Maximal ideal space of a commutative coefficient algebra

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Abstract

The basic notion of the article is a pair \((A, U)\), where \(A\) is a commutative \(C^*\)-algebra and \(U\) is a partial isometry such that \(A \ni a \rightarrow UaU^*\) is an endomorphism of \(A\) and \(U^*U \in \mathcal{A}'\). We give a description of the maximal ideal space of the smallest coefficient \(C^*\)-algebra \(E_*(\mathcal{A})\) of the algebra \(C^*(\mathcal{A}, U)\) generated by the system \((\mathcal{A}, U)\).

KEY WORDS: commutative \(C^*\)-algebra, endomorphism, partial isometry, coefficient algebra

1 Introduction. Extensions of \(C^*\)-algebras by partial isometries and coefficient algebras

The notion of a coefficient algebra was introduced in [1] in connection with the study of extensions of \(C^*\)-algebras by partial isometries.

Namely in [1] the authors investigated the following object.

Let \(H\) be a Hilbert space and \(A \subset L(H)\) be a certain \(*\)-algebra containing the identity \(1\) of \(L(H)\), the paper was devoted to the description of the \(C^*\)-extensions of \(A\) associated with the mappings

\[\delta(x) = UxU^*, \quad \delta_*(x) = U^*xU, \quad x \in L(H)\] (1)
where $U \in L(H), U \neq 0$. It is clear that $\delta$ and $\delta_\ast$ are linear and continuous ($\|\delta\| = \|\delta_\ast\| = \|U\|^2$) maps of $L(H)$ and $\delta(x^*) = \delta(x)^*, \delta_\ast(x^*) = \delta_\ast(x)^*$. When using the powers $\delta^k$ and $\delta_\ast^k, \ k = 0, 1, 2, \ldots$ we assume for convenience that $\delta^0(x) = \delta_\ast^0(x) = x$.

Observe that if $\delta : A \to L(H)$ is a morphism then we have

$$UU^* = \delta(1) = \delta(1^2) = \delta^2(1) = (UU^*)^2$$

and therefore $U$ is a partial isometry.

In [1] the authors studied the $C^*$-algebra $C^*(A, U)$ generated by $A$ and $U$ assuming additionally that $A$ is the coefficient algebra of $C^*(A, U)$, by this they meant that $A$ possessed the following three properties

\begin{align*}
\mathcal{A} & \ni a \to \delta(a) = UaU^* \in \mathcal{A}, & (2) \\
\mathcal{A} & \ni a \to \delta_\ast(a) = U^*aU \in \mathcal{A}, & (3) \\
Ua & = \delta(a)U, \quad a \in \mathcal{A} & (4)
\end{align*}

Algebras possessing these properties really play the role of ’coefficients’ in $C^*(A, U)$ which was shown in [1], Proposition 2.3, telling us that

if a $^*$-algebra $A$ and $U$ satisfy conditions (2), (3), (4) then the vector space consisting of finite sums

$$x = U^*a_\pi + \ldots + U^*a_\pi + a_0 + a_1U + \ldots + a_NU^N,$$

where $a_\pi, a_\pi \in A$ and $N \in \mathbb{N}$, is a uniformly dense $^*$-subalgebra of the $C^*$-algebra $C^*(A, U)$.

It is worth mentioning that property (4) can also be written in a different equivalent form which is stated in the next

**Proposition 1.1** ([1], Proposition 2.2.) Let $A$ is a $C^*$-algebra of $L(H), 1 \in A$ and $U \in L(H)$ then the following conditions are equivalent

\begin{enumerate}
  \item[(i)] $Ua = \delta(a)U, \quad a \in \mathcal{A}$,
  \item[(ii)] $U$ is a partial isometry and $U^*U \in \mathcal{A}'$ \quad (6)
  \begin{align*}
  \text{where } \mathcal{A}' \text{ is the commutant of } \mathcal{A},
  \item[(iii)] $U^*U \in \mathcal{A}'$ and $\delta : A \to \delta(A)$ is a morphism.
\end{enumerate}
Thus property (4) implies that the mapping \( \delta(a) = UaU^*, \ a \in \mathcal{A} \), is a morphism. Hence, every coefficient algebra is associated with an endomorphism \( \delta \) generated by a partial isometry \( U \).

In [1] the authors gave a construction of an algebra satisfying (2), (3) and (4) starting from an initial algebra that satisfies only some of these conditions or even does not satisfy any of them. Hereafter we present a part of this construction for the case of a commutative algebra \( \mathcal{A} \).

Let us denote by 
\[ \overline{\mathcal{E}}^*(\mathcal{A}) = \bigcup_{n=0}^{\infty} \delta^n_*(\mathcal{A}) \]

the \( C^* \)-algebra generated by \( \bigcup_{n=0}^{\infty} \delta^n_*(\mathcal{A}) \). It was proved in [1] (Proposition 4.1.) that the following statement is true.

