A scattering amplitudes approach to hard thermal loops

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Abstract. Inspired by recent progress on classical limits of scattering amplitudes, we show that hard thermal loops can be obtained from classical limits of off-shell currents. The classicality of hard thermal loops is made manifest by associating classical wavenumbers to soft particles. We compute the classical limit of these currents in QED, QCD and gravity. Our proposal does not involve the introduction of ghosts.
1. Introduction

In the past decades progress in the field of scattering amplitudes has improved our ability to make predictions in scattering experiments.\footnote{See Refs. [1–5] for reviews on the modern approach to scattering amplitudes and Ref. [6] for a review of the state of the art of multi-loop calculations.} Recently, techniques developed originally to tackle problems in perturbative quantum field theory are being applied to the study of classical observables and have led to insights about the classical limit, see e.g., [7–23]. We would like to single out two developments in scattering amplitudes. First, a generalisation of Feynman’s tree theorem [24] known as loop-tree duality [25–35], which relates multi-loop amplitudes and phase space integrals and whose integrand is a tree-level amplitude-like object.\footnote{A non-exhaustive sample references on the relation between loops and trees is [36–42].} The loop tree-duality can also be formulated in terms of response functions which can be obtained from the zero temperature Schwinger-Keldysh formalism [26, 43]. Second, the novel framework introduced by Kosower-Maybee-O’Connell (KMOC) [10] to study classical observables from amplitudes. The KMOC formalism is based on expectation values of operators between initial states and thus is akin to the zero temperature Schwinger-Keldysh formalism itself.

In this paper we are interested in relating these two ideas to thermal field theory. In their form however we cannot directly apply them since in thermal field theory one is usually interested in currents (rather than scattering amplitudes) from which the thermodynamic properties of the system can be derived. Although the methods we will develop can be applied to QED, QCD and gravity, our primary example is a non-abelian plasma in the high temperature limit.

The limit of high temperature $T$ in perturbative thermal QCD is useful for the description of collective phenomena in plasmas. This limit can be consistently incorporated into an effective theory known as Hard Thermal Loop (HTL) effective theory [44–48]. At the
core of HTL effective theory is the resummation of 1-loop diagrams with the property of having external soft momenta and internal hard loop momenta. Hard Thermal Loops (HTLs) are non-local currents which are not only gauge invariant but obey simple Ward identities. Assuming a small coupling $g$ the soft momenta are of order $gT$ and the hard momenta of order $T$. HTLs can be computed from the forward scattering of thermal particles [48] which is reminiscent of the loop-tree duality for scattering amplitudes at zero temperature [25,26]. On the other hand, HTLs can be reformulated in the language of classical kinetic theory [49,50] and can be obtained from solutions of kinetic equations [51].

The classicality of HTLs and their equivalence with solutions of kinetic equations raises the question of whether we can directly obtain them as a classical limit understood as the limit of $\hbar \to 0$ of a quantum current. Inspired by the KMOC approach to classical observables, we propose a map between classical limits of off-shell currents at zero temperature and HTL currents. This map is based on the simple observation that to extract classical limits of scattering amplitudes one should distinguish between the momentum $p$ of a particle and its wavenumber $\vec{p}$

$$p \equiv \hbar \vec{p},$$

which we interpret as the distinction between soft and hard in HTL currents. As we will see, taking the classical limit will correspond to the high temperature limit. This scaling is the same one requires to study the classical dynamics of gravitational waves from quantum field theory. Hence we will mostly inspired in the KMOC approach but other strategies to obtain the classical limit may be used as well. e.g., [11,19,52].

This paper is organised as follows. In Section 2 we discuss briefly the equivalence of HTLs and solutions of kinetic equations and introduce the map to classical limits of off-shell currents. In Section 3 we consider QED. In Section 4 we consider QCD and gravity. We discuss our results in Section 5.

2. From kinetic theory to classical limits of off-shell currents

The dynamics of non-abelian plasmas is described by QCD at finite temperature at distances $\hbar/T \lesssim d \lesssim \hbar/(gT)$. For the rest of the paper we will use units in which $k_B = c = 1$ but keeping $\hbar \neq 1$ since we are interested in the study of the classical limit. The leading order thermal effects can be studied employing relativistic kinetic theory, which is applicable at scales where the average distance between particles is of order $\hbar/T$ and such that the plasma is characterised by the Debye length $\hbar/(gT)$. In the microscopic approach to kinetic theory\footnote{This approach has been reviewed in [53,54], see also Ref. [55] for a recent review.}, we are interested in an ensemble of point-like particles characterised by a phase-space distribution $f \equiv f(x,k,c)$, where the colour degrees of freedom $c$ of the plasma are treated as continuous classical variables. The phase space variables obey
Wong equations [56]

\[ \frac{dk^\mu}{d\tau} = g c^a(\tau) F^{a\mu\nu}(x(\tau)) v_\nu(\tau), \]
\[ \frac{dc^a}{d\tau} = g f^{abc\mu}(\tau) A^b_\mu(x(\tau)) c^c(\tau), \]
\[ D^\mu E^{a\mu}(x) = J^a_\nu(x) = g \int d\Phi(k) dc k_\nu c^a f(x, k, c), \]

where the Lorentz invariant phase space measure is defined by

\[ d\Phi(k) \equiv \frac{d^4k}{(2\pi)^4} \Theta(k_0) 2\pi\delta(k^2 - m^2). \]

