Hopf’s lemmas for parabolic fractional Laplacians and parabolic fractional $p$-Laplacians

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Abstract

In this paper, we first establish Hopf’s lemmas for parabolic fractional equations and parabolic fractional $p$-equations. Then we derive an asymptotic Hopf’s lemma for antisymmetric solutions to parabolic fractional equations. We believe that these Hopf’s lemmas will become powerful tools in obtaining qualitative properties of solutions for nonlocal parabolic equations.

Keywords: Parabolic fractional Laplacians, parabolic fractional $p$-Laplacians, Hopf’s lemmas, sub-solutions, maximum principles.

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1. Introduction

Hopf’s lemma is a classic result in analysis, dating back to the discovery of the maximum principles for harmonic functions $[21]$, and it has become a fundamental and powerful tool in the study of partial differential equations $[1, 26, 31]$.

In the past few decades, elliptic equations involving either local or nonlocal operators have been extensively studied by many scholars, a number of systematic approaches have been established to explore qualitative properties of
solutions, such as the method of moving planes in integral forms \cite{8}, the extension method \cite{3}, the method of moving planes \cite{4, 5, 6, 11, 12, 2, 15, 16, 17, 23, 29, 30, 32, 33}, the method of moving spheres \cite{10, 24}, and the sliding methods \cite{28, 14, 18, 34}.

Hopf’s lemma is a powerful tool in carrying out the method of moving planes to derive symmetry, monotonicity, and non-existence of solutions for elliptic partial differential equations.

Recently, with the extensive study of fractional Laplacian and fractional p-Laplacian, some fractional version of Hopf’s lemma have been established. For instance, Li and Chen \cite{25} introduced a fractional version of a Hopf’s lemma for anti-symmetric functions which can be applied immediately to the method of moving planes to establish qualitative properties, such as symmetry and monotonicity of solutions for fractional equations. Jin and Li \cite{22} derived a Hopf’s lemma for a fractional p-Laplacian. Chen, Li and Qi \cite{9} obtained a Hopf’s type lemma for positive weak super-solutions of the fractional p-Laplacian equations with Dirichlet conditions.

So far as we aware, not much is known concerning Hopf type lemmas for parabolic equations involving nonlocal elliptic operators. This is a motivation of the present paper. Here, we investigate the following parabolic fractional p-equations

$$\frac{∂u}{∂t} + (-Δ)^s_p u(x, t) = f(t, u(x, t)), \quad (x, t) ∈ Ω × (0, ∞),$$

where $0 < s < 1$, $2 \leq p < ∞$, and $Ω$ is either a bounded or an unbounded domain in $\mathbb{R}^n$. For each fixed $t > 0$,

$$(-Δ)^s_p u(x, t) = C_{n, sp} P.V. \int_{\mathbb{R}^n} \frac{|u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t))}{|x - y|^{n+sp}} dy,$$

where $P.V.$ stands for the Cauchy principal value. It is easy to see that for $u ∈ C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{sp}$, $(-Δ)^s_p u$ is well defined, where

$$L_{sp} = \{u(\cdot, t) ∈ L^{p-1}_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x, t)|^{p-1}}{1 + |x|^{n+sp}} dx < +∞\}.$$

In the special case when $p = 2$, $(-Δ)^s_p$ becomes the well-known fractional Laplacian $(-Δ)^s$. We first obtain the following
Theorem 1.1. (A Hopf’s lemma for parabolic fractional Laplacians and fractional $p$-Laplacians) Assume that $u(x,t) \in (C^{1,1}_{loc}(\Omega) \cap L^{sp}) \times C^1((0,T])$ is a positive solution to

\[
\begin{cases}
\frac{\partial u}{\partial t} + (-\Delta)^s_p u(x,t) = f(t,u(x,t)), & (x,t) \in \Omega \times (0,T], \\
u(x,t) = 0, & (x,t) \in \Omega^c \times (0,T],
\end{cases}
\]

(1.1)

where $0 < s < 1$, $p \geq 2$ and $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary.

Assume that

\[f(t,0) = 0, \ t \in (0,T), \ f \text{ is Lipschitz continuous in } u \text{ uniformly for } t. \ (1.2)\]

Suppose that at a point $x_0 \in \partial \Omega$, tangent to which a sphere in $\Omega$ can be constructed. Then there exists a positive constant $c_0$, such that for any $t_0 \in (0,T)$ and for all $x$ near the boundary of $\Omega$, we have

\[u(x,t_0) \geq c_0 d^s(x),\]

where $d(x) = \text{dist}(x,\partial \Omega)$. It follows that

\[\frac{\partial u}{\partial \nu}(x_0,t_0) < 0 \ (\text{may possibly be } -\infty), \ \forall \ t_0 \in (0,T),\]

where $\nu$ is the outward normal of $\partial \Omega$ at $x_0$ and $\frac{\partial u}{\partial \nu}$ is the derivative of fractional order $s$.

One of the applications of this kind of Hopf’s lemma is in the process of moving planes. To obtain a priori estimates of the solutions on a boundary layer (a neighborhood of $\partial \Omega$), or to prove symmetry of solutions if $\Omega$ is symmetric, one effective tool is the method of moving planes. To ensure that the moving plane has a starting point, we need to show that the solution is monotone decreasing in the outward normal direction near the boundary. To this end, we can exploit an immediate conclusion from the above theorem:

\[\frac{\partial u(x_0,t)}{\partial \nu} = -\infty, \ x_0 \in \partial \Omega, \ \forall \ t \in (0,T).\]

It is expected that, through some regularity estimates, one would be able to obtain that

\[\frac{\partial u(x,t)}{\partial \nu} < 0, \ \forall \ x \text{ near } x_0 \in \partial \Omega, \ \forall \ t \in (0,T).\]
The readers can find more details at the end of Section 2.

Our second main result is the asymptotic Hopf’s lemma for antisymmetric functions which is an important in carrying out the method of moving planes. Before stating it, let’s first introduce some relevant background notation.

Choose any direction to be the \(x_1\) direction. Let

\[
T_\lambda = \{ x \in \mathbb{R}^n | x_1 = \lambda, \text{ for some given } \lambda \in \mathbb{R} \}
\]

be the moving planes,

\[
\tilde{\Sigma}_\lambda = \{ x \in \mathbb{R}^n | x_1 > \lambda \}
\]

be the region to the right of the plane,

\[
x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)
\]

be the reflection of \(x\) about the plane \(T_\lambda\), and

\[
u_\lambda(x, t) = u(x^\lambda, t) \text{ and } w_\lambda(x, t) = u_\lambda(x, t) - u(x, t).
\]

To study the asymptotic symmetry and monotonicity of solutions, we consider the well-known \(\omega\)-limit set of \(u\)

\[
\omega(u) := \{ \varphi \mid \varphi = \lim_{t \to \infty} u(\cdot, t_k) \text{ for some } t_k \to \infty \},
\]

with the limit in \(C_0(\mathbb{R}^n)\). Under the very mild assumptions that both \(u\) and \(f\) are bounded, from the regularity result of \cite{19}, one can derive that \(\omega(u)\) is a nonempty compact subset of \(C_0(\mathbb{R}^n)\) and

\[
\lim_{t \to \infty} \text{dist}_{C_0(\mathbb{R}^n)}(u(\cdot, t), \omega(u)) = 0.
\]

For each \(\varphi(x) \in \omega(u)\), denote

\[
\psi_\lambda(x) = \varphi(x^\lambda) - \varphi(x) = \varphi_\lambda(x) - \varphi(x).
\]

Obviously, it is the \(\omega\)-limit of \(w_\lambda(x, t)\).