**Proposition 1.2** Let \( \mathcal{A} \) be a commutative \( C^* \)-subalgebra of \( L(H) \) containing 1. Let \( \delta \) be an endomorphism of \( \mathcal{A} \) and let \( U^*U \in \mathcal{A}' \) then the \( C^* \)-algebra \( \overline{\mathcal{E}}^*(\mathcal{A}) = \bigcup_{n=0}^{\infty} \delta^n_*(\mathcal{A}) \) is the minimal commutative coefficient algebra for \( C^*(\mathcal{A}, U) \) and both the mappings \( \delta : \overline{\mathcal{E}}^*(\mathcal{A}) \to \overline{\mathcal{E}}^*(\mathcal{A}) \) and \( \delta_* : \overline{\mathcal{E}}^*(\mathcal{A}) \to \overline{\mathcal{E}}^*(\mathcal{A}) \) are endomorphisms.

This proposition leads to the natural problem of obtaining the description of the maximal ideal space of the coefficient algebra \( \overline{\mathcal{E}}^*(\mathcal{A}) \) in terms of the maximal ideal space of \( \mathcal{A} \) and the action of \( \delta \). Precisely the solution to this problem is the theme of the present article.

We have to mention that some concrete examples of the description of the maximal ideal space of \( \overline{\mathcal{E}}^*(\mathcal{A}) \) in the situation when \( \mathcal{A} = C[a, b] \) and \( \delta \) is generated by a continuous mapping \( \alpha : [a, b] \to [a, b] \) of a special form are given in [2].

The paper is organized as follows. In the second section we introduce a number of necessary for our future goals notions and notation and present some (mostly known) facts on the structure of endomorphisms of commutative algebras. Our main result — the description of the maximal ideal space of \( \overline{\mathcal{E}}^*(\mathcal{A}) \) is given in section 3.

## 2 Endomorphisms of commutative \( C^* \)-algebras

Our starting objects are a commutative \( C^* \)-algebra \( \mathcal{A} \) containing an identity 1 and an endomorphism
\[
\delta : \mathcal{A} \to \mathcal{A}.
\] (7)
Our first observation is that any endomorphism $\delta$ generates a continuous partial mapping on the maximal ideal space $M = M(A)$ of $A$ which is stated in theorem 2.1.

Remark. This theorem is a particular case of the general result on the description of endomorphisms of semisimple Banach algebras and we present its proof for the sake of completeness.

Since by the Gelfand-Naimark theorem the Gelfand transform defines an isomorphism $A \cong C(M)$ we shall identify $A$ and $C(M)$ in all further considerations.

**Theorem 2.1** Let $A$ be a commutative $C^*$-algebra. Let $1 \in A$ and $\delta$ be an endomorphism of $A$. Then there exists a subset $\Delta$ of the maximal ideal space $M$ such that

i) the set $\Delta$ is open and closed,

ii) the endomorphism $\delta$ is given by the formula

$$
(\delta f)(x) = \begin{cases} f(\alpha(x)), & x \in \Delta \\ 0, & x \notin \Delta \end{cases},
$$

where $f \in C(M)$ and $\alpha : \Delta \to M$ is a continuous mapping.

**Proof.** Let $\Delta \subseteq M$ be the set given by the condition

$$
\tau \in \Delta \iff \tau(\delta(1)) = 1.
$$

First let us observe the closedness and openness of $\Delta$. Note that $\delta(1) = \delta^2(1)$ and therefore the function $\delta(1) \in C(M)$ is an idempotent, so its values are either 0 or 1. This implies the closedness and openness of $\Delta$.

In terms of the Gelfand transform $a \to \hat{a}$ we can define the action of $\delta$ on $C(M)$ by means of the formula

$$
(\delta \hat{a})(\tau) = \tau(\delta(a)) = \hat{a}(\delta^*(\tau)), \quad \tau \in M,
$$

where $\delta^* : A^* \to A^*$ is the adjoint operator to $\delta$. Clearly

$$
\delta^*(\tau) = \tau \circ \delta
$$

is a multiplicative functional and $\delta^*(\tau)(1) = \tau(\delta(1))$. By the definition of the set $\Delta$ we have

$$
\tau \notin \Delta \Rightarrow \delta^*(\tau) \equiv 0,
$$

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Now defining the map $\alpha : \Delta \rightarrow M$ as the restriction of the map $\delta^*$

$$\alpha = \delta^*|_{\Delta}$$

we obtain the statement of the theorem.

**Theorem 2.2** Let the conditions of theorem 2.1 be satisfied and let the mapping $\alpha : \Delta \rightarrow M$ be given by (8). Then

i) if $\ker \delta = \{0\}$ then $\alpha : \Delta \rightarrow M$ is a surjection,

ii) if $\delta(1) = 1$ then $\Delta = M$.

**Proof.** Let $\ker \delta = \{0\}$. Then $\delta$ is an injection and there exists a left inverse to it

$$\varrho : \delta(A) \rightarrow A.$$

Therefore we have

$$\varrho(\delta(a)) = a, \quad a \in A.$$  \hspace{1cm} (15)

For any $\tau \in M$ the functional $\tau \circ \varrho$ being defined on $\delta(A)$ is nonzero and multiplicative. Therefore there exists its extension $\tau_1 \in M$ on $A$ (see [4], 2.10.2.) and thus

$$\tau_1 \circ \delta = \tau.$$  \hspace{1cm} (16)

Now (11) and (14) imply the surjectivity of $\alpha$.

