1. Introduction

Many phenomena of electrical conduction in small mesoscopic conductors have been successfully explained within the framework of the scattering approach [1]. The main emphasis of this work is an extension of this approach to time-dependent phenomena. Of particular interest are the basic requirements which a dynamic conductance of a mesoscopic conductor has to satisfy. There is a gap in the way dynamic conduction is treated in a large body of the physics literature and in more applied discussions based on simple circuit theory. For an electric network we know that its time-dependent behavior is crucially determined by its capacitive and inductive elements. In contrast to this, time-dependent phenomena are often treated by a response theory of non-interacting carriers. Unfortunately, this approach to ac-conductance which has been employed by some of the leading condensed matter physicists [2] is prevalent. We emphasize that in order to obtain a reasonable answer it is not sufficient to consider non-interacting electrons, a Fermi liquid or even a Luttinger liquid with short range interactions. An analysis based on an electric circuit model gives an answer which conserves the total charge and which has the property that all frequency dependent currents at the input and output nodes of such a circuit add up to zero. To obtain results which are charge and current conserving, it is necessary to consider the implication of the long range Coulomb interaction. This leads to a theory which can be considered to be an extension of the work of Pines and Nozières [3] and Martin [4] on bulk systems to mesoscopic conductors. Below we present a discussion [5] which for simplicity focuses on the low-frequency transport. The benefit of this restriction is its considerable generality, and that it can be used to analyze the self-consistency of the dc scattering-approach (Sect. 3), the leading order nonlinearities of the
Figure 1. Mesoscopic capacitor connected to contacts with electrochemical potentials $\mu_1$ and $\mu_2$ and electrostatic potentials $U_1$ and $U_2$. After Ref. 7.

current voltage characteristic (Sect. 4), and the low-frequency dynamical conductance (Sect. 5).

For a two-terminal conductor the low-frequency admittance is of the form

$$G(\omega) = G(0) - i\omega E_\mu + O(\omega^2).$$

(1)

where $G(0)$ is the dc-conductance and $E_\mu$ is called the emittance [5]. The emittance describes the current response in the leading order with respect to frequency and can be associated with the displacement charge passing a contact. For conductors with poor transmission as, e.g., a condenser or a metallic diffusive wire, the emittance is positive: the current follows the voltage as is characteristic of a capacitive response. On the other hand, it turns out that the emittance is negative in samples with good transmission. The current leads the voltage as is characteristic of circuits with an inductance.

To illustrate the physics contained in this response coefficient $E_\mu$, we present now a few results. Their detailed derivation is presented in Sect. 6.

Consider first the case of a mesoscopic condenser [6, 7] as shown in Fig. 1 with a geometrical capacitance $C_0$. The measured electrochemical capacitance $C_\mu$ is determined by the geometrical capacitance in series with quantum capacitances $e^2dN_1/dE$, $e^2dN_2/dE$. Here $dN_1/dE$ and $dN_2/dE$ are the total densities of states of the mesoscopic plates in the regions into which the electrical field penetrates. Thus the electrochemical capacitance of a mesoscopic condenser is [7]

$$C_\mu^{-1} = C_0^{-1} + (e^2dN_1/dE)^{-1} + (e^2dN_2/dE)^{-1}. \quad (2)$$
As it must be, in the macroscopic limit, where the densities of states diverge, one finds $E_\mu = C_0$. On the other hand, if the density of states at the surfaces of the plates are sufficiently small, the geometrical capacitance can be neglected. For the symmetric case this leads to the emittance $E_\mu = (1/4)e^2(dN/dE)$, where $dN/dE = dN_1/dE + dN_2/dE$ is the total density of states of the two surfaces.

Next consider a sample which permits transmission of carriers between the contacts. To be definite, first consider a metallic diffusive wire connecting two reservoirs (see Fig. 2). Experiments have been carried out by Pieper and Price [8] and related theoretical work can be found in Refs. [9, 10, 11, 12]. We determine the local potential with the help of Thomas-Fermi screening (a local charge-neutrality condition). In the presence of an applied dc-voltage, the ensemble averaged potential drops linearly. The total density of states in the volume over which the electric field is non-vanishing is denoted by $dN/dE$. We find that the emittance of such a metallic diffusive wire is positive like a capacitance and is given by [13]

$$E_\mu = \frac{1}{6}e^2(dN/dE) .$$

(3)

It is not the total density of states that counts! Without screening and in the presence of a dc potential-drop, one-half of the states would be filled. Screening, however, reduces the naively expected result $(1/2)(dN/dE)$ by a factor $1/3$. Such a $1/3$-reduction is familiar from the theory of shot noise of metallic diffusive conductors were it is found that the actual shot noise is $1/3$ of the full Poisson noise [14, 15, 16]. The physical origin of the factor $1/3$ in these two problems is quite different but from a purely mathematical
Next consider a ballistic wire of length $L$. Now, for dc transport the ensemble-averaged potential drops as we enter the wire from the reservoir, it is constant in the wire, and it drops again as we leave the wire. We find then a negative (inductive) emittance

$$E_\mu = -(1/4)e^2(dN/dE)$$

where $dN/dE$ is the total density of states of the wire in the volume where the potential is uniform.

As a further example, consider a resonant double barrier [17, 18, 19]. Suppose for simplicity that it is symmetric and suppose that it is reasonable to determine the potential in the well from a charge neutrality condition. If we denote by $T$, $R = 1 - T$, and $dN/dE$ the transmission probability, the reflection probability, and the density of states in the well, respectively, the emittance can be written in the form [19, 20]

$$E_\mu = (1/4)(R - T)e^2(dN/dE)$$

Note that Eq. (5) interpolates between the emittance of a symmetric condenser with large geometrical capacitance and a ballistic wire. The emittance is negative (inductive) at resonance $T = 1$ and crosses zero at $T = R = 1/2$ where the Fermi level is a half-width of the resonant level above (or below) the energy of the resonant state. If the Fermi level is further than a half-width away from the resonant level the emittance is positive (capacitive). The case of a double barrier which is asymmetric has been considered in Ref. [19]. A result for single tunneling barriers which goes beyond Thomas-Fermi screening will be discussed elsewhere [20].

Finally, consider a two-dimensional electron gas patterned into the shape of a Corbino disk (Fig. 3a) or a Hall bar (Fig. 3b) with two metallic contacts. In the range of magnetic fields over which the Hall conductance is quantized the dc-resistances can be evaluated solely by considering edge states [21]. In a non-interacting theory the edge states intercept the Fermi energy with a finite slope and have a local density of states per unit length given by $dn(s)/dE = 1/hv(s)$, where $s$ is a coordinate along the edge state and where $v(s)$ is the velocity of carriers at point $s$. The integrated density of states along the edge channel $k$ is thus $dN_k/dE = \int ds(dn_k/dE)$. Suppose for simplicity that the magnetic field is such that we have only one pair of edge states. The Corbino disk acts then as an insulator with vanishing dc conductance $G(0) = 0$. The long-range Coulomb interaction between the inner and the outer edge state can be described by a geometrical capacitance $C_0$ (which depends logarithmically on the width of the sample). As one expects, we find that the Corbino-disk exhibits an emittance which
equals the electrochemical capacitance [22] \( E_\mu = C_\mu \), where \( C_\mu \) is given by Eq. (2). A topologically different conductor of the same width and length but with its contacts arranged at its ends is shown in Fig. 3b. Now the two-terminal dc-conductance is quantized and given by \( G(0) = e^2/h \) for spin-split Landau levels. Obviously, the low frequency response of this conductor is not capacitive but dominated by transmission of carriers between different reservoirs. In particular, we find the inductive-like emittance [22] \( E_\mu = -C_\mu \). Thus the ac-response of a quantized Hall conductor is determined by the way the edge states are connected to reservoirs. In the Corbino disk each edge state returns to the reservoir from which it emanates. In contrast, in the two-terminal Hall-bar, each edge state connects to a reservoir that is different from the one from which it emanates. We do not discuss this example anymore in the sequel. A forthcoming publication [22] presents a general formula for the emittance which is applicable to a wide variety of samples with different edge-state topologies.

