Conditional independence testing via weighted partial copulas

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Abstract
This paper introduces the weighted partial copula function for testing conditional independence. The proposed test procedure results from these two ingredients: (i) the test statistic is an explicit Cramer-von Mises transformation of the weighted partial copula, (ii) the regions of rejection are computed using a bootstrap procedure which mimics conditional independence by generating samples from the product measure of the estimated conditional marginals. Under conditional independence, the weak convergence of the weighted partial copula process is established when the marginals are estimated using a smoothed local linear estimator. Finally, an experimental section demonstrates that the proposed test has competitive power compared to recent state-of-the-art methods such as kernel-based test.

1 Introduction
Let \((Y_1, Y_2, X)\) be a triple of real random variables. We say that \(Y_1\) and \(Y_2\) are conditionally independent given \(X\) if \(\forall (y_1, y_2, x) \in \mathbb{R}^3:\)
\[
\Pr(Y_1 \leq y_1, Y_2 \leq y_2 \mid X = x) = \Pr(Y_1 \leq y_1 \mid X = x) \Pr(Y_2 \leq y_2 \mid X = x). \tag{1}
\]
This is denoted by \(Y_1 \perp \! \! \! \perp Y_2 \mid X\) and roughly speaking, it means that for a given value of \(X\), the knowledge of \(Y_1\) does not provide any further information on \(Y_2\) (and vice versa). Determining conditional independence has become in the recent years a fundamental question in statistics and machine learning. For instance, it plays a key role in defining graphical models (Koller and Friedman, 2009; Bach and Jordan, 2003); see also Markowitz and Spang (2007) for a study specific to cellular networks. Moreover the concept of conditional independence lies at the core of sufficient dimension reduction methods (Li, 2018) and is useful to conduct variable selection in regression (Lee et al., 2016). Finally, conditional
independence is relevant in many application fields such as economy (Huber and Melly, 2015) or psychometry (Bell et al., 1988). This paper proposes new statistical tests to assess conditional independence.

The approach taken is related to the well-studied problem of (unconditional) independence testing, in which the most intuitive way to proceed is perhaps to compute a distance between the estimated joint distribution and the product of the estimated marginals (Hoeffding, 1948). Inspired by Kendall (1948), rank-based statistics have been extensively used in independence testing (Ruymgaart, 1974; Ruschendorf, 1976; Ruymgaart and van Zuijlen, 1978). Because rank-based statistics do not depend on the marginals, they have appeared as a key tool for modelling the joint distribution of random variables without being affected by their margins. This has led to the introduction of the copula function (Deheuvels, 1981), defined as the cumulative distribution function associated to the ranks. We refer to Fermanian et al. (2004); Segers (2012) for recent studies on the estimation of the copula function. The copula function, which in principle measures the dependency between random variables has been used with success in independence testing (Genest and Rémillard, 2004; Genest et al., 2006). Because the asymptotic distribution of the copula function is difficult to estimate, the related bootstrap estimate properties are of prime interest for inference (Fermanian et al., 2004; Rémillard and Scaillet, 2009; Bücher and Dette, 2010).

The conditional copula of $Y_1$ and $Y_2$ given $X$ is defined in the same way as the copula of $Y_1$ and $Y_2$ but uses the conditional distribution of $Y_1$ and $Y_2$ given $X$ in place of the joint distribution of $Y_1$ and $Y_2$. Compared to the copula, the conditional copula captures the conditional dependency between random variables and is thus useful to build conditional dependency measures (Gijbels et al., 2011). Therefore, as in the case of independence testing, the conditional copula appears to be a relevant tool for building statistical test of conditional independence. This has been pointed out as a an “interesting open issue” in (Veraverbeke et al., 2011, Section 4).

In this work, a new statistical test procedure, called the weighted partial copula test is investigated to assess conditional independence. The proposed approach follows from the use of an integrated criterion, the weighted partial copula, a function that equals 0 if and only if conditional independence holds. Given estimators of the conditional marginals of $Y_1$ and $Y_2$ given $X$, the empirical weighted partial copula is introduced to estimate the weighted partial copula and the test statistic results from a Cramer-von Mises transformation.

From a theoretical standpoint, the use of an “integrated” criterion enables to establish, in a general nonparametric framework, a convergence rate of order $n^{-1/2}$ for the empirical weighted partial copula. More precisely, by using a smoothed local linear estimator for the conditional marginals, we obtain the weak convergence of the empirical weighted partial copula rescaled by $n^{1/2}$. The rate of convergence $n^{-1/2}$, which is the same as the one derived in the (unconditional) independence test, is notable because conditional copula estimates are known (Veraverbeke et al., 2011) to converge at a slower rate, $(nh^3)^{-1/2}$ where $h$ is a smoothing parameter going to 0. Note finally that integrated criterion for testing
has been frequently used in the *conditional moment restrictions* literature (see Lavergne and Patilea (2013) and the reference therein).

Inspired by the independence testing literature (Beran et al., 2007; Kojadinovic and Holmes, 2009), the computation of the quantiles is made using a bootstrap procedure which generates bootstrap samples from the product of the marginal estimators to mimic the null hypothesis. Thanks to this bootstrap procedure, one is allowed to perform the weighted partial copula test using any marginal estimates as soon as one can generate from them.

**Related literature.** Testing for conditional independence has been considered only recently in the literature. Some of the existing approaches are based on comparing the (estimated) conditional distributions involved in the definition of conditional independence. The distributions can be compared using their conditional characteristic functions as in Su and White (2007), their conditional densities as proposed in Su and White (2008), or their conditional copulas as studied in Bouezmarni et al. (2012). Unfortunately, the estimation of these conditional quantities are subjected to the well-known curse of dimensionality, i.e., the convergence rates are badly affected by the dimension of the conditioning variable. As a consequence, the power of the previous tests rapidly deteriorates if the conditioning variable has a large dimension. Note also Bergsma (2010) that uses partial copulas to derive the test statistic. Unfortunately, partial copulas fail to capture the whole conditional distribution and lead to detect a null hypothesis much larger than conditional independence.

Other approaches rely on the characterization of conditional independence using cross-covariance operators defined on reproducing kernel Hilbert spaces (Fukumizu et al., 2004). Extending the Hilbert-Schmidt independence criterion proposed in Gretton et al. (2008), Zhang et al. (2012) defines a kernel-based conditional independence test (KCI-test) by estimating the cross-covariance operator. A surge of recent research (Doran et al., 2014; Runge, 2017; Sen et al., 2017) has focused on testing conditional independence using permutation-based tests. The seminal work of Candes et al. (2018) had led to many conditional independence tests depending on the availability of an approximation to the distribution of \( Y_1 | X \), such as the conditional permutation test (CPT) proposed in Berrett et al. (2019). In Sen et al. (2017), the authors propose to train a classifier (e.g., XGBoost) to distinguish between two samples, one is the original sample, another one is a bootstrap sample generated in a way that reflects conditional independence. According to the accuracy of the trained classifier the test rejects, or not, conditional independence. This is further referred to as the classifier based conditional independence test (CCI-test).

