SELF DUALITY AND CODINGS FOR EXPANSIVE GROUP AUTOMORPHISMS

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Abstract. Lind and Schmidt have shown that the homoclinic group of a cyclic \( \mathbb{Z}^k \) algebraic dynamical system is isomorphic to the dual of the phase group. We show that this duality result is part of an exact sequence if \( k = 1 \). The exact sequence is a well known algebraic object, which has been applied by Schmidt in his work on rigidity. We show that it can be derived from dynamical considerations only. The constructions naturally lead to an almost \( 1-1 \)-coding of certain Pisot automorphisms by their associated \( \beta \)-shift, generalizing similar results for Pisot automorphisms of the torus.

1. Introduction

In this paper we study discrete systems in which the phase space is a topological group and the action is induced by an expansive automorphism.

A common practice in algebra is to embed a ring \( R \) as a discrete subring of a locally compact ring \( K \) such that \( K/R \) is compact and is equal to the Pontryagin dual of \( R \) as an additive group. Standard examples of such embeddings are: \( R \) is the ring of integers of a number field \( \mathbb{Q}(\alpha) \) and \( K \) is the product of the archimedean completions of \( \mathbb{Q}(\alpha) \); or, \( R \) is a number field and \( K \) is the ad`ele ring. Pontryagin duality concerns groups and not rings, so the multiplicative structure on \( R \) is lost in \( K/R \). However it can be regained by considering \( R \) and \( K/R \) as dynamical systems. For instance, if \( R \) is equal to \( \mathbb{Z}[\beta, \beta^{-1}] \) then the multiplicative structure on \( R \) is determined by the action \( x \mapsto \beta x \) and so \( K/R \) is a compact group with a dual action; i.e., it is an algebraic dynamical system.

In this paper we build up the exact sequence

\[
0 \longrightarrow R \longrightarrow K \longrightarrow G \longrightarrow 0
\]

by dynamical considerations, starting from \( G \) rather than \( R \): \( K \) is the product of the stable and unstable group of \( G \), endowed with suitable topologies, and \( R \) is the homoclinic group. Our paper was motivated by work of Lind and Schmidt \[9\], who showed that the homoclinic group of an algebraic dynamical system is isomorphic to the Pontryagin dual, and by work of Schmidt on symbolic codings \[12\]. We only consider \( \mathbb{Z} \)-actions. Lind and Schmidt obtained their results in a more general case of ergodic \( \mathbb{Z}^k \) actions on compact abelian groups.

\[1991 \text{ Mathematics Subject Classification.} \] Primary 54 H15; Secondary 37 B10, 11 R04.

\textit{Key words and phrases.} algebraic dynamical system, symbolic coding, Pontryagin duality, algebraic numbers.
We give two applications. We interpret a number theoretic result of Pisot and Vijayaraghavan in dynamical terms and we find an almost $1-1$-coding of a class of Pisot automorphisms. Such codings were known to exist in the case the underlying group is a torus (see [12, 13, 14]), but our constructions apply to a more general class of groups. Our analysis of the coding also leads to an interpretation of the adic transformation associated to a $\beta$-shift as the return map of a naturally related flow.

2. Notation

We shall study certain rings of polynomials and power series that are a little unusual, so our notation is a little unusual. If $R$ is a ring and $x$ is an indeterminate, then $R[x]$ denotes the ring of Laurent polynomials $r_0 x^n + r_1 x^{n+1} + \ldots + r_k x^{n+k}$ with $r_i \in R$, $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. $R[x]$ denotes the ring of power series over $R$ of the form $r_0 x^n + r_1 x^{n+1} + \ldots + r_k x^{n+k} + \ldots$ for $k = 0, 1, 2, \ldots$. Similarly, $R[[x]]$ denotes the ring of power series over $R$ of the form $\ldots + r_k x^{n-k} + \ldots + r_1 x^{n-1} + r_0 x^n$. The ring of two-sided or Laurent power series over $R$ is denoted by $R[[x]]$.

3. Cyclic algebraic dynamical systems

A dynamical system is a group action on a topological space $G \times X \to X$. If the phase space $X$ is an abelian topological group and if the action $x \mapsto g \cdot x$ is an automorphism, then the system is called an algebraic dynamical system. In this case, $X$ is a $\mathbb{Z}[G]$ module. These systems have been well studied and more information on them can be found in Schmidt’s excellent monograph [11]. In our case, $G$ is $\mathbb{Z}$ and $X$ is a compact connected abelian group denoted $\Gamma$ and the group ring $\mathbb{Z}[G]$ is isomorphic to the ring $\mathbb{Z}[x]$ of Laurent polynomials $a_0 x^n + \ldots + a_k x^{n+k}$ for $n, a_i \in \mathbb{Z}$ and $k \in \mathbb{N}$.

We denote the dual group of the phase space $\Gamma$ by $G$. Then $G$ is a $\mathbb{Z}[x]$-module, as is $G$ under the dual action. Since $\Gamma$ is connected, $G$ is torsion-free. Hence, the annihilator of $G$ in $\mathbb{Z}[x]$ is a principal ideal, generated by a primitive Laurent polynomial $f$. If we require that $f = a_0 + \ldots + a_n x^n$ be a polynomial; i.e., that all powers of $x$ be positive, that $f(0) \neq 0$ and that $a_n > 0$, then $f$ is uniquely determined. This polynomial $f$ is the associated polynomial of the dynamical system on $\Gamma$.

The action of $\mathbb{Z}[x]$ on $\Gamma$ is generated by the automorphism $\gamma \mapsto x \cdot \gamma$. We denote this automorphism by $\alpha$ and we denote the algebraic dynamical system by $(\Gamma, \alpha)$. The stable group $\Omega^+ \subset \Gamma$ is defined as

$$\Omega^+ = \{g \in \Gamma : \alpha^n(g) \to 0 \text{ as } n \to \infty\}$$

The unstable group $\Omega^-$ is defined likewise for $n \to -\infty$. The homoclinic group is then $\Omega = \Omega^+ \cap \Omega^-$. An algebraic dynamical system $(\Gamma, \alpha)$ is expansive if there exists a neighborhood $U$ of the identity $e \in G$ such that

$$\bigcap \{\alpha^n(U) : n \in \mathbb{Z}\} = \{e\}$$

A polynomial is hyperbolic if all its roots in $\mathbb{C}$ are off the unit circle $|z| = 1$. The following classification theorem of expansive systems has apparently been proved first by Aoki and Dateyama [1].
**Theorem 1.** An algebraic system $(\Gamma, \alpha)$ is expansive if and only if the associated polynomial is hyperbolic and the dual group is a Noetherian $\mathbb{Z}[x]$-module.

An algebraic system on $\Gamma$ is cyclic or principal if the Pontryagin dual $G$ is a cyclic $\mathbb{Z}[x]$-module. We shall only consider algebraic systems that are expansive and cyclic. In this case $G$ is equivalent to the quotient module $\mathbb{Z}[x]/(f)$ as described by the exact sequence.

\begin{equation}
0 \longrightarrow \mathbb{Z}[x] \xrightarrow{g \mapsto fg} \mathbb{Z}[x] \longrightarrow G \longrightarrow 0
\end{equation}

Let $T$ denote the circle group. The Pontryagin dual of $\mathbb{Z}[x]$ is the group $T(x)$ of formal power series $\sum_{n \in \mathbb{Z}} t_n x^n$ with coefficients $t_n \in T$. The Pontryagin dual of sequence 1 is

\begin{equation}
0 \longleftarrow T(x) \xleftarrow{fg \mapsto g} T(x) \longleftarrow \Gamma \longleftarrow 0
\end{equation}

Some caution is required. If the action of the indeterminate $x$ on $G$ is given by multiplication by $x$, then the adjoint action on $T(x)$ is given by multiplication by $x^{-1}$. We have to choose for which of the dual groups $G$ and $\Gamma$ the action is by multiplication of $x$. We decide that the action of $\mathbb{Z}[x]$ on $\Gamma$ is by multiplication of $x$ and the action on $G$ is by multiplication of $x^{-1}$.

