Weighted minimal translation surfaces in the Galilean space with density

Abstract: Translation surfaces in the Galilean 3-space $G_3$ have two types according to the isotropic and non-isotropic plane curves. In this paper, we study a translation surface in $G_3$ with a log-linear density and classify such a surface with vanishing weighted mean curvature.

Keywords: Galilean space, Translation surface, Weighted manifold, Weighted mean curvature

MSC: 53A35, 53C25

1 Introduction

Constant mean curvature and constant Gaussian curvature surfaces are one of main objects which have drawn geometers' interest for a very long time. In particular, regarding the study of minimal surfaces, L. Euler found that the only minimal surfaces of revolution are the planes and the catenoids, and E. Catalan proved that the planes and the helicoids are the only minimal ruled surfaces in the Euclidean 3-space $E^3$. Also, H. F. Scherk in 1835 studied translation surfaces in $E^3$ defined as graph of the function $z(x, y) = f(x) + g(y)$ and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log |\cos(ax)| - \frac{1}{a} \log |\cos(ay)|,$$

where $a$ is a non-zero constant.

Translation surfaces having constant mean curvature, in particular zero mean curvature, in the Euclidean space and the Minkowski space are described in [1]. Other results for minimal translation surfaces were obtained in [2, 3] when the ambient spaces are the affine space and the hyperbolic space, respectively.

As a new category in geometry, manifold with density (called also a weighted manifold) appears in many ways in mathematics, such as quotients of Riemannian manifolds or as Gauss spaces. It was instrumental in Perelman’s proof of the Poincare conjecture [4]. A manifold with density is a Riemannian manifold $M$ with a positive density function $e^\phi$ used to weighted volume and area, that is, for a Riemannian volume $dV_0$ and a area $dA_0$ the new weighted volume $dV$ and area $dA$ are defined by

$$dV = e^\phi dV_0, \quad dA = e^\phi dA_0.$$

By using the first variation of the weighted area, the weighted mean curvature $H_\phi$ (also called $\phi$-mean curvature) of a surface in the Euclidean 3-space $E^3$ with density $e^\phi$, is given by

$$H_\phi = H - \frac{1}{2} \frac{d\phi}{dN}.$$ (1)
where $H$ is the mean curvature and $N$ is the unit normal vector of the surface. The weighted mean curvature $H_\phi$ of a surface in $\mathbb{E}^3$ with density $e^\phi$ was introduced by Gromov [5] and it is a natural generalization of the mean curvature $H$ of a surface.

A surface with $H_\phi = 0$ is called a weighted minimal surface or a $\phi$-minimal surface in $\mathbb{E}^3$. For more details about manifolds with density and some relative topics we refer to [6–12]. In particular, Hieu and Hoang [8] studied ruled surfaces and translation surfaces in $\mathbb{E}^3$ with density $e^2$ and they classified the weighted minimal ruled surfaces and the weighted minimal translation surfaces. Lopez [10] considered a linear density $e^{ax+by+cz}$ and he classified the weighted minimal translation surfaces and weighted minimal cyclic surfaces in the Euclidean 3-space $\mathbb{E}^3$. Also Belarbi and Belkhelfa [6] investigated properties of the weighted minimal graphs in $\mathbb{E}^3$ with a log-linear density.

In this article, we focus on a class of translation surfaces in the Galilean 3-space $G_3$. There are two types of translation surfaces according to a non-isotropic curve and an isotropic curve, called translation surfaces of type 1 and type 2, respectively. We classify the weighted minimal translation surfaces in $G_3$ with a log-linear density.

## 2 Preliminaries

In 1872, F. Klein in his Erlangen program proposed how to classify and characterize geometries on the basis of projective geometry and group theory. He showed that the Euclidean and non-Euclidean geometries could be considered as spaces that are invariant under a given group of transformations. The geometry motivated by this approach is called a Cayley-Klein geometry. Actually, the formal definition of Cayley-Klein geometry is pair $(G, H)$, where $G$ is a Lie group and $H$ is a closed Lie subgroup of $G$ such that the (left) coset $G/H$ is connected. $G/H$ is called the space of the geometry or simply Cayley-Klein geometry.

The Galilean geometry is the real Cayley-Klein geometry equipped with the projective metric of signature $(0, 0, +, +)$ in the Galilean 3-space $G_3$ consists of an ordered triple $(\omega, f, I)$, where $\omega$ is the ideal (absolute) plane, $f$ the line (the absolute line) in $\omega$ and $I$ the fixed elliptic involution of points of $f$.

