Covariant Information Theory and 
Emergent Gravity

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Abstract. Informational dependence between statistical or quantum subsystems can be described with Fisher information matrix or Fubini-Study metric obtained from variations of the sample/configuration space coordinates. Using these (non-covariant) objects as macroscopic constraints we consider statistical ensembles over the space of classical probability distributions (i.e. in statistical space) or quantum wave-functions (i.e. in Hilbert space). The ensembles are covariantized using (dual) field theories with either complex scalar field (identified with complex wave-functions) or real scalar field (identified with square root of probabilities). We construct space-time ensembles for which an approximate Schrodinger dynamics is satisfied by the dual field (we call infoton due to its informational origin) and argue that a full space-time covariance on the field theory side is dual to local computations (defined in terms of parallel computing) on the information theory side. We define a fully covariant information-computation tensor and show that it must satisfy conservation equations.

Then we switch to a thermodynamic description of the quantum/statistical systems and argue that the (inverse of) space-time metric tensor is a conjugate thermodynamic variable to the ensemble-averaged information-computation tensor. In the (local) equilibrium the entropy production vanishes and the metric is not dynamical, but away from equilibrium the entropy production gives rise to an emergent dynamics of the metric. This dynamics can be described by expanding the entropy production into products of generalized forces (derivatives of metric) and conjugate fluxes. Near equilibrium these fluxes are given by an Onsager tensor contracted with generalized forces and on the grounds of time-reversal symmetry the Onsager tensor is expected to be symmetric. We show that a particularly simple and highly symmetric form of the Onsager tensor gives rise to the Einstein-Hilbert term. This proves that general relativity is equivalent to a theory of non-equilibrium (thermo)dynamics of the metric, but the theory is expected to break down far away from equilibrium where the symmetries of the Onsager tensor are to be broken.
1 Introduction

Quantum gravity is an ongoing attempt to unify quantum mechanics and general relativity within a single theory. There are many approaches to unification that had been proposed, but perhaps the most ambitious of all is to start with nothing but quantum mechanics and then to derive everything from it, i.e. space-time, general relativity, etc. In all such attempts the first step should be a derivation of space-time from a state vector or density matrix in a Hilbert space (which describes state of the system) and a Hermitian matrix or Hamiltonian operator (which describes dynamics of the system). In the recent years a number of interesting proposal for deriving space were put forward, which includes tensor networks [1], multi-dimensional scaling [2] and information graph flow [3], but it is less clear how to derive space-time (see however Ref. [4] and [3] for some recent ideas) or how general relativity might emerge. In this paper we will try to answer these questions in context of either classical or quantum theories of information. We will show that general relativity is equivalent to a theory of non-equilibrium (thermo)dynamics of information. Note that the idea of viewing general relativity not as fundamental phenomena, but as an emergent phenomena is not new. The first attempt to derive Einstein equations from the laws of thermodynamics was given in Ref. [5] and more recently in Ref. [6].

But before we dive into the questions of quantum gravity we will first construct a covariant information theory (which in a physicist’s jargon should mean a “second quantization” of information.) In a sense we will be describing probability distributions over either classical probabilities or quantum wave-functions, but to make the notion more precise we will define statistical ensembles using dual (we shall call infoton) fields. The reason why our analysis is
Table 1. Mapping between information theory, field theory and thermodynamic variables.

| Information Theory | Field Theory | Thermodynamics |
|-------------------|-------------|----------------|
| sample/configuration space, \( \vec{x} \) | physical space, \( \vec{x} \) | physical space, \( \vec{x} \) |
| state vector \(|\psi\rangle\) or \(\vec{p}\) | | |
| square root of prob., \(\sqrt{p(\vec{x})}\) | real scalar field, \(\varphi\) | |
| quantum wave-function, \(\psi(\vec{x})\) | complex scalar field, \(\varphi\) | |
| probability density, \(\psi(\vec{x})^*\psi(\vec{x})\) | probability scalar, \(\varphi^*\varphi\) | particle number, \(n\) |
| information matrix, \(A_{ij}\) | information tensor, \(A_{\mu\nu}\) | information tensor, \(a_{ij}\) |
| mass squared, \(\lambda\) | | chemical potential, \(m\) |
| spatial metric, \(g_{ij}\) | spatial metric, \(g_{ij}\) | |
| free energy, \(-\hbar\log(Z)\) | free energy, \(f\) | |
| information-computation tensor, \(A_{\mu\nu}\) | information-computation tensor, \(a_{\mu\nu}\) | |
| space-time metric, \(g_{\mu\nu}\) | space-time metric, \(g_{\mu\nu}\) | entropy production, \(\frac{1}{\hbar}\frac{\partial S}{\partial t}\) |
| | | generalized forces, \(g_{\alpha\beta\mu}\) |
| | | Onsager tensor, \(\sigma^{\mu\nu\alpha\beta\gamma\delta}\) |

directly related to the theories of information (as opposed to just probability theory) is that the macroscopic constraints that we shall impose are generalizations and covariantizations of the Fisher information metric and of the Fubini-Study metric. In fact, we would like to stress that our entire approach to covariantization of information and also to (perhaps un-)quantization of gravity is based on an explicit assumption that most of the microscopic parameters, which specify either statistical or quantum states, are not observable, and one can only keep track of a very small number of macroscopic parameters. And these parameters (at least in this paper) are taken to describe informational dependence between subsystems.

The first attempt to analyze dynamics of wave-functions using statistical ensembles was made in [7], where the motivation was not coming from describing information, but from preforming efficient computations and from the action-complexity duality [8]. Since the ensembles of [7] are defined in both space and time, they are in a sense more general than what we intend to do here initially, but they will play a key role in understanding covariance that was not established in [7]. In Table (1) we summarize a mapping between information theory, field theory and thermodynamic variables that we shall use through the paper. It is worth mentioning that we will make an assumption that the space (that will be described by coordinate vector \(\vec{x}\)) had already emerged, which on itself is not a trivial task. As was already mentioned, it is not a priori clear how a low \(D\)-dimensional (say 3-dimensional)
space can emerge from a vector in a very high $N$ dimensional Hilbert/statistical space. In Ref. [3] we described one possibility (actually a large class of possibilities that we called the information graph flows) and to illustrate the procedure we used mutual information as a measure of informational dependence, however other definitions of information matrix such as Fisher information matrix could have been used instead. In this paper we will consider a generalization of Fisher information matrix (or Fubini-Study metric) which better captures quantum entanglements and also admits a covariant description. In fact, it might be useful (although not necessary) to consider the present discussion as the analysis of the final stages of the graph flow when a $D$ dimensional space had already emerged, but the space-time covariance and the emergent dynamics of metric is yet to be derived. However, instead of following the temporal dynamics of individual state vectors or individual information matrices as in Ref. [3] we will work directly with statistical ensembles as in Ref. [7] and in the later sections with thermodynamic variables.

The paper is organized as follows. In the next section we introduce (and motivate) the basic concepts of the (classical and quantum) information theories such as Fisher information metric and Fubini-Study metric. In Sec. 3 we define statistical ensembles for the infoton field which is identified with (or mapped to) either complex wave-functions or real square-roots of probability distributions. In Sec. 4 we construct space-time covariant ensembles and define a fully covariant information-computation tensor which satisfies conservation equations. In Sec. 5 we develop (first equilibrium and then non-equilibrium) thermodynamic description of quantum/statistical systems. In Sec. 6 we show how Einstein’s dynamics of the metric emerges due to a non-equilibrium production of entropy and argue that deviations from general relativity are expected far away from equilibrium. In Sec. 7 we summaries the main results of the paper.