## 4. Almost convergent series

The interest in the homoclinic group arose from ideas of Vershik on symbolic codings. Homoclinic points have exponential decay; i.e., $\lim_{|n| \to \infty} f^n(x) = 0$ exponentially, so that if $(a_n)$ is a bounded sequence of integers, then $\sum_{n \in \mathbb{Z}} a_n f^n(x)$ converges. This defines a conjugation between the shift on the symbolic sequences and the action of $f$ on the phase group. In Vershik’s approach, the symbolic sequences are the integral sequences in $l^\infty$, the Banach space of bounded sequences. It turns out that the homoclinic group corresponds to the subset of sequences for which $a_n = 0$ for all but finitely many $n$. The stable group and the unstable group correspond to sequences that have a tail of zeroes. Lind and Schmidt analyze the homoclinic group. For our purposes, showing that the stable and unstable group are rings, this setting is a little inconvenient. The multiplication of two stable elements $(a_n), (b_n)$ is well defined: take the product of the generating power series $\sum a_n x^n$ and $\sum b_n x^n$. However, the coefficients of this product are not necessarily bounded. Therefore we replace $l^\infty$ by a slightly larger space that is closed under the multiplication of stable elements, and we call this larger space the space of almost convergent power series.

The outer radius $R$ of a Laurent series $\sum_{n=-\infty}^{\infty} c_n x^n$ is defined as $R = \lim \inf_{n \to \infty} \sqrt[n]{|c_n|}$. The inner radius $r$ is defined as $r = \lim \sup_{n \to -\infty} 1/\sqrt[n]{|c_n|}$. Let $\mathcal{H}$ be the set of Laurent series with complex coefficients such that $r < 1 < R$. Then $\mathcal{H}$ represents the space of analytic functions that are defined on some domain around the unit circle. Thus $\mathcal{H}$ is a ring. Let $\mathcal{L}$ be the set of Laurent series with complex coefficients such that $r \leq 1 \leq R$. We say that $\mathcal{L}$ is the space of *almost convergent series*. By identifying $\sum c_i x^i$ with the sequence $(c_i) \in \mathbb{C}^\mathbb{Z}$, we endow $\mathcal{L}$ with the product topology, and we call this the *weak topology*. 
It is obvious that $\mathcal{L}$ is a $\mathbb{Z}[x]$-module. In accordance with the action on $\mathbb{T}(x)$, we define the action of $x$ on $\mathcal{L}$ as multiplication by $x$. It turns out that this action extends to $\mathcal{H}$.

**Lemma 2.** $\mathcal{L}$ is an $\mathcal{H}$-module.

**Proof.** Suppose that $g = \sum_{n=0}^{\infty} c_n x^n$ is an almost convergent series. Then $g^+ = \sum_{n>0} c_n x^n$ converges on $|x| < 1$ and $g^- = \sum_{n\leq 0} c_n x^n$ converges on $|x| > 1$. Since an element $h \in \mathcal{H}$ has no poles on the unit circle, $h \cdot g^+$ is analytic on a region $r < |x| < 1$, while $h \cdot g^-$ is analytic on a region $1 < |x| < R$. Thus, the outer radius of $f \cdot (g^- + g^+)$ is $\geq 1$ and the lower radius is $\leq 1$. In particular, $h \cdot g \in \mathcal{L}$. □

We shall denote almost convergent power series by the subscript $ac$. For example, $\mathbb{R}_{ac}(x)$ denotes the ring of almost convergent Laurent series with real coefficients. We shall denote convergent power series for which $r < 1 < R$ by the subscript $c$. For example, $\mathbb{R}_{c}(x)$ denotes the ring of Laurent series with real coefficients, convergent on an annulus around the unit circle. By Lemma 2, $\mathbb{R}_{ac}(x)$ is a $\mathbb{R}_{c}(x)$-module. Note that hyperbolic polynomials are units in $\mathbb{R}_{c}(x)$.

**Lemma 3.** Suppose that $f \in \mathbb{Z}[x]$ is hyperbolic. Then $f \cdot \mathbb{Z}_{ac}(x) \cap \mathbb{Z}_{ac}[x] = f \cdot \mathbb{Z}_{ac}[x]$.

**Proof.** Suppose that $m^+ \in f \cdot \mathbb{Z}_{ac}(x) \cap \mathbb{Z}_{ac}[x]$, so that $m^+ = f \cdot m$ for some $m \in \mathbb{Z}_{ac}(x)$. Since $m^+$ is holomorphic on the punctured disc and since $\frac{1}{f}$ is meromorphic without a pole on the unit circle, $m = \frac{1}{f} \cdot m^+$ has inner radius $r < 1$. Since $m \in \mathbb{Z}_{ac}(x)$, this implies that $m = \sum a_n x^n$ and that $a_n = 0$ for sufficiently small index. □

The projection $\mathbb{R}_{ac}(x) \to \mathbb{T}(x)$ is defined by reducing the coefficients modulo 1. It obviously is a surjection. We have the following commutative diagram of continuous group homomorphisms

\[\begin{array}{cccccc}
0 & \xleftarrow{\text{f,g→g}} & \mathbb{R}_{ac}(x) & \xrightarrow{f,g→g} & \mathbb{R}_{ac}(x) & \xleftarrow{0} \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \xleftarrow{\text{f,g→g}} & \mathbb{T}(x) & \xrightarrow{f,g→g} & \mathbb{T}(x) & \xleftarrow{\Gamma} & \xleftarrow{0}
\end{array}\]

(3)\]

in which the top row is invertible since $f$ is a unit in $\mathbb{R}_{c}(x)$. The image of $\Gamma$ in $\mathbb{T}(x)$ is equal to the projection of $\frac{1}{f} \cdot \mathbb{Z}_{ac}(x)$ on $\mathbb{T}(x)$. More specifically

\[\Gamma \cong \frac{1}{f} \cdot \mathbb{Z}_{ac}(x)/\mathbb{Z}_{ac}(x) \cong \mathbb{Z}_{ac}(x)/f \cdot \mathbb{Z}_{ac}(x)\]

(4)

as algebraic $\mathbb{Z}[x]$ modules. We identify $\Gamma$ with $\mathbb{Z}_{ac}(x)/f \cdot \mathbb{Z}_{ac}(x)$ and denote this quotient by $\mathbb{Z}_{ac}(x)/(f)$.

