GROUPOIDS, ROOT SYSTEMS AND WEAK ORDER II

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Abstract. This is the second introductory paper concerning structures called rootoids and protorootoids, the definition of which is abstracted from formal properties of Coxeter groups with their root systems and weak orders. The ubiquity of protorootoids is shown by attaching them to structures such as groupoids with generators, to simple graphs, to subsets of Boolean rings, to possibly infinite oriented matroids, and to groupoids with a specified preorder on each set of morphisms with fixed codomain; in each case, the condition that the structure give rise to a rootoid defines an interesting subclass of these structures. The paper also gives non-trivial examples of morphisms of rootoids and describes (without proof, and partly informally) some main ideas, results and questions from subsequent papers of the series, including the basic facts about principal rootoids and functor rootoids which together provide the raison d’être for these papers.

Introduction

This is the second introductory paper in a series of papers concerning structures called rootoids and protorootoids, the definition of which is abstracted from formal properties of Coxeter groups with their root systems and weak orders. The basic definitions appear in [11]. This paper provides further basic facts about and examples of rootoids and protorootoids, and formulates (often incompletely, in more limited generality or informally) many of the main results of subsequent papers. Assuming familiarity with (at least the introduction to) [11], its concerns may be detailed as follows.

Section 8 discusses a construction of a protorootoid from a groupoid \( G \) with a specified generating set \( S \) (assumed to be closed under inversion and to not contain any identity morphism). This generalizes a standard construction of the abstract root system of a Coxeter group which arises naturally in the theory of buildings. The pair \((G, S)\) is called a \( C_2 \)-system if the associated protorootoid is a rootoid. Some basic characterizations, examples and properties of \( C_2 \)-systems are mentioned in Section 8 without proof. In particular, their braid presentations and the fact that even \( C_2 \)-systems (defined as those with a sign character) correspond to abridged principal rootoids, up to isomorphism of each, are discussed. The braid relations may be specified by Coxeter matrices (the entries of which equal lengths of joins of pairs of simple generators, and determine lengths of the corresponding braid relation) together with local data which determine the successive simple generators on each side of the relation. For example, a cyclic group \( G = \langle x \rangle \) considered with generating set \( S = \{x, x^{-1}\} \) gives a \( C_2 \)-system \((G, S)\), which is even if and only if \( G \) is of even order or of infinite order.
Section 9 describes several other constructions of protorootoids from structures of various types. It attaches protorootoids to simple graphs, to certain edge-labelled simple graphs which are called rainbow graphs, to subsets of Boolean rings, and to possibly infinite oriented matroids. In each case, as also in Section 8, the condition that the resulting protorootoid is a rootoid (of some specified type) distinguishes an interesting class of these structures. Notions of covering rootoids and covering quotient rootoids mentioned in [11, 2.13 and 4.9] enable (connected) rootoids in various natural subclasses to be described in subsequent papers as covering quotients of (connected, simply connected) rootoids arising from the above structures, associated to a suitable group of automorphisms of the structure. For convenience of reference, a table showing standard notations introduced in this paper and [11] for protorootoids, set protorootoids and signed groupoid sets attached to various structures is provided in 9.10.

Section 10 describes results and questions related to our original motivations for this work. It first describes a protorootoid \( R \) attached to the set of initial sections of reflection orders of a Coxeter system. Conjectures from [12] imply that \( R \) is a complete, saturated, regular rootoid and we conjecture further that it is pseudoprincipal (see [11, Section 3] for the definitions). Section 10 also introduces notions of completions of rootoids suggested by this conjecture and asks to what extent they exist in general. The only non-trivial result of the section is the fact (which would be a weak consequence of existence of a completion) that the weak orders of a rootoid are isomorphic to order ideals of complete ortholattices.

Section 11 constructs a left adjoint \( Q \), mentioned already in [11, 5.8], to the forgetful functor \( \mathfrak{P} \) from the category of protorootoids to the category of groupoid-preorders. It also shows that the full subcategory of protorootoidal groupoid-preorders (those underlying a protorootoid) is equivalent to a full reflective subcategory of the category of protorootoids, a result which enables the development of an analogue of the theory of protorootoids and rootoids in terms of groupoid-preorders (though this is not done in this paper).

The notion of a square of a protorootoid \( R \) plays a crucial role in our main results, and is introduced in Section 12. A commutative square of a protorootoid \( R = (G, \Lambda, N) \) is defined to be a commutative diagram in \( G \) involving morphisms \( x, y, z, w \) with \( xy = zw \) such that \( x(N(y)) = N(z) \) and \( N(x)N(z) = 0 \). Equivalently, the requirement is that the morphisms at each corner, suitably oriented, are orthogonal i.e the pairs \( (x, z), (y, x^{-1}), (w^{-1}, y^{-1}) \) and \( (z^{-1}, w) \) are each orthogonal. In terms of signed groupoid-sets, the condition is that \( x(\Phi y) = \Phi z \). A trivial point which facilitates certain local to global comparisons is the following rigidity property of squares: in a faithful protorootoid, a square is determined by any two of its adjacent sides.

Section 13 gives examples of morphism of rootoids, which, when specialized to Coxeter groups, appear implicitly in [24] or [5]. The general proofs are deferred, but are given here in a particularly simple and instructive case which provides a prototype for proofs in subsequent papers. Namely, it is a well-known theorem of Tits that if \( G \) is a group of diagram automorphisms of a Coxeter system \((W, S), \)
and $W^G$ is the group of $G$-invariants of $W$, then $W^G$ is a Coxeter group. In [12,2] it is proved that the inclusion $\theta : W^G \to W$ satisfies the AOP and shown how AOP implies both that the simple generators $R$ of $W^G$ are the elements of the form $\theta^2(s)$ with $s \in S \cap \text{dom}(\theta^2)$, and that $W^G$ is preprincipal. Tits’ theorem is easily recovered from this. The proof of AOP for $\theta$ requires an expression of $\theta^\perp(w)$ (which, a priori, is defined as a meet) as a join; in the special case of $\theta^\perp(s)$, for $s$ as above, this expression amounts to the explicit description by Tits of the simple generators $R$.

The results in [24] generalize the above one of Tits to Coxeter subgroups of $W$ generated by longest elements of finite parabolic subgroups of $W$ and satisfying a length compatibility condition which, in more general contexts, is the main requirement in the definition of preprincipal rootoid. In [13,3] a generalization of these results to preprincipal rootoids is stated in terms of order-theoretic hypotheses; it amounts roughly to a description in semilocal terms of the possible images of atomic generators of preprincipal rootoids under rootoid local embeddings into a fixed preprincipal rootoid. The result is illustrated by examples with Coxeter groups in [13,3].

Subsections [13,5]-[13,7] discuss what are here called normalizer groupoids, which are obtained by a generalization of a construction in [5]. To illustrate the definition, let $(W, S)$ be a Coxeter system with root system $\Phi$ and simple roots $\Pi$. The normalizer groupoid $N$ is a groupoid, the objects of which identify with the subsets of $\{ \Phi_w \mid w \in W \}$, two such objects being in the same component if and only if they are in the same $W$-orbit for the natural action of $W$ on sets of subsets of the root system, and for which the vertex group at such a subset identifies with its setwise stabilizer (normalizer) in $W$. The groupoid $N$ has a subgroupoid (a union of components of $N$) with objects the subsets of $\{ \Phi_w \mid w \in S \} \cong \Pi$, the components of which were studied in [5]. The normalizer groupoid construction underlies a construction with protorootoids which preserves, amongst others, the classes of rootoids, interval finite rootoids, complete rootoids and preprincipal rootoids. Restricted to preprincipal rootoids, it provides natural examples of rootoid local embeddings as discussed in the previous paragraph.

The results about normalizer rootoids will be deduced in later papers from more general facts about functor rootoids which are stated without proof in Section [14]. The notion of functor rootoids was discussed in the introduction to [11] in relation to signed groupoid-sets. Here we provide further details, in terms of rootoids. For a rootoid $\mathcal{R} = (G, \Lambda, N)$ and a groupoid $H$, with $G$ and $H$ assumed non-empty and connected for simplicity, one has the category $G^H$ of functors from $H$ to $G$, with natural transformations as morphisms. This category is a groupoid, and it has a subgroupoid $G^H_{\Box}$ with all objects but only morphisms for which each commutative square in $G$ from the definition of natural transformation is a commutative square as defined above.

Fix a groupoid homomorphism $F : H \to G$ and let $G^H[F]$ be the component of $G^H[F]$ containing $F$. For any $b \in \text{ob}(G)$, one has trivially an “evaluation at $b$” groupoid morphism $\epsilon_b : G^H[F] \to G$, a pullback protorootoid $\epsilon_b^\sharp(\mathcal{R})$ with underlying groupoid $G^H[F]$ and a natural protorootoid morphism $\epsilon_b^\flat : \epsilon_b^\sharp(\mathcal{R}) \to \mathcal{R}$. The main result concerning functor rootoids is that $\epsilon_b^\flat$ is a morphism in the category of
rootoid local embeddings (in particular, the protorootoid $\epsilon^v_b(\mathcal{R})$ is a rootoid, called a functor rootoid). It follows that the abridged functor rootoid $(\epsilon^v_b(\mathcal{R}))^a$ is principal or complete if $\mathcal{R}$ is so; it turns out to be independent of choice of $b$ up to canonical isomorphism. The morphism $\epsilon_b$ above is a called a stable local embedding (of groupoids).

Consider the category of based, connected groupoids with objects the pairs $(G, a)$ of a connected groupoid $G$ and an object $a$ of $G$, with morphisms basepoint preserving groupoid homomorphisms. If we set $F(b) = a$, then the above constructs from a morphism $F: (H, b) \to (G, a)$ a morphism $\epsilon_b: (G^H[F], F) \to (G, a)$ which we call the dual of $F$. It will be shown that $F$ factors canonically through its double dual, and that the construction restricts to a duality on a suitable category of based stable local embeddings over $(G, a)$. The proofs of these results involve local descriptions of the construction in terms of certain symmetric Galois connections associated to the objects of $G$.

The fact that $(\epsilon^v_b(\mathcal{R}))^a$ is principal if $\mathcal{R}$ is principal is the most substantial result of these papers. It is comparatively straightforward to show that if $\mathcal{R}$ is a rootoid, then $\epsilon^v_b(\mathcal{R})$ is a rootoid, and then trivial that if $\mathcal{R}$ is also complete or interval finite, then so is $\epsilon^v_b(\mathcal{R})$. The crucial step is then to show that if $\mathcal{R}$ is preprincipal, then so is $\epsilon^v_b(\mathcal{R})$. This follows, by arguments similar to those appearing in regard to Tits' theorem (see [12.2]), from the fact that (whether or not $\mathcal{R}$ is preprincipal) $\epsilon^v_b$ is a rootoid local embedding. In turn, the key step in the proof that $\epsilon^v_b$ is a rootoid local embedding is the proof that it satisfies the AOP. This also follows the general lines of the argument for the corresponding fact in [12.2] the main point is expressing certain meets explicitly as joins. That is accomplished by a (finite for preprincipal rootoids, transfinite in general) repetition of certain “zig-zag” and “loop” constructions (defined in later papers in terms of basic combinatorics of squares) in the setting of the previously mentioned Galois connections. Similar but simpler arguments are involved in the proofs of other results showing that principal rootoids are preserved by natural constructions.

To oversimplify, the category of rootoids is defined so as to be able to express in natural terms the proof that $(\epsilon^v_b(\mathcal{R}))^a$ is principal if $\mathcal{R}$ is principal. It then turns out that the category of rootoids has all small limits, as does its full subcategory of complete rootoids, while the full subcategories of preprincipal and interval finite rootoids admit all limits from small categories with finitely many objects. These results may be viewed as general analogues of the above-mentioned theorem of Tits.

As an application of functor rootoids, Section [14] informally describes (the part involving underlying groupoids) of a construction which attaches to any rootoid $\mathcal{R}$ a structure $\mathcal{R}^{\omega}$, which we call a symmetric cubical $\omega$-rootoid. It has an underlying cubical $\omega$-groupoid in the sense of higher category theory, each of the $n$ groupoid $n$-compositions of which underlies a rootoid (reducing to $\mathcal{R}$ for $n = 1$). The supremum of the set of $n$ for which $\mathcal{R}^{\omega}$ has non-trivial $n$-morphisms provides a measure of the degree of non-triviality of the theory of functor rootoids for $\mathcal{R}$. For the rootoid $\mathcal{C}_{(W,S)}$ attached to a Coxeter system, it is the supremum of the ranks of finite standard parabolic subgroups.
It may be expected that study of functor rootoids and related structures will be of particular interest in examples such as Coxeter groups (and more particularly, finite and affine Weyl groups and the symmetric groups) and that significantly stronger properties than are given by the existing general theory may hold in such cases. For instance, underlying groupoids of functor rootoids from finite rank Coxeter systems \((W, S)\) inherit from \((W, S)\) not only root systems realizable in real vector spaces but also analogues of the "canonical automaton" ([I]) from \((W, S)\) (it is a natural question if the appropriate analogue of the finiteness of the set of elementary roots ([I]) of \((W, S)\) holds in them).

8. **C₂-systems**

8.1. **Protorootoid of a groupoid with generating set.** Considerable use is made in this section of the results of [II §5] and especially of the functors \(I, J, K, L\), and \(\mathcal{L}\) defined there (see [II 5.8]).

**Definition.** (a) A \(C₀\)-system is a pair \((G, S)\) where \(G\) is a groupoid and \(S\) is a set of generators of \(G\) such that \(S = S^*\) and \(S\) contains no identity morphism of \(G\).

(b) A \(C₀\)-system \((G, S)\) is **even** if the pair \((G, S)\) admits a sign character in the sense of [II 3.8].

8.2. Fix a \(C₀\)-system \((G, S)\) and let \(l = lₘ\) be the corresponding length function on \(G\). Let \(X\) be the left regular \(G\)-set; this is the \(G\)-set (functor \(G \to \text{Set}\)) with \(a X := aG\) and action map \(aG_b \times X \to aX\) identified with the multiplication map \(aG_b \times bG \to aG\) of \(G\), for all \(a, b \in \text{ob}(G)\). Let \(Y : G \to \text{BRng}\) with the forgetful functor \(\text{BRng} \to \text{Set}\), so \(aY = \varphi(aX)\) is the set of subsets of \(aX\). For any \(a \in \text{ob}(G)\) and \(s \in aS := S \cap aG\), define

\[
G^s := \{ g \in aG \mid l_s(s^*g) > l_s(g) \} \subseteq aY.
\]

Observe that

\[
1_a \in G^s, \quad s \notin G^s
\]
since by assumption, \(S\) contains no identity morphism. Hence

\[
\emptyset \subseteq G^s \subsetneq aX.
\]

Define \(\Psi_G = \Psi : G \to \text{Set}\) to be the \(G\)-subrepresentation of \(Y\) generated by all the elements \(G^s \subseteq aY\) for \(s \in aS\) and \(a \in \text{ob}(G)\). Thus, for \(a \in \text{ob}(G)\),

\[
aΨ := \{ g(G^s) \mid g \in aG_b, s \in bS, b \in \text{ob}(G) \}, \quad g(G^s) := \{ gx \mid x \in G^s \}
\]

and for \(g \in aG_b\) and \(Z \in bY\), one has \(Ψ(g)(Z) = gZ := \{ gz \mid z \in Z \} \subseteq aY\). For \(a \in \text{ob}(G)\), define

\[
aΨ_G = aΨ := \{ A : aΨ \mid 1_a \in A \} \subseteq aΨ.
\]

Viewing \(Ψ\) as a functor \(Ψ : G \to \text{Set}\) with \(Ψ(a) = aΨ\), there is a \(G\)-cocycle (in fact, a coboundary) \(N\) for \(Ψ_G(Ψ)\) defined by \(aN(g) := aΨ + g(bΨ') \subseteq \varphi(aΨ)\) for \(a, b \in \text{ob}(G)\) and \(g \in aG_b\). Note that for \(s \in aS\),

\[
\{ G^s, s(G^s) \} \subseteq aN(s), \quad G^s \in aΨ', \quad s(G^s) \notin aΨ'
\]
by \((8.2.2)\). Hence
\[
(8.2.7) \quad \left| N(s) \right| \geq 2, \quad s \in S.
\]
Simple direct calculation shows that for \(a \in \text{ob}(G)\), \(s \in _a S\) and \(g \in _a G\),
\[
(8.2.8) \quad G^*_s \in N(g) \iff l_S(s^*g) \leq l_S(g), \quad s(G^*_s) \in N(g) \iff l_S(s^*g) < l_S(g).
\]
Hence
\[
(8.2.9) \quad s(G^*_s) \in N(g) \implies G^*_s \in N(g).
\]

**Definition.** Let \((G, S)\) be a \(C_0\)-system and notation be as above.

(a) The set protorootoid and protorootoid corresponding to \((G, S)\) are \(\mathcal{J}_{(G,S)} := (G, \Psi, N)\) and \(\mathcal{J}_{(G,S)} := (G, \varphi_G(\Psi), N)\) respectively.

(b) The signed groupoid-set of \((G, S)\) is \(J_{(G,S)} := \mathcal{R}(\mathcal{J}_{(G,S)})\).

(c) The \(C_0\)-system \((G, S)\) is called a \(C_1\)-system or is said to satisfy the weak exchange condition (WEC) if \(\mathcal{J}_{(G,S)}\) is cocycle finite and for all \(a \in \text{ob}(G)\) and \(g \in _a G\), one has \(\text{rank}(N(g)) = 2l_S(g)\) in \(\varphi(a \Psi)\) (i.e. \(|N(g)| = 2l_S(g))\).

(d) The \(C_0\)-system \((G, S)\) is called a \(C_2\)-system if it is a \(C_1\)-system and \(\mathcal{J}_{(G,S)}\) is a rootoid.

8.3. Let \((G, S)\) be a \(C_0\)-system. Write \(R := J_{(G,S)} = (G, \Lambda), \mathcal{R} := \mathcal{J}_{(G,S)} = (G, \Psi, N), \mathcal{R} := \mathcal{J}_{(G,S)} = (G, \varphi_G(\Psi), N)\) as above. From the proof of \([11]\) Proposition 5.5, \(a \Lambda := a \Psi \times \{\pm\}\) with \(a \Lambda^+ = a \Psi \times \{+\}\). For \(g \in _a G_b\), \(\Lambda_g := a \Lambda^+ \cap g(a \Lambda^-)\). From \([11] 5.7\), \(|\Lambda(g)| = |N(g)|\) for any \(g \in \text{mor}(G)\).

**Lemma.**

(a) If \(g \in _a G_a\) and \(g(G^*_s) \in b \Psi\) for all \(s \in _a S\), then \(g = 1_a\).

(b) The protorootoid \(\mathcal{R} = \mathcal{J}_{(G,S)}\) is faithful.

(c) If \((G, S)\) is a \(C_1\)-system, then for \(a \in \text{ob}(G)\) and \(x, y \in _a G\), one has \(x \leq y\) if and only if \(l_S(y) = l_S(x) + l_S(x^*y)\).

**Proof.** Suppose that \(g \in _a G_a\) is as in (a). Let \(s \in _a S\). Then \(g(G^*_s) \in b \Psi\) i.e. \(1_b \in g(G^*_s)\) or equivalently \(g^* \in G^*_s\). This means that \(l_S(s^*g^*) > l_S(g^*)\) for all \(s \in _a S\), which implies that \(g = 1_a\) since \(S = S^*\) generates \(G\). This proves (a).

To prove (b), suppose that \(g \in _b G_a\) with \(N(g) = \emptyset\) i.e. \(_b \Psi = g_*(\Lambda^+\Psi)\). By \((8.2.6)\), \(g(G^*_s) \in _b \Psi\) for all \(s \in _a S\) and by (a), \(g = 1_a\).

Part (c) is immediate from the definition of \(C_1\)-system and \([11]\) Corollary 3.14(a). \(\square\)

8.4. This subsection provides, in the special case of even \(C_0\)-systems, simpler variants of the above conditions and constructions. Retain the assumptions and notation of \([8.3]\) but assume in addition that \((G, S)\) is even.

Note that for any \(s \in _a S_b\), \(g \in _a G\) one has \(l_S(s^*g) = l_S(g) \pm 1 \neq l_S(g)\) and therefore
\[
(8.4.1) \quad H^<_s := \{ g \in _a G \mid l_S(s^*g) < l_S(g) \} = _a G \setminus H^>_s.
\]
Also,
\[
(8.4.2) \quad H^<_s = s(\{ g \in _b G \mid l_S(s^g) > l_S(g) \}) = s(H^<_s) \in _a \Psi.
\]
Now for any \( h \in \alpha G_c \) and \( r \in _cG \),

\[
(8.4.3) \quad _\alpha G \setminus h(H_r^\alpha) = h(H_r^\alpha) \in _\alpha \Psi.
\]

The above implies that for each \( a \in \text{ob}(G) \), there is a definitely signed set \( _a\hat{\Psi} \) with underlying set \( _a\Psi \), such that \(-\alpha := _aG \setminus \alpha \) for all \( \alpha \in _a\Psi \) and the subset of positive elements of \( _a\hat{\Psi} \) is \( _a\hat{\Psi}_+ := _a\Psi' \). Hence there is a unique functor \( \hat{\Psi} : G \to \text{Set}_\pm \) with the following properties:

(i) The composite functor \( G \xrightarrow{\hat{\Psi}} \text{Set}_\pm \to \text{Set} \) of \( \hat{\Psi} \) with the forgetful functor \( \text{Set}_\pm \to \text{Set} \) is equal to \( \Psi \).

(ii) For \( a \in \text{ob}(G) \), \( \hat{\Psi}(a) = _a\hat{\Psi} \), as defined above, in \( \text{Set}_\pm \).

This defines a signed groupoid-set \( \hat{R} = (G, \hat{\Psi}) \). Let \( \hat{\mathcal{R}}' = (G, \hat{\Psi}, N) := \mathcal{L}(\hat{R}) \) be the associated set protorootoid. Let \( \hat{\mathcal{R}} := \mathcal{L}(\hat{\mathcal{R}}') = (G, \varphi_G(\hat{\Psi}), N) \) be the protorootoid corresponding to the set protorootoid \( \hat{\mathcal{R}}' \). To indicate dependence on \( (G, S) \), write \( \hat{\mathcal{R}} = \mathcal{L}_{(G,S)} \hat{\mathcal{R}} = \hat{\mathcal{R}}'_{(G,S)} \) and \( \hat{\mathcal{R}} = \mathcal{L}_{(G,S)} \hat{\mathcal{R}}' \).

For \( a \in \text{ob}(G) \), one has by definition that \( _a\hat{\Psi} = _a\hat{\Psi}/\{\pm\} \), the set of \( \{\pm\} \)-orbits on \( _a\hat{\Psi} \). There is a natural transformation \( \pi : \Psi \to \hat{\Psi} \) of functors \( G \to \text{Set} \) such that its component \( \pi_a \) at \( a \in \text{ob}(G) \) is the orbit map \( \alpha \mapsto \{\pm\alpha\} : _a\Psi \to _a\hat{\Psi} \).

**Proposition.** Let \( (G, S) \) be an even \( C_0 \)-system and let notation be as above. Then

(a) \( \mathcal{R}(\hat{\mathcal{R}}') \cong \hat{R} \) in \( \text{Gpd-Set}_\pm \).

(b) There is a morphism \( f := (\text{Id}_G, \pi) : \hat{\mathcal{R}}' \to \hat{\mathcal{R}} \) in \( \text{Set-Prd} \).

(c) Write \( \mathcal{I}(f) = (\text{Id}_G, \varphi_G(\pi)) : \hat{\mathcal{R}} \to \hat{\mathcal{R}}' \) for the morphism of protorootoids corresponding to \( f \). Then the abridgement \( \mathcal{A}(\mathcal{I}(f)) : \mathcal{A}(\hat{\mathcal{R}}') \to \mathcal{A}(\hat{\mathcal{R}}) \) is an isomorphism in \( \text{Prd}^\theta \).

