Adjoint optimal control problems for the RANS system

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Abstract. Adjoint optimal control in computational fluid dynamics has become increasingly popular recently because of its use in several engineering and research studies. However the optimal control of turbulent flows without the use of Direct Numerical Simulation is still an open problem and various methods have been proposed based on different approaches. In this work we study optimal control problems for a turbulent flow modeled with a Reynolds-Averaged Navier-Stokes system. The adjoint system is obtained through the use of a Lagrangian multiplier method by setting as objective of the control a velocity matching profile or an increase or decrease in the turbulent kinetic energy. The optimality system is solved with an in-house finite element code and numerical results are reported in order to show the validity of this approach.

1. Introduction

The optimization of industrial devices in which the flow is turbulent is one of the main problems in current engineering research and development issues. Turbulent flows are encountered commonly in engineering studies and when performing Computational Fluid Dynamics simulations there are several ways to deal with this problem. To perform studies at a very deep and precise level Direct Numerical Simulations are needed in order to resolve all the time and space scales of the turbulent flow. However these methods are computationally prohibitive for industrial applications and therefore several models have been developed to overcome this problem, with the main employed in the last years being Reynolds Averaged Navier Stokes models. Some interesting studies are currently being performed to use Large Eddy Simulations or hybrid LES-RANS models but common practice is still to use RANS models in industrial applications [1, 2].

Continuous adjoint optimal control is a very interesting and promising optimization technique that can be used in several research fields [3, 4]. In this method one defines an objective functional, the control and the constraints of the problem, which are the state equations. Then through a Lagrangian multiplier technique the adjoint system is derived and it can be solved to obtain a local optimal solution starting from a reference state. Many objectives can be set when studying optimal control problems such as velocity matching profile, lift enhancement or drag reduction, decrease of the viscous dissipation and many others. We should remark that continuous adjoint methods are not widely used in practice while discrete or automatic differentiation tools are preferred in industrial design, since they are available in many commercial and open-source CFD packages. This is also due to the fact that these optimization
methods are employed usually to improve the shape of devices, such as airplane wings or turbine blades. However, the continuous adjoint method that we study in this work could improve the design of industrial devices when the optimization is performed on other parameters than shape, such as the control of a source term in the domain or the value of a boundary condition as an inflow velocity and direction from a specified boundary [4, 5]. Another big advantage of adjoint methods over automatic differentiation tools that are very commonly used is that in order to obtain the sensitivities with respect to all the parameters one needs to solve only one adjoint system at the computational cost of a CFD solution [6, 7]. Moreover in this work we take into account the turbulence modeling without considering a frozen turbulent viscosity as it is usually done in optimization methods, see [6, 7, 8, 9] for a discussion on these subjects.

To summarize, in this work we study the optimal control problem with distributed and boundary control for a turbulent flow modeled with a two-equation RANS model [10]. Interesting features of this approach include the presence of turbulence modeling included explicitly in the derivation of the adjoint equations, the possibility of setting as objective of the control both a function of the velocity or of the turbulence intensity, which can be useful when a laminarization of the flow is sought, and the extension to boundary control with respect to the work already done in [8, 9]. The optimality system is derived for both the distributed and boundary control and numerical results obtained with the distributed control are reported. These results show the effectiveness of our approach in the optimization of turbulent flows.

2. Optimality system

For an incompressible fluid the conservation of mass and momentum is defined by the Navier-Stokes system. The RANS equations are very similar to the Navier-Stokes ones since they differ in one unknown term, the Reynolds Stress tensor, that needs to be modeled. These equations together with Wilcox $k$-$\omega$ turbulence model are [10]

\[
\nabla \cdot \mathbf{v} = 0, \\
\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \cdot (\mathbf{T} + \rho \mathbf{F}_t) + \mathbf{f}, \\
(\mathbf{v} \cdot \nabla)k = \nabla \cdot [(\nu + \sigma_k \nu_t) \nabla k] + P_k - \beta^* \omega k, \\
(\mathbf{v} \cdot \nabla)\omega = \nabla \cdot [(\nu + \sigma_\omega \nu_t) \nabla \omega] + \alpha \frac{\omega}{k} P_k - \beta \omega^2 + \frac{\sigma_d}{\omega} \nabla k : \nabla \omega,
\]