Let $x = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$ be vectors in $G_3$. A vector $x$ is called isotropic if $x_1 = 0$, otherwise it is called non-isotropic. The Galilean scalar product $\langle \cdot, \cdot \rangle$ of $x$ and $y$ is defined by (cf. [13])

\[
\langle x, y \rangle = \begin{cases} 
  x_1 y_2, & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0, \\
  y_1 y_2 + z_1 z_2, & \text{if } x_1 = 0 \text{ and } x_2 = 0.
\end{cases}
\]  

(2)

From this, the Galilean norm of a vector $x$ in $G_3$ is given by $||x|| = \sqrt{\langle x, x \rangle}$ and all unit non-isotropic vectors are the form $(1, y_1, z_1)$. For an isotropic vector $x_1 = 0$ holds. The Galilean cross product of $x$ and $y$ on $G_3$ is defined by

\[
x \times y = \begin{vmatrix} 
  0 & e_2 & e_3 \\
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2
\end{vmatrix}.
\]  

(3)

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Consider a $C^r$-surface $\Sigma$, $r \geq 1$, in $G_3$ parameterized by

\[
x(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).
\]

Let $\Sigma$ be a regular surface in $G_3$. Then the unit normal vector field $N$ of the surface $\Sigma$ is defined by

\[
N = \frac{1}{\omega}(0, x_{u_1} z_{u_2} - x_{u_2} z_{u_1}, x_{u_2} y_{u_1} - x_{u_1} y_{u_2}),
\]

where the positive function $\omega$ is given by

\[
\omega = \sqrt{\left(x_{u_1} z_{u_2} - x_{u_2} z_{u_1}\right)^2 + (x_{u_2} y_{u_1} - x_{u_1} y_{u_2})^2}.
\]

Here the partial derivatives of the functions $x$, $y$ and $z$ with respect to $u_i$ ($i = 1, 2$) are denoted by $x_{u_i}$, $y_{u_i}$ and $z_{u_i}$, respectively. On the other hand, the matrix of the first fundamental form $ds^2$ of a surface $\Sigma$ in $G_3$ is given by (cf.
where \( ds_1^2 = (g_1 du_1 + g_2 du_2)^2 \) and \( ds_2^2 = h_{11}du_1^2 + 2h_{12}du_1 du_2 + h_{22}du_2^2 \). Here \( g_i = x_{u_i} \) and \( h_{ij} = \langle \tilde{x}_{u_i}, \tilde{x}_{u_j} \rangle \) \( (i, j = 1, 2) \) stand for derivatives of the first coordinate function \( x(u_1, u_2) \) with respect to \( u_1, u_2 \), and for the Euclidean scalar product of the projections \( \tilde{x}_{u_k} \) of vectors \( x_{u_k} \) onto the \( yz \)-plane, respectively.

The Gaussian curvature \( K \) and the mean curvature \( H \) of a surface \( \Sigma \) are defined by means of the coefficients \( L_{ij}, i, j = 1, 2 \) of the second fundamental form, which are the normal components of \( x_{u_i u_j}, i, j = 1, 2 \), that is,

\[
L_{ij} = \frac{1}{g_1} (g_1 \tilde{x}_{u_i u_j} - g_{i,j} \tilde{x}_{u_1}, N) = \frac{1}{g_2} (g_2 \tilde{x}_{u_i u_j} - g_{i,j} \tilde{x}_{u_2}, N).
\]

Thus, the Gaussian curvature \( K \) of a regular surface is defined by

\[
K = \frac{L_{11}L_{22} - L_{12}^2}{\omega^2}
\]  

and the mean curvature \( H \) is given by

\[
H = \frac{1}{2\omega^2} (g_2^2L_{11} - 2g_1 g_2 L_{12} + g_1^2L_{22}).
\]

### 3 Translation surfaces in \( G_3 \)

In this section, we define translation surfaces in \( G_3 \) that are obtained by translating two planar curves. According to the planar curves, we have two types as follows [16]:

Type 1. a non-isotropic curve (having its tangent non-isotropic) and an isotropic curve.
Type 2. non-isotropic curves.

There are no motions of the Galilean space that carry one type of a curve into another, so we will treat them separately.

First, we construct translation surfaces of type 1 in the Galilean 3-space \( G_3 \).

Let \( \alpha(x) \) be a non-isotropic curve in the plane \( y = 0 \) and \( \beta(y) \) an isotropic curve in the plane \( x = 0 \). This means that

\[
\alpha(x) = (x, 0, f(x)), \quad \beta(y) = (0, y, g(y)).
\]

In this case, a translation surface of type 1 is parameterized by

\[
x(x, y) = (x, y, f(x) + g(y)),
\]

where \( f \) and \( g \) are smooth functions. The unit normal vector field \( N \) of the surface is

\[
N = \frac{1}{g'^2 + 1} (0, -g'(y), 1).
\]

By a straightforward computation, the mean curvature \( H \) is given by

\[
H = \frac{g''(y)}{2(1 + g'^2(y))^2}.
\]

Next, we construct translation surfaces of type 2 in the Galilean 3-space \( G_3 \).

