The metric structure of the formigram interleaving distance

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Abstract

Formigrams are a natural generalization of the notion of dendrograms. This notion has recently been proposed as a signature for studying the evolution of clusters in dynamic datasets across different time scales. Although its formulation is set-theoretic, the notion of formigram is deeply related to certain algebraic-topological methods used in topological data analysis, such as Reeb graphs and zigzag persistence modules. In this paper we give a self-contained study of the algebraic structure of formigrams and their interleaving distance. For a finite set $X$, we define a partial order on the collection of all formigrams and we show that every formigram over $X$ has a canonical decomposition into a join of simpler formigrams. This is analogous to the decomposition of persistence modules into direct sums of interval modules. Furthermore, we show that the interleaving distance between formigrams decomposes into a product metric of the interleaving distance between certain pre-cosheaves. This is analogous to the celebrated interleaving-bottleneck isometry theorem for persistence modules.

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1 Introduction

Topological data analysis (TDA) studies in a rigorous way the notion of ‘shape of a dataset’ [4]. The main paradigm of TDA, called persistent homology, studies the ‘evolution’ or ‘persistence’ of certain topological features (loops, holes, cavities) on a dataset, under a given filtration on the dataset [15]. The ‘persistence’ of these features is encoded in a persistence diagram [14]. This, involves considering, first a graded sequence of vector spaces from datasets, called a persistence module, usually yielded by either the Vietoris-Rips or the Čech filtration on datasets, and then considering their associated persistence diagram. Persistence modules are compared via the interleaving distance, and the persistence diagrams are compared via the bottleneck distance [8]. Arguably, the most important results in TDA are (i) the stability theorem, by Cohen-Steiner et al. [9], that shows that persistence diagrams are stable under small perturbations in the datasets, and (ii) the interleaving-bottleneck isometry theorem, by Chazal et al. [8] and Lesnick [20], which shows that there is an isometry between point-wise finite dimensional persistence modules and their persistence diagrams. Carlsson et al. observed that if we think of persistence modules as graded vector spaces, then we can alternatively think of the persistence diagram as a graded basis of a graded vector space. This way, we obtain the notion of barcode of a persistence module which is equivalent to the notion of persistent diagram [7]. Crawley-Boevey showed that any pointwise-finite dimensional persistence module has a unique decomposition (up to permuting the factors) into a direct sum of interval modules [10].

Formigrams One of the main tasks in data analysis is finding and studying clusters in datasets. A dataset is often given as a finite metric space. Some of the commonly used methods are: flat and hierarchical clustering [17]. Hierarchical clustering studies hierarchical families of partitions on datasets, called dendrograms. More generally, we can study dynamic metric spaces. W. Kim and F. Mémoli developed a generalization of dendrograms, called formigrams, which permits to modeling phenomena arising from dynamic data. For example: when data points may separate or
disband and then regroup at different parameter values. Formigrams arise naturally from dynamic graphs via the path connected functor $\pi_0$ [18, Defn. 5.15]. In particular, the map $\pi_0$ is 1-Lipschitz, meaning that formigrams are stable to small perturbations of dynamic graphs; see [18, Thm. 6.32]. A webpage dedicated to illustrating this theoretical framework via synthetic flocking models can be found at [https://research.math.osu.edu/networks/formigrams/](https://research.math.osu.edu/networks/formigrams/).

**Reeb graphs** Every formigram has an underlying *Reeb graph* [18]. de Silva et al. [12] have shown that Reeb graphs can be identified with certain Set-valued cosheaves on $\mathbb{R}$, and therefore, can be thought of as *generalized persistence modules* in the sense of Bubenik et al. [2]. Now, for a given Reeb graph, one can associate the *levelset persistent homology* and obtain a *zigzag persistence module* [18]. Then, because of the interval decomposition of zigzag persistence modules one can thus obtain a meaningful topological signature for Reeb graphs. However, by doing so, an important part of the information of the Reeb graph is lost. Recently, A. Stefanou showed that there exists also a canonical decomposition for Reeb graphs: if we fix a direction, then every Reeb graph admits a canonical coproduct decomposition into ordered Reeb trees [23].

**Our contribution** Inspired by the tree-decomposition of Reeb graphs (which in turn was inspired by interval-decomposition of persistence modules), and the interleaving-bottleneck isometry theorem for persistence modules, in this paper we show that analogous results also hold in the setting of formigrams. Our main result is

**Theorem 4.30** (Structural theorem for $d_F^I$). Let $\theta_X, \theta_Y$ be any two formigrams over $X$ and $Y$, respectively. We have:

$$d_F^I(\theta_X, \theta_Y) = \min_{R} \max_{(x,y) \in R} \max_{(x',y') \in R} d_H(\text{supp}(\widehat{\theta}_X\{x,x'\}), \text{supp}(\widehat{\theta}_Y\{y,y'\})),$$

where the minimum is taken over all tripods $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ between $X$ and $Y$.

**Organization** The paper is organized as follows:

- In Sec. 2 we recall the basic definitions from the theory of lattices, we define subpartitions of a set $X$ and show that the collection of all subpartitions of $X$ forms a lattice. Then, we define formigrams over $X$, and their interleaving distance.
- In Sec. 3 we study two canonical decompositions of subpartitions of $X$: the one given by single block subpartitions, and the other one given by restrictions, and then we utilize the second type of decomposition of subpartitions of $X$ to the setting of formigrams over $X$, to obtain a decomposition of formigrams into a join of simpler formigrams. This join-decomposition of formigrams can be seen as an analogue of the direct sum decomposition of persistence modules in the setting of formigrams. Furthermore, we show that joins of formigrams are stable in the interleaving distance.
- In Sec. 4 we define pre-cosheaves on $\mathbb{R}$ valued in the category of subpartitions of $X$, and we define the interleaving distance on these structures in the sense of Bubenik et al. [2]. In particular, by utilizing the decomposition of subpartitions of $X$ into single block subpartitions,
we show that the interleaving distance on pre-cosheaves has an associated product decomposition. This metric decomposition is an analogous to the interleaving-bottleneck isometry theorem. We show that the space of formigrams isometrically embeds into the space of pre-cosheaves valued in the category of subpartitions of $X$. To some extend, this is analogous to the way Reeb graphs are identified with certain Set-valued cosheaves, in the sense of de Silva et al [12]. This result allows us to obtain a closed formula for the formigram interleaving distance.

2 The space of formigrams

In this section we recall the basic definitions from the theory of lattices and subpartitions [16, 18]. Moreover, we show that the collection of all subpartitions of a fixed set $X$ forms a lattice. Then, we define the notion of formigrams over a set $X$ and their associated interleaving distance.

2.1 Lattices

First, we recall the notion of a poset.

Definition 2.1. A partially ordered set or poset $(\mathcal{P}, \leq)$ is any set $\mathcal{P}$ equipped with a relation $\leq$ on $\mathcal{P}$ satisfying the following properties

- (reflexive) For any $x \in \mathcal{P}$, we have: $x \leq x$.
- (anti-symmetric) For any $x, y \in \mathcal{P}$, we have: $x \leq y$ and $y \leq x \Rightarrow x = y$.
- (transitive) For any $x, y, z \in \mathcal{P}$, we have: $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

Remark 2.2. Any poset $(\mathcal{P}, \leq)$ can be thought of as a category whose objects are the elements $p \in \mathcal{P}$, and whose morphisms are the inequalities $x \leq y$ in $\mathcal{P}$. Also it is easy to check that a functor $F : (\mathcal{P}, \leq) \to (\mathcal{Q}, \leq)$ of posets is simply an order preserving map, i.e. $x \leq y \Rightarrow F(x) \leq F(y)$.

Let $(\mathcal{P}, \leq)$ be a poset which we fix from now on.

Definition 2.3. A subset $\mathcal{U}$ of $\mathcal{P}$ is said to be an upper set in $\mathcal{P}$, if for any $x, y \in \mathcal{P}$:

$$x \in \mathcal{U} \text{ and } x \leq y \Rightarrow y \in \mathcal{U}.$$ 

Remark 2.4. We can define a topology $\tau$ on the poset $\mathcal{P}$ by choosing the open sets to be the upper sets $\mathcal{U}$ in $\mathcal{P}$. This topology is known as the Alexandroff topology on $\mathcal{P}$ [1].

Definition 2.5. Let $x_1, x_2, \ldots, x_n$ in $\mathcal{P}$.

- A join of $x_1, \ldots, x_n$, is an element $y \in \mathcal{P}$ such that (i) $x_i \leq y$, for all $i = 1, \ldots, n$, and (ii) it satisfies the property: for any $w \in \mathcal{P}$, if $x_i \leq w$ for all $i = 1, \ldots, n$, then $y \leq w$.
- A meet of $x_1, \ldots, x_n$, is an element $z \in \mathcal{P}$ such that (i) $z \leq x_i$, for all $i = 1, \ldots, n$, and (ii) it satisfies the property: for any $w \in \mathcal{P}$, if $w \leq x_i$ for all $i = 1, \ldots, n$, then $w \leq z$. 


**Proposition 2.6** (See [16]). Let \( x_1, \ldots, x_n \) in \( \mathcal{P} \).

- If a join of \( x_1, \ldots, x_n \) exists in \( \mathcal{P} \), then it is unique and it is denoted by \( \bigvee_{i=1}^{n} x_i \).
- If a meet of \( x_1, \ldots, x_n \) exists in \( \mathcal{P} \), then it is unique and it is denoted by \( \bigwedge_{i=1}^{n} x_i \).

