On $H$-topological intersection graphs

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Abstract

Biró, Hujter, and Tuza (1992) introduced the concept of $H$-graphs, intersection graphs of connected subgraphs of a subdivision of a graph $H$. They are related to and generalize many important classes of geometric intersection graphs, e.g., interval graphs, circular-arc graphs, split graphs, and chordal graphs. Our paper starts a new line of research in the area of geometric intersection graphs by studying several classical computational problems on $H$-graphs: recognition, graph isomorphism, dominating set, clique, and colorability.

We negatively answer the 25-year-old question of Biró, Hujter, and Tuza which asks whether $H$-graphs can be recognized in polynomial time, for a fixed graph $H$. We prove that it is NP-complete if $H$ contains the diamond graph as a minor. On the positive side, we provide a polynomial-time algorithm recognizing $T$-graphs, for each fixed tree $T$. For the special case when $T$ is a star $S_d$ of degree $d$, we have an $O(n^{3.5})$-time algorithm.

We give FPT- and XP-time algorithms solving the minimum dominating set problem on $S_d$-graphs and $H$-graphs, parametrized by $d$ and the size of $H$, respectively. The algorithm for $H$-graphs adapts to an XP-time algorithm for the independent set and the independent dominating set problems on $H$-graphs.

If $H$ contains the double-triangle as a minor, we prove that the graph isomorphism problem is GI-complete and that the clique problem is APX-hard. On the positive side, we show that the clique problem can be solved in polynomial time if $H$ is a cactus graph. Also, when a graph has a Helly $H$-representation, the clique problem is polynomial-time solvable.

Further, we show that both the $k$-clique and the list $k$-coloring problems are solvable in FPT-time on $H$-graphs, parameterized by $k$ and the treewidth of $H$. In fact, these results apply to classes of graphs with treewidth bounded by a function of the clique number.

We observe that $H$-graphs have at most $n^{O(||H||)}$ minimal separators which allows us to apply the meta-algorithmic framework of Fomin, Todinca, and Villanger (2015) to show that for each fixed $t$, finding a maximum induced subgraph of treewidth $t$ can be done in polynomial time. In the case when $H$ is a cactus, we improve the bound to $O(||H||n^c)$.

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1 Introduction

An intersection representation $\mathcal{R}$ of a graph $G$ is a collection of sets $\{R_v : v \in V(G)\}$ such that $R_u \cap R_v \neq \emptyset$ if and only if $uv \in E(G)$. Many important classes of graphs arise from restricting the sets $R_v$ to geometric objects (e.g., intervals, circular-arcs, convex sets, planar curves). The study of these geometric representations has been motivated through various application domains. For example, intersection graphs of planar curves relate to circuit layout problems [Sin66, BS90], interval graphs relate to scheduling problems [Rob78] and can be used to model biological problems (see, e.g., [JMT92]), and intersection representations of convex sets relate to the study of wireless networks [HS95].

We study $H$-graphs, intersection graphs of connected subgraphs of a subdivision of a fixed graph $H$, introduced by Biró, Hujter, and Tuza [BHT92]. We answer their open question concerning the problem of recognition of $H$-graphs and further start a new line of research in the area of geometric intersection graphs, by studying $H$-graphs from the point of view of fundamental computational problems of theoretical computer science: recognition, graph isomorphism, dominating set, clique, and colorability. We begin by discussing several closely related graph classes.

Interval graphs (INT) form one of the most studied and well-understood classes of intersection graphs. In an interval representation, each set $R_v$ is a closed interval of the real line; see Fig. 1a. A primary motivation for studying interval graphs (and related classes) is the fact that many important computational problems can be solved in linear time on them; see for example [BL76, Cha98, LB79].

Chordal graphs (CHOR) were originally defined as the graphs without induced cycles of length greater than three. Equivalently, as shown by Gavril [Gav74b], a graph is chordal if and only if it can be represented as an intersection graph of subtrees of some tree; see Fig. 1b. This immediately implies that INT is a subclass of the chordal graphs.

The recognition problem can be solved in linear time for CHOR [RTL76], and such algorithms can be used to generate an intersection representation by subtrees of a tree. However, asking for special host trees can be more difficult. For example, when the desired tree $T$ is a part of the input, deciding whether $G$ is a $T$-graph is NP-complete [KKOS15]. Additionally, some other important computational problems, for example the dominating set [BJ82] and graph isomorphism [LB79], are harder on chordal graphs than on interval graphs.

One can ask related questions about having "nice" tree representations of a given chordal graph. For example, for a given graph $G$, if one would like to find a tree $T$ with the fewest leaves such that $G$ is a $T$-graph, it can be done in polynomial time [HS12], this is known as the leafage problem. However, for any fixed $d \geq 3$, if one would like to find a tree $T$ where $G$ is a $T$-graph and, for each vertex $v$, the subtree representing $v$ has at most $d$ leaves, the problem again

![Figure 1](image-url)
becomes NP-complete [CS14], this is known as the \textit{d-vertex leafage} problem. The minimum vertex leafage problem can be solved in \(n^{O(d)}\)-time via a somewhat elaborate enumeration of minimal tree representations of \(G\) with exactly \(\ell\) leaves where \(\ell\) is the leafage of \(G\) [CS14].

\textbf{Split graphs} (SPLIT) form an important subclass of chordal graphs. These are the graphs that can be partitioned into a clique and an independent set. Note that every split graph can be represented as an intersection graph of subtrees of a \textit{star} \(S_d\), where \(S_d\) is the complete bipartite graph \(K_{1,d}\).

\textbf{Circular-arc graphs} (CARC) naturally generalize interval graphs. Here, each set \(R_v\) corresponds to an arc of a circle. The \textit{Helly circular-arc graphs} form an important subclass of circular-arc graphs. A graph \(G\) is a \textit{Helly circular-arc graph} if the collection of circular-arcs \(\mathcal{R} = \{R_v\}_{v \in V(G)}\) satisfies the \textit{Helly property}, i.e., in each sub-collection of \(\mathcal{R}\) whose sets pairwise intersect, the common intersection is non-empty. Interestingly, it is NP-hard to compute a minimum coloring for Helly circular-arc graphs [Gav96].

1.1 \textit{H}-graphs

Biró, Hujter, and Tuza [BHT92] introduced \textit{H-graphs}. Let \(H\) be a fixed graph. A graph \(G\) is an \textit{intersection graph} of \(H\) if it is an intersection graph of connected subgraphs of \(H\), i.e., the assigned subgraphs \(H_v\) and \(H_u\) of \(H\) share a vertex if and only if \(uv \in E(G)\).

A \textit{subdivision} \(H'\) of a graph \(H\) is obtained when the edges of \(H\) are replaced by internally disjoint paths of arbitrary lengths. A graph \(G\) is a \textit{topological intersection graph} of \(H\) if \(G\) is an intersection graph of a subdivision \(H'\) of \(H\). We say that \(G\) is an \textit{H-graph} and the collection \(\{H'_v : v \in V(G)\}\) of connected subgraphs of \(H'\) is an \textit{H-representation} of \(G\). The class of all \(H\)-graphs is denoted by \(H\)-\text{GRAPH}. Alternatively, we can view \(H\)-graphs geometrically as intersection graphs of connected subregions of a one-dimensional simplicial complex (this is a topological definition of a graph). We have the following relations:

\[
\text{INT} = K_2\text{-GRAPH}, \quad \text{CARC} = K_3\text{-GRAPH},
\]

\[
\text{SPLIT} \subseteq \bigcup_{d=2}^{\infty} S_d\text{-GRAPH}, \quad \text{CHOR} = \bigcup_{\text{Tree } T} T^\ast\text{-GRAPH}.
\]

\textbf{Motivation}. It is easy to see that every graph \(G\) is an \(H\)-graph for an appropriate choice of \(H\) (e.g., by taking \(H = G\)). In this sense, the families of \(H\)-graphs provide a parameterized view through which we can study all graphs. We also mentioned that several important computational problems are polynomial on interval (the most basic class of \(H\)-graphs), but are hard on chordal graphs. This inspires the question of when we can use this parameterization to provide a refined understanding of computational problems. Of course, to approach this problem, we first need to observe some relations among the classes of \(H\)-graphs and related well-studied graph classes.

For any pair of (multi-)graphs \(H_1\) and \(H_2\), if \(H_1\) is a \textit{minor} of \(H_2\), then \(H_1\text{-GRAPH} \subseteq H_2\text{-GRAPH}\). Moreover, if \(H_1\) is a subdivision of \(H_2\), then \(H_1\text{-GRAPH} = H_2\text{-GRAPH}\). Specifically, we have an infinite hierarchy of graph classes between interval and chordal graphs since for every tree \(T\) with at least one edge, \(\text{INT} \subseteq T^\ast\text{-GRAPH} \subseteq \text{CHOR}\). This motivates the study of the above mentioned problems on \(T\)-graphs, for a fixed tree \(T\).

We note a dichotomy regarding computing a minimum coloring on \(H\text{-GRAPH}\). Namely, if \(H\) contains a cycle, then computing a minimum coloring on \(H\text{-GRAPH}\) is already NP-hard even
for the subclass of Helly $H$-graphs [Gav96]. On the other hand, when $H$ is acyclic, a minimum coloring can be computed in linear time since $H$-GRAPH is a subclass of CHOR.

Biró, Hujter, and Tuza originally introduced $H$-graphs in the context of the $(p,k)$ pre-coloring extension problem (PrColExt$(p,k)$). In this problem, the input is a graph $G$ together with a $p$-coloring of $W \subseteq V(G)$, and the goal is to find a proper $k$-coloring of $G$ extending this pre-coloring. Biró, Hujter, and Tuza [BHT92] provide an XP (in $k$ and $|H|$) algorithm to solve PrColExt$(k,k)$ on $H$-graphs. Biró, Hujter, and Tuza asked the following question which we answer negatively.

[Biró, Hujter, and Tuza [BHT92], 1992] Let $H$ be an arbitrary fixed graph. Is there a polynomial algorithm testing whether a given graph $G$ is an $H$-graph?

1.2 Our results

We give a comprehensive study of $H$-graphs from the point of view of several important problems of theoretical computer science: recognition, graph isomorphism, dominating set, clique, and colorability. We focus on five collections of classes of graphs. In particular, $S_d$-GRAPH, $T$-GRAPH, $C$-GRAPH, Helly $H$-GRAPH, and $H$-GRAPH, where $S_d$ is the star of degree $d$, $T$ is a tree, $C$ is a cactus, and $H$ is an arbitrary graph. Our results are displayed in Table 1. The following list provides a summary of our results and should help the reader to navigate through the paper:

- **Recognition.** In Section 3 we negatively answer the question of Biró, Hujter, and Tuza. We prove that recognizing $H$-graphs is NP-complete if $H$ is not a cactus (Theorem 1). Equivalently this means that $H$ contains the diamond graph as a minor. We do this by a reduction from the problem of testing whether the interval dimension of a partial order of height 2 is at most 3. On the positive side, in Section 4, we give an $O(n^{3.5})$-time algorithm for recognizing $S_d$-graphs (Theorem 3), and we give a polynomial-time algorithm for recognizing $T$-graphs (Theorem 4), for a fixed tree $T$.

- **Dominating set.** In Section 5, we solve the problem of finding a minimum dominating set for $S_d$-graphs in time $O((d(n+m)) + 2^d(d + 2^d)^{O(1)}$ (Theorem 5) and for $H$-graphs in $n^{O(|H|)}$-time (Theorem 6). The latter algorithm can be easily adapted to solve the maximum independent set problem and minimum independent dominating set problem in $n^{O(|H|)}$-time for $H$-graphs (Corollary 7).

- **Clique.** In Section 6, we study the clique problem. We show that if $H$ contains the double-triangle $\Delta_2$ (see Fig. 5a) as a minor, then the clique problem is APX-hard for $H$-graphs (Theorem 8). On the positive side, we solve the clique problem in polynomial time for Helly $H$-graphs (Theorem 9), and in the case when $H$ is a cactus (Theorem 10).

- **Graph isomorphism.** Theorem 8 also gives that if $H$ contains the double-triangle $\Delta_2$ (see Fig. 5a) as a minor, then graph isomorphism problem is GI-complete for $H$-graphs.

- **$k$-coloring and $k$-clique.** In Section 7, we use treewidth based methods to provide an FPT-time algorithm for finding a $k$-clique in an $H$-graphs (Theorem 11) and an FPT-time algorithm for $k$-coloring of $H$-graphs (Theorem 12). In fact, these results apply to more general graph classes formalized via the concept of a clique-treewidth property.
Table 1: The table of the complexity of different problems for the four considered classes. Our contributions are highlighted. Note: $A \preceq B$ denotes that $A$ is a minor of $B$, and $\Delta_2$ denotes the double-triangle (see Fig. 5).

(which is defined as in the parameter-treewidth properties of bidimensionality theory; see, e.g., [DFHT04]) and may be of independent interest.

