ON ROBUST SOLUTIONS TO UNCERTAIN LINEAR COMPLEMENTARITY PROBLEMS AND THEIR VARIANTS

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Abstract. Variational inequality and complementarity problems have found utility in modeling a range of optimization and equilibrium problems arising in engineering, economics, and the sciences. Yet, while there has been tremendous growth in addressing uncertainty in optimization, relatively less progress has been seen in the context of variational inequality problems, exceptions being efforts to solve variational inequality problems with expectation-valued maps \cite{16, 19, 23} as well as suitably defined expected residual minimization (ERM) problems \cite{9, 10}. Both avenues necessitate distributional information associated with the uncertainty and neither approach is explicitly designed to provide robust solutions. Motivated by this gap, we define a robust solution to a complementarity problem as one that minimizes the worst-case of the gap function over a prescribed uncertainty set and requires feasibility for every point in the uncertainty set. In what we believe is amongst the first efforts to comprehensively address such problems in a distribution-free environment, we present an avenue for obtaining robust solutions to uncertain linear complementarity problems. We show that robust solutions to such problems can be tractably obtained through the solution of a single convex program under a broad range of uncertainty sets. We also articulate uncertainty sets that allow for showing that robust solutions to non-monotone generalizations can be obtained tractably. In more general non-monotone regimes, we prove that robust counterparts are low-dimensional nonconvex quadratically constrained quadratic programs. By customizing an existing scheme to these problems, robust solutions to uncertain non-monotone LCPs can be provided. Importantly, these results can be extended to account for uncertainty in the underlying sets by generalizing the results to uncertain affine variational inequality problems defined over uncertain polyhedral sets as well as to hierarchical regimes captured by mathematical programs with uncertain complementarity constraints. Preliminary numerics on uncertain linear complementarity problems and a traffic equilibrium case study suggest that such avenues hold promise.

1. Introduction. The fields of robust control \cite{12} and robust optimization \cite{3} have grown immensely over the last two decades in an effort and are guided by the desire to provide solutions robust to parametric uncertainty. To provide a context for our discussion, we begin by defining a convex optimization problem

\[
\min_{x \in X} f(x; u), \quad (UOpt(u))
\]

where \(X \subseteq \mathbb{R}^n\), \(u \in \mathcal{U} \subseteq \mathbb{R}^L\), \(f : X \times \mathcal{U} \to \mathbb{R}\) is a convex function in \(x\) for every \(u \in \mathcal{U}\). The resulting collection of uncertain optimization problems is given by the following set:

\[
\left\{ \min_{x \in X} f(x; u) \right\}_{u \in \mathcal{U}}.
\]

Given such a set of problems, one avenue for defining a robust solution to this collection of uncertain problems is given by the solution to the following worst case problem:

\[
\min_{x \in X} \max_{u \in \mathcal{U}} f(x; u). \quad (ROpt)
\]

Robust optimization has grown into an established field and there has been particular interest in deriving tractable robust counterparts to (ROpt); in particular, can one formulate a single convex optimization problem whose solution lies in the set of solutions of (ROpt). Such questions have been investigated in linear, quadratic, and in more general convex regimes \cite{3, 5} while more recent efforts have considered integer programming problems \cite{25}.

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A particularly important class of problems that includes convex optimization problems is that of variational inequality problems \cite{13}. Recall that a variational inequality problem \(\text{VI}(X,F)\) requires an \(x \in X\) such that
\[
(y - x)^T F(x) \geq 0, \quad \forall y \in X, \quad (\text{VI}(X,F))
\]
where \(F : X \to \mathbb{R}^n\). Moreover when \(X\) is a cone, it is known \cite{13} that \(\text{VI}(X,F)\) is equivalent to the complementarity problem \(\text{CP}(X,F)\), that requires an \(x\) such that
\[
X \ni x \perp F(x) \in X^*, \quad (\text{CP}(X,F))
\]
where \(X^*\) denotes the dual cone defined as
\[
X^* \triangleq \{ y : y^T x \geq 0, \ x \in X \}
\]
for every \(i\). Such problems have grown increasingly important in control and optimization theory and find application in the modeling of convex Nash games in communication networks \cite{11,27}, traffic equilibrium problems \cite{17}, and spatial economic equilibrium problems. Naturally, in almost all of these settings, uncertainty represents a key concern. For instance, in Nash-Cournot games, the price function of the quantity being sold may have uncertain parameters while in traffic equilibrium problems, travel times are rarely known with certainty. Given such a challenge, one may articulate an uncertain \(\text{VI}(X,F(\bullet;u))\) that requires an \(x\) such that
\[
(y - x)^T F(x;u) \geq 0, \quad \forall y \in X. \quad (1.1)
\]
The resulting collection of uncertain variational inequality problems is given by the following:
\[
\{\text{VI}(X,F(\bullet;u))\}_{u \in U}. \quad (\text{UVI}(X,F,U))
\]
In this paper, we focus on instances where \(F(x;u) \triangleq M(u)x + q(u)\) where \(M(u) \in \mathbb{R}^{n \times n}, \ q(u) \in \mathbb{R}^n\), and \(X \triangleq \mathbb{R}^+_n\) and the resulting affine variational inequality problem is equivalent to the linear complementarity problem:
\[
0 \leq x \perp M(u)x + q(u) \geq 0. \quad (\text{LCP}(u))
\]
Before proceeding, we briefly touch upon earlier efforts in addressing this class of problems. In particular, much of the prior work has focused on the minimization of the expected residual function (cf. \cite{9,10,11} and the references therein). It may be recalled that the residual function \(h\) for \(\text{VI}(X,F)\) is a nonnegative function on a closed set \(D \supseteq X\) such that \(h(x) = 0\) if and only if \(x\) solves \(\text{VI}(X,F)\). Given such a random map \(F(x;\xi)\) where \(\xi : \Omega \to \mathbb{R}^d, \ F : X \times \mathbb{R}^d \to \mathbb{R}^n\), and an associated probability space \((\Omega, F, \mathbb{P})\), the expected residual minimization problem (ERM) problem utilizes the associated residual function \(f(x;\xi)\) and is given by the following:
\[
\min_{x \in X} \mathbb{E}[f(x;\xi)]. \quad (\text{ERM})
\]
Such avenues have derived such solutions for both monotone as well as more general stochastic variational inequality problems but are complicated by several challenges:
(i) First, such avenues necessitate the availability of a probability distribution \(\mathbb{P}\).
(ii) Second, the expected residual minimization problem, given by (ERM), leads to a possibly nonconvex stochastic optimization problem and much of the research has focused on providing estimators of local
solutions to such problems.

(iii) Third, this approach focuses on minimizing the average or expected residual and may be less capable of providing solutions that minimize worst-case residuals unless one employs risk-based variants.

In the spirit of robust approaches employed for the resolution of a range of optimization and control-theoretic problems, we consider an avenue that requires an uncertainty set $\mathcal{U}$. An alternative of immense importance not considered here is the scenario-based approach \[7,8\] and is left as the subject of future work. Furthermore, rather than minimizing the expected residual function, we consider the minimization of the worst-case residual over this uncertainty set. Specifically, we make the following contributions:

(a) First, in the context of stochastic linear complementarity problems with possibly asymmetric positive semidefinite matrices, we show that the robust counterpart is a single convex optimization problem under varying assumptions on the associated uncertainty set.

(b) Second, we observe that somewhat surprisingly, robust solutions to non-monotone regimes can also be obtained tractably under some conditions, revealing that such problems are characterized by hidden convexity. More generally, robust solutions to uncertain non-monotone LCPs are shown to lead to low-dimensional nonconvex quadratically constrained quadratic programs. We customize a recently presented branching scheme to allow for obtaining global solutions to the resulting QCQPs.

(c) Third, we extend our statements to two sets of generalizations: (i) Uncertain affine variational inequality problems over uncertain polyhedral sets; and (ii) Mathematical programs with uncertain complementarity constraints.

(d) These results are validated through preliminary numerics on a class of uncertain LCPs, both monotone and non-monotone, and a traffic equilibrium case study where we observe the differences between ERM solutions and the provided robust solutions.

The remainder of this paper is organized as follows. In Section 2, we motivate our study through two applications and provide an instance of a monotone complementarity problem with arbitrarily high price of robustness. In Section 3, we discuss the tractable solution of uncertain monotone LCPs and a subclass of uncertain monotone LCPs. Non-monotone generalizations and their global solution form the core of Section 4 and generalizations to variational inequality and hierarchical regimes are considered in Section 5. Preliminary numerics are provided in Section 6 and we conclude with a short set of remarks in Section 7.

2. Motivating examples and applications. In this section, we begin by providing an example of an uncertain linear complementarity problem and proceed to discuss two applications that motivate the study of uncertain linear complementarity problems.

A class of uncertain LCPs. Consider the simple LCP, denoted by $eLCP(u)$:

\[
0 \leq \begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} M & 0 \\ S(\xi, \eta) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -q_x \\ q(u) \end{pmatrix} \geq 0, \quad \forall (u, \xi, \eta) \in \mathcal{U}_u \times \mathcal{U}_\zeta \tag{2.1}
\]

where $M = \left( I - \frac{1}{(n+1)e^T}ee^T \right) \in \mathbb{R}^{n\times n}$, $S(\xi, \eta) = \xi S_1 + \eta S_2$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $q_x \in \mathbb{R}_+^n$, $q(u) = u e$, $u \in \mathcal{U}_u \triangleq [0,1]$. Furthermore, $S_1 = nI + e_n e_n^T$ and $S_2 = ee^T + e_n e_n^T$. $e$ denotes the column of ones, $e_n = (1, \ldots, n)^T$ and $\mathcal{U}_\zeta \triangleq \{ \zeta = (\xi, \eta) : \xi + \eta \leq 1, \xi \geq 0, \eta \geq 0 \}$. We begin by noting that a solution to the upper system

\[
0 \leq x \perp Mx - q_x \geq 0
\]
Let $d$ (uncertain) travel cost on equilibrium principle, users choose a minimum cost path between each O-D pair: $u$ vector of minimum travel costs between any OD pair, and the uncertainty connecting each

Additionally, the travel costs are related to demand satisfaction through this problem:

**A robust solution:** The robust solution of this problem is given by $(x^*, y_1)$ where $y_1 \geq 0$. Consequently, the worst-case residual given by $x^T (M x - q_x) + \max \{y^T S_1 y_1, y^T S_2 y_1\}$ could be arbitrarily high since $y_1$ is any nonnegative vector. In effect, a non-robust solution chosen under a single realization can have large worst-case residual.

A robust solution: The robust solution of this problem is given by $(x^*, 0)$ and achieves that worst-case residual equals to $x^T (M x^* - q_x)$.

**Uncertain traffic equilibrium problems.** A static traffic equilibrium model [13, 17] captures equilibrating (or steady-state) flows in a traffic network in which a collection of selfish users attempt to minimize travel costs. Here, we present a path-based formulation in $N$. $N$ denotes the network while $A$ represents the associated set of edges. Further, let $O$ and $D$ denote the set of origin and destination nodes, respectively while the set of origin-destination (OD) pairs is given by $W \subseteq O \times D$. Let $P_w$ denote the set of paths connecting each $w \in W$ and $P = \cup_{w \in W} P_w$. Let $h_p$ denote the flow on path $p \in P$ while $C_p(h; u)$, the associated (uncertain) travel cost on $p$, is a function of the entire vector of flows $h \equiv (h_p)$ and the uncertainty $u \in U$. Let $d_w(v; u)$ represent the uncertain travel demand between O-D pair $w$ and is a function of $v \equiv (v_w)$, the vector of minimum travel costs between any OD pair, and the uncertainty $u \in U$. Based on Wardrop user equilibrium principle, users choose a minimum cost path between each O-D pair:

$$0 \leq h_p \perp v_w - C_p(h) \geq 0, \quad \forall w \in W, \ p \in P_w.$$ (2.2)

Additionally, the travel costs are related to demand satisfaction through this problem:

$$0 \leq v_w \perp \sum_{p \in P_w} h_p - d_w(v) \geq 0, \quad \forall w \in W,$$ (2.3)

Static traffic user equilibrium problem is to solve a pair $(h, u)$ satisfying (2.2) and (2.3), compactly stated as the following uncertain complementarity problem:

$$0 \leq \begin{pmatrix} h \\ v \end{pmatrix} \perp \begin{pmatrix} C(h; u) - B^T v \\ Bh - d(v; u) \end{pmatrix} \geq 0, \quad \forall u \in U$$ (2.4)
where \( C(h; u) = (C_p(h; u) \mid p \in \mathcal{P}) \), \( d(v; u) = (d_w(v; u) \mid w \in \mathcal{W}) \) and \( B \) is the (OD pair, path)-incidence matrix \( (b_{wp}) \):

\[
b_{wp} \triangleq \begin{cases} 
1 & \text{if } p \in \mathcal{P}_w \\
0 & \text{otherwise}
\end{cases}
\]

This represents an uncertain collection of complementarity problems and we desire an equilibrium \((h, v)\) that is robust to uncertainty.

**Uncertain Nash-Cournot games.** Nash-Cournot models for competitive behavior find application in a variety of settings, including the context of networked power markets [18]. We describe an instance of a single node \( N \)-player Nash-Cournot game in which \( N \) players compete in the market for a single good. Suppose player \( i \)'s uncertain linear cost function given by \( c_i(u)x_i \) where \( x_i \) is her production level decision. Furthermore, the \( i \)th player’s capacity is denoted by \( \text{cap}_i \). We assume that sales of the good are priced using an (uncertain) price function dependent on aggregate sales \( X \) and denoted by \( p(X; u) \) where \( u \in \mathcal{U} \). We restrict our attention to settings where this price function is affine and defined as follows:

\[
p(X; u) \triangleq a(u) - b(u)X \quad \text{where} \quad a(u), b(u) > 0, \quad X \triangleq \sum_{i=1}^{N} x_i \quad \text{and} \quad u \in \mathcal{U}.
\]

The ith agent’s problem is given by the following:

\[
\begin{align*}
\min_{x_i} & \quad (c_i(u)x_i - p(X; u)x_i) \\
\text{subject to} & \quad x_i \leq \text{cap}_i, \quad (\lambda_i) \\
& \quad x_i \geq 0.
\end{align*}
\]

The sufficient equilibrium conditions of the Nash-Cournot game are given by the concatenation of the resulting optimality conditions:

\[
\begin{align*}
0 \leq x_i & \perp b(u)(X + x_i) + \lambda_i + c_i(u) - a(u) \geq 0, & \forall i, \\
0 \leq \lambda_i & \perp \text{cap}_i - x_i \geq 0, & \forall i.
\end{align*}
\]

The resulting uncertain LCP is given by the following:

\[
0 \leq z \perp M(u) + q(u) \geq 0, \quad \forall u \in \mathcal{U} \tag{2.5}
\]

where

\[
M(u) \triangleq \begin{pmatrix} b(u)(I + ee^T) & I \\ -I & 0 \end{pmatrix}, \quad q(u) \triangleq \begin{pmatrix} c(u) - a(u)e \\ \text{cap} \end{pmatrix},
\]

\(e\) denotes the column of ones, \(\text{cap}\) is the column of capacities, and \(I\) represents the identity matrix.

