OPTIMAL CONTROL AND STABILIZATION OF BUILDING MAINTENANCE UNITS BASED ON MINIMUM PRINCIPLE

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ABSTRACT. In this paper we present a mathematical model describing the physical dynamics of a building maintenance unit (BMU) equipped with reaction jets. The momentum provided by reaction jets is considered as the control variable. We introduce an objective functional based on the deviation of the BMU from its equilibrium state due to external high-wind forces. Pontryagin minimum principle is then used to determine the optimal control policy so as to minimize possible deviation from the rest state thereby increasing the stability of the BMU and reducing the risk to the workers as well as the public. We present a series of numerical results corresponding to three different scenarios for the formulated optimal control problem. These results show that, under high-wind conditions the BMU can be stabilized and brought to its equilibrium state with appropriate controls in a short period of time. Therefore, it is believed that the dynamic model presented here would be potentially useful for stabilizing building maintenance units thereby reducing the risk to the workers and the general public.

1. Introduction. In most of the highly populated modern cities around the world, due to high demand it has become a general practice to build high-rise buildings for both residential and business purposes. These facilities require regular maintenance for general upkeep and periodic inspection to maintain and guarantee structural integrity. Generally these activities are carried out by use of building maintenance units which are suspended from large cranes. In the event of large and gusty wind the system may be destabilized risking lives of the workers on the unit and the public near the buildings. To avoid such risk it is essential to equip the BMU with stabilizing controls so that in the presence of dangerous wind activities the controls are automatically activated and dangers are avoided. The term building maintenance can be interpreted in different ways. However it has been popularly referred to as any set of activities related to improving any facility (any part of a building or its services and surroundings) to an acceptable level and to maintaining the utility as well as its value [4, 17]. It is mentioned in [9] that there are long-term costly consequences without proper maintenance, and that maintenance should be carried out if necessary so as to ensure the continued, safe and profitable use of the building at acceptable levels of satisfaction. Therefore it is normally advised to carry

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out periodic maintenance for high-rise buildings. More importantly, periodic building maintenance can eliminate many of the potential uncertainties of the building structure, the expected lifetimes, the cost of specific measures, etc [12].

There are multiple techniques developed for the management of building maintenance, for example radio frequency identification [11], building information modeling [14], predictive maintenance management [7], etc. However, in order to carry out maintenance for high-rise buildings, building maintenance units, also called BMUs, are commonly employed in practice. They are widely used to complete certain projects, such as cleaning cycles, repairing damaged windows or facade panels, and regular building inspections [5]. Normally, a group of workers has to stand on a BMU to accomplish these tasks. Although it is fairly easy to operate the BMUs, we still have to face the challenge presented by high winds destabilizing the BMU and thereby putting both the workers and the general public at risk [10]. Therefore it is of great importance to study the stability and control of the BMU so that it can work safely and efficiently in the presence of high winds.

In this respect, finding an optimal control strategy that can stabilize the BMU under the impact of high winds would be of great significance. This can be achieved by developing a proper mathematical model of the BMU and then using optimal control theory [1, 2, 8, 15] to determine the stabilizing control policies. It appears from the current literature that not much work has been done in modeling the physical dynamics of a BMU. We believe this is the first time that a comprehensive dynamic model of a BMU is developed and that optimal control theory is used to determine the stabilizing controls for BMUs equipped with reaction jets. Therefore, the motivation of our work is to provide a straightforward but effective methodology to stabilize BMUs.

The rest of the paper is structured as follows. In Section 2, we develop a mathematical (dynamic) model for the building maintenance unit (BMU) and formulate an optimal control problem based on the model proposed. In Section 3, we introduce the general Pontryagin minimum principle and then apply it to address the problem formulated in the preceding section. In Section 4, we present a computational algorithm based on the minimum principle described in Section 3. In the same section, a series of numerical results is also presented to show the efficacy of the proposed methodology. Finally the paper is concluded in Section 5.

2. Mathematical modeling and problem formulation.

2.1. System dynamics. We consider a building maintenance unit which is approximated by a compact mass $m$ attached to the end of a light inextensible string with length $r$. The schematic of the BMU is shown in Figure 1. Suppose the body of the BMU is free to move in any direction in the spherical space and it is well equipped with reaction jets marked with $\times$ as shown in Figure 2. For the convenience of presentation, we shall introduce some necessary notations listed in Table 1.