- **Minimal Separators.** Finally, in Section 8, we show that each $H$-graph has $n^{O(\|H\|)}$ minimal separators (Theorem 13) and, when $H$ is a cactus, we improve this bound to $O(\|H\|n^2)$ (Theorem 15). Thus, by the algorithmic framework of Fomin, Todinca, and Villanger [FTV15], on $H$-graphs, we obtain a large class of problems (including, e.g., feedback vertex set) which can be solved in XP-time (parameterized by $\|H\|$) and polynomial time (in both $\|H\|$ and the size of the input graph) when $H$ is a cactus.

**Open problems.** Since all the sections are mostly self-contained, instead of including a separate section for open problems and conclusions, we decided to include the open problems and possible future research directions in the corresponding sections.

**Recent developments.** After the publication of the two conference articles [CTVZ17, CZ17] (which this paper includes and extends), there have already been further developments regarding $H$-graphs [FGR20, JKT20]. The results contained in these articles complement and build on our work regarding combinatorial optimization problems. For instance, to complement our XP-time algorithms for minimum dominating set and maximum independent set, Fomin, Golovach, and Raymond [FGR20] show that these problems are W[1]-hard, when parameterized by $\|H\|$ and the desired solution size. They additionally tighten our result regarding the fixed parameter tractability of the $k$-Clique problem on $H$-graphs by showing that this problem admits a polynomial size kernel in terms of both $\|H\|$ and the solution size. Jaffke, Kwon, and Telle [JKT20] adapt the W[1]-hardness proof from [FGR20] for maximum independent set to additionally show that feedback vertex set is also W[1]-hard. Only recently, the problem of testing isomorphism of $S_d$ graphs was solved in FPT time [AH20].
2 Preliminaries

We assume that the reader is familiar with the following standard and parameterized computational complexity classes: NP, XP, and FPT (see, e.g., [CFK+15] for further details).

Let \( G \) be an \( H \)-graph. For a subdivision \( H' \) certifying \( G \in H\text{-GRAPH} \), we use \( H'_c \) to denote the subgraph of \( H' \) corresponding to \( v \in V(G) \). The vertices of \( H \) and \( H' \) are called nodes. By \( \|H\| \) we denote the size of \( H \), i.e., \( \|H\| = |V(H)| + |E(H)| \).

We refer to the degree one nodes of \( H \) as leaves and the nodes degree at least three as branching points. Note that, while we sometimes speak of degree two nodes in \( H \), they are actually redundant since their presence or absence does not change \( H\text{-GRAPH} \). As such, by thinking of \( H \) as a multi-graph with loops one can nearly always avoid the need for any nodes of degree two (by contracting edges where one end point has degree two). The exception here is the case of \( H \) being a cycle which leads to the true \( H \) simply being a single vertex with one loop, i.e., this vertex has degree two. Of course, when \( H \) is a tree, this works without the need for \( H \) to be a multi-graph.

We have some special notation for the case when \( H \) is a tree. Let \( a, b \) be two nodes of \( H' \). By \( P_{a,b} \) we denote the path from \( a \) to \( b \). Further, we define \( P_{a,b} := P_{a,b} - a \), and \( P_{a,b}, \ P_{a,b} \) analogously.

Let \( S \subseteq G \). Then \( G[S] \) is the subgraph of \( G \) induced by \( S \), and \( G - S \) is the graph obtained from \( G \) by deleting the vertices in \( S \) (together with the incident edges). For a graph \( G \), we assume \( G \) has \( n \) vertices and \( m \) edges.

In 1965, Fulkerson and Gross proved the following fundamental characterization of interval graphs by orderings of maximal cliques. It is used implicitly in several proofs.

Lemma 2.1 (Fulkerson and Gross [FG65]). A graph \( G \) is an interval graph if and only if there exists a linear ordering \( \preceq \) of the maximal cliques of \( G \) such that for every \( u \in V(G) \) the maximal cliques containing \( u \) appear consecutively in \( \preceq \).

A remark on the size of subdivisions and membership in \( \text{NP} \). As membership in \( H\text{-GRAPH} \) is certified through the existence of an appropriate subdivision of \( H \), one might wonder just how large subdivision \( H' \) is necessary to ensure that any \( n \)-vertex \( H \)-graph \( G \) has a representation by connected subgraphs of \( H' \). Note that as long as the size of this subdivision is bounded by a polynomial in \( n \), \( H \)-graph recognition does indeed belong to \( \text{NP} \). We observe that it suffices to subdivide every edge of \( H \) \( 2n \) times to accommodate an \( n \)-vertex \( H \)-graph, i.e., without loss of generality the size of \( H' \) is at most \( |V(H)| + 4n|E(H)| \).

To see this, we consider an edge \( ab \) of \( H \), and its corresponding path \( a, c_1, \ldots, c_{\ell}, b \) in \( H' \). Observe that, for each vertex \( v \in V(G) \), \( H'_v \) has at most two leaves on this path. Thus, if \( \ell > 2n \), there must be a \( c_i \) which does not contain any leaf of any \( H'_v \). In particular, this \( c_i \) can be contracted into its neighbour on the path while preserving the representation of \( G \). Therefore, it suffices to consider subdivisions of size \( |V(H)| + 4n|E(H)| \) and, in particular, for every \( H \), recognition of \( H\)-graphs is in \( \text{NP} \).

3 Recognition is hard if \( H \) is not a cactus

In this section, we negatively answer a question posed by Biró, Hujter, and Tuza [BHT92] Problem 6.3]. Namely, we prove that testing whether a graph is an \( H \)-graph is \( \text{NP} \)-complete.
when the diamond graph\(^2\) \(D\) is a minor of \(H\). Note that this sharply contrasts the polynomial
time solvability of the recognition problem for circular-arc graphs (i.e., when \(H\) is a cycle). Before
getting to the hardness proof itself, we first establish a technical (though rather straightforward
to prove) lemma regarding the essentially unique (up to automorphism) \(H\)-representability of
the 3-subdivision \(H_3\) of \(H\) as an \(H\)-graph. Namely, \(H_3\) is obtained from \(H\) by subdividing each
edge exactly 3 times, that is, in \(H_3\) we have one vertex \(x_v\) for each vertex \(v\) of \(H\), and for each
edge \(e = uv\) of \(H\) we have the path \(x_u, x_{ue}, x_e, x_{ve}, x_v\).

**Lemma 3.1.** Let \(H\) be any multi-graph without vertices of degree 2, and let \(H_3\) be the 3-
subdivision of \(H\). The graph \(H_3\) is an \(H\)-graph and, for every subdivision \(H'\) certifying \(H_3 \in
\text{H-GRAPH}\) (via the representation \(\{H'_x : x \in V(H_3)\}\)), we have:

- For each non-leaf vertex \(v\) of \(H\), the representation \(H'_{x_v}\) of the corresponding vertex \(x_v\)
in \(H_3\) contains exactly one branching point \(p\) of \(H\) where the degree of \(v\) (and \(x_v\)) and \(p\)
coincide.
- For each edge \(e = uv\) of \(H\) and the corresponding path \(x_u, x_{ue}, x_e, x_{ve}, x_v\) in \(H_3\), the repre-
sentation \(H'_{x_e}\) of \(x_e\) is strictly contained within the subdivision of a single edge \(zz'\) of \(H\)
such that for distinct edges \(e, f\) of \(H\) with corresponding “middle” vertices \(x_e, x_f\) in \(H_3\),
(\(H'_{x_e}\) and \(H'_{x_f}\) are contained within subdivisions of distinct edges of \(H\).

Moreover, each \(H\)-representation of \(H_3\) defines an automorphism of \(H\).

**Proof.** We first note that this holds trivially for the case when \(H\) is \(K_1\) or \(K_2\).

We now observe that \(H_3\) is indeed an \(H\)-graph. Let \(H'\) be the 4-subdivision of \(H\), that is, in
\(H'\) the edge \(e = uv\) of \(H\) becomes the path \(y_u, y_{ue}, z_{ue}, z_{ve}, y_{ve}, y_v\). For each vertex \(v\) of \(H\) with
incident edges \(\{e_1, \ldots, e_k\}\), we represent \(x_v\) by the star \(H'_v = H'_{\{y_v, y_{ve_1}, y_{ve_2}, \ldots, y_{ve_k}\}}\). For
each edge \(uv\) of \(H,\) we represent:

- \(x_{ue}\) by \(H'_{x_{ue}} =\) the edge \(y_{ue}, z_{ue}\).
- \(x_e\) by \(H'_{x_e} =\) the edge \(z_{ue}, z_{ve}\), and
- \(x_{ve}\) by \(H'_{x_{ve}} =\) the edge \(z_{ve}, y_{ve}\).

It is easy to see that this collection of subgraphs of \(H'\) is indeed \(H\)-representation of \(H_3\).

So, we now consider an arbitrary \(H\)-representation \(\{H'_x : x \in V(H_3)\}\) of \(H_3\), where \(H'\) is the
subdivision of \(H\) and establish the claimed properties.

Suppose that there is a vertex \(v\) of \(H\) where \(v\) has degree at least three (with incident edges
\(e_1, \ldots, e_k\) and \(H'_{x_v}\) does not contain a branching point, i.e., all nodes in \(H'_{x_v}\) have degree at
most two. Now, since the neighborhood \(\{x_{ve_1}, \ldots, x_{ve_k}\}\) of \(x_v\) is an independent set (and \(k \geq 3\)),
this implies that (without loss of generality), \(H'_{x_{ve_1}}\) is contained within \(H'_{x_v}\). However, this now
makes it impossible to represent \(x_{ve_i}\) since \(H'_{x_{ve_i}}\) should intersect \(H'_{x_v}\) but should not intersect
\(H'_{x_v}\). Thus, for each vertex \(v\) of \(H\) with degree at least three, \(H'_{x_v}\) contains a branching point.
Note that no branching point can occur in two such \(H'_{x_v}\) and \(H'_{x_{ve_i}}\), thus, the vertices of degree
at least three are bijectively mapped to the branching points. Finally, since we now know that
\(H'_{x_v}\) contains exactly one branching point, we remark that the degree of this branching point
must match be at least the degree of \(x_v\) as otherwise some \(H'_{x_{ve_i}}\) would be contained in \(H'_{x_v}\)

\(^2\)The diamond graph is obtained by deleting an edge from a 4-vertex clique.
between two nodes $D$ in at most 3. In particular, a “middle” part of the three paths connecting the two degree 3 vertices to get a graph $G$.

For a given height 1 poset $P$, we construct its incomparability graph $H$. Part 1: We prove the essential case which shows this problem by $\text{IntDim}(1, 3)$. Theorem 1.

Consider a collection $I$ of closed intervals on the real line. A poset $P$ is a partial order (poset) with height one; has interval dimension at most three; shown by Yannakakis [Yan82]. We denote this problem by $\text{IntDim}(1, 3)$. Note that having height one means that every element of the poset is either minimal or maximal.

Our hardness proof stems from the NP-hardness of testing whether a partial order (poset) with height one has interval dimension at most three; shown by Yannakakis [Yan82]. We denote this problem by $\text{IntDim}(1, 3)$. Note that having height one means that every element of the poset is either minimal or maximal.

Testing if $H$ is an $H$-graph is NP-complete if the diamond graph $D$ is a minor of $H$.

Proof. The proof is split into two parts. In the Part 1, we prove the essential case which shows that testing whether $G$ is an $D$-graph is NP-hard. This argument is generalized in Part 2 to the case when $H$ contains $D$ as a minor.

Part 1: $H$ is the diamond. First, we summarize the idea behind our proof. As stated above, we will encode an instance $P$ of $\text{IntDim}(1, 3)$ as an instance of membership testing in $D$-GRAPH. For a given height 1 poset $P$, we construct its incomparability graph $G_P$, slightly augment $G_P$ to get a graph $G$, and show that $G$ is in $D$-GRAPH if and only if the interval dimension of $P$ is at least three vertices in $D$ will encode the three interval orders whose intersection is $P$.

Note that, we consider $H$ as the multi-graph consisting of three parallel edges $e_a, e_b, e_c$ between two nodes $v_{\text{min}}$ and $v_{\text{max}}$ To construct $G$, we use the graph $H_3$ which the 3-subdivision
of $H$. Namely, $H_3$ has two vertices $u_{\min}$ and $u_{\max}$ of degree three and nine vertices $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ of degree two where $u_{\min}, b_1, b_2, b_3, u_{\max}$ is a path for each $\mathbb{N} \in \{a, b, c\}$. Note that, by Lemma 3.1 without loss of generality, $H_3$ is an $H$-graph where in every $H$-representation of $H_3$, say on a subdivision $H'$ of $H$, we have:

- $H'_{u_{\min}}$ contains $v_{\min}$ and $H'_{u_{\max}}$ contains $v_{\max}$,
- For each $\mathbb{N} \in \{a, b, c\}$, $H'_{\mathbb{N}}$ is contained in the subdivision of $e_{\mathbb{N}}$.

It is within these $H'_{\mathbb{N}}$ paths that we will see the interval orders.