**3. Uncertain LCPs with tractable robust counterparts.** While optimization problems admit a natural objective function, such a function is not immediately available when considering variational inequality problems. However, one may define a gap function associated with \(\text{VI}(X, F)\). It may be recalled that a function \(g(x)\) is a gap function for \(\text{VI}(X, F)\) if (i) it is nonnegative restricted on \(X\); and (ii) \(x \in X\) solves \(\text{VI}(X, F)\) if and only if \(g(x) = 0\) (see [21]). When \(X\) is a cone, the problem reduces to a complementarity problem and the a gap function associated with \(\text{CP}(X, F)\) is given by the following:

\[
\theta_{\text{gap}}(x, u) \triangleq \begin{cases} 
F(x, u)^T x & \text{if } F(x, u) \in X^*, \\
+\infty & \text{otherwise}.
\end{cases}
\]
Throughout this section, we assume that the set $X$ is the nonnegative orthant $\mathbb{R}_+^n$, $F(x, u) \triangleq M(u)x + q(u)$, where $M(u) \in \mathbb{R}^{n \times n}$ and $q(u) \in \mathbb{R}^n$ for every $u \in \mathcal{U}$. Furthermore, throughout this paper, we utilize the gap function as the residual function in developing tractable (convex) and relatively low-dimensional robust counterparts of uncertain LCPs with monotone and non-monotone maps. Specifically, in Section 3.1, we consider settings where $q(v)$ is an uncertain vector and $M(u)$ is an uncertain positive semidefinite matrix with $v \in \mathcal{V}$ and $u \in \mathcal{U}$. We provide robust counterparts in regimes where $\mathcal{U}$ and $\mathcal{V}$ are either distinct (unrelated) or related under varying assumptions on the uncertainty sets. In Section 3.2, we provide tractable robust counterparts to regimes where the $M(u)$ is not necessarily positive semidefinite.

### 3.1. Uncertain monotone LCPs

Much of the efforts in the resolution of uncertain variational inequality problems has considered the minimization of the expected residual; instead, we pursue a strategy that has defined the field of robust optimization in that we consider the minimization of the worst-case gap function (residual) over a prescribed uncertainty set. While in its original form, such a problem is relatively challenging nonsmooth semi-infinite optimization problem. Yet, it can be shown that these problems are shown to be equivalent to tractable convex programs. By setting $f(x, u) = \theta_{\text{gap}}(x, u)$, (ROpt) can be recast as follows:

$$
\min_{x} \max_{u \in \mathcal{U}} F(x, u)^T x \\
\text{subject to } F(x, u) \in X^* \quad \forall u \in \mathcal{U}, \\
x \in X.
$$

(3.1)

Before proceeding, it is worth noting that the robust formulation attempts to find a solution that minimizes the maximal (worst-case) value taken by $F(x, u)^T x$ over the set of solutions that are feasible for every $u \in \mathcal{U}$. In fact, the following relationship holds between the optimization problem (3.1) and the original uncertain complementarity problem.

**Lemma 3.1.** Consider the problem given by (3.1). Then $x \in X$ solves

$$
X \ni x \perp F(x, u) \in X^* \quad \forall u \in \mathcal{U},
$$

if and only if $x$ is a solution of (3.1) with optimal value zero.

Unfortunately, it is unlikely that an $x$ exists that solves CP$(X, F(\bullet, u))$ for every $u \in \mathcal{U}$; instead, we focus on deriving tractable counterparts that produce global minimizers $x$ of the problem (3.1). (3.1) can be rewritten as follows:

$$
\min_{u \in \mathcal{U}} \max_{x} x^T (M(u)x + q(u)) \\
\text{subject to } \min_{u \in \mathcal{U}} M_{\bullet}(u)x + q_{\bullet}(u) \geq 0, \quad \forall i, \\
x \geq 0.
$$

(3.2)

or

$$
\min_{u \in \mathcal{U}} \max_{x} t \\
\text{subject to } x^T (M(u)x + q(u)) \leq t, \\
\min_{u \in \mathcal{U}} M_{\bullet}(u)x + q_{\bullet}(u) \geq 0, \quad \forall i, \\
x \geq 0.
$$

(3.3)

We first consider the development of tractable counterparts of (3.2) or (3.3) when $M(u) = M$ and $q$ is uncertain. Subsequently, we consider the more general setting when both $M$ and $q$ are uncertain but are
derived from unrelated uncertainty sets. Finally, we assume that both $M$ and $q$ are derived from related uncertainty sets.

3.1.1. Uncertainty in $q$. In this subsection, $q$ is subject to uncertainty and the problem can be reduced to obtaining a robust solution to an uncertain quadratic program (cf. [3]). We define $q(u)$ as follows:

$$q(u) \triangleq q_0 + \sum_{l=1}^{L} u_l q_l, \quad u \in \mathcal{U}. \quad (3.4)$$

Consequently, (3.2) may be rewritten as follows:

$$\begin{align*}
\min & \quad x^T (Mx + q_0) + \max_{u \in \mathcal{U}} \sum_{l=1}^{L} u_l (x^T q_l) \\
\text{subject to} & \quad M_i x + q_0 + \min_{u \in \mathcal{U}} \sum_{l=1}^{L} u_l |q_l|_i \geq 0, \quad \forall i \\
& \quad x \geq 0.
\end{align*} \quad (3.5)$$

We begin by considering three types of uncertainty sets: $\mathcal{U}_1, \mathcal{U}_\infty$, or $\mathcal{U}_2$, where

$$\mathcal{U}_1 \triangleq \{ u : \|u\|_1 \leq 1 \}, \quad \mathcal{U}_2 \triangleq \{ u : \|u\|_2 \leq 1 \}, \quad \text{and} \quad \mathcal{U}_\infty \triangleq \{ u : \|u\|_\infty \leq 1 \}. \quad (3.6)$$

The proof of our first tractability result is relatively straightforward is inspired by Examples 1.3.2 and 1.3.3 from [3].

PROPOSITION 3.2 (TRC for uLCP($M, q(u)$)). Consider the uncertain LCP given by (3.5) where $M$ is a positive semidefinite matrix. Let $\mathcal{U}$ be defined as $\mathcal{U}_1, \mathcal{U}_2$ or $\mathcal{U}_\infty$ as specified by (3.6). Then (3.5) can be reformulated as a convex program.

Proof.

(i) $\mathcal{U} := \mathcal{U}_\infty$: We begin by noting that $\|w\|_1 = \max_{\eta \in \mathcal{U}} \|u^T \eta\|_1 \leq 1$. Consequently, the term $\sum_{l=1}^{L} u_l x^T q_l$ has maximal value $\sum_{l=1}^{L} |x^T q_l|$. Furthermore, we have that

$$\min_{u \in \mathcal{U}_\infty} \sum_{l=1}^{L} u_l |q_l|_i = \min_{u \in \mathcal{U}_\infty} u^T q = - \max_{u \in \mathcal{U}_\infty} (-u^T q) = - \left\| \begin{pmatrix} q_1 \\ \vdots \\ q_L \end{pmatrix} \right\|_1 = - \sum_{l=1}^{L} |q_l|.$$

Consequently, (3.5) is equivalent to the following:

$$\begin{align*}
\min & \quad x^T (Mx + q_0) + \sum_{l=1}^{L} |q_l^T x| \\
\text{subject to} & \quad Mx + q_0 - \sum_{l=1}^{L} |q_l| \geq 0, \quad x \geq 0.
\end{align*} \quad (3.7)$$

Finally, by adding additional variables, (3.7) can be rewritten as a convex quadratic program (QP):

$$\begin{align*}
\min & \quad x^T (Mx + q_0) + \sum_{l=1}^{L} t_l \\
\text{subject to} & \quad t_l \geq q_l^T x \geq -t_l, \quad \forall l, \\
& \quad Mx + q_0 - \sum_{l=1}^{L} |q_l| \geq 0, \quad x \geq 0.
\end{align*}$$
(ii) \( \mathcal{U} := \mathcal{U}_1 \): We proceed in a fashion similar to (i) and begin by recalling that 
\[
\max_{\|\eta\|_1 \leq 1} \eta^T w = \|w\|_\infty,
\]
leading to the following simplification:
\[
\max_{u \in \mathcal{U}_1} \sum_{l=1}^L u_l x^T q_l = \left\| \begin{pmatrix} x^T q_1 \\ \vdots \\ x^T q_L \end{pmatrix} \right\|_\infty = \max_{l \in \{1, \ldots, L\}} |x^T q_l|.
\]
Similarly, \( \min_{\|\eta\|_1 \leq 1} \eta^T w = -\|w\|_\infty \) and \( \sum_{l=1}^L u_l(q_l)_i \) has minimal value \( \max_{l \in \{1, \ldots, L\}} |(q_l)_i| \). Consequently, the constraints reduce to
\[
M_i x + (q_0)_i - \max_{l \in \{1, \ldots, L\}} |(q_l)_i| \geq 0, \quad \forall i
\]
\[
\Leftrightarrow \quad M_i x + (q_0)_i - |(q_l)_i| \geq 0, \quad \forall i, l
\]
\[
\Leftrightarrow \quad M x + q_0 - |q_l| \geq 0, \quad \forall l.
\]
Similarly, the objective function can be stated as
\[
x^T (M x + q_0) + \max_{l \in \{1, \ldots, L\}} |x^T q_l|.
\]
By adding a variable \( t \), this problem may be reformulated as the following convex quadratic program:
\[
\min x^T (M x + q_0) + t
\]
subject to
\[
t \geq q_l^T x \geq -t, \quad \forall l
\]
\[
M x + q_0 - |q_l| \geq 0, \quad \forall l
\]
\[
x \geq 0.
\]
(iii) \( \mathcal{U} := \mathcal{U}_2 \): By leveraging Example 1.3.3 in [3], it is seen that
\[
\max_{\|\eta\|_2 \leq r} \eta^T w = r \frac{w^T w}{\|w\|_2} = r \|w\|_2.
\]
As a result, \( \sum_{l=1}^L u_l x^T q_l \) has maximal value \( r \sqrt{\sum_{l=1}^L (q_l^T x)^2} \) while
\[
\min_{\|\eta\|_2 \leq r} \eta^T w = -r \frac{w^T w}{\|w\|_2} = -r \|w\|_2,
\]
indicating that \( \sum_{l=1}^L u_l |q_l|_i \) has minimal value \( -r \sqrt{\sum_{l=1}^L |q_l|_i^2} \). Then (3.5) may be rewritten as:
\[
\min x^T (M x + q_0) + r \sqrt{\sum_{l=1}^L |q_l|_i^2}
\]
subject to
\[
M x + q_0 - rq \geq 0,
\]
\[
x \geq 0.
\]
where \( q_l = \sqrt{\sum_{i=1}^L |q_l|_i^2} \). By adding an extra variable, we obtain convex program with a quadratic objective and a conic quadratic inequality.
\[
\min x^T (M x + q_0) + t
\]
subject to
\[
M x + q_0 - rq \geq 0,
\]
\[
x \geq 0.
\]
Next, we present a more result where the uncertainty set is captured by a more general convex set. Specifically, \( \mathcal{U} := \mathcal{U}_c \) where
\[
\mathcal{U}_c \triangleq \{ u \in \mathbb{R}^L : \exists \nu \in \mathbb{R}^k : Pu + Q\nu + p \in K \subseteq \mathbb{R}^N \},
\]
(3.8)

\( K \) is a cone, \( P \) and \( Q \) are given matrices, and \( p \) is a given vector.

**Proposition 3.3** (RC of \( \text{uLCP}(M, q(u)) \) when \( \mathcal{U} := \mathcal{U}_c \)). Consider the uncertain LCP given by (3.5) where \( M \) is a positive semidefinite matrix. Let \( \mathcal{U} := \mathcal{U}_c \), where \( \mathcal{U}_c \) is defined as (3.8). Suppose one of the following holds:

(i) \( K \) is a polyhedral cone;

(ii) \( K \) is a convex cone and the following holds:
\[
\exists (\bar{u}, \bar{\nu}) \text{ such that } P\bar{u} + Q\bar{\nu} + p \in \text{int}(K).
\]
(3.9)

Then the robust counterpart of (3.5) is a tractable convex program.

The proof follows from Theorem 1.3.4 and Proposition 6.2.1 from [3] and is omitted.

**Remark:** If \( K \) is chosen to be the nonnegative orthant, the uncertainty set is a polyhedron given \( Q = 0 \). Both \( \mathcal{U}_1 \) and \( \mathcal{U}_\infty \) are included in this general case. If \( K \) is chosen to be the second-order cone, a special case of the uncertainty set is a ball. Under both circumstances, the problem is tractable. Notice that nonnegative orthants and Lorentz cones are self-dual. When \( K \) is chosen to be \( \mathbb{R}^N_+ \), (3.5) reduces to a convex quadratic program (QP) while if \( K \) is chosen to be \( \mathbb{L}_N \), (3.5) can be recast as a convex quadratically constrained quadratic program (QCQP).

### 3.1.2. Uncertainty in both \( M \) and \( q \) under an independence assumption.

Next, we consider the setting where both \( M \) and \( q \) are uncertain but the sources of uncertainty are independent. This is a somewhat more challenging problem and a direct application of the results from robust quadratic programming appears difficult.

