Injectivity radius of manifolds with a Lie structure at infinity

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Abstract

Using Lie groupoids, we prove that the injectivity radius of a manifold with a Lie structure at infinity is positive.

Introduction

Manifolds with a Lie structure at infinity were introduced by Ammann, Lauter and Nistor in [1], forming a class of non-compact complete Riemannian manifolds of infinite volume. In the same article, they conjectured that the injectivity radius of a (connected) manifold with Lie structure at infinity is positive. In this paper, we give a proof of this conjecture using the associated groupoid given by [5] and [6]. Together with the results from [1], this implies that manifolds with a Lie structure at infinity are of bounded geometry. In particular, the hypothesis of injectivity radius in [2] is now automatically satisfied, as well as in [3], where positivity of the injectivity radius is used to obtain uniform parabolic Schauder estimates. Bounded geometry also yields uniform elliptic Schauder estimates, see [4] for a recent application in this direction.

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1 Preliminaries

Following [1] and [8], we recall some definitions and facts.

Definition 1.1. A groupoid is a small category $G$ in which every morphism is invertible.

The objects of the category are also called units, and the set of units is denoted by $G^0$. The set of morphisms is denoted by $G^1$. The range and domain maps are denoted respectively $r, d : G^1 \to G^0$. The multiplication operator $\mu$ is defined on the set of composable pairs of morphisms by:

$$\mu : G^2 = G^1 \times_{G^0} G^1 = \{(g, h) : d(g) = r(h)\} \to G^1$$

The inversion operation is a bijection $\iota : g \mapsto g^{-1}$ of $G^1$. The identity morphisms give an inclusion $u : x \mapsto \text{id}_x$ of $G^0$ into $G^1$.

Definition 1.2. An almost differentiable groupoid $G = (G^0, G^1, d, r, \mu, u, \iota)$ is a groupoid such that $G^0$ and $G^1$ are manifolds with corners ([7]), the structural maps $d, r, \mu, u, \iota$ are differentiable, and the domain map $d$ is a submersion.

Consequently, for an almost differentiable groupoid, $\iota$ is a diffeomorphism, $r = d \circ \iota$ is a submersion and each fiber $G_x = d^{-1}(x) \subset G^1$ is a smooth manifold whose dimension $n$ is constant on each connected component of $G^0$.

Following the convention in [5, p. 578], we require $G^0$ and $d^{-1}(x)$ to be Hausdorff (for all $x \in G^0$), but not necessarily $G^1$ to avoid excluding important cases.

From now on, Lie groupoid will stand for almost differentiable groupoid, manifold will stand for manifold with corners and smooth manifold will stand for manifold without corners.

A Lie groupoid is called $d$-simply connected if its $d$-fibers $G_x = d^{-1}(x)$ are simply connected ([5]).

Definition 1.3. A Lie algebroid $A$ over a manifold $M$ is a vector bundle $A$ over $M$, together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of $A$ and a bundle map $\rho : A \to TM$, extended to a map $\rho_T : \Gamma(A) \to \Gamma(TM)$ between sections of these bundles, such that

1. $\rho_T([X, Y]) = [\rho_T(X), \rho_T(Y)]$
2. $[X, fY] = f[X, Y] + (\rho_T(X)f)Y$

for any smooth sections $X$ and $Y$ of $A$ and any smooth function $f$ on $M$.

There is a Lie algebroid $A(G)$ associated to a Lie groupoid $G$, constructed as follows: let $T_{\text{vert}}G = \ker d_\rho = \bigcup_{x \in G} T_{\text{vert}}G_x \subset T^1_G$ be the vertical bundle over $G^1$. Then $A(G) = T_{\text{vert}}G|_{G^0}$ is the structural bundle of the Lie algebroid over $G^0$. The anchor map is given by

$$r_s|_A : A \to TG^0$$

(2). The Lie bracket of $\Gamma(A)$ is the Lie bracket of $\Gamma(T_{\text{vert}}G)$ restricted to right invariant sections.

