AN ANALOGUE OF COWLING-PRICE’S THEOREM FOR THE 
Q-FOURIER-DUNKL TRANSFORM

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Abstract. The Q-Fourier-Dunkl transform satisfies some uncertainty principles in a similar way to the Euclidean Fourier transform. By using the heat kernel associated to the Q-Fourier-Dunkl operator, we have established an analogue of Cowling-Price, Miyachi and Morgan theorems on $\mathbb{R}$ by using the heat kernel associated to the Q-Fourier-Dunkl transform.

Keywords: Cowling-Price’s theorem; Miyachi’s theorem; Uncertainty Principles; Q-Fourier-Dunkl transform.

1. Introduction

There are many theorems which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot simultaneously be very small at infinity. This principle has several versions which were proved by M.G. Cowling and J.F. Price [3] and Miyachi [6]. In this paper, we will study an analogue of Cowling-Price’s theorem and Miyachi’s theorem for the Q-Fourier-Dunkl transform. Many authors have established the analogous of Cowling-Price’s theorem in other various setting of harmonic analysis (see for instance [5]) The outline of the content of this paper is as follows. Section 2 is dedicated to some properties and results concerning the Q-Fourier-Dunkl transform. In Section 3 we give an analogue of Cowling-Price’s theorem, Miyachi’s theorem, and Morgan’s theorem for the Q-Fourier-Dunkl transform. Let us now be more precise and describe our results. To do so, we need to introduce some notations. Throughout this paper $\alpha > -\frac{1}{2}$. Notice that if $\alpha = -\frac{1}{2}$ then the space is the classical Lebesgue one, we can follow in this case the procedures for similar transforms, such as the Fourier transform (see for example [3, 6]).

\begin{equation}
Q(x) = \exp\left(-\int_0^x q(t)dt\right), \quad x \in \mathbb{R}
\end{equation}

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where $q$ is a $C^\infty$ real-valued odd function on $\mathbb{R}$.

- $L^p_\alpha(\mathbb{R})$ the class of measurable functions $f$ on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where
  \[
  \|f\|_{p,\alpha} = \left( \int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx \right)^{\frac{1}{p}}, \quad \text{if} \quad 1 < p < \infty,
  \]
  and $\|f\|_{\infty,\alpha} = \|f\|_{\infty} = \text{esssup}_{x \in \mathbb{R}} |f(x)|$.

- $L^p_Q(\mathbb{R})$ the class of measurable functions $f$ on $\mathbb{R}$ for which $\|f\|_{p,Q} = \|Qf\|_{p,\alpha} < \infty$, where $Q$ is given by (1.1).

We consider the first singular differential-difference operator $\Lambda$ defined on $\mathbb{R}$
\[
\Lambda f(x) = f'(x) + \left( \alpha + \frac{1}{2} \right) \frac{f(x) - f(-x)}{x} + q(x)f(x) \tag{1.2}
\]
where $q$ is a $C^\infty$ real-valued odd function on $\mathbb{R}$. For $q = 0$ we regain the Dunkl operator $\Lambda_\alpha$ associated with reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by
\[
\Lambda_\alpha f(x) = f'(x) + \left( \alpha + \frac{1}{2} \right) \frac{f(x) - f(-x)}{x}.
\]

### 1.1. Q-Fourier-Dunkl Transform

The following statements are proved in [1]

**Lemma 1.1.** 1. For each $\lambda \in \mathbb{C}$, the differential-difference equation
\[
\Lambda u = i\lambda u, \quad u(0) = 1
\]
admits a unique $C^\infty$ solution on $\mathbb{R}$, denoted by $\Psi_\lambda$, given by
\[
\Psi_\lambda(x) = Q(x)e_\alpha(i\lambda x), \tag{1.3}
\]
where $e_\alpha$ denotes the one-dimensional Dunkl kernel defined by
\[
e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha + 1)} j_{\alpha+1}(z) \quad (z \in \mathbb{C}),
\]
and $j_\alpha$ being the normalized spherical Bessel function of index $\alpha$ given by
\[
j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z^2/4)^n}{n! \Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}). \tag{1.4}
\]

2. For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $n = 0, 1, \ldots$ we have
\[
|\frac{\partial^n}{\partial \lambda^n} \Psi_\lambda(x)| \leq Q(x)|x|^n e^{\lambda|x|}. \tag{1.5}
\]
In particular
\[
|\Psi_\lambda(x)| \leq Q(x)e^{\lambda|x|}. \tag{1.6}
\]
3. For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, we have the Laplace type integral representation

$$\Psi_{\lambda}(x) = a_{\alpha} Q(x) \int_{-1}^{1} (1 - t^2)^{\alpha - \frac{1}{2}} (1 + t) e^{i\lambda xt} dt,$$

where $a_{\alpha} = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})}$.

