Wormhole with varying cosmological constant

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Abstract

It has been suggested that the cosmological constant is a variable dynamical quantity. A class of solution has been presented for the spherically symmetric space time describing wormholes by assuming the erstwhile cosmological constant Λ to be a space variable scalar, viz., Λ = Λ(r). It is shown that the Averaged Null Energy Condition (ANEC) violating exotic matter can be made arbitrarily small.

In 1917, Einstein introduced cosmological constant Λ which is related to the energy of the space to maintain the stability of his cosmological model. Recent observations of high redshift Type Ia supernovae [1] suggest that Λ could be scalar variable dependent both on space and time coordinate rather a constant, which was believed earlier. Time dependence of Λ plays a significant role on cosmology whereas space dependence of Λ does effect on astrophysical problems. Narlikar et al [2] have suggested that space dependence of Λ should be included to study the nature of local massive objects like galaxies. Recently, some authors have shown significant effects of the space dependence Λ on energy density of the classical electron [3]. Tiwari et al [4] and Dymnikova [5] have discussed the contribution of space dependence Λ to the effective gravitational mass of the astrophysical system.

Wormholes are classical or quantum solutions for the gravitational field equations describing a bridge between two asymptotic manifolds. Classically, they can be interpreted as instantons describing a tunneling between two distant regions. In a pioneer work, Morris and Thorne [6] have shown that a wormhole geometry can only appear as a solution to the Einstein equation if the stress energy tensor violates the null energy condition. The matter that characterized stress energy tensor is known as exotic matter [7]. Several authors have discussed wormholes in scalar tensor theory of gravity in which scalar field may play the role of exotic matter [8]. Recently, authors are interested to know how much

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exotic matter is needed to get a traversable wormhole [9]. Peebles and Ratra [10] proposed that like all energy, cosmological constant $\Lambda$ has some contribution to the source term in Einstein’s gravitational field equations. It is believed through indirect evidences that 70 percent of the contents of the Universe is to be in the form of vacuum energy and cosmological constant turns to be a measure of the energy density of the vacuum. In an interesting paper Lemos et al have studied wormhole geometry in presence of $\Lambda$ where $\Lambda$ is a constant[11]. In this work, we are interested to discuss and provide a prescription for obtaining wormhole solution by assuming cosmological constant $\Lambda$ to be a space variable scalar. To our knowledge, wormhole solution under the assumption that the cosmological constant is spatially variable has not been proposed earlier.

The Einstein field equation for the anisotropic fluid distribution are given by

$$R_{ab} - \frac{1}{2}g_{ab} + \Lambda g_{ab} = -8\pi T_{ab} \quad (G = c = 1).$$

(1)

where matter momentum tensor is given by $T^b_a = (-\rho, p, p_t, p_t)$ and the related conservation law here is

$$8\pi [T^b_a]_b = -\Lambda_b$$

(2)

as the cosmological constant is assumed to be spatially varying i.e. $\Lambda = \Lambda(r)$.

[ The usual energy momentum tensor is modified by the addition of a term $T^{(vac)}_{ab} = -\Lambda(r)g_{ab}$. Hence, the new energy momentum tensor is $T^{(total)}_{ab} = T^{(matter)}_{ab} + \Lambda(r)g_{ab}$. Here energy conservation equation $T^b_{a;b} = 0$ implies equation (2) ]

Let us now consider the spherically symmetric line element

$$ds^2 = -e^\gamma dt^2 + e^\mu dr^2 + r^2 d\Omega^2,$$

(3)

where

$$e^{-\mu} = \left[1 - \frac{b(r)}{r}\right].$$

(4)

Here $\frac{\gamma(r)}{2}$ is the redshift function and $b(r)$ is the shape function determining the shape of the wormhole.

The field equations (1) corresponding to the above line element (3) are given by

$$e^{-\mu} \left[ -\frac{1}{r^2} + \frac{\mu'}{r} \right] + \frac{1}{r^2} = 8\pi \rho + \Lambda,$$

(5)
\( e^{-\mu}\left[ \frac{1}{r^2} + \frac{\gamma'}{r} \right] - \frac{1}{r^2} = 8\pi p - \Lambda, \) \( (6) \)

\[ \frac{1}{2} e^{-\mu} \left[ \gamma'' + \frac{1}{2}(\gamma')^2 - \frac{1}{2}\gamma'\mu' + \frac{\gamma' - \mu'}{r} \right] = 8\pi p_t - \Lambda, \] \( (7) \)

where \( p, p_t \) are radial and tangential pressures respectively and \( \rho \) is the matter energy density.

['\prime' refers to differentiation with respect to radial coordinate.]

