Toward a Gravitation Theory in Berwald–Finsler Space

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Abstract

Finsler geometry is a natural and fundamental generalization of Riemann geometry. The Finsler structure depends on both coordinates and velocities. It is defined as a function on tangent bundle of a manifold. We use the Bianchi identities satisfied by Chern curvature to set up a gravitation theory in Berwald-Finsler space. The geometric part of the gravitational field equation is nonsymmetric in general. This indicates that the local Lorentz invariance is violated. Nontrivial solutions of the gravitational field equation are presented.

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1 Introduction

The possible violation of Lorentz invariance have been proposed within several models of quantum gravity (QG) as well as the Very Special Relativity (VSR) [1]. A succinct list of QG includes: tensor VEVs originated from string field theory [2], cosmologically varying moduli scenarios [3], spacetime foam models [4], semiclassical spin–network calculations in Loop QG [5, 6], noncommutative geometry gravity [7, 8, 9, 10] and brane–world scenarios [11]. A common feature of these phenomenological studies on Planck scale physics is introducing of modified dispersion relations (MDR) for elementary particles. Girelli et al. [12] proposed a possible relation between MDR and Finsler geometry. Gibbons et al. [13] pointed out that VSR is Finsler geometry. In the VSR, CPT symmetry is preserved. VSR has radical consequences for neutrino mass mechanism. Lepton-number conserving neutrino masses are VSR invariant. The mere observation of ultra-high energy cosmic rays and analysis of neutrino data give an upper bound of $10^{-25}$ on the Lorentz violation [14].

The above facts imply that new physics may connected with Finsler geometry. In facts, in 1941 Randers [15] published his work on possible application of Finsler geometry in physics. Properties of Randers space have been investigated exhaustively by both mathematicians and physicists [16]–[20].

In a recent paper [21], Kostelecky studied the effect of gravitation in the Lorentz- and CPT-violating Standard Model Extension (SME). The incorporation of Lorentz and CPT violation into general relativity based on Riemann-Cartan geometry was discussed. It provided dominant terms in the effective low-energy action for the gravitational sector, thereby completing the formulation of the leading-order terms in the SME with gravity. It shows that a generalized geometric framework is helpful in constructing a unification theory of gravity and electromagnetism, weak and strong interaction.

Finsler geometry is a natural and fundamental generalization of Riemann geometry. The Finsler structure depends on both coordinates and velocities. It is defined as a mapping function from tangent bundle of a manifold to $\mathbb{R}$. S. S. Chern [22] proved that there is a unique connection in the Finsler manifold that is torsion free and almost $g$-compatibility. We use the Bianchi identities satisfied by Chern curvature to set up a gravitation theory in Berwald-Finsler space. The geometric part of the gravitational field equation is nonsymmetric in general. This indicates that the local Lorentz invariance is violated. Nontrivial solutions of the gravitational field equation are presented.

This paper is organized as follows. In Sec. 2, we briefly review basic concept and notations of Finsler geometry [23]. The torsion free Chern connection and corresponding curvature are introduced. The first and second Bianchi identities for curvature are presented. Sec. 3 is devoted to construct a gravitation theory in Berwald-Finsler space. Solutions of gravitational field equation are shown in Sec. 4. In the final, we give conclusion and remarks.
2 Finsler geometry

2.1 Finsler Manifold

Denote by $T_xM$ the tangent space at $x \in M$, and by $TM$ the tangent bundle of $M$. Each element of $TM$ has the form $(x, y)$, where $x \in M$ and $y \in T_xM$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) \equiv x$.

A Finsler structure of $M$ is a function $F : TM \to [0, \infty)$ with the following properties:

(i) Regularity: $F$ is $C^\infty$ on the entire slit tangent bundle $TM\setminus 0$.
(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.
(iii) Strong convexity: The $n \times n$ Hessian matrix $g_{ij} \equiv \frac{1}{2} F^2 y^i y^j$ is positive-definite at every point of $TM\setminus 0$, where we have used the notation $(\ )_{y^i} = \frac{\partial}{\partial y^i}(\ )$.

