Considering the Motion of the Elastic Lunar Pole

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Abstract. In connection with the development of the Japanese project ILOM on the study of the rotational motion of the Moon using a telescope mounted on the surface of the moon (near the north pole), there are increased demands on the accuracy of the analytical and numerical description of its physical librations. In particular, studies of the influence of the elastic properties of the Moon on its rotational motion with the high-precision description of its orbital motion and taking into account the resonant nature of motion are of a significant importance. In this work it is shown that the elastic properties of the Moon result in extending of the oscillations period of its pole by 8.0 days.

1. Introduction
The studies of the rotational and translational-rotational motion of the viscous-elastic Earth and the Moon are devoted to the work of many authors [1], [2]. Basically, these works were devoted to the study of the evolution of the Earth-Moon system. For modern studies of the Moon, it is also important to study the manifestations of the elastic properties of the Moon in its physical libration. In connection with the development of the Japanese project ILOM on the study of the rotational motion of the Moon using a telescope mounted on the surface of the Moon (near its North Pole), there are increased demands on the accuracy of the analytical and numerical description of its physical librations [3], [4]. In particular, studies of the influence of the liquid core of the Moon, its solid core, the elasticity of the mantle of the Moon on its rotational motion with a highly accurate description of the orbital motion of the Moon and taking into account the resonant nature of motion are of a significant interest. In this paper, we explore the dynamic role of the elasticity of the Moon in the movement of the pole of its axis of rotation. The influence of the liquid and solid core of the Moon on its libration is not taken into account in this work. To describe the rotational motion of the Moon the equations of the Liouville problem in Andoyer variables [5] and the perturbation theory [6] is used. The significant effects that must be taken into account when interpreting observational data using the lunar telescope are found. In this work it is shown that the elastic properties of the Moon lead to an extension in the oscillations period of its pole by 7-8 days.

2. Statement of the problem. The canonical equations of rotational motion of a deformable body in Andoyer variables.
We will consider the Moon as a weakly deformed body with deformations due to its own rotation and under the influence of the Earth gravitational force. We assume that its particles weakly deviate from their original positions during the motion process or are displaced in a known manner with respect of time with a small speed. It can also be assumed that the body has an internal solid shell, and we associate a Cartesian coordinate system $C_{ξηζ}$ with it (body axes), and an external deformable shell.
Let the Cartesian coordinate system \( C_{xyz} \) with the origin at the Moon’s center of mass maintain a constant orientation in space.

We introduce the Andoyer variables that are associated with the vector of the angular momentum \( \mathbf{G} \) of the Moon:

\[ \mathbf{G}, \theta, \rho, l, g, h \quad (1) \]

Let \( CG, G_1, G_2, G_3 \) be the intermediate coordinate system associated with the vector \( \mathbf{G} \). The axis \( CG \) is directed along the vector \( \mathbf{G} \), and the axis \( CG_1 \) is located in the \( C_{xy} \) plane of the main coordinate system and is directed along the line of intersection of the planes \( CG_1 G_2 \) and \( C_{xy} \) in the direction of the ascending node of the plane \( CG_1 G_2 \) (Figure 1). Let \( G = |\mathbf{G}| \) be the module of the angular momentum vector, \( \rho \) and \( h \) are angles that uniquely determine the orientation of the coordinate system \( CG_1 G_2 G_3 \) axes of the with respect to the coordinate system \( C_{xyz} \): \( \rho \) is an angle between the axis \( C_z \) and the angular momentum vector \( \mathbf{G} \), \( h \) is an angle between the positive directions of the coordinate axes \( C_x \) and \( CG_1 \) (\( h \) is the longitude of the ascending node of the intermediate plane \( CG_1 G_2 \)).

The orientation of the axes \( C_\xi, \eta, \zeta \) with respect to the intermediate coordinate system \( CG_1 G_2 G_3 \) is determined by the Euler angles \( l, g, \theta \) (Figure 1). Nutation angle \( \theta \) is the angle between the positive directions of the axes \( CG_1 \) and \( C_\xi \). The precession angle \( g \) is the angle between the axis \( CG_1 \) and the line of intersection of the coordinate planes \( CG_1 G_2 \) and \( C_\xi \). The angle of self rotation \( l \) is the angle between the positive direction to the ascending node of the \( C_\xi, \eta \) plane and the \( C_\xi \) axis.

