Adiabatic Limit and the Frölicher Spectral Sequence
Dan Popovici

Abstract. Motivated by our conjecture of an earlier work predicting the degeneration at the second page of the Frölicher spectral sequence of any compact complex manifold supporting an SKT metric (i.e. such that $\partial \bar{\partial} \omega = 0$), we prove degeneration at $E_2$ whenever the manifold admits a Hermitian metric whose torsion operator $\tau$ and its adjoint vanish on $\Delta''$-harmonic forms of positive degrees up to $\dim_{\mathbb{C}} X$. Besides the pseudo-differential Laplacian inducing a Hodge theory for $E_2$ that we constructed in earlier work and Demailly’s Bochner-Kodaira-Nakano formula for Hermitian metrics, a key ingredient is a general formula for the dimensions of the vector spaces featuring in the Frölicher spectral sequence in terms of the asymptotics, as a positive constant $h$ decreases to zero, of the small eigenvalues of a rescaled Laplacian $\Delta_h$, introduced here in the present form, that we adapt to the context of a complex structure from the well-known construction of the adiabatic limit and from the analogous result for Riemannian foliations of Álvarez López and Kordyukov.

1 Introduction

Let $X$ be a compact complex manifold of dimension $n$. It is well known that the existence of a Kähler metric $\omega$ on $X$ implies the degeneration at $E_1$ of the Frölicher spectral sequence that relates the complex structure of $X$ (encapsulated in the Dolbeault, i.e. the $\bar{\partial}$-, cohomology $H^{p,q}(X, \mathbb{C})$, the start page of this spectral sequence) to the differential structure of $X$ (encapsulated in the De Rham, i.e. the $d$-, cohomology $H^k(X, \mathbb{C})$, the limiting page of this spectral sequence). However, since Kähler metrics exist only rarely when $n \geq 3$, it is natural to search for weaker metric conditions on $X$ that ensure a (possibly weaker) degeneration property of the algebro-geometric object that is the Frölicher spectral sequence of $X$. The best we can hope for in the non-Kähler context is the degeneration at the second page. To this end, we proposed the following conjecture in [Pop16]:

Conjecture 1.1. If a compact complex manifold $X$ admits an SKT metric $\omega$ (i.e. a Hermitian metric $\omega$ such that $\partial \bar{\partial} \omega = 0$), the Frölicher spectral sequence of $X$ degenerates at $E_2$.

There is evidence that this ought to be true. The statement holds true on all the examples of compact complex manifolds that we are aware of, namely all the 3-dimensional nilmanifolds, the 3-dimensional solvmanifolds that are currently classified, the Calabi-Eckmann manifold $S^3 \times S^3$, etc. In [Pop16], we proved this statement under the extra assumption that the SKT metric $\omega$ which is supposed to exist has a small torsion in the sense that the upper bound of its torsion operator of type $(0, 0)$ (defined in a precise way) does not exceed a third of the spectral gap of the elliptic, self-adjoint and non-negative, differential operator $\Delta' + \Delta''$ in every bidegree $(p, q)$. As usual, $\Delta' = \Delta'_\omega = \partial \bar{\partial}_\omega^* + \bar{\partial}_\omega^* \partial$ and $\Delta'' = \Delta''_\omega = \partial \bar{\partial}_\omega^* + \bar{\partial}_\omega^* \partial$ are the $\partial$-, resp. $\bar{\partial}$-Laplacians on smooth differential forms on $X$.

While Conjecture 1.1 remains elusive at the moment, we give in this paper a different sufficient metric condition for degeneration at $E_2$ that does not assume the fixed Hermitian metric $\omega$ to be SKT. As usual (see e.g. [Dem84] or [Dem97, VII, §1]), we consider the torsion operator $\tau = \tau_\omega := [\Lambda_\omega, \partial \omega \wedge \cdot]$ of type $(1, 0)$ defined on smooth differential forms on $X$, where $\Lambda_\omega$ is the adjoint of
the multiplication by $\omega$ w.r.t. the inner product defined by $\omega$, while $[A, B] = AB - (-1)^{ab} BA$ is the graded commutator of any two endomorphisms $A, B$ of respective degrees $a, b$ of the bi-graded algebra $C^{\infty}_{\bullet, \bullet}$ of smooth differential forms on $X$. Specifically, we prove

**Theorem 1.2.** Let $(X, \omega)$ be a compact Hermitian manifold with dim$_C X = n$ such that the inclusion of kernels

$$\ker \Delta'' \subset \ker [\tau, \tau^*]$$

holds for the operators $\Delta'', [\tau, \tau^*] : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ in every degree $k \in \{1, \ldots, n\}$.

Then, the Frölicher spectral sequence of $X$ degenerates at the second page $E_2$.

Hypothesis (1) is of a qualitative nature and it is comparatively easy to check on concrete examples of compact Hermitian manifolds $(X, \omega)$ whether it holds or not. For example, $S^3 \times S^3$ equipped with the Calabi-Eckmann complex structure and the Iwasawa manifold do not satisfy it when they are given the natural non-Kähler metrics (easy verifications that are left to the reader). Intuitively, (1) requires the torsion of $\omega$ to be “small” since, for non-negative operators, the smaller one has a larger kernel. (We will use throughout the paper the usual order relation for linear operators $A, B$: $A \geq B$ will mean that $\langle\langle Au, u \rangle\rangle \geq \langle\langle Bu, u \rangle\rangle$ for all forms $u$, where $\langle\langle , \rangle\rangle$ stands for the $L^2$ inner product induced by the fixed Hermitian metric $\omega$ on $X$.) Hypothesis (1) is obviously satisfied if $\omega$ is Kähler since $\tau = 0$ in that case. We do not know whether there exist compact complex non-Kähler manifolds that satisfy hypothesis (1).

Inspired by the extensive literature on the adiabatic limit associated with a Riemannian foliation (see e.g. [Wi85], [MM90], [For95], [ALK00] and the references therein), we adapt that construction to the case of the splitting $d = \partial + \bar{\partial}$ defining the complex structure of $X$. Thus, for every constant $h > 0$ that is eventually let to converge to 0, we define in section §.2 two rescalings of the usual $d$-Laplacian $\Delta = dd^* + d^*d$ acting on the smooth differential forms on an arbitrary compact Hermitian manifold $(X, \omega)$:

$$\Delta_h := dh^* d_h + d_h^* dh, $$

where $d_h := h\partial + \bar{\partial}$ modifies $d$ by rescaling $\partial$ while keeping $\bar{\partial}$ fixed, but its formal adjoint $d_h^*$ is computed w.r.t. the given Hermitian metric $\omega$, and

$$\Delta_{\omega_h} := d_{\omega_h}^* d_h + d_h^* d_{\omega_h}, $$

where $d = \partial + \bar{\partial}$ is kept unchanged, but its formal adjoint $d_{\omega_h}^*$ is computed w.r.t. a rescaled metric $\omega_h$ that modifies the original $\omega$ by multiplying the pointwise inner product of $(p, q)$-forms by $h^{2p}$. So, the anti-holomorphic degree $q$ of $(p, q)$-forms does not contribute to the definition of $\omega_h$. Although strongly inspired by the adiabatic limit construction in the presence of a Riemannian foliation, this partial rescaling of a Hermitian metric seems to be new and to hold further promise for the future.

In section §.2, we study these two rescaled Laplacians and the relationships between them. As in the foliated case of [ALK00], $\Delta_h$ and $\Delta_{\omega_h}$ are seen to have the same spectrum and to have eigenspaces that are obtained from each other via a rescaling isometry.

A key ingredient in the proof of Theorem 1.2 is the following formula for the dimensions of the vector spaces featuring on each page of the Frölicher spectral sequence of $X$ in terms of the number of small eigenvalues of the rescaled Laplacian $\Delta_h$ (or, equivalently, $\Delta_{\omega_h}$). “Small” refers to
the eigenvalues’ decay rate to zero as $h \downarrow 0$. This result and its proof are strongly inspired by the analogous result for foliations proved by Alvarez Lópéz and Kordyukov in [ALK00]. However, to our knowledge, this particular form of the result in the context of the Frölicher spectral sequence seems new and is of independent interest.

**Theorem 1.3.** Let $(X, \omega)$ be a compact Hermitian manifold with $\dim \mathbb{C} X = n$. For every $r \in \mathbb{N}^*$ and every $k = 0, \ldots , 2n$, the following identity holds:

$$\dim \mathbb{C} E^k_r = \# \{ i \mid \lambda^k_i(h) \in O(h^{2r}) \text{ as } h \downarrow 0 \},$$

where $E^k_r := \oplus_{p+q=k} E^{p,q}_r$ is the direct sum of the spaces of total degree $k$ on the $r$th page of the Frölicher spectral sequence of $X$, while $0 \leq \lambda^k_1(h) \leq \lambda^k_2(h) \leq \cdots \leq \lambda^k_r(h) \leq \cdots$ are the eigenvalues, counted with multiplicities, of the rescaled Laplacian $\Delta_h : C^\infty_{p,q}(X, \mathbb{C}) \rightarrow C^\infty_{p,q}(X, \mathbb{C})$ ($= \text{those of } \Delta_{\omega_h} : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})$) acting on $k$-forms. As usual, $\#$ stands for the cardinal of a set.

The proof of this statement proceeds along the lines of the one given in [ALK00] for the analogous statement in the foliated case with some simplifications, adjustments and inevitable differences in detail. We spell it out in section \S 4. In the proof of Theorem 1.3, we also use our pseudo-differential Laplacian $\tilde{\Delta} = \partial\tau'' \partial^* + \partial^* p'' \partial + \Delta'' : C^\infty_{p,q}(X, \mathbb{C}) \rightarrow C^\infty_{p,q}(X, \mathbb{C})$ (where $p''$ is the orthogonal projection onto $\ker \Delta''$) constructed in every bidegree $(p, q)$ in [Pop16] and shown there to induce a Hodge isomorphism between its kernel and the space $E^p_{2,q}$ of bidegree $(p, q)$ featuring on the second page of the Frölicher spectral sequence.

Along with Theorem 1.3 and the pseudo-differential Laplacian $\tilde{\Delta}$, the third main ingredient in the proof of Theorem 1.2 is the following formula of the Bochner-Kodaira-Nakano type for Hermitian (not necessarily Kähler) metrics $\omega$ established by Demailly in [Dem84] (see also [Dem97, VII, § 1]):

$$\Delta'' = \Delta_\tau + [\Lambda, \Lambda], \quad [\Lambda, \partial \omega \wedge \cdot] - [\partial \omega \wedge \cdot, (\partial \omega \wedge \cdot)^*],$$

where $[\bullet, \bullet]$ is the usual graded commutator (see e.g. Notation 1.4 below), $\Lambda = \Lambda_\omega$ is the adjoint of the multiplication operator $\omega \wedge \cdot$, $\tau = \tau_\omega := [\Lambda, \partial \omega \wedge \cdot]$ is the torsion operator of $\omega$ and $\Delta'_\tau := [\partial + \tau, (\partial + \tau)^*]$. This formula enables us to compare various Laplacians and finish the proof of Theorem 1.2 in section \S 6.

This paper owes much to the ideas and techniques in our main source of inspiration [ALK00] and to the treatment given to the Leray spectral sequence in [MM90] and [For95], although the setting and the objectives are different.

In the Appendix, we give an estimate of the discrepancy between the Laplacians $\Delta'$ and $\Delta''$ under the SKT assumption on the metric $\omega$ (cf. Lemma 7.1). This is of independent interest and leads to the lower bound $-Ch^2$ for the operator $\Delta_h - h^2 \Delta$ for all $0 < h < 1$ when $\omega$ is SKT, where $C \geq 0$ is a constant independent of $h$ that can be chosen to be any upper bound of the non-negative bounded torsion operator $[\bar{\tau}, \bar{\tau}^*]$ (cf. Lemma 7.2). In view of Theorem 1.3 and some minor extra arguments, if the lower bound $-Ch^2$ could be improved to 0, Conjecture 1.1 would be solved, but at the moment we are unfortunately short of arguments to perform this improvement.

**Notation 1.4.** For a given Hermitian metric $\omega$ on a given compact complex manifold $X$, $\langle \langle \cdot , \cdot \rangle \rangle$ will stand for the $L^2$ inner product defined by $\omega$ on the spaces $C^\infty_{p,q}(X, \mathbb{C})$ (resp. $C^\infty_k(X, \mathbb{C})$) of smooth differential $(p, q)$-forms (resp. $k$-forms) on $X$, while $|| \cdot ||$ will denote the corresponding $L^2$-norm. For self-adjoint linear operators $A, B$ on the bi-graded algebra $\oplus_{p,q} C^\infty_{p,q}(X, \mathbb{C})$,
by $A \geq B$ we shall mean (as is the standard convention) that $\langle \langle Au, u \rangle \rangle \geq \langle \langle Bu, u \rangle \rangle$ for every form $u$ lying in the space on which $A$ and $B$ are defined. We shall also use the usual bracket $[A, B] := AB - (-1)^{ab} BA$ for graded linear operators $A, B$ of respective degrees $a, b$ on the algebra $\oplus_k \Lambda^k T^* X$ of differential forms on $X$.

**Acknowledgments.** The author wishes to thank L. Ugarte for useful discussions about the content of this paper and for suggestions for section 5. Thanks are also due to S. Rao and Q. Zhao for stimulating discussions.

## 2 Rescaled Laplacians

Let $X$ be a compact complex manifold with $\dim \mathbb{C} X = n$. We fix a Hermitian metric $\omega$ on $X$.

### 2.1 Rescaling the metric

The first operation we will consider is a partial rescaling of $\omega$ in a way that depends solely on the holomorphic degree of forms.

**Definition 2.1.** For all $p, q \in \{0, \ldots, n\}$, all $(p, q)$-forms $u, v$ and every constant $h > 0$, we define the following pointwise inner product

$$\langle u, v \rangle_{\omega_h} := h^{2p} \langle u, v \rangle_{\omega}$$

where $\langle, \rangle_{\omega}$ stands for the pointwise inner product defined by the original Hermitian metric $\omega$.

Note that, for every $h > 0$, we obtain in this way a Hermitian metric $\omega_h = 1/h^2 \omega$, $h > 0$, on the holomorphic tangent bundle $T^{1,0} X$ of vector fields of type $(1, 0)$, or equivalently, a rescaled $C^\infty$ positive-definite $(1, 1)$-form $\omega_h = h^{-2} \omega$ on $X$. This induces a $C^\infty$ positive volume form

$$dV_{\omega_h} := \frac{\omega^n_h}{n!} = \frac{1}{h^{2n}} \frac{\omega^n}{n!} = \frac{1}{h^{2n}} dV_{\omega}$$

on $X$, which in turn gives rise, in conjunction with the above pointwise inner product $\langle, \rangle_{\omega_h}$, to the following $L^2$ inner product
\[ \langle\langle u, v \rangle\rangle_{\omega_h} := \int_X \langle u, v \rangle_{\omega_h} \, dV_{\omega_h} = \frac{1}{h^{2n}} \int_X \langle \theta_h u, \theta_h v \rangle_{\omega} \, dV_{\omega} = \frac{1}{h^{2n}} \langle\langle \theta_h u, \theta_h v \rangle\rangle_{\omega} \]

for all forms \( u, v \in C_{p,q}^\infty(X, \mathbb{C}) \) and all bidegrees \((p, q)\).

**Formula 2.2.** For all \((p, q)\)-forms \( u, v \), we have

\[ \langle\langle u, v \rangle\rangle_{\omega_h} = \frac{1}{h^{2(n-p)}} \langle\langle u, v \rangle\rangle_{\omega}, \quad \text{hence} \quad ||u||_{\omega_h} = h^{-(n-p)} ||u||_{\omega}. \]

**Proof.** The formula follows at once from the last identity and from the fact that \( \theta_h u = h pu \) for all \((p, q)\)-forms \( u \). \( \square \)

**Definition 2.3.** Let \((X, \omega)\) be a compact Hermitian manifold with \( \dim_{\mathbb{C}} X = n \). For every \( k = 0, \ldots, 2n \) and every constant \( h > 0 \), we consider the \( d \)-Laplacian \( \omega_h \) \( \text{w.r.t.} \) the rescaled metric \( \omega_h \) acting on \( C^\infty \) \( k \)-forms on \( X \):

\[ \Delta_{\omega_h} : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}), \quad \Delta_{\omega_h} := dd^*_{\omega_h} + d^*_{\omega_h} d, \]

where \( d^*_{\omega_h} \) is the formal adjoint of \( d \) \( \text{w.r.t.} \) \( \langle\langle , \rangle\rangle_{\omega_h} \) and \( \langle\langle , \rangle\rangle_{\omega_h} \) has been extended from the spaces \( C_{p,q}^\infty(X, \mathbb{C}) \) to \( C^\infty_k(X, \mathbb{C}) = \bigoplus_{p+q=k} C_{p,q}^\infty(X, \mathbb{C}) \) by sesquilinearity and by imposing that \( \langle\langle u, v \rangle\rangle_{\omega_h} = 0 \) whenever \( u \in C_{p,q}^\infty(X, \mathbb{C}) \) and \( v \in C_{r,s}^\infty(X, \mathbb{C}) \) with \((p, q) \neq (r, s)\).

### 2.2 Rescaling the differential

The second operation we will consider is a **partial rescaling of \( d = \partial + \bar{\partial} \)** that applies solely to its component of type \((1, 0)\).

