THE LOCAL-GLOBAL PRINCIPLE FOR INTEGRAL POINTS ON STACKY CURVES

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ABSTRACT. We construct a stacky curve of genus 1/2 (i.e., Euler characteristic 1) over \( \mathbb{Z} \) that has an \( \mathbb{R} \)-point and a \( \mathbb{Z}_p \)-point for every prime \( p \) but no \( \mathbb{Z} \)-point. This is best possible: we also prove that any stacky curve of genus less than 1/2 over a ring of \( S \)-integers of a global field satisfies the local-global principle for integral points.

1. Introduction

Let \( k \) be a global field, i.e., a finite extension of either \( \mathbb{Q} \) or \( \mathbb{F}_p(t) \). For each nontrivial place \( v \) of \( k \), let \( k_v \) be the completion of \( k \) at \( v \). Let \( X \) be a smooth projective geometrically integral curve of genus \( g \) over \( k \). If \( X \) has a \( k \)-point, then of course \( X \) has a \( k_v \)-point for every \( v \). The converse holds if \( g = 0 \) (by the Hasse–Minkowski theorem), but there are well-known counterexamples of higher genus; in fact, counterexamples exist over every global field [Poo10]. This motivates the question: What is the smallest \( g \) such that there exists a counterexample of genus \( g \) over some global field? The answer is 1. Indeed, the first counterexample discovered was a genus 1 curve, the smooth projective model of \( 2y^2 = 1-17x^4 \) over \( \mathbb{Q} \) [Lin40, Rei42]. In fact, a positive proportion of genus 1 curves in the weighted projective space \( \mathbb{P}(1,1,2) \) given by \( z^2 = f(x,y) \), where \( f(x,y) \) is an integral binary quartic form, violate the local-global principle over \( \mathbb{Q} \) [Bha13].

Let us now generalize to allow \( X \) to be a stacky curve over \( k \). (See Sections 2 and 3 for our conventions.) Then the genus \( g \) of \( X \) — defined by the formula \( \chi = 2 - 2g \), where \( \chi \) is the topological Euler characteristic of \( X \) — is no longer constrained to be a natural number; certain fractional values are also possible. Therefore we may now ask: What is the smallest \( g \) such that there exists a stacky curve of genus \( g \) over some global field \( k \) violating the local-global principle? It turns out that if we formulate the local-global principle using rational points over \( k \) and its completions, then the answer is not interesting, because rational points are almost the same as rational points on the coarse moduli space of \( X \): see Section 4.

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Therefore we will answer our question in the context of a local-global principle for integral points on a stacky curve. Our first theorem gives a proper stacky curve of genus $1/2$ over $\mathbb{Z}$ that violates the local-global principle.

**Theorem 1.** Let $p, q, r$ be primes congruent to $7 \pmod{8}$ such that $p$ is a square $\pmod{q}$ and $\not\equiv \pmod{r}$, and $q$ is a square $\pmod{r}$. Let $f(x, y) = ax^2 + bxy + cy^2$ be a positive definite integral binary quadratic form of discriminant $-pqr$ such that $a$ is a nonzero square $\pmod{q}$ but a nonsquare $\pmod{r}$ and $\not\equiv \pmod{r}$. Let $\mathcal{Y} := \text{Proj} \mathbb{Z}[x, y, z]/(z^2 - f(x, y))$. Define a $\mu_2$-action on $\mathcal{Y}$ by letting $\lambda \in \mu_2$ act as $(x : y : z) \mapsto (x : y : \lambda z)$. Let $\mathcal{X}$ be the quotient stack $[\mathcal{Y}/\mu_2]$. Then

(a) the genus of $\mathcal{X}$ is $1/2$ (i.e., $\chi(\mathcal{X}) = 1$);

(b) $\mathcal{X}(\mathbb{Q}) \neq \emptyset$ for every rational prime $\ell$ and $\mathcal{X}(\mathbb{R}) \neq \emptyset$;

(c) $\mathcal{X}(\mathbb{Z}) = \emptyset$, and even $\mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset$.