**Definition 1.4.** A Lie algebroid $A$ over a manifold $M$ is said to be integrable if there exists a Lie groupoid $G$ such that $G^0 = M$ and $A$ is isomorphic to the Lie algebroid associated to $G$. $G$ is said to integrate $A$.

**Remark 1.5.** There might be more than one Lie groupoid integrating a Lie algebroid. However, by [5] Lie I], if a Lie algebroid over a smooth manifold is integrable, there is a unique $d$-simply connected Lie groupoid integrating it.

**Example 1.6.**

1. Any Lie group is a Lie groupoid with the set of units being a singleton.

2. ([8] Example 4, Section 4]) Let $M$ be a smooth manifold. Let $\tilde{M}$ be the universal covering of $M$. Let $H = (\tilde{M} \times \tilde{M})/\pi_1(M)$. Then $H$ is naturally a $d$-simply connected Lie groupoid with the set of units being $M$, and the associated Lie algebroid being $id : TM \to TM$. It is called the homotopy groupoid.

3. The space of continuous paths on a topological space modulo homotopy equivalence forms a groupoid which is called the fundamental groupoid.

We recall the definitions and basic properties of manifolds with Lie structures at infinity. For details and proofs, we refer to [1].

**Definition 1.7.** A structural Lie algebra of vector fields on a manifold $M$ (possibly with corners) is a subspace $\mathcal{V} \subset \Gamma(TM)$ of the real vector space of vector fields on $M$ with the following properties:

1. $\mathcal{V}$ is closed under Lie brackets;

2. $\mathcal{V}$ is a finitely generated projective $\Gamma(M)$-module;

3. The vector fields in $\mathcal{V}$ are tangent to all faces in $M$.

Denote by $\mathcal{V}_b(M) \subset \Gamma(TM)$ the subspace of vector fields tangent to all faces in $M$. This is a structural Lie algebra of vector fields, and any structural Lie algebra is a subspace of $\mathcal{V}_b(M)$ ([1] Example 2.5]).

By the Serre-Swan theorem, given a structural Lie algebra of vector fields $\mathcal{V}$ on $M$, there exists a vector bundle $A = A_{\mathcal{V}} \to M$ such that $\mathcal{V} \simeq \Gamma(A_{\mathcal{V}})$, and there exists a natural vector bundle map $\rho : A_{\mathcal{V}} \to TM$ such that the induced map $\rho_T : \Gamma(A_{\mathcal{V}}) \to \Gamma(TM)$ is identified with the inclusion map $\mathcal{V} \subset \Gamma(TM)$. The vector bundle $A_{\mathcal{V}}$ is then a Lie algebroid with anchor map $\rho$.

**Definition 1.8.** A Lie structure at infinity on a smooth manifold $M_0$ is a pair $(M, \mathcal{V})$, where

1. $M$ is a compact manifold, possibly with corners, and $M_0$ is the interior of $M$;

2. $\mathcal{V}$ is a structural Lie algebra of vector fields on $M$;

3. $\rho : A_{\mathcal{V}} \to TM$ induces an isomorphism on $M_0$, that is, $\rho|_{M_0} : A|_{M_0} \to TM_0$ is an isomorphism of vector bundles.

**Definition 1.9.** A Riemannian manifold with a Lie structure at infinity is a manifold with a Lie structure at infinity $(M, \mathcal{V})$ endowed with a bundle metric $g$ on $A = A_{\mathcal{V}}$. In particular, $g$ defines a Riemannian metric on $M_0$ via the anchor map.

A Riemannian manifold with a Lie structure at infinity has infinite volume ([1] Proposition 4.1]), bounded curvature ([1] Corollary 4.3]) and is complete ([1] Corollary 4.9]). Sufficient conditions for the positivity of the injectivity radius are given in [1] Theorem 4.14] and [1] Theorem 4.17].
2. Injectivity radius of a manifold with Lie structure at infinity

The following theorem is due to Deborg (66 Theorem 2, see also [5 Corollary 5.9]).