Recall that the map \( F \) is said to be monotone over a set \( X \) if the following holds:
\[
(F(x) - F(y))^T(x - y) \geq 0, \quad \forall x, y \in X.
\]

Additionally, \( Mx + q \) is monotone over \( \mathbb{R}^n_+ \) if and only if \( \frac{1}{2}(M + MT) \) is positive semidefinite (cf. [13]). Without loss of generality, we assume that \( M \) or \( M(u) \) is symmetric through this subsection, if not, we may always replace the matrices by their symmetrized counterparts. For the present, we assume that \( q \) is deterministic and reformulate (3.2) as follows:
\[
\begin{align*}
\min \quad & t \\
\text{subject to} \quad & x^T(M(u)x + q) \leq t, \quad \forall u \in \mathcal{U}, \\
& M(u)x + q \geq 0, \quad \forall u \in \mathcal{U}, \\
& x \geq 0,
\end{align*}
\]
(3.10)

where \( M(u) \) is defined as follows:
\[
M(u) \triangleq M_0 + \sum_{l=1}^{L} u_l M_l,
\]
(3.11)

\( M_0 \geq 0, \quad M_l \geq 0, \quad l = 1, \ldots, L. \)
Of course, $M_l, l = 0, \ldots, L$ have also been assumed to be symmetric. We now present a tractability result for nonnegative restrictions of $\mathcal{U}_l$ and $\mathcal{U}_\infty$ defined as follows:

$$\mathcal{U}_\infty \triangleq \{ u : \|u\|_\infty \leq 1, u \geq 0 \}$$

and

$$\mathcal{U}_l \triangleq \{ u : \|u\|_l \leq 1, u \geq 0 \}.$$  

Note that under the definitions of $\mathcal{U}_\infty$ and $\mathcal{U}_l$, $M(u)$ is always positive semidefinite. This implies that (3.10) is convex for each $u \in \mathcal{U}$.

**Proposition 3.4 (TRC for uLCP($q, M(u)$) for $\mathcal{U}_l, \mathcal{U}_\infty$).** Consider the problem (3.10) where $M(u)$ is defined by (3.11), and $\mathcal{U}$ is chosen either to be $\mathcal{U}_\infty$ or $\mathcal{U}_l$, both of which are defined in (3.12). Then the uncertain LCP has a tractable robust counterpart.

**Proof.**

(a) $\mathcal{U} := \mathcal{U}_\infty$: We first derive the robust counterpart of the following constraint:

$$x^TM_0x + \sum_{l=1}^{L} u_l x^TM_lx + x^Tq \leq t, \quad \forall u \in \mathcal{U}_\infty.$$  

But this can be equivalently stated as

$$x^TM_0x + \max_{u \in \mathcal{U}_\infty} \left[ \sum_{l=1}^{L} u_l x^TM_lx \right] + x^Tq \leq t,$$

We now evaluate the maximum in the right hand side:

$$\max_{u \in \mathcal{U}_\infty} \left[ \sum_{l=1}^{L} u_l x^TM_lx \right] = \sum_{l=1}^{L} \max_{u_l \in [0,1]} \left[ u_l x^TM_lx \right] = \sum_{l=1}^{L} \max_l x^TM_lx = \sum_{l=1}^{L} x^TM_lx,$$

where the last equality is a consequence of applying the positive semidefiniteness of $M_l$ for $l = 1, \ldots, L$. Consequently, the robust counterpart of (3.10) can be stated as follows:

$$\min_{u \in \mathcal{U}_\infty} \left[ \left( \sum_{l=1}^{L} u_l [M_l]_{\bullet \bullet} x \right) \right] + \left[ M_0 \right]_{\bullet \bullet} x + q_i \geq 0, \quad \forall i$$

$$x \geq 0.$$  

We may now simplify the second constraint as follows:

$$\min_{u \in \mathcal{U}_\infty} \left[ \sum_{l=1}^{L} u_l [M_l]_{\bullet \bullet} x \right] = \sum_{l=1}^{L} \min_{u_l \in [0,1]} \left[ M_l \right]_{\bullet \bullet} x = \sum_{l=1}^{L} u_l,$$

where

$$v_l = \begin{cases} [M_l]_{\bullet \bullet} x, & \text{if } [M_l]_{\bullet \bullet} x > 0 \\ 0, & \text{if } [M_l]_{\bullet \bullet} x \leq 0. \end{cases}$$

As a consequence, $v_l = \max(-[M_l]_{\bullet \bullet} x, 0)$ for $l = 1, \ldots, L$ and the TRC may be rewritten as follows:

$$\min_{u \in \mathcal{U}_\infty} \left[ t \right]$$

subject to

$$x^TM_0x + \sum_{l=1}^{L} M_l x + x^Tq \leq t,$$

$$- \sum_{l=1}^{L} \max(-[M_l]_{\bullet \bullet} x, 0) + [M_0]_{\bullet \bullet} x + q_i \geq 0, \quad \forall i$$

$$x \geq 0.$$  

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Through the addition of variables $z_1, \ldots, z_L$, we may eliminate the max. function, leading to the following quadratically constrained quadratic program (QCQP):

$$\begin{align*}
\min &\quad t \\
\text{subject to} &\quad x^T (M_0 + \sum_{l=1}^{L} M_l) x + x^T q \leq t, \\
&\quad M_0 x + q - \sum_{l=1}^{L} z_l \geq 0, \\
&\quad M_l x + z_l \geq 0, \quad \forall l, \\
&\quad z_l \geq 0, \quad \forall l, \\
&\quad x \geq 0.
\end{align*}$$

(b) $\mathcal{U} := \mathcal{U}^t$: In an analogous fashion, when $\mathcal{U} := \mathcal{U}_1$, we have the following sequence of equivalence statements for the quadratic constraint:

$$\begin{align*}
x^T M_0 x + \max_{u \in \mathcal{U}_1} \left[ \sum_{l=1}^{L} u_l x^T M_l x \right] + x^T q \leq t &\iff x^T M_0 x + \max_{l \in \{1, \ldots, L\}} \left[ \max(0, x^T M_l x) \right] + x^T q \leq t \\
&\iff x^T M_0 x + \max_{l \in \{1, \ldots, L\}} x^T M_l x + x^T q \leq t, \quad l = 1, \ldots, L,
\end{align*}$$

where the last equivalence statement follows from the positive semidefiniteness of $M_l$. The semi-infinite linear constraint can be reformulated as follows:

$$\begin{align*}
\min_{u \in \mathcal{U}_1} &\quad \left[ \sum_{l=1}^{L} u_l [M_l]_{i \bullet} x \right] + [M_0]_{i \bullet} x + q_i \geq 0, \quad \forall i \\
\iff &\quad - \max_{l \in \{1, \ldots, L\}} \max(-[M_l]_{i \bullet} x, 0) + [M_0]_{i \bullet} x + q_i \geq 0, \quad \forall i \\
\iff &\quad \max(-[M_l]_{i \bullet} x, 0) \leq [M_0]_{i \bullet} x + q_i, \quad \forall i, l \\
\iff &\quad \max(-M_l x, 0) \leq M_0 x + q, \quad \forall i, l \\
\end{align*}$$

Finally, by the addition of a variable $z$, we obtain the following QCQP:

$$\begin{align*}
\min &\quad t \\
\text{subject to} &\quad x^T (M_0 + M_l) x + x^T q \leq t, \quad \forall l \\
&\quad M_0 x + q - z \geq 0, \\
&\quad M_l x + z \geq 0, \quad \forall l \\
&\quad z \geq 0, \\
&\quad x \geq 0.
\end{align*}$$

Remark: Note that while we do not explicitly consider the case when $q$ is also uncertain, this may be easily introduced when the uncertainty set that prescribes $M(u)$ is unrelated to that producing $q(u)$. On this occasion, we may address each term individually, as in the prior subsection. Next, we consider the possibility that $M$ and $q$ are derived from the same uncertainty sets.
### 3.1.3. Uncertainty in $M$ and $q$ under a dependence assumption

Next, we extend the realm of applicability of the tractability result to accommodate uncertainty sets that are more general than (3.11). Specifically, we employ an uncertainty set that relies on computing the Cholesky Factorization of $M$, defined next as adopted in [5]:

$$
\mathcal{U}_A \triangleq \left\{ (M, q) \mid M = A^T A, A = A_0 + \sum_{l=1}^L \xi_l A_l, q = q_0 + \sum_{l=1}^L \xi_l q_l, \|\xi\|_2 \leq 1 \right\}. \tag{3.13}
$$

Consequently, (3.2) may be recast as follows:

$$
\begin{align*}
\min & \quad t \\
\text{subject to} & \quad x^T (Mx + q) \leq t, & \forall (M, q) \in \mathcal{U}_A, \\
& \quad Mx + q \geq 0, & \forall (M, q) \in \mathcal{U}_A, \\
& \quad x \geq 0.
\end{align*} \tag{3.14}
$$

For the sake of convenience, we write the first constraint as

$$
x^T M x + 2 x^T \left( \frac{1}{2} q \right) - t \leq 0, \quad \forall (M, q) \in \mathcal{U}_A \tag{3.15}
$$

The tractability of constraint (3.15) follows directly from Theorem 2.3 in [5] and is formalized next without a proof.

**Lemma 3.5.** Consider the constraint (3.15) where $\mathcal{U}_A$ is defined by (3.13). Then the tractable counterpart of this constraint is given by (3.16):

$$
\begin{pmatrix}
-q_0^T x + t - \tau & -\frac{1}{2} q_1^T x & \ldots & -\frac{1}{2} q_L^T x & (A_0 x)^T \\
-q_1^T x & \tau & & (A_1 x)^T \\
\vdots & & \ddots & \vdots \\
-q_L^T x & \tau & & (A_L x)^T \\
A_0 x & A_1 x & \ldots & A_L x & I
\end{pmatrix} \succeq 0. \tag{3.16}
$$

However, it is somewhat more challenging constructing a robust counterpart of the second constraint given by (3.17):

$$
M x + q \geq 0, \quad \forall (M, q) \in \mathcal{U}_A. \tag{3.17}
$$

In fact, this is the key departure from the result provided in [5]. For purposes of convenience and clarity, we rewrite $\mathcal{U}_A$ in terms of $A_0, q_0$ and $A_l, q_l, l = 1, \ldots, L$:

$$
\mathcal{U}_A \triangleq \left\{ (M, q) \mid M = A_0^T A_0 + \sum_{l=1}^L (A_l^T A_0 + A_0^T A_l) \xi_l + \sum_{l<m} (A_l^T A_m + A_m^T A_l) \xi_l \xi_m + \sum_{l=1}^L A_l^T A_l \xi_l^2, \\
q = q_0 + \sum_{l=1}^L q_l \xi_l, \quad \|\xi\|_2 \leq 1 \right\}. \tag{3.18}
$$

We may utilize [3, Lemma 14.3.7] in deriving the tractability of (3.17).

**Lemma 3.6.** Consider the constraint (3.17) where $\mathcal{U}_A$ is defined by (3.18). Then the semi-infinite constraint has a tractable robust counterpart, which will be presented as (3.24).
Proof. We begin by noticing that obtaining a feasible solution of (3.17) requires solving the following $i$th optimization problem for $i = 1, \ldots, n$:

$$\begin{align*}
\min & \quad \sum_{l=1}^{L} \left( [A_f^T A_0 + A^T_{d} A_i] \cdot x + [q]_l \right) \xi_l + \sum_{1 \leq m < l \leq L} [A^T_m A_m + A^T_m A_l] \cdot x_l \xi_m + \sum_{l=1}^{L} [A_f^T A_i] \cdot x_l \\
\text{subject to} & \quad \|\xi\|_2 \leq 1.
\end{align*}$$

(3.19)

We may compactly rewrite (3.19) as follows:

$$\begin{align*}
\min & \quad b_i(x)^T \xi + \xi^T C_i(x) \xi \\
\text{subject to} & \quad \|\xi\|_2 \leq 1,
\end{align*}$$

(3.20)

where $b_i : \mathbb{R}^n \to \mathbb{R}^L, C_i : \mathbb{R}^n \to \mathbb{R}^{L \times L}$ are all linear functions of $x$. We now define the following:

$$\begin{align*}
\hat{\xi} = \begin{pmatrix} \xi^T \\ \xi \end{pmatrix},
M_i(x) = \begin{pmatrix} \frac{1}{2} b_i(x) & \frac{1}{2} b_i^T(x) \\ C_i(x) & \end{pmatrix},
\text{and } Z = \left\{ \begin{pmatrix} \xi^T \\ \xi \end{pmatrix} \mid \|\xi\|_2 \leq 1 \right\}
\end{align*}$$

(3.21)

Then the QCQP (3.20) is equivalent to the following matrix optimization problem:

$$\begin{align*}
\min & \quad \langle \hat{\xi}, M_i(x) \rangle \\
\text{subject to} & \quad \hat{\xi} \in Z,
\end{align*}$$

(3.22)

where $(A, B) = Tr(A^T B)$. Since the objective function is linear in $\hat{\xi}$, we may extend the feasible region $Z$ to its convex hull $\tilde{Z}$ which is given by $\tilde{Z} = \text{conv}\{Z\}$. By Lemma 14.3.7 from [3], we have that

$$\tilde{Z} = \left\{ \begin{pmatrix} \xi \\ w^T \\ W \end{pmatrix} \in S^{L+1} | \begin{pmatrix} 1 & w^T \\ w & W \end{pmatrix} \succeq 0, Tr(W) \leq 1 \right\},$$

where $S^{N+1}$ represents the cone of symmetric positive semidefinite matrices. Using variable replacement, (3.22) is equivalent to the following semidefinite program:

$$\begin{align*}
\min & \quad \langle X, M_i(x) \rangle \\
\text{subject to} & \quad \left\langle X, \begin{pmatrix} 0 \\ I \end{pmatrix} \right\rangle \leq 1, \\
& \quad \left\langle X, \begin{pmatrix} 1 \\ I \end{pmatrix} \right\rangle = 1, \\
& \quad X \in S^{L+1}.
\end{align*}$$

(3.23)

By assumption, the feasible region of (3.23) has a nonempty relative interior. Therefore its dual optimum can be obtained and thus we can reformulate the $i$th constraint in (3.17) as the following SDP constraint:

$$\begin{align*}
y_{i,1} + y_{i,2} + a_i(x) & \geq 0, \\
y_{i,1} \begin{pmatrix} 0 \\ I \end{pmatrix} + y_{i,2} \begin{pmatrix} 1 \\ I \end{pmatrix} & \preceq M_i(x), \\
y_{i,1} & \leq 0.
\end{align*}$$

(3.24)

where $a_i(x) = [A_f^T A_0] \cdot x + [q]_l$. $\Box$
Our main result of this subsection can be seen to follow from Lemma 3.5 and Lemma 3.6.

