There is no anomaly in the nonlocality of two entangled qutrits

E. A. Fonseca and Fernando Parisio
Departamento de Física, Universidade Federal de Pernambuco, 50670-901, Recife, Pernambuco, Brazil

There is no controversy on the assertion that entanglement and nonlocality are distinct resources. It is acknowledged that a clear illustration of this fact is the discrepancy between maximally entangled states and states that maximally violate a Bell inequality (usually identified with maximally nonlocal states), for systems composed by two qutrits. We argue that this anomaly is an artefact of the measures that have been used to quantify Bell nonlocality. By reasoning that the numeric value of a Bell function is akin to a witness rather than a quantifier, we define a measure of nonlocality and show that, for two qutrits and two four-dimensional Hilbert spaces, maximal entanglement does correspond to maximal nonlocality. As a consequence of our proposal, the question whether violations in local causality grow with the dimension of Hilbert spaces is reopen.

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INTRODUCTION

Entanglement \( [1,2] \) is behind some of the most perplexing physical effects ever observed. In spite of this, it alone, may be regarded as a purely mathematical concept: the failure of a vector in a Hilbert space \( \mathcal{H} \) to be factorized as a single product of vectors in spaces \( \mathcal{H}_i \), that together form \( \mathcal{H} = \bigotimes_i \mathcal{H}_i \). For mixed states there is a corresponding definition in terms of convex sums of explicitly separable density operators \( \rho \). Nonlocality \( [3,6] \), in contrast, also refers to the experimental scheme that is used to investigate the entangled state and, thus, to angles of Stern-Gerlach apparatuses or to the spatial disposition of beam splitters, for instance. That is to say, entanglement happens in Hilbert spaces while nonlocality manifests itself in our ordinary \((3+1)D\) space (see however \( [7] \)). Therefore, from a physical perspective, to carefully quantify nonlocality is as important as to seek entanglement measures. In fact, a faulty estimation of the extent of nonlocality embodied by a physical situation (system + context) may lead to deceptive conclusions.

In an essay in honor of Abner Shimony, Gisin provides a list of questions related to Bell inequalities \( [8] \). The one closely related to our goal in this letter is “Why are almost all known Bell inequalities for more than 2 outcomes maximally violated by states that are not maximally entangled?” (for an exception see \( [9] \)). This fact originally reported in \( [10] \) and referred to as an anomaly \( [11] \) of nonlocality, has received a great deal of attention in the literature \( [3,12,10] \). To investigate this issue, we begin by addressing the tacit association often made between states that maximally violate a Bell inequality and maximally nonlocal states.

Given a particular Bell function \( I \), let us generically denote the associated inequality by

\[
I \leq \xi,
\]

(1)

\( \xi \) representing the bound imposed by local causality (we use this terminology instead of “local realism” \( [17] \)). Then, if a state satisfies \( \xi \) for all possible settings of the measurement apparatus, it is local with respect to the inequality. Otherwise, the state is said to be nonlocal. Recall that the Werner matrices \( [4] \), which are local with respect to the CHSH inequality, are nonlocal if more complex measurements are considered \( [18] \). A context independent definition of locality must be exhaustive: a state is said to be local, with no further qualifiers, if it is local with respect to all conceivable Bell inequalities.

So, the notion of Bell nonlocality is quite neatly defined. The sensitive question is about ordering. What should one imply by asserting that state \( \rho \) is more nonlocal than state \( \sigma \)? A common answer is that \( \rho \) is more nonlocal than \( \sigma \) if \( I_{\text{max}}(\rho) \) is larger than \( I_{\text{max}}(\sigma) \). The maximum being determined by scanning all possible settings. Although it is known that for any Bell function one can find another, equivalent function that arbitrarily increases the numerical value of the maximal violation \( [6] \), it is acknowledged that carefully normalised Bell inequalities may provide objective figures to quantify nonlocality. In the next section we reason against this view.