Specialising to the case of \( SU(3) \) the colour measure is defined by

\[ dc \equiv d^8c c_R \delta(c^a c^a - q_2)\delta(d_{abc} c^a c^b - q_3), \]

where the colour part ensures the conservation of the Casimir invariants \( q_i \) and \( c_R \) is a normalisation constant defined such that \( \int dc = 1^4 \). In the collision-less case the Vlasov-type equation for the distribution function \( f(x, k, c) \) is given by

\[ \frac{df}{d\tau} = k^\mu \left( \frac{\partial}{\partial x^\mu} - g f^{abc} A^b_\nu c^c \frac{\partial}{\partial c^a} - g c^a E^{a\mu} \frac{\partial}{\partial k_\nu} \right) f = 0. \]

It is well-known that both Wong and Vlasov equations can be derived from suitable classical limits [57]. The Vlasov equations arises from the classical limit of the corresponding equation for the Wigner operator [58–60].

In this paper however we are interested in the classical limits of currents leading to solutions of the kinetic equations rather than a derivation of kinetic equations themselves. Eq.(7) can be solved iteratively by expanding \( f(x, k, c) \) around the equilibrium state. In equilibrium the distribution function only depends on the energy \( k_0 \) of the system and we can expand it in powers of the coupling \( g \)

\[ f(x, k, c) = f^{(0)}(k_0) + \Delta^{(1)} f(x, k, c) + \Delta^{(2)} f(x, k, c) + \ldots, \]

where \( \Delta^{(i)} f(x, k, c) \) is of order \( O(g^i) \). We can then use the equilibrium distribution function to solve for \( \Delta^{(1)} f(x, k, c) \) and re-insert it to find \( \Delta^{(2)} f(x, k, c) \) and so on. From Eq.(4) we can write the total current as

\[ J^a_\mu(x) = \Delta^{(1)} J^a_\mu(x) + \Delta^{(2)} J^a_\mu(x) + \ldots. \]

The currents can be expressed in the form

\[ J^a_\mu(x) = \Pi^{ab}_\mu A^b_\nu + \frac{1}{2} \Pi^{abc}_\mu A^b_\nu A^c_\rho + \ldots, \]

\[ \text{Notice that the measure is defined in terms of the dimensionless constant } c^a/\hbar. \]

\[ \text{The term of order } O(g^n) \text{ vanishes due to the identity } \int dc c^a = 0. \]
where $\Pi^{\mu_1 \mu_2 \cdots \mu_n}_{\nu_1 \nu_2 \cdots \nu_m}$ are thermal currents, which can be matched against those obtained from the high temperature limit of thermal QCD (See e.g., Ref. [61]). One can thus conclude that HTL are classical [51, 62]. Notice that despite our choice of units we will keep our classical quantities independent of $\hbar$ as they should. The setup leading to Eq. (9) is reminiscent of the one to obtain perturbative solutions from classical equations of motion [52, 63–69] which can be matched to classical limits of scattering amplitudes. Within the KMOC formalism this is done by defining certain observables which are well-defined both classically and quantum mechanically [10, 15, 70]. Besides the distinction between the momentum $p$ of a particle and its wave-number $\tilde{p}$ the introduction of appropriate coherent states is required to take the classical limit. Since we are interested in off-shell currents rather than observables, we cannot directly apply the KMOC formalism but as we will see many of its characteristics remain.

The classical limit of amplitudes in the KMOC formalism is slightly different in QED and QCD due to the presence of colour in the latter. In QCD the dimensionless coupling $g$ scales as $\tilde{g}\sqrt{\hbar}$. In QED the dimensionless coupling $e$ scales as $\tilde{e}/\sqrt{\hbar}$ and similarly for gravity. The scaling in QCD is complemented by scaling of the colour factors so ultimately we can use the same procedure in Ref. [10] to obtain the classical limit. Adopting the conventions of Ref. [70] we have

$$[C^a, C^b] = i\hbar f^{abc}C^c,$$

(11)

emphasising that $C$ corresponds to an operator and

$$\langle p_i | C^a | p^j \rangle \equiv (C^a)^i_j = \hbar (T^a)^i_j.\tag{12}$$

The corresponding classical colour charge $c^a \equiv \langle \psi | C^a | \psi \rangle$ is obtained from appropriate coherent states $|\psi\rangle$. In the spinless case gravity does not bring any new ingredient other than complexity so the KMOC algorithm follows the QED one.