**Definition 2.4.** Let \( X \) be a compact complex manifold, \( \dim_{\mathbb{C}} X = n \). For every constant \( h > 0 \), let

\[ d_h := h\partial + \bar{\partial} : C^\infty_k(X, \mathbb{C}) \to C^\infty_{k+1}(X, \mathbb{C}), \quad k \in \{0, \ldots, 2n\}. \]

Some basic properties of the rescaled differential \( d_h \) are summed up in the following

**Lemma 2.5.** (i) The operators \( d \) and \( d_h \) are related by the identity

\[ d_h = \theta_h d \theta_h^{-1}. \]

(ii) \( d_h^2 = 0 \) and the \( d \)- and \( d_h \)-cohomologies are related by the **isomorphism**

\[ H^k_d(X, \mathbb{C}) \xrightarrow{\cong} H^k_{d_h}(X, \mathbb{C}), \quad \{u\}_d \mapsto \{\theta_h u\}_{d_h} \]

where \( H^k_d(X, \mathbb{C}) = H^k_{DR}(X, \mathbb{C}) \) are the usual De Rham cohomology groups, while \( H^k_{d_h}(X, \mathbb{C}) := \ker(d_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_{k+1}(X, \mathbb{C}))/\text{Im}(d_h : C^\infty_{k-1}(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})) \) are the \( d_h \)-cohomology groups.
Proof. (i) If \( u \) is a \((p, q)\)-form, we have
\[
(\theta_h d\theta_h^{-1})(u) = \theta_h d(h^{-p} u) = h^{-p} \theta_h(\partial u) + h^{-p} \theta_h(\bar{\partial} u) = h^{-p} h^{p+1} \partial u + h^{-p} h^p \bar{\partial} u = h \partial u + \bar{\partial} u = d_h u.
\]
Thus, \( d_h = \theta_h d\theta_h^{-1} \) on pure-type forms, so this identity extends to arbitrary forms by linearity.

(ii) On the one hand, \( d_h^2 = \theta_h d^2 \theta_h^{-1} = 0 \); on the other hand,
\[
d_h(\theta_h u) = \theta_h d u \quad \text{so we have the equivalence:} \quad \theta_h u \in \ker(d_h) \iff u \in \ker d;
\]
\[
\theta_h u = d_h v \iff u = d(\theta_h^{-1} v), \quad \text{so we have the equivalence:} \quad \theta_h u \in \im(d_h) \iff u \in \im d.
\]
These equivalences show that the linear map \( H^k_d(X, \mathbb{C}) \ni \{ u \} \mapsto \{ \theta_h u \} \in H^k_{d_h}(X, \mathbb{C}) \) is well defined and bijective.

In particular, the spectral sequences induced by the pairs of differentials \((\partial, \bar{\partial})\) and \((h\partial, \bar{\partial})\) are isomorphic, so degenerate at the same page. The first of them is the Frölicher spectral sequence of \( X \).

**Definition 2.6.** Let \((X, \omega)\) be a compact Hermitian manifold with \( \dim_{\mathbb{C}} X = n \). For every constant \( h > 0 \) and every degree \( k \in \{0, \ldots, 2n\} \), we consider the \( d_h \)-Laplacian w.r.t. the given metric \( \omega \) acting on \( C^\infty_k \)-forms on \( X \):
\[
\Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}), \quad \Delta_h := d_h^* d_h + d_h d_h^*;
\]
where \( d_h^* \) is the formal adjoint of \( d_h \) w.r.t. the \( L^2 \) inner product induced by \( \omega \).

### 2.3 Comparison of the two rescaled Laplacians

We now bring together the above two operations by comparing the corresponding Laplace-type operators. Note that \( \Delta_{\omega_h} \) was defined by the rescaled differential \( d_h \) and the original metric \( \omega \), while \( \Delta_h \) was induced by the rescaled metric \( \omega_h \) and the original differential \( d \).

**Lemma 2.7.** (i) If \( \theta_h^* \) and \( d_h^* \) stand for the formal adjoints of \( \theta_h \), resp. \( d_h \), w.r.t. the pointwise, resp. \( L^2 \), inner product induced by \( \omega \), we have
\[
\theta_h^* = \theta_h \quad \text{and} \quad d_h^* = \theta_h^{-1} d^* \theta_h.
\]

(ii) The adjoints \( \partial_{\omega_h}^* \), \( \bar{\partial}_{\omega_h}^* \) w.r.t. to the metric \( \omega_h \), as well as the adjoints \( \partial_{\omega}^* = \partial^* \), \( \bar{\partial}_{\omega}^* = \bar{\partial}^* \) w.r.t. to the metric \( \omega \), of \( \partial \), resp. \( \bar{\partial} \) are related by the formulae:
\[
\partial_{\omega_h}^* = h^2 \partial^* \quad \text{and} \quad \bar{\partial}_{\omega_h}^* = \bar{\partial}^*.
\]
Consequently, we get
\[
\Delta_{\omega_h} = h^2 \Delta' + \Delta'' + [\partial, \partial^*] + h^2 [\bar{\partial}, \partial^*] = h^2 \Delta' + \Delta'' - [\tau, \partial^*] - h^2 [\bar{\partial}, \tau^*] - h^2 [\bar{\partial}, \partial^*],
\]
and
\[
\Delta_h = h^2 \Delta' + \Delta'' + h[\partial, \partial^*] + h[\bar{\partial}, \partial^*] = h^2 \Delta' + \Delta'' - h[\tau, \partial^*] - h[\bar{\partial}, \partial^*] - h[\bar{\partial}, \tau^*],
\]

6
where the adjoints $\partial^*, \bar{\partial}^*, \tau^*$ and the Laplacians $\Delta', \Delta''$ are computed w.r.t. the metric $\omega$, while

$$\tau = \tau_{\omega} := [\Lambda_{\omega}, \partial_{\omega} \wedge \cdot] : C^\infty_{p,q}(X, \mathbb{C}) \to C^\infty_{p+1,q}(X, \mathbb{C})$$

is the torsion operator (of type $(1,0)$ and order zero, acting on smooth forms of any bidegree $(p, q)$, where $\Lambda_{\omega}$ is the adjoint of the multiplication operator $\omega \wedge \cdot$) associated with the metric $\omega$ as defined in [Dem84] (see also [Dem97, VII, §8.1]).

In particular, the second-order Laplacians $\Delta_{\omega h}$ and $\Delta_h$ are elliptic since the second-order Laplacians $\Delta'$ and $\Delta''$ are and the deviation terms $-\bar{\partial} \bar{\tau} - h^2[\bar{\tau}, \partial^*] - h\partial[\bar{\partial}, \tau^*] - h[\partial, \bar{\tau}^*]$ are only of order 1.

Note that $\langle \langle [\partial, \partial^*]u, u \rangle \rangle = \langle \langle [\bar{\partial}, \partial^*]u, u \rangle \rangle = 0$ whenever the form $u$ is of pure type and whatever metric is used to define $\langle \langle , \rangle \rangle$ (because pure-type forms of different bidegrees are orthogonal w.r.t. any metric), so

$$\langle \langle \Delta_{\omega h}u, u \rangle \rangle = \langle \langle \Delta_h u, u \rangle \rangle = h^2 \langle \langle \Delta' u, u \rangle \rangle + \langle \langle \Delta'' u, u \rangle \rangle \quad \text{for every pure-type form } u. \quad (4)$$

(This fails, in general, if $u$ is not of pure type, unless the metric $\omega$ is Kähler.)

(iii) The rescaled Laplacians $\Delta_{\omega h}$ and $\Delta_h$ are related by the formula

$$\Delta_h = \theta_h \Delta_{\omega h} \theta_h^{-1}. \quad (5)$$

Proof. (i) For any $k$-forms $u = \sum_{p+q=k} u^{p,q}$ and $v = \sum_{p+q=k} v^{p,q}$, we have

$$\langle \theta_h u, v \rangle_\omega = \sum_{p+q=k} \langle h^p u^{p,q}, v^{p,q} \rangle_\omega = \sum_{p+q=k} \langle u^{p,q}, h^p v^{p,q} \rangle_\omega = \langle u, \theta_h v \rangle_\omega, \quad \text{so } \theta_h^* = \theta_h.$$

The second identity in (i) follows by taking conjugates in $d_h = \theta_h d\theta_h^{-1}$.

(ii) For any forms $\alpha \in C^\infty_{p-1,q}(X, \mathbb{C})$ and $\beta \in C^\infty_{p,q}(X, \mathbb{C})$, we have

$$\langle \langle \alpha, \partial^*_h \beta \rangle \rangle_\omega = \langle \langle \partial \alpha, \beta \rangle \rangle_\omega = \int_X \langle \partial \alpha, \beta \rangle_\omega dV = \int_X \frac{1}{h^{2p}} \langle \partial \alpha, \beta \rangle_\omega h^{2n} dV_{\omega h} = h^{2(p-1)} \langle \langle \alpha, \partial^* \beta \rangle \rangle_\omega_h$$

$$= h^{2(p-1)} \langle \langle \alpha, \partial^*_h \beta \rangle \rangle_\omega_h = \frac{1}{h^{2n}} \int_X h^{2(p-1)} \langle \alpha, \partial^*_h \beta \rangle_\omega h \frac{1}{h^{2n}} dV_{\omega h} = \frac{1}{h^2} \langle \langle \alpha, \partial^*_h \beta \rangle \rangle_\omega.$$

We get $\partial^*_h = h^{-2} \partial^*_h$, which is the first identity under (ii).

The identity $\bar{\partial}^*_h = \bar{\partial}^*_h$ is proved in the same way by using the fact that $\partial$ acts only on the anti-holomorphic degree of forms which is unaffected by the change of metric from $\omega$ to $\omega_h$.

Using these formulae, we get

$$\Delta_{\omega h} = [\partial + \bar{\partial}, \partial^*_h + \bar{\partial}^*_h] = [\partial, h^2 \partial^*] + [\bar{\partial}, \bar{\partial}^*] + [\partial, \bar{\partial}^*] + [\bar{\partial}, h^2 \partial^*]$$

$$= h^2 \Delta' + \Delta'' + [\partial, \bar{\partial}^*] + h^2 [\bar{\partial}, \partial^*]$$

and

$$\Delta_h = [h \partial + \bar{\partial}, h \partial^* + \bar{\partial}^*] = h^2 [\partial, \partial^*] + [\bar{\partial}, \bar{\partial}^*] + h[\partial, \bar{\partial}^*] + h[\bar{\partial}, \partial^*]$$

$$= h^2 \Delta' + \Delta'' + h[\partial, \bar{\partial}^*] + h[\bar{\partial}, \partial^*].$$
On the other hand, we know from [Dem84] (or [Dem97, VII, §1]) that

\[ [\partial, \bar{\partial}^*] = -[\partial, \tau^*] = -[\tau, \bar{\partial}^*] \quad \text{and, by conjugation, we get} \quad [\bar{\partial}, \partial^*] = -[\bar{\partial}, \tau^*] = -[\tau, \partial^*]. \]

So, the terms measuring the deviations of \( \Delta_{\omega_h} \) and \( \Delta_h \) from \( h^2 \Delta' + \Delta'' \) are of order 1 and we get the alternative formulae for \( \Delta_{\omega_h} \) and \( \Delta_h \) spelt out in the statement.

(iii) For any smooth \((p, q)\)-form \( \alpha \), we have

\[
(\theta_h \Delta_{\omega_h} \theta_h^{-1}) \alpha = \frac{1}{h^p} \theta_h \Delta_{\omega_h} \alpha = \frac{1}{h^p} \theta_h (h^2 \Delta' \alpha) + \frac{1}{h^p} \theta_h (\Delta'' \alpha) + \frac{1}{h^p} \theta_h ([\partial, \bar{\partial}^*] \alpha) + \frac{1}{h^p} \theta_h (h^2 [\bar{\partial}, \partial^*] \alpha) \\
= \frac{h^2 h^p}{h^p} \Delta' \alpha + \frac{h^p}{h^p} \Delta'' \alpha + \frac{h^{p+1}}{h^p} [\partial, \bar{\partial}^*] \alpha + \frac{h^2 h^{p-1}}{h^p} [\bar{\partial}, \partial^*] \alpha \\
= h^2 \Delta' \alpha + \Delta'' \alpha + h[\partial, \bar{\partial}^*] \alpha + h[\bar{\partial}, \partial^*] \alpha = \Delta_h \alpha.
\]

Thus, \( \theta_h \Delta_{\omega_h} \theta_h^{-1} = \Delta_h \) on pure-type forms and this identity extends to arbitrary forms by linearity. \( \square \)

**Corollary 2.8.** Let \((X, \omega)\) be a compact Hermitian manifold with \( \dim_{\mathbb{C}} X = n \). For every constant \( h > 0 \) and every degree \( k \in \{0, \ldots, 2n\} \), the spectra of the rescaled Laplacians \( \Delta_h, \Delta_{\omega_h} : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C}) \) coincide, i.e.

\[ \text{Spec}(\Delta_h) = \text{Spec}(\Delta_{\omega_h}), \]

and their respective eigenspaces are obtained from each other via the rescaling isometry \( \theta_h \):

\[ \theta_h (E_{\Delta_{\omega_h}}(\lambda)) = E_{\Delta_h}(\lambda) \quad \text{for every} \quad \lambda \in \text{Spec}(\Delta_h) = \text{Spec}(\Delta_{\omega_h}), \]

where \( E_{\Delta_{\omega_h}}(\lambda) \), resp. \( E_{\Delta_h}(\lambda) \), stands for the eigenspace corresponding to the eigenvalue \( \lambda \) of the operator \( \Delta_{\omega_h} \), resp. \( \Delta_h \).

Thus, \( \Delta_h \) and \( \Delta_{\omega_h} \) have the same eigenvalues with the same multiplicities.

**Proof.** Let \( \lambda \in \text{Spec}(\Delta_{\omega_h}) \) and let \( \alpha \in E_{\Delta_{\omega_h}}(\lambda) \subset C^\infty_k(X, \mathbb{C}) \). So \( \Delta_{\omega_h} \alpha = \lambda \alpha \), hence

\[
\Delta_h(\theta_h \alpha) = (\theta_h \Delta_{\omega_h} \theta_h^{-1})(\theta_h \alpha) = \theta_h(\lambda \alpha) = \lambda (\theta_h \alpha).
\]

Thus, \( \lambda \in \text{Spec}(\Delta_h) \) and \( \theta_h \alpha \in E_{\Delta_h}(\lambda) \). These implications also hold in reverse order, so we get the equivalences:

\[
\lambda \in \text{Spec}(\Delta_h) \iff \lambda \in \text{Spec}(\Delta_{\omega_h}) \quad \text{and} \quad \alpha \in E_{\Delta_{\omega_h}}(\lambda) \iff \theta_h \alpha \in E_{\Delta_h}(\lambda).
\]

These equivalences amount to (6) and (7). \( \square \)

Another consequence of the above discussion is a Hodge Theory for the \( d_h \)-cohomology and the resulting equidimensionality of the kernels of \( \Delta \) and \( \Delta_h \) in every degree.

**Corollary 2.9.** Let \((X, \omega)\) be a compact Hermitian manifold with \( \dim_{\mathbb{C}} X = n \). For every constant \( h > 0 \) and every degree \( k \in \{0, \ldots, 2n\} \), the operator \( d_h : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C}) \) induces the following \( L^2_\omega \)-orthogonal direct-sum decomposition:
The vector space $H^k_{\Delta_h}(X, \mathbb{C})$ is the kernel of $\Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ and $\ker d_h = H^k_{\Delta_h}(X, \mathbb{C}) \oplus \text{Im } d_h$. The vector space $H^k_{\Delta_h}(X, \mathbb{C})$ is finite-dimensional, while $\text{Im } d_h$ and $\text{Im } d_h^*$ are closed subspaces of $C^\infty_k(X, \mathbb{C})$.

This, in turn, induces the Hodge isomorphism

$$H^k_{\Delta_h}(X, \mathbb{C}) \simeq H^{k}_{d_h}(X, \mathbb{C}), \quad \alpha \mapsto \{\alpha\}_{d_h}.$$ 

Since $H^k_d(X, \mathbb{C})$ and $H^k_{d_h}(X, \mathbb{C})$ are isomorphic (via $\theta$, see Lemma 2.5) and $H^k_{\Delta}(X, \mathbb{C}) \simeq H^k_{\Delta}(X, \mathbb{C})$ (by standard Hodge theory), we infer that $H^k_{\Delta}(X, \mathbb{C})$ and $H^k_{\Delta_h}(X, \mathbb{C})$ are isomorphic (although the isomorphism need not be defined by $\theta_h$). In particular,

$$\dim H^k_{\Delta_h}(X, \mathbb{C}) = \dim H^k_{\Delta}(X, \mathbb{C}) \quad \text{for all } h > 0.$$ 

Proof. Since $X$ is compact and $\Delta_h$ is elliptic and self-adjoint, a standard consequence of Gårding’s inequality (see e.g. [Dem97, VI]) yields the two-space orthogonal decomposition $C^\infty_k(X, \mathbb{C}) = H^k_{\Delta_h}(X, \mathbb{C}) \oplus \text{Im } \Delta_h$, while this, together with the integrability property $d_h^* = 0$, further induces the orthogonal splitting $\text{Im } \Delta_h = \text{Im } d_h \oplus \text{Im } d_h^*$. The same consequence of Gårding’s inequality ensures that $\ker \Delta_h$ is finite-dimensional and that the images in $C^\infty_k(X, \mathbb{C})$ of $d_h$ and $d_h^*$ are closed.

