A simple proof of uniqueness of the particle trajectories for solutions of the Navier–Stokes equations

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Abstract
We give a simple proof of the uniqueness of fluid particle trajectories corresponding to (1) the solution of the two-dimensional Navier–Stokes equations with an initial condition that is only square integrable and (2) the local strong solution of the three-dimensional equations with an $H^{1/2}$-regular initial condition, i.e. with the minimal Sobolev regularity known to guarantee uniqueness. This result was proved by Chemin and Lerner (1995 J. Diff. Eqns 121 314–28) using the Littlewood–Paley theory for the flow in the whole space $\mathbb{R}^d$, $d \geq 2$. We first show that the solutions of the differential equation $\dot{X} = u(X, t)$ are unique if $u \in L^p(0, T; H^{(d/2)-1})$ for some $p > 1$ and $\sqrt{t}u \in L^2(0, T; H^{(d/2)+1})$. We then prove, using standard energy methods, that the solution of the Navier–Stokes equations with initial condition in $H^{(d/2)-1}$ satisfies these conditions. This proof is also valid for the more physically relevant case of bounded domains.

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1. Introduction
We consider the Navier–Stokes equations

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \quad u(x, 0) = u_0, \quad u|_{\partial \Omega} = 0,$$

(1)

in which $x \in \Omega \subset \mathbb{R}^d$ with $d = 2, 3$, $\Omega$ is an open set with a sufficiently smooth boundary, $u(x, t)$ is the velocity vector field, $p(x, t)$ the pressure scalar function, $f(x, t)$ the body force and $\nu$ is the kinematic viscosity which is considered constant.
The minimal Sobolev regularity for the initial condition that is known to give rise to a unique solution is \( u_0 \in H^{d/2-1}(\Omega) \), and then the solution \( u \) is in \( L^\infty(0, T; H^{d/2-1}(\Omega)) \cap L^2(0, T; H^{d/3}(\Omega)) \), where \( H^s \) with real \( s > 0 \) is the standard Sobolev space of order \( s \) (we recall the characterization of fractional Sobolev spaces in section 3.2). That it is sufficient to take \( u_0 \in L^2(\Omega) \) for the two-dimensional domain was shown by Leray (1933) for the whole plane, and Lions and Prodi (1959) and Ladyzhenskaya (1958) for bounded domains, and that \( u_0 \in H^{1/2}(\Omega) \) suffices in a three-dimensional domain was shown by Fujita and Kato (1964).

For the two-dimensional Navier–Stokes equations the above unique solution is global in time, while in the three-dimensional case a unique solution exists on \([0, T_1]\) and the best available bound on the \( H^{1/2}\) -norm of \( u(t) \) tends to infinity as \( t \to T_1 \). We consider \( T < T_1 \) in the three-dimensional case. In this paper we denote by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) the inner product and norm in \( L^2(\Omega) \), respectively. The norm in \( H^s(\Omega) \) is denoted by \( \| \cdot \|_s \) and the norm in any other normed space \( E \), by \( \| \cdot \|_E \).

Corresponding to the solution \( u \) defined above, as long as it exists, the fluid particle trajectories are the solutions of

\[
\frac{dX}{dt} = u(X, t), \quad X(0) = a \in \Omega.
\]  

(2)

At least one solution to the above system exists. This can be shown (following Foias et al (1985)) by considering \( u_n \) to be the Galerkin approximations of \( u \) and defining \( X_n \) to be the solution of

\[
\frac{dX_n}{dt} = u_n(X_n, t), \quad \text{with } X_n(0) = a
\]

and then showing the uniform convergence of \( X_n \) to \( X \) in \([0, T]\) and strong convergence of \( u_n \) to \( u \) in \( L^1(0, T; L^\infty) \). We adapt and explain this argument in the proof of the existence of solutions in theorem 2.1.

The uniqueness of the solutions of (2) in the whole space is shown by Chemin and Lerner (1995) using the Littlewood–Paley theory. In this paper we present an alternative simpler proof which is valid in a general bounded domain as well.

1.1. Chemin and Lerner’s proof of uniqueness

The uniqueness of the solutions of (2) in the whole space is shown by Chemin and Lerner in their 1995 paper. They use the Littlewood–Paley theory to prove enough regularity for the solution of the Navier–Stokes equations in order to be able to apply a generalization of the Osgood criterion to (2). They denote by \( \mathcal{H}_{1,T}^{d/2+1} \) the space of functions \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) satisfying

\[
\left( \sum_{q \in \mathbb{N}} 2^{q(2d+d)} \left( \int_0^T \| \Delta_q u(t) \|_{L^2}^2 \right)^{1/2} \right) < \infty,
\]

where

\[
\Delta_q u = 2^{qd} \int_{\mathbb{R}^d} h(2^q y)u(x - y) \, dy,
\]

