A relation between the $Z_3$-graded Grassmann variables and parafermions is established. Coherent states are constructed as a direct consequence of such a relationship. We also give the analog of the Bargmann-Fock representation in terms of these Grassmann variables.
1 Introduction

Recently Kerner [1-3] investigated the use of $Z_3$-graded structures instead of $Z_2$-graded ones in physics. This leads to some interesting results especially when gauge theories are constructed using these structures.

One of these structures is the $Z_3$-graded analog of the Grassmann algebra. The generators of this algebra obey some ternary relations instead of the usual binary relations (anticommutation) for the generators of the conventional Grassmann algebra ($Z_2$-graded one).

In this letter we relate the $Z_3$-graded Grassmann variables to the parafermionic harmonic oscillator. We will use this result in the subsequent construction of coherent states related to this parafermionic oscillator. The construction of these coherent states as well as all the subsequent derivations are carried out in the spirit of what was achieved in the case of the fermionic harmonic oscillator, via the $Z_2$-graded Grassmann variables [4]. This will prove useful in understanding the physical meaning of such structures.

We start with a review of the $Z_3$-graded Grassmann algebra and its properties. In section 3 we review the parafermionic harmonic oscillator algebra. In section 4 we discuss the relationship between the Grassmann algebra of section 2 and the harmonic oscillator of section 3. This will allow us in section 5 to construct the coherent states for this system in a clear and unambiguous manner. In this construction we propose a new form for the resolution of the identity operator, different from the usual one. This will permit in section 6 to give an analog of the Bargmann-Fock representation space: The space of Grassmann representatives.

In section 7 we suggest how the supersymmetric formulation of this construction should look like.

2 $Z_3$-graded Grassmann algebra

$Z_3$ is the cyclic group of three elements. It can be represented in the complex plane as multiplication by the primary cubic root of unity $q = e^{i2\pi/3}$, $q^2$ and $q^3 = 1$. The analog of the $Z_2$-graded Grassmann algebra can be introduced as follows:

Consider an associative algebra spanned by $N$ generators $\xi_a$; $a = 0, 1, \ldots, N$, between which only ternary relations exist. By this we mean that the binary products of any couple of such elements are considered as independent entities, i.e. $\xi_a \xi_b$ are independent of $\xi_b \xi_a$ where $a, b = 0, 1, \ldots, N$. Instead, the analog of the anti-commutation in the $Z_2$-graded case is given by the following ternary relations:

$$\xi_a \xi_b \xi_c = q \xi_b \xi_c \xi_a = q^2 \xi_c \xi_a \xi_b \quad a, b, c = 0, 1, \ldots, N \quad (1)$$

Two important properties follow automatically:

• The third power (or higher) of any generator vanishes:

$$（\xi_a）^3 = 0 \quad (2)$$

• Any product of four or more generators also vanishes:

$$\xi_a \xi_b \xi_c \xi_d = 0 \quad a, b, c, d = 0, 1, \ldots, N \quad (3)$$

At this level one remarks that there is no symmetry between the grade-1 elements (the $\xi$'s) and the grade-2 elements (the $\xi \xi$'s). It seems normal that these elements should play a symmetric role with regard to $q$ and $q^2$. This symmetry is restored in the most natural way,
by adding \( N \) grade-2 generators \( \bar{\xi}_a \) (the duals of \( \xi_a \)). These extra elements satisfy the same relations as the \( \xi \)'s, but with \( q \) replaced by \( q^2 \):

\[
\bar{\xi}_a \bar{\xi}_b \bar{\xi}_c = q^2 \bar{\xi}_b \bar{\xi}_c \bar{\xi}_a \tag{4}
\]

and their binary products with the \( \xi \)'s satisfy:

\[
\xi_a \bar{\xi}_b = q \bar{\xi}_b \xi_a \tag{5}
\]

