A conditional result on exponential sums over primes in short intervals

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Abstract. The aim of this work is to illustrate a conditional result involving the exponential sums over primes in short intervals under the assumption that both the Generalized Riemann Hypothesis and the Density Hypothesis for Dirichlet $L$-functions are true.

1 Introduction

Let

$$S(\alpha) = \sum_{n \leq X} \Lambda(n)e(n\alpha)$$

be an exponential sum, where $\Lambda$ is the Von-Mangoldt function and $e(x) = e^{2\pi ix}$.

Then, we write $\alpha$ as $\alpha = \frac{a}{q} + \beta$, where $(a,q) = 1$.

The first attempt in trying to estimate this type of additive exponential sums was due to Vinogradov (\cite{11}, Chapter 9).

More recently, Vaughan found a new and simpler version of Vinogradov’s method (see \cite{9}, \cite{10} or \cite{3}, Chapters 24, 25). He proved the following result.

Theorem. (Vaughan)

If

$$|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}, \quad (a,q) = 1,$$

then the following estimate holds:

$$S(\alpha) \ll \left(Nq^{-\frac{1}{2}} + N\frac{1}{2} + N\frac{1}{4}q \frac{1}{2}\right) \cdot (\log N)^4.$$

This type of additive exponential sums was studied also by Montgomery (see \cite{5}, Chapter 16), who used the so called zero-density method to estimate $S(\alpha)$.

The estimates of $S(\alpha)$ depend on $q$ and are non trivial for those $q$ in the range $\log^B X \leq q \leq X \log^{-B} X$, where $B > 0$ is a suitable constant.

The method of Vinogradov and the zero-density method were used by Pan Chen-Dong \cite{6} in 1959 and, subsequently, by Chen Jing-Run \cite{2} in 1965 to find some estimates for exponential sums on short intervals, that is, exponential sums of the form

$$S(\alpha, X, Y) = \sum_{X < n \leq X+Y} \Lambda(n)e(n\alpha),$$

where $2 \leq Y \leq X$ or, equivalently, $Y = o(X)$. In particular, Chen’s estimates are non trivial only when $Y \gg X^{2/3+\varepsilon}$ and $q$ in a certain range.

Chen’s result was improved in 1986 by Balog and Perelli \cite{1}, who were able to extend the range of values of $Y$ for which one can have non trivial estimates for the exponential sum $S(\alpha, X, Y)$. More precisely, they found non trivial estimates for all $Y \gg X^{3/5+\varepsilon}$. In their method, Balog and Perelli didn’t use density results, but they obtained the following estimates using Heath-Brown’s Identity \cite{4}, combined with analytic techniques.

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Theorem. (Balog-Perelli) Let $1 \leq a < q$, $(a, q) = 1$ and $1 \leq Y \leq X$. Then
\[
S\left(\frac{a}{q}, X, Y\right) \ll \left(X^{1/2}q^{1/2} + Y^{-1/2} + X^{3/10}Y^{1/2}\right) \log^{100} X.
\]

However, a limit on Balog’s and Perelli’s method is that it is valid only for $\alpha$ rational. In 1994, a conditional result involving the exponential sum $S(\alpha, X, Y)$ was found by Puglisi [7]. He proved that, under the assumption of the so called Quasi-Riemann Hypothesis (Q.R.H.), some non trivial estimates for $S(\alpha, X, Y)$ hold, when $\alpha$ lies on certain ”minor arcs”. More precisely, he found the following result.

Theorem. (Puglisi) Suppose that $\zeta(s) \neq 0$ in the parallelogram
\[
\sigma > \theta, \quad |t| \leq 3T^2,
\]
with $1/2 \leq \theta < 1$ and $T \geq 2\pi$. Then the estimate
\[
\sum_{X < n \leq X + Y} \Lambda(n)e^{2\pi i \alpha n} \ll Y(\alpha X^\theta + X^{-1/8}) \log X + \left(\frac{X^{3/4} + 1}{\alpha}\right) \log^{29} X
\]
holds for all $\alpha, X, Y$ such that
\[
2\pi \alpha X = T, \quad \frac{1}{\alpha} \leq Y \leq X.
\]

Puglisi’s estimate is good when $Y \gg X^{3/4+\varepsilon}$ and $X^{-\theta-\varepsilon} \gg \alpha \gg Y^{-1+\varepsilon}$. Furthermore, if we assume the existence of a whole strip $\sigma > \theta$ free from zeros, the bound on the right above may be improved to
\[
Y\alpha X^\theta \log X + \frac{1}{\alpha} \log^2 X,
\]
with $\alpha^{-1} \leq Y \leq X$.