The interplay between soft and hard momenta is very generally linked to the classical limit, usually through the Eikonal approximation, and thus it is tempting to relate it the HTL approximation. However, operationally in the HTL approximation one integrates over hard momenta running in the loop $k \sim T$ and considers external soft momenta $p \sim gT$. The results are then expanded in powers of $|p|/|k|$, see Ref. [61] for a summary of the rules to compute HTLs. By associating classical wavenumbers to the external soft momenta such that they scale with $\hbar$ an expansion in powers of $\hbar$ will correspond to an expansion in high temperature. This formal equivalence can also be observed at the level of the dimensional reduced effective action. Physically this has the interpretation of matching the high temperature regime to the classical regime.

In momentum space the coefficients in Eq. (10) can be written as follows [48, 72]

$$\Pi^{a_1 a_2 \cdots a_n}_{\mu_1 \mu_2 \cdots \mu_n}(p) = \int d\Phi(k)N(k_0)H^{a_1 a_2 \cdots a_n}_{\mu_1 \mu_2 \cdots \mu_n}(k, p),\tag{13}$$

where $N(k_0)$ is some distribution function and $H_n \equiv H^{a_1 a_2 \cdots a_n}_{\mu_1 \mu_2 \cdots \mu_n}(k, p)$ is the integrand of the HTL current. If we are interested in fermions we will define the distribution function

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6The current has to be understood as a functional of the soft gauge fields.

7See Ref. [70] for a longer discussion about this point.

8See e.g., Chapter 38 in Ref. [71].
with a minus sign due to the presence of a fermion loop. We have used the short-hand notation \( p \) to express the dependence on the full set of external momenta \( p_1, \ldots, p_n \). A generating functional has been given in [47].

In thermal perturbation theory the integrands in Eq. (13) can be constructed by considering permutations of comb diagrams (See Fig. 1). This approach can be applied systematically to a variety of theories including gravity [43, 79–86]. For on-shell amplitudes at zero temperature Eq. (13) is the usual forward limit [25, 26, 43] and hence a current at zero temperature is a natural candidate to relate the classical limit and the high temperature limit. Therefore we consider the current represented in Fig. 2 which can be computed using the Feynman rules of the theory under consideration. These currents have the property of having \( n \)-particles off-shell while the momentum of the massive particle running in the loop is on-shell, that is \( k^2 = m^2 \).

The forward limit is in general singular therefore we must consider a regularisation scheme. We define \( \mathcal{F} \) as the set of all Feynman graphs and \( \mathcal{S} \) as the set of tadpole graphs — i.e. those graphs containing a zero-momentum internal edge (see Fig. 3). Suppressing colour and Lorentz indices, we consider the following regularised current

\[
A_n(k, p_1, \ldots, p_n, k) \equiv \sum_{G \in \mathcal{F} \setminus \mathcal{S}} f(G),
\]

where \( f(G) \) is a rational expression of the form \( N(G)/D(G) \). The regularised current is simply obtained by removing tadpole graphs. In the following we assume that the current is regulated. The classical limit of this current will be defined by

\[
\bar{A}_n(k, \bar{p}_1, \ldots, \bar{p}_n) \equiv \text{Tr} \left( \lim_{\hbar \to 0} A_n(k, \hbar \bar{p}_1, \ldots, \hbar \bar{p}_n, k) \right),
\]

which is obtained by performing a Laurent expansion in powers of \( \hbar \) after rescaling the momenta of the soft particles and couplings following the algorithms in Refs. [10, 15, 70]. The operator \( \bar{\text{Tr}} \) depends on whether the theory is coloured or not. We define it by

\[
\bar{\text{Tr}}(\bullet) \equiv \begin{cases} 
\hbar^{n-2} \text{Tr}(\bullet) & \text{QCD}, \\
\text{Id}(\bullet) & \text{QED and gravity},
\end{cases}
\]

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9See Ref. [73] for a simple derivation of the functional and Refs. [74–78] for the relation of the generating functional and Chern-Simons theory.

10See Ref. [32] for a general version of this idea within loop-tree duality.
Figure 2.: Off-shell current. The blob represents a sum over tree-level Feynman diagrams.

Off-shell gauge bosons can be photons, gluons or gravitons.

where $\text{Id}$ is the identity operator and the $\hbar^{n-2}$ is required on dimensional grounds. We adopt the convention that our classical results will depend on the dimensionless coupling $g = \tilde{g} \sqrt{\hbar}$ and $e = \tilde{e} / \sqrt{\hbar}$ and that the external momenta is associated to wavenumbers. If desired one may restore the dependence on momenta by dimensional analysis. In the case of QCD, this requires an overall inverse power of $\hbar$ in Eq. (16). The currents just defined are in general gauge dependent. Similar currents occur for example in the Berends-Giele recursion [87].

We propose to map the integrand in (13) in momentum space to the classical limit of the $n$-point forward current

$$\tilde{A}_n(k, \bar{p}_1, \ldots, \bar{p}_n) \leftrightarrow H_n(k, p_1, \ldots, p_n), \tag{17}$$

and similarly for QED and gravity.