The same conclusions hold if instead we define $\mathcal{X}$ as $[\mathcal{Y}/(\mathbb{Z}/2\mathbb{Z})]$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{Y}$ through the nontrivial homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$; this $\mathcal{X}$ is a Deligne–Mumford stack even over $\mathbb{Z}$.

**Remark 2.** The hypotheses in Theorem 1 can be satisfied. For example, let $p = 7$, $q = 47$, $r = 31$, and $f(x, y) = 3x^2 + xy + 850y^2$.

**Remark 3.** The reason for considering $\mathbb{Z}[1/(2pqr)]$ in (c) is that $\mathcal{X}$ is smooth over that base.

**Remark 4.** Section 8 of [DG95] can be interpreted as saying that the proper stacky curve

$$\left(\text{Spec} \frac{\mathbb{Z}[x, y, z]}{(x^2 + 29y^2 - 3z^3)} - \{x = y = z = 0\}\right)/\mathbb{G}_m$$

is a similar counterexample to the local-global principle, but of genus $2/3$.

Our second theorem shows that any stacky curve of genus less than $1/2$ over a ring of $S$-integers of a global field satisfies the local-global principle. Let $k$ be a global field, and let $k_v$ denote the completion of $k$ at $v$. Let $S$ be a finite nonempty set of places of $k$ containing all the archimedean places. Let $\mathcal{O}$ be the ring of $S$-integers in $k$; that is, $\mathcal{O} := \{x \in k : v(x) \geq 0 \text{ for all } v \notin S\}$. For each $v \notin S$, let $\mathcal{O}_v$ be the completion of $\mathcal{O}$ at $v$. For each $v \in S$, let $\mathcal{O}_v = k_v$.

**Theorem 5.** Let $\mathcal{X}$ be a stacky curve over $\mathcal{O}$ of genus less than $1/2$ (i.e., $\chi(\mathcal{X}) > 1$). If $\mathcal{X}(\mathcal{O}_v) \neq \emptyset$ for all places $v$ of $k$, then $\mathcal{X}(\mathcal{O}) \neq \emptyset$.

2. **Stacks**

By a stack, we mean an algebraic (Artin) stack $\mathcal{X}$ over a scheme $S$ [SP, Tag 0260]. For any object $T \in (\text{Sch}/S)_{fppf}$, we write $\mathcal{X}(T)$ for the set of isomorphism classes of $S$-morphisms...
$T \to \mathcal{X}$, or equivalently (by the 2-Yoneda lemma \cite{SP, Tag04SS}), the set of isomorphism classes of the fiber category $\mathcal{X}_T$. If $T = \text{Spec } A$, we write $\mathcal{X}(A)$ for $\mathcal{X}(T)$.

3. Stacky curves

Let $k$ be an algebraically closed field. Let $X$ be a \textit{stacky curve} over $k$, i.e., a smooth separated irreducible 1-dimensional Deligne–Mumford stack over $k$ containing a nonempty open substack isomorphic to a scheme. (This definition is slightly more general than \cite[Definition 5.2.1]{VZB19} in that we require only separatedness instead of properness, to allow punctures.)

By the Keel–Mori theorem \cite{KM97} in the form given in \cite{Con05} and \cite[Theorem 11.1.2]{Ols16}, $X$ has a morphism to a coarse moduli space $X_{\text{coarse}}$ that is a smooth integral curve over $k$. We have $X_{\text{coarse}} = \hat{X}_{\text{coarse}} - Z$ for some smooth projective integral curve $\hat{X}_{\text{coarse}}$ and some finite set of closed points $Z$. Moreover, by \cite[Theorem 11.3.1]{Ols16}, each $P \in X_{\text{coarse}}(k)$ has an étale neighborhood $U$ above which $X \to X_{\text{coarse}}$ has the form $[V/G] \to U$ for some possibly ramified finite $G$-Galois cover $V \to U$ (by a scheme), where $G$ is the stabilizer of $X$ above $P$. The stacky curve $X$ is called \textit{tame above} $P$ if $\text{char } k \nmid |G|$, and \textit{tame} if it is tame above every $P$. Let $\mathcal{P} \subset X_{\text{coarse}}(k)$ be the (finite) set above which the stabilizer is nontrivial; then the morphism $X \to X_{\text{coarse}}$ is an isomorphism above $X_{\text{coarse}} - \mathcal{P}$.

