A STUDY OF THE HILBERT-MUMFORD CRITERION FOR THE
STABILITY OF PROJECTIVE VARIETIES

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Abstract. We make a systematic study of the Hilbert-Mumford criterion for
different notions of stability for polarised algebraic varieties \((X, L)\); in particular
for K- and Chow stability. For each type of stability this leads to a concept of slope
\(\mu\) for varieties and their subschemes; if \((X, L)\) is semistable then
\(\mu(Z) \leq \mu(X)\) for all \(Z \subset X\). We give examples such as curves, canonical models and Calabi-
Yaus. We prove various foundational technical results towards understanding the
converse, leading to partial results; in particular this gives a geometric (rather
than combinatorial) proof of the stability of smooth curves.

1. Introduction; slope stability

Geometric Invariant Theory [GIT] has been very successful in forming moduli
spaces of (semi)stable coherent sheaves over polarised algebraic varieties \((X, L)\)
[HL]. Moduli space is constructed as a quotient of a subset of a Quot scheme
by a group, and the Hilbert-Mumford criterion is applied to 1-parameter subgroups.
Their weights are found to be dominated by positive linear combinations of weights
of particularly simple 1-parameter subgroups corresponding to a degeneration of a
sheaf \(E\) into a splitting \(F \oplus E/F\), for some subsheaf \(F \leq E\). Thus stability of sheaves
is governed by subsheaves; calculating the corresponding weights leads to the notion
of slope stability (the exact form of the slope depending on the linearisation used
on Quot).

Forming moduli spaces of (semi)stable varieties themselves using GIT has proved
much more difficult, and has mainly been accomplished for canonically polarised
varieties using the Chow linearisation, due to work of Mumford [GIT, Mu], Gieseker
[Gi], and, in a little more generality, Viehweg [V]. Roughly speaking one expects
varieties polarised by their canonical bundle \(L = K_X\) to be automatically stable
since their moduli functor is already separated because of the birational invariance
of spaces of sections of powers of the canonical bundle. In the general case no geo-
metric criterion for (in)stability has emerged. This is because the Hilbert-Mumford
criterion has not been successfully simplified or interpreted for varieties; instead
Viehweg proved deep positivity results to produce the group-invariant sections of
the appropriate line bundle directly (for varieties with semi-ample canonical bun-
dle). Kollar [Ko] and others have turned to other methods for producing moduli
of varieties, but new impetus to understanding stability has come from the link be-
tween K-stability and the existence of Kähler-Einstein and constant scalar curvature
Kähler metrics [Ti1, Do1] to which we apply our methods in [RT].
Our approach uses the Hilbert-Mumford criterion, as pioneered by Mumford [Mu]. He calculates the relevant weights in terms of blow-ups of $X \times \mathbb{C}$ in subschemes supported on thickenings of $X \times \{0\}$; one of our main results (Corollary 5.7) reduces the calculation to a sum of weights of blow-ups in the scheme-theoretic central fibre. (In a sense which is made clear by the proof the construction turns the horizontal thickenings of Mumford “vertical”, into the central fibre.)

Taking just one such blow-up in a subscheme $Z \subset X$ gives the “deformation to the normal cone of $Z$”, analogous to the simple 1-parameter subgroups that arise in the GIT of sheaves. This gives a numerical condition for $Z$ to destabilise $X$, and so a notion of “slope stability”, by analogy with the sheaf theory which we now review briefly.

For a sheaf $E$ over $(X, L)$, the reduced Hilbert polynomial $p_E$ is the monic version of the Hilbert polynomial $P_E(r) = \chi(E \otimes L^r) = a_0 r^n + a_1 r^{n-1} + \ldots$:

$$p_E(r) = \frac{\chi(E \otimes L^r)}{a_0} = r^n + \mu(E) r^{n-1} + \ldots,$$

and the slope of $E$ is its leading nontrivial coefficient,

$$\mu(E) = a_1 / a_0. \quad (1.1)$$

(This differs from the usual definition $\deg(E) / \text{rank}(E)$ by unimportant terms that depend only on the geometry of $(X, L)$.) Then we say that $E$ is semistable if for all proper coherent subsheaves $F \leq E$,

$$p_F \preceq p_E.$$

For Gieseker semistability, $\preceq$ means there exists an $r_0 > 0$ such that $p_F(r) \leq p_E(r)$ for all $r \geq r_0$; for slope semistability the inequality is at the level of the $r^{n-1}$ coefficients:

$$\mu(F) \leq \mu(E). \quad (1.2)$$

If the inequalities are all strict then $E$ is stable, and this agrees with the appropriate GIT notions for different choices of linearisation (i.e. choice of equivariant line bundle on the Quot scheme; in fact Jun Li’s line bundle [Li] is only semi-ample, but he extends GIT to this case.)

Due to different choices of linearisation and asymptotics there are also many notions of stability (Hilbert, Chow, asymptotic Hilbert, asymptotic Chow and K-stability) for varieties, defined in the next section, and so many versions of slope. We first describe the one relevant to K-stability.

Given a subscheme $Z$ of a polarised algebraic variety $(X, L)$, define

$$\epsilon(Z) = \sup \{ c \in \mathbb{Q} > 0 : L^r \otimes \mathcal{I}_Z^c \text{ is globally generated } \forall r \gg 0 \text{ with } cr \in \mathbb{N} \}.$$ 

The Hilbert-Samuel polynomial for fixed $x$,

$$\chi(L^r \otimes \mathcal{I}_Z^x) = a_0(x) r^n + a_1(x) r^{n-1} + \ldots, \quad r \gg 0, \; rx \in \mathbb{N}, \quad (1.3)$$

defines $a_i(x)$ which are polynomials in $x$ (see Section 3) and so extend by the same formulae to $x \in \mathbb{R}$. Then analogously to (1.1) we define the K-slope of $\mathcal{I}_Z$ (with
respect to $L$ and $c \in (0, \epsilon(Z)]$ to be
\[ \mu_c(\mathcal{I}_Z) = \mu_c(\mathcal{I}_Z, L) := \frac{\int_0^c (a_1(x) + \frac{a_0'(x)}{2}) dx}{\int_0^c a_0(x) dx}. \] (1.4)

Setting $Z = \emptyset$ defines the slope of $X$ (precisely: of $\mathcal{O}_X$ with respect to $L$ and $c$) as
\[ \mu(X) = \mu(X, L) = \frac{a_1}{a_0}, \] (1.5)

independently of $c$. Here $a_i$ are the coefficients of the Hilbert polynomial $\chi(L^r) = a_0 r^n + a_1 r^{n-1} + \ldots$, $a_0 = a_0(0)$ and, for $X$ normal, $a_1 = a_1(0)$ (4.21).

Setting $\mu(Z) := \sup_{0 < c \leq \epsilon(Z)} \mu_c(\mathcal{I}_Z)$, we say that $(X, L)$ is $K$-slope semistable if for all proper $Z \subset X$,
\[ \mu(Z) \leq \mu(X), \text{ i.e. } \frac{\int_0^c (a_1(x) + \frac{a_0'(x)}{2}) dx}{\int_0^c a_0(x) dx} \leq \frac{a_1}{a_0} \quad \forall c \in (0, \epsilon(Z)]. \]

Cf. (1.2). The definition of slope stability is slightly trickier (Definition 4.17); for this reason we work with $\mu_c(\mathcal{I}_Z)$ instead of $\mu(Z)$. Then (Theorem 4.18) $X$ is slope (semi)stable if it is $K$-(semi)stable.

The $a_0'/2$ “correction term” in the definition of slope arises from the difference between the Hilbert polynomial of a 2-component normal crossing variety and the sum of those of its components (whereas for sheaves, $\mathcal{P}_E = \mathcal{P}_F + \mathcal{P}_{E/F}$ for any $F \leq E$). Simon Donaldson pointed out that his analysis of stability of toric varieties [Do2] throws up a boundary term similar to our $(a_0(c) - a_0)/2 = \int_0^c a_0'/2$; we explain this in [RT].

Similarly for $X \subseteq \mathbb{P}^N = \mathbb{P}(H^0(X, L)^*)$ (embedded in projective space by the space of sections of its polarisation $L = \mathcal{O}_X(1)$ for ease of exposition; see Section 7 for the general case), a subscheme $Z \subset X$ and an integer $0 < c \leq \epsilon(Z)$, we define the Chow slope of $\mathcal{I}_Z$ to be
\[ Ch_c(\mathcal{I}_Z) := \frac{\sum_{i=1}^c h^0(\mathcal{I}_Z^i(1))}{\int_0^c a_0(x) dx}. \]

Setting $Z = \emptyset$ gives $Ch(\mathcal{O}_X) = Ch(X) := \frac{h^0(\mathcal{O}_X(1))}{a_0}$; then Chow (semi)stability implies Chow slope (semi)stability (Theorem 7.2):
\[ Ch_c(\mathcal{I}_Z) \leq (\leq) Ch(X). \]

Asymptotic Chow is more like Gieseker stability and so our slope criterion for that (4.33) is more complicated.

Section 3 describes the various stability notions uniformly, while Section 4 introduces slope stability via the deformation to the normal cone. Arbitrary 1-parameter subgroups are studied in Section 5, calculating their weights in terms of a sequence of simple blow ups in Corollary 5.7. In Section 6 these weights are shown to be those of a deformation to the normal cone under certain circumstances, giving a partial converse to “stability $\Rightarrow$ slope stability”. We used to think this could be done in general, but the failure of the thickenings of certain flat families of subschemes to themselves be flat prevents us from carrying out the programme in full. We study
when the thickenings are flat, and get round the problem with a basechange trick in some situations. This deals with the curve case completely, i.e. stability and slope stability are equivalent there, giving geometric proofs of the K- and asymptotic Chow stability of curves of genus $\geq 1$. As far as we know all previous proofs used analysis and combinatorics respectively. Section 7 is devoted to Chow stability and Section 8 to examples. Many more examples, such as projective bundles, appear in [RT], in particular showing that K- and slope stability are also equivalent for projective bundles over a curve.

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2. Notation

Throughout this paper $Z$ will denote a closed subscheme of a proper irreducible polarised scheme $(X, L)$ of dimension $n = \dim X$; for much of the paper $X$ will be a normal irreducible variety. By $jZ$ we mean the subscheme which is the $j$-fold thickening of $Z$ defined by its ideal sheaf $I_{jZ} := I_{jZ}$. We denote the blow up along $Z$ by $\pi: \tilde{X} \to X$, with exceptional divisor $E$. Then $\pi^* \mathcal{I}_Z^j = \mathcal{O}(-jE)$ and there exists a $p$ such that $\pi^* \mathcal{O}(-jE) = \mathcal{I}_Z^j$ for all $j \geq p$. For convenience we often suppress pullback maps, mix multiplicative and additive notation for line bundles, and use the same letter to denote a divisor and the associated line bundle. For example on $\tilde{X}$, we denote $(\pi^* \mathcal{L}(-E))^\otimes k$ by $L^k - kE$. Worse still, where it does not cause confusion, $L^k$ may also denote $c_1(\mathcal{L})^\cap k$.

For brevity we often denote sheaf cohomology on a space $X$ by $H^i_X$; this never means local cohomology.

Any vector space $V$ with a $\mathbb{C}^\times$-action splits into one-dimensional weight spaces $V = \bigoplus_i V_i$, where $t \in \mathbb{C}^\times$ acts on $V_i$ by $t^{w_i}$. The $w_i$ are the weights of the action, and $w(V) = \sum_i w_i$ is the total weight of the action; i.e. the weight of the induced action on $\Lambda_{\max} V$.

On any family over $\mathbb{C} = \text{Spec} \mathbb{C}[t]$, $t$ will denote the pullback of the standard coordinate on $\mathbb{C}$.

A line bundle $L$ is semi-ample [La] if its high powers are globally generated (i.e. basepoint free). In this paper all semi-ample line bundles will have the additional property that the contraction they define is birational; that is $L$ is the pull back of an ample line bundle from a birational scheme. (It is important that this contraction can be trivial, i.e. semi-ample includes ample.)

Given a (semi-)ample line bundle $L$ on $X$, the Seshadri constant of $Z$ is

$$\epsilon(Z) = \epsilon(Z, X, L) = \sup \{ c \in \mathbb{Q}_{\geq 0} : L^r \otimes \mathcal{I}_Z^{cr} \text{ is globally generated for } r \gg 0 \}$$

$$= \max \{ c \in \mathbb{Q}_{\geq 0} : L - cE \text{ is nef on } \tilde{X} \}. \quad (2.1)$$
Given a pair of ideals $J \subset I \subset \mathcal{O}_X$, we say that $J$ saturates $I$ if there exists $i > 0$ such that $JI^{i-1} = I^i$. Equivalently, on the blow up $p: \tilde{X} \to X$ of $X$ along $I$ with exceptional divisor $E$, the natural inclusion $p^*J \to \mathcal{O}(-E)$ is an isomorphism. (This equivalence is a tautology: the zero set of the section $s$ of $L$ is a scheme $(\otimes L)_{\text{subsheaf}}$ homogeneous ideal the saturation of $(\otimes i)_I$ if and only if the zero set is empty, if and only if the sections generate $\mathcal{O}(-E)$.)

Similarly, give a line bundle $L$ on $X$, the global sections of $L \otimes I$ generate a subsheaf $L \otimes J \subset L \otimes I$, defining an ideal $J$. We say that the global sections of $L \otimes I$ saturate $I$ if $J$ saturates $I$. Equivalently, the global sections of $L \otimes I$ generate the line bundle $L(-E)$ on $\tilde{X}$. This is weaker than (i.e. is implied by) $L \otimes I$ being globally generated.

3. Hilbert, Chow and K-stability, and test configurations

Fix a polynomial $\mathcal{P}$ of degree $n$ and consider any $n$-dimensional proper polarised scheme $(X, L)$ whose Hilbert polynomial equals $\mathcal{P}$:

$$\mathcal{P}(r) = \chi_X(L^r) = a_0 r^n + a_1 r^{n-1} + \ldots,$$

where $a_0 = \frac{1}{n!} \int_X c_1(L)^n = \frac{L^n}{n!}$ and, for smooth $X$,

$$a_1 = \frac{1}{2(n-1)!} \int_X c_1(X) c_1(L)^{n-1} = -\frac{K_X.L^{n-1}}{2(n-1)!}.$$

Fix $r > 0$ such that $L^r$ is both very ample on $X$, and

$$H^i(L^r) = 0 \quad \text{for } i > 0, \quad \Rightarrow \quad H^0(L^r)^* \cong \mathbb{C}^{\mathcal{P}(r)}. \quad (3.1)$$

Then $L^r$ embeds $X$ in $\mathbb{P} = \mathbb{P}^{\mathcal{P}(r)-1}$, defining a point of the Hilbert scheme $\text{Hilb} = \text{Hilb}_{\mathcal{P}, K}$ of subschemes of $\mathbb{P}$ with Hilbert polynomial $\mathcal{P}(K)' = \mathcal{P}(Kr)$. Then there is a $K_0$ such that for all points $\{X\}$ of Hilb and $K \geq K_0$ we have an exact sequence

$$0 \to H^0_\mathbb{P}(\mathcal{S}_\mathbb{P}(K)) \to S^K \mathbb{C}^{\mathcal{P}(r)*} \cong S^K H^0_X(L^r) \to H^0_X(L^{Kr}) \to 0. \quad (3.2)$$

This (see for example [V]) defines $\text{Hilb}$ as a closed subscheme of the Grassmannian $G = \text{Grass}(S^K \mathbb{C}^{\mathcal{P}(r)*}, \mathcal{P}(Kr))$.

So $(X, L^r)$ and a choice of isomorphism $H^0(L^r)^* \cong \mathbb{C}^{\mathcal{P}(r)}$ give us a point $x = x_{r,K}$ in $G$, with different choices of isomorphism corresponding (up to scale) to the orbits of $SL(\mathcal{P}(r), \mathbb{C})$ on $\mathbb{P}^{\mathcal{P}(r)-1}$. A $g \in SL(\mathcal{P}(r), \mathbb{C})$ acting on $\mathbb{C}^{\mathcal{P}(r)}$ induces an action

$$(S^K g^*)^{-1} \quad \text{on} \quad S^K \mathbb{C}^{\mathcal{P}(r)*}, \quad (3.3)$$

and so one on $G = \text{Grass}(S^K \mathbb{C}^{\mathcal{P}(r)*}, \mathcal{P}(Kr))$. It is this action that commutes with the action on $\text{Hilb} \subset G$ induced by that on $\mathbb{P}$. Points in the orbit of $x \in G$ correspond to the projective transformations of $X$.

From (3.2), the fibre over $x \in \text{Hilb}$ of the hyperplane bundle on $G$ is naturally isomorphic to

$$\mathcal{O}_G(1)|_x = \Lambda^{\text{max}} H^0_X(L^{Kr}) \otimes (\Lambda^{\text{max}} S^K H^0_X(L^r))^*. \quad (3.4)$$
Definition 3.5. \((X, L)\) is Hilbert (semi)stable with respect to \(r\) if the point \(x, K \in \text{Hilb}\) is GIT (semi)stable for the action of \(SL(P(r), \mathbb{C})\), linearised on \(3.4\), for all \(K \gg 0\).

Asymptotic Hilbert stability is defined to mean Hilbert stability for all sufficiently large \(r\). By picking a different line bundle on the Hilbert scheme (i.e. a different projective embedding of \(\text{Hilb}\) – the beautiful Chow embedding [Mu]) we also get the notion of Chow stability with respect to \(r\) and asymptotic Chow stability. We need the concept of a test configuration, as defined in the foundational paper [Do2].

Definition 3.6. A test configuration for a polarised variety \((X, l)\) is

1. A proper flat morphism \(\pi: \mathcal{X} \to \mathbb{C}\),
2. An action of \(\mathbb{C}^\times\) on \(\mathcal{X}\) covering the usual action of \(\mathbb{C}^\times\) on \(\mathbb{C}\),
3. An equivariant very ample line bundle \(\mathcal{L}\) on \(\mathcal{X}\),

such that the fibre \((\mathcal{X}_t, \mathcal{L}|_{\mathcal{X}_t})\) is isomorphic to \((X, l)\) for one, and so all, \(t \in \mathbb{C}\{0\}\).

A test configuration is called a product configuration if \(\mathcal{X} \cong X \times \mathbb{C}\), and a trivial configuration if in addition \(\mathbb{C}^\times\) acts only on the second factor.

We will often need a weaker concept which we call a semi test configuration where \(\mathcal{L}\) is just globally generated.

Proposition 3.7. In the situation of (3.1), a 1-parameter subgroup of \(GL(P(r), \mathbb{C})\) is equivalent to the data of a test configuration for \((X, L^r)\).

Proof. The action of a 1-parameter subgroup of \(GL(P(r), \mathbb{C})\) on \(X \subseteq \mathbb{P}(H^0_X(L^r)^*)\) clearly gives a test configuration \((\mathcal{X}, \mathcal{L})\) for \((X, L^r)\) with \(\mathcal{L}\) the pull back of the (equivariant) line bundle \(O_P(1)\).

Conversely the subgroup can be recovered from the test configuration via the induced \(\mathbb{C}^\times\)-action on the dual of the vector space \((\pi_*\mathcal{L})|_{\{0\}}\). Since \(\pi_*\mathcal{L}\) is torsion-free (by flatness) over the curve \(\mathbb{C}\) it is a vector bundle, and \(\mathcal{X}\) sits inside the projectivisation of its dual by the very ampleness of \(\mathcal{L}\). Its general fibre has dimension \(P(r)\), so \((\pi_*\mathcal{L})|_{\{0\}}\) does too.

Pick a trivialisation of \(\pi_*\mathcal{L}\) over \(\mathbb{C}\), identifying it with \((\pi_*\mathcal{L})|_{\{0\}} \times \mathbb{C}\). This has a diagonal \(\mathbb{C}^\times\)-action, inducing one on \(\mathbb{P}(\pi_*\mathcal{L})^* \supset \mathcal{X}\) which yields the original test configuration; thus these two operations are mutual inverses.

(Note that in fact \((\pi_*\mathcal{L})|_{\{0\}} \cong H^0_{\mathcal{X}}(\mathcal{L})/tH^0_{\mathcal{X}}(\mathcal{L})\). The map \(\leftarrow\) is just restriction; we need to define its inverse \(\to\). Any element of \((\pi_*\mathcal{L})|_{\{0\}}\) can be lifted to a meromorphic section \(s\) of \(\mathcal{L}\) on \(\mathcal{X}\) that is regular on \(\mathcal{X}_0\). The projection of its polar locus to \(\mathbb{C}\) does not contain \(\{0\}\) and so is a finite number of points in \(\mathbb{C}\). Choosing a polynomial \(p\) with high order zeros at these points such that \(p(0) = 1\), \(p, s\) is a holomorphic section with the same class as \(s\) in \((\pi_*\mathcal{L})|_{\{0\}}\) since \(p - 1 \in (t)\). Then \([p, s]\) defines the required class in \(H^0_{\mathcal{X}}(\mathcal{L})/tH^0_{\mathcal{X}}(\mathcal{L})\).)

Denote the weight of the induced \(\mathbb{C}^\times\)-action on \((\pi_*\mathcal{L}^K)|_{\{0\}} = H^0_{\mathcal{X}}(\mathcal{L}^K)/tH^0_{\mathcal{X}}(\mathcal{L}^K)\) by \(w(Kr)\). (We enumerate by \(k := Kr\) since \((\mathcal{X}, \mathcal{L}^K)\) is a test configuration for \((X, L^{Kr})\). The confused reader may set \(r = 1\) temporarily.) For \(K \gg 0\), \((\pi_*\mathcal{L}^K)|_{\{0\}}\) is \(H^0_{\mathcal{X}_0}(\mathcal{L}^K)\) and \(w(k) = w(Kr)\) is a polynomial of degree \(n + 1\) in \(k = Kr\) by the
equivariant Riemann-Roch theorem. (It is important that we do not modify \( w(k) \) to be this polynomial for small \( k \), so for instance \( w(r) \) really is the weight on \( (\pi_*,\mathcal{L})|_{\{0\}} \).)