**Definition 1.1.** The Q-Fourier-Dunkl transform associated with $\Lambda$ for a function in $L^1_Q(\mathbb{R})$ is defined by

$$F_Q(f)(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{-\lambda}(x) x^{2\alpha + 1} dx.$$

**Theorem 1.1.**

1. Let $f \in L^1_Q(\mathbb{R})$ such that $F_Q(f) \in L^1(\mathbb{R})$. Then for almost all $x \in \mathbb{R}$ we have the inversion formula

$$f(x) (Q(x))^2 = m_{\alpha} \int_{\mathbb{R}} F_Q(f)(\lambda) \Psi_{\lambda}(x)|\lambda|^{2\alpha + 1} d\lambda,$$

where

$$m_{\alpha} = \frac{1}{2^{2(\alpha + 1)}(\Gamma(\alpha + 1))^2}.$$

2. For every $f \in L^2_Q(\mathbb{R})$, we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 (Q(x))^2 |x|^{2\alpha + 1} dx = m_{\alpha} \int_{\mathbb{R}} |F_Q(f)(\lambda)|^2 |\lambda|^{2\alpha + 1} d\lambda.$$

3. The Q-Fourier-Dunkl transform $F_Q$ extends uniquely to an isometric isomorphism from $L^2_Q(\mathbb{R})$ onto $L^2(\mathbb{R})$.

The heat kernel $N(x,s)$, $x \in \mathbb{R}$, $s > 0$, associated with the Q-Fourier-Dunkl transform is given by

$$N(x,s) = m_{\alpha} \frac{e^{-\frac{x^2}{4s}}}{(2s)^{\alpha + \frac{1}{2}} Q(x)}.$$

Some basic properties of $N(x,s)$ are the following:

- $N(x,s) Q^2(x) = m_{\alpha} \int_{\mathbb{R}} e^{-sy^2} \Psi_{y}(x)|y|^{2\alpha + 1} dy$.
- $F_Q(N(.,s))(x) = e^{-sx^2}$.

we define the heat functions $W_l$, $l \in \mathbb{N}$ as

$$Q^2(x) W_l(x,s) = \int_{\mathbb{R}} y^l e^{-\frac{y^2}{4s}} \Psi_{y}(x)|y|^{2\alpha + 1} dy.$$
The intertwining operators associated with a Q-Fourier-Dunkl transform on the real line is given by

\[ X_Q(f)(x) = a_\alpha Q(x) \int_{-1}^{1} f(tx)(1 - t^2)^{\alpha - \frac{1}{2}} dt, \]

its dual is given by

\[ t^\alpha X_Q(f)(y) = a_\alpha \int_{\mathbb{R}} f(x) Q(x) \text{sgn}(x)(x^2 - y^2)^{\alpha - \frac{1}{2}} (x + y) dx \]

(1.12) \[ t^\alpha X_Q \] can be written as

\[ t^\alpha X_Q(f)(y) = a_\alpha \int_{\mathbb{R}} f(x) Q(x) d\nu_y(x), \]

where

\[ d\nu_y(x) = a_\alpha \chi_{\{|x| \geq |y|\}} \text{sgn}(x)(x^2 - y^2)^{\alpha - \frac{1}{2}} (x + y) dx \]

and \( \chi_{\{|x| \geq |y|\}} \) denote the characteristic function with support in the set \( \{x \in \mathbb{R} / |x| \geq |y|\} \).

**Proposition 1.1.** If \( f \in L^1_Q(\mathbb{R}) \) then \( t^\alpha X_Q(f) \in L^1(\mathbb{R}) \) and \( \| t^\alpha X_Q(f) \|_1 \leq \| f \|_{1,Q} \).