Using the imaginary-time formalism, where time is traded with the temperature and the structure of the propagator is similar to the zero temperature case, we can relate zero-temperature amplitudes and finite-temperature amplitudes by analytic continuation, see e.g., [88]. The currents we are proposing are amplitude-like objects and their computation is not based on sums over permutations of comb diagrams (see Fig. 1) as the usual forward scattering approach to thermal field theory [48]. Let us also remark that although we are considering the case where only massive particles appear in the loop, this case can also be utilised to study the case in which hard gauge particles run in the loop since the tensor structures are the same. Thus it is enough to discuss the case with matter loops. The pure gauge calculation with ghosts included appears e.g., in Ref. [43]. It is in this sense that ghosts will not be required. From an amplitudes point of view this is a simple consequence of our calculation being a tree-level one.

In order to model the non-abelian plasma we use the lagrangian

$$\mathcal{L} = \left[ (D^{\mu} \varphi^\dagger) D_{\mu} \varphi - \frac{m^2}{\hbar^2} \varphi^\dagger \varphi \right] - \frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu}, \tag{18}$$

where $D_\mu = \partial_\mu + ig A^a_\mu T^a$ and the field strength tensor $F_{\mu \nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A^b_\mu A^c_\nu$. For QED we employ the abelian version of this lagrangian. Let us remark that Eq. (7) and its abelian version describe a plasma for spinless particles and therefore this lagrangian is the appropriate to consider in the classical limit [12]. We use Feynman gauge for computations and adopt the following normalisation of the generators of the Lie algebra

$$\text{Tr} \left( C^a C^b \right) = \frac{\hbar^2}{2} \delta^{ab}. \tag{19}$$

11 We thank Fernando T. Brandt for this insight.
12 See Ref. [89] for the use of scalar QED as a toy model for gluonic QCD.
3. QED

As our first example we will consider a QED plasma where the equilibrium distribution is given by the Fermi-Dirac distribution

\[ N(k_0) \equiv -\frac{1}{\exp(k_0/T) + 1}, \tag{20} \]

where the sign is included to indicate that we are interested in matching the high temperature limit of \( n \)-point functions in QED with a fermion running in the loop. Recall that in our approach however the classical limit is taken from scalar QED since we are solving the spinless Vlasov equation. The regulated current is simply

\[ A_{\mu\nu}(k, \hbar \vec{p}_1, \cdots, \hbar \vec{p}_n, k), \tag{21} \]

where all photons are outgoing. Let us consider the simple case \( n = 2 \) whose contributing diagrams are shown in Fig. 3. Using the rescaled momenta a simple calculation leads to

\[
\begin{align*}
\mathcal{A}^{\mu\nu}(k, \hbar \vec{p}_1, \hbar \vec{p}_2, k) &= 2e^2 \left( \eta^{\mu\nu} + \frac{k^\mu (4\vec{p}_1^2 k^\mu - 4k \cdot \vec{p}_1 \vec{p}_1^\mu)}{4(k \cdot \vec{p}_1)^2 - \hbar^2 (\vec{p}_1^2)^2} - \frac{k^\nu \vec{p}_1^\mu + k^\mu \vec{p}_1^\nu}{k \cdot \vec{p}_1} \right),
\end{align*}
\]

where we have used \( p_2 = -p_1 \). Upon performing a Laurent expansion in powers of \( \hbar \) we obtain

\[
\begin{align*}
\tilde{\mathcal{A}}^{\mu\nu}(k, \vec{p}) &= 2e^2 \left( \eta^{\mu\nu} + \frac{\vec{p}_1^2 k^\mu k^\nu}{(k \cdot \vec{p}_1)^2} - \frac{k^\mu \vec{p}_1^\mu + k^\nu \vec{p}_1^\nu}{k \cdot \vec{p}_1} \right), \tag{22}
\end{align*}
\]

which recovers the \( \mathcal{O}(T^2) \) HTL 2-photon integrand.

Notice that the propagators considered for our currents do not have an \( i \epsilon \) term. In order to recover the retarded temperature dependent currents we consider the analytic continuation \( \vec{p}_n^0 \to \vec{p}_n^0 + i \epsilon \) and \( \vec{p}_i^0 \to \vec{p}_i^0 - i \epsilon \), for \( i = 1, \ldots, n-1 \), where we assume that the vertex corresponding to \( \vec{p}_n \) corresponds to the one with the largest time \[82\]. Finally, we can perform the radial integration by parametrising \( k^\mu = |k| K^\mu \) with \( K^\mu = (\sqrt{1 + m^2/|k|^2}, k/|k|) \) \[43, 79, 86\] (See Appendix A).

This simple example summarises our approach in general: first consider the \( n \)-point regulated current (15), then compute the current using Feynman graphs or Berends-Giele
recursions, etc, and calculate the classical limit with the replacement $\hbar \bar{p}_i$ for the gauge bosons. Physical propagators are obtained by analytic continuation.

