Forbidding Kuratowski Graphs as Immersions

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**Abstract:** Immersion is a containment relation on graphs that is weaker than topological minor. (Every topological minor of a graph is also its immersion.) The graphs that do not contain any of the Kuratowski graphs \(K_5\) and \(K_{3,3}\) as topological minors are exactly planar graphs. We give a structural characterization of graphs that exclude the Kuratowski graphs as immersions. We prove that they can be constructed from planar graphs that are subcubic or of branch-width at most 10 by repetitively applying \(i\)-edge-sums, for \(i \in \{1, 2, 3\}\). We also use this result to give a structural characterization of graphs that exclude \(K_{3,3}\) as an immersion.

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1. **INTRODUCTION**

A famous graph-theoretic result is the theorem of Kuratowski that states that a graph \(G\) is planar if and only if it does not contain \(K_5\) and \(K_{3,3}\) (also known as the Kuratowski graphs) as a topological minor, that is, if \(K_5\) and \(K_{3,3}\) cannot be obtained from the graph by applying vertex and edge removals and vertex dissolutions. It is well known that the topological minor relation defines a (partial) ordering on the class of graphs.

In a similar way, the immersion and the minor orderings can be defined in graphs if instead of vertex dissolutions we ask for edge lifts and edge contractions, respectively. (For detailed definitions see Section 2.) Notice that the topological minor ordering is stronger than the minor and the immersion orderings, in the sense, that if a graph \(G\) contains a graph \(H\) as a topological minor then it also contains it as an immersion and as a minor but the inverse direction does not always hold.

In the celebrated theory of Graph Minors, developed by Robertson and Seymour, it was proven that both the immersion and minor orderings are well-quasi-ordered, that is, there are no infinite sets of mutually noncomparable graphs \([17, 18]\) according to these orderings. This result has as a consequence the complete characterization of the graph classes that are closed under taking immersions or minors in terms of forbidden graphs. (A graph class is closed under taking immersions, respectively minors, if for any graph that belongs to the graph class all of its immersions, respectively minors, also belong to the graph class.) For example, by an extension of the Kuratowski theorem (also known as Wagner’s theorem), it is also known that a graph is planar if and only if it does not contain \(K_5\) and \(K_{3,3}\) as a minor.

Thus, a question that naturally arises is about the characterization of the structure of a graph \(G\) that excludes some fixed graph \(H\) as an immersion or as a minor. While this subject has been extensively studied for the minor ordering (see \([2, 3, 6, 9, 13, 15, 16, 19–21]\)), the immersion ordering only recently attracted the attention of the research community \([1, 4, 8, 10, 12]\). DeVos et al. \([4]\) proved that for every positive integer \(t\), every simple graph of minimum degree at least \(200t\) contains the complete graph on \(t\) vertices as a (strong) immersion and Ferrara et al., given a graph \(H\), provide a lower bound on the minimum degree of any graph \(G\) in order to ensure that \(H\) is contained in \(G\) as an immersion \([7]\). More recently, Wollan \([23]\) proved a structure theorem for graphs excluding complete graphs as immersions.
In terms of graph colorings, Abu-Khzam and Langston [1] provided evidence supporting the immersion ordering analog of Hadwiger’s Conjecture, that is, the conjecture stating that if the chromatic number of a graph $G$ is at least $t$, then $G$ contains the complete graph on $t$ vertices as an immersion, and proved it for $t \leq 4$. This conjecture is proven for $t = 5, 6$ and $t \leq 7$ by Lescure and Meyniel [14] and by DeVos et al. [5] independently. The most recent result on colorings is an approximation of the list coloring number on graphs excluding the complete graph as immersion [12].

Finally, in terms of algorithms, Grohe et al. gave a cubic time algorithm that decides whether a fixed graph $H$ is contained in the input graph $G$ as immersion [10] and Giannopoulou et al. provided sufficient conditions which, when given, make the computation of the minimal graphs not belonging to a graph class closed under immersions effective [8].

Going back to the subject of the structural characterization of the graphs that exclude some fixed graph $H$ as an immersion we notice that it is straightforward to find such characterizations in the cases where $H = K_{i,j}$, where $i \in [2]$ and $j \in [3]$. In particular, the graphs that exclude $K_{1,1}$ are exactly all edgeless graphs, and the graphs that exclude $K_{1,2}$ are disjoint unions of (possibly multiple) edges. It is also easy to verify, that the connected graphs that exclude $K_{1,3}$ are the subgraphs of the following graphs; the cycle of length 3, where some of its edges may be multiple, the cycles of length at least four that have no multiple edges and the paths on at least two vertices where only the edges that are incident to the endpoints of the path may appear multiple times (for this notice that every vertex of a graph that excludes $K_{1,3}$ as an immersion has at most two neighbors). Note here that the tree-width of graphs that exclude $K_{1,j}$ as an immersion is upper bounded by $j-1$.

Similarly, one can show that the tree-width of the graphs that exclude $K_{2,2}$ as an immersion is upper bounded by 2. The reason for this is that if a graph excludes $K_{2,2}$ as an immersion then it also has to exclude $K_{2,2}$, and thus $K_{4}$, as a (topological) minor. Furthermore, after careful inspection one can show that these graphs have a much more specific form; their biconnected components are either single edges (edges that have multiplicity exactly 1), triangles that may have multiple edges, or edges of multiplicity at least 2 and also no triangle shares a common vertex with a biconnected component unless it is an edge of multiplicity 1, and no edge of multiplicity at least 2 shares a vertex with more than one other edge of multiplicity at least 2.

Finally, for the case where a graph $G$ excludes $K_{2,3}$ as an immersion it is also easy to see that the tree-width of $G$ is upper bounded by 3. Indeed, first notice that the following four graphs contain $K_{2,3}$ as a minor; $K_5$, the pentagonal prism, the octahedron, and Wagner’s graph. This implies that if a graph $G$ contains either one of those graphs as a minor, then it also contains $K_{2,3}$ as a minor. As the maximum degree of $K_{2,3}$ is at most 3, a folklore result ensures that the graph $G$ also contains $K_{2,3}$ as a topological minor (and therefore as an immersion). Hence, any graph $G$ that excludes $K_{2,3}$ as an immersion, also excludes the four graphs mentioned above as minors. From a well-known result [2] it follows that the tree-width of $G$ is upper bounded by 3.

In this note we characterize the structure of the graphs that do not contain $K_5$ and $K_{3,3}$ as immersions. As these graphs already exclude Kuratowski graphs as topological minors, they are planar. Additionally, we show that they have a more special structure: they can be constructed by repetitively joining together simpler graphs, starting from either graphs of small decomposability or by planar graphs with maximum degree at most 3. In particular, we prove that a graph $G$ that contains neither $K_5$ nor $K_{3,3}$ as immersions
can be constructed by applying consecutive \(i\)-edge-sums, for \(i \in \{3\}\), to graphs that are planar subcubic or of branch-width at most 10.