Now let $\delta(1) = 1$. This equality means that for any $\tau \in M$ we have $\tau(\delta(1)) = 1$. So (8) implies $\Delta = M$. The proof is complete.

Now we return to the initial object of the article. To this end $A$ is a $C^*$-subalgebra of the algebra $L(H)$ containing the identity operator $1$ and $U \in L(H)$ is an operator such that the mapping

$$\delta(a) = UaU^*, \quad a \in A$$  \hspace{1cm} (17)

is an endomorphisms of $A$ (this implies in particular that $U$ is a partial isometry).

Note that applying endomorphism $\delta$ $n$ times one obtains

$$U^nU^* = \delta^n(1) = (\delta^n(1))^2 = (\delta^n(1)) = (U^nU^*)^2.$$

Which means that $U^n$ is a partial isometry and therefore $U$ is a power isometry.

In view of the form (17) of the endomorphism $\delta$ and relation (8) it is possible to rewrite theorem 2.1 for the objects considered in the following form
Theorem 2.3 Let \( A \subset L(H) \) be a commutative \( C^* \)-algebra containing the identity operator 1 and let the mapping \( \delta(a) = UaU^* \) be an endomorphism of the algebra \( A \). Then

i) The set \( \Delta = \{ \tau \in M : \tau(UU^*) = 1 \} \) is open and closed,

ii) On the maximal ideal space \( M \) the endomorphism \( \delta \) is given by the formula

\[
(\delta f)(x) = \begin{cases} f(\alpha(x)), & x \in \Delta \\ 0, & x \notin \Delta \end{cases}
\]

where \( f \in C(M) \), and \( \alpha : \Delta \rightarrow M \) is a continuous mapping.

Moreover theorem 2.2 implies

Theorem 2.4 Let the conditions of theorem 2.3 be satisfied and \( \alpha : \Delta \rightarrow M \) be given by (17). Then we have

i) if \( U \) is a unitary operator then \( \Delta = M \) and \( \alpha : M \rightarrow M \) is surjective,

ii) if \( U \) is an isometry then \( \alpha : \Delta \rightarrow M \) is surjective,

iii) if \( U^* \) is an isometry then \( \Delta = M \).

As the next theorem shows the situation described in theorem 2.3 simplifies considerably if along with endomorphism \( \delta \) (17) the mapping

\[
\delta_*(a) = U^*aU , \quad a \in A.
\]

is an endomorphism of \( A \) as well.

Theorem 2.5 Let \( A \) be a commutative \( C^* \)-subalgebra of \( L(H) \) containing the identity 1 and the mappings \( \delta, \delta_* \) given by formulae (17), (19) be endomorphisms of the algebra \( A \). Let \( M \) be the maximal ideal space of \( A \). Then

i) the sets \( \Delta_1 = \{ \tau \in M : \tau(UU^*) = 1 \} \) and \( \Delta_{-1} = \{ \tau \in M : \tau(U^*U) = 1 \} \) are open and closed,

ii) in terms of the algebra \( C(M) \) the endomorphism \( \delta \) is given by the formula

\[
(\delta f)(x) = \begin{cases} f(\alpha(x)), & x \in \Delta_1 \\ 0, & x \notin \Delta_1 \end{cases}
\]

where \( f \in C(M) \) and \( \alpha : \Delta_1 \rightarrow \Delta_{-1} \) is a homeomorphism.
iii) the endomorphism $\delta_*$ is given by the formula

$$(\delta_* f)(x) = \begin{cases} f(\alpha^{-1}(x)), & x \in \Delta_{-1} \\ 0, & x \notin \Delta_{-1} \end{cases}$$

(21)

where $f \in C(M)$.

Proof. By theorem 2.3 there exist closed and open sets $\Delta_1, \Delta_{-1} \subseteq M$ given by the relations

$$\tau \in \Delta_1 \iff \tau(UU^*) = 1,$$

(22)

$$\tau \in \Delta_{-1} \iff \tau(U^*U) = 1$$

(23)

and continuous mappings $\alpha : \Delta_1 \to M, \alpha' : \Delta_{-1} \to M$ for which $\delta$ and $\delta_*$ satisfy respectively. To finish the proof it is enough to show that $\alpha' = \alpha^{-1}$. This follows from the relations

$$\tau \in \Delta, a \in A \implies \tau(\delta(\delta_* (a))) = \tau(UU^*)\tau(a)\tau(UU^*) = \tau(a) \quad \text{(24)}$$

$$\tau \in \Delta_{-1}, a \in A \implies \tau(\delta_* (a)) = \tau(U^*U)\tau(a)\tau(U^*U) = \tau(a) \quad \text{(25)}$$

Or $(\alpha' \circ \alpha)(\tau) = (\alpha \circ \alpha')(\tau) = \tau$. The proof is complete.

In the next section we shall make use of the sets introduced hereafter.