For every \( f \in L^1_Q(\mathbb{R}) \)
(1.13) \[ F_Q = \mathcal{F} \circ t^\alpha X_Q(f), \]

where \( \mathcal{F} \) is the usual Fourier transform defined by

\[ \mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x} dx. \]

2. **Cowling-Price’s Theorem for the Q-Fourier-Dunkl Transform**

**Theorem 2.1.** Let \( f \) be a measurable function on \( \mathbb{R} \) such that

\[ \int_{\mathbb{R}} e^{apx^2} Q^p(x)|f(x)|^p |x|^{2\alpha + 1} dx < \infty \]

and

\[ \int_{\mathbb{R}} e^{bq\xi^2} |\mathcal{F}_Q(f)(\xi)|^q (1 + |\xi|)^{m} d\xi < \infty, \]

for some constants \( a, b > 0, \ k > 0, \ m > 1 \) and \( 1 \leq p, r \leq +\infty \).

i) If \( ab > \frac{1}{4} \), then \( f = 0 \) almost everywhere.
ii) If \( ab = \frac{1}{2} \), then \( f(x) = P(x)N(x, b) \) where \( P \) is a polynomial with 
\[ \deg P \leq \min \{ \frac{k}{p} + \frac{2\alpha + 1}{p'}, \frac{m-1}{r} \} \] 
Especially, if 
\[ k \leq 2\alpha + 2 + p \min \{ \frac{k}{p} + \frac{2\alpha + 1}{p'}, \frac{m-1}{r} \} \] 
then \( f = 0 \) almost everywhere. Furthermore, if \( m \in ]1, 1 + \tau \] and \( k > 2\alpha + 2 \), then \( f \) is a constant multiple of \( N(\cdot, b) \).

iii) If \( ab < \frac{1}{2} \), then for all \( \delta \in ]b, \frac{1}{2}a[ \) all functions of the form \( f(x) = P(x)N(x, \delta) \) satisfy (2.1) and (2.2).

Proof. It follows from (2.1) that \( f \in L^q_Q \) and \( F_Q(f)(\xi) \) exists for all \( \xi \in \mathbb{R} \). Moreover, it has an entire holomorphic extension on \( \mathbb{C} \) satisfying for some \( s > 0 \),
\[ |F_Q(f)(\xi)| \leq Ce^{\frac{a\pi^2}{4}}(1 + |Im\xi|)^s. \]

By (1.1) we have for all \( z = \xi + i\eta \in \mathbb{C} \),
\[ (2.3) \quad |F_Q(f)(\xi)| \leq \int_\mathbb{R} |f(x)||\Lambda(\xi)(x)||x|^{2\alpha + 1} \, dx \leq e^{\frac{a\pi^2}{4}} \int_\mathbb{R} e^{a\pi^2} Q(x) |f(x)| (1 + |x|)^{\frac{1}{2}} e^{-a(x - \frac{\pi}{2})^2} |x|^{2\alpha + 1} \, dx. \]

By Hölder inequality we have
\[ |F_Q(f)(\xi + i\eta)| \leq e^{\frac{a\pi^2}{4}} \left( \int_\mathbb{R} e^{a\pi^2} Q(x)^{\frac{1}{p'}} |f(x)|^{\frac{1}{p'}} |x|^{2\alpha + 1} \, dx \right)^{\frac{1}{p'}} \left( \int_\mathbb{R} \frac{1}{(1 + |x|)^k} \frac{e^{-a(x - \frac{\pi}{2})^2}}{x} \, dx \right)^{\frac{1}{p'}} \]
according to (2.1) we get that
\[ |F_Q(f)(\xi + i\eta)| \leq Ce^{\frac{a\pi^2}{4}} \left( \int_\mathbb{R} (1 + |x|)^{\frac{1}{p'}} e^{-a(x - \frac{\pi}{2})^2} |x|^{2\alpha + 1} \, dx \right)^{\frac{1}{p'}} \leq Ce^{\frac{a\pi^2}{4}} \left( \int_0^\infty (1 + |x|)^{\frac{1}{p'}} + 2\alpha + 1 e^{-a(x - \frac{\pi}{2})^2} \, dx \right)^{\frac{1}{p'}} \leq Ce^{\frac{a\pi^2}{4}} (1 + |\eta|)^{\frac{1}{2} + \frac{2\alpha + 1}{p'}}. \]

If \( ab = \frac{1}{2} \), then
\[ |F_Q(f)(\xi + i\eta)| \leq Ce^{\frac{a\pi^2}{4}} (1 + |\eta|)^{\frac{1}{2} + \frac{2\alpha + 1}{p'}.} \]

