Optimized $\delta$-Expansion in QCD; a challenge

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Abstract: We split the Yang-Mills Lagrangian into a free and an interaction part in such a way, that the free part is non-local and contains an arbitrary form factor. Manifest gauge invariance is guaranteed by connecting the field-strength tensors at different space-time points by a string. As a result the gluon propagator, which, due to the presence of the string, now contains many different contributions, comes out strictly transversal in one-loop order. We also calculate the ghost self-energy and the ghost-gluon vertex in one-loop order. Subsequently we discuss how one can determine the “optimal” form of the form factor. To apply well known principles like “fastest apparent convergence” or “principle of minimal sensitivity” one has first to solve some problems connected with divergences and renormalization. Here we concentrate on the calculation of the anomalous dimensions and the $\beta$-function. This is technically simpler because only the divergent contributions of the integrals have to be determined. The $\beta$-function becomes independent of the gauge parameter as it should. A puzzle with respect to the principle of minimal sensitivity shows up. Really interesting non-perturbative results are expected when applying the above principles directly to the propagator. For the $\beta$-function a two loop calculation would be required to obtain non-trivial results.

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1 Introduction

The “optimized $\delta$-expansion”, also called “linear $\delta$-expansion” or, more appropriately, “variation perturbation theory”, is a powerful method which combines the merits of perturbation theory with those of variational approaches. The underlying idea is simple. Generically, the Lagrangian is split into a free and an interacting part in such a way that an arbitrary parameter $\lambda$ (or more) is artificially introduced. The interacting part is multiplied by a factor $\delta$ which serves as formal expansion parameter and is put equal to one at the end. The exact solution should be independent of the parameter $\lambda$ while any approximate solution will, of course, depend on it. One way to fix the value of $\lambda$ is the “principle of minimal sensitivity” (PMS): It demands that the approximate solution should depend as little as possible on the parameter. This means that $\lambda$ should be chosen such that the quantity to be calculated has an extremum. In this way the result becomes non-perturbative because $\lambda$ becomes a non-linear function of the coupling constant. In every order of perturbation theory the optimal value of the parameter $\lambda$ has to be calculated again.

The method is now well established and it is therefore impossible to give all references. For the older literature we refer to Stevenson [1] and references therein, some more recent references can be found, e.g. in [2].

The field-theoretical applications have been rather modest up to now as far as the splitting of the Lagrangian into a free and an interacting part is concerned. Essentially only the mass parameter was used as variational parameter in the free part of the Lagrangian. The method is then also called “Gaussian effective potential”. Some references are given in [3]. In [2], on the other hand, the whole Lagrangian was scaled with a constant.

It has been suggested in the literature [4] that one should start the procedure with the most general free Lagrangian. This would lead to a non-local quadratic action containing an arbitrary form factor. It is also believed that the behavior in the ultraviolet region should be identical to that of usual perturbation theory if the form factor approaches one for large momenta. We will see that the situation is in fact more complicated in our case.

In the present paper we apply this idea to QCD, for simplicity without quarks for the present. The first central point, treated in sect. 2, is the construction of a Lagrangian which is gauge invariant for every $\delta$. This can be done by connecting the field tensors, now taken apart to different space time points in the non local Lagrangian, by a path ordered string. Gauge fixing can be done in analogy to the familiar case. The action becomes an infinite series in $\delta$ which coincides with the usual one for $\delta = 1$. For explicit calculations one expands the action up to the desired order.

In sect. 3 we give the general formula for the gluonic vacuum polarization. The transversality can be explicitly checked, thus confirming the manifest gauge invariance of the formalism. In sect. 4 we give the corresponding expressions for the ghost self-energy and the ghost-gluon vertex.

Up to this point the approach is quite general and the form factor, introduced when splitting the Lagrangian, essentially arbitrary. In sect 5. we discuss some conceptional questions which arise when one tries to fix the form factor in an optimal way. Due to
divergences this is a non-trivial task. In sect 6. we therefore specialize to a scale-invariant ansatz and concentrate on a calculation of the anomalous dimensions and the $\beta$-function. This is much simpler because only the divergent contributions have to be extracted.

Due to the complexity of the problem we have not yet achieved a break-through in the present paper. We believe, however, that we have demonstrated the feasibility and the potential of the method. With some optimism one may hope that it can become a new alternative approach to quantum field theories. In the conclusion we discuss, how non-perturbative results could be obtained in future applications.

2 The action

Our starting point is the classical Yang-Mills action

$$S^{(cl)} = -\frac{1}{4} \int F^a_{\mu\nu}(x) F^{\mu\nu a}(x) dx.$$  \hspace{1cm} (2.1)

The only modification at this point is that we replace the bare coupling constant $g_0$ in $F^a_{\mu\nu}$ by $\delta g_0$. The same replacement is made in the definition of gauge transformations. This is necessary in order to obtain gauge invariance for any $\delta$, which is of central importance. Thus we put

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \delta g_0 f^{abc} A^b_\mu A^c_\nu \equiv G^a_{\mu\nu} + \delta g_0 C^a_{\mu\nu}. \hspace{1cm} (2.2)$$

For later convenience we have introduced the notations $G^a_{\mu\nu}$ and $C^a_{\mu\nu}$ for the abelian part and the commutator term of $F^a_{\mu\nu}$, respectively.

In order to start the $\delta$-expansion with a general non-local free action we take the two factors $F^a_{\mu\nu}(x)$ in (2.1) apart to different space-time points $x, y$ and introduce a Lorentz invariant form factor $K(x - y)$. To do this in a gauge invariant way one has to connect the points $x, y$ by a path ordered octet exponential $U(x, y)$:

$$U(x, y) = \text{P}\exp[i\delta g_0 \int_x^y T^a A^a_\mu(z) dz^\mu], \hspace{1cm} (2.3)$$

with $(T^a)_{bc} = -if^{abc}$ the representation matrices in the octet representation. Technically this is much simpler than using two factors $U(x, y)$ and $U(y, x)$ in the triplet representation. So we rewrite (2.1) as

$$S^{(cl)}(x) = -\frac{1}{4} \int [(1 - \delta^2)K(x - y) + \delta^2 \delta(x - y)] F^a_{\mu\nu}(x) U^{ab}(x, y) F^{\mu\nu b}(y) dxdy = S_0^{(cl)} + S_1^{(cl)}, \hspace{1cm} (2.4)$$

with

$$S_0^{(cl)} = -\frac{1}{4} \int K(x - y) G^a_{\mu\nu}(x) G^{\mu\nu a}(y) dxdy. \hspace{1cm} (2.5)$$

Clearly (2.4) is gauge invariant for any $\delta$ and coincides with the original action (2.1) for $\delta = 1$. We used $\delta^2$ and not $\delta$ in the first two terms of (2.4) in order to keep the analogy with usual perturbation theory, where the first correction to the wave-function
renormalization is also quadratic in the coupling constant. Furthermore the Lagrangian (2.4) is symmetric with respect to the transformation $A_\mu \rightarrow -A_\mu$, $\delta \rightarrow -\delta$. The free action $S_0^{(cl)}$ is the part of $S^{(cl)}$ which survives for $\delta = 0$, while $S_I^{(cl)}$ denotes all the rest.

