Fast Spectral Low Rank Matrix Approximation

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Abstract

In this paper, we study subspace embedding problem and obtain the following results:

1. We extend the results of approximate matrix multiplication from the Frobenius norm to the spectral norm. Assume matrices $A$ and $B$ both have at most $r$ stable rank and $\tilde{r}$ rank, respectively. Let $S$ be a subspace embedding matrix with $l$ rows which depends on stable rank, then with high probability, we have

$$\|A^T S^T S B - A^T B\|_2 < \varepsilon \|A\|_2 \|B\|_2.$$

2. We develop a class of fast approximate generalized linear regression algorithms with respect to the spectral norm. We design a new least square regression algorithm in which subspace embedding matrix $S$ has $(\sqrt{\varepsilon/r}, \delta)$-JL moment property. Here $r$ is the stable rank $A$, which is never greater than rank of $A$. Let $x' = \arg\min_x \|S A x - S b\|_2$, we have

$$\|A x' - b\|_2 \leq (1 + \varepsilon) \min_x \|A x - b\|_2.$$

3. We give a concise proof and tighter error upper bound for the randomized SVD of Halko et al. (2011). Besides gaussian random projection and Subsample Randomized Hadamard Transform in Halko et al. (2011), we find that a large class of matrices which have oblivious $\ell_2$-subspace embedding property can be used in randomized SVD. We give a fast randomized SVD algorithm using sparse embedding matrix. We give a framework that composing different subspace embedding matrices still has the same relative error bound.

4. We design a fast low rank approximation algorithm with relative error based on spectral norm and the stable rank. For $A \in \mathbb{R}^{n \times d}$, given $k$, and $\varepsilon$, we get a decomposition of $A$ into $L$, $D$, $W$, such that

$$\|A - LDW^T\|_2 \leq (1 + \varepsilon) \|A - A_k\|_2,$$

and our algorithm runs in $\tilde{O}(n m) \varepsilon^{-1/2} + (n + d) r_1^2 / \varepsilon^2 + r_1 r_2^2 / \varepsilon^3$. $A_k$ and $A_{d/k}$ both have stable rank at most $r_1$. $SA$ and $A - A(SA)^+ SA$ both have stable rank at most $r_2$. And $S$ is a sparse subspace embedding matrix with $\tilde{O}(r_1 / \varepsilon)$ rows.

Keywords: Spectral norm, approximate SVD, subspace embedding, JL moment property, matrix product, linear regression, low rank approximation
1. Introduction

This paper studies fast approximate matrix algorithms. Singular value decomposition (SVD), linear regression and matrix products are basic problems in numerical linear algebra. How to compute them fast is challenging since they are widely used in various areas. For example, SVD is an important tool in data mining (Azar et al., 2001), information retrieval using Latent Semantic Indexing (Papadimitriou et al., 1998), spectral clustering, and projective clustering (Feldman et al., 2013). Besides, PCA widely used in statistics and machine learning is closely related to SVD. Many classification problems can be reduced to regularized regression problems (Drineas et al., 2006b). Text database querying is a matrix-vector products process.

The computation mentioned above is intensive when performed exactly. Dense SVD methods need $O(m^2n)$ time; similarly, matrix product is of the same order (Golub and Van Loan, 2012). Hence, much work comes out to approximate matrix operations with much faster speed (Clarkson and Woodruff, 2013; Cohen and Lewis, 1999; Drineas et al., 2006a, 2011; Sarlos, 2006; Drineas and Mahoney, 2005). The previous work (Drineas et al., 2006a; Sarlos, 2006; Cohen and Lewis, 1999; Magen and Zouzias, 2011) gave fast approximate matrix products. Much work (Drineas et al., 2006b, 2011; Nelson and Nguyen, 2013; Halko et al., 2011; Clarkson and Woodruff, 2013; Martinsson et al., 2011; Woolfe et al., 2008) focus on the fast $\ell_2$ regression and SVD problem. In the work of Clarkson and Woodruff (2013), they gave an approximate SVD with relative error with respect to the Frobenius norm in input sparsity time using sparse sketching method. On the other hand, the work of Halko et al. (2011) gave a fast randomized methods called randomized SVD to approximate SVD with relative error with respect to spectral norm.

Fast matrix products approximation with respect with spectral norm is studied in this paper. We extend the work with respect to the Frobenius norm Kane and Nelson (2014) to the spectral norm. As proved by Kane and Nelson (2014), most matrices with the Johnson-Lindenstrauss property have JL moment property. Based on the JL moment property, we find that all matrices having JL moment property can be used to accelerate matrix products as shown in Theorem 13. Besides, we give a tighter bound for the number of rows of subspace embedding matrix $S$. When $S$ is a gaussian matrix, we can prove that our bound is optimal up to a constant. For sparse subspace embedding matrices, our result is near optimal.

Generalized linear regression problems with respect to the spectral norm are studied. Our result is similar to the one with respect to the Frobenius norm except a slight difference that subspace embedding matrices have different properties. The difference leads to the difference of algorithms of approximate SVD between spectral norm and Frobenius norm. In Theorem 30, we give a faster linear regression algorithm in which subspace embedding matrix $S$ has $(\sqrt{\varepsilon/r}, \delta)$-JL moment property. Here $\varepsilon$ is relative error parameter, and $r$ is the stable rank of $A$, which is never greater than rank of $A$. To the best of our knowledge, the previous best result is that $S$ has to satisfy $(\sqrt{\varepsilon/r}, \delta)$-JL moment property where $\tilde{r}$ is the rank of $A$. Besides, our result is of interest since the stable rank of input matrix can be computed quickly contrast to the computation of rank of input matrix.

Several spectral low rank matrix approximation methods are raised. We give a tighter bound for randomized SVD in Halko et al. (2011). We reduce the relative error bound from
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$O(\sqrt{kt})$ to $O(\sqrt{r/k})$ for gaussian subspace embedding matrices. A fast randomized SVD method is given which is suitable for sparse input matrices. In the framework of our proof, we show that matrices composed by different kinds of subspace embedding matrices have additive relative error bound. Hence a large class of subspace embedding matrices can be used to construct randomized SVD and other spectral low rank matrix approximation.

A fast relative SVD algorithm with respect to spectral norm is raised. For a matrix $A \in \mathbb{R}^{n \times d}$, an approximate SVD satisfies $\|A - LDW^T\|_2 \leq (1 + \varepsilon)\|A - A_k\|_2$, and $L, D, W$ can be computed in $O(n m z(A) \varepsilon^{-1/2} + (n + d)r_1^2/\varepsilon^2 + r_1 r_2^3/\varepsilon^3)$. $A_k^T$ and $A_{d/k}$ both have stable rank at most $r_1$. $SA$ and $A - A(SA)^T SA$ both have stable rank at most $r_2$. And $S$ is a sparse subspace embedding matrix with $O(r_1/\varepsilon)$ rows. To the best of our knowledge, the best approximate SVD with respect to the spectral norm is based on a gaussian random projection method combining power method proposed by Halko et al. (2011), and improved in the work of Boutsidis et al. (2014). The algorithm outputs a rank $k$ orthonormal matrix $Z$ such that $\|A - ZZ^T A\|_2 \leq (1 + \varepsilon)\|A - A_k\|_2$, The algorithm can be implemented in $O(n m z(A) k \log(nd)/\varepsilon)$. Our algorithm can fast if the singular values of matrix decay quickly which is common in real application matrix. Besides, if the input matrix is of low coherence, our algorithm can run in input sparsity.

The remainder of the paper is organized as follows. After notation and preliminary which describes the basic fact about subspace embedding and related results, we give the result of approximate matrix products in Section 3. Based on the results in Section 3, we give our generalized linear regression results with respect to spectral norm in Section 4. The low rank approximation results are given in Section 5 where spectral low rank matrix approximation raised, a fast SVD approximation algorithm implemented and time complexity is analyzed.

2. Notation and Preliminaries

2.1 Matrix

Given a matrix $A \in \mathbb{R}^{m \times n}$ of rank $\rho$, the SVD is given as $A = U \Sigma V^T = U_k \Sigma_k V_k^T + U_{\rho-k} \Sigma_{\rho-k} V_{\rho-k}^T$, where $U_k$ and $U_{\rho-k}$ contain the left singular vector of $A$, and, similarly, $V_k$ and $V_{\rho-k}$ contain right singular vectors of $A$. It is well known that $A_k = U_k \Sigma_k V_k^T$ minimizes $\|A - X\|_F$ and $\|A - X\|$ over all matrix $X \in \mathbb{R}^{m \times n}$ of rank at most $k \leq \rho$. Besides, we define the stable rank of $A$ as $\text{sr}(A) = \|A\|_F^2/\|A\|_2^2$, and $\text{sr}(A) \leq \text{rank}(A)$ always holds. The orthogonal projector of a matrix $A$ onto the rowspace of a matrix $C$ is denoted by $PC(A) = AC^T$. And we define $PC_k$ as the best rank-$k$ approximation of the matrix $P_C$.

