Boundary form factors in the Smirnov–Fateev model with a diagonal boundary $S$ matrix

Michael Lashkevich,

Landau Institute for Theoretical Physics,
142432 Chernogolovka of Moscow Region, Russia

Abstract

The boundary conditions with diagonal boundary $S$ matrix and the boundary form factors for the Smirnov–Fateev model on a half line has been considered in the framework of the free field representation. In contrast to the case of the sine-Gordon model, in this case the free field representation is shown to impose severe restrictions on the boundary $S$ matrix, so that a finite number of solutions is only consistent with the free field realization.

1. Introduction

The form factors in quantum field theory provide a natural framework for calculation of large distance asymptotics of correlation functions. In the integrable quantum field theory the form factors can be, in principle, found exactly as solution to a system of linear functional equations, called form factor axioms, as soon as the spectrum and the $S$ matrix of the model has been found [1–3]. This construction admits a generalization to the case of an integrable model with a boundary. In this case the correlation functions are expressed in terms of both bulk and boundary form factors. The boundary form factors can be also found exactly [4, 5].

Here we consider a two-parametric family of integrable models proposed by Smirnov [6] with four charged particles $z_{\varepsilon \varepsilon'}$ ($\varepsilon = \pm, \varepsilon' = \pm$). It means that the space of internal states of a particle is

$$V = \mathbb{C}^2 \otimes \mathbb{C}^2. \quad (1.1)$$

The $S$ matrix of the model is factorizable, and the two-particle $S$ matrix $S_{p_1 p_2}(\theta)$ is given by

$$S_{p_1 p_2}(\theta) = -S_{p_1}(\theta) \otimes S_{p_2}(\theta), \quad (1.2)$$

where each tensor component acts on the tensor square of the corresponding tensor component $\mathbb{C}^2$ of the space $V$. The matrix $S_p(\theta)$ is the two-soliton $S$ matrix of the sine-Gordon model with the coupling constant $\beta_{SG} = 8\pi \frac{p}{p+1}$ [7]:

$$S_p(\theta)_+^+ = -e^{i\delta_p(\theta)}, \quad S_p(\theta)_+^- = -e^{i\delta_p(\theta)} \frac{\text{sh} \frac{\theta}{2}}{\text{sh} \frac{\theta}{p}}, \quad S_p(\theta)_+^- = -e^{i\delta_p(\theta)} \frac{\text{sh} \frac{\theta}{2}}{\text{sh} \frac{\theta}{p}}. \quad (1.3)$$

The Lagrangian description of this model was found by Fateev [8]. Consider three scalar fields $\varphi_i(x)$, $i = 1, 2, 3$ with the action

$$S = \int d^2 x \left( \frac{\left( \partial_\mu \varphi_1 \right)^2 + \left( \partial_\mu \varphi_2 \right)^2 + \left( \partial_\mu \varphi_3 \right)^2}{8\pi} + \frac{\mu}{\pi} \left( \cos(\alpha_1 \varphi_1 + \alpha_2 \varphi_2)e^{\beta \varphi_3} + \cos(\alpha_1 \varphi_1 - \alpha_2 \varphi_2)e^{-\beta \varphi_3} \right) \right) \quad (1.4)$$

with the parameters $\alpha_1, \alpha_2, \beta$ satisfying the integrability condition

$$\alpha_1^2 + \alpha_2^2 - \beta^2 = 1. \quad (1.5)$$
Let us recall the free field representation of the SF model \[9\]. Consider three families of bosonic operators realizations of the SF model and of the sine-Gordon model in more detail later. From that of the sine-Gordon model, where the whole one-parametric family of diagonal solutions to the functions that admit free field representation for the form factors. We see that the situation differs specially in this regime. First, all of them have the tensor product form $\rho_{\alpha_i}(x_1) \otimes \rho_{\alpha_j}(x_2)$, where $\rho_{\alpha_i}$ is a solution of the SF model, and $\rho_{\alpha_j}$ is a solution of the sine-Gordon model. Second, we have a finite number of solutions for these three functions that admit free field representation for the form factors. Hence, we shall think of the subscripts $\alpha_i, \alpha_j, Q_i$ etc. to belong the cyclic group $\mathbb{Z}_3$. Due to this symmetry, there are three types of charged particles $z_{\alpha\beta}$ with the topological charges $Q_i = 0, Q_i+1 = \varepsilon, Q_{i-1} = \varepsilon'$. There is also a set of bound states. In the regime I only one of these three families $z_{\alpha\beta}$ survives.

Here we consider the SF model with a boundary. From the bootstrap point of view we only need a solution $R(\theta): V \rightarrow V$ to the boundary Yang–Baxter equation with the S matrix (1.2):

$$R_2(\theta_2)S_{12}(\theta_1 + \theta_2)R_1(\theta_1)S_{21}(\theta_1 - \theta_2) = S_{12}(\theta_1 - \theta_2)R_1(\theta_1)S_{21}(\theta_1 + \theta_2)R_2(\theta_2),$$

where $R_i(\theta)$ acts on the space of internal states $V_i$ of the $i$th particle, and $S_{ij}(\theta)$ acts on the tensor product $V_i \otimes V_j$. We shall restrict ourselves by the particular case of diagonal boundary S matrices. In fact, we shall see that the free field representation provides solutions for the boundary S matrices of very special form. First, all of them has the tensor product form

$$R(\theta) = \rho(\theta) \begin{pmatrix} 1 & 0 \\ r_1(\theta) & r_2(\theta) \end{pmatrix},$$

with some functions $\rho(\theta), r_1(\theta), r_2(\theta)$. Second, we have a finite number of solutions for these three functions that admit free field representation for the form factors. We see that the situation differs from that of the sine-Gordon model, where the whole one-parametric family of diagonal solutions to the boundary Yang–Baxter equations admit the free field representation \[5\]. We shall compare the free field realizations of the SF model and of the sine-Gordon model in more detail later.

2. Free field representation for bulk asymptotic states

Let us recall the free field representation of the SF model \[9\]. Consider three families of bosonic operators $a_i(t) (i \in \mathbb{Z}_3)$, which depend on the real parameter $t$ and satisfy the commutation relations:

$$[a_i(t), a_j(t')] = t \frac{\sinh^2 \frac{\pi t}{2}}{\sinh \pi t \sinh \frac{\pi t}{2}} \delta(t + t') \delta_{ij}. $$

(2.1)
The 'bare' vertex operators are defined as 
\[ \phi_i(\theta; \nu) = \int_{-\infty}^{\infty} \frac{dt}{i} a_i(t) e^{i\theta t + \nu t/4}, \]  
(2.2a) 
\[ \tilde{\phi}_i(\theta; \nu) = \int_{-\infty}^{\infty} \frac{dt}{i} \text{sh} \frac{\pi t}{2} a_i(t) e^{i\theta t + \nu t/4}, \]  
(2.2b) 
\[ \phi_i^{(\pm)}(\theta; \nu) = 2 \int_{0}^{\infty} \frac{dt}{i} \text{sh} \frac{\pi t}{2} a_i(\pm t) e^{\pm i\theta t + \nu t/4}. \]  
(2.2c) 

and 
\[ \chi_i^{(\pm)}(\theta) = \phi_i^{(\pm)}(\theta; 2 - p_i) + \phi_i^{(\pm)}(\theta; p_{i+1} + 2) - \phi_i^{(\pm)}(\theta; p_{i+1} - p_i). \]  
(2.2d) 

Let us also introduce three central elements ('zero modes') \( \hat{k}_i, i = 1, 2, 3 \). Define the vacuum \( |0\rangle_{k_1, k_2, k_3} \):
\[ a_i(t)|0\rangle_{k_1, k_2, k_3} = 0 \quad \text{for} \ t \geq 0, \quad \hat{k}_i|0\rangle_{k_1, k_2, k_3} = k_i|0\rangle_{k_1, k_2, k_3}. \]  
(2.3) 

The Fock space \( \mathcal{F}_{k_1, k_2, k_3} \) is defined as the space spanned by the vectors 
\[ a_{i_1}(-t_1) \ldots a_{i_n}(-t_n)|0\rangle_{k_1, k_2, k_3}, \quad t_1, \ldots, t_n > 0, \quad n = 0, 1, 2, \ldots. \]  

