Probing the XYZ states through radiative decays

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I. INTRODUCTION

In the past decade, many charmonium-like and bottomonium-like states have been observed by the Belle, BaBar, CLEO-c, CDF, D0, CMS, LHCb and BESIII collaborations [1]. These states are sometimes denoted as XYZ states. Especially, some of them do not fit into the conventional charmonium or bottomonium spectrum in the quark model. Their underlying structure, production mechanism and decay pattern are very intriguing. Up to now, there has been extensive phenomenological study of these XYZ states (for a recent review see Refs. [2, 3]).

In fact, the observation of the XYZ states provides an ideal platform to investigate the exotic states since those charged XYZ states is a very loosely bound state of the proton and neutron. To a very good extent, the deuteron is a hadronic molecular state. Therefore, it is very natural to look for the other loosely bound systems composed of two hadrons. In the past several decades, there have been lots of investigations of the di-meson systems composed of two charmed/bottomed mesons. The existence of the loosely bound hadronic molecular states depends on the competition between the kinetic energy and potential in the Hamiltonian of the system. The presence of the heavy quarks lowers the kinetic energy while the chiral interaction between the two light quarks could still provide strong enough attraction.

Voloshin and Okun studied the interaction between a pair of the charmed mesons and the possible hadronic molecular states [4]. Rujula, Geogi and Glashow proposed that ψ(4040) was a D∗D∗ molecular state [5]. Törnqvist calculated the possible deuteron-like two-meson bound states such as DD∗ and D∗D∗ using the quark-pion interaction model [6, 7]. Later, Dubynskiy and Voloshin suggested that there might exist a possible new resonance at the D∗D∗ threshold [6, 8]. In the light quark sector, Weinstein and Isgur suggested the scalar resonances f0(980) and a0(980) as the KK molecular states [10–12].

The XYZ states stimulated new interest in the molecular states composed of a pair of heavy mesons. The possibility of X(3872) as the D∗D∗ molecular state was discussed in Refs. [13, 14]. Liu and Zhu proposed Y(3940) and Y(4140) as the D∗D∗ and D∗D∗ molecular candidates respectively [23, 24], which were further studied in Refs. [25, 31]. Y(4274) as the S-wave D∗D∗(2317) + h.c. molecular state was proposed in Refs. [32, 33]. In Refs. [34, 36], the authors discussed whether Z+(4430) is a loosely bound S-wave state of D∗D∗, or D∗D∗ with J P = 0−, 1−, 2−. Later, the observations of two charged charmonium-like states Z+(4051) and Z+(4248) also inspired the discussion of whether Z+(4051) and Z+(4248) can be the molecular states [37, 38]. Before the experimental observation of Zc(10610) and Zb(10650), it was pointed out that there probably exist loosely bound S-wave BB∗ or B′B∗ molecular states [19, 24]. Later, these two charged bottomonium-like states Zc(10610) and Zb(10650) attracted lots of attention in the molecular framework [39, 43]. In Refs. [41, 42], the possible charged charmonium-like molecular states were explored. Later, the observation Zc(3900) [44] again inspired lots of theoretical work [43, 45, 51]. In addition, Y(4260) was proposed as a candidate of the D1D molecular state [38, 52] while the newly observed Zc(4025) also as...
the hadronic molecule candidate \cite{53}.

Generally speaking, the dynamical calculation using the phenomenological models such as the one boson exchange model may answer whether there exists the loosely bound state. On the other hand, the production mechanism and decay pattern will also shed light on the inner structure of the $XYZ$ states. There were some discussions about the decay of $Z_c(3900)$ \cite{54}, and the decays and productions of $Z_b(10610)/Z_b(10650)$ using the re-coupling formulae of angular momentum \cite{55}. Liu studied the heavy quark spin selection rules in the meson-animeson states \cite{56}. In Ref. \cite{53}, the authors discussed the decay behavior of $Z_c(4025)$ as the $D^* \bar{D}^*$ molecular state.

In this work we will perform a comprehensive study of the radiative decay pattern of the heavy flavor molecular states. We choose the heavy flavor molecular states composed of a pair of $S$-wave and $P$-wave bottom mesons to illustrate the formalism. We use the notations $\Psi^{(0)}$ and $(\bar{b}b)$ to denote molecular state and bottomonium, respectively. We will consider the following three classes of radiative decays

\begin{align*}
\Psi^{(0)} &\rightarrow (\bar{b}b) + \gamma, \\
(\bar{b}b) &\rightarrow \Psi^{(0)} + \gamma, \\
\Psi^{(0)} &\rightarrow \Psi^{(0)} + \gamma
\end{align*}

corresponding to the electromagnetic transitions between one molecular state and bottomonium, one bottomonium and molecular state, and two molecular states respectively.

We also extend the same formalism to study the radiative decays of the molecular states with hidden charm. We will discuss the radiative decay patterns of the hidden-charm molecular states. These results are helpful to further test the molecular assignment of some charmonium-like states such as $X(3872)$, $Y(4260)$ and $Y(4360)$.

This paper is organized as follows. After introduction, we give the calculation details of the radiative decays of the heavy molecular states in Sec. II. In Sec. III we present the numerical results of the above three classes of radiative transitions associated with the hidden-bottom molecular states. We collect the numerical results of the radiative decays of the possible hidden-charm molecules in Sec. IV. The last section is the summary.

II. THE FORMALISM OF THE RADIATIVE DECAYS IN THE HEAVY QUARK LIMIT

Heavy quark symmetry plays an important role in studying the properties of hadrons containing heavy quarks. In the heavy quark limit $m_Q \rightarrow \infty$, heavy quarks can be seen as a static color source and only interact with gluons via its chromoelectric charge in Quantum Chromodynamics (QCD). This kind of interaction is invariant under the spin transformation of heavy quarks. The spin-dependent interaction is proportional to the chromomagnetic moment of heavy quarks. However, this potentially spin-flipping chromomagnetic interaction is suppressed by $1/m_Q$. When $m_Q \rightarrow \infty$, the spin of heavy quarks $S_H$ is conserved. Similar conclusions hold for the electromagnetic interaction of the heavy quarks in quantum electrodynamics (QED). The heavy quark spin symmetry is very useful in the study of heavy hadron decays.

For the heavy hadron, its total angular momentum $J$ can be decomposed into the sum of the heavy quark spin $S_H$ and the spin of the light degrees of freedom, which is defined as $S_I = J - S_H$. $S_I$ is also a conserved operator and includes the spin of light quark and all orbital angular momenta within a hadron. In the following we simply denote this operator as the "light spin" since it includes all degrees of freedom except the spin of heavy quarks, which is named as "heavy spin".

When considering the heavy flavor molecular state system, its light degrees of freedom are quite complicated. For instance, an $S$-wave molecular state $B_1\bar{B}$ is composed of a $P$-wave $b\bar{q}$ meson and an $S$-wave $b\bar{q}$ antimeson. Here, the spin of light quarks $q$ and $\bar{q}$, and the orbital angular momentum between the heavy quark $b$ and light quark $q$ are not conserved separately. However, their sum is the conserved light spin $S_I$.

Following the definition of the $C$-parity eigenstate of the molecular states in Ref. \cite{54}, we list all the hidden heavy molecular states covered in this work in Table I. In the following we will apply the heavy quark symmetry to discuss the radiative decays of the $XYZ$ states.

| $J^{PC}$ | States |
|----------|--------|
| $1^{--}$ | $\frac{1}{\sqrt{2}}(B_0\bar{B} - B^*\bar{B}_0) \frac{1}{\sqrt{2}}(B_1\bar{B} - B\bar{B}_1) \frac{1}{\sqrt{2}}(B_1\bar{B} - B\bar{B}_1)$ |
| $1^{--}$ | $\frac{1}{\sqrt{2}}(B_0\bar{B} + B^*\bar{B}_0) \frac{1}{\sqrt{2}}(B_1\bar{B} + B\bar{B}_1) \frac{1}{\sqrt{2}}(B_1\bar{B} + B\bar{B}_1)$ |
| $1^{++}$ | $\frac{1}{\sqrt{2}}(B\bar{B} + B^*\bar{B}) B^*\bar{B}$ |
| $0^{--}$ | $\frac{1}{\sqrt{2}}(B_0\bar{B} B\bar{B}_0) \frac{1}{\sqrt{2}}(B_1\bar{B} - B\bar{B}_1) \frac{1}{\sqrt{2}}(B_1\bar{B} - B\bar{B}_1)$ |
| $0^{--}$ | $\frac{1}{\sqrt{2}}(B_0\bar{B} B\bar{B}_0) \frac{1}{\sqrt{2}}(B_1\bar{B} + B\bar{B}_1) \frac{1}{\sqrt{2}}(B_1\bar{B} + B\bar{B}_1)$ |
| $0^{--}$ | $B\bar{B} B^*\bar{B}$ |
| $2^{--}$ | $\frac{1}{\sqrt{2}}(B_1\bar{B} - B^*\bar{B}_1) \frac{1}{\sqrt{2}}(B_1\bar{B} - B\bar{B}_1) \frac{1}{\sqrt{2}}(B_1\bar{B} - B\bar{B}_1)$ |
| $2^{--}$ | $\frac{1}{\sqrt{2}}(B_1\bar{B} + B^*\bar{B}_1) \frac{1}{\sqrt{2}}(B_1\bar{B} + B\bar{B}_1) \frac{1}{\sqrt{2}}(B_1\bar{B} + B\bar{B}_1)$ |
| $2^{--}$ | $B^*\bar{B}$ |

A. The spin structures of the hidden beauty molecular states and bottomonia

In the radiative decays of the hidden beauty molecules, the heavy spin, light spin and total angular momentum are good quantum numbers in the heavy quark limit, which are sepa-
rately conserved. Therefore we can decompose the total angular momentum of the initial and final states according to their heavy spin and light spin.

With the heavy quark spin symmetry, the heavy and light spins of molecular state can be re-coupled separately. We adopt the spin re-coupling formula with $6\cdot j$ or $9\cdot j$ symbols in analyzing the general spin structure. Let’s take the molecular states composed of an S-wave bottom antimeson and a P-wave bottom meson as an example. We explicitly write out its spin structure

\[
|B_{1(2)}B_{12}\rangle = \frac{1}{\sqrt{3}} \left[ \sum_{g=0}^{s} \sum_{m=0}^{g} \sum_{h=|-s|}^{|s|} (-1)^{s+g+m} (2s+1)(2m+1) \right]
\]
\[
\times (2K+1)(2L+1)(2g+1)(2h+1)^{1/2}
\]
\[
\times \sum_{s} \frac{1}{2} \left< \frac{1}{2} s \ K \right| \left( \frac{1}{2} 1 \ s \right| h \ \frac{1}{2} m \rangle
\]
\[
\times \left[ \left| \bar{b} g \otimes (|q\bar{q}|_{m} \otimes 1)_{h} \right| \right] (\bar{b}\bar{q})(b\bar{q})
\]
\[
= \sum_{s} \sum_{g=0}^{s} \sum_{m=0}^{g} \sum_{h=|-s|}^{|s|} C_{g,m,h}^{s,L,K,J} \left[ \left| \bar{b} g \otimes (|q\bar{q}|_{m} \otimes 1)_{h} \right| \right] (\bar{b}\bar{q})(b\bar{q}).
\]

In the above equation, $s$ and $K$ denote the light spin and total angular momentum of the P-wave bottom meson respectively. $L$ is the total angular momentum of the S-wave bottom meson. $J$ is the total angular momentum of the molecular state. We are only interested in the S-wave molecular systems in this work. The orbital angular momentum between the two bottom mesons is zero.

In addition, the indices $b, \bar{b}, q$ and $\bar{q}$ in the square brackets in Eq. 1 represent the corresponding quark spin wave functions. The notation $\left[ \left| \bar{b} g \otimes (|q\bar{q}|_{m} \otimes 1)_{h} \right| \right]$ means that the spins of the $\bar{b}$ and $b$ quarkes are coupled into the heavy quark spin $g$ and the spins of the $q$ and $\bar{q}$ quarkes are coupled into the light quark spin $m$. And then $m$ couples with the orbital angular momentum to form the light spin $h$. We need to emphasize that we explicitly include the flavor wave function $|\bar{b}q)(b\bar{q})$ in Eq. 1.

The spin structure of the $|B^{(s)}B_{1(2)}\rangle$ can be obtained by applying the C-parity transformation to $|B_{1(2)}B^{(s)}\rangle$, i.e.,

\[
|B^{(s)}B_{12}\rangle = \frac{1}{\sqrt{3}} \left[ \sum_{g=0}^{s} \sum_{m=0}^{g} \sum_{h=|-s|}^{|s|} (-1)^{s+g+m} (2s+1)(2m+1) \right]
\]
\[
\times (2K+1)(2L+1)(2g+1)(2h+1)^{1/2}
\]
\[
\times \sum_{s} \frac{1}{2} \left< \frac{1}{2} s \ K \right| \left( \frac{1}{2} 1 \ s \right| h \ \frac{1}{2} m \rangle
\]
\[
\times \left[ \left| \bar{b} g \otimes (|q\bar{q}|_{m} \otimes 1)_{h} \right| \right] (\bar{b}\bar{q})(b\bar{q})
\]
\[
= \sum_{s} \sum_{g=0}^{s} \sum_{m=0}^{g} \sum_{h=|-s|}^{|s|} C_{g,m,h}^{s,L,K,J} \left[ \left| \bar{b} g \otimes (|q\bar{q}|_{m} \otimes 1)_{h} \right| \right] (\bar{b}\bar{q})(b\bar{q}).
\]

where the factor $(-1)^{L+K-J}$ arises from the exchange of the two bottoms in the molecular state.