We establish
Theorem 1.2. (Asymptotic Hopf’s lemma for antisymmetric functions) Assume that $w_\lambda(x,t) \in (C^{1,1}_{\text{loc}}(\tilde{\Sigma}_\lambda) \cap L^2_{\text{loc}}) \times C^1((0,\infty))$ is bounded and satisfies

\[
\begin{aligned}
\frac{\partial w_\lambda}{\partial t} + (-\Delta)^s w_\lambda(x,t) &= c_\lambda(x,t)w_\lambda(x,t), \quad (x,t) \in \tilde{\Sigma}_\lambda \times (0,\infty), \\
w_\lambda(x^\lambda, t) &= -w_\lambda(x, t), \quad (x,t) \in \tilde{\Sigma}_\lambda \times (0,\infty), \\
\lim_{t \to \infty} w_\lambda(x, t) &\geq 0, \quad x \in \tilde{\Sigma}_\lambda,
\end{aligned}
\]

where $c_\lambda(x, t)$ is bounded uniformly for $t$. If $\psi_\lambda$ is nonnegative and $\psi_\lambda > 0$ somewhere in $\tilde{\Sigma}_\lambda$. Then

\[
\frac{\partial \psi_\lambda}{\partial \nu}(x) < 0, \quad x \in \partial \tilde{\Sigma}_\lambda,
\]

where $\nu$ is an outward normal vector on $\partial \tilde{\Sigma}_\lambda$.

This asymptotic Hopf’s lemma can be applied to the second step in the asymptotic method of moving planes instead of the asymptotic narrow region principle to directly obtain that the plane can move a little bit toward the right as we will illustrate at the end of Section 3.

This paper is organized as follows. In Section 2 we present proofs of the Hopf’s lemma for parabolic fractional Laplacians and fractional $p$-Laplacians based on maximum principles and proper construction of sub-solutions. Due to the full nonlinearity of fractional $p$-Laplacians, the proof is quite different than that of fractional Laplacians.

Section 3 is devoted to an asymptotic Hopf’s lemma for antisymmetric solutions to parabolic fractional equations.

We belief that both Hopf lemmas will become useful tools in the analysis of qualitative properties of solutions to parabolic fractional and fractional $p$ equations.

2. Hopf’s lemmas for parabolic fractional and fractional $p$-equations

In this section, we prove Theorem 1.1. Since the different natures of fractional Laplacian and the fractional $p$-Laplacian—the former is linear while the
latter is fully non-linear, the proofs are different. Hence we present them in two separate subsections.

2.1. A Hopf’s lemma for parabolic fractional Laplacian

In order to prove first part of Theorem 1.1 when $p = 2$, we first obtain

**Theorem 2.1.** (A maximum principle for parabolic fractional Laplacian) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Assume that $u(x, t) \in (C^{1,1}_{loc}(\Omega) \cap L^2) \times C^1([0, \infty))$ and is lower semi-continuous about $x$ on $\Omega$. If

\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) + (-\Delta)^s u(x, t) \geq 0, & (x, t) \in \Omega \times (0, \infty), \\
u(x, t) \geq 0, & (x, t) \in (\mathbb{R}^n \setminus \Omega) \times [0, T], \\
u(x, 0) \geq 0, & x \in \Omega,
\end{cases}
\]

then

\[
u(x, t) \geq 0 \text{ in } \Omega \times [0, T], \forall T > 0.
\]

Furthermore, (2.5) holds for unbounded region $\Omega$ if we further assume that for all $t \in [0, T]$

\[
\lim_{|x| \to \infty} u(x, t) \geq 0.
\]

Under the conclusion (2.5), if $u$ vanishes somewhere at $(x_0, t_0) \in \Omega \times (0, T]$, then

\[
u(x, t_0) = 0, \text{ almost everywhere for } x \in \mathbb{R}^n.
\]

**Proof.** Fix an arbitrary $T \in (0, \infty)$. If (2.5) does not hold, then the lower semi-continuity of $u$ on $\bar{\Omega}$ implies that there exists an $\bar{x} \in \Omega$ and $\bar{t} \in (0, T]$ such that

\[
u(\bar{x}, \bar{t}) := \min_{\Omega \times (0, T]} u(x, t) < 0.
\]

It follows that $\frac{\partial u}{\partial t}(\bar{x}, \bar{t}) \leq 0$. Furthermore, by a direct calculation,

\[
(-\Delta)^s u(\bar{x}, \bar{t}) = C_{n, s} P.V. \int_{\mathbb{R}^n} \frac{u(\bar{x}, \bar{t}) - u(y, \bar{t})}{|\bar{x} - y|^{n+2s}} dy
\]

\[
= C_{n, s} P.V. \int_\Omega \frac{u(\bar{x}, \bar{t}) - u(y, \bar{t})}{|\bar{x} - y|^{n+2s}} dy + C_{n, s} \int_{\Omega^c} \frac{u(\bar{x}, \bar{t}) - u(y, \bar{t})}{|\bar{x} - y|^{n+2s}} dy
\]

\[
\leq C_{n, s} \int_{\Omega^c} \frac{u(\bar{x}, \bar{t}) - u(y, \bar{t})}{|\bar{x} - y|^{n+2s}} dy
\]

\[
< 0,
\]
the second inequality from the bottom holds because
\[ u(y, \tilde{t}) \geq 0, \ y \in \Omega^c, \ \tilde{t} \in [0, T] \]
by (2.4) and \( u(\bar{x}, \tilde{t}) < 0 \). Hence
\[ \frac{\partial u}{\partial \tilde{t}}(\bar{x}, \tilde{t}) + (-\Delta)^s u(\bar{x}, \tilde{t}) < 0, \]
which contradicts the first inequality in (2.4). Hence (2.5) must be valid.

If \( \Omega \) is unbounded, then (2.6) guarantees that the negative minima of \( u(x, t) \)
in \( \Omega \times (0, T] \) must be attained at some points. Then one can follow the same
discussion as the proof of (2.5) to arrive at a contradiction.

Next we prove (2.6) based on (2.5). Suppose that there exists \((x_0, t_0) \in \Omega \times (0, T] \)
such that
\[ u(x_0, t_0) = 0. \]
It is obvious that \((x_0, t_0)\) is the minimum point of \( u(x, t) \). Hence, \( \frac{\partial u}{\partial t}(x_0, t_0) \leq 0 \).
Then from
\[ 0 \geq \frac{\partial u}{\partial t}(x_0, t_0) = -(-\Delta)^s u(x_0, t_0) = C_{n, s} \int_{\mathbb{R}^n} \frac{u(y, t_0)}{|x_0 - y|^{n+2s}} \, dy \]
and \( u(y, t) \geq 0 \) in \( \mathbb{R}^n \times [0, T] \), we obtain
\[ u(x, t_0) = 0 \text{ almost everywhere in } \mathbb{R}^n. \]

Next we will construct a subsolution to prove the Hopf’s lemma for parabolic fractional Laplacian. For reader’s convenience, we restate first part of Theorem 1.1 as follows.

**Theorem 2.2.** (A Hopf’s lemma for parabolic fractional Laplacian) Let \( u(x, t) \in (C^{1,1}_{\text{loc}}(\Omega) \cap L^2_{2s}) \times C^1([0, T]) \) be a positive solution to
\[ \begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^s u(x, t) = f(t, u(x, t)), & (x, t) \in \Omega \times (0, T], \\ u(x, t) = 0, & (x, t) \in \Omega^c \times (0, T], \end{cases} \tag{2.8} \]
where $0 < s < 1$ and $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary.