Let \( \alpha(x) \) be a non-isotropic curve in the plane \( y = 0 \) and \( \beta(y) \) an non-isotropic curve in the plane \( z = 0 \), that is,

\[
\alpha(x) = (x, 0, f(x)), \quad \beta(y) = (y, g(y), 0).
\]
Then the parametrization of the surface is given by

$$x(x, y) = (x + y, g(y), f(x)), \quad (10)$$

where \(f\) and \(g\) are smooth functions. The unit normal vector field \(N\) is

$$N = \frac{1}{\omega}(0, f'(x), g'(y)). \quad (11)$$

From (6) the mean curvature \(H\) of the surface is given by

$$H = \frac{1}{2\omega^3} (f''(x) g'(y) + f'(x) g''(y)). \quad (12)$$

where \(\omega^2 = f''(x) + g''(y)\).

In [16], Šipuš and Divjak classified minimal translation surfaces in the Galilean 3-space \(G_3\) and they proved the following theorems:

**Theorem 3.1.** A translation surface of type 1 of zero mean curvature in the Galilean 3-space \(G_3\) is congruent to a cylindrical surface with isotropic rulings.

**Theorem 3.2.** A translation surface of type 2 of zero mean curvature in the Galilean 3-space \(G_3\) is congruent to an isotropic plane or a non-cylindrical surface with isotropic rulings.

## 4 Weighted minimal translation surfaces of type 1

In this section, we classify translation surfaces of type 1 with zero weighted mean curvature in the Galilean 3-space. Let \(\Sigma_1\) be a translation surface of type 1 defined by

$$x(x, y) = (x, y, f(x) + g(y)).$$

Suppose that \(\Sigma_1\) is the surface in \(G_3\) with a linear density \(e^\phi\), where \(\phi = ax + by + cz, a, b, c\) not all zero. In this case, the weighted mean curvature \(H_\phi\) of \(\Sigma_1\) can be expressed as

$$H_\phi = H - \frac{1}{2}(N, \nabla \phi), \quad (13)$$

where \(\nabla \phi\) is the gradient of \(\phi\).

If \(\Sigma_1\) is the weighted minimal surface, then the weighted minimality condition \(H_\phi = 0\) with the help of (9) turns out to be

$$\frac{g''(y)}{2(1 + g''(y))^2} = \frac{1}{2(g'^2(y) + 1)^2}((0, -g'(y), 1), (a, b, c))$$

or equivalently,

$$g''(y) = (g'^2(y) + 1)((0, -g'(y), 1), (a, b, c)). \quad (14)$$

Let us distinguish two cases according to the value of \(a\).

### Case 1. \(a \neq 0\).

In this case the vector \((a, b, c)\) is non-isotropic and from (2) we get \(g''(y) = 0\). Therefore \(\Sigma_1\) is determined by

$$z(x, y) = f(x) + d_1 y + d_2$$

for some constants \(d_1, d_2\).

In other words, the obtained surface is a ruled surface with rulings having the constant isotropic direction \((0, 1, d_1)\) and it is a cylindrical surface.
Case 2. \( a = 0 \).

In this case the vector \((0, b, c)\) is isotropic. From (14) we have the following ordinary differential equation:

\[
g''(y) = (g'^2(y) + 1)(-bg'(y) + c). \quad (15)
\]

Subcase 2.1. \( b = 0 \). The general solution of (15) is given by

\[
g(y) = -\frac{1}{c} \ln |\cos(cy + d_1)| + d_2.
\]

where \( d_1, d_2 \) are constant.

Subcase 2.2. \( c = 0 \). In the case, equation (15) writes as

\[
g''(y) + b(g'^3(y) + g'(y)) = 0. \quad (16)
\]

In order to solve the equation, we put \( g'(y) = p(y) \). Then equation (16) can be rewritten as the form

\[
\frac{dp}{dy} = -bp(p^2 + 1). \quad (17)
\]

A function \( g(y) = d_1, d_1 \in \mathbb{R} \) is a solution of (16), and in that case the surface \( \Sigma_1 \) is given by \( z(x, y) = f(x) + d_1 \).