**Theorem 3.7 (TRC for uLCP (q(u), M(u)) for u ∈ U_A).** Consider the uncertain LCP (3.14). Then this semi-infinite program has a tractable robust counterpart.

**Remark:**

(i) When the uncertainty set ∥ξ∥₂ ≤ 1 is replaced by either ∥ξ∥₁ ≤ 1 or ∥ξ∥₂ ≤ 1, Lemma 3.5 does not hold. What we may do instead is to enlarge the uncertainty set to get a tractable robust counterpart.

In the case of ∥ξ∥₂ ≤ 1, [3] Lemma 14.3.9 provides a semidefinite representable set that contains \( \hat{Z} \).

On the other hand, we may enlarge ∥ξ∥₁ ≤ 1 or ∥ξ∥₂ ≤ 1 to their circumscribed spheres representing a scaling of \( U_2 \) allowing for the construction of tractable robust counterparts of (3.17).

(ii) We note that [26] claims a similiar result (Theorem 3.2) as Theorem 3.7. However, there appears to be a significant issue in that the tractability of (3.17) is not proved and does not seem to follow directly.

3.2. **Tractable uncertain non-monotone LCPs.** When the matrix \( \frac{1}{2}(M(u)^T + M(u)) \) is not necessarily positive semidefinite for every \( u \in U \), \( M(u)x + q(u) \) is no longer monotone for every \( u \). Consequently, the problem (3.10) no longer has convex constraints for every realization of \( u \). As we proceed to show, we may still obtain a tractable robust counterpart under a suitably defined uncertainty set on \( M(u) \) with the caveat that \( M(u) \) is unrelated to \( q(u) \). We begin by defining the uncertainty set for \( M(u) \).

\[
M(u) \triangleq M_0 + \sum_{l=1}^{L} u_l M_l + \sum_{k=1}^{K} u_{L+k} M_k',
\]

\[
\frac{1}{2}(M_0 + M_0^T) \succeq 0,
\]

\[
\frac{1}{2}(M_l + M_l^T) \succeq 0, \quad \forall l = 1, \ldots, L,
\]

\[
\frac{1}{2}(M_k' + M_k'^T) \preceq 0, \quad \forall k = 1, \ldots, K,
\]

Without loss of generality, we assume that \( M(u), M_0 \) and \( M_l \) are symmetric for \( l = 1, \ldots, L + K \); if not, we may always replace the matrices by their symmetrized counterparts. The tractability of the robust counterpart of the uncertain nonmonotone LCP is proved next.

**Proposition 3.8 (TRC for non-monotone uLCP (q, M(u))).** Consider the problem (3.10). Suppose \( M \) is defined by (3.25) and \( U \) is either \( U_\infty, U_1 \) or \( U_2 \). Then this problem admits a tractable robust counterpart.

**Proof.**

(a) \( U := \{ u : \|u\|_\infty \leq 1 \} \): We begin by determining the robust counterpart of the following constraint:

\[
x^T M_0 x + \sum_{l=1}^{L} u_l x^T M_l x + \sum_{k=1}^{K} u_{L+k} x^T M_k' x + x^T q \leq t, \quad \forall u \in U.
\]

(3.26)

This may be equivalently stated as

\[
x^T M_0 x + \max_{u \in U_\infty} \left[ \sum_{l=1}^{L} u_l x^T M_l x + \sum_{k=1}^{K} u_{L+k} x^T M_k' x \right] + x^T q \leq t.
\]

By noting that the summation can be written from \( l = 1, \ldots, K + L \), through the application of \( \max_{\|u\|_\infty \leq 1} \eta^T u = \|\eta\|_1 \), it follows that

\[
x^T M_0 x + \sum_{l=1}^{L} |x^T M_l x| + \sum_{k=1}^{K} |x^T M_k' x| + x^T q \leq t.
\]

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Since $M_l \geq 0$ or $M'_k \leq 0$ for every $l, k$, $|x^T M_l x| = x^T M_l x$, $|x^T M'_k x| = -x^T M'_k x$. Consequently, the robust counterpart of (3.26) can be stated as the convex constraint:

$$x^T (M_0 + \sum_{l=1}^{L} M_l - \sum_{k=1}^{K} M'_k) x + x^T q \leq t.$$ 

Similarly, the constraint $M(u)x + q \geq 0$, $\forall u \in U$ can be reformulated as follows:

$$M_0 x + q \geq 0,$$ 

$$\forall u \in U \quad \iff \quad M_0 x + \sum_{l=1}^{L} u_l M_l x + \sum_{k=1}^{K} u_{L+k} M'_k x + q \geq 0,$$ 

$$\forall u \in U \quad \iff \quad M_0 x + \min_{u \in U} \left[ \sum_{l=1}^{L} u_l M_l x + \sum_{k=1}^{K} u_{L+k} M'_k x \right] + q \geq 0,$$ 

$$\iff \quad M_0 x - \max_{u \in U} \left[ \sum_{l=1}^{L} u_l (-M_l x) + \sum_{k=1}^{K} u_{L+k} (-M'_k x) \right] + q \geq 0,$$ 

$$\iff \quad M_0 x - \sum_{l=1}^{L} |M_l x| - \sum_{k=1}^{K} |M'_k x| + q \geq 0.$$

Through the addition of variables, $z_1, \ldots, z_{L+K}$, the resulting robust counterpart can then be stated as the following convex QCQP:

$$\min \quad \frac{t}{x^T (M_0 + \sum_{l=1}^{L} M_l - \sum_{k=1}^{K} M'_k) x + x^T q \leq t,}$$ 

subject to:

$$M_0 x + q - \sum_{l=1}^{L} z_l \geq 0,$$

$$z_l \geq M_l x \geq -z_l, \quad l = 1, \ldots, L,$$

$$z_l \geq M'_{L-l} x \geq -z_l, \quad l = L+1, \ldots, L+K,$$

$$z_l \geq 0, \quad l = 1, \ldots, L+K.$$ 

(3.27)

(b) $U := U_1$ As in (a), we begin by determining the robust counterpart of (3.26):

$$x^T M_0 x + \max_{u \in U} \left[ \sum_{l=1}^{L} u_l x^T M_l x + \sum_{k=1}^{K} u_{L+k} x^T M'_k x \right] + x^T q \leq t.$$ 

By noting that the summation can be written from $l = 1, \ldots, K + L$, through the application of $\max_{\|u\|_1 \leq 1} \eta^T u = \|\eta\|_\infty$, it follows that

$$x^T M_0 x + \max_{l \in \{1, \ldots, L\}} |x^T M_l x|, \max_{k \in \{1, \ldots, K\}} |x^T M'_k x| + x^T q \leq t.$$ 

But $x^T M_l x \geq 0$ for all $x$ and $l = 1, \ldots, L$ while $x^T M_l x \leq 0$ for all $x$ and $l = L + 1, \ldots, L + K$ implying that this constraint can be rewritten as follows:

$$x^T M_0 x + \max_{l \in \{1, \ldots, L\}} x^T M_l x, \max_{k \in \{1, \ldots, K\}} [-x^T M'_k x] + x^T q \leq t.$$
Consequently, this constraint can be rewritten as two constraints:

\[ x^T M_0 x + \max_{l \in \{1, \ldots, L\}} x^T M_l x + x^T q \leq t \]
\[ x^T M_0 x + \max_{k \in \{1, \ldots, K\}} [-x^T M'_k x] + x^T q \leq t. \]

The max. function can be eliminated by replacing each constraint by a finite collection:

\[ x^T M_0 x + x^T M_l x + x^T q \leq t, \quad l = 1, \ldots, L \]
\[ x^T M_0 x - x^T M'_k x + x^T q \leq t, \quad k = 1, \ldots, K \]

Similarly, the semi-infinite constraint \( M(u)x + q \geq 0, \quad \forall u \in \mathcal{U} \) can be reformulated as follows:

\[ M(u)x + q \geq 0, \quad \forall u \in \mathcal{U} \]
\[ \iff M_0 x + \sum_{l=1}^{L} u_l M_l x + \sum_{k=1}^{K} u_{L+k} M'_k x + q \geq 0, \quad \forall u \in \mathcal{U} \]
\[ \iff M_0 x + \min_{u \in \mathcal{U}} \left[ \sum_{l=1}^{L} u_l M_l x + \sum_{k=1}^{K} u_{L+k} M'_k x \right] + q \geq 0, \]
\[ \iff M_0 x - \max_{u \in \mathcal{U}} \left[ \sum_{l=1}^{L} u_l [-M_l x] + \sum_{k=1}^{K} u_{L+k} [-M'_k x] \right] + q \geq 0, \]
\[ \iff M_0 x - \max \left( \max_{l \in \{1, \ldots, L\}} ||-M_l x||, \max_{k \in \{1, \ldots, K\}} ||-M'_k x|| \right) + q \geq 0, \]
\[ \iff \begin{cases} M_0 x - \max_{l \in \{1, \ldots, L\}} ||-M_l x|| + q \geq 0, \\ M_0 x - \max_{k \in \{1, \ldots, K\}} ||-M'_k x|| + q \geq 0, \\ M_0 x - z + q \geq 0, \\ z \geq M_l x \geq -z, \quad l = 1, \ldots, L \\ M_0 x - v + q \geq 0, \\ z \geq M'_k x \geq -z, \quad k = 1, \ldots, K \end{cases} \]
\[ \iff \begin{cases} M_0 x - z + q \geq 0, \\ z \geq M_l x \geq -z, \quad l = 1, \ldots, L \\ M_0 x - v + q \geq 0, \\ z \geq M'_k x \geq -z, \quad k = 1, \ldots, K \end{cases} \]

Consequently, the TRC is given by the following:

\[ \min_t \]
\[ \text{subject to} \quad x^T (M_0 + M_l)x + x^T q \leq t, \quad \forall l \]
\[ x^T (M_0 - M_k)x + x^T q \leq t, \quad \forall k \]
\[ M_0 x + q - z \geq 0, \]
\[ z \geq M_l x \geq -z, \quad \forall l \]
\[ M_0 x + q - v \geq 0, \]
\[ z \geq M'_k x \geq -z, \quad \forall k \]
\[ x, v, z \geq 0. \]
(c) $\mathcal{U} := \mathcal{U}_2$: We first consider constraint \ref{eq:3.26} which can be equivalently stated as follows:

\[
x^T M_0 x + \max_{u \in \mathcal{U}_2} \left[ \sum_{l=1}^L u_l M_l x + \sum_{k=1}^K u_{L+k} x^T M'_k x \right] + x^T q \leq t
\]

\[
\Leftrightarrow x^T M_0 x + \sqrt{\sum_{l=1}^L (x^T M_l x)^2 + \sum_{k=1}^K (x^T M'_k x)^2} + x^T q \leq t.
\]

Similarly, the constraint $M(u)x + q \geq 0$ for every $u \in \mathcal{U} = \mathcal{U}_2$ can be reformulated as follows:

\[
M(u)x + q \geq 0,
\]

\[
\Leftrightarrow M_0 x + \min_{u \in \mathcal{U}_2} \left[ \sum_{l=1}^L u_l M_l x + \sum_{k=1}^K u_{L+k} M'_k x \right] + q \geq 0,
\]

\[
\Leftrightarrow M_0 x - \max_{u \in \mathcal{U}_2} \left[ \sum_{l=1}^L u_l [-M_l x] + \sum_{k=1}^K u_{L+k} [-M'_k x] \right] + q \geq 0,
\]

\[
\Leftrightarrow [M_0]_{i\bullet} - \sqrt{\sum_{l=1}^L (|[M_l]_{i\bullet}|)^2 + \sum_{k=1}^K (|[M'_k]_{i\bullet}|)^2} + q_i \geq 0, \quad \forall i.
\]

Consequently, the robust counterpart of \ref{eq:3.10} can be stated as:

\[
\begin{align*}
\min & \quad \quad \quad \quad \quad \quad \quad t \\
\text{subject to} & \quad x^T M_0 x + \sqrt{\sum_{l=1}^L (x^T M_l x)^2 + \sum_{k=1}^K (x^T M'_k x)^2} + x^T q \leq t \\
& \quad [M_0]_{i\bullet} - \sqrt{\sum_{l=1}^L (|[M_l]_{i\bullet}|)^2 + \sum_{k=1}^K (|[M'_k]_{i\bullet}|)^2} + q_i \geq 0, \quad \forall i
\end{align*}
\]

By examining the second derivative of $f(x)$ defined as

\[
f(x) = \sqrt{\sum_{l=1}^L (x^T M_l x)^2 + \sum_{k=1}^K (x^T M'_k x)^2},
\]

it can be concluded that $f$ is a convex function. This result indicates that the the left hand side of the first constraint in \ref{eq:3.28} is a convex function, implying that resulting feasible region is convex. The $n$ remaining inequalities are in the form of second-order cone constraints and are therefore tractable convex constraints. It follows that \ref{eq:3.28} is a convex program.

\[\Box\]

To get a geometric understanding of the prior proposition, we consider the following example. Example: Consider the case when $M(u)$ and $q$ are defined as follows:

\[
M(u) = u \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, q = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \text{ and } \mathcal{U} = \{u \mid -1 \leq u \leq 1\}.
\]

It can be observed that the constraint: $x^T (M(u)x + q) \leq t$ is nonconvex when $u < 0$. Note that this constraint can be rewritten as $u(x_1^2 + 2x_2^2) + 2x_1 + 2x_2 \leq t$ and we defined $R_u$ as follows:

\[
R_u \triangleq \{(x_1, x_2, t) \mid u(x_1^2 + 2x_2^2) + 2x_1 + 2x_2 \leq t\}.
\]
Fig. 3.1: Hidden convexity in two and three dimensions

Then $R_{-1}$ denotes the region above the surface shown in Fig. 3.1 labeled $u = -1$, clearly a nonconvex set. Likewise, the feasible regions $R_1, R_0$ represent the regions above the surfaces presented in Fig. 3.1 labeled $u = 1$ and $u = 0$, respectively. Though the set $R_{-1}$ is nonconvex and appears to make the program challenging to solve, a better understanding emerges when we consider the intersection of $R_u$ over $u$, as given by

$$R \triangleq \bigcap_{-1 \leq u \leq 1} R_u.$$ 

It can be seen that $R_1 = R$. The figure on the left in Fig. 3.1 hints as to why this holds. The three surfaces intersect at a single point, namely $(0, 0)$ and the surface with greater index $u$ stays above that with the lower index. This implies that $R_1 \subseteq R_0 \subseteq R_{-1}$. Actually, $R_u$ is monotone in $u$ in that if $u_1 \leq u_2$, then $R_{u_2} \subseteq R_{u_1}$. When considering such constraints in higher dimensions, similar behavior emerges. Finally, there have been prior observations regarding the presence of hidden convexity in nonconvex programs (cf. [4]).