3 The differentials in the Frölicher spectral sequence

We begin by recalling the well-known construction of the Frölicher spectral sequence in order to fix the notation and to point out the key features for us.

Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}}X = n$. Recall that the zero-th page $E_0$ of the Frölicher spectral sequence of $X$ consists of the spaces $E^{p,q}_0 := C^\infty_{p,q}(X, \mathbb{C})$ of smooth pure-type forms on $X$ and of the type-$(0, 1)$ differentials $d_0 := \bar{\partial}$ forming the Dolbeault complex:

$$\cdots \xrightarrow{d_0} E^{p-1,q}_0 \xrightarrow{d_0} E^{p,q}_0 \xrightarrow{d_0} E^{p,q+1}_0 \xrightarrow{d_0} \cdots.$$ 

Thus, in every bidegree $(p, q)$, the inclusions $\text{Im } d^{p,q}_0 \subset \ker d^{p,q}_0 \subset E^{p,q}_0$ induce (infinitely many, non-canonical) isomorphisms

$$C^\infty_{p,q}(X, \mathbb{C}) \simeq \text{Im } d^{p,q}_0 \oplus E^{p,q}_1 \oplus (E^{p,q}_0/\ker d^{p,q}_0), \quad (8)$$

where $d_0 = d^{p,q}_0 : E^{p,q}_0 \to E^{p,q+1}_0$ is the differential $d_0$ acting in bidegree $(p, q)$ and the $E^{p,q} := \ker d^{p,q}_0/\text{Im } d^{p,q-1}_0 = H^{p,q}_\bar{\partial}(X, \mathbb{C})$ are the Dolbeault cohomology groups of $X$.

The first page $E_1$ of the Frölicher spectral sequence consists of the spaces $E^{p,q}_1$ (i.e. the cohomology of the zero-th page) and of the type-$(1, 0)$ differentials $d_1$:

$$\cdots \xrightarrow{d_1} E^{p-1,q}_1 \xrightarrow{d_1} E^{p,q}_1 \xrightarrow{d_1} E^{p+1,q}_1 \xrightarrow{d_1} \cdots$$

induced in cohomology by $\partial$ (i.e. $d_1(\alpha|_\partial) := [\partial \alpha|_\partial]$). Thus, in every bidegree $(p, q)$, the inclusions $\text{Im } d^{p-1,q}_1 \subset \ker d^{p,q}_1 \subset E^{p,q}_1$ induce (infinitely many, non-canonical) isomorphisms
\[ E_1^{p,q} \simeq \text{Im} \, d_1^{p-1,q} \oplus E_2^{p,q} \oplus (E_1^{p,q} / \ker d_1^{p,q}), \]  

(9)

where \( d_1^{p,q} \) is \( d_1 \) acting in bidegree \( (p, q) \), while the spaces \( E_2^{p,q} := \ker d_1^{p,q} / \text{Im} \, d_1^{p-1,q} \) form the cohomology of the page \( E_1 \).

The remaining pages are constructed inductively: the differentials \( d_r = d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1} \) are of type \( (r, -r + 1) \) for every \( r \), while the spaces \( E_r^{p,q} := \ker d_{r-1}^{p,q} / \text{Im} \, d_{r-1}^{p+r-1,q+r-2} \) on the \( r \)th page are defined as the cohomology of the previous page \( E_{r-1} \). On every page \( E_r \) and in every bidegree \( (p, q) \), the inclusions \( \text{Im} \, d_{r-1}^{p-r,q+r-1} \subset \ker \, d_r^{p,q} \subset E_r^{p,q} \) induce (infinitely many, non-canonical) isomorphisms

\[ E_r^{p,q} \simeq \text{Im} \, d_r^{p-r,q+r-1} \oplus E_{r+1}^{p,q} \oplus (E_r^{p,q} / \ker d_r^{p,q}), \]

(10)

where \( E_r^{p,q} := \ker d_r^{p,q} / \text{Im} \, d_r^{p-r,q+r-1} \).

It is worth stressing that (8), (9) and (10) only assert that the vector spaces on either side of \( \simeq \) are isomorphic, but no choice of preferred isomorphism is possible at this stage.

A classical result of Frölicher [Fro55] asserts that this spectral sequence converges to the De Rham cohomology of \( X \) and degenerates after finitely many steps. This means that there are (non-canonical) isomorphisms:

\[ H^k_{DR}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} E_{\infty}^{p,q}, \quad k = 0, \ldots, 2n, \]

(11)

where \( E_0^{p,q} = \cdots = E_{N+2}^{p,q} = E_{N+1}^{p,q} = E_{N}^{p,q} \) for all \( p, q \) and where \( N \geq 1 \) is the positive integer such that the spectral sequence degenerates at \( E_N \).

### 3.1 Identification of the \( d_r \)'s with restrictions of \( d \)

Summing up (8), (9), (10) over \( r = 0, \ldots, N-1 \), we get (infinitely many, non-canonical) isomorphisms

\[ C_{p,q}^{\infty}(X, \mathbb{C}) \simeq \bigoplus_{r=0}^{N-1} \text{Im} \, d_r^{p-r,q+r-1} \oplus E_{\infty}^{p,q} \oplus \bigoplus_{r=0}^{N-1} (E_r^{p,q} / \ker d_r^{p,q}) \]

for every bidegree \( (p, q) \). Note that the isomorphisms (8), (9), (10) identify the spaces \( \text{Im} \, d_r^{p-r,q+r-1} \), \( E_r^{p,q} \) (including for \( r = \infty \)) and \( E_r^{p,q} / \ker d_r^{p,q} \) with certain subspaces of \( C_{p,q}^{\infty}(X, \mathbb{C}) \). However, these subspaces have not been specified yet since multiple choices (and no canonical choice) are possible for the isomorphisms (8), (9), (10). These choices can only be made unique once a Hermitian metric has been fixed on \( X \). (See §3.2.)

Now, since \( C_k^{\infty}(X, \mathbb{C}) = \bigoplus_{p+q=k} C_{p,q}^{\infty}(X, \mathbb{C}) \) for all \( k = 0, \ldots, 2n \), we get

\[
\begin{align*}
C_k^{\infty}(X, \mathbb{C}) \simeq & \bigoplus_{0 \leq r \leq N-1} \text{Im} \, d_r^{p-r,q+r-1} \oplus \bigoplus_{p+q=k} E_{\infty}^{p,q} \oplus \bigoplus_{0 \leq r \leq N-1} (E_r^{p,q} / \ker d_r^{p,q}) \\
\downarrow d & \\
C_{k+1}^{\infty}(X, \mathbb{C}) \simeq & \bigoplus_{0 \leq r \leq N-1} \text{Im} \, d_r^{p-r,q+r-1} \oplus \bigoplus_{p+q=k} E_{\infty}^{p,q} \oplus \bigoplus_{0 \leq r \leq N-1} (E_r^{p,q} / \ker d_r^{p,q}) \\
& \bigoplus_{0 \leq r \leq N-1} \bigoplus_{p+q=k} (E_r^{p-r,q-r+1} / \ker d_r^{p-r,q-r+1}).
\end{align*}
\]
Thus, under these isomorphisms, the operator $d = d^{(k)} : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_{k+1}(X, \mathbb{C})$ identifies as

\[
d^{(k)} \simeq \bigoplus_{0 \leq r \leq N-1 \atop p+q=k} d^{p,q}_r,
\]

where the isomorphism $d^{p,q}_r : E^{p,q}_r/\ker d^{p,q}_r \rightarrow \text{Im} d^{p,q}_r$ is the restriction of $d_r = d^{p,q}_r : E^{p,q}_r \rightarrow E^{p+r,q-r+1}_r$ to the third piece on the r.h.s. of (10). The fact that $d_r$ is of type $(r, -r+1)$ will play a key role in the sequel.

On the other hand, summing up the splittings of $C^\infty_{p,q}(X, \mathbb{C})$ over $p \geq s$ for any given $s$, we get

\[
A_s^k := \bigoplus_{p \geq s} C^\infty_{p,q}(X, \mathbb{C}) \simeq \bigoplus_{p \geq s} \left[ \bigoplus_{r=0}^{N-1} \text{Im} d^{p-r,q+r-1}_r \oplus E^{p,q}_{\infty} \oplus \bigoplus_{r=0}^{N-1} (E^{p,q}_r/\ker d^{p,q}_r) \right].
\]

**Lemma 3.1.** (i) For every $r$ and every $k$, let $E^k_r := \bigoplus_{p+q=k} E^{p,q}_r$. Then

\[
dim E^k_r = \sum_{p+q=k} \dim E^{p,q}_r = b_k + m^{k-1}_r + m^k_r, \quad 0 \leq r \leq N, \quad 0 \leq k \leq 2n,
\]

where we set $m^k_r := \sum_{l \geq r} \sum_{p+q=k} \dim (E^{p,q}_l/\ker d^{p,q}_l)$.

(ii) For every $r$ and every $k$, let $L^k_r := \bigoplus_{l \geq r} (E^{p,q}_l/\ker d^{p,q}_l)$ and $L^k := \bigoplus_{p+q=k} L^{p,q}_r$. Then, $\dim L^k_r = m^k_r$ (obvious) and, under the identifications defined by the isomorphisms (8), (9), (10), the following inclusions hold:

\[
d(L^{p,q}_r) \subset A^{p+q+1}_{p+r}, \quad 0 \leq r \leq N, \quad 0 \leq p, q \leq n,
\]

where $d(L^{p,q}_r) := \bigoplus_{l \geq r} (E^{p,q}_l/\ker d^{p,q}_l)$ in keeping with identification (12).

**Proof.** (i) For every fixed $r$, summing up the splittings (10) with $l$ in place of $r$ over $l \geq r$ and then summing up over $p+q = k$ for every fixed $k$, we get

\[
E^k_r \simeq \bigoplus_{p+q=k} E^{p,q}_r \oplus \bigoplus_{l \geq r} \bigoplus_{p+q=k} \text{Im} d^{p-l,q+l-1}_l \oplus \bigoplus_{l \geq r} \bigoplus_{p+q=k} (E^{p,q}_l/\ker d^{p,q}_l).
\]

Since $\text{Im} d^{p-l,q+l-1}_l \simeq E^{p-l,q+l-1}_l/\ker d^{p-l,q+l-1}_l$ for all $p, q, l$, if we set $p' := p - l$ and $q' := q + l - 1$, we have $p' + q' = k - 1$ when $p + q = k$ and the above isomorphism translates to

\[
E^k_r \simeq \bigoplus_{p+q=k} E^{p,q}_r \oplus \bigoplus_{l \geq r} \bigoplus_{p' + q' = k-1} (E^{p',q'}/\ker d^{p',q'}_l) \oplus \bigoplus_{l \geq r} \bigoplus_{p+q=k} (E^{p,q}_l/\ker d^{p,q}_l)
\]

for every $k$. Now, $\dim \bigoplus_{p+q=k} E^{p,q}_\infty = b_k$ (the $k^{th}$ Betti number of $X$) thanks to (11), so taking dimensions in the above isomorphism, we get (13).

(ii) Since $d^{p,q}_l : E^{p,q}_l/\ker d^{p,q}_l \rightarrow \text{Im} d^{p,q}_l$ is an isomorphism of type $(l, -l+1)$ for all $l, p, q$, we get for all $l \geq r$:
under the identification of each space \( E_{l}^{p+l,q-l+1} \) with a subspace of \( C_{p+l,q-l+1}^{\infty} \) defined by the isomorphisms (8), (9), (10). This proves (14).

### 3.2 Explicit description of the above identifications

We take this opportunity to point out an explicit description of the differentials \( d_r \) in cohomology and of their unique realizations induced by a given Hermitian metric on \( X \).

**Lemma 3.2.** Let \( X \) be a compact complex manifold with \( \dim_{\mathbb{C}}X = n \).

(i) For every \( r \) and every bidegree \( (p, q) \), the vector space of type \( (p, q) \) featuring on the \( r \)-th page of the Frölicher spectral sequence of \( X \) can be explicitly described as the following set of multi-cohomology classes (i.e. each of these is the \( d \)-cohomology class of a \( d \)-class):

\[
E_{r}^{p,q} = \{ \ldots [[\alpha]_\delta]_{d_1} \ldots ]_{d_{r-1}} \mid \alpha \in C_{p,q}^{\infty}(X, \mathbb{C}) \text{ such that } \alpha \text{ satisfies condition } (P_{r}) \},
\]

where condition \( (P_{r}) \) on \( \alpha \) requires the existence of forms \( u_l \in C_{p+l,q-l}^{\infty}(X, \mathbb{C}) \) for \( l \in \{1, \ldots, r-1\} \) such that

\[
\bar{\partial} \alpha = 0, \quad \partial \alpha = \bar{\partial} u_1, \quad \partial u_1 = \bar{\partial} u_2, \ldots, \partial u_{r-2} = \bar{\partial} u_{r-1}.
\]

(ii) For every \( r \) and every bidegree \( (p, q) \), the differential \( d_r = d_{r,p,q} : E_{r}^{p,q} \to E_{r}^{p+r,q-r+1} \) featuring on the \( r \)-th page of the Frölicher spectral sequence of \( X \) is explicitly described as

\[
d_r \left( \ldots [[\alpha]_\delta]_{d_1} \ldots ]_{d_{r-1}} \right) = \ldots [\partial u_{r-1}]_\delta d_1 \ldots ]_{d_{r-1}},
\]

for every \( \ldots [[\alpha]_\delta]_{d_1} \ldots ]_{d_{r-1}} \in E_{r}^{p,q} \). Moreover, this description of \( d_r \) is independent of the choice of forms \( u_l \in C_{p+l,q-l}^{\infty}(X, \mathbb{C}) \) in (16) (which are unique only modulo \( \ker \bar{\partial} \)).

**Proof.** These facts are well-known (cf. [CFGU97]). We will only explain the well-definedness of formula (17) for \( d_r \). Let \( (u_1, \ldots, u_{r-1}) \) and \( (u_1 + \zeta_1, \ldots, u_{r-1} + \zeta_{r-1}) \) be two sets of forms satisfying (16), i.e. \( \bar{\partial} \alpha = 0, \partial \alpha = \bar{\partial} u_1 = \bar{\partial}(u_1 + \zeta_1) \) and

\[
\partial u_1 = \bar{\partial} u_2 \quad \text{and} \quad \partial(u_1 + \zeta_1) = \bar{\partial}(u_2 + \zeta_2), \ldots, \partial u_{r-2} = \bar{\partial} u_{r-1} \quad \text{and} \quad \partial(u_{r-2} + \zeta_{r-2}) = \bar{\partial}(u_{r-1} + \zeta_{r-1}).
\]

These identities imply the identities

\[
\bar{\partial} \zeta_1 = 0, \quad \partial \zeta_1 = \bar{\partial} \zeta_2, \ldots, \partial \zeta_{r-2} = \bar{\partial} \zeta_{r-1},
\]

which, in turn, imply that \( \zeta_1 \) satisfies condition \( (P_{r-1}) \) (hence defines a multi-cohomology class lying in \( E_{r-1}^{p+1,q-1} \)) and that

\[
d_{r-1}(\ldots [[\zeta_1]_\delta]_{d_1} \ldots ]_{d_{r-2}}) = \ldots [\partial \zeta_{r-1}]_\delta d_1 \ldots ]_{d_{r-2}} \in \text{Im} \, d_{r-1}.
\]

Consequently, \( \ldots [[\partial \zeta_{r-1}]_\delta]_{d_1} \ldots ]_{d_{r-2}} \] \( d_{r-1} = 0 \), so...
\[ \ldots [[\partial (u_{r-1} + \zeta_{r-1})] \delta] d_i \ldots ] d_r = \ldots [[\partial u_{r-1}] \delta] d_i \ldots ] d_r. \]

Thus, the result we get by formula (17) for \( d_r(\ldots [[\alpha] \delta] d_i \ldots ] d_r) \) is the same whether we work with the choices \((u_1, \ldots, u_{r-1})\) or \((u_1 + \zeta_1, \ldots, u_{r-1} + \zeta_{r-1})\). \(\square\)

Thus, \( d\alpha = \partial \alpha \) induces the multi-cohomology class \( d_r(\ldots [[\alpha] \delta] d_i \ldots ] d_r) \). This helps to explain that, intuitively, \( d \) acts as \( d_r \) on representatives of \( E_r \)-classes (cf. (12)).

Now, recall that infinitely many choices are possible for the isomorphisms (8), (9) and (10). However, any fixed Hermitian metric \( \omega \) on \( X \) selects a unique realisation of each of these isomorphisms and, implicitly, identifies each space \( E_{r}^{p,q} \) with a precise subspace \( H_{r}^{p,q} \) (depending on \( \omega \)) of \( C_{p,q}^\infty (X, \mathbb{C}) \) via an isomorphism \( E_{r}^{p,q} \simeq H_{r}^{p,q} \) depending on \( \omega \). These harmonic subspaces \( H_{r}^{p,q} \subset C_{p,q}^\infty (X, \mathbb{C}) \) are constructed by induction on \( r \geq 1 \) as follows.

