KÄHLER IDENTITIES FOR ALMOST COMPLEX MANIFOLDS

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ABSTRACT. We obtain a generalization, for a general compact almost complex manifold, of the well-known Kähler (or Hodge) identities for Kähler manifolds involving the commutators of the exterior differential and the Lefschetz operator and its adjoint. The main idea is to study the problem on the Clifford bundle via the Dirac operator, and then translate the results to the exterior bundle.

1. Introduction

On a Kähler manifold, the Kähler (or Hodge) identities give relationships between the commutator of the exterior differential $d$ and the Lefschetz operator on forms given by $L(\alpha) = \omega \wedge \alpha$, where $\omega$ is the fundamental form. Namely, we have

$$[d^*, L] = -d^c$$

$$[d, L] = 0,$$

where the bracket denotes the commutator, $d^* := -\ast d \ast$ is the formal adjoint of $d$, and $d^c := J^{-1} \circ d \circ J = i(\bar{\partial} - \partial)$, with $J$ being the extension of the complex structure $J$ as an algebra operator on the exterior algebra. These identities seem to have appeared first in [7, 8], and in modern notation in [13]. They are a fundamental tool to prove several important results for Kähler manifolds, such as the Lefschetz decomposition of complex DeRham cohomology or the fact that, on a Kähler manifold, the notions of $d$-harmonic, $\partial$-harmonic, and $\bar{\partial}$-harmonic forms coincide.

There are generalizations of the Kähler-Hodge identities: In [3], Demailly defines new operators

$$\lambda(\alpha) = d\omega \wedge \alpha$$

and shows that, on a complex manifold (i.e. integrable), the Kähler identities extend to

(1) $$[d^*, L] = -d^c - \tau^c$$

and $$[d, L] = \lambda.$$

For almost Kähler manifolds (i.e. non-integrable almost complex manifolds with a closed fundamental form), similar identities are proved in [2] and [1]. The proofs and generalizations we know of these identities rely on a local analytic study of the problem.

In this paper we prove the identities (1) for general almost complex manifolds (i.e. just a manifold with an almost complex structure $J$). Our approach does not rely on a local analytic study of the problem. Instead, we formulate the problem in the Clifford bundle of the manifold instead of the exterior bundle, following the ideas used in [12] for Kähler manifolds, and then we translate the results to the exterior algebra. This offers new possibilities of study due to the somewhat richer algebraic structure of the Clifford algebra, and we think that there still a lot to explore using this formulation.

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We make an effort to distinguish between the Clifford bundle and the exterior bundle, even though they are isomorphic via the metric. The aim is to emphasize the richer structure of the former and to separate the idea of a “Clifford cohomology” from the usual DeRham cohomology. While this separation requires additional notation, we believe that it makes the exposition more clear.

The paper is organized as follows: in Section 2 we make the necessary definitions, present the basic preliminary results, and fix the notation that will be used throughout the paper. In Section 3 we formulate the problem of finding an equivalent of the Kähler identities in the Clifford bundle, and we prove the Clifford version of these identities, which is surprisingly simple. We also prove other interesting commutation identities. Finally, Section 4 is devoted to translate these identities to the exterior algebra. This translation becomes rather technical and often requires computations using a local frame, but it is nevertheless a linear algebraic procedure.

2. Preliminaries and Notation

Let $M$ be a $2n$ dimensional manifold with an almost complex structure, that is, a smooth bundle map $J : TM \to TM$ satisfying $J^2 = -I$. We will use the same notation for the complex-linear extension $J : T_C M \to T_C M$, and will denote by $\check{J}$ the dual map $\check{J} : T^*_C M \to T^*_C M$. In general, the dual of an operator $P$ will be denoted by $\check{P}$, and its adjoint by $P^*$.

Let $g$ be a metric compatible with $J$, that is, $\langle X, Y \rangle = \langle JX, JY \rangle$. Then we can construct the Clifford bundle $\text{Cl}(M)$ over $TM$, and its complexification $\text{Cl}(M)$. With a slight abuse of notation, the space of smooth sections over these bundles will also be denoted by $\text{Cl}(M)$ and $\text{Cl}(M)$, respectively. Likewise, $\Lambda(M)$ (or $\Lambda_C(M)$) will denote both the exterior bundle (or complexified exterior bundle) and the space of smooth forms (or complex smooth forms).

The sesquilinear extension of the metric $g$ to the complexification will be also be denoted by $g$. As is well known, $\text{Cl}(M) (= \text{Cl}(TM))$ and $\Lambda(TM)$ are canonically isomorphic (see for example [11]), and by extension so are $\text{Cl}(M)$ and $\Lambda_C(TM)$. In turn, via the metric, $\Lambda_C(TM)$ is isomorphic to $\Lambda_C(M) \equiv \Lambda_C(T^*M)$ via the map

$$X \longrightarrow (Y \rightarrow \langle Y, \overline{X} \rangle),$$

where the bar denotes conjugation. The composition of the two isomorphisms will be denoted by

$$b : \text{Cl}(M) \to \Lambda_C(M), \quad \text{with inverse } \check{b} := b^{-1} : \Lambda_C(M) \to \text{Cl}(M).$$

We will often denote $b(X)$ by $X^\sharp$ and $\check{b}(\alpha)$ by $\alpha^\flat$. Note that, if $\{e_A\}_{A=1}^{2n}$ is an orthonormal basis with dual basis $\{\theta^A\}_{A=1}^{2n}$, then $b(e_A \cdot \ldots \cdot e_A) = \theta^{A_1} \wedge \ldots \wedge \theta^{A_k}$. We will often use the symbol “$\equiv$” to denote correspondence via these “musical” isomorphisms, so for example we could have written $e_A \cdot \ldots \cdot e_A \equiv \theta^{A_1} \wedge \ldots \wedge \theta^{A_k}$ instead.

The maps $J : TM \to TM$ and $\check{J} : T^*_M \to T^*_M$ can be extended to $\text{Cl}(M)$ and $\Lambda_C(M)$, respectively, in two different ways:

- As a derivation $J_d$: $J_d(X \cdot Y) = JX \cdot Y + X \cdot JY$ for $X, Y \in T_C M$ (and similarly for $\check{J}_d$).
- As an algebra map $J_a$: $J_a(X \cdot Y) = JX \cdot JY$ for $X, Y \in T_C M$ (and similarly for $\check{J}_a$).
Notice that $J_a$ and $J_d$ commute, and so do $\hat{J}_a$ and $\hat{J}_d$.

The algebra $A(M)$ is $\mathbb{Z}$-graded and $\text{Cl}(M)$ is $\mathbb{Z}_2$-graded. The eigenspaces of $\hat{J}_a$ in each degree induce a $\mathbb{Z} \times \mathbb{Z}$ bigrading of $A_C(M)$; the algebra $\text{Cl}(M)$ is also $\mathbb{Z} \times \mathbb{Z}$-bigraded (see [12]). Given a complex form $\psi$, its $(p,q)$ part in the bigrading $A_C(M) = \oplus A_C^{p,q}(M)$ will be denoted by $\psi^{p,q}$. Also, if $\psi \in A_C^k(M)$, we will use the same notation as in [5]:

$$\psi^+ := \psi^{1,2} + \psi^{2,1} \quad \text{and} \quad \psi^- := \psi^{0,3} + \psi^{3,0}.$$ 

An operator $P$ is said to have degree $p$, or bidegree $(p,q)$, if it takes elements of degree $r$ to elements of degree $r + p$, or elements of degree $(r,s)$ to elements of degree $(r + p, s + q)$, respectively.

Given operators $P$, $Q$ in $\text{Cl}(M)$ or $A_C(M)$ of total degree $p$ and $q$, respectively, $[P,Q]$ will denote the supercommutator of $P$ and $Q$, that is,

$$[P,Q] = PQ - (-1)^{pq}QP.$$

Notice that $[P,Q] = -(-1)^{pq}[Q,P]$, and if $P$ and $Q$ are derivations, then $[P,Q]$ is also a derivation.

The supercommutator satisfies the Jacobi identity

$$[P,[Q,R]] = [[P,Q],R] + (-1)^{pq}[Q,[P,R]].$$

As usual, given an operator $P$ in $\text{Cl}(M)$ or $A_C(M)$, the superindex $c$ will denote conjugation with $J_a$ (or $\hat{J}_a$), that is

$$P^c = J_a^{-1} \circ P \circ J_a \quad \text{(or} \quad P^c = \hat{J}_a^{-1} \circ P \circ \hat{J}_a).$$

Notice that if $P$ is an operator on $A_C(M)$ of bidegree $r,s$, then if $\phi \in A^{p,q}$,

$$P^c(\phi) = \hat{J}_a^{-1}(P(\hat{J}_a \phi)) = i^{p-q} \hat{J}_a^{-1}(P(\phi)) = i^{p-q} \phi^{q+p-r} P(\phi) = i^{s-r} P(\phi),$$

and therefore

$$P^c = i^{s-r} P \quad \text{if} \ P \ \text{has bidegree} \ (r,s)$$

All these objects satisfy the following elementary properties that will be used throughout.

