Fuzzy quantification for linguistic data analysis and data mining.

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Abstract—Fuzzy quantification is a subtopic of fuzzy logic which deals with the modelling of the quantified expressions we can find in natural language. Fuzzy quantifiers have been successfully applied in several fields like fuzzy, control, fuzzy databases, information retrieval, natural language generation, etc. Their ability to model and evaluate linguistic expressions in a mathematical way, makes fuzzy quantifiers very powerful for data analytics and data mining applications. In this paper we will give a general overview of the main applications of fuzzy quantifiers in this field as well as some ideas to use them in new application contexts.

Keywords: fuzzy quantification, theory of generalized quantifiers, data analysis, data mining
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I. INTRODUCTION

Fuzzy logic [1] is a subfield of artificial intelligence that deals with the management of vague and imprecise expressions. In fuzzy logic systems, the classical logic based on binary truth values is generalized to fuzzy truth values defined on the interval [0, 1].

Classical logical operators are substituted with families of fuzzy operators that generalize them to fuzzy truth values, and sets are generalized to ‘fuzzy sets’, where belongingness cease to be a ‘classic or crisp’ concept to become a fuzzy concept. In this way, fuzzy sets accept partial fulfillment of their elements. For example, the fuzzy set of ‘tall people’ can include people that is tall only to a partial degree.

A crucial concept in fuzzy logic is the concept of linguistic variable [2], which allows to divide the range of variation of a variable (e.g., ‘temperature’) by means of fuzzy linguistic labels (e.g. ‘very low’, ‘low’, ‘warm’, ‘hot’, ‘very hot’), see Figure 1.

This paper deals with the specific application of fuzzy quantifiers to data analytics and data mining. Fuzzy quantifiers were introduced by Zadeh [3] to model quantified linguistic expressions. In his approach, Zadeh distinguished two types of linguistic quantifiers: quantifiers of the first kind used to represent absolute quantities (defined by using fuzzy numbers on \( \mathbb{N} \)), and quantifiers of the second kind, used to represent relative quantities (defined by using fuzzy numbers on [0, 1]).

In this work, we will follow Glöckner’s approximation to fuzzy quantification [4]. In his approach, the author generalizes the concept of generalized classic quantifier [5] (second order predicates or set relationships) to the fuzzy case; that is, a fuzzy quantifier is a fuzzy relationship between fuzzy sets. After generalizing the concept of classic quantifier to the fuzzy case, he rewrote the fuzzy quantification problem as a problem of looking for possible mechanisms to convert semi-fuzzy quantifiers (quantifiers occupying a middle point between generalized classic quantifiers and fuzzy quantifiers) to fuzzy quantifiers. The author called these transformation mechanisms Quantifier Fuzzification Mechanism (QFMs). Following the linguistic Theory of Generalized Quantifiers (TGQ) [5], this approach is capable of handling most of the quantification phenomena of natural language.

In addition, it also allows the translation of most of the analysis that has been made in TGQ from a linguistic perspective to the fuzzy case, facilitating the definition and the test of adequacy properties. Glöckner has also defined a rigorous axiomatic framework to ensure the good behavior of QFMs. Models fulfilling this framework are called Determiner fuzzification schemes (DFSs) and they comply with a broad set of properties that guarantee a good behavior from a linguistic and fuzzy point of view. See the recent [6] or [4] for a comparison between Zadeh’s and Glöckner’s approaches.

The objective of this paper is to present some of the different roles that fuzzy quantification can play in data analytics and data mining. First, fuzzy quantification can be used in a ‘descriptive sense’. In this case, fuzzy quantifiers are simply used to model some linguistic expression that can be of utility in a particular domain. We will explicitly show some examples of the application of fuzzy quantifiers in the temporal domain, to prove the capacity of fuzzy quantifiers to model ‘quantified temporal expressions’ (e.g., “the temperature was low for most of the last minutes”).

Second, fuzzy quantifiers can be used in a ‘summarization sense’. In this case, we are interested in automatically computing a single quantifier or a set of quantified expressions to summarize a set of data (e.g. to infer that the quantifier ‘most’ is the one that better explained the amount of ‘warm temperatures in June’). Different problems arise in the summarization of data by means of fuzzy quantifiers, as we will see throughout the paper.