Let the pivot (fixed end of the string) be located at the origin of the inertial coordinate frame. The $x$ and $y$ axes are in the horizontal plane, and the $z$ axis lies collinearly in the direction of gravity. We also introduce a body-fixed coordinate frame for the BMU with the origin of this frame located at the pivot while the $z'$ axis coincides with the vector from the pivot to the center of mass of the BMU body. It is assumed that the BMU is axi-symmetric, meaning that two of the principal moments of inertia ($C_x$ and $C_y$) of the BMU are identical and that the pivot is
Table 1. Definition of Notations

| Notation | Description |
|----------|-------------|
| $x_1 = \omega_x$ | First component of the angular velocity |
| $x_2 = \omega_y$ | Second component of the angular velocity |
| $x_3 = \gamma_x$ | First component of the unit vertical vector $\gamma$ |
| $x_4 = \gamma_y$ | Second component of the unit vertical vector $\gamma$ |
| $x_5 = \gamma_z$ | Third component of the unit vertical vector $\gamma$ |
| $\psi$ | Costate vector (adjoint state) |
| $C$ | Constant inertia matrix |
| $I = [0, T]$ | Total operating period in seconds |
| $U$ | Control (decision) constraint set |
| $U_{ad}$ | Set of admissible controls |
| $u$ | Lower bound of the control variable |
| $\bar{u}$ | Upper bound of the control variable |
| $J(u)$ | Objective (cost) functional |
| $\ell$ | Integrando of running cost |
| $\Phi$ | Terminal cost |
| $H$ | Hamiltonian function |
| $V(x)$ | Lyapunov function candidate |
| $x^d$ | Desired state during the operating period |
| $\bar{x}$ | Target state at the terminal time |
| $x^e$ | Equilibrium state of the system |
| $\langle a, b \rangle$ | Scalar product of vectors $a$ and $b$ |
| $a \times b$ | Cross product of vectors $a$ and $b$ |
| $x^T$ | Transpose of vector $x$ |

located on the axis of symmetry of the BMU. Under standard assumptions, the dynamics of the BMU in the absence of external forces can be described by the Euler equation involving the moments due to gravity

$$C\dot{\omega} = C\omega \times \omega + mg \rho \times \gamma,$$

and the kinematics is given by

$$\dot{\gamma} = \gamma \times \omega,$$

where $C = \text{diag}(C_x, C_y, C_z)$ is the constant inertia matrix, $\omega = (\omega_x, \omega_y, \omega_z)$ is the vector of angular velocity, $\gamma = (\gamma_x, \gamma_y, \gamma_z)$ is the unit vertical vector (indirectly representing the attitude of the BMU), and $\rho = re_3$ is the body-fixed vector pointing from the pivot to the center of mass of the BMU with $e_3$ being the third unit vector in the inertial coordinate frame ($e_3 = (0, 0, 1)^T$). Note that all these variables are introduced relative to the body-fixed coordinate frame.
Similar representations of the dynamics of a pendulum have been proposed and can be found in [18, 24, 25]. We use this dynamic model and control theory (Pontryagin minimum principle) to develop optimal control strategies for building maintenance units. Equations (1) and (2) can be interpreted as the dynamics of a degenerate rigid body. For this interpretation, the constant inertia matrix is given by $C = \text{diag}(mr^2, mr^2, 0)$ by choosing the orthonormal fixed-body coordinate frame with the $z'$ axis aligned along the direction from the pivot to the BMU body. Thus it is clear that $\omega_z = 0$ because rotation about the $z'$ axis has no impact on the motion of the BMU. Therefore, the dynamics of the BMU, represented by equations (1) and (2), can be reduced to the following set of differential equations

$$
\dot{\omega}_x = -\frac{g}{r} \gamma_y,
\dot{\omega}_y = \frac{g}{r} \gamma_x,
\dot{\gamma}_x = -\omega_y \gamma_z,
\dot{\gamma}_y = \omega_x \gamma_z,
\dot{\gamma}_z = \omega_y \gamma_x - \omega_x \gamma_y.
$$

Equations (3) provide the state space model of the dynamic system. Let $x_1 = \omega_x$, $x_2 = \omega_y$, $x_3 = \gamma_x$, $x_4 = \gamma_y$ and $x_5 = \gamma_z$. Thus the system described
by equation (3) can be represented by the following system of equations

\[
\begin{align*}
\dot{x}_1 &= \frac{dx_1}{dt} = -\frac{g}{r}x_4, \\
\dot{x}_2 &= \frac{dx_2}{dt} = \frac{g}{r}x_3, \\
\dot{x}_3 &= \frac{dx_3}{dt} = -x_2x_5, \\
\dot{x}_4 &= \frac{dx_4}{dt} = x_1x_5, \\
\dot{x}_5 &= \frac{dx_5}{dt} = x_2x_3 - x_1x_4.
\end{align*}
\] (4)

Before discussing optimal control of the BMU, we shall present a stability result of the dynamic system in the absence of external forces. The result is presented in the following Lemma.

**Lemma 2.1.** Consider the dynamic system described by equation (4) with \( x = (x_1, x_2, x_3, x_4, x_5)^T \) denoting its state vector. Then the equilibrium state \( x^e = (0, 0, 0, 0, 1)^T \) of the system (4) is stable in the sense of Lyapunov.