**Theorem 2.1** (Deborg). Every almost injective Lie algebroid over a smooth manifold is integrable.

This has the following implication for Lie structures at infinity.

**Theorem 2.2.** Any Lie algebroid over a manifold with corners associated with a Lie structure at infinity is integrable.

**Proof.** This extension of Theorem 2.1 to manifolds with corners is well-known to experts. However, since no explicit proof seems to be available in the literature, we will provide one for the convenience of the readers.

Let $(M, V)$ be a Lie structure at infinity of $M_0$ and $A = AV_0$ be the corresponding structural vector bundle. Taking two copies of $M$ and gluing them along a maximal subset of disjoint boundary hypersurfaces, we obtain a compact manifold with corners $M_1$ with at least one hypersurface less. Repeating this operation finitely many times, we obtain a closed manifold $\bar{M}$ with a finite group $\Gamma$ acting on $\bar{M}$ such that $\bar{M}/\Gamma \simeq M$ topologically.

Now, by the Serre-Swan theorem, we have a vector bundle $\tilde{\mathbb{V}} \subset C^\infty(\bar{M})$ which is pull-back of the structural vector fields. For instance, if $V_0(M)$ is the space of vector fields tangent to the faces of $M$, then $\tilde{V}_0(M)$ is the space of vector fields on $\tilde{M}$ which are tangent to $q^{-1}(\partial M)$ (the union of some closed submanifolds of $M$).

Let $\tilde{\mathbb{V}}$ be a finitely generated projective $C^\infty(\bar{M})$-module. To see this, it suffices to show that $\tilde{V}$ is locally free of rank $k$ for some $k$. Given $p \in M$, then $\tilde{V}$ is locally free of rank $k$ for some $k$, there exist $v_1, \ldots, v_k \in V$ which locally and freely span $V$ near $q(p)$. This means $\tilde{V}$ is locally and freely spanned by $q^*v_1, \ldots, q^*v_n \in \tilde{V}$ near $p$, showing that $\tilde{V}$ is locally free of rank $k$ as claimed.

By the Serre-Swan theorem, we have a vector bundle $A_{\tilde{V}}$ over $\bar{M}$ with the smooth structure on $\tilde{M}$, and the smooth structure on $\tilde{M}$ becomes a Riemannian manifold for all $\tilde{M}$.

The quotient $\tilde{G}/\Gamma$ is then the desired $d$-simply connected Lie groupoid integrating $(\tilde{M}, \tilde{V})$.

Let $M_0$ be a connected smooth manifold with a Lie structure at infinity $(M, V)$. By Theorem 2.1 there exists a $d$-simply connected groupoid $G = (M, G^d, a, r, \mu, u, i)$ with units $M$ such that $A(G) \simeq A$ as Lie algebroids over $M$. Therefore $A(G)$ is equipped with an inner product also noted $g$. The anchor map is given by $r_* : A(G) \to TM$.

We have an isomorphism $r^*A(G) \simeq T_{vert}G$ where $r^*A(G)$ is the pull-back of $A(G)$ via the range map $r : G \to M$ ([2 (19)]). Explicitly, for $p \in G$, $(r^*A(G))_p = A(G)_{r(p)} = T_{r(p)}G_{r(p)}$. The vector bundle $r^*A(G)$ is equipped with a metric induced by the metric $g$ on $A(G)$, hence so is $T_{vert}G$. Therefore each $G_x$ becomes a Riemannian manifold for all $x \in M$.

Let $G^x = \{g \in G_x : r(g) = x\}$. For $x \in M_0$, $G^x$ is a discrete group since $T_xG^x$ is of dimension 0 (being the kernel of the map $r_* : A(G)_x \to T_xM_0$).