To make the \( \mathbb{C}^\times \)-action special linear on \( (\pi_*,\mathcal{L})|_{\{0\}} \) we first pull back the family by the cover \( C \to \mathbb{C} \), \( t \mapsto tp^r \), making the action of weight \( rP(w(r)) \). Composing with the trivial action which scales the \( \mathcal{L} \)-fibres with weight \(-rw(r)\) scales \( \Lambda^{\max}(\pi_*,\mathcal{L})|_{\{0\}} \) with weight \(-rP(w(r))\), cancelling out the previous action. (The extra factor of \( r \) is to make the formula (3.8) nicer, and is natural if \((X,\mathcal{L})\) is the \( r \)th twist of a test configuration for \((X,\mathcal{L})\).

Since this new action acts with zero total weight on \( SK(\pi_*,\mathcal{L})|_{\{0\}} \), it acts on the line \( O_G(1)|_{\mathbb{P}^0} = \Lambda^{\max}H^0_{\mathbb{P}^0}(\mathcal{L}^K) \otimes (\Lambda^{\max}S^K(\pi_*,\mathcal{L})|_{\{0\}}) \) (compare (3.4) which was for \((X,\mathcal{L})\) with no higher cohomology of \( \mathcal{L}^r \)) with normalised weight

\[
\tilde{w}_{r,K} = \tilde{w}_{r,k} = w(k)rP(r) - w(r)kP(k), \quad k := Kr. \tag{3.8}
\]

The normalised weight is a polynomial \( \sum_{i=0}^{n+1} e_i(r)k^i \) of degree \( n + 1 \) in \( k \) \( \gg 0 \), with coefficients which are also polynomial of degree \( n + 1 \) in \( r \) for \( r \gg 0 \):

\[
e_i(r) = \sum_{j=0}^{n+1} e_{i,j}r^j
\]

for \( r \gg 0 \). The normalisation means that the coefficient of \((kr)^{n+1}\) vanishes: \( e_{n+1,n+1} = 0 \). The Hilbert-Mumford criterion relates this weight to stability as follows.

**Theorem 3.9.** A polarised variety \((X,\mathcal{L})\) is stable if and only if

\[
\tilde{w}_{r,k} \gg 0 \quad \forall \text{ nontrivial test configurations for } (X,\mathcal{L}),
\]

where \( \gg \) and \( \forall \) mean the following for the different notions of stability:

- **Hilbert stable with respect to \( r \):** for any nontrivial test configuration for \((X,\mathcal{L})\), \( \tilde{w}_{r,k} > 0 \) for all \( k \gg 0 \),

- **Asymptotically Hilbert stable:** for all \( r \gg 0 \), any nontrivial test configuration for \((X,\mathcal{L})\) has \( \tilde{w}_{r,k} > 0 \) for all \( k \gg 0 \),

- **Chow stable with respect to \( r \):** for any nontrivial test configuration for \((X,\mathcal{L})\), the leading \( k^{n+1} \) coefficient \( e_{n+1}(r) \) of \( \tilde{w}_{r,k} \) is positive: \( e_{n+1}(r) > 0 \),

- **Asymptotically Chow stable:** for all \( r \gg 0 \), any nontrivial test configuration for \((X,\mathcal{L})\) has \( e_{n+1}(r) > 0 \),

- **K-stable:** for all \( r \gg 0 \), any nontrivial test configuration for \((X,\mathcal{L})\) has leading coefficient \( e_{n,n+1} \) of \( e_{n+1}(r) \) (the Donaldson-Futaki invariant of the test configuration) positive: \( e_{n,n+1} > 0 \).

Furthermore the result holds if we replace “stable” with “semistable” and strict inequalities with non strict inequalities throughout.

Finally \((X,\mathcal{L})\) is polystable if it is semistable and any destabilising test configuration (i.e. one which is not strictly stable) is a product configuration.

The increasing number of test configurations that have to be tested in the second, fourth and fifth definitions as \( r \to \infty \) currently prevent us from adding K-stability to the left of the following consequences of Theorem 3.9.

Asymptotically Chow stable \( \Rightarrow \) Asymptotically Hilbert stable \( \Rightarrow \) Asymptotically Hilbert semistable \( \Rightarrow \) Asymptotically Chow semistable \( \Rightarrow \) K-semistable.
The relation between K-stability and asymptotic Chow stability is analogous to the relation between slope stability and Gieseker stability for vector bundles, and a geometric criterion for asymptotic Chow stability would show it was implied by K-stability; we only have a necessary condition (Theorem 4.33).

The fact that Chow stability is controlled by the coefficient $e_{n+1}(r)$ is due to Mumford [Mu]. The definition of K-stability above is due to Donaldson, adapting Tian’s original differential-geometric definition to allow nonnormal central fibres $X_0$ (though what is called properly semistable in [Ti2] and stable in [Do2] is what we call K-polystable). Tian [Ti1] defines a line bundle (the “CM polarisation”) on Hilb such that K-stability is exactly stability in the sense of the Hilbert-Mumford criterion for this line bundle [PT]. K-stability is probably not a bona fide GIT notion: Tian’s polarisation may not be ample, and the number of test configurations increases as $r$ tends to infinity. However it is K-polystability that is conjecturally related to the existence of constant scalar curvature Kähler metrics; we apply our methods to these in [RT].

The definition of polystability says that any destabilising test configuration comes from a $\mathbb{C} \times$-action on $X$ with the appropriate weight 0 (i.e. Donaldson-Futaki invariant 0 in the K-polystability case). This corresponds to an orbit in Hilb which is closed in the semistable points but with possibly higher dimensional stabilisers; equivalently the orbit in the total space of the dual of the polarising line bundle over Hilb is closed. There seems to be no universally accepted name for this; we use polystability by analogy with bundles.

A test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L^r)$ can be twisted to get another, $(\mathcal{X}, \mathcal{L}^K)$, for $(X, L^{Kr})$; for $K \gg 0$ this will have no higher cohomology. Since the definition of K-stability is unchanged if $L$ is replaced by some power, for this notion we can allow $\mathcal{L}$ to be an ample $\mathbb{Q}$-line bundle in the definition of a test configuration.

Letting $F := e_{n,n+1}$ denote the Donaldson-Futaki invariant and writing the un-normalised weight $w(k)$ as $b_0k^{n+1} + b_1k^n + \ldots$, we see that

$$F = b_0a_1 - b_1a_0,$$

and $-a_0^{-2}F$ is the coefficient of $k^{-1}$ in the expansion of $w(k)/kP(k)$. When the central fibre $\mathcal{X}_0$ is smooth, $F = \frac{a_0}{4}\nu$, where $\nu$ is the usual Calabi-Futaki invariant of $c_1(L)$ and the vector field generated by the $S^1$-action [Do2].

Without loss of generality we now take $r = 1$. An arbitrary test configuration $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))$ for $(X, L)$ is, by definition, $\mathbb{C}^\times$-isomorphic to the trivial test configuration $(X \times \mathbb{C}, \mathcal{L})$ away from the central fibre. It is therefore $\mathbb{C}^\times$-birational to $(X \times \mathbb{C}, \mathcal{L})$ and so is dominated by a blow up $(\mathcal{X}, \mathcal{L})$ of $X \times \mathbb{C}$ in a $\mathbb{C}^\times$-invariant ideal $I$ supported on (a thickening of) the central fibre $X \times \{0\}$:

$$\begin{align*}
(\mathcal{X}, \mathcal{L}) &\overset{\phi}{\to} (\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1)) \\
\downarrow p &\quad \downarrow \\
(X \times \mathbb{C}, L) &\to (X \times \mathbb{C}, L).
\end{align*}$$

(3.11)
Here we use the canonical $\mathbb{C}^*$-action on $\text{Bl}_I(X \times \mathbb{C})$ inherited from that on $I$, and its linearisation on the line bundle $\mathcal{L} := L = p^*(L \otimes I)$, where $E$ denotes the exceptional divisor. $L - E = \phi^*\mathcal{O}_Y(1)$ and the horizontal arrow in (3.11) is an equivariant map of equivariant polarisations (although $L - E$ may be only semi-ample), whereas $p$ does not respect the polarisation.

Mumford ([Mu] section 3 of the proof of Theorem 2.9) essentially shows that any test configuration is of this form, where $I = I_r$ is of the form

$$I_r = \mathcal{I}_0 + t\mathcal{I}_1 + t^2\mathcal{I}_2 + \ldots + t^{r-1}\mathcal{I}_{r-1} + (t^r) \subset \mathcal{O}_X \otimes \mathbb{C}[t]. \quad (3.12)$$

Here the ideals $\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \ldots \subseteq \mathcal{I}_r \subseteq \mathcal{O}_X$ correspond to subschemes $Z_0 \supseteq Z_1 \supseteq \ldots \supseteq Z_{r-1}$ of $(X, L)$, so $I_r$ is $\mathbb{C}^*$-invariant and corresponds to a subscheme of $X \times \mathbb{C}$ supported on (the $r$-fold thickening of) the central fibre $X \times \{0\}$.

This can be proved by writing the $\mathbb{C}^*$-invariant ideal $I$ as a sum of weight spaces: defining $\mathcal{I}_j$ in terms of the weight-$j$ piece $t^j\mathcal{I}_j$ we get the weight space decomposition (3.12). Or, embedding a test configuration in some $\mathbb{P}^N \times \mathbb{C}$, Mumford’s result applies directly.

For example, given a $\mathbb{C}^*$-action on $X$ with “repulsive fixed point set” $Z$ (that part of the fixed point set with negative weight spaces in its normal cone), there is a blow up of $X \times \mathbb{C}$ supported on $Z \times \{0\}$ in which the proper transform of $X \times \{0\}$ can be blown down to give back $X \times \mathbb{C}$ but with a nontrivial $\mathbb{C}^*$-action.

Since $(\mathcal{I}, \mathcal{L})$ is a semi test configuration (and because later we will want to replace an ample $(X, L)$ with a semi-ample $(\widehat{X}, p^*L)$ for some resolution of singularities $p$) we will prove many results in the generality of semi test configurations.

General test configurations (3.11) will be studied in Section 5; next we look at the simplest case (beyond product configurations) of $r = 1$, $I = \mathcal{I}_Z + (t)$ in (3.11): the deformation of $X$ to the normal cone of $Z$.

4. Deformation to the normal cone and K-slope stability

Given any proper subscheme $Z \subset X$ we get a canonical test configuration $\mathcal{I}$, the blow up of $X \times \mathbb{C}$ along $Z \times \{0\}$ with exceptional divisor $P$. This deformation to the normal cone has central fibre $\mathcal{I}_0 = \widehat{X} \cup P$, where $\widehat{X}$ is the blow up of $X$ along $Z$ with exceptional divisor $E$. When $Z$ and $X$ are both smooth $P = \mathbb{P}(\nu \oplus \mathbb{C})$ is the projective completion of the normal bundle $\nu$ of $Z$ in $X$, glued along $E = \mathbb{P}(\nu)$ to the blow up of $X$. For more on the deformation to the normal cone see [Fu]; for a diagram see Section 5.

Let $\pi$ be the composite of the projections $\mathcal{I} \to X \times \mathbb{C} \to X$. For $L$ an ample line bundle on $X$ and $c$ a positive rational number let $\mathcal{L}_c$ be the $\mathbb{Q}$-line bundle $\pi^*L - cP$. This restricts to $L$ on the general fibre of $\mathcal{I} \to \mathbb{C}$, and is ample for $c$ sufficiently small.

**Proposition 4.1.** Fix $L$ an ample (respectively semi-ample) line bundle on $X$. For $c \in (0, \epsilon(Z)) \cap \mathbb{Q}$ (2.1), the $\mathbb{Q}$-line bundle $\mathcal{L}_c$ is ample (semi-ample) on $\mathcal{I}$. If $\epsilon(Z) \in \mathbb{Q}$ then $\mathcal{L}_{\epsilon(Z)}$ is nef. If in addition the global sections of $L^k \otimes \mathcal{I}^k(Z)^k$ saturate (see Section 2) for $k \gg 0$ then $\mathcal{L}_{\epsilon(Z)}$ is semi-ample.
Proof. Note that if $k, c \in \mathbb{N}$ and $L^k$ is globally generated and the sections of $L^k \otimes \mathcal{E}_Z^c$ saturate, then $\mathcal{L}_c^k$ is globally generated on $X \times \mathbb{C}$ by the sections $\pi^* H^0(L^k \otimes \mathcal{E}_Z^c) + t^c \pi^* H^0(L^k)$. Algebraically this is the statement that on $X \times \mathbb{C}$, $(\mathcal{E}_Z^c + (c^k))$ generates $(\mathcal{I}_Z + (t))^c$. Geometrically it says that the global sections saturating $L^k \otimes \mathcal{E}_Z^c$ generate $L^k - cP$ on $\mathcal{X}$ away from the zero section $\mathbb{P}(\mathcal{L} \rightarrow Z) \cong Z$ of the cone $P \rightarrow Z$, over which the sections $t^c H^0(L^k)$ generate.

Putting $c = \varepsilon$ now gives the third claim. For the first two we claim that we may assume that $L$ is ample by replacing $(X, L)$ by $(Y, \mathcal{O}_Y(1)) := \text{Proj} \bigoplus_k H^0(L^k)$ if necessary. For $k$ sufficiently large that $L^k \otimes \mathcal{E}_Z$ is globally generated, $H^0(L^k \otimes \mathcal{E}_Z) \subset H^0(L^k) = H^0(\mathcal{O}_Y(k))$ generates a subsheaf $\mathcal{O}_Y(k) \otimes \mathcal{E}_{Z_0} \subset \mathcal{O}_Y(k)$ and so a subscheme $Z_0 \subset Y$ such the pullback of $\mathcal{E}_{Z_0}$ to $X$ is $\mathcal{E}_Z$. Thus it is sufficient to prove the result for $Z_0 \subset (Y, \mathcal{O}_Y(1))$ and then pullback to get the result for $Z \subset (X, L)$.

On $\mathcal{X}$, $L - cE$ is in the ample cone for small $c$ and on its boundary for $c = \varepsilon(Z)$, so by Kleiman [Kl] $L - cE$ is ample for rational $c < \varepsilon(Z)$. So $L^k - cE$ is globally generated for $k \gg 0$; equivalently the global sections of $L^k \otimes \mathcal{E}_Z^c$ saturate $\mathcal{E}_Z^c$ (for $k$ sufficiently large that the pushdown of $\mathcal{O}(-cE)$ to $X$ is $\mathcal{E}_Z^c$). $L^k$ is also globally generated so again this implies that $\mathcal{L}_c^k$ is globally generated. Thus $\mathcal{L}_c$ is nef for $c \in (0, \varepsilon(Z))$, but it is ample for small $c$ since $\pi^* L$ is ample. Thus by Kleiman again, $\mathcal{L}_c$ is actually ample and $\mathcal{L}_c(\varepsilon(Z))$ is nef.

The obvious $\mathbb{C}^\times$-action on $X \times \mathbb{C}$ (acting trivially on the $X$ factor) lifts to an action on the deformation to the normal cone $\mathcal{X}$, which on the central fibre $\mathcal{X}_0 = \mathcal{X} \cup_p P$, is trivial on $\mathcal{X}$. When $Z$ and $X$ are both smooth, $\lambda \in \mathbb{C}^\times$ acts on $P = \mathbb{P}(\nu \oplus \mathbb{C})$ as $(\lambda, 1)$.

**Theorem 4.2.** Fix $L$ ample and $c \in (0, \varepsilon(Z)) \cap \mathbb{Q}$. Then $(\mathcal{X}, \mathcal{L}_c)$ is a flat family of polarised schemes, and, for all $k \gg 0$, $ck \in \mathbb{N}$, the total weight of the induced action on $H^0(\mathcal{X}_0, \mathcal{L}_c^k)$ is

$$w(k) = -\sum_{j=1}^{ck} j h^0(L^k \otimes (\mathcal{E}_Z^{ck-j} / \mathcal{E}_Z^{ck-j+1})) = \sum_{j=1}^{ck} \chi(L^k \otimes \mathcal{E}_Z^j) - ckh^0(L^k).$$

**Proof.** From the definition of the blow up of a subscheme, $\mathcal{X} = \text{Proj} \bigoplus_{k \geq 0} S_k$, where

$$S_k = (\mathcal{I}_{Z \times \{0\}})^k = (\mathcal{I}_Z + (t))^k = \bigoplus_{j=0}^{k-1} t^j \mathcal{E}_Z^{k-j} \oplus t^k \mathbb{C}[t] \mathcal{O}_X \subset \mathbb{C}[t] \otimes \mathcal{O}_X. \quad (4.3)$$

It follows that for all $k \gg 0$, the pushdown of $-kP$ to $X \times \mathbb{C}$ is $\bigoplus_{j=0}^{k-1} t^j \mathcal{E}_Z^{k-j} \oplus t^k \mathbb{C}[t] \mathcal{O}_X$, with the higher pushdowns zero. By the ampleness of $L - cP$ (4.1), for $k \gg 0$ we have the vanishing of

$$H^i_{\mathcal{X}}((L - cP)^k) = H^i_{\mathcal{X} \times \mathbb{C}}(\pi_* (L^k - cP)) = \bigoplus_{j=0}^{ck-1} t^j H^i_{\mathcal{X}}(L^k \otimes \mathcal{E}_Z^{ck-j}) \oplus t^k \mathbb{C}[t] H^i_{\mathcal{X}}(L^k),$$

for $i > 0$, and

$$H^0_{\mathcal{X}}(\mathcal{L}_c^k) = t^k \mathbb{C} \otimes \mathcal{O}_X.$$

Since $-kP$ is ample, the ampleness of $L - cP$ implies $H^i_{\mathcal{X}}((L - cP)^k) = 0$ for $i > 0$ and $k \gg 0$.
for \( i > 0 \). In particular then,
\[
H^i_X(L^k \otimes \mathcal{I}^{ck-j}_Z) = 0 \quad \text{for} \quad j = 0, \ldots, ck, \quad i > 0.
\] (4.4)

Now
\[
\mathcal{I}_0 = \text{Proj} \bigoplus_{k \geq 0} S_k/tS_k,
\]
where by (4.3),
\[
S_k/tS_k = \mathcal{I}^k_Z \oplus t\left(\mathcal{I}^{k-1}_Z/\mathcal{I}^k_Z\right) \oplus \cdots \oplus t^k(\mathcal{O}_X/\mathcal{I}_Z).
\] (4.5)

Replacing \( \mathcal{I}_Z \) by \( \mathcal{I}^k_Z \) does not change the blow up (just the corresponding exceptional line bundle) so that, for \( k \gg 0 \),
\[
H^0_{\mathcal{I}_0} (\mathcal{L}^k_c) = H^0_X (L^k \otimes \mathcal{I}^{ck}_Z) \oplus \bigoplus_{j=1}^{ck} H^0_X (L^k \otimes (\mathcal{I}^{ck-j}_Z/\mathcal{I}^{ck-j+1}_Z)).
\] (4.6)

The vanishing (4.4) of \( H^1(L^k \otimes \mathcal{I}^{ck-j+1}_Z) \) for \( 1 \leq j \leq ck \) means that the dimension of this is
\[
h^0_{\mathcal{I}_0} (\mathcal{L}^k_c) = h^0_X (L^k \otimes \mathcal{I}^{ck}_Z) + \sum_{j=1}^{ck} \left( h^0_X (L^k \otimes \mathcal{I}^{ck-j}_Z) - h^0_X (L^k \otimes \mathcal{I}^{ck-j+1}_Z) \right),
\]
which is \( h^0_X (L^k) \). By ([Ha] Theorem III.9.9) this proves flatness of the family.

Now (4.6) is the weight space decomposition with respect to the \( \mathbb{C}^\times \)-action: \( \mathbb{C}^\times \) acts on \( t \) with weight \(-1\) and trivially on \( X \) and so on \( \mathcal{I}_Z \). Therefore
\[
w(k) = - \sum_{j=1}^{ck} j h^0 (L^k \otimes (\mathcal{I}^{ck-j}_Z/\mathcal{I}^{ck-j+1}_Z))
\]
\[
= - \sum_{j=1}^{ck} \left( h^0 (L^k \otimes \mathcal{I}^{ck-j}_Z) - h^0 (L^k \otimes \mathcal{I}^{ck-j+1}_Z) \right)
\]
\[
= \sum_{j=1}^{ck} (ck - j + 1) h^0 (L^k \otimes \mathcal{I}^j_Z) - \sum_{j=0}^{ck-1} (ck - j) h^0 (L^k \otimes \mathcal{I}^j_Z)
\]
\[
= \sum_{j=1}^{ck} h^0 (L^k \otimes \mathcal{I}^j_Z) - ck h^0 (L^k) = \sum_{j=1}^{ck} \chi (L^k \otimes \mathcal{I}^j_Z) - ck h^0 (L^k),
\]
where the second and last equalities follow from the vanishing (4.4) of \( H^i(L^k \otimes \mathcal{I}^{ck-j+1}_Z) \) for \( 1 \leq j \leq ck \) and \( i > 0 \).

