The role of local and global geometry in quantum entanglement percolation

Gerald John Lapeyre Jr.
ICFO–Institut de Ciències Fotòniques, Mediterranean Technology Park, 08860 Castelldefels, Spain
(Dated: November 28, 2013)

PACS numbers: 03.67.Bg, 03.67.Hk, 64.60.ah, 03.67.Pp

I. INTRODUCTION

Distribution of quantum entanglement on networks has been studied vigorously over the past few years. This has been driven by the fact that entanglement is the fundamental resource in quantum information, but it is created locally via interaction, while it is often consumed in systems with widely separated components. In an ideal description, each node of the network represents a collection of qubits, and each edge or link represents entangled states of qubits in different nodes.

But, even in the case of transporting entanglement along a chain of partially entangled pure states, using perfect quantum operations, the resulting entanglement decays exponentially in the number of links. Unfortunately, technical and fundamental limits on effectively moving entanglement over even a single link further complicate the ideal picture and have led to elaborate protocols involving the distribution, storage, and purification of entangled states. The most direct approach is the quantum repeater which has been proposed to overcome these limitations on a one-dimensional chain of nodes [1–6]. There are examples of practical, deployed quantum networks, such as quantum key distribution networks. But the technical challenges in implementing quantum repeaters remain too great to be useful in contemporary quantum key distribution networks [7]. Typically, entanglement is established over only a single link, while at each node information is processed classically and re-encoded in a quantum state.

A different approach is to use the entire network, rather than a linear chain, to distribute entanglement. The availability of multiple paths is used to overcome the inevitable decay of entanglement. This leads to models that are immediately more interesting because it is not obvious how to prove which of two protocols is better, let alone which protocol is optimal. In fact percolation theory [8, 9] has provided powerful tools for evaluating protocols. The best protocols use quantum operations to transform the initial lattice into a different lattice [10–12].

As in the one-dimensional case, more realistic studies of multi-dimensional networks have been done, for instance by considering mixed states and imperfect quantum operations [13–17]. But sharp questions, say in the thermodynamic limit, are difficult to pose in these dirtier situations because of the decay of entanglement. Furthermore, questions about asymptotic behavior remain that are not only of intrinsic interest, but address fundamental limits on entanglement distribution. These are the questions that we address here.

This paper has two main goals. The first goal is to show that enhanced entanglement percolation (defined below) via lattice transformation is possible even if the coordination number of the transformed lattice decreases or the classical percolation threshold increases. The second goal is to introduce a new tool that we call partial entanglement swapping. In partial swapping, we simply stop the swapping procedure after the first step, the projection, and evaluate whether the output state and the new geometry may be more profitably used in a different operation. In fact, the usefulness of the tool is demonstrated by using it to accomplish the first goal. Although the idea behind partial swapping is simple, it introduces a complication. In previous entanglement percolation protocols, the Bell measurement in the computational basis is optimal. But, the optimal basis for partial swapping is not obvious and depends on the amount of initial entanglement.

II. ENTANGLEMENT PERCOLATION

Entanglement percolation is described in detail in several sources [10, 11, 18–20]. Here we give only a brief description. We consider the following class of entanglement percolation models. Each node consists of a collection of qubits. Each edge, or link, consists of a partially
Entangled pure state between two qubits, each on a different node. These states $|\alpha\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ are written in a Schmidt basis as

$$|\alpha\rangle = \sqrt{\alpha_0} |00\rangle + \sqrt{\alpha_1} |11\rangle,$$

where the Schmidt coefficients $\alpha_0, \alpha_1$ satisfy $\alpha_0 \geq \alpha_1$ and $\alpha_0 + \alpha_1 = 1$. If $\alpha_0 = \alpha_1 = 1/2$, the state is maximally entangled, and is called a Bell pair or singlet. If either $\alpha_0$ or $\alpha_1$ vanishes, then the state is separable and is useless for quantum information tasks. The smallest Schmidt coefficient may be used as a measure of entanglement with the amount of entanglement increasing with $\alpha_1$. The lattice is initialized with identical states $|\alpha\rangle$ on each link. So we have one free parameter $\alpha_1$. The goal is to design a protocol to maximally entangle two arbitrary nodes $A$ and $B$. The utility of the protocol is measured by the probability of success $\text{Pr}(A \leftrightarrow B)$ as the distance between $A$ and $B$ tends to infinity. We require that the protocols use only local operations and classical communication (LOCC) [21]. This means that quantum operations that include interaction between qubits on different nodes are not allowed. But classical communication between all nodes is allowed.