Finsler geometry has its genesis in integrals of the form

$$\int_s^r F(x^1, \ldots, x^n; \frac{dx^1}{dt}, \ldots, \frac{dx^n}{dt}) dt.$$ (1)

Throughout the paper, the lowering and raising of indices are carried out by the fundamental tensor $g_{ij}$ defined above, and its matrix inverse $g^{ij}$. Given a manifold $M$ and a Finsler structure $F$ on $TM$, the pair $(M, F)$ is called as a Finsler manifold. It is obvious that the Finsler structure $F$ is a function of $(x^i, y^i)$. In the case of $F$ depending on $x^i$ only, the Finsler manifold reduces to Riemannian Manifold.

The symmetric Cartan tensor can be defined as

$$A_{ijk} \equiv \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{F}{4} (F^2)_{y^iy^jy^k},$$ (2)

Cartan tensor vanishes if and only if $g_{ij}$ has no $y$-dependence. So that Cartan tensor is a measurement of deviation from Riemannian Manifold.

Using Euler's theorem on homogenous function, we can get useful properties of the fundamental tensor $g_{ij}$ and Cartan tensor $A_{ijk}$

$$g_{ij}l^i = F_{y^j},$$ (3)
$$g_{ij}l^i l^j = 1,$$ (4)
$$y^i \frac{\partial g_{ij}}{\partial y^k} = 0, \quad y^i \frac{\partial g_{ij}}{\partial y^j} = 0, \quad y^k \frac{\partial g_{ij}}{\partial y^k} = 0,$$ (5)
$$y^i A_{ijk} = y^j A_{ijk} = y^k A_{ijk} = 0,$$ (6)

where $l^i \equiv \frac{y^i}{F}$. 

2.2 Chern Connection

The nonlinear connection $N^i_j$ on $TM\setminus 0$ is defined as

$$N^i_j \equiv \gamma^i_{jk} y^k - \frac{A^i_{jk}}{F} y^r y^s,$$

where $\gamma^i_{jk}$ is the formal Christoffel symbols of the second kind

$$\gamma^i_{jk} \equiv \frac{g^{is}}{2} \left( \frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} \right).$$

The invariant connection under the transform $y \rightarrow \lambda y$ is of the form

$$\frac{N^i_j}{F} \equiv \gamma^i_{jk} y^k - \frac{A^i_{jk}}{F} \gamma^k_{rs} y^r y^s.$$

As usually, we define the covariant derivatives $\nabla \frac{\partial}{\partial x^i}$ and $\nabla dx^i$ as

$$\nabla \frac{\partial}{\partial x^i} \equiv \omega^j_i \frac{\partial}{\partial x^j},$$

$$\nabla dx^i \equiv -\omega^j_i dx^j,$$

where $\omega^j_i$ is the connection 1-forms. The operator $\nabla$ have the same linear property with the covariant derivatives defined on Riemannian manifold.

Here, we introduce the Chern connection that is torsion freeness

$$dx^j \wedge \omega^i_j = 0$$

and almost $g$-compatibility

$$dg_{ij} - g_{kj} \omega^k_i - g_{ik} \omega^k_j = 2 A_{ij} \delta y^s \frac{\delta y^s}{F}.$$

A theorem given by S. S. Chern guarantees the uniqueness of Chern connection. Theorem (Chern): Let $(M, F)$ be a Finsler manifold. The pulled-back bundle $\pi^*TM$ admits a unique linear connection, called the Chern connection. Its connection forms are characterized by the structural equations (12), (13).

We ignore the proof of the theorem, just give some consequence of it directly. Torsion freeness is equivalent to the absence of $dy^i$ terms in $\omega^i_j$; namely,

$$\omega^i_j = \Gamma^i_{jk} dx^k,$$

together with the symmetry

$$\Gamma^i_{jk} = \Gamma^i_{kj}.$$
And almost $g$-compatibility implies that

$$\Gamma^i_{jk} = \frac{g^{is}}{2} \left( \frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right),$$  \hspace{1cm} (16)

where

$$\frac{\delta}{\delta x^i} \equiv \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial x^j}. \hspace{1cm} (17)$$

The dual basis of $\frac{\partial}{\partial y^i}$ is

$$\delta y^i \equiv dy^i + N^j_i dx^j. \hspace{1cm} (18)$$

As before, we prefer to work with

$$\frac{\delta y^i}{F} = \frac{1}{F}(dy^i + N^j_i dx^j), \hspace{1cm} (19)$$

which is invariant under rescaling of $y$.

