Study of the influence of stochastic inhomogeneity of a pipe on stress-strain behavior

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Abstract. The paper is devoted to stress-strain behavior of an elastic heavy-wall pipe from incompressible material, which is under the influence of compression hydrostatic pressure. We have obtained the analyticity condition for the solution of the problem in small parameters close to zero. These parameters are independent random variables. They characterize the deviation of the cross-section outline from a circle, and the inhomogeneity of the pipe material. The solution has been found by perturbation method up to the second order of smallness. It has been concluded that in the case of experimental analysis of the pipe state under the load meeting the specified condition, the average values of stress and displacement will be close to the values obtained at axially symmetric state.

1. Introduction
The problems of deformable solid mechanics normally lead to boundary problems for partial equations [1, 2]. Hence, the consideration of various random factors impacting the deformation processes requires the use of random field theory apparatus (random functions for several variables). As a rule, the deterministic computation methods, which are normally used, provide for the first approximation, which is not enough for a number of cases. For instance, inaccuracy in strength calculation is offset by factors of safety, which are not always optimum. It results either in underutilized strength or in premature failure of the structures. There are a number of factors which are neglected and cannot be explained within the framework of deterministic methods.

The first publications dedicated to the use of statistical methods in deformable solid mechanics are dated by 1920s-1930s. M. Mayer proposed an idea based on statistical approach to factors of safety. N.S. Streletsky [3], A.R. Rzhyanitsyn [4] and V.V. Bolotin [5] made a substantial contribution to the development of these methods.

At the beginning only the probability methods based on random distribution characteristics were used. Further development and application of statistical methods in solid mechanics is associated with the use of the theory of stationary stochastic processes while calculating elastic systems oscillation under...
the influence of random forces (V.V. Bolotin [5], V.V. Yekimov [6] et al.). The works of V.V. Bolotin [5] are dedicated to numerous problems associated with structural reliability and the use of statistical methods in deformable solid mechanics.

A number of studies are dedicated to the exploration of microheterogeneous media (polycrystals and various composites) and determination of their effective elastic moduli and other characteristics. The earliest publications on this topic belong to I.M. Livshits and L.N. Rozentsveyg [7]. In 1960-s there appeared the first works in which statistical problems of mechanics were solved with the help of random vector and tensor field theory.

Among them we need to mention the works by V.A. Lomakin [7, 8], who considers the problems formulated for deformable media with random inhomogeneities, such as quasihomogeneous media with microheterogeneous structure (polycrystals, glass-fibre reinforced plastic etc.). The influence of random disturbances of materials’ mechanical characteristics on deformation and stress fields is considered in many works [7, 10-12].

The inhomogeneity of real material and the difference between the actual contours and the canonical shape influence the solids' deformation process. The existing methods of solving such problems are often based on numerical procedures and statistical testing method [11-14].

Analytical methods of solving stochastic boundary problems for structurally heterogeneous materials are well developed for linear elastic media [8]. The development of analytical methods for solving stochastic problems is associated with a number of obstacles, among which we should mention physical and statistical nonlinearity of the defining equations. One of the common methods used for analytical solution of stochastic boundary problems is perturbation method [15]. The use of small parameter method in the study of stress-strain behavior of elastoplastic bodies is described in the monograph by D.D. Ivlev and L.V. Yershov [16]. The essence of this method is that by expanding stress and strain tensor components into a series in small parameter, a statistically nonlinear problem can be represented as a sequence of statistically linear problems. The problem is that this approach is associated with computation difficulties, and for this reason the solutions of concrete stochastic problems are normally limited to the first approximation. This is justified for weakly inhomogeneous media [8-10].

In [17] you can find an analytical solution of a stochastic nonlinear boundary problem of steady-state creep for a heavy-wall pipe under the influence of internal pressure. Stochastic properties of the pipe are described by a random function of a single variable (r radius). The problem is generalized for a heavy-wall pipe where the properties of the material are described by a random function of two arguments. The boundary effects are studied through the solution of a stochastic boundary problem represented as a sum of two series on the basis of small parameter method first approximation.

The ways of solving stochastic boundary problems with higher order of expansion terms for multidimensional problems, as well as the problem of solution convergence, are still underexplored.

In [18] the author describes the conditions under which (for several spaces of solutions and given data [19-21]) the solution will be analytical in small parameters close to zero.