Next we perform gauge fixing and quantization with the help of the Faddeev-Popov procedure, slightly adopted to our case. We introduce the covariant gauge

\[ F[A^\alpha(x)] = \int K(x-y)[\partial^\mu A_\mu^\alpha(y) - \chi^\alpha(y)]dy. \] (2.6)

The ghost Lagrangian thus becomes

\[ S^{(ghost)} = -\int \bar{\omega}^a(x)K(x-y)\{\Box \omega^a(y) - \delta g_0 f^{abc}\partial^\mu [A^b_\mu(y)\omega^c(y)]\}dxdy. \] (2.7)

At the end of the gauge fixing procedure we integrate over the auxiliary fields $\chi^a$ with the weight function

\[ \exp[-\frac{i}{2\alpha} \int \chi^a(x)K(x-y)\chi^a(y)dxdy]. \] (2.8)

The result then combines with the terms involving $\partial^\mu \partial^\nu A_\nu^\alpha$ in the usual way. We thus arrive at the action $S = S_0 + S_I$ with

\[
S_0 = \frac{1}{2} \int A_\mu^\alpha(x)K(x-y)\{\Box A^{\alpha\mu}(y) + (1/\alpha - 1)\partial^\mu \partial_\nu A^\alpha_\nu(y)\}dxdy \\
- \int \bar{\omega}^a(x)K(x-y)\Box \omega^a(y)dxdy, \tag{2.9}
\]

\[
S_I = S_I^{(cl)} - \delta g_0 f^{abc} \int \partial^\mu \omega^a(x)K(x-y)A^b_\mu(y)\omega^c(y)dxdy. \tag{2.10}
\]

The free gluon propagator will therefore have the form

\[
D^a_{\mu\nu}(q) = \delta^{ab} D_{\mu\nu}(q) = \delta^{ab} D(q)(g_{\mu\nu} - \xi \frac{q_\mu q_\nu}{q^2}), \quad \text{with} \quad \xi = 1 - \alpha. \tag{2.11}
\]

Here

\[
D(q) = \frac{1}{(q^2 + i\epsilon)K(q)}, \tag{2.12}
\]

with $K(q)$ the Fourier transform of $K(x)$. The free ghost propagator is also given by $D(q)$. In fact the form factor in the ghost gluon vertex will cancel against the one in the ghost propagator in all ghost loops. This is due to the fact that our Faddeev-Popov determinant is just the usual one, multiplied with a field-independent term. The form (2.6) which leads to (2.7) has, however, the special virtue that the total action given below is invariant under the usual BRS transformation.

We mention that the special case $K = 1/\zeta$ with constant $\zeta$ would lead back to the ansatz in [4] which is, however, only useful if one wants to connect renormalized with un-renormalized quantities. If one expresses renormalized quantities through renormalized parameters, a constant $\zeta$ drops out.
For the explicit calculations we expand the string \( U(x, y) \) in powers of \( \delta \) and insert the series into (2.4). We thus obtain the following expansion of the action, which we will need only up to order \( \delta^2 \) here.

\[
S = S_0 + \delta S^{(1)} + \delta^2 S^{(2)} + \cdots.
\]  

(2.13)

\( S_0 \) was already given in (2.9), the interaction terms below are classified in an obvious notation, where the indices \( \text{ins}, C, S, \text{Gh} \) denote the origin from insertions (the term \(-\frac{i\rho}{4}\int[\delta(x - y) - K(x - y)]G^a_{\mu\nu}(x)G^\mu\nu a(y)dx dy \) present in (2.4)), commutator, string, or ghost interactions respectively. One finds

\[
S^{(1)}_C = -g_0f^{abc}\int \partial_\mu A^a_\nu(x)K(x-y)A^{\mu b}(y)A^{\nu c}(y)dx dy,
\]

\[
S^{(1)}_S = \frac{g_0}{2}\int f^{abc}\int \partial_\mu A^a_\nu(x)K(x-y)(x-y)^\rho A^b_\nu(s x + (1-s)y)[\partial^\rho A^{\nu c}(y) - \partial^\nu A^{\rho c}(y)]dx dy ds,
\]

\[
S^{(1)}_{\text{Gh}} = -g_0f^{abc}\int \partial^\mu \omega^a(x)K(x-y)A^b_\nu(y)\omega^c(y)dx dy.
\]

(2.14)

\[
S^{(2)}_{\text{ins}} = -\frac{1}{2}\int \partial_\mu A^a_\nu(x)[\delta(x-y) - K(x-y)][\partial^\mu A^{\nu a}(y) - \partial^\nu A^{\mu a}(y)]dx dy,
\]

\[
S^{(2)}_{CC} = -\frac{g_0^2}{4}\int f^{abc}f^{ade}\int A^b_\mu(x)A^c_\nu(x)K(x-y)A^{\mu d}(y)A^{\nu e}(y)dx dy,
\]

\[
S^{(2)}_{CS} = g_0^2\int f^{abc}f^{ade}\int A^b_\mu(x)A^c_\nu(x)K(x-y)(x-y)^\rho A^d_\nu(s x + (1-s)y)\partial^\mu A^{\nu e}(y)dx dy ds,
\]

\[
S^{(2)}_{SS} = -\frac{g_0^2}{2}\int f^{abc}f^{ade}\int \partial_\mu A^b_\nu(x)K(x-y)\Theta(s-s')(x-y)^\rho A^d_\nu(s x + (1-s)y)
\]

\[(x-y)^\sigma A^{\nu e}(s' x + (1-s')y)[\partial^\rho A^{\mu e}(y) - \partial^\nu A^{\rho e}(y)]dx dy ds ds'.
\]

(2.15)

Due to the strings, the action is in fact an infinite series in \( \delta \). This is a necessary consequence of the non-locality and the gauge invariance of the approach. In any finite order we will, of course, only need a finite number of terms.

The vertices of the action can be visualized by slightly modified Feynman graphs which are shown in fig. 1 in momentum space. The propagators refer to \( D(q) \) now, the presence of a thick line denotes a factor \( K(q) \). If such a factor appears in an internal line, it results in a total propagator \( D(q)K(q) = 1/(q^2 + i\epsilon) \), i.e. the usual free propagator. Thick lines with \( n \) gluon lines attached at the interior of the line denote the \( n \)-th order expansion of the string \( U(x, y) \). The graphs make the structure of the action more transparent, we did, however, not set up general modified Feynman rules here, but preferred the direct one-loop calculation.
3 Vacuum polarization

We start with the one particle irreducible contributions to the full gluon propagator which make up the vacuum polarization $\Pi_{\mu\nu}$. Due to the nonlocality of $S_0$, it is most convenient to write down the path integral representation for the propagator, expand it with respect to $\delta$, perform the Gaussian integrations and calculate the contractions in momentum space. The whole calculation can now be done using familiar methods. The result, written in $d$ dimensions and for $N_c$ colors, has the form

$$\Pi_{\mu\nu}(q) = \delta^2[K(q) - 1](q^2 g_{\mu\nu} - q_\mu q_\nu) - \frac{i \delta^2 g_0 N_c}{(2\pi)^d} \int \pi_{\mu\nu}(q, k) d^d k. \quad (3.1)$$

The first contribution at the rhs of (3.1) stems from the insertion $S_{\text{ms}}^{(2)}$ in (2.15). The integrand $\pi_{\mu\nu}(q, k)$ is a sum of 20 terms which we denote by $\pi_{\mu\nu}^{(j)}$. Terms (1) to (14) are loop terms arising from $S^{(1)} S^{(1)}$ in (2.14): (1)-(4) from $S^{(1)}_{C} S^{(1)}_{C}$, (5)-(9) from $S^{(1)}_{C} S^{(1)}_{S}$, (10)-(13) from $S^{(1)}_{S} S^{(1)}_{S}$, and (14) from $S^{(1)}_{Gh} S^{(1)}_{Gh}$.

