On the Kähler-Ricci flows near the Mukai-Umemura 3-fold.

Yuanqi Wang

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Abstract

In this short note, we show that given a special Kähler-Einstein degeneration with bounded geometry, for any noncentral fiber, there exists a Kähler-Ricci flow which converges to the Kähler-Einstein metric of the central fiber. As an example, Tian’s deformations [10] of the Mukai 3-fold admit Kähler-Ricci flows which converge to Donaldson’s Kähler-Einstein metric [4] over the Mukai 3-fold.

1 Introduction

This is a following up note of [9]. In 1982, R.Hamilton introduced the Ricci flow in [5]. In the Kähler setting, the Kähler condition is preserved under the flow. H-D. Cao([2]) first studied the Kähler-Ricci flow. Using Yau’s estimates for complex Monge-Ampère equations, Cao proved when $C_1(M)$ is negative or zero, the properly normalized Kähler-Ricci flow converges to a Kähler-Einstein metric, thus obtained another proof of the Calabi conjecture originally proved by Yau in [13].

When $C_1(M) > 0$, the situation is more complicated. A result announced by Perelman, and proved by Tian-Zhu in [11] says that if there exists a Kähler-Einstein metric in $C_1(M)$ with respect to the underline complex structure, then the Kähler-Ricci flow

$$\frac{\partial g}{\partial t} = -Ric + g$$

initiated from any metric in $C_1(M)$ converges to the Kähler-Einstein metric up to a biholomorphism. However, when there is no Kähler-Einstein metric with respect to the underlying complex structure, it’s very interesting to study where does the flow go. It’s widely believed that even the limit complex structure (as $t \to \infty$) could be different from the complex structure of the initial metric of the flow.

In this note, we show that there really exist such kind of examples of flows with jumping complex structures as $t \to \infty$. These examples live on certain
deformations of the well known Mukai-Umemura 3-fold (see [7, 8]). In the appendix we will say more about these 3-folds for the sake of a self-contained proof, but in this section we abbreviate everything to make it easiest to understand. Denote $M_{F_0}$ as the Mukai-Umemura 3-fold, and $M_{F_v}$ as the deformations (parametrized by tensors $v$) which are considered by Tian in [10]. The amazing history is that: on one hand, Tian proved that $M_{F_v}$ does not admit any Kähler-Einstein metric; on the other hand, Donaldson proved in [4] that $M_{F_0}$ admits a Kähler-Einstein metric. Moreover, Donaldson also showed there is a 2-parameter family $M_{\alpha,C}$ of deformations (of $M_{F_0}$) which do admit Kähler-Einstein metrics. Though $M_{F_0}$ is not isolated Kähler-Einstein, the following theorem holds.

**Theorem 1.1.** For any $v$ close enough to 0 (as a tensor), there exists a metric $\omega_v$ over Tian’s deformations $M_{F_v}$, such that the Kähler-Ricci flow initiated from $\omega_v$ converges to Donaldson’s Kähler-Einstein metric over the Mukai 3-fold $M_{F_0}$ (up to a biholomorphism). In particular, the limit complex structure (as $t \to \infty$) is different from the complex structure of the initial metric $\omega_v$.

Actually it’s well known that $M_{F_0}$ is a special example of the following degenerations.

**Definition 1.2.** We call the triple $(\mathcal{M},\varpi, B)$ a special Kähler-Einstein degeneration with bounded geometry if it satisfies the following.

1. $(\mathcal{M},\varpi, B)$ is a differentiable family of complex manifolds in the sense of Definition 4.1 in [6] ($\varpi$ is a map of full rank from $\mathcal{M}$ to $B$), $B$ is a ball in $\mathbb{R}^m$ and is centered at the origin.

2. For any $u \in B$, $\varpi^{-1}(u) \cong M_u$ is a Fano Kähler manifold. There exists a smooth tensor $J \in \Gamma(T^*\mathcal{M} \otimes T\mathcal{M})$, such that for any point $p \in \mathcal{M}$, $T_p M_{\varpi(p)}$ is an invariant subspace of $J$, and $J$ restricts to the complex structure of $M_{\varpi(p)}$. Moreover, there exists a smooth two-form $G$ over $\mathcal{M}$ such that restricted to each fiber $(M_u, TM_u)$, $G|_{(M_u, TM_u)} = \omega_u$ is a Kähler metric form in $C^1(M_u, J_u)$.