The aim of this work is to find a conditional estimate for exponential sums on short intervals of the form $S(\alpha, X, Y)$ when $\alpha$ lies on certain ”major arcs”. More precisely, we will prove the following result.

Theorem 1. Consider
\[
S(\alpha, X, Y) = \sum_{X < n \leq X + Y} \Lambda(n)e(n\alpha)
\]
where $2 \leq Y \leq X$, $2 \leq q \leq X$ and
\[
\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}, \quad (a, q) = 1.
\]

Suppose that the Generalized Q.R.H. holds, that is, it exists a constant $\theta$, with $1/2 \leq \theta < 1$, such that every Dirichlet $L$-function $L(s, \chi)$ belonging to an arbitrary Dirichlet character $\chi$ modulus $q$ has no zeros with real part greater than $\theta$. Furthermore, suppose that the Generalized Density Hypothesis is also true. Then, the following estimate
\[
S(\alpha, X, Y) \ll (\log(qX \log^3 X))^{A+1} \cdot \log X \cdot \left(q^{-\frac{1}{4}+\frac{1}{20}} + X^{\theta+\frac{1}{50}} \cdot Y^{-\frac{1}{2}+\frac{1}{30}}\right).
\]
holds, where $A$ is an effective constant.

Two immediate corollaries follow. First of all, if instead of the Generalized Q.R.H. we suppose that the Generalized Riemann Hypothesis is true, that is, $\theta = \frac{1}{2}$, we obtain the following result.
Corollary 1. If the Generalized Riemann Hypothesis is true, then, using the same notations as in Theorem [1], the estimate
\[ S(\alpha, X, Y) \ll (\log(qX \log^3 X))^{A+1} \cdot \log X \cdot \sqrt{qX} \]
holds.

Since the exponents of $Y, X, q$ are always positive in the range $\frac{1}{2} \leq \theta < 1$, another interesting case is the estimate we obtain when $\theta$ assumes the value for which the exponent $\theta + \frac{1}{2\theta} - 1$ of the power of $X$ in Theorem [1] reaches the minimum value, that is, for $\theta = \frac{1}{\sqrt{2}}$.

Corollary 2. Under the same notations and assumptions of Theorem [1] if $\theta = \frac{1}{\sqrt{2}}$, then the following estimate
\[ S(\alpha, X, Y) \ll (\log(qX \log^3 X))^{A+1} \cdot \log X \cdot \left( \sqrt{qX} + q^{-\frac{1}{2} + \frac{1}{\sqrt{2}}} \cdot X^{\sqrt{2}-1} \cdot Y^{-\frac{1}{\sqrt{2}}} \right) \]
holds.

2 Proof of Theorem [1]

We will use the following notations.

We put
\[ \alpha = \frac{a}{q} + \beta \text{ with } |\beta| \leq \frac{1}{q^2} \]
and $s = \sigma + it$.

Then, we suppose that
\[ |\beta| \leq \frac{1}{qQ} = \frac{P}{qN} \quad (1) \]
where
\[ P = (\log N)^B \]
and
\[ Q = \frac{N}{(\log N)^B}. \]

for a suitable constant $B > 0$ we will choose later.

First of all, using the following relation
\[ \frac{1}{\phi(q)} \sum_{\chi} \tau(\chi) \chi(h) = e \left( \frac{h}{q} \right) \]
where $(h, q) = 1$ and $e(x) := e^{2\pi ix}$, we have
\[ \sum_{X < n \leq X+Y \atop (n, q) = 1} \Lambda(n)e(n\alpha) = \sum_{X < n \leq X+Y \atop (n, q) = 1} \Lambda(n)e(n\beta) e \left( \frac{na}{q} \right) = \]
\[ = \sum_{X < n \leq X+Y} \Lambda(n) \cdot \frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi) \chi(na)e(n\beta) = \]
\[ = \frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi) \chi(a) \sum_{X < n \leq X+Y} \Lambda(n) \chi(n)e(n\beta). \]
Furthermore,

\[ S(\alpha, X, Y) = \sum_{X < n \leq X + Y} \Lambda(n)e(n\alpha) = \]

\[ = \sum_{X < n \leq X + Y} \Lambda(n)e(n\alpha) + \sum_{(n, q) > 1} \Lambda(n)e(n\alpha) = \]

\[ = \frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi)\chi(a) \sum_{X < n \leq X + Y} \Lambda(n)\chi(n)e(n\beta) + \sum_{(n, q) > 1} \Lambda(n)e(n\alpha) \]

where

\[ \sum_{X < n \leq X + Y} \Lambda(n)e(n\alpha) \]

is negligible, being the tail of a convergent series.