We put \( g(z) = e^{b\pi z} F_Q(f)(z) \), then
\[ |g(z)| \leq Ce^{b(Re z)^2} (1 + |Im z|)^{\frac{s}{2} + \frac{2\alpha + 1}{p'}}. \]

It follows from (2.2) that
\[ \int_\mathbb{R} \frac{|g(z)|^r}{(1 + |\xi|)^m} \, d\xi < \infty. \]
Lemma 2.1. Let $h$ be an entire function on $\mathbb{C}$ such that

$$|h(z)| \leq Ce^{a|\text{Re}z|^2}(1 + |\text{Im}z|)^l$$

for some $l > 0$, $a > 0$ and

$$\int_{\mathbb{R}} \frac{|h(x)|^r}{(1 + |x|)^m} |P(x)| dx < \infty$$

for some $r \geq 1$, $m > 1$ and $P$ is a polynomial with degree $m$. Then $h$ is a polynomial with $\deg h \leq \min\{l, m - \frac{M - 1}{r}\}$ and if $m \leq r + M + 1$, then $h$ is a constant.

From this Lemma, $g$ is a polynomial, we say $P_b$ with $\deg P_b \leq \min\{kp + 2\alpha + 1, \frac{m - 1}{q}\}$.

Then $F_Q(f)(x) = P_b(x)e^{-bx^2}$ then,

$$f(x) = Q_b(x)N(x, b)$$

where $\deg P_b = \deg Q_b$. Therefore, nonzero $f$ satisfies (1.10) provided that

$$k > 2\alpha + 2 + p \min\left\{\frac{kp}{p} + \frac{2\alpha + 1}{q}, \frac{m - 1}{r}\right\}.$$  

If $m < r + 1$, by Lemma 1 we have $g$ as a constant and $F_Q(f)(x) = Ce^{-bx^2}$ and $f(x) = CN(x, b)$. If $m > 1$ and $k > 2\alpha + 2$, these functions satisfy (2.1) and (2.2), which proves (ii).

If $ab > \frac{1}{4}$, then we can find positive constants $a_1$ and $b_1$ such that $a > a_1 = \frac{1}{3b_1} > \frac{4}{3b}$. Then $f$ and $F_Q(f)$ also satisfy (2.2) with $a$ and $b$ replaced by $a_1$ and $b_1$ respectively. Then $F_Q(f)(x) = P_{b_1}(x)e^{-b_1x^2}$. $F_Q(f)$ cannot satisfy (2.2) unless $P_{b_1} = 0$, which implies that $f = 0$, this proves (i). If $ab < \frac{1}{4}$, then for all $\delta \in ]b, \frac{1}{4b}[,$ the functions of the form $f(x) = P(x)N(x, \delta),$ where $P$ is a polynomial on $\mathbb{R}$, satisfy (2.1) and (2.2). This proves (iii).  

3. Mathematical Formulas

4. Miyachi’s Theorem for the Q-Fourier-Dunkl Transform

Theorem 4.1. Let $f$ be a measurable function on $\mathbb{R}$ such that

$$(4.1) \quad e^{ax^2}f \in L^p_Q(\mathbb{R}) + L^r_Q(\mathbb{R})$$

and

$$(4.2) \quad \int_{\mathbb{R}} \log^+ \frac{|F_Q(f)(\xi)e^{b\xi^2}|}{\lambda} d\xi < \infty,$$  

for some constants $a, b, \lambda > 0$ and $1 \leq p, r \leq +\infty$.  

(i) if \( ab > \frac{1}{4} \) then \( f = 0 \) almost everywhere.
(ii) if \( ab = \frac{1}{4} \) then \( f = cN(.,b) \) with \( |c| \leq \lambda \).
(iii) if \( ab > \frac{1}{4} \) then for all \( \delta \in ]b, \frac{1}{4}[ \), all functions of the form \( f(x) = P(x)N(x, \delta) \), where \( P \) is a polynomial on \( \mathbb{R} \) satisfy (2.1) and (2.2).

To prove this result, we need the following lemmas.

**Lemma 4.1.** [5] Let \( h \) be an entire function on \( \mathbb{C} \) such that

\[
|h(z)| \leq Ae^{B|Rez|^2},
\]

and

(4.3) \[
\int_{\mathbb{R}} \log^+ |h(y)|dy < \infty,
\]

for some constants \( A \) and \( B \). Then \( h \) is a constant.

