Monochromatic metrics are generalized Berwald

Nina Bartelmeß and Vladimir S. Matveev

Abstract

We show that monochromatic Finsler metrics, i.e., Finsler metrics such that each two tangent spaces are isomorphic as normed spaces, are generalized Berwald metrics, i.e., there exists an affine connection, possibly with torsion, that preserves the Finsler function.

Contents

1 Introduction

1.1 Definitions and main result ........................................ 1
1.2 History and motivation ........................................... 2

2 Proof of Theorem 1 .................................................. 3

2.1 Locally, one can choose $A_x$ such that it depends smoothly
on $x$. ..................................................................... 4
2.2 Construction of the associated connection ..................... 7

3 Existence of generalized Berwald metrics on 2- and 3-
dimensional closed manifolds ......................................... 9

1 Introduction

1.1 Definitions and main result

A Finsler metric on a smooth manifold $M$ of dimension $n \geq 2$ is a function $F: TM \to [0, \infty)$ such that for every point $x \in M$ the restriction $F_x = F|_{T_x M}$ is a Minkowski norm. That means that

1. $F_x(\lambda \xi) = \lambda F_x(\xi)$ for all $\lambda \geq 0, \xi \in T_x M$
2. $F_x(\xi + \eta) \leq F_x(\xi) + F_x(\eta)$ for all $\xi, \eta \in T_x M$
3. $F_x(\xi) = 0 \Rightarrow \xi = 0$

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We allow irreversibility and do not require strict convexity. We assume that $F$ is smooth on the slit tangent bundle $TM \setminus \{0\}$.

**Definition 1.1.** A Finsler metric $F$ on a connected manifold $M$ is said to be a *generalized Berwald metric* if there exists a smooth affine connection $\nabla$ on $M$, called *associated connection*, whose parallel transport preserves the Finsler function $F$.

That is, for every $x, y \in M$ and for any curve $\gamma: [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ the parallel transport $P_{\gamma}^\nabla: T_x M \to T_y M$ along this curve is an isomorphism of the normed spaces $(T_x M, F_x)$ and $(T_y M, F_y)$ in the sense

$$F_y(P_{\gamma}^\nabla(\xi)) = F_x(\xi) \text{ for all } \xi \in T_x M.$$ 

In the definition above, the connection may have a torsion.

**Definition 1.2.** A Finsler metric $F$ is called *monochromatic* if for every two points $x, y \in M$ there exists a linear isomorphism between the tangent spaces at these points which is an isometry with respect to $F_x$ and $F_y$.

Clearly, as one of these points one can take some fixed point $x_0$, so monochromacy of a Finsler metric is equivalent to the existence of a field of linear isomorphisms $A_x := A(x): T_{x_0} M \to T_x M$ such that

$$F(x, A_x(\xi)) = F(x_0, \xi) \text{ for all } \xi \in T_{x_0} M.$$ 

We do not assume that $A_x$ depends smoothly or even continuously on $x$. This definition is due to David Bao [3] and is motivated by a suggestion of Zhongmin Shen, who proposed to assign a unique color to each Minkowski norm. Generic Finsler spaces are then “multicolored” because different points of the manifold will generically correspond to different colors. Monochromatic manifolds are such that all points correspond to the same color. Our main result is:

**Theorem 1.** Let $(M, F)$ be a connected Finsler manifold. Then, $F$ is a generalized Berwald metric if and only if $F$ is monochromatic.

In Theorem 1 we assume that $M$ is at least of class $C^k$ and $F$ is at least of class $C^{k-1}$, $k \geq 2$. From the proof it will be clear that the associated connection will be at least of class $C^{k-2}$. At the end of the paper, in §3 we discuss the existence of non-Riemannian generalized Berwald metrics on closed manifolds of small dimensions.

**1.2 History and motivation**

Definitions obviously similar to the definition of monochromatic metrics appeared many times in the literature, possibly first time in a
commentary of Hermann Weyl on Riemann’s habilitation address [12]. There, Weyl suggested to consider Finsler manifolds such that all tangent spaces are isomorphic as normed spaces. It is not clear though whether he assumed that the field of isomorphisms $A_x$ in definition 1.2 depends smoothly on $x$.

In 1965 Detlef Laugwitz [8] referred to Weyl’s idea and suggested the following definition: he called a Finsler metric *metrically homogeneous* if, in our terminology, it is monochromatic and if in addition the field of linear isomorphisms $A_x$ in our definition of a monochromatic metric depends smoothly on $x$. An equivalent definition was given by Yoshimiro Ichijyo [6] who called such Finsler metrics *Finsler metrics modeled on a Minkowski space*. Other equivalent definitions exist in the literature; such Finsler metrics were called *1-form metrics* in e.g. [9] and *affine deformations of Minkowski spaces* in e.g. [13].