**Lemma 1.** *(Elementary properties.)* Let $\psi \in A_C^k(M)$.

(a) $\circ \circ J \circ \circ = - \hat{J}$.

(b) $(J_a)^\circ = (\hat{J})_a$ (that is, extending as an algebra map commutes with duality).

(c) $(J_d)^\circ = (\hat{J})_d$ (that is, extending as a derivation commutes with duality).

(d) $\circ \circ J_a \circ \circ = \circ \circ \hat{J}_a \psi$.

(e) $(\hat{J}_a)^\circ \circ = (\circ \circ) \circ \circ \hat{J}_a \psi$ and $\hat{J}_d^\circ = - \hat{J}_d$. (and the same identities for $J_a$ and $J_d$).

(f) $(\hat{J}_a)^\circ \circ = (\circ \circ) \circ \circ \hat{J}_a \psi$ and $\hat{J}_d^\circ = - \hat{J}_d$. (and the same identity for $J_a$).

(h) If $P$ is an operator of degree $p$ on $\Gamma \text{Cl}(M)$ or $A_C(M)$, then $(P^c)^* = (P^*)^c$ and $(P^c)^c = (-1)^p P$.

(i) If $P : \Gamma \text{Cl}(M) \to \Gamma \text{Cl}(M)$ is an operator of degree $p$ then $\circ \circ P^c \circ \circ = (-1)^p (\circ \circ P \circ \circ)^c$.

(j) If $P : \Gamma A_C(M) \to \Gamma A_C(M)$ is an operator of bidegree $(p,q)$ then $P^c = i^{s-p} P$. 
Proof. They are all elementary consequences of the definitions. For example, to prove (a), let $X \in T_C(M)$ and $\beta \in T^*_C(M)$. Since $\flat$ and $\sharp$ are isometries, 
\[
\langle X, J\beta \rangle = \langle JX, -\beta \rangle = \langle (JX)\flat, -\beta \rangle = -\beta(JX) = \langle X, (-\check{J}\beta)\flat \rangle,
\]
so $J\beta = (\check{J}\beta)\flat$, or $\nabla_X J \circ \sharp(\beta) = -\check{J}\beta$.
\qed

The fundamental form $\omega \in A^2(M)$ is defined by $\omega(X,Y) := \langle JX,Y \rangle$. In terms of a local adapted orthonormal coframe, i.e. an orthonormal coframe $\{\theta^A\}_{A=1}^{2n}$ such that $\check{J}\theta^i = -\theta^i + i, 1 \leq i \leq n$, $\omega$ can be written as 
\[
\omega = \sum_{i=1}^{n} \theta^i \wedge \theta^{i+n}
\]
(in general, uppercase sub- and superindices will run from 1 to $2n$, whereas lowercase sub- and superindices will run from 1 to $n$).

The Lee form $\theta$ is defined by (see for example [5]).
\[
\theta := \omega \downarrow d\omega = -\check{J}d^* \omega^+
\]
For forms $\alpha, \beta$, we write $\alpha \downarrow \beta$ to denote the adjoint of $\beta \rightarrow \alpha \wedge \beta$, or equivalently, contraction of $\beta$ with $\alpha^\sharp$.

The same symbol $\omega$ will be used to denote $\omega^\sharp \in \text{Cl}(M)$, which can be expressed, in terms of a frame, by 
\[
\omega = \sum_{i=1}^{n} e_i \cdot e_{i+n}.
\]

Following [12], we define the following operators in $\text{Cl}(M)$:

**Definition 1.** Let $\nabla$ denote the Levi-Civita connection on $TM$ or its extension to $\text{Cl}(M)$ or $A_C(M)$.

- **The Dirac operator** $D : \text{Cl}(M) \rightarrow \text{Cl}(M)$ is defined by 
  \[
  DX = \sum_{A=1}^{2n} e_A \cdot \nabla_{e_A} X,
  \]
  where $\{e_A\}_{A=1}^{2n}$ is an orthonormal basis (it is immediate to check that the definition is independent of the basis chosen).
- $\mathcal{H} : \text{Cl}(M) \rightarrow \text{Cl}(M)$ defined by (see [12], p. 1091) 
  \[
  \mathcal{H}(X) = \frac{1}{2i} (\omega \cdot X + X \cdot \omega).
  \]
- For $X \in \text{Cl}(M), L_X(Y) := X \cdot Y$ and $R_X := Y \cdot X$.

These objects have the following basic properties.

**Lemma 2.** We have
\begin{enumerate}[(a)]
  \item $J_d \omega = 0$ and $J_a \omega = \omega$.
  \item For $X \in \text{Cl}(M), J_d(X) = \frac{1}{2}(\omega \cdot X - X \cdot \omega)$. Therefore, $\mathcal{H} = iJ_d - iL_\omega$.
  \item $\nabla_X J_a$ and $J_a^{-1} \circ \nabla_X J_a$ are derivations of order 0, and they are anti-self adjoint.
  \item $\mathcal{H} J_a = J_d \mathcal{H}, \mathcal{H}^c = \mathcal{H}$, and $J_a^d = J_d$ (i.e. $J_d$ and $J_a$ commute).
\end{enumerate}
(e) $\nabla \psi^i = (\nabla \psi)^i$ (that is, $b \circ \nabla \circ \sharp = \nabla$).

(f) $J_a \nabla \omega = -\nabla \omega$ and $\tilde{J}_a \nabla \omega = -\nabla \omega$.

Proof. For (a), note that for an adapted frame, if $1 \leq i \leq n$, $J_a(e_i \cdot e_{i+n}) = e_{i+n} \cdot e_i - e_i \cdot e_i = 0$ and $J_a(e_i \cdot e_{i+n}) = -e_{i+n} \cdot e_i = e_i \cdot e_{i+n}$.

For (b), see [12], Lemma 2.5 (note though that our definition of $J$ is $i$ times their definition of $J$).

To show (c), by definition, $(\nabla_X J_d)(Y) = \nabla_X (J_d Y) - J_d \nabla_X Y = [\nabla_X, J_d](Y)$. Therefore $\nabla_X J_d$ is a derivation since it is the commutator of two derivations. To show that $J_a^{-1} \circ (\nabla_X J_d)$ is a derivation, note that $J_a^{-1} \circ (\nabla_X J_a) = J_a^{-1} \circ \nabla \circ J_a - \nabla$, which is a sum of derivations and therefore also a derivation. To show that $\nabla_X J_d$ is anti-self adjoint, let $X \in T_{\mathbb{C}} M, Y, Z \in \mathbb{C}l(M)$, recall that $J_a^* = -J_d$ (Lemma 1(f)), and compute

\begin{align*}
((\nabla_X J_d)Y, Z) &= (\nabla_X (J_d Y), Z) - (J_d \nabla_X Y, Z) \\
&= X(J_d Y, Z) - (J_d Y, \nabla_X Z) + (\nabla_X Y, J_d Z) \\
&= -X(Y, J_d Z) + (Y, J_d \nabla_X Z) + X(Y, J_d Z) - (Y, \nabla_X (J_d Z)) \\
&= -\langle Y, (\nabla_X J_d)Z \rangle.
\end{align*}

A similar computation shows that $J_a^{-1} \nabla J_a$ is anti-self adjoint.

Item (d) follows directly from (a), the definition of $\mathcal{H}$, and (b).

Part (e) is essentially the definition of $\nabla$ in $\mathcal{A}_C(M)$.

To show (f), use the definition of $\omega \in \Gamma A^2_\mathbb{C}(M)$ and the properties of $\nabla$. Let $X \in T_{\mathbb{C}} M$, and let $Y, Z$ be sections of $T_{\mathbb{C}} M$. Then

\begin{align*}
(\tilde{J}_a \nabla_X \omega)(Y, Z) &= \nabla_X \omega(Y, JZ) \\
&= X(\omega(Y, JZ)) - \omega(\nabla_X Y, JZ) - \omega(Y, \nabla_X JZ) \\
&= -X(Y, JZ) - J(\nabla_X Y, JZ) + (Y, \nabla_X JZ) \\
&= -X(Y, JZ) - X(JY, Z) + (JY, \nabla_X Z) + X(Y, JZ) - (\nabla_X Y, JZ) \\
&= -(X(\omega(Y, X) - \omega(Y, \nabla_X Z) - \omega(\nabla_X Y, Z)) \\
&= -(\nabla_X \omega)(Y, Z)
\end{align*}

The equivalent statement for $\omega \in \Gamma \mathbb{C}l(M)$ follows then from Lemma 3 (d) and (e).

\[ \square \]

3. Kähler identities in the Clifford bundle

The classical Kähler identities read

\begin{align*}
[d^\ast, L] &= -d^c \\
[d, L] &= 0 \\
[d, \Lambda] &= (d^c)^* \\
[d^\ast, \Lambda] &= 0,
\end{align*}

where $L(\phi) := \omega \wedge \phi$ is the Lefschetz operator and $\Lambda(\phi) := \sum_{i=1}^n \omega \cdot J_i \phi$ its adjoint. In order to find an equivalent formulation in the Clifford bundle, we need the following conversion between operators.
Proposition 1. There are the following equivalences between operators in $\mathcal{C}(M)$ and operators in $A_C(M)$.

