Likelihoods and Parameter Priors for Bayesian Networks

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Abstract
We develop simple methods for constructing likelihoods and parameter priors for learning about the parameters and structure of a Bayesian network. In particular, we introduce several assumptions that permit the construction of likelihoods and parameter priors for a large number of Bayesian-network structures from a small set of assessments. The most notable assumption is that of likelihood equivalence, which says that data cannot help to discriminate network structures that encode the same assertions of conditional independence. We describe the constructions that follow from these assumptions, and also present a method for directly computing the marginal likelihood of a random sample with no missing observations. Also, we show how these assumptions lead to a general framework for characterizing parameter priors of multivariate distributions.

Keywords: Bayesian network, learning, likelihood equivalence, Dirichlet, normal-Wishart.

Corrections to the original text in red are taken from J. Kuipers, G. Moffa, and D. Heckerman, Addendum on the scoring of Gaussian directed acyclic graphical models. *Annals of Statistics* 42, 1689-1691, Aug 2014. Other updates to the original are in blue.

1 Introduction

A Bayesian network is a graphical representation of a probabilistic model that most people find easy to construct and interpret (Howard, 1981; Pearl, 1988). Over the last decade, the Bayesian network has become a popular representation for encoding uncertain expert knowledge in expert systems (Heckerman et al., 1995a). More recently, researchers have developed methods for learning Bayesian networks from a combination of expert knowledge and data. In this paper, we describe methods for assigning likelihoods and parameter priors for this learning problem.

Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ be a set of random variables. We use $x_i$ to denote a value of $X_i$ and $y = (x_i)_{X_i \in \mathbf{Y}}$ to denote a value of $\mathbf{Y} \subseteq \mathbf{X}$. A Bayesian network for $\mathbf{X}$ is a graphical factorization of the joint probability distribution of $\mathbf{X}$. The representation consists of two components: a structure and a set of local probability distributions. The structure $S$ for $\mathbf{X}$ is a directed acyclic graph that represents a set of conditional-independence assertions about the variables $\mathbf{X}$. Namely, let $(X_1, \ldots, X_n)$ be an ordering of $\mathbf{X}$. For every variable $X_i$ in $\mathbf{X}$, there is a corresponding node $X_i$ in $S$, and a set $\text{Pa}_S^X \subseteq \{X_1, \ldots, X_{i-1}\}$ corresponding to the parents of $X_i$ in $S$.\footnote{We use the same symbol for a variable and its corresponding node.} The structure $S$
for $X$ represents the assertions that, for $i = 2, \ldots, n$, $X_i$ and $\{X_1, \ldots, X_{i-1}\} \setminus Pa_i$ are independent given $Pa_i^S$. That is,

$$p(x) = \prod_{i=1}^{n} p(x_i|Pa_i^S)$$

(1)

The local distributions associated with the Bayesian network are precisely those in Equation 1.

In this discussion, we assume that the local distributions depend on a finite set of parameters $\theta_s \in \Theta_s$. Thus, we rewrite Equation 1 as follows:

$$p(x|\theta_s) = \prod_{i=1}^{n} p(x_i|Pa_i^S, \theta_i)$$

(2)

where $\theta_s = (\theta_1, \ldots, \theta_n)$. We assume that $\Theta_s$ is absolutely continuous.

Let $S^h$ denote the assertion or hypothesis that the joint distribution of $X$ can be factored according to the structure $S$. That is, define $S^h$ to be true if there exists $\theta_s \in \Theta_s$, where $\Theta_s$ is absolutely continuous, such that Equation 2 holds. It will be useful to include this hypothesis explicitly in the factorization of the joint distribution. In particular, we write

$$p(x|\theta_s, S^h) = \prod_{i=1}^{n} p(x_i|Pa_i^S, \theta_i, S^h)$$

(3)

This notation often makes it unnecessary to use the superscript $S$ in the term $Pa_i^S$, and we use the simpler expression where possible.

Let us consider the situation where both the parameters $\theta_s$ and the structure hypothesis $S^h$ are uncertain. Given data $D = \{x_1, \ldots, x_m\}$, a random sample from $p(X|\theta_s, S^h)$ where $\theta_s$ and $S^h$ are the true parameters and structure hypothesis, respectively, we can compute the posterior probability of an arbitrary structure hypotheses $S^h$ using

$$p(S^h|D) = c p(S^h) p(D|S^h) = c p(S^h) \int p(D|\theta_s, S^h) p(\theta_s|S^h) d\theta_s$$

(4)

where $c$ is a normalization constant. We can then select a model (i.e., structure) that has a high posterior probability or average several good models for prediction. Methods for searching through the space of Bayesian-network structures are discussed by Cooper and Herskovits (1992), Aliferis and Cooper (1994), and Heckerman et al. (1995b).

A difficulty with this approach arises when many network structures are possible. In this case, we need to assign likelihoods, structure priors, and parameter priors to a large number of (if not all possible) structures to enable a search among these models. Buntine (1991) and Heckerman et al. (1995b) discuss methods for determining structure priors from a small number of direct assessments.

In this paper, we develop practical methods for assigning likelihoods and parameter priors to a large number of structures. In particular, we describe a set of assumptions under which likelihoods and parameter priors can be determined by a relatively small number of direct assessments. We show how likelihoods and priors are constructed and how marginal likelihoods $p(D|S^h)$ are computed from these assessments. Some of our assumptions are abstracted from those made previously by researchers who examined cases where the local likelihoods are unrestricted discrete distributions (Cooper and Herskovits, 1992; Spiegelhalter et al., 1993; Heckerman et al., 1995b) and linear-regression models (Geiger and Heckerman, 1994; Heckerman and Geiger, 1995). The most notable assumptions are global parameter independence, which says that the parameter variables $\Theta_1, \ldots, \Theta_n$ are mutually independent, and likelihood equivalence, which (roughly speaking) says that data can not help to discriminate structures that encode the same assertions of conditional independence.

An important outgrowth of our work is a framework for characterizing prior distributions for the parameters of multivariate distributions. In particular, for a given family of local likelihoods
the application of parameter independence and likelihood equivalence yields a functional equation, the solution to which delimits all allowed prior distributions under these assumptions. For the likelihoods that we have studied, these solutions correspond to well-known distribution families. Namely, when likelihoods are unrestricted discrete distributions, the solution to the functional equation is the Dirichlet distribution. When $X$ contains two variables and likelihoods are linear-regression models, the only solution to the functional equation is the bivariate normal-Wishart distribution.