**Outline.** In Section 2, we introduce the weighted partial copula test and provide implementation details including the mentioned bootstrap procedure. In Section 3, we state the main theorems (weak convergence results). In Section 4, the theory is illustrated by numerical experiments. Our approach is compared to the ones described in Zhang et al. (2012) when facing simulated datasets. The proofs are given in a supplementary material file, as well an additional study dealing with functional connectivity.
2 The weighted partial copula test

2.1 Set-up and definitions

Let \( f_{XY} \) be the density function (with respect to the Lebesgue measure) of the random triple \((X, Y) = (X_1, Y_1, X_2, Y_2) \in \mathbb{R}^d \times \mathbb{R}^2\). Let \( f_X \) and \( S_X = \{ x \in \mathbb{R} : f_X(x) > 0 \} \) denote the density and the support of \( X \), respectively. The conditional cumulative distribution function of \( Y \) given \( X = x \) is given by \( y \mapsto H(y \mid x) \) for \( x \in S_X \). The generalized inverse of a univariate distribution function \( F \) is defined as \( F^{-1}(u) = \inf \{ y \in \mathbb{R} : F(y) \geq u \} \), for all \( u \in [0, 1] \), with the convention that \( \inf \emptyset = +\infty \). Since \( H(y \mid x) \) is a continuous bivariate cumulative distribution function, its copula is given by the function

\[
C(u \mid x) = H \left( F_1^{-1}(u_1 \mid x), F_2^{-1}(u_2 \mid x) \mid x \right),
\]

for \( u = (u_1, u_2) \in [0, 1]^2 \) and almost every \( x \in S_X \), where \( F_1(\cdot \mid x) \) and \( F_2(\cdot \mid x) \) are the margins of \( H(\cdot \mid x) \). We are interested in testing the null hypothesis that \( Y_1 \) and \( Y_2 \) are conditionally independent given \( X \), that is,

\[
H_0 : Y_1 \perp \perp Y_2 \mid X.
\]

By definition (Dawid, 1979), \( H_0 \) is equivalent to \( H(y \mid x) = F_1(y_1 \mid x)F_2(y_2 \mid x) \), for every \( y \in \mathbb{R}^2 \) and almost every \( x \in S_X \). Using the conditional copula introduced before, it follows that

\[
H_0 \iff C(u \mid x) = u_1u_2, \quad \text{for every } u \in [0, 1]^2, \text{ and almost every } x \in S_X.
\]

Let \( w : \mathbb{R}^d \to \mathbb{R} \) be a measurable function. The \textit{weighted partial copula} is given by, for every \( u \in [0, 1]^2 \) and almost every \( t \in \mathbb{R} \),

\[
W(u, t) = E \left[ (C(u \mid X) - u_1u_2)w(t - X) \right].
\]

The proposed test follows from the observation, that \( H_0 \) is satisfied if and only if the function \( W \) is identically equal to 0 under a certain (mild) condition on \( w \). This is presented in the following lemma whose proof is given in the supplementary material.

\textbf{Lemma 1.} Suppose that \( w : \mathbb{R}^d \to \mathbb{R} \) is integrable with respect to the Lebesgue measure and with a Fourier transform being non-zero almost everywhere, then \( H_0 \) is equivalent to \( W(u, t) = 0 \), for every \( u \in [0, 1]^2 \) and almost every \( t \in \mathbb{R} \).

2.2 The test statistic

In the following, we define a general estimator of \( W \) relying on some empirical copula construction that works for any estimate of the marginals \( F_1 \) and \( F_2 \) (see Section 2.5 for a typical example). That is, we first compute sample based observations of \( F_j(Y_j \mid X) \), \( j = 1, 2 \), by estimating each marginal \( F_j \). Those are usually called pseudo-observations. Second we define an estimate of \( W \).
based on the ranks of the pseudo-observation. For the sake of generality, the estimator used for the conditional marginals is left unspecified in the subsequent development.

Let \((X_i, Y_{i1}, Y_{i2})\), for \(i \in \{1, \ldots, n\}\), be independent and identically distributed random vectors, with common distribution equal to the one of \((X, Y_1, Y_2)\). Estimate the conditional margins in some way, producing random functions \(F_{n,j}(r|x)\), \(j = 1, 2\), and then proceed with the pseudo-observations \(\hat{U}_{ij} = \hat{F}_{n,j}(Y_{ij} | X_i)\). Let \(G_{n,j}\), for \(j \in \{1, 2\}\), be the empirical distribution function of the pseudo-observations \((\hat{U}_{1j}, \ldots, \hat{U}_{nj})\), i.e. \(G_{n,j}(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\hat{U}_{ij} \leq u\}}\), for \(u \in [0, 1]\). From a conditioning argument, the weighted partial copula is given by

\[
W(u, t) = E[\mathbb{1}_{\{F_1(Y_1|X) \leq u_1\}} \mathbb{1}_{\{F_2(Y_2|X) \leq u_2\}} - u_1 u_2] w(t - X)]. \tag{2}
\]

The previous expression suggests the introduction of following so-called the empirical partial copula process, given by

\[
\hat{W}_n(u, t) = \frac{1}{n} \sum_{i=1}^{n} \left( \mathbb{1}_{\{\hat{U}_{i1} \leq \hat{G}_{n,1}(u_1)\}} \mathbb{1}_{\{\hat{U}_{i2} \leq \hat{G}_{n,2}(u_2)\}} - u_1 u_2 \right) w(t - X). \tag{3}
\]

The use of the transform \(\hat{G}_{n,1}\) and \(\hat{G}_{n,2}\) implies that \(\hat{W}_n\) depends on \(\hat{U}_{i1}\) and \(\hat{U}_{i2}\) only through their ranks. Indeed, because \(\hat{G}_{n,j}\) is a càdlàg function with jumps \(1/n\) at each \(\hat{U}_{ij}\), it holds that \(\hat{U}_{ij} \leq \hat{G}_{n,j}(u_j)\) is equivalent to \((\hat{R}_{ij} - 1)/n < u_j\), where \(\hat{R}_{ij} = n \hat{G}_{n,j}(U_{ij})\) is the rank of \(U_{ij}\) among the sample \((\hat{U}_{1j}, \ldots, \hat{U}_{nj})\).

Hence, we have

\[
\hat{W}_n(u, t) = \frac{1}{n} \sum_{i=1}^{n} \left( \mathbb{1}_{\{(\hat{R}_{i1} - 1) < nu_1\}} \mathbb{1}_{\{(\hat{R}_{i2} - 1) < nu_2\}} - u_1 u_2 \right) w(t - X). \tag{4}
\]

The test statistic is given by

\[
\hat{T}_n = \int_{[0,1]^2 \times \mathbb{R}^d} \hat{W}_n(u, t)^2 \, du \, dt. \tag{5}
\]

Remark 1. The test statistics \(\hat{T}_n\) is of Cramér-von Mises type, as opposed to the Kolmogorov-Smirnov type (which would be defined taking the sup instead of integrating). In Genest and Rémillard (2004) these two types of statistics are introduced in the context of (unconditional) independence testing. In our context, the Cramér-von Mises type is preferred over the Kolmogorov-Smirnov for practical reasons. Indeed, as we will see in the next section, a closed formula exists for \(\hat{T}_n\).