The matrix norms are defined as follows. $\|A\|_F = (\sum_{i,j} a_{ij}^2)^{1/2} = (\sum_i \sigma_i^2)^{1/2}$ is the Frobenius norm, $\|A\|_2 = \sigma_1$ is the spectral norm. $A^\dagger = V \Sigma^{-1} U^T \in \mathbb{R}^{n \times m}$ denotes the so-called Moore-Penrose pseudo-inverse of $A \in \mathbb{R}^{m \times n}$, i.e., the unique $n \times m$ satisfying all four properties: $A = AA^\dagger A$, $A^\dagger = A^\dagger AA^\dagger$, $(AA^\dagger)^T = AA^\dagger$, $(A^\dagger A)^T = A^\dagger A$. It is easy to check that, for all $i = 1, \ldots, \rho = \text{rank}(A) = \text{rank}(A^\dagger), \sigma_i(A) = 1/\sigma_{i+1}(A)$. Besides, for any matrix $A \in \mathbb{R}^{m \times n}$ with full row rank, then $AA^\dagger = I_n$. Similarly, if $A$ is of full column rank, then $A^\dagger A = I_n$. For all $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$: $(AB)^\dagger = B^\dagger A^\dagger$ if one of following three properties hold:$(1) A^T A = I_n; (2) B^T B = I_p; (3) \text{rank}(A) = \text{rank}(B) = n.$
2.2 Subspace embedding

Subspace embedding is an important tool in the following work. Using subspace embedding, a matrix can be projected to a much lower dimension, leading to much faster operation on matrix, and most part of property of the matrix is preserved. Now, we give its definition.

**Definition 1 (Woodruff (2014))** A $(1 \pm \varepsilon) \ell_2$-subspace embedding for the column space of an $n \times d$ matrix $A$ is a matrix $S$ for which for all $x \in \mathbb{R}^d$

$$\|SAx\|_2^2 = (1 \pm \varepsilon)\|Ax\|_2^2$$

There are many ways to construct an $\ell_2$-subspace embedding matrix. Oblivious embedding first introduced in Sarlos (2006) is a particularly useful form of $\ell_2$-subspace embedding.

**Definition 2 (Woodruff (2014))** Suppose $\Pi$ is a distribution on $r \times n$ matrices $S$, where $r$ is a function of $n, d, \varepsilon$ and $\delta$. Suppose that with probability at least $1 - \delta$, for any fixed $n \times d$ matrix $A$, a matrix $S$ drawn from distribution $\Pi$ has the property that $S$ is a $(1 + \varepsilon)$ $\ell_2$-subspace embedding for $A$. Then we call $\Pi$ an $(\varepsilon, \delta)$ oblivious $\ell_2$-subspace embedding.

For convenience, oblivious $\ell_2$-subspace embedding will be referred as $\ell_2$-subspace embedding. Johnson-Lindenstrauss transform has intrinsic subspace embedding property. Now, we give the definition of Johnson-Lindenstrauss transform.

**Definition 3 (Sarlos (2006))** A random matrix $S \in \mathbb{R}^{k \times n}$ forms a Johnson-Lindenstrauss transform with parameters $\varepsilon, \delta, f$, if with probability at least $1 - \delta$, for any $f$-element subset $V \subset \mathbb{R}^n$, for all $v, v' \in V$, $|\langle Sv, Sv' \rangle - \langle v, v' \rangle| \leq \varepsilon\|v\|_2\|v'\|_2$

When $v = v'$, we can get the usual notation that $\|Sv\|_2^2 = (1 + \varepsilon)\|v\|_2^2$. There is much work to construct Johnson-Lindenstrauss transform. Random Gaussian matrix is a simple way to form Johnson-Lindenstrauss transform.

**Theorem 4** Let $0 < \varepsilon, \delta < 1$ and $S = \frac{1}{\sqrt{k}}R \in \mathbb{R}^{k \times n}$, where the entries of $R$ are independent standard normal random variables. Then if $k = \Omega(f^2 \log(1/\delta))$, then $S$ is a JLT$(\varepsilon, \delta, f)$. And also for all vectors $\|x\|_2 = 1$,

$$P(\|Sx\|_2^2 - 1) > \varepsilon < 2e^{-\Omega(\varepsilon^2k)}$$  \hspace{1cm} (1)

It is easy to check that $S$ in Theorem 4 has the $\ell_2$-subspace embedding property. For Oblivious subspace embedding, $k$ can be reduced to $k = \Theta((d + \log(1/\delta))\varepsilon^2)$.

**Theorem 5 (Woodruff (2014))** Let $0 < \varepsilon, \delta < 1$ and $S = \frac{1}{\sqrt{k}}R \in \mathbb{R}^{k \times n}$, where the entries of $R$ are independent standard normal random variables. Then if $k = \Theta((d + \log(1/\delta))\varepsilon^2)$, for any fixed $n \times d$ matrix $A$, with probability $1 - \delta$, $S$ is a $(1 \pm \varepsilon) \ell_2$-subspace embedding, that is for all $x \in \mathbb{R}^d$, $\|SAx\|_2^2 = (1 \pm \varepsilon)\|Ax\|_2^2$.

In fact, the number of rows of $S$ in Theorem 5 is optimal up to a constant factor.

Following the work of Gaussian matrix, there are lots of work to construct $\ell_2$-subspace embedding matrix. Fast Johnson-Lindenstrauss transform is raised up by Ailon and Chazelle.
In the work of Ailon and Chazelle (2006), subspace embedding matrix constructed in $S = P \cdot H \cdot D$, where $D$ is a diagonal matrix with i.i.d entries that $D_{i,i} = 1$ with probability $\frac{1}{2}$ and $D_{i,i} = -1$ with probability $\frac{1}{2}$, $H$ is a Hadamard matrix which can be applied to an $n$-dimension vector in $O(n \log n)$ time complexity, $P$ is a an $k \times n$ coordinate sampling matrix. Fast Johnson-Lindenstrauss transform can be applied to a vector in $O(n \log n)$ time and to a $n \times d$ matrix in $O(nd \log n)$. In the following theorem, we give a slightly different version of Fast Johnson-Lindenstrauss transform called Subsample Randomized Hadamard Transform or SRHT for short. The work related to SRHT can be found in the work of (Ailon and Chazelle, 2006; Sarlos, 2006; Tropp, 2011; Ailon and Liberty, 2009).

**Theorem 6** Matrix $S = \sqrt{\frac{\ell}{d}} P \cdot H \cdot D$, where $D$ is an $n \times n$ diagonal matrix with i.i.d entries that $D_{i,i} = 1$ with probability $\frac{1}{2}$ and $D_{i,i} = -1$ with probability $\frac{1}{2}$, $H$ is an $n \times n$ Hadamard matrix, $P$ is a an $k \times n$ coordinate sampling matrix to choose $l$ rows uniformly at random and without replacement, where

$$l = \Omega(\varepsilon^{-2}(\log d/\delta)(\sqrt{d} + \sqrt{\log n})^2)$$

Then for any fixed $n \times d$ matrix $A$, with probability at least $1 - \delta$, such that,

$$\|SAx\|_2^2 = (1 \pm \varepsilon)\|Ax\|_2^2$$

And for any vector $x \in \mathbb{R}^n$, $Sx$ can be computed in $O(n \log k)$.

Besides the Fast Johnson-Lindenstrauss transform, to construct sparse subspace embedding matrix is an important research topic in subspace embedding (Clarkson and Woodruff, 2013; Kane and Nelson, 2014; Dasgupta et al., 2010). Clarkson and Woodruff (2013) constructed an oblivious $\ell_2$-subspace embedding matrix $S$ such that $SA$ can be computed in $O(nnz(A))$. Every column of $S$ only has one non-zero element which is uniformly randomly chosen from $\{-1, 1\}$, and the number of rows of $S$ is $O(d^2/\varepsilon^2 \text{poly}(\log(d/\varepsilon)))$. In the work of (Nelson and Nguyen, 2013; Meng and Mahoney, 2013), the number of rows of $S$ reduced to $O(d^2/(\delta \varepsilon^2))$.

**Theorem 7** For any $0 < \delta < 1$, $\varepsilon$ is the error parameter. $S$ is a sparse embedding matrix with $O(d^2/(\delta \varepsilon^2))$, then with probability at least $1 - \delta$, $S$ is a $(1 \pm \varepsilon)\ell_2$-subspace embedding matrix for any fixed matrix $A$, and $SA$ can be computed in $O(nnz(A))$.

After the work of Clarkson and Woodruff (2013), Nelson and Nguyen (2013) achieve fewer than $O(d^2/\varepsilon^2)$ rows for constant probability subspace embeddings at the cost of increasing the running time of applying the subspace embedding from $O(nnz(A))$ to $O(nnz(A)/\varepsilon)$.

**Theorem 8** (Nelson and Nguyen (2013); Woodruff (2014)) There is a $(1 \pm \varepsilon)$ oblivious $\ell_2$-subspace embedding for $A \in \mathbb{R}^{n \times d}$ with $l = d \cdot \text{poly}(\log(d/(\varepsilon \delta)))/\varepsilon^2$ rows and error probability $\delta$. Further $SA$ can be computed in $O(nnz(A)\text{poly}(\log(d/(\varepsilon \delta))/\varepsilon)$ time.