The definition of normal ordering \( \ldots \) is evident. The conjugate vacuum \( k_{1,2,3} |0\rangle \) is defined as 
\[ k_{1,2,3} |0\rangle_{k_1, k_2, k_3} = 0 \quad \text{for} \ t \geq 0, \quad k_{1,2,3} |0\rangle_{k_1, k_2, k_3} = k_i |0\rangle_{k_1, k_2, k_3} \quad k_{1,2,3} |0\rangle_{k_1, k_2, k_3} = 1. \]  
(2.4) 

The 'bare' vertex operators are defined as 
\[ V_i(\theta) = \exp(i\phi_{i+1}(\theta; p_{i+1}) + i\phi_{i+2}(\theta; -p_{i+2})); \quad (2.5a) \] 
\[ I_i^{(\pm)}(\theta) = \exp(-i\tilde{\phi}_i(\theta; p_i) \pm i\chi_i^{(\pm)}(\theta)); \quad (2.5b) \]

These operators satisfy the following relations:
\[ V_i(\theta')V_j(\theta) = g_{ij}(\theta - \theta'): V_i(\theta)V_j(\theta'); \quad (2.6a) \] 
\[ V_i(\theta')I_j^{(\pm)}(\theta) = w_{ij}^{(\pm)}(\theta - \theta'): V_i(\theta')I_j^{(\pm)}(\theta); \quad (2.6b) \] 
\[ I_j^{(\pm)}(\theta')V_i(\theta) = w_{ij}^{(\mp)}(\theta - \theta'): V_i(\theta)I_j^{(\pm)}(\theta'); \quad (2.6c) \] 
\[ I_j^{(A)}(\theta')I_j^{(B)}(\theta) = g_{ij}^{(AB)}(\theta - \theta'): I_j^{(A)}(\theta')I_j^{(B)}(\theta); \quad (2.6d) \]

The functions \( g_{ij}, w_{ij}^{(\pm)}, g_{ij}^{(AB)} \) can be found in the Appendix A.

The screening operators read
\[ S_i(k, \kappa|\theta) = c_i \int_{C_i} \frac{d\gamma}{2\pi i} (I_i^{(\pm)}(\gamma)e^{\kappa} - iI_i^{(-)}(\gamma)e^{-\kappa}) \frac{\pi e^{-k\gamma}}{\text{sh} \gamma - \theta - i\pi/2}, \]  
(2.7)

with some normalization constants \( c_i \) (see Appendix A). The contour \( C_i \) in this equation goes from \(-\infty\) to \(+\infty\) above the pole at the point \( \theta + i\pi/2 \). As for the poles related to other operators, the contour goes below all poles arising due to the operators standing to the left of the screening operator \( S_i \) and above the poles related to the operators standing to the right of \( S_i \). The screening operators commute
\[ [S_i(k_i, \kappa_i|\theta_1), S_j(k_j, \kappa_j|\theta_2)] = 0, \]  
(2.8)

subject to the condition
\[ \kappa_i = -\frac{i\pi}{4} (p_i k_i + p_{i+1} k_{i+1} - p_{i+2} k_{i+2}) \]  
(2.9)

for all \( i \in \mathbb{Z}_3 \). Let
\[ \tilde{k}_i = -\frac{i\pi}{4} (p_i \hat{k}_i + p_{i+1} \hat{k}_{i+1} - p_{i+2} \hat{k}_{i+2}) \]  
(2.10)
We also need an auxiliary algebra generated by two elements $\rho$ and $\omega$ with the relations
\[ \omega^2 = \rho^2 = 1, \quad \omega \rho = -\rho \omega, \quad \text{Tr} \rho = \text{Tr} \omega = 0. \tag{2.11} \]
The corner Hamiltonian and the vertex operators read\(^1\)
\[ H = \int_0^\infty dt \sum_{i=1}^3 \frac{\sin \pi t}{\pi t} \frac{\sin \pi t}{\pi t} a_i(-t)a_i(t), \tag{2.12a} \]
\[ Z_{++}^i(\theta) = \omega V_i(\theta)e^{(k_i+1+k_i+2)\theta/2}, \tag{2.12b} \]
\[ Z_{+-}^i(\theta) = \omega \rho V_i(\theta)S_{i+1}(k_i+1, k_{i+1}+1)\theta e^{(k_i+1+k_i+2)\theta/2}, \tag{2.12c} \]
\[ Z_{+-}^i(\theta) = -\omega \rho V_i(\theta)S_{i+2}(k_i+2, k_{i+2}+1)|\theta - \frac{2\mu_1+2}{i+i+2})e^{(k_i+1+k_i+2)\theta/2}, \tag{2.12d} \]
\[ Z_{--}^i(\theta) = -\omega V_i(\theta)S_{i+1}(k_i+1, k_{i+1}+1)\theta S_{i+2}(k_i+2, k_{i+2}+2)|\theta - \frac{2\mu_1+2}{i+i+2})e^{(k_i+1+k_i+2)\theta/2}. \tag{2.12e} \]

These operators satisfy the algebra
\[ [H, Z_{e,e'}_i^j(\theta)] = i\frac{d}{d\theta} Z_{e,e'}_i^j(\theta) - \text{i} \Omega_{e,e'}^i Z_{e,e'}_i^j(\theta), \quad \Omega_{e,e'}^i = \frac{\varepsilon k_i + 1 + \varepsilon' k_{i+2}}{2}, \tag{2.13a} \]
\[ Z_{e,e'}_i^j(\theta) = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3} S_{p_i+1}(\theta_1 - \theta_2) \varepsilon_{\varepsilon_1, \varepsilon_2} S_{p_{i+2}}(\theta_1 - \theta_2) \varepsilon_{\varepsilon_1, \varepsilon_2} Z_{e,e'}_i^j(\theta_2) Z_{e,e'}_i^j(\theta_1), \tag{2.13b} \]
\[ Z_{e,e'}_i^j(\theta) = \varepsilon \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3} S_{p_i+1}(\theta_1 - \theta_2) \varepsilon_{\varepsilon_1, \varepsilon_2} S_{p_{i+2}}(\theta_1 - \theta_2) \varepsilon_{\varepsilon_1, \varepsilon_2} Z_{e,e'}_i^j(\theta_2) Z_{e,e'}_i^j(\theta_1). \tag{2.13c} \]

Here $\tilde{S}_p(\theta) = i\hbar \left( \frac{1}{2} + \frac{\pi \mu_1}{4} \right) S_p \theta + \frac{\pi \mu_1}{4} \).

We see that the commutation relation for the operators $Z_{e,e'}(\theta) = Z_{e,e'}^3(\theta)$ contains just the $S$ matrix (1.2). The operators $Z_{e,e'}(\theta)$ describe the elementary particles in the model in the unitary regime I, while the whole set of operator $Z_{e,e'}(\theta), i \in \mathbb{Z}_3$, describes the set of elementary particles in the 'symmetric' regime II.

As it was clarified in [12,13], the products of vertex operators $Z_{e,e'}^1(\theta_1) \ldots Z_{e,e'}^N(\theta_N)$, being operators in the angular quantization scheme, are in the one-to-one correspondence with the $N$-particle eigenstates of the Hamiltonian of the system. The bulk form factors are given in terms of traces of such products.

Introduce a notation
\[ \langle X \rangle_{k_1, k_2, k_3} = \frac{\text{Tr}_{F_{k_1, k_2, k_3}}(e^{-2\pi H} X)}{\text{Tr}_{F_{k_1, k_2, k_3}}(e^{-2\pi H})}. \tag{2.14} \]

For short, we shall use the notations $I = (i, \varepsilon, \varepsilon')$, $I_N = (i_N, \varepsilon_N, \varepsilon'_N)$ etc. Besides, let $\tilde{I} = (i, -\varepsilon, -\varepsilon')$, i. e. the particle $z_{\tilde{I}}$ is the antiparticle to the particle $z_I$. Then the function
\[ F_{k_1, k_2, k_3}(\theta_1, \ldots, \theta_N)_{I_1 \ldots I_N} = \langle Z_{I_N}(\theta_N) \ldots Z_{I_1}(\theta_1) \rangle_{k_1, k_2, k_3} \tag{2.15} \]

satisfy the form factor axioms and we have
\[ \langle \theta'_1, j_1, \ldots, j_{N'}|J_{N'}|\varepsilon^1 \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3} k_{i_1, i_2, i_3} \varepsilon(0) |\theta_1 I_1, \ldots, \theta_N I_N \rangle = e^{i\sum_{j_1} \Omega_{j,j} N_{k_1, k_2, k_3} F_{k_1, k_2, k_3}(\theta_1 - \frac{i\mu_1}{2}, \ldots, \theta_N - \frac{i\mu_1}{2}, \theta'_1 \frac{i\mu_1}{2}, \ldots, \theta'_1 + \frac{i\mu_1}{2})_{I_1 \ldots I_N} j_1 \ldots j_{N'}, \tag{2.16} \]

for $\theta'_1 < \ldots < \theta'_{N'}$, $\theta_1 < \ldots < \theta_N$. Here
\[ \omega \sum_{n=1}^{N} \Omega_{I_n} - \sum_{n=1}^{N'} \Omega_{I_n}. \tag{2.17} \]

The function $N_{k_1, k_2, k_3}$ is the normalization factor found by Baseilhac and Fateev [14].