Suppose that \( p = g'(y) \neq 0 \). A direct integration of (17) yields

\[
p(y) = \pm \frac{1}{\sqrt{e^{2(b^2y + d_1)} - 1}}.
\]

Thus, the general solution of (16) appears in the form

\[
g(y) = \frac{1}{2b} \tan^{-1} \sqrt{e^{2(b^2y + d_1)} - 1} + d_2,
\]

where \( d_1, d_2 \) are constant.

Subcase 2.3. \( bc \neq 0 \). We put \( p = g'(y) \) in (15), then we have

\[
\frac{dp}{dy} = -(p^2 + 1)(bp - c).
\]

From this, the function \( p(y) \) satisfies the following equation:

\[
\frac{1}{b^2 + c^2} \left( \frac{b}{2} \ln(p^2 + 1) + c \tan^{-1} p - b \ln |bp - c| \right) + y + d_1 = 0.
\]

Thus the weighted minimal translation surface \( \Sigma_1 \) in \( G_3 \) with a linear density \( e^{by + cz} \) is given by \( z(x, y) = f(x) + g(y) \), where \( f(x) \) is any smooth function and a function \( g(y) \) satisfies

\[
b \ln(g'^2(y) + 1) + 2c \tan^{-1} g'(y) - 2b \ln |bg'(y) - c| + 2(b^2 + c^2)y + d = 0, \quad (19)
\]

where \( d \) is constant.

**Theorem 4.1.** Let \( \Sigma_1 \) be a translation surface of type 1 in the Galilean 3-space \( G_3 \) with a log-linear density \( e^{ax + by + cz} \). Suppose \( \Sigma_1 \) is weighted minimal. Then \( \Sigma_1 \) is parameterized as

\[
x(x, y) = (x, y, f(x) + g(y)),
\]

where either

1. \( g \) is constant, or
2. \( g(y) = d_1 y + d_2, \) or
3. \( g(y) = -\frac{1}{c} \ln (\cos(cy + d_1)) + d_2, \) or
4. \( g(y) = \frac{1}{2b} \tan^{-1} \sqrt{e^{2(b^2y + d_1)} - 1} + d_2, \) or
5. \( g \) satisfies (19).

**Remark.** In general, the normal vector of a surface in \( G_3 \) is always isotropic, therefore weighted minimal translation surfaces in \( G_3 \) with a log-linear density \( e^{ax + by + cz} \) for \( a \neq 0 \) is just the one in \( G_3 \) with density 1. Thus, such a surface is minimal in \( G_3 \) and classified by Šipuš and Divjak [16].
5 Weighted minimal translation surfaces of type 2

Let $\Sigma_2$ be a translation surface of type 2 defined by

$$x(x, y) = (x + y, g(y), f(x)),$$

where $f$ and $g$ are smooth functions.

If $\Sigma_2$ is a surface in $G_3$ with a linear density $e^\phi$, where $\phi = ax + by + cz$, $a, b, c$ not all zero, then the weighted minimality condition $H_\phi = 0$ becomes

$$H - \frac{1}{2\omega} \langle 0, f'(x), g'(y) \rangle, (a, b, c) = 0. \quad (20)$$

Let us distinguish two cases according to the value of $a$.

Case 1. $a \neq 0$.

In this case, $H = 0$ and from (12) we have

$$f''(x)g'(y) + f'(x)g''(y) = 0.$$

If $f$ or $g$ is constant, then $\Sigma_2$ is a plane. Now, we assume that $f'g' \neq 0$. Then, the above equation writes as

$$\frac{f''(x)}{f'(x)} = -\frac{g''(y)}{g'(y)}$$

which implies there exists a real number $m \in \mathbb{R}$ such that

$$\frac{f''(x)}{f'(x)} = -\frac{g''(y)}{g'(y)} = m. \quad (21)$$

If $m = 0$, then the functions $f$ and $g$ are linear functions, which generates an isotropic plane.

If $m \neq 0$, then the general solutions of (21) are given by

$$f(x) = \frac{1}{m} e^{mx} + d_1, \quad g(y) = -\frac{1}{m} e^{-my} + d_2,$$

where $d_1, d_2$ are constant.

Case 2. $a = 0$.

From (12) and (20) the weighted minimality condition $H_\phi = 0$ becomes

$$f''(x)g'(y) + f'(x)g''(y) = (f'^2(x) + g'^2(y))(bf'(x) + cg'(y)). \quad (22)$$

Taking the derivative with respect to $x$ and next with respect to $y$, we have the following ordinary differential equation:

$$f'''(x)g''(y) + f''(x)g'''(y) = 2hf''(x)g'(y)g''(y) + 2cf'(x)f''(x)g'(y). \quad (23)$$

If $f''(x) = 0$ or $g''(y) = 0$, then $f$ or $g$ is a linear function. Now we assume that $f''(x)g''(y) \neq 0$.

