SMOOTHNESS OF DERIVED CATEGORIES OF ALGEBRAS

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ABSTRACT. We prove smoothness in the dg sense of the bounded derived category of finitely generated modules over any finite-dimensional algebra over a perfect field, thereby answering a question of Iyama. More generally, we prove this statement for any algebra over a perfect field that is finite over its center and whose center is finitely generated as an algebra. These results are deduced from a general sufficient criterion for smoothness.

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1. Introduction

Many triangulated categories have dg enhancements (differential graded enhancements). If we consider triangulated categories of algebraic or geometric origin, e.g., derived categories of modules over an algebra or of sheaves on some space, it is natural to ask what properties their dg enhancements have and how these properties do or do not depend on properties of the algebra or the space. The focus of this article is on the smoothness of dg enhancements (see Definition 2.2), where we always work over a field $k$. Since dg enhancements are often essentially unique (see [LO10], [CS18]), we are a bit sloppy in this introduction and just say that a triangulated category is smooth when we mean that a certain natural dg enhancement has this property (cf. Definition 2.3 and Remark 2.5 for the choices used in this article).

For example, a quasi-projective scheme $X$ over a field $k$ is smooth in the sense of algebraic geometry if and only if the category $D_{\text{perf}}(X)$ of perfect complexes on $X$ is smooth in the dg sense (see [Lun10, LS16b]). However, if the field $k$ is perfect, the bounded derived category $D^b(\text{coh}(X))$ of coherent sheaves on $X$ is always smooth, regardless of $X$ being smooth or not (see [Lun10, LS16b]). This example illustrates the phenomenon that different triangulated subcategories of the unbounded derived

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category $D_{	ext{eq}}(X)$ naturally associated to $X$ may or may not detect smoothness of $X$.

If $A$ is a noetherian $k$-algebra (associative and unital, but not necessarily commutative) it is natural and interesting to ask whether the bounded derived category $D^b(\text{mod}(A))$ of finitely generated $A$-modules is smooth. (The category $\text{per}(A)$ of perfect complexes of $A$-modules, for an arbitrary $k$-algebra $A$, is not so interesting: it is smooth if and only if the $A \otimes_k A^{\text{op}}$-module $A$ has finite projective dimension.) If $A$ is commutative and finitely generated, the answer is clear from the above discussion by taking $X = \text{Spec } A$. Hence one may hope that $D^b(\text{mod}(A))$ is always smooth.

In this article, we extend the methods of [Lun10, LS16b] to prove the smoothness of bounded derived categories for some classes of noncommutative algebras.

**Theorem A** (see Theorem 3.7). Let $A$ be a finite-dimensional algebra over a field $k$ such that $\frac{A}{\text{rad}(A)}$ is separable over $k$ (this condition is automatic if $k$ is perfect). Then $D^b(\text{mod}(A))$ is smooth over $k$.

This theorem answers affirmatively a question of Osamu Iyama [Iya14]. In down-to-earth terms it says that the dg endomorphism algebra of a projective resolution of the direct sum of the simple $A$-modules is perfect as a bimodule over itself (see Remark 3.8).

We also prove the following partial generalization of Theorem A.

**Theorem B** (see Theorem 5.1). Let $A$ be an algebra over a perfect field $k$. Assume that $A$ is a finite module over its center $Z(A)$ and that the center $Z(A)$ is a finitely generated $k$-algebra. Then $D^b(\text{mod}(A))$ is smooth over $k$.

In fact, we prove a general result from which both Theorems A and B follow: Given a $k$-algebra $A$ (or, more generally, a $k$-linear category $A$) and a triangulated subcategory $\mathcal{T}$ of $D(A)$, Theorem 2.15 gives a sufficient condition for the smoothness of $\mathcal{T}$. We expect that this theorem could be used for example to prove the smoothness of $D^b(\text{mod}(A))$ for certain noetherian $k$-algebras $A$.

Let us mention two results of independent interest used in the proof of Theorem B. The first result concerns the existence of a classical generator of the bounded derived category of coherent modules over a noncommutative structure sheaf.

**Theorem C** (see Theorem 4.15). Let $X$ be a noetherian J-2 scheme (see Definition 4.9) and $A$ a coherent $\mathcal{O}_X$-algebra (which is assumed to be unital and associative, but not necessarily commutative). Then $D^b(\text{coh}(A))$ has a classical generator.

The J-2 condition in this result is actually very natural by the following interesting recent result [IT18, Prop. 2.8] by Iyengar and Takahashi: A commutative noetherian ring $R$ is J-2 if and only if $D^b(\text{mod}(A))$ has a classical generator for any finite commutative $R$-algebra $A$.

The proof of Theorem C is based on a Verdier localization sequence given by the following theorem and a technical result using Azumaya algebras (see Proposition 4.10).

**Theorem D** (see Proposition 4.4). Let $X$ be a noetherian scheme and $A$ a coherent $\mathcal{O}_X$-algebra. Let $U$ be an open subscheme of $X$ and $Z := X - U$ its closed complement. Then the sequence of triangulated categories $D^b_Z(\text{coh}(A)) \rightarrow D^b(\text{coh}(A)) \rightarrow D^b(\text{coh}(A|_U))$
is a Verdier localization sequence (see Definition 4.2) where the first arrow is the inclusion and the second arrow is restriction to $U$.

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1.2. Conventions. We fix a field $k$. Whenever $k$ is present, all categories and functors are $k$-linear. By a dg category we mean a $k$-linear dg category. Sometimes we write $\otimes$ instead of $\otimes_k$ or $\otimes_{\mathcal{O}_X}$.

Rings and algebras are assumed to be associative and unital, but not necessarily commutative. An algebra over a commutative ring $R$ is a ring $A$ together with a morphism $R \to A$ of rings landing in the center $Z(A)$ of $A$. Similarly, if $(X, \mathcal{O}_X)$ is a ringed space where $\mathcal{O}_X$ is a sheaf of commutative rings, an $\mathcal{O}_X$-algebra is a sheaf $A$ of rings together with a morphism $\mathcal{O}_X \to A$ of sheaves of rings landing in the center of $A$.

By a module we mean a right module. If $R$ is a ring, $\text{Mod}(R)$ denotes the category of $R$-modules and $\text{D}(R)$ its derived category. When we say that a ring is noetherian we mean that it is right noetherian. If $R$ is a noetherian ring, $\text{mod}(R)$ denotes the full subcategory of $\text{Mod}(R)$ of finitely generated $R$-modules, $\text{D(\text{mod}(R))}$ its derived category and $\text{D}^b(\text{mod}(R))$ its subcategory of objects with bounded cohomology. If $R$ is a finite-dimensional algebra over a field $k$, then $\text{mod}(R)$ is just the category of finite-dimensional $R$-modules.

A thick subcategory of a triangulated category $\mathcal{T}$ is a strictly full triangulated subcategory that is closed under taking direct summands in $\mathcal{T}$. Given an object $E$ of $\mathcal{T}$ we write $\text{thick}(E) = \text{thick}_\mathcal{T}(E)$ for the smallest thick subcategory of $\mathcal{T}$ containing $E$. The object $E$ is a classical generator of $\mathcal{T}$ if and only if $\text{thick}(E) = \mathcal{T}$.

2. Smoothness of derived categories of linear categories

This section is written in greater generality than needed in the rest of this article. The main result of this section, Theorem 2.15, concerns categories of modules over $k$-linear categories; we apply this theorem later on only to categories of modules over $k$-algebras.

2.1. Modules over $k$-linear categories. Let $\mathcal{A}$ be a $k$-linear category. The reader may think of a $k$-algebra which is the same thing as a $k$-linear category with precisely one object. In the rest of this article, the results of this section will only be applied in this special case.

A (right) $\mathcal{A}$-module is a functor $\mathcal{A}^{\text{op}} \to \text{Mod}(k)$, where $\text{Mod}(k)$ is the category of $k$-modules. Let $\text{Mod}(\mathcal{A})$ be the (abelian) category of $\mathcal{A}$-modules. Let

$\text{Yo}: \mathcal{A} \to \text{Mod}(\mathcal{A})$,

$A \mapsto \mathcal{A}(-, A)$,
be the Yoneda functor. The objects of the essential image of this functor are the representable \( \mathcal{A} \)-modules. An \( \mathcal{A} \)-module is finitely generated if it is a quotient of a finite coproduct of representable \( \mathcal{A} \)-modules. It is free if it is isomorphic to a coproduct of representable \( \mathcal{A} \)-modules. It follows that a finitely generated free \( \mathcal{A} \)-module is isomorphic to a finite coproduct of representable \( \mathcal{A} \)-modules.

Let \( \mathcal{C}(\mathcal{A}) := \mathcal{C}(\text{Mod}(\mathcal{A})) \) be the dg category of complexes of \( \mathcal{A} \)-modules. Let \( \mathcal{C}(\mathcal{A}) \) be the category with the same objects whose morphisms are the closed degree zero morphisms in \( \mathcal{C}(\mathcal{A}) \). Let \( \mathcal{D}(\mathcal{A}) \) be the derived category of \( \mathcal{A} \)-modules.

Since \( \mathcal{D}(\mathcal{A}) \) has arbitrary coproducts, it is Karoubian, i.e. idempotent complete. Therefore, a strictly full subcategory of \( \mathcal{D}(\mathcal{A}) \) is Karoubian if and only if it is closed under taking direct summands in \( \mathcal{D}(\mathcal{A}) \).

We say that an object of \( \mathcal{D}(\mathcal{A}) \) is pseudo-coherent if it is isomorphic to a bounded above complex of finitely generated free \( \mathcal{A} \)-modules (cf. \([ \text{SGA6}, \text{Exp. I}] \), \([ \text{TT90, Ch. 2}] \), \([ \text{SP18, 064N}] \)). Note that any bounded above complex of finitely generated projective \( \mathcal{A} \)-modules is pseudo-coherent. Let \( \mathcal{D}(\mathcal{A})_{\text{ps-coh}} \) be the full subcategory of \( \mathcal{D}(\mathcal{A}) \) of pseudo-coherent objects. It is a strictly full triangulated subcategory and Karoubian (see \([ \text{SP18, 064V, 064X}] \)).

We write \( \mathcal{D}(\mathcal{A})_{\text{pf}} \) for the full subcategory of \( \mathcal{D}(\mathcal{A}) \) of perfect complexes, i.e. of objects that are isomorphic to bounded complexes of finitely generated projective \( \mathcal{A} \)-modules. It is a strictly full Karoubian triangulated subcategory of \( \mathcal{D}(\mathcal{A}) \).

We write \( \mathcal{D}(\mathcal{A})^{-} \) for the full subcategory of \( \mathcal{D}(\mathcal{A}) \) of objects \( \mathcal{M} \) whose total cohomology \( \bigoplus_{n \in \mathbb{Z}} H^{n}(\mathcal{M}) \) is bounded above. Similarly, we define \( \mathcal{D}(\mathcal{A})^{+} \) and \( \mathcal{D}(\mathcal{A})_{\text{b}} \).

These categories are strictly full Karoubian triangulated subcategories of \( \mathcal{D}(\mathcal{A}) \). We have \( \mathcal{D}(\mathcal{A})_{\text{ps-coh}} \subset \mathcal{D}(\mathcal{A})^{-} \) and \( \mathcal{D}(\mathcal{A})_{\text{pf}} \subset \mathcal{D}(\mathcal{A})_{\text{b}} := \mathcal{D}(\mathcal{A})_{\text{ps-coh}} \cap \mathcal{D}(\mathcal{A})_{\text{b}} \).

\begin{remark}
Any algebra \( \mathcal{A} \) over a field \( k \) may be viewed as a \( k \)-linear category with one object, so all the notions just introduced may be used for \( \mathcal{A} \). For example, \( \mathcal{D}(\mathcal{A}) \) is the derived category of the abelian category \( \text{Mod}(\mathcal{A}) \) of \( \mathcal{A} \)-modules.

If we assume that \( \mathcal{A} \) is a noetherian \( k \)-algebra, then there is a canonical functor \( \mathcal{D}(\text{mod}(\mathcal{A})) \to \mathcal{D}(\text{Mod}(\mathcal{A})) \). This functor is obviously fully faithful on \( \mathcal{D}(\text{b}(\text{mod}(\mathcal{A}))) \), and the essential image of this category under this functor is the full subcategory \( \mathcal{D}(\text{mod}(\mathcal{A}))_{\text{b}} \) of \( \mathcal{D}(\mathcal{A}) \) of objects with bounded finitely generated cohomology. Hence we get an equivalence

\[ \mathcal{D}(\text{b}(\text{mod}(\mathcal{A}))) \xrightarrow{\sim} \mathcal{D}(\text{mod}(\mathcal{A}))_{\text{b}} \]

of \( k \)-linear triangulated categories; note also that

\[ \mathcal{D}(\text{mod}(\mathcal{A}))_{\text{b}} = \mathcal{D}(\text{b}(\text{mod}(\mathcal{A}))_{\text{ps-coh}}. \]
\end{remark}

2.2. \textbf{DG categories and smoothness.} Given a dg category \( \mathcal{E} \), we denote its homotopy category by \( [\mathcal{E}] \). The derived category of dg \( \mathcal{E} \)-modules is denoted by \( \mathcal{D}(\mathcal{E}) \). To avoid misunderstandings, we emphasize that the objects of \( \mathcal{D}(\mathcal{E}) \) are dg \( \mathcal{E} \)-modules. The full subcategory of \( \mathcal{D}(\mathcal{E}) \) of compact objects coincides with \( \text{thick}(\mathcal{E}) \) and is denoted \( \text{per}(\mathcal{E}) \). We remind the reader of the following definition.

\begin{definition}
A dg category \( \mathcal{E} \) is \textbf{smooth over} \( k \) if \( \mathcal{E} = \mathcal{E} \mathcal{E} \in \text{per}(\mathcal{E} \otimes_{k} \mathcal{E}^{\text{op}}) \).
\end{definition}

Since a dg algebra is a dg category with precisely one object we can also speak about \( k \)-smoothness of dg algebras.
2.3. **Smoothness for triangulated categories of modules.** We go back to the setting in section 2.1 and assume that \( \mathcal{A} \) is a \( k \)-linear category. Let \( \mathrm{hProj}(\mathcal{A}) \) resp. \( \mathrm{hInj}(\mathcal{A}) \) be the full dg subcategory of \( \mathcal{C}(\mathcal{A}) \) of h-projective resp. h-injective objects. They are dg enhancements of \( D(\mathcal{A}) \).

If \( T \subset D(\mathcal{A}) \) is a strictly full triangulated subcategory we write \( \mathrm{hProj}^{-}(\mathcal{A}) \) for the full dg subcategory of \( \mathrm{hProj}(\mathcal{A}) \) whose objects are in \( T \). This is a dg enhancement of \( T \). Similarly, we define the quasi-equivalent dg enhancement \( \mathrm{hInj}^{-}(\mathcal{A}) \) of \( T \).

**Definition 2.3.** A strictly full triangulated subcategory \( T \subset D(\mathcal{A}) \) is **smooth** over \( k \) if \( \mathrm{hProj}^{-}(\mathcal{A}) \) is a smooth dg \( k \)-category.

**Remark 2.4.** Let \( T \) be a strictly full triangulated subcategory of \( D(\mathcal{A}) \). Assume that \( P \) is an h-projective classical generator of \( T \). Then \( T \) is smooth over \( k \) if and only if the endomorphism dg algebra \( \underline{\mathcal{A}}(P, P) \) is smooth over \( k \) (see [LS16a, Prop. 2.18]).