Furthermore, we show that our main result can be employed to obtain a structural characterization for the graphs that only exclude \(K_{3, 3}\) as an immersion.

2. DEFINITIONS

For every integer \(n\), we let \([n] = \{1, 2, \ldots, n\}\). All graphs we consider are finite, undirected, and loopless but may have multiple edges. Given a graph \(G\) we denote by \(V(G)\) and \(E(G)\) its vertex set and edge set, respectively. Given a set \(F \subseteq E(G)\) (resp. \(S \subseteq V(G)\)), we denote by \(G \setminus F\) (resp. \(G \setminus S\)) the graph obtained from \(G\) if we remove the edges in \(F\) (resp. the vertices in \(S\) along with their incident edges). We denote by \(\mathcal{C}(G)\) the set of the connected components of \(G\). Given a vertex \(v \in V(G)\), we also use the notation \(G \setminus v = G \setminus \{v\}\). The neighborhood of a vertex \(v \in V(G)\), denoted by \(N_G(v)\), is the set of edges in \(G\) that are adjacent to \(v\). We denote by \(E_G(v)\) the set of the edges of \(G\) that are incident with \(v\). The degree of a vertex \(v \in V(G)\), denoted by \(\deg_G(v)\), is the number of edges that are incident with it, i.e. \(\deg_G(v) = |E_G(v)|\). Notice that, as we are dealing with multigraphs, \(|N_G(v)| \leq \deg_G(v)\). The minimum degree of a graph \(G\), denoted by \(\delta(G)\), is the minimum of the degrees of the vertices of \(G\), that is, \(\delta(G) = \min_{v \in V(G)} \deg_G(v)\). A graph is called subcubic if all its vertices have degree at most 3. We also denote by \(K_r\) the complete graph on \(r\) vertices and by \(K_r \times q\) the complete bipartite graph with \(r\) vertices in its one part and \(q\) in the other. Let \(P\) be a path and \(v, u \in V(P)\). We denote by \(P[v, u]\) the subpath of \(P\) with end-vertices \(v\) and \(u\).

We say that a graph \(H\) is a subgraph of a graph \(G\), denoted by \(H \subseteq G\), if \(H\) can be obtained from \(G\) by removing edges or vertices. An edge cut in a graph \(G\) is a nonempty set \(F\) of edges that belong to the same connected component of \(G\) and such that \(G \setminus F\) has more connected components than \(G\). If \(G \setminus F\) has one more connected component than \(G\) then we say that \(F\) is a minimal edge cut. Let \(F\) be an edge cut of a graph \(G\) and let \(G'\) be the connected component of \(G\) containing the edges of \(F\). We say that \(F\) is an internal edge cut if it is minimal and each of the two connected components of \(G' \setminus F\) contains at least two vertices. An edge cut is also called \(i\)-edge-cut if it has cardinality \(\leq i\).

In this article, we mostly deal with planar graphs, that is, graphs that are embedded in the sphere \(S_0\). We call such a graph, along with its embedding, \(\Sigma_0\)-embeddable graph. Let \(C_1, C_2\) be two disjoint cycles in a \(\Sigma_0\)-embeddable graph \(G\). Let also \(\Delta_i\) be the open disk of \(S_0\ \setminus C_i\) that does not contain points of \(C_{3-i}, i \in \{2\}\). The annulus between \(C_1\) and \(C_2\) is the set \(S_0 \setminus (\Delta_1 \cup \Delta_2)\) and we denote it by \(A[C_1, C_2]\). Notice that \(A[C_1, C_2]\) is a closed set. Let \(A = \{C_1, \ldots, C_r\}\) be a collection of cycles of a \(\Sigma_0\)-embeddable graph \(G\). We say that \(A\) is nested if for every \(i \in \{r - 2\}\), \(A[C_i, C_{i+1}] \cup A[C_{i+1}, C_{i+2}] = A[C_i, C_{i+2}]\).

Contractions and minors. The contraction of an edge \(e = \{x, y\}\) from \(G\) is the removal from \(G\) of all edges incident with \(x\) or \(y\) and the insertion of a new vertex \(v_e\) that is made adjacent to all the vertices of \((N_G(x) \setminus \{y\}) \cup (N_G(y) \setminus \{x\})\) such that edges corresponding to the vertices in \((N_G(x) \setminus \{y\}) \cap (N_G(y) \setminus \{x\})\) increase their multiplicity, that is, if there was a vertex \(v \in (N_G(x) \setminus \{y\}) \cap (N_G(y) \setminus \{x\})\), \(k\) edges joining \(v\) and \(x\) and, \(l\) edges joining \(v\) and \(y\) then in the resulting graph there will be \(k + l\) edges joining \(v\) with \(v_e\). Finally, remove any loops resulting from this operation. Given two graphs \(H\) and \(G\), we
say that $H$ is a contraction of $G$, denoted by $H \leq c G$, if $H$ can be obtained from $G$ after a (possibly empty) series of edge contractions. Moreover, $H$ is a minor of $G$ if $H$ is a contraction of some subgraph of $G$.

**Topological minors.** A subdivision of a graph $H$ is any graph obtained after replacing some of its edges by paths between the same endpoints. A graph $H$ is a topological minor of $G$ (denoted by $H \leq t G$) if $G$ contains as a subgraph some subdivision of $H$.

**Immersions.** The lift of two edges $e_1 = \{x, y\}$ and $e_2 = \{x, z\}$ to an edge $e$ is the operation of removing $e_1$ and $e_2$ from $G$ and then adding the edge $e = \{y, z\}$ in the resulting graph. We say that a graph $H$ can be (weakly) immersed in a graph $G$ (or is an immersion of $G$), denoted by $H \leq im G$, if $H$ can be obtained from a subgraph of $G$ after a (possibly empty) sequence of edge lifts. Equivalently, we say that $H$ is an immersion of $G$ if there is an injective mapping $f: V(H) \to V(G)$ such that, for every edge $\{u, v\}$ of $H$, there is a path from $f(u)$ to $f(v)$ in $G$ and for any two distinct edges of $H$ the corresponding paths in $G$ are edge-disjoint, that is, they do not share common edges. Additionally, if these paths are internally disjoint from $f(V(H))$, then we say that $H$ is strongly immersed in $G$ (or is a strong immersion of $G$). The injective mapping $f$ together with the edge-disjoint paths is called a model of $H$ in $G$ defined by $f$.

**Edge sums.** Let $G_1$ and $G_2$ be graphs, let $v_1, v_2$ be vertices of $V(G_1)$ and $V(G_2)$ respectively such that $\deg_G(v_1) = \deg_G(v_2)$, and consider a bijection $\sigma : E_{G_1}(v_1) \to E_{G_2}(v_2)$, where $E_{G_1}(v_1) = \{e_i^1 | i \in \{k\}\}$. We define the $k$-edge sum of $G_1$ and $G_2$ on $v_1$ and $v_2$ as the graph $G$ obtained if we take the disjoint union of $G_1$ and $G_2$, identify $v_1$ with $v_2$, and then, for each $i \in \{1, \ldots, k\}$, lift $e_i^1$ and $\sigma(e_i^1)$ to a new edge $e_i^0$ and remove the vertex $v_1$. (See Figs. 1 and 2.)