**Proof.** Consider the following function as the candidate for a Lyapunov function

\[
V(x) = \frac{1}{2}(\omega, \omega) - mgr(\gamma_z - 1). \tag{5}
\]

It is easily observed that \( V(x^e) = 0 \) and \( V(x) > 0 \) for \( x \neq x^e \). Further, the time derivative of the Lyapunov function is zero as shown below:

\[
\begin{align*}
\dot{V}(x) &= \omega^T C \dot{\omega} - mgr\gamma_z \\
&= mr^2 \omega_x \dot{\omega}_x + mr^2 \omega_y \dot{\omega}_y - mgr\gamma_z \\
&= mr^2 x_1(\frac{g}{r}x_4) + mr^2 x_2(\frac{g}{r}x_3) - mgr(x_2x_3 - x_1x_4) \\
&= -mgrx_1x_4 + mgrx_2x_3 - mgrx_2x_3 + mgrx_1x_4 \\
&= 0.
\end{align*}
\]

Therefore, the equilibrium state \( x^e = (0, 0, 0, 0, 1)^T \) is stable in the sense of Lyapunov.

Since we have introduced reaction jets which are rigidly mounted on the body of the BMU, appropriate momentum can therefore be provided to stabilize the BMU whenever it is displaced from its equilibrium state. The moments are applied collinearly to the angular momentum of the BMU body. And the reaction jets mounted on any surface of the BMU (Figure 2) have to be fired at the same time in order to enhance the stability of the BMU. These reaction jets can also be strategically placed inside a steel housing attached to the bottom of the BMU just below the platform where the workers stand. Hence, the dynamic equation (1) with forces generated by the firing of reaction jets can be rewritten as

\[
C \dot{\omega} = C\omega \times \omega + mgu \times \gamma + M, \tag{6}
\]

where \( M \) is the control moment produced by the reaction jets and takes the following form [18]

\[
M = mgu \times \gamma, \tag{7}
\]

where \( u = (u_x, u_y, u_z)^T \) represents the control (incremental displacement) vector due to momentum provided by the reaction jets. Note that it is also possible to
express the control moment \( M \) in other forms. However the expression given by (7) can maintain the symmetry [18] of the rigid BMU dynamics.

By applying the control described in (7), one can easily rewrite the system dynamics (4) including the control momentum as follows

\[
\begin{align*}
\dot{x}_1 &= f_1(x, u) = -\frac{g}{r}x_4 + \frac{g}{r^2}u_1x_5 - \frac{g}{r^2}u_2x_4, \\
\dot{x}_2 &= f_2(x, u) = \frac{g}{r}x_3 - \frac{g}{r^2}u_2x_5 + \frac{g}{r^2}u_1x_3, \\
\dot{x}_3 &= f_3(x, u) = -x_2x_5, \\
\dot{x}_4 &= f_4(x, u) = x_1x_5, \\
\dot{x}_5 &= f_5(x, u) = x_2x_3 - x_1x_4.
\end{align*}
\]

where \( x = (x_1, x_2, x_3, x_4, x_5)^T \in \mathbb{R}^5 \) is the state vector, and \( u = (u_x, u_y, u_z)^T \in \mathbb{R}^3 \) is the control (incremental displacement) vector determining the control moment \( M \) applied to the angular momentum. For the convenience of analysis of the dynamic system, we can write the system model in the following compact form

\[
\dot{x} = f(x, u),
\]

where \( f(x, u) = (f_1(x, u), f_2(x, u), f_3(x, u), f_4(x, u), f_5(x, u))^T \) and \( \dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5)^T \in \mathbb{R}^5 \).

**Remark 2.2.** The dynamics of the BMU (system) can be derived also from the Euler-Lagrange equations in spherical coordinates. However, that would introduce singularities when the vector \( \rho \) is collinear with the \( z \) axis because the equations describing the physical movement of the BMU in spherical coordinates have denominators vanishing under such situation [24]. Thus the dynamic model expressed in spherical coordinates does not fully represent the physics of the BMU, and is therefore not advised for computational purpose.

### 2.2. Problem formulation.

We denote the control constraint set \( U \) by

\[
U \equiv \{ u = (u_x, u_y, u_z)^T \in \mathbb{R}^3 : \underline{u} \leq u_x, u_y, u_z \leq \overline{u} \} \subset \mathbb{R}^3,
\]

where \( \underline{u} \) and \( \overline{u} \) are the lower and upper bounds of the control variables (incremental displacement variables), respectively.

Our objective is to stabilize the BMU for the safety of the workers and the public. This means that we must bring the BMU to its equilibrium state \( x^e = (0, 0, 0, 0, 1)^T \) as soon, and as closely, as possible in the presence of high winds. Based on such an objective, we define the following cost functional (also called objective functional) over the operating period \( I = [0, T] \),

\[
J(u) = \frac{1}{2} \int_I \langle Q(x(t) - x^d(t)), (x(t) - x^d(t)) \rangle dt + \frac{1}{2} \langle P(x(T) - \bar{x}), (x(T) - \bar{x}) \rangle,
\]

where \( \{x^d(t) \in \mathbb{R}^5, t \geq 0 \} \) is the desired state trajectory, and \( \bar{x} \in \mathbb{R}^5 \) is the desired target state at time \( T \). On the right-hand side of equation (11), the first term (known as running cost) stands for the weighted cost due to mismatch between the actual state and the desired state over the time period \( I \), and the second term (known as terminal cost) represents the deviation of the final state from the target state at the end of the operating period. In order to evaluate the performance of the system, we choose the symmetric positive definite matrices \( Q \) and \( P \) as

\[
Q = \text{diag}(w_1, w_2, w_3, w_4, w_5) \quad \text{and} \quad P = \text{diag}(v_1, v_2, v_3, v_4, v_5),
\]

respectively.
parameters \(\{w_i\}\) and \(\{v_i\}\) denote the weights assigned to specific terms in the objective functional \(J(u)\).