Heavy-wall steel pipes are one of the most widely spread types of pipes used in various fields of industry and construction. High strength and reliability of the pipes allow for the reduction of the structure weight, which is one of the priorities in aircraft building. Besides, high reliability of the pipelines is ensured. In shipbuilding the fact that the pipes are affected by highly aggressive seawater and marine air should be taken into consideration. Only the pipes whose walls are thick enough can cope with such a stress. Heavy-wall pipes are not only able to resist the high pressure of the medium carried, but also endure the contact with corrosive substances while staying leak-proof and durable. For this reason they are irreplaceable in chemical sector and laboratories. In this paper we offer a solution to the problem of a stochastically inhomogeneous elastic heavy-wall pipe based on the second approximation of small parameter method. Besides, a restriction of elastic strain smallness is introduced, which is considered to be negligible.
2. Problem description and the method of solution

Based on the results described in [16, 22], in [23] there are presented various mathematical models of boundary conditions in terms of stresses. The analysis shows that while studying continuous dependence of the solution on the given data (functions, parameters of the mathematical model), the statistical boundary conditions should be formulated on the boundary of a real body in strained state. In [18] special cases of differential operators used in the solutions of solids’ quasistatic deformation problems are described. The conditions for continuous dependence of the solution on the given data have been obtained for several Banach spaces with certain norms, which are normally used for solving the problems of continuum mechanics (Hölder spaces, Hilbert space, $C^2([a,b], R^m)$, $C^4([a,b], R^m)$ etc.), where Fréchet derivative of mapping is an isomorphism [19, 21].

Let us consider a heavy-wall pipe from elastic incompressible material under the influence of internal and external pressures. Its state is described by the solution of the following problem:

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau}{\partial \theta} + \frac{\partial \sigma_r}{\partial \theta} = 0
\]

\[
\frac{\partial \tau}{\partial r} + \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau}{r} = 0
\]

\[
r \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} + u = 0
\]

\[
\sigma_r - \sigma_\theta = 4G \frac{\partial u}{\partial \theta}
\]

\[
\tau = G \left( \frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial r} - \frac{v}{r} \right)
\]

\[
\sigma_r \big|_{r=\Psi_1(0)} = -p_1; \quad \sigma_\theta \big|_{r=\Psi_1(0)} = -p_2; \quad \tau \big|_{r=\Psi_1(0)} = \tau \big|_{r=\Psi_2(0)} = 0,
\]  

where $G = G_0(1 + \varepsilon_1 f_1(0))$, $\Psi_1(\theta)$, $\Psi_2(\theta)$ describes the cross-section shape of the pipe in strained state. While nonloaded, the contours are characterized by the following functions up to small parameters: $r = a + \varepsilon_2 f_2(\theta)$ and $r = b + \varepsilon_2 f_2(\theta)$. Here $\varepsilon_1$ and $\varepsilon_2$ are independent random variables ($|\varepsilon_i| << 1$)

Let us use the perturbation method to solve problems (1)-(4), then

\[
\sigma_p = \sum_{i,j=0}^{\infty} \sigma^0_{ij} \varepsilon_i \varepsilon_j; \quad \sigma_0 = \sum_{i,j=0}^{\infty} \sigma^0_{ij} \varepsilon_i \varepsilon_j; \quad \nu = \sum_{i,j=0}^{\infty} \nu^0 \varepsilon_i \varepsilon_j,
\]  

where zero approximation is represented by the solution of the problem (1)-(4) characterized by axially symmetric state

\[
\sigma_p^0 = \frac{2A}{\rho^2} + B; \quad \sigma_0^0 = -\frac{2A}{\rho^2} + B; \quad \tau^0 = \nu^0 = 0; \quad u^0 = -\frac{A}{\rho}
\]  

where $\rho = \frac{r}{b}$, components of displacement vector are also considered to be related to outer radius of section $b$, components of stress tensor — to shear modulus $G_0$. 


The series (5) will be convergent if the conditions described by the analyticity criterion in small parameters close to $\varepsilon = 0$ ($i = 1, 2$) [18, 20] are satisfied. It is evident that the requirement for analyticity of the problem definition in small parameters close to zero is met for each of the arguments, and we only have to analyze continuous dependence of the problem solution (1)-(4) on the given data where $\varepsilon = 0$ ($i = 1, 2$).

