ON POSITIVITY OF THE KADISON CONSTANT AND NONCOMMUTATIVE BLOCH THEORY

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ABSTRACT. In [V. Mathai, K-theory of twisted group C*-algebras and positive scalar curvature, Contemp. Math. 231 (1999) 203–225], we established a natural connection between the Baum-Connes conjecture and noncommutative Bloch theory, viz., the spectral theory of projectively periodic elliptic operators on covering spaces. We elaborate on this connection here and provide significant evidence for a fundamental conjecture in noncommutative Bloch theory on the non-existence of Cantor set type spectrum. This is accomplished by establishing an explicit lower bound for the Kadison constant of twisted group C*-algebras in a large number of cases, whenever the multiplier is rational.

INTRODUCTION

In this paper, we study noncommutative Bloch theory, which is the spectral theory of elliptic operators on covering spaces that are invariant under a projective action of the fundamental group. A central conjecture in noncommutative Bloch theory (cf. [Ma], [MM]) is the following:

Conjecture A. Suppose that $D : L^2(\tilde{M}, \tilde{E}) \to L^2(\tilde{M}, \tilde{E})$ is a $(\Gamma, \bar{\sigma})$-invariant, self-adjoint, elliptic operator on a normal covering space $\Gamma \to \tilde{M} \to M$ of a compact manifold $M$, where $\tilde{E}$ is a $\Gamma$-invariant Hermitian vector bundle over $\tilde{M}$. If $\sigma$ is rational, then there are only a finite number of gaps in the spectrum of $D$ that lie in any bounded interval $[a, b]$.

Here $\sigma$ is a multiplier on the discrete group $\Gamma$ (i.e., a normalized $U(1)$-valued group 2-cocycle on $\Gamma$) and $\bar{\sigma}$ is its conjugate multiplier. The $(\Gamma, \bar{\sigma})$-action is a projective action of $\Gamma$ that is defined by the multiplier $\bar{\sigma}$ (cf. Section 2). A multiplier $\sigma$ is said to be rational if $[\sigma] \in H^2(\Gamma, \mathbb{Q}/\mathbb{Z})$, where we have identified $\mathbb{R}/\mathbb{Z}$ with $U(1)$.

By a fundamental result of [BrSt], one knows that Conjecture A follows from the positivity of the Kadison constant $C_{\sigma}(\Gamma)$ (cf. Subsection 2.3) of the reduced twisted group C*-algebra $C_r^*(\Gamma, \sigma)$, which is a norm completion of the twisted group algebra (cf. Section 1). Recall that the Kadison constant is the smallest value of the trace evaluated on projections in $C_r^*(\Gamma, \sigma)$, which can be zero if there are infinitely many projections in the algebra. This is the case for instance whenever $\Gamma = \mathbb{Z}^2$ and $\sigma$ is not a rational multiplier, [Rieff]. In this paper, we restrict ourselves to the case when $\Gamma$ is

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a torsion-free discrete group and to the case when the multiplier $\sigma$ has trivial Dixmier-Douady invariant, $\delta(\sigma) = 0$. The method of proving the positivity of the Kadison constant consists of first defining a twisted analogue of the Baum-Connes map (or analytic assembly map or twisted Kasparov map [Ma]),

$$\mu_\sigma : K_j(B\Gamma) \to K_j(C^*_r(\Gamma, \sigma)), \quad j = 0, 1$$

which is done in Section 2. It is more convenient for our purposes than the definition given in [Ma], but is also equivalent to it. The twisted Baum-Connes conjecture asserts that $\mu_\sigma$ is an isomorphism whenever $\Gamma$ is torsion-free; the conjecture has to be modified using classifying spaces for proper $\Gamma$-actions when $\Gamma$ has torsion. Here $B\Gamma$ is the classifying space of the discrete torsion-free group $\Gamma$. In [Ma] it was shown that whenever the discrete group has the Dirac-Dual Dirac property, then the twisted Baum-Connes map $\mu_\sigma$ is split injective. For example, by results of Kasparov [Kas], discrete subgroups of connected Lie groups have the Dirac-Dual Dirac property. If $\Gamma$ is assumed to be $K$-amenable in addition, then it has been proved in [Ma] that $\mu_\sigma$ is an isomorphism. By results of Kasparov [Kas2], Julg-Kasparov [JuKas] and Higson-Kasparov [HiKa], all discrete amenable groups, all discrete subgroups of $SO_0(1, n)$ and of $SU_0(1, n)$ are $K$-amenable and have the Dirac-Dual Dirac property.

The canonical trace $\tau$ on the reduced twisted group $C^*$-algebra $C^*_r(\Gamma, \sigma)$ induces a homomorphism $[\tau] : K_0(C^*_r(\Gamma, \sigma)) \to \mathbb{R}$. Under the assumption that the twisted Baum-Connes map (1) is an isomorphism, we will use a twisted $L^2$ index theorem for covering spaces to compute in Section 2 the range of the trace on $K$-theory in terms of classical characteristic classes. This enables us to prove our main theorem supporting the conjecture above. We first recall that a $CW$ complex $X$ is said to have finite rational cohomological dimension if $H^j(X, \mathbb{Q}) = 0$ for all $j$ large. The cohomological dimension of $X$ is then the largest $j$ such that $H^j(X, \mathbb{Q}) \neq 0$. A discrete group $\Gamma$ is said to have finite rational cohomological dimension if its classifying space $B\Gamma$ has finite rational cohomological dimension.

**Theorem B (Positivity of the Kadison constant).** Suppose that $\Gamma$ is a torsion-free discrete group satisfying the twisted Baum-Connes conjecture, and that $\Gamma$ has finite rational cohomological dimension. Then the Kadison constant $C_\sigma(\Gamma)$ (cf. Subsection 2.3) of the reduced twisted group $C^*$-algebra $C^*_r(\Gamma, \sigma)$ is positive.

More precisely, suppose that $\sigma^q = 1$ for some $q > 0$ and $\ell$ is the cohomological dimension of $\Gamma$, then one has the following lower bound

$$C_\sigma(\Gamma) \geq c_\ell^{-1}q^{-\ell(\ell+1)/2} > 0,$$

where $c_\ell = \prod_{j \geq 1, i \geq 0}(2^j - 2^i) > 0$.

In particular, Conjecture A holds for such pairs $(\Gamma, \sigma)$.

Observe that by the discussion above, all discrete subgroups of $SO_0(1, n)$ and of $SU_0(1, n)$ satisfy the hypotheses of Theorem B, so do all discrete amenable groups of finite cohomological dimension. Theorem B was previously known in the case of two dimensions, when $\Gamma$ is a discrete cocompact subgroup of $PSL(2, \mathbb{R})$. Here, not only the Kadison constant was computed, but also the range of the trace on $K$-theory was calculated in [CHMM] when $\Gamma$ is torsion-free (see also [BC]), and when $\Gamma$ has torsion
in [MM]. The range of the trace on $K$-theory was computed in the case of $\mathbb{Z}^2$ in [Rieff]. For the relevance of noncommutative Bloch theory to physics, see [Bel].