We may only take \( c = \epsilon(Z) \) if the global sections of \( L^k \otimes \mathcal{I}^{(Z)}_c \) saturate for \( k \gg 0 \). In this case \( \mathcal{L}_c \) is semi-ample, pulled back from a contraction \( p \) from \((\mathcal{Y}, \mathcal{L}_c) \to \mathbb{C}\) to \( \text{Proj} \bigoplus_{k \gg 0} H^0_X (\mathcal{L}^k_c) \), which we call \( \mathcal{L}_0, \mathcal{O}_{\mathcal{L}_0}(1) \to \mathbb{C} \). By construction, \( p^* : H^0_Y (\mathcal{O}_{\mathcal{Y}}(k)) \to H^0_X (\mathcal{L}^k_c) \) is then an isomorphism. By Lemma 2.13 of [Mu], \( \mathcal{Y} \to \mathbb{C} \) is also a flat family, so \( \mathcal{L}_0, \mathcal{O}_{\mathcal{L}_0}(k) \to \mathbb{C} \) is a test configuration for \( k \gg 0 \).
Theorem 4.8. Let $Z$ be a proper subscheme of an irreducible polarised algebraic variety $(X, L)$, and suppose that $c = \epsilon(Z) \in \mathbb{Q}$ and the global sections of $L^k \otimes \mathcal{I}_Z^k$ saturate for $k \gg 0$, $ck \in \mathbb{N}$. Letting $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))$ be the contraction of $(\mathcal{X}, \mathcal{L}_c)$ constructed above, the weight of the action on $\Lambda^\max H^0_\mathcal{Y}(\mathcal{O}_\mathcal{Y}(k))$ is

\[ w(k) = \sum_{j=1}^{ck} \chi(L^k \otimes \mathcal{I}_Z^j) - ckh^0(L^k) = \sum_{j=1}^{ck} \chi(L^k \otimes \mathcal{I}_Z^j) - ckh^0(L^k) + O(k^{n-1}). \]

Proof. By (4.3), for $k \gg 0$,

\[ H^0_\mathcal{Y}(\mathcal{L}_c^k) = H^0_X(\mathbb{C} \otimes S_{ck}) \cong \bigoplus_{j=0}^{ck-1} t^j h^0_X(L^k \otimes \mathcal{I}_Z^{k-j}) \oplus t^{ck} \mathbb{C}[t] H^0_X(L^k), \]

yielding

\[ \frac{H^0_\mathcal{Y}(\mathcal{L}_c^k)}{t H^0_\mathcal{Y}(\mathcal{L}_c^k)} \cong H^0_X(L^k \otimes \mathcal{I}_Z^{ck}) \oplus \bigoplus_{j=1}^{ck} t^j \frac{H^0_X(L^k \otimes \mathcal{I}_Z^{ck-j})}{H^0_X(L^k \otimes \mathcal{I}_Z^{ck-j+1})}. \]  

(4.9)

By the isomorphism $p^*: H^0_\mathcal{Y}(\mathcal{O}_\mathcal{Y}(k)) \cong H^0_\mathcal{Y}(\mathcal{L}_c^k)$ this is the weight space decomposition of $H^0_\mathcal{Y}(\mathcal{O}_\mathcal{Y}(k))/t H^0_\mathcal{Y}(\mathcal{O}_\mathcal{Y}(k))$, which by flatness of $\mathcal{Y} \to \mathbb{C}$ and ampleness of $\mathcal{O}_\mathcal{Y}(1)$ is $H^0_\mathcal{Y}(\mathcal{O}_\mathcal{Y}(k))$. So the total weight is

\[ -\sum_{j=1}^{ck} j \left( h^0(L^k \otimes \mathcal{I}_Z^{ck-j}) - h^0(L^k \otimes \mathcal{I}_Z^{ck-j+1}) \right). \]

Just as in (4.7) this is $\sum_{j=1}^{ck} h^0(L^k \otimes \mathcal{I}_Z^j) - ckh^0(L^k)$. Then replacing $h^0$ by $\chi$ gives an error bounded by $\sum_{j=1}^{ck} h^1(L^k \otimes \mathcal{I}_Z^{ck-j+1})$, where $h^1 := \sum_{i=1}^{n} h^i$. So the result follows from Lemma 4.10 below.

Lemma 4.10. Fix $Z \subset (X, L)$ and $c \in (0, \epsilon(Z)) \cap \mathbb{Q}$, or $c \in (0, \epsilon(Z)) \cap \mathbb{Q}$ if $\epsilon(Z) \in \mathbb{Q}$ and the global sections of $L^k \otimes \mathcal{I}_Z^k$ saturate for $k \gg 0$. Then

\[ \sum_{j=0}^{ck} h^1(L^k \otimes \mathcal{I}_Z^j) = O(k^{n-1}). \]

Proof. Fix $\epsilon \in (0, c) \cap \mathbb{Q}$. Then $\mathcal{L}_{c-\epsilon}$ is ample on $\mathcal{X}$, while $\mathcal{L}_c$ is generated by its global sections and so nef. So we can apply Fujita vanishing ([La] Theorem 1.4.35) to give $N \gg 0$ such that $\mathcal{L}_{c-\epsilon}^N \otimes \mathcal{L}_c^k$ has no higher cohomology for any $p \geq 0$. For $k > N$, setting $p = k - N$ shows that $\mathcal{L}_{c-\epsilon N/k}^k$ has no higher cohomology. So for $k \gg 0$ we have

\[ 0 = H^i_\mathcal{X}((L - (c - \epsilon N/k)P)^k) = H^i_X(\pi_*(L^k - (ck - \epsilon N)P)) \]

\[ \bigoplus_{j=0}^{ck-\epsilon N-1} t^j H^i_X(L^k \otimes \mathcal{I}_Z^{ck-\epsilon N-j}) \oplus t^{ck-\epsilon N} \mathbb{C}[t] H^i_X(L^k), \]  

(4.11)
for \(i > 0\). That is, \(h^{\geq 1}(L^k \otimes \mathcal{J}_Z^{ck-\varepsilon N-j}) = 0\) for \(j = 0, \ldots, ck - \varepsilon N\). So the sum becomes
\[
\sum_{j=0}^{ck} h^1_X(L^k \otimes \mathcal{J}_Z^j) = \sum_{j=0}^{\varepsilon N-1} h^1_X(L^k \otimes \mathcal{J}_Z^{ck-j}) = \sum_{j=0}^{\varepsilon N-1} h^1_X(L^k - (ck - j)E),
\]
where we have taken \(k\) sufficiently large that the pushdown of \(\mathcal{O}_X(-iE)\) is \(\mathcal{J}_Z^i\) (and higher pushdowns are zero) for \(i > ck - \varepsilon N\). A corollary of Fujita vanishing is that \(h^i(\mathcal{F}(kd)) = O(k^{n-i})\) for any coherent sheaf \(\mathcal{F}\) and nef divisor \(D\) ([La] Theorem 1.4.40). Applying this on \(X\) in turn to \(\mathcal{F} = \mathcal{O}, \mathcal{O}(E), \ldots, \mathcal{O}((\varepsilon N - 1)E)\) and \(D = L - cE\) shows that each \(h^{\geq 1}_X(L^k - (ck - j + 1)E) = O(k^{n-1})\). Since \(N\) is fixed, then, we get a similar bound on the whole sum. \(\square\)

**Corollary 4.12.** For \(c \in (0, \varepsilon(Z)] \cap \mathbb{Q}\) define
\[
\hat{w}_{r,k}(c) = r\chi(L^r) \sum_{j=1}^{ck} h^0(L^k \otimes \mathcal{J}_Z^j) - k\chi(L^k)w(r) - ck\chi(L^k)r\chi(L^r),
\]
which, for \(r\) sufficiently large for fixed \(Z \subset X\), is
\[
\hat{w}_{r,k}(c) = r\chi(L^r) \sum_{j=1}^{ck} h^0(L^k \otimes \mathcal{J}_Z^j) - k\chi(L^k) \sum_{j=1}^{cr} h^0(L^r \otimes \mathcal{J}_Z^j).
\]
For \(cr \in \mathbb{Z}\), \(L^r\) globally generated and \(L^r \otimes \mathcal{J}_Z^{cr}\) saturated by its sections, \((X, L^r)\) is unstable if \(\hat{w}_{r,k}(c) \leq 0\), where \(\leq\) is defined as in Theorem 3.9, depending on the type of stability. We say that \((X, L^r)\) is destabilised by \(Z\). Similarly for strict instability and \(\prec\).

**Proof.** The conditions on \(r\) are just to ensure that \((\mathcal{F}, L^r)\) (or \((\mathcal{F}, \mathcal{O}_\mathcal{F}(r))\) if \(c = \varepsilon\)) is a genuine test configuration; for K-stability we are free to twist by higher \(r\) and use Proposition 4.1 to remove these conditions.

The result is a direct consequence of Theorems 4.2 and 4.8 and the identity \(\hat{w}_{r,k} = w(k)r\chi(L^r) - w(r)k\chi(L^k)\) (3.8).

Recall from the introduction the definition of the \(a_i\) and \(a_i(x)\) (1.3). Choose \(p\) so that \(\mathbf{R}\pi_*\mathcal{O}(-jE) = \mathcal{J}_Z^j\) for \(j \geq p\), then for \(k, xk \in \mathbb{N}, k \geq p/x,\)
\[
\chi_X(L^{k} - xkE) = \chi_X(L^{k} \otimes \mathcal{J}_Z^{xk}) = a_0(x)k^n + a_1(x)k^{n-1} + \ldots + a_n(x).
\]
(4.13)
But by Riemann-Roch, \(P(k, j) := \chi_X(L^{k} - jE)\) is a polynomial of total degree \(n\). Writing \(P = P_0 + \ldots + P_n\) where \(P_i\) is homogeneous of degree \(n - i\), \(a_i(x) = P_i(1, x)\) is therefore a degree \(n - i\) polynomial in \(x\) which can be extended to all real \(x\).

**Proposition 4.14.** For \(X\) irreducible the weights \(\sum_{j=1}^{ck} \chi(L^k \otimes \mathcal{J}_Z^j) - ck\chi(L^k)\) of Theorems 4.8 and 4.2 can be written
\[
w(k) = \left(\int_0^c a_0(x)dx - ca_0\right)k^{n+1} + \left(\int_0^c a_1(x) + \frac{a'_0(x)}{2}dx - ca_1\right)k^n + O(k^{n-1}).
\]
Proof. Using (4.13) for \( j > p \) we split up \( \sum_{j=1}^{ck} \chi(L^k \otimes \mathcal{F}_Z^j) \) as

\[
\sum_{j=1}^{p} \chi(L^k \otimes \mathcal{F}_Z^j) + \sum_{j=p+1}^{ck} \chi(L^k \otimes \mathcal{F}_Z^j)
\]

\[
= p \chi(L^k) - \sum_{j=1}^{p} \chi(L^k \otimes \mathcal{O}_{jZ}) + \sum_{j=p+1}^{ck} (a_0(j/k)k^n + a_1(j/k)k^{n-1} + \ldots)
\]

\[
= \int_0^{p/k} a_0(x)dx k^{n+1} + \sum_{j=p+1}^{ck} (a_0(j/k)k^n + a_1(j/k)k^{n-1}) + O(k^{n-1}), \quad (4.15)
\]

using the fact that \( jZ \) has dimension \( \leq n - 1 \) (since \( X \) is irreducible) and \( a_0(x) \) is a polynomial with \( a_0(0) = a_0 \).

For a smooth function \( f \), as \( k \to \infty \) with \( ck \in \mathbb{N} \), the trapezium rule gives [Hi]

\[
\sum_{j=1}^{ck} f(j/k) = \int_0^{c} \left( kf(x) + \frac{f'(x)}{2} \right) dx + O(k^{-1}). \quad (4.16)
\]

This can be proved by Taylor’s theorem, or directly for polynomials by noting that for \( f(x) = x^m \) we have the identity

\[
\sum_{j=1}^{ck} j^m = \frac{(ck)^{m+1}}{m+1} + \frac{(ck)^m}{2} + O(k^{m-1}).
\]

Applying this to \( f(x) = a_i(x), \ i = 1, 2, (4.15) \) now approximates \( \sum_{j=1}^{ck} \chi(L^k \otimes \mathcal{F}_Z^j) \) by

\[
\int_0^{p/k} a_0(x)dx k^{n+1} + \left( \int_0^{c} ka_0(x) + \frac{a_0'(x)}{2} \right) k^n + \left( \int_0^{c} ka_1(x)dx \right) k^{n-1} + O(k^{n-1})
\]

\[
= \left( \int_0^{c} a_0(x)dx \right) k^{n+1} + \left( \int_0^{c} a_1(x) + \frac{a_0'(x)}{2} \right) k^n + O(k^{n-1}),
\]

since \( \int_0^{p/k} a_1(x) + \frac{a_0'(x)}{2} \) \( dx \) \( k^n = O(k^{n-1}) \). Subtracting \( ckh^0(L^k) = ca_0 k^{n+1} + ca_1 k^n + O(k^{n-1}) \) gives the result. \( \square \)

Note that by Riemann-Roch on \( \hat{X} \), \( n!a_0(x) = (L - xE)^n > 0 \) for \( x \in (0, \epsilon(Z)) \) by the ampleness of \( L - xE \), so for \( c \in (0, \epsilon(Z)) \), \( \int_0^{c} a_0(x)dx > 0 \). Therefore we can define the slope (or K-slope) of \( \mathcal{F}_Z \) (1.4) by

\[
\mu_c(\mathcal{F}_Z) = \mu_c(\mathcal{F}_Z, L) = \frac{\int_0^{c} a_1(x) + \frac{a_0'(x)}{2}}{\int_0^{c} a_0(x)dx} = \frac{\int_0^{c} a_1(x)dx}{\int_0^{c} a_0(x)dx} + \frac{a_0(c) - a_0}{2 \int_0^{c} a_0(x)dx},
\]

and that of \( X \) (1.5),

\[
\mu(X) = \frac{a_1}{a_0} \quad \left( = -\frac{n}{2} \cdot \frac{K_X \cdot L^{n-1}}{L^n} \text{ for } X \text{ smooth} \right).
\]
Theorem 4.18. If a polarised variety \((X, L)\) is K-stable then it is slope stable. If it is K-polystable (respectively K-semistable) then it is slope polystable (semistable).

Proof. From the deformation to the normal cone of \(Z\) we get test configurations \((\mathcal{V}, L^k)\) (for \(c < \epsilon(Z)\), \(k \gg 0\)), or \((\mathcal{Y}, \mathcal{O}_\mathcal{Y}(k))\) (for \(c = \epsilon(Z)\) if \(\epsilon(Z) \in \mathbb{Q}\) and the global sections of \(L^k \otimes \mathcal{I}_Z^{\epsilon(Z)k}\) saturate). Its total weight \(w(k)\) is given by Theorems 4.2 and 4.8 respectively. Writing \(w(k) = b_0k^{n+1} + b_1k^n + O(k^{n-1})\), its Donaldson-Futaki invariant (3.10) is, by Proposition 4.14,

\[
F(Z) = b_0a_1 - b_1a_0 = \left( \int_0^c a_0(x)dx - ca_0 \right) a_1 - \left( \int_0^c a_1(x) + \frac{a_0'(x)}{2} dx - ca_1 \right) a_0 = a_0 \left( \mu(X) - \mu_c(\mathcal{I}_Z) \right) \int_0^c a_0(x)dx. \tag{4.19}
\]

This is a strictly positive multiple of \(\mu(X) - \mu_c(\mathcal{I}_Z)\), and K-stability (K-semistability) implies it is strictly positive (nonnegative). This gives the result so long as, in the semistable/polystable case, the test configuration is not trivial. But the central fibres of both \((\mathcal{V}, L^c)\) and \((\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))\) have nontrivial \(\mathbb{C}^\times\)-actions. \(\square\)

This result means that slope instability provides an obstruction to the existence of constant scalar curvature Kähler metrics (and so also Kähler-Einstein metrics); see [RT] where there are also numerous examples.

Letting \(\tilde{a}_1(x) = a_1 - a_0(x)\) be the coefficients of \(\chi(L^k \otimes \mathcal{O}_{\mathcal{V}kZ}) = \chi(L^k/(L^k \otimes \mathcal{I}_Z^k))\), we define the slope of \(\mathcal{O}_Z\) (in slightly misleading notation) to be

\[
\mu_c(\mathcal{O}_Z) = \frac{\int_0^c \tilde{a}_1(x) + \frac{\tilde{a}_0'(x)}{2} dx}{\int_0^c \tilde{a}_0(x)dx}. \tag{4.20}
\]

Notice that we can rephrase slope stability in the equivalent ways

\[
\mu_c(\mathcal{I}_Z) < \mu(X) \iff \mu(X) < \mu_c(\mathcal{O}_Z) \iff \mu_c(\mathcal{I}_Z) < \mu_c(\mathcal{O}_Z),
\]

due to the implications

\[
\frac{A}{B} < \frac{C}{D} \iff \frac{C}{D} < \frac{C - A}{D - B} \iff \frac{A}{B} < \frac{C - A}{D - B}
\]

for \(0 < B < D\), on setting \(B = \int_0^c a_0(x)dx\) and \(D = ca_0\) (so \(D-B = \int_0^c \tilde{a}_0(x)dx > 0\).
Theorem 4.21. On the blow up $\hat{X}$ we have the formula
\[ a_0(x) = \frac{1}{n!}(L - xE)^n, \quad (4.22) \]
and so $a_0(0) = a_0$, since the intersection $L^n$ can be calculated equally on $X$ or $\hat{X}$.

The $a_i(0)$ are the coefficients of the polynomial
\[ \chi_{\hat{X}}(L^k) = \chi_X((\mathbb{R}\pi_*\mathcal{O}_\hat{X}) \otimes L^k) = \sum_{i=0}^{n} (-1)^i h^0_X(R^i\pi_*\mathcal{O}_\hat{X} \otimes L^k) \quad \text{for} \ k \gg 0. \]

If $X$ is normal, then $\pi_*\mathcal{O}_\hat{X} = \mathcal{O}_X$ ([Ha] proof of Corollary III.11.4). $E$ has dimension $n - 1$ and $R^i\pi_*\mathcal{O}$ is supported on points over which the fibre has dimension $\geq i$, so the support of $R^i\pi_*\mathcal{O}_\hat{X}$ has codimension at least $i + 1$. Hence $\chi(R^i\pi_*\mathcal{O}_\hat{X} \otimes L^k) = O(k^{n-i})$, and $\chi_{\hat{X}}(\pi^*L^k) = \chi_X(L^k) + O(k^{n-2})$ so $a_1(0) = a_1$ also.

The same argument that if $Z$ has dimension $j$, then $a_i(0) = a_i$ for $i \leq \max\{n - j - 1, 0\}$ (and $i \leq \max\{n - j - 1, 1\}$ for $X$ normal). If $Z \subset X$ are both smooth in a neighbourhood of $Z$, then $\mathbb{R}\pi_*\mathcal{O}_\hat{X} = \mathcal{O}_X$ so $a_i(0) = a_i$ for all $i$.

Since $H^0(L^r \otimes \mathcal{J}_Z^m) \subset H^0(L^r \otimes \mathcal{J}_Z^{m'})$ for $x > y$, $a_0(x)$ is a decreasing function in $x$: $a'_0(x) \leq 0$. In fact from (4.22),
\[ a'_0(x) = -\frac{1}{(n-1)!} (L - xE)^{n-1}E < 0 \quad \text{for} \ x \in (0, \epsilon(Z)), \quad (4.23) \]
by the ampleness of $L - xE$. In particular $a_0(x) < a_0$ for $x \in (0, \epsilon(Z))$, showing that $\mu_c(\mathcal{O}_Z)$ (4.20) is finite.

For $X$ normal then,
\[ \mu_c(\mathcal{J}_Z) = \mu(X) + \frac{a'_0(0)}{2a_0} + O(c) \quad (4.24) \]
is strictly less than $\mu(X)$ for small $c$, and the slope inequality is automatically satisfied. In all of the examples we have considered [Ro, RT], one need only test the slope inequality at $c = \epsilon(Z)$. If this held in general it would simplify the definition of stability. Székelyhidi [Sz] has shown by example that for the modification of K-stability relevant to extremal metrics this is not the case.

Simplifying destabilising subschemes.

Proposition 4.25. Suppose that $Z \subset X$ is a strictly destabilising subscheme in the sense that it violates the slope inequality (4.17). Then at least one of the connected components of $Z$ strictly destabilises. Similarly if $Z$ is a thickening $Z = mZ'$ of $Z' \subset X$ then $Z'$ strictly destabilises.

Proof. Suppose $Z = Z_1 \cup Z_2$ with $Z_1$ and $Z_2$ disjoint. Then $\tilde{\alpha}_i^Z(x) = \tilde{\alpha}_i^{Z_1}(x) + \tilde{\alpha}_i^{Z_2}(x)$ (4.20) since $\mathcal{O}_{xkZ} = \mathcal{O}_{xkZ_1} \oplus \mathcal{O}_{xkZ_2}$. Suppose $Z$ is strictly destabilises. Then there is a $c \in (0, \epsilon(Z)] \cap \mathbb{Q}$ such that
\[ \mu_c(\mathcal{O}_Z) = \frac{\int_0^c \tilde{\alpha}_0^{Z_1}(x)dx \mu_c(\mathcal{O}_{Z_1}) + \int_0^c \tilde{\alpha}_0^{Z_2}(x)dx \mu_c(\mathcal{O}_{Z_2})}{\int_0^c \tilde{\alpha}_0^{Z_1}(x)dx + \int_0^c \tilde{\alpha}_0^{Z_2}(x)dx} < \mu(X). \quad (4.26) \]
This implies that for some \( j \in \{0,1\} \), \( \mu_c(\mathcal{O}_{Z_j}) < \mu(X) \), and by Lemma 4.27 \( c \leq \epsilon(Z_j) \), so \( Z_j \) is strictly destabilising.

Finally if \((\mathcal{E}^m, c)\) destabilises then so does \((\mathcal{I}_Z, mc)\) since \( \epsilon(mZ) = \frac{1}{m} \epsilon(Z) \) and
\[
\mu_c(\mathcal{E}^m) = \mu_{mc}(\mathcal{I}_Z) + (m-1) \int_0^{mc} a'_0(x) dx < \mu_{mc}(\mathcal{I}_Z),
\]
as \( a'_0(x) < 0 \) for \( x \in (0,mc) \) (4.23).

\[ \square \]

**Lemma 4.27.** If \( Z_1 \cap Z_2 = \emptyset \) then \( \epsilon(Z_1 \cup Z_2) \leq \min(\epsilon(Z_1), \epsilon(Z_2)) \).

