Adversarial Scheduling in Evolutionary Game Dynamics

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Abstract. Consider a system in which players at nodes of an underlying graph \(G\) repeatedly play Prisoner's Dilemma against their neighbors. The players adapt their strategies based on the past behavior of their opponents by applying the so-called win-stay lose-shift strategy. This dynamics has been studied in [Kit94,DGG+02,MR06]. With random scheduling, starting from any initial configuration with high probability the system reaches the unique fixed point in which all players cooperate. This paper investigates the validity of this result under various classes of adversarial schedulers. Our results can be summarized as follows:

- An adversarial scheduler that can select both participants to the game can preclude the system from reaching the unique fixed point on most graph topologies.
- A nonadaptive scheduler that is only allowed to choose one of the participants is no more powerful than a random scheduler. With this restriction even an adaptive scheduler is not significantly more powerful than the random scheduler, provided it is “reasonably fair”.

The results exemplify the adversarial scheduling approach we propose as a foundational basis for the generative approach to social science [Eps07].

Keywords: evolutionary games, self stabilization, discrete dynamical systems, adversarial analysis

1 Introduction

Evolutionary game theory [Wei95] and agent-based simulation in the social sciences [BW00,TJ06,GT05] share the object of study and a significant set of concerns, and are largely distinguished only by the method of choice (mathematical reasoning versus computational experiments). In particular, the two areas deal with fairly similar models (see, e.g. [Wil06,BEM05]). A particular class of such models assumes a large population of agents located at the vertices of a graph.

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Agents interact by playing a fixed game, and update their behavior based on the outcome of this interaction (according to a pre-specified rule).

How does one interpret properties of such systems, be they obtained through mathematical analysis or computational simulations? A possible answer is that results characterizing dynamical properties of such models provide insights (and possible explanations) for features observed in “real-world” social dynamics. For instance, the primary intuition behind the concept of *stochastically stable strategies* in evolutionary game theory is that a small amount of “noise” (or, equivalently, small deviations from rationality) in game dynamics can solve the equilibrium selection problem, by focusing the system on one particular equilibrium. Similarly, in agent-based social theory, Epstein [Eps07,Ep99] (see also [AE96]) has advocated a generative approach to social science. The goal is to explain a given social phenomenon by generating it using multiagent simulations. Related concerns have been recently voiced throughout analytical social science, with a particular emphasis on mechanism-based explanations [Hed05,HS06].

Given that game theory and agent-based simulation are emerging as tools for guiding political decision-making (see e.g. [BEM05,ECC+04,EGK+04,TRA]), it is important to make sure that the conclusions that we derive from these techniques are robust to small variations in model specification.

In this paper we consider the effect of one particular factor that can affect the robustness of results of agent-based simulations and evolutionary game theory: *agent scheduling*, i.e. the order in which agents get to update their strategies. Many models in the literature assume one of two popular alternatives:

- **synchronous update**: every player can update at every moment (in either discrete or continuous time); this is the model implicitly used by “large population” models in evolutionary game theory.
- **uniform matching**: agents are vertices of a (hyper)graph. At each step we choose a (hyper)edge uniformly at random and all players corresponding to this hyperedge are updated using the local update function.

Instead of postulating one of these two update mechanisms, we advocate the study of social dynamics under an approach we call *adversarial scheduling*. We exemplify adversarial scheduling by studying, in such a setting, the Iterated Prisoners’ Dilemma game with win-stay lose-shift strategy. This dynamics (originally motivated by the colearning model in [ST94]) have received substantial attention in the game-theoretic literature [Kit94,DGG+02,MR06].

We are, of course, far from being the first researchers to recognize the crucial role of scheduling/activation order on the properties of social dynamics (to

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4 cf. [Eps07], Chapter 1: “if you didn’t grow it [the social phenomenon, n.n.], you didn’t explain its emergence”.

5 The importance of stability has been previously recognized in the literature, for instance, in [AE96] (pp. 35), where emergent phenomena are defined as “stable macroscopic patterns arising from local of interaction of agents”.
only give two examples, see [HG93,Axt00]). However, what we advocate is a
(somewhat) more systematic approach, based on the following principles:

(i) Start with a "base case" result \( P \), stated under a particular scheduling model.
(ii) Identify several structural properties of the scheduling model that impact the
validity of \( P \). Ideally, these properties should be selected by careful exami-
nation of the proof of \( P \), which should reveal their importance.
(iii) Identify those properties (or combinations of properties) that are necessary/sufficient
for the validity of \( P \). Correspondingly, identify properties
that that are inconsequential to the validity of \( P \).
(iv) The process outlined so far can continue by recursively applying steps (i)-(iii). In the process
we may need to reformulate the original statement in a way that makes it hold under larger classes of schedulers,
thus making it more robust. The precise reformulation(s) normally arise from inspecting the
cases when the proof of \( P \) fails in an adversarial setting.

The intended benefits of the adversarial scheduling approach are multiple
(see [Ist06] for further discussions). The aim of this paper is to show that these
benefits do not come at the expense of mathematical tractability: at least for
one nontrivial dynamics, adversarial scheduling (as outlined in the five point
approach above) is feasible and can lead to interesting results.

2 Preliminaries

First we define two classes of graphs that we will frequently consider in this
paper. The \textit{line graph} \( L_n \), \( n \geq 1 \) consists of \( n \) vertices \( v_1, \ldots v_n \) and edges
\((v_i, v_{i+1})\), \( 1 \leq i \leq n-1 \). We will use \textit{Line} to denote the set of all line graphs.
A \textit{star graph} \( \text{Star}_n \), \( n \geq 1 \), is the complete bipartite graph \( K_{1,n} \). We will use
\textit{Star} to denote the set of all star graphs.

2.1 Basic Model: Prisoner’s Dilemma with Pavlov dynamics

Next we describe the basic mathematical model for Prisoner’s Dilemma with
Pavlov dynamics; see [Kit94,MR06,DGG+02] for additional discussion. We are
given an undirected graph \( G(V,E) \), \( |V| = n \) and \( |E| = m \). Each vertex \( v \in V \)
represents an agent. Each agent has a label from the set \{0, 1\}. These labels
denote the strategies that the players follow: 0 can also be equivalently viewed
as cooperation and 1 can be viewed as defection. Without loss of generality, we
assume that \( G \) is connected, otherwise the dynamics will reduce to independent
dynamics on the connected components of \( G \). We will also assume that the graph
contains at least two edges.

Time will be discrete. At time \( t = 0 \) all nodes are assigned a label from \{0, 1\}.
At each subsequent step, certain nodes/agents change their label (strategy) ac-
cordingly to the rules given below. We will use \( x_t(v) \) to denote the label of node
At each step $t + 1$ an edge $e = (u, v)$ is selected according to some rule and the states of $u$ and $v$ are updated as follows:

$$x_{t+1}(u) ← (x_t(u) + x_t(v)) \pmod{2}$$

$$x_{t+1}(v) ← (x_t(u) + x_t(v)) \pmod{2}$$

We will use $X_t$ to denote the vector $(x_t(v_1), ..., x_t(v_n))$ representing the states of the nodes $v_1, ..., v_n$. This will be sometimes referred to as the global configuration. Sometime we will omit the subscript $t$ for ease of exposition and its value will be clear from context. With this terminology, each step of the dynamics can be viewed as a global update function $F$. It takes as input an element $e = (v_i, v_j) \in E$, $X = (x(v_1), ..., x(v_n))$ and returns the next global $Y = (y(v_1), ..., y(v_n))$ given as follows: $\forall v_k, \text{s.t. } k \neq i \text{ and } k \neq j, y(v_k) = x(v_k)$; and $y(v_i) = y(v_j) = (x(v_i) + x(v_j)) \pmod{2}$. In this case $Y$ is said to be reachable from $X$ in one step. A global configuration $X$ is said to be a fixed point if $\forall e \in E, F(X, e) = X$. It is easy to see the dynamical system studied here has a unique fixed point $0 = (0, ..., 0)$. Following dynamical systems literature, a configuration $X$ is called a Garden of Eden configuration if the configuration is not reachable from any other configuration. For the rest of this paper, we will use $X, Y, ...$ to denote global configurations. An instance of the Prisoner’s Dilemma with Pavlov dynamics (PDPD) can thus be represented as a $(G, f)$, where $G$ is the underlying interaction graph and $f$ is the local function associated with each node. In the remainder of this paper, since $f$ is always fixed, an instance will be specified simply by $G$.