In Ref. [48] “super-leading” terms in the temperature expansion appear, which cancel after all permutations of graphs are considered. In our approach we are not considering permutations (e.g., in the $n = 2$ case we are only summing 3.a-3.c in Fig.3) so it is interesting to ask if “superleading” terms appear here too. In our approach these terms have the form $k^{\mu_1} \cdots k^{\mu_n}/(\hbar^{n-1}(k \cdot \bar{p}_1) \cdots (k \cdot \bar{p}_{n-1}))$ and hence they correspond to (if any) singular terms in the classical limit. In our general expression in Eq.(15) these terms cancel (QED) or vanish (QCD). The HTL currents truly arise as classical limits of the currents $\mathcal{A}_{\mu}(k, \hbar \bar{p}_1, \hbar \bar{p}_2, k)$. For a neutral QED plasma it is well-known that higher point functions are either vanishing or subleading, see e.g., [90]. We can however consider a non-neutral plasma which would still be described by the Vlasov equation and thus study generalisations of HTL for higher point functions in QED [86]. After taking the classical limit and keeping terms of $O(\hbar^0)$ it is straightforward to compute higher order corrections in the temperature. We have checked up to $n = 4$ that our methods reproduce those obtained from the thermal field theory approach [86]. A 3-point example is given in Appendix B.

4. QCD and gravity

The QED case illustrates the procedure of taking the classical limit for the kinematics. The new ingredient in QCD is colour. The lowest order contribution requires the calculation of a diagram with a three-gluon vertex as shown in Fig.3, which leads to a tadpole diagram and therefore removed.\(^{13}\) Notice that the first diagram in Fig.3 does not lead to a tadpole since the internal edge with zero momentum is not present. Computing the current and taking the classical limit leads to

$$
\mathcal{A}_{\mu_1 \mu_2}(k, \hbar \bar{p}_1, \hbar \bar{p}_2) = \frac{g^2}{\hbar} \left[ C^{a_1} \cdot C^{a_2} \left( -\frac{2k^{\mu_1} k^{\mu_2}}{kk \cdot \bar{p}_1} + \frac{\bar{p}_1^2 k^{\mu_1} k^{\mu_2}}{(k \cdot \bar{p}_1)^2} - \frac{k^{\nu} \bar{p}_1^\nu}{k \cdot \bar{p}_1} \frac{k^{\mu_1} \bar{p}_1^\mu}{k \cdot \bar{p}_1} + O(\hbar) \right) \\
+ C^{a_2} \cdot C^{a_1} \left( \frac{2k^{\mu_1} k^{\mu_2}}{kk \cdot \bar{p}_1} + \frac{\bar{p}_1^2 k^{\mu_1} k^{\mu_2}}{(k \cdot \bar{p}_1)^2} - \frac{k^{\mu_2} \bar{p}_1^{\mu_1}}{k \cdot \bar{p}_1} \frac{k^{\mu_1} \bar{p}_1^{\mu_2}}{k \cdot \bar{p}_1} + O(\hbar) \right) \right],
$$

(23)

where $C^a \cdot C^b \equiv (C^a)^i_j (C^b)^j_k$. At first sight this expression contains singular terms of the form $k^{\mu_1} k^{\mu_2}/(hk \cdot \bar{p}_1)$, which we mentioned previously. However using the Lie algebra (11) these produce classical terms with a vanishing trace. Therefore after tracing we obtain the well-known result for QCD

$$
\tilde{\mathcal{A}}_{\mu_1 \mu_2}(k, \bar{p}) = \delta^{\alpha_1 \alpha_2} \Pi^{\mu_1 \mu_2}(\bar{p}_1) \equiv \delta^{\alpha_1 \alpha_2} g^2 \left( \frac{\bar{p}_1^2 k^{\mu_1} k^{\mu_2}}{(k \cdot \bar{p}_1)^2} - \frac{k^{\mu_2} \bar{p}_1^{\mu_1}}{k \cdot \bar{p}_1} - \frac{k^{\mu_1} \bar{p}_1^{\mu_2}}{k \cdot \bar{p}_1} + O(\hbar) \right).
$$

(24)

Notice that the final result is expressed in terms of the dimensionless coupling $g \sqrt{\hbar}$. We now consider the three-point function. We decompose the current in a basis of colour

\(^{13}\)The contribution from three-gluon vertices typically neglected because it does not correspond to a one-particle irreducible graph.
factors as follows
\[ A_{\alpha_1 \alpha_2 \alpha_3}^{\mu_1 \mu_2 \mu_3}(k, h\vec{p}_1, h\vec{p}_2, h\vec{p}_3, k) = \sum_{\sigma \in S_3} C^{\alpha_1} \cdot C^{\alpha_2} \cdot C^{\alpha_3} J^{\mu_1 \mu_2 \mu_3}(\sigma_1, \sigma_2, \sigma_3). \]