(d) \((G, S)\) satisfies the WEC if and only if \( \hat{\mathcal{R}} \) is a principal protorootoid. In that case, \( S \) is the set of simple generators of \( \hat{\mathcal{R}} \).

(e) The underlying set representation of \( \Lambda \) is equivalent in \( \text{Set}^G \) to the coproduct (disjoint union) \( \Psi \coprod \Psi \) of two copies of \( \Psi \).

**Proof.** Part (a) follows immediately from the definitions and \[\text{Proposition 5.5}\]. By the definitions, for \( g \in _aG \), one has \( \hat{N}_g = N_g \cap _a\Psi_+ \), \( N_g = \{\alpha, -\alpha \mid \alpha \in \hat{\Psi}_g\} \) and \( \hat{N}_g = \{\{\pm\alpha\} \mid \alpha \in \hat{\Psi}_g\} \). Hence \( N_g = \pi_{a}^{-1}(\hat{N}_g) \), which proves (b). Also, \( \{\pm\} \) acts freely on \( N(g) \) with \( \hat{N}_g \) as set of orbit representatives and \( \hat{N}(g) \) as its set of orbits. Hence \( |N(g)| = 2|\hat{N}(g)| \), which implies (d) by the definitions. Note next that \( \pi_a \) is surjective, so \( \varphi_G(\pi)_a = \varphi(\pi_a) \) is an injective homomorphism of Boolean rings \( \varphi(a\hat{\Psi}) \to \varphi(\Psi) \), which we regard for the proof of (c) as an inclusion. With this identification, \( \hat{N}_g = N_g \) and (c) follows immediately by definition of abridgement.

Now for (e). For any \( a \in \text{ob}(G) \) and \( \alpha \in _a\Psi \), define the sign of \( \alpha \) to be

\[
\epsilon(\alpha) := \begin{cases} 
+ & \text{if } \alpha \in _a\Psi' \\
- & \text{otherwise}
\end{cases}
\]
By definition, _A_ = _a_Ψ × {±}. Using [11, (5.5.1)], the _G_-action is given by
\[ \Lambda(g)(\alpha, \eta) = (\Psi(g)(\alpha), \eta \epsilon(\Psi(g)(\alpha))), \quad \eta \in \{\pm\}, g \in G, \alpha \in _a\Psi, a \in \text{ob}(G). \]
The underlying set representation of _A_ is the disjoint union of two subrepresentations specified (imprecisely) by
\[ \{ (\alpha, \eta) \mid \eta \epsilon(\alpha) = + \}, \quad \{ (\alpha, \eta) \mid \eta \epsilon(\alpha) = - \}, \]
each of which is isomorphic by projection (\alpha, \eta) \mapsto \alpha to _Ψ_ (and which are interchanged by the action of \(- \in \{\pm\}\)). From this, it is straightforward to prove (e). □

Remarks. Although _A_ is not the coproduct in (\textbf{Set}_±)^G of two copies of ̂_Ψ_, the proof of (e) can easily be elaborated to an explicit description of (_G_, _Λ_ ) in terms of (_G_, ̂_Ψ_).

8.5. The constructions discussed in §8.1–8.4 are modelled on an interpretation, in the theory of buildings, of the abstract root system of a Coxeter system (_W_, _S_) in terms of a _W_-action on a set of subsets (called half-spaces) of _W_. Though the following is a corollary of both Theorems 8.8 and 8.14, which are proved in subsequent papers, it is instructive to include here a direct proof which makes the connection between the constructions of this section and half-spaces explicit.

**Proposition.** A Coxeter system is an even _C_2-system.

**Proof.** Suppose that (_G_, _S_) = (_W_, _S_) is a Coxeter system. Then (_W_, _S_) is certainly an even _C_0-system, so _T_(_W_, _S_) = (_W_, ̂_Ψ_) is defined. Write the standard signed groupoid-set of (_W_, _S_) (see [11, 6.5, 6.7]) as _C_(_W_, _S_) = (_W_, _Φ_). Regard ̂_Ψ_ = \{ _w(H_±) \mid s \in _S_, _w \in _W_ \} and \_Φ_ = _T_ × \{±\} as _W_ × \{±\}-sets with distinguished positive subsets of \{±\}-orbit representatives \{ _A_ ∈ ̂_Ψ_ \mid _1_W_ ∈ _A_ \} and _T_ × \{±\} respectively (see [11, 6.5]). We claim that there is a unique morphism of signed _W_ × \{±\}-sets \_Φ_ → ̂_Ψ_ such that (s, 1) → _G_±_s_ for all _s_ ∈ _S_, and that it induces an isomorphism
\[(8.5.1) \quad \Phi \cong ̂\Psi \]
between these _W_ × \{±\}-sets preserving their positive subsets. Though this is well known (see [11] Ch IV, §1, Exercise 16(i)), an alternative direct proof is sketched below. For _t_ ∈ _T_, define half-spaces
\[
\begin{align*}
W_{{(t,+)}} &:= \{ _w \in _W_ \mid l(t_w) > l(_w) \} \\
W_{{(t,-)}} &:= \{ _w \in _W_ \mid l(t_w) < l(_w) \} = _W_ \setminus W_{{(t,+)}}
\end{align*}
\]
Since _T_ × \{±\} (resp., ̂_Ψ_) is the union of _W_-orbits of the elements (s, +) (resp., _H_±_s_ for _s_ ∈ _S_), the claim follows from the claims (a)–(c) below:

(a) for _w_ ∈ _W_, _w(H_±_s_) = \{ _wx_ \mid x \in _W_, l(sx) > l(x) \} = _W_(_w(s,+)).

(b) for (_t_, _ε_) in _T_ × \{±\}, 1 ∈ _W_(_t_,_ε_) if and only if _ε_ = +.

(c) for (_t_, _ε_) and (_t′_, _ε′_) in _T_ × \{±\}, _W_(_t_,_ε_ = _W_(_t′_,_ε′_) implies (_t_, _ε_) = (_t′_, _ε′_).

The claim (a) can be proved by a simple calculation involving the fact that for _w_ ∈ _W_ and _s_ ∈ _S_, _l(wsx) > l(wx)_ if and only if _l(ws) = l(w) = l(sx)− _l(x)_; the fact itself follows from the cocycle property of the reflection cocycle of (_W_, _S_)
and [11, 6.7.1]. The claim (b) follows immediately from the definitions. To prove (c), suppose that \( W_{(\epsilon, \epsilon)} = W_{(\epsilon', \epsilon')} \). By (b), \( \epsilon = \epsilon' \). Taking complements in \( W \) shows that \( W_{(\theta, \epsilon)} = W_{(\theta, \epsilon')} \). Hence for any \( w \in W \), \( l(tw) > l(w) \) if and only if \( l(t'w) > l(w) \).

Choose (see [13]) a palindromic reduced expression \( t = s_1 s_2 \cdots s_n + 1 \cdots s_2 s_1 \) for \( t \), with \( s_i \in S \) and \( 2n + 1 = l(w) \). Let \( w_i = s_1 \cdots s_i \) for \( i = 0, 1, \ldots, n + 1 \), so \( l(w_i) = i \) and \( tw_n = w_{n+1} \). Note that \( l(tw_n) > l(w_n) \), \( l(w_n s_{n+1}) > l(w_n) \) and \( l(t w_n s_{n+1}) < l(w_n s_{n+1}) \). The same is therefore true with \( t \) replaced by \( t' \). The strong exchange condition (see [11, 6.3]) implies that \( t' w_n = w_n s_{n+1} = w_{n+1} \) and so \( t' = w_{n+1} w_n^{-1} = t \).

From the claim, it follows that \( \overline{R} = (W, \overline{\Psi}) \cong (W, \Phi) \) as signed groupoid-set. By [11, Theorem 6.7(b)], \( \overline{R} \) is therefore principal and rootoidal. So is \( \overline{\mathcal{R}} \) by the terminological conventions in [11, 5.6]. Proposition 8.3(d) implies that \( (W, S) \) satisfies WEC; that is, it is a \( C_1 \)-system. Also, by Proposition 8.4(c) and [11, Remark 4.3(2)], \( \mathcal{R} \) is a rootoid. Hence \( (W, S) \) is a \( C_2 \)-system as required. \( \square \)

8.6. The following subsections 8.7–8.16 survey, without proof and partly informally, basic properties and examples of \( C_2 \)-systems, especially even ones. In particular, Section 8.16 gives some simple examples. Full statements and proofs will be given in subsequent papers.

8.7. WEC. The following proposition lists some of many equivalent formulations of the WEC. Compare (d)–(e) with EC in [11, Proposition 6.3].

**Proposition.** The following five conditions on a \( C_0 \)-system \((G, S)\) are equivalent:

(i) \((G, S)\) satisfies the WEC i.e. for all \( g \in \text{mor}(G) \), \(|N(g)| = 2l_S(g)\).

(ii) For all \( s \in S \), \(|N(s)| = 2\).

(iii) For all \( a \in \text{ob}(G) \) and \( g \in \text{G}_{a} \), \(|N(g) \cap \text{G}_{a}^\prime| = l_S(g)\).

(iv) If \( g \in \text{mor}(G) \) and \( r, s \in S \) with \( l(gr) > l(g) \) and \( l(sgr) \leq l(sg) \), then \( g(H_{r}^{\geq}) = H_{s}^{\leq} \).

(v) If \( g \in \text{mor}(G) \) and \( r, s \in S \) are such that \( \exists gr \text{ such that } g(H_{r}^{\geq}) = H_{s}^{\leq} \text{ if and only if } l(gr) > l(g) \) and \( l(sgr) \leq l(sg) \).

8.8. Principal rootoids and even \( C_2 \)-systems. Abridged principal rootoids correspond bijectively to even \( C_2 \)-systems up to the natural notion of isomorphism of each. In particular, one has the following.

**Theorem.** (a) If \((G, S)\) is an even \( C_2 \)-system, then \( \mathfrak{A}(\mathcal{F}_{(G, S)}) \cong \mathfrak{A}(\overline{\mathcal{F}}_{(G, S)}) \) is a principal rootoid with simple generators \( S \).

(b) If \( \mathcal{R} = (G, \Lambda, N) \) is a principal rootoid with simple generators \( S \), then \((G, S)\) is an even \( C_2 \)-system, and \( \mathcal{R} \) and \( \overline{\mathcal{F}}_{(G, S)} \) have isomorphic abridgements.

The easier part (a) already follows using Proposition 8.4 and [11, Remark 4.3(2)]. Part (b) is deduced in a subsequent paper from a similar characterization of arbitrary \( C_2 \)-systems (which involves a condition related to (8.2.9)).
8.9. **Braid presentation.** An even $C_2$-system $(G, S)$ has a canonical presentation by generators $S$ subject to *trivial relations* and *braid relations*. The trivial relations are the relations $s^{-1} = s^*$ for $s \in S$ (recall that $S = S^*$). The braid relations may be specified by (uniquely determined) data consisting of a Coxeter matrix $M^a$ indexed by $\pi S$ attached to each $a \in \text{ob}(G)$, and certain maps $\{\pi_r\}_r \in S$ between subsets of the sets $\pi_a S$ for $a \in \text{ob}(G)$, as described below.

For $r, s \in \pi S$, the corresponding entry of the Coxeter $M^a$ is $m^{r, s}_{a} := l_S(r \lor s)$ (if the join exists in $\pi S$; if it doesn’t exist, one interprets the length as $\infty$). If $r \neq s$ and $n := m^{a}_{r,s} \neq \infty$, there are exactly two reduced expressions of $r \lor s$. One of them is of the form $r_1 \cdots r_n$ with $r_1 = r$ and the other is of the form $s_1 \cdots s_n$ with $s_1 = s$. The corresponding braid relation is

$$(8.9.1) \quad a[r_1, \ldots, r_n]_a = a[s_1, \ldots, s_n]_b$$

(using the more precise notation of [11, 2.6]) where $r \lor s \in \pi G_b$. A braid relation is defined as a relation arising in this way. A basic fact is that inverses and cyclic shifts of braid relations are again braid relations. For instance,

$$(8.9.2) \quad b[r^*_1, \ldots, r^*_1]_a = b[s^*_1, \ldots, s^*_1]_a$$

and

$$(8.9.3) \quad a'[s^*_1, r_1, \ldots, r_{n-1}]_b = a'[s_2, \ldots, s_n, r^*_1]_b$$

are braid relations, where $a' := \text{cod}(s_2)$ and $b' := \text{dom}(r_{n-1})$ (in fact, $(8.9.2)$ follows by repeated application of $(8.9.3)$). Note that it is obvious that $(8.9.2)$–$(8.9.3)$ are relations, but not that they are braid relations. The above also implies that for any braid relation $(8.9.1)$, one has

$$(8.9.4) \quad m^{a}_{r_1, s_1} = m^{b}_{r_2, s_2} = m^{a'}_{s_1, s_2}.$$  

Next, for all $r \in S$, say $r \in \pi S c$, there is one map $\pi_r = \pi_r^{(G, S)}$, giving a bijection (with inverse $\pi^{-1}$)

$$(8.9.5) \quad \pi_r : \{ t \in \pi S \mid m^{t, r}_r \neq \infty \} \xrightarrow{\cong} \{ s \in \pi S \mid m^{t, s}_r \neq \infty \}$$

It is defined by setting $\pi_r(r_2) := s_1$ for each braid relation $(8.9.1)$ with $r_1 = r$ and by setting $\pi_r(r^*) := r$. Using the above fact about cyclic shifts of braid relations, one easily sees that the braid relations are completely determined as stated by the data consisting of the family of Coxeter matrices $\{M^a\}_{a \in \text{ob}(G)}$ and partially defined maps $\{\pi_r\}_r \in S$: informally, the partially defined maps constitute the local data necessary to determine in turn successive simple generators appearing on the two sides of a braid relation, left to right, beginning with the leftmost simple generator, until each side of the relation reaches the length specified by the corresponding entry of the Coxeter matrix.

**Remarks.** An even, $C_2$-system $(G, S)$ is completely determined (up to isomorphism) by $\text{ob}(G)$, the set $S$ with its decomposition $S = \bigcup_{a, b \in \text{ob}(G)} a S_b$ and the map $s \mapsto s^* : S \rightarrow S$, the Coxeter matrices $M^a = (m^a_{r,s})_{r,s \in \pi S}$ for $a \in \text{ob}(G)$ and the bijections $\pi_r$ for $r \in \pi S b$. Simple necessary and sufficient conditions for arbitrarily specified data of this type to determine an even $C_2$-system are not known.
8.10. Groupoid representation on its simple generators. An interval finite rootoid \( \mathcal{R} = (G, \Lambda, N) \) with atomic generators \( A := A_\mathcal{R} \) is said to be \( n \)-complete, where \( n \in \mathbb{N} \), if for any \( a \in \text{ob}(G) \) and any elements \( r_1, \ldots, r_n \in _aA \), the join \( \bigvee_i r_i \) exists in \( _aG \) in its weak order. Note that \( n \)-completeness holds automatically for \( n = 0, 1 \), and that if \( \mathcal{R} \) is complete, it is \( n \)-complete for all \( n \in \mathbb{N} \). Let \( (G, S) \) be an even \( C_2 \)-system and \( \mathcal{R} := \mathcal{J}_{(G,S)} \). According to [11, Lemma 3.7] and Theorem 8.8, one has \( A_\mathcal{R} = S \). Define \( (G, S) \) to be \( n \)-complete (resp., complete) if \( \mathcal{R} \) is \( n \)-complete (resp., complete). Obviously if \( (G, S) \) is complete, it is \( n \)-complete for all \( n \).

Example. A Coxeter system \( (W, S) \) is \( n \)-complete, where \( n \in \mathbb{N} \), if and only if for all subsets \( J \) of \( S \) with \( |J| \leq n \), the standard parabolic subgroup \( W_J \) is finite. In particular, if \( W \) is of infinite rank \( |S| \) but all its standard parabolic subgroups \( W_J \) of finite rank are finite, then \( (W, S) \) is \( n \)-complete for all \( n \in \mathbb{N} \) but not complete.

Fix an even \( C_2 \)-system \( (G, S) \). Note that \( (G, S) \) is 2-complete if and only if there are no infinite entries in any of the Coxeter matrices \( M^a \) for \( a \in \text{ob}(G) \). This holds if and only if for all \( a, b \in \text{ob}(G) \) and \( r \in _aS_b \), the map \( \pi_r : \bigvee_r S \to _aS \). One defines \( (G, S) \) to be 5/2-complete if it is 2-complete and the maps \( \pi_r \) determine a representation (i.e. functor) \( \pi : G \to \text{Set} \) such that for all \( a \in \text{ob}(G) \), \( \pi(a) = _aS \) and for all \( r \in S \), \( \pi(r) = \pi_r \). If \( r \) exists, it is uniquely determined, and will be denoted as \( \pi_{(G,S)} \). Note that if \( G \) is connected and simply connected, then \( \pi \) is trivial (if it exists) by [11, Lemma 1.14].

Theorem. (a) A 3-complete, even \( C_2 \)-system is 5/2-complete.
(b) A 2-complete, even \( C_2 \)-system \( (G, S) \) with \( G \) finite is both complete and 5/2-complete.

Remarks. Define a 5/2-completion of an even \( C_2 \)-system \( (G, S) \) to be a 5/2-complete even \( C_2 \)-system \( (H, R) \) with the following additional properties:

(i) \( G \) is a subgroupoid of \( H \) and \( S \subseteq R \).
(ii) For \( a \in \text{ob}(G) \), \( _aG \) is an order ideal of \( _aH \) (where \( _aH \) has the weak order from the associated rootoid \( \mathcal{J}_{(H,R)} \)).
(iii) For \( a, b \in \text{ob} G \), \( r \in _aS_b \) and \( s \in _bS \) such that \( \pi_r^{(G,S)}(s) \in _aS \) is defined, one has \( \pi_r^{(G,S)}(s) = \pi_r^{(H,R)}(s) \).

Here, (iii) follows from (i)–(ii) but is stated explicitly for emphasis. It is an open question whether every even \( C_2 \)-system has a 5/2-completion. A special case, open for infinite \( G \), is the question of whether every 2-complete, even \( C_2 \)-system \( (G, S) \) is 5/2-complete. These questions have an obvious similarity to open questions involving notions of completions of rootoids as considered in Section 10 (which would not in general give \( C_2 \)-systems from \( C_2 \)-systems), but their relationship is unclear.

8.11. Coxeter groupoids and \( C_2 \)-systems. Coxeter (and Weyl) groupoids are classes of groupoids \( G \) with distinguished generating sets \( S \) (closed under inverses) defined in [20] and [7] (note that [20] uses a different but equivalent notion of groupoid to that used in these papers). They will be studied in relation to rootoids in subsequent papers, but it is appropriate to include some imprecise general remarks here to indicate in what respects the resulting pairs \( (G, S) \) are similar to,
and differ from, even $C_2$-systems. As a first comment in that regard, note that a Coxeter groupoid with one object is a Coxeter group, but many other groups arise as underlying groupoids of even $C_2$-systems (see [8,14,8,16]). Correspondingly, Coxeter groupoids share more properties in common with Coxeter groups than general even $C_2$-systems (but they also do not form a closed class with respect to basic constructions of these papers; see the comments on covering quotients below, and also [13,7]).

A Coxeter groupoid $G$ may be defined by a presentation by generators $S$ and relations specified in terms of certain defining data consisting of families of $I$-indexed Coxeter matrices, for a fixed index set $I$, related by additional combinatorial data. See [7] and [20] for precise definitions, which provide a specified indexing of the canonical generating set $S$ of $G$ by $I \times \text{ob}(G)$, inducing bijections $aS \cong I$ for $a \in \text{ob}(G)$. This indexing may not be completely determined by the pair $(G, S)$ alone, and provides additional structure, when the Coxeter matrices contain infinite entries.

The theory of Coxeter groupoids in [7] and [20] is developed assuming existence of a root system, realized in a family of real vector spaces, satisfying certain conditions in terms of the defining data. In particular, the root systems satisfy an integrality condition (roots are integral linear combinations of simple roots), though [20, Remark 5] raises the possibility of relaxing the condition to allow real linear combinations. It can be shown, however, that root systems even in the relaxed sense do not exist for all possible sets of defining data (i.e. the open question at the end of [20, Remark 5] has a negative answer), and necessary and sufficient conditions for their existence are not known. On the other hand, Remarks (2)–(3) in [9,6] suggest that there may be natural examples of Coxeter groupoids sharing many of the formal properties developed in [7] and [20], without root systems in the sense there, but with root systems in a more abstract sense than (even the relaxed sense) considered in [7] and [20] above.

Fix a Coxeter groupoid $G$ with canonical generators $S$ and, for simplicity in explanation here, with a root system $\Phi$ in the sense of [7] and [20]. View $\Phi$ as a representation of $G$ in the category $\text{Set}_\pm$ in a similar way as for root systems of Coxeter groups. Then the pair $(G, \Phi)$ turns out to be a principal, rootoidal signed groupoid-set with simple generators $S$, and the braid presentation of $(G, S)$ coincides with the defining presentation of $(G, S)$ as Coxeter groupoid. This can be proved by verifying WEC and SLC using properties of $G, \Phi, S$ established in [7] and [20], though in these papers we shall give the result assuming only existence of a root system in a more general sense. Further, as a consequence of the existence of the canonical indexing $S \cong \text{ob}(G) \times I$ as part of the definition of Coxeter groupoid, one has also the following fact:

(*) There exists a representation $\pi': G \to \text{Set}$ such that:

(i) For all $a \in \text{ob}(G)$, one has $\pi'(a) = aS$.

(ii) For all $a, b \in \text{ob} G$, $r \in aG_b$, $s \in bG$ with $\pi_r(s)$ defined, one has $\pi_r(s) = (\pi'(r))(s) \in aS$.

(iii) The restriction of $\pi'$ to each component of $G$ is a trivial representation.
Here, \( \pi_r = \pi_r^{(G,S)} \) as in [8.9]. Note that by [11] Lemma 1.14, the condition (iii) can be omitted if \( G \) is simply connected. These last comments are closely related to the following facts, which are stated imprecisely here: the class of even \( C_2 \)-systems is closed under taking both coverings and covering quotients, while the class of Coxeter groupoids is closed under coverings but not covering quotients (see Example 9.5(5)).

The condition \((\ast)\) depends only on \((G, S)\). The class of even \( C_2 \)-systems \((G, S)\) which satisfy \((\ast)\) may be regarded as a natural extension of the class of Coxeter groupoids with root system as in [7] and [20], to a class of Coxeter groupoids with abstract root systems. Note that explicitly specifying a representation \( \pi' \) as in \((\ast)\) provides additional structure beyond that from the pair \((G, S)\) itself, in general.

8.12. **Solvability of the word problem.** Tits solution of the word problem for Coxeter groups ([4, Ch 4, §1, Ex 13]) applies mutatis mutandis to even \( C_2 \)-systems. Slightly more precisely, in an even \( C_2 \)-system \((G, S)\), any non-reduced expression is braid equivalent to an obviously non-reduced one (i.e. one with a consecutive subexpression \( ss^* \) for some \( s \in S \)), and any two reduced expressions of the same element differ by a sequence of braid operations. Of course, as for Coxeter systems \((W, S)\), for this to give rise to an actual algorithm, finiteness of \( S \) is necessary.

8.13. **Standard parabolic subgroupoids.** For a rootoid \( \mathcal{R} = (G, \Lambda, N) \), a standard parabolic subgroupoid is defined as a subgroupoid \( H \) of \( G \) such that for each \( a \in \text{ob}(H) \), \( aH \) is a join-closed order ideal of \( _aG \) in weak right order i.e. an order ideal such if \( h_i \) is a non-empty family in \( aH \) such that \( g := \bigvee_i h_i \) exists in \( _aG \), then \( g \in aH \). A standard parabolic subgroupoid \( H \) is saturated if \( \text{ob}(H) = \text{ob}(G) \); any standard parabolic subgroup can be enlarged to a saturated one by possibly adjoining some trivial components.