and \( h \) is the inverse Fourier transform of some \( \phi \) that is an appropriate bump function supported on the annulus \( \{3/4 \leq |\xi| \leq 8/3\} \). They prove that the solution of the two-dimensional Navier–Stokes equations is an element of \( \mathcal{H}_{1,T}^{d/2+1} \) and then show that \( \mathcal{H}_{1,T}^{d/2+1} \subset L^1_{loc}(0, T; C_{\omega_1}(\mathbb{R}^d; \mathbb{R}^d)) \), where \( \omega_1(r) = r(1 - \log r)^{d+1}/2 \) and \( C_{\omega_1}(\mathbb{R}^d; \mathbb{R}^d) \) is a Banach space with the norm

\[
\|u\|_{\omega_1} = \|u\|_{L^\infty(\mathbb{R}^d)} + \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|u(x) - u(y)|}{\omega_1(|x - y|)}.
\]
They conclude the uniqueness of the flow by proving a generalization of the Osgood criterion, which states that if \( F \in L^1_{\text{loc}}(0, T; C_\omega) \) with \( \omega \) satisfying

\[
\int_0^1 \frac{dr}{\omega(r)} = +\infty,
\]

then the equation

\[
x(t) = x_0 + \int_0^t F(s, x(s)) \, ds
\]

has a unique solution over \([0, t_1]\) for some \( t_1 < T \).

1.2. The summary of our proof

We present an alternative proof of the same uniqueness result in the case of bounded two- and three-dimensional domains, which is also valid for the whole space \( \mathbb{R}^d \) and periodic domains.

The proof is in fact elementary. We denote by \( \eta(t) \) the Euclidean norm of the difference of two solutions of (2) at time \( t \), write down the differential equation satisfied by \( \eta(t) \) and derive an upper bound for this difference in terms of the vector field \( u \) and the value of \( \eta \) at some previous time \( s > 0 \). Letting \( s \to 0 \), however, is not straightforward. We outline the difficulty here and address it in the following sections.

Assuming that both \( X(t) \) and \( Y(t) \) satisfy (2), we have

\[
\frac{d}{dt} (X - Y) = u(X, t) - u(Y, t), \quad \text{with } X(0) - Y(0) = 0.
\]

Using a result of Zuazua (2002), we know that a vector field \( u \) over a \( d \)-dimensional domain \( \Omega \) satisfies

\[
|u(X) - u(Y)| \leq c\|u\|_{1+d/2} |X - Y| (-\log |X - Y|)^{1/2}
\]

for any \( X, Y \in \Omega \). This bound is obtained easily by considering the extension of \( u \) to \( \mathbb{R}^d \), \( E[u] \), and writing \( E[u] \) as the inverse of its Fourier transform. Once we have the above inequality we can write, for \( \eta(t) = |X(t) - Y(t)| \),

\[
\frac{d\eta}{dt} \leq c\|u\|_{1+d/2} \eta (-\log \eta)^{1/2},
\]

implying that

\[
\eta(t) \leq \exp \left( - \left( \left( \log(1/\eta(s)) \right)^{1/2} - c \int_s^t \|u\|_{1+d/2} \, d\tau \right)^2 \right)
\]

for \( 0 < s < t \). The uniqueness of the solutions of (2) follows by showing that the right-hand side of the above inequality converges to zero as \( s \to 0 \). If \( \int_0^t \|u\|_{1+d/2} \, d\tau < \infty \), then one could simply let \( s \to 0 \) to obtain the result. The problem is that it is not known whether \( u \in L^1(0, T; H^{1+d/2}) \) (in fact even for the heat equation this will not be true in general). To circumvent this, we bound \( \int_s^t \|u\|_{1+d/2} \, d\tau \) and \( \eta(s) \) from above and show that

\[
\lim_{s \to 0} \left( \left( \log(1/\eta(s)) \right)^{1/2} - c \int_s^t \|u\|_{1+d/2} \, d\tau \right) = \infty
\]

for small enough \( t \). We note that \( u \) is smooth for any \( t > 0 \) and therefore showing the uniqueness for an arbitrary small interval containing \( t = 0 \) is enough.

In section 2 we give sufficient conditions on the vector function \( u \) that result in appropriate bounds on the logarithmic and integral terms in the right-hand side of (3), as discussed above,
to ensure the uniqueness of the solution of the ordinary differential equation (2). We require that \( u \) satisfies
\[
u \in L^p(0, T; H^{(d/2)-1}(\Omega)) \quad \text{with} \quad p > 1, \quad \text{and} \quad \sqrt{t} u \in L^2(0, T; H^{1+d/2}(\Omega)). \tag{4}
\]

We then, in section 3, show that the solution of the Navier–Stokes equations (1) with \( u_0 \in H^{(d/2)-1}(\Omega) \) and \( f \in L^2(0, T; H^{(d/2)-1}(\Omega)) \) satisfies (4). This is straightforward in the two-dimensional case. For bounded three-dimensional domains, we need to use the fractional powers of \( A = -\Delta \) with \( D(A) = \{ u \in H^1(\Omega) : u|_{\partial \Omega} = 0 \} \). To deal with these, we need the equivalence of \( \| A^{1/2} u \| \) and \( \| u \|_s \) for any non-negative real number \( r \) and any \( u \in D(A^{1/2}) \).