2. The Scattering Approach

Consider a conductor connected to a number of contacts [23]. The contacts are labeled with the greek indices \( \alpha \). We assume that the distance between these contacts is so small that transmission from one contact to another one can be considered to be phase coherent. Thus we assume that scattering inside the conductor is purely elastic. For small deviations of the electrochemical potentials away from their equilibrium value the dc-current \( I_\alpha \) at
probe $\alpha$ is given by [23]

$$I_{\alpha} = \left(\frac{e}{h}\right) \int dE \left(-\frac{df}{dE}\right) \left((N_{\alpha} - R_{\alpha\alpha})\mu_{\alpha} - \sum_{\beta} T_{\alpha\beta}\mu_{\beta}\right). \quad (6)$$

Here, $N_{\alpha}$ is the number of quantum channels with thresholds below the equilibrium electrochemical potential in contact $\alpha$, and $f$ is the Fermi function. Carriers in the $N_{\alpha}$ incident channels have a total combined probability $R_{\alpha\alpha}$ for reflection back into contact $\alpha$. Carriers incident in contact $\beta$ have a total probability $T_{\alpha\beta}$ to traverse the sample into contact $\alpha$. Equation (6) is a quantum-mechanical Kirchhoff law. It states the conservation of the current at an arbitrary intersection of a mesoscopic wire and that the currents are only a function of voltage differences. In the present context these features are a consequence of the unitarity of the scattering matrix $S$ and its behavior under time reversal. The fact that the scattering matrix is unitary and that the microscopic equations are reversible implies that under a reversal of magnetic field the scattering matrix has the symmetry $S^T(B) = S(-B)$. The scattering matrix $S$ for the conductor can be arranged such that it is composed of sub-matrices $s_{\alpha\beta}$ which relate the incident current amplitudes in probe $\beta$ to the outgoing current amplitudes in probe $\alpha$. In terms of these scattering matrices the probabilities introduced above are simply $T_{\alpha\beta} = Tr(s_{\alpha\beta}^\dagger s_{\alpha\beta})$ if $\alpha \neq \beta$ and $R_{\alpha\alpha} = Tr(s_{\alpha\alpha}^\dagger s_{\alpha\alpha})$. The trace is over the quantum channels in each probe. As a consequence of the unitary properties of the scattering matrix, we have $N_{\alpha} = R_{\alpha\alpha} + \sum_{\beta} T_{\alpha\beta}$ and $N_{\alpha} = R_{\alpha\alpha} + \sum_{\beta} T_{\beta\alpha}$ which imply current conservation $\sum_{\alpha} I_{\alpha} = 0$. Furthermore, the currents depend only on differences of the electrochemical potentials.

Let us restate these properties in terms of conductance coefficients $G_{\alpha\beta} = I_{\alpha}/V_{\beta}$. Taking into account that in a reservoir the voltages and electrochemical potentials move in synchronism, $\delta\mu_{\alpha} = eV_{\alpha}$, we have [23]

$$G_{\alpha\alpha}(0) = \frac{e^2}{h} \int dE \left(-\frac{df}{dE}\right)(N_{\alpha} - R_{\alpha\alpha}) = \frac{e^2}{h} \int dE \left(-\frac{df}{dE}\right) \sum_{\beta} T_{\alpha\beta}. \quad (7)$$

for $\alpha = \beta$ and

$$G_{\alpha\beta}(0) = -\frac{e^2}{h} \int dE \left(-\frac{df}{dE}\right) T_{\alpha\beta}. \quad (8)$$

Current conservation means for the conductance matrix

$$\sum_{\alpha} G_{\alpha\beta}(0) = 0 \quad (9)$$
for any $\beta$. On the other hand, the statement that the currents can only depend on voltage differences leads to

$$\sum_{\beta} G_{\alpha\beta}(0) = 0 \quad . \quad (10)$$

We emphasize the simple properties (9) and (10) since later on we point out that they are also valid for $G_{\alpha\beta}(\omega)$.

Before proceeding we briefly comment on closely related work. In his work Landauer determined the voltage drop across an obstacle with the help of a local charge neutrality argument applied to the perfect sections of the conductor on either side of an obstacle [24]. For an obstacle with transmission probability $T$, this leads to a resistance proportional to $(1 - T)/T$. This resistance has the obviously correct property that it vanishes in the limit of perfect transmission. Already Engquist and Anderson pointed out that voltages measured at contacts are determined not by a charge neutrality condition but by adjusting the electrochemical potential of a voltage probe in such a manner that the voltmeter draws zero net current [25]. Due to the special geometry considered, the restriction to a one channel conductor, and the restriction to a phase-incoherent voltage measurement [26], the answer found by Engquist and Anderson is the same as that found by Landauer. The key notion that the resistance across an obstacle should vanish if the obstacle permits perfect transmission persisted long after the work of Engquist and Anderson. The task seemed, therefore, to be to find resistance formulae [27] of the type $(1 - T)/T$. In a work that largely centers on this notion, Imry [28] observed that if one considers in the discussion of Engquist and Anderson not the voltage drop across the obstacle but the voltage drop between the current source and sink one obtains a resistance that is proportional to $T^{-1}$, i.e. a conductance that is proportional to $T$. Equation (6), in contrast, was motivated by experiments by Benoit et al. [29] in which the role of the current and voltage source was exchanged. Such experiments imply that we need a formulation of electrical transport in which apriori all contacts are treated equivalently and on an equal footing [23]. In a specific arrangement of current and voltage sources the voltages are a posteriori determined by a zero current condition. This leads to conductances which are electrochemical quantities and which obey the basic requirements of a transport theory. These basic requirements include a reciprocity symmetry [23, 29, 30] (which is a consequence of microreversibility and the irreversibility of thermodynamic electron reservoirs) and include a fluctuation-dissipation theorem [31]. Below we discuss in more detail the electrochemical requirements which lead to Eq. (6).

It is sometimes implied in the theoretical literature, that the derivation of conductance formulae is just a question of applying formal linear response
theory correctly to this problem. This is not borne out by the history of this field: Depending on one’s preconceived notions quite different results can be derived [32]. But it is certainly true, that the derivation of these results from linear response has made them more acceptable. This is, however, neither a question of rigour nor depth but simply a consequence of the fact that even simple results are often only accepted if they are embellished by a sufficiently complicated derivation. For very concise linear response discussions of Eq. (6) and some still open related questions we refer the reader to Ref. [33, 34, 35].

3. Potential Distribution in a Mesoscopic Conductor

The scattering approach as discussed above seems like a simple black-box approach. If we know through which contact carriers leave the sample and know the current that is injected by a reservoir we can find the total currents. The issue which is not trivial is the fact that the voltages at the contacts must be well-defined in order for Eq. (7) and Eq. (8) to be the correct answer. In many articles on the subject one finds a conductance attributed to a wire which is strictly one-dimensional, or one finds pictures of mesoscopic conductors with leads that are narrower then the mesoscopic sample itself and are called reservoirs. Such geometries, as we now show, do in fact not lead to the conductances given above. This criticism is not novel: it has been aired in a number of papers by Landauer [36] and one of the authors [31] - unfortunately without much success. The following discussion is most closely related to work by Levinson [37] but in detail
follows Ref. [5].