Let $G$ be a graph, let $F$ be a minimal $i$-edge cut in $G$, and let $G'$ be the connected component of $G$ that contains $F$. Let also $C_1$ and $C_2$ be the two connected components of $G' \setminus F$. We denote by $C_i'$ the graph obtained from $G'$ after contracting all edges of $C_{3-i}$ to a single vertex $v_i$, $i \in \{2\}$. We say that the graph consisting of the disjoint union of the graphs in $C(G) \setminus \{C_1, C_2\} \cup \{C_1', C_2'\}$ is the $F$-split of $G$ and we denote it by $G|_F$. Notice that if $G$ is connected and $F$ is a minimal $i$-edge cut in $G$, then $G$ is the result of the $i$-edge
sum of the two connected components $G_1$ and $G_2$ of $C(G_F)$ on the vertices $v_1$ and $v_2$. From Menger’s Theorem we obtain the following.

**Observation 2.1.** Let $k$ be a positive integer. If $G$ is a connected graph that does not contain an internal $i$-edge cut, for some $i \in [k-1]$ and $v, v_1, \ldots, v_i \in V(G)$ are distinct vertices such that $deg_{C_i}(v) \geq i$ then there exist $i$ edge-disjoint paths from $v$ to $v_1, v_2, \ldots, v_i$.

**Lemma 2.2.** If $G$ is a $[K_5, K_{3,3}]$-immersion free connected graph and $F$ is a minimal internal $i$-edge cut in $G$, for $i \in [3]$, then both connected components of $G|_F$ are $[K_5, K_{3,3}]$-immersion free.

**Proof.** For contradiction assume that $G$ is a $[K_5, K_{3,3}]$-immersion free connected graph and one of the connected components of $G|_F$, say $C_1$, contains $K_5$ or $K_{3,3}$ as an immersion, where $F$ is a minimal internal $i$-edge cut in $G$, $i \in [3]$. Assume that $H \in \{K_5, K_{3,3}\}$ is immersed in $C_1$ and let $f : V(H) \to V(C'_1)$ be a model of $H$ in $C'_1$. Let also $v_1$ be the newly introduced vertex of $C'_1$. Notice that if $v_1 \not\in f(V(H))$ and $v_1$ is not an internal vertex of any of the edge-disjoint paths between the vertices in $f(V(H))$, then $f$ is a model of $H$ in $C_1$. As $C_1 \subseteq G$, $f$ is a model of $H$ in $G$, a contradiction to the hypothesis. Thus, we may assume that either $v_1 \in f(V(H))$ or $v_1$ is an internal vertex in at least one of the edge-disjoint paths between the vertices in $V(H)$. Note that, as neither $K_5$ nor $K_{3,3}$ contain vertices of degree 1, $deg_{C'_1}(v_1) = 2$ or $deg_{C'_1}(v_1) = 3$.

We first exclude the case where $v_1 \not\in f(V(H))$, that is, $v_1$ only appears as an internal vertex on the edge-disjoint paths. Observe that, as $deg_{C'_1}(v_1) \leq 3$, $v_1$ belongs to exactly one path $P$ in the model defined by $f$. Let $v^1_1$ and $v^2_1$ be the neighbors of $v_1$ in $P$. Recall that, by the definition of an internal $F$-split, there are vertices $v^1_2$ and $v^2_2$ in $C_2$ such that $\{v^1_1, v^1_2\}, \{v^2_1, v^2_2\} \in E(G)$. Furthermore, as $C_2$ is connected, there exists a $(v^1_1, v^2_2)$-path $P'$ in $C_2$. Therefore, be substituting the subpath $P[v^1_1, v^2_1]$ by the path defined by the union of the edges $\{v^1_1, v^1_2\}, \{v^2_1, v^2_2\} \in E(G)$ and the path $P'$ in $C_2$ we obtain a model of $H$ in $G$ defined by $f$, a contradiction to the hypothesis.

Thus, the only possible case is that $v_1 \in f(V(H))$. As $\delta(K_5) = 4$ and $deg_{C'_1}(v_1) \leq 3$, $f$ defines a model of $K_{3,3}$ in $C'_1$. Let $v^1_1, v^2_1,$ and $v^3_1$ be the neighbors of $v_1$ in $C'_1$. We claim that there is a vertex $v$ in $C_2$ and edge-disjoint paths from $v$ to $v^1_1, v^2_1, v^3_1$ in $G$, thus proving that there exists a model of $K_{3,3}$ in $G$ as well, a contradiction to the hypothesis. By the definition of an internal $F$-split, there are vertices $v^1_2, v^2_2,$ and $v^3_2$ in $C_2$ such that $\{v^1_1, v^1_2\}, \{v^2_1, v^2_2\} \in E(G)$, $i \in [3]$. Recall that $C_2$ is connected. Therefore, if for every vertex $v \in C_2$, $deg_{C_2}(v) \leq 2$, $C_2$ contains a path whose endpoints, say $u$ and $u'$ belong to $\{v^1_1, v^2_1, v^3_1\}$ and internally contains the vertex in $\{v^1_2, v^2_2, v^3_2\} \setminus \{u, u'\}$, say $u''$. Then it is easy to verify that $u''$ satisfies the conditions of the claim. Assume then that there is a vertex $v \in C_2$ of degree at least 3. Let $G'$ be the graph obtained from $G$ after removing all vertices in $V(C_1) \setminus \{v^1_1, v^2_1, v^3_1\}$ and adding a new vertex that we make it adjacent to the vertices in $\{v^1_1, v^2_1, v^3_1\}$. As $G$ does not contain an internal $i$-edge cut, $i \in [2]$. $G'$ does not contain an internal $i$-edge cut, $i \in [2]$. Therefore, from Observation 2.1 and the fact that $v \not\in \{v^1_1, v^2_1, v^3_1\}$, we obtain that there exist three edge-disjoint paths from $v$ to $v^1_1, v^2_1, v^3_1$ in $G'$ and thus in $G$. This completes the proof of the claim and the lemma follows.

Let $r$ and $q$ be integers such that $r \geq 3$ and $q \geq 1$. A $(r, q)$-cylinder, denoted by $C_{r,q}$, is the Cartesian product of a cycle on $r$ vertices and a path on $q$ vertices. (See, for
example, Fig. 3.) A \((r, q)\)-railed annulus in a graph \(G\) is a pair \((A, W)\) such that \(A\) is a collection of \(r\) nested cycles \(C_1, C_2, \ldots, C_r\) that are all met by a collection \(W\) of \(q\) paths \(P_1, P_2, \ldots, P_q\) (called rails) in such a way that the intersection of a rail and a path is always connected, that is, it is a (possibly trivial, that is, consisting of only one vertex) path. (See, e.g. Fig. 3.) Notice that given a graph \(G\) embedded in the sphere and a \((k, h)\)-cylinder ((\(r, q)\)-railed annulus, respectively) of \(G\), then any two cycles of the \((k, h)\)-cylinder ((\(r, q)\)-railed annulus, respectively) define an annulus between them.