Let

$$\Delta_n = \alpha^{-n}(M), \quad n = 0, 1, 2, ...$$

(26)

be the set on which $\alpha^n$ is defined and let

$$\Delta_{-n} = \alpha^n(\Delta_n), \quad n = 1, 2, ...$$

(27)

be the image of $\alpha^n$.

We have that

$$\alpha^n : \Delta_n \to \Delta_{-n}, \quad (\alpha^n(x)) = \alpha^n+m(x), \quad x \in \Delta_{n+m}.$$ 

(28)

In terms of the multiplicative functionals the sets $\Delta_n$ can be defined in the following form: for $n > 0$

$$\tau \in \Delta_n \iff \forall 0 < k \leq n \tau(U^kU^*) = 1,$$

(30)

$$\tau \in \Delta_{-n} \iff \exists \tau_n \in \Delta_n \tau_n \circ \delta^n = \tau.$$ 

(31)

Note however that the sequence of projections $U^kU^*$ is decreasing and therefore $\tau(U^nU^*) = 1$ implies $\tau(U^kU^*) = 1$ for $k < n$. So we can rewrite condition (30) in the form

$$\tau \in \Delta_n \iff \tau(U^nU^*) = 1.$$ 

(32)
Remark 2.6 In the situation considered in theorem 2.5 one can describe the sets $\Delta_{-n}$ as the sets on which the mapping $\alpha^{-n}$ is defined. Moreover in terms of the maximal ideals we have

$$\tau \in \Delta_n \iff \tau(U^n U^*) = 1,$$

(33)

$$\tau \in \Delta_{-n} \iff \tau(U^* U^n) = 1,$$

(34)

where $n \geq 0$

3 Maximal ideal space of a commutative co-efficient algebra

Throughout this section we fix a commutative $C^*$-subalgebra $A \subset L(H)$, $1 \in A$, and a partial isometry $U \in L(H)$ such that the mapping (17) is an endomorphism of $A$ and $U^* U \in A'$. The aim of the section is to give a description of the maximal ideal space $M(\overline{E_*(A)})$ of the coefficient $C^*$-algebra $\overline{E_*(A)} = \{ \bigcup_{n=0}^{\infty} \delta^*_n(A) \}$ in terms of the maximal ideal space $M = M(A)$ of $A$ and the action of $\delta$ given by (18).

To start with we introduce a number of objects and notation.

Let $\tilde{x} \in M(\overline{E_*(A)})$ be a linear and multiplicative functional on $\overline{E_*(A)}$. Let us consider a sequence of functionals $\xi^n_{\tilde{x}} : A \to \mathbb{C}$, $n = 0, 1, \ldots$, defined by the conditions

$$\xi^n_{\tilde{x}}(a) = \delta^*_n(a)(\tilde{x}), \quad a \in A.$$  

(35)

Since $\overline{E_*(A)} = \{ \bigcup_{n=0}^{\infty} \delta^*_n(A) \}$ it follows that the sequence $\xi^n_{\tilde{x}}$ determines $\tilde{x}$ in a unique way. On the other hand, since $\delta_*$ is an endomorphism of $\overline{E_*(A)}$ we have that the functionals $\xi^n_{\tilde{x}}$ are linear and multiplicative on $A$ (may be zero). So either

$$\xi^n_{\tilde{x}} = x_n \in M,$$

(36)

or

$$\xi^n_{\tilde{x}} = 0.$$  

(37)

Clearly the mapping

$$\tilde{x} \to (\xi^0_{\tilde{x}}, \xi^1_{\tilde{x}}, \ldots)$$

is an injection.

The next theorem is our first step in the description of the maximal ideal space $M(\overline{E_*(A)})$. 8
Theorem 3.1 Let $A \subset L(H)$ be a commutative $C^*$-subalgebra, $1 \in A$. Let $\delta(a) = UaU^*$ be an endomorphism of $A$, $U^*U \in A'$ and $\alpha : \Delta \to M(A)$ be a partial mapping given by formula (18) and $\Delta_n$ be the sets defined by (26), (27). Then the maximal ideal space $M(E_*(A))$ of $E_*(A)$ is homeomorphic to a subset of the countable sum of disjoint sets. Precisely the mapping (38) (along with observations (36) and (37) defines the topological embedding

$$M(E_*(A)) \hookrightarrow \bigcup_{N=0}^{\infty} M_N \cup M_\infty,$$

where $M_N$ are the sets of the form

$$M_N = \{ \tilde{x} = (x_0, x_1, ..., x_N, 0, ...) : x_n \in \Delta_n, \alpha(x_n) = x_{n-1}, n = 1, ..., N \},$$

and $M_\infty$ is given by

$$M_\infty = \{ \tilde{x} = (x_0, x_1, ...) : x_n \in \Delta_n, \alpha(x_n) = x_{n-1}, n \in \mathbb{N} \}.$$

The topology on the sets $M_N$, $N \in \mathbb{N} \cup \{0\}$, and $M_\infty$ is induced by the base of neighborhoods of points $\tilde{x} \in M_N$

$$O(a_1, ..., a_k, \varepsilon) = \{ \tilde{y} \in M_N : |a_i(x_N) - a_i(y_N)| < \varepsilon, i = 1, ..., k \}$$

and respectively $\tilde{x} \in M_\infty$

$$O(a_1, ..., a_k, n, \varepsilon) = \{ \tilde{y} \in \bigcup_{N=n}^{\infty} M_N \cup M_\infty : |a_i(x_n) - a_i(y_n)| < \varepsilon, i = 1, ..., k \}$$

where $\varepsilon > 0$, $a_i \in A$, and $k, n \in \mathbb{N} \cup \{0\}$.