For Eqs. (7) and (8) to be valid, the electric potential $U_\alpha$ in the reservoirs $\alpha$ must follow the electrochemical potential $\mu_\alpha$ in this contact. To generate transport we must consider a non-equilibrium situation. The electrochemical potentials in the contacts $\mu_\alpha$ must be allowed to differ from the equilibrium chemical potential $\mu_0$. Suppose that the increment of the electrochemical potential in contact $\alpha$ is $\delta\mu_\alpha = \mu_\alpha - \mu_0$. The electric potential changes from its equilibrium configuration $U_{eq}(r)$ to a new non-equilibrium configuration $U([\mu_\alpha], r)$. Here the argument $[\mu_\alpha]$ indicates that the non-equilibrium potential depends on the electrochemical potentials in the contacts. Equations (7) and (8) presuppose that deep in contact $\alpha$ the electrochemical and electric potential change in synchronism, i.e. $\delta\mu_\alpha = e\delta U([\mu_\alpha], r)$. This is a consequence of the charge neutrality deep in the reservoirs. The difference between the electrochemical potential and the electrostatic potential is the Fermi energy, $E_{F\alpha}(r) = \mu_\alpha - eU(r)$. This chemical potential $E_{F\alpha}$ determines the charge in the neighborhood of $r$. Deep in the reservoir the charge cannot change, even if we bias the conductor. Consequently we have $0 \equiv \delta E_{F\alpha}(r) = \delta\mu_\alpha - e\delta U([\mu_\alpha], r)$. Note that we have taken here the chemical potential to be space depended. However, if the quantum coherence of the wave functions is taken serious then also deep in the reservoir, i.e. in the wide wire, a single energy $E_{F\alpha}$ is all that is needed to specify the chemical potential. The energy dispersions in a wide wire are of the type $E_{an}(k) = \hbar^2k^2/2m + E_{an}^0 + eU_\alpha$. Here $\hbar^2k^2/2m$ is the longitudinal energy for motion along the wire, $E_{an}^0$ is the energy for transverse motion (the channel threshold) and $eU_\alpha$ is the equilibrium potential (the bottom of the conduction band) in contact $\alpha$. The energy dispersion depends in an explicit manner only on a single spatially independent constant $eU_\alpha$. Instead of the spatially dependent relationship between the electrochemical, chemical, and electrostatic potentials we have for coherent conductors the relation $E_{F\alpha} = \mu_\alpha - eU_\alpha$. In either case the differential relationship is spatially independent in the lead. In order to discuss the validity of Eqs. (6)-(8), we have thus to find the electrostatic potential and the conditions under which it changes synchronously with the electrochemical potential in the reservoirs.

3.1. CHARACTERISTIC POTENTIALS

The electrostatic potential $U([\mu_\alpha], r)$ for mesoscopic conductors is a function of the electrochemical potentials of the contacts, and a complicated function of position. Small increases in the electrochemical potentials $\delta\mu_\alpha$ bring the conductor to a new state (see Fig. 4) with an electrostatic potential $U([\mu_\alpha + \delta\mu_\alpha], r)$. The difference $\delta U$ between these two potentials can
be expanded in powers of the increment in the electrochemical potential. To linear order we have

\[ e \delta U([\mu_\alpha], r) = \sum_\alpha u_\alpha(r) \delta \mu_\alpha. \]  

(11)

Here, \( u_\alpha(r) = e \partial U([\mu_\alpha], r)/\partial \mu_\alpha|_{\mu_\alpha=\mu_0} \), with \( \alpha = 1, 2 \) are the characteristic potentials [5] which determine the electrostatic potential inside the sample in response to a variation of an electrochemical potential at a contact.

Suppose for a moment that we increase all electrochemical potentials simultaneously and by the same amount, \( \delta \mu_\alpha \equiv \delta \mu \). Both before and after the change the conductor is at equilibrium, hence the physical properties of the conductor remain unchanged.

Consequently, the shift of the electrochemical potentials must be accompanied by a shift \( e \delta U \equiv \delta \mu \) of the electrical potential. Imposing this condition on Eq. (11) implies that the sum of all characteristic potentials is equal to one at every space point [5],

\[ \sum_\alpha u_\alpha(r) \equiv 1. \]  

(12)

Equation (12) is a consequence of the long-range Coulomb interaction. It is the most important result of this work: the conservation of charge under the application of a dc or ac bias and the conservation of current are, as we will show, a consequence of Eq. (12).

Let us now return to our original problem and consider what happens if we increase just one electrochemical potential, say in reservoir \( \alpha \), by \( \delta \mu_\alpha \). Obviously, the condition that the electrochemical potential and the electrostatic potential move in synchronism deep inside reservoir \( \alpha \) implies that the characteristic function \( u_\alpha(r) = 1 \) for \( r \) deep inside reservoir \( \alpha \). Together with Eq. (11), this implies that Eqs. (6)-(8) are valid if and only if the characteristic potentials have the property that \( u_\alpha(r) = 1 \) for \( r \) deep in contact \( \alpha \) and \( u_\alpha(r) = 0 \) for \( r \) deep in any other contact.

The electrostatic potentials are determined by the charge distribution in the sample. As we increase the chemical potential of contact \( \alpha \) keeping all electrostatic potentials fixed, the additional charge

\[ \delta n(r) = (dn(r, \alpha)/dE) \delta \mu_\alpha \]  

(13)

enters the conductor. Here, \( dn(r, \alpha)/dE \) is the injectivity of contact \( \alpha \) into point \( r \) of the sample. With the help of the scattering states \( \Psi_{\alpha n}(r) \) which have unit amplitude in the incident channel \( n \) in lead \( \alpha \), the injectivity can be expressed as

\[ dn(r, \alpha)/dE = \sum_n (hv_{\alpha n})^{-1} |\Psi_{\alpha n}(r)|^2 \]  

(14)
where \( v_{\alpha \alpha} \) is the velocity of carriers at the Fermi energy in channel \( n \) in contact \( \alpha \). Equation (13) gives of course not the true density variation. The injected charges induce a change in the electrostatic potential which in turn implies an induced contribution to the density, \( \delta n_{\text{ind}} \), which has to be determined self-consistently. The total charge density is

\[
\delta n(\mathbf{r}) = \left( \frac{dn(\mathbf{r}, \alpha)}{dE} \right) \delta \mu_{\alpha} + \delta n_{\text{ind}}(\mathbf{r}) \ .
\]  

(15)

The induced charge density is connected to the electrostatic potential via the response function \( \Pi(\mathbf{r}, \mathbf{r}') \) (Lindhard function):

\[
\delta n_{\text{ind}}(\mathbf{r}) = - \int d^3 r' \Pi(\mathbf{r}, \mathbf{r}') e \delta U(\mathbf{r}') \ .
\]  

(16)

The response function can be expressed in terms of the scattering states (Green’s function of the Schrödinger equation). Note that the Lindhard function describes the variation of the charge density not only of the mobile electrons which can be reached from the contacts but also of the localized states. For the purpose of our discussion we simply assume that this response function has been calculated and is known. There is one property of the Lindhard function which is needed later on and which is a simple consequence of the invariance of the electrical system under a global potential shift. A simultaneous change in all electrochemical potentials injects a charge

\[
\delta n(\mathbf{r}) = \sum_{\alpha} \left( \frac{dn(\mathbf{r}, \alpha)}{dE} \right) \delta \mu_{\alpha} + \delta n_{\text{ind}}(\mathbf{r}) \equiv 0 \ .
\]  

(17)

Taking into account Eq. (16) and \( e \delta U = \delta \mu \), one concludes [5]

\[
\frac{dn(\mathbf{r})}{dE} = \int d^3 r' \Pi(\mathbf{r}, \mathbf{r}') \ .
\]  

(18)

The quantity on the left hand side of Eq. (18),

\[
\frac{dn(\mathbf{r})}{dE} = \sum_{\alpha} \left( \frac{dn(\mathbf{r}, \alpha)}{dE} \right)
\]  

(19)

is the local density of states which equals the sum of all injectivities. Equation (18) connects a chemical response quantity, the local density of states, to the Lindhard function \( \Pi \). It is in this regard similar to the Einstein relation between a diffusion constant and a conductivity.