We will work on two new natural local bases that are dual to each other:

$\{\frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i}\}$ for the tangent bundle of $TM\backslash 0$,

$\{dx^i, \frac{\delta y^i}{F}\}$ for the cotangent bundle of $TM\backslash 0$.

One can check that the transformation law of Chern connection on Finsler manifold is the same with Riemannian connection on Riemannian manifold. This fact is useful to guide us define the covariant derivative of a tensor.

Let $V \equiv V^j_i \frac{\partial}{\partial y^j} \otimes dx^i$ be an arbitrary smooth local section of $\pi^*TM \otimes \pi^*T^*M$. The definition (10), (11) and property of operator $\nabla$ imply that the covariant derivatives of $V$ is

$$\nabla V \equiv (\nabla V)^j_i \frac{\partial}{\partial x^j} \otimes dx^i, \hspace{1cm} (20)$$

where

$$(\nabla V)^j_i \equiv dV_i + V^k_i \omega^j_k - V^j_k \omega^k_i. \hspace{1cm} (21)$$

$\nabla V$ is a 1-form on $TM\backslash 0$. Thus, it can be expressed in terms of the natural basis $\{dx^i, \frac{\delta y^i}{F}\}$,

$$\nabla V^j_i \equiv V^j_i \frac{\partial}{\partial x^i} + V^j_i \frac{\delta y^i}{F}. \hspace{1cm} (22)$$

Using relation between the Chern connection and the connection 1-forms $\omega^i_j$ \cite{4}, we obtain the horizontal covariant derivative $V^j_i \mid_s$

$$V^j_i \mid_s = \frac{\delta V^j_i}{\delta x^s} + V^k_i \Gamma^j_k \mid_s - V^j_k \Gamma^k_i \mid_s, \hspace{1cm} (23)$$
and the vertical covariant derivative $V^j_{i:s}$

$$V^j_{i:s} = F \frac{\partial V^j_i}{\partial y^s}. \quad (24)$$

The treatment for tensor fields of higher rank is similar with the methods used on Riemannian manifold. Here, we give results of covariant derivatives of the fundamental tensor $g$ and the norm 1 vector $l$:

$$g_{ij|s} = g^i_j|_s = 0, \quad (25)$$
$$g_{ij:s} = 2 A_{ij:s} \quad \text{and} \quad g^i_j|_s = -2 A^i_{jj:s}, \quad (26)$$
$$l^i|_s = l_{i|s} = 0, \quad (27)$$
$$l^i_s = \delta^i_s - l^i l_s \quad \text{and} \quad l_{i;s} = g_{is} - l^i l_s. \quad (28)$$

### 2.3 Curvature

The curvature 2-forms of Chern connection are

$$\Omega^i_j \equiv d\omega^i_j - \omega^i_j \wedge \omega^k_k. \quad (29)$$

The expression of $\Omega^i_j$ in terms of the natural basis $\{dx^i, \delta y^i/F\}$ is of the form

$$\Omega^i_j \equiv \frac{1}{2} R^i_{jkl} dx^k \wedge dx^l + P^i_{jkl} dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q^i_{jkl} \delta y^k \wedge \frac{\delta y^l}{F}, \quad (30)$$

where $R$, $P$ and $Q$ are the $hh$, $hv$, $vv$-curvature tensors of the Chern connection, respectively. The following property is manifest

$$R^i_{jkl} = -R^i_{jlk}, \quad (31)$$
$$Q^i_{jkl} = -Q^i_{jkl}. \quad (32)$$

We are now at the position to demonstrate the Bianchi identities for the curvature.

Exterior differential of the structural equation (12) gives

$$dx^j \wedge d\omega^i_j = 0. \quad (33)$$

The combination of equations (33) and (12) shows that

$$dx^j \wedge \Omega^i_j = 0. \quad (34)$$

Substituting equation (34) into (30), we get

$$\frac{1}{2} R^i_{jkl} dx^j \wedge dx^k \wedge dx^l + P^i_{jkl} dx^k \wedge dx^l \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q^i_{jkl} dx^j \wedge \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F} = 0. \quad (35)$$
The three terms on the left side are completely independent. Thus, all of them should vanish. This gives identities

\[ R_{ij}^{i} + R_{k}^{i} + R_{i}^{i} = 0, \quad (36) \]
\[ P_{j}^{k} = P_{j}^{k}, \quad (37) \]
\[ Q_{j}^{k} = 0. \quad (38) \]