Insightful alternatives have been put forward in the last two decades. The tolerance of nonclassical correlations against noise has been considered as an operational measure of nonlocality \( [10] \), but this approach is not consensual \( [10] \). In \( [20] \), it is shown that optimal Bell tests occur for states that are neither maximally entangled nor maximally violating. The (statistically) optimal state found by the authors is the most suitable to unveil nonlocality, given that the experimentalist can only perform a finite number \( N \) of realisations, as is always the case. However, at least in principle, one should be allowed to think of the limit \( N \to \infty \), as we do with many other concepts in quantum theory. In this limit all nonlocal states can be safely devised. In a different framework, the communication cost for a local model to reproduce the quantum correlations has also been used as a task-based quantifier of nonlocality \( [21,22] \).

Recently, two measures have been experimentally implemented in \( [24] \); the first one is based on how far is the
state from the set of local correlations, and, the second is a measure also related to the amount of communication needed to establish correlations. In [23] a nonlocality quantifier has been defined, such that in certain scenarios it is inversely related to concurrence. Some other proposals to quantify nonlocality can be found in the literature, focusing on multipartite systems [26, 27] and presenting nonlocality as a concept derivable from a notion of “irreality” [28].

Common to most of these previous works is the fact that the different figures of merit associated (or even identified) to nonlocality attain their maxima for non-maximally entangled states. This, of course, does not violate any logical necessity, but, one should not refrain from a critical assessment of this, arguably, counterintuitive finding.

In this letter we define a measure of nonlocality which indicates that the anomaly that appears to exist for two entangled three- and four-level systems is, indeed, an artefact of the previous definitions. Our suggestion seems entangled three- and four-level systems is, indeed, an indication that the anomaly that appears to exist for two nonlocality attain their maxima for non-

**MEASURE OF BELL-NONLOCALITY**

A tenable reasoning about the quantification of nonlocality is that some clue might come from nonlocal hidden variable (NLHV) models capable of reproducing the quantum correlations. For example, one could say that a state is more nonlocal than another if the underlying NLHV model violates local causality in different degrees for these different states. This, however, cannot be inferred from these models in any obvious way. The distance between subsystems does not enter in the Bell functions, after all. As a first illustration we refer to the model developed by Bell in his seminal paper [29]. It is simply assumed that “the results of measurements with one magnet now depend on the setting of the distant magnet [...]”. The mutual influence between the subsystems being instantaneous, no matter the numerical value assumed by the Bell function. In a more general picture, consider the NLHV theory *par excellence*, Bohmian mechanics [31]. The so-called quantum potential does not react faster or slower, for different states, under a measurement on one of the subsystems. In particular, this holds for two entangled rotors of spin-1/2 [31]. The fact that, for a given Bell inequality, some of these instantaneous interactions are related to non violating states must be understood in the light of the generalised Gisin’s theorem: all bipartite $N \times N$ entangled states violate some Bell inequality [32, 33]. The action at a distance appears to be equally spooky for all nonlocal states within these NLHV models.

Even for theories relying on finite (superluminal) signalling speed, the relation $I_{\text{max}}(\rho) > I_{\text{max}}(\sigma) > \xi$ does not necessarily imply that $v_\rho > v_\sigma > c$, where $v$ is the signal velocity associated to each state and $c$ is the speed of light. This reasoning suggests that all violating states for a particular setting are equally nonlocal, that the essential information provided by a Bell inequality is of a seemingly Boolean nature, a state being either local or nonlocal with respect to that setting, with no gradations.

This apparently all-or-nothing picture, however, does not lead to a dead-end. On the contrary, it points to a conceptually simple solution.

Given a state and a specific Bell inequality, the most exhaustive experimental procedure one can go through is to investigate violations in local causality for all settings. Based on our previous discussion, it is meaningful to state that $\rho$ is more nonlocal than $\sigma$ if the former violates the inequality, *by any extent*, for a larger amount of settings than the latter. This statement can be cast in very simple statistical terms: $\rho$ is more nonlocal than $\sigma$ if, for an unbiased random choice of settings, the probability to obtain a violation is larger for $\rho$.