2 Information Theory

Consider a quantum state $|\psi\rangle$ or a probability measure $\vec{p}$ and a given tensor product factorization of respectively Hilbert space $\mathcal{H}$,

$$|x\rangle = \bigotimes_i |x_i\rangle$$

or statistical space $\mathcal{P}$,

$$\hat{x} = \bigotimes_i \hat{x}_i$$

where $i \in \{1, \ldots, D\}$. Note that $D$ is the dimensionality of a sample/configuration space $\Omega$ and the dimensionality of $\mathcal{H}$ or $\mathcal{P}$ will be denoted with $N$. If the sample space $\Omega$ is compact

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1The key idea in Ref. [3] was to try to approximate a state vector $|\psi\rangle$ or $\vec{p}$ by keeping track of only very few macroscopic parameters (such as information matrix) and neglecting microscopic parameters describing these vectors. If the dimensionality $N$ of the (Hilbert or statistical) space is very large, the number of parameters in a mutual information matrix is still very large $\sim N^2$, but one can employ the information graph flow procedure to reduce the number of parameters further. It was shown that for a suitable choice of the graph flow equations (with target dimensionality $D$) the number of (macroscopic) parameters in the information matrix (after the graph flow) can be as low as $\sim DN$. And then at the final stages of the graph flow it is more convenient to use the emergent spatial geometry to define a local information tensor instead of information matrix. At this point the microscopic degrees of freedom (e.g. described by (qu)bits) already arranged themselves on a low dimensional lattice and can be thought to form more macroscopic degrees of freedom described by positions of bits on the lattice. In the limit when the number of qubits is large the position variable can be assumed to take continues values which is what we intend to do here.
and approximated by a lattice, then \( N \) is finite and one can assume that \( |x\rangle \) and \( \hat{x} \) are normalized unit vectors with respect to inner products on \( \mathcal{H} \) and \( \mathcal{P} \) respectively. Then one can define representations for \( |\psi\rangle \) and \( \vec{p} \) known as a quantum wave-function

\[
\psi(x^1, ..., x^D) \equiv \langle x|\psi \rangle
\]  

(2.3)

and a classical probability distribution

\[
p(x^1, ..., x^D) \equiv \vec{p} \cdot \hat{x}.
\]  

(2.4)

The only difference between the two objects is that the probability distribution \( p(x^1, ..., x^N) \) is non-negative with normalization\(^2\)

\[
\int d^D x \ p(x^1, ..., x^D) = 1
\]  

(2.5)

and the wave-function \( \psi(x^1, ..., x^N) \) is complex valued with normalization

\[
\int d^D x \ \psi^*(x^1, ..., x^D)\psi(x^1, ..., x^D) = 1.
\]  

(2.6)

### 2.1 Classical Information

A classical probability distribution \( p(\vec{x}) \) of independent random variables \( x^1, x^2, ... , x^N \) can always be factorized as

\[
p(\vec{x}) = p(x^1)p(x^2)...p(x^D)
\]  

(2.7)

where \( \vec{x} = (x_1, ..., x_D) \) is a vector in sample space \( \Omega \) (and not in \( \mathcal{P} \)). In the information theory we are usually dealing with dependent random variables and then it is important to have a covariant (in a sense of transforming covariantly between different tensor product factorizations of \( \mathcal{H} \) and \( \mathcal{P} \)) measure of the amount by which the probability distribution \( p(\vec{x}) \) does not factorize, i.e.

\[
p(\vec{x}) \neq p(x^1)p(x^2)...p(x^D)
\]  

(2.8)

or

\[
\log(p(\vec{x})) \neq \log(p(x^1)) + \log(p(x^2)) + ... + \log(p(x^D)).
\]  

(2.9)

To make the notion more precise we can expand the left hand side around a global maxima of \( p(\vec{x}) \), i.e.

\[
\log(p(\vec{x})) \approx \log(p(\vec{y})) - \frac{1}{2}(x^i - y^i)\Sigma_{ij}(x^j - y^j) + ...
\]  

(2.10)

where

\[
\Sigma_{ij} \equiv -\left[ \frac{\partial^2}{\partial x^i \partial x^j} \log(p(\vec{x})) \right]_{\vec{x}=\vec{y}}
\]  

(2.11)

Note that the global maxima of \( p(\vec{x}) \) exists and the above expansion makes sense only when \( p(\vec{x}) \) is not too far from a Gaussian

\[
p(\vec{x}) = \sqrt{\frac{\mid\Sigma\mid}{(2\pi)^N}} e^{-\frac{1}{2}(x^i - y^i)\Sigma_{ij}(x^j - y^j)}
\]  

(2.12)

\(^2\)Whenever the sample/configuration space \( \Omega \) is approximated with a discrete lattice the integration should be interpreted as a summation over all lattice points.
with mean $y^i$ and covariance matrix $Σ^{ij}$ (which is the inverse of $Σ_{ij}$, i.e. $Σ^{ik}Σ_{kj} = δ^i_j$) and $Σ$ is the determinant of $Σ_{ij}$.

Evidently $Σ^{ij}$ describes (the amount of) informational dependence between $i$’s and $j$’s random variables (or subsystems), but we can do better than that. The idea is to promote a global quantity $Σ^{ij}$ to a local quantity $A_{ij}(\vec{x}) \equiv -\frac{1}{4} \int d^D x \ p(\vec{x}) \frac{∂^2}{∂x^i∂x^j} \log(p(\vec{x}))$ (2.13)

which need not be evaluated at $\vec{x} = \vec{y}$. First we note that the expansion of $\log(p(\vec{x}))$ around an arbitrary local maxima $y^{(J)}$ of $p$ is still given by

$$\log(p(\vec{x})) \approx \log(p(\vec{y}^{(J)})) - \frac{1}{2} \left( x^i - y^i_{(J)} \right) Σ^{(J)}_{ij} \left( x^j - y^j_{(J)} \right) + ...$$ (2.14)

and the entire probability distributions need not be Gaussian everywhere, but could be well approximated by products of Gaussians, i.e.

$$p(\vec{x}) \propto e^{-\frac{1}{2} \left( x^i - y^i_{(J)} \right) Σ^{(J)}_{ij} \left( x^j - y^j_{(J)} \right)}$$ (2.15)

where summation over $J$ is also assumed. But, now to obtain a single number (a measure) which represents informational dependence between $i$’s and $j$’s subsystems the quantity must be summed (or integrated) over different values with perhaps different weights. One useful choice is to weight the matrix $Σ^{ij}(\vec{x})$ by probabilities $p(\vec{x})$ and then in a continuum limit we get as a measure of informational dependence

$$A_{ij} \equiv \frac{1}{4} \int d^D x \ p(\vec{x}) Σ_{ij}(\vec{x})$$

$$= -\frac{1}{4} \int d^D x \ p(\vec{x}) \frac{∂^2}{∂x^i∂x^j} \log(p(\vec{x}))$$ (2.16)

where the factor of $1/4$ is introduced for future convenience. We will refer to $A_{ij}$ as a (classical) information matrix and will generalize it in the next subsection to better describe informational dependence in quantum systems.

Note that if one considers a family of probability distributions $p_\theta(\vec{x}; \theta^1, \theta^2, ...)$ parametrized by some parameters $\theta^i$’s then the following quantity is known as Fisher information matrix

$$F_{ij} = \int d^D x \ p_\theta(\vec{x}; \theta^1, \theta^2, ...) \frac{∂^2}{∂θ^i∂θ^j} \log(p_\theta(\vec{x}; \theta^1, \theta^2, ...)).$$ (2.17)

If we are to consider small shifts in the sample space coordinates, then this would generate a family of probability distributions parametrized by $D$ parameters $θ^i$’s, i.e.

$$p_\theta(\vec{x}; \theta^1, \theta^2, ...) \equiv p(\vec{x} + \vec{θ})$$ (2.18)

and then

$$F_{ij} = \int d^D x \ p(\vec{x} + \vec{θ}) \frac{∂^2}{∂θ^i∂θ^j} \log(p(\vec{x} + \vec{θ}))$$

$$= \int d^D x \ p(\vec{x} + \vec{θ}) \frac{∂^2}{∂x^i∂x^j} \log(p(\vec{x} + \vec{θ}))$$ (2.19)

Note that there might be other weight functions of probability density $w(p)$ that could have been used for defining information matrix, i.e. $A_{ij} = -\int d^D x \ w(p) \frac{∂^2}{∂x^i∂x^j} \log(p)$, but since the linear choice $f(p) = p$ seems to be the most natural (and also the simplest) we will use it throughout the paper.
This suggests that (up to a constant factor of 1/4) our information matrix \( A_{ij} \) is nothing but Fisher information matrix for the parameters \( \theta^j \)'s being the shifts in the sample space coordinates \( x^j \)'s, i.e.

\[
A_{ij} = \frac{1}{4} \left[ F_{ij}(\bar{\theta}) \right]_{\theta=0}.
\] (2.20)

One can also shown that this information matrix has non-negative elements,

\[
A_{ij} = \frac{1}{4} \int d^D \bar{x} \ p(\bar{x}) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \log(p(\bar{x}))
\]

\[
= \frac{1}{4} \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int d^D \bar{x} \ p(\bar{x}) \log \left( \frac{\bar{q}(\bar{x})}{p(\bar{x})} \right) \log \left( \frac{\bar{r}(\bar{x})}{p(\bar{x})} \right)
\]

\[
\geq \frac{1}{4} \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \log \left( \int d^D \bar{x} \ p(\bar{x}) \left( \frac{\bar{q}(\bar{x})}{p(\bar{x})} \right) \right) \log \left( \int d^D y \ p(y) \left( \frac{\bar{r}(y)}{p(y)} \right) \right)
\]

\[
\geq \frac{1}{4} \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \log (1) \log (1) = 0
\] (2.21)

where \( q(\bar{x}) \equiv p(x^1, \ldots, x^i + \epsilon, \ldots, x^N) \) and \( r(\bar{x}) \equiv p(x^1, \ldots, x^j + \epsilon, \ldots, x^N) \) are shifted probability distribution.

