AN ANSWER TO A QUESTION OF ZEILBERGER AND ZEILBERGER ABOUT
FRACTIONAL COUNTING OF PARTITIONS

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ABSTRACT. We answer a question of Zeilberger and Zeilberger about certain partition statistics.

1. INTRODUCTION

For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \), define \( w_\lambda = \lambda_1 \lambda_2 \cdots \lambda_l \) (this is the product of the parts of \( \lambda \)). Zeilberger and Zeilberger \cite{ZZ18} define two quantities:

\[
b(n) = \sum_{\lambda \vdash n} \frac{1}{w_\lambda}.
\]

and

\[
b(n, k) = \sum_{\lambda \vdash n, \lambda_1 = k} \frac{1}{w_\lambda}.
\]

The latter sum is over partitions of \( n \) whose largest part is equal to \( k \), so \( b(n) = \sum_{i=1}^{\lfloor xn \rfloor} b(n, k) \). They ask to determine

\[
f(x) = \lim_{n \to \infty} b(n, \lfloor xn \rfloor)
\]

as a function on \([0,1]\). To answer this question, we use two tools. Firstly, a recurrence for \( b(n, k) \) given by Zeilberger and Zeilberger \cite{ZZ18}:

\[
b(n, k) = \frac{1}{k} \sum_{i=1}^{k} b(n-k, i).
\]

Secondly, we use the asymptotic behaviour of \( b(n) \), first considered by Lehmer \cite{Leh72}.

**Theorem 1.1** (Lehmer). We have \( b(n) = e^{-\gamma} n (1 + o(1)) \) as \( n \to \infty \), where \( \gamma \) is Euler’s gamma.

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2. UNDERSTANDING \( b(n, k) \)

In this section \( x \) will be a number in \([0,1]\).

**Definition 2.1.** Let

\[
c(n, k) = e^\gamma b(n, k)
\]

and

\[
c(n) = e^\gamma b(n).
\]

Using this new function will make the following calculations cleaner. For example, \( \lim_{n \to \infty} c(n)/n = 1 \) according to our new convention. Note that \( c(n, k) \) satisfies the same recurrence identities as \( b(n, k) \).
Example 2.2. Suppose that \( x \in (1/2, 1] \). Then for \( n \) sufficiently large, we have

\[
c(n, \lfloor xn \rfloor) = \frac{1}{\lfloor xn \rfloor} \sum_{i=1}^{\lfloor xn \rfloor} c(n - \lfloor xn \rfloor, i) = \frac{c(n - \lfloor xn \rfloor)}{\lfloor xn \rfloor},
\]

because \( \lfloor xn \rfloor \geq n - \lfloor xn \rfloor \) for \( n \) sufficiently large. By Theorem 1.1, we may take the limit as \( n \to \infty \), and obtain \( \frac{1-x}{x} \).

Proposition 2.3. For \( r \in \mathbb{Z}_{>0} \), there exists a smooth function \( F_r(t) \) such that for \( x \in (\frac{1}{r+1}, \frac{1}{r}] \),

\[
c(n, \lfloor xn \rfloor) = F_r(x) + o(1)
\]
as \( n \to \infty \). Moreover, these \( F_r(x) \) are related via

\[
F_r(x) = \frac{1-x}{x} - \frac{1-x}{x} \left( \int_{\frac{1}{r+1}}^{1} F_{r-1}(t) dt + \sum_{s=1}^{r-2} \int_{\frac{s}{r+1}}^{\frac{s+1}{r+1}} F_s(t) dt \right).
\]

Proof. Example 2.2 demonstrated this for \( x \in (1/2, 1] \), where we obtained \( F_1(x) = \frac{1-x}{x} \); this forms the base case of an induction on \( r \). We now assume \( x \in (\frac{1}{r+1}, \frac{1}{r}] \);

\[
c(n, \lfloor xn \rfloor) = \frac{1}{\lfloor xn \rfloor} \sum_{i=1}^{\lfloor xn \rfloor} c(n - \lfloor xn \rfloor, i) = \frac{1}{\lfloor xn \rfloor} \left( c(n - \lfloor xn \rfloor) - \sum_{i=\lfloor xn \rfloor+1}^{n-\lfloor xn \rfloor} c(n - \lfloor xn \rfloor, i) \right).
\]