2 Examples

We illustrate the ideas in this paper using two standard probability distributions, which we review in this section. In the first case, each variable $X_i \in X$ is discrete, having $r_i$ possible values $x_i^1, \ldots, x_i^{r_i}$.

Each local likelihood is an unrestricted discrete distribution

$$p(x_i^k|\mathbf{pa}_i, \theta_i, S^h) = \theta_{x_i^k|\mathbf{pa}_i}^h \equiv \theta_{ijk}$$

(5)

where $\mathbf{pa}_i^1, \ldots, \mathbf{pa}_i^q \ (q_i = \prod_{X_j \in \text{pa}_i} r_j)$ denote the values of $\mathbf{pa}_i$. The local parameters are given by $\theta_i = ([\theta_{ijk}]_{k=1}^{r_i})_{j=1}^{|\mathbf{pa}_i|}$. We assume that each parameter $\theta_{ijk}$ is greater than zero.

In the second case, each variable $X_i \in X$ is continuous, and each local likelihood is the linear-regression model

$$p(x_i|\mathbf{pa}_i, \theta_i, S^h) = N(x_i|m_i + \sum_{j \in \mathbf{pa}_i} b_{ji}x_j, 1/v_i)$$

(6)

where $N(x_i|\mu, \tau)$ is a normal distribution with mean $\mu$ and precision $\tau > 0$. Given this form, a missing arc from $X_j$ to $X_i$ implies that $b_{ji} = 0$ in the full regression model. The local parameters are given by $\theta_i = (m_i, b_i, v_i)$, where $b_i$ is the column vector $(b_{i1}, \ldots, b_{i-1,i})$. We call a Bayesian network constructed with these likelihoods a Gaussian network after Shachter and Kenley (1989).

3 Simplifying Assumptions

In this section, we present assumptions that simplify the assessment of likelihoods and prior distributions. In this explication, we consider situations where all structure hypotheses for $X$ are possible—that is, $p(S^h) > 0$ for all $S$ for $X$.

The first assumption, already mentioned, is that $\Theta_1, \ldots, \Theta_n$ are mutually independent.

**Assumption 1 (Global Parameter Independence)** *Given any structure $S$ for $X$,*

$$p(\theta|S^h) = \prod_{i=1}^n p(\theta_i|S^h)$$

Spiegelhalter and Lauritzen (1990) introduced this assumption in the context Bayesian networks under the name global independence.

Roughly speaking, the next two assumptions capture the notion that the likelihoods and priors are modular in the sense that these quantities for variable $X_i \in X$ depend only the structure that is local to $X_i$—namely, the parents of $X_i$—and not on the entire structure.

**Assumption 2 (Likelihood Modularity)** *Given any structure $S$ for $X$,*

$$p(x_i|\mathbf{pa}_i^S, \theta_i, S^h) = p(x_i|\mathbf{pa}_i^S, \theta_i)$$

(7)

for all $X_i \in X$.

2When we refer to an arbitrary value of $X_i$, we drop the superscript.
Assumption 3 (Prior Modularity) Given any two structures \( S_1 \) and \( S_2 \) for \( X \) such that \( X \) has the same parents in \( S_1 \) and \( S_2 \),
\[
p(\theta_i | S_1^h) = p(\theta_i | S_2^h)
\]
Both assumptions have been used implicitly in the work of (e.g.) Cooper and Herskovits (1992), Spiegelhalter et al. (1993), and Buntine (1994). Heckerman et al. (1995b) made Assumption 3 explicit under the name parameter modularity.

The assumption of likelihood modularity holds in our examples (see Equations 5 and 6). To illustrate the assumption of prior modularity, consider the set of binary variables \( X = \{X, Y\} \). In both the structures \( S_{x\rightarrow y} \) (\( X \rightarrow Y \)) and \( S_{xy} \) (no arc between \( X \) and \( Y \)), the node \( X \) has the same parents (none). Consequently, by prior modularity, we have that \( p(\theta_x | S_{x\rightarrow y}^h) = p(\theta_x | S_{xy}^h) \).

The next two assumptions relate to the notion of structure equivalence. Consider the two structures \( X_1 \rightarrow X_2 \rightarrow X_3 \) and \( X_1 \leftarrow X_2 \leftarrow X_3 \). Both structures represent the assertion that \( X_1 \) and \( X_3 \) are conditionally independent given \( X_2 \) and no other assertions of independence. In general, we say that two structures for \( X \) are independence equivalent if they represent the same assertions of conditional independence. Independence equivalence is an equivalence relation, and induces a set of equivalence classes over the possible structures for \( X \). Verma and Pearl (1990) provides a simple characterization of independence-equivalent structures. Given a structure \( S \), a v-structure in \( S \) is an ordered node triple \((X_i, X_j, X_k)\) where \( S \) contains the arcs \( X_i \rightarrow X_j \) and \( X_j \leftarrow X_k \), and there is no arc between \( X_i \) and \( X_k \) in either direction.

Theorem 1 (Verma and Pearl, 1990) Two structures for \( X \) are independence equivalent if and only if they have identical edges and identical v-structures.

This characterization makes it easy to identify independence equivalent structures. The following characterization by Chickering (1995) is useful for proving technical claims about independence equivalence. An arc reversal is a transformation from one structure to another, in which a single arc between two nodes is reversed. An arc between two nodes is said to be covered if those two nodes would have the same parents if the arc were removed.

Theorem 2 (Chickering, 1995) Two structures for \( X \) are independence equivalent if and only if there exists a set of covered arc reversals that transform one structure into the other.

A concept related to that of independence equivalence is that of distribution equivalence. As is typically done is practice, we assume that the local likelihoods \( p(x_1 | pa_i, \theta_i, S^h) \) are restricted to some family of probability distributions \( \mathcal{F} \). Then, \( S_1 \) and \( S_2 \) are distribution equivalent with respect to (wrt) \( \mathcal{F} \) if the two structures represent the same set of distributions—that is, for every \( \theta_{i1} \), there exists a \( \theta_{i2} \) such that \( p(x_1 | \theta_{i1}, S_1^h) = p(x_1 | \theta_{i2}, S_2^h) \), and vice versa.