The kinematic coefficients \( J^{\mu_1 \mu_2 \mu_3}(\sigma_1, \sigma_2, \sigma_3) \) are straightforward to compute but lengthy and not presented here. The leading contribution in the classical limit is of order \( \mathcal{O}(1/\hbar) \), i.e., singular at first sight. However upon tracing we can bring the result into the form
\[ \tilde{A}_{\alpha_1 \alpha_2 \alpha_3}^{\mu_1 \mu_2 \mu_3}(k, \vec{p}_1, \vec{p}_2, \vec{p}_3, k) = 2\tilde{g}^2 \hbar \frac{\kappa^2}{k^{3/2}} \text{Tr} \left( C^{\alpha_1} \cdot C^{\alpha_2} \cdot C^{\alpha_3} \right) - \text{Tr} \left( C^{\alpha_1} \cdot C^{\alpha_2} \cdot C^{\alpha_3} \right) \frac{A_{\text{QED}}^{\mu_1 \mu_2 \mu_3}}{\hbar}, \tag{25} \]

where
\[ A_{\text{QED}}^{\mu_1 \mu_2 \mu_3} = \sum_{\sigma \in \text{Cyclic}} \left[ \frac{2k^{\mu_1} k^{\mu_2}}{k \cdot \vec{p}_3} \left( \frac{\vec{p}_{\sigma_1}^{\mu_3}}{k \cdot \vec{p}_{\sigma_1}} - \frac{\vec{p}_{\sigma_2}^{\mu_3}}{k \cdot \vec{p}_{\sigma_2}} \right) + k^{\mu_1} k^{\mu_2} k^{\mu_3} \left( \frac{\vec{p}_{\sigma_1}^2}{(k \cdot \vec{p}_{\sigma_1})^2} \left( \frac{1}{k \cdot \vec{p}_{\sigma_2}} - \frac{1}{k \cdot \vec{p}_{\sigma_3}} \right) \right) \right] , \tag{26} \]

where Cyclic is the set of cyclic permutations of \{1, 2, 3\}. Notice that in the commutative case Eq.(25) is simple telling us that singular terms vanish in the classical limit since the operator \( \text{Tr} \) is replaced by the identity operator. Using Eqs.(11) and (19) we find
\[ \tilde{A}_{\alpha_1 \alpha_2 \alpha_3}^{\mu_1 \mu_2 \mu_3}(k, \vec{p}) = ig^3 \bar{f}^{a_1 a_2 a_3} A_{\text{QED}}^{a_1 a_2 a_3}, \tag{27} \]

where the current satisfies the identity
\[ \bar{p}_{3\mu_3} \tilde{A}_{\alpha_1 \alpha_2 \alpha_3}^{\mu_1 \mu_2 \mu_3}(k, \vec{p}) = ig \bar{f}^{a_1 a_2 a_3} [\Pi^{\mu_1 \mu_3}(p_1) - \Pi^{\mu_1 \mu_2}(p_2)]. \tag{28} \]

Remarkably, we find that the kinematic structure of the current is encoded in the commutative QED part in a similar way as the classical colour and momentum impulse observables [70]. Although we can continue computing the next contributions in a similar fashion, it is well-known we can reconstruct higher point functions using Ward identities and the above relation [45, 46]. The expression (26) agrees with previously computed expressions in Refs. [84, 91].

There is no new ingredient in gravity regarding the classical limit and therefore we can apply our formalism for this case too. HTL for gravity, reviewed in Ref. [92], are relevant for the physics of the early universe. They have been investigated in Refs. [80, 93]. We expand in powers of the graviton coupling \( \kappa \) with \( \kappa^2 = 32\pi G \) and set \( g_{\mu_1 \mu_2} = \eta_{\mu_1 \mu_2} + \kappa \hbar_{\mu_1 \mu_2} \). The 2-point graviton function can be computed from the diagrams in Fig.3 with the gluon replaced by a graviton, where as in QCD we regulate the current by removing contributions from the 3-graviton vertex. Using the conventions in Ref. [94] the calculation is straightforward and leads to
\[ \tilde{A}_{\alpha_1 \beta_2 \gamma_3}^{\mu_1 \mu_2}(k, \vec{p}) = \frac{1}{2} \kappa^2 \left[ \frac{\bar{p}_1^2 k^\alpha k^\beta k^\gamma k^\delta}{(k \cdot \vec{p}_1)^2} - \frac{k^\beta k^\gamma k^\delta \bar{p}_1^\alpha + k^\alpha k^\gamma k^\delta \bar{p}_1^\beta + k^\alpha k^\beta k^\gamma \bar{p}_1^\delta - k^\alpha k^\beta k^\gamma \bar{p}_1^\delta}{k \cdot \vec{p}_1} \right. \]
\[ \left. + \bar{p}_2^\gamma k^\alpha k^\delta + \bar{p}_3^\beta k^\gamma k^\alpha + \bar{p}_1^\beta k^\gamma k^\delta + \bar{p}_1^\alpha k^\gamma k^\delta \right], \tag{29} \]
which agrees with the result in Ref. [80]. In general the 2-point graviton function depends on the representation of the graviton field but one can redefine it in terms of the above expression to obtain a 2-point function independent of the graviton parametrisation [80].

