Influence Maximization with Semi-Bandit Feedback

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Abstract

We study a stochastic online problem of learning to influence in a social network with semi-bandit feedback, individual observations of how influenced users influence others. Our problem combines challenges of partial monitoring, because the learning agent only observes the influenced portion of the network, and combinatorial bandits, because the cardinality of the feasible set is exponential in the maximum number of influencers. We propose a computationally efficient UCB-like algorithm for solving our problem, IMLinUCB, and analyze it on forests. Our regret bounds are polynomial in all quantities of interest; reflect the structure of the network; and do not depend on inherently large quantities, such as the reciprocal of the minimum probability of being influenced and the cardinality of the action set. To the best of our knowledge, these are the first such results. IMLinUCB permits linear generalization and therefore is suitable for large-scale problems. We evaluate IMLinUCB on several synthetic problems and observe that the regret of IMLinUCB scales as suggested by our upper bounds. A special form of our problem can be viewed as a linear bandit and we match the regret bounds of LinUCB in this case.

1 Introduction

Social networks have been playing an increasingly important role in the past decade as the media for spreading information, ideas, and influence among its members. A large body of literature has been dedicated to this field, and various models of how the influence spreads have been studied [13, 6, 9]. The best known and studied are the models of Kempe et al. [13], and in particular the independent cascade model. In this model, the social network is modeled as a graph, where each edge \((i, j)\) is associated with probability \(w(i, j)\). After the agent chooses a set of influencers (source nodes) \(S\), the independent cascade model defines an activation (influence) process as follows: at the beginning, all nodes in \(S\) are activated; subsequently, every node \(i\) can activate its neighbor \(j\) with probability \(w(i, j)\) once, independently of the history of the process. This process runs until no activations are possible. In an influence maximization (IM) problem, the goal of the agent is to maximize the expected number of the influenced (activated) nodes subject to a cardinality constraint on \(S\). This problem is NP-hard but can be efficiently solved approximately within the factor of \(1 - 1/e\) [13].

In many practical social networks, however, the activation probabilities are initially unknown, and the agent needs to learn to choose a good set of source nodes while interacting with the networks. This motivates the framework of IM bandits [22]. Depending on the feedback to the learner, the IM bandit can be (1) a full bandit, where the number of influenced nodes is observed; (2) a node semi-bandit, where the identity of influenced nodes is observed; or (3) an edge semi-bandit, where the identity of influenced edges is observed. In all models, the IM bandit combines challenges of partial monitoring,
because only the influenced portion of the network is observed, and combinatorial bandits, because the number of actions $S$ grows exponentially with the cardinality constraint on $S$.

Although several recent papers studied IM bandits [18, 7, 22, 21], many open challenges remain. One challenge is to identify reasonable complexity metrics, which depend on both the structure (topology) of the network and the activation probabilities of edges, and develop learning algorithms whose performances scale gracefully with these metrics. Another challenge is to develop efficient learning algorithms for large-scale IM bandits, which is increasingly important because social networks have millions or even billions of users.

In this paper, we suggest to overcome these two challenges in IM with edge semi-bandit feedback, where we observe for each activated node the neighbors that this node activated. Modern online social networks track activities of their users and these activation events can be often observed, for instance when the user retweets a tweet of another user. We refer to our model as an IM semi-bandit. We make three main contributions. First, we propose IMLinUCB, a linear UCB-like algorithm for IM semi-bandits that permits linear generalization and is suitable for large-scale problems. Second, we bound the regret of IMLinUCB when the structure of the network is a forest. Our regret bounds are polynomial in all quantities of interest; reflect the structure and activation probabilities of the network; and do not depend on inherently large quantities, such as the reciprocal of the minimum probability of being influenced and the cardinality of the action set. The forest is important in practice because influence maximization in general graphs is computationally expensive, and known scalable approximations use forests to evaluate only most influential paths, such as in the maximum influence arborescence (MIA) model [6]. Finally, we evaluate IMLinUCB on several synthetic problems and show that the regret of IMLinUCB scales as suggested by our upper bounds. A special form of our problem can be viewed as a linear bandit and we match the regret bounds of LinUCB in this case.

## 2 Influence Maximization Problem

We start by defining notation. Consider an undirected graph $G = (V, E)$ with a set $V$ of $L = |V|$ nodes and a set $E$ of edges, and an arbitrary binary weight function $w : E \rightarrow \{0, 1\}$. We say that nodes $v_1, v_2 \in V$ are connected under $w$ if there is a path $p = (e_1, e_2, \ldots, e_l)$ from $v_1$ to $v_2$ in $G$ satisfying $w(e_i) = 1$ for all $i = 1, 2, \ldots, l$. For a given source node set $S \subseteq V$ and we say that node $v \in V$ is influenced if $v$ is connected to at least one source node in $S$ under $w$; and denote the number of influenced nodes in $G$ by $f(S, w)$. By definition, the nodes in $S$ are always influenced.

The influence maximization (IM) problem is characterized by a triple $(G, K, \pi)$, where $G$ is a given undirected graph, $K \leq L$ is the cardinality of source nodes, and $\pi : E \rightarrow [0, 1]$ is a probability weight function mapping each edge $e \in E$ to a real number $\pi(e) \in [0, 1]$. The agent needs to choose a set of $K$ source nodes $S \subseteq V$ based on $(G, K, \pi)$. Then a binary weight function $w$ is sampled by independently sampling a Bernoulli random variable $w(e) \sim \text{Bern}(\pi(e))$ for each edge $e \in E$. The agent’s objective is to maximize the expected number of the influenced nodes:

$$\max_{S: |S| = K} f(S, \pi),$$

where $f(S, \pi) \triangleq \mathbb{E}_w [f(S, w)]$ is the expected number of influenced nodes when the source node set is $S$ and $w$ is sampled according to $\pi$. Note that this problem formulation is mathematically equivalent to the independent cascade model discussed in Section 1 when the activation probabilities are symmetric (i.e. the probability that node $v$ will activate node $u$ is the same as the probability that node $u$ will activate node $v$). We also notice that $f(S, \pi)$ is monotone and submodular in $S$, and monotone in $\pi$ in the sense that if $0 \leq \pi_1(e) \leq \pi_2(e) \leq 1$ for all $e \in E$, then $f(S, \pi_1) \leq f(S, \pi_2)$ for any $S$.

It is well-known that the IM problem is NP-hard [13], but can be approximately solved by approximation / randomized algorithms [6]. In this paper, we refer to such algorithms as oracles to distinguish them from the machine learning algorithms discussed in subsequent sections. Let $S^{\text{opt}}$ be the optimal solution of this problem, and $S^*$ = ORACLE($G$, $K$, $\pi$) be the (possibly random) solution of an oracle ORACLE. For any $\alpha, \gamma \in [0, 1]$, we say that ORACLE is an ($\alpha, \gamma$)-approximation oracle for a given $(G, K)$ if for any $\pi$, $f(S^*, \pi) \geq \gamma f(S^{\text{opt}}, \pi)$ with probability at least $\alpha$. Notice that this further implies that $\mathbb{E} [f(S^*, \pi)] \geq \alpha \gamma f(S^{\text{opt}}, \pi)$.

\[\text{Notice that the definitions of } f(S, \pi) \text{ and } f(S, w) \text{ are consistent in the sense that if } \pi \in \{0, 1\}^{|E|}, \text{ then } f(S, \pi) = f(S, w) \text{ with probability } 1.\]
3 Influence Maximization Semi-Bandit and Algorithm

In many practical IM problems, the social network provider is aware of the topology of the network, but has to learn the influence probabilities to maximize the influence spread. The network provider also observes all attempted influences from an influenced user, such as a tweet which is retweeted by other users. This motivates the framework of the influence maximization semi-bandit.

Specifically, an influence maximization semi-bandit is also characterized by a triple \((G, K, \omega)\), but \(\omega\) is initially unknown to the agent. The agent interacts with the influence maximization semi-bandit for \(n\) times. At each time \(t = 1, 2, \ldots, n\), the agent first adaptively chooses a source node set \(S_t \subseteq V\) with cardinality \(K\) based on its prior information and past observations; and then the binary weight function \(\omega_t\) is sampled by independently sampling \(\omega_t(e) \sim \text{Bern}(\omega(e))\). The agent receives a reward \(f(S_t, \omega_t)\) at time \(t\). For any edge \(e = (u_1, u_2) \in E\), the agent observes the realization of \(\omega_t(e)\) if and only if at least one of \(u_1\) and \(u_2\) is influenced under binary weight \(\omega_t\) with source node set \(S_t\). This feedback model is an example of the partial monitoring feedback in literature \([2, 3]\). The agent’s objective is to maximize the expected cumulative reward in the first \(n\) steps.