4. **General uncertain non-monotone LCPs.** In this section, we consider non-monotone uncertain LCPs in more general settings where tractable robust counterparts are unavailable. Instead, we examine when such problems result in finite or low-dimensional nonconvex programs. In Section 4.1, we discuss two avenues via which we may obtain nonconvex robust counterparts. While stationary points of such problems can be obtained by nonlinear programming solvers, global solutions require branching-based schemes. In Section 4.2, inspired by recent research Fampa et al. [14], we present a technique for obtaining global solutions of a nonconvex quadratically constrained quadratic program.

4.1. **Nonconvex robust counterparts.** In Section 3.2, we showed that the RC of uncertain nonmonotone LCP may be tractable under some assumptions. However, in general, this is not the case, particularly when $M(u)$ and $q(u)$ are defined on the same (or related) uncertainty sets. In this circumstance, we can show that the RC may still be reformulated as a nonconvex problem with finite number of constraints.

**Proposition 4.1.** For non-monotone uLCP($q(u), M(u)$), suppose $M(u)$ is defined as (3.25) and $q(u) = q_0 + \sum_{i=1}^{L+K} u_i q_i$ where $u \in \mathcal{U}$ and $\mathcal{U}$ is defined as $\mathcal{U}_1, \mathcal{U}_2$, or $\mathcal{U}_\infty$, defined in (3.6). Then (3.3) may be written as a single nonconvex program with a finite number of constraints.
Proof. We introduce an artificial variable \( w \) into a finite set of nonconvex quadratic equality constraints:

\[
\begin{align*}
  w_l &= x^T M_l x + q_l^T x, & l = 1,\ldots,L, \\
  w_{L+k} &= x^T M_k' x + q_{L+k}^T x, & k = 1,\ldots,K.
\end{align*}
\]  

(4.1)

Then the quadratic constraint in (3.3) can be written as follows:

\[
x^T M_0 x + q_0^T x + u^T w \leq t, \quad \forall u \in \mathcal{U}
\]  

(4.2)

This semi-infinite constraint can be equivalently written as follows:

\[
x^T M_0 x + q_0^T x + \max_{u \in \mathcal{U}} u^T w \leq t.
\]  

(4.3)

But, it is known that \( \max_{u \in \mathcal{U}} u^T w \) is given by \( \|w\|_1, \|w\|_\infty \) or \( \|w\|_2 \) if \( \mathcal{U} \) is given by \( \mathcal{U}_\infty, \mathcal{U}_1 \) or \( \mathcal{U}_2 \), respectively. Consequently, (4.2) may always be recast in a tractable fashion. Similarly, consider the second constraint in (3.3):

\[
M_0 x + q_0 + \sum_{l=1}^L u_l (M_l x + q_l) + \sum_{k=1}^K u_{L+k} (M_k' x + q_{L+k}) \geq 0, \forall u \in \mathcal{U}.
\]  

(4.4)

Based on Prop. 3.8, the constraint (4.4) can also be reformulated in a tractable fashion when \( \mathcal{U} = \mathcal{U}_1, \mathcal{U}_2 \) or \( \mathcal{U}_\infty \). This is demonstrated for \( \mathcal{U} = \mathcal{U}_\infty \) and omitted for \( \mathcal{U} = \mathcal{U}_1 \) and \( \mathcal{U}_2 \):

\[
\begin{align*}
  M(u) x + q(u) &\geq 0, \quad \forall u \in \mathcal{U} \\
  \iff M_0 x + q_0 + \sum_{l=1}^L u_l (M_l x + q_l) + \sum_{k=1}^K u_{L+k} (M_k' x + q_{L+k}) \geq 0, \quad \forall u \in \mathcal{U} \\
  \iff M_0 x + q_0 + \min_{u \in \mathcal{U}_\infty} \left[ \sum_{l=1}^L u_l (M_l x + q_l) + \sum_{k=1}^K u_{L+k} (M_k' x + q_{L+k}) \right] \geq 0, \\
  \iff M_0 x + q_0 - \max_{u \in \mathcal{U}_\infty} \left[ \sum_{l=1}^L u_l [-M_l x - q_l] + \sum_{k=1}^K u_{L+k} [-M_k' x - q_{L+k}] \right] \geq 0, \\
  \iff M_0 x + q_0 - \sum_{l=1}^L |M_l x - q_l| - \sum_{k=1}^K |M_k' x - q_{L+k}| \geq 0. \\
  \iff M_0 x + q_0 - \sum_{l=1}^L |M_l x + q_l| - \sum_{k=1}^K |M_k' x + q_{L+k}| \geq 0.
\end{align*}
\]

We note that when \( \mathcal{U} = \mathcal{U}_\infty \), (4.3) can be rewritten as

\[
x^T M_0 x + q_0^T x + \|w\|_1 \leq t.
\]  

(4.5)
Through the addition of variables, \( z_1, \ldots, z_{L+K} \), the resulting robust counterpart when \( U = U_\infty \) can then be stated as the following nonconvex optimization problem:

\[
\begin{align*}
\min & \quad t \\
\text{subject to} & \quad -\tau_l \leq x^T M_l x + q_l^T x \leq \tau_l, \quad l = 1, \ldots, L \\
& \quad -\tau_{L+k} \leq x^T M_{L+k} x + q_{L+k}^T x \leq \tau_{L+k}, \quad k = 1, \ldots, K \\
& \quad x^T M_0 x + q_0^T x + \sum_{i=1}^{L+K} \tau_i \leq t, \\
& \quad M_0 x + q_0 - \sum_{i=1}^{L+K} z_i \geq 0, \\
& \quad z_l \geq M_l x + q_l \geq -z_l, \quad l = 1, \ldots, L \\
& \quad z_{L+k} \geq M_{L+k} x + q_{L+k} \geq -z_{L+k}, \quad k = 1, \ldots, K \\
& \quad x \geq 0.
\end{align*}
\]

Consequently, the overall problem has \( L + K \) nonconvex equality constraints.

We now provide a corollary of this result when \( U := \mathcal{U}_c \).

**Corollary 4.2 (RC for non-monotone uLCP\((q(u), M(u))\) where \( U = \mathcal{U}_c \).** Suppose \( M(u) \) is defined as \((3.25)\) and \( q(u) = q_0 + \sum_{i=1}^{L+K} u_i q_i \) where \( u \in U \) and \( U \) is given by \( \mathcal{U}_c \), defined as \((3.8)\). Then \((3.2)\) can be represented as a finite-dimensional nonconvex program.

**Remark:** While stationary points of such problems may be computed through standard nonlinear programming schemes such as globalized sequential quadratic programming or interior point methods [24], our interest lies in obtaining global solutions of such problems. In the next subsection, we review several approaches for obtaining global solutions to QCQPs.

An alternate approach lies in utilizing the modified Cholesky factorization. In such a setting, we require an assumption on \( M(u) \).

**Assumption 1.** Suppose \( \epsilon \) is a positive scalar and \( M(u) \) satisfies the following for every \( u \in U \):

\[
\lambda_{\min}(M^*(u)) \geq -\epsilon, \quad \text{where } M^*(u) \triangleq \left( \frac{1}{2} (M(u) + M(u)^T) \right).
\]

Consequently, \( M^*(u) + \epsilon I \) is always a symmetric positive semidefinite matrix and admits a Cholesky factorization. We consider the following specification for the uncertainty set:

\[
M^*(u) = A(u)^T A(u) - \epsilon I, q(u) = q_0 + \sum_{l=1}^{L} u_l q_l, A(u) = A_0 + \sum_{l=1}^{L} u_l A_l, \|u\|_2 \leq 1.
\]

**Proposition 4.3.** Consider the uLCP\((q(u), M(u))\) and suppose Assumption 4 holds. Furthermore, let \( M(u) \) is defined as \((4.7)\) and \( q(u) = q_0 + \sum_{i=1}^{L+K} u_i q_i \) and \( \mathcal{U} = \mathcal{U}_2 \). Then \((3.2)\) can be reformulated as a nonconvex quadratically constrained quadratic program with a single nonconvex constraint.

**Proof.** We begin by considering the quadratic inequality constraint:

\[
x^T M^*(u)x + q(u)^T x \leq t, \quad \forall u \in \mathcal{U}.
\]
But this may be equivalently stated as follows:

\[
\begin{align*}
\{ x^T M(s)(u)x + q(u)^T x &\leq t, \\
&\quad \forall u \in U_2 \\
\} \iff \\
\{ x^T M(s)(u)x + \epsilon \|x\|^2 + q(u)^T x + w &\leq t, \\
&\quad \forall u \in U_2 \\
&\quad -\epsilon \|x\|^2 = w
\end{align*}
\]

Then (3.3) can be formulated as the following QCQP with one nonconvex constraint:

\[
\begin{align*}
\min_t & \\
\text{subject to} & x^T (M(s)(u) + \epsilon I)x + q(u)^T x + w &\leq t, \\
&\quad \forall u \in U_2 \\
& M(u)x + q(u) &\geq 0, \\
&\quad \forall u \in U_2 \\
& x &\geq 0 \\
& w + \epsilon \|x\|^2 & = 0.
\end{align*}
\]

Remark: (4.10) is the only nonconvex constraint in the optimization problem (4.9). If we could a priori guarantee that the optimal solution lay in a bounded set, then the spatial branch and bound scheme presented in the next section may be applied.

4.2. A branching scheme for resolving nonconvex QCQPs. Before presenting our scheme, we provide a brief review of global optimization schemes for resolving indefinite quadratic programs and their quadratically constrained generalizations. Such a class of problems has seen significant study \[2, 20\]. In \[2\], the authors combine reformulation-linearization-technique (RLT) with an SDP relaxation to tackle QCQP. In \[20\], a general framework is built for solving such problems. While branching schemes come in varied forms, Burer and Vandenbussche \[6\] employ SDP relaxations for addressing indefinite quadratic programming. We consider a spatial branch-and-bound approach inspired by Fampa et al. \[14\] developed for nonconvex quadratic programs. This approach uses secant inequalities for deriving a relaxation. We extend this approach to quadratically constrained variants. We emphasize that the focus of this paper lies in extending standard robust optimization techniques to allow for accommodating uncertain linear complementarity problems. While a comprehensive study of branching schemes is beyond the scope of the current paper, we show that at least one of the approaches can be readily adapted to this context.

We begin by noting that the QCQP can be recast as an optimization problem with a linear objective and quadratic constraints, some of which may be nonconvex. Throughout this subsection, we define \( M(u) \) by (3.25) and rewrite it as follows:

\[
M(u) = M_0 + \sum_{i=1}^{L+K} u_i M_i,
\]

where \( M_0, \ldots, M_L \) are positive semidefinite while \( M_{L+1}, \ldots, M_{L+K} \) are negative semidefinite. Furthermore, \( q(u) = q_0 + \sum_{i=1}^{L+K} u_i q_i \). We illustrate the scheme for the case when \( U := U_\infty \) and qualify the relaxations and the bounds by using the superscript \( \infty \). Suppose \( u \in U_\infty \). From Prop 4.1 the optimization problem
given by (3.3) may be reformulated as follows:

\[
\begin{align*}
\min 
& \
t \\
\text{subject to} 
& \
x^T M_0 x + q_0^T x + \sum_{i=1}^{L+K} |x^T M_i x + q_i^T x| \leq t, \\
& \
M_0 x + q_0 - \sum_{i=1}^{L+K} |M_i x + q_i| \geq 0, \\
& \
x \geq 0.
\end{align*}
\] (4.11)

The chief concern lies in the first constraint which can be decomposed into a linear constraint and 2\((L + K)\) quadratic constraints:

\[
\begin{align*}
x^T M_0 x + q_0^T x + \sum_{i=1}^{L+K} \tau_i & \leq t, \quad (4.12) \\
-\tau_i & \leq x^T M_i x + q_i^T x \leq \tau_i, \quad i = 1, \ldots, L + K.
\end{align*}
\]

Of these, the following constraints are nonconvex:

\[
\begin{align*}
-x^T M_i x - q_i^T x & \leq \tau_i, \quad i = 1, \ldots, L, \\
x^T M_i x + q_i^T x & \leq \tau_i, \quad i = L + 1, \ldots, L + K.
\end{align*}
\] (4.13)

Consequently, (4.13) has \(L + K\) nonconvex constraints in the following form:

\[
x^T M_i x + q_i^T x \leq \tau_i, \quad i = 1, \ldots, L + K,
\] (4.14)

where

\[
\mathcal{M}_i \triangleq \begin{cases} 
-M_i, & i = 1, \ldots, L \\
M_i, & i = L + 1, \ldots, L + K 
\end{cases}
\quad \text{and} \quad 
\bar{q}_i \triangleq \begin{cases} 
-q_i, & i = 1, \ldots, L \\
q_i, & i = L + 1, \ldots, L + K 
\end{cases}
\] (4.15)

where \(\mathcal{M} \leq 0\).