where $\rho$ is the fluid density, $\mathbf{f}$ a force acting on the flow and the viscous stress tensor $\mathbf{T}$ can be defined for a Newtonian fluid as

\[
T_{ij} := -\rho' \delta_{ij} + \mu S_{ij}, \quad S_{ij} := \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i},
\]

where $\rho'$ is the fluid pressure, $\mu$ the dynamic viscosity and $\mathbf{S}$ the deformation tensor. The production term of the turbulence kinetic energy in (3-4) is

\[
P_k = \frac{\nu_t}{2} \| \mathbf{S} \|^2.
\]

The unknown Reynolds stress tensor $\mathbf{T}_t$ can be computed by introducing the eddy viscosity $\nu_t$,

\[
T_{tki} = -\frac{2}{3} k \delta_{ki} + \nu_t S_{ki}.
\]

The eddy viscosity $\nu_t$ is defined in the turbulence model in the following form [10]

\[
\nu_t = \min \left[ \frac{k}{\omega}, \frac{k}{\omega \text{Clim}} \sqrt{\frac{\mathbf{S} : \mathbf{S}}{2\nu_t^2}} \right].
\]
Table 1. Coefficients employed in the turbulence model.

| \(\sigma_k\) | \(\sigma_\omega\) | \(\beta^*\) | \(\alpha\) | \(C_{lim}\) | \(\beta_0\) | \(\chi\) |
|---|---|---|---|---|---|---|
| 0.6 | 0.5 | 0.09 | 13/25 | 7/8 | 0.0708 | \(\frac{1 + 85 \chi}{1 + 100 \chi}\) |

We also report the other coefficients needed by the model in Table 1. In this Table \(\Omega_{ij}\) is the velocity rotation tensor. It should be noted that in a two-dimensional problem the function \(\chi\) vanishes. As we study only two-dimensional problems in this work we can neglect this term in the following. For details see [10] and citations therein.

The state equations presented above are considered the constraints in the optimal control setting. To complete the definition of the problem we define the objective functional on the computational domain \(\Omega\) and controlled surface \(\Gamma_c\)

\[
\mathcal{J}(v, k, f, g) = \frac{1}{2} \int_\Omega a (v - v_d)^2 d\Omega + \frac{1}{2} \int_\Omega b |k - k_d|^2 d\Omega + \frac{\beta}{2} \int_\Omega |f|^2 d\Omega + \frac{\beta_1}{2} \int_{\Gamma_c} \|g\|^2 d\Gamma + \frac{\lambda}{2} \int_{\Gamma_c} \|\nabla g\|^2 d\Gamma. \tag{9}
\]

In (9) \(f\) is the force that acts on the fluid as reported in (2) while \(g\) is a Dirichlet condition for the velocity, \(v = g\) on \(\Gamma_c\). The weight functions \(a\) and \(b\) can be used to better specify the region on which we want to obtain the velocity profile or turbulence kinetic energy [5]. Moreover they can be used to study a velocity matching problem by setting \(b = 0\) or a turbulence kinetic energy matching by setting \(a = 0\). We note that the distributed or boundary control can be studied by considering \(\beta_1 = 0\), \(\lambda = 0\) or \(\beta = 0\), respectively. The adjoint equations obtained are the same for the two types of control but the problem changes completely. The last terms of (9) are regularization terms needed to obtain a non-singular control \(f \in L^2(\Omega)\) and \(g \in H^1(\Gamma)\) [4]. The numerical values of \(\beta\), \(\beta_1\) and \(\lambda\) are important for the solution of the optimal control problem since high values can result in poor optimization results while low values may lead to convergence issues.