Let $\tilde{g}_{\text{coarse}}$ be the genus of $\hat{X}_{\text{coarse}}$; then the Euler characteristic $\chi(X_{\text{coarse}})$ is $(2 - 2\tilde{g}_{\text{coarse}}) - \#Z$. We now follow \cite{Kob20} to define $\chi(X)$ and $g(X)$. For $P, U, V, G$ as above, let $G_i \leq G$ be the ramification subgroups for $V \to U$ above $P$, and define

$$\delta_P := \sum_{i \geq 0} \left\lfloor \frac{|G_i| - 1}{|G|} \right\rfloor$$

(which simplifies to only the first term $(|G| - 1)/|G|$ if $X$ is tame above $P$). Then define the \textit{Euler characteristic} by

$$\chi(X) := \chi(X_{\text{coarse}}) - \sum_{P \in \mathcal{P}} \delta_P.$$ 

(This is motivated by the Riemann–Hurwitz formula. See \cite{VZB19,Kob20} for other motivation.) Finally, define the \textit{genus} $g = g(X)$ by $\chi(X) = 2 - 2g$.

\textbf{Lemma 6.} Let $X$ be a stacky curve over an algebraically closed field $k$ with $g < 1/2$. Then $X_{\text{coarse}} \simeq \mathbb{P}^1$ and $\#\mathcal{P} \leq 1$ and $X$ is tame.

\textit{Proof.} Since $g < 1/2$, we have $\chi(X) > 1$. For each $P \in \mathcal{P}$, note that $\delta_P \geq (|G| - 1)/|G| \geq 1/2$. Now

$$\chi(X) = 2 - 2\tilde{g}_{\text{coarse}} - \#Z - \sum_{P \in \mathcal{P}} \delta_P,$$

which is $\leq 1$ if $\tilde{g}_{\text{coarse}} \geq 1$ or $\#Z \geq 1$ or $\#\mathcal{P} \geq 2$. Thus $\tilde{g}_{\text{coarse}} = 0$, $\#Z = 0$, and $\#\mathcal{P} \leq 1$. Furthermore, if $X$ is not tame, then there exists $P \in \mathcal{P}$ with $\delta_P \geq (|G| - 1)/|G| + 1/|G| \geq 1$, which again forces $\chi(X) \leq 1$, a contradiction. \hfill \Box
Now let $k$ be any field. Let $\overline{k}$ be an algebraic closure of $k$, and let $k_n$ be the separable closure of $k$ in $k$. By a stacky curve over $k$, we mean an algebraic stack $X$ over $k$ such that the base extension $X_{\overline{k}}$ is a stacky curve over $\overline{k}$. Define $\chi(X) := \chi(X_{\overline{k}})$ and $g(X) := g(X_{\overline{k}})$.

**Lemma 7.** If $X$ is a tame stacky curve over $k$, then the set $\mathcal{P} \subset X_{\text{coarse}}(\overline{k})$ for $X_{\overline{k}}$ consists of points whose residue fields are separable over $k$.

**Proof.** Let $\bar{P} \in \mathcal{P}$. Let $P$ be the closed point of $X_{\text{coarse}}$ associated to $\bar{P}$. By working étale locally on $X_{\text{coarse}}$, we may assume that $X = [V/G]$ for a smooth curve $V$ over $k$ that is a $G$-Galois cover of $X_{\text{coarse}}$ totally tamely ramified above $P$. Analytically locally above $P$, the tame cover is given by the equation $y^n = \pi$ for some uniformizer $\pi$ at $P \in X_{\text{coarse}}$. After base change to $\overline{k}$, however, $\pi = u\pi^i$, where $u$ is a unit, $\pi$ is a uniformizer at $\bar{P}$, and $i$ is the inseparable degree of $k(P)/k$. Thus $V_{\overline{k}}$ is analytically locally given by $y^n = u\pi^i$. Since $V_{\overline{k}}$ is smooth, $i = 1$. Thus $k(P)/k$ is separable. □