### 2.2 The base-case result

The following property of the PDPD is easily seen to hold under random matching: for all interaction graphs $G$ with no isolated vertices the system converges with probability $1 - o(1)$ to the “all zeros” configuration (henceforth denoted $0$). With a slight abuse of convention, we will refer to this event as self-stabilization. This will the statement we will aim to study in an adversarial setting.

We will also be interested in the convergence time of the dynamics. Under random scheduling Dyer et al. [DGG+02] prove that the number of steps needed to self-stabilize is $O(n \log n)$ on $C_n$ (the simple cycle on $n$ nodes) and exponential in $n$ on $K_n$ (the complete graph on $n$ nodes). The convergence time was further investigated by Mossel and Roch [MR06].

### 2.3 Types of scheduler

A schedule $S$ is specified as an infinite string over $E$, i.e. $S \in E^*$. Given a schedule $S = (e_1, e_2, ..., e_t, ...)$, the graph $G$ and an initial configuration $I$, the dynamics of the system evolve as follows. At time $t = 0$ the system is in state $I$. At time $t$, we pick the $t^{th}$ edge from $S$. Call this edge $e_t$. If the configuration at the
beginning of time $t$ is $X$ then the configuration $Y$ at the beginning of time $t+1$ is given by $Y \leftarrow F(X, e_t)$. The iterated global transition function $F^*$ is defined as

$$F^*(I, (e_1, e_2, \ldots, e_t, \ldots)) \equiv F^*(F(I, e_1), (e_2, \ldots, e_t, \ldots))$$

We say that a system $G$ self stabilizes for a given initial configuration $I$ and a schedule $S = (e_1, \ldots, e_t, \ldots)$ if $\exists t \geq 1$ such that the system starting in $I$ reaches the (unique fixed point) configuration $0$ after $t$ time steps, i.e. $0 \leftarrow F(I, (e_1, \ldots, e_t))$. $G$ is said to self stabilize for a schedule $S$ if $G$ eventually reaches a fixed point when started at any initial configuration $I$, i.e. $\forall I, F(I, S) \rightarrow 0$. Conversely, a schedule $S$ can preclude self stabilization of $G$ if $\exists I$ such that $F(I, S)$ does not ever reach $0$. Given a set of schedules, a scheduler is simply an algorithm (possibly randomized) that chooses a schedule. The schedulers considered here are all polynomial time algorithms and the set of feasible schedules are described below.

Schedulers can be adaptive or non-adaptive. An adaptive scheduler decides on the next edge or node based on the current global configuration. A non-adaptive scheduler decides on a schedule in advance by looking at the graph (and possibly the initial configuration). This schedule is then fixed for remainder of the dynamic process. One particularly restricted class of non-adaptive schedules is a fixed permutation of nodes/edges repeated periodically and independent of the initial state of the system. For node updates, this is the model employed in sequential dynamical systems [BHM+03,BEM05].

There are several distinctions that can be made, concerning the power of a scheduler. The first one concerns the number of players that the scheduler is able to choose. There are two possibilities.

- An edge-daemon (or edge-scheduler) is able to choose both players of the interacting pair. In other words, an edge daemon constructs $S$ by selecting edges $e_i \in E$ in some order.
- A node-daemon (or node-scheduler) can choose only one of the players. We can let this player choose its partner. A natural model is to consider the case where partner is chosen uniformly at random among the neighbors of the first player. In such a case, we say that the node-scheduler model has random choice, or random-choice node-scheduler.

### 2.4 Summary of results

Our results can be summarized as follows:

- Not surprisingly, some amount of fairness is a necessary to extend self-stabilization in an adversarial setting.
- The power given to the scheduler makes a big difference in whether or not the system self stabilizes:

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Note that $e_i$ and $e_j$ in the sequence need not be distinct
(i) If the scheduler can *exogenously* choose both participants in the game then (Theorems 2,3) it can preclude convergence on most graphs, even when bounded by fairness constraints.

(ii) On the other hand schedulers that allow a limited amount of *endogeneity in agent interactions*, by only choosing one of the participants are no more powerful than the random scheduler (Theorem 4) when nonadaptive, and are not significantly more powerful (Theorem 5) when adaptive but “reasonably fair”.

We also investigate experimentally (in Section 7) the convergence time of the colearning dynamics for a few nonadaptive schedulers, in a case when the convergence time for the random scheduler is known rigorously.

2.5 Related work

Our approach is naturally related to the *theory of self-stabilization of distributed systems* [Dol00]. Multi-agent systems, like the ones considered in the evolutionary game dynamics, have many of the characteristics of a distributed system: a number of entities (the agents) capable of performing certain *computations* (changing their strategies) based on *local information*. Randomized models of this type (including the model we study in this paper) have been in fact recently considered in the context of self-stabilization [FMP05]. There are, however, a number of differences. First, in self-stabilization the computational entities (processors) are capable of executing a wide-range of activities (subject to certain constraints, for example the requirement that all processors run the same program, in the context of so-called uniform self-stabilizing systems). The goal of such systems is to achieve a certain goal (legal state) in spite of transient errors and malicious scheduling. In contrast, in our setup, there is no “goal”, the computations are fixed, and restricted to steps of the evolutionary dynamics. The only source of uncertainty arises from the scheduling model. A second difference is in the nature of the update rule. Usually, in self-stabilizing systems there is a difference between *enabled* processors, that intend to take a step, and those that indeed take it. Such a notion is not so natural in the context of game-theoretic models.

As mentioned earlier, the dynamics, can be easily recast in the context of Prisoner’s Dilemma: let 1 encode “defection” and 0 encode “cooperation”. Then the update rule corresponds to the so-called *win-stay, lose-shift* [Pos97], or Pavlov strategy. This rule specifies that agents defect on the next move precisely when the strategy they used in the last interaction was different from the strategy used by the other player. It was the object of much attention in the context of Iterated Prisoner’s Dilemma [Axe84], [NS93],[Axe97]. Related versions of the dynamics have an even longer history in the Psychology literature, where they...

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7 The importance of this property has been recognized [Vri06] in the agent-based simulation literature
were proposed to model emergence of cooperation in situations where players do not know precisely the payoffs of the game in which they are participating, and might even be unaware they are playing a game: Sidowski [Sid57] has proposed the “minimal social situation” (MSS), a two-person experiment representing an extremely simple form of interaction between two agents. MSS was first viewed as a game by Thibaut and Kelley [TK59] who called it “Mutual Fate Control”. An explanation for the empirical observations in [TK59] was proposed by Kelley, Thibaut, Radloff and Mundy [HKM62], that raised the possibility that players were acting according to the Pavlov dynamics. MSS was generalized to multiplayer games by Colman et al. [ACT91,Col05], who obtained mathematical characterization for the emergence of cooperation.