### A. Classical Entanglement Percolation

The simplest entanglement distribution protocol is called classical entanglement percolation (CEP). For some lattices, better protocols have been found, the so-called quantum entanglement percolation (QEP) protocols. The reason for this distinction and the relation between CEP and QEP will be made clear below. For now, we note that it is QEP that uses lattice transformation. We introduce the CEP and QEP using the kagome lattice shown in Fig. 1, because it allows a concise exposition. First we describe CEP. For the moment, consider choosing fixed $A$ and $B$. In step 1 we perform an LOCC operation on each link, optimally converting it with probability $p = 2\alpha_1$ to a Bell pair, and probability $1 - p$ to a separable state. This operation is called a singlet conversion and $p$ is the singlet conversion probability (SCP). After step 1, we have a lattice in which each link is either open (a Bell pair), or closed (separable). In step 2, we search for an unbroken path of open links between $A$ and $B$. If no such path exists, then $\text{Pr}(A \leftrightarrow B) = 0$. If a path does exist, then at each intermediate node we perform an entanglement swapping operation. Because the input links are singlets, each swap succeeds with probability 1 deterministic. This description corresponds exactly to classical bond percolation, with density of open bonds $p = 2\alpha_1$. The critical bond density for the kagome lattice is $p^\text{kag}_{\text{c}} \approx 0.52$. Thus $\text{Pr}(A \leftrightarrow B) = 0$ if $p < p^\text{kag}_{\text{c}}$ and $\text{Pr}(A \leftrightarrow B) > 0$ if $p > p^\text{kag}_{\text{c}}$.

### B. Quantum Entanglement Percolation

A QEP scheme for the kagome lattice is shown in Fig. 1. We first perform swapping on all pairs of qubits enclosed in loops. Each of the input states is $|\alpha\rangle$, so the probability of obtaining a singlet in the resulting vertical link is $p = 2\alpha_1$. We then perform a singlet conversion on the remaining horizontal bonds, resulting in a square lattice where each link is a Bell pair with probability $p$ and is separable otherwise. Finally we perform step 2 of CEP (swapping with singlets) on this square lattice. This is successful precisely when $p > p^\square_{\text{c}}$, where $p^\square_{\text{c}}$ is the critical density for bond percolation on the square lattice. Since $p^\square_{\text{c}} = 1/2$, it follows that long-distance entanglement on the kagome lattice is possible with this QEP scheme, but not with CEP, if $\alpha_1$ satisfies $p^\square_{\text{c}} < 2\alpha_1 < p^\text{kag}_{\text{c}}$.

CEP always gives an easily computable upper bound on the minimum initial entanglement required for long-distance entanglement. Thus, CEP serves as a benchmark to compare with any QEP protocol. Because we are not interested in QEPs that perform worse than CEP, we will call any advantageous QEP simply a QEP. However, the measure by which the QEP is advantageous may vary. We call the smallest value of $\alpha_1$ such that long-range entanglement is possible the lower threshold or percolation threshold $\alpha_\text{c}$. We call the smallest value of $\alpha_1$ such that

| Lattice     | $p_c$ for bond percolation |
|-------------|----------------------------|
| triangular  | $2\sin(\pi/18) \approx 0.347$ |
| square      | $1/2$                      |
| kagome      | $\approx 0.5244053$ MC estimate |
| hexagonal   | $1 - 2\sin(\pi/18) \approx 0.653$ |

TABLE I. $p_c$ for bond percolation on some lattices. All critical densities are exact [9] except for $p_c({\text{kagome}})$ [22].