### 2.3 Computation of the statistic

The following lemma provides a closed-formula for the test statistics \(\hat{T}_n\). The proof is left in the supplementary material.
Lemma 2. If $w : \mathbb{R}^d \rightarrow \mathbb{R}$ is an integrable function, then
\[ \hat{T}_n = n^{-2} \sum_{1 \leq i,j \leq n} M(\hat{G}_i, \hat{G}_j) w^*(X_i - X_j) \]

where $\hat{G}_i = (\hat{R}_{i1} - 1, \hat{R}_{i2} - 1)/n$, $w^* = w \ast w_s$ with $w_s(x) = w(-x)$, and
\[ M(u, v) = (1 - u_1 \lor v_1)(1 - u_2 \lor v_2) - \frac{1}{4} (1 - u_1^2)(1 - u_2^2) \]
\[ + \frac{1}{4} (1 - v_1^2)(1 - v_2^2) + \frac{1}{5}. \]

Remark 2. The function $w$ is left unspecified for the sake of generality. Examples include $w(t) = \exp(-t^2)$, $w(t) = \mathbb{1}_{|t| \leq 1}$ and other popular kernel functions such as the Epanechnikov kernel. In the simulations, we consider the Gaussian kernel as in this case, $w^*$ remains Gaussian. In line with the result stated in Proposition 1, empirical evidences suggest that it does not have a leading role in the performance of the test.

2.4 Bootstrap approximation

To compute the rejection regions of the test, we propose a bootstrap approach to generate new samples in a way that reflects the null hypothesis even when $H_0$ is not realized in the original sample. This has been notified as a guideline for bootstrap hypothesis testing in (Hall and Wilson, 1991) and it enables, in practice, to control for the level of the test and to obtain a sufficiently large power.

The proposed bootstrap follows from the estimated conditional marginals of $Y_1|X$ and $Y_2|X$, respectively $\hat{F}_{n,1}$ and $\hat{F}_{n,2}$, and from the estimated distribution of $X$, denoted by $\hat{F}_n$. First choose $X^*$ uniformly over the $(X_i)_{i=1,...,n}$, that is, $X^* \sim \hat{F}_n$. Then generate
\[ Y_1^* \sim \hat{F}_{n,1}(|X^*|), \quad \text{and} \quad Y_2^* \sim \hat{F}_{n,2}(|X^*|). \]

Execute the previous steps $n$ times until obtaining an independent and identically distributed bootstrap sample of size $n$. Denote by $(X_1^*, Y_{11}^*, Y_{12}^*), \ldots, Y_{2n}^*$ the obtained sample. Compute the test statistic based on this sample. We repeat this $B$ times and obtain $B$ realizations of the statistic under $H_0$, denoted by $(T_{n,1}^*, \ldots, T_{n,B}^*)$. Now define the cumulative distribution function of the bootstrap statistics $t \mapsto (1/B) \sum_{b=1}^B \mathbb{1}_{(T_{n,b}^* \leq t)}$, and denote by $\xi_n(\alpha)$ its quantile of level $\alpha \in (0, 1)$. The weighted partial copula test statistic with level $\alpha$ rejects $H_0$ as soon as $\hat{T}_n > \xi_n(\alpha)$.

2.5 A generic example using Nadaraya-Watson estimator

In this section, the aim is to illustrate the proposed test procedure when using the classical Nadaraya-Watson estimator for the margins $F_j$, $j \in \{1, 2\}$ when $d = 1$. 

Nadaraya-Watson estimator. Let $K : \mathbb{R} \to [0, \infty)$ be the standard Gaussian density function on $\mathbb{R}$. For $x \in \mathbb{R}$ and $h > 0$, put $K_h(x) = h^{-1} K(h^{-1} x)$. For $j \in \{1, 2\}$, the Nadaraya-Watson estimator of $F_j(\cdot | x)$ is given by

$$
\hat{F}_{n,j}(y | x) = \frac{\sum_{i=1}^{n} I(Y_{ij} \leq y) K_{b_j}(x - X_{ij})}{\sum_{i=1}^{n} K_{b_j}(x - X_{ij})}, \quad (y \in \mathbb{R}).
$$

The choice of the bandwidths $b_1$ and $b_2$ will be discussed below.

Cross-validation selection of the bandwidth. The bandwidths $b_1$ and $b_2$ have a critical effect on the shape of the resulting estimates, and thus on the performance of our test. Indeed, these estimates of the margins $\hat{F}_j(y_j | x)$ for $j \in \{1, 2\}$ are used in the computation of the test statistic $T_n$ as well as in the bootstrap procedure to simulate under the null (see Section 2.4). The idea is to assess the performance of each regression model $Y_1 | X$ and $Y_2 | X$ and to choose each bandwidth $b_1$ and $b_2$ accordingly. We randomly divide the set of observations into $K$ groups of nearly equal size. These groups are denoted by $\{I_k\}_{k=1}^{K}$. Define $\text{MSE}_{j,k}(b) = (1/|I_k|) \sum_{i \in I_k} (Y_{ij} - \hat{\theta}_{j,b}^{(k)}(X_i))^2$, where $\hat{\theta}_{j,b}^{(k)}$ stands for the Nadaraya-Watson estimate of the regression $Y_j | X$ computed on $\{1, \ldots, n\} \setminus I_k$ with bandwidth $b$. We choose $b_{n,j}$ as the minimizer of $(1/K) \sum_{k=1}^{K} \text{MSE}_{j,k}(b)$ over $b$.

The success of the approach in distinguishing $H_0$ from its contrary is illustrated on Figure 1 considering the generic post-nonlinear noise model as described in the supplementary material.

Remark 3. Though this example has been carried out using the Nadaraya-Watson estimate of the marginal distributions, other approaches to estimate the marginals can be used to conduct the weighted partial copula test. The only restriction on the employed marginal estimates comes from the bootstrap procedure in which the ability to generate according to the margins is required. For instance a $k$-nearest neighbours approach shall be considered in Section 4.
3 Weak Convergence

3.1 Smooth estimator of the margins

The theoretical results are provided in a general nonparametric setting, using a smoothed version of the local linear estimator (Fan and Gijbels, 1996) of the conditional marginals when \( d = 1 \) (see Remark 5 below). This estimate has been introduced in Portier and Segers (2018) and is a natural extension of the Nadaraya-Watson estimator (defined in the previous section). Such a nonparametric approach will result in mild assumptions on the distribution of \((X, Y_1, Y_2)\). Let \( K : \mathbb{R} \to [0, \infty) \) and \( L : \mathbb{R} \to [0, \infty) \) be two kernel functions, i.e., nonnegative, symmetric functions integrating to unity. Let \((b_{n,j})_{n \geq 1}\) and \((h_{n,j})_{n \geq 1}\), for \( j = 1, 2 \), be four bandwidth sequences that tend to 0 as \( n \to \infty \). For \((y, Y) \in \mathbb{R}^2 \) and \( h > 0 \), put

\[
\varphi_h(y, Y) = \int_{-\infty}^{y} L_h(t - Y) \, dt. \tag{7}
\]

with \( L_h(y) = h^{-1} L(h^{-1} y) \). For \( j \in \{1, 2\} \), we introduce the smoothed local linear estimator of \( F_j(y_j|x) \) defined by