**Remark 2.5.** If \( A \) is a noetherian algebra over a field \( k \) we also want to speak about smoothness of \( D^{b}(\text{mod}(\mathcal{A})) \). We say that \( D^{b}(\text{mod}(\mathcal{A})) \) is smooth over \( k \) if the equivalent category \( D^{b}_{\text{mod}}(\mathcal{A}) \) is smooth over \( k \) in the sense of the above definition (cf. equivalence (2.1)). An equivalent condition is that the standard projective dg enhancement of \( D^{b}(\text{mod}(\mathcal{A})) \) by bounded above complexes of finitely generated projective \( A \)-modules with bounded cohomology is a smooth dg \( k \)-category. Equivalently, we could use the standard injective dg enhancement by bounded below complexes of injective \( A \)-modules with bounded finitely generated cohomology modules.

**Remark 2.6.** If a strictly full triangulated subcategory \( T \subset D(\mathcal{A}) \) is smooth over \( k \) then \( T \) has a strong generator. This follows from [Lun10, Lemma 3.5, Lemma 3.6.(a)] and the fact that any smooth dg \( k \)-category has a classical generator; we do not prove the last statement here. For the categories \( D^{b}(\text{mod}(\mathcal{A})) \) appearing in Theorems 3.7 and 5.1 the existence of a classical generator is established in order to prove smoothness (see Lemma 3.1 and Theorem 4.15).

2.4. **Dualizing objects.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( k \)-linear categories. Consider the dg functor

\[
\otimes: \mathcal{C}(\mathcal{A}) \otimes \mathcal{C}(\mathcal{B}) \to \mathcal{C}(\mathcal{A} \otimes \mathcal{B}),
\]

\[(M, N) \mapsto M \otimes N,
\]

defined by \( (M \otimes N)(A, B) = M(A) \otimes N(B) \) on objects \((A, B) \in \mathcal{A} \otimes \mathcal{B}\). Define the dg functor

\[
\mathrm{Hom}_{\mathcal{B}}(-, -): \mathcal{C}(\mathcal{B}) \otimes \mathcal{C}(\mathcal{A} \otimes \mathcal{B}) \to \mathcal{C}(\mathcal{A}),
\]

\[(N, X) \mapsto \mathrm{Hom}_{\mathcal{B}}(N, X),
\]

in the obvious way such that

\[
\underline{\mathcal{A}}_{\mathcal{B}}(M \otimes N, X) \cong \underline{\mathcal{A}}(M, \mathrm{Hom}_{\mathcal{B}}(N, X))
\]

natural in \( M, N \) and \( X \) as above.
If we identify $C(A \otimes B) \cong C(B \otimes A)$ in the obvious way we obtain isomorphisms

$$C_A(M, \text{Hom}_B(N, X)) \cong C_{A \otimes B}(M \otimes N, X)$$

$$\cong C_{B \otimes A}(N \otimes M, X)$$

$$\cong C_B(N, \text{Hom}_A(M, X))$$

$$= C_{B^\text{op}}(\text{Hom}_A(M, X), N).$$

Hence we obtain an adjunction

$$\text{Hom}_A(-, X): C(A) \rightleftarrows C(B)^{\text{op}}: \text{Hom}_B(-, X)$$

of dg functors for each $X \in C(A \otimes B)$. The unit and counit of this adjunction are the obvious maps “into the bidual with respect to $X$”: the unit $M \to \text{Hom}_B(\text{Hom}_A(M, X), X)$ is the map sending $m \in M$ to the evaluation map sending $\mu \in \text{Hom}_A(M, X)$ to $\mu(m)$, and the counit has essentially the same description.

On the level of derived categories we obtain an adjunction

$$(2.4) \quad R\text{Hom}_A(-, X): D(A) \rightleftarrows D(B)^{\text{op}}: R\text{Hom}_B(-, X).$$

Its unit is the obvious natural transformation

$$\eta = \eta^X: \text{id} \to R\text{Hom}_B(R\text{Hom}_A(-, X), X)$$

between endofunctors of $D(A)$ and its counit is the obvious natural transformation

$$\varepsilon = \varepsilon^X: \text{id} \to R\text{Hom}_A(R\text{Hom}_B(-, X), X)$$

between endofunctors of $D(B)$ (strictly speaking, the counit is the transformation of endofunctors of $D(B)^{\text{op}}$ obtained by reversing the arrow).

**Lemma 2.7.** Let $A$, $B$, $X$, $\eta^X$, $\varepsilon^X$ be as above. Consider the following two full subcategories of $D(A)$ and $D(B)$ defined by

$$(2.5) \quad D(A)^X := \{M \in D(A) \mid \eta^X_M \text{ is an isomorphism}\},$$

$$D(B)^X := \{N \in D(B) \mid \varepsilon^X_N \text{ is an isomorphism}\}.$$ 

Then these two subcategories are thick, the adjunction $(2.4)$ restricts to an adjoint equivalence

$$R\text{Hom}_A(-, X): D(A)^X \rightleftarrows (D(B)^X)^{\text{op}}: R\text{Hom}_B(-, X),$$

and they form the biggest pair of subcategories on which $(2.4)$ restricts to an adjoint equivalence.

**Proof.** The subcategories are clearly thick. The other claims are a special case of the following categorical Lemma 2.8. \qed

**Lemma 2.8.** Let $(L, R, \eta, \varepsilon): C \to D$ be an adjunction of categories, given by functors $L: C \rightleftarrows D: R$ and unit $\eta: \text{id} \to RL$ and counit $\varepsilon: LR \to \text{id}$. Let $C'$ be the full subcategory of $C$ of objects of $C$ such that $\eta:C \to RLC$ is an isomorphism. Let $D'$ be the full subcategory of $D$ of objects of $D$ such that $\varepsilon:D \to LRD$ is an isomorphism. Then our adjunction restricts to an adjoint equivalence $L: C' \rightleftarrows D': R$. Moreover, if $C''$ and $D''$ are full subcategories of $C$ and $D$ such that our adjunction restricts to an adjoint equivalence $L: C'' \rightleftarrows D'': R$, then $C'' \subset C'$ and $D'' \subset D'$. 

Proof. For arbitrary objects \( C \in \mathcal{C} \) and \( D \in \mathcal{D} \) there are commutative diagrams

\[
\begin{array}{ccc}
LC & \xrightarrow{\nu_C} & LRLC \\
\downarrow{id_{LC}} & & \downarrow{\epsilon_{LC}} \\
LC & \xrightarrow{\eta_C} & RD \\
\end{array}
\]

For \( C \in \mathcal{C}' \) the first diagram implies \( LC \in \mathcal{D}' \). For \( D \in \mathcal{D}' \) the second diagram implies \( RD \in \mathcal{C}' \). This implies that \( L \) and \( R \) restrict to the subcategories \( \mathcal{C}' \) and \( \mathcal{D}' \), and these restrictions clearly form an adjoint equivalence. The last claim is obvious since \( \eta \) must be an isomorphism on all objects of \( \mathcal{C}'' \) and \( \epsilon \) must be an isomorphism on all objects of \( \mathcal{D}'' \).

\[ \Box \]

In the following Definition 2.9 we use the above construction in the special case that \( \mathcal{B} = \mathcal{A}^{\text{op}} \).

Definition 2.9. Let \( \mathcal{A} \) be a \( k \)-linear category and \( \mathcal{T} \subset \mathcal{D}(\mathcal{A}) \) a strictly full triangulated subcategory. A dualizing object (or dualizing bimodule or dualizing complex of bimodules) for \( \mathcal{T} \) is a complex \( \mathcal{D} \) of \( \mathcal{A} \otimes \mathcal{A}^{\text{op}} \)-modules such that every object of \( \mathcal{T} \) is contained in the category \( \mathcal{D}(\mathcal{A})^{\mathcal{D}} \) defined in (2.5): This just means that the unit

\[ T \xrightarrow{\sim} R\text{Hom}_{\mathcal{A}^{\text{op}}}(R\text{Hom}_{\mathcal{A}}(T, \mathcal{D}), \mathcal{D}) \]

is an isomorphism in \( \mathcal{D}(\mathcal{A}) \) for all objects \( T \in \mathcal{T} \). Given a dualizing object \( \mathcal{D} \) for \( \mathcal{T} \) we denote the essential image of \( T \) under the functor

\[ R\text{Hom}_{\mathcal{A}}(-, \mathcal{D}) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}^{\text{op}}) \]

by \( \mathcal{T}^\vee = \mathcal{T}^{\vee,\mathcal{D}} \) and call it the dual of \( \mathcal{T} \) (with respect to \( \mathcal{D} \)). Here we view \( \mathcal{T}^\vee \) as a full subcategory of \( \mathcal{D}(\mathcal{A}^{\text{op}}) \) (and not of its opposite category).

Remark 2.10. If, in the setting of Definition 2.9, \( \mathcal{T} \) is classically generated by an object \( E \), then \( \mathcal{D} \) is a dualizing object for \( \mathcal{T} \) if and only if

\[ E \to R\text{Hom}_{\mathcal{A}^{\text{op}}}(R\text{Hom}_{\mathcal{A}}(E, \mathcal{D}), \mathcal{D}) \]

is an isomorphism. This follows immediately from Lemma 2.7 because \( \mathcal{D}(\mathcal{A})^{\mathcal{D}} \) is a thick subcategory of \( \mathcal{D}(\mathcal{A}) \).

Remark 2.11. Let \( \mathcal{A} \) be a \( k \)-linear category and \( \mathcal{D} \) a dualizing object for a strictly full triangulated subcategory \( \mathcal{T} \subset \mathcal{D}(\mathcal{A}) \) with dual \( \mathcal{T}^\vee \). Then, by Lemma 2.7, the adjunction (2.4) (for \( X = \mathcal{D} \) and \( \mathcal{B} = \mathcal{A}^{\text{op}} \)) restricts to an adjoint equivalence

\[ R\text{Hom}_{\mathcal{A}}(-, \mathcal{D}) : \mathcal{T} \xrightarrow{\sim} (\mathcal{T}^\vee)^{\text{op}} : R\text{Hom}_{\mathcal{A}^{\text{op}}}(-, \mathcal{D}) \]

Remark 2.12. Let \( \mathcal{A} \) be a \( k \)-linear category and let \( \mathcal{T} \subset \mathcal{D}(\mathcal{A}) \) and \( \mathcal{S} \subset \mathcal{D}(\mathcal{A}^{\text{op}}) \) be strictly full triangulated subcategories. Let \( \mathcal{D} \) be a complex of \( \mathcal{A} \otimes \mathcal{A}^{\text{op}} \)-modules such that the adjunction (2.4) (for \( X = \mathcal{D} \) and \( \mathcal{B} = \mathcal{A}^{\text{op}} \)) restricts to an adjoint equivalence

\[ R\text{Hom}_{\mathcal{A}}(-, \mathcal{D}) : \mathcal{T} \xrightarrow{\sim} \mathcal{S}^{\text{op}} : R\text{Hom}_{\mathcal{A}^{\text{op}}}(-, \mathcal{D}) \]

Then \( \mathcal{D} \) is a dualizing object for \( \mathcal{T} \) and \( \mathcal{S} \) is the dual of \( \mathcal{T} \), i.e. \( \mathcal{S} = \mathcal{T}^\vee \). This follows immediately from Lemma 2.7.
Lemma 2.13. Let $\mathcal{A}$ and $\mathcal{B}$ be $k$-linear categories. Let $X \in C(\mathcal{A} \otimes \mathcal{B})$ be an h-injective complex. Then the functor $\Hom_{\mathcal{B}}(-,X) : C(\mathcal{B}) \to C(\mathcal{A})$ lands in the subcategory of h-injective complexes and preserves acyclic complexes.

Proof. Let $N \in C(\mathcal{B})$ be any object. We first show that $\Hom_{\mathcal{B}}(N,X)$ is h-injective. Given any acyclic object $M \in C(\mathcal{A})$ we need to see that

$$C_{\mathcal{A}}(M, \Hom_{\mathcal{B}}(N,X)) \overset{(2.3)}{=} C_{\mathcal{A} \otimes \mathcal{B}}(M \otimes N, X)$$

is acyclic. Since $X$ is h-injective it suffices to show that $M \otimes N$ is acyclic. But this is true since $k$ is a field.

Now assume that $N$ is acyclic. We show that $\Hom_{\mathcal{B}}(N,X)$ is acyclic. Given an arbitrary object $A \in \mathcal{A}$ we need to see that

$$\Hom_{\mathcal{B}}(N,X)(A) = C_{\mathcal{A}}(\text{Yo}(A), \Hom_{\mathcal{B}}(N,X)) \overset{(2.3)}{=} C_{\mathcal{A} \otimes \mathcal{B}}(\text{Yo}(A) \otimes N, X)$$

is acyclic. Since $X$ is h-injective it suffices to see that $\text{Yo}(A) \otimes N$ is acyclic. But this is true since $N$ is acyclic and $k$ is a field. □

2.5. Tensor product and dg homomorphisms. We prove a key technical result which uses the notion of pseudo-coherence.

Proposition 2.14. Let $\mathcal{A}$ and $\mathcal{B}$ be $k$-linear categories. Let $M \in C(\mathcal{A})$ and $N \in C(\mathcal{B})$. Let $I \in C(\mathcal{A})$ and $J \in C(\mathcal{B})$ be h-injective objects and let $\kappa : I \otimes J \to K$ be a quasi-isomorphism to an h-injective object $K \in C(\mathcal{A} \otimes \mathcal{B})$. Assume that one of the following two conditions is satisfied.

(a) $M \in D(\mathcal{A})_{\text{pf}}$ and $N \in D(\mathcal{B})_{\text{pf}}$;
(b) $M \in D(\mathcal{A})_{\text{ps-coh}}$ and $N \in D(\mathcal{B})_{\text{ps-coh}}$ and $I \in D^{+}(\mathcal{A})$ and $J \in D^{+}(\mathcal{B})$.

Then the composition

$$(2.6) \quad C_{\mathcal{A}}(M, I) \otimes C_{\mathcal{B}}(N, J) \xrightarrow{\otimes} C_{\mathcal{A} \otimes \mathcal{B}}(M \otimes N, I \otimes J) \xrightarrow{\kappa \cdot} C_{\mathcal{A} \otimes \mathcal{B}}(M \otimes N, K)$$

in $C(k)$ is a quasi-isomorphism.

Proof. Note that $C_{\mathcal{C}}(-, L)$ preserves quasi-isomorphisms if $L \in C(\mathcal{C})$ is an h-injective complex of modules over a k-linear category $\mathcal{C}$, that $(I' \otimes -) : C(\mathcal{B}) \to C(\mathcal{A} \otimes \mathcal{B})$ and $(- \otimes J') : C(\mathcal{A}) \to C(\mathcal{A} \otimes \mathcal{B})$ preserve quasi-isomorphisms, for $I' \in C(\mathcal{A})$ and $J' \in C(\mathcal{B})$, and that tensoring over $k$ preserves quasi-isomorphisms, since $k$ is a field. Hence we can replace $M$ and $N$ by isomorphic objects in $D(\mathcal{A})$ and $D(\mathcal{B})$, respectively.

If (a) holds, $M$ and $N$ can be assumed to be bounded complexes of finitely generated projective modules. Using brutal truncation, shifts, and the fact that any finitely generated projective module is a direct summand of a finitely generated free module we reduce to the case that $M = \text{Yo}(A) = \mathcal{A}(-, A)$ for some $A \in \mathcal{A}$ and $N = \text{Yo}(B) = \mathcal{B}(-, B)$ for some $B \in \mathcal{B}$. But in this case (2.6) is isomorphic to the quasi-isomorphism $I(A) \otimes J(B) \to I(A) \otimes J(B) \to K(A, B)$.