**Example.** The saturated standard parabolic subgroupoids of \( G := \mathcal{C}_{(W,S)} \), where \((W,S)\) is a Coxeter system, are the standard parabolic subgroups \( W_J \) for \( J \subseteq S \), where \( W_J \) is the standard parabolic subgroup of \( W \) generated by \( J \) (both \( W_J \) and \( W \) being regarded as one-object groupoids). The only non-saturated standard parabolic subgroupoid is the empty subgroupoid.

Certain subtleties arise, for instance with the notions of rank of standard parabolic subgroupoids (see the examples of [8.16] below). However, basic results about longest elements, standard parabolic subgroups and shortest coset representatives of Coxeter groups extend mutatis mutandis to even \( C_2 \)-systems \((G, S)\). (The results about cosets apply only to saturated standard parabolic subgroups.) Some closely related results in the special case of Weyl groupoids can be found in [19], but the notion of standard parabolic subgroupoid used there is not the same as the restriction to Weyl groupoids of the notion here.

8.14. **Reflection systems.** Reflection systems \((G, X)\) in the sense of [13] are groups \( G \) with presentations by generators \( X \) satisfying relations obtained by suitably mixing the relations occurring in standard presentations of Coxeter groups, and coproducts of cyclic groups in the categories of groups and abelian groups. The results of this subsection and 8.15 imply that an extensive class of (even) \( C_2 \)-systems may be
constructed from reflection systems; the proofs, given in subsequent papers, involve verifying the WEC and SLC.

To define reflection systems precisely, denote an alternating product $aba \cdots$ of length $m$ of elements $a$ and $b$ in a group as $(aba \cdots)_m$. Let $X$ be a set and suppose given $m_x \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $x \in X$, $m_{x,y} = m_{y,x} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $x \neq y$ in $X$ such that $m_{x,y} \notin \{2, \infty\}$ implies $m_x = m_y = 2$. A reflection system is a pair of a group $G$ and subset $X \subseteq G$ such that for some $m_x, m_{x,y}$ satisfying the above conditions,

\[(8.15.1) \quad G \cong \{X \mid x^{m_x} = 1 \text{ if } x \in X, m_x \neq \infty, \quad (xyx \cdots)_{m_{x,y}} = (yxy \cdots)_{m_{x,y}} \text{ if } x, y \in X, x \neq y \text{ and } m_{x,y} \neq \infty\}\]

(more precisely, the relations on the right hold in $G$ and the canonical map from the group with presentation on the right to $G$ is an isomorphism).

These conditions imply that the canonical map $X \to G$ is injective, that $m_x = \text{ord}(x)$ for $x \in X \subseteq G$, and that for distinct $x, y \in X$, $m_{x,y}$ is the minimum $n \in \mathbb{N}_{\geq 2}$ such that $(xyx \cdots)_n = (yxy \cdots)_n$ if such an $n$ exists and otherwise $m_{x,y} = \infty$.

**Theorem.** Let $(G, X)$ be a reflection system in the sense of [13] and set $S := X \cup X^*$. Then $(G, S)$ is a $C_2$-system, which is even if and only if there exists no $x \in X$ for which $m_x$ is a finite, odd integer.

**Remarks.** In particular, arbitrary coproducts $G$ in the category of groups (resp., in the category of abelian groups) $G$ of cyclic groups $\langle x_i \rangle$ of even or infinite order, for $i \in I$ give even $C_2$-systems $(G, S)$ with $S = \{x_i, x_i^* \mid i \in I\}$. On the other hand, if one allows odd order cyclic groups $\langle x_i \rangle$ as well above, the Theorem gives only that $(G, S)$ is a $C_2$-system. The class of general $C_2$-systems seems to be a natural extension of the class of the even ones, but its theory is substantially more involved and we leave it largely open. Note that while the condition that $(G, S)$ be a $C_2$-system entails no restriction on the groupoid $G$ (see Example 8.16(a)), it is still a stringent restriction on the pair $(G, S)$.

8.15. **Semidirect products.** Let $G$ be a groupoid and $H$ be a group. Assume that there is a given group homomorphism $\alpha : H \to \text{Aut}_{\text{Gpd}}(G)$ denoted as $h \mapsto \alpha_h$. For $h \in H$ and any object or morphism $x$ of $G$, write $h(x) := \alpha_h(x)$. Then the semidirect product groupoid $K := G \rtimes_h H$ is defined as follows. One has

\[(8.15.1) \quad \text{ob}(K) = \text{ob}(G), \quad \text{mor}(K) = \text{mor}(G) \times H,\]

\[(8.15.2) \quad \text{cod}(\delta, h) = \text{cod}(\delta), \quad \text{dom}(\delta, h) = h^*(\text{dom}(\delta))\]

and

\[(8.15.3) \quad (\delta, h)(\gamma, h') = (\delta h(\gamma), hh')\]

when defined, where $\delta, \gamma \in \text{mor}(G)$ and $h, h' \in H$.

Now assume in addition that $(G, S)$ and $(H, R)$ are $C_0$-systems (regarding $H$ as one-object groupoid) and that the action by $H$ on $G$ preserves $S$. We say that $(H, R)$ acts on $(G, S)$. Let

\[(8.15.4) \quad T := \{(\delta, 1_H) \mid \delta \in S\} \cup \{(1_u, r) \mid a \in \text{ob}(G), r \in R\} \subseteq \text{mor}(K).\]
Then \((K, T)\) is a \(C_0\)-system called the semidirect product of \((G, S)\) and \((H, R)\) and denoted as \((K, T) = (G, S) \rtimes (H, R)\).

**Theorem.** Let \(i \in \{0, 1, 2\}\). If \((G, S)\) and \((H, R)\) are \(C_i\)-systems such that \((H, R)\) acts on \((G, S)\), then the semidirect product \((G, S) \rtimes (H, R)\) is a \(C_i\)-system, which is even if and only if \((G, S)\) and \((H, R)\) are both even.

**Remarks.** Similarly, \(C_2\)-systems are closed under natural notions of infinite direct sums (special subgroups of infinite products) and coproducts (disjoint union). Those with underlying groupoid equal to a group are also closed under constructions giving free products (in the category of groups) and restricted wreath products of underlying groups. Several of these constructions have versions for general rootoids, which are compatible with those for (even) \(C_2\)-systems; the compatibility amounts to a non-trivial closure property of rootoids from (even) \(C_2\)-systems under such general constructions. The study of \(C_2\)-systems in relation to amalgamated free products and HNN extensions, and, more generally, arboreal group theory (fundamental groups and groupoids of graphs of groups, etc; see [26]) may be of interest.

8.16. To finish this section, here are examples of \(C_0\)-systems with their underlying groupoid equal to a group. They can be checked to be \(C_2\)-systems either directly or using the results quoted without proof above. The braid presentations, and in most cases the maps \(\pi_r\), of the even \(C_2\)-systems are also given.

**Example.** (1) Let \(G\) be any groupoid and let \(S\) be the set of all non-identity morphisms of \(G\). Then \((G, S)\) is a \(C_2\)-system. For \(a \in \text{ob}(G)\), the weak order \(\leq_a\) on \(aG\) has minimum element \(1_a\), and the elements of \(aG \setminus \{1_a\}\) are incomparable.

(2) Let \(G\) be a non-trivial cyclic group with generator \(x\), and let \(S := \{x, x^*\} \subseteq G\). Then \((G, S)\) is a \(C_2\)-system, which is even if and only if the order \(m\) of \(x\) is even or infinite; the corresponding rootoid \(\mathcal{J}_{(G, S)}\) is complete if and only if \(x\) is of finite, even order. Suppose from now that \((G, S)\) is even. If \(m = 2\), the Coxeter matrix at the (unique) object of \(G\) is specified by the Coxeter graph \(\circ\) and the braid presentation \(\langle x, x^* | x^{-1} = x \rangle\) involves only a single relation, which is trivial. Henceforward assume that \(m \geq 4\). Then the Coxeter matrix of \((G, S)\) is specified by a Coxeter graph \(\circ \cdots \circ\). The braid presentation is

\[
\langle x, x^* | x^{-1} = x^*, (xx \cdots)_{m/2} = (x^*x^* \cdots)_{m/2} \rangle
\]

where in the relation involving \(m/2\), there are \(m/2\) factors in each product if \(m\) is finite and the relation is omitted if \(m = \infty\). One has \(\pi_x(x) = x^*\) and \(\pi_{x^*}(x^*) = x\), while \(\pi_x(x^*)\) and \(\pi_{x^*}(x)\) are defined if and only if \(m\) is finite, in which case \(\pi_x(x^*) = x\) and \(\pi_{x^*}(x) = x^*\). Note that although \((G, S)\) is an even \(C_2\)-system, in this case \(S\) is not a minimal generating set of \(G\).

(3) Let \((W, S)\) be a dihedral Coxeter system i.e. one with \(|S| = 2\). Write \(S = \{r, s\}\). Let \(x = rs\), so \(x^* = sr\). Let \(R = \{x, x^*, r = r^*\}\), which is another set of generators of \(W\) as group. Then \((W, R)\) is a \(C_2\)-system; it is even if and only if the order \(m\) of \(rs\) in \(W\) is even or infinite. The corresponding rootoid \(\mathcal{J}_{(W, R)}\) is complete if and only if \(m\) is finite and even. If \(m = 2\), then \((W, R)\) is a dihedral
Coxeter system. Assume \( m > 3 \) is even. The Coxeter graph of the unique object of \((W, R)\) is then
\[
x^{m/2} \xrightarrow{r} x^* \xrightarrow{r} \quad \text{and the braid presentation is}
\]
\[
\langle x, x^*, r \mid x^{-1} = x^*, \quad r^{-1} = r, \quad rx = x^*r, \quad r x^* = x r \rangle = (x x^* \cdots x^*)^{m/2}.
\]
The description of the maps \( \pi_r \), for \( r \in R \), is left to the reader in this example.

(4) Suppose in (3) that \( W \) is dihedral of order 8, and let \( t = r s r \in W \). Set \( V := \langle r, s, t \rangle \). One can check that \((W, V)\) is an even \( C_2 \)-system. The unique weak order of \( (W, V) \) is Boolean of rank 3 i.e. isomorphic to the lattice of subsets of \( \{r, s, t\} \). The Coxeter graph at the unique object consists of three isolated vertices
\[
r \quad s \quad t
\]
but it still corresponds to a non-trivial braid presentation
\[
\langle r, s, t \mid r^{-1} = r, \quad s^{-1} = s, \quad t^{-1} = t, \quad ts = st, \quad rt = sr, \quad rs = tr \rangle
\]

since the representation \( \pi^{W,V} \) of \( W \) on \( V \) is non-trivial \((\pi_s = \pi_t = \text{Id}_V\) and \( \pi_r = (s, t) \) is the transposition on \( V \) which interchanges \( s \) and \( t \)\). Since \((W, V)\) is not a Coxeter system, this shows that WEC does not imply EC. Also, a \( C_2 \)-system \((G, S)\) with \( G \) a group and in which \( S \) consists of involutions need not be a Coxeter system.

(5) For a Coxeter system \((W, S)\), the Coxeter matrix of \((W, S)\) as \( C_2 \)-system coincides with the Coxeter matrix of \((W, S)\) as a Coxeter system. Denote the Coxeter matrix as \((m_{r,s})_{r,s \in S}\). The braid presentations of \((W, S)\) as \( C_2 \)-system and as Coxeter system also coincide. One has, for \( r, s \in S \), that \( \pi_r(s) = s \) if \( m_{r,s} \neq \infty \) and \( \pi_r(s) \) is undefined if \( m_{r,s} = \infty \).

(6) The example (4) (and also (5)) is a special case of the following one. Let \((W, S)\) be a Coxeter system. Suppose that \( S = I \cup J \) where no element of \( I \) is conjugate to any element of \( J \). Let \( \tilde{J} := \{ wsw^{-1} \mid w \in W_I, s \in J \} \) and \( \tilde{W} \) be the subgroup of \( W \) generated by \( \tilde{J} \). It is shown in [15] and in [3] that \((\tilde{W}, \tilde{J})\) is a Coxeter system and \( W \) is the semidirect product \( W = W_I \rtimes \tilde{W} \). Set \( V := I \cup \tilde{J} \). It follows from Theorem [15] that \((W, V)\) is an even \( C_2 \)-system. The braid relations are the braid relations of \((W_I, I)\), those of \((\tilde{W}, \tilde{J})\) and the relations \( rs = s r \) for \( r \in I \), \( s, \tilde{s} \in \tilde{J} \) with \( \tilde{s} = r s r^{-1} \).

Let \((m_{r,s})_{r,s \in S}\) be the Coxeter matrix of \((W, S)\), let \((\tilde{m}_{r,s})_{r,s \in \tilde{J}}\) be that of \((\tilde{W}, \tilde{J})\) and let \((m'_{r,s})_{r,s \in V}\) be the Coxeter matrix of \((W, V)\) as \( C_2 \)-system. For any \( r, s \in V \), one has \( m'_{r,s} = 2 \) if one of \( r, s \) is in \( I \) and the other is in \( J \), while \( m'_{r,s} = m_{r,s} \) if \( r, s \) are both in \( I \) and \( m'_{r,s} = \tilde{m}_{r,s} \) if both \( r, s \) are in \( \tilde{J} \). Note that one could have \( r \in I, s \in J \) with \( m'_{r,s} = 2 \) but \( rs \) of infinite order as an element of the group \( W \). Recall that for \( r, s \in V \), \( \pi_r(s) \) is defined if and only if \( m'_{r,s} \neq \infty \). The maps \( \pi_r \) for \( r \in V \) may be specified as restrictions of maps \( \pi'(r) \) where \( \pi' \) is the permutation representation of the group \( W \) on the set \( V \) defined as follows First, the extended representation \( \pi' \) is
trivial when restricted to $\tilde{W}$, so it factors through a representation of $W_I \cong W/\tilde{W}$. On $V = I \cup J$, $W_I$ acts trivially on $I$ and by conjugation on $J$.

9. General constructions of protorootoids

9.1. Protorootoid of a simple graph. We fix some notation and terminology concerning simple (undirected) graphs. A simple graph $\Gamma$ is by definition a pair $(V, E)$ where $V$ is a set (the vertex set of $\Gamma$) and $E$ (the edge set of $\Gamma$) is a set of two-element subsets of $V$ (called edges of $\Gamma$). A path in $\Gamma$ is a sequence $(v_n, \ldots, v_0)$ in $V$, where $n \in \mathbb{N}$, such that $\{v_i, v_{i-1}\} \in E$ for $i = 1, \ldots, n$. This path is called a path of length $n$ from $v_0$ to $v_n$. The path is called a cycle (of length $n$) if $v_0 = v_n$, and a simple cycle (of length $n$) if it is a cycle, $n \geq 3$ and $v_i \neq v_j$ for $0 \leq i < j < n$.

The graph $\Gamma$ is said to be connected if it is non-empty and for any two vertices $x, y \in V$, there is a path from $x$ to $y$. An isomorphism $(V, E) \to (V', E')$ of simple graphs is a bijection $\sigma: V \to V'$ such that for distinct $x, y \in V$, one has $\{x, y\} \in E$ if and only if $\{\sigma(x), \sigma(y)\} \in E$. A subgraph of $(V, E)$ is a graph $(V', E')$ with $V' \subseteq V$ and $E' \subseteq E$. The subgraph $(V', E')$ is full if $E' = E \cap \{ \{x, y\} \mid x, y \in V', x \neq y \}$. A component of $(V, E)$ is a full, connected subgraph $(V', E')$ with $V' \neq \emptyset$ maximal under inclusion.

A forest is a simple graph with no simple cycles. A tree is a connected forest. Any simple graph $\Gamma$ has, by Zorn’s lemma, a maximal subforest $F$ i.e. a subgraph $F$ of $\Gamma$ which is a forest such that its sets of vertices and sets of edges are both maximal (under inclusion) amongst those of subgraphs of $\Gamma$ which are forests. It is easily seen (see e.g. [26]) that the vertex set of a maximal subforest $F$ of $\Gamma$ coincides with that of $\Gamma$.

Attach to the simple graph $\Gamma = (V, E)$ the (unique up to isomorphism) simply connected groupoid $G = G_\Gamma$ with $\text{ob}(G) = V$ in which there is a morphism $b \to a$ in $G$ if and only if there is a path from $b$ to $a$ in $\Gamma$. Concretely, we set

$$(9.1.1) \quad \text{Hom}_G(b, a) = \begin{cases} \{(a, b)\}, & \text{if there is a path from } b \text{ to } a \text{ in } \Gamma \\ \emptyset, & \text{otherwise} \end{cases}$$

with composition $(a, b)(b, c) = (a, c)$ when $(a, b)$ and $(b, c)$ are morphisms of $G$. The groupoid $G$ has a generating set $S := \{(a, b) \mid a, b \in V, \{a, b\} \in E\}$. Also, the groupoid $G$ is simply connected, and it is connected if and only if $\Gamma$ is connected. More generally, the components of $G$ naturally correspond to those of $\Gamma$.

The definitions of Section 8 for the $C_0$-system $(G, S)$ can be transferred to the graph $\Gamma$ as follows.

**Definition.**

(a) For $i \in \{0, 1, 2\}$, the graph $\Gamma$ is an (even) $C_i$-graph if $(G, S)$ is an (even) $C_i$-system.

(b) The protorootoid (resp., set protorootoid, signed groupoid set) of $\Gamma$ is $\mathcal{J}_\Gamma := \mathcal{J}_{(G, S)}$ (resp., $\mathcal{J}'_\Gamma := \mathcal{J}'_{(G, S)}$, $J_\Gamma := J_{(G, S)}$).

(c) For an even $C_0$-graph $\Gamma$, define $\overline{\mathcal{J}}_{\Gamma} := \overline{\mathcal{J}}_{(G, S)}$, $\overline{\mathcal{J}}'_{\Gamma} := \overline{\mathcal{J}}'_{(G, S)}$ and $\overline{J}_{\Gamma} := \overline{J}_{(G, S)}$. 


Note that $C_0$-graphs are just simple graphs, and saying that a $C_1$-graph is even means it has no odd length cycles.

**Example.** (1) Suppose that $n \in \mathbb{N}_{\geq 3}$ and the simple graph $\Gamma = (V, E)$ is an $n$-cycle. Write $V = \{v_1, \ldots, v_n\}$ and $E = \{ \{v_i, v_{i+1}\} \mid i = 1, \ldots, n \}$ where $v_{n+1} := v_1$. Then (see Examples 9.2(3)–(4)) $\Gamma$ is a $C_2$-graph, which is even if and only if $n$ is even. The corresponding rootoid $\mathcal{J}_\Gamma$ is complete if and only if $n$ is even. The rootoids $\mathcal{J}_\Gamma$ from the graphs here are easily seen to be the universal covering rootoids of the rootoids of finite cyclic groups of order $n \geq 3$ in Example 9.10(2). (The universal covering rootoid of the rootoid from the infinite cyclic group is $\mathcal{J}$ where $\Gamma$ is a doubly infinite path).

(2) If $\Gamma$ is a forest then it is an even $C_2$-graph; further, in that case, $\mathcal{J}_\Gamma$ is complete if and only if $\Gamma$ contains no distinct edges with a common vertex.

(3) More generally than (1), let $(G, S)$ be a $C_0$-system such that $G$ is connected. Define the Cayley graph $(V, E)$ of $(G, S)$ as follows. Fix $a \in \text{ob}(G)$. Set $V := aG$ and

$$E := \{ \{g, gs\} \mid b \in \text{ob}(G), g \in aG_b, s \in bS \}.$$  

(9.1.2)

It is easy to see that, up to isomorphism of simple graphs, this is independent of the choice of $a$. Then for $i \in \{0, 1, 2\}$, $(G, S)$ is an (even) $C_i$-system if and only if its Cayley graph $(V, E)$ is an (even) $C_i$-graph. (The easy details of the argument will be given in subsequent papers). This fact implies that the conditions for a general $C_0$-system $(G, S)$ to be a $C_i$-system amount to certain metric conditions (with respect to the path length metric) on the vertices of the Cayley graphs of the components of $G$.

9.2. **Protorootoid attached to a rainbow graph.** A *rainbow graph* $X$ is defined to be a quadruple $X = (V, E, c, B)$ where $\Gamma := (V, E)$ is a simple graph, $B$ is a Boolean ring and $c : E \to B$ is a function, subject to the requirement that for any cycle $(v_n, \ldots, v_0)$ in the graph, $c(e_1) + \ldots + c(e_n) = 0_B$ where $e_i := \{v_{i-1}, v_i\}$.

The rainbow graph $X$ is said to be defined over $B$. A particular case which motivates the terminology, is when $B = \wp(Y)$ for some set $Y$. The set $Y$ is called the color set of $X$ and elements of $Y$ are called colors. Then $c$ determines a labelling of each edge $e \in E$ by a set $c(e)$ of colors subject to the requirement that the symmetric difference of color labels of the edges in any cycle is empty i.e. each color appears an even number of times in the color sets attached to the edges of the cycle.

Let $G = G_\Gamma$ be the groupoid and $S$ its set of generators attached to the simple graph $\Gamma$ as in 9.2. Let $\Lambda$ be the constant functor $G \to \text{BRng}$ with constant value $B$. Define a cocycle $N$ of $G$ for $\Lambda$ as follows; for any $a, b \in G$ which are joined by a path in $\Gamma$ with successive vertices $a = a_n, a_{n-1}, \ldots, a_0 = b$, set $a \Lambda((a, b)) := c(e_1) + \ldots + c(e_n) \in a\Lambda$ where $e_i := \{a_{i-1}, a_i\} \in E$ (here as in 9.1 (a, b) denotes the unique morphisms from $b$ to $a$ in $G$). This is well-defined and gives a cocycle, as required by the definition of rainbow graph.

**Definition.** (a) The protorootoid of the rainbow graph $R$ is defined as $\mathcal{J}_R := (G, \Lambda, N)$.

(b) The rainbow graph $R$ is rootoidal if $\mathcal{J}_R$ is a rootoid.
Note that if $X$ is defined over a Boolean algebra $B = \varphi(Y)$ of sets, then $\mathcal{J}_X$ is the associated protorootoid $\mathcal{J}_X = \mathcal{J}(\mathcal{J}_X)$ of a set protorootoid $\mathcal{J}_X := (G, Y', N)$ where $Y': G \to \text{Set}$ is the constant functor with value $Y$. In that case, one also has an associated signed groupoid-set $J_X := \mathfrak{R}(\mathcal{J}_X)$.

9.3. Though the following result is equivalent to a well known description of the cycle space of a simple graph, details of its proof are given for completeness. It provides a convenient means of specifying examples of rainbow graphs, but is not useful in characterizing the more interesting subclass of rootoidal rainbow graphs.

**Proposition.** Let $\Gamma = (V, E)$ be a simple graph, $B$ be a Boolean ring and $T = (V, E')$ be a maximal subforest of $\Gamma$, with edge set $E' \subseteq E$. Then any function $c': E' \to B$ extends uniquely to a function $c: E \to B$ such that $(V, E, c, B)$ is a rainbow graph.

**Proof.** Let $\Gamma$ be as above. Let $U$ be a vector space with $E$ as $\mathbb{F}_2$-basis, and $U'$, $U''$ be the subspaces of $U$ spanned by $E'$, $E \setminus E'$ respectively, so $U = U' \oplus U''$. The cycle space of $\Gamma$ is the subspace $C$ of $U$ spanned by all its elements $e_1 + \ldots + e_n$ such that there is a cycle $(v_0, \ldots, v_{n-1})$ in $\Gamma$ satisfying $e_i = \{v_{i-1}, v_i\}$. For any function $c: E \to B$, let $\hat{c}: U \to B$ be its unique extension to a $\mathbb{F}_2$-linear map (regarding $B$ as possibly non-unital $\mathbb{F}_2$-algebra in the natural way). From the definitions, $(V, E, c, B)$ is a rainbow graph if and only if $\hat{c}$ vanishes on $C$, or equivalently, if and only if it vanishes on $Z$ where $Z$ is some chosen basis of $C$.

