ABSTRACT. A class of generalized conditional gradient algorithms for the solution of optimization problem in spaces of Radon measures is presented. The method iteratively inserts additional Dirac-delta functions and optimizes the corresponding coefficients. Under general assumptions, a sub-linear $O(1/k)$ rate in the objective functional is obtained, which is sharp in most cases. To improve efficiency, one can fully resolve the finite-dimensional subproblems occurring in each iteration of the method. We provide an analysis for the resulting procedure: under a structural assumption on the optimal solution, a linear $O(\zeta^k)$ convergence rate is obtained locally.

1. Introduction

In this paper we consider generalized conditional gradient methods for sparse optimization problems, where the optimization variable lies in a space of measures. These problems arise in different contexts, and they are intrinsically related to certain optimization problems in terms of the spatial location parameters and associated coefficient variables: For the purposes of this paper, we want to find a “sparse” measure, which consists of a sum of Dirac delta functions,

$$u = u[x, u] = \sum_{i=1}^{N} u_i \delta_{x_i}. \quad (1.1)$$

It can be expressed in terms of a finite number of distinct points $x_i$, $i = 1, \ldots, N$, from a (continuous) candidate set $\Omega \subset \mathbb{R}^d$, $d \geq 1$ and corresponding coefficients $u_i$ in a Hilbert space $H$ (for instance, $\mathbb{R}$, $\mathbb{C}^M$, $M \in \mathbb{N}$, etc.), and $N \geq 0$ the cardinality of the support. Additionally, we will consider restrictions on the coefficients $u_i \in C$, where $C \subset H$ is a cone (for instance $u_i \geq 0$ for $H = \mathbb{R}$). It should be emphasized that neither the number of points, nor the coefficients are subject to any further restrictions. Usually, the measure $u$ has a physical interpretation as a number of point-wise sources or sensors in a physics-based model. There are many applications, where one is interested to choose $x$ and $u$ to minimize a functional of the form:

$$j(u[x, u]) = F\left(\sum_{n=1}^{N} k(x_i, u_i)\right) + G\left(\sum_{n=1}^{N} \|u_i\|_H\right).$$

Here, $F$ is a suitable design functional or quality criterion for the variable $y = \sum k(x, u)$ (which we will also refer to as observation variable), which is given in terms of the kernel function $k: \Omega \times H \rightarrow Y$, and evaluates the response of a model to the optimization variables $x$ and $u$. The second term, which is expressed in terms of the sum of the norms of the coefficients (the $\ell^1(H)$ norm of $u$) models either the cost of the coefficient variable, or is added as a regularization term to ensure that the coefficients are sufficiently small.

Often, the functionals $F$ and $G$ are convex, but $k$ is linear only the coefficients $u$, but not in the location parameters $x$. Thus, the corresponding optimization problem,

$$\text{Minimize } j(u[x, u]) \quad \text{for } x \in \Omega^N, u \in C^N \subseteq H^N, N \geq 0, \quad (1.2)$$

is not convex. Moreover, it has a combinatorial aspect, since $N$ is not fixed. However, by embedding this problem into a more general formulation, a convex formulation can be obtained. Concretely, the sparse measure (1.1) can be considered as an element of the space of regular
vector-measures $\mathcal{M}(\Omega, H)$. Requiring $k$ to be continuous in the coefficients, we can introduce the (convolution) operator $K$ and the total variation norm as

$$Ku = \sum_{n=1}^{N} k(x_i, u_i), \quad \|u\|_{\mathcal{M}(\Omega, H)} = \sum_{n=1}^{N} \|u_i\|_H. \quad (1.3)$$

We refer to section 2 for the rigorous definitions in the case of a general measure from the space of vector measures. Now, we can formulate the following generalized convex optimization problem:

$$\text{Minimize } F(Ku) + G\left(\|u\|_{\mathcal{M}(\Omega, H)}\right) \quad \text{for } u \in \mathcal{M}(\Omega, C). \quad (1.4)$$

Note that the formulation (1.4) is more general than (1.2), since not all vector measure are of the form (1.1) (in particular, the Lebesgue space $L^1(\Omega, H)$ is contained in $\mathcal{M}(\Omega, H)$). However, in many cases, the solutions of (1.4) have the desired discrete sparsity structure. In particular, if $Y$ is a finite-dimensional space, sparse solutions with $N \leq \dim Y$ can always be found. This then renders both problem formulations essentially equivalent.

Let us give two examples for problems of considerable practical interest. The first, which arises in the context of inverse source location [8, 48], optimal control [14, 29, 37, 38], or compressed sensing [3, 9, 23], is of the form:

$$\text{Minimize } \frac{1}{2}\|Ku - y_d\|_Y^2 + \alpha \|u\|_{\mathcal{M}(\Omega, H)} \quad \text{for } u \in \mathcal{M}(\Omega, H). \quad (P_{\text{source}})$$

Here, $u$ encodes a collection of vector valued signals originating from a number of source locations $x \in \Omega$, and $K$ models the signal that will be received by a measurement setup. The data vector $y_d$ contains (potentially noisy) observations obtained in practice, and the first term measures the misfit of the data.

The second example arises in the theory of optimal design, going back to the concept of approximate designs by Kiefer and Wolfowitz [27, 36, 49]. It is given by

$$\text{Minimize } \Psi(Iu) \quad \text{for } u \in \mathcal{M}(\Omega), \quad \text{subject to } u \geq 0, \quad \|u\|_{\mathcal{M}(\Omega)} \leq M. \quad (P_{\text{sensor}})$$

Here, $u$ encodes a number of pointwise sensors at locations $x_i$ with the reciprocal of the sensor error variance given by the the scalar coefficients $u_i \geq 0$, and $Iu = I(x, u)$ is the corresponding Fisher information matrix. Here, the information criterion $\Psi$ encodes the quality of the measurement setup as a function of the information matrix. In this case we choose $C = \mathbb{R}_{+} \subset \mathbb{R} = H$ and $G$ as the convex indicator function, i.e. $G(m) = 0$ for $m \leq M$ and $G(m) = +\infty$ for $m > M$, in order to incorporate the constraints into the general formulation (1.4). In this context, $\|u\|_{\mathcal{M}(\Omega)} = \sum_i u_i$ describes the overall cost of the measurement setup.

**Accelerated GCG methods.** The objective of this paper is to analyze certain sequential point insertion and coefficient optimization methods as efficient solution algorithms for sparse optimization problems of the form (1.4). We refer to [5, 8] for a description and analysis of the method applied to special instances of the general problem (1.4). Starting from a sparse initial measure $u^0$ of the form (1.1), these type of algorithms generates a sequence of sparse iterates $u^k$, $k = 1, 2, \ldots$, by the iterative procedure

$$u^{k+1} = u^k + s^k (v^k - u^k), \quad v^k = \hat{u}^k \delta_{\hat{x}^k}, \quad s^k \in [0, 1], \quad (1.5)$$

where $\hat{x}^k$ maximizes a certain continuous function over the set $\Omega$, which is computed from the previous iterate $u^k$; see Algorithm 1 below. The new source location $\hat{x}^k$ and the coefficient function $\hat{u}$ are chosen such that $v^k$ corresponds to a descent direction in a generalized conditional gradient method (GCG) – also known as Frank-Wolfe algorithm [28] – applied to an equivalent reformulation of (1.4). We also point to different variations of the Fedorov-Wynn algorithm [26, 45, 54–57], developed in the context of approximate design theory, which can be interpreted in this framework.

While the practical implementation of the GCG algorithm is fairly simple, it suffers from slow asymptotic convergence. Several works [5, 8, 24, 46] derive a sublinear $O(1/k)$ convergence
rate for the objective functional values of the iterates under mild assumptions on the problem and several choices of the step size $s^k$. Numerical experiments (e.g., [46]) confirm that this convergence is also observed in practice. Therefore, it is impractical to solve the problem to high precision, which motivates the introduction of additional acceleration steps. Moreover, the absence of point removal steps leads to undesirable clustering effects: The support size of the iterate grows monotonically with $k$ and, in later iterations, new support points are inserted very close to existing ones. As a remedy, one is interested to incorporate additional sparsification steps which can iteratively remove support points without increasing the objective functional values. In the present work, we consider additional optimization steps based on the sparse representation of the iterates in terms of their support points $x$ and coefficients $u$ according to (1.1). Defining the updated support corresponding to (1.5) as $x_i^{k+1} = x_i^k$ for $n = 1,\ldots,N_k$ and $x_{N_k+1}^{k+1} = \hat{x}^k$, where $N_{k+1} = N_k + 1$, we improve the coefficients of the next iterate by approximately solving the coefficient optimization problem

$$\text{Minimize } j(u[x^{k+1}, u]) \text{ for } u \in \mathcal{C}^{N_{k+1}} \subseteq H^{N_{k+1}}. \quad (1.6)$$

Note that this is a convex minimization problem on the Hilbert space $H^{N_{k+1}}$ due to the linearity of the kernel $k$ in the argument $u$. In fact, (1.6) has the same structure as (1.4); it is simply its restriction to the space $\mathcal{M}(\mathcal{A}_k, H)$, with the active set $\mathcal{A}_k = \{ x_i^{k+1} \mid n = 1,\ldots,N_{k+1} \}$. Since it is also a sparse optimization problem, some coefficients of the associated optimal solution may be zero. In the next iteration, we can thus exclude the corresponding support point from the representation of the measure (1.1), which also serves as a sparsification step. In [8] the authors suggest to improve the algorithm by performing several steps of a proximal gradient for (1.6) starting from the current coefficients as initial guess. Acceleration of GCG by fully resolving the coefficient optimization problem (1.6) in each iteration of the method has been proposed in [5, 24, 48, 55]. Alternatively to coefficient optimization, point moving strategies have been suggested, which we do not consider in this work. Let us briefly comment on this issue. Here, we fix the coefficients $u^{k+1} \in H^{N_{k+1}}$ (obtained either from (1.5) or (1.6)) and approximately solve the problem

$$\text{Minimize } j(u[x, u^{k+1}]) \text{ for } x \in \Omega^{N_{k+1}}. \quad (1.7)$$

We note that this is a finite-dimensional, generally non-convex optimization problem subject to bound constraints. For instance, the authors in [8] propose to move the support points according to the gradient flow of the smooth part $x \mapsto F(Ku[x, u^{k+1}])$. In [5] it is advocated to solve (1.7) by general purpose optimization methods based on derivatives with respect to $x$. We also mention the recent work [16], where the authors propose to include steps which simultaneously optimize the positions and coefficients of the current iterate, i.e. to fully resolve a local optimum of (1.2) in each iteration. This method, under a non-degeneracy condition on the optimal solution, is shown to converge in finitely many iterations to a global minimizer. Note that all of these approaches require the kernel function $k$ to be continuously differentiable with respect to the position $x$ and the derivatives to be efficient to evaluate in practice, which is not required for coefficient optimization. Moreover, the computational complexity of the nonconvex (and also nonsmooth, if both coefficients and positions are optimized) subproblems is an open issue.

In the present work we focus only on coefficient optimization and do not consider acceleration based on point moving. Besides the complications arising from non-convexity of (1.7), one particular reason for this decision is our interest in sparse minimization problems which require further discretization. For example, the operator $K$ could correspond to the solution operator of a partial differential equation [11, 37, 38, 48] or otherwise involve such quantities [46]. To solve (1.4) in practice, we thus replace the operator $K$ by an approximation employing finite elements. Note that the most commonly employed Lagrangian finite elements are continuous, but not continuously differentiable and thus the objective function in (1.7) is no longer $C^1$ with respect to the positions. This prevents a straightforward algorithmic solution of the
point moving problem by derivative based methods, whereas coefficient optimization can be implemented in a straightforward fashion.

**Contribution.** One of the main contributions of this paper is to analyze the procedure resulting from combining point insertion steps (1.5) with subsequent full resolution of (1.6), which is summarized in Algorithm 2. Note that the method can be interpreted as an active set method, where new points are added to the active set at the global maxima of a dual variable, and points are removed if their primal coefficients are set to zero (by resolving (1.6)), we also refer to this method as Primal-Dual-Active-Point strategy. This is motivated by the similarities to the Primal-Dual-Active-Set method [33].

Since the coefficient optimization steps are carried out in addition to the point insertion steps, the $O(1/k)$ convergence rate for GCG is also valid for the accelerated methods. We derive this convergence result in Theorem 4.7 for the general problem formulation (1.4). In comparison to existing results, we relax certain assumptions; in particular, $F$ does not need to be finite on the whole space $Y$ and Lipschitz continuity of its gradient is only required on sublevel sets. These minor technical refinements are crucial in order to be able to include sensor placement problems in the general framework; cf. Section 3.1. Concerning the improved convergence behavior of methods combining point insertion and coefficient optimization over GCG – as reported in [46,48] – we are not aware of any improved theoretical results. However, in this paper, we prove a linear convergence rate $O(\zeta^k)$ for $0 \leq \zeta < 1$; see Theorem 5.16. Note that, since the improved result is local in character, we still have to rely on the general $O(1/k)$ convergence result mentioned above, to ensure that some iterate $u^k$ is sufficiently close to an optimal solution. In order to obtain the improved linear convergence result, we impose a non-degeneracy condition on the optimal solution; see Assumptions 5.2 and 5.3. This enables us to derive further convergence results for the location parameters $x^k$ and the coefficients $u^k$. In particular, we show that the support points of the iterate asymptotically converge towards the support points of the optimal solution, again at a linear rate; see Proposition 5.18. This also gives theoretical evidence for the sparsifying effect of the coefficient optimization steps, since it shows that support points far away from the optimal locations eventually will be removed from the iterate measure. Moreover, we derive convergence estimates for the coefficients. Here, we need to account for the fact that multiple support points of $u^k$ can be close to the each optimal location. Lumping together the corresponding coefficients, we again obtain a linear convergence rate; see Theorem 5.23. Together, this results in a linear convergence rate of the iterate measure $u^k$ in the dual space $C^{0,1}(\Omega, H)^*$; see Theorem 5.24.

We note that the improved convergence rate proved here also requires additional regularity assumptions. In particular, we need second derivatives of the kernel function in $x$, which may not be available if discrete approximations to $K$ are employed in practice. We point out that these assumptions are only of technical nature: The computation of the derivatives of the kernel function with respect to the position is not required in the algorithm. However, this means that the derived fast convergence results do not apply directly to the discrete problems. In practice, the algorithm behaves similar on meshes of different fineness; in particular the residual converges with the rate $O(\zeta^k)$, where $\zeta$ and the constant appear to be independent of the mesh. This suggests that the behavior is dictated by the properties of the underlying continuous problem. For numerical evidence we refer to [46,48].

**Related work.** The design of efficient algorithms for (1.4) is a challenging task since the space of vector-valued Borel measures is in general non-reflexive. Moreover, it lacks useful properties such as strict convexity and smoothness which are desirable for the convergence analysis of many optimization methods. Consequently, a direct extension of most well-known optimization routines to the present setting is not possible.

**Discretization-based methods.** A first approach to the solution of (1.4) for a continuous candidate set is to replace $\Omega$ by a approximating sequence of finite sets with $\Omega_h \subset \Omega$ for a sequence
of mesh parameters $h > 0$. For example, $\Omega_h$ may be chosen as the nodal set of a triangulation $T_h$ of $\Omega$. Since $\Omega_h$ consists of finitely many points, every $u \in \mathcal{M}(\Omega_h, H)$ is of the form $u = \sum_{x_i \in \Omega} u_i \delta_{x_i}$. Substituting the space of regular Borel measures in (1.4) by the discretized space $\mathcal{M}(\Omega_h, H)$ yields the convex minimization problem for the coefficient functions $u_i \in H$ discussed above. While the resulting problem remains non-smooth due to the appearance of the total variation norm, it can be solved by a large number of well-studied algorithms. For examples we point to semi-smooth Newton methods [44], the fast iterative shrinkage-thresholding algorithm (FISTA) [4], and the alternating direction of multipliers method [6]. However, this philosophy of discretize then optimize harbors the danger of yielding mesh dependent solution methods. While a particular algorithm may be efficient for the solution of the discrete problem associated to a fixed discretization parameter, its convergence behaviour can critically depend on $h$. This usually is the case for the aforementioned methods. For methods based on iterative point insertion and coefficient optimization and sparsification steps, such problems only have to be solved on a very small candidate set.

**Regularization based methods.** A different approach to circumvent the non-reflexivity of the space $\mathcal{M}(\Omega, H)$ can be based on path-following strategies. Here the original problem is replaced by a sequence of regularized ones

$$\min_{u \in \mathcal{M}(\Omega, H)} \left[ F(Ku) + G(\|u\|_{L^1(\Omega, H)}) + \frac{\varepsilon}{2} \|u\|_{L^2(\Omega, H)}^2 \right],$$

over the Hilbert space $L^2(\Omega, H)$. Note that the appearance of the $L^1(\Omega, H)$ norm in the objective functional still promotes optimal solutions which are nonzero only on small subsets of $\Omega$. Furthermore in the limiting case for $\varepsilon \to 0$ the regularized solutions approximate solutions to the original one; see, e.g., [47]. For fixed $\varepsilon > 0$ those problems are amenable to efficient function space based solution methods such as semi-smooth Newton, [31, 51]. While these methods behave mesh independent in principle, the convergence behavior deteriorates for small values of $\varepsilon$. In the practical realization it is therefore necessary to start at a large value of $\varepsilon$ and to alternate between decreasing the regularization parameter and a (possibly inexact) solution of the regularized problem initialized at the previous iterate. Thus, a complete analysis of path-following methods requires a quantitative convergence analysis of the method used for the solution of the regularized problem in dependence of $\varepsilon$, a quantification of the additional regularization error and sophisticated update strategies for the parameter; see, e.g. [34].

**Existing convergence results for conditional gradient methods.** Conditional gradient methods (see, e.g. [42]) have been originally proposed by Frank and Wolfe [28]. They constitute a simple iterative scheme for computing a minimizer of a smooth convex function over compact subsets of a Banach space. Since norm balls in $\mathcal{M}(\Omega, H)$ are weak* compact, the general problem formulation fits into this setting for the choice of the convex indicator function $G(m) = I_{m \leq M}$. Feasibility of the iterates is ensured by taking the new iterate $u^{k+1}$ as a convex combination between the previous iterate $u^k$ and a descent direction $v^k$, which is obtained by minimizing a linearization of the objective functional around $u^k$ over the admissible set. A sublinear rate for the convergence of the objective functional values towards its minimum can be proven for various choices of the step size $s^k$. For an overview we refer to [19–21]. The sublinear rate is tight even for strongly convex objective functionals [10]. An improved rate of convergence can only be derived in more restrictive settings: For problems on infinite dimensional spaces, a linear rate of convergence is provided in [15, 42] if the gradient of the objective functional is uniformly bounded away from zero on a strongly convex admissible set. The papers [19, 20] yield the same rate if the linearized objective functional fulfills a certain growth condition on the admissible set. We emphasize that, apart from trivial cases, none of the mentioned results is directly applicable to the problem at hand. Moreover, we point out that, on finite dimensional spaces, accelerated conditional gradient methods, such as Wolfe’s away-step conditional gradient [53], eventually yield a linear rate of convergence [1, 39]. In infinite dimensions, where the candidate set $\Omega$ is not finite, we are not aware of similar results. Last we point out that
for $H = \mathbb{R}$, $C = \mathbb{R}_+$ and $G(m) = I_{m \leq M}$ Algorithm 2 corresponds to the fully-corrective conditional gradient method [35]. For finite-dimensional observation space $Y$, this particular algorithm can be related to an exchange method [32] on the semi-infinite convex dual problem of (1.4). We are also not aware of convergence results comparable to those provided in this work for these type of methods.

**Plan of the paper.** The paper is organized as follows. In Section 2, we fix some basic notation and provide the functional analytic background used for the rest of the work. Section 3 introduces the optimization problem and some basic results on the existence and structure of optimal solutions are derived. We also discuss how different practically relevant problems fit into the general framework. In Section 4 we formulate the optimization algorithms and prove the subsequential convergence of the generated iterates as well as a sublinear worst-case convergence rate for the objective functional values. Under additional structural assumptions on the problem, an improved local linear rate of convergence is established in Section 5. Moreover, quantitative convergence results for the support points and the coefficients of the iterates are presented.

**2. Notation**

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be compact and denote by $H$ a separable Hilbert space with respect to the norm $\| \cdot \|_H$ induced by the inner product $(\cdot, \cdot)_H$. In the following, $H$ is identified with its dual space using the Riesz representation theorem. A countably additive mapping $u: \mathcal{B}(\Omega) \to H$ is called a vector measure, where $\mathcal{B}(\Omega)$ denote the Borel sets of $\Omega$. Associated to $u$ we define its total variation measure $|u|: \mathcal{B}(\Omega) \to \mathbb{R}_+$ as

$$|u|(O) = \sup \left\{ \sum_{i=1}^{\infty} \| u(O_i) \|_H \mid O_i \in \mathcal{B}(\Omega), \text{ disjoint partition of } O \right\}.$$ 

for each $O \in \mathcal{B}(\Omega)$. By $|u|(\Omega)$ we denote the total variation of $u$. The space of vector measures with finite total variation is now denoted by

$$\mathcal{M}(\Omega, H) = \{ u : \mathcal{B}(\Omega) \to H \mid u \text{ countably additive, } |u|(\Omega) < \infty \}.$$

For each vector measure $u \in \mathcal{M}(\Omega, H)$ we thus clearly have $|u| \in \mathcal{M}^+(\Omega)$, the space of positive Borel measures on $\Omega$. The support of $u$ is defined as the support of the corresponding total variation measure

$$\text{supp } u = \text{supp } |u| = \Omega_u \setminus \left( \bigcup \{ O \in \mathcal{B}(\Omega_u) \mid O \text{ open, } u(O) = 0 \} \right).$$

The space $\mathcal{M}(\Omega, H)$ is a Banach space with respect to the norm

$$\| u \|_{\mathcal{M}} := |u|(\Omega) = \| |u| \|_{\mathcal{M}(\Omega)} = \int_{\Omega} d|u|.$$ 

For a reference see the discussion in [41, Chapter 12.3]. Furthermore for $u \in \mathcal{M}(\Omega, H)$ it is easy to see that

$$\| u(O) \|_H \leq |u|(O) \quad \forall O \in \mathcal{B}(\Omega).$$

In particular this implies that $u$ is absolutely continuous with respect to $|u|$, i.e. there holds

$$|u|(O) = 0 \Rightarrow \| u(O) \|_H = 0 \quad \forall O \in \mathcal{B}(\Omega).$$

Moreover there exists a unique function

$$u' \in L^\infty(\Omega, |u|; H) \quad \text{with} \quad \| u'(x) \|_H = 1 \quad \text{for } |u|\text{-almost all } x \in \Omega,$$

such that $u$ can be decomposed as

$$u(O) = \int_O du = \int_O u' d|u| \quad \forall O \in \mathcal{B}(\Omega).$$

We point out to [40, Chapter 12.4] for a reference. The function $u'$ is called the Radon-Nikodým derivative of $u$ with respect to $|u|$; see [17]. We refer to this splitting of $u$ in terms of
its Radon-Nikodým derivative $u'$ and its total variation measure $|u|$ as its polar decomposition. For abbreviation we write $\,du = u'\,|u|$ in the following.