\[
\hat{F}_{n,j}(y_j|x) = \hat{a}_{n,j}, \tag{8}
\]

where \( \hat{a}_{n,j} \) is the first component of the random pair

\[
(\hat{a}_{n,j}, \hat{b}_{n,j}) = \arg \min_{(a,b) \in \mathbb{R}^2} \sum_{i=1}^{n} \left\{ \varphi_{h_{n,j}}(y_j, Y_{ij}) - a - b(X_i - x) \right\}^2 \left( \frac{x - X_i}{b_{n,j}} \right), \tag{9}
\]

where \( \varphi_h \) in (7) serves to smooth the indicator function \( y \mapsto \mathbb{I}_{\{Y \leq y\}} \). The kernels \( K \) and \( L \) do not have the same role: \( L \) is concerned with “smoothing” over \( Y_1 \) and \( Y_2 \) whereas \( K \) “localises” the variable \( X \) at \( x \in S_X \). For this reason, we purposefully use two different bandwidth sequences \((b_{n,j})_{n \geq 1}\) and \((h_{n,j})_{n \geq 1}\). We shall see that the conditions on the bandwidth \( h_{n,j} \) for the \( y \)-directions are weaker than the ones for the bandwidth \( b_{n,j} \) for the \( x \)-direction. The assumptions related to the two kernels and bandwidth sequences are stated in (G3) and (G4) below. Note that if the previous optimization would be carried out only over \( a \) we would recover the Nadaraya-Watson estimator with a smoothed indicator function.

3.2 Weak convergence of the weigh partial copula

We rely on the following Hölder regularity class. Let \( \delta \in (0, 1) \), \( k \in \mathbb{N} \), and \( M > 0 \) be scalars and let \( S \subset \mathbb{R} \) be non-empty, open and convex. Let \( \mathcal{C}_{k+\delta,M}(S) \) be the space of functions \( S \to \mathbb{R} \) that are \( k \) times differentiable and whose derivatives (including the zero-th derivative, that is, the function itself) are uniformly bounded by \( M \) and such that every mixed partial derivative of order
\[ l \leq k, \text{ say } f^{(l)}, \text{ satisfies the Hölder condition} \]
\[
\sup_{z \neq \tilde{z}} \frac{|f^{(l)}(z) - f^{(l)}(\tilde{z})|}{|z - \tilde{z}|^{\alpha}} \leq M, \quad (10)
\]

where \( | \cdot | \) in the denominator denotes the Euclidean norm. In particular, \( C_{1,M}(\mathbb{R}) \) is the space of Lipschitz functions \( \mathbb{R} \to \mathbb{R} \) bounded by \( M \) and with Lipschitz constant bounded by \( M \).

(G1) The law \( P \) admits a density \( f_{X,Y} \) on \( S_X \times \mathbb{R}^2 \) such that \( S_X \) is a nonempty, bounded, open interval. For some \( M > 0 \) and \( \delta > 0 \), the functions \( F_1(\cdot \, | \cdot) \) and \( F_2(\cdot \, | \cdot) \) belong to \( C_{3+\delta,M}(\mathbb{R} \times S_X) \) and the function \( f_X \) belongs to \( C_{2,M}(S_X) \). There exists \( b > 0 \) such that \( f_X(x) \geq b \) for every \( x \in S_X \). For any \( j \in \{1, 2\} \) and any \( \gamma \in (0, 1/2) \), there exists \( b_n \) such that, for every \( y_j \in [F_j^{-}(\gamma|x), F_j^{-}(1 - \gamma|x)] \) and every \( x \in S_X \), we have \( f_j(y_j|x) \geq b_\gamma \).

(G2) The function \( w : \mathbb{R} \to \mathbb{R} \) is of bounded variation.

(G3) The kernels \( K \) and \( L \) are bounded, nonnegative, symmetric functions on \( \mathbb{R} \), supported on \((-1, 1)\), and such that \( \int L(u) \, du = \int K(u) \, du = 1 \). The function \( L \) is continuously differentiable on \( \mathbb{R} \) and its derivative is a bounded real function of bounded variation. The function \( K \) is twice continuously differentiable on \( \mathbb{R} \) and its second-order derivative is a bounded real function of bounded variation.

(G4) There exists \( \alpha > 0 \) such that for any \( j = 1, 2 \), the bandwidth sequences \( b_{n,j} > 0 \) and \( h_{n,j} > 0 \) satisfy, as \( n \to \infty \),
\[
\begin{align*}
nb_{n,j}^8 & \to 0, & nh_{n,j}^8 & \to 0, & b_{n,j}^{1-\alpha/2}h_{n,j}^2 & \to 0, \\
\frac{nb_{n,j}^{3+\alpha}}{|\log b_{n,j}|} & \to \infty, & \frac{nh_{n,j}^{1+\alpha}}{|\log h_{n,j}|} & \to \infty.
\end{align*}
\]

Let \( \mathbb{P} \) denote the probability measure on the underlying probability space associated to the whole sequence \( (X_i, Y_i)_{i=1,2,...} \). Let \( \ell^\infty(T) \) denote the space of bounded real functions on the set \( T \), the space being equipped with the supremum distance. Define \( U_{i1} = F_1(Y_{i1}|X_i) \), \( U_{i2} = F_2(Y_{i2}|X_i) \), for any \( i = 1, \ldots, n \), and
\[
\hat{W}_n(u, t) = \hat{Z}_n(u, t) - (f_X * w)(t)(u_1 \hat{Z}_{n,2}(u_2) + u_2 \hat{Z}_{n,1}(u_1)), \quad (11)
\]
for any \( u \in [0, 1]^2 \), \( t \in \mathbb{R} \), with
\[
\hat{Z}_n(u, t) = n^{-1} \sum_{i=1}^n \left\{ w(t - X_i)(1_{\{U_{i1} \leq u_1, U_{i2} \leq u_2\}} - u_1 u_2) \right\}, \quad (12)
\]
and \( \hat{Z}_{n,j}(u_j) = n^{-1} \sum_{i=1}^n \left\{ 1_{\{U_{ij} \leq u_j\}} - u_j \right\} \). Our main result is now stated. Its proof is provided in the supplementary material.
Theorem 3. Assume that (G1), (G2), (G3) and (G4) hold. If $H_0$ holds, then for any $\gamma \in (0, 1/2)$, we have when $n \to \infty$

$$\sup_{u \in [\gamma, 1-\gamma]^2, t \in \mathbb{R}} |\hat{W}_n(u, t) - \tilde{W}_n(u, t)| = o_p(n^{-1/2}).$$

In addition, the process $\left\{n^{1/2}\hat{W}_n(u, w)\right\}_{u \in [\gamma, 1-\gamma]^2, t \in \mathbb{R}}$ converges weakly in $\ell_\infty([\gamma, 1-\gamma]^2 \times \mathbb{R})$ to a certain Gaussian process.

Remark 4. Theorem 3 is a nontrivial extension of Theorem 2 in Portier and Segers (2018). By taking $w_t = 1$ we would recover their result.