Choose the basis $Z$ of $C$ according to the following well-known procedure involving $T$. For each edge $f \in E \setminus E'$, there is a simple cycle of the graph $(V, E' \cup \{f\})$, by maximality of $T$. The set $\{e_{f,1}, \ldots, e_{f,n_f}\}$ of edges of this cycle contains $f$, say $f = e_{f,1} \in E \setminus E'$ and $e_{f,i} \in E'$ for $i > 1$ without loss of generality. Define $z_f := e_{f,1} + \ldots + e_{f,n_f} \in C$. Since $z_f + U' = f + U' \in U/U' \cong U''$ and the elements $f \in E \setminus E'$ are pairwise distinct and the set $Z := \{ z_f \mid f \in E \setminus E' \}$ is a basis of $C$ over $\mathbb{F}_2$. It is also well known and easily shown (see for instance [9]) that $Z$ spans $C$, so it is a basis of $C$. Hence $Z \cup E'$ is a basis of $U$.

It follows immediately that if $(V, E, c, B)$ is a rainbow graph, then $c$ is uniquely determined by its restriction to a function $c': E' \to B$ (since $\hat{c}$ vanishes on $Z$). Conversely, an arbitrary function $c': E' \to B$ extends uniquely to a linear map $c'' : U \to B$ which is zero on $Z$. Let $c$ denote the restriction of $c''$ to a function $c: E \to B$. Then $(V, E, c, B)$ is a rainbow graph such that $c$ extends $c'$. \hfill \Box

9.4. Let $\Gamma := (V, E)$ be a simple graph. The associated protorootoid $\mathcal{J}_\Gamma$ (and, if $\Gamma$ is even, $\mathcal{J}_\Gamma$) have as components the associated protorootoids $\mathcal{J}_\Gamma_i$ (and, if $\Gamma$ is even, $\mathcal{J}_\Gamma_i$) for the components $\Gamma_i$ of $\Gamma$. Assume henceforward in this subsection that $\Gamma$ is connected. Rainbow graphs which afford the associated protorootoid(s) are described below.

Let $(G, S)$ be the $C_0$-system associated to $\Gamma$, as defined in [31]. One has $\text{ob}(G) = V$, $\text{mor}(G) = \{(a, b) \mid a, b \in \text{ob}(G)\}$ and $S = \{(a, b) \mid \{a, b\} \in E\}$. Hence for $(a, b) \in S$ $l_S(a, b) := l_S((a, b))$ is the minimum length of a path in $\Gamma$ from $a$ to $b$. 

For all \( s = (a, b) \in S \), define
\[
X_s := \{ x \in V \mid l_S(b, x) > l_S(a, x) \} \subseteq V
\]
Note that \( a \in X_s \) and \( b \notin X_s \). Let \( X := \{ X_r \mid r \in S \} \) and \( B := \mathcal{P}(X) \). Define an edge labelling \( c : E \to B \) as follows: for an edge \( \{a, b\} \in E \), \( c(\{a, b\}) \) is the set of all elements of \( X \) which contain exactly one of the vertices of the edge i.e.
\[
c(\{a, b\}) := \{ A \in X \mid |\{a, b\} \cap A| = 1 \}.
\]
Also, if \( \Gamma \) is even, note that for \( s = (a, b) \in S \), \( Y_s := \{ X_s, X_{s'} \} \) is a partition of \( V \) into two disjoint non-empty sets. Set \( Y := \{ Y_s \mid s \in S \} \) and \( B' := \mathcal{P}(Y) \). Define an edge labelling \( c' : E \to B' \) as follows: for an edge \( \{a, b\} \in E \), \( c'(\{a, b\}) \) is the subset of all elements \( A = \{A_1, A_2\} \) of \( Y \) such that \(|A_i \cap \{a, b\}| = 1\) for \( i = 1, 2\).

**Proposition.**  
(a) The quadruple \( R = (V, E, c, B) \) is a rainbow graph whose associated protorootoid \( \mathcal{J}_R \) is isomorphic to the protorootoid \( \mathcal{J}_\Gamma \).

(b) If \( \Gamma \) is even, \( R' = (V, E, c', B') \) is a rainbow graph whose associated protorootoid \( \mathcal{J}_{R'} \) is isomorphic to the protorootoid \( \mathcal{J}_\Gamma' \).

**Proof.** We prove (a). In addition to the above notation, use notation concerning \( \mathcal{J}_\Gamma = \mathcal{J}_{(G, S)} = (G, \Psi, N) \) as in Section 8. Suppose that \( s = (a, b) \in S \) and \( g = (e, a) \in \text{mor}(G) \). By definition,
\[
G^2_s = \{ (a, x) \in aG \mid l_S(b, x) > l_S(a, x) \} = \{ (a, x) \mid x \in X_s \}
\]
and therefore
\[
g(G^2_s) = X_{e, s} := \{ (e, x) \mid x \in X_s \}.
\]
Hence \( e \Psi = \{ X_{e, s} \mid s \in S \} \) and \( e \Psi' = \{ X_{e, s} \mid s \in S, e \in X_s \} \). So for \( h = (e, d) \in \text{mor}(G) \), \( N(h) := e\Psi' + h(e\Psi') \) is given by
\[
N(h) = \{ X_{e, s} \mid s \in S, \ ((\{e\} + \{d\}) \cap X_s) = 1 \}
\]
Since \( N \) is a cocycle, for any path \( (e = e_n, \ldots, e_0 = d) \) from \( d \) to \( e \), \( N(e, d) \) consists of the elements \( X_{e, s} \) such that \( X_s \in M((e, d)) := c(e_n, e_{n-1}) + \ldots + c(e_1, e_0) \).

Let \( \Theta \) be the constant functor \( G \to \text{Set} \) with constant value \( X \). The above implies that \( M \) is a cocycle for \( \mathcal{P}_G(\Theta) \), since \( N \) is a cocycle. Therefore \( R \) is a rainbow graph. The set protorootoids underlying \( \mathcal{J}_\Gamma \) and \( \mathcal{J}_R \) are \( \mathcal{J}_\Gamma = (G, \Psi, N) \) and \( \mathcal{J}_R = (G, \Theta, M) \). It follows readily from the above that there is an isomorphism \((\text{Id}_G, \nu) : \mathcal{J}_\Gamma \to \mathcal{J}_R \) of set protorootoids where \( \nu : a\Theta \to a\Psi \) is the isomorphism \( X_s \mapsto X_{a, s} : X \to a\Psi \). Then \( \mathfrak{J}((\text{Id}_G, \nu)) : \mathcal{J}_\Gamma \to \mathcal{J}_R \) is an isomorphism as required to prove (a). The proof of (b) is similar from the above and the definitions, and it is omitted.  

9.5. Here are examples illustrating the above constructions.
Example. (1) Let $\Gamma = (V, E)$ be the even simple graph with vertex set $V = \{p, q, r, s, t\}$ indicated schematically on the left.

This gives rise to the rainbow graph $R' = (V, E', B')$ shown schematically on the right above where $Y := \{u, v, w, x, y, z\}$, $B' := \wp(Y)$ and the label “uvw” on the edge $\{p, q\}$ means that $c'(\{p, q\}) = \{v, w, x\}$, etc. In fact, in the notation of [9,4]

$$
\begin{align*}
X_{(p,q)} &= \{p, r, s\} & X_{(q,p)} &= \{q, t\} & X_{(p,s)} &= \{p, r, q\} & X_{(s,p)} &= \{s, t\} \\
X_{(p,r)} &= \{p, s, q\} & X_{(r,p)} &= \{r, t\} & X_{(t,q)} &= \{t, r, s\} & X_{(q,t)} &= \{p, q\} \\
X_{(t,s)} &= \{t, r, q\} & X_{(s,t)} &= \{p, s\} & X_{(t,r)} &= \{t, s, q\} & H_{(r,t)} &= \{r, p\}
\end{align*}
$$

As color set, take the set $Y$ of partitions of $V$ where

$$
\begin{align*}
x &= \{\{p, r, s\}, \{q, t\}\} & y &= \{\{p, r, q\}, \{s, t\}\} & z &= \{\{p, s, q\}, \{r, t\}\} \\
u &= \{\{t, r, s\}, \{p, q\}\} & v &= \{\{t, r, q\}, \{p, s\}\} & w &= \{\{t, s, q\}, \{r, p\}\}.
\end{align*}
$$

This gives the rainbow graph $R'$ with edges labelled by subsets of $Y$ as above e.g. the partitions from $Y$ in which $p, q$ belong to complementary sets of the partition are precisely $v$, $w$ and $x$. Since the color labels aren’t all singleton sets, $\Gamma$ is not a $C_1$-graph.

(2) Similarly, the even graph $\Gamma$ at the left below gives rise to the rainbow graph $R'$ at the right.

Here, $Y = \{x, y, z\}$ where

$$
x = \{\{p, s, u\}, \{q, r, t, v\}\} & y = \{\{p, q, s, t\}, \{r, v, u\}\} & z = \{\{p, q, r\}, \{s, t, u, v\}\}
$$

Regarding $\Gamma$ as a subgraph of the cube graph obtained by deleting one vertex and the incident edges, edges receive the same label if and only if they correspond to parallel edges of the cube. Write $R = \mathcal{F}_\Gamma = (G, \Lambda, N)$. For a morphism $(a, b)$ of $G$, $l_S(a, b)$ is the minimum length of a path from $a$ to $b$, whereas $l_N(a, b)$ is the number of elements of $X$ which appear an odd number of times as edge-label in some (equivalently, every) path from $a$ to $b$. One easily checks $l_N(a, b) = l_S(a, b)$ for all vertices $a, b$, so $G$ is an (even) $C_1$-graph.

The Hasse diagram of weak order at a vertex $a \in V = \{p, q, r, s, t, u, v\}$ is just the graph $\Gamma$ directed by edge-distance $(v \mapsto l_S(a, v))$ from the vertex $a$. There are three
isomorphism types of weak orders, corresponding to the three orbits \( \{t\}, \{p, r, u\}, \{q, s, v\} \), of the automorphism group of \( \Gamma \) on \( V \). The corresponding Hasse diagrams are

respectively, so the weak orders are (complete) meet semilattices. However, \( \mathcal{R} \) is not a rootoid (and equivalently, \( \Gamma \) is not a \( C_{2^2} \)-graph) since JOP (more precisely, its consequence [11, Proposition 4.5(b)]) fails; in the third diagram, the middle atom is below the join of the leftmost atom and the rightmost atom. On the other hand, the edge graph of a cube (or any hypercube) is an even \( C_{2^n} \)-graph.

(3) Consider the even graph \( \Gamma \) (a 6-cycle) shown at the left in the diagram below.

The corresponding rainbow graphs \( R \) and \( R' \) are shown to the right, using as colors for \( R \) the set \( X = \{x', y', z', x'', y'', z''\} \) of subsets of \( V := \{p, q, r, s, t, u\} \) where

\[
x' = \{q, r, s\} \quad y' = \{r, s, t\} \quad z' = \{s, t, u\}
x'' = \{t, u, p\} \quad y'' = \{u, p, q\} \quad z'' = \{p, q, r\}
\]

and for \( R' \) the color set \( X' = \{x, y, z\} \) where

\[
x = \{x', x''\} \quad y = \{y', y''\} \quad z = \{z', z''\}
\]

The Hasse diagram of weak order at any object (vertex of \( \Gamma \)) is isomorphic to \( \Gamma \) directed by distance from that vertex:

It is easy to see that \( \Gamma \) is an even \( C_{2^n} \)-graph. The corresponding rootoids \( \mathcal{J}_\Gamma, \mathcal{J}_\Gamma' \) are complete, connected and simply connected, and preprincipal, and \( \mathcal{J}_\Gamma \) is principal. The case of an arbitrary cycle \( \Gamma \) of even length \( m \geq 4 \) is analogous.
(4) Consider the graph $\Gamma$ (a 5-cycle) shown at left in the diagram below:

The corresponding rainbow graph $R$ is shown to the right with color set $X = \{x, y, z, v, w\}$ where

$$x = \{q, r\}, \quad y = \{r, s\}, \quad z = \{s, t\}, \quad u = \{t, p\}, \quad v = \{p, q\}$$

Since $\Gamma$ is not even, the rainbow graph $R'$ is not defined. The Hasse diagram of weak order at any object (vertex of $\Gamma$) is

It is easy to see that $\Gamma$ is a $C_2$-graph. The corresponding rootoid $J_\Gamma$ is connected and simply connected, but not preprincipal and not complete. The case of an arbitrary cycle of odd length $m \geq 3$ is analogous.

(5) Let $\mathcal{T}$ be the universal covering of the rootoid $G_{(W,S)}$ of a dihedral Coxeter system of order $2m$ ($m \geq 2$). As is easily checked, $\mathcal{T} \cong \overline{\mathcal{F}}_\Gamma$, where the simple graph $\Gamma$ is a cycle of length $2m$ if $m$ is finite and $\Gamma$ is a doubly infinite path if $m = \infty$. From the definitions in [7] and [20], it is easily seen that $\mathcal{T}$ has a Coxeter groupoid as underlying groupoid (with canonical generating set as Coxeter groupoid given by the simple generators of $\mathcal{T}$ as principal rootoid). In turn, $\mathcal{T}$ has a covering quotient isomorphic to the principal protorootoid $\overline{\mathcal{F}}_{(G,X)}$ where $G$ is a cyclic group $G = \langle x \rangle$ of order $2m$ and $X = \{x, x^{-1}\}$. However $G$ is not a Coxeter groupoid.

9.6. Protorootoid of an oriented matroid. A (possibly infinite) oriented matroid in the sense of [6] is a triple $\langle E, *, c \rangle$ where $E$ is a set, $x \mapsto x^*: E \to E$ is a function and $c: \varphi(E) \to \varphi(E)$ is a closure operator, satisfying certain axioms which are not repeated here (in particular, $c$ is a convex closure operator in a suitable sense for signed sets). For a motivating example, see Remark (1) below.

Let $M = \langle E, *, c \rangle$ be an oriented matroid in the sense of [6]. Assume for simplicity that the following extra conditions hold: $E \neq \emptyset$, $c(\emptyset) = \emptyset$, and $x^* \neq x$ and $c(x) = \{x\}$ for $x \in E$. The axioms and extra conditions imply that setting $-x := x^*$ defines a free action of the sign group on $E$, making $E$ an indefinitely signed set. A subset $H$ of $E$ is called a hemispace if $H = c(H) = E \setminus (-H)$. It is shown in op. cit. that hemispaces exist.

Define a signed groupoid-set $(G, \Phi)$ as follows. Let $G$ be the connected, simply connected groupoid with the set of hemispaces as its set of objects. For $H, K$ in $\text{ob}(G)$, let $h\bar{f}_{HK} \in hG_K$ be the unique morphism $K \to H$ in $G$. Set $h\Phi := E$ regarded as definitely signed set with the above action of $\{\pm\}$ and with the hemispace $h\Phi_+ := H$ as its subset of positive elements. Define $\Phi: G \to \text{Set}_\pm$ to be a functor
with $\Phi(H) := _H\Phi$ for $H \in \text{ob}(G)$, and such that for any $x \in K\Phi = E$, one has $(\Phi_H f_K)(x) = x$. Thus, the composite $\Phi: G \to \text{Set}_\pm \to \text{Set}$ of $\Phi$ with the forgetful functor $\text{Set}_\pm \to \text{Set}$ is the constant functor with value $E$.

Note that $\Phi_H f_K = H \cap K \subseteq H = _H\Phi_+$. It is easily seen that this defines a faithful, complemented signed groupoid-set $J_M := (G, \Phi)$. Since $J_M$ is complemented, if it is rootoidal then it is complete. The associated set protorootoid and protorootoid are denoted as $J'_M := \mathcal{L}(J_M)$ and $J_M := \mathcal{I}(\mathcal{L}(J_M))$.

Remarks. (1) Given a subset $E$ of a real vector space closed under multiplication by $-1$, define $\ast$ by $x \ast = -x$, for $x \in E$, and $c$ by $c(X) := \mathbb{R}_{\geq 0}X \cap E$ where $\mathbb{R}_{\geq 0}X$ is the cone of non-negative linear combinations of elements of $X$; then $M = (E, \ast, c)$ is an oriented matroid. The additional conditions hold if $E \neq \emptyset$, $0 \not\in E$ and $x, \lambda x \in E$ with $\lambda \in \mathbb{R}_{> 0}$ implies $\lambda = 1$. The above construction therefore attaches a signed groupoid-set $J_M$ to $(V, E)$ when these conditions all hold.

(2) Oriented geometries (see [2]) are special oriented matroids $M$ with finite underlying sets $E$. Examples include the oriented matroid $M = (E, \ast, c)$ naturally associated (as in (1)) to the (finite) set $E$ of unit normals to the hyperplanes of a real, central hyperplane arrangement $\mathcal{H}$ as in [11, 6.11]; oriented geometries of this type are called realizable. Using facts established in [2], it can be shown by an argument similar to that in [11, 6.11] that the signed groupoid-set $J_M$ attached to a (non-empty) oriented geometry $M$ is rootoidal if and only if the oriented geometry is simplicial.

(3) If the non-empty oriented geometry $M$ is simplicial, then $J_M$ is a finite, connected, simply connected, complete, principal, signed groupoid-set. Also, any signed groupoid-set $(G, \Phi)$ with all these properties can be shown to give a Coxeter groupoid $G$ with root system $\Phi$ of a more general type than considered in [20] and [7] (see [8, 11]); further, for $a \in \text{ob}(G)$, $a\Phi_+$ admits a natural convex closure operator in the sense of [17]. Note these are closure operators for unsigned sets and are not known to all be induced by a single oriented matroid closure operator. It is an open question whether every finite, connected, simply connected, complete, abridged, principal, signed groupoid-set $(G, \Phi)$ is isomorphic to $J_M$ for some oriented simplicial geometry $M$ (i.e. whether $\Phi$ is realizable in oriented matroids).

For non-realizable simplicial oriented geometries $M$, it is expected that in general the root system of $J_M$ is not realizable in real vector spaces in any natural way (even more generally than as in [11, 6.6 or 6.11] or in [20], [7]). Finding an example to show this remains a non-trivial open problem, however, because the most natural notions of realizability for root systems do not exactly correspond to that of realizability of oriented matroids. Detailed discussion and proof of the claims in Remarks (2)–(3) is deferred to subsequent papers.

(4) In the setting of [11, Example 6.12(2)], there is an infinite oriented matroid $M = (E, \ast, c)$ naturally associated as in (1) to the unit sphere $E = S \subseteq V$. The hemispaces of $M$ are precisely the subsets of $S$ which are the positive elements of $S$ with respect to some vector space total ordering of $V$. It follows readily that the protorootoid $J_M$ is isomorphic to the universal cover of $J_{O(V)}$ and hence is a rootoid by [25] and [11, Lemma 4.9]. The class of possibly infinite oriented matroids $M$ for
which \( \mathcal{J}_M \) is a (necessarily complete) rootoid may therefore be regarded as a non-vacuous extension of the class of simplicial oriented geometries. Examples of such \( M \) satisfying stringent additional conditions (e.g. such that \( \mathcal{J}_M \) is pseudoprincipal, saturated and regular, see [11, Definition 3.3]) are known to exist, and should be particularly interesting.

9.7. **Protorootoid of a protomesh.** Let \( P = (R, L) \) be a protomesh (see [11, 2.8] for the definition). Assume \( L \neq \emptyset \) to avoid trivialities. There is an associated rainbow graph \( T = T_P = (V, E, c, R) \) as follows. The graph \( \Gamma = (V, E) \) is the complete graph on vertex set \( L \) i.e. \( V := L \) and \( E := \{ \{A, B\} \mid A, B \in L, A \neq B \} \). The labelling \( c: E \to R \) is given by \( c(\{A, B\}) = A + B \) where + denotes addition in \( R \). Define the protorootoid \( \mathcal{J}_T = (G, \Lambda, N) \) of the rainbow graph \( T \).

Concretely, \( \mathcal{J}_T = (G, \Lambda, N) \) where \( G \) is a connected, simply connected groupoid with \( \text{ob}(G) = L \), \( \Lambda \) is the constant functor \( G \to \text{BRng} \) with constant value \( R \), and \( N \) is the \( G \)-cocycle (in fact, coboundary) for \( \Lambda \) such that for \( f \in \Lambda G_B \), where \( A, B \in L \), one has \( N(f) = A + B \). It is obvious from this description that \( \mathcal{J}_T \) is faithful.

**Definition.**
(a) The protorootoid of the protomesh \( P = (R, L) \) is defined to be \( \mathcal{J}_P := \mathcal{J}_T_P = (G, \Lambda, N) \) as above.
(b) The protomesh \( P \) is called a **mesh** if \( \mathcal{J}_P \) is a rootoid.

We say that the protomesh \( P = (R, L) \) is defined over \( R \). If \( R \) is the Boolean ring of subsets of a set, one also defines the set protorootoid \( \mathcal{J}_{P'} := \mathcal{J}_{T_P} \) and signed groupoid-set \( J_M := J_{T_P} \) of \( P \). Terminology for rootoids and protorootoids may be extended to meshes and protomeshes as in [11, 5.6], by saying that a protomesh \( P \) has some property of rootoids if \( \mathcal{J}_P \) has that property. For example, a mesh \( P \) is said to be complete (resp., principal) if \( \mathcal{J}_P \) is a complete (resp., principal) rootoid.

9.8. **Characterization of meshes.** The following is obvious from the definitions.

**Proposition.**
(a) A protomesh \( (R, L) \) is a mesh if and only if for all \( \Gamma \in L \),
\[ \Gamma + L \text{ is a complete meet semilattice (in the order induced by the natural order of } R \text{) and the protomesh } (R, \Gamma + L) \text{ satisfies the JOP}. \]
(b) A mesh \( (R, L) \) is complete if for each \( \Gamma \in L \), \( \Gamma + L \) has a maximum element i.e. is a complete lattice.

**Example.** Let \( R \) be a complete Boolean algebra. By [11, Lemma 4.1], the protomesh \( (R, R) \) satisfies the JOP. Let \( L \) be an ideal of \( R \) as ring. It is easy to see that the protomesh \( (R, L) \) inherits the JOP from \( (R, R) \). Since \( \Gamma + L = L \) for all \( \Gamma \in L \), the proposition implies that that \( (R, L) \) is a mesh, and so \( \mathcal{J} := \mathcal{J}_{(R,L)} \) is a rootoid.

It is trivial that \( \mathcal{J} \) is regular and saturated, and that it is complete if \( L = R \). It is also pseudoprincipal. To prove this, it suffices to check that if \( A, B \in L \) with \( A \neq \emptyset \), there exists \( X \in L \) with \( \emptyset \subseteq X \subseteq A \) and either \( X \subseteq B \) or \( X \cap B = \emptyset \). Since \( \emptyset \neq A = (A \cap B) \cup (A \cap B^c) \), one may take \( X \) to be whichever of \( A \cap B, A \cap B^c \) is non-empty.
9.9. The definitions and [11, Proposition 2.9] immediately imply the following.

**Proposition.** Let $\mathcal{R} = (G, \Lambda, N)$ be a faithful protorootoid with big weak order $L$. Then the following conditions are equivalent:

(i) $\mathcal{R}$ is a (complete) rootoid.
(ii) For each $a \in \text{ob}(G)$, $(a\Lambda_a, L)$ is a (complete) mesh.
(iii) For all $b \in \text{ob}(G)$, there exists $a \in \text{ob}(G[b])$ such that $(a\Lambda_a, L)$ is a (complete) mesh.