### 2.2 Quantum Information

In the case of periodic or vanishing boundary conditions (an assumption that we are going to make from now on) one can integrate by parts (2.16) to obtain

\[
A_{ij} = -\frac{1}{4} \int d^D \bar{x} \ p(\bar{x}) \frac{\partial^2}{\partial x_i \partial x_j} \log(p(\bar{x}))
\]

\[
= \int d^D \bar{x} \ \frac{\partial \sqrt{p(\bar{x})}}{\partial x^i} \frac{\partial \sqrt{p(\bar{x})}}{\partial x^j}
\]

\[
= \int d^D \bar{x} \ \frac{\partial \psi(\bar{x})}{\partial x^i} \frac{\partial \psi(\bar{x})}{\partial x^j}
\] (2.22)

where \( \psi \) is a real wave-function defined as

\[
\psi(\bar{x}) \equiv \sqrt{p(\bar{x})}.
\] (2.23)

For a complex wave-function \( \psi(\bar{x}) \) one might also like to construct a matrix whose elements represent the informational dependence between subsystems. Where by informational dependence (once again) we understand the amount by which the complex wave-function fails to be factorized as in a Hartee-Fock approximation, i.e.

\[
\psi(\bar{x}) \neq \psi_1(x^1) \psi_2(x^2) \ldots \psi(x^N).
\] (2.24)

Then we could have defined an information matrix as

\[
A_{ij} = \frac{1}{4} \int d^D x \psi(\bar{x})^* \psi(\bar{x}) \frac{\partial^2}{\partial x_i \partial x_j} \log(\psi^*(\bar{x}) \psi(\bar{x}))
\]

\[
= \int d^D x \ \frac{\partial |\psi(\bar{x})|^2}{\partial x^i} \frac{\partial |\psi(\bar{x})|^2}{\partial x^j}
\] (2.25)
where
\[ p_q(x) \propto \psi(x)^* \psi(x) \frac{\partial^2}{\partial x^i \partial x^j} \log(\psi^*(x) \psi(x)) \] (2.26)
is the so-called quantum pressure tensor. This, however, does not completely capture all that we want, as there might still be non-factorization of the phase which leads to quantum entanglements between subsystems. Perhaps a better object which also carries information about the phase is a straightforward generalization of (2.22), i.e.
\[ A_{ij} = \int d^Dx \frac{\partial \psi^*(x)}{\partial x^i} \frac{\partial \psi(x)}{\partial x^j}. \] (2.27)

As was already mentioned, in the case of probability distributions our (classical) information matrix is related to the Fisher matrix and in the case of quantum states it is closely related to the Fubini-Study metric. Indeed, if we now consider small shifts in the configuration space coordinates \( \vec{x} \), then this would generate a family of wave-functions parametrized by \( D \) parameters \( \theta^i \)'s, i.e.
\[ \psi_\theta(x; \vec{\theta}) \equiv \psi(x + \vec{\theta}) \] (2.28)
In this case the Fubini-Study metric is given by
\[ S_{ij} = \int d^Dx \frac{\partial \psi^*(x + \vec{\theta})}{\partial \theta^i} \frac{\partial \psi(x + \vec{\theta})}{\partial \theta^j} - \left( \int d^Dx \frac{\partial \psi^*(x + \vec{\theta})}{\partial \theta^i} \psi(x + \vec{\theta}) \right) \left( \int d^Dy \psi^*(y + \vec{\theta}) \frac{\partial \psi(y + \vec{\theta})}{\partial \theta^j} \right) = \int d^Dx \frac{\partial \psi^*(x + \vec{\theta})}{\partial x^i} \frac{\partial \psi(x + \vec{\theta})}{\partial x^j}. \] (2.29)
where the second term drops out for the shifts which preserve normalization (2.6) and thus
\[ A_{ij} = \left[ S_{ij}(\vec{\theta}) \right]_{\theta^i = 0}. \] (2.30)
Note that we could have consider other deformations of either probability distributions \( \vec{p}(x) \) or wave-functions \( \psi(x) \) that are not necessarily defined though shifts of either sample space or configuration space coordinates. In this (more general) case the Fisher information matrix and the Fubini-Study metric can be thought of as covariant metric tensors on the respectively statistical manifold \( P \) and in Hilbert space \( H \). Instead of going this route and trying to construct covariant quantities in \( P \) and \( H \), we will concentrate on the covariance in the sample/configuration space \( \Omega \).

### 3 Statistical Ensembles

It is convenient to introduce a (dual) scalar field \( \varphi(x) \) in the sample/configuration space \( \Omega \) defined (up to a constant factor) as
\[ \varphi(x) \propto \begin{cases} \sqrt{\hbar} p(x) & \text{for probability measures } \vec{p} \\ \sqrt{\hbar} \psi(x) & \text{for quantum states } |\psi\rangle \end{cases} \] (3.1)
and then the information matrix (from now on) is defined as
\[ A_{ij} = \frac{1}{\hbar} \int d^Dx \frac{\partial \varphi^*(x)}{\partial x^i} \frac{\partial \varphi(x)}{\partial x^j}. \] (3.2)
The parameter $\hbar$, which usually plays the role of Planck constant in quantum field theories and of temperature in statistical mechanics, need not to be either of the two in the context of the information theory considered so far. We will refer to the scalar field $\varphi$ as *infoton* due to its informational origin and to $A_{ij}$ as *information matrix* for both probability measures $\vec{p}$ and quantum states $|\psi\rangle$.

The information matrix $A_{ij}$ is a global and non-covariant object in $\Omega$, but the integral (3.2) suggests that one might be able to define a local and covariant quantity. For that we will consider probability distributions (we shall call second-probability distributions) over probability measures or quantum states and so it might be useful to think of this as a “second quantization”. More precisely, we will construct a statistical ensemble $P[\varphi]$ over the infoton field $\varphi$ which would define second-probabilities of probability measures through equation

$$P[\vec{p}] = \int_{\varphi^2 \propto \vec{p}} D\varphi P[\varphi]$$

(3.3)

or second-probabilities of quantum states through equation

$$P[|\psi\rangle] = \int_{\varphi \propto \sqrt{\hbar}\psi} D\varphi D\varphi^* P[\varphi]$$

(3.4)

In both Eqs. (3.3) and (3.3) we only demand that the (real or complex) wave function is proportional to the (real or complex) infoton field $\varphi$, but an approximate normalization of $\varphi$ will be imposed later on and so it is useful to think of the infoton as either real or complex wave-function (up to a constant factor of order one).

In the language of statistical mechanics equations (3.3) and (3.4) describe what can be called “micro-canonical” ensembles with integration taken over all configurations of the infoton field $\varphi$ subject to “microscopic” constraint $\varphi^2 \propto \hbar\vec{p}$ or $\varphi \propto \sqrt{\hbar}\psi$. In the following section we will define instead a “canonical” ensemble with “macroscopic” constrains set by expected values of information matrix, $\bar{A}_{ij}$. Since the case of a complex infoton field is more general, for the most part we will assume that $\varphi$ is complex and the case of a real $\varphi$ can be recovered by setting $\varphi = \varphi^*$.

### 3.1 Canonical Ensemble

The (second-probability) distributions over pure quantum states certainly contain more information about the system than density matrices (See, for example, Ref. [9]), but that is not the main motivation to introduce them. What we want is a machinery to define distributions over (microscopic) quantum states subject to (macroscopic) constraints such as information matrix. More precisely, our next task will be to define an ensemble over $\varphi$, and thus (second-probability) distribution over $|\psi\rangle$’s, such that an expected information matrix is

$$\bar{A}_{ij} = \langle A_{ij} \rangle$$

(3.5)

for a given hermitian matrix $\bar{A}_{ij}$.