3 Fairness in scheduling

A necessary restriction on schedulers we will be concerned with is fairness. In self-stabilization this is usually taken to mean that each node is updated infinitely often in an infinite schedule. We will also consider notions of bounded fairness. A natural definition is the following:

**Definition 1.** Let $b \geq 1$. A scheduler that can choose one item among a set of $m$ elements is (worst-case) $b$-fair if for every agent $x$, no other agent is scheduled more than $b$ times between two consecutive times that $x$ is scheduled.

It is easy to see that a 1-fair edge scheduler chooses a fixed permutation of edges uses this as a periodic schedule. A 1-fair node daemon selects a fixed permutation of nodes and for each node selects a random neighbor and repeats the same permutation (with possibly different partners) periodically.

The fact that we want to investigate properties of the random scheduler suggest investigating fairness of probabilistic schedulers. For such schedulers, the worst-case fairness in Definition 1 is far too restrictive.

**Definition 2.** A probabilistic scheduler is weakly fair if for any node $x$ and any initial schedule $y$, the probability that $x$ will eventually be scheduled, given that the scheduler selected nodes according to $y$ is positive.

The random scheduler is weakly fair. Not every scheduler is weakly fair, and a scheduler need not be weakly fair to make the system self-stabilize. On the other hand, the base-case result does not extend to the adversarial setting when weak fairness is not required:

**Theorem 1** The following are true:

(i) There exists an edge scheduler that is not weakly fair and that makes the system self-stabilize no matter what its starting configuration is.
(ii) For any graph $G$ there exists an edge/node scheduler that is not weakly fair and that prevents the system from self-stabilizing on some initial configuration.

Another restatement of Theorem 1 is that weak fairness is necessary to preclude some "degenerate" schedulers like the ones we construct for the proof of point (ii).

Proof. (i) Consider an edge scheduler that works as follows:
- Choose an edge $e$ that has not yet self-stabilized, i.e. at least one of its endpoints is 1.
- Turn nodes of $e$ to 0 by playing $e$ twice.
- The scheduler never schedules $e$ subsequently.
(ii) Consider a node (edge) scheduler that repeatedly schedules the same node (edge). It is easy to see that the system does not self-stabilize unless the graph consists of a single edge (a star in the case of a node scheduler when the center node is the scheduled one).

Definition 3. A probabilistic scheduler is $O(f(n))$-node fair w.h.p. if the following condition is satisfied while the system has not reached the fixed point: For any schedule $W$ (call its last scheduled node $x$), every node $y$ and every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that, with probability at least $1 - \epsilon$ node $y$ will be scheduled at most $C_\epsilon \cdot f(n)$ times before $x$ is scheduled again.

A scheduler is boundedly node fair w.h.p. if it is $O(f(n))$-node fair w.h.p. for some function $f(n)$.

We emphasize the fact that Definition 2 applies to edge schedulers as well, when a node $x$ is considered to occur at stage $t$ if some edge containing $x$ is scheduled at that step.

With these definitions we have:

Theorem 2 Let $S$ be a (node or edge) weakly fair probabilistic scheduler such that the following result holds: for any initial configuration, the probability that the system self-stabilizes tends to one. Then the scheduler is boundedly node fair.

Proof. We will consider both node and edge schedulers at the same time. Let $\epsilon > 0$ and let $T = T(\epsilon, n)$ be an integer such that, no matter in which configuration $S$ we start the system, the probability that the system does not self-stabilize (taken over all the coin tosses of the scheduler) is at most $\epsilon$.

Consider any state $S$ of the system after a node $x$ has been scheduled, and assume that $S$ is not the absorbing state 0. Run the system for $T$ steps. The probability that the system does not self-stabilize is at most $\epsilon$. On the other hand, if some node $y$ is not played at all during the $T$ steps then the system has no chance to self-stabilize. It follows that the maximum number of times a given node can be scheduled before $x$ is scheduled again is at most $T - 1$.
4 The Power of Edge-Schedulers

From now on we will restrict ourselves to boundedly fair schedulers. This section aims to show that edge-schedulers are too powerful. Indeed, it is easy to show that there exist graphs on which even 1-fair edge-schedulers can prevent self-stabilization. The following two results provide a modest improvement, showing that even 2-fair edge-daemons on any graph are too strong:

**Theorem 3** Let $G$ be an instance of PDPD. Then there exists an initial configuration $I$ and a 2-fair edge-scheduler $S$ that precludes self-stabilization on $G$ starting in configuration $I$.

**Proof.** Consider a sequence of edges $e_0, \ldots, e_k$ (with repetitions allowed) such that

- every edge of $G$ appears in the list.
- for every $i = 0, \ldots, k$, $e_i$ and $e_{i+1}$ have exactly one vertex in common (where $e_{k+1} = e_0$).
- Every edge appears in the sequence at most twice.

We will show that an enumeration $F(G)$ with these properties can be found for any connected graph with more than one edge. Such a sequence specifies in a natural (via its periodic extension) a 2-fair edge daemon. It is easy to see that the only states that can lead to the fixed point are the fixed point and states leading to it in one step: such a state has exactly two (adjacent) ones. But such a state cannot be reached from any other state according to the previously described scheduler, since the edge that would have been “touched” immediately before has unequal labels on its extremities, which cannot be the case after updating it.

We now have to show how to construct the enumeration $F(G)$. First we give the enumeration in the case graph $G$ is a tree. In this case we perform a walk on $G$, listing the edges as follows: suppose the root $r$ is connected, via vertices $v_1, \ldots, v_k$ to subtrees $T_1, \ldots, T_k$. Then define recursively

$$F(G) = (v, t_1)F(T_1)(t_1, v)(v, t_2)F(T_2)(t_2, v) \ldots (t_{k-1}, v)(v, t_k)F(T_k)(t_k, v),$$

where, if the list of edges thus constructed contains two consecutive occurrences of the same edge we eliminate the second occurrence.

Consider now the general case of a connected graph $G$, and let $S(G)$ be a spanning tree in $G$. Edges of $G$ belong to two categories:

(i) Edges of the spanning tree $S(G)$.
(ii) Edges in $E(G) \setminus E(S(G))$.

Define $F(G)$ as the list of edges obtained from $F(S(G))$ in the following way: whenever $F(S(G))$ first touches a new vertex $w$ in $G$ insert the edges in
$E(G) \setminus E(S(G))$ adjacent to $w$ (in some arbitrary order); continue then with $F(S(G))$. It is easy to see that each edge $e$ is listed at most twice in $F(G)$. To prove this consider the two cases, $e \in E(S(G))$ and $e \in E(G) \setminus E(S(G))$. The statement follows in the first case by the recursive definition 1. In the second case it follows by construction, since a nontree edge is visited only when one of its endpoints is first touched in $E(S(G))$.

Even the most restricted edge-schedulers, 1-fair edge-schedulers, are able to preclude self stabilization on a large class of graphs. To see that define

(i) $\mathcal{G}_1$ to be the class of graphs $G$ that contain a cycle of length at least four.
(ii) $\mathcal{G}_2$ to be the class of graphs $G$ that contain no cycles of length at least four and $m$, the number of edges of $G$ is even.
(iii) $\mathcal{G}_3$ to be the class of trees with $n = 4k$ vertices.

**Theorem 4** Let $G$ be a connected graph in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. Then there exists an initial configuration on $G$ and a 1-fair edge-schedule $S$ that is able to forever preclude self-stabilization on $G$.

In other words, connected graphs for which the system self-stabilizes for all 1-fair schedulers have an odd number of edges and all their cycles (if any) have length 3.

**Proof.** Define $\Delta_{i,j} = (d^{(i,j)}_{k,l})$ a $n \times n$ matrix over $\mathbb{Z}_2$ by

$$d^{(i,j)}_{k,l} = \begin{cases} 1 , & \text{if } (k,l) = (i,j) \\ 0 , & \text{otherwise}. \end{cases}$$

(2)

Suppose we represent configurations of the system as vectors in $\mathbb{Z}_2^n$.