Because of Prop. 2.6, we will refer to the join and the meet of \( x_1, \ldots, x_n \) whenever they exist.

**Remark 2.7.** From the viewpoint of category theory, the notions of ‘meet’ and ‘join’ correspond to the categorical notions of a product and coproduct respectively [21].

**Definition 2.8.** A poset \((\mathcal{P}, \leq)\) that admits all finite meets and all finite joins is said to be a lattice. In particular we write \((\mathcal{P}, \leq, \lor, \land)\).

**Proposition 2.9** (See [16]). Any lattice \((\mathcal{P}, \leq, \lor, \land)\) satisfies the following properties

- If \( x_i \leq y_i, i = 1, \ldots, n \Rightarrow \bigvee_{i=1}^{n} x_i \leq \bigvee_{i=1}^{n} y_i \).
- If \( x_i \leq y_i, i = 1, \ldots, n \Rightarrow \bigwedge_{i=1}^{n} x_i \leq \bigwedge_{i=1}^{n} y_i \).

**Example 2.10.** Consider the space \( \mathbb{R}^m \) of all real \( m \)-tuples, \((a_1, \ldots, a_m)\). We define the partial order \( \leq_m \) on \( \mathbb{R}^m \) given by:

\[
(a_1, \ldots, a_m) \leq_m (b_1, \ldots, b_m) \iff a_i \leq b_i, \text{ for all } i = 1, \ldots, m.
\]

One can easily verify that the poset \((\mathbb{R}^m, \leq_m)\) is a lattice, because it admits all finite joins and all finite meets. Namely, if \( x_i = (a_1^{(i)}, \ldots, a_m^{(i)}) \in \mathbb{R}^m \), \( i = 1, \ldots, n \), is a set of \( m \)-tuples, then:

- \( \bigvee_{i=1}^{n} x_i = \left( \max_{1 \leq i \leq n} a_1^{(i)}, \ldots, \max_{1 \leq i \leq n} a_m^{(i)} \right) \).
- \( \bigwedge_{i=1}^{n} x_i = \left( \min_{1 \leq i \leq n} a_1^{(i)}, \ldots, \min_{1 \leq i \leq n} a_m^{(i)} \right) \).

### 2.2 Subpartitions

Let \( X \) be a finite nonempty set which we fix from now on.

**Definition 2.11** (See [16]). A partition of \( X \) is a collection \( P \) of nonempty subsets \( B \subset X \), called blocks, such that

- (cover) the union of the blocks \( B \) is equal to \( X \), i.e. \( \bigcup_{B \in P} B = X \).
- (pairwise disjoint) no element \( x \in X \) lies in two different blocks, i.e. \( B \neq B' \Rightarrow B \cap B' = \emptyset \).
Remark 2.12. It is important to recall here that there is a natural one-to-one correspondence between partitions of $X$ and equivalence relations on $X$. Namely, every partition $P$ of $X$ induces the equivalence relation given by: $x \sim_P x' \iff x, x' \in B$, for some block $B \in P$. Dually, every equivalence relation $\sim$ on $X$ induces a partition $X/\sim$ of $X$ given by: $B \in X/\sim \iff B = [x]$, for some $x \in X$, where $[x] = \{x' \in X|x' \sim x\}$.

Definition 2.13 ([18] Defn. 5.1). A subpartition $P$ of $X$ is a partition $P$ of some subset $X'$ of $X$. The only partition of $\emptyset$ is $\emptyset$. The subpartition $\emptyset$ of $X$ is said to be the empty subpartition. We denote the set of all subpartitions of $X$ by $\text{SubPart}(X)$.

Definition 2.14. Let $P_1, P_2$ be two subpartitions of $X$. $P_1$ is said to be a refinement of $P_2$, if for any block $B_1 \in P_1$, there exists a block $B_2 \in P_2$ such that $B_1 \subset B_2$.

We write $P_1 \leq P_2$ whenever the subpartition $P_1$ is a refinement of the subpartition $P_2$ of $X$.

Remark 2.15. Note that since $\emptyset$ has no blocks, then for any subpartition $P$ of $X$, we have $\emptyset \leq P$.

If $P_1 \leq P_2$, then $P_1$ is said to be finer than $P_2$, and $P_2$ is said to be coarser than $P_1$.

Proposition 2.16. $\text{SubPart}(X)$ is a poset.

Proof. The proof is straightforward from the definition of refinement of subpartitions and so, omitted.

Proposition 2.17. The poset of subpartitions over $X$ forms a lattice $(\text{SubPart}(X), \leq, \lor, \land)$.

Proof. First, we claim that the poset $(\text{SubPart}(X), \leq)$ admits finite joins. Let $P_1, \ldots, P_m$ be subpartitions of $X$ with underlying sets $X_1, \ldots, X_m \subset X$, respectively. Let

$$X' = \bigcup_{k=1}^{m} X_k \subset X$$

be the union of their underlying sets. Consider the equivalence relation $\sim$ generated by the relation

$$R = \{(x, x') \in X' \times X'| x, x' \in B \in P_k, \text{ for some } k = 1, \ldots, m\}.$$ 

Let $J$ be the partition of $X' \subset X$—and thus a subpartition of $X$—yielded by $\sim$. By definition of $J$, we note that $P_k \leq J$ for all $k = 1, \ldots, m$. Now we claim that $J = P_1 \lor \cdots \lor P_m$. Indeed: let $Q$ be any subpartition of $X$. Assume that $P_k \leq Q$, for all $k = 1, \ldots, m$. It suffices to show that there exists an inequality $J \leq Q$: that way every inequality $P_k \leq Q$ factors through it, i.e. $P_k \leq J \leq Q$. For all $k = 1, \ldots, m$, since $P_k \leq Q$ then, for all $x, x' \in X'$ we have $x \sim_{P_k} x' \Rightarrow x \sim_Q x'$. By definition, the equivalence relation $\sim_J$ associated to $J$ is equal to the equivalence relation generated by the relations $x \sim_{P_k} x', k = 1, \ldots, m$ in $X'$. Thus $x \sim_J x' \Rightarrow x \sim_Q x'$. In other words $J \leq Q$.

Next we show that $\text{SubPart}(X)$ admits finite meets. Consider

$$X'' = \bigcap_{k=1}^{m} X_k \subset X$$

the intersection of the underlying sets of $P_1, \ldots, P_m$, and

$$R' = \{(x, x') \in X'' \times X'' | x, x' \text{ belong to the same block in } P_k, \text{ for every } k = 1, \ldots, m\}.$$ 

Then, the subpartition $X''/\sim_{R'}$ is the meet of $P_1, \ldots, P_m$. 

Definition 2.18. Let $P_1, \ldots, P_m$ be a finite set of subpartitions of $X$. The join $P_1 \lor \cdots \lor P_m$ is said to be the finest common coarsening of $P_1, \ldots, P_m$. Dually, $P_1 \land \cdots \land P_m$ is said to be the coarsest common refinement of $P_1, \ldots, P_m$. 


2.3 Formigrams

The formigrams is a mathematical tool for modeling ‘time varying clusters’ in dynamic metric spaces. Here, by ‘time varying clusters’, we mean ‘time varying subpartitions of a given set $X$’. We define the poset of formigrams and its associated interleaving distance.

Let $X$ be a finite nonempty set which we fix from now on.

**Definition 2.19 ([19, Sec. 3.1]).** A formigram over $X$ is a function $\theta : \mathbb{R} \rightarrow \text{SubPart}(X)$ such that:

- (Tameness) the set $\text{crit}(\theta)$ of discontinuity of $\theta$ is locally finite. We call the elements of $\text{crit}(\theta)$ the **critical points** of $\theta$.
- (Interval lifespan) for every $x \in X$ the set $I^\theta_x = \{ t \in \mathbb{R} | x \in B \in \theta(t) \}$ is a nonempty closed interval of $\mathbb{R}$, called the **lifespan** of $x$.
- (Comparability) for every point $c \in \mathbb{R}$ it holds that $\theta(c - \varepsilon) \leq \theta(c) \geq \theta(c + \varepsilon)$ for all sufficiently small $\varepsilon > 0$.

Since $\text{SubPart}(X)$ is a poset, the collection of all formigrams on $X$ forms a poset on its own, with the inequality given by

$$\theta \leq \theta' \iff \theta(t) \leq \theta'(t), \text{ for all } t \in \mathbb{R}. \quad (1)$$

**Notation 2.20.** We denote by $\text{Formi}(X)$ the poset of all formigrams over $X$.

**Example 2.21.** Let $X = \{x_1, x_2, x_3, x_4\}$. Consider the formigram $\theta : \mathbb{R} \rightarrow \text{SubPart}(X)$ defined as

$$\theta(t) = \begin{cases} 
\{\{x_1, x_2, x_3\}\}, & \text{if } t < t_0 \\
\{\{x_1, x_2, x_3\},\{x_4\}\}, & \text{if } t_0 \leq t < t_1 \\
\{X\}, & \text{if } t_1 \leq t \leq t_2 \\
\{\{x_1, x_2\},\{x_3, x_4\}\}, & \text{if } t_2 \leq t \leq t_3 \\
\{\{x_1\},\{x_2\},\{x_3, x_4\}\}, & \text{if } t_3 \leq t < t_4 \\
\{\{x_1, x_2\},\{x_3\},\{x_4\}\}, & \text{if } t_4 \leq t < t_5 \\
\{X\}, & \text{if } t \geq t_5,
\end{cases}$$

and depicted as in Fig. [7].