By $\mathcal{C}(\Omega,H)$ we further denote the space of bounded and continuous functions on $\Omega$ which assume values in $H$. It is a separable Banach space when endowed with the usual supremum norm
\[ \| \varphi \|_{\mathcal{C}} = \max_{x \in \Omega} \| \varphi(x) \|_H \]
for any $\varphi \in \mathcal{C}(\Omega,H)$; see e.g. [2, Lemma 3.85]. By Singer’s representation theorem (see, e.g., [30]) its topological dual space is identified with $\mathcal{M}(\Omega,H)$ where the associated duality paring is given by
\[ \langle \varphi, u \rangle = \int_{\Omega} (\varphi(x), u'(x))_H \, d|u|(x) \quad \forall \varphi \in \mathcal{C}(\Omega,H), \ u \in \mathcal{M}(\Omega,H). \]
As a consequence we conclude
\[ \| u \|_{\mathcal{M}} = \sup_{\| \varphi \|_{\mathcal{C}} \leq 1} \langle \varphi, u \rangle = \sup_{\| \varphi \|_{\mathcal{C}} \leq 1} \int_{\Omega} (\varphi(x), u'(x))_H \, d|u|(x). \]
A sequence $\{u^k\}_{k \in \mathbb{N}} \subset \mathcal{M}(\Omega,H)$ is called weak* convergent with limit $u \in \mathcal{M}(\Omega,H)$ if
\[ \langle \varphi, u^k \rangle \to \langle \varphi, u \rangle \quad \forall \varphi \in \mathcal{C}(\Omega,H). \]
We denote this by $u^k \rightharpoonup^* u$. Given a closed and convex cone $C \subset H$ we further define
\[ \mathcal{M}(\Omega,C) = \{ u \in \mathcal{M}(\Omega,H) \mid u(O) \in C \quad \text{for all } O \in B(\Omega) \} . \tag{2.1} \]
The set $\mathcal{M}(\Omega,C)$ is a weak* closed convex cone and there holds
\[ \mathcal{M}(\Omega,C) = \{ u \in \mathcal{M}(\Omega,C) \mid u'(x) \in C \quad \text{for all } x \in \Omega \} . \]
For a proof of these statements we refer to [52, Section 6.3.1]. We refer to the polar cone of $C$ by $C^o$. The $H$-projections onto $C$ and $C^o$ are denoted by $\pi_C$ and $\pi_{C^o}$, respectively. There holds
\[ u = \pi_C(u) + \pi_{C^o}(u), \quad (\pi_{C^o}(u), \pi_C(u))_H = 0 \quad \forall u \in H, \]
Moreover, the projections are Lipschitz continuous with constant one. Last we define
\[ \mathcal{M}_{N}(\Omega,C) = \left\{ u \in \mathcal{M}(\Omega,C) \mid u = \sum_{i=1}^{N} u_i \delta_{x_i}, \ N \in \mathbb{N}, \ u_i \in C, \ x_i \in \Omega, \ i = 1, \ldots, N \right\}. \]
Given $u \in \mathcal{M}(\Omega,C)$ there exists a sequence $u^k \in \mathcal{M}_{N}(\Omega,C)$ fulfilling $\| u^k \|_{\mathcal{M}} \leq \| u \|_{\mathcal{M}}$ and $u^k \rightharpoonup^* u$. In particular this implies $\mathcal{M}_{N}(\Omega,C)^* = \mathcal{M}(\Omega,C)$, where $\rightharpoonup^*$ denotes the closure with respect to the weak* topology.

Given any two Banach spaces $X$ and $Y$ with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ as well as a linear mapping $B: X \to Y$, we define the operator norm of $B$ as usual as
\[ \| B \|_{\mathcal{L}(X,Y)} = \sup_{\| \varphi \|_X = 1} \| B\varphi \|_Y. \]
The vector space $\mathcal{L}(X,Y) := \{ B \mid B: X \to Y \text{ linear}, \ |B|_{\mathcal{L}(X,Y)} < \infty \}$ forms a Banach space together with the operator norm $\| \cdot \|_{\mathcal{L}(X,Y)}$.

Finally, let $Y$ be another Hilbert space and $k: \Omega \times H \to Y$ be a continuous function, which is linear in the second argument. Now, we define the operator $K: \mathcal{M}(\Omega,H) \to Y$ for each argument $u \in \mathcal{M}(\Omega,H)$ by the relation
\[ (Ku, v)_Y = \int_{\Omega} \langle k(x, u'(x)), v \rangle_Y \, d|u|(x) \quad \forall v \in Y. \]
Additionally, we define the formal adjoint $K^*: Y \to \mathcal{C}(\Omega,H)$ by
\[ K^* y = \varphi, \quad \text{where} \quad (\varphi(x), u)_H = \langle k(x, u), y \rangle_Y \quad \forall x \in \Omega, u \in H. \]
It is easy to see that \(( Ku, v )_Y = \langle u, K^* v \rangle\) for all \( u \in \mathcal{M}( \Omega, H )\) and \( y \in Y\), using the definitions. Moreover, \( K^* \) is a linear and bounded operator with norm
\[
\| K^* \|_{\mathcal{L}(Y, \mathcal{C}(\Omega, H))} = \sup_{x \in \Omega, \| u \|_H = 1} \| k(x, u) \|_Y < \infty.
\]
Thus, \( K \) is the Banach space adjoint of \( K^* \) and thus also linear and bounded with the same norm bound. For the same reason, \( K \) is sequentially weak*-to-strong continuous. Note that \( K^* \) is not the Banach space adjoint of \( K \), since \( \mathcal{M}( \Omega, H )^* \neq \mathcal{C}(\Omega, H) \). It can be understood as the adjoint in the sense of topological vector spaces, if \( \mathcal{M} \) is endowed with the weak* topology, but we will not need this property in the following.

3. Sparse minimization problems

We now turn to the study of sparse minimization problems. Our aim is to solve the non-smooth convex optimization problem
\[
\min_{u \in \mathcal{M}(\Omega, C)} [ F(Ku) + G(\| u \|_\mathcal{M}) ] .
\] (P)

Here, the design functional \( F : Y \to \mathbb{R} \cup \{ +\infty \} \) is a convex (extended real valued) functional with open domain \( \text{dom} \ F = \{ y \in Y \mid F(y) < +\infty \} \). The convex cost functional \( G : \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \) is assumed to be monotone on \( \mathbb{R}^+ \).

In order to write \((P)\) as an unconstrained problem, we introduce the convex indicator function \( I_{\mathcal{M}(\Omega, C)} \) of the convex cone \( \mathcal{M}(\Omega, C) \). Then the problem can be considered as the unconstrained minimization of the functional \( j \) defined as
\[
j(u) := F(Ku) + G(\| u \|_\mathcal{M}) + I_{\mathcal{M}(\Omega, C)}(u) \quad (3.1)
\]

We note that its domain is given by \( \text{dom} \ j = \{ u \in \mathcal{M}(\Omega, C) \mid \| u \|_\mathcal{M} \in \text{dom} \ G, \ Ku \in \text{dom} \ F \} \). In order to ensure well-posedness of this problem the following standing assumptions are made.

**Assumption 3.1.** Let the following assumptions hold:

**A1** The function \( G : \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \) is proper, convex, lower semi-continuous, and monotonically increasing on \( \mathbb{R}^+ \) with \( G(m) \to +\infty \) for \( m \to \infty \). Without loss of generality we set \( G(m) = +\infty \) for \( m < 0 \).

**A2** The domain of the functional \( j \) is nonempty and \( j \) is radially unbounded.

**A3** The function \( F : Y \to \mathbb{R} \cup \{ +\infty \} \) is convex and lower semi-continuous. Moreover, \( \text{dom} \ F \) is open in \( Y \), and \( F \) is continuously Fréchet differentiable on \( \text{dom} \ F \).

The (Hilbert-space) Fréchet derivative of \( F \) at \( y \in \hat{Y}_{ad} \) will be denoted by \( \nabla F(y) \). For later use, we also define the smooth part of the reduced cost functional as
\[
f(u) := F(Ku) .
\]
We also define the gradient of \( f \) for every \( u \in \text{dom} \ j \). From Assumption **A3**, the linearity of \( K \) as well as the chain rule we conclude that \( f \) is Gâteaux differentiable at \( u \). By a simple computation we have
\[
f'(u)(\delta u) = (\nabla F(Ku), K\delta u)_Y = \langle K^* \nabla F(Ku), \delta u \rangle \quad \forall \delta u \in \mathcal{M}(\Omega, H).
\]

and thus the gradient of \( f \) can be defined as
\[
\nabla f : \text{dom} \ j \to \mathcal{C}(\Omega, H), \quad u \mapsto K^* \nabla F(Ku)
\]
Moreover, due to the weak*-to-strong continuity of \( K \), the gradient is sequentially weak*-to-strong continuous.

3.1. **Examples.** To motivate the general assumptions, we briefly discuss how the examples mentioned in the introduction fit into the general framework.
Sparse inverse problems. For inverse problem applications, one seeks to recover an unknown signal originating from unknown locations in space (and/or time). The kernel $k$ encodes the (indirect) measurements obtained from a given signal by a measurement setup. Often, such models involve trigonometric polynomials or other analytically given functions, [3, 9, 23]. More complicated models involve partial differential equations, [8, 12, 48]. Here, $k(x, u)$ could correspond to (possibly pointwise) observations of the PDE solution corresponding to the signal. Then, the continuity properties of the kernel functions (the mapping properties of $K$) have to be deduced from the regularity of the PDE.

Concerning the signals, besides the scalar case $H = \mathbb{R}$, the space $H$ could also be given by $\mathbb{C}^M \simeq \mathbb{R}^{2M}$, [3, 48], corresponding to discrete frequencies, but also $L^2(0, T)$, [37], corresponding to time-dependent signals. Here, one often has that $F$ is given by a quadratic tracking functional on the observation space, thus clearly all of the assumptions are fulfilled. Moreover, for sparse inverse problems we usually set $G(m) = \alpha m$ for some appropriately chosen regularization parameter $\alpha > 0$ for any $m \geq 0$.

Optimal sensor placement. We consider a problem of selecting the locations $x_i$ and sensor qualities $u \in \mathbb{R}^+$ in the linear model: Find $q \in \mathbb{R}^N$: $z_i = g(x_i)^\top q + \varepsilon_i$. Here, the parameter vector $q$ should be identified from pointwise observations of the model, given by a vector of continuous functions $g \in \mathcal{C}(\Omega, \mathbb{R}^N)$. The noise terms $\varepsilon_i$ are independent Gaussian random variables, and $\text{Var}[\varepsilon_i] = 1/u_i$, which we suppose to be able to chose freely (by the choice of an appropriate sensor).

To evaluate the quality of the sensor distribution, we introduce the Fisher information matrix as $\mathcal{I}(u) = Ku \in \text{Sym}(N_q)$, where the kernel is chosen as

$$k(x, u) = u g(x) g(x)^\top \in Y = \text{Sym}(N_q)$$

Roughly speaking, the Fisher information matrix is formed as a convolution of rank-one products of the vector $g(x)$, [36]. Thus, the set $Y$ is the space of symmetric matrices (endowed with the Frobenius inner product). The quality of the sensor distribution is then determined by the function $F = \Psi$. For instance, we can consider the $A$-criterion, which is given by

$$\Psi(\mathcal{I}) = \begin{cases} \text{Trace} \left( \mathcal{I}^{-1} \right) & \text{if } \mathcal{I} \text{ is positive definite}, \\ +\infty & \text{else}. \end{cases}$$

This particular criterion evaluates the variance of the parameter vector recovered from the linear model by least-squares regression. Note that A3 can be easily verified. Additionally, we interpret $\|u\|_\mathcal{M}$ as the total cost of the measurement setup. To restrict its size, additional constraints are imposed and we can choose $G$ as the convex indicator function of the interval $[0, M]$ for some $M > 0$ (classically, $M = 1$). Concerning A2, we note that $\mathcal{I}(u)$ is generally only a positive semi-definite matrix, but the existence of a admissible $u \in \text{dom } j$ follows by imposing the assumption of linear independence on the vectors $g_n \in \mathcal{C}(\Omega)$, $n = 1, \ldots, N_q$, [46, Proposition 3.4].

3.2. Existence of minimizers and optimality conditions. From Assumption 3.1 as well as the convexity and weak* closedness of $\mathcal{M}(\Omega, C)$ we conclude the radial unboundedness and lower weak* continuity of the functional $j$ on $\mathcal{M}(\Omega, C)$. The existence of at least one global minimizer to $(P)$ thus follows immediately from standard arguments.

Proposition 3.1. Let Assumption 3.1 hold. There exists at least one optimal solution $\bar{u} \in \mathcal{M}(\Omega, C)$ to $(P)$. The set of optimal solutions is bounded.

Remark 3.1. We emphasize that the previous proposition does not yield the existence of a sparse minimizer $\bar{u} \in \mathcal{M}_s(\Omega, C)$. In fact, this cannot be guaranteed for general sparse optimization problems. However, the practically desired sparse structure of minimizers can be ensured in a relevant special case: Let $Y$ be finite-dimensional. Then, given $u \in \mathcal{M}(\Omega, C)$,
there exists \( u \in \mathcal{M}_N(\Omega, C) \) with
\[
K\tilde{u} = Ku, \quad \|\tilde{u}\|_\mathcal{M} \leq \|u\|_\mathcal{M}, \quad \#\supp|\tilde{u}| \leq \dim Y.
\]

The proof of this statement can be based on the Carathéodory lemma. For references see [48].

Clearly, from this statement we also conclude the existence of a sparse minimizer to (P) if \( Y \) is finite dimensional. Moreover, for general \( Y \), we point out that every minimizer \( \tilde{u} \in \mathcal{M}(\Omega, C) \) of (P) can be approximated by sparse measures up to arbitrary accuracy. In fact, there exists a sequence \( \{u^k\}_{k \in \mathbb{N}} \subset \mathcal{M}_N(\Omega, C) \) with
\[
u^k \rightharpoonup^{*} \tilde{u}, \quad j(u^k) \to j(\tilde{u}), \quad \|u^k\|_\mathcal{M} \leq \|\tilde{u}\|_\mathcal{M}.
\]

This particularly implies \( \dom j \cap \mathcal{M}(\Omega, C) \neq \emptyset \).

The following variational characterization of global minimizers to (P) can be obtained by standard results from convex subdifferential calculus.

**Proposition 3.2.** Let \( \tilde{u} \in \dom j \) be given. Set \( \bar{p} = -\nabla f(\tilde{u}) \in C(\Omega, H) \). The measure \( \tilde{u} \) is an optimal solution to (P) if and only if
\[
\langle \bar{p}, u - \tilde{u} \rangle + G(\|\tilde{u}\|_\mathcal{M}) \leq G(\|u\|_\mathcal{M}) \quad \forall u \in \mathcal{M}(\Omega, C).
\]

(3.2)

Throughout the rest of the paper we will refer to \( \bar{y} = K\tilde{u} \) as the optimal observation and to the continuous function \( \bar{p} \) as the dual variable associated to \( \tilde{u} \). Let us turn to a structural characterization of minimizers obtained from (P).

**Theorem 3.3.** Let \( \tilde{u} \in \dom j \) be given. Then (3.2) holds if and only if
\[
\langle \bar{p}, \tilde{u} \rangle = \|\pi_C(\bar{p})\|_C \|\tilde{u}\|_\mathcal{M}, \quad \|\pi_C(\bar{p})\|_C \in \partial G(\|\tilde{u}\|_\mathcal{M})
\]

(3.3)

\[
\langle \bar{p}, u \rangle = \int_\Omega (\bar{p}(x), u'(x))_H \, dx \leq \int_\Omega (\pi_C(\bar{p}(x)), u'(x))_H \, dx \leq \|\pi_C(\bar{p})\|_C \|u\|_\mathcal{M}.
\]

Putting everything together yields
\[
\langle \bar{p}, u - \tilde{u} \rangle + G(\|\tilde{u}\|_\mathcal{M}) = -\|\pi_C(\bar{p})\|_C \|\tilde{u}\|_\mathcal{M} + \langle \bar{p}, u \rangle + G(\|\tilde{u}\|_\mathcal{M}) \leq \|\pi_C(\bar{p})\|_C (\|u\|_\mathcal{M} - \|\tilde{u}\|_\mathcal{M}) + G(\|\tilde{u}\|_\mathcal{M}) \leq G(\|u\|_\mathcal{M}).
\]

Since \( u \in \mathcal{M}(\Omega, C) \) was chosen arbitrary the variational inequality (3.2) follows.

Conversely assume that (3.2) holds. First let \( \tilde{u} \neq 0 \) hold. From the monotonicity of \( G \) we infer
\[
\langle \bar{p}, u - \tilde{u} \rangle \leq 0 \quad \forall u \in \mathcal{M}(\Omega, C), \quad \|u\|_\mathcal{M} \leq \|\tilde{u}\|_\mathcal{M}
\]
or, equivalently,
\[
\bar{p} \in \partial \left[ I_{\mathcal{M}(\Omega, C)}(\cdot) + I_{\|\cdot\|_\mathcal{M} \leq \|\tilde{u}\|_\mathcal{M}}(\cdot) \right](\tilde{u}).
\]

Here, the set on the right hand side denotes the convex subdifferential of \( I_{\mathcal{M}(\Omega, C)}(\cdot) + I_{\|\cdot\|_\mathcal{M} \leq \|\tilde{u}\|_\mathcal{M}}(\cdot) \).

Utilizing Proposition 5.1 and Corollary 5.2 from [25] we obtain
\[
\left[ I_{\mathcal{M}(\Omega, C)} + I_{\|\cdot\|_\mathcal{M} \leq \|\tilde{u}\|_\mathcal{M}} \right]^*(\bar{p}) = \sup_{u \in \mathcal{M}(\Omega, C)} \langle \bar{p}, u \rangle = \langle \bar{p}, \tilde{u} \rangle.
\]

For an arbitrary measure \( u \in \mathcal{M}(\Omega, C), \quad \|u\|_\mathcal{M} \leq \|\tilde{u}\|_\mathcal{M} \), we readily obtain
\[
\langle \bar{p}, u \rangle \leq \int_\Omega (\pi_C(\bar{p}(x)), u'(x))_H \, dx \leq \|\pi_C(\bar{p}(x))\|_C \|\tilde{u}\|_\mathcal{M}.
\]

(3.4)

Let \( \hat{x} \in \Omega \) with \( \|\pi_C(\bar{p}(\hat{x}))\|_H = \|\pi_C(\bar{p})\|_C \) be given and define
\[
\tilde{u} = \|\tilde{u}\|_\mathcal{M} \begin{cases} 0 & \pi_C(\bar{p}) = 0 \\ \pi_C(\bar{p}) \delta_{\hat{x}} & \pi_C(\bar{p}) \neq 0 \in \mathcal{M}(\Omega, C). \end{cases}
\]
We claim that ū achieves equality in (3.4). If \( \pi_C(\bar{p}) = 0 \) this trivially holds. In the second case we compute

\[
\langle \bar{p}, \bar{u} \rangle = \| \bar{u} \|_M \left( \pi_C(\bar{p}(\bar{x})) + \pi_C^\ast(\bar{p}(\bar{x})), \pi_C(\bar{p}(\bar{x})) \right)_H = \| \bar{u} \|_M \pi_C(\bar{p})|_C,
\]

where we used \( (\pi_C(\bar{p}(x)), \pi_C(\bar{p}(x)))_H = 0, x \in \Omega \). Consequently we conclude

\[
\langle \bar{p}, \bar{u} \rangle = \| \bar{u} \|_M \pi_C(\bar{p})|_C.
\]

In a similar way we get

\[
\sup_{u \in M(\Omega, C)} \langle \bar{p}, u \rangle = m \pi_C(\bar{p})|_C \quad \forall m \in \mathbb{R}_+.
\]

Combining these results the variational inequality (3.2) can be reformulated as

\[
\| \pi_C(\bar{p})|_C \| m - \| \bar{u} \|_M + G(\| \bar{u} \|_M) \leq G(m) \quad \forall m \in \mathbb{R}_+
\]

By definition of the subdifferential and \( \text{dom } G \subset \mathbb{R}_+ \) this yields the second condition in (3.3). The case \( \bar{u} = 0 \) follows by similar arguments finishing the proof.

The first condition in (3.3) can be equivalently expressed through a sparsity condition on the total variation measure \( |\bar{u}| \) and a projection formula for the Radon-Nikodým derivative \( \bar{u}' \).

