mu$ is differentiable at $x$ for $\Omega$-a.e $\mu$ with the change of variable formula,

$$\bar{f}(x) = f_\mu(T_\mu(x)) \det DT_\mu(x).$$

Moreover, wherever the above equation holds and $f(x)$ is non zero, $f_\mu(T_\mu(x))$ is clearly nonzero as well. Multiply the change of variables equation by the coefficients $\alpha_{\lambda_x}(T_\mu(x))/f_\mu(T_\mu(x))$, take the resulting expression to the $1/n$ power and integrate to obtain

$$\bar{f}^{1/n}(x) \int_{P(M)} \frac{\alpha^{1/n}_{\lambda_x}(T_\mu(x))}{(f_\mu(T_\mu(x)))^{1/n}} d\Omega(\mu) = \int_{P(M)} \alpha^{1/n}_{\lambda_x}(T_\mu(x)) \det^{1/n} DT_\mu(x) d\Omega(\mu).$$

The righthand side is less than 1 by Theorem 4.6, completing the proof. 

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Using this and Proposition 2.9 we get an analogue of [42, Corollary 19.5]:

**Corollary 6.4 (Density estimates under \( \text{Ric} \geq 0 \)).** Assume the conditions in Theorem 6.1 that \( \Omega \)-a.e. \( \mu \) is absolutely continuous and that \( \text{Ric} \geq 0 \). Then \( \| \bar{f} \|_{L^\infty} \leq \| \| f_\mu \|_{L^\infty(M)} \|_{L^\infty(\Omega)} \), where \( \| \cdot \|_{L^\infty(\Omega)} \) is the \( L^\infty \)-norm on \( P(M) \) with respect to the measure \( \Omega \in P(P(M)) \).

## 7 Convexity over Wasserstein barycenters

The goal of this section is to establish a geometric version of Jensen’s inequality for measures \( \Omega \) on \( P(M) \) with respect to a class of functionals on \( P(M) \) that are studied extensively in the literature. In fact, we establish two forms of Jensen’s inequality on \( P(M) \), which we will call Wasserstein Jensen’s inequalities. See Theorems 7.11 and 7.14 below for the results and recall the historical discussion of Jensen’s inequality in Section 1.

Our proofs below use in a crucial way the absolute continuity of \( \bar{\mu} \) of the Wasserstein barycenter of \( \Omega \in P(P(M)) \), as given in Theorem 6.1. First, it guarantees the existence and uniqueness of optimal maps \( T_\mu \) from \( \bar{\mu} \) to each \( \mu \in \text{spt} \Omega \), via the Brenier-McCann theorem (Theorem 2.3). Moreover, many of the functionals we consider are defined only on the subset \( P_{ac}(M) \subseteq P(M) \) of absolutely continuous measures. In addition, the absolute continuity of \( \bar{\mu} \) allows us to use the first and second order balance conditions (with respect to \( \Omega \)) of \( T_\mu \) given in Theorem 4.4 as well as the associated Jacobian determinant equation for \( T_\mu \).

Finally, let us note that our Wasserstein Jensen’s inequalities on \( P(M) \) with a curved underlying space \( M \) cannot be easily established by adapting the techniques from the Euclidean case, \( M = \mathbb{R}^n \). In the Euclidean case, i.e. in \( P(\mathbb{R}^n) \), the result was proven by Agueh-Carlier [1] (when the support of the measure \( \Omega \) on \( P(\mathbb{R}^n) \) is finite), using so-called generalized geodesics, which amount to exponentiation of the linear interpolation of \( c \)-convex functions. A key ingredient in their proof is the fact that, in \( \mathbb{R}^n \) with \( c = |x - y|^2/2 \), the linear interpolant of two \( c \)-convex functions is again \( c \)-convex. However, their method is restrictive, since \( c \)-convexity of functions is not preserved under linear interpolation on more general spaces. In fact, as shown in [14], this property holds only on so-called nonnegatively cross curved spaces, which are spaces satisfying a certain fourth order condition on the metric; this fourth order condition is a strengthened variant of the celebrated Ma-Trudinger-Wang condition [27,40] arising in the regularity theory of optimal transport maps (see also [19,24]). Nonnegatively cross curved spaces include, for example, Euclidean space, the round sphere [25] and its small perturbations in two dimensions [11,15], products and quotients of these [20], and some other symmetric spaces [10], but exclude many other spaces, including any manifold with negative sectional curvature anywhere [24], as well as some manifolds with everywhere nonnegative sectional curvature [18].
7.1 Functionals on $P(M)$

In this subsection, we recall certain classes of functionals on $P(M)$, which are widely studied in the optimal transport literature, and their basic properties. These are entropy type functionals (also known as internal energy functionals), potential energy functionals, and interaction energy functionals (see e.g., [2, 29, 41, 42]). Due to the notions of displacement interpolation and displacement convexity, uncovered by McCann [29] and discussed here in the introduction, optimal transport plays an important role to the study of these functionals. Our notation below is mostly borrowed from the book of Villani [42].