We are now ready to construct our graph $G$ from $H_3$ and the graph $G_P$ of a given height one poset $P = (P, <)$, recall that $K_{\min}$ and $K_{\max}$ denote cliques on the minima and maxima of $P$ respectively. Let $V_{\min} = \{u_{\min}, a_1, a_2, b_1, b_2, c_1, c_2\}$ and let $V_{\max} = \{u_{\max}, a_3, a_2, b_3, b_2, c_3, c_2\}$. The graph $G$ is the union of $G_P$ and $H_3$ where, additionally, each vertex of $K_{\min}$ is adjacent to each vertex of $V_{\min}$ and each vertex of $K_{\max}$ is adjacent to each vertex of $V_{\max}$.

**Claim 3.1.** $P$ has interval dimension at most 3 if and only if $G$ is an $H$-graph.

**Proof.** For the reverse direction, consider an $H$-representation of $G$ on a subdivision $H'$ of $H$. As remarked above, by Lemma 3.1 $H'_{u_{\min}}$ contains the node $v_{\min}$ and $H'_{u_{\max}}$ contains the node $v_{\max}$. The minimal elements of $P$ are not adjacent to the vertices of $u_{\max}$. Therefore, for each $x \in K_{\min}$, $H'_x$ cannot contain $v_{\max}$, i.e., $H'_{x}$ is a subtree of $H' - \{v_{\max}\}$. In particular, for each of the three ($v_{\min}, v_{\max}$) paths $A, B, C$ in $H'$, $H'_x$ defines one (possibly empty) subpath/interval (originating in $v_{\min}$). Similarly, for each $y \in K_{\max}$, $H'_y$ cannot contain $v_{\min}$ and as such $H'_y$ defines, for each of $A, B, C$, one subpath (originating in $v_{\max}$). It is easy to see that these intervals provide the interval orders $P_{I_A}, P_{I_B}$, and $P_{I_C}$ such that $P_{I_A} \cap P_{I_B} \cap P_{I_C} = P$.

For the forward direction, let $I_1, I_2, I_3$ be sets of intervals such that $P = P_{I_1} \cap P_{I_2} \cap P_{I_3}$. We assume that each interval in $I_i$ is labelled according to the corresponding element of $P$. Further, we assume that the intervals corresponding to the minimal elements have their left endpoints at 0 and their right endpoints are integers in the range $[0, n - 1]$. Similarly, we assume that the intervals corresponding to the maximal elements have their right endpoints at $n$ and their left endpoints are integers in the range $[1, n]$. With this in mind, for each minimal element $x$ and each $i \in \{1, 2, 3\}$, we use $x_i$ to denote the right endpoint of its interval in $I_i$, and for each

![Figure 2](image-url)

Figure 2: (a) A partially ordered set $P = (P, <)$ of height 1, interval dimension 3, but not 2. We define the following interval orders: $I_1 = l_a b c r d e a_2 d_1 r e f f$, $I_2 = l_a b c d e r f f r d e f$, and $I_3 = l_a b c d e r d f f r d e f f$, where $[l, r]$ represents an interval corresponding to $\alpha \in P$. Note that $P_{I_1} \cap P_{I_2} \cap P_{I_3} = P$. (b) An illustration of part of the $D$-representation. Here, $T_a$ and $T_b$ indicate the subgraphs representing the elements $a$ and $b$. 

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maximal element $y$ and each $i \in \{1, 2, 3\}$, we use $y_i$ to denote the left endpoint of its interval in $I_i$.

Let $H'$ be the subdivision of $H$ obtained by subdividing the three $v_{\text{min}}v_{\text{max}}$ edges $n + 5$ times. We label the three $(v_{\text{min}}, v_{\text{max}})$-paths in $H'$ as follows:

- $v_{\text{min}}, \alpha_{\text{min}}, \alpha'_{\text{min}}, \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha'_{\text{max}}, \alpha_{\text{max}}, v_{\text{max}},$
- $v_{\text{min}}, \beta_{\text{min}}, \beta'_{\text{min}}, \beta_0, \beta_1, \ldots, \beta_n, \beta'_{\text{max}}, \beta_{\text{max}}, v_{\text{max}},$ and
- $v_{\text{min}}, \gamma_{\text{min}}, \gamma'_{\text{min}}, \gamma_0, \gamma_1, \ldots, \gamma_n, \gamma'_{\text{max}}, \gamma_{\text{max}}, v_{\text{max}}.$

We are now ready to describe an $H$-representation of $G$ on $H'$. Each minimal element $x$ is represented by the minimal subtree of $H'$ which includes the nodes $v_{\text{min}}, \alpha_{x1}, \beta_{x2}, \gamma_{x3}$. Similarly, each maximal element $y$ is represented by the minimal subtree of $H'$ which includes the nodes $v_{\text{max}}, \alpha_{y1}, \beta_{y2}, \gamma_{y3}$. We can now see that the comparable elements of $\mathcal{P}$ are represented by disjoint subgraphs of $H'$ and that the incomparable elements map to intersecting subgraphs. Finally, the vertices of $H_3$ are represented as follows:

- $u_{\text{min}}$ is represented by the subtree induced by $v_{\text{min}}, \alpha_{\text{min}}, \beta_{\text{min}},$ and $\gamma_{\text{min}}$; analogously, $u_{\text{max}}$ is represented by the subtree induced by $v_{\text{max}}, \alpha_{\text{max}}, \beta_{\text{max}},$ and $\gamma_{\text{max}}$.
- $\alpha_1, b_1,$ and $c_1$ are represented by the edges $\alpha_{\text{min}}\alpha'_{\text{min}}, \beta_{\text{min}}\beta'_{\text{min}},$ and $\gamma_{\text{min}}\gamma'_{\text{min}}$, respectively; analogously, $\alpha_3, b_3,$ and $c_3$ are represented by the edges $\alpha_{\text{max}}\alpha'_{\text{max}}, \beta_{\text{max}}\beta'_{\text{max}},$ and $\gamma_{\text{max}}\gamma'_{\text{max}}$, respectively, and
- $\alpha_2$ is represented by the path $\alpha'_{\text{min}}, \alpha_0, \ldots, \alpha_n, \alpha'_{\text{max}},$ and
- $\beta_2$ is represented by the path $\beta'_{\text{min}}, \beta_0, \ldots, \beta_n, \beta'_{\text{max}},$ and
- $c_2$ is represented by the path $\gamma'_{\text{min}}, \gamma_0, \ldots, \gamma_n, \gamma'_{\text{max}}$.

Clearly, in this construction, the graph $H_3$ is correctly represented. Moreover, the subtree corresponding to every minimal element includes all of the nodes $v_{\text{min}}, \alpha_{\text{min}}, \alpha'_{\text{min}}, \beta_{\text{min}}, \beta'_{\text{min}}, \gamma_{\text{min}}, \gamma'_{\text{min}}$, but none of the opposite max-nodes. Thus, each minimal element is universal to $V_{\text{min}}$ and non-adjacent to the vertices of $V_{\text{max}} \setminus \{a_2, b_2, c_3\}$, as needed. Symmetrically, each maximal element is universal to $V_{\text{max}}$ and non-adjacent to the vertices of $V_{\text{min}} \setminus \{a_2, b_2, c_3\}$. It follows that $G$ is an $H$-graph.

This completes the first part of the proof.

**Part 2: $H$ contains the diamond graph $D$ as a minor.** The argument here follows very similarly to the proof shown in Part 1. We again use the 3-subdivision $H_3$ of $H$ which, by Lemma 3.1, canonically “covers” $H$. Again, $H_3$ will be used as part of the graph $G$ we will construct from $G_P$ so that $G \in \mathcal{G}$ if and only if $\mathcal{P}$ has interval dimension at most 3. Importantly, $H_3$ also allows us to, with a careful choice of $V_{\text{min}}$ and $V_{\text{max}}$, appropriately restrict the representations of the minima and maxima to only use a chosen diamond minor of $H$ (up to automorphism of course) as before.

Observe that, since $H$ contains $D$ as a minor (and the maximum degree of $D$ is three), a subdivision of $D$ (or $D$ itself) is a subgraph of $H$. Let $D^*$ be a subgraph of $H$ that is a subdivision of $D$. In particular, $D^*$ consists of two nodes $d_{\text{min}}$ and $d_{\text{max}}$ of degree 3 and three $(d_{\text{min}}, d_{\text{max}})$-paths $A, B, C$ that are edge disjoint and whose internal vertices are of degree 2. Let
that every branching point is “contained” in some maximal clique of $T$.

We present a polynomial-time algorithm testing whether $G \in \text{H-GRAPH}$ if and only if $P$ has interval dimension three. Recall that $K_{\text{min}}$ and $K_{\text{max}}$ denote cliques on the minima and maxima of $P$ respectively. As in Part 1, we let $V_{\text{min}} = \{z_{\text{min}}, a_1, a_2, b_1, b_2, c_1, c_2\}$. Similarly to Part 1, we let $V_{\text{max}}$ be the vertex set of the minimal subgraph of $D^*_3$ containing $\{z_{\text{max}}, a_2, b_2, c_2\}$. In other words $V_{\text{max}} = V(D^*_3) \setminus V_{\text{min}} \cup \{a_2, b_2, c_2\} = V(D^*_3) \setminus \{z_{\text{min}}, a_1, b_1, c_1\}$. Now, as in Part 1, the graph $G$ is the union of $G_P$ and $H_3$ where, additionally, each vertex of $K_{\text{min}}$ is adjacent to each vertex of $V_{\text{min}}$ and each vertex of $K_{\text{max}}$ is adjacent to each vertex of $V_{\text{max}}$.

The completion of the proof now follows nearly identically to the proof of the claim in Part 1. Namely, by Lemma 3.1, $H_3$ has a unique up to automorphism $H$-representation, and the vertices of $K_{\text{min}}$ and $K_{\text{max}}$ can essentially only be represented on the $D^*$ part of $H$ (due to their adjacency with the vertices of $H_3$). Moreover, within the three edges $\alpha, \beta, \gamma$, there will be the representations of $a_2, b_2, c_2$ and within these representations we will indeed have the (at most) 3 interval models.

The next section gives a positive answer for the following problem in the case when $H$ is a tree. Also, recall that when $H$ is a single cycle, $H$-GRAPH is the class of circular-arc graphs and as such can be recognized in linear time. This leaves the following problem.

**Problem 1.** For a non-tree fixed cactus graph $H$ (other than a single cycle), is there a polynomial-time algorithm testing whether $G$ is an $H$-graph?

## 4 Polynomial-time recognition algorithms

We present an $O(n^{3.5})$-time algorithm recognizing $S_d$-graphs and an $XP$-time algorithm recognizing $T$-graphs (parametrized by the size of the tree $T$). We begin with a lemma that motivates our approach. It implies that if $G$ is a $T$-graph, then there exists a representation of $G$ such that every branching point is “contained” in some maximal clique of $G$.

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$^3$While the representation of a vertex of $G_P$ might “reach out” beyond $D^*$ onto an incident edge, it can never traverse all of such an edge because, by Lemma 3.1, there is a vertex $x_e$ of $H_3$ occupying the “middle” of that edge and, by construction, $x_e$ is not adjacent to any vertex of $G_P$. 

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Lemma 4.1. For any $T$-graph $G$ and $T$-representation $R$, of $G$, $R$ can be modified such that for every branch node $b \in V(T')$, we have $b \in \bigcap_{v \in C} V(T'_v)$, for some maximal clique $C$ of $G$.

Proof. For every node $x$ of the subdivision $T'$, let $V_x = \{ u \in V(G) : x \in V(T'_u) \}$ be the set of vertices of $G$ corresponding to the subtrees passing through $x$. Let $b$ be a branching point such that $V_b$ is not a maximal clique.

We pick a maximal clique $C$ with $C \supseteq V_b$. Since $R$ satisfies the Helly property, there is a node $a \in \bigcap \{ V(T'_v) : v \in C \}$. Note that for every node $x$ of $P_{[a,b]}$, we have $V_x \supseteq V_b$. Let $x$ be the node of $P_{[b,a]}$ closest to $b$ such that $V_x \neq V_b$. Then, for each $v \in V_x \setminus V_b$, we update $T'_v$ to be $T'_v \cup P_{[b,x]}$. Thus, we obtain a correct representation of $G$ with $V_b = V_x$.

We repeat the process described in the previous paragraph until $V_b$ is a maximal clique. □

Remark on subdivisions. For convenience, we assume throughout the whole section that we already have a sufficiently large subdivision $T'$ of $T$. At the end, it will be clear that a subdivision $T' \subseteq T$ with $|V(T')| \leq cn + |V(T)|$, for some constant $c$, suffices. In fact, it suffices to have $c = 3$.

General idea. It is well-known that chordal graphs, and therefore also $T$-graphs, have at most $n$ maximal cliques and that they can be listed in linear time. Let $B$ be the set of branching points of $T$ and let $C$ be the set of all maximal cliques of $G$. The main part of our algorithm attempts, for a given $f : B \rightarrow C$, to construct a $T$-representation satisfying $V_b = \bigcap_{v \in f(b)} V(T'_v)$, for every $b \in B$, where $V_b = \{ u \in V(G) : b \in V(T'_u) \}$. By Lemma 4.1 there always exists such a representation.

