A Theory of Quantized Fields Based on Orthogonal and Symplectic Clifford Algebras

Matej Pavšič
Jožef Stefan Institute, Jamova 39, SI-1000, Ljubljana, Slovenia;
email: matej.pavsic@ijs.si

Abstract

The transition from a classical to quantum theory is investigated within the context of orthogonal and symplectic Clifford algebras, first for particles, and then for fields. It is shown that the generators of Clifford algebras have the role of quantum mechanical operators that satisfy the Heisenberg equations of motion. For quadratic Hamiltonians, the latter equations are obtained from the classical equations of motion, rewritten in terms of the phase space coordinates and the corresponding basis vectors. Then, assuming that such equations hold for arbitrary path, i.e., that coordinates and momenta are undetermined, we arrive at the equations that contains basis vectors and their time derivatives only. According to this approach, quantization of a classical theory, formulated in phase space, is replacement of the phase space variables with the corresponding basis vectors (operators). The basis vectors, transformed into the Witt basis, satisfy the bosonic or fermionic (anti)commutation relations, and can create spinor states of all minimal left ideals of the corresponding Clifford algebra. We consider some specific actions for point particles and fields, formulated in terms of commuting and/or anticommuting phase space variables, together with the corresponding symplectic or orthogonal basis vectors. Finally we discuss why such approach could be useful for grand unification and quantum gravity.

1 Introduction

During last few decades we are faced with the persisting problems of quantum gravity and the unification of fundamental interactions. The situation reveals the need to reformulate the conceptual foundations of physics and to employ a more evolved mathematical formalism. It has turned out that Clifford algebras provide very promising tools for description [1] and generalization of geometry and physics [2, 3, 4]. There exist two kinds of Clifford algebras, orthogonal and symplectic [5]. In orthogonal Clifford algebras, the symmetric product of two basis vectors $\gamma_a$ is the inner product and it gives the orthogonal metric, while the antisymmetric product gives a basis bivector:

\[
\gamma_a \cdot \gamma_b \equiv \frac{1}{2}(\gamma_a \gamma_b + \gamma_b \gamma_a) = g_{ab} \quad \text{(orthogonal metric)}
\]

\[
\gamma_a \wedge \gamma_b \equiv \frac{1}{2}(\gamma_a \gamma_b - \gamma_b \gamma_a) \quad \text{(bivector)}
\]

(1)

In symplectic Clifford algebras, the antisymmetric product of two basis vectors $q_a$ is the inner product and it gives the symplectic metric, whilst the symmetric product
gives a basis bivector:

\[ q_a \wedge q_b \equiv \frac{1}{2}(q_a q_b - q_b q_a) = J_{ab} \quad \text{(symplectic metric)} \]

\[ q_a \cdot q_b \equiv \frac{1}{2}(q_a q_b + q_b q_a) \quad \text{(bivector)} \]

The generators of an orthogonal Clifford algebra can be transformed into a basis (the Witt basis) in which they behave as fermionic creation and annihilation operators. The generators of a symplectic Clifford algebra behave as bosonic creation and annihilation operators. We will show how both kinds of operators can be united into a single structure so that they form a basis of a ‘superspace’. We will consider an action for a point particle in such superspace. It contains the well known spinning particle action as a particular case [6]. After quantization we obtain, in particular, the Dirac equation. Spinors can be constructed in terms of the orthogonal basis vectors rewritten in the Witt basis. So a spinor of each minimal left ideal of an orthogonal Clifford algebra is an element of the Fock space, whose basis elements are products of creation operators acting on a vacuum which in turn is the product of all annihilation operators [7]–[10]. The role of creation and annihilation operators together with the corresponding vacuum can be interchanged, and so we obtain [10] as many different vacua—and thus different kinds of spinors—as there are different minimal left ideals of the Clifford algebra. This property has been exploited in order to explain the ‘internal’ degrees of freedom behind gauge theories of fundamental forces [11, 13, 14, 10]. Other approaches to the unification by Clifford algebras have been investigated in Refs. [12]–[16].

Instead of finite dimensional spaces, we can consider infinite dimensional spaces. Then we have a description of field theory in terms of fermionic and bosonic creation and annihilation operators. The latter operators can be considered as being related to the basis vectors of the corresponding infinite dimensional space.

Our approach provides a fresh view on quantization. ‘Quantization’ is replacement of the phase space coordinates with the corresponding basis vectors. The latter vectors are quantum mechanical operators. We point out that, in the symplectic case, the Poisson brackets between phase space coordinates are equal to the wedge products (i.e., to \( \frac{1}{2} \) times the commutators) of the corresponding basis vectors. We show that, for quadratic Hamiltonians, the latter vectors satisfy the Heisenberg equations of motion, under the assumption that the classical trajectories in phase space are arbitrary, and not necessarily solutions to the classical equations of motion. The analogous holds in the orthogonal case.

Such novel insight on the basis vectors, namely that they give the metric of spacetime (a subspace of a phase space), and at the same time they have the role of quantum mechanical operators from which one can create, e.g., spinors, could be exploited in the development of quantum gravity.
In Sec. 2 we review spaces with orthogonal and symplectic forms and their role in quantization. In Sec. 3 we present a relation between the classical equations of motion and the Heisenberg equations for operators. We consider the latter relation for a generic ‘Hamiltonian’ that is quadratic in coordinates and momenta. In Sec. 4 we formulate a theory of a point particle in the space of commuting and anticommuting (Grassmann) coordinates, the action being a generalization of the spinning particle action [6] and of the local Sp(2) symmetric action considered in Refs. [17]. In Sec. 5 we discuss the representation of spinors in terms of Grassmann coordinates. In Sec. 6 we present a theory of quantized bosonic and fermionic fields in terms of symplectic and orthogonal Clifford algebras. We show that there exist many possible definitions of vacua and fermionic creation and annihilation operators, and how this fact could be exploited for resolution of the cosmological constant problem. Finally, in Sec. 7 we discuss what are the prospects of such theory for grand unification and quantum gravity.

2 Spaces with orthogonal and symplectic forms

2.1 The inner product and metric

I. Orthogonal case

The inner product of vectors $a$ and $b$ is given by

$$(a, b)_g = (a^a \gamma_a, b^b \gamma_b)_g = a^a (\gamma_a, \gamma_b)_g b^b = a^a g_{ab} b^b.$$  

(3)

The metric is given by the inner product of two basis vectors:

$$(\gamma_a, \gamma_b)_g = g_{ab},$$  

(4)

where $g_{ab}$ is a symmetric tensor. For basis vectors we can take the generators of the orthogonal Clifford algebra satisfying

$$(\gamma_a, \gamma_b)_g = \gamma_a \cdot \gamma_b \equiv \frac{1}{2} (\gamma_a \gamma_b + \gamma_b \gamma_a) = g_{ab}.$$  

(5)

Here vectors are Clifford numbers. The inner product of the vectors $a$ and $b$ is given by the symmetric part of the Clifford product

$$(a, b)_g = \frac{1}{2} (ab + ba) = a \cdot b.$$  

(6)

II. Symplectic case

The inner product of vectors $z$ and $z'$ is given by

$$(z, z')_J = (z^a q_a, z'^b q_b)_J = z^a (q_a, q_b)_J z'^b = z^a J_{ab} z'^b.$$  

(7)
where \((q_a, q_b)_J = J_{ab}\) \(\text{(8)}\) is the symplectic metric.

For a symplectic basis we can take generators of the symplectic Clifford algebra:
\[(q_a, q_b)_J = q_a \wedge q_b \equiv \frac{1}{2}(q_a q_b - q_b q_a) = J_{ab}\] \(\text{(9)}\)

Vectors are now symplectic Clifford numbers. The inner product of symplectic vectors \(z\) and \(z'\) is given by the antisymmetric product
\[(z, z')_J = \frac{1}{2}(z z' - z' z) = z \wedge z'. \quad \text{(10)}\]

Dimension of the symplectic vector space is even. Physically it is realized as the phase space. We can split a symplectic vector according to
\[z = z^a q_a = x^\mu q_\mu^{(x)} + p^\nu q_\nu^{(p)}, \quad \mu = 1, \ldots, n. \quad \text{(11)}\]

Then we have
\[(z, z')_J = z^a J_{ab} z^b = (x^\mu p^\nu - p^\mu x^\nu) g_{\mu\nu}, \quad \text{(12)}\]

with
\[J_{ab} = \begin{pmatrix} 0 & g_{\mu\nu} \\ -g_{\mu\nu} & 0 \end{pmatrix}, \quad \text{(13)}\]

where, depending on signature, the \(n \times n\) block \(g_{\mu\nu}\) is the euclidean, \(g_{\mu\nu} = \delta_{\mu\nu}\), or the Minkowski metric, \(g_{\mu\nu} = \eta_{\mu\nu}\).

Relations
\[\frac{1}{2}[q_a, q_b] = J_{ab} \quad \text{(14)}\]

give
\[\frac{1}{2}[q_\mu^{(x)}, q_\nu^{(x)}] = 0, \quad \frac{1}{2}[q_\mu^{(p)} , q_\nu^{(p)}] = 0, \quad \text{(15)}\]

\[\frac{1}{2}[q_\mu^{(x)} , q_\nu^{(p)}] = g_{\mu\nu}, \quad \text{(16)}\]

which are the Heisenberg commutation relations.

\subsection*{2.2 Poisson bracket}

In symplectic case, the Poisson bracket between functions \(f(z)\) and \(g(z)\) of phase space coordinates \(z^a\) is given by
\[
\{f, g\}_\text{PB} \equiv \frac{\partial f}{\partial z^a} J^{ab} \frac{\partial g}{\partial z^b}, \quad \text{(17)}
\]
where
\[
J^{ab} = \begin{pmatrix} 0 & -g^{\mu\nu} \\ g^{\mu\nu} & 0 \end{pmatrix}
\] (18)
is the inverse symplectic metric.

By introducing the symplectic basis vectors, we can rewrite the above expression as the wedge product \( \frac{\partial f}{\partial z^a}q^a \wedge \frac{\partial g}{\partial z^b}q^b \). i.e.,
\[
\{ f, g \}_{PB} = \frac{1}{2} i \left[ \frac{\partial f}{\partial z^a}q^a, \frac{\partial g}{\partial z^b}q^b \right] = \frac{\partial f}{\partial z^a} J^{ab} \frac{\partial g}{\partial z^b}
\] (19)
If we take
\[
f = z^c, \quad g = z^d
\] (20)
we obtain
\[
\{ z^c, z^d \}_{PB} = \frac{1}{2} [q^c, q^d] = J^{cd}
\] (21)
These are the Heisenberg commutation relations for ‘operators’ \( q^c, q^d \).

Usually by ‘quantization’ we mean the replacement of the classical phase space coordinates \( z^a = (x^\mu, p^\mu) \) by operators \( \hat{z}^a = (\hat{x}^\mu, \hat{p}^\mu) \) that satisfy the Heisenberg commutation relations. The above derivation reveals that the quantum mechanical operators \( \hat{z}^a = (\hat{x}^\mu, \hat{p}^\mu) \) are in fact the symplectic basis vectors \( q^a \). This is true up to the factor \( i \) in front of the ‘momentum’ part \( q^{\mu(p)} \), a factor that is necessary in order to have Hermitian momentum operator \( \hat{p}^\mu \). We also see that the Poison bracket of the phase space coordinates is equal (apart from factor \( \frac{1}{2} \)) to the commutator of the corresponding basis vectors.

In orthogonal case, the Poisson bracket between functions \( f(\lambda) \) and \( g(\lambda) \) of phase space coordinates \( \lambda^a \) is given by
\[
\{ f, g \}_{PB} \equiv \frac{\partial f}{\partial \lambda^a} g^{ab} \frac{\partial g}{\partial \lambda^b},
\] (22)
where
\[
g^{ab} = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & g^{\mu\nu} \end{pmatrix}
\] (23)
is the inverse orthogonal metric. In Sec. 4 we point out that in a phase space with orthogonal metric the coordinates \( \lambda^a \) must be Grassmann valued.

By introducing the basis vectors we can rewrite the above expression as the dot product \( \left( \frac{\partial f}{\partial \lambda^a} \gamma^a \right) \cdot \left( \frac{\partial g}{\partial \lambda^b} \gamma^b \right) \), i.e.,
\[
\{ f, g \}_{PB} = \frac{1}{2} \left\{ \frac{\partial f}{\partial \lambda^a} \gamma^a, \frac{\partial g}{\partial \lambda^b} \gamma^b \right\} = \frac{\partial f}{\partial \lambda^a} g^{ab} \frac{\partial g}{\partial \lambda^b}
\] (24)
If we take
\[
f = \lambda^c, \quad g = \lambda^d
\] (25)
then we obtain
\[ \{\lambda^c, \lambda^d\}_{\text{PB}} = \frac{1}{2}(\gamma^c \gamma^d + \gamma^d \gamma^c) = g^{cd} \] (26)

These are the anticommutation relations for ‘operators’ \( \gamma^c, \gamma^d \). The generators \( \gamma^a \) of an orthogonal Clifford algebra are thus ‘quantized’ \( \lambda^a \).