**Definition 3.3.** Let \( H_{r}^{p,q} \subset C_{p,q}^\infty (X, \mathbb{C}) \) be the orthogonal complement for the \( L_\omega^2 \)-norm of \( \text{Im} \, d_0^{p,q-1} \) in \( \ker d_0^{p,q} \). Due to (8), \( H_{1}^{p,q} \) is isomorphic to \( E_{1}^{p,q} \). In every bidegree \((p, q)\), the linear map \( d_1^{p,q} : E_{1}^{p,q} \to E_{1}^{p+1,q} \) induces a linear map (denoted by the same symbol) \( d_1^{p,q} : H_{1}^{p,q} \to H_{1}^{p+1,q} \) via the isomorphisms \( H_{1}^{p,q} \simeq E_{1}^{p,q} \) and \( H_{1}^{p+1,q} \simeq E_{1}^{p+1,q} \). Let \( H_{r}^{p,q} \subset H_{1}^{p,q} \subset C_{p,q}^\infty (X, \mathbb{C}) \) be the orthogonal complement for the \( L_\omega^2 \)-norm of \( \text{Im} \, d_1^{p,q-1} \) in \( \ker d_1^{p,q} \) (viewed as subspaces of \( H_{1}^{p,q} \)). Due to (9), \( H_{2}^{p,q} \) is isomorphic to \( E_{2}^{p,q} \). Continuing inductively, when the linear maps \( d_r^{p,q} : E_{r}^{p,q} \to E_{r+1}^{p+r,q-r+1} \) have induced counterparts (denoted by the same symbol) \( d_r^{p,q} : H_{r}^{p,q} \to H_{r+1}^{p+r,q-r+1} \) between the already constructed subspaces \( H_{r}^{p,q} \subset C_{p,q}^\infty (X, \mathbb{C}) \) and \( H_{r+1}^{p+r,q-r+1} \subset C_{p+1,q-r+1}^\infty (X, \mathbb{C}) \), we let \( H_{r+1}^{p,q} \subset H_{r}^{p,q} \subset C_{p,q}^\infty (X, \mathbb{C}) \) be the orthogonal complement for the \( L_\omega^2 \)-norm of \( \text{Im} \, d_r^{p,q-1} \) in \( \ker d_r^{p,q} \) (viewed as subspaces of \( H_{r}^{p,q} \)). Due to (10), \( H_{r+1}^{p,q} \) is isomorphic to \( E_{r+1}^{p,q} \).

Note that we have

\[
\begin{align*}
H_{1}^{p,q} &= \ker (\tilde{\Delta}'' : C_{p,q}^\infty (X, \mathbb{C}) \to C_{p,q}^\infty (X, \mathbb{C})) = \{ u \in C_{p,q}^\infty (X, \mathbb{C}) \mid \tilde{\partial} u = 0 \text{ and } \tilde{\partial}^* u = 0 \}, \\
H_{2}^{p,q} &= \ker (\tilde{\Delta} : C_{p,q}^\infty (X, \mathbb{C}) \to C_{p,q}^\infty (X, \mathbb{C})) \\
&= \{ u \in C_{p,q}^\infty (X, \mathbb{C}) \mid \tilde{\partial} u = 0, \tilde{\partial}^* u = 0, p''(\partial u) = 0 \text{ and } p''\partial^* u = 0 \},
\end{align*}
\]

where \( \tilde{\Delta} = \partial'' \partial^* + \partial^* p'' \partial + \Delta'' \) is the pseudo-differential Laplacian constructed in [Pop16].

Also note that standard Hodge theory (for the elliptic differential operator \( \Delta'' \)) is used to ensure that \( \text{Im} \, d_0^{p,q-1} \) is closed in \( C_{p,q}^\infty (X, \mathbb{C}) \) and that \( H_{1}^{p,q} \) is finite-dimensional. However, all the other images \( \text{Im} \, d_r^{p-r,q+r-1} \) are automatically closed since they are (necessarily finite-dimensional) vector subspaces of a finite-dimensional vector space. It is also possible to construct pseudo-differential operators \( \tilde{\Delta}_r : C_{p,q}^\infty (X, \mathbb{C}) \to C_{p,q}^\infty (X, \mathbb{C}) \) whose kernels are isomorphic to the spaces \( H_{r}^{p,q} \) (cf. forthcoming joint work of the author with L. Ugarte, where the Hodge theory found in [Pop16] for the second page of the Frölicher spectral sequence is extended to all the pages), making these spaces into harmonic spaces for these pseudo-differential Laplacians, but the mere spaces \( H_{r}^{p,q} \) suffice for our purposes in this paper.

When the vector space \( C_{p,q}^\infty (X, \mathbb{C}) \) is endowed with the \( L_2^2 \)-norm induced by \( \omega \), every subspace \( H_{r}^{p,q} \) inherits the restricted norm. On the other hand, every space \( E_{r}^{p,q} \) has a quotient norm induced by the \( L_\omega^2 \)-norm. The isomorphisms \( E_{r}^{p,q} \simeq H_{r}^{p,q} \) constructed above are isometries when \( E_{r}^{p,q} \) and \( H_{r}^{p,q} \) are endowed with the quotient, resp. \( L^2 \) norms.
Conclusion 3.4. Let $X$ be a compact complex manifold and let $\omega$ be any Hermitian metric on $X$. Let $\cdots \subset \mathcal{H}_{p+1}^{r} \subset \mathcal{H}_{p}^{r} \subset \cdots \subset \mathcal{H}_{p}^{1} \subset C_{0}^{\infty}(X, \mathbb{C})$ be the subspaces of Definition 3.3 induced by $\omega$.

For every $r$ and every bidegree $(p, q)$, each class $[\ldots[[\alpha]]d_{1}\ldots]d_{r-1} \in E_{r}^{p, q}$ contains a unique representative $\alpha \in \mathcal{H}_{r}^{p, q}$ (necessarily satisfying condition $(P_{r})$). For $l \in \{1, \ldots, r-1\}$, let $u_{l} \in C_{\rho+l, q-l}^{\infty}(X, \mathbb{C})$ be the unique solutions with minimal $L_{\omega}^{2}$-norms of the equations

$$\bar{\partial}\alpha = 0, \quad \partial\alpha = \bar{\partial}u_{1}, \quad \bar{\partial}u_{1} = \partial u_{2}, \ldots, \partial u_{r-2} = \bar{\partial}u_{r-1}$$

constructed inductively from one another. The well-known Neumann formula yields

$$u_{1} = \Delta''^{-1}\bar{\partial}^* (\partial\alpha) \quad \text{and} \quad u_{l} = \Delta''^{-1}\bar{\partial}^* (\partial u_{l-1}) \quad \text{for} \quad l \in \{2, \ldots, r-1\}.$$ 

In particular, the maps $\alpha \mapsto u_{1}$ and $u_{l-1} \mapsto u_{l}$ are linear.

For all $r, p, q$, we define the linear operator

$$T_{r} = T_{r}^{p, q} : \mathcal{H}_{r}^{p, q} \longrightarrow C_{\rho+r, q-r+1}^{\infty}(X, \mathbb{C}), \quad \alpha \mapsto T_{r}(\alpha) := \partial u_{r-1}.$$ 

Since $\mathcal{H}_{r}^{p, q}$ is finite-dimensional, $T_{r}$ is bounded, so there exists a constant $C_{r}^{p, q} > 0$ such that

$$||T_{r}(\alpha)|| = ||\partial u_{r-1}|| \leq C_{r}^{p, q} ||\alpha|| \quad \text{for all} \quad \alpha \in \mathcal{H}_{r}^{p, q}.$$ 

It is easy to see that $T_{r}(\alpha)$ need not belong to $\mathcal{H}_{r}^{p+r, q-r+1}$ when $\alpha \in \mathcal{H}_{r}^{p, q}$. If we let $P_{r}^{p, q} : C_{p, q}^{\infty}(X, \mathbb{C}) \longrightarrow \mathcal{H}_{r}^{p, q}$ be the $L_{\omega}$-orthogonal projection onto $\mathcal{H}_{r}^{p, q}$, we get

$$||(P_{r}^{p, q} \circ T_{r})(\alpha)|| = ||P_{r}^{p, q}(\partial u_{r-1})|| \leq ||\partial u_{r-1}|| \leq C_{r}^{p, q} ||\alpha|| \quad \text{for all} \quad \alpha \in \mathcal{H}_{r}^{p, q}.$$ 

4 Use of the rescaled Laplacians in the study of the Fröhlicher spectral sequence

In this section, we prove Theorem 1.3.

As in [ES89], [GS91], [ALK00], we consider the spectrum distribution function associated with any of the rescaled Laplacians $\Delta_{h}, \Delta_{\omega_{h}}$ in our context. Its definition and its study are made far simpler in this setting than in those references by the manifold $X$ being compact and by the Laplacians $\Delta'$, $\Delta''$ being elliptic.

Definition 4.1. Let $(X, \omega)$ be a compact Hermitian manifold with $\dim_{\mathbb{C}}X = n$. For every $k \in \{0, \ldots, n\}$ and every constant $\lambda \geq 0$, let $N_{h}^{k}(\lambda)$ stand for the number of eigenvalues (counted with multiplicities) of $\Delta_{h}$ that are $\leq \lambda$.

Replacing $\Delta_{h}$ with $\Delta_{\omega_{h}}$ does not change the spectrum distribution function $N_{h}^{k} : [0, +\infty) \longrightarrow \mathbb{N}$ since $\Delta_{h}$ and $\Delta_{\omega_{h}}$ have the same eigenvalues with the same multiplicities (cf. Corollary 2.8). Theorem 1.3 can be reworded as ensuring the existence of a constant $C > 0$ independent of $h$ such that, for all $r$ and $k$, we have

$$\dim E_{r}^{k} = N_{h}^{k}(Ch^{2r}) \quad \text{when} \quad 0 < h \ll 1.$$ 

(19)
4.1 The Efremov-Shubin variational principle

The main technical ingredient we will need is the following variant of the variational principle proved in a more general context in [ES89] and used extensively thereafter (e.g. [GS91], [ALK00]) in settings different from ours. We adapt to our situation the result of [ES89].

Proposition 4.2. (see e.g. Efremov-Shubin [ES89]) Let \((X, \omega)\) be a compact Hermitian manifold with \(\dim C^\infty X = n\). For every \(k = 0, \ldots, 2n\) and every \(\lambda \geq 0\), the following identity holds

\[
N^k_h(\lambda) = F^k_{h-1}(\lambda) + b_k + F^k_h(\lambda),
\]

where \(b_k\) is the \(k^{th}\) Betti number of \(X\) and the function \(F^k_h : [0, +\infty) \rightarrow \mathbb{N}\) is defined by

\[
F^k_h(\lambda) = \sup_L \dim L,
\]

where \(L\) ranges over the closed vector subspaces of the quotient space \(C^\infty_k(X, \mathbb{C})/\ker d\) on which the operator \(d : C^\infty_k(X, \mathbb{C})/\ker d \rightarrow C^\infty_{k+1}(X, \mathbb{C})\) induced by \(d : C^\infty(X, \mathbb{C}) \rightarrow C^\infty_{k+1}(X, \mathbb{C})\) satisfies the following \(L^2_{\omega_h}\)-norm estimate:

\[
||d\zeta||_{\omega_h} \leq \sqrt{\lambda} ||\zeta||_{\omega_h}, \quad \text{for every } \zeta \in L.
\]

(The understanding is that \(||d\zeta||_{\omega_h}\) stands for the usual \(L^2\)-norm induced by the metric \(\omega_h\), while \(||\zeta||_{\omega_h}\) stands for the quotient norm induced on \(C^\infty_k(X, \mathbb{C})/\ker d\) by the \(L^2_{\omega_h}\)-norm.)

We will present a detailed proof of this statement along the lines of [ES89] with a few minor simplifications afforded by our special setting where the manifold \(X\) is compact and the operator \(\Delta_h\) is elliptic. While a more general version for unbounded operators on \(L^2\) spaces was needed in [ALK00], we stress that, in this context, we can confine ourselves to the case of operators on spaces of \(C^\infty\) differential forms.

The main step is the following statement (a version of the classical Min-Max Principle) that was proved in a more general setting in [ES89].

Proposition 4.3. Let \((X, \omega)\) be a compact Hermitian manifold with \(\dim C^\infty X = n\). For an arbitrary \(k \in \{0, \ldots, 2n\}\), let \(P : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})\) be an elliptic, self-adjoint and non-negative differential operator of order \(\geq 1\).

Then, for every \(\lambda \geq 0\), the spectrum distribution function \(N_k\) of \(P\) (i.e. \(N_k(\lambda)\) is defined to be the number of eigenvalues of \(P\), counted with multiplicities, that are \(\leq \lambda\)) is given by the following identities (in which the suprema are actually maxima):

\[
N_k(\lambda) = \sup_{L \in \mathcal{L}^{(k)}_k} \dim L = \sup_{E \in \mathcal{P}^{(k)}_k} \text{Tr} E,
\]

where \(\mathcal{L}^{(k)}_k\) stands for the set of closed vector subspaces \(L \subset C^\infty_k(X, \mathbb{C})\) such that

\[
\langle\langle Pu, u \rangle\rangle \leq \lambda ||u||^2 \quad \text{for all } u \in L,
\]

while \(\mathcal{P}^{(k)}_k\) stands for the set of all bounded linear operators \(E : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})\) satisfying the conditions:
(i) \( E^2 = E = E^* \) (i.e. \( E \) is an orthogonal projection w.r.t. the \( L_2^k \) inner product); 
(ii) \( \langle \langle Pu, u \rangle \rangle \leq \lambda ||u||^2 \) for all \( u \in \text{Im} E \).

(In other words, \( E \) is the orthogonal projection onto one of the subspaces \( L \in \mathcal{L}_\lambda^{(k)} \), so \( L = \text{Im} E \) for some \( L \in \mathcal{L}_\lambda^{(k)} \).)

Proof. The second identity in (23) follows at once from the fact that the dimension of any closed subspace \( L \subset C^\infty_k(X, \mathbb{C}) \) equals the trace of the orthogonal projection onto \( L \). So, we only have to prove the first identity in (23).

Since \( X \) is compact and \( P \) is elliptic, self-adjoint and non-negative, the spectrum of \( P \) is discrete and consists of non-negative eigenvalues, while there exists a countable orthonormal (w.r.t. the \( L_2^k \)-inner product) basis of \( C^\infty_k(X, \mathbb{C}) \) (and of the Hilbert space \( L_2^k(X, \mathbb{C}) \) of \( L^2 \) \( k \)-forms) consisting of eigenvectors of \( P \). For every \( \mu \geq 0 \), let \( E_\mu \subset C^\infty_k(X, \mathbb{C}) \) be the eigenspace of \( E \) corresponding to the eigenvalue \( \mu \) (with the understanding that \( E_\mu = \{0\} \) if \( \mu \) is not an actual eigenvalue). The spaces \( E_\mu \) are finite-dimensional and consist of \( C^\infty \) forms since \( P \) is assumed to be elliptic (hence also hypoelliptic) and \( X \) is compact.

For every \( \lambda \geq 0 \), let \( L_\lambda := \bigoplus_{0 \leq \mu \leq \lambda} E_\mu \subset C^\infty_k(X, \mathbb{C}) \). Thus, \( L_\lambda \) is finite-dimensional and \( \dim L_\lambda = N_k(\lambda) \), while \( \langle \langle Pu, u \rangle \rangle \leq \lambda ||u||^2 \) for all \( u \in L_\lambda \). Hence \( L_\lambda \in \mathcal{L}_\lambda^{(k)} \), so \( N_k(\lambda) \leq \sup_{L \in \mathcal{L}_\lambda^{(k)}} \dim L \).

To prove the reverse inequality, let \( \lambda \geq 0 \) and let \( L \in \mathcal{L}_\lambda^{(k)} \). The existence of an orthonormal basis of eigenvectors implies the orthogonal direct-sum decomposition
\[
C^\infty_k(X, \mathbb{C}) = \bigoplus_{0 \leq \mu \leq \lambda} E_\mu \oplus \bigoplus_{\mu > \lambda} E_\mu.
\]
In particular, \( \bigoplus_{\mu > \lambda} E_\mu = \ker E_\lambda \), where \( E_\lambda \) is the orthogonal projection onto \( \bigoplus_{0 \leq \mu \leq \lambda} E_\mu \).

Now, \( \langle \langle Pu, u \rangle \rangle > \lambda ||u||^2 \) for all \( u \in \bigoplus_{\mu > \lambda} E_\mu \setminus \{0\} \), while \( \langle \langle Pu, u \rangle \rangle \leq \lambda ||u||^2 \) for all \( u \in L \). So, \( L \cap \ker E_\lambda = L \cap \bigoplus_{\mu > \lambda} E_\mu = \{0\} \). This implies that the restriction
\[
E_{\lambda|L} : L \rightarrow \text{Im} E_\lambda = \bigoplus_{0 \leq \mu \leq \lambda} E_\mu
\]
is injective. In particular, \( \dim L \leq \dim \bigoplus_{0 \leq \mu \leq \lambda} E_\mu = N_k(\lambda) \). Since \( L \) has been chosen arbitrarily in \( \mathcal{L}_\lambda^{(k)} \), we conclude that \( \sup_{L \in \mathcal{L}_\lambda^{(k)}} \dim L \leq N_k(\lambda) \) and we are done. \( \square \)

The second step towards proving Proposition 4.2 is the standard 3-space decomposition used in Hodge theory. For every \( k = 0, \ldots, 2n \), the operator \( \Delta_{\omega_h} : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C}) \) is elliptic and since the manifold \( X \) is compact and \( d^2 = 0 \), we have the \( L_2^\omega \)-orthogonal decomposition:
\[
C^\infty_k(X, \mathbb{C}) = \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}) \oplus E_k(X, \mathbb{C}) \oplus E^*_k(X, \mathbb{C}), \quad \text{where} \quad \ker d = \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}) \oplus E_k(X, \mathbb{C}), \quad (24)
\]
and where \( \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}) \) is the kernel of \( \Delta_{\omega_h} : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C}) \), \( E_k(X, \mathbb{C}) := \text{Im} \langle d : C^\infty_{k-1}(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C}) \rangle \) and \( E^*_k(X, \mathbb{C}) := \text{Im} \langle d^*_{\omega_h} : C^\infty_{k+1}(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C}) \rangle \).