Now, by requiring that the grades of factors add up (modulo 3) under multiplication and that grade-0 elements commute with all other elements, and that grade-1 with grade-2 elements satisfy the same relation as in (5), then many additional terms must vanish (terms like \( \xi_a \xi_b \xi_c \) for example). The resulting algebra contains only the elements of the form:

- **Grade-0**: \( I, \xi \bar{\xi}, \xi \xi \bar{\xi}, \bar{\xi} \)
- **Grade-1**: \( \xi, \bar{\xi} \)
- **Grade-2**: \( \bar{\xi}, \xi \xi \)

and its dimension is:

\[
D = \frac{3 + 4N + 9N^2 + 2N^3}{3}
\]

The case \( N = 1 \) was considered in detail in [5], however the author imposed there also binary relations between elements of the same grade, and the arguments for this are not too convincing. In the following sections we are going to investigate the same case, however, without imposing such binary relations. This will be more consistent with what we announced previously based on Kerner’s works [1, 2, 3].

### Parafermionic Harmonic Oscillator

The most natural way to introduce a parafermionic harmonic oscillator which can be related to the \( Z_3 \)-graded Grassmann algebra is through the deformed harmonic oscillator algebra generated by the operators \( a, a^+, N \). These operators shall satisfy the following commutation relations:

\[
\begin{align*}
aa^+ - qa^+ a &= q^{-N} \\
Na - aN &= -a \\
Na^+ - a^+ N &= a^+ \\
q^N a^+ &= a^+ q^{N+1} \\
q^N a &= aq^{N-1}
\end{align*}
\tag{6}
\]

where \( q \) is an arbitrary complex parameter of deformation.

Now, when \( q \) is the primitive \( k^{th} \)-root of unity (i.e. \( q = e^{\frac{2\pi i}{k}} \)), one can prove that the annihilation and creation operators \( a \) and \( a^+ \) are nilpotent of degree \( k \):

\[
(a)^k = 0, \quad (a^+)^k = 0 \tag{7}
\]

A comparison of the above relation with (2) suggests that \( Z_3 \)-graded Grassmann variables in section 2 can be used as a representation of this oscillator (for \( k = 3 \)). This is indeed the case, as we shall see in the next section.

The Fock space representation of this algebra is given by:

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5 its dual, in the sense that the product of a \( \xi \) and a \( \bar{\xi} \) is a grade-0 element, i.e. it behaves like a scalar.

6 This means that no more than \( k - 1 \) parafermions are allowed to occupy the same state. This is a straightforward generalization of the Pauli exclusion principle.
\[ a|n> = \sqrt{|n|}|n-1> \\
\]
\[ a^+|n> = \sqrt{|n+1|}|n+1> \\
N|n> = n|n> \tag{8} \]

where \(|n>: n = 0, 1, ..., k\) is the usual Fock space orthonormal basis, and
\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{9}
\]

At this point we should stress the fact that for the case we are considering (\(q=\) root of unity) \(a^+\) is still the usual hermitian conjugate of \(a\). This reflects itself in the representation (8), where \([n] = [\bar{n}]\) which is not the case for a generic \(q\). One then has to introduce two other operators in the algebra, a creation (hermitian conjugate to \(a\)) and an annihilation operator (hermitian conjugate to \(a^+\)), but we do not intend to consider this possibility here.

### 4 Z₃-graded Grassmann variables and the parafermionic harmonic oscillator:

Before we start to construct the coherent states associated to the \(Z₃\)-graded harmonic oscillator we should define the relationship between the \(Z₃\) grassmannian variables introduced in section 2 and the \(Z₃\) harmonic oscillator’s operators introduced in section 3.

According to [5]:
\[ |0> \text{ behaves like a grade-0, } |1> \text{ like a grade-2 element, while } |2> \text{ behaves like a grade-1 element. This, combined with the equations (8), defines the algebra completely:} \]

\[ a^+|0> = |1> \\
a|1> = |0> \tag{10} \]

or
\[ a^+a^+|0> = \sqrt{|2|}|2> \\
a|2> = \sqrt{|2|}|1> \tag{11} \]

This fact suggests to interpret \(a^+\) as being of grade-2 and \(a\) as being of grade-1.