Proof. As we have already observed the mapping (38) is an injection. Considering the righthand part of (38) one can come across the following two possibilities.

1) Let us assume first that some of the functionals $\xi^n_\tilde{x}$ are equal to zero. Let $N$ be the least number such that

$$\xi_{\tilde{x}}^{N+1} \equiv 0.$$

Note that for an arbitrary $n \in \mathbb{N}$ we have

$$\xi^n_{\tilde{x}} \neq 0 \iff \tilde{x}(U_*^n U^n) = 1$$

and the family $\{U_*^n U^n\}_{n \in \mathbb{N}}$ forms a commutative decreasing family of projections (see [1], Proposition 3.5), that is for $i \leq j$

$$U_*^i U_*^j U_*^j U_*^i = U_*^j U_*^j U_*^i U_*^i = U_*^j U_*^j.$$
Hence, for each $n > N$ we obtain

$$\bar{x}(U^n U^n) = \bar{x}(U^{N+1} U^{N+1} U^n U^n) = \bar{x}(U^{N+1} U^{N+1} U^n) \bar{x}(U^n U^n) = 0,$$

that is $\xi^n_{x} \equiv 0$ for $n > N$.

Since for every $0 \leq n \leq N$ we have $\xi^n_{x} \neq 0$ it follows that there exists $x_n \in M(A)$ such that

$$\xi^n_{x}(a) = a(x_n), \quad a \in A, \quad (43)$$

and the mapping (38) (along with (36) and (37)) takes the form $\bar{x} \mapsto (x_0, x_1, ..., x_N, 0, ...)$. Furthermore, since the projections $U^i U^j, U^j U^i$ commute we have that

$$U^n U^n U^{n-1} = U^{n-1}(U^i U)(U^{n-1} U^{n-1}) = (U^{n-1} U^{n-1} U^{n-1}) U^i U = U^n U,$$

$$U^{n-1} U^n U^n = (U^n U^{n-1})(U^i U) U^n = U^i U (U^{n-1} U^{n-1} U^{n-1}) = U^n U.$$ 

This along with the fact that $\bar{x}(U^n U^n) = 1$, for each $n = 1, ..., N$, implies that for an arbitrary $a \in A$ and $0 < n \leq N$ we have

$$a(x_{n-1}) = \xi^n_{x}(a) = \bar{x}(\delta^{n-1}_{x}(a)) = \bar{x}(U^n U^n) \bar{x}(\delta^{n-1}_{x}(a)) \bar{x}(U^n U^n) =$$

$$= \bar{x}(U^n U^n U^{n-1} a U^{n-1} U^n U^n) = \bar{x}(U^n U a U^n U^n) = \bar{x}(\delta^n_{x}(a)) =$$

$$= \frac{\xi^n_{x}}{\alpha} \delta(a(n)) = \delta(a(x_n)) = a(\alpha(x_n))$$

where in the two final equalities we used formulae (43) and (48). Since $\mathcal{A}$ separates points of $M(A)$ we obtain

$$\alpha(x_n) = x_{n-1}, \quad n \in \mathbb{N}, \quad (44)$$

Hence we have

$$x_n \in \Delta_n, \quad 0 \leq n \leq N, \quad (45)$$

and thus

$$\bar{x} \mapsto (x_0, x_1, ..., x_N, 0, ...) \in M_N.$$ 

2) Now let us assume that for an arbitrary $n \in \mathbb{N}$ we have

$$\xi^n_{x} \neq 0,$$

that is $\xi^n_{x} \in M(A)$. Thus, taking $x_n \in M(A)$ such that (43) is true we see that the mapping (38) takes the form $\bar{x} \mapsto (x_0, x_1, x_2, ...)$. By the same argument as presented in case 1) we obtain that

$$\alpha(x_n) = x_{n-1}, \quad n \geq 1, \quad (46)$$

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Hence in view of the definition of $\Delta_n$, $n \in \mathbb{Z}$ we have

$$x_n \in \Delta_n, \quad n \in \mathbb{N}.$$  

(47)

And therefore

$$\tilde{x} \mapsto (x_0, x_1, x_2, ...) \in M_\infty.$$  

We recall that the mapping $\delta$ is an injection and as we have already proved its right hand side belongs to $M_N$ or $M_\infty$ (depending on $\tilde{x}$). This means that $\delta$ defines an embedding $\delta$. Finally recall that the topology $M(E_s(A))$ is *-weak. That is why a fundamental system of neighborhoods of a point $\tilde{x} = (x_0, x_1, ...) \in M_\infty$ is the family of sets of the form

$$O(b_1, ..., b_k, \varepsilon) = \{ \bar{y} \in M(E_s(A)) : |b_i(\tilde{x}) - b_i(\bar{y})| < \varepsilon, \ i = 1, ..., k \}$$  