(a) $D \cong d + d^*$.
(b) $\mathcal{H} \cong i(\Lambda - L)$.
(c) $[D, \mathcal{H}] \cong i[d + d^*, \Lambda - L]$.
(d) $D^c \cong -(d^c + d^c)$.

Proof. For (a) see for example [11], Theorem 5.12. For (b) see [12], page 1118. Part (c) follows immediately from (a) and (b). Part (d) follows immediately from (a) and Lemma 1 (i), since $D$ has degree 1.

Thus, as a guide, to find an equivalent formulation of the Kähler identities it seems sensible to find an expression for $[D, \mathcal{H}]$ which contains a term involving $D^c$. Miraculously, one obtains a nice expression once a suitable operator is defined. More precisely, we have

Lemma 3. Let $\{e_A\}$ an adapted frame as before. Then

$$[D, \mathcal{H}] = -iD^c - iL_{D\omega} + i \sum_{A=1}^{2n} e_A \cdot (\nabla_{e_A} J_d + J_d^{-1} \circ \nabla_{J e_A} J_a).$$

Proof. Since $\mathcal{H} = iJ_d - iL_{\omega}$, we first compute the commutator with each of the two terms and compare the result with $D^c$.

$$[D, J_d](X) = \sum_{A=1}^{2n} (e_A \cdot \nabla_{e_A} (J_d X) - J_d(e_A \cdot \nabla_{e_A} X))$$

$$= \sum_{A=1}^{2n} (e_A \cdot (\nabla_{e_A} J_d) X + e_A \cdot J_d \nabla_{e_A} X - J e_A \cdot \nabla_{e_A} X - e_A \cdot J_d \nabla_{e_A} X)$$

$$= \sum_{A=1}^{2n} e_A \cdot (\nabla_{e_A} J_d) X - \sum_{A=1}^{2n} J e_A \cdot \nabla_{e_A} X.$$

Also,

$$[D, L_{\omega}](X) = \sum_{A=1}^{2n} (e_A \cdot \nabla_{e_A} (\omega \cdot X) - \omega \cdot e_A \cdot \nabla_{e_A} X)$$

$$= \sum_{A=1}^{2n} (e_A \cdot \nabla_{e_A} \omega \cdot X + e_A \cdot \omega \cdot \nabla_{e_A} X - \omega \cdot e_A \cdot \nabla_{e_A} X)$$

$$= L_{D\omega}(X) - 2 \sum_{A=1}^{2n} J e_A \cdot \nabla_{e_A} X \quad \text{using Lemma 4 (b)}.$$

Thus, since $\mathcal{H} = iJ_d - iL_{\omega}$,

$$[D, \mathcal{H}] + iL_{D\omega}(X) = i \sum_{A=1}^{2n} e_A \cdot (\nabla_{e_A} J_d) X + i \sum_{A=1}^{2n} J e_A \cdot \nabla_{e_A} X.$$
On the other hand,

\[ D^c(X) = J_a^{-1} \left( \sum_{A=1}^{2n} e_A \cdot \nabla_{e_A} J_a X \right) \]

\[ = \sum_{A=1}^{2n} \left( -Je_A \cdot J_a^{-1}((\nabla_{e_A} J_a)X + J_a \nabla_{e_A} X) \right) \]

\[ = \sum_{A=1}^{2n} e_A \cdot J_a^{-1}(\nabla_{Je_A} J_a)X - \sum_{A=1}^{2n} Je_A \cdot \nabla_{e_A} X, \]

where the first term in the last equality comes from the fact that if \( 1 \leq A \leq n \), then \( Je_A = e_{A+n} \), and if \( n+1 \leq A \leq 2n \), then \( Je_A = e_{A-n} \). Therefore,

\[ [D, \mathcal{H}] + iL_{D\omega} + iD^c = i \sum_{A=1}^{2n} e_A \cdot (\nabla_{e_A} J_d + J_a^{-1} \circ \nabla_{Je_A} J_a), \]

which proves the claim.

This motivates the definition of the following operators.

**Definition 2.** Let

(a) For \( X \in T_{CM} \), define \( \sigma_X : \text{Cl}(M) \to \text{Cl}(M) \) by

\[ \sigma_X = \nabla_X J_d + J_a^{-1} \circ \nabla_{JX} J_a. \]

(b) For \( Y \in \text{Cl}(M) \), let

\[ D_\sigma = \sum_{A=1}^{2n} e_A \cdot \sigma_{e_A}(Y). \]

We will see below that \( \sigma \) and \( D_\sigma \) have remarkable properties. But first we have

**Theorem 1.** (Generalized Kähler identities in the Clifford bundle.)

\[ [D, \mathcal{H}] = -iD^c + iD_\sigma - iL_{D\omega}. \]

*Proof.* Immediate from the previous lemma and definition.

In the next section we will see that the conversion of this identity to the exterior bundle gives a generalization of the classical identities. Next we study some of the properties of the operators \( \sigma_X \) and \( D_\sigma \) defined above.

**Lemma 4.** The operator \( \sigma \) has the following properties.

(a) \( \sigma_X \) is a derivation of degree 0.

(b) \( \sigma_X \) is anti-self adjoint.

(c) If \( Y \in T_{CM} \), then \( \sigma_J(Y) = J\sigma_X(Y) = -\sigma_X(JY) \).
Then substitute the expression for $\sum \psi$ for $\sum 2\sigma_{JX}$.

Proof. (a) and (b) follow immediately from Lemma 1 (c).

For (c), let $X, Y \in T_{C}M$ and note that, on $T_{C}M$, $J_{a} \equiv J_{d} \equiv J$. Then

$$\sigma_{JX}(Y) = (\nabla_{JX}J)(Y) - J^{-1}(\nabla_{X}J)(Y) = J((\nabla_{X}J_{d})(Y) + J_{a}^{-1}(\nabla_{JX}J_{a})(Y)) = -J\sigma_{X}Y,$$

and

$$\sigma_{X}(JY) = (\nabla_{X}J_{d})(JY) + J_{a}^{-1}(\nabla_{JX}J_{a})(JY)$$

$$= -\nabla_{X}Y - J\nabla_{X}JY - J^{-1}\nabla_{JX}Y - J_{X}JY$$

$$= -J(\nabla_{X}JY - J\nabla_{X}Y + J^{-1}\nabla_{JX}JY - \nabla_{JX}Y)$$

$$= -J((\nabla_{X}J_{d})Y + J_{a}^{-1}(\nabla_{JX}J_{a})Y)$$

$$= -J\sigma_{X}Y.$$

To prove (d), given $Y \in T_{C}M$, use (c) to obtain $(\sigma_{X})^{c}(Y) = -J(\sigma_{X}(JY)) = -J(-J\sigma_{X}Y) = -\sigma_{X}Y$. Thus, $(\sigma_{X})^{c} = -\sigma_{X}$ on vectors and scalars (note that $\sigma \equiv 0$ on scalars), and since both sides are derivations of degree 0, they must agree in all of $\mathbb{C}l(M)$.

For (e), if $X, Y \in T_{C}M$, $[\sigma_{X}, J_{d}](Y) = \sigma_{X}(J_{d}(Y)) - J_{d}(\sigma_{X}(Y)) = -2\sigma_{JX}Y$ by part (c). Thus, $[\sigma_{X}, J_{d}]$ and $-2\sigma_{JX}$ are derivations that agree on scalars and vectors, so they must be equal.

To prove further properties of $\sigma$ we need the following technical result that appears in [5]. It will also be used in the next section when we translate the operator $\sigma_{X}$ into the exterior algebra.

Lemma 5. For 3-forms in an almost complex manifold $M$ we have:

- If $\psi \in A_{C}^{2,1}(M) \oplus A_{C}^{1,2}(M)$ and $X, Y, Z \in T_{C}M$. Then

  (a) $\psi(X, Y, Z) = \psi(JX, JY, Z) + \psi(JX, Y, JZ) + \psi(X, JY, JZ)$.

