Abstract

The Collatz variations pattern seems not to have any recurrence relation between numbers. But knowing that there is at least a natural number that converges after several iterations we construct a function \( f_{X,Y} \) that is equal to the value of convergence for all convergent sequences. A canonical decomposition can be expressed for such numbers.

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1 Introduction

The Collatz conjecture has many denominations. It is also known as the Syracuse problem or the 3N+1 problem. The problem was first stated by the German mathematician Lothar Collatz in the 1930’s [1]. The conjecture is summarized as follows. Take any natural number \( n \) not equal to zero. If \( n \) is even divide by 2. If \( n \) is odd multiply it by 3 and add 1. Repeat the process to infinity. Does the sequence created reaches 1 for every initial number \( n \)? The Collatz sequence \((C_p)_{p \in \mathbb{N}}\) started with a natural number \( n \) different of zero is called convergent when after \( k \) iterations the sequence is equal to 1. The total stopping time \( \sigma_\infty(n) = \inf \{ k : T^k(n) = 1 \} \) [2]; \( k \) is the finite least iterations before \((C_p)\) converges. Consider the function:

\[
g(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 3n + 1, & \text{if } n \text{ is odd} \end{cases}
\]

Form the sequence by performing an infinite operation of the function. Notation:

\[
C_p = \begin{cases} n, & \text{for } p = 0 \\ g(C_{p-1}), & \text{for } p > 0 \end{cases}
\]
$C_p$ is the value of $g$ applied to $n$ recursively $p$ times: in notation $C_p = g^p(n)$. The smallest $p$ such that $C_p = 1$ is nothing than $\sigma_\infty(n)$ defined earlier as the total stopping time ($p = k$).

A divergent sequence isn’t yet found. The divergence would consist of a total stopping being infinity. In notation: $\sigma_\infty(n) = \infty$ [2]. Even though computational method had proven the convergence of all natural number $n < 20 \cdot 2^{58}$ [3], does not totally prove the Collatz conjecture. But it tells us the existence of several convergent numbers (The partition set of the convergent numbers in $\mathbb{N}$ is not empty).

This document is intended to prove that all convergent numbers have their convergence same as a function $f_{X,Y}$. In general, $g^k(n) = f_{X,Y} = 1$. This paper also includes properties of convergent numbers by the Collatz sequence and a generalisation of the idea that the set of convergent $n$ is never empty to an infinite set.

## 2 The odd and even iterations $X$ and $Y$ at convergence

### 2.1 The $k$−tuple associated to a Collatz sequence at the total stopping time $\sigma_\infty(n) = k$

**Definition 2.1**: The $k$−tuple associated to $n$ after $k$ total Collatz iterations is the chain constituted of all values $C_p$ when $p$ varies from 0 to $p − 1$ ($p = 0, 1, \ldots, k − 1$). Then the $k$−tuple is $(C_0, C_1, \ldots, C_{k−1})$.

**Examples 2.1**

For $n = 6$, $C_8 = 1$. The $8$−tuple associated to 6 is $(6, 3, 10, 5, 16, 8, 4, 2)$.

Another example is $n = 19$, it takes 20 iterations before it gets to 1. $C_{20} = 1$ and its $20$−tuple is $(19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2)$.

### 2.2 The smallest odd and even iterations $X$ and $Y$

Consider the $k$−tuple $(C_0, C_1, \ldots, C_{k−1})$ associated to a convergent sequence of $n$. Let us make a set $E$ of all the element in the $k$−tuple chain and 2 subsets $E_1$, $E_2$ defined respectively as set of all the odd and all even numbers of $E$.

$$E = \{C_0, C_1, \ldots, C_{k−1}\}. \quad (1)$$

**Definition 2.2**: The iterations on odd numbers $X$ is the cardinal of the set $E_1$ and $Y$, the iterations on even number is the cardinal of the set $E_2$.

$$Card\{E_1\} = X \quad (2)$$
and,
\[ \text{Card} \{ E_2 \} = Y. \] \hspace{1cm} (3)

**Remark 2.2**
\[
\text{Card} \{ E \} = \text{Card} \{ E_1 \} + \text{Card} \{ E_2 \}, \\
\{ E \} = E_1 \cup E_2, \\
\sigma_\infty(n) = k = X + Y.
\]
By convenience we'll note a convergent sequence of \( n \) after \( X \) and \( Y \) iterations \( n = n(X,Y) \), and we'll denote by \( \mathcal{N} \) the set of convergent natural numbers.
By definition of the Collatz conjecture \( n \neq 0 \), so \( \mathcal{N} \subseteq \mathbb{N}^* \).