In statistical mechanics, ensembles are conveniently defined using partition function

$$Z = \int D\varphi D\varphi^* \exp \left( -\frac{S[\varphi]}{\hbar} \right)$$

(3.6)

where the (Euclidean) action $S$ is an arbitrary functional of the infoton field $\varphi$. If the theory is local then the action $S$ is given by an integral over a local function $\mathcal{L}$ of fields and its
derivatives, e.g.

\[ S = \int d^Dx \ L \left( \varphi, \frac{\partial \varphi(x)}{\partial x^1}, ..., \frac{\partial \varphi(x)}{\partial x^D} \right) \]  

(3.7)

A simple example of a local theory is described by

\[ S = \int d^Dx \ \left( g^{ij} \frac{\partial \varphi^*(\vec{x})}{\partial x^i} \frac{\partial \varphi(\vec{x})}{\partial x^j} + \lambda \varphi^*(\vec{x}) \varphi(\vec{x}) \right) \]  

(3.8)

where the values of \( g^{ij} \) do not (yet) depend on \( \vec{x} \) and the “mass-squared” constant \( \lambda \) must be chosen so that the infoton field (which is identified with wave-functions) is on average normalized. The partition function defined by action (3.8) is Gaussian and so it can be easily evaluated as

\[ Z = \int D\varphi D\varphi^* \exp \left( -\frac{1}{\hbar} \int d^Dx \ \varphi^* \left( \lambda - g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) \varphi \right) = \det \left( \frac{1}{\hbar} - \frac{g^{ij}}{\hbar} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right)^{-1} \]  

(3.9)

The determinant is divergent and should be regularized, which is possible if \( \lambda > 0 \) and \( \det(g^{ij}) > 0 \). The corresponding free energy is

\[ F \equiv -\hbar \log(Z) = \hbar \log \left( \det \left( \lambda - g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) \right) = \text{Tr} \left( \hbar \log \left( \frac{1}{\hbar} - \frac{g^{ij}}{\hbar} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) \right). \]  

(3.10)

By differentiating the free energy with respect to (Hermitian) \( g^{ij} \) we get expected information matrix

\[ \frac{1}{\hbar} \frac{\partial F}{\partial g^{ij}} = \left( \frac{1}{\hbar} \int d^Dx \ \frac{\partial}{\partial x^i} \varphi^* \frac{\partial}{\partial x^j} \varphi \right) = \bar{A}_{ij} \]  

(3.11)

and expected normalization

\[ \frac{1}{\hbar} \frac{\partial F}{\partial \lambda} = \left( \frac{1}{\hbar} \int d^Dx \ \varphi^* \varphi \right) = 1. \]  

(3.12)

Then \( N^2 + 1 \) parameters in \( g^{ij} \) and \( \lambda \) are to be chosen to satisfy \( N^2 + 1 \) equations (3.11) and (3.12). Thus one can view the corresponding ensemble as a “canonical” ensemble of (either quantum \( |\psi\rangle \) or statistical \( \vec{p} \)) microscopic states subject to macroscopic constraints imposed by the information matrix \( \bar{A}_{ij} \).

### 3.2 Information Tensor

The next step is to restrict ourselves to only real (and symmetric) \( g_{ij} \), but to promote it to depend on coordinates \( (x_1, x_2, ..., x_N) \) and thus to play the role of a metric in sample/configuration space \( \Omega \) (and not in \( \mathcal{P} \) nor \( \mathcal{H} \) as what Fisher and Fubini-Study metrics intend to do). Moreover we will add \( \sqrt{|g|} \) to the volume integral and replace partial derivatives with covariant derivatives, i.e.

\[ S = \int d^Dx \sqrt{|g|} \left( g^{ij}(\vec{x}) \nabla_i \varphi^*(\vec{x}) \nabla_j \varphi(\vec{x}) + \lambda \varphi^*(\vec{x}) \varphi(\vec{x}) \right) \]  

(3.13)
There are two advantages of using this new partition function. The first one is that the ensemble is covariant and does not depend on the choice of coordinates or, in other words, on a tensor product factorization of either Hilbert space $\mathcal{H}$ or statistical space $\mathcal{P}$. The second one is that it allows us to introduce a local (in a sample space $\Omega$) notion of informational dependence using variational derivatives of the free energy with respect to the metric.

Indeed, if we define a covariant information tensor as

$$A_{ij}(\vec{x}) \equiv \frac{1}{\hbar} \nabla_i \varphi^* \nabla_j \varphi. \quad (3.14)$$

and a covariant probability scalar

$$N(\vec{x}) \equiv \frac{1}{\hbar} \varphi^*(\vec{x}) \varphi(\vec{x}), \quad (3.15)$$

then these two quantities can be used to express the stress tensor for the infoton field as

$$T_{ij}(\vec{y}) = \nabla_i \varphi^* \nabla_j \varphi + g_{ij} \left( g^{kl} \nabla_k \varphi^* \nabla_l \varphi + \lambda \varphi^* \varphi \right)$$

$$= 2\hbar A_{(ij)} + h g_{ij} \left( g^{kl} A_{kl} + \lambda N \right) \quad (3.16)$$

where symmetrization (and for later use anti-symmetrization) of indices is defined with normalization, i.e.

$$A_{[\mu\nu]} \equiv \frac{1}{2} (A_{\mu\nu} - A_{\nu\mu})$$

$$A_{(\mu\nu)} \equiv \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu}). \quad (3.18)$$

Then for a given (ensemble averaged) information tensor $\bar{A}_{(ij)}$ the parameters $g_{ij}(\vec{x})$ and $\lambda$ are to be chosen such that

$$\int d^D x \sqrt{|g(\vec{x})|} \langle N(\vec{x}) \rangle = \int d^D x \frac{1}{\hbar} \frac{\delta \mathcal{F}[g, \lambda]}{\delta \lambda} = - \int d^D x \frac{\delta \log (Z[g, \lambda])}{\delta \lambda} = 1 \quad (3.19)$$

and

$$\langle T_{ij}(\vec{y}) \rangle = \frac{2}{\sqrt{|g(\vec{y})|}} \frac{\delta \mathcal{F}[g, \lambda]}{\delta g^{ij}(\vec{y})} = 2\hbar \bar{A}_{(ij)} + h g_{ij} \left( g^{kl} A_{kl} + \lambda \langle N \rangle \right). \quad (3.20)$$

One can also obtain an expected equation for the infoton field by extremizing the action, i.e.

$$\frac{\delta \mathcal{S}[\varphi]}{\delta \varphi^*} = (\lambda - g^{ij} \nabla_i \nabla_j) \varphi = 0. \quad (3.21)$$

4 Computation Theory

Consider a probability distribution $p(\vec{x})$ or a wave-function $\psi(\vec{x})$ which evolves in time, i.e. it depends on $D$ spatial coordinates $x^i$ and one time coordinate $x^0$. As before we are interested to see how these functions fail to factor into products, i.e.

$$p(\vec{x}) \neq p(x^0, x^1)p(x^0, x^2)\ldots p(x^0, x^D) \quad (4.1)$$

$$\psi(\vec{x}) \neq \psi(x^0, x^1)\psi(x^0, x^2)\ldots\psi(x^0, x^D), \quad (4.2)$$
where by $\vec{x}$ we now denote an $D+1$ vector in sample/configuration space-time, i.e. $(x^0, x^1, ..., x^D)$. What is however different is that an approximate normalization of the infoton field must be imposed independently at all times $x^0$. This can be accomplished with the following (non-covariant) action

$$\tilde{S} = \int d^{N+1}x \left( g^{ij}(\vec{x}) \frac{\partial \varphi^*(\vec{x})}{\partial x^i} \frac{\partial \varphi(\vec{x})}{\partial x^j} + \lambda(x^0) \varphi^*(\vec{x}) \varphi(\vec{x}) \right)$$  \hspace{1cm} (4.3)

where the tildes are introduced to indicate that we are now working with space-time quantities. As before the first term defines statistical ensemble for desired information tensor, but the second term is placed to ensure that the probabilities are conserved at all times $x^0$, i.e.

$$\frac{1}{\hbar} \delta \tilde{F} = \left\langle \frac{1}{\hbar} \int d^Dx \varphi^* \varphi \right\rangle = 1.$$  \hspace{1cm} (4.4)

Upon variation of the action with respect to the infoton field we get expected equations of motion

$$\left( \lambda(x^0) - g^{ij}(\vec{x}) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) \varphi(\vec{x}) = 0.$$  \hspace{1cm} (4.5)

which is nothing but a time-independent Schrodinger equation and the evolution is entirely due to the time dependence of $\lambda(x^0)$, i.e. external source.