Branch decompositions. A branch decomposition of a graph \(G\) is a pair \(B = (T, \tau)\), where \(T\) is a ternary tree and \(\tau : E(G) \to L(T)\) is a bijection of the edges of \(G\) to the leaves of \(T\), denoted by \(L(T)\). Given a branch decomposition \(B\), we define \(\sigma_B : E(T) \to \mathbb{N}\) as follows.

Given an edge \(e \in E(T)\) let \(T_1\) and \(T_2\) be the trees in \(T \setminus \{e\}\). Then \(\sigma_B(e) = |\{v \mid \text{there exist } e_i \in \tau^{-1}(L(T_i)), i \in [2], \text{such that } e_1 \cap e_2 = \{v\}\}|\). The width of a branch decomposition \(B\) is \(\max_{e \in E(T)} \sigma_B(e)\) and the branch-width of a graph \(G\), denoted by \(bw(G)\), is the minimum width over all branch decompositions of \(G\). When \(|V(T)| \leq 1\) the width of the branch decomposition is defined to be 0.

Theorem 2.3 ([11]). If \(G\) is a planar graph and \(k, h\) are integers with \(k \geq 3\) and \(h \geq 1\) then \(G\) either contains the \((k, h)\)-cylinder as a minor or has branch-width at most \(k + 2h - 2\).

We now prove the following.

Lemma 2.4. If \(G\) is a planar graph of branch-width at least 11, then \(G\) contains a \((4, 4)\)-railed annulus as a subgraph.

Proof. Let \(G\) be a planar graph of branch-width at least 11. Then by Theorem 2.3, \(G\) contains \((4, 4)\)-cylinder as a minor. By the definition of the minor relation, \(G\) contains a \((4, 4)\)-railed annulus as a subgraph. \(\square\)

Confluent paths. Let \(G\) be a graph embedded in some surface \(\Sigma\) and let \(x \in V(G)\). A disk around \(x\) is an open disk \(\Delta_x\) with the property that each point in \(\Delta_x \cap G\) is either \(x\) or belongs to the edges incident with \(x\). Let \(P_1\) and \(P_2\) be two edge-disjoint paths in \(G\) and \(x\) be a vertex of \(V(P_1) \cap V(P_2)\) that is not an endpoint of \(P_1\) or \(P_2\). From now on, we restrict
the disks \( \Delta_x \) to be such that \( \Delta_x \setminus P_1 \) and \( \Delta_x \setminus P_2 \) have exactly two connected components each. We say that \( P_1 \) and \( P_2 \) are confluent if for every \( x \in \overline{V(P_1)} \cap V(P_2) \), that is not an endpoint of \( P_1 \) or \( P_2 \), and for every disk \( \Delta_x \) around \( x \), one of the connected components of the set \( \Delta_x \setminus P_1 \) does not contain any point of \( P_2 \). We also say that a collection of paths is confluent if the paths in it are pairwise confluent.

Moreover, given two edge-disjoint paths \( P_1 \) and \( P_2 \) in \( G \) we say that a vertex \( x \in V(P_1) \cap V(P_2) \) that is not an endpoint of \( P_1 \) or \( P_2 \) is an overlapping vertex of \( P_1 \) and \( P_2 \) if there exists a \( \Delta_x \) around \( x \) such that both connected components of \( \Delta_x \setminus P_1 \) contain points of \( P_2 \). For a family of paths \( \mathcal{P} \), a vertex \( v \) of a path \( P \in \mathcal{P} \) is called an overlapping vertex of \( P \) if there exists a path \( P' \in \mathcal{P} \) such that \( v \) is an overlapping vertex of \( P \) and \( P' \).

Finally, given two paths \( P_1 \) and \( P_2 \) that share a common endpoint \( v \), we say that they are well-arranged if their common vertices appear in the same order in both paths.

3. PRELIMINARY RESULTS ON THE CONFLUENCE OF PATHS

Lemma 3.1. Let \( G \) be a graph and \( v, v_1, v_2 \in V(G) \) such that there exist edge-disjoint paths \( P_1 \) and \( P_2 \) from \( v \) to \( v_1 \) and \( v_2 \), respectively. If the paths \( P_1 \) and \( P_2 \) are not well arranged then there exist edge-disjoint paths \( P_1' \) and \( P_2' \) from \( v \) to \( v_1 \) and \( v_2 \) respectively such that \( E(P_1') \cup E(P_2') \subsetneq E(P_1) \cup E(P_2) \).

Proof. Let \( U = V(P_1) \cap V(P_2) = \{v, u_1, u_2, \ldots, u_k\} \), where \( (v, u_1, u_2, \ldots, u_k) \) is the order that the vertices in \( U \) appear in \( P_1 \), and \( (v, u_{i_1}, u_{i_2}, \ldots, u_{i_k}) \) is the order that they appear in \( P_2 \). As the paths are not well arranged there exists \( \lambda \in [k] \) such that \( u_{i_\lambda} \neq u_{i_\lambda} \). Without loss of generality assume that \( \lambda \) is the smallest such integer. Also, without loss of generality assume that \( u_{i_\lambda} < u_{i_{\lambda+1}} \). We define

\[
P_1' = P_1[v, u_{i_{\lambda-1}}] \cup P_2[u_{i_{\lambda-1}}, u_{i_\lambda}] \cup P_1[u_{i_\lambda}, v_1]
\]

and observe that \( P_1' \) and \( P_2' \) satisfy the desired properties. (For an example, see Fig. 4.)

Before proceeding to the statement and proof of the next proposition we need the following definition. Given a collection of paths \( \mathcal{P} \) in a graph \( G \), we define the function \( f_\mathcal{P} : \bigcup_{P \in \mathcal{P}} V(P) \to \mathbb{N} \) such that \( f_\mathcal{P}(x) \) is the number of pairs of paths \( P, P' \in \mathcal{P} \) for which \( x \) is an overlapping vertex. Let

\[
g(\mathcal{P}) = \sum_{x \in \bigcup_{P \in \mathcal{P}} V(P)} f_\mathcal{P}(x).
\]

Notice that \( f_\mathcal{P}(x) \ge 0 \) for every \( x \in \bigcup_{P \in \mathcal{P}} V(P) \) and thus \( g(\mathcal{P}) \ge 0 \). Observe also that \( g(\mathcal{P}) = 0 \) if and only if \( \mathcal{P} \) is a confluent collection of paths.

Lemma 3.1 allows us to prove the main result of this section. We state the result for general surfaces as the proof for this more general setting does not have any essential difference than the case where \( \Sigma \) is the sphere \( \mathbb{S}_0 \).

