Copulas and time series with long-ranged dependences

Rémy Chicheportiche and Anirban Chakraborti

Chaire de finance quantitative, École Centrale Paris, 92295 Châtenay-Malabry, France

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We review ideas on temporal dependences and recurrences in discrete time series from several areas of natural and social sciences. We revisit existing studies and redefine the relevant observables in the language of copulas (joint laws of the ranks). We propose that copulas provide an appropriate mathematical framework to study non-linear time dependences and related concepts — like aftershocks, Omori law, recurrences, waiting times. We also critically argue using this global approach that previous phenomenological attempts involving only a long-ranged autocorrelation function lacked complexity in that they were essentially mono-scale.

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I. INTRODUCTION

A thorough understanding of the occurrences and statistics of extreme events is crucial in fields like seismology, finance, astronomy, physiology, etc. The analyses of extreme events play a pivotal role every time an addressed problem has a stochastic nature, since the rare extreme events can have rather strong or drastic consequences — making it widely useful. One theoretical motivation for studying extreme events in a particular field like finance, is to account for the observed fat tails of log-returns (deviation from the Normal distribution in the tails) of stock prices. A more practical motivation is that the extreme events such as “market crashes” or “shocks”, pose a substantial risk for investors, even though these events are rare and do not provide enough data for reliable statistical analyses. It has been observed that common financial shocks are relatively smaller in magnitudes of volatility, the duration, and the number of stocks affected. However, the extremely large and infrequent financial crashes, such as the Black Monday crash, have significant “aftershocks” that can last for many months. This observation is very similar to the “dynamic relaxation” of the aftershock cascade following an earthquake. Hence, it is meaningful to also ask the general scientific question: How is the dynamics of a “complex” system, such as an earthquake fault or a financial market, affected when the system undergoes an extreme event? The statistics of return intervals between extreme events is a powerful tool to characterize the temporal scaling properties of the observed time series and to estimate the risk for such hazardous events like earthquakes or financial crashes. Evaluating the return time statistics of extreme events in a stochastic process, is one of the classical problems in probability theory.

Earlier, from an analysis of the probability density functions (PDF) of waiting times for earthquakes, Bak et al. had suggested a unified scaling law combining the Gutenberg-Richter law, the Omori law, and the fractal distribution law in a single framework. This global approach was later extended by Corral, who proposed the existence of a universal scaling law for the PDF recurrence times between earthquakes in a given region. This is useful because, due to the scaling properties, it is possible to analyse the statistics of return intervals for different thresholds by studying only the behavior of small fluctuations occurring very frequently, which have much better statistics and reliability than those of the rare extreme large fluctuations. It also reveals a spatiotemporal organization of the seismicity, as suggested by Saichev and Sornette.

In this paper, we review the ideas on temporal dependences and recurrences in discrete time series from several areas of earthquakes, etc. (natural sciences) and financial markets (social sciences). We revisit the existing studies, cited above, and redefine the relevant observables in the mathematical language of “copulas”. We propose that copulas is a very general and appropriate framework to study non-linear time dependences and related concepts — like aftershocks, Omori law, recurrences, waiting times. Our overall aim is to study several properties of recurrence times and the statistic of other observables (waiting times, cluster sizes, records, aftershocks) described in terms of the diagonal copula. We hope that these studies can shed light on the n-points properties of the process. We also critically argue that that previous phenomenological attempts involving only a long-ranged autocorrelation function, lacked complexity in that they were essentially mono-scale.

The copula

As a tool to study the — possibly highly non-linear — correlations between random variables, “copulas”, i.e. joint distributions of the ranks (see formal definition below), have long been used in actuarial sciences and finance to describe and model cross-dependences of assets, often in a risk management perspective. Al-
though the widespread use of simple analytical copulas to model multivariate dependences is more and more criticized [23, 24], copulas remain useful as a tool to investigate empirical properties of multivariate data [24]. More recently, copulas have also been studied in the context of serial dependences in univariate time series, where they find yet another application range: just as Pearson’s \( p \) coefficient is commonly used to measure both linear cross-dependences and temporal correlations, copulas are well-designed to assess non-linear dependences both transversally or serially [25, 27] — we will speak of “self-copulas” in the latter case.

### Notations

We consider a time series \( \{X_t\}_{t=1..T} \) of length \( T \), as a realization of a discrete stochastic process. The joint cumulative distribution function (CDF) of \( n \) occurrences (\( 1 \leq t_1 < \ldots < t_n < T \)) of the process is

\[
F_{t_1,...,t_n}(x) = P[X_{t_1} < x_{t_1}, \ldots, X_{t_n} < x_{t_n}] .
\] (1)

We assume that the process is stationary with a distribution \( F \), and a translational-invariant joint distribution \( F \) with long-ranged dependences, as is typically the case e.g. for seismic and financial data.

