Quadrupole interaction of relativistic quantum particle with external fields

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Abstract

We consider the motion of a spinning relativistic particle with an arbitrary value of spin in external electromagnetic and gravitational fields, to first order in the external field. We use the noncovariant description of spin. An explicit expression is obtained for the interaction of second order in spin. The value of the quadrupole moment is found for which this interaction decreases when the energy grows.

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1 Introduction

The problem of motion of a particle with internal angular momentum (spin) in external fields has been considered in many papers (see [1] and references therein). The problem has various applications, ranging from the accelerator physics to the dynamics of rotating stars or black holes. Our interest to the problem originated from the question about evolution of polarisation of nuclei in storage rings.

In the present paper we discuss in detail quadratic in spin effects. We consider only the terms proportional to first derivatives of the field strength, neglecting higher derivatives. Thus electric quadrupole interaction for arbitrary particle velocities will be found. The linear in spin interaction (magnetic dipole one) is well-known (see, e.g., [4], §41), so we will not write it down here.

We find the interaction of a spinning relativistic particle with external gravitational field to the same approximations.

We use here the approach described in Refs. [1], [2]. It allows one to obtain equations of motion of a particle in external fields, to an arbitrary order in spin. However we confine neither to semiclassical approximation, in contrast to [1], nor nonrelativistic approximation, in contrast to [3].

2 Spinning particle in electromagnetic field

The Lagrangian of the spin interaction with an external field can be derived from the elastic scattering amplitude

\[ -eJ^\mu A_\mu \]  

of a particle with spin \( s \) on a vector potential \( A_\mu \) [1]. The matrix element \( J_\mu \) of the electromagnetic current operator between states with momenta \( k \) and \( k' \) can be written (under \( P \) and \( T \) invariance) as follows (see [1-3]):

\[ J_\mu = \frac{1}{\sqrt{\epsilon_{k'k}}} \tilde{\psi}(k') \left\{ p_\mu F_e + \frac{1}{2} \Sigma_{\mu\nu} q^\nu F_m \right\} \psi(k). \]  

Here \( p_\mu = (k' + k)_\mu / 2 \), \( q_\mu = (k' - k)_\mu \). The wave function of a particle with an arbitrary spin \( \psi \) can be written (see, for instance, Ref. [4], §31) as

\[ \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \]  

Both spinors,

\[ \xi = \{ \xi_{\alpha_1 \alpha_2 ... \alpha_p} \} \]

and

\[ \eta = \{ \eta_{\beta_1 \beta_2 ... \beta_q} \}, \]
are symmetric in the dotted and undotted indices separately, and
\[ p + q = 2s. \]

For a particle of a half-integer spin one can choose
\[ p = s + \frac{1}{2}, \quad q = s - \frac{1}{2}. \]

In the case of an integer spin it is convenient to put
\[ p = q = s. \]

The spinors \( \xi \) and \( \eta \) are chosen in such a way that under reflection they go over into each other (up to a phase). At \( p \neq q \) they are different objects which belong to different representations of the Lorentz group. If \( p = q \), these two spinors coincide. Nevertheless, we will use the same expression (3) for the wave function of any spin, i.e., we will also introduce formally the object \( \eta \) for an integer spin, keeping in mind that it is expressed in terms of \( \xi \). This will allow us to perform calculations in the same way for the integer and half-integer spins.

In the rest frame both \( \xi \) and \( \eta \) coincide with a nonrelativistic spinor \( \xi_0 \), which is symmetric in all indices; in this frame there is no difference between dotted and undotted indices. The spinors \( \xi \) and \( \eta \) are obtained from \( \xi_0 \) through the Lorentz transformation:
\[
\xi = \exp\{\Sigma \phi/2\}\xi_0; \quad \eta = \exp\{-\Sigma \phi/2\}\xi_0.
\]

Here the vector \( \phi \) is directed along the velocity, \( \tanh \phi = v \);
\[
\Sigma = \sum_{i=1}^{p} \sigma_i - \sum_{i=p+1}^{p+q} \sigma_i,
\]
and \( \sigma_i \) acts on the \( i \)th index of the spinor \( \xi_0 \) as follows:
\[
\sigma_i \xi_0 = (\sigma_i)_{\alpha_i\beta_i} (\xi_0)_{\beta_i\ldots}. \tag{5}
\]

In the Lorentz transformation (4) for \( \xi \), after the action of the operator \( \Sigma \) on \( \xi_0 \) the first \( p \) indices are identified with the upper undotted indices and the next \( q \) indices are identified with the lower dotted indices. The inverse situation takes place for \( \eta \).