Taking one step of the dynamics on an arbitrary configuration $X$ with the scheduled edge being $(i,j)$ leads to configuration $\mathbf{X} = A_{i,j} \cdot X$, where matrix $A_{i,j}$ is given by

$$A_{i,j} = I_n + \Delta_{i,j} + \Delta_{j,i}. \quad (3)$$

Indeed, the only nondiagonal elements of matrix $A_{i,j}$ that are nonzero are in positions $(i,j)$ and $(j,i)$. This means that all elements of a configuration $X$ in positions other than $i,j$ are preserved under multiplication with $A_{i,j}$. It is easy to see that labels in positions $i,j$ change according to the specified dynamics.

Consider a graph $G$ with $m$ edges, $E(G) = \{(i_1,j_1), \ldots, (i_m,j_m)\}$. The action of a 1-fair edge schedule $S$ (specified by permutation $\pi$ of $\{1, \ldots, m\}$) on a configuration $X$ corresponds to multiplication of $X$ by

$$\pi(S) = A_{i_{\pi[1]},j_{\pi[1]}} \cdot A_{i_{\pi[2]},j_{\pi[2]}} \cdot \ldots \cdot A_{i_{\pi[m]},j_{\pi[m]}}. \quad (4)$$

An edge-schedule cannot prevent self-stabilization implies that starting at any initial configuration, the system reaches a fixed point. In other words, $\forall I \in$
\[ Z^n \exists k \in \mathbb{N} \ s.t. [\pi(S)]^k \cdot I = 0. \] Since the number of vectors \( I \) is finite, this is equivalent to saying that \( \exists k_1 \in \mathbb{N}, \forall I, \ [\pi(S)]^{k_1} \cdot X = 0. \) Equivalently this means that \( [\pi(S)]^k = 0, \) i.e. matrix \( \pi[S] \) is nilpotent. Thus, what we want to show is

**Lemma 4.** For any graph \( G \) there exists a schedule \( S \) such that the corresponding matrix \( \pi[S] \) is not nilpotent.

Consider now an arbitrary ordering of vertices in \( G \) and let \( \pi \) be the permutation corresponding to the induced lexicographic ordering of edges of \( G \) (where an edge is seen as an ordered pair, with the vertex of lower index appearing first).

It is easy to see that

\[ \Delta_{i,j} \cdot \Delta_{k,l} = \begin{cases} \Delta_{i,l}, & \text{if } j = k \\ 0, & \text{otherwise.} \end{cases} \tag{5} \]

Indeed, if \( \Delta_{i,j} = (d_{m,n})_{m,n \geq 1} \), then the only way for some element \( c_{m,n} \) of the product \( \Delta_{i,j} \cdot \Delta_{k,l} \) to be nonzero is that there exists at least one term \( d_{m,p} \cdot d_{p,n} \) that is nonzero. But this is only possible for \((i,j) = (m,p)\) and \((p,n) = (k,l)\), in other words for \(m = i, n = l\) and \(p = j = k\), which immediately yields equation (5).

Let us now consider the integer matrix \( \Pi[S] \) obtained by interpreting equations (3) and (4) as equations over integers. Because integer addition and multiplication commute with taking the modulo 2 value, matrix \( \pi[S] \) can be obtained by applying reduction modulo 2 to every element of \( \Pi[S] \).

Let \( P_{i,j} = \Delta_{i,j} + \Delta_{j,i} \). From the definition of matrix \( \pi(S) \) in equation (4) and the definition of matrices \( A_{i,j} \) in equation (3) we see that \( \pi[S] \) is a sum of products, each term in a product corresponding to either a \( P_{i,j} \) or to the identity matrix. Thus

\[ \Pi[S] = I + \sum_{\emptyset \neq S \subseteq \{1, \ldots, m\}} (\prod_{k \in S} P_{\pi[k], \pi[k]}), \tag{6} \]

Consider the directed graph \( \overline{G} \) obtained from \( G \) by duplicating every edge \( \{i,j\} \) of \( G \) into two directed edges \( (i,j), (j,i) \) in \( \overline{G} \). Label every edge \( e \in E(G) \) by the (unique) integer \( k \) such that \( e = \{i_{\pi[k]}, j_{\pi[k]}\} \), and apply the same labelling to the two oriented versions of edge \( e \) in \( \overline{G} \). Then equation (5) shows that nonzero products of matrices \( \Delta \) are in bijective correspondence to directed paths of length two with increasing labels, when read from the starting to the end node of the path. Inductively generalizing these observations to all sets \( S \) we see that products in (6) are nonzero exactly when they specify a directed path in \( \overline{G} \) (i.e. a path in \( G \)) from a vertex \( k \) to a vertex \( l \) with increasing labels when read from \( k \) to \( l \), in which case they are equal to \( \Delta_{k,l} \).
Therefore $\Pi[S] = I + C$, where $C = (c_{i,j})$ is given by

$$c_{i,j} = \begin{cases} \text{# of paths from } i \text{ to } j \text{ with increasing labels,} & \text{if such paths exist} \\ 0, & \text{otherwise.} \end{cases}$$  

(7)

The matrix $\pi[S]$ is, of course, obtained by reducing modulo 2 the elements of $\Pi[S]$. A well known result in linear algebra\footnote{justified as follows: a classical result states that the characteristic and the minimal polynomial of a matrix have the same roots (with different multiplicities). But it is easy to see that the minimal polynomial of a nilpotent matrix is $x^k$ for some $k \leq n.$} is that the characteristic polynomial of a nonzero nilpotent matrix $A$ is $x^n$. Thus, one strategy to show that a given matrix $\pi[S]$ is not nilpotent is to make sure that for some $p$, $0 \leq p \leq n$, the sum $s_p$ of its principal minors of order $p$ is non-zero. This is equivalent to making sure that the sum of the corresponding minors of the associated integer matrix is odd. The proof consists of three cases:

**Case (a) $G \in \mathcal{G}_1$:** We will prove the following

**Lemma 5.** There exist two permutations $\sigma_1$ and $\sigma_2$ with corresponding matrices over integers $A_1 = \Pi[\sigma_1]$ and $A_2 = \Pi[\sigma_2]$ such that

$$\text{trace}(A_2) \equiv (\text{trace}(A_1) + 1) \pmod{2}.$$  

Given Lemma 4, the proof of Lemma 3 for Case (a) follows since matrices $\pi[\sigma_1]$ and $\pi[\sigma_2]$ cannot both be nilpotent. This is true since the trace of matrix $A_1$ and trace of $A_2$ have different parities. We will prove Lemma 4 using a multistep argument, combining the conclusions of Lemmas 3-5 below. Consider first the following “basic” graphs: $K_4$, $K_3 \triangle K_3$ (the graph obtained by merging two triangles on a common edge), and $C_n$, $n \geq 4$.

**Lemma 6.** The conclusion of Lemma 4 is valid for the “basic” graphs.

**Proof.** By the previous result on the value of coefficients $c_{i,j}$ the value of the trace of a matrix $A$ can be easily computed from the number of cycles with increasing labels. Also, note the following:

- Any cycle $C$ contributes a one to at most one $c_{i,i}$, for some vertex $i$ appearing in $C$. This is because of the restriction on the increasing labels, who might be verified for at most one node of the cycle.
- Moreover, any triangle contributes a 1 to exactly one $c_{i,i}$. Therefore, in considering the trace of matrix $A$ triangles add the same quantity irrespective of permutation, and can thus be ignored.