It is easy to show and was independently done in [8, Exercise 15.4.1], [6, Theorem 2] and, quite recently, in [13, Theorem 1], that (locally) such metrics are generalized Berwald metrics. We essentially repeat their proof at the end of the proof of Theorem 1.

Many special cases of Theorem 1 were proved before. For example, [2] proved it for left-invariant Finsler metrics and [14] proves it for $(\alpha,\beta)$-metrics such that the $\alpha$-norm of $\beta$ is constant (in this paper it is also assumed that the metric should satisfy the so called sign property, but actually by Theorem 1 this additional assumption can be omitted), see also [15].

Recent interest to generalized Berwald spaces and monochromatic metrics is due to their relation to the Landsberg unicorn problem, see e.g. [16, 17], and because for these metrics one obtains relatively simple formulas for different curvature-type invariants, see e.g. [1, 4, 5, 9].

Recently, a 2-dimensional version of our Theorem 1 was independently proved and applied in [7].

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**2 Proof of Theorem 1**

The direction “$\Rightarrow$” is easy and was done many times before, in particular in the above mentioned [6, 8]. Indeed, if $F$ is a generalized Berwald metric on a connected manifold, then for every two points $x, y \in M$ the parallel transport $P^v_x$ along any curve connecting $x$ and $y$ gives us an isomorphism of $(T_xM, F_x)$ and $(T_yM, F_y)$.

The proof in the direction “$\Leftarrow$” goes as follows: we first show that, though we do not require *a priori* that $A_x$ depends smoothly on $x$,
one can locally choose it such that it depends smoothly on $x$. As we explained before, the case when $A_x$ depends smoothly on $x$ was solved before by Laugwitz [5] and Ichiyjo [6]. To make this work self-contained we repeat their proof. The last step is the transition from local to global, it uses a standard trick using the partition of unity argument.

2.1 Locally, one can choose $A_x$ such that it depends smoothly on $x$.

Let $F$ be a smooth Finsler metric. We assume that it is not a Riemannian metric; the case of Riemannian metrics is trivial since they are automatically Berwald. We will work in a sufficiently small neighborhood of a point $p$, let $x = (x_1, ..., x_n)$ be a local coordinate system in this neighborhood.

We consider the Riemannian Binet-Legendre metric $g = g_F$ corresponding to $F$. The definition and properties of $g_F$ are in [10]. We will work in a local orthonormal frame $e_1(p), ..., e_n(p)$ with respect to this metric (i.e., $g_F(e_i, e_j) = \delta_{ij}$). The coordinates of tangent vectors will be coordinates in this frame. The local existence of such an orthonormal frame is known and immediately follows from the Gram-Schmidt orthogonalization process.

Next, we consider the Minkowski norm $F_p$ on $T_pM$, $m \in \mathbb{N}$ and vectors $\xi_1, ..., \xi_m \in T_pM$ such that the following conditions hold:

(I) The differentials of the functions $\psi_1, ..., \psi_m : SO(n) \to \mathbb{R}$, $\psi_i(B) = F_p(B\xi_i)$ are linearly independent at $B = \text{Id}$.

(II) The number $m$ is a maximal number with the above property.

The notation $B\xi_i$ simply means the multiplication of a matrix $B \in SO(n)$ with the vector $(\xi_1^i, ..., \xi_n^i) \in \mathbb{R}^n$, where $\xi_1^i, ..., \xi_n^i$ are coordinates of $\xi_i$ in the orthonormal frame $e_1(p), ..., e_n(p)$. The resulting element of $\mathbb{R}^n$ will be identified with a vector of $T_pM$ (later also with a vector of $T_xM$) via the basis $e_1(p), ..., e_n(p)$ (later, via the basis $e_1(x), ..., e_n(x)$).

The existence of such a number $m$ and the vectors $\xi_1, ..., \xi_m$ is trivial. Indeed, $m$ such that (I) holds is bounded from above by $\frac{n(n+1)}{2}$ and because our Finsler metric is not a Riemannian metric the function $B \mapsto F_p(B\xi)$ is not a constant (for each fixed $\xi \neq 0$) which implies the existence of at least one $\xi$ with property (I).