5. Discussion

In this paper we have shown that HTLs arise from classical limits of off-shell currents. The classical nature of hard-thermal loop amplitudes was made manifest by relating the momenta of the soft particles to wavenumbers. The classical limit is then obtained following the KMOC algorithm. In this way, the high temperature limit is formally equivalent to an expansion in powers of $\hbar$ thus allowing a map between HTL amplitudes and classical limits of off-shell currents. The off-shell currents encode the information of permutation of comb diagrams and can be easily computed from Feynman diagrams or Berends-Giele recursions. Since our off-shell currents in the classical limit are gauge invariant and satisfy Ward identities their “on-shell” properties would be interesting to study, in particular the Britto-Cachazo-Feng-Witten [95] recursion, the colour-kinematics duality and the double copy [96–98]. For general off-shell currents these properties are generally more difficult to make manifest than for amplitudes [99–101]. At the classical level the double copy is much more flexible but the idea of a formal replacement between colour and kinematics is preserved [102]. This classical double copy is the appropriate to relate gravity and QCD in the high temperature limit.

On the more phenomenological side the inclusion of spin and collisions\footnote{Collision functions in the context of classical transport theory have been studied e.g., in [103–105].} would be relevant for a more complete description of a non-abelian plasma from kinetic theory. Vlasov equations with spin can be obtained from first principles using Wigner functions and thus it would be interesting to describe currents with spin. See e.g., [106, 107] for a recent application of Wigner functions in the case with collisions. The KMOC formalism has been applied to the spin cases in Ref. [15]. In principle, the spin case would require considering classical limits of spinor wavefunctions. Collision functions in kinetic theory can be obtained from Wigner functions, which can be computed using scattering amplitudes [108]. Perhaps one can study the inclusion of collisions considering classical limits of scattering amplitudes within Wigner functions.

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A. Temperature dependence

The temperature can be recovered following the procedure in Ref. [86], where the reader can check details. For simplicity, let us consider the 2-point QED example and consider Eq. (13). Using our map, we have

$$\Pi^{\mu\nu}(\bar{p}) = \int \frac{d^4k}{(2\pi)^4} \theta(k_0) 2\pi \delta(k^2 - m^2) N(k_0) \tilde{A}_2^{\mu\nu}(k, \bar{p}),$$

which leads to

$$\Pi^{\mu\nu}(\bar{p}) = \frac{1}{(2\pi)^3} \int d^3k \frac{N(\sqrt{k^2 + m^2})}{2\sqrt{k^2 + m^2}} \tilde{A}_2^{\mu\nu}(k, \bar{p})\bigg|_{k_0 = \sqrt{k^2 + m^2}}.$$  (31)

Let us define \( k_0 = |k|K_0 \) with \( K_0 > 0 \) and use spherical coordinates

$$\Pi^{\mu\nu}(\bar{p}) = \frac{1}{(2\pi)^3} \int d|k| |k|^2 N(|k|K_0) \int d\Omega \tilde{A}_2^{\mu\nu}(k, \bar{p})\bigg|_{k_0 = |k|K_0}.$$  (32)

We can simplify the above expression by introducing the unit vector vector \( \hat{K} = k/|k| \) and defining the four vector

$$K^\mu = (k_0/|k|, k/|k|),$$  (33)

such that \( k^\mu = |k|K^\mu \). Since \( \tilde{A}_2^{\mu\nu}(k, \bar{p}) \) is homogeneous function of degree zero in \( k \) we can then write

$$\Pi^{\mu\nu}(\bar{p}) = \frac{1}{(2\pi)^3} \int_0^\infty d|k| |k|^2 N(|k|K_0) \int d\Omega \tilde{A}_2^{\mu\nu}(K, \bar{p})$$  (34)

Using \( K_0 = \sqrt{1 + m^2/|k|^2} \) and the change of variables \( x = |k|/T \) leads to

$$\Pi^{\mu\nu}(\bar{p}) = -\frac{1}{(2\pi)^3} I(T, m) \int d\Omega \tilde{A}_2^{\mu\nu}(K, \bar{p}),$$  (35)

where we have included a minus sign to indicate that we are interested in the classical limit where a fermion is running in the loop. The temperature dependence is then obtained from

$$I(T, m) \equiv T^2 \int_0^\infty dx \frac{x^2}{2\sqrt{x^2 + m^2}} \frac{1}{\exp(\sqrt{x^2 + m^2}) + 1}.$$  (36)

B. 3-point QED example

We calculate the 3-point function for QED. The current requires the computation of 12 graphs. We will write the current in terms of the following bases of tensor structures