3 The function \( f_{X,Y} \) associated to the convergence value \( C_k(n) \) of \( n \)

### 3.1 The value at the convergence

**Definition 3.1:** The sequence \( (C_p)_{p \in \mathbb{N}} \) is called convergent when after \( p = k \) iterations \( C_k(n) = 1 \). The value at the convergence of any Collatz sequence started with \( n \) non-zero positive integer is the limit taken at the total stopping time. In terms of limit notation:

\[
\lim_{p \to \sigma_\infty} (C_p) = 1.
\]

### 3.2 The function \( f_{X,Y} \)

**Definition 3.2:** Let \( \mathcal{N} \) be the set of the convergent \( n = n(X,Y) \), with the couple \( (X,Y) \) associated to the its respective \( n \). By a function at the convergence of \( n \) we mean a map
\[
f_{X,Y} : \mathcal{N} \to \mathbb{N}
\]
where:
\[
f_{X,Y}(n) = \left\lceil \frac{3^X(2n+1)-1}{2^Y+1} \right\rceil.
\] \hspace{1cm} (4)

**Lemma 3.1:** Let \( Z_n = \frac{3^X(2n+1)-1}{2^Y+1} \). \( \forall n = n(X,Y) \in \mathcal{N}, \exists \varepsilon \) such that
\[
0 \leq \varepsilon < \frac{1}{3}, \text{ for which,}
\]
\[
Z_n = 1 - \varepsilon.
\] \hspace{1cm} (5)

**Proof:** For \( n(0,i) = 2^i \ (i \in \mathbb{Z}^*_+) \), then \( Z_n = \frac{2^i}{2^i} = 1 \) where \( \varepsilon = 0 \).
Let \( n \neq n(0,i) \). Proceed by ABSURD i.e we suppose that \( \exists n = n(X,Y) \), and \( \varepsilon > \frac{1}{3} \) such that \( Z_n \neq 1 - \varepsilon \)
If \( Z_n < 1 - \varepsilon \)
\[ Z_n < 1 - \varepsilon \implies \exists \varepsilon' \text{ such that } \frac{1}{3} < \varepsilon' < 1 \text{ and } Z_n = 1 - \varepsilon'. \text{ That’s ABSURD.} \]

If \( Z_n > 1 - \varepsilon \)
\[ Z_n > 1 - \varepsilon \implies \exists \varepsilon' \text{ such that } 0 < \varepsilon' < \frac{1}{3} \text{ and } Z_n = 1 - \varepsilon' \text{ (ABSURD),} \]

or
\[ \exists L > 1 \text{ and } \varepsilon'', \text{ for which } Z_n = L - \varepsilon'' \text{ with } \frac{1}{3} < \varepsilon'' < 1 \text{ (} L \in \mathbb{N} \text{).} \]

\[ 3Z_n + 1 = 3L - 3\varepsilon'' + 1 \]
\[ = (3L + 1) - 3\varepsilon'' \]

\( n \) converges, \( n = n(X, Y) \implies Z_n = 1 - \varepsilon'' \text{ and } 0 < \varepsilon'' < \frac{1}{3} \left(n \neq n(0, i)\right). \)

\[ 3Z_n + 1 = 3 - 3\varepsilon'' + 1 \]
\[ = 4 - 3\varepsilon'' \]

For the same \( n = n(X, Y) \), we have 2 values of \( 3Z_n + 1 \) where one’s function of \( L \).

Since a number cannot differ from itself, \( L \) must be equal to 1 and \( 0 < \varepsilon'' < \frac{1}{3} \).

There is a contradiction meaning that there is no such \( L \) greater than 1 and there is no \( \varepsilon'' \), such that \( \frac{1}{3} < \varepsilon'' < 1 \) for which \( Z_n = L - \varepsilon'' \).

**Conclusion:** \( \forall n = n(X, Y) \in \mathcal{N}, \exists \varepsilon \text{ such that } 0 \leq \varepsilon < \frac{1}{3} \text{ for which } Z_n = 1 - \varepsilon. \)

We can now prove the following theorem:

**Theorem 3.1:** For all \( n = n(X, Y) \in \mathcal{N}, \) there is a function \( f_{X,Y}(n) = \left\lceil \frac{3^X (2n + 1) - 1}{2^Y + 1} \right\rceil \) which is equal to the value of convergence \( C_k \) of \( n \).

\[ \forall n = n(X, Y) \in \mathcal{N}, f_{X,Y}(n) = 1. \tag{6} \]