Next, let $\mathcal{O}$ be a ring of $S$-integers in a global field $k$. By a stacky curve $\mathcal{X}$ over $\mathcal{O}$, we mean a separated finite-type algebraic stack over Spec $\mathcal{O}$ such that $\mathcal{X}_k$ is a stacky curve. (To be as general as possible, we do not impose Deligne–Mumford, tameness, smoothness, or properness conditions on the fibers above closed points of Spec $\mathcal{O}$.) Define $\chi(\mathcal{X}) := \chi(\mathcal{X}_{\overline{k}})$ and $g(\mathcal{X}) := g(\mathcal{X}_{\overline{k}})$.

4. **LOCAL–GLOBAL PRINCIPLE FOR RATIONAL POINTS**

We now explain why the local-global principle for rational points is not so interesting.

**Proposition 8.** Let $k$ be a global field. Let $X$ be a stacky curve over $k$ with $g < 1$. If $X(k_v) \neq \emptyset$ for all nontrivial places $v$ of $k$, then $X(k) \neq \emptyset$.

**Proof.** We have $0 < \chi(X) \leq 2 - 2\tilde{g}_{\text{coarse}}$, so $\tilde{g}_{\text{coarse}} = 0$. Thus $X_{\text{coarse}}$ is a smooth geometrically integral curve of genus 0. Because of the morphism $X \to X_{\text{coarse}}$, we have $X_{\text{coarse}}(k_v) \neq \emptyset$ for every $v$. By the Hasse–Minkowski theorem, $X_{\text{coarse}}(k) \neq \emptyset$, so $X_{\text{coarse}}$ is a dense open subscheme of $\mathbb{P}^1_k$. In particular, $X_{\text{coarse}}(k)$ is Zariski dense in $X_{\text{coarse}}$, and all but finitely many of these $k$-points correspond to $k$-points on $X$. □

Because of Proposition 8, our main theorems are concerned with the local-global principle for integral points.

5. **PROOF OF THEOREM D** COUNTEREXAMPLE TO THE LOCAL–GLOBAL PRINCIPLE

(a) Since $(\mathcal{X}_Q)_{\text{coarse}}$ is dominated by the genus 0 curve $\mathcal{Y}_Q$, we have $\tilde{g}_{\text{coarse}} = 0$. The action of $\mu_2$ on $\mathcal{Y}_Q$ fixes exactly two $\overline{Q}$-points, namely those with $z = 0$; thus $\mathcal{P} = 2$, and $\delta_P = 1/2$ for each $P \in \mathcal{P}$. Hence $\chi(\mathcal{X}) = (2 - 2 \cdot 0) - (1/2 + 1/2) = 1$. (Alternatively, $\chi(\mathcal{X}) = \chi(\mathcal{Y})/2 = 2/2 = 1$.)
(b) Let $R$ be a principal ideal domain. By definition of the quotient stack, a morphism $\text{Spec } R \to X$ is given by a $\mu_2$-torsor $T$ equipped with a $\mu_2$-equivariant morphism $T \to Y$. The torsors are classified by $H^1_{\text{fppf}}(R, \mu_2)$, which is isomorphic to $R^\times/R^\times2$, since $H^1_{\text{fppf}}(R, \mathbb{G}_m) = \text{Pic } R = 0$. Explicitly, if $t \in R^\times$, the corresponding $\mu_2$-torsor is $T_t := \text{Spec } R[u]/(u^2 - t)$. Define the twisted cover

$$Y_t := \text{Proj } R[x, y, z]/(tz^2 - f(x, y))$$

with its morphism $\pi_t : Y_t \to X$. To give a $\mu_2$-equivariant morphism $T_t \to Y$ is the same as giving a morphism $\text{Spec } R \to Y_t$. Thus we obtain

