The Einstein equations and multipole moments at null infinity

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Received 26 November 2021, revised 11 April 2022
Accepted for publication 28 April 2022
Published 17 May 2022

Abstract
We consider vacuum metrics admitting conformal compactification which is smooth up to the scrt \( \scr{I}^+ \). We write metric in the Bondi–Sachs form and expand it into power series in the inverse affine distance \( 1/r \). Like in the case of the luminosity distance, given the news tensor and initial data for a part of metric the Einstein equations define coefficients of the series in a recursive way. This is also true in the stationary case however now the news tensor vanishes and the role of initial data is taken by multipole moments which are equivalent to moments of Thorne. We find an approximate form of metric and show that in the case of vanishing mass the mass dipole may be different from zero. Then the known result about the Kerr like behaviour of a stationary metric is violated. Finally we find an approximate (up to the quadrupole moment) Bondi–Sachs form of the Kerr metric.

Keywords: multipole, moments, null, infinity

1. Introduction

In 1958 Trautman [24, 25] began an era of theoretical description of gravitational waves in the full nonlinearized Einstein theory. He defined outgoing radiation conditions and showed that total energy cannot increase in time what was interpreted as an effect of the radiation. In 1960 Bondi [5] presented his approach based on a foliation of spacetime by null surfaces \( u = \text{const.} \) imitating null cones in the Minkowski space. Complete description of the axially symmetric case was presented by Bondi, van der Burg and Metzner [6] and generalization to nonsymmetric metrics was given by Sachs [22]. In this formalism metric has a special form related to the null foliation. Metric coefficients are expanded into inverse powers of a radial
coordinate \( r \). The total energy at \( u = \text{const.} \) is defined as an integral of so called mass aspect which generalizes the mass parameter in the Schwarzschild metric. The energy diminishes in time in a rate given by the square of the Bondi news function defined by low order metric coefficients. This fact is interpreted as an effect of gravitational radiation.

In the Bondi–Sachs approach one uses an intuitive notion of the null infinity available in the limit \( r \to \infty \) when \( u \) stays bounded. Its geometrical definition was proposed by Penrose [20] who considered spacetimes with metric \((\hat{M}, \hat{g})\) admitting a conformal compactification to \((\hat{M}, r^{-2} \hat{g})\) with a boundary containing the future null infinity \( \mathcal{I}^+ \). Originally the conformal metric was assumed to be smooth up to \( \mathcal{I}^+ \). In 1983 Friedrich [13] and in 1985 Winicour [26] noticed that this assumption may be too strong. Anderson and Chrusciel [1] suggested the so called polyhomogeneous expansion of metric admitting logarithmic terms \( r^{-n} \log^k r \).

In 1995 Chruścier, MacCallum and Singleton [8] generalized the Bondi–Sachs formalism to polyhomogeneous expansions. Still the class of smooth conformal metrics is sufficiently big in many situations (see e.g. [2]).

In this paper we reexamine the vacuum Einstein equations for metrics \( \hat{g} \) admitting smooth \( \mathcal{I}^+ \). We put \( \hat{g} \) into the Bondi–Sachs form but, for geometrical reasons, we use the affine distance along null geodesics instead of the luminosity one. We expand the conformal metric and the Ricci tensor \( \hat{R}_{\mu\nu} \) into power series in \( 1/r \). The low order Einstein equations have the most interesting consequences as noticed by Bondi, Sachs and others (see [19] for a review in the luminosity gauge). We present a method of recursive solving of the equations summarized in theorem 2.1. This is not the existence theorem because to have this status one should prove convergence of the obtained series (note that for natural data for the considered situation existence theorems are yet unknown). Still the recursive solution can be useful for numerical computations (see [27] for known results and perspectives). In section 3 we consider stationary metrics. First we prove that these metrics undergo equations from section 2 (this is not a trivial observation since these equations are obtained for a specific choice of the conformal factor). From the low order equations we obtain restrictions which, for nonvanishing mass \( M \), allow to write \( \hat{g} \) as an approximate Kerr metric. Solutions of higher order equations are given up to multipole moments which are equivalent to those of Thorne [23] and Geroch [14] and Hansen [16]. The main results are summarized in theorem 3.1. As an example we find approximate Bondi–Sachs coordinates for the Kerr metric and we write this metric up to terms defined by quadrupole moments.

2. The Einstein equations near conformal boundary

In this section we reexamine the Einstein equations at null infinity in spirit of the Bondi–Sachs formalism combined with the Penrose conformal approach. The main difference between our results and those obtained by many authors (see [19] and references therein) is that our calculations are performed in the affine gauge. Theorem 2.1 should be considered not as a completely new result but rather as a way to systemize recursive solving of the Einstein equations under assumption that solution with smooth scri does exist.

Following the Penrose compactification method we assume that spacetime \( \hat{M} \) with metric \( \hat{g} \) can be partially compactified in a conformal way to \((M, g)\) with a future boundary \( \mathcal{I}^+ \) which can be foliated by surfaces diffeomorphic to the two-dimensional sphere, \( \mathcal{I}^+ = R \times S^2 \). We assume that metric \( g \) is smooth in a neighbourhood \( U \) of \( \mathcal{I}^+ \). In \( U \) we introduce a system of the Bondi–Sachs coordinates in the following way. First we define coordinates \( u, x^A \) (with \( A = 2, 3 \)) on \( \mathcal{I}^+ \) such that \( u = \text{const.} \) on leaves of the foliation and vector field \( \partial_u \) is orthogonal to the foliation. Now from each point \( p \in \mathcal{I}^+ \) we emit a null geodesic with the tangent vector \( v \) orthogonal to the foliation and such that \( g(v, \partial_u) = -1 \). We propagate coordinates \( u, x^A \) along
these geodesics and choose the fourth coordinate $\Omega$ to be the affine parameter along geodesics such that $\Omega = 0$ and $\partial_\Omega = v$ on $\mathcal{I}^+$. In the coordinates $x^0 = u, x^1 = \Omega$ and $x^A$ the compactified metric takes the form

$$g = du(g_{00}du - 2df + 2g_{0A}dx^A) + g_{AB}dx^Adx^B,$$  

(1)

where

$$\hat{g}_{0A} = 0$$  

(2)

(the hat denotes value on $\mathcal{I}^+$).