Moreover, each of the three subspaces into which \( C^\infty_k(X, \mathbb{C}) \) splits in (24) is \( \Delta_{\omega_h} \)-invariant, i.e. \( \Delta_{\omega_h}(\mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C})) \subset \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}) \), etc.
\[ \Delta_{\omega_h}(\mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C})) \subset \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}), \quad \Delta_{\omega_h}(E_k(X, \mathbb{C})) \subset E_k(X, \mathbb{C}), \quad \Delta_{\omega_h}(E^*_k(X, \mathbb{C})) \subset E^*_k(X, \mathbb{C}) \]

because \( \Delta_{\omega_h} \) commutes with \( d \) and with \( d^*_{\omega_h} \). The invariance implies that an \( L^2_{\omega_h} \)-orthonormal basis \( \{e^k_i(h)\}_{i \in \mathbb{N}^*} \) of \( C^\infty_k(X, \mathbb{C}) \) consisting of eigenvectors for \( \Delta_{\omega_h} \) (whose existence follows from the standard elliptic theory) can be chosen such that each \( e^k_i(h) \) belongs to one and only one of the subspaces \( \mathcal{H}^k_{\Delta_{\omega_h}}(X, \mathbb{C}), E_k(X, \mathbb{C}) \) and \( E^*_k(X, \mathbb{C}) \). Let \( 0 \leq \lambda^k_i(h) \leq \cdots \leq \lambda^k_i(h) \leq \cdots \) be the corresponding eigenvalues, counted with multiplicities, of the rescaled Laplacian \( \Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}) \) (= those of \( \Delta_{\omega_h} : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C}) \)). Thus, \( \Delta_{\omega_h} e^k_i(h) = \lambda^k_i(h) e^k_i(h) \) for all \( i \).

Consequently, we can define functions \( F^k_h : [0, +\infty) \to \mathbb{N} \) and \( G^k_h : [0, +\infty) \to \mathbb{N} \) by

\[
F^k_h(\lambda) := \# \{ i \mid e^k_i(h) \in E^k_h(X, \mathbb{C}) \text{ and } \lambda^k_i(h) \leq \lambda \}
\]

and

\[
G^k_h(\lambda) := \# \{ i \mid e^k_i(h) \in E_k(X, \mathbb{C}) \text{ and } \lambda^k_i(h) \leq \lambda \}.
\]

These definitions of \( F^k_h \) and \( G^k_h(\lambda) \) are independent of the choice of orthonormal basis \( \{e^k_i(h)\}_{i \in \mathbb{N}^*} \) of \( C^\infty_k(X, \mathbb{C}) \) satisfying the above properties.

**Lemma 4.4.** The functions \( F^k_h \) and \( G^k_h(\lambda) \) are the spectrum distribution functions of the restrictions \( \Delta_{\omega_h|E^k_h(X, \mathbb{C})} : E^k_h(X, \mathbb{C}) \to E^k_h(X, \mathbb{C}) \), resp. \( \Delta_{\omega_h|E_k(X, \mathbb{C})} : E_k(X, \mathbb{C}) \to E_k(X, \mathbb{C}) \).

In other words, they are described as follows:

\[
F^k_h(\lambda) = \sup_{L \in \mathcal{L}''(\lambda)} \dim L, \quad (25)
\]

\[
G^k_h(\lambda) = \sup_{L \in \mathcal{L}'(\lambda)} \dim L
\]

where \( \mathcal{L}''(\lambda) \) stands for the set of closed vector subspaces \( L \subset E^k_h(X, \mathbb{C}) \) such that

\[
\|du\|_{\omega_h}^2 \leq \lambda \|u\|_{\omega_h}^2 \quad \text{for all } u \in L.
\]

and \( \mathcal{L}'(\lambda) \) stands for the set of closed vector subspaces \( L \subset E_k(X, \mathbb{C}) \) such that

\[
\|d^*_h u\|_{\omega_h}^2 \leq \lambda \|u\|_{\omega_h}^2 \quad \text{for all } u \in L.
\]

**Proof.** This is an immediate application of the *variational principle* of Proposition 4.3 to the restrictions \( \Delta_{\omega_h|E^k_h(X, \mathbb{C})} : E^k_h(X, \mathbb{C}) \to E^k_h(X, \mathbb{C}) \) and \( \Delta_{\omega_h|E_k(X, \mathbb{C})} : E_k(X, \mathbb{C}) \to E_k(X, \mathbb{C}) \). Estimates (26) and (27) are consequences of the identity \( \langle \langle \Delta_{\omega_h} u, u \rangle \rangle_{\omega_h} = \|du\|_{\omega_h}^2 + \|d^*_h u\|_{\omega_h}^2 \) and of the fact that \( d^*_h u = 0 \) whenever \( u \in E^k_h(X, \mathbb{C}) \) (since \( \text{Im} \ d^*_h \subset \ker d_{\omega_h} \)) and that \( du = 0 \) whenever \( u \in E_k(X, \mathbb{C}) \) (since \( \text{Im} \ d \subset \ker d \)). \( \square \)

The last ingredient we need is the following very simple observation.

**Lemma 4.5.** For every \( \lambda \geq 0 \) and every \( k \in \{-1, 0, \ldots, 2n\} \), we have

\[
F^k_h(\lambda) = G^{k+1}_h(\lambda) \quad \text{with the understanding that } \quad F^{-1}_h(\lambda) = G^{2n+1}_h(\lambda) = 0.
\]
Proof. We know from the orthogonal decompositions (24) that the restriction of $d$ to $E_k^*(X, \mathbb{C})$ is injective, so

$$d|_{E_k^*(X, \mathbb{C})} : E_k^*(X, \mathbb{C}) \rightarrow E_{k+1}(X, \mathbb{C})$$

is an isomorphism. Moreover, $d\Delta_{\omega_h} = \Delta_{\omega_h} d$, so whenever $\Delta_{\omega_h} u_i = \lambda_i^h(h) u_i$, we get $\Delta_{\omega_h}(du_i) = \lambda_i^h(h) (du_i)$. Combined with the above isomorphism, with the invariance of $E_k^*(X, \mathbb{C})$ under $\Delta_{\omega_h}$ and with the definitions of $F_k^h(\lambda)$ and $G_k^h_{k+1}(\lambda)$, this implies the contention. \[ \square \]

Proof of Proposition 4.2. Putting together (24), the definitions of $F_k^h(\lambda)$ and $G_k^h(\lambda)$ and the fact that the Hodge isomorphism $H^k_{\Delta_{\omega_h}} \simeq H^k_{DR}(X, \mathbb{C})$ (which follows at once from (24)) implies $b_k = \dim \mathcal{H}^k_{\Delta_{\omega_h}}$, we get

$$N^k_h(\lambda) = b_k + G^k_h(\lambda) + F^k_h(\lambda)$$

for all $k$ and all $\lambda \geq 0$. Using Lemma 4.5, this is equivalent to (20).

On the other hand, the descriptions (25) and (26) of $F^k_h(\lambda)$ coincide with the descriptions (21) and (22) thanks to the isomorphism $E_k^*(X, \mathbb{C}) \simeq C^\infty_k(X, \mathbb{C})/\ker d$, which is another consequence of the decompositions (24). \[ \square \]

4.2 Metric independence of asymptotics

Although the following statement has no impact on either the statement of Theorem 1.3 or its proof, we pause briefly to show, exactly as in the foliated case of [ALK00], that the asymptotics of the eigenvalues $\lambda_i^h(h)$ and of the spectrum distribution function $N^k_h$ as $h \downarrow 0$ depend only on the complex structure of $X$. The proof is an easy application of the Variational Principle of Proposition 4.2.

Proposition 4.6. The asymptotics of the $\lambda_i^h(h)$’s and of $N^k_h$ as $h \downarrow 0$ are independent of the choice of Hermitian metric $\omega$.

Proof. We adapt to our setting the proof of the corresponding result in [ALK00]. Let $\omega$ and $\omega'$ be two Hermitian metrics on $X$. They induce, respectively, rescaled metrics $(\omega_h)_{h>0}$ and $(\omega'_h)_{h>0}$. Let $N^k_h(\lambda) = F^k_{\omega'_h}(\lambda) + b_k + F^k_h(\lambda)$ be the spectrum distribution function associated with the rescaled Laplacian $\Delta^2_{\omega_h} : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})$, written as in (20).

Since $X$ is compact, there exists a constant $C > 0$ such that the respective $L^2$-norms satisfy the following inequalities in every bidegree $(p, q)$:

$$\| | \omega \| \leq C \| | \omega' \| \leq C \| | \omega \|,$$

hence

$$\frac{1}{C} \| | \omega_h \| \leq C \| | \omega_h' \| \leq C \| | \omega_h \| \quad \text{on $L^2_{p,q}(X, \mathbb{C})$ for every $h > 0$.}$$

The constant $C$ is independent of $h > 0$ thanks to Formula 2.2.

Hence, for every $\zeta \in C^\infty_k(X, \mathbb{C})/\ker d$ such that $|d\zeta|_{\omega_h} \leq \sqrt{\lambda} |\zeta|_{\omega_h}$, we get $|d\zeta|_{\omega_h'} \leq \sqrt{C^4 \lambda} |\zeta|_{\omega_h'}$. Thanks to Proposition 4.2, this implies that

$$F^k_h(\lambda) \leq F^k_{\omega'_h}(C^4 \lambda), \quad \lambda \geq 0, \quad h > 0.$$

By symmetry, we also get $F^k_{\omega'_h}(\lambda) \leq F^k_h(\lambda)$, so putting the last two inequalities together, we get

$$F^k_{\omega'_h}(C^4 \lambda) \leq F^k_h(\lambda) \leq F^k_{\omega'_h}(C^4 \lambda), \quad \lambda \geq 0, \quad h > 0.$$

The proof is complete. \[ \square \]
4.3 Proof of the inequality “≤” in Theorem 1.3

We are now in a position to prove the following

**Theorem 4.7.** Let \((X, \omega)\) be a compact Hermitian manifold with \(\dim C X = n\). For every \(r \) and every \(k = 0, \ldots, 2n\), the following inequality holds:

\[
\dim E_r^k \leq \mathbb{Z}\{i \mid \lambda_i^k(h) \in O(h^{2r}) \text{ as } h \downarrow 0\}. \tag{28}
\]

**Proof.** We have to prove the existence of a uniform constant \(C > 0\) such that \(\dim E_r^k \leq N^k_h(C h^{2r})\) for all \(r, k\) and all \(0 < h \ll 1\). Recall the following facts:

(i) \(\dim E_r^k = b_k + m_r^{k-1} + m_r^k\), where \(m_r^k := \dim L_r^k\) and \(L_r^k := \bigoplus_{p+q=k} L_r^{p,q} = \bigoplus_{p+q=k} (E_r^{p,q} / \ker d_r^{p,q})\)

(proved in (13) of Lemma 3.1);

(ii) \(N_h^k(\lambda) = b_k + F_h^{k-1}(\lambda) + F_h^k(\lambda)\) for all \(\lambda \geq 0\)

(cf. (20) of Proposition 4.2).

Thus, it suffices to prove that

\[
m_r^k \leq F_h^k(C h^{2r}) \quad \text{for all } 0 < h \ll 1, \tag{29}
\]

for a uniform constant \(C > 0\) and for all \(r \) and \(k\).

Now, thanks to the definition (21) of \(F_h^k\), to prove (29) it suffices to prove that \(L_r^k\) is one of the subspaces of \(C_k^\infty(X, \mathbb{C})/\ker d\) contributing to the definition of \(F_h^k(C h^{2r})\) for some uniform constant \(C > 0\). In other words, it suffices to prove that there exists \(C > 0\) such that

\[
||d\zeta||_{\omega_h} \leq \sqrt{C} h^r ||\zeta||_{\omega_h}, \quad \text{for all } \zeta \in L_r^k \text{ and all } 0 < h \ll 1. \tag{30}
\]

Meanwhile, every \(\zeta \in L_r^k = \bigoplus_{p+q=k} L_r^{p,q}\) splits uniquely as \(\zeta = \sum_{p+q=k} \zeta^{p,q}\) with \(\zeta^{p,q} \in L_r^{p,q}\) for all \(p, q\).

Thus, it suffices to prove that, for a uniform constant \(C > 0\), we have

\[
||d\zeta^{p,q}||_{\omega_h} \leq \sqrt{C} h^r ||\zeta^{p,q}||_{\omega_h}, \quad \text{for all } p, q, \text{ all } \zeta^{p,q} \in L_r^{p,q} \text{ and all } 0 < h \ll 1. \tag{31}
\]

This holds mainly because \(d_r\) is of type \((r, -r + 1)\), so \(d_r\) increases the holomorphic degree by \(r\) and thus the norm \(|| \cdot ||_{\omega_h}\) brings out an extra factor \(h^r\). Specifically, for every \(\zeta^{p,q} \in L_r^{p,q}\), (14) of Lemma 3.1 yields \(d\zeta^{p,q} \in d(L_r^{p,q}) \subset A_r^{p+q-1}\). Therefore, the holomorphic degree of \(d\zeta^{p,q}\) is \(\geq p + r\), so from Formula 2.2 we get

\[
||d\zeta^{p,q}||_{\omega_h} \leq \frac{h^{p+r}}{h^n} ||d\zeta^{p,q}||_{\omega} \quad \text{for all } p, q, \text{ all } \zeta^{p,q} \in L_r^{p,q} \text{ and all } 0 < h < 1.
\]

Now, \(L_r^{p,q}\) is a finite-dimensional vector subspace of \(C_k^\infty(X, \mathbb{C})/\ker d\), so there exists a constant \(C_r > 0\) (depending on \(r, p, q\), but independent of \(h\)) such that \(||d\zeta^{p,q}||_{\omega} \leq C_r ||\zeta^{p,q}||_{\omega_h}\) for all \(\zeta^{p,q} \in L_r^{p,q}\). Meanwhile, Formula 2.2 tells us again that \(||\zeta^{p,q}||_{\omega} = (h^n/h^p)||\zeta^{p,q}||_{\omega_h}\), so putting the last three relations together, we get

\[
||d\zeta^{p,q}||_{\omega_h} \leq C_r h^r ||\zeta^{p,q}||_{\omega_h} \quad \text{for all } p, q, \text{ all } \zeta^{p,q} \in L_r^{p,q} \text{ and all } 0 < h < 1.
\]
This proves (31) after setting $C := \max_{0 < r \leq N} C_r^2 > 0$.

The proof is complete. \hfill \Box

Note that $L^k_r$ is a vector space of classes of cohomology classes, rather than of differential forms, so what is meant by $L^k_r$ in the above proof is its image in $C^\infty_k(X, \mathbb{C})/\ker d$ under the isometries explained in §3.2. We can use these isometries, the identification of $d$ acting on $\mathcal{H}^{p,q}$ with $d_r$ and Conclusion 3.4 in the following way to make the above proof even more explicit. If we choose $\zeta^{p,q}$ to be the $\omega_h$-harmonic representative of its class (also denoted by $\zeta^{p,q}$) and to play the role of $\alpha$ of Conclusion 3.4, we can re-write the above inequalities in a more detailed form as follows:

$$||d\zeta^{p,q}||_{\omega_h} = ||(P(\partial u_{r-1})||_{\omega_h} \leq \frac{h^{p+r}}{h^n} ||(P \circ T)(\zeta^{p,q})||_\omega$$

$$\leq \frac{h^{p+r}}{h^n} C_r ||\zeta^{p,q}||_\omega = C_r h^r ||\alpha||_{\omega_h},$$

where $P$ and $T$ are the linear maps $P^{p,q}$ and $T^{p,q}$ (with indices removed) of Conclusion 3.4 that was used above, while $|| \cdot ||_{\omega_h}$ stands for the $L^2_{\omega_h}$-norm when applied to a form and for the induced quotient norm when applied to a class.

### 4.4 Preliminaries to the proof of the inequality “≥” in Theorem 1.3

We will need a few simple observations.