**Proof.** Let \( \pi: \tilde{X} \to X \) be the blowup of \( X \) along \( Z_1 \cup Z_2 \) with exceptional divisor \( E = E_1 \cup E_2 \), where \( E_i \) is the subset of \( E \) sitting over \( Z_i \). Let \( \epsilon = \epsilon(Z_1 \cup Z_2) \), so by definition \( \pi^*L - \epsilon E \) is nef. If \( C \) is an irreducible curve contained in \( E_2 \) then \( (\pi^*L - \epsilon E_1).C \geq 0 \) by ampleness of \( L \) and the fact that \( E_1.C = 0 \). If \( C \) is not contained in \( E_2 \) then \( (\pi^*L - \epsilon E_1).C = (\pi^*L - \epsilon E).C + \epsilon E_2.C \geq 0 \). Hence by the Kleiman criterion, \( \pi^*L - \epsilon E_1 \) is nef. But this line bundle is the pullback of \( L - \epsilon E_1 \) from \( \text{Bl}_{Z_1}X \), so the latter line bundle is also nef ([La] Example 1.4.4(ii)), proving that \( \epsilon \leq \epsilon(Z_1) \). \[ \square \]

**Lemma 4.28.** Let \( (X,L) \) be a smooth polarised variety of dimension \( n \) and \( \epsilon = \epsilon(p,L) \) be the Seshadri constant of some point \( p \) in \( X \). Then \( p \) strictly destabilises if and only if
\[
((-K_X).L^{n-1})\epsilon(p,L) > (n+1)L^n.
\]

**Proof.** We have \( \tilde{a}_0(x) = a_0 - a_0(x) = \frac{1}{n!(L^n - (L-xE)^n) = -\frac{(-x)^n}{n!}E^n = \frac{x^n}{n!} \) since \( c_1(L)^j, j > 0 \) can be represented by a cycle on \( X \) missing \( p \), so by a cycle on \( \tilde{X} \) missing \( E \). Similarly using \( K_{\tilde{X}} = K_X + (n-1)E \),
\[
\tilde{a}_1(x) = a_1 - a_1(x) = -\frac{K_XL^{n-1} - (K_X + (n-1)E)(L-xE)^{n-1}}{2(n-1)!},
\]
yielding (4.20),
\[
\mu_c(\mathcal{O}_p,L) = \int_0^{c} \frac{(n-1)x^{n-1}}{2(n-1)!} \frac{x^n}{n!} dx = \frac{n(n+1)}{2c}.
\]

So \( p \) strictly destabilises if and only if \( n(n+1)a_0 < 2ca_1 \). As this is linear in \( c \) it holds for some \( c \leq \epsilon \) if and only if it holds for \( c = \epsilon \). Substituting \( a_0 = \frac{1}{n}L^n \) and \( 2a_1 = -\frac{1}{(n-1)!}K_XL^{n-1} \) gives the result. \[ \square \]

One of course also calculate the slope of a smooth point more directly by working locally with \( H^0(L^k \otimes \mathcal{O}_{ckp}) \cong \bigoplus_{j=0}^{k-1} S^jT^*_p X \). We now get

**Theorem 4.29.** If \( X \) is smooth then no point strictly destabilises.

**Proof.** For any line bundle \( A \) we say that \( H^0(A) \) generates \( s \)-jets at \( p \) if \( H^0(A) \to A \otimes (\mathcal{O}_p/\mathcal{I}_p^{s+1}) = A_{\mid(s+1)\{p\}} \)
Remark 4.31. $B L - \epsilon$ as slope stability is invariant under rescaling. So

Then by the identity ([De] Lemma 7.6)

\[ \{ \text{Definition 4.17, and the deformation to the normal cone of } X \} \]

\[ \text{such that on the blow up } \]

\[ \text{Thus by (4.30), (4.32)} \]

\[ \text{we have the exact formula} \]

\[ \chi( (kA) \otimes \mathcal{J}_Z^k ) = a_0(x) k^n + a_1(x) k^{n-1} + \ldots + a_n(x). \]

Letting $B_i$ denote the Bernoulli numbers, define $\beta_0 = 1$, $\beta_1 = \frac{1}{2}$ and $\beta_i = \frac{B_i}{i!}$ for $i \geq 2$. Then we define the asymptotic Chow slope $\eta_c(\mathcal{J}_Z)$ of $\mathcal{J}_Z$ for $c \leq \epsilon(Z)$ to be

\[ \eta_c(\mathcal{J}_Z) = \eta_c(\mathcal{J}_Z, L, r) = \sum_{j=0}^{n+1} c_j r^{n+1-j} = r^{n+1} + \mu_c(\mathcal{J}_Z)r^n + \ldots + c_{n+1}, \]
where
\[ c_j = \frac{\sum_{i=0}^{j} \beta_i a_{j-i}(x) dx}{\int_0^c a_0(x) dx}. \]
I.e. \( c_0 = 1, \quad c_1 = \frac{\int_0^c a_1(x) + a_0(x) dx}{\int_0^c a_0(x) dx} = \mu_c(\mathcal{I}Z), \)
\[ c_2 = \frac{1}{\int_0^c a_0(x) dx} \int_0^c a_2(x) + \frac{a'_1(x)}{2} + \frac{a''_0(x)}{12} dx, \quad \text{etc.} \]
Defining the asymptotic Chow slope of \( X \) to be the slope of the empty subscheme,
\[ \eta(X)(r) = \eta(O_X, L, r) = \frac{r\chi(L^r)}{a_0} = r^{n+1} + \mu(X)r^n + \frac{a_2}{a_0}r^{n-1} + \ldots + \frac{a_n}{a_0}r, \]
we say (with great difficulty) that \( X \) is asymptotically Chow slope strictly destabilised by \( Z \) if, for all \( r \gg 0 \),
\[ \eta_c(\mathcal{I}Z, r) > \eta_X(r). \]

**Theorem 4.33.** If a polarised variety is asymptotically Chow slope strictly destabilised by \( Z \) satisfying (4.32) then it is asymptotically Chow strictly unstable.

**Proof.** The proof is almost the same as for Theorem 4.18. We calculate the coefficient \( e_{n+1}(r) \) of \( k^{n+1} \) in \( \tilde{w}_{r,k} \) (3.8),
\[ e_{n+1}(r) = b_0 r \chi(L^r) - a_0 w(r) = (b_0 + ca_0)r \chi(L^r) - a_0\left(w(r) + cr \chi(L^r)\right) \]
(instead of its \( r^n \) coefficient the Donaldson-Futaki invariant) for the deformation to the normal cone. By Theorem 3.9, if \( e_{n+1}(r) < 0 \) then \( (X, L) \) is Chow unstable with respect to \( r \). By the continuity of \( \eta_c \) we may take \( c < \epsilon(Z) \), so that \( w(r) \) is calculated by Theorem 4.2 and \( b_0 \) by Proposition 4.14, giving
\[ e_{n+1}(r) < 0 \implies \int_0^c a_0(x) dx r \chi(L^r) < a_0 \sum_{j=1}^{cr} \chi(L^r \otimes \mathcal{I}_{Z,j}). \]
Instead of the trapezium rule (4.16) we use the fact [Hi] that for any polynomial \( f \),
\[ \sum_{j=1}^{cr} f(j/r) = \int_0^c \sum_{i=0}^n \beta_i \frac{f(i)(x)}{r^{i-1}} dx. \]
(The case \( f(x) = x^m \) follows from the definition of \( \beta_i \), implying the general case by linearity.) The theorem follows by applying this to the polynomial \( f(x) = a_0(x)r^n + a_1(x)r^{n-1} + \ldots + a_n(x) = \chi(L^r \otimes \mathcal{I}_{Z,r^n}). \)  \( \square \)

5. **Simplifying arbitrary test configurations**

We begin with an important technical result allowing us to calculate weights of a test configuration \( (\mathcal{Y}, O_{\mathcal{Y}}(k)) \) in terms of a semi test configuration \( (\mathcal{X}, \mathcal{L}^k) \) that dominates it.
So let \((\mathcal{X}, \mathcal{O}_\mathcal{X}(1))\) be an equivariantly polarised flat \(\mathbb{C}^\times\)-family with general fibre \((X, L)\), and fix another flat \(\mathbb{C}^\times\)-family \((\mathcal{Y}, \mathcal{L})\) → \(\mathbb{C}\) with a birational \(\mathbb{C}^\times\)-equivariant map

\[ p : (\mathcal{X}, \mathcal{L}) \rightarrow (\mathcal{Y}, \mathcal{O}_\mathcal{X}(1)) \quad \text{such that} \quad \mathcal{L} = p^*\mathcal{O}_\mathcal{X}(1) \quad \text{(equivariantly).} \]

**Proposition 5.1.** If \(X\) is normal then there exists an \(a \geq 0\) such that

\[ w(H^0_{\mathcal{Y}_0}(\mathcal{O}_\mathcal{X}(k))) = w\left( H^0_{\mathcal{Y}}(\mathcal{L}^k)/tH^0_{\mathcal{Y}}(\mathcal{L}^k) \right) = ak^n + O(k^{n-1}). \]

**Proof.** \(p\) factors through the its Stein factorisation ([Ha] Corollary III.11.5) as

\[ (\mathcal{X}, \mathcal{L}) \rightarrow (\mathcal{Y}, \mathcal{O}_\mathcal{X}(1)) = \text{Proj} \bigoplus_k H^0_{\mathcal{X}}(\mathcal{L}^k), \mathcal{O}_{\mathcal{X}}(1) \rightarrow (\mathcal{Y}, \mathcal{O}_\mathcal{X}(1)). \]

\(q_*\) induces an equivariant isomorphism between \(H^0_{\mathcal{X}}(\mathcal{L}^k)\) and \(H^0_{\mathcal{Y}}(\mathcal{O}_\mathcal{X}(k))\) for \(k \gg 0\), and \(\mathcal{X}'\) is flat over \(\mathbb{C}\) since \(H^0_{\mathcal{X}}(\mathcal{L}^k)\) has no \(t\)-torsion by the flatness of \(\mathcal{X}\). Thus replacing \(\mathcal{X}\) by \(\mathcal{X}'\) and \(p\) by \(p'\) if necessary, we may assume that \(p\) is finite and so \(\mathcal{L} = p^*\mathcal{O}_\mathcal{X}(1)\) is ample. Moreover, the general fibre of \(\mathcal{X}'\) is \((X, L)\) by the normality of \(X\), so \((\mathcal{X}, \mathcal{L}^k)\) is now a genuine test configuration for \((X, L^k)\).

So we have a \(\mathbb{C}^\times\)-equivariant morphism of polarised families \((\mathcal{X}, \mathcal{L}) \rightarrow (\mathcal{Y}, \mathcal{O}_\mathcal{X}(1)) \rightarrow (\mathcal{Y}, \mathcal{O}_\mathcal{X}(1)), \mathcal{O}_\mathcal{Y}(1))\), and we wish to relate the total weights of the \(\mathbb{C}^\times\)-actions on \(H^0_{\mathcal{Y}_0}(\mathcal{O}_\mathcal{X}(k))\) and \(H^0_{\mathcal{Y}}(\mathcal{L}^k)/tH^0_{\mathcal{Y}}(\mathcal{L}^k)\); but the latter is now isomorphic to \(H^0_{\mathcal{Y}_0}(\mathcal{L}^k)\) for \(k \gg 0\) by ampleness and flatness. \(p^*\) induces an exact sequence

\[ 0 \rightarrow \mathcal{O}_\mathcal{Y} \rightarrow p_*\mathcal{O}_\mathcal{X} \rightarrow Q \rightarrow 0, \]

for some cokernel \(Q\) supported on \(\mathcal{Y}_0\) (since \(p\) is an isomorphism away from the central fibres). So \(t^kQ = 0\) for some \(s \geq 0\). We first give the argument for \(s = 1\) to illustrate the more technical general case.

Tensoring with \(\mathcal{O}_\mathcal{X}(k)\) and pushing down to \(\mathcal{C}\) gives an exact sequence on \(\mathcal{C}\),

\[ 0 \rightarrow \pi_*(\mathcal{O}_\mathcal{X}(k)) \rightarrow \Pi_*(\mathcal{L}^k) \rightarrow Q_k \rightarrow 0, \]

where \(Q_k = \pi_*(Q \otimes \mathcal{O}_\mathcal{X}(k))\) satisfies \(tQ_k = 0\). Therefore its restriction to \(0 \subset \mathcal{C}\) is also isomorphic to \(Q_k\), and \(\text{Tor}_1(Q_k, \mathcal{O}_0) = \ker (Q_k \xrightarrow{\times t} Q_k) \cong Q_k\), giving the exact diagram

\[
\begin{array}{cccccc}
0 & \to & \pi_*(\mathcal{O}_\mathcal{X}(k)) & \xrightarrow{p^*} & \Pi_*(\mathcal{L}^k) & \to & Q_k & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Q_k & \xrightarrow{\mathcal{H}^0_{\mathcal{Y}_0}(\mathcal{O}_\mathcal{X}(k))} & \mathcal{H}^0_{\mathcal{Y}_0}(\mathcal{L}^k) & \to & Q_k|_0 & \to & 0
\end{array}
\]

where the vertical arrows are restriction to \(0 \subset \mathcal{C}\), and the flatness of \(\pi, \Pi\) and ampleness of \(\mathcal{O}_\mathcal{X}(1), \mathcal{L}\) give the two central terms and ensure that \(\text{Tor}_1(\Pi_*(\mathcal{L}^k), \mathcal{O}_0) = 0\).

These are maps of \(\mathbb{C}^\times\)-modules, except that the left hand \(Q_k\) has weight shifted by \(-1\) (i.e. is isomorphic to \(Q_k \otimes (t)\) as a \(\mathbb{C}^\times\)-module). We see this by exhibiting an explicit weight\(-1\) isomorphism \(\delta\) from \(Q_k\) to the kernel of the lower \(p^*\). Given \(q \in Q_k\), choose a lift \(\tilde{q} \in \Pi_*(\mathcal{L}^k)\). \(t\tilde{q}\) has zero image in \(Q_k\) so is in the image of some
Remark 5.2. In particular if $X$ is normal then for K-stability we need only consider normal test configurations. This is because any test configuration $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))$ is dominated by its normalisation, which also has general fibre $X$ if $X$ is normal. The pullback of $\mathcal{O}_\mathcal{Y}(1)$ is ample, so some twist gives another test configuration which is less stable than $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))$, in the sense that it has the same weight to leading order and a smaller Donaldson-Futaki invariant, by Proposition 5.1 above.

Similarly for Chow stability we may compute the Chow weight of any test configuration on its normalisation. Of course most of our test configurations have nonnormal central fibre, however.

Given any test configuration $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))$ (3.11) we now build inductively the semi test configuration $(\text{Bl}_{I_1}(X \times \mathbb{C}), L - E)$ (3.12) that dominates it, starting from the deformation to the normal cone $(\text{Bl}_{I_1}(X \times \mathbb{C}), L - P)$ of $Z_0$.

So let $\mathcal{X}^1 \to X \times \mathbb{C}$ denote $\text{Bl}_{I_1}(X \times \mathbb{C})$, i.e. the blow up in $Z_0 \times \{0\}$:

$$
\mathcal{X}^1 = \text{Proj} \bigoplus_k S_k, \quad S_k := (\mathcal{O}_0 + (t))^k.
$$
Recall that the central fibre of $\mathcal{X}^1$ is $\hat{X} \cup_e P$, where $\hat{X}$ is the blow up of $X$ along $Z_0$ with exceptional divisor $e$, and $P$ is the exceptional divisor of $\pi^1$. $P$ is a projective cone over $Z_0$ (Proj of the graded algebra over $Z_0$ with $k$th graded piece $\bigoplus_{i=0}^{k-1} \mathcal{I}_0/i \mathcal{I}_0^i$) – the projective completion of the normal cone to $Z_0 \subset X$. Its zero section $Z_0^i$ is a copy of $Z_0$ which fits into a flat family with the $Z_0 \times \{t\}$ in each fibre $X_t$, which we see as follows.

The proper transform $\overline{Z_0 \times C}$ is defined by the graded sheaf of ideals generated by $\mathcal{I}_0 \subset S_1 = \mathcal{I}_0 + (t)$ in the graded sheaf of algebras $\bigoplus_k S_k$. That is, $\mathcal{O}(P) \otimes \mathcal{I}_{Z_0 \times C}$ is generated by the sections of $\mathcal{I}_0 \subset S_1$. It is abstractly isomorphic to the blow up of $Z_0 \times C$ along its intersection with $Z_0 \times \{0\}$, but this is a divisor in $Z_0 \times C$, so $\overline{Z_0 \times C} \cong Z_0 \times C$. The central fibre $Z_0' \cong Z_0$ is defined by the graded sheaf of ideals generated by $\mathcal{I}_0 + tS_1 = \mathcal{I}_0 + (t^2) \subset S_1$.

Similarly the proper transform of $Z_1 \times C$ is the blow up of $Z_1 \times C$ along its intersection $Z_1 \times \{0\}$ with $Z_0 \times \{0\}$; that is $Z_1 \times C \cong Z_1 \times C$. It is defined by the graded sheaf of ideals generated by $\mathcal{I}_0 + \mathcal{I}_1 S_1 = \mathcal{I}_0 + t \mathcal{I}_1 \subset S_1$, with central fibre $Z_1' \cong Z_1 \times C$ and defined by

$$\mathcal{I}_0 + (\mathcal{I}_1 + (t))S_1 = \mathcal{I}_0 + t \mathcal{I}_1 + (t^2) \subset S_1.$$ 

We now form $\mathcal{X}^2$ by blowing up $\mathcal{X}^1$ in $Z_1'$. Since $\mathcal{I}_0 + t \mathcal{I}_1 + (t^2)$ is just $I_2$ (3.12), we have basically shown that $\mathcal{X}^2$ dominates $\text{Bl}_{I_2}(X \times C)$. Precisely, we have maps $\mathcal{X}^2 \rightarrow \mathcal{X}^1 \rightarrow X \times C$, and set $E_1 := (\pi^1)^* P$, $E_2$ to be the exceptional divisor of $\pi^2$, and $E$ to be the exceptional divisor of $\text{Bl}_{I_2}(X \times C)$. While $\mathcal{O}(-cE_1 - cE_2)$ is relatively ample for $0 < e < c$, it is only semi-ample for $e = c$:

**Proposition 5.4.** $\mathcal{X}^2 \rightarrow X \times C$ factors through a map $p^2 : \mathcal{X}^2 \rightarrow \text{Bl}_{I_2}(X \times C)$. Under this map, $(p^2)^*(\mathcal{O}(-E)) = \mathcal{O}(-E_2 - E_1).$
The central fibre of the proper transform of $X$ is defined by the ideal $(\mathcal{J}_0 + t \mathcal{J}_1 + (t^2))$, which we are recycling the symbol $S_k$ and shall do so again below. The coordinate ring $\bigoplus_k S_k$ is generated by $(\mathcal{J}_0 + t \mathcal{J}_1 + (t^2))^k$. Since this is the largest ideal that localises to $\mathcal{J}_1$ when we invert $t$ and work on $X \times \mathbb{C}^\infty$. Thus $\mathcal{J}_Z$ is generated by $\mathcal{J}_0 + t \mathcal{J}_1 + t(\mathcal{J}_0 + t \mathcal{J}_1 + (t^2)) = \mathcal{J}_0 + t \mathcal{J}_1 + (t^3)$. So there is also a $Z'_2 \subset Z'_1$, isomorphic to $Z_2$, inside it. $Z_2 \times \mathbb{C}$ has ideal generated by $\mathcal{J}_0 + t \mathcal{J}_1 + t \mathcal{J}_2 \subset S_1 = \mathcal{J}_0 + t \mathcal{J}_1 + (t^2)$, since this is the largest ideal that localises to $\mathcal{J}_2$ when we invert $t$ and work on $X \times \mathbb{C}^\infty$. Thus its central fibre $Z''_2$ is defined by the ideal

$$\mathcal{J}_0 + t \mathcal{J}_1 + t^2 \mathcal{J}_2 + tS_1 = \mathcal{J}_0 + t \mathcal{J}_1 + t^2 \mathcal{J}_2 + (t^3).$$

(5.5)

Blowing it up gives $\mathcal{X}^3$, with exceptional divisor $E_3$ and the pullbacks to $\mathcal{X}^3$ of $E_1$, $E_2$ by the same notation). Then, just as in Proposition 5.4, the pushdown of $\mathcal{O}_{\mathcal{X}^3}(-E_3-E_2-E_1)$ to $X \times \mathbb{C}$ is generated by $I_3 = \mathcal{J}_0 + t \mathcal{J}_1 + t^2 \mathcal{J}_2 + (t^3)$ by (5.5), so $\mathcal{X}^3 \to X \times \mathbb{C}$ factors through $\text{Bl}_{I_3}(X \times \mathbb{C})$.

Inductively we obtain $\mathcal{X}^s \to X \times \mathbb{C}$ as the blow up $\pi^s$ along $Z_{s-1}^{(s-1)} \subset \mathcal{X}^{s-1}$, the central fibre of the proper transform of $Z_{s-1} \times \mathbb{C}$. The coordinate ring of $\mathcal{X}^{s-1}$ over $X \times \mathbb{C}$ has $k$th graded piece $S_k = (\mathcal{J}_0 + t \mathcal{J}_1 + \ldots + t^{s-2} \mathcal{J}_{s-2} + (t^{s-1}))^k$, and the ideal of the proper transform of $Z_{s-1} \times \mathbb{C}$ is generated by $\mathcal{J}_0 + \ldots + t^{s-2} \mathcal{J}_{s-2} + t^{s-1} \mathcal{J}_{s-1} \subset S_1 = \mathcal{J}_0 + \ldots + t^{s-2} \mathcal{J}_{s-2} + (t^{s-1})$, since this is the largest ideal that localises to $\mathcal{J}_{s-1}$ when we invert $t$ and work on $X \times \mathbb{C}^\infty$. Therefore the ideal of its central fibre $Z_{s-1}^{(s-1)}$ is generated by

$$\mathcal{J}_0 + \ldots + t^{s-2} \mathcal{J}_{s-2} + t^{s-1} \mathcal{J}_{s-1} + tS_1 = \mathcal{J}_0 + \ldots + t^{s-2} \mathcal{J}_{s-2} + t^{s-1} \mathcal{J}_{s-1} + (t^s).$$

Thus the pushdown of $\mathcal{O}_{\mathcal{X}^s}(-E_s - \ldots - E_1)$ contains $\mathcal{J}_0 + t \mathcal{J}_1 + \ldots + t^{s-2} \mathcal{J}_{s-2} + t^{s-1} \mathcal{J}_{s-1} + (t^s)$, giving the following, as in Proposition 5.4.

**Theorem 5.6.** $\mathcal{X}^s \to X \times \mathbb{C}$ factors through a map $p^s: \mathcal{X}^s \to \text{Bl}_{I_s}(X \times \mathbb{C})$, where $I_s = \mathcal{J}_0 + t \mathcal{J}_1 + \ldots + t^s \mathcal{J}_{s-1} + (t^s)$ (3.12). Under this map, $(p^s)^*(\mathcal{O}(-E)) = \mathcal{O}(-E_s - \ldots - E_1)$. □

In turn any test configuration $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))$ is dominated by a map $\phi$ from some $\text{Bl}_{L_r}(X \times \mathbb{C})$ (3.11), giving $\rho^r := \phi_\circ p^r: \mathcal{X}^r \to \mathcal{Y}$. Denote by $L_r$ the semi-ample line
Corollary 5.7. The total weight on (the kth twist of) an arbitrary test configuration \((\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))\) can be calculated on \((\mathcal{X}, L_r)\) by

\[
w(H^0_{\mathcal{Y}}(k)) = w \left( H^0_{\mathcal{X}}(L^k_r) / t H^0_{\mathcal{X}}(L^k_r) \right) - ak^n + O(k^{n-1}).
\]

Corollary 5.8. The Seshadri constant \(c(I_r)\) of the ideal \(I_r\) (3.12) is \(\min\{\epsilon(Z_i)\}_{i=1}^{r-1}\).

