On choosability with separation of planar graphs with lists of different sizes

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Abstract

A \((k, d)\)-list assignment \(L\) of a graph \(G\) is a mapping that assigns to each vertex \(v\) a list \(L(v)\) of at least \(k\) colors and for any adjacent pair \(xy\), the lists \(L(x)\) and \(L(y)\) share at most \(d\) colors. A graph \(G\) is \((k, d)\)-choosable if there exists an \(L\)-coloring of \(G\) for every \((k, d)\)-list assignment \(L\). This concept is also known as choosability with separation.

It is known that planar graphs are \((4, 1)\)-choosable but it is not known if planar graphs are \((3, 1)\)-choosable. We strengthen the result that planar graphs are \((4, 1)\)-choosable by allowing an independent set of vertices to have lists of size 3 instead of 4.

Our strengthening is motivated by the observation that in \((4, 1)\)-list assignment, vertices of an edge have together at least 7 colors, while in \((3, 1)\)-list assignment, they have only at least 5. Our setting gives at least 6 colors.

1 Introduction

Given a graph \(G\), a list assignment \(L\) is a mapping assigning to each vertex \(v \in V(G)\) a list of colors \(L(v)\). An \(L\)-coloring is a vertex coloring \(\varphi\) such that \(\varphi(v) \in L(v)\) for each vertex \(v\) and \(\varphi(x) \neq \varphi(y)\) for each edge \(xy\). A graph \(G\) is said to be \(k\)-choosable if there is an \(L\)-coloring for each list assignment \(L\) where \(|L(v)| \geq k\) for each vertex \(v\). The minimum such \(k\) is called the list chromatic number or choice number of \(G\), denoted \(\chi(L(G))\). A graph \(G\) is said to be \((k, d)\)-choosable if there is an \(L\)-coloring for each list assignment \(L\) where \(|L(v)| \geq k\) for each vertex \(v\) and \(|L(x) \cap L(y)| \leq d\) for each edge \(xy\).

This concept is called choosability with separation, since the second parameter may force the lists of adjacent vertices to be somewhat separated. If \(G\) is \((k, d)\)-choosable, then \(G\) is

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also \((k', d')\)-choosable for all \(k' \geq k\) and \(d' \leq d\). A graph is \((k, k)\)-choosable if and only if it is \(k\)-choosable. Clearly, all graphs are \((k, 0)\)-choosable for \(k \geq 1\). Thus, for a graph \(G\) and each \(1 \leq k < \chi(G)\), there is some threshold \(d \in \{0, \ldots, k - 1\}\) such that \(G\) is \((k, d)\)-choosable but not \((k, d + 1)\)-choosable.

The concept of choosability with separation was introduced by Kratochvíl, Tuza, and Voigt [4]. They used the following, more general definition. A graph \(G\) is \((k, d)\)-choosable, if for every list assignment \(L\) of \(G\), there is some threshold \(d \in \{0, \ldots, k - 1\}\) such that \(G\) is \((k, d)\)-choosable for all \(d\). Voigt [7] also constructed a non-3-choosable triangle-free planar graph. Škrekovski [5] observed that there are examples of triangle-free planar graphs that are not \((3, 2)\)-choosable, and posed:

**Question 1** ([4]). Are all planar graphs \((4, 2)\)-choosable?

Voigt [7] also constructed a non-3-choosable triangle-free planar graph. Škrekovski [5] observed that there are examples of triangle-free planar graphs that are not \((3, 2)\)-choosable, and posed:

**Question 2** ([5]). Are all planar graphs \((3, 1)\)-choosable?

Kratochvíl, Tuza and Voigt [4] showed that every triangle-free planar graph is \((3, 1)\)-choosable.

Choi et al [1] proved that every planar graph without 4-cycles is \((3, 1)\)-choosable and that every planar graph without 5-cycles and 6-cycles is \((3, 1)\)-choosable.