Assume that

$$ f(t,0) = 0, \ t \in (0,T), \ f \text{ is Lipschitz continuous in } u \text{ uniformly for } t. \quad (2.9) $$

Suppose that at a point $x_0 \in \partial \Omega$, tangent to which a sphere in $\Omega$ can be constructed. Then there exists a positive constant $c_0$, such that for any $t_0 \in (0,T)$ and for all $x$ near the boundary of $\Omega$, we have

$$ u(x,t_0) \geq c_0 d^s(x), $$

where $d(x) = \text{dist}(x,\partial \Omega)$. It follows that

$$ \frac{\partial u(x,t_0)}{\partial \nu} < 0, \ \forall \ x \in \partial \Omega, \ \forall \ t_0 \in (0,T), $$

where $\nu$ is the outward normal of $\partial \Omega$ at $x_0$.

**Proof.** In the following Figure 1, consider the cylinder

$$ E = \Omega \times (0,T], \quad B = B_1(0) \times [t_0-1,t_0+1] $$

Let $\partial B_\delta(\bar{x})$ be a sphere in $\Omega$ that is tangent to $x_0$. Use this, we construct a small circular cylinder

$$ B := B_\delta(\bar{x}) \times [t_0-\epsilon,t_0+\epsilon] \subset \Omega \times (0,T), \ \bar{x} \in \Omega, \ t_0 \in (0,T), $$

where $\delta$ and $\epsilon$ are small positive constants. By a translation and a rescaling, for simplicity of notation, we may assume that $B := B_1(0) \times [t_0-1,t_0+1]$. 
Let \( \hat{B} = D \times [t_0 - 1, t_0 + 1] \) be a compact subset of \( E \) which has a positive distance from \( B \). Since \( u \) is positive and continuous, we have

\[
u(x, t) \geq c_1 > 0, \ (x, t) \in \hat{B}, \quad (2.10)\]

for some constant \( c_1 \).

To obtain a lower bound of \( u \) in \( B \), we construct a subsolution. Set

\[
u(x, t) = \chi_D(x)u(x, t) + \varepsilon(1 - |x|^2)^{\ast}_+ \eta(t),
\]

where \( \varepsilon \) is a positive constant to be determined later,

\[
\chi_D(x) = \begin{cases} 
1, & x \in D, \\
0, & x \notin D^c,
\end{cases}
\quad (2.11)
\]

and \( \eta(t) \in C^\infty_0([0, T]) \) satisfies

\[
\eta(t) = \begin{cases} 
1, & t \in [t_0 - \frac{1}{2}, t_0 + \frac{1}{2}], \\
0, & t \notin (t_0 - 1, t_0 + 1).
\end{cases}
\quad (2.12)
\]

By the definition of fractional Laplacian and (2.10), we derive that for each fixed \( t \in [t_0 - 1, t_0 + 1] \) and for any \( x \in B_1(0) \)

\[
(-\Delta)^s[\chi_D(x)u(x, t)] = C_{n,s}P.V. \int_{\mathbb{R}^n} \frac{0 - \chi_D(y)u(y, t)}{|x - y|^{n+2s}} dy
\]

\[
= C \int_D \frac{-u(y, t)}{|x - y|^{n+2s}} dy
\quad (2.13)
\]

\[
\leq -C_1,
\]

where \( C_1 \) is a positive constant.

By (2.13), we obtain that for \( (x, t) \in B_1(0) \times [t_0 - 1, t_0 + 1] \)

\[
\frac{\partial u}{\partial t} + (-\Delta)^s u(x, t)
\]

\[
= \varepsilon \eta'(t)(1 - |x|^2)^{\ast}_+ + (-\Delta)^s[\chi_D(x)u(x, t)] + \varepsilon(-\Delta)^s[(1 - |x|^2)^{\ast}_+] \eta(t)
\]

\[
\leq -C_1 + \varepsilon[\eta'(t)(1 - |x|^2)^{\ast}_+ + a\eta(t)],
\]

where the last inequality is due to (2.10)

\[
(-\Delta)^s(1 - |x|^2)^{\ast}_+ = a, \ x \in B_1(0), \ a \text{ is a positive constant.}
\]
Denote 

\[ w(x,t) = u(x,t) - \underline{u}(x,t). \]

Then \( w(x,t) \) satisfies

\[
\frac{\partial w}{\partial t} + (-\Delta)^s w(x,t) \geq f(t,u) + C_1 - \varepsilon(\eta'(t)(1 - |x|^2)^s + a\eta(t))
\]

\[
= C(x,t)u(x,t) + C_1 - \varepsilon(\eta'(t)(1 - |x|^2)^s + a\eta(t)),
\]

where \( C(x,t) = \frac{f(t,u) - f(t,0)}{u(x,t) - \underline{u}(x,t)} \) is bounded by (2.9). From \( u(x,t) = 0 \), \( (x,t) \in \Omega^c \times (0,T) \)

and the continuity of \( u(x,t) \), one knows that \( u(x,t) \) is small when \( x \in \Omega \) is sufficiently close to \( \partial \Omega \) and \( t \in [t_0 - \varepsilon, t_0 + \varepsilon] \). Hence taking \( \varepsilon \) sufficiently small, we have

\[
\frac{\partial w}{\partial t} + (-\Delta)^s w(x,t) \geq 0, \ (x,t) \in B_1(0) \times [t_0 - 1, t_0 + 1].
\]

By the definition of \( \underline{u}(x,t) \), we have

\[
w(x,t) \geq 0, \ (x,t) \in B_1^c(0) \times [t_0 - 1, t_0 + 1]
\]

and

\[
w(x,t_0 - 1) \geq 0, \ x \in B_1(0).
\]

Now, by applying Theorem 2.1 we derive

\[
w(x,t) \geq 0, \ (x,t) \in B_1(0) \times [t_0 - 1, t_0 + 1].
\]

Therefore

\[
 u(x,t) \geq \underline{u}(x,t) = \varepsilon(1 - |x|^2)^s \eta(t), \ \text{in} \ B_1(0) \times [t_0 - 1, t_0 + 1].
\]

In particular, fixed \( \varepsilon \), by (2.12), for \( x \in B_1(0) \) one has

\[
u(x,t_0) \geq \varepsilon(1 - |x|^2)^s \eta(t) = \varepsilon(1 - |x|)^s(1 + |x|)^s \eta(t) = \text{cod}^s(x),
\]

where \( d(x) = \text{dist}(x,\partial\Omega) \) and \( c_0 \) is a positive constant. Consequently,

\[
\lim_{x \to \partial\Omega} \frac{u(x,t_0)}{d^s(x)} \geq c_0 > 0, \ \forall \ t_0 \in (0,T).
\]
It follows that if $\nu$ is the outward normal of $\partial \Omega$ at $x_0$, then
\[
\frac{\partial u(x_0)}{\partial \nu} < 0, \ \forall \ x \in \partial \Omega, \ \forall \ t_0 \in (0, T),
\]
since $u(x_0, t_0) = 0$.