To this end, we try find interval representations of the connected components of $G - \bigcup_{b \in B} f(b)$ on the paths $T' - B$ such that the following conditions hold:

(i) If interval representations of the connected components $X_1, \ldots, X_k$ are on a path $P_{[b,l]}$, where $b \in B$ and $l$ is a leaf of $T'$, then the induced subgraph $G[f(b) \cup V(X_1) \cup \cdots \cup V(X_k)]$ has an interval representation on $P_{[b,l]}$ in which $f(b)$ is the leftmost clique.

(ii) If interval representations of the connected components $X_1, \ldots, X_k$ are on a path $P_{[b,b']}$, where $b, b' \in B$, then the induced subgraph $G[f(b) \cup V(X_1) \cup \cdots \cup V(X_k) \cup f(b')]$ has an interval representation on $P_{[b,b']}$ in which $f(b)$ and $f(b')$ are the rightmost and leftmost cliques, respectively.

4.1 Recognition of $S_d$-graphs

In the case when $T = S_{2d}$, we have $B = \{ b \}$ and $V(T) = \{ b \} \cup \{ l_1, \ldots, l_d \}$. The number of mappings $f : \{ b \} \rightarrow C$ is exactly the same as the number of maximal cliques of $G$, which is at most $n$ (otherwise it is not an $S_d$-graph). For every maximal clique $C$ of $G$, we try to construct a $T$-representation $R$ such that $b \in \bigcap_{v \in C} V(T'_v)$.

Assume that $G$ has such an $S_d$-representation, for some maximal clique $C$. Then the connected components of $G - C$ are interval graphs and each connected component can be represented on one of the paths $P_{[b,l_i]}$, which is a subdivision of the edge $bl_i$; see Fig. 3a and 3b. However, some pairs of connected components of $G - C$ cannot be placed on the same path $P_{[b,l_i]}$, since their “neighborhoods” in $C$ are not “compatible”. The idea is to define a partial order $\triangleright$ on the components of $G - C$ such that for every linear chain $X_1 \triangleright \cdots \triangleright X_k$, the induced subgraph $G[C, V(X_1), \ldots, V(X_k)]$ can be represented on some path $P_{[b,l]}$; see Fig. 3b.

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We define \( N_C(u) \) and \( N_C(X) \) to be the neighbourhoods of the vertex \( u \) in \( C \) and of the components \( X \) in \( C \), respectively. Formally,

\[
N_C(u) = \{ v \in C : vu \in E(G) \} \quad \text{and} \quad N_C(X) = \bigcup \{ N_C(u) : u \in V(X) \}.
\]

Note that, if we have two components \( X \) and \( X' \) on the same branch where \( N_C(X') \subseteq N_C(u) \) for every \( u \in V(X) \), then \( X \) must be closer to \( C \) than \( X' \) if they are represented on the same path \( P_{[b,l]} \).

We say that components \( X \) and \( X' \) are equivalent, \( X \sim X' \), if there is a subset \( C' \) of \( C \) such that \( N_C(u) = C' \) for every \( u \in V(X) \) and \( N_C(u') = C' \) for every \( u' \in V(X') \). Note that equivalent components \( X \) and \( X' \) can be represented in an interval representation of \( G[C, V(X), V(X')] \) in an arbitrary order and they can be treated as one component. We denote the set of the equivalence classes \( G - C/\sim \) by \( \mathcal{X} \). For \( X, X' \in \mathcal{X} \), we put:

\[
X \triangleright X' \quad \text{if for every } u \in V(X), N_C(X') \subseteq N_C(u) \text{ or if } X = X'.
\]  

(1)

**Lemma 4.2.** The relation \( \triangleright \) is a partial ordering on \( \mathcal{X} \).

**Proof.** The relation \( \triangleright \) is reflexive by definition. Suppose that \( X \triangleright X' \) and \( X' \triangleright X \). For every \( u \in V(X) \) and \( u' \in V(X') \), we have

\[
N_C(u') \subseteq N_C(X') \subseteq N_C(u) \quad \text{and} \quad N_C(u) \subseteq N_C(X) \subseteq N_C(u').
\]

Therefore, \( N_C(u) = N_C(u') \) for every \( u \in V(X) \) and \( u' \in V(X') \) and \( X \) and \( X' \) are equivalent. We assume that \( \mathcal{X} \) contains only non-equivalent components. So, \( X = X' \) and the relation \( \triangleright \) is asymmetric. It can be easily checked that \( \triangleright \) is also transitive. \( \square \)
Lemma 4.3. Let $X_1, \ldots, X_k \in \mathcal{X}$. Then the induced subgraph $G[C, V(X_1), \ldots, V(X_k)]$ has an interval representation with $C$ being the leftmost clique if and only if $X_1 \triangleright \cdots \triangleright X_k$ and each $G[C, X_i]$ has an interval representation with $C$ being the leftmost clique.

Proof. Suppose that there is an interval representation $\mathcal{R}$ of $G[C, V(X_1), \ldots, V(X_k)]$ with $C$ being the leftmost maximal clique. Since each $X_i$ is a connected components of $G - C$, their representations in $\mathcal{R}$ cannot overlap. Without loss of generality, we assume that the components $X_1, \ldots, X_k$ are ordered such that $i < j$ if and only if $X_i$ is placed closer to $C$ in $\mathcal{R}$ than $X_j$. Let $u \in V(X_i)$ and $v \in N_C(X_j)$. The vertex $v$ is adjacent to at least one vertex of $X_j$. Therefore, the representation of $v$ covers the whole component $X_i$ in $\mathcal{R}$, i.e., we have $v \in N_C(u)$ and $X_i \triangleright X_j$.

For the converse, we assume that $X_1, \ldots, X_k$ form a chain in $\triangleright$ and every $G[C, X_i]$ has an interval representation $\mathcal{R}_i$ with $C$ being the leftmost clique. Since $X_i \triangleright X_j$, for $i < j$, every vertex in $N_C(X_j)$ is adjacent to every vertex of $X_i$. We now construct an interval representation of $G[C, V(X_1), \ldots, V(X_k)]$. We first place the interval representations of all $X_i$’s (i.e., we use $\mathcal{R}_i$ restricted to the intervals of $V(X_i)$) on the real line according to $\triangleright$, with $X_1$ being the leftmost. Let $x_1, \ldots, x_{k+1} \in \mathbb{R}$ be the points of the real line such that $X_i$ is represented on the interval $(x_i, x_{i+1}) \subseteq \mathbb{R}$.

It remains to construct a representation for every vertex $v \in C$. Let

$$C_k = N_C(X_k) \text{ and } C_i = N_C(X_i) \setminus \bigcup_{j=i+1}^{k} N_C(X_j), i = 0, \ldots, k - 1 \text{ where } X_0 = C.$$

Let $x_0 \in \mathbb{R}$ be a point left of $x_1$. All the vertices in $C_0$ are represented by the interval $[x_0, y]$, for some $y < x_1$. The intervals representing vertices in $C_i$ are constructed inductively, for $i = k, k - 1, \ldots, 1$. For $i \leq k$, we assume that we constructed the representations of vertices in $C_{i+1}, \ldots, C_k$. Note, if $X_j \triangleright X_i$, then for every $u \in V(X_j)$, we have $N_C(X_i) \subseteq N_C(u)$. Therefore, every vertex in $C_i$ is represented by an interval of the form $[x_i, z]$, where $z \in (x_i, x_{i+1})$ is a suitable point given by the representation $\mathcal{R}_i$ of $G[C, X_i]$.

The following theorem gives a characterization of $S_d$-graphs. It generalizes the characterization of interval graphs due to Fulkerson and Gross; see Lemma 2.1.

Theorem 2 (Characterization of $S_d$-graphs). A graph $G$ is an $S_d$-graph if and only if there is a maximal clique $C$ of $G$ such that the following hold:

(i) For every connected component $X$ of $G - C$, the induced subgraph $G[C, X]$ has an interval representation with $C$ being the leftmost clique.

(ii) The partial order $\triangleright$ on $\mathcal{X} = G - C / \sim$ has a chain cover of size at most $d$.

Proof. Suppose that $G$ is an $S_d$-graph with a representation satisfying $b \in \bigcap_{c \in C} V(T^*_b)$; such a representation always exists by Lemma 4.1. The representation of a connected component $X \in \mathcal{X}$ can not pass through the node $b$ since otherwise $C$ would not be a maximal clique. Clearly, the conditions (i) is satisfied. The representations of every two components in $\mathcal{X}$ have to be placed on non-overlapping parts of the subdivided $S_d$. By Lemma 4.3, we have that the components placed on some path $P_{[b, l]}$ of the subdivided $S_d$ form a linear chain in $\triangleright$. Therefore, the partial order $\triangleright$ has a chain cover of size at most $d$ and the condition (ii) is satisfied; see Fig. 3.
Suppose that the conditions (i) and (ii) are satisfied. We put the components in $\mathcal{X}$ on the paths $P_{[b, l]}[1], \ldots, P_{[b, l]}[d]$ according to the chain cover of the partial order $\triangleright$. For every chain of $\triangleright$, we can find an interval representation of the graph $G[C, V(X_1), \ldots, V(X_k)]$ with $C$ being the leftmost maximal clique.

Algorithm. By combining Lemmas 4.3 and Theorem 2, we obtain an algorithm for recognizing $S_d$-graphs. For a given graph $G$ and its maximal clique $C$, we do the following:

1. We delete the maximal clique $C$ and construct the partial order $\triangleright$ on the set of non-equivalent connected components $\mathcal{X}$.
2. We test whether the partial order $\triangleright$ can be covered by at most $d$ chains.
3. For each linear chain $X_1 \triangleright \cdots \triangleright X_k$, $1 \leq i \leq d$, we construct an interval representation $\mathcal{R}_i$ of the induced subgraph $G[C, V(X_1), \ldots, V(X_k)]$, with $C$ being the leftmost maximal clique, on one of the paths of the subdivided $S_d$.
4. We complete the whole representation by placing each $\mathcal{R}_i$ on the path $P_{[b, l]}$ so that $b \in \bigcap_{v \in C} V(T_v)$.

Theorem 3. Recognition of $S_d$-graphs can be solved in $O(n^{3.5})$ time.

Proof. Every chordal graph has at most $n$ maximal cliques, where $n$ is the number of vertices, and they can be listed in linear time $[RTL76]$. For every clique $C$, our algorithm tries to find an $S_d$-representation with $b \in \bigcap_{v \in C} V(T_v)$. The partial order $\triangleright$ can be constructed in time $O(n^2)$. By forgetting the orientation in the partial order $\triangleright$, we get a comparability graph, and every clique in the comparability graph induces a linear chain in $\triangleright$. A relatively simple algorithm finds a minimum clique-cover of a comparability graph in time $O(n^3) [Gol77]$. An algorithm that runs in time $O(n^{2.5})$ can be obtained by a combination of [Ful86] and [HK73]. Testing whether $G[C, V(X_1), \ldots, V(X_k)]$ has an interval representation with $C$ being the leftmost maximal clique can be done in linear time. Thus, the overall time complexity of our algorithm is $O(n^{3.5})$.

Problem 2. Can we recognize $S_d$-graphs in time $O(n^{2.5})$? In particular, can we find the clique that can be placed in the center of $S_d$ efficiently?

4.2 Recognition of T-graphs

The algorithm for recognizing $T$-graphs is a generalization of the algorithm for recognizing $S_d$-graphs described above. Let $f : B \to \mathcal{C}$ be an fixed assignment of cliques.

Assumption (connectedness of $G$). Suppose that $G$ is disconnected. Then it can be written as a disjoint union of some $X$ and $\tilde{G}$, where $X$ is a connected component of $G$. Let $\mathcal{C}_X$ and $\hat{\mathcal{C}}$ be the maximal cliques of $X$ and $\tilde{G}$, respectively. The sets $f^{-1}(\mathcal{C}_X)$ and $f^{-1}(\hat{\mathcal{C}})$ induce subtrees $T_X$ and $\hat{T}$ of $T$ separated by the branch $ab$, where $a \in V(T_X)$ and $b \in V(\hat{T})$ (otherwise $f$ is invalid). We subdivide the branch $ab$ by nodes $c_1$ and $c_2$. Then we try to find a representation of $X$ on the tree $T_X \cup ac_1$ and a representation of $\tilde{G}$ on $\hat{T} \cup c_2 b$. Therefore, we may assume that $G$ is connected.
Assumption (injectiveness of $f$). Suppose that $f$ is not injective, i.e., $f(b) = f(b')$. Then for every branching point $b''$ which lies on the path from $b$ to $b'$, we must have $f(b) = f(b'') = f(b')$ (otherwise $f$ is invalid). For $C \in f(B)$, the branching points in $f^{-1}(C)$, together with the paths connecting them, have to form a subtree $T_C$ of $T$. In this case the whole subtree $T_C$ can be contracted into a single node $a$. Note that if there is a component $X$ of $G - \bigcup_{b \in B} f(b)$ where every vertex of $X$ adjacent to every vertex of $C$, then $X$ can be represented on any branch incident to $a$ by subdividing it appropriately. Thus, we may assume that $f$ is injective.