With the above approach, we obtain the re-coupled spin structures of these hidden beauty molecular states, where the coefficients $B_{g,m,h}^{s,L,K,J}$ and $C_{g,m,h}^{s,L,K,J}$ in Eqs. (1)-2 are collected in Table III (see Table III for more details).

Besides the spin structures of these hidden beauty molecular states, we also need the spin structures of the bottomonia relevant to the radiative decays, i.e.,

\[
|\eta_b(1^{1}S_{0})\rangle = |(0^{0}_{H} \otimes 0^{-}_{l})^{++}_{0}(b\bar{b})\rangle,
\]
\[
|\Upsilon_{b}(1^{3}S_{1})\rangle = |(1^{0}_{H} \otimes 1^{0}_{l})^{++}_{0}(b\bar{b})\rangle,
\]
\[
|\eta_b(1^{3}P_{1})\rangle = |(0^{0}_{H} \otimes 1^{1}_{l})^{++}_{0}(b\bar{b})\rangle,
\]
\[
|\chi_{b0}(1^{3}P_{0})\rangle = |(1^{0}_{H} \otimes 1^{+}_{l})^{++}_{0}(b\bar{b})\rangle,
\]
\[
|\chi_{b1}(1^{3}P_{1})\rangle = |(1^{+}_{H} \otimes 1^{1}_{l})^{++}_{0}(b\bar{b})\rangle,
\]
\[
|\chi_{b2}(1^{3}P_{2})\rangle = |(1^{+}_{H} \otimes 2^{1}_{l})^{++}_{0}(b\bar{b})\rangle,
\]
\[
|\eta_{b2}(1^{3}D_{2})\rangle = |(0^{+}_{H} \otimes 2^{0}_{l})^{+}_{++}(b\bar{b})\rangle,
\]
\[
|\Upsilon_{b}(1^{3}D_{1})\rangle = |(1^{1}_{H} \otimes 2^{1}_{l})^{+}_{++}(b\bar{b})\rangle,
\]
\[
|\Upsilon_{b}(1^{3}D_{2})\rangle = |(1^{1}_{H} \otimes 2^{1}_{l})^{+}_{++}(b\bar{b})\rangle,
\]
\[
|\Upsilon_{b}(1^{3}D_{3})\rangle = |(1^{1}_{H} \otimes 2^{1}_{l})^{+}_{++}(b\bar{b})\rangle,
\]

where we use the subscripts $H$ and $l$ to distinguish the heavy and light spins of bottomonia. The superscripts $+$ and $-$ inside the parentheses denote the positive and negative parity of the corresponding parts, respectively, while the superscripts $++$ and subscripts $0,1,2,3$ correspond to the quantum numbers $PC$ and $J$ of $j^{PC}$ of the bottomonium. When calculating the radiative decay, we also need the bottomonium flavor wave function $|(b\bar{b})\rangle \equiv \frac{1}{\sqrt{3}}(b\bar{b}) + |b\bar{b})\rangle$, which is invariant under the C transformation. The C parity of the bottomonium is reflected through its spin and orbital wave function, i.e., $C = (-1)^{S_{+}+S_{-}}$. 
TABLE II: The coefficients $B_{j,p}^{i,h,k}$ and $C_{j,p}^{i,h,k}$ in Eqs. (11)-(12), which depend on the values $[g,m,h]$ and the corresponding molecular states. The combination $[g,m,h]$ corresponds to the subscripts in $[\bar{q}q] \otimes ([q\bar{q}]_m \otimes 1)_h$.

| $J^p = 0^+$ | $J^p = 1^+$ | $J^p = 2^+$ |
|------------|------------|------------|
| $[0,0,0]$  | $[0,0,0]$  | $[0,0,0]$  |
| $B^* B^*$  | $B^* B^*$  | $B^* B^*$  |

For the spin wave function, the orthogonalization requires

$$
\langle (a_{\gamma} \otimes b_L)_{p,\gamma}' \mid (c_{\gamma} \otimes d_L)_{p,\gamma}' \rangle = \delta_{a,c} \delta_{b,d} \delta_{l,l'} \delta_{p,p'} \delta_{\gamma,\gamma'},
$$

(13)

where the superscripts $p^{(\gamma)}$ and $c^{(\gamma)}$ denotes the parity and $C$ parity, respectively. The requirement of the orthogonalization of the spin wave function results in the conservation of the parity, $C$ parity, the total angular momentum, heavy spins, and light spins respectively.

In addition, the orthogonalization of the flavor wave function requires

$$
\langle (\bar{b} q)(\bar{b} q) \mid (\bar{b} q)(\bar{b} q) \rangle = 1, \quad \langle (\bar{b} q)(\bar{b} q) \mid (\bar{b} q)(\bar{b} q) \rangle = 0,
$$

which are adopted in the calculation of the electromagnetic transitions between two hidden beauty molecular states.

In the following sections, we will employ the above spin structures and flavor wave functions of the hidden beauty molecular states and bottomonia to calculate the radiative decays. The spin structure of the photon will be given later.

**B. Reduced matrix elements of the radiative decays**

In Sec. 12A we present the definitions of the spin structures of the initial and final hadrons in the radiative decays. In the following, we give the general expression of the reduced radiative decay matrix element. In the heavy quark limit, both the heavy and light spins are conserved. Thus, it is convenient to calculate the corresponding process with the uncoupled representation. Denote the total angular momentum of the initial state hadron as $j$ and the third component of $j$ as $j_z$. We have

$$
|j, j_z\rangle = \sum_{S_H, S_L} \langle S_H, S_H; S_L, S_L | j, j_z \rangle \langle S_H, S_H | S_L, S_L \rangle,
$$

which is a unitary transformation between the coupled and uncoupled representations. For the final state hadron, we denote the total angular momentum of the final state hadron as $j'$ and its third component as $j_z'$. In addition, $Q$ is the light spin of the photon and $Q_z$ is its third component. Thus, the matrix
element for the radiative decay $A \to B + \gamma$ is written as

$$M[A(j, j_c) \to B(j', j_c') + \gamma(Q, Q_c)] = \langle\gamma(Q, Q_c); j', j_c'|H_{eff}|j, j_c\rangle = \sum_{S_H, S_l, S'_H, S'_l} \langle\gamma(Q, Q_c); S'_H, S'_l; S_l; S_H; S, j, j_c|H_{eff}|S_H; S_l; S, j, j_c\rangle \times \langle S'_H, S'_l; S_l; S_H|j', j_c'|S_H; S_l; S, j, j_c\rangle = \sum_{S_H, S_l} \langle S_H, S_H; S_l, S_l|j, j_c\rangle \langle\gamma(Q, Q_c); S'_l|H_{eff}|S_l\rangle \times \langle S_H, S_H; S'_l|j, j_c\rangle \langle S_l, S_l|H_{eff}|S_l\rangle \times (-1)^{Q_S + S_H + j} \sqrt{(2j + 1)(2j' + 1)} \left\{ \begin{array}{c} Q_S \ S'_l \ S_l \\ S_H \ j \ j' \end{array} \right\} \langle\gamma(Q, Q_c); j, j_c|H_{eff}|j, j_c\rangle = (-1)^{Q + S_H + j} \sqrt{(2j + 1)(2j' + 1)} \left\{ \begin{array}{c} Q \ S'_l \ S_l \\ S_H \ j \ j' \end{array} \right\} \langle\gamma(Q, Q_c); j, j_c|H_{eff}|j, j_c\rangle. \tag{14}$$

where $S_H$ and $S'_H$ stand for the heavy spin of initial and final hadrons, respectively. The third components of $S_H$ and $S'_H$ are $S_l$ and $S'_l$, respectively. $S_l$ and $S'_l$ denote the light spins of the initial and final hadrons, respectively. The third components of $S_l$ and $S'_l$ are $S_l$ and $S'_l$, respectively.

The effective Hamiltonian $H_{eff}$ conserves both the heavy and light spins and can be decomposed into

$$H_{eff} = H_{eff}^{spatial} \otimes H_{eff}^{flavor}, \tag{15}$$

where $H_{eff}^{spatial}$ and $H_{eff}^{flavor}$ denote the spatial and flavor parts, respectively. With the help of the Wigner-Eckart theorem, we get

$$\langle\gamma(Q, Q_c); S'_H, S'_l; S_l, S_H|H_{eff}|S_H, S_l, S_l, S_H\rangle = \langle\gamma(Q, Q_c); S'_l|H_{eff}|S_l\rangle \delta_{S_H, S'_H, S_l, S'_l},$$

where $\langle\gamma(Q, Q_c); S'_l|H_{eff}|S_l\rangle$ is the reduced matrix element. The $\gamma$ symbol and the factors in the front of $\gamma$ symbol in Eq. 14 come from the spin rearrangement of the initial and final states.

Let’s consider two different radiative decay channels with the same final state. If the initial state hadrons happen to have the same radial and orbital quantum numbers, the ratio of the decay widths of these two channels can be expressed as

$$\frac{\Gamma(H_1 \to H_2 + \gamma)}{\Gamma(H_2 \to H_2 + \gamma)} \approx \frac{|c_1\langle\gamma(Q, Q_c); S'_l|H_{eff}|S_l\rangle + c_2\langle\gamma(Q, Q_c); S'_l|H_{eff}|S_l\rangle + \cdots|^2}{|d_1\langle\gamma(Q, Q_c); S'_l|H_{eff}|S_l\rangle + d_2\langle\gamma(Q, Q_c); S'_l|H_{eff}|S_l\rangle + \cdots|^2}. \tag{16}$$

where $c_1, c_2, d_1$ and $d_2$ represent the corresponding expansion coefficients of the spin configuration. We also consider two different radiative decay channels with the same initial state. If the two final state hadrons have the same radial and orbital quantum numbers, we can obtain similar expressions as Eq. 16. We need to specify that the ratio listed in Eq. 16 does not include the phase space contribution.

C. $\Upsilon \to (b\bar{b}) + \gamma$

For the radiative decay $\Upsilon \to (b\bar{b}) + \gamma$, where the initial state is a hadronic molecule and final state is a bottomonium, the photon is from the $Q\bar{Q}$ annihilation. Now we need to introduce the spin structure of the involved photon. We assume that the photon has a so-called “flavor” wave function equivalently, which is denoted by $|\gamma\rangle$ in the following. Its spin structure is defined as

$$|\gamma(E1)\rangle = |(0_H \otimes 1_{L}^-)\rangle|\gamma\rangle, \tag{17}$$

$$|\gamma(M1)\rangle = |(0_H \otimes 1_{L}^+)\rangle|\gamma\rangle, \tag{18}$$

$$|\gamma(E2)\rangle = |(0_H \otimes 2_{L}^+\rangle|\gamma\rangle, \tag{19}$$

which correspond to the E1, M1 and E2 transitions, respectively. Here, the notation of subscripts and subscripts are similar to those in Eqs. 5-12. The transition matrix elements related to the flavor wave functions are

$$\langle\gamma; b\bar{b}|H_{eff}^{flavor}|(\bar{b}q)(b\bar{q})\rangle = 1, \quad \langle\gamma; b\bar{b}|H_{eff}^{flavor}|(\bar{b}q)(b\bar{q})\rangle = 0,$$

$$\langle\gamma; b\bar{b}|H_{eff}^{flavor}|(b\bar{q})(b\bar{q})\rangle = 1, \quad \langle\gamma; b\bar{b}|H_{eff}^{flavor}|(b\bar{q})(b\bar{q})\rangle = 0.$$

With the above preparation, we can calculate the rearranged spin structures of the final states in the $\Upsilon \to (b\bar{b}) + \gamma$ decays. The general expression is

$$|Bottomonium\rangle \otimes |\gamma\rangle = \left[ \left( (b\bar{b})_g \otimes L \right) \otimes Q \right]_J (\langle\bar{b}\bar{b}\rangle)_{L, Q, J}, \tag{20}$$

where $g$ and $L$ denote the heavy and light spins of the bottomonium, respectively. $Q$ stands for the heavy spin of the photon. The indices $b$, $\bar{b}$ and $\gamma$ in the square brackets represent the corresponding spin wave functions. We collect the coefficients $D_{g, L, Q, J}^{K}$ in Table III

In this work, we mainly focus on the ratios of the decay widths in the heavy quark limit, where the spatial matrix elements of some decays are the same. For example, the molecular state $\Upsilon(1S)\Upsilon(1S)$ can decay into $\Upsilon(1S)\Upsilon(1S)$, which are typical M1 transitions and have the same spatial matrix elements. In this case, there exist some model independent predictions.