Now assume that (b) holds. Observe that $I$ and $J$ are homotopy equivalent to bounded below complexes of injective modules. Using this it is easy to see that we can assume that $I$, $J$ and $K$ are bounded below complexes of injective modules. We can also assume that $M$ and $N$ are bounded above complexes of finitely generated free modules since $M$ and $N$ are pseudocoherent.
When checking that the composition in (2.6) induces an isomorphism on the \( n \)-th cohomology, for a fixed \( n \in \mathbb{Z} \), only finitely many components of \( M \) and \( N \) matter. Hence we can assume that \( M \) and \( N \) are bounded complexes of finitely generated free modules. Then \( M \) and \( N \) are perfect and we can use (a). \( \square \)

2.6. A sufficient condition for smoothness. We now state the main theorem of this article.

**Theorem 2.15.** Let \( \mathcal{A} \) be a \( k \)-linear category where \( k \) is a field. Then a strictly full triangulated subcategory \( \mathcal{T} \subset \text{D}(\mathcal{A}) \) is smooth over \( k \) if there is a dualizing object \( \mathcal{D} \in \text{D}(\mathcal{A} \otimes \mathcal{A}^{\text{op}}) \) for \( \mathcal{T} \) such that the following two conditions are satisfied:

\( (A1) \) \( \mathcal{T} \) is classically generated by an object of \( \text{D}^b(\mathcal{A})_{\text{ps-coh}} \) whose image under the equivalence

\[
\text{RHom}_{\mathcal{A}}(-, \mathcal{D}) : \mathcal{T} \xrightarrow{\sim} (\mathcal{T}^\vee)^{\text{op}}
\]

from Remark 2.11 is in \( \text{D}^b(\mathcal{A}^{\text{op}})_{\text{ps-coh}} \); 

\( (A2) \) \( \mathcal{D} \) is contained in the thick subcategory of \( \text{D}(\mathcal{A} \otimes \mathcal{A}^{\text{op}}) \) generated by the essential image of the functor

\[
\otimes : \mathcal{T} \times \mathcal{T}^\vee \rightarrow \text{D}(\mathcal{A} \otimes \mathcal{A}^{\text{op}}).
\]

**Remark 2.16.** Condition \( (A1) \) clearly implies \( \mathcal{T} \subset \text{D}^b(\mathcal{A})_{\text{ps-coh}} \) and \( \mathcal{T}^\vee \subset \text{D}^b(\mathcal{A}^{\text{op}})_{\text{ps-coh}} \) because the image of a classical generator of \( \mathcal{T} \) under the equivalence is a classical generator of \( \mathcal{T}^\vee \).

If \( E \) is a classical generator of \( \mathcal{T} \) and \( F \) is a classical generator of \( \mathcal{T}^\vee \), then condition \( (A2) \) is clearly equivalent to:

\( (A2)' \) \( \mathcal{D} \) is contained in the thick subcategory of \( \text{D}(\mathcal{A} \otimes \mathcal{A}^{\text{op}}) \) generated by \( E \otimes F \).

**Proof.** We first reduce to the case that \( \mathcal{T} \) is Karoubian. Obviously, \( \mathcal{D} \) is also a dualizing object for \( \text{thick}(\mathcal{T}) = \text{thick}_{\text{D}(\mathcal{A})}(\mathcal{T}) \), and the assumptions \( (A1) \) and \( (A2) \) are also satisfied by the pair \( (\text{thick}(\mathcal{T}), \mathcal{D}) \). Moreover, a classical generator of \( \mathcal{T} \) is certainly a classical generator of \( \text{thick}(\mathcal{T}) \), so \( \mathcal{T} \) is smooth over \( k \) if and only if \( \text{thick}(\mathcal{T}) \) is smooth over \( k \), by Remark 2.4.

Hence, by replacing \( \mathcal{T} \) by \( \text{thick}(\mathcal{T}) \) we can and will assume in the following that \( \mathcal{T} \) is Karoubian. Without loss of generality we assume that \( \mathcal{D} \) is an h-injective complex of \( \mathcal{A} \otimes \mathcal{A}^{\text{op}} \)-modules.

Consider the adjunction

\[
(-)^\vee := \text{Hom}_{\mathcal{A}}(-, \mathcal{D}) : \mathcal{C}(\mathcal{A}) \rightleftharpoons \mathcal{C}(\mathcal{A}^{\text{op}})^{\text{op}} : (-)^\vee := \text{Hom}_{\mathcal{A}^{\text{op}}}(-, \mathcal{D})
\]

of dg functors. Both functors preserve quasi-isomorphisms, by Lemma 2.13, and therefore our adjunction descends straightforwardly to an adjunction

\[
(-)^\vee = \text{Hom}_{\mathcal{A}}(-, \mathcal{D}) : \text{D}(\mathcal{A}) \rightleftharpoons \text{D}(\mathcal{A}^{\text{op}})^{\text{op}} : (-)^\vee = \text{Hom}_{\mathcal{A}^{\text{op}}}(-, \mathcal{D})
\]

of triangulated functors. This is (up to unique isomorphism) the adjunction (2.4) (for \( \mathcal{B} = \mathcal{A}^{\text{op}} \) and \( X = \mathcal{D} \) there).

Since \( \mathcal{D} \) is a dualizing object for \( \mathcal{T} \), this adjunction restricts by Remark 2.11 to an adjoint equivalence

\[
(-)^\vee = \text{Hom}_{\mathcal{A}}(-, \mathcal{D}) : \mathcal{T} \rightleftharpoons (\mathcal{T}^\vee)^{\text{op}} : (-)^\vee = \text{Hom}_{\mathcal{A}^{\text{op}}}(-, \mathcal{D}).
\]

**Claim:** For any h-projective object \( Q \in \mathcal{T} \) and any \( R \in \mathcal{T} \) the morphism

\[
\mathcal{C}(Q, R) \xrightarrow{(-)^\vee} \mathcal{C}^{\text{op}}_{\mathcal{A}^{\text{op}}}(Q^\vee, R^\vee) = \mathcal{C}_{\mathcal{A}^{\text{op}}}(R^\vee, Q^\vee)
\]
is a quasi-isomorphism.

Indeed, the induced map on the $n$-th cohomology is given by

$$[\mathcal{C}_A](Q, [n]R) \xrightarrow{(-)^\vee} [\mathcal{C}_A^{[n]}](([n]R)^\vee, Q^\vee).$$

Since $Q$ is h-projective and $Q^\vee = \text{Hom}_A(Q, \mathcal{O})$ is h-injective, by Lemma 2.13, this map is identified with the map

$$D_A([n]R) \xrightarrow{(-)^\vee} D_A^{[n]}(([n]R)^\vee, Q^\vee)$$

which is an isomorphism by the equivalence (2.7). This proves the claim.

Let $P \in D^b(A)_{\text{ps-coh}}$ be a classical generator of $T$ such that $P^\vee \in D^b(A^{\text{op}})_{\text{ps-coh}}$; such an object exists by assumption (A1). Additionally, we can assume that $P$ is h-projective. By Remark 2.4 we need to show $k$-smoothness of the dg algebra $E := \mathcal{C}_A(P, P)$.

The above claim shows that

$$E = \mathcal{C}_A(P, P) \xrightarrow{(-)^\vee} \mathcal{C}_A^{[n]}(P^\vee, P^\vee)$$

is a quasi-isomorphism of dg algebras.

Note that the obvious map

$$P \to (P^\vee)^\vee$$

is a quasi-isomorphism in $C(A)$ because it becomes an isomorphism in $D(A)$ by the assumption that $\mathcal{O}$ is a dualizing object. If we apply $\mathcal{C}(P, -)$ to this quasi-isomorphism we obtain a quasi-isomorphism

$$E = \mathcal{C}_A(P, P) \to \mathcal{C}_A(P, (P^\vee)^\vee)$$

of dg $E$-modules because $P$ is h-projective. Here $E$ acts from the right. There is also a natural left action of the dg algebra $E$ on $\mathcal{C}(P, (P^\vee)^\vee)$, the action on the latter coming from the morphism

$$E = \mathcal{C}(P, P) \xrightarrow{((-)^\vee)^\vee} \mathcal{C}((P^\vee)^\vee, (P^\vee)^\vee)$$

of dg algebras. It is easy to check that (2.9) is compatible with these actions and hence a quasi-isomorphism of dg $E \otimes E^{\text{op}}$-modules.

Let $P \to I$ be a quasi-isomorphism to an h-injective object $I \in C(A)$. Let $\kappa: I \otimes P^\vee \to K$ be a quasi-isomorphism to an h-injective object $K \in C(A \otimes A^{\text{op}})$. Then

$$\lambda: P \otimes P^\vee \to I \otimes P^\vee \xrightarrow{\kappa} K$$

is a quasi-isomorphism because $\kappa$ is a field. Now consider the following commutative diagram with obvious maps where $\beta$ is defined so that the triangle containing $\beta$ is commutative.
Its vertical arrows are quasi-isomorphisms: this uses the quasi-isomorphism (2.8), the fact that $P$ is $h$-projective, and the fact that $k$ is a field. The composition of the lower row is a quasi-isomorphism by Proposition 2.14(b); we use here that $I$ and $P^\vee$ are $h$-injective (by Lemma 2.13) and that $I \xrightarrow{\sim} P \in D^b(A)_{\text{ps-coh}}$ and $P^\vee \in D^b(A)_{\text{ps-coh}}$ by assumption (A1). Hence the composition $\lambda_* \circ \beta$ is a quasi-isomorphism.

We now apply [LS16b, Lemma B.1.(a)] to the dg category $C(A \otimes A^{\text{op}})$, its full pretriangulated dg subcategory $\text{hInj}_S(A \otimes A^{\text{op}})$ where $S = \text{thick}(P \otimes P^\vee)$, the quasi-isomorphism $\lambda: P \otimes P^\vee \to K$ in $C(A \otimes A^{\text{op}})$, and the morphism $\beta$ of dg algebras from the above diagram. This yields the equivalence

$$\text{res}_{E \otimes E^{\text{op}}} \circ C_{A \otimes A^{\text{op}}}(P \otimes P^\vee, -): [\text{hInj}_S(A \otimes A^{\text{op}})] \xrightarrow{\sim} \text{per}(E \otimes E^{\text{op}})$$

of triangulated categories.

By assumption (A2), the object $\mathcal{D}$ is in $S$. We have assumed that $\mathcal{D}$ is $h$-injective, so $\mathcal{D} \in \text{hInj}_S(A \otimes A^{\text{op}})$. Hence

$$C_{A \otimes A^{\text{op}}}(P \otimes P^\vee, \mathcal{D}) \in \text{per}(E \otimes E^{\text{op}})$$

by the above equivalence. The isomorphisms

$$C_{A \otimes A^{\text{op}}}(P \otimes P^\vee, \mathcal{D}) = C_{A \otimes A^{\text{op}}}(P \otimes \text{Hom}_A(P, \mathcal{D}), \mathcal{D})$$

$$\cong C_A(P, \text{Hom}_A(P, \mathcal{D}), \mathcal{D})) \quad \text{(by (2.3))}$$

$$= C_A(P, (P^\vee)^\vee)$$

of dg $E \otimes E^{\text{op}}$-modules and the quasi-isomorphism

$$E = C_A(P, P) \xrightarrow{(2.9)} C_A(P, (P^\vee)^\vee)$$

of dg $E \otimes E^{\text{op}}$-modules then show $E \in \text{per}(E \otimes E^{\text{op}})$, i.e. $E$ is smooth over $k$. \qed

### 3. Smoothness for finite dimensional algebras

We denote the Jacobson radical of a ring $R$ by $\text{rad}(R)$. Recall that a ring is semisimple if and only if it is Artinian and its Jacobson radical is zero (see e.g. [FD93, Thm. 2.2]). In particular, for a finite-dimensional algebra $A$ over a field $\frac{A}{\text{rad}(A)}$ is semisimple.

Given a finite-dimensional $k$-algebra $A$ remember that $\text{mod}(A)$ is the abelian category of finite-dimensional $A$-modules and that $D^b(\text{mod}(A))$ is the full subcategory of its derived category $D(\text{mod}(A))$ of objects with bounded cohomology.

**Lemma 3.1.** Let $A$ be a finite-dimensional algebra over a field $k$. Then $D^b(\text{mod}(A))$ has a classical generator, for example the direct sum of representatives of the set of simple $A$-modules up to isomorphism, or $\frac{A}{\text{rad}(A)}$.

**Proof.** Any object of $D^b(\text{mod}(A))$ is built from its finitely many non-zero cohomology modules, and each such module has a finite filtration with simple subquotients (a composition series). Since each simple $A$-module appears in such a composition series of $A$, there are, up to isomorphism, only finitely many simple $A$-modules, say $S_1, \ldots, S_r$. Then $\bigoplus_{i=1}^r S_i$ is a classical generator of $D^b(\text{mod}(A))$. Since $\frac{A}{\text{rad}(A)}$ is a semisimple $A$-module which contains each $S_i$ with positive multiplicity, $\frac{A}{\text{rad}(A)}$ is a classical generator as well. \qed
Recall the Artin–Wedderburn theorem saying that every semisimple ring $A$ is isomorphic to a finite product $\prod_{i=1}^{r} M_{n_i}(D_i)$ of matrix rings over division rings $D_i$, for suitable $n_i \in \mathbb{N}_{>0}$. In particular, the center $Z(A)$ of $A$ is then isomorphic to the product $\prod_{i=1}^{r} Z(D_i)$ of fields. If $A$ is an algebra over the field $k$, we get field extensions $k \rightarrow Z(D_i)$; these field extensions are unique up to isomorphism and order.

**Definition 3.2** ([FD93, Def. on page 89]). Let $k$ be a field. A $k$-algebra $A$ is separable (over $k$) if and only if $A$ is a finite-dimensional semisimple $k$-algebra such that each field extension $k \rightarrow Z(D_i)$ is separable if we fix an isomorphism $A \cong \prod_{i=1}^{r} M_{n_i}(D_i)$ as above.

**Remark 3.3.** In particular, if $k$ is perfect, then a $k$-algebra is separable if and only if it is finite-dimensional and semisimple.

**Remark 3.4.** There is a general definition of a separable algebra over a commutative ring, see [AG60] or [KO74, Ch. III]. For algebras over a field this general definition is equivalent to Definition 3.2, by [KO74, Thm. III.3.1]. Note also that Definition 3.2 generalizes the usual definition of a separable field extension.

**Proposition 3.5.** Let $A$ and $B$ be finite-dimensional $k$-algebras. Assume that at least one of $\frac{A}{\text{rad}(A)}$ and $\frac{B}{\text{rad}(B)}$ is a separable $k$-algebra. Then $\frac{A}{\text{rad}(A)} \otimes_k \frac{B}{\text{rad}(B)}$ is a semisimple $k$-algebra and

$$\frac{A}{\text{rad}(A)} \otimes_k \frac{B}{\text{rad}(B)} = \frac{A \otimes_k B}{\text{rad}(A \otimes_k B)}$$

(3.1) canonically. The displayed $k$-algebra is separable if both $\frac{A}{\text{rad}(A)}$ and $\frac{B}{\text{rad}(B)}$ are separable over $k$.

**Proof.** It is well-known that the tensor product of a separable $k$-algebra with a finite-dimensional semisimple $k$-algebra is again semisimple, see [FD93, Prop. 3.9]. This shows that the tensor product $\frac{A}{\text{rad}(A)} \otimes_k \frac{B}{\text{rad}(B)}$ is semisimple if one of the factors is separable. If both factors are separable over $k$, then so is the tensor product $\frac{A}{\text{rad}(A)} \otimes_k \frac{B}{\text{rad}(B)}$ by [AG60, Prop. 1.5] (for $R = R_1 = R_2 = k$ there) (using Remark 3.4).

We now deduce equality (3.1) from semisimplicity of

$$\frac{A}{\text{rad}(A)} \otimes_k \frac{B}{\text{rad}(B)} = \frac{A \otimes_k B}{\text{rad}(A \otimes_k B)}.$$ 

If we can show that

$$I := \text{rad}(A) \otimes_k B + A \otimes_k \text{rad}(B)$$

is a nilpotent ideal, then $I = \text{rad}(A \otimes_k B)$ by [ARS97, Prop. I.3.3], yielding equality (3.1). Observe that

$$I^n = \sum_{i=0}^{n} \text{rad}(A)^i \otimes_k \text{rad}(B))^{n-i}$$

for any $n \in \mathbb{N}$. Since $\text{rad}(A)$ and $\text{rad}(B)$ are nilpotent two-sided ideals, by [ARS97, Prop. I.3.1], we see that $I$ is nilpotent as well. This proves the proposition.