(b) $\sum_{A,B=1}^{2n} (\psi(e_{A}, Z, e_{B}) - \psi(e_{A}, JZ, e_{B})) \theta^{A} \wedge \theta^{B} = \frac{1}{2} \sum_{A,B=1}^{2n} (\psi(e_{A}, Z, e_{B}) + \psi(Je_{A}, Z, e_{B})) \theta^{A} \wedge \theta^{B}$.

- If $\xi \in A_{C}^{3,0}(M) \oplus A_{C}^{0,3}(M)$, $X, Y, Z \in T_{C}M$. Then $\xi(JX, Y, Z) = \xi(X, JY, Z) = \xi(X, Y, JZ)$.

Proof. These results can be proved by straightforward computations. For (a), note that if $\psi \in A_{C}^{2,1}(M) \oplus A_{C}^{1,2}(M)$, then $\psi^{2,1}(X, Y, Z) = \psi(X^{1,0}, Y^{1,0}, Z^{0,1}) + \psi(X^{1,0}, Y^{0,1}, Z^{1,0}) + \psi(X^{0,1}, Y^{1,0}, Z^{1,0})$, and similarly for $\psi^{1,2}$. Comparing this with $\psi(JX, JY, Z) + \psi(JX, Y, JZ) + \psi(X, JY, JZ)$ gives the result.

For item (b), use part (a) and compute to obtain

$$\sum_{A,B=1}^{2n} (\psi(e_{A}, Z, e_{B}) \theta^{A} \wedge \theta^{B} - \psi(Je_{A}, Z, e_{B}) \theta^{A} \wedge \theta^{B}) = 2 \sum_{A,B=1}^{2n} \psi(e_{A}, JZ, e_{B}) \theta^{A} \wedge \theta^{B}.$$

Then substitute the expression for $\sum_{A,B=1}^{2n} \psi(e_{A}, JZ, e_{B}) \theta^{A} \wedge \theta^{B}$ on the right hand side of (a).
Lemma 6. The operator $\sigma$ has the following properties.

(a) For $Y \in T_C M$, $\sigma_X(Y) = \sum_{B=1}^{2n} (d\omega^+(X, Y, e_B) - d\omega^+(X, JY, Je_B))e_B$.

(b) $\sigma_X^\flat(\varphi) = -\nabla_X J\varphi + J^{-1} \circ \nabla_X \tilde{J}_a$, and therefore it is a derivation by Lemma 2 (c).

Further, if $\alpha$ is a 1-form, $\sigma_X^\flat(\alpha) = \sum_{B, C=1}^{2n} (d\omega^+(X, e_C, e_B) - d\omega^+(X, Je_C, Je_B))\alpha(e_C)\theta^B$.

(c) $\sum_{A=1}^{2n} \sigma(e_A) = -2(\tilde{J}\theta)^\sharp$, where $\theta := \omega_\omega d\omega^+ = -\tilde{J}d^*\omega^+$ is the Lee form defined in (3).

Proof. (a): Using Proposition 4.2 in page 148 of [10] we have, adjusting the constants to our definitions, that for vectors $X, Y, Z \in T_C M$,

(4) \[ 2\langle \nabla_X JY, Z \rangle = d\omega(X, Y, Z) - d\omega(X, JY, JZ) + 4\langle JX, N(Y, Z) \rangle, \]

where $N$ is the Nijenhuis tensor defined by

\[ N(X, Y) = \frac{1}{4} ([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]). \]

This implies, using the fact that $N(Y, JZ) = -JN(Y, Z)$,

(5) \[ 2\langle J^{-1}(\nabla_X J)Y, Z \rangle = d\omega(JX, Y, JZ) + d\omega(JX, JY, Z) - 4\langle JX, N(Y, Z) \rangle, \]

and adding (4) and (5),

\[ 2\langle \nabla_X J + J^{-1}\nabla_J X J \rangle Y, Z \rangle = d\omega(X, Y, Z) - d\omega(X, JY, JZ) + d\omega(JX, JY, JZ) + d\omega(JX, JY, Z) = d\omega^+(X, Y, Z) - d\omega^+(X, JY, JZ) + d\omega^+(JX, JY, JZ) + d\omega^+(JX, JY, Z) = 2(d\omega^+(X, Y, Z) - d\omega^+(X, JY, JZ)) \]

by Lemma 5. Thus, $\langle \sigma_X(Y), Z \rangle = d\omega^+(X, Y, Z) - d\omega^+(X, JY, JZ)$, which proves (a).

(b): The first identity follows from the fact that, using Lemma 1 (d), (e) and Lemma 2 (e), if $\psi \in A^0_C(M)$, then

\[ b \circ (\nabla J_d) \circ \varphi(\psi) = b(\nabla J_d \psi^\sharp - J_d \nabla \psi^\sharp) = -\nabla J_d \psi + J_d \nabla \psi = (-\nabla \tilde{J}_d) \psi, \]

and

\[ b \circ J_a^{-1} \circ (\nabla J_a) \circ \varphi(\psi) = b(J_a^{-1}(\nabla J_a \psi^\sharp) - \nabla \psi^\sharp) = (-1)^{k+k} J_a^{-1}(\nabla J_a \psi^\sharp - \nabla \psi^\sharp) = J_a^{-1}(\nabla J_a) \circ \psi^\sharp. \]

The second identity follows directly from (a).
(c): From (a) we have

\[ \sum_{A=1}^{2n} \sigma e_A = \sum_{A,B=1}^{2n} (d\omega^+(e_A, e_B) - d\omega^+(e_A, Je_A, Je_B)) e_B \]

\[ = -\sum_{B=1}^{2n} \left( (\sum_{A=1}^{2n} \theta_A \wedge \tilde{J} \theta_A) \cdot d\omega^+ \right) (Je_B) e_B \]

\[ = -2\sum_{B=1}^{2n} \tilde{J} (\omega \cdot d\omega^+) (e_B) e_B \quad \text{(since } \sum_{A=1}^{2n} \theta_A \wedge \tilde{J} \theta_A = 2\omega) \]

\[ = -2(\tilde{J} \theta)^2. \]

The commutator between \( \mathcal{H} \) and \( D_\sigma - L_{D\omega} \) will add information about commutators in the exterior algebra and will be used in the next section. We find it here.

**Lemma 7.** The following identities hold:

(a) \( L_{D\omega}^c = L_{D^c\omega} \).

(b) \([L_{D\omega}, \mathcal{H}] = iJ_dL_{D\omega} \) and \([L_{D^c\omega}, \mathcal{H}] = iJ_dL_{D^c\omega}\).

(c) \( D_\sigma \omega = -J_dD\omega + 3D^c\omega \) and \( D^c_\sigma \omega = -J_dD^c\omega - 3D\omega \).

(d) \([D_\sigma, J_d] = D^c_\sigma \).

(e) \([D^c_\sigma, J_d] = -D_\sigma \).

(f) \([D_\sigma, \mathcal{H}] = i(3D^c_\sigma - L_{D_\sigma \omega}) \).

(g) \([D^c_\sigma, \mathcal{H}] = -i(3D_\sigma + L_{D^c_\sigma \omega}) \).

(h) \([D_\sigma - L_{D\omega}, \mathcal{H}] = 3i(D^c_\sigma - L_{D^c\omega}) \).

(i) \([D^c_\sigma - L_{D^c\omega}, \mathcal{H}] = 3i(D_\sigma - L_{D\omega}) \).

(j) \((D_\sigma - L_{D\omega}) \) and \((D^c_\sigma - L_{D^c\omega}) \) are self-adjoint.