Thus, if the angles \( h \) and \( \rho \) set the orientation of the intermediate coordinate system \( CG_1 G_2 G_3 \) with respect to the main coordinate system, and the Euler angles \( l, g, \theta \) set the orientation of the connected coordinate axes with respect to the intermediate coordinate system \( CG_1 G_2 G_3 \).

![Figure 1. Basic coordinate systems and Andoyer variables.](image-url)
In the figure 1 we denote the unit vectors of the corresponding coordinate systems $C_{xyz}$, $CG_1G_2G_3$, and $C_{\xi\eta\zeta}$ respectively:

$$i_x, j_y, k_z; \quad i_G, j_G, k_G; \quad i_b, j_b, k_b,$$

and also the unit vectors $e_{a_x}, e_{a_y}$ and $e_{a_z}$ are directed along the intersection lines of the corresponding coordinate planes $C_{xy}$, $C_{\xi\eta}$, $C_{xy}$, $CG_1G_2$ and $CG_1G_2$, $C_{\xi\eta}$ (figure 1). In other words Andoyer variables determine the magnitude and orientation of the angular momentum vector $G$ of the body (Moon) in two coordinate systems: in the basic ecliptic coordinate system $C_{xyz}$, in the intermediate coordinate system $CG_1G_2G_3$ associated with the vector $G$ and the axes $C_{\xi\eta\zeta}$ associated with the celestial body (see Figure 1) [6], [7].

We define three more Andoyer variables: $L$, $G$, $H$. $L$ is the projection of the vector $G$ onto the polar axis of the body $C_{\zeta}$, $H$ is the projection of the vector $G$ onto the axis $C_z$ and $G$ is the magnitude of the vector $G$.

Obviously,

$$L = G \cdot k_b, \quad G = G \cdot k_G, \quad H = G \cdot k_x$$

or

$$L = G \cos \theta, \quad G = |G|, \quad H = G \cos \rho.$$

The vector of the angular momentum $G$ of the rotational motion of the body in the coordinate system $C_{xyz}$ is determined by the formula

$$G = G_x i_b + G_y j_b + G_z k_b,$$

and its projections of the vector on the coordinate axes of the body are:

$$G_x = Ap - Fq - Er + P,$$
$$G_y = Fp + Bq - Dr + Q,$$
$$G_z = -Ep - Dq + Cr + R.$$

Here $A$, $B$, $C$ and $D$, $E$, $F$ are the axial and centrifugal moments of inertia of the body (by assumption, they are some known functions of time); $P$, $Q$ and $R$ are the projections of the angular momentum vector of the relative motion of the particles of the body on the coordinate axes $C_{\xi}$, $C_{\eta}$ and $C_{\zeta}$. According to the assumptions of the problem these 9 dynamic characteristics are given functions of time and, in general, they can be represented as the sum of two terms:

$$A = A_0 + \delta A(t), \quad B = B_0 + \delta B(t), \quad C = C_0 + \delta C,$$
$$D = D_0 + \delta D(t), \quad E = E_0 + \delta E(t), \quad F = F_0 + \delta F(t),$$

where $A_0$, $B_0$, $C_0$, $D_0$, $E_0$, $F_0$ are some constant (initial or unperturbed) values of the components of the body inertia tensor, for example, corresponding to the non-deformed state of the body, and

$$\delta A(t), \delta B(t), \delta C(t), \delta D(t), \delta E(t), \delta F(t)$$
are the perturbing additives to them due to the displacement of body particles and the corresponding density variations. We consider that the terms in (7) and the components of the angular momentum of the particles’ relative motions \( P, Q \) and \( R \) are the given functions of time. They can be obtained sometimes from the satellite observations of variations of the Earth gravitational field or other celestial bodies.