**Lemma 4.2.** Let \( r \in [1, +\infty[ , a > 0 \). Then for \( g \in L^r_Q(\mathbb{R}) \) there exist \( c > 0 \) such that

\[
\| e^{ax^2} tX_Q(e^{-ay^2}g) \|_r \leq c \| g \|_{r,Q}.
\]

**Proof.** From the hypothesis, it follows that \( e^{-ay^2} \) belongs to \( L^r_Q(\mathbb{R}) \). Then by Proposition 1.1, \( tX_Q(e^{-ay^2}g) \) is defined almost everywhere on \( \mathbb{R} \). Here we consider two cases:

i) If \( r \in [1, +\infty[ \) then

\[
\| e^{ax^2} tX_Q(e^{-ay^2}g) \|_r \leq \int_{\mathbb{R}} e^{arx^2} \left( \int_{\mathbb{R}} Q(y)e^{-ay^2}|g(y)|\nu_x(y)^r dy \right)^{\frac{1}{r'}} dx
\]

\[
\leq \int_{\mathbb{R}} e^{arx^2} \left( \int_{\mathbb{R}} |Q(y)g(y)|^r \nu_x(y)^r dy \right)^{\frac{1}{r'}} \left( \int_{\mathbb{R}} e^{-ar'y^2} \nu_x(y)^r dy \right)^{\frac{1}{r'}} dx
\]

where \( r' \) is the conjugate exponent for \( r \). Since

(4.4) \[
\int_{\mathbb{R}} e^{-ry^2} \nu_x(y) = Ce^{-rx^2},
\]

for \( r > 0 \) it follows from (4.4) that

\[
\| e^{ax^2} tX_Q(e^{-ay^2}g) \|_r \leq C \int_{\mathbb{R}} tX_Q(|g'|)(x)dx,
\]

\[
= C \int_{\mathbb{R}} |g'(x)||x|^{2\alpha+1}dx < \infty.
\]

ii) If \( r = \infty \) then it follows from (4.4) that

\[
\| e^{ax^2} tX_Q(e^{-ay^2}g) \|_r \leq e^{ax^2} tX_Q(e^{-ay^2})(x)||g||_{Q,\infty}
\]

\[
= C||g||_{Q,\infty}.
\]
Lemma 4.3. Let $r, p \in [1, +\infty]$ and let $f$ be a measurable function on $\mathbb{R}$ such that
\begin{equation}
 e^{ax^2}f \in L_Q^p(\mathbb{R}) + L_Q^r(\mathbb{R})
\end{equation}
for some $a > 0$. Then for all $z \in \mathbb{C}$, the integral
\[
 F_Q(f)(z) = \int_{\mathbb{R}} f(x)\Lambda_{z,Q}(-x)|x|^{2\alpha+1}dx
\]
is well defined. $F_Q(f)(z)$ is entire and there exists $C > 0$ such that for all $\xi, \eta \in \mathbb{R}$,
\begin{equation}
 |F_Q(f)(\xi + i\eta)| \leq Ce^{\frac{\eta^2}{4a}}.
\end{equation}
Proof. From (5) and Hölder’s inequality we have the first assertion. For (4.6) using (4.5) we have $f \in L_Q^1(\mathbb{R})$ and $tX_Q(f) \in L^1(\mathbb{R})$. For all $\xi, \eta \in \mathbb{R}$,
\[
 F_Q(f)(\xi + i\eta) = \int_{\mathbb{R}} tX_Q(f)(x)e^{-ix(\xi+i\eta)}dx
\]
\[
 |F_Q(f)(\xi + i\eta)| \leq e^\frac{\pi^2}{4} \int_{\mathbb{R}} e^{ax^2} \left| tX_Q(f)(x) \right| e^{-ax^2 + x\eta - \frac{\alpha^2}{4}}dx
\]
\[
 \leq e^\frac{\pi^2}{4} \int_{\mathbb{R}} e^{ax^2} \left| tX_Q(f)(x) \right| e^{-a(x-\frac{\eta}{2})^2}dx.
\]
From (4.5) we can deduce that there exists $u \in L_Q^p(\mathbb{R})$ and $v \in L_Q^r(\mathbb{R})$ such that
\[
 f(x) = e^{-ax^2}u(x) + e^{-ax^2}v(x),
\]
by Lemma 4 we have
\[
 \int_{\mathbb{R}} e^{ax^2} \left| tX_Q(f)(x) \right| e^{-a(x-\frac{\eta}{2})^2}dx \leq C \left( \|u\|_{p,Q} + \|v\|_{r,Q} \right) < \infty,
\]
which proves the Lemma.  