Finally, fuzzy quantifiers can be used in combination with other techniques in machine learning problems. We will show some specific examples of the application of fuzzy quantifiers for learning fuzzy quantified constraint networks and fuzzy quantified systems of rules. Some of the examples we will present in this paper have not been theoretically developed yet, and they are presented just as an idea of the power of fuzzy quantifiers to be combined with other techniques.

II. THE FUZZY QUANTIFICATION FRAMEWORK

Most approaches to fuzzy quantification follow the concept of fuzzy linguistic quantifier, which was proposed to represent absolute or proportional fuzzy quantities. Following Zadeh [3], quantifiers of the first kind are the adequate mean to represent absolute quantities (by using fuzzy numbers on \( \mathbb{N} \)), whilst quantifiers of the second kind are the adequate mean to represent relative quantities (by using fuzzy numbers on [0, 1]).

As we mentioned before, in this paper, we will follow the approximation to fuzzy quantification proposed in [4]. Let us introduce now some definitions:

Definition 1: A two valued (generalized) quantifier on a base set \( E \neq \emptyset \) is a mapping \( Q : \mathcal{P}(E)^n \to \mathbb{R} \), where \( n \in \mathbb{N} \) is the arity (number of arguments) of \( Q \), \( \mathbb{R} = [0, 1] \) denotes the set of crisp truth values, and \( \mathcal{P}(E) \) is the powerset of \( E \).

Here we present two examples of classic quantifiers:

\[
\text{all}(Y_1, Y_2) = Y_1 \subseteq Y_2
\]

\[
\text{at least 60%}(Y_1, Y_2) = \left\{ \begin{array}{ll}
\frac{|Y_1 \cap Y_2|}{|Y_1|} \geq 0.60 & Y_1 \neq \emptyset \\
1 & Y_1 = \emptyset
\end{array} \right.
\]
In a fuzzy quantifier, inputs and outputs can be fuzzy. They assign a gradual result to each choice of \(X_1, \ldots, X_n \in \mathcal{P}(E)\), where by \(\mathcal{P}(E)\) we denote the fuzzy powerset of \(E\) (i.e., the set of all possible fuzzy sets of \(E\)).

**Definition 2:**\(^4\) An \(n\)-ary fuzzy quantifier \(Q\) on a base set \(E \neq \emptyset\) is a mapping \(Q : \mathcal{P}(E)^n \rightarrow I\) that could be interpreted as a middle ground between classic quantifiers and fuzzy quantifiers. In \([4]\) this problem is faced by introducing least sixty percent’\(\), which are depicted in Figure 2. For each pair of crisp sets \((X_1, \ldots, X_n)\), \(Q(X_1, \ldots, X_n)\) is just a trapezoidal function of parameters \((a, b, c, d)\) whilst \(S_{\alpha, \gamma}\) is defined as:

\[
S_{\alpha, \gamma}(x) = \begin{cases} 
0 & x < \alpha \\
\frac{(x-\alpha)^2}{(c-\alpha)(c-\gamma)} & \alpha < x \leq \frac{\alpha + \gamma}{2} \\
\frac{(x-\alpha)^2}{(d-\alpha)(d-\gamma)} & \frac{\alpha + \gamma}{2} < x \leq \gamma \\
1 & x > \gamma
\end{cases}
\]

\(^1\)\(T_{a,b,c,d}(x)\) is just a trapezoidal function of parameters \((a, b, c, d)\) whilst \(S_{\alpha, \gamma}\) is defined as:

\[
S_{\alpha, \gamma}(x) = \begin{cases} 
0 & x < \alpha \\
\frac{(x-\alpha)^2}{(c-\alpha)(c-\gamma)} & \alpha < x \leq \frac{\alpha + \gamma}{2} \\
\frac{(x-\alpha)^2}{(d-\alpha)(d-\gamma)} & \frac{\alpha + \gamma}{2} < x \leq \gamma \\
1 & x > \gamma
\end{cases}
\]

solve that, in [4] definition 2.8] mechanisms to transform semi-fuzzy quantifiers into fuzzy quantifiers were proposed (i.e., mappings with domain in the universe of semi-fuzzy quantifiers and range in the universe of fuzzy quantifiers):

**Definition 4:**\(^4\) A quantifier fuzzification mechanism (QFM) \(\mathcal{F}\) assigns to each semi-fuzzy quantifier \(Q : \mathcal{P}(E)^n \rightarrow I\) a corresponding fuzzy quantifier \(\mathcal{F}(Q) : \mathcal{P}(E)^n \rightarrow I\) of the same arity \(n \in \mathbb{N}\) and on the same base set \(E\).