**Proof.** Let \( X \in \mathcal{C}(M) \). For (a), note that since \( D\omega \) has odd degree, \( J^{-1}_dD\omega = -J_dD\omega \) by Lemma 1 (g), and then \( L_{D\omega}^c X = J^{-1}_d(D\omega \cdot J_d X) = -L_{J_dD\omega} X \). For (b), note that if \( Y \in \mathcal{C}(M) \), using the definition of \( \mathcal{H} \) and Lemma 2 (b),

\[ [L_Y, \mathcal{H}] X = -\frac{i}{2} (Y \cdot (\omega \cdot X + X \cdot \omega) - (\omega \cdot Y \cdot X + Y \cdot X \cdot \omega)) = -\frac{i}{2} (Y \cdot \omega - \omega \cdot Y) \cdot X = iL_{J_dY} X. \]
We compute $D_\sigma \omega$ directly in order to prove (c). Since $J_d \omega = \omega$ and $J_d \omega = 0$, and $\sum_{A=1}^{2n} e_A \cdot \nabla_{e_A} X = -\sum_{A=1}^{2n} J e_A \cdot \nabla_{e_A} X$ for any $X \in \text{Cl}(M)$,

\[
D_\sigma \omega = \sum_{A=1}^{2n} e_A \cdot \sigma_A(\omega) = \sum_{A=1}^{2n} (e_A \cdot \nabla_{e_A} J_d \omega - e_A \cdot J_d \nabla_{e_A} \omega + e_A \cdot J_a^{-1}(\nabla_{e_A} J_a \omega) - e_A \cdot \nabla_{J_a \omega})
\]

\[
= \sum_{A=1}^{2n} (-J_d(e_A \cdot \nabla_{e_A} \omega) + J e_A \cdot \nabla_{e_A} \omega + J_a^{-1}(J e_A \cdot \nabla_{J_a} J_a \omega) + J e_A \cdot \nabla_{e_A} \omega)
\]

\[
= -J_d D \omega + D^c \omega + 2 \sum_{A=1}^{2n} J e_A \cdot \nabla_{e_A} \omega
\]

\[
= -J_d D \omega + D^c \omega + 2 \sum_{A=1}^{2n} J_a^{-1}(e_A \cdot \nabla_{e_A} \omega) \quad \text{(by Lemma 2 (f))}
\]

\[
= -J_d D \omega + 3D^c \omega
\]

The statement about $D^c_\sigma \omega$ follows easily conjugating the last expression with $J_a$ and using Lemma 1 (h).

To prove (d), compute

\[
[D_\sigma, J_d]X = \sum_{A=1}^{2n} (e_A \cdot [\sigma_{e_A} J_d - J_d e_A] X).
\]

\[
= \sum_{A=1}^{2n} (e_A \cdot [\sigma_{e_A}, J_d] X - J_d e_A \cdot \sigma_{e_A} X)
\]

\[
= \sum_{A=1}^{2n} (-2e_A \cdot \sigma_{J_a e_A} X - J_d e_A \cdot \sigma_{e_a} X) \quad \text{(by Lemma 4 (e))}
\]

\[
= \sum_{A=1}^{2n} J e_A \cdot \sigma_{e_A} X.
\]

On the other hand, since $\sigma_X^{e} = -\sigma_X$ (see Lemma 4 (d)),

\[
(6) \quad D^e_\sigma X = J_a^{-1} \left( \sum_{A=1}^{2n} e_A \cdot \sigma_{e_A} J_a X \right) = -\sum_{A=1}^{2n} J e_A \cdot J_a^{-1} \sigma_{e_A} J_a X = -\sum_{A=1}^{2n} J e_A \cdot \sigma_{e_A} X = \sum_{A=1}^{2n} J e_A \cdot \sigma_{e_A} X,
\]

which proves (d).

(e) follows from (d) and Lemma 1 (h).
For (f), recall (Lemma 1 (b)) that \( \mathcal{H} = i(J_d - L_\omega) \), so we only have to find \([D_\sigma, L_\omega]\):

\[
[D_\sigma, L_\omega] X = \sum_{A=1}^{2n} (e_A \cdot \sigma_{e_A}(\omega \cdot X) - \omega \cdot \sigma_{e_A}X)
\]

\[
= \sum_{A=1}^{2n} (e_A \cdot \sigma_{e_A} \omega \cdot X + e_A \cdot \omega \cdot \sigma_{e_A}X - \omega \cdot e_A \cdot \sigma_{e_A}X)
\]

\[
= L_{D_\omega}X - 2 \sum_{A=1}^{2n} J e_A \cdot \sigma_{e_A}X \quad \text{by Lemma 2 (b)}
\]

\[
= L_{D_\omega}X - 2 D_\sigma^\ast X \quad \text{by (6)}.
\]

Thus, \([D_\sigma, \mathcal{H}] = i(3D_\sigma - L_{D_\omega})\).

(g) follows from the previous one and Lemma 1 (h).

(h) and (i) follow immediately from (b), (c), (f), and (g).

To prove (j) we will use the fact that if \( Z \in T_c M \), then \((LZ)^\ast = -LZ\), and if \( Z \in Cl(M) \) is such that \( Z^\flat \in A_3^C(M) \), then \((LZ)^\ast = LZ\) (see Propositions 9.27 and 9.29 of [6]). If we let \( X, Y \in Cl(M) \),

\[
\langle D_\sigma X, Y \rangle = \sum_{A=1}^{2n} (e_A \cdot \sigma_{e_A}X, Y)
\]

\[
= - \sum_{A=1}^{2n} \langle \sigma_{e_A}X, e_A \cdot Y \rangle
\]

\[
= \sum_{A=1}^{2n} \langle X, \sigma_{e_A}(e_A \cdot Y) \rangle \quad \text{by Lemma 4 (b)}
\]

\[
= \sum_{A=1}^{2n} \langle X, (\sigma_{e_A}e_A) \cdot Y + e_A \cdot \sigma_{e_A}Y \rangle \quad \text{by Lemma 4 (a)}
\]

\[
= \left\langle X, \left( \sum_{A=1}^{2n} \sigma_{e_A}e_A \right) \cdot Y + D_\sigma Y \right\rangle \quad \text{by Lemma 4 (c)}.
\]

Thus,

\[ D_\sigma^\ast = D_\sigma - 2L_{(J_\theta)^\flat}. \]

On the other hand, since \((D\omega)^\flat = d\omega + d^\ast \omega\) (see Proposition 1(a)), we can decompose \( D\omega \) as \((d\omega)^\sharp + (d^\ast \omega)^\sharp\), and therefore \( L_{D\omega} = L_{(d\omega)^\sharp} + L_{(d^\ast \omega)^\sharp} \) and \((L_{D\omega})^\ast = L_{(d\omega)^\sharp} - L_{(d^\ast \omega)^\sharp} \) since \( d\omega \in A_3^C(M) \) and \( d^\ast \omega \in A_1^C(M) \). Thus, using \( d^\ast \omega = J_\theta \),

\[ (L_{D\omega})^\ast = L_{D\omega} - 2L_{(J_\theta)^\flat}, \]

and subtracting the last displayed equations we obtain

\[ (D_\sigma - L_{D\omega})^\ast = D_\sigma - L_{D\omega}, \]

as desired. The other expression in (j) follows immediately from this one and Lemma 1 (h).
Next, we convert the identities obtained in this section to familiar objects in the exterior algebra. Most of the proofs will involve straightforward linear algebra computations.

4. Conversion to the Exterior Algebra

The formula $[D, \mathcal{H}] = -iD^c + iD_\sigma - iL_{D\omega}$ proved in Section 2 summarizes all the Kähler identities once it is translated to the exterior algebra. In this section we make this translation.

Recall that the operator $d$ decomposes into 4 parts, which we will denote by $\mu, \partial, \bar{\partial}$ and $\bar{\mu}$, of degrees $(2, -1), (1, 0), (0, 1)$ and $(-1, 2)$, respectively. That is,

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}.$$  

From Proposition 1 we know that $D^c \equiv -(d^c + d^{*c})$. In order to translate $D_\sigma$ and $L_{D\omega}$ we need to extend the definition of some well-known operators in the exterior algebra (see for example [3]) to the almost-complex case.

**Definition 3.** For $\phi \in A_C(M)$, let

- $L\phi := \omega \wedge \phi$ (the usual Lefschetz operator), $\Lambda := L^* = \omega \cdot \phi$, $H := [L, \Lambda]$.
- $\lambda_0(\phi) := \partial \omega \wedge \phi$, $\lambda_\bar{\partial}(\phi) := \bar{\partial} \omega \wedge \phi$ (usually denoted $\lambda$ and $\bar{\lambda}$, respectively, in the literature).
- $\lambda_\mu(\phi) := \mu \omega \wedge \phi$, $\lambda_\bar{\mu}(\phi) := \bar{\mu} \omega \wedge \phi$.
- $\lambda_+ := \lambda_0 + \lambda_\bar{\partial}$, $\lambda_- := \lambda_\mu + \lambda_\bar{\mu}$, $\lambda := \lambda_+ + \lambda_-.$

- $\tau_0 := [\Lambda, \lambda_0]$, $\tau_\bar{\partial} := [\Lambda, \lambda_\bar{\partial}]$ (usually denoted $\tau$ and $\bar{\tau}$, respectively, in the literature).
- $\tau_\mu := [\Lambda, \lambda_\mu]$, $\tau_\bar{\mu} := [\Lambda, \lambda_\bar{\mu}]$.
- $\tau_+ := \tau_0 + \tau_\bar{\partial}$, $\tau_- := \tau_\mu + \tau_\bar{\mu}$. $\tau = \tau_+ + \tau_-$. *Note that $\tau(1) = \theta$ (the Lee form).*

To translate $L_{D\omega}$ to an operator in the exterior bundle, first note that $D\omega \equiv d\omega + d^c\omega \in A^1_C(M) \oplus A^3_C(M)$. If $\alpha \in A^1(M)$, it is well known that

$$\langle \alpha^\sharp \cdot \phi \rangle^\flat = \alpha \wedge \phi - \alpha^\sharp \cdot \phi.$$  

The equivalent statement for $\xi \in A^3(M)$ is as follows.