(48)

where $b_i \in E_s(A)$, $\varepsilon > 0$. Since $E_s(A) = \bigcup_{n=0}^{\infty} \delta_s^n(A)$ it is enough to take $b_i = \delta_s^n(a_i)$, $a_i \in A$, $i = 0, ..., k$, in (48) and then we have that

$$O(b_1, ..., b_k, \varepsilon) = \{ \bar{y} \in M(E_s(A)) : |\delta_s^n(a_i)(\tilde{x}) - \delta_s^n(a_i)(\bar{y})| < \varepsilon, \ i = 1, ..., k \} =$$

$$\{ \bar{y} = (y_0, y_1, ...) \in M(E_s(A)) : |a_i(x_n) - a_i(y_n)| < \varepsilon, \ i = 1, ..., k \}.$$  

If we set $O(a_1, ..., a_k, n, \varepsilon) = O(b_1, ..., b_k, \varepsilon) \cap (\bigcup_{n=0}^{\infty} M_N \cup M_\infty)$ then we obtain a fundamental system of neighborhoods from the thesis.

In the case of a point $\tilde{x} = (x_0, x_1, ..., x_N, 0, ...) \in M_N$ we set

$$O(a_1, ..., a_k, \varepsilon) = O(a_1, ..., a_k, N, \varepsilon) \cap M_N.$$  

Thus formulae (49) and (50) define a base of neighborhoods of a point $\tilde{x} \in M(E_s(A)) \hookrightarrow \bigcup_{N=0}^{\infty} M_N \cup M_\infty$ and the proof is complete.

**Remark 3.2** A very useful fact is that the set of operators of the form $b = a_0 + \delta_s(a_1) + ... + \delta_s^n(a_N)$, where $a_0, a_1, ..., a_N \in A$, is dense in $E_s(A)$ (see [1], Proposition 3.7.) and by definition (43) of functionals creating the sequence $\tilde{x} = (x_0, ...)$ we have that

$$b(\tilde{x}) = a_0(x_0) + a_1(x_1) + ... + a_N(x_N),$$  

when $\tilde{x} = (x_0, ...) \in \bigcup_{n=0}^{\infty} M_n \cup M_\infty$ and

$$b(\tilde{x}) = a_0(x_0) + a_1(x_1) + ... + a_n(x_k),$$  

when $\tilde{x} = (x_0, ..., x_k, 0, ...) \in \bigcup_{n=0}^{N} M_n$. 

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Theorem just proved can be considered as an "upper estimate" for \( M(E^*(A)) \). The next theorem presents the corresponding "lower estimate". Before passing to its statement let us stress that the previous theorem tells us that any element \( \tilde{x} \in M(E^*(A)) \) generates (defines uniquely) either an element of \( M_N \) or an element of \( M_\infty \), on the other hand given a sequence \((x_0, x_1, ..., x_N, 0, ...) \) \in M_N or \((x_0, x_1, ..., x_N, ...) \) \in M_\infty one cannot say in advance that it is generated by a certain element \( \tilde{x} \in M(E^*(A)) \) (this will have place only if this sequence is of the form \((38)\)). Theorem 3.3 tells that all the sequences from the subset \( \hat{M}_N \subset M_N \) (which in general can be essentially smaller than \( M_N \)) and all the sequences from \( M_\infty \) are really generated by certain elements \( \tilde{x} \in M(E^*(A)) \).

**Theorem 3.3** Let \( A \subset L(H) \) be a commutative \( C^* \)-algebra, \( 1 \in A \), and let \( U \in L(H) \) be a partial isometry such that \( U^*U \in \mathcal{A}' \). In view of theorem 3.1 we can treat \( M(E^*(A)) \) as a subset of \( \bigcup_{N \geq 0} M_N \cup M_\infty \) on the other hand the following inclusion holds

\[
\bigcup_{N=0}^\infty \hat{M}_N \cup M_\infty \subset M(E^*(A)) \tag{49}
\]

where \( \hat{M}_N \) are the sets of the form

\[
\hat{M}_N = \{ \tilde{x} = (x_0, x_1, ..., x_N, 0, ...) : x_N \in \Delta_N, x_N \notin \Delta_{-1}, \alpha(x_n) = x_{n-1} \}
\]

and \( M_\infty = \{ \tilde{x} = (x_0, x_1, ...) : x_n \in \Delta_n, \alpha(x_n) = x_{n-1}, n \geq 1 \} \).

**Proof.** Let us prove first that \( M_\infty \subset M(E^*(A)) \). Let \((x_0, x_1, ...)\) be an arbitrary sequence of elements of \( M(A) \) satisfying the condition \( \alpha(x_n) = x_{n-1} \) for each \( n \geq 1 \). We shall show that there exists a linear and multiplicative functional \( \tilde{x} \in M(E^*(A)) \) that generates this sequence (that is all the equations \((33)\), \( n = 0, 1, 2, ... \) where \( \xi^a_x \) are given by \((35)\) are satisfied).