Now, we focus our attention on the quantity

\[ \frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi)\chi(a) \sum_{X < n \leq X + Y} \Lambda(n)\chi(n)e(n\beta). \]

Since \( \tau(\chi_0) = \mu(q) \), we have

\[ \frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi)\chi(a) \sum_{X < n \leq X + Y} \Lambda(n)\chi(n)e(n\beta) = \]

\[ = \frac{1}{\phi(q)} \tau(\chi_0)\chi_0(a) \sum_{X < n \leq X + Y} \Lambda(n)\chi_0(n)e(n\beta) + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi)\chi(a) \sum_{X < n \leq X + Y} \Lambda(n)\chi(n)e(n\beta) \ll \]

\[ \ll \frac{\mu(q)}{\phi(q)} \sum_{X < n \leq X + Y} \Lambda(n)\chi_0(n)e(n\beta) + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi)\chi(a) \sum_{X < n \leq X + Y} \Lambda(n)\chi(n)e(n\beta) \ll \]

\[ \ll \frac{1}{q} \sum_{X < n \leq X + Y} \Lambda(n)\chi_0(n)e(n\beta) + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi)\chi(a) \sum_{X < n \leq X + Y} \Lambda(n)\chi(n)e(n\beta). \]

We start considering all terms involving the Dirichlet characters \( \chi \neq \chi_0 \) and we analyze the inner sum

\[ \sum_{X < n \leq X + Y} \Lambda(n)\chi(n)e(n\beta). \]

We know that, for \( \chi \neq \chi_0 \), the following equality

\[ \psi_0(x, \chi) = \sum_{n < x} \chi(n)\Lambda(n) + \begin{cases} \frac{1}{2}\chi(x)\Lambda(x) & \text{if } x \in \mathbb{N} \\ 0 & \text{if } x \notin \mathbb{N} \end{cases} \]

holds, and

\[ \psi(x, \chi) = \sum_{n \leq x} \chi(n)\Lambda(n). \]

Following Puglisi’s method in [7], we define the following two variables:

\[ x^* = \max \left( x - \frac{1}{2}, X \right) \]

and

\[ x^{**} = \min \left( x + \frac{1}{2}, X + Y \right). \]
Furthermore, we know that

\[
\int_{u-\frac{1}{2}}^{u+\frac{1}{2}} e^{2\pi i \beta x} \, dx = \frac{\sin \pi \beta}{\pi \beta} \cdot e^{2\pi i \beta u} = \frac{\sin \pi \beta}{\pi \beta} e(u\beta).
\]

Hence, it follows that

\[
\frac{\sin \pi \beta}{\pi \beta} \sum_{X < n \leq X + Y} \Lambda(n) \chi(n) e(n \beta) = \sum_{X < n \leq X + Y} \Lambda(n) \chi(n) \cdot \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} e^{2\pi i \beta x} \, dx =
\]

\[
= \int_{X-\frac{1}{2}}^{X+Y+\frac{1}{2}} e^{2\pi i \beta x} (\psi(x^*, \chi) - \psi(x^*, \chi)) \, dx =
\]

\[
= \int_{X-\frac{1}{2}}^{X+Y+\frac{1}{2}} e^{2\pi i \beta x} (\psi_0(x^*, \chi) - \psi_0(x^*, \chi)) \, dx + O(\log X),
\]

since

\[
\psi(x^*, \chi) = \psi_0(x^*, \chi)
\]

and

\[
\psi(x^{**}, \chi) = \psi_0(x^{**}, \chi)
\]

for almost all values of \(x\) such that \(X - \frac{1}{2} \leq x \leq X + Y - \frac{1}{2}\).

Now, we set

\[
c = 1 + \frac{1}{\log X}.
\]

We have

\[
\psi_0(x^{**}, \chi) - \psi_0(x^*, \chi) = \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} L'(s, \chi) ((x^{**})^s - (x^*)^s) - \frac{1}{s} ds + R(x, T_0)
\]

where

\[
R(x, T_0) = O \left( \frac{x \log^2 x}{T_0} \right) + \log x \left\{ \min \left( 1, \frac{x}{T_0 \|x^*\|} \right) + \min \left( 1, \frac{x}{T_0 \|x^{**}\|} \right) \right\},
\]

with \(T_0 \geq 2\) and \(\|y\| = \min\{\|y - m\|; m \in \mathbb{N}\}\).