Proposition 3.2. Let \( r \) be a positive integer. If \( G \) is a graph embedded in a surface \( \Sigma \), \( v, v_1, v_2, \ldots, v_r \in V(G) \) and \( \mathcal{P} \) is a collection of \( r \) edge-disjoint paths from \( v \) to
v_1, v_2, \ldots, v_r in G, then G contains a confluent collection \( \mathcal{P}' \) of \( r \) well-arranged edge-disjoint paths from \( v \) to \( v_1, v_2, \ldots, v_r \) where \(|\mathcal{P}'| = |\mathcal{P}| \) and such that \( E(\bigcup_{P \in \mathcal{P}} P) \subseteq E(\bigcup_{P \in \mathcal{P}} P) \).

**Proof.** Let \( \hat{G} \) be the spanning subgraph of \( G \) induced by the edges of the paths in \( \mathcal{P} \) and let \( G' \) be a minimal spanning subgraph of \( \hat{G} \) that contains a collection of \( r \) edge-disjoint paths from \( v \) to \( v_1, v_2, \ldots, v_r \). Let also \( \mathcal{P}' \) be the collection of \( r \) edge-disjoint paths from \( v \) to \( v_1, v_2, \ldots, v_r \) in \( G' \) for which \( g(\mathcal{P}') \) is minimum. It is enough to prove that \( g(\mathcal{P}') = 0 \).

For a contradiction, we assume that \( g(\mathcal{P}') > 0 \) and we prove that there exists a collection \( \tilde{\mathcal{P}} \) of \( r \) edge-disjoint paths from \( v \) to \( v_1, v_2, \ldots, v_r \) in \( G' \) for which \( g(\mathcal{P}) \) is minimum. We will now prove that \( g(\mathcal{P}) < g(\mathcal{P}') \).

First notice that if \( x \neq u \), then \( f_{\hat{P}}(x) = f_{\hat{P}'}(x) \). Thus, it is enough to prove that \( f_{\hat{P}}(u) < f_{\hat{P}'}(u) \). Observe that if \( \{P, P'\} \subseteq \mathcal{P}' \setminus \{P_1, P_2\} \) and \( u \) is an overlapping vertex of \( P \) and \( P' \) then \( u \) is also an overlapping vertex of \( \hat{P} \) and \( \hat{P}' \). Furthermore, while \( u \) is an overlapping vertex in the case where \( \{P, P'\} = \{P_1, P_2\} \), it is not an overlapping vertex of \( \hat{P}_1 \) and \( \hat{P}_2 \). It remains to examine the case where \(|\{P, P'\} \cap \{P_1, P_2\}| = 1 \). In other words, we examine the case where one of the paths \( P \) and \( P' \), say \( P' \), is \( P_1 \) or \( P_2 \), and \( P \in \mathcal{P}' \setminus \{P_1, P_2\} \). Let
FIGURE 5. The paths $P$ (black), $P_1$ (red - dotted) and $P_2$ (blue - dashed) and the paths $\tilde{P}_1$ (blue - dashed) and $\tilde{P}_2$ (red - dotted).

FIGURE 6. The paths $P$ (black), $P_1$ (red - dotted) and $P_2$ (blue - dashed) and the paths $\tilde{P}_1$ (blue - dashed) and $\tilde{P}_2$ (red - dotted).

$\Delta_u$ be a disk around $u$ and $\Delta_1, \Delta_2$ be the two distinct disks contained in the interior of $\Delta_u$ after removing $P$. We distinguish the following cases.

Case 1. $u$ is neither an overlapping vertex of $P_1$ and $P$, nor of $P_2$ and $P$ (see Fig. 5).

Then it is easy to see that the same holds for the pairs of paths $\tilde{P}_1$ and $P$ and, $\tilde{P}_2$ and $P$. Indeed, notice that for every $i \in [2]$, $P_i$ intersects exactly one of $\Delta_1$ and $\Delta_2$. Furthermore, as $u$ is an overlapping vertex of $P_1$ and $P_2$, both paths intersect the same disk. From the observation that $P_1 \cup P_2 = \tilde{P}_1 \cup \tilde{P}_2$, we obtain that $u$ is neither an overlapping vertex of $\tilde{P}_1$ and $P$ nor of $\tilde{P}_2$ and $P$.

Case 2. $u$ is an overlapping vertex of $P_1$ and $P$ but not of $P_{3-i}$ and $P$, $i \in [2]$ (see Fig. 6).

Notice that exactly one of the following holds.

- $P_i[v, u] \cup P_{3-i}[v, u]$ intersects exactly one of the disks $\Delta_1$ or $\Delta_2$, say $\Delta_1$. Then $P_i[u, z_i]$ intersects $\Delta_2$ and $P_{3-i}[u, z_{3-i}]$ intersects $\Delta_1$. Therefore, it is easy to see that, $u$ is not an overlapping vertex of $P_i$ and $P$ anymore but becomes an overlapping vertex of $\tilde{P}_{3-i}$ and $P$.
- $P_i[u, z_i] \cup P_{3-i}[u, z_{3-i}]$ intersects exactly one of the disks $\Delta_1$ or $\Delta_2$, say $\Delta_1$. Then $P_i[v, u]$ intersects $\Delta_2$ and $P_{3-i}[v, u]$ intersects $\Delta_1$. Therefore, it is easy to see that $u$ remains an overlapping vertex of $\tilde{P}_i$ and $P$ and does not become an overlapping vertex of $P_{3-i}$ and $P$.

Case 3. $u$ is an overlapping vertex of both $P_1$ and $P$ and, $P_2$ and $P$ (see Fig. 7).

As above, exactly one of the following holds.

\[ \text{Journal of Graph Theory DOI 10.1002/jgt} \]
FIGURE 7. The paths $P$ (black), $P_1$ (red - dotted) and $P_2$ (blue - dashed) and the paths $\tilde{P}_1$ (blue - dashed) and $\tilde{P}_2$ (red - dotted).

From the above cases we obtain that $f(\tilde{P}(u)) < f(P')(u)$ and therefore $g(\tilde{P}) < g(P')$, contradicting the choice of $P'$. This completes the proof of the proposition.

\[\Box\]

4. A DECOMPOSITION THEOREM

In this section, we give a decomposition theorem for $(K_5, K_{3,3})$-immersion free graphs and use it to obtain as a corollary a decomposition theorem for $K_{3,3}$-immersion free graphs.

A. The structure of $(K_5, K_{3,3})$-immersion free graphs

We first prove the following decomposition theorem for $(K_5, K_{3,3})$-immersion free graphs.

Theorem 4.1. If $G$ is a graph not containing $K_5$ or $K_{3,3}$ as an immersion, then $G$ can be constructed by applying consecutive $i$-edge sums, for $i \in [3]$, to graphs that are planar and are subcubic or have branch-width at most 10.