For any $t \in \mathbb{R}$ and any $\varepsilon \geq 0$ we denote the closed interval $[t - \varepsilon, t + \varepsilon]$ of $\mathbb{R}$ by $[t]^\varepsilon$.

**Definition 2.22 ([18, Defn. 6.23]).** Let $\theta : \mathbb{R} \rightarrow \text{SubPart}(X)$ be a formigram over $X$ and let $\varepsilon \geq 0$. We define the $\varepsilon$-smoothing $S_\varepsilon(\theta) : \mathbb{R} \rightarrow \text{SubPart}(X)$ of $\theta$ as

$$S_\varepsilon(\theta)(t) := \bigvee_{s \in [t]^\varepsilon} \theta(s), \text{ for } t \in \mathbb{R}.$$
Figure 1: The formigram \( \theta \) with \( \text{crit}(\theta) = \{t_0, t_1, t_2, t_3, t_4, t_5\} \).

**Remark 2.23.** Because every formigram over \( X \), \( \theta \), is ‘tame’, i.e. satisfies the first condition in Defn. 2.19, and because \( X \) is finite, the join of infinite subpartitions \( \bigvee_{s \in [t]} \theta(s) \) is actually a join of finitely many distinct subpartitions. Therefore, since \( \text{SubPart}(X) \) is a lattice, \( \bigvee_{s \in [t]} \theta(s) \) exists in \( \text{SubPart}(X) \). Thus, the map \( t \mapsto \text{SubPart}(\theta)(t) \) is well defined.

**Proposition 2.24 ([18, Prop. 6.24]).** Let \( \epsilon \geq 0 \). If \( \theta \) is a formigram over \( X \), then the \( \epsilon \)-smoothing \( \text{S}_{\epsilon}\theta \) of \( \theta \) is also a formigram over \( X \).

**Proposition 2.25 ([18, Prop. 6.25]).** Let \( \theta, \theta' \) be a pair of formigrams over \( X \). Then:

- (semi-group) For any \( a, b \geq 0 \), we have: \( \text{S}_{a}(\text{S}_{b}(\theta)) = \text{S}_{a+b}(\theta) \).
- (order-preserving) For any \( \epsilon \geq 0 \), we have: \( \theta \leq \theta' \Rightarrow \text{S}_{\epsilon}(\theta) \leq \text{S}_{\epsilon}(\theta') \).

**Remark 2.26.** One can check that Prop. 2.25 shows that the \( \mathbb{R} \)-indexed family \( \text{S} = (\text{S}_{\epsilon})_{\epsilon \geq 0} \) of maps forms a *strict flow* on the poset \( \text{Formi}(X) \) (viewed as a category) in the sense of de Silva et al. [13], or equivalently, a *linear family of translations* on the poset \( \text{Formi}(X) \) in the sense of Bubenik et al. [2].

**Definition 2.27 (Intrinsic formigram interleaving distance).** Let \( \theta, \theta' \) be two formigrams over \( X \). \( \theta, \theta' \) are said to be \( \epsilon \)-interleaved if \( \theta \leq \text{S}_{\epsilon}(\theta') \) and \( \theta' \leq \text{S}_{\epsilon}(\theta) \). We define the *intrinsic formigram interleaving distance* between \( \theta \) and \( \theta' \) as

\[
d_{F}(\theta, \theta') := \inf \{ \epsilon \geq 0 | \theta, \theta' \text{ are } \epsilon\text{-interleaved} \}.
\]

If \( \theta, \theta' \) are not \( \epsilon \)-interleaved for every \( \epsilon \), then we declare that \( d_{F}(\theta, \theta') \) is \( \infty \).

**Remark 2.28.** The *intrinsic* interleaving distance \( d_{F} \) in Defn. 2.27 is different from the interleaving distance \( d_{I} \) in [18 Defn. 6.26] (see Defn. 4.28). In brief, the relationship between \( d_{F} \) and \( d_{I} \) is analogous to the relationship between the Hausdorff distance \( d_{H} \) and the Gromov-Hausdorff distance \( d_{GH} \) [3]: \( d_{F} \) measures the structural difference between formigrams over the same underlying set, whereas \( d_{I} \) measures the structural difference between formigrams over (possibly) underlying sets. Furthermore, it is easy to check the inequality \( d_{I} \leq d_{F} \) for formigrams on the same underlying set, which is analogous to the inequality \( d_{GH} \leq d_{H} \) for closed subsets of a certain metric space.

**Proposition 2.29.** \( d_{F} : X \times X \rightarrow [0, \infty] \) is an extended metric on formigrams. Namely, it satisfies the properties: For all \( \theta, \theta', \theta'' \) in \( \text{Formi}(X) \),

- \( d_{F}(\theta, \theta) = 0 \),
- \( d_{F}(\theta, \theta') = d_{F}(\theta', \theta) \),
- \( d_{F}(\theta, \theta') \leq d_{F}(\theta, \theta'') + d_{F}(\theta'', \theta') \),
- \( d_{F}(\theta, \theta') = 0 \) if and only if \( \theta = \theta' \).

If \( \theta, \theta' \) are not \( \epsilon \)-interleaved for every \( \epsilon \), then we declare that \( d_{F}(\theta, \theta') \) is \( \infty \).
\[ d_F(\theta, \theta') = d_F(\theta', \theta), \]
\[ d_F(\theta, \theta'') \leq d_F(\theta, \theta') + d_F(\theta', \theta''), \]
\[ d_F(\theta, \theta') = 0 \Rightarrow \theta = \theta'. \]

**Proof.** The first two properties are straightforward and so they are omitted.

For the triangle inequality: assume that \( \theta, \theta' \) are \( \varepsilon \)-interleaved and \( \theta', \theta'' \) are \( \delta \)-interleaved. Then, \( \theta \leq S_\varepsilon(\theta') \) and \( \theta' \leq S_\varepsilon(\theta) \), and also \( \theta' \leq S_\delta(\theta'') \) and \( \theta'' \leq S_\delta(\theta) \). So, we have \( \theta \leq S_\varepsilon(\theta') \) and \( \theta'' \leq S_\delta(\theta) \), and in particular \( S_\varepsilon(\theta') \leq S_\varepsilon(S_\delta(\theta'')) \) and \( S_\delta(\theta') \leq S_\delta(S_\varepsilon(\theta)) \), by applying the second bullet of Prop. 2.25 to the inequalities \( \theta' \leq S_\delta(\theta'') \) and \( \theta'' \leq S_\varepsilon(\theta) \). By the transitive property of \( \leq \) and the first bullet of Prop. 2.25 we obtain: \( \theta \leq S_{\varepsilon+\delta}(\theta'') \) and \( \theta'' \leq S_{\varepsilon+\delta}(\theta) \). Thus, \( \theta, \theta'' \) are \((\varepsilon + \delta)\)-interleaved.

For the fourth property: assume that \( d_F(\theta, \theta') = 0 \). Then for any \( \varepsilon > 0 \), the formigrams \( \theta, \theta' \) are \( \varepsilon \)-interleaved. We claim that \( \theta = \theta' \). It suffices to show that \( \theta' \leq \theta \); the proof of the inequality \( \theta \leq \theta' \) is the same: Let \( t \in \mathbb{R} \). Because of the third condition in the definition of formigrams, for \( c = t \) we can find a \( \delta > 0 \) small enough, such that \( \theta(t-r) \leq \theta(t) \) and \( \theta(t+r) \leq \theta(t) \), for all \( 0 < r \leq \delta \), respectively. Since \( \theta, \theta' \) are \( \varepsilon \)-interleaved for any \( \varepsilon > 0 \), then in particular, \( \theta, \theta' \) are \( \delta \)-interleaved. Thus, we have

\[
\theta'(t) \leq \bigvee_{s \in [t-\delta,t+\delta]} \theta(s) \quad \text{(since \( \theta, \theta' \) are \( \delta \)-interleaved)}
\]

\[
= \left( \bigvee_{s \in [t-\delta,t]} \theta(s) \right) \lor \left( \theta(t) \right) \lor \left( \bigvee_{s \in (t,t+\delta]} \theta(s) \right)
\]

\[
= \left( \bigvee_{r \in (0,\delta]} \theta(t-r) \right) \lor \left( \theta(t) \right) \lor \left( \bigvee_{r \in (0,\delta]} \theta(t+r) \right)
\]

\[
\leq \left( \bigvee_{r \in (0,\delta]} \theta(t) \right) \lor \left( \theta(t) \right) \lor \left( \bigvee_{r \in (0,\delta]} \theta(t) \right)
\]

\[
= (\theta(t)) \lor (\theta(t)) \lor (\theta(t))
\]

\[
= \theta(t).
\]

\[\Box\]

**Example 2.30.** Fix \( \delta > 0 \). Consider the formigrams \( \theta, \theta' : \mathbb{R} \rightarrow \text{SubPart}(\{1, 2\}) \) given as:

\[
\theta(t) = \{\{1, 2\}\}, \quad \text{for all } t \in \mathbb{R},
\]

\[
\theta'(t) = \begin{cases} 
\{\{1, 2\}\}, & \text{if } t \leq -\delta \\
\{\{1\}, \{2\}\}, & \text{if } -\delta < t < \delta \\
\{\{1, 2\}\}, & \text{if } t \geq \delta
\end{cases}
\]

(see Fig. 2). The formgram \( \theta \) has no critical point, whereas the formgram \( \theta' \) has two critical points \(-\delta, \delta\). Let \( \varepsilon \geq 0 \). Because the partition \( \{\{1\}, \{2\}\} \) of \( \{1, 2\} \) always refines the partition \( \{\{1, 2\}\} \)
of \{1,2\}, we have \(\theta' \leq S_\varepsilon(\theta)\). Let us assume \(\varepsilon < \delta\): Then, we have \(S_\varepsilon(\theta')(0) := \bigvee_{s \in [0,\varepsilon]} \theta'(s) = \{1\} \cup \{2\}\) and \(\theta(0) = \{1,2\}\). Thus, \(\theta' \not\leq S_\varepsilon(\theta)\) and so \(\theta, \theta'\) are not \(\varepsilon\)-interleaved. On the other hand, it is easy to check that \(S_\varepsilon(\theta') = \theta\) for \(\varepsilon \geq \delta\). Hence, \(\theta, \theta'\) are \(\varepsilon\)-interleaved if and only if \(\varepsilon \geq \delta\). Therefore, \(d_F(\theta, \theta') = \delta\). If we view the loop as a ‘cycle’, then the \(d_F(\theta, \theta')\) is equal to the ‘radius of the cycle’, \(\delta\).