We start with entropy type functionals, which, among other uses, model internal energy of gases:

**Definition 7.1 (Entropy type functionals; see, e.g. [2, 41, 42].)** Let $U : [0, \infty) \to \mathbb{R}$ be a differentiable function, and $\nu$ a probability measure on $M$ with $d\nu(x) = e^{-U(x)}d\text{vol}(x)$. Define the functional $U_\nu : P_{ac}(M) \to \mathbb{R}$ by

$$U_\nu[\mu] := \int_M U\left( f_\nu^\mu(x) \right) d\nu(x),$$

where $f_\nu^\mu$ is the density of $\mu$ with respect to $\nu$, i.e. $d\mu(x) = f_\nu^\mu(d
nu(x)$. For measures $\mu$ which are not absolutely continuous with respect to $\nu$, we define $U_\nu(\mu) = +\infty$. Also, for $r \in [0, \infty)$ and a semi-convex function $u : M \to \mathbb{R}$, we define

$$p(r) = rU''(r) - U(r) \quad (7.1)$$

$$Lu = \Delta u - \nabla V \cdot \nabla u.$$

Note that $Lu$ is well defined wherever $u$ is twice differentiable; for semi-convex functions $u$, the Laplacian $\Delta$ in $L$ should be understood to be the Alexandrov Laplacian (defined Lebesgue a.e.), or equivalently the absolutely continuous part of the distributional Laplacian.

Before giving further conditions on these functionals, let us first recall McCann’s displacement interpolation (we consider only the special form where the first endpoint measure $\mu_0$ is absolutely continuous):

**Definition 7.2 (Displacement interpolation [29]).** For any absolutely continuous measure $\mu_0 \in P_{ac}(M)$, any other measure $\mu \in P(M)$, and $0 \leq t \leq 1$, we call the measure $\mu_t = (T_t)_\#\mu_0$, $t \in [0, 1]$ the displacement interpolant between $\mu_0$ and $\mu$. Here, the map $T_t$ is the optimal transport map given by $T_t(x) = \exp_x t\nabla u_\mu(x)$ for $\mu_0$-a.e. $x$, for a c-convex function $u_\mu$.

Part of the importance of this notion is that the curve $t \mapsto \mu_t$ coincides with the minimal geodesic segment in $(P(M), W_2)$ between the two measures $\mu_0$ and $\mu$ [29]; see [2, 41, 42] for an extensive review.

There are two main conditions we will impose on the entropy type functionals $U_\nu$. 

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Condition 7.3 (Conditions on entropy type functionals). Our conditions on $U_\nu$ are:

1. $U(0) = 0$ and $r \mapsto p(r)$ is a continuous function, and $p \geq 0$.

2. (see [42, equation (23.29)]) For the displacement interpolation $\mu_t$ as given in Definition 7.2 for absolutely continuous measure $\mu_0$ with $d\mu_0(x) = f^{\nu}_{\mu_0}(x)\,d\nu(x)$,

$$\liminf_{t \to 0^+} \frac{U_\nu(\mu_t) - U_\nu(\mu_0)}{t} = -\int_M p(f^{\nu}_{\mu_0}(x))Lu_\mu(x)d\nu(x). \quad (7.2)$$

(This holds for a large class of functionals $U_\nu$, see, for example, the [42, proof of (23.29), page 667].)

Other important types of functionals are given below:

Definition 7.4 (Potential energy functionals and interaction energy functionals; see e.g., [2, 41, 42]). For $\tilde{V} : M \to \mathbb{R}$ and $\tilde{W} : M \times M \to \mathbb{R}$, the corresponding potential energy functional and interaction energy functional $V, W : P(M) \to \mathbb{R} \cup \{+\infty\}$ are defined, respectively, as

$$\mu \mapsto V(\mu) = \int_M \tilde{V}(x)d\mu(x),$$

$$\mu \mapsto W(\mu) = \int_M \int_M \tilde{W}(x,y)d\mu(x)d\mu(y).$$

### 7.2 First order balance for functionals on $P(M)$ at Wasserstein barycenters

Our proof of the Wasserstein Jensen’s inequalities will rely on the following lemma, which we believe are interesting their own. Our proof of the lemma exploits the first and second order balance conditions (with respect to $\Omega$) given in Theorem 4.4.