The conservation equation \((2)\) becomes,

\[ \frac{d}{dr} \left( p - \frac{\Lambda}{8\pi} \right) = -(p + \rho)\gamma' \frac{r}{2} + \frac{2}{r}(p_t - p). \] \( (8) \)

In this work, we are not interested in discussing the traversability constrains mentioned by Morris and Thorne \([6]\). We assume a zero tidal force as seen by the stationary observer, \( \frac{\gamma(r)}{2} = 0 \), to make the problem simpler. We suppose also that the pressures are anisotropic and

\[ p_t = np. \] \( (9) \)

( \( n \) is an arbitrary constant )

Now, from the field equation \((6)\), one finds,

\[ e^{-\mu} = 1 - \frac{b(r)}{r} = 1 + 8\pi pr^2 - \Lambda r^2, \] \( (10) \)

where

\[ b(r) = \Lambda r^3 - 8\pi pr^3. \] \( (11) \)

Equation \((8)\) yields

\[ \frac{d}{dr} \left( p - \frac{\Lambda}{8\pi} \right) = \frac{2(n - 1)}{r} p. \] \( (12) \)
Since the vacuum energy (which is equivalent to $\Lambda$) can be thought as a contributor of the anisotropic fluid distribution, we impose the condition, $\frac{\Lambda}{8\pi} \propto p$ for simplicity and this implies

\[
\frac{\Lambda}{8\pi} = ap. \tag{13}
\]

[a is the proportional constant]

By solving Eq. (12), one obtains,

\[
p = Ar^{-B}, \tag{14}
\]

where

\[
B = \frac{2(n - 1)(a - 1)}{(a - 1)}, \tag{15}
\]

and $A$ is the integration constant.

The expression for $b(r)$ is

\[
b(r) = 8\pi A(a - 1)r^{(-B+3)}, \tag{16}
\]

where $a > 1$.

Since the space time is asymptotically flat i.e. $\frac{b(r)}{r} \to 0$ as $|r| \to \infty$, the Eq. (16) is consistent only when $B > 2$ i.e. $n > a > 1$.

Eq. (5), after some rearrangement, reduces to

\[
\rho = A(B - 3 - aB + 2a)r^{-B} \tag{17}
\]

Using eqs. (14) and (17), one can find that

\[
p + \rho = -2(n - a)Ar^{-B} < 0 \tag{18}
\]

since, $n > a > 1$.

Thus, null energy condition is violated.
Now, we will check whether the wormhole geometry in principle, supported by arbitrary amount of Averaged Null Energy Condition (ANEC) violating exotic matter. The ANEC violating matter can be quantified by the integrals [9]

\[ I = \oint (p_i + \rho) dV \] (19)

In this model, we assume that the ANEC violating matter is related only to \( p \) (radial pressure), not to the transverse components [as one can see from eqs. (9), (14) and (17), that the sign of \( p_i + \rho \) is not fixed but depends on the values of the parameters].

Now, if one assumes, \( n = a + \epsilon \), then \( n - a = \epsilon \), in other words, the integral (19) tends to zero as \( \epsilon \to 0 \). Hence the ANEC violating matter can be made arbitrarily small.

The throat of the wormhole occurs at

\[ r_0 = \left[ 8\pi A(a - 1) \right]^{\frac{1}{(a-2)}}. \] (20)

The axially symmetric embedded surface \( z = z(r) \) shaping the Wormhole’s spatial geometry is a solution of

\[ \frac{dz}{dr} = \pm \frac{1}{\sqrt{b(r) - 1}}. \] (21)

One can note from the definition of Wormhole that at \( r = r_0 \) (the wormhole throat) Eq. (21) is divergent i.e. embedded surface is vertical there.

The embedded surface (solution of Eq. (21)) in this case is [we assume \( B = 4 \)],

\[ z = \sqrt{8\pi A(a - 1)} \cosh^{-1} \frac{r}{\sqrt{8\pi A(a - 1)}}. \] (22)

One can see embedding diagram of this wormhole in Fig. 1. The surface of revolution of this curve about the vertical z axis makes the diagram complete (Fig. 2).
Figure 1: The embedding diagram of the wormhole

Figure 2: The full visualization of the surface generated by the rotation of the embedded curve about the vertical z axis.
According to Morris and Thorne [6], the 'r' co-ordinate is ill-behaved near the throat, but proper radial distance

\[ l(r) = \pm \int_{r_0^+}^{r} \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \]  \hspace{1cm} (23)

must be well behaved everywhere i.e. we must require that \( l(r) \) is finite throughout the space-time.

In this model (for \( B = 4 \)),

\[ l(r) = \pm \sqrt{r^2 - 8\pi A(a - 1)}. \]  \hspace{1cm} (24)

Due to the simple expression for \( l(r) \), one can rewrite the metric tensor in terms of this proper radial distance,

\[ ds^2 = -dt^2 + dl^2 + r^2(l) d\Omega^2, \]  \hspace{1cm} (25)

where

\[ r^2(l) = l^2 + 8\pi A(a - 1). \]  \hspace{1cm} (26)

This is a well behaved coordinate system. The radial distance is positive above the throat (our Universe) and negative below the throat (other Universe). At very large distance from the throat, the embedding surface becomes flat \( \frac{dz}{dl}(l \rightarrow \pm \infty) = 0 \) corresponding to the two asymptotically flat regions \( l \rightarrow +\infty \) and \( l \rightarrow -\infty \), which the wormhole connects.

In conclusion, our aim in this work has been to provide a prescription for obtaining wormhole in presence of variable cosmological constant. The most striking features of our model is that if we choose the parameters, 'n' is very close to 'a', then ANEC violating matter can be made arbitrarily small. Our wormhole can be visualized by the surface of revolution of the curve \( r = \sqrt{8\pi A(a - 1)} \cosh \left[ \frac{z}{\sqrt{8\pi A(a - 1)}} \right] \). Though this research work is mostly theoretical in nature, the outcome of the result may be of interest to the researchers working in this field. The traversable wormhole opens up several possible interesting physical applications and we hope to report on this elsewhere.
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