A realization of \( X_t \) at date \( t \) will be called an “event” when its value exceeds a threshold \( X(\pm) \): “negative event” when \( X_t < X(\pm) \), and “positive event” when \( X_t > X(\pm) \). The probability \( p_- \) of such a ‘negative event’ is \( F(X(\pm)) \), and similarly, the probability that \( X_t \) is above a threshold \( X(+) \) is the tail probability \( p_+ = 1 - F(X(+)) \).

If a unique threshold \( X(+) = X(\pm) \) is chosen, then obviously \( p_+ = 1 - p_- \). This is appropriate when the distribution is one-sided, typically for positive only signals, and one wishes to distinguish between two regimes: extreme events (above the unique threshold), and regular events (below the threshold). This case is illustrated schematically in Fig. 1(b) When the distribution is two-sided, it is more convenient to define, \( X(+) \) as the \( q \)-th quantile of \( F \), and \( X(\pm) \) as the \((1-q)\)-th quantile, for a given \( q \in [\frac{1}{2}, 1] \), so that \( p_+ = p_- = 1 - q \). This allows to investigate persistence and reversion effects in signed extreme events, while excluding a neutral zone of regular events between \( X(-) \) and \( X(+) \), see Fig. 1(a)

When the threshold for the recurrence is defined in terms of quantiles like above (a relative threshold), stationarity is not needed theoretically but much wanted empirically as already said, otherwise the height of the threshold might change every time. In contrast, when the threshold is set as a number (an absolute threshold), there’s no issue on the empirical side, but the theoretical discussion makes sense only under stationary marginal.

The next section recalls several two-points and many-points properties of stationary processes, and discusses associated measures of dependence in light of the copula. This rather theoretical content is followed in Section III by applications to financial data. The definition

\[
\rho_\ell = \mathbb{E}[X_t X_{t+\ell}] - \mathbb{E}[X_t] \mathbb{E}[X_{t+\ell}]
\] (2)

is rooted in the analysis of Gaussian processes, as those are completely characterized by their covariances, and multi-linear correlations are reducible to all combinations of 2-points expectations, according to Isserli’s theorem. Some non-linear dependences, like the tail-dependence for example [21, 22], are however not expressed in terms of simple correlations, but involve the whole bivariate copula:

\[
C_\ell(u,v) = F_{t,t+\ell}(F^{-1}(u), F^{-1}(v)),
\] (3)

where \( (u,v) \in [0,1]^2 \). \( C_\ell \) can be understood as the distribution of the marginal ranks \( U = F(X_t), V = F(X_{t+\ell}) \), and contains the full information on bivariate dependence that is invariant under increasing transformations of the marginals. For example, the conditional probability

\[
p_{\ell(+)} = \mathbb{P}[X_{t+\ell} > X(+)| X_t > X(+)],
\] (4)

which is a measure of persistence of the “positive” events, can be written in terms of copulas, together with all three

\[
F(X(\pm)) = 1 - F(X(\pm)) = q
\]

FIG. 1. Two possible definitions of events: either \( p_- \) and \( p_+ \) are probabilities of extremes (negative and positive, respectively), or only \( p_+ \) is a probability of extreme and \( p_- = 1 - p_+ \), and some properties of copulas are recalled in appendix, and the Gaussian case with long-ranged correlations is treated.

II. DEPENDENCES IN DISCRETE-TIME PROCESSES

We consider the case where the discrete times \( t_n \) in the definition (1) are equidistant (“regularly sampled”).

A. Two-points dependence measures

Typical measures of dependences in stationary processes are two-points expectations that only involve one parameter: the lag \( \ell \) separating the points in time. For example, the usefulness of the linear correlation function

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where values of $p$ is power-law: earthquakes occurrences \[30\]
the average frequency of events occurring
other cases of conditioning
\begin{align}
\rho_{+}^{(\ell)} &= \frac{2p_+ - 1 + C_\ell (1-p_+, 1-p_+)}{p_+}, \\
\rho_{-}^{(1)} &= C_\ell (p_+, p_+ - 1)/p_+, \\
\rho_{+}^{(2)} &= [p_+ - C_\ell (1-p_+, 1-p_+)]/p_+, \\
\rho_{-}^{(1)} &= [p_- - C_\ell (1-p_-, 1-p_-)]/p_-,
\end{align}
where $\rho_{\pm}^{(\ell)}$ and $\rho_{\pm}^{(1)}$ are defined similarly to Eq. (5) with obvious inequality sign choices. When $X^{(+)} = X^{(-)} = 0$ and $\ell = 1$, this is exactly the definition of Boguná and Masoliver [28], with accordingly $p_+ = p_- = F(0)$, see Fig. 1. Note also that $\rho_{\pm}^{(\ell)}$ and $\rho_{\pm}^{(1)}$ are straightforwardly related to the so-called ‘tail dependence coefficients’ [29].