For the convenience of notation, we will denote the integrand of the running cost by
\[
\ell(t, x(t), u(t)) \equiv \frac{1}{2} \left( Q \left( x(t) - x^d(t) \right), (x(t) - x^d(t)) \right) \\
= \frac{1}{2} \left\{ w_1 \left( x_1(t) - x_1^d(t) \right)^2 + w_2 \left( x_2(t) - x_2^d(t) \right)^2 \\
+ w_3 \left( x_3(t) - x_3^d(t) \right)^2 + w_4 \left( x_4(t) - x_4^d(t) \right)^2 + w_5 \left( x_5(t) - x_5^d(t) \right)^2 \right\}, \tag{12}
\]
and the terminal cost by
\[
\Phi(x(T)) \equiv \frac{1}{2} \left( P(x(T) - \bar{x}), (x(T) - \bar{x}) \right) \\
= \frac{1}{2} \left\{ v_1 \left( x_1(T) - \bar{x}_1 \right)^2 + v_2 \left( x_2(T) - \bar{x}_2 \right)^2 \\
+ v_3 \left( x_3(T) - \bar{x}_3 \right)^2 + v_4 \left( x_4(T) - \bar{x}_4 \right)^2 + v_5 \left( x_5(T) - \bar{x}_5 \right)^2 \right\}. \tag{13}
\]
Therefore, the objective functional given by (11) can be written as
\[
J(u) = \int_I \ell(t, x(t), u(t)) dt + \Phi(x(T)), \tag{14}
\]
where on the righthand side, the first term and the second term represent the running cost and the terminal cost, respectively.

The problem is to minimize the objective (performance) functional \(J(u)\) given by (11) (or (14)) subject to the dynamic constraints (8) (or (9)), and the control constraint set \(U\). It is believed that, under windy conditions, BMUs can be better stabilized if the solution proposed here is implemented. The methodology proposed can be generally applied to various types of BMUs. Therefore, the solution of the problem presented here is of great significance to the safety of both the workers on the BMU and the general public.

3. Minimum principle applied to BMU.

3.1. Pontryagin minimum principle. Optimal control has been applied to various fields, such as bio-medical science [3], energy economics [13, 19], transportation engineering [21, 22], among many others. One of the very powerful techniques in optimal control is the well-known Pontryagin minimum principle. It is widely used to find the best possible control in order to drive a dynamic system from one state to another, especially in the presence of state/control constraints. Consider the \(n\)-dimensional (\(n = 5\) in the case of BMU) dynamic system given by
\[
\dot{x} = f(t, x, u), \quad t \in I, \quad x(0) = x_0, \tag{15}
\]
where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}^d\) is the control vector, and \(x_0\) is the given initial state of the system. The general objective functional is expressed by equation (14). The control constraint set \(U \subset \mathbb{R}^d\) is a closed bounded set (compact) meaning that the controls are allowed to take values only from \(U\). Hence, the problem is to find the control policy \(u^*(t), \quad t \in I\) such that the functional \(J(u)\) given by (14) is minimized. In order to address this problem, Pontryagin and
his colleagues [16] introduce the Hamiltonian function $H$ given by the following expression

$$H(t, x, u, \psi) \equiv \langle f(t, x, u), \psi \rangle + \ell(t, x, u). \quad (16)$$

Then a pair of canonical differential equations in terms of the Hamiltonian is introduced

\[
\begin{align*}
\dot{x} &= H_x = f(t, x, u), \quad t \in I, \ x(0) = x_0, \\
\dot{\psi} &= -H_x = -f^T_x(t, x, u)\psi - \ell_x(t, x, u), \quad t \in I, \ \psi(T) = \Phi^T_x(x(T)),
\end{align*}
\]

where $f_x$ is the Jacobian matrix of the vector $f$ and $f^T_x$ is its transpose. Equation (17) is the original system dynamics while equation (18) is the well-known adjoint (or costate) equation.

Let $U_{ad} \subset \mathcal{M}(I, U)$ denote the class of measurable functions defined on the time interval $I$ and taking values from the set $U$. We are now prepared to state the celebrated Pontryagin minimum principle as follows.