**Corollary 3.6.** Let $A$ and $B$ be finite-dimensional $k$-algebras such that at least one of $\frac{A}{\text{rad}(A)}$ and $\frac{B}{\text{rad}(B)}$ is a separable $k$-algebra. If $E$ and $F$ are classical generators
of $\mathcal{D}^b(\text{mod}(A))$ and $\mathcal{D}^b(\text{mod}(B))$, respectively, then $E \otimes_k F$ is a classical generator of $\mathcal{D}^b(\text{mod}(A \otimes_k B))$.

Proof. Recall that $\overline{A} := \frac{A}{\text{rad}(A)}$ and $\overline{B} := \frac{B}{\text{rad}(B)}$ are classical generators of $\mathcal{D}^b(\text{mod}(A))$ and $\mathcal{D}^b(\text{mod}(B))$, respectively, by Lemma 3.1. From $\overline{A} \in \text{thick}(E)$ we obtain $\overline{A} \otimes_k F \in \text{thick}(E \otimes_k F)$. From $\overline{B} \in \text{thick}(F)$ we obtain $\overline{A} \otimes_k \overline{B} \in \text{thick}(\overline{A} \otimes_k F) \subset \text{thick}(E \otimes_k F)$. Hence it is sufficient to show that $\overline{A} \otimes_k \overline{B}$ is a classical generator of $\mathcal{D}^b(\text{mod}(A \otimes_k B))$. But $\frac{\overline{A} \otimes \overline{B}}{\text{rad}(A \otimes B)}$ is a classical generator of this category, by Lemma 3.1, and $\overline{A} \otimes_k \overline{B} = \frac{\overline{A} \otimes \overline{B}}{\text{rad}(A \otimes B)}$ by Proposition 3.5.

Theorem 3.7. Let $A$ be a finite-dimensional algebra over a field $k$ such that $\frac{A}{\text{rad}(A)}$ is separable over $k$ (this condition is automatic if $k$ is perfect). Then $\mathcal{D}^b(\text{mod}(A))$ is smooth over $k$ (in the sense defined in Remark 2.5).

The idea of proof is as follows. The standard equivalence “take the $k$-linear dual” $\mathcal{D}^b(\text{mod}(A)) \xrightarrow{\sim} (\mathcal{D}^b(\text{mod}(A^{\text{op}})))^{\text{op}}$ is equivalently obtained from the dualizing bimodule $A = _AA_A$. This bimodule is an object of $\mathcal{D}^b(\text{mod}(A \otimes_k A^{\text{op}}))$, and this category coincides, by our separability assumption, with its thick subcategory generated by all tensor products of objects of $\mathcal{D}^b(\text{mod}(A))$ and $\mathcal{D}^b(\text{mod}(A^{\text{op}}))$. Moreover, $\mathcal{D}^b(\text{mod}(A))$ has a classical generator. This shows that the sufficient condition for smoothness of Theorem 2.15 is satisfied for $\mathcal{D}^b(\text{mod}(A))$.

Proof. Remember that $\mathcal{D}^b(\text{mod}(A))$ is equivalent to the category

$$(3.2) \quad \mathcal{T} := \mathcal{D}^b_{\text{mod}(A)}(A) = \mathcal{D}^b(A)_{\text{ps-coh}}$$

(see equivalence (2.1) and equality (2.2) in Remark 2.1). By our definition of $k$-smoothness of $\mathcal{D}^b(\text{mod}(A))$ in Remark 2.5 we need to prove that $\mathcal{T}$ is $k$-smooth. We will use the sufficient condition for smoothness of Theorem 2.15.

If $M$ is a right $A$-module, then its $k$-linear dual $M^* := \text{Hom}_k(M, k)$ is a left $A$-module, i.e. a right $A^{\text{op}}$-module, and similarly the $k$-linear dual $^*N$ of an $A^{\text{op}}$-module is an $A$-module. More precisely we have an adjunction of exact functors

$$(3.3) \quad (\rightarrow)^* : \text{Mod}(A) \rightleftarrows (\text{Mod}(A^{\text{op}}))^{\text{op}} : ^*\leftarrow$$

between abelian categories whose unit and counit are the obvious maps into the bidual. It induces an adjunction $D(A) \rightleftarrows (D(A^{\text{op}}))^{\text{op}}$ on (unbounded) derived categories which restricts to an adjoint equivalence

$$(3.4) \quad (\rightarrow)^* : \mathcal{T} = \mathcal{D}^b_{\text{mod}(A)}(A) \rightleftarrows (\mathcal{D}^b_{\text{mod}(A^{\text{op}})}(A^{\text{op}}))^{\text{op}} : ^*\leftarrow$$

because any object of either category is isomorphic to a bounded complex of finite-dimensional modules over $A$ or $A^{\text{op}}$, respectively.

Note that $A = _AA_A$ is both a left $A$-module and a right $A$-module. Hence $D := \text{Hom}_k(A, k) \in \text{mod}(A \otimes_k A^{\text{op}})$. This is the natural candidate bimodule to induce the equivalence (3.4). Let us check that it indeed induces this equivalence. If $M$ is a right $A$-module, then

$$M^* = \text{Hom}_k(M, k) = \text{Hom}_k(M \otimes_A A_A, k)$$

$$= \text{Hom}_A(M, \text{Hom}_k(A_A, k)) = \text{Hom}_A(M, D)$$
as left \( A \)-modules natural in \( M \). Similarly, if \( N \) is a left \( A \)-module, then
\[
* N = \text{Hom}_k(N, k) = \text{Hom}_k(A A \otimes_A N, k) = \text{Hom}_{A^{op}}(N, \text{Hom}_k(A A, k)) = \text{Hom}_{A^{op}}(N, \mathcal{D})
\]
as right \( A \)-modules natural in \( N \). Hence the functors \((-)^*\) and \(^*(-)^*\) in the adjunction (3.3) and in the adjoint equivalence (3.4) may be written as
\[
(-)^* = \text{Hom}_A(-, \mathcal{D}) \quad \text{and} \quad ^*(-) = \text{Hom}_{A^{op}}(-, \mathcal{D}).
\]
Moreover, the unit and counit of the adjunction (3.3) correspond to the unit and counit of the adjunction obtained from \( \mathcal{D} \), cf. (2.4). Since (3.4) is an adjoint equivalence, Remark 2.12 shows that \( \mathcal{D} \) is a dualizing object for \( \mathcal{T} \) and that
\[
\mathcal{T}^\vee = D_{\text{mod}(A^{op})}^b(A^{op}) = D^b(A^{op})_{\text{ps-coh}}
\]
where the last equality comes from (2.2).

Now it is easy to check conditions (A1) and (A2) from Theorem 2.15 in our situation.

Condition (A1) is obviously satisfied since \( \mathcal{T} = D_{\text{mod}(A)}^b(A) \cong D^b(\text{mod}(A)) \) has a classical generator, by Lemma 3.1, and since the equalities (3.2) and (3.5) hold.

In order to check condition (A2), let \( E \) be a classical generator of \( \mathcal{T} \). Then its dual \( E^* \) is a classical generator of \( \mathcal{T}^\vee \). We may assume without loss of generality that \( E \) and \( E^* \) are bounded complexes of finite-dimensional modules. Since \( \frac{A}{\text{rad}(A)} \) is a separable \( k \)-algebra and the opposite of any separable algebra is separable we see that \( (\frac{A}{\text{rad}(A)})^{op} = \frac{A^{op}}{\text{rad}(A^{op})} \) is separable over \( k \). Therefore \( E \otimes_k E^* \) is a classical generator of \( D^b(\text{mod}(A \otimes_k A^{op})) \), by Corollary 3.6, and also of the equivalent category \( D^b_{\text{mod}(A \otimes_k A^{op})}(A \otimes_k A^{op}) \). Since \( \mathcal{D} \) obviously lies in this category we see that condition (A2) is satisfied; more precisely, we have checked the equivalent condition \( (A2)^\vee \) in Remark 2.16. Now Theorem 2.15 applies and shows that \( D^b(\text{mod}(A)) \) is smooth over \( k \).

\begin{proof}
Remark 3.8. Let \( A \) be a finite-dimensional algebra over a field \( k \) such that \( \frac{A}{\text{rad}(A)} \) is separable over \( k \). Then smoothness over \( k \) of \( D^b(\text{mod}(A)) \) (see Theorem 3.7) has the following down-to-earth interpretation, by Remark 2.4: Let \( S = \bigoplus_{i=1}^r S_i \) be a finite direct sum of simple \( A \)-modules such that each simple \( A \)-module is isomorphic to one of the \( S_i \). Let \( P \) be a projective resolution of \( S \) in \( \text{mod}(A) \). Then the dg algebra \( \bigwedge_A(P, P) \) of endomorphisms of \( P \) is \( k \)-smooth.

Remark 3.9. Let \( A \) be a finite-dimensional algebra over a field \( k \) such that \( \frac{A}{\text{rad}(A)} \) is separable over \( k \) (this is automatic if \( k \) is perfect), and assume that \( A \) has finite global dimension. Then \( A \) is a classical generator of \( D^b(\text{mod}(A)) \). We have proven that this category is \( k \)-smooth, see Theorem 3.7. This means that \( A \) itself is \( k \)-smooth, by Remark 2.4. This also follows from [Rou08, Lemma 7.2].

4. Existence of a Classical Generator of \( D^b(\text{coh}(A)) \)

The goal of this section is Theorem 4.15: Given a coherent \( O_X \)-algebra \( A \) on a noetherian J-2 scheme \( X \) (for example a scheme of finite type over a field or over the integers), the bounded derived category \( D^b(\text{coh}(A)) \) of coherent \( A \)-modules has a classical generator. We also prove Theorem 4.18 which says that the boxproduct
of classical generators is a classical generator for finite type schemes over a perfect field.

In contrast to our convention in the rest of this article, we work with left modules in this section because this seems to be the standard choice in commutative algebra and algebraic geometry. By a noetherian ring we mean a left noetherian ring.

4.1. Noncommutative structure sheaves. Let \( \mathcal{A} \) be a sheaf of (possibly non-commutative) rings on a topological space. Quasi-coherent and coherent (left) \( \mathcal{A} \)-modules are defined in the usual way (cf. [SP18, 01BE, 01BV] for modules over a sheaf of commutative rings). Every coherent \( \mathcal{A} \)-module is quasi-coherent (cf. [SP18, 01BW]). Let \( \text{Qcoh}(\mathcal{A}) \) and \( \text{coh}(\mathcal{A}) \) denote the corresponding full subcategories of the abelian category \( \text{Mod}(\mathcal{A}) \) of all \( \mathcal{A} \)-modules. Recall that \( \text{coh}(\mathcal{A}) \) is a full abelian subcategory of \( \text{Mod}(\mathcal{A}) \) (cf. [SP18, 01BY]).

If \( X \) is a scheme, the category \( \text{Qcoh}(\mathcal{O}_X) \) is a full abelian subcategory of \( \text{Mod}(\mathcal{O}_X) \) (see [SP18, 01LA]).

**Lemma 4.1.** Let \( X \) be a scheme, \( \mathcal{A} \) an \( \mathcal{O}_X \)-algebra (not necessarily commutative) and \( M \) an \( \mathcal{A} \)-module.

(a) Assume that \( \mathcal{A} \) is \( \mathcal{O}_X \)-quasi-coherent. Then \( M \) is \( \mathcal{A} \)-quasi-coherent if and only if \( M \) is \( \mathcal{O}_X \)-quasi-coherent. In particular, \( \text{Qcoh}(\mathcal{A}) \) is a full abelian subcategory of \( \text{Mod}(\mathcal{A}) \).

(b) Assume that \( X \) is locally noetherian and that \( \mathcal{A} \) is \( \mathcal{O}_X \)-coherent. Then \( M \) is \( \mathcal{A} \)-coherent if and only if \( M \) is \( \mathcal{O}_X \)-coherent.

**Proof.** (a) Assume that \( M \) is \( \mathcal{A} \)-quasi-coherent. Then any \( x \in X \) has an open neighborhood \( U \) in \( X \) such that there is an exact sequence \( \bigoplus_j \mathcal{A}|_U \to \bigoplus_i \mathcal{A}|_U \to M|_U \to 0 \) of \( \mathcal{A} \)-modules. Since \( \mathcal{A} \) is \( \mathcal{O}_X \)-quasi-coherent, the first two terms of this sequence are \( \mathcal{O}_U \)-quasi-coherent. But then \( M|_U \) is \( \mathcal{O}_U \)-quasi-coherent as a cokernel of a morphism between quasi-coherent \( \mathcal{O}_U \)-modules. This shows that \( M \) is a quasi-coherent \( \mathcal{O}_X \)-module.

Conversely, assume that \( M \) is \( \mathcal{O}_X \)-quasi-coherent. Then any \( x \in X \) has an open neighborhood \( U \) in \( X \) such that there is an epimorphism \( \bigoplus_j \mathcal{O}_U \to M|_U \) of \( \mathcal{O}_U \)-modules. By adjunction we get an epimorphism morphism \( \bigoplus_j \mathcal{A}|_U \to M|_U \) of \( \mathcal{A}|_U \)-modules. Let \( N \) be its kernel, an \( \mathcal{A}|_U \)-module. Since \( \bigoplus_j \mathcal{A}|_U \) and \( M|_U \) are \( \mathcal{O}_X \)-quasi-coherent, so is \( N \). Repeating the above argument and possibly replacing \( U \) by a smaller open neighborhood of \( x \), we find an epimorphism \( \bigoplus_j \mathcal{A}|_U \to N \) and hence an exact sequence \( \bigoplus_j \mathcal{A}|_U \to \bigoplus_i \mathcal{A}|_U \to M|_U \to 0 \) of \( \mathcal{A}|_U \)-modules. This shows that \( M \) is \( \mathcal{A} \)-quasi-coherent.

Since \( \text{Qcoh}(\mathcal{O}_X) \) is a full abelian subcategory of \( \text{Mod}(\mathcal{O}_X) \) we deduce that \( \text{Qcoh}(\mathcal{A}) \) is a full abelian subcategory of \( \text{Mod}(\mathcal{A}) \).

(b) Assume that \( M \) is \( \mathcal{A} \)-coherent. Then any \( x \in X \) has an open neighborhood \( U \) in \( X \) such that there is an exact sequence \( \bigoplus_{j=1}^n \mathcal{A}|_U \to \bigoplus_{i=1}^n \mathcal{A}|_U \to M|_U \to 0 \) of \( \mathcal{A} \)-modules. Since the first two objects are \( \mathcal{O}_U \)-coherent and since \( \text{coh}(\mathcal{O}_U) \) is a full abelian subcategory of \( \text{Mod}(\mathcal{O}_U) \) we see that \( M|_U \) is \( \mathcal{O}_U \)-coherent. This implies that \( M \) is \( \mathcal{O}_X \)-coherent.