2.2. A Hopf’s lemma for parabolic fractional $p$-Laplacian

We first prove

**Theorem 2.3.** (A maximum principle for parabolic fractional $p$-Laplacian) Let $p > 2$, $\Omega$ be a bounded domain in $\mathbb{R}^n$. Assume that
\[
u(x,t), u(x,t) \in (C^{1,1}_{loc}(\Omega) \cap L^\infty) \times C^1([0, \infty))
\]
and $w(x,t) = u(x,t) - u(x,t)$ is lower semi-continuous about $x$ on $\bar{\Omega}$. If
\[
\begin{align*}
\frac{\partial w}{\partial t}(x,t) + (-\Delta)^p_s u(x,t) - (-\Delta)^p_s w(x,t) &\geq 0, \ (x,t) \in \Omega \times (0, \infty), \\
w(x,t) &\geq 0, \quad (x,t) \in (\mathbb{R}^n \setminus \Omega) \times [0, T], \\
w(x,0) &\geq 0, \quad x \in \Omega,
\end{align*}
\]
then
\[
w(x,t) \geq 0 \text{ in } \bar{\Omega} \times [0, T], \ \forall \ T > 0. \tag{2.16}
\]
Furthermore, (2.16) hold for unbounded region $\Omega$ if we further assume that for all $t \in [0, T]$
\[
\lim_{|x| \to \infty} u(x,t) \geq 0. \tag{2.17}
\]
Under the conclusion (2.16), if $w = 0$ at some point $(x_0, t_0) \in \Omega \times (0, T]$, then
\[
w(x_0, t_0) = 0, \ \text{almost everywhere in } \mathbb{R}^n. \tag{2.18}
\]

**Remark 2.1.** Here, the main difference between parabolic fractional $p$-Laplacian and parabolic fractional Laplacian is that the former is a non-linear operator, hence we need to use $(-\Delta)^p_s u(x,t) - (-\Delta)^p_s w(x,t)$ instead of $(-\Delta)^p_s w(x,t)$, and this makes the proof different.
Proof. Fix an arbitrary $T \in (0, \infty)$. If (2.16) does not hold, then the lower semi-continuity of $w$ on $\bar{\Omega}$ implies that there exists an $\bar{x} \in \Omega$ and a $\bar{t} \in (0, T]$ such that

$$w(\bar{x}, \bar{t}) := \min_{\Omega \times (0,T]} w(x, t) < 0.$$ 

Hence

$$\frac{\partial w}{\partial t}(\bar{x}, \bar{t}) \leq 0.$$ 

Denote $G(z) = |z|^{p-2}z$. Obviously, $G(z)$ is strictly increasing, and $G'(z) = (p - 1)|z|^{p-2} \geq 0$. Through a direct calculation,

$$(\Delta)^{sp}_\Omega u(\bar{x}, \bar{t}) - (\Delta)^{sp}_{\bar{\Omega}} \bar{u}(\bar{x}, \bar{t})$$

$$= C_{n,sp} P.V. \int_{\mathbb{R}^n} \frac{G(u(\bar{x}, \bar{t}) - u(y, \bar{t})) - G(\bar{u}(\bar{x}, \bar{t}) - \bar{u}(y, \bar{t}))}{|\bar{x} - y|^{n+sp}} dy$$

$$= C_{n,sp} P.V. \int_{\Omega} \frac{G(u(\bar{x}, \bar{t}) - u(y, \bar{t})) - G(\bar{u}(\bar{x}, \bar{t}) - \bar{u}(y, \bar{t}))}{|\bar{x} - y|^{n+sp}} dy$$

$$+ C_{n,sp} \int_{\mathbb{R}^n \setminus \Omega} \frac{G(u(\bar{x}, \bar{t}) - u(y, \bar{t})) - G(\bar{u}(\bar{x}, \bar{t}) - \bar{u}(y, \bar{t}))}{|\bar{x} - y|^{n+sp}} dy$$

$$= I_1 + I_2. \quad (2.19)$$

In $I_1$, $y \in \Omega$, we derive

$$G(u(\bar{x}, \bar{t}) - u(y, \bar{t})) - G(\bar{u}(\bar{x}, \bar{t}) - \bar{u}(y, \bar{t})) \leq 0,$$

due to the monotonicity of $G$, $(\bar{x}, \bar{t})$ is the minimum point of $w$ and the fact that

$$[u(\bar{x}, \bar{t}) - u(y, \bar{t})] - [\bar{u}(\bar{x}, \bar{t}) - \bar{u}(y, \bar{t})] = w(\bar{x}, \bar{t}) - w(y, \bar{t}) \leq 0.$$ 

Hence

$$I_1 \leq 0. \quad (2.20)$$

For $I_2$, noticing that $w(y, \bar{t}) \geq 0$, $y \in \Omega^c$ and $w(\bar{x}, \bar{t}) < 0$, we obtain

$$u(\bar{x}, \bar{t}) - u(y, \bar{t}) - [\bar{u}(\bar{x}, \bar{t}) - \bar{u}(y, \bar{t})] = w(\bar{x}, \bar{t}) - w(y, \bar{t}) < 0, \quad y \in \Omega^c.$$ 

Then

$$I_2 < 0 \quad (2.21)$$

follows from the strict monotonicity of $G(\cdot)$.
Combining (2.19), (2.20), and (2.21), we arrive at
\[ (-\Delta)_p^s u(\bar{x}, \bar{t}) - (-\Delta)_p^sw(\bar{x}, \bar{t}) < 0. \]
Consequently
\[ \frac{\partial w}{\partial t}(\bar{x}, \bar{t}) + (-\Delta)_p^s u(\bar{x}, \bar{t}) - (-\Delta)_p^sw(\bar{x}, \bar{t}) < 0, \]
which contradicts the first inequality of (2.15). Hence (2.16) holds.

If \( \Omega \) is unbounded, (2.17) guarantees that the negative minimum of \( w(x, t) \) must be attained at some point, then one can follow the same discussion as in the proof of (2.16) to arrive at a contradiction.

Next we prove (2.18) based on (2.16). Suppose that there exists \((x_0, t_0) \in \Omega \times (0, T)\) such that
\[ w(x_0, t_0) = 0. \]
It is obvious that \((x_0, t_0)\) is the minimum point of \( w(x, t) \). Hence, \( \frac{\partial w}{\partial t}(x_0, t_0) \leq 0. \)
Then by (2.19), (2.20) and (2.21), one has
\[ \frac{\partial w}{\partial t}(x_0, t_0) + (-\Delta)_p^s u(x_0, t_0) - (-\Delta)_p^sw(x_0, t_0) \leq C \int_{\mathbb{R}^n} \frac{-(w(y, t_0)^{p-1}}{|x_0 - y|^{n+sp}} dy. \]
(2.22)
Here the last inequality was derived from the following

**Lemma 2.1.** [33] For \( G(z) = |z|^{p-2}z \), \( p > 2 \), there exists a constant \( C > 0 \) such that
\[ G(z_2) - G(z_1) \geq C(z_2 - z_1)^{p-1}, \]
for arbitrary \( z_2 \geq z_1 \).