This observation leads to a simple solution when the underlying graph is a simple cycle $C_n$, $n \geq 4$, or the graph $K_3 \triangle K_3$. Note that these graphs contain a unique cycle $C$ of length at least 4. Consider an ordering of the edges of this cycle, corresponding to moving around the cycle. We will create two labellings...
corresponding to this ordering. The first one assigns labels 1 to $|C|$ in this order. The other labelling assigns labels $1, 2, \ldots, |C| - 2, |C|, |C| - 1$ in this order. It is easy to see that the first ordering contributes a 1 to exactly one diagonal element, while the second one does not contribute a 1 to any element. Hence the traces of the corresponding matrices differ by exactly 1.

For graph $K_4$ we first label the diagonal edges by 5 and 6. There are three cycles of length 4 in the graph $K_4$ – one that uses no diagonal edges, the other two using them both. For the outer cycle consisting of no diagonal edges, we consider the two orderings described in the previous case on the outer cycle $C_4$. This as before shows that the traces of the corresponding matrices differ exactly by 1. Next, note that irrespective of the labelling of the non-diagonal edges, the two cycles containing the diagonal edges cannot be traversed in increasing label order, so they do not contribute to the trace of the associated matrix. Therefore, the result follows for graph $K_4$ as well. This completes the proof of Lemma 3.

**Lemma 7.** Let $G$ be a graph and let $G_2$ be a subgraph of $G$ induced by a subset of the vertices in $G$. If the conclusion of Lemma 3 holds for $G_2$, then it holds for $G$.

**Proof.** Extend a permutation of the edges in $G_2$ to a permutation of the edges in $G$ via a fixed labelling of the edges in $E(G) \setminus E(G_2)$ such that

(i) The index of any edge with both ends in $G_2$ is strictly smaller than the index of all edges not in this class.

(ii) The index of any edge with exactly one end in $G_2$ is strictly larger than the index of any edge not in this class.

The trace of the resulting matrix is determined by the cycles with strictly increasing labels. There are several types of such cycles:

(i) Cycles in $G \setminus G_2$. Whether such a cycle can be traversed in increasing label order does not depend on the precise labelling on edges of $G_2$ as long as the conditions of the extension are those described before.

(ii) Cycles containing some edges in $G_2$, as well as additional edges from $G \setminus G_2$.

Because of the restriction we placed on the labelings, the only such cycles that can have increasing labels are the triangles with two vertices in $G_2$ and one vertex in $G \setminus G_2$. Since there exist an unique way to “read” a triangle in the increasing order of the edge labels, their contribution to the total trace is equal to the number of such triangles, and does not depend on the precise labeling of edges in $G_2$, as long as the restriction of the labeling is met.

(iii) Cycles entirely contained in $G_2$.

Let now $\sigma_1, \sigma_2$ be labelings on $G_2$ verifying the conclusion of Lemma 2 and $\overline{\sigma_1}, \overline{\sigma_2}$ extensions to $G$ verifying the stated restriction. The conclusion of the previous analysis analysis is that a difference in the parity of the traces of matrices corresponding to the labelings $\sigma_1$ on $G_2$ directly translates into a difference in the parity of the traces of matrices corresponding to labelings $\overline{\sigma_1}$ on $G$. 
Finally, we reduce the case of a general graph to that of a base case graph via the following result.

**Lemma 8.** Let $G$ be a graph that contains a cycle of length $\geq 4$ and is minimal (any induced subgraph $H$ does not contain a cycle of length $\geq 4$ any more). Then $G$ is one of the “basic” graphs from Lemma 3.

**Proof.** Let $G$ be minimal with the property that it contains a cycle of length $\geq 4$, let $n$ be the number of nodes in $G$ and let $C$ be a cycle of length $\geq 4$ in $G$. Because of minimality, $C$ contains all the vertices of $G$ (otherwise $G$ would not be minimal, since one could eliminate nodes outside $C$). Thus $C$ is a Hamiltonian cycle. If no other edge is present then we get the cycle $C_n$. Moreover, no other edge can be present unless $n = 4$ (otherwise $G$ would contain a smaller cycle of length $\geq 4$ and thus would not be minimal). In this case the two possibilities are $K_4$ and $K_3 \triangle K_3$.

**Case (b) $G \in \mathcal{G}_2$:** Consider the ordering $<_{\text{sum}}$ on the edges of $G$ so that

$$\{i, j\} <_{\text{sum}} \{k, l\}$$

when either

$$i + j < k + l$$

or

$$i + j = k + l \text{ and } \min\{i, j\} < \min\{k, l\}.$$  

In this case $c_{k,k} = 0$ for every $k$, except when $k$ is the middle-index vertex of a triangle (i.e. a triangle with vertex labels $i, j, k$ such that $i < k < j$).

We infer that $s_1 = \text{trace}(A)$ is congruent (mod 2) to $n$ plus the number of triangles in $G$, that is to $m + 1$ (mod 2) (where $m$ is the number of edges of $G$).

**Case (c) $G \in \mathcal{G}_3$:** Consider the sum $s_2$ of principal minors of size 2 of $A$. In a tree there can be at most one path between two nodes. Since we are counting paths with increasing labels, the only way for $a_{i,j} = a_{j,i} = 1$ to hold is that vertices $i$ and $j$ be adjacent. But in this case the corresponding minor is zero. It follows that $s_2$ is the number of sets of different nonadjacent vertices in $G$, that is

$$s_2 = \binom{n}{2} - (n - 1) = \frac{(n-1)(n-2)}{2} = 1 \mod 2$$

if $n = 4k$.

One might suspect that Theorem 2 (ii) extends to all graphs, thus strengthening the statement of Theorem 1 to 1-fair daemons. This is not the case: Let the line graph $L_6$ be an instance of PDPD. Then for all 1-fair edge-schedulers $S$ and for all initial configurations $I$ the system self-stabilizes starting at $I$. We verified this statement via computer simulation, by running PDPD for all $6!$ 1-fair daemons. A result that rendered this experiment computationally feasible is the state-reduction technique highlighted in the proof of (ii): to prove self-stabilization we only needed to consider those initial configurations with exactly
one. Thus we had to run $6 \times 6!$ simulations. It is an open problem to find all graphs for which this happens. However, Theorem 2 (ii) shows that the class of such graphs is really limited.

5 Nonadaptive node-schedulers

As we saw, even 1-fair edge schedulers are able to prevent self-stabilization. What if we only allow the scheduler to choose one of the nodes? In this section we study the Prisoners dilemma with Pavlov dynamics when adversaries are 1-fair node-schedulers. Because only one of the nodes of the scheduled edge is chosen by the adversary and the other one is chosen randomly, the self-stabilization of the system is a stochastic event.

**Theorem 5** Let $\text{Star}_n$ be an instance of PDPD. Then $S_n$ self-stabilizes with probability 1 against any 1-fair scheduler.

*Proof.* One can assume, without loss of generality, that the first node to be scheduled is the center (labeled 0) and the rest of the nodes are scheduled in the order $1, 2, \ldots, n$. Indeed, if the center was scheduled later in the permutation of nodes, it is enough to prove self-stabilization from the configuration that corresponds to first running the system up to just before the center is scheduled, and then viewing the run as initialized at the new configuration, and with a new periodic schedule (that now starts with node 0). As for the order in which the other nodes get scheduled, by relabelling the nodes we may assume without loss of generality that this is $1, 2, \ldots, n$.

Let $a_0, a_1, \ldots, a_n$ be the labels of nodes at the beginning of the process. It is useful to first consider a deterministic version of the dynamics in question, specified as a game between two players

- The first player is choosing one node to be scheduled. It is required that the sequence of nodes chosen by this player forms a periodic sequence $\pi$. The goal of the first player is to prevent self-stabilization.
- Given a node choice by the first player, the second player is responding with a choice of the second node to be scheduled. Unlike the first player, the sequence of nodes chosen by the second player can potentially vary between successive repetitions of the permutation $\pi$. The goal of the second player is to make the system converge to state $0$.