**Lemma 4.** The stable group $\Omega^+ \subset \Gamma$ is algebraically isomorphic to $\mathbb{Z}_{ac}(x)/(f)$.

**Proof.** A power series $g = \sum r_i x^i$ in $\frac{1}{f} \cdot \mathbb{Z}_{ac}(x)$ projects onto a stable element of $\mathbb{T}(x)$ if and only if $r_i \bmod 1$ converges to 0 as $i \to -\infty$. Without changing the projection of $g$ onto $\mathbb{T}(x)$ we may choose the $r_i$ such that $\lim_{i \to -\infty} r_i = 0$. If $f \cdot g = \sum s_i x^i$ then, since $f$ is a polynomial, $\lim_{i \to -\infty} s_i = 0$ as well. Since $f \cdot g \in \mathbb{Z}_{ac}(x)$ the coefficients
Lemma 6. For a natural number $N$ and $s_i$ are integers and so $s_i = 0$ for sufficiently small $i$. In other words, $f \cdot g \in \mathbb{Z}_{ac}(x)$ and by Lemma 5, we see that
\begin{equation}
\Omega^+ \cong \frac{1}{f} \cdot \mathbb{Z}_{ac}[x]/\mathbb{Z}_{ac}[x] \cong \mathbb{Z}_{ac}[x]/f \cdot \mathbb{Z}_{ac}[x].
\end{equation}

The following corollary is a special case of a result of Lind and Schmidt characterizing the homoclinic group, which they have derived for certain ergodic $\mathbb{Z}^k$-actions.

Corollary 5. The homoclinic group is isomorphic to the dual group. Even more so, the two are equivalent as algebraic dynamical systems.

Proof. $\Omega = \Omega^- \cap \Omega^+ = (\mathbb{Z}_{ac}(x) \cap \mathbb{Z}_{ac}[x])/(f) = \mathbb{Z}_{ac}[x]/(f) = \mathbb{Z}[x]/(f)$. Not only is the homoclinic group isomorphic to the dual group, but it is equivalent as a dynamical system as well. The action on the dual group is multiplication by $x^{-1}$. The action on the homoclinic group is multiplication by $x$. 

Notice that this representation of the homoclinic group also shows that it possesses a fundamental homoclinic point; that is, a homoclinic point whose iterates span the homoclinic group. So we have found good algebraic descriptions of $\Gamma$ and its dynamically defined subgroups. In particular, we have an exact sequence of algebraic groups
\begin{align*}
0 & \longrightarrow \Omega \xrightarrow{x \mapsto (x,x)} \Omega^- \times \Omega^+ \xrightarrow{(x,y) \mapsto x+y} \Gamma \longrightarrow 0
\end{align*}

To see that this sequence is exact, observe that if $x + y = 0$ for $(x, y) \in \Omega^- \times \Omega^+$ then $(x, y)$ can be represented by $(g, h) \in \mathbb{Z}_{ac}(x) \times \mathbb{Z}_{ac}[x]$ such that $g + h = f \cdot k$ for some $k \in \mathbb{Z}_{ac}(x)$. Let $k_1$ be the sum of the positive powers of $k$ and let $k_2$ be sum of the positive powers. Then $g - f \cdot k_1 = -(h - f \cdot k_2)$ and $g - f \cdot k_1 \in \mathbb{Z}_{ac}(x)$ while $g - f \cdot k_2 \in \mathbb{Z}_{ac}[x]$. So $p = g - f \cdot k_1$ is a polynomial and $(g, h) = (p, -p) \mod (f)$.

Now we have to find good topological descriptions as well. $\Gamma$ is not topologically isomorphic to $\mathbb{Z}_{ac}(x)/(f)$ if this quotient is endowed with the quotient topology of $\mathbb{Z}_{ac}(x)$. The problem is that $g \mapsto f \cdot g$ is algebraically invertible, but the inverse is not continuous in the weak topology. To remedy this, we shall endow $\mathbb{Z}_{ac}(x)$ with a topology that is stronger than the weak topology.

5. Strong topologies on the stable group and the unstable group

For a natural number $N$ define the subset $B(N) \subset \mathbb{Z}_{ac}(x)$ by
$$B(N) = \left\{ \sum a_i x^i \in \mathbb{Z}_{ac}(x) : 0 \leq a_i \leq N \text{ for all } i \right\},$$

Lemma 6. For every $f \in \mathbb{Z}[x]$ there exists an $N$ such that $B(N) + f \cdot \mathbb{Z}_{ac}(x) = \mathbb{Z}_{ac}(x)$.

Proof. Let $N$ be equal to the sum of the absolute values of the coefficients of $f$. Suppose that $g \in \mathbb{Z}_{ac}(x)$ and that $\frac{1}{f} \cdot g = \sum r_i x^i$ with $r_i \in \mathbb{R}$. For $h = \sum [r_i] x^i$ one verifies that $g - f \cdot h \in B(N - 1)$ and that $h \in \mathbb{Z}_{ac}(x)$. □
By this lemma, the projection $B(N) \to \mathbb{Z}_{ac}(x)/(f)$ is onto provided that $N$ is sufficiently large. We define the strong topology on $\mathbb{Z}_{ac}(x)/(f)$ as the quotient topology induced by $B(N)$ for some $N$ that is sufficiently large. In particular, $\mathbb{Z}_{ac}(x)/(f)$ is compact. Moreover, the choice of $N$ does not alter this topology.

**Theorem 7.** $\Gamma$ is isomorphic to $\mathbb{Z}_{ac}(x)/(f)$ with the strong topology as a $\mathbb{Z}[x]$-module.

**Proof.** It suffices to show that the inverse map $g \mapsto \frac{1}{f} \cdot g$ restricted to $B(N)$ is continuous. Since $f$ is a hyperbolic polynomial, it has an inverse $\frac{1}{f} = \sum q_i x^i$ in $\mathbb{R}[x]$. For every $\epsilon > 0$ there exists an $n$ such that $\sum_{|i|>n} |q_i| < \epsilon/N$. Let $0 \in U \subset B(N)$ be the neighborhood defined by $U = \{ \sum a_i x^i : a_i = 0 \text{ if } |i| \leq n \}$. Then all elements $\sum r_i x^i \in \frac{1}{f} \cdot U$ have coefficient $|r_0| < \epsilon$. By choosing a larger $n$, we can restrict arbitrarily many coefficients $r_k$ by $\epsilon$.

The stable group $\Omega^+ \subset \Gamma$ is a dense subgroup, provided $f$ is not a unit in $\mathbb{Z}_{ac}(x)$. So, in general, the stable group is not locally compact. To remedy that, we endow the stable group with a stronger topology. For a natural number $N$ define the subset $B_k(N) \subset \mathbb{Z}_{ac}[x]$ by

$$B_k(N) = \left\{ \sum a_i x^i : |a_i| \leq N \text{ for all } i \text{ and } a_i = 0 \text{ for } i < -k \right\},$$

Clearly, $B_k(N)$ is a compact set in the weak topology. Let $B_{\infty}(N)$ be the union of all $B_k(N)$.

**Lemma 8.** For every $f \in \mathbb{Z}[x]$ there exists an $N$ such that $B_{\infty}(N) + f \cdot \mathbb{Z}_{ac}[x] = \mathbb{Z}_{ac}[x]$.

**Proof.** Let $N$ be the sum of all the absolute values of the coefficients of $f$. Suppose that $g \in \mathbb{Z}_{ac}[x]$ and that $\frac{1}{f} \cdot g = \sum r_i x^i$ with $r_i \in \mathbb{R}$. Then $\lim_{i \to -\infty} |r_i| = 0$. Let $[r_i]$ be the integer that is nearest to $r_i$. For $h = \sum [r_i] x^i$ one verifies that $g - f \cdot h \in B_{\infty}(N)$ and that $h \in \mathbb{Z}_{ac}[x]$.

Let $N$ be a sufficiently large integer so that $B_{\infty}(N)$ projects onto $\Omega^+$. We define the strong topology on $\Omega^+$ by the filtration $B_k(N)$. In particular, $U \subset \Omega^+$ is closed if and only if its preimage in $\tilde{U} \subset B_{\infty}(N)$ intersects each $B_k(N)$ in a closed subset. By symmetry, we endow $\Omega^-$ with the strong topology, and we endow $\Omega$ with the discrete topology.