In this paper we give a strengthening of the result that every planar graph is \((4, 1)\)-choosable by allowing some vertices to have lists of size three. In a \((4, 1)\)-list assignment \(L\) on \(G\), for every \(uv \in E(G)\) holds that \(|L(u) \cup L(v)| \geq 7\). In a \((3, 1)\)-list assignment \(L\), for every \(uv \in E(G)\) holds that \(|L(u) \cup L(v)| \geq 5\). An intermediate step is to investigate the case where for every \(uv \in E(G)\) holds that \(|L(u) \cup L(v)| \geq 6\).

A \((*, 1)\)-list assignment is a list assignment \(L\) where \(|L(v)| \geq 1\) and \(|L(u) \cap L(v)| \leq 1\) for every pair of adjacent vertices \(u, v\).

The main result of this paper is the following theorem.

**Theorem 3.** Let \(G\) be a planar graph and \(I \subseteq V(G)\) be an independent set. If \(L\) is a \((*, 1)\)-list assignment such that \(|L(v)| \geq 3\) for every \(v \in I\) and \(|L(v)| \geq 4\) for every \(v \in V(G) \setminus I\) then \(G\) has an \(L\)-coloring.

The following theorem shows it is not possible to strengthen Theorem 3 by allowing \(|L(v)| \geq 2\) for every vertex \(v \in V(G)\) and requiring that \(|L(u) \cup L(v)| \geq 6\) for every \(uv \in E(G)\).
Theorem 4. For every \( k \) there exists a planar graph \( G \) and a \((\ast, 1)\)-list assignment \( L \) such that \(|L(v)| \geq 2\) for every \( v \in V(G) \), \(|L(u) \cup L(v)| \geq k\) for every \( uv \in E(G) \), and \( G \) is not \( L\)-colorable.

We first give some notation. In the next section, we prove Theorem 3 using Thomassen’s precoloring extension method. In the last section we show a construction proving Theorem 4.

1.1 Notation

Given a graph \( G \) and a cycle \( K \subset G \), an edge \( uv \) of \( G \) is a chord of \( K \) if \( u, v \in V(K) \), but \( uv \) is not an edge of \( K \). If \( G \) is a plane graph, then let \( \text{Int}_K(G) \) be the subgraph of \( G \) consisting of the vertices and edges drawn inside the closed disc bounded by \( K \), and let \( \text{Ext}_K(G) \) be the subgraph of \( G \) obtained by removing all vertices and edges drawn inside the open disc bounded by \( K \). In particular, \( K = \text{Int}_K(G) \cap \text{Ext}_K(G) \). Finally, denote the characteristic function of a set \( S \) by \( \iota_S \). So \( \iota_S(x) = 1 \) if \( x \in S \); else \( \iota_S(x) = 0 \).

2 Main theorem

In this section, we prove Theorem 3 by proving a slightly stronger theorem that is more amenable to induction. Observe that any list assignment satisfying the assumptions of Theorem 3 also satisfies the conditions of the following theorem.

Theorem 5. Let \( G \) be a plane graph with outer face \( F \) and let \( P \) be a subpath of \( F \) containing at most two vertices. Let \( I \subseteq V(G - P) \) be an independent set. If \( L \) is a \((\ast, 1)\)-list assignment satisfying the following conditions:

\[
\begin{align*}
(i) & \quad |L(v)| \geq 4 - \iota_I(v) - \iota_{V(F)}(v) - 2\iota_{V(P)}(v) \quad \text{for } v \in V(G), \\
(ii) & \quad P \text{ is } L\text{-colorable}, \\
(iii) & \quad \text{for every } v \in I \text{ there is at most one } p \in N(v) \cap V(P) \text{ with } (L(p) \cap L(v)) \neq \emptyset,
\end{align*}
\]

then \( G \) is \( L\)-colorable.