D. $(b\bar{b}) \to \Upsilon + \gamma$

In the radiative decay $(b\bar{b}) \to \Upsilon + \gamma$, a bottomonium state emits a photon and decays into a hidden beauty molecule state,
TABLE III: The coefficient $Z_{g,h}^{J_{f,g}}$ in Eq. \[20\] corresponding to different combinations of $[g,h]$.

| $J = 0$     | $J = 1$     | $J = 2$     |
|------------|------------|------------|
| $[0,0]$    | $[0,1]$    | $[0,2]$    |
| $[1,1]$    | $[1,0]$    | $[1,1]$    | $[1,2]$    |
| $[1,1]$    | $[1,1]$    | $[1,1]$    | $[1,3]$    |

\[
|\eta_{0}(1^1S_0)\gamma(E1/M1)\rangle | \eta_{1}(1^1P_1)\gamma(E1/M1)\rangle | \eta_{0}(1^1P_0)\gamma(E1/M1)\rangle | \eta_{0}(1^1D_2)\gamma(E1/M1)\rangle | \eta_{0}(1^1D_1)\gamma(E1/M1)\rangle | \eta_{0}(1^1S_1)\gamma(E2)\rangle |

\[
|\langle 0,0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

where a $q\bar{q}$ pair is created from the vacuum. Thus, this process can be expressed as

\[
|\text{Heavy Quarkonium}\rangle \otimes |\text{VAC}\rangle \\
\rightarrow \frac{1}{\sqrt{2}}(|B_{1,2}B_{1,2}^{(*)}\rangle \pm |B_{1,2}^{(*)}B_{1,2}\rangle) \otimes (0_f^+ \otimes \bar{I}^+_f)\gamma),
\]

where $|\text{VAC}\rangle$ is the hadronic vacuum, which has the spin structure

\[
|\text{VAC}\rangle = |(0_f^+ \otimes 0_f^+)\rangle \langle q\bar{q}|,
\]

where $|(q\bar{q})\rangle = \frac{1}{\sqrt{2}}((q\bar{q}) + |\bar{q}q\rangle)$ is the so-called “flavor” wave function.

The flavor part of the final states in $(b\bar{b}) \rightarrow \gamma \eta + \gamma$ can be written as

\[
\langle \gamma | (\bar{b}q)(b\bar{q}) \rangle = \langle (\bar{b}q)(b\bar{q})\rangle + \langle (b\bar{q})(b\bar{q})\rangle + \langle (\bar{b}q)(b\bar{q})\rangle + \langle (b\bar{q})(b\bar{q})\rangle.
\]

Similarly, we list the spin structure of the final states in $(b\bar{b}) \rightarrow \gamma \eta + \gamma$ from the flavor wave functions only are

\[
\langle (b\bar{q})(b\bar{q})|H_{\text{eff}}^{\text{flavor}}|b\bar{b}\rangle|\bar{q}q\rangle = 1,
\]

\[
\langle (b\bar{q})(b\bar{q})|H_{\text{eff}}^{\text{flavor}}|b\bar{b}\rangle|\bar{q}q\rangle = 1,
\]

\[
\langle (b\bar{q})(b\bar{q})|H_{\text{eff}}^{\text{flavor}}|b\bar{b}\rangle|\bar{q}q\rangle = 1,
\]

\[
\langle (b\bar{q})(b\bar{q})|H_{\text{eff}}^{\text{flavor}}|b\bar{b}\rangle|\bar{q}q\rangle = 1,
\]

\[
\langle (b\bar{q})(b\bar{q})|H_{\text{eff}}^{\text{flavor}}|b\bar{b}\rangle|\bar{q}q\rangle = 1.
\]
\( M + \gamma: \)

\[
|B_{12}B^{(s)}\rangle \otimes |\gamma\rangle
= \left[|\bar{b} \otimes (q \otimes 1)_{Lk} \otimes |b \otimes \bar{q}\rangle_{Ll}\right] \otimes (0_f^\gamma \otimes 1_{l}^f)
\]

\[
= \sum_{x=0}^{1} \sum_{m=0}^{s} \sum_{h=0}^{m} (-1)^{h+x} \gamma_{m+h}
\times \sqrt{(2s+1)(2m+1)(2K+1)(2L+1)(2h+1)}
\times (2g+1) \sqrt{3(2J+1)(2h_0+1)}
\times 0, 1 \quad \left\{ \begin{array}{c}
\frac{1}{2} 1 \\ \frac{1}{2} s \end{array} \right\} \\
\times 0, 1 \quad \left\{ \begin{array}{c}
\frac{1}{2} 1 \\ \frac{1}{2} L \end{array} \right\}
\times [\bar{b}b]_{g} \otimes [(qq\bar{q})_{m} \otimes 1_{h_0} \otimes 1_{h_0}]_{h_0},
\]

which will be applied in the calculation.

For the radiative transition between the two hidden beauty molecular states, its dynamics is different from that of \((b\bar{b}) \rightarrow M + \gamma\). The photon is emitted from either \(q/\bar{q}\) quark or \(b/\bar{b}\) quark. The spin structure of the photon is similar to that of \(b/\bar{b}\) quark or \(q/\bar{q}\) quark. The spin structure of the photon is similar to that given in Eqs. (17-19).

The nonzero transition matrix elements of \(M \rightarrow M' + \gamma\) from the flavor wave functions are

\[
\langle (b\bar{q} \gamma)(\bar{b}q) \rangle_{H_{off}^{1}}^{\text{flavor}} = 1, \\
\langle (b\bar{q}) (b\gamma)(\bar{q}) \rangle_{H_{off}^{1}}^{\text{flavor}} = 1, \\
\langle (b\bar{q}) (b\gamma)(\bar{q}) \rangle_{H_{off}^{1}}^{\text{flavor}} = 1, \\
\langle (b\bar{q}) (b\gamma)(\bar{q}) \rangle_{H_{off}^{1}}^{\text{flavor}} = 1, \\
\langle (b\bar{q}) (b\gamma)(\bar{q}) \rangle_{H_{off}^{1}}^{\text{flavor}} = 1, \\
\langle (b\bar{q}) (b\gamma)(\bar{q}) \rangle_{H_{off}^{1}}^{\text{flavor}} = 1.
\]

\section{Numerical Results}

With the above preparation, we are ready to calculate the typical ratios of the radiative decays relevant to the hidden beauty molecular states. In the following, we present the results of \(M \rightarrow (b\bar{b}) + \gamma, (b\bar{b}) \rightarrow M + \gamma\) and \(M \rightarrow M' + \gamma\).

\subsection{\(M \rightarrow (b\bar{b}) + \gamma\)}

Using the rearranged spin structures and the orthogonalization of the spin and flavor wave functions, we first obtain some typical ratios of the \(M \rightarrow (b\bar{b}) + \gamma\) decays. We only consider the final states containing the S-wave, P-wave and D-wave bottomonia.

If the bottomonia belong to the same spin multiplet, the spatial matrix elements of these radiative decays are the same, which leads to quite simple ratios between their decay widths. We collect the typical ratios in Tables [LV] and [V] where the values in the brackets are the values with the phase space correction. Since the \(B_0\) meson and D-wave bottomonia are still absent experimentally, we do not consider the contribution of the phase space factors when calculating the corresponding ratios.

There are six hidden beauty molecular states with \(J^{PC} = 1^{--}\), which can decay into \(\chi_{b1}(1^3P_J)\gamma\) \((J = 0, 1, 2)\). Except \(\frac{1}{\sqrt{2}} (B_0^* B^* + B^* B_0^*)\), the remaining five molecular states have the same ratio \(\Gamma(\chi_{b1}(1^3P_J)\gamma) = 1\). Each \(B_0\) meson and D-wave bottomonia are still absent experimentally, we do not consider the contribution of the phase space factors. This phenomena can be understood well since these decay widths are related to the spin configuration \((1^+_H \otimes 1^+_f)_{J_{1=1}}\). For \(\frac{1}{\sqrt{2}} (B_0^* B^* + B^* B_0^*)\), its decays into \(\chi_{b1}(1^3P_J)\gamma\) are strongly suppressed due to heavy quark symmetry.

Both \(\frac{1}{\sqrt{2}} (B_0^* B^* + B^* B_0^*)\) and \(B_0^* B^*\) decay into \(\chi_{b1}(1^3P_J)\gamma\) with the typical ratios \(\Gamma(\chi_{b1}(1^3P_J)\gamma) = 1\). Each \(\chi_{b1}(1^3P_J)\gamma\) via the \(E_1\) transition, where the spin configuration \((1^+_H \otimes 0^+_f)_{J_{1=1}}\). We notice that the ratio \(1:3:5\) is consistent with that given in Refs. [53, 55].

The typical ratios \(\Gamma(1^3D_J)\gamma\) and \(\Gamma(1^3D_J)\gamma\) via the \(E_1\) transition, where the spin configuration \((1^+_H \otimes 1^+_f)_{J_{1=1}}\). Its ratio \(\Gamma(\chi_{b1}(1^3D_J)\gamma) = 1\). This ratio \(\Gamma(\chi_{b1}(1^3D_J)\gamma) = 1\) is also 1:3.

When the initial state is \(\frac{1}{\sqrt{2}} (B_0^* B^* + B^* B_0^*)\), \(\frac{1}{\sqrt{2}} (B_0^* B^* + B^* B_0^*)\) and \(\frac{1}{\sqrt{2}} (B_0^* B^* + B^* B_0^*)\) with \(J^{PC} = 1^{--}\), we find that their decays into \(\chi_{b1}(1^3D_J)\gamma\) depend on the two spin configurations \((1^+_H \otimes 1^+_f)_{J_{1=1}}\) and \((1^+_H \otimes 2^+_f)_{J_{1=1}}\). Thus, we define

\[
x = \frac{H_{22}(M_1)}{H_{21}(M_1)}, \quad x' = \frac{H_{22}(E_2)}{H_{21}(E_2)}, \quad \alpha = \frac{H_{22}(E_2)}{H_{20}(E_2)}.
\]
TABLE IV: The typical ratios of the $\mathcal{B}\rightarrow (bb) + \gamma$ decay widths. The parameters $x, x', \alpha$ are defined as $x = H_{22}(M1)/H_{22}(M1)$, $x' = H_{22}(E2)/H_{22}(E2)$, and $\alpha = H_{22}(E2)/H_{22}(E2)$, respectively.