Proof. If \(c \in (0, \min\{\epsilon(Z_i)\}_{i=1}^{r-1}) \cap \mathbb{Q}\) then for \(k \gg 0\), the sections

\[
H^0_{\mathcal{X}}(L^k \otimes \mathcal{I}^0_0) \oplus t^ck H^0_{\mathcal{X}}(L^k \otimes \mathcal{I}^c_1) \oplus \ldots \oplus t^{(r-1)ck} H^0_{\mathcal{X}}(L^k \otimes \mathcal{I}^{c(r-1)}_{r-1}) \oplus t^{rc} H^0_{\mathcal{X}}(L^k)
\]

saturate \(I^k_r\) (3.12). Therefore the Seshadri constant of \(I_r\) (which is \(\geq 1\) by the semi-ampleness of \(L_E\) in (3.11)) is at least \(\min\{\epsilon(Z_i)\}_{i=1}^{r-1}\) and \(c \leq \epsilon(Z_r)\).

To show the former, we claim that since \(L - c(E_1 + \ldots + E_{r+1})\) is nef on \(\mathcal{X}^{r+1}\), \(L - c(E_1 + \ldots + E_r)\) is nef on \(\mathcal{X}^r\). Given any irreducible proper curve \(C \subset \mathcal{X}^r\) not entirely contained in \(Z_r^{(r)}\), let \(\overline{C}\) denote its proper transform in \(\mathcal{X}^{r+1}\). Then \([L - c(E_1 + \ldots + E_r)], C \geq [L - c(E_1 + \ldots + E_{r+1})], \overline{C} \geq 0\) since \(\overline{C}, E_{r+1} \geq 0\). On the other hand if \(C \subset Z_r^{(r)}\), then there is an isomorphic copy \(C' \subset Z_r^{(r+1)} \subset \mathcal{X}^{r+1}\) such that \([L - c(E_1 + \ldots + E_r)], C = [L - c(E_1 + \ldots + E_{r+1})], C' \geq 0\). So indeed \(L - c(E_1 + \ldots + E_r)\) is nef on \(\mathcal{X}^r\) and so \(c \leq \min\{\epsilon(Z_i)\}_{i=1}^{r-1}\) by induction.

Secondly, fix \(c\) such that \(L - c(E_1 + \ldots + E_r)\) is nef on \(\mathcal{X}^r\). \(Z_r^{(r)} \cong Z_r\) lies in the central fibre \((\mathcal{X}^r)_0\), fitting into a flat family with \(Z_r \times \mathbb{C}^x\) away from the central fibre. Seshadri constants are lower semicontinuous in polarised families, so the Seshadri constant (with respect to \(L - c(E_1 + \ldots + E_r)\)) of \(Z_r^{(r)}\) inside the central fibre \((\mathcal{X}^r)_0\) is at most \(\epsilon(Z_r)\) (with respect to \(L\), since this is \(L - c(E_1 + \ldots + E_r)\) restricted to a general fibre). The Seshadri constant of \(Z_r^{(r)}\) inside the whole of \(\mathcal{X}^r\) can only be smaller still; therefore if \(L - c(E_1 + \ldots + E_r) - cE_{r+1}\) is nef then \(c \leq \epsilon(Z_r)\). \(\square\)

We could contract each \(\mathcal{X}^s\) using \(L_s\); this would give an isomorphism in a neighbourhood of \(Z_s^{(s)}\) by ([Ha] Proposition II.7.3) since \(L_s|_{Z_s^{(s)}} \cong L|_{Z_s}\) is ample and \(L^k_s \otimes \mathcal{I}^0_{Z_s^{(s)}}\) is globally generated (by sections of \(L^k_s \otimes (\mathcal{I}_0 + t\mathcal{I}_1 + \ldots + t^s\mathcal{I}_s + (t^{s+1}))\) on \(X\)). Thus we could proceed inductively with these contracted \(\mathcal{X}^s\)s with ample line bundles on them, but since we cannot currently seem to get significantly better estimates by working with ample bundles we proceed with the semi-ample \((\mathcal{X}^s, L_s)\) and contract at the last, rth stage. By Corollary 5.7 we lose nothing by ignoring this contraction and simply calculating the weight on \(\mathcal{X}^r\).
So, modulo the contraction, we have exhibited any test configuration \((3.11)\) as a finite number of blow ups (starting with \(X \times \mathbb{C}\)) in subschemes \(Z^{(i)}_r\) supported in the scheme theoretic central fibre that themselves sit in flat families with the \(Z_i \subset X\). We calculate the weight on such a blow up in Theorem 5.26, for which we need two preliminary results.

The following Proposition is the appropriate generalisation to general test configurations of the case \(Z \times \mathbb{C} \subset X \times \mathbb{C}\) used in \((4.5), (4.9)\) and \((4.10)\). We will apply it to the flat families \(Z = Z_s \times \mathbb{C} \subset X_s \to \mathbb{C}\) and their thickenings \(kZ\), when these thickenings are also flat.

**Proposition 5.9.** Fix flat families \(Z \subset X \to \mathbb{C}\) with central fibres \(Z' \subset X_0\), such that the thickenings \(kZ \subset X \to \mathbb{C}\) are also flat over \(\mathbb{C}\). Let \(\mathcal{I}_Z'\) (respectively \(\mathcal{I}_Z\)) denote the ideal sheaf of \(Z' \subset X\) (\(Z' \subset X_0\)). Then

\[
\begin{align*}
\frac{\mathcal{I}_Z \cap \mathcal{I}_Z'}{\mathcal{I}_Z} &\cong \mathcal{I}_Z' \oplus \bigoplus_{j=1}^k \mathcal{I}_Z^{k-j} \mathcal{I}_Z^{k-j+1}.
\end{align*}
\]

**Proof.** Flatness of \(X \to \mathbb{C}\) and \(jZ \to \mathbb{C}\) imply the exactness of the bottom two rows of the following, from which follows the exactness of the top row.

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \mathcal{I}_Z & \mathcal{I}_Z' \\
\downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}_X & \mathcal{O}_X' \\
\downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}_Z & \mathcal{O}_Z' \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

Chasing through either the top two rows or the first two columns then shows that in \(\mathcal{O}_X\), \((t) \cap \mathcal{I}_Z = t\mathcal{I}_Z\). Also by flatness there is a similar diagram with \(t\) replaced by \(t^i\) (and the right hand column suitably modified) showing that in fact

\[
(t^i) \cap \mathcal{I}_Z = t^i \mathcal{I}_Z^i.
\] (5.10)

Applying \(H^0_X(L \otimes \cdot)\) gives a similar exact diagram without the right hand and lower zeros, so the same argument shows that

\[
(t^i) \cap H^0_X(L \otimes \mathcal{I}_Z^i) = t^i H^0_X(L \otimes \mathcal{I}_Z^i).
\] (5.11)

The top row also gives \(\mathcal{I}_Z^i/t\mathcal{I}_Z = \mathcal{I}_Z^i\), which implies that

\[
\frac{\mathcal{I}_Z^i}{\mathcal{I}_Z^{i+1} + t\mathcal{I}_Z^i} = \frac{\mathcal{I}_Z^i}{\mathcal{I}_Z^{i+1}}.
\] (5.12)
\[ \mathcal{I}_{\mathcal{Z} \times \mathbb{C}} = \mathcal{I}_Z + (t), \text{ so } \mathcal{I}_Z^{k} \mathcal{I}_{\mathcal{Z} \times \mathbb{C}} = \sum_{j=0}^{k} t^j \mathcal{I}_Z^{k-j} \] and

\[
\frac{\mathcal{I}_Z^{k} \mathcal{I}_{\mathcal{Z} \times \mathbb{C}}}{t \mathcal{I}_Z^{k} \mathcal{I}_{\mathcal{Z} \times \mathbb{C}}} = \frac{\sum_{j=0}^{k} t^j \mathcal{I}_Z^{k-j}}{\sum_{j=0}^{k} t^j \mathcal{I}_Z^{k-j} + 1} = \sum_{j=0}^{k} \frac{t^j \mathcal{I}_Z^{k-j}}{t \mathcal{I}_Z^{k-j} + 1} = \sum_{i=0}^{k} t^i \mathcal{I}_Z^{k-j} + 1, \tag{5.13}
\]

where we use the fact that in an abelian category, for \( A, B, C \subset V \) with \( C \subset A + B \), we have \( (A + B)/C = A/(A \cap C) + B/(B \cap C) \) in \( V/C \). We claim that

\[
\sum_{i=0}^{k} t^{i+1} \mathcal{I}_Z^{k-i} = t^j \mathcal{I}_Z^{k-j} + t^{j+1} \mathcal{I}_Z^{k-j}, \tag{5.14}
\]

except for \( j = 0 \) when the right hand side becomes \( t \mathcal{I}_Z^{k} \). The inclusion \( \supseteq \) is clear. For \( \subseteq \), consider the left hand side:

\[
t^j \mathcal{I}_Z^{k-j} \cap \left( (t \mathcal{I}_Z^{k} + \ldots + t^i \mathcal{I}_Z^{k-j} + \ldots + (t^{i+1})) \right) \subseteq t^j \mathcal{I}_Z^{k-j} \cap \left( t \mathcal{I}_Z^{k-j} + (t^{i+1}) \right).
\]

An element of this can be written as \( t^j f = tg + t^{j+1}h \), that is \( t^{j-1} f - g = t^j h \), where \( f \in \mathcal{I}_Z^{k-j}, g \in \mathcal{I}_Z^{k-j+1} \) and so \( t^{j-1} f - g \in \mathcal{I}_Z^{k-j} \). So \( t^j h \in (t^i) \cap \mathcal{I}_Z^{k-j} \), which by (5.10) is \( t^j \mathcal{I}_Z^{k-j} \). Thus \( h \) may be taken to lie in \( \mathcal{I}_Z^{k-j} \). Similarly \( g \in (t^{-i}) \cap \mathcal{I}_Z^{k-j+1} = t^{-1} \mathcal{I}_Z^{k-j+1} \). Therefore \( t^j f = tg + t^{j+1}h \in t^j \mathcal{I}_Z^{k-j} \), proving the inclusion.

So by (5.14) and (5.12), equation (5.13) has become

\[
\frac{\mathcal{I}_Z^{k} \mathcal{I}_{\mathcal{Z} \times \mathbb{C}}}{t \mathcal{I}_Z^{k} \mathcal{I}_{\mathcal{Z} \times \mathbb{C}}} = \frac{\mathcal{I}_Z^{k}}{t \mathcal{I}_Z^{k} + \sum_{j=0}^{k} t^j \mathcal{I}_Z^{k-j} + 1} = \mathcal{I}_Z^{k} + \sum_{j=0}^{k} \frac{t^j \mathcal{I}_Z^{k-j}}{t \mathcal{I}_Z^{k-j} + 1}.
\]

To check that the sum is direct, intersect the \( j \)th numerator with the others:

\[
\sum_{p \neq j} t^p \mathcal{I}_Z^{k-p} \subseteq t^j \mathcal{I}_Z^{k-j} \cap \left( \mathcal{I}_Z^{k-j+1} + (t^{j+1}) \right).
\]

By the same methods as before this lies in \( t^j \mathcal{I}_Z^{k-j} + t^{j+1} \mathcal{I}_Z^{k-j} \), which is the \( j \)th denominator, as required.

To apply the above result inductively to the \( \mathcal{X}^i \) requires the flatness of the thickenings \( k(\mathcal{Z}_i \times \mathbb{C}) \) in \( \mathcal{X}^i \) of the proper transforms of the \( \mathcal{Z}_i \times \mathbb{C} \subset X \times \mathbb{C} \). This is only automatic for \( k = 1 \), but also holds for arbitrary \( k \) if the \( \mathcal{Z}_i \) and \( X \) are all smooth, or, we shall show, if:

\( X \) is reduced, each \( \mathcal{Z}_{r-1} \subseteq \ldots \subseteq \mathcal{Z}_0 \) is a Cartier divisor in \( X \), and any irreducible component common to any pair \( \mathcal{Z}_i, \mathcal{Z}_j \) has the same multiplicity in each. \tag{5.15}

This odd looking condition is clearly satisfied if, for instance, each \( \mathcal{Z}_i \) is reduced.

**Proposition 5.16.** Suppose that \( X \) and the \( \mathcal{Z}_i \) satisfy (5.15). Then \( k(\mathcal{Z}_i \times \mathbb{C}) \subset \mathcal{X}^i \) is flat over \( \mathbb{C} \) for each \( j \in \mathbb{N} \).
Proof. Firstly, consider $j(Z_i \times \mathbb{C})$. Its ideal is defined by those functions which, on restriction to $t \neq 0$, lie in $\mathcal{I}_{Z_i \times \mathbb{C}}^j$. Since $t$ is invertible there, this implies that if $tf \in \mathcal{I}_{j(Z_i \times \mathbb{C})}^j$ then $f \in \mathcal{I}_{j(Z_i \times \mathbb{C})}^j$; therefore the structure sheaf of $j(Z_i \times \mathbb{C})$ has no $t$-torsion. It is also flat away from $t = 0$ (where it is $j(Z_i \times \mathbb{C}^\times)$); thus $j(Z_i \times \mathbb{C})$ is automatically flat over $\mathbb{C}$.

Firstly, consider the restriction to $\mathbb{C}$. Using this trivialisation identifies the functions on $X \times \mathbb{C}$ with the $\mathcal{O}_X \otimes \mathbb{C}[t]$-module structure, corresponding to the projection to $X \times \mathbb{C}$.

There are many ways to see this. One is to note that, away from $(\mathcal{X}^i_{-1})_0$, the map to Bl$_t(X \times \mathbb{C})$ of Theorem 5.6 is an isomorphism. This is because, in the notation of that section, sections of $\mathcal{O}(-E_1 - \ldots - E_i)$ over $X \times \mathbb{C}$ do not contract the exceptional divisor $E_i$ of the blow up of $\mathcal{X}^i_{-1}$ in $Z_{i-1}$ (as noted in the remarks following the proof of Corollary 5.8). Let $E$ denote the exceptional divisor of Bl$_t(X \times \mathbb{C})$, and $s_E \in H^0(\mathcal{O}(E))$ the canonical section vanishing on $E$. For $k \gg 0$, the sections of $\mathcal{O}(-kE)$ are the sections of $I_k^j/s_E^k$ (that is, pull back sections of $I_k$ from $X \times \mathbb{C}$ to Bl$_t(X \times \mathbb{C})$ and divide by $s_E^k$) to get a regular section of $\mathcal{O}(-kE)$. $t^i \in I_i = \mathcal{I}_0 + tI_1 + \ldots + t^{i-1}I_{i-1} + (t^i)$ defines the section $t^i/s_E$ of $\mathcal{O}(-E)$ which trivialises $\mathcal{O}(-E)$ over $\mathcal{X}^i_{\text{aff}}$ -- the complement of its zero locus $(\mathcal{X}^i_{-1})_0$. Using this trivialisation identifies the functions on $\mathcal{X}^i_{\text{aff}}$ which have poles of order $\leq k$ on $(\mathcal{X}^i_{-1})_0$ with

$$\frac{I_k}{t^k} = \left(\mathcal{O} + \frac{I_{i-1}}{t} + \ldots + \frac{I_0}{t^i}\right)^k;$$

taking the limit as $k \to \infty$ gives the regular functions (5.18).

Alternatively, we can work inductively with the $\mathcal{X}^j$. A similar analysis as above shows the coordinate ring of the complement of $\mathcal{X}_0$ in the blow up of a family $\mathcal{X}$ over $\mathbb{C}$ in an ideal $I + (t)$ is

$$\mathcal{O}_\mathcal{X} + \frac{I}{t} + \frac{I^2}{t^2} + \frac{I^3}{t^3} + \ldots \tag{5.19}$$
Thus we find the coordinate ring of $\mathcal{R}_{\text{aff}}^1 = \mathcal{R}^{-1} \setminus X \times \{0\}$ is

$$O_X \otimes \mathbb{C}[t] + \frac{\mathcal{I}}{t} + \frac{\mathcal{I}_0}{t^2} + \frac{\mathcal{I}_0^3}{t^3} + \ldots$$

Inside this the ideal of $\mathcal{Z}_1 \times \mathbb{C}$ is $\mathcal{J}_1 + \frac{\mathcal{J}_1^2}{t} + \frac{\mathcal{J}_1^3}{t^2} + \ldots$ (as this is the largest ideal that localises to $\mathcal{J}_1 \otimes \mathbb{C}[t, t^{-1}]$ on $t \neq 0$); applying (5.19) to this ideal $I$ gives the coordinate ring of $\mathcal{R}_{\text{aff}}^2 = \mathcal{R}^{-2} \setminus (\mathcal{I}^{-1})_0$ as

$$O_X \otimes \mathbb{C}[t] + \frac{\mathcal{I}_1}{t} + \frac{\mathcal{I}_1^2 + \mathcal{I}_0}{t^2} + \frac{\mathcal{I}_1^3 + \mathcal{I}_0 \mathcal{I}_1}{t^3} + \ldots$$

But this is the $i = 2$ case of (5.18), and inductively we recover it for all $i$. In (5.18) we have the ideal

$$\mathcal{J}_i(Z_1 \times \mathbb{C}) = \sum_{a_0, a_1, \ldots, a_{i-1} \geq 0} \frac{\mathcal{J}_i \cap (\mathcal{I}_0 \ldots \mathcal{I}_1 \ldots \mathcal{I}_{i-1}^{a_i-1})}{t^{ia_0}(i-1)a_1 \ldots t^{a_{i-1}}},$$

(5.20)

as this is the largest ideal that localises to $\mathcal{J}_i \otimes \mathbb{C}[t, t^{-1}]$ on $t \neq 0$. In the $j = 1$ case, $\mathcal{J}_i \subset \mathcal{I}_0 \ldots \mathcal{I}_1 \ldots \mathcal{I}_{i-1}$ unless $a_j = 0$ for all $j$, so $\mathcal{J}_i(Z_1 \times \mathbb{C})$ differs from (5.18) only in the first term $\mathcal{J}_i \subset O_X \otimes \mathbb{C}[t]$:

$$\mathcal{J}_i(Z_1 \times \mathbb{C}) = \mathcal{J}_i + \sum_{a_j \geq 0, \sum_{j=0}^{i-1} a_j \geq 1} \frac{\mathcal{J}_i \cap (\mathcal{I}_0 \ldots \mathcal{I}_1 \ldots \mathcal{I}_{i-1}^{a_i-1})}{t^{ia_0}(i-1)a_1 \ldots t^{a_{i-1}}}.$$

(5.21)

By (5.17) we are left with showing that each term of (5.20) is contained in the $j$th power of (5.21):

$$\mathcal{J}_i(Z_1 \times \mathbb{C})^j = \sum_{p=0}^j \left( \sum_{a_j \geq 0, \sum_{j=0}^{i-1} a_j \geq p} \mathcal{J}_i^{j-p} \cdot \frac{\mathcal{J}_i \cap (\mathcal{I}_0 \ldots \mathcal{I}_1 \ldots \mathcal{I}_{i-1}^{a_i-1})}{t^{ia_0}(i-1)a_1 \ldots t^{a_{i-1}}} \right).$$

(5.22)

We now work locally, where the conditions (5.15) on $Z_1 \subset Z_{i-1} \subset \ldots \subset Z_0$ imply that $\mathcal{J}_i = (f_i)$ and $\mathcal{J}_j = (g_j f_i)$, $j \leq i - 1$ for some $g_j$ which do not divide $f_i$. Therefore, for any $a_j \geq 0$ with $p := \sum_{j=0}^{i-1} a_j \leq j$, we have

$$\mathcal{J}_i \cap (\mathcal{I}_0 \ldots \mathcal{I}_{i-1}^{a_i-1}) = (f_i^j) \cap (f_i^p g_0 \ldots g_{i-1}) = (f_i^j g_0 \ldots g_{i-1})$$

$$= (f_i^{j-p} f_i^p g_0 \ldots g_{i-1}) = \mathcal{J}_i^{p-a} \cdot \mathcal{J}_0 \ldots \mathcal{J}_{i-1}^{a_i-1},$$

using the fact the $O_X$ is torsion-free. This gives the desired inclusion. 

To apply Proposition 5.16 will involve replacing $X$ by a blow up on which the pullbacks of the $Z_i$ are divisors. Pulling back the polarisation, we find we are forced to work with a semi-ample line bundle. To this end, for any $Z \subset X$ and semi-ample $L \to X$, we define

$$w_k(Z) := \sum_{j=1}^{ck} h^j_X(L \otimes \mathcal{J}_j) - ck h^0_X(L^k)$$

(5.23)
Again by flatness and ampleness, this has the same dimension as for the general fibre (Proj $X$ flatness of $H^r$ with sections for $\epsilon > 0$.

**Proof.** Consider the blow up of $X \times \mathbb{P}^1$ in $Z \times \{0\}$ with exceptional divisor $P$ and line bundle $\mathcal{O}_S(ck) \boxtimes k - ckP$. This is semi-ample, generated by $t^{ck} \boxtimes H^0_X(L^k) + s^{ck} \boxtimes H^0_X(L^k \otimes \mathcal{J}^{ck})$, where $s, t \in H^0(\mathcal{O}_S(1))$ are the sections vanishing at $\infty$. So its higher cohomology has total dimension bounded by $\mathcal{O}(k^n)$. But by pushing down first to $X \times \mathbb{P}^1$, then to $X$, this higher cohomology can be computed as $\otimes_{j=0}^{ck} s^{ck-j} \boxtimes H^1_X(L^k \otimes \mathcal{J}^{ck-j})$, of total dimension $\mathcal{O}(k^n)$. □

Fix $(\mathcal{X}, L) \rightarrow \mathbb{C}$ such that for $k \gg 0$, $(\mathcal{X}, L)$ is a semi test configuration for $(X, L_k)$. Given $Z \subset \mathcal{X}$ a $\mathbb{C}^\times$-invariant subscheme, denote its general fibre by $Z \subset X$ and central fibre $Z' \subset \mathcal{X}_0$. Let $(\text{Bl}_{Z'}(\mathcal{X}), \mathcal{L}_c) \rightarrow (\mathcal{X}, L)$ denote the blow up of $\mathcal{X}$ along $Z'$, with exceptional divisor $E$ and line bundle $\mathcal{L}_c = \pi^*L - cE$. (As usual $c \in \mathbb{C}(Z')$ or $c = \epsilon(Z')$ if $L^k \otimes \mathcal{J}^{c}(Z')$ is saturated by global sections for $k \gg 0$.) Thus $\mathcal{L}_c$ is semi-ample by Proposition 4.1, making $(\text{Bl}_{Z'}(\mathcal{X}), \mathcal{L}_c)$ a semi test configuration for $(X, L_k)$ for $k \gg 0$.