Physical metric $\tilde{g}$ is related to $g$ via a conformal factor $\Omega' = f\Omega$, where $f$ is a function nonvanishing and regular up to the boundary $\mathcal{I}^+$. In the coordinates $u, x^A$ and $r = \Omega^{-1}$ it is given by

$$\tilde{g} = du(\tilde{g}_{00}du + 2f^{-2}dr + 2\tilde{g}_{0A}dx^A) + \tilde{g}_{AB}dx^Adx^B,$$  

(3)

where $\tilde{g}_{00}, \tilde{g}_{AB}$ are of the order $r^2$ and $\tilde{g}_{00} = O(r)$. Taking coordinate $r' = \int f^{-2}dr$ instead of $r$ leads to elimination of $f$

$$\tilde{g} = du(\tilde{g}_{00}du + 2dr' + 2\tilde{g}_{0A}dx^A) + \tilde{g}_{AB}dx^Adx^B.$$

(4)

Now, as new compactified metric we take $r'^{-2}\tilde{g}$, which has the form (1) with $1/r'$ as the new coordinate $\Omega$. Thus, without a loss of generality we can assume that unphysical metric is (1) and the physical metric is given by

$$\tilde{g} = \Omega^{-2}g.$$  

(5)

It follows from (5) that the Einstein vacuum equations

$$\tilde{R}_{\mu\nu} = 0$$  

(6)

can be written in the form

$$R_{\mu\nu} - 2Y_{\mu\nu} - Yg_{\mu\nu} = 0,$$

(7)

where

$$Y_{\mu\nu} = -\frac{1}{\Omega} \Omega_{\mu\nu} + \frac{1}{2\Omega^2} \Omega_\alpha\Omega^\alpha g_{\mu\nu}, \quad Y = Y^\alpha_\alpha$$

(8)

and $\gamma_{\mu\nu}$ denotes the covariant derivative related to $g$. In the Bondi–Sachs coordinates nonvanishing components of $g^{\mu\nu}$ are given by

$$g^{01} = -1, \quad g^{11} = -g_{00} + g_{0A}g_0^A, \quad g^{1A} = g_0^A, \quad g^{AB},$$

(9)

where $g_0^A = g^{AB}g_{0B}$ and $g^{AB}$ is an inverse matrix to $g_{AB}$. Tensor $Y_{\mu\nu}$ and its trace take the form

$$Y_{\mu\nu} = \frac{1}{\Omega} \Gamma^1_{\mu\nu} + \frac{1}{2\Omega^2} g^{11}g_{\mu\nu},$$

(10)

and

$$Y = -\frac{1}{\Omega \sqrt{|g|}} (\sqrt{|g|} g^{1\alpha} \gamma^\alpha + 2 \Omega^2 g^{11}).$$

(11)
where $\Gamma$'s denote the Christoffel symbols and

$$ |g| = \det g_{AB}. \quad (12) $$

We will expand metric $g_{\mu\nu}$ into the Taylor series in $\Omega$ and study equation (7) in all orders $\Omega^k$. Even if they can be solved the resulting series does not have to be convergent. Nevertheless a cutoff of this series approximates a true solution of the Einstein equations. We begin with a rather mild assumption that $\bar{R}_{\mu\nu}$ is finite on the boundary $\mathcal{I}^+$. Then $Y_{\mu\nu}$ must be also finite. In order to avoid a second order pole at $\Omega = 0$ one has to assume the following expansion of $g_{00}$

$$ g_{00} = a\Omega + b\Omega^2 - 2M\Omega^3 + O(\Omega^4). \quad (13) $$

Now, components $Y_{1A}$ have no first order poles provided

$$ g_{0A} = q_A\Omega^2 + 2L_A\Omega^3 + O(\Omega^4) \quad (14) $$

and regularity of $Y_{AB}$ is equivalent to

$$ \hat{g}_{AB} = a\hat{g}_{AB}. \quad (15) $$

All remaining components of $Y_{\mu\nu}$ are nonsingular at $\Omega = 0$ under conditions (13)–(15).

Equation (15) implies that $\hat{g}_{AB}$ is proportional to an $u$-independent two-dimensional metric which, thanks to the uniformization theorem and freedom of transformation of coordinates $x^A$, is proportional to the standard metric $s_{AB}$ of the two-dimensional sphere $S^2$. Thus,

$$ \hat{g}_{AB} = -\gamma^2 s_{AB}, \quad a = 2(\ln \gamma)_0, \quad \gamma > 0, \quad (16) $$

where $\gamma$ is a function of all variables. Now, we can choose new coordinates $\Omega'$ and $u'$ such that

$$ \frac{\Omega'}{\Omega} = \gamma, \quad u' = \int \gamma du. \quad (17) $$

This transformation leads to $\gamma' = 1$ and $a' = 0$. Hence, we can assume without loss of generality that

$$ g_{00} = b\Omega^2 - 2M\Omega^3 + O(\Omega^4) \quad (18) $$

and

$$ g_{AB} = -s_{AB} + n_{AB}\Omega + p_{AB}\Omega^2 + O(\Omega^3). \quad (19) $$

Still metric $s_{AB}$ is defined up to the conformal group of the sphere, which together with ‘supertranslations’ of $u$ form the Bondi–Metzner–Sachs group of asymptotic symmetries. These transformations can be also combined with a shift of $r$

$$ r' = r + h(u, x^A) + O(\Omega) \quad (20) $$

which can be used e. g. to obtain

$$ n = 0, \quad (21) $$

where

$$ n = s^{AB}n_{AB}. \quad (22) $$
Before we start a more advanced analysis of the Einstein equations we will reduce their number by means of the Bianchi identity

$$\tilde{\nabla}_\mu \tilde{G}^{\mu} = 0$$  \hspace{1cm} (23)

(we do it in a slightly different way than that of Bondi and Sachs, see [26] for a comparison). To this end we write the identity in the form

$$2\Omega^2 (\sqrt{|g|} \Omega^{-4} \tilde{G}^\alpha)_{,\alpha} + \sqrt{|g|} g^{\alpha\beta} \tilde{R}_{\alpha\beta} + (\sqrt{|g|} \Omega^{-2})_{,\rho} \tilde{R} = 0.$$  \hspace{1cm} (24)

Note that $$\alpha, \beta \neq 0$$ in the middle term and

$$\Omega^{-2} \tilde{R} = -2\tilde{R}_{01} + g^{11} \tilde{R}_{11} + 2g^{1A} \tilde{R}_{1A} + g^{AB} \tilde{R}_{AB},$$

$$\tilde{G}_{01} = \frac{1}{2} (g^{11} \tilde{R}_{11} + 2g^{1A} \tilde{R}_{1A} + g^{AB} \tilde{R}_{AB})$$  \hspace{1cm} (25)

$$\tilde{G}_{11} = \tilde{R}_{11}, \quad \tilde{G}_{1A} = \tilde{R}_{1A}.$$  

In equation (24) with $$\nu = 1$$ function $$\tilde{R}_{01}$$ appears only in the last term. One obtains

$$\tilde{R}_{01} = \tilde{G}_{01} + f \left( (\sqrt{|g|} \Omega^{-4} \tilde{G}^\alpha)_{,\alpha} + \frac{1}{2} \Omega^{-2} (\sqrt{|g|} g^{\alpha\beta} \tilde{R}_{\alpha\beta}) \right)$$  \hspace{1cm} (26)

provided that

$$f = ((\sqrt{|g|} \Omega^{-2})_{,\Omega})^{-1}$$  \hspace{1cm} (27)

is well defined. In a neighbourhood of the scri there is

$$f = -\frac{\Omega^3}{2 \sqrt{|g|}} (1 + O(\Omega))$$  \hspace{1cm} (28)

and from (26) it follows that fulfillment of equations

$$\tilde{R}_{11}^{(l)} = \tilde{R}_{1A}^{(l)} = \tilde{R}_{AB}^{(l)} = 0, \quad l \leq k$$  \hspace{1cm} (29)

guarantees $$\tilde{R}_{01}^{(k)} = 0$$, where (k) denotes the kth coefficient in the Taylor expansion in $$\Omega$$. If $$\nu = A$$ functions $$\tilde{R}_{01}$$ appear in (24) only in the expression $$-2\Omega^2 (\sqrt{|g|} \Omega^{-2} \tilde{R}_{0A})_{,1}$$ which vanishes if (29) is satisfied. Hence $$\tilde{R}_{0A}^{(k)} = 0$$ for all k except $$k = 2$$. We obtain a similar result taking $$\nu = 0$$. Thus, in order to solve the Einstein equations up to the order $$k \geq 2$$ it is sufficient to consider (29) and

$$\tilde{R}_{00}^{(2)} = \tilde{R}_{01}^{(2)} = 0.$$  \hspace{1cm} (30)