Then, the curvature 2-forms can be simplified as

\[ \Omega_{i}^{j} \equiv \frac{1}{2} R_{j}^{i} dx^k \land dx^l + P_{j}^{i} dx^k \land \frac{\delta y^l}{F}. \quad (39) \]

T eros but straightforward manipulation of exterior differential on the structural equation (13) gives

\[ \Omega_{ij} + \Omega_{ji} = -2(\nabla A)_{ijk} \land \frac{\delta y^k}{F} - 2A_{ijk} \left[ d(\frac{\delta y^k}{F}) + \omega_{i}^k \land \frac{\delta y^l}{F} \right]. \quad (40) \]

It can be rewritten into

\[ \frac{1}{2}(R_{ijkl} + R_{jkil})dx^k \land dx^l + (P_{ijkl} + P_{jkil})dx^k \land \frac{\delta y^l}{F} = -A_{iju}R_{kl}dx^k \land dx^l - 2(A_{iju}P_{kl} + A_{ijlk})dx^k \land \frac{\delta y^l}{F} \]
\[ + 2(A_{ijkl} - A_{ijlk}) \frac{\delta y^k}{F} \land \frac{\delta y^l}{F}, \quad (41) \]

where we have used the abbreviations

\[ R_{j}^{i} \equiv \nu_{j} R_{j}^{i}, \quad (42) \]
\[ P_{j}^{i} \equiv \nu_{P} P_{j}^{i}. \quad (43) \]

Equalization of three different types of terms at two sides of equation (41) shows identities

\[ R_{ijkl} + R_{klij} = -2A_{iju}R_{kl}, \quad (44) \]
\[ P_{ijkl} + P_{klij} = -2(A_{iju}P_{kl} + A_{ijlk}), \quad (45) \]
\[ A_{ijkl} - A_{ijlk} = A_{ijlk} - A_{ijkl}. \quad (46) \]

The formula (31) and identities (36),(44) enable us get the fourth property of \( hh \)-curvature,

\[ R_{klji} - R_{ijkl} = (B_{klji} - B_{ijkl}) + (B_{kijl} + B_{ijkl}) + (B_{ijlk} - B_{ijkl}), \quad (47) \]

where, for convenient, we have used the notation \( B_{ijkl} \equiv -A_{iju}R_{kl}. \) On Riemannian manifold, the Cartan tensor vanish. This means that \( B_{ijkl} = 0 \) on Riemannian manifold. The familiar properties of Riemannian curvature

\[ \tilde{R}_{ijkl} + \tilde{R}_{ijlk} = 0, \]
\[ \tilde{R}_{ijkl} + \tilde{R}_{klij} = 0, \]
\[ \tilde{R}_{ijkl} + \tilde{R}_{jikl} = 0, \]
\[ \tilde{R}_{ijkl} - \tilde{R}_{klij} = 0, \]
can be deduced directly from the four properties of $hh$-curvature (31), (36), (44) and (47). Making use of the identity (45) and equations (6), (27), we may get a constituent relation for $P_{ijkl}$,

$$P_{ijkl} = -(A_{ijkl} + A_{klij}) + A_{ij} A_{ukl} + A_{jkl} A_{uil} + A_{kil} A_{ujl},$$  

(48)

where

$$\dot{A}_{ijkl} = A_{ijkl}.$$

(49)

Contracting $P_{ijkl}$ with $l^i$ in equation (48), we obtain an important relation

$$P_{jkl} = -\dot{A}_{jkl}.$$

(50)

The expression of $R$ and $P$ can be got by substituting the formula (29) into (39),

$$R_{ijkl} = \frac{\delta \Gamma_{i}^{j}}{\delta x^{k}} - \frac{\delta \Gamma_{i}^{j}}{\delta x^{l}} + \Gamma_{hk}^{i} \Gamma_{jl}^{k} - \Gamma_{hl}^{i} \Gamma_{jk}^{k},$$

(51)

$$P_{ijkl} = -F \frac{\partial \Gamma_{i}^{j}}{\partial y^{l}}.$$

(52)