**Lemma 8.** Let $\xi \in A^3_C(M)$. For $\phi \in A_C(M)$, let $r_\xi(\phi) := \sum_{A=1}^{2n} (e_A \cdot \xi^\flat) \wedge (e_A \cdot \phi)$. Then

(a) $\langle \xi^\sharp \cdot \phi \rangle^\flat = \xi \wedge \phi + r_\xi(\phi) + (r_\xi)^*(\phi) + \xi \cdot \phi$.

(b) If $\xi$ has bidegree $(r, s)$, then the operator $r_\xi$ has bidegree $(r-1, s-1)$.

**Proof.** To prove part (a) we can assume, by linearity, that $\phi \in A^k_C(M)$ for some $k$ and that $\xi = \theta^A \wedge \theta^B \wedge \theta^C$, with $A < B < C$ (so $\xi^\sharp = e_A \cdot e_B \cdot e_C$). Using the formula (7) repeatedly it is easy to see that the operator $\bigcirc L_{\bar{\xi}} \circ \xi$ has 4 components, of degrees $-3, -1, 1, and 3$ respectively. On the other hand, $\bigcirc L_{\xi} \circ \xi$ is self-adjoint, which follows immediately from Propositions 9.27 and 9.29 of [6]. This implies that the components of degree $-3$ and $-1$ of $\bigcirc L_{\xi} \circ \xi$ are the adjoints of the components of degree $3$ and $1$, respectively.
Thus, we only need to compute the components of degrees 3 and 1. Using the formula (7) three times, one obtains

\[
(e_A \cdot e_B \cdot e_C \cdot \phi^*)^b = \theta^A \wedge \theta^B \wedge \theta^C \wedge \phi + \theta^A \wedge \theta^B \wedge (e_B \cdot \phi) - \theta^A \wedge \theta^B \wedge (e_C \cdot \phi) - \theta^A \wedge \theta^B \wedge (e_A \cdot \phi) + \text{the adjoint of the previous terms}
\]

\[
= \xi \wedge \phi - (e_B \cdot \xi) \wedge (e_B \cdot \phi) - (e_C \cdot \xi) \wedge (e_C \cdot \phi) - (e_A \cdot \xi) \wedge (e_A \cdot \phi) + \text{the adjoint of the previous terms}
\]

\[
= \xi \wedge \phi + \epsilon^j (\phi) + \epsilon^j (\phi) + \xi \cdot \phi.
\]

To prove part (b), for \(1 \leq j \leq n\), let

\[
\epsilon_j := \frac{e_j - ie_{j+n}}{2} \quad \text{and} \quad \bar{\epsilon}_j := \frac{e_j + ie_{j+n}}{2},\text{ so that } e_j = \epsilon_j + \bar{\epsilon}_j \text{ and } e_{j+n} = i(\epsilon_j - \bar{\epsilon}_j).
\]

Then \(\epsilon_j\) and \(\bar{\epsilon}_j\) are \((1,0)\) and \((0,1)\) vectors, respectively, and if \(\psi \in A^p_q(M)\),

\[
r^*_\xi(\psi) = \sum_{A=1}^{2n} (e_A \cdot \xi) \wedge (e_A \cdot \psi)
\]

\[
= \sum_{j=1}^{n} \left( ((\epsilon_j + \bar{\epsilon}_j) \cdot \xi) \wedge (\epsilon_j + \bar{\epsilon}_j) \cdot \psi) - ((\epsilon_j - \bar{\epsilon}_j) \cdot \xi) \wedge (\epsilon_j - \bar{\epsilon}_j) \cdot \psi \right)
\]

\[
= 2 \sum_{j=1}^{n} \left( (\epsilon_j \cdot \xi) \wedge (\bar{\epsilon}_j \cdot \psi) + (\bar{\epsilon}_j \cdot \xi) \wedge (\epsilon_j \cdot \psi) \right),
\]

which has bidegree \((p + r - 1, q + s - 1)\). Therefore, \(r^*_\xi\) has bidegree \((r - 1, s - 1)\).

\[\square\]

In view of this lemma, the definition of the following operator \(\rho\) becomes quite natural.

**Definition 4.** For \(\phi \in A_{C}(M)\), let

\[
\rho_\theta(\phi) := - \sum_{A=1}^{2n} (e_A \cdot \partial \omega) \wedge (e_A \cdot \phi), \quad \rho^*_\theta(\phi) := - \sum_{A=1}^{2n} (e_A \cdot \bar{\partial} \omega) \wedge (e_A \cdot \phi),
\]

\[
\rho_\mu(\phi) := - \sum_{A=1}^{2n} (e_A \cdot \mu \omega) \wedge (e_A \cdot \phi), \quad \rho^*_\mu(\phi) := - \sum_{A=1}^{2n} (e_A \cdot \bar{\mu} \omega) \wedge (e_A \cdot \phi).
\]

\[
\rho_+ := \rho_\theta + \rho^*_\theta, \quad \rho_- := \rho_\mu + \rho^*_\mu; \quad \rho := \rho_+ + \rho_-. \]

With these definitions, the following proposition is almost trivial. For convenience, we use \(E_\phi\) to denote exterior multiplication by \(\phi\) and \(I_\phi\) to denote interior multiplication by \(\phi^*\) (i.e. the adjoint of \(E_\phi\)).

**Proposition 2.** \(L_{D\omega} \cong \lambda + \rho + E_{j\theta} + \rho^* + \lambda^* - I_{j\theta}\).
Proof. Since \((D\omega)^h = d\omega + d^*\omega\), with \(d\omega \in A^3_C(M)\) and \(d^*\omega \in A^1_C(M)\), Lemma 8 and formula (7) give, for \(\phi \in A_C(M)\),

\[
\flat \circ L_{D\omega} \circ \sharp(\phi) = (D\omega \cdot \phi^\sharp)^h
\]

\[
= ((d\omega)^\sharp \cdot \phi + (d^*\omega)^\sharp \cdot \phi)^h
\]

\[
= d\omega \wedge \phi + r_{d\omega}(\phi) + (r_{d\omega})^* (\phi) + d\omega \cdot \phi + d^*\omega \wedge \phi - d^*\omega \cdot \phi
\]

\[
= \lambda \phi + \rho \phi + \lambda^* \phi + \rho^* \phi + E_\mathcal{I}_\lambda \phi - I_\mathcal{I}_\phi.
\]

□

The operators \(\rho\), while quite natural, seem a bit mysterious. At first it seems that they can be written in terms of the other operators in Definition 3, and in fact this is partly the case: it turns out that \(\tau_\mu = i\rho_\mu\). To show this, we first prove a general statement that will also be used later.

**Lemma 9.** Let \(\xi\) be an odd-degree form and \(\psi\) any form. Then

\[
[L_\xi, L_\xi](\psi) = L_\Lambda \xi \psi + \sum_{C=1}^{2n} (e_C \cdot \xi) \wedge (Je_C \cdot \psi).
\]

In particular, \([L_\xi, L_\xi] - L_\Lambda \xi\) is a derivation.

**Proof.** Using an adapted basis \(\{e_A\}_{A=1}^{2n}\),

\[
[L_\xi, L_\xi](\psi) = \omega \cdot (\xi \wedge \psi) - \xi \wedge (\omega \cdot \psi)
\]

\[
= \sum_{i=1}^{n} (e_{i+n} \cdot e_i \cdot (\xi \wedge \psi)) - \xi \wedge (\omega \cdot \psi)
\]

\[
= \sum_{i=1}^{n} (e_{i+n} \cdot ((e_i \cdot \xi) \wedge \psi) - e_{i+n} \cdot (\xi \wedge (e_i \cdot \psi)) - \xi \wedge (e_{i+1} \cdot e_i \cdot \psi))
\]

\[
= \sum_{i=1}^{n} (e_{i+n} \cdot e_i \cdot \xi) \wedge \psi + (e_i \cdot \xi) \wedge (e_{i+n} \cdot \psi) - (e_{i+n} \cdot \xi) \wedge (e_i \cdot \psi)
\]

\[
= \Lambda \xi \wedge \psi + \sum_{C=1}^{2n} (e_C \cdot \xi) \wedge (Je_C \cdot \psi),
\]

Then \([L_\xi, L_\xi] - L_\Lambda \xi\) is a derivation since \(\cdot\) is and \(\cdot e_C \cdot \xi\) has even degree.