The simplest one from this reduced set of equations is $$\tilde{R}_{11} = 0$$. It reads

$$-\frac{1}{2} (\ln |g|)_{,11} + \frac{1}{4} g_{AB,1} g_{AB}^{AB} = 0.$$  \hspace{1cm} (31)

In the order $$k \geq 1$$ it takes the form

$$\frac{1}{2} (k + 1)(k + 2) s^{AB}_{g_{AB}} s^{(k+2)}_{AB} + \frac{1}{2} (k + 1)^2 s^{AB}_{g_{AB}} = \langle s_{AB}^{(l)}, l \leq k \rangle, \quad k \geq 1,$$  \hspace{1cm} (32)
where indices $A, B$ in $n_{AB}$ are raised by means of $s_{AB}$ and $\langle \ldots \rangle$ denotes an expression depending on variables in the bracket. For $k = 0$ equation (31) yields

$$p = -\frac{1}{4} n_{AB} n^{AB},$$

(33)

where $p = p_A^A$. Thus, for all values $k \geq 0$ one obtains

$$s_{AB} g^{(k+2)}_{AB} = \langle s_{AB}^{(l)} \rangle, \quad k \geq 0.$$  

(34)

Equations $\bar{R}_{1A} = 0$ in the order $k = 0$ define

$$q_{A} = \frac{1}{2} n_{A}^{B} - \frac{1}{2} n_{B},$$

(35)

where symbol $|A|$ denotes the covariant derivative with respect to $s_{AB}$. Taking into account (31) for $k \geq 1$ one obtains

$$\bar{R}^{(k)}_{1A} - \frac{1}{k} \bar{R}^{(k-1)}_{11A} = -\frac{1}{2} (k-1)(k+2) s^{(k+2)}_{10A} - \frac{1}{2} (k-1) n_{A}^{B} s^{(k+1)}_{0B}$$

(36)

$$\frac{n}{4} (k+1) s^{(k+1)}_{0A} - \frac{1}{2} (k+1) (s^{(k+1)}_{AB} + s^{(l)}_{AB}, l \leq k) = 0, \quad k \geq 1.$$

For $k \neq 1$ it follows from (35) and (36) that

$$g^{(k+2)}_{0A} = \langle s^{(l)}_{AB}, l \leq k+1 \rangle, \quad k \geq 0, \quad k \neq 1.$$  

(37)

Thanks to (33), (35) and the following identity in dimension 2

$$n_{AC} n^{C}_{B} = n n_{AB} + \frac{1}{2} (n_{CD} n^{CD} - n^2) s_{AB}$$

(38)

equation (36) with $k = 1$ reads

$$\left( p^{A} - \frac{1}{4} n n^{A}_{A} \right)_{B} + \frac{1}{8} (n_{B}^{CD} n^{CD} - n^2)_{A} = 0.$$  

(39)

Hence

$$p_{AB} = \frac{1}{8} (n^2 - n_{AB} n^{AB}) s_{AB} - \frac{1}{4} n n_{AB} + \bar{p}_{AB},$$

(40)

where $\bar{p}_{AB}$ is a symmetric TT-tensor

$$\bar{p}^{A}_{A} = 0, \quad \bar{p}^{B}_{A|B} = 0$$

(41)

on the sphere. In terms of the complex stereographic coordinates $\zeta, \bar{\zeta}$ solutions of (41) are given by $\bar{p}_{AB}^B A d x^A d x^B = \text{Re}(h(\zeta) d \zeta^2)$, where $h$ is a holomorphic function. The only regular solution is $\bar{p}_{AB} = 0$. Due to this one obtains

$$p_{AB} = \frac{1}{8} (n^2 - n_{AB} n^{AB}) s_{AB} - \frac{1}{4} n n_{AB}.$$  

(42)
In order to analyse the remaining Einstein equations we need a more explicit form of (11). For $k \geq 1$ one obtains
\[ \gamma^{(k)} + \frac{1}{k(k+1)} \tilde{R}^{(k-1)}_{11,0} = k g^{(k+2)}_{00} + \frac{n}{2} g^{(k+1)}_{00} + (g^{(k+1)}_{\mu\nu}, l \leq k), \quad k \geq 1. \] (43)

Equation $\tilde{R}^{(k)}_{AB} = 0$ with $k \geq 2$ can be splitted into its trace (with respect to $s_{AB}$) and a traceless part. The trace part
\[ s^{AB} \tilde{R}^{(k)}_{AB} = 0, \quad k \geq 2 \] (44)
allows to obtain $g^{(k+2)}_{00}$ in terms of lower order coefficients
\[ g^{(k+2)}_{00} = \langle g^{(l)}_{\mu\nu}, l \leq k+1 \rangle, \quad k \geq 2. \] (45)

The traceless part is equivalent to the equation
\[ \tilde{R}^{(k)}_{AB} + \left(-\frac{1}{2} s^{CD} \tilde{R}^{(k)}_{CD} + \frac{1}{k+1} \tilde{R}^{(k-1)}_{11,0} \right) s_{AB} = 0, \quad k \geq 2 \] (46)
which yields a simple differential condition for $g^{(k+1)}_{AB}$
\[ g^{(k+1)}_{AB,0} = \langle g^{(l)}_{0\mu}, g^{(l)}_{\mu0}, l \leq k \rangle, \quad k \geq 2. \] (47)

Now equation $\tilde{R}^{(k-1)}_{11} = 0$ plays a role of a constraint which is preserved by (47). Otherwise speaking, equation $\tilde{R}^{(k-1)}_{11} = 0$ defines the trace of $g^{(k+1)}_{AB}$ with respect to $s_{AB}$, whereas (47) is an equation for the traceless part of $g^{(k+1)}_{AB}$.