In Andoyer variables, the components of the angular momentum (5)-(6) are determined by simple formulas [6]:

\[
\begin{align*}
G_z &= G k_b = G k_c i_b = G \sin \theta \sin l = \sqrt{G^2 - L^2} \sin l, \\
G_y &= G j_b = G k_c j_b = G \sin \theta \cos l = \sqrt{G^2 - L^2} \cos l, \\
G_x &= G k_b = G k_c k_b = G \cos \theta = L.
\end{align*}
\]

To solve practical problems of studying forced variations in the rotation of specific celestial bodies (Earth, Moon, etc.) instead of the moments of inertia, it is sometimes convenient to use the second harmonic coefficients of the gravitational potential. Their standard non-normalized values are related to the moments of inertia (7) by simple relations [8]:

\[
\begin{align*}
J_2 &= -C_{20} = 2C - A - B, \\
C_{22} &= \frac{B - A}{4mr_0^2}, \\
S_{22} &= \frac{F}{2mr_0^2}, \\
C_{21} &= \frac{E}{mr_0^2}, \\
S_{21} &= \frac{D}{mr_0^2}.
\end{align*}
\]

Accordingly, the differences of the axial moments of inertia of the planet and the product of inertia are expressed in terms of the geopotential coefficients by the formulas:

\[
\begin{align*}
A - B &= -4C_{22} mr_0^2, \\
A - C &= (C_{20} - 2C_{22}) mr_0^2, \\
B - C &= (C_{20} + 2C_{22}) mr_0^2, \\
F &= 2S_{22} mr_0^2, \\
E &= C_{21} mr_0^2, \\
D &= S_{21} mr_0^2.
\end{align*}
\]

If we introduce the polar dimensionless moment of inertia \( I \) by the formula \( C = I \cdot mr_0^2 \), then from formulas (11) we get the expressions for the moments of inertia

\[
\begin{align*}
A &= (I + C_{20} - 2C_{22}) mr_0^2, \\
B &= (I + C_{20} + 2C_{22}) mr_0^2, \\
C &= I \cdot mr_0^2.
\end{align*}
\]

Let the axes of inertia that are associated with the body are the main ones in the absence of temporal variations of the moments of inertia (in the undeformed state). In this case, the constant parameters of the problem are related by the following relations:

\[
\begin{align*}
J_2^{(0)} &= -C_{20}^{(0)} = \frac{2C_0 - A_0 - B_0}{2mr_0^2}, \\
C_{22}^{(0)} &= \frac{B_0 - A_0}{4mr_0^2}, \\
S_{22}^{(0)} &= \frac{F_0}{2mr_0^2} = 0, \\
C_{21}^{(0)} &= \frac{E_0}{mr_0^2} = 0, \\
S_{21}^{(0)} &= \frac{D_0}{mr_0^2} = 0.
\end{align*}
\]

Due to the deformations of the planet and, in general, due to any variations of its mass geometry, the geopotential parameters (10) experience temporal variations, for which similar relations take place:
Near-surface changes and redistributions of masses (for example, fluid masses for the Earth) make the greatest contribution to the temporal variations of the moments of inertia and coefficients of the geopotential. In these cases, in practice, an additional relation is used between variations of the moments of inertia,

$$\delta A + \delta B + \delta C = 0.$$  

The relation (15) follows in fact from the definition of the spherical moment of inertia about the center of the planet, which does not change with arbitrary surface redistributions of masses in a thin spherical layer. From formulas (14) and (15) it is easy to get expressions for variations of the moments of inertia - components of the inertia tensor:

$$\frac{\delta A}{C} = -\frac{2}{J} \delta C_{22} - \frac{1}{3J} \delta J_2,$$

$$\frac{\delta B}{C} = \frac{2}{J} \delta C_{22} - \frac{1}{3J} \delta J_2,$$

$$\frac{\delta C}{C} = \frac{2}{3J} \delta J_2,$$

$$\frac{\delta F}{C} = \frac{2}{J} \delta S_{22},$$

$$\frac{\delta E}{C} = \frac{1}{J} \delta C_{21},$$

$$\frac{\delta D}{C} = \frac{1}{J} \delta S_{21}.$$  

(16)