Distribution equivalence wrt some \( \mathcal{F} \) implies independence equivalence, but the converse does not hold. Nonetheless, if all structures that differ by a single arc reversal are distribution equivalent wrt \( \mathcal{F} \), then by Theorem 2, independence equivalence implies distribution equivalence. We adopt this assumption formally as follows.

Assumption 4 (Covered-Arc-Reversal Equivalence) Given local likelihoods restricted to \( \mathcal{F} \), any two structures for \( X \) that differ by a single covered arc reversal are distribution equivalent wrt \( \mathcal{F} \).

Assumption 4 holds trivially in the discrete case. Shachter and Kenley (1989) show that Assumption 4 holds in the linear-regression case. A case where Assumption 4 does not hold is one where \( X \) consists of three or more binary variables and the local likelihoods are restricted to the sigmoid function
\[
p(x_i | pa_i, \theta_i, S^h) = \frac{1}{1 + \exp \left(a_i + \sum_{j \in pa_i} b_{ji} x_j \right)}
\]
where $\theta_i = (a_i, b_i)$. For example, with $X = \{X_1, X_2, X_3\}$, suppose $S_1$ is the structure with arcs $X_1 \rightarrow X_2$, $X_1 \rightarrow X_3$, and $X_2 \rightarrow X_3$, and $S_2$ is the structure with arcs $X_1 \rightarrow X_2$, $X_1 \rightarrow X_3$, and $X_3 \rightarrow X_2$. Then, $S_1$ and $S_2$ differ by the reversal of a covered arc between $X_2$ and $X_3$, but, given the sigmoid restriction, there are certain joint likelihoods that can be represented by one structure, but not the other.

We assume that the parameters $\theta_i$ are uniquely determined given the local likelihood $p(x_i|pa^S_i, \theta_i)$ (i.e., all parameters are identified). Consequently, given two structures $S_1$ and $S_2$ that are distribution-equivalent wrt $\mathcal{F}$, for every $\theta_{s_1}$, there exists a unique $\theta_{s_2}$ such that $p(x|\theta_{s_1}, S^h_1) = p(x|\theta_{s_2}, S^h_2)$. That is, there is a one-to-one mapping from $\theta_{s_1}$ to $\theta_{s_2}$, which we write $\theta_{s_2} = \Theta_{s_2}(\theta_{s_1})$. Also, we assume that, the Jacobian $|\partial \theta_{s_1}/\partial \theta_{s_2}|$ exists and is non-zero for all values of $\theta_{s_1}$. These technical assumptions hold for our examples.

Given Assumption 4 and these technical assumptions, we can make the following assumption.

**Assumption 5 (Marginal Likelihood Equivalence)** Given any two independence-equivalent structures $S_1$ and $S_2$ for $X$,

$$p(x|\theta_{s_2}(\theta_{s_1}), S^h_2) = p(x|\theta_{s_1}, S^h_1)$$

and

$$p(\theta_{s_2}|S^h_2) = \frac{|\partial \theta_{s_1}|}{\partial \theta_{s_2}} p(\theta_{s_1}|S^h_1)$$

for all values of $\theta_{s_1}$.

An immediate consequence of this assumption is that, given any two independence-equivalent structure $S_1$ and $S_2$, $p(D|S^h_1) = p(D|S^h_2)$—hence, the name “marginal likelihood equivalence.” For the sake of brevity, we often refer to this assumption as likelihood equivalence.

Given our definition of $S^h$ in the introduction, likelihood equivalence follows from covered-arc-reversal equivalence. In particular, by this definition, $S^h$ is nothing more than a constraint on the possible joint likelihoods. Furthermore, given Assumption 4, whenever $S_1$ and $S_2$ are independence equivalent, $S^h_1$ and $S^h_2$ correspond to the same constraint on the possible joint likelihoods. Thus, if $S^h_1$ and $S^h_2$ are independence equivalent, then $S^h_1 = S^h_2$. This property, which we call hypothesis equivalence, implies likelihood equivalence.

Nonetheless, some researchers give Bayesian-network structure a causal interpretation (e.g., Spirtes et al., 1993; Pearl, 1995). In this case, we can modify the definition of $S^h$ to include the assertion that if $X_i \rightarrow X_j$ in $S$, then $X_j$ is a direct cause of $X_i$. Consequently, hypothesis equivalence does not hold. Nonetheless, the weaker assumption of likelihood equivalence is sometimes reasonable. For a detailed discussion of this point, see Heckerman (1995). To allow for the causal interpretation of structure, we take likelihood equivalence to be an assumption.

We close this section with a few observations about the acausal interpretation of structure where hypothesis equivalence holds. Given this property, we can think of a model as an equivalence class of structures rather than an individual structure. Thus, for example, we can search for good models by searching through the space of equivalence classes. Spirtes and Meek (1995) and Chickering (1995) describe such search methods. Also, when we make a prediction by averaging models, we can average over equivalence classes.

Model averaging raises another important point. To average model predictions, the structure hypotheses should be mutually exclusive. Without the assumption that parameters are absolutely continuous, however, hypotheses are not mutually exclusive. For example, given $X = \{X, Y\}$, both structure hypotheses $S^h_{x \rightarrow y}$ and $S^h_{x \leftarrow y}$ include the case where $X$ and $Y$ are independent. In fact, given $S^h_{x \rightarrow y}$, if we assign non-zero priors only to those parameter values that encode independence between $X$ and $Y$, then $S^h_{x \rightarrow y}$ implies $S^h_{x \leftarrow y}$. Fortunately, our technical assumption implies that, if $S_1$ and $S_2$ are not independence equivalent, then $S^h_1$ and $S^h_2$ are mutually exclusive, that is $p(S^h_1, S^h_2) = 0$. 


In particular, assume that $S_1$ and $S_2$ are not independence equivalent. Given the definition of structure hypothesis, we have that

$$p(S_2^h|S_1^h, \theta_{s1}) = \begin{cases} 1 & \exists \theta_{s2} \text{ s.t. } p(x|\theta_{s2}, S_2^h) = p(x|\theta_{s1}, S_1^h) \\ 0 & \text{otherwise} \end{cases}$$

If $p(S_2^h|S_1^h, \theta_{s1}) = 0$ almost everywhere in $\Theta_{s1}$, then, because $\Theta_{s1}$ is absolutely continuous, we can construct the prior

$$p(S_2^h|S_1^h) = \int p(S_2^h|\theta_{s1}, S_1^h) p(\theta_{s1}|S_1^h) d\theta_{s1} = 0$$

Thus, mutually exclusivity holds because, by assumption, $p(S_2^h|S_1^h, \theta_{s1}) > 0$. If $p(S_2^h|S_1^h, \theta_{s1}) \neq 0$ almost everywhere in $\Theta_{s1}$, then the independencies encoded by $S_2$ must be a proper subset of those encoded by $S_1$. (E.g., $S_1 = S_{xy}$ and $S_2 = S_{x \rightarrow y}$.) In this case, given $S_1^h$, the parameters $\Theta_{s2}$ cannot be absolutely continuous, and $p(S_2^h|S_1^h) = 0$.