**Lemma 4.8.** Let $(X, \omega)$ be a compact Hermitian manifold with $\dim \mathbb{C}X = n$. For every bidegree $(p, q)$ and every $(p, q)$-form $u$ on $X$, the following identities hold:

$$\langle \langle \Delta_h u, u \rangle \rangle_\omega = h^{2(n-p)} \langle \langle \Delta_{\omega_h} u, u \rangle \rangle_\omega = h^{2(n-p)} (||du||_{\omega_h}^2 + ||d^*_w u||_{\omega_h}^2).$$

(32)

**Proof.** The latter identity is obvious, so we will only prove the former one. Since $u$ is of pure type, (4) yields the first identity below, while the second identity follows from Formula 2.2:

$$\langle \langle \Delta_h u, u \rangle \rangle_\omega = h^2 \langle \langle \Delta' u, u \rangle \rangle_\omega + \langle \langle \Delta'' u, u \rangle \rangle_\omega = h^2 h^{2(n-p)} \langle \langle \Delta' u, u \rangle \rangle_\omega + h^{2(n-p)} \langle \langle \Delta'' u, u \rangle \rangle_\omega$$

$$= h^{2(n-p)} \langle \langle \Delta_{\omega_h} u, u \rangle \rangle_\omega.$$

The last identity followed again from (4). \hfill \Box

**Lemma 4.9.** Let $u \in C^\infty_{p,q}(X, \mathbb{C})$ be an arbitrary form. Considering the splitting $d = d^{(k)} = \bigoplus_{0 \leq r \leq N-1, p+q=k} d^p_r : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_{k+1}(X, \mathbb{C})$ of the operator $d$ (see (12)) and the splitting

$$u = \sum_{r=0}^{N-1} u_r + \ker d, \quad \text{implying} \quad du = \sum_{r=0}^{N-1} d_r u_r,$$

with $u_r \in E^p_{r,q}/\ker d^p_r$ (see §3 and recall that $d_r : E^p_{r,q}/\ker d^p_r \rightarrow \text{Im} d^p_r \subset C^\infty_{p+r,q-r+1}(X, \mathbb{C})$ is an isomorphism), the following identity holds:

$$h^{2(n-p)} ||du||^2_{\omega_h} = \sum_{r=0}^{N-1} h^{2r} ||d_r u_r||^2_{\omega} \quad \text{for all } h > 0.$$  \hfill (33)
Proof. Since $d_r$ is of type $(r, -r + 1)$, $d_r u_r$ is of type $(p + r, q - r + 1)$, so the $d_r u_r$’s are mutually orthogonal (w.r.t. any metric) when $r$ varies. We get

$$
||du||^2_{\omega_h} = \sum_{r=0}^{N-1} ||d_r u_r||^2_{\omega_h} = \sum_{r=0}^{N-1} \frac{h^{2r}}{h^{2n}} ||d_r u_r||^2_{\omega},
$$

where for the last identity we used Formula 2.2.

\[ \square \]

**Lemma 4.10.** For every $r$ and every bidegree $(p, q)$, the formal adjoints of $d_r$ w.r.t. the metrics $\omega_h$ and $\omega$ compare as follows:

$$(d_r)^*_\omega = h^{2r} (d_r)^*_\omega.$$  \hspace{1cm} \text{(34)}

Consequently, for every form $u \in C^\infty_{p,q}(X, \mathbb{C})$, the following counterpart of Lemma 4.9 for the adjoints holds. Considering the splitting $(d^k)^*_\omega_h = \bigoplus_{0 \leq r \leq N-1} (d^{p,q})^*_{\omega_h} : C^\infty_{k+1}(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})$ of the operator $d^*$ and the splitting

$$u = \sum_{r=0}^{N-1} v_r + \ker d^*_\omega_h,$$

implying

$$d^*_\omega_h u = \sum_{r=0}^{N-1} (d_r)^*_\omega_h v_r,$$

with $v_r \in \text{Im} d^{p-r, q+r-1}$ (see §3.1), the following identity holds:

$$h^{2(n-p)} ||d^*_\omega_h u||^2_{\omega_h} = \sum_{r=0}^{N-1} \frac{h^{2p}}{h^{2n}} ||(d_r)^*_\omega_h v_r||^2_{\omega} \quad \text{for all } h > 0.$$  \hspace{1cm} \text{(35)}

\textbf{Proof.} For every $(p, q)$-form $v$ and every $(p - r, q + r - 1)$-form $u$, we have

$$\frac{h^{2(p-r)}}{h^{2n}} \langle (d_r)^*_{\omega_h} v, u \rangle_{\omega} = \langle (d_r)^*_{\omega_h} v, u \rangle_{\omega_h} = \langle v, d_r u \rangle_{\omega_h} = \frac{h^{2p}}{h^{2n}} \langle v, d_r u \rangle_{\omega} = \frac{h^{2p}}{h^{2n}} \langle (d_r)^*_\omega v, u \rangle_{\omega}.$$

This proves (34). Using the mutual orthogonality of the $(d_r)^*_\omega v_r$’s (due to bidegree reasons) and Formula 2.2, we get

$$\begin{align*}
||d^*_\omega_h u||^2_{\omega_h} &= \sum_{r=0}^{N-1} ||(d_r)^*_\omega_h v_r||^2_{\omega_h} = \sum_{r=0}^{N-1} \frac{h^{2(p-r)}}{h^{2n}} ||(d_r)^*_\omega_h v_r||^2_{\omega} = \sum_{r=0}^{N-1} \frac{h^{2(p-r)}}{h^{2n}} h^{4r} ||(d_r)^*_\omega v_r||^2_{\omega}.
\end{align*}$$

This proves (35). \[ \square \]

Putting together (32), (33) and (35), we get

**Corollary 4.11.** Let $(X, \omega)$ be a compact Hermitian manifold with $\text{dim}_{\mathbb{C}} X = n$. For every bidegree $(p, q)$ and every $(p, q)$-form $u$ on $X$, the following identity holds:

$$\langle \Delta_h u, u \rangle_{\omega} = \sum_{r'=0}^{N-1} \frac{h^{2r'}}{h^{2n}} ||d_r u_{r'}||^2_{\omega} + \sum_{r'=0}^{N-1} \frac{h^{2r'}}{h^{2n}} ||(d_r)^*_\omega v_{r'}||^2_{\omega},$$

21
where $u$ splits uniquely (cf. §3.1) as

$$u = \sum_{r' = 0}^{N-1} u_{r'} + \ker d = \sum_{r' = 0}^{N-1} v_{r'} + \ker d^* = \sum_{r' = 0}^{N-1} u_{r'} + \sum_{r' = 0}^{N-1} v_{r'} + w$$

with $u_{r'} \in E^{p,q}_{r'} / \ker d^{p,q}_{r'}$, $v_{r'} \in \text{Im} d^{p-r',q+r'-1}$ and $w \in E^{\infty q}_\infty$.

4.5 Proof of the inequality “$\geq$” in Theorem 1.3

Following again the analogy with the foliated case of [ALK00], we will actually prove a stronger statement from which the following result will follow as a corollary.

Theorem 4.12. Let $(X, \omega)$ be a compact Hermitian manifold with $\dim \mathbb{C} X = n$. For every $r$ and every $k = 0, \ldots, 2n$, the following inequality holds:

$$\dim E^k_r \geq \sharp \{ i \mid \lambda^k_i(h) \in O(h^{2r}) \text{ as } h \downarrow 0 \}. \quad (36)$$

The first main ingredient we will use is the pseudo-differential Laplacian

$$\tilde{\Delta} = \partial p'' \partial^* + \partial^* p'' \partial + \Delta'' : C^\infty_{p,q}(X, \mathbb{C}) \rightarrow C^\infty_{p,q}(X, \mathbb{C})$$

defined in arbitrary bidegree $(p, q)$ and introduced in [Pop16], where $p'': C^\infty_{p,q}(X, \mathbb{C}) \rightarrow \ker \Delta''$ is the orthogonal projection (w.r.t. the $L^2_\omega$-norm) onto the $\Delta''$-harmonic subspace of $C^\infty_{p,q}(X, \mathbb{C})$. The pseudo-differential Laplacian $\tilde{\Delta}$ gives a Hodge theory for the second page of the Frölicher spectral sequence in the sense that there is a Hodge isomorphism

$$E^p_{2q} \xrightarrow{\sim} H^q_{\tilde{\Delta}}(X, \mathbb{C}) := \ker(\tilde{\Delta} : C^\infty_{p,q}(X, \mathbb{C}) \rightarrow C^\infty_{p,q}(X, \mathbb{C})) \text{ for all } p, q = 0, \ldots, n. \quad (37)$$

Note that $(p'')^2 = p''$, so $\partial p'' \partial^* = (p'' \partial^*)(p'' \partial)$ and $\partial^* p'' \partial = (p'' \partial)(p'' \partial)$. Thus, $\tilde{\Delta}$ is a sum of non-negative operators, so its kernel is the intersection of the respective kernels. Since $\ker(A^* A) = \ker A$ for any operator $A$, we get

$$\ker \tilde{\Delta} = \ker(p'' \partial) \cap \ker(p'' \partial^*) \cap \ker \partial \cap \ker \partial^*.$$ 

The second main ingredient we will use is the following lower estimate of the rescaled Laplacian $\Delta_h$. It is the analogue in our context of a result in [ALK00].

Lemma 4.13. Let $(X, \omega)$ be a compact Hermitian manifold with $\dim \mathbb{C} X = n$. There exists a constant $C > 0$ such that the following inequality of linear operators (cf. Notation 1.4) holds on differential forms of any degree $k = 0, \ldots, 2n$:

$$\Delta_h \geq \frac{3}{4} \Delta'' + h^2 \Delta' - Ch^2 \quad \text{for all } h > 0,$$

where $\Delta'' = \bar{\partial}\partial^* + \partial^* \bar{\partial}$ and $\Delta' = \partial\partial^* + \partial^* \partial$ are the usual $\bar{\partial}$- and $\partial$-Laplacians.
The coefficients 3/4 and 1 are not optimal, but they suffice for our purposes and the proof provided below shows that they can be made optimal if this is desired.

**Proof of Lemma 4.13.** We know from (ii) of Lemma 2.7 that

$$\Delta_h = \Delta'' + h^2 \Delta' - h([\tau, \tilde{\tau}^*] + [\tau^*, \tilde{\partial}]),$$

where $\tau = \tau_\omega := [\Lambda, \partial \omega \wedge \cdot]$ is the zero-th order torsion operator of type $(1,0)$ associated with $\omega$

For any form $u$, the first-order terms on the r.h.s. of (38) are easily estimated using the Cauchy-Schwarz inequality as follows:

$$h |\langle [\tau, \tilde{\tau}^*]u + [\tau^*, \tilde{\partial}]u, u \rangle| = h |\langle \tilde{\tau}^* u, \tau u \rangle + \langle \tau u, \tilde{\partial} u \rangle + \langle \tau^* u, \tilde{\partial} u \rangle + \langle \tau u, \tau^* u \rangle|$$

$$\leq 2h|\tau u||\tilde{\partial} u| + 2h|\tau^* u||\tilde{\partial} u|$$

$$\leq \frac{1}{4} (||\tilde{\partial} u||^2 + ||\tilde{\partial}^* u||^2) + 4h^2 (||\tau u||^2 + ||\tau^* u||^2)$$

$$\leq \frac{1}{4} (\langle \Delta'' u, u \rangle) + Ch^2 ||u||^2,$$

where the constant $C > 0$ exists because the linear operators $\tau$ and $\tau^*$ are of order zero, hence bounded. In particular, we get the operator inequality $-h([\tau, \tilde{\tau}^*] + [\tau^*, \tilde{\partial}]) \geq -\frac{1}{4} \Delta'' - C h^2$ which, alongside (38), proves the contention. \( \square \)

We are now ready to state and prove a general result that will imply Theorem 4.12.

**Theorem 4.14.** Let $(X, \omega)$ be a compact Hermitian manifold with $\dim \mathcal{C} X = n$. Let $k \in \{0, \ldots, 2n\}$ and $r \geq 1$ be fixed integers. Suppose there exists a sequence $(h_i)_{i \in \mathbb{N}}$ of constants $h_i > 0$ such that $\frac{1}{n} \downarrow 0$ and a sequence $(u_i)_{i \in \mathbb{N}}$ of $k$-forms $u_i \in C^\infty_k(X, \mathbb{C})$ such that $||u_i||_\omega = 1$ for every $i$ and

$$\langle \langle \Delta_h u_i, u_i \rangle \rangle_\omega \in o(h_i^{2(r-1)}) \quad \text{as } i \to +\infty. \quad (39)$$

Then, there exists a subsequence $(u_{i_j})_{j \in \mathbb{N}}$ of $(u_i)_{i \in \mathbb{N}}$ such that $(u_{i_j})_{j \in \mathbb{N}}$ converges in the $L^2_\omega$-topology to some $k$-form $u \in \mathcal{H}_r^k := \bigoplus_{p+q=k} \mathcal{H}_{p,q}^r \simeq E_k^r$, where the $\mathcal{H}_{p,q}^r \subset C^\infty_{p,q}(X, \mathbb{C})$ are the "harmonic" vector subspaces of Definition 3.3 induced by the metric $\omega$.

**Proof.** *Case $r = 1$.* In this case, Hypothesis (39) means that $\langle \langle \Delta_h u_i, u_i \rangle \rangle_\omega \to 0$ as $i \to +\infty$. Then also $\langle \langle \Delta_h u_i, u_i \rangle \rangle_\omega + Ch_i^2 \to 0$ as $i \to +\infty$. Since, by Lemma 4.13, we have

$$\langle \langle \Delta_h u_i, u_i \rangle \rangle_\omega + Ch_i^2 \geq \frac{3}{4} \langle \langle \Delta'' u_i, u_i \rangle \rangle_\omega + h_i^2 \langle \langle \Delta' u_i, u_i \rangle \rangle_\omega \geq 0 \quad \text{for all } i \in \mathbb{N},$$

we get

(i) $\langle \langle \Delta'' u_i, u_i \rangle \rangle_\omega \to 0$ as $i \to +\infty$ and (ii) $h_i^2 \langle \langle \Delta' u_i, u_i \rangle \rangle_\omega \to 0$ as $i \to +\infty. \quad (40)$

Meanwhile, the $\tilde{\partial}$-Laplacian $\Delta''$ is elliptic and the manifold $X$ is compact, so the Gårding inequality yields constants $\delta_1, \delta_2 > 0$ such that the first inequality below holds:

$$\delta_2 ||u_i||_{W^{1}} \leq \langle \langle \Delta'' u_i, u_i \rangle \rangle_\omega + \delta_1 ||u_i||_\omega \leq C_1, \quad \text{for all } i \in \mathbb{N},$$

23
where $|| \cdot ||_{W^1}$ stands for the Sobolev norm $W^1$ induced by the metric $\omega$. The second inequality above holds for some constant $C_1 > 0$ since the quantity $\langle \Delta'' u_i, u_i \rangle_\omega$ converges to zero (cf. (40)), hence is bounded, and $||u_i||_\omega = 1$ by the hypothesis of Theorem 4.14.

Consequently, the sequence $(u_i)_{i \in \mathbb{N}}$ is bounded in the Sobolev space $W^1$ (a Hilbert space), so by the Banach-Alaoglu Theorem there exists a subsequence $(u_{i_l})_{l \in \mathbb{N}}$ that converges in the weak topology of $W^1$ to some $k$-form $u \in W^1$. In particular, the following convergences hold in the weak topology of distributions:

$$\bar{\partial} u_{i_l} \rightharpoonup \bar{\partial} u \quad \text{and} \quad \bar{\partial}^* u_{i_l} \rightharpoonup \bar{\partial}^* u \quad \text{as} \quad l \to +\infty.$$ 

On the other hand, $||\bar{\partial} u_{i_l}||^2 + ||\bar{\partial}^* u_{i_l}||^2 = \langle \Delta'' u_{i_l}, u_{i_l} \rangle_\omega \to 0$ as $i \to +\infty$, so $\bar{\partial} u_{i_l} \to 0$ and $\bar{\partial}^* u_{i_l} \to 0$ in the $L^2$-topology as $i \to +\infty$. Comparing this with the above convergences in the weak topology of distributions, we get

$$\bar{\partial} u = 0 \quad \text{and} \quad \bar{\partial}^* u = 0,$$

which, by (18), is equivalent to $u \in \ker(\Delta'') : C^\infty_c(X, \mathbb{C}) \hookrightarrow C^\infty_c(X, \mathbb{C})) = \mathcal{H}^k \simeq E^k$.

Note that by the Rellich Lemma (asserting the compactness of the inclusion $W^1 \hookrightarrow L^2$), the convergence of $(u_{i_l})_{l \in \mathbb{N}}$ to $u$ in the weak topology of $W^1$ implies that $(u_{i_l})_{l \in \mathbb{N}}$ also converges in the $L^2$-topology to $u$. Moreover, the ellipticity of $\Delta''$ and the relation $u \in \ker \Delta''$ imply that $u$ is $C^\infty$.