**Remark 9** Bourgain and Nelson (2013) showed that if $A$ has low coherence, then a sparse embedding matrix $S$ in Theorem 8 providing a $(1 \pm \varepsilon)$ $\ell_2$-subspace embedding for $A$. And the number of nonzero entry of each entry remains 1 as in Theorem 7.
Remark 10 One can achieve $1 - \delta$ success probability bounds in which the sparsity and dimension depend on $(\log 1/\delta)$ as Theorem 5, Theorem 6 and Theorem 8. As we can see in Theorem 7, the number of rows of $S$ depends on the error probability $1/\delta$ linearly. However, on can repeat the entire procedure $O(\log 1/\delta)$ times and take the best solution found, such as in regression or low rank matrix approximation (Woodruff, 2014).

3. Matrix multiplication

Given matrices $A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{n \times p}$, it is well known that the complexity of computing $A^T B$ is $O(mnp)$. Approximate matrix product problem is to output a matrix $C$ that $\|A^T B - C\| \leq \varepsilon\|A\|\|B\|$ with $o(mnp)$ time complexity. Much work (Drineas et al., 2006a; Clarkson and Woodruff, 2013; Drineas et al., 2011; Sarlos, 2006; Kane and Nelson, 2014) has been done to get $o(mnp)$ computing complexity for matrix multiplication for $\|\cdot\|_F$ norm. Much fewer work on spectral matrix multiplication approximation. Results for spectral norm were shown in the work of Magen and Zouzias (2011) and Magdon-Ismail (2011).

In this section, we give some results with respect to spectral norm based on JL moment property (Kane and Nelson, 2014), subspace embedding and stable rank.

First, we give the definition of JL moment property.

Definition 11 (Kane and Nelson (2014)) A distribution $D$ on matrices $S \in \mathbb{R}^{m \times n}$ has the $(\varepsilon, \delta, l)$-JL moment property if for all $x \in \mathbb{R}^d$ with $\|x\|_2 = 1$

$$E_{S \sim D}[\|S\|_2^2 - 1]^l \leq \varepsilon^l \delta$$

(2)

For convenience, sometimes we just write $(\varepsilon, \delta)$-JL moment property, omitting $l$. Using this definition, we prove the matrix multiplication result with respect to spectral norm. First we give an important work of approximate matrix multiplication based on Frobenius norm.

Theorem 12 (Kane and Nelson (2014)) For $\varepsilon, \delta \in (0, 1/2)$, let $D$ be a distribution over the matrix with $d$ columns that satisfies the $(\varepsilon, \delta, l)$-JL moment property for some $l \geq 2$. Then for $A, B$ matrices each with $d$ rows.

$$\mathbb{P}_{S \sim D}[\|A^T S^T SB - A^T B\|_F > 3\varepsilon\|A\|_F \|B\|_F] < \delta$$

(3)

Based on the above theorem, we give our result based on spectral norm.

Theorem 13 For $\varepsilon, \delta \in (0, 1/2)$, $k_1, k_2$ are stable rank of $A, B$ respectively. Let $D$ be a distribution over the matrix with $d$ columns that satisfies the $(\varepsilon/\sqrt{k_1 k_2}, \delta, l)$-JL moment property. Then for $A, B$ matrices each with $d$ rows.

$$\mathbb{P}_{S \sim D}[\|A^T S^T SB - A^T B\|_2 > 3\varepsilon\|A\|_2 \|B\|_2] < \delta$$

(4)

Proof w.l.o.g, assuming that $\|A\|_2 = \|B\|_2 = 1$, it always holds that $\mathbb{P}_{S \sim D}[\|A^T S^T SB - A^T B\|_2 \geq 3\varepsilon] \leq \mathbb{P}_{S \sim D}[\|A^T S^T SB - A^T B\|_F \geq 3\varepsilon] \leq \delta$. Then, we have

$$\mathbb{P}_{S \sim D}[\|A^T S^T SB - A^T B\|_F \geq 3\varepsilon] = \mathbb{P}_{S \sim D}[\|A^T S^T SB - A^T B\|_F \geq \frac{3\varepsilon}{\|A\|_F \|B\|_F} \|A\|_F \|B\|_F] = \mathbb{P}_{S \sim D}[\|A^T S^T SB - A^T B\|_F \geq \frac{3\varepsilon}{\sqrt{k_1 k_2}} \|A\|_F \|B\|_F]$$
hence, when $D$ satisfies the $(\varepsilon/\sqrt{k_1k_2}, \delta, l)$-JL moment property, and combining Theorem 12, then $\mathbb{P}_{S \sim D}\left[\|AT^TSA - A^TB\|_2 > 3\varepsilon\|A\|_2\|B\|_2\right] \leq \mathbb{P}_{S \sim D}\left[\|AT^TSA - A^TB\|_F > 3\varepsilon\right] \leq \delta$. 

**Corollary 14** For $\varepsilon, \delta \in (0, 1/2)$, $k$ is stable rank of $A$. And let $D$ be a distribution over the matrix with $d$ columns that satisfies the $(\varepsilon/k, \delta)$-JL moment property. Then

$$\mathbb{P}_{S \sim D}\left[\|AT^TSA - A^TB\|_2 > 3\varepsilon\|A\|_2\|B\|_2\right] < \delta$$

**Proof** Let $B = A$ in Theorem 13, we get the result. 

Using Corollary 14, we can prove that $S$ in Theorem 7 just needs $O(r^2/\varepsilon^2\delta)$ rows, where $r$ is the stable rank of $A$. And $\|SA\|^2 = (1 \pm \varepsilon)\|A\|^2$ holds with probability at least $1 - \delta$.

In the following work, we bring up spectral matrix multiplication approximation based on subspace embedding.

**Lemma 15** Given $A \in \mathbb{R}^{m \times n}$, $r$ is the stable rank of $A$. If $k > r$, then $\|A - A_k\|^2 \leq \frac{r}{k}\|A\|^2$.

**Proof** It is easy to check that $\|A - A_k\|^2 \leq \frac{1}{k}\frac{\|A\|^2}{\|A\|^2} = \frac{r}{k}\|A\|^2$. Hence, we have

$$\|A - A_k\|^2 \leq \frac{1}{k}\frac{\|A\|^2}{\|A\|^2} = \frac{r}{k}\|A\|^2$$

**Theorem 16** Given $A \in \mathbb{R}^{m \times n}$, and $B \in \mathbb{R}^{m \times p}$. Let $S \in \mathbb{R}^{l \times m}$ is a gaussian subspace embedding matrix in Theorem 5. Let $\bar{r} = \max\{\text{rank}(A), \text{rank}(B)\}$ and $r = \max\{\text{srank}(A), \text{srank}(B)\}$. If $l = O(\frac{(r + 2\log(\bar{r})/\delta)^2 + \log^2 \bar{r}/\epsilon}{\epsilon^2})$, for convenience $l = O(\frac{(r + 2\log(\bar{r})/\delta)^2 + \log^2 \bar{r}/\epsilon}{\epsilon^2})$, then

$$\mathbb{P}\left[\|AT^TSB - A^TB\|_2 > \varepsilon\|A\|_2\|B\|_2\right] < \delta$$

(5)

**Proof** w.l.o.g. $\|A\|_2 = \|B\|_2 = 1$. Let $A_{ir}$ be the best $ir$ rank approximate to $A$ with respect to $A$, where $i = 1, 2, \ldots, \bar{r}$. Set $A_{ir} = A_{i+1} - A_{i}$ and $A_{(i+1)r} = A_{i} - A_{i+1}$. We have $A = A_{ir} + A_{r}$ and $B = B_{ir} + B_{r}$. It is easy to check that $A_{ir}A_r = 0$ and $B_{ir}B_r = 0$. First we use $B = B_{ir} + B_{r}$, we have

$$\|AT^TSB - A^TB\|_2 = \|AT^TS(B_{ir} + B_{r}) - A^TB_{ir} + B_{r}\|_2$$

$$= \|(A^TS^TS - A^T)B_{ir} - (A^TBS - A^T)B_{r}\|_2$$

$$\leq \|(A^TBS - A^T)B_{ir}\|_2 + \|(A^TBS - A^T)B_{r}\|_2$$

(6)
The inequality 6 is because $B_{/r}^r B_r = 0$. Using $A = A_{/r} + A_r$ to equation 6, we similarly have

\[
\|A^T S^T B - A^T B\|_2^2 \leq \|(A^T S^T S - A^T)B_r\|_2^2 + \|(A^T S^T S - A^T)B_{/r}\|_2^2 \\
\leq \|A^T_{/r} S^T S B_r - A^T_{/r} B_r\|_2^2 \\
+ \|A^T_{/r} S^T S B_{/r} - A^T_{/r} B_{/r}\|_2^2 \\
+ \|A^T_r S^T S B_r - A^T_r B_r\|_2^2 \\
+ \|A^T_r S^T S B_{/r} - A^T_r B_{/r}\|_2^2
\]