Now we come back to the case where the voltage is changed only in contact \( \alpha \). By inserting Eq. (15) into Poisson’s equation and taking \( e \delta U(\mathbf{r}) = u_{\alpha}(\mathbf{r}) \delta \mu_{\alpha} \) into account, we find that the characteristic potential
$u_\alpha(r)$ is the solution of a field equation with a non-local screening kernel and a source term given by the *injectivity* of contact $\alpha$,

$$-\Delta u_\alpha(r) + 4\pi e^2 \int d^3r' \Pi(r, r') u_\alpha(r') = 4\pi e^2 (dn(r, \alpha)/dE) .$$  \hspace{0.5cm} (20)

We define the Green’s function $g(r, r_0)$ as the solution of Eq. (20) with the source term $e \, dn(r, \alpha)/dE$ replaced by a localized test charge $e \delta(r - r_0)$ at point $r_0$. The characteristic potential $u_\alpha(r)$ can then be written in the form

$$u_\alpha(r) = \int d^3r' \, g(r, r') \, (dn(r', \alpha)/dE) .$$  \hspace{0.5cm} (21)

Using Eqs. (12) and (19), a summation over $\alpha$ implies for Green’s function the property [5]

$$\int d^3r' g(r, r') \sum_\alpha (dn(r', \alpha)/dE) = \int d^3r' g(r, r') \,(dn(r')/dE) \equiv 1 .$$  \hspace{0.5cm} (22)

The same relationship follows from the condition that the sum of all induced charge densities plus the test charge is zero.

Now we find the condition for the electrical self-consistency of Eqs (6). According to Eq. (22) the characteristic potential is equal to unity if the Green’s function is convoluted with the local density of states. Therefore, we must have that the injectivity $dn(r, \alpha)/dE$ deep in contact $\alpha$ is equal to the local density of states $dn(r)/dE$. This requires that nearly all (in a thermodynamic sense) electrons approaching the contact $\alpha$ be reflected into the reservoir. If the conductor and the reservoir consist of the same material then the reservoir must be wide compared to the mesoscopic conductor. In semiconductor samples with metallic contacts, on the other hand, the contact might be actually narrow compared to the dimensions of the semiconductor since the density of states at the Fermi energy of the metal is much larger then that of the semiconductor. This is the case, for instance, in Ga/As samples used to measure the quantized Hall effect [21].

Our emphasis that self-consistency requires geometries which are of a wide-narrow-wide geometry deserves further discussion: the notion that a portion of length $L$ of a purely one-dimensional conductor has a conductance $G = (e^2/h)T$ (per spin) seems to be widely accepted. In contrast, from the point of view taken here a strictly one-dimensional conductor cannot be characterized by a conductance.

3.2. CHARGE CONSERVATION

Let us now demonstrate that if the conditions of self-consistency are met, then the application of an external bias to the conductor preserves the
total charge in the system. To be more precise, imagine a volume $\Omega$ which encloses the entire conductor including a portion of the reservoirs which is so large that at the place where the surface of $\Omega$ intersects the reservoir all the characteristic potentials are either zero or unity. We demonstrate charge conservation for the case that we increase $\mu_1$ by a small value $\delta\mu_1$ above the equilibrium chemical potential. The change in density is given by Eq. (15). Expressing the induced charge with the help of the Lindhard function $\Pi(r,r')$ and using the Green’s function to relate the characteristic potentials to the injected density gives

$$\delta n(r) = \left(\frac{dn(r,1)}{dE} - \int d^3r'd^3r''\Pi(r,r')g(r',r'')\frac{dn(r'',1)}{dE}\right)\delta\mu_1 \quad (23)$$

The total variation in charge is

$$\int_{\Omega} d^3r \delta n(r) = \int_{\Omega} d^3r' \left(\frac{dn(r',1)}{dE} - \int_{\Omega} d^3r''\frac{dn(r')}{dE}g(r',r'')\frac{dn(r'',1)}{dE}\right)\delta\mu_1 \quad (24)$$

where we used Eq. (18) and the symmetry property $\Pi(r,r') = \Pi(r',r)$ of the Lindhard function. Equation (22) then implies that the total variation in charge inside the volume $\Omega$ vanishes. This can be understood in the following way. According to the law of Gauss the charge included in a volume $\Omega$ is $\int_{\partial\Omega} E dS = 4\pi Q$. The charge in $\Omega$ is conserved if the electric flux through the surface of $\Omega$ vanishes. This means that for this conductor all electric field-lines which are generated when we bias the sample have their sources and sinks within the volume $\Omega$. Application of a bias voltage to an electrical conductor results in a redistribution of the charge within our sample but not in an overall change of the charge. If the conductor is poor, i. e. nearly an insulator, the reservoirs act like plates of capacitors. In this case long-range fields exist which run from one reservoir to the other and from a reservoir to a portion of the conductor. But if we chose the volume $\Omega$ to be large enough then all field lines stay within this volume.

In all these considerations we have implicitly assumed that our conductor and the reservoirs is all that counts. Such a situation might be realized for the metallic mesoscopic structures fabricated on insulating substrates. But this picture is certainly not complete if we deal with modern mesoscopic semiconductor structures which are often defined with the help of a number of nearby gates. In such a case we must take a broader view and include inside our volume $\Omega$ not only the conductor of interest but also all of the nearby gates [5]. From an electrostatic point of view, we deal then not only with the mesoscopic object of interest but we have to take into account the nearby electrical bodies used to define this object. In such a case the overall charge is still conserved, even though the total charge
on the mesoscopic conductor of interest varies with the applied bias. The theory presented here can easily be extended to this case [5].

4. Nonlinear I-V Characteristic

As an application of the discussion given above, let us consider the nonlinear I-V characteristic. The discussion presented here can also be carried out in terms of an external and internal response and provides a nice illustration of these concepts [5]. Nonlinearities in metallic mesoscopic samples have been analyzed by Al'tshuler and Khmelnitskii [38] using diagrammatic techniques without a self-consistent potential. For transmission through a tunnel contact the effect of a potential which changes with increasing applied bias has been investigated by Frenkel [39]. Landauer has pointed to the necessity of a self-consistent treatment of the internal potential [40].

In this section we derive the current-voltage characteristic taking into account nonlinearities which are a consequence of the increase of the external electrochemical potential differences as well as the changing internal (self-consistent) potential distribution. We focus on the leading nonlinear correction of the low-voltage ohmic behavior of the sample. The I-V characteristic of a mesoscopic sample is in general rectifying, i.e. \( I(V) \neq -I(-V) \). Furthermore, since rectification also depends on the internal potential and since the internal potential in conductor \( k \) depends on the charge distribution of other nearby conductors, the rectification properties of a small sample are dependent on its entire electric environment. Nevertheless, we consider for simplicity a conductor which is in electric isolation. Reference [5] presents a more general result being valid if there are additional nearby conductors like gates or capacitors.

To proceed we view the scattering matrices as a functional of the potential distribution \( s_{\alpha\beta}(E, U([\mu_\alpha], r)) \) and expand \( U \) away from the equilibrium potential-distribution. The scattering matrix in the neighborhood of the equilibrium reference-state (index 0) is \( s_{\alpha\beta}(E, eU(r)) = s_{\alpha\beta}^0(E) + \int d^3r' (s_{\alpha\beta}^0/E)e\delta U(r')e\delta U(r') \). Here, \( e\delta U(r') \) can be expressed in terms of the characteristic potentials and the electrochemical potentials of the reservoirs. The total current at probe \( \alpha \) can be found by the same considerations that lead to Eq. (6). However, we stop short of linearizing the resulting expression in the electrochemical potentials. The neglect of any inelastic scattering in the presence of large applied voltages is of course a limitation, but since we only focus on the quadratic term in the voltages this limitation might be not be so serious. The current is [31]

\[
I_\alpha = (e/h) \sum_\beta \int dEf_\beta Tr \left[ 1_{\alpha\delta\alpha\beta} - s_{\alpha\beta}^\dagger(E, U(r))s_{\alpha\beta}(E, U(r)) \right],
\]