3. $M_0$ admits a Kähler-Einstein metric $\omega_{KE}$.

4. There exists a smooth vector field $\sigma$ over $\mathcal{M}$, such that $\varpi \cdot \sigma = -\sum u_i \frac{\partial}{\partial u_i}$. Furthermore, the diffeomorphism $\sigma_a$ generated by $\sigma$ is a biholomorphism from $M_u$ to $M_{e^{-a}u}$.

With respect to Definition (1.2), the following more general theorem is true.

**Theorem 1.3.** Let $(\mathcal{M},\varpi, B)$ be a special Kähler-Einstein degeneration with bounded geometry. For any fiber $M_u$, $u \in B$, $u \neq 0$, there exists a metric $\omega_{\phi,u}$, such that the Kähler-Ricci flow initiated from $\omega_{\phi,u}$ converges in
Cheeger-Gromov sense with polynomial rate to the Kähler-Einstein metric $\omega_{KE}$ of $M_0$, up to a biholomorphism.

Remark 1.4. Actually Theorem 1.3 implies more than Theorem 1.1 on the behavior of Kähler-Ricci flows near $M_{F_0}$, starting from the 4th kind of orbits as in page 44 of [4]. Since these are exactly parallel to Theorem 1.1, for the sake of brevity we only consider Tian’s deformations. Though holomorphic degenerations are considered most often, Theorem 1.3 holds more generally for smooth degenerations. This is consistent with the real setting of Theorem 1.1 in [9]. The bounded geometry assumption is satisfied in most of the existing examples of degenerations. Similar topic is also discussed in Tian-Zhu’s work [12].

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2 Proof of Theorem 1.3.

The proof of Theorem 1.1 and Theorem 1.3 are by simple observations based on the work of Sun and the author [9], Donaldson [4], Tian [10], and Chen-Sun [3]. The point is to construct metrics which satisfy the requirements of Theorem 1.1 in [9]. First we have the following lemma which is well known and essentially due to Kodaira [6].

Lemma 2.1. Suppose $(\mathfrak{M}, \varpi, B)$ is a special Kähler-Einstein degeneration with bounded geometry. Then for any $u = (u_1, \ldots, u_m) \in B$, there is a family of smooth diffeomorphisms $\psi_u$ from $M_0$ to $M_u$ with the following property. For any $k > 0$ we have

$$\lim_{u \to 0} |\psi_u^{-1} \circ J_u \circ \psi_u - J_0|_{C_k,g_0,M_0} = 0,$$

and

$$\lim_{u \to 0} |\psi_u g_u - g_0|_{C_k,g_0,M_0} = 0.$$  

Moreover, there is a smooth family of smooth diffeomorphisms $\psi_{s,u}$ from $\mathfrak{M}$ to itself such that

$$\varpi_*(\frac{d\psi_{s,u}}{ds}) = \sum_{\beta=1}^m u_\beta \frac{\partial}{\partial t_\beta};\ \psi_{1,u}|_{M_0} = \psi_u.$$

Proof. of Lemma 2.1 We work in the real setting. As the terminology of [6], given any point $p \in \mathfrak{M}$, we can choose local coordinates

$$(x_b, t_\alpha), \ 1 \leq b \leq 2n, \ 1 \leq \alpha \leq m.$$
Suppose \( u = (u_1, \ldots, u_m) \in B \). The we can lift the vector field
\[
\sum_{\beta=1}^{m} u_{\beta} \frac{\partial}{\partial t_{\beta}}
\]
up to a vector field \( V_u \) over \( \mathfrak{M} \), as in the proof of Theorem 2.4 in [6]. Since \( \varpi_* V_u = \sum_{\beta=1}^{m} u_{\beta} \frac{\partial}{\partial t_{\beta}} \) over \( B \), then \( V_u \) generates a smooth family of diffeomorphisms \( \psi_{s,u} \) from \( M_0 \) to \( M_s u \). Namely,
\[
V_u = \sum_{\beta=1}^{m} u_{\beta} \sum_{k=1}^{k_0} \rho_k \left( \frac{\partial}{\partial t_{\beta}} \right)_k,
\]
(4)
where \( \rho_k \) is the partition of unity subject to a open cover \( U_k, 1 \leq k \leq k_0 \). Moreover \( \psi_u \triangleq \psi_{1,u} \) is a diffeomorphism from \( M_0 \) to \( M_u \).