**Theorem 9** (Unfolding). The sequence of locally compact abelian groups

$$0 \longrightarrow \Omega \xrightarrow{x \mapsto (-x,x)} \Omega^- \times \Omega^+ \xrightarrow{(x,y) \mapsto x+y} \Gamma \longrightarrow 0$$

is a self-dual exact sequence.

**Proof.** We have already seen that the sequence is algebraically exact. It is easy to verify that all maps are continuous. To prove that the maps are open, it suffices to show that $\Omega$ embeds as a discrete subgroup of $\Omega^- \times \Omega^+$. Since the image of $\Omega$ is equal to the kernel of the continuous projection $\Omega^- \times \Omega^+ \to \Gamma$, this image is closed, hence locally compact. Since $\Omega$ is countable, this image is countable. A countable and locally compact group is necessarily discrete.

By Corollary the homoclinic group $\Omega$ is isomorphic to the dual of $\Gamma$. So to establish self duality of the sequence, we need to show that $\Omega^+$ and $\Omega^-$ are self-dual. We claim that the dual of $\Omega^+$ is stable under the adjoint action. To see this, note that all
elements of $\Omega^+$ converge to 0 under iteration of $x$. By continuity, a character of $\Omega^+$ converges to the zero character under the adjoint action. Hence the dual group of $\Omega^+$ is stable under the adjoint action and, by symmetry, the dual group of $\Omega^-$ is unstable. Since $\Omega$ embeds discretely, both in $\Omega^- \times \Omega^+$ and in its dual, $\Omega^- \times \Omega^+$ is locally isomorphic to its dual. Let $U \subset \Omega^- \times \Omega^+$ be a neighborhood for which the local isomorphism is defined and let $V$ be its image. The stable elements of $V$ are in the dual of $\Omega^+$. We extend the local isomorphism to an isomorphism between $\Omega^+$ and its dual, as follows. For a stable element $s \in \Omega^+$, let $x^n \cdot s$ be the first element in $U$ under forward iteration of $x$ and let $t$ be its image in $V$. Define the image of $s$ as $x^{-n} \cdot t$, where $x$ now denotes the adjoint action on the dual group. This defines a homomorphism of $\Omega^+$ to its dual group and its inverse can be defined in the same way. So $\Omega^+$ is self-dual and by symmetry so is $\Omega^-$. □

In the next section we describe $\Omega^+$ and $\Omega^-$ by algebraic methods.

6. Stable prime divisors

We do not use our special notation for rings of power series when we add determinates: $\mathbb{Z}[\beta]$ denotes the ring generated by $\mathbb{Z}$ and $\beta$, as usual. It is different from $\mathbb{Z}[\beta, \beta^{-1}]$. We stick to our notation for indeterminates, and so our $\mathbb{Z}[x]$ is usually denoted by $\mathbb{Z}_x$. $x^{-1}$.

In this section, we first assume that the associated polynomial $f \in \mathbb{Z}[x]$ is irreducible. The case of a reducible $f$ follows from the Chinese remainder theorem, which we deal with at the end of this section. So for now, the homoclinic group $\mathbb{Z}[x]/(f)$ is a ring without zero divisors. Let $Q$ be its field of fractions and let $K_P$ be the completion of $Q$ with respect to a prime divisor $P$. The standard method to discretize $\mathbb{Z}[x]/(f)$ is by embedding it in a product of $K_P$ for a certain choice of prime divisors. For instance, if $f$ is a monic polynomial with unit constant coefficient, then all elements of $\mathbb{Z}[x]/(f)$ are algebraic integers and $P$ ranges over all archimedean prime divisors. In the general case, one just needs a few prime-divisors more.

From now on, let $\beta$ denote a root of $f$, so that $\mathbb{Z}[x]/(f)$ is isomorphic to $\mathbb{Z}[\beta, \beta^{-1}]$ and the field of fractions is isomorphic to $\mathbb{Q}(\beta)$. We say that a prime-divisor $P$ of a number field $\mathbb{Q}(\beta)$ is stable if and only if $\lim_{n \to \infty} \beta^n = 0$ in the $P$-topology. Likewise, when $\lim_{n \to -\infty} \beta^n = 0$, we say $P$ is unstable. Since the associated polynomial $f$ is hyperbolic, all archimedean prime-divisors are either stable or unstable. If $P$ is a stable prime divisor, then we say that $\beta$ is a stable root with respect to $P$. More specifically, we say that $\beta$ is $P$-stable if and only if $\beta$ is an element of $O_P \subset \mathbb{Q}(\beta)$, the ring of integers with respect to $P$.

We denote the set of all stable prime-divisors of $\mathbb{Q}(\beta)$, archimedean or non-archimedean, by Stable. It is a finite set. By the Newton polygon, the non-archimedean prime-divisors in Stable are extensions of the rational primes that divide $f(0)$, the constant coefficient.

**Lemma 10.** Suppose that $a^{-1} \in O_P$. Then $1/(a - x) \in O_P[x]$.

**Proof.** The inverse of $a - x$ is $\sum_{n=0}^{\infty} a^{-1-n} x^n$. □
Let $K_P$ be the completion of $\mathbb{Q}(\beta)$ with respect to $P$. If $P$ is stable, then the evaluation map $\sum a_n x^n \mapsto \sum a_n \beta^n$ induces a homomorphism $e_P : O_P[x] \to K_P$. Since $\mathbb{Z}_{ac}[x] \subset O_P[x]$ whenever $P$ is stable, the product of the evaluation maps defines a homomorphism from $\mathbb{Z}_{ac}[x]$ to the product of $K_P$, where $P$ ranges over the stable non-archimedean prime-divisors. Since the radius of convergence of a power series in $\mathbb{Z}_{ac}[x]$ is at least 1, the evaluation map remains well defined for stable archimedean prime divisors as well. Moreover, it is continuous with respect to the strong topology. We are going to show that the product of the evaluation maps $e_P$ for $P \in \text{Stable}$ gives a surjective homomorphism with kernel $(f)$.

**Lemma 11.** Let $P$ be a non-archimedean stable prime-divisor. If $e_P(g) = 0$ for $g \in O_P[x]$, then $g$ is divisible by $\beta - x$.

**Proof.** Let $| \cdot | \in P$ be a valuation and let $g = \sum a_n x^n$ be a power series such that $e_P(g) = 0$. Then the partial sums $s_N = \sum_{n < N} a_n \beta^n$ converge to zero in the $P$-topology and have additive inverse $\sum_{n \geq N} a_n \beta^n$. Since $|a_n| \leq 1$ and since $P$ is non-archimedean, $|s_m| = |\sum_{n \geq m} a_n \beta^n| \leq |\beta|^m$. If $h = \sum (s_n/\beta^n)x^{n-1}$, then $|s_n/\beta^n| \leq 1$ and $g = (\beta - x)h$. \hfill $\square$

**Lemma 12.** Suppose that $g \in \mathbb{Z}[x]$ and that $g \in \ker(e_P)$ for all stable non-archimedean $P$. Then $g$ is divisible by $f$ in $\mathbb{Z}[x]$.