The following theorem is an immediate consequence of Theorem B and the main theorem in [BrSu] (cf. Theorem 2.8), and is a quantitative version of the application of Theorem B to Conjecture A.

**Theorem C.** Let $D : L^2(\tilde{M}, \tilde{E}) \to L^2(\tilde{M}, \tilde{E})$ be a $(\Gamma, \sigma)$-invariant self-adjoint elliptic differential operator which is bounded below, where $\Gamma$ and $\sigma$ satisfy the hypotheses of Theorem B. If $\psi(\lambda)$ denotes the number of gaps in the spectrum of $D$ that lie in the $1/2$-line $(-\infty, \lambda]$, then one has the asymptotic estimate

\[
\lim_{\lambda \to +\infty} \sup \left\{ \frac{(2\pi)^n \psi(\lambda) \lambda^{-n/d}}{c_\ell q^{-\ell(\ell+1)/2} \text{vol}(M)} \right\} \leq 1,
\]

where $w_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, $\text{vol}(M)$ is the volume of $M$, $n = \dim M$, $d$ is the degree of $D$, $\ell$ is the cohomological dimension of $\mathcal{B}\Gamma$ and $c_\ell = \prod_{\ell \geq j, i \geq 0} (2^j - 2^i)^{\ell} > 0$.

Theorem C gives an asymptotic upper bound for the number of gaps in the spectrum of such operators. In [MS] we prove the existence of an arbitrarily large number of spectral gaps in the spectrum of magnetic Schrödinger operators with Morse-type potentials on covering spaces.

Recall that an ICC group is a group such that every nontrivial conjugacy class in the group is infinite. We also obtain the following theorem in Section 2.4.

**Theorem D.** Suppose that $\Gamma$ and $\sigma$ satisfy the hypotheses of Theorem B, and suppose in addition that $\Gamma$ is an ICC group. Then there are only a finite number of projections in the twisted group $C^*$-algebra $C_r^*(\Gamma, \sigma)$, up to Murray-von Neumann equivalence in the enveloping von Neumann algebra.

In fact, if $\sigma^q = 1$ for some $q > 0$ and $\ell$ is the cohomological dimension of $\mathcal{B}\Gamma$, then there are at most $c_\ell q^{\ell(\ell+1)/2}$ non-trivial projections in the twisted group $C^*$-algebra $C_r^*(\Gamma, \sigma)$, up to Murray-von Neumann equivalence in the enveloping von Neumann algebra, where $c_\ell = \prod_{\ell \geq j, i \geq 0} (2^j - 2^i)^{\ell} > 0$.

We also prove the following useful theorem, which says in particular that if the Dixmier-Douady invariant $\delta(\sigma) = 0$, then $K_0(C_r^*(\Gamma, \sigma)) \cong K_0(C_r^*(\Gamma))$, i.e., the $K$-theory in this case can be computed from the (untwisted) Baum-Connes conjecture.

**Theorem E.** Let $\Gamma$ be a discrete group and $\sigma^0, \sigma^1$ be multipliers on $\Gamma$ such that $\delta(\sigma^0) = \delta(\sigma^1)$. Then there is an isomorphism

\[
\lambda_1 : K_0(C_r^*(\Gamma, \sigma^1)) \xrightarrow{\cong} K_0(C_r^*(\Gamma, \sigma^0)).
\]

Section 1 contains the preliminary material. The twisted Baum-Connes map is defined in Section 2, and Theorems B, C and D are also proved there. Theorem E is proved in Section 3.

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1. Preliminaries

1.1. Twisted group $C^*$-algebras. We begin by reviewing the concept of twisted crossed product $C^*$-algebras. Let $A$ be a $C^*$-algebra and $\Gamma$ a discrete group. Let $\text{Aut}(A)$ denote the group of $*$-automorphisms of $A$. Let $\alpha : \Gamma \to \text{Aut}(A)$ be a representation. Then $A$ is called a $\Gamma$-$C^*$-algebra. Let $\sigma : \Gamma \times \Gamma \to \text{U}(1)$ be a multiplier satisfying the following conditions:

1. $\alpha_1 = 1$, $\sigma(1, s) = \sigma(s, 1) = 1$ for all $s \in \Gamma$,
2. $\alpha_s \alpha_t = \alpha_{st}$,
3. $\sigma(s, t) \sigma(st, r) = \sigma(s, tr) \sigma(t, r)$ for all $r, s, t \in \Gamma$.

Then $(A, \Gamma, \alpha, \sigma)$ is called a twisted dynamical system. From such a twisted dynamical system, one can construct a twisted crossed product algebra as follows. Let $A(\Gamma, \alpha, \sigma)$ denote the finitely supported, $A$-valued functions on $\Gamma$ and with multiplication given by

$$\left( \sum_{\gamma \in \Gamma} a_\gamma \gamma \right) \left( \sum_{\gamma' \in \Gamma} b_{\gamma'} \gamma' \right) = \sum_{\gamma \in \Gamma} \sum_{\gamma' \in \Gamma} a_\gamma \alpha_\gamma(b_{\gamma'}) \sigma(\gamma, \gamma') \gamma \gamma', $$

where $a_\gamma, b_{\gamma'} \in A$, and with adjoint or involution given by

$$\left( \sum_{\gamma \in \Gamma} a_\gamma \gamma \right)^* = \sum_{\gamma \in \Gamma} \alpha_\gamma(\overline{a_\gamma}) \sigma(\gamma, \gamma)^{-1} \gamma.$$

Recall that a covariant representation $\pi$ of $(A, \Gamma, \alpha, \sigma)$ consists of a pair of unitary representations $\pi_A$ of $A$ and $\pi_\Gamma$ of $\Gamma$ on a Hilbert space $\mathcal{H}$ satisfying

$$\pi_\Gamma(\gamma) \pi_\Gamma(\gamma') = \sigma(\gamma, \gamma') \pi_\Gamma(\gamma \gamma')$$

for all $\gamma, \gamma' \in \Gamma$ and

$$\pi_\Gamma(\gamma) \pi_A(a) \pi_\Gamma(\gamma)^{-1} = \pi_A(\alpha_\gamma(a))$$

for all $\gamma \in \Gamma, a \in A$. It gives rise to an involutive representation $\tilde{\pi}$ of $A(\Gamma, \alpha, \sigma)$ as follows:

$$\tilde{\pi} \left( \sum_{\gamma \in \Gamma} a_\gamma \gamma \right) = \sum_{\gamma \in \Gamma} \pi_a(\gamma) \pi_\Gamma(\gamma).$$

Define the following norm on $A(\Gamma, \alpha, \sigma)$,

$$||f|| = \sup \{||\tilde{\pi}(f)||; \pi \text{ a covariant representation of } (A, \Gamma, \alpha, \sigma)\}.$$

The the completion of $A(\Gamma, \alpha, \sigma)$ in this norm is called the full twisted crossed product and is denoted by $A \rtimes_{\alpha, \sigma} \Gamma$. Consider the Hilbert $\Gamma$-module

$$\ell^2(\Gamma, A) = \left\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma; \sum_{\gamma \in \Gamma} a_\gamma^* a_\gamma \text{ converges in } A \right\}.$$

Then $A(\Gamma, \alpha, \sigma)$ acts on $\ell^2(\Gamma, A)$ and its norm completion is called the reduced twisted crossed product and is denoted by $A \rtimes_{\alpha, \sigma, r} \Gamma$. We refer to [Pe] for more information on these constructions.