Proof. It is easy to check that the statement of the theorem is true for graphs on at most three vertices. Let \( G = (V, E) \) and \( L \) be a counterexample where \(|V| + |E|\) is as small as possible. Moreover, assume that the sum of the sizes of the lists is also as small as possible subject to the previous condition. Define \( L(uv) = L(u) \cap L(v) \) if \( uv \in E \); else \( L(uv) = \emptyset \). Since \( G \) is minimal, we have:

Claim 1. The following is true:

\[
\begin{align*}
(1) & \quad \text{for every } uv \in E \setminus E(P), |L(uv)| = 1; \\
(2) & \quad \text{for every } u \in V, L(u) = \bigcup_{v \in N(u)} L(uv); \text{ and} \\
(3) & \quad \text{for every triangle } uvw \text{ such that } uv, vw, uw \in E \setminus E(P), L(uw) = L(vw) \text{ implies } L(uv) = L(uw).
\end{align*}
\]
Proof. For (1), note that $|L(uv)| \leq 1$, and if $L(uv) = \emptyset$ then it suffices to $L$-color $G - uv$, which is possible by minimality. For (2), the definitions imply $L(u) \supseteq \bigcup_{v \in N(u)} L(uv)$, and if $\gamma \in L(u) \setminus \bigcup_{v \in N(u)} L(uv)$ then $L$-coloring $G - u$, and then coloring $u$ with $\gamma$ yields an $L$-coloring of $G$. Finally consider (3). By (1), there exists a color $\gamma$ with $L(uv) = \{\gamma\} = L(vw)$. Thus $\gamma \in L(u) \cap L(w)$, so by definition and (1), $L(uw) = \{\gamma\}$.

Claim 2. $G$ is 2-connected. In particular, $F$ is a cycle.

Proof. Suppose not. Then there exists $v \in V$ and two induced connected subgraphs $G_1$ and $G_2$ of $G$ where $G_1 \cap G_2 = v$ and $G_1 \cup G_2 = G$. Moreover, both $G_1$ and $G_2$ have at least two vertices. By symmetry assume that $P \subseteq G_1$. By the minimality of $G$, there exists an $L$-coloring $\varphi$ of $G_1$. Let $L_2$ be a list assignment on $V(G_2)$ such that $L_2(v) = \{\varphi(v)\}$ if $v = u$, and $L_2(u) = L(u)$ otherwise. Let $I_2 = (I - v) \cap V(G_2)$, which is an independent set and $P_2 = \{v\}$, which is a subpath of the outer face of $G_2$. Since $L_2$, $I_2$, $P_2$ and $G_2$ satisfy the assumptions of Theorem 5, there exists an $L_2$-coloring $\psi$ of $G_2$. Colorings $\varphi$ and $\psi$ coincide on $v$; hence $\varphi \cup \psi$ is an $L$-coloring of $G$, a contradiction.

Claim 3. (1) $|N(v) \cap V(P)| \leq 1$ for all $v \in I$, and (2) $V(F) \setminus (I \cup V(P)) \neq \emptyset$.

Proof. Claim (1) and condition (ii) imply (1). Using Claim (2), $F - P$ is a path. Since $I$ is independent, if $V(F) \subseteq I \cup V(P)$ then $|I \cap V(F)| = 1$, contradicting (1).

Claim 4. $G$ does not contain a separating triangle with a vertex in $I$.

Proof. Let $T = xyz$ be a separating triangle in $G$ and let $x \in I$. Assume that $P \subseteq \text{Ext}_T(G)$ and $|V(\text{Int}_T(G))| \geq 4$. By the minimality of $G$, there exists an $L$-coloring $\varphi$ of $\text{Ext}_T(G)$.