3. Continuous dependence of the solution describing stress-strain behavior of the band on the given data

As we know from the implicit function theorem, to analyze this continuous dependence we need to formulate the following problem for auxiliary functions $\zeta_i$ ($q_i = p_i / G_p$):

$$
\frac{\partial (\sigma^0 + \zeta_1)}{\partial \rho} + \frac{1}{\rho} \frac{\partial (\tau^0 + \zeta_3)}{\partial \theta} + \sigma^0 + \zeta_1 - \sigma^0 - \zeta_2 = 0
$$

$$
\frac{\partial (\tau^0 + \zeta_3)}{\partial \rho} + \frac{1}{\rho} \frac{\partial (\sigma^0 + \zeta_2)}{\partial \theta} + 2\frac{\partial (\tau^0 + \zeta_3)}{\partial \rho} = 0
$$

$$
\frac{\partial (u^0 + \zeta_1)}{\partial \rho} + \frac{\partial (v^0 + \zeta_2)}{\partial \rho} + (u^0 + \zeta_4) = 0
$$

$$
\sigma^0 + \zeta_1 - \sigma^0 - \zeta_2 = 4 \frac{\partial (u^0 + \zeta_4)}{\partial \rho}
$$

$$
\tau^0 + \zeta_3 = \frac{\partial (v^0 + \zeta_5)}{\partial \rho} + \frac{1}{\rho} \frac{\partial (u^0 + \zeta_4)}{\partial \rho} - \frac{(v^0 + \zeta_5)}{\rho}
$$

$$
\left(\sigma^0_n + \zeta_n\right)_{\rho = 0, \Phi_1(\theta)} = -q_1;
\left(\tau^0_n + \zeta_n\right)_{\rho = 0, \Phi_1(\theta)} = 0,
\left(\sigma^0_n + \zeta_n\right)_{\rho = 0, \Phi_2(\theta)} = -q_2;
\left(\tau^0_n + \zeta_n\right)_{\rho = 0, \Phi_2(\theta)} = 0.
$$

For example, as it appears from [18], function $\Psi_1(\theta)$ in (4), up to the terms of the first order of smallness, is written as

$$
\Psi_1(\theta) = \alpha - \frac{A}{\alpha} + \left(1 + \frac{A}{\alpha^2}\right)f_1(\theta) + u^{00}(\theta, \alpha) + u^{01}(\theta, \alpha)
$$

Accordingly, function $\Phi_1(\theta)$ in (8), up to the terms of the first order of smallness, will be written as

$$
\Phi_1(\theta) = \alpha - \frac{A}{\alpha} + \zeta_4(\theta, \alpha)
\Phi_2(\theta) = 1 - A + \zeta_4(0, \alpha)
$$

The problem linearized to $\zeta_i$, which corresponds to the problem (7)–(10), given that (6) is the solution to the problem (1)-(4) where $f_i \equiv 0$, will be written as:
\[
\frac{\partial \zeta_1}{\partial \rho} + \frac{1}{\rho} \frac{\partial \zeta_3}{\partial \theta} + \frac{\zeta_1}{\rho} - \zeta_2 = 0
\]
\[
\frac{\partial \zeta_3}{\partial \rho} + \frac{1}{\rho} \frac{\partial \zeta_2}{\partial \theta} + \frac{2\zeta_3}{\rho} = 0
\]
\[
\rho \frac{\partial \zeta_4}{\partial \rho} + \frac{\partial \zeta_5}{\partial \theta} + \zeta_4 = 0
\]
\[
\zeta_1 - \zeta_2 = 4 \frac{\partial \zeta_4}{\partial \rho}
\]
\[
\zeta_3 = \frac{\partial \zeta_5}{\partial \rho} + \frac{1}{\rho} \frac{\partial \zeta_4}{\partial \theta} - \zeta_3
\]

The boundary conditions (8) can be represented as

\[
\begin{align*}
\left( \zeta_1 + \frac{d\sigma^0}{d\rho} \zeta_4 (\theta, \alpha) \right)_{\rho = \alpha - A} & = \left[ \zeta_3 + \frac{(\sigma^0 - \sigma_0^0) \partial \zeta_4}{\rho} (\theta, \alpha) \right]_{\rho = \alpha - A} = 0 \\
\left( \zeta_1 + \frac{d\sigma^0}{d\rho} \zeta_4 (\theta, 1) \right)_{\rho = 1 - A} & = \left[ \zeta_3 + \frac{(\sigma^0 - \sigma_0^0) \partial \zeta_4}{\rho} (\theta, 1) \right]_{\rho = 1 - A} = 0
\end{align*}
\]