To describe the rotational motion of a weakly deformable moon in a potential force field, we use canonical equations in Andoyer variables. A detailed derivation of these equations is given in [6]. They have the form:

$$\frac{dl}{dt} = \frac{\partial K}{\partial l}, \quad \frac{dL}{dt} = -\frac{\partial K}{\partial l},$$

$$\frac{dg}{dt} = \frac{\partial K}{\partial G}, \quad \frac{dG}{dt} = -\frac{\partial K}{\partial g},$$

$$\frac{dh}{dt} = \frac{\partial K}{\partial H}, \quad \frac{dH}{dt} = -\frac{\partial K}{\partial h},$$

$$K = \frac{1}{2} G^2 \left\{ (a \sin^2 l + b \cos^2 l - f \sin 2l) \sin^2 \theta + c \cos^2 \theta - \sin 2\theta (e \sin l + d \cos l) \right\} - \frac{G}{U} \left( \Omega \sin l + \Omega \cos l \right) \sin \theta + \Omega \cos \theta \right\} - U (L, G, H, l, g, h, t).$$  

(18)

Here entered the notation:

$$a = \frac{BC - D^2}{\Delta}, \quad b = \frac{AC - E^2}{\Delta}, \quad c = \frac{AB - F^2}{\Delta},$$

$$d = -\frac{ED + FC}{\Delta}, \quad e = -\frac{DF + BE}{\Delta}, \quad f = -\frac{FE + AD}{\Delta},$$

$$\Delta = ABC - AD^2 - BE^2 - CF^2 - 2DEF.$$  

(19)
The resulting formulas (19) are exact and, with the above assumptions, can be replaced by fairly simple approximate formulas.

The force function \( U \) in (18) should be represented as a function of canonical variables (1) and time. The latter problem is usually solved using known representations of the direction cosines \( a_{ij} \) of the coordinate axes \( C\xi\eta\zeta \) of the body in the main coordinate system \( Cxyz \): \( a_{ij}(\theta, \rho, l, g, h) \). In the formulas for the Hamiltonian (18) and direction cosines we have:

\[
\sin \theta = \frac{G^2 - L^2}{G}, \quad \cos \theta = \frac{L}{G}, \quad \sin \rho = \frac{\sqrt{G^2 - H^2}}{G}, \quad \cos \rho = \frac{H}{G}.
\]

(20)

3. Elongation of the period of free movement of the pole of the moon due to its elasticity.

To estimate the period of oscillations of the pole of a weakly deformable Moon due to its rotational deformation, we will use the method proposed in [6], [9] to estimate the periods of Chandler oscillations of the poles of the Earth and Mars. According to this method, the rotational motion of a solid moon and a weakly deformable moon can be approximately described by the same Hamiltonian in shape (written in Andoyer variables), if we change the main moments of inertia of a solid model to close values with a special correction. If in the first case (rigid Moon) the Hamiltonian of the rotational motion has the form:

\[
H = \frac{G^2 - L^2}{2} \left( \frac{\sin^2 l}{A} + \frac{\cos^2 l}{B} \right) + \frac{L^2}{2C} - U,
\]

(21)

then in the case of a weakly deformable Moon, the Hamiltonian retains its overall shape,

\[
H_0 = \frac{G^2 - L^2}{2} \left( \frac{\sin^2 l}{A} + \frac{\cos^2 l}{B} \right) + \frac{L^2}{2C},
\]

(22)

where

\[
A = A + 3D_{\omega}, \quad B = B + 3D_{\omega}, \quad C = C \quad \text{and} \quad D_{\omega} = k_2 \frac{a_e^5 \omega_0^2}{9f}.
\]

(23)

In expression (23) \( a_e \) is the equatorial radius of the Moon, \( \omega_0 \) is the unperturbed value of the angular velocity of rotation of the Moon, \( k_2 \) is the Love number, \( f \) is the gravitational constant. In other words in the second case the axial moments of inertia of the body experience certain constant increments. Thus in the coordinate system \( C\omega_1\omega_2\omega_3 \) associated with the vector of angular velocity (the axis \( C\omega_3 \) is directed along the vector \( \omega \)), the components of the inertia tensor of the equatorial thickenings of the Earth, caused by its rotational deformation, are determined by the formulas [10]:

\[
A_{\omega} = B_{\omega} = D_{\omega}, \quad C_{\omega} = 2D_{\omega}.
\]