Lemma 7.5 (First order balance for entropy type functionals at Wasserstein barycenters). Let $U_\nu$ be an entropy type functional given in Definition 7.1. Assume that $U_\nu$ satisfies Condition 7.3. Let $\bar{\mu}$ be the Wasserstein barycenter of $\Omega$, which is assumed to be absolutely continuous. Use the notation in Condition 7.3, letting $\mu_0 = \bar{\mu}$. Then,

$$\int_{P(M)} \liminf_{t \to 0^+} \frac{U_\nu[\mu_t] - U_\nu[\bar{\mu}]}{t} d\Omega(\mu) \geq 0.$$

Proof. Apply part 2, i.e. (7.2), of Condition 7.3 to get

$$\liminf_{t \to 0^+} \frac{U_\nu[\mu_t] - U_\nu[\bar{\mu}]}{t} = -\int_M p(f^{\nu}_{\bar{\mu}}(x))Lu_\mu(x)d\nu(x)$$

$$= -\int_M p(f^{\nu}_{\bar{\mu}}(x))\Delta u_\mu(x)d\nu(x) + \int_M p(f^{\nu}_{\bar{\mu}}(x))\nabla V(x) \cdot \nabla u_\mu(x)d\nu(x).$$

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Due to Lemma 4.1 (which applies due to the absolute continuity of \( \bar{\mu} \)) \( u_\mu(x) \) is twice differentiable in \( x \) for \( \Omega \)-a.e. \( \mu \), for a fixed \( x \), \( \nu \)-a.e. Therefore, we can integrate the above expression over \( P(M) \) with respect to the probability measure \( \Omega \). Now, observe that due to semi-convexity of \( u_\mu \) as well as the continuity and nonnegativity of \( p \), the negative part of the integrand involving \( \Delta u_\mu \) is uniformly bounded (recall that \( M \) is compact). Similarly, the integrand involving \( \nabla u_\mu \) is uniformly bounded. Therefore, we can use the Fubini-Tonelli theorem to exchange the order of integrals \( \int_M d\nu \) and \( \int_{P(M)} d\Omega(\mu) \), and get

\[
- \int_M p(f_\mu^\nu(x)) \Delta u_\mu(x) d\nu(x) = -\int_M p(f_\mu^\nu(x)) \left[ \int_{P(M)} \Delta u_\mu(x) d\Omega(\mu) \right] d\nu(x) \tag{7.3}
\]

and

\[
\int_M p(f_\mu^\nu(x)) \nabla V(x) \cdot \nabla u_\mu(x) d\nu(x) = \int_M p(f_\mu^\nu(x)) \nabla V(x) \cdot \left[ \int_{P(M)} \nabla u_\mu(x) d\Omega(\mu) \right] d\nu(x) \tag{7.4}
\]

Finally, apply the first and second order balance properties (4.5) and (4.6) to the integrals of \( \Delta u_\mu \) and \( \nabla u_\mu \) with respect to \( \Omega \), and use \( p(f_\mu^\nu) \geq 0 \). We see that the right-hand side of (7.3) is nonnegative while the right-hand side of (7.4) vanishes. This completes the proof. \( \square \)

**Remark 7.6** (Remark on the proof of Lemma 7.5). Note that we could prove the above lemma using only the first order balance condition (4.5) (that is, without relying on the second order balance condition (4.6)), if we assumed \( p(f_\mu^\nu) \in W^{1,1}_{loc}(M) \). In that case, we can use the result in [42, (23.42)]:

\[
- \int_M p(f_\mu^\nu(x)) \Delta u_\mu(x) d\nu(x) \geq \int_M \nabla u_\mu(x) \cdot \nabla p(f_\mu^\nu(x)) d\nu(x).
\]

However, it is not known at present whether \( p(f_\mu^\nu) \in W^{1,1}_{loc}(M) \), even under the assumption that \( \Omega \)-a.e. \( \mu \) is smooth. It is almost certainly not true in general under our much weaker sufficient condition for absolute continuity of \( \bar{\mu} \), \( \Omega(A_L) > 0 \).