**Case $r = 2$.** In this case, Hypothesis (39) means that $\langle \Delta h, u_i \rangle_\omega \in o(h_2^2)$ as $i \to +\infty$. Since $\langle \Delta h, u_i \rangle_\omega = ||d_h u_i||^2 + ||d^*_h u_i||^2 = ||\Delta h u_i + \bar{\partial} u_i||^2 = ||h_2 \bar{\partial} u_i + \bar{\partial} u_i||^2$, this implies that

$$\bar{\partial} u_i + \frac{1}{h_i} \bar{\partial} u_i \to 0 \quad \text{and} \quad \bar{\partial}^* u_i + \frac{1}{h_i} \bar{\partial}^* u_i \to 0 \quad \text{in the } L^2\text{-topology, as } i \to +\infty. \quad (41)$$

Since the orthogonal projection $p''$ onto $\ker \Delta''$ is continuous w.r.t. the $L^2$-topology and since $p'' \bar{\partial} = 0$ and $p'' \bar{\partial}^* = 0$ (because $\text{Im} \bar{\partial} \perp \ker \Delta''$ and $\text{Im} \bar{\partial}^* \perp \ker \Delta''$), an application of $p''$ to (41) yields

$$p'' \bar{\partial} u_i \to 0 \quad \text{and} \quad p'' \bar{\partial}^* u_i \to 0 \quad \text{in the } L^2\text{-topology, as } i \to +\infty. \quad (42)$$

On the other hand, we know from the discussion of the case $r = 1$ (whose weaker assumption is still valid in the case $r = 2$) that there exists a subsequence $(u_{i_l})_{l \in \mathbb{N}}$ that converges in the weak topology of $W^1$ to some $k$-form $u \in W^1$. Thus, $\partial u_{i_l} \rightharpoonup \partial u \in L^2$ in the weak topology of $L^2$ as $l \to +\infty$. This means that

$$\langle \partial u_{i_l}, v \rangle_\omega \rightharpoonup \langle \partial u, v \rangle_\omega \quad \text{for all } v \in L^2, \quad \text{hence} \quad \langle \partial u_{i_l}, p'' v \rangle_\omega \rightharpoonup \langle \partial u, p'' v \rangle_\omega \quad \text{for all } v \in L^2,$$

as $l \to +\infty$. (The second convergence follows from the fact that $||p'' v|| \leq ||v||$ for all $v \in L^2$, so $p''(L^2) \subset L^2$.) Now, $p''$ is self-adjoint, so the last convergence translates to

$$\langle p'' \partial u_{i_l}, v \rangle_\omega \rightharpoonup \langle p'' \partial u, v \rangle_\omega \quad \text{as } l \to +\infty, \quad \text{for all } v \in L^2.$$

This means that $p'' \partial u_{i_l}$ converges to $p'' \partial u$ in the weak topology of $L^2$ as $l \to +\infty$. However, we know from (42) that $p'' \partial u_{i_l}$ converges to 0 in the $L^2$-topology. Hence $p'' \partial u = 0$. The same argument run with $\bar{\partial}^*$ in place of $\partial$ yields that $p'' \bar{\partial}^* u = 0$. On the other hand, we know from the discussion of the case $r = 1$ that $u \in \ker \bar{\partial} \cap \ker \bar{\partial}^* = \ker \Delta''$, so we get

$$u \in \ker(p'' \partial) \cap \ker(p'' \bar{\partial}^*) \cap \ker \bar{\partial} \cap \ker \bar{\partial}^* = H^k \simeq E^k$$
after remembering the description (18) of the spaces $\mathcal{H}_2^{p,q}$ and that $\mathcal{H}_2^k = \bigoplus_{p+q=k} \mathcal{H}_2^{p,q}$.

- **Case** $r \geq 3$. Using the information from the first two cases and from subsection §4.4, this last case can easily be dealt with as follows.

  For each of the $k$-forms $u_i$ given by the hypotheses of Theorem 4.14, we consider the splitting

  $$u_i = \sum_{r'=0}^{N-1} u_{i,r'}^{(i)} + \sum_{r'=0}^{N-1} v_{i,r'}^{(i)} + w_i,$$

  with $u_{i,r'}^{(i)} \in E_{r'}^{p,q} / \ker d_{r'}^{p,q}$, $v_{i,r'}^{(i)} \in \text{Im} d_{r'}^{p-r',q+r'-1}$ and $w_i \in E_\infty^{p,q}$, and the corresponding splitting

  $$\langle \langle \Delta_h u_i, u_i \rangle \rangle = \sum_{r'=0}^{N-1} h_i^{2r'} \| d_{r'} u_{i,r'}^{(i)} \|_\omega^2 + \sum_{r'=0}^{N-1} h_i^{2r'} \| (d_{r'}^*)_{\omega} v_{i,r'}^{(i)} \|_\omega^2$$

  obtained in Corollary 4.11.

  On the other hand, (39) ensures that $\langle \langle \Delta_h u_i, u_i \rangle \rangle \in o(h_i^{2(r-1)})$ as $i \to +\infty$. Together with the above identity, this implies the following convergences in the $L^2$-norm as $i \to +\infty$:

  $$d_{r'} u_{i,r'}^{(i)} \to 0 \quad \text{and} \quad (d_{r'})^*_{\omega} v_{i,r'}^{(i)} \to 0 \quad \text{for every} \quad r' \in \{0, \ldots, r-1\}.$$

  We even get

  $$\frac{1}{h_i^{r-r'-1}} d_{r'} u_{i,r'}^{(i)} \to 0 \quad \text{and} \quad \frac{1}{h_i^{r-r'-1}} (d_{r'})^*_{\omega} v_{i,r'}^{(i)} \to 0 \quad \text{for every} \quad r' \in \{0, \ldots, r-1\}.$$

  Defining in an ad hoc way a “formal” Laplacian by $\Delta_{r'}^{\text{formal}} := d_{r'} (d_{r'}^*)_{\omega} + (d_{r'})^*_{\omega} d_{r'}$, we get that the limit $u$ of a subsequence of $(u_i)_{i \in \mathbb{N}}$ lives in

  $$\ker \left( \Delta_{r-1}^{\text{formal}} : \bigoplus_{p+q=k} E_{r-1}^{p,q} \to \bigoplus_{p+q=k} E_{r-1}^{p,q} \right) \simeq \mathcal{H}_r^k \simeq E_r^k$$

  and we are done.

*Proof of Theorem 4.12.* It is an immediate consequence of Theorem 4.14. Indeed, fix any $r \in \mathbb{N}^*$ and $k \in \{0, \ldots, 2n\}$ and suppose that inequality (36) does not hold. Then, the reverse strict inequality holds, so there exists a sequence $(h_i)_{i \in \mathbb{N}}$ of positive constants such that $h_i \downarrow 0$ when $i \to +\infty$ and a sequence $(u_i)_{i \in \mathbb{N}}$ of eigenvectors for the Laplacians $\Delta_{h_i}$ acting on $k$-forms such that $\|u_i\|_\omega = 1$, $u_i \perp \mathcal{H}_r^k$ for all $i$ and $\langle \langle \Delta_{h_i} u_i, u_i \rangle \rangle \in o(h_i^{2(r-1)})$ as $i \to +\infty$.

  Thanks to Theorem 4.14, there exists a subsequence $(u_{i_j})_{j \in \mathbb{N}}$ of $(u_i)_{i \in \mathbb{N}}$ such that $(u_{i_j})_{j \in \mathbb{N}}$ converges in the $L^2_\omega$-topology to some $k$-form $u \in \mathcal{H}_r^k \simeq E_r^k$. However, the form $u$ is orthogonal to $\mathcal{H}_r^k$ since $u_i \perp \mathcal{H}_r^k$ for all $i$ and the orthogonality property is preserved in the limit. Since $\|u\|_\omega = 1$ (because $\|u_i\|_\omega = 1$ for all $i$), $u \neq 0$, so $u$ cannot be at once orthogonal to and a member of $\mathcal{H}_r^k$. This is a contradiction. 

\[\square\]
5 Consequences of Theorem 1.3

The following consequences of Theorem 1.3 are of independent interest.

**Proposition 5.1.** Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For every $r \in \mathbb{N}^*$ and every $k = 0, \ldots, 2n$, the following identity (a kind of numerical Poincaré duality extended to all the pages of the spectral sequence) holds:

$$\dim_{\mathbb{C}} E^k_r = \dim_{\mathbb{C}} E^{2n-k}_r,$$

where, as usual, $E^k_r = \sum_{p+q=k} E^{p,q}_r$ is the direct sum of the spaces of total degree $k$ on the $r^{th}$ page of the Frölicher spectral sequence of $X$.

This is an immediate consequence of Theorem 1.3 and of the following

**Proposition 5.2.** Let $(X, \omega)$ be a compact complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary constant $h > 0$.

(i) If $d^*_h$, resp. $\ast$, are the formal adjoint of $d_h$, resp. the Hodge star operator induced by $\omega$, then

$$d^*_h = - \ast d_h \ast.$$

(ii) If, for every $h > 0$, every $k = 0, \ldots, 2n$ and every $\lambda \geq 0$, $E^k_{\Delta_h}(\lambda)$ stands for the $\lambda$-eigenspace of $\Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$, the linear map

$$E^k_{\Delta_h}(\lambda) \to E^{2n-k}_{\Delta_h}(\lambda), \quad u \mapsto \ast \bar{u},$$

is well defined and an isomorphism.

In particular, the operators $\Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})$ and $\Delta_h : C^\infty_{2n-k}(X, \mathbb{C}) \to C^\infty_{2n-k}(X, \mathbb{C})$ have the same spectra and their corresponding eigenvalues have the same multiplicities for all $h > 0$ and all $k = 0, \ldots, 2n$.

**Proof.** (i) We have $d^*_h = h \partial^* + \bar{\partial}^* = -h \ast \bar{\partial} \ast - \ast \partial \ast = -\ast (h \bar{\partial} + \partial) \ast = -\ast d_h \ast$ thanks to the standard formulae $\partial^* = -\ast \bar{\partial} \ast$ and $\bar{\partial}^* = -\ast \partial \ast$.

(ii) Using the formula under (i) and $\ast \ast = (-1)^k$ on $k$-forms, we get the following equivalences:

$$u \in E^k_{\Delta_h}(\lambda) \iff -d_h \ast d_h \ast u - \ast d_h \ast d_h u = \lambda u$$

$$\iff (- \ast d_h \ast) d_h (\ast \bar{u}) - (-1)^{\deg u} \ast d_h \ast d_h \ast \bar{u} = \lambda (\ast \bar{u})$$

$$\iff d^*_h d_h (\ast \bar{u}) + d_h d^*_h (\ast \bar{u}) = \lambda (\ast \bar{u}) \iff \ast \bar{u} \in E^{2n-k}_{\Delta_h}(\lambda),$$

where (a) was obtained by conjugating and then applying the isomorphism $\ast$.

This shows the well-definedness of the linear map under consideration. Both the conjugation and $\ast$ are isomorphisms, hence so is that linear map.

**Proof of Proposition 5.1.** By Theorem 1.3, $\dim_{\mathbb{C}} E^k_r$, resp. $\dim_{\mathbb{C}} E^{2n-k}_r$, is the number of eigenvalues $\lambda_i^k(h) \in \mathcal{O}(h^{2r})$, resp. $\lambda_i^{2n-k}(h) \in \mathcal{O}(h^{2r})$, counted with multiplicities, of $\Delta_h$ in degree $k$, resp. $2n - k$. Since, by Proposition 5.2, $\lambda^k_i(h) = \lambda_{i}^{2n-k}(h)$ for all $i \in \mathbb{N}^*$ and all $h > 0$, the statement follows.

The last consequence of Theorem 1.3 that we notice in this section is the following degeneration criterion for the Frölicher spectral sequence.
Proposition 5.3. Let \((X, \omega)\) be a compact complex Hermitian manifold with \(\dim_C X = n\). For every constant \(h > 0\), let \(\delta_h^{(k)} > 0\) be the smallest positive eigenvalue of \(\Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})\).

Then, for every \(r \in \mathbb{N}^*\), the Frölicher spectral sequence of \(X\) degenerates at \(E_r\) if and only if

\[
\lim_{h \to 0} \frac{\delta_h^{(k)}}{h^{2r}} = +\infty, \quad \text{for all} \quad k \in \{1, \ldots, n\}.
\]

Proof. The multiplicity of 0 as an eigenvalue of \(\Delta_h : C^\infty_k(X, \mathbb{C}) \to C^\infty_k(X, \mathbb{C})\) is the \(k^{th}\) Betti number \(b_k\) of \(X\) (cf. Corollary 2.9), so the degeneration at \(E_r\) of the Frölicher spectral sequence (known to be equivalent to the identities \(b_k = \dim E^k_r\) for all \(k = 0, 1, \ldots, 2n\)) amounts, thanks to Theorem 1.3, to \(\delta_h^{(k)}\) converging to zero (if it does converge to zero at all as \(h \downarrow 0\)) strictly less fast than \(Ch^{2r}\) for all \(k = 0, 1, \ldots, 2n\). On the other hand, the numerical duality statement of Proposition 5.1 reduces the verification of this property to the cases \(k = 1, \ldots, n\). \(\square\)

6 Degeneration at \(E_2\) of the Frölicher spectral sequence

In this section, we prove Theorem 1.2.

We start off by noticing a lower estimate for \(\Delta_h - h^2\Delta\) that holds for any Hermitian metric.

Lemma 6.1. Let \((X, \omega)\) be a compact complex manifold. For every \(0 < h < 1\), the following inequality of operators holds on smooth differential forms of all degrees:

\[
\Delta_h - h^2\Delta \geq (1 - h)h \left( \Delta'' - h[\tau, \tau^*] \right). \tag{43}
\]

Proof. We know from Lemma 2.7 that \(\Delta_h = h^2\Delta' + \Delta'' - h[\tau, \tau^*] - h[\bar{\partial}, \tau^*]\) for any Hermitian metric \(\omega\), while \(\Delta = [\partial + \bar{\partial}, \partial^* + \bar{\partial}^*] = \Delta' + \Delta'' - [\tau, \tau^*] - [\bar{\partial}, \tau]\). Thus, we get

\[
\Delta_h - h^2\Delta = (1 - h^2) \Delta'' + h(h - 1) ([\bar{\partial}, \tau^*] + [\bar{\partial}^*, \tau])
\]

\[
= (1 - h) \left( (1 + h) \Delta'' - h [\bar{\partial}, \tau^*] - h [\bar{\partial}^*, \tau] \right). \tag{44}
\]

We shall now estimate the signless terms on the r.h.s. of (44). For any form \(u\), we have

\[
\langle \langle [\bar{\partial}, \tau^*] u, u \rangle \rangle + \langle \langle [\bar{\partial}^*, \tau] u, u \rangle \rangle = \langle \langle \tau^* u, \bar{\partial}^* u \rangle \rangle + \langle \langle \bar{\partial} u, \tau u \rangle \rangle + \langle \langle \tau u, \bar{\partial} u \rangle \rangle + \langle \langle \bar{\partial}^* u, \tau^* u \rangle \rangle
\]

\[
= \text{2Re} \langle \langle \bar{\partial}^* u, \tau^* u \rangle \rangle + \text{2Re} \langle \langle \bar{\partial} u, \tau u \rangle \rangle.
\]

Thus, for any Hermitian metric \(\omega\), we have

\[
h |\langle \langle [\bar{\partial}, \tau^*] + [\bar{\partial}^*, \tau] u, u \rangle \rangle| \leq 2h \langle \langle \bar{\partial} u, \tau u \rangle \rangle + 2h \langle \langle \bar{\partial}^* u, \tau^* u \rangle \rangle
\]

\[
\leq (||\bar{\partial} u||^2 + ||\bar{\partial}^* u||^2) + h^2 (||\tau u||^2 + ||\tau^* u||^2)
\]

\[
= \langle \langle \Delta'' u, u \rangle \rangle + h^2 \langle \langle [\tau, \tau^*] u, u \rangle \rangle.
\]

Using this last estimate in (44), we get \(\Delta_h - h^2\Delta \geq (1 - h) (h\Delta'' - h^2[\tau, \tau^*])\) in the sense of operators. This is precisely (43).
Note that we can also write \(|\langle \langle [\tilde{\partial}, \tau^*] + [\tilde{\partial}^*, \tau] \rangle u, u \rangle | \leq \langle \langle \Delta u, u \rangle \rangle + \langle \langle [\tau, \tau^*] u, u \rangle \rangle\) for every form \(u\), which, alongside (44), yields \(\Delta_h - h^2 \Delta \geq (1 - h)(\Delta - h[\tau, \tau^*])\). This is slightly better than (43) if the r.h.s. is non-negative, but worse otherwise.

\[\square\]

We shall now give a sufficient condition for the r.h.s. of (43) to be non-negative.

**Lemma 6.2.** Let \((X, \omega)\) be a compact Hermitian manifold with \(\dim_{\mathbb{C}} X = n\) such that the inclusion of kernels

\[\ker \Delta'' \subset \ker [\tau, \tau^*]\]

holds for the operators \(\Delta'' : C^\infty_k(X, \mathbb{C}) \longrightarrow C^\infty_k(X, \mathbb{C})\) in a fixed degree \(k \in \{1, \ldots, n\}\).

Then, there exists a constant \(h_0(k) \in (0, 1)\) such that the following inequality of operators holds in degree \(k\):

\[\Delta'' \geq h [\tau, \tau^*] \quad \text{for all } 0 < h < h_0(k)\]

**Proof.** Let \(\delta''_k > 0\) be the smallest positive smallest eigenvalue of the elliptic, self-adjoint and non-negative differential operator \(\Delta'' : C^\infty_k(X, \mathbb{C}) \longrightarrow C^\infty_k(X, \mathbb{C})\).

On the other hand, the operator \([\tau, \tau^*] : C^\infty_k(X, \mathbb{C}) \longrightarrow C^\infty_k(X, \mathbb{C})\) is of order zero, hence bounded, so the constant \(C_k := \sup_{||u|| \leq 1} \langle \langle [\tau, \tau^*] u, u \rangle \rangle\) is finite.

We put \(h_0(k) := \min\{\delta''_k / C_k, 1\}\) and will prove that \(\langle \langle \Delta''u, u \rangle \rangle \geq h \langle \langle [\tau, \tau^*] u, u \rangle \rangle\) for all \(u \in C^\infty_k(X, \mathbb{C})\) and all \(h \in (0, h_0(k))\). Let us fix a form \(u \in C^\infty_k(X, \mathbb{C})\).