### 2.3 Representation of Clifford numbers

I. **Orthogonal Clifford algebra** \( Cl(2n) \)

In even dimensions we can split the generators \( \gamma_a \) according to
\[ \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu), \quad \mu = 1, 2, ..., n \] (27)

We can introduce the Witt basis:
\[ \theta_\mu = \frac{1}{\sqrt{2}}(\gamma_\mu + i \bar{\gamma}_\mu) \]
\[ \bar{\theta}_\mu = \frac{1}{\sqrt{2}}(\gamma_\mu - i \bar{\gamma}_\mu) \] (28)

Then the anticommutation relations (5) become
\[ \theta_\mu \cdot \bar{\theta}_\nu \equiv \frac{1}{2}(\theta_\mu \bar{\theta}_\nu + \bar{\theta}_\nu \theta_\mu) = \eta_{\mu\nu}, \quad \theta_\mu \cdot \theta_\nu = 0, \quad \bar{\theta}_\mu \cdot \bar{\theta}_\nu = 0. \] (29)

The generators \( \gamma_\mu, \bar{\gamma}_\mu, \theta_\mu, \bar{\theta}_\mu \) can be represented:
1) as matrices,
2) in terms of Grassmann coordinates:
\[ \theta^\mu \rightarrow \sqrt{2} \xi^\mu, \quad \bar{\theta}_\mu \rightarrow \sqrt{2} \frac{\partial}{\partial \xi^\mu} \] (30)
\[ \xi^\mu \xi^\nu + \xi^\nu \xi^\mu = 0. \] (31)

II. **Symplectic Clifford algebra** \( Cl_S(2n) \)

The generators are:
\[ q_a = (q_\mu^{(x)}, q_\mu^{(p)}), \quad \mu = 1, 2, ..., n \] (32)

Commutation relation (14) or (15),(16) can be represented:
1) as 4 x 4 matrices, in which case the operators cannot be cast into the Hermitian form,
2) in terms of commuting coordinates:
\[ q_\mu^{(x)} \rightarrow \sqrt{2} x^\mu, \quad q_\mu^{(p)} \rightarrow \sqrt{2} \frac{\partial}{\partial x^\mu} \] (33)
\[ x^\mu x^\nu - x^\nu x^\mu = 0 \] (34)
3 Heisenberg equations as equations of motion for basis vectors

3.1 General considerations

Let us now consider a particle whose motion is described by the following phase space action:

\[ I = \frac{1}{2} \int d\tau \left( \dot{z}^a J_{ab} \dot{z}^b + z^a K_{ab} \dot{z}^b \right), \]  

(35)

where

\[ \frac{1}{2} z^a K_{ab} \dot{z}^b = H. \]  

(36)

Here \( K_{ab} \) is a symmetric \( 2n \times 2n \) matrix. By varying the latter action with respect to \( z^a \) we obtain the equations of motion

\[ \dot{z}^a = J_{ab} \frac{\partial H}{\partial z^b}. \]  

(37)

Let us consider trajectories \( z^a(\tau) \) as components of an infinite dimensional vector

\[ z = z^a(\tau) q_a(\tau) \equiv \int d\tau \ z^a(\tau) q_a(\tau), \]  

(38)

where \( q_a(\tau) \) are basis vectors satisfying

\[ q_a(\tau) \wedge q_b(\tau') = J_{ab} \delta(\tau - \tau'), \]  

(39)

and write the action (35) in the form

\[ I = \frac{1}{2} \dot{z}^a(\tau) J_{ab(\tau')} z^b(\tau') + \frac{1}{2} z^a(\tau) K_{a(\tau)b(\tau')} \dot{z}^b(\tau'). \]  

(40)

We have introduced

\[ 1 \int d\tau \ z^a(\tau) K_{ab} \dot{z}^b(\tau) = \frac{1}{2} z^a(\tau) K_{a(\tau)b(\tau')} \dot{z}^b(\tau') \equiv H, \]  

(41)

where

\[ K_{a(\tau)b(\tau')} = K_{b(\tau')a(\tau)} = K_{ab} \delta(\tau - \tau'). \]  

(42)

The action (40) gives the following form of the equations of motion:

\[ \dot{z}^a(\tau) = J_{ab(\tau')} \frac{\partial H}{\partial z^b(\tau')} = J_{ac(\tau')} K_{c(\tau')b(\tau')} \dot{z}^b(\tau'), \]  

(43)

where \( \partial/\partial z^a(\tau) \equiv \delta/\delta z^a(\tau) \) denotes functional derivative. Multiplying both sides of the latter equation by the basis vectors \( q_a(\tau) \) we obtain an equivalent equation

\[ \dot{z}^a(\tau) q_a(\tau) = -q^a(\tau) K_{a(\tau)b(\tau')} \dot{z}^b(\tau'). \]  

(44)
Let us now use the following relation:

\[ \dot{z}^a(\tau) q_a(\tau) \equiv \int d\tau \dot{z}^a(\tau) q_a(\tau) = -\int d\tau z^a(\tau) \dot{q}_a(\tau) \equiv -z^a(\tau) \dot{q}_a(\tau), \]  

(45)

that holds, if the boundary term vanishes, which we will assume is the case. Then Eq. (44) becomes

\[ z^b(\tau') \dot{q}_b(\tau') = q^a(\tau) K_{a(b)} z^b(\tau'). \]  

(46)

Eq. (46) is equivalent to the classical equations of motion derived from the action (35). It holds for a classical trajectory \( z^b(\tau') \equiv z^b(\tau) \) satisfying the minimal action principle.

Let us now explore what happens if we drop the requirement that Eq. (46) must hold for a trajectory that satisfies the minimal action principle associated with (35), and make the assumption that Eq. (46) is satisfied for an arbitrary trajectory (path) \( z^b(\tau') \). Then we have

\[ \dot{q}_b(\tau') = q^a(\tau) K_{a(b)} \]  

(47)

which is an equation of motion for the basis vectors (operators) \( q_a(\tau) \equiv q_a(\tau) \). From Eqs. (42) and (47) we have

\[ \dot{q}_b(\tau) = q^a(\tau) K_{ab}. \]  

(48)

The right hand side of the latter equation can be written as

\[ K_{ab} q^b = [q_a, \hat{H}], \quad \hat{H} = \frac{1}{2} q^a K_{ab} q^b \]  

(49)

i.e., as the inner product (up to the factor 2) of a symplectic vector \( q^a \) with the symplectic bivector \( \hat{H} \). Remember that in the symplectic case the inner product is given by the commutator. So we obtain

\[ \dot{q}_a = [q_a, \hat{H}], \]  

(50)

which are the Heisenberg equations of motion.

We have found that the basis vectors of phase space satisfy the Heisenberg equations of motion. This is in agreement with the finding of Sec. 2 that the quantum mechanical operators \( \hat{z}^a = (\hat{x}^\mu, \hat{p}^\mu) \) are in fact the symplectic basis vectors \( q^a \). We have arrived at the Heisenberg equations (50) from Eq. (46) in which we assumed that \( z^b(\tau) \equiv z^b(\tau) \) was arbitrary. Arbitrary \( z^b(\tau) = (x^\mu(\tau), p^\mu(\tau)) \) means that coordinates and momenta are undetermined. In other words, if in the classical equations of motion (46) we assume that coordinates and momenta are undetermined, then we obtain the operator equations (50). This sheds new light on ‘quantization’.

8
3.2 Particular physical cases

The general form of the action (35) contains particular cases that depend on choice of $K_{ab}$ and the interpretation of coordinates $x^\mu$ and parameter $\tau$.

(i) One possibility is to interpret $x^\mu$, $\mu = 1, 2, ..., n$, as coordinates of a non relativistic point particle, and $\tau$ as time $t$, the signature being $(+ + + + ...)$. Then (35) describes a non relativistic harmonic oscillator in $n$-dimensions.

(ii) Another possibility is to interpret $x^\mu$, $\mu = 0, 1, 2, ..., n-1$, as coordinates in $n$-dimensional spacetime with signature $(+ - - - ...)$, $\tau$ as an arbitrary monotonically increasing parameter along a particle’s worldline, take

\[ K_{ab} = \left( \begin{array}{cc} 0_{n \times n} & 0 \\ 0 & \lambda(\tau)\eta_{\mu\nu} \end{array} \right), \tag{51} \]

and assume that $\lambda$ is a Lagrange multiplier. Then we have a phase space action for a massless relativistic particle in a higher dimensional spacetime:

\[ I[x^\mu, p_\mu] = \frac{1}{2} \int d\tau \left( \dot{x}^\mu \eta_{\mu\nu} \dot{p}^\nu - x^\mu \eta_{\mu\nu} \ddot{p}^\nu - \lambda p^\mu \delta_{\mu\nu} \right) = \int d\tau \left( p_\mu \dot{x}^\mu - \frac{\lambda}{2} p_\mu p^\mu \right) \tag{52} \]

In a 4-dimensional subspace with signature $(+ - - -)$, such particle behaves as a massive relativistic particle.

More generally, we can take

\[ K^a_b \equiv K^{i\mu}_{j\nu} = A^i_j \delta^\mu_{\nu} = \left( \begin{array}{cc} A^1_{11} & A^1_{12} \\ A^2_{11} & A^2_{22} \end{array} \right) \delta^\mu_{\nu}, \tag{53} \]

and consider $A^i_j(\tau)$ as Lagrange multipliers that give three independent constraints. Then (35) becomes the Bars action [17]. Functions $A^i_j(\tau)$ have the role of compensating gauge field that render the action (35) invariant under local symplectic transformations of Sp(2). Here indices $i, j = 1, 2$, occurring in double indices $a \equiv i\mu, b \equiv j\nu$, distinguish $x^\mu$ from $p^\mu$. Some more explanation can be found in Sec. 4.3, where it is also shown that the Bars action is a special case of a super phase space action.

4 Point particle in ‘superspace’

4.1 A generalized space spanned over an orthogonal and symplectic basis

Let us introduce the generalized vector space whose elements are:

\[ Z = z^A q_A \tag{54} \]
where
\[ z^A = (z^a, \lambda^a), \quad z^a = (x^\mu, \bar{x}^\mu), \quad \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu) \] (55)

\[ q_A = (q_a, \gamma_a), \quad q_a = (q_\mu, \bar{q}_\mu), \quad \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu) \] (56)

The scalar part of the product of two such basis elements gives the metric
\[ \langle q_A q_B \rangle_S = G_{AB} = \begin{pmatrix} J_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix} \] (57)

where
\[ q_a \wedge q_b = J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix} \] (58)

\[ \gamma_a \cdot \gamma_b = g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} \] (59)

Let us consider a particle moving in such space. Its worldline is given by:
\[ z^A = Z^A(\tau) \] (60)

where \( \tau \) is a parameter on the worldline.

### 4.2 Examples of possible actions

A possible action is
\[ I = \frac{1}{2} \int d\tau \langle \dot{Z}^A q_A q_B \dot{Z}^B \rangle_S = \frac{1}{2} \int d\tau \dot{Z}^A G_{AB} \dot{Z}^B \] (61)

Inserting the metric (57) and the coordinates (55), we have

\[ I = \frac{1}{2} \int d\tau \left( \dot{z}^a J_{ab} \dot{z}^b + \dot{\lambda}^a g_{ab} \dot{\lambda}^b \right) \] (62)

Since \( J_{ab} = -J_{ba} \) and \( g_{ab} = g_{ba} \), the first term differs from zero, if \( z^a \) are Grassmann (anticommuting) coordinates, whilst the second term differs from zero if \( \lambda^a \) are commuting coordinates.