Then,
\[
\bar{\psi} = \psi^\dagger \gamma_0 = \psi^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix};
\]
here \( I \) is the sum of unit \( 2 \times 2 \) matrices acting on all indices of the spinors \( \xi \) and \( \eta \). The components of the matrix \( \Sigma_{\mu\nu} = -\Sigma_{\nu\mu} \) are:
\[
\Sigma_{\alpha\beta} = \begin{pmatrix} -\Sigma_{\alpha} & 0 \\ 0 & \Sigma_{\beta} \end{pmatrix}; \tag{6}
\]
\[ \Sigma_{mn} = -2i\epsilon_{mnk} \begin{pmatrix} s_k & 0 \\ 0 & s_k \end{pmatrix}; \]

\[ s = \frac{1}{2} \sum_{i=1}^{2a} \sigma_i. \]

The scalar operators \( F_{e,m} \) depend on two invariants, \( t = q^2 \) and \( \tau = (Sq)^2 \), here \( S^\mu = (i/4)\epsilon^{\mu\nu\kappa\lambda}\Sigma_{\kappa\lambda}p_\nu/m \) is the spin 4-vector. In the expansion in the electric multipoles

\[ F_e(t, \tau) = \sum_{n=0}^{N_e} f_{e,2n}(t) \tau^n \]

the highest power \( N_e \) equals obviously to \( s \) and \( s - 1/2 \) for an integer and half-integer spin, respectively. In the magnetic multipole expansion

\[ F_m(t, \tau) = \sum_{n=0}^{N_m} f_{m,2n}(t) \tau^n \]

the highest power \( N_m \) constitutes \( s - 1 \) and \( s - 1/2 \) for an integer and half-integer spin. It can be easily seen that

\[ f_{e,0} = 1, \quad f_{m,0} = \frac{g}{2}. \]

Here and below we confine to the consideration of formfactors at zero momentum transfer, \( f_{e,0} = f_{e,0}(0), f_{m,0} = f_{m,0}(0) \).

In our consideration we will discuss terms linear and quadratic in the momentum transfer momentum \( q \) only, i.e., proportional to the field strength and its derivatives. We will write down quadratic in spin terms only, because linear in spin interaction is well-known (see, for instance, [4], §41).

We will start with the contribution of the convection term

\[ -\frac{e}{\sqrt{\epsilon_k\epsilon_{k'}}} \tilde{\psi}(k')\psi(k) p^\mu A_\mu. \]

The product of exponents in the expression

\[ \tilde{\psi}(k')\psi(k) = \frac{1}{2}\xi_0^\dagger [\exp\{\Sigma\phi'/2\} \exp\{-\Sigma\phi'/2\} + \exp\{-\Sigma\phi'/2\} \exp\{\Sigma\phi'/2\}] \xi_0 \]

can be calculated analogously to [1]. To our accuracy it equals to

\[ \exp\{\Sigma\phi'/2\} \exp\{-\Sigma\phi'/2\} = \exp \left\{ \frac{1}{2m} \Sigma q - \frac{\Sigma p(pq)}{2m(\epsilon + m)} + i \frac{s[p \times q]}{m(\epsilon + m)} \right\}, \]

where \( \epsilon = \sqrt{m^2 + p^2} = m\gamma \). Thus wave functions scalar product takes the form

\[ \tilde{\psi}(k')\psi(k) = \xi_0^\dagger \left\{ 1 + i \frac{s[p \times q]}{m(\epsilon + m)} - \frac{(s[p \times q])^2}{2m^2(\epsilon + m)^2} + \frac{1}{8m^2} \left( \Sigma q - \frac{\Sigma p(pq)}{\epsilon(\epsilon + m)} \right)^2 + O(q^3) \right\} \xi_0. \]
The contribution to the Lagrangian of the term from (11), which contains $\Sigma$ and therefore vanishes in the classical limit, is

$$ e \frac{\Lambda}{2m^2} \left[ (s\nabla) - \frac{\gamma}{\gamma + 1} (vs)(v\nabla) \right] \left[ (sE) - \frac{\gamma}{\gamma + 1} (sv)(vE) + (s[v \times B]) \right], \quad (12) $$

where

$$ \Lambda = \begin{cases} 
1/(2s-1), & \text{integer spin,} \\
1/(2s), & \text{half-integer spin.} 
\end{cases} \quad (13) $$