The simultaneous equations (11) can be solved in the following way [6, 7]:

\[
\begin{align*}
\zeta_1 & = 4(c_1 \rho + c_2 \rho^{-3}) \cos \theta \\
\zeta_2 & = 4(3c_1 \rho - c_2 \rho^{-3}) \cos \theta \\
\zeta_3 & = 4(c_1 \rho + c_2 \rho^{-3}) \sin \theta \\
\zeta_4 & = -(c_1 \rho^2 + c_2 \rho^{-2}) \cos \theta \\
\zeta_5 & = -(3c_1 \rho^2 - c_2 \rho^{-2}) \sin \theta
\end{align*}
\]

As it appears from (6), (13), the first two conditions in (12) are analogous, and the other two conditions also analogous. Hence, to derive the arbitrary constants \( c_1 \) and \( c_2 \) from (12), (13) we need the following simultaneous equations:

\[
\left( \alpha - A \alpha^{-1} \right)^4 + \alpha^2 A \left[ c_1 + \left( 1 + \alpha^{-3} A \right) c_2 \right] = 0,
\]
\[
\left( 1 - A \right)^4 + A \left[ c_1 + \left( 1 + A \right) c_2 \right] = 0
\]

As it appears from (13), (14), the problem (10), (11) has a nontrivial solution when the following condition is met:

\[
\Delta = \left[ \left( \alpha - A \alpha^{-1} \right)^4 + \alpha^2 A \right] \left( 1 + A \right) - \left( 1 + \alpha^{-3} A \right) \left[ \left( 1 - A \right)^4 + A \right] = 0
\]

Let us denote by \( q_- \) the greatest negative root, and by \( q_+ \) the smallest positive root of the simultaneous equations (14) and the following equation:

\[
q = 2A \left[ \frac{1}{1 - A} \right]^2 - \left( \frac{\alpha}{\alpha^2 - A} \right)^2
\]
Then the condition for a nontrivial solution of the problem (11), (12) can be represented as follows:

\[
q_1 - q_2 = q_*
\]
\[
q_1 - q_2 = q_{**}
\]

We have deduced that if the compression force parameters belong to the domain bounded by (16), then the solution will be analytical functions close to \( \varepsilon_i = 0 \) \( (i = 1,2) \) In figure 1 the pressure parameter plain shows the analyticity region as a band for a special case of a cross-section.

\[
\begin{align*}
q_2 & \quad q_* \\
-q_* & \quad q_{**}
\end{align*}
\]

\[ q_{**} \]

Figure 1. Analyticity region where \( \alpha = 0.8 \); \( q_* \approx -0.502 \); \( q_{**} \approx 0.034 \).

4. Finding the approximation components

To find the expansion components (5), we have formulated problems in which equilibrium equations, incompressibility condition are analogous to (1) and (2), and the rheological relationships and boundary conditions can be represented in the following way:

For components with index “10”

\[
\begin{align*}
\sigma^0_{10} - \sigma^0_0 &= 4G_0 \frac{\partial u^0_{10}}{\partial \rho}; \quad \tau^0_{10} = G_0 \left( \frac{\partial v^0_{10}}{\partial \rho} - \frac{v^0_{10}}{\rho} + \frac{1}{\rho} \frac{\partial u^0_{10}}{\partial \theta} \right) \\
\sigma^0_{10} + \frac{d\sigma^0_{10}}{d\rho} \left[ \left( 1 - u^0(1) \right) f_2(\theta) + u^0(\theta,1) \right] &= 0; \\
\tau^0_{10} + \frac{\sigma^0_{10} - \sigma^0_0}{\rho} \left[ \left( 1 - u^0(1) \right) \frac{df_2}{d\theta} + \frac{\partial u^0(\theta,1)}{\partial \theta} \right] &= 0
\end{align*}
\]

for \( \rho = 1 + u^0(1) \)

\[
\begin{align*}
\sigma^0_{10} + \frac{d\sigma^0_{10}}{d\rho} \left[ \left( \alpha - u^0(\alpha) \right) f_3(\theta) + u^0(\alpha,\theta,\alpha) \right] &= 0; \\
\tau^0_{10} + \frac{\sigma^0_{10} - \sigma^0_0}{\rho} \left[ \left( 1 - u^0(\alpha) \right) \frac{df_3}{d\theta} + \frac{\partial u^0(\alpha,\theta,\alpha)}{\partial \theta} \right] &= 0
\end{align*}
\]

for \( \rho = \alpha + u^0(\alpha) \) \( (\alpha = a/b) \)