**Proof:** From Lemma 3.1 for all \( n = n(X, Y) \) there is always \( \varepsilon \) satisfying the condition \( 0 \leq \varepsilon < \frac{1}{3} \) and we have \( Z_n = 1 - \varepsilon. \)

\[ Z_n = 1 - \varepsilon \implies \left\lfloor Z_n \right\rfloor = 1, \]
\[ f_{X,Y}(n) = \left\lfloor Z_n \right\rfloor \implies f_{X,Y}(n) = 1. \]

**Corollary 3.1:** \( \forall n = n(X, Y) \in \mathcal{N}, f_{X,Y}(n) \) is a constant function.
4 Canonical decomposition \( n' \) of \( n = n(X, Y) \)

**Definition 4.1:** For \( n \) in \( \mathcal{N} \) with its respective couple \((X, Y)\), the expression \( n' \) in \( \mathbb{R}_+^\ast \) of \( n \) is:

\[
n' = 2^Y (3^{-X} - \varepsilon_n);
\]  
(7)

\( n' \) is called by definition the canonical decomposition of \( n \).

**Lemma 4.1:** For all convergent \( n = n(X, Y) \), \( 0 < \frac{1}{2}(1 - 3^{-X}) < \frac{1}{2} \).

*Proof:* \( \forall X \in \mathbb{N} \),

\[
0 < 3^{-X} < 1
-1 < -3^{-X} < 0
0 < 1 - 3^{-X} < 1
0 < \frac{1}{2}(1 - 3^{-X}) < \frac{1}{2}
\]

**Lemma 4.2:** If \( \{n'\} \) is the fractionnal part of \( n' \) then, \( \{n'\} = \frac{1}{2}(1 - 3^{-X}) \) and \( \varepsilon_n = 3^{-X}\varepsilon \).

*Proof:* Let \( n(X, Y) \in \mathcal{N} \), then from (6) we know that \([Z_n] = 1\).

\[
\left\lceil Z_n \right\rceil = 1 \iff Z_n = 1 - \varepsilon
\]

\[
\frac{3^X (2n + 1) - 1}{2^{Y+1}} = 1 - \varepsilon
\]

\[
3^X (2n + 1) - 1 = 2^{Y+1}(1 - \varepsilon)
= 2^{Y+1} - 2^{Y+1}\varepsilon
\]

\[
2 \cdot 3^X n + 3^X - 1 = 2^{Y+1} - 2^{Y+1}\varepsilon
\]

\[
2 \cdot 3^X n = 1 - 3^X + 2^{Y+1} - 2^{Y+1}\varepsilon
\]

\[
n = \frac{1}{2}(3^{-X}) - \frac{1}{2} + 3^{-X} \cdot 2^Y - (3^{-X}\varepsilon) \cdot 2^Y
= \frac{1}{2}(3^{-X} - 1) + 3^{-X} \cdot 2^Y - 2^Y\varepsilon_n
= \frac{1}{2}(3^{-X} - 1) + 2^Y(3^{-X} - \varepsilon_n)
= 2^Y(3^{-X} - \varepsilon_n) - \frac{1}{2}(1 - 3^{-X})
= n' - \frac{1}{2}(1 - 3^{-X})
\]

\[
n = n' - \frac{1}{2}(1 - 3^{-X}).
\]  
(8)
The number $n$ is a natural number written as the difference of 2 real numbers which are positive. Also $0 < \frac{1}{2}(1 - 3^{-X}) < \frac{1}{2}$ and $n' > \frac{1}{2}(1 - 3^{-X})$. The relation $n = n' - \frac{1}{2}(1 - 3^{-X})$ is true if and only if $\frac{1}{2}(1 - 3^{-X})$ is the fractional part of $n'$; i.e $\{n'\} = \frac{1}{2}(1 - 3^{-X})$.

**Lemma 4.3:** If $n'$ is the canonical decomposition of $n$ then $0 \leq \varepsilon_n < \frac{1}{3^{X + 1}}$.

**Proof:** $\varepsilon_n = 3^{-X}\varepsilon$ and $0 \leq \varepsilon < \frac{1}{3}$.

**Remarks 4.1:** For the same iterations $X$ and $Y$ at the convergence of $n$ and $m$, $\varepsilon_n = \varepsilon_m$, For $n = n(0, i)$ (or $n = 2^i$, $i \in \mathbb{Z}_+^*$), $\varepsilon_n = 0$.