(7)

(8)

(9)

(10)

$A_r$ and $B_r$ both have rank $r$. Let $l = O\left(\frac{r + \log 1/\delta}{\epsilon^2}\right)$, then $\|A^T_{/r} S^T S B_r - A^T_{/r} B_r\|_2^2 \leq \epsilon^2$. Now we begin to bound Equation 9. Using $B_{/r} = B_{/r} + B_{br}$ and $B_{/r}^r B_{/r} = 0$, we have

\[
\|A^T_{/r} S^T S B_{/r} - A^T_{/r} B_{/r}\|_2^2 \leq \|A^T_{/r} S^T S B_{/br} - A^T_{/r} B_{br}\|_2^2 + \|A^T_{/r} S^T S B_{br} - A^T_{/r} B_{br}\|_2^2.
\]

(11)

We add all the subproblem bound, we have

\[
\|A^T S^T S B - A^T B\|_2^2 \leq \epsilon^2 \left(1 + \sum_{i=1}^{\hat{r}/r} \frac{1}{i} (\log \frac{\hat{r}}{r} - \log i) + 1/i^2\right)
\]

\[
\leq \epsilon^2 \left(1 + \int_1^{\hat{r}/r} \frac{1}{x} (\log \frac{\hat{r}}{r} - \log x) + \frac{1}{x^2} dx\right)
\]

\[
\leq \epsilon^2 \left(2 + \frac{1}{2} \log^2 \frac{\hat{r}}{r}\right)
\]

\[
= O(\epsilon^2\left(2 + \log^2 \frac{\hat{r}}{r}\right)||A||_2^2||B||_2^2
\]

There are $O((\hat{r}/r)^2)$ items needed to bound by $S$. Using probability union bound, and let

\[
e^2 = \frac{\epsilon^2}{2 + \log^2 \frac{\hat{r}}{r}},\text{ then } l = O\left(\frac{r + 2 \log (\hat{r}/r \delta)}{(2 + \log^2 \frac{\hat{r}}{r})\epsilon^2}\right).
\]

\[\]
values and the other singular values are zero. \( \| A^{T} S^{T} S A - A^{T} A \|_{2}^{2} \leq \varepsilon \| A \|_{2}^{2} \) can be reduced to \( \| U^{T}_{r} S^{T} U_{r} - U^{T}_{r} U_{r} \|_{2}^{2} \leq \varepsilon \), which means \( S \) is an \((1 \pm \varepsilon)\) \( \ell_2 \)-subspace embedding matrix for \( r \)-dimension subspace. And \( l = O\left(\frac{r+\log 1/\delta}{\varepsilon^2}\right) \) is optimal for \( S \) is an \((1 \pm \varepsilon)\) \( \ell_2 \)-subspace embedding matrix for \( r \)-dimension subspace due to the work of Nelson and Nguyen (2014). ■

Now we give similar bound for some kinds of important subspace embedding matrices. The following theorems share similar proof with Theorem 16.

**Theorem 18** Let \( A, B \) share the same property as in Theorem 16. And \( S \) is an SRHT matrix which has \( l = \Omega\left(\varepsilon^{-2}(\log^{2} \frac{r}{\varepsilon})(\sqrt{r} + \sqrt{\log m})^{2}\right) \) rows. Then Equation 5 holds.

**Theorem 19** Let \( A, B \) share the same property as in Theorem 16. And \( S \) is a sparse subspace embedding matrix as in Theorem 7. If \( S \) has \( O\left(\varepsilon^{-2} r^{2} \log^{2} \frac{\tilde{r}}{\varepsilon}\right) \) rows, then Equation 5 holds.

**Theorem 20** Let \( A, B \) share the same property as in Theorem 16. And \( S \) is a sparse subspace embedding matrix as in Theorem 8. If \( S \) has \( O\left(\varepsilon^{-2} r \log^{2} \frac{\tilde{r}}{\varepsilon}\right) \text{poly}(\log(\frac{\tilde{r}}{(r\varepsilon\delta)})) \) rows, then Equation 5 holds.

In fact, using JL moment property, Theorem 19 has a tighter bound.

**Theorem 21** \( S \) is sparse subspace embedding matrix in Theorem 7. \( r_1 \) and \( r_2 \) are stable rank of \( A \) and \( B \) respectively. Then with \( l = O\left(\frac{r_1 r_2}{\varepsilon^2 \delta}\right) \) rows, Equation 5 holds.

**Proof** The result is due to Theorem 7, Theorem 25 and Theorem 13. ■

Theorem 19, Theorem 20 and Theorem 21 is near optimal. It can be easily proved using the similar argument of Theorem 17 and the result of Nelson and Nguyen (2013).

**Theorem 22** Theorem 19, Theorem 20 and Theorem 21 is near optimal.

In fact, when \( A = B \), there exist a tighter bound for sparse subspace embedding matrix.

**Theorem 23** \( S \) is a sparse subspace embedding matrix in Theorem 7 with \( l \) rows. \( A \) is the input matrix. And \( r \) is the stable rank of \( A \). If \( l = O\left(\frac{r^2}{\varepsilon^2 \delta}\right) \), then \( \| S A \|_{2}^{2} = (1 \pm \varepsilon) \| A \|_{2}^{2} \) holds with probability at least \( 1 - \delta \).

**Proof** Corollary 14 means that when \( S \) has \((\varepsilon/3r, \delta)\)-JL moment property, then \( \| S A \|_{2}^{2} = (1 \pm \varepsilon) \| A \|_{2}^{2} \) holds with probability at least \( 1 - \delta \). And sparse subspace embedding matrix need \( l = O\left(\frac{r^2}{\varepsilon^2 \delta}\right) \) rows to satisfy \((\varepsilon/3r, \delta)\)-JL moment property due to Theorem 25. ■

Since JL moment property is very important for matrix product problem, now we give two lemmas describing how to construct matrices satisfying \((\varepsilon, \delta, l)\)-JL moment property.

**Lemma 24** (Kane and Nelson (2014)) \( S \in \mathbb{R}^{k \times d} \) is constructed based on a JL distribution \( D \) over \( k \times d \), that is for all \( x \) with \( \| x \|_{2} = 1 \) and for all \( 0 < \varepsilon < 1 \),

\[
P_{S \sim D}(\| S x \|_{2}^{2} - 1 > \varepsilon) < e^{-\Omega(\varepsilon^{2 k} + \varepsilon k)}
\]
Then, any such distribution automatically satisfies the $\left(\varepsilon, e^{-\Omega(\varepsilon^2 k + \varepsilon k)}, \min(\varepsilon^2 k, \varepsilon k)\right)$-JL moment property.

The following theorem describes the relation between sparse subspace embedding and JL moment property.

**Theorem 25 (Thorup and Zhang (2012))** If $S$ is a sparse embedding matrix with at least $\frac{2}{\varepsilon^2 \delta}$ rows. Then $S$ satisfies the $(\varepsilon, \delta, 2)$-JL moment property.

4. Generalized Regression

The generalized regression problem based on spectral norm is

$$\min_X \|AX - B\|_2$$

where $X$ and $B$ are matrices rather than vectors. By multiplying a subspace embedding matrix $S$ which can guarantee regression accuracy, it makes the problem become

$$\min_{X'} \|SAX' - SB\|_2$$

The problem above is much easier than the original one if the dimension of $SA$ and $SB$ have much lower dimensions than $A$ and $B$.

In this section, we give main condition and results for generalized regression with respect to spectral norm. Similar work in Frobenius norm can be found in the work of Clarkson and Woodruff (2013). It is important base work for the low rank approximation with respect to spectral norm and also of independent interest.

**Lemma 26 (Woodruff (2014))** If $X^* = \arg\min_X \|A - ZX\|_2$, where $Z^TZ = I$, then $X^*$ satisfies $ZX^* = ZZ^TA$

**Lemma 27** Given $n \times d$ matrix $C$, and $n \times d'$ matrix $D$ consider the regression problem

$$\min_{X \in \mathbb{R}^{d \times d'}} \|CX - D\|_2$$

Then $X^* = C^D$ is a solution to this regression problem. Moreover, $C^T(CX^* - D) = 0$, and

$$\|CX - D\|_2^2 \leq \|C(X - X^*)\|_2^2 + \|CX^* - D\|_2^2$$

**Proof** Let $Z$ is orthonormal basis for the column space of $C$, then there exits $Y$ such that $CX = ZY$. Using Lemma 26, $Y^* = Z^TD$ is a solution since $Y^*$ has the property that $ZY^* = ZZ^TD$. Also $CX^* = CC^TD = ZZ^TD$, hence, $X^* = C^TD$ is a solution to this regression problem.