Step 1 (components between branching points). The first step of our algorithm is to find for $b, b' \in E(T)$, which components have to be represented on the path $P_{(b,b')}$ of $T'$.

Lemma 4.4. Let $X$ be a connected component of $G - \bigcup_{b \in B} f(b)$ and $bb' \in E(T)$. If the sets $$(f(b) \setminus f(b')) \cap N_{f(b)}(X) \neq \emptyset \text{ and } (f(b') \setminus f(b)) \cap N_{f(b')}(X) \neq \emptyset,$$
then $X$ has to be represented on $P_{(b,b')}$ of $T'$.

Proof. Let $v \in (f(b) \setminus f(b')) \cap N_{f(b)}(X)$ and $u \in (f(b') \setminus f(b)) \cap N_{f(b')}(X)$. Since $v \notin f(b')$, we have $b' \notin V(T'_u)$. Similarly we have $b \notin V(T'_u)$. Putting it together, we have that $b \in V(T'_u)$ and $b' \notin V(T'_u)$ and $b' \in V(T'_u)$. Since $X$ is adjacent to both $u$ and $v$, the only possible path where $X$ can be represented is $P_{(b,b')}$; see Fig. 4a and 4b. \hfill \Box

We do the following for each $b, b' \in B$ such that $bb' \in E(T)$. Let $X_{b,b'}$ be the disjoint union of the components satisfying the conditions of Lemma 4.4. If the induced interval subgraph $G[C \cup V(X_{b,b'})] \cup C'$ has a representation such that the cliques $C$ and $C'$ are the leftmost and the rightmost, respectively, then we can represent $X_{b,b'}$ in the middle of the path $P_{(b,b')}$. If no such representation exists, then $G$ does not have $T$-representation for this particular $f: B \to C$. This means that the representation of $X_{b,b'}$ is constructed on a proper subpath of $P_{(b,b')}$ – recall that we are assuming that the subdivision $T'$ is sufficiently large.

Next, we do the following for every $b \in B$. Let $l_1, \ldots, l_p$ and $b_1, \ldots, b_q$ be the leaves of $T$ and the branching points of $T$, respectively, such that $bl_i \in E(T)$, for every $i = 1, \ldots, p$, and $bb_j \in E(T)$, for every $j = 1, \ldots, q$. Let $a_1, \ldots, a_p$ and $a'_1, \ldots, a'_q$ be the points of the paths $P_{(b_1,b_1)}, \ldots, P_{(b_q,b_q)}$, respectively, such that $X_{b,b_1}$ is represented on the subpath $P_{(a_i,a'_i)}$. We define $S(b)$ to be the subdivided star consisting of the paths $P_{(b_1,l_1)}, \ldots, P_{(b_p,l_p)}, P_{(b_1,a_1)}, \ldots, P_{(b_q,a_q)}$. Note that if a vertex $u \in V(X_{b,b_1})$ is adjacent to a vertex $v$ in $f(b)$, then the representation of $T'_u$ of $v$ contains the whole subpath $P_{(b,a_1)}$. This means that a component $X$, which does not satisfy the condition of Lemma 4.4, can be represented on $P_{(a_i,a'_i)}$ only if $N_{f(b)}(X_{b,b_1}) \subseteq N_{f(b)}(X)$. We remove the subpaths $P_{(a_i,a'_i)}$ together with the representations of $X_{b,b_1}$ and we are left with disjoint subdivided stars with restrictions; see Fig 4b.

Step 2 (disjoint stars with restrictions). We reduced the problem of recognizing $T$-graphs to the following problem. Let $H$ be a fixed graph formed by the disjoint union of $k$ stars $S(b_1), \ldots, S(b_k)$ with branching points $b_1, \ldots, b_k$. On the input we have a graph $G$, an injective mapping $f: \{b_1, \ldots, b_k\} \to C$, and for every edge of $S(b_i)$ a subset of $f(b_i)$, called restrictions. We want to find a representation of $G$ on $H$ such that $b_i \in \bigcap_{e \in f(b_i)} V(H'_e)$, and for every connected component $X$ of $G - \bigcup_{i=1}^k f(b_i)$, the vertices $V(X)$ have to be adjacent to every vertex in the restrictions corresponding to the path on which $X$ is represented.
Theorem 4. Is there an FPT algorithm for recognizing $T$-graphs?
4.3 Bounded list coloring of co-comparability graphs

Here, we provide a polynomial time algorithm for the problem of bounded list coloring of co-comparability graphs. Our result can be seen as a generalization of the polynomial time algorithm of Enright, Stewart, and Tardos [EST14] for bounded list coloring on a class which includes both interval graphs and permutation graphs. However, they [EST14] explicitly state that their approach does not extend to co-comparability graphs. To prove this also for co-comparability graphs, we slightly modify the approach in [BMO11].

In [BMO11], the problem of capacitated coloring is solved for a more general class of graphs, so called \(k\)-thin graphs. A graph \(G\) is \(k\)-thin if there exists an ordering \(v_1, \ldots, v_n\) of \(V(G)\) and a partition of \(V(G)\) into \(k\) classes \(V^1, \ldots, V^k\) such that, for each triple \(p, q, r\) with \(p < q < r\), if \(v_p, v_q\) belong to the same class and \(v_r v_p \in E(G)\), then \(v_r v_q \in E(G)\). Such ordering and partition are called consistent. The minimum \(k\) such that \(G\) is \(k\)-thin is called the thinness of \(G\). Graphs with bounded thinness were introduced in [MORC07] as a generalization of interval graphs. Note that interval graphs are exactly the 1-thin graphs.

Recall that a graph \(G\) is a comparability graph if there exits an ordering \(v_1, \ldots, v_n\) of \(V(G)\) such that, for each triple \(p, q, r\) with \(p < q < r\), if \(v_p v_q\) and \(v_q v_r\) are edges of \(G\), then so is \(v_p v_r\). Such an ordering is a comparability ordering.

**Lemma 4.5** (Theorem 8, [BMO11]). Let \(G\) be a co-comparability graph. Then the thinness of \(G\) is at most \(\chi(G)\), where \(\chi\) is the chromatic number. Moreover, any vertex partition given by a coloring of \(G\) and any comparability ordering for its complement are consistent.

Let \(G\) be \(k\)-thin graph, and let \(v_1, \ldots, v_n\) and \(V^1, \ldots, V^k\) be an ordering and a partition of \(V(G)\) which are consistent. Note that the ordering induces an order on each class \(V^j\). For each vertex \(v_r\) and class \(V^j\), let \(N(v_r, j)_<\) be the set of neighbors of \(v_r\) in \(V^j\) that are smaller than \(v_r\), i.e., \(N(v_r, j)_<= V^j \cap \{v_1, \ldots, v_{r-1}\} \cap N(v_r)\). For each class \(V^j\) let \(\Delta(j)_<\) be the maximum size of \(N(v_r, j)_<\) over all vertices \(v_r\). The following lemma gives an alternative definition of \(k\)-thin graphs.

**Lemma 4.6** (Fact 7, [BMO11]). For each vertex \(v_r \in \{v_1, \ldots, v_n\}\) and each \(j \in \{1, \ldots, k\}\), the set \(N(v_r, j)_<\) is such that:

- the vertices in \(N(v_r, j)_<\) are consecutive, with respect to the order induced on \(V^j\).
- if \(N(v_r, j)_< \neq \emptyset\), then it includes the vertex with largest index in \(V^j \cap \{v_1, \ldots, v_{r-1}\}\).

**Bounded List Coloring On \(k\)-thin Graphs.** In [BMO11], the problem of capacitated coloring is reduced to a reachability problem on an auxiliary acyclic digraph.\(^4\) We obtain an algorithm for bounded list coloring on \(k\)-thin graphs by slightly modifying the algorithm for capacitated coloring in [BMO11]. The only difference is that we do not have a restriction on how many times we can use a particular color and for every vertex we can only use the colors from the list assigned to it. Otherwise, everything is the same as in [BMO11]. We include it here for the sake of completeness.

Let \(G\) be a \(k\)-thin graph with an ordering \(v_1, \ldots, v_n\) and a partition \(V^1, V^2, \ldots, V^k\) of \(V(G)\). Let \(S\) be a set of colors, \(s = |S|\), and \(L : V(G) \rightarrow \mathcal{P}(S)\) be a function that assigns a list of allowed colors to a vertex. Consider an instance \((G, L)\) of list coloring. We reduce the problem

\(^4\)Note that this is just a representational convenience for dynamic programming.
to a reachability problem on an auxiliary acyclic digraph $D(N,A)$. We will refer to the elements of $N$ and $A$ as nodes and arcs while the elements of $V(G)$ and $E(G)$ will be referred to as vertices and edges (as we did so far).

The digraph $D$ will be layered, i.e., the set $N$ is the disjoint union of subsets (layers) $N_0,N_1,\ldots,N_n$ and all arcs in $A$ have the form $(u,w)$ with $u\in N_r$ and $w\in N_{r+1}$, for some $0\leq r\leq n-1$. Note for each vertex $v_r \in V$, there is a layer $N_r$ with $r \neq 0$. We denote by $j(r)$ the class index $q$ such that $v_r \in V^q$.

We first describe the set of nodes in each layer. The first layer consists of colors which can be assigned to the first vertex, i.e., $N_0 = L(v_1)$. For the layers $N_1, \ldots, N_{n-1}$, there is a one-to-one correspondence between nodes at layer $N_r$ and $(sk+1)$-tuples $(r,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k})$ with $0 \leq \beta_i^j \leq \Delta(j)_c$, for each $i,j$. The last layer $N_n$ has only one node $t$ corresponding to the tuple $(n,0,\ldots,0)$.

We associate with each node $u \notin N_0$ a suitable list coloring problem with additional constraints, that we call the constrained sub-problem associated with $u$. As we show in the following, $u$ is reachable from a node $z \in N_0$ if and only if this constrained sub-problem has a solution. Namely, we will show that the following property holds:

\begin{equation}
(\ast) \text{ a node } (r,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k}) \text{ is reachable from a node } z \in N_0 \text{ if and only if the induced subgraph } G[v_1,\ldots,v_r] \text{ admits a list coloring with the lists given by } L \text{ and with additional constraint that, for each } i=1,\ldots,s \text{ and } j=1,\ldots,k, \text{ color } i \text{ is forbidden for the last } \beta_i^j \text{ vertices in } V_j \cap \{v_1,\ldots,v_r\}. \nonumber
\end{equation}

In this case, $G$ admits a list coloring if and only if the node $t$ is reachable from a node $z \in N_0$.

Property \([\ast]\) will follow from the definition of the set of arcs $A$ given as follows. Let $u = (r,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k})$. Note that the problem associated with $u$ has a solution where the vertex $v_r$ gets color $i$ only if $\beta_i^j(r) = 0$. Let $C(u) = \{i \in L(r) : \beta_i^j(r) = 0\}$. We will make exactly $|C(u)|$ arcs entering into $u$, and give each such arc a color $i \in C(u)$ (exactly one color from $C(u)$ per arc). Each arc $(u',u) \in A$, with $u' \in N_{r-1}$ and $i \in C(u)$, will then have the following meaning: if the constrained sub-problem associated with $u'$ has a solution, i.e., a coloring $\varphi'$, then we can extend $\varphi'$ into a solution $\varphi$ to the constrained sub-problem associated with $u$ by giving color $i$ to vertex $v_r$.

We now give the formal definition of the set $A$. We start with the arcs from $N_0$ to $N_1$. Let $u = (1,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k}) \in N_1$. There is an arc from $z_i$ (where $i \in L(1)$), to $u$ if and only if $i \in C(u)$; moreover, the color of its arc is $i$. We now deal with the arcs from $N_{r-1}$ to $N_r$, with $2 \leq r \leq n$. Let $u = (r,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k}) \in N_r$. As we discussed above, for each $i^* \in C(u)$, there will be an arc from a node $u_{i^*} \in N_{r-1}$ to $u$, with color $i^*$. Namely, $u_{i^*} = (r-1,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k})$, where:

\begin{equation}
\beta_i^j = \begin{cases} \max\{|N(v_{r},j)_c|,\beta_i^j\} & i = i^* \\ \max\{0,\beta_i^j - 1\} & i \neq i^*, j = j(r) \\ \beta_i^j & i \neq i^*, j \neq j(r) \end{cases} 
\end{equation}

Note that $u_{i^*}$ is indeed a node of $N_{r-1}$, as the $(sk+1)$-tuple $(r-1,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k})$ is such that $0 \leq \beta_i^j \leq \Delta(j)_c$, for each $i,j$ (in fact, $\beta_i^j \leq \Delta(j)_c$, since $u$ is a node of $N_r$).

**Lemma 4.7.** $G$ admits an $L$ list coloring if and only if $D$ contains a directed path from a node $z \in N_0$ to $t$. Moreover, if such a path exists, then a list coloring of $G$ can be obtained by assigning each node $v_r$ ($r \in \{1,\ldots,n\}$) the color of the arc of the path entering into layer $N_r$. 