Indeed, let us consider the following sets

\[
\tilde{X}^n = \{ \tilde{x} \in M(E^*(A)) : \forall a \in A \, \xi^a_x(a) = a(x_n) \}, \quad n = 0, 1...
\]

where \( \xi^a_x \) are given by \((35)\). We have that

1) \( \tilde{X}^n \neq \emptyset \). This is due to the fact that \((50)\) is the set of all extensions of a certain multiplicative functional from \( \delta^a_x(A) \) up to \( \overline{E^*(A)} \). It is known that for any multiplicative functional on a commutative \( C^* \)-subalgebra there exists an extension up to a multiplicative functional on a larger commutative \( C^* \)-algebra.

2) \( \tilde{X}^n \) is closed. This follows from the definition of weak* convergence.
3) \( \tilde{X}^0 \supset \tilde{X}^1 \supset \ldots \supset \tilde{X}^n \supset \ldots \). Indeed, if \( \tilde{x} \in \tilde{X}^n \) then

\[
\alpha(x_{n-1}) = \alpha(x_n) = \delta(a)(x_n) = \xi^0_x(\delta(a)) = \xi^{n-1}_x(a), \quad a \in A,
\]

that is \( \tilde{x} \in \tilde{X}^{n-1} \).

Hence the family of sets \( \tilde{X}^n \) forms a decreasing sequence of nonempty compact sets and thus \( \bigcap_{n=0}^{\infty} \tilde{X}^n \neq \emptyset \). From the definition of \( \tilde{X}^n \) it follows that any point \( \tilde{x} \in \bigcap_{n=0}^{\infty} \tilde{X}^n \) generates the same given sequence

\[
(\xi^0_x, \xi^1_x, \xi^2_x, \ldots) = (x_0, x_1, x_2, \ldots).
\]

But we have already observed that the mapping

\[
\tilde{x} \rightarrow (\xi^0_x, \xi^1_x, \xi^2_x, \ldots)
\]

is injective and therefore

\[
\bigcap_{n=0}^{\infty} \tilde{X}^n = \{ \tilde{x} \}
\]

consists of a single point.

Now, let \((x_0, x_1, \ldots, x_N, 0, \ldots) \in \tilde{M}_N\). Let us consider the sets \( \tilde{X}^n \) defined by (50), but only for \( n = 0, 1, \ldots, N \). The above argument shows that these sets are decreasing and nonempty. Let \( \tilde{x} \in \tilde{X}^N \). To identify \( \tilde{x} \) with the sequence \((\xi^0_x, \xi^1_x, \xi^2_x, \ldots) = (x_0, x_1, \ldots, x_N, 0, \ldots) \) given by (38) it is enough to show that

\[
\xi^{N+1}_x \equiv 0.
\]

Let us assume the opposite, that is \( \xi^{N+1}_x \neq 0 \). Then from the first part of the proof of theorem 3.1 we obtain that \( \tilde{x} = (x_0, x_1, \ldots, x_N, x_{N+1}, \ldots) \) where \( \alpha(x_{N+1}) = x_N \), which contradicts the fact that \( x_N \notin \Delta_{-1} = \alpha(\Delta_1) \). The proof is complete.

In general in the preceding theorems 3.1, 3.3 none of the inclusions (39) or (49) can be replaced by the equality. However, as the next result tells in the situation when \( A \) contains \( \delta_*(1) = U^*U \) we have the equality in formula (49) and therefore it gives the full description of the maximal ideal space of \( M(E_*(A)) \).

**Theorem 3.4** Let \( A \subset \mathcal{L}(H) \) be commutative \( C^* \)-algebra containing the identity. Let \( \delta(\cdot) = U(\cdot)U^* \) be an endomorphism of \( A \). Moreover, let

\[
U^*U \in A.
\]
Then the maximal ideal space \(\hat{M}(E_\ast (A))\) of the algebra \(E_\ast (A)\) is homeomorphic to the countable sum of disjoint closed and open sets \(\hat{M}_N\) and the closed set \(M_\infty\) (we admit empty sets)

\[
M(\hat{E}_\ast (A)) = \bigcup_{N=0}^\infty \hat{M}_N \cup M_\infty
\]  

(51)

where

\[
\hat{M}_N = \{ \bar{x} = (x_0, x_1, \ldots, x_N, 0, \ldots) : x_N \in \Delta_N, x_N \notin \Delta_{-1}, \alpha(x_n) = x_{n-1} \},
\]

\[
M_\infty = \{ \bar{x} = (x_0, x_1, \ldots) : x_n \in \Delta_n \cap \Delta_{-\infty}, \alpha(x_n) = x_{n-1}, n \geq 1 \}.
\]

The topology on \( \bigcup_{N=0}^\infty \hat{M}_N \cup M_\infty \) is defined by a fundamental system of neighborhoods of points \( \bar{x} \in \hat{M}_N \)

\[
O(a_1, \ldots, a_k, \varepsilon) = \{ \bar{y} \in \hat{M}_N : |a_i(x_N) - a_i(y_N)| < \varepsilon, i = 1, \ldots, k \}
\]

(52)

and respectively \( \bar{x} \in M_\infty \)

\[
O(a_1, \ldots, a_k, n, \varepsilon) = \{ \bar{y} \in \bigcup_{N=n}^\infty \hat{M}_N \cup M_\infty : |a_i(x_n) - a_i(y_n)| < \varepsilon, i = 1, \ldots, k \}
\]

where \( \varepsilon > 0, a_i \in A \) and \( k, n \in \mathbb{N} \cup \{0\} \).