### 4 The Construction of Likelihoods Parameter Priors

Given Assumptions 1 through 5 and the technical assumptions, we can construct the likelihoods and parameter priors for all structures for $X$ from a small number of assessments. In this section, we describe these constructions. An important concept in this approach is that of a complete structure: one that has no missing arcs. All complete structures for a given $X$ are independence equivalent.

First, let us consider likelihoods.

**Theorem 3** Given Assumptions 2, 4, and 5, we can construct the likelihood $p(x|\theta_x, S^h)$ for any structure $S$ given $p(x|\theta_{sc}, S^h_c)$ for any complete structure $S_c$.

**Proof:** Given any structure $S$ for $X$, we construct the likelihoods $p(x_i|\pa_i, \theta_i, S^h)$ as follows. For each $i = 1, \ldots, n$, we find a complete structure $S_{ci}$ such that $\pa_i^S = \pa_i^{S_{ci}}$. Then, we use likelihood equivalence to compute $p(x_i|\theta_{sc}, S^h_{ci})$ and hence $p(x_i|\pa_i, \theta_i, S^h_{ci})$ from $p(x_i|\theta_{sc}, S^h_c)$. Using likelihood modularity, we obtain $p(x_i|\pa_i, \theta_i, S^h) = p(x_i|\pa_i, \theta_i, S^h_{ci})$. \hfill \Box

The construction of parameter priors is similar.

**Theorem 4** Given Assumptions 1, 3, 4, and 5, we can construct the prior $p(\theta_x|S^h)$ for any structure $S$ given $p(\theta_{sc}|S^h_c)$ for any complete structure $S_c$.

**Proof:** Given any structure $S$ for $X$, we determine $p(\theta_x|S^h)$ by computing $p(\theta_x|S^h_i)$, $i = 1, \ldots, n$, and then applying global parameter independence. To determine $p(\theta_i|S^h)$, we identify a complete structure such that $\pa_i^S = \pa_i^{S_{ci}}$. Then, we use likelihood equivalence to compute $p(\theta_{sc}|S^h_c)$ from $p(\theta_{sc}|S^h_i)$. Next, we apply global parameter independence to obtain $p(\theta_i|S^h_{ci})$. Finally, we use prior modularity, which gives $p(\theta_i|S^h) = p(\theta_i|S^h_{ci})$. \hfill \Box

Given likelihood equivalence, we can compute $p(x|\theta_{sc}, S^h_c)$ and $p(\theta_{sc}|S^h_c)$ for one complete structure from the likelihood and prior for another complete structure. In so doing, we are simply performing coordinate transformations between parameters for different variable orderings in the factorization of the joint likelihood. Thus, likelihood equivalence raises the possibility of defining a unique joint likelihood $p(x|\theta_x, S^h)$ whose parameters $\theta_x$ are variable-order independent. Given this likelihood and the corresponding prior $p(\theta_x|S^h)$, we can compute $p(x|\theta_{sc}, S^h_c)$ and $p(\theta_{sc}|S^h_c)$ for any complete structure and, using the techniques described previously in this section, the likelihoods and parameters priors for any structure. Note that the choice of $p(x|\theta_{sc}, S^h_c)$ must be consistent with global parameter independence and likelihood equivalence. We address this issue in Section 7.

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3To simplify the presentation, we omit explicit reference to the technical assumptions in the remainder of the paper.
Let us consider this approach for our discrete and linear-regression examples. In the discrete case, the joint likelihood is the $n$-dimensional multivariate-discrete distribution:

$$p(x_1, \ldots, x_n | \theta_x, S^h_x) = \theta_{x_1, \ldots, x_n}$$

where $\theta_x = (\theta_{x_1, \ldots, x_n})_{(x_1, \ldots, x_n) \in X}$. The one-to-one mapping from $\theta_{xc}$ for the complete structure with ordering $(X_1, \ldots, X_n)$ to $\theta_x$ is given by

$$\theta_{x_1, \ldots, x_n} = \prod_{i=1}^{n} \theta_{x_i|x_1, \ldots, x_{i-1}}$$

(8)

The Jacobian for this mapping (and its inverse) exist and are non-zero for all allowed values of $\theta_{xc}$. The Jacobian for the inverse mapping, derived by Heckerman et al. (1995b) is given by

$$\left| \frac{\partial \theta_x}{\partial \theta_{xc}} \right| = \prod_{i=1}^{n-1} \prod_{x_i = x_1, \ldots, x_{i-1}} [\theta_{x_i|x_1, \ldots, x_{i-1}} | \Pi_{j=i+1}^{n} r_j ]^{-1}$$

(9)

In the linear-regression example, the joint likelihood is the $n$-dimensional multivariate-normal distribution with mean $\mu$ and symmetric positive definite precision matrix $W$

$$p(x | \theta_x, S^h_x) = N_n(x | \mu, W)$$

The one-to-one mapping from $\theta_{sc}$ for the complete structure with ordering $(X_1, \ldots, X_n)$ to $\theta_x = \{ \mu, W \}$ is given by

$$\mu_i = m_i + \sum_{j=1}^{i-1} b_{ij} \mu_j$$

(10)

and the recursive formula

$$W(1) = \frac{1}{v_1}$$

$$W(i + 1) = \left( W(i) + \frac{b_{i+1,i} + b_{i+1}'}{v_{i+1}} - \frac{b_{i+1}}{v_{i+1}} \right)$$

(11)

where $W(i)$ is the $i \times i$ upper left submatrix of $W$ (e.g., Shachter and Kenley, 1989), and the prime symbol denotes transpose. The Jacobian for this mapping (and its inverse) exist and are non-zero for all allowed values of $\theta_{sc}$. Let $m = (m_1, \ldots, m_n)$, $v = (v_1, \ldots, v_n)$ and $B = (b_1, \ldots, b_n)$. The Jacobian for the mapping from $\mu$ to $m$ for a given $B$ is

$$\left| \frac{\partial \mu}{\partial m} \right| = 1$$

(12)