As an example, consider the Gaussian bivariate copula of the pair $(X_1, X_{t+\ell})$, whose whole $\ell$-dependence is in the linear correlation coefficient $\rho(\ell)$. Fig. 2(a) illustrates the conditional probabilities [5] as a function of the threshold, when $p_+ = p_- = 1 - q$. A similar plot for the Student copula (with $\nu = 5$ degrees of freedom) is shown in Fig. 2(b). The fatter tails of the joint distribution are responsible for the abnormal behavior of the conditional probabilities in the region $q = 1$. When $q = 0.5$, the coefficients [3] are all equal to
\[\frac{1}{2} + \frac{1}{\pi} \arcsin \rho(\ell)\]
for any elliptical copula [29].

\section*{Aftershocks}

Omori’s law characterizes the $\ell$ dependence of $\rho_{++}^{(\ell)}$, i.e. the average frequency of events occurring $\ell$ time steps after a main event. It was first stated in the context of earthquakes occurrences \[30\], where this time dependence is power-law:
\[\rho_{++}^{(\ell)} = \lambda \cdot \ell^{-\alpha}.
\]
Notice that any dependence on the threshold must be hidden in $\lambda$ according to this description. The average cumulated number $N_\ell$ of these aftershocks until $\ell$ is thus
\[\langle N_\ell \rangle_+ = \lambda \cdot \ell^{1-\alpha}/1-\alpha,
\]
with in fact $\lambda \equiv p_+$ since, when $\alpha \to 0$, $N_\ell$ has no time-dependence, i.e. it counts independent events (white noise), and $\rho_{++}^{(\ell)}$ must thus tend to the unconditional probability.

In order to give a phenomenological grounding to this empirical law also later observed in finance \[15\] [31], Lillo and Mantegna [32] model the aftershock volatilities in financial markets as a decaying scale $\sigma(\ell)$ times an independent stochastic amplitude $r_\ell$ with CDF $\phi$. As a consequence, $p_{++}^{(\ell)} \sim 1 - \phi(X^{(+)} / \sigma(\ell))$ and the power-law behavior of Omori’s law results from (i) power-law marginal $\phi(r) \sim r^{-\gamma}$, and (ii) scale decaying as power-law $\sigma(\ell) \sim \ell^{-\beta}$, so that relation (6) is recovered with $\alpha = 3\gamma$. The non-stationarity described by $\sigma$ is only introduced in a conditional sense, and might be appropriate for aging systems or financial markets, but we believe that Omori’s law can be accounted for in a stationary setting and without necessarily having power-law distributed amplitudes.

The scaling of $p_{++}^{(\ell)}$ with the magnitude of the main shock is encoded in the prefactor $\lambda \equiv p_+$, which, for example, accounts for the exponentially distributed amplitudes of earthquakes (Gutenberg-Richter law \[33\]). The linear dependence of $p_{+}^{(\ell)}$ on $p_+$ shall be reflected in the diagonal of the underlying copula:
\[C_\ell (p, p) = p^2 \ell^{-\alpha},
\]
a prediction that can be tested empirically.

Note that Omori’s law is a measure involving only the two-points probability. In the next subsection, we show what additional information many-points probability can reflect.

\section*{B. Multi-points dependence measures}

Although the $n$-points expectations of Gaussian processes reduce to all combinations of 2-points expectations \[2\], their full dependence structure is not reducible to the bivariate distribution, unless the process is also Markovian (i.e. only in the particular case of exponential correlation). Furthermore, when the process is not Gaussian, even the multi-linear correlations are irreducible. In the general case, the whole multivariate CDF is needed, but many measures of dependence that we introduce below only involve the diagonal $n$-points copula [34]:
\[C_n (p) = F_{t+1|n} (F^{-1}(p), \ldots, F^{-1}(p)),
\]
which measures the joint probability that all $n \geq 1$ consecutive variables $X_{t+1}, \ldots, X_{t+n}$ are below the upper $p$-th quantile of the stationary distribution $p \in [0, 1]$. turn, accounts for the exponentially distributed amplitudes of earthquakes (Gutenberg-Richter law \[33\]). The linear dependence of $p_{+}^{(\ell)}$ on $p_+$ shall be reflected in the diagonal of the underlying copula:
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\[C_\ell (p, p) = p^2 \ell^{-\alpha},
\]
and \( t + [1, n] \) is a shorthand for \( \{t+1, \ldots, t+n\} \). Clearly, \( C_1(p) = p \) and we set by convention \( C_0(p) \equiv 1 \).