Now, using the conventions cited at the end of section 2, one is led to the following relations:
\[ \xi a^+ = qa^+\xi \]
\[ \bar{\xi} a = \bar{q}a\xi \tag{12} \]

To be consistent with what we announced in section 2, no relations are imposed on the products \(\xi a\) and \(a\xi\) (the same stands for \(\xi a^+\) and \(a^+\bar{\xi}\)). Furthermore no analogs of the ternary relation (1) can be imposed on products of the form \(a\xi a^+\); this is due to the non-conventional commutation relations in (6). In order to evaluate such products, one should use the relations (6) and (12).

We notice that in [5] the author chose \(\xi\) and \(\bar{\xi}\) to commute with \(a\) and \(a^+\). This does not seem consistent, since even in the fermionic case (i.e. \(Z₂\) case) that we are supposed to generalize, the \(\xi\)’s do anti-commute with the \(a\)’s!

\footnote{Of course we should suppose that the operators \(a\) and \(a^+\) obey, in addition to (6), the same equations but with \(\bar{q}\) instead of \(q\), for instance \(a\bar{a}^+ - \bar{q}a^+a = q^N\)}
5 Coherent States

In what follows, we shall drop the rule cited at the end of section 2 (the grades adding up mod-3 under multiplication, the grade-1 elements behaving as the ξ's, grade-2 elements as the \( \bar{\xi} \)'s, grade-0 elements as scalars.). As a matter of fact, it is interesting to note that unlike in the \( \mathbb{Z}_2 \)-graded case, where such a rule follows automatically\(^8\), in the \( \mathbb{Z}_3 \)-graded case this is no more true, so that this rule have to be explicitly imposed.

We can now proceed towards the construction of coherent states:

Considering the possibilities that are allowed in order to construct coherent states for the \( \mathbb{Z}_3 \)-graded harmonic oscillator introduced in section 3, it is easy to see that the only consistent combination is given by:

\[
|\xi> = f(a^+ \xi)|0> = |0> + a^+ \xi |0> - a^+ \xi a^+ \xi |0> \tag{13}
\]

where \( f(a^+ \xi) = 1 + a^+ \xi - a^+ \xi a^+ \xi \) generalizes the function \((1 + a^+ \xi)\) in the fermionic case \(^4\).

Using (12) this state can be rewritten as:

\[
|\xi> = |0> + q^2 \xi |1> - \sqrt{2} \xi |2> \tag{14}
\]

In what follows, we shall demonstrate that these states satisfy the usual coherence criteria and therefore can represent genuine coherent states for the parafermionic harmonic oscillator.

First of all, using (6), (8), (12) and the rules cited at the end of section 4, it’s easy to see that the states (13) are indeed eigenstates of the annihilation operator

\[
a|\xi> = \xi |\xi> \tag{15}
\]

One can also compute the scalar product of two such states using the same relations and the orthonormality of the Fock space basis, the result being then as follows :

\[
<\xi_1|\xi_2> = 1 + q^2 \xi_1 \xi_2 - q \xi_1 \xi_2 \xi_1 \xi_2 = g(\xi_1 \xi_2) \tag{16}
\]

where

\[
<\xi_1| = |0> + q <1| - \sqrt{2} |2> \tag{17}
\]