Let $G' := \text{Int}_T(G) - z$, $I' := I \setminus V(\text{Ext}_T(G))$ and $P' = xy$. Define a list assignment $L'$ on vertices $u \in V(G')$ in the following way:

$$L'(u) = \begin{cases} \{\varphi(u)\} & \text{if } u \in \{x, y\}, \\ L(u) \setminus \{\varphi(z)\} & \text{if } uz \in E(G - P'), \\ L(u) & \text{otherwise}. \end{cases}$$

Since $x \in I$, no neighbor of $x$ is in $I'$. Thus condition (iii) of Theorem 5 is satisfied for $G'$, $I'$, $P'$ and $L'$. Condition (iii) is witnessed by $\varphi$. Since each vertex $u \in N_{G'}(z)$ is on the outer face of $G'$, but not $G$, it is straightforward to check that (iii) is satisfied. Hence $G'$ has an $L'$-coloring $\varphi$. The coloring $\varphi \cup \psi$ is an $L$-coloring of $G$, a contradiction.

Claim 5. If $xy$ is a chord of $F$ then neither $x$ nor $y$ is in $V(P)$, and there exists $z \in I \cap V(F)$ such that $|L(z)| = 2 = d(z)$, $L(zx) \neq L(zy)$, and $xyz \subseteq F$.

Proof. Suppose $xy \in E$ is a chord of $F$. Let $G_1$ and $G_2$ be subgraphs of $G$ where $G_1 \cap G_2 = xy$ and $G_1 \cup G_2 = G$. Since $xy$ is a chord, both $G_1$ and $G_2$ have at least three vertices. By symmetry assume that $P \subseteq G_1$.

First suppose $G_2$ contains exactly three vertices, say $x, y, z$. Using Claim (1), $2 \leq |L(z)| = |L(zx) \cup L(zy)| \leq d(z) \leq 2$. So $|L(z)| = 2 = d(z)$ and $L(zx) \neq L(zy)$. By condition (i),
\[|L(z)| = 2 \] implies \( z \in I \cap V(F) \). Thus \( xy \subseteq F \), since \( x \) and \( y \) are the only possible neighbors of \( z \). Finally, since \( L(xz) \neq L(zy) \), Claim 13 implies \( L(xy) \not\subseteq L(xz) \cup L(zy) \). Thus \( |L(x)|, |L(y)| \geq 2 \), and so \( x, y \not\in P \).

Now suppose for a contradiction that \( G_2 \) has at least four vertices. Define \( G'_1 \) in the following way. If there exists a vertex \( v \in V(G_2) \cap I \) such that \( v \) is adjacent to both \( x \) and \( y \) and \( L(xv) \neq L(yv) \) then \( G'_1 \) is obtained from \( G_1 \) by adding a new vertex \( v' \) adjacent to \( x \) and \( y \) to the outer face of \( G_1 \). Moreover, let \( I' = (I \cap V(G_1)) \cup \{v'\} \) and let \( L' \) be an extension of \( L \) by defining \( L'(v') = L(xv) \cup L(yv) \). Notice that \( v \) is unique if it exists, since Claim 4 implies \( G \) has no separating triangles that contain a vertex of \( I \). If no such \( v \) exists, let \( G'_1 = G_1 \), \( L' = L \), and \( I' = I \cap V(G_1) \). If \( G'_1 \) contains \( v' \), neither \( x \) nor \( y \) is in \( I \). Hence \( I' \) is indeed an independent set. Using that \( v' \in I' \) is on the outer face, \( L' \) satisfies conditions (iii). By the minimality of \( G \), there exists an \( L' \)-coloring \( \varphi \) of \( G'_1 \) which gives an \( L \)-coloring of \( G_1 \).