Now combining (2.15), (2.22) and \( w(y, t) \geq 0 \) in \( \mathbb{R}^n \times [0, T] \), we obtain
\[ w(x, t_0) = 0 \text{ almost everywhere in } \mathbb{R}^n. \]

For reader’s convenience, we restate Theorem 1.1 before its proof as follows.
Theorem 2.4. (A Hopf’s lemma for parabolic fractional $p$-Laplacian) Assume that $u(x,t) \in (C_{\text{loc}}^{1,1}(\Omega) \cap L_{\text{sp}}(\Omega)) \times C^1((0,T])$ is a positive solution to

$$\begin{cases}
\frac{\partial u}{\partial t} + (-\Delta)^s_p u(x,t) = f(t,u(x,t)), & (x,t) \in \Omega \times (0,T], \\
u(x,t) = 0, & (x,t) \in \Omega^c \times (0,T],
\end{cases}$$

where $0 < s < 1$, $p > 2$ and $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume that $f$ satisfies (1.2). Suppose that at a point $x_0 \in \partial \Omega$, tangent to which a sphere in $\Omega$ can be constructed. Then there exists a positive constant $c_0$, such that for any $t_0 \in (0,T)$ and for all $x$ near the boundary of $\Omega$, we have

$$u(x,t_0) \geq c_0 d^s(x),$$

where $d(x) = \operatorname{dist}(x,\partial \Omega)$. It follows that

$$\frac{\partial u(x,t_0)}{\partial \nu^s} < 0, \forall x \in \partial \Omega, \forall t_0 \in (0,T),$$

where $\nu$ is the outward normal of $\partial \Omega$ at $x_0$.

Proof. Let $\partial B_\delta(\bar{x})$ be a sphere in $\Omega$ that is tangent to $x_0$. Use this, we construct a small circular cylinder

$$B := B_\delta(x) \times [t_0 - \epsilon, t_0 + \epsilon] \subset \Omega \times (0,T), \bar{x} \in \Omega, \ t_0 \in (0,T),$$

where $\delta$ and $\epsilon$ are small positive constants. By translation and rescaling, for simplicity, we may assume that $B := B_1(0) \times [t_0 - 1, t_0 + 1]$. Also see Figure 1.

Let $\dot{B} = D \times [t_0 - 1, t_0 + 1]$ be a compact subset of $\Omega \times (0,T)$ which has a positive distance from $B$. Since $u$ is positive and continuous, we have (2.10).

Next, we construct a subsolution in $B_1(0) \times [t_0 - 1, t_0 + 1]$. Set

$$\underline{u}(x,t) = \chi_D(x)u(x,t) + \varepsilon \psi(x,t),$$

where $\varepsilon$ is a positive constant,

$$\psi(x,t) = (1 - |x|^2)^{s}_+ \eta(t),$$

$\chi_D(x)$ be as defined in (2.11) and $\eta(t) \in C_0^\infty([0,T])$ satisfies (2.12). By a result in [27], we have

$$(-\Delta)^s_p (1 - |x|^2)^{s}_+ \leq C, \ x \in B_1(0),$$
Combining (2.23) and (2.25), for 
\[ (-\Delta)_p^s \psi(x, t) = \eta^{p-1}(t)(-\Delta)^{s}_p(1 - |x|^2)^s_+ \]
\[ \leq C\eta^{p-1}(t), \quad (x, t) \in B_1(0) \times [t_0 - 1, t_0 + 1], \]
where \( C \) is a constant and \( 0 \leq \eta(t) \leq 1 \).

By the definition of \((-\Delta)_p^s\), (2.24) and lemma 2.1, for each fixed \( t \in [t_0 - 1, t_0 + 1] \) and for any \( x \in B_1(0) \), we obtain
\[ (-\Delta)_p^s u(x, t) = (-\Delta)_p^s [\chi_D(x)u(x, t) + \varepsilon \psi(x, t)] \]
\[ = C_{n, sp} P.V. \int_{\mathbb{R}^n} \frac{G(\varepsilon \psi(x, t) - \chi_D(y)u(y, t) - \varepsilon \psi(y, t))}{|x - y|^{n+sp}} dy \]
\[ = C_{n, sp} P.V. \int_{B_1(0)} \frac{G(\varepsilon \psi(x, t) - \varepsilon \psi(y, t))}{|x - y|^{n+sp}} dy + \int_{D} \frac{G(\varepsilon \psi(x, t) - u(y, t))}{|x - y|^{n+sp}} dy \]
\[ + \int_{\mathbb{R}^n \setminus (B_1(0) \cup D)} \frac{G(\varepsilon \psi(x, t))}{|x - y|^{n+sp}} dy + \int_{D} \frac{G(\varepsilon \psi(x, t))}{|x - y|^{n+sp}} dy - \int_{D} \frac{G(\varepsilon \psi(x, t))}{|x - y|^{n+sp}} dy \]
\[ = (-\Delta)_p^s (\varepsilon \psi(x, t)) + \int_{D} \frac{C_{u^{p-1}}(y, t)}{|x - y|^{n+sp}} dy \]
\[ \leq (-\Delta)_p^s (\varepsilon \psi(x, t)) - \int_{D} \frac{Cu^{p-1}(y, t)}{|x - y|^{n+sp}} dy \]
\[ \leq C\varepsilon^{p-1} \eta^{p-1}(t) - C_1, \]
(2.25)

where \( C_1 \) is a positive constant.

For \((x, t) \in B_1(0) \times [t_0 - 1, t_0 + 1] \), we have
\[ u(x, t) \geq \chi_D(x)u(x, t) = \underline{u}(x, t). \]

It follows from (2.12) that
\[ u(x, t_0 - 1) \geq \underline{u}(x, t_0 - 1), \quad x \in B_1(0). \]

Denote
\[ u(x, t) = u(x, t) - \underline{u}(x, t). \]

Combining (2.23) and (2.25), for \((x, t) \in B_1(0) \times [t_0 - 1, t_0 + 1] \), we have
\[ \frac{\partial w}{\partial t} + (-\Delta)_p^s u(x, t) - (-\Delta)_p^s \underline{u}(x, t) \]
\[ \geq f(t, u) - \varepsilon \eta'(t)(1 - |x|^2)^s_+ - C\varepsilon^{p-1} \eta^{p-1}(t) + C_1 \]
\[ = C(x, t)u - \varepsilon \eta'(t)(1 - |x|^2)^s_+ - C\varepsilon^{p-1} \eta^{p-1}(t) + C_1, \]

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where $C(x,t) = \frac{I(t,u)}{u(x,t)}$ is bounded by \([1,2]\).

Similarly to the proof of the case when $p = 2$ in Theorem 1.1. Taking $\varepsilon$ small, we obtain

$$
\frac{\partial w}{\partial t} + (-\Delta)^{s} u(x,t) - (-\Delta)^{s} u(t,u) \geq 0, \quad (x,t) \in B_1(0) \times [t_0 - 1, t_0 + 1].
$$

Hence

$$
\begin{cases}
\frac{\partial w}{\partial t} + (-\Delta)^{s} u(x,t) - (-\Delta)^{s} u(x,t) \geq 0, & (x,t) \in B_1(0) \times [t_0 - 1, t_0 + 1], \\
w(x,t) \geq 0, & (x,t) \in B_1(0) \times [t_0 - 1, t_0 + 1], \\
w(x,t_0 - 1) \geq 0, & x \in B_1(0).
\end{cases}
$$

By Theorem 2.3 we derive

$$
w(x,t) \geq 0, \text{ in } B_1(0) \times [t_0 - 1, t_0 + 1].
$$

Therefore

$$
u(x,t) \geq \varepsilon \psi(x,t) = \varepsilon (1 - |x|^2)^{s} \eta(t), \text{ in } B_1(0) \times [t_0 - 1, t_0 + 1]. \quad (2.26)
$$

In particular,

$$
u(x,t_0) \geq \varepsilon (1 - |x|^2)^{s} = \nu_0 d^s(x), \quad x \in B_1(0),
$$

where $d(x) = dist(x,\partial \Omega)$ and $\nu_0$ is a positive constant.