where \( u^0(1) \) and \( u^0(\alpha,\theta,\alpha) \) denote the components of the jump at the boundary in the direction of the coordinate. \( f_i(\theta) \) are functions of the overall normal. The coefficients \( G_0 \) are determined by the material properties.
For components with index “01”

\[ \sigma^0_{\rho} - \sigma^0_{\theta} = 4 \left( \frac{\partial \sigma^0_{\theta}}{\partial \rho} + f_2(\theta) \frac{\partial u^0_{\theta}}{\partial \rho} \right); \]

\[ \tau^0_{\rho} = \frac{\partial \sigma^0_{\rho}}{\partial \rho} - \frac{\partial \sigma^0_{\theta}}{\partial \theta} + \frac{1}{\rho} \frac{\partial u^0_{\theta}}{\partial \theta} + f_2(\theta) \tau^0 \]  \hspace{2cm} (19)

for \( \rho = g_0 = \rho_0 + u^0_1(\rho_0) \) \( (\rho_0 = 1 \text{ и } \rho_0 = \alpha) \)

\[ \sigma^0_{\rho} + \frac{d \sigma^0_{\rho}}{d \rho} u^0_1(\theta, \rho_0) = 0; \]

\[ \tau^0_{\rho} + \frac{\sigma^0_{\rho} - \sigma^0_{\theta}}{\rho} \frac{\partial u^0_{\theta}(\theta, \rho_0)}{\partial \theta} = 0 \]  \hspace{2cm} (20)

The form of the boundary conditions (18) with neglect of \( u^{10} \) corresponds to those shown in [16], while the form of the condition (20) is analogous to those obtained in [22] up to the designations.

For components with index “20” the rheological relationships are analogous to those in (17), while the boundary conditions for \( \rho = g_0 = \rho_0 + u^0_1(\rho_0) \) \( (\rho_0 = 1 \text{ и } \rho_0 = \alpha) \) can be represented as:

\[ \sigma^{20}_{\rho} + \frac{d \sigma^{20}_{\rho}}{d \rho} g^{10} + \frac{\partial \sigma^{20}_{\rho}}{\partial \rho} g^{10} + \frac{1}{2} \frac{d \sigma^{20}_{\rho}}{d \rho} \left( g^{10} \right)^2 + \left( \sigma^{20}_{\rho} - \sigma^{20}_{\theta} \right) \left( \frac{\dot{g}^{10}}{g^{10}} \right)^2 - 2 \tau^{20} \frac{\dot{g}^{10}}{g^{10}} = 0; \]

\[ \tau^{20} + \left( \frac{\partial \left( \sigma^{20}_{\rho} - \sigma^{20}_{\theta} \right)}{\partial \rho} \frac{\dot{g}^{10}}{g^{10}} + \frac{\partial \tau^{10}}{\partial \rho} \right) g^{10} + \left( \sigma^{20}_{\rho} - \sigma^{20}_{\theta} \right) \left( \frac{\dot{g}^{10}}{g^{10}} \right)^2 - \left( \sigma^{10}_{\rho} - \sigma^{10}_{\theta} \right) \frac{\ddot{g}^{10}}{g^{10}} = 0 \]  \hspace{2cm} (21)

Here the point on top signifies differentiation by \( \theta \).

For approximation “02” physical relationships can be represented as (19), where zero approximation should be replaced by components with index “01”, and indices “01” have to be replaced by “02” respectively. In this case the boundary conditions will be analogous to (21).

For components with index “11” we get:

\[ \sigma^{11}_{\rho} - \sigma^{11}_{\theta} = 4 \left( \frac{\partial \sigma^{11}_{\theta}}{\partial \rho} + f_2(\theta) \frac{\partial u^{11}_{\theta}}{\partial \rho} \right); \]

\[ \tau^{11}_{\rho} = \frac{\partial \sigma^{11}_{\rho}}{\partial \rho} - \frac{\partial \sigma^{11}_{\theta}}{\partial \theta} + \frac{1}{\rho} \frac{\partial u^{11}_{\theta}}{\partial \theta} + f_2(\theta) \tau^{10} \]

for \( \rho = g_0 = \rho_0 + u^0_1(\rho_0) \) \( (\rho_0 = 1 \text{ и } \rho_0 = \alpha) \)

\[ \sigma^{11}_{\rho} + \frac{d \sigma^{11}_{\rho}}{d \rho} g^{11} + \frac{d^2 \sigma^{11}_{\rho}}{d \rho^2} g^{10} + \frac{\partial \sigma^{11}_{\rho}}{\partial \rho} g^{10} + \frac{\partial \sigma^{11}_{\theta}}{\partial \rho} g^{10} = 0; \]

\[ \tau^{11} + \frac{\partial \tau^{11}}{\partial \rho} g^{10} + \frac{\partial \tau^{10}}{\partial \rho} g^{10} = 0 \]