| $J^{PC}$ | Initial state | Final state |
|----------|---------------|-------------|
| $1^{--}$ | $1/\sqrt{2} (B_0 \bar{B}^* - B \bar{B}_0)$ | $\Gamma(\chi_{10}\gamma(E1)) : \Gamma(\chi_{10}\gamma(E1)) : \Gamma(\chi_{10}\gamma(E1))$ |
|         | $1/\sqrt{2} (B' \bar{B} - B B'_1)$ | $4 : 3 : 5$ |
|         | $1/\sqrt{2} (B_1 \bar{B} - B B'_1)$ | $4 : 3 : 5 (1.4 : 1 : 1.6)$ |
|         | $1/\sqrt{2} (B'_1 \bar{B}^* + B' \bar{B}'_1)$ | $0 : 0 : 0$ |
|         | $1/\sqrt{2} (B_1 \bar{B}^* + B \bar{B}_1)$ | $4 : 3 : 5 (1.4 : 1 : 1.6)$ |
|         | $1/\sqrt{2} (B_2 \bar{B}^* - B \bar{B}_2)$ | $4 : 3 : 5 (1.4 : 1 : 1.6)$ |
| $1^{--}$ | $1/\sqrt{2} (B B' - B' B)$ | $1 : 3 : 5 (1 : 2.6 : 4.1)$ |
|         | $B' B'$ | $1 : 3 : 5 (1 : 2.7 : 4.1)$ |
| $1^{++}$ | $1/\sqrt{2} (B B' + B' B)$ | $1 : 3$ |
| $1^{--}$ | $1/\sqrt{2} (B B' - B' B)$ | $0 : 0$ |
|         | $B' B'$ | $0 : 0$ |
| $1^{++}$ | $1/\sqrt{2} (B_0 \bar{B} + B \bar{B}_0)$ | $9(1 - \sqrt{3} \alpha)^2 : 5(\sqrt{3} - \alpha)^2 : 7(\sqrt{3} + 2 \alpha)^2$ |
|         | $1/\sqrt{2} (B'_1 \bar{B} + B B'_1)$ | $9(1 + \sqrt{3} \alpha)^2 : 5(\sqrt{3} + \alpha)^2 : 7(\sqrt{3} - 2 \alpha)^2$ |
|         | $1/\sqrt{2} (B_1 \bar{B} + B B_1)$ | $9(\sqrt{3} + \sqrt{35} x')^2 : 5(1 + \sqrt{105} x')^2 : 7(\sqrt{3} - \sqrt{105} x')^2$ |
|         | $1/\sqrt{2} (B'_1 \bar{B} - B' B_1)$ | $3 : 5 : 7$ |
|         | $1/\sqrt{2} (B_1 \bar{B} - B B_1)$ | $9(3 \sqrt{5} - \sqrt{35} x')^2 : 15(\sqrt{3} + \sqrt{35} x')^2 : 4(3 \sqrt{5} + \sqrt{105} x')^2$ |
|         | $1/\sqrt{2} (B_2 \bar{B} + B B_2)$ | $9(5 \sqrt{5} + \sqrt{35} x')^2 : 25(\sqrt{5} - \sqrt{21} x')^2 : 4(5 \sqrt{5} - \sqrt{105} x')^2$ |
| $1^{++}$ | $1/\sqrt{2} (B_1 \bar{B}^* - B' \bar{B}'_1)$ | $1 : 3 (1 : 2.9)$ |
|         | $1/\sqrt{2} (B'_1 \bar{B}^* + B \bar{B}'_1)$ | $1 : 3 (1 : 2.8)$ |
|         | $1/\sqrt{2} (B_2 \bar{B}^* + B B'_2)$ | $1 : 3 (1 : 2.9)$ |
|         | $1/\sqrt{2} (B_2 \bar{B}^* - B' \bar{B}'_2)$ | $1 : 3 (1 : 2.9)$ |
| $2^{++}$ | $B' B'$ | $1 : 15 : 84$ |
| $2^{++}$ | $1/\sqrt{2} (B'_1 \bar{B}' + B B'_1)$ | $1 : 15 : 84$ |
|         | $1/\sqrt{2} (B_1 \bar{B}' + B B_1)$ | $9(1 - 3 \sqrt{5} x')^2 : 3(\sqrt{5} + 25 x')^2 : 28(3 - \sqrt{5} x')^2$ |
|         | $1/\sqrt{2} (B_2 \bar{B}' + B B_2)$ | $3(1 + \sqrt{5} x')^2 : (3 \sqrt{5} + 25 x')^2 : 28(3 - \sqrt{5} x')^2$ |
|         | $1/\sqrt{2} (B_2 \bar{B}' - B B_2)$ | $27(1 - \sqrt{5} x')^2 : (9 \sqrt{5} + 25 x')^2 : 28(9 + \sqrt{5} x')^2$ |
| $2^{++}$ | $1/\sqrt{2} (B'_1 \bar{B}' + B B'_1)$ | $9 : 35 : 56$ |
|         | $1/\sqrt{2} (B_1 \bar{B}' + B B_1)$ | $9(\sqrt{5} - 3 \sqrt{35} x')^2 : (\sqrt{105} + 45 x')^2 : 8(\sqrt{21} + 9 \sqrt{5} x')^2$ |
|         | $1/\sqrt{2} (B_1 \bar{B}' - B B_1)$ | $(3 + \sqrt{105} x')^2 : (\sqrt{35} + 5 \sqrt{3} x')^2 : 8(\sqrt{7} - \sqrt{15} x')^2$ |
|         | $1/\sqrt{2} (B_2 \bar{B}' - B B_2)$ | $(3 \sqrt{5} - \sqrt{35} x')^2 : (\sqrt{105} + 5 x')^2 : 8(\sqrt{21} + \sqrt{5} x')^2$ |
| Initial state | Final state | $\eta y(M1)$ | $\chi x0y(E1)$ | $\chi x1y(E1)$ | $\chi x2y(E1)$ | $\eta y(1^{1}D_{2})y(M1)$ | $\eta y(1^{1}S_{0})y(E2)$ | $\eta y(1^{1}D_{2})y(E2)$ |
|---------------|-------------|---------------|---------------|---------------|---------------|------------------|------------------|------------------|
| $1^{-}$       | $\eta y(M1)$ | $\chi x0y(E1)$ | $\chi x1y(E1)$ | $\chi x2y(E1)$ | $\eta y(1^{1}D_{2})y(M1)$ | $\eta y(1^{1}S_{0})y(E2)$ | $\eta y(1^{1}D_{2})y(E2)$ |
| $2^{-}$       | $\eta y(E1)$ | $\chi x0y(M1)$ | $\chi x1y(E1)$ | $\chi x2y(M1)$ | $\eta y(1^{1}D_{2})y(E1)$ |
| $1^{+}$       | $\eta y(M1)$ | $\chi x0y(E1)$ | $\chi x1y(E1)$ | $\chi x2y(E1)$ | $\eta y(1^{1}D_{2})y(M1)$ | $\eta y(1^{1}S_{0})y(E2)$ | $\eta y(1^{1}D_{2})y(E2)$ |
| $2^{+}$       | $\eta y(E1)$ | $\chi x0y(M1)$ | $\chi x1y(E1)$ | $\chi x2y(M1)$ | $\eta y(1^{1}D_{2})y(E1)$ |
| $1^{1+}$      | $\eta y(M1)$ | $\chi x0y(E1)$ | $\chi x1y(E1)$ | $\chi x2y(E1)$ | $\eta y(1^{1}D_{2})y(M1)$ | $\eta y(1^{1}S_{0})y(E2)$ | $\eta y(1^{1}D_{2})y(E2)$ |
| $2^{1+}$      | $\eta y(E1)$ | $\chi x0y(M1)$ | $\chi x1y(E1)$ | $\chi x2y(M1)$ | $\eta y(1^{1}D_{2})y(E1)$ |
| $0^{+}$       | $\eta y(M1)$ | $\chi x0y(E1)$ | $\chi x1y(E1)$ | $\chi x2y(E1)$ | $\eta y(1^{1}D_{2})y(M1)$ | $\eta y(1^{1}S_{0})y(E2)$ | $\eta y(1^{1}D_{2})y(E2)$ |
| $0^{1+}$      | $\eta y(E1)$ | $\chi x0y(M1)$ | $\chi x1y(E1)$ | $\chi x2y(M1)$ | $\eta y(1^{1}D_{2})y(E1)$ |

TABLE V: The typical ratios $\frac{\Gamma(1^{1}S_{0} \rightarrow (1^{1}D_{2}) \gamma)}{\Gamma(1^{1}S_{0} \rightarrow (1^{1}P_{1}) \gamma)}$, where the initial molecular states are different while the final states are same. The parameters are defined as $x = H_{22}(M1)/H_{22}(M1)$, $x' = H_{22}(E2)/H_{22}(E2)$, and $\alpha = H_{22}(E2)/H_{22}(E2)$. 
where $H_{22}(M1) = \langle 1, 2 || H_{eff}(M1) || 1 \rangle$ and $H_{22}(M1) = \langle 1, 2 || H_{eff}(M1) || 2 \rangle$ are the reduced matrix elements for the M1 transition. $H_{02}(E2) = \langle 2, 0 || H_{eff}(E2) || 0 \rangle$, $H_{21}(E2) = \langle 2, 2 || H_{eff}(E2) || 1 \rangle$ and $H_{22}(E2) = \langle 2, 2 || H_{eff}(E2) || 2 \rangle$ are the reduced matrix elements for the E2 transition.

The $E2$ decay modes $\chi_{b1} (1^3 P_J^j) \gamma (J = 1, 2)$ of the two states $\frac{1}{\sqrt{2}} (B^- B^- \bar{B}^-) \otimes (B^+ B^+ \bar{B}^+)$ and $B^- B^- \bar{B}^- B^+ B^+ \bar{B}^+$ with $J^{PC} = 1^{-+}$ depend on the spin configurations $(0_H \otimes 1^+_J)_{j_{z=1}}$ and $(1_H \otimes 0^+_J)_{j_{z=1}}$. From Table III we conclude that these $\chi_{b1} \gamma(E2)$ and $\chi_{b2} \gamma(E2)$ modes are suppressed due to heavy quark symmetry.

We can see from Table III the $B^- B^- \bar{B}^- B^+ B^+$ molecular states with $J = 1$ contain the spin configurations $(0_H \otimes 1^+_J)_{j_{z=1}}$, $(0_H \otimes 1^+_J)_{j_{z=1}}$ and $(1_H \otimes 0^+_J)_{j_{z=1}}$. On the other hand, we know from Table III that the recoupled final state $\gamma(1^3 S_1) \gamma(M1)$ with $J = 1$ only contains the component of $(1_H \otimes 1^+_J)_{j_{z=1}}$. So both $B^- B^- \bar{B}^-$ and $B^- B^+ \bar{B}^+$ components with $J = 1$ can independently decay into $\gamma(1^3 S_1) \gamma(M1)$. Unfortunately, these two parts have the opposite relative phase for the component of $(1_H \otimes 1^+_J)_{j_{z=1}}$. When they constitute the C-parity eigenstate $\frac{1}{\sqrt{2}} (B^- B^- \bar{B}^- B^+)$, the radiative decay into $\gamma(1^3 S_1) \gamma(M1)$ is suppressed in the heavy quark limit, as listed in Table V.

Since the $0^{-+}$ states $\frac{1}{\sqrt{2}} (B_0^- B^- \bar{B}^-)$ and $\frac{1}{\sqrt{2}} (B^- B^+ \bar{B}^- \bar{B}^+)$ do not contain the spin configuration $(0_H \otimes 2^+_J)_{j_{z=1}}$, their decays into $\eta_{b2}(1^D_2) \gamma(E2)$ are suppressed. Similarly, the $0^{+-}$ states $\frac{1}{\sqrt{2}} (B_0^- B^+ \bar{B}^-)$ and $\frac{1}{\sqrt{2}} (B^+ B^+ \bar{B}^- \bar{B}^+)$ do not contain the spin configuration $(0_H \otimes 0^+_J)_{j_{z=1}}$. Thus, their decays into $\eta_{b2}(1^D_2) \gamma(E2)$ are also suppressed.

From Table IV we see that the ratio $\Gamma(\chi_{b2}(1^3 P_J^j) \gamma(E1)) : \Gamma(\chi_{b2}(1^3 P_J^j) \gamma(E1)) = 1 : 3$ is the same for all the $2^-$ molecular states. These decay widths are due to the contribution of the spin configuration $(1_H \otimes 1^+_J)_{j_{z=1}}$.

Since $\frac{1}{\sqrt{2}} (B_1 B^- B^+ \bar{B}^+)$, $\frac{1}{\sqrt{2}} (B_1 B^- B^- \bar{B}^+)$ and $\frac{1}{\sqrt{2}} (B_1 B^- B^+ \bar{B}^- \bar{B}^+)$ do not contain the spin configuration $(0_H \otimes 2^+_J)_{j_{z=1}}$, their decays into $h_{b2}(1^1 P_1) \gamma(E1)$ are suppressed in the heavy quark limit.

We need to specify that the ratios shown in Tables IV and V are also suitable for the radiative decays involving the higher radially excited bottomonium.

B. $(b\bar{b}) \rightarrow 3\gamma + \gamma$

Adopting the same formalism, we obtain the typical ratios of the $(b\bar{b}) \rightarrow 3\gamma + \gamma$ decay widths, which are listed in Tables VI and VII.

Among all the possible hidden beauty molecular states, the mass of the lowest state $\chi_{b2}(1^3 P_J^j)$ is around 10616 MeV. At present, $\gamma(10860)$ is the heaviest bottomonium state. Thus we can only study the production of the hidden beauty molecular states via the radiative decays of the higher radial excitations of bottomonium.

In Table VII we define

$$ y = \frac{H_{22}(M1)}{H_{12}(M1)} \gamma = \frac{H_{22}(E2)}{H_{12}(E2)} \beta = \frac{H_{12}(E2)}{H_{22}(E2)} $$

where $H_{12}(M1) = \langle 1, 1 || H_{eff}(M1) || 1 \rangle$ and $H_{22}(M1) = \langle 1, 2 || H_{eff}(M1) || 2 \rangle$ are the reduced matrix elements for the M1 transition. $H_{02}(E2) = \langle 2, 0 || H_{eff}(E2) || 0 \rangle$, $H_{21}(E2) = \langle 2, 2 || H_{eff}(E2) || 1 \rangle$ and $H_{22}(E2) = \langle 2, 2 || H_{eff}(E2) || 2 \rangle$ are the reduced matrix elements for the E2 transition. If comparing the results shown in Tables VI and VII we notice the similarity between these ratios. For example, the $\frac{1}{\sqrt{2}} (B_0 B^- B^- \bar{B}^- \bar{B}^+)$ molecular state can decay into $\chi_{b2}(1^3 P_J^j) \gamma (J = 0, 1, 2)$. In fact, if there exist the following relations between the relevant reduced matrix elements

$$ H_{21}(M1) = -\sqrt{\frac{5}{3}} H_{12}(M1), \quad H_{21}(E2) = -\sqrt{\frac{5}{3}} H_{12}(E2), $$

$$ H_{20}(E2) = \sqrt{5} H_{12}(E2), $$

the obtained ratio $\Gamma(\chi_{b2}(E1)) : \Gamma(\chi_{b2}(E1)) : \Gamma(\chi_{b2}(E1))$ is the same as that listed in the second row in Table VII where $\frac{1}{\sqrt{2}} (B_0 B^- B^- \bar{B}^- \bar{B}^+)$ is produced by the $E1$ radiative decays of $\chi_{b2}(1^3 P_J^j)$ ($J = 0, 1, 2$). This phenomenon reflects the crossing symmetry. In the above discussion we do not consider the effect from the phase space factor of the corresponding process.