\(^1\)We could remove all $\tilde{k}_i, \tilde{k}_i$ from these formulas by redefining the fields: $-i\frac{k_i}{2} \theta + \phi_i(\theta; v) \rightarrow \phi_i(\theta; v), -i\frac{k_i}{2} \theta + \tilde{\phi}_i(\theta; v) \rightarrow \tilde{\phi}_i(\theta; v), -i\frac{k_i}{2} + \phi(\pm)(\theta; v) \rightarrow \phi(\pm)(\theta; v)$, but it is instructive and convenient to write them separately from the fields.
We have to put corner transfer matrices and vertex operators in between. Let us introduce the states vector \( | \theta_1 I_1 \ldots \theta_N I_N \rangle \) in the basis of eigenstates of the half line Hamiltonian:

\[
| \theta_1 I_1 \ldots \theta_N I_N \rangle_{b_1} \quad t \quad b_2
\]

on the half line, and the boundary form factors are matrix elements of a boundary operator \( \mathcal{O} \) of the form:

\[
\langle \theta_1 I_1 \ldots \theta_N I_N \rangle_{b_1} \quad t \quad b_2
\]

First a model with a time-like boundary (Fig. 1). In this picture we consider the evolution of the system two ways: with a time-like boundary condition and with a space-like boundary condition [10]. Consider now let us discuss the notion of boundary form factors. A model with a boundary can be formulated in a number of ways, with both time-like and space-like boundary conditions [10]. Consider

Now let us consider the form factors in the angular quantization picture. Let us look again at Fig. 2.

The right half of the \( x \) axis with the boundary condition \( b_1 \) is associated to an angular boundary ket-vector \( | b_1 \rangle_{\mathcal{O}} \). The left half line is described by a bra-vector \( \mathcal{O} | b_2 \rangle \). Both vectors depend on the operator \( \mathcal{O} \).

Consider the functions in the l. h. s. of (3.1), (3.2) as analytic functions of complex rapidities. Then

\[
F_{\mathcal{O}} \left( \frac{\theta_1}{b_1}, \ldots, \frac{\theta_N}{b_1} \right)_{b_2, J_1 \ldots J_N} = b_2 \langle \theta_1 J_1 \ldots \theta_N J_N | \mathcal{O}(0) | \theta_1 I_1 \ldots \theta_N I_N \rangle_{b_1}
\]

for \( \theta'_1 < \ldots < \theta'_N, \theta_1 < \ldots < \theta_N \). The boundary conditions below and above the point \( t = 0 \), denoted as \( b_1, b_2 \) may be different, and the set of admissible operators \( \mathcal{O} \) depends on them.

Consider the functions in the l. h. s. of (3.1), (3.2) as analytic functions of complex rapidities. Then

\[
F_{\mathcal{O} b_2 b_1} \left( \theta_1, \ldots, \theta_N \right)_{I_1 \ldots I_N} = \langle B | \mathcal{O}_{b_2 b_1}(0) | \theta_1 I_1 \ldots \theta_N I_N \rangle.
\]

for \( \theta_1 < \ldots < \theta_N \). Here the boundary conditions \( b_1, b_2 \) are the result of the right action of the operator \( \mathcal{O}_{b_2 b_1}(0) \) to the boundary bra-vector \( | B \rangle \).

Consider the functions in the l. h. s. of (3.1), (3.2) as analytic functions of complex rapidities. Then

\[
F_{\mathcal{O}} \left( \frac{\theta_1}{b_1}, \ldots, \frac{\theta_N}{b_1} \right)_{b_2, J_1 \ldots J_N} = e^{i \frac{\pi}{2} \omega} F_{\mathcal{O} b_2 b_1} \left( \theta_1 - \frac{i \pi}{2}, \ldots, \theta_N - \frac{i \pi}{2}, \theta'_1, \ldots, \theta'_N + \frac{i \pi}{2}, \ldots, \theta'_1 + \frac{i \pi}{2} \right)_{I_1 \ldots I_N - J_N \ldots - J_1}
\]

for \( \theta_1 < \ldots < \theta_N \). Here the boundary conditions \( b_1, b_2 \) may be different, and the set of admissible operators \( \mathcal{O} \) depends on them.

The quantity \( \omega \) is defined according to (2.17) with \( \Omega_I \) being mutual locality indeces related to the bulk operator \( \mathcal{O}_{b_2 b_1} \).

Now let us consider the form factors in the angular quantization picture. Let us look again at Fig. 2. The right half of the \( x \) axis with the boundary condition \( b_1 \) is associated to an angular boundary ket-vector \( | b_1 \rangle_{\mathcal{O}} \). The left half line is described by a bra-vector \( \mathcal{O} | b_2 \rangle \). Both vectors depend on the operator \( \mathcal{O} \).

We have to put corner transfer matrices and vertex operators in between. Let us introduce the states

\[
| b_1 \rangle_{\mathcal{O}} = e^{-\frac{\pi}{2} \mathcal{O}} | b_1 \rangle, \quad \mathcal{O} | b_2 \rangle = \mathcal{O} | b_2 \rangle e^{-\frac{\pi}{2} \mathcal{O}}.
\]

Do not mix these states in the angular quantization picture with the boundary state \( | B \rangle \) on the line. Let

\[
\langle X | \mathcal{O} b_2 b_1 = \frac{\mathcal{O} | b_2 \rangle \langle X | b_1 \rangle}{\sqrt{\mathcal{O} | b_2 \rangle | b_2 \rangle} \sqrt{\mathcal{O} | b_1 \rangle | b_1 \rangle}. \]

Figure 1. The time-like boundary. 
\( b_1, b_2 \) are boundary conditions.

Figure 2. The space-like boundary.

3. Systems with boundary. General description

Now let us discuss the notion of boundary form factors. A model with a boundary can be formulated in two ways: with a time-like boundary condition and with a space-like boundary condition [10]. Consider first a model with a time-like boundary (Fig. 1). In this picture we consider the evolution of the system on the half line, and the boundary form factors are matrix elements of a boundary operator \( \mathcal{O}(t) \) at \( t = 0 \) in the basis of eigenstates of the half line Hamiltonian:

\[
F_{\mathcal{O}} \left( \frac{\theta_1}{b_1}, \ldots, \frac{\theta_N}{b_1} \right)_{b_2, J_1 \ldots J_N} = b_2 \langle \theta_1 J_1 \ldots \theta_N J_N | \mathcal{O}(0) | \theta_1 I_1 \ldots \theta_N I_N \rangle_{b_1}
\]

for \( \theta'_1 < \ldots < \theta'_N, \theta_1 < \ldots < \theta_N \). The boundary conditions below and above the point \( t = 0 \), denoted as \( b_1, b_2 \) may be different, and the set of admissible operators \( \mathcal{O} \) depends on them.