**Theorem 3.1** (Necessary conditions of optimality). Let $u^o \in U_{ad}$, and $x^o \in C(I, \mathbb{R}^n)$ be the solution to equation (17) corresponding to the control $u^o$. Then, for the control $u^o$ to be optimal, it is necessary that there exists a costate $\psi^o$ such that the following inequality and equations hold:

1. $H(t, x^o(t), u^o(t), \psi^o(t)) \leq H(t, x^o(t), u(t), \psi^o(t))$, for $\forall u \in U_{ad}$, $t \in I$,
2. $\dot{x}^o = H_{x}^o = f(t, x^o(t), u^o(t))$, $x^o(0) = x_0$, $t \in I$,
3. $\dot{\psi}^o = -H_x^o = -f^T_x(t, x^o(t), u^o(t))\psi^o - \ell_x(t, x^o(t), u^o(t))$, $\psi^o(T) = \Phi^T_x(x^o(T))$, $t \in I$.

**Proof.** The proof is classic and can be found in [2, Theorem 6.2.1, p. 252]. Any reader interested in the proof may also refer to the following books Pontryagin [16], Ahmed [1] and Teo [20].

3.2. Minimum principle applied to BMU control. Now we are ready to apply the minimum principle to our problem, where the BMU is required to reach the desiredattitude at the end of the operating time while the difference between the actual state and the desired one is minimized over the period. This problem involves both the terminal cost and the running cost. Since our objective is to bring the BMU to its equilibrium state $x^e = (0, 0, 0, 0, 0)^T$ as closely as possible in the presence of high winds, the cost (objective) functional and the corresponding Hamiltonian for this case are given by equations (19) and (20), respectively

$$J(u) = \frac{1}{2} \int_0^T \left\{ w_1 x_1^2(t) + w_2 x_2^2(t) + w_3 x_3^2(t) + w_4 x_4^2(t) + w_5 \left( x_5(t) - 1 \right)^2 \right\} dt$$

$$H(t, x, u, \psi) \equiv f^T(t, x, u, \psi) + \ell(t, x, u)$$

$$= \left( -\frac{g}{r} x_3 + \frac{g}{r^2} u_2 x_5 - \frac{g}{r^2} u_3 x_4 \right) \psi_1 + \left( \frac{g}{r} x_2 - \frac{g}{r^2} u_1 x_5 + \frac{g}{r^2} u_3 x_3 \right) \psi_2$$

$$- x_2 x_5 \psi_3 + x_1 x_5 \psi_4 + \left( x_2 x_3 - x_1 x_4 \right) \psi_5$$

$$+ \frac{1}{2} \left\{ w_1 x_1^2(t) + w_2 x_2^2(t) + w_3 x_3^2(t) + w_4 x_4^2(t) + w_5 \left( x_5(t) - 1 \right)^2 \right\}. \quad (19)$$

$$H(t, x, u, \psi) \equiv f^T(t, x, u, \psi) + \ell(t, x, u)$$

$$= \left( -\frac{g}{r} x_3 + \frac{g}{r^2} u_2 x_5 - \frac{g}{r^2} u_3 x_4 \right) \psi_1 + \left( \frac{g}{r} x_2 - \frac{g}{r^2} u_1 x_5 + \frac{g}{r^2} u_3 x_3 \right) \psi_2$$

$$- x_2 x_5 \psi_3 + x_1 x_5 \psi_4 + \left( x_2 x_3 - x_1 x_4 \right) \psi_5$$

$$+ \frac{1}{2} \left\{ w_1 x_1^2(t) + w_2 x_2^2(t) + w_3 x_3^2(t) + w_4 x_4^2(t) + w_5 \left( x_5(t) - 1 \right)^2 \right\}. \quad (20)$$
Algorithm 1 Computational Algorithm (●)

Require:
Choose the appropriate initial state \(x(0)\);
Set the length of time horizon \(T \in \mathbb{R}^+\) and the number of subintervals (of equal length) \(N \in \mathbb{Z}^+\);
Set step size \(\epsilon\), stopping criterion \(\tau\), maximum number of iterations \(K\) and control bounds \(u, \overline{u}\).

Ensure:
Optimal cost \(J^o\);
Optimal state trajectory \(x^o\);
Optimal control trajectory \(u^o\).

1: Subdivide equally the time horizon \(I = [0, T]\) into \(N\) subintervals and assume the control function is piecewise-constant. That is, \(u^n(t) = u^n(t_i)\), for \(t \in [t_i, t_{i+1})\), \(i = 0, 1, \ldots, N - 1\), where \(u^n(t), t \in I\) is the control (decision) policy at the \(n\)th iteration (starting from \(n = 0\)).

2: Integrate the state equations from 0 to \(T\) with initial state \(x(0) = x_0\) and the assumed controls \(u^{(n)} = u^n(t), t \in I\), store the obtained state trajectory \(x^{(n)}\) and the control vector \(u^{(n)}\).

3: Use \(x^{(n)}\) and \(u^{(n)}\) to integrate the adjoint equations backward in time starting from the costate \(\psi^{(n)}(T)\) at the terminal time. The terminal costate is given by \(\psi^{(n)}(T) = \Phi_x(x^{(n)}(T))\) where \(\Phi\) is the terminal cost.