The game above is an example of the scheduler-luck games from the self-stabilization literature [DIM95]. We will provide a strategy for the second player that (when applied) will turn any configuration into the “all zeros” configuration. But a winning strategy for the second player in the scheduler-luck game will be played with positive probability in any round of the scheduler. Thus with probability going to one (as the number of rounds goes to infinity) this strategy will be played at least in one round, making the system converge to state $0$. 
The crux of the strategy is to carefully use the “partner node” of node 0, when
this is scheduled, to create a segment of nodes 1, 2, . . . , i (with i nondecreasing,
and eventually reaching n) with labels zero at the beginning of a round of
scheduling.

This is simple to do at the very beginning: if node 0 plays node 1 (when 0 is
scheduled), then the labels of the two node will be identical, thus when node 1
is scheduled (and plays again node 0) the label of node 1 will be zero.

If at the beginning of a round the label of node 0 is 1, we make it play (when
scheduled at the beginning of a round) the node of smallest positive index (i + 1)
still labelled 1. This will turn the labels of both nodes to 0. Further scheduling
of nodes 1 to i + 1 will not change this, and at the end of the round, nodes 1 to
i + 1 will still be labelled 0.

If, on the other hand at the beginning of the round node 0 is labelled 0, we
make it keep this label (and, thus, not affect the zero labels of nodes 1 to 1) by
making it play (when scheduled) against another node labelled 0 (say node 1).

To complete the argument it remains to show that for any configuration
x0, . . . , xn different from the “all zeros” configuration, in a finite number of rounds
we will reach a configuration where the first case applies, and thus the length of
the “all zero” initial segment increases.

Indeed, assume that x(0) = 0 and it stays that way throughout the process.
Then, denoting by Yt = (x(1)t, . . . , x(n)) the labels of the nodes 1 to n at the
beginning of the t’th round, it is easy to see that the dynamics of the system is
described by the recurrence

\[ Y_{t+1} = B \cdot Y_t, \]

with B = (bi,j) is a matrix of order n over Z2 specified by

\[ b_{i,j} = \begin{cases} 1, & \text{if } i \geq j \\ 0, & \text{otherwise.} \end{cases} \]

Consider now B as a matrix over Z, rather than Z2. It is immediate to show
by induction that Bk = (b(k) i,j) given by

\[ b(k)_{i,j} = \begin{cases} \binom{i+j+k-1}{k-1}, & \text{if } i \geq j \\ 0, & \text{otherwise.} \end{cases} \]

The k’th power over Z2 is obtained, of course, by reducing these values mod
2. In particular define St to be the sum x(1)t + . . . x(n)t. It is easy to see that
St = x(0)t+1 thus by our hypothesis St has to be zero. On the other hand a
consequence of the previous result is that

\[ S_t = \left[ n - i + t - 1 \right] \cdot x_i \pmod{2}. \]

In particular

\[ \Delta x_t := S_{t+1} - S_t = \left[ n - i + t - 1 \right] \cdot x_i \pmod{2}. \]
By induction and algebraic manipulation we generalize this to higher order of iterated differences $\Delta^k x_t = \Delta(\Delta^{k-1} x_t)$ as:

$$
\Delta^k x_t = \left[ \sum_{i=1}^{n-k} \left( \frac{n-i+t-1}{t+k-1} \right) \cdot x_i \right] \pmod{2}.
$$

Let $i_0$ be the smallest index such that $x(i_0) = 1$. Then, by the previous relation

$$
\Delta^{n-i_0} x_t = \left[ \sum_{i=1}^{i_0} \left( \frac{n-i+t-1}{t+k-1} \right) \cdot x_i \right] = \left( \frac{n-i_0+t-1}{n-i_0+t-1} \right) \cdot x_{i_0} = 1 \pmod{2}.
$$

But this contradicts the fact that $S_t = 0$ for every value of $t$ and completes the proof.

A 1-adaptive scheduler keeps repeating the nodes to choose according to a fixed permutation. Thus, for a fixed scheduler and interaction graph we can talk about the probability of stabilization in the limit. Also, for a given fixed scheduler, the event that this limit is one is a deterministic statement. Consequently we can talk of the probability that this event happens when the interaction graph is sampled from a class of random graphs. As noted, for a random scheduler the condition that $G$ has no isolated vertices is necessary and sufficient to guarantee self-stabilization with probability 1. This is also true for the adversarial model in the case of non-adaptive (1-fair) daemons. This is in case with the case of an edge daemon, when even non-adaptive daemons could preclude stabilization.

**Theorem 6** Let $G$ be an instance of PDPD such that $G$ has no isolated vertices. Then for any 1-fair node-scheduler and any initial configuration the system $G$ reaches state 0 with probability 1.

The results of Theorem 5 should be contrasted with the corresponding result for edge schedulers, for which, as we showed, even non-adaptive daemons could preclude stabilization.

The proof consists of the following three components:

(i) our earlier result that guarantees a winning strategy for scheduler-luck game associated to the dynamics when the underlying graph is $S_n$,
(ii) the partition of a spanning forest of $G$ into node disjoint stars and
(iii) the fact that the existence of such a winning strategy is a monotone graph property w.r.t to edge insertions. This is formally stated in the following

**Lemma 9.** Suppose $H$ is a graph such that a winning strategy $W$ exists for the scheduler-luck game on a graph $H$. Let $e \notin E(H)$, and let $L = H \cup \{e\}$. Then $W$ is also a winning strategy for any graph $L$. 
Proof. Given any node choice by the first player, the second player can choose the corresponding node according to strategy \( W \) (thus never scheduling the additional edge \( e \)). The outcome of the game is, therefore, identical on \( H \) and \( L \).

**Proof of Theorem 5.** By Lemma 6 it is enough to show the existence of a winning strategy for the second player in the scheduler-luck game on a graph \( G \), when \( G \) is a tree. We decompose tree \( G \) into a set \( \{ S_1, \ldots, S_p \} \) of node-disjoint stars as follows.

– Root \( G \) at an arbitrary node \( r \).
– Consider the star formed by the root and its children. Call it \( S_1 \).
– Remove the nodes in \( S_1 \) and all edges with one end point incident on nodes in \( S_1 \).
– Recursively apply the procedure on each forest created by the above operation.

Now consider a 1-fair schedule \( \pi \) on graph \( G \), corresponding to a strategy of the first player in the schedule-luck game on \( G \). For every star \( S_i \), the projection \( \pi_i \) of the schedule on the nodes of \( S_i \) (that amounts to only considering scheduled nodes that belong to \( S_i \)) specifies a 1-fair schedule on \( S_i \). According to Theorem 5, the second player has a winning strategy \( W_i \) for the scheduler-luck game on \( S_i \) when the first player acts according to the schedule \( \pi_i \).

Next, we devise a strategy \( W \) for the scheduler-luck game on graph \( G \), by “composing” the winning strategies \( W_i \). Specifically if the node chosen by the first player belongs to star \( S_i \), strategy \( W \) will employ \( W_i \) to choose the corresponding second node. Since on each star \( S_i \) the labels of the node will eventually be \( 0 \), \( W \) is a winning strategy for the second player in the scheduler-luck game on \( G \). \( \square \)

6 Adaptive node schedulers

Nonadaptive schedulers could not preclude self-stabilization. In contrast, as the following theorem shows, 3-fair nonadaptive node-schedulers are still powerful enough to preclude self-stabilization with complete certainty, and so are 2-fair adaptive\(^9\) schedulers.