A resolution of the identity is also possible in terms of the states (13) or alternatively (14). In fact, since the three eigenvectors \(|0>, |1>, |2>\) and the rules cited at the end of section 4, it’s easy to see that the states (13) are indeed eigenstates of the annihilation operator

\[
I = |0> <0| + |1> <1| + |2> <2| \tag{18}
\]

and using the integrals defined by Majid \(^5\):

\[
\int 1 d\xi = \int \xi d\xi = 0
\]

\[
\int 1 d\bar{\xi} = \int \bar{\xi} d\bar{\xi} = 0
\]

\[
\int \xi^2 d\xi = \int \bar{\xi}^2 d\bar{\xi} = 1 \tag{19}
\]

then the identity operator, in terms of the \(|\xi>\)'s is given by:

\[^8\text{In fact in this case, the anticommutation of the generators of the algebra and the associativity, imply automatically that all grade-0 elements commute with all the other elements (scalar behavior), and the grade-1 elements anti-commute with each other (ξ’s behavior)}\]
\[ \int d\xi \, d\bar{\xi} \, w(\xi\bar{\xi}) \, |\xi><\xi| = I \] (20)

where the weight function is defined as:

\[ w(\xi\bar{\xi}) = -q + \xi\bar{\xi} + \xi\xi\bar{\xi}\bar{\xi} \] (21)

This completes our proof of the fact that the states (13) are indeed coherent states. What we feel particularly not comfortable with in this construction is the fact that the different functions involved (in the process of building up these coherent states), does not seem to generalize the usual exponential function which permits the construction of fermionic coherent states. In fact none of the functions \( f, g \) or \( w \) resemble to any known deformation or generalization of the exponential function.

Nevertheless, and apart from aesthetic reasons, this will not constrain us from pushing further the analogy between these states and the conventional fermionic coherent states. Indeed, in addition to properties (15),(16) and (20) (which constitute the essential defining properties of a coherent state), we shall show in the next section that these states provide us with a Bargmann-Fock representation analog. To do this we have to express the identity operator; using states (14); in a very special manner:

\[ I = \int |\xi> d\bar{\xi} d\xi \, w(\xi\bar{\xi}) \, <\bar{\xi}| \] (22)

The choice of this form will become clear in the next section.

This resolution of the identity is not equivalent to the one given in (20). In contrast to the fermionic (or even bosonic) case where this two resolutions are equivalent, so it is sufficient to give one form to recover the other.

Note that the weight function \( w \), involved in the two forms (20) and (22), is the same.

For completeness we shall also define the behavior of \( d\xi \) and \( d\bar{\xi} \). Indeed, in order to be consistent with what precedes we can prove that \( d\xi \) should behave like the \( \xi \), and \( d\bar{\xi} \) like the \( \bar{\xi} \):

\[ d\xi \, d\bar{\xi} = q\bar{\xi} \, d\xi \, , \quad \xi \, d\bar{\xi} = q\bar{\xi} \, \xi \, , \quad d\xi \, d\bar{\xi} = q\bar{\xi} \, d\xi \, \ldots \]

6 Grassmann representatives of state vectors

One of the most interesting features of coherent states is that they permit the construction of the so called Bargmann-Fock representation space. In this section we shall prove that the states constructed (13) do also exhibit this feature and one can construct an analog of this representation space.

In analogy with the fermionic case, for any state vector \(|\psi>\) in the Fock space, we can now define its Grassmann representative by:

\[ \psi(\bar{\xi}) = <\bar{\xi}|\psi> \] (23)

and its adjoint:

\[ \bar{\psi}(\xi) = <\psi|\xi> \] (24)

Note that the Grassmann representatives defined in this manner depend on the Grassmann variable \( \xi \). This is just a matter of convention, we could have defined the coherent states in (13) using \( \bar{\xi} \) instead of \( \xi \), the Grassmann representatives will then be expressed in terms of the \( \xi \)'s.