Define a list assignment \( L_2 \) on \( V(G_2) \) by \( L_2(u) = \{\varphi(u)\} \) if \( u \in \{x, y\} \), else \( L_2(u) = L(u) \). We wish to use \( xy \) as \( P \). Conditions (iii) of Theorem 5 hold since \( G \) satisfies them. For (iii), consider a vertex \( w \in I \) with \( \{x, y\} \subseteq N(w) \). As remarked above, \( w = v \). If \( G' \) contains \( v' \) then since \( L'(v') = L(xv) \cup L(yv) \), and \( \varphi \) is an \( L' \)-coloring of \( G'_1 \), there exists \( u \in \{x, y\} \) with \( \varphi(v') \in L(uv) \). Then \( \varphi(v') \neq \varphi(u) \) implies \( \varphi(u) \notin L(v) \), and (iii) holds. If \( G' \) does not contain \( v' \), then \( L(xv) = L(yv) \) and \( \varphi(x) \neq \varphi(y) \) imply there exists \( u \in \{x, y\} \) such that \( \varphi(u) \notin L(v) \), and (iii) holds. By the minimality of \( G \), there exists an \( L_2 \)-coloring \( \psi \) of \( G_2 \).

Colorings \( \varphi \) restricted to \( G_1 \) and \( \psi \) coincide on \( xy \); hence \( \phi \cup \psi \) is an \( L \)-coloring of \( G \), a contradiction.

By the minimality of the sum of the sizes of the lists, we can assume \( |V(P)| \geq 1 \). Let \( F = v_0v_1v_2v_3 \ldots v_t \), where \( v_0 \in V(P) \subseteq \{v_0, v_1\} \), identifying index \( i \) with index \( i + t + 1 \). Choose \( v_i \in V(F) \setminus (I \cup V(P)) \) with minimum index \( i \). Such an index exists by Claim 3. Claim 5 implies \( v_i, v_{i-2} \) is not a chord, and condition (ii) implies \( L(v_i) - L(v_i, v_{i-1}) - L(v_i, v_{i+1}) \neq \emptyset \).

Select a set \( X \subseteq \{v_i, v_{i+1}, v_{i+2}\} \) and an \( L \)-coloring \( \varphi \) of \( X \) by the following rules:

(X1) If \( v_i, v_{i+2} \) is not a chord then set \( X = \{v_i\} \) and pick \( \varphi(v_i) \in L(v_i) \setminus (L(v_{i-1}) \cup L(v_{i+1})) \).

(X2) Else, if there is \( c \in L(v_i) \setminus (L(v_{i-1}) \cup L(v_{i+1}) \cup L(v_{i+2})) \), then set \( X = \{v_i\} \) and \( \varphi(v_i) = c \).

(X3) Else set \( X = \{v_i, v_{i+1}, v_{i+2}\} \). Pick:

(a) \( \varphi(v_{i+2}) \in L(v_{i+2}) \setminus (L(v_{i+3}) \cup L(v_{i+4})) \) if \( v_{i+2}, v_{i+4} \in E(G) \)

\[ \varphi(v_{i+2}) \in L(v_{i+2}) \setminus L(v_{i+3}) \] otherwise;

(b) \( \varphi(v_i) \in L(v_i, v_{i+2}) \) if \( \varphi(v_{i+2}) \notin L(v_{i+2}) \); else \( \varphi(v_i) \in L(v_i, v_{i+1}) \);

(c) \( \varphi(v_{i+1}) \in L(v_{i+1}) - \varphi(v_i) - \varphi(v_{i+2}) \).

See Figure 1 for an illustration of these rules. Observe that exactly one of (X1), (X2), or (X3) applies and \( X \) is well defined. Also, in cases (X2) and (X3), Claim 5 implies \( v_{i+1} \in I \), and so \( v_{i+2} \not\in I \). Thus the sizes of their lists are as claimed in Figure 1. In (X3) either \( \varphi(v_i) \in L(v_i, v_{i+2}) \) or \( \varphi(v_{i+2}) \in L(v_i, v_{i+2}) \). Also, by Claims 13 and 5, \( L(v_i, v_{i+2}) \not\subseteq L(v_{i+1}) \).