![FIG. 1. Transformation of kagome to square lattice. Circles represent qubits. Lines represent partially entangled bipartite states. Left) Full entanglement swapping is performed for each pair of links marked with a (blue) loop. Right) The result is the square lattice, where the vertical (dashed) links are the outcome of the swap and the horizontal links remain in the state $|\alpha\rangle$. The remainder of the QEP protocol is described in the text.](image-url)
long-range entanglement is achieved with probability 1 the upper threshold $\alpha_c$. Note that $\alpha_c$ marks a phase transition, but $\alpha_c^*$ does not. For every lattice, CEP gives $\alpha_c^* = 1/2$. We call a QEP robust if it satisfies at least one of two conditions. 1) that it lowers the percolation threshold $\alpha_c$, and 2) that the upper threshold satisfies $\alpha_c^* < 1/2$. We are interested in isolating the effect of the geometry of the transformed lattice on the performance of the QEP. We therefore emphasize that we will compare the geometry of the classical transformed lattice to that of the the initial lattice with no reference to quantum states.

C. Lattice structure and entanglement distribution

For any lattice, CEP is defined and the relevant quantities can be taken directly from percolation theory. But there is no generic prescription for constructing a QEP. In previous work, QEP protocols have been identified by choosing a lattice and searching for good lattice transformations.

In the example above, the initial lattice was transformed into one new lattice. However, in general, transformations may take the initial lattice $L$ to multiple, decoupled lattices $\{L'_i\}$ [11]. It is reasonable to search for $\{L'_i\}$ that are more highly connected than $L$. In fact, in all of the examples of QEP given in refs [10–12, 18] one $L'_i$ has average coordination number greater than or equal to that of $L$ (condition i). Furthermore, one $L'_i$ has a classical percolation threshold that is less than or equal to that of $L$ (condition ii). In Refs. [10, 11, 18] this is easy to see because the lattices involved are well-known[23]. The protocols in Ref [12] generate multi-partite entanglement from the initial bi-partite states. The multi-partite swapping was explicitly designed to increase connectivity. These protocols give the best performance to date, and often result in less common or unclassified lattices including non-planar graphs and lattices whose sites have different coordination numbers.

Given that all known protocols satisfy conditions i and ii, a natural question is whether this must always be the case. Must these properties, one local and one global, that are associated with high connectivity, be non-decreasing in an advantageous QEP? In the following section we present a counter-example demonstrating that the answer to this question is "no". In fact both conditions are violated, and the improvement is robust. To achieve this, we introduce a new ingredient into the lattice transformation protocols.

III. QEP for the Triangular Lattice

CEP on the triangular lattice corresponds to classical bond percolation. With CEP, long-range entanglement is only possible for $\alpha_1 > \alpha_c(\text{CEP}) = p_c^\Delta/2 \approx 0.1736$, and deterministic long-range entanglement is only possible for maximally entangled initial states, i.e $\alpha_1 = \alpha_c^*(\text{CEP}) = 1/2$. Here we present a QEP that transforms the triangular lattice into the hexagonal lattice on which a singlet can be created between any two nodes with probability 1 if $\alpha_1 > 0.3246$. That is, the upper threshold $\alpha_c^*$ is lowered. This is possible even though, classically, the hexagonal lattice has larger critical density $p_c$ and smaller coordination number than the triangular lattice.

A. Partial entanglement swapping

In order to show the counter-example promised in the introduction, it is enough to consider one of the outcomes from the same swapping measurement used in previous studies on entanglement percolation. However, we consider here more general measurements that allow us to optimize for certain figures of merit. In this paper we consider entanglement swapping using Bell measurements on two qubits, one from each pair, as shown in Fig. 2. For brevity, we omit referring to any necessary local unitaries. We call the usual entanglement swapping in any of these bases full entanglement swapping. We shall always assume that the two input pairs are in the same state $|\alpha\rangle$. Following Ref. [18], we define an orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}_j$ for each qubit $j = 1, 2$

$$
\begin{pmatrix}
|\uparrow\rangle \\
|\downarrow\rangle
\end{pmatrix}_j = U_j \begin{pmatrix}
|0\rangle \\
|1\rangle
\end{pmatrix}_j, \quad U_j \in U(2),
$$

and the Bell vectors

$$
|\Phi^\pm\rangle = \frac{|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle}{\sqrt{2}} \quad \text{and} \quad |\Psi^\pm\rangle = \frac{|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle}{\sqrt{2}}.
$$