Remark 5. The approach employed follows from an approximation of the process $W_n$ by an oracle version of $W_n$ where the estimated marginals are replaced by the true ones. In doing this, a crucial step consists in an embedding of some functions class involving estimated conditional quantiles into a Donsker class (van der Vaart and Wellner, 1996). First, because the estimated quantiles are difficult to control near the boundary of $[0, 1]$, we need to restrict the proof to the interval $[\gamma, 1-\gamma]^2$. We believe that the extension to the whole interval is an interesting avenue for further research. Second, the regularity properties of the local linear estimate defined in (8) are essential to obtain that the resulting quantile functions are sufficiently smooth to be contained in a Donsker class. Third, as noticed in Portier and Segers (2018), the extension to higher dimensions, though feasible, is not straightforward and represents an avenue for further research. In the case $d = 1$, covered by Theorem 3, the rate of convergence, $n^{-1/2}$, is not affected by the size of the different bandwidths. We conjecture that this remains true in multiple dimensions with the same rate of order $n^{-1/2}$.

As a corollary of the previous weak convergence result, we obtain (invoking the continuous mapping theorem) the weak convergence, under $H_0$, of a slightly modified version of $\hat{T}_n$.

Corollary 4. Assume that (G1), (G2), (G3) and (G4) hold. If $H_0$ holds, then for any $\gamma \in (0, 1/2)$ and any finite measure $\mu$ on $\mathbb{R}$, we have that

$$n \int_{[\gamma, 1-\gamma]^2 \times \mathbb{R}} \hat{W}_n(u, t)^2 dud\mu(t)$$

converges weakly to a tight nonnegative random variable as $n \to \infty$.

4 Numerical experiments

In this section, we apply the proposed copula test to synthetic data to evaluate its performance based on the nominal level and the power of the test. We compare it with the KCI-test Zhang et al. (2012) presented in the related literature section. Since the level $\alpha$ is hard to set for the CCI-test of Sen et al. (2017), this approach will only be considered when the proportions of correct decision will be computed.
Figure 2: Simulation results for the linear model. Figures 2(a), 2(b) show the probability of acceptance (i.e. the type II error rate), plotted against the constant \(a\) and \(n\). Figure 2(c) shows the probability of rejection (type I error) against \(d\). The plots show the average probabilities with standard error bars.

In all the experiments, the function \(w\) is a Gaussian kernel given by \(w(t) = \exp(-t^2)\) and the estimate of the marginals is the \(k\)-nearest neighbors version of (6) using the cross-validation approach of Section 2.5 to tune \(k\).

We use two datasets, respectively a linear model and a probabilistic graphical model for testing causality detection. We also test our method on the post-nonlinear noise model in the supplementary material. Finally, we apply our test in a practical setting, using the movie watching based brain development dataset Richardson et al. (2018). This last test can also be found in the supplementary material. In all of our simulations we set \(\alpha = 5\%\) as the desired type-I error rate. All results are averaged over 300 trials, and we used \(B = 200\) bootstrap realizations. The average CPU time taken by the tests in competition for \(n = 1500, d = 1\) copulas: 18.3 s , KCI-test: 17.7 s, CCI-test: 19.7 s.

### 4.1 Linear model

Consider the joint distribution given by \(Y_1 = X^T\beta_1 + \epsilon_1, Y_2 = X^T\beta_2 + \epsilon_2\), where \(X \sim \mathcal{N}(0, I_d), \beta_1\) and \(\beta_2\) are two constant vectors of \([0, 1]^d\), and \(\epsilon_1, \epsilon_2\) are two standard Gaussian variables with \(\text{Cov}(\epsilon_1, \epsilon_2) = a\). When \(a = 0\), \(\mathcal{H}_0\) is true. It is false when \(a > 0\). We examine the effect of the constant \(a > 0\), and the size of the dataset \(n\) on the type-II error rate. We also examine the type-I errors when the dimension of the variable \(X\) increases, in a setting where the null hypothesis \(\mathcal{H}_0\) holds. Figure 2 shows the attractive performance of our test compared to the KCI-test. Notably, we can see that in high dimensions, our test is more accurate with respect to the level set \(\alpha\) than the KCI-test.
4.2 Causality Discovery

Hereinafter we consider a particular type of DAG called "Latent cause" model. To draw samples from the alternative hypothesis, we break the conditional independence by adding an edge between the nodes $Y_1$ and $Y_2$. For the "Latent cause" model of interest we have $X \sim \mathcal{N}(0, 1)$, $Y_1 | X \sim \mathcal{N}(X, 1)$, and $Y_2 | X, Y_1 \sim \mathcal{N}(X + aY_1, 1)$. It is easy to verify that $H_0$ is true when $a = 0$, and false otherwise. It can be seen in Figure 3 that for large sample size $n$, our test outperforms the ones in competition. Furthermore, our test is slightly more powerful than the KCI-test across a range of values of $a$ but overall shows fairly similar performance.
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Appendix A Additional Numerical Experiments

A.1 Post-Nonlinear Noise

We apply the proposed test on the post-nonlinear noise model. We first examine the effect of the constant $a > 0$ on the probability of type-I and type-II error of our test. The results are shown in Figure 4. As expected, larger values of $a$ yields lower type-II error probabilities. For every value $a$, we observe that the type-I error probability is closed to $\alpha$. The performances of the tests are also compared when the sample size $n$ changes. The role of $n$ is critical and the results are shown in Figure 5. We note that the type-I error probability is again closed to $\alpha$ and that the type-II quickly vanishes when $n$ increases. In this experiment, the proposed procedure outperforms the KCI-test.
A.2 Classification of age groups using functional connectivity

In this paragraph, we apply our test in a practical setting, using the movie watching based brain development dataset Richardson et al. (2018) obtained from the OpenNeuro database\textsuperscript{1}. The dataset consists in 50 patients (10 adults and 40 children). The fMRI data consists in measuring the brain activity in 39 Region of Interest (ROI). For every patient, 168 measurements are provided for each ROI. We denote for $j \in \{1, \ldots, 39\}$ by $X_j$ the variable that represents the $j^{th}$ region signal value. Given two ROI $j$ and $j'$, we seek to test the null hypothesis that $X_j$ and $X_{j'}$ are conditionally independent given $X_{\{j,j'\}}$. The decisions given by our test allow us to obtain a map of connections between all the ROI, called connectome, given in Figure 6. In this figure, a line is drawn between two ROI whenever our test rejects the null for these two ROI. Here, due to the high dimension of the conditional variable, the margins are no longer estimated using a Gaussian kernel as in Section 2.5, but using a $k$-nearest neighbours approach. For a given $x$, the mapping $F_{n,j}(y|x)$ is estimated for every $y \in \mathbb{R}$ as the proportion of samples $i$ amongst the $k$-nearest nearest neighbours of $x$ which satisfy $Y_{ij} \leq y$. The integer $k$ is select by cross-validation.

As a sanity check, our connectome is used as an input feature of a classifier (Linear Support Vector Classifier (SVC)) in order to distinguish children from adults. We estimate the classification accuracy of our classifier using $k$-fold. The obtained accuracy is 97.4\% which is close to the so-called tangent method (98.9\%) which is known to be fitted for this task Dadi et al. (2019).