Terms (15)-(20) originate from the second-order part $S^{(2)}$ in (2.15): Term (15) is the tadpole from the commutator term $S^{(2)}_{C C}$, (16),(17) stem from $S^{(2)}_{C S}$, and (18)-(20) from $S^{(2)}_{S S}$. Wherever possible, we simplified the expressions by using relation (2.12). The contributions are shown in graphical representation in fig. 2. The explicit forms read:

\[
\begin{align*}
\pi_{\mu\nu}^{(1)} &= \{q^2 g_{\mu\nu} - q_\mu q_\nu - \frac{\xi}{k^2}[qk(qk g_{\mu\nu} - q_\mu k_\nu) + (q^2 k_\mu - qk q_\mu)k_\nu] \} \\
&\quad + \frac{\xi^2 q(q - k)}{k^2(q - k)^2} k_\mu [q^2 k_\nu - qk q_\nu] D(k) D(q - k) K^2(q), \\
\pi_{\mu\nu}^{(2)} &= 2\{qk g_{\mu\nu} - q_\mu k_\nu - \frac{k(q - k)}{(q - k)^2}[q(q - k) g_{\mu\nu} - q_\mu(q - k)k_\nu] \} D(q - k) \frac{K(q)}{k^2}, \\
\pi_{\mu\nu}^{(3)} &= -(d - 1)k_\mu(q - k) \nu \frac{1}{k^2(q - k)^2}, \\
\pi_{\mu\nu}^{(4)} &= \{k^2 g_{\mu\nu} + (d - 2) k_\mu k_\nu - \frac{\xi}{(q - k)^2}[(k(q - k))^2 g_{\mu\nu} - 2k(q - k)q_\mu k_\nu] \} D(q - k) \frac{K(k)}{k^2}, \\
&\quad + (q^2 - k^2) k_\mu k_\nu \} D(q - k) \frac{K(k)}{k^2}, \\
\pi_{\mu\nu}^{(5)} &= \{q^2 k_\mu - qk q_\mu \} D(k) D(q - k) K(q) \int K_\nu(k - sq)ds, \\
\pi_{\mu\nu}^{(6)} &= 2\{[q(q - k) g_{\mu\nu} - q_\mu(q - k)k_\nu][q_\rho - \xi k^2 k_\rho] - (q^2 k_\mu - qk q_\mu)(g_{\nu\rho} - \xi k_\nu k_\rho) \} \\
&\quad \times D(k) D(q - k) K(q) \int K_\rho(q - sk)ds, \\
\pi_{\mu\nu}^{(7)} &= -2\{(d - 2)k(q - k)k_\mu + k^2(q - k)_\mu \} \frac{D(q - k)}{k^2} \int K_\nu(k - sq)ds, \\
\pi_{\mu\nu}^{(8)} &= 2\{[q(q - k) g_{\mu\nu} - q_\mu(q - k)k_\nu][q(q - k)_\rho - \xi k(q - k) k_\rho] \} \\
&\quad - (q - k)_\mu[q(q - k)g_{\nu\rho} - (q - k)_{\nu q_\rho}]
\end{align*}
\]
\(-\frac{\xi}{k^2}(q - k)_\mu[qkq_\nu - q^2k_\nu]k_\rho\frac{D(k)}{(q - k)^2}\int K^\rho(q - sk)ds,\)

\[\pi^{(9)}_{\mu\nu} = 2(q - k)_\mu\{kqg_{\nu\rho} - k_\nu q_\rho\}\frac{D(q - k)}{k^2}\int K^\rho(q - sk)ds,\]

\[\pi^{(10)}_{\mu\nu} = \frac{1}{2}\{(d - 2)(k(q - k))^2 + k^2(q - k)^2\}D(k)D(q - k)\int\int K_\mu(sq - k)K_\nu(s'q - k)dsds',\]

\[\pi^{(11)}_{\mu\nu} = -2\{k(q - k)[(q - k)g_{\nu\rho} - (q - k)_\nu q_\rho] + [q^2(q - k)_\mu - q(q - k)_\mu](q - k)_\nu\}D(k)D(q - k)\int\int K^\rho(q - sk)K^\sigma(1 - s')q + s'k)dsds',\]

\[\pi^{(12)}_{\mu\nu} = -\{kqg_{\nu\rho} - k_\mu q_\rho\}\{(q - k)g_{\nu\rho} - q_\mu(q - k)_\nu\}D(k)D(q - k)\int\int K^\rho(q - sk)K^\sigma(q - s'k)dsds',\]

\[\pi^{(13)}_{\mu\nu} = \frac{k_\mu(q - k)_\nu}{k^2(q - k)^2};\]

\[\pi^{(14)}_{\mu\nu} = \frac{k_\mu(q - k)_\nu}{k^2(q - k)^2};\]

\[\pi^{(15)}_{\mu\nu} = -\{(d - 1 - \xi)g_{\mu\nu} + \xi\frac{(q - k)_\mu(q - k)_\nu}{(q - k)^2}\}D(q - k)K(k),\]

\[\pi^{(16)}_{\mu\nu} = -2\{g_{\mu\nu}q_\rho - g_{\mu\rho}q_\nu - \frac{\xi}{k^2}(qkq_{\mu\nu} - q_\mu k_\nu)k_\rho\}D(k)\int K^\rho(q + sk)ds,\]

\[\pi^{(17)}_{\mu\nu} = -2(d - 1)k_\mu D(k)\int K_\nu(k + sq)ds,\]

\[\pi^{(18)}_{\mu\nu} = -[q^2g_{\mu\nu} - q_\mu q_\nu]\{g_{\rho\sigma} - \frac{\xi k_\rho k_\sigma}{k^2}\}D(k)\int\int \Theta(s - s')K^\rho(q + (s - s')k)dsds',\]

\[\pi^{(19)}_{\mu\nu} = -(d - 1)k^2 D(k)\int\int \Theta(s - s')K_{\mu\nu}(k + (s - s')q)dsds',\]

\[\pi^{(20)}_{\mu\nu} = 2[kqg^\rho_\mu - k_\mu q^\rho_\nu]D(k)\int\int \Theta(s - s')K_{\mu\nu}((1 - s')q + sk)dsds'.\]

The integrations over \(s\) or \(s'\) in the string terms, run from 0 to 1. We introduced the short hand notations \(K^\mu(k) = \partial^\mu K(k)\) and \(K_{\mu\nu}(k) = \partial^\mu \partial^\nu K(k)\). The string terms contain only derivatives of \(K\) because they have to vanish for the usual local form \(K = const\).