3 Join-decomposition of a formigram

In this section we consider two canonical join-decompositions for subpartitions of a given set \(X\).

3.1 Join-decompositions of subpartitions

Let \(X\) be a finite nonempty set which we fix from now on.

**Definition 3.1.** Let \(P\) be a subpartition of \(X\). For any \(x,x' \in X\) we denote by \(P_{\{x,x'\}}\) the subpartition of \(X\) given by

\[
P_{\{x,x'\}} := \begin{cases} \{\{x,x'\}\}, & \text{if } x, x' \in B \in P \\ \emptyset, & \text{otherwise.} \end{cases}
\]

**Proposition 3.2.** For any pair \(P, Q\) of subpartitions on \(X\) we have:

\[P \leq Q \iff P_{\{x,x'\}} \leq Q_{\{x,x'\}}, \text{ for all } x, x' \in X.\]

**Proof.** We have

\[
P \leq Q \\
\iff \text{For any } B \in P, \text{ there exists a } B' \in Q, \text{ such that } B \subset B'. \\
\iff \text{For any } x, x' \in X : x \sim_P x' \Rightarrow x \sim_Q x'. \\
\iff \text{For any } x, x' \in X : x, x' \in B \in P \Rightarrow x, x' \in B' \in Q. \\
\iff \text{For any } x, x' \in X : P_{\{x,x'\}} \leq Q_{\{x,x'\}}.
\]
**Theorem 3.3** (Decomposition into single blocks). Let $P$ be a subpartition of $X$. Then

$$P = \bigvee_{x,x' \in X} P_{\{x,x'\}}.$$ 

**Proof.** First, we prove the inequality $\geq$, and then the inequality $\leq$.

- By definition, for all $x, x' \in X$, we have $P_{\{x,x'\}} \leq P$. By definition of joins, we get:
  $$\bigvee_{x,x' \in X} P_{\{x,x'\}} \leq P.$$

- Let $x, x' \in X$. Assume $x \sim_P x'$. Then, $x \sim_{P_{\{x,x'\}}} x'$. In particular, $x \sim \bigvee_{y,y' \in X} P_{\{y,y'\}} x'$. Therefore, $P \leq \bigvee_{x,x' \in X} P_{\{x,x'\}}$.

**Example 3.4.** Consider $X = \{1, 2, 3, 4, 5, 6, 7\}$ and the subpartition $P = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$ of $X$. Let $\sim_P$ be the equivalence relation corresponding to $P$. Then $\sim_P$ is generated by the relations $1 \sim_P 2$, $2 \sim_P 3$, and $4 \sim_P 5$. In particular, $\sim_P$ is generated by the relations $1 \sim_P 2$, $2 \sim_P 3$, $1 \sim_P 3$, $4 \sim_P 5$, and $j \sim_P j$ for all $j = 1, \ldots, 6$. Also, by definition of $P$, we have $P_{\{i,7\}} = P_{\{7,i\}} = \emptyset$, for any $i = 1, \ldots, 7$. Thus, we get

$$\bigvee_{i,j \in \{1,\ldots,7\}} P_{\{i,j\}} = \bigvee_{i \sim_P j} \{i, j\}$$

$$= \{\{1,2\}\} \cup \{\{1,3\}\} \cup \{\{2,3\}\} \cup \{\{4,5\}\} \cup \bigvee_{i=1,\ldots,6} \{\{i\}\}$$

$$= \{\{1,2,3\}, \{4,5\}, \{6\}\}$$

$$= P.$$

Next, we define restrictions on subpartitions.

**Definition 3.5.** Let $P$ be any subpartition of $X$. Let $A$ be any subset of $X$. We define the *restriction* $P|_A$ of $P$ on $A$ to be the subpartition of $X$ given by

$$P|_A := \{B \cap A : B \in P\}.$$

**Proposition 3.6.** For any pair $P, Q$ of subpartitions of $X$ and any $A \subset X$ we have: $P \leq Q \Rightarrow P|_A \leq Q|_A$.

**Proof.** Assume $P \leq Q$. Then, for any $B \in P$ there exists a $B' \in Q$ such that $B \subset B'$. By taking intersections with $A \subset X$ we get $B \cap A \subset B' \cap A$. Thus, for any block $B \cap A \in P|_A$ there exists a block $B' \cap A \in Q|_A$ such that $B \cap A \subset B' \cap A$. 


Remark 3.7. Let $P$ be a subpartition of $X$. For any $x, x' \in X$, the subpartition $P_{\{x, x'\}}$ is given by the formula

$$P_{\{x, x'\}} = \{ B \cap \{ x, x' \} : B \in P \} = \begin{cases} \{ \{ x, x' \} \}, & \text{if } x, x' \in B \in P \\ \{ \{ x \} \}, & \text{if } x \in B \in P, \text{ and } x' \notin B', \text{ for all } B' \in P' \\ \{ \{ x' \} \}, & \text{if } x' \in B' \in P', \text{ and } x \notin B, \text{ for all } B \in P \\ \emptyset, & \text{otherwise}. \end{cases}$$

(2)

**Theorem 3.8** (Decomposition into restrictions). Let $X$ be a finite set. Let $P$ be a subpartition of $X$. Then

$$P = \bigvee_{(x, x') \in X \times X} P_{\{x, x'\}}.$$

**Proof.** The proof of Thm. 3.8 is similar to that of Thm. 3.3 and thus we omit it. \qed

**Proposition 3.9.** For any pair $P, Q$ of subpartitions on $X$ we have:

$$P \leq Q \iff P_{\{x, x'\}} \leq Q_{\{x, x'\}}, \text{ for all } x, x' \in X.$$

**Proof.**

- Assume $P \leq Q$. By Prop. 3.6, for $A = \{ x, x' \}$, we get $P_{\{x, x'\}} \leq Q_{\{x, x'\}}$, for all $x, x' \in X$.
- Assume $P_{\{x, x'\}} \leq Q_{\{x, x'\}}$, for all $x, x' \in X$. By Prop. 2.9 we obtain

$$\bigvee_{x, x' \in X} P_{\{x, x'\}} \leq \bigvee_{x, x' \in X} Q_{\{x, x'\}}.$$

By Thm. 3.8 we obtain $P \leq Q$. \qed

### 3.2 Join-decomposition of a formigram

Let $X$ be a finite nonempty set. Recall that $\text{Formi}(X)$ is a poset.

**Theorem 3.10** (Join-decomposition). Let $\theta : \mathbb{R} \rightarrow \text{SubPart}(X)$ be a formigram over $X$. For each pair $x, x' \in X$, consider the function

$$\theta_{\{x, x'\}} : \mathbb{R} \rightarrow \text{SubPart}(\{ x, x' \})$$

$$t \mapsto \theta(t)_{\{x, x'\}},$$

given point-wisely by the restriction of the subpartition $\theta(t)$ of $X$ to the set $\{ x, x' \}$.

Then:

- For any $x, x' \in X$, $\theta_{\{x, x'\}}$ is a formigram over $\{ x, x' \}$.
\[ \theta = \bigvee_{x,x' \in X} \theta_{\{x,x'\}}. \]

**Proof.** The join decomposition is yielded, point-wisely, by Thm. 3.8. Now, we need to show that \( \theta_{\{x,x'\}} \) is a formigram over \( \{x, x'\} \); namely it satisfies the three conditions of Defn. 2.19.

- (Tameness) This is straightforward from the definition of \( \theta_{\{x,x'\}} \).
- (Interval lifespan) We claim that the lifespans of \( x \) and \( x' \) are nonempty closed intervals. It suffices to show that the lifespan of \( x \) is indeed a nonempty closed interval; the proof for the other case is similar. Indeed, we have:

\[
I^\theta_{\{x,x'\}}(x) = \{ t \in \mathbb{R} | x \in B \in \theta(t) \}_{\{x,x'\}} \\
= \{ t \in \mathbb{R} | x \in B \cap \{x, x'\}, \text{ and } B \in \theta(t) \} \\
= \{ t \in \mathbb{R} | x \in B \in \theta(t) \} \\
= I^\theta_x.
\]

Thus, \( I^\theta_{\{x,x'\}}(x) \) is a nonempty closed interval, because \( I^\theta_x \) is.

- (Comparability) This follows directly by combining Prop. 3.9 and the fact that \( \theta \) itself satisfies the comparability condition.

\[ \square \]

**Remark 3.11.** One can view Thm. 3.10 as an analogue of the decomposition theorem for persistence modules [10, Thm. 1.1], in the setting of formigrams.