(25)
where $f_\beta$ is the Fermi function belonging to reservoir $\beta$. The sum over all currents at all terminals is still zero due to the unitarity of the scattering matrix, hence the current is conserved. In order that the current depends on voltage differences only, it is necessary to treat the potentials self-consistently. We expand Eq. (25) in powers of the electrochemical potential deviations $\delta \mu_\alpha = eV_\alpha$.

$$I_\alpha = \sum_\beta g_{\alpha\beta} V_\beta + (1/2) \sum_{\beta\gamma} g_{\alpha\beta\gamma} V_\beta V_\gamma .$$  \hspace{1cm} (26)

The terms linear in the electrochemical potentials are determined by the deconductances $g_{\alpha\beta} = (e^2/h) \int (-df/dE) Tr \left[ 1_{\alpha\beta} - s^\dagger_{\alpha\beta}(E) s_{\alpha\beta}(E) \right]$ which are a functional of the equilibrium reference-potential only. The leading order nonlinear terms are given by transport coefficients which are composed of an external and an internal response:

$$g_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}^e + g_{\alpha\beta\gamma}^i .$$  \hspace{1cm} (27)

The external response arises from the expansion of the Fermi functions in powers of the electrochemical potentials and is given by [5]

$$g_{\alpha\beta\gamma}^e = -\frac{e^3}{h} \delta_{\beta\gamma} \int dE \left( -\frac{df_0}{dE} \right) Tr \left[ \frac{ds_{\alpha\beta}}{dE} s^\dagger_{\alpha\beta} + \frac{ds^\dagger_{\alpha\beta}}{dE} s_{\alpha\beta} \right]$$  \hspace{1cm} (28)

where $f_0$ is the equilibrium Fermi function. Let us examine the external response for the case that we have a two-terminal conductor. From the unitarity of the scattering matrix we find that all the non-vanishing second-order conductance coefficients are equal in magnitude,

$$g_{111}^e = -g_{122}^e = -g_{211}^e = g_{222}^e .$$  \hspace{1cm} (29)

Furthermore, these coefficients can also be expressed just as the energy derivative of the transmission probability,

$$g_{111}^e = \frac{e^3}{h} \int dE \left( -\frac{df_0}{dE} \right) \frac{dT}{dE}$$  \hspace{1cm} (30)

Thus up to the second order the contribution of the external response to the current is

$$I_1 = -I_2 = g(V_1 - V_2) + (1/2) g_{111}^e (V_1^2 - V_2^2) .$$  \hspace{1cm} (31)

Despite the fact that currents are conserved this is an unphysical result. The quadratic term depends not only on the voltage difference but on the
individual voltages. Equation (31) would predict that we should observe a
different current depending on whether we rise the voltage of the left con-
tact by $\delta \mu_1 = eV_1$ or whether we decrease the voltage on the right contact
by $\delta \mu_2 = -eV_1$ as compared to the equilibrium value of the electrochemical
potential. This is a simple example which demonstrates why the calcula-
tion of a nonlinear current voltage characteristic without the self-consistent
adjustment of the electrostatic potential makes no sense.

Let us now consider the internal response. The internal response is a
consequence of the change in the potential distribution and is given by [5]

$$ g_{\alpha \beta \gamma}^i = -\frac{e^3}{h} \int dE \left( -\frac{d f_0}{dE} \right) \int d^3 r T r \left[ s_{\alpha \beta}^\dagger \delta s_{\alpha \beta} + \frac{\delta s_{\alpha \beta}^\dagger}{\delta eU(r)} s_{\alpha \beta} \right] u_\gamma(r) $n

$$ -\frac{e^3}{h} \int dE \left( -\frac{d f_0}{dE} \right) \int d^3 r T r \left[ s_{\alpha \gamma}^\dagger \delta s_{\alpha \gamma} + \frac{\delta s_{\alpha \gamma}^\dagger}{\delta eU(r)} s_{\alpha \gamma} \right] u_\beta(r). \quad (32) $$

Note that the internal response contributes only to quadratic order in the
voltage. The linear conductance is a purely external response. If we now
add external and internal response, take into account Eq. (12) and that the
integral over $\Omega$ of an internal response term with $u = 1$ is equal to minus
the external response with the functional derivative replaced by an energy
derivative, we find [5]

$$ \sum_\alpha g_{\alpha \beta \gamma} = \sum_\beta g_{\alpha \beta \gamma} = \sum_\gamma g_{\alpha \beta \gamma} = 0. \quad (33) $$

For a two-terminal conductor the second-order conductance coefficients
obey $g_{111} = -g_{121} = g_{211} = -g_{221} = g_{122} = -g_{212}$ = $g_{222}$. Consequently, the currents are

$$ I_1 = -I_2 = g(V_1 - V_2) + (1/2)g_{111}(V_1 - V_2)^2 + ... \quad . \quad (34) $$

Now, the current depends only on the voltage difference as it must be. In
contrast to the external response which could simply be expressed in terms
of energy derivatives of the transmission probability, the total response de-
dpends on the charge distribution inside the conductor.

As a simple example, we consider an asymmetric resonant double bar-
rier. The long lived state has a decay width $\Gamma_1$ to the left and $\Gamma_2$ to the
right. For simplicity, assume that the potential in the well is determined
by a local charge-neutrality argument (see Sec. 6). The characteristic po-
tentials in the well are $u_1 = \Gamma_1/\Gamma$ and $u_2 = \Gamma_2/\Gamma$, where $\Gamma = \Gamma_1 + \Gamma_2$ is the decay width. From Eqs. (28) and (32) we find for the second order conductance coefficient $g_{111} = (e^3/h)(dT/dE)(1 - 2u_1)$ and hence [13, 20]

$$ g_{111} = (e^3/h)(dT/dE)(\Gamma_2 - \Gamma_1)/\Gamma. \quad (35) $$
In summary, we emphasize that the nonlinearity cannot be discussed without a concern for the way the potential drops in the interior of the conductor.

5. Frequency Dependent Conductance

5.1. EXTERNAL RESPONSE

We are interested in the dynamical response of the conductor. A time-dependent voltage \( \propto \exp(-i\omega t) \) can be applied across two terminals, between a terminal and a nearby gate, or between two nearby gates. We want to know the currents which appear as a consequence of these oscillating voltages at the contacts of the conductor or at the contacts to the nearby gates. We are seeking the admittance matrix

\[
G_{\alpha\beta}(\omega) = \frac{I_\alpha(\omega)}{V_\beta(\omega)}. \tag{36}
\]

Again we consider for simplicity the case of a two-terminal conductor in electrical isolation. The case where there are a number of nearby gates (capacitors) or other conductors has been the subject of a number of discussions [5, 46]. Thus the indices \( \alpha, \beta \) take the values 1 and 2 for the left and right contact, respectively. If there are no other nearby electrical conductors then all electric field lines which emanate from the conductor also return to the conductor or to the reservoirs. Even in the presence of time-dependent voltages applied to this conductor the reservoirs remain locally charge neutral, i.e. the electric field lines emanate from a reservoir only in the region where the transition to the conductor occurs. If the conductor is very short then the reservoirs act like the plates of a capacitor and, due to long range Coulomb interaction, field lines connect the surfaces of the two reservoirs facing each other. As in the dc-case there exists, therefore, a volume \( \Omega \) which is so large that there is no electric flux through its surface. Consequently, if we include all components of the system within \( \Omega \) then the total charge within this volume is zero, i.e. all currents at the terminals must add up to zero. Furthermore, since a potential which is uniform over the entire volume \( \Omega \) is of no physical consequence the resulting currents must depend on the potential differences only. Therefore, Eqs. (9) and (10) hold also for the dynamic conductance: the rows and columns of the dynamic conductance matrix \( G_{\alpha\beta}(\omega) \) must add up to zero. We call such a discussion of the ac-transport a charge and current conserving theory. Below we illustrate the features of such a theory for the case of low frequencies only. But the extension to a larger range of frequencies and to nonlinearities must follow the very same line of thought.
5.2. DECOMPOSITION OF THE DENSITY OF STATES

The total density of states in the conductor inside the volume $\Omega$ is a sum of four contributions [34],

$$dN/dE = \sum_{\alpha\beta} (dN_{\alpha\beta}/dE),$$

where

$$dN_{\alpha\beta}/dE = \frac{1}{4\pi i} \text{Tr} \left[ s_{\alpha\beta}^\dagger \frac{ds_{\alpha\beta}}{dE} - \frac{ds_{\alpha\beta}^\dagger}{dE} s_{\alpha\beta} \right].$$

are the partial densities of states. Fig. 5 gives a schematic representation of the partial density of states. The partial density of states $dN_{11}/dE$ consists of carriers that originate in contact 1 and return to contact 1. The partial density of states $dN_{21}/dE$ consists of carriers that originate in contact 1 and are transmitted to contact 2. It turns out that the external response is determined exactly by these four partial densities of states. The partial densities of states represent a decomposition of the total density of states both with respect to the origin of the carriers (injecting contact, right index) and the final destination of the carriers (emitting contact, left index).