Our procedure in the last section guarantees that \(\Pi_{\mu\nu}\) is transversal. We check this by contracting \(\Pi_{\mu\nu}\) with \(q^\mu q^\nu\). We denote the corresponding integrands by \(\pi^{(1)}_i, \ldots, \pi^{(20)}_i\). The non vanishing terms are

\[\pi^{(3)}_i = -(d - 1)qkq(q - k)\frac{1}{k^2(q - k)^2};\]

\[\pi^{(4)}_i = \{q^2k^2 + (d - 2)(qk)^2 - \frac{k^2[q^2k^2 - (qk)^2]}{(q - k)^2}\}D(q - k)K(k)\]
\begin{align*}
\pi_i^{(7)} &= -2\{(d-2)q_kk(q-k) + k^2q(q-k)\} \frac{D(q-k)}{k^2} \int q^\nu K_{\nu}(k - sq)ds \\
&= -2\{(d-2)q_kk(q-k) + k^2q(q-k)\}\left[\frac{D(q-k)K(k)}{k^2} - \frac{1}{k^2(q-k)^2}\right], \\
\pi_i^{(10)} &= \frac{1}{2}\{(d-2)(k(q-k))^2 + k^2(q-k)^2\}D(k)D(q-k) \\
&\quad \int \int q^\mu K_{\mu}(sq-k)q^\nu K_{\nu}(s'q-k)dsds' \\
&= \{(d-2)(k(q-k))^2 + k^2(q-k)^2\}\left[\frac{D(q-k)K(k)}{k^2} - \frac{1}{k^2(q-k)^2}\right].
\end{align*}

\begin{align*}
\pi_i^{(14)} &= \frac{q_kq(q-k)}{k^2(q-k)^2}, \\
\pi_i^{(15)} &= -\{(d-1 - \xi)q^2 + \xi\frac{(q(q-k))^2}{(q-k)^2}\}D(q-k)K(k), \\
\pi_i^{(17)} &= -2(d-1)q_kD(k) \int q^\nu K_{\nu}(k + sq)ds, \\
&= 2(d-1)k^2q(q-k)\left[\frac{D(q-k)K(k)}{k^2} - \frac{1}{k^2(q-k)^2}\right], \\
\pi_i^{(19)} &= -(d-1)k^2D(k) \int \int \Theta(s-s')q^\mu q^\nu K_{\mu\nu}(k + (s-s')q)dsds' \\
&= -(d-1)k^2(q-k)^2\left[\frac{D(q-k)K(k)}{k^2} - \frac{1}{k^2(q-k)^2}\right].
\end{align*}

(3.3)

The second forms for the string terms \(\pi_i^{(7)}, \pi_i^{(10)}, \pi_i^{(14)}, \pi_i^{(15)}, \pi_i^{(17)}, \pi_i^{(19)}\) which do no longer contain integrations over \(s, s'\) are obtained as follows: E.g. in \(\pi_i^{(19)}\) use \(q^\mu q^\nu K_{\mu\nu}(k + (s-s')q) = -(d/ds)(d/ds')K(k + (s-s')q)\), perform the integrations over \(s\) and \(s'\), and substitute \(k \rightarrow k - p\) where appropriate. The cancellation of the sum \(\sum_j \pi_i^{(j)}\) now happens in the following way: All the \(\pi_i^{(j)}\) contain either the factors \(D(q-k)K(k)/k^2\) or the products of the free propagators \(1/k^2(q-k)^2\). The terms of the first type add up to zero, the sum of the latter ones may be written as

\begin{equation}
-(d-2)\left\{\frac{q(q-k)}{(q-k)^2} + \frac{q_k}{k^2}\right\}.
\end{equation}

(3.4)

Obviously this expression vanishes after integration over \(k\).

Due to the transversality of \(\Pi_{\mu\nu}\) we may write

\begin{equation}
\Pi_{\mu\nu}(q) = (q^2g_{\mu\nu} - q_{\mu}q_{\nu})\Pi(q^2).
\end{equation}

(3.5)

\(\Pi(q^2)\) has the form

\begin{equation}
\Pi(q^2) = \delta^2[K(q) - 1] - \frac{i\delta^2 g_0^2 N_c}{(2\pi)^d} \int \pi(q, k)d^4k.
\end{equation}

(3.6)

It is easy to obtain the integrands \(\pi^{(j)}(q, k)\) appearing in \(\Pi(q^2)\). Since we have already checked the transversality for the sum in (3.2), one can simply calculate the trace: \(\pi = \sum_j \pi^{(j)} = \sum_j \pi_{\mu}^{(j)\nu}/((d-1)q^2)\).
Proceeding further in the usual way one gets for the gluon propagator

\[ \delta^{ab} \Delta_{\mu\nu}(q) = \delta^{ab} \{ D_{\mu\nu}(q) + D_{\mu\rho}(q) \Pi^\alpha(q) D_{\lambda\nu}(q) + D \Pi D \Pi D + \cdots \}. \]  

(3.7)

In matrix notation the geometrical series becomes

\[ (\Delta) = [(D)^{-1} - (I)]^{-1}. \]  

(3.8)

This gives

\[ \Delta_{\mu\nu}(q) = \Delta(q)[g_{\mu\nu} - \tilde{\xi}(q^2) \frac{q_{\mu}q_{\nu}}{q^2}], \]  

(3.9)

with

\[ \Delta(q) = \frac{1}{(q^2 + i\epsilon)[K(q) - \Pi(q)]} \]  

(3.10)

and a modified \( q^2 \)-dependent \( \tilde{\xi}(q^2) \) which is of no interest here. The gluon wave-function renormalization constant \( Z_{\text{gluon}} \), defined at the renormalization scale \( -Q^2 > 0 \), is obtained from \( \Delta(-Q^2) = Z_{\text{gluon}}/(-Q^2) \). In order \( \delta^2 \) this gives

\[ Z_{\text{gluon}} = [1 + \Pi(-Q^2)/K(-Q^2)]/K(-Q^2). \]  

(3.11)

4 Ghost self-energy and vertex function

A calculation of the renormalized coupling constant from the gluon three-point function would be rather complicated. For this one would need the expansion of the action up to order \( \delta^3 \). Furthermore many mixing terms between \( S(1) \) and \( S(2) \) would show up. Therefore we will use the ghost-gluon vertex instead, which is much simpler.

We start with the ghost self-energy \( \Sigma(q^2) \). It consists only of the graph (21) (fig. 3) and reads

\[ \Sigma(q)/q^2 = -\frac{i\delta^2 q_0^2 N_c}{(2\pi)^d} \int \sigma^{(21)}(q, k)d^dk, \]  

(4.1)

with

\[ \sigma^{(21)}(q, k) = -[q(q - k) - \xi \frac{q k(q - k)k}{k^2}] \frac{K(q)D(k)}{q^2(q - k)^2}. \]  

(4.2)

It is connected to the ghost propagator \( \Delta_{\text{ghost}}(q) \) by

\[ \Delta_{\text{ghost}}(q) = \frac{1}{(q^2 + i\epsilon)[K(q) - \Sigma(q)/q^2]}. \]  

(4.3)

Defining the ghost wave-function renormalization constant \( Z_{\text{ghost}} \) by \( \Delta_{\text{ghost}}(-Q^2) = Z_{\text{ghost}}/(-Q^2) \) one has in order \( \delta^2 \)

\[ Z_{\text{ghost}} = [1 + \Sigma(-Q^2)/(-Q^2) K(-Q^2)]/K(-Q^2). \]  