**Proposition 3.4.** Let \( \varphi \in C(\Omega, H) \) and \( u \in M(\Omega, C) \) with polar decomposition \( du = u'd|u| \) be given. Then the following two statements are equivalent:

- There holds

  \[
  \langle \varphi, u \rangle = \| \pi_C(\varphi)\|_C \| u \|_M. \tag{3.5}
  \]

- There holds

  \[
  \text{supp} \, |u| \subset \{ x \in \Omega \mid \| \pi_C(\varphi(x))\|_H = \| \pi_C(\varphi)\|_C \} , \tag{3.6}
  \]

  as well as

  \[
  u'(x) = \frac{1}{\| \pi_C(\varphi)\|_C} \pi_C(\varphi(x)) \quad \text{if } \| \pi_C(\varphi)\|_C \neq 0 \quad \text{if } \| \pi_C(\varphi)\|_C = 0 \quad |u|-\text{a.e. } x \in \Omega. \tag{3.7}
  \]

**Proof.** Assume that (3.5) holds. If \( \| \pi_C(\varphi)\|_C = 0 \) the support condition in (3.6) becomes trivial and

\[
\langle \varphi, u \rangle = \langle \pi_C^\ast(\varphi), u \rangle = \int_{\Omega} \langle \pi_C^\ast(\varphi(x)), u'(x) \rangle_H d|u|(x) = 0
\]

Since the integrand is non-positive it vanishes \( |u| \)-almost everywhere. This yields (3.7) in this case. Let \( \| \pi_C(\varphi)\|_C \neq 0 \). We readily observe that

\[
\| \pi_C(\varphi)\|_C \| u \|_M = \langle \varphi, u \rangle \leq \langle \pi_C(\varphi), u \rangle \leq \| \pi_C(\varphi)\|_C \| u \|_M.
\]

Therefore there holds

\[
\langle \pi_C(\varphi), u \rangle = \| \pi_C(\varphi)\|_C \| u \|_M.
\]

Rearranging this equality and writing out the duality paring yields

\[
\int_{\Omega} \left[ \langle \pi_C(\varphi(x)), u'(x) \rangle_H - \| \pi_C(\varphi)\|_C \right] d|u|(x) = 0. \tag{3.8}
\]

By estimating

\[
\langle \pi_C(\varphi(x)), u'(x) \rangle_H \leq \| \pi_C(\varphi(x))\|_H \| u'(x) \|_H \leq \| \pi_C(\varphi)\|_C, \tag{3.9}
\]

it follows that the integrand in (3.8) is non-positive and thus vanishes for \( |u|\)-a.e. \( x \in \Omega \). Accordingly there holds

\[
\langle \pi_C(\varphi)(x), u'(x) \rangle_H = \| \pi_C(\varphi)\|_C \quad |u|-\text{a.e. } x \in \Omega.
\]
In perspective of (3.9) this can only be valid if
\[ \|\pi_C(\varphi)(x)\|_H = \|\pi_C(\varphi)\|_C, \quad u'(x) = \frac{1}{\|\pi_C(\varphi)\|_C} \pi_C(\varphi)(x), \]
for \(|u|\) almost all \(x \in \Omega\). Therefore (3.7) holds. It remains to show the inclusion for \(\text{supp}|u|\) in (3.6). W.l.o.g assume \(u \neq 0\). To this end we note that the function
\[ h: \Omega \to \mathbb{R}, \quad h(x) = \|\pi_C(\varphi(x))\|_H - \|\pi_C(\varphi)\|_C, \]
is continuous, non-negative and its integral with respect to \(|u|\) vanishes. Let an arbitrary point \(\hat{x} \in \Omega\) with \(h(\hat{x}) < 0\) be given. Since \(h\) is continuous this holds in a whole neighborhood \(B_\delta(\hat{x})\). Let an arbitrary nonnegative function \(y \in C_0(\Omega(\hat{x}))\) be given. Then there exists \(t > 0\) small enough such that \(h + ty \leq 0\) on \(\Omega\). We conclude
\[ 0 \geq \langle h + ty, u \rangle = t(y, u) \geq 0. \]
Due to the arbitrary choice of \(y\) this implies \(|u||_0 = 0\) and \(B_\delta(\hat{x}) \subset \Omega \setminus \text{supp}|u|\).

Conversely let (3.6) and (3.7) hold. If \(\|\pi_C(\varphi)\|_C = 0\) we immediately get
\[ \langle \varphi, u \rangle = \int_\Omega (\pi_C(\varphi(x)), u'(x))_H \, d\|u\|_x = 0 = \|\pi_C(\varphi)\|_C \|u\|_M. \]
In the second case, for \(\|\pi_C(\varphi)\|_C \neq 0\), we split the integral to obtain
\[ \langle \varphi, u \rangle = \int_\Omega (\pi_C(\varphi(x)), u'(x))_H \, d\|u\|_x + \int_\Omega (\pi_C(\varphi(x)), u'(x))_H \, d\|u\|_x \]
\[ = \frac{1}{\|\pi_C(\varphi)\|_C} \int_\Omega (\pi_C(\varphi(x)), \pi_C(\varphi(x)))_H \, d\|u\|_x = \|\pi_C(\varphi)\|_C \|u\|_M. \]
Here we again used that \((\pi_C(\varphi(x)), \pi_C(\varphi(x)))_H = 0\) for \(|u|\)-almost every \(x \in \Omega\). This concludes the proof.

Throughout the following discussions we will restrict ourselves to optimal vector measures \(\bar{u} \neq 0\) with non-degenerate dual variable \(\bar{p}\), i.e \(\|\pi_C(\bar{p})\|_C \neq 0\). As a consequence of the previous proposition the optimality of \(\bar{u} \in M(\Omega, C)\) is characterized by conditions on its polar decomposition.

**Theorem 3.5.** Let \(\bar{u}\) be an optimal solution to (P) with polar decomposition \(d\bar{u} = \bar{u}'|\bar{u}|\) and \(\pi_C(\bar{u}) \neq 0\). Then we have
\[ \|\pi_C(\bar{p})\|_C \in \partial G(\|\bar{u}\|_M), \]
as well as
\[ \text{supp}|\bar{u}| \subset \{ x \in \Omega \mid \|\pi_C(\bar{p}(x))\|_H = \|\pi_C(\bar{p})\|_C \}, \]
\[ \bar{u}'(x) = \frac{1}{\|\pi_C(\bar{p})\|_C} \pi_C(\bar{p}(x)) \quad |\bar{u}|\text{-a.e. } x \in \Omega. \]

**Proof.** The statement follows immediately by combining Propositions 3.1 and 3.4. 

Based on the previous theorem, we can identify characteristic special cases.

**Corollary 3.6.** Let a minimizer \(\bar{u}\) to (P) be given and assume that \(\|\pi_C(\bar{p}(x))\|_H\) achieves its maximum in a finite collection of points:
\[ \{ x \in \Omega \mid \|\pi_C(\bar{p}(x))\|_H = \|\pi_C(\bar{p})\|_C \} = \{ \bar{x}_i \}_{i=1}^N. \]
Then \(\bar{u}\) is given as a sum of Dirac delta functions, i.e. there holds
\[ \bar{u} = \frac{1}{\|\pi_C(\bar{p})\|_C} \sum_{i=1}^N \bar{\mu}_i \pi_C(\bar{p}(\bar{x}_i)) \delta_{\bar{x}_i}, \]
for some \(\bar{\mu}_i \in \mathbb{R}_+\), \(i = 1, \ldots, N\).
Proof. From the inclusion condition on $\text{supp}\, |\bar{u}|$ we infer $|\bar{u}| = \sum_{i=1}^{N} \bar{c}_i \delta_{\bar{x}_i}$ for some $\bar{c}_i \in \mathbb{R}_+$, $i = 1, \ldots, N$. The claim now directly follows from the characterization of the Radon-Nikodým derivative yielding

$$\bar{u} = \sum_{i=1}^{N} \bar{\mu}_i \bar{u}^i(\bar{x}_i) \delta_{\bar{x}_i}, \quad \bar{u}'(\bar{x}_i) = \frac{1}{\|\pi_C(\bar{p})\|_C} \pi_C(\bar{p}(\bar{x}_i)).$$

\[\square\]

Corollary 3.7. Assume that $F$ is strictly convex on its domain. Then the optimal observation $\bar{y}$ and dual variable $\bar{p}$ are the same for every minimizer to $(P)$. Furthermore assume that $(3.10)$ holds and that the set

$$\{ K(\pi_C(\bar{p}(\bar{x}_i))\delta_{\bar{x}_i}) \mid i = 1, \ldots, N \} \subset Y,$$

is linearly independent. Then $(P)$ admits a unique minimizer $\bar{u} \in M(\Omega, C)$.

Proof. The prove for the uniqueness of the optimal observation is standard: assume that there are two optimal solutions $\bar{u}_1$, $\bar{u}_2$ to $(P)$ with $K\bar{u}_1 \neq K\bar{u}_2$. Set $u_s = u_1 + s(u_2 + u_1)$ for $s \in (0, 1)$. Then $u_s$ is also a minimizer of $(P)$. Since $F$ is strictly convex we conclude

$$\min_{u \in M(\Omega, H)} j(u) = j(u_s) < (1 - s)j(u_1) + sj(u_2) = j(u_s).$$

This gives a contradiction. The uniqueness of the dual variable follows now due to $\bar{p} = -\nabla f(\bar{u})$. Assume that $(3.10)$ holds and that the set in $(3.11)$ is linear independent. Moreover define the operator

$$\hat{K}: \mathbb{R}^N \rightarrow Y, \quad v \mapsto \frac{1}{\|\pi_C(\bar{p})\|_C} \sum_{i=1}^{N} v_i K(\pi_C(\bar{p}(\bar{x}_i))\delta_{\bar{x}_i}).$$

Following Corollary 3.6 every minimizer $\bar{u}$ to $(P)$ is of the form

$$\bar{u} = \frac{1}{\|\pi_C(\bar{p})\|_C} \sum_{i=1}^{N} \|\bar{u}_i\|_H \bar{p}(\bar{x}_i) \delta_{\bar{x}_i}, \quad \|\bar{u}_i\|_H \in \mathbb{R}_+.$$

Obviously the vector $\bar{\mu} = (\|\bar{u}_1\|_H, \ldots, \|\bar{u}_N\|_H)^T$ is an optimal solution to

$$\min_{\mu \in \mathbb{R}^N} \left[ F(\hat{K}\mu) + G(\|\mu\|_H) \right].$$

(3.12)

Since the set in $(3.11)$ is linearly independent we conclude that the operator $\hat{K}$ is injective. Thus the composite functional $F \circ \hat{K}$ is strictly convex on its domain in $\mathbb{R}^N$ and $(3.12)$ admits a unique solution. Combining all previous considerations yields the uniqueness of the minimizer to $(P)$. \[\square\]

4. Algorithmic solution

In this section we elaborate on the solution of $(P)$. The presentation is split into three parts. First, in Section 4.1 we formulate an algorithm relying on finitely supported iterates and the sequential insertion of single Dirac delta functions based on the method discussed in. We draw parallels between the proposed procedure and a generalized conditional gradient method; see e.g. [7]. Moreover, we provide all necessary results to prove the subsequential convergence of the generated measures towards minimizers of $(P)$ together with a sublinear convergence rate for the objective function values in Section 4.2. Finally, we propose an accelerated version of the method in Section 4.3 which aims to improve the sparsity pattern of the iterates as well as the convergence of the algorithm. It will be based on alternating between the insertion of single Dirac deltas and the optimization of the associated coefficient functions.
4.1. A generalized conditional gradient method. Similar to [8], the method to solve (P) will rely on an equivalent surrogate problem. We recall that the norms of elements in the set of solutions to (P) are bounded; see Proposition 3.1. Let $M_0 > 0$ be an upper bound on the solution set. Consider the norm-constrained problem

$$\min_{\|u\|_M \leq M_0} \left[ F(Ku) + G(\|u\|_M) + I_{\mathcal{M}(\Omega,C)}(u) \right].$$

(CP)$_{M_0}$

Clearly, by choice of $M_0$, the problems (CP)$_{M_0}$ and (P) admit the same global minimizers. Associated to this auxiliary problem we define the primal-dual gap $\Phi$: $\text{dom } j \to \mathbb{R}_+$ by

$$\Phi(u) = \max_{v \in \mathcal{M}(\Omega,C)} \left[ (\nabla f(u), u - v) + G(\|u\|_M) - G(\|v\|_M) \right].$$

(4.1)

The following proposition relates $\Phi$ to the residual of $j$ given by

$$r_j: \mathcal{M}(\Omega,H) \to \mathbb{R}_+ \cup \{+\infty\}, \quad u \mapsto j(u) - \min_{\tilde{u} \in \mathcal{M}(\Omega,H)} j(\tilde{u}).$$

Proposition 4.1. For every $u \in \text{dom } j$ there holds

$$r_j(u) \leq \Phi(u).$$

(4.2)

A measure $\tilde{u} \in \text{dom } j$ is a solution to (P) if and only if $\Phi(\tilde{u}) = 0$.

Proof. Let $u \in \text{dom } j$ and a solution $\tilde{u}$ to (P) be given. From the convexity of $f = F \circ K$ we readily obtain

$$r_j(u) = j(u) - j(\tilde{u}) \leq (\nabla f(u), u - \tilde{u}) + G(\|u\|_M) - G(\|\tilde{u}\|_M).$$

The right hand side is estimated by

$$(\nabla f(u), u - \tilde{u}) + G(\|u\|_M) - G(\|\tilde{u}\|_M) \leq \max_{v \in \mathcal{M}(\Omega,C)} \left[ (\nabla f(u), u - v) + G(\|u\|_M) - G(\|v\|_M) \right] = \Phi(u),$$

using $\|\tilde{u}\|_M \leq M_0$. This yields (4.2). It remains to prove the second claim. Let $\tilde{u} \in \text{dom } j$ be given. Clearly, if $\Phi(\tilde{u}) = 0$ we also get $r_j(\tilde{u}) = 0$. Thus $\tilde{u}$ is also a global minimizer of (P).

Conversely assume that $\tilde{u}$ is a solution to (P). According to Proposition 3.1 there holds

$$\langle \nabla f(\tilde{u}), \tilde{u} - v \rangle + G(\|\tilde{u}\|_M) - G(\|v\|_M) \leq 0 \quad \forall v \in \mathcal{M}(\Omega,C).$$

Maximizing on the left hand side over all $v \in \mathcal{M}(\Omega,C)$ with $\|v\|_M \leq M_0$ we conclude

$$\Phi(\tilde{u}) = \max_{v \in \mathcal{M}(\Omega,C)} \left[ (\nabla f(\tilde{u}), \tilde{u} - v) + G(\|\tilde{u}\|_M) - G(\|v\|_M) \right] \leq 0.$$

Since $\Phi(\tilde{u}) \in \mathbb{R}_+$ this yields $\Phi(\tilde{u}) = 0$. 

We now propose to compute a solution of (CP)$_{M_0}$, and thus also of (P), by the method described in Algorithm 1. Starting from a sparse initial measure $u^0 \in \mathcal{M}_0(\Omega,C)$, the procedure generates a sequence of sparse iterates $\{u^k\}_{k \in \mathbb{N}} \subset \mathcal{M}_S(\Omega,C)$ by the sequential insertion of single Dirac delta functions. Following Proposition 4.1, convergence of the method can be monitored by the primal-dual gap $\Phi(u^k)$ We give a detailed description of the individual steps and their derivation in the following.

In steps 1–5 of the $k$-th iteration in Algorithm 1, an intermediate iterate $u^{k+1/2}$ is obtained as convex combination between the old iterate $u^k$ and $v^k = \tilde{u}_k \delta_{x_k} \in \mathcal{M}(\Omega,C)$. The position $\tilde{x}_k$ of the new Dirac delta corresponds to a global maximizer of $\|\pi_C(p^k(x))\|_H$, where $p^k = -\nabla f(u^k)$. Depending on $\|\pi_C(p^k)\|_C$, its coefficient function $\hat{u}_k \in C$ is either zero or given by a scalar multiple of the signum $\sigma_C(p^k(\tilde{x}_k))/\|\pi_C(p^k)\|_C$.

In the following proposition we relate the definition of $v^k$ to the computation of a descent direction in the context of a generalized conditional gradient method for the auxiliary problem (CP)$_{M_0}$. 

Algorithm 1 Generalized conditional gradient method for vector measures

\begin{algorithm}
\textbf{while} $\phi(u^k) \geq \text{TOL}$ \textbf{do}
\begin{enumerate}
\item Compute $p^k = -\nabla f(u^k) = -K^*\nabla F(Ku^k)$.
\item Determine
\begin{equation*}
\hat{x}^k \in \arg \max_{x \in \Omega} \|\pi_C(p^k(x))\|_C.
\end{equation*}
\item Compute a constant $\|v^k\|_M \leq M_0$ with
\begin{equation*}
\|v^k\|_M \in \begin{cases}
\{0\} & \text{if } \|\pi_C(p^k)\|_C < \inf \partial G(0), \\
\partial G^\ast(\|\pi_C(p^k)\|_C) & \text{if } \|\pi_C(p^k)\|_C > \sup \partial G(M_0), \\
\{M_0\} & \text{else}.
\end{cases}
\end{equation*}
\item Set $v^k = v^k_M \begin{cases}
0 \quad & \text{if } \pi_C(p^k) = 0, \\
\|\pi_C(p^k)\|_C K_\delta^k & \text{else}.
\end{cases}$
\item Select a stepsize $s^k \in [0, 1]$ and set $u^{k+1/2} = u^k + s^k(u^k - u^k)$.
\item Define $A_k = \supp |u^k| \cup \{\hat{x}^k\}$. Choose $u^{k+1} \in \mathcal{M}(\Omega, C)$, with $\supp |u^{k+1}| \subset A_k$, $j(u^{k+1}) \leq j(u^{k+1/2})$, and $\|u^{k+1}\|_M \leq M_0$.
\end{enumerate}
\textbf{end while}
\end{algorithm}

Proposition 4.2. Let $u^k \in \text{dom } j$ be given and set $p^k = -\nabla f(u^k)$. Choose a point $\hat{x}^k \in \Omega$ with $\|\pi_C(p^k(\hat{x}^k))\|_H = \|\pi_C(p^k)\|_C$ and a constant $\|v^k\|_M \leq M_0$ with
\begin{equation}
\|v^k\|_M \in \begin{cases} 
\{0\} & \|\pi_C(p^k)\|_C < \inf \partial G(0), \\
\partial G^\ast(\|\pi_C(p^k)\|_C) & \|\pi_C(p^k)\|_C \in \bigcup_{m \in [0, M_0]} \partial G(m). \\
\{M_0\} & \|\pi_C(p^k)\|_C > \sup \partial G(M_0). \end{cases} \tag{4.3}
\end{equation}

Then the measure
\begin{equation}
v^k = v^k_M \begin{cases} 
0 & \pi_C(p^k) = 0, \\
\|\pi_C(p^k)\|_C K_\delta^k & \pi_C(p^k) \neq 0, \end{cases} \tag{4.4}
\end{equation}
is a minimizer of
\begin{equation}
\min_{v \in \mathcal{M}(\Omega, C)} \left\{ -p^k(v) + G(\|v\|_M) \right\} \tag{4.5}
\end{equation}

Proof. We note that with the substitution $v = m\tilde{v}$ for $m \in [0, M_0]$ and $\tilde{v} \in \mathcal{M}(\Omega, H)$, $\|\tilde{v}\|_M \leq 1$, the problem (4.5) can be decomposed into
\begin{equation*}
\min_{m \in [0, M_0]} \min_{\tilde{v} \in \mathcal{M}(\Omega, C)} \left\{ -m\langle p^k, \tilde{v} \rangle + G(m) \right\}
\end{equation*}

Due to the non-negativity of $m$ we estimate
\begin{equation*}
m\langle -p^k, \tilde{v} \rangle = -m \int_{\Omega} \langle p^k(x), \tilde{v}'(x) \rangle_H d|\tilde{v}|(x)
\geq -m \int_{\Omega} \langle \pi_C(p^k(x)), \tilde{v}'(x) \rangle_H d|\tilde{v}|(x) \geq -m \|\pi_C(p^k)\|_C.
\end{equation*}

for every $\tilde{v} \in \mathcal{M}(\Omega, C)$, $\|\tilde{v}\|_M \leq 1$. Accordingly a solution to the inner problem is given by
\begin{equation*}
\hat{v} = \begin{cases} 
0 & \pi_C(p^k) = 0, \\
\|\pi_C(p^k)\|_C K_\delta^k & \pi_C(p^k) \neq 0, \end{cases} \quad \hat{x} \in \arg \max_{x \in \Omega} \|\pi_C(p^k(x))\|_H.
\end{equation*}
To solve the outer problem it thus suffices to consider
\[
\min_{m \in [0,M_0]} [-m\|\pi_C(p^k)\|_C + G(m)].
\]
By standard arguments, \(\bar{m} \in [0,M_0]\) is optimal if and only if
\[
\|\pi_C(p^k)\|_C \in \partial(G(\cdot) + I_{[0,M_0]}(\cdot))(\bar{m}).
\]
Since \(I_{[0,M_0]}\) is continuous on the interior of its domain we can split the subdifferential to obtain
\[
\|\pi_C(p^k)\|_C \in \partial G(\bar{m}) + \partial I_{[0,M_0]}(\bar{m}).
\]
Distinguishing between the three different cases in (4.3) completes the proof. \(\square\)

In particular, it is immediate that \(v^k\) realizes the maximum in the definition of \(\Phi(u^k)\) i.e. there holds
\[
\Phi(u^k) = \langle p^k, v^k - u^k \rangle + G(||u^k||_M) - G(||v^k||_M)
\]
\[
= \langle -p^k, u^k \rangle + G(||u^k||_M) + \|\pi_C(p^k)\|_C ||v^k||_M - G(||v^k||_M).
\]
Moreover, \(\Phi(u^k)\) can be cheaply computed as a by-product of steps 1–3 in Algorithm 1.

The step size \(s^k \in [0,1]\) in step 5 of Algorithm 1 is chosen according to the following generalization of the well-known Armijo-Goldstein condition.