These are counterparts of the Riemannian curvature expressed in terms of the Christoffel symbols $\tilde{\Gamma}_{jk}^{i}$

$$\tilde{R}_{ijkl} = \frac{\partial \tilde{\Gamma}_{i}^{j}}{\partial x^{k}} - \frac{\partial \tilde{\Gamma}_{i}^{j}}{\partial x^{l}} + \tilde{\Gamma}_{hk}^{i} \tilde{\Gamma}_{jl}^{k} - \tilde{\Gamma}_{hl}^{i} \tilde{\Gamma}_{jk}^{k}.$$

(53)

Before ending the section, we present the second Bianchi identity. Exterior differential of the Chern connection (29) gives

$$d\Omega_{ij} - \omega_{j}^{h} \wedge \Omega_{k}^{i} + \omega_{k}^{i} \wedge \Omega_{j}^{k} = 0.$$

(54)

Substituting (39) into the above equation, we obtain

$$\frac{1}{2}dR_{ijkl} \wedge dx^{k} \wedge dx^{l} + dP_{ijkl} \wedge dx^{k} \wedge \frac{\delta y^{l}}{F} - P_{ijkl} dx^{k} \wedge d\left(\frac{\delta y^{l}}{F}\right) = \frac{1}{2} R_{ijkl} \omega_{r}^{i} \wedge dx^{k} \wedge dx^{l} - \frac{1}{2} R_{ijkl} \omega_{r}^{i} \wedge dx^{k} \wedge dx^{l} + P_{ijkl} \omega_{r}^{i} \wedge dx^{k} \wedge \frac{\delta y^{l}}{F} - P_{ijkl} \omega_{r}^{i} \wedge dx^{k} \wedge \frac{\delta y^{l}}{F}.$$

(55)

To evaluate $d\left(\frac{\delta y^{l}}{F}\right)$, we first rewrite $\frac{\delta y^{l}}{F}$ as

$$\frac{\delta y^{l}}{F} dl^{l} + \Gamma_{jkl}^{i} dx^{l} + \frac{dF}{F} l^{l}.$$

(56)
Then, one has
\[ d\left(\frac{\delta y^i}{F}\right) = dl^i \wedge \omega^j_l + l^i d\omega^j_l + dl^i \wedge \frac{dF}{F} \]
\[ = l^i \Omega^j_l + l^i \wedge \omega^k_j + \omega^l_j + \left(\frac{\delta y^j}{F} - \omega^k_j l^i - l^i \frac{dF}{F}\right) \wedge \omega^l_j + \left(\frac{\delta y^j}{F} - \omega^k_j l^i\right) \wedge \frac{dF}{F} \]
\[ = l^i \Omega^j_l + \frac{\delta y^j}{F} \wedge (\omega^l_j - l^i \frac{\delta y^j}{F}), \quad (57) \]
here we have used the identity
\[ l^i \frac{\delta y^i}{F} = \frac{dF}{F}, \quad (58) \]
to get the third equal.
Substituting formula \((57)\) into \((54)\) and noticing the torsion freeness of the Chern connection, we obtain
\[ \frac{1}{2} \nabla R^i_{jkl} dx^k \wedge dx^l + \nabla P^i_{jkl} \wedge \frac{\delta y^l}{F} \]
\[ = P^i_{jkl} dx^k \wedge \left(\frac{1}{2} R^i_{t rs} dx^r \wedge dx^s + P^i_{t rs} dx^r \wedge \frac{\delta y^s}{F}\right) - P^i_{jkl t_r} dx^k \wedge \frac{\delta y^l}{F} \wedge \frac{\delta y^r}{F}, \quad (59) \]
In natural basis, we can rewrite equation \((59)\) into the form
\[ \frac{1}{2} \left( R^i_{jkl t} - P^i_{j ku R^u_{lt}} \right) dx^k \wedge dx^l \wedge dx^t \]
\[ + \frac{1}{2} \left( R^i_{j klt} - 2 P^i_{j klt} + 2 P^i_{j ku A^u_{lt}} \right) dx^k \wedge dx^l \wedge \frac{\delta y^t}{F} \]
\[ + \left( P^i_{j klt} - P^i_{j klt t} \right) dx^k \wedge \frac{\delta y^l}{F} \wedge \frac{\delta y^t}{F} = 0, \quad (60) \]
The three terms in the left side are completely independent. Then, we get the following identities
\[ R^i_{j kl t} + R^i_{j l t k} + R^i_{j t k l} = P^i_{j ku A^u_{lt}} + P^i_{j su R^u_{lt}} + P^i_{j tu R^u_{kt}}, \quad (61) \]
\[ R^i_{j klt} = P^i_{j klt} - P^i_{j lt k} - (P^i_{j ku A^u_{lt}} - P^i_{j tu A^u_{kt}}), \quad (62) \]
\[ P^i_{j kl t} - P^i_{j klt} = P^i_{j klt} - P^i_{j klt t}. \quad (63) \]