Proof. Observe first that a $(K_5, K_{3,3})$-immersion-free graph is also $(K_5, K_{3,3})$-topological-minor-free, therefore, from Kuratowski’s theorem, $G$ is planar. Applying Lemma 2.2, we may assume that $G$ is a $(K_5, K_{3,3})$-immersion-free graph $G$ without any internal $i$-edge cut, $i \in [3]$. It is now enough to prove that $G$ is either subcubic or has branch-width at most 10. For a contradiction, we assume that $bw(G) \geq 11$ and that $G$ contains some vertex $v$ of degree at least 4. Our aim is to prove that $G$ contains $K_{3,3}$ as an immersion. First, let $G'$ be the graph obtained from $G$ after subdividing all of its edges once. Notice that $G'$ contains $K_{3,3}$ as an immersion if and only if $G$ contains $K_{3,3}$ as an immersion. Hence, from now on, we want to find $K_{3,3}$ in $G'$ as an immersion.

From Lemma 2.4, $G$ and thus $G'$, contains a $(4, 4)$-railed annulus as a subgraph. Observe then that $G'$ also contains as a subgraph a $(2, 4)$-railed annulus such that the vertex $v$ of degree at least 4 does not belong to the annulus between its cycles. (Fig. 8
FIGURE 8. The (4, 4)-railed annulus and the vertex v.

depicts the case where v is inside the annulus between the second and the third cycle.)

We denote by $C_1$ and $C_2$ the nested cycles and by $R_1$, $R_2$, $R_3$, and $R_4$ the rails of the aforementioned $(2, 4)$-railed annulus. Let $A$ be the annulus between $C_1$ and $C_2$. Without loss of generality we may assume that $C_1$ separates $v$ from $C_2$ and that $A$ is edge-minimal, that is, there is no other annulus $A'$ such that $|E(A')| < |E(A)|$ and $A' \subseteq A$.

Let now $G_1$, $G_2$, $G_p$ be the connected components of $A \setminus (C_1 \cup C_2)$.

Claim 1. For every $i \in [p]$ and every $j \in [2]$, $|N_{G_s}(V(G_i)) \cap V(C_j)| \leq 1$.

Proof of Claim 1. Assume the contrary. Then there is a cycle $C_j'$ such that $C_j'$ and $C_j \mod 2 + 1$ define an annulus $A'$ with $A' \subseteq A$ and $|E(A')| < |E(A)|$; a contradiction to the edge-minimality of the annulus $A$.

For every $i \in [p]$, we denote by $u_i^1$ and $u_i^2$ the unique neighbor of $G_i$ in $C_1$ and $C_2$, respectively (whenever they exist). We call the connected components of $A \setminus (C_1 \cup C_2)$ that have both a neighbor in $C_1$ and a neighbor in $C_2$ substantial. Let

$$C = \{ \widehat{G}_i = G[V(G_i)] \cup \{u_i^1, u_i^2\} \mid G_i \text{ is a substantial connected component} \}.$$ 

That is, $C$ is the set of graphs induced by the substantial connected components and their neighbors in the cycles $C_1$ and $C_2$. Note that every edge of $G$ has been subdivided in $G'$ and thus every edge $e \in G$ for which $e \cap C_1 \neq \emptyset$ and $e \cap C_2 \neq \emptyset$ corresponds to a substantial connected component in $C$.

We now claim that there exist four confluent edge-disjoint paths $P_1$, $P_2$, $P_3$, and $P_4$ from $v$ to $C_2$ in $G'$. Indeed, recall first that $G$, and hence $G'$, does not contain an internal $i$-edge cut, $i \in [3]$. Moreover, $C_2$ contains at least four vertices, say $w_i$, $i \in [4]$. Then, as $\deg_{G'}(v) \geq 4$, Observation 2.1 yields that there exist four edge-disjoint paths $P_1$, $P_2$, $P_3$, and $P_4$ from $v$ to $w_1$, $w_2$, $w_3$, and $w_4$. Finally, from Proposition 3.2, we may assume that $P_1$, $P_2$, $P_3$, and $P_4$ are confluent.

Let $P_i'$ be the subpath $P_i[v, z_i]$ of $P_i$, where $z_i$ is the vertex in $V(P_i) \cap V(C_2)$ whose distance from $v$ in $P_i$ is minimum, $i \in [4]$. Recall that all edges of $G$ have been subdivided

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in $G'_i$. This implies that there exist four (possibly not disjoint) graphs in $C$, say $\widehat{G}_1$, $\widehat{G}_2$, $\widehat{G}_3$, and $\widehat{G}_4$ such that $z_i = u'_i$, $i \in [4]$. We distinguish two cases.

**Case 1.** The graphs $\widehat{G}_1$, $\widehat{G}_2$, $\widehat{G}_3$, and $\widehat{G}_4$ are vertex disjoint.

This implies that the endpoints of $P'_1$, $P'_2$, $P'_3$, and $P'_4$ in $C_2$ are disjoint. Let $G'$ be the graph induced by the cycles $C_1$, $C_2$, and the paths $P'_1$, $P'_2$, $P'_3$, $P'_4$ and let $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$, and $\widehat{P}_4$ be confluent edge-disjoint paths from $v$ to $u'_1$, $u'_2$, $u'_3$, and $u'_4$ in $G'$ such that

(i) $\sum \{ e \mid e \in \bigcup_{i \in [4]} E(\widehat{P}_i) \setminus E(A) \}$ is minimum, that is, the number of the edges of the paths that are outside of $A$ is minimum, and

(ii) subject to (i), $\sum \{ e \mid e \in \bigcup_{i \in [4]} E(\widehat{P}_i) \}$ is minimum.

Let also $\widehat{G}$ be the graph induced by $C_1$, $C_2$, $\widehat{P}_1$, $\widehat{P}_2$, $\widehat{P}_3$, and $\widehat{P}_4$. From now on we work toward showing that $\widehat{G}$ contains $K_{3,3}$ as an immersion. For every $i \in [4]$ we call a connected component of $\widehat{P}_i \cap C_1$ nontrivial if it contains at least an edge.

**Claim 2.** For every $i \in [4]$, $\widehat{P}_i \cap C_1$ contains at most one nontrivial connected component $Q_i$ and $u'_i$ is an endpoint of $Q_i$.