Next we express the complex structure \( J \) of \( \mathfrak{M} \) in a coordinate neighborhood \( U \) as the following.
\[
J = \sum_{b,c=1}^{2n} J_{bc} dx_b \otimes \frac{\partial}{\partial x_c} + \sum_{b=1}^{2n} \sum_{\beta=1}^{m} J_{bt_{\beta}} dx_b \otimes \frac{\partial}{\partial t_{\beta}} + \sum_{\alpha,\beta=1}^{m} J_{t_{\alpha}t_{\beta}} dt_{\alpha} \otimes \frac{\partial}{\partial t_{\beta}} + \sum_{b=1}^{2n} \sum_{\beta=1}^{m} J_{bt_{\beta}} dt_{\beta} \otimes \frac{\partial}{\partial x_b}.
\]
Since \( T_p M_{\varpi(p)} \) is an invariant subspace of \( J \) for any \( p \), we have
\[
J_{bt_{\beta}} = 0, \text{ for any } \beta, b.
\]
Then restricted over \( M_0 \) and \( TM_0 \), using \( g_0 \) as our reference metric, we consider
\[
D = \psi_u^{-1} \circ J_u \circ \psi_u \circ \ast - J_0.
\]
Let the coordinates representation of \( D \) be
\[
D = \sum_{b,c=1}^{2n} D_{bc} dx_b \otimes \frac{\partial}{\partial x_c}.
\]
Let \( w_b = x_b \circ \psi_u \). Then for any \( p \in M_0 \) and \( 1 \leq b,c \leq 2n \), we have
\[
D_{bc}(p) = \sum_{\alpha,\beta=1}^{2n} \frac{\partial w_{\alpha}}{\partial x_b}(p) \frac{\partial x_c}{\partial w_{\beta}}(\psi_u(p)) J_{\alpha\beta}(\psi_u(p)) - J_{bc}(p).
\]
(5)
Using (4), we deduce
\[
\lim_{u \to 0} |V_u|_{C^k, g_0, M_0} = 0.
\]
Hence
\[
\lim_{u \to 0} |\psi_u - id|_{C^k, \mathfrak{M}} = 0, \text{ id is the identity diffeomorphism from } \mathfrak{M} \text{ to } \mathfrak{M}.
\]
(6)
(6) implies
\[
\lim_{u \to 0} |D_{bc}|_{C^k,U} = \lim_{u \to 0} |\Sigma_{\alpha,\beta=1}^{2n} \partial w_\alpha (p) \frac{\partial x_c}{\partial u_\beta} (\psi_u(p)) J_{\alpha \beta} (\psi_u(p)) - J_{bc}(p)|_{C^k,U} = 0. \tag{7}
\]

(7) implies (2). Similarly, (3) is true. \qed

Lemma 2.2. Assumptions and setting as in Lemma 2.1. Let
\[
\tilde{J}_u = \psi_u^{-1} \circ J_u \circ \psi_u, \tilde{\omega}_u = \psi_u^* \omega_u \in C_1(M_0, J_0).
\]
For any \(\delta\), there is an \(\epsilon\) such that if \(|u| < \epsilon\), then there exists a 2-form \(\omega_{\phi,u} \in C_1(M_0, J_0)\) with the following properties.

- \(\omega_{\phi,u}(\cdot, J_u\cdot)\) is a Kähler metric with respect to \(\tilde{J}_u\);
- \(|\omega_{\phi,u} - \omega_{KE}\mid_{C^{k-1}, M_0, g_0} \leq C\delta, \omega_{KE}\) is the Kähler-Einstein metric of the central fiber (gauge fixed).