3.1 Linear Generalization of Weight Function

We also assume that there is a linear generalization model for the probability weight function \(\omega\). Specifically, each edge \(e \in E\) is associated with a known feature vector \(x_e \in \mathbb{R}^d\), where \(d\) is the dimension of the feature vector, and there is an unknown coefficient vector \(\theta^* \in \mathbb{R}^d\) s.t.

\[
\omega(e) \approx x_e^T \theta^* \quad \forall e \in E.
\]  

Similar to the existing literature in linear bandits, we exploit the linear generalization architecture to develop a learning algorithm for the influence maximization bandit. Without loss of generality, we assume that \(\|x_e\|_2 \leq 1\) for all \(e \in E\). Moreover, we use \(X \in \mathbb{R}^{|E| \times d}\) to denote the feature matrix, i.e. the row of \(X\) associated with edge \(e\) is \(x_e^T\). We also refer to the special case \(X = I \in \mathbb{R}^{|E| \times |E|}\) as the tabular case. As we will see in Section \(4\), tighter regret bounds can be achieved in this case.

3.2 \textit{IMLinUCB} Algorithm

Our proposed algorithm, Influence Maximization Linear UCB (IMLinUCB), is detailed in Algorithm 1 where \(\text{Proj}_{[0, 1]}(\cdot)\) projects a real number into interval \([0, 1]\) to ensure it is a probability. Notice that \(\text{IMLinUCB}\) represents its past observations as a positive-definite matrix (gram matrix) \(M_t \in \mathbb{R}^{d \times d}\) and a vector \(B_t \in \mathbb{R}^d\). Specifically, let \(X_t\) be a matrix whose rows are the feature vectors of all observed edges in \(t\) steps and \(Y_t\) be a binary column vector encoding the realizations of all observed edges in \(t\) steps. Then \(M_t = I + \sigma^{-2} X_t^T X_t\) and \(B_t = X_t^T Y_t\).

At each time \(t\), \(\text{IMLinUCB}\) operates in three steps: first, it computes an upper confidence bound \(U_t(e)\) for each edge \(e \in E\). Second, it chooses a set of source nodes based on the given \(\text{ORACLE}\) and \(U_t\), which is also a probability weight function. Finally, it receives the edge semi-bandit feedback and uses it to update \(M_t\) and \(B_t\). It is worth emphasizing that \(\text{IMLinUCB}\) is computationally efficient as long as \(\text{ORACLE}\) is computationally efficient. Specifically, at each time \(t\), the computational complexities of both Step 1 and 3 of \(\text{IMLinUCB}\) are \(O(|E|d^2)\).

3.3 Performance Metrics

Recall that the agent’s objective is to maximize the expected cumulative reward, which is equivalent to minimizing the expected cumulative regret. Similar to the algorithms proposed in \([7]\), \(\text{IMLinUCB}\) also needs to call an oracle \(\text{ORACLE}\) for solving an IM problem at each time \(t\). Since the IM problem is NP-hard, to ensure that \(\text{IMLinUCB}\) is computationally efficient, we allow \(\text{ORACLE}\) to be an approximation/randomized oracle. Obviously, this will lead to \(O(n)\) cumulative regret, since at each time \(t\) there is a non-diminishing reward loss due to the approximation/randomized nature of \(\text{ORACLE}\). To analyze the performance of \(\text{IMLinUCB}\) in such cases, we define a more generalized performance metric, the scaled cumulative regret, as \(R^o(n) = \sum_{t=1}^n \mathbb{E}[R_t^o]\), where \(n\) is the number

\[R_t^o = M_t^{-1} \left( M_t^{-1} x_e x_e^T - \frac{M_t^{-1} x_e x_e^T M_t^{-1} \sigma^2}{x_e^T M_t^{-1} x_e + \sigma^2} \right).
\]

Notice that in a practical implementation, we store \(M_t^{-1}\) instead of \(M_t\). Moreover, \(M_t \leftarrow M_t + \sigma^{-2} x_e x_e^T\) is equivalent to \(M_t^{-1} \leftarrow \frac{M_t^{-1} x_e x_e^T M_t^{-1}}{x_e^T M_t^{-1} x_e + \sigma^2} \).
Algorithm 1 IMLinUCB: Influence Maximization Linear UCB

**Input:** graph $G$, source node cardinality $K$, oracle ORACLE, feature vector $x_e$’s, and algorithm parameters $\sigma, c > 0$.

**Initialization:** $B_0 \leftarrow 0 \in \mathbb{R}^d$, $M_0 \leftarrow I \in \mathbb{R}^{d \times d}$

for $t = 1, 2, \ldots, n$ do

1. set $\bar{\theta}_{t-1} \leftarrow \sigma^{-2} N^{-1}_{t-1} B_{t-1}$ and the UCBs as

$$U_t(e) \leftarrow \text{Proj}_{[0,1]} \left( x_e^T \bar{\theta}_{t-1} + c \sqrt{x_e^T N^{-1}_{t-1} x_e} \right) \quad \forall e \in \mathcal{E}$$

2. choose $S_t \in \text{ORACLE}(G, K, U_t)$, and observe the edge-level semi-bandit feedback

3. update statistics:

(a) initialize $M_t \leftarrow M_{t-1}$ and $B_t \leftarrow B_{t-1}$

(b) for all observed edges $e \in \mathcal{E}$, update $M_t \leftarrow M_t + \sigma^{-2} x_e x_e^T$ and $B_t \leftarrow B_t + x_e w_t(e)$

end for

end Algorithm

of steps, $\eta > 0$ is the scale, and $R^\eta_t = f(S^{opt}, w_t) - \frac{1}{\eta} f(S_t, w_t)$ is the $\eta$-scaled realized regret $R^\eta_t$ at time $t$. Obviously, when $\eta = 1$, $R^\eta_t(n)$ reduces to the standard expected cumulative regret $R_t(n)$.

4 Analysis

In this section, we develop a regret bound for IMLinUCB for the case when (1) the graph $G = (\mathcal{V}, \mathcal{E})$ is a forest and (2) $\pi(e) = x_e^T \theta^*$ for all $e \in \mathcal{E}$ (i.e. the linear generalization is perfect). Our regret bound depends on a new complexity metric, maximum observed relevance, which depends on both the topology of $G$ and the probability weight function $\pi$, and is defined in Section 4.1.

4.1 Maximum Observed Relevance

We start by defining some terminologies. For a given forest $G = (\mathcal{V}, \mathcal{E})$ and a given source node set $S \subseteq \mathcal{V}$, for any node $v \in \mathcal{V} \setminus S$, we say a source node $s \in S$ is a relevant source node for $v$ if there is a (unique) path $p(s, v)$ from $s$ to $v$ in $G$ and $p(s, v)$ does not contain another source node. We use $S_v$ to denote the set of relevant source nodes for $v$. Obviously, the collection of the paths $\{p(s, v) : s \in S_v\}$ forms a subtree of $G$. We refer to this subtree as the relevant subtree of $v$ and denote it as $T_{S,v} = (V_{S,v}, E_{S,v})$ to emphasize its dependence on both $S$ and $v$. Notice that if $|S| = 1$, for any node $v \in \mathcal{V} \setminus S$, its relevant subtree is the path from the unique source node to $v$. The notion of relevant subtree is illustrated in Figure 1a. We adopt the convention that $v$ is fixed as the root of $T_{S,v}$. Under this convention, the leaves of $T_{S,v}$ are the relevant source nodes $S_v$ for node $v$.

Now we define maximum observed relevance. Recall that each node $v \in \mathcal{V} \setminus S$ has its own relevant subtree $T_{S,v}$. For each edge $e \in \mathcal{E}$, we define $N_{S,e}$ as the number of relevant subtrees with $e$, and $P_{S,e}$ as the conditional probability that $e$ is observed given $S$,

$$N_{S,e} = \sum_{v \in \mathcal{V} \setminus S} 1 \{ e \in E_{S,v} \} \quad \text{and} \quad P_{S,e} = \mathbb{P}(e \text{ is observed } | S).$$

Notice that $P_{S,e}$ depends on both the topology of $G$ and the probability weight $\pi$. The maximum observed relevance is defined as

$$C_{\pi} = \max_{S:|S|=K} \sqrt{\sum_{e \in \mathcal{E}} N_{S,e}^2 P_{S,e}}.$$  

As is detailed in the proof of Lemma 2 in Appendix A, this complexity metric arises in the step where we apply Cauchy-Schwarz inequality. Note that $C_{\pi}$ depends on both the topology of $G$ and the probability weight function $\pi$. We also note that by definition, $C_{\pi} \leq \max_{S:|S|=K} \sqrt{\sum_{e \in \mathcal{E}} N_{S,e}^2} < (L - K) \sqrt{n} = O(L^{3/2})$. That is, $C_{\pi}$ can be bounded from above only based on the topology of $G$ or the size of the problem. Note that there exists $(G, \pi)$ such that $C_{\pi} = \Theta(L^{3/2})$. One such example is when $G$ is a line graph with $L$ nodes and $\pi(e) = L/(L + 1)$ for all edges $e$ in this graph.