After integration by parts the action of (4.3) can be rewritten as

$$\tilde{S} = \int d^{N+1}x \left( g^{ij}(\vec{x}) \frac{\partial \varphi^*(\vec{x})}{\partial x^i} \frac{\partial \varphi(\vec{x})}{\partial x^j} - 2\lambda^{(1)}(x^0) \Re \left( \varphi^*(\vec{x}) \frac{\partial}{\partial x^0} \varphi(\vec{x}) \right) \right)$$  \hspace{1cm} (4.6)

where

$$\lambda^{(1)}(x^0) \equiv \int_0^{x^0} d\tau \lambda(\tau).$$  \hspace{1cm} (4.7)

This form suggests another statistical ensemble that we can construct by simply replacing $\Re$ with $\Im$ in the action, i.e.

$$\tilde{S} = \int d^{N+1}x \left( g^{ij}(\vec{x}) \frac{\partial \varphi^*(\vec{x})}{\partial x^i} \frac{\partial \varphi(\vec{x})}{\partial x^j} - 2\lambda^{(1)}(x^0) \Im \left( \varphi^*(\vec{x}) \frac{\partial}{\partial x^0} \varphi(\vec{x}) \right) \right)$$

$$= \int d^{N+1}x \left( g^{ij}(\vec{x}) \frac{\partial \varphi^*(\vec{x})}{\partial x^i} \frac{\partial \varphi(\vec{x})}{\partial x^j} + i\lambda^{(1)}(x^0) \left( \varphi^*(\vec{x}) \frac{\partial}{\partial x^0} \varphi(\vec{x}) - \varphi(\vec{x}) \frac{\partial}{\partial x^0} \varphi^*(\vec{x}) \right) \right)$$

Upon variation the expected equations of motion are now given by

$$\left( i\lambda^{(1)}(x^0) \frac{\partial}{\partial x^0} - g^{ij}(\vec{x}) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) \varphi(\vec{x}) = 0.$$  \hspace{1cm} (4.8)

which is nothing but a (time-dependent) Schrodinger equation. Now the dynamics is not only due to time-dependence of $\lambda^{(1)}$, but also due to internal dynamics of the infoton field $\varphi$.

If we try to promote the action (4.8) to a covariant from we encounter a problem which was first noticed by Dirac: the action has terms with only one time derivative, but two spatial derivative. We are not going to “fix” it using introduction of Dirac spinors and instead we will define a different space-time ensemble for the infoton field.
4.1 Information-Computation Tensor

Up to a boundary term the action (4.8) can be rewritten as

\[ \tilde{S} = \int d^{N+1}x \left( g^{ij}(\vec{x}) \frac{\partial \varphi^*}{\partial x^i} \frac{\partial \varphi}{\partial x^j} - i\lambda^{(2)}(x^0) \left( \varphi^*(\vec{x}) \frac{\partial^2}{\partial (x^0)^2} \varphi(\vec{x}) - \varphi(\vec{x}) \frac{\partial^2}{\partial (x^0)^2} \varphi^*(\vec{x}) \right) \right) \]

\[ = \int d^{N+1}x \left( g^{ij}(\vec{x}) \frac{\partial \varphi^*}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + 2\lambda^{(2)}(x^0) \Im \left( \varphi^*(\vec{x}) \frac{\partial^2}{\partial (x^0)^2} \varphi(\vec{x}) \right) \right) \]  

where

\[ \lambda^{(2)}(x^0) \equiv \int_0^{x^0} d\tau \lambda^{(1)}(\tau) = \int_0^{x^0} d\tau \int_0^\tau d\sigma \lambda(\sigma) \] (4.10)

Then by (once again) replacing \( \Im \) with \( \Re \) we define a new ensemble described by action

\[ \tilde{S} = \int d^{N+1}x \left( g^{ij}(\vec{x}) \frac{\partial \varphi^*}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + 2\lambda^{(2)}(x^0) \Re \left( \varphi^*(\vec{x}) \frac{\partial^2}{\partial (x^0)^2} \varphi(\vec{x}) \right) \right) \]  

\[ = \int d^{N+1}x \left( g^{ij}(\vec{x}) \frac{\partial \varphi^*}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + \lambda^{(2)}(x^0) \left( \varphi^*(\vec{x}) \frac{\partial^2}{\partial (x^0)^2} \varphi(\vec{x}) + \varphi(\vec{x}) \frac{\partial^2}{\partial (x^0)^2} \varphi^*(\vec{x}) \right) \right) \]  

which has the desired property of having the same number of spatial and time derivatives and so can be covariantized,

\[ \tilde{S} = \int d^{N+1}x \sqrt{|g|} \ g^{\mu\nu} \nabla_\mu \varphi^* \nabla_\nu \varphi(\vec{x}) \] (4.12)

where \( g^{00} = -2\lambda^{(2)} \), \( g^{0i} \) and \( g^{ij} \) are all promoted to be functions on sample space and also time. If \( \lambda \) is positive, both \( \lambda^{(1)} \) and \( \lambda^{(2)} \) are positive and so the metric has Lorentzian signature and therefore the corresponding partition function must be complex, i.e.

\[ Z = \int \mathcal{D}\psi \mathcal{D}\psi^* \ \exp \left( i \frac{S[\varphi]}{\hbar} \right) \] (4.13)

An important observation is that this partition function describes the simplest example of a dual field theory of quantum computation discussed in Ref. [7].\(^4\) The main difference, however, is that in the case of Ref. [7] the physical space consisted of a (deformed) Hamming cube lattice with periodic boundary conditions, but in a more general case considered here it can be arbitrary large and arbitrary curved. With this respect the theory described by (4.12) should be viewed as a dual field theory of computation where the fundamental units of information need not be qu-bits, but can be qu-trits and more generally what we shall call qu-inf. In the next subsection we will discuss this point in more details by constructing an explicit example of a quantum computational system with qu-inf.

Upon variation of the action (4.12) (with possibly potential term) with respect to the infoton field \( \varphi \) we obtain a fully covariant equation,

\[ \nabla_\mu \nabla^\mu \varphi = \frac{V(|\varphi|)}{\partial \varphi^*}. \] (4.14)

\(^4\)Note that in Ref. [7] the dual theory had to be generalized further to include additional fields (e.g. an Abelian field) in order to cure the so-called \( Z \)-problem. We will consider such theories in Sec 6, but the additional (emergent) field will be the metric tensor \( g_{\mu\nu} \).
As was already known to Dirac, not all of the solutions of this equation can be interpreted as wave-functions (or square-roots of probabilities), which is not a big problem. In our analysis we view $\varphi$ not as a wave-function, but as a dual field whose role is to describe an ensemble from which an approximate dynamics of wave-functions can be obtained. On the other hand it is well known that in a non-relativistic limit the solutions of $\varphi$ would be describe by Schrodinger equation and so can be interpreted as wave-functions/square-roots of probabilities (See Ref. [7] for discussion of this point in context of dual theories of quantum computation).

Now that we have a fully covariant action (which describes a dual theory of information and also computation) we can look at a covariant generalization of the information tensor, i.e.

$$A_{\mu\nu} \equiv \frac{1}{\hbar} \nabla_\nu \varphi^* \nabla_\mu \varphi. \quad (4.15)$$

We had already seen that the space-space components, i.e. $A_{ij}$, provide a good measure of informational dependence between quantum or statistical subsystems, but now there are also time-time, i.e. $A_{00}$, and space-time components, i.e. $A_{0i}$, that should also have a dual meaning. In particular, the time-time component measures temporal changes of the infoton field that can only take place due to computations on the information side, but these computations cannot occur unless the wave-function (represented by infoton field) has spatial gradients. This suggests that the space-time component $A_{0i}$(which is a product of a spatial derivative and a temporal derivative of the infoton field) measures the amount that a given random variable $i$ is contributing to the computation in $A_{00}$. Thus it is useful to think of $A_{00}$ as a density of computations and of $A_{0i}$ as a flux of computations which together with information tensor $A_{ij}$ form a generally covariant tensor $A_{\mu\nu}$ that we shall call information-computation tensor. In the next subsection we will describe the flow of information and also computations which is encoded in the information-computation tensor by constructing a network of quantum computers arranged on a $D$ dimensional lattice which run in parallel and the memory is shared between neighboring computers.

As such, the tensor $A_{\mu\nu}$ is not conserved but it is related to the the energy momentum tensor of the infoton field

$$T_{\mu\nu} = -2\hbar A_{(\mu\nu)} + \hbar g_{\mu\nu} \left( g^{\alpha\beta} A_{\alpha\beta} \right) \quad (4.16)$$

which is conserved, i.e.

$$\nabla^\mu T_{\mu\nu} = 0. \quad (4.17)$$

This implies that the information-computation tensor should satisfy the following (conservation) equation

$$\nabla^\nu \left( A_{(\mu\nu)} - \frac{1}{2} A g_{\mu\nu} \right) = 0 \quad (4.18)$$

or

$$\nabla^\nu A_{(\mu\nu)} = \frac{1}{2} \nabla_\mu A. \quad (4.19)$$

where $A \equiv g^{\alpha\beta} A_{\alpha\beta}$ is the trace.