The Jacobian for the mapping from $W$ to $(v, B)$ is easily obtained from Equation 11:

$$\left| \frac{\partial W}{\partial v, B} \right| = \prod_{i=1}^{n} v_i^{-(i+1)}$$

(13)

To illustrate our techniques for constructing priors, consider again the simple two-binary-variable case $X = \{X, Y\}$. Suppose $p(\theta_x | S^h_{x \rightarrow y})$ is the Dirichlet distribution

$$p(\theta_x | S^h_{x \rightarrow y}) = c \prod_{xy} \theta_{xy}^{\alpha p(xy | S^h_{x \rightarrow y})}$$

7
where \( \alpha \) is an effective sample size and \( p(xy|S^h_{x\rightarrow y}) = \int \theta_{xy} p(\theta_{x}|S^h_{x\rightarrow y})d\theta_x \). That is, \( p(xy|S^h_{x\rightarrow y}) \) is the marginal likelihood that \( X = x \) and \( Y = y \) in a one-sample random sample. Given this prior, we compute \( p(\theta_{xy}|S^h_{x\rightarrow y}) \) for the structure containing no arc between \( X \) and \( Y \) as follows. First, we use Equations 8 and 9 to change variables to \( \theta \) obtaining

\[
p(\theta_{x\rightarrow y}|S^h_{x\rightarrow y}) = c \prod_x \theta_i^{\alpha p(x|S^h_{x\rightarrow y})} \prod_{xy} \theta_{xy}^{\alpha p(y|x,S^h_{x\rightarrow y})}
\]

Note that this prior satisfies global parameter independence. Integrating out all parameters except \( \theta \) and using prior modularity, we get

\[
p(\theta_{x}|S^h_{xy}) = c \prod_x \theta_i^{\alpha p(x|S^h_{x\rightarrow y})}
\]

Likewise, changing variables to \( \theta_{x\rightarrow y} \), integrating, and applying prior modularity, we have

\[
p(\theta_{y}|S^h_{xy}) = c \prod_y \theta_i^{\alpha p(y|S^h_{x\rightarrow y})}
\]

Equations 14 and 15 together with global parameter independence yield the desired prior.

## 5 Computation of the Marginal Likelihood for Complete Data

Given a structure \( S \), we can use the techniques developed in the last section to construct the likelihood and parameter prior, and then apply Equation 4 to compute the marginal likelihood \( p(D|S^h) \). In this section, we present a short cut to this approach for situations where the random sample \( D \) is complete (i.e., contains no missing observations). In particular, we derive a formula for the marginal likelihood of a complete random sample that bypasses the explicit construction of likelihoods and priors.

For a given \( X \), consider any structure \( S \) and complete random sample \( D \). Assuming global parameter independence, the parameters remain independent given complete data. That is,

\[
p(\theta_{i}|D,S^h_{i}) = \prod_{i=1}^n p(\theta_{i}|D,S^h_{i})
\]

In addition, assuming global parameter independence, likelihood modularity, and prior modularity, the parameters remain modular given complete data. In particular, if \( X_i \) has the same parents in \( S_1 \) and \( S_2 \), then

\[
p(\theta_{i}|D,S^h_{i}) = p(\theta_{i}|D,S^h_{i})
\]

Also, for any \( Y \subseteq X \), define \( D^Y \) to be the random sample \( D \) restricted to observations of \( Y \). For example, if \( X = \{X_1, X_2, X_3\}, \ Y = \{X_1, X_2\} \), and \( D = \{x_1 = \{x_{11}, x_{12}, x_{13}\}, x_2 = \{x_{21}, x_{22}, x_{23}\}\} \), then we have \( D^Y = \{\{x_{11}, x_{12}\}, \{x_{21}, x_{22}\}\} \). Let \( Y \) be a subset of \( X \), and \( S_c \) be a complete structure for any ordering where the variables in \( Y \) come first. Then, assuming global parameter independence and likelihood modularity, it is not difficult to show that

\[
p(Y|D,S^h_{c}) = p(Y|D^Y,S^h_{c})
\]

Given these observations, we can compute the marginal likelihood as follows.

**Theorem 5** Given any complete structure \( S_c \) for \( X \), any structure \( S \) for \( X \), and any complete random sample \( D \), Assumptions 1 through 5 imply

\[
p(D|S^h) = \prod_{i=1}^n \frac{p(D_{Pa_i} \cup \{X_i\}|S^h_{i})}{p(D_{Pa_i}|S^h_{c})}
\]
Proof: From the rules of probability, we have

\[ p(D|h) = \prod_{i=1}^{m} \int p(x_i|\theta_s, S^h) \, p(\theta_s|D_i, S^h) \, d\theta_s \]  

(20)

where \( D_i = \{x_1, \ldots, x_{i-1}\} \). Using Equations 3 and 16 to rewrite the first and second terms in the integral, respectively, we obtain

\[ p(D|h) = \prod_{i=1}^{m} \prod_{i=1}^{n} p(x_{id}|pa_{id}, \theta_i, S^h) \, p(\theta_i|D_i, S^h) \, d\theta_s \]

Using likelihood modularity and Equation 17, we get

\[ p(D|h) = \prod_{i=1}^{m} \prod_{i=1}^{n} p(x_{id}|pa_{id}, \theta_i, S^h_{ci}) \, p(\theta_i|D_i, S^h_{ci}) \, d\theta_s \]  

(21)

where \( S_{ci} \) is a complete structure with variable ordering \( Pa_i, X_i \) followed by the remaining variables. Decomposing the integral over \( \theta_i \) into integrals over the individual parameter sets \( \theta_i \), and performing the integrations, we have

\[ p(D|h) = \prod_{i=1}^{m} \prod_{i=1}^{n} p(x_{id}|pa_{id}, D_i, S^h_{ci}) \]

Using Equation 18, we obtain

\[ p(D|h) = \prod_{i=1}^{m} \prod_{i=1}^{n} \frac{p(x_{id}|pa_{id}, D_i, S^h_{ci})}{p(pa_{id}|D_i, S^h_{ci})} \]

\[ = \prod_{i=1}^{m} \prod_{i=1}^{n} \frac{p(x_{id}|pa_{id}, D_i, S^h_{ci})}{p(D^h_{pa_i}|S^h_{ci})} \]

(22)

By likelihood modularity and likelihood equivalence, we have that \( p(D|h_{ci}) = p(D|h) \), \( i = 1, \ldots, n \). Consequently, for any subset \( Y \) of \( X \), we obtain \( p(D^Y|h_{ci}) = p(D^Y|h) \) by summing over the variables in \( D^X \setminus Y \). Applying this result to Equation 22, we get Equation 19. □

To apply Equation 19, we assume a prior for \( p(\theta_x|S^h_{ci}) \) that is consistent with global parameter independence and likelihood equivalence, and use this prior to compute the individual terms in Equation 19. In the remainder of this section, we illustrate this approach for the discrete and linear-regression models. We demonstrate consistency in Section 7.