**Proof.** Let $O \subset \mathbb{Q}(\beta)$ be the ring of algebraic integers. It is equal to the intersection of all non-archimedean $O_P$. The two previous lemmas imply that $g = (\beta - x)h$ with $h \in O[x]$. Let $K_f$ be the splitting field of $f$ over $\mathbb{Q}$ and let $\beta_i$ be a conjugate of $\beta$ in $K_f$. By applying Lemma 11 to $\mathbb{Q}(\beta_i)$ we see that $g = (\beta_i - x)\sigma(h)$ in $O_f(x)$, the ring of power series over the algebraic integers in $K_f$. So in $O_f(x)$, $g$ is divisible by all conjugates $x - \beta_i$. Since $f = c(x - \beta_1) \ldots (x - \beta_d)$ for some integer $c$, this implies that $g = (f/c)m$ for $m \in O_f[x]$. On the other hand, since $g$ and $f$ are integral, $g/f \in \mathbb{Q}[x]$, and so $m \in \mathbb{Z}[x]$. Since $fm = 0 \mod c$ and since $f$ is primitive, $m = 0 \mod c$. Hence $m/c \in \mathbb{Z}[x]$. \hfill $\square$

**Lemma 13.** Suppose that the associated polynomial $f$ is irreducible. Then

$$\mathbb{Z}_{ac}[x] \supset (f) = \bigcap \{\ker(e_P) : P \in \text{Stable}\}.$$ 

**Proof.** It is obvious that $(f)$ is contained in the intersection of these kernels, so we only need to prove that the opposite inclusion holds. Suppose that $e_P(g) = 0$ for all $P \in \mathcal{G}$. By the corollary above $\frac{f}{g} \in \mathbb{Z}[x]$. As a complex function, $g$ is holomorphic on the unit disc and it has zeroes at the stable roots. Therefore $\frac{f}{g} \in \mathbb{Z}_{ac}[x]$ is holomorphic on the unit disc. \hfill $\square$

We shall say that the product of the evaluation maps $e_P$ for $P \in \text{Stable}$ is the stable evaluation, since it evaluates an element of $\mathbb{Z}_{ac}[x]$ at all the stable roots of the associated polynomial. We denote the stable evaluation by $e$.

**Theorem 14.** Suppose that the associated polynomial is irreducible. Then the stable evaluation is a topological isomorphism.
Proof. We identify \( \mathbb{Q}(\beta) \) with its image in \( \prod \{ K_P : P \in \text{Stable} \} \). Let \( \mathcal{S}_\infty \subset \text{Stable} \) be the subset of archimedean primes and let \( \mathcal{S}_p \subset \text{Stable} \) be the subset of non-archimedean primes. Let \( K_\infty = \prod_{P \in \mathcal{S}_\infty} K_P \) and \( K_p = \prod_{P \in \mathcal{S}_p} K_P \). Both \( K_\infty \) and \( K_p \) are locally compact.

We argue that the image of \( e \) projects onto \( K_\infty \). The image of \( e \) contains \( \mathbb{Z}[\beta, \beta^{-1}] \). The algebraic integers in \( \mathbb{Z}[\beta, \beta^{-1}] \) form a lattice in the product of \( K_P \) if \( P \) ranges over the archimedean prime-divisors. So the image of \( \mathbb{Z}[x] \) under \( e \) contains a lattice \( L \) of full rank in \( K_\infty \). Let \( U \) be a fundamental domain for \( L \) and let \( F \) be a finite set of lattice points such that the translates \( F + U \) cover \( \beta^{-1} \cdot U \).

For an arbitrary \( x \in K_\infty \) we have to show that there exists a \( g \in \mathbb{Z}_{ac}[x] \) such that \( e(g) \) projects onto \( x \). By the definition of \( U \) there is a \( v \in L \) such that \( u_0 = x - v \in U \). If \( u_0 = 0 \) then \( x \in L \) and since \( L \) is in the projection of the image of \( e \), we are done. If \( u_0 \neq 0 \) then continue dividing \( u_0 \) by \( \gamma \) until \( u_0 / \beta^{n_0} \notin U \), which is possible since \( \beta \) is stable. Then \( u_0 / \beta^{n_0} \in U + f_0 \) for some \( f_0 \in F \). Define \( u_1 = u_0 / \beta^{n_0} - f_0 \). Continue by induction and put \( u_{i+1} = u_i / \beta^{n_i} - f_i \) and truncate the sequence if \( u_{i+1} = 0 \). Then \( x = v + \beta^{n_0} f_0 + \beta^{n_1} f_1 + \ldots \) is a sum, possibly finite, that converges to \( u_0 \). Since \( F \subset L \) each \( f_i \) is a projection onto \( K_\infty \) of \( e(p_i) \) for some polynomial \( p_i \). Since \( F \) is finite there exists a \( g \in \mathbb{Z}_{ac}[x] \) such that \( e(g) \) projects onto \( x \).

We argue that the image of \( e \) projects onto \( K_p \). Let \( R_0 \subset \mathbb{Z}[\beta, \beta^{-1}] \) be the subring of algebraic integers and let \( \gamma = k\beta \) be a multiple of \( \beta \) such that \( \gamma \in R_0 \). Let \( R_k = \gamma^k R_0 \). Then \( R_0 \supset R_1 \supset \ldots \) forms a descending chain that intersects in \( \{ 0 \} \) since \( \gamma^k \) converges to zero. Let \( F \in R_0 \) be a finite set of representatives of \( R_0 / R_1 \). The closure of \( R_0 \) in \( K_p \) is closed and open. For an arbitrary \( x \in K_p \) we have to show that there exists a \( g \in \mathbb{Z}_{ac}[x] \) such that \( e(g) \) projects onto \( x \). Multiply \( x \) by a power of \( \gamma \) such that \( u_0 = \gamma^k x \) is in the closure of \( R_0 \). There exists an \( f_0 \in F \) such that \( u_0 = \gamma u_1 + f_0 \) and we can construct an infinite sum that converges to \( u_0 \) in the same way as in the archimedean case.

So the image of the stable evaluation \( e : \mathbb{Z}_{ac}[x]/(f) \to K_\infty \times K_p \) projects onto both factors. By local compactness and by Baire’s property, both projections are open, and thus \( e \) is an open map. Therefore, the image \( I \) of the stable evaluation is a locally compact subgroup of \( K_\infty \times K_p \), hence \( I \) is closed. The factor group \( K_\infty \times K_p / I \) is isomorphic to \( K_\infty / (I \cap K_\infty) \), hence it is connected. It is isomorphic to \( K_p / (I \cap K_p) \) as well, hence it is totally disconnected. So \( e \) has to be a surjection. The results above imply that the kernel of the stable evaluation \( e \) is equal to \( (f) \). \( \square \)

Summarizing these results, the stable group \( \Omega^+ \) is isomorphic to \( \prod \{ K_P : P \in \text{Stable} \} \) and the isomorphism is induced by the evaluation of a power series at the stable roots of the associated polynomial. By symmetry \( \Omega^- \) is isomorphic to \( \prod \{ K_P : P \in \text{Unstable} \} \) under the evaluation at the unstable roots. So algebraically, the exact sequence \( \square \) is an embedding of the ring \( \mathbb{Z}[\beta, \beta^{-1}] \) in a product of completions of \( \mathbb{Q}(\beta) \), such that its cokernel is its Pontryagin dual. In other words, it is the standard algebraic discretization of a ring. In the dynamic setting, it is not easy to find a fundamental domain for \( \Omega \) in \( \Omega^- \times \Omega^+ \), but in the algebraic setting, it is.
Theorem 15. Let $R$ be the ring of algebraic integers in $\mathbb{Z}[\beta, \beta^{-1}]$. Let $K_a$ be the product of the archimedean completions of $\mathbb{Q}(\beta)$. Let $K_{na}$ be the product of the non-archimedean completions that are stable or unstable. Let $U$ be a fundamental domain of $R$ in $K_a$ and let $O_{na}$ be the ring of integers in $K_{na}$. Then $O_{na} \times U$ is a fundamental domain of $\mathbb{Z}[\beta, \beta^{-1}]$ in the unfolding.