9.10. This section concludes with the following Table 1 which summarizes previously introduced notation for objects of the categories $\text{Prd}$, $\text{Set-Prd}$ and $\text{Gpd-Set}_\pm$ attached to various special structures.

**Table 1.** Notation for protorootoids, set protorootoids and signed groupoid sets attached to special structures.

| Structure                        | $\text{Prd}$ | $\text{Set-Prd}$ | $\text{Gpd-Set}_\pm$ |
|----------------------------------|-------------|------------------|----------------------|
| Coxeter system                   | $(W, S)$    | $\mathcal{C}_{(W, S)}$ | $\mathcal{C}_{(W, S)}$ | $C_{(W, S)}$ |
| Real central simplicial arrangement | $\mathcal{H}$ | $\mathcal{C}_\mathcal{H}$ | $\mathcal{C}_\mathcal{H}$ | $C_\mathcal{H}$ |
| Compact real orthogonal group    | $O(V)$      | $\mathcal{O}(V)$ | $\mathcal{O}(V)$ | $C_{O(V)}$ |
| $C_0$-system                     | $(G, S)$    | $\mathcal{J}_{(G, S)}$ | $\mathcal{J}_{(G, S)}$ | $J_{(G, S)}$ |
| Even $C_0$-system                | $(G, S)$    | $\mathcal{J}_{(G, S)}'$ | $\mathcal{J}_{(G, S)}'$ | $J_{(G, S)}'$ |
| $C_0$-graph                      | $\Gamma$   | $\mathcal{J}_{\Gamma}$ | $\mathcal{J}_{\Gamma}$ | $J_{\Gamma}$ |
| Even $C_0$-graph                 | $\Gamma$   | $\mathcal{J}_{\Gamma}$ | $\mathcal{J}_{\Gamma}$ | $J_{\Gamma}$ |
| Rainbow graph                    | $X$         | $\mathcal{J}_{X}$ | $\mathcal{J}_{X}$ | $(J_X)$ |
| Protomesh                        | $P$         | $\mathcal{J}_{P}$ | $(\mathcal{J}_{P})'$ | $(J_P)$ |
| Possibly infinite oriented matroid | $M$         | $\mathcal{J}_{M}$ | $\mathcal{J}_{M}$ | $J_{M}$ |

Those objects denoted by notation of the form $\mathcal{C}_U$, $\mathcal{O}_U$, $C_U$ are rootoidal for any structure $U$ of the indicated type, while those denoted $\mathcal{J}_U$, $\mathcal{J}_U'$, $J_U$, $\mathcal{F}_U$, $\mathcal{F}_U'$, $\mathcal{F}_U$ or $\mathcal{J}_U$ may or may not be rootoidal depending on the specific structure $U$. The abridgement of a protorootoid denoted $\mathcal{J}_U$ is isomorphic to the abridgement of $\mathcal{J}_U$. Other regularities in the notation should be apparent from the table. The bracketed entries for a rainbow graph or protomesh are only defined if it is defined over a Boolean ring $\mathcal{P}(Y)$ of sets.

10. **Completion**

10.1. **Conjectural complete rootoid of a Coxeter system.** Suppose $(W, S)$ is a Coxeter system and $\mathcal{C} = \mathcal{C}_{(W, S)} = (W, \Lambda, N)$ is the associated rootoid. The $W$-set corresponding to the functor $\Lambda$ is $\varphi(T)$, with $W$-action induced by the action of $W$ on $T$ by conjugation. The (unique) weak order is $L = \{ N(w) \mid w \in W \}$. 
One therefore has an associated mesh $P := (\wp(T), L)$. In turn, this mesh has an associated rootoid $R := J_P$, which is complete if and only if $W$ is finite. The rootoid $R$ is easily seen to be the universal covering rootoid of $\mathcal{C}$. Let $\widehat{L} \subseteq \wp(T)$ be the set of all initial sections of reflection orders of the reflections $T$ of $(W, S)$, as defined in [10]. Then from [10],

$$L = \{ \Gamma \in \widehat{L} \mid |\Gamma| \text{ is finite} \}.$$  

One has $\widehat{L} = L$ if and only if $W$ is finite. Define the protomesh $\hat{P} := (\wp(T), \widehat{L})$ and the corresponding protorootoid $\hat{R} := J_P$. The identity map $i := \text{Id}_{\wp(T)}$ is a morphism of protomeshes $i : P \to \hat{P}$.

**Conjecture.** The protorootoid $\hat{R}$ is a complete, regular, saturated, pseudoprincipal rootoid. Equivalently, $\hat{P}$ is a complete, regular, saturated, pseudoprincipal mesh.

Except for the statement that $\hat{R}$ is pseudoprincipal, all parts of the conjecture follow from the unproven conjectures [12, (2.2) and (2.5)], which motivated many of the concerns of these papers. The conjecture holds trivially for finite $(W, S)$ from the results in [11] since then $\hat{R} = R$, and it is not difficult to check it holds if $(W, S)$ is infinite dihedral. It is not known for any other irreducible Coxeter systems. Subsequent papers will show that complete, regular, saturated, pseudoprincipal rootoids have some very favorable properties.

**10.2. Completion of rootoids.** Let $R = (G, \Lambda, N)$ be a rootoid. A completion of $R$ is a complete rootoid $\mathcal{F} = (H, \Gamma, M)$ together with a morphism $(\alpha, \nu) : R \to \mathcal{F}$ of rootoids such that the following conditions (i)–(iii) hold:

(i) $\alpha = \alpha' \alpha''$ where $\alpha' : G' \to H$ is the inclusion morphism of a subgroupoid $G'$ of $H$ and $\alpha'' : G \to G'$ is an isomorphism.

(ii) For all $a \in \text{ob}(G)$, $\nu_a : \alpha \Lambda \to \alpha(a) \Gamma$ is injective.

(iii) For each $a \in \text{ob}(G)$ the inclusion map $\alpha a : aG \to a(a)H$ induced by $\alpha$ is an embedding of the weak right order of $G$ at $a$ as an order ideal of the weak right order of $H$ at $\alpha(a)$.

**Remarks.** A (different) definition of completion, more similar to standard notions of completions of semilattices, would be obtained by replacing “order ideal” by “join-closed meet subsemilattice” in (iii).

**10.3. Completions of meshes.** A variant for meshes of the notion of completion seems particularly simple. Define a completion of a mesh $P = (\Lambda, \mathcal{L})$ to be a morphism $i : P \to \hat{P}$ of meshes satisfying the conditions (i)–(iii) below:

(i) $\hat{P} = (\hat{\Lambda}, \hat{\mathcal{L}})$ is a complete mesh

(ii) $i$ is injective as a function $i : \Lambda \to \hat{\Lambda}$. Let $\mathcal{L}' := \{ i(\Gamma) \mid \Gamma \in \mathcal{L} \} \subseteq \hat{\mathcal{L}}$.

(iii) For each $\Gamma \in \mathcal{L}'$, the subset $\Gamma + \mathcal{L}'$ of $\hat{\Lambda}$ is an order ideal of $\Gamma + \hat{\mathcal{L}}$ (both in the order induced by the Boolean ring $\hat{\Lambda}$).

It is easily seen that a completion $i$ as above induces a completion $J_P \to J_{\hat{P}}$ of the associated rootoids.
Example. (1) Let $P = (\Lambda, \mathcal{L})$ where $\Lambda = \wp(\{x, y, z\})$ and $\mathcal{L} := \emptyset, \{x\}, \{y\}, \{z\}$. It is easy to see that $P$ is a mesh which has no completion $i: P \to \hat{P}$ such that the map of Boolean rings underlying $i$ is an isomorphism. However, let $\hat{\Lambda} := \wp(\{x, y, z, w\})$, 

$$\hat{\mathcal{L}} := \mathcal{L} \cup \{ \{x, y, z, w\} \setminus \Gamma \mid \Gamma \in \mathcal{L} \}$$

and $i: \Lambda \to \hat{\Lambda}$ be the natural inclusion homomorphism of Boolean rings (which doesn’t preserve identity elements, but isn’t required to). It is easily checked that $\hat{P} := (\hat{L}, \hat{\mathcal{L}})$ is a mesh and $i: P \to \hat{P}$ is a completion of $P$.

(2) It is not even conjectural whether, for general Coxeter systems $(W, S)$, $C(W,S)$ has a completion. However, Conjecture [10.1] and (10.1.1) imply that the morphism $i: P \to \hat{P}$ defined in [10.1] is a completion of $P$ (with the additional stringent property that the map of Boolean rings underlying $i$ is an isomorphism). In particular, the conjecture implies that the universal covering rootoid $J_P$ of $C(W,S)$ has a completion.

Remarks. (1) Conjecture [10.1] would imply that for any non-empty tree $\Gamma$, $\overline{J}\Gamma$ has a completion, by embedding $\Gamma$ as a subtree of the Cayley graph of a universal Coxeter system $(W, S)$ (i.e. one with no braid relations) of sufficiently high rank $|S|$ and taking a completion of the universal covering rootoid $J_P$ of $C(W,S)$. Note that the rootoids $\overline{J}\Gamma$ for non-empty trees $\Gamma$ are the “least complete” of connected, simply connected, principal rootoids, in the sense that the only non-empty subsets of their weak orders which have upper bounds are finite chains (totally ordered subsets). These observations suggest the question of whether an arbitrary principal mesh $P$ has a completion $\hat{P}$.

(2) If rootoid completions as defined above do no exist in useful generality, it may be an indication that a weaker or different notion of completion is more appropriate or even that the axioms for rootoids should be strengthened (in some unknown way).

(3) Subsequent papers will make use of a construction which will be called the pseudocompletion of a rootoid, the definition of which is suggested by the above notions of completions. In general, the pseudocompletion of a rootoid is not a rootoid, but only a protorootoid with self-dual weak orders; however, it enables one to use for general rootoids a weak version of a useful partial duality in the theory of rootoids which admit a completion.

10.4. By definition, the weak orders of a rootoid $\mathcal{R}$ with a completion $\mathcal{F}$ embed as order ideals of weak orders of $\mathcal{F}$, which by [11 Proposition 7.1], are complete ortholattices. In [10.8] it will be shown that the weak orders of any rootoid $\mathcal{R}$ embed as order ideals of complete ortholattices. The proof uses constructions which are described in [10.5–10.7]. The first of these glues two complete lattices related by a Galois connection to give a complete lattice, which is a complete ortholattice under a stringent symmetry condition.

Recall that a Galois connection between two posets $X$ and $Y$ is a pair $\alpha, \beta$ of order reversing maps $\alpha: X \to Y$ and $\beta: Y \to X$ such that for all $x \in X, y \in Y$ one has $y \leq \alpha(x)$ if and only if $x \leq \beta(y)$. This is equivalent to saying that, regarding $X$, $Y$ as categories and $\alpha, \beta$ as (covariant) functors between $X$ and $Y^{op}$ in the natural
way, $\alpha$ is left adjoint to $\beta$. The subset $\mathcal{X}' := \{ x \in X \mid \beta \alpha(x) = x \}$ is called the set of stable elements of $X$; similarly, one defines the set $\mathcal{Y}'$ of stable elements of $Y$.

The following well-known facts concerning such a Galois connection are recorded here for convenience of reference (additional properties holding by virtue of the symmetry interchanging $X$ and $Y$, and $\alpha$ and $\beta$, are not listed).

**Lemma.** Let $x, x' \in X$ and $y \in Y$.

(a) $x \leq x'$ implies $\alpha(x') \leq \alpha(x)$.
(b) $y \leq \alpha(x)$ implies $x \leq \beta(y)$.
(c) $x \leq \beta \alpha(x)$ and $\alpha(x) = \beta \alpha(x)$.
(d) $\mathcal{X}' = \{ \beta(u) \mid u \in Y \}$.
(e) If $X, Y$ are complete lattices and $x_i \in X$ for $i \in I$, then $\alpha(\bigvee x_i) = \bigwedge \alpha(x_i)$.
(f) If $X, Y$ are complete lattices, then $\mathcal{X}'$ and $\mathcal{Y}'$ are complete lattices in the induced order (with meet given by meet in $X, Y$ respectively) and $\alpha, \beta$ restrict to order reversing bijections between $\mathcal{X}'$ and $\mathcal{Y}'$.

10.5. Suppose that $\alpha : X \to Y$ is an order reversing map between posets $X$ and $Y$. Choose a set $V = V_0 \cup V_1$, where $V_0$ (resp., $V_1$) is in bijection with $X$ (resp., $Y$) by a map $i_0 : X \to V_0$ (resp., $i_1 : Y \to V_1$). Define a relation $\leq$ on $V$ by the following conditions: for all $x, x' \in X$ and $y, y' \in Y$,

(i) $i_0(x) \leq i_0(x')$ if and only if $x \leq x'$ in $X$.
(ii) $i_1(y) \leq i_1(y')$ if and only if $y' \leq y$ in $Y$.
(iii) $i_2(y) \leq i_0(x)$.
(iv) $i_0(x') \leq i_1(y)$ if and only if $y \leq \alpha(x')$ in $Y$.

One readily checks that $\leq$ is a partial order on $V$. Obviously, $V_0$ is an order ideal of $V$ isomorphic to $X$ and $V_1$ is an order coideal of $V$ order isomorphic to $Y^{op}$.

**Lemma.** Suppose above that $X, Y$ are complete lattices and the map $\alpha$ is part of a Galois connection $(\alpha, \beta)$ between $X$ and $Y$. Then

(a) The poset $V$ is a complete lattice.
(b) Suppose further that $X = Y$, $\alpha = \beta$ and $x \land \alpha(x) = 0_X$ for all $x \in X$.

Then $V$ becomes a complete ortholattice with orthocomplement defined by

$$(i_0(x))^\perp := i_1(x)$$

and $$(i_1(x))^\perp := i_0(x)$$ for all $x \in X$.

**Proof.** For any non-empty subset of $V_1$, its join in $V_1$ is also its join in $V$, since $V_1$ is an order coideal of $V$. We claim that for any non-empty subset $A$ of $V_0$, its join in $V_0$ is also its join in $V$. To see this, write $A = \{ i_0(a) \mid a \in A' \}$ where $A' \subseteq X$ has join $x := \bigvee A'$. It will suffice to show that for any $y$ in $Y$ with $i_0(a) \leq i_1(y)$ for all $a \in A'$, one has $i_0(x) \leq i_1(y)$. By definition, $i_0(a) \leq i_1(y)$ if and only if $y \leq \alpha(a)$ in $Y$. This holds if and only if $a \leq \beta(y)$. By $a \leq \beta(y)$ for all $a \in A'$ if and only if $x = \bigvee_{a \in A'} a \leq \beta(y)$, which in turn holds if and only if $y \leq \alpha(x)$ if and only if $i_0(x) \leq i_1(y)$. Hence any non-empty subset of $V_0$ (resp., $V_1$) has a join in $V$, lying in $V_0$ (resp., $V_1$). To show that $V$ is a complete lattice, it will therefore suffice to show that the join $i_0(x) \lor i_1(y)$ exists in $V$ for any $x \in X$ and $y \in Y$. By the definitions, for $z \in Y$, $i_0(x) \leq i_1(z)$ if and only if $z \leq \alpha(x)$ in $Y$. This holds if and only if $i_1(\alpha(x)) \leq i_1(z)$ in $V_1$. Writing $i_1(z) = w$, this means that $i_1(\alpha(x)) = \min(\{ w \in V_1 \mid i_0(x) \leq w \}$.
Since any upper bound of \( \{i_0(x), i_1(y)\} \) in \( V \) must be in \( V_1 \), it follows that in \( V \),
\[
i_0(x) \lor i_1(y) = i_1(\alpha(x)) \lor i_1(y) = i_1(\alpha(x) \land y).
\]
This proves (a).

Next, make the assumptions as in (b) and define \( z^\mathcal{L} \) for \( z \in V \) as there. The definition of \( V \) and \([10.4](a)-(b)\) readily imply that \( z \mapsto z^\mathcal{L} \) is order-reversing and \((z^\mathcal{L})^\mathcal{L} = z \) for \( z \in V \). Further, \( z \lor z^\mathcal{L} = 1_V \) since for all \( x \in X \),
\[
i_0(x) \lor i_1(x) = i_1(x \land \alpha(x)) = i_1(0_X) = 1_V
\]
from above. This implies that \( z \land z^\mathcal{L} = (z^\mathcal{L} \lor z)^\mathcal{L} = 0_V \). This completes the proof. □

10.6. The above gluing construction will be applied with \( X = Y = L' \) where \( L' \) is the following lattice completion of a complete meet semilattice \( L \). Let \( L' \) be the set of all non-empty join-closed order ideals of \( L \), and partially order \( L' \) by inclusion. Let \( \iota': L \to L' \) map each element of \( L \) to the principal order ideal of \( L \) it generates i.e. \( \iota'(y) := \{ x \in L \mid x \leq y \} \) for all \( y \in L \).

**Lemma.** Ordered by inclusion, \( L' \) is a complete lattice and \( \iota' \) is an order isomorphism of \( L \) with an order ideal of \( L' \).

**Proof.** To avoid trivialities, assume \( L \neq \emptyset \). By definition, \( L' \) consists of the order ideals \( I \neq \emptyset \) of \( L \) such that if a subset \( X \) of \( I \) has a join \( \bigvee X \) in \( L \), then \( \bigvee X \in I \). The intersection of any family of elements of \( L' \) is an element of \( L' \). It follows that \( L' \), ordered by inclusion, is a complete lattice with minimum element \( \{0_L\} \) and maximum element \( L \). This map \( \iota' \) is clearly an order-isomorphism onto its image \( \iota'(L) \), regarded as subposet of \( L' \). Moreover, \( \iota'(L) \) is an order ideal of \( L' \). For suppose that \( x \in L \) and \( y \in L' \) with \( y \subseteq \iota'(x) \). Then as a subset of \( L \), \( y \) is bounded above by \( x \), so \( u := \bigvee y \) exists in \( L \), \( u \in y \) since \( y \) is join-closed, and necessarily, \( y = \{ z \in L \mid z \subseteq u \} = \iota'(u) \) since \( y \) is an order ideal of \( L \).

10.7. Let \( L \) be a non-empty complete meet semilattice. For the purposes of this subsection, a relation \( P \) on \( L \) (i.e. a subset \( P \) of \( L \times L \)) is called an orthogonality relation on \( L \) if it satisfies the conditions (i)–(iii) below:

(i) As a relation on \( L \), \( P \) is symmetric i.e. \( (x, y) \in P \) implies \( (y, x) \in P \).

(ii) For any \( x \in L \), \( \{ y \in L \mid (x, y) \in P \} \) is a non-empty join-closed order ideal of \( L \).

(iii) If \( a \in L \) with \( (a, a) \in P \), then \( a = 0_L \).

For example, if \( L \) is a non-empty order ideal of a complete ortholattice with orthocomplementation \( z \mapsto z^\mathcal{L} \), then the relation \( P \) on \( L \) defined by \( (x, y) \in P \) if \( x, y \in L \) and \( x \leq y^\mathcal{L} \) is an orthogonality relation.

**Theorem.** Let \( L \) be a non-empty complete meet semilattice with an orthogonality relation \( P \) as above. Then there is a canonically associated order isomorphism of \( L \) with an order ideal of a complete ortholattice \( V \).

**Proof.** Let \( \theta: \wp(L) \to \wp(L) \) be the function defined by setting, for all \( z \subseteq L \),
\[
\theta(z) = z^\dagger := \{ b \in L \mid (a, b) \in P \text{ for all } a \in z \}.
\]
The definitions immediately imply that \((\theta, \theta)\) gives a Galois connection from \(\wp(L)\) to itself. Note that (ii) states that \(\{x\}^\dagger \subseteq L\) for all \(x \in L\), where \(L'\) as in Lemma 10.6. By Lemma 10.4 for any \(z \subseteq L, z^\dagger = (\cup_{x \in z}\{x\})^\dagger = \cap_{x \in z}\{x\}^\dagger \in L',\) since meet in \(L'\) is given by intersection. This proves that \(\theta\) restricts to a map \(\theta': L' \to L'\). The definitions readily imply that \((\theta', \theta')\) define a Galois connection from \(L'\) to itself. We assert that for \(x \in L'\), 
\[x \wedge \theta'(x) = 0_{L'} = \{0\} \in L'\] since \(0_{L'} \in x \wedge \theta'(x)\), the assertion is proved.

Now apply Lemma 10.5 with \(X = Y = L'\) (which is a complete meet semilattice by Lemma 10.6) and \(\alpha = \beta = \theta'\). It gives a complete ortholattice \(V\) with a canonical embedding of \(L'\) as an order ideal of \(V\). Since \(L\) canonically embeds as an order ideal of \(L'\), the theorem follows.

10.8. Finally, we may prove the following, in which (c) is just one of many possible illustrative applications of (b).

**Corollary.**

(a) Let \((R, L)\) be a standard protomesh such that \(L\) is a complete meet semilattice and \((R, L)\) satisfies the JOP. Then there is a complete ortholattice \(\hat{L}\) and an order-isomorphism \(i\) of \(L\) with an order ideal of \(\hat{L}\).

(b) Any weak right order of a rootoid \(\mathcal{R}\) is order isomorphic to an order ideal of a complete ortholattice.

(c) The weak right order of a Coxeter system \((W, S)\) is order isomorphic to an order ideal of a complete ortholattice.

**Remarks.** The conclusion of (b) is much weaker than that which would follow from existence of a completion \(\hat{\mathcal{R}}\) of \(\mathcal{R}\), which would give compatible embeddings of the weak orders of \(\mathcal{R}\) in complete ortholattices, and additional complete ortholattices arising as weak orders of objects of \(\hat{\mathcal{R}}\) which don’t come from objects of \(\mathcal{R}\).

**Proof.** Take
\[P := \{(x, y) \in L \times L \mid x \cap y = \emptyset\}\]
in Theorem 10.7. The condition 10.7(i) on \(P\) is trivial. In 10.7(ii), the indicated set is an order ideal because for any \(x, y, z \in R\) with \(z \subseteq y, x \cap y = \emptyset\) implies \(x \cap z = \emptyset\), and it is join closed by the JOP. Further, if \((x, y) \in P\), then \(x \wedge y \subseteq x \cap y = \emptyset = 0_L\) in \(L\). Taking \(x = y\) shows that 10.7(iii) holds. This proves the hypotheses of Theorem 10.7, and (a) follows. Part (b) follows by applying (a) to the protomeshes \((G, \Lambda_\alpha, L')\) of a rootoid \(\mathcal{R} = (G, \Lambda, N)\) with big weak order \(L\), and (c) follows from (b) by taking \(\mathcal{R} := \mathcal{C}(W, S)\). □

11. Protorootoids and groupoid-preorders

11.1. Unital completion. This subsection and the next recall some additional background about Boolean rings which will be used in several places subsequently in these papers.

The inclusion functor \(I: \text{BAlg} \to \text{BRng}\) has a left adjoint \(U: \text{BRng} \to \text{BAlg}\), which sends each Boolean ring to its *unital completion* \(U(B)\). Concretely, for a
Boolean ring $B$, $U(B) := B \oplus \mathbb{F}_2$ as abelian group with multiplication

$$(b, \epsilon)(b', \epsilon') = (bb' + \epsilon b' + \epsilon' b, \epsilon \epsilon')$$

(using that $B$ is naturally a possibly non-unital $\mathbb{F}_2$-algebra). For any morphism $f : B \to C$ in $\text{BRng}$, $f' := U(f) : U(B) \to U(C)$ is given by $f'(b, \epsilon) = (f(b), \epsilon)$. Identify $B$ as an ideal of $U(B)$ via $b \mapsto (b, 0)$ and let $1_{U(B)} := (0_B, 1)$.

Every element of $U(B)$ is either in $B$, or of the form $b^2 = 1_{U(B)} + b$ for unique $b \in B$. This implies that $B$ is an order ideal of $U(B)$ (in the natural order of $U(B)$) and $B^e := \{ b^2 \mid b \in B \}$ is an order coideal of $U(B)$.