Then we have the following generalisation of Theorems 4.2 and 4.8.

**Theorem 5.26.** In the above situation, suppose that the thickenings $jZ \subset \mathcal{X}$ are flat over $\mathbb{C}$ for all $j \in \mathbb{N}$, and the $\mathbb{C}^\times$-action on $H^0_{\mathcal{X}_0}(L_k)$ has only weights which lie between $-ck$ and $0$, for some $c > 0$. Then

$$w(H^0_{\mathcal{X}_0}(L_k)) = w(H^0_{\mathcal{X}}(L^k)/tH^0_{\mathcal{X}}(L^k)) + O(k^n),$$

$$H^0_{\text{Bl}_{Z'}(\mathcal{X})_0}(\mathcal{L}_c^k)$$

has only weights which lie between $-(c + c)k$ and $0$, and

$$w(H^0_{\text{Bl}_{Z'}(\mathcal{X})_0}(\mathcal{L}_c)) = w(H^0_{\mathcal{X}_0}(L_k)) + w(Z) + O(k^n).$$

If $L$ is ample then the first $O(k^n)$ correction vanishes, and if in addition either $c < \epsilon(Z')$ or the $\mathbb{C}^\times$-action on $H^0_{\mathcal{X}_0}(L_k)$ is trivial then the second correction is $O(k^{n-1})$.

**Proof.** $(\mathcal{V} = \text{Proj } \bigoplus_k H^0_{\mathcal{X}}(L^k), \mathcal{O}_\mathcal{V}(1))$ is a polarised family over $\mathbb{C}$ with general fibre $(\text{Proj } \bigoplus_k H^0_X(L^k), \mathcal{O}(1))$. It is flat because $H^0_{\mathcal{X}}(L^k)$ has no $t$-torsion, by the flatness of $\mathcal{X}$. For $k \gg 0$, by the flatness of $\mathcal{V}$ and ampleness of $\mathcal{O}(1)$, the central fibre has sections $H^0_{\mathcal{V}_0}(O(k)) = H^0_{\mathcal{X}}(O(k))/tH^0_{\mathcal{X}}(O(k)) = H^0_{\mathcal{X}}(L^k)/tH^0_{\mathcal{X}}(L^k)$.

Again by flatness and ampleness, this has the same dimension as for the general fibre, which is $h^0_{\mathcal{X}}(L^k)$, which equals $\chi_X(L^k) + O(k^{n-1})$ by semi-ampleness. In turn by
flatness and semi-ampleness, this equals $\chi_{\mathcal{X}_0}(L^k) + O(k^{n-1}) = h^0_{\mathcal{X}_0}(L^k) + O(k^{n-1})$. Therefore the inclusion
\[ \frac{H^0_{\mathcal{X}_0}(L^k)}{tH^0_{\mathcal{X}_0}(L^k)} \subseteq H^0_{\mathcal{X}_0}(L^k) \]
has codimension $O(k^{n-1})$. This inclusion is $\mathbb{C}^\times$-equivariant, and all weights on the right hand side lie between $-ck$ and 0 by assumption. Therefore the total weights on the two vector spaces differ by at most $O(k^n)$, as claimed. If $L$ is ample then by cohomology vanishing $H^0_{\mathcal{X}_0}(L^k) = H^0_{\mathcal{X}}(L^k) / \text{tH}^0_{\mathcal{X}}(L^k)$ and the correction vanishes.

To streamline notation we fix the convention in the proof of the second result that $ck + 1 = \infty$ so that, for instance, $\mathcal{I}^{ck - 1} / \mathcal{I}^{ck + 1}$ means $\mathcal{I}^{ck}$ and we can deal with all of the terms in Proposition 5.9 uniformly. For $k \gg 0$, this gives
\[ H^0_{(\text{Bl}_Z(\mathcal{X}))_0}(\mathcal{L}_c^k) = H^0_{\mathcal{X}_0}(L^k \otimes \mathcal{I}^{ck - i} / t\mathcal{I}^{ck - i + 1}) = \bigoplus_{i=0}^{ck} t^i H^0_{\mathcal{X}_0}(L^k \otimes \mathcal{I}^{ck - i}) \bigoplus_{i=0}^{ck} t^i H^0_{\mathcal{X}_0}(L^k \otimes \mathcal{I}^{ck - i + 1}) \].

(5.27)

Since the weight on $t^i$ is $-i$ and lies between $-ck$ and 0, this shows the weights on $H^0_{(\text{Bl}_Z(\mathcal{X}))_0}(\mathcal{L}_c^k)$ indeed lie between $-(C + c)k$ and 0. Included in this $\mathbb{C}^\times$-module is
\[ \bigoplus_{i=0}^{ck} t^i \frac{H^0_{\mathcal{X}_0}(L^k \otimes \mathcal{I}^{ck - i})}{tH^0_{\mathcal{X}_0}(L^k \otimes \mathcal{I}^{ck - i + 1})} \],

(5.28)
of dimension $h^0_{\mathcal{X}_0}(L^k)$, which we have already noted above is the same as $h^0_X(L^k)$ to $O(k^{n-1})$. The same working (applied to $(\text{Bl}_Z(\mathcal{X}), \mathcal{L}_c)$ instead of $(\mathcal{X}, L)$) shows that $h^0_{(\text{Bl}_Z(\mathcal{X}))_0}(\mathcal{L}_c^k)$ also equals $h^0_X(L^k) + O(k^{n-1})$. Thus the $-(C + c)k$-bound on weights means we can instead calculate the weight on (5.28) at the expense of an $O(k^n)$ error. If $L$ is ample and $c < \epsilon(Z')$ then $L_c$ is also ample, so $h^0_{(\text{Bl}_Z(\mathcal{X}))_0}(\mathcal{L}_c^k)$ $= h^0_X(L^k) = h^0_{\mathcal{X}_0}(L^k)$ and we can calculate the weight on (5.28) without error. Finally if $c = \epsilon(Z')$ and $L$ is ample then as in Proposition 4.10, (5.27) and (5.28) agree for all but $i = 0, \ldots, eN$ (independent of $k$) so if the $\mathbb{C}^\times$-action on $H^0_{\mathcal{X}_0}(L^k)$ is trivial then their weights differ by $\leq \sum_{i=0}^{eN} ih^1(L^k \otimes \mathcal{I}^{ck - i + 1}) \leq cN O(k^{n-1}) = O(k^{n-1})$.

Define $V^0 := H^0_{\mathcal{X}_0}(L^k)$, and
\[ V^p := H^0_{\mathcal{X}_0}(L^k \otimes \mathcal{I}^{p+1}) \subseteq V^{p-1} \subseteq \ldots \subseteq V^0. \]

(5.29)

Let $V^0 = \bigoplus_{j=-Ck}^{0} V^{0,j}$ be its weight space decomposition. Given such a splitting $\bigoplus_j V^{0,j}$ of a vector space $V^0$, the generic subspace $V^p \subset V^0$ is not generated by the pieces $V^{p,j} := V^p \cap V^{0,j}$; i.e. $V^p \not\supseteq \bigoplus_j V^{p,j}$. But if $V^0$ has a $\mathbb{C}^\times$-action whose weight space decomposition is $\bigoplus_j V^{0,j}$, and each $V^p \subset V^0$ is $\mathbb{C}^\times$-invariant, then indeed
\[ V^p = \bigoplus_j t^j V^{p,j} \quad \text{and} \quad \frac{V^p}{V^p+1} \cong \bigoplus_j t^j \frac{V^{p,j}}{V^{p+1,j}}. \]

(5.30)

This holds here since $\mathcal{I}^{p+1}_Z$ is $\mathbb{C}^\times$-invariant.
We can replace \(h\) by (5.24). And if \(O\) of \(I\) reduced divisors \(\{F\}\) of \(Z\) equals \(O\) trivial then the correction is (3.11) as sums of \(w\) \(\dim\) \(\chi\) pass to a blow up \(p: \hat{X} \to X\) help achieve this by applying Proposition 5.16 we assume that \(h\) \(\hat{D}\) \(\cap\) \(\mathcal{I}_{\hat{Z}}\) equals \(\dim\) \(\chi\) \(\psi\) \(\chi\) high normal crossing (snc) support. That is there are smooth reduced divisors \(\{F\}\) \(\subset \hat{X}\) such that \(F = \cup F\) has simple normal crossing (snc) support. That is \(\dim\) \(\chi\) \(\hat{Z}\) \(\cap\) \(\mathcal{I}_{\hat{Z}}\) equals \(\dim\) \(\chi\) trivial then the correction is \(O(k^n)\).

6. Towards a converse

To apply Theorem 5.26 to Corollary 5.7 to express weights of test configurations (3.11) as sums of \(w_k(Z)\) \(s\) requires flatness of the thickenings \(j(z_2 \times \mathcal{C})\) in \(X\). To help achieve this by applying Proposition 5.16 we assume that \(X\) is reduced and pass to a blow up \(p: \hat{X} \to X\) on which the (pullbacks of the) \(Z_i\) are Cartier divisors \(D_i\): \(p^* \mathcal{I}_{Z_i} = \mathcal{O}(-D_i)\). In fact by Hironaka’s resolution of singularities we may take the \(D_i\) to have simple normal crossing (snc) support. That is there are smooth reduced divisors \(\{F\}\) \(\subset \hat{X}\) such that \(F = \cup F\) has simple normal crossings and \(D_i = \bigcup_{j} m_{ij}F_j\) for each \(i\) and some nonnegative integers \(m_{ij}\). We pull \(L\) back to \(\hat{X}\), and construct the families \((\hat{\mathcal{X}}^s, L_s) \to \mathcal{C}\) as before, using \(D_s \subset \hat{X}\) in place of \(Z_s \subset X\). There are equivariant surjective maps \(\hat{\mathcal{X}}^s \to \mathcal{X}^s\) which identify (by
pullback) the $L_s$ line bundles, defined by pulling back sections of $L^k \otimes \mathcal{J}^j_i$ on $X$ to sections of $L^k \otimes \mathcal{O}(-jm,D_i)$ on $\hat{X}$.

**Theorem 6.1.** Suppose that $X$ is normal, and fix an arbitrary test configuration $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))$ (3.11) with associated subschemes $Z_0 \subseteq \ldots \subseteq Z_{r-1} \subset X$ (3.12). Suppose that there is a resolution $D_0 \subseteq \ldots \subseteq D_{r-1} \subset \hat{X}$ with divisors $D_i$ satisfying condition (5.15). (For instance if the connected components of the snc divisors $D_i$ have the same multiplicities locally, i.e. $m_{ij} \in \{0, m\}$ for some locally constant $m$ and all $i$. E.g. if the $D_i$ are all reduced, which is the $m = 1$ case.) Then

$$w(H^0_{\mathcal{O}_\mathcal{Y}}(k))) = \sum_{i=0}^{r-1} w_k(Z_i) + O(k^n).$$

**Proof.** $(\mathcal{X}^r, L_r)$ dominates $(\mathcal{X}^r, L_r)$ which in turn dominates $(\text{Bl}_{I_r}(X \times \mathbb{C}), \mathcal{L}_1)$ and so $(\mathcal{Y}, \mathcal{O}_\mathcal{Y}(1))$, and the space of sections of the $k$th power of the line bundle on the general fibre is $h^0_X(L^k)$ for all three, by the normality of $X$. Therefore by Proposition 5.1 and the first part of Theorem 5.26 we may calculate $w(H^0_{\mathcal{O}_\mathcal{Y}}(L^k))$ at the expense of an $O(k^n)$ error.

We apply Theorem 5.26 inductively to $\mathcal{X} = \mathcal{X}^i$ with $c = 1$ (which is $\leq \epsilon(Z_i) = \epsilon(D_i) = \epsilon(D^i_{Z_i})$ by Corollary 5.8) and $\mathcal{Z} = D_i \times \mathbb{C}$ with central fibre $D^i_{Z_i}$. The condition (5.15) on the $D_i$ guarantees that each $j(D_i \times \mathbb{C})$ is flat over $\mathbb{C}$ by Proposition 5.16.

The induction starts with $\hat{X} \times \mathbb{C}$, for which all weights are zero so trivially satisfy the $-Ck$ bound. Theorem 5.26 then ensures the induction continues to compute the weight as

$$\sum_{i=0}^{r-1} w_k(D_i) + O(k^n).$$

Since $X$ is normal, all sections of $p^*L$ on $\hat{X}$ are pullbacks from $X : H^0_X(L^k) \cong H^0_{\hat{X}}(L^k)$. Thus the same is true of those sections vanishing on the pullback of $Z_i : H^0_X(L^k \otimes \mathcal{J}^j_i) \cong H^0_{\hat{X}}(L^k \otimes \mathcal{J}^j_{Z_i})$. Thus $w_k(D_i) = w_k(Z_i)$ by their definition (5.23), as required. \hfill $\square$

In particular we can now calculate the leading order term of the weight if the snc divisors $D_i$ in a resolution of singularities $\hat{X}$ of $(X, Z_i)$ are reduced. The next case we consider is when they have multiplicities which can vary with $i$ but are still locally constant over their snc support. Here the relevant flatness result does not hold, but we will find that it does after performing a basechange and normalisation.

So we consider the case when $I_r$ (3.12) is locally of the form

$$\mathcal{F}^p \subset \mathcal{F}^p + t \mathcal{F}^p + \ldots + (t^r),$$

for some reduced snc divisor $D$. These $p_i$ may vary over the different connected components of $D$, but since the total weight is a sum over contributions from each connected component we can calculate the weight of each separately.
Pick a local function $z$ generating the ideal $\mathcal{I}_D$, so that (6.2) is $(z^{p_1}) + t(z^{p_2}) + \ldots + (t^r)$. In the concave hull of the points $(p_i, i), i = 1, \ldots, r$ in the $(z, t)$-plane, we choose extremal vertices $(k_i, \rho_i), i = 1, \ldots, l$ so that they form a concave set with the same concave hull (and $(k_1, \rho_1) = (p_1, 0), (k_l, \rho_l) = (0, r))$. This defines a new ideal

$$(z^{k_1}) + t^{\rho_2}(z^{k_2}) + \ldots + (t^{\rho_l}),$$

(6.3)

with the same integral closure as (6.2) since in this situation taking integral closures corresponds to taking concave hulls, and an ideal saturates its integral closure. In the next theorem we decorate $w_k(D)$ (5.23) as $w_k(D, c)$ with the value $c$ that determines the line bundle $L_c = L - cE$ on the blow up.

**Theorem 6.4.** Set $m_i = \frac{p_i + 1 - \rho_i}{k_{i-1} - k_i + 1}, i = 1, \ldots, l$, and $m_0 = 0$. If $L$ is ample, then the total weight of the blow up in the ideal (6.2) is the sum over the connected components $D$ of

$$- \sum_{i=1}^{l-1} (m_i - m_{i-1})w_k(D, k_i) - ak^n + O(k^{n-1}),$$

for some $a \geq 0$. If $L$ is semi-ample, the above expression is correct to $O(k^n)$.

**Proof.** We start by proving the weaker estimate for $L$ semi-ample. First blow up $X \times \mathbb{C}$ in $D \times \{0\}$ (i.e. locally in the ideal $(z) + (t)$), then in $D'$ (recall this is the central fibre of $\overline{D \times \mathbb{C}}$ in the blow up), then $D'' = D^{(2)}$, etc. up to $D^{(j-1)}$. As in Theorem 5.6 we denote by $E_i$ the pullback of the exceptional divisor of the $i$th blow up, and by $s_i$ the canonical section of $\mathcal{O}(E_i)$ vanishing on $E_i$. Then we claim that the pushdown of $\mathcal{O}(-p_1E_1 - \ldots - p_jE_j), p_1 \geq p_2 \geq \ldots \geq p_j$, to $X \times \mathbb{C}$ is the ideal

$$(z^{p_1}) + \ldots + t^{p_1 + p_2 - 2p_2}(z^{p_2}) + \ldots + t^{p_1 + p_2 + p_3 - 3p_3}(z^{p_3})$$

$$+ \ldots + \ldots + \ldots + t^{p_1 + \ldots + p_j - jp_j}(z^{p_j}) + (t^{p_1 + \ldots + p_j}).$$

(6.5)

Here “…” means “all convex combinations in between”, i.e. the integral closure of the ideal generated by the named terms. (So $(f) + \ldots + (g)$ includes all monomials $h$ such that there exist $\lambda, \mu \in \mathbb{N}$ with $h^{\lambda + \mu} = f^{\lambda}g^{\mu}$.) That is, we claim the sections of $\mathcal{O}(-p_1E_1 - \ldots - p_jE_j)$ over $X \times \mathbb{C}$ are $s_1^{p_1} \ldots s_j^{-p_j}$ times by the above ideal. This is standard but fiddly to prove, and is best done by Newton diagram. We give an unenlightening proof; the reader is advised to skip straight to the Newton diagram (Figure 1) for the special case below.

We prove (6.5) inductively alongside the claim that the complement of the divisor $t/s_j = 0$ (which is the proper transform, in the $j$th blow up, of the central fibre of the $(j-1)$th blow up) is affine over $X \times \mathbb{C}$ with coordinate ring

$$\mathcal{O}_{X \times \mathbb{C}} \left[ \frac{z}{t^{s_j}} \right].$$

(6.6)

(In particular, at a smooth point of $D$ where $z$ is a local coordinate in an analytic coordinate system $(y_1, \ldots, y_{n-1}, z)$ for $X$, we see that this part of the $j$th blow up is $\text{Spec } \mathbb{C}[y_1, \ldots, y_{n-1}, z, t, z/t^{s_j}] = \text{Spec } \mathcal{O}_{X \times \mathbb{C}}[[y_1, \ldots, y_{n-1}, Z, t]]$, where $Z = z/t^{s_j}$, and so is locally isomorphic to $X \times \mathbb{C}$ with the proper transform of $(z^k = 0)$ being $(z^k = 0)$.)
For the first blow up, the sections of $O(-kE_1)$ are $(z,t)^k s_1^{-k}$ (there are no more because the exceptional divisor is a $\mathbb{P}^1$-bundle over $D$). This is in agreement with (6.5), and the proper transform of the central fibre is $t/s_1 = 0$. On its complement, $t^k/s_1^k$ trivialises $O(-kE_1)$; dividing by it identifies the sections $((z+t)^k s_1^{-k})$ with $O_{X \times \mathbb{C}}((z/t)^k + (z/t)^{k-1} + \ldots + (z/t) + 1)$. Taking the limit as $k \to \infty$ gives the coordinate ring $O_{X \times \mathbb{C}}[\frac{z}{t}]$ claimed (6.6). So the induction starts at $j = 1$.

At the $j$th stage the coordinate ring (6.6) shows that $D^{(j)}$ has ideal $(Z,t)$, where $Z = z/t^j$. Let $\pi$ denote the $(j + 1)$th blow up. By the first step of the induction, \{t/s_{j+1} \neq 0\} has coordinate ring augmented by $Z/t = z/t^{j+1}$; i.e. by induction $O_{X \times \mathbb{C}}[\frac{z}{t}] [\frac{z}{t^{j+1}}] = O_{X \times \mathbb{C}}[\frac{z}{t^{j+1}}].$

$\pi_* O(-p_{j+1}E_{j+1})$ is the ideal $((Z) + (t))^{p_{j+1}} = t^{-j}p_{j+1}(z^{p_{j+1}}) + \ldots + (t^{p_{j+1}})$, i.e. the sections of $O(-p_{j+1}E_{j+1})$ over $\mathcal{X}^j$ are $s_{j+1}^{-p_{j+1}}$ times by sections of this ideal.

Multiplying by the trivialising section $t^{p_1}s_1^{-p_1}\ldots t^{p_j}s_j^{-p_j}$ of $O(-p_1E_1 - \ldots - p_jE_j)$ shows that over our affine piece,

$$\pi_* O(-p_1E_1 - \ldots - p_{j+1}E_{j+1}) = (\pi_* O(-p_{j+1}E_{j+1})) \otimes O(-p_1E_1 - \ldots - p_jE_j)$$

is the ideal

$$[\tau^{p_1+\ldots+p_j-jp_{j+1}}(z^{p_{j+1}}) + \ldots + (\tau^{p_1+\ldots+p_j+p_{j+1}})] \cdot s_1^{-p_1}\ldots s_j^{-p_j} \subseteq O(-p_1E_1 - \ldots - p_jE_j).$$

Thus the sections of $O(-p_1E_1 - \ldots - p_{j+1}E_{j+1})$ are the sections of $O(-p_1E_1 - \ldots - p_jE_j)$ which lie in the above ideal, i.e. the intersection of (6.5) with the ideal $t^{p_1+\ldots+p_j-jp_{j+1}}(z^{p_{j+1}}) + \ldots + (t^{p_1+\ldots+p_j+p_{j+1}})$. But this is (6.5) with $j$ replaced by $j + 1$, completing the induction.

We first assume that the $m_i$ are all integers. Then the ideal (6.3) has integral closure of the form (6.5), on taking $p_1, \ldots, p_{m_1}$ all equal to $k_1$, then $p_{m_1+1}, \ldots, p_{m_2}$ all equal to $k_2$, and so on, up to $p_{m_{N-1}+1}, \ldots, p_{m_N}$ all equal to $k_N$. This is illustrated in Figure 1, the Newton diagram of the $(z,t)$ plane, with the $m_i$s being (minus) the gradients of the bold lines. Taking the integral closure, i.e. including the monomials "..." in (6.5), corresponds to including all integral points both on and above the line to lie in the ideal. Replacing $z$ by $Z = z/t^j$ and multiplying by the trivialising section $t^{p_1+\ldots+p_j}$ in the above working corresponds to the integral affine transformation that locally takes one corner of the bold line into the $(z,t)$-axes.

So we may calculate the weight of this sequence of blow ups, $m_1$ times in the central fibre of $D \times \mathbb{C}$ with weight $c = k_1$, then $(m_2 - m_1)$ times in the central fibre of $D \times \mathbb{C}$ with weight $c = k_2$, etc. (The weight here just means the coefficient of the exceptional divisor in the line bundle $L - cE$ we use, as usual.) The flatness criterion (5.15) is trivially satisfied so that by Theorem 5.26 we may calculate the weight to be that claimed. This differs from the weight of the blow up in (6.2) by a $-ak^n + O(k^{n-1})$ correction by Proposition 5.1 since the blow up in the integral closure (6.3) is the normalisation of the blow up in (6.2).