Empirically, the \( n \)-points probabilities are very hard to measure due to the large noise associated with such rare joint occurrences. However, there exist observables that embed many-points properties and are more easily measured, such as the length of sequences (clusters) of thresholded events, and the recurrence times of such events, that we study next.

**Recurrence intervals**

The probability \( \pi(\tau) \) of observing a recurrence interval \( \tau \) between two events is the conditional probability of observing a sequence of \( \tau - 1 \) “non-events” bordered by two events:

\[
\pi(\tau) = P[X_{t+\tau}^{(+)}|X_0 > X^{(+)}],
\]

where

\[
X_{t+\tau}^{(+)} = \{X_{t+1, t+\tau} < X^{(+)}, X_{t+\tau} > X^{(+)}\}
\]

designates a sequence of ‘non-events’ starting in \( t \) and terminated by a ‘positive event’ at \( t + \tau \). (We focus on positive events, but the recurrence of negative events can be studied with the substitution \( X \rightarrow -X \), and the case of recurrence in amplitudes with the substitution \( X \rightarrow |X| \).) After a simple algebraic transformation flipping all ‘>’ signs to ‘<’, it can be written in the language of copulas as:

\[
\pi(\tau) = \frac{C_{\tau-1}(1-p_+) - 2C_{\tau}(1-p_+) + C_{\tau+1}(1-p_+)}{p_+}. \tag{12}
\]

The cumulative distribution

\[
\Pi(\tau) = \sum_{n=1}^{\tau} \pi(n) = 1 - \frac{C_{\tau}(1-p_+)}{p_+} + C_{\tau+1}(1-p_+) \tag{13}
\]

is more appropriate for empirical purposes, being less sensitive to noise. These exact expressions make clear — almost straight from the definition — that (i) the distribution of recurrence times depends only on the copula of the underlying process and not on the stationary law, in particular its domain or its tails (this is because we take a relative definition of the threshold as a quantile); (ii) non-linear dependences are highly relevant in the statistics of recurrences, so that linear correlations can in the general case by no means explain alone the properties of \( \pi(\tau) \); and (iii) recurrence intervals have a long memory revealed by the \((\tau+1)\)-points copula being involved, so that only when the underlying process \( X_t \) is Markovian will the recurrences themselves be memoryless. Hence, when the copula is known (Eq. (A1) in appendix for Gaussian processes), the distribution of recurrence times is characterized by the exact expression in Eq. (12).

The average recurrence time is found straightforwardly by summing the series

\[
\mu_\pi = \langle \tau \rangle = \sum_{\tau=1}^{\infty} \tau \pi(\tau) = \frac{1}{p_+}, \tag{14}
\]

and is universal whatever the dependence structure. This result was first stated and proven by Kac in a similar fashion. It is intuitive as, for a given threshold, the whole time series is the succession of a fixed number \( p_+T \) of recurrences whose lengths \( \tau_i \) necessarily add up to the total size \( T \), so that \( \langle \tau \rangle = \sum \tau_i/(p_+T) = 1/p_+ \). Note that Eq. (14) assumes an infinite range for the possible lags \( \tau \), which is achieved either by having an infinitely long time series, or more practically when the translational-invariant copula is periodic at the boundaries of the time series, as is typically the case for artificial data which are simulated using numerical Fourier Transform methods. Introducing the copula allows to emphasize the validity of the statement even in the presence of non-linear long-term dependences, as Eq. (14) means that the average recurrence interval is copula-independent.

More generally, the \( m \)-th moment can be computed as well by summing \( \tau^m \pi(\tau) \) over \( \tau \):

\[
\langle \tau^m \rangle = \frac{1 + \sum_{\tau=1}^{\infty} [\tau+1] m - 2\tau^m + |\tau-1| m]}{p_+}. \tag{15}
\]

In particular, the variance of the distribution is

\[
\sigma^2_\pi = \langle \tau^2 \rangle - \mu^2_\pi = \frac{2}{p_+} \sum_{\tau=1}^{\infty} C_\tau(1-p_+) - \frac{1-p_+}{p_+^2}, \tag{15}
\]

It is not universal, in contrast with the mean, and can be related to the average unconditional waiting time, see below. Notice that in the independent case the variance \( \sigma^2_\pi = (1-p_+)/p_+^2 \) is not equal to the mean \( \mu_\pi = 1/p_+ \), as would be the case for a continuous-time Poisson process, because of discreteness effects.

It is important to notice that the main ingredient in the distribution of recurrence times (13) is the copula (i.e. the serial dependence structure) rather than the stationary distribution \( F \), a finding already noted by Olla, but which the current description highlights. The sensitivity to the extreme statistics of the process is in fact hidden in \( p_+ \), but what matters more is the (possibly multi-scale) dependence structure \( C_\tau \).