It is clear that the number $\frac{n(n+1)}{2} - m = \dim(SO(n)) - m$ is the dimension of the group of endomorphisms of $T_pM$ preserving $F_p$. Let us consider a local coordinate system $B = (b_1, ..., b_{\frac{n(n+1)}{2}})$ on $SO(n)$ in a small neighborhood of the neutral element $\text{Id}$. In this coordinates, $d\psi_i$ is simply the $\frac{n(n+1)}{2}$-tuple $\left(\frac{\partial \psi_i}{\partial b_1}, ..., \frac{\partial \psi_i}{\partial b_{\frac{n(n+1)}{2}}}\right)$ and the condition
that the differentials of $\psi_i$ are linearly independent means that the matrix

$$\left( \frac{\partial \psi_i}{\partial b_j} \right)_{i=1,\ldots,m; j=1,\ldots,n(n+1)/2}$$

has rank $m$. Without loss of generality we will assume that the coordinates $b_1, \ldots, b_{n(n+1)/2}$ are chosen such that the last $m$ columns of the matrix (1) form a nondegenerate matrix.

Let us now show that, possibly in a smaller neighborhood of $p$, one can choose the field $A_x$ in definition 1.2 such that it depends smoothly on $x$. We will use the Implicit Function Theorem; though it is well known, we formulate it below in order to fix the terminology.

**Implicit Function Theorem.** Consider $W \subseteq \mathbb{R}^{k+m}$ with coordinates $(X,Y) = (X_1,\ldots,X_k,Y_1,\ldots,Y_m)$. Let $\Phi: \mathbb{R}^{k+m} \to \mathbb{R}^m$ be a continuously differentiable mapping and $(\hat{X},\hat{Y}) \in \mathbb{R}^{k+m}$ a point with $\Phi(\hat{X},\hat{Y}) = c$, where $c \in \mathbb{R}^m$. If the $m \times m$-matrix

$$\left( \frac{\partial \Phi_i}{\partial y_j} \right)_{i,j=1,\ldots,m}(\hat{X},\hat{Y}) \neq 0$$

is nondegenerate, then there exists a non-empty open set $U \subset \mathbb{R}^k$ containing $\hat{X}$, an open set $V \subset \mathbb{R}^m$ containing $\hat{Y}$ and a unique continuously differentiable mapping $g: U \to V$ such that for all $X \in U$ and such that for all points $(X,Y) \in U \times V$ with $\Phi(X,Y) = c$ we have that $Y = g(X)$. Moreover, if $\Phi$ is of class $C^\ell$, $\ell \geq 1$, then $g$ is also of class $C^\ell$.

Let us now apply this theorem to our situation. Set $k = n + \left( \frac{n(n+1)}{2} - m \right)$. The first $n$ coordinates of $X = (X_1,\ldots,X_k)$ will be denoted by $x = (x_1,\ldots,x_n)$, one should think about them as about local coordinates in a neighborhood of $p$. The remaining $\left( \frac{n(n+1)}{2} - m \right)$ coordinates of $X = (X_1,\ldots,X_k)$ will be denoted by $B_1 = (b_1,\ldots,b_{n(n+1)/2-m})$, one can think about them as about the first portion of the local coordinates on $SO(n)$ as discussed above. The coordinates $y = (y_1,\ldots,y_m) = (B_2)$ should be viewed as the remaining portion of the local coordinates on $SO(n)$ as discussed above.

Next, consider the mapping $\Phi: \mathbb{R}^{k+m} \to \mathbb{R}^m$, whose $j$-th component is given by

$$\Phi_j(x,B) = F(x,B\xi_j).$$

The vectors $\xi_j$ are precisely the vectors $\xi_1,\ldots,\xi_m$ described above. As $(\hat{X},\hat{Y})$ we take the point corresponding to $(x_1,\ldots,x_n) = p, B = \text{Id}$. 

5
The differential of the mapping $\Phi$ at $(\hat{X}, \hat{Y})$ is given by the following $m \times k$-matrix in which all partial derivatives are taken at the point $(X,Y) = (\hat{X}, \hat{Y})$:

$$
\left. d\Phi \right|_{B=\text{Id}} = \begin{pmatrix}
\frac{\partial F(x, B\xi)}{\partial x} & \frac{\partial F(x, B\xi)}{\partial B_1} & \cdots & \frac{\partial F(x, B\xi)}{\partial B_k}
\end{pmatrix}.
$$