We remark that when $\alpha$ is trivial and when $A = \mathbb{C}$, then $A \rtimes_{\alpha, \sigma, r} \Gamma = C^*_r(\Gamma, \sigma)$ is the reduced twisted group $C^*$-algebra of $\Gamma$. The canonical trace $\tau : C^*_r(\Gamma, \sigma) \to \mathbb{C}$ is defined as $\tau \left( \sum_{\gamma \in \Gamma} a_\gamma \gamma \right) = a_1$. Similar remarks apply to the full twisted crossed products.
In this paper, we will only discuss the reduced twisted crossed product $C^*$-algebras, but many of our results also apply to the full $C^*$-algebra.

1.2. Dixmier-Douady invariant. Let $\sigma$ be a multiplier on a discrete group $\Gamma$. Consider the short exact sequence of coefficient groups

$$1 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{e^{2\pi i}} U(1) \to 1,$$

which gives rise to a long exact sequence of cohomology groups (the change of coefficient groups sequence)

$$\cdots \to H^2(\Gamma, \mathbb{Z}) \xrightarrow{i_*} H^2(\Gamma, \mathbb{R}) \xrightarrow{e^{2\pi i}} H^2(\Gamma, U(1)) \xrightarrow{\delta} H^3(\Gamma, \mathbb{Z}) \xrightarrow{i_*} H^3(\Gamma, \mathbb{R}) \to \cdots .$$

Then one definition of the Dixmier-Douady invariant of $\sigma$ is $\delta(\sigma) \in H^3(\Gamma, \mathbb{Z})$ [DD]. Notice that the Dixmier-Douady invariant of $\sigma$ is always a torsion element.

1.3. Classifying space for discrete groups. Let $\Gamma$ be a discrete group, and $E\Gamma \to B\Gamma$ be a locally trivial, principal $\Gamma$-bundle such that $E\Gamma$ is contractible and $B\Gamma$ is paracompact. Then $B\Gamma$ is unique up to homotopy and is called the classifying space of $\Gamma$. For example, if $\Gamma$ is a discrete, torsion-free subgroup of a connected Lie group $G$, then $E\Gamma = G/K$, where $K$ is a maximal compact subgroup of $G$, and $B\Gamma = G/K$. For groups with torsion, it turns out (cf. [BCH]) that it is more convenient to consider instead $E\Gamma\hat{\Gamma}$, which is defined to be the classifying space for proper $\Gamma$-actions (as opposed to free actions). Then $E\Gamma\hat{\Gamma}$ is unique up to $\Gamma$-homotopy and coincides with $E\Gamma$ when $\Gamma$ is torsion-free. For example, if $\Gamma$ is a discrete subgroup of a connected Lie group $G$, then $E\Gamma\hat{\Gamma} = G/K$.

1.4. Topological $K$-homology. We now give a brief description of the Baum-Douglas version of $K$-homology. The basic objects are $K$-cycles. A $K$-cycle on a topological space is a triple $(M, E, \phi)$, where $M$ is a compact Spin$^c$ manifold, $E \to M$ is a complex vector bundle on $M$, and $\phi : M \to X$ is a continuous map. Two $K$-cycles $(M, E, \phi)$ and $(M', E', \phi')$ are said to be isomorphic if there is a diffeomorphism $h : M \to M'$ such that $h^*(E') \cong E$ and $h^*\phi' = \phi$. Let $\Pi(X)$ denote the collection of all $K$-cycles on $X$.

- **Bordism**: $(M_i, E_i, \phi_i) \in \Pi(X), i = 0, 1$, are said to be bordant if there is a triple $(W, E, \phi)$, where $W$ is a compact Spin$^c$ manifold with boundary $\partial W$, $E$ is a complex vector bundle over $W$ and $\phi : W \to X$ is a continuous map, such that $(\partial W, E|_{\partial W}, \phi|_{\partial W})$ is isomorphic to the disjoint union $(M_0, E_0, \phi_0) \cup (-M_1, E_1, \phi_1)$. Here $-M_1$ denotes $M_1$ with the reversed Spin$^c$ structure.
- **Direct sum**: Suppose that $(M, E, \phi) \in \Pi(X)$ and that $E = E_0 \oplus E_1$. Then $(M, E, \phi)$ is isomorphic to $(M, E_0, \phi) \cup (M, E_1, \phi)$.
- **Vector bundle modification**: Let $(M, E, \phi) \in \Pi(X)$ and $H$ be an even dimensional Spin$^c$ vector bundle over $M$. Let $\tilde{M} = S(H \oplus 1)$ denote the sphere bundle of $H \oplus 1$. Then $\tilde{M}$ is canonically a Spin$^c$ manifold. Let $S$ denote the bundle of spinors on $H$. Since $H$ is even dimensional, $S$ is $\mathbb{Z}_2$-graded,

$$S = S^+ \oplus S^-$$

into bundles of 1/2-spinors on $M$. Define $E = \pi^*(S^{++} \otimes E)$, where $\pi : \tilde{M} \to M$ is the projection. Finally, $\hat{\phi} = \pi^*\phi$. Then $(\tilde{M}, E, \hat{\phi}) \in \Pi(X)$ is said to be obtained from $(M, E, \phi)$ and $H$ by vector bundle modification.
Let $\sim$ denote the equivalence relation on $\Pi(X)$ generated by the operations of bordism, direct sum and vector bundle modification. Notice that $\sim$ preserves the parity of the dimension of the $K$-cycle. Let $K_0(X)$ denote the quotient $\Pi^{\text{even}}(X)/\sim$, where $\Pi^{\text{even}}(X)$ denotes the set of all even dimensional $K$-cycles in $\Pi(X)$, and let $K_1(X)$ denote the quotient $\Pi^{\text{odd}}(X)/\sim$, where $\Pi^{\text{odd}}(X)$ denotes the set of all odd dimensional $K$-cycles in $\Pi(X)$.

1.5. $K$-theory of $C^*$-algebras. We briefly recall the definition and some useful facts concerning the $K$-theory of $C^*$-algebras. Let $A$ be a unital $C^*$-algebra. Recall that two projections $P$ and $Q$ in $A$ are said to be Murray-von Neumann equivalent if there is an element $U \in A$ such that $UU^* = P$ and $U^*U = Q$. Then the set of Murray-von Neumann equivalence classes of projections in $A \otimes \mathcal{K}$ is an Abelian semigroup, where $\mathcal{K}$ denotes the algebra of compact operators on a separable Hilbert space. Then the associated Grothendieck group is an Abelian group which is denoted by $K_0(A)$. We will make use of the following lemma in this paper.