Now consider a model with a space-like boundary (Fig. 2). From the point of view of the functional integral, this picture differs from the first one just by a rotation in the corresponding Euclidean space. Nevertheless, the Hamiltonian description is quite different. We have to consider the evolution of the system on the whole line, but it inevitably ends with a special boundary state \( | B \rangle \). Hence, the form factors are matrix elements of some bulk operator \( \mathcal{O}_{b_2 b_1}(0) \) between the eigenstates of the bulk Hamiltonian and the boundary state:

\[
F_{\mathcal{O} b_2 b_1} \left( \theta_1, \ldots, \theta_N \right)_{I_1 \ldots I_N} = \langle B | \mathcal{O}_{b_2 b_1}(0) | \theta_1 I_1 \ldots \theta_N I_N \rangle.
\]

for \( \theta_1 < \ldots < \theta_N \). Here the boundary conditions \( b_1, b_2 \) are the result of the right action of the operator \( \mathcal{O}_{b_2 b_1}(0) \) to the boundary bra-vector \( | B \rangle \).
The $F^B$ function is given by
\[
F^B_{\mathcal{O}_{b_2b_1}}(\theta_1, \ldots, \theta_N)_{I_1} \ldots I_N = N^B_{\mathcal{O}_{b_2b_1}} \langle Z_{I_N}(\theta_N) \ldots Z_{I_1}(\theta_1) \rangle_{\mathcal{O}_{b_2b_1}}. \tag{3.6}
\]
Here $N^B_{\mathcal{O}_{b_2b_1}}$ is the normalization constant. Similarly,
\[
F^B_{\mathcal{O}} \left( \theta'_1, \ldots, \theta'_{N'} \right)_{b_1, I_1} \ldots I_{N'} = N^B_{\mathcal{O}_{b_2b_1}} \langle e^{zH} Z_{J_1}(\theta'_1) \ldots Z_{J_{N'}}(\theta'_{N'}) e^{-\pi H} Z_{I_N}(\theta_N) \ldots Z_{I_1}(\theta_1) e^{zH} \rangle_{\mathcal{O}_{b_2b_1}}. \tag{3.7}
\]

The states $|0\rangle_{\mathcal{O}}, \langle 0|_{\mathcal{O}}$ satisfy the relations
\[
Z_I(\theta)|b\rangle_{\mathcal{O}} = \sum_j R_b(\theta)^j_I Z_J(-\theta)|b\rangle_{\mathcal{O}},
\langle b|Z_I(-\theta) = \sum_j \langle b|Z_J(\theta) R_b(\theta)^j_J. \tag{3.8}
\]

The boundary $S$ matrix $R_b(\theta)^j_I$ depends on the boundary condition $b$. With given $Z_I(\theta)$ these equations can be used to find the bosonization of the vectors $|b\rangle_{\mathcal{O}}, \langle 0|_{\mathcal{O}}$. Up to now, it is only known how to do it in the case of diagonal boundary $S$ matrix. \(^2\)

4. Free field representation for boundary states

Following the guidelines of [16] let us search the state $|b\rangle_{k_1,k_2,k_3}$ in the form of a coherent state:
\[
|b\rangle_{k_1,k_2,k_3} = e^F|0\rangle_{k_1,k_2,k_3}, \quad F = \sum_{i=1}^3 \int_0^\infty \frac{dt}{t} \left( -\frac{1}{2} K_i(t) a_i^2(-t) + \beta_i(t) a_i(-t) \right) \tag{4.1}
\]
with some functions $K_i(t), \beta_i(t)$. For shorthand, we often omit the subscript $k_1, k_2, k_3$ below. The corresponding bra-vector is defined as
\[
k_1,k_2,k_3 \langle b| = k_1,k_2,k_3 \langle 0| e^{F^*}, \tag{4.2}
\]
where the star means the antiautomorphism
\[
z^* = \bar{z} \quad (z \in \mathbb{C}), \quad \hat{k}_i^* = \hat{k}_i, \quad a_i^*(t) = -a_i(-t), \tag{4.3}
\]
where bar means complex conjugate.

We expect that $Z_{++}^i(\theta)|0_b\rangle = R_i(\theta)^{++}_+ Z_{++}^i(-\theta)|0_b\rangle$. Since $Z_{++}^i$ is an exponent of free fields, the functions $K_i(t)$ must be chosen in such a way that $a_i(t)|b\rangle = (-a_i(-t) + \ldots)|b\rangle$, where dots mean a c-number function of $t$. This fixes $K_i(t)$ uniquely:
\[
K_i(t) = \frac{\text{sh} \pi t \text{sh} \frac{B_i t}{2}}{\text{sh}^2 \frac{B_i t}{2}}. \tag{4.4}
\]

With this definition we have
\[
\exp \left( \int_{-\infty}^\infty \frac{dt}{t} f(t) a_i(t) \right) |b\rangle = g_i[f](\theta) \exp \left( \int_{-\infty}^\infty \frac{dt}{t} f(-t) a_i(t) \right) |b\rangle, \tag{4.5}
\]
where
\[
\log g_i[f](\theta) = \int_0^\infty \frac{dt}{t} \frac{f(t) - f(-t)}{K_i(t)} \left( \beta_i(t) - \frac{f(t) + f(-t)}{2} \right)
= \int_0^\infty \frac{dt}{t} \left( \frac{\beta_i(t)(f(t) - f(-t))}{K_i(t)} - \frac{f^2(t/2) - f^2(-t/2)}{2K_i(t/2)} \right). \tag{4.6}
\]
\(^2\)In the lattice theory the correlation functions and form factors of some models with nondiagonal boundary $R$ and $S$ matrices can be calculated by means of the vertex-face correspondence [15].
if \( f(t) - f(-t) = O(t) \) as \( t \to 0 \).

Roughly speaking, the reflection at the vector \( |b\rangle_{k_1,k_2,k_3} \) is of the form: \( \phi_i(\theta) \to \phi_i(-\theta) + \ldots, \phi_i(\theta) \to \phi_i(-\theta) + \ldots \). Hence

\[
\begin{align*}
V_i(\theta)|b\rangle_{k_1,k_2,k_3} &= \rho_i(\theta)V_i(-\theta)|b\rangle_{k_1,k_2,k_3} \\
I_i^{(+)}(\theta)|b\rangle_{k_1,k_2,k_3} &= \tilde{\rho}_i(\theta)I_i^{(+)}(-\theta)|b\rangle_{k_1,k_2,k_3}.
\end{align*}
\] (4.7a, 4.7b)

In particular,

\[
\log \tilde{\rho}_i(\theta) = \sum_{j=0}^{2} \int_0^\infty \frac{dt}{t} ((A_{ij}(t)\beta_{i+j}(t) + B_{ij}(t))e^{i\theta t} + (C_{ij}(t)\beta_{i+j}(t) + D_{ij}(t))e^{-i\theta t})
\]

with some functions \( A_j(t), \ldots, D_j(t) \) listed in the Appendix. To get reasonable reflection of the screening operators \((2.7)\) we demand

\[
(I_i^{(+)}(\theta)e^{\kappa i} - iI_i^{(-)}(\theta)e^{-\kappa i})|b\rangle_{k_1,k_2,k_3} = (\tilde{\rho}_i(\theta)I_i^{(+)}(-\theta)e^{\kappa i} - i\tilde{\rho}_i^{-1}(-\theta)e^{-\kappa i})|b\rangle_{k_1,k_2,k_3}
\]

to coincide with

\[
i e^{3\kappa i}\tilde{\rho}_i(\theta)I_i^{(+)\theta} - i e^{3\kappa i}\tilde{\rho}_i^{-1}(-\theta)e^{-\kappa i} |b\rangle_{k_1,k_2,k_3}.
\]

Therefore

\[
i e^{3\kappa i}\tilde{\rho}_i(\theta) = -i\tilde{\rho}_i^{-1}(-\theta)e^{-\kappa i}
\]
or

\[
\tilde{\rho}_i(\theta)\tilde{\rho}_i(-\theta) = -e^{-4\kappa i}.
\]

This is only consistent with (4.8), if

\[ e^{4\kappa i} = -1 \]

and

\[
\log \tilde{\rho}_i(\theta) = 2i \int_0^\infty \frac{dt}{t} E_i(t) \sin \theta t
\]

with some function \( E_i(t) \). We have a system of equations for \( \beta_i(t) \):

\[
\sum_{j=0}^{2} (A_{ij}(t)\beta_{i+j}(t) + B_{ij}(t)) = E_i(t),
\]

\[
\sum_{j=0}^{2} (C_{ij}(t)\beta_{i+j}(t) + D_{ij}(t)) = -E_i(t).
\]

(4.10a, 4.10b)

Take the sum of these two equations:

\[
\sum_{j=0}^{2} (A_{ij}(t) + C_{ij}(t))\beta_{i+j}(t) + \sum_{j=0}^{2} (B_{ij}(t) + D_{ij}(t)) = 0.
\]

It is easy to check that this equation is non-degenerate and its only solution reads:

\[
\beta_i(t) = -\mathrm{ch} \frac{\pi p_i t}{4}, \quad E_i(t) = 0, \quad \tilde{\rho}_i(\theta) = 1 \quad (i = 1, 2, 3).
\]