**Remark 3.12.** Note that, to prove Thm. 3.10, we applied point-wisely the decomposition of subpartitions into restrictions (Thm. 3.8) rather than decomposition into single blocks (Thm. 3.3). Choosing this type of decomposition is necessary, because a decomposition of a formigram that uses single block subpartitions will not always yield functions that are formigrams. For instance, consider the formigram over \( \{1, 2\}, \theta' \), in Ex. 2.30. It is easy to see, that if we choose the function \( \theta_{\{1,2\}} : t \mapsto (\theta(t))_{\{1,2\}} \), then the lifespans of \( x = 1 \) (and also of \( x = 2 \)) in the formigram \( \theta_{\{1,2\}} \) are not intervals (as they supposed too), but a disjoint union of intervals. So, the second bullet of Defn. 2.19 is not satisfied. Hence, the induced function \( \theta_{\{1,2\}} : t \mapsto \theta(t)_{\{1,2\}} \) is not a formigram.

**Remark 3.13 (Dendrograms).** A dendrogram over a finite set \( X \) is a special type of formigram satisfying several properties: Namely, a formigram \( \theta : R \rightarrow \text{SubPart}(X) \) is a dendrogram if (1) \( \theta(t) = \emptyset \) for \( t \in (-\infty, 0) \), (2) \( \theta(0) = \{ \{x\} : x \in X \} \), and (3) \( \theta(t_1) \leq \theta(t_2) \) for every pair \( t_1 \leq t_2 \) in \( \mathbb{R} \), and (4) there exists \( T > 0 \) such that \( \theta(t) = \{X\} \) for \( t \in [T, \infty) \) (note that by Conditions (2) and (3), for each positive \( t \), \( \theta_X(t) \) is a partition of \( X \), not just a subpartition). Given a dendrogram \( \theta \) over \( X \) and \( x, x' \in X \), we define

\[
u_\theta(x, x') := \min \{ \varepsilon \in \mathbb{R} : x \text{ and } x' \text{ belong to the same block in } \theta(\varepsilon) \},
\]
which is an ultrametric [5] on \( X \). For any \( x, x' \in X \), note that \( \theta_{\{x,x'\}} \) is defined as

\[
\theta_{\{x,x'\}}(t) = \begin{cases} 
\emptyset, & t \in (-\infty, 0), \\
\{\{x\}, \{x'\}\}, & t \in [0, \nu_\theta(x, x')), \\
\{\{x, x'\}\}, & t \in [\nu_\theta(x, x'), \infty). 
\end{cases}
\]
Figure 3: Example of the join-decomposition of a formigram. The elements 1, 2, 3 of X are colored blue, red, and green, respectively.

**Example 3.14.** Let $X = \{1, 2, 3\}$. Consider the formigram $\theta$ over $X$ as in Fig. 3 (where the real line $\mathbb{R}$ can be thought of as seating vertically next to each formigram). The formigrams $\theta_{\{1\}}, \theta_{\{2\}}$, and $\theta_{\{3\}}$ correspond to the blue, red, and green colored line segments respectively. The formigram $\theta_{\{1,2\}}$ has a loop, and the formigrams $\theta_{\{1,3\}}, \theta_{\{2,3\}}$ have no loops. These are called *treegrams* [19] [22].

**Theorem 3.15** (Stability of joins). Let $J$ be any finite index set. Let $(\theta_j)_{j \in J}, (\theta'_j)_{j \in J}$ be two $J$-indexed families of formigrams over $X$. Then

$$d_F \left( \bigvee_{j \in J} \theta_j, \bigvee_{j \in J} \theta'_j \right) \leq \max_{j \in J} d_F(\theta_j, \theta'_j).$$

**Proof.** Assume that $\theta_j, \theta'_j$ are $\varepsilon$-interleaved, for all $j \in J$, for some $\varepsilon \geq 0$. Then

$$\theta_j \leq S_\varepsilon(\theta'_j) \text{ and } \theta'_j \leq S_\varepsilon(\theta_j), \text{ for all } j \in J. \quad (3)$$

Let $i \in J$. Then, clearly

$$\theta'_{\bigvee i} \leq \bigvee_{j \in J} \theta'_j \text{ and } \theta_{\bigvee i} \leq \bigvee_{j \in J} \theta_j.$$

By Prop. 2.25 we obtain

$$S_\varepsilon(\theta'_{\bigvee i}) \leq S_\varepsilon \left( \bigvee_{j \in J} \theta'_j \right) \text{ and } S_\varepsilon(\theta_{\bigvee i}) \leq S_\varepsilon \left( \bigvee_{j \in J} \theta_j \right).$$
By combining with Eqn. (3), we thus obtain

\[ \theta_i \leq S_\varepsilon \left( \bigvee_{j \in J} \theta'_j \right) \quad \text{and} \quad \theta'_i \leq S_\varepsilon \left( \bigvee_{j \in J} \theta_j \right), \text{ for all } i \in J. \]

By definition of joins we get

\[ \bigvee_{j \in J} \theta_j \leq S_\varepsilon \left( \bigvee_{j \in J} \theta'_j \right) \quad \text{and} \quad \bigvee_{j \in J} \theta'_j \leq S_\varepsilon \left( \bigvee_{j \in J} \theta_j \right). \]

Therefore, \( \bigvee_{j \in J} \theta_j, \bigvee_{j \in J} \theta'_j \) are \( \varepsilon \)-interleaved.

4 Metric structure of the formigram interleaving distance

In this section, we define pre-cosheaves valued in \( \text{SubPart}(X) \), and their associated interleaving distance. In particular, we show that this distance decomposes into a product metric on pre-cosheaves valued in \( \text{SubPart}(\{x, x'\}) \), for all \( x, x' \in X \). Moreover, we show that formigrams isometrically embed into the category of pre-cosheaves valued in \( \text{SubPart}(X) \). This embedding combined with the product decomposition of the interleaving distance on pre-cosheaves, yields a closed formula for the formigram interleaving distance.

4.1 Pre-cosheaves valued in \( \text{SubPart}(X) \)

Let \( X \) be a finite nonempty set which we fix from now on.

**Definition 4.1.** Define \( \text{Int} = (\text{Int}, \subset) \) the poset whose elements are the open intervals \( I \) of \( \mathbb{R} \), and the partial order \( \subset \) is given by set inclusions.

**Remark 4.2.** Geometrically, we can visualize \( \text{Int} \) as the subset of \( \mathbb{R}^2 \):

\[ \text{Int} = \{(a, b) \in \mathbb{R}^2 | a < b\}, \]

given by the upper half of the plane, strictly above the diagonal \( y = x \). For any pair of intervals, \( (a, b), (a', b') \), we have \( (a, b) \subset (a', b') \iff a \geq a' \text{ and } b \leq b' \). Hence, geometrically, we have an inclusion \( (a, b) \subset (a', b') \) if and only if there is an arrow joining the two points heading ‘up’ and to the ‘left’.

**Definition 4.3.** A **pre-cosheaf valued in** \( \text{SubPart}(X) \) or simply a **pre-cosheaf**, \( \alpha \), consists of an \( \text{Int} \)-indexed sequence \( \{(\alpha(I))_{I \in \text{Int}}\} \) of subpartitions of \( X \), such that \( \alpha(I) \leq \alpha(J) \) whenever \( I \subset J \).

In terms of category theory, a pre-cosheaf, \( \alpha \), is simply a functor

\[ \alpha : (\text{Int}, \subset) \to (\text{SubPart}(X), \leq). \]

**Definition 4.4 (Support).** Let \( \alpha : \text{Int} \to \text{SubPart}(X) \) be a pre-cosheaf. We define the support of \( \alpha \) to be the set

\[ \text{supp}(\alpha) := \{ I \in \text{Int} | \alpha(I) \neq \emptyset \}. \]
Proposition 4.5. For any pre-cosheaf \( \alpha : \text{Int} \to \text{SubPart}(X) \), its support \( \text{supp}(\alpha) \) is an upper set in the poset \((\text{Int}, \subset)\).

Proof. Assume that \( I \in \text{supp}(\alpha) \) and \( I \subset J \). Then, \( \alpha(I) \neq \emptyset \), and \( \alpha(I) \leq \alpha(J) \), because \( \alpha \) is a pre-cosheaf. Thus, we have \( \alpha(J) \neq \emptyset \) and in turn \( J \in \text{supp}(\alpha) \). \( \square \)

Definition 4.6. For every \( \varepsilon \geq 0 \), the poset \( \text{Int} \) is equipped with the \( \varepsilon \)-thickening map

\[
\Omega_{\varepsilon} : \text{Int} \to \text{Int}
\]

\[
(a, b) \mapsto (a - \varepsilon, a + \varepsilon).
\]

Notation 4.7. We denote the \( \varepsilon \)-thickening \( \Omega_{\varepsilon}(I) \) of a nonempty open interval \( I \) of \( \mathbb{R} \) by \( I^\varepsilon \) for simplicity.

Definition 4.8 (\([2, 11]\)). Let \( \alpha, \alpha' : \text{Int} \to \text{SubPart}(X) \) be a pair of pre-cosheaves. Then, \( \alpha, \alpha' \) are said to be \( \varepsilon \)-interleaved if for all \( I \in \text{Int} \) we have \( \alpha(I) \leq \alpha'(I^\varepsilon) \) and \( \alpha'(I) \leq \alpha(I^\varepsilon) \). We define the pre-cosheaf interleaving distance as

\[
d_I(\alpha, \alpha') := \inf \{ \varepsilon \geq 0 | \alpha, \alpha' \text{ are } \varepsilon \text{-interleaved} \}.
\]

If \( \alpha, \alpha' \) are not \( \varepsilon \)-interleaved for all \( \varepsilon \geq 0 \), we declare \( d_I(\alpha, \alpha') = \infty \).