**Constructing a relaxation:** Akin to the approach employed in [14], we use the eigenvalue decomposition of \(\mathcal{M}_i\), defined as

\[
\mathcal{M}_i = -\sum_{j=1}^{J_i} \lambda_{i,j} \nu_{i,j} \nu_{i,j}^T,
\]

where \(\lambda_{i,j} > 0\) for all \(j = 1, \ldots, J_i, \forall i = 1, \ldots, L + K\). By defining \(y_{i,j} = \sqrt{\lambda_{i,j}} \nu_{i,j}^T x\), inequality (4.13) may be rewritten as

\[
-\sum_{j=1}^{J_i} y_{i,j}^2 + \bar{q}_i^T x \leq \tau_i, \quad i = 1, \ldots, L + K.
\] (4.16)

Suppose \(l_{i,j} \leq y_{i,j} \leq u_{i,j}\) for \(j = 1, \ldots, J_i\) and \(i = 1, \ldots, L + K\). Then we may use a secant inequality for providing a relaxation to the (4.16) in the form of the following:

\[
-\sum_{j=1}^{J_i} \left( (y_{i,j} - l_{i,j}) \left( \frac{u_{i,j}^2 - l_{i,j}^2}{u_{i,j} - l_{i,j}} + l_{i,j}^2 \right) \right) + \bar{q}_i^T x \leq \tau_i, \quad i = 1, \ldots, L + K.
\] (4.17)
When $\mathcal{U} := \mathcal{U}_\infty$, the resulting relaxed problem is denoted by $(P_\infty(l, u))$ and is defined as follows:

$$
\begin{align*}
\min & \quad x^T M_0 x + q_0^T x + \sum_{i=1}^{L+K} \tau_i \\
\text{subject to} & \quad x^T M_i x + q_i^T x \leq \tau_i, \quad i = 1, \ldots, L \\
& \quad -x^T M_i x - q_i^T x \leq \tau_i, \quad i = L + 1, \ldots, L + K \\
& \quad \tau_i \geq 0, \quad i = 1, \ldots, L + K \\
& \quad \tau_i \leq \tau, \quad i = 1, \ldots, L + K
\end{align*}
$$

(4.18) (P_\infty(l, u))

Obtaining upper and lower bounds for $y_{i,j}$: Crucial to this scheme is the need for obtaining upper and lower bounds on $y_{i,j}$ given by $l_{i,j}$ and $u_{i,j}$, respectively. For this purpose, we consider the first constraint in (4.11). Recall that this constraint can be recast as a set of $2(L + K)$ convex constraints and one linear constraint given by (4.12). We pursue a relaxation by including only the $L + K$ convex constraints and the linear constraint given by the following:

$$
\begin{align*}
\min & \quad x^T M_0 x + q_0^T x + \sum_{i=1}^{L+K} \tau_i \\
\text{subject to} & \quad x^T M_i x + q_i^T x \leq \tau_i, \quad i = 1, \ldots, L \\
& \quad -x^T M_i x - q_i^T x \leq \tau_i, \quad i = L + 1, \ldots, L + K
\end{align*}
$$

(4.19)

By eliminating $\tau$, it follows that (4.18) may be reformulated as the following:

$$
\begin{align*}
\min & \quad x^T M_0 x + q_0^T x + \sum_{i=1}^{L+K} \tau_i \\
\text{subject to} & \quad x^T M_i x + q_i^T x \leq \tau_i, \quad i = 1, \ldots, L \\
& \quad -x^T M_i x - q_i^T x \leq \tau_i, \quad i = L + 1, \ldots, L + K
\end{align*}
$$

If $\left(M_0 + \sum_{i=1}^{L} M_i - \sum_{i=L+1}^{L+K} M_i\right)$ is full rank, for a given $t := t_0$, the set $X_\infty(t_0)$ is bounded where

$$
X_\infty(t_0) := \left\{ x \mid x^T \left(M_0 + \sum_{i=1}^{L} M_i - \sum_{i=L+1}^{L+K} M_i\right) x + \left(q_0 + \sum_{i=1}^{L} q_i - \sum_{i=L+1}^{L+K} q_i\right) x \leq t_0, \quad M_0 x + q_0 - \sum_{i=1}^{L+K} |M_i x + q_i| \geq 0, \quad x \geq 0 \right\}.
$$

The upper and lower bound for $y_{i,j}$ can then be obtained as follows:

$$
\begin{align*}
\min / \max & \quad \sqrt{\lambda_{i,j}^2 u_{i,j}^T x} \\
\text{subject to} & \quad x \in X_\infty(t_0).
\end{align*}
$$

(123)

Note that when the uncertainty set is either $\mathcal{U}_1$ or $\mathcal{U}_2$, the relaxation and the upper/lower bounds have to be derived in an analogous fashion. A formal outline of the branching scheme is provided in Algorithm 1.
to the regime of affine variational inequality problems over polyhedral sets. Next, we demonstrate how our

Algorithm 1 Spatial branch and bound

1: Init.i: \( k := 1 \); \( \texttt{terminate} := 0 \); choose \( t_0, \epsilon > 0 \);
2: Init.ii: \( M_i = - \sum_{j=1}^{d_i} \lambda_{i,j} \nu_{i,j} \nu_{i,j}^T \) for \( i = 1, \ldots, L + K \);
3: Init.iii: For all \( i,j \) compute \( t_{i,j}^\alpha(t_0) \) and \( u_{i,j}^\alpha(t_0) \);
4: Init.iv: Let \( P_k := P_\infty(t, u); (x_1^*, \tau_1^*, y_1^*, t_1^*) \in \argmin P_k \);
5: Init.v: Assign bounds: \( \text{glb}_{lib} := t_1^*; \text{glb}_{ub} := x_1^T M_0 x_1^* + q_0^T x_1^* + \sum_{i=1}^{L+K} |x_1^T M_i x_1^* + q_i^T x_1^*| \);
6: Init.vi: Update list: \( \text{list} := \{(P_1, \text{glb}_{lib}, \text{glb}_{ub})\} \).
7: while \( \text{terminate} == 0 \) do
8: Branching index: For \( P_k \), choose index pair \( (\tilde{i}, \tilde{j}) := \arg\max_{(i,j)} (u_{i,j} - l_{i,j}) \); \( \phi_{i,j} = (u_{i,j} - l_{i,j})/2 \);
9: Update bounds:

\[
\tilde{u}_{i,j} = \begin{cases} 
\phi_{i,j}, & (i,j) = (\tilde{i}, \tilde{j}) \\
(u_{i,j}, & \text{otherwise.}
\end{cases}
\]

\[
\tilde{l}_{i,j} = \begin{cases} 
\phi_{i,j}, & (i,j) = (\tilde{i}, \tilde{j}) \\
l_{i,j}, & \text{otherwise.}
\end{cases}
\]

10: Construct leaves: \( P_k^l := P_\infty(u, \tilde{i}); P_k^u := P_\infty(\tilde{u}, \tilde{l}) \).
11: Upper and lower bounds for \( P_k^l; (x_1^l, \tau_1^l, y_1^l, t_1^l) \in \argmin(P_k^l) \):

\[
\text{lb}^l := t_1^l; \quad \text{ub}^l := (x_1^l)^T M_0 x_1^l + q_0^T x_1^l + \sum_{i=1}^{L+K} |(x_1^l)^T M_i x_1^l + q_i^T x_1^l|.
\]

12: Upper and lower bounds for \( P_k^u; (x_1^u, \tau_1^u, y_1^u, t_1^u) \in \argmin(P_k^u) \):

\[
\text{lb}^u := t_1^u; \quad \text{ub}^u := (x_1^u)^T M_0 x_1^u + q_0^T x_1^u + \sum_{i=1}^{L+K} |(x_1^u)^T M_i x_1^u + q_i^T x_1^u|.
\]

13: Delete \( P_k \) from list: \( \text{list} := \text{list} \setminus P_k \);
14: Append list by \( (P_k^l, \text{lb}^l, \text{ub}^l) \):
If \( t_1^l < \text{glb}_{ub} \), then \( \text{list} := \text{list} \cup (P_k^l, \text{lb}^l, \text{ub}^l) \); If \( \text{ub}^l < \text{glb}_{ub} \), then \( \text{glb}_{ub} := \text{ub}^l \) and \( P^l := P_k^l \);
15: Append list by \( (P_k^u, \text{lb}^u, \text{ub}^u) \):
If \( t_1^u < \text{glb}_{ub} \), then \( \text{list} := \text{list} \cup (P_k^u, \text{lb}^u, \text{ub}^u) \); If \( \text{ub}^u < \text{glb}_{ub} \), then \( \text{glb}_{ub} := \text{ub}^u \) and \( P^u := P_k^u \);
16: Termination test: If \( \text{glb}_{lib} - \text{glb}_{ub} < \epsilon \), then \( \text{terminate} := 1 \); Output \( P^\ell \) and its solution.
17: Choose \( (P, \text{lb}, \text{ub}) \) from list such that the the associated lower bound \( 1b \) is the smallest in the list and set the global lower bound \( \text{glb}_{lb} = 1b \). Let \( P_{k+1} := P \).
18: \( k := k + 1 \);
19: end while

5. Extensions to uncertain VIs and MPCCs. In this section, we consider two key generalizations of the uncertain monotone linear complementarity problem. In Section 4.2, we extend this framework to the regime of affine variational inequality problems over polyhedral sets. Next, we demonstrate how our
framework can address a subclass of stochastic mathematical programs with equilibrium constraints (MPECs) (cf. [22]), given by a stochastic quadratic program with (uncertain) linear complementarity constraints.

5.1. Uncertain affine monotone polyhedral VIs. Two shortcomings immediately come to the fore when considering the model (3.2):

(i) The set $X$ is a cone;

(ii) The underlying set is deterministic in that it is uncorrupted by uncertainty.

In this subsection, we show that examining uncertain polyhedral sets can also be managed within the same framework. Specifically, we begin by considering an uncertain affine variational inequality problem over a polyhedral set of the form given by (1.1) wherein $X(u)$ and $F(x,u)$ are defined as

$$X(u) \triangleq \{x : C(u)x \geq b(u), x \geq 0\} \text{ and } F(x,u) \triangleq M(u)x + q(u),$$

respectively. From [13, Prop. 1.3.4], if Abadie’s constraint qualification holds at $x$ (this CQ is not needed since the VI is linearly constrained), then $x$ solves $VI(X(u),F(u))$ if and only if there exists a vector $\lambda \in \mathbb{R}^m$ such that

$$0 \leq x \perp M(u)x - C(u)^T\lambda + q(u) \geq 0$$

$$0 \leq \lambda \perp C(u)x - b(u) \geq 0.$$  \hspace{1cm} (5.2)

In short, when $F(x,u)$ is an affine map and $X(u)$ is a polyhedral set, the affine variational inequality problem is equivalent to a linear complementarity problem over a larger space of primal and dual variables. This can be more compactly stated as the following monotone linear complementarity problem:

$$0 \leq z \perp B(u)z + d(u) \geq 0,$$  \hspace{1cm} (5.3)

where

$$B(u) \triangleq \begin{pmatrix} M(u) & -C(u)^T \\ C(u) & 0 \end{pmatrix} \text{ and } d(u) \triangleq \begin{pmatrix} q(u) \\ -b(u) \end{pmatrix},$$

respectively. It is relatively easy to see that $B(u)$ is a positive semidefinite matrix since $z^TB(u)z = x^TM(u)x \geq 0$ if $M(u)$ is a positive semidefinite matrix. This allows for making the following tractability claim when $\mathcal{U} = \mathcal{U}_2$. Naturally, we may also extend other statements drawn from the regime of uncertain linear complementarity problems but leave that for future work.

**Proposition 5.1.** Consider an uncertain variational inequality problem denoted by $\{VI(X(u),F(\bullet;u))\}_{u \in \mathcal{U}}$ where $\mathcal{U} = \{u \mid \|u\|_2 \leq 1\}$, $X(u)$ and $F(x,u)$ are defined in (5.1), where

$$M(u) = S^T(u)S(u), S(u) = \sum_{l=1}^{L} u_lS_l + S_0, q = q_0 + \sum_{l=1}^{L} u_lq_l,$$

$$C(u) = C_0 + \sum_{l=1}^{L} u_lC_l, b(u) = b_0 + \sum_{l=1}^{L} u_lb_l, u \in \mathcal{U}.$$  

Then a robust solution of this problem is given by a solution to a tractable convex program.
Proof. Recalling that $z = (x, \lambda)$, the robust counterpart of (5.3) is given by the following:

$$\min t$$
subject to
$$z^T (B(u)z + d(u)) \leq t, \quad \forall u \in U,$$  \hspace{1cm} (5.4)
$$M(u)x - C(u)^T \lambda + q(u) \geq 0, \quad \forall u \in U,$$  \hspace{1cm} (5.5)
$$C(u)x - b(u) \geq 0, \quad \forall u \in U,$$  \hspace{1cm} (5.6)
$$x, \lambda \geq 0.$$  

We begin by considering constraint (5.4) which can be recast as

$$z^T (B(u) + B(u)^T) z + 2z^T d(u) \leq 2t.$$  \hspace{1cm} (5.7)

Analogous to Theorem 3.7, this problem can be rewritten as a linear matrix inequality and some linear inequalities. It follows that (5.5) can be rewritten as a collection of $n$ linear matrix inequalities and a bunch of linear inequalities.

Finally, constraint (5.6) can be rewritten as the following set of constraints:

$$[C_0] \cdot x - [b_0] + \sum_{l=1}^{L} u_l ([C_l] \cdot x - [b_l]) \geq 0, \quad \forall u \in U, \quad i = 1, \ldots, n.$$  

This set of semi-infinite constraints is equivalent to a finite set of convex constraints in the form of second order cone constraints, which is discussed in Example 1.3.3. from [3].