III. SOME PROBABILISTIC QFMS

In this section we will present some QFMs that can be interpreted from a probabilistic point of view. In [7] a thoroughly comparison of these models with other of the main QFMs that have been presented in the literature can be consulted. The work [9] reviews the main approaches to fuzzy quantification that have been proposed, comparing them against a list of criteria that do not include some of the properties that have been used in [7].

A. Alpha-cut based QFMs \(\mathcal{F}^I\) and \(\mathcal{F}^{MD}\)

In this section we will present two QFMs which are based on alpha-cuts (see below) of the input sets. Both of them admit a probabilistic interpretation of fuzzy sets:

**Definition 5:**\(^9, 10\) Let \(Q : \mathcal{P}(E)^n \rightarrow I\) be a semi-fuzzy quantifier over a base set \(E\). The QFM \(\mathcal{F}^{MD}\) is defined as:

\[
\mathcal{F}^{MD}(Q)(X_1, \ldots, X_n) = \int_0^1 Q \left( (X_1)_{\geq \alpha}, \ldots, (X_n)_{\geq \alpha} \right) d\alpha
\]

for every \(X_1, \ldots, X_n \in \mathcal{P}(E)\), where \((X_i)_{\geq \alpha} = \{ e \in E : \mu_X(e) \geq \alpha \}\) is the alpha-cut of level \(\alpha\) of \(X_i\).

For normalized fuzzy sets, \(\mathcal{F}^{MD}\) coincides with the quantification model \(GD\) defined in \([11, 12]\) for quantified expressions following the Zadeh’s framework.

The definition of the \(\mathcal{F}^I\) model is presented now:

**Definition 6:**\(^[13], 9, 10\) Let \(Q : \mathcal{P}(E)^n \rightarrow I\) be a semi-fuzzy quantifier over a base set \(E\). The QFM \(\mathcal{F}^I\) is defined as:

\[
\mathcal{F}^I(Q)(X_1, \ldots, X_n) = \int_0^1 \int_0^1 \cdots \int_0^1 Q \left( (X_1)_{\geq \alpha_1}, \ldots, (X_n)_{\geq \alpha_n} \right) d\alpha_1 \cdots d\alpha_n
\]

for every \(X_1, \ldots, X_n \in \mathcal{P}(E)\).

B. QFM \(\mathcal{F}^A\)

The QFM \(\mathcal{F}^A\) fulfills the axiomatic framework presented in [4], although it does not belong to the class of ‘standard DFSS’ proposed by the author, being a ‘non-standard DFS’. This model can also be interpreted in a probabilistic way, although it also accepts a definition purely based on fuzzy operators, without reference to the probability theory [10, 7, 13].

**Definition 7:** Let \(X \in \mathcal{P}(E)\) be a fuzzy set, \(E\) finite. The probability of the crisp set \(Y \in \mathcal{P}(E)\) of being a representative of the fuzzy set \(X \in \mathcal{P}(E)\) is defined as

\[
m_X(Y) = \prod_{e \in Y} \mu_X(e) \prod_{e \in E \setminus Y} (1 - \mu_X(e))
\]

Previous definition is used to define the \(\mathcal{F}^A\) DFS:

**Definition 8:**\(^[15]\) Let \(Q : \mathcal{P}(E)^n \rightarrow I\) be a semi-fuzzy quantifier, \(E\) finite. The DFS \(\mathcal{F}^A\) is defined as

\[
\mathcal{F}^A(Q)(X_1, \ldots, X_n) = \sum_{Y_1 \in \mathcal{P}(E)} \cdots \sum_{Y_n \in \mathcal{P}(E)} m_{X_1}(Y_1) \cdots m_{X_n}(Y_n) Q(Y_1, \ldots, Y_n)
\]

for all \(X_1, \ldots, X_n \in \mathcal{P}(E)\).
IV. FUZZY QUANTIFICATION TO MODEL QUANTIFIED PATTERNS IN THE TEMPORAL DOMAIN