At constant electrostatic potential the total charge injected into the conductor under a simultaneous and equal increase of the chemical potentials at its contacts $\delta E_{F1} = \delta E_{F2} = e\delta\mu$ is $\delta Q = e \sum_{\alpha\beta} (dN_{\alpha\beta}/dE) \delta\mu$. To find the current at contact $\alpha$ we need to know which portion of this
charge enters or leaves the conductor through contact $\alpha$, i.e. how the total charge is partitioned on the two contacts. The answer to this question was found by one of the authors, Pretre and Thomas [34] using a linear response calculation. The following simple argument leads to the same result. The scattering matrix $S_{\alpha\beta}$ determines the current amplitudes of the outgoing waves in contact $\alpha$ as a function of the current amplitudes of the incident waves in contact $\beta$. The charge which is injected by an increase of the Fermi energy at contact 1 is $(edN_{11}/dE + edN_{21}/dE)eV_{1}(\omega)$. Only the additional charge $\delta Q_{1}^{c}(\omega) = e(dN_{11}/dE)eV_{1}(\omega)$ leads to a current at contact 1, whereas $\delta Q_{2}^{c}(\omega) = e(dN_{21}/dE)eV_{1}(\omega)$ is determined by carriers which leave the conductor through contact 2. Therefore, a variation of the Fermi levels of the contacts $\beta$ causes the current

$$\delta I_{\alpha}^{c}(\omega) = -i\omega e^{2}\sum_{\beta}(dN_{\alpha\beta}/dE)V_{\beta}(\omega)$$

(39)

at contact $\alpha$. Since direct transmission between contact 1 and 2 is possible, an oscillating voltage causes in addition at these contacts a current determined by the dc-conductance. Thus the leading external low-frequency current response to an oscillating chemical potential $eV_{\alpha}(\omega)$ is given by [34]

$$G_{\alpha\beta}^{e}(\omega) = G_{\alpha\beta}(0) - i\omega e^{2}(dN_{\alpha\beta}/dE) .$$

(40)

This external response is not current conserving. Since the dc-conductances satisfy $\sum_{\alpha}G_{\alpha\beta}(0) = \sum_{\beta}G_{\alpha\beta}(0) = 0$, one finds that to leading order in frequency $\sum_{\alpha}G_{\alpha\beta}^{e}(\omega)$ is proportional to the total charge injected from contact $\beta$ into the conductor. The injected charges create an internal, time-dependent electric potential $\delta U(\mathbf{r}, t)$ which in turn causes additional currents. In the following subsection, we investigate the response to such an internal electrostatic potential. It turns out that this requires a detailed knowledge of the charge distribution.

The density of states (37) has been obtained assuming a perturbation which is (in a mathematical sense) asymptotically far away from the sample. The derivative with respect to the energy $E$ is a consequence of the asymptotic nature of this perturbation. Physically what counts is the density of states in a finite volume $\Omega$. To obtain these densities it is better to first calculate the local densities corresponding to Eq. (38) and to integrate these local densities over the volume $\Omega$. As shown below the local densities are not given by energy derivatives of the scattering matrix but by functional derivatives with respect to the local potential $eU(\mathbf{r})$. The densities determined by integration of such local densities of states differ in general from a simple energy derivative by a quantum correction [41, 42]. The difference vanishes in the semi-classical (WKB) limit. The same is of course valid for the derivatives in Eq. (28).
We conclude this section with a remark on localized states. Equation (37) is not complete. A conductor might also contain a contribution to the density of states from localized states, in addition to the extended scattering states considered so far. For the external response the localized states play no role, but later when we consider the screening the localized states are important. To be brief, however, we do not discuss here their role in detail.

5.3. RESPONSE TO AN OSCILLATING ELECTROSTATIC POTENTIAL

To investigate the self-consistency of dc-transport we already discussed the local charge distribution inside the conductor. In Eq. (13) we introduced a local partial density of states which we called the injectivity (see Fig. 6). Now we introduce additional local partial densities of states which permit eventually to write the dynamical conductivity in a simple and transparent manner. We are interested in the currents generated at the contacts of a sample in the presence of an oscillating potential $\delta U(r, t)$. We can Fourier transform this potential with respect to time and consider a perturbation of the form

$$
\delta U(r, t) = u(r)(U_{+\omega} \exp(-i\omega t) + U_{-\omega} \exp(+i\omega t)) .
$$

(41)

Since the potential is real we have $U_{-\omega} = U_{+\omega}^*$. The response to such a potential can be treated using a scattering approach [19]: due to the oscillating internal potential a carrier incident with energy $E$ can gain or lose modulation energy $\hbar\omega$ during reflection at the sample or during transmission through the sample. The amplitude of an outgoing wave is a superposition of carriers incident at energy $E$ and at the side-band energies, $E \pm \hbar \omega$. In the low-frequency limit the amplitudes of the outgoing waves can be obtained by considering the scattering matrix $s_{\alpha\beta}(U(r, t), E)$ to be a slowly varying function of the potential $U(r, t)$. Since the deviations of the actual potential away from the (time-independent) equilibrium potential $U_{eq}(r)$ are small, we can expand the scattering matrix in powers of $\delta U(r, t) = U(r, t) - U_{eq}(r)$ to linear order

$$
s_{\alpha\beta}(U(r, t), E) = s_{\alpha\beta}(U_{eq}(r), E) + (\delta s_{\alpha\beta}/\delta U(r, t))\delta U(r, t) .
$$

(42)

Evaluation of the current at contact $\alpha$ gives [19]

$$
\delta I_\alpha(\omega) = ie^2\omega \int d^3r (dn(\alpha, r)/dE)u(r)U_{+\omega} .
$$

(43)

Here we have introduced the local partial density of states [5, 19]

$$
\frac{dn(\alpha, r)}{dE} = -\frac{1}{4\pi i} \sum_\beta Tr \left[ s_{\alpha\beta}^\dagger \frac{\delta s_{\alpha\beta}}{e\delta U(r)} - \frac{\delta s_{\alpha\beta}^\dagger}{e\delta U(r)}s_{\alpha\beta} \right]
$$

(44)
which we call the *emissivity* (see Fig. 7). It describes the local density of states of carriers at point \( \mathbf{r} \) which are emitted by the conductor at probe \( \alpha \). A more detailed derivation of Eq. (43) can be found in Ref. [19]. It is useful to express the response to the internal potential in the form of a conductance defined as \( \delta I^i_\alpha(\omega) = G^i_\alpha(\omega)U_\omega \). Comparison with Eq. (40) gives for the internal conductances

\[
G^i_\alpha(\omega) = i\varepsilon^2 \omega \int d^3r \left( \frac{dn(\alpha, \mathbf{r})}{dE} \right) u(\mathbf{r}).
\]  

(45)

Below we use this internal response to complete the calculation of the total current. Before doing this it seems useful to pause for a moment and to discuss in more detail the local density of states which determine the internal response.