To formalise this idea, we define the space $\mathcal{X} = \{x_1, \ldots, x_n\}$ of all possible settings for a given (preferably tight) Bell inequality, $n$ being the number of independent parameters. For a particular state $\rho$, let $\Gamma_{\rho} \subset \mathcal{X}$ be the set of points that lead to violation and $V(\rho)$ be proportional to the volume of $\Gamma_{\rho}$. We say that if $V(\rho) > V(\sigma)$, then $\rho$ is more nonlocal than $\sigma$, with

$$V(\rho) \equiv \frac{1}{N} \int_{\Gamma_{\rho}} d^nx,$$  \hspace{1cm} (2)

where $\mathcal{N}$ is a normalisation constant. We call $V$ the *volume of violation*. Hereafter we focus on the important case where the experimental settings are such that $\mathcal{X}$ is a bounded set. We remark that the numeric calculations needed to determine the volume of violation are the paradigmatic problem for which Monte Carlo methods are intended [34]. Notice that the above definition has no relation to the volume of the set of separable states defined in [35]. The volume of violation is an integration over settings for each state and each Bell inequality.

As an initial test, we consider the CHSH inequality [36] for two entangled qubits in pure and mixed states. In this case the Bell function depends on four unit vectors: $I_{\text{CHSH}}(\hat{a}, \hat{b}, \hat{c}, \hat{d}) = E(\hat{a}, \hat{b}) - E(\hat{a}, \hat{d}) + E(\hat{c}, \hat{d}) + E(\hat{c}, \hat{b})$, with $E$ being a correlation function defined for a pair of directions. We can write it more explicitly in terms of eight angular parameters, $I_{\text{CHSH}} = I(\theta_a, \varphi_a, \theta_b, \varphi_b, \theta_c, \varphi_c, \theta_d, \varphi_d)$, $\mathcal{X}$ corresponding to the cartesian product of four unit spheres, yielding $d^n x = d\Omega_a d\Omega_b d\Omega_c d\Omega_d$, with $\Omega_a = \sin \theta_a d\theta_a d\varphi_a$. We found that the maximally entangled state maximises, both, $I$ and $V$. In Fig. 2 (a) we show these quantities along with the entropy of entanglement for the family of pure states

$$|\psi_\alpha\rangle = \alpha |00\rangle + \sqrt{1-\alpha^2} |11\rangle,$$  \hspace{1cm} (3)
as functions of \( \alpha \). The volume \( V \) is rather sensitive to variations of \( \alpha \), presenting the steepest descent from its maximum at \( \alpha = 1/\sqrt{2} \). In Fig. 1(b) we plot the concurrence \( C(\alpha) \) and \( V(\alpha) \equiv V(\rho_\alpha) \) of the noisy state \( \rho_\alpha = (1 - F) |\psi_\alpha\rangle \langle \psi_\alpha| + F 1/4 \), where \( F \) is the \( 4 \times 4 \) identity operator and \( F \) is the white noise fraction. Note that the volume of violation is more fragile against noise than entanglement. Around a noise fraction of \( F \approx 0.3 \), nonlocality, as rendered by \( V \), completely disappears.

![FIG. 1: (color online) In panel (a) we show the entropy of entanglement (circles), \( I_{\max} \) (triangles), and \( V \) (squares) as functions of \( \alpha \) for the pure state \( |\psi_\alpha\rangle \). In (b) noise is considered and the plots correspond to \( V(\alpha) \) compared to the concurrence \( C(\alpha) \) for different values of the noise fraction \( F \).](image)

So far, the volume of violation gives no sensible new information in comparison to the maximum of the Bell-CHSH function, yet, it is consistent with our expectations on what should be a nonlocality measure in the safe terrain of two entangled qubits. This agreement between \( V \) and \( I_{\max} \) ceases to happen when two higher dimensional systems are considered, even in the pure case.