Conversely, assume that \( M \) is \( \mathcal{O}_X \)-coherent. Then \( M \) is of finite type over \( \mathcal{O}_X \) and a fortiori of finite type over \( \mathcal{A} \). Let \( U \subset X \) be open and let \( \bigoplus_{j=1}^n \mathcal{A}|_U \to M|_U \) be a morphism of \( \mathcal{A} \)-modules. Let \( N \) be its kernel. Since \( \bigoplus_{j=1}^n \mathcal{A}|_U \) and \( M|_U \) are \( \mathcal{O}_U \)-coherent, so is \( N \). In particular \( N \) is of finite type over \( \mathcal{O}_U \) and a fortiori of finite type over \( \mathcal{A} \). This shows that \( M \) is a coherent \( \mathcal{A} \)-module. \( \square \)
4.1.1. Inverse and direct image. Let $f: Y \to X$ be a morphism of schemes and let $A$ be an $\mathcal{O}_X$-algebra. Then $B := f^*A$ is an $\mathcal{O}_Y$-algebra and

\begin{equation}
 f^*: \text{Mod}(A) \rightleftarrows \text{Mod}(B): f_*
\end{equation}

is an adjunction where inverse image $f^*M = B \otimes_{f^{-1}A} f^{-1}M$ and direct image $f_*$ are defined in the usual way. This adjunction is compatible with the usual adjunction

\begin{equation}
 f^*: \text{Mod}(\mathcal{O}_X) \rightleftarrows \text{Mod}(\mathcal{O}_Y): f_*
\end{equation}

in the sense that $f^*$ and $f_*$ commute with the forgetful functors $\text{Mod}(A) \to \text{Mod}(\mathcal{O}_X)$ and $\text{Mod}(B) \to \text{Mod}(\mathcal{O}_Y)$.

Assume that $f$ is quasi-compact and quasi-separated. If $A$ is $\mathcal{O}_X$-quasi-coherent, then $B$ is $\mathcal{O}_Y$-quasi-coherent and (4.1) restricts to an adjunction

\begin{equation}
 f^*: \text{Qcoh}(A) \rightleftarrows \text{Qcoh}(B): f_*
\end{equation}

by Lemma 4.1 because $f_*$ maps $\mathcal{O}_Y$-quasi-coherent modules to $\mathcal{O}_X$-quasi-coherent modules by our assumptions on $f$ (see [SP18, 01LC]). If $X$ and $Y$ are locally noetherian and $A$ is $\mathcal{O}_X$-coherent, then $B$ is $\mathcal{O}_Y$-coherent and $f^*$ restricts to $f^*: \text{coh}(A) \to \text{coh}(B)$, by [SP18, 01XZ, 01BQ] and Lemma 4.1. The direct image of a coherent $B$-module is in general not $A$-coherent.

4.1.2. The affine situation. Let $R$ be a commutative ring and $X = \text{Spec } R$. Serre’s theorem states that taking global sections is an equivalence

\begin{equation}
 \text{Qcoh}(\mathcal{O}_X) \xrightarrow{\sim} \text{Mod}(R)
\end{equation}

between abelian categories which is compatible with tensor products [SP18, 011B, 0118].

Let $A$ be a quasi-coherent $\mathcal{O}_X$-algebra and $A = A(X)$ the corresponding $R$-algebra. Then an $A$-module corresponds under Serre’s equivalence to an $\mathcal{O}_X$-quasi-coherent $A$-module which is, by Lemma 4.1, the same thing as a quasi-coherent $A$-module. Hence we obtain an equivalence

\begin{equation}
 \text{Qcoh}(A) \xrightarrow{\sim} \text{Mod}(A)
\end{equation}

of abelian categories.

If $R$ is noetherian, Serre’s equivalence restricts to an equivalence

\begin{equation}
 \text{coh}(\mathcal{O}_X) \xrightarrow{\sim} \text{mod}(R)
\end{equation}

where $\text{mod}(R)$ is the category of finite (= finitely generated) $R$-modules [SP18, 01XZ]. Let $A$ be a coherent $\mathcal{O}_X$-algebra (= an $\mathcal{O}_X$-algebra that is coherent as an $\mathcal{O}_X$-module) and $A = A(X)$ the corresponding finite (and hence noetherian) $R$-algebra (= $R$-algebra that is finite as an $R$-module). The same argument as above yields an equivalence

\begin{equation}
 \text{coh}(A) \xrightarrow{\sim} \text{mod}(A)
\end{equation}

of abelian categories.

If $f: Y = \text{Spec } S \to X = \text{Spec } R$ is a morphism of affine schemes, the adjunction (4.2) corresponds to the usual adjunction

\begin{equation}
 B \otimes_A -: \text{Mod}(A) \rightleftarrows \text{Mod}(B): \text{res}^B_A
\end{equation}

between extension and restriction of scalars along $A \to B$ where $B := (f^*A)(Y) = S \otimes_R A$. 

4.2. Verdier localization sequences and classical generators.

Definition 4.2. We say that a sequence
\[ S \xrightarrow{i} T \xrightarrow{q} Q \]
of triangulated categories and functors is a Verdier localization sequence if the composition \( q \circ i \) is zero, \( i \) is fully faithful, and the induced functor from the Verdier quotient \( T / \text{Im}(i) \) to \( Q \) is an equivalence where \( \text{Im}(i) \) is the essential image of \( i \).

Proposition 4.3. Let \( S \xrightarrow{i} T \xrightarrow{q} Q \) be a Verdier localization sequence. If \( S \) and \( Q \) have classical generators, then \( T \) has a classical generator.

More precisely, if \( E \) is a classical generator of \( S \) and \( F \) is an object of \( T \) such that \( q(F) \) is a classical generator of \( Q \), then \( i(E) \oplus F \) is a classical generator of \( T \).

Proof. We assume without loss of generality that \( S \) is a strictly full triangulated subcategory of \( T \) and that \( q \) is the Verdier quotient functor \( T \to T / S = Q \).

Recall from [Ver96, Prop. II.2.3.1, items d), c)bis, d)bis] that the obvious map defines a bijection between the set of thick subcategories of \( T \) containing \( S \) (and hence its thick closure in \( T \)) and the set of thick subcategories of \( T / S = Q \).

Let \( U \) be the thick subcategory of \( T \) generated by \( E \oplus F \). It contains \( S \) since \( E \) is a classical generator of \( S \). In order to show \( U = T \) it is enough to see, by the above reminder, that the image of \( U \) in \( Q = T / S = Q \) is all of \( Q \). But this is true because this image is a thick subcategory of \( Q \) that contains the classical generator \( q(F) \). \( \square \)

4.3. Verdier localization sequence for \( D^b(\text{coh}(A)) \).

If \( X \) is a locally noetherian scheme and \( A \) is a coherent \( \mathcal{O}_X \)-algebra, we let \( D(\text{coh}(A)) \) be the derived category of the abelian category \( \text{coh}(A) \) of coherent \( A \)-modules. Its full subcategory \( D^b(\text{coh}(A)) \) of objects \( M \) whose total cohomology \( \bigoplus_{n \in \mathbb{Z}} H^n(M) \) is bounded is a Karoubian subcategory (see [LC07]). If \( Z \subset X \) is a closed subset let \( D^b_Z(\text{coh}(A)) \) denote the full subcategory of \( D^b(\text{coh}(A)) \) of objects whose cohomology sheaves have support in the set \( Z \). It is a thick subcategory.

Theorem 4.4. Let \( X \) be a noetherian scheme and \( A \) a coherent \( \mathcal{O}_X \)-algebra. Let \( U \) be an open subscheme of \( X \) and \( Z := X - U \) its closed complement. Then the sequence of triangulated categories
\[
D^b_Z(\text{coh}(A)) \to D^b(\text{coh}(A)) \to D^b(\text{coh}(A_U))
\]
is a Verdier localization sequence where the first arrow is the inclusion and the second arrow is restriction to \( U \).

Proof. We abbreviate \( A_U := A|_U \). During the proof we assume without loss of generality that all objects of \( D^b(\text{coh}(A)) \) are bounded complexes of coherent \( A \)-modules, and similarly for \( D^b_Z(\text{coh}(A)) \) and \( D^b(A_U) \).

Let \( j : U \to X \) be the open embedding. Then \( A_U = j^*A \) and \( j^* : \text{coh}(A) \to \text{coh}(A_U) \) is exact. We denote the induced functor \( j^* : D^b(\text{coh}(A)) \to D^b(\text{coh}(A_U)) \) by the same symbol. This functor is the second functor in (4.4). Clearly, its kernel is the subcategory \( D^b_Z(\text{coh}(A)) \). Let
\[
V := \frac{D^b(\text{coh}(A))}{D^b_Z(\text{coh}(A))}
\]
be the Verdier quotient and

\[ \Phi: \mathcal{V} \rightarrow D^b(\text{coh}(\mathcal{A}_U)) \]

the induced triangulated functor. We need to prove that \( \Phi \) is an equivalence. We first prove a useful fact.

**Observation:** If \( M \) is a bounded complex of quasi-coherent \( \mathcal{A} \)-modules whose restriction \( j^* M \) consists of coherent \( \mathcal{A}_U \)-modules, then there is a subcomplex \( K \subset M \) of coherent \( \mathcal{A} \)-modules such that \( j^* K = j^* M \).

Recall that every quasi-coherent \( \mathcal{O}_X \)-module is the directed colimit (or union) of its coherent submodules, see [SP18, 01XZ, 01PG] where we use that \( X \) is quasi-compact and quasi-separated [SP18, 01OY]. The same statement is true for \( \mathcal{A} \)-modules: indeed, if \( G \) is a coherent \( \mathcal{O}_X \)-submodule of a quasi-coherent \( \mathcal{A} \)-module \( F \), then the image of \( \mathcal{A} \otimes_{\mathcal{O}_X} G \rightarrow F \) is a coherent \( \mathcal{A} \)-submodule of \( F \) containing \( G \).

We deduce that every complex \( M \) of quasi-coherent \( \mathcal{A} \)-modules is the directed colimit of its subcomplexes of coherent \( \mathcal{A} \)-modules: indeed, each component \( M^n \) is the directed colimit of its coherent \( \mathcal{A} \)-submodules, and if we are given coherent \( \mathcal{A} \)-submodules \( N^n \) of \( M^n \), for each \( n \in \mathbb{Z} \), there is a subcomplex of \( M \) with coherent components which contains all \( N^n \): just take \( N^n + d(N^{n-1}) \) in degree \( n \).

To prove the observation, let \( M \) be a bounded complex of quasi-coherent \( \mathcal{A} \)-modules such that \( j^* M \) has \( \mathcal{A}_U \)-coherent components. Write \( M = \text{colim} M_i \) as a directed colimit of subcomplexes \( M_i \) of coherent \( \mathcal{A} \)-modules. Then \( j^* M = j^* \text{colim} M_i = \text{colim} j^* M_i \). In particular, the \( n \)-th component \( (j^* M)^n = \text{colim} j^* \mathcal{M}^n_i \) is a directed colimit of coherent \( \mathcal{A}_U \)-submodules and is itself \( \mathcal{A}_U \)-coherent by assumption. Hence \( (j^* M)^n = j^* (\mathcal{M}^n_i) \) for some \( i \) by [SP18, 01Y8] and Lemma 4.1. Since \( M \) is bounded there is some \( i \) such that \( (j^* M)^n = j^* (\mathcal{M}^n_i) \) for all \( n \in \mathbb{Z} \), i.e. \( M_i \subset M \) satisfies \( j^* M_i = j^* M \). This proves the observation.

In the following we will often use the adjunction

\[ j^*: \text{Qcoh}(\mathcal{A}) \rightleftarrows \text{Qcoh}(\mathcal{A}_U): j_* \]

from (4.2) where we use the fact that \( j \) is quasi-compact and quasi-separated as a map between noetherian schemes (see [SP18, 02IK, 01OY, 01KV, 03GI]). Note that its counit is an isomorphism \( j^* j_* \xrightarrow{\sim} \text{id} \).

**\( \Phi \) is essentially surjective:** Let \( N \) be a bounded complex of coherent \( \mathcal{A}_U \)-modules. Then \( j_* N \) is a bounded complex of quasi-coherent \( \mathcal{A} \)-modules which satisfies \( j^* j_* N \xrightarrow{\sim} N \). Hence our observation yields a subcomplex \( K \subset j_* N \) with coherent components such that \( j^* K = j^* j_* N \xrightarrow{\sim} N \). This shows that \( \Phi \) is essentially surjective.

**\( \Phi \) is faithful:** Let \( g: M \rightarrow M' \) be any morphism in \( \mathcal{V} \). Then \( g \) can be represented by a roof \( M \xrightarrow{g'} M'' \xleftarrow{u} M' \) of morphisms in \( D^b(\text{coh}(\mathcal{A})) \) where \( u \) has cone in \( D^b(\text{coh}(\mathcal{A})) \), i.e. \( g = u^{-1} g' \). Similarly, \( g' \) can be represented by a roof \( M \xrightarrow{g''} N \xleftarrow{u'} M'' \) of morphisms in the homotopy category \( K^b(\text{coh}(\mathcal{A})) \) where \( u' \) is a quasi-isomorphism and \( N \) is a bounded complex of coherent \( \mathcal{A} \)-modules, i.e. \( g' = u'^{-1} g'' \) in \( D^b(\text{coh}(\mathcal{A})) \). Then \( \Phi(g) = j^*(u)^{-1} j^*(g') = j^*(u)^{-1} j^*(u')^{-1} j^*(g'') \) in \( D^b(\text{coh}(\mathcal{A}_U)) \). For faithfulness of \( \Phi \) we need to prove that \( \Phi(g) = 0 \) implies \( g = 0 \). Equivalently, we need to prove that \( j^*(g'') = 0 \) in \( D^b(\text{coh}(\mathcal{A}_U)) \) implies \( g'' = 0 \) in \( \mathcal{V} \).
Hence the proof of faithfulness of $\Phi$ is reduced to the following claim: Let $f: M \to N$ be a morphism in the category $C^b(coh(A))$ of bounded complexes of coherent $A$-modules such that $j^*(f) = 0$ in $D^b(coh(A_U))$. Then $f = 0$ in $V$.

The assumption $j^*(f) = 0$ in $D^b(coh(A_U))$ shows that the roof $j^*M \xrightarrow{j^*(f)} j^*N \xleftarrow{id} j^*N$ in the homotopy category $K^b(coh(A_U))$ is equivalent to the roof $j^*M \xleftarrow{0} j^*N \xrightarrow{id} j^*N$. Hence there are a bounded complex $L$ of coherent $A_U$-modules and a quasi-isomorphism $s: j^*N \to L$ in $C^b(coh(A_U))$ such that the composition

$$j^*M \xrightarrow{j^*(f)} j^*N \xrightarrow{s} L$$

is homotopic to zero, i.e. there is a homotopy $h: j^*M \to L[1]$ such that $s \circ j^*(f) = d(h) = dh + hd$. Let $s': N \to j_*L$ and $h': M \to j_*L[1]$ correspond to $s$ and $h$ under the adjunction. Then the composition

$$M \xrightarrow{f} N \xrightarrow{s'} j_*L$$

is homotopic to zero via the homotopy $h'$.

Note that the image $s'(N) \subset j_*L$ is a subcomplex of coherent $A$-modules. Similarly, $h'(M[-1]) \subset j_*L$ is a graded submodule of coherent $A$-modules, and the subcomplex $h'(M[-1]) + d(h'(M[-1])) \subset j_*L$ it generates is a subcomplex of coherent $A$-modules. Let $K \subset j_*L$ be a subcomplex of coherent $A$-modules which contains these two subcomplexes and has the property that $j^*K = j^*j_*L$; it exists by the observation made above using $j^*j_*L \xrightarrow{\sim} L$.

By construction, $s'$ and $h'$ factor as $s': N \xrightarrow{s''} K \xleftarrow{} j_*L$ and $h': M \xrightarrow{h''} K[1] \xrightarrow{} j_*L[1]$, respectively, and the composition

$$M \xrightarrow{f} N \xrightarrow{s''} K$$

is homotopic to zero via the homotopy $h''$, i.e. $s'' \circ f = 0$ in $D^b(coh(A))$.