Proof. Assume for the moment that $\mathbb{Z}[\beta, \beta^{-1}]$ is dense in $K_{na}$. Then for every $\kappa \in K_a \times K_{na}$ there exists a $q \in \mathbb{Z}[\beta, \beta^{-1}]$ such that $\kappa - q \in K_a \times O_{na}$. By translation of an $r \in R$ we can move $\kappa - q$ to an element $v \in U \times O_{na}$. The element $v$ is unique, for if $v - v' \in \mathbb{Z}[\beta, \beta^{-1}]$ for some $v'$ then $v - v' \in O_P$ for all non-archimedean $P$ that are stable or unstable and since $\mathbb{Z}[\beta, \beta^{-1}] \subset O_P$ for all other non-archimedean prime-divisors.

It remains to show the validity of our assumption. By the approximation theorem it suffices to show that $\mathbb{Z}[\beta, \beta^{-1}]$ is dense in $\mathbb{Q}(\beta) \subset K_{na}$. The closure of the ring of algebraic integers in $\mathbb{Z}[\beta, \beta^{-1}]$ is open in $\mathbb{Q}(\beta)$. So the closure of $\mathbb{Z}[\beta, \beta^{-1}]$ contains a neighborhood of 0. The sequence $1/(\beta^n + \beta^{-n})$ converges to 0. Hence, for every $x \in \mathbb{Q}(\beta)$ there exists an $n$ such that $x/(\beta^n + \beta^{-n})$ is in the closure of $\mathbb{Z}[\beta, \beta^{-1}]$. \Box

So far we only considered irreducible associated polynomials, but it is not difficult to extend the result to general associated polynomials. By the Chinese remainder theorem the stable group is isomorphic to the direct sum of $\mathbb{Z}_{ac}[x]/(f_i)$ over all primary factors $f_i$ of the associated polynomial, so we need to extend the results to primary polynomials only.

Theorem 16. Suppose that the associated polynomial is primary $f = g^n$ for an irreducible polynomial $g$. Then $\mathbb{Z}_{ac}[x]/(f)$ is isomorphic to $\prod \{K^n_P : P \in \text{Stable}\}$ as a topological group.

Proof. Let $m^{(j)}$ denote the $j$-th derivative of the power series $m$. Obviously, if $m \in (g^n)$ then $m \in (g^{n-1})$ and $m^{(n)} \in (g)$. Conversely, if $m = g^n \cdot m_1$ for some power series $m_1$ and if $g$ divides $m^{(n)}$, then $g$ divides $(g^n) \cdot m_1$. By the irreducibility of $g$ there exist polynomials $h_1, h_2$ such that $h_1(g')^n + h_2g = 1$. Therefore $g$ divides $m_1$. So $m \in (g^n)$. By induction, $m \in (g^n)$ if and only if $m^{(j)} \in (g)$ for all $j < n$. In particular, the map $m \mapsto (m, m', \ldots, m^{(n-1)})$ induces an injective group homomorphism $\varphi : \mathbb{Z}_{ac}[x]/(\bar{f}) \to \bigoplus_{i=1}^n \mathbb{Z}_{ac}[x]^+/(\bar{g})$. This map is also surjective. To prove this, let $h_k$ be a polynomial such that $h_k(g')^k = 1 \mod (g)$ and such that $m = h_kg_k$. Then $m^{(i)} \in (g)$ for $i < k$ and $m^{(k)} = 1 \mod (g)$, so for a polynomial $h$, $(h \cdot m)^{(i)} \in (g)$ for $i < k$ and $(h \cdot m)^{(k)} = h \mod (g)$. The composition of $\varphi$ and the product of the evaluation map $e$ is clearly continuous. \Box

Corollary 17. The associated polynomial is monic (up to a sign) if and only if all unstable prime-divisors are archimedean.

Proof. Let $f = a_dx^d + \ldots + a_0$. By the Newton polygon non-archimedean unstable primes-divisors extend the primes that divide $a_d$ while the unstable prime divisors extend the primes that divide $a_0$. So $a_0$ is a unit if and only if $\mathbb{Q}(\beta)$ has no unstable non-archimedean prime-divisors. \Box

If the expansive automorphism is defined on the torus $\mathbb{T}^n$, then $a_d$ and $a_0$ are units, so all stable and unstable prime-divisors are archimedean. The exact sequence of $\mathbb{T}^n$
is simply the universal factorization through the universal covering space $\mathbb{R}^n$. This is the case that has been considered by Vershik.

7. The path-component of the identity

We show that if the associated polynomial $f$ is irreducible, then the homoclinic group $\Omega$ is irreducible in the following way: its intersection with the path-component of the identity is either $\{0\}$ or $\Omega$.

We denote the union $\text{Stable} \cup \text{Unstable}$ by $\mathcal{P}$. For any subset $\mathcal{Q} \subseteq \mathcal{P}$ we denote the factor ring $\prod_{P \in \Omega} K_P \times \prod_{P \in \mathcal{P} \setminus \Omega} \{0\}$ by $\mathcal{U}_\Omega$. In particular, $\Omega^+ \times \Omega^-$ is identical to $\mathcal{U}_\Omega$.

Since $\mathcal{U}_\mathcal{Q}$ is a factor of $\mathcal{U}_\mathcal{P}$ it projects onto $\Gamma$.

Theorem 18. Suppose that $(\Gamma, \alpha)$ is a cyclic expansive system such that the associated polynomial $f$ is irreducible. Let $\Gamma_0 \subset \Gamma$ be the path-component of the identity. The following statements are equivalent:

1. $\Omega \cap \Gamma_0 \neq \{0\}$
2. $\Omega \subset \Gamma_0$
3. $f(x)$ or $f(1/x)$ is monic (up to a sign).

Proof. (2) $\implies$ (1) is trivial.

(1) $\implies$ (3) Let $\mathcal{Q} \subset \mathcal{P}$ be the subset of archimedean prime-divisors. Then $\mathcal{U}_\mathcal{Q}$ is the path-component of the identity of $\Omega^- \times \Omega^+$ and the projection of $\mathcal{U}_\mathcal{Q}$ onto $\Gamma$ is equal to $\Gamma_0$. Suppose that $\Omega \cap \Gamma_0 \neq \{0\}$. By the exact sequence $\mathcal{P}$ the preimage of $\Omega \subset \Gamma$ in $\Omega^- \times \Omega^+$ is equal to $\Omega \times \Omega$. So $\mathcal{U}_\mathcal{Q}$ intersects $\Omega \times \Omega$ in a point outside the origin. This point is represented by $(g, h)$ for two Laurent polynomials $g$ and $h$. Let $P$ be a prime-divisor in $\mathcal{P} \setminus \mathcal{Q}$ and to fix our ideas, suppose that $P$ is stable. Since $(g, h) \in \mathcal{U}_\mathcal{Q}$ necessarily $h(\beta) = 0 \in K_P$, hence $f$ divides $h$, so $(g, h)$ is equivalent to $(g, 0)$. Then $g \neq 0$ and all unstable prime-divisors are archimedean. By Corollary 17 $f$ is monic up to a sign. By symmetry, if $\mathcal{Q}$ contains all unstable prime-divisors, then $f(1/x)$ is monic.