For \( 0 < r < 2 \), a concrete characterization of \( D(A^{1/2}) \) is known (Fujiwara 1967) which gives the equivalence of \( \| A^{1/2} u \| \) and \( \| u \|_r \), for \( 0 < r < 2 \). But, to our knowledge, such a characterization does not exist for \( r > 2 \). The equivalence of the norms in \( D(A^{1/2}) \) and \( H^r \) for any \( r \geq 0 \), however, can be concluded almost immediately from an interpolation theorem of Lions and Magenes (1972). We show this in section 3.2 and use it to prove the validity of (4) for the solutions of the three-dimensional Navier–Stokes equations.

2. A general uniqueness result

In this section we prove a uniqueness result for general ordinary differential equations. It will then be shown in the next section that the uniqueness condition of this theorem is satisfied by the solution of the Navier–Stokes equations.

**Theorem 2.1.** Let \( \Omega \) be the whole space \( \mathbb{R}^d \), \( d \geq 2 \), a periodic \( d \)-dimensional domain or an open bounded subset of \( \mathbb{R}^d \) with a sufficiently smooth boundary. Consider
\[
u \in L^p(0, T; H^{(d/2)-1}(\Omega)) \quad \text{with} \quad p > 1, \quad \text{and} \quad \sqrt{t} u \in L^2(0, T; H^{1+d/2}(\Omega)),
\]
with \( u = 0 \) on \( \partial \Omega \) when \( \Omega \) is a bounded domain. Then the differential equation
\[
\frac{dX(t)}{dt} = u(X(t), t), \quad X(0) = a \in \Omega \tag{5}
\]
has a unique solution over \([0, T]\).

We note that in the case of a bounded domain \( \Omega \), assuming \( \partial \Omega \) to satisfy the uniform \( C^1 \)-regularity condition (Adams 1975) is sufficient. This ensures that the trace operator, mapping \( u \) to \( u|_{\partial \Omega} \), makes sense (Adams 1975, theorem 7.53), and also the extension operator used to derive the Sobolev embedding results on \( \Omega \) from those on \( \mathbb{R}^d \) is well-defined (Adams 1975, theorem 4.32).

**Proof.**

To prove the existence, we first show the integrability of \( \| u \|_{L^\infty(\Omega)} \) over \((0, T)\). By Agmon’s inequality we have
\[
\| u \|_{L^\infty(\Omega)} \leq c \| u \|_{(d/2)\cdot 1}^{1/2} \| u \|_{(d/2)\cdot 1}^{1/2}
\]
and therefore we can write
\[
\int_0^t \| u \|_{L^\infty(\Omega)} \, dt \leq c \int_0^t \tau^{(3p-2)/2} \| u \|_{(d/2)\cdot 1}^{1/2} \| u \|_{(d/2)\cdot 1}^{1/2} \, dt \leq c \left( \int_0^t \tau^{-p(3p-2)} \, dt \right)^{1/2-p(3p-2)/2} \left( \int_0^t \| u \|_{(d/2)\cdot 1}^{2} \, dt \right)^{1/2}
\]
\[
\leq c \left( \int_0^t \tau^{-p} \, dt \right)^{1/2} \left( \int_0^t \| u \|_{(d/2)\cdot 1}^{2} \, dt \right)^{1/2}.
\]


For a bounded or periodic domain $\Omega$, we let $u_n$ be the $n$-dimensional Galerkin approximation of $u$ (the image of $u$ under the projection in $L^2$ onto the space spanned by the first $n$ eigenfunctions of $A = -\Delta$ with Dirichlet boundary conditions when $\Omega$ is bounded and periodic boundary conditions for a periodic domain). For the whole space $\mathbb{R}^d$, we consider $u_n$ to be a sequence of mollified versions of $u$ with $1/n$ the parameter of mollification. Following Foias et al (1985) we consider $X_n$ to be the solution of

$$\frac{dX_n}{dt} = u_n(X_n, t), \quad \text{with} \quad X_n(0) = a.$$ 

Now, $u_n$ is continuous in $X_n$ and also has the same regularity properties of $u$ and is therefore integrable with respect to $t$. Hence a solution of the above differential system exists over $[0, T]$ unless, in the case of $\Omega$ a bounded domain, the particle leaves $\Omega$ at some time less than $T$. This is not possible since $u_n$ is zero on $\partial\Omega$ and hence (2) is solvable over $[0, T]$. By (7), we also have

$$|X_n(t)| \leq \int_0^t |u_n(X_n, t)| \, dt < c$$

and

$$|X_n(t) - X_n(s)| \leq \int_s^t \|u(\tau)\|_{L^\infty(\Omega)} \, d\tau \leq c |t - s|^{\frac{d-1}{2}},$$

implying that $\{X_n\}$ is equicontinuous. Therefore, by the Arzelà–Ascoli theorem, $X_n$ has a subsequence (which we label by $n$ again) that converges uniformly to some $X(t)$. To prove that $X(t)$ is the solution of (6) we need to show that $u_n(X_n, t) \to u(X, t)$ in $L^1(0, T)$. We can write

$$\int_0^T |u_n(X_n, t) - u(X, t)| \, dt \leq \int_0^T |u_n(X_n, t) - u(X_n, t)| \, dt + \int_0^T |u(X_n, t) - u(X, t)| \, dt.$$

The first term in the right-hand side converges to zero since we know that $u_n \to u$ in $L^1(0, T; L^\infty)$. For the second term in the right-hand side of (8) we note that for almost every strictly positive $t$, $u(t) \in H^{d/2+1}$ and is therefore continuous in $x$, implying that $u(X_n, t) \to u(X, t)$ as $n \to \infty$ for almost every $t \in (0, T)$. Since $u(x, t)$ is integrable over $(0, T)$, the result follows by the Lebesgue dominated convergence theorem.