Proof. of Lemma 2.2: Denote \(\omega_{KE} \circ \tilde{J}_u\) as the tensor
\[
(\omega_{KE} \circ \tilde{J}_u)(X, Y) = \omega_{KE}(X, \tilde{J}_u(Y)).
\]
Consider the antisymmetric part of \(\omega_{KE} \circ \tilde{J}_u\) as
\[
(AS \omega_{KE} \circ \tilde{J}_u)(X, Y) = -\omega_{KE}(X, \tilde{J}_u Y) + \omega_{KE}(Y, \tilde{J}_u X).
\]
Since \(\omega_{KE} \circ J_0\) is symmetric, we have
\[
-\omega_{KE}(X, \tilde{J}_u Y) + \omega_{KE}(Y, \tilde{J}_u X) = -\omega_{KE}[X, (\tilde{J}_u - J_0)Y] + \omega_{KE}(Y, (\tilde{J}_u - J_0)X). \tag{8}
\]
Using Lemma 2.1 by letting \(\epsilon\) to be sufficiently small we obtain
\[
|\tilde{J}_u - J_0|_{C^{k-1}, M_0} \leq \delta, \text{ for all } |u| < \epsilon.
\]
Hence
\[
|AS \omega_{KE} \circ \tilde{J}_u|_{C^{k-1}, M_0} \leq C\delta. \tag{9}
\]
Then we complexify \(\omega_{KE} \circ \tilde{J}_u\) with respect to \(\tilde{J}_u\). Using the fact that \(\omega_{KE}^{2,0} \oplus \omega_{KE}^{0,2}\) is precisely the antisymmetric part of \(\omega_{KE} \circ \tilde{J}_u\), we get
\[
|\omega_{KE}^{2,0}|_{C^{k}, M_0} + |\omega_{KE}^{0,2}|_{C^{k}, M_0} \leq C\delta. \tag{10}
\]
Since \(\omega_{KE} = \tilde{\omega}_u + d\theta\), we have
\[
\text{Re} \omega_{KE}^{2,0} = \text{Re} (\omega_{KE} - \tilde{\omega}_u)^{2,0} = \text{Re} (d\theta)^{2,0}. \tag{11}
\]
Then,
\[ \omega_{KE}^{0.2} = \bar{\partial}_u \theta^{0.1}, \quad \omega_{KE}^{2.0} = \partial_u \theta^{1.0}. \] (12)

By (10), (11), and (12), we get
\[ |\bar{\partial}_u \theta^{0.1}|_{C^k, g_0, M_0} + |\partial_u \theta^{1.0}|_{C^k, g_0, M_0} \leq C\delta. \] (13)

We want to find a \((0, 1)\)-form \(\phi\) (with respect to \(\tilde{J}_u\)) such that
\[ |\phi|_{C^k, g_0, M_0} \leq C\delta, \quad \text{and} \quad \bar{\partial}_u \phi = \omega_{KE}^{0.2}. \] (14)

This is straightforward by considering
\[ \bar{\partial}_u^* \theta^{0.1} = -\bar{\omega}^i u_j \theta_{i,j}. \] (15)

By solving the Dirichlet problem
\[ \Delta u f = \bar{\partial}_u^* \theta^{0.1}, \] (16)
we get a \(f\) such that
\[ \bar{\partial}_u^*(\theta^{0.1} - \bar{\partial}_u f) = 0. \] (17)

Let \(\phi = \theta^{0.1} - \bar{\partial}_u f\), then
\[ \partial_u \phi = \bar{\partial}_u \theta^{0.1} = \omega_{KE}^{0.2}, \quad \bar{\partial}_u^* \phi = 0. \] (18)

Claim 2.3. \(|\phi|_{C^k, g_0, M_0} \leq C\delta.\)

The claim easily follows from the existence of \(G\), which is actually a smooth family of metrics. For the sake of a self-contained note, we include the detail here. We first show \(|\phi|_0 \leq C\delta\). Were this not true, there exists a sequence \((\phi_i, u_i)\) such that
\[ u_i \to u_{\infty}, \quad \bar{\partial}_{u_i} \phi_i \to 0 \quad \text{in the sense of} \quad (C^k, g_0), \quad \bar{\partial}_{u_i}^* \phi_i = 0, \quad \text{but} \quad |\phi_i|_{C^0} = 1. \] (19)

Hence
\[ \Delta_{H, u_i} \phi_i \to 0 \quad \text{in the sense of} \quad (C^{k-1}, g_0). \] (20)

The Bochner formula for any metric and 1-form \(\eta\) reads as:
\[ \Delta_g \eta = \Delta_{H, g} \eta - \text{Ric}_g \odot \eta, \quad \odot \text{ is some algebraic tensor product.} \] (21)

By standard Schauder estimates (with respect to \(g_0\)), and the fact that \(|\text{Ric}_u|_{C^k, g_0, M_0} \leq C\delta\) (from item 2 in Definition 1.2), we deduce from (19), (20), and (21) that
\[ |\phi_i|_{C^k, g_0, M_0} \leq C. \] (22)

Then \(\phi_i \to \phi_{\infty}\) in \(C^{k-1}\)-topology such that
\[ |\phi_{\infty}|_0 = 1, \quad |\phi|_{C^{k-1}, g_0, M_0} \leq C, \quad \Delta_{u_{\infty}} \phi_{\infty} = 0. \] (23)
This contradict the simply connectness of Fano-manifolds (there should be no nontrivial harmonic form). Therefore

\[ |\phi|_0 \leq C\delta. \] (24)