Note that $s$ is the composition

$$j^*N \xrightarrow{j^*(s'')} j^*K = j^*j_*L \xrightarrow{\sim} L$$

of morphisms in $C^b(coh(A))$. Since $s$ is a quasi-isomorphism, so is $j^*(s'')$. In particular, the mapping cone of $s''$ has cohomology supported in $Z$. Hence $s''$ becomes invertible in $V$. Since $s'' \circ f = 0$ in $D^b(coh(A))$ this implies $f = 0$ in $V$. This finishes the proof that $\Phi$ is faithful.

$\Phi$ is full: Let $M, N$ be bounded complexes in $coh(A)$. We need to show that any morphism $f: j^*M \to j^*N$ in $D^b(coh(A_U))$ comes from a morphism in $V$.

We first prove this statement under the more restrictive assumption that $f: j^*M \to j^*N$ is a morphism in $C^b(coh(A_U))$.

Consider the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\eta_M} & j_*j^*M \\
\downarrow{j_*f} & & \downarrow{j_*f} \\
N & \xrightarrow{\eta_N} & j_*j^*N
\end{array}
$$

in $C^b(Qcoh(A))$ where the horizontal arrows are the respective adjunction units. The images of the morphisms $\eta_N$ and $j_*f \circ \eta_M$ are subcomplexes of $j_*j^*N$ whose
components are coherent \( A \)-modules. Since \( j^*j_*j^*N \cong j^*N \) is a complex of coherent \( A_U \)-modules, there is a subcomplex \( K \subset j_*j^*N \) consisting of coherent \( A \)-modules which contains these two images and satisfies \( j^*K = j^*j_*j^*N \); this follows from our observation.

By construction we obtain the morphisms \( \kappa \) and \( \mu \) in the following left diagram turning it into a commutative diagram; the commutative diagram on the right is obtained from it by restriction to \( U \) and by using formal properties of the adjunction \((j^*, j_*)\) where \( \varepsilon \) is the adjunction counit.

\[
\begin{array}{ccc}
M & \xrightarrow{\eta M} & j_*j^*M \\
\downarrow{\kappa} & & \downarrow{\eta_N} \\
N & \xrightarrow{\mu} & j_*j^*N
\end{array}
\quad
\begin{array}{ccc}
j^*M & \xrightarrow{\sim} & j^*j_*j^*M \\
\downarrow{\sim} & & \downarrow{\sim} \\
j^*N & \xrightarrow{\sim} & j^*j_*j^*N
\end{array}
\quad
\begin{array}{ccc}
\quad & \xrightarrow{\eta M} & \\
\xrightarrow{id} & & \\
\xrightarrow{id} & & \\
\sim & \xrightarrow{id} & \sim
\end{array}
\]

The diagram on the right shows \( j^*(\kappa) \circ f = j^*(\mu) \) and that \( j^*(\kappa) = j^*(\eta_N) \) is an isomorphism, so \( \kappa \) becomes invertible in \( \mathcal{V} \). Hence \( \kappa^{-1} \circ \mu \) is a morphism in \( \mathcal{V} \) such that \( \Phi(\kappa^{-1} \circ \mu) = (j^*(\kappa))^{-1} \circ j^*(\mu) = f \) in \( D^b(\text{coh}(A_U)) \).

Now assume that \( f : j^*M \to j^*N \) is an arbitrary morphism in \( D^b(\text{coh}(A_U)) \). It can be represented by a roof \( j^*M \xrightarrow{\mu} j^*N' \xrightarrow{\kappa} j^*N \) in \( C^b(\text{coh}(A_U)) \) where \( q \) is a quasi-isomorphism, i.e. \( f = q^{-1}g \); here we can assume without loss of generality that the apex of our roof has the form \( j^*N' \) where \( N' \) is a bounded complex of coherent \( A \)-modules, as follows from the proof of essential surjectivity of \( \Phi \).

As seen above, there are morphisms \( \tilde{g} : M \to N' \) and \( \tilde{q} : N \to N' \) in \( \mathcal{V} \) such that \( \Phi(\tilde{g}) = g \) and \( \Phi(\tilde{q}) = q \).

Note that \( \Phi(\text{Cone}(\tilde{q})) \cong \text{Cone}(\Phi(\tilde{q})) \cong \text{Cone}(q) = 0 \) since \( q \) is an isomorphism in \( D^b(\text{coh}(A_U)) \). Since we already know that \( \Phi \) is faithful we get \( \text{Cone}(\tilde{q}) \cong 0 \) and hence \( \tilde{q} \) is an isomorphism. (Abstractly, we have used that a faithful triangulated functor reflects isomorphisms.) But then \( \Phi(\tilde{q}^{-1}g) = \Phi(\tilde{q})^{-1}\Phi(g) = q^{-1}g = f \). This finishes the proof that \( \Phi \) is full. \( \square \)

**Remark 4.5.** In the setting of Theorem 4.4, the sequence of abelian categories

\[
(4.5) \quad \text{coh}_Z(A) \to \text{coh}(A) \to \text{coh}(A|_U)
\]

is a **Serre localization sequence** where \( \text{coh}_Z(A) \) is the full subcategory of \( \text{coh}(A) \) of objects with support in \( Z \), the first arrow is the inclusion of this subcategory, and the second arrow is restriction to \( U \). Here the term **Serre localization sequence** means that the obvious functor from \( \text{coh_Z}(A) \) to \( \text{coh}(A|_U) \) is an equivalence. The proof is an obvious variation of the proof of Theorem 4.4 and actually easier than this proof.

### 4.4. Classical generators and open-closed decompositions.

If \( i : Z \to X \) is the inclusion of a closed subscheme into a noetherian scheme, then \( i_* : \text{Mod}(\mathcal{O}_Z) \to \text{Mod}(\mathcal{O}_X) \) is exact and preserves coherence. In particular, if \( A \) is a coherent \( \mathcal{O}_X \)-algebra, then \( i_* : \text{coh}(i^*A) \to \text{coh}(A) \) is well-defined and exact (cf. Lemma 4.1). We use the same symbol for the induced functor \( i_* : D^b(\text{coh}(i^*A)) \to D^b(\text{coh}(A)) \).
Proposition 4.6. Let $Z$ be a closed subscheme of a noetherian scheme $X$ and let $i: Z \to X$ be the inclusion. Let $\mathcal{A}$ be a coherent $\mathcal{O}_X$-algebra and set $\mathcal{A}_Z := i^* \mathcal{A}$. Assume that $E$ is a classical generator of $D^b(\text{coh}(\mathcal{A}_Z))$. Then $i_* E$ is a classical generator of $D^b_Z(\text{coh}(\mathcal{A}))$.

Proof. Note that $i_* E$ is an object of $D^b_Z(\text{coh}(\mathcal{A}))$. Let $\mathcal{U}$ be the thick subcategory of $D^b_Z(\text{coh}(\mathcal{A}))$ generated by $i_* E$. We need to show $\mathcal{U} = D^b_Z(\text{coh}(\mathcal{A}))$.

Since any object of $D^b_Z(\text{coh}(\mathcal{A}))$ is build up from its finitely many non-zero cohomology modules, it is enough to show that any coherent $\mathcal{A}$-module $M$ with support in $Z$ is in $\mathcal{U}$. Note that $M$ is $\mathcal{O}_X$-coherent by Lemma 4.1.

Let $\mathcal{I} \subset \mathcal{O}_X$ be the ($\mathcal{O}_X$-coherent) ideal sheaf of $Z$. Then there is some $n \in \mathbb{N}$ such that $\mathcal{I}^n M = 0$ (see [SP18, 01Y9]). Hence $M$ has a finite filtration
\[ 0 = \mathcal{I}^n M \subset \mathcal{I}^{n-1} M \subset \cdots \subset \mathcal{I} M \subset M \]
by coherent $\mathcal{A}$-modules. All subquotients $\mathcal{I}^n M/\mathcal{I}^{n+1} M$ are coherent $\mathcal{A}$-modules that are annihilated by $\mathcal{I}$. Hence all these subquotients are in $\mathcal{U}$ (cf. proofs of [SP18, 087T, 01QY]) and so is $M$. \qed

Proposition 4.7. Let $X$ be a noetherian scheme and $\mathcal{A}$ a coherent $\mathcal{O}_X$-algebra. Let $U$ be an open subscheme of $X$ and let $Z$ be a closed subscheme of $X$ such that $Z := X - U$ as sets. Let $\mathcal{A}_Z := i^* \mathcal{A}$ where $i: Z \to X$ is the inclusion. If $D^b(\text{coh}(\mathcal{A}_Z))$ and $D^b(\text{coh}(\mathcal{A}|_U))$ each have a classical generator, then $D^b(\text{coh}(\mathcal{A}))$ has a classical generator.

More precisely, if $E$ is a classical generator of $D^b(\text{coh}(\mathcal{A}_Z))$ and $F$ is a classical generator of $D^b(\text{coh}(\mathcal{A}|_U))$, then $i_* E \oplus \hat{F}$ is a classical generator of $D^b(\text{coh}(\mathcal{A}))$ where $\hat{F}$ is any object of $D^b(\text{coh}(\mathcal{A}))$ with $\hat{F}|_U \cong F$ (such an object $\hat{F}$ exists by Theorem 4.4).

Proof. Note that $i_* E$ is a classical generator of $D^b_Z(\text{coh}(\mathcal{A}))$ by Proposition 4.6. Hence the proposition follows from Proposition 4.3 applied to the Verdier localization sequence from Theorem 4.4. \qed

4.5. Local existence of a classical generator. Given a ring $\mathcal{A}$ (unital, associative, but not necessarily commutative), we denote its center by $Z(\mathcal{A})$.

Definition 4.8 (cf. e.g. [KO74, p. 95], [MR87, 13.7.6]). An Azumaya algebra (over its center) is a ring $\mathcal{A}$ that satisfies the following two conditions:

(a) As a $Z(\mathcal{A})$-module, $\mathcal{A}$ is finitely generated projective.

(b) The ring map
\[ \eta_\mathcal{A}: \mathcal{A} \otimes_{Z(\mathcal{A})} \mathcal{A}^{\text{op}} \to \text{End}_{Z(\mathcal{A})}(\mathcal{A}), \]
\[ a \otimes b \mapsto (x \mapsto axb), \]

is an isomorphism where $\text{End}_{Z(\mathcal{A})}(\mathcal{A})$ is the ring of $Z(\mathcal{A})$-module endomorphisms of $\mathcal{A}$.

We refer the reader to [KO74, Thm. III.5.1] or [AG60, Thm. 2.1] for equivalent conditions characterizing Azumaya algebras; for example, a ring is an Azumaya algebra if and only if it is separable as an algebra over its center.

If $X$ is a scheme, we denote its regular locus by $X_{\text{reg}}$.

Definition 4.9 ([SP18, 07R3]). A scheme $X$ is called $\textbf{J}$-$\mathbf{2}$ if it is locally noetherian and if for every morphism $Y \to X$ locally of finite type the set $Y_{\text{reg}}$ is open in $Y$. 

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We refer the reader to \[\text{SP18, 07R2}\] for basic properties and examples of J-2 schemes. Important examples of J-2 schemes are schemes locally of finite type over a field or schemes locally of finite type over the integers.

Recall that a commutative ring \(R\) is called regular if it is noetherian and all localizations \(R_p\) at prime ideals \(p\) are regular local rings (see \[\text{SP18, 00OD}\]).

The following proposition is the key for several later results we prove by noetherian induction.

**Proposition 4.10.** Let \(X\) be a non-empty J-2 scheme. Let \(\mathcal{A}\) be a coherent \(O_X\)-algebra (which is not assumed to be commutative). Then there exist a non-empty affine open subset \(U\) of \(X\) and a nilpotent two-sided ideal \(I\) of \(\mathcal{A}(U)\) and \(O_X(U)\)-algebras \(A_1, \ldots, A_r\), for some \(r \in \mathbb{N}\), such that each \(A_i\) is an Azumaya algebra over its center \(Z(A_i)\), each center \(Z(A_i)\) is a regular ring, and there is an isomorphism

\[\mathcal{A}(U)/I \cong A_1 \times \cdots \times A_r\]

of \(O_X(U)\)-algebras.

**Proof.** Without loss of generality we may and will impose the following additional assumptions.

(i) \(X\) is irreducible.

Indeed, any J-2 scheme is by definition locally noetherian, so we may assume that \(X\) is noetherian. Then \(X\) has only finitely many irreducible components. Choose one irreducible component and consider the complement in \(X\) of the union of the other irreducible components. This is an irreducible open subscheme of \(X\), and we may replace \(X\) by this irreducible open subscheme and \(\mathcal{A}\) by its restriction to this open subscheme.

(ii) \(X\) is affine, say \(X = \text{Spec } R\) where \(R\) is a J-2 ring.

Indeed, just replace \(X\) by a non-empty affine open subscheme (which is the spectrum of a J-2 ring, by \[\text{SP18, 07R4}\]) and \(\mathcal{A}\) by its restriction to this open subscheme.

Set \(A = \mathcal{A}(X)\). Then \(A\) is a finite algebra\(^1\) over the noetherian ring \(R\), and \(\mathcal{A}\) is the corresponding coherent \(O_X\)-algebra. In the following, we often work with \(R\) and \(A\) instead of \(X\) and \(\mathcal{A}\) and use results from 4.1.2 without mentioning this explicitly. If we say that the statement of the proposition is true for the pair \((R, A)\) we mean that it is true for \(\text{Spec } R\) and the \(O_X\)-algebra associated to \(A\).

Without loss of generality we may and will impose the following additional assumptions.

(iii) \(X = \text{Spec } R\) is reduced (and hence integral), i.e. \(R\) is an integral domain.

Indeed, let \(\text{nil}(R)\) be the nilradical of \(R\). It is a nilpotent ideal because \(R\) is noetherian. The two-sided ideal \(\langle \text{nil}(R) \rangle\) generated by \(\text{nil}(R)\) in \(A\) is then also nilpotent. Hence, if the proposition is true for the reduced ring \(R/\text{nil}(R)\) (corresponding to the underlying reduced scheme \(X_{\text{red}} = \text{Spec } R/\text{nil}(R)\)) and the \(R/\text{nil}(R)\)-algebra \(A/\langle \text{nil}(R) \rangle = A \otimes_R (R/\text{nil}(R))\), then it is also true for \(R\) and the \(R\)-algebra \(A\).

(iv) \(A\) is a finite free \(R\)-module.

Indeed, by generic freeness, \(X = \text{Spec } R\) contains a non-empty open subscheme such that the restriction of \(\mathcal{A}\) to this subscheme is finite free as

\footnote{Note that \(A\) is not assumed to be commutative, but we extend the usual commutative algebra terminology in the obvious way: \(A\) is an \(R\)-algebra which is finitely generated as an \(R\)-module.}
a module over the structure sheaf [GW10, Lemma 10.81]. Hence there is a non-zero element \( s \in R \) such that \( A_s \) is a free \( R_s \)-module, and we can replace the pair \((R, A)\) by the pair \((R_s, A_s)\).

(v) The structure morphism \( R \to A \) is injective, i.e. \( R \subset A \).

Indeed, all claims of the proposition are obvious if \( A = 0 \). Otherwise, the structure morphism is injective since \( A \) is a free module over the integral domain \( R \).

Let \( K = \text{Quot}(R) \) be the field of fractions of \( R \). Geometrically, it is the stalk of the structure sheaf \( \mathcal{O}_X \) at the generic point of the integral scheme \( X = \text{Spec} R \). Then \( A_K := A \otimes_R K \) is a finite-dimensional \( K \)-algebra.

Note that \( K \) is the localization of the integral domain \( R \) at \( S := R - \{0\} \), that \( A_K \) is the localization of \( A \) at the central subset \( S \), and that \( A \) is torsion-free over \( R \). Hence we have the following commutative diagram of inclusions.

\[
\begin{array}{c}
A \\ \cup \\ R
\end{array} \quad \begin{array}{c}
\subset \\ \cup \\ \subset \\
A_K \\ \cup \\ K
\end{array}
\]

Without loss of generality we may and will impose the following additional assumptions.