In order to analyse equations $\tilde{R}_{AB} = 0$ in the order $k = 0, 1$ we need the following expansion of $\det(g_{AB})$
\[ |g| = |\hat{g}| \left(1 - n\Omega + \left(-p - \frac{1}{2} n^2 - \frac{1}{2} n_{AB}^{(A)} \Omega^2 \right) + O(\Omega^2) \right). \] (48)

Equation $\tilde{R}^{(0)}_{00} = 0$ reads
\[ \hat{R}_{00} - \left( b + \frac{1}{2} n_{00} \right) s_{00} = 0, \] (49)

hence
\[ b = 1 - \frac{1}{2} n_{00}. \] (50)

Using the standard identity in two dimensions
\[ \hat{R}'_{AB} = \frac{1}{2} R'_{g_{AB}}, \] (51)
where $R'_{AB}$ is the Ricci tensor of $g_{AB}$, shows that equation $\tilde{R}^{(1)}_{AB} = 0$ does not carry any new information (it coincides with the $u$-derivative of (42)).

The last equations to consider are $\tilde{R}^{(2)}_{00} = 0$ and $\tilde{R}^{(2)}_{11} = 0$. The first one takes the form
\[ M_{0} = \langle n_{AB} \rangle. \] (52)
This equation is responsible for diminishing of the gravitational energy if time \( u \) increases. Equation \( \tilde{R}^{(2)}_{0A} = 0 \) yields

\[
L_{A,0} = -\frac{1}{3}M_A + \langle n_{AB} \rangle. \tag{53}
\]

We summarize consequences of the vacuum Einstein equations in the affine gauge in the following theorem.

**Theorem 2.1.** Vacuum metric with smooth scri \( \mathcal{I}^+ \) can be transformed to the form

\[
\tilde{g} = du(\tilde{g}_{00} du + 2dr + \tilde{g}_{0A} dx^A) + \tilde{g}_{AB} dx^A dx^B, \tag{54}
\]

\[
\tilde{g}_{00} = 1 - \frac{1}{2}n_{00} - \frac{2M}{r} + \sum_{k=2}^{\infty} \delta_{00}^{(k+2)} r^{-k}, \tag{55}
\]

\[
\tilde{g}_{0A} = q_A + \frac{2L_A}{r} + \sum_{k=2}^{\infty} \delta_{0A}^{(k+2)} r^{-k}, \tag{56}
\]

\[
\tilde{g}_{AB} = -r^2 s_{AB} + r n_{AB} + p_{AB} + \sum_{k=2}^{\infty} \delta_{AB}^{(k+2)} r^{-k} \tag{57}
\]

with coefficients defined recursively in the following steps:

- Tensor \( n_{AB} \) can be arbitrary up to a gauge condition e.g. \( n = 0 \) (and unknown convergence conditions). It defines \( q_A \) and \( p_{AB} \) according to (35) and (42).
- Coefficients \( M \) and \( L_A \) are defined in quadratures by equations (52) and (53), respectively.
- Trace \( s^A_B \tilde{g}^{(3)}_{AB} \) is given by (34) with \( k = 1 \) and the traceless part of \( g^{(3)}_{AB} \) is defined in quadratures by equation (47) with \( k = 2 \).
- For \( l \geq 4 \) components \( \tilde{g}^{(l)}_{00}, \tilde{g}^{(l)}_{0A} \) and \( s^A_B \tilde{g}^{(l)}_{AB} \) follow directly from equations (34), (37) and (44), respectively. Then the traceless part of \( s^{(l)}_{AB} \) is defined in quadratures by equation (47).

This analysis of the Einstein equations is equivalent to that in the luminosity gauge (see [19]). The traceless part of \( n_{AB,0} \) corresponds to the Bondi news function. Free data consist of 2 arbitrary functions on the boundary (the traceless part of \( n_{AB} \)) and initial values of \( M, L_A \) and the traceless parts of \( g^{(l)}_{AB} \) with \( l \geq 3 \) on a section \( u = u_0 \) of the scri. The latter fields are coefficients of an expansion of the traceless part of \( \tilde{g}_{AB} \) restricted to the three-dimensional null surface \( u = u_0 \) approaching the scri. Unfortunately, we are not able to find conditions which guarantee convergence of series describing metric.

Above free data are given on the outgoing null surface and on the part of the null scri in the future of the surface. This combination differs from that assumed in mathematically sophisticated existence theorems of Kannar [17], Chrusciel and Paetz [9] and others, where evolution of data is considered in the future of two intersecting null surfaces (method of Rendall [21]) or in the future of a lightcone (method of Dossa [11]). All these theorems are based on the Friedrich formulation of the conformal Einstein equations [12]. They assume free data for functions different from those in the Bondi–Sachs formulation. For above reasons they are not very helpful in solving the existence problem in our case.

In order to obtain the total energy at \( u = \text{const.} \) one should pass from the affine gauge to the luminosity one. It means that we should replace coordinate \( r \) by

\[
r_\mathcal{B} = \left( \frac{\det \tilde{g}_{AB}}{\det s_{AB}} \right)^{\frac{1}{2}}. \tag{58}
\]
Practically it is sufficient to consider an approximate formula
\[
\rho_B = r - \frac{1}{4} n - \frac{1}{8} r n_{AB} n^{AB} + O(r^{-2}),
\]
(59)
which leads to the Bondi mass aspect
\[
M_B = M - \frac{1}{16} \left( n_{AB} n^{AB} - \frac{1}{2} r^2 \right),
\]
(60)
The total energy is given by
\[
E(u) = \frac{1}{4\pi} \int_{S_0} M_B d\sigma.
\]
(61)
Equation (52) assures that \( E_{,0} \leq 0 \) what is interpreted as a loss of energy due to emission of gravitational waves.

3. Stationary metrics

If metric admits the smooth null scri and a timelike Killing vector \( K \) then there are coordinates in which metric takes the form (54) with \( u \)-independent coefficients and \( K = \partial_u \). In order to show this let us first observe that any timelike vector must be null on the scri, so the Killing vector \( K \) coincides with the null generator \( \partial_u \) on the scri. Let us fix a null surface \( \Sigma_0 \) intersecting \( \mathcal{I}^+ \) along a spherical surface \( S_0 \). We endow \( \Sigma_0 \) into the following coordinates: \( \Omega \) (affine distance from \( S_0 \) along null geodesics forming the surface) and \( x^A \) (spherical coordinates transported from \( S_0 \) along the geodesics). Using the one-parameter group of motion \( \phi_u \) related to \( K \) we can generate from \( \Sigma_0 \) foliation of a neighbourhood \( U \) by surfaces \( u = \text{const} \).

If we write the physical metric in coordinates \( (u, r = 1/\Omega, x^A) \) and transform \( r \) appropriately we obtain (54) with coefficients independent of \( u \). The Killing field is given by \( K = \partial_u \) and we can choose the conformal factor to be \( \Omega = 1/r \). The unphysical metric is given by (1) and (2) with \( u \)-independent coefficients.