This section deals with the use of fuzzy quantification in a ‘descriptive sense’, where the objective of fuzzy quantifiers is simply to check the degree of fulfillment of a particular linguistic expression. The different examples we will consider are aimed to model quantified temporal expressions, as they represent one of the best examples of the use of fuzzy quantifiers for modelling the semantics of natural language. Before proceeding, we should take into account that to express the semantics of quantified expressions, we will only need to define the convenient semi-fuzzy quantifiers (which take as inputs binary arguments), as we will rely on QFMs like the ones presented in section III to convert these semi-fuzzy quantifiers into fully operational fuzzy quantifiers.

- **Proportional temporal case**, which are useful to evaluate expressions fitting the pattern “Q in T are Y” or the pattern “Q in T fulfilling Y1 are Y2”, where Q is a proportional semi-fuzzy quantifier, T is a temporal reference and Y, Y1, Y2 are binary time series. An example of an expression fitting the first pattern is “most days in the last weeks were hot”, whilst “most hot days in the last weeks were associated to high humidity values” fits the second pattern. This situation can be modeled by means of the following semi-fuzzy quantifiers:

\[
Q(T, Y) = \begin{cases} 
  f_Q \left( \frac{\text{\#Y} \cap T}{\text{\#T}} \right) & T \neq \emptyset \\
  1 & T = \emptyset 
\end{cases}
\]

\[
Q(T, Y_1, Y_2) = \frac{f_n \left( \frac{\text{\#} (Y_1 \cap Y_2) \cap T}{\text{\#} (Y_1 \cup Y_2) \cap T} \right)}{1}
\]

where \( f_Q \) is a proportional fuzzy number, like the one presented in Figure 2 b). In Figure 3 two fuzzy signals and a fuzzy temporal reference are depicted. Semi-fuzzy quantifiers as the ones previously defined, could be applied to these inputs after transforming them into fuzzy quantifiers by means of a QFM.

- **Similarity temporal case**: which are useful to evaluate expressions fitting the pattern “In T, Y1 and Y2 are Q similar”. An example of an expression fitting this pattern is “in the last weeks, hot temperatures and high humidity values happened together about the 80% or more of the days”. This situation can be modelled by means of the following semi-fuzzy quantifier:

\[
Q(T, Y_1, Y_2) = \frac{f_n \left( \frac{\text{\#} (Y_1 \cap Y_2) \cap T}{\text{\#} (Y_1 \cup Y_2)} \right)}{1}
\]

where \( f_Q \) is also a proportional fuzzy number.

More complex semi-fuzzy quantifiers could be defined for other situations.

In practical problems, it is common that we would like to check a quantified temporal pattern for the whole temporal axis. To deal with this situation we will introduce some notation that will allow us to define a quantified temporal pattern relative to a moving temporal window.

### Example

Let \( FT \) be a relative temporal fuzzy number defined with respect to a temporal point \( 0 \) (e.g., the temporal reference in Figure 3). The idea of \( FT \) being relative is to work as a reference fuzzy number that we can displace over one or several temporal signals. On the basis of the fuzzy number \( FT \), we define the temporal fuzzy number \( FT^{t_0} \) relative to the instant \( t_0 \) as:

\[
FT^{t_0}(t) = FT(t - t_0)
\]

Now, let us suppose \( \tilde{Q} : \mathbb{P}(E)^n \rightarrow [0, 1] \) is a fuzzy quantifier like the ones defined before, where we are supposing the first argument refers to the temporal constraint. We define the application of the fuzzy quantified pattern to the temporal axis as:

\[
R_{S_1, \ldots, S_n}^{\tilde{Q}, FT}(t_i) = \tilde{Q}(FT^{t_i}, S_1, \ldots, S_n)
\]

where \( FT^{t_i} \) is the displacement of \( FT \) by \( t_i \) towards the right, and \( S_1, \ldots, S_n : T \rightarrow [0, 1] \) are fuzzy signals.