To increment the polar moment of inertia of the Moon, due to its rotational deformation, we can use the relation

\[
2D_{\omega} = \Delta C_M = \frac{2}{9} k_2 \frac{a_e^5 \omega_0^2}{f} = \frac{2}{9} k_2 \frac{m_M r_M^2}{m_M f} \frac{a_e^2}{r_M^2} \omega_0^2 a_e^3.
\]

(24)

According to modern observations [11], [12] we have the following values of the parameters of the Moon:
Here \(a_e\) is the average equatorial radius of the Moon. The angular velocity is comparable to its unperturbed value \(\omega_0 = n_F\), \(n_F\) is the angular velocity of the change in the argument of lunar orbital motion. The corresponding period and the indicated rotation correspond to the motion of the Moon according to the laws of Cassini. Now, using formulas (24), (25), we find the variation of the polar moment of inertia of the Moon due to its rotational deformation:

\[
\Delta C_M = \frac{2}{9} k_2 C_m \frac{100171619}{0.39349} \cdot \frac{1}{m_M} \frac{a_e^2}{r_m^2} = 1.352754 \cdot 10^{-7} C_m = 1.181612 \cdot 10^{35} \text{ g cm}^2.
\]

It can be said that the elastic Moon, deformed by its own rotation, rotates as some fictitious (additional) solid, but with changed moments of inertia.

So the elastic Moon (deformed by its own rotation) rotates according to the Euler-Poinsot laws, but like some other free solid body whose moments of inertia slightly differ from the main moments of inertia of the Moon itself \(A, B\) and \(C\). We omit the intermediate calculations and give the final expressions for the Chandler period of motion of the pole of a weakly deformable Moon without taking into account the gravitational attraction of the central celestial body (Earth), and taking into account the gravitational attraction of the Earth taking into account the resonant properties of the perturbed orbital motion of the Moon.

\[
T_q^* = \frac{T_F}{\sqrt{\frac{B}{A} - 1}}, \quad T_q = \frac{T_F}{\sqrt{\frac{B}{A} - 1 - \frac{1}{B \Omega^2} \frac{\partial^2 \langle U \rangle}{\partial l^2} \left( \frac{B}{C} - 1 - B \frac{\partial^2 \langle U \rangle}{\partial l^2} \right)}}.
\]

Here \(T_F\) is the period of the daily rotation of a celestial body (around its axis). The first of these formulas in (27) is obtained for a free celestial body, i.e. not subject to gravitational influence from external celestial bodies. The second formula takes into account the gravitational influence of the central external body (for the Moon it is primarily the Earth, for the Earth and Mars it is the Sun) by means of the averaged value of the force function \(\langle U \rangle\) of its Newtonian interaction with the planet in question. In this case, disturbances in the orbital motion of the perturbing body are taken into account. The values of the gravitational components in the expression of the period of motion of the pole were calculated for the high-precision orbit of the Moon and they are according to our estimates:

\[
\frac{1}{B \Omega^2} \frac{\partial^2 \langle U \rangle}{\partial l^2} = -1830.440218 \cdot 10^{-6}, \quad B \frac{\partial^2 \langle U \rangle}{\partial l^2} = 0.184727 \cdot 10^{-6}.
\]

When calculating the dimensionless quantities (28), the coefficients of the second harmonic of the selenium potential were used according to its modern model obtained during the implementation of the Japanese project Selenium [13].

Table 1 compares the Chandler periods calculated by the formulas of our theory for the Earth, Mars and the Moon for comparison. The Euler period \(T_{Eu}\) and the Chandler period \(T_{Ch}\) for the Earth, Mars and the Moon (first, second and third row) are calculated using the first formula from (27) for the observed inertia moments \(A, B\) and \(C\), and for reduced inertia moments \(A, B\) and \(C\) respectively. In these calculations, the gravitational influence of the central body on the movement of the pole is not
taken into account. The fourth line contains the values of the periods $T_{Eu}$ and $T_{Ch}$ and already taking into account the gravitational contribution to the motion of the pole. The values of the indicated periods are given in days, the differences between the periods in days and in percentages are given in relation to the values $T_{Eu}$ and $T_{Ch}$ of the periods for the inelastic planet (or the Moon).