The total Lagrangian of this problem, neglecting for simplicity the limiting values on \(\nu_t\), can be written as

\[
\mathcal{L} = \mathcal{J}(v, k, f, g) + \int_\Omega p_n \nabla \cdot v d\Omega + \int_\Omega v_a \cdot \left[ (v \cdot \nabla) v - \nabla \cdot \left( \frac{T}{\rho} + T_k \right) - f \right] d\Omega + \int_{\Gamma_c} \tau \cdot (v - g) d\Gamma + \int_\Omega k_a [(v \cdot \nabla) k - \nabla \cdot [(\nu + \sigma_k \nu_t) \nabla k] - P_k + \beta^* k \omega \cdot d\Omega + \int_\Omega \omega_a \left[ (v \cdot \nabla) \omega - \nabla \cdot [(\nu + \sigma_\omega \nu_t) \nabla \omega] - \alpha \omega \cdot P_k + \beta \omega^2 - \frac{\sigma_\omega}{\omega} \nabla k \cdot \nabla \omega \right] d\Omega. \tag{10}
\]

The total Lagrangian (10) is composed of the objective functional (9) and the RANS equations with the turbulence model (1-4), each of them multiplied by a proper Lagrange multiplier. The term with \(\tau\) is present only if we choose a boundary control so that the velocity can vary on the controlled boundary \(\Gamma_c\) during the optimization process. The first order necessary
conditions needed to obtain the optimal solution can be set by considering the total variation of the Lagrangian null, $\delta C = 0$. By taking the Fréchet derivatives of (10) with respect to all the variables involved and setting them to zero one obtains the optimality system [4, 8, 9]. When the derivatives are taken with respect to the Lagrangian multipliers we obtain back the state system (1-4) in weak form, while when we varies with $\delta$ we get

$$
\int_{\Omega} \nabla \cdot v_a \delta p \, d\Omega = 0 \quad \forall \delta p \in L^2(\Omega),
$$

(11)

$$
\int_{\Omega} \left[ - \nabla p_a \cdot \delta v + v_a \cdot (\delta v \cdot \nabla) v - [(v \cdot \nabla)v_a] \cdot \delta v - \nabla \cdot [(\nu + \nu_t)S(v_a)] \cdot \delta v + k_a (\delta v \cdot \nabla)k + \omega \frac{\delta v}{\omega} \nabla \omega + a (v - v_d) \cdot \delta v + 2 \nabla \cdot [(k_a \nu_t + \alpha \omega_a)S(v)] \cdot \delta v \right] \, d\Omega = 0 \quad \forall \delta v \in H^1(\Omega),
$$

(12)

and with $\delta k$

$$
\int_{\Omega} \left[ - (v \cdot \nabla)k_a - \nabla \cdot [(\nu + \sigma k \nu_t)\nabla k_a] + \beta^* k_a \omega + \nabla \cdot \left[ \sigma_d \frac{\omega_a}{\omega} \nabla \omega \right] + \frac{\nu_t}{\omega} + b (k - k_d) \right] \, \delta k \, d\Omega = 0 \quad \forall \delta k \in H^1(\Omega),
$$

(13)

and finally for $\delta \omega$

$$
\int_{\Omega} \left[ - (v \cdot \nabla)\omega_a - \nabla \cdot [(\nu + \sigma \omega_t)\nabla \omega_a] + \beta^* k_a \delta \omega + 2 \beta \omega_a \omega + \sigma_d \nabla \cdot \left[ \frac{\omega_a}{\omega} \nabla k \right] + \frac{\sigma_d \omega_a}{\omega^2} \nabla k \cdot \nabla \omega - \frac{\nu_t k}{\omega^2} \right] \, \delta \omega \, d\Omega = 0 \quad \forall \delta \omega \in H^1(\Omega).
$$

(14)

These are the adjoint equations of the state system (1-4) written in weak form. The boundary conditions are already incorporated in this formulation. The interested reader can find a complete derivation and discussion on the boundary conditions to be set for this problem in [8, 9]. By considering the variations with respect to the turbulent viscosity $\nu_t$ one obtains an algebraic equation in $\nu_t$ that is solved by setting

$$
\nu_t = k_a S : S - \nabla v_a : S(v) - \sigma k \nabla k_a \cdot \nabla k - \sigma \omega \omega_a \cdot \nabla \omega.
$$

(15)

Therefore this expression can be substituted in the equations (13-14).