Since \(\Delta''\) is elliptic and preserves bidegrees, the following orthogonal splitting

\[C^\infty_k(X, \mathbb{C}) = \ker \Delta'' \oplus \text{Im } \Delta''\]

holds and induces a unique splitting \(u = u_h + u_{h^\perp}\) with \(u_h \in \ker \Delta''\) and \(u_{h^\perp} \in \text{Im } \Delta''\). In particular, \(u_h \in \ker [\tau, \tau^*]\) thanks to our assumption.

We get

\[\langle \langle \Delta''u, u \rangle \rangle = \langle \langle \Delta''u_{h^\perp}, u_h + u_{h^\perp} \rangle \rangle = \langle \langle \Delta''u_{h^\perp}, u_{h^\perp} \rangle \rangle \geq \delta''_k ||u_{h^\perp}||^2\] \hspace{1cm} (45)

since \(u_{h^\perp} \perp \ker \Delta''\), so \(u_{h^\perp}\) lies in the orthogonal direct sum of the eigenspaces of \(\Delta''\) corresponding to positive eigenvalues (= eigenvalues \(\geq \delta''_k\)).

On the other hand,

\[
\langle \langle [\tau, \tau^*] u, u \rangle \rangle \overset{(a)}{=} \langle \langle [\tau, \tau^*] u_{h^\perp}, u_h + u_{h^\perp} \rangle \rangle \overset{(b)}{=} \langle \langle u_{h^\perp}, [\tau, \tau^*] u_h \rangle \rangle + \langle \langle [\tau, \tau^*] u_{h^\perp}, u_{h^\perp} \rangle \rangle \overset{(c)}{=} \langle \langle [\tau, \tau^*] u_{h^\perp}, u_{h^\perp} \rangle \rangle \overset{(d)}{\leq} C_k ||u_{h^\perp}||^2, \hspace{1cm} (46)
\]

where for (a) we used the fact that \(u_h \in \ker [\tau, \tau^*]\), for (b) we used the self-adjointness of \([\tau, \tau^*]\), (c) follows from \(u_h \in \ker [\tau, \tau^*]\), while (d) follows from the definition of \(C_k\).

Since \(h_0(k) = \min\{\delta''_k / C_k, 1\}\), inequalities (45) and (46) imply that

\[h \langle \langle [\tau, \tau^*] u, u \rangle \rangle \leq C_k h ||u_{h^\perp}||^2 \leq \frac{C_k h}{\delta''_k} \langle \langle \Delta''u, u \rangle \rangle \leq \langle \langle \Delta''u, u \rangle \rangle\]

for all \(h \in (0, h_0(k))\). \[\square\]
Corollary 6.3. Let \((X, \omega)\) be a compact Hermitian manifold such that \(\ker \Delta'' \subset \ker [\tau, \tau^*]\) in a fixed degree \(k\). Then, there exists a constant \(h_0(k) \in (0, 1]\) such that the following inequality of operators holds in degree \(k\):
\[
\Delta_h \geq h^2 \Delta \quad \text{for all } 0 < h < h_0(k).
\]

Proof. This is an immediate consequence of Lemmas 6.1 and 6.2. \(\square\)

We can now prove the spectral sequence degeneration statement of this paper.

Proof of Theorem 1.2. Let us fix an arbitrary \(k \in \{1, \ldots, n\}\). Hypothesis (1) and Corollary 6.3 imply that \(\ker \Delta_h \subset \ker \Delta\) for all \(0 < h < h_0(k)\) since \(\langle \langle \Delta u, u \rangle \rangle \geq 0\) for every \(u\) and \(u \in \ker \Delta\) if and only if \(\langle \langle \Delta u, u \rangle \rangle = 0\). Meanwhile, we know from Corollary 2.9 that \(\ker \Delta_h\) and \(\ker \Delta\) are finite-dimensional vector spaces of equal dimensions, so for all \(0 < h < h_0(k)\) we get
\[
\ker \Delta_h = \ker \Delta. \tag{47}
\]

For every \(h > 0\), let \(\delta_h^{(k)} > 0\) be the smallest positive eigenvalue of the elliptic operator \(\Delta_h : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})\) and let \(u_h \in C^\infty_k(X, \mathbb{C})\) be a corresponding unitary eigenvector, i.e.
\[
||u_h|| = 1 \quad \text{and} \quad \Delta_h u_h = \delta_h^{(k)} u_h.
\]

Now, \(u_h\) is orthogonal to \(\ker \Delta_h\), hence, thanks to (47), \(u_h\) is also orthogonal to \(\ker \Delta\) for every \(0 < h < h_0(k)\). Consequently, \(\langle \langle \Delta u_h, u_h \rangle \rangle \geq \delta_k ||u_h||^2 = \delta_k\), where \(\delta_k > 0\) is the smallest positive eigenvalue of \(\Delta : C^\infty_k(X, \mathbb{C}) \rightarrow C^\infty_k(X, \mathbb{C})\).

Using this and Corollary 6.3, we get
\[
\delta_h^{(k)} = \langle \langle \Delta_h u_h, u_h \rangle \rangle \geq h^2 \langle \langle \Delta u_h, u_h \rangle \rangle \geq \delta_k h^2 \quad \text{for all } 0 < h < h_0(k).
\]

In particular, \(\lim_{h \to 0} (\delta_h^{(k)}/h^4) = +\infty\).

As in the proof of Proposition 5.3, this and Theorem 1.3 imply that \(\dim E_2^k = b_k\) for the degree \(k \in \{1, \ldots, n\}\) that was arbitrarily fixed in the beginning. By the duality statement of Proposition 5.1, this also yields \(\dim E_2^{2n-k} = b_k = b_{b_{2n-k}}\). Since this holds for all \(k \in \{1, \ldots, n\}\), the Fröhlicher spectral sequence of \(X\) degenerates at \(E_2\). \(\square\)

7 Appendix: Comparison of Laplacians when the metric is SKT

In this section, we come within an \(\varepsilon = C h^2\) of solving Conjecture 1.1 as an application of Theorem 1.3 and of a comparison of the Laplacians \(\Delta'\) and \(\Delta''\) defined by an arbitrary SKT metric \(\omega\) supposed to exist on a given compact complex manifold \(X\). Recall that an SKT metric \(\omega\) is a \(C^\infty\) positive definite \((1, 1)\)-form \(\omega\) such that \(\partial \bar{\partial} \omega = 0\) on \(X\).

Lemma 7.1. Let \(X\) be a compact complex manifold on which an SKT metric \(\omega\) exists.
(i) The usual $\partial$- and $\bar{\partial}$-Laplacians $\Delta' = [\partial, \partial^*]$ and $\Delta'' = [\bar{\partial}, \bar{\partial}^*]$ induced by $\omega$ satisfy the following inequalities on differential forms of all bidegrees:

$$(1 + \delta) \Delta'' + \left(1 + \frac{1}{\delta} \right) [\bar{\tau}, \tau^*] \geq \Delta' \geq \frac{1}{1 + \delta} \Delta'' - \frac{1}{\delta} [\bar{\tau}, \tau^*], \quad \text{for all } \delta > 0, \quad (48)$$

where $\tau = \tau_\omega := [\Lambda_\omega, \partial \omega \wedge \cdot]$ is the torsion operator of type $(1, 0)$ and $\bar{\tau}^*$ is the formal adjoint w.r.t. the $L^2_\omega$-inner product of its complex conjugate.

(ii) The following inequality also holds:

$$\Delta'' \geq h \Delta' + \left( h \overline{X}_\omega - \frac{h}{1 - h} [\bar{\tau}, \tau^*] \right), \quad \text{for all } 0 < h < 1, \quad (49)$$

where $X_\omega := [\partial \omega \wedge \cdot, (\partial \omega \wedge \cdot)^*]$. Implicitly, we have

$$\Delta_h - h\Delta \geq h \left( 1 - h \right) \overline{X}_\omega - [\bar{\tau}, \tau^*], \quad \text{for all } 0 < h < 1. \quad (50)$$

Since $\overline{X}_\omega$ and $[\bar{\tau}, \tau^*]$ are zero-th order operators, they are bounded, so (50) implies the existence of a constant $C > 0$ independent of $h$ such that

$$\Delta_h - h\Delta \geq -Ch, \quad \text{for all } 0 < h < 1. \quad (51)$$

Proof. (i) Demailly’s formula (cf. [Dem84] or [Dem97, VII, §1]) of the Bochner-Kodaira-Nakano type for arbitrary Hermitian metrics $\omega$ reads

$$\Delta' = \Delta'' - \overline{X}_\omega + [\Lambda_\omega, [\Lambda_\omega, \frac{i}{2} \partial \bar{\partial} \omega]],$$

where $\Delta'' := [\partial + \bar{\tau}, (\partial + \bar{\tau})^*]$ and $\overline{X}_\omega := [\partial \omega \wedge \cdot, (\partial \omega \wedge \cdot)^*]$. The last term on the r.h.s. above vanishes if $\omega$ is SKT, so we get

$$\Delta'' + ([\bar{\partial}, \tau^*] + [\bar{\tau}, \partial^*]) + [\bar{\tau}, \tau^*] = \Delta' + \overline{X}_\omega \quad \text{if } \partial \bar{\partial} \omega = 0. \quad (52)$$

Now, the signless terms can be easily estimated using the elementary inequality $2|ab| \leq \delta a^2 + (1/\delta) b^2$ for arbitrary $a, b \in \mathbb{C}$ and $\delta > 0$. For every differential form $u$ of any degree, we get:

$$|\langle \langle \bar{\partial}, \tau^* \rangle \rangle u, u \rangle| + |\langle \langle \bar{\tau}, \partial^* \rangle \rangle u, u \rangle| = 2\text{Re} \langle \langle \bar{\partial} u, \tau u \rangle \rangle + 2\text{Re} \langle \langle \bar{\partial}^* u, \tau^* u \rangle \rangle \leq 2|\langle \langle \bar{\partial} u, \tau u \rangle \rangle| + 2|\langle \langle \bar{\partial}^* u, \tau^* u \rangle \rangle| \leq \delta ||\bar{\partial} u||^2 + \frac{1}{\delta} ||\tau u||^2 + \delta ||\bar{\partial}^* u||^2 + \frac{1}{\delta} ||\tau^* u||^2$$

$$= \delta \langle \langle \Delta'' u, u \rangle \rangle + \frac{1}{\delta} \langle \langle \tau, \tau^* \rangle \rangle u, u \rangle.$$
\[(1 + \delta) \Delta'' + \left(1 + \frac{1}{\delta}\right) [\bar{\tau}, \bar{\tau}^*] - X_\omega \geq \Delta' \geq \frac{1}{1 + \delta} \Delta'' + \frac{1}{1 + \delta} X_\omega - \frac{1}{\delta} [\bar{\tau}, \bar{\tau}^*], \tag{53}\]

for all \(\delta > 0\). Since \(X_\omega\) and \(X_\omega\) are non-negative operators, ignoring them weakens these inequalities to (48).

(ii) After dividing by \(1 + \delta\), the l.h.s. inequality in (53) translates to

\[\Delta'' \geq \frac{1}{1 + \delta} \Delta' + \frac{1}{1 + \delta} X_\omega - \frac{1}{\delta} [\bar{\tau}, \bar{\tau}^*].\]

This is precisely (49) if we put \(h := \frac{1}{1 + \delta} \in (0, 1)\) since in this case \(\delta = \frac{1}{h}\).

To get (50) from (49), it suffices to notice that \(\Delta_h - h\Delta = h(h - 1) \Delta' + (1 - h) \Delta'' = (1 - h)(\Delta'' - h \Delta').\)

We now observe an analogue of inequality (50) for \(\Delta_h - h^2 \Delta\).

**Lemma 7.2.** Let \(X\) be a compact complex manifold on which an SKT metric \(\omega\) exists. The following inequalities of operators hold:

\[\Delta_h - h^2 \Delta \geq h^2 \left((1 - h) \overline{X_\omega} - [\bar{\tau}, \bar{\tau}^*]\right) \geq -C h^2, \quad \text{for all } 0 < h < 1, \tag{54}\]

where \(\overline{X_\omega} := [\partial \omega \wedge \cdot, (\partial \omega \wedge \cdot)^*]\) and \(C \geq 0\) is a constant independent of \(h\).

**Proof.** Since \(\Delta_h = h^2 \Delta' + \Delta'' + hA\) and \(\Delta = \Delta' + \Delta'' + A\), where \(A := [\partial, \partial^*] + [\bar{\partial}, \bar{\partial}^*]\), we get

\[\Delta_h - h^2 \Delta = (1 - h) ((1 + h) \Delta'' + hA).\]

On the other hand, the signless operator \(A\) can be estimated in the same way as a similar operator was estimated in the proof of Lemma 7.1. We get \(\|\langle\langle Au, u\rangle\rangle\| = 2 \text{Re} \langle\langle \partial u, \partial^* u\rangle\rangle + 2 \text{Re} \langle\langle \partial^* u, \partial^* u\rangle\rangle\), hence

\[h \|\langle\langle Au, u\rangle\rangle\| \leq h^2 \|\partial u\|^2 + \|\partial^* u\|^2 + h^2 \|\partial^* u\|^2 + \|\partial^* u\|^2 = h^2 \langle\langle \Delta' u, u\rangle\rangle + \langle\langle \Delta'' u, u\rangle\rangle\]

for any form \(u\). Consequently, \((1 + h) \Delta'' + hA \geq h \Delta'' - h^2 \Delta'\) as operators, so we get

\[\Delta_h - h^2 \Delta \geq h(1 - h) (\Delta'' - h \Delta').\]

(Note that we can also write \(\|\langle\langle Au, u\rangle\rangle\| \leq \langle\langle \Delta' u, u\rangle\rangle + \langle\langle \Delta'' u, u\rangle\rangle\) and we get \(\Delta_h - h^2 \Delta = (1 - h) ((1 + h) \Delta'' + hA) \geq (1 - h) (\Delta'' - h \Delta')\) for every form \(u\).)

Meanwhile, from (49) we know that \((1 - h) (\Delta'' - h \Delta') \geq h \left((1 - h) \overline{X_\omega} - [\bar{\tau}, \bar{\tau}^*]\right)\) for all \(0 < h < 1\). Together with the last inequality, this proves the first inequality in (54).

The second inequality in (54) follows at once from the first since \(\overline{X_\omega} \geq 0\) and the non-negative operator \([\bar{\tau}, \bar{\tau}^*]\) is of order zero, hence bounded, so we can choose \(C := \sup_{\|u\| = 1} \langle\langle [\bar{\tau}, \bar{\tau}^*] u, u\rangle\rangle < +\infty\).

(Using the alternative lower estimate \(\Delta_h - h^2 \Delta \geq (1 - h) (\Delta'' - h \Delta')\) noticed above, the inequalities in (54) get replaced by \(\Delta_h - h^2 \Delta \geq h ((1 - h) \overline{X_\omega} - [\bar{\tau}, \bar{\tau}^*]) \geq -Ch\).)
If the lower bound $-Ch^2$ in (54) could be improved to 0, then we would have $\Delta h \geq h^2 \Delta$ for all $0 < h \ll 1$ (as in Corollary 6.3) and Conjecture 1.1 would follow by the argument spelt out at the end of section §6.

References.

[ALK00] J.A. Álvarez López, Y.A. Kordyukov — Adiabatic Limits and Spectral Sequences for Riemannian Foliations — Geom. Funct. Anal. 10 (2000), no. 5, 977–1027.

[CFGU97] L.A. Cordero, M. Fernandez, A. Gray, L. Ugarte — A General Description of the Terms in the Frölicher Spectral Sequence — Diff. Geom. and its Applic. 7 (1997), 75–84.

[Dem 84] J.-P. Demailly — Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne — Séminaire d'analyse P. Lelong, P. Dolbeault, H. Skoda (editors) 1983/1984, Lecture Notes in Math., no. 1198, Springer Verlag (1986), 88-97.

[Dem 97] J.-P. Demailly — Complex Analytic and Algebraic Geometry—http://www-fourier.ujf-grenoble.fr/ demailly/books.html

[ES89] D.V. Efremov, M.A. Shubin — Spectrum Distribution Function and Variational Principle for Automorphic Operators on Hyperbolic Space —

[For95] R. Forman — Spectral Sequences and Adiabatic Limits — Commun. Math. Phys. 168, 57-116 (1995).

[Fro55] A. Frölicher — Relations between the Cohomology Groups of Dolbeault and Topological Invariants — Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 641–644.

[GS91] M. Gromov, M.A. Shubin — Von Neumann Spectra near Zero — Geom. Funct. Anal. 1 (1991), no. 4, 375–404.

[MM90] R. R. Mazzeo, R. B. Melrose — The Adiabatic Limit, Hodge Cohomology and Leray’s Spectral Sequence — J. Diff. Geom. 31 (1990) 185-213.

[Pop16] D. Popovici — Degeneration at $E_2$ of Certain Spectral Sequences — International Journal of Mathematics 27, no. 14 (2016), DOI: 10.1142/S0129167X16501111.

[Wi85] E. Witten — Global Gravitational Anomalies — Commun. Math. Phys, 100, 197-229 (1985).

Université Paul Sabatier, Institut de Mathématiques de Toulouse,
118 route de Narbonne, 31062 Toulouse, France
Email: popovici@math.univ-toulouse.fr