(4.11)

Hence, for the reflection functions \( \rho_i(\theta) \) in (4.7a) we have

\[
\log \rho_i(\theta) = -2i \int_0^\infty \frac{dt}{t} \frac{\mathrm{sh} \frac{\pi t}{2} \mathrm{sh} \frac{3\pi t}{2} \mathrm{sh} \frac{2p_i + 3}{2}}{\mathrm{sh} \frac{\pi p_i t}{2} \mathrm{sh} \frac{2p_i + 3}{2}} \sin \theta t.
\]

(4.12)

Since \( Z_i^{++}(\theta) \sim V_i(\theta) \), it gives the ++ entries for the boundary \( S \) matrices. To get other entries let us consider action of the screening operators on the boundary state: \( S_i(k,\kappa_i)|b\rangle_{k_1,k_2,k_3} \). There are
Let all four cases it reads: \( r - \theta \pm (\theta) = (0 \pm \theta) + i \theta \pm (\theta) \) for brevity. Consider, for example, the first case:

\[
S_i(0, -\pm |\theta|)b = \int \frac{d\gamma}{2\pi} (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \]

\[
= - \int \frac{d\gamma}{2\pi} (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \]

\[
= \int \frac{d\gamma}{2\pi} (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma + \theta + i \pi/2} \]

Taking a half of the sum of the expressions in the first and third line, we obtain the final expression. For all four cases it reads:

\[
S_i(0, -\pm |\theta|)b = ch \frac{\theta + i \pi/2}{p_i} \int \frac{d\gamma}{2\pi} V_3(\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

\[
S_i(0, +\pm |\theta|)b = sh \frac{\theta + i \pi/2}{p_i} \int \frac{d\gamma}{2\pi} V_3(\theta) (I_{i}^{+}(\gamma) + I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

\[
S_i(\pm \frac{1}{p_i}, -\pm |\theta|)b = e^{\mp \theta + i \pi/2} \int \frac{d\gamma}{2\pi} V_3(-\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

Using the commutation relations (A.6) one can find the corresponding reflection equations for the products that enter \( Z_{\theta} \). For example,

\[
V_3(\theta) S_i(0, -\pm |\theta|)b = ch \frac{\theta + i \pi/2}{p_i} \int \frac{d\gamma}{2\pi} V_3(\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

\[
= ch \frac{\theta + i \pi/2}{p_i} \int \frac{d\gamma}{2\pi} V_3(\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

\[
= \rho_3(\theta) ch \frac{\theta + i \pi/2}{p_i} \int \frac{d\gamma}{2\pi} V_3(-\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

\[
= \rho_3(\theta) ch \frac{\theta + i \pi/2}{p_i} \int \frac{d\gamma}{2\pi} V_3(-\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{-1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

Similarly we can treat other products. As a result the relation (3.8) takes the form

\[
Z_{\theta} S_i(0, -\pm |\theta|)b = \rho_3(\theta e^{(i1 + i2)\theta}) \frac{1}{r_{i1}(\theta)} \int \frac{d\gamma}{2\pi} V_3(-\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

\[
= \rho_3(\theta e^{(i1 + i2)\theta}) \frac{1}{r_{i1}(\theta)} \int \frac{d\gamma}{2\pi} V_3(-\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

Let

\[
r_i^\pm(\theta) = \frac{ch \frac{\theta + i \pi/2}{p_i}}{ch \frac{\theta + i \pi/2}{p_i}} \]

\[
r_i^\pm(\theta) = \frac{sh \frac{\theta + i \pi/2}{p_i}}{sh \frac{\theta + i \pi/2}{p_i}} \]

\[
r_i^\pm(\theta) \equiv e^{\pm 2\theta/p_i} \]

Then

\[
R_i(k_1, k_2; \theta) = \rho_3(\theta e^{(i1 + i2)\theta}) \frac{1}{r_{i1}(\theta)} \int \frac{d\gamma}{2\pi} V_3(-\theta) (I_{i}^{+}(\gamma) - I_{i}^{-}(\gamma)) \frac{i^{1/2} \pi c_i}{sh \gamma - \theta - i \pi/2} \frac{\gamma}{p_i} \]

with

\[
r_i^{i-1}(\theta) = \begin{cases} r_i^1(\theta) & \text{for } k_i = 0, \kappa_i = -\frac{i}{4}, \\ r_i^2(\theta) & \text{for } k_i = 0, \kappa_i = +\frac{i}{4}, \\ r_i^2(\theta) & \text{for } k_i = \frac{1}{p_i}, \kappa_i = +\frac{i}{4}, \\ r_i^1(\theta) & \text{for } k_i = \frac{1}{p_i}, \kappa_i = -\frac{i}{4}. \end{cases} \]
The values of parameters \( k_i, \kappa_i \) \( (i = 1, 2, 3) \) must satisfy the relation (2.9). With this restriction we obtain seven types of admissible boundary conditions, which we denote as \( A_i, B_i \) \( (i = 1, 2, 3) \), \( C \):

\[
A_i : |A_i \rangle = |b \rangle_{k_1,k_2,k_3} \text{ with } k_i = p_i^{-1}, \ k_{i \pm 1} = 0,
\]

\[
 r_i^{+1}(\theta) = r_i^{+ -1}(\theta), \quad r_i^{-1}(\theta) = r_i^{- -1}(\theta), \quad r_i^{+0}(\theta) = r_i^{+0}(\theta), \quad r_i^{-0}(\theta) = r_i^{-0}(\theta);
\]

\[
 B_i : |B_i \rangle = |b \rangle_{k_1,k_2,k_3} \text{ with } k_i = p_i^{-1}, \ k_{i \pm 1} = -p_i^{-1},
\]

\[
 r_i^{+1}(\theta) = r_i^{-}(\theta), \quad r_i^{-1}(\theta) = r_i^{+1}(\theta) = r_i^{-1}(\theta);
\]

\[
 C : |C \rangle = |b \rangle_{p_1^{-1},p_2^{-1},p_3^{-1}},
\]

\[
r_i^{+1}(\theta) = r_i^{+1}(\theta) \quad (i = 1, 2, 3).
\]

Generally, these boundary conditions are nonunitary, \( R_i(-\theta)^{\dagger} \neq R_i(-\theta)^{\dagger} \), due to the factors \( r_i^{+}(\theta) \). The only exception is the boundary condition \( A_3 \) in the unitary regime (1.8a). In this case the particles with \( i = 3 \) only survive and their boundary \( S \) matrix is unitary.

Consider now the bra-vectors:

\[
A^*_i : k_{1,2,3} |A^*_i \rangle = k_{1,2,3} |b \rangle \text{ with } k_i = p_i^{-1}, \ k_{i \pm 1} = 0;
\]

\[
 B^*_i : k_{1,2,3} |B^*_i \rangle = k_{1,2,3} |b \rangle \text{ with } k_i = p_i^{-1}, \ k_{i \pm 1} = -p_i^{-1};
\]

\[
 C^* : k_{1,2,3} |C^* \rangle = p_i^{-1},p_2^{-1},p_3^{-1} |b \rangle.
\]

The boundary conditions \( A^*_i, B^*_i, C^* \) may differ from the boundary conditions \( A_i, B_i, C \), and their boundary \( S \) matrices are related with the ‘starless’ boundary \( S \) matrices as

\[
 R_i(-\theta)^{\dagger} = R_i(-\theta)^{\dagger}. \quad (4.20)
\]

We conclude that

\[
 A^*_i \neq A_i, \quad B^*_i = B_i, \quad C^* = C. \quad (4.21)
\]

Since any vectors corresponding to different eigenvalues of the zero mode operators \( \hat{k}_1, \hat{k}_2, \hat{k}_3 \) are orthogonal, the operators \( \mathcal{O} \) corresponding to the form factors obtained in such a way change the boundary condition according to the rule: \( b_i = b_i^* \).