We now prove uniqueness. In section 1, we showed that $\eta(t) = |X(t) - Y(t)|$ satisfies

$$\eta(t) \leq \exp \left(- \left( \left( \log(1/\eta(s)) \right)^{1/2} - c \int_s^t \|u\|_{L^{1+d/2}} \, d\tau \right)^2 \right)$$

for $0 < s < t$. We need to obtain appropriate lower and upper bounds for the logarithmic and integral term, respectively, in the right-hand side of the above inequality as $s \to 0$. To bound $\log(1/\eta(s))$, we write

$$\frac{d\eta}{dt} \leq |u(X, t) - u(Y, t)| \leq 2\|u\|_{L^\infty(\Omega)}.$$

By (7) we have

$$\eta(s) \leq c \int_0^s \|u\|_{L^\infty(\Omega)} \, d\tau \leq c s^{\frac{d-1}{2}},$$

which implies that

$$\log \frac{1}{\eta(s)} \geq c \log \frac{1}{s},$$

for $s$ sufficiently small.
For the integral term in the right-hand side of (9) we write
\[ \int_s^t \|u\|_{1+d/2} \, d\tau \leq \int_s^t \tau^{-1/2} \tau^{1/2} \|u\|_{1+d/2} \, d\tau \]
\[ \leq \left( \int_s^t \tau^{-1} \, d\tau \right)^{1/2} \left( \int_s^t \tau \|u\|_{1+d/2}^2 \, d\tau \right)^{1/2} \]
\[ \leq c (\log t - \log s)^{1/2} \left( \int_s^t \tau \|u\|_{1+d/2}^2 \, d\tau \right)^{1/2} \]
and therefore for \( s \) small enough
\[ \int_s^t \|u\|_{1+d/2} \, d\tau \leq c K_d(t) \left( \log t + \log \frac{1}{s} \right)^{1/2} \]
with
\[ K_d^2(t) = \int_0^t \tau \|u\|_{1+d/2}^2 \, d\tau. \]

Having the above bounds, we go back to (9), fix \( t \) and let \( s \to 0 \) in the right-hand side to write
\[ \eta(t) \leq \exp \left( -\lim_{s \to 0} \log(1/s) (1 - c K_d(t))^2 \right). \]
Since \( K_d^2(t) \) is the integral of an integrable function, it is absolutely continuous (Priestley 1997) and therefore we can choose \( T^* \) small enough so that
\[ 1 - c K_d(T^*) > 0. \]
Hence \( \eta(t) \to 0 \) for any \( t \leq T^* \) as \( s \to 0 \). This gives the result for \( t \in [0, T^*] \). The uniqueness over \([T^*, T]\) then follows easily from (9) since \( u \in L^1([T^*, T] ; H^{d/(2r+1)}) \).

We note that the assumption \( \sqrt{t} \, u \in L^2(0, T ; H^{d/(2r+1)}) \) implies that \( u \in L^r(0, T ; H^{d/(2r+1)}) \) for any \( r < 1 \).

Using an argument similar to the proof of the existence part of the above theorem, one can show that if a sequence \( \sqrt{t} \, u_n \) is uniformly bounded in \( L^2(0, T ; H^{d/(2r+1)}) \) and \( u_n \to u \) strongly in \( L^p(0, T ; H^{d/(2r+1)}(\Omega)) \) with \( p > 1 \), then the solution of \( X_n = u_n(X_n, t) \) converges uniformly to the solution of \( \dot{X} = u(X, t) \) (Dashti and Robinson 2009). Using this in the case of the Navier–Stokes equations, one can show that the map \( u_0 \mapsto X(t) \) is continuous from \( H^{d/(2r+1)}(\Omega) \) into \( C^0([0, T); \mathbb{R}^d) \).

3. The estimates for the solutions of the Navier–Stokes equations

In this section we show that \( u \), a unique solution of the Navier–Stokes equations (1) with \( u_0 \in H^{d/(2r+1)}(\Omega) \), satisfies the conditions of theorem 2.1 and thereby prove the following theorem:

**Theorem 3.1.** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^d \), \( d = 2, 3 \), with a sufficiently smooth boundary. Consider \( u \in L^\infty(0, T; H^{d/(2r+1)}(\Omega)) \cap L^2(0, T; H^{d/(2r+1)}(\Omega)) \), a unique solution of the Navier–Stokes equations (1) with \( u_0 \in H^{d/(2r+1)}(\Omega) \) and \( f \in L^2(0, T; H^{d/(2r+1)}(\Omega)) \). Then the ordinary differential system
\[ \frac{dX(t)}{dt} = u(X(t), t), \quad \text{with } X(0) = a, \]
has a unique solution.
We note that for the above result to be true, assuming $\partial \Omega$ to satisfy uniform $C^1$-regularity condition is sufficient. For such a domain both the boundary trace embedding theorem (Adams 1975, theorem 5.22) that we need for obtaining the estimates of lemmas 3.2 and 3.4, and the result of theorem 2.1 hold.