Hence if $\nu$ is the outward normal of $\partial \Omega$ at $x_0$, we obtain

$$
\frac{\partial u(x,t_0)}{\partial \nu} < 0, \quad \forall \ x \in \partial \Omega, \forall \ t_0 \in (0,T),
$$

since $u(x_0,t_0) = 0$.

Finally, we briefly explain how to apply Hopf’s lemma for parabolic Laplacian and parabolic $p$-Laplacian in the first step of the method of moving planes. Take $\Omega = B_1(0)$ as an example. Let

$$
\Omega_\lambda = \{x \in \Omega \mid x_1 < \lambda\}.
$$
To obtain the radial symmetry of positive solutions to
\[
\begin{cases}
    \frac{\partial u}{\partial t} + (-\Delta)^s u(x, t) = f(t, u(x, t)), & (x, t) \in \Omega \times (0, T], \\
u(x, t) = 0, & (x, t) \in \Omega^c \times (0, T],
\end{cases}
\]
where \(0 < s < 1\), \(2 \leq p < \infty\), the first step is to show that for the plane \(T_\lambda\) sufficiently close to the left end of \(\Omega\), i.e., for \(\lambda\) sufficiently close to
\[
\inf\{x_1 \mid x \in \Omega\},
\]
we have
\[
u(x_\lambda, t) > u(x, t), \; x \in \Omega_\lambda, \; t \in (0, T).
\]
This will provide a starting point to move the plane.

By applying the Hopf’s lemma (Theorem 1.1), we have
\[
\frac{\partial u}{\partial x_1}(\bar{x}, t) > 0, \; \bar{x} \in \partial \Omega, \; t \in (0, T).
\]
As a consequence,
\[
\frac{\partial u}{\partial x_1}(\bar{x}, t) = +\infty, \; \bar{x} \in \partial \Omega, \; t \in (0, T).
\]
Then by the continuity of \(\frac{\partial u}{\partial x_1}\) in some proper sense, it is natural to expect that
\[
\frac{\partial u}{\partial x_1}(x, t) > 0, \; \text{for } x \text{ sufficiently close to } \bar{x} \in \Omega, \; t \in (0, T),
\]
which implies (2.27) immediately.

We will prove (2.27) in a near future. While at this moment, the following two simple examples may shed some light on its validity.

It is wellknown that
\[
(-\Delta)^s(x_1)_+^s = 0 \; \text{in the right half space where } x_1 > 0,
\]
and
\[
(-\Delta)^s(1 - |x|^2) = \text{constant}, \; \text{in the unit ball } |x| < 1.
\]
Here \(0 < s < 1\), and
\[
(x_1)_+^s = \begin{cases}
x_1^s, & x_1 > 0, \\
0, & x_1 \leq 0.
\end{cases}
\]
Let \( \eta(t) \) be any positive smooth function.

(a) Consider

\[ \Psi_1(x, t) = (x_1)_+^s \eta(t). \]

It satisfies the equation

\[
\frac{\partial \Psi_1}{\partial t} + (-\Delta)^s \Psi_1(x, t) = f_1(t, \Psi_1(x, t))
\]

for some function \( f_1 \).

For \( x = (x_1, x_2, \cdots, x_n) \in \{ x \in \mathbb{R}^n \mid x_1 > 0 \}, t \in (0, T) \), we derive

\[
\frac{\partial \Psi_1(x, t)}{\partial x_1} = \frac{\partial (x_1)_+^s \eta(t)}{\partial x_1} = \frac{s \eta(t)}{x_1^{1-s}} \to +\infty, \text{ as } x_1 \to 0.
\]

And obviously, for \( x_1 > 0 \), we have

\[
\frac{\partial \Psi_1(x, t)}{\partial x_1} > 0.
\]

(b) Then consider

\[ \Psi_2(x, t) = (1 - |x|^2)_+^s \eta(t), \]

then it satisfies the equation

\[
\frac{\partial \Psi_2}{\partial t} + (-\Delta)^s \Psi_2(x, t) = f_2(t, \Psi_2(x, t))
\]

for some function \( f_2 \).

For \( x \in B_1(0) \) close to the left end of the region, i.e. \( x_1 \) close to \(-1\), we have

\[
\frac{\partial \Psi_2(x, t)}{\partial x_1} = \frac{\partial (1 - |x|^2)_+^s \eta(t)}{\partial x_1} = \frac{-2sx_1 \eta(t)}{(1 - |x|^2)^{1-s}} \to +\infty, \text{ as } |x| \to 1,
\]

and at these points \( x \),

\[
\frac{\partial \Psi_2(x, t)}{\partial x_1} > 0.
\]

3. Asymptotic Hopf’s lemma for antisymmetric solutions

We start this section with the proof of the following
Theorem 3.1. (Asymptotic Hopf’s lemma for antisymmetric functions) Assume that \( w_\lambda(x,t) \in (C^{1,1}_{\text{loc}}(\tilde{\Sigma}_\lambda) \cap L^2_{\text{loc}}(\tilde{\Sigma}_\lambda)) \times C^1((0,\infty)) \) is bounded and satisfies
\[
\begin{cases}
\frac{\partial w_\lambda}{\partial t} + (-\Delta)^s w_\lambda(x,t) = c_\lambda(x,t)w_\lambda(x,t), & (x,t) \in \tilde{\Sigma}_\lambda \times (0,\infty), \\
w_\lambda(x^\lambda,t) = -w_\lambda(x,t), & (x,t) \in \tilde{\Sigma}_\lambda \times (0,\infty), \\
\lim_{t \to \infty} w_\lambda(x,t) \geq 0, & x \in \tilde{\Sigma}_\lambda,
\end{cases}
\]
where \( c_\lambda(x,t) \) is bounded uniformly for \( t \). If \( \psi_\lambda \) is nonnegative and \( \psi_\lambda > 0 \) somewhere in \( \tilde{\Sigma}_\lambda \). Then
\[
\frac{\partial \psi_\lambda}{\partial \nu}(x) < 0, \quad x \in \partial \tilde{\Sigma}_\lambda,
\]
where \( \nu \) is an outward normal vector on \( \partial \tilde{\Sigma}_\lambda \).

Recall that \( \tilde{\Sigma}_\lambda \) is the region to the right of the plane \( T_\lambda \).

Proof. Without loss of generality, we may assume that \( \lambda = 0 \), and it suffices to show that \( \frac{\partial \psi_0}{\partial x_1}(0) > 0 \).

For any \( \varphi(x) \in \omega(u) \), there exists \( t_k \) such that \( w_\lambda(x,t_k) \to \psi_\lambda(x) \) as \( t_k \to \infty \).