To give more intuition, in the rest of this subsection, we illustrate how $C_{\pi}$ varies with three graph topologies in Figure 1b: bar, star, and ray. We fix the node set $\mathcal{V} = \{1, 2, \ldots, L\}$ for all graphs. The bar graph (Figure 1b) is a graph where nodes $i$ and $i + 1$ are connected when $i$ is odd. The star graph
Moreover, if the feature matrix \( X = I \) (i.e. the tabular case), we have
\[
R^{\alpha \gamma}(n) \leq \frac{2cC_*}{\alpha \gamma} \sqrt{n|\mathcal{E}| \log_2 (1 + n)} + 1 = \widetilde{O}(nC_* \sqrt{n}) .
\]
We now briefly discuss the regret bounds in Theorem 1. First, notice that for the tabular case with feature matrix \( X = I \) and \( d = |E| \), a \( \tilde{O}(\sqrt{d/\alpha}) \) tighter regret bound is obtained in Equation \( 7 \). Second, we would like to point out that our regret bound in Equation \( 6 \) matches the regret bound of the classic LinUCB algorithm in the tabular case illustrated in Figure 1 with \( K = 1 \) and \( L = 2A \). Notice that with perfect linear generalization, this problem is equivalent to a linear bandit problem with \( A \) arms and feature dimension \( d \). Since \( E_s = C_s = 1 \) in this case, the regret bound in Equation \( 6 \) reduces to a \( \tilde{O}(d/\sqrt{n}) \) regret bound, which matches the known regret bound of LinUCB [11].

4.3 Proof Sketch

We now briefly sketch the proof for Theorem 1 and the detailed proof is available in Appendix A. Let \( \mathcal{H}_t \) be the “history” by the end of time \( t \), by definition of \( R_t^{\alpha, \gamma} \), we have

\[
\mathbb{E}[R_t^{\alpha, \gamma} | \mathcal{H}_{t-1}] = f(S_{\text{opt}}, \overline{w}) - \frac{1}{\alpha \gamma} \mathbb{E}[f(S_t, \overline{w}) | \mathcal{H}_{t-1}],
\]

where the expectation is over the possible randomness in \( \text{ORACLE} \). For any \( t \leq n \), we define event \( \mathcal{F}_{t-1} = \{ (|e^T \overline{\theta}_{t-1} - \theta^T|) \leq \sqrt{\frac{d}{n} M_{\tau-1}^2} \forall e \in E, \forall \tau \leq t \} \), and \( \overline{\mathcal{F}}_{t-1} \), the complement of \( \mathcal{F}_{t-1} \), as the complement of \( \mathcal{F}_{t-1} \). Notice that \( \mathcal{F}_{t-1} \) and \( \overline{\mathcal{F}}_{t-1} \) are \( \mathcal{H}_{t-1} \)-measurable. Based on the worst-case bound \( f(S_{\text{opt}}, \overline{w}) - \frac{1}{\alpha \gamma} f(S_t, \overline{w}) \leq L - K \), we can decompose the regret as

\[
\mathbb{E}[R_t^{\alpha, \gamma}] \leq \mathbb{P}(\mathcal{F}_{t-1}) \mathbb{E}[f(S_{\text{opt}}, \overline{w}) - f(S_t, \overline{w})/(\alpha \gamma)]|\mathcal{F}_{t-1}] + \mathbb{P}(\overline{\mathcal{F}}_{t-1}) [L - K].
\]

Notice that by definition of \( \mathcal{F}_{t-1} \), \( \overline{w}(e) \leq U_t(e) \), \( \forall e \in \mathcal{E} \) under event \( \mathcal{F}_{t-1} \). Thus we have

\[
f(S_{\text{opt}}, \overline{w}) \leq f(S_{\text{opt}}, U_t) \leq \max_{S: |S| = K} f(S, U_t) \leq \frac{1}{\alpha \gamma} \mathbb{E}[f(S_t, U_t)] | \mathcal{H}_{t-1}],
\]

where the first inequality follows from the monotonicity of \( f \) in the probability weight, and the last inequality follows from the fact that \( \text{ORACLE} \) is an \( (\alpha, \gamma) \)-approximation algorithm. Thus, we have

\[
\mathbb{E}[R_t^{\alpha, \gamma}] \leq \frac{\mathbb{P}(\mathcal{F}_{t-1})}{\alpha \gamma} \mathbb{E}[f(S_t, U_t) - f(S_t, \overline{w})]|\mathcal{F}_{t-1}] + \mathbb{P}(\overline{\mathcal{F}}_{t-1}) [L - K].
\]

To simplify the exposition, for any source node set \( S \subseteq \mathcal{V} \), any probability weight function \( w : \mathcal{E} \to [0, 1] \), and any node \( v \in \mathcal{V} \), we define \( f(S, w, v) \) as the probability that node \( v \) is influenced if the source node set is \( S \) and the probability weight is \( w \). From the additivity of expectation, we have

\[
f(S_t, U_t) - f(S_t, \overline{w}) = \sum_{v \in \mathcal{V} \setminus S_t} [f(S_t, U_t, v) - f(S_t, \overline{w}, v)].
\]

We also define event \( O_t(e) = \{ \text{edge } e \text{ is observed at time } t \} \), which depends on both \( \overline{w} \) and \( S_t \). We then have the following lemma, which bounds \( f(S_t, U_t, v) - f(S_t, \overline{w}, v) \) by the edge-level gap \( U_t(e) - \overline{w}(e) \) on the observed edges in the relevant subtree \( \mathcal{T}_{S_t,v} \) for node \( v \).

Lemma 1 For any \( t \), any “history” \( \mathcal{H}_{t-1} \) and \( S_t \) s.t. \( \mathcal{F}_{t-1} \) holds, and any \( v \in \mathcal{V} \setminus S_t \), we have

\[
f(S_t, U_t, v) - f(S_t, \overline{w}, v) \leq \sum_{e \in \mathcal{E}_{S_t,v}} \mathbb{E}[1 \{O_t(e)\} | U_t(e) - \overline{w}(e)] | \mathcal{H}_{t-1}, S_t],
\]

where \( \mathcal{E}_{S_t,v} \) is the edge set of the relevant tree \( \mathcal{T}_{S_t,v} \).

Please refer to Appendix A.2 for the proof of Lemma 1 which is based on a tree structure recursion in the relevant subtree \( \mathcal{T}_{S_t,v} \). Here, we just show why Lemma 1 holds for the special case when \( K = 1 \) (i.e. \( S_t = \{s\} \) is a singleton) to illustrate the idea. Notice that if there is no path \( p \) from \( s \) to \( v \), then this lemma holds trivially. Otherwise, if there is a path \( p = (e_1, e_2, \ldots, e_l) \) from \( s \) to \( v \), where \( e_k \)'s are edges on \( p \) and \( l \) is the length of \( p \), then we have

\[
f(S_t, U_t, v) - f(S_t, \overline{w}, v) = \prod_{k=1}^{l} U_t(e_k) - \prod_{k=1}^{l} \overline{w}(e_k)
\]

\[
= \sum_{e_k} \{ \prod_{i=1}^{k-1} \overline{w}(e_i) [U_t(e_k) - \overline{w}(e_k)] \prod_{j=k+1}^{l} U_t(e_j) \}
\]

\[
\leq \sum_{e_k} \{ \prod_{i=1}^{k-1} \overline{w}(e_i) [U_t(e_k) - \overline{w}(e_k)] \}
\]

\[
= \sum_{e_k} \mathbb{E}[1 \{O_t(e_k)\} | U_t(e_k) - \overline{w}(e_k)] | \mathcal{H}_{t-1}, S_t],
\]

(11)
where the inequality follows from the fact that $0 \leq U_t(e) \leq 1$, $\forall e \in E$ and the last equality follows from the fact $E[1 \{O_t(e)\} | \mathcal{H}_{t-1}, \mathcal{S}_t] = \prod_{i=1}^{k-1} \mathbb{E}(e_i)$.

Recall that under event $\mathcal{F}_{t-1}$, $U_t(e) - \mathbb{E}(e) \leq 2c \sqrt{x_e^TM_{t-1}^{-1}x_e}$ for all $e \in E$. Thus, by Lemma 1

$$E[R_t^\gamma] \leq \frac{2c}{\sqrt{n}} \mathbb{E}(\mathcal{F}_{t-1}) E \left[ \sum_{e \in V \setminus S} \sum_{e \in S_{t,e}} 1 \{O_t(e)\} \sqrt{x_e^TM_{t-1}^{-1}x_e} \right] + \mathbb{P} (\mathcal{F}_{t-1}) |L - K|.$$

With a little bit algebra, which is detailed in Appendix A, we have

$$R_t^\gamma(n) \leq \frac{2c}{\sqrt{n}} \mathbb{E}(\mathcal{F}_{t-1}) E \left[ \sum_{i=1}^{n} \sum_{e \in E} 1 \{O_t(e)\} N_{S_{t,e}} \sqrt{x_e^TM_{t-1}^{-1}x_e} \right] + [L - K] \sum_{i=1}^{n} \mathbb{P} (\mathcal{F}_{t-1}), \quad (12)$$

where $N_{S_{t,e}}$ is defined in (3). In Appendix A, we bound $\mathbb{P} (\mathcal{F}_{t-1})$ (Lemma 3) and give a worst-case bound on $\sum_{i=1}^{n} \sum_{e \in E} 1 \{O_t(e)\} N_{S_{t,e}} \sqrt{x_e^TM_{t-1}^{-1}x_e}$ (Lemma 2). This concludes the proof.