Remark 4.9. Note that the pre-cosheaf interleaving distance in Defn. 4.8 is a special case of an interleaving distance on generalized persistent modules in the sense of Bubenik et al. \([2]\).

Definition 4.10 (Formigrams induce pre-cosheaves). Given any formigram \( \theta \) over \( X \), we associate the map

\[
\hat{\theta} : \text{Int} \to \text{SubPart}(X)
\]

\[
I \mapsto \bigvee_{s \in I} \theta(s).
\]

By definition of joins, if \( I \subset J \) in \( \text{Int} \), then \( \hat{\theta}(I) \leq \hat{\theta}(J) \). Thus, \( \hat{\theta} \) is a pre-cosheaf.

Remark 4.11. Again, following the same reasoning as in Rem. \([2, 23]\), we easily see that the join of subpartitions \( \bigvee_{s \in I} \theta(s) \) (although seems infinite) is actually a join of finitely many distinct subpartitions of \( X \). Because \( \text{SubPart}(X) \) is lattice, then \( \bigvee_{s \in I} \theta(s) \) exists, i.e. it is indeed a subpartition of \( X \). Thus, the pre-cosheaf \( \hat{\theta} : I \mapsto \hat{\theta}(I) \) is well defined.

Theorem 4.12 (Isometric embedding). If \( \theta, \theta' \) is a pair of formigrams over \( X \), then

\[
d_F(\theta, \theta') = d_I(\hat{\theta}, \hat{\theta'}).
\]

Proof. First we prove the inequality \( \geq \), and then the inequality \( \leq \).
• Assume that $\theta, \theta'$ are $\varepsilon$-interleaved, for some $\varepsilon \geq 0$. Let $I$ be an open interval of $\mathbb{R}$. We compute:

$$\hat{\theta}(I) = \bigvee_{t \in I} \theta(t)$$
$$\leq \bigvee_{t \in I} \left( \bigvee_{s \in [t]_{\varepsilon}} \theta'(s) \right)$$
$$= \bigvee_{s \in [I]_{\varepsilon}} \theta'(s)$$
$$= \hat{\theta'}(I_{\varepsilon}).$$

Similarly $\hat{\theta'}(I) \leq \hat{\theta}(I_{\varepsilon})$. Hence, $\hat{\theta}, \hat{\theta'}$ are $\varepsilon$-interleaved, and so, $d_1(\hat{\theta}, \hat{\theta'}) \leq d_1(\theta, \theta').$

• Assume that $d_1(\hat{\theta}, \hat{\theta'}) = \varepsilon$. Then, by definition of the interleaving distance $d_1$:

For all $\delta > 0$, the pre-cosheaves $\hat{\theta}, \hat{\theta'}$ are $(\varepsilon + \delta)$-interleaved. (4)

Pick any $\zeta > 0$. We claim that the formigrams $\theta, \theta'$ are $(\varepsilon + 2\zeta)$-interleaved. Let $t \in \mathbb{R}$. Then we compute

$$\theta(t) \leq \bigvee_{t-\zeta < s < t+\zeta} \theta(s)$$
$$= \hat{\theta}((t - \zeta, t + \zeta))$$
$$\leq \hat{\theta'}((t - \zeta - (\varepsilon + \zeta), t + \zeta + (\varepsilon + \zeta))), \text{ by taking } \delta = \zeta \text{ in Eqn. (4)}$$
$$= \bigvee_{t-(\varepsilon+2\zeta) < s < t+(\varepsilon+2\zeta)} \theta'(s)$$
$$\leq \bigvee_{s \in [t]_{\varepsilon+2\zeta}} \theta'(s)$$
$$= S_{\varepsilon+2\zeta}(\theta')(t).$$

Hence, $\theta \leq S_{\varepsilon+2\zeta}(\theta')$ and similarly we get $\theta' \leq S_{\varepsilon+2\zeta}(\theta)$. Namely, $\theta, \theta'$ are $(\varepsilon + 2\zeta)$-interleaved. Therefore, we have $d_F(\theta, \theta') \leq \varepsilon + 2\zeta$. If we let $\zeta \to 0$, then we obtain $d_F(\theta, \theta') \leq \varepsilon.$

\[ \square \]

4.2 Metric decomposition of the pre-cosheaf interleaving distance

We show that the pre-cosheaf interleaving distance $d_1$ between a pair of pre-cosheaves valued in $\text{SubPart}(X)$ is actually equal to the product metric between certain pre-cosheaves valued in $\text{SubPart}({x, x'})$, over all pairs $x, x' \in X$. To construct these pre-cosheaves, we will apply the decomposition of subpartitions that uses single block subpartitions (see Defn. 3.1 and Thm. 3.3).
Remark 4.13. Note that if we apply the decomposition of subpartitions that uses restrictions, we could still obtain a metric decomposition. However, working with the decomposition of subpartitions that uses single blocks has computational advantages over the other.

Definition 4.14. Let $\alpha : \text{Int} \to \text{SubPart}(X)$ be a pre-cosheaf. Then, for each $x, x' \in X$, we define the pre-cosheaf $\alpha_{\{x,x'\}} : \text{Int} \to \text{SubPart}(\{x, x'\})$, given interval-wisely by the formula

$$\alpha_{\{x,x'\}}(I) := \alpha(I)_{\{x,x'\}} = \begin{cases} \{\{x, x'\}\}, & \text{if } x, x' \in B \in \alpha(I) \\ \emptyset, & \text{otherwise.} \end{cases}$$

Theorem 4.15 (Metric decomposition of pre-cosheaves valued in $\text{SubPart}(X)$). Let $\alpha, \alpha' : \text{Int} \to \text{SubPart}(X)$ be a pair of pre-cosheaves. Then

$$d_1(\alpha, \alpha') = \max_{x,x' \in X} d_1(\alpha_{\{x,x'\}}, \alpha'_{\{x,x'\}}).$$

Proof. We claim that: $\alpha, \alpha'$ are $\varepsilon$-interleaved $\iff \alpha_{\{x,x'\}}, \alpha'_{\{x,x'\}}$ are $\varepsilon$-interleaved, for all $x, x' \in X$. Indeed:

$\alpha, \alpha'$ are $\varepsilon$-interleaved.

$\iff$ For any $I \in \text{Int} : \alpha(I) \leq \alpha'(I^\varepsilon)$ and $\alpha'(I) \leq \alpha(I^\varepsilon)$.

$\iff$ For any $I \in \text{Int}$, and any $x, x' \in X : \alpha(I)_{\{x,x'\}} \leq \alpha'(I^\varepsilon)_{\{x,x'\}}$ and $\alpha'(I)_{\{x,x'\}} \leq \alpha(I^\varepsilon)_{\{x,x'\}}$.

$\iff$ For any $x, x' \in X : \alpha_{\{x,x'\}}(I) \leq \alpha'_{\{x,x'\}}(I^\varepsilon)$ and $\alpha'_{\{x,x'\}}(I) \leq \alpha_{\{x,x'\}}(I^\varepsilon)$.

$\iff$ For any $x, x' \in X : \alpha_{\{x,x'\}}, \alpha'_{\{x,x'\}}$ are $\varepsilon$-interleaved.

The equivalence $(\ast)$ holds by Prop. 3.2.

Remark 4.16. Note that, formally, if we replace the pair $(\text{Int}, \Omega)$ with an arbitrary pair $(P, F)$ consisting of any poset $P$ with a linear family of translations $F$, then Thm. 4.15 still holds. So, the theorem is true for arbitrary pre-cosheaves valued in $\text{SubPart}(X)$.

The next step is to compute the interleaving distance $d_1(\alpha_{\{x,x'\}}, \alpha'_{\{x,x'\}})$, for each $x, x' \in X$, individually.

Definition 4.17. Let $\star$ be any set. Any upper set $A$ in $\text{Int}$ has an associated pre-cosheaf $I^A : \text{Int} \to \text{SubPart}(\star)$, given point-wisely by the formula:

$$I^A((a, b)) = \begin{cases} \{\star\}, & \text{if } (a, b) \in A \\ \emptyset, & \text{otherwise.} \end{cases}$$

Remark 4.18. Consider a pre-cosheaf $\alpha : \text{Int} \to \text{SubPart}(X)$. Let $x, x' \in X$ and let $\alpha_{\{x,x'\}} : \text{Int} \to \text{SubPart}(\{x, x'\})$ be the associated pre-cosheaf as in Defn. 4.14. Then, by definition, if we consider $\star = \{x, x'\}$, we obtain $I^{\text{supp}(\alpha_{\{x,x'\}})} = \alpha_{\{x,x'\}}$. 

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Let $\text{Int}$ be the set of all nonempty open intervals, viewed as a subset of $\mathbb{R}^2$ as in Rem. 4.2. Now, consider the metric space structure $(\text{Int}, || \cdot ||_\infty)$ on $\text{Int}$, yielded by $(\mathbb{R}^2, || \cdot ||_\infty)$. Let $d_H$ denote the associated Hausdorff distance on the set of all nonempty subsets of the metric space $(\text{Int}, || \cdot ||_\infty)$. We have the following result.