C. $3\gamma \rightarrow 3\gamma' + \gamma$

In this subsection, we discuss the $E1$ and $M1$ radiative transition between two molecular states. We only consider the hidden beauty molecular states with $J^{PC} = 1^{-+}, 1^{++}$ and $1^{++}$. The typical ratios of the $3\gamma \rightarrow 3\gamma' + \gamma$ decay widths depend on the following parameters

$$ A = \frac{H_{10}(M1)}{H_{11}(M1)}, \quad B = \frac{H_{12}(M1)}{H_{11}(M1)}, $$

$$ C = \frac{H_{01}(M1)}{H_{11}(M1)}, \quad D = \frac{H_{21}(M1)}{H_{11}(M1)}, $$

which are related to the reduced matrix elements of the $M1$ transitions, where $H_{10}(M1) = \langle 1, 1 || H_{eff}(M1) || 0 \rangle$, $H_{01}(M1) = \langle 1, 0 || H_{eff}(M1) || 1 \rangle$, $H_{11}(M1) = \langle 1, 1 || H_{eff}(M1) || 1 \rangle$, $H_{12}(M1) = \langle 1, 2 || H_{eff}(M1) || 2 \rangle$, and $H_{21}(M1) = \langle 2, 1 || H_{eff}(M1) || 1 \rangle$.

The ratios of the $E1$ transitions between two molecular states depend on the following parameters

$$ A' = \frac{H_{01}(E1)}{H_{11}(E1)}, \quad C' = \frac{H_{10}(E1)}{H_{11}(E1)}, \quad (21) $$

where $H_{10}(E1) = \langle 1, 1 || H_{eff}(E1) || 0 \rangle$, $H_{11}(E1) = \langle 1, 1 || H_{eff}(E1) || 1 \rangle$, and $H_{01}(E1) = \langle 1, 0 || H_{eff}(E1) || 1 \rangle$.

If the reduced matrix elements satisfy the following rela-
TABLE VI: The ratios between the $(b\bar{b}) \to 3\gamma + \gamma$ decay widths. The parameters $\gamma, \gamma', \beta$ are defined as $y = H_{23}(M1)/H_{12}(M1)$, $\gamma' = H_{23}(E2)/H_{12}(E2)$, and $\beta = H_{23}(E2)/H_{03}(E2)$, respectively.

| $^{jPC}$ | Final state | Initial state |
|---------|-------------|---------------|
| $^{1^+}$ | $\frac{1}{\sqrt{2}}(B_0 \bar{B}^* - B' \bar{B}_0)\gamma(E1)$ | $\chi_{60}(n^3 P_0) : \chi_{61}(n^3 P_1) : \chi_{62}(n^3 P_2)$ |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} - B \bar{B}_0')\gamma(E1)$ | 4 : 3 : 5 |
| $^{1^-}$ | $\frac{1}{\sqrt{2}}(B_0 \bar{B} + B' \bar{B}_0)\gamma(E1)$ | 4 : 3 : 5 |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} + B \bar{B}_0')\gamma(E1)$ | 0 : 0 : 0 |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B}^* + B' \bar{B}_1)\gamma(E1)$ | 4 : 3 : 5 |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B} - B' \bar{B}_1)\gamma(E1)$ | 4 : 3 : 5 |
| $^{1^+}$ | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} - B' \bar{B}_0)\gamma(M1)$ | $\chi_{60}(n^3 P_0) : \chi_{61}(n^3 P_1) : \chi_{62}(n^3 P_2)$ |
|         | $B' \bar{B}^* \gamma(M1)$ | 1 : 3 : 5 |
| $^{1^-}$ | $\frac{1}{\sqrt{2}}(B_0 \bar{B} + B' \bar{B}_0)\gamma(M1)$ | $\gamma(n^2 D_1) : \gamma(n^2 D_2)$ |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} + B \bar{B}_0')\gamma(M1)$ | 1 : 3 |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B}^* - B' \bar{B}_1)\gamma(M1)$ | $(\frac{1+3\sqrt{3}y}{3(1+\sqrt{3}y)})$ |
|         | $\frac{1}{\sqrt{2}}(B'_1 \bar{B} + B \bar{B}_1')\gamma(M1)$ | 0 : 0 |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B}^* - B' \bar{B}_1)\gamma(M1)$ | $(\frac{1+\sqrt{3}y}{3(1+\sqrt{3}y)})$ |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} + B \bar{B}_0')\gamma(M1)$ | $(\frac{1\sqrt{3}y}{3(1+\sqrt{3}y)})$ |
| $^{1^+}$ | $\frac{1}{\sqrt{2}}(B_1 \bar{B}^* + B' \bar{B}_1)\gamma(E1)$ | $\chi_{60}(n^3 P_1) : \chi_{62}(n^3 P_2)$ |
|         | $\frac{1}{\sqrt{2}}(B'_1 \bar{B} + B \bar{B}_1')\gamma(E1)$ | 1 : 3 |
| $^{1^-}$ | $\frac{1}{\sqrt{2}}(B_0 \bar{B}^* - B' \bar{B}_0)\gamma(E2)$ | $\gamma(n^2 D_1) : \gamma(n^2 D_2)$ |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} - B \bar{B}_0')\gamma(E2)$ | 0 : 0 |
|         | $B' \bar{B}^* \gamma(E2)$ | 0 : 0 |
| $^{1^+}$ | $\frac{1}{\sqrt{2}}(B_0 \bar{B} + B' \bar{B}_0)\gamma(E2)$ | $\gamma(n^2 D_1) : \gamma(n^2 D_2)$ |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} + B \bar{B}_0')\gamma(E2)$ | 27(1 + $\beta^2$) : 5($3 + \beta^2$) : 7($3 + 2\beta^2$) |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B}^* - B' \bar{B}_1)\gamma(E2)$ | 27(1 - $\beta^2$) : 5($3 - \beta^2$) : 7($3 + 2\beta^2$) |
|         | $\frac{1}{\sqrt{2}}(B'_1 \bar{B} + B \bar{B}_1')\gamma(E2)$ | 27(1 - $\beta^2$) : 5($1 - 3\sqrt{3}y'$) : 4($\sqrt{3} + 3y'$) |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B}^* + B' \bar{B}_1)\gamma(E2)$ | $3 : 5 : 7$ |
|         | $\frac{1}{\sqrt{2}}(B'_1 \bar{B} - B \bar{B}_1')\gamma(E2)$ | $3(1 + \sqrt{3}y'$) : 5($1 - 3\sqrt{3}y'$) : 4($\sqrt{3} - y'$) |
|         | $\frac{1}{\sqrt{2}}(B_0 \bar{B}^* - B' \bar{B}_0)\gamma(E2)$ | $27(5 - \sqrt{3}y') : 5(\sqrt{3} + 3\sqrt{3}y') : 4(5\sqrt{3} + 3\sqrt{3}y')$ |
| $^{2^-}$ | $\frac{1}{\sqrt{2}}(B'_1 \bar{B} + B \bar{B}_1')\gamma(E1)$ | $\chi_{60}(n^3 P_1) : \chi_{62}(n^3 P_2)$ |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} - B \bar{B}_0')\gamma(E1)$ | 1 : 3 |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B}^* - B' \bar{B}_1)\gamma(E1)$ | 1 : 3 |
|         | $\frac{1}{\sqrt{2}}(B'_1 \bar{B} - B \bar{B}_1')\gamma(E1)$ | 1 : 3 |
| $^{2^+}$ | $B' \bar{B}^* \gamma(E1)$ | $\gamma(n^2 D_1) : \gamma(n^2 D_2)$ |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B}^* + B \bar{B}_0')\gamma(M1)$ | $\gamma(n^2 D_1) : \gamma(n^2 D_2)$ |
|         | $\frac{1}{\sqrt{2}}(B'_1 \bar{B}^* - B' \bar{B}_0)\gamma(M1)$ | 1 : 15 : 84 |
|         | $\frac{1}{\sqrt{2}}(B_0 \bar{B} + B' \bar{B}_0)\gamma(M1)$ | 1 : 15 : 84 |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} + B \bar{B}_0')\gamma(M1)$ | $1 + 9\sqrt{3}y)$ : $15(\sqrt{3} + 3\sqrt{3}y') : 8(\sqrt{3} + 3\sqrt{3}y')$ |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B}^* + B' \bar{B}_1)\gamma(M1)$ | $(1 - 3\sqrt{3}y') : 5(\sqrt{3} + 3\sqrt{3}y') : 28(\sqrt{3} + y')$ |
|         | $\frac{1}{\sqrt{2}}(B'_1 \bar{B} + B \bar{B}_1')\gamma(M1)$ | $27(1 + \sqrt{3}y') : 5(\sqrt{3} + 3\sqrt{3}y') : 28(\sqrt{3} - 3\sqrt{3}y')$ |
| $^{2^-}$ | $\frac{1}{\sqrt{2}}(B_0 \bar{B}^* - B' \bar{B}_0)\gamma(E2)$ | $9 : 35 : 56$ |
|         | $\frac{1}{\sqrt{2}}(B'_0 \bar{B} + B \bar{B}_0')\gamma(E2)$ | $9(1 + 3\sqrt{3}y') : 5(\sqrt{3} + 3\sqrt{3}y') : 8(\sqrt{3} - 3\sqrt{3}y')$ |
|         | $\frac{1}{\sqrt{2}}(B_1 \bar{B} + B' \bar{B}_1)\gamma(E2)$ | $9(1 - \sqrt{3}y') : 5(\sqrt{3} - 3\sqrt{3}y') : 8(\sqrt{3} + 3\sqrt{3}y')$ |
|         | $\frac{1}{\sqrt{2}}(B'_1 \bar{B} - B \bar{B}_1')\gamma(E2)$ | $(3 + \sqrt{3}y') : 5(\sqrt{3} + y') : 8(\sqrt{3} - y')$ |
TABLE VII: The typical ratios of the \((b\bar{b}) \rightarrow 3\eta + \gamma\) decay widths. The parameters \(y, y', \beta\) are defined as \(y = H_{23}(M1)/H_{12}(M1), y' = H_{23}(E2)/H_{12}(E2), \) and \(\beta = H_{15}(E2)/H_{00}(E2), \) respectively.

| Final state | Initial state | \(\eta_0(n^1P_0)\) | \(\eta_0(n^3P_0)\) | \(\chi_{II}(n^1P_0)\) | \(\chi_{II}(n^3P_0)\) | \(\chi_{II}(n^3P_1)\) | \(\chi_{II}(n^3P_2)\) |
|-------------|---------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:2 | 1:2 | 4:0 | 4:0 | 4:0 |
| 1\(^+\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:5 | 1:5 | 9:5 | 9:5 | 9:5 |
| 1\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:2 | 1:2 | 4:0 | 4:0 | 4:0 |
| 2\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 2:1 | 2:1 | 1:1 | 1:1 | 1:1 |
| 2\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 2:5 | 2:5 | 2:5 | 2:5 | 2:5 |
| 0\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 0:0 | 0:0 | 3:1 | 3:1 | 3:1 |
| 1\(^+\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:2 | – | 16:0 | 16:0 | 16:0 |
| 1\(^+\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:5 | – | 9:5 | 9:5 | 9:5 |
| 1\(^+\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:2 | – | 16:0 | 16:0 | 16:0 |
| 2\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 2:1 | – | 2:9 | 2:9 | 2:9 |
| 2\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:1 | – | 1:1 | 1:1 | 1:1 |
| 2\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 2:5 | – | 2:5 | 2:5 | 2:5 |
| 0\(^+\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:3 | – | 3:1 | 3:1 | 3:1 |
| 0\(^+\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 0:0 | – | 3:1 | 3:1 | 3:1 |
| 1\(^+\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 1:1 | – | 3:1 | 3:1 | 3:1 |
| 1\(^+\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | 2:3 | – | 3:1 | 3:1 | 3:1 |
| 2\(^-\)     | \(\frac{1}{\gamma} \frac{B_{B\bar{B}}^{y}(y)}{(B_{B\bar{B}}^{y})}\) | – | 3:2 | – | – | – |
TABLE VIII: The typical ratios between the $(b\bar{b})\rightarrow \gamma \gamma$ decay widths. The parameters $\gamma', \beta'$ are defined as $\gamma = H_{22}(M1)/H_{12}(M1)$, $\gamma' = H_{22}(E2)/H_{12}(E2)$, and $\beta = H_{12}(E2)/H_{10}(E2)$, respectively.

| Initial state | $\Gamma(n^1D_1)$ | $\Gamma(n^1D_2)$ | $\Gamma(n^3D_1)$ |
|---------------|-----------------|-----------------|-----------------|
| $1^{-+}$      | $1:2:1:2$       | $1:2$           | $1:2$           |
| $2^{-+}$      | $1:2:1:2$       | $1:2$           | $1:2$           |
| $1^{-+}$      | $1:2:1:2$       | $1:2$           | $1:2$           |
| $2^{-+}$      | $1:2:1:2$       | $1:2$           | $1:2$           |

The ratios satisfy the crossing symmetry, which acts as an important test of our calculation. We collect the obtained typical ratios of the $\gamma \gamma \rightarrow \gamma \gamma'$ + $\gamma$ decay widths in Tables IX and X.