There is a one to one correspondence between the space of Grassmann representatives and the Fock space. In fact a state \(|\psi>\)can be uniquely defined when its representative \( \psi(\bar{\xi}) \) is given.
Now, we shall use the resolution of the identity (22) to define the inner product in this space (of Grassmann representatives). We proceed as in the usual case, i.e. sandwiching eq(22) between two states $<\psi|$ and $|\phi>$:

$$<\psi|\phi> = \int \bar{\psi}(\xi) \, d\xi \, d\bar{\xi} \, \phi(\xi) \tag{25}$$

Taking this equation as a definition for the inner product in the space, permits to describe this representation space as an analog of the Bargmann-Fock representation space. The role of the weight function $exp(\bar{z}z)$ in this last one is played, in our case by $w(\bar{\xi}\xi)$ given in (21).

Next we demonstrate that the scalar product (25) yield the same result as the conventional scalar product in the Fock space.

The basis vectors $|n>$ are represented in this Bargmann-Fock space analog by:

$$<\bar{\xi}|n> = c_n \bar{\xi}^n \quad \text{a complex parameter} \tag{26}$$

as a matter of fact, the orthonormality of this basis should be translated in this representation:

$$<n|m> = c^*_n c_m \int \xi^n \, d\bar{\xi} \, d\xi \, w(\bar{\xi}\xi) \bar{\xi} = \delta_{n,m} \tag{27}$$

We show this for two representative cases:

$$<0|0> = 1:$$

we have

$$<0|\xi> = <\bar{\xi}|0> = 1$$

$$<0|0> = \int d\bar{\xi} \, d\xi \, w(\bar{\xi}\xi) = \int d\bar{\xi} \, d\xi \, (-q + \bar{\xi}\xi + \xi\bar{\xi}\xi)$$

$$= \int d\bar{\xi} \, d\xi \, \xi^2 \bar{\xi}^2 = 1 \tag{28}$$

$$<0|1> = 0$$

$$<\xi|1> = q <1|\bar{\xi}> = q\bar{\xi}$$

$$<0|1> = \bar{q} \int d\bar{\xi} \, d\xi \, w(\bar{\xi}\xi) \xi$$

$$= \bar{q} \int d\bar{\xi} \, d\xi \, (-q + \bar{\xi}\xi + \xi\bar{\xi}\xi)\xi = 0 \tag{29}$$

the calculations are done in the same way for the other examples.

### 7 Z₃-graded supersymmetric coherent states

In addition to the parafermionic harmonic oscillator related to the $Z_3$-Grassmann algebra, we need a bosonic harmonic oscillator, in order to construct a $Z_3$-graded supersymmetric one.

The bosonic harmonic oscillator algebra is generated by the triplet $\{M, b, b^+\}$, satisfying the usual commutation relations.

$$[b, b^+] = 1 \quad [b, M] = -b \quad [b^+, M] = b^+$$

acting on the usual Fock space basis $\{|m>, m = 0, 1, ...\}$ as:

$$b|m> = m|m> \quad b^+|m> = (m+1)|m+1> \quad M|m> = m|m>$$

The bosonic coherent states are given for a complex $z$ by:
\[ |z\rangle = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} |m\rangle \]  

As in the case of \(Z_2\)-graded supersymmetry, a \(Z_3\)-graded supersymmetric coherent state is obtained by coupling this states with the states (13):

\[
|z, \xi\rangle = |z\rangle \otimes |\xi\rangle = D(z, \xi) |0\rangle \otimes |0\rangle
\]

where \(D(z, \xi) = e^{zb^+} f(\xi a^+)\)

We believe that the results obtained in this paper constitute a non-trivial physical application of the \(Z_3\)-graded structures. As a matter of fact, taking the view according to which the \(Z_3\)-graduation is a natural generalization of conventional non-commutative geometries [2, 3], we can use the above results to construct the generalized quantum plane associated to it, then investigate its properties and try to apply them to physical problems. Another area where we can use these results is of course supersymmetry. A deeper investigation of the states defined in section 6, in the light of what has been achieved in \[9\], remains to be performed.

As for the Bargmann-Fock representation analog, this will prove useful to construct a path integral approach in terms of the Grassmann variables for the description of this parafermions. The results concerning these points will be soon available in the forthcoming study.

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