5 Experiments

We evaluate IMLinUCB on two graph topologies, star and ray (Figure 1), and validate that its regret grows with the number of nodes $L$ and the maximum observed relevance $C_*$ as predicted in Section 4. We focus on the case of $|S| = 1$, where the IM problem can be solved exactly. We vary the number of nodes $L$; edge weight $\omega$, which is the same for all edges; and also features $X$ (Figure 2). We experiment with two kinds of features: $X = I$ and $X = X_4 \in \{0, 1\}^{2 \times 4}$, where each row of $X_4$ contains exactly one random non-zero entry. When $X = I$, each edge weight is learned independently. When $X = X_4$, linear generalization ties edge weights together through four parameters. We run IMLinUCB for $n = 10^4$ steps and verify that it converges to the optimal solutions in each experiment. We report the $n$-step regret of IMLinUCB for $8 \leq L \leq 32$ in Figure 2.

First, we study the star graph with $\omega = 0.8$ and $X = I$. In this graph, $C_* = O(L)$ (Section 4.1), and therefore we expect that $R(n) = \tilde{O}(L^2)$. We numerically estimate the growth of regret in $L$, the exponent of $L$, in the log-log space of $L$ and regret. In particular, since $\log(f(L)) = p \log(L) + \log(c)$ for any $f(L) = cL^p$ and $c > 0$, both $p$ and $\log(c)$ can be estimated by linear regression in the new space. Our estimated growth is $O(L^{2.046})$, which is close to the expected $\tilde{O}(L^2)$. Next we study the ray graph with $\omega = 0.8$ and $X = I$. In this graph, $C_* = O(L^{\frac{2}{3}})$ (Section 4.1), and therefore we expect that $R(n) = \tilde{O}(L^{\frac{4}{3}})$. Indeed, the regret in Figure 2 grows faster than in the star graph and our estimated growth is $O(L^{2.467})$, which is again close to the expected $\tilde{O}(L^{\frac{4}{3}})$.

Now we show how the regret depends on $\omega$. When $\omega$ decreases to 0.7, $C_*$ decreases in both graph topologies and for all $L$, and so does the regret. The regret decreases significantly faster than $C_*$ and we believe that this may be due to the looseness of our upper bounds as $\omega$ decreases. Notice that when $\omega$ is small, any strategy influences only a single node, which makes the problem trivial and precludes any scaling with the graph structure. Our estimated growths of regret are $O(L^{2.056})$ and $O(L^{2.467})$, and are close to the expected $\tilde{O}(L^2)$ and $\tilde{O}(L^{\frac{4}{3}})$, respectively.
Finally, we evaluate the effect of changing features from $I$ to $X_d$. The regret decreases significantly in both graph topologies and for all $L$, increases with $L$, and grows faster in the ray graph than in the star graph. This is consistent with the shape of our bounds in Theorem 1. Unfortunately, the dependence on $L$ is not as clear as in the first two comparisons. We estimate that the regret is $O(L^{0.295})$ and $O(L^{0.793})$, while our upper bound suggests $\tilde{O}(L^{0.2})$ and $\tilde{O}(L^{4})$, respectively. This indicates that our regret bounds may not be tight when the number of features $d$ is small.

6 Related Work

IM semi-bandits were studied in several recent papers [18][7][22]. Lei et al. [18] proposed several learning algorithms for the same feedback model as ours. The algorithms are not analyzed and cannot solve large-scale problems because they estimate each edge weight independently. Our problem can be viewed as a stochastic combinatorial semi-bandit with a submodular reward function and stochastically observed edges [7]. Chen et al. [7] proposed an algorithm for these problems and bounded its regret. Both of their gap-dependent and gap-free bounds are problematic because they depend on the reciprocal of the minimum observation probability $p^*$ of an edge. Consider a line graph with $L$ edges where all edge weights are 0.5. Then $1/p^*$ is $2^{L-1}$. Varwani et al. [22] proposed a learning algorithm for a different and more challenging feedback model, where the learning agent observes influenced nodes but not the edges. They do not prove any regret bounds. Carpenter and Valko [5] give a minimax optimal algorithm for IM bandits but only consider a local model of influence, a model when the source node can only influence its neighbors and a cascade of influences never happens. In related networked bandits [10], the learner chooses a node and its reward is the sum of the rewards of the chosen node and its neighborhood. This setting is different from ours because the influence model is local and the feedback model is different. Our work addresses two shortcomings of recent papers on IM bandits. We propose a learning algorithm that scales to large problems and shows that its regret is polynomial in all quantities of interest.

Our problem can be naïvely formulated as a multi-armed bandit where any set of source nodes is an arm and solved by UCB1 [3]. Both the computational complexity and regret of this solution are exponential in the maximum number of source nodes $K$, and therefore it is impractical even for small $K$. The bounds in our Theorem 1 are tighter than those of this naïve solution for any $K > 3$ and any network topology. Our problem can be solved by a variant of Thompson sampling (Appendix C) and analyzed as in Gopalan et al. [12]. This analysis is $O(1/\Delta^2)$, where $\Delta$ is the minimum gap between the optimal and best suboptimal solutions, and cannot lead to a $O(1/\Delta)$ bound as in Section 1. Lin et al. [20] proposed an algorithm for learning to maximize greedily in the bandit setting. The variant of our problem where $X = I$ can be solved by this algorithm. Its regret bound is also $O(1/\Delta^2)$ and cannot lead to a $O(1/\Delta)$ bound. It is unclear if any of the above solutions can be analyzed to reflect the structure of the network as in Theorem 1.

Our problem can be also viewed as an instance of partial monitoring [2][4] where the learning agent observes only the influenced portion of the network. General algorithms for partial monitoring [2][4] are not practical for combinatorial action sets, such as choosing a small set of influencers out of many. Lin et al. [19] and Kveton et al. [16] studied combinatorial partial monitoring but their feedback models are different from ours, and hence their algorithms cannot solve our problem. Our problem is also related to the so-called cascading bandits [15][8][24], where the learning agent observes a sequence of no-clicks of the user up to the first click. Roughly speaking, we generalize the cascading feedback from line graphs to more complex topologies. Our problem is also related to combinatorial bandits and semi-bandits since the action space is combinatorial. However, except [7], most existing works in this field focus on modular objective [11][14][17][23] and hence are not applicable to this problem.

7 Conclusions

In this paper, we proposed a computationally efficient algorithm, IMLinUCB, for IM semi-bandits, and analyzed its performance in forests. Our regret bounds scale with maximum observed relevance, a metric that reflects both the topology of the network and activation probabilities. We empirically evaluated the scaling of the regret of IMLinUCB in some canonical graphs, such as stars and rays.

We conclude this paper by making a few comments: first, in this paper we assume that the graph is undirected (and hence the activation probabilities are symmetric) to simplify the exposition. It is
straightforward to extend our proposed algorithm and analysis to directed graphs where the activation probabilities are asymmetric. Second, our results are based on linear generalization across \( \mathbb{W}(e) \)'s. It is straightforward to extend the algorithm and the analysis results to contextual IM semi-bandits, where the influence probability \( \mathbb{W}(e, y) \) depends on both the edge \( e \) and the context \( y \), which follows an exogenous Markov process. Third, we have only analyzed IMLinUCB for the forest case. The analysis for the general graph case is non-trivial and we leave it to future work. However, there are non-forest graphs that are easy to analyze. One such example is a circle, in which a node is influenced via at most two paths from source nodes. By properly adapting our analysis techniques for Lemma 1, we can derive a similar regret bound in the circle case (see Appendix B). Finally, it is also straightforward to propose a Thompson sampling algorithm, IMLinTS, for IM semi-bandit based on linear Thompson sampling (see Appendix C). IMLinTS might outperform IMLinUCB in practice, but we leave the comparisons to future work.

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Appendices

A Proof for Theorem 1

A.1 Proof for Theorem 1

Proof: To simplify the exposition, for any source node set \( S \subseteq V \), any probability weight function \( w: E \rightarrow [0, 1] \), and any node \( v \in V \), we define \( f(S, w, v) \) as the probability that node \( v \) is influenced if the source node set is \( S \) and the probability weight function is \( w \). We also define event \( O_t(e) = \{ \text{edge } e \text{ is observed at time } t \} \). Let \( \mathcal{H}_t \) be the “history” by the end of time \( t \), we have

\[
\mathbb{E}[R_t^{\alpha \gamma} | \mathcal{H}_{t-1}] = f(S^{opt}, \overline{w}) - \frac{1}{\alpha \gamma} \mathbb{E}[f(S_t, \overline{w}) | \mathcal{H}_{t-1}],
\]

(13)

where the expectation is over the possible randomness in ORACLE. For any \( t \leq n \), we define event \( \mathcal{F}_{t-1} \) as

\[
\mathcal{F}_{t-1} = \left\{ \left| x_e^{T} (\overline{\mathcal{F}}_{t-1} - \theta^*) \right| \leq c \sqrt{x_e^T M_{t-1}^{-1} x_e}, \forall e \in E, \forall \tau \leq t \right\},
\]