**Definition 4.1.** For \(s \in [0,1]\) set \(u^k_s = u^k + s(v^k - u^k)\). Let \(\gamma \in (0,1), \alpha \in (0,1/2]\). The step size \(s^k\) is chosen according to the Quasi-Armijo-Goldstein condition if \(s^k = \gamma^n k\) where \(n_k \in \mathbb{N}\) is the smallest integer with
\[
\alpha \gamma^n k \Phi(u^k) \leq j(u^k) - j(u^k_{\gamma^n k}). \tag{4.6}
\]

The following lemma illustrates that this choice of the step size is always possible if \(u^k\) is not optimal.

**Lemma 4.3.** Let an arbitrary measure \(u^k \in \text{dom} j\) be given. Assume that \(\Phi(u^k) > 0\) and denote by \(v^k \in M(\Omega,C)\) a solution of the associated partially linearized problem (4.5). Define \(u^k_s = u^k + s(v^k - u^k)\) and the extended real-valued function
\[
W : [0,1] \to \mathbb{R} \cup \{-\infty\} \quad W(s) = \frac{j(u^k) - j(u^k_s)}{s\Phi(u^k)}.
\]
The function \(W\) is upper semi-continuous on \((0,1]\) and \(\lim \inf_{s \to 0} W(s) = 1\).

**Proof.** Due to \(u^k \in \text{dom} j\), \(v^k \in M(\Omega,C)\) and **A3** there holds \(u^k_s \in \text{dom} j\) for all \(s\) small enough. By definition of \(v^k\) we have
\[
W(s) = \frac{j(u^k) - j(u^k_s)}{s\Phi(u^k)} = \frac{j(u^k) - j(u^k_s)}{s(\langle \nabla f(u^k), u^k - v^k \rangle + G(||u^k||_M) - G(||v^k||_M))}.
\]
From the mean value theorem we get the existence of \(\zeta_s \in [0,1]\) and \(\tilde{u}^k_s = u^k + \zeta_s (u^k_s - u^k) \in \text{dom} j\) with
\[
W(s) = \frac{s(\langle \nabla f(\tilde{u}^k_s), u^k - v^k \rangle + G(||u^k||_M) - G(||u^k||_M))}{s(\langle \nabla f(u^k), u^k - v^k \rangle + G(||u^k||_M) - G(||v^k||_M))}.
\]
Using the convexity of \(G \circ || \cdot ||_M\), we estimate
\[
\frac{s(\langle \nabla f(\tilde{u}^k_s), u^k - v^k \rangle + G(||u^k||_M) - G(||u^k||_M))}{s(\langle \nabla f(u^k), u^k - v^k \rangle + G(||u^k||_M) - G(||v^k||_M))} \geq \frac{s(\langle \nabla f(\tilde{u}^k_s), u^k - v^k \rangle + G(||u^k||_M) - G(||v^k||_M))}{s(\langle \nabla f(u^k), u^k - v^k \rangle + G(||u^k||_M) - G(||v^k||_M))}.
\]
Since $\zeta_s$ is bounded independently of $s$, there holds $\hat{u}^k_s \rightharpoonup^* u^k$ for $s \to 0$. Due to the weak*-to-strong continuity of $\nabla f$, the right hand side of the inequality tends to 1 yielding $\liminf_{s \to 0} W(s) \geq 1$. The upper semi-continuity of $W$ on $(0, 1)$ follows directly from $u^k_s \in \mathcal{M}(\Omega, C)$ for all $s \in (0, 1]$ and from the lower weak* semi-continuity of $j$ on $\mathcal{M}(\Omega, C)$.

We point out that the choice of $s_k$ according to the Quasi-Armijo-Goldstein condition ensures the monotonicity of the objective function values, i.e. we have $j(u^{k+1}) \leq j(u^{k+1/2}) < j(u^k)$ if $u^k$ is not a minimizer of $(\mathcal{P})$. It is however important to note that the GCG step only allows for a removal of points in the unlikely case that $s_k = 1$, i.e. $u^k$ is replaced by the solution $v^k$ to the linearized problem. In particular, if $(\mathcal{P})$ admits a unique sparse minimizer $\bar{u}$ each of its Dirac delta functions may be approximated by an ever growing number of delta functions in the iterate $u^k$. This leads to undesired clustering of Dirac delta functions around the optimal positions.

To mitigate these effects we include the black box point removal step 6 into the method. In order to discuss these additional optimization steps we consider an ordered set of distinct points $\mathcal{A} = \{ x_i \in \Omega \mid i = 1, \ldots, N \}$ and the associated parametrization $U_{\mathcal{A}}$ defined by

$$U_{\mathcal{A}}: H^N \to \mathcal{M}(\Omega, H), \quad u \mapsto \sum_{i=1}^{N} u_i \delta_{x_i}.$$ (4.7)

The point removal procedure in step 6 of Algorithm 1 is now based on the approximate solution of an auxiliary problem on the Hilbert space $H^{#\mathcal{A}}$

$$\min_{u \in C^{#\mathcal{A}}} j(U_{\mathcal{A}}(u)) = F(KU_{\mathcal{A}}(u)) + G(\|U_{\mathcal{A}}(u)\|_\mathcal{M}).$$ (\mathcal{P}(\mathcal{A}))

where the set $\mathcal{A}$ is chosen in the algorithm as $\mathcal{A}_k = \text{supp} |u^k| \cup \{\hat{x}^k\}$. We point out that

$$\|U_{\mathcal{A}}(u)\|_\mathcal{M} = \#\mathcal{A} \sum_{i=1}^{#\mathcal{A}} \|u_i\|_H.$$ 

Thus, loosely speaking, we fix the positions of the Dirac delta functions in the current iterate $u^k$ and approximately optimize their coefficient functions while ensuring descent $j(u^{k+1}) \leq j(u^{k+1/2})$ and $\|u^{k+1}\|_\mathcal{M} \leq M_0$. In particular, this choice implies that all Dirac delta functions for which the corresponding coefficient functions are set to zero will be removed from the iterate due to the choice of the set $\mathcal{A}_k = \text{supp}$.

4.2. Worst-case convergence analysis. In this section we address the convergence of the method described in Algorithm 1. To this end, given $u_0 \in \text{dom} j$, define the sublevel set

$$E_j(u_0) = \{ u \in \mathcal{M}(\Omega, H) \mid j(u) \leq j(u_0) \} \subset \text{dom} j$$

as well as the image set

$$KE_j(u_0) := \{ Ku \mid u \in E_j(u_0) \} \subset \text{dom} F.$$ 

In order to obtain quantifiable estimates for the descent in the objection function values we impose additional regularity requirements on the gradient of $F$ until the end of this section.

Assumption 4.1. For every $u_0 \in \text{dom} j$ the gradient $\nabla F$ is Lipschitz continuous on the image set of $E_j(u_0)$ under $K$: There exists a constant $L_{Ku_0}$ only depending on $j(u_0)$ with

$$\|\nabla F(y_1) - \nabla F(y_2)\|_{\mathcal{Y}} \leq L_{u_0}\|y_1 - y_2\|_{\mathcal{Y}} \quad \forall y_1, y_2 \in KE_j(u_0).$$

Clearly, Assumption 4.1 implies Lipschitz continuity of $\nabla f$ on $E_j(u_0)$.

Lemma 4.4. Let Assumption 4.1 hold. Given $u_0 \in \text{dom} j$ we define $L_{u_0} = L_{Ku_0}\|K^*\|_{\mathcal{L}(\mathcal{Y}, \mathcal{C}(\Omega, H))}^2$. There holds

$$\|\nabla f(u_1) - \nabla f(u_2)\|_{\mathcal{C}} \leq L_{u_0}\|u_1 - u_2\|_{\mathcal{M}} \quad \forall u_1, u_2 \in E_j(u_0).$$
where the right-hand side simplifies to

\[ E \]

Note that

\[ \text{Due to the convexity of the sublevel set} \]

Proof. Let \( u_1, u_2 \in E_j(u_0) \) be given. Then there holds

\[
\| \nabla f(u_1) - \nabla f(u_2) \|_C = \| K^* \nabla F(Ku_1) - K^* \nabla F(Ku_2) \|_C \\
\leq \| K^* \|_{\mathcal{L}(Y,L(H))} \| \nabla F(Ku_1) - \nabla F(Ku_2) \|_Y \\
\leq L_{Kw} \| K^* \|_{\mathcal{L}(Y,L(H))}^2 \| u_1 - u_2 \|_M.
\]

Here we used Assumption 4.1 in the last inequality. Since \( u_1, u_2 \in E_j(u_0) \) were chosen arbitrarily this observation yields the desired result. \( \square \)

Let us now proceed to the convergence analysis of Algorithm 1. The following growth estimate for \( j \) at \( u^k \) in the search direction \( v^k \) is obtained.

Lemma 4.5. Fix an index \( k \in \mathbb{N} \). Let \( u^k, v^k \) be generated by Algorithm 1. Further let a step size \( s \in [0,1] \) with \( u^k_s = u^k + s(v^k - u^k) \in E_j(u^0) \) be given. Then there holds

\[
j(u^k_s) - j(u^k) \leq -s\Phi(u^k) + \frac{L_{u^0}}{2} \left( s\|u^k - v^k\|_M \right)^2. \tag{4.8}
\]

Proof. Due to the convexity of the sublevel set \( E_j(u^0) \) we may expand

\[
j(u^k_s) - j(u^k) = -s\langle \nabla f(u^k), u^k - v^k \rangle + G(\|u^k_s\|_M) - G(\|u^k\|_M) + R(u^k)
\]

where the remainder term is given by

\[
R(u^k) = \int_0^s \langle \nabla f(u_s) - \nabla f(u^k), v^k - u^k \rangle \ d\sigma.
\]

Note that \( u_\sigma = u^k + \sigma(v^k - u^k) \in E_j(u^0) \) for \( \sigma \in [0,s] \). Using the convexity of \( g \), \( \|\tilde{u}\|_M \leq M_0 \) and the definition of \( v^k \) we obtain

\[
-s\langle \nabla f(u^k), u^k - v^k \rangle + G(\|u^k_s\|_M) - G(\|u^k\|_M) \\
\leq -s \left( \langle \nabla f(u^k), u^k - v^k \rangle + G(\|u^k\|_M) - G(\|v^k\|_M) \right),
\]

where the right-hand side simplifies to \( -s\Phi(u^k) \). Due to the Lipschitz continuity of \( \nabla f(u^k) \) on \( E_j(u^0) \) we get

\[
R(u^k) \leq \|v^k - u^k\|_M \int_0^s \| \nabla f(u_s) - \nabla f(u^k) \|_C \ d\sigma \\
\leq L_u \|v^k - u^k\|_M^2 \int_0^s \sigma \ d\sigma = \frac{L_u}{2} \left( s\|v^k - u^k\|_M \right)^2.
\]

Combining both estimates yields the result. \( \square \)

Due to the possibly open domain of \( F \) in \( M(\Omega,C) \) we also need the following technical lemma concerning the continuity properties of the function \( W \) which was introduced in Lemma 4.3.

Lemma 4.6. Let \( u^k \in \text{dom} j \) with \( \Phi(u^k) > 0 \) be given and denote by \( v^k \in M(\Omega,C) \) a solution to the associated linearized problem (4.5). If \( u^k \in \text{dom} j \) we have \( W \in C((0,1)) \). Otherwise there exists \( \tilde{s} \in (0,1] \) with \( W \in C((0,\tilde{s})) \) and \( \lim_{s \to \tilde{s}^-} W(s) = -\infty \).

Proof. Since \( u^k \) is not optimal the function \( W \) is proper. Set \( u^k_s = u^k + s(v^k - u^k) \) and define the convex auxiliary function

\[
j: [0,1] \rightarrow \mathbb{R}, \quad s \mapsto j(u^k_s),
\]

Since \( \Phi(u^k) > 0 \) there exists \( s \in (0,1] \) with \( j(s) \in \mathbb{R} \). We further conclude

\[
(0,\tilde{s}) \subset \text{dom}_{(0,1]} j, \quad \tilde{s} = \sup \text{dom}_{[0,1]} j \in (0,1].
\]

Note that \( j \) is continuous on \( (0,\tilde{s}) \), see [25, Proposition 2.5]. Let us distinguish two cases. If \( v^k \in \text{dom} j \) there holds \( \tilde{s} = 1 \). From its definition we thus get \( W \in C((0,1)) \). In the second case if \( v^k \not\in \text{dom} j \) there holds

\[
\tilde{s} \not\in \text{dom}_{[0,1]} j, \quad \lim_{s \to \tilde{s}^-} j(u^k_s) = +\infty,
\]

where the remainder term is given by
due to the openness assumption on the domain of $F$. Hence we conclude
\[
W \in C((0, \tilde{s})), \quad \lim_{s \to \tilde{s}} W(s) = -\infty,
\]
which finishes the proof. \hfill $\square$

Collecting all the previous results we can prove a sublinear rate of convergence for the residuals of the iterates generated by Algorithm 1.

**Theorem 4.7.** Let $F$, $K$ and $G$ fulfill Assumptions 3.1 and 4.1. Let the sequence $\{u^k\}_{k \in \mathbb{N}}$ be generated by Algorithm 1 where the stepsize is chosen according to the Quasi-Armijo-Goldstein condition with $\gamma \in (0, 1)$, $\alpha \in (0, 1/2]$. Furthermore denote by $L_{K,u^0} > 0$ the Lipschitz constant of $\nabla F$ on $KE_j(u^0)$. Then $\{u^k\}_{k \in \mathbb{N}}$ is a minimizing sequence for $j$ and there holds
\[
\eta_j(u^k) \leq \eta_j(u^k) \leq \gamma (1 - \alpha) r(u^0) \frac{\gamma (1 - \alpha) r(u^0)}{2L_{K,u^0} \sqrt{\|
abla F(u^0)\|_{\mathcal{M}}^2}}, \quad (4.9)
\]
Moreover $\{u^k\}_{k \in \mathbb{N}}$ admits at least one weak* convergent subsequence and each weak* accumulation point $u$ of $\{u^k\}_{k \in \mathbb{N}}$ is a minimizer of $j$ over $\mathcal{M}(\Omega, H)$.

**Proof.** By the definition of the step size $s^k$ as well as (4.2) there holds
\[
\alpha s^k \eta_j(u^k) \leq \alpha s^k \Phi(u^k) \leq \eta_j(u^k) - \eta_j(u^{k+1/2}),
\]
which yields
\[
\eta_j(u^{k+1/2}) \leq (1 - \alpha) \eta_j(u^k).
\] (4.10)

Since $\Phi(u^k) > 0$ we obtain $s^k \neq 0$ for all $k$. Two cases have to be distinguished. If $s^k$ is equal to one we immediately arrive at
\[
\eta_j(u^{k+1}) \leq \eta_j(u^{k+1/2}) \leq (1 - \alpha) \eta_j(u^k) \leq \eta_j(u^k) - \frac{1}{\gamma} \frac{r_j(u^k) \|\delta u^k\|_{\mathcal{M}}^2}{\Phi(u^k)}.
\]
In the second case, if $s^k < 1$, there exists $\tilde{s}^k \in [s^k, s^k/\gamma]$ with
\[
\alpha = \frac{j(u^k) - j(u^k + \tilde{s}^k(u^k - u^k))}{\tilde{s}^k \Phi(u^k)}.
\]
using Lemma 4.6 and applying the intermediate value theorem to $W$. Consequently, $u^k + s(u^k - u^k) \in E_j(u^0)$ for all $0 \leq s \leq \tilde{s}^k$ due to the convexity of $j$. Because of the Lipschitz-continuity of $\nabla f$ on $E_j(u^0)$, Lemma 4.5 can be applied and, defining $\delta u^k = v^k - u^k$, there holds
\[
\alpha = \frac{j(u^k) - j(u^k + \tilde{s}^k \delta u^k)}{\tilde{s}^k \Phi(u^k)} \geq 1 - \frac{L_{w_0} \tilde{s}^k \|\delta u^k\|_{\mathcal{M}}^2}{2} \Phi(u^k) \geq 1 - \frac{L_{w_0} s^k \|\delta u^k\|_{\mathcal{M}}^2}{2\gamma} \Phi(u^k).
\]
The last estimate is true because of $\tilde{s}^k \leq s^k/\gamma$. Note that we have $\delta u^k \neq 0$ since $\Phi(u^k) > 0$. Reordering and using (4.2) yields
\[
1 \geq s^k \geq 2\gamma (1 - \alpha) \frac{\|\delta u^k\|_{\mathcal{M}}^2}{L_{w_0} \|v^k - u^k\|_{\mathcal{M}}^2} \geq 2\gamma (1 - \alpha) \frac{r_j(u^k)}{L_{w_0} \|v^k - u^k\|_{\mathcal{M}}^2}.
\]
Combining the estimates in both cases and using $r_j(u^{k+1}) \leq r_j(u^{k+1/2})$, the inequality
\[
0 \leq \frac{r_j(u^{k+1})}{r_j(u^k)} \leq \frac{r_j(u^{k+1/2})}{r_j(u^k)} - \frac{r_j(u^k)}{r_j(u^0)} - q_k \left(\frac{r_j(u^k)}{r_j(u^0)}\right)^2 \quad \forall k \in \mathbb{N}
\] (4.11)
holds, where the constant $q_k$ is given by
\[
q_k = r_j(u^0) \alpha \min \left\{ \frac{2\gamma (1 - \alpha)}{L_{w_0} \|v^k - u^k\|_{\mathcal{M}}^2}, \frac{1}{r_j(u^k)} \right\} \geq \alpha \min \left\{ \frac{2\gamma (1 - \alpha) r_j(u^0)}{4L_{w_0} (M_0)^2}, 1 \right\} =: q_k,
\]
if $s^k < 1$ and $q_k = \alpha$ otherwise. The claimed convergence rate (4.9) now follows directly from the recursion formula (4.11), see [20, Lemma 3.1], and the definition of $L_{w_0}$. Since $\{u^k\}_{k \in \mathbb{N}}$ is
bounded it admits at least one weak* accumulation. Since \( j \) is weak* lower semicontinuous and \( r(u^k) \to 0 \) we conclude that every such point is a global minimizer of \( j \).

4.3. Acceleration. The remainder of this section is focused on a fully corrective variant of Algorithm 1, where the new coefficient vector \( u^{k+1} \) in step 4 is chosen as a minimizer of the coefficient optimization problem \((P(A))\) on the point set \( A_k = \text{supp}|u^k| \cup \{\hat{x}^k\} \). The resulting method is described in Algorithm 2. In comparison to Algorithm 1 we may drop the intermediate conditional gradient step since we have \( \text{supp}|u^{k+1/2}| \subset A_k \) and all subproblems are solved up to optimality. However the computation of the solution \( v^k \in M(\Omega, C) \) to the linearized problem is still necessary for the exact evaluation of the termination criterion \( \Phi(u^k) \).

From this perspective the resulting algorithm can be also interpreted as a method acting on a sequence of active sets \( A_k \) containing a finite number of points. Recall that the support points of an optimal measure \( \tilde{u} \) align themselves with global maximizers of the dual certificate

\[
\|\pi_C(\tilde{p})\|_H: \Omega \to \mathbb{R}_+, \quad x \mapsto \|\pi_C(\tilde{p}(x))\|_H.
\]

In the \( k \)-th iteration of Algorithm 2 we greedily add a new point \( \hat{x}^k \) to the active set which maximizes the violation of this constraint by the current dual certificate \( \|\pi_C(p(x))\|_H \)

\[
\hat{x}^k \in \arg\max_{x \in \Omega} \left[ \|\pi_C(p^k(x))\|_H - \max_{\tilde{x} \in \text{supp}|u^k|} \|\pi_C(p^k(\tilde{x}))\|_H \right].
\]

The coefficient optimization problem \((P(A_k))\) can then be seen as a solution of the original problem \((P)\) on the reduced cone \( M(A_k, C) \). Again we emphasize that the iterates are pruned in each iteration by removing all Dirac delta functions with zero coefficient function. Due to

\[
\text{Algorithm 2 Primal-Dual-Active-Point strategy}
\]

\[
\text{while } \Phi(u^k) \geq \text{TOL do}
\]

1. Calculate \( p^k = -\nabla f(u^k) = -K^*\nabla F(Ku^k) \). Determine \( \hat{x}^k \in \arg\max_{x \in \Omega} \|\pi_C(p^k(x))\|_H \).

2. Set \( A_k = \text{supp}|u^k| \cup \{\hat{x}^k\} \).

3. Compute a solution \( u^{k+1} \in C\#A_k \) of \((P(A))\) with \( A = A_k \).

4. Set \( u^{k+1} = U_{A_k}(u^{k+1}) \).

end while

the choice of the position \( \hat{x}^k \) of the new Dirac delta function Algorithm 2 can be interpreted as a particular instance of the generalized conditional gradient method described in Algorithm 1. Therefore the following worst-case convergence results hold.

**Theorem 4.8.** Let \( \{u^k\}_{k \in \mathbb{N}} \) be generated by Algorithm 2 and let Assumption 4.1 hold. Then we have

\[
r_j(u^k) \leq \frac{r_j(u^0)}{1 + qk}, \quad q = \frac{1}{2} \min \left\{ \frac{\gamma(1 - \alpha)r(u^0)}{8L_K \omega\|K^*\|_{E(L, C(\Omega, H))}^2 M_0^2}, 1 \right\},
\]

where

\[
M_0 = \sup \{ \|u\|_{M_1} \mid u \in E_j(u^0) \}.
\]

**Proof.** First note that \( M_0 < \infty \) since \( j \) is radially unbound. Clearly, \( M_0 \) bounds the norms of elements in the solution set to \((P)\) and, by construction, we have \( j(u^{k+1}) \leq j(u^k) \leq j(u^0) \). Thus, there holds \( \|u^k\|_{M_1} \leq M_0 \) for all \( k \in \mathbb{N} \). Moreover, we observe that the choices of the new position \( \hat{x}^k \) as well as of the set

\[
A_k = \text{supp}|u^k| \cup \{\hat{x}^k\},
\]

coincide in Algorithms 1 and 2. The claim now follows from Theorem 4.7 setting \( \alpha = \gamma = 0.5 \) since \( \tilde{u} \in C^N \) is chosen as a global minimizer of \( j(U_{A_k}(.)) \).

\[\square\]
In the following proposition first order necessary optimality conditions for solutions $\bar{u} \in C^{\# A}$ to the coefficient optimization problem ($P(A)$) are presented.