**Conditional recurrence intervals, clustering**

The dynamics of recurrence times is as important as their statistical properties, and in fact impacts the empirical determination of the latter. It is now clear, both from empirical evidences and analytically from the discussion on Eq. (12), that recurrence intervals have a long memory. In dynamic terms, this means that their
occurrences show some clustering. The natural question is then: “Conditionally on an observed recurrence time, what is the probability distribution of the next one?” This probability of observing an interval $\tau'$ immediately following an observed recurrence time $\tau$ is

$$\mathbb{P}[X^{(+)\tau'}|X^{(+)\tau}, X_0 > X^{(+)\tau}].$$

(16)

Again, flipping the ‘>' to '<' allows to decompose it as

$$\frac{C_{\tau-1;\tau'} - C_{\tau;\tau' - 1} - C_{\tau-1;\tau'} + C_{\tau;\tau'}}{C_{\tau-1} - 2C_{\tau} + C_{\tau+1}} = \frac{\pi(\tau + \tau')}{\pi(\tau)},$$

where the $(\tau + \tau')$-points probability

$$C_{\tau;\tau'}(p) = \mathcal{F}_{[0;\tau+\tau]\setminus\{\tau\}}(F^{-1}(p), \ldots, F^{-1}(p))$$

shows up. Of course, this exact expression has no practical use, again because there is no hope of empirically measuring any many-points probabilities of extreme events with a meaningful signal-to-noise ratio. We rather want to stress that non-linear correlations and multi-points dependences are relevant, and that a characterization of clustering based on the autocorrelation of recurrence intervals is an oversimplified view of reality.

**Waiting times**

The conditional mean residual time to next event, when sitting $\tau$ time steps after a (positive) event, is

$$\langle w|\tau \rangle = \sum_{w=1}^{\infty} w \pi(\tau+w) = \frac{1}{p_+}C_\tau(1-p_+).$$

(17)

One is often more concerned with unconditional waiting times, which is equivalent to asking what the size $w$ of a sequence of ‘non-events’ starting now will be, regardless of what happened previously. The distribution $\phi(w) = \mathbb{P}[X^{(+)w+1}_0]$ of these waiting times is equal to

$$\phi(w) = C_w(1-p_+) - C_{w+1}(1-p_+),$$

(18)

and its expected value is

$$\mu_\phi = \langle w \rangle = \sum_{w=1}^{\infty} C_w(1-p_+),$$

(19)

consistently to what would be obtained by averaging $\langle w|\tau \rangle$ over all possible elapsed times in Eq. (17). From Eq. (15), we have the following relationship between the variance of the distribution $\pi$ of recurrence intervals, and the mean waiting time:

$$\sigma_\pi^2 = \mu_\pi \left[2\mu_\phi + 1\right] - \mu_\pi^2$$

(20)

### III. FINANCIAL SELF-COPULAS

We illustrate some of the quantities introduced above on series of daily index returns. The properties of the time series used are summarized in Tab. I.

| $\pi_\Phi(\tau)$ | $(\tau)$ | $\phi_1(w)$ | $(w)$ | $\psi_0(n)$ | $(n)^2$ | $R(t)$ |
|------------------|----------|-------------|------|------------|-------|-------|
| $(1-q)q^{\tau-1}$ | $\frac{1}{1-q}$ | $(1-q)q^n$ | $\frac{n}{1-q}$ | $q(1-q)^{n-1}$ | $\frac{1}{q}$ | $\frac{1}{t}$ |

**TABLE I.** Different probabilities introduced, with thresholds defined as $F(X^{(+)\tau}) = q = 1 - F(X^{(+)\tau})$, for the White Noise process.

**Sequences lengths**

The serial dependence in the process is also revealed by the distribution of sequences sizes. The probability that a sequence of consecutive negative events [39], starting just after a ‘non-event’, will have a size $n$ is

$$\psi(n) = \frac{C_n(p_-) - 2C_{n+1}(p_-) + C_{n+2}(p_-)}{p_-(1-p_-)},$$

(21)

and the average length of a random sequence

$$\mu_\psi = \langle n \rangle = \sum_{n=1}^{\infty} n \psi(n) = \frac{1}{1-p_-}$$

(22)

is universal, just like the mean recurrence time. This property rules out the analysis of Boguná and Masoliver [28] who claim to be able to distinguish the dependence in processes according to the average sequence size.