**Lemma 1.1** ([Pc]). Let $P$ and $Q$ be two projections in a $C^*$-algebra $A$ such that $\|P - Q\| < 1$. Then they are Murray-von Neumann equivalent. In particular, suppose that $I \ni t \mapsto P(t)$ is a norm continuous path of projections in $A$, where $I$ is an interval in $\mathbb{R}$. Then $P(t)$ are mutually Murray-von Neumann equivalent.

2. On positivity of the Kadison constant

2.1. Twisted Baum-Connes map. In this section, we will define the twisted Baum-Connes map for an arbitrary torsion-free discrete group $\Gamma$ and for an arbitrary multiplier $\sigma$ on $\Gamma$ with trivial Dixmier-Douady invariant.

From the previous section, elements of $K_0(B\Gamma)$ are equivalence classes of even dimensional $K$-cycles $(M,E,f)$, where $M$ is a compact Spin$^C$ manifold, $E \to M$ is a complex vector bundle on $M$, and $\phi : M \to B\Gamma$ is a continuous map. Let $\tilde{\varphi}_E^C : L^2(M, S^+ \otimes E) \to L^2(M, S^- \otimes E)$ denote the Spin$^C$ Dirac operator with coefficients in $E$. Let $\Gamma \to \tilde{M} \xrightarrow{p} M$ be the covering space of $M$ such that $\tilde{M} = \phi^*(ET)$. Let $\tilde{\varphi}_E^C$ be the lift of $\varphi_E^C$ to $\tilde{M}$,

$$
\tilde{\varphi}_E^C : L^2(\tilde{M}, \tilde{S}^+ \otimes \tilde{E}) \to L^2(\tilde{M}, \tilde{S}^- \otimes \tilde{E}).
$$

Note that $\tilde{\varphi}_E^C$ commutes with the $\Gamma$ action on $\tilde{M}$.

Since the Dixmier-Douady invariant $\delta(\sigma) = 0$, there is an $\mathbb{R}$-valued cohomology class $c$ on $\Gamma$ such that $e^{2\pi ic} = [\sigma]$. Let $\omega$ be a closed 2-form on $\tilde{M}$ such that the cohomology class of $\omega$ is equal to $\phi^*(c)$, where $c$ is as before. Note that $\tilde{\omega} = p^*\omega = d\eta$ is exact, since $ET$ is contractible. Define $\nabla = d + i\eta$. Then $\nabla$ is a Hermitian connection on the trivial line bundle over $\tilde{M}$, and the curvature of $\nabla$, $(\nabla)^2 = i\tilde{\omega}$. Then $\nabla$ defines a projective action of $\Gamma$ on $L^2$ spinors as follows:

First of all, observe that since $\tilde{\omega}$ is $\Gamma$-invariant, $0 = (\gamma^{-1})^*\tilde{\omega} - \tilde{\omega} = d((\gamma^{-1})^*\eta - \eta)$ for all $\gamma \in \Gamma$. So $(\gamma^{-1})^*\eta - \eta$ is a closed 1-form on the manifold $\tilde{M}$ whose cohomology class is the pullback of a cohomology class on $ET$, therefore

$$(\gamma^{-1})^*\eta - \eta = d\phi_\gamma \quad \text{for all } \gamma \in \Gamma$$
where $\phi_\gamma$ is a smooth function on $\tilde{M}$ satisfying in addition,

$$\phi_\gamma(x_0) = 0 \text{ for some } x_0 \in \tilde{M} \text{ and all } \gamma \in \Gamma.$$ 

Then $\tilde{\sigma}(\gamma, \gamma') = \exp(i \phi_\gamma(\gamma'^{-1} \cdot x_0))$ defines a multiplier on $\Gamma$, i.e., $\tilde{\sigma} : \Gamma \times \Gamma \to U(1)$ satisfies the following identity for all $\gamma, \gamma', \gamma'' \in \Gamma$:

$$\tilde{\sigma}(\gamma, \gamma') \tilde{\sigma}(\gamma, \gamma'') = \tilde{\sigma}(\gamma', \gamma'') \tilde{\sigma}(\gamma', \gamma'').$$

This is verified by observing that $\phi_\gamma(\gamma'^{-1} \cdot x) + \phi_{\gamma'}(x) - \phi_{\gamma \gamma'}(x)$ is independent of $x \in \tilde{M}$ for all $\gamma, \gamma' \in \Gamma$.

For $u \in L^2(\tilde{M}, \tilde{S} \otimes \tilde{E})$, let $S_\gamma u = e^{i \phi_\gamma} u$, $U_\gamma u = \gamma^{-1} u$, and $T_\gamma = U_\gamma S_\gamma$ be the composition, for all $\gamma \in \Gamma$. Then $T$ defines a projective $(\Gamma, \tilde{\sigma})$-action on $L^2$-spinors, i.e.,

$$T_\gamma T_{\gamma'} = \sigma(\gamma, \gamma') T_{\gamma \gamma'}.$$ 

**Lemma 2.1.** The twisted Spin$^C$ Dirac operator on $\tilde{M}$,

$$\tilde{\theta}_E^C \otimes \nabla : L^2(\tilde{M}, \tilde{S}^+ \otimes \tilde{E}) \to L^2(\tilde{M}, \tilde{S}^- \otimes \tilde{E})$$

commutes with the projective $(\Gamma, \tilde{\sigma})$-action.

**Proof.** Let $D_\eta = \tilde{\theta}_E^C \otimes \nabla$. Then $D_\eta = D + ic(\eta)$, where $D = \tilde{\theta}_E^C$ and $c(\eta)$ denotes Clifford multiplication by the one-form $\eta$. An easy computation establishes that $U_\gamma D_\eta = D_{\gamma^{-1} \cdot \eta} U_\gamma$ and that $S_\eta D_{\gamma^{-1} \cdot \eta} = D_\eta S_{\gamma}$ for all $\gamma \in \Gamma$. Then $T_\gamma D_\eta = D_\eta T_\gamma$, where $T_\gamma = U_\gamma S_\gamma$ denotes the projective $(\Gamma, \tilde{\sigma})$-action. \(\square\)

Let $D^+$ denote the twisted Spin$^C$ Dirac operator on $\tilde{M}$, $\tilde{\theta}_E^C \otimes \nabla$ and $(D^+)^* = D^-$ its adjoint $\tilde{\theta}_E^C \otimes \nabla$. Then for $t > 0$, using the standard Gaussian off-diagonal estimates for the heat kernel one sees that the heat kernels $e^{-tD^-D^+}$ and $e^{-tD^+D^-}$ are elements in $C_*(\Gamma, \sigma) \otimes \mathcal{K}$, cf. [BrSu]. Define the idempotent $e_t(D) \in M_2(C_*(\Gamma, \sigma) \otimes \mathcal{K}) \cong C_*(\Gamma, \sigma) \otimes \mathcal{K}$ as follows:

$$e_t(D) = \begin{pmatrix} e^{-tD^-D^+} & e^{-tD^-D^+} (1 - e^{-tD^-D^+}) D^- \\ e^{-tD^+D^-} D^+ & 1 - e^{-tD^+D^-} \end{pmatrix}.$$ 