For potential energy and interaction energy functionals, a similar result can be proved, in an easier way, using only the first order balance condition (4.5):

**Lemma 7.7** (First order balance for potential energy and interaction energy functionals at Wasserstein barycenters). For the potential energy and interaction energy functionals \( V \) and \( W \) given in Definition 7.4, assume that the functions \( V : M \rightarrow \mathbb{R} \), \( W : M \times M \rightarrow \mathbb{R} \) are Lipschitz.
Let $\bar{\mu}$ be the Wasserstein barycenter of $\Omega$, and assume it is absolutely continuous. Use the notation in Definition 7.2, letting $\mu_0 = \bar{\mu}$. Then,
\[
\int_{P(M)} \liminf_{t \to 0^+} \frac{\mathcal{V}[\mu_t] - \mathcal{V}[\bar{\mu}]}{t} d\Omega(\mu) = 0;
\]
\[
\int_{P(M)} \liminf_{t \to 0^+} \frac{\mathcal{W}[\mu_t] - \mathcal{W}[\bar{\mu}]}{t} d\Omega(\mu) = 0.
\]

**Proof.** We recall the well known fact that (see, for example, [2, 41])
\[
\liminf_{t \to 0^+} \frac{\mathcal{V}[\mu_t] - \mathcal{V}[\bar{\mu}]}{t} = \int_M \nabla \tilde{V}(x) \cdot \nabla u_\mu(x) d\bar{\mu}(x);
\]
\[
\liminf_{t \to 0^+} \frac{\mathcal{W}[\mu_t] - \mathcal{W}[\bar{\mu}]}{t} = \int [\nabla_x \tilde{W}(x, y) \cdot \nabla u_\mu(x) + \nabla_y W(x, y) \cdot \nabla u_\mu(y)] d\bar{\mu}(x) d\bar{\mu}(y).
\]
In both cases, after integrating against $\Omega$ and using Fubini’s theorem, the right-hand sides vanish due to the first order balance condition (4.5).

### 7.3 $k$-displacement convex functionals

In this subsection, we first recall the notion of $k$-displacement convexity [29], and then some key examples of $k$-displacement convex functionals in the Riemannian setting. Although potential energy and interaction energy functionals also have important applications, here we will only consider, for simplicity, examples of entropy type functionals; these have been key tools in applications of optimal transport theory to Riemannian geometry; see [42] for an extensive review.

**Definition 7.8 ($k$-displacement convexity [29]).** A functional $F : \text{dom}(F) \subset P(M) \to \mathbb{R} \cup \{+\infty\}$ is said to be $k$-displacement convex for $k \in \mathbb{R}$, if for each displacement interpolation $\mu_t$ between endpoint measures $\mu_0$ and $\mu_1$ (Definition 7.2), we have
\[
F(\mu_t) \leq (1 - t) F(\mu_0) + t F(\mu_1) - \frac{k}{2} t(1 - t) W_2^2(\mu_0, \mu_1).
\]

Examples of entropy type functionals $U_\nu$ satisfying parts 1 of Condition 7.3 and the $k$-displacement convexity are found on metric measure spaces satisfying the so-called $CD(K, N)$ condition [26,36,37,42]; part 2 of Condition 7.3 in those examples is satisfied if the domain $M$ is a smooth manifold (as we assume in this paper). To demonstrate some of these examples, let us consider

**Definition 7.9 ($CD(K, N)$ condition; see. e.g [42, Ch. 14]).** A (complete) $n$-dimensional Riemannian manifold $M$ equipped with a reference measure $\nu = e^{-V} \text{vol}$, where $V \in C^2(M)$, is said to satisfy a $CD(K, N)$ condition for $N \in (n, \infty)$, $K \in \mathbb{R}$ if $\text{Ric}_{N, \nu} \geq K$. Here,
\[
\text{Ric}_{N, \nu} := \text{Ric} + \nabla^2 V - \frac{1}{N - n} \nabla V \otimes \nabla V,
\]
where, $\nabla V \otimes \nabla V : T^2 M \to \mathbb{R}$ is defined as

$$(\nabla V \otimes \nabla V)_x(v, w) = (\nabla V(x) \cdot v)(\nabla V(x) \cdot w).$$

We note that when $N = n$, the $CD(K, n)$ condition is simply that the Ricci curvature $\text{Ric} \geq K$ is bounded below by $K$.

