MULTIDIMENSIONAL BLACK HOLE SOLUTIONS IN MODEL WITH PERFECT FLUID

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A family of black-hole solutions in the model with 1-component perfect fluid is obtained. The metric of any solution contains \((n-1)\) Ricci-flat “internal space” metrics and for certain equations of state \((p_i = \pm \rho)\) coincides with the metric of black brane (or black hole) solution in the model with antisymmetric form. Certain examples (e.g. imitating \(M2\) and \(M5\) black branes) are considered. The post-Newtonian parameters \(\beta\) and \(\gamma\) corresponding to the 4-dimensional section of the metric are calculated.

1 Introduction

Black brane solutions (see, for example, [1] and references therein) defined on product manifolds \(\mathbb{R} \times M_0 \times \ldots \times M_n\) are widely interested now. The solutions appear in the models with antisymmetric forms and scalar fields. These and more general \(p\)-brane cosmological and spherically symmetric solutions are usually obtained by the reduction of the field equations to the Lagrange equations corresponding to Toda-like systems [3]. An analogous reduction for the models with multicomponent perfect fluid was done earlier in [5]. For cosmological-type models with antisymmetric forms without scalar fields any \(p\)-brane is equivalent to an anisotropic perfect fluid with the equations of state:

\[
p_i = -\rho \quad \text{or} \quad p_i = \rho,
\]

when the manifold \(M_i\) belongs or does not belong to the brane worldvolume, respectively (here \(p_i\) is the pressure in \(M_i\) and \(\rho\) is the density).

The aim of this paper is to find explicitly the analogues of black brane (or black hole) solutions in the model with 1-component perfect fluid and extend it to more general equations of state.

The paper is organized as follows. In Section 2 the model is formulated. In Section 3 the general black hole solutions are presented. Section 4 deals with certain analogues of black brane solutions, e.g. the Reissner-Nordström solution and \(M2\)- and \(M5\)- black brane solutions. In Section 5 the post-Newtonian parameters for the 4-dimensional section of the metric are calculated.

2 Model

In this paper we consider a family of spherically symmetric solutions to Einstein equations with a perfect-fluid matter source

\[
R^M_N - \frac{1}{2} \delta^M_N R = k T^M_N
\]

defined on the manifold

\[
M = \mathbb{R} \times (M_0 = S^{d_0}) \times (M_1 = \mathbb{R}) \times \ldots \times M_n,
\]

with the block-diagonal metrics

\[
ds^2 = e^{2\gamma(u)} du^2 + \sum_{i=1}^{n} e^{2\chi^i(u)} h^{(i)}_{m,n} dy^m dy^n.
\]
Here $R = (a, b)$ is interval. The manifold $M_i$ with the metric $h^{(i)}$, $i = 1, 2, \ldots, n$, is a Ricci-flat space of dimension $d_i$:
\[ R_{m,n_i}[h^{(i)}] = 0, \]
and $h^{(0)}$ is standard metric on the unit sphere $S^{d_0}$
\[ R_{m_0,n_0}[h^{(0)}] = (d_0 - 1)h^{(0)}_{m_0,n_0}, \]
u is radial variable, $\kappa$ is the gravitational constant, $d_1 = 1$ and $h^{(1)} = -dt \otimes dt$.

The energy-momentum tensor is adopted in the following form
\[ (T^M_N) = \text{diag}(-\hat{\rho}, \hat{p}_0 \delta_{m_0}^m \delta_{n_0}^n, \hat{p}_1 \delta_{m_1}^m \delta_{n_1}^n, \ldots, \hat{p}_n \delta_{m_n}^m \delta_{n_n}^n), \]
where $\hat{\rho}$ and $\hat{p}_i$ are "effective" density and pressures, respectively, depending upon the radial variable $u$. We also impose the following equation of state
\[ \hat{p}_i = \left(1 - \frac{2U_i}{d_i} \right) \hat{\rho}, \]
where $U_i$ are constants, $i = 0, 1, 2, \ldots, n$.

The physical density and pressures are related to the effective ("hat") ones by formulas
\[ \rho = -\hat{\rho}, \quad p_u = -\hat{\rho}, \quad p_i = \hat{p}_i, \quad (i \neq 1). \]

In what follows we put $\kappa = 1$ for simplicity.