IV. THE RADIATIVE DECAYS OF THE HIDDEN-CHARM MOLECULES

In the previous sections, we focus on the radiative decays of the hidden beauty molecular states. However, we can easily extend our approach to study the radiative decays of the hidden charm molecular states. Readers should keep in mind that the heavy quark symmetry is expected to work well for the hidden beauty molecular system since the bottom quark mass is much larger than $\Lambda_{QCD}$. For the hidden charm molecular states, the $1/m_Q$ correction may be non-negligible in certain cases since the charm quark mass is not so heavy compared to the bottom mass. Especially the $1/m_Q$ correction plays the dominant role in those radiative channels which are forbidden in the heavy quark limit. For the other allowed channels, one may naively expect that the decay pattern and ratios of the radiative decays of the hidden charm molecular states are roughly the same as those in the heavy quark limit.

As mentioned in Sec. I some observed charmonium-like states were considered as the candidates of the hidden-charm molecular states. In the following, we discuss the radiative decays of several charmonium-like states under the molecular assignment.

A. $Y(4260)\text{ and } Y(4360)$

The $J^{PC} = 1^{-+}$ state $Y(4260)$ was reported by the BaBar Collaboration in the $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ mode [57]. Assuming $Y(4260)$ is the $(D_1\bar{D}_1 - \bar{D}_1D_1)$ molecular state [2, 38, 52], we can write down the rearranged spin structure of $Y(4260)$,

$$|Y(4260)⟩ = \frac{\sqrt{6}}{6}(|0^-_H \otimes 1^+_l⟩|j_1 = 1⟩ - \frac{\sqrt{3}}{6}(|0^-_H \otimes 1^+_l⟩|j_1 = 1⟩$$

$$+ \frac{\sqrt{3}}{6}(|1^-_H \otimes 1^+_l⟩|j_1 = 1⟩ - \frac{\sqrt{6}}{12}(|1^-_H \otimes 1^+_l⟩|j_1 = 1⟩$$

After performing the spin rearrangement, the allowed final states can be expressed as

$$|\chi_{c0}\gamma(E1)⟩ = \frac{1}{3}(|1^-_H \otimes 0^+_l⟩|j_{z1} = 1⟩ - \frac{\sqrt{3}}{3}(|1^-_H \otimes 1^+_l⟩|j_{z1} = 1⟩$$

$$+ \frac{\sqrt{3}}{3}(|1^-_H \otimes 2^+_l⟩|j_{z1} = 1⟩)[(c\bar{c})]|\gamma⟩,$$

$$|\chi_{c1}\gamma(E1)⟩ = \left[ - \frac{\sqrt{3}}{3}(|1^-_H \otimes 0^+_l⟩|j_{z1} = 1⟩ + \frac{1}{2}(|1^-_H \otimes 1^+_l⟩|j_{z1} = 1⟩$$

$$+ \sqrt{3}(|1^-_H \otimes 2^+_l⟩|j_{z1} = 1⟩)[(c\bar{c})]|\gamma⟩,$$

$$|\chi_{c2}\gamma(E1)⟩ = \left[ \frac{\sqrt{3}}{3}(|1^-_H \otimes 0^+_l⟩|j_{z1} = 1⟩ - \frac{\sqrt{3}}{6}(|1^-_H \otimes 1^+_l⟩|j_{z1} = 1⟩$$

$$+ \frac{1}{6}(|1^-_H \otimes 2^+_l⟩|j_{z1} = 1⟩)[(c\bar{c})]|\gamma⟩.$$
TABLE IX: The ratios of the $\gamma(M1)$ and $\gamma(E1)$ transitions between two molecular states.

| Initial state | Final state 1$^-$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ |
|---------------|-------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\gamma(M1)$  | $\gamma(E1)$      | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ |
| $\gamma(M1)$  | $\gamma(E1)$      | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ | $\gamma(M1)$ | $\gamma(E1)$ |
TABLE X: The ratios of the $\gamma(M1)$ and $\gamma(E1)$ transitions between two molecular states.

| Initial state | Final state 1<sup>−</sup> | Final state 1<sup>+</sup> | Final state 1<sup>−</sup> | Final state 1<sup>+</sup> | Final state 1<sup>−</sup> | Final state 1<sup>+</sup> | Final state 1<sup>−</sup> | Final state 1<sup>+</sup> | Final state 1<sup>−</sup> | Final state 1<sup>+</sup> |
|---------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ |
| $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ |
| $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ | $\frac{\gamma(B\bar{B}^* + B\bar{B}_1)}{\gamma(B\bar{B}^* + B\bar{B}_2)}$ |
In the heavy quark limit, we get the ratio of the $E1$ transitions $Y(4260) \rightarrow \chi_{cJ}\gamma (E1) \ (J = 0, 1, 2)$,

$$\Gamma(\chi_{c0}\gamma(E1)) : \Gamma(\chi_{c1}\gamma(E1)) : \Gamma(\chi_{c2}\gamma(E1)) = 4 : 3 : 5,$$

where we ignore the phase space factor. If considering the difference of the phase space of the three decay channels, we have

$$\Gamma(\chi_{c0}\gamma(E1)) : \Gamma(\chi_{c1}\gamma(E1)) : \Gamma(\chi_{c2}\gamma(E1)) = 1.8 : 1 : 1.4.$$

The above ratios can be used to test the molecular assignment of $Y(4260)$ if the BES collaboration can measure these decay widths. $Z(4248)$ may be the isovector molecular partner of $Y(4260)$ \cite{2}. The radiative decay pattern of its neutral component is the same as that of $Y(4260)$.

2. $Y(4360) \rightarrow \chi_{cJ}\gamma$

$Y(4360)$ was observed by the Belle collaboration in the $\psi(2S)\pi^+\pi^- \pi^0$ invariant mass spectrum in the $e^+e^- \rightarrow \psi(2S)\pi^+\pi^- \pi^0$ process \cite{23}. Since the mass of $Y(4360)$ is close to the threshold of $D_1D^*_1$, $Y(4360)$ may be a $\frac{1}{\sqrt{2}}(D_1D^*_1 + D^*D_1)$ molecular state \cite{2}. If so, its spin structure can be written as

$$|Y(4360)\rangle = \left[\frac{\sqrt{3}}{6}(0_H^- \otimes 1^+_7)|j_{z=1}^- + \frac{\sqrt{6}}{12}(0_H^- \otimes 1^+_1)|j_{z=1}^- + \frac{\sqrt{6}}{4}(1_H^- \otimes 1^+_7)|j_{z=1}^- \right] |\bar{c}q; \bar{c}q\rangle$$

$$+ \left[\frac{1}{2}(0_H^+ \otimes 1^-_7)|j_{z=1}^- - \frac{\sqrt{6}}{12}(0_H^+ \otimes 1^-_1)|j_{z=1}^- + \frac{\sqrt{6}}{4}(1_H^- \otimes 1^-_7)|j_{z=1}^- \right] |\bar{c}q; \bar{c}q\rangle.$$

$Y(4360)$ can decay into $\chi_{cJ}$ with the following ratio

$$\Gamma(\chi_{c0}\gamma(E1)) : \Gamma(\chi_{c1}\gamma(E1)) : \Gamma(\chi_{c2}\gamma(E1)) = 4 : 3 : 5 (1.8 : 1 : 1.4),$$

where the value listed in the bracket is the result considering the phase space correction.

$Y(4660)$ may be the hidden-strangeness partner of $Y(4360)$ which is composed of $D_{1s}D^*_s$ \cite{6}. Its radiative decay pattern is the same as that of $Y(4360)$.

3. Ratios between the radiative decay widths of $Y(4260)$ and $Y(4360)$

$Y(4260)$ and $Y(4360)$ can also decay into $\eta_s$ and $\eta_s(1D_2)$ via the $M1$ transition. The spin-rearranged final state are

$$|\eta_s(1D_1)\gamma(M1)\rangle = (0_H^+ \otimes 1^-_1)|j_{z=1}^-|((c\bar{c}))|\gamma\rangle,$$

$$|\eta_s(1D_2)\gamma(M1)\rangle = (0_H^+ \otimes 1^-_1)|j_{z=1}^-|((c\bar{c}))|\gamma\rangle.$$

Under the above molecular assumption, $Y(4260)$ and $Y(4360)$ have the same spatial wave functions. Therefore, we obtain the following ratios

$$\frac{\Gamma(Y(4260) \rightarrow \eta_s\gamma(M1))}{\Gamma(Y(4360) \rightarrow \eta_s\gamma(M1))} = 2 : 1,$$

$$\frac{\Gamma(Y(4260) \rightarrow \eta_s(1D_2)\gamma)}{\Gamma(Y(4360) \rightarrow \eta_s(1D_2)\gamma)} = 2 : 1,$$

$$\frac{\Gamma(Y(4260) \rightarrow \chi_{c0}\gamma(E1))}{\Gamma(Y(4360) \rightarrow \chi_{c0}\gamma(E1))} = 2 : 9,$$

where the $J = 0, 1, 2$. Considering the phase factors, we have

$$\frac{\Gamma(Y(4260) \rightarrow \eta_s\gamma(M1))}{\Gamma(Y(4360) \rightarrow \eta_s\gamma(M1))} = 1.7 : 1,$$

$$\frac{\Gamma(Y(4260) \rightarrow \chi_{c0}\gamma(E1))}{\Gamma(Y(4360) \rightarrow \chi_{c0}\gamma(E1))} = 1 : 5.8,$$

$$\frac{\Gamma(Y(4260) \rightarrow \chi_{c0}\gamma(E1))}{\Gamma(Y(4360) \rightarrow \chi_{c0}\gamma(E1))} = 1 : 6.1,$$

$$\frac{\Gamma(Y(4260) \rightarrow \chi_{c0}\gamma(E1))}{\Gamma(Y(4360) \rightarrow \chi_{c0}\gamma(E1))} = 1 : 6.2.$$

B. $X(3872)$

There are extensive discussions of the possibility of $X(3872)$ as a $DD^*$ molecular state with $J^{PC} = 1^{++}$ \cite{13, 22}. Under this molecular state assignment, the spin structure of $X(3872)$ reads

$$|X(3872)\rangle = \left[\frac{1}{2}(0_H^+ \otimes 1^-_7)|j_{z=1}^- - \frac{1}{2}(1_H^- \otimes 0^+_1)|j_{z=1}^- + \frac{1}{\sqrt{2}}(1_H^- \otimes 1^-_7)|j_{z=1}^- \right] |\bar{c}q; \bar{c}q\rangle$$

$$+ \left[\frac{1}{2}(0_H^- \otimes 1^-_7)|j_{z=1}^- + \frac{1}{2}(1_H^- \otimes 0^+_1)|j_{z=1}^- + \frac{1}{\sqrt{2}}(1_H^- \otimes 1^-_7)|j_{z=1}^- \right] |\bar{c}\bar{q}; c\bar{q}\rangle.$$

If we ignore the heavy quark symmetry, the radiative decay modes of $X(3872)$ are $\psi(1^3D_1)\gamma(E1), \phi(1^3D_2)\gamma(E1),$
$J/\psi\gamma(E1), h_c\gamma(M1)$ and $h_c\gamma(E2)$, whose spin structures are

$$\begin{align*}
|\psi(1^3D_1)\gamma(E1)\rangle &= \left[ -\frac{1}{2}(1^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle
+\frac{\sqrt{3}}{2}(1^+_H \otimes 2^-_i)|j_{z=1}^+=1\rangle \right]|(c\bar{c})\rangle|\gamma\rangle,

|\psi(1^3D_2)\gamma(E1)\rangle &= \left[ \frac{\sqrt{3}}{2}(1^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle
+\frac{1}{2}(1^+_H \otimes 2^-_i)|j_{z=1}^+=1\rangle \right]|(c\bar{c})\rangle|\gamma\rangle.
\end{align*}$$

However, the decay modes $h_c\gamma(M1)$ and $h_c\gamma(E2)$ are suppressed while the other three ones are allowed in the heavy quark limit. The decay widths of $X(3872) \rightarrow J/\psi\gamma(E1)$ and $X(3872) \rightarrow \psi\gamma(E1)$ depend on the reduced matrix element $|[1,0]|H_{eff}(E1)|1\rangle|^2$. For the decay modes $\psi(1^3D_1)\gamma(E1)$ and $\psi(1^3D_2)\gamma(E1)$, we have the ratio

$$\Gamma(\psi(1^3D_1)\gamma(E1)) : \Gamma(\psi(1^3D_2)\gamma(E1)) = 1 : 3 (1 : 2.9) \quad (22)$$

without and with the phase space correction, respectively.