**Record statistic**

We conclude this theoretical section on multi-points non-linear dependences by mentioning that the diagonal $n$-points copula $C_n$ can be alternatively understood as the distribution of the maximum of $n$ realizations of $X$ in a row, since

$$\mathbb{P}\left[\max_{\tau \leq n} X_\tau < F^{-1}(p)\right] = \mathbb{P}\left[X_{[1,n]} < F^{-1}(p)\right]$$

is equal to $C_n(p)$. Thus, studying the statistics of such “local” maxima in short sequences (see e.g. [40]) can provide information on the multi-points properties of the underlying process. The CDF of the running maximum, or record, is $C_t(F(x))$ and the probability that $t > 1$ will be a record-breaking time is the joint probability

$$R(t) = \mathbb{P}\left[\max_{\tau < t} X_\tau < X_t\right],$$

which is irrepresentative of the marginal law!
A. Conditional probabilities and 2-points dependences

We reproduce the study of Boguná and Masoliver [28] on the statistic of price changes conditionally on previous return sign, and extend the analysis to any threshold $|X|^\pm \geq 0$ and to remote lags. In addition to the time series of the five stock indices presented in Tab. II, we look at electroencephalogram (EEG) data from [41]. We first illustrate on Fig. 3(f) the conditional probabilities $p_{\pm \pm}^{(\ell)}$ (filled symbols) and $p_{\pm \mp}^{(\ell)}$ (empty symbols) with varying threshold $q = F(X(+)) = 1 - F(X(-))$, for $\ell = 1$. To study the departure from the independent case, it is more convenient to subtract the White Noise contribution, to get the corresponding excess probabilities.

First, the EEG data, Fig. 3(f), exhibit a very strong and symmetric persistence: reversion on the other side is shut down for extreme events (like for WN), and is more suppressed than WN for intermediate values. As of the plots relative to financial indices, several features can be immediately observed: positive events (upward triangles) trigger more subsequent positive (filled) than negative (empty) events; negative events (downward triangles) trigger more subsequent negative (filled) than positive (empty) events, except in the far tails $q \gtrsim 0.9$ where reversion is stronger than persistence after a negative event. Both these effects dominate the WN benchmark, but the latter effect is however much stronger. This overall behavior is similar for the time series of returns of all the stock indices studied. The shapes of $p_{\pm \pm}$ and $p_{\pm \mp}$ versus $q$ are not compatible with the Student copula benchmarks (correlation $\rho = 0.05$ and d.o.f. $\nu = 5$) shown in dashed and dotted lines, respectively. Notice that, due to its non-trivial tail-correlations, see Ref. [44], the Student copula does generate increased persistence with respect to WN, lower reversion in the core and higher reversion in the tails. But empirically the reversion is asymmetric and typically stronger when conditioning on negative events rather than on positive events, a property reminiscent of the leverage effect which cannot be accounted for by a pure (symmetric) Student copula. Some of the indices exhibit more pronounced reversion and persistence effects. Interestingly, the CAC-40 returns have a regime $0.5 \leq q \lesssim 0.9$ close to a white noise (with, in particular, a value of $p_{\pm \pm}^{(1)} = p_{\pm \mp}^{(1)}$ very close to 0 at $q = 0.5$, revealing an inefficient conditioning, i.e. as many positive and negative returns immediately follow-

ing positive or negative returns), but the extreme positive events $q \gtrsim 0.9$ show a very strong persistence, and the extreme negative events a very strong reversion.

Chicheportiche and Bouchaud [12] study in detail the $p$- and $\ell$- dependence of $[C_t(p,p) - p^2]$ and $[C_t(1-p,1-p) - p(1-p)]$ — which are straightforwardly related to $p_{\pm \pm}$ and $p_{\pm \mp}$, respectively — and find that the self-copula of stock returns can be modeled with a high accuracy by a log-normal volatility with log-decaying correlation, in agreement with multifractal volatility models. We give an overview of the results in Fig. 4 for the aggregated copula of all stocks in the S&P500 in 2000-2004. It is possible to show precisely how every kind of dependence present in the underlying process (discussed in [43]) reflects itself in $p_{\pm \pm}^{(\ell)}$ for different $q$’s: short ranged linear anti-correlation accounts for the central part ($p \approx 0.5$) departing from the WN prediction, long-ranged amplitude clustering is responsible for the “M” and “W” shapes that reveal excess persistence and suppressed reversion, and the leverage effect can be observed in the asymmetric heights of the “M” and “W”.

B. Recurrence intervals and many-points dependences

Even the simple, two-points measures of self dependence studied up to now show that non-linearities and multi-scaling are two ingredients that must be taken into account when attempting to describe financial time series; we now examine their many-points properties. As an example, we compute the distribution of recurrence times of returns above a threshold $X^{(+)} = F^{-1}(1-p_+)$.