$$X(R) = \prod_{t \in R^\times} \pi_t(Y_t(R)).$$

For any $\ell \notin \{p, q, r\}$, the rank 3 form $z^2 - f(x, y)$ has good reduction at $\ell$, so $Y(\mathbb{F}_\ell) \neq \emptyset$, and Hensel’s lemma yields $Y(\mathbb{Z}_\ell) \neq \emptyset$. Since the discriminant of $f(x, y)$ is divisible only by $p$ and not $p^2$, the form is not identically 0 modulo $p$, so there exist $\bar{a}, \bar{b} \in \mathbb{F}_p$ with $f(\bar{a}, \bar{b}) \in \mathbb{F}_p^\times$. Lift $\bar{a}, \bar{b}$ to $a, b \in \mathbb{Z}_p$, so $f(a, b) \in \mathbb{Z}_p^\times$. Then $Y_{f(a, b)}(\mathbb{Z}_p) \neq \emptyset$. The same argument applies at $q$ and $r$. Since $f$ is positive definite, $Y(\mathbb{R}) \neq \emptyset$. Thus $X(\mathbb{Z}_\ell) \neq \emptyset$ for all primes $\ell$, and $X(\mathbb{R}) \neq \emptyset$.

(c) We now show that $X(\mathbb{Z}[1/(2pqr)]) = \emptyset$, i.e., that $Y_t(\mathbb{Z}[1/(2pqr)]) = \emptyset$ for all $t \in \mathbb{Z}[1/(2pqr)]^\times$, or equivalently, that the quadratic form $f(x, y)$ does not represent any element of $\mathbb{Z}[1/(2pqr)]^\times$ times a square in $\mathbb{Z}[1/(2pqr)]$.

Completing the square shows that $f$ is equivalent over $\mathbb{Q}$ to the diagonal form $[a, apqr]$. If we use $u = u_v$ to denote a unit nonresidue in $\mathbb{Z}_v$, then

- over $\mathbb{Q}_p$, the form $f$ is equivalent to $[u, up]$ and represents the squareclasses $u, up$;
- over $\mathbb{Q}_q$, the form $f$ is equivalent to $[1, uq]$ and represents the squareclasses $1, uq$;
- over $\mathbb{Q}_r$, the form $f$ is equivalent to $[u, ur]$ and represents the squareclasses $u, ur$.

Therefore,

- $f$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- $-f$ takes square values in $\mathbb{Q}_q$ and $\mathbb{Q}_r$, but not in $\mathbb{R}$ and $\mathbb{Q}_q$.

It follows that $f$ and $-f$ together represent squares locally at all places, but do not globally represent squares.

We now further check that $sf$, for every factor $s$ of $pqr$, fails to globally represent a square (by quadratic reciprocity, $r$ is not a square (mod $p$) and (mod $q$), and $q$ is not a square (mod $p$)):

- $pf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.
- $qf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- $rf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- $pqf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
• prf takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
• qrf takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.
• pqrf takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.

Since $2$ is a square in $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Q}_q$, and $\mathbb{Q}_r$, multiplying each of the sf’s in the above statements by $2$ would not change the truth of any these statements. Meanwhile, since $-1$ and $-2$ are nonsquares in $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Q}_q$, and $\mathbb{Q}_r$, multiplying the sf’s in the statements above by $-1$ or $-2$ would simply reverse all the conditions (in particular, all would fail to represent squares in $\mathbb{R}$).

We conclude that $\mathcal{Y}_t(\mathbb{Z}[1/(2pqr)]) = \emptyset$ for all $t \in \mathbb{Z}[1/(2pqr)]^\times$, i.e., $\mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset$, as claimed.

The same arguments apply to $\mathcal{X}' := [\mathcal{Y}/(\mathbb{Z}/2\mathbb{Z})]$; in particular,
$$\mathcal{X}'(\mathbb{Z}[1/(2pqr)]) = \mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset,$$
because the homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$ is an isomorphism over $\mathbb{Z}[1/2]$ and hence over $\mathbb{Z}[1/(2pqr)]$.