By construction the last $m$ columns of the matrix form a nondegenerate $m \times m$ matrix, which is precisely the nondegenerate submatrix of matrix (1). Thus, all assumptions of the Implicit Function Theorem are satisfied and therefore there exists the smooth mapping $B_2(x, B_1)$ such that for $j = 1, \ldots, m$ holds

$$
\Phi(x, B_1, B_2(x, B_1)) = \Phi(\hat{X}, \hat{Y}).
$$

Next, we construct a family $B_x \in SO(n)$ which depends smoothly on $x$ and which should be viewed as a field of isomorphisms $B(x) = B_x : T_pM \to T_xM$. In the local coordinates $B$ this family is given by

$$
B_x := (B_1, B_2(x, B_1)).
$$

Here $B_1$ is composed of the first $\frac{n(n+1)}{2} - m$ components of Id. By construction of $B_x$, we have

$$
F(x, B_x(\xi_i)) = F(p, \xi_i) \text{ for } i = 1, \ldots, m.
$$

Our next goal is to prove that (4) holds for all $\xi \in T_pM$. In order to do this, we consider the following two subsets of $SO(n)$:

- $U = \{u \in SO(n) \mid \forall \xi \in T_pM : F(p, \xi) = F(p, u(\xi))\}$
- $U' = \{u \in SO(n) \mid F(p, \xi_i) = F(p, u(\xi)) \text{ for } i = 1, \ldots, m\}$

Both subsets are compact, $U$ is a Lie subgroup of $SO(n)$ and we have $U \subseteq U'$. Let us now show that $U' \setminus U$ is also compact. It is sufficient to show that $U' \setminus U$ is closed, i.e., we need to show the nonexistence of a sequence $u_1, \ldots, u_n, \ldots \in U' \setminus U$ such that it converges to an element of $U$.

We will use that the elements $u \in U'$ such that they are sufficiently close to Id $\in SO(n)$ automatically lie in $U$. Indeed, in a small neighborhood of Id both $U$ and $U'$ are submanifolds of $SO(n)$ of the same dimension $\frac{n(n+1)}{2} - m$ and $U \subseteq U'$.

Suppose a sequence $\{u_l\}_{l \in \mathbb{N}} \in U' \setminus U$ converges to $u \in U$. We consider the sequence $\{u^{-1}u_l\}_{l \in \mathbb{N}}$. It converges to $u^{-1}u = \text{Id}$. Clearly,
all elements of the sequence \( \{u^{-1}u_l\}_{l \in \mathbb{N}} \) lie in \( U' \). Then, the elements of this sequence with sufficiently big indices also lie in \( U' \). But this would imply that the corresponding elements of the sequence \( \{u_l\}_{l \in \mathbb{N}} \) lie in \( U \), which contradicts the assumption. Thus, \( U' \setminus U \) is compact.

Now we can show that for each \( x \in M \) which is sufficiently close to \( p \), the linear isomorphisms \( B_x \) constructed above are isomorphisms of the normed spaces \( (T_pM, F_p) \) and \( (T_xM, F_x) \). Since our Finsler metric \( F \) is monochromatic, there exists a linear isomorphism \( A_x : T_pM \to T_xM \) such that

\[
F(x, A_x(\xi)) = F(p, \xi) \quad \text{for all } \xi \in T_pM.
\]

Consider \( A_x^{-1}B_x \) which lies in \( U' \) by construction. In order to show that (4) holds for \( B_x \), it is sufficient to show that \( A_x^{-1}B_x \in U \). Let \( \phi : SO(n) \to \mathbb{R} \) be the following function:

\[
\phi(u) = \int_K |F(p, \xi) - F(p, u(\xi))| d\text{vol}_{g_F},
\]

where \( K \) is the unit ball and \( \text{vol}_{g_F} \) the volume form of the Binet-Legendre metric \( g_F \). The function \( \phi \) is continuous and nonnegative. Moreover, \( \phi(u) = 0 \) if and only if \( u \in U \). Then, because of the compactness of \( U' \setminus U \), there exists an \( \epsilon > 0 \) such that

\[
\phi|_{U' \setminus U} > \epsilon.
\]

Let us now consider \( \phi(A_x^{-1}B_x) \):

\[
\phi(A_x^{-1}B_x) = \int_K |F(p, \xi) - F(p, A_x^{-1}B_x(\xi))| d\text{vol}_{g} = \int_K |F(p, \xi) - F(p, B_x(\xi))| d\text{vol}_{g} = \phi(B_x)
\]

But \( \phi(B_p) = \phi(\text{Id}) = 0 \), so that for \( x \) sufficiently close to \( p \) we have \( \phi(A_x^{-1}B_x) < \epsilon \), which implies that \( A_x^{-1}B_x \in U \) and hence \( B_x \in U \). Thus, locally one can find a field \( B_x : T_pM \to T_xM \) of isomorphisms of the normed spaces \( (T_pM, F_p) \) and \( (T_xM, F_x) \).