**Lemma 2.3.** ([2 page 733]) If $A \to TM$ is the Lie algebroid associated with a Lie structure at infinity and $G$ is the corresponding $d$-simply connected Lie groupoid, then for all $x \in M_0, r : G_x \to M_0$ is a covering map with group $G^x$.

**Proof.** By [5 Proposition 1.1], for all $x \in M_0, r(G_x) \subset M_0$ (which is the leaf of the singular foliation of $A$ passing by $x$). On the other hand, $G|_{M_0}$ is the unique $d$-simply connected Lie groupoid which integrates $TM_0$, and therefore it isomorphic to the homotopy groupoid $(M_0 \times M_0)/\pi_1(M_0)$. Consequently, $M_0 = r(G_x)$ for all $x \in M_0$.

Now, by definition of a Lie structure at infinity, $r_* : T_yG_x \to T_{r(y)}M_0$ is an isomorphism. This means that $r : G_x \to M_0$ is a local diffeomorphism. Moreover, $g_1, g_2 \in G_x$ with $r(g_1) = r(g_2)$ if and only if there exists $h \in G^-1 \cdot g_2$ such that $g_2 = g_1h$. That is, $r : G_x \to M_0$ is a covering map with group $G^x$.

}\end{proof}
Theorem 2.4. Let $M_0$ be a connected smooth manifold with a Lie structure at infinity $(M, V)$. Then for any Riemannian metric $g$ on $A$, the injectivity radius of $(M_0, g)$ is positive.

Proof. We prove the theorem by contradiction. Suppose that the injectivity radius of $(M_0, g)$ is zero, then, as the curvature is bounded, there is a sequence of geodesic loops $c_i : [0, a_i] \to M_0$, parametrized by arc-length, with $a_i \to 0$. By compactness of $M$, we can suppose that $c_i(0)$ converges to a point $p \in M$.

We have $p \in \partial M$ since the injectivity radius is positive in any compact region of $M_0$.

Let $U$ be a local chart of $M$ containing $p$ such that $U$ is contractible.

Lemma 2.5. There exists a number $N > 0$ such that $\forall n > N$, the loop $c_n$ is contained in $U$.

Proof. Let $(x_1, \ldots, x_k, y_1, \ldots, y_l)$ be a set of local coordinates centered at the point $p$ with $x_i \geq 0$ for all $i$ and $p = (0, \ldots, 0)$. Let $g_b = \sum_{i=1}^k dx_i^2 + \sum_{i=1}^l dy_i^2$ be a local $b$-metric and $g_0 = \sum_{i=1}^k dx_i^2 + \sum_{i=1}^l dy_i^2$ be a local metric with boundary. Since the structural vector fields are tangential vector fields ($V \subset V_b$), taking $U$ smaller if needed, there exist constants $C, K > 0$ such that $g_0 \geq CGb \geq CKg_0$ in $U \cap M_0$. Let $l^i(c_i), l^j(c_i), l^0(c_i)$ denote the lengths of the segment $[c_i(0), c_i(t)]$ (of the geodesic loop $c_i$) with respect to the metric $g$, the local $b$-metric $g_b$ and the local metric with boundary $g_0$ respectively (suppose that the segment is contained in $U$). Let $\varepsilon > 0$ be such that $B_0(p, \varepsilon) = \{x \in \mathbb{R}^k \times \mathbb{R}^l : d_0(x, p) < \varepsilon\} \subset U$ (where $d_0$ is the distance with respect to the metric $g_0$, well-defined on $B_0(p, \varepsilon)$). Since $a_i \to 0$, there exists $N_1$ such that $a_i < \min\left(\frac{\varepsilon}{4}, CK\frac{1}{2}\right)$ for all $i > N_1$. Since $c_i(0) \to p$, there exists $N_2$ such that $d_0(p, c_i(0)) < \frac{\varepsilon}{2}$ for all $i > N_2$. Let $N = \max(N_1, N_2)$.