3.1. The two-dimensional case

The following result is due to Temam (1977).

**Lemma 3.2.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^2$ with a sufficiently smooth boundary. Consider $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, a unique solution of the two-dimensional Navier–Stokes equations (1) with $u_0 \in L^2(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$. Then

$$
\int_0^T t \|u\|_2^2 \, dt \leq C, 
$$

where the constant $C$ depends on $\Omega$, $\nu$, $\|u_0\|$ and $\int_0^T \|f\|_2^2 \, dt$.

**Proof.**

Let $A = \Pi(-\Delta)$ be the Stokes operator, where $\Pi$ is the orthogonal projection from $L^2(\Omega)$ onto

$$
H = \{u \in L^2(\Omega) : \nabla \cdot u = 0 \text{ and } u \cdot n = 0 \text{ in the trace sense},
$$

with $n$ the outward normal vector on the boundary. We take the inner product of (1) with $tAu$.

Since $(\nabla p, Au) = 0$, we can write

$$
\frac{1}{2} \frac{d}{dt} (t \|A^{1/2}u\|^2) - \|A^{1/2}u\|^2 + \nu t \|Au\|^2 
\leq t \left( (u \cdot \nabla)u, Au \right) + t (f, Au) 
\leq c t \|u\|_2^{1/2} \|u\|_1 \|u\|_2^{3/2} + t \|f\| \|Au\| 
\leq c t \|u\|_2^2 \|u\|_1^4 + c t \|f\|^2 + \frac{\nu}{2} \|Au\|^2 
\leq c t \|u\|_2^2 \|u\|_1^4 \|A^{1/2}u\|^2 + c t \|f\|^2 + \frac{\nu}{2} \|Au\|^2,
$$

since $\|u\|_1 \leq c \|A^{1/2}u\|$ (see, for example, Robinson (2001, proposition 6.18)). Therefore,

$$
\frac{d}{dt} (t \|A^{1/2}u\|^2) + \nu t \|Au\|^2 \leq (c \|u\|_2^2 \|u\|_1^4) t \|A^{1/2}u\|^2 + c t \|f\|^2 + c \|u\|_1^2.
$$

Multiplying both sides of the above inequality with $E_2(t) = \exp \left( -c \int_0^t \|u\|_2^2 \|u\|_1^4 \, dx \right)$, we obtain

$$
\frac{d}{dt} (t E_2(t) \|A^{1/2}u\|^2) + \nu t E_2(t) \|Au\|^2 \leq c t \|f\|^2 + c \|u\|_1^2.
$$

Now integrating the above inequality from 0 to $T$ (which can be made rigorous using the Galerkin approximation), we obtain

$$
\int_0^T t \|u\|_2^2 \, dt \leq c e^{c \|u\|_2^2} \int_0^T \|u\|_1^4 + \|f\|^2 \, dt,
$$

since $\|u\|_2 \leq c \|Au\|$. The result follows. \qed
This, by theorem 2.1, proves the uniqueness of the particle trajectories.

3.2. The three-dimensional case

In this case when obtaining the bound on \( \int_0^T \|u\|_{5/2} \, d\tau \) we will need to use the equivalence of \( \|u\|_{r} \) and \( \|A^{1/2} u\| \) where \( A = -\Delta \) with Dirichlet boundary conditions, and \( r \) is any non-negative real number. For \( 0 \leq r \leq 2 \), this equivalence follows from Fujiiwara’s (1967) characterization of \( D(A^r) \). For \( r > 2 \), to our knowledge, such a characterization does not exist. However, noting that \( \|u\|_{r} \) and \( \|A^{1/2} u\| \) are equivalent when \( r \) is a non-negative integer (Gilbarg and Trudinger 1983, Robinson 2001), the result for positive real \( r \) follows from an interpolation theorem of Lions and Magenes (1972). We could not find this equivalence result stated explicitly in the literature and therefore we think it would be worthwhile to show it here.

We first note that we consider the fractional Sobolev spaces \( H^s \) characterized by the following norm, for \( s = m + \sigma \) with \( m \in \mathbb{Z} \) and \( 0 < \sigma < 1 \) (Adams 1975)

\[
\|u\|^2_s = \|u\|^2_m + \sum_{|\alpha| = m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{d+2\sigma}} \, dx \, dy.
\]

However, we assume \( \Omega \) bounded with \( \partial \Omega \) uniformly \( C^1 \)-regular and therefore the above characterization is equivalent to the definition based on the real interpolation method (Adams 1975, theorem 7.48; Lions and Magenes 1972, theorem 9.1 of chapter 1).