(14)

and \( \overline{\mathcal{F}}_{t-1} \) as the complement of \( \mathcal{F}_{t-1} \). Notice that \( \mathcal{F}_{t-1} \) is also \( \mathcal{H}_{t-1} \)-measurable. Hence we have

\[
\mathbb{E}[R_t^{\alpha \gamma}] \leq \mathbb{P}(\mathcal{F}_{t-1}) \mathbb{E}[f(S^{opt}, \overline{w}) - f(S_t, \overline{w})/(\alpha \gamma)|\mathcal{F}_{t-1}] + \mathbb{P}(\overline{\mathcal{F}}_{t-1}) [L - K].
\]

Notice that under event \( \mathcal{F}_{t-1}, \overline{w}(e) \leq U_t(e), \forall e \in E, \forall t \leq n \), thus we have

\[
f(S^{opt}, \overline{w}) \leq f(S^{opt}, U_t) \leq \max_{S:|S|=K} f(S, U_t) \leq \frac{1}{\alpha \gamma} \mathbb{E}[f(S_t, U_t)|\mathcal{H}_{t-1}],
\]

where the first inequality follows from the monotonicity of \( f \) in the probability weight, and the last inequality follows from the fact that ORACLE is an \((\alpha, \gamma)\)-approximation algorithm. Thus, we have

\[
\mathbb{E}[R_t^{\alpha \gamma}] \leq \frac{\mathbb{P}(\mathcal{F}_{t-1})}{\alpha \gamma} \mathbb{E}[f(S_t, U_t) - f(S_t, \overline{w})|\mathcal{F}_{t-1}] + \mathbb{P}(\overline{\mathcal{F}}_{t-1}) [L - K].
\]

(15)

Notice that we have

\[
f(S_t, U_t) - f(S_t, \overline{w}) = \sum_{v \in V \setminus S_t} [f(S_t, U_t, v) - f(S_t, \overline{w}, v)].
\]

We have the following lemma:

Lemma 1 For any \( t \), any “history” \( \mathcal{H}_{t-1} \) and \( S_t \) s.t. \( \mathcal{F}_{t-1} \) holds, and any \( v \in V \setminus S_t \), we have

\[
f(S_t, U_t, v) - f(S_t, \overline{w}, v) \leq \sum_{e \in E_{S_t,v}} \mathbb{E}[1 \{ O_t(e) \} | U_t(e) - \overline{w}(e)|\mathcal{H}_{t-1}, S_t],
\]

where \( E_{S_t,v} \) is the edge set of the relevant tree \( T_{S_t,v} \).

Please refer to Section A.2 for the proof of Lemma 1. Generally speaking, Lemma 1 is proved based on a tree structure recursion. We also notice that under event \( \mathcal{F}_{t-1} \), we have \( U_t(e) - \overline{w}(e) \leq 2c \sqrt{x_e^T M_{t-1}^{-1} x_e} \) for all \( e \in E \). So we have

\[
\mathbb{E}[R_t^{\alpha \gamma}] \leq \frac{2c}{\alpha \gamma} \mathbb{P}(\mathcal{F}_{t-1}) \mathbb{E} \left[ \sum_{v \in V \setminus S_t} \sum_{e \in E_{S_t,v}} 1 \{ O_t(e) \} \sqrt{x_e^T M_{t-1}^{-1} x_e} | \mathcal{F}_{t-1} \right] + \mathbb{P}(\overline{\mathcal{F}}_{t-1}) [L - K]
\]

\[
\leq \frac{2c}{\alpha \gamma} \mathbb{E} \left[ \sum_{v \in V \setminus S_t} \sum_{e \in E_{S_t,v}} 1 \{ O_t(e) \} \sqrt{x_e^T M_{t-1}^{-1} x_e} \right] + \mathbb{P}(\overline{\mathcal{F}}_{t-1}) [L - K]
\]

\[
= \frac{2c}{\alpha \gamma} \mathbb{E} \left[ \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e} \sqrt{x_e^T M_{t-1}^{-1} x_e} \right] + \mathbb{P}(\overline{\mathcal{F}}_{t-1}) [L - K],
\]

(16)
where \( N_{S_t,e} = \sum_{e \in V \setminus S} 1 \{ e \in E_{S_t,e} \} \) is defined in Eqn 3. Thus we have
\[
R^{\alpha \gamma}(n) \leq \frac{2c}{\alpha \gamma} \mathbb{E} \left[ \sum_{t=1}^{n} \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e} \sqrt{x_e^T M_{t-1}^{-1} x_e} \right] + [L - K] \sum_{t=1}^{n} \mathbb{P} ( \mathcal{F}_{t-1} ). \tag{17}
\]

We have the following worst-case bound on \( \sum_{t=1}^{n} \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e} \sqrt{x_e^T M_{t-1}^{-1} x_e} \):

**Lemma 2** For any time \( t = 1, 2, \ldots, n \), we have
\[
\sum_{t=1}^{n} \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e} \sqrt{x_e^T M_{t-1}^{-1} x_e} \leq \sqrt{\left( \sum_{t=1}^{n} \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e}^2 \right) \mathbb{E} [ L \| \theta \|_2^2 ] / \log (1 + \frac{nE_*}{d})}. \tag{18}
\]

Moreover, if \( X = I \in \mathbb{R}^{|E| \times |E|} \), then we have
\[
\sum_{t=1}^{n} \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e} \sqrt{x_e^T M_{t-1}^{-1} x_e} \leq \sqrt{\left( \sum_{t=1}^{n} \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e}^2 \right) |E| \log (1 + \frac{n}{d}) / \log (1 + \frac{1}{\sigma})}. \tag{19}
\]

Please refer to Section A.3 for the proof of Lemma 2. Finally, notice that for any \( t \),
\[
\mathbb{E} \left[ \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e}^2 \right] = \sum_{e \in E} N_{S_t,e} \mathbb{E} [ 1 \{ O_t(e) \} | S_t ] \leq C_*^2,
\]
thus
\[
\mathbb{E} \left[ \sum_{t=1}^{n} \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e}^2 \right] \leq \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{e \in E} 1 \{ O_t(e) \} N_{S_t,e}^2 \right] \leq C_* \sqrt{n}.
\]
So we have
\[
R^{\alpha \gamma}(n) \leq \frac{2cC_*}{\alpha \gamma} \sqrt{dnE_* \log (1 + \frac{nE_*}{d})} + [L - K] \sum_{t=1}^{n} \mathbb{P} ( \mathcal{F}_{t-1} ). \tag{18}
\]

For the special case when \( X = I \), we have
\[
R^{\alpha \gamma}(n) \leq \frac{2cC_*}{\alpha \gamma} \sqrt{n|E| \log (1 + \frac{n}{d})} + [L - K] \sum_{t=1}^{n} \mathbb{P} ( \mathcal{F}_{t-1} ). \tag{19}
\]

Finally, we need to bound \( \sum_{t=1}^{n} \mathbb{P} ( \mathcal{F}_{t-1} ) \). We have the following bound on \( \mathbb{P} ( \mathcal{F}_{t-1} ) \):

**Lemma 3** For any time \( t = 1, 2, \ldots, n \), any \( \sigma > 0 \), any \( \delta \in (0, 1) \), and any
\[
c \geq \frac{1}{\sigma} \sqrt{d \log \left( 1 + \frac{nE_*}{d\sigma^2} \right) + 2 \log \left( \frac{1}{\delta} \right) + \| \theta^* \|_2^2},
\]
we have \( \mathbb{P} ( \mathcal{F}_{t-1} ) \leq \delta \).

Please refer to Section A.4 for the proof of Lemma 3. From Lemma 3 for a known upper bound \( M \) on \( \| \theta^* \|_2 \), if we choose \( \sigma = 1 \) and \( c = \sqrt{d \log \left( 1 + \frac{nE_*}{d} \right) + 2 \log \left( n(L + 1 - K) \right) + M} \), which corresponds to \( \delta = \frac{1}{n(L + 1 - K)} \) in Lemma 3, then we have
\[
[L - K] \sum_{t=1}^{n} \mathbb{P} ( \mathcal{F}_{t-1} ) < 1.
\]

This concludes the proof for Theorem \[ \] □
A.2 Proof for Lemma 1

Proof: We fix the “target node” v throughout this proof. Recall that the relevant subtree for node v is $T_{S_t,v} = (V_{S_t,v}, E_{S_t,v})$, v is the root of $T_{S_t,v}$, and $S_{t,v}$, the set of relevant source nodes for v, are leaves of $T_{S_t,v}$. For any node $u \in V_{S_t,v}$ in the relevant subtree $T_{S_t,v}$, we define the subtree $T_{u,v}$ as the subtree of the relevant subtree $T_{S_t,v}$ rooted at u.

Remark 1 To further clarify the notions defined above, let us consider the forest $G$ illustrated in Figure 3 where $S_t = \{1, 2, 7, 11, 12, 14, 15\}$ (the orange nodes) and v = 5 (the gray node). Notice that by definition, the relevant subtree $T_{S_t,5}$ for node v = 5 includes nodes 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and the edges connecting them. Specifically,

- node 1 $\notin V_{S_t,5}$ since it is blocked by another source node 2;
- node 3, 4 $\notin V_{S_t,5}$ since there is no source node in their direction;
- node 15, 16 $\notin V_{S_t,5}$ since there is no path connecting them and node v = 5.