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Proof. The proof is analogous to the proof of Lemma 10 in [BMOH] and we omit it here. □

Lemma 4.8. Suppose that for a \((k\text{-thin})\) graph \(G\) with \(n\) vertices we are given an ordering and a partition of \(V(G)\) into \(k\) classes that are consistent. Further consider an instance \((G, L)\) of the list coloring problem. Let \(s = \bigcup_{v \in V(G)} L(v)\). Then \((G, L)\) can be solved in \(O(n s^2 k \prod_{j=1,\ldots,k} \Delta(j)^s)\)-time, i.e., \(O(n^{ks+1}s^2k)\)-time.

Proof. By definition, for \(r = 1, \ldots, n - 1\), \(|N_r| = \prod_{i=1,\ldots,k} (\Delta(j)^s + 1)^s\). Note that each node of \(D\) has at most \(s\) incoming arcs, and each arc can be built in \(O(sk)\)-time. Therefore, \(D\) can be built in \(O(n s^2 k \prod_{i=1,\ldots,k} (\Delta(j)^s + 1)^s)\)-time. Since \(D\) is acyclic, the reachability problem on \(D\) can be solved in linear time. Therefore the list coloring problem on \(G\) can be solved in \(O(n s^2 k \prod_{i=1,\ldots,k} \Delta(j)^s)\)-time, that is \(O(n^{ks+1}s^2k)\)-time. □

Lemma 4.9. Let \(G\) be a co-comparability graph and \((G, L)\) an instance of the list coloring problem with the total number of colors \(s \geq 2\). Then \((G, L)\) can be solved in \(O(n^{s^2+1}s^3)\)-time, i.e., polynomial time when \(s\) is fixed.

Proof. By Lemma 4.5, the graph \(G\) is \(k\)-thin. It can be tested in \(O(n^3)\) time whether \(G\) is \(s\)-colorable [Gol77]. If it is \(s\)-colorable, then by Lemma 4.5 we get a comparability ordering and a \(k\)-partition of \(V(G)\). Moreover, by Lemma 4.5 we know that \(k \leq s\). Thus, by Lemma 4.8 we can solve the problem in time \(O(n^3 + n^{s^2+1}s^3) = O(n^{s^2+1}s^3)\). □

5 Minimum dominating Set

In this section, we discuss the minimum dominating set problem on \(H\)-\text{graph}. The basic idea behind our algorithms is to reduce the minimum dominating set problem for \(H\)-graphs to several minimum dominating set problems on interval graphs, obtained as induced subgraphs of the original graph.

We start with a useful tool (Lemma 5.1) which states that one can compute a dominating set of an interval graph \(G\) which is minimum subject to including one or two of certain special vertices of \(G\). This lemma is an essential tool for both of our dominating set algorithms presented in the subsequent subsections.

Lemma 5.1. Let \(G = (V, E)\) be an interval graph and let \(C_1, \ldots, C_k\) be the left-to-right ordering of the maximal cliques in an interval representation of \(G\).

1. For every \(x \in C_1\), a dominating set of \(G\) which is minimum subject to including \(x\) can be found in linear time.
2. For every \(x \in C_1\) and \(y \in C_k\), a dominating set of \(G\) which is minimum subject to including both \(x\) and \(y\) can be found in linear time.

Proof. We provide the proof for the part 1 (the proof of the part 2 follows analogously). We construct a new graph \(G' = (V', E')\) where \(V' = V \cup \{u, u'\}\) and \(E' = E \cup \{ux, u'x\}\). Clearly, \(G'\) is an interval graph as certified by the following linear order of its maximal cliques \(\{u, x\} = C_0, C_0' = \{u', x\}, C_1, \ldots, C_k\). Furthermore, to dominate both \(u\) and \(u'\) without using \(x\), we would need to include both \(u\) and \(u'\). Thus, every minimum dominating set of \(G'\) includes \(x\), i.e., we can find such a dominating set in linear time using the standard greedy algorithm [Gol04]. □
5.1 Dominating sets in $S_d$-graphs

Here, we solve the minimum dominating set problem on $S_d$-GRAPH in FPT-time, parameterized by $d$.

**Theorem 5.** For an $S_d$-graph $G$, a minimum dominating set of $G$ can be found in $O(dn(n + m)) + 2^d(d + 2^d)^{O(1)}$ time when an $S_d$-representation is given. (If such a representation is not given, we can compute one in $O(n^{3.5})$ time by Theorem 3.)

**Proof.** Let $G$ be an $S_d$-graph and let $S'$ be a subdivision of the star $S_d$ such that $G$ has an $S'$-representation. Let $b$ be the central branching point of $S'$ and let $l_1, \ldots, l_d$ be the leaves of $S'$. Recall that, by Lemma 4.1 we may assume $b \in \bigcap\{S'_v : v \in C\}$, for some maximal clique $C$ of $G$. Let $C_{i,1}, \ldots, C_{i,k_i}$ be the maximal cliques of $G$ as they appear on the branch $P_{b,l_i}$, for $i = 1, \ldots, d$.

For each $G_i = G[C_{i,1}, \ldots, C_{i,k_i}]$, we use an interval graph greedy algorithm [Gol04] to find the size $d_i$ of a minimum dominating set in $G_i$. Let $B_i$ be the set of vertices of $C$ that can appear in a minimum dominating set of $G_i$. By Lemma 5.1, a minimum dominating set $D_i^x$ containing a vertex $x \in C$ can be found in linear time. Note that $x \in B_i$ if and only if $|D_i^x| = d_i$. Therefore, every $B_1, \ldots, B_d$ can be found in $O(d \cdot n \cdot (n + m))$ time. Let $B = \{B_1, \ldots, B_d\}$.

If $B_i$ is empty, then no minimum dominating set of $G_i$ contains a vertex from $C$. So for $G_i$, we pick an arbitrary minimum dominating set $D_i$. Note that $D_i$ dominates $C \cap C_{i,1}$ regardless of the choice of $D_i$. Thus, if $\bigcup_{i=1}^d D_i$ dominates $C$, then it is a minimum dominating set of $G$. Otherwise, $\{x\} \cup \bigcup_{i=1}^d D_i$ is a minimum dominating set of $G$ where $x$ is an arbitrary vertex of $C$.

Let us assume now that the $B_i$’s are nonempty (every branch with an empty $B_i$ can be simply ignored). Let $H$ be a subset of $C$ such that $H \cap B_i$ is not empty, for every $i = 1, \ldots, d$, and $|H|$ is smallest possible. For every branch $P_{b,b_i}$, we pick a minimum dominating set $D_i$ of $G_i$ containing an arbitrary vertex $x_i \in H \cap B_i$. Now, the union $D_1 \cup \cdots \cup D_d$ is a minimum dominating set of $G$. It remains to show how to find the set $H$ in time depending only on $d$.

Finding the set $H$ can be seen as a set cover problem where $B$ is the ground set. Namely, we have one set for each vertex $x$ in $C$ where the set of $x$ is simply its subset of $B$, and our goal is to cover $B$. Note, if two vertices cover the same subset of $B$ it suffices to keep just one of them for our set cover instance, i.e., giving us at most $2^d$ sets over a ground set of size $d$. Such a set cover instance can be solved in $2^d(d + 2^d)^{O(1)}$ time (see Theorem 6.1 [CFK13]).

Thus, we spend $O(dn(n + m)) + 2^d(d + 2^d)^{O(1)}$ time in total. \hfill $\square$

5.2 Dominating sets in $H$-graphs

We turn to $H$-GRAPH, for general fixed $H$. There we solve the problem in XP-time, parameterized by $|H|$. This latter result can be easily adapted to also obtain XP-time algorithms to find a maximum independent set and minimum independent dominating set on $H$-GRAPH (these algorithms are also parameterized by $|H|$); see Corollary 7.

**Theorem 6.** For an $H$-graph $G$ the minimum dominating set problem can be solved in $n^{O(|H|)}$ time when an $H$-representation is given as part of the input.

**Proof.** Recall that, when $H$ is a cycle, $H$-GRAPH = CARC, i.e., minimum dominating sets can be found efficiently [Cha98]. Thus, we assume $H$ is not a cycle.
To introduce our main idea, we need some notation. Consider $G \in H$-GRAPH and let $H'$ be a subdivision of $H$ such that $G$ has an $H'$-representation $\{H'_v : v \in V(G)\}$. We distinguish two important types of nodes in $H'$: namely, $x \in V(H')$ is called high degree when it has at least three neighbors and $x$ is low degree otherwise. As usual, the high degree nodes play a key role. In particular, if we know the sub-solution which dominates the high degree nodes of $H'$, then the remaining part of the solution must be strictly contained in the low degree part of $H'$. Moreover, since $H$ is not a cycle, the subgraph $H'_{\leq 2}$ of $H'$ induced by its low degree nodes is a collection of paths. In particular, the vertices $v$ of $G$ where $H'_v$ only contains low degree nodes, induce an interval graph $G_{\leq 2}$ and, as such, we can efficiently find minimum dominating sets on them. Thus, the general idea here is to first enumerate the possible sub-solutions on the high degree nodes, then efficiently (and optimally) extend each sub-solution to a complete solution.

Consider a high degree node $x$ of $G$ such that $x \in D_{\geq 3}$. For each edge $xx'$ in $H$, let $x = x_1, \ldots, x_k = x'$ be the corresponding path in $H'$. We assign a single vertex $a$ in $D$ to the ordered pair $(x, x')$ such that $H'_a$ contains the longest subpath of $x_1, \ldots, x_k$ including $x = x_1$. Notice that each ordered pair receives precisely one element of $D$. However, if some element $v$ of $D_{\geq 3}$ was not assigned to an ordered pair, then it is easy to see that $D$ is not a minimum dominating set (since all adjacencies achieved by this element are already achieved by the elements we have charged to ordered pairs).

By Claim 5.1, there are at most $n^{2|E(H)|}$ possible sets $D_{\geq 3}$. We now fix one such $D_{\geq 3}$ and describe how to compute a minimum dominating set of $G$ containing it. Notice that, there can be some difficult decisions we might need to make in this process. In particular, suppose there is a high degree node $x$ of $H'$ where no vertex from $V_x$ is in $D_{\geq 3}$. It is not clear how we might be able to efficiently choose from “nearby” $x$ to dominate these vertices. To get around this case, we simply enumerate more vertices. Specifically, for each path $P_{[x,y]} = (x, x_1, \ldots, x_k, y)$ in $H'$ where $x$ and $y$ are high degree nodes (or where $x$ is high degree and $y$ is a leaf), and the $x_i$'s are low degree, we will pick a “first” and “last” vertex among the vertices $v$ of $G$ where $H'_v$ is contained in the subpath $(x_1, \ldots, x_k)$ of $P_{[x,y]}$. That is, for a given $D_{\geq 3}$ we enumerate all possible subsets of size $2 \cdot |E(H)|$ from among the vertices of $G_{\geq 2}$ to act as the “first” and “last” vertices of each path $P_{[x,y]}$. Clearly, there are at most $O(n^{2|E(H)|})$ such subsets. We fix one such subset $D_{\leq 2}$.

We now have our candidate sub-solutions $D^* = D_{\geq 3} \cup D_{\leq 2}$. There are just some simple sanity checks we must make on $D^*$ to test if it is a good candidate to be extended to a dominating set. First, by the definition of $D_{\geq 3}$, it must already dominate every vertex of $G_{\geq 3}$. Second, if there is some path $P_{[x,y]}$ where $D_{\leq 2}$ contains fewer than two vertices from $P_{[x,y]}$, then $D^*$ must already dominate every vertex contained in this path. And finally, for every path $P_{[x,y]}$, for every $v$ with $H'_v$ contained strictly between $x$ and the “left-end” of the “first” chosen vertex, then $v$ must be dominated by $D_{\geq 3}$. If one of these conditions is violated, we discard this candidate $D^*$ and go to the next one.
Finally, what remains to be dominated consists of a collection of disjoint interval graphs where possibly some sequence of “left-most” and “right-most” maximal cliques have already been dominated by $D^*$. Observe that the partially constructed dominating set will consist of one vertex which reaches the farthest in from the right and one which does the same from the left. Namely, we can apply Lemma [5.1] to construct a minimum dominating set for each such interval graph subject to the inclusion of these two special vertices and as such compute a minimum dominating set of $G$ which contains our candidate partial dominating set.

This completes the description of the algorithm. From the discussion, we can see that the algorithm is correct and that the total running time is dominated by the enumeration of the possible sets $D^*$ plus some additional polynomial factors. In particular, the algorithm runs in $n^O(||H||)$ time.

We further remark that the above approach can also be applied to solve the maximum independent set and minimum independent dominating set problems in $n^O(||H||)$ time. This approach is successful since these problems can be solved efficiently on interval graphs.

**Corollary 7.** For an $H$-graph $G$, the maximum independent set problem and minimum independent dominating set problem can both be solved in $n^O(||H||)$ time.