4: Use the triple \(\{u^{(n)}, x^{(n)}, \psi^{(n)}\}\) to compute the gradient \(g_n(t) = \frac{\partial H}{\partial u^{(n)}}(x^{(n)}, u^{(n)}, \psi^{(n)}; x^{(n)}, t^{(n)}, \psi^{(n)})\) and store this vector.

5: Compute the cost functional \(J^{(n)}(u)\) using equation (19) and store this value.

6: If \(\|g_n\| < \tau\) then set \(u^n = u^{(n)}, J^n = J^{(n)}\), return.
Otherwise, go to Step 7.

7: Construct the control policy for the next iteration as \(u^{(n+1)}(t) = u^n(t) - \epsilon g_n(t), t \in I\) by choosing an appropriate \(\epsilon \in (0, 1)\) such that \(u^{(n+1)} \in U\). For the chosen \(\epsilon\), if \(u^{(n+1)} > \overline{u}\) set \(u^{(n+1)} = \overline{u}\); if \(u^{(n+1)} < u\) set \(u^{(n+1)} = u\).

8: If \(n < K\) then set \(n = n + 1\), go to Step 2.
Otherwise, display “Stopped before required residual is obtained”.

Hence the adjoint (costate) equations are given by
\[
\begin{align*}
\dot{\psi}_1 &= -H_{x_1} = -x_5\psi_4 + x_4\psi_5 - w_1x_1, \\
\dot{\psi}_2 &= -H_{x_2} = x_5\psi_3 - x_3\psi_5 - w_2x_2, \\
\dot{\psi}_3 &= -H_{x_3} = -\left(\frac{g}{r} + \frac{g}{r^2}u_3\right)\psi_2 - x_2\psi_5 - w_3x_3, \\
\dot{\psi}_4 &= -H_{x_4} = \left(\frac{g}{r} + \frac{g}{r^2}u_3\right)\psi_1 + x_1\psi_5 - w_4x_4, \\
\dot{\psi}_5 &= -H_{x_5} = -\frac{g}{r^2}u_2\psi_1 + \frac{g}{r^2}u_1\psi_2 + x_2\psi_3 - x_1\psi_4 - w_5(x_5 - 1).
\end{align*}
\]
4. Numerical algorithm and simulation results.

4.1. Computational algorithm. In this section, we present the computational algorithm (●) based on the minimum principle introduced above. A similar algorithm can also be found in [1, section 8.5, p. 283]. The basic algorithm is based on gradient method [6] and generates a sequence of control policies \( \{u^{(n)}\} \) along which the objective function \( J \) monotonically converges (decreases) to its minimum with an appropriate choice of step size \( \epsilon \). The proposed algorithm is implemented on MATLAB [23] and the key steps are described in Algorithm 1.

4.2. Numerical results. In this section, we conduct a series of simulation experiments involving three different scenarios. In scenario 1, we study the BMU dynamics with initial state far from the equilibrium and without any control applied. In scenario 2, the BMU dynamics is investigated with control under the same initial condition as that of scenario 1. In scenario 3, we consider the BMU system hit by high winds for a short but continuous period of time.

For numerical simulations, we assume the initial state of the BMU to be
\[
x(0) = (0.6, 0.4, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2})^T.
\]
The initial control (incremental displacement) vector is chosen as \( u(0) = (0, 0, 0)^T \) while the control constraint set is defined as
\[
U = \{u = (u_x, u_y, u_z)^T \in \mathbb{R}^3 : -3.0 \leq u_x, u_y, u_z \leq 3.0\}.
\]
The symmetric positive definite matrices \( Q \) and \( P \) are chosen as \( Q = \text{diag}(10, 10, 10, 10, 10) \) and \( P = \text{diag}(200, 200, 200, 200, 200) \), respectively. The overall system performance is evaluated over the time period \( I = [0, 80] \) for scenarios 1 and 2, while it is chosen as \( I = [0, 100] \) for scenarios 3. We run the simulation for 1000 iterations and present comprehensive numerical results in the following three parts. Note that the simulation results are obtained on the basis of the parameters described above and they are presented only for the purpose of illustration.

A. Scenario 1. In this part, we simulate the BMU dynamics with a given initial condition without any control applied. The angular velocity and the attitude vector of the BMU are presented in Figure 3a and 3b, respectively.

![Figure 3. Simulation results of scenario 1.](image)

Figure 3 shows the state trajectories of the BMU. It is observed that both the angular velocity and the attitude of the BMU oscillate periodically but does neither grow nor decay. This is due to the fact that the system without control is conservative and hence stable in the sense of Lyapunov as shown in Lemma 2.1. The cost of this scenario is 393.7665 which can be easily obtained by equation (11).
B. Scenario 2. In this part, we use reaction jets to produce control torques applied to the BMU so as to stabilize it under the same initial condition as in the first scenario. We present two sets of numerical results corresponding to two different control constraint sets, namely the larger set $U_1 = \{ u = (u_x, u_y, u_z)^T \in \mathbb{R}^3 : -3.0 \leq u_x, u_y, u_z \leq 3.0 \}$ and the smaller one $U_2 = \{ u = (u_x, u_y, u_z)^T \in \mathbb{R}^3 : -0.5 \leq u_x, u_y, u_z \leq 0.5 \}$. The results corresponding to $U_1$ and $U_2$ are shown in Figure 4 and 5, respectively.