Another possible action, more precisely, a part of a phase space action, is
\[ I = \frac{1}{2} \int d\tau \langle \dot{Z}^A q_A Z^B \rangle_S = \frac{1}{2} \int d\tau \dot{Z}^A G_{AB} Z^B \] (63)

or
\[ I = \frac{1}{2} \int d\tau \left( \dot{z}^a q_a \dot{z}^b + \dot{\lambda}^a \gamma_a \dot{\lambda}^b \right) \] (64)

Here, in order to have non vanishing terms, \( z^a \) must be commuting, and \( \lambda^a \) Grassmann (anticommuting) coordinates.
The corresponding canonical momenta are
\[
p_a^{(z)} = \frac{\partial L}{\partial \dot{z}^a} = \frac{1}{2} J_{ab} \dot{z}^b, \tag{65}
\]
\[
p_a^{(\lambda)} = \frac{\partial L}{\partial \dot{\lambda}^a} = \frac{1}{2} g_{ab} \dot{\lambda}^a \tag{66}
\]
Inserting expressions (58) and (59) for \(J_{ab}\) and \(g_{ab}\), the action (64) becomes
\[
I = \frac{1}{2} \int d\tau \left( \dot{\bar{x}}^\mu \eta_{\mu\nu} \bar{x}^{\nu} - \dot{x}^\mu \eta_{\mu\nu} \dot{x}^{\nu} + \dot{\lambda}^\mu \eta_{\mu\nu} \lambda^{\nu} + \dot{\bar{\lambda}}^\mu \eta_{\mu\nu} \bar{\lambda}^{\nu} \right) \tag{67}
\]
where
\[
[x^\mu, x^\nu] = 0, \quad [\bar{x}^\mu, \bar{x}^\nu] = 0, \tag{68}
\]
\[
\{\lambda^\mu, \lambda^\nu\} = 0, \quad \{\bar{\lambda}^\mu, \bar{\lambda}^\nu\} = 0. \tag{69}
\]
The canonical momenta now read
\[
p_{\mu}^{(x)} = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2} \eta_{\mu\nu} \dot{x}^{\nu}, \quad p_{\mu}^{(x)} = \frac{\partial L}{\partial \dot{x}^{\mu}} = -\frac{1}{2} \eta_{\mu\nu} \dot{x}^{\nu} \tag{70}
\]
\[
p_{\mu}^{(\lambda)} = \frac{\partial L}{\partial \dot{\lambda}^\mu} = \frac{1}{2} \eta_{\mu\nu} \dot{\lambda}^{\nu}, \quad p_{\mu}^{(\lambda)} = \frac{\partial L}{\partial \dot{\lambda}^{\mu}} = \frac{1}{2} \eta_{\mu\nu} \dot{\lambda}^{\nu} \tag{71}
\]
Upon quantization, the coordinates and momenta become operators,
\[
x^\mu, \ p_{\mu}^{(x)} \to \hat{x}^\mu, \ \hat{p}_{\mu}^{(x)} \tag{72}
\]
\[
\lambda^\mu, \ p_{\mu}^{(\lambda)} \to \hat{\lambda}^\mu, \ \hat{p}_{\mu}^{(\lambda)} \tag{73}
\]
satisfying
\[
[\hat{x}^\mu, \hat{p}_{\nu}^{(x)}] = i \delta^\mu_\nu, \quad [\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{p}_{\mu}^{(x)}, \hat{p}_{\nu}^{(x)}] = 0, \tag{74}
\]
\[
\{\hat{\lambda}^\mu, \hat{p}_{\nu}^{(\lambda)}\} = i \delta^\mu_\nu, \quad \{\hat{\lambda}^\mu, \hat{\lambda}^\nu\} = 0, \quad \{\hat{p}_{\mu}^{(\lambda)}, \hat{p}_{\nu}^{(\lambda)}\} = 0. \tag{75}
\]
Similar relations hold for barred quantities. Altogether, we have
\[
z^a, \ p_a^{(z)} \to \hat{z}^a, \ \hat{p}_a^{(z)} \tag{76}
\]
\[
\lambda^a, \ p_a^{(\lambda)} \to \hat{\lambda}^a, \ \hat{p}_a^{(\lambda)} \tag{77}
\]
where the operators satisfy
\[
[\hat{z}^a, \hat{p}_b^{(z)}] = i \delta^a_b \tag{78}
\]
\[
\{\hat{\lambda}^a, \hat{p}_b^{(\lambda)}\} = i \delta^a_b. \tag{79}
\]
If we insert
\[
\hat{p}_a^{(z)} = \frac{1}{2} J_{ab} \hat{z}^b, \quad \hat{p}_a^{(\lambda)} = \frac{1}{2} g_{ab} \hat{\lambda}^a, \tag{80}
\]
we obtain
\[ \frac{1}{2} [\hat{z}^a, \hat{z}^b] = i J^{ab}, \quad \frac{1}{2} \{ \hat{\lambda}^a, \hat{\lambda}^b \} = i g^{ab}. \]

But we see that the above operator equations are just the relations for the basis vectors of the orthogonal and symplectic Clifford algebras, provided that we identify:
\[ \hat{z}^a = (q^\mu, i \bar{q}^\mu), \quad \hat{\lambda}^a = (\gamma^\mu, i \bar{\gamma}^\mu) \]

We see that ‘quantization’ is in fact the replacements of the coordinates \( z^a, \lambda^a \) with the corresponding basis vectors. The only difference is in the factor \( i \) in front of \( \bar{q}_\mu \). Basis vectors, entering the action, are ‘quantum operators’, apart from the \( i \) in the relations (82).

Instead of coordinates \( \lambda^a \) and basis vectors \( \gamma^a \) it is convenient to introduce new coordinates and new basis vectors
\[ \lambda'^a \equiv \xi^a = (\xi^\mu, \bar{\xi}^\mu), \quad \xi^\mu = \frac{1}{\sqrt{2}} (\lambda^\mu - i \bar{\lambda}^\mu) \]
\[ \bar{\xi}^\mu = \frac{1}{\sqrt{2}} (\lambda^\mu + i \bar{\lambda}^\mu) \]
\[ \gamma'_a \equiv \theta_a = (\theta_\mu, \bar{\theta}_\mu), \quad \theta_\mu = \frac{1}{\sqrt{2}} (\gamma_\mu + i \bar{\gamma}_\mu) \]
\[ \bar{\theta}_\mu = \frac{1}{\sqrt{2}} (\gamma_\mu - i \bar{\gamma}_\mu) \]

In the new coordinates we have
\[ g'_{ab} = \gamma'_a \cdot \gamma'_b \equiv \theta_a \cdot \theta_b = \left( \begin{array}{c} 0 \\ \eta_{\mu\nu} \end{array} \right) \]
and the action (64) becomes
\[ I = \frac{1}{2} \int d\tau \left( \dot{z}^a J_{ab} z^b + \xi^a g'_{ab} \xi^b \right) = \frac{1}{2} \int d\tau \left( \dot{z}^a J_{ab} z^b + \dot{\xi}^\mu \eta_{\mu\nu} \bar{\xi}^\nu + \xi^\mu \eta_{\mu\nu} \xi^\nu \right) \]

Then the canonical momenta are
\[ p^{(x)}_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2} \eta_{\mu\nu} \dot{x}^\nu, \quad p^{(x)}_{\bar{\mu}} = \frac{\partial L}{\partial \dot{\bar{x}}^\mu} = -\frac{1}{2} \eta_{\mu\nu} \dot{\bar{x}}^\nu \]
\[ p^{(\xi)}_\mu = \frac{\partial L}{\partial \dot{\xi}^\mu} = \frac{1}{2} \eta_{\mu\nu} \dot{\xi}^\nu, \quad p^{(\xi)}_{\bar{\mu}} = \frac{\partial L}{\partial \dot{\bar{\xi}}^\mu} = \frac{1}{2} \eta_{\mu\nu} \dot{\bar{\xi}}^\nu \]

We see that now the canonically conjugate variables are \( (\xi^\mu, \frac{1}{2} \bar{\xi}^\mu) \) and \( (\bar{\xi}^\mu, \frac{1}{2} \xi^\mu) \). In old coordinates and basis vectors the situation was somewhat unfortunate, because the canonically conjugate variables were \( (\lambda^\mu, \frac{1}{2} \lambda_\mu) \) and \( (\bar{\lambda}^\mu, \frac{1}{2} \bar{\lambda}_\mu) \), i.e., the canonical momenta were essentially the same as the variables which they were conjugated to.
The commuting coordinates \( z^a = (x^\mu, \bar{x}^\mu) \equiv (x^\mu, p^\mu) \) and the symplectic basis vectors \( q_a = (q_\mu, \bar{q}_\mu) \) span a subspace that we will call the **bosonic subspace**. The Grassmann (anticommuting) coordinates \( \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu) \) and the orthogonal basis vectors \( \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu) \), or equivalently, the Grassman coordinates \( \xi^a = (\xi^\mu, \bar{\xi}^\mu) \) and the Witt basis vectors \( \theta_a = (\theta_\mu, \bar{\theta}_\mu) \), span a subspace that we will call the **fermionic subspace**. In Sec. 2.3 we pointed out that \((q^\mu, \bar{q}_\mu)\) can be represented by \((x^\mu, \partial/\partial x^\mu)\), whilst \((\theta^\mu, \bar{\theta}_\mu)\) can be represented by \((\xi^\mu, \partial/\partial \xi^\mu)\).

### 4.3 Completing the super phase space action

The action considered in the previous subsection is not complete. An additional term is needed. Let the \( \tau \)-derivative in the action

\[
I = \frac{1}{2} \int d\tau \dot{Z}^A G_{AB} Z^B \tag{89}
\]

be replaced with the covariant derivative

\[
\dot{Z}^A \rightarrow \dot{Z}^A + A^A_{\ B C} Z^C, \tag{90}
\]

where \( A^B_{\ C} \) depend on \( \tau \). So we obtain

\[
I = \frac{1}{2} \int d\tau (\dot{Z}^A + A^A_{\ B C} Z^C) G_{AB} Z^B \tag{91}
\]

which is invariant under \( \tau \)-dependent, i.e., **local**, ‘rotations’ of \( Z^A \), and where \( A^A_{\ B C} (\tau) \) are the corresponding compensating gauge fields. We will take into account only particular local ‘rotations’ of \( Z^A \), namely those that rotate between the canonically conjugate pairs of the (super) phase space variables \( Z^A \). In the bosonic subspace the latter local ‘rotations’ manifest themselves as local symplectic transformations of \( \text{Sp}(2) \). Then the corresponding compensating gauge fields are \( A^{ij}_{\ \mu \nu} = A^i_j \delta^\mu_{\ \nu} \), where the \( i, j = 1, 2 \) distinguishes \( x^\mu \) from \( \bar{x}^\mu \), since \( z^a = z^\mu \equiv (x^\mu, \bar{x}^\mu) \). There are only three independent gauge fields \( A^i_j (\tau) \); they represent a choice of gauge, as discussed in refs [17], and have thus the role of Lagrange multipliers. Analogous considerations hold in the fermionic subspace. No kinetic term for the gauge fields \( A^A_{\ B C} \) so designed is necessary.

The action (91) is the Bars action [17], generalized to the superspace (see also Ref. [18]). In particular, the extra term gives

\[
A^A_{\ B C} Z^C G_{AB} Z^B = \alpha p_\mu p^\mu + \beta \lambda^\mu p_\mu + \gamma \bar{\lambda}^\mu p_\mu, \tag{92}
\]

where \( \alpha, \beta, \gamma \) are Lagrange multipliers contained in \( A^A_{\ B C} \). Other choices of Lagrange multipliers \( A^A_{\ B C} \) are possible, and they give expressions that are different from Eq. (92). For the bosonic subspace, this was discussed in Refs. [17].
For the case (92), the action (91) gives the following constraints:

\[ p_\mu p^\mu = 0, \quad \lambda^\mu p_\mu = 0, \quad \bar{\lambda}^\mu p_\mu = 0. \]  

(93)

Mass comes from extra dimensions. It was shown by Bars [17] that the consistency of the constraints resulting from the action (91) requires one extra time like and one extra space like dimension of the bosonic subspace. Consequently, also the dimensionality of the fermionic subspace is adequately enlarged. So in this theory there are in fact extra dimensions that allow for the occurrence of mass in four dimensions.

Upon quantization, the classical constraints \( \lambda^\mu p_\mu = 0 \) and \( \bar{\lambda}^\mu p_\mu = 0 \) become two copies of the Dirac equation

\[ \hat{\lambda}^\mu \hat{p}_\mu \Psi = 0, \quad \text{and} \quad \hat{\bar{\lambda}}^\mu \hat{\bar{p}}_\mu \Psi = 0 \]  

(94)

where

\[ \hat{\lambda}^\mu = \gamma^\mu, \quad \text{and} \quad \hat{\bar{\lambda}}^\mu = i \bar{\gamma}^\mu. \]  

(95)

The state \( \Psi \) can be represented:

1) as a column \( \psi^\alpha(x) \),
2) as a function \( \psi(x^\mu, \xi^\mu) \).

The \( \hat{\lambda}^\mu = \gamma^\mu \) can be represented can be represented

1) as matrices \( (\gamma^\mu)^\alpha_\beta \),
2) as \( \xi^\mu + \frac{\partial}{\partial \xi^\nu} \).

We also have \( \hat{\bar{\lambda}}^\mu = i \bar{\gamma}^\mu \), which can be represented

1) as matrices \( i(\bar{\gamma}^\mu)^\alpha_\beta \),
2) as \( i(\xi^\mu - \frac{\partial}{\partial \xi^\nu}) \),

where \( \xi^\mu \) are Grassmann coordinates (see Eq.(83)).