**Proof.** In view of theorems 3.1 and 3.3 to prove the equality (51) it is enough to prove that for an arbitrary \( \bar{x} \in M(\hat{E}_\ast (A)) \) we have

\[
\bar{x} \in M_N \implies \bar{x} \in \hat{M}_N.
\]

Suppose on the contrary that \( \bar{x} \in M_N \) and \( \bar{x} \notin \hat{M}_N \). Then \( \bar{x} = (x_0, x_1, \ldots, x_N, 0, \ldots) \), \( x_n \in M(A) \) and there exists \( x_{N+1} \in \Delta_1 \subset M(A) \) such that \( \alpha(x_{N+1}) = x_N \). By definition (33) of functionals \( x_n \) and definition (35) of functionals \( \xi_{x_n} \) we have that \( \bar{x}(U^{n}aU^{n}) = a(x_n) \), for each \( a \in A \) and \( n = 0, \ldots, N \), and \( \bar{x}(U^{n}aU^{n}) = 0 \) when \( n > N \). In particular we have that

\[
\bar{x}(U^{n}U^{N}) = 1 \quad \text{and} \quad \bar{x}(U^{N+1}U^{N+1}) = 0.
\]

By formula (18) we obtain that

\[
\bar{x}(U^{n}aU^{n}) = a(x_N) = a(\alpha(x_{N+1})) = \delta(a)(x_{N+1}), \quad a \in A.
\]

Setting \( a = U^*U \in A \) in the above formula we have that \( \delta(U^*U)(x_{N+1}) = 0 \). On the other hand for \( \delta(U^*U)(x_{N+1}) \) we have

\[
\delta(U^*U)(x_{N+1}) = x_{N+1}(UU^*UU^*) = x_{N+1}(UU^*) \cdot x_{N+1}(UU^*) = 1.
\]

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Thus we arrived at a contradiction and the proof of (51) is complete.

The closedness and openness of the sets $\widehat{M}_N$ follows from the relation

$$\bar{x} \in M_N \iff \bar{x} \in \widehat{M}_N \iff \{\bar{x} (U^* U^N) = 1 \text{ and } \bar{x} (U^* U^{N+1}) = 0 \}.$$ 

Further the openness of the sets $\widehat{M}_N$ implies the closedness of $M_\infty$ which finishes the proof of the theorem.

**Remark 3.5** In fact the theorem just proved gives us the key to obtain the complete description of $M(E_*(A))$ in the general situation. Indeed, if $U^* U \notin A$ then one can consider the $C^*$-algebra $A_1 = \langle A, U^* U \rangle$ generated by $A$ and $U^* U$. Since

$$\delta(U^* U) = UU^* UU^* = \delta(1)\delta(1) \in \mathcal{A}$$

we have that $\delta : A_1 \to A$. By applying the foregoing theorem to the algebra $A_1$ and the operator $U$ one obtains the full description of $M(E_*(A_1))$. However, as

$$\overline{E_*(A_1)} = E_*(A)$$

we have

$$M(\overline{E_*(A_1)}) = M(\overline{E_*(A)}).$$

The preceding results often can be improved (simplified) when we know some specific features of a partial isometry $U$. For instance, if $U$ is an isometry, i.e. $U^* U = 1$, then by theorem 2.4 the mapping $\alpha$ generated by the endomorphism $\delta$ is surjective and hence all the sets $\widehat{M}_N$ are empty. Thus by theorem 3.4 we obtain the following

**Corollary 3.6** Let $A \subset L(H)$ be a commutative $C^*$-algebra containing the identity and $U$ be an isometry such that $\delta(\cdot) = U(\cdot) U^*$ is an endomorphism of $A$. Then the mapping $\alpha : \Delta \to M(A)$ defined by (13) is a surjection and the spectrum $M(\overline{E_*(A)})$ of $E_*(A)$ has the form

$$M(\overline{E_*(A)}) = \{\bar{x} = (x_0, x_1, \ldots) : x_n \in \Delta_n, \alpha(x_{n+1}) = x_n, n \geq 0\} \quad (53)$$

A fundamental system of neighborhoods of a point $\bar{x}$ is family of sets

$$O(a_1, \ldots, a_k, n, \varepsilon) = \{\bar{y} \in M(\overline{E_*(A)}) : |a_i(x_n) - a_i(y_n)| < \varepsilon, \ i = 1, \ldots, k\}$$

where $n \in \mathbb{N} \cup \{0\}$, $\varepsilon > 0$ and $a_i \in \mathcal{A}$, $1 \leq i \leq k$. 

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