Figure 4a and 4b show the velocity and attitude trajectories, respectively, with the control constraint set being $U_1$. It is observed that the angular velocities as well as the attitude components ($x_3$ and $x_4$) reduce to zero while the last component of the attitude vector $x_5$ reaches 1. This means that the optimal control forces the BMU to its equilibrium state $x^e = (0, 0, 0, 0, 1)^T$ which is what is desired. Figure 4c shows that the optimal control oscillates and eventually decays to zero as the equilibrium state is approached. In this case, the final optimal cost after 1000 iterations is 34.4595 as shown in Figure 4d.

For comparison, we simulate the same dynamic system with the same parameters but with a smaller control constraint set $U_2$. It is clear that the results shown in Figure 5 have similar pattern as seen in Figure 4. However, because of limited control energy, it takes a bit longer time to bring the system to the desired rest state $x^e$. In Figure 5c, it is observed that the control variable stays at $\pm 0.5$ (maximum magnitude) for some time due to the fact that $U_2$ is smaller than $U_1$ and $U_2$ is bounded by $\pm 0.5$. In this case, the final optimal cost after 1000 iterations is 50.6926 as shown in Figure 5d. This is larger than the optimal cost corresponding to $U_1$.
since the running cost is greater in this case. It is clear that the optimal cost corresponding to either \( U_1 \) or \( U_2 \) is much smaller than the cost of scenario 1 because there is no control applied at all in the first scenario.

\section*{C. Scenario 3}

In this part, we study the dynamics of the BMU system in the presence of high winds for a particular but short period of time. For the purpose of numerical experiments, we assume that this period is given by \( \Delta = [20, 30] \). The two components of the wind force are represented by \( F_1 = 0.3 \) and \( F_2 = 0.2 \).

That is

\[
F_1(t) = \begin{cases} 
0.3, & t \in \Delta, \\
0, & \text{otherwise}. 
\end{cases}
\]

\[
F_2(t) = \begin{cases} 
0.2, & t \in \Delta, \\
0, & \text{otherwise}. 
\end{cases}
\]

\section*{Case 1}

For case 1 in this scenario, we study the BMU dynamics in the presence of continuous high winds without any control applied. We conduct numerical experiments with the same initial condition as before. In this case the operating period is chosen as \( I = [0, 100] \). The simulation results are shown in Figure 6. It is observed that the state of the system is significantly disturbed by the high winds. The magnitude of the state variable increases, meaning that the BMU violently oscillates and tends to become unstable putting the workers and the public at higher risk. Similarly, by use of equation \( (11) \), one can easily obtain the cost 1016.1 of this case. It is much larger than that of scenario 1 (without control applied) due to
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Figure 6. Simulation results of case 1 in scenario 3.

Figure 7. Simulation results of case 2 in scenario 3.

the fact that the BMU has experienced a significant displacement under high-wind conditions with no control applied.

Case 2. For case 2 in this scenario, we investigate optimal control of the BMU in the presence of continuous high winds in order to stabilize it and thereby reducing the risk to the workers and the public. In this case, with high winds taking effect,
the controlled system dynamics given by equation (8) is rewritten as
\[
\begin{align*}
\dot{x}_1 &= f_1(x,u) + F_1, \\
\dot{x}_2 &= f_2(x,u) + F_2, \\
\dot{x}_3 &= f_3(x,u), \\
\dot{x}_4 &= f_4(x,u), \\
\dot{x}_5 &= f_5(x,u).
\end{align*}
\]
(21)

Based on the characters of the high winds described above, we carry out simulation experiments with the control constraint set \(U\) and the same initial condition as used before. The same operating period \(I = [0, 100]\) is used as in the previous case. The numerical results are shown in Figure 7. It is observed from Figure 7a and 7b that the system can be brought to the equilibrium state after a certain period of time. We can also observe that the state, in particular the attitude of the BMU (from Figure 7b), experiences a disturbance during \(\Delta = [20, 30]\) when the high winds take place. This is marked with green rectangles in Figure 7a and 7b. However, it is observed that the system can still be stabilized and brought to its equilibrium state though it takes a longer period of time. The final (after 1000 iterations) optimal cost of this case is 55.2655 as shown in Figure 7d. It is much smaller than the cost of case 1 (1016.1) since there is no control applied in case 1. However, the optimal cost of case 2 is larger than that of scenario 2. This is expected because the system has experienced a disturbance due to high winds.

5. Conclusion. In this paper we have presented a dynamic model for optimal control of a BMU (building maintenance unit) in order to stabilize it in the presence of high winds. The system dynamics is given by a set of nonlinear differential equations describing the physics of the BMU. Based on the system model presented, we formulate an optimization problem with the objective of stabilizing the BMU and thereby reducing the risk to the workers and the public. In order to address the problem formulated, we have developed a computational algorithm based on Pontryagin minimum principle, and used it to determine the optimal controls. It is clear that optimal controls depend on multiple factors, such as the control constraint set \(U\), the total operating period \(I\), and the weighting matrices \(Q\) and \(R\), etc. By carrying out a series of numerical simulations for three different scenarios, we have demonstrated that the system can be stabilized and brought to its equilibrium state in a reasonable period of time by using optimal controls. Thus it is possible to maintain stability of the system even in the presence of high winds.