□

**Lemma 10.** \(\tau_\mu = i\rho_\mu\) (and \(\tau_\bar{\mu} = -i\rho_\bar{\mu}\)). Therefore, \(\rho_- = -\tau_-\).

**Proof.** Using \(\xi = \mu \omega\) in Lemma 9, and since \(\Lambda(\mu \omega) = 0\) because \(\mu \omega\) has bidegree \((3, 0)\) and \(\Lambda\) has bidegree \((-1, -1)\), we find that

\[
\tau_\mu(\psi) = \sum_{C=1}^{2n} (e_C \cdot \mu \omega) \wedge (Je_C \cdot \psi).
\]
Hence $\tau_\mu$ is a derivation of degree 1, and therefore it suffices to prove the identity on scalars and 1-forms. For scalars, it is clear that both sides are 0. If $\alpha$ is a 1-form,

$$J_a(\tau_\mu(\alpha)) = J_a \left( \sum_{C=1}^{2n} (e_C \cdot \mu \omega) \alpha(J e_C) \right) = - \sum_{C=1}^{2n} (e_C \cdot \mu \omega) J_\alpha(e_C) = \rho_\mu(J_\alpha),$$

where we have used that $J_a(e_C \cdot \mu \omega) = -e_C \cdot \mu \omega$ because $\mu \omega$ is a $(3,0)$-form and therefore $e_C \cdot \mu \omega$ is a $(2,0)$-form. Thus, $\tau_\mu = \rho_\mu^c = i\rho_\mu$ by Lemma 1 (j) since $\rho_\mu$ has bidegree $(2,-1)$ (see Lemma 8 (b)). This also gives $\tau^-_\mu = -i\rho_\mu$ and therefore $\tau^-_\mu = -\rho_-.$

\[ \square \]

In order to translate the operator $D_\sigma$ into the exterior algebra in terms of $\rho$ and $\tau$ we need the following technical lemmas.

**Lemma 11.** The operators $\tau_+^c$ and $\tau_+^c$ can be expressed locally as follows:

(a) If $\psi$ is any form, $\tau_+(\psi) = \theta \wedge \psi + \sum_{C=1}^{2n} (e_C \cdot \omega^+) \wedge (J e_C \cdot \psi)$, where $\theta$ is the Lee form.

(b) If $\alpha$ is a 1-form, $\tau_+^c(\alpha) = -J \theta \wedge \alpha + \frac{1}{2} \sum_{A,B,C=1}^{2n} \omega^+(J e_A, e_C, J e_B) \alpha(e_C) \theta^A \wedge \theta^B.$

**Proof.** For part (a), note that $\tau_+ = [\Lambda, \lambda_+] = [\Lambda, L_{\omega^+}]$ and use Lemma 9, observing that $\Lambda \omega^+ = \theta$ by definition.

In particular, when $\alpha$ is a 1-form,

$$\tau_+(\alpha) = \theta \wedge \alpha + \frac{1}{2} \sum_{A,B,C=1}^{2n} \omega^+(e_C, e_A, e_B) \alpha(J e_C) \theta^A \wedge \theta^B.$$

Using this, part (b) is then a simple computation using that $\tau_+^c := J_a^{-1} \circ \tau_+ \circ J_a.$

\[ \square \]

**Lemma 12.** Let $D_\sigma^{\text{ext}} \psi := \sum_{A=1}^{2n} \theta^A \wedge \sigma^{\text{ext}} e_A \psi$ and $D_\sigma^{\text{int}} \psi := \sum_{A=1}^{2n} e_A \wedge \sigma^{\text{int}} e_A \psi.$ Then

- $D_\sigma^{\text{ext}} = \rho_+ + \tau_+^c + E_{j\theta}$
- $D_\sigma^{\text{int}} = -(D_\sigma^{\text{ext}})^* + 2I_{j\theta}.$

**Proof.** For the first statement, since $\sigma^{\text{ext}}_{e_A}$ is a derivation of degree 0 by Lemma 4 (b), $D_\sigma^{\text{ext}}$ is a derivation of degree 1. On the other hand, $\rho_+ + \tau_+^c + E_{j\theta}$ is also a derivation of degree 1 since $\rho_+$ and $\tau_+^c + E_{j\theta}$ are.

Thus, it suffices to check the identity on 0- and 1-forms. Both sides vanish for 0-forms. If $\alpha$ is a 1-form let us use Lemma 4 (b) to find an expression for $D_\sigma^{\text{ext}} \alpha$. Using Lemma 5 (b) we have

$$D_\sigma^{\text{ext}} \alpha = \sum_{A,B,C=1}^{2n} \left( \omega^+(e_A, e_C, e_B) \alpha(e_C) \theta^A \wedge \theta^B - \omega^+(e_A, J e_C, e_B) \alpha(e_C) \theta^A \wedge \theta^B \right)$$

$$= \sum_{A,B,C=1}^{2n} \left( \omega^+(e_A, e_C, e_B) \alpha(e_C) \theta^A \wedge \theta^B + \omega^+(J e_A, e_C, e_B) \alpha(e_C) \theta^A \wedge \theta^B \right)$$

$$= \frac{1}{2} \sum_{A,B,C=1}^{2n} \left( \omega^+(e_A, e_C, e_B) \alpha(e_C) \theta^A \wedge \theta^B + \omega^+(J e_A, e_C, e_B) \alpha(e_C) \theta^A \wedge \theta^B \right)$$
The first term can be written as
\[ \sum_{A,B,C=1}^{2n} d\omega^+(e_A, e_C, e_B) \alpha(e_C) \theta^A \wedge \theta^B = -2 \sum_{C=1}^{2n} (e_C \wedge d\omega^+) \wedge (e_C \wedge \alpha) = 2\rho_+ (\alpha). \]

From Lemma 11 (b), the second term is
\[ \sum_{A,B,C=1}^{2n} d\omega^+(J e_A, e_C, J e_B) \alpha(e_C) \theta^A \wedge \theta^B = 2(\tau^c_+ (\alpha) + E_{j\theta} (\alpha)). \]

Thus,
\[ D_{\sigma^*}^\ext \alpha = \rho_+ (\alpha) + \tau^c_+ (\alpha) + E_{j\theta} (\alpha), \]
as desired.

For the second statement, we calculate the adjoint directly. Let \( \phi, \psi \in A^\ext_C (M) \). Then
\[
\langle D_{\sigma^*}^{\int} \phi, \psi \rangle = \left\langle \sum_{A=1}^{2n} e_A \wedge \sigma^b_{e_A} (\phi), \psi \right\rangle \\
= \sum_{A=1}^{2n} \langle \sigma^b_{e_A} (\phi), \theta^A \wedge \psi \rangle \\
= -\sum_{A=1}^{2n} \langle \phi, \sigma^b_{e_A} (\theta^A \wedge \psi) \rangle \quad \text{by Lemma 4 (b)} \\
= -\sum_{A=1}^{2n} \langle \phi, \sigma^b_{e_A} (\theta^A) \wedge \psi + \theta^A \wedge \sigma^b_{e_A} (\psi) \rangle \quad \text{by Lemma 6 (b)} \\
= \langle \phi, 2J\theta \wedge \psi - D_{\sigma^*}^\ext (\psi) \rangle \quad \text{by Lemma 6 (c)}. 
\]

Therefore, \( D_{\sigma^*}^{\int} = -D_{\sigma^*}^\ext + 2E_{j\theta} \rangle^* = -(D_{\sigma^*}^\ext)^* + 2I j\theta. \)

\[ \square \]

**Proposition 3.** \( D_\sigma \cong \rho_+ + \tau^c_+ + E_{j\theta} + \rho^*_+ + \tau^c_+ - I_{j\theta}, \) and therefore \( D_\sigma - L_{D\omega} \cong \tau^c - \lambda + \tau^c - \lambda^* \) and \( D_{\sigma^*}^\ext - L_{D\omega}^\ext \cong \tau + \lambda^c + \tau^* + \lambda^c. \)

**Proof.** Using the definitions and the expression of the Clifford multiplication in terms of exterior and interior product, we have
\[
D_\sigma \psi^* = \sum_{A=1}^{2n} e_A \cdot \sigma_{e_A} \psi^* \cong \sum_{A=1}^{2n} \theta^A \wedge \sigma^b_{e_A} \psi = \sum_{A=1}^{2n} \theta^A \wedge \sigma^b_{e_A} \psi = D_{\sigma^*}^\ext \psi - D_{\sigma^*}^{\int} \psi.
\]

Using Lemma 12 we find
\[
D_\sigma \psi^* \cong D_{\sigma^*}^\ext + (D_{\sigma^*}^\ext)^* - 2I_{j\theta} \\
= \rho_+ + \tau^c_+ + E_{j\theta} + \rho^*_+ + \tau^c - \tau^* + I_{j\theta} - 2I_{j\theta} \\
= \rho_+ + \tau^c_+ + E_{j\theta} + \rho^*_+ + \tau^c - \tau^* - I_{j\theta}
\]
as claimed.