Again, following Puglisi’s method in [7], one has

\[
\min \left( 1, \frac{x}{T_0 \|x^*\|} \right) = 1 \quad \text{if} \quad X - \frac{1}{2} \leq x \leq X + \frac{1}{2},
\]

\[
\min \left( 1, \frac{x}{T_0 \|x^{**}\|} \right) = 1 \quad \text{if} \quad X + Y - \frac{1}{2} \leq x \leq X + Y + \frac{1}{2}
\]

and

\[
\int_{X-\frac{1}{2}}^{X+Y+\frac{1}{2}} e^{2\pi i \beta x} \cdot R(x, T_0) \, dx \ll \left( Y \left( \frac{X}{T_0} \right)^{1/2} + 1 \right) \log X
\]

with

\[
T_0 > X \log^2 X \quad \text{and} \quad c = 1 + \frac{1}{\log X}.
\]

Again, as in [7], the following relation holds:

\[
\frac{1}{s} \int_{X-\frac{1}{2}}^{X+Y+\frac{1}{2}} e^{2\pi i \beta x} ((x^{**})^s - (x^*)^s) \, dx = \frac{\sin \pi \beta}{\pi \beta} \int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} \, du.
\]
It follows that
\[
\sum_{X < n \leq X + Y} \Lambda(n) \chi(n) e(n\beta) = \int_X^{X + Y} e^{2\pi i \beta x} \left( \psi_0(x^*, \chi) - \psi_0(x, \chi) \right) dx + O(\log X) = \\
= \int_X^{X + Y} e^{2\pi i \beta x} \cdot \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \frac{L'}{L}(s, \chi) ((x^*)^s - (x)^s) \cdot \frac{1}{s} ds + \\
+ \int_X^{X + Y} e^{2\pi i \beta x} R(x, T_0) dx + O(\log X) = \\
= \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \frac{L'}{L}(s, \chi) ds \cdot \frac{\pi \beta}{\pi} \int_X^{X + Y} u^{s-1} e^{2\pi i \beta u} du + O \left( \left( Y \left( \frac{X}{T_0} \right)^{1/2} + 1 \right) \log X \right) + \\
+ O(\log X)
\]
or, equivalently,
\[
\sum_{X < n \leq X + Y} \Lambda(n) \chi(n) e(n\beta) = \int_X^{X + Y} u^{s-1} e^{2\pi i \beta u} du.
\]
Now, we focus our attention to the quantity
\[
\int_X^{X + Y} u^{s-1} e^{2\pi i \beta u} du.
\]
First of all, we have
\[
\int_X^{X + Y} u^{s-1} e^{2\pi i \beta u} du = \int_X^{X + Y} u^{\sigma-1} \cdot u^t e^{2\pi i \beta u} du = \\
= \int_X^{X + Y} u^{\sigma-1} e^{it \log u + 2\pi i \beta u} du = \\
= \int_X^{X + Y} G_\sigma(u) e^{iF_t(u)} du,
\]
where
\[
G_\sigma(u) = u^{\sigma-1}
\]
and
\[
F_t(u) = 2\pi \beta u + t \log u.
\]
Now, we use (1) with \( N = X \) and \( N = X + Y \). Under these assumptions, we have, for \( u \in [X, X + Y] \), that
\[
\frac{(\log(X + Y))^B}{q(X + Y)} \leq |\beta| \leq \frac{(\log X)^B}{qX},
\]
where \( B \) is chosen suitably, such that
\[
\frac{12\pi (\log X)^B}{q} \leq 1.
\]
Lemma 1. Given \( c = 1 + \frac{1}{\log X} \), the relation
\[
\int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du \ll \min \left( \frac{1}{|\beta|}, \frac{X^\sigma}{|t|} \right)
\]
holds for every \(-1 \leq \sigma \leq c\) and for every \(1 \leq |t| \leq T_0\).

Proof. We have already seen that
\[
\int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du = \int_X^{X+Y} G_\sigma(u) e^{iF_t(u)} du
\]
where
\[G_\sigma(u) = u^{\sigma-1}\]
and
\[F_t(u) = 2\pi \beta u + t \log u.\]
As Puglisi observed in [7], the following relation holds:
\[
0 < G_\sigma(u) \leq \begin{cases} 
(X + Y)^{\sigma-1} & \text{if } 1 \leq \sigma \leq c \\
X^{\sigma-1} & \text{if } -1 \leq \sigma \leq 1 
\end{cases}
\]
Now, we shift our attention to the function \( F_t(u) \).
We will prove that, given its first derivative
\[
F'_t(u) = 2\pi \beta + \frac{t}{u},
\]
the relation
\[
F'_t(u) \begin{cases} 
\geq \frac{|t|}{X+Y} & \text{if } t \geq 0 \\
\leq -\frac{|t|}{X} & \text{if } t \leq 0 
\end{cases}
\]
holds for every \(1 \leq |t| \leq T_0\).
To do that, we consider the cases \( t \geq 0 \) and \( t \leq 0 \) separately.