In order to apply the technique proposed here, a number of reaction jets should be properly mounted on the BMU body so that appropriate control torques can be provided whenever it is necessary. We believe that this methodology would be very useful for stabilizing the BMU thereby reducing the risk to the workers and the general public.

REFERENCES

[1] N. U. Ahmed, Dynamic Systems and Control with Applications, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
[2] N. U. Ahmed, Elements of Finite Dimensional Systems and Control Theory, Pitman Monographs and Surveys in Pure and Applied Mathematics, 37, John Wiley & Sons, Inc., New York, 1988.
[3] T. Ahmed and N.U. Ahmed, Optimal Control of Antigen-Antibody Interactions for Cancer Immunotherapy, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms*, 26 (2019), 135–152.

[4] D. Allen, *What is building maintenance?*, *Facilities*, 11 (1993), 7–12.

[5] *Building Maintenance Units*, Report of Alimak Group AB. Available from: [https://alimakservice.com/building-maintenance-units/](https://alimakservice.com/building-maintenance-units/).

[6] T. R. Chandrupatla, A. D. Belegundu and T. Ramesh, et al., *Introduction to Finite Elements in Engineering*, Prentice Hall, 2002.

[7] J. C. P. Cheng, W. Chen and Y. Tan, et al., A BIM-based decision support system framework for predictive maintenance management of building facilities, 16th International Conference on Computing in Civil and Building Engineering, Osaka, Japan, 2016, 711–718.

[8] T. Glad and L. Ljung, *Control Theory*, CRC Press, London, 2014.

[9] R. M. W. Horner, M. A. El-Haram and A. K. Munns, Building maintenance strategy: A new management approach, *J. Quality Maintenance Engineering*, 3 (1997), 273–280.

[10] *Instability of Building Maintenance Units*, WorkSafe Victoria, 2018. Available from: [https://www.worksafe.vic.gov.au/safety-alerts/instability-building-maintenance-units](https://www.worksafe.vic.gov.au/safety-alerts/instability-building-maintenance-units).

[11] C. H. Ko, RFID-based building maintenance system, *Automat. Construction*, 18 (2009), 275–284.

[12] H. Lind and H. Muyingo, Building maintenance strategies: Planning under uncertainty, *Property Management*, 30 (2012), 14–28.

[13] P. Maryam, N.U. Ahmed and M.C.E. Yagoub, Optimum Decision Policy for Replacement of Conventional Energy Sources by Renewable Ones, *International Journal of Energy Science*, 3 (2013), 311–319.

[14] I. Motawa and A. Almarshad, A knowledge-based BIM system for building maintenance, *Automat. Construction*, 29 (2013), 173–182.

[15] K. Ogata and Y. Yang, *Modern Control Engineering*, Prentice Hall, New Jersey, 2002.

[16] L. S. Pontryagin, V. G. Boltyanski and R. V. Gamkrelidze, et al., *The Mathematical Theory of Optimal Processes*, The Macmillan Co., New York, 1964.

[17] I. H. Seeley, *Building Maintenance*, Building and Surveying Series, Palgrave, London, 1987.

[18] J. Shen, A. K. Sanyal and N. A. Chaturvedi, et al., *Dynamics and control of a 3D pendulum*, 43rd IEEE Conference on Decision and Control, Atlantis, Bahamas, 2004, 323–328.

[19] M.M. Suruz, N.U. Ahmed and M. Chowdhury, Optimum policy for integration of renewable energy sources into the power generation system, *Energy Economics*, 34 (2012), 558–567.

[20] K. L. Teo, C. J. Goh and K. H. Wong, *A Unified Computational Approach to Optimal Control Problems*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 55, John Wiley & Sons, Inc., New York, 1991.

[21] S. Wang and N.U. Ahmed, Dynamic model of urban traffic and optimum management of its flow and congestion, *Dynamic Systems and Applications*, 26 (2017), 575–588.

[22] S. Wang, N.U. Ahmed and T.H. Yeap, Optimum management of urban traffic flow based on a stochastic dynamic model, *IEEE Transactions on Intelligent Transportation Systems*, 20 (2019), 4377–4389.

[23] X. Wang, Solving optimal control problems with MATLAB: Indirect methods, ISE Dept., NCSU, Raleigh, NC, 2009.

[24] D. V. Zenkov, *On Hamel's equations*, *Theoret. Appl. Mechanics*, 43 (2016), 191–220.

[25] D. V. Zenkov, M. Leok and A. M. Bloch, *Hamel's formalism and variational integrators on a sphere*, 51st IEEE Conference on Decision and Control, Hawaii, 2012, 7504–7510.

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