The following theorem gives the main result of generalized regression with respect to spectral norm. There is a similar result for generalized regression in Frobenius norm (Clarkson and Woodruff, 2013).
**Theorem 28** Suppose $A$ and $B$ are matrices with $n$ rows, $r_1$ and $r_2$ are stable rank of $A$ and $B - AA^\dagger B$. Suppose $S$ is a $t \times n$ matrix. $S$ satisfies $(\sqrt{\epsilon/(2r_1r_2)}, \delta)$-JL moment property and assume that the event in Theorem 13 occurs. Or $S$ has the property that

\[
P[\|A^T S^T S (B - AA^\dagger B) - A^T (B - AA^\dagger B)\|_2 > \sqrt{\epsilon/2}\|A\|_2\|B - AA^\dagger B\|_2 < \delta]
\]

(11)

Also $S$ is a subspace embedding for $A$ with error parameter $\epsilon_0 \leq 1/\sqrt{2}$. Then if $\tilde{Y}$ is the solution to

\[
\min_Y \|S(AY - B)\|_2
\]

and $Y^*$ is the solution to

\[
\min_Y \|AY - B\|_2
\]

then

\[
\|A\tilde{Y} - B\|_2 \leq (1 + \epsilon)\|AY^* - B\|_2
\]

**Proof** Using Lemma 29 and Lemma 27,

\[
\|A\tilde{Y} - B\|_2^2 = \|A\tilde{Y} - AY^* + AY^* - B\|_2^2
\]

\[
\leq \|AY^* - B\|_2^2 + \|A(\tilde{Y} - Y^*)\|_2^2
\]

\[
\leq (1 + 2\epsilon)\|AY^* - B\|_2^2
\]

\[
\leq (1 + \epsilon^2)\|AY^* - B\|_2^2
\]

Taking square roots get the result. □

**Lemma 29** Suppose $S$, $A$, $B$, $\tilde{Y}$ and $Y^*$ as in Theorem 28, Then

\[
\|A(\tilde{Y} - Y^*)\|_2 \leq \sqrt{2}\epsilon\|B - AY^*\|_2
\]

**Proof** Let $A = U\Sigma V^T$ be the thin SVD of $A$, then $A(\tilde{Y} - Y^*) = U\Sigma V^T(\tilde{Y} - Y^*) = U\Sigma_1 (\tilde{X} - X^*)$, where $\Sigma_1 = \Sigma/\delta(A)$ and $\tilde{X} = \delta(A) \cdot V^T \tilde{Y}$ and $X^* = \delta(A) \cdot V^T Y^*$, then $\|U\Sigma_1\|_2 = 1$, $U\Sigma_1 \tilde{X} = A\tilde{Y}$ and $U\Sigma_1 X^* = AY^*$. We first bound $\|\beta\|_2$ where $\beta \equiv \tilde{X} - X^*$. We have

\[
\Sigma_1 U^T S^T S (U\Sigma_1 \tilde{X} - B) = \Sigma_1 U^T S^T S (A\tilde{Y} - B) = 0
\]

To bound $\|\beta\|_2$, we bound $\|A^T S^T S A\beta\|_2$, and then show that this implies that $\|\beta\|_2$ is small. We have

\[
\Sigma_1 U^T S^T S U \Sigma_1 \beta = \Sigma_1 U^T S^T S U \Sigma_1 (\tilde{X} - X^*) = \Sigma_1 U^T S^T S U \Sigma_1 (\tilde{X} - X^*) + \Sigma_1 U^T S^T S (B - U\Sigma_1 \tilde{X}) = \Sigma_1 U^T S^T S (B - U\Sigma_1 X^*)
\]

Using the fact that $\Sigma_1 U^T (B - U\Sigma_1 X^*) = \delta(A) V^T A^T (B - AY^*) = 0$,

\[
\|\Sigma_1 U^T S^T S U \Sigma_1 \beta\|_2 = \|\Sigma_1 U^T S^T S (B - U\Sigma_1 X^*)\|_2 \leq \sqrt{\epsilon/2}\|U\Sigma_1\|_2\|B - AY^*\|_2 = \sqrt{\epsilon/2}\|B - AY^*\|_2
\]
The first inequality holds is due to $S$ satisfies ($\sqrt{\epsilon/(2r_1r_2)}, \delta, l$)-JL moment property and Theorem 13, Or Equation 11. To show that this bound implies that $\|\beta\|_2$ is small, we use the subadditivity and submultiplicity of $\|\cdot\|_2$, to obtain

$$\|\beta\|_2 \leq \|\Sigma_1 U^T S^T S \Sigma_1 \beta\|_2 + \|\Sigma_1 U^T S^T S \Sigma_1 - I\|_2 \|\beta\|_2$$

By hypothesis, $\|S \Sigma_1 x\|_2^2 = (1 \pm \epsilon_0)\|\Sigma_1 x\|_2^2$ for all $x$, so that $\Sigma_1 U^T S^T S \Sigma_1 - I$ has eigenvalue bounded in magnitude by $\epsilon_0^2$. Thus

$$\|\beta\|_2 \leq \sqrt{\epsilon/2} \|B - AY^*\|_2 + \epsilon_0^2 \|\beta\|_2,$$

since $\epsilon_0^2 \leq 1/2$. Using the submultiplicity of $\|\cdot\|_2$ and $\|\Sigma_1\|_2 = 1$ we have

$$\|A(\hat{Y} - Y^*)\|_2 = \|\Sigma_1 (\hat{X} - X^*)\|_2$$

$$= \|\Sigma_1 \beta\|_2 \leq \|\Sigma_1\|_2 \|\beta\|_2$$

$$\leq \sqrt{2\epsilon} \|B - AY^*\|_2.$$

Theorem 28 gives an improved result in least square regression.

**Theorem 30** Given matrix $A \in \mathbb{R}^{n \times d}$ with stable rank $r$, and error parameter $\epsilon$ and probability parameter $\delta$. If $S$ has $(O(\sqrt{\epsilon/r}), \delta)$-JL moment property, then with probability at least $1 - \delta$, the solution

$$\min_{x'} \|SAx' - Sb\|_2 \leq (1 + \epsilon) \min_x \|Ax - b\|$$ (12)

**Proof** The result can be easily followed from Theorem 28. The stable rank of $b - Ax^*$ is 1, hence, $(\sqrt{\epsilon/(2r)}, \delta$)-JL moment property meet the need.

**Corollary 31** Given matrix $A \in \mathbb{R}^{n \times d}$ with stable rank $r$ where $n > d$, and error parameter $\epsilon$. $S$ is a sparse subspace embedding matrix in Theorem 7 with $r/\epsilon$ rows. Regression problem $\min_{x'} \|SAx' - Sb\|_2$ can be solved up to $\epsilon$ relative error with probability at least 0.99, in time $O(nnz(A) + rd^2/\epsilon)$

**Proof** First, we need to compute the stable rank $r$ which costs $O(nnz(A))$. $SA$ can be computed in $O(nnz(A))$ time. And reduced regression problem $\min_y \|SAy - Sb\|_2$ can be computed in $O(rd^2)$ time using standard methods. Combining Theorem 30 and Theorem 25, we get the conclusion.

Theorem 30 is of interest in real application since stable rank is easy to compute and never bigger than rank. To the best of our knowledge, the previous best result is that $S$ has to satisfy $(O(\sqrt{\epsilon/r}), \delta)$-JL moment property where $r$ is the exact rank of $A \in \mathbb{R}^{n \times d}$. The
time complexity to compute the rank of input matrix $A$ is $O(nd^2)$ where $n > d$ which has the same time complexity to solving least square regression. There are probabilistic methods to determine the rank of $A$. Cheung et al. (2013) gave a method to determine the rank of $A$ which costs $O(nnz(A) \log d + \tilde{r}^3)$ with probability at least $1 - O(\log d/d^{1/3})$. The method of Cheung et al. (2013) is still costly. Hence, it is common to relax the rank of $A$ to $d$. However, the stable rank of input matrix can be computed quickly, because $\|A\|_F^2 = \sum a_{i,j}^2$, it can be computed in $O(nnz(A))$. Besides, $\|A\|_2$ can be computed by power method which also runs in $O(nnz(A))$.

5. Low rank approximation

There is a large body work on low rank matrix approximations (Clarkson and Woodruff, 2013; Halko et al., 2011; Magen and Zouzias, 2011; Martinsson et al., 2011; Nelson and Nguyen, 2013; Mark Rudelson, 2005; Nam H. Nguyen, 2009). Most of these work focus on fast approximation algorithms with respect to Frobenius norm, and several work handle spectral norm problem (Mark Rudelson, 2005; Magen and Zouzias, 2011; Halko et al., 2011; Nam H. Nguyen, 2009). In this section we will give several low rank approximation algorithms with respect to spectral norm. First, we give a method to construct a $(1 + \epsilon)$ approximation to $A \in \mathbb{R}^{n \times d}$ in the rowspace of $SA$ in spectral norm, where $S$ is a subspace embedding matrix. Next, tighter bounds for randomized SVD in the work of Halko et al. (2011) is proved. And a randomized SVD method is constructed using sparse subspace embedding matrix share the same error upper bound. Then, a low rank approximation algorithm with respect to spectral norm is given with relative error. Finally, a fast SVD algorithm is given based on the low rank approximation algorithm. Using sparse subspace embedding matrices, the fast SVD algorithm can be computed in $O(nnz(A)\varepsilon^{-1/2} + (n + d)r_1^2\varepsilon^2 + r_1r_22/\varepsilon^3)$. A similar result with respect to Frobenius norm can be found in the work of Clarkson and Woodruff (2013); Woodruff (2014).