3 Gravitation theory in Berwald space

Einstein proposed successfully his general relativity in Riemannian space to describe gravity. It is interest to investigate the behaviors of gravitation in a more general Finsler spaces. Let us briefly recall the setup way of the Einstein field equation on Riemannian manifold. One starts from the second Bianchi identities on Riemannian manifold
\[ \tilde{R}^i_{j kl t} + \tilde{R}^i_{j l t k} + \tilde{R}^i_{j t k l} = 0. \quad (64) \]
The metric-compatibility
\[ \tilde{g}_{ij,k} = 0 \quad \text{and} \quad \tilde{g}^{ij} |_k = 0, \] (65)
and contraction of (64) with \( \tilde{g}^{jl} \) gives that
\[ \tilde{R}^{ji}_{\ kllj} + \tilde{R}^{j}_{l|kl} - \tilde{R}^{ji}_{\ kll} = 0, \] (66)
where \( \tilde{R}^{ji}_{\ } \equiv \tilde{R}^{ji}_{\ j} \) is the Ricci tensor. Lowering the index \( i \) and contracting with \( \tilde{g}^{ik} \), we obtain
\[ \tilde{R}^j_{\ lj} + \tilde{R}^j_{\ lj} - \tilde{S} = 0, \] (67)
where \( \tilde{S} = \tilde{g}^{ij} \tilde{R}^{ij}_{\ } \) is the scalar curvature. An equivalent but more familiar form is
\[ (\tilde{R}^{jl}_j - \frac{1}{2} \tilde{g}^{jl} \tilde{S})_j = 0. \] (68)

In the weak field limit, gravitation theory should reduce to the Newtonian theory. Einstein suggested his gravitational field equation of the form
\[ \tilde{R}^{jl}_j - \frac{1}{2} \tilde{g}^{jl} \tilde{S} = 8\pi G T^{jl}, \] (69)
where \( T^{jl} \) is the energy–momentum tensor and \( G \) is the Newton’s constant.

In the paper, we use similar approach to discuss gravitation on Finsler manifold. Let us introduce first two notions for Ricci curvature: the Ricci scalar \( \text{Ric} \) and the Ricci tensor \( \text{Ric}_{ij} \).

The Ricci scalar is defined as
\[ \text{Ric} = g^{ik} R_{ik}, \] (70)
where \( R_{ik} \equiv l^j R_{jikl} \) is symmetric. The Ricci tensor on Finsler manifold was first introduced by Akbar-Zadeh [24]
\[ \text{Ric}_{ik} \equiv (\frac{1}{2} F^2 \text{Ric})_{g^i g^k}, \] (71)
which is manifestly symmetric and covariant. Expanding \( y \) derivatives in the defining formula for Ricci tensor \( \text{Ric}_{ik} \), we get
\[ \text{Ric}_{ik} = \frac{1}{4} (\text{Ric}_{i;k} + \text{Ric}_{j;k;i}) + \frac{3}{4} (l_i \text{Ric}_{:k} + l_k \text{Ric}_{:i}) + g_{ik} \text{Ric}. \] (72)
Substituting the defining formula for Ricci scalar \( \text{Ric} \) into the above equation, we obtain
\[ \text{Ric}_{ik} = \frac{1}{4} (R^s_{k \ si} + R^s_{i \ sk}) + \frac{1}{4} l^l (R^s_{j \ st;ki} + R^s_{j \ st;ik}) - \frac{1}{4} l^l (l_i R^s_{j \ st;kl} + l_k R^s_{j \ st;li}) \]
\[ + \frac{1}{2} l^l (R^s_{i \ sj;kl} + R^s_{j \ si;kl} + R^s_{k \ sj;li} + R^s_{j \ sk;li}) \]
\[ = \frac{1}{2} (R^s_{k \ si} + R^s_{i \ sk}) + E_{ik}, \] (73)
\[ = \frac{1}{2} (R^s_{k \ si} + R^s_{i \ sk}) + E_{ik}, \] (74)
where we introduced the notation
\[
E_{ik} \equiv \frac{1}{4} l^j l^l (R^i_j s_{ik} + R^i_j s_{ik}) - \frac{1}{4} l^j l^l (l^k R^i_j s_{ik} + l_k R^i_j s_{ik}) + \frac{1}{2} l^l (R^i_j s_{ik} + R^i_j s_{ik} + R^i_k s_{ji} + R^i_j s_{ki}).
\] (75)