Appendix B Proofs of the basic lemmas of Section 2

B.1 Proof of Lemma 1

The “only if” part is obvious. Suppose that the function $W = 0$ and let $u \in [0,1]^2$. Define $g(x) = C(u \mid x) - u_1 u_2 f_X(x)$. We have $(g \ast w) = 0$, a.e. on $\mathbb{R}^d$, where $\ast$ stands for the standard convolution product with respect to the Lebesgue measure. Applying the Fourier transform gives that $\mathcal{F}(g) \mathcal{F}(w) = 0$ which, by assumption, yields $\mathcal{F}(g) = 0$.

\textsuperscript{1}Accession number ds000228.
By the Fourier inversion theorem we obtain that $g = 0$ a.e. on $\mathbb{R}^d$. That is for any $u \in [0, 1]^2$ and any $x \in S_X$, $C(u \mid x) = u_1 u_2$. 

\[ \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{1j} < u_1 \hat{G}_{2j} < u_2 \text{du} = \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{2j} < u_2 \text{du}. \]

Thus we obtain

\[ \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{1j} < u_1 \hat{G}_{2j} < u_2 \text{du} = \left( \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{j1} < u_1 \hat{G}_{j2} < u_2 \text{du} \right) \left( \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{1j} < u_1 \hat{G}_{2j} < u_2 \text{du} \right). \]

Now let derive the second integral term of the right hand side, the third term will follow directly.

\[ \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{1j} < u_1 \hat{G}_{2j} < u_2 \text{du} = \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{1j} < u_1 \hat{G}_{2j} < u_2 \text{du} \]

\[ = \frac{1}{4} \left( 1 - \hat{G}_{i1}^2 \right) \left( 1 - \hat{G}_{i2}^2 \right). \]

By combining (13) and (14) we obtain the desired result. 

\[ \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{1j} < u_1 \hat{G}_{2j} < u_2 \text{du} = \left( \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{j1} < u_1 \hat{G}_{j2} < u_2 \text{du} \right) \left( \int_{[0,1]^2} \hat{G}_i \left( \hat{G}_{i1} < u \right) \hat{G}_{1j} < u_1 \hat{G}_{2j} < u_2 \text{du} \right). \]

\[ \text{Appendix C Proof of Theorem 3} \]

We use notation from empirical process theory. Let $P_n = n^{-1} \sum_{i=1}^{n} \delta(X_i, Y_i)$ denote the empirical measure. For a function $f$ and a probability measure $Q$, write $Q f = \int f \text{d}Q$. The empirical process is

\[ \mathbb{G}_n = n^{1/2}(P_n - P). \]

For any pair of cumulative distribution functions $F_1$ and $F_2$ on $\mathbb{R}$, put $F(y) = (F_1(y_1), F_2(y_2))$ for $y = (y_1, y_2) \in \mathbb{R}^2$ and $F^- (u) = (F^-_1(u_1), F^-_2(u_2))$ for $u = (u_1, u_2) \in [0, 1]^2$. 

\[ \]
C.1 Sketch of the proof

We introduce an oracle copula estimator, defined as the empirical copula based on the unobservable random pairs \((F_1(Y_{ij}|X_i), F_1(Y_{ij}|X_i))\), \(i \in \{1, \ldots, n\}\). Let \(G_{n,j}^{(or)}\) be the empirical distribution function of the uniform random variables \(F_j(Y_{ij}|X_i)\), \(i \in \{1, \ldots, n\}\), i.e.,

\[
\hat{G}_{n,j}^{(or)}(u_j) = P_n\{1_{F_j \leq u_j}\}, \quad u_j \in [0, 1].
\]

Let \(\hat{G}_{n,j}^{(or)}\) be its generalized inverse. The oracle estimator of \(W\) is then

\[
\hat{W}_n^{(or)}(u, t) = P_n\left\{w(1_{F \leq \hat{G}_{n,j}^{(or)}(u)} - u_1 u_2)\right\},
\]

with \(w(x) = w(t - x)\). A crucial result is that the processes \(\hat{W}_n\) and \(\hat{W}_n^{(or)}\) are asymptotically equivalent as stated in the following lemma.

**Lemma 5.** Assume that \((G1), (G2), (G3) and (G4) hold. If \(H_0\) holds, then for any \(\gamma \in (0, 1/2)\), we have when \(n \to \infty\),

\[
\sup_{u \in [\gamma, 1-\gamma]^2, t \in \mathbb{R}} n^{1/2}\{\hat{W}_n(u, t) - \hat{W}_n^{(or)}(u, t)\} = o_p(1). \tag{15}
\]

Using the notation from empirical process theory, introduced below, we have

\[
\hat{Z}_n(u, t) = P_n\left\{w_t(1_{F \leq u} - u_1 u_2)\right\}, \quad u \in [0, 1]^2, t \in \mathbb{R},
\]

\[
\hat{Z}_{n,j}(u_j) = \hat{G}_{n,j}^{(or)}(u_j) - u_j, \quad u_j \in [0, 1],
\]

\[
\hat{W}_n(u, t) = \hat{Z}_n(u, t) - P\{w_t\left(u_1\hat{Z}_{n,2}(u_2) + u_2\hat{Z}_{n,1}(u_1)\right)\}, \quad u \in [0, 1]^2, t \in \mathbb{R}.
\]

A second crucial result is the following one, where it is shown that \(\hat{W}_n^{(or)}\) is asymptotically equivalent to \(\hat{W}_n\).

**Lemma 6.** Assume that \((G1), (G2), (G3) and (G4) hold. If \(H_0\) holds, we have when \(n \to \infty\),

\[
\sup_{u \in [0, 1]^2, t \in \mathbb{R}} \left|\hat{W}_n^{(or)}(u, t) - \hat{W}_n(u, t)\right| = o_p(n^{-1/2})
\]

Based on Lemma 5 and 6, we deduce that

\[
\sup_{u \in [\gamma, 1-\gamma]^2, t \in \mathbb{R}} \left|\hat{W}_n(u, t) - \hat{W}_n(u, t)\right| = o_p(n^{-1/2}).
\]

Invoking the Slutsky’s Lemma, the process \(\{\hat{W}_n(u, t)\}_{u \in [\gamma, 1-\gamma]^2, t \in \mathbb{R}}\) and \(\{\hat{W}_n(u, t)\}_{u \in [\gamma, 1-\gamma]^2, t \in \mathbb{R}}\) have the same weak limit in \(L^\infty([\gamma, 1-\gamma]^2 \times \mathbb{R})\). Now note that \(\{x \mapsto w_t(x) : t \in \mathbb{R}\}\) is a Euclidean or VC class with constant envelop \(C_w = \sup_{x \in \mathbb{R}} |w(x)|\) (Nolan and Pollard, 1987, Lemma 22, (ii)), i.e., the covering numbers are polynomials. Moreover, the class of indicator functions is also Euclidean (van der Vaart and Wellner, 1996, Example 2.5.4). This implies that both classes have finite entropy integrals and therefore are Donsker (van der Vaart and Wellner, 1996, Chapter 2.1, equation (2.1.7)). Using the preservation of the Donsker property through products and sums (van der Vaart and Wellner, 1996, Example 2.10.7 and 2.10.8), the class \(\{(y, x) \mapsto \hat{w}_t(x) 1_{F(y,x) \leq u} : t \in \mathbb{R}, u \in [0, 1]^2\}\) is Donsker. As a result, the process \(\{\hat{W}_n(u, t)\}_{u \in [\gamma, 1-\gamma]^2, t \in \mathbb{R}}\) converges weakly to a tight Gaussian process in \(L^\infty([\gamma, 1-\gamma]^2 \times \mathbb{R})\).
C.2 Proof of Lemma 5

Our proof is adapted from the proof of Theorem 1 in Portier and Segers (2018). For the sake of clarity, we start by recalling some of the results established in Portier and Segers (2018) that will be used further in our proof. Apart from this, the proof is self-consistent.