### 3 Black hole solutions

We will make some natural assumptions:

1. $U_0 = 0 \iff \hat{p}_0 = \hat{\rho}$,
2. $U_1 = 1 \iff \hat{p}_1 = -\hat{\rho}$,
3. $(U, U) = U_i G^{ij} U_j > 0$,

where
\[ G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}, \]
are components of the matrix inverse to the matrix of the minisuperspace metric
\[ (G_{ij}) = (d_i \delta_{ij} - d_i d_j), \]
and $D = 1 + \sum_{i=0}^{n} d_i$ is the total dimension. (It may be proved that the restriction 3 follows from the first ones: 1 and 2.)

It follows from (2.7) and restriction 1 that
\[ \rho = -\hat{\rho}, \quad p_u = -\hat{\rho}, \quad p_0 = \rho. \]

Under the relations (2.7) and (3.1) imposed we obtained the following black-hole solutions to the Einstein equations (2.1):
\[ ds^2 = J_0 \left( \frac{dr^2}{1 - \frac{2\rho}{r_d}} + r^2 d\Omega_{d_0}^2 \right) - J_1 \left( 1 - \frac{2\mu}{r_d} \right) dt^2 + \sum_{i=2}^{n} J_i h^{(i)}_{m_i,n_i} dy^{m_i} dy^{n_i}, \]
\[ \rho = \frac{d^2 v^2 P(P + 2\mu)}{2H^2 J_0 r^{2d_0}}, \]
that may be verified by analogy with the $p$-brane solution (the detailed treatment will be in a separate publication). Here $d = d_0 - 1$, $d\Omega_{d_0}^2 = h^{(0)}_{m_0,n_0} dy^{m_0} dy^{n_0}$ is spherical element, the metric factors
\[ J_i = H^{-2v^2 U_i}, \quad H = 1 + \frac{P}{r_d}. \]
\( P > 0, \mu > 0 \) are integration constants and

\[
U^i = G^{ij}U_j = \frac{U_i}{a_i} + \frac{1}{2 - D} \sum_{j=0}^{n} U_j, \quad \nu = (U, U)^{-1/2}.
\]

(3.7)

(3.8)

Using (3.7) and the first assumption from (3.1) one can rewrite (3.5) as follows

\[
ds^2 = J_0 \left\{ \frac{dr^2}{1 - \frac{2\mu}{r}} + r^2 d\Omega_2 - H^{-2\nu^2} \left( 1 - \frac{2\mu}{r} \right) dt^2 + \sum_{i=2}^{n} H^{-2\nu^2} U_{i/d} h_{m,i}^m dy^m dy^n \right\}.
\]

(3.9)

4 Imitation of black brane solutions

Here we consider certain examples of solutions with metrics of charged black hole and \( M \)-branes.

4.1 Reissner-Nordström solution

Let us consider the 4-dimensional space-time manifold \( \mathbb{R} \times S^2 \times \mathbb{R} \). The metric and the density from (3.9) and (3.5) read

\[
ds^2 = H^2 \left\{ \frac{dr^2}{1 - \frac{2\mu}{r}} + r^2 d\Omega_2 - H^{-2} (1 - \frac{2\mu}{r}) dt^2 \right\}, \quad \rho = \frac{\rho^3}{H^{3r^4}}.
\]

(4.1)

(4.2)

By changing the variable \( r = r' - P \) we obtain a standard Reissner-Nordström metric with the charge squared \( Q^2 = P(P + 2\mu) \) and the gravitational radius \( GM = P + \mu \).

4.2 Analogues of \( M \)-brane solutions.

Here we consider the case \( D = 11 \) and \( n = 3 \).