We also could suppose some kind of temporal displacement between the application of the temporal constraint for the different input signals. For example, for the proportional case with two signals and a temporal reference:

\[
R_{S_1, S_2}^{\tilde{Q}, FT^d}(t_i) = \tilde{Q}(FT^{t_i}, S_1, S_2^d)
\]

where by \( S_2^d(t) \) we are representing the displacement of \( S \) by \( d \) temporal units.

Let us show now an example of the application of a fuzzy quantified expression to a time series. In Figure 4 the daily world oil production in the period 1965-2006 is represented. For this example, we will evaluate the quantified expression “in most of the last five years, increments in oil production were negative or only slightly superior to 0”, which could be modeled by means of the following expression:

\[
R_{\text{\%oil}_{\text{\%oil}}}^{\text{\%oil}_{\text{\%oil}}}(t) = \text{\%oil}_{\text{\%oil}}(\text{\%oil}_{\text{\%oil}}^t, \text{\%oil}_{\text{\%oil}}, \text{\%oil}_{\text{\%oil}}, \text{\%oil}_{\text{\%oil}}^t, \text{\%oil}_{\text{\%oil}}^t, \text{\%oil}_{\text{\%oil}}^t)
\]

where most is a binary proportional fuzzy quantifier defined as \( \mathcal{F}^A \) (most), being most the semi-fuzzy quantifier:

\[
\text{most}(T, S) = \begin{cases}
  S_{0.7, 0.9} \left( \frac{FT^{t_i} \cap T}{FT^{t_i}} \right) & T \neq \emptyset \\
  1 & T = \emptyset
\end{cases}
\]

where \( \text{last_five_years} \) is a fuzzy set defining the temporal constraint:

\[
\text{last_five_years}(x) = T \rightarrow x \rightarrow 5.0, 5.0, x \rightarrow [0, 1]
\]

and \( \text{nsoil}(t) \) is the fuzzy signal that results of applying the fuzzy number \( S_{1, 4}(t) \) to the percentage variations in oil production.

\[
n_{\text{soil}}(t) = S_{1, 4}^{\text{\%oil}} \left( 100 \cdot \frac{\text{oil}(t) - \text{oil}(t - 1)}{\text{oil}(t) - \text{oil}(t - 1)} \right)
\]
We show in Figure 5 the result of evaluating $\delta$ for the dataset in Figure 3. A threshold value 0.8 is depicted for indicating the years fulfilling the expression. Since, for example, for the year 1995 the threshold is surpassed, we could interpret that “in most of the years preceding 1995, increments in oil production were negative or only slightly positive”.

In [16], the use of linguistic quantified patterns was also presented with two other objectives: differentiation and aggregation. Let us consider a set of elements $E = \{e_1, \ldots, e_n\}$ for which a specific temporal quantified pattern can be applicable to some of the properties of the $e_i$s. For example, $E$ could be a set of clients of an energy company and the quantified pattern: “in nearly all the days of last month, energy consumption in the morning was higher than consumption in the afternoon for client $e_i$”. By means of differentiation, we try to detect elements that do not follow the common behavior. For example, if most of the elements in $E$ fulfill the quantified pattern with a high degree, then we will search for specific $e_i$s with a low degree of fulfillment. Symmetrically, we could look for $e_i$s fulfilling the pattern with a high degree of fulfillment when most elements do not fulfill the pattern. This could be useful to identify anomalous patterns expressed as linguistic quantified statements.

In the case of aggregation we also consider a series of elements $E$ and a quantified pattern that can be evaluated for some property of the $e_i$s. In this case, the objective is to summarize the general fulfillment of the pattern by the elements of $E$ (i.e., “for most $e_i$s the pattern is fulfilled”). For example, to deal with linguistic expressions like “almost all the days of the last month, consumption in the morning was higher than consumption in the afternoon for most clients of the company”.