11.2. **Free Boolean rings and algebras.** The forgetful functor $F : \text{BRng} \to \text{Set}$ has a left adjoint $G : \text{Set} \to \text{BRng}$ sending each set $X$ to the free Boolean ring $G(X)$ on $X$. This gives rise to a diagram

$$\text{BAlg} \xrightarrow{\imath} \text{BRng} \xrightarrow{F} \text{Set}$$

in which each functor to the left is left adjoint to the functor to the right which is above it in the diagram.

It is convenient to describe $G(X)$ as a subring of its unital completion $U(G(X))$, which is the free Boolean algebra $U(G(X))$ on $X$. Here, $U(G(X))$ naturally identifies with the quotient of the (commutative) polynomial ring $\mathbb{F}_2[X]$ on $X$ by the ideal $J$ generated by all elements $x^2 + x$ for $x \in X$. The natural map $X \to U(G(X))$ given by $x \mapsto x + J$ is injective, and we use it to identify $X$ with a subset of $U(G(X))$. Then $U(G(X))$ has a $\mathbb{F}_2$-basis consisting of the monomials $x_1 \cdots x_n$ for $n \in \mathbb{N}$ and pairwise distinct $x_i \in X$. Two such monomials are equal only if they contain the same factors up to order, and the empty monomial is the identity element 1 of $U(G(X))$. The ring $G(X)$ naturally identifies with the subring (not containing 1) of $U(G(X))$ generated by $X$. Note that the monomials as above with $n > 0$ form a $\mathbb{F}_2$-basis of $G(X)$. An arbitrary function $X \to B$ where $B$ is a Boolean ring (resp., Boolean algebra) has a unique extension to a ring homomorphism (resp., unital ring homomorphism) $G(X) \to B$ (resp., $U(G(X)) \to B$). It is easy to see that the map $G$ just defined on objects extends naturally to a functor $\text{Set} \to \text{BRng}$; the left adjointness of $G$ to $F$ (and of $UG$ to $FI$) follows using these universal properties.

Note that if $X$ is finite, of cardinality $n$, then $G(X)$ (resp., $U(G(X))$) is a finite Boolean ring of cardinality $2^{2^n}$ (resp., $2^{2^n-1}$) with atoms equal to the products

$$(11.2.1) \quad e_Y := \left( \prod_{y \in Y} y \right) \left( \prod_{y \in X \setminus Y} (1 + y) \right) = \sum_{Z : Y \subseteq Z \subseteq X} \left( \prod_{z \in Z} z \right)$$

for $Y \subseteq X$ (resp., $\emptyset \subseteq Y \subseteq X$). In particular, $G(X)$ has an identity element

$$(11.2.2) \quad 1_{G(X)} = \sum_{Y : \emptyset \subseteq Y \subseteq X} e_Y = 1 + e_{\emptyset} = \sum_{Y : \emptyset \subseteq Y \subseteq X} \left( \prod_{y \in Y} y \right)$$

if $X$ is finite, since $1 = \sum_{Y : Y \subseteq X} e_Y$. 


Remarks. The Boolean ring \( \langle X \mid R \rangle \) generated by generators \( X \) subject to relations \( R \) is by definition the quotient \( B/I \) of the free Boolean ring \( B = G(X) \) on the set \( X \), as constructed above, by the ideal \( I \) of \( B \) generated by the subset \( R \) of \( B \).

11.3. Recall from [11, 2.15 and 5.8] the definition of the functors \( \mathfrak{P} : \Prd \to \Gpd\text{-PreOrd} \) and \( \mathfrak{P}' : \Prd^a \to \Gpd\text{-PreOrd} \).

**Theorem.** The functors \( \mathfrak{P} \) and \( \mathfrak{P}' \) have left adjoints \( \Omega : \Gpd\text{-PreOrd} \to \Prd \) and \( \Omega' : \Gpd\text{-PreOrd} \to \Prd^a \) respectively.

**Proof.** Let \((G, \leq)\) be an object of \( \Gpd\text{-PreOrd} \). Attach to \((G, \leq)\) the following protorootoid \( \mathcal{R} := (G, \Lambda, N) \). The Boolean ring \( _a\Lambda \) for \( a \in \mathrm{ob}(G) \) is defined to be the Boolean ring generated by a set \( X_a \) of indeterminates subject to relations \( R_a \), both specified below. One takes \( X_a := \{ (a, x, y) \mid x \in _aG_b, y \in _bG \} \) where one regards the indeterminate \((a, x, y)\) as corresponding to \((\Lambda(x))(N(y))\). The relations \( R_a \) are of the following types:

(i) for \( x \in _aG_b, y \in _bG_c, z \in _cG_d \), \((a, x, yz) + (a, x, y) + (a, xy, z) = 0\).

(ii) for \( x \in _aG_b, y \in _bG_c \) with \( y \leq z \), \((a, x, y)(a, x, z) + (a, x, y) = 0\).

In this, (i) (resp., (ii)) is chosen to correspond to the result of acting by \( \Lambda(x) \) on the cocycle condition \( N(yz) = N(y) + \Lambda(y)N(z) \) (resp., the order relation \( N(y) \leq N(z) \) i.e. \( N(y)N(z) = N(y) \)). Formally, the Boolean ring \( _a\Lambda \) defined above is the evident quotient \( _a\Lambda := B(X_a)/I_a \) of the free Boolean ring \( B(X_a) \) generated by \( X_a \) by the ideal \( I_a \) of \( B(X_a) \) generated by the left hand sides of the relations of type (i)–(ii) above. The image in the quotient \( _a\Lambda \) of an element \((a, x, y) \in X_a \leq B(X_a) \) will still be written as \((a, x, y)\).

There is evidently a representation \( \Lambda \) of \( G \) determined by \( \Lambda(a) := _a\Lambda \) for \( a \in \mathrm{ob}(G) \) and \( \Lambda(g)(a, x, y) = (b, gx, y) \) for \( g \in _bG_a \). There is a cocycle \( N \in Z^1(G, \Lambda) \) given by \( N(g) = (a, 1_a, g) \in \Lambda(a) \) for \( g \in _bG \). This defines the protorootoid \( \mathcal{R} = (G, \Lambda, N) \). Set \( \Omega(G, \leq) := \mathcal{R} \). It is straightforward to check that \( \mathrm{Id}_G \) defines a morphism \( I_{(G, \leq)} : (G, \leq) \to \mathfrak{P} \Omega(G, \leq) \) in \( \Gpd\text{-PreOrd} \).

By a standard criterion for existence of left adjoints, there is a unique extension of the map \( \Omega \) on objects to a functor \( \Omega \) left adjoint to \( \mathfrak{P} \), with \( I_{(G, \leq)} \) as component at \((G, \leq)\) of its unit, provided that \( I_{(G, \leq)} \) is universal from \((G, \leq)\) to \( \mathfrak{P} \) i.e. provided that for any protorootoid \( \mathcal{R} = (H, \Gamma, M) \) and for every morphism \( \theta : (G, \leq) \to \mathfrak{P}(\mathcal{R}) \) (specified by the groupoid homomorphism \( \theta : G \to H \)), there is a unique morphism \( f = (\alpha, \nu) : (G, \Lambda, N) \to (H, \Gamma, M) \) in \( \Prd \) such that \( \theta = \mathfrak{P}(f)I_{(G, \leq)} \). It is easy to check that there is a unique such \( f \) as required, given by \( \alpha = \theta : G \to H \) and \( \nu_a((a, x, y)) = (\Gamma(\theta(x)))(M_{\theta(y)}) \).

The image of \( \Omega \) is contained in \( \Prd^a \), so \( \Omega \) factorizes uniquely as \( \Omega = \mathfrak{B}\Omega' \) as in [11, 5.8], with \( \Omega' \) clearly left adjoint to \( \mathfrak{P}' = \mathfrak{P}\mathfrak{B} \). \( \square \)

11.4. Let \((G, \leq)\) be a groupoid-preorder in which \( G \) is (empty or) a connected, simply connected groupoid. Assume that the big weak right preorder is an anti-chain i.e. none of the elements of any of the weak right preorders of \( G \) are comparable. The proposition below provides a concrete description, which will be useful in a subsequent paper, of a protorootoid \( \mathcal{R} \) which is isomorphic to \( \Omega(G, \leq) \).
Define $𝒜 := (G, Λ', N')$ as follows. Let $V := \text{ob}(G)$ and for $x, y \in V$, denote the unique morphism in $\mathcal{G}_y$ as $(x, y)$. Let $\hat{B}$ be the Boolean ring freely generated by $V$, and let $B$ denote the (possibly non-unital) subring of $\hat{B}$ generated by all the elements $x + y$ of $\hat{B}$ where $x, y \in V \subseteq \hat{B}$. Let $\Lambda': G \to \mathbf{BRng}$ be the constant functor with value $B$, and let $N \in Z^1(G, \Lambda')$ be the cocycle determined by $N'((x, y)) = x + y \in \mathcal{G}_y$ for all $(x, y) \in \text{mor}(G)$.

**Proposition.** For $(G, \leq)$ and $𝒜$ as above, there is an isomorphism $\mathfrak{Q}(G, \leq) \xrightarrow{\cong} 𝒜$ in $\mathbf{Prd}$.

**Proof.** Use notation for $\mathfrak{Q}(G, \leq) = (G, \Lambda, N)$ as in the proof of Theorem 11.3. It will be shown that there is an isomorphism of protorootoids $(\text{Id}_G, \nu) : \mathfrak{Q}(G, \leq) \xrightarrow{\cong} 𝒜$ such that for $a, b, c \in \text{ob}(G)$, $g := (a, b) \in a\mathcal{G}_b$ and $h := (b, c) \in b\mathcal{G}_c$, one has

$$\nu_a((a, g, h)) = \nu_a(a, (a, b), (b, c)) := b + c \in a\Lambda' = B.$$ 

To prove this, note first that for fixed $a \in \text{ob}(G)$, there is a bijective correspondence $(a, (a, b), (b, c)) \mapsto (b, c)$, for $b, c \in \text{ob}(G)$, from the set of generators of $\Lambda(a)$ to $\text{mor}(G)$. So $\Lambda(a)$ is naturally isomorphic to the Boolean ring $B$ generated by $\text{mor}(G)$ subject to relations $(b, d) + (b, c) + (c, d)$ for $b, c, d \in V$ (these correspond to the relations of type 11.b(i) for $\Lambda(a)$, and by the assumption on $(G, \leq)$, there are no other relations). These relations immediately imply that $(b, b) = 0$ (take $c = d = b)$ and then $(b, c) = (c, b)$ (take $d = b$). Let $\Gamma := (V, E)$ be the complete graph on $V$, with vertex set $E := \{\{b, c\} \mid b, c \in V, b \neq c\}$. The above implies that $\Lambda(a)$ identifies with the Boolean ring $B'$ generated by $E$ subject to relations $e_1 + e_2 + e_3 = 0$ for the edges $e_1, e_2, e_3$ of any triangle in $\Gamma$, by an isomorphism $\alpha : \Lambda(a) \to B'$ with $\alpha(a, (a, b), (b, c)) = \{b, c\} \in E \subseteq B'$ if $b \neq c$ and $\alpha(a, (a, b), (b, c)) = 0$ otherwise.

One can readily see (compare Proposition 9.3) that for any fixed $a \in V$, $B'$ is the Boolean ring freely generated by the set $\{e \in E \mid a \in e\}$ of edges of $\Gamma$ which have $a$ as an endpoint. Similarly, for fixed $a$, $B$ identifies naturally with the Boolean ring freely generated by its subset $\{a + b \mid b \in V, b \neq a\}$. Hence there is an isomorphism $\beta : B' \to B$ such that $\{a + b\} \mapsto a + b$ for all $b \in V \setminus \{a\}$. By the relations for $B'$, $\beta$ satisfies $\beta((b, c)) = b + c$ for all $(b, c) \in E$. Clearly $\nu_a := \beta\alpha$ is an isomorphism of Boolean rings $\Lambda(a) \to \Lambda'(a) = B$ satisfying (11.4.1). It is now trivial to check that $(\text{Id}_G, \nu_a)$ is an isomorphism of protorootoids as required. \[\Box\]

11.5. Let $\eta : \text{Id} \to \mathfrak{P} \mathfrak{Q}$ and $\epsilon : \mathfrak{Q} \mathfrak{P} \to \text{Id}$ denote the unit and counit respectively of the adjoint pair $(\mathfrak{Q}, \mathfrak{P})$. Recall the definition of the category $\mathbf{Gpd-PreOrd}_p$ of protorootoidal groupoid-preorders from [11.2.15]. Also, denote the full subcategory of $\mathbf{Prd}$ with objects $𝒜$ such that $\epsilon_𝒜$ is an isomorphism as $\mathbf{Prd}_F$. Part (a) of the following corollary provides a characterization of protorootoidal groupoid-preorders which will be made more explicit in a subsequent paper.

**Corollary.**

(a) An object $a$ of $\mathbf{Gpd-PreOrd}$ is in $\mathbf{Gpd-PreOrd}_F$ if and only if $\eta_a$ is an isomorphism in $\mathbf{Gpd-PreOrd}$.

(b) $\mathfrak{Q}$ and $\mathfrak{P}$ restrict to an adjoint equivalence between $\mathbf{Gpd-PreOrd}_p$ and $\mathbf{Prd}_F$. 
Proof. To prove (a), let \( a \) be an object of \( \mathbf{Gpd-PreOrd} \). Suppose first that \( \eta_a : a \to \mathfrak{P}_{\Omega}(a) \) is an isomorphism. Then \( a \cong \mathfrak{P}(b) \) where \( b := \Omega(a) \), so \( a \) is an object of \( \mathbf{Gpd-PreOrd} \). On the other hand, suppose that \( a = \mathfrak{P}(b) \) where \( b \) is a protorootoid. By one of the triangular identities for the adjunction, the composite

\[
(11.5.1) \quad \mathfrak{P}(b) \xrightarrow{\eta(b)} \mathfrak{P}_{\Omega}(b) \xrightarrow{\varphi(b)} \mathfrak{P}(b)
\]

is the identity. Writing \( \mathfrak{P}(b) = (G, \leq) \) and \( \mathfrak{P}_{\Omega}(b) = (G, \leq') \), this means that the composite \( (G, \leq) \to (G, \leq') \to (G, \leq) \) is \( \text{Id}_{(G, \leq)} \), which as a morphism of groupoids is \( \text{Id}_G \). From the proof of Theorem 11.3, \( \eta_G = \text{Id}_G \) has \( \text{Id}_G \) as underlying groupoid morphism. Hence \( \mathfrak{P}(\epsilon_b) \) also has \( \text{Id}_G \) as underlying groupoid morphism. This shows that \( \text{Id}_G \) induces a preorder isomorphism between \((G, \leq)\) and \((G, \leq')\), so the preorders \( \leq \) and \( \leq' \) on \( G \) coincide. Hence in (11.5.1), the two maps are inverse isomorphisms, and \( \eta_a : a \cong \mathfrak{P}_{\Omega}(a) \) is an isomorphism. This proves (a).

If \( F : A \to B \) and \( G : B \to A \) are functors with \( F \) left adjoint to \( G \), with unit \( \eta \) and counit \( \epsilon \), then, using the triangular identities for the adjunction, \( F \) and \( G \) restrict to an adjoint equivalence between the full subcategory of \( A \) with objects \( a \) such that \( \eta_a \) is an isomorphism, and the full subcategory of \( B \) with objects \( b \) such that \( \epsilon_b \) is an isomorphism. Part (b) follows from (a) by applying this remark to the adjoint pair \((\Omega, \mathfrak{P})\), and (c) follows from (b).

\[ \square \]

11.6. Let \( \mathcal{R} = (G, \Lambda, N) \) be a protorootoid. Denote the component at \( \mathcal{R} \) of the counit of the adjunction in Theorem 11.3 as \( \epsilon_{\mathcal{R}} = (\alpha, \nu) : \mathcal{R}' \to \mathcal{R} \). Write \( \mathcal{R}' = (G, \Lambda', N') \) and let \( \mathcal{L}', \mathcal{L} \) denote the big weak orders of \( \mathcal{R} \) and \( \mathcal{R}' \) respectively.

Corollary. (a) \( \mathcal{R}' \) is abridged.

(b) \( \mathfrak{P}(\epsilon_{\mathcal{R}}) : \mathfrak{P}(\mathcal{R}') \to \mathfrak{P}(\mathcal{R}) \) is an identity map in \( \mathbf{Gpd-PreOrd} \), with underlying groupoid map \( \alpha = \text{Id}_G \).

(c) For any \( a \in \text{ob}(G) \), the weak right preorders of \( \mathcal{R} \) and \( \mathcal{R}' \) at \( a \) are equal.

(d) For \( a \in \text{ob}(G) \) \( \nu_a : a^\Lambda' \to a^\Lambda \) restricts to an order isomorphism \( a^\mathcal{L}' \to a^\mathcal{L} \).

(e) If \( \mathcal{R} \) is abridged, then for \( a \in \text{ob}(G) \), \( \nu_a \) is a surjection of Boolean rings.

Proof. By construction, for each \( a \in \text{ob}(G) \), \( a^\Lambda' \) is generated as ring by the elements \( (a, g, h) \) for \( g \in aG_b \) and \( h \in bG \). Since \( (a, g, h) = (N(g))(N'(h)) = N'(gh) + N'(g) \), (a) follows. Part (b) follows from the proof of Corollary 11.3(a) (taking \( b = \mathfrak{P}(\mathcal{R'}) \)) and (c) follows from (b). For (d), observe that the map \( a^\mathcal{L}' \to a^\mathcal{L} \) given by restriction of \( \nu_a \) identifies with the induced map (of associated partial orders) of the preorder isomorphism \( \text{Id}_G : aG \to aG \) from (c), where the left (resp., right) side has the weak right preorder from \( \mathcal{R}' \) (resp., \( \mathcal{R} \)).

Finally (e) follows from surjectivity of the map \( a^\mathcal{L}' \to a^\mathcal{L} \) in (d), because \( a^\mathcal{L} \) generates \( a^\Lambda \) since \( \mathcal{R} \) is abridged and \( a^\mathcal{L}' \) generates \( a^\Lambda' \) by (a).

\[ \square \]
11.7. This subsection shows that the right hand part of the diagram in [11, 5.8] can be refined to the following diagram by inserting the category $\text{Gpd-PreOrd}_p$:

\[
\begin{array}{c}
\text{Prd} \\
\text{Prd}^a \\
\text{Gpd-PreOrd}_p \\
\text{Gpd-PreOrd.}
\end{array}
\]

In this, $\mathcal{A}$, $\mathcal{P}'$, $\Omega'$, $\mathcal{B}$ have been previously defined (the functors $\mathcal{P} = \mathcal{P}'\mathcal{A}$ and $\Omega = \mathcal{B}\Omega'$ from [11, 5.8] are not shown here). Let $\iota: \text{Gpd-PreOrd}_p \to \text{Gpd-PreOrd}$ denote the inclusion functor. Note that $\mathcal{P}'$ uniquely factorizes as $\mathcal{P}' = \iota \mathcal{P}'$ for some functor $\mathcal{P}'$ as shown, by definition of $\text{Gpd-PreOrd}_p$. Set $\Omega'_p := \Omega' \iota$. A straightforward argument using Theorem 11.3 and the fact that $\iota$ is full and faithful shows that $\Omega'_p$ is left adjoint to $\mathcal{P}'$. Since $\mathcal{B}$ is left adjoint to $\mathcal{A}$, it follows that $\Omega_p := \mathcal{B}\Omega'_p$ is left adjoint to $\mathcal{P}_p := \mathcal{P}'\mathcal{A}$. Therefore each explicitly named functor except $\iota$ in the diagram is part of an adjoint pair, with the lower functor left adjoint to the symmetrically corresponding upper functor.

**Remarks.** Note that if $\mathcal{R}$ is an object of $\text{Prd}_F$, then $\mathcal{R}$ is abridged by Corollary 11.6(a). By Corollary 11.5, the functor $\Omega'_p$ (resp., $\Omega_p$) defines an equivalence of $\text{Gpd-PreOrd}_p$ with the full coreflective subcategory $\text{Prd}_F$ of $\text{Prd}^a$ (resp., of $\text{Prd}$). It will be shown in a later paper that abridged principal rootoids are objects of $\text{Prd}_F$. It is an open problem to determine conditions for an abridged rootoid or protorootoid to be in $\text{Prd}_F$.

12. Squares

12.1. Let $(W, S)$ be a Coxeter system. Suppose that $x \in W$ and $r, s \in S$ are such that $l(sx) > l(x)$ and $l(sxr) \leq l(xr)$. Then EC implies that $sxr = x$ so, setting $v := rx^s = x^r$, one has $vsx = 1$, and each of the expressions $xr, sx, vs$ and $rv$ is a compatible expression:

\[
\begin{array}{c}
\xymatrix{ & x \\
& v \\
& \downarrow \quad \downarrow \\
s \ar@{-}[r] & y \ar@{-}[u] &}
\end{array}
\]

12.2. The following simple notion, of which the previous subsection provides natural examples, plays a fundamental role in our main results.

**Definition.** Let $\mathcal{R} = (G, L, N)$ be a protorootoid.

(a) A oriented square of $\mathcal{R}$ is a quadruple of morphisms $(v, y, x, w)$ of $G$ as in the diagram

\[
\begin{array}{c}
\xymatrix{ & x \\
& w \\
& \downarrow \quad \downarrow \\
v \ar@{-}[r] & y \ar@{-}[u] &}
\end{array}
\]

such that the composite $vyxw$ is defined and equal to an identity morphism of $G$ and each expression $vy, yx, xw$ and $wv$ is compatible for $\mathcal{R}$. 
(b) A commutative square diagram in $G$

\[
\begin{array}{ccc}
 & x & \\
w & & z \\
& u & \end{array}
\]

is said to be a **commutative square of** $R$ if $(x, w, u^*, z^*)$ is an oriented square of $R$.

If $(v, y, x, w)$ is an oriented square of $R$, then $(y, x, w, v)$ and $(w^*, x^*, y^*, v^*)$ are also oriented squares of $R$, so there is a natural action of the dihedral group of order 8 on the set of oriented squares of $R$. This implies that the notion of commutative square of $R$ is well-defined, since in (b), $(x, w, u^*, z^*)$ is an oriented square of $R$ if and only if $(z, u, w^*, x^*)$ is an oriented square of $R$.

Sometimes the phrase “square of $R$” (or even just “square”) will be used refer to either an oriented square of $R$ or a commutative square of $R$ when context makes it clear which is meant. Note that the sets of oriented and commutative squares of $R$ depend only on the underlying groupoid-preorder of $R$.

12.3. The remainder of this section gives basic properties of squares, and examples.

**Lemma.** Let $R = (G, \Lambda, N)$ be a protorootoid. Fix a quadruple $s = (x, w, v, y)$ of morphisms of $G$ with $xwvy$ defined and equal to an identity morphism of $G$. Then:

(a) $s$ is an oriented square of $R$ if and only if both $x(N_w) = N_y$, and $N(w^*) \cap N(x^*) = \emptyset$.

(b) If $s$ is an oriented square of $R$, it is uniquely determined by $(x, w, v)$.

(c) If $R$ is faithful, and $s$ is an oriented square, then $s$ is uniquely determined by $(x, w)$ (this property will be referred to as rigidity of squares).

(d) Suppose that $R$ is the protorootoid attached to a signed groupoid-set $(G, \Phi)$. Then $s$ is an oriented square of $R$ if and only if $x(\Phi_w) = \Phi_y$.

**Remarks.** (1) Many results involving squares have an analogue involving a dual notion of cosquares, in which, for instance, the condition $x(N_w) = N_y$, and $N(w^*) \cap N(x^*) = \emptyset$ is replaced by $x(\Phi_w^C) = \Phi_y^C$ where for $z \in aG$, $\Phi_z^C := a\Phi_+ \setminus \Phi_z$ (see Remark 10.3(3), and [12] for some examples in Coxeter groups).