It is sometimes known as the Wasserman idempotent. Then the **twisted Baum-Connes map** is

$$\mu_\sigma : K_0(\mathcal{B}\Gamma) \to K_0(C_*(\Gamma, \sigma))$$

$$\mu_\sigma([M, E, \phi]) = [e_t(D)] - [E_0]$$

where $t > 0$ and $E_0$ is the idempotent

$$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

**Lemma 2.2.** The twisted Baum-Connes map is well-defined.
Proof. We need to show that $\mu_\sigma([M, E, \phi])$ defined in Equation (2.1) is independent of $t > 0$. By Lemma 1.1, it suffices to show that $\mathbb{R}^+ \ni t \to e_t(D)$ is a norm continuous family of projections in the $C^*$-algebra $M_2(C^*_r(\Gamma, \sigma) \otimes \mathcal{K})$. By Duhamel’s principle, one has the following identity whenever $t > \epsilon > 0$,

$$e^{-(t\pm\epsilon)D-D^+} - e^{-tD-D^+} = \frac{\epsilon}{t} \int_0^t e^{-sD-D^+} D^- D^+ e^{-(t-s)(D-D^+\pm\epsilon/tD-D^+)} ds.$$  

It follows that there is a constant $C$ independent of $\epsilon$ such that

$$||e^{-(t\pm\epsilon)D-D^+} - e^{-tD-D^+}|| < C\epsilon.$$  

Using Duhamel’s principle repeatedly, one sees that the family of projections $\mathbb{R}^+ \ni t \to e_t(D)$ is continuous in the norm topology.

**Twisted Baum-Connes conjecture.** Suppose that $\Gamma$ is a torsion-free discrete group and that $\sigma$ is a multiplier on $\Gamma$ with trivial Dixmier-Douady invariant. Then the twisted Baum-Connes map

$$\mu_\sigma : K_j(B\Gamma) \to K_j(C^*_r(\Gamma, \sigma)), \quad j = 0, 1,$$

is an isomorphism.

It turns out that the twisted Baum-Connes conjecture is a special case of the Baum-Connes conjecture with coefficients (cf. [Ma] for details). The following result is in [Ma], where the twisted Baum-Connes map is called the twisted Kasparov map.

**Theorem 2.3 ([Ma]).** Suppose $\Gamma$ is a torsion-free discrete group that has the Dirac-Dual Dirac property, and $\sigma$ is a multiplier on $\Gamma$ with trivial Dixmier-Douady invariant. Then the twisted Baum-Connes map

$$\mu_\sigma : K_j(B\Gamma) \to K_j(C^*_r(\Gamma, \sigma)), \quad j = 0, 1,$$

is split injective. If in addition, $\Gamma$ is $K$-amenable, then $\mu_\sigma$ is an isomorphism.

For example, by results of Kasparov [Kas], discrete subgroups of connected Lie groups have the Dirac-Dual Dirac property. By results of Kasparov [Kas2], Julg-Kasparov [JuKas] and Higson-Kasparov [HiKa], all discrete amenable groups, all discrete subgroups of $\text{SO}_0(1, n)$ and of $\text{SU}_0(1, n)$ are $K$-amenable and also have the Dirac-Dual Dirac property.

2.2. **Characteristic classes.** We recall some basic facts about some well-known characteristic classes that will be used in this paper, cf. [Hir].

Let $E \to M$ be a Hermitian vector bundle over the compact manifold $M$ that has dimension $n = 2m$. The *Chern classes* of $E$, $c_j(E)$, are by definition integral cohomology classes. The *Chern character* of $E$, $\text{Ch}(E)$, is a rational cohomology class

$$\text{Ch}(E) = \sum_{r=0}^m \text{Ch}_r(E),$$

where $\text{Ch}_r(E)$ denotes the component of $\text{Ch}(E)$ of degree $2r$. Then $\text{Ch}_0(E) = \text{rank}(E)$, $\text{Ch}_1(E) = c_1(E)$ and in general

$$\text{Ch}_r(E) = \frac{1}{r!} P_r(E) \in H^{2r}(M, \mathbb{Q}),$$
where \( P_r(E) \in H^{2r}(M, \mathbb{Z}) \) is a polynomial in the Chern classes of degree less than or equal to \( r \) with integral coefficients, that is determined inductively by the Newton formula

\[
P_r(E) - c_1(E)P_{r-1}(E) \ldots + (-1)^{r-1}c_{r-1}(E)P_1(E) + (-1)^r r c_r(E) = 0
\]

and by \( P_0(E) = \text{rank}(E) \). The next two terms are \( P_1(E) = c_1(E), P_2(E) = c_1(E)^2 - 2c_2(E) \).

The Todd-genus characteristic class of the Hermitian vector bundle \( E \) is a rational cohomology class in \( H^{2\bullet}(M, \mathbb{Q}) \),

\[
\text{Todd}(E) = \sum_{r=0}^{m} \text{Todd}_r(E),
\]

where \( \text{Todd}_r(E) \) denotes the component of \( \text{Todd}(E) \) of degree \( 2r \). Then \( \text{Todd}_r(E) = B_r Q_r(E) \), where \( Q_r(E) \) is a polynomial in the Chern classes of degree less than or equal to \( r \), with integral coefficients, and \( B_r \neq 0, B_r \in \mathbb{Q} \) are the Bernoulli numbers.

2.3. Range of the canonical trace. Here we will present some consequences of the twisted Baum-Connes conjecture above and the twisted \( L^2 \) index theorem for covering spaces (cf. Appendix).

The canonical trace \( \tau \) on \( C^*_r(\Gamma, \sigma) \) induces a linear map

\[
[\tau] : K_0(C^*_r(\Gamma, \sigma)) \to \mathbb{R},
\]

which is called the trace map in \( K \)-theory. Explicitly, first \( \tau \) extends to matrices with entries in \( C^*_r(\Gamma, \sigma) \) as (with Trace denoting matrix trace):

\[
\tau(f \otimes r) = \text{Trace}(r)\tau(f).
\]

Then the extension of \( \tau \) to \( K_0 \) is given by \( [\tau]([e] - [f]) = \tau(e) - \tau(f) \), where \( e \) and \( f \) are idempotent matrices with entries in \( C^*_r(\Gamma, \sigma) \). The following result is in [Ma] (see also [BC]).

**Theorem 2.4** (Range of the trace theorem). Suppose that \((\Gamma, \sigma)\) satisfies the twisted Baum-Connes conjecture and \( \delta(\sigma) = 0 \). Then the range of the canonical trace on \( K \)-theory is

\[
[\tau](K_0(C^*_r(\Gamma, \sigma))) = \left\{ c_0 \int_M \text{Todd}(M) \wedge e^i \wedge \text{Ch}(E); \text{ for all } (M, E, \phi) \in \Pi^{\text{even}}(B\Gamma) \right\}.
\]

Here \( c_0 = 1/(2\pi)^{n/2} \) is a universal constant determined by the relevant index theorem, \( n = \dim M \), Todd and Ch denote the Todd-genus and the Chern character respectively.