V. SUMMARIZING DATA WITH FUZZY QUANTIFIERS

The objective of summarizing data with fuzzy quantifiers is to compute a single quantifier or a set of quantified expressions that adequately summarize a set of data. Let us suppose there is a set of data $E = \{e_1, \ldots, e_m\}$ (e.g., students) and a set of numerical attributes $a_1, \ldots, a_r$ (e.g., ‘age’, ‘height’) that can be applied to the elements in $E$. For each attribute $a_j$ we will assume that a linguistic variable $L_j = \{l_{j,1}, \ldots, l_{j,p_j}\}$ has been defined to adapt the numerical attributes to linguistic values. Moreover, in some cases we will also suppose there is a predefined fuzzy quantified partition $FQ = \{f_{Q_0}, \ldots, f_{Q_{W-1}}\}$ of the proportional universe $[0, \ldots, 100\%]$ (e.g., ‘nearly none’, ‘a few’, ‘several’, ‘many’, ‘nearly all’).

Several options arise to build a summary of the input data based on fuzzy quantification, depending on the structure of the summaries and the consideration of a possible predefined quantified partition. For example, we could be interested in building a summary composed of a unique quantified expression, limiting us to summarize the fulfillment of an specific label (e.g. “most people are young”) or to explain the data by means of several quantified expressions dealing simultaneously with several labels (e.g. “some people are young and some are old”). In the following sections, we will present some of the proposals that have been previously published to handle this problem.

A. Computing a unique quantifier to summarize a set of data

We will consider two different options to compute a quantified label summarizing a set of data.

1) Computing the best quantifier within an existing quantified partition: Let us suppose first that there exist a predefined quantified partition $FQ$ in which we should base our summary and that we want to summarize the data with respect to the fulfillment of a fixed set of linguistic labels $l_1, \ldots, l_n$, where label $l_i$ is applied to the attribute $a_i$. For example, considering two properties ‘age’ and ‘height’ and a binary proportional quantifier, $l_1$ could be ‘tall’ applied to heightness and $l_2$ could be ‘normal’ applied to weightness.

In this case, the most reasonable option is to return the quantified label which provides the greater degree of fulfillment. For example, if for a given set of students we obtained the higher degree of fulfillment for the fuzzy quantifier ‘most’ for the fixed set of labels ‘tall’ and ‘heavy’, we could summarize the data as “most tall students are heavy”.

Summarizing a set of data by means of a unique quantified label can be inadequate in the case there is not a unique quantified label with a high degree of fulfillment, or if several quantified labels share a similar degree of fulfillment. In this case, a convenient answer could be to indicate that none of the quantified labels is adequate to summarize the data.

2) Computing the optimal quantifier to summarize a set of data: In this case, we will not constraint us to a set of predefined labels, being the objective to compute automatically the quantified label that better summarize the data. This problem was addressed in [17] where an algorithm solution was provided to compute the optimal crisp proportional quantifier following the expression:

$$rate_{[r_1, r_2]}(Y_1, Y_2) = \begin{cases} 1 & Y_1 \neq \emptyset \land \frac{|Y_1 \cap Y_2|}{|Y_1|} \in [r_1, r_2] \\ 0 & \text{else} \end{cases}$$

(3)

The semantics of the previous quantifier is associated to expressions like “between $r_1$ and $r_2$ percent of the $X_1$’s are $X_2$’s”. When $r_2 = 100\%$, the semantics is “at least $r_1$ percent of the $X_1$’s are $X_2$’s”.

The model proposed in [17] uses a predefined parameter $\delta_{\text{max}}$ to restrict the amplitude of the semi-fuzzy proportional quantifier (i.e. $\delta_{\text{max}} = 0.2$ limits $r_2 - r_1$ to 0.2, or in proportional terms a percentage range equal or inferior to 20%). Constrained by this parameter it computes the optimal (in the sense of producing the highest evaluation degree) semi-fuzzy quantifier following expression 3. This algorithm was developed for the $MC_X$ DFS proposed by the author, although it is also valid for the class of ‘standard DFSs’ [4]. We cannot present the details of the algorithm for lack of space as it depends heavily on the properties of the $MC_X$ DFS. Basically, given two fuzzy sets $X_1$ and $X_2$, and a supposed parameter $\delta_{\text{max}} = 0.2$, the proposal in [17] permits the computation of expressions like “between 62.5% and 75% of the $X_1$’s are $X_2$’s”, where ‘between 62.5% and 75%’ is the rate quantifier which produces the highest evaluation degree for this $\delta_{\text{max}}$.