### TWO QUTRITS

Now we consider two entangled qutrits, i.e., a composite system with Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), \( \dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 3 \). Let \( \{|0\rangle, |1\rangle, |2\rangle\} \) be an orthonormal basis in \( \mathcal{H}_i \) (\( i = 1, 2 \)) and consider the three-outcome observables \( A_{i,a} \) for system 1, and \( B_b \) for system 2 (\( a, b = 1, 2 \)). It has been shown in \([30]\) that local hidden variable models must satisfy the tight inequality

\[
I_3 = P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1) - P(A_1 = B_1 - 1) - P(B_1 = A_2) - P(B_2 = A_1 - 1) \leq 2 \quad (4)
\]

where the arguments of the probabilities above are taken modulo 3, e.g., \( P(B_1 = A_2 + 1) = P(B_1 = 0, A_2 = 1) + P(B_1 = 1, A_2 = 2) + P(B_1 = 2, A_2 = 0) \). As in \([10]\) let us focus on the family of pure states

\[
|\Psi_\gamma\rangle = \frac{1}{\sqrt{2 + 3 \gamma^2}} (|00\rangle + \gamma |11\rangle + |22\rangle) \quad (5)
\]

The maximal entropy of entanglement is, naturally, given by \( \gamma = 1 \), while it was shown that \( \theta \) is maximally violated by the state with \( \gamma \equiv \gamma \approx 0.792 \) \([11]\). This, arguably, unexpected result is cause of puzzlement for, at least, part of the community working on quantum foundations. Entanglement and Bell nonlocality are, indeed, physically distinct, but, the fact is that the former constitutes the sole source of the latter (we exclusively refer to Bell nonlocality \([40]\)). Thus, it is not unreasonable to expect that, for two sufficiently close states, equal entanglement should lead to equal Bell nonlocality.

Let us now apply measure \([2]\) to this problem. Note carefully that general Stern-Gerlach-type measurements on a pair of spin-1 particles only demand eight parameters. However, these measurements do not reveal the whole richness of the Hilbert space \([19]\). For this reason, in order to calculate \( V(\gamma) \equiv V(|\Psi_\gamma\rangle \langle \Psi_\gamma|) \) one must perform an integration in a twelve-dimensional space, as we will see. Completely general unitary operations are achievable in the laboratory via multiport beam splitters \([41, 42]\). In this optical context the whole space of parameters can be visited by varying the reflectivity of beam splitters and the angle of phase shifters, for instance. From the family of states \([5]\), with these linear optical elements \([14]\), we can get:

\[
|\Psi_\gamma\rangle = \frac{1}{3} \sum_{j,k,l=0}^{2} \alpha_j e^{i(\phi_1(j)+\phi_2(j))} e^{i\frac{2\pi}{3}j(k+l)} |k l\rangle \quad (6)
\]

with \( a, b = 1, 2 \), and \( \alpha_0 = \alpha_2 = 1, \alpha_1 = \gamma \). The optimal set of parameters for violations of \([43]\) by the maximally entangled state has been determined in \([30, 43]\) and reads \( \phi_1(j) = 0, \phi_2(j) = \pi j/3, \varphi_1(j) = \pi j/6, \) 

![FIG. 2: (color online) Entropy of entanglement (circles), \( I_{3\max} \) (triangles), and \( V \) (squares) as functions of \( \gamma \) for state \([5]\). All quantities are normalised such that their maximal value is 1. The inset shows a zoom in of the region marked by the rectangle in dashed lines.](image)
and $\phi_2(j) = -\pi j/6$. The aforementioned surprise arises when, for these settings, one realises that the maximal violation as a function of $\gamma$ has a peak at $\gamma = \tilde{\gamma}$.

In Fig. 2 we compare our numerical calculations of $V(\gamma)$ to the normalised entropy of entanglement $E$ and to the maximum of $I_3$. The maxima of $E$ and $V$ coincide exactly at $\gamma = 1$, as can be seen in the inset, while $I_{3\text{max}}$ attains its maximum at $\gamma = \tilde{\gamma}$. This shows that the anomaly in the nonlocality of two entangled qutrits does not exist, if one adopts the volume of violation as the measure of nonlocality.