For the discrete model, we assume that \( p(\theta_x|S^h_{ci}) \) is the Dirichlet distribution:

\[ p(\theta_x|S^h_{ci}) = \prod_{x_1, \ldots, x_n} \theta^{(\alpha^p(x_1, \ldots, x_n)|S^h_{ci})-1} \]  

(23)

where \( \alpha \) is an effective sample size, and the probabilities are defined as they were in our two-variable example. It follows that, for any \( Y \subseteq X \), the parameter set \( \theta_Y \) also has a Dirichlet distribution:

\[ p(\theta_Y|S^h_{ci}) = \prod_{Y} \theta^{(\alpha^p(y)|S^h_{ci})-1} \]  

(24)
(e.g., DeGroot, 1970, p. 50). Furthermore, the marginal likelihood for \( D^Y \) is given by

\[
p(D^Y | S^h) = \frac{\Gamma(\alpha)}{\Gamma(\alpha + m)} \prod_y \frac{\Gamma(\alpha \cdot p(y|S^h_y) + N_y)}{\Gamma(\alpha \cdot p(y|S^h_y))}
\]

(25)

where \( m \) is the number of samples in \( D \), and \( N_y \) is the number of samples in \( D \) where \( Y = y \). Combining Equations 19 and 25, we obtain

\[
p(D | S^h) = \prod_{i=1}^n \prod_{j=1}^{q_i} \frac{\Gamma(\alpha_{ij})}{\Gamma(\alpha_{ij} + N_{ij})} \prod_{k=1}^{r_i} \frac{\Gamma(\alpha_{ijk} + N_{ijk})}{\Gamma(\alpha_{ijk})}
\]

(26)

where

\[
\alpha_{ijk} = \alpha \cdot p(x^h_i, p\alpha^j|S^h_c), \quad \alpha_{ij} = \sum_{k=1}^{r_i} \alpha_{ijk} = \alpha \cdot p(p\alpha^j|S^h_c)
\]

(27)

\( N_{ijk} \) is the number of samples where \( X_i = x^h_i \) and \( p\alpha^j = p\alpha^j \), and \( N_{ij} = \sum_{k=1}^{r_i} N_{ijk} \). Equations 26 and 27 were originally derived in Heckerman et al. (1994).

For the linear-regression model, we assume that \( p(\mu, W|S^h) \) is a normal-Wishart distribution. In particular, we assume that \( p(\mu|W, S^h) \) is a multivariate-normal distribution with mean \( \mu_0 \) and precision matrix \( \alpha \mu W \) (\( \mu > 0 \)); and that \( p(W|S^h) \) is a Wishart distribution with \( \alpha W \) degrees of freedom (\( \alpha W > n - 1 \)) and precision matrix \( T_0 \). Thus, the posterior \( p(\mu, W|D, S^h) \) is also a normal-Wishart distribution. In particular, \( p(\mu|W, D, S^h) \) is multivariate normal with mean vector \( \mu_m \) given by

\[
\mu_m = \frac{\alpha\mu\mu_0 + m\bar{x}_m}{\alpha\mu + m}
\]

(28)

and precision matrix \( (\alpha\mu + m)W \), where \( \bar{x}_m \) is the sample mean of \( D \); and \( p(W|D, S^h) \) is a Wishart distribution with \( \alpha W + m \) degrees of freedom and matrix \( T_m \) given by

\[
T_m = T_0 + S_m + \frac{\alpha\mu m}{\alpha\mu + m}((\mu_0 - \bar{x}_m)(\mu_0 - \bar{x}_m)^\prime)
\]

(29)

where \( S_m \) is the scatter matrix, given by \( \sum_{i=1}^m (X_i - \bar{x}_m)(X_i - \bar{x}_m)^\prime \) (e.g., DeGroot, 1970, p. 178). From these equations, we see that \( \alpha\mu \) and \( \alpha W \) can be thought of as effective sample sizes for the normal and Wishart components of the prior, respectively.

Given \( Y \subseteq X \) (\( |Y| = l \)), and vector \( z = (z_1, \ldots, z_n) \), let \( z^Y \) denote the vector formed by the components \( z_i \) of \( z \) such that \( X_i \in Y \). Similarly, given matrix \( M \), let \( M^{YY} \) denote the submatrix of \( M \) containing elements \( m_{ij} \) such that \( X_i, X_j \in Y \). If \( p(\mu, W|S^h) \) is a normal-Wishart distribution as we have described, then \( p(\mu^Y, ((W^{-1})^{YY})^{-1}|S^h) \) is also a normal–Wishart distribution with constants \( \mu_0^Y \), \( \alpha\mu^Y \), \( T_0^{YY} \), and \( W, Y = \alpha W + n + l \) (e.g., see Press, 1971, Theorems 5.1.3 and 5.1.4). Thus, we obtain the terms in Equation 19:

\[
p(D^Y | S^h) = \pi^{lm/2} \left( \frac{\alpha\mu}{\alpha\mu + m} \right)^{l/2} \frac{c(l, \alpha W, Y + m)}{c(l, \alpha W, Y)} |T_0^{YY}|^{-\alpha W Y + m} |T_m^{YY}|^{-\alpha W Y + m}
\]

(30)

where

\[
c(l, \alpha) = \prod_{i=1}^l \Gamma \left( \frac{\alpha + 1 - i}{2} \right)
\]

(31)

(See Geiger and Heckerman, 1994, for a derivation when \( l = n \).)
6 Priors from a Prior Bayesian Network

Whether we construct priors explicitly or use the short cut described in the previous section, we require \( p(\theta_x | S^h_c) \) to compute marginal likelihoods. In this section, we discuss the assessment of this distribution for our example models. We describe one of multiple alternatives that makes use of what we call a prior Bayesian network.