**Proposition 4.9.** Let $A = \{ x_i \in \Omega \mid i = 1, \ldots, N \}$ be given and denote by $\bar{u} \in C^N$ an optimal solution to ($P(A)$). Set $u = U_A(\bar{u})$ and $p = -\nabla f(u)$. Then there holds

$$\max_{x \in A} \| \pi_C(p(x)) \| \in \partial G(\| u \|_M), \quad \langle p, u \rangle = \max_{x \in A} \| \pi_C(p(x)) \|_H \| u \|_M.$$

If $\max_{x \in A} \| \pi_C(p(x)) \|_H \neq 0$ this is equivalent to

$$\max_{x \in A} \| \pi_C(p(x)) \|_H \in \partial G(\| u \|_M),$$

as well as

$$\bar{u}_i \neq 0 \Rightarrow \| \pi_C(p(x_i)) \|_H = \max_{x \in A} \| \pi_C(p(x)) \|_H, \quad \bar{u}_i \|_H = \frac{\pi_C(p(x_i))}{\max_{x \in A} \| \pi_C(p(x)) \|_H}.$$

**Proof.** These statements are obtained from the results in Theorem 3.3 and Proposition 3.4. To this end note that

$$\mathcal{M}(A, H) \simeq (H^{\# A}, \| \cdot \|_{\ell^1(H)}) \simeq (H^{\# A}, \| \cdot \|_{\ell^\infty(H)})^* \simeq C(A, H)^*,$$

where the $\ell^\infty(H)$ and $\ell^1(H)$ norms of $u \in H^{\# A}$ are given by

$$\| u \|_{\ell^\infty(H)} = \max_{i=1, \ldots, \# A} \| u_i \|_H, \quad \| u \|_{\ell^1(H)} = \sum_{i=1}^{\# A} \| u_i \|_H.$$

The cone $\mathcal{M}(A, C)$ is readily identified with $C^{\# A}$. Moreover the operator $K$ can be restricted to a linear continuous operator

$$K|_A : \mathcal{M}(A, H) \to Y, \quad U_A(u) \mapsto \sum_{i=1}^{\# A} K(u_i \delta_{x_i}),$$

whose adjoint operator is given by

$$(K|_A)^* : Y \to C(A, H), \quad [(K|_A)^* y](x) = [K^* y](x),$$

for $y \in Y$ and $x \in A$. \hfill \Box

Algorithm 2 terminates if the active sets in two subsequent iterations coincide. This is shown in the next corollary. Additionally, this implies convergence in finitely many iterations if $\Omega$ is discrete.

**Corollary 4.10.** Let $\{ u^k \}_{k \in \mathbb{N}}$ be generated by Algorithm 2. Assume that $A_k = A_{k+1}$ for some $k > 1$. Then $u^{k+1} \in \mathcal{M}(\Omega, C)$ is a global minimizer of ($P$).

**Proof.** Let $k > 1$ with $A_k = A_{k+1}$ be given. Then there holds

$$\hat{x}^{k+1} \in A_k, \quad \| \pi_C(p^{k+1}(\hat{x})) \|_H = \| \pi_C(p^{k+1}) \|_C = \max_{x \in A_k} \| \pi_C(p^k(x)) \|_H.$$

Since $u^{k+1} = U_{A_k}(u^{k+1})$ we conclude

$$\| p^{k+1} \|_C \in \partial G(\| u^{k+1} \|_M),$$

as well as

$$\langle p^{k+1}, u^{k+1} \rangle = \max_{x \in A_k} \| \pi_C(p^k(x)) \|_H \| u^{k+1} \|_M = \| \pi_C(p^{k+1}) \|_C \| u^{k+1} \|_M.$$

from Proposition 4.9. Invoking Theorem 3.3 it follows that $u^{k+1}$ is a solution to ($P$). \hfill \Box

**Corollary 4.11.** Assume that $\Omega = \{ x_i \in \mathbb{R}^d \mid i = 1, \ldots, N \}$ for some $N \in \mathbb{N}$. Then there exists $k \in \mathbb{N}$ such that $u^k$ is a solution to ($P$).
Proof. Since the subproblems in step 2 of Algorithm 2 are solved up to optimality and \( j(u^{k+1}) < j(u^k) \) if \( \Phi(u^k) > 0 \) we have
\[
\sup |u^{k+1}| \in P(\Omega) \setminus \bigcup_{i=1}^{k} \{ \sup |u^i| \}.
\]
Here \( P(\Omega) \) denotes the power sets of \( \Omega \). Since \( \Omega \) only contains only finitely many points, Algorithm 2 converges after at most \( k = \#P(\Omega) \) iterations. \( \square \)

We further derive the following estimates for the primal-dual gap \( \Phi(u^k) \).

Lemma 4.12. Assume that the sequence \( \{u^k\}_{k \in \mathbb{N}} \) is generated by Algorithm 2. Set \( p^k = -\nabla f(u^k) \) and \( \lambda^k = \max_{x \in \text{supp}|u^k|} \|\pi_C(p^k(x))\|_H \). Then there holds
\[
\|u^k\|_M \left( \|\pi_C(p^k)\|_C - \lambda^k \right) \leq \Phi(u^k) \leq \|u^k\|_M \left( \|\pi_C(p^k)\|_C - \lambda^k \right),
\]
where \( v^k \) is determined according to Proposition 4.2. In particular, we have
\[
\Phi(u^k) \leq M_0 \left( \|\pi_C(p^k)\|_C - \lambda^k \right).
\]

Proof. By construction of \( v^k \) and \( u^k \) there holds
\[
\Phi(u^k) = (-p^k, u^k) + G(\|u^k\|_M) + \langle p^k, v^k \rangle - G(\|v^k\|_M)
\]
\[
= -\lambda^k \|u^k\|_M + G(\|u^k\|_M) + \|\pi_C(p^k)\|_C \|v^k\|_M - G(\|v^k\|_M).
\]
Since \( v^k \) is a solution of the partially linearized problem and \( \|u^k\|_M \leq M_0 \) we further obtain
\[
-\|\pi_C(p^k)\|_C \|v^k\|_M + G(\|v^k\|_M) \leq -\|\pi_C(p^k)\|_C \|u^k\|_M + G(\|u^k\|_M),
\]
which gives the first inequality. Using \( \lambda^k \in \partial G(\|u^k\|_M) \), see Proposition 4.9, we estimate
\[
G(\|v^k\|_M) \geq G(\|u^k\|_M) + \lambda^k (\|u^k\|_M - \|u^k\|_M),
\]
which provides the second inequality. The last inequality is a consequence of \( \|v^k\|_M \leq M_0 \). \( \square \)

5. IMPROVED CONVERGENCE ANALYSIS

This part of the paper is devoted to an improved convergence analysis for Algorithm 2 method under additional structural assumptions on the sparse minimization problem (\( P \)). To this end we first fix some additional notation and function spaces. Associated to the sequence \( u^k \) of iterates generated by Algorithm 2 we consider the sequences of observations \( y^k = Ku^k \), dual variables \( p^k = -\nabla f(u^k) \) and dual certificates \( P^k = \|\pi_C(p^k)\|_H \). Furthermore we define \( \lambda^k = \max_{x \in \text{supp}|u^k|} |p^k(x)| \) for all \( k \in \mathbb{N} \). If \( \bar{u} \) is a weak* accumulation point of \( \{u^k\}_{k \in \mathbb{N}} \) we set
\[
\bar{y} = Ku, \quad \bar{p} = -\nabla f(u), \quad \bar{P} = \|\pi_C(\bar{p})\|_H, \quad \bar{\lambda} = \max_{x \in \text{supp}|u^k|} \bar{P}(x).
\]
Moreover given an open set \( \Omega_R \subset \Omega \) we denote by \( C^2(\Omega_R, H) \) (resp. \( C^2(\Omega_R) \)) the spaces of \( H \)-valued (resp. scalar-valued) two times continuously differentiable functions on \( \Omega_R \) whose derivatives can be continuously extended up to the boundary of \( \Omega_R \). Analogously we define the space of Lipschitz continuous functions on its closure as
\[
C^{0,1}(\Omega_R, H) = \left\{ \varphi \in C(\Omega_R, H) \left| \|\varphi\|_{\text{Lip}} = \sup_{x_1, x_2 \in \Omega_R, x_1 \neq x_2} \frac{\|\varphi(x_1) - \varphi(x_2)\|_H}{|x_1 - x_2|_{\mathbb{R}^d}} < \infty \right. \right\},
\]
which is a Banach space with respect to the norm
\[
\|\varphi\|_{C^{0,1}(\Omega_R, H)} = \|\varphi\|_{C(\Omega_R, H)} + \|\varphi\|_{\text{Lip}}.
\]
Throughout this part of the paper we make the following additional assumptions on the smooth part \( f = F \circ K \) of \( j \) and the set of admissible controls. We restrict the following considerations to the special case of \( C = H \). A discussion of the derived results in the presence of additional constraints on the vector measures is given in Section 5.3.
The functional \( F \) is differentiable on \( \text{dom} \ F \). Moreover it is uniformly convex around the optimal observation \( \bar{y} \in \text{dom} \ F \), i.e. there exists a neighbourhood \( N(\bar{y}) \subset \text{dom} \ F \) of \( \bar{y} \) in \( Y \) and a constant \( \gamma_0 > 0 \) with
\[
(\nabla F(y_1) - \nabla F(y_2), y_1 - y_2)_Y \geq \gamma_0 \|y_1 - y_2\|_Y^2 \quad \forall y_1, y_2 \in N(\bar{y}).
\]

Note that the smoothness assumption on \( F \) implies Lipschitz continuity of its gradient \( \nabla F \) on the image of the sublevel set \( F_\gamma \) for some \( \gamma > 0 \).

**Proposition 5.1.** Let \( u_0 \in \text{dom} \ f \) be given. Then \( \nabla F : \text{dom} \ F \to Y \) is Lipschitz continuous on \( K E_j(u_0) \): there exists \( L_{u_0} > 0 \) with
\[
\|\nabla F(y_1) - \nabla F(y_2)\|_Y \leq L_{u_0} \|y_1 - y_2\|_Y \quad \forall y_1, y_2 \in K E_j(u_0).
\]

**Proof.** Due to the weak*-to-strong continuity of \( K \) the set \( K E_j(u_0) \) is compact in \( Y \). Thus the statement follows from the continuous differentiability of \( \nabla F \).

In the following we derive improved local convergence results for Algorithm 2 provided that several structural assumptions on the unique dual variable \( \bar{u} \in C(\bar{\Omega}, H) \) as well as the dual certificate \( \bar{P} \in C(\bar{\Omega}) \) are fulfilled. For a better illustration of the intuition behind these additional requirements we split them in two parts. First recall that the support points of the total variation measure \( |\bar{\mu}| \) associated to a minimizer \( \bar{u} \in \mathcal{M}(\bar{\Omega}, H) \) align themselves with global maximizers of the dual certificate \( \bar{P} \). Moreover the Radon-Nikodým derivative \( \bar{u}' \) is completely characterized by the dual variable \( \bar{p} \), see Theorem 3.3.

**Assumption 5.2.** The dual certificate \( \bar{P} \in C(\bar{\Omega}) \) fulfills
\[
\|\bar{P}\|_{C(\bar{\Omega})} > 0, \quad \{ x \in \bar{\Omega} \mid \bar{P}(x) = \bar{\lambda} \} = \{ \bar{x}_i \}_{i=1}^N \subset \text{int} \ \bar{\Omega}.
\]
Moreover the set
\[
\{ K(\bar{p}(\bar{x}_i)\delta_{\bar{x}_i}) \mid i = 1, \ldots, N \} \subset Y;
\]
is linearly independent and there exists a radius \( R > 0 \) with
\[
\Omega_R := \bigcup_{i=1}^N B_R(\bar{x}_i) \subset \text{int} \ \bar{\Omega}, \quad B_R(\bar{x}_i) \cap B_R(\bar{x}_j) = \emptyset, \quad i \neq j
\]
as well as
\[
K^* \in \mathcal{L}(Y, C(\bar{\Omega}, H)) \cap \mathcal{L}(Y, C^2(\bar{\Omega}_R, H)).
\]

This assumption has two important implications. On the one hand the minimizer \( \bar{u} \) to (P) is unique and given by a finite sum of Dirac delta functions
\[
\bar{u} = \sum_{i=1}^N \bar{u}_i \delta_{\bar{x}_i}, \quad \bar{u}_i = \|\bar{u}_i\|_H \frac{\bar{p}(\bar{x}_i)}{\bar{\lambda}}, \quad \bar{\lambda} \in G(||\bar{u}\|_\mathcal{M}),
\]
where \( ||\bar{u}_i||_H \in \mathbb{R}_+, \ i = 1, \ldots, N \), see Corollary 3.7. On the other hand this implies \( \bar{p} \in C^2(\bar{\Omega}_R, H) \) and, since we have \( \bar{\lambda} > 0 \), \( R \) may be chosen small enough to ensure \( \bar{P} \in C^2(\bar{\Omega}_R) \), see Lemma A.1, and \( P^k \in C^2(\Omega_R) \) for all \( k \in \mathbb{N} \) large enough following Lemma A.3. In particular this yields
\[
\nabla \bar{P}(\bar{x}_i) = 0, \quad i = 1, \ldots, N.
\]

Secondly we now assume that the curvature of \( \bar{P} \) around its global maximizers does not degenerate.

**Assumption 5.3.** There holds \( \text{supp} |\bar{u}| = \{\bar{x}_i\}_{i=1}^N \), i.e. \( ||\bar{u}_i||_H > 0 \) for \( i = 1, \ldots, N \). Furthermore we have
\[
-(\xi, \nabla^2 \bar{P}(\bar{x}_i)\xi)_{\mathbb{R}^d} \geq \theta_0 |\xi|_{\mathbb{R}^d}^2 \quad \forall \xi \in \mathbb{R}^d,
\]
for some \( \theta_0 > 0 \) and all \( i \in \{1, \ldots, N\} \).
Lemma 5.3. In the context of super-resolution the conditions in this last assumption (for the case of $H = \mathbb{R}$) are referred to as non-degenerate source condition for the measure $\bar{u}$, see [22, 23]. Furthermore we recall the connection of sparse minimization problems to state constrained optimization, cf. [13]. From this point of view the equality condition on $\text{supp}|u^k|$ corresponds to a strict complementarity assumption on the Lagrange multiplier associated to the state constraint. Moreover in this case the definiteness assumption on the Hessian of $P$ can be interpreted as a condition on the curvature of the optimal state around those points in which it touches the constraint. Both of these conditions are well-established in the field of semi-infinite optimization. We refer e.g. to [43] where similar assumptions are used to derive finite element error estimates. In [50] the author imposes comparable conditions to derive second order optimality conditions for semi-infinite optimization problems.

In order to make the following presentation more transparent we state the main result of this section beforehand. The following theorem yields improved local convergence rates for the residual $r_j(u^k)$ associated to the sequence $\{u^k\}_{k \in \mathbb{N}}$ generated by Algorithm 2. Moreover since both, the iterates $u^k$ as well as the minimizer $\bar{u}$, are sparse we may quantify the convergence of $\{u^k\}_{k \in \mathbb{N}}$ through convergence rates for the support points of the iterates as well as their coefficient functions.

**Theorem 5.2.** Let the sequence $\{u^k\}_{k \in \mathbb{N}}$ be generated by Algorithm 2 started at $u^0$. Assume that Assumptions 5.1, 5.2 and 5.3 hold. Then $\{u^k\}_{k \in \mathbb{N}}$ is a minimizing sequence for $j$ and there holds

$$u^k \to^* \bar{u}, \quad r_j(u^k) \leq \frac{c_1}{1 + qk}, \quad (5.1)$$

for all $k \in \mathbb{N}$ and some constants $c_1$, $q > 0$ which only depend on the initial residual $r_j(u^0)$ and problem dependent quantities but are otherwise independent of $\{u^k\}_{k \in \mathbb{N}}$ and $\bar{u}$. Moreover there exist $R_1 > 0$, $\bar{k} \in \mathbb{N}$ and $\zeta \in (0, 1)$ with

$$\text{supp}|u^k| \subseteq \bigcup_{i=1}^{N} \bar{B}_{R_1}(\bar{x}_i), \quad \text{supp}|u^k| \cap \bar{B}_{R_1}(\bar{x}_i) \neq \emptyset, \quad i = 1, \ldots, N,$$

as well as, for all $k \geq \bar{k}$, it holds

$$r_j(u^k) + \max_{i=1, \ldots, N} \max_{x \in \text{supp}|u^k| \cap \bar{B}_{R_1}(\bar{x}_i)} |x - \bar{x}_i|_{\mathbb{R}^d} + \max_{i=1, \ldots, N} \|\bar{u}_i - u^k(\bar{B}_{R_1}(\bar{x}_i))\|_H \leq c_2 \zeta^k, \quad (5.2)$$

**Proof.** For the convergence rate in (5.1) we refer to Theorem 4.7. Moreover this yields sequential weak* convergence of $\{u^k\}_{k \in \mathbb{N}}$ towards minimizers of $\langle P \rangle$. Since the minimizer $\bar{u}$ is unique this implies weak* convergence of the whole sequence. The claim on the localization of the support points will follow from Corollary 5.10. The improved convergence results of (5.2) are found in Theorem 5.16, Proposition 5.18 and Theorem 5.23.

5.1. **Rates for the residual.** In the following $c > 0$ always denotes a constant which is independent of the iteration index $k$. As an immediate consequence of Assumption 3.1 we obtain the following estimates.

**Lemma 5.3.** Given $u_1, u_2 \in \mathcal{M}(\Omega, H)$ with $Ku_1$, $Ku_2 \in N(\bar{y})$, there holds

$$j(u_1) - j(u_2) \geq \gamma_0 \|K(u_1 - u_2)\|_{\mathcal{M}}^2 - \Phi(u_2).$$

**Proof.** Due to Assumption 5.1 there holds

$$j(u_1) = F(Ku_1) + G(\|u_1\|_{\mathcal{M}})$$

$$\geq F(Ku_2) + \gamma_0 \|K(u_1 - u_2)\|_{\mathcal{M}}^2 + \langle \nabla F(Ku_2), K(u_1 - u_2) \rangle_{\mathcal{M}} + G(\|u_1\|_{\mathcal{M}})$$

$$= j(u_2) + \gamma_0 \|K(u_1 - u_2)\|_{\mathcal{M}}^2 - \langle \nabla f(u_2), u_2 - u_1 \rangle - G(\|u_2\|_{\mathcal{M}}) + G(\|u_1\|_{\mathcal{M}})$$

$$\geq j(u_2) + \gamma_0 \|K(u_1 - u_2)\|_{\mathcal{M}}^2 - \Phi(u_2).$$
Corollary 5.4. Given \( u \in \mathcal{M}(\Omega, H) \) with \( Ku \in N(\bar{y}) \) we have
\[
\gamma_0 \|K(u - \bar{u})\|_Y^2 \leq j(u) - j(\bar{u}) = r_j(u)
\] (5.3)

Proof. By optimality of \( \bar{u} \) there holds \( \Phi(\bar{u}) = 0 \). The statement now follows directly from the previous Lemma.

In particular the quadratic growth of \( j \) implies the following convergence rates for the observations \( y^k = Ku^k \in Y \) and dual variables \( p^k = -\nabla f(u^k) \in \mathcal{C}(\Omega, H) \).

Lemma 5.5. For all \( k \in \mathbb{N} \) large enough there holds
\[
\|y^k - \bar{y}\|_Y + \|p^k - \bar{p}\|_C \leq c\sqrt{r_j(u^k)}.
\]

Proof. Let us first proof the claimed estimated for the iterated observations \( y^k \). Due to the weak* convergence of \( \{u^k\}_{k \in \mathbb{N}} \) towards \( \bar{u} \) and the weak*-to-strong continuity of \( K \) there holds \( y^k \in N(\bar{y}) \) for all \( k \in \mathbb{N} \) large enough. Thus we have
\[
\gamma_0 \|y^k - \bar{y}\|_Y^2 \leq j(u^k) - j(\bar{u}) = r_j(u^k).
\]
Taking the square root yields the first estimate. The estimates for the dual variables can be concluded by the same arguments since
\[
\|p^k - \bar{p}\|_C = \left\|K^* (\nabla F(Ku^k) - \nabla F(K\bar{u}))\right\|_C
\]
\[
\leq L_{u_0} \|K^* \|_{\mathcal{L}(Y^*;C)}\|y^k - \bar{y}\|_Y.
\]
This finishes the proof.

Since the subproblems in step 2 of Algorithm 2 are solved up to optimality we conclude the following characterization of the iterates \( u^k \).

Corollary 5.6. For all \( k \) large enough there holds \( u^k \neq 0 \). Let the \( k \)-th iterate in Algorithm 2 be supported on \( \{x_i\}_{i=1}^{N_k} \). Then we have
\[
\langle p^k, u^k \rangle = \lambda_k \|u^k\|_{\mathcal{M}}, \quad \lambda_k = \max_{x \in \text{supp} |u^k|} P^k(x) \in \partial G(\|u^k\|_{\mathcal{M}}).
\]
For all \( k \) large enough there holds \( \lambda_k > 0 \) and thus
\[
u^k = \sum_{i=1}^{N_k} u^k_i \delta_{x^k_i} = \frac{1}{\lambda_k} \sum_{i=1}^{N_k} \|u^k_i\|_{\mathcal{M}} \delta_{x^k_i}, \quad (5.4)
\]

Proof. We only prove the statement on the positivity of \( \lambda_k \). The remaining claims follow from Proposition 4.9 and supp \( |u^k| \subset \mathcal{A}_{k-1} \). From the weak* convergence of \( \{u^k\}_{k \in \mathbb{N}} \), the strong convergence of \( p^k \) and the weak* lower semicontinuity of the norm we readily obtain
\[
\lambda_k \|u^k\|_{\mathcal{M}} = \langle p^k, u^k \rangle \to \langle \bar{p}, \bar{u} \rangle = \bar{\lambda} \|\bar{u}\|_{\mathcal{M}}, \quad \|u^k\|_{\mathcal{M}} \geq \|\bar{u}\|_{\mathcal{M}}/2,
\]
for all \( k \in \mathbb{N} \) large enough. This yields \( \lambda_k > 0 \) for all \( k \) large enough.