4.4 Building up spinors from basis vectors

We have seen that upon quantization the classical Grassmann coordinates \( \lambda^\mu = (\lambda^\mu, \bar{\lambda}^\mu) \) become operators \( \hat{\lambda}^\mu = (\hat{\lambda}^\mu, \hat{\bar{\lambda}}^\mu) = (\gamma^\mu, i \bar{\gamma}^\mu) \), where \( \gamma^\mu, \bar{\gamma}^\mu \) are generators (basis vectors) of \( Cl(2n) \), or in general of \( Cl(p,q) \), \( p + q = 2n \). In the Witt basis (28) the basis vectors satisfy the fermionic anticommutation relations (29). Using the basis vectors \( \theta_\mu, \bar{\theta}_\mu \) we can build up spinors by taking a ‘vacuum’

\[ \Omega = \prod_\mu \bar{\theta}_\mu \quad \text{that satisfies} \quad \bar{\theta}_\mu \Omega = 0 \]  

(96)

and acting on it by the ‘creation’ operators \( \theta_\mu \). So we obtain a ‘Fock space’ basis for spinors, that contains \( 2^n \) independent elements:

\[ s_\alpha = (1_\Omega, \theta_\mu \Omega, \theta_\mu \theta_\nu \Omega, \theta_\mu \theta_\nu \theta_\rho \Omega, \theta_\mu \theta_\nu \theta_\rho \theta_\sigma \Omega, ...) \]  

(97)
in terms of which any state can be expanded:

$$Ψ_Ω = \sum \psi^α s_α$$  \hspace{1cm} (98)

With operators $θ_μ, \bar{θ}_μ$, defined above, we can construct spinors as the elements of a minimal left ideal of $Cl(p,q)$, $p + q = 2n$, where $n$ is dimension of spacetime. Notice that in the case of 4-dimensional spacetime, i.e., when $n = 4$, the symplectic or bosonic phase space is 8-dimensional. Therefore, if with every commuting phase space coordinate $z^α = (x^μ, \bar{x}^μ)$ one associates a corresponding Grassmann phase space coordinate $λ^α = (λ^μ, \bar{λ}^μ)$ (i.e., $ξ^α = (ξ^μ, \bar{ξ}^μ)$, in the Witt basis), then also the 'orthogonal' or fermionic phase space is 8-dimensional. In the usual theory of the spinning particle, only four Grassmann coordinates $λ^μ$ (or $ξ^μ$) are considered, which then leads to the spinors of $Cl(1,3)$. The usual theory thus imposes an asymmetry between the bosonic phase space, which has eight dimensions, and the fermionic phase space, which has four dimensions.

Definition (96) is just one of the possible definitions of vacuum. In general, vacuum can be defined as the product of $n$ factors, some of which are $θ_μ$ and some are $\bar{θ}_μ$:

$$Ω = θ_μ_1θ_μ_2...θ_μ_r\bar{θ}_{μ_{r+1}}\bar{θ}_{μ_{r+2}}...\bar{θ}_{μ_n}, \quad r = 0, 1, 2, ..., n$$  \hspace{1cm} (99)

The number of different vacua, $2^n$, is equal to the number of left minimal ideals of $Cl(2n)$ (more precisely, $Cl(p,q)$, $p + q = 2n$). With each vacuum so constructed, we can associate a different Fock space basis for spinors of the corresponding left ideal. The direct sum of all those different spinor spaces is the Clifford algebra $Cl(p,q)$, $p + q = 2n$, whose generic element, denoted $Ψ$, can be expanded according to

$$Ψ = ψ^{αβ}s_{αβ} ≡ ψ^{A^\dagger}s_{A^\dagger}.$$  \hspace{1cm} (100)

Here the first index, $α$, denotes the spinor components of a given ideal, which is denoted by the second index, $β$. It is convenient to denote the double index $αβ$ by a single index $A$. In the case, when we start from the 4-dimensional Minkowski spacetime with signature $(1,3)$, the signature of phase space is $(2,6)$, and we obtain $Cl(2,6)$.

With the Witt basis [84], we obtain the spinor spaces as subspaces of $Cl(2,6)$, but those spinor spaces do not contain the ordinary spinors of $M_{1,3}$. The latter spinors are constructed [8] [10] in terms of a different Witt basis, which contains the elements

$$θ_1 = \frac{1}{\sqrt{2}}(γ_0 + γ_3), \quad θ_2 = \frac{1}{\sqrt{2}}(γ_1 + iγ_2)$$

$$\bar{θ}_1 = \frac{1}{\sqrt{2}}(γ_0 - γ_3), \quad \bar{θ}_2 = \frac{1}{\sqrt{2}}(γ_1 - iγ_2).$$  \hspace{1cm} (101)
Then the full basis of the 8-dimensional phase space contains, in addition to the above elements, also the following elements:

\[
\begin{align*}
\theta_3 &= \frac{1}{\sqrt{2}}(\gamma_0 + \gamma_3), \\
\theta_4 &= \frac{1}{\sqrt{2}}(\gamma_1 + i \gamma_2) \\
\bar{\theta}_3 &= \frac{1}{\sqrt{2}}(\bar{\gamma}_0 - \bar{\gamma}_3), \\
\bar{\theta}_4 &= \frac{1}{\sqrt{2}}(\bar{\gamma}_1 - i \bar{\gamma}_2).
\end{align*}
\]  

(102)

Taking the scalar product of \(\Psi\) with the basis elements, we obtain the components:

\[
\langle \bar{s}^\dagger \Psi \rangle_S = \psi^\dagger, 
\]

(103)

where \(\langle \rangle_S\) denotes the scalar part of a Clifford algebra valued object. Operation \(\dagger\) reverses the order of vectors in a Clifford product and performs complex conjugation. More details can be found in Ref. [14].

Components \(\psi^\dagger\) represent a generic element \(\Psi\) of the Clifford algebra. If we take the scalar product of \(\Psi\) with basis elements of one left ideal, say, \(\alpha_1 \equiv \alpha\),

\[
\langle \bar{s}^\dagger \Psi \rangle_S = \psi^\alpha, 
\]

(104)

then the components \(\psi^\alpha\) represent a spinor state of this chosen left ideal.

A state \(\Psi\) can be represented by its components \(\psi^\dagger\), which are projections of \(\Psi\) onto the basis elements. Alternatively, since it is a Clifford algebra valued object, it can be represented by matrices

\[
\langle \bar{s}^\dagger \Psi \bar{s}_B \rangle_S = (\Psi)^\dagger_B. 
\]

(105)

These are \(2^n \times 2^n\) matrices, and they belong to a reducible representation of the Clifford algebra \(\text{Cl}(2n)\). If instead of the full basis \(\bar{s}_\alpha \equiv s_{\alpha\beta}\), we take the basis elements of a chosen left ideal, e.g., \(s_{\alpha 1} \equiv s_\alpha\), then we have

\[
\langle \bar{s}^\dagger \Psi \bar{s}_\beta \rangle_S = (\Psi)^\alpha_\beta = \psi^\dagger(\bar{s}_\alpha s_{\bar{s}_\beta})S = \psi^\dagger(s_{\bar{s}_\alpha})^\alpha_\beta. 
\]

(106)

This shows how every element of a Clifford algebra can be represented as an irreducible \(2^n \times 2^n\) matrix. For instance, \(\gamma_\mu\) can be represented by a matrix \((\gamma_\mu)^\alpha_\beta\), and \(\theta_\mu\) by \((\theta_\mu)^\alpha_\beta\).

5 On the representation of spinors in terms of the Grassmann coordinates

Operators \(\theta_\mu\), \(\bar{\theta}_\mu\), which are in fact nothing but the generator of the Clifford algebra, expressed in the Witt basis (28) or (84), can be represented either as matrices or in

1 For the sake of simplicity, we will use the simplified notation \(\text{Cl}(2n)\) for \(\text{Cl}(p, q), \ p + q = 2n\).
terms of the Grassmann coordinates and their derivatives:

$$\theta^\mu \to \sqrt{2}\xi^\mu, \quad \bar{\theta}_\mu \to \sqrt{2} \frac{\partial}{\partial \xi^\mu}$$  \hspace{1cm} (107)

A spinor state is then represented by a wave function $\psi(\xi^\mu)$. Since $\xi^\mu$ are Grassmann, anticommuting, coordinates, the Taylor expansion of $\psi(\xi^\mu)$ has a finite number of terms, namely $2^n$, which is the same as the number of components of a spinor in an $2n$-dimensional space.

The definition of vacuum $\bar{\theta}_\mu \Omega = 0$, $\mu = 1, 2, \ldots, n$, is now represented by the equation

$$\frac{\partial}{\partial \xi^\mu} \Omega(\xi^\mu) = 0$$  \hspace{1cm} (108)

whose solution is a constant, e.g., $\Omega(\xi^\mu) = 1$. A state (108) can then be represented as

$$\Psi_\Omega \rightarrow \psi(\xi^\mu) = \sum_{r=0}^{r=n} \psi_{\mu_1\mu_2\ldots\mu_r} \xi^{\mu_1} \xi^{\mu_2} \ldots \xi^{\mu_r}.$$  \hspace{1cm} (109)

Some other definition of vacuum, e.g., $\bar{\theta}_1 \bar{\theta}_2 \bar{\theta}_3 \ldots \bar{\theta}_n \Omega = 0$, is represented by the equation

$$\xi^1 \xi^2 \frac{\partial}{\partial \xi^3} \frac{\partial}{\partial \xi^4} \ldots \frac{\partial}{\partial \xi^n} \Omega(\xi^\mu) = 0,$$  \hspace{1cm} (110)

which has for a solution $\Omega(\xi^\mu) = \xi^1 \xi^2$. The corresponding state is then represented as

$$\Psi_\Omega \rightarrow \psi(\xi^\mu) = \left( \sum_{r=0}^{r=n} \psi_{\mu_1\mu_2\ldots\mu_r} \xi^{\mu_1} \xi^{\mu_2} \ldots \xi^{\mu_r} \right) \xi^1 \xi^2.$$  \hspace{1cm} (111)

In general, a vacuum state is given by (99), which can be represented by function

$$\Omega(\xi^\mu) = \xi^{\mu_1} \xi^{\mu_2} \ldots \xi^{\mu_s},$$  \hspace{1cm} (112)

and the corresponding spinor state is represented as

$$\Psi_\Omega \rightarrow \psi(\xi^\mu) = \left( \sum_{r=0}^{r=n} \psi_{\mu_1\mu_2\ldots\mu_r} \xi^{\mu_1} \xi^{\mu_2} \ldots \xi^{\mu_r} \right) \xi^{\mu_1} \xi^{\mu_2} \ldots \xi^{\mu_s}.$$  \hspace{1cm} (113)

The spinor states of every minimal left ideal of $\text{Cl}(p,q)$, $p + q = n$, can be thus represented in terms of a function of $n$ Grassmann coordinates $\xi^\mu$, $\mu = 1, 2, \ldots, n$.

So far we have considered functions of Grassmann coordinates $\xi^\mu$. Those functions can be considered as components of vectors (states) $\Psi_\Omega$ with respect to a basis, say $h(\xi)$:

$$\Psi_\Omega = \int d\xi^1 d\xi^2 \ldots d\xi^n \psi(\xi) h(\xi).$$  \hspace{1cm} (114)
The transformation coefficients

\[ \psi^{(\xi)} = \psi^{i(\xi)} h_{(\xi)}, \]  \tag{115} 

where \( \psi^{(\xi)} \equiv \psi(\xi) \equiv \psi(\xi^\mu) \), and \( h_{(\xi)} \equiv h(\xi) \equiv h(\xi^\mu) \). Here \( (\xi) \) is written as an index denoting a component of the vector \( \Psi_\Omega \), and if the index is repeated, then the integration over \( \xi \) in the sense of Eq. (114) is implied. Components \( \psi^{(\xi)} \) can be complex valued; then also the basis vectors are complex (i.e., consisting of two real components, as described in more detail later).