Following same setup process for gravitational field equation in Riemannian space, we start from the second Bianchi identities \([61]\), contracting it with \(g^{ji}\), lowering the index \(i\), and contracting again with \(g^{ik}\), we get
\[
R^{ji}_{il|j} + R^{ji}_{lj|i} + R^{ji}_{ji|l} = g^{ji} g^{ik} (P_{ijkl} R^u_{lt} + P_{jitu} R^u_{tk} + P_{jitu} R^u_{kl}).
\] (76)

Using the first Bianchi identity \([44]\) and formula \([47]\), we can divide the left side of the above equation into symmetric part labeled by \([\ ]\) and nonsymmetric part labeled by \(\{\}\)
\[
R^{ji}_{il|j} + R^{ji}_{lj|i} + R^{ji}_{ji|l} = \left(\text{Ric}^j_i + \frac{1}{2} B^j_k l - E^j_i\right)_{lj} + \left(2B^j_k l + \text{Ric}^j_i + \frac{1}{2} B^j_k l - E^j_i\right)_{lj} - \delta^j_i (S - E)_{lj}
\] (77)

where \(E \equiv g^{ji} E_{ij}\) and \(S = g^{ji} \text{Ric}_{ij}\). Using the constituent relation of the \(hv\)-curvature tensor \([48]\), we rewrite the right side of identity \((76)\) as
\[
g^{ji} g^{ik} (P_{ijkl} R^u_{lt} + P_{jitu} R^u_{tk} + P_{jitu} R^u_{kl})
= 2(A^j_{il} - A^j_{it} \hat{A}_{riu}) R^u_{ji} + 2(A^j_{il} + A^j_{ki} \hat{A}_{rij} - A^j_{i} \hat{A}_{rju}) R^u_{ji}.
\] (78)

where \(A_{r} \equiv g^{ji} A_{ijr}\).

Finally, we get the equivalent form of the identity \([76]\)
\[
\left[\left(\text{Ric}^j_i - \frac{1}{2} g^{ji} S\right) - \left(E^j_i - \frac{1}{2} g^{ji} E\right)\right]_{lj} + \left\{\frac{1}{2} B^j_k l + B^j_k l\right\}_{lj}
= (A^j_{il} - A^j_{it} \hat{A}_{riu}) R^u_{ji} + (A^j_{il} - A^j_{hi} \hat{A}_{riu} - A^j_{i} \hat{A}_{rju}) R^u_{ji}.
\] (79)

A Finsler structure \(F\) is said to be of Berwald type if the Chern connection coefficients \(\Gamma^i_{jk}\) in natural coordinates have no \(y\) dependence. A direct proposition on Berwald space is that \(hv\)-part of the Chern curvature vanishes identically
\[
P^i_{jkl} = 0,
\] (80)

and the \(hh\)-part of the Chern connection reduce to
\[
R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{hk} \Gamma^h_{jl} - \Gamma^i_{hl} \Gamma^h_{jk}.
\] (81)
So that, in Berwald space the identity (79) reduces as

\[
\left[ Ric^{jl} - \frac{1}{2} g^{jl} S \right]_{ij} + \left\{ \frac{1}{2} B^k_{j}^{kl} + B^k_{jik} \right\} = 0. \tag{82}
\]

Thus, the counterpart of the Einstein’s field equation on Berwald space takes the form

\[
\left[ Ric^{jl} - \frac{1}{2} d jl S \right]_{ij} + \left\{ \frac{1}{2} B^k_{j}^{kl} + B^k_{jik} \right\} = 8\pi GT_{jl}. \tag{83}
\]

The gravitational field equation on Berwald space is obvious different from the Einstein’s field equation. The geometric part contains nonsymmetric term. Thus, in general, the energy–momentum tensor \( T_{jl} \) is not symmetric. It means that local Lorentz invariance is violated in general.