Let us consider low order Einstein equation (7) in the stationary case. The regularity of the physical Ricci tensor \( \hat{R}_{AB} \) on \( \mathcal{I}^+ \) (see equations (13)–(15)) implies
\[
g_{00} = b\Omega^2 - 2M\Omega^3 + O(\Omega^4)
\]
(62)
and
\[
g_{0A} = qA\Omega^2 + 2LA\Omega^3 + O(\Omega^4)\]
(63)
In order to obtain \( b = 1 \) we consider equation \( \hat{R}^{(2)}_{00} = 0 \). It takes the form
\[
\hat{\Delta} b = 0,
\]
(64)
where \( \hat{\Delta} \) is the Laplace operator related to the metric \( \hat{g}_{AB} \) on the sphere. Multiplying (64) by \( b \) and integrating over the sphere shows that \( b = \text{const} \). A rescaling of \( u \) and \( r \) allows to obtain
\[
b = 1.
\]
(65)
Then equation \( \hat{R}^{(0)}_{AB} = 0 \) (see (49)) yields \( \hat{R} = -2 \), hence
\[
\hat{g}_{AB} = -s_{AB}.
\]
(66)
Due to (65) and (66) metric $\tilde{g}$ is asymptotically Minkowskian in coordinates adapted to the Killing field $K$. The remaining gauge freedom consists of transformations of coordinates $x^A$ preserving $s_{AB}$ (rotations of the sphere) and supertranslations $u' = u + f(x^A)$ combined with a shift of the radial coordinate $r' = r + h(x^A)$. For a later convenience we write the latter two transformations in the linear approximation in $\Omega$

$$u' = u + f - \frac{1}{2} f^A f_A \Omega, \quad r' = r + h - f^A \left( \frac{1}{2} f - h \right) \Omega, \quad x'^A = x^A - f^A \Omega. \quad (67)$$

They induce the following transformation of $n_{AB}$

$$n_{AB}' = n_{AB} + 2 f_{[AB]} + 2 h s_{AB}. \quad (68)$$

Let us continue our analysis of the low order equations from the system (29) and (30). Equation $\tilde{R}_{BA}^{(2)} = 0$ reads

$$\frac{1}{3} M_A = - q_B[A] B. \quad (69)$$

It can be written in the form

$$\frac{1}{3} dM = -^{*} d\alpha, \quad (70)$$

where

$$\alpha = \eta^{AB} q_{A[B]} . \quad (71)$$

$\eta_{AB}$ is the Levi-Civita tensor and star denotes the Hodge dual on the sphere. Since a differential form which is simultaneously exact and coexact on $S_2$ must vanish one obtains

$$M = \text{const.} \quad (72)$$

and

$$\alpha = \text{const.} \quad (73)$$

Definition of $\alpha$ is equivalent to

$$d(q_A d^A) = -2 \alpha \eta, \quad (74)$$

where $\eta$ is the volume form corresponding to $s_{AB}$. Due to (73) integration of (74) over the sphere implies

$$\alpha = 0. \quad (75)$$

Hence

$$q_A = q_A. \quad (76)$$

where $q$ is a function. Summarizing this part, equation $\tilde{R}_{BA}^{(2)} = 0$ is equivalent to (72) and (76).

Consider now equation (35) in the gauge $n = 0$

$$q_A = \frac{1}{2} n^A_{A[B]} . \quad (77)$$
In order to find consequences of (77) we prove the following lemma.

**Lemma 3.1.** Every traceless tensor $n_{AB}$ on $S_2$ admits functions $F, H$ such that

$$n_{AB} = F_{[AB} - \frac{1}{2} \Delta F_{AB} + \eta^C_{[A} H_{B]C},$$

(78)

where $\Delta$ is the standard Laplace operator on the sphere. Functions $F$ and $H$ are defined up to $c^m Y_{lm} + c$, where $c^m$ and $c$ are constants and $Y_{lm}$ are spherical harmonics.

**Proof.** Every tensor $n_{AB}$ defines the following form $\omega$ on $S_2$

$$\omega = n^{B}_{A[B} d\alpha^A,$$

(79)

which can be decomposed into an exact and coexact form

$$n^{B}_{A[B} = \tilde{F}_{A} + \eta^C_{A} \tilde{H}_{C}.$$  

(80)

We will show that given $\tilde{F}$ and $\tilde{H}$ there is solution $n_{AB}'$ of (80) of the form (78).

If $\tilde{H}$ = const. we assume

$$n_{AB}' = F_{[AB} - \frac{1}{2} \Delta F_{AB},$$

(81)

Then equation (80) yields

$$(\Delta + 2) F = 2 \tilde{F} + 2c.$$ 

(82)

Expanding $F$ and $\tilde{F}$ into the spherical harmonics $Y_{lm}$ shows that only the dipole part of $\tilde{F}$ has no counterimage in $F$. However, a direct analysis of equation (80) with $\tilde{F} = c^m Y_{lm} \neq 0$ and $\tilde{H}$ = const. shows that regular solutions $n_{AB}$ are not admitted in this case. It means that $\tilde{F}$ cannot contain harmonics $Y_{lm}$ and solution $F$ of (82) always exists.

If $\tilde{F}$ = const. equation (80) can be written in the form

$$^{*}n^{B}_{A[B} = \tilde{H}_{A},$$

(83)

where $^{*}n_{AB} = \eta^C_{A} n_{CB}$ is also traceless and symmetric. Equation (83) is satisfied by

$$^{*}n_{AB}' = H_{[AB} - \frac{1}{2} \Delta H_{AB}$$

(84)

with $H$ satisfying

$$(\Delta + 2) H = 2 \tilde{H} + 2c.$$ 

(85)

It follows from (84) that $n_{AB}' = \eta^C_{A} H_{[B]C}$. Thus, functions $\tilde{F}$ and $\tilde{H}$ defined by $n_{AB}$ can be also obtained from $n_{AB}'$ of the form (78). Tensor $n_{AB}' - n_{AB}$ is a (trivial) TT-tensor on $S_2$, so $n_{AB}' = n_{AB}$. Note that solutions $F$ and $H$ of equations (82) and (85) are defined up to $c^m Y_{lm} + c$. □

Let us introduce the following symmetric, traceless operators on the sphere

$$\nabla_{AB} = \nabla_{A} \nabla_{B} - \frac{1}{2} \nabla_{AB} \Delta, \quad ^{*}\nabla_{AB} = \eta^C_{A} \nabla_{CB}.$$ 

(86)
Due to lemma 3.1 every symmetric traceless tensor on $S^2$ can be decomposed into linear combination of

$$\nabla_{AB} Y_{lm}, \quad \ast \nabla_{AB} Y_{lm}. \tag{87}$$

Tensors (87) can be called tensor harmonics (compare with those in [23] and references therein). In the same spirit every vector field on the sphere can be decomposed into

$$\nabla_A Y_{lm}, \quad \ast \nabla_A Y_{lm} \tag{88}$$

which can be called vector harmonics.

Now we return to the Einstein equations. It follows from (77) and the proof of lemma 3.1 that $n_{AB}$ has form (81). Such $n_{AB}$ can be gauged away by means of transformation (67) with $f = -\frac{1}{2} F$ and $h = \frac{1}{4} \Delta F$. Thus, there are coordinates adapted to the symmetry $\partial_u$ such that

$$n_{AB} = 0. \tag{89}$$

Now, it follows from (42), (77) and (89) that

$$p_{AB} = 0, \quad q_A = 0. \tag{90}$$

Moreover, lemma 3.1 shows that transformations (67) reduce to the case

$$f = e^\theta Y_1 + c, \quad h = e^\theta Y_1. \tag{91}$$

Note that these residual transformations play a role of the translation subgroup of the BMS group. Together with the group of rotations preserving $s_{AB}$ it forms the group of Euclidean motions of $R^3$ completed by time translations (constant $c$).