We consider the motion of particles in empty space (for example, in an accelerator), so in the derivation of (12) we used sourceless Maxwell equations. Here and below we use the orthogonality condition $pq = 0$, which means that initial and final particles are on mass-shell. In the nonrelativistic approximation the contribution (13) has been derived in [2] (including the additional so-called Darwin term corresponding to the contact interaction with external sources).

The remaining part of the convection contribution to the Lagrangian is

$$ - \frac{e}{2m^2} \frac{\gamma}{\gamma + 1} (s [v \times \nabla]) \left[ \left( 1 - \frac{1}{\gamma} \right) (sB) - \frac{\gamma}{\gamma + 1} (sv)(vB) - \frac{\gamma}{\gamma + 1} (s [v \times E]) \right]. \quad (14) $$

This expression which has nonvanishing classical limit was derived in [1].

Let us go over to the terms proportional to $g$-factor. Their classical part also was found in [1]. The part vanishing in the classical limit is

$$ - eg \frac{\Lambda}{2m^2} \left[ (s\nabla) - \frac{\gamma}{\gamma + 1} (vs)(v\nabla) \right] \left[ (sE) - \frac{\gamma}{\gamma + 1} (sv)(vE) + (s[v \times B]) \right]. \quad (15) $$

At last, the "bare" quadrupole interaction, which enters the expressions (11) and (12), is

$$ - \frac{e}{\sqrt{\epsilon_k \epsilon_k'}} f e_{\alpha} \tau p^\mu A_\mu. \quad (16) $$

The corresponding contribution of the "bare" quadrupole interaction to the Lagrangian is

$$ - e f e_{\alpha} \left[ (s\nabla) - \frac{\gamma}{\gamma + 1} (vs)(v\nabla) \right] \left[ (sE) - \frac{\gamma}{\gamma + 1} (sv)(vE) + (s[v \times B]) \right]. \quad (17) $$

The general expression for the quadratic in spin interaction can be presented as follows

$$ L_{2s} = - e \left( f e_{\alpha} + \frac{\Lambda (g-1)}{2m^2} \right) \left[ (s\nabla) - \frac{\gamma}{\gamma + 1} (vs)(v\nabla) \right] \left[ (sE) - \frac{\gamma}{\gamma + 1} (sv)(vE) + (s[v \times B]) \right] $$

$$ + \frac{e}{2m^2} \frac{\gamma}{\gamma + 1} (s [v \times \nabla]) \left[ \left( g - 1 + \frac{1}{\gamma} \right) (sB) - (g - 1) \frac{\gamma}{\gamma + 1} (sv)(vB) \right]. $$
\begin{align}
- \left( g - \frac{\gamma}{\gamma + 1} \right) (s [v \times E]) \right].
\end{align}

(18)

The obtained expression for the quadratic in spin interaction differs from the corresponding semiclassical formula in [1] by the following substitution only

\begin{align}
f_{e,2} \to f_{e,2} + \frac{\Lambda(g - 1)}{2m^2}.
\end{align}

(19)

There is a value of $f_{e,2}$, for which the quadratic in spin interaction $L_2 s$ decreases for $\gamma \to \infty$. An analogous phenomenon exists for the linear in spin interaction: for $g = 2$ it decreases when the particle energy grows. The corresponding preferred value of the formfactor $f_{e,2}$ can be easily obtained by substituting (19) into the corresponding expression in [1]:

\begin{align}
f_{e,2} = (1 - \Lambda) \frac{g - 1}{2m^2}; \quad \text{for} \quad g = 2 \quad f_{e,2} = \frac{1 - \Lambda}{2m^2}.
\end{align}

(20)

Let us write down the preferred value of the quadrupole moment:

\begin{align}
Q = Q_{zz} \big|_{s_z = s} = -2 e \left( f_{e,2} + \frac{\Lambda(g - 1)}{2m^2} \right) s(2s - 1) = -\frac{e s(2s - 1)}{m^2}.
\end{align}

(21)

The same preferred value of the quadrupole moment was recently derived in [3] from different approach – based on the supersymmetric sum rules. This good high-energy behavior of the interaction is a necessary (but insufficient) condition of renormalizability. Indeed, charged vector boson in the renormalizable electroweak theory has the quadrupole moment equal to

\begin{align}
Q = -\frac{e}{m^2}.
\end{align}

(22)

in accordance with the expression (21).