### 5.4. Decomposition of the Local Density of States

In Eq. (14) we have expressed the injectivity (13) with the help of scattering states. Now we give an expression of the injectivity in terms of derivatives of the scattering matrix. Expressions which relate wave functions to functional derivatives are known from the discussion of the characteristic times occurring in tunneling processes [43, 44]. Consider for a moment a one-dimensional scattering problem with a potential \( V(x) \) in an interval \((-a, a)\). The scattering matrices are \( s_{\alpha\beta} \) where \( \alpha \) and \( \beta \) take the values 1 and 2 to designate left and right, respectively. Of interest is the time a particle *dwell* s in this region irrespective of whether it is ultimately transmitted or whether it is ultimately reflected. There are two dwell times \( \tau_{D\alpha} \),
for the particles arriving from the left or from the right. In terms of the
scattering states $\Psi_\alpha(x)$ and the incident current $I$, the dwell time is given
by [44]

$$\tau_{D\alpha} = \int_x^{x+a} dx |\Psi_\alpha(x)|^2 / I .$$

(46)

To find the time a particle dwells in an interval $(x, x + a)$ an infinitesimal
uniform perturbation $dV$ is added in this region to the potential $V(x)$. It
is found that the dwell time is then related to the scattering matrix via the
following relationship [43, 44]

$$\tau_{D\alpha} = \hbar \text{Im} \left( |s_{1\alpha}|^2 \frac{d \ln s_{1\alpha}}{dV} + |s_{2\alpha}|^2 \frac{d \ln s_{2\alpha}}{dV} \right) .$$

(47)

where Im denotes the imaginary part. For a plane-wave scattering state
with wave vector $k$ the current is $v = \hbar k/m$. Thus a comparison with Eq.
(14) shows that for a single quantum channel the dwell time is related to
the injectivity by

$$\tau_{D\alpha}/\hbar = \int_x^{x+a} dx \left( \frac{dn(x, \alpha)}{dE} \right) .$$

(48)

This means that their exists a simple relationship between local density of
states and derivatives of the scattering matrix with respect to potentials. It
is easy to extend this relation to the case of an arbitrary space dependent
potential and to an arbitrary number of channels. The final result is that
the injectivity is given by [5, 19]

$$dn(r, \alpha)/dE = -\frac{1}{4\pi i} \sum_{\beta} Tr \left[ s_{1\alpha}^\dagger \frac{\delta s_{2\alpha}}{\delta U(r)} - \frac{\delta s_{2\alpha}^\dagger}{\delta U(r)} s_{1\alpha} \right] .$$

(49)

The injectivity contains information about the origin of the particles: it is
important through which contact the carriers enter the conductor. Consequently, the summation is over the first index of the scattering matrices.
In contrast, the emissivity defined in Eq. (44) contains information about
the future of the carriers: it is important through which contact the carri-
ers leave the sample. The summation is thus over the second index of the
scattering matrix. The sum of all the injectivities (see Fig. 6) or the sum
of all the emissivities (see Fig. 7) is equal to the local density of states,

$$dn(r)/dE = \sum_\alpha dn(\alpha, r)/dE = \sum_\alpha dn(r, \alpha)/dE .$$

(50)
Equation (50) represents a decomposition of the local density of states into emissivities and injectivities. We mention that the emissivities and injectivities are not independent of one another. In fact, in the absence of a magnetic field they are equal, \( dn(\alpha, r)/dE = dn(r, \alpha)/dE \). In the presence of a magnetic field the microreversibility of the scattering matrix implies that the emissivity into contact \( \alpha \) in magnetic field \( B \) is equal to the injectivity of contact \( \alpha \) if the magnetic field is reversed,

\[
dn(\alpha, r; B)/dE = dn(r, \alpha; -B)/dE. \tag{51}
\]

While the local density of states at equilibrium is an even function of the magnetic field, i. e. \( dn(r; B)/dE = dn(r; -B)/dE \), the injectivities and emissivities are in general not even functions of \( B \). This has some peculiar physical consequences, as has been shown recently by a low-frequency measurement of capacitances in a quantum Hall system [45].

5.5. COMBINED EXTERNAL AND INTERNAL RESPONSE

We need to find an expression of the current response generated by the electric potential oscillations caused by the external potentials \( \delta \mu_\alpha(\omega) \exp(-i\omega t) \). We are interested in the response to first order in \( \omega \), and since the currents in Eqs. (39) and (43) are proportional to \( \omega \) it is sufficient to know the quasistatic nonequilibrium state discussed in Sect. 3. We express the deviation of the potential away from the equilibrium potential with the help of the
characteristic potentials defined in Eq. (11). From Eq. (43) we find
\[ \delta I_{\alpha}^i(\omega) = i e \omega \sum_{\beta} \int d^3r \frac{dn(\alpha, r)}{dE} u_\beta(r) \delta \mu_\beta(\omega). \] (52)

Thus the induced potentials give rise to a conductance
\[ G_{\alpha \beta}^i(\omega) = i e^2 \omega \int d^3r \frac{dn(\alpha, r)}{dE} u_\beta(r), \] (53)
which can be written in terms the Green's function and the injectivity,
\[ G_{\alpha \beta}^i(\omega) = i e^2 \omega \int d^3r \int d^3r' \frac{dn(\alpha, r)}{dE} g(r, r') \frac{dn(r', \beta)}{dE}. \] (54)

Eq. (54) tells us that the internal response \( G_{\alpha \beta}^i(\omega) \) is a consequence of the charge injected from contact \( \beta \) which generates a potential determined by the Green’s function and that this potential in turn generates a current at \( \alpha \) determined by the emissivity into that contact. The total response is the sum of the external response (40) and the internal response (54),
\[ G_{\alpha \beta}(\omega) = G_{\alpha \beta}^e(\omega) + G_{\alpha \beta}^i(\omega). \] (55)

We express it in the form
\[ G_{\alpha \beta}(\omega) = G_{\alpha \beta}(0) - i \omega E_{\alpha \beta} + O(\omega^2) \] (56)
and call \( E_{\alpha \beta} \) the (screened) emittance of the conductor. It is given by [5]
\[ E_{\alpha \beta} = e^2 \frac{dN_{\alpha \beta}}{dE} - e^2 \int d^3r \int d^3r' \frac{dn(\alpha, r)}{dE} \frac{dn(r', \beta)}{dE} g(r, r'). \] (57)

Before we apply this result in the next section to a few simple problems, we want to demonstrate that \( G_{\alpha \beta} \) is indeed current conserving, i. e. that the rows and columns of the emittance matrix add up to zero. Consider the first column. If we add \( E_{11} \) and \( E_{21} \) the first terms in the emittance give the total charge \( dN_{11}/dE + dN_{21}/dE \) injected from contact 1. In the second term the two emissivities add to give the local density of states. Now Eq. (22) is used. What remains is the integral over the entire volume of the injectivity which is just the total injected charge. Thus for a two terminal conductor in electrical isolation the emittance matrix satisfies \( E_{\mu} = E_{11} = -E_{12} = -E_{21} = E_{22}. \)

6. Examples
6.1. EMITTANCE OF A METALLIC DIFFUSIVE CONDUCTOR

Let us consider a mesoscopic metallic conductor connecting two reservoirs. In a metallic conductor charge accumulations are screened over a Thomas-Fermi screening length (apart from miniscule and more subtle Friedel-like
long-range effects [37]). If we assume in addition that the density varies not to rapidly then the local potential is directly determined by the local density. The local potential \( \delta U(\mathbf{r}) \) generated by an injected charge \( \delta n_{in}(\mathbf{r}) \) is determined by \( (dn(\mathbf{r})/dE)e\delta U(\mathbf{r}) = \delta n_{in}(\mathbf{r}) \). This corresponds to a Green’s function which is a delta function in space and with a weight inversely proportional to the local density of states \( g(\mathbf{r}, \mathbf{r}') = (dn(\mathbf{r})/dE)^{-1}\delta(\mathbf{r} - \mathbf{r}') \).