The author has not extended this proposal to other kinds of semi-fuzzy quantifiers. However, the same ideas presented in the previous
reference can be used to adapt the algorithm to other kinds of semi-fuzzy quantifiers which will allow us to search for other relationships between the data (e.g. comparative quantifiers, etc.).

Although a similar proposal have not been presented for the probabilistic models in section III it is possible to approximate the optimal rate\(_{(p_1, p_2)}\) quantifier for a given amplitude \(\delta_{\text{max}}\) evaluating a series of rate quantifiers starting in \(rate(0, \delta_{\text{max}})\), and displacing this semi-fuzzy quantifier over the proportional axis following the pattern \(rate(0, \delta_{\text{max}} + h)\). The summary will be constructed using the quantifier for which the greatest evaluation value was obtained.

We have sketched some ideas for summarizing data by means of crisp proportional quantifiers. However, the adaptation of previous ideas to learn non crisp quantifiers, have not been dealt with in the literature to our knowledge.

B. Computing a set of compatible quantified expressions to summarize the data

In [18] a method was proposed to summarize a set of temporal data by means of several compatible quantified expressions using the \(F^A\) DFS. Let us consider the existence of a linguistic variable (like the one in Figure 1) and a Ruspini unary proportional quantified partition of the quantification universe. In their proposal, the authors computed the evaluation results of each possible pair of label/quantifier. Pairs of label/quantifiers with a high evaluation result, the authors computed the evaluation results of each possible pair of quantified partition of the quantification universe for which the greatest evaluation value was obtained.

A. Fuzzy quantification for data mining of temporal constraint networks

In this section we will present two specific examples which prove the utility of fuzzy quantifiers in machine learning applications.

A fuzzy quantification for data mining of temporal constraint networks

In this section we will make a proposal to integrate fuzzy quantification into temporal constraint data mining. This proposal is hypothetical, and it has not been implemented yet.

2The linguistic variable represented in Figure 1 is a Ruspini partition as the membership degrees of the different labels adds to 1 for each point in the x axis. A Ruspini quantified partition follows a similar pattern for fuzzy quantifiers.

Temporal constraint networks are temporal structures whose aim is to represent the temporal occurrence of events constrained by some temporal metric between them. Temporal constraint networks are represented by means of graphs, where nodes represent the occurrence of events whilst arcs represent temporal distances between nodes.

We will follow [19] to introduce the idea. In this reference, a specific proposal to mine fuzzy constraint networks inspired in the Apriori algorithm for detecting association rules was proposed. The mining process operates over a set of observables \(O = \{o_1, \ldots, o_n\}\), or entities of the domain for which there exists an observation procedure. Observation procedures identify the presence of an observable in a temporal point. This abstract definition will allow us to introduce fuzzy quantifiers as observables, in order to propose an idea to mine fuzzy constraint networks were temporal entities are modelled by means of fuzzy quantifiers.

In previous proposal two types of temporal entities are considered:

- An event is a tuple \((O_i, a = v, t)\) where \(O_i \in O\) is an observable, \(a\) is an attribute with value \(v \in V(a)\) and \(t \in \mathbb{N}\) is a time instant.

- An episode is a tuple \((O_i, a = v, t_0, t_e)\) where \(O_i \in O\) is an observable, \(a\) is an attribute with value \(v \in V(a)\) and \(t_0, t_e \in \mathbb{N}\) denote respectively the begin and the end of the episode. In practice, episodes can be represented by an initial and an ending event.

As we introduced before, fuzzy quantification can be introduced into temporal data mining playing the role of observation procedures. Let \(\mathbb{T} = \{t_{p_1}, \ldots, t_{p_n}\}\) be a finite set of temporal patterns defined over our set of input signals \(S_1(t), \ldots, S_G(t) \in S\). For example, a temporal pattern \(t_{p_k}\) could follow the scheme:

\[
 tp_k(t_l) = \bar{Q} \left( FT^{t_l}, S_1 \right)
\]

for a unary quantifier, or

\[
 tp_k(t_l) = \bar{Q} \left( FT^{t_l}, S_1, S_2^d \right)
\]

for a binary proportional quantifier with displacement \(d\). These patterns could be predefined by the expert guiding the mining process or generated by means of an automatic procedure.