\section{Spinning particle in gravitational field}

The equations of motion in an external gravitational field to any order in spin can be obtained from the equations of motion in an electromagnetic field by simple substitution.

The elastic scattering amplitude in a weak external gravitational field $h_{\mu\nu}$ is

\begin{align}
- \frac{1}{2} T_{\mu\nu} h^{\mu\nu}
\end{align}

(23)

(in due time we will go over to a generally covariant form). The matrix element $T_{\mu\nu}$ of the energy-momentum tensor between the states of momenta $k$ and $k'$ can be written as [1]:

\begin{align}
T_{\mu\nu} = \frac{1}{4 \sqrt{\epsilon_k \epsilon_{k'}}} \bar{\psi}(k') \left\{ 4 p_{\mu} p_{\nu} F_1 + (p_{\mu} \Sigma_{\nu\lambda} + p_{\nu} \Sigma_{\mu\lambda}) q^\lambda F_2 
+ (\eta_{\mu\nu} q^2 - q_{\mu} q_{\nu}) F_3 + [S_{\mu} S_{\nu} q^2 - (S_{\mu} q_{\nu} + S_{\nu} q_{\mu})(S q) + \eta_{\mu\nu}(S q)^2] F_4 \right \} \psi(k).
\end{align}

(24)
The scalar operators $F_i$ in this expression are also expanded in powers of $\tau = (Sq)^2$:

$$F_i(t, \tau) = \sum_{n=0}^{N_i} f_{i,2n}(t)\tau^n. \quad (25)$$

Since we are interested in the equations of motion in a sourceless field, the terms proportional to $F_3$ and $F_4$ in the expansion (24) will be omitted, because when rewritten in the covariant form, they are proportional to the scalar curvature and Ricci tensor, respectively. So, the amplitude (23) can be presented in the following form

$$-\frac{1}{2} \sqrt{\epsilon_k \epsilon_{k'}} \tilde{\psi}(k') \left\{ p_\mu F_1 + \frac{1}{2} \Sigma_{\mu \lambda} q^\lambda F_2 \right\} \psi(k) h^{\mu \nu} p_\nu. \quad (26)$$

Clearly, (26) differs (1), (2) by the following substitution only:

$$e A_\mu \to \frac{1}{2} h_{\mu \nu} P^\nu. \quad (27)$$

With this substitution in (18), one can obtain quadratic in spin interaction of a particle with a gravitational field.

Let us now consider the preferred values of the gravitational formfactors. The first coefficients in the expansion (25) $f_{1,0} = 1$ and $f_{2,0} = 1$ are fixed by the general covariance [6,7], and they coincide with the corresponding electromagnetic formfactors at zero momentum transfer if $g = 2$. Due to analogous structure of the amplitudes (see (27)) and to the coincidence of the first coefficients in the formfactors expansion, the preferred values of formfactors coincide for electromagnetic and gravitational interactions: $f_{e,2n} = f_{1,2n}$ and $f_{m,2n} = f_{2,2n}$. In particular, it follows from (20) that

$$f_{1,2} = \frac{1 - \Lambda}{2m^2}. \quad (28)$$

Let us note that the Lagrangian corresponding to the interaction with $f_{1,2}$, can be rewritten in the generally covariant form [1], [8]:

$$L_{gm} = \frac{\kappa}{8m} R_{abcd} S^{ab} S^{cd}. \quad (29)$$

Here the dimensionless parameter $\kappa$ (the gravimagnetic ratio [1]) is related to $f_{1,2}$ as follows:

$$\kappa = \frac{2m^2 f_{1,2}}{1 - \Lambda}. \quad (30)$$

So, the preferred value of the parameter $\kappa$ is

$$\kappa = 1. \quad (31)$$

The preferred value of the gravimagnetic ratio $\kappa = 1$ was also derived in [1,8,9] and, on the basis of the supersymmetric sum rules, in [10].

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