4.2 Parallel Computing

The fact that (4.12) admits a space and also time covariance can be understood by recalling a condition/restriction that was imposed on the dynamics of wave-functions on a computation.
(or information) side in Ref. [7]. The condition was a suppression of the dynamics of wave-function which involves non-local interactions/computations. To illustrate this point (as well as the role of the information-computation tensor) we consider a quantum state $|\psi\rangle$ of a computational system of $D$ qutrits or more generally quinf{s}. The $qu$ stands here for quantum, and $inf$ stands here for a unit of information. The Hilbert space of a single quinf has an arbitrary number $K$ of dimensions that are assumed to have a cyclic ordering. For $K = 2$ quinf is just a qubit, for $K = 3$ quinf is a qutrit and the (cyclic) ordering of dimensions is not required, but for $K > 3$ it is essential for what we are going to do next. If one is not yet comfortable with the new concept of quinfs it is safe to assume (for starters) that $K = 3$ and the necessity of ordering dimensions and of introduction of quinfs will become clear shortly.

Then the configuration space $\Omega$ for the system can be thought of as a $D$ dimensional lattice with $K^D$ lattice points (e.g. $3^D$ for qutrits) and periodic boundary conditions, i.e. a $D$ dimensional torus (See Fig. 1.)

Indeed, the computational basis are given by

$$|n\rangle = \bigotimes_i |n_i\rangle$$

where $n_i \in \{1, 2, ..., K\}$ and thus $n$ is an integer in base $K$, i.e. $n \in \{1, 2, ..., K\}^D$ and then the wave-function is defined as

$$\psi(n) \equiv \langle n|\psi\rangle.$$  \hspace{1cm} (4.21)

In the dual field theory description the system is described with infoton field

$$\varphi(n) \propto \sqrt{n}\psi(n)$$ \hspace{1cm} (4.22)
where the integer $n$ describes lattice coordinates. Note that for qutrits (or $K = 3$) the labeling of digits with 1, 2 and 3 has no significance, and the (dual) description should be invariant under arbitrary permutations of these labels. To accommodate this symmetry (on the dual field theory side) we assume that the $D$ dimensional lattice is periodic or it is a $D$-torus. Then, by construction, $n^i = 1$ is related to $n^i = 2$ in exactly the same way as $n^i = 2$ related to $n^i = 3$ or as $n^i = 3$ related to $n^i = 1$. For $K > 3$ the (cyclic) ordering of dimensions in our definition of quinfs is a necessary additional ingredient that allows us to use the integer $n \in \{1, 2, ..., K\}^D$ as coordinates on a $D$-dimensional periodic lattice which defines the physical space for a dual field theory. If the ordering was not available and $K > 3$ we might still be able to define the dual physical space, but its dimensionality would have to be larger than $D$.

Our next task is to figure out what kind of dynamics of the wave-functions has a dual description in terms of local dynamics of the infoton field. As one might expect such dynamics of wave-functions will be even more restrictive than allowing only a few (one or two) quinf gates. Indeed even applications of a single quinf gate would generically effect all lattice points. However, if we are restricted to only local changes of the infoton field, then we are to constrain ourselves to a small subset of quantum gates on quinfs. We will call these gates local gates, but it is still expected that these gates will form a universal set of quantum gates. To understand what these gates are let us zoom in on a single $D$-dimensional lattice hypercube of our $D$-dimensional lattice (See Fig. 1). Since the lattice is invariant under shifts by any of the lattice vectors we can (without loss of generality) take the hypercube to have vertices with coordinates $n \in \{1, 2\}^D$ (highlighted with solid bold lines on Fig. 1). This is just a hypercube of the very same type that was used in Ref. [7] with the difference that the boundary conditions are not periodic. In other words if you are to move along $i$’s dimension from $n^i = 1$ to $n^i = 2$ and then continue the motion, then you will not come back to $n^i = 1$ until after you pass through $n^i = 3, ..., K$. But now if you are interested in all possible transformations of the infoton on this (local) hypercube, this is exactly the realm of quantum computing for a system of $D$ qubits. It is well known that to describe all such transformations we can restrict ourselves to only one and two qubit gates (which do from a universal set of gates), and those are exactly the local gates that on a dual field theory side are described by a local Lagrangian of a complex scalar field (with possibly potential terms and other fields such as Abelian field as in [7] or an (emergent) metric field as in Sec. 6).

So in summary, the computations in each of the $K^D$ hypercubes are described by (local one or two) qubit gates but these computations are not isolated from each other and not only run in parallel, but can communicate with each other by exchanging information and also results of computations. For a reader familiar with parallel computing it might be more insightful to think of these hypercubes as $K^D$ quantum computers that are arranged on a lattice $D$-dimensional lattice with periodic boundary conditions that run in parallel and share memory with their neighbors. For example a quantum computer described by a $D$-dimensional hypercube $n \in \{1, 2\}^D$ and a neighboring quantum computer described by a $D$-dimensional hypercube $n \in \{1, 2\}^{D-1} \times \{2, 3\}$ share memory (or Hilbert (sub)space) described by a $D-1$-dimensional hypercube $n \in \{1, 2\}^{D-1} \times \{2\}$. Then the flow of information and computations between these Hamming hypercubes is exactly what the information-computation tensor encodes $A_{\mu \nu}$. Clearly, the information cannot propagate instantaneously across the physical space as is guaranteed by a fully covariant dual field theory developed above (4.12) or a more general theory with additional fields such as metric that we shall consider next.
5 Thermodynamics

Now that we obtained a fully generally covariant description of information (and also computation) in terms of dual field theories (what can be called second quantization of information), we could proceed to the third, forth and so on quantizations. We are not going to do it here and instead we will switch to discussing the emergent dynamics of metric.

Let us go back to the action described by (3.13) with only spatial covariance. Since the corresponding free energy only depends on $g^{ij}(\vec{x})$, $\lambda(\vec{x})$ and $\hbar$ it can be expanded as

$$F \approx \int d^D x \sqrt{|g|} \left( g^{ij} \langle A_{ij} \rangle + \lambda \langle N \rangle \right) + \hbar \frac{\partial F}{\partial \hbar}.$$ (5.1)

This free energy provides an approximate description of a “canonical” ensemble of states, but if we turn on (a unitary, but perhaps unknown) dynamics for wave-functions then the dual infoton field should also evolve accordingly. As a result the ensemble and expectation values $\langle A_{ij} \rangle$ and $\langle N \rangle$ will change in some way. However, if we still want to keep the form of the ensemble to remain the same, then what must change are the macroscopic parameters $g^{ij}$ and $\lambda$ in (3.13). And if so, would it be possible to describe the emergent dynamics of $g^{ij}$ using dynamical equations, e.g. Einstein equations? In Sec. 6 we will attempt to answer this question, but instead of considering a specific unitary evolution for the wave-function (and the corresponding approximate evolution for the infoton field) we will consider random evolutions. Then one should be able to apply the ideas of thermodynamics to obtain effective equations for the metric.

5.1 Local Equilibrium

To emphasize that we are now dealing with (local) thermodynamic variables we define the following quantities:

- **entropy scalar** $s \equiv -\partial F / \partial \hbar$ (5.2)
- **temperature scalar** $t \equiv \hbar$ (5.3)
- **information tensor** $a_{ij} \equiv \langle A_{ij} \rangle$ (5.4)
- **metric tensor** $g^{ij} \equiv g^{ij}$ (5.5)
- **particle number scalar** $n \equiv \langle N \rangle$ (5.6)
- **chemical potential scalar** $m \equiv \lambda$ (5.7)
- **free energy scalar** $f \equiv (g^{ij} a_{ij} + mn) - ts$ (5.8)

and then the total free energy can be expressed as

$$F = \int d^D x \sqrt{|g|} f.$$ (5.9)

Equation (5.8) is the Euler equation (for thermodynamics) and then by combining it with the first law of thermodynamics

$$0 = d(mn + g^{ij} a_{ij}) - td\Sigma$$ (5.10)

we get the Gibbs-Duhem Equation

$$df = -sd\Sigma.$$ (5.11)
Note that the quantity in the parenthesis in (5.10) is usually interpreted as internal energy scalar,

\[ u \equiv mn + g^{ij}a_{ij}. \] (5.12)

but in our case we not only know that it is a scalar as a whole, but we also know how it is partitioned between different components, i.e. \( mn, g^{11}a_{11}, g^{12}a_{12}, \) etc.

At the level of statistical ensembles (discussed above) changes in temperature scalar \( t = \hbar \) can always be absorbed into \( g \) and \( m \) and so without loss of generality we can set \( t = \hbar = 1 \) (aka Planck units). Then the three thermodynamics equations (defined locally) become

Euler Equation: \[ s + f = mn + g^{ij}a_{ij}. \] (5.13)

Frist Law of Thermodynamics: \[ ds = d(mn + g^{ij}a_{ij}) \] (5.14)

Gibbs-Duhem Equation: \[ df = 0 \] (5.15)

Evidently, the free energy scalar \( f \) is an extensive quantity, but the total free energy \( F \) is neither extensive nor intensive as it involves a larger number of (local) thermodynamic system which are not in a (global) equilibrium with each other. Of course, the expectation is that when the system is equilibrated the total free energy would become an extensive quantity.