(4.4)
We next consider the ghost-gluon vertex function $\Gamma_{\mu}^{abc}(p, q)$, where $p$ and $q$ denote the momenta of the incoming and outgoing ghost line with color indices $a$ and $b$. We write it in the form

$$\Gamma_{\mu}^{abc}(p, q) = f^{abc} \Gamma_{\mu}(p, q) = f^{abc} \left[ K(q)q_{\mu} - \frac{i\delta^2 g_0^2 N_c}{(2\pi)^d} \int \gamma_{\mu}(p, q, k) d^d k \right]. \quad (4.5)$$

There are seven contributions (fig. 4). Graphs (22) - (25) are generalizations of the usual vertex graphs, (26) - (28) are string terms. For general momenta the expressions are rather lengthy. We therefore specialize to the case $p = q$, i.e. vanishing gluon four momentum. This is sufficient for our purpose. The vertex function simplifies to

$$\Gamma_{\mu}(q, q) = q_{\mu} \Gamma(q)$$

with

$$\Gamma(q) = K(q) - \frac{i\delta^2 g_0^2 N_c}{(2\pi)^d} \int \gamma(q, k) d^d k. \quad (4.6)$$

Here $\gamma(q, k) = g_{\mu} \gamma_{\mu}(q, q, k)/q^2$ is a sum of seven terms $\gamma^{(j)}(q, k)$, only four are different from zero. They read:

$$\gamma^{(22)} = [q(q - k) - \xi \frac{q k}{k^2}] \frac{q(q - k) D(k) K(q)}{2 q^2 (q - k)^4},$$

$$\gamma^{(24)} = \xi [q^2 k^2 - (q k)^2] \frac{q(q - k) D(q - k) K(q)}{2 q^2 (q - k)^4},$$

$$\gamma^{(25)} = [q^2 k^2 - (q k)^2] \left\{ 1 + \frac{k(q - k)}{(q - k)^2} \right\} \frac{D(q - k) K(q)}{2 q^2 k^2 (q - k)^2},$$

$$\gamma^{(28)} = -[q^2 k^2 - (q k)^2] \frac{D^2(q - k) K(q) q_{\mu} K_{\mu}(q - k)}{2 q^2 k^2},$$

$$\gamma^{(23)} = \gamma^{(26)} = \gamma^{(27)} = 0. \quad (4.7)$$

The vertex renormalization constant $\tilde{Z}_{\text{vertex}}$ is directly related to (4.6):

$$\tilde{Z}_{\text{vertex}}^{-1} = \Gamma(-Q^2). \quad (4.8)$$

## 5 Conceptual questions

The derivation of the formulae in the last section, though somewhat tedious, was essentially straightforward. The real problem starts, when the expressions are to be evaluated and, in particular, when one has to find a principle for the “optimal” choice of the form factor $K(k)$. We will discuss some of the problematics here before entering more detailed calculations.

The evaluation can be performed in the following way. We assume that both $K(k)$ and $D(k) = 1/(k^2 + i\epsilon)$ $K(k)$ satisfy spectral representations with roughly the same behavior for large $k^2$ as in the free case where $K(k) = 1, D(k) = 1/(k^2 + i\epsilon)$. Therefore we write down a once subtracted Källen-Lehmann representation for $K$, with the subtraction
point chosen at infinity for convenience. For $D$ we assume an unsubtracted dispersion relation. Thus put

$$K(k^2) = 1 + \int \frac{\bar{\kappa}(\mu^2) d\mu^2}{k^2 - \mu^2 + i\epsilon}, \quad D(k^2) = \int \frac{\bar{\rho}(\mu^2) d\mu^2}{k^2 - \mu^2 + i\epsilon}. \quad (5.1)$$

Due to (2.12) the spectral functions $\bar{\kappa}$ and $\bar{\rho}$ are not independent. One could scale $K(k)$ with a constant $C$ and $D(k)$ with $1/C$, respectively. The normalization constant $C$ cancels, however, in all the loop graphs contributing to the gluon vacuum polarization, the ghost self-energy, and the vertex function. It only enters in the insertion to the vacuum polarization (first term on the rhs in (3.1)). In the following the normalization constant $C$ will be of no importance, therefore we choose it equal to 1.

The momentum integrations $\int \cdots d^dk$ in the expressions of the previous sections can now be performed in the usual way, although this becomes somewhat ugly for some of the string terms. At the end one is left with some integrations over the spectral functions $\bar{\kappa}$ and $\bar{\rho}$. After having calculated, say the vacuum polarization, the ghost self-energy, and the ghost-gluon vertex in this way, one has to decide how to choose the “optimal” spectral function $\bar{\kappa}$, i.e. the input function $K(x - y)$ in our ansatz (2.4). Two well known and successful principles suggest themselves: The principle of fastest apparent convergence (FAC) postulates that the considered quantity $Q$ does not change when going to a higher order of perturbation theory, i.e. $Q_n \overset{!}{=} Q_{n-1}$. The principle of minimal sensitivity (PMS) requires that the quantity be stationary with respect to an arbitrary parameter $\lambda$, i.e. $\partial Q_n / \partial \lambda \overset{!}{=} 0$. In our case PMS would not simply lead to an extremal problem but to a variational problem, because, instead of a single parameter $\lambda$, we have an arbitrary function $\bar{\kappa}$ at our disposal.

In a finite theory one could apply FAC or PMS to, say, the propagator and thus obtain a non-perturbative solution. In fact this can be done very easily in a toy model like four-dimensional $\Phi^3$ theory (I thank I. Solovtsov for suggesting this simple exercise and N. Brambilla and A. Vairo for an enlightening discussion of the result). FAC leads to an integral equation of the Dyson-Schwinger type which can easily be solved by iteration. PMS, on the other hand, is not applicable in one-loop order, because the propagator becomes independent of the spectral function.

Fundamental conceptional questions arise, however, if one tries to apply these ideas to field theories with divergences as in the present case. It appears natural to apply the above principles (FAC or PMS, respectively) to renormalized quantities, expressed by renormalized parameters. But the situation is more subtle here than in usual perturbation theory. The reason is, that renormalized quantities are not necessarily finite for a general function $K$! This is rather obvious, e.g. from the expressions in (3.2): The divergent terms have different $q^2$-dependent factors in front and thus will not cancel in the differences appearing, say, in renormalized propagators. Technically, the reason is that our action (2.9), (2.10), though gauge invariant for every $\delta$, is not renormalizable if $\delta$ is considered as the coupling constant.

One way to overcome the problem of divergences would be to simply drop the divergent contributions in renormalized quantities. From ordinary perturbation theory we know that they have to be absent, so also in our case they have to cancel when the whole perturbation series in $\delta$ is summed up. The removal of the divergent terms is certainly
not unique, analogous to the ambiguities of renormalization schemes. In finite orders the results will depend on the detailed prescription, the exact result should, however, be independent of it. The remaining finite quantities could then be determined by using FAC or PMS. We shall not investigate this possibility here but proceed with a calculation of the anomalous dimensions and the $\beta$-function, where the problem of removing divergences does not show up.