Proof of Theorem

- If $ab > \frac{1}{4}$. Let $h$ be a function on $\mathbb{C}$ defined by
\[
 h(z) = e^{\frac{\pi^2}{4}F_Q(f)(z)}.
\]
h is entire function on $\mathbb{C}$, it follows from (4.6) that
\begin{equation}
 \forall \xi \in \mathbb{R}, \forall \eta \in \mathbb{R} \ |h(\xi + i\eta)| \leq Ce^{\frac{\pi^2}{4}}.
\end{equation}
On the other hand, we have

\[
\int_{\mathbb{R}} \log^+ |h(y)| \, dy = \int_{\mathbb{R}} \log^+ \left| \frac{e^{by^2} \mathcal{F}_Q(f)(y)}{\lambda} \right| \, dy \\
\leq \int_{\mathbb{R}} \log^+ \left| \frac{e^{by^2} \mathcal{F}_Q(f)(y)}{\lambda} \right| \, dy + \int_{\mathbb{R}} \lambda e^{(\frac{a}{4} - b)y^2} \, dy
\]

because \( \log^+(cd) \leq \log^+(c) + d \) for all \( c, d > 0 \). Since \( ab > \frac{1}{4} \), (2.2) implies that

\[
\int_{\mathbb{R}} \log^+ |h(y)| \, dy < \infty.
\]

A combination of (4.7), (4.8) and Lemma 3 shows that \( h \) is a constant and

\[
\mathcal{F}_Q(f)(y) = Ce^{-\frac{y^2}{4}a}.
\]

Since \( ab > \frac{1}{4} \), (2.2) holds whenever \( C = 0 \) and the injectivity of \( \mathcal{F}_Q \) implies that \( f = 0 \) almost everywhere.

- If \( ab = \frac{1}{4} \). We deduce from previous case that \( \mathcal{F}_Q(f) = Ce^{-\frac{y^2}{4}a} \). Then (2.2) holds whenever \( |C| \leq \lambda \). Hence \( f = CN(a, b) \) with \( |C| \leq \lambda \).

- If \( ab < \frac{1}{4} \). If \( f \) is a given form, then \( \mathcal{F}_Q(f)(y) = Q(y)e^{-by^2} \) for some \( Q \).

In the continuation, we will give an analogue of Hardy’s theorem \([?]\) for the Q-Fourier-Dunkl transform.

**Theorem 4.2.** Hardy Let \( N \in \mathbb{N} \). Assume that \( f \in L^2_Q(\mathbb{R}) \) is such that

\[
|f(x)| \leq M e^{-\frac{x^2}{4a}} \text{ a.e. }, \forall y \in \mathbb{R}, |\mathcal{F}_Q(f)(y)| \leq M(1 + |y|)^N e^{-by^2},
\]

for some constants \( a > 0, b > 0 \) and \( M > 0 \). Then,

i) If \( ab > \frac{1}{4} \), then \( f = 0 \) a.e.

ii) If \( ab = \frac{1}{4} \), then the function \( f \) is of the form

\[
f(x) = \sum_{|\sigma| \leq N} a_\sigma W_\sigma \left( \frac{1}{4a}, x \right) \text{ a.e. }, a_\sigma \in \mathbb{C}.
\]

iii) If \( ab < \frac{1}{4} \), there are infinitely many nonzero functions of \( f \) satisfying the conditions (4.9).

**Proof.** The first condition of (4.9) implies that \( f \in L^1_Q(\mathbb{R}) \). So by Proposition 1.1, the function \( ^tX_Q(f) \) is defined almost everywhere. By using the relation (1.13), we deduce that for all \( x \in \mathbb{R} \),

\[
| ^tX_Q(f)(x)| \leq M_0 e^{-ax^2},
\]
where $M_0$ is a positive constant. So
\begin{equation}
|\mathcal{T}_Q(f)(x)| \leq M_0 (1 + |x|)^N e^{-ax^2},
\end{equation}