We can use the definition of free energy to express a partition of unity

\[ 1 = \int D\varphi D\varphi^* \exp (-S[g,m,\varphi]) \] (5.16)

where

\[ S[g,m,\varphi] \equiv \int d^Dx \sqrt{|g|} \ (L(\varphi,g,m) - f(g,m)). \] (5.17)

is the entropy functional which (in the equilibrium) does not depend on \( m \) nor \( g \), but is a function of \( \varphi \). However, according to (5.13), on average

\[ \langle S[g,m,\varphi] \rangle = \int d^Dx \sqrt{|g|} \ (mn + g^{ij}a_{ij} - f) = \int d^Dx \sqrt{|g|} \ s \] (5.18)

which is independent on \( \varphi \). Moreover, the dependence also vanishes near field configurations which satisfy classical equations of motion, i.e.

\[ \frac{\delta S[g,m,\langle \varphi \rangle]}{\delta \langle \varphi \rangle} = 0. \] (5.19)

### 5.2 Non-equilibrium Thermodynamics

The next step if to understand the non-equilibrium dynamics [10]. In that case instead of dealing directly with entropy (5.17) we are interested in entropy production, i.e.

\[ \frac{d}{dt} S[g,m,\varphi] = \int d^Dx \sqrt{|g|} \ \frac{d}{dt} (L(\varphi,g,m) - f(g,m)). \] (5.20)

or in a more (but not completely) covariant from

\[ S[g,m,\varphi] = \int d^Dxdt \sqrt{|g|} \ (v^\alpha \nabla_\alpha (L(\varphi,g,m) - f(g,m)) + \Lambda) \] (5.21)
where $v_\nu$ defines a local rest frame of the fluid and $\Lambda$ is some integration constant. The main idea behind non-equilibrium thermodynamics is that the dynamics of systems near equilibrium is such that the entropy production is minimized. Apart from the integration constant $\Lambda$ there are two terms in the entropy functional: the first one represents the dynamics of the infoton field, i.e.

$$\int d^{D+1}x \sqrt{|g|} v^\alpha \nabla_\alpha \mathcal{L}(\varphi, g, m)$$

and the other one is supposed to describe the dynamics of the metric, i.e.

$$- \int d^{D+1}x \sqrt{|g|} v^\alpha \nabla_\alpha f(g, m).$$

In the previous section we argued that a full space-time covariance on a field theory side arises from restricting to only local computations on the information theory side. In that case the corresponding (Euclidean space-time) free energy can also be used to partition unity, i.e.

$$1 = \int \mathcal{D}\varphi \mathcal{D}\varphi^* \exp \left( -\tilde{S}[\tilde{g}, \varphi] \right)$$

where

$$\tilde{S}[\tilde{g}, \varphi] = \int d^{D+1}x \sqrt{|\tilde{g}|} \left( \tilde{\mathcal{L}}(\varphi, \tilde{g}) - \tilde{f}(\tilde{g}) \right)$$

and all the tildes are to remind us that these are space-time quantities. In particular $\tilde{g}$ is a space-time metric which (as was argued in the previous section) combines both the spatial metric $g$ and (integrals of) $m$. Thus, if we are to consider a random dynamics of quantum $|\psi\rangle$ (or statistical $\tilde{p}$) states constrained to only local computations, then the entropy/action functional must be covariantized, i.e.

$$S[g, m, \varphi] \rightarrow \tilde{S}[\tilde{g}, \varphi].$$

This suggests that we should make the following identifications

$$v^\alpha \nabla_\alpha \mathcal{L}(\varphi, g, m) \rightarrow \tilde{\mathcal{L}}(\varphi, \tilde{g})$$

and

$$v^\alpha \nabla_\alpha f(g, m) + \Lambda \rightarrow \tilde{f}(\tilde{g})$$

and so the only local term which so far does not have a space-time covariant description is the entropy production term

$$v^\alpha \nabla_\alpha f(g, m) \rightarrow \frac{1}{2\kappa} \mathcal{R}(\tilde{g}) \equiv \tilde{f}(\tilde{g}) - \Lambda.$$

In the equilibrium $df = v^\alpha \nabla_\alpha f$ vanishes (5.15) and so $\frac{1}{2\kappa} \mathcal{R}(\tilde{g})$ is due entirely to non-equilibrium processes. Note that $1/2$ and $\kappa$ are introduced for future convenience and at this point have no physical significance.

### 6 Emergent Gravity

So far we have only concluded that a fully space-time covariant entropy/action functional is expected to have the following form

$$S[g, \varphi] \equiv \int d^{D+1}x \sqrt{|g|} \left( \mathcal{L}(\varphi, g) - \frac{1}{2\kappa} \mathcal{R}(g) + \Lambda \right)$$
where we dropped all of tildes, but it is assumed that all of the quantities are defined in space-time. The term $\frac{1}{2\kappa} \mathcal{R}(g)$ is responsible for non-equilibrium entropy production, but it is now convenient to think of this terms as (a yet to be derived) action for the metric degrees of freedom, although as we have seen the metric $g_{\mu\nu}$ is not a fundamental field, but a thermodynamic variable.

By following the standard prescription we expand the entropy production around equilibrium, i.e.

$$\frac{1}{2\kappa} \mathcal{R} = g_{\alpha\beta,\mu} \gamma^{\alpha\beta}$$

(6.2)

where the generalized forces are (for starters) taken to be

$$g_{\alpha\beta,\mu} \equiv \frac{\partial g_{\alpha\beta}}{\partial x^\mu}$$

(6.3)

and fluxes are denoted by $j^{\alpha\beta}$. Then we can expand fluxes around local equilibrium (i.e. when they vanish) to the linear order in generalized forces

$$\gamma^{\alpha\beta} = \mathcal{L}^{\mu\nu\alpha\beta\gamma\delta} g_{\gamma\delta,\nu}.$$  

(6.4)

and thus

$$\frac{1}{2\kappa} \mathcal{R} = \mathcal{L}^{\mu\nu\alpha\beta\gamma\delta} g_{\alpha\beta,\mu} g_{\gamma\delta,\nu}.$$  

(6.5)

But then the Onsager reciprocity relations [11] force us to only consider Onsager tensors $\mathcal{L}^{\mu\nu\alpha\beta\gamma\delta}$ that are symmetric under exchange $(\mu, \alpha, \beta) \leftrightarrow (\nu, \gamma, \delta)$, i.e.

$$\mathcal{L}^{\mu\nu\alpha\beta\gamma\delta} = \mathcal{L}^{\nu\mu\beta\alpha\delta\gamma}$$

(6.6)

To illustrate the procedure, let us first consider a tensor

$$\mathcal{L}^{\mu\nu\alpha\beta\gamma\delta} = \frac{1}{2\kappa} \left( g^{\alpha\nu} g^{\beta\gamma} g^{\mu\delta} + g^{\alpha\gamma} g^{\beta\nu} g^{\mu\delta} - g^{\alpha\gamma} g^{\beta\delta} g^{\mu\nu} \right)$$

(6.7)

for which the flux can be rewritten as

$$j^{\mu\nu\alpha\beta} = \frac{1}{\kappa} g^{\alpha\gamma} g^{\beta\delta} \Gamma^{\mu}_{\gamma\delta}$$

(6.8)

where

$$\Gamma^{\mu}_{\gamma\delta} = \frac{1}{2} g^{\mu\nu} (g_{\nu\gamma,\delta} + g_{\nu\delta,\gamma} - g_{\gamma\delta,\nu})$$

(6.9)

are the Christoffel symbols and $\kappa$ is some phenomenological constant which measures the strength of the emergent gravitational effects. If we inset it back into the entropy functional