First, we give the following fact that two subspace embedding matrices can be composed to get a new subspace embedding matrix. Besides, the property in matrix multiplication approximation of new matrix still holds with additive error.

**Lemma 32** Let $S \in \mathbb{R}^{k \times n}$ approximates matrix products as in Theorem 13 and is subspace embedding with error $\epsilon$ and failure probability $\delta_S$. $\Pi \in \mathbb{R}^{k_1 \times k}$ approximates matrix products and is subspace embedding with error $\epsilon$ and failure probability $\delta_{\Pi}$, Then IS approximate matrix products with error $O(\epsilon)$ and failure probability is at most $\delta_S + \delta_{\Pi}$.

**Proof** Using Theorem 13 and $S$ has the property that $\|SA\|_2 = \max_{\|x\|_2 = 1} \|SAx\|_2 \leq (1 + \epsilon)\|Ax\|_2 \leq (1 + \epsilon)\|A\|_2$, then

\[
\|A^T S^T \Pi^T ISB - A^T B\|_2 \\
\leq \|A^T S^T \Pi^T ISB - A^T S^T SB\|_2 + \|A^T S^T SB - A^T B\|_2 \\
\leq \epsilon \|SA\|_2 \|SB\|_2 + \epsilon \|A\|_2 \|B\|_2 \\
\leq (\epsilon + \epsilon)^2 \|A\|_2 \|B\|_2 + \epsilon \|A\|_2 \|B\|_2 \\
= O(\epsilon)\|A\|_2 \|B\|_2
\]
Now we give our low rank approximation result in the following work.

**Theorem 33** Let \( S \) be an \( \ell_2 \)-subspace embedding for any fixed \( k \) dimensional subspace \( M \) with probability at least \( \delta \). And \( \epsilon_0 \) is the error parameter, so that \( \| Sy \|_2 = (1 \pm \epsilon_0) \| y \|_2 \) for all \( y \in M \). For any fixed matrix \( A \in \mathbb{R}^{n \times d} \), \( A_k \), \( \Sigma_k \), \( V_k \) is the best rank \( k \) approximation matrix of \( A \) and \( A_{d/k} = A - A_k \). Besides, \( r = \max\{\text{rank}(A_k^2), \text{rank}(A - A_k)\} \) and \( \tilde{r} = \max\{\text{rank}(A_k^2), \text{rank}(A - A_k)\} \). If \( S \) also has the following property,

\[
\mathbb{P}[\| U_k \Sigma_k (U_k \Sigma_k)^T S \Sigma(A_{d/k}) - U_k \Sigma_k (U_k \Sigma_k)^T (A_{d/k}) \|_2 > \sqrt{\epsilon} \| \Sigma_k \|_2^2 \| A_{d/k} \|_2] < \delta \quad (13)
\]

then the rowspace of \( SA \) contains a \( (1 + \epsilon) \) rank-\( k \) approximation to \( A \), i.e.

\[
\| A - P_{SA,k}(A) \|_2^2 \leq (1 + \epsilon) \| A - A_k \|_2^2
\]

**Proof** Consider the quantity

\[
\| (U_k \Sigma_k S U_k \Sigma_k)^T S (U_k \Sigma_k^T S A) - A \|_2
\]

(14)

The goal is to show quantity 14 is at most \( (1 + \epsilon) \| A - A_k \|_2 \). Note that this implies the lemma, since \( U_k \Sigma_k (S U_k \Sigma_k)^T S A \) is a rank-\( k \) matrix inside of the rowspace of \( SA \).

\[
\| U_k \Sigma_k (S U_k \Sigma_k)^T S A - A \|_2^2 \\
= \| U_k \Sigma_k (S U_k \Sigma_k)^T S A - A_k \|_2^2 \\
\leq \| U_k \Sigma_k (S U_k \Sigma_k)^T S A - A_k \|_2^2 + \| (A - A_k) \|_2^2
\]

The last inequality follows that \( (U_k \Sigma_k (S U_k \Sigma_k)^T S A - A_k)^T (A - A_k) = 0 \). It is sufficient to show \( \| U_k \Sigma_k (S U_k \Sigma_k)^T S A - A_k \|_2^2 = O(\epsilon) \| A - A_k \|_2^2 \). And \( (S U_k \Sigma_k)^T (S U_k \Sigma_k) = I \), since \( S U_k \Sigma_k \) is of full column rank. \( S \) is an \( \ell_2 \)-subspace embedding matrix for \( k \)-dimensional space, hence, \( S U_k \Sigma_k \) has the same rank with \( U_k \Sigma_k \) which is of full column rank.

\[
\| U_k \Sigma_k (S U_k \Sigma_k)^T S (U_k \Sigma_k V_k^T + A_{d/k}) - A_k \|_2^2 \\
= \| U_k \Sigma_k (S U_k \Sigma_k)^T S U_k \Sigma_k V_k^T + U_k \Sigma_k (S U_k \Sigma_k)^T S A_{d/k} - A_k \|_2^2 \\
= \| U_k \Sigma_k (S U_k \Sigma_k)^T S A_{d/k} \|_2^2
\]

Using the fact that if \( SU_k \Sigma_k \) has full column rank then \( (SU_k \Sigma_k)^T S U_k \Sigma_k = (U_k \Sigma_k)^T S U_k \Sigma_k \) and \( S \) is an \( \ell_2 \)-subspace embedding matrix with parameters \( \epsilon_0 \) and \( \delta \), hence, with probability at least \( 1 - \delta \), \( \| SU_k \Sigma_k \|_2 = (1 \pm \epsilon_0) \| U_k \Sigma_k \|_2 \). So, \( \| (SU_k \Sigma_k)^T S U_k \Sigma_k \|_2 \leq 1/((1 - \epsilon_0) \| U_k \Sigma_k \|_2)^2 \)

\[
\| U_k \Sigma_k (S U_k \Sigma_k)^T S A_{d/k} \|_2^2 \leq \frac{1}{((1 - \epsilon_0) \| U_k \Sigma_k \|_2)^2} \cdot \| U_k \Sigma_k (U_k \Sigma_k)^T S S A (A - A_k) \|_2^2 \quad (15)
\]

\[
\leq \frac{1}{((1 - \epsilon_0) \| U_k \Sigma_k \|_2)^4} \cdot (1 - \epsilon_0)^4 \cdot \epsilon \cdot \| \Sigma_k \|_2^2 \| A - A_k \|_2^2
\]

\[
= \epsilon \| A - A_k \|_2^2
\]
Using Theorem 33, if $S$ a rescaled sign matrix, our result guarantees the same error bound but has less rows than Theorem 3.4 in Magen and Zouzias (2011). In Magen and Zouzias (2011), it needs $\Omega(\text{rank}(A)/\epsilon^2)$ rows. Our result just needs $\tilde{O}(\text{srank}(A - A_k)/\epsilon^2)$ row.

**Theorem 34** Let $A$ share the same properties and notations as the one in Theorem 33. $S$ is a rescaled sign matrix with $\tilde{O}(r/\epsilon^2)$ rows, then

$$\|A - P_{SA,k}(A)\|_2^2 \leq (1 + \epsilon)\|A - A_k\|_2^2$$

holds with probability at least $1 - \delta$.

**Proof** Because rescaled sign matrix share the same subspace embedding properties, combining Theorem 33 and Theorem 18 leads to the result.

Theorem 33 has close relation to randomized SVD in the work of Halko et al. (2011), where matrix $A \in \mathbb{R}^{m \times n}$ multiplies a gaussian matrix or Subsample Randomized Hadamard Transform matrix to realize dimension reduction. In the following work, first we give a relative error randomized SVD construction method using gaussian random matrix in Theorem 35. Theorem 35 can be easily transformed to randomized SVD. Then we give a proof with tighter bound for randomized SVD related to stable rank where subspace embedding matrix $S$ is a gaussian matrix.

**Theorem 35** Given $A \in \mathbb{R}^{n \times d}$, $k$ is the target rank. And $0 < \epsilon < 1$ is the error parameter. $0 < \epsilon_0 < 1$ is the error parameter for subspace embedding. $0 < \delta < 1$ is the failure rate. $A_k = U_k \Sigma_k V^T_k$ is the best rank $k$ approximation matrix of $A$ and $A_{d/k} = A - A_k$. Besides, $r_1$ and $r_2$ are stable rank of $A^2_k$ and $A_{d/k}$ respectively. Let $r = \max\{r_1, r_2\}$ and $\tilde{r} = \max\{\text{rank}(A_{d/k}), k\}$. Let $l = O(\frac{r + 2 \log(\tilde{r}/(r\delta))) \log^2 \tilde{r}/r}{\epsilon})$, $S$ is a gaussian subspace embedding matrix which has $l$ rows, then with probability at least $\delta$

$$\min_X \|XSA - A\|_2 \leq (1 + \epsilon)\|A - A_k\|_2$$

**Proof** This theorem is an immediate result of Theorem 33 and Theorem 16. Equation 13 needs $l = O(\frac{(r + 2 \log(\tilde{r}/(r\delta))) \log^2 \tilde{r}/r}{\epsilon})$ rows due to Theorem 16.