Recall that the root of $T_{S_t,5}$ is node v = 5 (the gray node), and leaves of $T_{S_t,5}$ are node 2, 7, 11, 12, 14. Note that all the leaves of $T_{S_t,5}$ are source nodes. Note that by definition, the subtree $T_{5,9}$ for node $u = 9$ includes node 9, 11, 12 and the edges connecting them. Similarly, the subtree $T_{5,10}$ for node $u = 10$ includes node 10, 13, 14 and the edges connecting them; the subtree $T_{5,8}$ for node $u = 8$ includes node 8, 9, 10, 11, 12, 13, 14 and the edges connecting them.

Before proceeding, we define the following notion:

Definition 1 For any node $u \in V_{S_t,v}$, any probability weight function $w : E \to [0, 1]$, $h(u, w)$ is defined as the probability that u is influenced by at least one leaf node of $T_{u,v}$ under weight function $w$.

Remark 2 Notice that $w$ can be any probability weight function. In particular, $w$ can be $w$ or $U_t$. For example, in Figure 3, $h(9, w)$ is the probability that node $u = 9$ is influenced by source node 11 or 12 (leaf nodes of $T_{5,9}$) under weight function $w$; $h(10, w)$ is the probability that node $u = 10$ is influenced by source nodes 14 (leaf nodes of $T_{5,10}$) under weight function $w$; $h(8, w)$ is the probability that node $u = 8$ is influenced by source nodes 11, 12 or 14 (leaf nodes of $T_{5,8}$) under weight function $w$.

Recall that $f(S_t, w, v)$ is defined as the probability that v is influenced by all the source nodes in $S_t$. Then, by definition of the relevant subtree $T_{S_t,v}$, we have

$$h(v, w) = f(S_t, w, v),$$

for the root node v. On the other hand, however, for an arbitrary $u \in V_{S_t,v}$, we have

$$h(u, w) \leq f(S_t, w, u).$$
This is because \( f(S_t, w, u) \) is the probability that \( u \) is influenced by \( S_t \) and \( h(u, w) \) is the probability that \( u \) is influenced by the leaf nodes of \( T_w^u \), which is a subset of \( S_t \).

**Remark 3** To see it, notice that in Figure 3 we have

\[
h(5, w) = f(S_t, w, 5) = \mathbb{P} \left( \text{node 5 is influenced by source nodes 2, 7, 11, 12, 14|} S_t; w \right).
\]

On the other hand,

\[
h(9, w) = \mathbb{P} \left( \text{node 9 is influenced by source nodes 11, 12|} S_t; w \right),
\]

but

\[
f(S_t, w, 9) = \mathbb{P} \left( \text{node 9 is influenced by source nodes 2, 7, 11, 12, 14|} S_t; w \right) \geq h(9, w).
\]

Notice that \( h(u, w) \)'s can be computed recursively based on the tree structure of \( T_{S_t,v} \). Specifically, for any \( u \in V_{S_t,v} \), we use \( C(u) \) to denote the set of its children in \( T_{S_t,v} \). For instance, in Figure 3, \( C(9) = \{11, 12\} \) and \( C(8) = \{9, 10\} \) (notice that node 8 is the parent of node 9 and node 5 is the parent of node 8). Then we have the following recursive equation:

\[
h(u, w) = \begin{cases} 1 & \text{if } u \text{ is a leaf} \\ 1 - \prod_{u' \in C(u)} [1 - w(u, u')h(u', w)] & \text{otherwise} \end{cases}
\]

(22)

where \( w(u, u') \) is the probability associated with the edge connecting \( u \) and \( u' \).

Recall that both \( \overline{v} \) and \( U_t \) are probability weight functions. Consider a non-leaf node \( u \) in \( T_{S_t,v} \), assume its children in \( C(u) \) are ordered as \( u_1', u_2', \ldots, u_{|C(u)|}' \), then we have

\[
h(u, U_t) - h(u, \overline{v}) = \sum_{k=1}^{|C(u)|} [\prod_{i=1}^{k-1} b_{i}] [b_k - a_k] \left( \prod_{j=k+1}^{l} a_j \right) \leq \sum_{k=1}^{l} [b_k - a_k],
\]

(24)

where inequality (a) follows from the following technical property: Assume \( 0 \leq a_k \leq b_k \leq 1 \) for all \( k = 1, \ldots, l \), then we have

\[
\prod_{k=1}^{l} b_k - \prod_{k=1}^{l} a_k = \sum_{k=1}^{l} \left( \prod_{i=1}^{k-1} b_{i} \right) [b_k - a_k] \left( \prod_{j=k+1}^{l} a_{j} \right) \leq \sum_{k=1}^{l} [b_k - a_k],
\]

where the last inequality follows from \( 0 \leq a_k \leq b_k \leq 1 \) for all \( k = 1, \ldots, l \). Inequality (a) above is obtained by choosing \( l = |C(u)|, b_k = 1 - \overline{w}(u, u_k' h(u_k', \overline{w}) \) and \( a_k = 1 - U_t(u, u_k' h(u_k', U_t)) \). Note that Equality (b) is simply a change of notations and Inequality (c) follows from \( 0 \leq U_t(u, u') \leq 1 \).

Another key observation is that

\[
h(u', \overline{w}) \leq f(S_t, \overline{w}, u') \leq \mathbb{E} \left[ \mathbb{1} \{O_t(u, u')\} | \mathcal{H}_{t-1}, S_t \right],
\]

where \( O_t(u, u') = \{ \text{edge } (u, u') \text{ is observed} \} \). Notice that the first inequality follows from Equation 21 and the second inequality follows from the fact that \( u' \) is influenced by a sufficient (but not necessary) condition that edge \( (u, u') \) is observed (notice that edge \( (u, u') \) is observed if and only if node \( u \) or node \( u' \) is influenced). So we have

\[
h(u, U_t) = h(u, \overline{v}) \leq \sum_{u' \in C(u)} \left( h(u', U_t) - h(u', \overline{v}) + [U_t(u, u') - \overline{w}(u, u')] \mathbb{1} \{O_t(u, u')\} | \mathcal{H}_{t-1}, S_t \right)
\]
for any non-leaf \( u \). Notice that \( h(u, U_t) - h(u, \overline{w}) = 0 \) if \( u \) is a leaf in the relevant tree \( T_{s_t, v} \), then by induction, we have

\[
f(S_t, U_t, v) - f(S_t, \overline{w}, v) = h(v, U_t) - h(v, \overline{w}) \leq \sum_{e \in E_{s_t, v}} [U_t(e) - \overline{w}(e)] \left[ 1 \{O_t(e)\} | H_{t-1}, S_t \right] \tag{25}
\]

\[\square\]

### A.3 Proof for Lemma 2

**Proof:** To simplify the exposition, we define \( z_{t,e} = \sqrt{x_e^T M_{t-1}^{-1} x_e} \) for all \( t = 1, 2, \ldots, n \) and all \( e \in E \), and use \( E_t^o \) denote the set of edges observed at time \( t \). Recall that

\[
M_t = M_{t-1} + \frac{1}{\sigma^2} \sum_{e \in E} x_e x_e^T 1 \{O_t(e)\} = M_{t-1} + \frac{1}{\sigma^2} \sum_{e \in E_t^o} x_e x_e^T. \tag{26}
\]

Thus, for all \((t, e)\) such that \( e \in E_t^o \) (i.e., edge \( e \) is observed at time \( t \)), we have that

\[
\det [M_t] \geq \det \left[ M_{t-1} + \frac{1}{\sigma^2} x_e x_e^T \right] = \det \left[ M_{t-1}^2 \left( I + \frac{1}{\sigma^2} M_{t-1}^{-\frac{1}{2}} x_e x_e^T M_{t-1}^{-\frac{1}{2}} \right) M_{t-1}^{\frac{1}{2}} \right]
\]

\[
= \det [M_{t-1}] \det \left[ I + \frac{1}{\sigma^2} M_{t-1}^{-\frac{1}{2}} x_e x_e^T M_{t-1}^{-\frac{1}{2}} \right] = \det [M_{t-1}] \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right).
\]

Thus we have

\[
(\det [M_t])^{E_t^o} \geq (\det [M_{t-1}])^{E_t^o} \prod_{e \in E_t^o} \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right).
\]

**Remark 4** Notice that when the feature matrix \( X = I \), \( M_t \)'s are always diagonal matrices, and we have

\[
\det [M_t] = \det [M_{t-1}] \prod_{e \in E_t^o} \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right),
\]

which will lead to a tighter bound in the tabular \((X = I)\) case.