If $L$ is ample then we can improve the estimate. In the first blow up, $\mathcal{X} = X \times \mathbb{C}$ is the trivial product configuration, so the $\mathbb{C}^\times$-action on $H^0_{\mathcal{O}_X}(L^k)$ is trivial and we can use the better estimate of Theorem 5.26. Next we group the first $m_1$ blow ups...
together as one blow up in \(((z) + (t^{m_1}))^{k_1}\); this has the advantage that we do no blowing down (in fact the previous \(m_1\) blow ups blow down to this). Since this blow up is the \(t \mapsto t^{m_1}\) basechange of the blow up in the ideal \(((z) + (t))\) (with \(c = k_1\)), it has weight \(m_1 w_k(D, k_1)\), the same as the sum of the \(m_1\) blow ups we performed above. Similarly we group the next \(m_2\) blow ups together, blowing up with \(c = k_2\) in the ideal generated by \(t^{m_2}\) and the ideal of \(D \times \mathbb{C}\), and calculate its weight as the \(t \mapsto t^{m_2}\) basechange of a blow up we already know. Inductively we end up with the same formula for the weights, but with the added \(O(k^{n-1})\) accuracy of Theorem 5.26 coming from the fact that (after the first blow up) there are no blow downs, so \(c\) is less than the Seshadri constant of the relevant \(D^{(i)}\) at each stage.

Finally, if the \(m_i\) are not integers, we simply replace \(t\) by \(t^M\) in (6.2), i.e. we basechange our test configuration, where \(M\) clears the denominators of all the \(m_i\). This replaces all the \(m_i\) by the integers \(Mm_i\) while multiplying the \(\mathbb{C}^\times\)-weight by \(M\). Substituting the \(Mm_i\) into our formula for the weights gives the weight of this new test configuration with a \(-ak^n + O(k^{n-1})\) correction (coming from taking the integral closure of this new ideal; replacing the test configuration with its normalisation and using Proposition 5.1); dividing by \(M\) gives the weight of the original test configuration for any \(m_i\).

This just leaves the case of where the divisors \(E_j\) in the snc divisors \(D_i\) in a resolution \(\tilde{X}\) intersect with differing multiplicities (the simplest example being \(\mathcal{I}_{D_0} = (x^2y)\) and \(\mathcal{I}_{D_1} = (x)\) locally). This of course cannot happen for curves, so we have

\[ t^{p_{N+1}} \]

\[ (m_3 - m_2) \]

\[ (m_2 - m_1) \]

\[ t^{k_1} \]

\[ z^{k_1} \]

\[ z^{k_N}t^{p_N} \]

\[ z^{k_{p_2}}t^{p_2} \]

\[ z^{k_{p_3}}t^{p_3} \]

\[ z^{k_{p_4}}t^{p_4} \]
Corollary 6.7. A smooth curve $(X, L)$ is $K$-(semi/poly)stable if and only if it is slope (semi/poly)stable.

Proof. Smooth curves are normal, and for the resolution of singularities $\hat{X}$ we of course take $X$ itself, so we can apply the stronger form of Theorem 6.4. Since the Donaldson-Futaki invariant only uses the coefficients $b_0$ and $b_1$ of $w(k) = b_0k^{n+1} + b_1k^n + O(k^{n-1})$, this implies that the Futaki invariant of an arbitrary test configuration (3.11) is $\geq$ a positive linear combination of Futaki invariants of (the deformation to the normal cone of) subschemes. Slope stability implies that these are all positive. □

This result makes it trivial to understand $K$-stability for smooth curves; see Theorem 8.10.

7. Chow stability

Mumford’s notion of Chow (semi)stability [Mu] of $(X, \mathcal{O}_X(1)) \subseteq \mathbb{P}^N$, for fixed $N$, is the simplest form of stability to calculate (as opposed to asymptotic Chow stability (4.33), which is second only to asymptotic Hilbert stability in difficulty). It is also useful in algebro-geometric applications since it is a genuine GIT notion giving projective (and so proper and separated) moduli spaces.

For any subscheme $Z \subseteq X$, we define the polynomial $a_0(x)$ by $\chi_X(\mathcal{I}_Z^k(1)) = a_0(x)k^n + a_1(x)k^{n-1} + \ldots$ for $k \gg x^{-1} > 0$ as before. We also define $a_0$ by $h^0(\mathcal{O}_X(k)) = a_0k^n + a_1k^{n-1} + \ldots$; by (4.22), $a_0 = a_0(0)$.

For any subscheme $Z \subseteq X \subseteq \mathbb{P}^N$ and integer $0 < c \leq c(Z)$, we define the Chow slope of $\mathcal{I}_Z$ to be

$$Ch_c(\mathcal{I}_Z) := \sum_{i=1}^c \frac{h^0_{\mathbb{P}^N}(\mathcal{I}_Z^i(1))}{\int_0^a a_0(x)dx}, \quad Ch(\mathcal{I}_Z) = \max_{N \geq c \leq c(Z)} (Ch_c(\mathcal{I}_Z)) \in [-\infty, \infty].$$

(Here we define max of the empty set to be $-\infty$, and division by 0 to give $\infty$.)

Setting $Z = \emptyset$ gives $Ch(X) = Ch(\mathcal{O}_X)$ as

$$Ch(X) := \frac{h^0_{\mathbb{P}^N}(\mathcal{O}(1))}{a_0} = \frac{N + 1}{a_0}.$$

If $X \hookrightarrow \mathbb{P}^N$ is the Kodaira embedding of $X$ in $\mathbb{P}(H^0(\mathcal{O}_X(1))^*)$, i.e. if $H^0_{\mathbb{P}^N}(\mathcal{O}(1)) \to H^0_X(\mathcal{O}(1))$ is an isomorphism, then $h^0_{\mathbb{P}^N}(\mathcal{I}_Z(1)) = h^0_X(\mathcal{I}_Z(1))$ for all $Z \subseteq X$. This can be arranged, for fixed $(X, L)$, by taking a sufficiently large multiple $L^k =: \mathcal{O}_X(1)$ of the polarisation and setting $\mathbb{P}^N = \mathbb{P}(H^0_X(L^k)^*)$. In this situation the above slopes can be written intrinsically in terms of $(X, \mathcal{O}_X(1))$ as

$$Ch_c(\mathcal{I}_Z) = \frac{\sum_{i=1}^c h^0_X(\mathcal{I}_Z^i(1))}{\int_0^a a_0(x)dx} \quad \text{and} \quad Ch(X) = \frac{h^0_X(\mathcal{O}(1))}{a_0}.$$
We say that $X \subset \mathbb{P}^N$ is Chow slope stable if $Ch(\mathcal{I}_Z) < Ch(X)$ for all nonempty $Z \subset X$, and Chow slope semistable if $Ch(\mathcal{I}_Z) \leq Ch(X)$. We have

$$Ch_c(\mathcal{I}_Z) < Ch(X) \iff Ch(X) < Ch_c(\mathcal{O}_Z) := \frac{\sum_{i=1}^c (N+1-h^0(\mathcal{I}_Z^i(1)))}{\int_0^c a_0(x) dx}. \tag{7.1}$$

**Theorem 7.2.** If $X \subset \mathbb{P}^N$ is Chow (semi)stable then it is Chow slope (semi)stable.

**Proof.** Choose a basis of $H^0_{\mathbb{P}^N}(\mathcal{O}(1))$ compatible with the filtration $H^0_{\mathbb{P}^N}(\mathcal{I}_Z^c(1)) \subset H^0_{\mathbb{P}^N}(\mathcal{I}_Z^{c-1}(1)) \subset \ldots \subset H^0_{\mathbb{P}^N}(\mathcal{I}_Z(1)) \subset H^0_{\mathbb{P}^N}(\mathcal{O}(1))$, so that the first $p_i := h^0_{\mathbb{P}^N}(\mathcal{I}_Z(1))$ elements are contained in $H^0_{\mathbb{P}^N}(\mathcal{I}_Z^i(1))$. The corresponding hyperplanes $H_1, H_2, \ldots, H_{N+1}$ and subschemes $Z_i := X \cap H_1 \cap \ldots \cap H_i$ therefore satisfy $Z_{p_i} \supseteq iZ$ (i.e. $\mathcal{I}_Z^i \subseteq \mathcal{I}_Z^{p_i}$).

Choose weights $\rho_1 = 0 = \ldots = \rho_{p_c}$, $\rho_{p_c+1} = 1 = \ldots = \rho_{p_{c-1}}$, $\ldots$, $\rho_{p_1+1} = c = \ldots = \rho_0$ so that ideal on $X \times \mathbb{C}$,

$$t^{p_1}\mathcal{I}_{Z_1} + t^{p_2}\mathcal{I}_{Z_2} + \ldots + t^{p_N}\mathcal{I}_{Z_N} + (t^{p+1}) \tag{7.3}$$

equals

$$I = \mathcal{I}_{Z_{p_c}} + t\mathcal{I}_{Z_{p_{c-1}}} + \ldots + t^{c-1}\mathcal{I}_{Z_{p_1}} + (t^c) \tag{7.4}$$

which by construction is contained in the ideal

$$\mathcal{I}_Z + t\mathcal{I}_Z^{c-1} + \ldots + t^{c-1}\mathcal{I}_Z + (t^c) = (\mathcal{I}_Z + (t^c)) \tag{7.5}$$

$X \subset \mathbb{P}^N$ is Chow stable for the $\mathbb{C}^\times$-action which has weight $\rho_i$ on the $i$th vector of our basis of $H^0_{\mathbb{P}^N}(\mathcal{O}(1))$. Mumford ([Mu] Theorem 2.9) shows that this is equivalent to the inequality

$$-a < \frac{a_0}{N+1} \sum_{i=1}^{N+1} \rho_i, \tag{7.6}$$

where $a$ is the Chow weight of the blow up of $X \times \mathbb{C}$ in the ideal $I$ (7.4), i.e. $ak^{n+1} + O(k^n)$ is the total weight of the induced $\mathbb{C}^\times$-action on $H^0_{(Bl_j(X \times \mathbb{C}))}(L^k(-kE))$. (This is only one half of Mumford’s result, the harder part being his computation of $a$ in terms of ideals on $X \times \mathbb{C}$, which we do not use. The reader wishing to compare our conventions with Mumford’s should rewrite the above as $-n!a_0(\rho_n) \sum_{i=1}^{n+1} \rho_i$, replace $n$ with $r$, $N$ with $n$, and $\rho_i$ with $\rho_{N-i}$. Finally the sign arises from our convention (3.3) that if $g$ acts on $V$ then the induced action on its functions $S^KV*$ is by $(S^Kg^*)^{-1}$; Mumford calculates the weight of $S^Kg^*$. Notice this is Theorem 3.9 applied to (3.8) on setting $r = 1$, $h^0(L^r) = N+1$ and $w(1) = \sum \rho_i$.

Since $c \geq c(Z)$, we know the Chow weight on the blow up in $(\mathcal{I}_Z + (t^c))$ (Proposition 4.14). But $I$ is contained in $(\mathcal{I}_Z + (t^c))$ (7.5), so the weight on the latter is more negative (for instance Mumford’s formula ([Mu] Theorem 2.9) for the Chow weight shows this), giving the inequality

$$a \leq \int_0^c a_0(x) dx - ca_0.$$
Thus (7.6) gives
\[
\frac{N + 1}{a_0} \left( c a_0 - \int_0^c a_0(x) dx \right) < \sum_{i=1}^{N+1} \rho_i = 1(p_{c-1} - p_c) + 2(p_{c-2} - p_{c-1}) + \ldots + c(p_0 - p_1) = -p_c - p_{c-1} - \ldots - p_1 + c p_0 = - \sum_{i=1}^c h_0^0 (\mathcal{Z}_i^j(1)) + c(N + 1),
\]
which is
\[
\frac{\sum_{i=1}^c h_0^0 (\mathcal{Z}_i^j(1))}{\int_0^c a_0(x) dx} < \frac{N + 1}{a_0}.
\] (7.7)
Semistability is similar, replacing (7.6) by the non strict inequality.

\[\square\]

Remarks 7.8. Setting \( c = 1 \) gives the inequality \( \frac{N + 1}{a_0} \int_0^1 a_0(x) dx > h^0(\mathcal{Z}(1)) \), and replacing \( Z \) by \( kZ \) then gives \( \frac{N + 1}{a_0} \int_0^1 a_0(x) dx > h^0(\mathcal{Z}_k^j(1)) \). We could have gotten this alternative slope-type inequality directly from stab ility for the \( \mathbb{C}^\times \)-action that had all of the above \( \rho_i \) equal to zero (for \( i \leq p_k \)) or one (\( i > p_k \)). However, it is strictly weaker than our slope inequality \( \frac{N + 1}{a_0} \int_0^1 a_0(x) dx > \sum_{i=1}^k h^0(\mathcal{Z}_k^j(1)) \) (7.2) since this last sum is clearly \( \geq k h^0(\mathcal{Z}_k^j(1)) \).

For \( X \) semistable, taking \( Z = X \), \( c = 1 \) (so \( a_0(x) \equiv 0 \)) shows that \( h_0^0 (\mathcal{X}(1)) = 0 \), i.e. \( X \) is not contained in any hyperplane and \( H_0^0 (\mathcal{O}(1)) \) injects into \( H_0^0 (\mathcal{O}(1)) \). To make this injection an isomorphism requires the assumption that \( X \subseteq \mathbb{P}^N \) is a Kodaira embedding.

Notice that (7.7) is more-or-less the \( kn+1 \) coefficient of (4.12) (with \( r = 1 \) and rearranged). There can be higher cohomology corrections since \( I \) (7.4) is not the same as \( (\mathcal{Z} + (t))^n (7.5) \); these disappear for \( r \gg 0 \).

We now want to understand to what extent slope Chow stability should imply Chow stability. This involves demonstrating the inequality (7.6) for all linearly independent sequences of hyperplanes \( H_1, H_2, \ldots \) and all choices of weights \( 0 = \rho_1 \leq \rho_2 \leq \ldots \leq \rho_{N+1} \). As before we set \( Z_i = X \cap H_1 \cap \ldots \cap H_i \), \( \mathcal{I}_i := \mathcal{I}_{Z_i} \).

The idea is to relate the weight of the associated \( \mathbb{C}^\times \)-action (with weight \( \rho_i \) on the \( i \)th vector of our basis of \( H_0^0 (\mathcal{O}(1)) \)) to a sum of weights of \( \mathbb{C}^\times \)-actions of the standard form considered in Theorem 7.2. The problem is that we have noted that we can only express weights as a sum of weights of the deformation to the normal cone of subschemes if either condition (5.15) holds or the multiplicities of \( \mathcal{I}_i \) are locally constant (at least on some resolution where they are ncds). But in either of these good cases we can demonstrate the general procedure of passing from slope stability to stability.

So let us assume for illustration that each \( \mathcal{I}_i \) has the local form \( O(-k_i D) \) for some reduced Cartier divisor \( D \) and number \( k_i \) constant on \( D \). (We have seen, as in Theorem 6.1, one can also pass to a resolution \( \tilde{X} \) if necessary to make the \( Z_i \) ncds to calculate Chow weights (the \( k^{n+1} \)-coefficient of Hilbert weights), so the assumption is not so restrictive, and is sufficient to deal with curves.) That is,
\[ I = t^{\rho_1} \mathcal{I}_1 + t^{\rho_2} \mathcal{I}_2 + \ldots + (t^{\rho_{N+1}}) \] has the local form

\[ I = \mathcal{I}_{D}^{k_1} + t^{\rho_2} \mathcal{I}_{D}^{k_2} + \ldots + (t^{\rho_1}), \quad (7.9) \]

with \( 0 = \rho_1 \leq \rho_2 \leq \ldots \leq \rho_{N+1} \), and \( k_1 \geq k_2 \geq \ldots \geq k_l = 0 = k_{l+1} = \ldots = k_{N+1} \) (so \( l \leq N + 1 \) is the smallest number with \( k_l = 0 \)). These \( k_i \) and \( l \) may vary as \( D \) ranges over the connected components of \( Z_1 \).

**Theorem 7.10.** If \((X, \mathcal{O}(1))\) is slope Chow stable then the weight \( a \) (7.6) satisfies

\[-\frac{N+1}{a_0} a < \sum_{i=1}^{N+1} \rho_i + \delta, \quad \text{where} \quad \delta = \sum_{i=1}^{N} (\rho_{i+1} - \rho_i) h^1(\mathcal{I}_i(1)).\]

This is the inequality \(-\frac{N+1}{a_0} a < \sum_{i=1}^{N+1} \rho_i \) required for Chow stability (7.6), modulo some \( h^1 \) corrections (which we estimated away in the K-stability analogue). In (7.14) the result is strengthened slightly and the correction estimated on curves to prove their asymptotic Chow stability.

**Proof.** The \( k^{n+1} \)-coefficient \( a \) of the weight of our \( \mathbb{C}^\times \)-action is the same as that on the blow up of \( \mathcal{X} \times \mathbb{C} \) in \( I \) (7.9). In Theorem 6.4 we calculated this to be a sum \( a = \sum_D a_D \) over the connected components \( D \) of its support, where

\[ a_D = -\sum_{i=1}^{l-1} t \left( \frac{\bar{\rho}_{i+1} - \bar{\rho}_i}{k_i - k_{i+1}} - \frac{\bar{\rho}_i - \bar{\rho}_{i-1}}{k_{i-1} - k_i} \right) \int_0^{k_i} d^D_0(x) dx. \quad (7.11) \]

Here \( \bar{\rho}_i \) is defined uniquely by requiring that \((k_i, \bar{\rho}_i)\) lies on the boundary of the concave hull of the set of points \((k_i, \rho)_i \) in the \((k, \rho)\)-plane. Thus the \( \bar{\rho}_i \) need not be integers but, for instance, \( \bar{\rho}_1 = \rho_1 \) and \( \bar{\rho}_i = \rho_i \) for all \( i \geq l \). More generally, \( \bar{\rho}_i \leq \rho_i \) for all \( i \). Concavity of the \((k_i, \bar{\rho}_i)\) ensures that any term in the above sum with a zero in the denominator also has zero in the numerator; the prime \( ' \) on the summation sign signifies that we ignore these \( \frac{0}{0} \) terms (the terms with \( k_i = k_{i+1} \)) in the sum; equivalently we set \( \frac{0}{0} = 0 \). In the first term we set \( \bar{\rho}_0 := 0 \).

The Seshadri constant of \( D \) is \( \geq k_i \) for all \( i \) since \( \mathcal{I}_{D}^{k_i} \) is locally the intersection of a sequence of hyperplanes, so \( \mathcal{I}_{D}^{k_l}(1) \) is globally generated near \( D \), so \( \mathcal{I}_{D}^{k_i}(r) \) is too. Therefore Chow slope stability for \( D \subset X \) (7.1) gives the inequalities

\[ \frac{N + 1}{a_0} \int_0^{k_i} d^D_0(x) dx < \sum_{i=1}^{k_i} \left( N + 1 - h^0(\mathcal{I}_i(1)) \right). \]

Since all of the integrals in (7.11) have coefficients which are \( \geq 0 \) by the concavity of the \((k_i, \bar{\rho}_i)\), we obtain

\[ -\frac{N + 1}{a_0} a_D < \sum_{i=1}^{l-1} t \left( \frac{\bar{\rho}_{i+1} - \bar{\rho}_i}{k_i - k_{i+1}} - \frac{\bar{\rho}_i - \bar{\rho}_{i-1}}{k_{i-1} - k_i} \right) \sum_{j=1}^{k_i} \left( N + 1 - h^0(\mathcal{I}_j(1)) \right) \]

\[ = \sum_{i=1}^{l-1} t \frac{\bar{\rho}_{i+1} - \bar{\rho}_i}{k_i - k_{i+1}} \sum_{j=k_{i+1}+1}^{k_i} \left( N + 1 - h^0(\mathcal{I}_j(1)) \right) \]

\[ \leq \sum_{i=1}^{N} (\bar{\rho}_{i+1} - \bar{\rho}_i) \left( N + 1 - h^0(\mathcal{I}_i(1)) \right), \quad (7.12) \]
where in the last line we have added back in the \( k_i = k_{i+1} \) terms since they are positive. Since the last sum is also \( \sum_{i=2}^{N+1} \tilde{\rho}_i (h^0(\mathcal{D}_D^k(1)) - h^0(\mathcal{D}_D^{k_{i+1}}(1))) \), with a positive coefficient for each \( \tilde{\rho}_i \), we can replace \( \tilde{\rho}_i \) by \( \rho_i \geq \tilde{\rho}_i \) to give

\[
-\frac{N+1}{a_0} a_D < \sum_{i=1}^{N}(\rho_{i+1} - \rho_i)h^0(\mathcal{O}_{k_i D}(1)).
\]

Sum over the connected components \( D \) and use \( h^0(\mathcal{I}_i(1)) \geq i \) to give

\[
-\frac{N+1}{a_0} a < \sum_{i=1}^{N}(\rho_{i+1} - \rho_i)(N+1 - h^0(\mathcal{I}_i(1)) + h^1(\mathcal{I}_i(1)))
\]

\[
\leq \sum_{i=1}^{N}(\rho_{i+1} - \rho_i)(N+1 - i) + \sum_{i=1}^{N}(\rho_{i+1} - \rho_i)h^1(\mathcal{I}_i(1))
\]

\[
\leq \sum_{i=1}^{N+1} \rho_i + \delta.
\]

(In the above, any sum from \( i \) to \( j \) with \( j < i \) is to be interpreted as zero, and in passing from the first line to the second we have used the fact that \( \sum_{j=1}^{k_i} = 0 \) since \( k_l = 0 \).)

The Chow slope inequality (7.1) can be rewritten

\[
\epsilon + \frac{N+1}{a_0} = \epsilon + Ch(X) < Ch_c(\mathcal{O}_Z) = \frac{\sum_{i=1}^c (N+1 - h^0(\mathcal{I}_i^Z(1)))}{\int_0^c \tilde{a}_0(x)dx}
\]

for some small \( \epsilon > 0 \). This implies the weaker inequality (which is all that we shall require for curves)

\[
\left( \frac{N+1}{a_0} + \epsilon \right) \int_0^c \tilde{a}_0(x)dx < \sum_{i=1}^c h^0(\mathcal{O}_{i Z}(1)). \tag{7.13}
\]

If \( \epsilon \) can be chosen uniformly in (7.13) for all \( Z \subset X \) we call \( X \) uniformly Chow slope stable with constant \( \epsilon \). This will help us to deal with the correction \( \delta \).