The expansion components of the function describing the section outline of a strained pipe are the following (given that \( v^0 = 0, \dot{g}_0 = 0 \)):
We have found a solution

\[ g^{i0}(\theta, \rho_0) = u^{i0} + \left( \frac{du^0}{d\rho} + 1 \right) f_z(\theta); \quad g^{0i}(\theta, \rho_0) = u^{0i}; \]

\[ g^{20}(\theta, \rho_0) = u^{20} + \frac{d^2 u^0}{d\rho^2} \left( f_z(\theta) \right)^2 + f_z(\theta) \frac{d u^0}{d\rho} + \frac{1}{2 g_0} \left( (v^{10})^2 - (u^{10})^2 - 2 u^{10} \left( \frac{d u^0}{d\rho} + 1 \right) f_z(\theta) \right) \]

\[ g^{02}(\theta, \rho_0) = u^{02} + \frac{(v^{01})^2 - 2 v^{01} g^{01}}{2 g_0} \]

\[ g^{11}(\theta, \rho_0) = u^{11} + \frac{v^{01} v^{10} - v^{01} g^{10} - v^{10} g^{01}}{g_0} \]

It is problematic to find an analytical solution for rheological relationships and boundary conditions. For this reason we will confine ourselves to the widely spread case where \( A/\rho_0 \ll 1 \).

For the functions describing the deviations of the properties of the pipe material from homogeneity, and the deviations of the section outline from a circle \( f_z(\theta) = \cos \theta \quad (i = 1, 2) \) we have found a solution up to the second order of smallness.

\[ \sigma_\rho = \sigma_\rho^{10} + \varepsilon_1 \sigma_\rho^{10} + \varepsilon_2 \sigma_\rho^{01} + \varepsilon_1^2 \sigma_\rho^{20} + \varepsilon_2^2 \sigma_\rho^{02} + \varepsilon_1 \varepsilon_2 \sigma_\rho^{11}; \]

\[ \sigma_\rho^{10} = 4 A \rho^{-3} \cos \theta; \quad \sigma_\rho^{01} = -4 A \rho^{-3} \sin \theta; \quad \sigma_\rho^{20} = 4 \left( C_1^{01} \rho + C_2^{01} \rho^{-3} \right) \cos \theta; \quad \sigma_\rho^{02} = 4 \left( C_1^{01} \rho + C_2^{01} \rho^{-3} \right) \sin \theta; \]

\[ \sigma_\rho^{11} = (\alpha^2 + \alpha + 1) \alpha C_1^{01} = -\frac{4 A (\alpha^2 + \alpha + 1) \alpha}{3 (\alpha^2 + 1)(\alpha + 1)} \]

\[ \sigma_\rho^{20} = 4 \left( C_1^{20} \rho^{-2} - C_2^{20} + 3 C_3^{20} \rho^{-4} \right) \cos 2\theta - \frac{3 A}{4} \rho^{-2}; \quad \sigma_\rho^{20} = 4 \left( C_2^{20} + 6 C_2^{20} \rho^{-2} - 3 C_3^{20} \rho^{-4} \right) \cos 2\theta + \frac{3 A}{4} \rho^{-2}; \]

\[ \tau^{20} = 4 \left( C_1^{20} \rho^{-2} + C_2^{20} + 3 C_3^{20} \rho^{-4} + 3 C_4^{20} \rho^{-2} \right) \sin 2\theta; \]

\[ u^{20} = -2 \left( C_1^{20} \rho^{-1} + C_2^{20} + C_3^{20} \rho^{-3} + C_4^{20} \rho^{-3} \right) \cos 2\theta + \frac{3 A}{4} \rho^{-1}; \quad v^{20} = (2 C_2^{20} \rho - 2 C_3^{20} \rho^{-3} + 3 C_4^{20} \rho^{-3}) \sin 2\theta; \]