**Example 7.10** ($k$-displacement convex functionals; see, e.g., [42, Ch. 17]). A representative functional for displacement convexity is the following:

$$U_N(r) = \begin{cases} -N(r^{1-1/N} - r) & (1 < N < \infty) \\ r \log r & (N = \infty) \end{cases},$$

with the convention that $U_\infty(0) = 0$. For $M$ satisfying the $CD(K, N)$ condition, $U_\nu$ defined as $\int_M U_N(f_\mu)d\nu$, is $k$-displacement convex where the constant $k$ depends on $N, K$ and $\sup_{t \in [0,1]} \| \frac{d\mu_t}{d\nu} \|_\infty$. In this case, $k$ and $K$ have the same sign. (See [42, Theorem 17.15, (17.11) and Exercise 17.23]). Moreover, $U_\nu$ satisfies Condition 7.3.

### 7.4 Wasserstein Jensen’s inequalities for $k$-convexity

We now present one of the main results of this section, which follows easily from the first order balance for functionals given in Lemmas 7.5 and 7.7:

**Theorem 7.11** (Wasserstein Jensen’s inequality for $k$-displacement convex functionals). Let $U_\nu : P_{ac}(M) \to \mathbb{R}$ be the entropy type functional given in Definition 7.1, satisfying Condition 7.3. Let $V, W : P(M) \to \mathbb{R} \cup \{-\infty\}$ be the potential energy and the interaction energy functional, respectively, given in Definition 7.4, with Lipschitz functions $\tilde{V} : M \to \mathbb{R}$ and $\tilde{W} : M \times M \to \mathbb{R}$.

Let $\Omega$ be a probability measure on $P(M)$ and assume that $\Omega(\mathcal{A}_L) > 0$ for some $L < \infty$. Let $\bar{\mu}$ be the Wasserstein barycenter of $\Omega$.

For $F = U_\nu, V, \text{ or } W$, suppose $F$ is $k$-displacement convex as in Definition 7.8. Then, we have

$$F(\bar{\mu}) \leq \int_{P(M)} F[\mu]d\Omega(\mu) - \frac{k}{2} \int_{P(M)} W_2^2(\bar{\mu}, \mu)d\Omega(\mu). \tag{7.5}$$

**Proof.** Note that $\bar{\mu}$ is absolutely continuous from $\Omega(\mathcal{A}_L) > 0$ and Theorem 6.4. Use the notation in Definition 7.2 letting $\mu_0 = \bar{\mu}$. Then, from $k$-displacement convexity, which is translated to $k$-convexity of $t \mapsto F(\mu_t)$, we get

$$F(\mu) \geq F(\bar{\mu}) + \liminf_{t \to 0^+} \frac{F[\mu_t] - F[\bar{\mu}]}{t} + \frac{k}{2} W_2^2(\bar{\mu}, \mu).$$

To finish the proof, integrate this against $\Omega$ and use Lemma 7.5 for $F = U_\nu$, Lemma 7.4 for $F = V, W$. □

This theorem can be interpreted as a Wasserstein Jensen’s inequality for a variety of functionals on Riemannian manifolds, including those listed in Examples 7.10. In particular, we note the following immediate consequence:
Corollary 7.12 (Wasserstein Jensen’s inequality on $\text{Ric} \geq 0$). Assume $\text{Ric} \geq 0$. Let

$$U[\mu] := \int_{M} U(f_{\mu}(x))d\text{vol}(x),$$

where $r \mapsto r^{n}U(r^{-n})$ is convex nonincreasing. Then, letting $\bar{\mu}$ be the barycenter of $\Omega$, and assuming $\Omega(\mathcal{A}_{L}) > 0$, we have

$$U(\bar{\mu}) \leq \int_{P(M)} U[\mu]d\Omega(\mu). \quad (7.6)$$

Remark 7.13. The preceding Corollary also follows from a different geometric version of Jensen’s inequality, (see Theorem 7.14 below), which is similar to the distorted displacement convexity in [9] and [42, Theorem 17.37]. This gives a less clean looking version of convexity, as distortion coefficients show up inside the integrals, but eliminates the additional $k$ term.

7.5 Distorted Wasserstein Jensen’s inequality

In this subsection, we offer an alternate geometric version of Jensen’s inequality, which relies less on the $k$-convexity of the functional than the Wasserstein Jensen’s inequality in Theorem 7.11 but yields a less clean convexity result. This is similar to the distorted convexity of [9] and those in [42, Theorem 17.37]. For simplicity, we deal only with the simplest class of functionals.