Constraining the fuzzy patterns by means of some threshold (e.g., assuming the pattern is fulfilled in \(t\) if its degree of fulfillment is superior to 0.8, and that is not fulfilled in other case) we can introduce them in constraint data mining algorithms as binary observables. For example, a temporal pattern could be “the temperature was extremely high in the last five minutes” or “most high pressure values in the last half an hour were associated to extremely high temperature values”.

In Figure 6 we show a hypothetical example of a possible quantified constraint temporal network that could be obtained with this kind of approximation. As future work, we will analyze the interest of this proposal.

B. Systems of quantified fuzzy rules for classification and regression

In [20] the learning of fuzzy controllers in mobile robotics by means of quantified fuzzy rules was proposed. In fuzzy control, a fuzzy controller is composed of a set of rules fulfilling the pattern:

(R1) If \(x_1\) is \(l_{1,1}\) and \(x_2\) is \(l_{1,2}\) and ... and \(x_n\) is \(l_{1,n}\) then \(y\) is \(O_1\)

(R2) If \(x_1\) is \(l_{2,1}\) and \(x_2\) is \(l_{2,2}\) and ... and \(x_n\) is \(l_{2,n}\) then \(y\) is \(O_2\)

(Rm) If \(x_1\) is \(l_{m,1}\) and \(x_2\) is \(l_{m,2}\) and ... and \(x_n\) is \(l_{m,n}\) then \(y\) is \(O_m\)

where \(x_i\) are inputs (e.g. signal values), \(y\) is the output and \(l_{i,j}\) and \(O_i\) are linguistic labels. An example of a possible fuzzy rule could be
Fig. 6. Example of a hypothetical constraint network between quantified temporal expressions.

“If the temperature is low and the pressure is high then the velocity should be high”. The idea of fuzzy control is that if we observe the input values “\(x_1 = P_1\) and \(x_2 = P_2\) and ... and \(x_n = P_n\)” that do not fit exactly any of the rules of the system, but we can guarantee a partial match with, let us suppose, the rule \(R_i\), we still can make some kind of affirmation about the output based on the partial fulfillment. In fuzzy control systems, several rules can be partially active at the same time. Different aggregation procedures are available to integrate the output of the rules of the system and retrieve an specific output value. Moreover, polynomial outputs (Takagi-Sugeno systems) are a relevant variant of fuzzy controllers.

Fuzzy quantifiers can be used in fuzzy rule systems to introduce a new aggregation level. In previous example, if we were considering fuzzy signals, each atom (e.g., \(x_i = l_{ij}\)) would be applied to an specific instant. But by means of fuzzy quantifiers we can substitute simple atoms by quantified ones, allowing expressions like “most temperatures were high in the last minutes”.

This approach was followed in [20] for the automatic learning of fuzzy controllers in mobile robotics. The idea of the solution proposed in the author’s approach, was to use fuzzy quantifiers as a mean to aggregate ‘low level input variables’ (variables with a small single contribution to the system, as the distance of several laser beams). Given the complexity of learning a complete set of rules involving quantifiers, a genetic approach was proposed in which each individual codified a single rule. The general idea of the author’s approach was to learn the rule system rule by rule, incorporating new rules based on different criteria.

The possibility of learning fuzzy rule system in fuzzy control proves the capacity of fuzzy quantifiers to be integrated in fuzzy rule systems in regression and classification problems. Learning fuzzy control systems is an example of a regression procedure, in which input values are used to predict an output value. As we are dividing the output axis by means of a linguistic variable, classification can be associated to the selection of a specific fuzzy label (e.g., the one that better includes the output value of the fuzzy rule system).

VII. CONCLUSION

In this paper we presented some of the different roles that fuzzy quantification can play in data analytics and data mining. After introducing the field of fuzzy quantification, we showed some uses of fuzzy quantifiers in a ‘descriptive sense’, with focus in the modelling of temporal expressions. We continued presenting the application of fuzzy quantifiers for summarizing sets of data by means of linguistic quantified expressions. Finally, two applications of fuzzy quantifiers in machine learning, specifically in temporal constraint networks and fuzzy systems of rules, were presented.

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