Basis vectors are assumed to satisfy

\[ h_{(\xi)}h_{(\xi')} = \frac{1}{2} (h_{(\xi)}h_{(\xi')} + h_{(\xi')}h_{(\xi)}) = \delta^{(\xi)}(\xi') \]  \tag{116} 

where \( \delta^{(\xi)}(\xi') \equiv \delta(\xi - \xi') \) is the delta ‘function’ in the Grassmann space, satisfying

\[ \delta^{(\xi)}(\xi') f(\xi') = f(\xi) \]  \tag{117} 

Indices are lowered and raised, respectively, by the metric \( \rho(\xi)(\xi') = h_{(\xi)}h_{(\xi')} \) and its inverse \( \rho^\dagger(\xi)(\xi') \). If \( h_{(\xi)} \) are complex, then \( h_{(\xi)} \) actually means \( h_{\mu(\xi)} = (h_{1(\xi)}, h_{2(\xi)}) \) or \( h_{\bar{\nu}(\xi)} = (h_{\bar{1}(\xi)}, h_{\bar{2}(\xi)}) \). Analogously, \( h_{i(\xi)} \) means \( h_{ii(\xi)} = (h_{11(\xi)}, h_{22(\xi)}) \) or \( h_{ii(\xi)} = (h_{\bar{1}\bar{1}(\xi)}, h_{\bar{2}\bar{2}(\xi)}) \). Metric \( \rho(\xi)(\xi') \) thus has an implicit index \( i = 1, 2 \).

In Eq. (115) we can perform the expansion in terms of \( \xi^\mu \), and use the properties \( \int d\xi = 0 \), \( \int d\xi = 1 \), which leads to

\[
\psi^{(\xi)} h_{(\xi)} = \int d^n \xi \left( \psi_0 + \frac{\partial \psi}{\partial \xi^\mu}_0 \right) \xi^\mu + \frac{\partial^2 \psi}{\partial \xi^\mu \partial \xi^\nu}_0 \xi^\mu \xi^\nu + \ldots + \frac{\partial^n \psi}{\partial \xi^\mu_0 \ldots \partial \xi^{n\mu}_0} \xi^{\mu_1} \ldots \xi^{\mu_n} \\
\times \left( h_0 + \frac{\partial h}{\partial \xi^\mu}_0 \right) \xi^\mu + \frac{\partial^2 h}{\partial \xi^\mu \partial \xi^\nu}_0 \xi^\mu \xi^\nu + \ldots + \frac{\partial^n h}{\partial \xi^\mu_0 \ldots \partial \xi^{n\mu}_0} \xi^{\mu_1} \ldots \xi^{\mu_n} \\
= \psi_0 h_0 + \frac{\partial \psi}{\partial \xi^\mu}_0 \left. \frac{\partial^{n-1} h}{\partial \xi^\mu \ldots \partial \xi^n_0} \right|_0 + \ldots + \frac{\partial^n h}{\partial \xi^1_0 \ldots \partial \xi^n_0} \xi^1 \ldots \xi^n_0 \\
= \psi^\alpha h_\alpha, \quad \alpha = 1, 2, \ldots, 2^n. \]  \tag{118} 

Here \( h_\alpha \) are discrete basis vectors spanning the \( 2^n \)-dimensional space \( \mathbb{C}^{2^n} \) in which spinors of one minimal left ideal live. The basis vectors \( h_\alpha \in \mathbb{C}^{2^n} \) are used here instead of the basis spinors \( s_\alpha \in Cl(2n) \) defined in Eq. (97).

One can get the contact with the usual language of quantum theory by the correspondence \( h_{(\xi)} = |\xi\rangle \), \( h^{(\xi)} = \langle \xi| \), and \( h^{(\alpha)} = |\alpha\rangle \), \( h^{\alpha} = \langle \alpha| \). Then

\[ \Psi_\Omega = \psi^{(\xi)} h_{(\xi)} = \int |\xi\rangle d^n \xi \langle \xi| \Psi_\Omega \rangle = \sum_\alpha |\alpha\rangle \langle \alpha| \xi d^n \xi \langle \xi| \Psi_\Omega \rangle = h_\alpha c^{\alpha(\xi)} \psi^{(\xi)} \]  \tag{119} 

The transformation coefficients \( c^{\alpha(\xi)} \equiv \langle \alpha| \xi \rangle \) can be read from Eq. (118).
Let us now consider a generic element \( \Psi \in Cl(2n) \). If we project \( \Psi \) onto the basis \( h^{(\xi)} \), then we obtain the wave function of Grassmann coordinates,

\[
h^{(\xi)} \cdot \Psi = h^{(\xi)} \cdot (\psi(\xi') h_{(\xi')}) = \psi^{(\xi)}.
\]

This is one possible representation of a state \( \Psi \) that is analogous to (104). In fact, it is a projection of \( \Psi \) onto one ideal. In analogy with Eq. (106), we can put \( \Psi \) into a sandwich between \( h^{(\xi)} \) and \( h^{(\xi')} \), and consider matrices

\[
\langle h^{(\xi)} \Psi h^{(\xi')} \rangle_s = (\Psi)^{(\xi)}_{(\xi')}
\]

According to Eqs. (97), (98), a state \( \Psi \) is expressed in terms of basis vectors \( \theta_\mu, \bar{\theta}_\mu \), which, in turn can be expressed as linear combinations of \( \gamma_\mu, \bar{\gamma}_\mu \) (generators of \( Cl(2n) \)). Therefore, the following matrix elements are of particular interest:

\[
\begin{align*}
\theta_\mu & \rightarrow \langle h^{(\xi)} \theta_\mu h^{(\xi')} \rangle_s = \xi_\mu \delta(\xi - \xi') \\
\bar{\theta}_\mu & \rightarrow \langle h^{(\xi)} \bar{\theta}_\mu h^{(\xi')} \rangle_s = \frac{\partial}{\partial \xi_\mu} \delta(\xi - \xi').
\end{align*}
\]

They can be used as basic blocks for building up a matrix (121) that represents a generic state \( \Psi \), spanned over a spinor basis of all \( 2^n \) minimal left ideals of \( Cl(2n) \).

Whereas a matrix \( (\Psi)^{(\xi)}_{(\xi')} \) can represent any element \( \Psi \) of \( Cl(2n) \), components \( \psi^{(\xi)} \) represent a spinor \( \Psi_\Omega \) of one ideal only. From a 1st rank spinor, i.e., a spinor of one left minimal ideal,

\[
\Psi_\Omega = \psi^\alpha h_\alpha = \psi^{(\xi)} h_{(\xi)},
\]

and its reverse, interpreted as a spinor of a right minimal ideal

\[
\Psi^\dagger_\Omega = \psi^{*\alpha} h^\dagger_\alpha = \psi^{*(\xi)} h^\dagger_{(\xi)},
\]

we can pass to a 2nd rank spinor by taking the tensor product

\[
\Psi_\Omega \otimes \Psi^\dagger_\Omega = \psi^{*\alpha} \psi'^{\beta} h^\dagger_\alpha \otimes h^\dagger_\beta = \psi^{(\xi)} \psi'^{*(\xi')} h_{(\xi)} \otimes h^\dagger_{(\xi')}.
\]

Once we have the bases \( h_\alpha \otimes h^\dagger_\beta \) and \( h_{(\xi)} \otimes h^\dagger_{(\xi')} \), we can span over them the space of objects whose components are \( \psi^{*\alpha} \) and \( \psi'^{*(\xi')} \), respectively. Components \( \psi^{*\alpha} \) represent a generic element of \( Cl(2n) \). Similarly, also components \( \psi^{(\xi)} \) represent a generic element of \( Cl(2n) \). Instead of the double indices we can use the single indices, and write \( \psi^{\alpha\beta} \equiv \psi^\tilde{A} \) (as we did in Eq. (100)), or \( \psi^{(\xi)} \equiv \psi^{(\xi)} \). Instead of the tensor product basis \( h_\alpha \otimes h^\dagger_\beta \) we can take the basis \( s_{\alpha\beta} \equiv s_{\tilde{A}} \), used in Eq. (100), and instead

\footnote{We define reversion so to include complex conjugation \( \ast \).}
of \( h(\xi) \otimes h^\dagger(\xi') \) we can take another basis, denoted \( h(\xi)(\xi') \equiv h(\xi,\xi') \), which spans the space of Grassmann functions. An element of the latter space is

\[
\Psi = \psi(\xi,\xi') h(\xi,\xi')
\]

(126)

Whereas a spinor of one left ideal is described by a function \( \psi(\xi) \equiv \psi(\xi) \) of \( n \) Grassmann coordinates \( \xi \equiv \xi^\mu, \mu = 1, 2, ..., n \), a generic element of \( Cl(2n) \) is described by a function \( \psi(\xi,\xi') \equiv \psi(\xi,\xi') \) of \( 2n \) Grassmann coordinates \( (\xi,\xi') \equiv (\xi^\mu,\xi'^\mu) \), \( \mu,\nu = 1, 2, ..., n \). If we perform an expansion, analogous to that of Eq. (118), we obtain \( 2^{2n} \) independent components.

In general, a wave function depends on commuting coordinates \( x^\mu \) as well, because it has to form a representation, not only of the orthogonal basis vectors \( (\gamma_\mu, \bar{\gamma}_\mu) \) or \( (\theta_\mu, \bar{\theta}_\mu) \), but also of the symplectic basis vectors \( (q_\mu, \bar{q}_\mu) \equiv (q_\mu^{(x)}, q_\mu^{(p)}) \) according to Eq. (33). Therefore, instead of Eq. (115), we have

\[
\Psi_\Omega = \psi(x,\xi) h(x,\xi),
\]

(127)

where

\[
h(x,\xi) \cdot h(x',\xi') = \delta(x,\xi)(x',\xi').
\]

(128)

A state \( \Psi_\Omega \) is now an element of an infinite dimensional space spanned over a basis \( h(x,\xi) \), the components being a wave function \( \psi(x,\xi) \equiv \psi(x,\xi) \). In the next section we will formulate foundations of the field theory based on the orthogonal and symplectic Clifford algebras.

6 Description of fields

6.1 Bosonic fields

In Sec. 3, Eq. (35), we considered the phase space action, \( I[z^\alpha], z^\alpha = (x^\mu, p^\mu) \), of a point particle. The latter action is a functional of coordinates and momenta. If, by the equations of motion, the momenta can be expressed in terms of coordinates, then we can obtain an equivalent action, \( I[x^\mu] \), which is a functional of coordinates only. For instance, instead of the phase space action (52), we obtain

\[
I[x^\mu] = \frac{1}{2} \int d\tau \frac{\dot{x}^\mu \dot{x}_\mu}{\lambda}.
\]

(129)

We can interpret the coordinates \( x^\mu, \mu = 1, 2, ..., n \), in several different ways, as we did in Sec.3.2. For instance, the coordinates \( x^\mu \) can be interpreted to denote position (of a non relativistic particle) in \( n \)-dimensional space, \( \mathbb{R}^n \). The latter space
is a vector space, spanned over the set of \(n\)-basis vectors, \(\gamma_\mu\), that can be generators of the Clifford algebra \(\text{Cl}(n)\), satisfying

\[
\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \delta_{\mu\nu},
\]

(130)

so that the action (129) reads

\[
I[x^\mu] = \frac{1}{2} \int d\tau \frac{\dot{x}^\mu \gamma_\mu \gamma_\nu \dot{x}^\nu}{\lambda}
\]

(131)

Position in \(\mathbb{R}^n\) is thus described by a vector \(x = x^\mu \gamma_\mu\) of the orthogonal Clifford algebra \(\text{Cl}(n)\). Instead of the configuration space, and an action \(I[x^\mu]\), such as (129), we can consider the corresponding phase space, whose points are described by symplectic vectors \(z = z^a q_a\), considered in Sec. 2, the action \(I[x^\mu, p^\mu]\) being, e.g., Eq. (52) or, in general, Eq. (35).

Analogously, we can consider a state vector \(\Phi = \phi(x) h(x)\), spanned over an infinite set of basis vectors \(h(x)\) that satisfy the relations of a generalized orthogonal infinite dimensional Clifford algebra

\[
h(x) \cdot h(x') = \delta(x(x')).
\]

(132)

The latter vector is an infinite dimensional analog of a vector \(x = x^\mu \gamma_\mu\) considered in previous paragraph. Equations of “motion” for \(\phi(\tau, x) \equiv \phi(x(\tau))\) can be derived from an action functional

\[
I[\phi(\tau, x)] = \int d\tau \mathcal{L}(\phi, \partial_\tau \phi, \partial_\mu \phi),
\]

(133)

which can be any one known in the field theory, e.g., the scalar field action

\[
I[\phi] = \frac{1}{2} \int d\tau d^n x \left( (\partial_\tau \phi)^2 + \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right).
\]

(134)

where \(x^\mu\) are coordinates of space \(\mathbb{R}^n\), \(\partial_\tau \equiv \partial/\partial \tau\), and \(\partial_\mu \equiv \partial/\partial x^\mu\). The latter action can be written in the form

\[
I[\phi] = \frac{1}{2} \int d\tau \left( \partial_\tau \phi(x) h(x) h(x') \partial_\tau \phi(x') + \partial_\mu \phi(x) h(x) h(x') \partial^\mu \phi(x') - m^2 \phi(x) h(x) h(x') \phi(x') \right),
\]

(135)

which is an infinite dimensional analog of the action (131).