- Let \( t \geq 0 \) be in the interval \( 1 \leq |t| \leq T_0 \).
  Being \( t \geq 0 \), we are working with those \( \beta \geq 0 \). We have
  \[
  F'_t(u) = 2\pi \beta + \frac{t}{u} \geq 2\pi \beta + \frac{t}{X+Y} \geq \frac{t}{X+Y}.
  \]

- Let \( t \leq 0 \) be in the interval \( 1 \leq |t| \leq T_0 \).
  We have
  \[
  F'_t(u) = 2\pi \beta + \frac{t}{u} \leq 2\pi \beta - \frac{|t|}{|u|} \leq 2\pi |\beta| - \frac{|t|}{X+Y} \leq 2\pi \frac{(\log X)^B}{qX} - \frac{|t|}{X+Y} \leq -\frac{|t|}{3X}.
  \]
As a result, the relation (3) holds. Furthermore, considering again the interval \( 1 \leq |t| \leq T_0 \), one has also the following relation:

\[
F_t' (u) \begin{cases} 
\geq 2\pi\beta & \text{if } t \geq 0 \\
\leq -\frac{\pi|\beta|X}{2(X+1)} & \text{if } t \leq 0
\end{cases}
\]

At this point, using the same argument as Puglisi in [7], we can conclude that the uniform estimate

\[
\int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du \ll \frac{1}{|\beta|}
\]

holds in the interval stated in the hypothesis and Lemma 4.3 in [8] imply the relation

\[
\int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du \ll \min \left( \frac{1}{|\beta|}, \frac{X\sigma}{|t|} \right)
\]

for every \(-1 \leq \sigma \leq c\) and for every \(1 \leq |t| \leq T_0\).

Now, we shift our attention to the case with the principal character \( \chi = \chi_0 \). We know that

\[
\psi(x, \chi_0) = \psi(x) + O(\log^2(qx)),
\]

where

\[
\psi(x) = \sum_{n \leq x} \Lambda(n).
\]

Furthermore, as we have already proved before, the following equality holds:

\[
\frac{1}{\phi(q)} \cdot \tau(\chi_0) \chi_0(a) \sum_{X < n \leq X+Y} \Lambda(n) \chi_0(n)e(n\beta) = \mu(q) \frac{\phi(q)}{\phi(q)} \sum_{X < n \leq X+Y} \Lambda(n) \chi_0(n)e(n\beta).
\]

Following the same argument we used for the case \( \chi \neq \chi_0 \), one has

\[
\frac{\sin \pi \beta}{\pi \beta} \sum_{X < n \leq X+Y} \Lambda(n) \chi_0(n)e(n\beta) = \\
= \sum_{X < n \leq X+Y} \Lambda(n) \chi_0(n) \cdot \int_{n-\frac{T_0}{2}}^{n+\frac{T_0}{2}} e^{2\pi i \beta x} dx = \\
= \int_X^{X+Y} e^{2\pi i \beta x} \left( \psi(x^{**}, \chi_0) - \psi(x, \chi_0) \right) dx = \\
= \int_X^{X+Y} e^{2\pi i \beta x} \left( \psi(x^{**}) - \psi(x) \right) dx + O(\log^2(qX)) = \\
= \int_X^{X+Y} e^{2\pi i \beta x} \left( \psi_0(x^{**}) - \psi_0(x) \right) dx + O(\log X) + O(\log^2(qX)) = \\
= \int_X^{X+Y} \frac{e^{2\pi i \beta x}}{2\pi i} \left( \psi(s^{**}) - \psi(s) \right) ds + O(\log X) + O(\log^2(qX)) = \\
= \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \frac{z^s}{\zeta(s)} ds \cdot \frac{\sin \pi \beta}{\pi \beta} \int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du + O \left( Y \left( \frac{X}{T_0} \right)^{1/2} \log X \right) + O(\log^2(qX)).
\]
It follows that
\[
\sum_{X < n \leq X+Y} \Lambda(n) \chi_0(n)e(n\beta) =
\]
\[
= \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \zeta'(s) ds \cdot \int_{X}^{X+Y} u^{s-1} e^{2\pi i \beta u} du + O \left( \left( Y \left( \frac{X}{T_0} \right)^{1/2} + 1 \right) \log X \right) + \text{(4)}
\]
\[
+ O(\log^2(qX)).
\]

Now, we consider the rectangle with vertexes
\[
\begin{align*}
&c - iT_0, \ c + iT_0, \ -U + iT_0, \ -U - iT_0,
\end{align*}
\]
where \( U > 0 \) is a constant arbitrarily large.