Equations $\tilde{R}^{(k)}_{AB} = 0$ with $k = 0, 1$ are already exploited. For $k \geq 2$ in the nonstationary case they were used to define $s^{(k+2)}_{00}$ and $s^{(k+1)}_{AB,0}$. Since now $s^{(k+1)}_{AB,0} = 0$ let us write these equations in more detail

$$\left[ (k - 1)s^{(k+2)}_{00} + (s^{(k+1)}_{0c})^C - \frac{1}{2} R^{(k)} \right] s_{AB} = (k - 1)s^{(k+1)}_{0(A|B)} \tag{92}$$

$$+ \frac{1}{2} k(k - 1)s^{(k)}_{AB} = \langle s^{(l)}_{0\nu}, s^{(l+1)}_{AB}, l \leq k \rangle, \quad k \geq 2.$$

For $k = 2$ the trace of (92) yields

$$s^{(4)}_{00} = -L_C^{|C}, \tag{93}$$

where $L_C = \frac{1}{2} s^{(3)}_{0c}$, and the traceless part is

$$L_{(A|B)} = \alpha s_{AB}, \tag{94}$$

where $\alpha$ is a function. It follows from (94) that $L_A \partial_A$ is the conformal Killing field of the spherical metric. In terms of the position vector on the sphere $r \in S^2 \subset R^3$ the general smooth solution of (94) is

$$L_A dx^A = f (r \times dr) + Ddr, \tag{95}$$
where $\vec{J}$ and $\vec{D}$ are constant vectors. Function $D = D\hat{r}$ is a composition of harmonics $Y_{lm}$ so there is a chance to gauge it away by means of a transformation given by (67) and (91). This transformation induces the following change

$$L'_A = L_A + M f_A.$$  

(96)

Hence, for $M \neq 0$ we can eliminate $D$ by taking $f = -D\hat{r}/M$. Then we can use the rotation freedom to direct $\vec{J}$ along $z$-axis. Thus, in the adapted spherical coordinates $\theta$ and $\varphi$, for $M \neq 0$ one obtains

$$L_A d\xi^A = J \sin^2 \theta d\varphi$$  

(97)

An inspection of the Kerr metric shows that constant $J$ is the total angular momentum.

In the case $M = 0$ we can use only the rotation group to reduce the number of free parameters in $\vec{J}, \vec{D}$ to 3. For instance one can obtain

$$L_A d\xi^A = J \sin^2 \theta d\varphi + dD$$  

(98)

where

$$D = D_1 \sin \theta \cos \varphi + D_2 \sin \theta \sin \varphi + D_3 \cos \theta$$  

(99)

and either $D_1 = 0$ or $D_2 = 0$. Here again $J$ is the total angular momentum (see a discussion after prolongation (123)–(126) of metric to the spacelike infinity).

From (97) and (98) one obtains

$$L'_C = 0 \quad \text{if} \quad M \neq 0$$  

(100)

$$L'_C = -2D \quad \text{if} \quad M = 0.$$  

(101)

At this stage the physical metric is given by (54) with components

$$\tilde{g}_{00} = 1 - \frac{2M}{r} + \frac{2D}{r^2} + O \left( \frac{1}{r^3} \right), \quad M = \text{const.},$$  

(102)

$$\tilde{g}_{AB} = -r^2 \tilde{s}_{AB} + O \left( \frac{1}{r} \right)$$  

(103)

and

$$\tilde{g}_{0A} d\xi^A = \frac{2J}{r} \sin^2 \theta d\varphi + O \left( \frac{1}{r^2} \right), \quad D = 0 \quad \text{if} \quad M \neq 0$$  

(104)

or

$$\tilde{g}_{0A} d\xi^A = \frac{1}{r}(2J \sin^2 \theta d\varphi + 2dD), \quad \text{if} \quad M = 0$$  

(105)

with $D$ given by (99).

Let us consider now equation (92) with $k \geq 3$. Its trace defines $g^{(k+2)}_{00}$

$$g^{(k+2)}_{00} = \frac{1}{(k-2)(k+1)} \tilde{s}^{(k)}_{CD} \langle \tilde{s}^{(k)}_{0D}, \tilde{s}^{(l)}_{0C}, l \leq k-1 \rangle, \quad k \geq 3$$  

(106)
and the traceless part reads

\[ -\tilde{g}^{(k+1)}_{00(A)B} + \frac{1}{2}g^{(k+1)}_{0C}S_{AB} + \frac{1}{2}k\tilde{g}^{(k)}_{AB} = \langle \tilde{g}^{(k)}_{0\mu}, \tilde{g}^{(k)}_{\mu}, 1 \leq k \leq 1 \rangle, \]  

(107)

where the tilde denotes the traceless part of a tensor,

\[ \tilde{g}^{(k)}_{AB} = g^{(k)}_{AB} - \frac{1}{2}g^{0C}S_{CD}g^{(k)}_{AB}. \]  

(108)

Still coefficients \( g^{(k+1)}_{0A} \) can be expressed in terms of \( g^{(k)}_{AB} \) due to (36)

\[ g^{(k+1)}_{0A} = -\frac{k}{(k-2)(k+1)}g^{(k)}_{AC} + \langle g^{(k)}_{\mu0}, I \leq k - 1 \rangle, \quad k \geq 3. \]  

(109)

Substituting (109) into (107) and eliminating \( g^{(k)}_{0\mu} \) via (109) and (45) and trace of \( g^{(k)}_{AB} \) via (44) yields

\[ -\tilde{g}^{(k)}_{CA} (B) + \frac{1}{2}g^{(k)}_{CD} S_{AB} - \frac{1}{2}(k-2)(k+1)\tilde{g}^{(k)}_{AB} = \langle g^{(k)}_{\mu\nu}, I \leq k - 1 \rangle. \]  

(110)

Equation (110) can be written in a simpler form

\[ (\Delta + k^2 - k - 4)\tilde{g}^{(k)}_{AB} = \langle g^{(k)}_{\mu\nu}, I \leq k - 1 \rangle \]  

(111)

due to an identity following from (52)

\[ -\tilde{g}^{(k)}_{CA} (B) + \frac{1}{2}g^{(k)}_{CD} S_{AB} + \frac{1}{2}g^{(k)}_{ABC} (C) - \tilde{g}^{(k)}_{AB} = \langle g^{(k)}_{\mu\nu}, I \leq k - 1 \rangle, \]  

(112)

however (110) is better for further proceeding. Note that \( \Delta \) in (111) is the covariant Laplace operator on the sphere and it mixes indices in \( \tilde{g}^{(k)}_{AB} \).