**Remarks 2.5.** The set

\[
\left\{ c_0 \int_M \text{Todd}(M) \wedge e^i \wedge \text{Ch}(E); \text{ for all } (M, E, \phi) \in \Pi^{\text{even}}(B\Gamma) \right\}
\]

is a countable discrete subgroup of \( \mathbb{R} \). Note that it is not in general a subgroup of \( \mathbb{Z} \).
Remarks 2.6. When \( \Gamma \) is the fundamental group of a compact Riemann surface of positive genus, it follows from \cite{Rieff} in the genus one case, and \cite{CHMM} in the general case, that the set
\[
\left\{ c_0 \int_M \text{Todd}(M) \wedge e^\alpha \wedge \text{Ch}(E); \text{ for all } (M, E, \phi) \in \Pi^{\text{even}}(B\Gamma) \right\}
\]
reduces to the countable discrete group \( \mathbb{Z} + \theta \mathbb{Z} \), where \( \theta \in [0, 1) \) corresponds to the multiplier \( \sigma \) under the isomorphism \( H^2(\Gamma; \mathbb{C}^\times) \cong \mathbb{R}/\mathbb{Z} \).

Proof of Theorem 2.4. By hypothesis, the twisted Baum-Connes map is an isomorphism. Therefore to compute the range of the trace map on \( K_0(C^*_r(\Gamma, \sigma)) \), it suffices to compute the range of the trace map on elements of the form
\[
\mu_\sigma([M, E, \phi]), \quad [M, E, \phi] \in K_0(B\Gamma).
\]
Here \( (M, E, \phi) \) is an even parity \( K \)-cycle over \( B\Gamma \). By the twisted analogue of the \( L^2 \) index theorem of Atiyah \cite{At} and Singer \cite{Si} for elliptic operators on a covering space that are invariant under the projective action of the fundamental group defined by \( \sigma \), and which was stated and used by Gromov \cite{Gr1} (cf. Appendix), one has
\[
[\tau](\mu_\sigma([M, E, \phi])) = c_0 \int_M \text{Todd}(M) \wedge e^\alpha \wedge \text{Ch}(E)
\]
as desired. \( \square \)

2.4. The Kadison constant and the number of projections in \( C^*_r(\Gamma, \sigma) \). We begin with a key elementary lemma.

Lemma 2.7. Suppose that \( P(t) = \sum_{j=0}^\ell a_j t^j \) is a polynomial having the property that there is \( t_0 \in \mathbb{N} \) for which \( P(rt_0) \in \mathbb{Z} \) for all \( r \in \mathbb{Z} \). Then \( P(1) \in c_\ell^{-1} t_0^{-\ell(\ell+1)/2} \mathbb{Z} \), where \( c_\ell = \prod_{i \geq j \geq 0} (2^{2^i} - 2^j) \).

Proof. We observe that
\[
\begin{align*}
a_0 + a_1 t_0 + \ldots + a_\ell t_0^\ell &= P(t_0) \in \mathbb{Z}, \\
a_0 + a_1 2t_0 + \ldots + a_\ell 2^\ell t_0^\ell &= P(2t_0) \in \mathbb{Z}, \\
& \vdots \\
a_0 + a_1 2^\ell t_0 + \ldots + a_\ell 2^\ell t_0^\ell &= P(2^\ell t_0) \in \mathbb{Z}.
\end{align*}
\]
That is
\[
TA = Z \in \mathbb{Z}^\ell,
\]
where
\[
T = \begin{pmatrix}
1 & t_0 & t_0^2 & \ldots & t_0^\ell \\
1 & 2t_0 & 2^2 t_0^2 & \ldots & 2^\ell t_0^\ell \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2^\ell t_0 & 2^{2\ell} t_0^2 & \ldots & 2^{\ell^2} t_0^\ell
\end{pmatrix}
\]
and
\[
A = \begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_\ell
\end{pmatrix}, \quad Z = \begin{pmatrix}
P(t_0) \\
P(2t_0) \\
\vdots \\
P(2^\ell t_0)
\end{pmatrix}.
\]
Observe that $\det(T) = t_0^{\ell(\ell+1)}/2 \det(T')$, where $T'$ is the Vandermonde matrix

$$T' = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 2 & 2^2 & \ldots & 2^\ell \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 2^\ell & 2^{2\ell} & \ldots & 2^{\ell^2}
\end{pmatrix}.$$ 

Its determinant is given by the formula $\det(T') = \prod_{\ell > j > 0} (2^j - 2^i) > 0$. Therefore, for all $\ell$, one has $\det(T) = t_0^{\ell(\ell+1)/2} c_\ell > 0$, where $c_\ell = \prod_{\ell > j > 0} (2^j - 2^i)$. Therefore, $T$ is an invertible matrix, and one has

$$A = T^{-1}Z = \det(T)^{-1}SZ$$ 

where $S$ is a matrix with entries in $\mathbb{Z}$. Therefore $a_j \in c_\ell^{-1}t_0^{-\ell(\ell+1)/2} \mathbb{Z}$ for $j = 0, \ldots, \ell$. Therefore $P(1) = \sum_{j=0}^\ell a_j \in c_\ell^{-1}t_0^{-\ell(\ell+1)/2} \mathbb{Z}$ as desired.

We will now recall the definition of the Kadison constant of a twisted group $C^*$-algebra. The Kadison constant of $C_r^*(\Gamma, \sigma)$ is defined by:

$$C_\sigma(\Gamma) = \inf \{ \tau(P) ; P \text{ is a non-zero projection in } C_r^*(\Gamma, \sigma) \otimes \mathcal{K} \}.$$ 

**Proof of Theorem B.** The proof uses Theorem 2.4 and an analysis of the denominators of the relevant characteristic classes as discussed in Subsection 2.2. By assumption, there is a rational cohomology class $c$ such that $e^{2\pi ic} = [\sigma]$. Therefore there is a positive integer $q \in \mathbb{N}$ such that $q[c]$ is an integral cohomology class. If $(M, E, \phi)$ is a $K$-cycle on $B\Gamma$, then the cohomology class of $rq\omega$ is integral, where $q[\omega] = q\phi^* [c]$. Note also that $rq\omega$ is integral for all $r \in \mathbb{Z}$. It follows that there is a line bundle $L$ and connection $\nabla^L$ such that its curvature $(\nabla^L)^2 = rq\omega$, and also that the induced connection on the line bundle $L^{\sigma r}$ has curvature $irq\omega$. In particular, we see that

$$c_0 \int_M \text{Todd}(M) \land e^{rq\omega} \land \text{Ch}(E) = \text{index}(\phi^{EC}_{E \otimes L^{\sigma r}}) \in \mathbb{Z}.$$ 

Observe that since $[\omega] = \phi^*(c)$, it follows that $[\omega^j] = \phi^*(c \cup c \ldots \cup c) = 0$ if $j > \frac{1}{2} \text{cohdim}(B\Gamma) = \ell$, where $\text{cohdim}(B\Gamma)$ denotes the cohomological dimension of $B\Gamma$. Therefore, for all $r \in \mathbb{Z}$, one has

$$\mathbb{Z} \ni c_0 \int_M \text{Todd}(M) \land e^{rq\omega} \land \text{Ch}(E) = c_0 \sum_{j=0}^\ell \frac{r^j q^j}{j!} \int_M \text{Todd}(M) \land \omega^j \land \text{Ch}(E)$$ 

By Lemma 2.4, it follows that

$$c_0 \int_M \text{Todd}(M) \land e^{\omega} \land \text{Ch}(E) \in c_\ell^{-1}q^{-\ell(\ell+1)/2} \mathbb{Z},$$ 

where $c_\ell = \prod_{\ell > j > 0} (2^j - 2^i) > 0$. By Theorem 2.4, it follows that the Kadison constant $C_\sigma(\Gamma) \geq c_\ell^{-1}q^{-\ell(\ell+1)/2} > 0$.