Now let us discuss the problem of the identification of these form factors with the particular operators in the field theory. Since the boundary condition \( A_i \) is not realized in the free field representation, we consider any matrix element

\[
f_B^B(\theta_1, \ldots, \theta_N)_{I_1 \ldots I_N} = \frac{1}{|b\rangle |c^{-} \rangle} \langle b| e^{-\frac{\pi}{2} H} Z_{I_N}(\theta_N) \ldots Z_{I_1}(\theta_1) e^{-\frac{\pi}{2} H} |b\rangle \quad (4.22)
\]

with \( b = A_i, B_i, C \). Let us calculate the values \( q_i \) \( (i = 1, 2, 3) \) of the three topological charges \( Q_i \). Let \( I_k = (\bar{i}_k, \varepsilon_k, \varepsilon'_k) \). Then

\[
 q_j = \sum_{j=1}^{n} \begin{cases} 0, & j = i_k, \\ \varepsilon_k, & j = i_k - 1, \\ \varepsilon'_k, & j = i_k + 1. \end{cases} \quad (4.23)
\]

Consider the set form factors (4.22) with given values \( q_1, q_2, q_3 \) of the topological charges. This set can be identified with the operator

\[
 \mathcal{O}_{q_1,q_2,q_3}(x^0) = (N_{q_1,q_2,q_3})^{-1} e^{i \sum_{i=1}^{N} \hat{\varphi}(x)} \quad (4.24)
\]

where \( \hat{\varphi}(x) \) is the dual fields, \( \partial^\mu \hat{\varphi}(x) = \varepsilon^{\mu
u} \partial_\nu \hat{\varphi}(x) \), and \( N_{q_1,q_2,q_3} \) is some normalization factor.
5. Comparison with the sine-Gordon model

Recall the free field representation for the sine-Gordon model [17, 18, 5, 20]. Let \( a(t) \) be a family of bosonic operators with the commutation relations

\[
[a(t), a(t')] = i \frac{\text{sh} \frac{\pi t}{2} \text{sh} \frac{\pi (t+1)}{2} \delta(t + t')}{\text{sh} \pi t \text{sh} \frac{\pi t}{2}}.
\]  

(5.1)

The parameter \( p \) is just the parameter entering the \( S \) matrix (1.3). Let

\[
\phi(\theta) = \int_{-\infty}^{\infty} dt \frac{i t}{\text{sh} \pi t} a(t) e^{i \theta t},
\]

(5.2)

\[
\bar{\phi}(\theta) = \int_{-\infty}^{\infty} dt \frac{\text{sh} \pi t}{\text{sh} \frac{\pi t}{2}} a(t) e^{i \theta t} = \phi(\theta + i \pi/2) + \phi(\theta - i \pi/2).
\]

(5.3)

Let \( \hat{k} \) be a ‘zero mode’ operator. Let \( |0\rangle_k \) be a bosonic vacuum:

\[
a(t)|0\rangle_k = 0 \quad (t \geq 0), \quad \hat{k}|0\rangle_k = k|0\rangle_k.
\]

(5.4)

Define the ‘bare’ vertex operator and the screening current as

\[
V(\theta) = :e^{i \phi(\theta)}:, \quad I(\theta) = :e^{-i \bar{\phi}(\theta)}:.
\]

(5.5)

(5.6)

The screening operator is defined as

\[
S(k|\theta) = c \int_C \frac{d\gamma}{2\pi} I(\gamma) \frac{\pi e^{-k\gamma}}{\text{sh} \frac{\gamma - \theta - i\pi/2}{p}}.
\]

(5.7)

with the contour \( C \) going from \(-i\infty\) to \(+i\infty\) with a twist so that the point \( \theta + i\pi/2 \) is below it and point \( \theta - i\pi/2 \) is above it. The constant \( c \) is given in Appendix A.

The corner Hamiltonian \( H \) and the vertex operators \( Z_\varepsilon(\theta) \) are given by

\[
H = \int_0^\infty dt \frac{\text{sh} \pi t \text{sh} \frac{\pi pt}{2}}{\text{sh} \frac{\pi t}{2} \text{sh} \frac{\pi (p+1)t}{2}} a(-t)a(t),
\]

(5.8a)

\[
Z_+ (\theta) = V(\theta)e^{k\theta/2}, \quad Z_- (\theta) = V(\theta)S(k|\theta)e^{k\theta/2}.
\]

(5.8b) \quad (5.8c)

Let

\[
\langle X \rangle_k = \frac{\text{Tr}_{\mathcal{F}_k}(e^{-2\pi H} X)}{\text{Tr}_{\mathcal{F}_k}(e^{-2\pi H})}.
\]

Then the bulk form factors of the operator \( e^{\imath \alpha \varphi(x)} \) are given by

\[
F_a(\theta_1, \ldots, \theta_N \varepsilon_1 \ldots \varepsilon_N) = N_a \langle X \rangle_k = N_a \langle Z_{\varepsilon N}(\theta_N) \ldots Z_{\varepsilon 1}(\theta_1) \rangle_{2a/\beta_{SG}}.
\]

(5.9)

Here the normalization constant \( N_a \) is the vacuum expectation value found in \[19\].

Let us again search the boundary states in the form

\[
|b\rangle_k = e^{F}|0\rangle_k, \quad F = \int_0^\infty \frac{dt}{t} \left( -\frac{1}{2}K(t) a^2(-t) + \beta(t)a(-t) \right).
\]

(5.10)

Here

\[
K(t) = \frac{\text{sh} \pi t \text{sh} \frac{\pi pt}{2}}{\text{sh} \frac{\pi t}{2} \text{sh} \frac{\pi (p+1)t}{2}}.
\]

(5.11)
Let

\[ I(\theta)|b_k\rangle = \bar{\rho}(\theta) I(-\theta)|b_k\rangle, \quad \bar{\rho}(\theta) = (-1)^s \frac{\text{ch} \frac{\theta - \lambda}{p}}{\text{ch} \frac{\theta + \lambda}{p}}, \quad s \in \mathbb{Z}. \quad (5.12) \]

The equation for \( \beta(t) \) is simple:

\[ A(t)\beta(t) + B(t) = E(t), \]

where

\[ E(t) = -\frac{\text{sh} \lambda t}{\text{sh} \frac{\pi t}{4}} + s \quad (5.13) \]

and

\[ A(t) = -\frac{\text{sh} \frac{\pi(p+1)t}{2}}{\text{sh} \frac{\pi p t}{4}}, \]

\[ B(t) = \frac{\text{sh} \frac{\pi t}{4} \text{sh} \frac{\pi(p+1)t}{4}}{2 \text{sh} \frac{\pi t}{4} \text{sh} \frac{\pi p t}{4}}. \quad (5.14) \]

Unlike the situation in the SF model, there is just one equation for one function \( \beta(t) \) for whose solution is unique for any function \( E(t) \). This is the consequence of the fact that the screening current here does not contain any nonsymmetric in the parameter \( t \) fields like \( \chi_1(\pm)(\theta) \) for the SF model. For \( E(t) \) given by (5.13) we have

\[ \beta(t) \equiv \beta_{s,\lambda}(t) = \frac{\text{sh} \lambda t - s \cdot \text{sh} \frac{\pi p t}{4}}{\text{sh} \frac{\pi(p+1)t}{4}} + \frac{\text{sh} \frac{\pi t}{4} \text{ch} \frac{\pi p t}{4}}{\text{sh} \frac{\pi t}{4} \text{ch} \frac{\pi(p+1)t}{4}}. \quad (5.15) \]

Denote the corresponding operator \( F \) as \( F_{s,\lambda} \).

Now we should check that

\[ S(\theta)|b_k\rangle = \int \frac{d\gamma}{2\pi} \frac{\chi_{k,s,\lambda}(\theta) \psi_{k,s,\lambda}(\gamma)}{\text{sh} \frac{\gamma - \theta - \text{i} \pi / 2}{p} \text{sh} \frac{\gamma + \theta + \text{i} \pi / 2}{p}} I(\gamma)|b_k\rangle \quad (5.16) \]

with any \( \gamma \)-independent function \( \chi_{k,s,\lambda}(\theta) \) and any \( \theta \)-independent function \( \psi_{k,s,\lambda}(\gamma) \). The function \( \bar{\rho}(\theta) \) of the form (5.12) is consistent with this assumption in three cases:

\[ X_\lambda : |X_\lambda\rangle = e^{F_{1,\lambda}}|0\rangle_0, \quad \lambda \text{ is arbitrary}; \quad (5.17a) \]

\[ Y_\pm : |Y_\pm\rangle = e^{F_{1,\pm}}|0\rangle_{\pm p^{-1}}. \quad (5.17b) \]

The case \( k = 0, s = 0 \pmod{2}, \lambda = 0 \), which also satisfies the condition (5.16), is equivalent to the case \( k = 0, s = 1 \pmod{2}, \lambda = \pm \text{i} \pi / 2 \). Note, that the boundary conditions \( Y_\pm \) are nonunitary, while the boundary condition \( X_\lambda \) is unitary for real values of the parameter \( \lambda \).