Using lemma 3.1 let us represent tensor \( \tilde{g}^{(k+1)}_{AB} \) by scalar potentials \( \tilde{Q}^{(k)} \) and \( \tilde{P}^{(k)} \) such that

\[ \tilde{g}^{(k+1)}_{AB} = \nabla_{AB}\tilde{Q}^{(k)} + *\nabla_{AB}\tilde{P}^{(k)}. \]  

(113)

Equation (110) splits into two equations

\[ (\Delta + k(k+1))\tilde{Q}^{(k)} = -2\tilde{Q}^{(k)} \]  

(114)

\[ (\Delta + k(k+1))\tilde{P}^{(k)} = -2\tilde{P}^{(k)}, \]  

(115)

where \( \tilde{Q}^{(k)} \) and \( \tilde{P}^{(k)} \) are potentials related to the rhs of (110). A lengthy analysis of an explicit form of equation (110) shows that \( \tilde{Q}^{(k)} \) and \( \tilde{P}^{(k)} \) do not contain \( Y_{lm} \) with \( I \geq k \). Hence, solutions \( \tilde{Q}^{(k)} \) and \( \tilde{P}^{(k)} \) exist and are given up to \( e^m Y_{lm} \). These new parameters \( e^m \) are then implemented in \( \tilde{g}^{(k+2)}_{0A} \) and \( \tilde{g}^{(k+3)}_{00} \) via (109) and (117).

Below we summarize results of this section in terms of the physical metric \( \tilde{g} \). Note that the covariant derivatives, the Laplace operator \( \Delta \), the Levi-Civita tensor \( \eta_{AB} \) and the Hodge dual * are related to the spherical metric \( S_{AB} \).

**Theorem 3.1.** Every stationary vacuum metric with a smooth conformal boundary \( \mathcal{I}^+ = R \times S^2 \) can be transformed to the following form in a neighbourhood of \( \mathcal{I}^+ \)

\[ \tilde{g} = du(\tilde{g}_{00}du + 2dr + 2\tilde{g}_{0A}(dx^A)) + \tilde{g}_{AB}dx^Adx^B, \]  

(116)
\[
\tilde{g}_{00} = 1 - \frac{2M}{r} + \frac{2D}{r^2} + \sum_{k=2}^{\infty} \frac{k(k+1)}{r^{k+1}}(Q^k + 1.0),
\]
(117)

\[
\tilde{g}_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{r^2} (2J \sin^2 \theta d\varphi + 2dD) + \sum_{k=2}^{\infty} \frac{k+1}{r^k} [d(Q^k + 1.0) + d(P^k + 1.0)],
\]
(118)

\[
\tilde{g}_{AB} = -r^2 s_{AB} + \sum_{k=2}^{\infty} \frac{1}{r^{k+1}} [\nabla_{AB}(Q^k + 1.0) + \nabla_{AB}(P^k + 1.0) + (1.0)s_{AB}],
\]
(119)

\[
Q^k = \sum_{m=-k}^{m=k} Q_{lm} Y_{lm}, \quad P^k = \sum_{m=-k}^{m=k} P_{lm} Y_{lm}.
\]
(120)

and 1.o. (lower order) denotes terms spanned by \(Y_{lm}\) with \(l < k\) and depending on \(M, a, Q^m\) and \(P^m\) with \(l < k\) (no such terms for \(k = 2\)). If \(M = 0\) then \(D\) is given by (99), otherwise \(D = 0\).

**Remark.** The assumption of smoothness of the scri in theorem 3.1 may be replaced by the assumption that \(\Omega, \tilde{g}(K)\) and \(g\) are of the class \(C^2\) in a neighbourhood \(U\) of \(\mathcal{I}^+\). Indeed, under these assumptions all considerations in this section up to equations (102)–(105) are still valid. We can introduce coordinate \(t\) via \(u = t - r\) and continue metric analytically in \(t\) to all values of \(t\). For large values of \(r\) surfaces \(t = \text{const.}\) are spacelike and metric \(\tilde{g}\) is stationary and asymptotically flat at spacelike infinity. A generalization of results of Beig and Simon [3] by Kundu [18] assures analyticity of the Ernst potential and compactified three-dimensional metric in harmonic coordinates if \(M \neq 0\). Then one can introduce spherical coordinates based on the normal coordinates at point at infinity. Arguments of Damour and Schmidt (see appendix in [10]) show that four-dimensional spacetime metric \(\tilde{g}\) should admit an analytic compactification up to the scri \(\mathcal{I}^+\). Knowing that \(\Omega\) and \(g\) are analytic we can construct analytic foliation of \(U\) and the Bondi–Sachs coordinates along lines in the beginning of this section. Then all components of \(g\) should be analytic functions.

Constants \(Q^m\) and \(P^m\) are multipole moments of metric. The easiest way to find \(Q^m\) is to integrate (117) with spherical harmonics. Moments \(P^m\) arise if expression (118) is integrated with \(dY_{lm}\). For each \(k \geq 2\) up to \(4k + 2\) multipole moments are admitted. They are restricted by yet unknown convergence conditions. In the axially symmetric case only 2 parameters for each \(k\) can appear \((Q^k\) and \(P^k)\).

In order to identify \(Q^m\) and \(P^m\) with multipole moments introduced by Thorne [23] let us replace coordinate \(u\) by \(t\) defined by

\[
u = t - \int \frac{dr}{\tilde{g}_{00}}.
\]
(121)

Note that \(t\) is given up to a function on the sphere. For some choice of this function expansion of (121) takes the form

\[
u = t - r - 2M \ln \left( \frac{r}{2M} - 1 \right) - \frac{2D}{r} - \sum_{k=2}^{\infty} \frac{k+1}{r^k} (Q^k + 1.0).
\]
(122)
Let \( \tilde{g}'_{\mu\nu} \) denote components of metric in coordinates \( t, r, x^A \). Substituting (122) into equations (117)–(119), or their counterparts for \( M = 0 \), yields

\[
\tilde{g} = \tilde{g}_{00} dt^2 + 2 \tilde{g}_{0A}' dr dx^A - \frac{dr^2}{\tilde{g}_{00}} + 2 \tilde{g}_{1A}' dr dx^A + \tilde{g}_{AB}' dx^A dx^B, \tag{123}
\]

where

\[
\tilde{g}_{00}' dx^A = \frac{2J}{r} \sin^2 \theta \, d\varphi + \sum_{k=2}^{\infty} \frac{k+1}{r^k} [d(1.o.) + d l Y_k^m], \tag{124}
\]

\[
\tilde{g}_{1A}' dx^A = - \tilde{g}_{00}' dx^A + \sum_{k=2}^{\infty} \frac{1}{r^k} [d(1.o.) + d l Y_k^m], \tag{125}
\]

\[
\tilde{g}_{AB}' = \tilde{g}_{AB} + \sum_{k=3}^{\infty} \frac{1}{r^{k-1}} [d(1.o.) + d l Y_k^m], \tag{126}
\]

and l.o. means harmonics of order smaller than \( k \). It follows from (123)–(126) that \( t \) is a timelike coordinate for sufficiently big \( r \). The exterior curvature form \( K_{ij} \) of surface \( t = \) const. satisfies

\[
K_{11} = 0(r^{-5}), \quad K_{1A} dx^A = \frac{1}{r} J \sin^2 \theta \, d\varphi, \quad K_{AB} = 0(r^{-1}). \tag{127}
\]

The ADM formula for the linear momentum \( P^i \) shows that \( P^i = 0 \). Using proposition 2.2 in [7] allows to identify \( J \) as the total angular momentum.