6. Stacks over local rings

This section contains some results to be used in the proof of Theorem [5].

Proposition 9. Let $A$ be a noetherian local ring. Let $X$ be an algebraic stack of finite type over $A$. Let $x \in X(A)$. Then there exists a finite-type algebraic space $U$ over $A$, a smooth surjective morphism $f: U \to X$, and an element $u \in U(A)$ such that $f(u) = x$.

Proof. By definition, there exists a finite-type $A$-scheme $V$ and a smooth surjective morphism $V \to X$. Taking the $2$-fiber product with $\text{Spec } A \to X$ yields an algebraic space $V_x \to \text{Spec } A$. Then $V_x \to \text{Spec } A$ is smooth, so it admits étale local sections. Thus we can find a Galois étale extension $A'$ of $A$, say with group $G$, such that $x$ lifts to a morphism $\text{Spec } A' \to V$ equipped with a compatible system of isomorphisms between the conjugates of $v$.

Let $n = \#G$. Let $V^n_X$ be the $2$-fiber product over $X$ of $n$ copies of $V$, indexed by $G$. The left translation action of $G$ on $G$ induces a right $G$-action on $V^n_X$ respecting the morphism $V^n_X \to X$, and there is also a right $G$-action on $\text{Spec } A'$. Therefore we may twist $V^n_X$ to obtain a new algebraic space $U$ lying over $X$ (a quotient of $V^n_X \times_A A'$ by a twisted action of $G$) such that the element of $V^n_X(A')$ given by the conjugates of $v$ and the isomorphisms between them descends to an element of $U(A)$. \qed

Remark 10. Atticus Christensen, combining a variant of our proof with other arguments, has extended Proposition [5] to other rings $A$, such as arbitrary products of complete noetherian local rings, and adèle rings of global fields [Chr20, Theorem 7.0.7 and Propositions 12.0.5 and 12.0.8].
For any valued field \( K \), let \( \hat{K} \) denote its completion.

**Proposition 11.** Let \( A \) be an excellent henselian discrete valuation ring. Let \( K = \text{Frac} \, A \). Let \( U \) be a separated finite-type algebraic space over \( K \).

(a) The set \( U(K) \) has a topology inherited from the topology on \( K \).
(b) If \( U \) is smooth and irreducible, then any nonempty open subset of \( U(K) \) is Zariski dense in \( U \).

**Proof.**
(a) In fact, much more is true: if \( K = \hat{K} \), then the analytification of \( U \) exists as a rigid analytic space \([\text{CT09}, \text{Theorem 1.2.1}]\). If \( K \neq \hat{K} \), equip \( U(K) \) with the subspace topology inherited from \( U(\hat{K}) \).
(b) If \( K = \hat{K} \), this follows from the fact that a nonzero power series in \( n \) variables over \( K \) cannot vanish on a nonempty open subset of \( K^n \). If \( K \neq \hat{K} \), use Artin approximation: any point of \( U(\hat{K}) \) can be approximated by a point of \( U(K) \). \(\square\)

**Proposition 12.** Let \( A \) be an excellent henselian discrete valuation ring. Let \( K = \text{Frac} \, A \). Let \( U \) be a separated finite-type algebraic space over \( A \). Then \( U(A) \) is an open subset of \( U(K) \).