(3) $\implies$ (2) Suppose that $f$ is monic. Then all unstable prime-divisors are archimedean, and so $\Omega^- \subset \Gamma_0$. Hence $\Omega \subset \Gamma_0$. □

In the proof of the theorem we have used only that $\mathcal{Q}$ is a set of archimedean prime-divisors. So the proof remains valid if $\mathcal{Q}$ is a subset of the archimedean prime-divisors. In other words, if the image of $\mathcal{U}_\mathcal{Q}$ contains a non-trivial homoclinic point, then $\mathcal{Q}$ contains all stable or all unstable prime-divisors. This corollary of the proof is a form of the Pisot-Vijayaraghavan theorem.

Corollary 19 (weak form of the Pisot-Vijayaraghavan Theorem, cmp. [4]). Suppose that $\beta > 1$ is a real algebraic number and suppose that $\beta$ is hyperbolic. If for some $t \in \mathbb{R}$ the sequence $t\beta^n \bmod 1$ converges to $0$ as $|n| \to \infty$. Then $\beta$ is an algebraic integer and the absolute value of all conjugates of $\beta$ is $< 1$.

Proof. Let $f$ be the minimum polynomial of $1/\beta$. The formal power series $g = \sum (t\beta^n)x^n \in \mathbb{T}(x)$ is annihilated under multiplication by $f$. So we may consider $g$ as a non-trivial homoclinic point in the compact abelian group $\Gamma$ with associated polynomial $f$. Even more so, $g$ is contained in the arcwise-connected subgroup $\{(r\beta^n)_{n \in \mathbb{Z}} : r \in \mathbb{R}\}$ of $\Omega^-$. By the theorem above $\beta$ is an algebraic integer.
Let $\lfloor x \rfloor$ denote the integer part of the real number $x$. A preimage of $g$ in $R_{ac}(x)$ is given by

$$\sum ((t\beta)^n - \lfloor (t\beta)^n \rfloor) x^n.$$ 

Under multiplication by $f$ it is mapped onto $-f \cdot \sum \lfloor (t\beta)^n \rfloor x^n$. Note that $\lfloor t\beta^n \rfloor = 0$ if $n$ is sufficiently small. One verifies that the coefficients of the formal power series

$$(x - 1/\beta) \cdot \sum \lfloor t\beta^n \rfloor x^n = \sum \frac{\beta \lfloor t\beta^n \rfloor - \lfloor t\beta^{n+1} \rfloor}{\beta} x^{n+1}$$

are bounded by $\max\{1, |t|\}$, so it represents a holomorphic function in the unit disc.

In particular, $-f \cdot \sum \lfloor t\beta^n \rfloor x^n$ is equal to zero at all stable roots of $f(1/x)$ other than $1/\beta$. In particular, there is only one stable prime-divisor $P$ of $Q(1/\beta)$ for which $e_P(-f \cdot \sum \lfloor t\beta^n \rfloor x^n) \neq 0$. Hence, if we use $\{P\}$ for $\Omega$, then we see that the image of $\mathcal{U}_Q$ contains a non-trivial homoclinic point. So, as a corollary of the proof of the previous theorem, $P$ is the only stable prime-divisor.

Our corollary is a weak form of the Pisot-Vijayaraghavan Theorem since we need the additional (and superfluous) assumption that $\beta$ is hyperbolic. The Pisot-Vijayaraghavan Theorem also gives an explicit description of $t$, but we could have given an explicit description as well, since $(t\beta^n)$ is in the homoclinic group.

An algebraic integer $\beta$ for which all conjugates have absolute value $< 1$ is called a Pisot-Vijayaraghavan number, or simply a Pisot number.

8. Symbolic codings

By a coding of an automorphism $\alpha$ of $\Gamma$, we mean a continuous mapping from a closed, shift invariant subset $S$ of $\{0, \ldots, N\}^\mathbb{Z}$ onto $\Gamma$ commuting the shift map and $\alpha$. We call an element of $S$ finite if all but finitely many terms are 0. Arithmetic codings, which have the property that homoclinic points are coded by finite symbolic sequences, seem to have first appeared in [2] and were more fully developed by Vershik [17] and Sidorov [13] [14]. One may verify that if a coding is arithmetic, then the stable group and the unstable group are coded by symbolic sequences that have a tail of zeroes.

Let $\beta > 1$ be a real number. The one-sided $\beta$-shift $Z_\beta \subset \{0, \ldots, \lfloor \beta \rfloor\}^\mathbb{Z}$ is conjugate to $x \mapsto \beta x \mod 1$ under the projection $(s_n) \mapsto \sum_{n=1}^\infty s_n \beta^{-n}$, see [3] for a survey. If $\beta$ is a Pisot number, then $Z_\beta$ is a sofic set and the projection $Z_\beta \to [0, 1]$ is 1–1 on the complement of a countable set. The two-sided $\beta$-shift $S_\beta$ is obtained by augmenting every element of $Z_\beta$ with a left tail of zeroes and taking the closure of the orbits under the shift. If $\beta$ is a Pisot unit, its minimal polynomial is associated to an expansive toral automorphism, that is naturally coded by $S_\beta$. The coding is given by

$$(s_n)_{n \in \mathbb{Z}} \mapsto \lim_{N \to \infty} \left( \sum_{n=-N}^{N} s_n \beta^{-n} \right) \mathbf{e} \mod \mathbb{Z}^n$$

where $\mathbf{e}$ is a point of $\mathbb{R}^n$ representing a homoclinic point of the automorphism. Vershik and Sidorov gave a detailed analysis of this coding, analyzing the effect of the choice of the homoclinic point $\mathbf{e}$, which amounts to a choice of lattice.
A symbolic coding \( S \to \Gamma \) is called *almost* \( 1 - 1 \) if there exists a subset \( \Gamma_0 \subset \Gamma \) of full Haar measure such that the fibers over \( \Gamma_0 \) are singletons. If the polynomial associated to an automorphism is irreducible with a Pisot root \( \beta \), Schmidt [12] showed that \( S_\beta \) codes the automorphism bounded-to-one, and for a class of Pisot units he was also able to show that the coding is almost \( 1 - 1 \). This led to his conjecture that one could always find an almost \( 1 - 1 \) coding in the case of a Pisot unit. Sidorov and Vershik found that for a wider class of Pisot units (including all quadratic units), \( S_\beta \) gives an almost \( 1 - 1 \) coding of toral automorphisms. Sidorov has reduced the conjecture to algebraic considerations [14], [15].

We call an automorphism \( \alpha \sim f(x) \) a *Pisot automorphism* if \( f(x) \) or \( f(1/x) \) is (up to a sign) monic and irreducible with a Pisot root \( \beta \). And in this context, \( \beta \) is the *associated Pisot number*. If an automorphism is not Pisot, then one cannot expect \( S_\beta \) to code the automorphism almost \( 1 - 1 \) since the entropies in these cases do not necessarily match. For example, consider \( f(x) = 2x - 3 \). The entropy of \( S_{3/2} \) is \( \log(3/2) \), while the associated automorphism has entropy \( \log(3) \) by Yuzvinskii’s formula, (see, e.g., [10]).

One is naturally led to the following generalization of Schmidt’s conjecture.

**Conjecture 20.** A Pisot automorphism with associated Pisot number \( \beta \) admits an almost \( 1 - 1 \) coding by \( S_\beta \).