**Lemma 3.3.** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \), \( d \geq 2 \), with a sufficiently smooth boundary. Then for any non-negative real number \( r \)

\[
c_2 \|A^{1/2} u\| \leq \|u\| \leq c_1 \|A^{1/2} u\|,
\]

for all \( u \in D(A^{1/2}) \). (11)

**Proof.**

To prove \( \|u\| \leq c_1 \|A^{1/2} u\| \), let \( \{X_1, X_2\} \) and \( \{Y_1, Y_2\} \) be two pairs of normed spaces with \( X_1 \) and \( Y_1 \) dense subsets of \( X_2 \) and \( Y_2 \), respectively, with continuous injections. Define the interpolation space \( [X_1, X_2]_\theta \) with \( 0 < \theta < 1 \) as

\[
[X_1, X_2]_\theta = D(A^{1-\theta}),
\]

where \( A \) is a self adjoint, positive and unbounded operator in \( X_2 \), with domain \( X_1 \) and satisfying \( (u, v)_{X_1} = (Au, Av)_{X_2} \), for any \( u, v \in X_1 \). Also let

\[
[Y_1, Y_2]_\theta = D(S^{1-\theta}),
\]

with \( S \) having similar properties as \( A \). The interpolation result of Lions and Magenes (1972, chapter I, theorem 5.1) states that if a map \( \pi \) is a continuous linear operator of \( X_1 \) into \( Y_1 \) and also of \( X_2 \) into \( Y_2 \), then it is a continuous linear operator from \( [X_1, X_2]_\theta \) into \( [Y_1, Y_2]_\theta \).

For our purpose here, let \( X_1 = D(A^m) \), \( X_2 = D(A^0) = H^0 \), \( Y_1 = H^{2m} \) and \( Y_2 = H^0 \), with \( m \) a non-negative integer number. Then \( X_1 \) and \( Y_1 \) are dense subsets of \( X_2 \) and \( Y_2 \), respectively, with continuous injections. We note that for any real \( r \) by definition of \( H^r(\Omega) \) (Lions and Magenes 1972), we have

\[
H^r(\Omega) = [H^{2m}(\Omega), H^0(\Omega)]_\theta \quad \text{with} \quad \theta = 1 - r/(2m).
\]

Also for \( X_1 \) and \( X_2 \), letting \( \Lambda = A^m \) implies that

\[
D(A^{1/2}) = [D(A^m), H^0(\Omega)]_\theta \quad \text{with} \quad \theta = 1 - r/(2m).
\]

Now since for an integer \( m \), \( D(A^m) \subset H^{2m} \), the identity operator is a linear continuous operator from \( D(A^m) \) into \( H^{2m} \) and also obviously from \( H^0(\Omega) \) into \( H^0(\Omega) \). Therefore, by
the result mentioned above, it is a continuous operator of \( D(A^{1/2}) = [D(A^m), H^0(\Omega)]_0 \) into \( H'(\Omega) = [H^2m(\Omega), H^0(\Omega)]_0 \), implying that \( D(A^{1/2}) \subset H'(\Omega) \) and therefore
\[
\|u\|_r \leq c_1 \|A^{1/2}u\|.
\]

It remains to show that \( \|A^{1/2}u\| \leq \|u\|_r \). For any \( r \geq 0 \), there exists an integer \( m \geq 0 \) such that \( r = 2m + \hat{r} \) with real \( 0 \leq \hat{r} < 2 \). Noting that by the result of Fujiwara (1967)
\[
\|A^{1/2}u\| \leq \|u\|_r,
\]
we can write
\[
\|A^{1/2}u\| = \|A^{m+\hat{r}/2}u\| = \|A^{1/2}A^m u\|
\leq \sum_{|\alpha_j|=2i, 1 \leq j \leq m} \|D^{\alpha_1+\cdots+\alpha_m}u\|_r
\leq \|u\|_{2m+\hat{r}} = \|u\|_r,
\]
and the result follows.

\[ \square \]

Having (11), we can also show that \( u \), the local unique solution of (1) with \( u_0 \in H^{1/2} \) and \( f \in L^2(0, T; H^{1/2}) \), is bounded in \( C((0, T); D(A^{3/2})) \). Consider \( \{w_1, w_2, \ldots, w_m\} \) to be the first \( m \) eigenfunctions of \( A \) with corresponding eigenvalues \( \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \). Then since the Galerkin approximation \( u_m \in \text{span}\{w_1, \ldots, w_m\} \) satisfies \( \sum_{j=1}^m |\lambda_j|^2 (u_m, w_j)^2 < \infty \) for any finite \( r \) and therefore is in \( D(A^r) \), it can be shown in a similar way to the proof of lemma 4.2 of Temam (1995) that \( u \in C((0, T); D(A^{3/2})) \) and therefore \( \|A^{3/2}u\| < \infty \) for almost every \( t \in (0, T) \). The fact that \( u(t) \in D(A^{3/2}) \) for almost every \( t > 0 \) is used in the proof of the next lemma where we not only require enough Sobolev regularity for \( u \), but also need it to be in \( D(A^{3/2}) \) to be able to use (11).