The last terms that need to be considered are the variations with respect to the controls $f$ and $g$. If we are considering a distributed control the equation in the variation $\delta f$ is

$$
\int_{\Omega} (\beta f - v_a) \cdot \delta f \, d\Omega = 0 \quad \forall \delta f \in L^2(\Omega),
$$

(16)

so that the force $f$ on the fluid is equal to the adjoint velocity $v_a$ scaled by the parameter $\beta$. On the other hand, if a boundary control is chosen the variation in $\delta g$ gives rise to a differential equation on the controlled boundary $\Gamma_c$

$$
\int_{\Gamma_c} [\lambda \nabla g : \nabla \delta g + \beta_1 g \cdot \Delta g - [(\nu + \nu_t)S(v_a) \cdot n] \cdot \delta g] \, d\Gamma = 0 \quad \forall \delta g \in H^1(\Gamma),
$$

(17)

that needs to be solved in order to obtain the controlled velocity on the boundary $\Gamma_c$ as $v = g$. 

Some remarks can be done on this optimality system. First we notice that the source terms in the adjoint equations are the terms that we have defined in the objective functional as the differences \((v - v_d)\) and \((k - k_d)\). So the driving forces of the adjoint variables are big if the objective is far from being achieved while are null if the objective is reached, as it can be expected. Moreover the control, being it distributed or boundary, is a function of the adjoint velocity or of its derivatives. In the following we study two problems with distributed control, one to obtain a matching velocity and the other to control the value of the turbulent kinetic energy in some specific region of the domain. This can be achieved by setting properly the weight functions \(a\) and \(b\).

3. Numerical Results

In this Section we report the numerical results obtained by implementing the optimality system as described in the previous Section in an in-house finite element code. This code is a parallel mpi-based code which implements a multigrid solver and which is being developed at the laboratory of Montecuccolino of the University of Bologna [11]. It is based on several open-source libraries for parallel computing and linear algebra, namely the openmpi, PETSc and MED libraries [12, 13, 14]. We use standard quadratic-linear finite elements for the velocity-pressure solution for both the state and adjoint systems in order to fulfill the inf-sup condition. The turbulent variables \(k\)-\(\omega\) and their adjoints are approximated with quadratic elements. We use a steepest descent gradient based algorithm to solve the optimality system in a segregated way because a one-shot solver is not affordable for such a complex system [3, 8, 9].

We have studied a backward-facing step problem with Reynolds number \(Re = 10000\) based on the inlet diameter. In the results we report the same color scale and iso-lines are used for all the variables among different cases in order to better compare the effect of the optimization with respect to the reference case. In Figure 1 the geometry of the test case is reported together with the flow field in the reference case. With reference to Figure 3 the segment AB is the inlet, EF the outlet and the other boundaries are solid walls. The total length of the channel is 6 and the inlet and the step CD are 1 in non-dimensional length. The length of the entrance channel before the step BC is 2. From the flow field as depicted in Figure 1 one can see that the vortex that develops after the step has a reattachment point after around 2.5 in non-dimensional length. In Figure 2 the reference results for the turbulent kinetic energy are reported with iso-lines. One can see that the main source of turbulence is the step. The high values of turbulent kinetic energy produced here diffuse along the channel before exiting from the outlet.

We now report the results obtained by studying the distributed optimal control of this test case with three different objectives. The first is to control the velocity in a specific region of the domain near the outlet. In the second and the third we aim at increasing or decreasing

![Figure 1. Backward facing step geometry and velocity streamlines, reference case.](image-url)
the turbulence intensity in the region after the step where the vortex develops. We study these problems with $\beta = 0.01$.