- **M2 black brane.** For \( U_2 = d_2 = 2, U_3 = 0 \) we get from (3.9):

\[
ds^2 = H^2 \left\{ \frac{dr^2}{1 - \frac{2\mu}{r}} + r^2 d\Omega_2^2 - H^{-1} (1 - \frac{2\mu}{r}) dt^2 + H^{-1} h_{m_2}^{(2)} dy^m dy^n + h_{m_3}^{(3)} dy^m dy^n \right\}.
\]

(4.3)

This relation corresponds to the metrics of the electric \( M2 \) black brane solution in 11-dimensional supergravity [8,9]. The density (3.5) has the following form:

\[
\rho = \frac{d^2 P(P + 2\mu)}{4H^{7/3} r^{2d_0}}.
\]

(4.4)

- **M5 black brane.** Let us consider another example in \( D = 11 \) with \( U_2 = d_2 = 5, U_3 = 0 \). The metric reads

\[
ds^2 = H^2 \left\{ \frac{dr^2}{1 - \frac{2\mu}{r}} + r^2 d\Omega_2^2 - H^{-1} (1 - \frac{2\mu}{r}) dt^2 + H^{-1} h_{m_2}^{(2)} dy^m dy^n + h_{m_3}^{(3)} dy^m dy^n \right\}.
\]

(4.5)

and the density is as follows

\[
\rho = \frac{d^2 P(P + 2\mu)}{4H^{7/3} r^{2d_0}}.
\]

(4.6)

The metric coincides with that of well-known \( M5 \) solution [8,9].
5 Physical parameters

5.1 Gravitational mass and post-Newtonian parameters

Here we put \( d_0 = 2 \) \((d = 1)\). Let us consider the 4-dimensional space-time section of the metric (3.9). Introducing a new radial variable by the relation:

\[
    r = R \left(1 + \frac{\mu}{2R}\right)^2,
\]

we rewrite the 4-section in the following form:

\[
    ds^2_{(4)} = g^{(4)}_{\mu\nu} dx^\mu dx^\nu = H^{-2\nu^2 U^0} \times \left[-H^{-2\nu^2} \left(1 + \frac{\mu}{2R}\right)^2 dt^2 + \left(1 + \frac{\mu}{2R}\right)^4 \delta_{ij} dx^i dx^j \right],
\]

\(i, j = 1, 2, 3\). Here \( R^2 = \delta_{ij} x^i x^j \).

The parametrized post-Newtonian (Eddington) parameters are defined by the well-known relations

\[
    g^{(4)}_{00} = -(1 - 2V + 2\beta V^2) + O(V^3), \quad \beta = 0 \quad \text{(5.3)}
\]

\[
    g^{(4)}_{ij} = \delta_{ij} (1 + 2\gamma V) + O(V^2), \quad \gamma = 0 \quad \text{(5.4)}
\]

\(i, j = 1, 2, 3\). Here \( V = \frac{GM}{R} \) is the Newtonian potential, \( M \) is a gravitational mass and \( G \) is the gravitational constant. From (5.2)-(5.4) we obtain:

\[
    GM = \mu + \nu^2 P(1 + U^0) \quad \text{(5.5)}
\]

and

\[
    \beta - 1 = \frac{\nu^2 P (P + 2\mu)}{2 (GM)^2} (1 + U^0), \quad \gamma - 1 = -\frac{\nu^2 P}{GM} (1 + 2U^0), \quad \text{(5.6)}
\]

For fixed \( U_i \) the parameter \( \beta \) is proportional to the ratio of two physical parameters: the perfect fluid density parameter \(|A| = \frac{1}{2} \nu^2 P(P + 2\mu)\), and the gravitational radius squared \((GM)^2\).

5.2 Hawking temperature

The Hawking temperature of the black hole may be calculated using the relation from \[7\] and has the following form:

\[
    T_H = \frac{d}{4\pi (2\mu)^{1/d}} \left( \frac{2\mu}{2\mu + P} \right)^{\nu^2}. \quad \text{(5.8)}
\]

6 Conclusions

We have obtained a family of black-hole solutions in the model with 1-component perfect fluid with the equation of state (2.7) and the relations (3.1) imposed. The metric of the solutions contains \((n - 1)\) Ricci-flat “internal” space metrics. For certain equation of state (with \( p_i = \pm \rho \)) the metric of solution may coincide with that of black brane (or black hole) solution (in the model with antisymmetric forms without dilatons). Here we suggested certain examples imitating 4-dimensional charged black hole and \( M2, M5 \) black brane solutions in \( D = 11 \) supergravity.

Here we have calculated the post-Newtonian parameters \( \beta \) and \( \gamma \) corresponding to the 4-dimensional section of the metric. The parameter \( \beta \) is written in terms of ratios of the physical parameters: the perfect fluid parameter \(|A|\) and the gravitational radius squared \((GM)^2\).

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