### 4 Solutions of gravitational field equation

At this section, we present examples of Berwald-Finsler space. Kikuchi\(^{[25]}\) proved that in a Randers space of Berwald type, one has

\[
\tilde{b}_{ij} \equiv \tilde{b}_{i,j} - \tilde{b}_k \tilde{\gamma}^k_{ij} = 0, \tag{84}
\]

where \( \tilde{\gamma}^k_{ij} \) is the Christoffel symbols of Riemannian metric \( \tilde{a} \equiv \tilde{a}_{ij} dx^i \otimes dx^j \). In Randers space, one can derive straightforwardly the expression of the geodesic spray coefficients as

\[
G^i \equiv \gamma^i_{jk} y^j y^k = (\tilde{\gamma}^i_{jk} + l^i \tilde{b}_{jk}) y^j y^k + (\tilde{a}^{ij} - l^i \tilde{b}^j)(\tilde{b}_{jik} - \tilde{b}_{kij}) a y^k, \tag{85}
\]

and the Chern connection as

\[
\tilde{\Gamma}^i_{jk} = (N^i_{j}) y^k + \frac{1}{2} g^{it} y_s (N^s_{l}) y^j y^k. \tag{86}
\]

It is not difficult to check that the geodesic spray coefficients satisfy that

\[
\frac{1}{2} \frac{\partial G^i}{\partial y^j} = N^i_{j}. \tag{87}
\]

Thus in Randers spaces of Berwald type, the geodesic spray coefficients reduce to

\[
G^i = \tilde{\gamma}^i_{jk} y^j y^k. \tag{88}
\]

The Chern connection reduces to

\[
\tilde{\Gamma}^i_{jk} = \tilde{\gamma}^i_{jk}. \tag{89}
\]

Then, the \( hh \)–curvature takes the form

\[
R^i_{jkl} = \frac{\partial \tilde{\gamma}^i_{jl}}{\partial x^k} - \frac{\partial \tilde{\gamma}^i_{jk}}{\partial x^l} + \tilde{\gamma}^i_{hk} \tilde{\gamma}^{h}_{jl} - \tilde{\gamma}^i_{hl} \tilde{\gamma}^{h}_{jk}. \tag{90}
\]
In 4-dimensional Randers space, the Robertson-Walker metric

\[
\tilde{a}_{ij} = \text{diag}\{1, -\frac{a^2(t)}{1 - kr^2}, -a^2(t)r^2, -a^2(t)r^2\sin^2 \theta}\}
\]

(91)

and the constraint

\[
\dot{a}^2 + k = 0
\]

(92)
gives nontrivial solution of the gravitation in the Berwald-Finsler space.

A possible solution of (83) for Berwald-Finsler space with one extra dimension is of the form

\[
\tilde{a}_{ij} = \text{diag}\{1, -\frac{a^2(t)}{1 - kr^2}, -a^2(t)r^2, -a^2(t)r^2\sin^2 \theta, 0\},
\]

(93)

\[
\tilde{b}_i = \{0, 0, 0, 0, c\},
\]

(94)

where \(c\) is constant.

\section{Conclusion and remarks}

In this paper, we have setup a gravitation theory in a torsionfreeness Berwald-Finsler space. The geometric part of the gravitational field equation is, in general, nonsymmetric. This fact indicates that the local Lorentz invariance is violated in the Finsler manifold. This is in good agreement with discussions on special relativity in Finsler space\[13, 12, 20\]. Nontrivial solutions of gravitation in Berwald-Finsler space were presented.

However, problems still remain. How to construct a gravitation in general Finsler space is still a open question. It is well-known that in Riemannian space the sign of section curvature \(K(x)\) determine the type of geometry near \(x\) (hyperbolic, flat or spherical). In the landscape of Finslerian, the sign of \(K(x, y)\) depend on the direction \(y\) of our line of sight. This make it possible to encounter all three types of geometry during a survey. In such a cosmology model, one may wish to find a natural explanation for why the early universe is asymptotic flat.

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