We can now show the bound on \( \int_0^T t \|u\|_{5/2}^2 \, dt \):

**Lemma 3.4.** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^3 \) with a sufficiently smooth boundary. Consider \( u \in L^\infty(0, T; H^{1/2}(\Omega)) \cap L^2(0, T; H^{3/2}(\Omega)) \), a unique solution of the three-dimensional Navier–Stokes equations (1) with \( u_0 \in H^{1/2}(\Omega) \) and \( f \in L^2(0, T; H^{1/2}(\Omega)) \). Then
\[
\int_0^T t \|u\|_{5/2}^2 \, dt \leq C,
\]
where \( C \) depends on \( v, \Omega, \|u_0\|_{1/2} \) and \( \int_0^T \|f\|_{1/2}^2 \, dt \).

**Proof.**
We take the inner product of (1) with \( tA^{3/2}u \) to obtain
\[
\frac{1}{2} \frac{d}{dt} (t \|A^{3/2}u\|^2) = \|A^{3/4}u\|^2 + v t \|A^{5/4}u\|^2 \leq t \left| (u \cdot \nabla)u, A^{3/2}u \right| + t \left| (f, A^{3/2}u) \right|.
\]
\[
\text{(13)}
\]
The highest derivative exponent in the right-hand side is bigger than that of the left-hand side. Therefore, we need to integrate by parts in the right-hand side, which is why in the three-dimensional case we have to take the inner product of (1) with \( A^{3/2}u \) rather than \( A^{3/2}u \) (noting that \( A = \Pi(-\Delta) \neq -\Delta \) in the bounded domains).
For the second term in the right-hand side of (13), by integration by parts and appropriate use of the Sobolev embeddings, we obtain (noting that \( u|_{\partial \Omega} = 0 \))
\[
\left( u \cdot \nabla \right) u, A^{1/2} u \right| \\
\leq \left\| D u \right\|_{L^1(\Omega)} \left\| D A^{1/2} u \right\|_{L^1(\Omega)} + \left\| u \right\|_{L^1(\Omega)} \left\| D^2 u \right\|_{L^1(\Omega)} \\
\leq \left\| D u \right\|_{L^1(\Omega)}^2 + \left\| D A^{1/2} u \right\|_{L^1(\Omega)} + \left\| u \right\|_{L^1(\Omega)} \left\| D^2 u \right\|_{L^1(\Omega)} \\
\leq c \left\| u \right\|_{L^2(\Omega)}^2 + c \left\| u \right\|_{L^2(\Omega)} \left\| u \right\|_{L^2(\Omega)} \\
\leq c \left\| u \right\|_{L^2(\Omega)}^2 + c \left\| u \right\|_{L^2(\Omega)} \left\| u \right\|_{L^2(\Omega)}.
\]

For the term containing the pressure, again by integration by parts, we can write
\[
\left| \left( \nabla \cdot (u \cdot \nabla) u \right) \right| \\
\leq \left\| D^2 p \right\|_{L^1(\Omega)} \left\| D A^{1/2} u \right\|_{L^1(\Omega)} + \left\| D p \right\|_{L^1(\Omega)} \left\| A^{1/2} u \right\|_{L^1(\Omega)}.
\]

Since applying the divergence operator to (1) gives (assuming that \( f \) is also divergence-free)
\[
- \Delta p = \nabla \cdot ((u \cdot \nabla) u) = \sum_{i,j=1}^3 \partial_i u_i \partial_j u_j
\]

and also \( \left\| A p \right\|_{L^q} \leq \left\| D^2 p \right\|_{L^q} \leq c \left\| A p \right\|_{L^q} \) for any \( q > 1 \) (Gilbarg and Trudinger 1983, lemma 9.17), we can write
\[
\left\| D^2 p \right\|_{L^1(\Omega)} \left\| D A^{1/2} u \right\|_{L^1(\Omega)} \leq \left\| D^2 p \right\|_{L^1(\Omega)} \left\| D A^{1/2} u \right\|_{L^1(\Omega)} \\
\leq c \left\| D u \right\|_{L^2(\Omega)} \left\| u \right\|_{L^2(\Omega)} \leq c \left\| u \right\|_{L^2(\Omega)}^2 \left\| u \right\|_{L^2(\Omega)}^2
\]

and
\[
\left\| D p \right\|_{L^1(\Omega)} \left\| D A^{1/2} u \right\|_{L^1(\Omega)} \leq c \left\| D p \right\|_{L^1(\Omega)} \left\| u \right\|_{L^2(\Omega)} \leq c \left\| D u \right\|_{L^2(\Omega)} \left\| u \right\|_{L^2(\Omega)}^2 \\
\leq c \left\| u \right\|_{L^2(\Omega)}^3 \left\| u \right\|_{L^2(\Omega)}^2.
\]

Substituting these in (13) and using (11) (which holds for \( r \leq 5/2 \) and almost every \( t > 0 \), since \( u(t) \in D(A^{5/4}) \) for almost every \( t > 0 \), we conclude that
\[
\frac{d}{dt} \left( t \| A^{3/4} u \|^2 \right) + v t \| A^{5/4} u \|^2 \\
\leq c \| u \|_{L^2(\Omega)}^2 (1 + \| u \|_{L^2(\Omega)}^2) t \| A^{3/4} u \|^2 + c \| u \|_{L^2(\Omega)}^3 + c t \| A^{1/4} f \|^2.
\]