By the Schauder-estimate of \( g_0 \) again and the fact \( |Ric_{\omega}|_{C^k,g_0,M_0} \leq C\delta \), we get the following estimate with a \( C \) independent of \( u \).

\[ |\phi|_{C^k,g_0,M_0} \leq C\delta. \] (25)

Then \( \phi \) satisfies (14). Moreover, we have \( \partial \bar{\partial} \phi = \omega_{KE}^2 \) and \( |\bar{\phi}|_{C^k,g_0,M_0} \leq C\delta \).

The proof is complete. \( \square \)

Proof. of Theorem 1.3: Without loss of generality we assume \( B \) is the unit ball. Fix a finite cover \( (U_j, 1 \leq j \leq C) \) of \( M_0 \) in \( \mathcal{M} \) with nontrivial intersection with \( M_0 \). It suffices to show there exists a \( \omega_{\phi,u} \) as in Theorem 1.3 for any \( u \in B(r) \), where \( r \) is sufficient small such that \( B(r) \) is contained in the \( B \)-slice for all \( U_j \), and every \( u \in B(r) \) satisfies

\[ |\omega_{\phi,u} - \omega_{KE}|_{C^{k-1},g_0,M_0} \leq C\delta. \] (26)

guaranteed by Lemma 2.1. Then \( \tilde{J}_u = \psi_{u,*}^{-1} \circ J_u \circ \psi_{u,*} \) and \( \tilde{g}_u = \psi_{u,*}^* g_u \) satisfy the assumptions of Lemma 2.2. Notice that since \( \omega_{\phi,u} \) is a complex structure, \( \tilde{J}_u \) converges to \( J_0 \), and \( \tilde{g}_u \) converges to \( g_0 \in C_1(M_0,J_0) \), we have

\[ C_1(M_0,\tilde{J}_u) = C_1(M_0,J_0). \]

Hence, Lemma 2.2 produces a metric \( \omega_{\phi,u} \) which satisfies the assumptions in Theorem 1.1 of [9].

Step 1. We briefly show the Kähler-Ricci flow produces a degeneration of \( \tilde{J}_u \) to a Kähler-Einstein complex structure. Actually, Theorem 1.1 of [9] says the Kähler-Ricci flow initiated from \( \omega_{\phi,u} \) converges to a Kähler-Einstein metric \( (\tilde{\omega}_{KE},\tilde{J}_{KE}) \). In particular, there exists a smooth time-dependent family of smooth diffeomorphisms \( \psi_t \) such that \( \psi_t^* \tilde{J}_u \) converges to \( \tilde{J}_{KE} \), in the sense of the setting of Theorem 1.5 in [3].

Step 2. In this step we point out that \( \tilde{J}_u \) degenerates to \( J_0 \), which is also Kähler-Einstein. To be precise, suppose \( \sigma_a M_u = M_{u(a)} \), then \( (M_{u(a)},J_{u(a)}) \) is biholomorphic to \( (M_u,J_u) \) via \( \sigma_a^{-1} \). We consider the following diffeomorphism

\[ \Psi_a = (\psi_u^{-1} \circ \sigma_a^{-1} \circ \psi_{u(a)})^{-1} : M_0 \to M_0. \] (27)
Then it’s straightforward to deduce from Lemma 2.1 and (27) that
\[
\lim_{a \to \infty} |\Psi_{a,*} \circ \tilde{J}_a \circ \Psi_{a,*}^{-1} - J_0|_{C^k,g_0,M_0} = \lim_{a \to \infty} |\psi_{u(a),*}^{-1} \circ J_{u(a)} \circ \psi_{u(a),*} - J_0|_{C^k,g_0,M_0} = 0,
\]
which means \( \tilde{J}_a \) degenerates to \( J_0 \). Finally, by Theorem 1.5 in [3] on uniqueness of Kähler-Einstein degeneration and the uniqueness of Kähler-Einstein metric due to Bando and Mabuchi [1], we conclude
\[
\tilde{\omega}_{KE}(\text{limit of the Kähler-Ricci flow}) = \omega_{KE}(\text{the original Kähler-Einstein metric in the central fiber}),
\]
up to a biholomorphism. 