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5Since $g^{\alpha\beta}$ is conjugate to information it is closely related to temperature. In the simplest case of the heat flow the generalized force is given by a gradient of $(1/T)$ and so it makes sense to treat partial derivatives of $g_{\alpha,\beta}$ (or linear combinations of them) as generalized forces.
we get

$$
\int d^{D+1} x \sqrt{|g|} \frac{1}{2\kappa} R = \frac{1}{\kappa} \int d^{D+1} x \sqrt{|g|} g^{\alpha\beta,\mu} g^{\alpha\gamma} g^{\beta\delta} \Gamma_{\mu}^{\gamma \delta}
$$

$$
= -\frac{1}{\kappa} \int d^{D+1} x \sqrt{|g|} \partial_{x \mu} \Gamma_{\mu}^{\gamma \delta}
$$

$$
= \frac{1}{\kappa} \int d^{D+1} x \sqrt{|g|} \left( g^{\alpha\gamma} \partial_{x \mu} \Gamma_{\mu}^{\gamma \alpha} + g^{\alpha\gamma} \Gamma_{\mu}^{\gamma \mu} \frac{1}{\sqrt{|g|}} \partial_{x \mu} \left( \sqrt{|g|} \right) \right)
$$

$$
= \frac{1}{\kappa} \int d^{D+1} x \sqrt{|g|} \left( g^{\alpha\gamma} \partial_{x \mu} \Gamma_{\mu}^{\gamma \alpha} + g^{\alpha\gamma} \Gamma_{\mu}^{\gamma \mu} \frac{1}{\sqrt{|g|}} \partial_{x \mu} \left( \sqrt{|g|} \right) \right)
$$

$$
= \frac{1}{\kappa} \int d^{D+1} x \sqrt{|g|} \partial_{x \mu} g^{\mu\nu} \left( \Gamma_{\mu \nu \alpha \beta} + \Gamma_{\mu \nu \alpha \beta} \right) = (6.10)
$$

where

$$
\Gamma_{\mu \nu \beta} = \frac{\partial}{\partial x^\beta} \Gamma_{\mu \nu}.
$$

However, this action is not covariant unless we anti-symmetrize it with respect to lower indices $\alpha$ and $\mu$ which would produce the Einstein-Hilbert action, i.e.

$$
\int d^{D+1} x \sqrt{|g|} \frac{1}{2\kappa} R = \frac{1}{\kappa} \int d^{D+1} x \sqrt{|g|} g^{\mu\nu} \left( \Gamma_{\mu \nu \alpha \beta} + \Gamma_{\mu \nu \alpha \beta} \right) = (6.12)
$$

where the anti-symmetrization (and also symmetrization) was defined in Eq. (3.18).

From Onsager’s relation (6.6) we know that $\mathcal{L}^{\mu \nu \alpha \beta \gamma \delta}$ should be symmetric under exchange of indices $(\mu, \alpha, \beta) \leftrightarrow (\nu, \gamma, \delta)$, but there are also other (trivial) symmetries that one should impose $(\alpha) \leftrightarrow (\beta), (\gamma) \leftrightarrow (\delta)$, due to symmetries of the metric, i.e.

$$
\mathcal{L}^{\mu \nu \alpha \beta \gamma \delta} = L^{\mu \nu (\alpha \beta)} (\gamma \delta) = (6.13)
$$

The overall space of such tensors is still pretty large, but it turns out that a very simple choice leads to general relativity, i.e.

$$
L^{\mu \nu \alpha \beta \gamma \delta} = \frac{1}{8\kappa} \left( g^{\alpha \nu} g^{\beta \delta} g^{\mu \gamma} + g^{\alpha \gamma} g^{\beta \nu} g^{\mu \delta} - g^{\alpha \gamma} g^{\beta \delta} g^{\mu \nu} - g^{\alpha \beta} g^{\gamma \delta} g^{\mu \nu} \right).
$$

(6.14)

The new Onsager tensor has more symmetries and as a result of these symmetries we are led to a covariant theory of general relativity described with entropy production given by Einstein-Hilbert action. After integrating by parts, neglecting the boundary terms and collecting all other terms we get

$$
\int d^{D+1} x \sqrt{|g|} \frac{1}{2\kappa} R = \int d^{D+1} x \sqrt{|g|} g^{\mu \nu} \frac{1}{\kappa} \left( \Gamma_{\mu \nu \alpha \beta} + \Gamma_{\mu \nu \alpha \beta} \right) = (6.15)
$$

$$
= \int d^{D+1} x \sqrt{|g|} \frac{1}{8\kappa} \left( g^{\alpha \nu} g^{\beta \delta} g^{\mu \gamma} + g^{\alpha \gamma} g^{\beta \nu} g^{\mu \delta} - g^{\alpha \gamma} g^{\beta \delta} g^{\mu \nu} - g^{\alpha \beta} g^{\gamma \delta} g^{\mu \nu} \right) \partial_{\alpha \beta, \mu} g_{\gamma \delta, \nu}.
$$

Upon varying the full action (6.1) with respect to the metric (which corresponds to extremization of entropy production) we arrive at the Einstein equations

$$
\mathcal{R}_{\mu \nu} - \frac{1}{2} \mathcal{R} g_{\mu \nu} + \Lambda g_{\mu \nu} = \kappa \langle T_{\mu \nu} \rangle
$$

(6.16)
where

$$R_{\mu\nu} \equiv 2 \left( \Gamma^\alpha_{\nu[\mu,\alpha]} + \Gamma^\beta_{\nu[\mu} \Gamma^\alpha_{\alpha]\beta} \right)$$

(6.17)

is the Ricci tensor. Of course, the expectations are that this result would only hold near equilibrium, and there should be deviations from general relativity when some of the symmetries in the Onsager tensor are broken. It would be interesting to see if one can attribute the breaking of symmetries to, for example, dark matter.

In passing we note that in equation (6.1) in addition to the Einstein-Hilbert term we also get a cosmological constant term which is nothing but integration constant of the entropy functional that first appeared in (5.21). It would be interesting to see if this integration constant (which has thermodynamic origin) could be responsible for the observed accelerated expansion of the Universe or dark energy. Moreover, if general relativity is indeed a non-equilibrium process then it would be interesting to analyse other outstanding cosmological problems (e.g. entropy problem, measure problem, etc.) along the lines of ideas described in Ref. [12] where it was also argued that the (higher order than Onsager) symmetries described by the fluctuation-dissipation theorem might be observable in the CMB.

7 Conclusion

In this paper we achieved two main results: constructed a fully covariant information (and also computation) theory (Secs. 3, 4) and then developed (equilibrium and non-equilibrium) thermodynamic description of information (Secs. 5, 6) for either statistical or quantum systems. Although the first result motivated the development of the second, it is important to note that these two results can be used independently. In fact, the situation is not very different from the usual (effective) field theories that are useful at small temperatures and thermodynamic models that are useful at high temperatures.

In our case the role of temperature is played by the (inverse) metric tensor and so if the quantities, such as curvature, remain small we expect that the covariant information theory would provide a good description of the dynamics of information. However, when the curvature becomes large its dynamics cannot be ignored and the covariant information theory description becomes less useful. In this regime one can get a lot more insight into dynamics by considering a thermodynamic description developed in this paper. This does not mean that one should give up on the notion of effective fields, but these fields (such as metric) are no longer fundamental.

In particular, we described how general relativity (which is a field theory) provides a good (but only approximate) description of non-equilibrium (thermo) dynamics of the metric for a particularly simple and highly symmetric form of the Onsager tensor. But further away from equilibrium it is expected that some of the symmetries of the Onsager tensor would be broken which should give rise to deviations from general relativity.

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References

[1] B. Swingle, “Entanglement Renormalization and Holography,” Phys. Rev. D 86, 065007 (2012)
[2] C. Cao, S. M. Carroll and S. Michalakis, “Space from Hilbert Space: Recovering Geometry from Bulk Entanglement,” Phys. Rev. D 95, 024031 (2017)
[3] V. Vanchurin, “Information Graph Flow: a geometric approximation of quantum and statistical systems,” arXiv:1706.02229 [hep-th].
[4] M. Noorbala, “Space-Time from Hilbert Space: Decompositions of Hilbert Space as Instances of Time,” arXiv:1609.01295 [hep-th]
[5] T. Jacobson, “Thermodynamics of space-time: The Einstein equation of state,” Phys. Rev. Lett. 75, 1260 (1995)
[6] E. P. Verlinde, “On the Origin of Gravity and the Laws of Newton,” JHEP 1104, 029 (2011)
[7] V. Vanchurin, “Dual Field Theories of Quantum Computation,” JHEP 1606, 001 (2016)
[8] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle and Y. Zhao, “Complexity Equals Action,” arXiv:1509.07876 [hep-th]
[9] S. Weinberg, “Collapse of the State Vector,” Phys. Rev. A 85, 062116 (2012)
[10] L. D. Landau and E. M. Lifshitz, “Fluid Mechanics” (Butterworth-Heinemann, 1987).
[11] Onsager, L. “Reciprocal relations in irreversible processes, I”. Physical Review. 37 (4) 405-426 (1931)
[12] V. Vanchurin, “Dynamical systems of eternal inflation: a possible solution to the problems of entropy, measure, observables and initial conditions,” Phys. Rev. D 86, 043502 (2012)