(vi) \( A_K \) is a semisimple \( K \)-algebra.

Indeed, let \( \text{rad}(A_K) \) be the Jacobson radical of \( A_K \). This is a nilpotent two-sided ideal of \( A_K \), and \( \frac{A_K}{\text{rad}(A_K)} \) is a semisimple \( K \)-algebra. Moreover, \( A \cap \text{rad}(A_K) \) is a nilpotent two-sided ideal of \( A \), and we may expand the above diagram to the following commutative diagram.

\[
\begin{array}{c}
A' := \frac{A}{\text{rad}(A_K)} \\ \cup \\ A
\end{array} \quad \begin{array}{c}
\subset \\ \cup \\ \subset \\
\frac{A_K}{\text{rad}(A_K)} \\ \cup \\ K
\end{array}
\]

Note that \( A'_{K'} := A' \otimes_R K = \frac{A_K}{\text{rad}(A_K)} \) canonically as \( K \)-algebras. Hence, if the proposition is true for the pair \((R, A')\), it is also true for the pair \((R, A)\). Replacing \( A \) by \( A' \) may however destroy the assumption (iv).

Before taking care of this, let us consider assumption (v). Since \( R \) is an integral domain, its intersection with the nilpotent ideal \( \text{rad}(A_K) \) is \( \{0\} \), so we may view \( R \) as a subring of \( A' \), i.e. (v) holds for the \( R \)-algebra \( A' \) mutatis mutandis.

Let us explain how to deal with assumption (iv). Generic freeness provides a non-zero element \( s \in R \) such that \( (A')_s \) is a finite free \( R_s \)-module. Observe that

\[
(A')_s = \left( \frac{A}{(A \cap \text{rad}(A_K))} \right)_s = \frac{A_s}{(A \cap \text{rad}(A_K))_s}
\]

and that \( (A \cap \text{rad}(A_K))_s = A_s \cap \text{rad}(A_K) \) is a nilpotent two-sided ideal of \( A_s \). Moreover, \( (A')_{K'} = A'_{K'} = \frac{A_K}{\text{rad}(A_K)} \) is a semisimple \( K \)-algebra.
Hence we may replace the pair \((R, A)\) without loss of generality by the pair \((R_s, (A')_s)\).

Since \(A_K\) is a finite-dimensional, semisimple \(K\)-algebra, it decomposes as a finite product

\[
A_K = B_1 \times \cdots \times B_r
\]

of finite-dimensional, simple \(K\)-algebras \(B_1, \ldots, B_r\), for some \(r \in \mathbb{N}\).

Without loss of generality we may and will impose the following additional assumption.

(vii) The decomposition \((4.6)\) comes from a decomposition

\[A = A_1 \times \cdots \times A_r\]

of rings by extension of scalars along \(R \to K\), i.e. \(A_i \otimes_R K = B_i\) for all \(i \in \{1, \ldots, r\}\).

Indeed, let \(e_i\) be the central idempotent of \(A_K\) such that \(B_i = e_iA_K\). These idempotents satisfy \(e_ie_j = \delta_{ij}e_i\) and \(\sum e_i = 1\). Then \(e_i = \frac{r_i}{\sum r_i}\) for some elements \(r_i \in A\) and \(s_i \in R - \{0\}\). Setting \(s = s_1 \cdots s_r \neq 0\) we obtain the decomposition \(A_s = e_1A_s \times \cdots \times e_rA_s\) as rings. By construction, the scalar extension of this decomposition is the decomposition \((4.6)\). Hence we may replace the pair \((R, A)\) by the pair \((R_s, A_s)\).

Note that \(Z(A) = Z(A_1) \times \cdots \times Z(A_r)\) and hence geometrically

\[\text{Spec } Z(A) = \bigsqcup \text{Spec } Z(A_i) \to \text{Spec } R.\]

The \(Z(A)\)-module \(A\), viewed as a coherent module on \(\text{Spec}(Z(A))\), decomposes accordingly: its restriction to each \(\text{Spec } Z(A_i)\) is the \(Z(A_i)\)-module \(A_i\).

Since \(A\) is a finite algebra over the noetherian ring \(R\), its center \(Z(A)\) is a finite commutative \(R\)-algebra. Since \(R\) is J-2, the regular locus \((\text{Spec } Z(A))_{\text{reg}}\) is open in \(\text{Spec } Z(A)\), and \((\text{Spec } Z(A_i))_{\text{reg}}\) is open in \(\text{Spec } Z(A_i)\), for each \(i \in \{1, \ldots, r\}\).

Without loss of generality we may and will impose the following additional assumption.

(viii) The center \(Z(A_i)\) of \(A_i\) is a regular ring and \(A_i\) is a finite free \(Z(A_i)\)-module, for all \(i \in \{1, \ldots, r\}\).

Indeed, since the finite free \(R\)-module \(A\) is torsion-free over \(R\), we have

\[
Z(A) \subset Z(A) \otimes_R K = Z(A \otimes_R K) = Z(A_K)
\]

and hence \(Z(A_i) \subset Z(B_i)\) for each \(i\). We fix \(i\) for a moment. Since \(B_i\) is a matrix ring over a division ring, its center \(Z(B_i)\) is a field, so its subring \(Z(A_i)\) is an integral domain. In particular, the generic point of \(\text{Spec } Z(A_i)\) is regular. This shows that the open regular locus \((\text{Spec } Z(A_i))_{\text{reg}}\) of \(\text{Spec } Z(A_i)\) is non-empty.

By generic freeness, there is a non-empty open subset \(V_i\) of \((\text{Spec } Z(A_i))_{\text{reg}}\) such that \(A_i|_{V_i}\) is a finite free \(Z(A_i)\)-module. Let \(f\) : \(\text{Spec } Z(A_i) \to \text{Spec } R\) be the finite morphism corresponding to the finite ring morphism \(R \to Z(A_i)\) and consider the proper closed subset \(C_i := (\text{Spec } Z(A_i)) - V_i\) of \(\text{Spec } Z(A_i)\). Since finite morphisms are closed and the generic point of \(\text{Spec } Z(A_i)\) is the only point of \(\text{Spec } Z(A_i)\) whose image under \(f\) is the generic point of \(\text{Spec } R\), by [AM69, Cor. 5.9], the set \(f(C_i)\) is a proper closed subset of \(\text{Spec } R\). Choose \(s_i \in R - \{0\}\) such that \(\text{Spec } R_{s_i} \subset \)
Its scalar extension

Using identifies (note that $A_i$ with $B_i$) which have the property that every finitely generated module has finite projective dimension; an example is due to Nagata, see [KO74, Thms. III.3.1 and III.5.1]). This means that $\eta_{A_i} = \eta_{A_i \otimes_R K}$ is an isomorphism.

Note that source and target of $\eta_{A_i}$ are finite free $Z(A_i)$-modules. In particular, $\eta_{A_i}$ may be viewed as a morphism of coherent $\mathcal{O}_{\text{Spec} A}$-modules which is an isomorphism at the generic point. Hence it is already an isomorphism on an open neighborhood of the generic point, i.e. there is an element $s_i \in R - \{0\}$ such that $\eta_{A_i} \otimes_R R_{s_i}$ is an isomorphism. Since we already know that $(A_i)_{s_i}$ is a finite free module over $Z((A_i)_{s_i}) = Z(A_i)_{s_i}$, we see that $(A_i)_{s_i}$ is an Azumaya algebra over its center.

As above, we may set $s = s_1, \ldots, s_r$ and replace $R$ by $R_s$ and $A$ by $A_s$ without destroying our previous assumptions. Then $A = A_1 \times \cdots \times A_r$ as $R$-algebras where each $A_i$ is an Azumaya algebra over its center $Z(A_i)$ which is a regular ring. This finishes the proof of the proposition. \qed

**Lemma 4.11.** Let $A$ be an Azumaya algebra whose center $Z(A)$ is a regular ring. If $M$ is any finitely generated $A$-module then its projective dimension $\operatorname{pdim}_A M$ is finite.

**Proof.** Let $M$ be a finitely generated $A$-module. Then $\operatorname{pdim}_A M \leq \operatorname{pdim}_{Z(A)} M$ by [AG60, Thms. 1.8 and 2.1] (with $\Delta = R$ there). Note that $M$ is a finitely generated $Z(A)$-module since, by definition of an Azumaya algebra, $A$ is a finitely generated $Z(A)$-module. Now observe that this implies $\operatorname{pdim}_{Z(A)} M < \infty$ since $Z(A)$ is a regular ring (as explained in the proof of [BLS16, Prop. A.2]). \qed

**Remark 4.12.** There are regular commutative rings of infinite global dimension which have the property that every finitely generated module has finite projective dimension; an example is due to Nagata, see [AM69, Exercise 11.4], [MR87, Example 7.7.2].
Lemma 4.13. Let $A$ be a noetherian ring (not assumed to be commutative). If $I \subset A$ is a nilpotent two-sided ideal such that each finitely generated $A/I$-module has finite projective dimension over $A/I$, then $A/I$ is a classical generator of $\text{D}^b(\text{mod}(A))$.

Proof. Let $\mathcal{S}$ be the thick subcategory of $\text{D}^b(\text{mod}(A))$ generated by $A/I$. By assumption, every finitely generated $A/I$-module has a finite resolution by finitely generated projective $A/I$-modules. Hence every finitely generated $A$-module that is annihilated by $I$ is contained in $\mathcal{S}$.

Let $M$ be any finitely generated $A$-module. Since $I$ is nilpotent, say $I^n = 0$ for some $n \in \mathbb{N}$, $M$ has a finite filtration $0 = I^nM \subset I^{n-1}M \subset \cdots \subset IM \subset M$ by submodules. Each subquotient $I^iM/I^{i+1}M$ is annihilated by $I$ and finitely generated as a module over the noetherian ring $A$. Hence $I^iM/I^{i+1}M \in \mathcal{S}$ for all $i$ and therefore $M \in \mathcal{S}$.

Since any object of $\text{D}^b(\text{mod}(A))$ is built up from its finitely many non-zero cohomology modules, which are finitely generated $A$-modules, we deduce that $\text{D}^b(\text{mod}(A)) = \mathcal{S}$. □

Proposition 4.14. Let $X$ be a non-empty $J$-2 scheme. Let $A$ be a coherent $\mathcal{O}_X$-algebra (which is not assumed to be commutative). Then there exists a non-empty affine open subset $U$ of $X$ and a nilpotent two-sided ideal sheaf $\mathcal{I} \subset A|_U$ such that $A|_U/\mathcal{I}$ is a classical generator of $\text{D}^b(\text{coh}(A|_U))$.

Proof. Proposition 4.10 provides a non-empty open subset $U$ of $X$ and a nilpotent two-sided ideal $I$ of $A(U)$ such that $A(U)/I$ is isomorphic to a finite product of Azumaya algebras whose centers are regular. By Lemma 4.11, any finitely generated $A(U)/I$-module has finite projective dimension. Lemma 4.13 therefore shows that $A(U)/I$ is a classical generator of $\text{D}^b(\text{mod}(A(U)))$. Now transfer this statement to $\text{D}^b(\text{coh}(A|_U))$ using the equivalence $\text{coh}(A|_U) \cong \text{mod}(A(U))$ (cf. (4.3)). □

4.6. Global existence of a classical generator.

Theorem 4.15. Let $X$ be a noetherian $J$-2 scheme and $A$ a coherent $\mathcal{O}_X$-algebra. Then $\text{D}^b(\text{coh}(A))$ has a classical generator.

Proof. By noetherian induction we may assume that the category $\text{D}^b(\text{coh}(A|_Z))$ has a classical generator for all proper closed subschemes $Z$ of $X$; note that any such $Z$ is again noetherian $J$-2. Obviously, we may assume that $X \neq \emptyset$.

Proposition 4.14 yields a non-empty open subset $U$ of $X$ such that $\text{D}^b(\text{coh}(A|_U))$ has a classical generator. Equip $Z := X - U$ with the reduced scheme structure. By noetherian induction we know that $\text{D}^b(\text{coh}(A|_Z))$ has a classical generator. Proposition 4.7 then shows that $\text{D}^b(\text{coh}(A))$ has a classical generator. □

Remark 4.16. Theorem 4.15 shows in particular that $\text{D}^b(\text{coh}(\mathcal{O}_X))$ has a classical generator if $X$ is a noetherian $J$-2 scheme. We refer the reader to [Rou08, Thm. 7.38] (concerning strong generation) and [Lun10, Prop. 6.8] for related statements for separated schemes of finite type over a field. There are more general recent results by Neeman concerning strong generation (see [Nee17]).
Remark 4.17. In the setting of Theorem 4.15, the category $\text{coh}(\mathcal{A})$ even has a generator as defined in [IT18, 2.3] (this certainly implies Theorem 4.15). The proof of this result is a straightforward variation of the results leading to Theorem 4.15; instead of working on the triangulated level one works on the abelian level and uses the Serre localization sequence (4.5) instead of the Verdier localization sequence (4.4).

4.7. Boxproduct of classical generators. If $X$ and $Y$ are schemes over a field $k$ we write $X \times Y$ instead of $X \times_k Y$. If $E$ is an $\mathcal{O}_X$-module and $F$ is an $\mathcal{O}_Y$-module we abbreviate $E \boxtimes F := p^* E \otimes_{\mathcal{O}_{X \times Y}} q^* F$ where $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ are the projections.

If $X$ and $Y$ are affine, say $X = \text{Spec} R$ and $Y = \text{Spec} S$, and $E$ and $F$ are quasi-coherent, then $E \boxtimes F$ corresponds to the $R \otimes_k S$-module $E(X) \otimes_k F(Y)$.

If $\mathcal{A}$ is an $\mathcal{O}_X$-algebra and $\mathcal{B}$ is an $\mathcal{O}_Y$-algebra, then $\mathcal{A} \boxtimes \mathcal{B}$ is a an $\mathcal{O}_{X \times Y}$-algebra. Given an $\mathcal{A}$-module $E$ and a $\mathcal{B}$-module $F$, then $E \boxtimes F$ is in the obvious way an $\mathcal{A} \boxtimes \mathcal{B}$-module. If $X$ and $Y$ are affine and $\mathcal{A}$, $\mathcal{B}$, $E$ and $F$ are quasi-coherent over the structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ respectively, then the $\mathcal{A} \boxtimes \mathcal{B}$-module $E \boxtimes F$ corresponds to the $\mathcal{A}(X) \otimes_k \mathcal{B}(Y)$-module $E(X) \otimes_k F(Y)$.

Theorem 4.18. Let $X$ and $Y$ be noetherian J-2 schemes over a perfect field $k$ such that $X \times Y$ is Noetherian (these assumptions are for example satisfied if $X$ and $Y$ are schemes of finite type over the perfect field $k$). Let $\mathcal{A}$ be a coherent $\mathcal{O}_X$-algebra and let $\mathcal{B}$ be a coherent $\mathcal{O}_Y$-algebra. Consider the coherent $\mathcal{O}_{X \times Y}$-algebra $\mathcal{A} \boxtimes \mathcal{B}$. Then the following two statements are true:

(a) There exist a classical generator $E$ of $D^b(\text{coh}(\mathcal{A}))$ and a classical generator $F$ of $D^b(\text{coh}(\mathcal{B}))$ such that $E \boxtimes F$ is a classical generator of $D^b(\text{coh}(\mathcal{A} \boxtimes \mathcal{B}))$.

(b) For each classical generator $E$ of $D^b(\text{coh}(\mathcal{A}))$ and each classical generator $F$ of $D^b(\text{coh}(\mathcal{B}))$ the object $E \boxtimes F$ is a classical generator of $D^b(\text{coh}(\mathcal{A} \boxtimes \mathcal{B}))$.

Proof. The obvious analog of the argument used at the beginning of the proof of Corollary 3.6 shows that (a) implies (b).