C. $Y(4260) \rightarrow X(3872)\gamma$

There may also exist the $E1$ transition between the two molecular candidates $Y(4260)$ and $X(3872)$. The spin-recoupled final state can be expressed as

$$\begin{align*}
|X(3872)\gamma(E1)\rangle &= \left[ \frac{1}{2}(0^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle
-\frac{1}{2}(1^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle \right] \frac{1}{2}(0^+_H \otimes 0^-_i)|j_{z=1}^+=1\rangle
-\frac{\sqrt{3}}{6}(1^-_H \otimes 0^+_i)|j_{z=1}^+=1\rangle
+\frac{\sqrt{2}}{4}(1^-_H \otimes 1^+_i)|j_{z=1}^+=1\rangle
\right] |(c\bar{c})\rangle|\gamma\rangle

+\left[ -\frac{1}{2}(0^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle
+\frac{1}{2}(1^-_H \otimes 1^-_i)|j_{z=1}^+=1\rangle \right] \frac{1}{2}(0^+_H \otimes 0^-_i)|j_{z=1}^+=1\rangle
-\frac{\sqrt{3}}{6}(1^-_H \otimes 0^+_i)|j_{z=1}^+=1\rangle
+\frac{\sqrt{2}}{4}(1^-_H \otimes 1^+_i)|j_{z=1}^+=1\rangle
\right] |(c\bar{c})\rangle|\gamma\rangle.
\end{align*}$$

Clearly the decay width depends on the reduced matrix element $|[1,1]|H_{eff}(E1)|1\rangle$ and does not vanish in the heavy quark limit.

D. $Z_c(3900)$ and $Z_c(4020)$

$Z_c(3900)$ was first observed in the channel $e^+e^- \rightarrow J/\psi\pi^+\pi^-$ at $\sqrt{s} = 4.26$ GeV by BESIII [44]. $Z_c(3900)$ or $Z_c(3885)$ may be the candidate of the charged $D\bar{D}^*$ molecular state with $J^P = 1^+ (\ast)$, $Z_c(4020)$ was reported in the $h_c\pi^\pm$ invariant mass spectrum of $e^+e^- \rightarrow h_c\pi^\pm\pi^\mp$ at $\sqrt{s} = 4.26$ GeV [59]. The similar state $Z_c(4025)$ was observed in the $e^+e^- \rightarrow (D^*\bar{D}^\ast)^{\pm}\pi^\mp$ at $\sqrt{s} = 4.26$ GeV [60]. $Z_c(4020)$ or $Z_c(4025)$ may be the $D^*\bar{D}^*$ molecular state. The quantum number of the neutral partner of $Z_c(3900)$ and $Z_c(4020)$ is $I^GJ^P = 1^+1^-$ [53], whose spin structures are

$$\begin{align*}
|Z_c(3900)\rangle &= \left[ \frac{1}{2}(0^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle
-\frac{1}{2}(1^-_H \otimes 0^-_i)|j_{z=1}^+=1\rangle \right]|(c\bar{c})\rangle|\gamma\rangle

-\left[ -\frac{1}{2}(0^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle
+\frac{1}{2}(1^-_H \otimes 0^-_i)|j_{z=1}^+=1\rangle \right] |(c\bar{c})\rangle|\gamma\rangle

+\frac{1}{\sqrt{2}}(1^-_H \otimes 1^-_i)|j_{z=1}^+=1\rangle |(c\bar{c})\rangle|\gamma\rangle.
\end{align*}$$

The $M1$ transitions of $Z_c(3900)$ and $Z_c(4020)$ into $\chi_{cJ}$ result in the simple ratio $\Gamma(\chi_{c0}\gamma(E1)) : \Gamma(\chi_{c1}\gamma(E1)) : \Gamma(\chi_{c2}\gamma(E1)) = 1 : 3 : 5$ if we ignore the phase space correction [53, 55].

The $E2$ transition modes $\chi_{c1}\gamma(E2), \chi_{c2}\gamma(E2)$ of $Z_c(3900)$ and $Z_c(4020)$ are suppressed in the heavy quark limit, which is manifest from their spin structures

$$\begin{align*}
|\chi_{c1}\gamma(E2)\rangle &= \left[ -\frac{1}{2}(1^-_H \otimes 1^-_i)|j_{z=1}^+=1\rangle
+\frac{\sqrt{3}}{2}(1^-_H \otimes 2^-_i)|j_{z=1}^+=1\rangle \right]|(c\bar{c})\rangle|\gamma\rangle,

|\chi_{c2}\gamma(E2)\rangle &= \left[ \frac{\sqrt{3}}{2}(1^-_H \otimes 1^-_i)|j_{z=1}^+=1\rangle
+\frac{1}{2}(1^-_H \otimes 2^-_i)|j_{z=1}^+=1\rangle \right]|(c\bar{c})\rangle|\gamma\rangle.
\end{align*}$$

Their $E1$ modes $\eta_c\gamma(E1)$ and $\eta_c\gamma(1^1D_2)\gamma(E1)$ are allowed due to their spin structure

$$\begin{align*}
|\eta_c\gamma(E1)\rangle &= (0^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle |(c\bar{c})\rangle|\gamma\rangle,

|\eta_c\gamma(1^1D_2)\gamma(E1)\rangle &= (0^+_H \otimes 1^-_i)|j_{z=1}^+=1\rangle |(c\bar{c})\rangle|\gamma\rangle,
\end{align*}$$

which leads to the ratio $\Gamma_{Z_c(3900)}(\eta_c\gamma(E1)) : \Gamma_{Z_c(4200)}(\eta_c\gamma(E1)) = 1 : 1 (1 : 1.30)$.

E. $Y(4274)$

$Y(4274)$ was reported by CDF in the $J/\psi\phi$ invariant mass spectrum [61]. It was proposed to be a $S$-wave $D\bar{D}_{s0}(2317)$
molecular state with $J^{PC}=0^{-+}$ \cite{32}. Its spin structure is

$$|Y(4274)\rangle = \left[ -\frac{1}{2}(0_H^+ \otimes 0_i^-)|j_{J=0}^- + \frac{1}{2}(1_H^- \otimes 1_i^+)|j_{J=0}^+ + \frac{\sqrt{3}}{2}(1_H^- \otimes 1_i^-)|j_{J=0}^- \right] |\bar{c}q; \bar{c}\bar{q}\rangle$$

With the heavy quark spin symmetry, the $h_c\gamma(E1)$ is suppressed while $J/\psi\gamma(M1)$, $\psi(1^3D_1)\gamma(M1)$ and $\psi(1^3D_2)\gamma(E2)$ are allowed, whose spin structures are

$$|J/\psi\gamma(M1)\rangle = (1_H^- \otimes 1_i^-)|j_{J=0}^- (|\bar{c}c\rangle)|\gamma\rangle,$$
$$|h_c\gamma(E1)\rangle = (0_H^+ \otimes 0_i^+)|j_{J=0}^+ (|\bar{c}c\rangle)|\gamma\rangle,$$
$$|\psi(1^3D_1)\gamma(M1)\rangle = (1_H^- \otimes 1_i^-)|j_{J=0}^- (|\bar{c}c\rangle)|\gamma\rangle,$$
$$|\psi(1^3D_2)\gamma(E2)\rangle = (1_H^- \otimes 1_i^-)|j_{J=0}^- (|\bar{c}c\rangle)|\gamma\rangle.$$

The decay widths depend on the reduced matrix elements $|\langle 1,0|H_{eff}(M1)|1\rangle|$, $|\langle 1,2|H_{eff}(M1)|1\rangle|$, $|\langle 2,2|H_{eff}(E2)|1\rangle|$ respectively.

F. $Z(4430)$

$Z(4430)$ was suggested as a $D_1^*\bar{D}^*$ molecular state with $J^P = 0^-$ or a $D'_1\bar{D}^*$ with $J^P = 0^+, 1^-$. In the case that the neutral partner of $Z(4430)$ is a $D'_1\bar{D}^*$ molecular state with $2^+$, its spin structure is

$$|Z(4430)\rangle = \left[ \frac{\sqrt{3}}{3}(1_H^- \otimes 1_i^+)|j_{J=2}^- + \frac{\sqrt{6}}{3}(1_H^- \otimes 1_i^-)|j_{J=2}^- \right] |\bar{c}q; \bar{c}\bar{q}\rangle$$

In the heavy quark spin symmetry limit, the allowed decay modes are

$$|J/\psi\gamma(M1)\rangle = (1_H^- \otimes 1_i^-)|j_{J=2}^- (|\bar{c}c\rangle)|\gamma\rangle,$$
$$|\psi(1^3D_1)\gamma(M1)\rangle = \left[ \frac{1}{10}(1_H^- \otimes 1_i^-)|j_{J=2}^- - \frac{\sqrt{15}}{10}(1_H^- \otimes 2_i^-)|j_{J=2}^+ \right] |(\bar{c}c)\rangle|\gamma\rangle,$$
$$|\psi(1^3D_2)\gamma(M1)\rangle = \left[ -\frac{\sqrt{15}}{10}(1_H^- \otimes 1_i^-)|j_{J=2}^- + \frac{5}{6}(1_H^- \otimes 2_i^-)|j_{J=2}^+ \right] |(\bar{c}c)\rangle|\gamma\rangle.$$

We obtain the following $M1$ and $E2$ transition ratios

$$\Gamma(\psi(1^3D_1)\gamma(M1)) : \Gamma(\psi(1^3D_2)\gamma(M1)) : \Gamma(\psi(1^3D_3)\gamma(M1)) = 1 : 15 : 84,$$
$$\Gamma(\psi(1^3D_1)\gamma(E2)) : \Gamma(\psi(1^3D_2)\gamma(E2)) : \Gamma(\psi(1^3D_3)\gamma(E2)) = 9 : 35 : 56.$$

Y(3940) and Y(4140)

Y(3940) was announced by BaBar collaboration \cite{62} in $B \rightarrow K\phi\psi$ while Y(4140) was observed by CDF collaboration \cite{63}. Due to their similarity, Y(3940) and Y(4140) were proposed as the candidates of the $D^*\bar{D}^*$ and $D'_1\bar{D}^*$ molecular states \cite{23,24}, respectively, where their quantum numbers are either $J^{PC} = 0^{++}$ or $2^{++}$ \cite{23,24}.
If the quantum numbers of $Y(3940)$ and $Y(4140)$ are $0^+$, their spin structures are

$$|\psi(3940)\rangle = \left[\frac{\sqrt{3}}{2}\langle 0^+_H \otimes 0^+_I | j^0_{J=0} + \frac{1}{2}\langle 1^-_H \otimes 1^-_I | j^0_{J=0}\right]_{|c\bar{q}; \bar{c}q\rangle},$$

$$|\psi(4140)\rangle = \left[\frac{\sqrt{3}}{2}\langle 0^+_H \otimes 0^+_I | j^0_{J=0} + \frac{1}{2}\langle 1^-_H \otimes 1^-_I | j^0_{J=0}\right]_{|c\bar{s}; \bar{c}s\rangle},$$

in the heavy quark limit, the allowed decay modes are

$$|J/\psi\gamma(1^+)\rangle = (1^-_H \otimes 1^-_I)_{j^0_{J=0}}(\bar{c}\bar{c})\gamma),$$

$$|h_1\gamma(M1)\rangle = (0^-_H \otimes 0^-_I)_{j^0_{J=0}}(\bar{c}\bar{c})\gamma),$$

$$|\psi(1^3D_2)\gamma(E1)\rangle = (1^-_H \otimes 1^-_I)_{j^0_{J=0}}(\bar{c}\bar{c})\gamma).$$

We have

$$\Gamma_{Y(3940)}(J/\psi\gamma(1^+)) : \Gamma_{Y(4140)}(J/\psi\gamma(1^+)) = 1 : 1 (1 \pm 1.6),$$

$$\Gamma_{Y(3940)}(h_1\gamma(M1)) : \Gamma_{Y(4140)}(h_1\gamma(M1)) = 1 : 1 (1 \pm 2.8),$$

$$\Gamma_{Y(3940)}(\psi(1^3D_2)\gamma(E1)) : \Gamma_{Y(4140)}(\psi(1^3D_2)\gamma(E1)) = 1 : 1.$$