Since \( L_{D\omega} \cong \lambda + \rho + E_{j\theta} + \rho^* + \lambda^* - I_{j\theta} \) (Proposition 2), and since \( \rho = \rho_+ - \tau^c_+ \) by Lemma 10, we obtain the second expression. For the last expression, use Lemma 1 (h) and (i).
Putting together all the construction in this section, together with Theorem 1, we immediately obtain the result we want.

**Theorem 2.** (Kähler identities for almost complex manifolds)

\[
[d + d^*, \Lambda - L] = d^c + \tau^{c*} - \lambda + d^{c*} + \tau^{e*} - \lambda^*.
\]

Or, expanding and collecting degrees,

\[
[d, L] = \lambda \quad [d, \Lambda] = d^{c*} + \tau^{c*} \\
[d^*, L] = -d^e - \tau^e \quad [d^*, \Lambda] = -\lambda^*.
\]

**Proof.** Immediate from Theorem 1 and Propositions 1 and 3. □

**Remark 1.** For a complex manifold (i.e. integrable), note that these are exactly the identities in [3], Théorème 1.1.

**Theorem 3.** The following identity holds:

\[
[\tau^e - \lambda + \tau^{c*} - \lambda^*, \Lambda - L] = 3(\tau + \lambda + \tau^* + \lambda^{c*}).
\]

Or, expanding and collecting degrees,

\[
[\lambda, L] = 0 \quad [\lambda, \Lambda] = -\tau \\
[\tau, L] = -3\lambda \quad [\tau, \Lambda] = -2\tau^{c*}
\]

**Proof.** Follows immediately from Proposition 1 (b), Lemma 10 (h) and Proposition 3. □

**Corollary 1.** The following identities, together with their adjoints and conjugates, hold:

\[
[\mu, \Lambda] = i(\mu^* + \tau^*_\mu) \quad [\tau_{\mu}, \Lambda] = -2i\tau^*_\mu \quad [\lambda_{\mu}, \Lambda] = -\tau_{\mu} \\
[\mu, L] = \lambda_{\mu} \quad [\tau_{\mu}, L] = -3\lambda_{\mu} \quad [\lambda_{\mu}, L] = 0 \\
[\partial, \Lambda] = -i(\partial^* + \tau^*_\partial) \quad [\tau_{\partial}, \Lambda] = 2i\tau^*_\partial \quad [\lambda_{\partial}, \Lambda] = -\tau_{\partial} \\
[\partial, L] = \lambda_{\partial} \quad [\tau_{\partial}, L] = -3\lambda_{\partial} \quad [\lambda_{\partial}, L] = 0
\]

**Proof.** All the identities follow from Theorem 2 and Theorem 3 by separating by bidegrees. □

**Remark 2.** If $M$ is almost Kähler (that is, $d\omega = 0$), then $\rho, \lambda$ and $\tau$ are all 0, and (8) reduce to

\[
[\mu, \Lambda] = i\mu^* \quad [\mu, L] = 0 \quad [\partial, \Lambda] = -i\partial^* \quad [\partial, L] = 0
\]

(together with their adjoints and conjugates). These identities appear in [1], Proposition 3.1, and [2], p. 1295.

For reference, we give the following table of commutators of the common operators with $L$, $\Lambda$, and $H$. 

\[
\begin{array}{ccc}
\mu, \Lambda & = & i(\mu^* + \tau^*_\mu) \\
\mu, L & = & \lambda_{\mu} \\
\partial, \Lambda & = & -i(\partial^* + \tau^*_\partial) \\
\partial, L & = & \lambda_{\partial}
\end{array}
\]
**Remark 3.** The commutators $[\rho_\partial, L] = i\lambda_\partial$ and $[\rho_\partial, \Lambda] = -i\rho_\partial^* + \tau_\partial^*$ on the table above do not follow immediately from the results above, but we include them for reference. They were found using the methods developed in [4, 9].
## Appendix

| $q \setminus p$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $4$             |     |     |     |     | $\lambda_\mu$ | $[\mu, L]$ |     |     | $\lambda_\overline{\mu}$ | $[\overline{\mu}, L]$ |
| $3$             |     |     |     | $\mu, \tau_\mu$ | $[\mu^*, L]$ | $[\tau^*_\mu, L]$ |     |     | $\lambda_\overline{\partial}$ | $[\overline{\partial}, L]$ | $[\tau^*_\overline{\partial}, L]$ | $[\lambda_\overline{\partial}, L]$ |
| $2$             | $\mu^*, \tau^*_\mu$ | $[\lambda^*_\mu, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda_\partial$ | $[\partial, L]$ | $[\tau^*_\partial, L]$ | $[\lambda_\partial, L]$ |     |
| $1$             | $\mu^*, \tau^*_\mu$ | $[\lambda^*_\mu, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda_\partial$ | $[\partial, L]$ | $[\tau^*_\partial, L]$ | $[\lambda_\partial, L]$ |     |
| $0$             | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\partial^* \tau^*_\partial$ | $[\lambda^*_\partial, L]$ | $[\partial, L]$ | $[\tau^*_\partial, L]$ | $[\lambda_\partial, L]$ | $\mu, \tau_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ |
| $-1$            | $[\lambda^*_\mu, L]$ | $\lambda^*_\partial$ | $[\partial^*, L]$ | $[\partial, L]$ | $[\tau^*_\partial, L]$ | $\lambda^*_\partial$ | $[\partial^*, L]$ | $[\partial, L]$ | $[\tau^*_\partial, L]$ | $[\lambda_\partial, L]$ | $\mu, \tau_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ |
| $-2$            | $[\lambda^*_\partial, L]$ | $\lambda^*_\partial$ | $[\partial^*, L]$ | $[\partial, L]$ | $[\tau^*_\partial, L]$ | $\lambda^*_\partial$ | $[\partial^*, L]$ | $[\partial, L]$ | $[\tau^*_\partial, L]$ | $[\lambda_\partial, L]$ | $\mu, \tau_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ |
| $-3$            | $[\lambda^*_\partial, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $[\lambda_\mu, L]$ | $\mu, \tau_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ |
| $-4$            | $[\lambda^*_\mu, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $[\lambda_\mu, L]$ | $\mu, \tau_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ | $\lambda^*_\mu$ | $[\mu^*, L]$ | $[\mu, L]$ | $[\tau^*_\mu, L]$ |

**Figure 2**: Table of bidegrees $(p, q)$ of operators
References

1. Joana Cirici and Scott O. Wilson, *Topological and geometric aspects of almost Kähler manifolds via harmonic theory*, Selecta Math. (N.S.) 26 (2020), no. 3, Paper No. 35, 27. MR 4110721

2. Paolo de Bartolomeis and Adriano Tomassini, *On formality of some symplectic manifolds*, Internat. Math. Res. Notices (2001), no. 24, 1287–1314. MR 1866746

3. Jean-Pierre Demailly, *Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne*, Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984, Lecture Notes in Math., vol. 1198, Springer, Berlin, 1986, pp. 88–97. MR 874763

4. Luis Fernandez and Sam Hosmer, *The Dirac operator of the clifford bundle in almost complex manifolds*, Preprint.

5. Paul Gauduchon, *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B (7) 11 (1997), no. 2, suppl., 257–288. MR 1456265

6. F. Reese Harvey, *Spinors and calibrations*, Perspectives in Mathematics, vol. 9, Academic Press, Inc., Boston, MA, 1990. MR 1045637

7. W. V. D. Hodge, *Harmonic Integrals Associated with Algebraic Varieties*, Proc. London Math. Soc. (2) 39 (1935), no. 4, 249–271. MR 1576901

8. , *The Theory and Applications of Harmonic Integrals*, Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1941. MR 003947

9. Samuel Hosmer, *Clifford Harmonics*, ProQuest LLC, Ann Arbor, MI, 2021, Thesis (Ph.D.)–City University of New York. MR 4326632

10. Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996, Reprint of the 1969 original, A Wiley-Interscience Publication. MR 1393941

11. H. Blaine Lawson, Jr. and Marie-Louise Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR 1031992

12. M. L. Michelsohn, *Clifford and spinor cohomology of Kähler manifolds*, Amer. J. Math. 102 (1980), no. 6, 1083–1146. MR 595007

13. André Weil, *Introduction à l'étude des variétés kähleriennes*, Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind. no. 1267, Hermann, Paris, 1958. MR 0111056

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