**Theorem 7** The following are true:

(i) Let the star graph \( S_n \ (K_{1,n}) \) be an instance of PDPD. Then there exists an initial configuration \( I \) and a 3-fair nonadaptive scheduler that precludes self-stabilization on \( \text{Star}_n \) starting in \( I \).

(ii) Let the triangle \( K_3 \) be an instance of PDPD. Then there exists an initial configuration \( I \) and a 2-fair adaptive scheduler that precludes self-stabilization on \( K_3 \) starting in \( I \).

\(^9\) obviously, there are no 1-fair adaptive schedulers.
Proof. (i) Consider the star graph $S_n (K_{1,n})$, with the center labeled 0 and the rest of the nodes labeled 1, 2, \ldots, $n$. We have to provide an example of a 3-fair scheduler that precludes self-stabilization on some initial configuration. This initial configuration has two 1’s, at nodes 1 and 2. The scheduler repeats the schedule \([0, 1, 1, 3, 4, \ldots n - 2, 2, 1, n - 1, n - 1]\). After the scheduling of 0 1 1 the effect is that both nodes have label 0. Thus the scheduling of nodes 3, 4, \ldots, $n - 2$ does not change any label. With node 2 the label of node 0 will change to 1, thus changing in the next step the label of node 1 back to 1. Finally scheduling the node $n - 1$ twice turns back the label of node 0 to 0, thus yielding the initial configuration. It is easy to see that the scheduler is 3-fair.

(ii) Start with configuration $I$ consisting of all ones. The scheduler will adaptively schedule the nodes, in sequences of three, so that at the end of such a 3-block the system is guaranteed to be in configuration $I$ again. Figure 1 describes the strategy of the scheduler, assuming that node 1 is scheduled first.

\[\text{Fig. 1. A round of the 2-fair adaptive scheduler}\]

Elements in the rectangle represent the state of the system, followed by the scheduled node (in square brackets), choices. Labels on the edges represent the possible probabilistic choices of the partner node, with multiple (inconsequential) choices separated by a comma. Note that the scheduler has similar strategies if one of the nodes 2, 3 is scheduled first. Also, note that a 3-block consists of either a permutation of a node, or two nodes, with the initial and the final node in the block being identical. The scheduler proceeds now to create an infinite schedule consisting of 3-blocks according to the following rule:
(a) If a given block $B$ is a permutation then start the next block with the same starting element as $B$.
(b) Otherwise if the given block $B$ is missing node $z$, start the next block by first scheduling $z$.

It is easy to see that the scheduler we constructed is 2-adaptive and precludes self-stabilization.

Although formal definition of the probability of self-stabilization is more complicated in this case, we can talk of the probability of self-stabilization for adaptive daemons as well. However, as we have seen, the result of Theorem 5 is no longer true: on stars, 1-fairness is stronger than 2-fair adaptive scheduling.

It would seem that this result shows that nonadaptiveness is important for self-stabilization. However, we will see that the class of network topologies where this happens is reasonably limited. Indeed, we next study self-stabilization on Erdős-Renyi random graphs $G(n, p)$. We will choose $p$ in such a way that with high probability a random sample from $G(n, p)$ has no isolated vertices$^{10}$. In other words, we require that necessary condition on the topology of $G$ holds with probability $1 - o(1)$. Call a graph $G$ to be good if for any scheduler of bounded fairness and any starting configuration $I$, $G$ starting at $I$ converges to $0$ with probability $1 - o(1)$, as the number of steps goes to infinity.

**Theorem 8** Let $p$ be s.t. $np - \log n \to \infty$. Then with probability $1 - o(1)$ a random graph $G \in G(n, p)$ is good.

Of course, a natural question is whether such a weakening of the original result, from any graph topology satisfying a given condition to a generic random graph satisfying the same condition, is reasonable. We are, however, not the first ones to propose such an approach. Indeed, except for a handful of cases the network that a given social dynamics takes place on is not known in its entirety. Instead, a lot of recent work (see e.g. [PSV07,NBW06,Dur06] for presentations) has resorted to the study of generic properties of random network models that share some of the observable properties of a fixed network (such as the Internet or the World Wide Web.

**Proof.** The plan of the proof is similar to that for 1-fair schedulers. Define a round of a $b$-fair scheduler to consist of a consecutive sequence of $b(n-1) + 1$ steps.

(i) We prove that for graphs from a class $\mathcal{B}$ of “base case” graphs (Lemma 7 below) the second player has a winning strategy in the scheduler-luck game associated to any scheduler of bounded fairness, where the games corresponds to a finite number of rounds of the scheduler.

(ii) We use the monotonicity of the existence of a strategy.

$^{10}$ A random sample from $G(n, p)$ has no isolated vertices with probability $1 - o(1)$ when $[JLR00] np - \log n \to \infty$. 
We show that with probability $1 - o(1)$ the vertices of a random sample graph $G$ from the graph process can be partitioned such that all the induced subgraphs are isomorphic with one graph in $G$.

**Lemma 10.** The following are true:

(i) Let $G$ be a graph with a perfect matching. Then for any scheduler $S$ of bounded fairness, and any initial configuration on $G$, the second player has a winning strategy for the scheduler-luck game corresponding to one round of the scheduler.

(ii) Let the line $L_n$, $n \geq 6$ be an instance of PDPD. Then for any node-scheduler $S$ of bounded fairness and every initial configuration $I$, the second player has a winning strategy in the scheduler-luck game associated with two consecutive rounds of $S$.

**Proof.** (i) A perfect matching $M$ of $G$ specifies a winning strategy in a scheduler-luck game: each node plays (when scheduled) against its partner in $M$. Since every node is scheduled at least once in a round of scheduling, every edge of $M$ is played at least twice. Therefore, irrespective of the initial configuration, the final configuration is $0$.

(ii) The lemma only needs to be proved in the case when $n$ is odd (in the other case the strategy based on perfect matchings applies). In this case the winning strategy is specified as follows:

(a) In the first round turn the leftmost $L_4$ portion of $L_n$ into the all zero state by playing the matching based winning strategy.

(b) In the second round nodes in the leftmost $L_3$ will only choose to play against each other when scheduled, thus remaining at 0. The remaining nodes form a graph isomorphic to $L_{n-3}$, and in this round we use the perfect matching based strategy for this graph.

We now note now that the statement of Lemma 6 extends to scheduler-luck games associated to node daemons of bounded fairness. The proof is similar (a strategy for the game on $G$ is also a strategy for the game on a graph with a larger set of edges). Theorem 7 immediately follows if $n$ is even: a classical result in random graph theory (see e.g. [JLR00] pp. 82-85) asserts that with probability $1 - o(1)$ $G$ will have a perfect matching.

To complete the proof of Theorem 7 we only need to deal with the case when $n$ is odd. For a graph $G$ and a set of vertices $V$ denote by $G|_V$ the subgraph induced by vertex set $V$.

**Lemma 11.** With probability $1 - o(1)$ $G$ can be partitioned into $V = V_1 \cup V_2$ such that:

(i) $G|_{V_1}$ contains $L_7$ as an edge-induced subgraph.

(ii) $G|_{V_2}$ is a graph with a perfect matching.
Theorem 7 follows from Lemma 8, since for both $G|_{V_1}$ and $G|_{V_2}$ the second player has a winning strategy in the scheduler-luck game. A winning strategy for the corresponding game on $G$ proceeds by using the winning strategy for $G|_{V_1}$, when the scheduled node is in $V_1$ and the winning strategy for $G|_{V_2}$ when the scheduled node is in $V_2$.