(2) Much of the combinatorics involving squares and cosquares in subsequent papers applies in the framework of pairs $(R, L)$ where $R = (G, \Lambda, N)$ is a protorootoid and $L = (a, L)_{a \in \text{Ob}(G)}$, $a L \subseteq \Lambda(a)$ is a subset containing $0_{\Lambda(a)}$ of $\Lambda(a)$, endowed with the induced partial order, and the family $L$ is stable under the dot $G$-action in the natural sense.

**Proof.** For (a), first suppose that $s$ is a square of $R$. From $N(y) \cap N(v^*) = \emptyset$, it follows that $N(xw) = N_x + x(N_w) = N(y^*v^*) = N_y \cup y^*(N_{v^*})$.

Using $N_x \cap N_y = \emptyset$, this implies that $N_y \subseteq x(N_w)$. By symmetry, $N_w \subseteq x^*(N_y^*)$. One has $N_w \cap N_x = \emptyset$ by definition, and the “only if” implication in (a) follows. To prove the “if” in (a), it will suffice by symmetry to show that if $x(N_w) = N_y$,
and $N(w) \cap N(x^*) = \emptyset$, then $y(N_x) = N_{x^*}$ and $N_x \cap N_{x^*} = \emptyset$ (for then two more repetitions of the same argument shows that the defining conditions for $s$ to be an oriented square are satisfied). Now the assumptions imply that

$$N(xw) = N_x \cup x(N_w) = N_x \cup N_{x^*}.$$ 

But also, $N(xw) = N(y^*v^*) = N_{y^*} + y^*(N_{v^*})$. Comparing the last two equations shows that $N_x = y^*(N_{x^*})$ as required. This completes the proof of (a). Part (b) holds since $xwvy = 1_a$ where $a := \text{cod}(x)$. If also $R$ is faithful, then $y$ is uniquely determined by the condition $N_{y^*} = x(N_w)$ from (a). Then $v$ is determined by (b), since $(y, x, w, v)$ is a square of $R$. This proves (c).

For (d), write $L(G, \Phi) = (G, \Xi, N)$ where $\Xi : G \to \text{Set}$ with $\mathcal{P}_G(\Xi) = \Lambda$. From the constructions in the proof of [11] Proposition 5.5, it follows that $x(N_w) = N_{y^*}$ if and only if $x(\Phi_w)$ is a set of $\{\pm\}$-orbit representatives on $\Phi_{y^*} \cup - (\Phi_{y^*})$. Also, $N_w \cap N_{x^*} = \emptyset$ if and only if $\Phi_w \cap \Phi_{x^*} = \emptyset$ if and only if $x(\Phi_w) \subseteq \Phi_\pm$ where $a := \text{cod}(x)$. Hence $x(\Phi_w) = \Phi_{y^*}$ if and only if $(x(N_w) = N_{y^*}$ and $N_w \cap N_{x^*} = \emptyset$). This shows that (d) follows from (a). □

12.4. Consider a commutative diagram in $G$ as follows:

\[(12.4.1)\]

\[
\begin{array}{ccc}
  d & \downarrow a & e \\
  c & \downarrow b & f \\
  e & \downarrow g & f \\
\end{array}
\]

Define the quadruples $s_1 := (a, b, c^*, d^*)$, $s_2 := (e, f, g^*, b^*)$ and $s_3 := (ae, f, g^*c^*, d^*)$ of morphisms of $G$.

**Lemma.** If any two of $s_1, s_2, s_3$ are oriented squares of $R$, then they are all oriented squares of $R$.

**Remarks.** If $R$ is preorder-isomorphic to the rootoid attached to a signed groupoid-set, this is obvious from Lemma [12.2.3](d). Although the general case can be deduced from this special case, a direct proof is indicated for completeness.

**Proof.** Suppose for example that $s_1$ and $s_3$ are squares of $R$. Then using Lemma [12.2.3](a) it follows that

$$a(e(N_f)) = (ae)(N_f) = N_d = a(N_b)$$

and hence $e(N_f) = N_b$. Also, $N(e) \cap N(b) = \emptyset$ since

$$a(N(e) \cap N(b)) = aN(e) \cap aN(b) = (N(ce) + N(a)) \cap N(d) =
(N(ce) \cap N(d)) + (N(a) \cap N(d)) = \emptyset + \emptyset = \emptyset.$$ 

This shows that $s_2$ is a square of $R$ by Lemma [12.2.3](a) again. The other cases can be proved similarly, or deduced from the case treated using the symmetry properties of oriented squares. □
12.5. Examples of squares. This subsection gives examples of squares from Coxeter groups.

Example. Let \((W, S)\) be a Coxeter system, \(\mathcal{R} := C(W, S)\) and \(\Phi\) be the abstract root system of \(\Phi\).

(1) Assume that \((W, S)\) is of type \(A_3\), with \(S = \{r, s, t\}\) and Coxeter graph \(r \leftarrow s \rightarrow t\). Concretely, \(W = S_4\), the symmetric group on four letters, with \(r = (1, 2), s = (2, 3)\) and \(t = (3, 4)\) (adjacent transpositions). In the commutative cubical diagrams below, each face gives a commutative square of \(\mathcal{R}\).

Applying the previous Lemma to the above diagrams also gives some other squares of \(\mathcal{R}\), such as the oriented square \((sr, tsrt, st, srst)\).

(2) Assume that \(W_J\) is a finite standard parabolic subgroup of \((W, S)\). Denote the longest element of \((W_J, J)\) as \(w_J\). It is well known that \(w_J^2 = 1\) and that for any \(z \in W_J\), the expression \(zz'\) is compatible where \(z' := z^\ast w_J\). Hence if \(x \in W_J\), then \((x, x^\ast w_J, w_J x w_J, w_J x^\ast)\) is an oriented square of \(\mathcal{R}\).

(3) Suppose \(J, K \subseteq S\) and \(d \in W\) with \(d J d^{-1} = K\) and \(l(dr) = l(d) + 1\) for all \(r \in J\) (equivalently, \(l(sd) = l(d) + 1\) for all \(s \in K\)). Let \(x \in W_J\) and \(y = d x d^\ast \in W_K\). It is well known and easily seen (see [4] or [23]) that \(d(\Phi_x) = \Phi_y\). It follows that \((d^\ast, y^\ast, d, x)\) is an oriented square of \(\mathcal{R}\).

13. Examples of morphisms of rootoids

This section gives examples of morphisms of rootoids. Familiarity with the definitions of the categories \(\text{Rd}\) and \(\text{RdE}\) (and in particular, with JOP, AOP and the notation \(\theta^\perp\) for the partially defined adjoint of \(\theta\) as in [11] 4.6–4.8) is assumed.

13.1. Let \(\mathcal{R} = (G, \Lambda, N)\) be a prorootoid, and \(i : H \to G\) be the inclusion morphism for a subgroupoid \(H\) of \(G\). Recall the notion of the restriction \(\mathcal{R}_H\) of \(\mathcal{R}\) to \(H\) (see [11] 2.11–2.12]). Trivial examples show that \(\mathcal{R}_H\) need not be a rootoid if \(\mathcal{R}\) is a rootoid (and so, a fortiori, pullbacks of rootoids by arbitrary groupoid homomorphisms need not be rootoids either). This subsection discusses properties of the natural morphism \(i^\#: \mathcal{R}_H \to \mathcal{R}\) in two very simple examples.

Example. (1) Suppose \((W, S)\) is a dihedral Coxeter system of order \(2n\) where \(n \in \mathbb{N}_{\geq 2} \cup \{\infty\}\). Let \(\mathcal{R} := C(W, S) = (W, \Lambda, N)\). Write \(S = \{r, s\}\). Let \(x := rs, \) \(G = \langle x \rangle\) be the rotation subgroup of \(W\) and let \(i : G \to W\) be the inclusion. Note that \(G\) is a cyclic group of order \(n\). Set \(X := \{x, x^\ast\} \subseteq G\). Then \((G, X)\) is a \(C_\mathcal{R}\)-system discussed in Example 8.16(2), and there is an associated rootoid.
$\mathcal{I} := \mathcal{J}_{(G,X)}$ defined there. There is also the protorootoid $\mathcal{R}_G := \hat{i}(\mathcal{R})$ obtained by restriction of $\mathcal{R}$ to $G$. One can easily check that $\mathcal{R}_G \cong \mathcal{I}$, and that $\hat{i}$ is a morphism in $\text{Rd}$. It is a morphism in $\text{RdE}$ if and only if $n$ is even or infinite, since only then is $G$ join-closed in $W$. If this holds, then $\mathcal{R}_G$ is a preprincipal rootoid (and otherwise, it is not).

(2) Suppose $(W, S)$ is a Coxeter system and $J \subseteq S$. Write $\mathcal{R} := \mathcal{C}_{(W, S)} = (W, \Lambda, N)$ and let $i: W_J \to W$ be the inclusion. Recall that $(W_J, J)$ is a Coxeter system. Then $\hat{i}^\circ(\mathcal{R})$ is a principal rootoid, the abridgement of which is easily seen to be isomorphic to the abridgement of $\mathcal{C}_{(W_J, J)}$. The unique morphism $\theta$ of weak right orders induced by $i^\circ$ identifies with $i$ (as a map of sets) and induces an order isomorphism between $W_J$ (in weak right order) and its image $W_J$ in $W$, which is a join closed order ideal of $W$ in its weak right order. The adjoint $\theta^\perp$ is defined only on $W_J$ and reduces to the identity map $W_J \to W_J$. It follows that $\hat{i}$ is a morphism in $\text{Rd}$ and in $\text{RdE}$.

13.2. Fixed subgroups of automorphism groups of Coxeter groups. Let $(W, S)$ be a Coxeter system and $G$ be a group of automorphisms of $(W, S)$. It is a well-known Theorem of Tits that the fixed subgroup

\begin{equation}
W' := W^G = \{ w \in W \mid \sigma(w) = w \text{ for all } \sigma \in G \}
\end{equation}

of $G$ on $W$ is a Coxeter group. Its set $S'$ of standard Coxeter generators is the set of longest elements $w_J$ of finite standard parabolic subgroups $W_J$ of $W$ such that $J$ is a single $G$-orbit on $S$. It is known how to associate a root system of $(W', S')$ to one of $(W, S)$, either as a linearly realized root system (see e.g. \cite{[21]}) or abstractly (\cite{[14]–[15]}).

**Proposition.** Let $\mathcal{R} := \mathcal{C}_{(W, S)}$ denote the standard rootoid attached to $(W, S)$, let $i: W' \to W$ be the inclusion homomorphism and set $\mathcal{I} := \mathcal{R}_{W'} = \hat{i}(\mathcal{R})$. Then $\mathcal{I}$ is a preprincipal rootoid with simple generators $S'$, and $\hat{i}: \mathcal{I} \to \mathcal{R}$ is a morphism in $\text{RdE}$.

**Proof.** Let $\preceq$ denote the weak right order of $\mathcal{R}$ and $\preceq'$ denote that of $W'$ from $\mathcal{I}$. Let $I$ denote the order ideal of $W$ generated by $W'$. Meets and joins in $(W, \preceq)$ are denoted by $\vee$ and $\wedge$ respectively.

By \cite{[11]} Lemma 2.10, $i: W' \to W$ is order preserving. Moreover, since $i$ is injective, the definitions imply that $W'$ has the induced order as a subposet of $W$. Note that $G$ acts as a group of order automorphisms of $(W, \preceq)$. Hence any join or meet which exists in $W$ of a subset of $W'$ is itself in $W'$ and so $W'$ is a join-closed meet subsemilattice of $W$. This already implies that $\mathcal{I}$ is a rootoid (the JOP for $\mathcal{I}$ is inherited trivially from that for $\mathcal{R}$), that $\hat{i}$ is a morphism in $\text{rd}$, and that $\hat{i}$ is a morphism in $\text{RdE}$ provided that it is one in $\text{Rd}$ i.e. if $\hat{i}$ satisfies the AOP. Recall that for $w \in I$,

\begin{equation}
i^\perp(w) := \bigwedge\{ u \in W' \mid w \preceq u \}
\end{equation}
is the minimum of the elements $u$ of $W'$ (in either $\leq'$ or $\leq$) such that $w \leq i(u)$. There is an expression for $i^+(w)$ as a join instead of a meet as follows:

$$i^+(w) = \bigvee_{\sigma \in G} \sigma(w).$$

To prove this, note first that since $w \leq i^+(w)$, it follows that $\sigma(w) \leq \sigma(i^+(w)) = i^+(w)$ for all $\sigma \in G$. Therefore $u := \bigvee_{\sigma} \sigma(w) \leq i^+(w)$ as well. Clearly, one has $w \leq u$. Since $\{\sigma(w) \mid \sigma \in G\}$ is $G$-stable, its join $u$ is in $W'$ and so $i^+(w) \leq u$. This proves (13.2.3).

Now AOP requires that for $w' \in W'$ and $w \in I$ such that $N(i(w')) \cap N(w) = 0$, one has $N(w') \cap N(i^+(w)) = \emptyset$. But for all $\sigma \in G$, $N(w') \cap N(\sigma(w)) = \emptyset$ implies that $N(w') \cap N(\sigma(w)) = \emptyset$ and hence

$$N(w') \cap N(i^+(w)) = N(w') \cap N(\bigvee_{\sigma \in G} \sigma(w)) = \emptyset$$

by the JOP for $\mathcal{R}$. Hence $\hat{i}$ is a morphism in $RdE$.

Next, note that $W'$ is interval finite since intervals in $(W', \leq')$ are subsets of intervals in $(W, \leq)$ which are finite. By [11, Lemma 3.4], the set $A'$ of atoms of $W'$ in its weak order generates $W'$. We claim that

$$A' = \{ i^+(s) \mid s \in I \cap S \}.$$

To see this, suppose first that $s' \in A'$. Then there is some $s \in S$ with $s \leq s'$, since $W$ is interval finite with atoms $S$. Note that $s \in I$. Then $1_W < i^+(s) \leq s'$ and hence $1_W < i^+(s) \leq s'$, $s'$. This implies that $s' = i^+(s)$ since $s'$ is an atom of $W'$.

Conversely, suppose that $r \in S \cap I$ and let $r' := i^+(r)$. There is an $s' \in A'$ with $s' \leq r'$, since $W'$ is interval finite. From the argument above, $s' = i^+(s)$ for some $s \in I \cap S$. Since $W$ is preprincipal (in fact, principal), either $N(r) \cap N(s') = \emptyset$ or $N(s') \subseteq N(r)$. In the first case, AOP implies that $N(r') \cap N(s') = \emptyset$ contrary to $N(s') \cap N(r') = N(s')$ (from $s' \leq r'$). In the second case, $r' = i^+(r') \leq s'$ which implies $r' \leq s'$. This gives $r' = s' \in A'$.

To prove that $\mathcal{F}$ is preprincipal, consider $w \in W'$ and $r' \in A'$. It is required to show that either $N(r') \subseteq N(w)$ or $N(r') \cap N(w) = \emptyset$. Write $r' = i^+(r)$ where $r \in I \cap S'$. Since $\mathcal{F}$ is preprincipal, either $N(r) \subseteq N(w)$ or $N(r) \cap N(w) = \emptyset$. In the first case, $N(r') \subseteq N(w)$ by definition of $r' = i^+(r)$. In the contrary case, $N(r') \cap N(w) = \emptyset$ by AOP. Hence $\mathcal{F}$ is preprincipal as claimed. From (13.2.3) and (13.2.5), it follows that $A' = S'$.

Remarks. (1) In [24], it is shown that $\mathcal{F}$ preprincipal implies (in the circumstances above) that $(W', S')$ is a Coxeter system. It can also be shown from [15] that $\mathcal{F}^a \cong (\mathcal{G}_{(W', S')})^a$.

(2) The proposition and its proof extend mutatis mutandis to the context of a group of automorphisms of any preprincipal rootoid (but Theorem 14.1 is more general).
13.3. We now state without proof a generalization of Proposition 13.2 (quite different from [14.1]). Let $\mathcal{R}$ be a preprincipal rootoid. Denote $\mathcal{R}$ as $(W, \Lambda, N)$ and its set of atomic morphisms as $S := A_\mathcal{R}$ for notational consistency with the examples below. Let $W'$ be a groupoid and $\theta : W' \to W$ be a groupoid homomorphism such that for each $a \in \text{ob}(W')$, the induced map $a : W' \to \theta(a) W$ is injective (in examples below, $W$ is a Coxeter group and $\theta$ is the inclusion of a subgroup).

Suppose that $R = R^*$ is a set of non-identity morphisms generating $W'$. Write $\mathcal{T} := \theta^*(\mathcal{R}) = (W', \Psi, M)$. The assumptions on $\theta$ imply that $\mathcal{T}$ inherits faithfulness and interval finiteness from $\mathcal{R}$. Consider the following conditions:

(i) For each $a \in \text{ob}(W')$, $w \in_a W'$ and $r \in R_a$, either the expression $\theta(r)\theta(w)$ is compatible in $\mathcal{R}$ or the expression $\theta(r^*)\theta(w')$ is compatible in $\mathcal{R}$ where $w' := rw$.

(ii) For any $a \in \text{ob}(W')$ and distinct $r, s \in a R$ for which $\theta(r) \vee \theta(s)$ exists in weak right order of $_a W$, one has $\theta(r) \vee \theta(s) = \theta(w)$ for some $w \in _a W'$.

(iii) Let $a \in \text{ob}(W')$ and $s \in \theta(a) S$ be such that there is an element $w \in _a W'$ with $N(s) \subseteq N(\theta(w))$. Then there is an element $\hat{s} \in _a R$ with the following properties (1)–(2): (1) if $w \in _a W'$ with $N(s) \subseteq N(\theta(w))$, one has $N(s) \subseteq N(\theta(s)) \subseteq N(\theta(w))$ (2) if $w \in _a W'$ with $N(s) \cap N(\theta(w)) = \emptyset$, then $N(\theta(w)) \cap N(\theta(s)) = \emptyset$.

For $a, s$ as in (iii), the element $\hat{s}$ is uniquely determined. Let $R'$ be the subset of $R$ consisting of all elements $\hat{s}$ arising as in (iii) from $a$ and $s$ satisfying the conditions there.

**Theorem.** If the conditions (ii)–(iii) hold, then $\mathcal{T}$ is a preprincipal rootoid, $R'$ is the set of atomic generators of $\mathcal{T}$, (i)–(iii) hold with $R$ replaced by $R'$, and $\theta^*$ is a morphism in $\text{RdE}$. Further, if $R$ satisfies (i)–(iii) or if no proper subset $R_0 = R^*_0$ of $R$ generates $W'$ and satisfies (ii)–(iii), then $R' = R$.

**Remarks.** Theorems 13.3 and 14.2 provide extensive classes of (special) morphisms in $\text{RdE}$. The underlying morphism $\theta : W' \to W$ of groupoid-preorders of any morphism in $\text{RdE}$ between preprincipal rootoids arises as Theorem 13.3 with $R = R'$ equal to the atomic generators of $W'$. The version of the theorem proved in subsequent papers includes also another characterization of such morphisms with weaker hypotheses which are more amenable to verification in interesting cases (e.g. for Coxeter groups, using (generalizations of) the canonical automaton [1], Section 4]).

13.4. In this subsection, the theorem is used to describe additional examples of morphisms in $\text{RdE}$ with the rootoid $\mathcal{R} = \mathcal{C}_{(W, S)}$ of a Coxeter system $(W, S)$ as codomain. Fix $R = R^* \subseteq W \setminus \{1\}$, let $W'$ be the subgroup generated by $R$ and let $\theta$ denote the inclusion $\theta : W' \to W$.

**Example.** (1) Suppose above that $R$ consists of longest elements of non-trivial finite standard parabolic subgroups of $(W, S)$, so the elements of $R$ are involutions. Conditions of various degrees of generality (and ease of verification) under which (i) is satisfied can be found in [24], [14], [15] and [16]. Assume (i) holds. Then the sets of atoms in $W$ of weak right intervals $[1, r]$ for $r \in R$ are pairwise disjoint subsets of
Given another object $B$ as follows. The objects of $L$ to results of [5].

The remainder of this section describes another class of morphisms in $\text{RdE}$ and, in the case of Coxeter groups, indicates its connection to results of [5].

13.5. Normalizer rootoids. The remainder of this section describes another class of morphisms in $\text{RdE}$ and, in the case of Coxeter groups, indicates its connection to results of [5].

Fix a (faithful, for simplicity) protorootoid $\mathcal{R} = (G, \Lambda, N)$. Define a groupoid $L$ as follows. The objects of $L$ are pairs $A = (a, X)$ where $a \in \text{ob}(G)$ and $X \subseteq \Lambda G$. Given another object $B = (b, Y)$ of $L$, a morphism $A \rightarrow B$ in $G$ is defined to be a triple $(B, g, A)$ where $g \in \delta G_a$ has the following property:
there is a bijection \( \sigma: X \to Y \) such that for each \( x \in X \), there is a commutative square of \( R \) of the form

\[
\begin{array}{ccc}
    x & \xrightarrow{g} & \sigma(x) \\
    \downarrow & & \downarrow \\
    & & \\
\end{array}
\]

(note \( \sigma \) is uniquely determined, by rigidity of squares).

The composition in \( L \) is induced by that in \( G \), according to the obvious formula

\[
(C, h, B)(B, g, A) := (C, hg, A).
\]

It follows easily from Lemma 12.4 that \( L \) is a groupoid as claimed. There is a natural functor \( \theta: L \to G \) defined as follows. On objects, \( \theta(a, X) = a \) and on morphisms \( \theta(B, g, A) = g \). Define the protorootoid \( N := \theta^\flat(R) \) and the morphism \( \theta^\flat: N \to R \) in \( \Prd \).

**Theorem.** Assume above that \( R \) is a rootoid. Then

(a) \( N \) is a rootoid.

(b) If \( R \) is complete (resp., interval finite, cocycle finite or preprincipal), then \( N \) has that same property.

(c) \( \theta^\flat \) is a morphism in \( \RdE \).

In particular, if \( R \) is a principal rootoid, then \( N^\alpha \) is also a principal rootoid.

**Remarks.**

(1) Suppose that \( R \) is the protorootoid of a (strongly faithful) signed groupoid-set \( (G, \Phi) \). Let \( A = (a, X) \), \( B = (b, Y) \) be objects of \( L \). Then

\[
B^L_A = \{ (B, g, A) \mid g \in \_G_a, \{ g(\Phi_x) \mid x \in X \} = \{ \Phi_y \mid y \in Y \} \}.
\]

In particular, the vertex group \( a^L_A \) is isomorphic to the normalizer (i.e. setwise stabilizer) in the vertex group \( \_G_a \) of \( \{ \Phi_h \mid h \in X \} \). For this reason, \( L \) (resp., \( N \)) will be referred to as the normalizer groupoid (resp., normalizer rootoid) of \( R \). An analogous result applies to similarly defined centralizer rootoids which are coverings of the normalizer rootoids, and for which the vertex groups are the centralizers (pointwise stabilizers) of sets \( \{ \Phi_h \mid h \in X \} \) for \( X \subseteq \_aG \) and \( a \in \ob(G) \).

(2) In addition to the above construction, there is a dual version involving a rootoid \( N' \) with underlying groupoid groupoid \( L' \) where \( \ob(L') = \ob(L) \) and with composition given by the same formula as for \( L \), but in which the analogue of (13.5.1) is

\[
B'^L_A = \{ (B, g, A) \mid g \in \_G_a, \{ g(\Phi^C_x) \mid x \in X \} = \{ \Phi^C_y \mid y \in Y \} \}
\]

where \( \Phi^C_z := \_z\Phi \setminus \Phi^C_z \) for \( z \in \_G \). These matters for Coxeter groups are closely related to our initial motivations in [12] for this work and probably more generally to the questions on completions raised in [10.3]. Similar remarks to those here apply, for instance, to Theorem 14.2.