Theorem 7.14 (Distorted Wasserstein Jensen’s inequality). Let

$$U[\mu] := \int_{M} U(f_{\mu}(x))d\text{vol}(x),$$

where $r \mapsto r^{n}U(r^{-n})$ is convex nonincreasing and $f_{\mu}(x)$ is the density of the measure $\mu$, with respect to volume. Assume that $\Omega$ almost every $\mu$ is absolutely continuous with respect to volume and that $\Omega(\mathcal{A}_{L}) > 0$ for some $L$. Then $U$ is displacement convex over barycenters with distortion coefficients $\alpha$; that is,

$$U(\bar{\mu}) \leq \int_{P(M)} \int_{M} U\left(\frac{f_{\mu}(x)}{\alpha \lambda_{t_{\mu}^{-1}(x)}(x)}\right)\alpha \lambda_{t_{\mu}^{-1}(x)}(x)d\text{vol}(x)d\Omega(\mu)$$

where $\bar{\mu}$ is the barycenter of $\Omega$ and $T_{\mu}$ is the optimal map from $\bar{\mu}$ to $\mu$.

Remark 7.15. Notice that, like Theorem 7.11, Theorem 7.14 implies Corollary 7.12. It is not clear to us whether the two upper bounds on $U[\mu]$ in Theorems 7.11 and 7.14 are comparable; even for doubly supported measures $\Omega = (1 - t)\delta_{\mu_{0}} + t\delta_{\mu_{1}}$, this remains open (see [42, Open Problem 17.39]).

Proof of Theorem 7.14. In the following proof, the determinant estimates in Theorem 4.6 play a key role. Note that this result applies as, from the assumption $\Omega(\mathcal{A}_{L}) > 0$ and Theorem 6.1, the Wasserstein barycenter $\bar{\mu}$ is absolutely continuous.
For each $\mu$, let $T_\mu(x) = \exp_x(\nabla u_\mu(x))$ be the optimal map pushing $\bar{\mu}$ forward to $\mu$. For almost every point $x$, the map $T_\mu$ satisfies the following Jacobian equation:

$$|\det DT_\mu(x)| = \frac{f_\bar{\mu}(x)}{f_\mu(T_\mu(x))}.$$ 

By a change of variables, this implies

$$\int_M U \left( \frac{f_\mu(x)}{\alpha_{\lambda_\mu^{-1}(x)}(x)} \right) \alpha_{\lambda_\mu^{-1}(x)}(x) d\text{vol}(x)$$

$$= \int_M U \left( \frac{f_\mu(x)}{\alpha_{\lambda_\mu(T_\mu(x))} \det DT_\mu(x)} \right) \alpha_{\lambda_\mu}(T_\mu(x)) |\det DT_\mu(x)| d\text{vol}(x).$$

Integrating this equation against $\Omega$, using the Fubini-Tonelli theorem as in the proof of Theorem 7.11 (the ordinary) Jensen’s inequality and the convexity of $r \mapsto r^n U(r^{-n})$, we obtain

$$\int_{P(M)} \int_M U \left( \frac{f_\mu(x)}{\alpha_{\lambda_\mu^{-1}(x)}(x)} \right) \alpha_{\lambda_\mu^{-1}(x)}(x) d\text{vol}(x) d\Omega(\mu)$$

$$\geq \int_M U \left( \frac{f_\mu(x)}{r^n(x)} \right) r^n(x) d\text{vol}(x),$$

where

$$r(x) = \int_{P(M)} \alpha_{\lambda_\mu}(T_\mu(x))^{1/n} |\det DT_\mu(x)|^{1/n} d\Omega(\mu).$$

The monotonicity of $r \mapsto r^n U(r^{-n})$ and Theorem 4.6 then yield the desired result. \hfill \square

### 8 Curved random Brunn-Minkowski inequality

We complete this paper with an application: the proof of a random Brunn-Minkowski inequality on smooth metric measure spaces, Theorem 8.1. Recall the classical Brunn-Minkowski inequality states that for any two bounded measurable sets $X, Y \subset \mathbb{R}^n$,

$$|X + Y|^{1/n} \geq |X|^{1/n} + |Y|^{1/n}$$

where $X + Y = \{x + y \mid x \in X, y \in Y\}$ and $|X|$ denotes the Euclidean volume.

Optimal transport techniques have been used to extend this inequality to Riemannian manifolds, as well as more general (not necessarily smooth) metric measure spaces, satisfying an appropriate condition on the curvature, known as the curvature dimension condition $CD(K,N)$. This direction of research, on smooth manifolds, was initiated in [9] and an extensive discussion can be found in [12]; see, for instance, Theorem 30.7 therein.
On the other hand, one can easily use induction to extend the Euclidean version to the Minkowski sum of several sets $X_1, X_2, ..., X_m$, obtaining

$$ \left| \sum_{i=1}^{m} X_i \right|^{1/n} \geq \sum_{i=1}^{m} |X_i|^{1/n}. $$

One can go one step further, and obtain a version for infinitely many (or random) sets on Euclidean space. Unlike the extension to finitely many sets, the extension to random sets is nontrivial and is known as Vitale’s random Brunn-Minkowski inequality [43]; see [43] for a precise statement.