**Proof of Theorem D.** Let $P$ be a projection in $C_r^*(\Gamma, \sigma)$. Then $1 - P$ is also a projection in $C_r^*(\Gamma, \sigma)$, and one has

$$1 = \tau(1) = \tau(P) + \tau(1 - P).$$
Each term in the above equation is non-negative. By the Theorem B and by hypothesis, it follows that 
\( \tau(P) \) belongs to the finite set
\[
\{0, c^{-1}_\ell q^{-\ell(\ell+1)/2}, 2c^{-1}_\ell q^{-\ell(\ell+1)/2}, \ldots, 1\}.
\]
Since \( \Gamma \) is an ICC group, therefore the enveloping von Neumann algebra of \( \tilde{C}_\sigma^*(\Gamma, \sigma) \) is a factor, cf. \cite{CHMM}. It follows that for each value of the trace, there is a unique projection up to Murray-von Neumann equivalence in the enveloping von Neumann algebra. Therefore there are at most \( c_eq^{\ell(\ell+1)/2} \) non-trivial projections in \( \tilde{C}_\sigma^*(\Gamma, \sigma) \), up to Murray-von Neumann equivalence in the enveloping von Neumann algebra. \( \square \)

2.5. Applications to the spectral theory of projectively periodic elliptic operators. In this section, we discuss some quantitative results on the spectrum of projectively periodic self-adjoint elliptic operators on covering spaces. In particular, we formulate a generalization of the Bethe-Sommerfeld conjecture.

Let \( D : L^2(\tilde{M}, \tilde{E}) \to L^2(\tilde{M}, \tilde{E}) \) be a self-adjoint elliptic differential operator that commutes with the \((\Gamma, \tilde{\sigma})\)-action defined earlier. We begin with some basic facts about the spectrum of such an operator. Recall that the discrete spectrum of \( D \), \( \text{spec}_{\text{disc}}(D) \) consists of all the eigenvalues of \( D \) that have finite multiplicity, and the essential spectrum of \( D \), \( \text{spec}_{\text{ess}}(D) \) consists of the complement \( \text{spec}(D) \setminus \text{spec}_{\text{disc}}(D) \). That is, \( \text{spec}_{\text{ess}}(D) \) consists of the set of accumulation points of the spectrum of \( D \), \( \text{spec}(D) \). It can be shown that the discrete spectrum of \( D \) is empty (cf. \cite{VIM}). It follows that \( \text{spec}_{\text{ess}}(D) = \text{spec}(D) \), and in particular that \( \text{spec}_{\text{ess}}(D) \) is unbounded.

Since \( \text{spec}(D) \) is a closed subset of \( \mathbb{R} \), its complement \( \mathbb{R} \setminus \text{spec}(D) \) is an open subset of \( \mathbb{R} \), and so it is the countable union of disjoint open intervals. Each such interval is called a gap in the spectrum of \( D \). By the previous discussion, these also correspond to gaps in \( \text{spec}_{\text{ess}}(D) \). Therefore one can ask the following fundamental question: How many gaps are there in the spectrum of \( D \)?

This question had been studied previously by Brüning and Sunada, \cite{BrSu}, and we will now discuss some of their results.

Note that in general the spectral projections \( E_\lambda = \chi_{(-\infty, \lambda]}(D) \) of \( D \), do not belong to \( \tilde{C}_\sigma^*(\Gamma, \sigma) \otimes K \). However it is a result of \cite{BrSu} that if \( \lambda_0 \notin \text{spec}(D) \), then \( E_{\lambda_0} \in \tilde{C}_\sigma^*(\Gamma, \sigma) \otimes K \). The proof uses the off-diagonal decay of the heat kernel of \( D^2 \). The following quantitative estimate on the number of gaps in the spectrum of \( D \) in terms of the Kadison constant \( C_\sigma(\Gamma) \) is the main theorem in \cite{BrSu}.

**Theorem 2.8** (\cite{BrSu}). Let \( D : L^2(\tilde{M}, \tilde{E}) \to L^2(\tilde{M}, \tilde{E}) \) be a \((\Gamma, \tilde{\sigma})\)-invariant self-adjoint elliptic differential operator which is bounded below. If \( \psi(\lambda) \) denotes the number of gaps in the spectrum of \( D \) that lie in the \( 1/2 \)-line \((-\infty, \lambda]\), then one has the asymptotic estimate

\[
\lim_{\lambda \to +\infty} \sup \left\{ \frac{(2\pi)^n C_\sigma(\Gamma) \psi(\lambda) \lambda^{-n/d}}{w_n \text{vol}(M)} \right\} \leq 1,
\]

where \( w_n \) is the volume of the unit ball in \( \mathbb{R}^n \), \( \text{vol}(M) \) is the volume of \( M \), \( n = \text{dim } M \) and \( d \) is the degree of \( D \).

This theorem together with Theorem B immediately establishes Theorem C. That is, we have shown that under the conditions of Theorem B, the spectrum of a projectively periodic elliptic operator has countably many gaps which can only accumulate
at infinity. In particular, setting $\tilde{\sigma} = 1$, Theorem C gives evidence that the following generalization of the Bethe-Sommerfeld conjecture is true, [Ma], [MM] (see also [KaPd]).

**The Generalized Bethe-Sommerfeld conjecture.** Suppose that $\Gamma$ satisfies the Baum-Connes conjecture. Then the spectrum of any Hamiltonian $H_V = \Delta + V$ on $L^2(\tilde{M})$ has only a finite number of gaps, where $\Delta$ denotes the Laplacian acting on $L^2$ functions on $\tilde{M}$ and $V$ is a smooth $\Gamma$-invariant function on $\tilde{M}$.

We remark that the Bethe-Sommerfeld conjecture has been proved by Skriganov [Skr] when $\tilde{M}$ is the Euclidean plane. See [Grn] for related aspects of noncommutative Bloch theory.

### 3. Deformation Quantization and the Proof of Theorem E

Theorem E is established using a key result of Rosenberg on the rigidity of $K$-theory under deformation quantization. We will just sketch the relevant modifications needed to Rosenberg’s argument.