Since \( 1) \det [M_t] \geq \det [M_{t-1}] \) from Equation 26 and \( 2) |E_t^o| \leq E_* \), where \( E_* \) is the maximum cardinality of reachable edges defined in Section 4.2, we have

\[
(\det [M_t])^{E_*} \geq (\det [M_{t-1}])^{E_*} \prod_{e \in E_t^o} \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right).
\]

So we have

\[
(\det [M_n])^{E_*} \geq (\det [M_0])^{E_*} \prod_{t=1}^{n} \prod_{e \in E_t^o} \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right) = \prod_{t=1}^{n} \prod_{e \in E_t^o} \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right),
\]

since \( M_0 = I \). On the other hand, we have that

\[
\text{trace} (M_n) = \text{trace} \left( I + \frac{1}{\sigma^2} \sum_{t=1}^{n} \sum_{e \in E_t^o} x_e x_e^T \right) = d + \frac{1}{\sigma^2} \sum_{t=1}^{n} \sum_{e \in E_t^o} \|x_e\|^2 \leq d + \frac{nE_*}{\sigma^2},
\]

where the last inequality follows from the fact that \( \|x_e\|_2 \leq 1 \) and \( |E_t^o| \leq E_* \). From the trace-determinant inequality, we have \( \frac{1}{d} \text{trace} (M_n) \geq [\det (M_n)]^{\frac{1}{d}} \), thus we have

\[
\left[ 1 + \frac{nE_*}{d\sigma^2} \right]^{dE_*} \geq \left[ \frac{1}{d} \text{trace} (M_n) \right]^{dE_*} \geq [\det (M_n)]^{E_*} \geq \prod_{t=1}^{n} \prod_{e \in E_t^o} \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right),
\]
Taking the logarithm on the both sides, we have

\[
dE_s \log \left[ 1 + \frac{nE_s}{\sigma^2} \right] \geq \sum_{t=1}^{n} \sum_{e \in E_t} \log \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right).
\]  

(27)

Notice that \(z_{t,e}^2 = x_e^TM_{t-1}x_e \leq x_e^TM_0^{-1}x_e = \|x_e\|^2 \leq 1\), thus we have \(z_{t,e}^2 \leq \frac{\log \left(1 + \frac{z_{t,e}^2}{\sigma^2} \right)}{\log \left(1 + \frac{1}{\sigma^2} \right)}\). Hence we have

\[
\sum_{t=1}^{n} \sum_{e \in E_t} z_{t,e}^2 \leq \frac{1}{\log \left(1 + \frac{1}{\sigma^2} \right)} \sum_{t=1}^{n} \sum_{e \in E_t} \log \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right) \leq \frac{dE_s \log \left[ 1 + \frac{nE_s}{\sigma^2} \right]}{\log \left(1 + \frac{1}{\sigma^2} \right)}.
\]  

(28)

**Remark 5** When the feature matrix \(X = I\), we have \(d = |E|\).

\[
\det [M_n] = \prod_{t=1}^{n} \prod_{e \in E_t} \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right), \text{ and } |E| \log \left[ 1 + \frac{nE_s}{|E|\sigma^2} \right] \geq \sum_{t=1}^{n} \sum_{e \in E_t} \log \left( 1 + \frac{z_{t,e}^2}{\sigma^2} \right).
\]

This implies that

\[
\sum_{t=1}^{n} \sum_{e \in E_t} z_{t,e}^2 \leq \frac{|E| \log \left[ 1 + \frac{nE_s}{|E|\sigma^2} \right]}{\log \left(1 + \frac{1}{\sigma^2} \right)},
\]  

(29)

since \(E_s \leq |E|\).

Finally, from Cauchy-Schwarz inequality, we have that

\[
\sum_{t=1}^{n} \sum_{e \in E_t} \mathbf{1}\{O_t(e)\} N_{S_t,e} \sqrt{x_e^TM_{t-1}x_e} = \sum_{t=1}^{n} \sum_{e \in E_t} N_{S_t,e} z_{t,e} \leq \sqrt{\left( \sum_{t=1}^{n} \sum_{e \in E_t} N_{S_t,e}^2 \right) \left( \sum_{t=1}^{n} \sum_{e \in E_t} z_{t,e}^2 \right)} = \sqrt{\left( \sum_{t=1}^{n} \sum_{e \in E_t} \mathbf{1}\{O_t(e)\} N_{S_t,e}^2 \right) \left( \sum_{t=1}^{n} \sum_{e \in E_t} z_{t,e}^2 \right)}.
\]  

(30)

Combining this inequality with the above bounds on \(\sum_{t=1}^{n} \sum_{e \in E_t} z_{t,e}^2\) (see Equations 28 and 29), we obtain the statement of the lemma.

### A.4 Proof for Lemma 3

**Proof:** We use \(E_t^n\) denote the set of edges observed at time \(t\). The first observation is that we can order edges in \(E_t^n\) based on breadth-first search (BFS) from the source nodes \(S_t\), as described in Algorithm 2 where \(\pi_t(S_t)\) is an arbitrary conditionally deterministic order of \(S_t\). We also assume that there is a fixed order of neighbors for a given node \(v \in V\).

Let \(J_t = |E_t^n|\). Based on Algorithm 2, we order the observed edges in \(E_t^n\) as \(a_1^t, a_2^t, \ldots, a_{J_t}^t\). We start by defining some useful notation. For any \(t = 1, 2, \ldots\), any \(j = 1, 2, \ldots, J_t\), we define

\[
e_{t,j} = w_t(a_j^t) - \overline{w}(a_j^t).
\]

Notice that for any \(y \in [0, 1]\), we have \(y \leq \frac{\log \left(1 + \frac{y}{\log \left(1 + \frac{1}{\sigma^2} \right)} \right)}{\log \left(1 + \frac{1}{\sigma^2} \right)} = h(y)\). To see it, notice that \(h(y)\) is a strictly concave function, and \(h(0) = 0\) and \(h(1) = 1\).
where the first inequality follows from the Cauchy-Schwarz inequality and the second inequality as we will see later, we define V

Algorithm 2

Breadth-First Sort of Observed Edges

Input: graph \( \mathcal{G} \), \( \pi_i(S_t) \), and \( w_t \)

Initialization: node queue \( \text{queueN} \leftarrow \pi_i(S_t) \), edge queue \( \text{queueE} \leftarrow \emptyset \), dictionary of influenced nodes \( \text{dictN} \leftarrow S_t \)

while \( \text{queueN} \) is not empty do
  node \( v \leftarrow \text{queueN}.\text{dequeue()} \)
  for all neighbor \( u \) of \( v \) do
    if edge \((v, u)\) is not in queue \( \text{queueE} \) then
      \( \text{queueE}.\text{enqueue}((v, u)) \)
    if \( w_t(v, u) == 1 \) and \( u \notin \text{dictN} \) then
      \( \text{queueN}.\text{enqueue}(u) \) and \( \text{dictN} \leftarrow \text{dictN} \cup \{u\} \)

Output: edge queue \( \text{queueE} \)

One key observation is that \( \eta_{t,j} \)'s form a Martingale difference sequence (MDS). Moreover, \( \eta_{t,j} \)'s are bounded in \([-1, 1]\) and hence they are conditionally sub-Gaussian with constant \( R = 1 \). We further define that

\[
V_t = \sigma^2 M_t = \sigma^2 I + \sum_{\tau=1}^{t} \sum_{j=1}^{J_t} x_{a_{\tau}^j} x_{a_{\tau}^j}^T, \quad \text{and}
\]

\[
Y_t = \sum_{\tau=1}^{t} \sum_{j=1}^{J_t} x_{a_{\tau}^j} \eta_{t,j} = B_t - \sum_{\tau=1}^{t} \sum_{j=1}^{J_t} x_{a_{\tau}^j} \bar{w}(a_{\tau}^j) = B_t - \left[ \sum_{\tau=1}^{t} \sum_{j=1}^{J_t} x_{a_{\tau}^j} x_{a_{\tau}^j}^T \right] \theta^*.
\]

As we will see later, we define \( V_t \) and \( Y_t \) to use the self-normalized bound developed in [1] (see Algorithm 1 of [1]). Notice that

\[
M_t \bar{\theta}_t = \frac{1}{\sigma^2} B_t = \frac{1}{\sigma^2} Y_t + \frac{1}{\sigma^2} \left[ \sum_{\tau=1}^{t} \sum_{j=1}^{J_t} x_{a_{\tau}^j} x_{a_{\tau}^j}^T \right] \theta^* = \frac{1}{\sigma^2} Y_t + [M_t - I] \theta^*,
\]

where the last equality is based on the definition of \( M_t \). Hence we have

\[
\bar{\theta}_t - \theta^* = M_t^{-1} \left[ \frac{1}{\sigma^2} Y_t - \theta^* \right].
\]

Thus, for any \( e \in \mathcal{E} \), we have

\[
\left| \langle x_e, \bar{\theta}_t - \theta^* \rangle \right| = \left| x_e^T M_t^{-1} \left[ \frac{1}{\sigma^2} Y_t - \theta^* \right] \right| \leq \| x_e \|_{M_t^{-1}} \| \frac{1}{\sigma^2} Y_t - \theta^* \|_{M_t^{-1}}
\]

\[
\leq \| x_e \|_{M_t^{-1}} \left[ \frac{1}{\sigma^2} \| Y_t \|_{M_t^{-1}} + \| \theta^* \|_{M_t^{-1}} \right],
\]

where the first inequality follows from the Cauchy-Schwarz inequality and the second inequality follows from the triangle inequality. Notice that \( \| \theta^* \|_{M_t^{-1}} \leq \| \theta^* \|_{M_0^{-1}} = \| \theta^* \|_2 \), and \( \frac{1}{\sigma^2} \| Y_t \|_{M_t^{-1}} = \frac{1}{\sigma} \| Y_t \|_{V_t^{-1}} \) (since \( M_t^{-1} = \sigma^2 V_t^{-1} \)), so we have

\[
\left| \langle x_e, \bar{\theta}_t - \theta^* \rangle \right| \leq \| x_e \|_{M_t^{-1}} \left[ \frac{1}{\sigma} \| Y_t \|_{V_t^{-1}} + \| \theta^* \|_2 \right].
\]