6 Anomalous dimensions and $\beta$-function

The $\beta$-function describes the scaling behavior of the quantized field theory which differs from the naive expectations from classical scale invariance. In order to maintain the classical scale invariance in every order of the $\delta$-expansion we make a scale-invariant ansatz for the spectral functions $\bar{\kappa}$ (dimension 0) and $\bar{\rho}$ (dimension -2). The only scale available is the external momentum $q$ in the propagators or the vertex function, respectively. With $Q^2 = -q^2 > 0$ the euclidean squared momentum, we therefore put

$$\bar{\kappa}(\mu^2) = \kappa(\mu^2/Q^2) \equiv \kappa(m^2).$$

(6.1)

The function $\kappa$ as well as the integration variable $m^2 = \mu^2/Q^2$ are now dimensionless. From (5.1) we get

$$K(k^2) = 1 + \int \frac{\kappa(m^2)}{k^2/Q^2 - m^2 + i\epsilon} dm^2.$$  

(6.2)

In particular we have

$$K(-Q^2) = 1 - \int \frac{\kappa(m^2)}{1 + m^2} dm^2,$$  

(6.3)

i.e. $K(-Q^2)$ becomes independent of $Q^2$. Therefore the corresponding factors present e.g. in $\pi^{(1)}, \pi^{(2)}, \cdots$ are not differentiated when calculating the $\beta$-function and the divergent contributions disappear as usual.

From simple dimensional analysis in $d = 4 - 2\epsilon$ dimensions the integrals appearing in (3.6),(4.1),(4.6) are proportional to $(Q^2)^{-\epsilon} = 1 - \epsilon \ln Q^2$. Only the divergent terms $\sim 1/\epsilon$ survive after applying the operator $Q^2d/dQ^2$. By naive power counting in $k$ one would conclude that all the string contributions in the vacuum polarization (3.2), with the exception of $\pi^{(7)}, \pi^{(17)}, \pi^{(19)}$ should be convergent. This is, however not true. The string parameter $s$ has to be integrated from 0 to 1, and for $s = 0$ most of the terms are divergent again by power counting. In fact, a careful investigation allows to extract the divergent contribution arising from the region of small $s$. In the appendix we give some technical details, how this can be done in a rather straightforward way.

The propagators $D$ will be treated by expressing $D$ through $K$ by using (2.12), introducing (5.1) (or (6.2)) for $K$ and expanding with respect to the integral:

$$D(k^2) = [(k^2 + i\epsilon)K(k^2)]^{-1} = \frac{1}{(k^2 + i\epsilon)}[1 - \int \frac{\kappa(\mu^2)d\mu^2}{k^2 - \mu^2 + i\epsilon} + O(1/k^6)].$$  

(6.4)

The terms of order $1/k^6$ lead to finite integrals and are therefore irrelevant.
We write the renormalization constants, derived in (3.11), (4.4), (4.8) in the form

\[
Z_{\text{gluon}} = \frac{1}{K} \left[ 1 + \delta^2 (K - 1) / K + \frac{\delta^2 g_0^2 N_c}{(4\pi)^2 K} \left( \frac{1}{\epsilon} - \ln Q^2 \right) \hat{\pi} \right],
\]
\[
Z_{\text{ghost}} = \frac{1}{K} \left[ 1 + \frac{\delta^2 g_0^2 N_c}{(4\pi)^2 K} \left( \frac{1}{\epsilon} - \ln Q^2 \right) \hat{\sigma} \right],
\]
\[
\tilde{Z}_{\text{vertex}}^{-1} = K \left[ 1 + \frac{\delta^2 g_0^2 N_c}{(4\pi)^2 K} \left( \frac{1}{\epsilon} - \ln Q^2 \right) \hat{\gamma} \right].
\]

In the above equations we have abbreviated

\[
K \equiv K(-Q^2).
\]

We have only written down the divergent terms \( \sim (1/\epsilon - \ln Q^2) \) of the integrals \( \pi, \sigma, \gamma \), the factors in front have been denoted by \( \hat{\pi}, \hat{\sigma}, \hat{\gamma} \). The renormalized coupling constant becomes

\[
g = \delta g_0 Z_{\text{gluon}}^{1/2} Z_{\text{ghost}} \tilde{Z}_{\text{vertex}}^{-1} = \frac{\delta g_0}{\sqrt{K}} \left[ 1 + \frac{\delta^2}{2K} (K - 1) + \frac{\delta^2 g_0^2 N_c}{(4\pi)^2 K} \left( \frac{1}{\epsilon} - \ln Q^2 \right) \hat{\beta} \right],
\]

with \( \hat{\beta} = \hat{\pi} / 2 + \hat{\sigma} + \hat{\gamma} \). In lowest order one has

\[
g = \delta g_0 / \sqrt{K}.
\]

The various contributions to \( \hat{\pi}, \hat{\sigma}, \hat{\gamma}, \hat{\beta} \) all have the form

\[
h + \int l(m^2) \kappa(m^2) dm^2 + \int \int K(m^2, m^2') \kappa(m^2) \kappa(m'^2) dm^2 dm'^2,
\]

i.e. they are at most quadratic in the spectral function \( \kappa(m^2) \), higher powers only contribute finite terms. Below we give the non-vanishing contributions, where the indices correspond to the numbers in (3.2), (4.1), and (4.7) as well as to the graphs in the figures. All calculations can be easily done analytically, with the exception of the quadratic contributions of the string-string graphs (12), (13). We leave them unspecified here.

It is important to note, however, that graph (13) also has a \( \xi \)-dependent term which can be calculated analytically. This will be important for the cancellation of the \( \xi \)-dependence in the \( \beta \)-function. The non-vanishing contributions read:

\[
\begin{align*}
    h^{(1)} &= 1 - \xi / 2, \\
    h^{(2)} &= 1 + \xi / 2, \\
    h^{(3)} &= -1 / 2, \\
    h^{(4)} &= \xi / 2, \\
    h^{(14)} &= 1 / 6, \\
    h^{(21)} &= (1 + \xi / 2) / 2, \\
    h^{(22)} &= (1 - \xi) / 8, \\
    h^{(25)} &= 3 (1 - \xi) / 8. 
\end{align*}
\]
\[ l^{(1)} = -\frac{2(1 - \xi/2)}{1 + m^2}, \]
\[ l^{(2)} = -\frac{(1 + \xi/2)}{1 + m^2}, \]
\[ l^{(6)} = -\frac{2m^2 + \xi}{1 + m^2} + 2m^2 \ln \frac{1 + m^2}{m^2}, \]
\[ l^{(7)} = 4, \]
\[ l^{(8)} = -\frac{2m^2 + \xi}{1 + m^2} + 2m^2 \ln \frac{1 + m^2}{m^2}, \]
\[ l^{(9)} = \frac{1 + 2m^2}{1 + m^2} - 2m^2 \ln \frac{1 + m^2}{m^2}, \]
\[ l^{(17)} = -4, \]
\[ l^{(18)} = \frac{2 + \xi}{1 + m^2}, \]
\[ l^{(21)} = -\frac{1 + \xi/2}{2(1 + m^2)}, \]
\[ l^{(22)} = -\frac{1 - \xi}{8(1 + m^2)}, \]
\[ l^{(25)} = -\frac{3(1 - \xi)}{8(1 + m^2)}, \]  

\[ K^{(1)} = \frac{1 - \xi/2}{(1 + m^2)(1 + m'^2)}, \]
\[ K^{(6)} = (\frac{2m^2 + \xi}{1 + m^2} - 2m^2 \ln \frac{1 + m^2}{m^2}) \frac{1}{2(1 + m'^2)} + (m^2 \leftrightarrow m'^2), \]
\[ K^{(12)} = \hat{K}^{(12)}(m^2, m'^2), \]
\[ K^{(13)} = \hat{K}^{(13)}(m^2, m'^2) - \frac{\xi}{2(1 + m^2)(1 + m'^2)}. \]  