**Proof of Theorem E.** To prove this, we will define an auxiliary algebra, which is a formal deformation quantization of $C^*_r(\Gamma, \sigma^0)$. Since $\delta(\sigma^0) = \delta(\sigma^1)$, there is an $\mathbb{R}$-valued 2-cocycle $c$ on $\Gamma$ such that $\sigma^1 = \sigma^0 e^{2\pi i c}$. Define $C^*_r(\Gamma, \sigma^0)[[\hbar]]$ as being the associative algebra of formal power series over the ring $\mathbb{C}[[\hbar]]$ of power series over $\mathbb{C}$, where the multiplication in $C^*_r(\Gamma, \sigma^0)[[\hbar]]$ is defined as

$$a \star b = ab + \sum_{j=1}^{\infty} \hbar^j \phi_j(a, b),$$

where

$$\phi_j(a, b)(\gamma) = \frac{(2\pi i)^j}{j!} \sum_{\gamma_1 \gamma_2 = \gamma} a(\gamma_1)b(\gamma_2)c^j(\gamma_1, \gamma_2)$$

are $\mathbb{C}$-bilinear maps $C^*_r(\Gamma, \sigma^0) \times C^*_r(\Gamma, \sigma^0) \to C^*_r(\Gamma, \sigma^0)$. Then we observe that $C^*_r(\Gamma, \sigma^0)[[\hbar]]/(\hbar - s) \simeq C^*_r(\Gamma, \sigma^s)$ as algebras, where $\sigma^s = \sigma^0 e^{2\pi i s}$ and $(\hbar - s)$ denotes the ideal generated by $\hbar - s$, $s \in \mathbb{R}$. Therefore the canonical projection map

$$e_s : C^*_r(\Gamma, \sigma^0)[[\hbar]] \to C^*_r(\Gamma, \sigma^s)$$

is a homomorphism of algebras for all $s \in \mathbb{R}$. By adapting the proof in [Ros], we see that the induced map in $K$-theory is an isomorphism,

$$(e_s)_* : K_0(C^*_r(\Gamma, \sigma^0)[[\hbar]]) \xrightarrow{\cong} K_0(C^*_r(\Gamma, \sigma^s)) \text{ for all } s \in \mathbb{R}.$$  

Then $\lambda_s = (e_0)_*(e_s)^{-1} : K_0(C^*_r(\Gamma, \sigma^s)) \to K_0(C^*_r(\Gamma, \sigma^0))$ is an isomorphism for all $s$. Setting $s = 1$, we obtain the desired isomorphism. $\square$
Appendix: A twisted $L^2$ index theorem for covering spaces

We use the notation of the previous section. Let $P_+$ and $P_-$ denote the orthogonal projections onto the nullspace of $D^+ = \tilde{\theta}_E^+ \otimes \nabla$ and $D^- = \tilde{\theta}_E^- \otimes \nabla$, respectively. Then $(D^+)^* = D^-$ and one has

$$D^+ P_+ = 0, \quad \text{and} \quad D^- P_- = 0$$

By elliptic regularity, it follows that the Schwartz kernels of $P_+$ and $P_-$ are smooth. Since the operators $D^+$ and $D^-$ commute with the $(\Gamma, \sigma)$-action, the same is true for the spectral projections $P_+$ and $P_-$. One can define a semi-finite von Neumann trace on $(\Gamma, \bar{\sigma})$-invariant bounded operators $T$ on the Hilbert space of $L^2$-sections of $\tilde{S} \otimes \tilde{E}$ on the universal cover, similar to how Atiyah did in the untwisted case [At],

$$\tau(T) = \int_M \text{tr}(T(x, x)) \, dx,$$

if the Schwartz kernel of $T$ is smooth. It is well defined since $e^{-i\phi_\gamma(x)}T(\gamma x, \gamma y)e^{i\phi_\gamma(y)} = T(x, y)$ for all $x, y \in \tilde{M}$ and $\gamma \in \Gamma$, where we have identified the fiber at $x$ with the fiber at $\gamma x$ via the isomorphism induced by $\gamma$. In particular, $\text{tr}(T(x, x))$ is a $\Gamma$-invariant function on $\tilde{M}$. The $L^2$-index of $D^+$ is by definition

$$\text{index}_{L^2}(D^+) = \tau_s(P),$$

where $P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}$ and $\tau_s$ denotes the graded trace, i.e. the composition of $\tau$ and the grading operator. Let $k_+(t, x, y)$ and $k_-(t, x, y)$ denote the heat kernel of the Dirac operators $D^-D^+$ and $D^+D^-$, respectively. By general results in [CGT], [Roe], one knows that the heat kernels $k_+(t, x, y)$ and $k_-(t, x, y)$ converge uniformly over compact subsets of $\tilde{M} \times \tilde{M}$ to $P_+(x, y)$ and $P_-(x, y)$, respectively, as $t \to \infty$. Therefore if $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$, then $e^{-tD^2} = \begin{pmatrix} e^{-tD^-} & 0 \\ 0 & e^{-tD^+} \end{pmatrix}$ and one has

$$\lim_{t \to \infty} \tau_s(e^{-tD^2}) = \lim_{t \to \infty} \int_M \text{tr}(k_+(t, x, x)) \, dx - \lim_{t \to \infty} \int_M \text{tr}(k_-(t, x, x)) \, dx$$

$$= \int_M \text{tr}(P_+(x, x)) \, dx - \int_M \text{tr}(P_-(x, x)) \, dx$$

$$= \tau_s(P)$$

$$= \text{index}_{L^2}(D^+).$$

Observe that

$$\frac{\partial}{\partial t} \tau_s(e^{-tD^2}) = -\tau_s(D^2 e^{-tD^2})$$

$$= -\tau_s([D, D e^{-tD^2}])$$

$$= 0,$$
since $D$ is an odd operator. Therefore one also has the analogue of the McKean-Singer formula in this context, that is, for $t > 0$,
\[
\text{index}_{L^2}(D^+) = \tau_s(e^{-tD^2}) = \lim_{t \to \infty} \tau_s(e^{-tD^2}).
\]
By the discussion below Lemma 2.1, we have
\[
\tau(\mu_\sigma([M, E, \phi])) = \tau(e_t(D)) - \tau(E_0) = \tau_s(e^{-tD^2}).
\]
Combining the observations above, one has
\[
\tau(\mu_\sigma([M, E, \phi])) = \text{index}_{L^2}(D^+).
\]
The following twisted analogue of the $L^2$ index theorem for covering spaces due to Atiyah [At] and Singer [Si], was stated in [Gr1] and proved for instance in the appendix of [Ma]. It is established using the local index index theorem (cf. [Get]) and the observations above.

**Theorem 3.1** ($L^2$ Index Theorem for $(\Gamma, \sigma)$-invariant Dirac type operators). The $L^2$ index theorem for elliptic operators which are of Dirac type and which are invariant under the projective $(\Gamma, \sigma)$-action is
\[
\text{index}_{L^2}(\tilde{\phi}_E^+ \otimes \nabla) = \int_M \text{Todd}(\Omega) e^{\omega} \text{tr}(e^{RE}),
\]
where $\text{Todd}(\Omega)$ denotes the Todd genus of the Spin$^C$ manifold $M$ and $\text{tr}(e^{RE})$ is the Chern character of the Hermitian vector bundle $E$ over $M$.

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