**Proof.** Since \( U \) is separated over \( A \), the map \( U(A) \to U(K) \) is injective. Let \( u \in U(A) \). Choose a separated \( A \)-scheme \( V \) with an étale surjective morphism \( f: V \to U \). Then \( u \) lifts to some \( v \in V(A') \) for some finite étale \( A \)-algebra \( A' \). Let \( K' = \text{Frac} \, A' \). Since \( V \) is a separated \( A \)-scheme, \( V(A') \) is an open subset of \( V(K') \). If \( A \) is complete, then the étale morphism \( V \to U \) induces an étale morphism of analytifications \([\text{CT09}, \text{Theorem 2.3.1}]\), so \( V(K') \to U(K') \) is a local homeomorphism; in particular, it defines a homeomorphism from a neighborhood \( N_U \) of \( u \) in \( V(K') \) to a neighborhood \( N_U \) of \( u \) in \( U(K') \), and we may assume that \( N_U \subseteq V(A') \). In the general case, a given point of \( V(\hat{K}') \) maps to some point of \( U(K') \) if and only if it is in \( V(K') \), so the homeomorphism for \( \hat{K}' \)-points restricts to a homeomorphism for \( K' \)-points, which we again denote \( N_U \to N_U \). If \( u_1 \in N_U \cap U(K) \), then \( u_1 \) lies in the image of \( N_U \subseteq V(A') \), so \( u_1 \in U(A') \); now \( u_1 \in U(A') \cap U(K) \), which is \( U(A) \) since \( U \) is a sheaf on \( (\text{Spec} \, A)_{\text{fppf}} \). Hence \( U(A) \) is open in \( U(K) \). \(\square\)

7. **Proof of Theorem 5**

By Lemma \([\text{6}]\) we have \( (\mathcal{X}_\mathbb{C})_{\text{coarse}} \simeq \mathbb{P}^1_{\mathbb{K}} \), and hence \( (\mathcal{X}_k)_{\text{coarse}} \) is a smooth proper curve of genus 0. Since \( \mathcal{X} \) has an \( \mathcal{O}_v \)-point for every \( v \), the stack \( \mathcal{X}_k \) has a \( k_v \)-point for every \( v \), so \( (\mathcal{X}_k)_{\text{coarse}} \) has a \( k_v \)-point for every \( v \). Thus \( (\mathcal{X}_k)_{\text{coarse}} \simeq \mathbb{P}^1_k \).

If \( \mathcal{X}_k \to (\mathcal{X}_k)_{\text{coarse}} \) is not an isomorphism, then by Lemma \([\text{6}]\) there is a unique \( \overline{k} \)-point above which it fails to be an isomorphism, and by Lemma \([\text{7}]\) it is a \( k_s \)-point, and that point
must be \( \text{Gal}(k_\pi/k) \)-stable, hence a \( k \)-point of \( \mathbb{P}^1 \), which we may assume is \( \infty \). Thus \( \mathcal{X}_k \) contains an open substack isomorphic to \( \mathbb{A}^1_k \).

Since all the stacks are of finite presentation, the isomorphism just constructed extends above some affine open neighborhood of the generic point in \( \text{Spec} \mathcal{O} \). That is, there exists a finite set of places \( S' \supseteq S \) such that if \( \mathcal{O}' \) is the ring of \( S' \)-integers in \( k \), then the stack \( \mathcal{X}_{\mathcal{O}'} \) contains an open substack isomorphic to \( \mathbb{A}^1_{\mathbb{A}^1_{\mathcal{O}'}} \).

Let \( v \in S' - S \). Let \( \mathcal{O}_v(\mathcal{O}) \) be the localization of \( \mathcal{O} \) at \( v \), and let \( \mathcal{O}_{v,h} \) be its henselization in \( \mathcal{O}_v \), so \( \mathcal{O}_{v,h} \) is the set of elements of \( \mathcal{O}_v \) that are algebraic over \( k \). Let \( k_{v,h} = \text{Frac} \mathcal{O}_{v,h} \). We are given \( x \in \mathcal{X}(\mathcal{O}_v) \). Let \( U, f, u \) be as in Proposition \( \mathbb{A}^1 \) with \( A = \mathcal{O}_v \). By Proposition \( \mathbb{A}^1 \), \( U(\mathcal{O}_v) \) is open in \( U(k_v) \). Let \( U_0 \) be the connected component of \( U(k_v) \) containing \( u \), so \( U_0(k_v) \) is open in \( U(k_v) \). The morphisms \( U_0 \to U_{k_v} \to \mathcal{X}_{k_v} \to \text{Spec} k_v \) are smooth, so \( U_0 \) is smooth and irreducible. Therefore, by Proposition \( \mathbb{A}^1 \), the set \( U(\mathcal{O}_{v,h}) \cap U_0(k_v) \) is Zariski dense in \( U_0 \). On the other hand, \( U_0 \) dominates \( \mathcal{X}_{k_v} \) since \( U_0 \to \mathcal{X}_{k_v} \) is smooth and \( \mathcal{X}_{k_v} \) is irreducible.