Fig. 5 shows the tail cumulative distribution $1 - \Pi(\tau)$ of the recurrence intervals of DAX returns, at several thresholds $p_+ = 1/(\langle \tau \rangle) = 1/(\langle \tau \rangle)$ — the distribution for other indices is very similar. In the log-log representation used, an exponential distribution (corresponding to independent returns) would be concave and rapidly decreasing, while a power-law would decay linearly. The empirical distributions fit neither of those, and Ladetscher et al. [44] suggested a parametric fit of the form

$$1 - \Pi(\tau) = [1 + b(a-1)\tau]^{(2-a)/(a-1)}. \quad (23)$$

However, important deviations are present in the tail regions for thresholds at $X^{(+)} \gtrsim F^{-1}(0.9)$, i.e. $\langle \tau \rangle \gtrsim 1/(1-0.9) = 10$: as a consequence, there is no hope that the curves for different threshold collapse onto a single curve after a proper rescaling [45], as is the case e.g. for seismic data. A more fundamental determination of the form of $\Pi(\tau)$ should rely on Eq. (13) and a characterization of the $\tau$-points copula.

Similarly to the statistic of the recurrence intervals, their dynamics must be studied carefully. We have shown that the conditional distribution of recurrence intervals after a previous recurrence is very complex and involves

| Stock Index | Country | From | To | $T$ |
|-------------|---------|------|----|-----|
| S&P-500     | USA     | Jan. 02, 1970 | Dec. 23, 2011 | 10615 |
| KOSPI-200   | S. Korea| Jan. 03, 1990 | Dec. 26, 2011 | 5843  |
| CAC-40      | France  | Jul. 09, 1987 | Dec. 23, 2011 | 6182  |
| DAX-30      | Germany | Jan. 02, 1970 | Dec. 23, 2011 | 10564 |
| SMI-20      | Switzerland | Jan. 07, 1988 | Dec. 23, 2011 | 5902  |

TABLE II. Description of the dataset used: time series of returns of daily closing prices of international stock indices.
long-ranged non-linear dependences, so that a simple assessment of recurrence times auto-correlation may not be informative enough for a deep understanding of the mechanisms at stake.

**IV. DISCUSSION**

**A. Conditional Expected Shortfall**

In addition to caring for frequencies of conditional events, one can try to characterize their magnitudes. This of course does no longer fit in the framework of copulas (that “count” joint events) but can instead be quantified by a multivariate generalization of the Expected Shortfall (or Tail Conditional Expectation). For a single random variable with cdf $F$, the Expected Shortfall is the average loss when conditioning on large events:

$$\text{ES}(p_-) = \mathbb{E}[X_t | X_t < X^{(-)}] = \frac{1}{p_-} \int_{-\infty}^{F^{-1}(p_-)} x \, dF(x) = \frac{1}{p_-} \int_{0}^{p_-} F^{-1}(p) \, dp$$

In the same spirit, for bivariate distributions, the mean return conditionally on preceding return ‘sign’ is defined:

$$\langle X \rangle^{(-)}_\ell = \mathbb{E}[X_t | X_{t-\ell} < X^{(-)}] \quad (24a)$$
$$\langle X \rangle^{(+)}_\ell = \mathbb{E}[X_t | X_{t-\ell} > X^{(+)}]. \quad (24b)$$

As an example, consider the Gaussian bivariate pair $(X_t, X_{t+\ell})$, whose whole $\ell$-dependence is in the linear correlation coefficient $\rho(\ell)$. Fig. [3] shows the conditional Expected Shortfall that can be computed exactly from Eqs. (24), and is proportional to the inverse Mill ratio:

$$\langle X \rangle_\pm = \pm \rho(\ell) \frac{\Phi'(X^{(\pm)})}{p_\pm},$$

where $\Phi$ denotes the CDF of the univariate standard normal distribution.

This Gaussian prediction is to be compared with an empirical assessment of the same quantity for series of returns of stock indices. Fig. [7] displays the behavior of $\langle X \rangle_\pm$ versus $q$ (we also show the median $\text{med}(X)_\pm$ at lags corresponding to one day ($\ell = 1$), one week ($\ell = 5$) and one month ($\ell = 20$). The conditional amplitudes $\langle X \rangle_\pm$ measure “how large” a realization will be on average after an event at a given threshold, whereas the conditional probabilities $p_{\pm \ell}$ and $p_\ell$ quantify “how often” repeated such events occur. Mind the unconditional mean and median values, both above zero and distinct from each other. At $\ell = 1$, the reversion of extreme events is revealed again by the change of monotonicity from $q \approx 0.8$ on, and more strongly for $q > 0.9$ where
\( \langle X \rangle \) \ has an opposite sign than the preceding return; this corroborates the observation made on conditional probabilities above. Beyond the next day, the general picture is that dependences tend to vanish and the empirical measurements get more concentrated around the WN prediction. However, tail effects are strongly present, with unexpectedly a typical behavior opposite to that of \( \ell = 1 \): weekly, monthly reversion of extreme positive jumps. See the caption for a detailed discussion of the specificities of each stock index at every lag \( \ell \).