3 Appendix: Mukai 3-fold and proof of Theorem 1.1

It suffices to check the Mukai 3-folds form a special degeneration in the sense of Definition 1.2. This is already done by the work of Tian [10] and Donaldson [4]. Actually the Mukai 3-folds form a holomorphic degeneration, which is much more special than the differential family in Definition 1.2. To give a self-contained proof, we very briefly introduce the necessary aspects of Mukai-Umemura 3-folds for our application, where we use the convention employed by Tian in [10]. Let \( G(4,7) \) be the Grassmannian manifold consisting of the 4-planes in \( \mathbb{C}^7 \). For any 3-plane \( F \) in \( \wedge^2 \mathbb{C}^7 \), we consider the subvariety \( M_F \) as
\[
M_F = \{ p \in G(4,7) | F|_p = 0 \}.
\]
Consider the 3-plane \( F_0 = \{ f_1, f_2, f_3 \} \), where
\[
\begin{align*}
f_1 &= 3e_1 \wedge e_6 - 5e_2 \wedge e_5 + 6e_3 \wedge e_4; \\
f_2 &= 3e_1 \wedge e_7 - 2e_2 \wedge e_6 + e_3 \wedge e_5; \\
f_3 &= e_2 \wedge e_7 - e_3 \wedge e_6 + e_4 \wedge e_5.
\end{align*}
\]

The Lie algebra of holomorphic vector fields over \( M_{F_0} \), denoted as \( \eta(M_{F_0}) \), is isomorphic to \( sl(2, \mathbb{C}) \). As a subalgebra of \( sl(7, \mathbb{C}) \) (\( sl(7, \mathbb{C}) \) naturally acts on 3-planes in \( \wedge^2 \mathbb{C}^7 \) via its fundamental action), \( \eta(M_{F_0}) \) is given by
\[
H_1 = 
\begin{bmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0
\end{bmatrix};
\]
Now we consider the deformation $F_v$ of $F_0$ as $F_v = \{f_1^v, f_2^v, f_3^v\}$, such that

\[ f_i^v = f_i + \sum_{j+k\geq 7+i} v_{ijk} e_j \wedge e_k, \quad i = 1, 2, 3. \]

In [10], Tian showed that when $v \neq 0$ and $|v|$ is small, the deformed Fano 3-fold $M_{F_v}$ does not admit any Kähler-Einstein metric. On the other hand, Donaldson showed $M_{F_0}$ admits a Kähler-Einstein metric. Moreover, Donaldson showed there is at least a 2-parameter family of small deformations $M_{F, \alpha, C}$ of $M_{F_0}$ such that $M_{F, \alpha, C}$ also admit Kähler-Einstein metric for all $\alpha, C$ small. Thus $M_{F_0}$ is not isolated Kähler-Einstein. Nevertheless, $M_{F_0}$ is a Kähler-Einstein degeneration of $M_{F_v}$. To be precise, we consider the one real-parameter group generated by $H_1$.

\[
\psi(s) = \begin{bmatrix}
    s^{-3} & 0 & 0 & 0 & 0 & 0 \\
    0 & s^{-2} & 0 & 0 & 0 & 0 \\
    0 & 0 & s^{-1} & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & s & 0 \\
    0 & 0 & 0 & 0 & s^2 & 0 \\
    0 & 0 & 0 & 0 & 0 & s^3 \\
\end{bmatrix}.
\]

It’s easy to check that

\[
|s\psi(s)f_1^v - f_1| + |\psi(s)f_2^v - f_2| + |s^{-1}\psi(s)f_3^v - f_3| \leq C|s|. \tag{30}
\]

Hence

\[
\lim_{s\to 0} \psi(s)M_{F_v} = M_{F_0}. \tag{31}
\]

Thus $\psi_{s\in[-1,1]}(s)M_{F_v}$ is a smooth submanifold of $G(4, 7) \times \mathbb{R}$. Moreover, since $C_1(M_{F_v}) = C_1(Q)|_{M_{F_v}}$ ($Q$ is the universal quotient bundle of $G(4, 7)$), we can take a closed positive (1, 1)-form $\Omega \in C_1(Q)$ over $G(4, 7)$. Then the restricted form $\Omega|_{M_v}$ plays the role of the $G$ in item 2 of Definition 1.2.
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Yuanqi Wang, Department of Mathematics, University of California at Santa Barbara, Santa Barbara, CA, USA; wangyuanqi@math.ucsb.edu.