Now let $n$ be any number greater than $N$. Suppose that the loop $c_n$ is not contained in $U$. Then it is not contained in $B_0(p, \frac{\varepsilon}{2})$. Thus there exists $x \in [0, a_n]$ minimal such that $d_0(c_n(t), p) = \frac{\varepsilon}{2}$. Then we have $d_0(c_n(0), c_n(t)) \geq \max\{d_0(c_n(t), p) - d_0(c_n(0), p)\} \geq \frac{\varepsilon}{4}$, which implies $a_i = l(c_i) \leq l^i(c_i) \geq CKd_0(c_n(0), c_n(t)) \geq CK\frac{\varepsilon}{2}$, which is a contradiction. Therefore the loop $c_n$ is contained in $U$.

The lemma is proven. \hfill \Box

Hence, without loss of generality, we can suppose that the loops are contained in $U$.

Denote by $G = (M, G^1, d, r, \mu, u, \iota)$ the $d$-simply connected groupoid integrating $A_V \to TM$. Since $U$ is contractible, the fundamental class of each loop $c_i$ is trivial, therefore by Lemma 2.3 we can lift $c_i$ to a geodesic loop $\tilde{c}_i$ in $G_{c_i(0)}$ (i.e. $\tilde{c}_i : [0, a_i] \to r^{-1}(U) \cap G_{c_i(0)})$ such that the base points are $\tilde{c}_i(0) = \tilde{c}_i(a_i) = c_i(0)$. Let $S(T_{vert}G) = \{x \in T_{vert}G : \|x\| = 1\}$. We have a natural projection $\pi : S(T_{vert}G) \to G^1$. On $S(T_{vert}G)$ we have a flow $\Psi$ which, over each $d$-fiber $G_x$ of $d : G^1 \to G^0$, corresponds to the geodesic flow of $G_x$. The local geodesics on $G_x$ correspond to segments $[P_i, Q_i]$ of the flow $\Psi$ on $S(T_{G_x}G)$ (with $Q_i = \Psi_{a_i}(P_i)$). We have two sequences $P_i = (\tilde{c}_i(0), \tilde{c}_i(t))$ and $Q_i = (\tilde{c}_i(a_i), \tilde{c}_i(a_i))$ in $S(A) \subset S(T_{vert}G)$. By compactness of $S(A)$ and $M$, there exists a subsequence such that $P_i \to P \in S(TG_p)$ and $Q_i \to Q \in S(TG_p)$.

Since $a_i \to 0$, we have $P = Q$. In a local chart, we can write $(\frac{Q-P}{a_i}, c_i(0)) \to (w, p)$. Since $a_i \to 0$, $w = \tilde{\Psi}(P)$. Since $P_i, Q_i \in S(A), c_i(0)$ for all $i$, $w$ is tangent to the fiber $S(A)_p = S(TG_p)$, which is a contradiction (for the flow $\Psi$ on the geodesic flow over $G_p$).

Remark 2.6. In [1], a flow $\Phi$ is defined on $S(A)$ extending the geodesic flow on $S(TM_0)$. However, $\Phi$ itself is not quite a geodesic flow since typically it has fixed points at the boundary. Our approach does not seem to work with this flow. Indeed, to each geodesic loop $c_i : [0, a_i] \to M_0$, we have a corresponding segment $\Phi_t : [0, a_i] \to S(A)$. By considering a convergent subsequence, the limit of $(c_i(0), \tilde{c}_i(0))$ is a point $v$ contained in $\partial S(A) = S(A)_{\partial M}$. The limit of $c_i(0)$ is a point $p = \pi(v) \in \partial M$. In the notations of [1], we have $(\pi \circ r_{\iota})(H_v(v)) = 0$ and $r_{\iota}(v) = 0$. In particular, the flow $\Phi$ at $v$ is stationary: $\forall t, \Phi_t(v) = v$. This, however, is not sufficient to obtain a contradiction, since at the boundary, $\Phi$ may have some fixed points as mentioned above.

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