\[ \sigma_\rho^{02} = (-4 C_1^{02} + 8 C_2^{02} \rho^{-2} + 12 C_2^{02} \rho^{-4} - \frac{4 A}{3} C_1^{01} \rho + \frac{4 A}{3} C_2^{01} \rho^{-3} + \frac{4 A}{3} \rho^{-2} \ln \rho + \frac{5 A}{3} \rho^{-2}) \cos 2\theta + \]

\[ + 4 C_1^{02} \rho + \frac{4 A}{3} C_2^{01} \rho^{-3} + \frac{4 A}{3} \rho^{-2}; \]

\[ \sigma_\rho^{02} = (4 C_1^{02} + 24 C_2^{02} \rho^2 - 12 C_2^{02} \rho^{-2} + \frac{4 A}{3} C_1^{01} \rho - \frac{4 A}{3} C_2^{01} \rho^{-3} + \frac{4 A}{3} \rho^{-2}) \cos 2\theta + 8 C_1^{01} \rho - \]

\[ - \frac{8}{3} C_2^{01} \rho^{-3} - \frac{4 A}{3} \rho^{-2}; \]
\[ t^{02} = (4C_{1}^{02} + 4C_{2}^{02} \rho^{-2} + 12C_{3}^{02} \rho^{-4} + 12C_{4}^{02} \rho^{-4} + \frac{16}{15} C_{1}^{01} \rho + \frac{16}{15} C_{2}^{01} \rho^{-3} + \frac{2}{3} \rho^{-2} (\ln \rho + 1)) \sin 2\theta; \]
\[ u^{02} = -2(C_{1}^{02} \rho + C_{2}^{02} \rho^{-1} + C_{3}^{02} \rho^{-3} + C_{4}^{02} \rho^{-3} - \frac{2}{15} \rho^{2} C_{1}^{01} \rho^{-2} - \frac{2}{15} C_{2}^{01} \rho^{-2} + \frac{2}{3} \rho^{-4} \ln \rho) \cos 2\theta; \]
\[ v^{02} = (2C_{1}^{02} \rho + 4C_{3}^{02} \rho^3 - 2C_{4}^{02} \rho^{-3} - \frac{2}{3} C_{1}^{01} \rho^2 + \frac{2}{15} C_{2}^{01} \rho^2 + \frac{2}{3} \rho^{-4}) \sin 2\theta; \]
\[ \sigma_{\rho}^{11} = (-4C_{1}^{11} + 8C_{3}^{11} \rho^{-2} + 12C_{4}^{11} \rho^{-4} + \frac{2}{3} \Delta \rho^{-3}) \cos 2\theta + \frac{2}{3} \Delta \rho^{-3}; \]
\[ \sigma_{\rho}^{00} = (4C_{1}^{11} + 24C_{3}^{11} \rho^2 - 12C_{4}^{11} \rho^{-4} - \frac{8}{15} \Delta \rho^{-3}) \cos 2\theta - \frac{8}{15} \Delta \rho^{-3}; \]
\[ \tau^{11} = (4C_{1}^{11} + 4C_{2}^{11} \rho^{-2} + 12C_{3}^{11} \rho^2 + 12C_{4}^{11} \rho^{-4} + \frac{16}{15} \Delta \rho^{-3}) \sin 2\theta; \]
\[ u^{11} = -2(C_{1}^{11} \rho + C_{2}^{11} \rho^{-1} + C_{3}^{11} \rho^{-3} + C_{4}^{11} \rho^{-3} - \frac{2}{15} \Delta \rho^{-2} \cos 2\theta; \]
\[ v^{11} = (2C_{1}^{11} \rho + 4C_{3}^{11} \rho^3 - 2C_{4}^{11} \rho^{-3} + \frac{2}{3} \Delta \rho^{-2} \sin 2\theta; \]

The expressions for the constants \( C_{k}^{ij} \) for the second approximation are too long to be given in this article. The representation of the approximation components “10” and “20” is analogous to the one shown in [16].

The random variables \( \varepsilon_{1} \) and \( \varepsilon_{2} \) are independent, so

\[
\langle \sigma_{\rho} \rangle = \sigma_{\rho}^{00} + \langle \varepsilon_{1} \rangle \sigma_{\rho}^{02} + \langle \varepsilon_{2} \rangle \sigma_{\rho}^{02};
\]

\[
\langle v \rangle = v^{00} + \langle \varepsilon_{1} \rangle v^{02} + \langle \varepsilon_{2} \rangle v^{02};
\]

(23)

5. Conclusion

We have obtained a restriction on force parameters for a pipe with cross-section close to a circle and modulus of elasticity close to a constant. If the force values stay within the domain obtained, then the solution of the problem (1)-(4) will be analytical in small parameters. Here the parameters are independent. To evaluate the possible error of the solution we can use one of the Taylor series evaluations.