Using the residue theorem on the above rectangle for the integrals in (4) and in (2) and
the relation $|\tau(\chi)| \leq \sqrt{q}$ for $\chi \neq \chi_0$, we have

\[
\frac{1}{q} \sum_{X < n \leq X+Y} \Lambda(n) \chi_0(n) e(n\beta) + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) \sum_{X < n \leq X+Y} \Lambda(n) \chi(n) e(n\beta) =
\]

\[
= \frac{1}{q} \left( \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \frac{\zeta'(s)}{\zeta(s)} ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i\beta u} du +
\right.
\]

\[
+ O \left( \left( Y \left( \frac{X}{T_0} \right)^{1/2} + 1 \right) \log X \right) + O\left( \log^2(qX) \right) +
\]

\[
+ \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) \left( \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} -\frac{L'}{L}(s, \chi) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i\beta u} du +
\right.
\]

\[
+ O \left( \left( Y \left( \frac{X}{T_0} \right)^{1/2} + 1 \right) \log X \right) =
\]

\[
= \frac{1}{q} \left( \frac{1}{2\pi i} \int_{-U-iT_0}^{c+iT_0} \frac{\zeta'(s)}{\zeta(s)} ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i\beta u} du +
\right.
\]

\[
+ \frac{1}{q} \left( \frac{1}{2\pi i} \int_{-U-iT_0}^{c+iT_0} \frac{\zeta'(s)}{\zeta(s)} ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i\beta u} du -
\right.
\]

\[
- \frac{1}{q} \left( \frac{1}{2\pi i} \int_{c-iT_0}^{-U-iT_0} \frac{\zeta'(s)}{\zeta(s)} ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i\beta u} du +
\right.
\]

\[
+ \frac{1}{q} \sum_{|\gamma_\chi| < T_0} \int_X^{X+Y} u^{\rho-1} e^{2\pi i\beta u} du +
\]

\[
+ O \left( \frac{1}{q} \left( Y \left( \frac{X}{T_0} \right)^{1/2} + 1 \right) \log X \right) +
\]

\[
+ \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) \left( \frac{1}{2\pi i} \int_{-U-iT_0}^{c+iT_0} -\frac{L'}{L}(s, \chi) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i\beta u} du +
\right.
\]

\[
+ \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) \left( \frac{1}{2\pi i} \int_{-U-iT_0}^{c+iT_0} -\frac{L'}{L}(s, \chi) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i\beta u} du -
\right.
\]

\[
- \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) \left( \frac{1}{2\pi i} \int_{c-iT_0}^{-U-iT_0} -\frac{L'}{L}(s, \chi) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i\beta u} du +
\right.
\]

\[
+ \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) \sum_{|\gamma_\chi| < T_0} \int_X^{X+Y} u^{\rho-1} e^{2\pi i\beta u} du +
\]

\[
+ R_1(X, q) + O \left( \sqrt{q} \left( Y \left( \frac{X}{T_0} \right)^{1/2} + 1 \right) \log X \right) \ll
\]
Lemma 2. The relation

\[ \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) 1 \int_{-U-iT_0}^{c-iT_0} \frac{\zeta'(s)}{\zeta(s)} ds \cdot \int_{X}^{X+Y} u^{s-1} e^{2\pi i \beta u} du \]

holds, provided that \( T_0 > X \log^2 X \).
Proof. Using the fact that $|\tau(\chi)| \leq \sqrt{q}$ and Lemma 1 we have

$$\frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) \frac{1}{2\pi i} \int_{c+iT_0}^{c-iT_0} -\frac{L'}{L}(s, \chi) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du \ll$$

$$\ll \frac{\sqrt{q}}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \frac{1}{2\pi i} \int_{-U+iT_0}^{c+iT_0} -\frac{L'}{L}(s, \chi) ds \cdot \frac{X^\sigma}{|t|} \ll$$

$$\frac{\sqrt{q}}{\phi(q)} \cdot \frac{1}{T_0} \cdot \frac{1}{2\pi i} \cdot \log^2(qT_0) \int_c X^\sigma d\sigma \ll$$

$$\ll \log^2(qT_0) \cdot \frac{\sqrt{q}}{T_0} \ll \log^2(qT_0) \cdot \sqrt{q}$$

where, in the last estimate, we used the assumption $T_0 > X \log^2 X$. \qed

Similarly, the estimate found in Lemma 2 is valid also for the term

$$\frac{1}{\phi(q)} \sum_{\chi \neq \chi_0 \mod q} \tau(\chi) \chi(a) \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} -\frac{L'}{L}(s, \chi) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du.$$

Lemma 3. The relation

$$\frac{1}{q} \cdot \frac{1}{2\pi i} \int_{-U+iT_0}^{c+iT_0} \zeta'(s) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du \ll$$

$$\ll \frac{1}{q} \cdot \log^2 T_0$$

holds, provided that $T_0 > X \log^2 X$.