**Proof of Claim 2.** First, notice that any path from $v$ to $z_i$ in $\widehat{G}$ contains $u'_i$ and thus, $u'_i \in V(\widehat{P}_i)$. Observe now that $\widehat{P}_i[u'_i, u'_j]$ is a subpath of $\widehat{P}_i$ whose internal vertices do not belong to $C_i$, thus if $u'_i$ belongs to a nontrivial connected component $Q_i$ of $\widehat{P}_i \cap C_i$, then $u'_i$ is an endpoint of $Q_i$. We will now prove that any nontrivial connected component of $\widehat{P}_i \cap C_1$ contains $u'_i$. Assume to the contrary that there exists a nontrivial connected component $P$ of $\widehat{P}_i \cap C_1$ that does not contain $u'_i$. Let $u$ be the endpoint of $P$ for which $\text{dist}_{\widehat{P}_i}(u, u'_i)$ is minimum. Let also $u'$ be the vertex in $V(\widehat{P}_i[u'_i, u'_j] \cap C_1) \setminus \{ u \}$ such that $\text{dist}_{\widehat{P}_i}(u, u')$ is minimum. Let $P'$ be the subpath of $C_1$ with endpoints $u, u'$ such that $\widehat{P}_i[u, u'] \cup P'$ is a cycle $C$ with $C \cap P = \{ u \}$. We further assume that the interior of $\widehat{P}_i[u, u'] \cup P'$ is the open disk that does not contain any vertices of $\widehat{P}_i$. We will prove that for every path $\widehat{P}_j$, $j \in [4]$, $\widehat{P}_j \cap P' \subseteq \{ u, u' \}$. As this trivially holds for $j = i$ we will assume that $j \neq i$. Observe that, for every $j \in [4]$, $\widehat{P}_j[v, u'_i] \cap A \subseteq C_j$ as for every connected component $H$ of $A \setminus (C_1 \cup C_2)$ it holds that $|N_G(V(H)) \cap V(C_j)| \leq 1$. Furthermore, observe that $\widehat{P}_j[u, u'] \cup P'$ is a separator in $\widehat{G}$. This implies that $v$ does not belong to the interior of $\widehat{P}_j[u, u'] \cup P'$. Thus, if there is a vertex $z'$ such that $z' \in \widehat{P}_j \cap (P' \setminus \{ u, u' \})$, $j \neq i$, there is a vertex $z' \in \widehat{P}_j \cap \widehat{P}[u, u']$, a contradiction to the confluence of the paths. We may then replace $\widehat{P}[u, u']$ by $P'$, a contradiction to (i).

Let us denote by $v_i$ the endpoint of $Q_i$ that is different from $u'_i$ if $Q_i$ is a nontrivial connected component of $\widehat{P}_i \cap C_i$, $i \in [4]$. Observe that $\widehat{P}_i = \widehat{P}_i[v_i] \cup Q_i \cup \widehat{P}[u'_i, u'_i]$, where we let $Q_i = \emptyset$ in the case where $\widehat{P}_i \cap C_i$ is edgeless, $i \in [4]$. We denote by $T_i$ the subpath of $C_i$ with endpoints $u'_i$ and $u'_i \mod 4 + 1$ such that $T_i \cap \{ u'_1, u'_2, u'_3, u'_4 \} \setminus \{ u'_i, u'_i \mod 4 + 1 \} = \emptyset$, $i \in [4]$. From the confluence of the paths $\widehat{P}_i$ and the fact that $u'_i$ is an endpoint of $Q_i$ it follows that $Q_i \subseteq T_i$, or $Q_i \subseteq T_{i-1}$, $i \in [4]$ where $T_{i-1} = T_{3 + i \mod 4}$ if $i - 1 \notin [4]$.

**Claim 3.** There exists an $i_0 \in [4]$ such that $T_{i_0} \cap (Q_{i_0} \cup Q_{i_0 \mod 4 + 1}) \neq T_{i_0}$.

**Proof of Claim 3.** Toward a contradiction assume that for every $i \in [4]$, it holds that $T_i \cap (Q_i \cup Q_{i \mod 4 + 1}) = T_i$. It follows that either $Q_i = T_i = \widehat{P}_i[v_i, u'_i]$, $i \in [4]$, or $Q_{i \mod 4 + 1} = T_i$, $i \in [4]$. Notice then that either $v_i = u'_i \mod 4 + 1$, $i \in [4]$. This implies that there exist four (possibly not disjoint) graphs in $C$, say $\widehat{G}_1$, $\widehat{G}_2$, $\widehat{G}_3$, and $\widehat{G}_4$ such that $z_i = u'_i$, $i \in [4]$. We distinguish two cases.
Let $G'$ be the graph induced by the cycles $C_1$ and $C_2$ and the graphs in $\mathcal{C}'$. We will show that $G'$ contains $K_{3,3}$ as an immersion. First recall that the common vertices of $\widehat{G}_{i_1}$ and $\widehat{G}_{i_2}$ lie in at least one of the cycles $C_1$ and $C_2$. Without loss of generality assume that they have a common vertex in $C_1$. Recall that, as every edge of $G$ has been subdivided in $G'$, there does not exist an edge $e \in G'$ such that $e \cap C_j = \emptyset$, $j \in [2]$. This observation and the fact that there exist four rails between $C_1$ and $C_2$ imply that there exist at least four graphs in $\mathcal{C}'$ that are vertex disjoint. It follows that there exist three vertex-disjoint graphs, say $\widehat{G}_{i_1}, \widehat{G}_{i_2}, \widehat{G}_{i_3}$, in $\mathcal{C}'$ with the additional properties that $\widehat{G}_{i_{1,r}} \cap \widehat{G}_{i_{2,r}} \cap C_1 = \emptyset$, $r \in [3]$, and that at most one of the $\widehat{G}_{i_1}, \widehat{G}_{i_2}, \widehat{G}_{i_3}$ has a common vertex with one of the $\widehat{G}_{i_1}, \widehat{G}_{i_2}$.

\textbf{Case 2.} There exist $i_1, i_2 \in [4]$ such that $\widehat{G}_{i_1}$ and $\widehat{G}_{i_2}$ are not vertex disjoint.

It is now easy to see that $\widehat{G}$, and thus $G$, contains $K_{3,3}$ as an immersion. Indeed, first remove all edges of $C_1 \setminus T_0$ that do not belong to any path $\widehat{P}_i$, $i \in [4]$. Then lift the paths $\widehat{P}_i$ to a single edge where $i \neq i_0, i_0 \mod 4 + 1$. Now let $u_{i_0} (u_{i_0} \mod 4 + 1$, respectively) be the vertex of $T_0$ that belongs to $\widehat{P}_{i_0}$ ($\widehat{P}_{i_0} \mod 4 + 1$, respectively) whose distance from $v$ in $\widehat{P}_{i_0}$ ($\widehat{P}_{i_0} \mod 4 + 1$, respectively) is minimum and lift the paths $\widehat{P}_{i_0}[u, u_{i_0}]$ and $\widehat{P}_{i_0} \mod 4 + 1[v, u_{i_0} \mod 4 + 1]$ to single edges. Notice now that $\widehat{G}$ contains the graph $H_2$ depicted in Fig. 9 as an immersion. Thus, we get that $\widehat{G}$ contains $K_{3,3}$ as an immersion.
B. A decomposition theorem for \( K_{3,3} \)-immersion free graphs

In this subsection, we show how we can obtain a decomposition theorem for \( K_{3,3} \)-immersion free graphs from the decomposition theorem of the previous subsection. We first need the following definition.