Summing up contributions (1) - (20) for \( \hat{\pi} \), (21) for \( \hat{\sigma} \), and (22) - (25) for \( \hat{\gamma} \), we obtain

\[ h_\pi = \frac{5}{3} + \xi/2, \quad l_\pi = -\frac{2m^2 + \xi/2}{1 + m^2} + 2m^2 \ln \frac{1 + m^2}{m^2}, \]
\[ K_\pi = \frac{1 + m^2 + m'^2}{(1 + m^2)(1 + m'^2)} - \frac{m^2}{1 + m^2} \ln \frac{1 + m^2}{m^2} - \frac{m'^2}{1 + m'^2} \ln \frac{1 + m'^2}{m'^2} + \hat{K}^{(12)}(m^2, m'^2) + \hat{K}^{(13)}(m^2, m'^2), \]
\[ h_\sigma = (1 + \xi/2)/2, \quad l_\sigma = -\frac{1 + \xi/2}{2(1 + m^2)}, \quad K_\sigma = 0, \]  

13
\[ h_\gamma = (1 - \xi)/2, \quad l_\gamma = -\frac{1 - \xi}{2(1 + m^2)}, \quad K_\gamma = 0. \]  

(6.17)

Furthermore we have for \( \hat{\beta} = \hat{\pi}/2 + \hat{\sigma} + \hat{\gamma} \):

\[ h_\beta = 11/6, \quad l_\beta = m^2 \ln \frac{1 + m^2}{m^2} - 1, \quad K_\beta = \frac{1}{2}K_\pi. \]  

(6.18)

The anomalous dimensions and the \( \beta \)-function become

\[ \gamma_{\text{gluon}} = \frac{1}{Z_{\text{gluon}}} Q^2 \frac{dZ_{\text{gluon}}}{dQ^2} = -\frac{g^2 N_c}{(4\pi)^2} \left[ \frac{5}{3} \frac{\xi}{2} + \int l_\pi(m^2)\kappa(m^2)dm^2 \right. \]

\[ + \left. \int \int K_\pi(m^2, m'^2)\kappa(m^2)\kappa(m'^2)dm^2dm'^2 \right], \]  

(6.19)

\[ \gamma_{\text{ghost}} = \frac{1}{Z_{\text{ghost}}} Q^2 \frac{dZ_{\text{ghost}}}{dQ^2} = -\frac{g^2 N_c}{(4\pi)^2} \left[ \frac{1}{2} + \frac{\xi}{4} + \int l_\sigma(m^2)\kappa(m^2)dm^2 \right], \]  

(6.20)

\[ \beta = 2Q^2 \frac{dg}{dQ^2} = -\frac{2g^3 N_c}{(4\pi)^2} \left[ \frac{11}{6} + \int l_\beta(m^2)\kappa(m^2)dm^2 \right. \]

\[ + \left. \int \int K_\beta(m^2, m'^2)\kappa(m^2)\kappa(m'^2)dm^2dm'^2 \right]. \]  

(6.21)

We used (6.10) to replace \( \delta g_0/\sqrt{K} \) by the renormalized coupling constant \( g \). For \( \kappa(m^2) \equiv 0 \) we recover the well-known results of ordinary perturbation theory in one-loop order. Note further that the \( \beta \)-function is independent of the gauge parameter \( \xi \) as it should. This is a further test of the manifest gauge invariance of the approach.

Let us next discuss whether the principle of fastest apparent convergence (FAC) or the principle of minimal sensitivity (PMS) can be applied, say, to \( \beta \) in (6.21). An application of FAC is clearly impossible because the present one-loop calculation is the lowest non-trivial contribution; there is no lower order with which one could compare. An attempt to apply PMS also fails. The variation with respect to \( \kappa(m^2) \) gives no solution at all in (6.20) which is linear in \( \kappa \). In (6.19),(6.21), on the other hand, the presence of the linear term \( f l(m^2)\kappa(m^2)dm^2 \) leads to an extremum which is not situated at \( \kappa = 0 \). Therefore one would not reproduce the results of ordinary perturbation theory for small \( g \).

This is not yet a problem. We know from simple toy models that the lowest order usually gives no extremum. One needs the lowest and, at least, the next order to get a balance between these contributions and to find a relevant extremum. In both cases, FAC or PMS, one should therefore go to order \( \delta^4 \) for the anomalous dimensions and to order \( \delta^5 \) for the \( \beta \)-function. A calculation of the two loop contributions \( \sim \delta^5g^5 \) to the \( \beta \)-function is not feasible at present, but one can easily look for the effect of the contributions \( \sim \delta^5g^3 \) which, of course, are the most important ones for small coupling. These contributions arise from two sources: First we have to consider the insertion \( \delta^2(K - 1)/K \) in \( Z_{\text{gluon}} \), when expressing \( \delta g_0 \) by \( g \). This leads to
\[ g = \frac{\delta g_0}{\sqrt{K}}[1 + \frac{\delta^2}{2}(1 - 1/K) + O(\delta^2 g_0^2)]. \]  
(6.22)

Inverting this equation, one obtains for an arbitrary power \( n \)

\[ \left( \frac{\delta g_0}{\sqrt{K}} \right)^n = g^n[1 + \frac{n\delta^2}{2}(1/K - 1) + \cdots]. \]  
(6.23)

Thus, for \( \delta = 1 \), instead of (6.10), one should now replace

\[ \frac{\delta^2 g_0^2}{K} \Rightarrow \frac{g^2}{K}, \quad \frac{\delta^3 g_0^3}{K^{3/2}} \Rightarrow g^3 \left( \frac{3}{2K} - \frac{1}{2} \right). \]  
(6.24)

Different replacement rules for different powers look a bit strange but appear as a direct consequence of the expansion in \( \delta \).

The second effect is that we now have to consider insertions into the internal gluon lines in the graphs (1)-(28). Such an insertion replaces the original propagator \( D_{\mu\nu}(k) = D(k^2)(g_{\mu\nu} - \xi k_{\mu} k_{\nu}/k^2) \) by \( (1 - 1/K(k^2))D(k^2)(g_{\mu\nu} - k_{\mu} k_{\nu}/k^2) \). This is transversal, therefore it is most easily discussed in the Landau gauge \( \xi = 1 \) where one simply gets a factor \( 1 - 1/K(k^2) = \int d\mu^2 \kappa(\mu^2)/(k^2 - \mu^2 + i\epsilon) + O(1/k^4) \) which multiplies the original propagator. Obviously one has a suppression by an additional power of two in the denominator, therefore only insertions into the graphs (3),(4),(15) lead to divergent contributions, all the others become convergent. In Landau gauge we find an additional contribution of -4 to \( l^{(3)} \), -4 to \( l^{(4)} \), and 12 to \( l^{(15)} \).

An immediate result is, that now \( \gamma_{\text{ghost}} \) becomes independent of the spectral function \( \kappa(m^2) \); because of the relation \( l_\sigma = -h_\sigma/(1 + m^2) \) in (6.16) the square bracket in (6.20) is proportional to \( 1 - \int dm^2 \kappa(m^2)/(1 + m^2) = K \) which cancels against the \( 1/K \) in front which now survives when \( \delta^2 g_0^2/K \) is replaced by \( g^2/K \) according to (6.24). This is quite welcome, because the unpleasant linear contribution in \( \kappa \) has thus disappeared. The same is, however, not true for \( \gamma_{\text{gluon}} \) and for \( \beta \). This result will also hold in higher orders, because higher order insertions make all graphs finite.