When input matrix is sparse, then a sparse subspace embedding matrix is more attractive since two sparse matrices multiplication can be computed quickly.

**Theorem 36** Input matrix $A$ shares the same properties and notations as the one in Theorem 35. $S$ is a sparse embedding matrix described in Theorem 8, with $O(\epsilon^{-1}r \log^2 \frac{r}{\epsilon}) \text{poly}(\log(\tilde{r}/(r\epsilon\delta)))$ rows, then with probability at least $1 - \delta$

$$\min_X \|XSA - A\|_2 \leq (1 + \epsilon)\|A - A_k\|_2$$

and $SA$ can be computed in $O(nnz(A) \text{poly}(\log(d/(\epsilon\delta))))\epsilon^{-1/2}$.
Proof The result directly follows from Theorem 33 and Theorem 20.

Remark 37 When input matrix is dense, then an SRHT matrix is useful. And similar bound as in Theorem 35 and Theorem 36 can be proved using Theorem 33 and Theorem 18.

In the following theorem, we give a proof with tighter bound for randomized SVD. However, in our proof, the settings for gaussian matrix are stronger than Theorem 10.8 in Halko et al. (2011). First we give an Lemma that will be used in the following proof.

Theorem 38 Let $S$ be an $l_2$-subspace embedding for any fixed $k$ dimensional subspace $M$ with probability at least $1 - \delta$, and the error parameter $\epsilon_0$, so that $\|S_{\epsilon}y\|_2 = (1 \pm \epsilon_0)\|y\|_2$ for all $y \in M$. For any fixed matrix $A \in \mathbb{R}^{n \times d}$, then with probability at least $1 - \delta$

$$\min_{X} \|XSA - A\|_2 \leq \sqrt{1 + \left(\frac{\|S(A - A_k)\|_2}{1 - \epsilon_0}\right)^2} \tag{16}$$

Proof Let $U_k$ denote the $n \times k$ matrix of top $k$ left singular vectors of $A$. We have

$$\|U_k(SU_k)^{\dagger}SA - A\|_2^2$$

$$= \|U_k(SU_k)^{\dagger}SA - A_k - (A - A_k)\|_2^2$$

$$\leq \|U_k(SU_k)^{\dagger}SA - A_k\|_2^2 + \|(A - A_k)\|_2^2$$

$$= \|(SU_k)^{\dagger}SA - \Sigma_k V_k^T\|_2^2 + \|(A - A_k)\|_2^2$$

Now, we begin to bound $\|(SU_k)^{\dagger}SA - \Sigma_k V_k^T\|_2^2$. Since $(SU_k)^{\dagger}(SU_k) = I$, we have

$$\|(SU_k)^{\dagger}SA - \Sigma_k V_k^T\|_2^2$$

$$= \|(SU_k)^{\dagger}(S(A - A_k) + SU_k \Sigma_k V_k^T) - \Sigma_k V_k^T\|_2^2$$

$$= \|(SU_k)^{\dagger}(S(A - A_k) + SU_k \Sigma_k V_k^T - \Sigma_k V_k^T)\|_2^2$$

$$= \|(SU_k)^{\dagger}(S(A - A_k) + \Sigma_k V_k^T - \Sigma_k V_k^T)\|_2^2$$

$$= \|(SU_k)^{\dagger}(S(A - A_k))\|_2^2$$

$S$ is an $l_2$-subspace embedding for the column space of $U_k$, that is $\|SU_k z\|_2 = (1 \pm \epsilon_0)\|U_k z\|_2$ for all $z$. Since $U_k$ has orthonormal columns, this implies that all of the singular values of $SU_k$ are in the range $[1 - \epsilon_0, 1 + \epsilon_0]$. Hence, the singular values of $(SU_k)^{\dagger}$ are in $[1/(1 + \epsilon_0), 1/(1 - \epsilon_0)]$.

$$\|(SU_k)^{\dagger}(S(A - A_k))\|_2^2 \leq (1/(1 - \epsilon_0))^2 \cdot \|S(A - A_k)\|_2^2 \tag{17}$$

Proof Theorem 39 Given $A \in \mathbb{R}^{n \times d}$ is the target rank. And $0 < \epsilon < 1$ is the error parameter. $0 < \epsilon_0 < 1$ is the error parameter for subspace embedding. $0 < \delta < 1$ is the failure rate. $A_{\epsilon} = U_k \Sigma_k V_k^T$ is the best rank $k$ approximation matrix of $A$ and $A_{d/k} = A - A_{\epsilon}$.
Let $\tilde{r}$ and $r$ are stable rank of $A_{d/k}$ and rank of $A_{d/k}$ respectively. Let $\varepsilon_0^{-2} = 1.2$, and $l = k + p$, where $p = 0.2k + 1.2 \log \frac{\tilde{r}}{k}$. $S$ is a gaussian subspace embedding matrix which has $l$ rows, then with probability at least $1 - \delta$

$$\min_X \|XSA - A\|_2 \leq \sqrt{1 + 25((1 + \log \frac{\tilde{r}}{k})\frac{r}{k})\|A - A_k\|_2} = O(\sqrt{\frac{l}{k}}\|A - A_k\|_2)$$

**Proof** The proof is similar to the proof of Theorem 16. Let $A_{(i+1)bk}$ denote $A_{d/(i)k} - A_{d/(i+1)k}$, where $i = 1, 2, \ldots, \tilde{r}/k$. Then $A - A_k = \sum_{i=1}^{\tilde{r}/k} A_{(i+1)bk}$ and $A_{(i+1)bk}^T A_{jbk} = 0$ when $i \neq j$. Then we have

$$\|S(A - A_k)\|_2^2 \leq \sum_{i=1}^{\tilde{r}/k} \|SA_{(i+1)bk}\|_2^2$$

$$= \sum_{i=1}^{r/k} \|SA_{(i+1)bk}\|_2^2 + \sum_{i=r/k+1}^{\tilde{r}/k} \|SA_{(i+1)bk}\|_2^2$$  \hspace{1cm} (18)

When $i \leq r/k$, $\|SA_{(i+1)bk}\|_2 \leq \|A - A_k\|_2$. Hence, the first item in Equation 18 has the property that $\sum_{i=1}^{r/k} \|SA_{(i+1)bk}\|_2^2 \leq \frac{l}{k} \|SA_{(i+1)bk}\|_2^2 \leq (r/k - 1)(1 + \varepsilon_0)\|A - A_k\|_2^2$. For the second item in Equation 18, using Lemma 15, we have

$$\sum_{i=r/k+1}^{\tilde{r}/k} \|SA_{(i+1)bk}\|_2^2 \leq \frac{r}{k} \sum_{i=r/k+1}^{\tilde{r}/k} \frac{1}{i} \|SA_{(i+1)bk}\|_2^2$$

$$\leq (1 + \varepsilon_0)\frac{r}{k} \sum_{i=r/k+1}^{\tilde{r}/k} \frac{1}{i} \|A - A_k\|_2^2$$

$$\leq (1 + \varepsilon_0)\frac{r}{k} \log \frac{\tilde{r}}{k} \|A - A_k\|_2^2$$

combining the bound of first and second item in Equation 18, we have

$$\|S(A - A_k)\|_2^2 \leq (1 + \varepsilon_0)((1 + \log \frac{\tilde{r}}{k})\frac{r}{k})\|A - A_k\|_2^2$$  \hspace{1cm} (19)

Using Equation 16 and Equation 19, replacing $\varepsilon_0^{-2} = 1.2$, we have

$$\min_X \|XSA - A\|_2 \leq \sqrt{1 + 25((1 + \log \frac{\tilde{r}}{k})\frac{r}{k})\|A - A_k\|_2} = O(\sqrt{\frac{l}{k}}\|A - A_k\|_2)$$

There are $\tilde{r}/k$ items using $S$ as subspace embedding matrix, Using probability union rule, $\log(\tilde{r}/(k\delta))$ is needed. \hfill \blacksquare
The bound in Theorem 10.8 in Halko et al. (2011) can be represented in the form using $r$ which is the stable rank of $A - A_k$. We give the transformed form.

$$\min_X \|XSA - A\|_2 \leq [(1 + t \cdot \sqrt{\frac{3k}{p+1}}) + t \cdot \frac{e \sqrt{(k+p)r}}{p+1} + ut \cdot \frac{e \sqrt{k+p}}{p+1}] \|A - A_k\|_2$$

$$= O(\sqrt{kr} \|A - A_k\|_2) \quad (20)$$

Compare Equation 20 with Theorem 39, Theorem 39 reduce the upper bound of randomized SVD using gaussian matrix from $O(\sqrt{kr} \|A - A_k\|_2)$ to $O(\sqrt{r/k} \|A - A_k\|_2)$.