We can now state the following theorem;

**Theorem 4.1:** Let $n'$ be the canonical decomposition of $n = n(X, Y)$ in $\mathbb{R}_+^*$. The expression of $n$ in function of $n'$ is:

$$n = \lfloor n' \rfloor. \quad (9)$$

**Proof:** From (8) we have the equality $n = n' - \frac{1}{2}(1 - 3^{-X})$

$$n = n' - \frac{1}{2}(1 - 3^{-X})$$

$$n' = n + \frac{1}{2}(1 - 3^{-X})$$

$$\lfloor n' \rfloor = \lfloor n + \{n'\} \rfloor$$

$$\lfloor n' \rfloor = \lfloor n \rfloor$$

$$\lfloor n' \rfloor = n.$$

**Corollary 4.1:** $\forall n \in \mathcal{N}, n = n(X, Y) \iff n = \lfloor n' \rfloor$.

### 4.1 Properties

Consider $\mathcal{N}$, $n'$ the canonical decomposition of $n = n(X, Y)$, and $a$ and $b$ be 2 elements of $\mathcal{N}$. We consider the following strong properties arising from the canonical decomposition:

**Unicity of the couple $(X, Y)$:** $a = a(X, Y)$ and $b = b(X, Y)$ iff $a = b$. 

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Proof: Let \( a = a(X, Y) \) and \( b = b(X, Y) \) then \( a = \lfloor a' \rfloor \) and \( b = \lfloor b' \rfloor \)
\[
a - b = \lfloor a' \rfloor - \lfloor b' \rfloor, \\
a - b = \lfloor 2^Y(3^{-X} - \varepsilon_a) \rfloor - \lfloor 2^Y(3^{-X} - \varepsilon_b) \rfloor,
\]
It’s known from remark 4.1 that for the same iterations \( X \) and \( Y \) at the convergence of \( n \) and \( m \), \( \varepsilon_n = \varepsilon_m \) then:
\[
\varepsilon_a = \varepsilon_b, \\
a - b = 0, \\
a = b.
\]

The \( a + b \) addition: If \( a \) and \( b \) converge so does \( a + b \): i.e \( a = a(X, Y) \) and \( b = b(X', Y') \), then \( \exists (X'', Y'') \) such that \( a + b = [a + b](X'', Y'') \)

Proof: Let \( a = a(X, Y) \) and \( b = b(X, Y) \) then \( a = \lfloor a' \rfloor \) and \( b = \lfloor b' \rfloor \)
\[
a + b = \lfloor a' \rfloor + \lfloor b' \rfloor, \\
a + b = \lfloor a' + b' \rfloor.
\]
Because \( \{a'\} \) and \( \{b'\} \) are both less than \( \frac{1}{2} \). (See Lemma 4.1).

5 Algebra of the set \( \mathcal{N} \)

5.1 Equipotence to \( \mathbb{N} \)

Definition 5.1: \( \mathcal{N} \) is equipotent to \( \mathbb{N} \) or countably infinite when there exist a function bijective from \( \mathcal{N} \) to \( \mathbb{N} \).

Lemma 5.1: There is a bijection from \( \mathbb{N} \) to its infinite subsets especially to \( \mathcal{N} \).

Proof: \( \mathcal{N} \subseteq \mathbb{N} \); Let consider an order relation \( \leq \) on \( \mathcal{N} \) and let the set be finite,
\[
\exists M \in \mathcal{N}|M = \{k|\forall x \in \mathcal{N}, x < k\} \\
M + 1 \notin \mathcal{N}
\]
From properties above the addition of 2 convergent numbers is convergent so must \( M + 1 \) also be in \( \mathcal{N} \) i.e also convergent. We arrive at a contradiction. \( \mathcal{N} \) is not majored and not a finite set.
The application which to every single element of \( \mathcal{N} \) associate their perfect square in \( \mathbb{N} \) is bijective.
5.2 Total order relation in $\mathbb{N}$

The order relation $\leq$ is total in $\mathbb{N}$. By definition $\leq$ is a total relation order when $\forall a$ and $b$ in the set such that $a \leq b$, there is also $c$ in the set such $a + c = b$.

In $\mathbb{N}$ this relation is verified. In fact, if there is $M$ in $\mathbb{N}$, $M + 1$ also is in $\mathbb{N}$ leading to state that two elements $a$ and $b$ in the set are always comparable: $a \leq b$ or $b \leq a$.

5.3 Conclusion

We recall some basic properties: Any partition of $\mathbb{N}$ different from the empty set ($\emptyset$) has a least element. The least element to converge in $\mathbb{N}$ is $1$.

The addition in $\mathbb{N}$ is an internal law of composition.

So we can assure these following inclusions:

$$\mathbb{N} \subset \mathbb{N}^*, \text{and } \mathbb{N}^* \subset \mathbb{N}$$ (10)

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