Corollary 5.7. There holds
\[
\lim_{k \to \infty} |\bar{\lambda} - \|p^k\|_C| + |\lambda_k - \|p^k\|_C| = 0.
\]

Proof. Observe that
\[
|\bar{\lambda} - \|p^k\|_C| = ||\bar{p}|_C - \|p^k\|_C| \leq ||\bar{p} - p^k\|_C \leq c\sqrt{r_j(u^k)} \to 0,
\]
for \( k \) going to infinity. Since \( \|\bar{u}\|_{\mathcal{M}} > 0 \) there exists \( c > 0 \) such that \( \|u^k\|_{\mathcal{M}} > c \) for all \( k \) large enough. We consequently obtain
\[
0 \leq c(\|p^k\|_C - \lambda_k) \leq \Phi(u^k),
\]
from Lemma 4.12. The statement now directly follows due to \( \liminf_{k \to 0} \Phi(u^k) = 0 \).
Following Lemma A.2 quadratic growth of the optimal dual certificate $\tilde{P}$ in a vicinity of its global maximizers can be concluded based on Assumption 5.3. The next perturbation result states that a similar behaviour also holds true for the iterated dual certificates $P^k$.

**Lemma 5.8.** There exists $R_1 > 0$ such that for all $k$ large enough and all $i \in \{1, \ldots, N\}$ the function $P^k$ assumes a unique local maximum $\hat{x}^k_i$ on $B_{R_1}(\hat{x}_i)$. Furthermore there holds

$$|\hat{x}^k_i - \bar{x}_i|_{\mathbb{R}^d} \leq c\sqrt{r_j(u^k)}, \quad i = 1, \ldots, N.$$  \hfill (5.5)

Additionally there exists $R_2 > 0$ with

$$P^k(x) + \frac{\theta_0}{8} |x - \hat{x}^k_i|_{\mathbb{R}^d}^2 \leq P^k(\hat{x}^k_i) \quad \forall x \in B_{R_2}(\hat{x}^k_i),$$  \hfill (5.6)

for all $i = 1, \ldots, N$.

**Proof.** Following Lemma A.3, $R > 0$ and $\delta > 0$ may be chosen small enough such that the mapping

$$\mathcal{F}: \Omega R \times B_{\delta}(\bar{y}) \to \mathbb{R}^d, \quad (x, y) \mapsto \frac{\partial}{\partial x} [\|K^* \nabla F(y)(x)\|_H].$$

is well-defined and continuously Fréchet differentiable. Moreover, there holds

$$\mathcal{F}(\hat{x}_i, \bar{y}) = \nabla \tilde{P}(\hat{x}_i) = 0, \quad \frac{\partial}{\partial x} \mathcal{F}(\hat{x}_i, \bar{y}) = \nabla^2 \tilde{P}(\hat{x}_i) \geq \theta_0 \text{Id}, \quad i = 1, \ldots, N.$$

Thus we can apply the implicit function theorem to get the existence of $0 < R_1 < R$ and $0 < \delta \leq \delta$ such that for all $y \in Y$ with $\|y - \bar{y}\|_Y < \delta$ and each $i \in \{1, \ldots, N\}$ there exists a unique $\hat{x}_i(y) \in B_{R_1}(\hat{x}_i)$ with

$$\mathcal{F}(\hat{x}_i(y), y) = 0, \quad |\hat{x}_i(y) - \hat{x}_i|_{\mathbb{R}^d} \leq c\|y - \bar{y}\|_Y,$$

for some $c > 0$. Note that $y^k = Ku^k \in B_{\delta}(y)$ for all $k$ large enough due to $u^k \rightharpoonup \bar{u}$. Setting $\hat{x}^k_i = \hat{x}_i(y^k)$ and applying Lemma 5.5 we obtain

$$|\hat{x}^k_i - \bar{x}_i|_{\mathbb{R}^d} \leq c\|y - \bar{y}\|_Y \leq c\sqrt{r_j(u^k)}.$$  

Next we prove that $\hat{x}^k_i$ is a local maximum of $P^k$. Let an arbitrary but fixed $i \in \{1, \ldots, N\}$ be given. Note that there holds

$$-\nabla^2 P^k(\hat{x}^k_i) \geq \left( -\|\nabla^2 P^k - \nabla^2 \tilde{P}_{C(\Omega R, R^d \times \mathbb{R}^d)}\|_{C(\Omega R, R^d \times \mathbb{R}^d)} - \|\nabla^2 \tilde{P}(\hat{x}_i) - \nabla^2 \tilde{P}(\hat{x}^k_i)\|_{R^d \times \mathbb{R}^d} + \theta_0 \right) \text{Id}_{R^d}.$$  

Due to the continuity of $\nabla^2 \tilde{P}$, the uniform convergence of $P^k$ in $C^2(\Omega R)$ and (5.5) there holds

$$\|\nabla^2 P^k(\hat{x}_i) - \nabla^2 \tilde{P}(\hat{x}_i)\|_{C(\Omega R, R^d \times \mathbb{R}^d)} \leq \frac{\theta_0}{2},$$

for all $k$ large enough. Thus for every $i$, $\hat{x}^k_i$ is a strict local maximum of $P^k$. The growth estimate for $P^k$ in the vicinity of its maxima can be derived analogously to Lemma A.2. This concludes the proof. \hfill \square

Following these preceding results the support points of $u^k$ are located in a vicinity of the optimal positions $\{\bar{x}_i\}_{i=1}^N$ if $k \in \mathbb{N}$ is large enough. Moreover the new support point $\hat{x}^k$ determined in step 1 of Algorithm 2 is chosen from $\{\hat{x}^k_i\}_{i=1}^N$.

**Corollary 5.9.** There exists $\sigma > 0$ with

$$\tilde{P}(x) \leq \bar{\lambda} - \sigma \quad \forall x \in \Omega \setminus \bigcup_{i=1}^{N_d} B_{R_1}(\bar{x}_i)$$  \hfill (5.7)

and, for all $k$ large enough, there holds

$$P^k(x) \leq \lambda^k - \frac{\sigma}{2} \quad \forall x \in \Omega \setminus \bigcup_{i=1}^{N_d} B_{R_1}(\hat{x}_i).$$  \hfill (5.8)
Proof. By assumption the function $\bar{P}$ does not achieve its maximum outside of $\bigcup_{i=1}^{N} B_{R_i}(\bar{x}_i)$. The existence of $\sigma > 0$ fulfilling (5.7) follows by a continuity argument. Let an arbitrary point $x \in \Omega \setminus \bigcup_{i=1}^{N} B_{R_i}(\bar{x}_i)$ be given. We estimate
\[
P^k(x) \leq \bar{P}(x) + \|\bar{p} - p^k\|_C \leq \lambda - \sigma + \|\bar{p} - p^k\|_C \leq \lambda^k + |\lambda^k - \lambda| + \|\bar{p} - p^k\|_C - \sigma.
\]
Choosing $k$ large enough such that
\[|\lambda^k - \lambda| + \|\bar{p} - p^k\|_C \leq \frac{\sigma}{2}\]
yields (5.8) and finishes the proof. \hfill \Box

Corollary 5.10. For all $k$ large enough there holds
\[\operatorname{supp} |u^k| \subset \bigcup_{i=1}^{N} \bar{B}_{R_i}(\bar{x}_i) \quad \text{supp} |u^k| \cap \bar{B}_{R_i}(\bar{x}_i) \neq \emptyset
\]
for all $i = 1, \ldots, N$. Furthermore the new support point $\hat{x}^k$ determined in step 1 of Algorithm 2 fulfills
\[\hat{x}^k \in \{ \hat{x}^k_i \}_{i=1}^{N} \subset \bigcup_{i=1}^{N} \bar{B}_{R_i}(\bar{x}_i).
\]

Proof. Let $x \in \operatorname{supp} |u^k|$ be arbitrary. Then there holds $P^k(x) = \lambda^k$. Consequently we have $x \in \bigcup_{i=1}^{N} B_{R_i}(\bar{x}_i)$, see (5.8). Fix now an arbitrary index $i \in \{1, \ldots, N\}$ and denote by $u_i^k$ the restriction of $u^k$ to $\bar{B}_{R_i}(\bar{x}_i)$. Invoking Urysohn’s lemma there exists a cut-off function $\chi_i \in C(\Omega)$ with $\chi_i = 1$ on $\bar{B}_{R_i}(\bar{x}_i)$ and $\chi_i = 0$ on $\bar{B}_{R_i}(\bar{x}_j)$ for $j \neq i$. The weak* convergence of the iterates and the strong convergence of the dual variables yield
\[\lambda_k\|u_i^k\|_M = \langle \chi_i p^k, u^k \rangle \to \langle \chi_i \bar{p}, \bar{u} \rangle = \hat{\lambda}\|u_i\|_H > 0.
\]
Since $\lambda^k \to \bar{\lambda}$ we conclude $\|u_i^k\|_M = \|u_i^k\|_M(\bar{\Omega}) \neq 0$ for all $k$ large enough. The statement on the position of the new Dirac delta function follows directly since $P^k < \lambda^k$ outside of $\bigcup_{i=1}^{N} \bar{B}_{R_i}(\bar{x}_i)$ and
\[\operatorname{arg max}_{x \in \bigcup_{i=1}^{N} \bar{B}_{R_i}(\bar{x}_i)} P^k(x) \subset \{ \hat{x}^k_i \}_{i=1}^{N}. \quad \Box
\]

In the following corollary we show, loosely speaking, that the newly added support point $\hat{x}^k$ is also contained in the support of $u^{k+1}$.

Corollary 5.11. Denote by $\hat{x}^k$ the new support point determined in step 1 of Algorithm 2. Then there holds $\hat{x}^k \in \operatorname{supp} |u^{k+1}|$ for all $k \in \mathbb{N}$.

Proof. Since the algorithm does not converge after finitely many steps we have $j(u^{k+1}) < j(u^k)$ and
\[\operatorname{supp} |u^{k+1}| \subset \operatorname{supp} |u^k| \cup \{ \hat{x}^k \}
\]
for all $k \in \mathbb{N}$. Assume now that $\hat{x}^k \notin \operatorname{supp} |u^{k+1}|$. Then there holds $\operatorname{supp} u^{k+1} \subset \operatorname{supp} u^k$ and $j(u^{k+1}) = j(u^k)$ since the subproblems in step 2 are solved up to optimality. This gives a contradiction. \hfill \Box

We obtain the following estimates for the support points of $|u^k|$.

Lemma 5.12. Let an arbitrary index $i \in \{1, \ldots, N\}$ be given. For all $k$ large enough there holds
\[\max_{x \in \operatorname{supp} |u^k| \cap \bar{B}_{R_i}(\bar{x}_i)} |x - \bar{x}_i|_{R^4} \leq c \left( \sqrt{|\lambda^k - \lambda|} + \sqrt[j]{r_j(u^k)} \right).
\]

(5.9)
Furthermore for $k$ large enough there holds $\text{supp} \ u^k \subset \bigcup_{i=1}^{N_d} B_{R_2}(\tilde{x}_i^k)$ and

$$\max_{x \in \text{supp} \ u^k \cap B_{R_1}(\tilde{x}_i)} |x - \tilde{x}_i^k|_{\mathbb{R}^d} \leq c \sqrt{P^k(\tilde{x}_i^k) - \lambda^k}.$$ 

**Proof.** Given an arbitrary $i \in \{1, \ldots, N\}$ we first observe that $\text{supp} \ u^k \cap B_{R_1}(\tilde{x}_i) \neq \emptyset$, see Corollary 5.10. Let $x \in \text{supp} \ u^k \cap B_{R_1}(\tilde{x}_i)$. Using (A.1) we obtain

$$|x - \bar{x}_i|_{\mathbb{R}^d} \leq c \sqrt{\lambda - P(x)} \leq c \left( \sqrt{\lambda - P^k(x)} + \sqrt{\|p^k - \bar{p}\|_{C}} \right)$$

$$\leq c \left( \sqrt{|\lambda - \lambda^k| + \frac{1}{j} r_j(u^k)} \right),$$

for some constant $c > 0$ independent of $x$. Here we used $P^k(x) = \lambda^k$ for all $x \in \text{supp} \ u^k$ as well as Lemma 5.5. Taking the maximum over all $x \in \text{supp} \ u^k \cap B_{R_1}(\tilde{x}_i)$ yields the first statement. For the second estimate we observe that for every $x \in \text{supp} \ u^k \cap B_{R_1}(\tilde{x}_i)$ there holds

$$|x - \tilde{x}_i^k|_{\mathbb{R}^d} \leq |x - \bar{x}_i|_{\mathbb{R}^d} + |\bar{x}_i - \tilde{x}_i^k|_{\mathbb{R}^d}$$

$$\leq \max_{x \in \text{supp} \ u^k \cap B_{R_2}(\tilde{x}_i)} |x - \bar{x}_i|_{\mathbb{R}^d} + \sqrt{r_j(u^k)}.$$ 

Due to (5.9) and $\lambda^k \to \bar{\lambda}$ we get $\text{supp} \ u^k \subset \bigcup_{i=1}^{N_d} B_{R_2}(\tilde{x}_i^k)$ for all $k$ large enough. Consequently we obtain for all $i \in \{1, \ldots, N_d\}$ and $x \in \text{supp} \ u^k \cap B_{R_1}(\tilde{x}_i)$ that there holds

$$|x - \tilde{x}_i^k|_{\mathbb{R}^d} \leq c \sqrt{P^k(\tilde{x}_i^k) - \lambda^k}$$

using (5.6). Since the constant $c > 0$ is again independent of $x$ we finish the proof by maximizing on both sides.

With these auxiliary estimates at hand we now proceed to improve on the sublinear convergence rate for the residual $r_j(u^k)$. To this end fix an arbitrary index $k \in \mathbb{N}$ large enough such that all previous results hold and recall the definition of the intermediated iterate $u^{k+1/2}$ in the generalized conditional gradient method, see Algorithm 1,

$$u_s^{k+1/2} = u^k + s \Delta_s^k, \quad \Delta_s^k = v^k - u^k, \quad v^k = \|u^k\|_{\mathcal{M}} \frac{p^k(\tilde{x}_i^k)}{\|p^k\|_C}$$

for an appropriate choice of the stepsize $s \in [0, 1]$ and $\|u^k\|_{\mathcal{M}}$ chosen according to (4.3). Obviously we have $j(u^{k+1}) \leq j(u_s^{k+1/2})$ for all $s \in [0, 1]$. In fact this observation for the intermediate iterates $u_s^{k+1/2}$ remains true if we allow for more general descent directions $\Delta^k$:

$$j(u^{k+1}) \leq j(u_s^{k+1/2}), \quad u_s^{k+1/2} = u^k + s \Delta_s^k, \quad \text{supp} \ \Delta_s^k \subset \text{supp} \ u^k \cup \{\tilde{x}_i^k\}$$

and $s \in [0, 1]$ since the subproblems in Algorithm 2 are solved up to optimality.

In the following we will construct a descent direction $\Delta^k$ and a stepsize $s^k$ such that the residuals $r_j(u_s^{k+1/2}), u_s^{k+1/2} = u^k + s^k \Delta^k$, converge linearly for all $k \in \mathbb{N}$ large enough. From Corollary 5.10 we conclude the existence of an index $i \in \{1, \ldots, N\}$ with $\tilde{x}_i = \tilde{x}_i^k \in B_{R_1}(\tilde{x}_i)$.

Define the locally lumped measure $\hat{u}_i^k \in \mathcal{M}(\Omega, H)$ by

$$\hat{u}_i^k = u^{k}_{|\tilde{B}_{R_1}(\tilde{x}_i)} + u^{k}_{|\tilde{B}_{R_1}(\tilde{x}_i)} \|\frac{p^k(\tilde{x}_i^k)}{\|p^k\|_C} \delta_{\tilde{x}_i^k},$$

where $\tilde{B}_{R_1}(\tilde{x}_i) = \Omega \setminus \tilde{B}_{R_1}(\tilde{x}_i)$. The following statements establish the weak* convergence of $\hat{u}_i^k$ towards $\hat{u}$.

**Proposition 5.13.** For all $k \in \mathbb{N}$ large enough there holds

$$G(\|\hat{u}_i^k\|_{\mathcal{M}}) = G(\|u^k\|_{\mathcal{M}}), \quad \langle p^k, \hat{u}_i^k - u^k \rangle = \|u^k_{|\tilde{B}_{R_1}(\tilde{x}_i)}\|_{\mathcal{M}} \left( \|p^k\|_C - \lambda^k \right).$$
Proof. Since the sets \( \tilde{B}_R(x) \) are disjoint we note that
\[
\|u^k\|_\mathcal{M} = \sum_{i=1}^N \|u^k_{B_R(x)}\|_\mathcal{M} = \sum_{i \in \{1,\ldots,N\} \setminus \{i\}} \|u^k_{B_R(x)}\|_\mathcal{M} + \|u^k_{B_R(x)}\|_\mathcal{M}
\]
and consequently \( G(\|u^k\|_\mathcal{M}) = G(\|u^k\|_\mathcal{M}) \). Furthermore by construction there holds
\[
\langle p^k, u^k_i - u^k \rangle = \|u^k_{B_R(x)}\|_\mathcal{M}\|p^k\|_c - \|u^k_{B_R(x)}\|_\mathcal{M}\lambda^k
\]
yielding the result. \( \square \)

Lemma 5.14. For \( k \) large enough there holds
\[
\|K(\hat{u}^k - u^k)\|_Y \leq c\|u^k_{B_R(x)}\|_\mathcal{M}\sqrt{\|p^k\|_c - \lambda^k}.
\]
Proof. Let an arbitrary \( x \in \text{supp } u^k \cap \tilde{B}_R(x) \) be given and denote by \( u \in H \), \( u \neq 0 \) the coefficient of the associated Dirac delta function. Given \( \varphi \in Y \) there holds
\[
\left( K \left( \frac{p^k(\hat{x})}{\|p^k\|_c} \delta_{x^k} - \frac{u}{\|u\|_H} \delta_x \right), \varphi \right)_Y = \left( K^* \varphi, \frac{p^k(\hat{x})}{\|p^k\|_c} \delta_{x^k} - \frac{p^k(x)}{\lambda^k} \delta_x \right)_Y
\]
for \( \frac{K^* \varphi}{\|K^* \varphi\|_c} ) \|_{\bar{\mathcal{M}}} \|p^k\|_c - \lambda^k\right)_H \]
Using the properties of \( K^* \) and Lemma 5.12 the first term is estimated by
\[
\|K^* \varphi\|_{c^{0.1}(\Omega_R,H)}|\hat{x}^k - x|_{\bar{\mathcal{M}}} \leq c\|\varphi\|_Y \sqrt{\|p^k\|_c - \lambda^k},
\]
with a constant \( c > 0 \) independent of \( x \). For the second term we use \( \|p^k(\hat{x})\|_H = \|p^k\|_c \) to estimate
\[
\left( K \left( \frac{p^k(\hat{x})}{\|p^k\|_c} \delta_{x^k} - \frac{u}{\|u\|_H} \delta_x \right), \varphi \right)_Y \leq c\sqrt{\|p^k\|_c - \lambda^k \|\varphi\|_Y},
\]
and consequently
\[
\left( K \left( \frac{p^k(\hat{x})}{\|p^k\|_c} \delta_{x^k} - \frac{u}{\|u\|_H} \delta_x \right), \varphi \right)_Y \leq c\sqrt{\|p^k\|_c - \lambda^k}.
\]
Using \( \|u_{jB}^k(\tilde{x}_i)\|_M = \sum_{x_i^k \in \text{supp} \{u^k \cap B_{R_i}(\tilde{x}_i)\}} \|u_i\|_H \), we rewrite
\[
K(\hat{u}_i^k - u^k) = \sum_{x_i^k \in \text{supp} \{u^k \cap B_{R_i}(\tilde{x}_i)\}} \|u_i\|_H \mathcal{K}(\frac{p^k(\delta_x^k)}{|p^k|c} \delta_x^k - \frac{u_i}{\|u_i\|_H} - \hat{\delta}_x^k).
\]
Applying the estimate for all \( x_i^k \in \text{supp} \{u^k \cap B_{R_i}(\tilde{x}_i)\} \) we arrive at
\[
\|K(\hat{u}_i^k - u^k)\| \leq c\|u^k|_{B_{R_i}(\tilde{x}_i)}\|_M \sqrt{|p^k|c - \lambda^k},
\]
completing the proof.

**Corollary 5.15.** There holds
\[
\hat{u}_i^k \rightarrow^* \hat{u}, \quad j(\hat{u}_i^k) \rightarrow j(\hat{u}).
\]

**Proof.** We readily obtain
\[
0 \leq j(\hat{u}_i^k) - j(\hat{u}) \leq |j(u^k) - j(\hat{u})| + |F(K\hat{u}_i^k) - F(Ku^k)|.
\]
The first term tends to 0 since \( \{u^k\}_{k \in \mathbb{N}} \) is a minimizing sequence for \( j \) and the second vanishes due to Lemma 5.14. Thus \( \hat{u}_i^k \) gives a minimizing sequence for \( j \). Since \( \hat{u} \) is the unique minimizer of \( j \) the claim on the weak* convergence follows.

Finally, we show that \( \Delta^k = \hat{u}_i^k - u^k \) yields a search direction that achieves a linear decrease in the objective functional.