In the discrete case, we can assess \( p(\theta_x | S^h_c) \) by assessing (1) \( p(x | S^h_c) \) and (2) the effective sample size \( \alpha \). Methods for assessing \( \alpha \) are discussed in (e.g.) Heckerman et al. (1995b). To assess \( p(x | S^h_c) \), we construct a Bayesian network for \( x \) given \( S^h_c \), a prior Bayesian network. We can then derive the \( \alpha_{ijk} \) using Equation 27.

For the linear-regression case, we directly assess the effective sample sizes \( \alpha_{\mu} \) and \( \alpha_W \), and indirectly assess \( \mu_0 \) and \( T_0 \). For the latter assessments, we start with the observation that when \( p(\mu, W | S^h_c) \) is normal–Wishart as we have described, then \( p(x | S^h_c) \) is a multivariate \( t \) distribution with \( \alpha_W + n + 1 \) degrees of freedom, location vector \( \mu_0 \), and precision matrix \( \alpha_\mu (\alpha_W - n - 1) T_0^{-1} \). This result can be derived by first integrating over \( \mu \) using Equation 6 on p. 178 of DeGroot with sample size equal to one, and then integrating over \( W \) following an approach similar to that on pp. 179–180. Next, when \( \alpha_W > n + 1 \), it follows that

\[
E(x | S^h_c) = \mu_0 \quad \text{Cov}(x | S^h_c) = \frac{\alpha_\mu + 1}{\alpha_\mu \alpha_W - n - 1} \frac{1}{T_0} \quad (32)
\]

(e.g., DeGroot, 1970, pp. 61). Thus, a person can assess a prior linear-regression Bayesian network for \( E(x | S^h_c) \) and \( \text{Cov}(x | S^h_c) \), and then compute \( \mu_0 \) and \( T_0 \) using Equations 32.

In both cases, the unusual aspect of this assessment is the conditioning hypothesis \( S^h_c \) (see Heckerman et al. [1995b] for a discussion).

7 Consistency of the Dirichlet and Normal-Wishart Assumptions

In this section, we show that the Dirichlet and normal-Wishart priors \( p(\theta_x | S^h_c) \) are consistent with the assumptions of global parameter independence and likelihood equivalence. To see the potential for inconsistency, consider again the construction of parameter priors in our two-binary-variable example. Using \( x \) and \( \bar{x} \) as a shorthand for the values \( x^1 \) and \( x^2 \), and a similar shorthand for the values of \( Y \), suppose we choose the prior

\[
p(\theta_x | S^h_{x \rightarrow y}) = p(\theta_{xy}, \theta_{xy}, \theta_{xy} | S^h_{x \rightarrow y}) = \frac{c}{(\theta_{xy} + \theta_{xy})(1 - (\theta_{xy} + \theta_{xy}))} = \frac{c}{\theta_x (1 - \theta_x)}
\]

Using likelihood equivalence and Equations 8 and 9, we obtain

\[
p(\theta_y, \theta_{x|y}, \theta_{x|y} | S^h_{x \rightarrow y}) = \frac{c \cdot \theta_y (1 - \theta_y)}{\theta_x (1 - \theta_x)} = \frac{c \cdot \theta_y (1 - \theta_y)}{(\theta_y \theta_y (1 - \theta_y) \theta_{x|y} (1 - \theta_y \theta_{x|y} + (1 - \theta_y) \theta_{x|y}))}
\]

which does not satisfy global parameter independence.

When \( p(\theta_x | S^h_c) \) is Dirichlet for some complete structure \( S_c \), however, likelihood equivalence implies that global parameter independence holds for all complete structures. We demonstrated this fact for our two-variable example in Section 4. Heckerman et al. (1995b, Theorem 3) prove the general case, which we summarize here.\(^4\)

\(^4\)Note that the theorem is stated in a way that presupposes likelihood equivalence.
Theorem 6 If the parameters $\theta_x$ have the Dirichlet distribution
\[
p(\theta_x) = c \cdot \prod_{x_1, \ldots, x_n} [\theta_{x_1, \ldots, x_n}]^{\alpha_{x_1, \ldots, x_n} - 1}
\] (33)
then, for any complete structure $S_c$ for $X$, the distribution $p(\theta_{sc})$ satisfies global and local parameter independence. In particular,
\[
p(\theta_{sc}) = c \cdot \prod_{i=1}^{n} \prod_{x_i \in S_c} [\theta_{x_i = x_1, \ldots, x_{i-1}}]^{\alpha_{x_i = x_1, \ldots, x_{i-1}} - 1}
\] (34)
where $\alpha_{x_i = x_1, \ldots, x_{i-1}} = \sum_{x_{i+1}, \ldots, x_n} \alpha_{x_1, \ldots, x_n}$.

Proof: The result follows by multiplying the right-hand-side of Equation 33 by the Jacobian Equation 9, using the relation $\theta_{x_1, \ldots, x_n} = \prod_{i=1}^{n} \theta_{x_i | x_1, \ldots, x_{i-1}}$, and collecting powers of $\theta_{x_i | x_1, \ldots, x_{i-1}}$. □

Consistency for the linear-regression case is shown in the next theorem.

Theorem 7 If $(\mu, W)$ has a normal–Wishart distribution, then
\[
p(m, v, B) = \prod_{i=1}^{n} p(m_i, v_i, b_i)
\]