The additional contributions of order \( \delta^5 g^3 \) just discussed (remember that we did not take into account terms of order \( \delta^5 g^5 \)) all vanish if \( \kappa(m^2) = 0 \). Comparing order \( \delta^3 \) and \( \delta^5 \), FAC would now trivially give the solution \( \kappa \equiv 0 \) which is just what we expect. Inclusion of the two-loop contributions \( \sim \delta^5 g^5 \) would shift this to a non-trivial solution for \( \kappa \) and finally result in a non-perturbative solution for the \( \beta \)-function.

For PMS, on the other hand, there is really a problem. The contributions \( \sim \kappa \) are still there, and the optimized \( \delta \)-expansion would not reproduce the lowest order perturbative result!

7 Conclusions

At present we cannot offer a convincing explanation for the results obtained above. Of course we cannot exclude the possibility of a calculational error in our formulae, although we have carefully checked them. Assuming that they are correct, it remains a puzzle why PMS fails to reproduce the lowest order perturbative result. This is particularly
confusing because the puzzle persists to any order in $\delta$ if $g$ is small. An encouraging result would have consisted in a cancellation of all the terms linear in $\kappa(m^2)$ appearing in the $\beta$-function, i.e. $l_\beta = 0$. The quadratic terms would then lead to an extremum at $\kappa(m^2) \equiv 0$, thus reproducing the results of one-loop ordinary perturbation theory. In a two-loop calculation, which might be feasible with more intensive computer help, one would then expect a non-trivial solution for $\kappa(m^2)$ and a non-perturbative determination of the $\beta$-function. But this was not what we obtained. On the other hand, FAC could lead to interesting non-perturbative results in two-loop order which include ordinary perturbation theory when expanded with respect to $g$.

Let us now discuss a different possible approach which was already briefly mentioned in sect. 5. One could calculate the renormalized gluon propagator in terms of the general spectral function $\tilde{\kappa}$ introduced in (5.1) and remove the remaining divergent contributions by a definite prescription. This would be complementary to - and of course much more complicated than - the calculations in sect. 6 where we concentrated on the divergent terms. Finally one could determine $\tilde{\kappa}$ using FAC or PMS.

Note, however, that our ansatz in (5.1) implies $K(k^2) \to 1$ and, accordingly, $D(k^2) \to 1/k^2$ for $k^2 \to \infty$. The renormalized propagator, as determined by PMS, could nevertheless show the correct asymptotic behavior as obtained from the renormalization group, namely $\Delta(k^2) \sim [\ln k^2]^{-(10+3\xi)/44}/k^2$. (See e.g. [5]). If one would apply FAC, on the other hand, the bare and the renormalized propagator would be identical by definition of the method. In this case one should either use the special gauge $\xi = -10/3$ for which $\gamma_{\text{gluon}} = 0$ and $\Delta(k^2) \sim 1/k^2$, or, more generally, write down a twice subtracted dispersion relation for $K(k^2)$ with a finite subtraction point. A calculation of this type, though complicated, appears possible and will be undertaken in the future.

Finally we would like to mention a conceptional problem. The connection between the bare and the renormalized coupling constant will always start with $g \sim \delta g_0$. When $g_0$ is eliminated, higher powers $(\delta g_0)^n$ become proportional to $g^n$, i.e. become independent of $\delta$. So the clear bookkeeping of powers of $\delta$ is somehow blurred by the renormalization procedure.

The present approach appears complicated. We interpret this as a reflection of the fact that QCD is complicated and that non-perturbative results can only be obtained with considerable effort. We believe that interesting information can be extracted from the general expressions presented here, and even more interesting information from a two-loop calculation which might be feasible. The puzzle in connection with PMS is not understood at present. Any suggestions are welcome.

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A Appendix

We give here some technicalities how the divergent contributions of the string terms can be extracted. The problematic terms are those where the internal momentum $k$ is multiplied with a string parameter $s$ which has to be integrated from 0 to 1. For $s \approx 0$ there is no suppression by a power of $k$ in the denominator, therefore a more detailed analysis is necessary.

In a first step introduce the spectral representations (5.1) or (6.2), (6.4) for $K$ and $D$, as well as the resulting representations for $K^\mu$ and $K^{\mu\nu}$, and rotate to euclidean space as usual. In all denominators without string parameters $s$ or $s'$ expand with respect to $1/k^2$, i.e.

$$\frac{1}{(k-q)^2 + \mu^2} = \frac{1}{k^2} + \frac{2qk}{k^4} + \frac{4(qk)^2}{k^6} - \frac{q^2 + \mu^2}{k^4} + \cdots \quad (1.1)$$

up to the order where the further terms become ultraviolet finite. The apparent infrared singularities arising from the expansion are spurious.

In terms with only one string, e.g. (8), we have a further denominator of the form

$$\frac{1}{[(q-sk)^2 + \mu^2]^2}$$

arising from the spectral representation of $K^\rho(q-sk)$. The substitution $k = k'/s$ makes the integration over $k'$ finite. But for every power $k^{-n}$ in the integrand a power $s^{n-d} = s^{n-4+2\epsilon}$ appears in $d = 4 - 2\epsilon$ dimensions. The $s$-integration gives a factor $(n - 3 + 2\epsilon)^{-1}$ and thus a divergent contribution if $n = 3$.

Next we discuss the terms with two string integrations over $s, s'$. Terms (10),(11) are finite, in (12),(13) substitute $s = r(1-t), s' = rt$. This gives a factor $r$ from the Jacobian. Next substitute $k = k'/r$ and perform the integral over $r$ from 0 to 1 as before. This gives again some divergent factors, the remaining integral is finite. The second order term in (19) is finite, in (18) and (20) substitute $t = s - s', t' = (s + s' - 1)/2$. The $t'$-integration can be trivially performed and gives a factor $(1-t)$, then substitute $k = k'/t$ as before. After integration over $k'$ the various terms in (20) cancel, so that this graph, contrary to what one would expect, does not give a divergent contribution.
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Figure Captions

Fig. 1: Interaction terms in momentum space. All momenta are incoming. A thick line denotes the presence of a factor $K$. Thick lines with gluons attached at the interior of the lines arise from the expansion of the string $U(x, y)$. They are associated with a factor $K^{\mu} = \partial^{\mu}K$ or $K^{\mu\nu} = \partial^{\mu}\partial^{\nu}K$, respectively. String parameters $s, s'$ are integrated from 0 to 1.

Fig. 2: Contributions to the gluon vacuum polarization.

Fig. 3: Ghost self-energy.

Fig. 4: Contributions to the ghost gluon vertex.
$S_C^{(1)}$

$S_S^{(1)}$

$S_{Gh}^{(1)}$

$k$

$1 - K(k)$

$S_{ins}^{(2)}$

$S_{CC}^{(2)}$

$k$

$q$

$p$

$q$

$p$

$q$

$-(k + q)^\lambda$

$(k - p)^\rho$

$\Theta(s - s')K_{\lambda\rho}[sq - (1 - s')p + (s - s')k]$
Fig. 2a
Fig. 2b