13.6. This subsection discusses in more detail applications of the preceding theorem in the case in which \( R = \mathcal{E}_{(W,S)} \) is the standard (principal) rootoid of a Coxeter system \( (W, S) \). The groupoid \( L \) may be described as in Remark 13.5 with \( (G, \Phi) = (W, \Phi) \). The full subgroupoid \( K \) of \( L \) containing all objects \( (a, J) \) of \( L \) with \( J \subseteq S \) (rather than \( J \subseteq W \)) is clearly a union of components of \( L \). Identify subsets of \( S \) with
subsets of the simple roots \( \Pi = \bigcup_{s \in S} \Phi_s \) in the obvious way, by \( J \mapsto J \Pi := \bigcup_{s \in J} \Phi_s \). Then \( K \) may be described (up to isomorphism) as follows. The objects of \( K \) are subsets \( \Gamma \) of \( \Pi \). For \( \Gamma, \Delta \subseteq \Pi \), one has \( \Gamma \Delta = \{ (\Gamma, w, \Delta) \mid w(\Delta) = \Gamma \} \) and the composition is given by \( (\Gamma, w, \Delta)(\Delta, w', \Lambda) = (\Gamma, ww', \Lambda) \). The vertex groups of \( K \) are isomorphic to the setwise stabilizers in \( W \) of subsets of simple roots.

The groupoid \( K \) (or its components) has been studied implicitly in [22], [8] and explicitly in [5]. Natural groupoid generators were found for \( K \) if \( W \) is finite in [22], and in general in [8]. A natural presentation for \( K \) with these groupoid generators was described in [5]. The generators and relations are both defined in terms of finite standard parabolic subgroups of \( W \).

Since \( R \) is preprincipal, Theorem 13.5 implies that \( N \) is a preprincipal rootoid and hence the restriction \( N_K \) is also preprincipal. For each object \( a \) of \( L \), one has the map \( _a \theta : _a L \to W \) and its partially defined adjoint \( _a \theta^\perp \). The atomic generators of \( L \) (resp., \( K \)) are those elements \( _a \theta^\perp(s) \) for \( s \in S \cap \text{dom}(\theta^\perp) \) and \( a \in \text{ob}(L) \) (resp., \( a \in \text{ob}(K) \)). The construction of these simple generators for \( L \) in general involves repeatedly taking joins of certain recursively defined families of elements. However, in the special case of the simple generators of \( K \), it reduces to constructions with joins of elements of \( S \). These are the longest elements of finite parabolic subgroups, and the construction is seen to reduce to that in [8]. Thus, the generators for \( K \) in [8] turn out to be the atomic generators of \( N_K \) (or equivalently, the simple generators \( S \) of the abridgement). Similarly, the presentation obtained in [5] is the braid presentation discussed in [8].

The parts of the general theory described in this paper now imply additional facts about \( K \). For example, \( K \) has a natural abstract root system which may be described in several ways (e.g. as that of \( \mathfrak{T}_{(K,S)} \)), the number of positive roots made negative by a groupoid element is its length, the weak orders of \( K \) are complete meet semilattices which embed naturally in the weak orders of \( W \) satisfying AOP, and have the simple generators as their atoms, the solution by Tits of the word problem for Coxeter groups extends to \( K \), etc. Additionally, all these general consequences apply not only to \( K \), but also to the (in general much larger) groupoid \( L \); similar conclusions hold for any preprincipal rootoid in place of that from a Coxeter system.

13.7. The rootoids \( L, K \) also have additional special properties which can be established with the aid of the general theory, but can’t be proved entirely within it. Consider first the case in which \( W \) is finite. Then (components of) \( K \) turn out to be Coxeter groupoids with root systems realizable in real vector spaces as in [20] and [7] (in a way similar to that of Coxeter groups). Once one knows this, the presentations of components of \( K \) from [5] can be alternatively written as Coxeter groupoid presentations as in [20] and [7] (in terms of a family of Coxeter matrices attached to the objects and an object change diagram). Note that components of \( K \) (or \( L \)) need not simply be coverings of Coxeter groups. For example, taking \( (W,S) \) of type \( D_4 \), the component \( K' \) of \( K \) with objects the pairs \( (1_W, \{ s \}) \) for \( s \in S \) is a (connected) Coxeter groupoid with four objects such that each of its four stars has thirty elements, contains three atomic generators and has a longest element of length seven.
It is not known for finite $W$ whether all components of $L$ are Coxeter groupoids or whether their abstract root systems have realizations similar to those for $K$ in real vector spaces; in fact, it is not even known whether the self-composable atomic generators of $K$ are involutions.

If $W$ is infinite, then components of $K$ (and a fortiori, $L$) need not be Coxeter groupoids (since simple examples show that the cardinalities of the sets of simple generators with codomains in a fixed component of $K$ need not be constant). However, the natural geometric realizations in real vector spaces of the abstract root system of $W$ induces a realization $\Psi$ in a real vector space of the abstract root systems of $L$ (and therefore $K$) in a more general, and weaker, sense; for example, simple generators need not act as pseudoreflections, and a component $a\Pi$ for $a \in \text{ob}(L)$ of the simple roots may span a proper subspace of the span of the corresponding component $a\Phi_+$ of the positive roots.

**Remarks.**

1. Preservation of realizability in this weak sense of root systems of principal rootoids under several natural constructions is a general phenomenon, as will be shown in a subsequent paper. Realizability of root systems in the stronger sense (similar to that in \cite{7,20}) is only preserved in far more restricted circumstances, but has stronger consequences.

2. A final point worth emphasizing here is that Theorem 13.5 not only applies to produce, say, a preprincipal rootoid $\theta(R)$ from a preprincipal rootoid $R$, but the theorem may be applied again to $\theta(R)$ instead of $R$ to obtain, for example, the analogue for $\theta(R)$ of many results of \cite{3}.

14. **Functor rootoids**

14.1. **Limits of rootoids.** After describing in previous sections some background and examples of rootoids and morphisms between them, we finally discuss the main results of these papers, which involve the categories $\mathbf{Rd}$ and $\mathbf{RdE}$. An easily stated first result is the following.

**Theorem.**

(a) The category $\mathbf{Rd}$ has all small limits, as does its full subcategory of complete rootoids.

(b) The full subcategories of $\mathbf{Rd}$ with preprincipal rootoids as objects has all limits of functors from small categories with only finitely many objects. In particular, it has all small limits, and coequalizers of arbitrary families of parallel morphisms

**Remarks.** The categories in (b) do not have infinite products. This is similar to the fact that an infinite product of Coxeter groups is not necessarily a Coxeter group.

14.2. **Functor rootoids.** The construction of functor rootoids discussed below significantly extends the construction of normalizer rootoids from \cite{13.5}, puts it in a more conceptual framework and suggests the ubiquity of rootoid local embeddings.

Let $\mathcal{R} = (G, \Lambda, N)$ be a protorootoid and $H$ be a groupoid. For simplicity, assume that $\mathcal{R}$ is faithful and connected and $H$ is connected. Let $G^H$ denote the functor groupoid with objects the functors $F: H \to G$ and in which a morphism $\nu: F \to F'$ in $G^H$ is a natural transformation. Composition is the usual composition of natural
transformations: if $\mu: F^l \to F^n$ is another morphism, then $\mu\nu$ has components $(\mu\nu)_a = \mu_a\nu_a$ for all $a \in \text{ob}(G)$.

Let $G^H_H$ denote the subgroupoid of $G^H$ on all objects, but containing only morphisms $\nu: F \to F'$ in $G^H$ such that for each morphisms $g \in aG_b$ of $G$, the commutative diagram

$$
\begin{array}{ccc}
F(b) & \xrightarrow{\nu_b} & F'(b) \\
\downarrow F(g) & & \downarrow F'(g) \\
F(a) & \xrightarrow{\nu_a} & F'(a)
\end{array}
$$

in $G$ from the defining property of the natural transformation $\nu$ is a commutative square of $\mathcal{R}$. Using Lemma 12.4 one easily checks that this defines a groupoid $G^H_H$ as claimed.

For any $b \in \text{ob}(H)$, there is an evaluation functor $\epsilon_b: G^H_H \to G$ sending each functor $F': H \to G$ to $F'(b)$ and each morphism $\nu: F^l \to F^n$ in $G^H_H$ to its component $\nu_b: F'(b) \to F''(b)$ as natural transformation.

Recall that for any groupoid $K$ and object $a$ of $K$, $K[a]$ denotes the component of $K$ containing $a$. Choose base points $F \in \text{ob}(G^H_H)$ (fixed throughout the following discussion) and $b \in \text{ob}(H)$ (which will be allowed to vary). Let $\rho := \rho_b: G^H_H[F] \to G$ denote the restriction of the previously defined evaluation functor. Define the protorootoid $\mathcal{F} = \mathcal{F}_b = \rho_b^*(\mathcal{R})$ and the morphism $(\rho_b)^\flat: \mathcal{F}_b \to \mathcal{R}$ in $\text{Prd}$ (these depend only on $b$ and $G^H_H[F]$, not on $F$ itself).

**Theorem.** Assume above that $\mathcal{R}$ is a rootoid. Then

(a) $\mathcal{F}_b$ is a rootoid.

(b) If $\mathcal{R}$ is complete (resp., interval finite, cocycle finite or preprincipal), then $\mathcal{F}_b$ is complete (resp., interval finite, cocycle finite, or preprincipal).

(c) $(\rho_b)^\flat$ is a morphism in $\text{RdE}$.

(d) The abridgement $(\mathcal{F})^a$ is independent of the choice of $b$, up to canonical isomorphism.

If $\mathcal{R}$ is a principal rootoid, one obtains a well-defined (up to isomorphism) principal rootoid $(\mathcal{F})^a$ from any groupoid homomorphism from a connected groupoid to the underlying groupoid of $\mathcal{R}$. The rootoids $\mathcal{F}_b$ (resp., $(\mathcal{F}_b)^a$) are called *functor rootoids* (resp., abridged functor rootoids) of $\mathcal{R}$.

14.3. The proof of Theorem 13.5 is reduced to that of Theorem 14.2 by the following technical lemma (the main point in the proof of which is rigidity of squares) and generalities (Lemma 4.9) about covering protorootoids. The statement uses the notation of 13.5 and 14.2.

**Lemma.** Let $\mathcal{R}$ be a faithful protorootoid and $A = (a, X)$ be an object of the normalizer groupoid $L$ with $X \neq \emptyset$. Fix a connected, simply connected groupoid $H$ and an object $b$ of $H$ such that there is a bijection $\phi: H \xrightarrow{\sim} X$.

(a) There is a unique groupoid homomorphism $F: H \to G$ such that, if $H$ is non-empty, $F(b) = a$ and $F(h) = \phi(h) \in aG$ for all $h \in bH$. 

(b) If $b$ is complete (resp., interval finite, cocycle finite, or preprincipal), then $F$ is complete (resp., interval finite, cocycle finite, or preprincipal).

(c) $(\rho_b)^\flat$ is a morphism in $\text{RdE}$.

(d) The abridgement $(\mathcal{F})^a$ is independent of the choice of $b$, up to canonical isomorphism.
(b) Let $\theta': L[A] \to G[a]$ and $\rho'_b : G^H[F] \to G[a]$ denote the evident restrictions of $\theta$ and $\rho_b$. Then there exists a unique groupoid homomorphism $\tau : G^H[F] \to L[A]$ such that $\tau(F) = A$ and the diagram

$$
\begin{array}{ccc}
L[A] & \xrightarrow{\theta'} & G[a] \\
\uparrow{\tau} & & \downarrow{\rho'_b} \\
G^H[F] & & \\
\end{array}
$$

commutes. Moreover, $\tau$ is a covering.

(c) The protorootoid $R$ is a covering protorootoid of the restriction $N_{L[A]}$. Hence every component of $N$ is a covering quotient of a functor protorootoid.

Remarks. The component $G^H[F]$ may be alternatively regarded as a component of a centralizer rootoid as in Remark 13.5(1). The covering $\tau$ in the above lemma comes from the group of automorphisms of $H$ which fix the object $b$ and permute $H$ arbitrarily. Similarly, suitable automorphism groups of an arbitrary connected groupoid $H$ give rise to covering quotients of associated functor rootoids.

14.4. Duality. Define a based connected groupoid to be a pair $(H, e)$ consisting of a connected groupoid $H$ and a specified object $e$ of $H$. A morphism of based connected groupoids is defined to be a base-point preserving groupoid morphism.

With composition of underlying groupoid morphisms as composition, this defines a category $\text{CGpd}_\bullet$ of based connected groupoids.

The above construction may be interpreted as follows. Fix a connected rootoid $\mathcal{R}$ and a base point $a$ of its underlying groupoid $G$. Consider the slice category $C = \text{CGpd}_\bullet/(G, a)$. The construction attaches to any object $F : (H, b) \to (G, a)$ of $C$ another object $e(F) := (G^H[F], \rho_b)$ of $C$. An object of $C$ isomorphic to an object $e(F)$ will be called a stable object of $C$.

Theorem. (a) The map $e$ from $\text{ob}(C)$ to $\text{ob}(C)$ defined above naturally extends to a contravariant functor $e : C \to C$ (induced by the contravariant functor $\text{Gpd} \to \text{Gpd}$ given on objects by $H \mapsto G^H$).

(b) There are natural isomorphisms

$$
\text{Hom}_C(P_1, e(P_2)) \cong \text{Hom}_C(P_2, e(P_1))
$$

for all $P_1, P_2 \in \text{ob}(C)$, where each Hom-set has at most one element. More precisely, letting $e' : C \to C^{op}$ and $e'' : C^{op} \to C$ be the two covariant functors naturally identified with $e$, the functor $e'$ is left adjoint to $e''$.

(c) Let $D$ be the full subcategory of $C$ consisting of stable objects. For all $P \in \text{ob}(D)$, the component at $P$ of the unit (and counit) of the adjunction in (b) is an isomorphism.

(d) The restriction of $e$ to a contravariant functor $D \to D$ is a contravariant equivalence i.e $e'$ restricts to a category equivalence $D \to D^{op}$.

(e) Let $F$ be in $\text{ob}(C)$ and $F'' := e(e(F)) \in \text{ob}(C)$. Regard $F, F''$ as morphisms in $\text{CGpd}_\bullet$. Then $F''$ is a monomorphism and $F = F''F'$ for a (unique) morphism $F'$ in $\text{CGpd}_\bullet$. 

The stable morphism $F''$ in (e) is called the hull of $F$. Part (e) shows that any morphism $F$ in $C$ factors uniquely through its hull. Moreover, $e(F)$ and $e(F'')$ are canonically isomorphic, and $(F'')^*$ is a morphism in $\text{RdE}$ with $F''$ as underlying groupoid morphism. By (d), every hull has a dual. Loosely speaking, this means that in considering components $\mathcal{R}$ of functor rootoids, one may for most purposes restrict $F$ to be a stable local embedding. There is a category of stable local embeddings of rootoids not discussed here.

14.5. It is a well known fact that a connected groupoid is determined up to isomorphism by the isomorphism type of its vertex groups and the cardinality of the set of its objects. The proof of Theorems 14.2 and 14.4 make use of a more canonical version of this fact, described in the proposition below.

Define a category $\text{Dtm}_*$ with objects the triples $(A, X, x)$ where $A$ is a group, $X$ is a free (left) $A$-set and $x \in X$. A morphism $(A, X, x) \to (B, Y, y)$ in $\text{Dtm}_*$ is a pair $(\rho, \theta)$ where $\rho: A \to B$ is a group homomorphism and $\theta: X \to Y$ is a function satisfying $\theta(x) = y$ and $\theta(ax') = (\rho(a))\theta(x')$ for $a \in A$, $x' \in X$. Composition is componentwise: $(\rho, \theta)(\phi, \theta') := (\rho\phi, \theta\theta')$.

**Proposition.** The natural functor $\mathfrak{F}: \text{CGpd}_* \to \text{Dtm}_*$ defined on objects by mapping $(G, a) \mapsto (\chi G_{a_1} a G, 1_a)$ is a category equivalence.

The objects of $\text{Dtm}_*$ are accordingly called based connected groupoid datums.

14.6. Let $\mathcal{R}$ be a connected rootoid with underlying groupoid $G$ and fix $a \in \text{ob}(G)$. Let $C$ be as in 14.4. Define the coproduct (disjoint union) $\mathfrak{X} := \bigsqcup_a G$ of sets. Let $F: (H, b) \to (G, a)$ be a morphism in $C$ i.e. $F: H \to G$ is a groupoid homomorphism with $F(b) = a$. Write $\mathfrak{F}(F) = (\rho, \theta)$ where $\rho$ (resp., $\theta$) is the group homomorphism (resp., function) $\rho H_b \to \chi a G_a$ (resp., $\theta H_a \to \chi a G$) induced by $F$. Define $\chi(F) := \text{Im}(\rho) \bigsqcup \text{Im}(\theta) \subseteq \chi \mathfrak{X}$.

To any relation $R \subseteq X \times Y$ from a set $X$ to a set $Y$, there is an associated Galois connection determined by a pair of order-reversing maps between $\varphi(X)$ and $\varphi(Y)$ as follows. The map $\varphi(X) \to \varphi(Y)$ is given by

\[(14.6.1) \quad A \mapsto A^\dagger \ := \{ y \in Y \mid (x, y) \in R \text{ for all } x \in A \}.\]

and the map $\varphi(Y) \to \varphi(X)$ is defined by symmetry. We shall be concerned below only with the case that $R$ is symmetric (so $X = Y$), in which case these two order-reversing maps coincide and are denoted $A \mapsto A^\dagger$. Recall from 10.4 that a subset $A$ of $X$ is called stable if it is of the form $A = B^\dagger$ for some $B \subseteq X$, or equivalently, if $A = A^{\dagger\dagger}$. The stable subsets form a complete lattice and the map $A \mapsto A^\dagger$ is an order-reversing involution of this lattice.

**Theorem.** There is a (explicitly known) symmetric relation $R$ on $X = \mathfrak{X}$ such that the associated Galois connection has the following properties:

(a) The $R$-stable subsets of $X$ are precisely the sets $\chi(F)$ for stable objects $F \in \text{ob}(C)$.

(b) For any $F$ in $\text{ob}(C)$, $\chi(e(F)) = \chi(F)^\dagger$. 

The definition of \( R \cap (aG \times aG_a) \) is generalizes in an obvious way that of the relation on a Coxeter group described in [12], and the other intersections \( R \cap (aG_a \times aG) \), \( R \cap (aG \times aG_a) \), and \( R \cap (aG \times aG) \) have similar descriptions. The precise details are omitted here but are important for the proofs of Theorems 14.2 and 14.4, and provide a basis for calculation with functor rootoids. The main part of the proofs involves the study of combinatorics of squares in relation to these Galois connections for varying \( a \in \text{ob}(G) \). The overall strategy is similar to that in the proof of Proposition 13.2 but the key step of expressing certain meets as joins is more intricate, involving repetition (finite for interval finite rootoids, transfinite in general) of certain “zig-zag” and “loop” constructions to produce squares.

**Remarks.** Significant parts of the formalism mentioned above apply in the context of faithful protorootoids (giving functor protorootoids), and much of that formalism in turn extends to a general categorical context (e.g. as provided by [11, Remark 5.5(3)]). It seems very likely that non-trivial parts of the theory of rootoids have a natural extension in which underlying groupoids are replaced by suitable subcategories of groupoids. An interesting open question is whether the notion of rootoid has a natural extension (e.g. in the framework suggested by [11, Remark 5.5(3)]) allowing underlying general categories (instead of only groupoids or subcategories of groupoids).

**14.7. \( n \)-cubes.** The final subsections of this section contain only vague indications of results which will be formulated and proved elsewhere. As Example 12.5(1) suggests, the notion of squares of a rootoid \( \mathcal{R} = (G, \Lambda, N) \) extends naturally to higher dimensions. Informally, a commutative \( n \)-cube of \( \mathcal{R} \) is defined to be a commutative \( n \)-cubical diagram in \( G \) such that all of its 2-dimensional faces are commutative squares of \( \mathcal{R} \). The \( n \)-cubes for all \( n \) naturally constitute a cubical set as in algebraic topology. The \( n \)-cubes (of \( \mathcal{R} \)) admit \( n \) natural compositions; the \( i \)-th is defined by stacking two \( n \)-cubes together in the \( i \)-th direction along a common \((n-1)\)-face in an obvious way. For example, in [12.4] the first (horizontal) 1-composite of the two small commutative squares is the larger (exterior) square; the second composition of (composable) 2-squares given by stacking them vertically. These structures make the families of \( n \)-cubes into a cubical \( \omega \)-groupoid in the sense of higher category theory.

**14.8.** It can be shown, using the theory of functor rootoids, that the groupoid with morphisms given by a fixed one of the \( n \)-composition of \( n \)-cubes \((n \geq 1)\) in \( G^\omega \) can be given a natural structure of rootoid. This gives rise to what we call a symmetric cubical \( \omega \)-rootoid \( \mathcal{R}_\omega \) (with \( G^\omega \) as underlying cubical \( \omega \)-groupoid) associated to the rootoid \( \mathcal{R} \); each groupoid defined by an \( n \)-composition in \( G^\omega \) is the underlying groupoid of a rootoid, in a compatible way. For \( n = 1 \), the unique 1-composition underlies the original rootoid \( \mathcal{R} \). The term “symmetric” refers to a natural action of the hyperoctahedral group (Coxeter group of type \( B_n \)) on the set of \( n \)-cubes for each \( n \), which is compatible with the various rootoid structures.
14.9. Non-triviality. For certain rootoids $\mathcal{R}$ (e.g. $\mathcal{J}(G,X,X^\ast)$ for a free group $G$ on a set $X$ or $\mathcal{J}_\Gamma$ where $\Gamma$ is a tree) the theory discussed in this section largely reduces to trivialities.

That this is not the case in general can be seen in several ways. Firstly, for Coxeter systems $(W,S)$, the theory of functor rootoids of $\mathcal{C}(W,S)$ extends the results in [5], from which one sees that non-trivial functor rootoids certainly exist whenever $(W,S)$ has non-trivial braid relations.

The existence of non-trivial stable objects in 14.4 can be seen from the following fact: for $\mathcal{C}(W,S)$ with $W$ finite, there is a natural injection from $W$ to the set of $R$-stable subsets of $a \mathcal{X}$ (where $a$ is the unique object of $W$) in Theorem 14.6 (The dual of this fact, in a sense related to Remark 13.5(2), actually holds for arbitrary $(W,S)$; for finite $W$, the fact is self-dual). This injection is a bijection in the case of finite dihedral groups, but not in general; for example, for $W$ of type $A_3$, with $|W| = 24$, there are 26 $R$-stable subsets.

Finally, a crude quantitative measure of richness of the theory of functor rootoids associated to a general rootoid $\mathcal{R}$ may be given as follows. Say that an $n$-cube of $\mathcal{R}$ is non-trivial if it has no identity morphism assigned to any of its edges. Let $N$ be the largest integer $n$ for which non-trivial $n$-cubes exist (or $N = \infty$ if there is no largest such $n$). If $\mathcal{R} = \mathcal{C}(W,S)$, then $N$ turns out to be the supremum of the ranks of finite standard parabolic subgroups of $(W,S)$ (compare Example 12.5(1)) and it has a similar (but slightly more subtle) description for principal rootoids in general.

Remarks. (1) The quantity $N$-defined above is the sup of the $n \in \mathbb{N}$ for which $\mathcal{R}^\omega$ has non-trivial $n$-morphisms in the sense of higher category theory.

(2) Many structures similar to $\mathcal{R}^\omega$ could be constructed by taking components of suitable families of functor rootoids. Although the resulting structures are genuinely higher categorical in nature, their definition involves only two-dimensional structures (squares) satisfying conditions determined by a 1-cocycle. It is not known if there is an extension of these ideas incorporating, for instance, $n$-cocycles for $n > 1$.

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