The proof of Lemma 8 goes along lines similar to that of the proof of the existence of a perfect matching in a random graph (see [JLR00] pp. 82-85). The first step is to show that w.h.p. $G$ does not contain a set of distinct vertices $x_0, x_1, x_2, x_3, x_4$ such that:

- $\deg_G(x_0) = \deg_G(x_4) = 1$.
- For every $i = 0, 3$, $x_i$ and $x_{i+1}$ are adjacent.

This is easy to see, since the expected number of such structures is $O(n^5p^4(1-p)^2(n-1)) = O(n^5p^4e^{-2np}) = o(1)$, since $p = \Theta(\log(n)/n)$.

Consider now a random graph $G$, conditioned on not containing such a structure, and a vertex $x_0$ in $G$ of degree one. Since this information only exposes information on the edges with one endpoint at $x_0$, with probability $1 - o(1)$ graph $H = G \setminus \{x_0\}$ has a perfect matching. Let $x_1$ be the node in $G \setminus \{x_0\}$ adjacent to $x_0$, and let $x_2$ be the node matched to $x_1$ in $H$. With probability $1 - o(1)$ $x_2$ has another neighbor $x_3$ in $H$, (otherwise $G$ would contain a cherry, i.e. two vertices of degree one at distance exactly 2 in $G$ (see Figure 2). But (see [JLR00] pp. 86) a random sample from $G(n, p)$ only contains a cherry with probability $o(1)$). Let $x_4$ be the node $x_3$ is matched to in $H$. Again, with probability $1 - o(1)$ $x_4$ has a neighbor $x_5$ in $H$ different from $x_3$ (otherwise the five vertices $x_0$ to $x_4$ would form a structure we have conditioned on not occurring in $G$). Finally let $x_6$ be the node matched to $x_5$ in $H$. Then the restriction of $G$ to the set $V_1 = \{x_0, \ldots, x_6\}$ contains a copy of $L_7$, and $G$ restricted to $V_2 = V \setminus V_1$ contains a perfect matching (induced by the perfect matching on $H$).

[Fig. 2. A cherry in a graph (with bold lines)]
7 Speed of convergence

The previous theorems have shown that results concerning convergence to a fixed point can be studied in (and extend to) an adversarial framework. Perhaps what is not preserved as well in the adversarial framework is results on the computational efficiency of convergence to equilibrium. Such results include, for instance, the above mentioned $O(n \log n)$ bound of [DGG+02]. The proof of this theorem displays an interesting variation on the idea of a potential function. It uses such a function, but in this case the value of the function only diminishes “on the average”, rather than for every possible move. Therefore bounding the convergence time seems to critically use the “global” randomness introduced in the dynamics by random matching, and does not trivially extend to adversarial versions. On the other hand, the proof of Theorem 4 only guarantees an exponential upper bound on expected convergence time.

We have investigated experimentally the convergence time on $C_n$ for some classes of 1-fair schedulers (permutations). Some of our results are presented in Figure 3, where we present the average number of rounds, rather than steps, over 1000 samples at each point. The symbol id denotes the identity permutation $(12 \ldots n)$, $p3$ is the permutation $\sigma[i] = 3i \pmod{n}$, $(13)$ refers to permutations with pattern $(13)_{24}5768 \ldots$, and rd refers to the maximum average number of rounds, taken over $10$ random permutations. In all cases the convergence time is consistent with the above-mentioned $O(n \log n)$ result.

| $\pi/n$ | 4  | 8  | 16 | 32 | 64 | 128 |
|--------|----|----|----|----|----|-----|
| id     | 2.486 | 4.225 | 6.401 | 8.33 | 10.498 | 13.135 |
| p3     | 2.469 | 4.039 | 5.807 | 7.662 | 9.639 | 11.718 |
| rd     | 2.289 | 4.499 | 6.527 | 8.781 | 11.161 | 14.151 |
| (13)   | 2.168 | 4.656 | 7.069 | 9.837 | 12.653 | 14.859 |

| $\pi/n$ | 256 | 512 | 1024 |
|--------|-----|-----|------|
| id     | 16.091 | 17.954 | 20.331 |
| p3     | 14.323 | 16.054 | 19.826 |
| rd     | 17.342 | 20.518 | 22.336 |
| (13)   | 18.504 | 20.346 | 20.392 |

Fig. 3. Rounds on $C_n$ under 1-fair scheduling.

We are unable to obtain such a result (and leave it as an interesting open problem)\textsuperscript{11}. It is even more interesting to study the dependency of the mixing time of the dynamics [DGG+02] on the underlying network topology. While there are superficial reasons for optimism (for some models in evolutionary game theory, e.g. [Mor00], the impact of network topology on the convergence speed

\textsuperscript{11} A promising approach is outlined in [FM05]
of a given dynamics is reasonably well understood), the reader is directed to
[MR06] (especially the concluding remarks) for a discussion on the difficulties
of connecting network topology and convergence speed for the specific dynamics
we study.

8 Discussion of Results and Conclusions

We have advocated the study of evolutionary game-theoretic models under ad-
versarial scheduling, similar to the ones in the theory of self-stabilization. As an
illustration we studied the Iterated Prisoners’Dilemma with the win-stay lose-
shift strategy.

Our results are an illustration of the adversarial approach as follows:

(i) Start with some result $P$, valid under random scheduling. The original
statement is presented in Subsection 2.2.

(ii) Identify several structural properties of a random scheduler that
impact the validity of $P$. The random scheduler is

- \emph{fair} more precisely $O(n \log n)$ fair w.h.p. by the Coupon Collector Lemma.
- \emph{endogeneous}, since the next edge to be scheduled is not fixed in advance.
- \emph{nonadaptive}, since the next edge to be scheduled does not depend on the
configuration of the system.

(iii) Identify those properties (or combinations of properties) that are
\emph{necessary/sufficient} for the validity of $P$.

Theorem 1 shows that fairness is a \emph{necessary condition} for the extension of
the original result to adversarial settings. Next, the definition of node and
edge schedulers illustrates another important property of random schedulers: \emph{endogeneity of agent interactions}: an edge scheduler completely specifies the
dynamics of interaction. In contrast, node schedulers provide perhaps the
weakest possible form of endogeneity: the underlying social network is still
fixed, but the agents can choose a neighbor among his neighbors to interact
with (or simply play a random one).

Theorems 4 and 5 show that, in contrast with the case of edge schedulers,
even this limited amount of endogeneity is sufficient to recover the original
result for random scheduler. Moreover, the proofs illuminate the role of en-
dogeneity, that was somewhat obscured in the (trivial) original proof that the
Pavlov dynamics (under random matching) converges with high probability
to the “all zeros” fixed-point. This proof implicitly relies on the fact that
from every state there exists a sequence of “right” moves, that “funnels” the
system towards the fixed point. For \emph{node} schedulers the existence of a such
a set of right moves is proved by explicit construction and is more difficult
in the adversarial setting. The existence of such a set of moves is precisely
what exogeneous choice of agents is able to preclude.
Correspondingly, identify those properties that are inessential to the validity of $P$. In the process one can reformulate (if needed) the original statement in a way that makes it more robust.

Theorem 6 shows that, if we allow schedulers to be adaptive, then network topology becomes important, and can invalidate the original result in an adversarial setting. However adaptiveness (or, equivalently, the amount of fairness) is inessential if we require the convergence result to only hold generically with respect to the class of network topologies described by Erdős-Rényi random graphs.

The results we proved also highlight a number of techniques from the theory of self-stabilization that might be useful in developing a general theory:

– the concept of scheduler-luck game.
– composition of strategies by partitioning the interaction topology.
– monotonicity and “generic preservation” via threshold properties.

Obviously, a reconsideration of more central game-theoretic models under adversarial scheduling is required (and would be quite interesting).

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