We prove (a) in several steps. Proposition 4.10 will play a key role in the proof and motivates the following ad hoc terminology.

A pair $(U, \mathcal{R})$ consisting of an affine J-2 scheme $U$ over $k$ and a coherent $\mathcal{O}_U$-algebra $\mathcal{R}$ is called nice if there is a nilpotent two-sided ideal $I \subset \mathcal{R}(U)$ such that $\mathcal{R}(U)/I \cong A_1 \times \cdots \times A_r$ as $\mathcal{O}_U(U)$-algebras for suitable Azumaya algebras $A_i$ whose centers $Z(A_i)$ are regular rings. Note that each $Z(A_i)$ is a Noetherian ring since it is a finite algebra over the noetherian ring $\mathcal{O}_U(U)$.

Given a nice pair $(U, \mathcal{R})$ we do not distinguish between coherent $\mathcal{R}$-modules and finitely generated $\mathcal{R}(U)$-modules (cf. (4.3)).

Claim 1. Statement (a) is true if $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are nice.

Let $I \subset \mathcal{A}(X)$ and $J \subset \mathcal{B}(Y)$ be nilpotent two-sided ideals such that $\mathcal{A}(X)/I \cong A_1 \times \cdots \times A_r$, $\mathcal{B}(Y)/J \cong B_1 \times \cdots \times B_s$ where all $A_i$ and $B_j$ are Azumaya algebras with regular center. Then, by Lemma 4.13, $\mathcal{A}(X)/I$ and $\mathcal{B}(Y)/J$ are classical generators of $D^b(\text{mod}(\mathcal{A}(X)))$ and $D^b(\text{mod}(\mathcal{B}(Y)))$, respectively.
Note that

\[
\frac{A(X)}{I} \otimes_k \frac{B(Y)}{J} \cong \prod_{i,j} A_i \otimes_k B_j.
\]

Each factor \(A_i \otimes_k B_j\) is an Azumaya algebra over its center

\[
Z(A_i \otimes_k B_j) = Z(A_i) \otimes_k Z(B_j),
\]

by [AG60, Prop. 1.5] (for \(R_1 = Z(A_i), R_2 = Z(B_j)\) and \(R = k\)) and [KO74, Thm. III.5.1]), and this center \(Z(A_i \otimes_k B_j)\) is a regular ring because it is noetherian as a finite algebra over the noetherian ring \(O_X(X) \otimes_k O_Y(Y) = O_{X \times Y}(X \times Y)\) and the field \(k\) is perfect, so we obtain regularity from [TY03, Thm. 6.(e)]. Note that \(I \otimes_k B(Y) + A(X) \otimes_k J\) is a nilpotent two-sided ideal in \(A(X) \otimes_k B(Y)\) with quotient

\[
\frac{A(X) \otimes_k B(Y)}{I \otimes_k B(Y) + A(X) \otimes_k J} = \frac{A(X)}{I} \otimes_k \frac{B(Y)}{J}.
\]

These facts imply, by Lemma 4.13, that \(\frac{A(X)}{I} \otimes_k \frac{B(Y)}{J}\) is a classical generator of \(D^b(\text{mod}(A(X) \otimes_k B(Y)))\). Using the above description of the \(\boxtimes\)-product in terms of modules if \(X\) and \(Y\) are affine, this proves claim 1.

**Claim 2.** Statement (a) is true if at least one of \((X, A)\) and \((Y, B)\) is nice.

Assume without loss of generality that \((Y, B)\) is nice. By noetherian induction on \(X\) we can assume that for all proper closed subschemes \(Z \subset X\) with inclusion morphism \(i: Z \subset X\) there is a classical generator \(E_Z\) of \(D^b(\text{coh}(i^*A))\) and a classical generator \(F\) of \(D^b(\text{coh}(B))\) such that \(E_Z \boxtimes F\) is a classical generator of \(D^b(\text{coh}(i^*A \boxtimes B))\). Here we implicitly use that any such closed subscheme \(Z\) is noetherian J-2 and has the property that \(Z \times Y\) is noetherian.

By assumption, \(X\) is a noetherian J-2 scheme. We may assume that \(X\) is non-empty. Then Proposition 4.10 yields a non-empty affine open subset \(U\) of \(X\) such that \((U, A|_U)\) is nice; this uses that any affine open subscheme of a J-2 scheme is J-2. Note also that \(U \times Y\) is noetherian.

By claim 1 there are a classical generator \(E_U\) of \(D^b(\text{coh}(A|_U))\) and a classical generator \(F\) of \(D^b(\text{coh}(B))\) such that \(E_U \boxtimes F\) is a classical generator of \(D^b(\text{coh}(A|_U \boxtimes B))\).

Equip \(Z := X - U\) with the reduced scheme structure and let \(i: Z \subset X\) be the inclusion morphism. By noetherian induction (and the observation that (a) implies (b)) there is a classical generator \(E_Z\) of \(D^b(\text{coh}(i^*A))\) such that \(E_Z \boxtimes F\) is a classical generator of \(D^b(\text{coh}(i^*A \boxtimes B))\).

By Theorem 4.4 there is an object \(\hat{E}_U\) of \(D^b(\text{coh}(A))\) such that \(\hat{E}_U|_U \cong E_U\). Then \(i_*E_Z \oplus \hat{E}_U\) is a classical generator of \(D^b(\text{coh}(A))\), by Proposition 4.7. The same proposition shows that

\[
(i_*E_Z \oplus \hat{E}_U) \boxtimes F \cong (i_*E_Z \boxtimes F) \oplus (\hat{E}_U \boxtimes F) \cong (i \times \text{id})_*((E_Z \boxtimes F) \oplus (\hat{E}_U \boxtimes F))
\]

is a classical generator of \(D^b(\text{coh}(A \boxtimes B))\) because \((\hat{E}_U \boxtimes F)|_U \cong E_U \boxtimes F\). This proves the claim.

**Claim 3.** Statement (a) is true for arbitrary \((X, A)\) and \((Y, B)\).

To prove this we proceed as in the proof of claim 2 but do of course not assume that \((Y, B)\) is nice; the only other difference is that we invoke claim 2 at the place where we invoke claim 1 in the proof of claim 2. □
5. Smoothness for some algebras which are finite over their center

**Theorem 5.1.** Let $A$ be an algebra (not assumed to be commutative) over a perfect field $k$. Assume that $A$ is a finite module over its center $Z(A)$ and that the center $Z(A)$ is a finitely generated (commutative) $k$-algebra. Then $\mathcal{D}^b(\text{mod}(A))$ is smooth over $k$ (in the sense defined in Remark 2.5).

This result is a generalization of the version of Theorem 3.7 where $k$ is a perfect field. The strategy of proof is very similar.

**Proof.** Remember that $\mathcal{D}^b(\text{mod}(A))$ is equivalent to the category

\[(5.1) \quad \mathcal{T} := \mathcal{D}^b_{\text{mod}(A)}(A) = \mathcal{D}^b(A)_{\text{ps-coh}}\]

because $A$ is noetherian (see equivalence (2.1) and equality (2.2) in Remark 2.1). By our definition of $k$-smoothness of $\mathcal{D}^b(\text{mod}(A))$ in Remark 2.5 we need to prove that $\mathcal{T}$ is $k$-smooth. We will use the sufficient condition for smoothness of Theorem 2.15.

We first need to find a dualizing bimodule for $\mathcal{T}$ (in the sense of Definition 2.9). This will use the notion of a dualizing complex from commutative algebra, see [SP18, 0A7A].

We abbreviate $R := Z(A)$. This is a finitely generated $k$-algebra by assumption. Hence $R$ has a dualizing complex $\omega$, by [SP18, 0A7K]. In particular,

\[(5.2) \quad R\text{Hom}_R(-, \omega): \mathcal{D}^b_{\text{mod}(R)}(R) \rightleftarrows \mathcal{D}^b_{\text{mod}(R)}(R)^{\text{op}}: R\text{Hom}_R(-, \omega)\]

is an adjoint equivalence, by [SP18, 0A7C]. Note that $\omega \in \mathcal{D}^b_{\text{mod}(R)}(R)$. We can and will assume in the following that $\omega$ is a bounded below complex of injective $R$-modules and hence an $h$-injective complex of $R$-modules; this means that we can replace $R\text{Hom}$ by $\text{Hom}$ in the above adjunction and assume that unit and counit of the adjunction are the obvious maps into the biduals with respect to $\omega$.

If $M \in C(A)$ is a complex of (right) $A$-modules, then $\text{Hom}_R(A, \omega) \in C(A^{\text{op}})$ is a complex of left $A$-modules. Similarly, if $N \in C(A^{\text{op}})$ is a complex of left $A$-modules, then $\text{Hom}_R(A, \omega) \in C(A)$ is a complex of (right) $A$-modules. Moreover, the unit $M \to \text{Hom}_R(\text{Hom}_R(M, \omega), \omega)$ and counit $N \to \text{Hom}_R(\text{Hom}_R(N, \omega), \omega)$ are morphisms in $C(A)$ and $C(A^{\text{op}})$, respectively. Hence the adjunction in the lower row of the following diagram gives rise to the adjunction in its upper row (note that the functors need not be decorated with a derived symbol).

\[
\begin{array}{ccc}
\mathcal{D}(A) & \xrightarrow{\text{Hom}_R(-, \omega)} & \mathcal{D}(A^{\text{op}})^{\text{op}} \\
\text{Hom}_R(-, \omega) & & \text{Hom}_R(-, \omega) \\
\mathcal{D}(R) & \xleftarrow{\text{Hom}_R(-, \omega)} & \mathcal{D}(R)^{\text{op}} \\
\text{Hom}_R(-, \omega) & & \text{Hom}_R(-, \omega)
\end{array}
\]

The vertical arrows are the restriction functors along $R = Z(A) \to A$. The diagram is commutative if we ignore the two horizontal arrows pointing to the left or the two horizontal arrows pointing to the right.

An $A$-module is finite over $A$ if and only if it is finite over $R$, because $A$ is a finite $R$-module. Hence an object $M \in \mathcal{D}(A)$ is in $\mathcal{D}^b_{\text{mod}(A)}(A)$ if and only if its restriction $M|_R \in \mathcal{D}(R)$ is in $\mathcal{D}^b_{\text{mod}(R)}(R)$. Since we know that the lower adjunction restricts to the adjoint equivalence (5.2), we deduce that the upper adjunction restricts to
an adjoint equivalence

\( \text{Hom}_R(-, \omega): \mathcal{T} = \text{D}_{\text{mod}(A)}^b(A) \cong \text{D}_{\text{mod}(A^{op})}^b(A^{op})^{op} \colon \text{Hom}_R(-, \omega). \)

We will see that this adjoint equivalence originates from a complex of bimodules. The natural candidate is

\[ \mathcal{D} := \text{Hom}_R(A, \omega) = \text{Hom}_R(A \otimes_k A, \omega) \]

which is a complex of \( A \otimes_R A^{op}\)-modules and may also be viewed as a complex of \( A \otimes_k A^{op}\)-modules. We have

\[
\text{Hom}_A(M, \mathcal{D}) = \text{Hom}_A(M, \text{Hom}_R(A \otimes_k A, \omega)) = \text{Hom}_R(M \otimes_A A \otimes_k A, \omega) = \text{Hom}_R(M, \omega)
\]

in \( C(A^{op}) \) natural in \( M \in C(A) \) and

\[
\text{Hom}_{A^{op}}(N, \mathcal{D}) = \text{Hom}_{A^{op}}(N, \text{Hom}_R(A \otimes_k A, \omega)) = \text{Hom}_R(A \otimes_k A \otimes_k A, N, \omega) = \text{Hom}_R(N, \omega)
\]

in \( C(A) \) natural in \( N \in C(A^{op}) \). Hence the two functors in the adjoint equivalence (5.3) may be written as

\[
\text{Hom}_R(-, \omega) = \text{Hom}_A(-, \mathcal{D}) \quad \text{and} \quad \text{Hom}_R(-, \omega) = \text{Hom}_{A^{op}}(-, \mathcal{D}).
\]

Moreover, the unit and counit of the adjunction (5.3) correspond to the unit and counit of the adjunction obtained from \( \mathcal{D} \), cf. (2.4). Since (5.3) is an adjoint equivalence, Remark 2.12 shows that \( \mathcal{D} \) is a dualizing object for \( \mathcal{T} \) and that

\[
\mathcal{T}^\vee = \text{D}_{\text{mod}(A^{op})}^b(A^{op}) = \text{D}_{\text{ps-coh}}^b(A^{op})
\]

where the last equality comes from (2.2).

It remains to check conditions (A1) and (A2) from Theorem 2.15 in our situation.

Let \( X = \text{Spec} R \) and let \( A \) be the coherent \( \mathcal{O}_X \)-algebra corresponding to the finite \( R \)-algebra \( A \). Then we have equivalences of categories

\[
\mathcal{T} = \text{D}_{\text{mod}(A)}^b(A) \cong \text{D}_{\text{mod}(A)}^b(A) \cong \text{D}_{\text{mod}(A)}^b(A)
\]

(cf. the equivalence (4.3)). Since \( X = \text{Spec} R = \text{Spec} \mathcal{O}_X(\mathcal{A}) \) is of finite type over \( k \) it is a noetherian \( J\)-2 scheme, and hence Theorem 4.15 shows that \( \text{D}_{\text{mod}(A)}^b(A) \) and hence \( \mathcal{T} \) have a classical generator. This together with the equalities (5.1) and (5.4) shows that condition (A1) is satisfied.

Let \( E \) be a classical generator of \( \mathcal{T} \cong \text{D}_{\text{mod}(A)}^b(A) \). Its dual \( F := \text{Hom}_A(E, \mathcal{D}) \) is then a classical generator of \( \mathcal{T}^\vee \cong \text{D}_{\text{mod}(A^{op})}^b(A^{op}) \). In order to check condition (A2), or rather the equivalent condition (A2)' in Remark 2.16, we need to prove that the thick subcategory of \( \text{D}(A \otimes_k A^{op}) \) generated by \( E \otimes_k F \) contains \( \mathcal{D} \). Obviously we have

\[
\mathcal{D} = \text{Hom}_R(A, \omega) \in \text{D}_{\text{mod}(A \otimes_k A^{op})}^b(A \otimes_k A^{op}) \cong \text{D}_{\text{mod}(A \otimes_k A^{op})}^b(A \otimes_k A^{op}) \cong \text{D}_{\text{mod}(A \otimes_k A^{op})}^b(A \otimes_k A^{op}) \cong \text{D}_{\text{mod}(A \otimes_k A^{op})}^b(A \otimes_k A^{op})
\]

where \( A \otimes_k A^{op} \) is the coherent \( \mathcal{O}_{X \times_k X} \)-algebra corresponding to the finite \( (R \otimes_k R) \)-algebra \( A \otimes_k A^{op} \). If we view \( E \) as an object of \( \text{D}_{\text{mod}(A)}^b(A) \) and \( F \) as an object of \( \text{D}_{\text{mod}(A^{op})}^b(A^{op}) \) it is therefore enough to show that \( E \otimes F \) is a classical generator of \( \text{D}_{\text{mod}(A \otimes_k A^{op})}^b(A \otimes_k A^{op}) \). But this is true by Theorem 4.18 since \( k \) is assumed to be perfect. \( \square \)
Remark 5.2. This is the analog to Remark 3.9. Let \( A \) be an algebra over a perfect field \( k \) as in Theorem 5.1, i.e. \( A \) is a finite module over its center \( Z(A) \) and the center \( Z(A) \) is a finitely generated \( k \)-algebra. Assume in addition that \( A \) is right-regular in sense of [MR87, 7.7.1]: Any finitely generated module has finite projective dimension. Then \( A \) is a classical generator of \( D^{b}(\text{mod}(A)) \). Hence \( A \) is \( k \)-smooth, by Theorem 5.1 and Remark 2.4.

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