Hence \( \varphi \) is also well defined. Moreover, \( d(v_{i+1}) = 2 \), and so \( N(v_{i+1}) = \{v_i, v_{i+2}\} \).
Let $G' = G - X$, $I' = I \setminus X$, and $L'$ be the list assignment on $V(G')$ defined by

$$L'(v) = L(v) \setminus \{\varphi(x) : x \in N(v) \cap X\}.$$ 

It suffices to show that $G'$, $L'$, $I'$ and $P$ satisfy the assumptions of Theorem 5. Then by the minimality of $G$, there is an $L'$-coloring $\psi$ of $G'$, and by the choice of $L'$, the function $\psi \cup \varphi$ is an $L$-coloring of $G$, a contradiction.

Now we verify that $G'$, $L'$, $I'$ and $P$ satisfy the assumptions of Theorem 5. Since $I$ is an independent set, so is $I'$. Let $M = \{v \in V(G') : L'(v) \neq L(v)\}$. Clearly condition (i) holds for vertices in $V \setminus M$. By Claim 5, all chords have the form $v_jv_{j+2}$. Thus $\varphi$ was chosen so that $M \cap V(F) = \emptyset$. Hence the condition (i) is satisfied for $v \in V(F)$. Condition (ii) holds since $P$ did not change. Since $I' \subseteq I$, Claim 3(1) implies condition (iii).

It remains to show that every $v \in M$ satisfies condition (i). Let $F'$ be the outer face of $G'$. Since each vertex of $M$ has a neighbor in $X \subseteq F$, $M \subseteq F' \setminus F$. Thus it suffices to show that $|L'(v)| \geq |L(v)| - 1$. If $|N(v) \cap X| = 1$ then $|L'(v)| \geq |L(v)| - 1$. Otherwise $|N(v) \cap X| \geq 2$. Then $v$ is handled by rule (X3). So $N(v) \cap X = \{v_i, v_{i+2}\}$ and $L(v_iv_{i+2}) \subseteq C := \{\varphi(v_i), \varphi(v_{i+2})\}$. If $L(v_iv_i) \neq L(v_iv_{i+2})$ then Claim 1(3) implies $L(v_i, v_{i+2}) \not\subset L(v)$. Anyway, $|L(v) \cap C| \leq 1$, and we are done.

Figure 1: Rules (X1), (X2), and (X3). A black circle is a vertex with arbitrary list size, a white circle is a vertex with list size two, and a triangle is a vertex with list size at least three. The dashed box indicates $X$ and a label on an edge is the common color of lists of its endpoints.
3 Lists of size 3 are necessary

In this section we give a proof of Theorem 4. The construction is analogous to the construction that bipartite graphs are not 2-choosable.

Proof of Theorem 4. Let \( k \) be given. Let \( G \) be a complete bipartite graph with part \( X \) of size \((k - 1)^2\) and another part of size 2 formed by vertices \( a \) and \( b \). Let \( L \) be a list assignment assigning to \( a \) a list of colors \( \{a_1, \ldots, a_{k-1}\} \) and to \( b \) a list of colors \( \{b_1, \ldots, b_{k-1}\} \). To the other vertices, \( L \) assigns distinct lists of form \( \{a_i, b_j\} \) where \( 1 \leq i, j \leq k - 1 \). There are \((k - 1)^2\) such lists which is exactly the size of \( X \). Notice that \( |L(u) \cup L(v)| = k \) for every edge \( uv \). See Figure 2 for a sketch of \( G \) and \( L \).

\[
\begin{align*}
\{a_1, a_2, \ldots, a_{k-1}\} \\
\{a_1, b_1\} & \cdots & \{a_{k-1}, b_{k-1}\} \\
\{b_1, b_2, \ldots, b_{k-1}\}
\end{align*}
\]

Figure 2: Construction from Theorem 4

Suppose that there is an \( L \)-coloring of \( G \). It assigns colors \( a_i \) to \( a \) and \( b_j \) to \( b \) for some \( 1 \leq i, j \leq k - 1 \). However, there is a vertex with list \( \{a_i, b_j\} \), a contradiction. Hence \( G \) is not \( L \)-colorable.

\[\square\]

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