On the other hand from (1.13) and (4.9) we have for all $x \in \mathbb{R}$,
\begin{equation}
|\mathcal{F}(\mathcal{T}_Q(f))(y)| \leq M(1 + |y|)^N e^{-b|y|^2},
\end{equation}
The relations (4.10) and (4.11) show that the conditions of Proposition 3.4 of [2], p.36, are satisfied by the function $\mathcal{T}_Q(f)$. Thus we get:
i) If $ab > \frac{1}{4}$, $\mathcal{T}_Q(f) = 0$ a.e. Using (1.13) we deduce
\[
\forall y \in \mathbb{R}, \mathcal{F}_Q(f)(y) = \mathcal{F} \circ (\mathcal{T}_Q(f))(y) = 0.
\]
Then by the injectivity of $\mathcal{F}_Q$ we have $f = 0$ a.e.

ii) If $ab = \frac{1}{4}$, then $\mathcal{T}_Q(f)(x) = P(x)e^{-ax^2}$, where $P$ is a polynomial of degree lower than $N$. Using this relation and (1.13), we deduce that
\[
\forall x \in \mathbb{R}, \mathcal{F}_Q(f)(y) = \mathcal{F} \circ \mathcal{T}_Q(f)(y) = \mathcal{F}(P(x)e^{-ax^2})(y).
\]
but
\[
\forall x \in \mathbb{R}, \mathcal{F}(P(x)e^{-ax^2})(y) = S(y)e^{-\frac{a}{4}x^2},
\]
with $S$ a polynomial of degree lower than $N$.
Thus from (1.11), we obtain
\[
\forall x \in \mathbb{R}, \mathcal{F}_Q(f)(y) = \mathcal{F}_Q \left( \sum_{|s| < \frac{N-1}{2}} a_s W_s(\frac{1}{4a}, x) \right)(y).
\]
The injectivity of the transform $\mathcal{F}_Q$ implies
\[
f(x) = \sum_{|s| \leq N} a_s W_s(\frac{1}{4a}, x) \text{ a.e.}
\]

iii) If $ab < \frac{1}{4}$, let $t \in [a, \frac{1}{4}]$ and $f(x) = Ce^{-tx^2}$ for some real constant $C$, these functions satisfy the conditions (4.9).

In the next part, we will give an analogue of Morgan’s theorem [7] for the Q-Fourier-Dunkl transform.

**Theorem 4.3.** Morgan. Let $1 < p < 2$ and $r$ be the conjugate exponent of $p$. Assume that $f \in L^2_Q(\mathbb{R})$ satisfies
\begin{equation}
\int_{\mathbb{R}} e^{\frac{a}{2}|x|^p} |f(x)||x|^{2a+1}dx < +\infty, \text{ and } \int_{\mathbb{R}} e^{\frac{b}{2}|y|^r} |\mathcal{F}_Q(f)(y)|dy < +\infty,
\end{equation}
for some constants $a > 0$, $b > 0$.
Then if $ab > |\cos(\frac{2\pi}{n})|^\frac{2}{p}$, we have $f = 0$ a.e.
Proof. The first condition of (4.12) implies that $f \in L^1_Q(\mathbb{R})$. So by Proposition 1.1, the function $t^\alpha X_Q(f)$ is defined almost everywhere. By using the relation (4.12) and Proposition 1.1, we deduce that:

$$\int_{\mathbb{R}} |t^\alpha X_Q(f)(x)| e^{\frac{2\pi}{p}|x|^p} dx \leq \int_{\mathbb{R}} e^{\frac{2\pi}{p}|x|^p} |f(x)||x|^{2\alpha+1} dx < +\infty.$$  

So

$$\int_{\mathbb{R}} |t^\alpha X_Q(f)(x)| e^{\frac{2\pi}{p}|x|^p} dx < +\infty \tag{4.13}$$

On the other hand, from (1.13) and (4.12) we have:

$$\int_{\mathbb{R}} e^{\frac{2\pi}{p} |y|^p} |\mathcal{F}_Q(f)(y)| dy = \int_{\mathbb{R}} e^{\frac{2\pi}{p} |y|^p} |\mathcal{F}(t^\alpha X_Q(f))(y)| dy < +\infty. \tag{4.14}$$

The relations (4.13) and (4.14) are the conditions of Theorem 1.4, p.26 of [2], which are satisfied by the function $t^\alpha X_Q(f)$. Thus we deduce that if $ab > |\cos(\frac{\pi}{2})|^\frac{1}{2}$ we have $t^\alpha X_Q(f) = 0$ a.e. Using the same proof as in the end of Theorem 4, we have obtained $f(y) = 0$, a.e. $y \in \mathbb{R}$. \hfill \square

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