2. $J^P = 2^+$

If $Y(3940)$ and $Y(4140)$ are the molecular candidates with $J^P = 2^+$, their spin structures are

$$|\psi(3940)\rangle = (1^+_H \otimes 1^-_I)_{j^0_{J=2}}(\bar{c}\bar{q}; \bar{c}q\rangle),$$

$$|\psi(4140)\rangle = (1^-_H \otimes 1^-_I)_{j^0_{J=2}}(\bar{c}\bar{s}; \bar{c}s\rangle),$$

respectively. In the heavy quark symmetry limit, the allowed decay modes are $J/\psi\gamma(1^+), \psi(1^3D_2)\gamma(E1), \psi(1^3D_2)\gamma(E1)$ and $\psi(1^3D_2)\gamma(E1)$. Their spin structures are

$$|J/\psi\gamma(1^+)\rangle = (1^-_H \otimes 1^-_I)_{j^0_{J=2}}(\bar{c}\bar{c})\gamma),$$

$$|\psi(1^3D_1)\gamma(E1)\rangle = \frac{1}{10}(1^-_H \otimes 1^-_I)_{j^0_{J=2}} + \frac{\sqrt{15}}{10}(1^-_H \otimes 1^-_I)_{j^0_{J=2}} + \frac{\sqrt{15}}{5}(1^-_H \otimes 1^-_I)_{j^0_{J=2}}(\bar{c}\bar{c})\gamma),$$

$$|\psi(1^3D_2)\gamma(E1)\rangle = \frac{1}{10}(1^-_H \otimes 1^-_I)_{j^0_{J=2}} + \frac{5}{6}(1^-_H \otimes 1^-_I)_{j^0_{J=2}} + \frac{35}{15}(1^-_H \otimes 1^-_I)_{j^0_{J=2}}(\bar{c}\bar{c})\gamma),$$

$$|\psi(1^3D_3)\gamma(E1)\rangle = \frac{1}{10}(1^-_H \otimes 1^-_I)_{j^0_{J=2}} - \frac{\sqrt{15}}{10}(1^-_H \otimes 1^-_I)_{j^0_{J=2}} + \frac{1}{15}(1^-_H \otimes 1^-_I)_{j^0_{J=2}}(\bar{c}\bar{c})\gamma).$$

We obtain the following $E1$ transition ratio of $Y(3940)$ and $Y(4140)$

$$\Gamma(\psi(1^3D_1)\gamma(E1)) : \Gamma(\psi(1^3D_2)\gamma(E1)) : \Gamma(\psi(1^3D_3)\gamma(E1)) = 1 : 15 : 84.$$

V. SUMMARY

In the past decade many charmonium-like and bottomonium-like states were reported experimentally. These states are sometimes called the XYZ states. Many XYZ states are very close to the open-charm or open-bottom threshold. Some of them are even charged. Right now, it is difficult to accommodate these XYZ states within the simple quark model spectrum. Especially those charged states can not be a charmonium or bottomonium.

Many theoretical speculations were proposed to explain the inner structures of these XYZ states. Among them, the hadronic molecular picture becomes quite popular due to the closeness of these XYZ states to the open-charm or open-bottom threshold. The molecule scheme seems quite natural since we all know that the deuteron is a very loosely bound molecular state composed of a proton and neutron. It is very intriguing to explore whether the loosely bound di-meson molecular states exist or not. The dynamical calculation of the di-meson system within the framework of the meson exchange model may partly answer whether there exists attraction between the two mesons and whether the interaction is strong enough. The decay and production processes will provide additional information of these XYZ states.

In the heavy quark limit, the interaction of the heavy quark with both the gluon and photon does not flip the heavy quark spin. The conservation of the heavy quark spin greatly simplifies the analysis of the decay and production processes of heavy flavored hadrons. In this work we employ the heavy quark symmetry and adopt the spin rearrangement approach to study the radiative decay pattern of the possible molecular states composed of a pair of heavy mesons. We use the hidden-beauty molecules to illustrate the formalism. We have extensively investigated three classes of the radiative decays: $\gamma \rightarrow (\bar{b}b) + \gamma, (\bar{b}b) \rightarrow \gamma \gamma, \gamma \gamma \rightarrow (\gamma \gamma + \gamma)$, corresponding to the electromagnetic transitions between one molecular state and bottomonium, one bottomonium and molecular state, and two molecular states respectively. We have also extended the same formalism to study the radiative decays of the molecular states with hidden charm.

If the initial or final states belong to the same spin flavor multiplet, their spatial wave functions are the same. Then we can derive some model independent ratios of the radiative decay widths between different channels in the heavy quark limit. These ratios are different under different assumptions of the underlying structures of these XYZ states. Experimental measurement of the radiative decay ratios of the XYZ states may test different theoretical scenarios and help unveil their inner structures.

There exist speculations that some of the XYZ states may arise from either the pure kinematical effect or final state inter-
action (FSI). For example, if $Z_c(3900)$ or $Z_c(4020)$ is a kinematical artifact, their neutral component will not emit a photon and decay into the charmonium. If $Z_c(3900)$ or $Z_c(4020)$ arises from the final state interaction through the triangle diagram where a charmed meson is exchanged, the spin decomposition of this FSI signal may be different from the spin decomposition of the initial two mesons or $Z_c(3900)$ or $Z_c(4020)$ as the molecular candidates. The resulting radiative pattern of this FSI signal will deviate from that of the molecular states discussed in this work.

In short summary, the radiative decay ratios of the $XYZ$ states encode important information of their underlying structures. Systematical experimental measurement of these ratios will help judge various theoretical interpretations of the $XYZ$ states. Hopefully the present extensive investigations will be useful to the understanding of the future radiative data.

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[1] J. Beringer et al. [Particle Data Group Collaboration], Phys. Rev. D 86, 010001 (2012).
[2] W. Chen, W.-Z. Deng, J. He, N. Li, X. Liu, Z.-G. Luo, Z.-F. Sun and S.-L. Zhu, arXiv:1311.3763 [hep-ph].
[3] X. Liu, arXiv:1312.7408 [hep-ph].
[4] M. B. Voloshin and L. B. Okun, JETP Lett. 23, 333 (1976) [Pisma Zh. Eksp. Teor. Fiz. 23, 369 (1976)].
[5] A. De Rujula, H. Georgi and S. L. Glashow, Phys. Rev. Lett. 38, 317 (1977).
[6] N. A. Tornqvist, Nuovo Cim. A 107, 2471 (1994) [hep-ph/9310225].
[7] M. Suzuki, Phys. Rev. D 59, 074022 (2004) [hep-ph/0309253].
[8] M. B. Voloshin, Phys. Lett. B 578, 119 (2004) [hep-ph/0309307].
[9] C.-Y. Wong, Phys. Rev. C 69, 055202 (2004) [hep-ph/0311088].
[10] D. Weinberg and N. Isgur, Phys. Rev. D 41, 2236 (1990).
[11] F. E. Close and P. R. Page, Phys. Lett. B 312, 198 (2005) [arXiv:0801.3540 [hep-ph]].
[12] E. S. Swanson, Phys. Rev. D 78, 057702 (2008) [arXiv:0709.0950 [hep-ph]].
[13] N. A. Tornqvist, Phys. Lett. B 590, 209 (2004) [hep-ph/0402227].
[14] M. Suzuki, Phys. Rev. D 72, 114013 (2005) [hep-ph/0508258].
[15] X. Liu, arXiv:0805.3653 [hep-ph].
[16] I. W. Lee, A. Faessler, T. Gutsche and V. E. Lyubovitskij, Phys. Rev. D 80, 094005 (2009) [arXiv:0909.1009 [hep-ph]].
[17] N. Li and S.-L. Zhu, Phys. Rev. D 80, 074022 (2012) [arXiv:1207.3954 [hep-ph]].
[18] X. Liu and S.-L. Zhu, Phys. Rev. D 79, 094026 (2009) [arXiv:0903.2529 [hep-ph]].
[19] X. Liu, Z. G. Luo, Y. R. Liu and S. L. Zhu, Eur. Phys. J. C 61, 411-428 (2009) [arXiv:0808.0073 [hep-ph]].
[20] N. Mahajan, Phys. Lett. B 679, 228 (2009) [arXiv:0903.3107 [hep-ph]].
[21] T. Brant, T. Gutsche and V. E. Lyubovitskij, Phys. Rev. D 80, 054019 (2009) [arXiv:0903.5424 [hep-ph]].
[22] R. M. Albuquerque, M. E. Bracco and M. Nielsen, Phys. Lett. B 678, 186 (2009) [arXiv:0903.5540 [hep-ph]].
[23] G. J. Ding, Eur. Phys. J. C 64, 297 (2009) [arXiv:0904.1782 [hep-ph]].
[24] J. -R. Zhang and M. Q. Huang, arXiv:0905.4178 [hep-ph].
[25] Z. -F. Sun, J. He, X. Liu, Z. -G. Luo and S. -L. Zhu, Phys. Rev. D 80, 014001 (2009) [arXiv:0810.1598 [hep-ph]].
[26] X. Liu and H. W. Ke, Phys. Rev. D 80, 034009 (2009) [arXiv:0907.1349 [hep-ph]].
[27] X. Liu, Z.-G. Luo and S.-L. Zhu, Phys. Rev. D 80, 034003 (2008) [arXiv:0711.0494 [hep-ph]].
[28] X. Liu, Y.-R. Liu, W.-Z. Deng and S.-L. Zhu, Phys. Rev. D 77, 094015 (2008) [arXiv:0803.1295 [hep-ph]].
[29] Y.-R. Liu and Z.-Y. Zhang, Phys. Rev. C 80, 015208 (2009) [arXiv:0810.1598 [hep-ph]].
[30] G.-J. Ding, Phys. Rev. D 79, 014001 (2009) [arXiv:0809.4818 [hep-ph]].
[31] A. Bondar, A. Garmash, A. I. Milstein, R. Mizuk and M. B. Voloshin, Phys. Rev. D 84, 054010 (2011) [arXiv:1105.4473 [hep-ph]].
[32] J.-R. Zhang, M. Zhong and M.-Q. Huang, Phys. Lett. B 704, 312 (2011) [arXiv:1105.5472 [hep-ph]].
[33] Z.-F. Sun, J. He, X. Liu, Z.-G. Luo and S.-L. Zhu, Phys. Rev. D 84, 054002 (2011) [arXiv:1106.2968 [hep-ph]].
[34] Z.-F. Sun, Z.-G. Luo, J. He, X. Liu and S.-L. Zhu, Chin. Phys. C 36, 194 (2012).
[35] F.-K. Guo, C. Hidalgo-Duque, J. Nieves and M. P. Valderrama, Phys. Rev. D 88, 094007 (2013) [arXiv:1303.6305 [hep-ph]].
[36] A. Bondar, A. Dolgov, A. Poluektov and V. V orobiev, Eur. Phys. J. C 73, 2476 (2013) [arXiv:1303.6305 [hep-ph]].
[37] C.-Y. Cui, Y.-L. Liu, W.-B. Chen and M.-Q. Huang, Chin. Phys. C 36, 194 (2012) [arXiv:1303.6305 [hep-ph]].

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[48] J.-R. Zhang, Phys. Rev. D 87, 116004 (2013) [arXiv:1304.5748 [hep-ph]].
[49] J. M. Dias, F. S. Navarra, M. Nielsen and C. M. Zanetti, Phys. Rev. D 88, no. 1, 016004 (2013) [arXiv:1304.6433 [hep-ph]].
[50] Z.-G. Wang and T. Huang, [arXiv:1310.2422 [hep-ph]].
[51] Lu Zhao, Li Ma, Shi-Lin Zhu, [arXiv:1403.4043 [hep-ph]].
[52] Q. Wang, M. Cleven, F.-K. Guo, C. Hanhart, U.-G. Meissner, X.-G. Wu and Q. Zhao, Phys. Rev. D 89, 034001 (2014) [arXiv:1309.4303 [hep-ph]]; Q. Wang, [arXiv:1403.2243 [hep-ph]].
[53] J. He, X. Liu, Z.-F. Sun and S.-L. Zhu, Eur. Phys. J. C 73, 2635 (2013) [arXiv:1308.2999 [hep-ph]].
[54] E. Braaten, [arXiv:1305.6905 [hep-ph]].
[55] S. Ohkoda, Y. Yamaguchi, S. Yasui and A. Hosaka, Phys. Rev. D 86, 117502 (2012) [arXiv:1210.3170 [hep-ph]].
[56] Y.-R. Liu, Phys. Rev. D 88, 074008 (2013) [arXiv:1304.7467 [hep-ph]].
[57] B. Aubert et al. [BaBar Collaboration], Phys. Rev. Lett. 95, 142001 (2005) [hep-ex/0506081].
[58] B. Aubert et al. [BaBar Collaboration], Phys. Rev. Lett. 98, 212001 (2007) [hep-ex/0610057].
[59] M. Ablikim et al. [BESIII Collaboration], Phys. Rev. Lett. 111, 242001 (2013) [arXiv:1309.1896 [hep-ex]].
[60] M. Ablikim et al. [BESIII Collaboration], [arXiv:1308.2760 [hep-ex]].
[61] K. Yi [CDF Collaboration], PoS ICHEP 2010, 182 (2010) [arXiv:1010.3470 [hep-ex]].
[62] K. Abe et al. [Belle Collaboration], Phys. Rev. Lett. 94, 182002 (2005) [hep-ex/0408126].
[63] T. Aaltonen et al. [CDF Collaboration], Phys. Rev. Lett. 102, 242002 (2009) [arXiv:0903.2229 [hep-ex]].