For \( X \) a curve we can improve the estimates of Theorem 7.10 slightly and use Clifford’s theorem to bound the \( h^1 \) terms:

**Theorem 7.14.** If a smooth curve \( (X, L) \) of genus \( g \) and \( d = \deg L > 2g - 2 \) is uniformly Chow slope stable (7.13) with constant \( \epsilon \geq (1 + \frac{1}{g-2})(g - \frac{1}{2}) \cdot \frac{1}{\delta} \) then it is Chow stable.

**Proof.** We follow the same proof as above with \( \frac{N+1}{a_0} \) replaced by \( \frac{N+1}{a_0} + \epsilon \) throughout, up to (7.12), at which point we use estimates specific to curves to better bound the
$h^1$ terms. That is, $D$ is locally a smooth point \{p\} in $X$, and for those $i$ with $k_i \neq k_{i+1}$ (i.e. those involved in the sum $\sum'$),

$$\frac{1}{k_i - k_{i+1}} \sum_{j = k_i + 1}^{k_i} h^0(\mathcal{O}_{\mathcal{D}^j_{\{p\}}}(1)) = \frac{1}{2} (h^0(\mathcal{O}_{\mathcal{D}^{k_i}_{\{p\}}}(1)) + h^0(\mathcal{O}_{\mathcal{D}^{k_{i+1}}_{\{p\}}}(1)) + 1).$$

Therefore (7.12) becomes

$$- \left( \frac{N + 1}{a_0} + \epsilon \right) a_p < \sum_{i=1}^{l-1} (\tilde{\rho}_{i+1} - \tilde{\rho}_i) \frac{1}{2} (h^0(\mathcal{O}_{\mathcal{D}^{\tilde{\rho}_{i}}_{\{p\}}}(1)) + h^0(\mathcal{O}_{\mathcal{D}^{\tilde{\rho}_{i+1}}_{\{p\}}}(1)) + 1).$$

We can now add back in those $i$ with $k_i = k_{i+1}$, as all terms are positive. Each $\tilde{\rho}_i$ appears with positive coefficient in the result, since it can be rearranged as $\sum_{i=2}^{N} \tilde{\rho}_i (h^0(\mathcal{O}_{\mathcal{D}^{\tilde{\rho}_{i-1}}_{\{p\}}}(1)) - h^0(\mathcal{O}_{\mathcal{D}^{\tilde{\rho}_{i+1}}_{\{p\}}}(1))) + \rho_{N+1} (h^0(\mathcal{O}_{\mathcal{D}^{\tilde{\rho}_{N}}_{\{p\}}}(1)) + 1).$ So replacing $\tilde{\rho}_i$ by $\rho_i \geq \tilde{\rho}_i$, summing over $p$ in the support of $Z_1$, and using $h^0(\mathcal{I}(1)) \geq i$, gives

$$- \left( \frac{N + 1}{a_0} + \epsilon \right) a$$

$$< \sum_{i=1}^{N} (\rho_{i+1} - \rho_i) \frac{1}{2} (h^0(\mathcal{O}_{\mathcal{Z}^i}(1)) + h^0(\mathcal{O}_{\mathcal{Z}^{i+1}}(1)) + 1)$$

$$\leq \sum_{i=1}^{N} (\rho_{i+1} - \rho_i) \left[ N + 1 - h^0(\mathcal{I}(1)) + h^1(\mathcal{I}(1)) + 
$$

$$N + 1 - h^0(\mathcal{I}_{i+1}(1)) + h^1(\mathcal{I}_{i+1}(1)) + 1 \right]$$

$$\leq \sum_{i=1}^{N} (\rho_{i+1} - \rho_i) [N + 1 - i] + \frac{1}{2} \sum_{i=1}^{N} (\rho_{i+1} - \rho_i) (h^1(\mathcal{I}(1)) + h^1(\mathcal{I}_{i+1}(1)))$$

$$= \sum_{i=1}^{N+1} \rho_i + \delta,$$

$$\delta := \frac{1}{2} \sum_{i=1}^{N} (\rho_{i+1} - \rho_i) (h^1(\mathcal{I}(1)) + h^1(\mathcal{I}_{i+1}(1)))$$

i.e.

$$- \left( 1 + \epsilon \frac{a_0}{N + 1} \right) \frac{N + 1}{a_0} a < \left( 1 + \frac{\delta}{\sum_{i=1}^{N+1} \rho_i} \right) \sum_{i=1}^{N+1} \rho_i. \quad (7.15)$$

This $\delta$ is a tiny improvement over the one in Theorem 7.10, but can be bounded using Clifford’s theorem, which for our purposes says that $h^1(L) \leq \max \{1 + g - h^0(L), 0\}$. Thus $h^1(\mathcal{I}_i(1)) \leq 1 + g - i$ for $i \leq g$ and vanishes for $i > g$. This yields

$$\delta \leq \sum_{i=1}^{g} (\rho_{i+1} - \rho_i) \frac{1}{2} (1 + g - i + 1 + g - (i + 1)) =$$

$$\sum_{i=1}^{g} (\rho_{i+1} - \rho_i) \left( g - i + \frac{1}{2} \right) = \sum_{i=2}^{g} \rho_i + \frac{1}{2} \rho_{g+1},$$
since $\rho_1 = 0$. Since the $\rho_i$ are monotonic in $i$,

$$
\frac{2}{N - g + \frac{1}{2}} \left( \sum_{i=2}^{g} \rho_i + \frac{1}{2} \rho_{g+1} \right) \leq \frac{g - \frac{1}{2}}{N - g + \frac{1}{2}} \rho_{g+1} \leq \left( g - \frac{1}{2} \right) \frac{1}{2} \rho_{g+1} + \sum_{i=g+2}^{N+1} \rho_i.
$$

Adding $\frac{g-1}{N-g+\frac{1}{2}} \left( \sum_{i=2}^{g} \rho_i + \frac{1}{2} \rho_{g+1} \right)$ to both sides gives

$$
\frac{N}{N - g + \frac{1}{2}} \left( \sum_{i=2}^{g} \rho_i + \frac{1}{2} \rho_{g+1} \right) \leq \frac{g - \frac{1}{2}}{N - g + \frac{1}{2}} \sum_{i=2}^{N+1} \rho_i,
$$

and so

$$
\delta \leq \sum_{i=2}^{g} \rho_i + \frac{1}{2} \rho_{g+1} \leq \frac{g - \frac{1}{2}}{N} \sum_{i=2}^{N+1} \rho_i.
$$

Combined with (7.15) we find that uniform slope stability implies that

$$
- \left( 1 + \frac{a_0}{N + 1} \right) \frac{N + 1}{a_0} a < \left( 1 + \frac{g - \frac{1}{2}}{N} \right) \sum_{i=2}^{N+1} \rho_i,
$$

which implies the inequality $- \left( \frac{N+1}{a_0} \right) a < \sum_{i=1}^{N+1} \rho_i$ required (7.6) if $\epsilon \geq \frac{N+1}{a_0} \frac{g - \frac{1}{2}}{N}$. The condition $d > 2g - 2$ implies that $h^1(\mathcal{O}(1)) = 0$ so $N + 1 = d + 1 - g$ and $a_0 = d$. Therefore the inequality is $\epsilon \geq \left( 1 + \frac{1}{d-g} \right) \left( g - \frac{1}{2} \right) \frac{1}{d}$.

Theorem 7.16. Smooth curves $(X, \mathcal{O}(1))$ of genus $g \geq 1$ are uniformly slope stable (7.13) with any constant $\epsilon < \frac{a_0}{d}$, and so are asymptotically Chow stable.

Proof. We need only demonstrate the inequality (7.13) in the case of $Z = \{p\}$ a single reduced point, as the inequality for a multiple of $\{p\}$ is weaker and the inequality for arbitrary $Z$ follows from adding the inequalities for its connected components. By Riemann-Roch, $\tilde{a}_0(x) = x$, so that $\int_0^c \tilde{a}_0(x) = c^2/2$ while $\sum_{i=1}^{c} h^0(\mathcal{O}_iZ(1)) = \sum_{i=1}^{c} i = c(c+1)/2$. Therefore

$$
\left( \frac{d + 1 - g}{d} + \epsilon \right) \frac{c^2}{2} = \left( \frac{N + 1}{a_0} + \epsilon \right) \int_0^c \tilde{a}_0(x)dx < \sum_{i=1}^{c} h^0(\mathcal{O}_iZ(1)) = \frac{c^2 + c}{2},
$$

so long as $\epsilon < \frac{1}{c} + \frac{a_0}{d}$. Of course $c \leq d$, so $(X, L)$ is uniformly Chow slope stable with any constant $\epsilon < \frac{a_0}{d}$.

Theorem 7.14 then gives Chow stability, so long as $\frac{g}{d} > \left( 1 + \frac{1}{d-g} \right) \left( g - \frac{1}{2} \right) \frac{1}{d}$ which is true for $g \geq 1$ and sufficiently large $d$.

Remark 7.17. Mumford [Mu] proves the sharper result that $(X, \mathcal{O}(1))$ is Chow stable for $\deg \mathcal{O}(1) > 2g$ using a combinatorial argument.

8. Examples

Our remaining examples all deal with K-slope stability. Many more examples, calculations and applications are given in [RT].
8.1. Varieties with nonnegative canonical bundle.

Suppose that $X$ has at worst canonical canonical singularities. That is $X$ is normal, there is an integer $m$ such that $mK_X$ is Cartier, and given any resolution of singularities $\pi_1: \tilde{X} \to X$ we have

$$mK_X = \pi_1^*(mK_X) + \sum \alpha Fi \quad \text{with} \quad \alpha_i \geq 0,$$

where the $F_i$ are the irreducible components of the exceptional set of $\pi_1$. We can define intersection with the canonical class of $X$ on $\tilde{X}$ by $K_{\tilde{X}}(\cdot) := \frac{1}{m}(mK_{\tilde{X}})(\cdot)$.

For any subscheme $Z \subset X$ let

$$\tilde{X} \to \tilde{X} \to X$$

be a resolution of singularities of $\tilde{X}$, the blow up of $X$ along $Z$. $\pi_1 = \pi_\circ \pi_2: \tilde{X} \to X$ is a resolution of singularities of $X$, so (8.1) holds. Letting $F = \pi_2^*E$, $-L - EF = \pi_2^*(L - xE)$ is nef on $\tilde{X}$ for $0 \leq x \leq \epsilon(Z)$.

We wish to compute $a_i(x)$ on $\tilde{X}$ instead of $\tilde{X}$. Since $O_\tilde{X} \subseteq \pi_2^*O_X$ with quotient supported in codimension one, we have an inclusion

$$H^0_X((L - xE)^k) \subset H^0_X((L - xF)^k),$$

with cokernel of dimension $\leq O(k^{n-1})$ by Fujita vanishing, for $0 \leq x \leq \epsilon(Z)$. As in (4.21), for $k \gg 0$, $h^0_X((L - xE)^k) = h^0_X((L - xE)^k \otimes R^i\pi_2O) = O(k^{n-1-i})$ since the support of $R^i\pi_2O$ has codimension at least $i + 1$. Therefore $\frac{\chi}{\chi_{\tilde{X}}((L - xE)^k)} = \frac{\chi}{\chi_{\tilde{X}}((L - xF)^k)} - a(k^{n-1} + O(k^{n-2}))$ for some $a \geq 0$. That is, for $0 \leq x \leq \epsilon(Z)$, we have

$$a_0(x) = \frac{1}{n!(L - xE)^n},$$

and

$$a_1(x) \leq -\frac{1}{2(n-1)!}K_{\tilde{X}}(L - xE)^{n-1} \leq -\frac{1}{2(n-1)!}K_X(L - xF)^{n-1},$$

where the second inequality follows from (8.1) and the fact that $L - xF$ is nef. Again as in (4.21), the first two terms of the Euler characteristic of $L$ are the same as those of $\pi_1^*L$, so equality holds in (8.2) when $x = 0$. That is $a_0 = \frac{1}{n!}L^n$ and

$$a_1 = -\frac{1}{2(n-1)!}K_{\tilde{X}}L^{n-1} = -\frac{1}{2(n-1)!}K_XL^{n-1},$$

since $L$ is trivial along the $F_i$. With these preliminaries, it becomes easy to prove the following.

Theorem 8.4. Calabi-Yaus and canonical models.

Let $X$ be an irreducible variety with at worst canonical singularities.

- If $K_X$ is numerically trivial then $(X, L)$ is slope stable for all polarisations $L$.
- If $K_X$ is ample then the canonical polarisation $(X, K_X)$ is slope stable.

Proof. In both cases, $K_X \sim \alpha L$ is numerically equivalent to a nonnegative multiple $\alpha \geq 0$ of the polarisation. So by (8.3), $\mu(X) = a_1/a_0 = -n\alpha/2$. By (8.2),

$$-\mu(X)a_0(x) + a_1(x) \leq \frac{\alpha}{2(n-1)!}(L - xF)^n - \frac{1}{2(n-1)!}(\alpha L)(L - xF)^{n-1},$$
Theorem 8.5. Suppose that \( \mu \) which rearranges to give slope stability:

\[
\left( \text{is nef and big. Then} \right)
\]

This has the same sign as \( e \) which equals \( -44 \). J. ROSS AND R. P. THOMAS

More precisely,

\[
\mu(X) = \int_0^c a_0(x)dx + \int_0^c a_1(x) + \frac{d_0'(x)}{2}dx < 0 \quad \text{for} \ c \in (0, e(Z)],
\]

which rearranges to give slope stability: \( \mu_c(\mathcal{F}_Z) < \mu(X) \).

With more work, this can be generalised as follows.

Theorem 8.5. Suppose that \((X, L)\) has at worst canonical singularities, and \(K_X\) is nef and big. Then \((X, L)\) is slope stable for \( L \) ample and sufficiently close to \( K \). More precisely,

- For any divisor \( G \) there is a \( \delta_0 > 0 \) such that if \( 0 \leq \delta < \delta_0 \) and \( L = K + \delta G \) is ample then \((X, L)\) is slope stable.
- If \( 2\mu(X, L)L + nK_X \) is nef then \((X, L)\) is slope stable.
- If \(-2\mu(X, L)L - nK\) is nef then \((X, L)\) is slope stable.

In [RT] we prove that no smooth \( Z \) can slope destabilise a smooth \((X, L)\) with these properties, and the proof extends to general \( Z \) and \( X \) with canonical singularities using the preliminaries (8.2) and (8.3).

These results are to be expected due to the deep and difficult related results of Viehweg [V], and the expectation that the minimal model programme can be carried out in all dimensions (and so can be done in families) [Ka]. In fact Theorem 8.4 can be proved in a round about way for smooth varieties with no holomorphic vector fields by the stability results of [Do1, Zh] applied to the Kahler-Einstein metrics of [Au, Y] on such varieties. Similarly, if \( X \) has no holomorphic vector fields and \( L \) is ample and sufficiently close to \( K_X \) then an implicit function theorem argument applied to the Kahler-Einstein metric provides a constant scalar curvature Kahler metric in \([c_1(L)]\). But the quick proofs above demonstrate that using slope stability it could become much easier to produce and compactify moduli of varieties with semi-ample canonical bundle [V].

8.2. Irreducible Curves.

Proposition 8.6. Let \((\Sigma, L)\) be an irreducible polarised curve with arithmetic genus \( g \geq 2 \). The Hilbert-Samuel polynomial of a subscheme \( Z \subset \Sigma \) can be written

\[
i^0(L^k / (L^k \otimes \mathcal{F}_Z^k)) = e(Z)k - \rho(Z) \quad \text{for} \ k \gg 0,
\]

and \( Z \) destabilises \( \Sigma \) if and only if it strictly destabilises, if and only if \( 2\rho(Z) > e(Z) \).

Proof. The assumption on the genus implies that \( \mu(\Sigma) < 0 \). As \( \dim \Sigma = 1 \), \( \tilde{a}_0(x) \) is a degree 1 polynomial vanishing at the origin, while \( \tilde{a}_1(x) \) has degree 0. So writing \( \tilde{a}_0(x) = e(Z)x \) and \( \tilde{a}_1(x) = -\rho(Z) \),

\[
\mu_c(\mathcal{O}_Z) = \int_0^c \tilde{a}_1(x) + \frac{\tilde{a}_1'(x)}{2}dx = \frac{e(Z) - 2\rho(Z)}{e(Z)c}.
\]

This has the same sign as \( e(Z) - 2\rho(Z) \) for all \( c > 0 \), and tends to \( \pm \infty \) as \( c \to 0 \). Thus \( p \) destabilises if and only if it strictly destabilises if and only if this sign is negative.
Theorem 8.8. Let $(\Sigma, L)$ be a polarised irreducible curve of arithmetic genus $g \geq 2$.

- If $\Sigma$ has a point of multiplicity $e \geq 3$ then $(\Sigma, L)$ is not slope stable.
- If $\Sigma$ has at worst ordinary double points then $(\Sigma, L)$ is slope stable.

Proof. For the first statement suppose $Z = \{p\}$ is a point of $\Sigma$ with multiplicity $e \geq 3$. Northcott has shown ([No] Lemma 1) that $\rho \geq e - 1$. Hence $2\rho \geq 2e - 2 > e$ and $Z$ strictly destabilises.

For the second statement suppose for a contradiction that $Z$ is a destabilising subscheme of $\Sigma$. From Proposition 4.25 (especially (4.26)) and Lemma 4.27 we see that $\mu_c(\mathcal{I}_{Z_0}) \geq \mu(\Sigma)$ for some connected component $Z_0$ of $\Sigma$ and $c \leq \epsilon(Z_0)$ (we are not saying that $Z_0$ destabilises since we may not be allowed to take $c = \epsilon(Z_0)$). If the support $p \in \Sigma$ of $Z_0$ is a smooth point then $Z_0 = m\{p\}$ must by a thickened point, which by Proposition 4.25 shows that $\mu_{mc}(\mathcal{I}_{\{p\}}) \geq \mu(X)$. But in the notation of (8.7), $e(\{p\}) = 1$, which by Theorem 3.2 of [KM] implies that $\rho(\{p\}) = 0$, so by Proposition 8.6 $\mu_{mc}(\mathcal{I}_{\{p\}})$ is in fact $< \mu(X)$.

Hence $Z$ must be supported at one of the singular points, and we can reduce to looking at the local analytic model Spec $R$, $R = \mathbb{C}[X, Y]/(XY)$. Let $I$ be an ideal of $R$ which is supported at $(X, Y)$. (By abuse of notation we shall not distinguish between a polynomial in two variables and its class in $R$.) We can pick a finite number of generators of $I$ of the form $f_i = a_i X^p + b_i Y^q$, with $a_i, b_i \in \mathbb{C}$.

Let $p = \min \{p_j : a_j \neq 0\}$, $q = \min \{q_j : a_j \neq 0\}$; $p, q \geq 1$ since $I$ is supported at the origin. Pick an $i$ such that $p_i = p$ and $a_i \neq 0$; then $X^{p+1}_k = \frac{X}{a_i}(a_i X^p + b_i Y^q)^k \in I^k$; similarly $Y^{q+1}_k \in I^k$. Therefore $R/I^k$ is spanned by $\{1, X, \ldots, X^p, Y, \ldots, Y^q\}$. By the definition of $p$ and $q$, $I^k$ is spanned by $\{X^i, Y^j : i \geq pk, j \geq qk\}$, so the vectors $\{1, X, \ldots, X^{p-1}, Y, \ldots, Y^{q-1}\}$ in $R/I^k$ are linearly independent. Thus

$$(p + q)k - 1 \leq \dim R/I^k \leq (p + q)k + 1.$$ 

Writing $\dim R/I^k = ek - p$ we have $-1 \leq \rho \leq 1$. Hence $2\rho \leq 2 \leq p + q = e$, so by Proposition 8.6 $I$ does not destabilise. \hfill $\square$

Remark 8.9. Eisenbud and Mumford [Mu] analyse the effect of singular points on Chow stability of higher dimensional varieties. It would be interesting to know if their results can be seen using slope stability.

These results combined with Theorem 4.18 imply that curves with singularities of multiplicity greater than two are strictly K-unstable. We cannot deduce positive results about K-stability from the results of Section 6, however, since $\Sigma$ need not be normal. Unless, that is, $\Sigma$ is smooth:

Theorem 8.10. Any smooth polarised curve $(\Sigma, L)$ of genus $g$ is K-stable if $g \geq 1$ and strictly K-polystable if $g = 0$.

Proof. By Corollary 6.7 it is equivalent to prove the results for slope (poly)stability. Instead of using previous results it is now easier to proceed directly. Any nonempty subscheme $Z$ is a divisor of degree $d > 0$, so

$$\chi(L^k \otimes \mathcal{I}_Z^k) = k \deg L - xdk + 1 - g$$
shows that $\bar{a}_0(x) = xd$ and $\bar{a}_1(x) = 0$. Thus $\mu_c(\mathcal{O}_Z) = \frac{\epsilon d}{\epsilon d} = \frac{1}{c} > 0 \geq \frac{1 - \frac{g}{\deg L}}{\deg L} = \mu(X)$ for $g \geq 1$, proving slope stability.

For $g = 0$, $c$ may take values up to and including $\epsilon(Z) = \deg L/d$, since $L^d \otimes \mathcal{I}_Z = \mathcal{O}_{\mathbb{P}^1}(d \deg L - d \deg L) = \mathcal{O}_{\mathbb{P}^1}$ is globally generated. Thus $\mu_c(\mathcal{O}_Z) \geq \frac{d \deg L}{\deg L} \geq \mu(X)$ with equality (strict semistability) only for $d = 1$, i.e. $Z$ a single point, and $c = \epsilon(Z)$. Since the deformation to the normal cone of a single point on $\mathbb{P}^1$ blows down to $\mathbb{P}^1 \times \mathbb{C}$ (with a nontrivial $\mathbb{C}^\times$-action) from which the relevant line bundle $L_\epsilon$ pulls back, we find $\mathbb{P}^1$ is in fact slope polystable. □

This can also be proved using the constant curvature metric on $\Sigma$ and analysis of the Mabuchi functional, but this seems to be the first direct algebraic proof.

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