Proof. We have

$$\frac{1}{q} \cdot \frac{1}{2\pi i} \int_{-U+iT_0}^{c+iT_0} \zeta'(s) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du \ll$$

$$\ll \frac{1}{q} \cdot \frac{1}{2\pi i} \int_{-U+iT_0}^{c+iT_0} \zeta'(s) ds \cdot \frac{X^\sigma}{|t|} \ll$$

$$\ll \frac{1}{qT_0} \cdot \log^2 T_0 \int_{-\infty}^{c} X^\sigma d\sigma \ll$$

$$\ll \log^2 T_0 \cdot \frac{X}{qT_0} \ll$$

$$\ll \frac{1}{q} \cdot \log^2 T_0$$

where, in the last passage, we used the assumption $T_0 > X \log^2 X$. \qed

Similarly, the estimate found in Lemma 3 is true also for the term

$$\frac{1}{q} \cdot \frac{1}{2\pi i} \int_{-U+iT_0}^{c+iT_0} \zeta'(s) ds \cdot \int_X^{X+Y} u^{s-1} e^{2\pi i \beta u} du.$$

Now, it remains to study the term

$$\frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi) \chi(a) \sum_{|\gamma| < T_0} \int_X^{X+Y} u^{\rho-1} e^{2\pi i \beta u} du.$$
Lemma 4. The relation
\[
\frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(q) \sum_{|\gamma| < T_0} \int_X^{X+Y} u^{\beta - 1} e^{2\pi i \beta u} du \ll \\
\ll \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^A \cdot \log X \cdot \left( \sqrt{qX} + q^{1/2 + 1/\theta} \cdot X^{\theta + 1/\theta} - 1 \cdot Y^{1 - 1/\theta} \right),
\]
holds, provided that \( T_0 > X \log^2 X \).

Proof. Let \( T \) be a parameter we will choose later such that \( 0 \leq T \leq T_0 \). We have
\[
\sum_{|\gamma| < T_0} \int_X^{X+Y} u^{\beta - 1} e^{2\pi i \beta u} du = \\
= \sum_{|\gamma| < T} \int_X^{X+Y} u^{\beta - 1} e^{2\pi i \beta u} du + \sum_{T \leq |\gamma| < T_0} \int_X^{X+Y} u^{\beta - 1} e^{2\pi i \beta u} du \ll \\
\ll T \cdot X^{\theta - 1} \cdot Y + \sum_{T \leq |\gamma| < T_0} \frac{X^{\beta}}{T_0} \ll \\
\ll T \cdot X^{\theta - 1} \cdot Y + \log \left( \frac{T_0}{T} \right) \max_{T \leq U < T_0/2} \frac{1}{U} \sum_{U < \gamma < 2U} X^{\beta} \ll \\
\ll T \cdot X^{\theta - 1} \cdot Y + \log \left( \frac{T_0}{T} \right) \max_{T \leq U < T_0/2} \frac{1}{U} \left( X^{1/2} \cdot N \left( \frac{1}{2}, U, \chi \right) + \log X \int_{1/2}^{\theta} X^\sigma N(\sigma, U, \chi) d\sigma \right).
\]
We recall that the Generalised Density Hypothesis states that, for \( \frac{1}{2} \leq \sigma \leq 1 \), the relation
\[
\sum_{\chi \mod q} \chi N(\sigma, U, \chi) \ll (qU)^{2(1-\sigma)} (\log(qU))^A
\]
holds, where \( A \) is an effective constant.
If we use the Generalized Density Hypothesis and the inequality \( |\tau(\chi)| \leq \sqrt{q} \), we conclude that
\[
\frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi)\chi(a) \sum_{|\gamma| < T_0} \int_X^{X+Y} \alpha_0^{-1}e^{2\pi i \beta u} du \ll \\
\ll \frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi)\chi(a) \cdot TX^{\theta-1}Y + \\
+ \frac{1}{\phi(q)} \sum_{\chi \mod q} \tau(\chi)\chi(a) \cdot \log \left( \frac{T_0}{T} \right) \max_{T \leq U \leq T_{0}/2} \frac{1}{U} \left\{ X^{1/2}N \left( \frac{1}{2}, U, \chi \right) + \log X \int_{1/2}^{\theta} X^\sigma N(\sigma, U, \chi) d\sigma \right\} \ll \\
\ll \sqrt{q} \cdot \phi(q)TX^{\theta-1}Y + \\
+ \frac{1}{\sqrt{q}} \cdot \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^A \max_{T \leq U \leq T_{0}/2} \frac{1}{U} \left\{ X^{1/2}(qU)^{2(1-1/2)} + \log X \int_{1/2}^{\theta} X^\sigma(qU)^{2(1-\sigma)} d\sigma \right\} \ll \\
\ll \sqrt{q} \cdot T \cdot X^{\theta-1}Y + \\
+ \frac{1}{\sqrt{q}} \cdot \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^A \cdot \log X \max_{T \leq U \leq T_{0}/2} \frac{1}{U} \left( X^{1/2}qU + (qU)^2 \int_{1/2}^{\theta} \left( \frac{X}{(qU)^2} \right)^\sigma d\sigma \right) \ll \\
\ll \sqrt{q} \cdot T \cdot X^{\theta-1}Y + \\
+ \frac{1}{\sqrt{q}} \cdot \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^A \cdot \log X \max_{T \leq U \leq T_{0}/2} \frac{1}{U} \left( X^{1/2}qU + \frac{X^\theta}{q^2q-2, U^2q-1} \right) \ll \\
\ll \sqrt{q} \cdot T \cdot X^{\theta-1}Y + \frac{1}{\sqrt{q}} \cdot \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^A \cdot \log X \cdot \left( X^{1/2}qU + \frac{X^\theta}{q^2q-2 \cdot T^2q-1} \right) \ll \\
\ll \sqrt{q} \cdot T \cdot X^{\theta-1}Y + \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^A \cdot \log X \cdot \left( \sqrt{qX} + \frac{X^\theta}{q^2q-1 \cdot T^2q-1} \right) \cdot (7)
\]