Recall that we are only considering cyclic automorphisms: the above conjecture is false in general even for tori. Although we have not been able to prove this conjecture, we can establish it in certain cases that generalize previous results for Pisot units. As in [6], let \( \text{Fin}(\beta) \) denote the set of those numbers \( x \geq 0 \) admitting a finite \( \beta \) expansion. We call \( \beta \) *finitary* if \( \text{Fin}(\beta) = (\mathbb{Z}[\beta^{-1}])_+ \). The Pisot root of a polynomial of the form \( f(x) = x^n - a_{n-1}x^{n-1} + \cdots - a_0 \) with \( a_{n-1} \geq a_{n-2} \geq \cdots \geq a_0 \geq 1 \) is finitary [6], but it is not clear which other \( \beta \) are finitary.

**Theorem 21.** If \( \alpha \) is a Pisot automorphism with an associated finitary Pisot number \( \beta \), then the natural coding \( S_\beta : \Gamma \to \Gamma \)

\[
(s_n)_{n \in \mathbb{Z}} \mapsto \sum_{n=0}^{\infty} s_n x^n + (f)
\]

is almost \( 1 - 1 \).

**Proof.** We treat the case that \( f(1/x) \) is monic up to a sign, the other case being handled in a similar way. In this case, the stable manifold \( \Omega^+ \) has just one stable divisor, and it is archimedean. We identify the stable manifold with \( \mathbb{R} \) via the stable evaluation \( e \), which then conjugates the action of the automorphism on \( \Omega^+ \) with multiplication by \( \beta \). If we restrict \( e \) to those elements of \( S_\beta \) with a left tail of zeros, we obtain the “positive” ray in the stable manifold. We now use the unfolding of Theorem 1 to analyze the coding. For \( (s_n) \in S_\beta \), let \( c^+((s_n)) = \sum_{n=1}^{\infty} s_n x^n + (f) \in \Omega^+ \) and \( c^-((s_n)) = \sum_{n=0}^{\infty} s_n x^n + (f) \in \Omega^- \). Then the natural coding corresponds to adding \( c^+((s_n)) \) and \( c^-((s_n)) \) in \( \Gamma \).

Let \( S^-_\beta \) consist of those elements \( (s_n) \in S_\beta \) for which \( s_n = 0 \) for \( n > 0 \). For a class of Pisot numbers including the finitary, it has been shown (see, [10], [7], [6]) that \( S^-_\beta \) admits a group action that extends (measure-theoretically) the odometer or adic
transformation. This transformation maps points with an infinite tail of zeros to the left to their immediate successor in the lexicographic ordering. Notice that when $\beta$ is an integer, this odometer map corresponds to addition by 1 when one identifies sequences with their evaluations. In general it does not correspond to addition by 1 in this sense: sequences in $S_{-\beta}^-$ with a left tail of zeroes do not typically evaluate to an integer. However, when $\beta$ is finitary this map extends to $S_{-\beta}^-$ as a minimal map with purely discrete spectrum. We now relate this map to the return map of a flow on the two sided shift $S_{\beta}$ which has purely discrete spectrum and whose associated action extends the addition of evaluations of sequences with left tails of zeros.

For finitary $\beta$, the “carrying over” of digits when adding two elements of $\text{Fin}(\beta)$ has a uniform bound [6, Proposition 2]. Thus, there is a uniform gap length $G$ so that two elements of $S_{\beta}$ which have infinitely many (to the right and to the left) matching blocks of 0’s of length at least $G$ can be added by treating each pair of sequences between successive blocks of 0’s as an element of $\text{Fin}(\beta)$. The sum then corresponds to the sequence formed by concatenating all the resulting finite expansions of sums, see [13]. This group structure then coincides with addition of expansions and is defined on a subset of $S_{\beta} \times S_{\beta}$ of full measure, provided only that the measure is shift-invariant and positive on cylinders. One can define an additive action of $\mathbb{R}$ (a flow) on $S_{\beta}$ by identifying points in $\mathbb{R}$ with their greedy $\beta$-expansion and adding according to this group structure. We shall call this the additive flow, which is defined on a set of full measure in $\mathbb{R} \times S_{\beta}$. The orbits of the additive flow are dense and the group operation is compatible with the flow. Hence, the additive flow has purely discrete spectrum. Under the natural coding, the additive flow commutes with the continuous action in $\Gamma$ given by $(t, g + (f)) \mapsto e^{-1}(t) + g + (f)$, which has the stable manifold and its translates as orbits. (Here $e^{-1}(t)$ is understood to mean the point in the stable manifold that evaluates as $t$.) Since the group structure is compatible with this flow, it also has purely discrete spectrum. Notice that the image under the unstable evaluation of $c^{-1}(S_{\beta})$ then is a compact cross-section of this flow and the return map to this cross-section coincides with the odometer or adic transformation.

Thus, the natural coding is a (measure theoretic) homomorphism of one compact group onto another. Hence, the coding must be $k - to - 1$ for some finite $k$ off a set of measure 0. The collection of points of $S_{\beta}$ corresponding to the kernel of the coding are shift invariant and finite in number. Thus, each such point can be represented by a periodic sequence. Let $(p_n)$ then be any periodic sequence in the kernel. Recall that all points in the kernel of

$$\Omega^- \times \Omega^+ \to \Gamma$$

from the unfolding have homoclinic points in each factor. In terms of the coding, this means that the stable evaluation of $c^+((p_n))$ must then be a point in $(\mathbb{Z}[\beta^-])_+ = \text{Fin}(\beta)$. Thus, the positive portion of $(p_n)$ must have the same evaluation as a number with a finite $\beta$-expansion. The only alternative admissible expansion of a finite sequence is a sequence ending with the right tail of the infinite expansion of 1. Thus, if $(p_n)$ is not the zero sequence, it must be a sequence each right tail of which coincides with a tail of the infinite expansion of 1. But each such sequence (by considerations of continuity) must represent the identity of the group. $\square$
It might be objected that the above described coding is abstract. However, by identifying the polynomial 1 with a specific fundamental homoclinic point of $\Gamma$, one can easily form a geometric interpretation of the coding. The additive flow maps the identity to the chosen homoclinic point in one unit of time, and the remainder of the coding is determined by continuity.

**Example 22.** If $f(x) = 2x - 1$, then $\beta = 2$ and $S_\beta = \{0, 1\}^\mathbb{Z}$. In this case 2 is the only unstable prime divisor, the additive flow is the suspension of the standard adding machine on the $2^\mathbb{Z}$ integers, which form the unstable cross-section of the flow. The zero element of $S_\beta$ is also represented by the sequence of all 1's.

**Example 23.** If $f(x) = x^2 + nx - 1$ ($n > 1$); $f(1/x) \approx x^2 - nx - 1$. The additive flow is an irrational flow on a torus and the return time to the unstable cross-section is a step function with two steps. The zero element of $S_\beta$ is also represented by the periodic sequences composed of all n0's.

**Example 24.** If $f(x) = 2x^2 + nx - 1$ ($n > 2$); $f(1/x) \approx x^2 - nx - 2$. Then $S_\beta$ is a subshift of finite type and there is both an archimedean and a non-archimedean unstable divisor. The unstable cross-section is one-dimensional but disconnected, and the return map to the cross-section is the product of an interval exchange and a $2^\mathbb{Z}$-adic adding machine. The zero element of $S_\beta$ is also represented by the periodic sequences composed of all n1's.

9. **Acknowledgement**

This paper would not have been written without the help of Hendrik Lenstra. This paper was written while the first author was visiting Delft University, supported by an NWO visitor’s grant.

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