Let \( E_3(t) = \exp(-c \int_0^t \| u \|_{L^2(\Omega)}^3 (1 + \| u \|_{L^2(\Omega)}^2) \, dt) \) and multiply both sides of the above inequality by \( E_3(t) \) to obtain
\[
\frac{d}{dt} \left( t E_3(t) \| A^{3/4} u \|^2 \right) + v t E_3(t) \| A^{5/4} u \|^2 \leq c \| u \|_{L^2(\Omega)}^3 + c t \| A^{1/4} f \|^2.
\]

Integrating the above inequality between 0 and \( t \) (noting that this can be made rigorous using the Galerkin approximations of \( u \)) gives
\[
\int_0^t t \| A^{3/4} u \|^2 \, dt \leq \frac{c}{E_3(T)} \int_0^T (\| u \|_{L^2(\Omega)}^3 + \| f \|_{H^1(\Omega)}^2) \, dt,
\]

and the result follows. \( \square \)
We therefore conclude the uniqueness of particle trajectories in the three-dimensional case as well.

4. Conclusion

We considered the two- and three-dimensional Navier–Stokes equations with the initial conditions that have minimal Sobolev regularity required to give rise to a unique solution, and presented a much simpler proof than that of Chemin and Lerner (1995) for the uniqueness of particle trajectories associated with such solutions.

For a particular weak solution of the three-dimensional Navier–Stokes equations, it is shown by Foias et al (1985) that at least one continuous solution of (2) exists. The uniqueness of these solutions, however, is not known. Finding an extra condition on a weak solution that can lead to the uniqueness of the solution of (2) is the subject of the recent work by Robinson and Sadowski (2008).

We note that the above result and also the continuity and differentiability properties of the fluid particle trajectories with respect to the initial velocity field are useful in showing that the posterior measure in certain data assimilation problems is well-defined (Cotter et al 2009). It can be shown (Cotter et al 2009) that the two-dimensional trajectories are Lipschitz continuous and also differentiable with respect to an initial velocity field which is $H^s$-regular for some $s > 0$. The continuity of the trajectories with respect to an only $H^{(d/2)−1}$-regular initial condition follows from a general continuity result for ordinary differential equations proved in Dashti and Robinson (2009).

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References

Adams R A 1975 Sobolev Spaces (Pure and Applied Mathematics vol 65) (New York-London: Academic Press)
Chemin J-Y and Lerner N 1995 Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes J. Diff. Eqns 121 314–28
Cotter S L, Dashti M, Robinson J C and Stuart A M 2009 Data assimilation problems in fluid mechanics: Bayesian formulation in function space Inverse Probl. Submitted
Dashti M and Robinson J C 2009 The uniqueness of Lagrangian trajectories in Navier–Stokes flows Partial Differential Equations Fluid Mechanics ed J C Robinson and J L Rodrigo (Cambridge: Cambridge University Press) at press
Foias C, Guillopé C and Temam R 1985 Lagrangian representation of a flow J. Diff. Eqns 57 440–9
Fujita H and Kato T 1964 On the Navier–Stokes initial value problem: I. Arch. Ration. Mech. Anal. 16 269–315
Fujikawa D 1967 Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order Proc. Japan Acad. 43 82–6
Gilbarg D and Trudinger N S 1983 Elliptic Partial Differential Equations of Second Order (Berlin: Springer)
Ladyzhenskaya O A 1958 Solution in the large to the boundary-value problem for the Navier–Stokes equations in two-space variables Sov. Phys. Dokl. 123 1128–31
Leray J 1933 Étude de diverses équations intégrales non lineaires et de quelques problèmes que pose l’hydrodynamique J. Math. Pures Appl. 12 1–82
Lions J-L and Magenes E 1972 *Non-homogeneous Boundary Value Problems and Applications* vol I (New York: Springer)

Lions J-L and Prodi G 1959 Un thorme d’existence et unicité dans les quations de Navier–Stokes en dimension 2. (French) *C. R. Acad. Sci. Paris* 248 3519–21

Priestley H A 1997 *Introduction to Integration.* (New York: Oxford University Press)

Robinson J C 2001 *Infinite-Dimensional Dynamical Systems (Cambridge Texts in Applied Mathematics)* (Cambridge: Cambridge University Press)

Robinson J C and Sadowski W 2008 A criterion for uniqueness of Lagrangian trajectories for weak solutions of the 3d Navier–Stokes equations *Commun. Math. Phys.* submitted

Temam R 1995 *Navier–Stokes Equations and Nonlinear Functional Analysis* (Philadelphia: SIAM)

Temam R 1977 *Navier–Stokes Equations* (Providence, RI: AMS Chelsea Publishing)

Zuazua E 2002 Log-Lipschitz regularity and uniqueness of the flow for a field in \( W_{k}^{n/p+1,p} (\mathbb{R}^{n}) \) \(^{e}\) *C. R. Math. Acad. Sci. Paris* 335 17–22