Choosing

\[
T = X^{1/2\theta} Y^{1/2\theta} q^{1-1/2\theta}
\]

the estimate (7) becomes

\[
\ll \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^A \cdot \log X \cdot \left( \sqrt{qX} + q^{-\frac{1}{2} + \frac{1}{2\theta}} \cdot X^{\theta} + \frac{1}{2\theta} \cdot Y^{1-\frac{1}{2\theta}} \right).
\]
Finally, if we combine Lemma 2, Lemma 3 and Lemma 4 we conclude that

\[ S(\alpha, X, Y) \ll \sqrt{q} \cdot \log^2(qT_0) + \frac{1}{q} \cdot \log^2 T_0 + \]

\[ + \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^4 \cdot \log X \cdot \left( \sqrt{qX} + q^{\frac{1}{2} + \frac{1}{2\theta}} \cdot X^{\theta + \frac{1}{2\theta} - 1} \cdot Y^{1 - \frac{1}{2\theta}} \right) + \]

\[ + \sqrt{q} \cdot Y \left( \frac{X}{T_0} \right)^{1/2} \cdot \log X \ll \]

\[ \ll \log \left( \frac{T_0}{T} \right) \cdot (\log(qT_0))^4 \cdot \log X \cdot \left( \sqrt{qX} + q^{\frac{1}{2} + \frac{1}{2\theta}} \cdot X^{\theta + \frac{1}{2\theta} - 1} \cdot Y^{1 - \frac{1}{2\theta}} \right) + \]

\[ + \sqrt{q} \cdot Y \left( \frac{X}{T_0} \right)^{1/2} \cdot \log X \ll \]

\[ \ll (\log(qT_0))^{A+1} \cdot \log X \cdot \left( \sqrt{qX} + q^{\frac{1}{2} + \frac{1}{2\theta}} \cdot X^{\theta + \frac{1}{2\theta} - 1} \cdot Y^{1 - \frac{1}{2\theta}} \right) + \sqrt{q} \cdot Y \left( \frac{X}{T_0} \right)^{1/2} \cdot \log X \]

with \( T_0 > X \log^2 X \).

Now, we observe that the exponents \(-\frac{1}{2} + \frac{1}{2\theta}, \theta + \frac{1}{2\theta} - 1\) and \(1 - \frac{1}{2\theta}\) of the variables \(q, X\) and \(Y\) respectively in the last estimate in (8) are always positive in the range \(1/2 \leq \theta < 1\). Hence, the quantity

\[ (\log(qT_0))^{A+1} \cdot \log X \cdot q^{-\frac{1}{2} + \frac{1}{2\theta}} \cdot X^{\theta + \frac{1}{2\theta} - 1} \cdot Y^{1 - \frac{1}{2\theta}} \]

is always greater than the quantity

\[ \sqrt{q} \cdot Y \left( \frac{X}{T_0} \right)^{1/2} \cdot \log X, \]

where there is a negative exponent for the variable \(X\), being \(T_0 > X \log^2 X\).

As a result, if we take

\[ T_0 = X \log^3 X, \]

we can conclude that

\[ S(\alpha, X, Y) \ll (\log(qX \log^3 X))^{A+1} \cdot \log X \cdot \left( \sqrt{qX} + q^{\frac{1}{2} + \frac{1}{2\theta}} \cdot X^{\theta + \frac{1}{2\theta} - 1} \cdot Y^{1 - \frac{1}{2\theta}} \right). \]

\[ \text{On behalf of all authors, the corresponding author states that there is no conflict of interest.} \]

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