Using this in Eq. (57) gives an emittance in terms of densities only [19],

\[
E_{\alpha\beta}(\omega) = e^2 \frac{dN_{\alpha\beta}}{dE} - e^2 \int d^3 r \frac{dn(\alpha, \mathbf{r})}{dE} \left( \frac{dn(\mathbf{r})}{dE} \right)^{-1} \frac{dn(\mathbf{r}, \beta)}{dE} .
\]

(58)

There are no electric field lines outside the conductor.

The wire with cross-section \( A \) ranges from \( x = -L/2 \) to \( x = L/2 \). The mean distance between the impurities is \( l \). The reflection and the transmission probability per channel are \( R = 1 - l/L \) and \( T = l/L \), respectively. The partial densities of states of reflected carriers are \( dN_{11}/dE = dN_{22}/dE = (1/2)(1 - l/L)(dN/dE) \) where \( dN/dE \) is the total density of states in the volume \( \Omega = AL \). The partial densities of states of transmitted carriers are \( dN_{12}/dE = dN_{21}/dE = (1/2)(l/L)(dN/dE) \). The diffusion equation for the diffusive metallic conductor implies the ensemble averaged and over the cross section averaged injectivities

\[
dn(x, 1)/dE = (1/2L)(dN/dE)(1 - 2x/L) \quad \text{and} \quad dn(x, 2)/dE = (1/2L)(dN/dE)(1 + 2x/L).
\]

In the absence of a magnetic field, the emissivities are given by the same expressions. The linear dependence of the injectivities gives an ensemble averaged potential which drops also linearly. With the help of these expressions, we find for the emittance [13]

\[
E_{\mu} = (1/6 - l/2L)(dN/dE) .
\]

(59)

Already for metallic wires longer then \( 3l \) the emittance is a positive quantity. Since typically \( l/L \) is much smaller than \( 1/3 \) the metallic diffusive wire at low frequencies responds like a capacitor with an ensemble averaged capacitance given by Eq. (3).

6.2. EMITTANCE OF A PERFECT BALLISTIC WIRE

Consider a ballistic wire of length \( L \) with \( N \) quantum channels. We apply again Thomas-Fermi screening. However, this is not very well justified and permits to obtain an estimate only. We also ignore the variation of the potentials near the contacts, which in a more realistic treatment might well give us a capacitive contribution. Each quantum channel contributes with a density of states \( 2L/hv_n \), were \( v_n \) is the channel velocity evaluated at the Fermi energy. The total density of states per spin is \( dN/dE = \sum_n 2L/hv_n \). In the absence of backscattering it holds \( dN_{11}/dE = dN_{22}/dE = 0 \) and
\[
dN_{21}/dE = dN_{12}/dE = (1/2) dN/dE. \quad \text{The injectivities } \quad dn(x,1)/dE = (1/2L)(dN/dE) \quad \text{and} \quad dn(x,2)/dE = (1/2L)(dN/dE) \quad \text{are independent of the space coordinate. This corresponds to a potential in the ballistic wire which is constant and midway between the electrochemical potentials at the contacts. Hence, using Eqs. (57) we find that an ideal perfect wire has a negative emittance given by Eq. (4). A ballistic wire responds like a conductor which classically is represented by a resistance and an inductance in series, and where the emittance can be viewed as a kinetic inductance.}
\]

### 6.3. EMITTANCE OF A RESONANT DOUBLE BARRIER

As a third example we consider a resonant double barrier [19]. The scattering matrix is
\[
s_{\alpha\beta} = \left( \delta_{\alpha\beta} - i\Gamma_{\alpha}\Gamma_{\beta}/\Delta \right) \exp(i\delta_{\alpha} + i\delta_{\beta}),
\]
where \(\delta_{\alpha\beta}\) is the Kronecker symbol and the \(\delta_{\alpha}\) are phases whose energy dependence can be neglected compared to the rapid variation of the resonant denominator \(\Delta = E^2 - E^2_r - e\delta U - i\Gamma/2\) with \(\Gamma = \Gamma_1 + \Gamma_2\). Here, \(E_r\) is the resonant energy at equilibrium and \(e\delta U\) is the deviation of the electrostatic potential away from its equilibrium value at the site of the long lived state in the presence of transport. With the help of the injectances \(dN_{\alpha}/dE = (\Gamma_{\alpha}/2\pi|\Delta|^2)\) and the total density of states \(dN/dE = (\Gamma/2\pi|\Delta|^2)\) the partial densities of state can be expressed in the following manner: For \(\alpha = \beta\) we find
\[
dN_{\alpha\alpha}/dE = R^2 dN/dE \pm \frac{1}{2} (dN_1/dE - dN_2/dE) \quad (60)
\]
where the plus and the minus sign correspond to \(\alpha = 1\) and \(\alpha = 2\), respectively. Here, \(R = 1 - T\) is the reflection probability. For \(\alpha \neq \beta\), one has
\[
dN_{\alpha\beta}/dE = T dN/dE. \quad (61)
\]

The unscreened injectances (emittances) are found by integrating the injectivities (emissivities) over the volume of the localized state, i.e. over the well. They are given by
\[
dN_{1\alpha}/dE + dN_{2\alpha}/dE = dN_{\alpha}/dE. \quad (62)
\]

In a Thomas-Fermi approach the characteristic potentials \(u_{\alpha}\) in the well are determined by \((dN/dE)u_{\alpha} = dN_{\alpha}/dE\). This gives \(u_{\alpha} = \Gamma_{\alpha}/\Gamma\). Using Eq. (58) gives an emittance [19]
\[
E_\mu = -e^2 \frac{(dN_1/dE)(dN_2/dE)}{dN/dE} \left( \frac{\Gamma^2/2 - |\Delta|^2}{|\Delta|^2} \right) \quad (63)
\]

For a symmetric resonant tunneling barrier Eq. (63) simplifies and is given by Eq. (5). At resonance the emittance is negative reflecting kinetic (inductive) behavior, it is zero at half-width of the resonance, and it is positive
ELECTRICAL CONDUCTION

(capacitive) if the Fermi level is more than a half width above (below) the resonant energy. Clearly, Thomas-Fermi screening is not very realistic for such a conductor. Moreover, the quantization of charge in the well might play a decisive role. Nevertheless, these considerations indicate the character of the results that a more realistic treatment might yield and hopefully stimulate work in that direction.

7. Summary

We have developed a self-consistent discussion of mesoscopic electrical conduction. The determination of the electrical potential in the presence of a dc-current, although unimportant for the discussion of the dc-conductances itself, permits to discuss the conditions under which the dc-conductance formulae are valid, it permits to calculate the first nonlinear corrections of the purely linear response, and permits to find the ac-conductances to first order in frequency. We applied the results to some simple examples.

We have emphasized the case of two terminal conductors in electric isolation. The theory permits, however, also to discuss the effect of nearby capacitors and gates and in fact provides a mesoscopic description of capacitances [5, 46]. Some implications of a mesoscopic theory of capacitance, like Aharonov-Bohm oscillations in capacitance coefficients, the gate voltage dependence of persistent currents have already been the subject of recent works [47, 48].

The self-consistent nature of electrical transport is a consequence of the long-range Coulomb interaction of carriers. A self-consistent description must, therefore, tackle an interacting many-particle problem. Consequently, the results of such a theory depend somewhat on the sophistication that is used to treat the many-particle problem. Here we have used a simple Hartree approach. Since density-functional theory is nothing but an improved Hartree theory it gives results which look formally very similar to the results presented here [49]. A stronger modification of the results discussed here can be expected in situations where one must take the quantization of charge into account [48, 50].

The theory presented here, demonstrates that interesting results can be obtained by investigating nonlinearities and ac-conductances. The theory demonstrates that it is necessary to treat nonlinearities and the ac-response self-consistently to conserve both charge and current. We are confident that experiments will eventually demonstrate the close connection between electrostatic questions and nonlinearities and ac-response.
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