**Theorem 4.19.** Let $A, B$ be two upper sets in $\text{Int} \subset \mathbb{R}^2$ and let $I^A, I^B : \text{Int} \to \text{SubPart}(\{\star\})$ be their associated pre-cosheaves. Then,

$$d_I(I^A, I^B) = d_H(A, B).$$

**Proof.** Recall that

$$d_H(A, B) = \inf \{ \varepsilon \geq 0 | A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon \},$$

where

$$A^\varepsilon = \{ (a', b') \in \text{Int} | \max \{|a - a'|, |b - b'|\} \leq \varepsilon, \text{ for some } (a, b) \in A \},$$

$$B^\varepsilon = \{ (a', b') \in \text{Int} | \max \{|a - a'|, |b - b'|\} \leq \varepsilon, \text{ for some } (a, b) \in B \}.$$

We prove the two inequalities, $\geq$ and $\leq$, respectively.

- Assume that $I^A, I^B$ are $\varepsilon$-interleaved. Let $(a, b) \in A$. Then, $I^A((a, b)) = \{ \star \}$. Also, $I^A((a, b)) \leq I^B((a - \varepsilon, b + \varepsilon))$. Thus, $I^B((a - \varepsilon, b + \varepsilon)) = \{ \star \}$. This means $(a - \varepsilon, b + \varepsilon) \in B$ and hence $(a, b) \in B^\varepsilon$. Therefore, $A \subset B^\varepsilon$. Similarly we can show $B \subset A^\varepsilon$.

- Assume that $A \subset B^\varepsilon$ and $B \subset A^\varepsilon$. Let $(a, b) \in \text{Int}$. We claim that $I^A((a, b)) \leq I^B((a - \varepsilon, b + \varepsilon))$. If $I^A((a, b)) = \emptyset$, then this is trivially true. Assume that $I^A((a, b)) = \{ \star \}$ Then, equivalently we have $(a, b) \in A$. Since $A \subset B^\varepsilon$, there exists a $(a', b') \in B$ such that $\max \{|a - a'|, |b - b'|\} \leq \varepsilon$. Hence, $a - \varepsilon \leq a'$ and $b' \leq b + \varepsilon$. Equivalently, $(a', b') \leq (a - \varepsilon, b + \varepsilon)$. Because $(a', b') \in B$ and $B$ is an upper set in $\text{Int}$, then $(a - \varepsilon, b + \varepsilon) \in B$. This means $I^B((a - \varepsilon, b + \varepsilon)) = \{ \star \}$. Therefore, $I^A((a, b)) \leq I^B((a - \varepsilon, b + \varepsilon))$ as we wanted. Similarly, we can show that $I^B((a, b)) \leq I^A((a - \varepsilon, b + \varepsilon))$. Thus, the pre-cosheaves $I^A, I^B$ are $\varepsilon$-interleaved. \hfill \Box

By Rem. 4.18 and Thm. 4.19, we have:

**Corollary 4.20.** Let $x, x' \in X$ and let $\alpha, \alpha' : \text{Int} \to \text{SubPart}(\{x, x'\})$ be two pre-cosheaves. Let $\alpha_{\{x,x'\}}$ and $\alpha'_{\{x,x'\}}$ be their associated pre-cosheaves as in Defn. 4.14 Then,

$$d_I(\alpha_{\{x,x'\}}, \alpha'_{\{x,x'\}}) = d_H(\text{supp}(\alpha_{\{x,x'\}}), \text{supp}(\alpha'_{\{x,x'\}})).$$

**Remark 4.21.** Note that the quantity on the right side of Eqn. (5) is equal to the $\infty$-product metric of the sequences $(\alpha_{\{x,x'\}}(x,x'))_{(x,x') \in X \times X}$, $(\alpha'_{\{x,x'\}}(x,x'))_{(x,x') \in X \times X}$. Also, because of Cor. 4.20 and Rem. 4.18, we can think of the $X \times X$-indexed family $(\text{supp}(\alpha_{\{x,x'\}}(x,x')))_{(x,x') \in X \times X}$ of upper sets in $\text{Int}$ as an analogue of the barcode of a persistence module [24]. Hence, the $\infty$-product metric in the RHS of Eqn. (5) can naturally be thought of as the analogue of the bottleneck distance in the setting of pre-cosheaves valued in $\text{SubPart}(X)$. Thus, Thm. 4.15 forms an analogue to the interleaving-bottleneck isometry theorem [8, 20].

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4.3 Structural theorem for the formigram interleaving distance

Let $\theta$ be a formigram over $X$ and let $\hat{\theta}$ be its associated pre-cosheaf. Let $x, x' \in X$. By definition of $\hat{\theta}$ and Defn. 4.14, the pre-cosheaf $\hat{\theta} \{x, x'\} : \text{Int} \to \text{SubPart}(X)$ is given interval-wisely by the formula

$$
\hat{\theta} \{x, x'\}(I) := \hat{\theta}(I)_{\{x,x'\}} = \begin{cases} 
\{\{x, x'\}\}, & \text{if } x, x' \in B \in \bigvee_{s \in I} \theta(s) \\
\emptyset, & \text{otherwise}.
\end{cases}
$$

By combining Thm. 4.12, Thm. 4.15, and Cor. 4.20, we obtain:

**Theorem 4.22** (Structural theorem for $d_F$). If $\theta, \theta'$ are formigrams over $X$, then

$$
d_F(\theta, \theta') = \max_{x, x' \in X} d_H\left(\text{supp}(\hat{\theta} \{x, x'\}), \text{supp}(\hat{\theta}' \{x, x'\})\right).
$$

**Remark 4.23.** The RHS in formula (6) is reminiscent of the distance between persistent structures of [6, Formula (1), page 7].

**Remark 4.24.** Thm. 4.22 implies that computing $d_F$ between two formigrams $\theta, \theta'$ over $X$ with $|X| = n$ can be done via computing $O(n^2)$ Hausdorff distances between subsets of $(\mathbb{R}^2, \| - \|_{\infty})$. Assuming that $|\text{crit} (\theta)| =: m$, the set $\text{supp}(\hat{\theta} \{x, x'\})$ for each $x, x' \in X$ has the descriptive complexity of $O(m)$. Also, at worst $O(m^3)$ join computations between two subpartitions are required for obtaining $\hat{\theta}$ from $\theta$ (this follows from the subsequent observation: When $I \in \text{Int}$ contains $k$ critical points of $\theta$, the subpartition $\bigvee_{s \in I} \theta(s)$ can be obtained via computing $k$ number of join operations between two subpartitions). From $\hat{\theta}$, we can directly read off $\text{supp}(\hat{\theta} \{x, x'\})$ for each $x, x' \in X$.

**Remark 4.25** ($d_F$ between dendrograms). Recall the notion of dendrograms and the ultrametrics induced by dendrograms (Rem. 3.13). In Defn. 4.22, assume that $\theta$ and $\theta'$ are dendrograms over $X$. Then, for each $x, x' \in X$, $\text{supp}(\hat{\theta} \{x, x'\})$ and $\text{supp}(\hat{\theta}' \{x, x'\})$ are described as in Fig. 4. Now observe that $d_H(\text{supp}(\hat{\theta} \{x, x'\}), \text{supp}(\hat{\theta}' \{x, x'\}))$ is equal to $|u_\theta(x, x') - u_{\theta'}(x, x')|$. Therefore, Eqn. (6) can be re-expressed as

$$
d_F(\theta, \theta') = \max_{x, x' \in X} |u_\theta(x, x') - u_{\theta'}(x, x')|.
$$
Example 4.26. Consider the formigrams $\theta, \theta'$ from Ex. 2.30. Let us compute the formigram interleaving distance $d_I(\theta, \theta')$ using Thm. 4.22. By definition of $\theta, \theta'$ we easily see that $\text{supp}(\hat{\theta}_{\{1\}}) = \text{Int}$, and so $d_I(\text{supp}(\hat{\theta}_{\{1\}}), \text{supp}(\hat{\theta}'_{\{1\}})) = 0$. The same is true, if we replace 1 with 2. Thus, we have $d_F(\theta, \theta') = d_I(\text{supp}(\hat{\theta}_{\{1, 2\}}), \text{supp}(\hat{\theta}'_{\{1, 2\}}))$. Then, it is straightforward from Fig. 5 that the Hausdorff distance between these supports is $\delta$. Therefore: $d_F(\theta, \theta') = \delta$.

Example 4.27. Let $X = \{1, 2, 3\}$. Consider the formigram $\theta$ on $X$, as in the left side of Fig. 6. Let $\theta' = S_{\delta/2}(\theta)$ be the formigram yielded by $\theta$ via the $\delta/2$-smoothing, appeared on the right side of Fig. 6. The supports of the pre-cosheaves $\hat{\theta}_{\{i,j\}}, \hat{\theta}'_{\{i,j\}}, i, j \in \{1, 2, 3\}$, and their interleaving distances are computed in Fig. 7. Eventually, by Thm. 4.22 we obtain

$$d_I(\theta, \theta') = \max_{1 \leq i, j \leq 3} d_I(\hat{\theta}_{\{i,j\}}, \hat{\theta}'_{\{i,j\}}) = \delta/2.$$

Comparison of formigrams over different underlying sets. Let $X, Z$ be two sets, let $P \in \text{SubPart}(X)$, and let $\varphi : Z \to X$ be a surjective map. The pullback of $P$ via $\varphi$ is the subpartition of $Z$ defined as $\varphi^*P := \{\varphi^{-1}(B) \subset Z : B \in P\}$. Let $\theta_X$ be a formigram over $X$. The pullback of $\theta_X$ via $\varphi$ is the formigram $\varphi^*\theta_X : \mathbb{R} \to \text{SubPart}(Z)$ defined as

$$(\varphi^*\theta_X)(t) := \varphi^*(\theta_X(t)).$$

Let $X$ and $Y$ be any two nonempty sets. A tripod $R$ between $X$ and $Y$ is a pair of surjections $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ from another set $Z$ onto $X$ and $Y$. For $x \in X$ and $y \in Y$, we write $(x, y) \in R$ when there exists $z \in Z$ such that $x = \varphi_X(z)$ and $y = \varphi_Y(z)$.