**Fact 1.** On a sequence of events whose probabilities tend to one, it holds that for every $u_j \in [\gamma, 1 - \gamma]$ and every $(y_j, x) \in \mathbb{R} \times S_X$,

$$F_{n,j}(y_j|x) \leq u_j \Leftrightarrow y_j \leq F_{n,j}^{-1}(u_j|x).$$

This is shown page 170 in Portier and Segers (2018).

For $u_j \in [\gamma, 1 - \gamma]$, $x \in S_X$, and $j \in \{1, 2\}$, define

$$\hat{\Delta}_{n,j}(u_j|x) = F_j(\hat{F}_{n,j}^{-1}(\hat{G}_{n,j}^{-1}(u_j)|x)|x) - \hat{G}_{n,j}^{-1}(u_j).$$

**Fact 2.** We have for any $j = 1, 2$,

$$\sup_{u_j \in [\gamma, 1 - \gamma]} \left| n^{1/2} \int \hat{\Delta}_{n,j}(u_j|x) f_x(x) dx \right| = o_P(1).$$

This is shown page 171 in Portier and Segers (2018).

**Fact 3.** As established in (Portier and Segers, 2018, page 172), for each $j = 1, 2$,

$$\left\{ x \mapsto \hat{F}_{n,j}^{-1}(\hat{G}_{n,j}^{-1}(u_j)|x) : u_j \in [\gamma, 1 - \gamma] \right\} \subset C_{1+\delta_1,M_1}(S_X),$$

$$\left\{ x \mapsto \hat{F}_{j}^{-1}(\hat{G}_{n,j}^{-1}(u_j)|x) : u_j \in [\gamma, 1 - \gamma] \right\} \subset C_{1+\delta,M_2}(S_X),$$

with probability going to 1.

We are based on Theorem 2.1 stated in (van der Vaart and Wellner, 2007) and reported below; for a proof see for instance van der Vaart and Wellner (1996, Lemma 3.3.5).

Let $\xi_1, \xi_2, \ldots$ be independent and identically distributed random elements of a measurable space $(X, \mathcal{A})$ and with common distribution equal to $P$. Let $\mathbb{P}$ denote the probability measure on the probability space on which the sequence $\xi_1, \xi_2, \ldots$ is defined.

Let $\xi_n$ be the empirical process associated to the sample $\xi_1, \ldots, \xi_n$. Let $\mathcal{E}$ and $\mathcal{V}$ be sets and let $\{m_{v,\eta} : v \in \mathcal{V}, \eta \in \mathcal{E}\}$ be a collection of real-valued, measurable functions on $X$.

**Theorem 7** (Theorem 2.1 in (van der Vaart and Wellner, 2007)). Let $\hat{\eta}_n$ be random elements in $\mathcal{E}$. Suppose there exist $\eta_0 \in \mathcal{E}$ and $\mathcal{E}_0 \subset \mathcal{E}$ such that the following three conditions hold:

(i) $\sup_{v \in \mathcal{V}} P(m_{v,\eta_0} - m_{v,\eta_0})^2 = o_P(1)$ as $n \to \infty$;

(ii) $P(\hat{\eta}_n \in \mathcal{E}_0) \to 1$ as $n \to \infty$;

(iii) $\{m_{v,\eta} - m_{v,\eta_0} : v \in \mathcal{V}, \eta \in \mathcal{E}_0\}$ is $P$-Donsker.

Then it holds that

$$\sup_{v \in \mathcal{V}} |\hat{G}_{\xi,n}(m_{v,\hat{\eta}_n} - m_{v,\eta_0})| = o_P(1), \quad n \to \infty.$$
Theorem 7. Therefore we apply Theorem 7 with $w \in \{1, \ldots, v\}$. Moreover, the quantities $\eta_n$ and $\eta_n$ are given by, for every $u \in [\gamma, 1 - \gamma]^2$ and $x \in S_X$, $\eta_u(u, x) = (F^-(u|x), F^-(u|x))$, $\eta_u(u, x) = \left(F_n - (G_n^-(u|x)), F^-(u|x)\right)$. Identifying $v \in V$ with $(u, t) \in \gamma, 1 - \gamma]^2 \times \mathbb{R}$ and $\eta \in E$ with $(\eta_1, \eta_2)$, where $\eta_j, j \in \{1, 2\}$, are valued in $\mathbb{R}^2$, the map $m_{v, \eta} : \mathbb{R}^2 \times S_X \to \mathbb{R}$ is given by $m_{v, \eta}(y, x) = |w_t(1 \{y \leq G_n^-(u|x)\}) - 1 \{y \leq F^-(G_n^-(u|x))\}|$. Finally, the space $E_0$ is the collection of those elements $\eta = (\eta_1, \eta_2)$ in $E$ such that

$$
\{x \mapsto \eta_1(u, x) : u \in [\gamma, 1 - \gamma]^2 \subset (C_{1+\delta, M_1}(S_X))^2,
\{x \mapsto \eta_2(u, x) : u \in [\gamma, 1 - \gamma]^2 \subset (C_{1+\delta, M_2}(S_X))^2,
$$

where $M_2$ depends only on $b_u$ and $M$. In the following we check each condition of Theorem 7.

Verifying Condition (i) in Theorem 7. Because the indicator function and $w_t$ are bounded, we have

$$
\int \left|w_t(x)\right| \left|1 \{y \leq F_n^-(G_n^-(u|x))\} - 1 \{y \leq F^-(G_n^-(u|x))\}\right|^2 f(x, y) \, dx \, dy 
\leq C_w \sup_{j, k} \int \left|1 \{y_j \leq F_n^-(G_n^-(u_j|x))\} - 1 \{y_j \leq F^-(G_n^-(u_j|x))\}\right|^2 f(x, y_j) \, dx \, dy_j.
$$

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where we can focus on each margin separately. Recall that if the random variable $U$ is uniformly distributed on $(0, 1)$, then $E(I_{(U \leq u_1)} - I_{(U \leq u_2)})^2 = |u_1 - u_2|$. Writing $\hat{a}_{n,x}(u_j) = \hat{F}_{n,j}^{-}(\hat{G}_{n,j}(u_j)|x|)$, we have

$$\int \left| \int \left( \mathbb{1}_{(y_j \leq \hat{a}_{n,x}(u_j))} - \mathbb{1}_{(y_j \leq F_j(\hat{G}_{n,j}(u_j)|x|))} \right)^2 f(x,y) \, d(x,y) \right| \, dx$$

$$= \int \left| \int \left( \mathbb{1}_{(F_j(y_j|x) \leq F_j(\hat{a}_{n,x}(u_j)|x|))} - \mathbb{1}_{(F_j(y_j|x) \leq G_{n,j}(u_j))} \right)^2 f(x,y) \, d(x,y) \right| \, dx$$

$$= \int \hat{g}_{n,j}(u_j|x) \, f(x) \, dx$$

where $\hat{F}_{n,j}$ has been defined in (16). Result 2 permits to conclude.