Set
\[
w_k(x,t) = w_\lambda(x,t + t_k - 1).
\]

Then
\[
\frac{\partial w_k}{\partial t} + (-\Delta)^s w_k(x,t) = c_k(x,t)w_k(x,t), \quad (x,t) \in \tilde{\Sigma}_\lambda \times [0,\infty),
\]
where \( c_k(x,t) = c_\lambda(x,t + t_k - 1) \) is bounded uniformly for \( t \). From regularity theory for parabolic equations \[19\], we conclude that there is a subsequence of \( w_k(x,t) \) (still denoted by \( w_k(x,t) \)) which converges uniformly to a function \( w_\infty(x,t) \) in \( \tilde{\Sigma}_\lambda \times [0,2] \) and
\[
\frac{\partial w_k}{\partial t}(x,t) + (-\Delta)^s w_k(x,t) \to \frac{\partial w_\infty}{\partial t}(x,t) + (-\Delta)^s w_\infty(x,t),
\]
\[
c_k(x,t) \to c_\infty(x,t), \quad \text{as } k \to \infty.
\]

Hence
\[
\frac{\partial w_\infty}{\partial t} + (-\Delta)^s w_\infty(x,t) = c_\infty(x,t)w_\infty(x,t), \quad (x,t) \in \tilde{\Sigma}_\lambda \times [0,2]. \quad (3.31)
\]
In particular,

\[ \dot{w}(x, t) = \dot{w}(x, 1) \rightarrow w_\infty(x, 1) = \psi_\lambda(x) \text{ as } k \rightarrow \infty. \]

Let

\[ \ddot{w}(x, t) = e^{mt}w_\infty(x, t), \quad m > 0. \]

Since \( c_\infty \) is bounded, we can choose \( m \) such that

\[ m + c_\infty(x, t) \geq 0. \quad (3.32) \]

By the third condition in (3.30), we have

\[ w_\infty(x, t) \geq 0, \quad (x, t) \in \tilde{\Sigma}_\lambda \times [0, 2]. \quad (3.33) \]

Combining (3.31), (3.32) and (3.33), one has

\[ \frac{\partial \ddot{w}}{\partial t} + (-\Delta)\dot{w}(x, t) = (m + c_\infty(x, t)) \dot{w}(x, t) \geq 0, \quad (x, t) \in \tilde{\Sigma}_\lambda \times [0, 2]. \quad (3.34) \]

Since \( \psi_\lambda > 0 \) somewhere in \( \tilde{\Sigma}_\lambda \), by continuity, there exists a set \( D \subset \subset \tilde{\Sigma}_\lambda \) such that

\[ \psi_\lambda(x) > c > 0, \quad x \in D, \quad (3.35) \]

with positive constant \( c \). We may assume \( \text{dist}(T_0, \partial D) > 2\varepsilon \), where \( \varepsilon \) is a positive constant. Let \( B_\varepsilon(0) \) and \( B_{2\varepsilon}(0) \) be balls with the origin as the center and radius of \( \varepsilon \) and \( 2\varepsilon \), respectively.

By the continuity of \( w_\infty(x, t) \) (see [19]), there exist \( 0 < \varepsilon_\alpha < 1 \), such that

\[ w_\infty(x, t) > c/2, \quad (x, t) \in D \times [1 - \varepsilon_\alpha, 1 + \varepsilon_\alpha]. \]

For simplicity of notation, we may assume that

\[ w_\infty(x, t) > c/2, \quad (x, t) \in D \times [0, 2]. \quad (3.36) \]

Let \( D_\lambda \) be the reflection of \( D \) about the plane \( T_\lambda \) for any time \( t \). For convenience, for each \( t \in [0, 2] \), we show the following Figure 2.
Denote $g(x) = x_1 \zeta(x)$, where

$$
\zeta(x) = \zeta(|x|) = \begin{cases} 
1, & |x| < \varepsilon, \\
0, & |x| \geq 2\varepsilon,
\end{cases}
$$

and

$$
0 \leq \zeta(x) \leq 1, \quad \zeta(x) \in C_0^\infty(B_{2\varepsilon}(0)).
$$

Obviously, $g(x)$ is an anti-symmetric function with respect to plane $T_0$, i.e.

$$
g(-x_1, x_2, \cdots x_n) = -g(x_1, x_2, \cdots x_n).
$$

Denote

$$
\bar{w}(x, t) = \chi_{D \cup D_\lambda}(x)\hat{w}(x, t) + \delta \eta(t)g(x),
$$

where

$$
\chi_{D \cup D_\lambda}(x) = \begin{cases} 
1, & x \in D \cup D_\lambda, \\
0, & x \notin D \cup D_\lambda,
\end{cases}
$$

and $\eta(t) \in C_0^\infty([1 - \varepsilon_o, 1 + \varepsilon_o])$ satisfies

$$
\eta(t) = \begin{cases} 
1, & t \in [1 - \frac{\varepsilon_o}{2}, 1 + \frac{\varepsilon_o}{2}], \\
0, & t \notin [1 - \varepsilon_o, 1 + \varepsilon_o].
\end{cases}
$$

(3.37)

Since $g(x)$ is a $C_0^\infty(B_{2\varepsilon}(0))$ function, we have

$$
|(-\Delta)^s g(x)| \leq C_0.
$$

(3.38)
By the definition of fractional Laplacian and (3.36), we derive that for each fixed \( t \in [0, 2] \) and for any \( x \in B_{2\varepsilon}(0) \cap \tilde{\Sigma}_\lambda \)

\[
(-\Delta)^s(\chi_{D\cup D_\lambda}\tilde{w}(x,t)) \\
= C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{\chi_{D\cup D_\lambda}(x)\tilde{w}(x,t) - \chi_{D\cup D_\lambda}(y)\tilde{w}(y,t)}{|x-y|^{n+2s}} \, dy \\
= C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{-\chi_{D\cup D_\lambda}(y)\tilde{w}(y,t)}{|x-y|^{n+2s}} \, dy \\
= C_{n,s} P.V. \int_D \frac{-\tilde{w}(y,t)}{|x-y|^{n+2s}} \, dy + \int_D \frac{-\tilde{w}(y^\lambda,t)}{|x-y^\lambda|^{n+2s}} \, dy \\
= C_{n,s} P.V. \int_D \left( \frac{1}{|x-y^\lambda|^{n+2s}} - \frac{1}{|x-y|^{n+2s}} \right) \tilde{w}(y,t) \, dy \\
\leq -C_1, \quad (3.39)
\]

where \( C_1 \) is a positive constant. For \((x,t) \in (B_{2\varepsilon}(0) \cap \tilde{\Sigma}_\lambda) \times [0, 2]\), by (3.38) and (3.39), we obtain

\[
\frac{\partial w}{\partial t} + (-\Delta)^s w(x,t) = \delta \eta'(t)g(x) + (-\Delta)^s(\chi_{D\cup D_\lambda}\tilde{w}(x,t)) + \delta \eta(t)(-\Delta)^s g(x) \\
\leq \delta \eta'(t)g(x) - C_1 + \delta \eta(t)C_0.
\]

Hence, taking \( \delta \) sufficiently small, we derive

\[
\frac{\partial w}{\partial t} + (-\Delta)^s w(x,t) \leq 0, \quad (x,t) \in (B_{2\varepsilon}(0) \cap \tilde{\Sigma}_\lambda) \times [0, 2]. \quad (3.40)
\]

Set

\[
v(x,t) = \tilde{w}(x,t) - w(x,t).
\]

Obviously, \( v(x,t) = -v(x^\lambda,t) \). From (3.34) and (3.40), we derive that \( v(x,t) \) satisfies

\[
\frac{\partial v}{\partial t}(x,t) + (-\Delta)^s v(x,t) \geq 0, \quad (x,t) \in (B_{2\varepsilon}(0) \cap \tilde{\Sigma}_\lambda) \times [0, 2]. \quad (3.41)
\]

Also, by the definition of \( w(x,t) \), we have

\[
v(x,t) \geq 0, \quad (x,t) \in (\tilde{\Sigma}_\lambda \setminus (B_{2\varepsilon}(0) \cap \tilde{\Sigma}_\lambda)) \times [0, 2]
\]

and

\[
v(x,0) \geq 0, \quad x \in \tilde{\Sigma}_\lambda.
\]