Definition 4.28 (Interleaving distance between formigrams over (possibly) different underlying sets [18, 19]). Let $\theta_X, \theta_Y$ be any two formigrams over $X$ and $Y$, respectively. We define
Figure 6: 1, 2, 3 are colored blue, red and green respectively. On the left is the formigram $\theta$, and on the right is the formigram $\theta' = S_{\delta/2}(\theta)$.

$$d_I^F(\theta_X, \theta_Y) := \min_R d_F(\varphi_X^* \theta_X, \varphi_Y^* \theta_Y),$$

where the minimum is taken over all tripods between $X$ and $Y$ (because $X$ and $Y$ are finite, the minimum is always achieved by a certain tripod $R$).

**Remark 4.29** (About Defn. 4.28). Defn. 4.28 might look different from [18, Defn. 6.26] at first glance. However, given a tripod $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ and any $\varepsilon \in [0, \infty)$, the condition $\varphi_X^* \theta_X \leq S_\varepsilon (\varphi_Y^* \theta_Y)$ (Defn. 2.22 and Eqn. 1)) is equivalent to the condition $\theta_X \xrightarrow{R} S_\varepsilon \theta_Y$ in [18], and thus Defn. 4.28 coincides with [18 Defn. 6.26].

Given two formigrams $\theta_X$, $\theta_Y$ over $X$ and $Y$ respectively, both have “bags”

\[
\{ \text{supp } (\hat{\theta}_{X(x,x')}^\theta) : x, x' \in X \}\text{ and } \{ \text{supp } (\hat{\theta}_{Y(y,y')}^\theta) : y, y' \in Y \}\text{ of their features, which are multisets of subsets of } \text{Int}. \text{ We can compute } d_I^F(\theta_X, \theta_Y) \text{ by utilizing these bags and the Hausdorff distance between subsets of } \text{Int}: 
\]

**Theorem 4.30** (Structural theorem for $d_I^F$). Let $\theta_X, \theta_Y$ be any two formigrams over $X$ and $Y$, respectively. We have:

$$d_I^F(\theta_X, \theta_Y) = \min_R \max_{(x,y) \in R} d_H(\text{supp } (\hat{\theta}_{X(x,x')}^\theta), \text{supp } (\hat{\theta}_{Y(y,y')}^\theta)), $$

where the minimum is taken over all tripods $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ between $X$ and $Y$.

**Remark 4.31** (Two observations). (a) In Thm. 4.30 assume that $\theta_X$ and $\theta_Y$ are dendrograms over $X$ and $Y$, respectively (Rem. 3.13). Then, by Rem. 4.25, we have that

$$d_I^F(\theta_X, \theta_Y) = \min_R \max_{(x,y) \in R} |u_{\theta_X}(x, x') - u_{\theta_Y}(y, y')| = 2 \cdot d_{GH}((X, u_{\theta_X}), (Y, u_{\theta_Y})), $$

Therefore, Thm. 4.30 can be viewed as a generalization of [19, Thm. 2].
(a) $d_1(\text{supp}(\hat{\theta}_{\{1,1\}}), \text{supp}(\hat{\theta}'_{\{1,1\}})) = 0$.

(b) $d_H(\text{supp}(\hat{\theta}_{\{2,2\}}), \text{supp}(\hat{\theta}'_{\{2,2\}})) = 0$.

(c) $d_H(\text{supp}(\hat{\theta}_{\{3,3\}}), \text{supp}(\hat{\theta}'_{\{3,3\}})) = 0$.

(d) $d_H(\text{supp}(\hat{\theta}_{\{1,2\}}), \text{supp}(\hat{\theta}'_{\{1,2\}})) = \delta/2$.

(e) $d_H(\text{supp}(\hat{\theta}_{\{2,3\}}), \text{supp}(\hat{\theta}'_{\{2,3\}})) = \delta/2$.

(f) $d_H(\text{supp}(\hat{\theta}_{\{1,3\}}), \text{supp}(\hat{\theta}'_{\{1,3\}})) = \delta/2$.

Figure 7: Each gray region is the support of a pre-cosheaf. The diagonal $y = x$ is colored black.
(b) In Thm. [4.30] assume that $Y$ is the one point space together with the trivial formigm: $Y = \{\ast\}$ and $\theta_Y(t) = \{\{\ast\}\}$ for all $t \in \mathbb{R}$. Then, we have

$$d_1^F(\theta_X, \theta_Y) = \max_{x,x' \in X} d_{H} \left( \text{supp } \left( \hat{\theta}_X \{x,x'\} \right), \text{Int} \right)$$

If there exists $x_0 \in X$ such that the lifespan of $x_0$ is not $\mathbb{R}$, then $d_{H} \left( \text{supp } \left( \hat{\theta}_X \{x_0,0\} \right), \text{Int} \right) = \infty$, and thus $d_1^F(\theta_X, \theta_Y) = \infty$. Assume that every $x$ has the full lifespan, i.e. $\Phi^\theta_x = \mathbb{R}$. Then, for each pair $x, x'$ ($x \neq x'$) in $X$, $d_H \left( \text{supp } \left( \hat{\theta}_X \{x,x'\} \right), \text{Int} \right)$ is equal to the half of the maximal length of an interval $I \subset \text{Int}$ where $x$ and $x'$ do not belong to the same block in $\theta_X(t)$ for every $t \in I$.

**Proof of Theorem 4.30** Fix a tripod $R : X \leftrightarrow Z \rightarrow Y$. By Thm. [4.22] we have:

$$d_F(\varphi_X^*, \theta_X, \varphi_Y^*, \theta_Y) = \max_{z,z' \in Z} d_H \left( \text{supp } \left( \varphi_X^* \theta_X \{z,z'\} \right), \text{supp } \left( \varphi_Y^* \theta_Y \{z,z'\} \right) \right).$$

(7)

Pick $z_0, z_0' \in Z$, and let $\varphi_X(z_0) =: x_0$ and $\varphi_X(z_0') =: x_0'$. We will show that

$$\text{supp } \left( \varphi_X^* \theta_X \{z_0, z_0'\} \right) = \text{supp } \left( \hat{\theta}_X \{z_0, z_0'\} \right).$$

We show the inclusion $\subseteq$. Assume that an interval $I \subset \mathbb{R}$ belongs to $\text{supp } \left( \varphi_X^* \theta_X \{z_0, z_0'\} \right)$. This is equivalent to that $z_0, z_0'$ belong to the same block in the subpartition $\varphi X \theta X(I)$ of $Z$. By the definition of $\varphi X \theta X(I)$ (Defn. 4.10), this is equivalent to that there exist a sequence $z_0 = z_1, \ldots, z_\ell = z_0'$ in $Z$ and another sequence $t_1, \ldots, t_\ell$ in the interval $I$ such that $z_i, z_{i+1}$ belong to the same block of $\varphi X \theta X(t_i)$ for $i = 1, \ldots, \ell - 1$. This implies that $\varphi X(z_i), \varphi X(z_{i+1})$ belong to the same block in the subpartition $\theta X(t_i)$ of $X$ for each $i = 1, \ldots, \ell - 1$. Therefore, $\varphi X(z_1) = x_0$ and $\varphi X(z_\ell) = x_0'$ belong to the same block in $\hat{\theta}_X(I) = \bigvee_{s \in I} \theta X(s)$, and in turn $\hat{\theta}_X \{z_0, z_0'\}(I) = \{x_0, x_0'\}$. Hence, $I$ belongs to $\text{supp } \left( \hat{\theta}_X \{z_0, z_0'\} \right)$. By invoking that $\varphi X$ is surjective, the containment $\supseteq$ can be similarly proved. Also, by the same argument, we have

$$\text{supp } \left( \varphi_Y^* \theta_Y \{z_0, z_0'\} \right) = \text{supp } \left( \hat{\theta}_Y \{y_0, y_0'\} \right),$$

where $\varphi Y(z_0) =: y_0$ and $\varphi Y(z_0') =: y_0'$. Therefore, the RHS of Eqn. (7) is equal to

$$\max_{z,z' \in Z} d_H \left( \text{supp } \left( \hat{\theta}_X \{\varphi_X(z), \varphi_X(z')\} \right), \text{supp } \left( \hat{\theta}_Y \{\varphi_Y(z), \varphi_Y(z')\} \right) \right).$$

Observe that the set

$$\left\{ d_H \left( \text{supp } \left( \hat{\theta}_X \{x,x'\} \right), \text{supp } \left( \hat{\theta}_Y \{y,y'\} \right) \right) \in \mathbb{R} \mid (x, y), (x', y') \in R \right\}$$

is identical to

$$\left\{ d_H \left( \text{supp } \left( \hat{\theta}_X \{\varphi_X(z), \varphi_X(z')\} \right), \text{supp } \left( \hat{\theta}_Y \{\varphi_Y(z), \varphi_Y(z')\} \right) \right) \in \mathbb{R} \mid z, z' \in Z \right\}.$$ 

This implies that the RHS of Eqn. (7) is equal to

$$\max_{(x,y) \in R} d_H \left( \text{supp } \left( \hat{\theta}_X \{x,x'\} \right), \text{supp } \left( \hat{\theta}_Y \{y,y'\} \right) \right),$$

completing the proof. □
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