Verification of Condition (iii) in Theorem 7. This is given by Result 3.

Verification of Condition (ii) in Theorem 7. It is enough to show that the class of functions

$$\left\{ (y,x) \mapsto w_t(x)(\mathbb{1}_{(y \leq g_t(x))} - \mathbb{1}_{(y \leq g_2(x))}) : t \in \mathbb{R}, \quad (g_1,g_2) \in (C_{1+\delta,M}(S_X))^2 \times (C_{1+\delta,M}(S_X))^2 \right\}$$

is $P$-Donsker. Since the sum and the product of two bounded Donsker classes is Donsker (van der Vaart and Wellner, 1996, Example 2.10.8), it suffices to show that both classes

$$\left\{ \mathbb{1}_{(y \leq g(x))} : g \in C_{1+\delta,M}(S_X) \right\} \quad \text{and} \quad \left\{ w_t : t \in \mathbb{R} \right\}$$

are Donsker. For any $\delta > 0$ and $M > 0$, the first one is Donsker since the class of subgraphs of $C_{1+\delta,M}(S_X)$, under (G1), has a sufficiently small entropy (van der Vaart and Wellner, 1996, Corollary 2.7.3). The second one has been shown to be Donsker in Section C.1.

Second step: We show that

$$\sup_{u \in [\gamma,1-\gamma]^2, t \in \mathbb{R}} |\hat{A}_{n,2}(u,t)| = o_p(1), \quad n \to \infty$$

Under $H_0$, we have, for every $u \in [0,1]^2$, $t \in \mathbb{R}$,

$$P \left\{ w_t(\mathbb{1}_{(\hat{F}_n \leq \hat{G}_{n,1}(u))}) \right\} = \int w_t(x) \mathbb{1}_{(y \leq \hat{F}_n^{-}(\hat{G}_{n,1}(u)|x|))} f(x,y) \, d(x,y)$$

$$= \int w_t(x) F_1(\hat{F}_{n,1}^{-}(\hat{G}_{n,1}(u_1)|x|) \mid x) F_2(\hat{F}_{n,2}^{-}(\hat{G}_{n,2}(u_2)|x|) \mid x) f(x) \, dx$$

Consequently, using the bound $F_2 \leq 1$ and $\hat{G}_{n,1}^{(or)} \leq 1$,

$$|\hat{A}_{n,2}(u,t)| = \left| \int w_t(x) \mathbb{1}_{(y \leq \hat{F}_n^{-}(\hat{G}_{n,1}(u)|x|) \mid x) F_2(\hat{F}_{n,2}^{-}(\hat{G}_{n,2}(u)|x|) \mid x) - G_{n,1}^{(or)}(u)\hat{G}_{n,2}^{(or)}(u) f(x) \, dx \right|$$

$$\leq C_w \int \left| F_1(\hat{F}_{n,1}^{-}(\hat{G}_{n,1}(u)|x|) \mid x) F_2(\hat{F}_{n,2}^{-}(\hat{G}_{n,2}(u)|x|) \mid x) - G_{n,1}^{(or)}(u)\hat{G}_{n,2}^{(or)}(u) \right| f(x) \, dx$$

$$= C_w \int \left| \Delta_{n,1}(u_1|x) F_2(\hat{F}_{n,2}^{-}(\hat{G}_{n,2}(u_2)|x|) \mid x) + \hat{G}_{n,1}^{(or)}(u_1)\Delta_{n,2}(u_2|x) \right| f(x) \, dx$$

$$= 2C_w \max_{j=1,2} \sup_{u \in [\gamma,1-\gamma]} \int \Delta_{n,j}(u|x) f(x) \, dx$$
It remains to use Result 2 to obtain the conclusion.

C.3 Proof of Lemma 6

Recall the definition of $\hat{W}_n^{(or)}(u, t)$ and $\hat{Z}_n(u, t)$ that are given in Section C.1 and that under $\mathcal{H}_0$, in virtue of Lemma 1, $P\{w_t(1_{\{F\leq u\}} - u_1u_2)\} = 0$. Notice that

\[
\hat{W}_n^{(or)}(u, t) - \hat{Z}_n(u, t) = P_n\{w_t(1_{\{F \leq \hat{G}_n^{(or)}(u)\}} - 1_{\{F \leq u\}})\} \\
= n^{-1/2}G_n\{w_t(1_{\{F \leq \hat{G}_n^{(or)}(u)\}} - 1_{\{F \leq u\}})\} + P\{w_t(1_{\{F \leq \hat{G}_n^{(or)}(u)\}} - 1_{\{F \leq u\}})\} \\
= R_{n,1}(u, t) + \left(\hat{G}_n^{(or)}(u) - u_1\right)\hat{G}_n^{(or)}(u_2) - u_1u_2 \right) P\{w_t\} \\
= R_{n,1}(u, t) + \left(\hat{G}_n^{(or)}(u_1) - u_1\right)\left(\hat{G}_n^{(or)}(u_2) - u_2\right) + u_2(\hat{G}_n^{(or)}(u_1) - u_1) \right) P\{w_t\},
\]

with

\[
R_{n,1}(u, t) = n^{-1/2}(\hat{Z}_n(\hat{G}_n^{(or)}(u), t) - \hat{Z}_n(u, t)), \\
R_{n,2}(u) = (\hat{G}_n^{(or)}(u_1) - u_1)(\hat{G}_n^{(or)}(u_2) - u_2).
\]

Now just recall the definition of $\hat{W}_n$ to obtain that

\[
\hat{W}_n^{(or)}(u, t) - \hat{W}_n(u, t) = R_{n,1}(u, t) + P\{w_t\}R_{n,2}(u) + P\{w_t\} (u_1\rho_{n,2}(u_2) + u_2\rho_{n,1}(u_1)),
\]

with

\[
\rho_{n,j}(u_j) = (\hat{G}_n^{(or)}(u_j) - u_j) + (\hat{G}_n^{(or)}(u_j) - u_j).
\]

From Vervaat’s Lemma (Segers, 2015, Lemma 4.3), we have that

\[
\sup_{u_j \in [0, 1]} |\rho_{n,j}(u_j)| = o_P(n^{-1/2}), \\
\sup_{u \in [0, 1]^2} |R_{n,2}(u)| = O_P(n^{-1}).
\]

Because the class of functions $\{(y, x) \mapsto w_t(x)1_{\{F \leq y\}} : t \in \mathbb{R}, u \in [0, 1]^2\}$ is Donsker (as demonstrated in Section C.1), the process $\hat{Z}_n$ is asymptotically equicontinuous. This implies that

\[
\sup_{u \in [0, 1]^2} |R_{n,1}(u)| = o_P(n^{-1/2}).
\]

Consequently, each quantity in the above decomposition of $\hat{W}_n^{(or)}(u, t) - \hat{W}_n(u, t)$ is $o_P(n^{-1/2})$, uniformly over $u \in [0, 1]^2$ and $t \in \mathbb{R}$, and so comes the conclusion.