Introducing the momentum \(\Pi(x) = \delta \mathcal{L}/\delta \dot{\phi} = \dot{\phi}\) and the Hamiltonian \(H = \int d^n x (\Pi(\tau, x) \dot{\phi}(\tau, x) - \mathcal{L})\) we obtain the phase space action

\[
I[\phi, \Pi] = \int d\tau \left[ \int d^n x \Pi \dot{\phi} - H \right] = \int d\tau d^n x \left[ \Pi \dot{\phi} - \dot{\Pi} \phi - (\Pi^2 - \partial^\mu \phi \partial_\mu \phi + m^2 \phi^2) \right].
\]

(136)
The above equations also hold for complex fields, if an implicit index, \( c = 1, 2 \), denoting, e.g., the real and imaginary component, is assumed, with understanding that \( \phi_1^2 \equiv \phi_1^* \phi_1 = \phi_1^1 \phi_1^1 + \phi_1^2 \phi_1^2 \), or \( \phi_2^2 \equiv \phi_2^* \phi_2 = \frac{1}{2} (\phi_2^* \phi_2 + \phi_2^* \phi_2) \).

The action \( \mathcal{A} \) is a particular case of a generic phase space action for bosonic fields that we will consider in the following.

### 6.1.1 A generic phase space action for bosonic fields and its quantization

Let us consider a vector \( \Phi = \phi^i(x) k_i(x) \), \( i, j = 1, 2 \), in an infinite dimensional space, and assume that the basis vectors \( k_i(x) \) satisfy the relations of a symplectic Clifford algebra:

\[
k_i(x) \wedge k_j(x') \equiv \frac{1}{2} (k_i(x) k_j(x') - k_j(x') k_i(x)) = J_{i(x)j(x')}. \tag{137}\]

The symplectic metric has the following form:

\[
J_{i(x)j(x')} = \begin{pmatrix} 0 & \delta_{ij}(x-x') \\ -\delta_{ij}(x-x') & 0 \end{pmatrix}, \tag{138}\]

where \( \delta_{ij}(x-x') \equiv \delta(x-x') \). Shortly, \( J_{i(x)j(x')} = \epsilon_{ij} \delta(x-x') \), with \( \epsilon_{ij} = -\epsilon_{ji} \).

Components \( \phi^i(x) = (\phi^1(x), \phi^2(x)) \equiv (\phi(x), \Pi(x)) \) are analogous to \( z^a = (x^\mu, p^\mu) \) of Secs. 2 and 3, i.e., to coordinates and momenta.

The symplectic vector reads explicitly

\[
\Phi = \phi^i(x) k_i(x) = \phi^1(x) k_1(x) + \phi^2(x) k_2(x) = \phi^j(x) k_j(x) + \Pi^j(x) k_j(x) \tag{139}\]

The action is now

\[
I = \frac{1}{2} \int d\tau \left( \dot{\phi}^i(x) J_{i(x)j(x')} \phi^{j(x')} + \phi^i(x) J_{i(x)j(x')} \dot{\phi}^{j(x')} \right), \tag{140}\]

where the fields \( \phi^i(x) \) are assumed to be functions of \( \tau \), so that \( \phi^i(x) \) means \( \phi^i(x)(\tau) \equiv \phi^i(\tau, x) \). It gives the following equations of motion:

\[
\dot{\phi}^i(x) = J^{i(x)j(x')} \frac{\partial H}{\partial \phi^{j(x')}}, \tag{141}\]

where \( H = \phi^i(x) K_{i(x)j(x')} \phi^{j(x')} \equiv \int d^nx d'^nx' \phi^i(x) K_{ij}(x, x') \phi^j(x') \), and \( \partial/\partial \phi^{j(x')} \equiv \delta/\delta \phi^j(x') \). If we now follow the analogous procedure as in Eqs. \( \text{[40]} \) - \( \text{[50]} \), we arrive at the equations of motion for the operators:

\[
\hat{k}^{i(x)} k_{i(x)j(x')} = [k_{i(x)j(x')}, \hat{H}], \tag{142}\]

where

\[
\hat{H} = \frac{1}{2} k^{i(x)} K_{i(x)j(x')} k^{j(x')}. \tag{143}\]
The latter operator equation of motion can be derived from the action

\[ I = \frac{1}{2} \int d\tau \left( \dot{k}^i(x) J_{ij}(x') k^j(x') + k^i(x) K_{ij}(x') k^j(x') \right). \]  

(144)

We obtain

\[
\dot{k}^i(x) = \{ k^i(x), \hat{H} \}_{\text{P.B.}} = \frac{\partial k^i(x)}{\partial k^m(x')} J^{m(n)} \frac{\partial \hat{H}}{\partial k^n(x')},
\]

\[ = J^{i(n)}(x') \frac{\partial \hat{H}}{\partial k^n(x')} = [k^i(x), \hat{H}]. \]

(145)

In the above calculation we have assumed that \( \frac{\partial k^i(x)}{\partial k^m(x')} = \delta^i_m(x'). \) Indices are lowered and raised by the symplectic metric \( J^i(x) J_j(x') \) and its inverse \( J^i(x) J_j(x'). \)

### 6.1.2 Some particular cases

The general form of the action (140) or its quantum version (144) contains particular cases that depend on choice of \( K_{ij}(x', x) \), and on the space the coordinates \( x \) are associated with. Let us consider some of the cases:

(i) For instance, let \( x \equiv x^\mu, \mu = 1, 2, 3, \) be three spatial coordinates, and \( \tau = t \) the non relativistic time. Instead of \( x^\mu \), we will now write \( x^r, r = 1, 2, 3, \) so to have a closer contact with the conventional notation.

a) If we take

\[
K_{ij}(x,x') = \begin{pmatrix}
(m^2 + \partial_r \partial_r) \delta(x-x') & 0 \\
0 & \delta(x-x')
\end{pmatrix},
\]

(146)

then the action (140) becomes the scalar field phase space action (136).

b) If we take

\[
K_{ij}(x,x') = \left( -\frac{1}{2m} \partial_r \partial_r + V(x) \right) \delta(x-x') g_{ij}, \quad g_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

(147)

then the action (140) becomes

\[
I = \int dt d^3x \left[ \frac{1}{2} i \dot{\phi}^i(t,x) \epsilon_{ij} \phi^j(t,x) + i \phi^i(t,x) \left( -\frac{1}{2m} \partial_r \partial_r + V(x) \right) g_{ij} \phi^j(t,x) \right],
\]

(148)

where \( \phi^i(t,x) = (\phi(t,x), \Pi(t,x)) = (\phi(t,x), i\phi^*(t,x)) \). In Eq. (148) we have the action for the classical Schrödinger field.

Similarly, using (144), the action for operators becomes

\[
I = \int dt d^3x \left[ \frac{1}{2} \dot{k}^i(x) \epsilon_{ij} k^j(x) + i k^i(x) \left( -\frac{1}{2m} \partial_r \partial_r + V(x) \right) g_{ij} k^j(x) \right],
\]

(149)
which is the action for the quantized Schrödinger field.

(ii) Alternatively, let \( x \equiv x^\mu, \mu = 0, 1, 2, 3, \) be four coordinates of spacetime, and \( \tau \) a Lorentz invariant evolution parameter. Then the choice

\[
K_{i(x)j(x')} = \left( -\frac{1}{2\Lambda} \partial^\mu \partial_\mu \right) g_{ij} \delta(x - x'), \quad g_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]  

for a constant \( \Lambda \), gives the Stueckelberg action \([19]\)

\[
I = \int d\tau d^4x \left\{ \frac{1}{2} \dot{\phi}^i(x) \epsilon_{ij} \phi^j(x) + i\phi^i(x) \left( -\frac{1}{2\Lambda} \partial^\mu \partial_\mu \right) g_{ij} \phi^j(x) \right\},
\]

and the analogous action for the quantized field \( k^i(x) \).

### 6.2 Fermionic fields

Let us now consider the vector \( \Psi = \psi^i(x) h_i(x), i = 1, 2, \) and assume that the basis vectors \( h_i(x) \) satisfy the relations of an orthogonal Clifford algebra

\[
h_i(x) \cdot h_j(x') = \frac{1}{2} (h_i(x) h_j(x') + h_j(x') h_i(x)) = \rho_{i(x)j(x')},
\]

where \( \rho_{i(x)j(x')} \) is an orthogonal metric. Its explicit form depends on a chosen basis. In a particular basis the metric can be

\[
\rho'_{i(x)j(x')} = \delta_{ij} \delta(x(x')), \quad \rho_{i(x)j(x')} = \begin{pmatrix} 0 & \delta(x(x')) \\ \delta(x(x')) & 0 \end{pmatrix}.
\]

This can be transformed into another basis, namely the Witt basis, in which

\[
\rho_{i(x)j(x')} = \begin{pmatrix} 0 & \delta(x(x')) \\ \delta(x(x')) & 0 \end{pmatrix}.
\]

We assume that an implicit spinor index, \( \alpha = 1, 2, ..., 2^n \), occurs in the expressions. Thus, \( \Psi = \psi^i(x) h_i(x) \equiv \psi^i(x) h_{i\alpha}(x) \). Components \( \psi^i(x) = (\psi^1(x), \psi^2(x), ...), \psi^i(x) = (\psi^1(x), \psi^2(x), ... \psi^n(x), \pi(x)) \) are Grassmann valued phase space variables, and are analogous to \( \xi^a \equiv \xi^{i\mu} = (\xi_i, \xi^\mu) \) considered in Sec. 4.

The fermionic field action that corresponds to the bosonic field action \((140)\) is

\[
I = \int d\tau \left( \psi^i(x) \rho_{i(x)j(x')}(\dot{\psi}^j(x') + \psi^j(x) H_{i(x)j(x')} \psi^j(x')) \right),
\]

If we now repeat a procedure, analogous to that in Eqs. \((140) - (145)\), then we obtain the following equations of motion for the basis vectors

\[
\dot{h}^i(x) = \{h^i(x), \hat{H}\},
\]
where \( \hat{H} = h^{i(x)}H_{i(x)j(x')}h^{j(x')} \), and brace means the anticommutator.

Let, in particular, be \( x \equiv x' \), \( r = 1, 2, 3 \), and \( \tau = t \). If \( \rho_{i(x)j(x')} \) is given by Eq. (154), and

\[
H_{i(x)j(x')} = \begin{pmatrix} 0 & -(\alpha^r \partial_r + im)\delta(x - x') \\ (\alpha^r \partial_r + im)\delta(x - x') & 0 \end{pmatrix},
\]

where \( \alpha^r = \gamma^0 \gamma^r \) are hermitian matrices in the spinorial indices, i.e., \( (\alpha^r)^\beta_\alpha = (\alpha^r)_{\alpha\beta} \), then the action (155) becomes

\[
I = \int dt \, \mathrm{d}^3x \left[ \frac{1}{2}(\pi \psi - \bar{\pi} \psi) + \pi \gamma^0 \gamma^r \partial_r \psi - m\pi \psi \right],
\]

where we have taken into account the anticommutativity of \( \psi \) and \( \pi \). If we write \( \pi = i\psi^\dagger \) and omit the total derivative \( (\mathrm{d}/\mathrm{d}t)(\pi \psi) \), we obtain the usual phase space action for the Dirac field. The latter field can be the usual 2\(^{n/2}\)-component Dirac field in \( n \)-dimensional spacetime, or it can be a field with \( 2^n \) components, considered in Secs. 4.4 and 5.

Instead of a discrete spinor index \( \alpha \) we can take the Grassmann coordinates \( \xi \) and write the vector \( \Psi \) in the form \( \Psi = \psi^{i(x,\xi)}h_{i(x,\xi)} \), where the basis vectors satisfy

\[
h_{i(x,\xi)} : \hat{h}_{j(x',\xi')} = \rho_{i(x,\xi)j(x',\xi')} = \delta_{ij} \delta(x(x') \delta(\xi(\xi')).
\]

If, as a model, we take two Grassmann coordinates only, \( \xi \equiv (\xi^1, \xi^2) \), then \( \Psi \) is a usual 4-component spinor (see Eq. (101)). A generic fermionic field action can be written as

\[
I = \frac{1}{2} \int \mathrm{d}\tau \left( \psi^{i(x,\xi)}h_{i(x,\xi)}j^{j(x',\xi')}j^{j(x',\xi')} + \psi^{i(x,\xi)}H_{i(x,\xi)j(x',\xi')} \psi^{j(x',\xi')} \right).
\]

In particular, for \( H_{i(x,\xi)j(x',\xi')} \) we can take a matrix, analogous to (157), in which objects \( \gamma^\mu \) are represented in terms of \( (\xi \pm \partial/\partial\xi)\delta(\xi - \xi') \).