5.2. Uncertain mathematical programs with complementarity constraints. Over the last two decades, the mathematical program with equilibrium constraints (MPECs) has found utility in modeling a range of problems, including Stackelberg equilibrium problems, structural design problems, bilevel programming problems, amongst others. A comprehensive description of the models, theory, and the associated algorithms may be found in the monograph by Luo et al. [22]. When the lower-level problem is given by a complementarity problem, then the MPEC reduces to a mathematical program with complementarity constraints (MPCC). We consider the uncertain counterpart of MPCC defined as follows:

$$\min f(x, y)$$
subject to
$$h(x, y) \geq 0,$$  \hspace{1cm} (MPCC)
$$0 \leq y \perp F(x, y) \geq 0.$$

The MPCC is an ill-posed nonconvex program in that it lacks an interior. In fact, standard constraint qualifications (such as LICQ or MFCQ) fail to hold at any feasible point of such a problem. We define an uncertain MPCC as a collection of MPCCs given by

$$\{\text{MPCC}(f, h, F)\}_{u \in U},$$
in which $f, h$ and $F$ are parametrized by $u$ where $u \in \mathcal{U}$:

$$\begin{align*}
\min & \quad f(x, y, u) \\
\text{subject to} & \quad h(x, y, u) \geq 0, \\
& \quad 0 \leq y \perp F(x, y, u) \geq 0.
\end{align*}$$

(uMPCC)

We may then define a robust counterpart of this problem as (rMPCC1) as follows:

$$\begin{align*}
\min & \quad \max_{u \in \mathcal{U}} f(x, y, u) \\
\text{subject to} & \quad h(x, y, u) \geq 0, \quad \forall u \in \mathcal{U} \\
& \quad 0 \leq y \perp F(x, y, u) \geq 0, \quad \forall u \in \mathcal{U}.
\end{align*}$$

(rMPCC1)

This problem is a nonconvex semi-infinite program. By utilizing the framework developed earlier, we may reformulate (rMPCC1) as a finite-dimensional MPCC. Unfortunately, the semi-infinite complementarity constraint given by

$$0 \leq y \perp F(x, y, u) \geq 0, \quad \forall u \in \mathcal{U}$$

need not admit a solution. Instead, we recast the uncertain complementarity constraint as the following:

$$\begin{align*}
\min & \quad t \\
\text{subject to} & \quad f(x, y, u) \leq t, \quad \forall u \in \mathcal{U} \\
& \quad h(x, y, u) \geq 0, \quad \forall u \in \mathcal{U} \\
& \quad y \text{ solves } \begin{cases} \\
\min \max_{y \in \mathcal{U}} y^T F(x, y, u) \\
F(x, y, u) \geq 0, \quad \forall u \in \mathcal{U} \\
y \geq 0
\end{cases}.
\end{align*}$$

(rMPCC2)

A natural question is whether a low-dimensional counterpart of rMPCC2 is available. Under convexity assumptions on $f(x, y, u)$ and concavity assumptions on $h(x, y, u)$ in $x$ and $y$ for every $u$, and some assumptions on the uncertainty set, tractable counterparts may be constructed for the first two constraints in (rMPCC2). By the findings of the prior sections, under some conditions, a robust counterpart of an uncertain LCP can be cast as a single convex program. The following result presented without a proof provides a set of assumptions under which the lower-level problem can be recast as a convex program:

**Proposition 5.2.** Suppose $F(x, y, u)$ is an affine map given by $F(x, y, u) = Ax + M(u)y + q(u)$ and $M(u) = M_0 + \sum_{l=1}^{L} u_l M_l, q(u) = q, M_l \succeq 0, \forall l = 0, \ldots, L$, then the third constraint of rMPCC2 can be replaced by the optimality conditions of a convex program if $\mathcal{U} = \mathcal{U}_1, \mathcal{U}_2$ or $\mathcal{U}_\infty$.

6. Numerical results. In Section 6.1, we begin by examining accuracy and scalability benefits of the presented tractable robust counterparts over direct approximations of the semi-infinite problem. The performance benefits of the branching scheme are examined in Section 6.2 and we conclude with a case study on uncertain traffic equilibrium problems in Section 6.3 where we compare robust solutions with the ERM solutions investigated in the literature.

6.1. Monotone uncertain LCPs. In this subsection, we consider two sets of monotone uncertain LCPs.
Consider the constructed uncertain LCP defined in Section 2 for which the solution is known a priori. Table 6.1 shows that the presented techniques allow for obtaining the robust solution $x^{\text{rob}}$ and this corresponds closely with the analytically available solution $x^{\text{analyt}}$. Furthermore, an arbitrarily chosen scenario-specific solution, given as $x^{n-\text{rob}}$, leads to large deviation from the analytical optimal solution and significantly higher residual.

\[
\|x^{\text{rob}} - x^{\text{analyt}}\|_2 = \text{residual of } x^{\text{rob}} \quad \|x^{n-\text{rob}} - x^{\text{analyt}}\|_2 = \text{residual of } x^{n-\text{rob}}
\]

| n  | $\|x^{\text{rob}} - x^{\text{analyt}}\|_2$ | residual of $x^{\text{rob}}$ | $\|x^{\text{n-rob}} - x^{\text{analyt}}\|_2$ | residual of $x^{\text{n-rob}}$ |
|-----|-------------------------------------|------------------|-------------------------------------|------------------|
| 10  | 3.9e-08                             | 2.0e-07          | 0.4e+03                             | 5.0e+07          |
| 20  | 4.7e-08                             | 3.6e-07          | 0.7e+03                             | 1.0e+09          |
| 40  | 1.8e-07                             | 2.2e-06          | 1.6e+03                             | 4.3e+10          |
| 80  | 5.1e-07                             | 5.2e-06          | 3.9e+03                             | 3.9+12           |
| 160 | 1.6e-05                             | 5.3e-04          | 5.5e+03                             | 2.8e+14          |

Table 6.1: Robust vs non-robust solutions

(ii): Next, we consider uncertain LCP($M(u),q$) in which $M(u) \triangleq M_0 + uM_1$, whose robust counterpart is defined as follows:

\[
\begin{align*}
\min \quad & t \\
\text{subject to} \quad & x^T(M_0 + uM_1)x + q^Tx \leq t, \quad \forall u \in \mathcal{U} \\
& (M_0 + uM_1)x + q \geq 0, \quad \forall u \in \mathcal{U} \\
& x \geq 0,
\end{align*}
\]

where $q = De$, $e$ denotes the column of ones, $D, M_0$ and $M_1$ are defined as

\[
D = \begin{pmatrix} A^T & B \\ -B & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} A^T & -B \\ B & 0 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 2A^TA & 0 \\ 0 & 0 \end{pmatrix},
\]

$A$ and $B$ are $k$-dimensional square matrices whose elements $a_{ij}$ and $b_{ij}$ are iid random variables generated from $U(0,5)$ and $N(0,1)$, respectively. Furthermore, $\mathcal{U}$ is the discretized variant of the interval $[-1,1]$ defined as

\[
\mathcal{U} \triangleq \left\{ \frac{i}{n} \mid i = -n,-(n-1),\ldots,n-1,n \right\}.
\]

The tractable robust counterpart (TRC) of (6.1) is given by the following:

\[
\begin{align*}
\min \quad & t \\
\text{subject to} \quad & x^TM_0x + |x^TM_1x| + q^Tx \leq t, \\
& M_0x - |M_1x| + q \geq 0, \\
& x \geq 0.
\end{align*}
\]

Accuracy of TRC: Since $\mathcal{U}$ has been taken to be a finite set, the robust counterpart is a finite dimensional nonlinear program. The first observation is that the semi-infinite program enumerates the quadratic constraint $x^T(M_0 + uM_1)x + q^Tx \leq t$ for every $u \in \mathcal{U}$. However for some values of $u$, this is a nonconvex constraint and the resulting semi-infinite approach leads to a nonconvex quadratically constrained linear program which is solved by fmincon, a Matlab-solver for nonlinear programs. However, the TRC is a convex QCQP. Table 6.2 shows that for randomly generated matrices $A$ and $B$, while the TRC provides similar values to that obtained via the direct approach, for at least one problem (case 5), the TRC provides a superior optimal value. It can also been seen that a direct approach takes between 10 to 50 times the time...
Table 6.2: Accuracy of TRC for randomly generated problem sets with $k = 20$ and $n = 500$

| #  | Tractable reformulation | Semi-infinite program |
|----|-------------------------|-----------------------|
|    | time(s) | Opt. val. | time(s) | Opt. val. |
| 1  | 1.2168   | 66522         | 19.12   | 66522       |
| 2  | 0.2155   | 48716         | 30.74   | 48716       |
| 3  | 0.2079   | 68670         | 26.15   | 68670       |
| 4  | 0.1440   | 62586         | 31.76   | 62587       |
| 5  | 0.2435   | 43688         | 33.64   | 43935       |

Table 6.3: Scalability of TRC for randomly generated problem sets with $k = 30$

| n  | Tractable reformulation | Semi-infinite program | Ratio |
|----|-------------------------|-----------------------|-------|
|    | time (s) | Opt. val. | time (s) | Opt. val. |
| 25 | 0.24     | 59000.14  | 4.57     | 59000.14  | 19.24 |
| 50 | 0.21     | 59000.14  | 7.82     | 59000.14  | 36.69 |
| 75 | 0.22     | 59000.14  | 13.51    | 59000.14  | 62.83 |
| 100| 0.24     | 59000.14  | 13.92    | 59000.14  | 56.92 |
| 125| 0.24     | 59000.14  | 22.29    | 59035.38  | 91.07 |
| 150| 0.23     | 59000.14  | 20.62    | 59000.14  | 88.23 |
| 175| 0.27     | 59000.14  | 28.92    | 59370.78  | 108.68 |
| 200| 0.23     | 59000.14  | 27.63    | 59369.95  | 120.16 |
| 225| 0.22     | 59000.14  | 29.61    | 59000.14  | 133.48 |
| 250| 0.24     | 59000.14  | 26.02    | 59000.14  | 120.14 |
| 275| 0.24     | 59000.14  | 34.71    | 59000.14  | 142.04 |
| 300| 0.25     | 59000.14  | 45.16    | 59000.14  | 181.04 |
| 325| 0.26     | 59000.14  | 39.75    | 59000.14  | 153.76 |
| 350| 0.24     | 59000.14  | 41.98    | 59000.14  | 176.78 |
| 375| 0.26     | 59000.14  | 40.15    | 59000.14  | 157.14 |
| 400| 0.24     | 59000.14  | 64.78    | 59000.14  | 269.35 |
| 425| 0.23     | 59000.14  | 66.04    | 59420.38  | 283.82 |
| 450| 0.24     | 59000.14  | 60.60    | 59000.14  | 254.60 |
| 475| 0.21     | 59000.14  | 75.03    | 60335.63  | 358.45 |
| 500| 0.25     | 59000.14  | 76.15    | 59042.41  | 303.04 |
| 525| 0.26     | 59000.14  | 66.74    | 59000.14  | 258.88 |
| 550| 0.23     | 59000.14  | 83.19    | 59000.14  | 363.72 |
| 575| 0.27     | 59000.14  | 72.06    | 59000.14  | 282.95 |
| 600| 0.25     | 59000.14  | 73.92    | 59000.14  | 299.09 |
| 625| 0.24     | 59000.14  | 77.79    | 59000.14  | 319.98 |

6.2. Non-monotone uncertain LCPs. We now consider a non-monotone LCP($M(u), q(u)$) whose robust counterpart is given by the following:

$$\begin{align*}
\min_t & \quad t \\
\text{subject to} & \quad x^T (u_1 S_1 - u_2 S_2) x + (u_1 q_1 + u_2 q_2)^T x \leq t, \quad \forall u \in \mathcal{U}, \\
& \quad (u_1 S_1 - u_2 S_2) x + u_1 q_1 + u_2 q_2 \geq 0, \quad \forall u \in \mathcal{U}, \\
& \quad x \geq 0.
\end{align*}$$

(6.3)
where \( e_n^T = (1, \ldots, n) \), \( S_1 = e_n e_n^T \succeq 0 \), \( S_2 = 10^4 \times B^T B \succeq 0 \), \( \mathcal{U} = \{(u_1, u_2) \mid 0 \leq u_1, u_2 \leq 1\} \), \( B \) is a randomly generated matrix with elements drawn from \( N(0,1) \), \( q_1 = -e_n \) and \( q_2 = \frac{10}{n(n+1)} \times S_2 e_n \). It should be emphasized that our analysis allows for deriving the robust counterpart of this problem as a relatively low-dimensional nonconvex QCQP. In the absence of such an analysis, a direct approach would require solving an approximate nonconvex QCQP whose size is of the order of magnitude of the discretization.

Table 6.4 provides a comparison of the performance of three solvers on the RC on a set of test problems for increasing matrix dimension \( n \): (i) our branching scheme, (ii) the commercial global optimization solver \texttt{baron} and (iii) the multi-start solver from \texttt{Matlab}.

| Size | Branching scheme | \texttt{baron} | \texttt{matlab} | \( \|x_{\text{branch}} - x_{\text{baron}}\|_1 + \|x_{\text{baron}}\|_1 \) | \( \|x_{\text{branch}} - x_{\text{matlab}}\|_1 + \|x_{\text{matlab}}\|_1 \) |
|------|----------------|---------------|----------------|-------------------|-------------------|
| 6    | 1.32 0.1000    | 0.00 0.61    | 0.00 0.0998    | 3.52 2.54          | 0.00 0.33          |
| 7    | 2.21 0.0035    | 0.00 1.81    | 0.00 0.0164    | 4.54 2.53          | 0.00 0.21          |
| 8    | 2.29 0.1648    | 0.00 0.87    | 0.00 0.1682    | 5.07 1.38          | 0.00 0.33          |
| 9    | 14.19 0.0072   | 0.00 0.89    | 0.00 0.0076    | 6.40 3.65          | 0.00 0.23          |
| 10   | 9.80 0.0040    | 0.00 1001.7  | 0.00 0.0125    | 6.93 0.74          | 0.00 0.22          |
| 11   | 72.74 0.0036   | 0.00 2.94    | 0.00 0.0026    | 6.09 14.31         | 0.00 0.22          |
| 12   | 83.73 0.1998   | 0.00 1.79    | 0.00 0.1990    | 8.22 10.69         | 0.00 0.13          |

Table 6.4: Global optimization of nonconvex QCQPs: CPU 3.40Ghz RAM 16.0 GB

The results from Table 6.4 suggest the following. First, our branching scheme provides reasonably accurate solutions by comparing with the commercial solver \texttt{baron}, sometimes even better with respect to the optimal value \( z \). Furthermore, the performance is significantly superior in terms of optimal value to the solutions provided by \texttt{Matlab}. Third, \texttt{baron}’s performance in terms of time is superior to that provided by our \texttt{Matlab}-based branching solver is not altogether surprising, given that it uses extensive pre-processing and has been developed on C/C++.