Now, we apply the following lemma to \( v(x,t) \).
Lemma 3.1. (Maximum principle for antisymmetric functions) \cite{13} Let $\Omega$ be a bounded domain in $\Sigma_{\lambda}$. Assume that $w_{\lambda}(x,t) \in (C^{1,1}_{\text{loc}}(\Omega) \cap L_{2s}) \times C^1([0,\infty))$ is lower semi-continuous in $x$ on $\bar{\Omega}$ and satisfies

\begin{align}
\frac{\partial w_{\lambda}}{\partial t}(x,t) + (-\Delta)^s w_{\lambda}(x,t) &\geq c_{\lambda}(x,t)w_{\lambda}(x,t), \quad (x,t) \in \Omega \times (0,\infty), \\
w_{\lambda}(x,0) &\geq 0, \quad (x,t) \in (\Sigma_{\lambda}\setminus\Omega) \times [0,\infty).
\end{align}

(3.42)

If $c_{\lambda}(x,t)$ is bounded from above, then

$$w_{\lambda}(x,t) \geq 0, \quad (x,t) \in \Omega \times [0,T], \quad \forall \ T > 0.$$ (3.43)

As an immediate consequence of this lemma with $c_{\lambda}(x,t) = 0$, we obtain

$$v(x,t) \geq 0, \quad (x,t) \in (B_{2\varepsilon}(0) \cap \tilde{\Sigma}_{\lambda}) \times [0,2].$$

It implies that

$$e^{mt}w_{\infty}(x,t) - \delta g(x)\eta(t) \geq 0, \quad (x,t) \in (B_{2\varepsilon}(0) \cap \tilde{\Sigma}_{\lambda}) \times [0,2].$$

In particular,

$$w_{\infty}(x,1) \geq e^{-m}\delta g(x), \quad x \in B_{2\varepsilon}(0) \cap \tilde{\Sigma}_{\lambda}$$

and

$$w_{\infty}(x,1) \geq e^{-m}\delta x_1, \quad x \in B_{\varepsilon}(0) \cap \tilde{\Sigma}_{\lambda}.$$

Since $w_{\infty}(x,1) \equiv 0$, $x \in T_0$, particular, $w_{\infty}(0,1) = 0$. Hence

$$\frac{w_{\infty}(x,1) - 0}{x_1 - 0} \geq e^{-m}\delta > 0, \quad x \in B_{\varepsilon}(0) \cap \tilde{\Sigma}_{\lambda}.$$

Obviously, no matter how small $\varepsilon$ is, we have

$$\frac{\partial \psi_{\lambda}}{\partial x_1}(0) > 0.$$

Therefore,

$$\frac{\partial \psi_{\lambda}}{\partial \nu}(x) < 0, \quad x \in \partial \tilde{\Sigma}_{\lambda}.$$
Finally, to illustrate how the asymptotic Hopf’s lemma for anti-symmetric functions can be employed in the second step of the asymptotic method of moving planes, we consider the following example:

\[
\begin{cases}
\frac{\partial u}{\partial t} + (-\Delta)^s u(x,t) = f(t, u(x,t)), & (x,t) \in \mathbb{R}^n \times (0, \infty), \\
u(x,t) > 0, & (x,t) \in \mathbb{R}^n \times (0, \infty).
\end{cases}
\] (3.44)

Under certain conditions on \(f\) and \(\lim_{|x| \to \infty} u(x,t) = 0\), we want to show that positive bounded solutions are asymptotically symmetric about some point in \(\mathbb{R}^n\). That is, all \(\varphi(x) \in \omega(u)\) are radially symmetric and decreasing about some point in \(\mathbb{R}^n\).

To compare the values of \(u(x,t)\) with \(u(x^\lambda, t)\), let

\[w_\lambda(x,t) = u(x^\lambda, t) - u(x,t).\]

Obviously, \(w_\lambda(x,t)\) satisfies

\[\frac{\partial w_\lambda}{\partial t}(x,t) + (-\Delta)^s w_\lambda(x,t) = c_\lambda(x,t)w_\lambda(x,t), \quad x \in \Sigma_\lambda.\]

For each \(\varphi(x) \in \omega(u)\), denote

\[\psi_\lambda(x) = \varphi(x^\lambda) - \varphi(x) = \varphi_\lambda(x) - \varphi(x),\]

which is an \(\omega\)-limit of \(w_\lambda(x,t)\).

To obtain the asymptotic symmetry of solutions to (3.44) in the whole space, the first step is to show that for \(\lambda\) sufficiently close to either \(-\infty\) or \(\infty\), we have

\[\psi_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.\] (3.45)

This provides a starting position to move the plane.

In the second step, we move the plane \(T_\lambda\) to the right as long as inequality (3.45) holds to its rightmost limiting position \(T_{\lambda^-}\), where

\[\lambda^- = \sup\{\lambda \mid \psi_\mu(x) \geq 0, \quad \forall \ x \in \Sigma_\mu, \ \mu \leq \lambda\}.\]

To show that there is at least one \(\varphi \in \omega(u)\) which is symmetric about the plane \(T_{\lambda^-}\), or

\[\psi_{\lambda^-}(x) \equiv 0, \quad x \in \Sigma_{\lambda^-},\] (3.46)
one usually uses a contradiction argument. Suppose (3.46) is false, then for all $\varphi \in \omega(u)$,

$$
\psi_{\lambda_0^-}(x) > 0, \text{ somewhere in } \Sigma_{\lambda_0^-}.
$$

It follows from the asymptotic Hopf lemma for anti-symmetric functions, we have

$$
\frac{\partial \psi_{\lambda_0^-}}{\partial \nu}(x^0) < 0 \quad (3.47)
$$

for any point $x^0$ on the boundary of $\Sigma_{\lambda_0^-}$.

Furthermore, applying an asymptotic strong maximum principle for anti-symmetric function (Theorem 6 in [13]), we have

$$
\psi_{\lambda_0^-}(x) > 0, \quad \forall x \in \Sigma_{\lambda_0^-}, \quad \forall \varphi \in \omega(u). \quad (3.48)
$$

On the other hand, by the definition of $\lambda_0^-$, there exists a sequence $\lambda_k \searrow \lambda_0^-$, $x^k \in \Sigma_{\lambda_k}$, and $\psi_{\lambda_k}^k$ (corresponding to $\varphi^k \in \omega(u)$), such that

$$
\psi_{\lambda_k}^k(x^k) = \min_{\Sigma_{\lambda_k}} \psi_{\lambda_k}(x) < 0 \text{ and } \nabla \psi_{\lambda_k}^k(x^k) = 0. \quad (3.49)
$$

Under the assumption of $f_u(t, \cdot) < -\sigma(> 0)$, by asymptotic maximum principle near infinity (Theorem 4 in [13]), one knows that $\{x^k\}$ is bounded. Then it implies that $\{x^k\}$ converges to some point $x^0$. Due to the compactness of $\omega(u)$ in $C_0(\mathbb{R}^n)$, there exists $\psi_{\lambda_0^-}^0$ (corresponding to some $\varphi^0 \in \omega(u)$), such that

$$
\psi_{\lambda_k}^k(x^k) \to \psi_{\lambda_0^-}^0(x^0) \text{ as } k \to \infty.
$$

Hence from (3.49), we have

$$
\psi_{\lambda_0^-}^0(x^0) \leq 0 \quad (3.50)
$$

and

$$
\nabla \psi_{\lambda_0^-}^0(x^0) = 0. \quad (3.51)
$$

It follows from (3.48) and (3.50) that $x^0 \in \partial \Sigma_{\lambda_0^-}$. Now (3.51) contradicts (3.47). Therefore (3.46) must be valid.
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