Finally, as we have stated in Section [1.2] in a recent manuscript [FGR20], $W[1]$-hardness has been shown for both the minimum dominating set problem and the maximum independent set problem. Moreover, both of these results concern parameterization by both $||H||$ and the solution size. Thus, this classifies the computational complexity for both of these problems. It would be interesting to also have $W[1]$-hardness for the minimum independent dominating set problem. Additionally, one could make a more fine-grained examination of the running time and look for lower bounds via ETH.

**Problem 4.** Is the minimum independent dominating set problem $W[1]$-hard on $H$-graphs (parametrized by $||H||$ and the solution size)?

**Problem 5.** Can we obtain some interesting lower bounds using ETH?

### 6 Finding cliques in $H$-graphs

We discuss computational aspects of the maximum clique problem for $H$-graphs, parametrized by $||H||$. Let $\Delta_2$ be the double-triangle (see Fig. 5). First, we show that the maximum clique problem is APX-hard for $H$-graphs if $H$ contains $\Delta_2$ as a minor (Theorem 8). In other words, the maximum clique problem is para-NP-hard when parameterized only by $||H||$. As a consequence of our reduction, we also show that if $\Delta_2 \preceq H$, then $H$-GRAPH is GI-complete (the graph isomorphism on $H$-GRAPH is as hard as the general graph isomorphism problem). We then turn to cases where the clique problem can be solved efficiently. Namely, we consider two cases: one where we have a “nice” representation but $H$ is arbitrary, and the other where we restrict $H$ to be a cactus.

#### 6.1 Clique (and isomorphism) hardness results

To obtain our hardness results we show that there are graphs $H$ such that the complement of a 2-subdivision of every graph is an $H$-graph. The 2-subdivision $G_2$ of a graph $G$ is the result of
that the complement of its set \( \{v_1, v_2, v_3, v_4, v_5\} \) on \( H \).

As already mentioned, we prove the theorem by showing

\[ \text{Theorem 8.} \]

If \( \Delta_2 \preceq H \), then the maximum clique problem is APX-hard for \( H \)-graphs and \( H \)-GRAPH is GI-complete.

\textbf{Proof.} As already mentioned, we prove the theorem by showing \( \text{SUBD}_2 \subseteq H\text{-GRAPH} \). Since \( \Delta_2 \preceq H \), the graph \( H \) can be partitioned into three connected subgraphs \( H_1, H_2, H_3 \) such that there are at least two edges connecting \( H_i \) and \( H_j \), for each \( i \neq j \). For every graph \( G \), we show that the complement of its 2-subdivision has an \( H \)-representation.

The construction proceeds similarly to the constructions used by Francis et al. [FGO13], and we borrow their convenient notation. Let \( G \) be a graph with vertex set \( \{v_1, \ldots, v_n\} \) and edge set \( \{e_1, \ldots, e_m\} \). If \( e_k \in E(G) \) and \( e_k = v_iv_j \) where \( i < j \), we define \( l(k) = i \) and \( r(k) = j \) (as if \( v_i \) and \( v_j \) were respectively the \textit{left} and \textit{right} ends of \( e_k \)). In the 2-subdivision \( G_2 \) of \( G \), the edge \( e_k \) of \( G \) is replaced by the path \( (v_{l(k)}, a_k, b_k, v_{r(k)}) \); see Fig. 5(a) and Fig. 5(b).

Note that \( \overline{G_2} \) can be covered by three cliques, i.e., \( G_v = \{v_1, \ldots, v_n\}, G_a = \{a_1, \ldots, a_m\}, \) and \( G_b = \{b_1, \ldots, b_m\} \). We now describe a subdivision \( H' \) of \( H \) which admits an \( H \)-representation \( \{H'_v : v \in V(\overline{G_2})\} \) of \( \overline{G_2} \). We obtain \( H' \) by subdividing the six edges connecting \( H_1, H_2, \) and \( H_3 \). Specifically:

- We \( n \)-subdivide the edges connecting \( H_1 \) to \( H_2 \) to obtain two paths \( P_{12} = (a_0, a_1, \ldots, a_n, a_{n+1}) \), \( Q_{12} = (\beta_0, \beta_1, \ldots, \beta_n, \beta_{n+1}) \) where \( a_0, \beta_0 \in H_1 \) and \( a_{n+1}, \beta_{n+1} \in H_2 \).

- We \( n \)-subdivide the edges connecting \( H_1 \) to \( H_3 \) to obtain two paths \( P_{13} = (\gamma_0, \gamma_1, \ldots, \gamma_n, \gamma_{n+1}) \), \( Q_{13} = (\eta_0, \eta_1, \ldots, \eta_n, \eta_{n+1}) \) where \( \gamma_0, \eta_0 \in H_1 \) and \( \gamma_{n+1}, \eta_{n+1} \in H_2 \).
Problem 7. Let $H$ be a fixed graph such that $\Delta_2 \not\subseteq H$. What is the complexity of the graph isomorphism problem on $H$-graphs?

6.2 Tractable cases

Here, we consider two restrictions which allow polynomial-time algorithms for the maximum clique problem. First, we discuss the case when the $H$-representation satisfies the Helly property. This is followed by a discussion of the case when $H$ is a cactus. In both situations, we obtain polynomial-time algorithms.

**Helly $H$-graphs.** A Helly $H$-graph $G$ has an $H$-representation $\{H_v^i : v \in V(G)\}$ such that the collection $\mathcal{H} = \{V(H_v^i) : v \in V(G)\}$ satisfies the Helly property, i.e., for each sub-collection of $\mathcal{H}$ whose sets pairwise intersect, their common intersection is non-empty. Notice that, when $H$ is a tree, every $H$-representation satisfies the Helly property. When a graph $G$ has a Helly $H$-representation, we obtain the following relationship between the size of $H$ and the number of maximal cliques in $G$.

**Lemma 6.1.** Each Helly $H$-graph $G$ has at most $|V(H)| + |E(H)| \cdot |V(G)|$ maximal cliques.
Proof. Let \( H' \) be a subdivision of \( H \) such that \( G \) has a Helly \( H \)-representation \( \{ H'_v : v \in V(G) \} \). Note that, for each maximal clique \( C \) of \( G \), \( \bigcap_{v \in C} V(H'_v) \neq \emptyset \), i.e., \( C \) corresponds to a node \( x_C \) of \( H' \).

For every edge \( xy \in E(H) \), we consider the path \( P = P_{[x,y]} = (x, x_1, \ldots, x_k, y) \) in \( H' \). Let \( G_P \) be the subgraph of \( G \) formed by the union of the maximal cliques \( C \) of \( G \) such that \( x_C \in V(P) \).

Claim 6.1. The graph \( G_P \) is a Helly circular-arc graph.

Proof. Note that if a restriction of \( H'_v \) for \( v \in V(G_P) \), to \( P \) is disconnected, then it is a disjoint union of two paths containing the end-vertices \( x \) and \( y \), respectively. Let \( C \) by cycle obtained from \( P \) by adding the edge \( xy \). We construct a \( C \)-representation of \( G_P \). If the restriction of \( H'_v \) to \( P \) is a subpath of \( P \), then we let \( C_v \) to be this subpath. Otherwise, we let \( C_v \) to be the restriction of \( H'_v \) to \( P \) together with the edge \( xy \). Clearly, this is a Helly \( C \)-representation.

Now, since Helly circular-arc graphs have at most linearly many maximal cliques \cite{Gav74a}. \( G \) has at most \( |V(H)| + |E(H)| \cdot |V(G)| \) maximal cliques.

We can now use Lemma 6.1 to find the largest clique in \( G \) in polynomial time. In fact, we can do this without needing to compute a representation of \( G \). In particular, the maximal cliques of a graph can be enumerated with polynomial delay \cite{MU04}. Thus, since \( G \) has at most linearly many maximal cliques, we can simply list them all in polynomial time and report the largest, i.e., if the enumeration process produces too many maximal cliques, we know that \( G \) has no Helly \( H \)-representation. This provides the following theorem.

Theorem 9. The clique problem is solvable in polynomial time on Helly \( H \)-graphs.

Note that some co-bipartite circular-arc graphs have exponentially many maximal cliques and these graphs are not contained in Helly \( H \)-GRAPH, for any fixed \( H \). However, the clique problem is solvable for circular-arc graphs in polynomial time \cite{Hsu85}.

Cactus-graphs. The clique problem is efficiently solvable on chordal graphs \cite{Gol04} and circular-arc graphs \cite{Hsu85}. In particular, when \( H \) is either a tree or a cycle, the clique problem can be solved in polynomial-time, independent of \( |H| \). In Theorem 10 we observe that these results easily generalize to the case when \( G \) is a \( C \)-graph, for some cactus graph \( C \). We define,

\[
\text{CACTUS-GRAPH} = \bigcup_{\text{Cactus } C} \text{C-GRAPH}.
\]

To prove the result we will use the clique-cutset decomposition, which is defined as follows. A clique-cutset of a graph \( G \) is a clique \( K \) in \( G \) such that \( G - K \) has more connected components than \( G \). An atom is a graph without a clique-cutset. An atom of a graph \( G \) is a maximal induced subgraph \( A \) of \( G \) which is an atom. A clique-cutset decomposition of \( G \) is a set \( \{ A_1, \ldots, A_k \} \) of atoms of \( G \) such that \( G = \bigcup_{i=1}^k A_i \) and for every \( i, j \), \( V(A_i) \cap V(A_j) \) is either empty, or induces a clique in \( G \). Algorithmic aspects of clique-cutset decompositions were studied by Whitesides \cite{Whi84} and Tarjan \cite{Tar85}. In particular, if \( k \leq n \), then for any graph \( G \) a clique-cutset decomposition \( \{ A_1, \ldots, A_k \} \) of \( G \) can be computed in \( O(n^2 + nm) \) \cite{Tar85}. Additionally, to solve the clique problem on a graph \( G \) it suffices to solve it for each atom of \( G \) from a clique-cutset decomposition \cite{Whi84, Tar85}. Theorem 10 now follows from the following easy lemma and the fact that the clique problem can be solved in polynomial time for circular-arc graphs \cite{Hsu88}.
Lemma 6.2. Let $C$ be cactus and let $G \in \text{C-GRAPH}$. Then each atom $A$ of $G$ is a circular-arc graph.

Proof. Consider an $C$-representation $\{C_v' : v \in V(G)\}$ of $G$. Now, let $C|_A = \bigcup_{u \in V(A)} C_u'$. Clearly, if $C|_A$ is a path or a cycle, then we are done. Otherwise, $C|_A$ must contain a cut-node $x$. Let $X_1, \ldots, X_i$ be the components of $H|_A - \{x\}$, and let $S$ be the vertices of $A$ whose representations contain $x$. Note that $S$ is a clique in $A$. Moreover, since $A$ is an atom, $S$ is not a clique-cutset. Thus, there is a component $X_j$ such that the subgraph $C^*$ of $C$ induced by $V(X_j) \cup \{x\}$ provides a representation of $A$. In particular, if $C^*$ is either a cycle, or a path we are again done. Moreover, when $C^*$ is neither a path, nor a cycle, repeating this argument on $C^*$ provides a smaller subgraph of $C$, on which $A$ can be represented, i.e., this eventually produces either a path, or cycle. □

Theorem 10. The clique problem can be solved in polynomial time on CACTUS-GRAPH.

7 FPT results via clique-treewidth graph classes

The concept of treewidth was introduced by Robertson and Seymour [RS84]. A tree decomposition of a graph $G$ is a pair $(X, T)$, where $T$ is a tree and $X = \{X_i | i \in V(T)\}$ is a family of subsets of $V(G)$, called bags, such that (1) for all $v \in V(G)$, the set of nodes $T_v = \{i \in V(T) | v \in X_i\}$ induces a non-empty connected subtree of $T$, and (2) for each edge $uv \in E(G)$ there exists $i \in V(T)$ such that both $u$ and $v$ are in $X_i$. The maximum of $|X_i| - 1$, $i \in V(T)$, is called the width of the tree decomposition. The treewidth, $tw(G)$, of a graph $G$ is the minimum width over all tree decompositions of $G$.

An easy lower bound on the treewidth of a graph $G$ is the size of the largest clique in $G$, i.e., its clique number $\omega(G)$. This follows from the fact that each edge of $G$ belongs to some bag of $T$ and that a collection of pairwise intersecting subtrees of a tree must have a common intersection (i.e., they satisfy the Helly property). With this in mind, we say that a graph class $\mathcal{G}$ has the clique-treewidth property\footnote{In our prior work [CZ17], we referred to this as being treewidth-bounded, but have changed the name to be consistent with other parameter-treewidth bounds given in bidimensionality theory [DFHT04].} if there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $G \in \mathcal{G}$, $tw(G) \leq f(\omega(G))$. This concept generalizes the idea of $\mathcal{G}$ being $\chi$-bounded, namely, that the chromatic number $\chi(G)$ of every graph $G \in \mathcal{G}$ is bounded by a function of the clique number of $G$. In particular, the chromatic number of a graph $G$ is bounded by its treewidth since a tree decomposition $(X, T)$ of $G$ is a $T$-representation of a chordal supergraph $G'$ of $G$ where $\omega(G') = tw(G) + 1$, i.e., $\chi(G') = tw(G) + 1$ since chordal graphs are perfect. It was recently shown that the graphs which do not contain even holes (i.e., cycles of length $2k$ for any $k \geq 2$) and pans (i.e., cycles with a single pendant vertex attached) as induced subgraphs have their treewidth bounded by $f(\omega) = 3\omega/2 - 1$ [CCH13]. For a function $f : \mathbb{N} \rightarrow \mathbb{N}$, we use $\mathcal{G}_f$ to denote the class of graphs $G$ where $tw(G) \leq f(\omega(G))$. Each class $H$-GRAPH is known to be a subclass of $\mathcal{G}_f$ for certain linear functions $f$, as in the following lemma.