**B. Conclusion**

We report several properties of recurrence times and the statistic of other observables (waiting times, cluster sizes, records, aftershocks) in light of their description in terms of the diagonal copula, and hope that these studies can shed light on the \( n \)-points properties of the process by assessing the statistics of simple variables rather than positing an \textit{a priori} dependence structure.

The exact universality of the mean recurrence interval imposes a natural scale in the system. A scaling relation in the distribution of such recurrences is only possible in absence of any other characteristic time. When such additional characteristic times are present (typically in the non-linear correlations), no such scaling is expected, in contrast with time series of earthquake magnitudes.

We also stress that recurrences are intrinsically multi-points objects related to the non-linear dependences in the underlying time-series. As such, their autocorrelation is not a reliable measure of their dynamics, for their conditional occurrence probability is much history dependent.

Ultimately, recurrences may be used to characterize risk in a new fashion. Instead of — or in addition to — caring for the amplitude and probability of adverse events at a given horizon, one should be able to characterize the risk in a dynamical point of view. In this sense, an asset \( A_1 \) could be said to be “more risky” than another asset \( A_2 \) if its distribution of recurrence of adverse events has such and such “bad” properties that \( A_1 \) does not share. This amounts to characterizing the disutility by “When?” shocks are expected to happen, in addition to the usual “How often?” and “How large?”.

It would be interesting to study many-points dependences in continuous-time processes, where the role of the
A Hawkes process. A typical financial application could correspond to an underlying continuous process crossing a threshold, or more generally, the occurrence of an event during a random time interval. The value at $q = 0.5$ is $\sqrt{2/\pi} \rho(t)$.

$\rho = 0$ $\rho = 0.05$ $\rho = 0.1$ $\rho = 0.2$ $\rho = 0.3$

$E[X|X_0 < F^{-1}(1-q)]$, $E[X|X_0 > F^{-1}(q)]$

$\ell - 0.5 0.0 0.5$

random variables $X_0, X_1, \ldots, X_n$ are independent standard normal, and each decision-maker $i$ is characterized by a parameter $\rho_i$, which can be interpreted as a correlation coefficient. This is expressed mathematically by Eq. (3) for bivariate copula $C_n(p_1, \ldots, p_n) = \Phi_{\rho_1}(p_1), \ldots, \Phi_{\rho_n}(p_n)$, where $\Phi_{\rho_i}$ is the univariate inverse CDF, and $\Phi_{\rho}$ denotes the multivariate CDF with $n$ variables.

FIG. 6. Conditional Expected Shortfall of a Gaussian pair $(X_0, X_t)$ for different values of $\rho(t)$. The value at $q = 0.5$ is $\sqrt{2/\pi} \rho(t)$.

n-points copula is played by a counting process. The events to be counted can either be triggered by an underlying continuous process crossing a threshold, or more directly be modeled as a self-exciting point process, like a Hawkes process. A typical financial application could be found in transaction times in a Limit Order Book.

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Appendix A: Simple copulas and Sklar’s theorem

Sklar’s theorem [46] states that any multivariate distribution $F_{[1],n}(x_1, \ldots, x_n)$ can be written in terms of univariate marginal distribution functions $F_i(x_i)$ ($i = 1, \ldots, n$) and a ‘copula’ function $C(u_1, \ldots, u_n)$ on $[0,1]^n$ which, by definition, characterizes the dependence structure between the variables. In practice, constructing the copula is achieved letting $u_i = F_i(x_i)$ for every variable $i$. This is expressed mathematically by Eq. (3) for bivariate distributions, and can be generalized straightforwardly (see Eq. (3) for the diagonal of the $n$-points copula).

As an example, the Gaussian diagonal copula is

$$C_n(p) = \Phi_{\rho} \left( \Phi^{-1}(p_1), \ldots, \Phi^{-1}(p_n) \right) \qquad (A1)$$

where $\Phi^{-1}$ is the univariate inverse CDF, and $\Phi_{\rho}$ denotes the multivariate CDF with $(n \times n)$ covariance matrix $\rho$, which is Toeplitz with symmetric entries

$$\rho_{tt'} = \rho(|t - t'|), \quad t, t' = 1, \ldots, n. \quad (A2)$$

The White Noise (WN) product copula $C_n(p) = p^n$ is recovered in the limit of vanishing correlations $\rho(t) = \delta_0$, and other examples include the exponentially correlated Markovian Gaussian noise, the logarithmically correlated multi-fractal Gaussian noise, and the power-law correlated (thus scale-free) fractional Gaussian noise.

Fig. 8 displays $C_n(p_+ = 0.7)$ versus $n$ for different Hurst indices $H = 0.5, 0.7, 0.9$. The asymptotic behaviour at large $n$ cannot be displayed here because of numerical restrictions, but the small $n$ properties are more relevant for characterizing short-time conditional dynamics.

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