Similarly, define the corresponding bra-vectors

\[ X_\lambda : \langle X_\lambda| = 0|e^{F_{1,-\lambda}}\rangle, \quad \lambda \text{ is arbitrary}; \quad (5.18a) \]

\[ Y_\pm : \langle Y_\pm| = \pm p^{-1}|e^{F_{1,0}}\rangle. \quad (5.18b) \]

The reflection property of the operator \( V(\theta) \) reads

\[ V(\theta) = \rho(\theta)V(-\theta), \quad \log \rho_{s,\lambda}(\theta) = 2i \int_0^{\infty} \frac{dt}{t} \left( \frac{\beta_{s,\lambda}(t)}{K(t)} - \frac{1}{2K(t/2)} \right) \sin \theta t. \quad (5.19) \]

The boundary \( S \) matrix have the form

\[ R_{k,s,\lambda}(\theta) = \rho_{s,\lambda}(\theta)e^{k\theta} \left( \frac{1}{r_{k,s,\lambda}(\theta)} \right), \quad r_{k,s,\lambda}(\theta) = \frac{\chi_{k,s,\lambda}(\theta)}{\chi_{k,s,\lambda}(-\theta)}. \quad (5.20) \]

Surely, the function \( r_{s,\lambda}(\theta) \) is only defined for the values of \( s \) and \( \lambda \) defined in (5.17). It reads

\[ X_\lambda : r_{0,1,\lambda}(\theta) = \frac{\text{ch} \frac{\pi/2 - \lambda + \theta}{p}}{\text{ch} \frac{\pi/2 - \lambda - \theta}{p}}; \quad (5.21a) \]
\[
Y_\pm : r_{\pm p^{-1}, 1, 0}(\theta) = e^{\mp 2\beta \theta / p}.
\]

The family \(X_\lambda\) corresponds to the family of Dirichlet boundary condition described in [10] with

\[
\beta_{SG}(t, x = 0) = \frac{2\lambda - \pi}{p + 1}, \quad |\beta_{SG}(t, x = 0)| \leq \pi.
\]

The boundary conditions \(Y_\pm\) from the point of view of the boundary \(S\) matrix correspond to the limits \(\lambda \to \pm i\infty\). Nevertheless, we want to separate them from the family \(X_\lambda\) due to two reasons. First, they do not correspond to any known boundary conditions. Second, the free field representation provides finite and rather explicit expressions for boundary form factors with these boundary conditions. A peculiarity of these expression is a non-zero value of \(k\). As we have seen, in the case of the SF model such kind of boundary conditions appear inevitably.

Identification of the form factors is similar to the SF case. Consider the function

\[
f^{B}_{\phi_{\mu_{1}}(\theta_{1}, \ldots, \theta_{N})} = \frac{(b)e^{-\pi H}Z_{\epsilon_{N}}(\theta_{N})\ldots Z_{\epsilon_{1}}(\theta_{1})e^{-\pi H}|b)}{(b)e^{-\pi H}|b)}, \quad q = \sum_{n=1}^{N} \epsilon_{n}.
\]

In terms of the dual field \(\tilde{\phi}(x), \partial^{\mu}\tilde{\phi}(x) = \epsilon^{\mu\nu}\partial_{\nu}\phi(x)\), the operator \(\mathcal{O}^{q}_{\phi_{\mu}^{b}}\) can be identified as

\[
\mathcal{O}^{q}(x^{0}) = e^{i\frac{q}{\pi} \beta_{SG}\tilde{\phi}(x^{0})}
\]

with the appropriate change of the boundary condition at the point \(x^{0}\).

6. Conclusion

A free field representation for boundary form factors of some boundary fields in the Smirnov–Fateev model with a boundary has been found. This representation is limited to the boundary conditions with a diagonal boundary \(S\) matrix. It turns out that the consistency condition of the free field representation restricts the admissible boundary conditions to a finite number. This contrasts to the situation in the sine-Gordon model, where the admissible (from the point of view of the free field representation) boundary conditions form a one-parameter family. Note that this restriction is not due to the boundary Yang–Baxter equation, which only demands that

\[
r_{i}^{+1}(\theta) = \frac{\cosh i\theta - \theta}{\cosh i\theta + \theta}, \quad r_{i}^{-1}(\theta) = \frac{\sinh i\theta}{\sinh i\theta}, \quad (i = 1, 2, 3)
\]

with some values of the parameters \(x_{1}, x_{2}, x_{3}\). The described free field representation only admits the solutions with either the two of these parameters being equal to \(-\pi/2\) and the third tending to \(-i\infty\) (the \(A_{i}\) and \(A_{i}^{*}\) boundary conditions) or with all three tending to \(\pm i\infty\) (the \(B_{i}\) and \(C\) boundary conditions). It is not clear, if this restriction is physical, or it is a limitation of the free field technique. Probably, a study of consistency of higher quantum conserved currents of the model with the boundary conditions along the guidelines of [21] could shed light on this problem.

Another problem to be solved is identification of the boundary \(S\) matrices for the cases \(A_{i}, B\) with the particular conditions in the Lagrangian form. Note, that it would be interesting to do the same for the solutions denoted above as \(Y_{\pm}\) in the case of the sine-Gordon model. The solution to this problem could be found by studying nonlocal integrals of motion following the guidelines of [22].

Acknowledgments

I am grateful to P. Baseilhac for his hospitality during my stay at the University of Tours and for interesting discussions. My last visit there was supported by the program ENS–Landau. I am also grateful to T. Miwa, M. Jimbo and J. Shiraishi for their hospitality at the Kyoto University and the University of Tokyo. The work was supported, in part, by the Russian Foundation of Basic Research under the grants RFBR 05–01–01007, 05–01–02934 and by the Program of Support for the Leading Scientific Schools under the grant No. 6358.2006.2.
Appendix A. The functions in Eqs. (2.6), (2.7), (5.7)

The functions $g_{ij}(\theta)$ are defined as follows ($i, j$ are understood modulo 3):

$$g_{ii}(\theta) = G^{-1}(p_{i+1}, \theta)G^{-1}(p_{i+2}, \theta), \quad G(p, \theta) = \exp \int_{0}^{\infty} \frac{dt}{t} \frac{\sinh^{2} \frac{\pi \theta}{2}}{\sinh \pi t \sinh \frac{\pi p}{2}} e^{-i\theta t}, \quad (A.1a)$$

$$g_{ij}(\theta) = G_{1}^{-1}(p_{k}, \theta) \quad (i \neq j, \ k \neq i,j), \quad G_1(p, \theta) = \exp \int_{0}^{\infty} \frac{dt}{t} \frac{\sinh^{2} \frac{\pi \theta}{2}}{\sinh \pi t \sinh \frac{\pi p}{2}} e^{-i\theta t}. \quad (A.1b)$$

Here the integrals of the form

$$\int_{0}^{\infty} dt f(t)$$

with $f(t)$ having a pole at $t = 0$ are understood as [11]

$$\int_{C_0} \frac{dt}{2\pi i} f(t) \log(-t)$$

with the contour $C_0$ going from $+\infty + i0$ above the real axis, then around zero, and then below the real axis to $+\infty - i0$.