It is not obvious that these extensions hold on curved spaces; in particular, the barycenter operation is not associative on a Riemannian manifold, and so the simple induction argument implying the multi-set version on Euclidean space does not carry over to curved spaces. In this section, we use our Wasserstein Jensen’s inequality from Theorem 7.11 to extend the Brunn-Minkowski inequality to interpolations between both several and infinitely many or random sets on smooth Riemannian manifolds. Our main result in this direction requires a bit of terminology; the definitions below are direct extensions from Euclidean space to Riemannian manifolds of the nomenclature of Vitale [43], where the barycenter operation replaces the Euclidean average.

Let $X$ be a random measurable set on $M$: that is, a measurable mapping $X: (\mathcal{P}, \Omega) \to 2^M$ from a probability space $\mathcal{P}$ (equipped with the probability measure $\Omega$) to the set of measurable subsets of $M$, equipped with the Hausdorff distance. We say a measurable mapping $S: \mathcal{P} \to M$ is a selection of $X$ if $S(\omega) \in X(\omega)$, for $\Omega$-a.e $\omega$, i.e. $S \in X$, almost surely. For each selection $S$, let $BC(S) \subseteq M$ be the set of barycenters of the measure $S_{\#}\Omega$ on $M$; that is, the set of minimizers of $z \mapsto \int_{\mathcal{P}} d^2(z, S(\omega))d\Omega(\omega)$. Note that $BC(S)$ is itself a random set. We define an analogue of the Minkowski sum for a random set by

$$ Z(X) = \{ z : z \in BC(S) \text{ for some selection } S \text{ of } X \}. $$

Note that $Z(X) \subset M$ is simply a subset of $M$; that is, it is not a random set.

If $M$ satisfies a $CD(K,N)$ condition (recall Definition 7.9), we also define, for a given random set $X$,

$$ \alpha(X) = \begin{cases} \inf_S \inf_{z \in M} \int_{\mathcal{P}} d^2(z, S) d\Omega(S) & \text{if } K \geq 0, \\ \sup_S \inf_{z \in M} \int_{\mathcal{P}} d^2(z, S) d\Omega(S) & \text{otherwise}, \end{cases} $$

where $\inf_S$ and $\sup_S$ denote infimum and supremum over all selections $S$ of $X$, respectively.

**Theorem 8.1 (Curved Random Brunn-Minkowski on smooth metric measure spaces).** Let $M$ be a compact, smooth Riemannian manifold equipped with a reference probability measure $\nu$ with $d\nu(x) = e^{-V(x)}d\vol(x)$, $V \in C^2(M)$. Assume that $(M, \nu)$ satisfies a $CD(K,N)$ curvature-dimension condition (see Definition 7.9). Let $X$ be a random measurable set, and assume that almost surely $\nu(X) > 0$. Then,
1. If \( N = \infty \),
\[
\log[\nu(Z(X))] \geq \mathbb{E}[\log[\nu(X)]] + \frac{k}{2} \alpha(X)
\]
where \( k \) is the same constant as in Example 7.10.

2. If \( N < \infty \) and \( K \geq 0 \),
\[
[\nu(Z(X))]_{1/N} \geq \mathbb{E}[\nu(X)]_{1/N}
\]  
(8.1)

where \( \mathbb{E} \) denotes the expectation with respect to the probability measure \( \Omega \).

**Remark 8.2.** This result easily extends to a complete, non-compact manifold \( M \), provided that the random set \( X \subset B \) is contained in a fixed, bounded set \( B \) almost surely, but our proof does not cover the case where \( X \) satisfies the weaker hypothesis of almost sure compactness.

**Proof.** We present only the proof of the assertion 1; the proof of the second assertion is very similar and is omitted.