It is easy to understand what is going on. Although $|\Psi_3\rangle$ presents a more pronounced maximum of $I_{3\text{max}}$ in comparison to $|\Psi_1\rangle$, the nonlocality of the former is less robust, for, as we get farther away from the optimal setting in $\mathcal{X}$, $I_3(\Psi_3)$ falls off faster than $I_3(\Psi_1)$. This effect on the volume of violation is clearly illustrated in Fig. 3, where two-dimensional sections $[\phi_1(0) - \phi_2(2)]$ of $\Gamma$ are shown for $|\Psi_1\rangle$ (a) and for $|\Psi_3\rangle$ (b). The other parameters are set as $\phi_2(0) = \phi_2(1) = \pi j/6$, $\varphi(j) = 0$, the remaining angles keeping the optimal values. In this particular example the violation area for $\gamma = 1$ is about 14% larger than that of $\gamma = \tilde{\gamma}$. The scales are identical in both figures.

When white noise is present, the density operator reads

$$\rho_\gamma = (1 - F)|\Psi_\gamma\rangle\langle \Psi_\gamma| + F \frac{\mathbf{I}}{9}.$$  

We verified that the maxima of $I_{3\text{max}}$ and $V$ do not change their positions, with respect to the pure case, as the noise fraction is increased up to $F \approx 0.3$. For $F > 0.3$, again, $V$ vanishes leading the death of nonlocality.

Finally, to be sure that this conciliation between the maxima of entanglement and nonlocality is not an unlikely coincidence, we addressed the problem of two four-dimensional Hilbert spaces. We considered the following family of entangled states

$$|\Psi_{\lambda_1,\lambda_2}\rangle = \frac{1}{\Lambda}(100) + \lambda_1|11\rangle + \lambda_2|22\rangle + |33\rangle),$$

with $\Lambda = \sqrt{2 + \lambda_1^2 + \lambda_2^2}$. The CGLMP inequality is maximally violated by a state that is not maximally entangled. This state is given by $\lambda_1 = \lambda_2 \approx 0.739$, corresponding to $I_4 \approx 2.973$. We surveyed the volume of violation associated to $I_4$ in the region $(\lambda_1, \lambda_2) \in [0.6, 1.2] \times [0.6, 1.2]$. Once again, $V$ is maximal for $\lambda_1 = \lambda_2 = 1$ among all investigated states. In particular, the ratio of the volumes $V$ of the maximally entangled and maximally violating states is around 1.24.

**CONCLUSION AND PERSPECTIVES**

The consideration that entanglement and nonlocality are different resources has been pushed to a point that, possibly, does not correspond to the actual state of affairs. We argue that, given a state, a Bell inequality, and a particular setting, there should be no gradations of nonlocality, the inequality functioning as a witness. However, by “tracing over the settings”, attributing equal weight to all those that violate the inequality and weight zero to those that do not lead to violations, we showed that it is possible to quantify Bell nonlocality in a consistent way. In particular, within the context of our proposal, there is no discrepancy between maximally entangled and maximally nonlocal states, at least for entangled qutrits and also for systems composed of four-level subsystems.

Looking in perspective, one can go back to the meaning of the number $I_{\text{max}}$. It is, of course, reasonable to suppose that if $I_{\text{max}} - \xi$ is large then we should have a large volume of violation. Thus, the value at maximal violation may be seen as a quick indicator of the volume of violation itself, but not one to be blindly trusted.

**FIG. 3:** (color online) Sections $\phi_1(0) - \phi_2(2)$ of the 12-dimensional space $\mathcal{X}$. Some of the parameters were set away from the optimal values. The area of violation of the maximally entangled state (a) is about 14% larger than that of the state with $\gamma = 0.792$ (b).
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∗parisio@df.ufpe.br

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