By the previous two sentences, there exists \( u_0 \in U(\mathcal{O}_{v,h}) \cap U_0(k_v) \) mapping into the subset \( \mathbb{A}^1(k_v) \) of \( \mathcal{X}(k_v) \). By Artin approximation, we may replace \( u_0 \) by a nearby point to assume also that \( u_0 \in U(\mathcal{O}_{v,h}) \).

Let \( U_1 \) be the inverse image of \( \mathbb{A}^1_{k_{v,h}} \) under \( U_{k_v,h} \to \mathcal{X}_{k_v,h} \). By Proposition \( \mathbb{A}^1 \), \( U(\mathcal{O}_{v,h}) \) is open in \( U(k_{v,h}) \), so \( U(\mathcal{O}_{v,h}) \cap U_1(k_{v,h}) \) is an open neighborhood of \( u_0 \) in \( U_1(k_{v,h}) \). Since \( U_1 \to \mathbb{A}^1_{k_{v,h}} \) is smooth, the image of this neighborhood is a nonempty open subset \( B_v \) of \( \mathbb{A}^1(k_{v,h}) \). By construction, \( B_v \) is contained in the image of \( U(\mathcal{O}_{v,h}) \to \mathcal{X}(\mathcal{O}_{v,h}) \subseteq \mathcal{X}(k_{v,h}) \), so \( B_v \subseteq \mathcal{X}(\mathcal{O}_{v,h}) \).

By strong approximation, there exists \( x \in \mathbb{A}^1(\mathcal{O}') \) such that \( x \in B_v \) for all \( v \in S' - S \). For each \( v \in S' - S \), since \( B_v \subseteq \mathcal{X}(\mathcal{O}_{v,h}) \), there exists \( x_v \in \mathcal{X}(\mathcal{O}_{v,h}) \) such that \( x \) and \( x_v \) become equal in \( \mathcal{X}(k_{v,h}) \). Finally, the following lemma shows that \( x \) comes from an element of \( \mathcal{X}(\mathcal{O}) \).

**Lemma 13.** If \( x \in \mathcal{X}(\mathcal{O}') \) and \( x_v \in \mathcal{X}(\mathcal{O}_{v,h}) \) for each \( v \in S' - S \) are such that the images of \( x \) and \( x_v \) in \( \mathcal{X}(k_{v,h}) \) are equal for every \( v \in S' - S \), then there exists an element of \( \mathcal{X}(\mathcal{O}) \) mapping to \( x \) in \( \mathcal{X}(\mathcal{O}') \) and to \( x_v \) in \( \mathcal{X}(\mathcal{O}_{v,h}) \) for each \( v \in S' - S \).

**Proof.** Since \( \mathcal{X} \) is of finite presentation over \( \mathcal{O} \), the element \( x_v \) comes from an element \( \tilde{x}_v \) of some finitely generated \( \mathcal{O} \)-subalgebra \( A_v \) of \( \mathcal{O}_{v,h} \). The schemes \( \text{Spec} A_v \) together with \( \text{Spec} \mathcal{O}' \) form an fppf covering of \( \text{Spec} \mathcal{O} \), so the stack property of \( \mathcal{X} \) shows that \( x \) and the \( \tilde{x}_v \) come from an element of \( \mathcal{X}(\mathcal{O}) \). \( \square \)

**Remark 14.** Inspired by an earlier draft of our article, Christensen has found a natural way to define a topology on the set of adelic points of a finite-type algebraic stack, and has proved a strong approximation theorem for a stacky curve with \( \chi > 1 \) [Chr20, Theorem 13.0.6]. His argument can substitute for the three paragraphs before Lemma \( \mathbb{A}^1 \) and hence give a partially independent proof of Theorem \( \mathbb{A}^1 \).
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