**Clique-sums.** Let \( G_1 \) and \( G_2 \) be two graphs with disjoint vertex sets, let \( k \geq 0 \) be an integer, and let \( X_i \subseteq V(G_i) \) be a set of pairwise adjacent vertices in \( G_i \) of size \( k, i \in [2] \). Let \( G'_i \) be the graph obtained from \( G_i \) after deleting a (possibly empty) set of edges whose both endpoints belong to \( X_i \). If \( f : X_1 \to X_2 \) is a bijection, the graph \( G = G'_1 \oplus_k G'_2 \) obtained from the union of \( G'_1 \) and \( G'_2 \) by identifying \( x \) with \( f(x) \), \( x \in X_1 \), is called a \( k \)-clique-sum of \( G_1 \) and \( G_2 \).

**Theorem 4.2** ([22]). A graph \( G \) does not contain \( K_{3,3} \) as a minor if and only if it can be constructed from planar graphs and \( K_5 \) by applying \( i \)-clique-sums, \( i \in \{0 \} \cup [2] \).

Let us note here that the analog of Lemma 2.2 for \( K_{3,3} \)-immersion free graphs and \( i \)-clique-sums, \( i \in \{0 \} \cup [2] \), also holds.

**Lemma 4.3.** If \( G \) is a \( K_{3,3} \)-immersion free graph such that there exist \( G'_1 \) and \( G'_2 \) with \( G = G'_1 \oplus_i G'_2, i \in [2] \), that is, if \( G \) can be obtained from \( G'_1 \) and \( G'_2 \) by applying an \( i \)-clique-sum, \( i \in \{0 \} \cup [2] \), then there also exist \( K_{3,3} \)-immersion free graphs \( G_1 \) and \( G_2 \) such that \( G = G_1 \oplus_i G_2, i \in [2] \).

**Proof.** Notice first that the graph \( G \) is not 3-connected as the vertices occurring in the \( i \)-clique-sum, \( i \in \{0 \} \cup [2] \) form a separator of \( G \). Notice also that the lemma trivially holds in the case where \( G \) is not connected as it can be considered as the 0-clique-sum of the graph induced by exactly one of its connected components and the graph induced by the rest of its connected components. Thus, notice that it is enough to prove the lemma for the case where \( G \) is either 1-connected or biconnected. We assume first that \( G \) is 1-connected and \( x \) is a cut-vertex of \( G \). Let \( C_1, C_2, \ldots, C_l \) be the connected components of \( G \setminus \{x\} \) and notice that \( G = G_1 \oplus_i G_2, \) where \( G_1 = G[V(C_1) \cup \{x\}] \) and \( G_2 = G[V(C_2) \cup \{x\}] \). We claim that \( G_1 \) and \( G_2 \) satisfy the requirements of the lemma. Indeed, if \( K_{3,3} \) is an immersion of \( G_1 \) or \( G_2 \) then, as \( G_1 \) and \( G_2 \) are subgraphs of \( G \), \( K_{3,3} \) is also an immersion of \( G \).

Finally, let us consider the case where the graph \( G \) is biconnected. We denote by \( x \) and \( y \) the vertices of a separator of \( G \) of minimum size. Let \( C_1, C_2, \ldots, C_l \) be the connected components of \( G \setminus \{x, y\} \). Let also \( G_1 \) be the graph induced by \( V(C_1) \cup \{x, y\} \) and containing the edge \( \{x, y\} \) (if \( \{x, y\} \notin E(G) \)) and \( G_2 \) be the graph induced by \( \bigcup_{i=2}^l V(C_i) \cup \{x, y\} \) again containing the edge \( \{x, y\} \) (if \( \{x, y\} \notin E(G) \)). We claim that the lemma holds for the graphs \( G_1 \) and \( G_2 \). Indeed, to the contrary, let us assume that \( G_1 \) contains \( K_{3,3} \) as an immersion. Observe that if there is a model \( h \) of \( K_{3,3} \) in \( G_1 \), where \( G_1^* \) is the graph obtained from \( G_1 \) after the removal of the edge \( \{x, y\} \), then as \( G_1^* \) is a subgraph of \( G \), \( h \) is also a model of \( K_{3,3} \) in \( G \), a contradiction to the hypothesis that \( G \) does not contain \( K_{3,3} \) as an immersion. Therefore, every model \( h \) of \( K_{3,3} \) in \( G_1 \) uses the edge \( \{x, y\} \). This implies that \( \{x, y\} \notin E(G) \) as otherwise \( G_1 \) is a subgraph of \( G \) and therefore, \( K_{3,3} \) is also an immersion of \( G \). However, as \( \{x, y\} \) is a minimal separator of \( G \), there exists an \((x, y)\)-path \( P \) in \( G[V(C_2) \cup \{x, y\}] \). It follows that by replacing \( \{x, y\} \) with the path \( P \) in \( h \) we obtain a model \( h' \) of \( K_{3,3} \) in \( G \), a contradiction to the hypothesis. Similarly,
$G_2$ does not contain $K_{3,3}$ as an immersion. This completes the proof of the claim and the lemma.

**Theorem 4.4.** If $G$ is a $K_{3,3}$-immersion free graph then it can be constructed by applying $i$-edge-sums, $i \in [3]$, and $j$-clique-sums, $j \in \{0\} \cup [2]$, to a (possibly empty) set of disjoint copies of $K_5$ and planar graphs that are sub-cubic or have branch-width at most 10, with the further restriction that no 2-clique-sum is applied on two edges that belong to two disjoint copies of $K_5$.

**Proof.** Let $G$ be a graph that does not contain $K_{3,3}$ as an immersion. Then, it also does not contain $K_{3,3}$ as a topological minor. Furthermore, as the maximum degree of $K_{3,3}$ is upper bounded by 3, from a folklore result, it follows that $G$ does not contain $K_{3,3}$ as a minor as well. Combining Lemmata 4.2 and 4.3 we may also assume that $G$ is isomorphic to $K_5$ or planar. Applying Lemma 2.2, we may further assume that $G$ does not contain any internal $i$-edge-cut, $i \in [3]$. Therefore, $G$ is internally 4-edge-connected and is either isomorphic to $K_5$ or planar. Hence, by following along the lines of the proof of Theorem 4.1, we obtain that $G$ is either $K_5$ or it is a planar graph which is either subcubic or has branch-width at most 10. Finally, in order to see that no 2-clique-sums are applied on edges of two disjoint copies of $K_5$, it is enough to observe that $K_5 \oplus_2 K_5$ contains $K_{3,3}$ as an immersion.

**Remark 4.5.** It is easy to verify that our results hold for both the weak and strong immersion relations.

We believe that the upper bound on the branch-width of the building blocks of Theorem 4.1 can be further reduced, especially if we restrict ourselves to simple graphs. There is an infinite family of graphs that are not subcubic and have branch-width 3; two of them are depicted in Figure 10. However, we have not been able to find any simple nonsubcubic graph of branch-width greater than 3 that does not contain $K_5$ or $K_{3,3}$ as an immersion.

Finally, let us mention here that finding an exact structural characterization of the graphs that do not contain $K_5$ as a topological minor (which would also imply a structural characterization of the graphs that exclude $K_5$ as an immersion) is a long-standing open problem.

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