Lemma 7.1 (Bíró, Hujter, and Tuza [BHT92]). For every $G \in H$-GRAPH, $tw(G) \leq (tw(H) + 1) \cdot \omega(G) - 1$, i.e., $H$-GRAPH is a subclass of $\mathcal{G}_{f_H}$, where $f_H(\omega) = (tw(H) + 1) \cdot \omega - 1$.

We leverage any clique-treewidth-property (e.g., as in Lemma 7.1 together with some existing algorithms to to classify the k-coloring and k-clique problems as FPT on the $\mathcal{G}_f$ classes (e.g., on $H$-GRAPH classes as well). We first consider the $k$-clique problem.
**Theorem 11.** For any monotone computable function $f : \mathbb{N} \rightarrow \mathbb{N}$, the $k$-clique problem can be solved in $2^{O(f(k))} \cdot n$ time for $G \in \mathcal{G}_f$. Thus, for $H$-GRAPH, the $k$-clique problem can be solved in $2^{O(tw(H) \cdot k)} \cdot n$ time.

Proof. To test if $G$ contains a $k$-clique, we first try to generate a tree decomposition of $G$ with width roughly $f(k)$ via a recent algorithm [BDD+16], which, for any given graph $G$ and number $t$, provides a tree decomposition of width at most $5 \cdot t$ or states that the treewidth of $G$ is larger than $t$. This algorithm runs in $2^{O(t)} \cdot n$ time. If this algorithm does not produce a tree decomposition, then $G$ must contain a $k$-clique, and we are done. Otherwise, we obtain a tree decomposition $(X, T)$ of $G$ of width $5 \cdot f(k)$. Note that, an easy property of tree decompositions is that, for every clique $K$, there is a bag which contains the vertices of $K$. In particular, to check if $G$ has a $k$-clique it suffices to check whether each the subgraph induced by a bag of $G$ contains a $k$-clique. This can obviously be done in $2^{O(f(k))} \cdot n$ time by brute-force. Thus, we have $2^{O(f(k))} \cdot n$ time in total as needed.

For each fixed $k \geq 3$, it is known that testing $(k, k)$-pre-colouring extension (see Section 1.1 for a definition) for $G \in H$-GRAPH can be done in XP time [BI192]. The authors combine Lemma 7.1 together with a simple argument to obtain the result. We use a similar argument together with a more recent result regarding bounded treewidth graphs to observe that an even more general problem, list $k$-coloring (where each list is a subset of $\{1, \ldots, k\}$), is FPT on graph class satisfying the clique-treewidth property, and therefore, also on $H$-GRAPH.

**Theorem 12.** For any monotone computable function $f : \mathbb{N} \rightarrow \mathbb{N}$, the list-$k$-coloring problem can be solved in $k^{O(f(k))} \cdot n$ time for $G \in \mathcal{G}_f$. Thus, for $H$-GRAPH, the list-$k$-coloring problem can be solved in $k^{O(tw(H) \cdot k)} \cdot n$ time.

Proof. For fixed $k$, clearly, if $G$ contains a clique of size $k + 1$ then $G$ has no list-$k$-coloring, i.e., no list-$k$-coloring, regardless of the lists. We use Theorem 11 to test for such a clique, and reject if one is found. Otherwise, we have a $5 \cdot f(k)$-width tree decomposition, and this time we use it to solve the list-$k$-coloring problem via the known $O(k^{f(k)+2} \cdot n)$-time algorithm when given a width $t$ tree decomposition [JS07]. Thus, list-$k$-coloring can be solved in $(2^{O(f(k))} + k^{O(f(k))}) \cdot n$ time on $\mathcal{G}_f$.

Some further natural open questions remain regarding these results. For example, what other problems can be approached on graph classes with the clique-treewidth property? Can we obtain polynomial-size kernels for the $k$-clique or list-$k$-coloring problems on $H$-GRAPH or more generally on graph classes with the clique-treewidth property? The kernelization question has already been partially answered for the $k$-clique problem. Namely, on $H$-graphs, it was recently shown [FGR20] that the $k$-clique admits a polynomial kernel in terms of $\|H\|$ and $k$, but the kernelization requires an $H$-representation to be given as part of the input. In contrast, our FPT algorithm for $k$-clique (while also parameterized by both $\|H\|$ and $k$) does not need an $H$-representation.

**Problem 8.** Can the kernelization for $k$-clique be done without an $H$-representation a part of the input?

**Problem 9.** Can we obtain polynomial-size kernel for the list-$k$-coloring problem on $H$-graphs?
8 Minimal separators

For a connected graph $G$, a subset $S$ of $V(G)$ is a minimal separator when $G$ has vertices $u$ and $v$ belonging to distinct components of $G - S$ such that no proper subset of $S$ disconnects $u$ and $v$ – here, we say that $S$ is minimal $(u,v)$-separator. We denote the set of all minimal separators in $G$ by $S(G)$. Minimal separators are a commonly studied aspect of many graph classes [BT01, GM18, Gol04, KKW98]. Two particularly relevant cases include the fact that chordal graphs have at most $n$ minimal separators [Gol04], and that circular-arc graphs have at most $2n^2 - 3n$ minimal separators [KKW98].

Recently, several algorithmic results have been developed, where the runtime depends on the number of minimal separators in the input graph. The main result in this direction is the one by Fomin, Todorov, and Villanger [FTV15], which is phrased in terms of potential maximal cliques, but can also be phrased in terms of minimal separators since the number of potential maximal cliques in a graph $G$ is bounded by $n|S(G)|^2$ (see Proposition 2.8 in [FTV15]). Roughly, in [FTV15] the authors show that a large class of problems can be solved in time polynomial in the number of minimal separators of the input graph. These problems include several standard combinatorial optimization problems, e.g., maximum independent set and maximum induced forest.\

The class of problems considered in [FTV15] is formalized as follows. Consider a fixed graph $H$, the goal is to find a maximum size subset $X \subseteq V(G)$ satisfying: there is $F \subseteq V(G)$ such that $X \subseteq F$, the subgraph $G[F]$ has treewidth at most $t$, and the structure $G[F]$, $X$ models $\varphi$. The graph $G[X]$ is called maximum induced subgraph of treewidth $\leq t$ satisfying $\varphi$. The main result of [FTV15] is that this problem can be solved in time $O(|S|^2 n^{t+5} f(t, |\varphi|))$ where $f$ is a computable function.

Now, we prove that each $H$-graph has $n^{O(|H|)}$ minimal separators; see Theorem 13. We obtain Corollary 14 by applying the meta-algorithmic result of Fomin, Todorov, and Villanger. Subsequently, we consider the case of $H$-graphs when $H$ is a cactus and observe a much smaller bound on the number of minimal separators, in particular, $O(|H|^2 n^2)$; see Theorem 15. Similarly, by applying the meta-algorithmic result we obtain Corollary 16 for cactus-graphs, the maximum induced subgraph of treewidth $t$ modelling $\varphi$ can be solved in polynomial time.

Theorem 13. Let $G$ be a connected $H$-graph. Then $G$ has $n^{O(|E(H)|)}$ minimal separators. \(^8\)

Proof. We show that each minimal separator arises from vertices of $G$ such that their representations contain a small number of edges of the subdivision $H'$. Then we count all such subsets of edges of $H'$.

Let $H'$ be a subdivision of $H$ certifying that $G$ is an $H$-graph. Let $H^*$ be the subgraph of $H'$ formed by the union of the representations of the vertices of $G$, i.e.,

\[
H^* = \bigcup_{x \in V(G)} H'_x.
\]

\(^8\)A similar result with a slightly better bound is given in a recent manuscript, see [FGR20]. Our proof and theirs seem to follow similar reasoning, but have been obtained independently, as also noted in [FGR20].
Observe that, since \( G \) is connected, \( H^* \) must be also connected. Moreover, for any minimal \((u,v)\)-separator \( S \), the graph \( H^*_S = \bigcup_{x \in V(G) \setminus S} H'_x \) is not connected. Now, since \( S \) is an \((u,v)\)-separator, there are distinct components \( Z^*_u \) and \( Z^*_v \) of \( H^*_S \) such that \( H'_u \) is a subgraph of \( Z^*_u \) and \( H'_v \) is a subgraph of \( Z^*_v \).

Observe that, since \( S \) is minimal, then if \( x \in S \), then the representation \( H'_x \) contains an edge \( ab \) of \( H^* \) such that either \( a \in V(Z^*_u) \) and \( b \notin V(Z^*_u) \), or \( a \in V(Z^*_v) \) and \( b \notin V(Z^*_v) \). Namely, there is a set \( E_S \) of edges of \( H^* \) such that \( S \) is precisely the set of vertices \( x \) of \( G \) where \( H'_x \) contains an edge of \( E_S \). Moreover, for each edge of \( H \), at most two edges from its path in \( H' \) occur in \( E_S \).

To bound the number of all possible minimal separators in \( G \), it suffices to enumerate all possible subsets \( E \) of \( E(H') \) where, for each edge of \( H \), we pick at most two edges from its path in \( H' \). Hence, the candidate separator \( S \) would simply be all vertices \( x \) of \( G \) for which \( H'_x \) contains an edge of \( E \). Thus, since each edge of \( H \) will be subdivided at most \( 2n - 1 \) times (since \( 2n \) nodes are sufficient to accommodate any circular-arc representation), we obtain that the number of minimal separators in \( G \) is at most

\[
\left( \binom{2n}{2} + \binom{2n}{1} + \binom{2n}{0} \right)^{|E(H)|} = n^O(|E(H)|).
\]

\[\square\]

**Corollary 14.** Let \( H \) be a fixed graph. For every \( H \)-graph \( G \), \( t \geq 0 \), and every CMSO formula \( \varphi \), a maximum induced subgraph of treewidth \( \leq t \) modelling \( \varphi \) can be found in time \( O(n^{c|E(H)|} n^{t+5} f(t, \varphi)) \), where \( c \) is a constant and \( f \) is a computable function.

**Theorem 15.** Let \( G \) be a connected \( C \)-graph, where \( C \) is a cactus graph. Then \( G \) has at most \( |E(C)| (2n^2 + n) \) minimal separators.

**Proof.** The reasoning here follows similarly to the proof of Theorem 13. Namely, if we consider a minimal \((u,v)\)-separator, we again find the components \( Z^*_u \) and \( Z^*_v \) in the subdivision \( H' \). However, since \( H' \) is a cactus, we can now look more closely at the edges which are incident to \( Z^*_u \) and \( Z^*_v \) but contained in neither. In particular, it is easy to see that among all such edges incident to \( Z^*_u \), there are at most two edges \( e_1, e_2 \) which are actually important to ensure that there is no path from \( H'_u \) to \( H'_v \). In other words, our set \( E_S \) consists of at most two edges of \( H' \). Moreover, these two edges must belong to the same cycle of \( H' \). Finally, since each cycle of \( H' \) forms a circular-arc graph, it never needs to contain more than \( 2n \) nodes, i.e., also \( 2n \) edges. Thus, since \( H \) contains at most \( |E(H)| \) cycles, the number of minimal separators in \( G \) is at most

\[
|E(H)| \left( \binom{2n}{2} + \binom{2n}{1} \right) \leq |E(H)| (2n^2 + n).
\]

\[\square\]

**Corollary 16.** Let \( C \) be a cactus. For every \( G \in C\text{-GRAPH} \), \( t \geq 0 \) and every CMSO formula \( \varphi \), a maximum induced subgraph of treewidth \( \leq t \) modelling \( \varphi \) can be found in time \( O(|E(C)|^2 n^{t+9} f(t, \varphi)) \), where \( f \) is a computable function.

As we have mentioned, two recent manuscripts [FGR20, JKT20] have obtained \( W[1] \)-hardness results for both the maximum independent set problem and the minimum feedback vertex set problem (respectively) when parameterized by \( |H| \) and the solution size. In both results, the graphs \( H \) which are used have progressively larger clique minors. These indicate that the XP-time results of Corollary 14 are extremely unlikely to be improved to FPT-time, even when adding the solution size as an additional parameter. On the other hand, as in Corollary 16.
when $H$ is a cactus (i.e., diamond-minor free), these problems (and many more) can be solved in polynomial time in both $\|H\|$ and the size of the input graph.

**Problem 10.** For which classes $\mathcal{H}$ (besides the cacti), can one similarly bound the number of minimal separators by a polynomial in terms of $\|H\|$ and $\|G\|$ where $H \in \mathcal{H}$ and $G$ is an $H$-graph?

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