For \( M \neq 0 \) formulas (123)–(126) have the form (11.4) in [23] (note that \( Y_{l,lm}^m dx^j \) in [23] coincides with \( 'd Y_{lm}^m \)), hence

\[
l(l + 1)Q_{lm} = \frac{1}{2} (2l - 1)!! \left( \frac{2l(l - 1)}{(l + 1)(l + 2)} \right)^{1/2} P_{lm}, \tag{128}
\]

\[
(l + 1)P_{lm} = - \frac{1}{2} (2l - 1)!! \left( \frac{2l(1 - l)}{l + 2} \right)^{1/2} S_{lm}, \tag{129}
\]

where \( Q_{lm} \) and \( S_{lm} \) are multipole moments of Thorne. For \( M = 0 \) our expression for the square of the Killing vector \( \tilde{g}_{00} = K^2 \) contains the dipole mass moment \( D \) (this term is assumed to be constant in [23]). This is not in contradiction to [23] since the case \( M = 0 \) is not considered there.

Since \( P_{lm} \) and \( S_{lm} \) are equivalent to the STF moments of Thorne (see equations (11.2) and (11.3) in [23]) and the latter were proved [15] to be equivalent to moments of Geroch [14] and Hansen [16] all properties related to these moments can be translated into our formalism. The only exception is the case \( M = 0 \). As we have shown then the nontrivial mass dipole moment \( D \) is admitted. The known statement about asymptotic Kerr like behaviour of stationary metrics (see [4, 23] and references therein) should be clarified. We can rephrase this statement in the following way:

**Proposition 3.1.** Every asymptotically flat stationary vacuum metric with \( M \neq 0 \) or \( K^2 = 1 + 0(r^{-3}) \) tends to the Kerr metric in the order \( r^{-2} \) with respect to asymptotically Minkowskian coordinates.

Below we show how to compute the Bondi–Sachs coordinates for the Kerr metric with accuracy sufficient to find the quadrupole moments. We start with the standard Boyer–Lindquist
coordinates. In order to define foliation \( u = \text{const.} \) we consider generalization of the Eddington–Finkelstein retarded time. A correction of the order \( O(1) \) is removable by a supertranslation, so as the function \( u \) we take

\[
u = t - r - 2M \ln \left( \frac{r}{2M} - 1 \right) - \frac{A(\theta)}{r} + O \left( \frac{1}{r^2} \right).
\]

(130)

Condition \( u^\alpha u_{\alpha} = 0 \) implies

\[A = \frac{1}{2} a^2 \sin^2 \theta.\]

(131)

New coordinates \( r', \theta', \varphi' \) should satisfy \( u^{\alpha'} r'_{\alpha'} = 1, u^{\alpha'} \theta'_{\alpha} = 0, u^{\alpha'} \varphi'_{\alpha} = 0 \. Assuming that they are given by \( r, \theta, \varphi \) plus corrections one obtains

\[
r' = r + \frac{A}{r} + O \left( \frac{1}{r^2} \right), \quad \theta' = \theta + O \left( \frac{1}{r^2} \right), \quad \varphi' = \varphi + O \left( \frac{1}{r^2} \right).
\]

(132)

Inverting relations (130) and (132) yields

\[
t = u + r' + 2M \ln \left( \frac{r'}{2M} - 1 \right) + O \left( \frac{1}{r^2} \right), \quad r = r' - \frac{a^2 \sin^2 \theta'}{2r'} + O \left( \frac{1}{r^2} \right), \quad (133)
\]

\[
\theta = \theta' + O \left( \frac{1}{r^2} \right), \quad \varphi = \varphi' + O \left( \frac{1}{r^2} \right). \quad (134)
\]

Now we substitute (133) and (134) into the Kerr metric. It is easy to check that there is no term in \( \tilde{g}'_{AB} \) linear in \( r' \). Thus, condition (89) is satisfied in the new coordinates. The only term of the order \( 1/r^2 \) in \( \tilde{g}'_{0A} \) follows from \( dr'^2 \). Hence it must be an exact form, so

\[P_{20} = 0.\]

(135)

Component \( Q^{(2)} \) can be easily computed from the original component \( g_{00} = K^2 \) of the Kerr metric. Hence

\[Q^{(2)} = \frac{1}{6} Ma^2 (3 \cos^2 \theta - 1) \]

(136)

and

\[Q^{20} = \frac{2}{3} \sqrt{\frac{\pi}{5}} Ma^2, \quad Q^{2A} = P^{20} = P^{2A} = 0.\]

(137)

Thanks to theorem 3.1 we can write the quadrupole approximation of the Kerr metric in the Bondi–Sachs coordinates without knowledge of further terms in transformations (130) and (132).
\[
d u \left[ \left(1 - \frac{2M}{r'} + \frac{M a^2}{2r'^3} \left( \cos^2 \theta' - \frac{1}{3} \right) \right) du + 2dr' \right.
\]
\[
+ \frac{4Ma}{r'} \sin^2 \theta' d\phi' - \frac{3Ma^2}{2r'^2} \sin 2\theta' d\theta'
\]
\[
- \left(r'^2 + \frac{Ma^2}{2r'} \sin^2 \theta' \right) (d\theta'^2 + \sin^2 \theta' d\phi'^2) + \frac{Ma^2}{r'} \sin^2 \theta' d\theta'^2.
\] (138)

4. Summary

In section 2 we investigated the vacuum Einstein equations for metrics admitting the smooth null scri \(\scri^+\). We wrote these metrics in the Bondi–Sachs form using the affine gauge (\(\tilde{g}_{01} = 1\)) instead of the luminosity gauge (58). We defined a minimal set of independent equations (29) and (30). Expanding metrics and equations into powers of \(1/r\) led us to a hierarchy of equations which can be solved recursively (see theorem 2.1). This is an approach parallel to the standard one using the luminosity gauge. The problem of convergence of resulting series is still unsolved.

In section 3 we assumed that metrics admit additionally a timelike Killing vector \(\partial_u\). Using the low order Einstein equations we showed that the approach from section 2 is still applicable with omitted dependence on \(u\). The main results are given in theorem 3.1 describing an asymptotic form of metric. In agreement with the asymptotic analysis at spacelike infinity every solution with nonvanishing mass \(M\) tends to the Kerr metric. Multipole moments of two kinds appear in consecutive orders of \(1/r\). A relation between them and one set of moments of Thorne is given by (128) and (129). Our results on metrics with \(M = 0\) led us to a slight clarification (see proposition 3.1) of statement about the Kerr like behaviour of stationary metrics. In the last part of section 3 we found the approximate Bondi–Sachs coordinates for the Kerr metric and we wrote this metric up to first terms including the quadrupole moments.

Acknowledgments

I am grateful to Piotr Chruściel and Walter Simon for useful discussions and pointing out references [3, 10]. This work was partially supported by Project OPUS 2017/27/B/ST2/02806 of Polish National Science Centre (NCN).

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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