Table 1. Influence of elasticity on the period of motion of the pole of the Moon and planets.

| Planet  | $T_{Eu}$ (day) | $T_{Ch}$ (day) | $T_{Ch}$ (day) | $T_{Ch}$ % $T_{Eu}$ |
|---------|----------------|----------------|----------------|---------------------|
| Earth   | 304.300        | 447.086        | 142.786        | 46.92 % $T_{Eu}$    |
| Mars    | 191.816        | 218.049        | 26.233         | 13.68 % $T_{Eu}$    |
| Moon*   | 53922.70       | 53944.95       | 22.70          | 0.042 % $T_{Eu}$    |
| Moon    | 27267.65       | 27275.62       | 7.97           | 0.029 % $T_{q}$     |

It is important to emphasize that in the case of the Moon, the formula for the period $T_q^*$ in (27) is not at all applicable for estimating the period of free movement of the Moon’s pole. Since the additional terms in the expressions of the period $T_q$ (27) are comparable in magnitude with the first terms and actually lead to a decrease in the period of the pole movement by about 2 times [14]. This is well illustrated by the values of the period of the pole oscillations without taking into account the force interaction (line 3) and taking that into account (line 4). For illustration, here are the values of the periods (for solid and elastic models) for the Earth and Mars. They increase, respectively, by 47% and 14%. The latter values are obtained by the first formula of (27), in which the attraction of the central body is not taken into account. Thus, the elasticity of the planet most significantly affects the period of motion of the pole of the axis of rotation, and in the case of the Moon (synchronous satellite), this influence is small, but significant for a highly accurate description of its rotation, which is significant and significant for the implementation of the project for studying the orientation of the Moon using a telescope mounted on the surface of the Moon.

References
[1] Vilke V G, Markov Yu G 1988 Evolution of translator-rotational motion of an viscoelastic planet in the gravitational field Astr. Journ., Vol. 65 №4 pp 861–867 [in Russian]
[2] Vilke V G, Shatina A V 2001 Motion evolution of a double planet Cosmic Research Vol. 39. No 3. pp 295–305.
[3] Hanada H et al. 2005 Application of a PZT telescope to In situ Lunar orientation Measurements (ILOM). Proc. Int. Ass. Geod. (Springer) pp 163-168.
[4] Barkin Yu, Matsumoto K, Hanada H, Sasaki S, Petrova N, Barkin M 2012 The influence of elastic properties of the Moon on its pole motion Book of abstracts of the 118th Meeting of the Geodetic Society of Japan, Sendai, Japan, (29 October – 2 November 2012, Sendai) pp 149-150.
[5] Barkin Yu V, Dyomin V G 1984 Dynamics of the system of rigid and variable celestial bodies Book of abstracts of the 8th Republic interuniversity mathematics and mechanics conference (4-6 of September 1984) Part III. Theoretical and applied mechanics, Alma-Ata pp 10. [in Russian]
[6] Barkin Yu V 2000 Perturbated rotational motion of weakly deformable celestial bodies Astronomical and Astrophysical Transactions Vol.19. Issue 1, pp 19-65. doi: 10.1080/10556790008241350

[7] Arkhangelsky Yu A 1977 Analytical dynamics of a rigid body (Moscow: Nauka) p 328. [in Russian]

[8] Aksyonov E P 1977 Theory of artificial satellites’ motion (Moscow: Nauka) p 360. [in Russian]

[9] Barkin Yu, Ferrandiz J 2010 Elliptical Chandler pole motions of the Earth and Mars EGU General Assembly 2010, held 2-7 May, 2010 in Vienna, Austria, p. 2936

[10] Munk W, MacDonald G 1960 The rotation of the Earth (Cambridge University Press), p 348.

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