$$\tilde{A}^{\mu_1\mu_2\mu_3}(k, \bar{p}) = \frac{2e^3}{(k \cdot \bar{p}_1)^2} \sum_{i=1}^{23} T_i a_i,$$  (37)
where
\[
\begin{align*}
    a_1 &= \frac{\bar{p}_2^2 \bar{p}_1 \cdot \bar{p}_3}{(k \cdot \bar{p}_2)^2} + \frac{\bar{p}_3^2 \bar{p}_1 \cdot \bar{p}_2}{(k \cdot \bar{p}_3)^2} - \frac{2 \bar{p}_1 \cdot \bar{p}_2 \bar{p}_1 \cdot \bar{p}_3}{k \cdot \bar{p}_2 k \cdot \bar{p}_3}, \\
    a_2 &= -\bar{p}_1^2, \\
    a_3 &= -\frac{\bar{p}_2^2 (k \cdot \bar{p}_1)^2}{(k \cdot \bar{p}_2)^2}, \\
    a_4 &= a_{12} = -k \cdot \bar{p}_1, \\
    a_5 &= \frac{(k \cdot \bar{p}_1)^2}{k \cdot \bar{p}_2}, \\
    a_6 &= a_7 = \frac{\bar{p}_2^2 k \cdot \bar{p}_1}{(k \cdot \bar{p}_3)^2} + \frac{\bar{p}_1 \cdot \bar{p}_2 k \cdot \bar{p}_1}{k \cdot \bar{p}_2 k \cdot \bar{p}_3}, \\
    a_8 &= \frac{\bar{p}_2^2 k \cdot \bar{p}_3}{(k \cdot \bar{p}_2)^2} + \frac{\bar{p}_3^2}{k \cdot \bar{p}_2} - \frac{2 \bar{p}_1 \cdot \bar{p}_2}{k \cdot \bar{p}_2}, \\
    a_9 &= a_{10} = \frac{k \cdot \bar{p}_1}{k \cdot \bar{p}_2}, \\
    a_{14} &= a_{21} = a_{22} = \frac{(k \cdot \bar{p}_1)^2}{k \cdot \bar{p}_2 k \cdot \bar{p}_3}, \\
    a_{15} &= \frac{k \cdot \bar{p}_1}{k \cdot \bar{p}_2}, \\
    a_{16} &= \frac{\bar{p}_2^2}{k \cdot \bar{p}_3} + \frac{\bar{p}_3^2 k \cdot \bar{p}_2}{(k \cdot \bar{p}_3)^2} - \frac{2 \bar{p}_1 \cdot \bar{p}_3}{k \cdot \bar{p}_3}, \\
    a_{17} &= a_{18} = a_{23} = \frac{k \cdot \bar{p}_1}{k \cdot \bar{p}_3}, \\
    a_{19} &= \frac{(k \cdot \bar{p}_1)^2}{k \cdot \bar{p}_3}.
\end{align*}
\]

and
\[
\begin{align*}
    T_1 &= k^{\mu_1} k^{\mu_2} k^{\mu_3}, \\
    T_2 &= \eta^{\mu_2 \mu_3} k^{\mu_1}, \\
    T_3 &= \eta^{\mu_1 \mu_3} k^{\mu_2}, \\
    T_4 &= \eta_{\mu_2 \mu_3} \bar{p}_2^{\mu_1}, \\
    T_5 &= \eta^{\mu_1 \mu_3} \bar{p}_2^{\mu_2}, \\
    T_6 &= k^{\mu_2} k^{\mu_3} \bar{p}_2^{\mu_1}, \\
    T_7 &= k^{\mu_1} k^{\mu_3} \bar{p}_2^{\mu_2}, \\
    T_8 &= k^{\mu_1} k^{\mu_2} \bar{p}_2^{\mu_3}, \\
    T_9 &= k^{\mu_2} \bar{p}_2^{\mu_1} \bar{p}_2^{\mu_3}, \\
    T_{10} &= k^{\mu_1} \bar{p}_2^{\mu_2} \bar{p}_2^{\mu_3}, \\
    T_{11} &= \eta^{\mu_1 \mu_3} k^{\mu_2}, \\
    T_{12} &= \eta^{\mu_2 \mu_3} k^{\mu_1}, \\
    T_{13} &= \eta_{\mu_2 \mu_3} \bar{p}_2^{\mu_1}, \\
    T_{14} &= \eta_{\mu_1 \mu_3} \bar{p}_2^{\mu_2}, \\
    T_{15} &= \eta^{\mu_1 \mu_3} \bar{p}_2^{\mu_2}, \\
    T_{16} &= k^{\mu_1} \bar{p}_3^{\mu_2}, \\
    T_{17} &= k^{\mu_3} \bar{p}_2^{\mu_1} \bar{p}_3^{\mu_2}, \\
    T_{18} &= k^{\mu_3} \bar{p}_3^{\mu_1} \bar{p}_3^{\mu_2}, \\
    T_{19} &= \eta^{\mu_1 \mu_2} \bar{p}_3^{\mu_3}, \\
    T_{20} &= k^{\mu_1} \bar{p}_3^{\mu_2} \bar{p}_3^{\mu_3}, \\
    T_{21} &= k^{\mu_2} \bar{p}_3^{\mu_1} \bar{p}_3^{\mu_3}, \\
    T_{22} &= k^{\mu_3} \bar{p}_3^{\mu_1} \bar{p}_3^{\mu_2}, \\
    T_{23} &= k^{\mu_3} \bar{p}_3^{\mu_2} \bar{p}_3^{\mu_3}.
\end{align*}
\]

We have checked that this result is equivalent to the high temperature limit computed in Ref. [86]. The equivalence between these functions and solutions of the Abelian version of Eq.(7) is discussed Ref. [86].

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