The functions $w_{ij}^{(\pm)}(\theta)$ can be expressed in terms of the gamma-functions:

$$w_{ii}^{(+)}(\theta) = w_{ii}^{(-)}(\theta) = 1, \quad (A.2a)$$

$$w_{i-1,i}^{(+)}(\theta) = w(p_i, 0|\theta), \quad w_{i-1,i}^{(-)}(\theta) = w(p_i, 1|\theta), \quad (A.2b)$$

$$w_{i+1,i}^{(+)}(\theta) = w_{i+1,i}^{(-)}(\theta) = w(p_i, 1/2|\theta), \quad (A.2c)$$

where

$$w(p, z|\theta) = r_p^{-1} \frac{\Gamma \left( \frac{\alpha + \xi}{\pi p} - \frac{1}{2p} + \frac{z}{2} \right)}{\Gamma \left( \frac{\alpha + \xi}{\pi p} + \frac{1}{2p} + \frac{z}{2} \right)}, \quad r_p = e^{(C_E + \log \pi p)/p} \quad (A.2d)$$

with $C_E$ being the Euler constant. Note, that all these functions have one series of poles at the points $\theta = -i\pi + i\pi pn$ or $\theta = -i\pi + i\pi p(n+1/2)$ ($n = 0, 1, 2, \ldots$) and one series of zeros at the points $\theta = i\pi - i\pi pn$ or $\theta = i\pi - i\pi p(n+1/2)$. The functions $\bar{g}_{ij}^{(AB)}(\theta)$ ($A, B = \pm$) read

$$\bar{g}_{ii}^{(+)}(\theta) = \bar{g}(p_1, 0, 0|\theta), \quad \bar{g}_{ii}^{(-)}(\theta) = \bar{g}_{ii}^{(+)}(\theta) = \frac{i\theta}{\pi p_i}, \quad \bar{g}_{ii}^{(-)}(\theta) = \bar{g}(p_1, 1, 1|\theta), \quad (A.3a)$$

$$\bar{g}_{i,i+1}^{(+)}(\theta) = \bar{g}_{i,i+1}^{(-)}(\theta) = 1, \quad \bar{g}_{i,i+1}^{(-)}(\theta) = \bar{g}_{i,i+1}^{(+)}(\theta) = \frac{\theta - i\pi(p_i + 1/2)}{\theta - i\pi p_i + 1/2}, \quad \bar{g}_{k+1,k+1}^{(AB)}(\theta) = \bar{g}_{k+1,k+1}^{(BA)}(-\theta) \quad (A.3b)$$

with

$$\bar{g}(p, z_1, z_2|\theta) = r_p^2 \frac{\Gamma \left( \frac{\alpha + \xi}{\pi p} + \frac{1}{2p} + z_1 \right)}{\Gamma \left( \frac{\alpha + \xi}{\pi p} - \frac{1}{2p} + z_2 \right)}. \quad (A.3c)$$

The constants $c_i$ are given by

$$c_i = \frac{e^{2(C_E + \log \pi p_i)}/p_i}{\pi^{3/2}} \frac{\Gamma(1 + 1/p_i)}{\Gamma(-1/p_i)} G(p_i, -i\pi). \quad (A.4)$$

The constant in the expression (5.7) for the sine-Gordon model is given by

$$c = \frac{e^{2(C_E + \log \pi p)/p}}{\pi^{2}p_2} \frac{\Gamma(1 + 1/p)}{\Gamma(-1/p)} \exp \int_{0}^{\infty} \frac{dt}{t} \frac{\sinh^{2} \frac{\pi t}{2}}{\sinh \pi t \sinh \frac{\pi p}{2}} e^{-\pi t}. \quad (A.5)$$
We also need the commutation relations, that follow from Eqs. (2.6b–2.6d), (A.2), (A.3b):

\[
V_i(\theta_1) I_{i+1}^{(A)}(\theta_2) = \frac{\sin \frac{\theta_2 - \theta_i - i \pi / 2}{2}}{\sin \frac{\theta_2 - \theta_i + i \pi / 2}{2}} I_{i+1}^{(A)}(\theta_2) V_i(\theta_1),
\]

(A.6a)

\[
V_i(\theta_1) I_{i-1}^{(A)}(\theta_2) = \frac{\cosh \frac{\theta_2 - \theta_i - i \pi / 2}{2}}{\cosh \frac{\theta_2 - \theta_i + i \pi / 2}{2}} I_{i-1}^{(A)}(\theta_2) V_i(\theta_1),
\]

(A.6b)

\[
I_i^{(A)}(\theta_1) I_{i+1}^{(B)}(\theta_2) = I_{i+1}^{(B)}(\theta_2) I_i^{(A)}(\theta_1).
\]

(A.6c)

**Appendix B. The functions in Eq. (4.8)**

The functions \(A_i(t), \ldots, D_i(t)\) are given by

\[
A_0(t) = -2K_i^{-1}(t) e^{-\frac{\pi p_i t}{4}} \cosh \frac{\pi(1 - p_i)t}{2}, \quad B_0(t) = -2K_i^{-1}(t/2) e^{-\frac{\pi p_i t}{4}} \cosh \frac{\pi(1 - p_i)t}{4},
\]

\[
C_0(t) = 2K_i^{-1}(t) e^{-\frac{\pi p_i t}{4}} \cosh \frac{\pi t}{2}, \quad D_0(t) = 2K_i^{-1}(t/2) e^{-\frac{\pi p_i t}{4}} \cosh \frac{\pi t}{4},
\]

\[
A_1(t) = 2K_{i+1}^{-1}(t) e^{-\frac{\pi (p_i + 1 - p_i)t}{4}} \sinh \frac{\pi p_{i+1} t}{2}, \quad B_1(t) = -2K_{i+1}^{-1}(t/2) e^{-\frac{\pi (p_i + 1 - p_i)t}{4}} \sinh \frac{2\pi p_{i+1} t}{4},
\]

\[
C_1(t) = 0, \quad D_1(t) = 0,
\]

\[
A_2(t) = -2K_{i+2}^{-1}(t) e^{-\frac{\pi (p_{i+1} - p_i)t}{4}} \sinh \frac{\pi p_{i+2} t}{2}, \quad B_2(t) = -2K_{i+2}^{-1}(t/2) e^{-\frac{\pi (p_{i+1} - p_i)t}{4}} \sinh \frac{2\pi p_{i+2} t}{4},
\]

\[
C_2(t) = 0, \quad D_2(t) = 0.
\]

**References**

[1] M. Karowski and P. Weisz, *Nucl. Phys.* B139 (1978) 455.

[2] F. A. Smirnov, *J. Phys.* A17 (1984) L873–L878.

[3] F. A. Smirnov, *Form factors in completely integrable models of quantum field theory*, World Scientific, Singapore (1992).

[4] M. Jimbo, R. Kedem, H. Konno, T. Miwa and R. Weston, *Nucl. Phys.* B448 (1995) 429 [arXiv:hep-th/9502060].

[5] B. Y. Hou, K. J. Shi, Y. S. Wang and W. L. Yang, *Int. J. Mod. Phys.* A12 (1997) 1711 [arXiv:hep-th/9905197].

[6] F. A. Smirnov, *Int. J. Mod. Phys.* A9 (1994) 5121–5144 [hep-th/9312039].

[7] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Annals Phys.* 120 (1979) 253.

[8] V. A. Fateev, *Nucl. Phys.* B473 [FS] (1996) 509–538.

[9] V. A. Fateev and M. Lashkevich, *Nucl. Phys.* B696 (2004) 301 [arXiv:hep-th/0402082].

[10] S. Ghoshal and A. B. Zamolodchikov, *Int. J. Mod. Phys.* A9 (1994) 3841 [Erratum-ibid. A9 (1994) 4353] [arXiv:hep-th/9306002].

[11] M. Jimbo, H. Konno, and T. Miwa, Massless XXZ model and degeneration of the elliptic algebra \(A_{q,p}(sl_2)\), arXiv:hep-th/9610079.

[12] S. L. Lukyanov, *Phys. Lett.* B325 (1994) 409 [arXiv:hep-th/9311189].

[13] M. Jimbo and T. Miwa, *Algebraic analysis of solvable lattice models*, CBMS Regional Conference Series in Mathematics 85, AMS (1994).

[14] P. Baseilhac and V. A. Fateev, *Nucl. Phys.* B532 (1998) 567 [hep-th/9906010].

[15] Y. Hara, *Nucl. Phys.* B572 (2000) 574 [arXiv:math-ph/9910046].

[16] M. Jimbo, R. Kedem, T. Kojima, H. Konno, and T. Miwa, *Nucl. Phys.* B441 (1995) 437 [arXiv:hep-th/9411112].

[17] S. L. Lukyanov, *Commun. Math. Phys.* 167 (1995) 183 [arXiv:hep-th/9307196].

[18] S. L. Lukyanov, *Mod. Phys. Lett.* A12 (1997) 2543 [arXiv:hep-th/9703190].
[19] S. L. Lukyanov and A. B. Zamolodchikov, *Nucl. Phys.* B493 (1997) 571 [arXiv:hep-th/9611238].
[20] T. Kojima, *Int. J. Mod. Phys.* A17 (2002) 487 [arXiv:nlin/0101001].
[21] S. Penati and D. Zanon, *Phys. Lett.* B358 (1995) 63 [arXiv:hep-th/9501105].
[22] G. W. Delius and N. J. MacKay, *Commun. Math. Phys.* 233 (2003) 173 [arXiv:hep-th/0112023].