Proof: To prove the theorem, we factor $p(m|v, B)$ and $p(v, B)$ separately. By assumption, we know that $p(\mu|W)$ is a multivariate-normal distribution with mean $\mu_0$ and precision matrix $\alpha_\mu W$. Transforming this result to local distributions for $\mu_i$, we obtain
\[
p(\mu_i | \mu_1, \ldots, \mu_{i-1}, v, B) = \left(\frac{\alpha_\mu}{2\pi v_i}\right)^{1/2} \exp \left\{ \frac{\left(\mu_i - \mu_{0i} - \sum_{j=1}^{i-1} b_{ji}(\mu_j - \mu_{0j})\right)^2}{2v_i / \alpha_\mu} \right\}
\] (35)
for $i = 1, \ldots, n$. Using $m_{0i} = \mu_{0i} - \sum_{j=1}^{i-1} b_{ji}\mu_{0j}$, collecting terms for each $i$, and using Equation 12, we have
\[
p(m|v, B) = \prod_{i=1}^{n} N(m_i | m_{0i}, \alpha_\mu / v_i)
\] (36)
In addition, by assumption, $W$ has the Wishart distribution
\[
p(W) = c |W|^{(\alpha_W-n-1)/2} e^{-1/2tr(TW)}
\] (37)
From Equation 11, we have
\[
|W(i)| = \frac{1}{v_i} |W(i-1)| = \prod_{i=1}^{n} v_i^{-1}
\]
so that the determinant in Equation 37 factors as a function of $i$. Also, Equation 11 implies (by induction) that each element $w_{ij}$ in $W$ is a sum of terms each being a function of $b_i$ and $v_i$. Consequently, the exponent in Equation 37 factors as a function of $i$. Thus, given the Jacobian in Equation 13, which also factors as a function of $i$, we obtain
\[
p(v, B) = \prod_{i=1}^{n} p(v_i, b_i)
\] (38)
Equations 36 and 38 imply the theorem. □
8 From Consistency to Necessity

According to Equation 34, when \( p(\theta_x|S^h) \) is Dirichlet, not only are the parameters for each variable independent, but also the parameters corresponding to each instance of every variable’s parents are independent. We call this additional independence local parameter independence, again after Spiegelhalter and Lauritzen (1990). From our discussion in the previous section, we see that the Dirichlet assumption is consistent with likelihood equivalence and both global and local parameter independence.

It is interesting to ask whether there are any other choices for \( p(\theta_x|S^h) \) that are consistent in this sense. In our two-binary-variable example, using likelihood equivalence and Equation 9, we obtain

\[
p(\theta_x, \theta_y|x, \theta|x) = \frac{\theta_x(1 - \theta_x)}{\theta_y(1 - \theta_y)} p(\theta_y, \theta|x) = \frac{\theta_x(1 - \theta_x)}{\theta_y(1 - \theta_y)} \cdot p(\theta_x, \theta_y|x)\]

where

\[
\theta_x = \theta_x \theta_y + (1 - \theta_x) \theta_y, \quad \theta_y = \theta_x \theta_y + (1 - \theta_x) \theta_x, \quad \theta_x(1 - \theta_x) = \frac{\theta_y(1 - \theta_y)}{1 - \theta_x \theta_y + (1 - \theta_x) \theta_y} \]

Applying global and local parameter independence to both sides of Equation 39, we get

\[
f_x(\theta_x) f_y(\theta_y|x) f_y(\theta_y|x) = \frac{\theta_x(1 - \theta_x)}{\theta_y(1 - \theta_y)} f_x(\theta_x) f_y(\theta_y|x) f_y(\theta_y|x)\]

where \( f_x, f_y, f_y|x, f_y|x, f_y|x, f_y|x \) are unknown pdfs. Equations 40 and 41 define a functional equation. Geiger and Heckerman (1995a) show that the only pdf solutions to Equations 40 and 41 (and their generalizations for non-binary variables) are those where \( p(\theta_x|S^h) \) is a Dirichlet distribution. Heckerman et al. (1995b) generalize this result to n-variable models. Thus, in the discrete case, global and local parameter independence and likelihood equivalence provide a characterization of the Dirichlet distribution.

Geiger and Heckerman (1995b) obtain an analogous result for the two-variable linear-regression case. In particular, Let \( \{m_1, v_1, m_{21}, b_{12}, v_{21}\} \) and \( \{m_2, v_2, m_{12}, b_{21}, v_{12}\} \) denote the parameters for the structures \( X_1 \rightarrow X_2 \) and \( X_1 \leftarrow X_2 \), respectively. As we demonstrated in the previous section, if \( p(\theta_x|S^h) \) is a bivariate normal-Wishart distribution, then global parameter independence holds. Conversely, assuming global parameter independence and likelihood equivalence, and using the Jacobians in Equations 12 and 13, we obtain the functional equation

\[
f_1(m_1, v_1) f_{21}(m_{21}, b_{12}, v_{21}) = \frac{v_1^2 v_2^3}{v_2^2 v_1^3} f_2(m_2, v_2) f_{12}(m_{12}, b_{21}, v_{12})\]

where \( f_1, f_{21}, f_2, \) and \( f_{12} \) are arbitrary pdfs, and, from 10 and Equations 11,

\[
v_2 = v_2 + v_1 b_{12} \quad b_{21} = \frac{b_{12} v_1}{v_2} \quad v_{12} = \frac{v_2 v_1}{v_2}\]

\[
m_2 = m_{21} + b_{12} m_1 \quad m_{12} = m_1 + b_{21} m_2\]

The only pdf solutions to this functional equation are those where \( p(\mu, W|S^h) \) is a bivariate normal-Wishart distribution times an arbitrary pdf \( f(w_{12}) \), where \( w_{12} \) is the off-diagonal element of \( W \). Given the additional assumption of local parameter independence, which in this context says that each set of standardized parameters \( \{m^*_1, v^*_1\}, \{m_{21}^*, b_{12}^*, v_{21}^*\}, \{m_1^*, v_2^*\}, \) and \( \{m_{12}^*, b_{21}^*, v_{12}^*\} \) are mutually independent, the function \( f(\cdot) \) must be a constant.

In general, given likelihoods that satisfy covered-arc-reversal equivalence, the assumptions of parameter independence in combination with likelihood equivalence will yield a functional equation. As in these examples, the solutions to the equations may provide characterizations of well-known distributions. Alternatively, there may be no solutions or new distribution classes may be revealed.
9 Discussion

Parameter independence, covered-arc-reversal equivalence, and likelihood equivalence together yield strong constraints on priors. In the discrete case, someone who adopts these assumptions can have only one effective sample size for all variables. That is, this person must be equally confident in his or her knowledge about each variable. Similarly, in the linear-regression case, these assumptions permit the use of only two effective sample sizes: one for the normal component of the prior and one for the Wishart component. Nonetheless, if one learns about a portion of domain by reading or through word of mouth, or simply by applying common sense, then one or two effective sample sizes will likely be inadequate for an accurate expression of priors.

Thus, these assumptions should be checked when applying them to any real-world problem. If the assumptions are incorrect, then a sensitivity analysis should be done to see if the violation of the assumptions has much effect on the conclusions. If there is a large effect, then the assumptions should be relaxed. A proposal for doing so is given in Heckerman et al. (1995b).

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