The proof is a direct application of the \( k \)-displacement convexity of the functional \( U_\nu \) from Example 7.10 with \( U_\infty(r) = r \log r \) (for assertion 2, we use instead, \( U_N(\rho) = -N(\rho^{1-1/n} - r) \)), together with Theorem 7.11. We associate with the random set \( X \) the (random) measure \( \mu_X := \frac{1}{\nu(X)} \nu(X) \). Note that \( \mu_X \) is well defined and absolutely continuous almost surely, by the assumption \( \nu(X) > 0 \) a.s. We will apply our Wasserstein Jensen’s inequality (Theorem 7.11) to the push forward of \( \Omega \) by the map \( \omega \mapsto \mu_X(\omega) \), which is a probability measure on \( P(M) \). Let \( \bar{\mu} \) be the Wasserstein barycenter of this measure, which exists uniquely by Theorem 3.1. By Theorem 6.1, \( \bar{\mu} \) is absolutely continuous with respect to volume, and hence with respect to \( \nu \) as well; we denote its density with respect to \( \nu \) by \( f(X) := \frac{1}{\nu(X)} \). Now, from the Wasserstein Jensen’s inequality (Theorem 7.11) we have
\[
\int_{\text{spt}(\bar{\mu})} f(x) \log f(x) d\nu(x) \leq \int_{P} \int_{X(\omega)} \frac{1}{\nu(X(\omega))} \log \left( \frac{1}{\nu(X(\omega))} \right) d\nu(x) d\Omega(\omega)
\]
(8.2)

\[
- \frac{k}{2} \int_{P} W_2^2(\bar{\mu}, \mu_X(\omega)) d\Omega(\omega)
\]

Applying the (ordinary) Jensen’s inequality for the convex function \( r \to r \log r \) to the measure \( \frac{\nu(X)}{\nu(\text{spt}\bar{\mu})} \) and noting that \( \int_M f(x) d\nu(x) = 1 \), the left-hand side is bounded from below by \( \log(\frac{1}{\nu(\text{spt}\bar{\mu})}) \):
\[
\log \left( \frac{1}{\nu(\text{spt}\bar{\mu})} \right) \leq \int_{\text{spt}\bar{\mu}} f(x) \log f(x) d\nu(x).
\]
(8.3)

Next, we claim that \( \text{spt}(\bar{\mu}) \subseteq Z(X) \); this will imply that \( \nu(\text{spt}\bar{\mu}) \leq \nu(Z(X)) \) and therefore
\[
\log \left( \frac{1}{\nu(Z(X))} \right) \leq \log \left( \frac{1}{\nu(\text{spt}\bar{\mu})} \right).
\]
(8.4)
Now fix any \( z \in \text{spt}(\bar{\mu}) \). Let \( T_{\mu X(\omega)} \) be the optimal map from \( \bar{\mu} \) to \( \mu X(\omega) \), for any \( \omega \in \mathcal{P} \) and define the selection \( S : \mathcal{P} \to M \) by \( S(\omega) := T_{\mu X(\omega)}(z) ) \# \Omega = S \# \Omega \), \( \bar{\mu} \) almost surely; by definition, this means that \( z \in BC(S) \), which in turn shows \( z \in Z(X) \), \( \bar{\mu} \) almost surely. Thus, \( \text{spt}\bar{\mu} \subset Z(X) \), implying (8.4).

Moreover,
\[
\inf_S \inf_{z \in M} \int_P d^2(z, S) d\Omega(S) \leq \int_P W^2_2(\bar{\mu}, \mu X) d\Omega(X) \leq \sup_S \inf_{z \in M} \int_P d^2(z, S) d\Omega(S).
\]

The result now follows easily, by combining the preceding inequality with (8.2), (8.3) and (8.4).

As an immediate consequence, we get the following multi-set Brunn-Minkowski inequality on \( M \).

**Corollary 8.3 (Multi-set Brunn-Minkowski on spaces of nonnegative Ricci curvature).** Let \( M \) be a Riemannian manifold with a reference measure \( \nu \), satisfying the CD\((K, N)\) condition with \( K \geq 0 \). Let \( A_i \subseteq M \), \( i = 1, 2, \ldots, m \), be bounded sets and set \( Z = \{ bc_{\lambda}(x_1, \ldots, x_m) : x_i \in A_i \} \) be the set of barycenters of points in the \( A_i \). Then,

- if \( N < \infty \),
  \[
  \nu(Z)^{1/N} \geq \sum_{i=1}^{m} \lambda_i \nu(A_i)^{1/N};
  \]

- if \( N = \infty \),
  \[
  \log \nu(Z) \geq \sum_{i=1}^{m} \lambda_i \log \nu(A_i).
  \]

Note that on the Euclidean space (with \( \nu = \text{vol} \)), this corollary follows easily by iterating the classical Brunn-Minkowski inequality \( m \) times, as the barycenter operation \( bc_{\lambda}(x_1, \ldots, x_m) = \sum_i \lambda_i x_i \) is associative; however, this is not the case on curved spaces.

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