2.2 Construction of the associated connection

Let us now show that our metric (still in a small neighborhood of the point \( p \)) is a generalized Berwald metric. Our proof repeats, in slightly different notations, the proofs of Laugwitz [8] and Ichijyo [6].

Take a basis \( b_1 = \frac{\partial}{\partial x_1}, \ldots, b_n = \frac{\partial}{\partial x_n} \) on \( T_pM \) and for each point \( x \) of our small neighborhood consider the basis \( b_1(x) = B_x b_1, \ldots, b_n(x) = \ldots \)
Let us now construct an affine connection, possibly with torsion, such that its parallel transport along any curve connecting \( p \) and \( x \) maps \( b_i(p) \) to \( b_i(x) \). Clearly, this condition on the connection is equivalent to the condition

\[
\nabla b_j(x) = 0.
\]

In coordinates, (5) means that for all \( i \) we have

\[
0 = \nabla_i b^j_k = \partial_i b(x)^j + \Gamma^j_{si} b(x)^s.
\]

By construction, in coordinates, \( b_1(x), \ldots, b_n(x) \) are just columns of the matrix \( B(x) \). Next, consider the matrices

\[
\Gamma_i = \begin{pmatrix}
\Gamma^1_{1i} & \cdots & \Gamma^1_{ni} \\
\vdots & \ddots & \vdots \\
\Gamma^n_{1i} & \cdots & \Gamma^n_{ni}
\end{pmatrix}.
\]

In this notation, (5) reads

\[
\partial_i B(x) = -\Gamma_i B(x)
\]

and clearly has a solution

\[
\Gamma_i = -\partial_i B(x) B(x)^{-1}.
\]

This gives us Christoffel symbols such that the parallel transport associated to this connection along any curve connecting \( p \) and \( x \) is given by \( B_x \), and therefore is an isometry of the normed spaces \( (T_p M, F^p_p) \) and \( (T_x M, F_x) \). We therefore proved the local version of Theorem 1.

Finally, we want to prove that there exists an affine connection on the whole \( M \) which preserves \( F \). For each point \( x \in U \), consider a neighborhood \( U(x) \) such that in this neighborhood there exists an affine connection \( \tilde{\nabla} \) on it which is an associated connection of \( F \). Consider a partition of unity \( f_x \) subordinated to this open cover, its existence is standard. Next, set \( \nabla = \sum_x f_x \tilde{\nabla} \) (though the set of points \( x \) is infinite, at each sufficiently small neighborhood only finitely many terms in the sum are unequal to zero, guaranteed by the definition of the partition of unity; so the sum is well-defined). It is known and can easily be checked directly that this formula indeed defines an affine connection; it is easy to check that the parallel transport in this connection preserves our Finsler metric \( F \).
3 Existence of generalized Berwald metrics on 2- and 3-dimensional closed manifolds

We start with dimension 2. Clearly, the torus and the Klein bottle have non-Riemannian generalized Berwald metrics (induced by a Minkowski metric on the universal cover). Let us show that the other closed 2-dimensional manifolds can not have non-Riemannian generalized Berwald metrics. In order to prove this, observe that the group $SO(2)$ is one-dimensional and thereby each of its proper connected subgroup is discrete. Then, the local holonomy group of the associated affine connection of a non-Riemannian generalized Berwald metric is trivial, which implies, by the Ambrose-Singer Theorem, that the associated affine connection is flat. Then, by [1], the surface has Euler characteristic equal to zero.

In higher dimension we can prove the following:

**Theorem 2.** Every closed manifold with Euler characteristic zero admits a non-Riemannian generalized Berwald metric.

**Proof.** It is known that on manifolds with Euler characteristic zero there exists a non-vanishing vector field $V$. Take now a Riemannian metric $g$ on $M$ and normalize $V$ to the length $\frac{1}{2}$ with respect to $g$. Consider now the Randers metric $F_R$ on $T_x M$ generated by $g$ and $V$, i.e., the unit balls of $F_R$ are the $V$-translations of the unit balls of $g$. Those convex balls in every tangent space $T_x M$ are isomorphic to each other. By our Theorem 1 we obtain a generalized Berwald metric. □

**Corollary 3.** In dimension 3 every closed manifold has Euler characteristic zero and thus admits a non-Riemannian generalized Berwald metric.

It is an interesting problem to understand in all dimensions what manifolds can admit non-Riemannian generalized Berwald metrics.

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Institute of Mathematics, FSU Jena, 07737 Jena Germany, nina.bartelmess@uni-jena.de, vladimir.matveev@uni-jena.de