We can consider, for instance, the case (101), and represent

\[
\theta^1 \rightarrow \xi^1 \delta(\xi - \xi'), \quad \theta^2 \rightarrow \xi^2 \delta(\xi - \xi'), \quad \theta_1 \rightarrow \frac{\partial}{\partial\xi^1} \delta(\xi - \xi'), \quad \theta_2 \rightarrow \frac{\partial}{\partial\xi^2} \delta(\xi - \xi'),
\]

and invert the relations (101) so to express \( \gamma^\mu \) in terms of \( \theta^1, \theta^2, \theta_1, \theta_2 \). Then the action (160) is equivalent to the usual action for the 4-component spinor field.

Alternatively, we can consider the case (84), in which the number of vectors, spanning the fermionic phase space is the same as the number of vectors spanning the bosonic phase space, namely eight. Then \( \xi \equiv \xi^\mu, \mu = 0, 1, 2, 3, \) and a vector \( \Psi = \psi^{i(x,\xi)}h_{i(x,\xi)} \), for a fixed \( i \), represents a 16-component spinor field.
6.3 Poisson brackets

In Sec. 2.2 we have observed that in the symplectic case the Poisson bracket of the phase space coordinates is equal to the wedge product (i.e., to one half times the commutator) of the corresponding symplectic basis vectors. Similarly, in the orthogonal case, the Poisson bracket of the phase space variables is equal to one half of the anticommutator of the corresponding orthogonal basis vectors. Analogous holds for fields.

In **symplectic case** we have

\[
\{ f(\phi^i(x)), g(\phi^j(x)) \}_{PB} = \frac{\partial f}{\partial \phi^i(x)} j^i(x) j^j(x') \frac{\partial g}{\partial \phi^j(x')} = \frac{\partial f}{\partial \phi^i(x)} \frac{1}{2} [k^i(x), k^j(x')] \frac{\partial g}{\partial \phi^j(x')} \]  

(162)

In particular, if \( f(\phi^i(x)) = \phi^k(x'') \) and \( g(\phi^j(x)) = \phi^\ell(x'''') \), then

\[
\{ \phi^k(x''), \phi^\ell(x''') \}_{PB} = J^{k(\ell')(x''')} = \frac{1}{2} [k^k(x''), k^\ell(x''')] \]  

(163)

The Poisson bracket of two classical fields is equal to the symplectic metric which, in turn, is equal to the wedge product (i.e., 1/2 times the commutator) of two symplectic basis vectors. The latter vectors are bosonic field operators, and from Eq. (163) we see that the canonical commutation relations are in fact the relations (137) of a symplectic Clifford algebra.

In **orthogonal case** we have

\[
\{ f(\psi^i(x)), g(\psi^j(x)) \}_{PB} = \frac{\partial f}{\partial \psi^i(x)} \rho^i(x) j^j(x') \frac{\partial g}{\partial \psi^j(x')} = \frac{\partial f}{\partial \psi^i(x)} \frac{1}{2} \{ h^i(x), h^j(x') \} \frac{\partial g}{\partial \psi^j(x')} \]  

(164)

In particular, if \( f(\psi^i(x)) = \psi^k(x'') \) and \( g(\psi^i(x)) = \psi^\ell(x'''') \), then

\[
\{ \psi^k(x''), \psi^\ell(x''') \}_{PB} = \rho^{k(\ell')(x''')} = \frac{1}{2} \{ k^k(x''), k^\ell(x''') \} \]  

(165)

The Poisson bracket of two classical fields is equal to the orthogonal metric. On the other hand, the orthogonal metric is equal to the symmetrized product (given by the anticommutator) of two orthogonal basis vectors. The latter vectors are fermionic field operators, and Eq. (165) shows that the canonical anticommutation relations for fermionic fields are in fact the relations (152) of an orthogonal Clifford algebra.

Spinor indices \( \alpha, \beta \) are not explicitly displayed in Eqs. (164),(165). Another possibility is to rewrite the latter equations by replacing \( \psi^i(x) \equiv \psi^{i\alpha}(x) \) with \( \psi^i(x, \xi) \).
6.4 Generalization to ‘superfields’

Both actions (140), (155) can be unified into a single action by introducing a ‘superfield’

\[ \Psi = \psi^A h_A \]  \hspace{1cm} (166)

where \( \psi^A = (\phi^i(x), \psi^i(x)) \) and \( h_A = (k_i(x), h_i(x)) \). So we have

\[ I[\psi^A] = \frac{1}{2} \int d\tau (\dot{\psi}^A G_{AB} \psi^B + \psi^A H_{AB} \psi^B) \]  \hspace{1cm} (167)

where

\[ \langle h_A h_B \rangle_S = G_{AB} = \begin{pmatrix} J_{i(x)j(x')} & 0 \\ 0 & \rho_{i(x)j(x')} \end{pmatrix} \]  \hspace{1cm} (168)

If

\[ H_{AB} = \begin{pmatrix} K_{i(x)j(x')} & 0 \\ 0 & H_{i(x)j(x')} \end{pmatrix} \]  \hspace{1cm} (169)

then the action (168) is exactly the sum of the bosonic field action (140) and the fermionic field action (155). In general, \( H_{AB} \) may have non vanishing off diagonal terms which are responsible for a coupling between the fermionic and the bosonic fields.

Again we kept the spinor indices hidden, so that \( \psi^i(x) \) actually meant \( \psi^{i\alpha(x)} \). If, instead, we take \( \psi^i(x,\xi) \), then the discrete spinor components arise from the expansion of \( \psi^i(x,\xi) \equiv \psi^i(x,\xi) \) in terms of the Grassmann coordinates \( \xi \equiv \xi^\mu \). Our superfield can then be written as

\[ \Psi = \phi^i(x) k_i(x) + \psi^i(x,\xi) h_i(x,\xi) \]  \hspace{1cm} (170)

But in such expression for a superfield there is an asymmetry between the bosonic and the fermionic part. A more symmetric expression is

\[ \Psi = \phi^i(x,\xi) k_i(x,\xi) + \psi^i(x,\xi) h_i(x,\xi) \]  \hspace{1cm} (171)

so that both parts contains the commuting coordinates \( x \) and the anticommuting (Grassmann) coordinates \( \xi \). Then both fields, the commuting \( \phi^i(x,\xi) \) and the anticommuting \( \psi^i(x,\xi) \), can form a representation for the symplectic basis vectors \( (q^\mu, \bar{q}_\mu) \rightarrow (x^\mu, \partial/\partial x^\mu) \) and for the orthogonal basis vectors \( (\theta^\mu, \bar{\theta}_\mu) \rightarrow (\xi^\mu, \partial/\partial \xi^\mu) \). The action is then that of Eq. (167) in which the matrices \( G_{AB} \) and \( H_{AB} \) are given by suitably generalized Eqs. (168) and (169), in which instead of \( K_{i(x)j(x')} \) and \( H_{i(x)j(x')} \) we have \( K_{i(x,\xi)(x',\xi')} \) and \( H_{i(x,\xi)(x',\xi')} \), respectively.

But isn’t such a theory in conflict with the connection between spin and statistic that comes from the requirement of microcausality? Not necessarily. Had we taken for the submatrix \( K_{i(x,\xi)(x',\xi')} \) in Eq. (169) a “Dirac equation like matrix” such as (157), then the action (167) would contain a part with a bosonic field described by
the Dirac action. This would be problematic. But a scalar field like matrix (146) would pose no problem. The fact that the bosonic field depends also on Grassmann coordinates, $\xi^\mu$, means that we have a number of bosonic fields, coming from the expansion of $\phi^i(x^\mu, \xi^\mu)$ in terms of $\xi^\mu$. Each field is in agreement with microcausality, i.e., their commutators at different times all vanish outside the light cone.

Such ‘superfield’ (171) with the action (167) seems to be a natural generalization of the point particle in superspace considered in Sec. 4. A deeper investigation of this topics is beyond the scope of the present paper.

6.5 Fock space states

We will now explicitly show how the basis vectors, $h_i(x,\xi)$ and $k_i(x)$, that span, respectively, an orthogonal and a symplectic phase space, behave as creation and annihilation operators of a quantum field theory.

6.5.1 Fermionic fields: Generators of orthogonal Clifford algebras

Let

$$\Psi = \psi^i(x,\xi) h_i(x,\xi) = \psi^{\alpha\alpha}(x) h_{\alpha\alpha}(x), \quad i = 1, 2, \quad \alpha = 1, 2, \ldots, 2^n,$$

(172)

be a vector of the fermionic subspace of the total (super) phase space, where the basis vectors satisfy

$$h_i(x,\xi) \cdot h_j(x',\xi') = \delta_{ij} \delta(x)(x') \delta(\xi)(\xi'),$$

(173)

or

$$h_{\alpha\alpha}(x) \cdot h_{\beta\beta}(x') = \delta_{\alpha\beta} \delta(x)(x') \delta(\xi)(\xi').$$

(174)

Introducing the Witt basis,

$$h_{\alpha}(x) = \frac{1}{\sqrt{2}} (h_{1\alpha}(x) + ih_{2\alpha}(x)), \quad \bar{h}_{\alpha}(x) = \frac{1}{\sqrt{2}} (h_{1\alpha}(x) - ih_{2\alpha}(x)),$$

(175)

(176)

the anticommutation relations (173), (174) become the familiar relations for fermionic creation and annihilation operators:

$$h_{\alpha}(x) \cdot \bar{h}_{\beta}(x') = \delta_{\alpha\beta} \delta(x)(x'),$$

(177)

$$h_{\alpha}(x) \cdot h_{\beta}(x') = 0, \quad \bar{h}_{\alpha}(x) \cdot \bar{h}_{\beta}(x') = 0.$$  

(178)

A possible vacuum state is the product of all operators $\bar{h}_{\alpha}(x)$:

$$\Omega = \prod_{\alpha,x} \bar{h}_{\alpha}(x),$$

(179)

$$\bar{h}_{\alpha}(x) \Omega = 0.$$  

(180)
A basis of the Fock space is then
\[ \{ h_{\alpha_1}(x_1) h_{\alpha_2}(x_2) \cdots h_{\alpha_r}(x_r) \Omega \}, \quad r = 0, 1, 2, 3, \ldots \]  
(181)

For a fixed point \( x \) we have
\[ \{ h_{\alpha_1}(x) h_{\alpha_2}(x) \cdots h_{\alpha_r}(x) \Omega \}, \quad r = 0, 1, 2, \ldots, D, \]  
(182)
or, more explicitly,
\[ \{ \Omega, h_{\alpha_1}(x) \Omega, h_{\alpha_1}(x) h_{\alpha_2}(x) \Omega, \ldots, h_{\alpha_1}(x) h_{\alpha_2}(x) \cdots h_{\alpha_D}(x) \Omega \}. \]  
(183)

Here \( D = 2^n \) is the number of the basis vectors \( h_{\alpha}(x) \), i.e., the creation operators \( [175] \) that arise from expansion of the basis vectors \( h(x,\xi) \) in terms of \( n \) Grassmann coordinates \( \xi \equiv \xi^\mu, \quad \mu = 1, 2, \ldots, n \). We now also consider the case in which the number \( n \) of anticommuting (Grassmann) coordinates is different from the number \( n \) of the commuting coordinates, \( x^\mu, \quad \mu = 1, 2, \ldots, n \). This is distinguished by using normal and bold symbols. In particular, \( D = 16 \), if \( n = 4 \) (which was our choice), and \( D = 4 \), if \( n = 2 \) (which is the usual choice). The dimension of the Fock space spanned over the basis (183) is \( 2^D \). In particular,
\[ 2^D = \begin{cases} 2^{16} & \text{if } n = 4 \\ 2^4 & \text{if } n = 2 \end{cases} \]  
(184)

Besides the choice of vacuum (179), there are other possible choices, analogous to those considered in Sec. 4.4, for example,
\[ \Omega = \prod_{\alpha,x} h_{\alpha}(x) \Omega, \quad h_{\alpha(x)} \Omega = 0. \]  
(185)

More generally, we have
\[ \Omega = \left( \prod_{\alpha \in R_1, x} \bar{h}_{\alpha(x)} \right) \left( \prod_{\alpha \in R_2, x} h_{\alpha(x)} \right). \]  
(186)