In Figure 3 the weight function $a$ is reported on the whole domain with the values $a = 0$ and $a = 1$ for the velocity matching case. The desired velocity is set to $\mathbf{v}_d = (8, 0)$ in the region near the outlet. In Figure 4 the adjoint velocity is reported with streamlines and magnitude. By considering that this is the force that is applied to the fluid in order to control the fluid velocity, one can see that the control is pushing downward the flow in order to obtain a constant

![Figure 2](image2.png)

**Figure 2.** Iso-lines of the turbulent kinetic energy in the reference case.

![Figure 3](image3.png)

**Figure 3.** Velocity matching case, representation of the $a$ weight function with values $a = 0$ (blue) and $a = 1$ (red). AB is the inlet, EF the outlet, the other boundaries are solid walls.

![Figure 4](image4.png)

**Figure 4.** Velocity matching case, adjoint velocity streamlines and magnitude.
horizontal velocity in this region. In Figure 5 the controlled velocity is reported with streamlines and magnitude after the control is applied to the fluid. This result should be compared with the reference velocity reported in Figure 1. As one can see in the optimal solution the flow is horizontal and with a constant velocity of $(8, 0)$ in the region near the outlet where the objective functional was set.

We report in Figure 6 the weight function $b$ as set for the two turbulence control problems. The region with $b = 1$ is set after the step. In the second test case of turbulence increase we set $k_d = 20$. In this case we show the adjoint turbulent kinetic energy which is the main adjoint variable in this setting. In Figure 7 this variable is reported for the optimized case with iso-lines. One can see that $k_a$ is negative where the turbulence is too high while is positive near the wall where the turbulence cannot be increased too much. In Figure 8 the turbulent kinetic energy is reported in the optimized case with iso-lines. One can compare this result with the reference case in Figure 2 in order to better understand the optimized result obtained. This very strong increase in the turbulent kinetic energy is achieved with the formation of a vortex by the force $f$ applied to the fluid, as it is shown in Figure 9 where the streamlines and the magnitude of the fluid velocity in this increase turbulence test case are reported.

We now report the results obtained for the decrease of turbulence test case that has been studied by setting $k_d = 0$. The weight function $b$ is the same as reported for the previous test case in Figure 6. The contours and the magnitude of the turbulent kinetic energy are reported in Figure 10 for this test case. One can see that the turbulence intensity decreases in the region where $b = 1$ and that the plume of turbulence produced by the step is not diffusing in this test case.
region but it is advected further on the channel. This result is achieved by changing the velocity pattern as reported in Figure 11. The formation of the vortex after the step is now shifted in a further region with respect to the main flow, as the reattachment length becomes different.

Finally in Table 2 we report the values of the objective functionals as computed from the definition (9) for the three test cases considered. The reference case is always reported for a proper comparison of the obtained values. As one can see, by applying the optimal control a strong reduction of the objective functional is achieved.
Figure 10. Turbulence decrease case, turbulent kinetic energy iso-lines and magnitude.

Figure 11. Turbulence decrease case, velocity streamlines and magnitude.

Table 2. Objective functional values computed for the three test cases of velocity matching, turbulence increase and decrease.

| Test case            | Velocity matching | Turbulence increase | Turbulence decrease |
|----------------------|-------------------|---------------------|---------------------|
| Reference            | 15.64             | 50.18               | 191.63              |
| Optimal              | 2.42              | 21.06               | 35.16               |

4. Conclusion
In this work we studied the adjoint optimal problem for the RANS system closed with a two-equation turbulence model. The optimality system was derived for a distributed and a boundary control and two objectives were considered, namely a matching velocity and a matching turbulent kinetic energy. We studied the backward facing step problem with distributed control and three different objective definitions. The first is to control the value of the velocity field in a specific region and the other two to increase or decrease the turbulence intensity in the vortex region after the step. The results obtained have shown that the method is reliable and that the objective functional can be decreased even of one order of magnitude in this test case. For these reasons this approach could be of great interest for the study and optimization of industrial devices in which the flow is turbulent and CFD-RANS simulations can be performed on the system. More work needs to be done in order to test this method in practical applications where the flow pattern is more complex but these results on the backward facing step problem are very promising.
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