**Theorem 5.16.** There exists an index \( k \in \mathbb{N} \), a constant \( c_k > 0 \) and \( \zeta_1 \in (0,1) \) with
\[
\rho_j(u^k) \leq c_k \zeta_1 \forall k \geq k.
\]

**Proof.** For \( s \in [0,1] \) define
\[
u_s^k = u^k + s(\hat{u}_i^k - u^k) = (1 - s)u^k + s\hat{u}_i^k.
\]
Since \( j(\hat{u}_i^k) \rightarrow j(\hat{u}) \) we conclude \( u_s^k \in E_j(u^0) \) for all \( s \) and all \( k \) large enough. Let in the following \( k \) be big enough. Along the lines of proof in Lemma 4.5 it follows that
\[
j(u_s^k) = F(Ku_s^k) + G(\|u_s^k\|_M)
\leq F(Ku^k) + s(\nabla F(Ku^k), K(\hat{u}_i^k - u^k))_Y + \frac{s^2L_{u^0}}{2}\|K(\hat{u}_i^k - u^k)\|^2 + G(\|u_s^k\|_M)
\leq j(u^k) + s\left(\langle -p^k, \hat{u}_i^k - u^k \rangle + G(\|\hat{u}_i^k\|_M) - G(\|u^k\|_M)\right) + \frac{s^2L_{u^0}}{2}\|K(\hat{u}_i^k - u^k)\|^2,
\]
where \( L_{u^0} \) denotes the Lipschitz constant of \( \nabla F \) on \( KE_j(u^0) \). Now, by Proposition 5.13 and Lemma 5.14, we derive the estimate
\[
j(u_s^k) \leq j(u^k) - s\|u_s^k|_{B_{R_i}(\tilde{x}_i)}\|_M \left(\|p^k\|c - \lambda^k\right) + \frac{s^2c_1}{2}\|u_s^k|_{B_{R_i}(\tilde{x}_i)}\|^2 \left(\|p^k\|c - \lambda^k\right).
\]
Minimizing for \( s \in [0,1] \), we obtain
\[
j(u_s^k) \leq j(u^k) - \frac{1}{2}\min\left\{\frac{1}{c_1}, 1\right\}\left(\|p^k\|c - \lambda^k\right),
\]
where \( \delta^k = \min\{1, 1/(c_1\|u^k|_{B_{R_i}(\tilde{x}_i)}\|)\} \) and \( c_1 > 0 \) is the square of the constant from Lemma 5.14. Defining the constant \( c_2 > 0 \) by
\[
c_2 = (1/(2M_0)) \min_{i=1,...,N} \min\left\{\frac{\lambda}{c_1\|\tilde{u}|_{B_{R_i}(\tilde{x}_i)}\|}, 1/c_1\right\} < 1/2,
\]
we have with Lemma 4.12 that
\[
j(u_s^k) \leq j(u^k) - c_2M_0 \left(\|p^k\|c - \lambda^k\right) \leq j(u^k) - c_2\Phi(u^k) \leq j(u^k) - c_2r_j(u^k).
\]
Subtracting \( j(\hat{u}) \) from both sides, it follows
\[
r_j(u^{k+1}) \leq r_j(u_s^k) \leq (1 - c_2)r_j(u^k).
\]
Denote by $\tilde{k} \in \mathbb{N}$ an arbitrary but fixed index such that all previous results hold for all $k$ greater than $\tilde{k}$. By induction we get

$$r_j(u^k) \leq (1 - c_2)^{k - \tilde{k}} r_j(u^\tilde{k}).$$

Setting $\zeta_1 = (1 - c_2)$ and $c_{\tilde{k}} = r(u^\tilde{k})/\zeta_1^{\tilde{k}}$ yields the result. $\square$

To close this section we elaborate on the geometric intuition behind the construction of the new search direction $\Delta^k_2 = \hat{u}^k - u^k$ and the differences to the GCG direction $\Delta^k_1 = v^k - u^k$. We consider the special case of $G(||u||_\mathcal{M}) = \beta ||u||_\mathcal{M}$ for $\beta > 0$. A schematic comparison between both is given in Figure 1. Let us recall that by Corollary 5.10 the support of $u^k$ can be divided into $N$ nonempty and disjoint clusters around the optimal positions $\{\bar{x}_i\}_{i=1}^N$ for $k$ large enough.

First we consider the intermediate iterate $u^{k+1/2}_s$ given by $\Delta^k_1$. This yields

$$u^{k+1/2}_s = u^k + s \Delta^k_1 = (1 - s)u^k + sv^k = (1 - s)u^k + s M_0 \frac{p^k(\bar{x}^k)}{\|p^k\|_C} \delta_{\bar{x}^k}.$$  

Thus the GCG search direction adds a single point source in one of the clusters but, by forming the convex combination, the values of $u^k$ are changed globally. Additionally it is readily verified that every weak* accumulation point $\bar{v}$ of $\{v^k\}_{k \in \mathbb{N}}$ is given by $\bar{v} = M_0 \bar{p}(\bar{x}_i)/\bar{\lambda} \delta_{\bar{x}_i}$ for some $i = 1, \ldots, N$. In particular for every sequence of step sizes $\{s^k\}_{k \in \mathbb{N}}$ we necessarily have

$$u^{k+1}_s = (1 - s^k)u^k + s^k \delta_{\bar{x}_i} \rightharpoonup \bar{u} \Rightarrow s^k \to 0,$$

as $k \to \infty$ if $\bar{u}$ consists of more than one Dirac delta function. This results in the sublinear convergence of the residual. In contrast, choosing $\Delta^k_2$ gives

$$u^{k+1/2}_s = u^k + s \Delta^k_2 = (1 - s)u^k + s \hat{u}^k$$

$$= u^k_{|B_{R_1}(\bar{x}_i)} + u^k_{|B_{R_1}(\bar{x}_i)} + s \left( \frac{p^k(\hat{x}^k)}{\|p^k\|_C} \delta_{\hat{x}^k} - u^k_{|B_{R_1}(\bar{x}_i)} \right).$$

Here we still add a single Dirac delta function to one of the clusters. However, in contrast to the GCG search direction, the norm of its coefficient is determined by moving mass from the neighbouring Dirac delta functions in the same cluster to the new one. The values of $u^k$ on the remaining clusters remain unchanged. Moreover note that if $s = 1$ the new search direction replaces all Dirac delta functions in the cluster by the new one. Differently from the sequence $\{v^k\}_{k \in \mathbb{N}}$, the locally lumped measures $\hat{u}^k$ weak* converge to the minimizer $\bar{u}$. This allows to choose a sequence of step sizes $\{s^k\}_{k \in \mathbb{N}}$ which is uniformly bounded from below and thus yields the improved linear convergence rate for the residual.
5.2. Rates for the iterates. This section is devoted to quantitative convergence results for the sequence of iterates \( \{u^k\}_{k \in \mathbb{N}} \). While norm convergence towards the minimizer cannot be expected in general the weak* convergence of the iterates implies convergence of the support points of \( u^k \) towards those of \( \bar{u} \) as well as convergence of the coefficient functions.

5.2.1. Rates for the support points. We first provide an estimate for the difference between the maximum value of \( P \) and \( \lambda^k \).

**Lemma 5.17.** For all \( k \) large enough there exists \( c > 0 \) with

\[
|\bar{\lambda} - \lambda^k| \leq c \sqrt{r_j(u^{k-1})}.
\]

**Proof.** If we choose \( k \) large enough there exists \( \bar{x}^k \in \text{supp } |u^k| \) and an index \( i_k \) with

\[
\bar{x}^k \in \text{arg max}_{x \in \Omega} P^{k-1}(x),
\]

for some \( c > 0 \), see Corollary 5.11 and Lemma 5.8. Consequently we have

\[
|\bar{\lambda} - \lambda^k| = |\bar{P}(\bar{x}_{i_k}) - P(\bar{x}^k)| \leq |\bar{P}(\bar{x}_{i_k}) - P(\bar{x}^k)| + \|\bar{p} - p^k\|_C
\]

\[
\leq c \left( \|\bar{p}\|_{C^1(\bar{B}_R)} |\bar{x}_{i_k} - \bar{x}^k|_{\mathbb{R}^d} + \sqrt{r_j(u^{k-1})} \right) \leq c \sqrt{r_j(u^{k-1})},
\]

due to the monotonicity of \( r_j(u^k) \) and Lemma 5.5.

Putting everything together we obtain the following convergence results for the support points of the iterate \( u^k \).

**Proposition 5.18.** There exists a constant \( c > 0 \) with

\[
\max_{i=1,\ldots,N} \max_{x \in \text{supp } |u^k| \cap B_{R_1}(\bar{x}_i)} |x - \bar{x}_i|_{\mathbb{R}^d} \leq c \xi^k_2,
\]

for some \( 0 < \xi_2 < 1 \) and for all \( k \) large enough.

**Proof.** From Lemma 5.12 we get

\[
\max_{i=1,\ldots,N} \max_{x \in \text{supp } |u^k| \cap B_{R_1}(\bar{x}_i)} |x - \bar{x}_i|_{\mathbb{R}^d} \leq c \left( \sqrt{|\lambda^k - \bar{\lambda}|} + \sqrt{r_j(u^k)} \right).
\]

Due to the monotonicity of \( r_j(u^k) \), Lemma 5.16 and 5.17 there exists \( 0 < \xi_1 < 1 \) with

\[
\sqrt{|\lambda^k - \bar{\lambda}|} + \sqrt{r_j(u^k)} \leq c \sqrt{r_j(u^{k-1})} \leq c \xi^k_1.
\]

By setting \( \xi_2 = \sqrt{\xi_1} \) we conclude (5.10).

5.2.2. Rates for the coefficients. Let \( k \) be large enough such that all previous results hold. For \( i \in \{1,\ldots,N\} \) denote by \( u^k_i \) the restriction of \( u^k \) to \( B_R(\bar{x}_i) \). Due to the optimality conditions for \( \bar{u} \) and \( u^k \) respectively we get

\[
\bar{u} = \frac{1}{\lambda} \sum_{i=1}^{N} \|\bar{u}_i\|_H \bar{p}(\bar{x}_i) \delta_{\bar{x}_i}, \quad u^k_i = \frac{1}{\lambda^k} \sum_{x_i \in \text{supp } |u^k| \cap B_{R_1}(\bar{x}_i)} |u^k|((x_i)) p^k(x_i) \delta_{x_i}.
\]

Recall that the iterates \( \{u^k\}_{k \in \mathbb{N}} \) only converge with respect to the weak* topology on \( \mathcal{M}(\Omega, H) \). Therefore a single Dirac delta function in the optimal solution \( \bar{u} \) is in general approximated by several spikes in the iterate \( u^k \), i.e. \( \# \text{supp } |u^k| > 1 \) for \( i = 1,\ldots,N \). In particular this implies that the optimal coefficient function \( \bar{u}_i \) of the Dirac delta at \( \bar{x}_i \) should be approximated by

\[
u^k_i(\bar{B}_{R_1}) = \frac{1}{\lambda^k} \sum_{x_i \in \text{supp } |u^k| \cap B_{R_1}(\bar{x}_i)} |u^k|((x_i)) p^k(x_i).
\]
The aim of this section is to provide a quantitative confirmation of this observation. In detail we will prove
\[
\max_{i=1,\ldots,N} \left\| \bar{u}_i - u^k(\bar{B}_R(x_i)) \right\|_H + \max_{i=1,\ldots,N} \left\| \bar{u}_i \right\|_H - |u^k|([\bar{B}_R(x_i)]) \leq c\zeta_2^k,
\]
with \(\zeta_2 \in (0, 1)\) as in the previous section. In the following the generic constant \(c > 0\) may depend on the number of Dirac delta functions \(N\) in the minimizer \(\bar{u}\). We start by providing several auxiliary results.

**Lemma 5.19.** Let \(x \in \text{supp} |u^k| \cap \bar{B}_R(x_i)\) be given. Then there holds
\[
\left\| \frac{\bar{p}(x_i)}{\lambda} - \frac{p^k(x)}{\lambda^k} \right\|_H \leq c\zeta_2^k,
\]
for some constant \(c > 0\) independent of \(i\) and \(x\).

**Proof.** We split the error into three parts
\[
\left\| \frac{\bar{p}(x_i)}{\lambda} - \frac{p^k(x)}{\lambda^k} \right\|_H \leq \left\| \frac{\bar{p}(x_i)}{\lambda} - \frac{\bar{p}(x)}{\lambda^k} \right\|_H + \left\| \frac{\bar{p}(x)}{\lambda^k} - \frac{\bar{p}(x)}{\lambda^k} \right\|_H + \left\| \bar{p}(x) - \frac{p^k(x)}{\lambda^k} \right\|_H.
\]
For the first term we use Lemma 5.17 to obtain
\[
\left\| \frac{\bar{p}(x_i)}{\lambda} - \frac{\bar{p}(x)}{\lambda^k} \right\|_H \leq \|\bar{p}\|_C \frac{\lambda - \lambda^k}{\lambda \lambda^k} \leq c\zeta_2^k,
\]
due to (5.11) and since \(\lambda^k \lambda\) is bounded away from zero. From the Lipschitz continuity of \(\bar{p}\) and the uniform convergence of \(p^k\) the remaining terms are estimated by
\[
\left\| \frac{\bar{p}(x)}{\lambda^k} - \bar{p}(x) \right\|_H + \left\| \bar{p}(x) - \frac{p^k(x)}{\lambda^k} \right\|_H \leq \frac{c}{\lambda^k} \left( |x_i - x|_{\mathbb{R}^d} + \|\bar{p} - p^k\|_C \right).
\]
Using (5.10) and \(\|\bar{p} - p^k\|_C \leq \sqrt[n]{(n^{k-1})}\) for all \(k\) large enough we obtain
\[
|x_i - x|_{\mathbb{R}^d} + \|\bar{p} - p^k\|_C \leq c\zeta_2^k,
\]
independent of \(x\), see again (5.11). Adding both estimates yields the proof. \(\square\)

First we provide the convergence rate for the norms of the localized measures \(u^k_i\), \(i = 1, \ldots, N\). Therefore define the auxiliary operator
\[
\hat{K}: \mathbb{R}^N \to Y \quad v \mapsto \frac{1}{\lambda} \sum_{i=1}^N v_i K(\bar{p}(x_i)) \delta_{\hat{E}_i}.
\]
Due to the linear independence assumption in Assumption 5.1 the operator \(\hat{K}\) is injective. Thus the matrix \(\hat{K}^* \hat{K} \in \mathbb{R}^{N \times N}\) is invertible. We arrive at the following corollary.

**Corollary 5.20.** For \(v_1, v_2 \in \mathbb{R}^N\) there exists \(c > 0\) with
\[
|v_1 - v_2|_{\mathbb{R}^N} \leq c \|\hat{K}(v_1 - v_2)\|_Y.
\]

**Proof.** There holds
\[
|v_1 - v_2|_{\mathbb{R}^N} \leq \|\hat{K}^* \hat{K}^{-1}\|_{\mathbb{R}^{N \times N}} \|\hat{K}^* \hat{K}(v_1 - v_2)\|_{\mathbb{R}^N}
\leq \|\hat{K}^* \hat{K}^{-1}\|_{\mathbb{R}^{N \times N}} \|\hat{K}^*\|_{L^2(\mathbb{R}^N)} \|\hat{K}(v_1 - v_2)\|_Y.
\]

**Lemma 5.21.** Let an arbitrary but fixed index \(i \in \{1, \ldots, N\}\) be given. Then there exists \(c > 0\), independent of \(i\) with
\[
\left\| K \left( \left\| \tilde{u}_i \right\|_M \frac{\tilde{p}(x_i)}{\lambda} \delta_{\hat{E}_i} - u^k_i \right) \right\|_Y \leq c\zeta_2^k,
\]
for all \(k\) large enough.
Proof. The proof follows similar steps as in Lemma 5.14. Let \( x \in \text{supp} |u^k| \cap \bar{B}_{R_1}(\bar{x}_i) \) with coefficient function \( u \in H \), \( u \neq 0 \) be given. For \( \varphi \in Y \) we obtain

\[
\left( K \left( \frac{\bar{p}(\bar{x}_i)}{\lambda} \delta_{\bar{x}_i} - \frac{u}{\|u\|_H} \delta_x \right), \varphi \right)_Y = \left( K^* \varphi, \frac{\bar{p}(\bar{x}_i)}{\lambda} \delta_{\bar{x}_i} - \frac{p^k(x)}{\lambda^k} \delta_x \right)_H
\]

\[
= \left( |K^* \varphi| (\bar{x}_i), \frac{\bar{p}(\bar{x}_i)}{\lambda} \right) - \left( |K^* \varphi| (x), \frac{p^k(x)}{\lambda^k} \right)_H
\]

\[
\leq \|K^* \varphi\|_{c_0,1(B_{R_1},Y)}|\bar{x}_i - x|_{\mathbb{R}^d} + \|K^* \varphi\| c \left( \frac{\|\bar{p}(\bar{x}_i)\|}{\lambda} - \frac{\|p^k(x)\|}{\lambda^k} \right)_H \leq c\|\varphi\|_Y \zeta^k_2,
\]

for some constant \( c > 0 \) independent of \( x \) and \( i \), see Proposition 5.18 and Lemma 5.21. Thus we conclude

\[
\|K \left( \frac{\bar{p}(\bar{x}_i)}{\lambda} \delta_{\bar{x}_i} - \frac{u}{\|u\|_H} \delta_x \right)\|_Y \leq c\zeta^k_2.
\]

By observing that \( \|u^k_i\|_M = \sum_{x_j \in \text{supp} |u^k| \cap \bar{B}_{R_1}(\bar{x}_i)} \|u_j\|_H \) there holds

\[
\|K \left( \|u^k_i\|_M \frac{\bar{p}(\bar{x}_i)}{\lambda} \delta_{\bar{x}_i} - u^k_i \right)\|_Y \leq \sum_{x_j \in \text{supp} |u^k| \cap \bar{B}_{R_1}(\bar{x}_i)} \|u_j\|_H \|K \left( \frac{\bar{p}(\bar{x}_i)}{\lambda} \delta_{\bar{x}_i} - \frac{u_j}{\|u_j\|_H} \delta_{x_j} \right)\|_Y \leq c \|u^k_i\|_M \zeta^k_2 \leq c M_0 \zeta^k_2.
\]

The following proposition characterizes the convergence behavior of \( |u^k|(\bar{B}_{R_1}(\bar{x}_i)) = \|u^k\|_M \).

**Proposition 5.22.** There exists a constant \( c > 0 \) such that, for all \( k \) large enough,

\[
\max_{i=1,...,N} \|\vec{u}_i\|_H - \|u^k_i\|_M \leq c \zeta^k_2.
\]

**Proof.** Define the vectors \( \vec{v}, v^k \in \mathbb{R}^N \) with \( \vec{v}_i = \|u_i\|_M \) and \( v^k_i = |u^k|(\bar{B}_{R_1}(\bar{x}_i)) = \|u^k_i\|_M \). Using Corollary 5.20 we obtain

\[
\max_{i=1,...,N} \|\vec{u}_i\|_H - \|u^k\|_M \leq |\vec{v} - v^k| \leq c \left\| \tilde{K} \left( \vec{v} - v^k \right) \right\|_Y.
\]

We further estimate

\[
\left\| \tilde{K} \left( \vec{v} - v^k \right) \right\|_Y \leq \left\| K \left( \vec{u} - u^k \right) \right\|_Y + \sum_{i=1}^N \left\| K \left( \|u^k_i\|_M \frac{\vec{u}_i}{\|\vec{u}_i\|_H} \delta_{\bar{x}_i} - u^k_i \right) \right\|_Y.
\]

For the first term we get

\[
\left\| K \left( \vec{u} - u^k \right) \right\|_Y \leq \sqrt{r(u^k)} \leq c\zeta^k_2,
\]

for all \( k \) large enough, see Lemma 5.5. Due to Lemma 5.21 we conclude

\[
\sum_{i=1}^N \left\| K \left( \|u^k_i\|_M \frac{\vec{u}_i}{\|\vec{u}_i\|_H} \delta_{\bar{x}_i} - u^k_i \right) \right\|_Y = \sum_{i=1}^N \left\| K \left( \|u^k_i\|_M \frac{\bar{p}(\bar{x}_i)}{\lambda} \delta_{\bar{x}_i} - u^k_i \right) \right\|_Y \leq cN \zeta^k_2.
\]

Summarizing all previous estimates we arrive at the following theorem.

**Theorem 5.23.** There exists a constant \( c > 0 \) such that, for all \( k \) large enough,

\[
\max_{i=1,...,N} \|\vec{u}_i - u^k(\bar{B}_{R_1}(\bar{x}_i))\|_H \leq c \zeta^k_2.
\]
Proof. Let an arbitrary but fixed index \( i \in \{1, \ldots, N\} \) be given. By decomposing the norm as
\[
\|u_k\|_\mathcal{M} = \sum_{x_j \in \text{supp}(u^k \cap \overline{B}_{R_1}(\bar{x}_i))} \|u_j\|_H
\]
and using Lemma 5.19 as well as Proposition 5.22 we get
\[
\|\bar{u}_i - u^k(\overline{B}_{R_1}(\bar{x}_i))\|_H \\
= \left\| \int_{\overline{B}_{R_1}(\bar{x}_i)} \frac{\tilde{p}(\bar{x}_i)}{\lambda} \ d|\bar{u}|(x) - \int_{\overline{B}_{R_1}(\bar{x}_i)} \frac{p^k(x)}{\lambda} \ d|u^k|(x) \right\|_H \\
\leq |\bar{u}|(\overline{B}_{R_1}(\bar{x}_i)) - |u^k|(\overline{B}_{R_1}(\bar{x}_i)) + \left\| \int_{\overline{B}_{R_1}(\bar{x}_i)} \frac{\tilde{p}(\bar{x}_i)}{\lambda} - \frac{p^k(x)}{\lambda} \ d|u^k|(x) \right\|_H \\
\leq \|\bar{u}_i\|_H - \|u^k\|_\mathcal{M} + \sum_{x_j \in \text{supp}(u^k \cap \overline{B}_{R_1}(\bar{x}_i))} \|u_j\|_H \left\| \frac{\tilde{p}(\bar{x}_i)}{\lambda} - \frac{p^k(x)}{\lambda} \right\|_H \\
\leq cM_0 \zeta_2^k,
\]
with a constant \( c > 0 \) independent of \( i \). Maximizing with respect to \( i = 1, \ldots, N \) on both sides of the inequality finishes the proof.