We have also obtained a solution (22) describing the state of the pipe up to the second order of smallness for a special case of cross-section outline deviation, and material inhomogeneity. It follows from (23) that in the case of experimental analysis the average values of stress and displacement will be different from the values obtained at axially symmetric state (6) by the value of the second order of smallness \( \varepsilon_{1} \) and \( \varepsilon_{2} \) on condition that external impact parameters belong to the band (16).

References

[1] Pflüger A 1950 Stabilitätsprobleme der Elastostatik (Springer-Verlag)
[2] Green A E and Zerna W 1968 Theoretical Elasticity (Oxford, London: Univ. Press) p 457
[3] Streletsky N S 1947 The Basic of Statistical Accounting for the Factors of Structure Safety (Stroyizdat Publ) [in Russian]
[4] Rzhantysyn A R 1954 Structural Analysis with Regard to Materials’ Plastic Properties. (Stroyizdat Publ) [in Russian]
[5] Bolotin V V 1965 Statistical Methods in Structural Mechanics (M.: Publishing House for Construction Industry) p 208 [in Russian]
[6] Yekimov V V 1966 Probability Methods in Naval Structural Mechanics (“Sudostroyeniye”) [in Russian]
[7] Livshits I M, Rozentsvevg L N 1946 Journal of Experimental and Theoretical Physics Reflections on the Theory of Polycrystals Elastic Properties 16(II) [in Russian]
[8] Lomakin V A 1970 Statistical Problems of Deformable Solid Mechanics (M.: Nauka Publ) p 137 [in Russian]
[9] Lomakin V.A. 1970 Bulletin of USSR Academy of Sciences. Solid Mechanics Statistical Characteristics of Stress Fields in a Random Elastic Plane 4 124 [in Russian]
[10] Potapov V D 1999 Stability of stochastic elastic and viscoelastic systems (Chichester, England ; New York . John Wiley, (c)) p 275
[11] Shen L, Starzewski M O and Porcu E 2015 Journal of Engineering Mechanics Elastic rods and shear beams with random field properties under random field loads: Fractal and Hurst effects 141(7) 4015002 [04015004].
[12] Nishanth L, Sri Namachchivaya N, Singha P, Hoong Chieh Yeong 2016 Procedia Engineering Random Perturbations of Aeroelastic and Mechanical Systems 144 859
[13] Mihai L A, Goriely A Proc. R. Soc How to characterize a nonlinear elastic material? A review on nonlinear constitutive parameters in isotropic finite elasticity A 473 20170607
[14] Allaire G and Dapogny C 2014 Math. Models Methods Appl. Sci. A linearized approach to worst-case design in parametric and geometric shape optimization 24(11) 2199
[15] Nayfeh A H 2000 Perturbation Methods (New York: John Wiley and Sons) p 456
[16] Ivlev D D and Ershov L V 1978 The Perturbation Method in the Theory of Elasto-Plastic Body (Moscow: Nauka) p 208 [in Russian]
[17] Radchenko V P and Popov N N 2013 Chuvash State Pedagogical University Bulletin Series: Limit-State Mechanics Small Parameter Method Application for Solving Stochastic Nonlinear Problems in Steady-state Creep Theory 1(15) 185 [in Russian]
[18] Minaeva N V 2002 Perturbation method in Deformable Solid mechanics (Moscow: Nauchnaya kniga) p 236 [in Russian]
[19] Zachepa V R and Sapronov Y I 2002 Local Analysis of Fredholm Equations (Voronezh: Voronezh State Univ.) p 185 [in Russian]
[20] Krasnoselsky M A and Zabreiko P P 1975 Geometric Methods of Nonlinear Analysis (Moscow: Nauka) p 512 [in Russian]
[21] Darinsky B M, Sapronov Y I and Tsarev S L 2007 J. Math. Sci. 145 5311
[22] Ishlinsky A.Y. 1954 Ukrains’kyi Matematychnyi Zhurnal [Ukrainian Mathematical Journal]. - On the Problem of Elastic Bodies Equilibrium Stability in Mathematical Theory of Elasticity 6(2) 140
[23] Minaeva N V 2008 Mechanics of Solids Application of the perturbation method in mechanics of deformable solids 43(1) 31