Notice that the above inequality always holds. We now provide a high-probability bound on \( \| Y_t \|_{V^{-1}_{t}} \) based on self-normalized bound proved in [1]. From Theorem 1 of [1], we know that for any \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), we have

\[
\| Y_t \|_{V^{-1}_{t}} \leq \left( \frac{2 \log \left( \frac{\det(V_t)^{1/2} \det(V_0)^{-1/2}}{\delta} \right)}{\delta} \right) \forall t = 0, 1, \ldots
\]

Notice that the notion of “time” is indexed by the pair \((t, j)\), and follows the lexicographical order.
Notice that $\det(V_0) = \det(\sigma^2 I) = \sigma^{2d}$. Moreover, from the trace-determinant inequality, we have

$$\frac{[\det(V_t)]^{1/d}}{d} \leq \frac{\text{trace}(V_t)}{d} = \sigma^2 + \frac{1}{d} \sum_{\tau=1}^{t} \sum_{j=1}^{J_\tau} \|x_{\alpha_{\tau j}}\|^2_2 \leq \sigma^2 + \frac{iE_*}{d} \leq \sigma^2 + \frac{\kappa E_*}{d},$$

where the second inequality follows from the assumption that $\|x_{\alpha_{\tau j}}\|^2_2 \leq 1$ and the fact $J_\tau = |E_t^o| \leq E_\tau$, and the last inequality follows from $t \leq n$. Thus, with probability at least $1 - \delta$, we have

$$\|Y_t\|_{V_t^{-1}} \leq \sqrt{d \log \left(1 + \frac{\kappa E_*}{\sigma^2 \delta^2} \right) + 2 \log \left(\frac{1}{\delta}\right)} \quad \forall t = 0, 1, \ldots, n - 1.$$

That is, with probability at least $1 - \delta$, we have

$$|\langle x_e, \hat{y}_t - \theta^*\rangle| \leq \|x_e\|_{M_{t-1}} \left[\frac{1}{\sigma} \sqrt{d \log \left(1 + \frac{\kappa E_*}{\sigma^2 \delta^2} \right) + 2 \log \left(\frac{1}{\delta}\right)} + \|\theta^*\|_2\right]$$

for all $t = 0, 1, \ldots, n - 1$ and $\forall e \in E$. Recall that by definition of event $F_{t-1}$, the above inequality implies that, for any $t = 1, 2, \ldots, n$, if

$$c \geq \frac{1}{\sigma} \sqrt{d \log \left(1 + \frac{\kappa E_*}{\sigma^2 \delta^2} \right) + 2 \log \left(\frac{1}{\delta}\right)} + \|\theta^*\|_2,$$

then $P(F_{t-1}) \geq 1 - \delta$. That is, $P(\bar{F}_{t-1}) \leq \delta$. □

**B Circle Case**

In this section, we outline the analysis of IMLinUCB for the circle case. We start by defining some terminologies. For a given circle $G = (V, E)$ and a given source node set $S \subseteq V$, for any node $v \in V \setminus S$, we say a source node $s \in S$ is a relevant source node for $v$ if there is a path from $s$ to $v$ that does not include another source node. Notice that in a circle, a node $v \in V \setminus S$ has at most two relevant source nodes. Specifically, $v$ has one relevant source node if and only if $|S| = 1$; otherwise, it has two relevant source nodes, one in the clockwise direction and the other in the counter-clockwise direction. The notion of relevant source node in the circle case is illustrated in Figure 4.

Figure 4: Illustration of the notion of relevant source nodes in the circle case, where the orange nodes are the source nodes. Note that in Figure (a), $|S| = 1$ and the only relevant source node for node $v = 5$ is node 1. In Figure (b), $K > 1$ and the two relevant source nodes for node $v = 5$ is node 3 and node 7.

We now define the notion of the relevant segment $T_{S,v}$ for a node $v \in V \setminus S$ with the given source nodes $S$. If $K = 1$, then the relevant segment $T_{S,v}$ for node $v$ is the whole circle; otherwise, it is the segment of the circle between its two relevant source nodes and including itself. For instance, in Figure 4(b), the relevant segment for node $v = 5$ includes node 3, 4, 5, 6, 7 and the edges connecting them.
For any source node set $S \subseteq V$, any probability weight function $w : E \rightarrow [0, 1]$, and any node $v \in V$, let $f(S, w, v)$ be the probability that node $v$ is influenced if the source node set is $S$ and the probability weight function is $w$. Recall that $H_{t-1}$ is the “history” by the end of time $t-1$, $S_t$ is the source node set chosen by $\text{IMLinUCB}$ at time $t$, and $\mathcal{F}_{t-1}$ is defined in Equation [14]. Then we have the following lemma if we apply $\text{IMLinUCB}$ to the circle case:

Lemma 4 For any $t$, any “history” $H_{t-1}$ and $S_t$ s.t. $\mathcal{F}_{t-1}$ holds, and any $v \in V \setminus S_t$, we have

$$f(S_t, U_t, v) = f(S_t, \bar{w}, v) \leq \sum_{e \in E_{S_t, v}} \mathbb{E} [1 \{O_t(e)\} | \mathcal{F}_{t-1}, S_t],$$

where $E_{S, v}$ is the edge set of the relevant segment $T_{S, v}$ for node $v$.

The proof of Lemma 4 is similar to that of Lemma 1 and is omitted. The remainder of the analysis is the same as the tree case. Specifically, if we also define

$$N_{S, e} = \sum_{v \in V \setminus S} 1 \{e \in E_{S, v}\} \quad \text{and} \quad P_{S, e} = \mathbb{P}(e \text{ is observed } | S)$$

and maximum observed relevance as

$$C_s = \max_{S: |S| = K} \left\{ \sum_{e \in E} N_{S, e}P_{S, e} \right\},$$

then we have the following theorem for the circle case:

Theorem 2 Assume that (1) graph $G = (V, E)$ is a circle, (2) $\bar{w}(e) = x_e^T \theta^*$ for all $e \in E$, and (3) $\text{ORACLE}$ is an $(\alpha, \gamma)$-approximation algorithm. Let $M$ be a known upper bound on $\|\theta^*\|_2$. If we apply $\text{IMLinUCB}$ with $\sigma = 1$ and

$$c = \sqrt{d \log \left(1 + \frac{nE_s}{d}\right) + 2 \log (nL(1 - K)) + M},$$

then we have

$$R^{\alpha \gamma}(n) \leq \frac{2cC_s}{\alpha \gamma} \sqrt{nE_s \log_2 \left(1 + \frac{nE_s}{d}\right) + 1} = \tilde{O} \left(dC_s \sqrt{E_s n} \right).$$

Moreover, if $X = I$ (the tabular case), we have

$$R^{\alpha \gamma}(n) \leq \frac{2cC_s}{\alpha \gamma} \sqrt{n|E| \log_2 (1 + n)} + 1 = \tilde{O} \left(|E|C_s \sqrt{n} \right).$$

C IMLinTS Algorithm

In this section, we propose Influence Maximization Linear Thompson Sampling (IMLinTS), a Thompson sampling algorithm for IM semi-bandits. The algorithm is detailed in Algorithm [3].

**Algorithm 3 IMLinTS: Influence Maximization Linear Thompson Sampling**

**Input:** graph $G$, number of source nodes $K$, oracle $\text{ORACLE}$, feature vector $x_e$’s, and algorithm parameters $\sigma > 0$.

**Initialization:** $B_0 \leftarrow 0 \in \mathbb{R}^d$, $M_0 \leftarrow I \in \mathbb{R}^{d \times d}$

for $t = 1, 2, \ldots, n$ do

1. compute $\theta_{t-1} \leftarrow \sigma^{-2} M_{t-1}^{-1} B_{t-1}$, sample $\theta_t \sim N \left(\theta_{t-1}, M_{t-1}^{-1}\right)$, and set

$$\bar{w}_t(e) \leftarrow \text{Proj}_{[0, 1]} \left(x_e^T \theta_t\right) \quad \forall e \in E$$

2. choose $S_t \in \text{ORACLE}(G, K, \bar{w}_t)$, and observe the edge-level semi-bandit feedback

3. update statistics:

(a) initialize $M_t \leftarrow M_{t-1}$ and $B_t \leftarrow B_{t-1}$

(b) for all the observed edges $e \in E$, update

$$M_t \leftarrow M_t + \sigma^{-2} x_e x_e^T \quad \text{and} \quad B_t \leftarrow B_t + x_e w_t(e)$$

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