In the latter sum, the ratio \( \frac{i}{n - \lfloor xn \rfloor} \) is minimised when \( i = \lfloor xn \rfloor + 1 \), and the resulting quantity is a weakly decreasing function of \( x \). Because \( x > \frac{1}{r+1} \), we conclude

\[
\frac{i}{n - \lfloor xn \rfloor} \geq \frac{n}{n - \lfloor \frac{n}{r+1} \rfloor} \geq 1/r.
\]

We may therefore apply the induction hypothesis to the terms in the sum.

\[
c(n, \lfloor xn \rfloor) = \frac{1}{\lfloor xn \rfloor} \left( c(n - \lfloor xn \rfloor) - \left( \sum_{i=\lfloor xn \rfloor+1}^{n-\lfloor xn \rfloor} F_{r-1} \left( \frac{i}{n - \lfloor xn \rfloor} \right) + o(1) \right) \right)
\]

Each term is a Riemann sum converging to an integral of the corresponding \( F_s \). We note that although each \( o(1) \) error term is summed \( O(n) \) times, this is accounted for by the leading factor of \( 1/\lfloor xn \rfloor \), so these still vanish in the limit \( n \to \infty \). Note that we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lfloor \frac{n}{r+1} \rfloor + 1}^{\lfloor \frac{n}{r+1} \rfloor} F_s \left( \frac{i}{n - \lfloor xn \rfloor} \right) = \int_{\frac{1}{r+1}}^{1} F_s \left( \frac{t}{1-x} \right) dt = (1-x) \int_{\frac{1}{x+1}}^{\frac{1}{x}} F_s(t) dt.
\]

We conclude that

\[
\lim_{n \to \infty} c(n, \lfloor xn \rfloor) = \frac{1-x}{x} - \frac{1-x}{x} \left( \int_{\frac{1}{r+1}}^{1} F_{r-1}(t) dt + \sum_{s=1}^{r-2} \int_{\frac{s}{r+1}}^{\frac{s+1}{r+1}} F_s(t) dt \right).
\]

For \( x \in (\frac{1}{r+1}, \frac{1}{r}] \), it is this quantity which we define to be \( F_r(x) \), and the above limit is exactly the statement of the proposition. We conclude that \( \lim_{n \to \infty} c(n, [nx]) \) is smooth for \( x \not\in \{1/n \mid n \in \mathbb{Z}_{>0}\} \).

□

Example 2.4. We may compute

\[
F_2(x) = \frac{1-x}{x} - \frac{1-x}{x} \left( \int_{\frac{1}{x+1}}^{1} \frac{1-t}{t} dt \right) = \frac{2-3x}{x} \frac{1-x}{x} \log \left( \frac{1-x}{x} \right).
\]
Remark 2.5. We may differentiate the expression for $F_r(x)$ to obtain a differential equation satisfied by $F_r(x)$:

$$\frac{d}{dx} \left( \frac{x}{1-x} F_r(x) \right) = \frac{1}{(1-x)^2} F_{r-1} \left( \frac{x}{1-x} \right)$$

Finally, we obtain our result.

Corollary 2.6. Because $c(n, k)$ and $b(n, k)$ differed only by rescaling, and the above relations are linear in the $F_r$, we have

$$\lim_{n \to \infty} b(n, \lfloor xn \rfloor) = e^{-\gamma} F_r(x)$$

whenever $x \in (\frac{1}{r+1}, \frac{1}{r}]$.

Remark 2.7. Suppose we assemble all the functions $F_r(x)$ into a single function $F(x)$ on $(0, 1]$ (and say $F(x) = 0$ for $x > 1$). Let $G(x) = F(1/x)$. Then, the differential equation becomes

$$G(x) - (x - 1)G'(x) = G(x - 1).$$

The upshot of this is that the current equation is well adapted for a Laplace transform. Writing $\hat{G}(t)$ for the Laplace transform of $G(x)$, we obtain:

$$\hat{G}(t) + (t\hat{G}(t) - G(0)) + \frac{d}{dt}(t\hat{G}(t) - G(0)) = e^{-t}\hat{G}(t),$$

using the boundary condition $G(0) = 0$, this becomes

$$\frac{d}{dt}\hat{G}(t) = \frac{e^{-t} - t - 2}{t}\hat{G}(t).$$

We may solve this explicitly:

$$\hat{G}(t) = Kt^{-2} \exp(Ei(-t) - t),$$

where $Ei$ is the exponential integral, and $K$ is a constant.

References

[Leh72] D Lehmer. On reciprocally weighted partitions. Acta Arithmetica, 21:379–388, 1972.

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