Here \( R = R_1 \cup R_2 \) is the set of indices \( \alpha = 1, 2, \ldots, D \), where \( R_1 \) and \( R_2 \) are subsets of indices, e.g., \( R_1 = \{ 1, 4, 7, \ldots, D \} \) and \( R_2 = \{ 2, 3, 5, 6, \ldots D - 1 \} \). Altogether, with respect to all such arrangements of indices \( \alpha \), there are \( 2^D \) possible vacua, giving \( 2^D \times 2^D = 2^{2D} \) basis states. Since we consider operators \( h_{\alpha}(x) \bar{h}_{\alpha}(x) \) at a fixed point \( x \), we can factor out from \( \Omega \) the part due to the product over \( x \), and so those \( 2^{2D} \) states span a Clifford algebra \( Cl(2D) \). In particular, we have \( Cl(32) \), if \( n = 4 \), and \( Cl(8) \), if \( n = 2 \). To sum up, a Clifford algebra \( Cl(2D) \) is generated at every point \( x \) by a set of \( 2D \) basis vectors, \( (h_{\alpha}(x), \bar{h}_{\alpha}(x)) \), or equivalently, by \( (h(x,\xi), \bar{h}(x,\xi)) \), which are fermionic creation and annihilations operators.
If we do not factor out from $\Omega$ the part due to the product over points $x \in \mathbb{R}^n$, then it turns out that there are many other possible definitions of vacuum, such as

$$\Omega = \left( \prod_{\alpha, x \in \mathcal{R}_1} \bar{h}_\alpha(x) \right) \left( \prod_{\alpha, x \in \mathcal{R}_2} h_\alpha(x) \right),$$  \hspace{1cm} (187)$$

depending on a partition of $\mathbb{R}^n$ into two domains $\mathcal{R}_1$ and $\mathcal{R}_2$ so that $\mathbb{R}^n = \mathcal{R}_1 \cup \mathcal{R}_2$. Instead of the configuration space, we can take the momentum space, and consider, e.g., positive and negative momenta $p$. If $\mathbb{R}^n$ is the Minkowski spacetime, then we can have a vacuum of the form

$$\Omega = \left( \prod_{\alpha, p^0 > 0} \bar{h}_\alpha(p, 0) \right) \left( \prod_{\alpha, p^0 < 0} h_\alpha(p, 0) \right),$$  \hspace{1cm} (188)$$

which is annihilated according to

$$\bar{h}_\alpha(p^0 > 0) \Omega = 0, \quad h_\alpha(p^0 < 0) \Omega = 0,$$  \hspace{1cm} (189)$$

whereas one particle states are created according to

$$h_\alpha(p^0 > 0) \Omega, \quad \bar{h}_\alpha(p^0 < 0) \Omega.$$  \hspace{1cm} (190)$$

With respect to the above vacuum (188), one kind of particles are created by positive energy unbarred operators $h_\alpha(p^0 > 0, p)$, whilst the other kind of particles are created by negative energy barred operators $\bar{h}_\alpha(p^0 < 0, p)$. The vacuum with reversed properties can also be defined, besides many other possible vacua. All those vacuum definitions participate in a description of the interactive processes of elementary particles. What we take into account in our current quantum field theory calculations seem to be only a part of a larger theory that has been neglected. It could be that some of the difficulties (e.g., infinities) that we have encountered, are partly due to neglection of such a larger theory. For instance, the vacuum (189) is considered by Jackiw et al. \[20\] within the context of a 2-dimensional field theory with signature $(+−)$. In Refs. \[21\] it is shown how with such definition of vacuum we can obtain vanishing zero point energy, and yet the Casimir and other such effects remain intact. This could be a resolution \[21\] of the problem of the huge cosmological constant predicted by the ordinary quantum field theory. Another application is in string theory which, as shown in Ref. \[22\], can be formulated in non critical dimensions

6.5.2 Bosonic fields: Generators of symplectic Clifford algebras

Let us now consider a vector

$$\Phi = \phi^{i(x)} k_i(x), \quad i = 1, 2$$  \hspace{1cm} (191)$$
of the bosonic subspace of the total (super) phase space, where the basis vectors satisfy relation (137) of a symplectic Clifford algebra. In the basis

\[ k'_{1(x)} \equiv k_{(x)} = \frac{1}{\sqrt{2}}(k_{1(x)} + k_{2(x)}) , \]  
\[ k'_{2(x)} \equiv \bar{k}_{(x)} = \frac{1}{\sqrt{2}}(k_{1(x)} - k_{2(x)}) \]  

relations (137) become

\[ k_{(x)} \wedge k_{(x')} = 0, \quad \bar{k}_{(x)} \wedge \bar{k}_{(x')} = 0, \]  
\[ \bar{k}_{(x)} \wedge k_{(x')} = \delta_{(x)(x')}, \]  

which, apart from a factor 1/2 that enters definition of the wedge product, are the commutation relations for bosonic creation and annihilation operators.

In the case of fermionic operators, a possible vacuum was defined as the product of all annihilation operators. For boson operators such definition does not work. At the moment it is not clear to me whether a bosonic vacuum can be defined in terms of creation and annihilation operators. Formally, we define

\[ \bar{k}_{(x)} \Omega = 0, \]  

where \( \Omega \) now denotes a bosonic vacuum. The basis

\[ \{ k_{(x_1)}k_{(x_2)}...k_{(x_r)} \} , \quad r = 0, 1, 2, ... \]  

spans a Fock space, whose vectors are

\[ \Phi_F = \sum_{r=0}^{\infty} \phi^{(x_1)(x_2)...(x_r)} k_{(x_1)}k_{(x_2)}...k_{(x_r)} \Omega , \]  

where components \( \phi^{(x_1)(x_2)...(x_r)} \) are symmetric in \((x_1)(x_2)...(x_r)\).

On the other hand, a vector (191) can be generalized to an element of a symplectic Clifford algebra:

\[ \Phi_C = \sum_{r=0}^{\infty} \phi^{i_1(x_1)i_2(x_2)...i_r(x_r)} k_{i_1(x_1)}k_{i_2(x_2)}...k_{i_r(x_r)}, \]  

which contain both kinds of operators, \( k'_{1(x)} \equiv k_{(x)} \) and \( k'_{2(x)} \equiv \bar{k}_{(x)} \). It remains to be explored whether the vectors \( \Phi_F \) of the form (198) belong to a subspace of the space whose vectors are \( \Phi_C \) of Eq. (199). This is equivalent to the question of whether \( \Omega \) can be defined in terms of \( \bar{k}_{(x)} \).
7 Discussion

7.1 Prospects for unification

We started from the super phase space action (91) in which the number of commuting variables \( z^a(\tau) = (x^\mu(\tau), p^\mu(\tau)) \) is the same as the number of anticommuting variables \( \lambda^a(\tau) = (\lambda^\mu(\tau), \bar{\lambda}^\mu(\tau)) \), \( \mu = 1, 2, ..., n \). This arises, if we consider superfields \( Z^a(\tau, \zeta) \) which depend on a commuting parameter \( \tau \), and on an anticommuting parameter \( \zeta \). So we have \( Z^a(\tau, \zeta) = z^a(\tau) + \zeta \lambda^a(\tau) \).

After (first) quantization we arrived at the action (155) or (160) for a vector field \( \Psi = \psi_i(x, \xi) h_i(x, \xi) = \psi_i^\alpha(x) h_i^\alpha(x) \), \( \alpha = 1, 2, ..., 2^N \), \( i = 1, 2 \). The latter index denotes a field and its canonically conjugate field. The basis vectors \( h_i^\alpha(x) \), satisfying relations (174), are generators of an infinite dimensional Clifford algebra, and they have the role of quantized fields. If we transform them into another basis according to (175), (176), then the new operators \( h_\alpha(x), \bar{h}_\alpha(x) \) satisfy the anticommutation relations (177), (178) for fermionic fields. If we start from 4-dimensional spacetime, then the index \( \alpha \) assume \( 2^4 = 16 \) values. Therefore, the set of of fields \( \{ h_\alpha(x), \bar{h}_\alpha(x) \} \) is bigger than it is necessary for description of a spin \( \frac{1}{2} \) particle and its antiparticle, in which case four values of \( \alpha \) are sufficient. We now have a possibility that \( \{ h_\alpha(x), \bar{h}_\alpha(x) \} \) describe electron, neutrino, and, e.g., the corresponding mirror particles, as shown in Ref. [10].

In order to describe other particles and gauge interactions of the standard model, one has to extend the theory. One possibility [13, 14, 23] is to replace spacetime with Clifford space [2, 4], which is a manifold of dimension \( N = 2^n \), whose tangent space at any point \( X \) is a Clifford algebra \( Cl(n) \). Clifford space is a quenched configuration space associated with \( p \)-branes [24, 23]. One can then proceed as we did in this paper, and arrive at the set of fermionic field operators \( \{ h_\alpha(X), \bar{h}_\alpha(X) \} \), where \( \alpha \) now runs from 1 to \( 2^N \), because we replaced \( n \)-dimensional spacetime with \( N \)-dimensional Clifford space. The theory then becomes analogous to a unification in the presence of higher dimensions. Another possibility is to exploit the \( 2D \)-dimensional Fock space spanned over the basis (183), and take into account the fact that the latter Fock space is a left minimal ideal of a Clifford algebra \( Cl(2D) \), and is thus the space of spinors. Various approaches to the unification of fundamental particles and forces by Clifford algebras have been explored in Refs. [11], [16], [10].

7.2 Prospects for quantum gravity

We have seen in Sec. 2.3 that the generators \( \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu) \), \( \mu = 1, 2, ..., n \), of an orthogonal Clifford algebra \( Cl(2n) \) can be rewritten in terms of new generators, \( \theta_a = (\theta_\mu, \bar{\theta}_\mu) \), which satisfy the fermionic anti commutation relations (29). In Sec. 4.4. we
then show that $\theta_\mu, \bar{\theta}_\mu$ act as fermionic creation and annihilation operators, from which we can build basis spinors, $s_\bar{A}$, of all minimal left ideals of $C(2n)$. A generic element is $\Psi = \psi^\bar{A}s_\bar{A}$, where the coefficients $\psi^\bar{A}$ may depend on spacetime position $x^\mu$. An interesting object to consider is

$$\langle \gamma_\mu \rangle_1 = \langle \Psi^\dagger(x) \gamma_\mu \Psi(x) \rangle_1,$$

(200)
i.e., the vector part of the expectation value of a vector $\gamma_\mu$. Since $\Psi$ is a generic element of Clifford algebra, a “Clifford aggregate” or polyvector, the expectation value is a linear superposition of vectors $\gamma_\mu'$:

$$\langle \gamma_\mu \rangle_1 = e_{\mu \nu'}(x) \gamma_{\nu'}.$$

(201)
Here $\gamma_{\nu'}$ are orthogonal vectors, whilst $\langle \gamma_\mu \rangle_1$ need not be orthogonal, and $e_{\mu \nu'}$ may serve the role of vielbein. There is a possibility that the vectors $\langle \gamma_\mu \rangle_1$ are tangent vectors to manifold with non vanishing curvature, so that their inner product

$$\langle \gamma_\mu \rangle_1 \cdot \langle \gamma_\nu \rangle_1 = g_{\mu \nu}(x)$$

(202)
gives a metric that cannot be transformed into $\eta_{\mu \nu}$ at every point $x$. If it is indeed the case that the curvature can be different from zero, then curved space(time) can be generated from position dependent spinors. This remains to be explored, and if it turns to be true, this will have implications for quantum gravity.

8 Conclusion

In this work we have pointed out how ‘quantization’ can be seen from yet another perspective. We reformulated and generalized the theory of quantized fields. An action for a physical system, such as a point particle or a field, can be written in the phase space form, and it contains either the symplectic or the orthogonal form (or both). The corresponding basis vectors satisfy either the fermionic anticommutation relations or the bosonic commutation relations. If we take a Hamiltonian that is quadratic in the phase space variables, derive the classical equations of motion and then assume that coordinates and momenta are undetermined, it turns out that the basis vectors satisfy the Heisenberg equations of motion. Quantum mechanical operators are just the basis vectors included in the phase space action. They can be expressed as creation and annihilation operators acting on a vacuum which is the product of annihilation operators (in the fermionic case). In the finite dimensional case this gives the Fock basis for spinors. If we consider not only one vacuum, but all possible vacua, then we obtain the Fock basis for a Clifford algebra [9, 10]. In infinite dimensional case, i.e., in the case of fields, we obtain a more general Fock
basis and many possible vacua that go beyond those usually considered in quantum field theories. It would be interesting to explore whether such a generalized quantum field theory is free of the difficulties, such as infinities and the cosmological constant problem.

As a particular model we considered a point particle described in terms of commuting and an equal number, \( n \), of anticommuting (Grassmann) phase space variables. The phase space action contains a form that consists of the latter variables and the corresponding basis vectors—the generators of an orthogonal, \( Cl(2n) \), and a symplectic Clifford algebra, \( Cl_S(2n) \). If we start from a 4-dimensional spacetime, i.e., if we take \( n = 4 \), then we obtain the spinor states—created from the basis vectors—of sufficiently high dimensionality that they can be considered in our attempts for grand unification. Finally, we showed how the fact that the basis vectors on the one hand are quantum mechanical operators, and on the other hand they give metric, could be exploited in the development of quantum gravity.

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