In the work of Halko et al. (2011), they give SRHT matrix to construct randomized SVD when input matrix is dense. In our work, we give a fast randomized SVD when input matrix is sparse.

**Theorem 40** Let input matrix $A$ has the same properties as the one in Theorem 39. Set error parameter $\varepsilon_0^2 = 1.2$. $S$ is the sparse subspace embedding matrix in Theorem 8 with $l = 1.2k \cdot \text{poly}(\bar{r}/\delta)$ rows. $SA$ can be computed in $O(\text{nnz}(A) \text{poly}(\bar{r}/\delta))$.

$$\min_X \|XSA - A\|_2 \leq \sqrt{1 + 25((1 + \log \frac{\bar{r}}{k}) \|A - A_k\|_2^2)} = O(\sqrt{r} \|A - A_k\|_2)$$

holds with probability at least $1 - \delta$.

**Proof** A similar proof of Theorem 39 and combining Theorem 8 lead to the result. \hfill \qed

**Remark 41** When input matrix is dense, then an SRHT matrix is useful to construct randomized SVD as Halko et al. (2011) did. And similar bound as in Theorem ?? and Theorem 40 can be proved using Theorem 33 and Theorem 18.

As we can see, beyond a gaussian matrix or Subsample Randomized Hadamard Transform matrix, any matrix with subspace embedding property can be used to construct randomized SVD algorithm. And it is better to choose the subspace embedding matrix according to the structure of the input matrix, for example, if the input matrix is sparse, then a sparse subspace embedding matrix is a good choice. Even, we can compose of two different subspace embedding matrix to construct randomized SVD due to Lemma 32.

Now, we give our result of low rank matrix approximation.

**Theorem 42** Given $A \in \mathbb{R}^{n \times d}$, let $r_1 = \max\{\text{srank}(A_k^2), \text{srank}(A - A_k)\}$ and $\bar{r}_1 = \max\{k, \text{rank}(A - A_k)\}$. Let $S$ be a matrix satisfies the property described in Theorem 33. Let $r_2 = \max\{\text{srank}(SA), \text{srank}(A - A(SA)^TSA)\}$ and $\bar{r}_2 = \max\{\text{rank}(SA), \text{ank}(A - A_k)\}$. Let $R$ be a $(1 \pm \sqrt{1/2})$-approximation $L_2$-subspace embedding for the row space of $SA$, and $R$ has the property as in Theorem 28 with Equation 11 holding. Then

$$\|AR^T(SA)^TSA - A\|_2 \leq (1 + \varepsilon)\|A - A_k\|_2$$

holds with high probability.
Proof Theorem 33 implies that

$$\min_X \|XSA - A\|_2 \leq (1 + \varepsilon)\|A - A_k\|_2$$

One minimizer of problem \( \min_X \|XSA - A\|_2 \) is \( X = AR^T(SA^TR)^\dagger \). Using Theorem 28, we have

$$\|AR^T(SA^TR)^\dagger SA - A\|_2 \leq (1 + \varepsilon)\min_X \|XSA - A\|_2 \leq (1 + \varepsilon)^2\|A - A_k\|_2$$

which implies the equation 21.

Subspace embedding matrices \( S \) and \( R \) can be chosen sparse embedding matrices.

**Theorem 43** Input matrix \( A \) share the same properties and notations as the ones in Theorem 42. \( S \) and \( R \) are sparse subspace embedding matrices. \( S \) has \( \tilde{O}(r_1/\varepsilon) \) rows. \( R \) is of \( \tilde{O}(r_2/\varepsilon) \) rows. Then

$$\|AR^T(SA^TR)^\dagger SA - A\|_2 \leq (1 + \varepsilon)\|A - A_k\|_2$$

holds with high probability. And \( SA \) and \( AR^T \) can be computed in \( \tilde{O}(\text{nnz}(A)\varepsilon^{-1/2}) \) time. Computation of \( (SA^TR)^\dagger \) costs \( \tilde{O}((\text{nnz}(A)^{1/2}(n + d)r_1^2/\varepsilon^2 + r_1r_2r_2^2/\varepsilon^3)) \)

Now we give a fast low rank approximation algorithm, the running time depends input sparsity and distribution of singular values.

**Theorem 44** Let input \( A \in \mathbb{R}^{n \times d} \) share the same properties and notations as in Theorem 43. \( k \) and \( \varepsilon \) are input parameters. Let \( S \) and \( R \) be sparse subspace embedding matrices satisfying the property described in Theorem 42. Then there is an algorithm that computes an approximate SVD that finds \( L, D, W \), such that \( \|A - LDW^T\|_2 \leq (1 + \varepsilon)\|A - A_k\|_2 \) with probability at least \( 1 - \delta \), and the factorization can be computed in \( \tilde{O}(\text{nnz}(A)\varepsilon^{-1/2} + (n + d)r_1^2/\varepsilon^2 + r_1r_2r_2^2/\varepsilon^3) \).

**Proof** Now, we give the approximate SVD algorithm in following,

1. Compute QR decomposition of \( SA \) in rowspace, and get \( V^T \), where \( S \) has the property as the one in Theorem 42.
2. Compute \( V^TR^T \), where \( R \) has the property as the one in Theorem 42.
3. Compute SVD \( LDW_1^T \) of \( AR^T(V^TR^T)^\dagger \)
4. Return \( L, D \) and \( W = VW_1 \)

Let \( S \) and \( R \) be sparse subspace embedding matrices in Theorem 8. Hence the number of row of \( S \) need to be \( \tilde{O}(\varepsilon^{-1}r_1) \). And \( SA \) computes in \( \tilde{O}(\text{nnz}(A)\varepsilon^{-1/2}) \) time. Similarly, \( R \) needs \( \tilde{O}(\varepsilon^{-1}r_2) \) rows. Computation of \( AR^T \) costs \( \tilde{O}(\text{nnz}(A)\varepsilon^{-1/2}) \). Computing QR decomposition of \( SA \) costs \( \tilde{O}(dr_1/\varepsilon^2) \) time. Computing \( V^TR^T \) costs \( \tilde{O}(\text{nnz}(V)\varepsilon^{-1/2}) \) time and pseudo inverse of \( V^TR^T \) costs \( \min\{\tilde{O}(r_1^2r_2^2/\varepsilon^3), \tilde{O}(r_1^2r_2/\varepsilon^3)\} \).
Computing SVD of $\mathbf{AR}^T (\mathbf{VR}^T)^\dagger$ costs $O(n(r_1/\varepsilon)^2)$, as does computing $\mathbf{VW}_1$. Hence all the cost of algorithm is $\tilde{O}(\text{nnz}(\mathbf{A})\varepsilon^{-1/2} + (n + d)r_1^2/\varepsilon^2 + r_1 r_2^2/\varepsilon^3)$.

Next, we prove the correctness of the approximate SVD algorithm. Theorem 33 guarantees that

$$\min_{\mathbf{X}} \|\mathbf{XSA} - \mathbf{A}\|_2 \leq (1 + \varepsilon)\|\mathbf{A} - \mathbf{A}_k\|_2 \tag{22}$$

Let $\mathbf{SA} = \mathbf{PV}^T$ is the QR decomposition of $\mathbf{SA}$, and $\bar{\mathbf{X}} = \mathbf{XP}$, then Equation 22 can be transform to

$$\min_{\mathbf{X}} \|\bar{\mathbf{X}}\mathbf{V}^T - \mathbf{A}\|_2 \leq (1 + \varepsilon)\|\mathbf{A} - \mathbf{A}_k\|_2 \tag{23}$$

By Lemma 26,

$$\min_{\mathbf{X}} \|\bar{\mathbf{X}}\mathbf{V}^T\mathbf{R}^T - \mathbf{AR}^T\|_2 \leq (1 + \varepsilon)\min_{\mathbf{X}} \|\mathbf{XSA} - \mathbf{A}\|_2 \leq (1 + \varepsilon)^2\|\mathbf{A} - \mathbf{A}_k\|_2 \tag{24}$$

$\bar{\mathbf{X}}^* = \mathbf{AR}^T (\mathbf{V}^T\mathbf{R}^T)^\dagger = \arg\min_{\mathbf{X}} \|\bar{\mathbf{X}}\mathbf{V}^T\mathbf{R}^T - \mathbf{AR}^T\|_2$, and $\mathbf{LDW}_1^T$ is the SVD of $\mathbf{AR}^T (\mathbf{V}^T\mathbf{R}^T)^\dagger$, hence $\|\mathbf{LDW}^T - \mathbf{A}\|_2 = \|\bar{\mathbf{X}}^*\mathbf{V}^T - \mathbf{A}\|_2 \leq \|\mathbf{A} - \mathbf{A}_k\|_2 \quad \square$

If the input matrix has good property that has low coherence, then the sparsity of $\mathbf{S}$ and $\mathbf{R}$ can be reduced to 1. Hence Theorem 44 can be computed in $O(\text{nnz}(\mathbf{A})) + \tilde{O}((n + d)r_1^2/\varepsilon^2 + r_1 r_2^2/\varepsilon^3)$ time.

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