6.3. Case study: Uncertain traffic equilibrium problems.

2-node and 5 link network. Consider the uncertain traffic equilibrium of the form described in Section 6.3 sourced from [13]. Suppose the associated network has two vertices A, B and five arcs \( D_1, D_2, D_3, U_1, U_2 \). Let \( \xi \) denote the flow over these five paths and \( T(u)\xi + t \) represent the travel associated travel times, where \( T(u) \) is an uncertain \( 5 \times 5 \) matrix and \( t \in \mathbb{R}^5 \) is a constant vector. Suppose \( B \) represents the path-OD pair incidence matrix and \( d(u) \in \mathbb{R}^2 \) represents the uncertain demand. Let \( \tau \) represent the minimum travel time
for each direction. Recall that the equilibrium point is given by a solution to the following:

\[
0 \leq x \perp M(u)x + q(u) \geq 0, \quad \forall u \in \mathcal{U}
\]

where \(x = (\xi, \tau)\), \(M(u)\) and \(q(u)\) are defined as

\[
M(u) = \begin{pmatrix}
T(u) & -B^T \\
B & 0
\end{pmatrix},
q(u) = \begin{pmatrix}
t \\
-d(u)
\end{pmatrix},
B = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix},
t = \begin{pmatrix}
1000 \\
950 \\
3000 \\
1000 \\
1300
\end{pmatrix},
\]

and \(T(u)\) is defined as

\[
T(u) = \begin{pmatrix}
40\alpha(u) & 0 & 0 & 20\beta(u) & 0 \\
0 & 60\beta(u) & 0 & 0 & 20\beta(u) \\
0 & 0 & 80\beta(u) & 0 & 0 \\
8\alpha(u) & 0 & 0 & 80\alpha(u) & 0 \\
0 & 4\beta(u) & 0 & 0 & 100\beta(u)
\end{pmatrix},
d(u) = \begin{pmatrix}
260 - 100(\alpha(u) + \beta(u)) \\
170 - 100(\alpha(u) + \beta(u))
\end{pmatrix},
\]

\(\alpha(u) = \frac{1}{2}u(u - 1)\) and \(\beta(u) = u(2 - u)\). Suppose \(\mathcal{U} \triangleq \{u_1, u_2, u_3\}\) where \(u_1, u_2,\) and \(u_3\) denote a sunny, windy, and a rainy day respectively. In an effort to compare the obtained solutions with that obtained from the ERM model \[15\], we assume that these events occur with probability \(\frac{1}{2}, \frac{1}{4},\) and \(\frac{1}{4}\) (Note that our model does not require a probability distribution). Corresponding to this problem, the ERM solution is denoted by \(x^{\text{erm}}\) while the robust solution is \(x^{\text{rob}}\). Furthermore, non-robust scenario-specific solutions are denoted by \(x^1, x^2\) and \(x^3\). Table \[6.5\] compares the optimality and feasibility of such points with respect to the robust counterpart. In this table, the infeasibility function is defined as \(\max_{u \in \mathcal{U}} (e^T \max(-M(u)x - q(u), 0))\) while the complementarity residual is defined as: \(\max_{u \in \mathcal{U}} x^T (M(u)x + q(u))\). It is seen that the robust solution and \(x^1\) are feasible for every \(u\). Notably, the “sunny day” design is feasible but leads to a large complementarity residual. Note that if both the feasibility and the complementarity metric is zero, this implies that the solution is an equilibrium for every \(u \in \mathcal{U}\). The robust solution minimizes the worst complementarity residual among all possible scenarios and from that standpoint, it is seen to be superior to \(x^1\), the solution that minimizes the residual for the first scenario. Furthermore, \(x^{\text{erm}}\) might have a superior complementarity residual but such a solution may be rendered infeasible for certain realizations. Table \[6.3\] compares the value of parametrized gap function \(G(x, u)\), defined as

\[
G(x, u) \triangleq \sup_{y \geq 0} (x - y)^T (M(u)x + q(u)).
\]

| solution | infeasibility | complementarity |
|----------|---------------|----------------|
| \(x^1\) | (0, 260, 0, 170, 0, 950, 1000) | 0 | 4.25E+06 |
| \(x^2\) | (159.2, 0.83, 0, 70, 0, 1000, 1000) | 250 | 1.71E+06 |
| \(x^3\) | (0, 160, 0, 3.75, 66.25, 950, 1300) | 500 | 2.22E+06 |
| \(x^{\text{erm}}\) | (84, 84, 21, 80, 20, 975, 1000) | 166 | 1.09E+06 |
| \(x^{\text{rob}}\) | (117.7, 89.5, 52.8, 90.5, 79.5, 950, 1000) | 0 | 1.84E+06 |
Table 6.6: Evaluation of $G(x^*, u)$

| $x^{rob}$ | $u = u_1$ | $u = u_2$ | $u = u_3$ |
|-----------|-----------|-----------|-----------|
| 1.4e+5   | 1.8e+6   | 1.8e+6   |           |

OD pair | possible demand | $x^{RO}$ | $x^{ERM}$ | $x_1$ | $x_2$ | $x_3$
|--------|----------------|---------|---------|-------|-------|-------|
| AB     | 260,160        | 260     | 189     | 260   | 160   | 160   |
| BA     | 170,70         | 170     | 100     | 170   | 70    | 70    |

Table 6.7: Flow of each OD pair

![Traffic Network](image)

Fig. 6.2: Traffic Network

The lowest value of $G(x^*, u)$ is achieved by $x^{erm}$ when $u = u_2$. However, $G(x^{erm}, u_1) = G(x^{erm}, u_3) = +\infty$, a consequence of infeasibility. However, $G(x^{rob}, u) < \infty$ for every $u \in U$. In Table 6.3, we consider how the robust solution satisfies demand requirements (ensuring feasibility) while the ERM solution may not satisfy demand for all realizations (leading to infeasibility).

**5-node and 7-link network:** We now consider a larger traffic network considered in Section 2 with 7 links and 6 paths. Figure 6.2 represents a 7-link network with 6-paths sourced from 10 and A → D and A → E represent two origin-destination (OD) pairs. The OD pair A → D is connected by paths $p_1 = \{1, 3\}, p_2 = \{1, 7, 6\}, p_3 = \{2, 6\}$ while the OD pair A → E is connected by paths $p_1 = \{1, 5\}, p_5 = \{1, 7, 4\}, p_6 = \{2, 4\}$. The demand along every OD pair is denoted by $d(u) \in \mathbb{R}^2$ where $u \in U$ while the link capacity is captured by the vector $c(u) \in \mathbb{R}^7, u \in U$. Let vector $x \in \mathbb{R}^6$ denote the assignment of flows to all path from $p_1$ to $p_6$ and $f \in \mathbb{R}^7$ denote the assignment of flows to all links 1, ..., 7. Then the relationship between $x$ and $f$ is presented by: $f = \Delta x$, where $\Delta = (\delta_{i,j})$ is the link-path incidence matrix. The entry $\delta_{i,j}$ is set at 1 if and only if link $i$ lies in path $j$. Let $B = (b_{i,j})$ denote the OD-path incidence matrix and $b_{i,j} = 1$ if and only if path $j$ connects the $i$th OD pair. In this case, the two matrices are given as follows:

$$
\Delta = \begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

and

$$
B = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$

(6.4)

Following a generalized bureau of public roads (GBPR) function, the multivalue link cost function $C(f, u)$ is defined as:

$$
C_i(f, u) = c_i^0 \left(1.0 + 0.15 \left(\frac{f_i}{c_i(u)}\right)^{n_i}\right), i = 1, \ldots, 7
$$

(6.5)
where $c_i^0$ and $n_i$ are known parameters. Let $n_i = 1$ for all $i$, then the travel cost function is given as:

$$F(u, x) = \eta \Delta^T C(\Delta x, u) = 0.15 \eta \Delta^T \text{diag} \left( \frac{c_i^0}{c_i(u)} \right) \Delta x + \eta \Delta^T c^0 \triangleq M(u)x + q.$$  \hfill (6.6)

Let $w \in \mathbb{R}^2$ denote the minimum travel cost of each OD pair. Last, by Wardrop’s user equilibrium, the uncertain VI formulation is given by the following:

$$0 \leq \begin{pmatrix} x \\ w \end{pmatrix} \perp \begin{pmatrix} M(u) & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} q \\ -d(u) \end{pmatrix} \geq 0, \quad \forall u \in \mathcal{U}. \hfill (6.7)$$

Tables 6.8 and 6.9 show the comparison between different solutions of the LCP given by (6.7). $x^{\text{rob}}$ denotes the robust solution of (6.7) in that it minimizes the worst case of the gap function $G(z, u)$, defined as

$$G(z, u) = \sup_{y \geq 0} (z - y)^T (A(u)z + b(u)), z = \begin{pmatrix} x \\ w \end{pmatrix}, A(u) = \begin{pmatrix} M(u) & -B^T \\ B & 0 \end{pmatrix}, b(u) = \begin{pmatrix} q \\ -d(u) \end{pmatrix}.$$  

We consider a case when $\mathcal{U} = \{ u \mid -1 \leq u \leq 1 \}, \frac{1}{c_i(u)} = \hat{c}_i, \forall i$, where $\hat{c}_0$ and $\hat{c}_1$ are defined as follows:

$$\hat{c}_0 = -\hat{c}_1 = (1/40, 1/40, 1/20, 1/20, 1/20, 1/20, 1/20), c^0 = (3, 5, 6, 4, 6, 4, 1),$$

$$d(u) = d_0 + ud_1, d_0 = (200; 220), d_1 = (50; 40).$$

The ERM solution $x_{\text{ERM}}$ is constructed as follows. Let $x_{\text{ERM}} = (y; w)$ where $y$ is obtained by

$$y = (I - B^1 B)x^* + B^1 \mathbb{E}[d(u)], \quad \text{where} \quad B^1 = B^T (BB^T)^{-1}.$$

Note that $x^*$ is a minimizer of $\phi(x)$ over the set $D$, where

$$\phi(x) = \mathbb{E}[f(x, u)], f(x, u) = z(x, u)^T F(z(x, u), u), z(x, u) = (I - B^1 B)x + B^1 d(u),$$

$$F(z, u) = M(u)z + q, Q(z, u) = \max \left\{ -y^T F(z, u) \mid By = d(u), y \geq 0 \right\} = \min \left\{ y^T d(u) \mid B^Ty + F(z, u) \geq 0 \right\},$$

$$D = \left\{ x \mid B^1 Bx \leq c, c_i = \min_{u \in \mathcal{U}} (B^1 d(u))_i \right\},$$

as per the recent work by Chen, Wets and Zhang [10]. Note that an estimator the minimizer of $\mathbb{E}[f(x, u)]$ is obtained via sample-average approximation schemes while $w$ is acquired by taking the minimum of the average costs of paths in each OD-pair. Let the average costs of paths be captured by a vector $v = \mathbb{E}[M(u)y + q]$, then $v_1 = \min\{v_1, v_2, v_3\}, w_2 = \min\{v_4, v_5, v_6\}$. $x_1, \ldots, x_5$ are the solutions of the program

$$\min_{A(u)x + b(u) \geq 0} x^T (A(u)x + b(u))), \quad \text{when} \quad u = -1, -0.5, 0, 0.5, 1, \text{respectively.}$$

Table 6.8 shows the traffic flow between two OD pairs. Again, the robust solution satisfies the largest possible demand while the ERM solution does not satisfy demand for all possible realizations. When we compare the residual function for a particular $x$ and $u$ as seen in Table 6.9, while the robust solution $x^{\text{rob}}$ does not provide the best function value for every scenario, it minimizes the worst case. In fact, for the non-robust solutions, except $x^4$, every solution displays an infinite residual function for some $u$. Notably, the ERM solution also have infinite residuals for some realizations of $u$.

| OD pair | range of possible demand | $x^{\text{rob}}$ | $x^{\text{erm}}$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---------|-------------------------|------------------|------------------|-------|-------|-------|-------|-------|
| AD      | 150-250                 | 250              | 200              | 150   | 175   | 225   | 250   | 200   |
| AE      | 180-260                 | 260              | 220              | 180   | 200   | 240   | 260   | 220   |

Table 6.8: Flow across two OD pairs
Table 6.9: Residual function value at different sample points

| $u$  | $x^*_{138}$ | $x^*_{206}$ | $x^*$  | $x^*$  | $x^*$  | $x^*$  |
|------|-------------|-------------|--------|--------|--------|--------|
| -1   | 10343       | 4340        | 6488   | 7922   | 18322  | 19000  | 4329   |
| -0.5 | 7863        | 2176        | Inf    | 6772   | 14671  | 14250  | 2165   |
| 0.0  | 5382        | Inf         | Inf    | 11021  | 9500   | 0.000449 |
| 0.5  | 2901        | Inf         | Inf    | 7370   | 4750   | Inf    |
| 1.0  | 421         | Inf         | Inf    | Inf    | 1.170e-05 | Inf    |

Table 7.1: Characterization of robust counterparts under varying assumptions.

| $\mathcal{U}$ | $M, q(u)$ | $M(u), q$ | $M(u), q(u)$ | $A^T(u)A(u), q(u)$ |
|---------------|-----------|-----------|--------------|--------------------|
| $\mathcal{U}_\infty$ | convex QP | convex QP | nonconvex QCQP | / |
| $\mathcal{U}_t$ | convex QP | convex QCQP | nonconvex QCQP | / |
| $\mathcal{U}_c$ | convex QCQP | convex program | nonconvex program | SDP |
| $\mathcal{U}_r$ | QP with conic constraints | / | / | / |
| $\mathcal{U}_r'$ | / | convex QCQP | / | / |
| $\mathcal{U}_r^\infty$ | / | convex QP | / | / |

7. Concluding remarks. In this paper, we consider the resolution of finite-dimensional monotone complementarity problems corrupted by uncertainty. A distinct thread in the literature has considered the minimization of the expected residual. This avenue relies on the availability of a probability distribution and the solution of a stochastic, and possibly nonconvex, program. Instead, we consider an avenue that relies on the availability of an uncertainty set. By leveraging findings from robust convex programming, we show that uncertain monotone linear complementarity problems can be tractably resolved as a single convex program. In fact, when the uncertain linear complementarity problem is not necessarily monotone, under some conditions on the uncertainty set, the tractable robust counterpart of this problem can be shown to be convex, a consequence of leveraging hidden convexity in the problem. More generally, the robust counterpart is a nonconvex quadratically constrained quadratic program. We adapt and present a recently presented branching scheme to accommodate such problems. Table 7.1 provides a compact representation of the tractability statements and the nature of the uncertainty sets that correspond to these statements. The columns of this table correspond to different assumptions of uncertainty on $M$ and $q$. Note that $M(u) = M_0 + \sum_{l=1}^{L} u_l M_l$, $M_l \succeq 0$ or $\preceq 0$ for $l = 1, \ldots, L$, $q(u) = q_0 + \sum_{l=1}^{L} u_l q_l$, and $A(u) = A_0 + \sum_{l=1}^{L} u_l A_l$. We further observe that such statements can be utilized to show that:

1. The tractable robust counterparts of an uncertain affine variational inequality problem (uncertain AVIs) over uncertain polyhedral sets are SDPs under some assumptions on the uncertainty set.
2. Robust counterparts of mathematical programs with uncertain linear complementarity constraints (uncertain MPCC) can be reformulated as deterministic low-dimensional mathematical programs with complementarity constraints.

Finally, our preliminary numerical investigations suggest that such tractable counterparts provide tremendous computational benefits in contrast with the naive discretized formulations. Furthermore, the branching-based procedure for non-monotone uncertain LCPs is shown to be effective. Finally, robust formulations provide qualitatively different solutions from their ERM counterparts in the context of traffic equilibrium problems.

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