Convergence rates in weaker norms. As already pointed out the norm convergence of \( \{u^k\}_{k \in \mathbb{N}} \) towards the unique minimizer \( \bar{u} \) in \( \mathcal{M}(\Omega, H) \) cannot be expected in general. However norm convergence results can still be obtained by resorting to weaker spaces. In particular since the space of Lipschitz continuous functions embeds compactly into \( C(\Omega, H) \) weak* convergence on \( \mathcal{M}(\Omega, H) \) implies strong convergence with respect to the canonical norm on the topological dual space of \( C^{0,1}(\Omega, H) \). To this end we note that
\[
\|u\|_{C^{0,1}(\Omega, H)^*} = \sup_{\|\varphi\|_{C^{0,1}(\Omega, H)} \leq 1} \langle \varphi, u \rangle,
\]
for all \( u \in \mathcal{M}(\Omega, H) \). The results of the following theorem give a quantitative description of this observation.

Theorem 5.24. There exists a constant \( c > 0 \) such that, for all \( k \) large enough,
\[
\|u^k - \bar{u}\|_{C^{0,1}(\Omega, H)^*} \leq c \zeta_2^k, \tag{5.13}
\]
for all \( k \) large enough.

Proof. Let \( \varphi \in C^{0,1}(\Omega, H) \) with \( \|\varphi\|_{C^{0,1}(\Omega, H)} \leq 1 \) be given. We estimate
\[
|\langle \varphi, u^k - \bar{u} \rangle| \leq \sum_{i=1}^{N} \left| \int_{\overline{B}_{R_1}(\bar{x}_i)} \varphi \ d\bar{u}(x) - \int_{\overline{B}_{R_1}(\bar{x}_i)} \varphi \ du^k(x) \right|.
\]
Fix an arbitrary index \( i \in \{1, \ldots, N\} \) and split the error on the right hand side of the last inequality as
\[
\left| \int_{\overline{B}_{R_1}(\bar{x}_i)} \varphi \ d\bar{u}(x) - \int_{\overline{B}_{R_1}(\bar{x}_i)} \varphi \ du^k(x) \right| \\
= \left| \left( \varphi(\bar{x}_i), \bar{u}_i - u^k(\overline{B}_{R_1}(\bar{x}_i)) \right) \right|_H + \left| \left( \varphi(\bar{x}_i), u^k(\overline{B}_{R_1}(\bar{x}_i)) - \int_{\overline{B}_{R_1}(\bar{x}_i)} \varphi \ du^k(x) \right) \right|_H
\]
The first term is bounded by
\[
\left| \left( \varphi(\bar{x}_i), \bar{u}_i - u^k(\overline{B}_{R_1}(\bar{x}_i)) \right) \right|_H \leq \|\varphi(\bar{x}_i)\|_H \|\bar{u}_i - u^k(\overline{B}_{R_1}(\bar{x}_i))\|_H \leq c \|\varphi\|_{C^{0,1}(\Omega, H)} \zeta_2^k
\]
for some constant $c > 0$ independent of $i$ following Theorem 5.23. For the second term we use the Lipschitz continuity of $\varphi$ to obtain
\[
\left| (\varphi(\bar{x}_i), u^k(\bar{B}_R(\bar{x}_i))))_H - \int_{\bar{B}_R(\bar{x}_i)} \varphi \, du^k(x) \right| 
\leq \|\varphi\|_{\text{Lip}} \max_{x \in \text{supp}|u^k| \cap \bar{B}_R(\bar{x}_i)} |x - \bar{x}_i| E \|u^k_t\|_M \leq c \zeta^k_2,
\]
from the convergence results on the support points in Proposition 5.18. Again, the constant $c > 0$ can be chosen independent of the index $i$. Combining all previous observations we conclude
\[
\langle \varphi, u^k - \bar{u} \rangle \leq c c_2^k,
\]
for some constant $c > 0$ independent of $\varphi$. Taking the supremum over all Lipschitz continuous functions $\varphi \in C^{0,1}(\Omega, H)$, $\|\varphi\|_{C^{0,1}(\Omega, H)} \leq 1$, on both sides of the inequality yields the claimed statement.

5.3. Conic constraints. In this last section we comment on improved convergence results for Algorithm 2 in the case of conic constraints i.e. $C \neq H$. Let Assumptions 5.1, 5.2, 5.3 hold and denote by $\bar{u} = \sum_{i=1}^N \bar{u}_i \delta_{\bar{x}_i}$ the unique minimizer to (P). By $\bar{p}$, $\bar{P}$ and $p^k$, $P^k$ we refer to the dual variables and dual certificates associated to $\bar{u}$ and $u^k$, respectively. Let us first recall the unconstrained case, i.e. $C = H$. In this situation we based our proof on the local smoothness of the dual variables around the optimal support points. Moreover, since $\bar{p}(\bar{x}_i) \neq 0$, this regularity also transfers to the dual certificates which, together with Assumption 5.3, allowed to establish the perturbation results of Lemma 5.8. Obviously such reasoning fails in the constrained situation $C \neq H$ since
\[
\varphi \in C^2(\bar{\Omega}_H, H) \neq \pi_C(\varphi) \in C^2(\bar{\Omega}_H, H),
\]
in general. This is for example the case if there exists an index $i \in \{1, \ldots, N\}$ such that $\pi_C(\varphi(\bar{x}_i))$ lies at the boundary of $C$.

While this observation prevents a direct adaptation of the presented results to the general constrained case the aforementioned difficulty can be bypassed if the optimal dual variable $\bar{p}$ maps locally into the interior of $C$ in $H$. To this end let us assume that $\text{int} C \neq \emptyset$. In particular this encompasses the important case of positive scalar-valued measures. Furthermore assume that $\bar{p}(\bar{x}_i) \in \text{int} C$ for $i = 1, \ldots, N$. Due to the projection formula for the optimal coefficient functions $\bar{p}(\bar{x}_i)/\|\bar{p}\|_C = \bar{u}_i/\|\bar{u}_i\|_H$ this is equivalent to $\bar{u}_i \in \text{int} C$. Since $\bar{p}$ is continuous the set $\Omega_R$ can be chosen small enough such that $\bar{p}(x) \in \text{int} C$ for all $x \in \Omega_R$. Thus we obtain $\bar{P}(x) = \|\pi_C(\bar{p}(x))\|_H = \|\bar{p}(x)\|_H$ on $\bar{\Omega}_R$. This yields $\bar{P} \in C^2(\bar{\Omega}_R)$ following Lemma A.1. Furthermore arguing as in Lemma A.3 gives $\|\|K^*\nabla F(y)\|_\gamma\|_H \in C^2(\bar{\Omega}_R)$ for all $y$ in a neighborhood of $\bar{y}$ and, in particular, $P^k \in C^2(\bar{\Omega}_R)$ for all $k \in \mathbb{N}$ large enough.

The remaining improved convergence results are now obtained by repeating the presented arguments. In particular note that the intermediate iterates $u^{k+1/2}_s$, $s \in [0, 1]$ in the proof of Theorem 5.16 are admissible since $\pi_C(p^k(x^k_s)) = p^k(x^k_s)$ for all $k \in \mathbb{N}$ large enough and
\[
\begin{align*}
u^{k+1/2}_s &= u^k + s \Delta_2^k = (1 - s)u^k + su^k_c \\
u^{k+1/2}_s &\in B_{\mathcal{R}_1}(\hat{x}_i) + s \left( u^k \in B_{\mathcal{R}_1}(\bar{x}_i) + s \left( \|u^k\|_{B_{\mathcal{R}_1}(\bar{x}_i)} \right) \right) \in \mathcal{M}(\Omega, C),
\end{align*}
\]
due to $u^k \in B_{\mathcal{R}_1}(\bar{x}_i)$, $u^k \in B_{\mathcal{R}_1}(\bar{x}_i)$, $\pi_C(p^k(x^k_s)) \in \mathcal{M}(\Omega, C)$.

As a consequence of these considerations we conclude the following convergence result in the case of additional conic constraints.

**Theorem 5.25.** Let $C \subset H$ be a closed and convex cone with nonempty interior in $H$. Let Assumptions 5.1, 5.2, 5.3 hold and denote by $\bar{u} \in M(\Omega, C)$ the unique minimizer to (P). Further assume that $\bar{p}(x) \in \text{int} C$ for all $x \in \text{supp} |\bar{u}|$. Then Theorem 5.2 applies to $\{u^k\}_{k \in \mathbb{N}}$. 

**References.**

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Remark 5.2. Please note that for the important case of scalar measures with positivity constraints, i.e. $C = \mathbb{R}_+$, the additional condition $\bar{\mu}(\bar{x}_i) \in \text{int} \mathbb{R}_+$ is redundant since we assume that strict complementarity, $\text{supp} \{\bar{u}\} = \{\bar{x}_i\}_{i=1}^N$, holds.

5.4. Multiple point insertion. To close on the discussions of this section we emphasize that all of the presented results remain valid for more general choices of the active set $A_k$ provided that

$$\text{supp} \{u^k\} \cup \{\hat{x}^k\} \subset A_k, \quad \#A_k < \infty,$$

for all $k \in \mathbb{N}$. To this end recall that under the stated assumptions and for all $k \in \mathbb{N}$ large enough, the new Dirac delta position $\hat{x}^k$ in Algorithm 2 is taken from a finite set $\{\hat{x}^k_i\}_{i=1}^N$ where each point $\hat{x}^k_i \in B_{R_1}(\bar{x}_i)$ is given by the unique local minimizer of $P^k$ in a vicinity of the optimal point $\bar{x}_i$. Points outside of these neighborhoods should be not considered as new positions since $P^k$ is strictly smaller than $\lambda^k$ on $\Omega \setminus \bigcup_{i=1}^N B_{R_1}(\bar{x}_i)$.

If $k \in \mathbb{N}$ is sufficiently large these considerations suggest to update the active set as

$$A_k = \text{supp} \{u^k\} \cup \left\{ x \in \Omega \mid x \in \{\hat{x}^k_i\}_{i=1}^N, \quad P^k(x) \geq \lambda^k \right\}.$$ 

Thus instead of only adding one global maximizer of the dual certificate to the active set we now put in all points corresponding to sufficiently large local maxima of $P^k$. Due to the localization of $\text{supp} \{u^k\}$ around the optimal positions this can also be interpreted as adding up to one new Dirac delta function to each cluster in the current iterate. Intuitively this new update rule should lower the number of iterations to reduce the residual below a given threshold and improve the scalability of the method with respect to the support size of the minimizer $\bar{u}$.

This intuition is backed up by the following formal reasoning. Let the active set be updated by adding the global minimizer $\hat{x}^k$ in each iteration. Assume that $\text{supp} \{\bar{u}\} \cap \text{supp} \{u^k\} = \emptyset$ for all $k \in \mathbb{N}$ i.e. none of the optimal positions is contained in any of the iterated supports. Fix an arbitrary index $i = 1, \ldots, N$. By assumption there holds

$$\min_{x \in \text{supp} \{u^k\} \cap B_{R_1}(\bar{x}_i)} |x - \bar{x}_i| > 0, \quad k \in \mathbb{N}, \quad \max_{x \in \text{supp} \{u^k\} \cap B_{R_1}(\bar{x}_i)} |x - \bar{x}_i| \to 0.$$

As the movement of Dirac delta functions in $u^k$ is not possible this means that at some point a new Dirac delta function will be inserted in the vicinity of $\bar{x}_i$. Since the index $i$ was arbitrary and only a single point is inserted we conclude that Algorithm 2 eventually visits each of the $N$ Dirac delta clusters in a separate iteration. The new definition of the active set now aims to mitigate this cycling behavior of the point insertion step by inserting new points simultaneously in all clusters. In this context we also recall that a point insertion step is always connected to one solution of $(P(A))$. From this perspective we may also reduce the overall number of necessary solves for the coefficient optimization problems by inserting multiple points.

However these considerations are far from being conclusive and we have not been able to provide additional improved convergence results for this choice of $A_k$. Moreover note that these observations are of limited practical use since all arguments are only valid in the asymptotic regime i.e. for all $k \in \mathbb{N}$ large enough and if the structural assumptions from the beginning of this section hold.

**Appendix A. Auxiliary results**

In this section we summarize some technical auxiliary results that we needed in the preceding arguments but were postponed until now to avoid distraction.

**Lemma A.1.** Assume that Assumption 5.2 holds. Let $\bar{p} = -\nabla f(\bar{u}) \in C(\Omega, H)$ be given. Define the function

$$\bar{P}: \Omega \to \mathbb{R}_+, \quad x \mapsto \|\bar{p}(x)\|_H.$$ 

Then $R > 0$ may be chosen small enough such that $\bar{P} \in C^2(\bar{\Omega}_R)$. 

Proof. By Assumption 5.2 we have $\tilde{p} \in \mathcal{C}^2(\tilde{\Omega}_R, H)$ and $\tilde{P}(\tilde{x}_i) = \|\tilde{p}(\tilde{x}_i)\|_H = \tilde{\lambda} > 0$, $i = 1, \ldots, N$. In the following we denote by $\partial_x \tilde{p}$, $\partial_{x,i} \tilde{p} \in \mathcal{C}(\tilde{\Omega}_R, H)$, $i, j \in \{1, \ldots, d\}$, the first and second order partial derivatives of $\tilde{p}$. Note that $\tilde{P} \in \mathcal{C}(\Omega)$ due to the continuity of $\tilde{p}$. By continuity we may assume that $R > 0$ is chosen small enough such that $\tilde{P}(x) > \tilde{\lambda}/2$ for all $x \in \bigcup_{i=1}^N B_R(\tilde{x}_i)$. Using the chain rule we conclude that $\tilde{P}$ is two times continuously differentiable in each $x \in \bigcup_{i=1}^N B_R(\tilde{x}_i)$ with

$$\nabla \tilde{P}(x)_i = \frac{\langle \tilde{p}(x), \partial_x \tilde{p}(x) \rangle_H}{\tilde{P}(x)}$$

$$\nabla^2 \tilde{P}(x)_{ij} = \frac{\langle \partial_{x,i} \tilde{p}(x), \partial_{x,j} \tilde{p}(x) \rangle_H}{\tilde{P}(x)} + \frac{\langle \tilde{p}(x), \partial_{x,i} \tilde{p}(x) \rangle_H}{\tilde{P}(x)} - \frac{\langle \tilde{p}(x), \partial_{x,j} \tilde{p}(x) \rangle_H}{\tilde{P}(x)}$$

for all $i, j \in \{1, \ldots, d\}$. Obviously these derivatives can be continuously extended up to the boundary yielding $\tilde{P} \in \mathcal{C}^2(\Omega_R)$.

Lemma A.2. There exists $R_1 > 0$ such that for all $i \in \{1, \ldots, N\}$ the quadratic growth condition

$$\tilde{P}(x) + \frac{\theta_0}{4} |x - \tilde{x}_i|^2 \leq \tilde{P}(\tilde{x}_i) \quad \forall x \in \bar{B}_{R_1}(\tilde{x}_i)$$

is satisfied.

Proof. Let an arbitrary but fixed $i \in \{1, \ldots, N\}$ be given. By Taylor expansion we obtain for $x \in \bar{B}_{R_1}(\tilde{x}_i)$,

$$\tilde{P}(x) = \tilde{P}(\tilde{x}_i) + \langle \nabla \tilde{P}(\tilde{x}_i), x - \tilde{x}_i \rangle_{\mathbb{R}^d} + \frac{1}{2} \langle x - \tilde{x}_i, \nabla^2 \tilde{P}(x_\zeta)(x - \tilde{x}_i) \rangle_{\mathbb{R}^d}$$

where $x_\zeta = (1 - \zeta) x + \zeta \tilde{x}_i \in \Omega_R$ for some $\zeta \in (0, 1)$. Note that $\nabla \tilde{P}(\tilde{x}_i) = 0$ by Assumption 5.3. Using the coercivity of $\nabla^2 \tilde{P}(\tilde{x}_i)$ the second order term is estimated by

$$\langle x - \tilde{x}_i, \nabla^2 \tilde{P}(x_\zeta)(x - \tilde{x}_i) \rangle_{\mathbb{R}^d} \leq \langle x - \tilde{x}_i, \nabla^2 \tilde{P}(x_\zeta)(x - \tilde{x}_i) \rangle_{\mathbb{R}^d} + \langle x - \tilde{x}_i, \nabla^2 \tilde{P}(x_\zeta) - \nabla^2 \tilde{P}(\tilde{x}_i)(x - \tilde{x}_i) \rangle_{\mathbb{R}^d}$$

$$\leq \left( \| \nabla^2 \tilde{P}(x_\zeta) - \nabla^2 \tilde{P}(\tilde{x}_i) \|_{\mathbb{R}^{d \times d}} - \theta_0 \right) |x - \tilde{x}_i|^2$$

Since $\nabla^2 \tilde{P}$ is uniformly continuous on $\tilde{\Omega}_R$ there exists $R_1 \leq R$, independent of $i \in \{1, \ldots, N_d\}$ such that

$$|x - \tilde{x}_i|_{\mathbb{R}^d} \leq R_1 \Rightarrow \| \nabla^2 \tilde{P}(x) - \nabla^2 \tilde{P}(\tilde{x}_i) \|_{\mathbb{R}^{d \times d}} \leq \frac{\theta_0}{2}.$$ 

Consequently, for every $x \in \bar{B}_{R_1}(\tilde{x}_i)$ we obtain

$$\tilde{P}(x) \leq \tilde{P}(\tilde{x}_i) - \frac{\theta_0}{4} |x - \tilde{x}_i|^2_{\mathbb{R}^d},$$

proving (A.1) since $i$ was arbitrary.

Lemma A.3. Define the mapping

$$\mathcal{P}: \text{dom} \ F \rightarrow \mathcal{C}(\Omega) \quad y \mapsto \|K^* \nabla F(y)(\cdot)\|_H$$

Furthermore let $\bar{y} = K\bar{u}$. Then there exists $\delta > 0$ such that $\mathcal{P} \in \mathcal{C}^1(B_\delta(\bar{y}), \mathcal{C}^2(\tilde{\Omega}_R))$. In particular the mapping

$$\mathcal{F}: \tilde{\Omega}_R \times B_\delta(\bar{y}) \rightarrow \mathbb{R}^d, \quad (x, y) \mapsto \frac{\partial}{\partial x} \|K^* \nabla F(y)(x)\|_H,$$

is continuously Fréchet differentiable.
Proof. Due to the continuity of $K^*$, $\nabla F$ and the norm there exists $\delta > 0$ such that

$$[P(y)](x) > \frac{\lambda}{4} \quad \forall x \in \bigcup_{i=1}^{N_d} B_R(\bar{x}_i)$$

for all $y$ with $\|y - \bar{y}\|_Y \leq \delta$. Arguing as in Lemma A.1 we conclude $P(y) \in C^2(\Omega_R)$. As for $P$ we can derive formulas for the gradient $[\nabla P(y)]$ and the Hessian $[\nabla^2 P(y)]$ which depend differentiable on $y$ since $F$ is two times continuously Fréchet differentiable and $K^*$ maps continuously into $C^2(\Omega_R)$. In particular we obtain

$$\nabla [P(y)](x)_i = F(x,y)_i = \frac{\left( [K^*\nabla F(y)](x), \frac{\partial}{\partial x_i} [K^*\nabla F(y)](x) \right)_H}{P(x,y)}$$

for $i = 1, \ldots, d$. Thus the partial derivatives of $F$ with respect to $x$ and $y$ exist on $\Omega_R \times B_\delta(\bar{y})$ and are continuous. Continuous Fréchet differentiability of the mapping in (A.2) now follows from Proposition 3.2.18 and Remark 3.2.19 in [18].

\begin{lemma}
Let a compact set $\Omega \subset \mathbb{R}^d$ be given and assume that $K^*: Y \mapsto C^{0,1}(\Omega,H)$ is linearly and continuous. Let $u_1, u_2 \in H$, $x_1, x_2 \in \Omega$ be given. Then there exists $c > 0$ only depending on $K$ with

$$\|K(u_1 \delta_{x_1}) - K(u_1 \delta_{x_2})\|_Y \leq c\|u_1\|_H|x_1 - x_2|_{\mathbb{R}^d}$$

$$\|K(u_1 \delta_{x_1}) - K(u_2 \delta_{x_1})\|_Y \leq c\|u_1 - u_2\|_H.$$  

Proof. For $\varphi \in Y \backslash \{0\}$ we obtain

$$(K(u_1 \delta_{x_1}) - K(u_1 \delta_{x_2}), \varphi)_Y = (u_1(\delta_{x_1} - \delta_{x_2}), [K^*\varphi]) \leq \|u_1\|_H\|[K^*\varphi](x_1) - K^*\varphi(x_2)\|_H$$

$$\leq \|u_1\|_H\|[K^*\varphi]_{C^{0,1}(\Omega,H)}|x_1 - x_2|_{\mathbb{R}^d}$$

$$\leq \|u_1\|_H\|[K^*\varphi]_{L(C^{0,1}(\Omega,H))}\|\varphi\|_Y|x_1 - x_2|_{\mathbb{R}^d}.$$  

Analogously we get

$$(K(u_1 \delta_{x_1}) - K(u_2 \delta_{x_1}), \varphi)_Y \leq \|K^*\varphi\|c\|u_1 - u_2\|_H$$

$$\leq \|K^*\|_{L(C^{0,1}(\Omega,H))}\|\varphi\|_Y\|u_1 - u_2\|_H$$

Dividing both sides of the inequalities by $\|\varphi\|_Y$ and taking the supremum over all $\varphi \in Y \backslash \{0\}$ we conclude both estimates. \qed

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