Subcase 2.1. $b = 0$. Dividing (23) by $f''(x)g''(y)$, we have

$$\frac{f'''(x)}{f''(x)} - 2cf'(x) = -\frac{g'''(y)}{g''(y)}$$

Then there exists a real number $m \in \mathbb{R}$ such that

$$\frac{f'''(x)}{f''(x)} - 2cf'(x) = -\frac{g'''(y)}{g''(y)} = m.$$

If $m = 0$, $g(y) = \frac{1}{2}d_1 y^2 + d_2 y + d_3$ with $d_1, d_2, d_3 \in \mathbb{R}$. Substituting the function $g(y)$ into (22), equation (22) is a polynomial in $y$ with functions of $x$ as coefficients. Thus, all the coefficients must be zero. The coefficient of the highest degree $y^3$ in (22) is $ca_1^3$, thus it follows $d_1 = 0$ and $g''(y) = 0$, a contradiction.
Suppose that \( m \neq 0 \). The solution of the ODE \( g'''(y) + mg''(y) = 0 \) is
\[
g(y) = m^2 e^{-my+d_1} + d_2 y + d_3.
\]
Substituting the function \( g(y) \) into (22), we get a polynomial on \( e^{-my+d_1} \) with functions of \( x \) as coefficients. In the polynomial we can obtain the coefficient of \( e^{3(-my+d_1)} \) and it is \(-m^2 c\), a contradiction.

Subcase 2.2. \( c = 0 \). This subcase is similar to the previous one. Thus, there are no weighted minimal translation surfaces of type 2 in \( G_3 \) with a linear density \( e^{by} \).

Subcase 2.3. \( bc \neq 0 \). Dividing (23) by \( f''(x)g''(y) \), we get
\[
\frac{f'''(x)}{f''(x)} - 2cf'(x) = 2bg'(y) - \frac{g'''(y)}{g''(y)}.
\]
Hence, we deduce the existence of a real number \( m \in \mathbb{R} \) such that
\[
\frac{f'''(x)}{f''(x)} - 2cf'(x) = 2bg'(y) - \frac{g'''(y)}{g''(y)} = m.
\]
(24)

If \( m = 0 \), the general solution of (24) is given by
\[
\begin{align*}
  f(x) &= -\frac{1}{c} \ln |c(x+d_2)| - \frac{d_1}{c}, \\
  g(y) &= -\frac{1}{b} \ln |b(y+d_4)| - \frac{d_3}{b},
\end{align*}
\]
(25)
where \( d_i (i = 1, \cdots, 4) \) are constant. Substituting the function (25) into (22) we can obtain the equation:
\[
b^4 \frac{1}{(x+d_2)^3} + c^4 \frac{1}{(y+d_4)^3} = 0,
\]
which is impossible.

Suppose \( m \neq 0 \). Equation (24) can be written as
\[
\begin{align*}
  f'''(x) - c(f'(x))^2' &= mf''(x), \\
  g'''(y) - b(g'(y))^2' &= -mg''(y).
\end{align*}
\]
(26)

First, in order to solve the first equation of (26) we put \( p = f'(x) \). Then we have
\[
\frac{dp}{dx} = p(cp + m)
\]
and its solution is
\[
p = \frac{m}{e^{-m(x+d_1)} - c}.
\]
Thus a first integration implies
\[
f(x) = m^2 \int \frac{1}{e^{-m(x+d_1)} - c}.
\]
(27)

By using the similar method, the function \( g \) satisfying the second equation of (26) is given by
\[
g(y) = -m^2 \int \frac{1}{e^{m(y+d_1)} - b}.
\]
(28)

When substituting (27) and (28) into (22), equation (22) appears a polynomial in \( e^{-m(x+d_1)} \) and \( e^{m(y+d_1)} \). The coefficient of \( e^{-3m(x+d_1)} \) in the polynomial is \(-cm^3\), which is impossible because \( m \) and \( c \) are non-zero constant.

Thus we have:

**Theorem 5.1.** Let \( \Sigma_2 \) be a translation surface of type 2 in the Galilean 3-space \( G_3 \) with a log-linear density \( e^{ax+by+cz} \). If \( \Sigma_2 \) is weighted minimal, then \( \Sigma_2 \) is an isotropic plane or parameterized as \( x(x, y) = (x + y, 1/m e^{-my} + d_1, 1/m e^{nx} + d_2) \) with \( m \neq 0, d_1, d_2 \in \mathbb{R} \).
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