![Fig. 2. Entanglement swapping. (a) a measurement is performed on qubits 1 and 2. (b) after the measurement, qubits 3 and 4 are in one of four states $\{|\phi_m\rangle\}$, which are partially or maximally entangled. In full entanglement swapping, a singlet conversion is performed on the pair (3, 4) which results in either a maximally entangled state, or a separable state. In partial entanglement swapping, only the measurement is performed. (c) A distillation is then performed on the output state together with another entangled pair, here in the state $|\alpha\rangle$, to produce (d) either a more highly entangled pair, or a separable state.](image)
The four measurement outcomes are
\[ \{ |\phi_m\rangle\} \text{ with probabilities } \{ p_m \}. \]
Furthermore, \( p_{\text{min}} = \min \{ p_m \} \) and \( p_{\text{max}} = \max \{ p_m \} \)
are given by
\[ p_{\text{min}} = \alpha_0 \alpha_1 \quad \text{and} \quad p_{\text{max}} = \frac{1}{2} - \alpha_0 \alpha_1. \]

There is a bijective mapping between the probabilities \( \{ p_m \} \) and \( \{ U_j \} \). In particular, every (orderless) choice of \( p_m \) satisfying \( p_{\text{min}} \leq p_m \leq p_{\text{max}} \) and \( \sum_m p_m = 1 \) corresponds to a Bell measurement. The smallest Schmidt coefficients of the output states are given by
\[ \lambda_m = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{\alpha_0^2 \alpha_1^2}{p_m^2}} \right). \]

In full entanglement swapping, we first perform the Bell measurement, and then perform a singlet conversion on the output state. Since a singlet conversion succeeds with probability equal to twice the smallest Schmidt coefficient, the average SCP for full entanglement swapping is given by \( S_M = 2 \sum_m p_m \lambda_m \).

However, in partial entanglement swapping, we perform the Bell measurement only, and not the singlet conversion. Instead of immediately doing a singlet conversion we take advantage of the new geometry of the output state. We attempt to distill a singlet from the output state and another entangled pair. Although this is a simple idea, it is quite useful, and it has not been used in previous work on entanglement distribution. From majorization theory [24, 25], we find that we can distill a Bell pair from two partially entangled pairs with optimal probability
\[ p_{\text{distill}} = \min \{ 1, 2 [1 - (1 - \beta_1)(1 - \gamma_1)] \}, \]
where \( \beta_1 \) and \( \gamma_1 \) are the smallest Schmidt coefficients of the input states [11]. In the example below, the second state used in the distillation will be \( |\alpha\rangle \). Thus, the input states to the distillation have \( \beta_1 = \alpha_1 \) and \( \gamma_1 = \lambda_m \). The average SCP from combining the partial swapping with distillation is then
\[ S_M = \sum_m p_m \min \left\{ 1, 2 - \alpha_0 \left( 1 + \sqrt{1 - \frac{\alpha_0^2 \alpha_1^2}{p_m^2}} \right) \right\}. \]

1. **Swapping in ZZ basis**

Suppose the measurement is in the ZZ basis, \( U_1 = U_2 = \mathbb{I}_2 \). This is the measurement that maximizes the average SCP in full swapping. Thus, it is the one used in all previous entanglement percolation schemes (with a modified version for multi-partite entanglement percolation). In this case, \( p_1 = p_2 = p_{\text{min}} \) and \( p_3 = p_4 = p_{\text{max}} \), with corresponding smallest Schmidt coefficients
\[ \lambda(p_{\text{max}}) = \frac{\alpha_1^2}{\alpha_0^2 + \alpha_1^2}, \quad \lambda(p_{\text{min}}) = \frac{1}{2}. \]

Two of the outcomes are already singlets. Each of the other two may be distilled together with \( |\alpha\rangle \) into a singlet with probability
\[ p = \min \left\{ 1, 2 \left( 1 - \frac{\alpha_0^3}{\alpha_0^2 + \alpha_1^2} \right) \right\}, \]
given by (1). The average SCP using partial swapping in the ZZ basis is then
\[ S_{ZZ} = \alpha_0 \alpha_1 + (1 - 2 \alpha_0 \alpha_1) \min \left\{ 1, 2 \left( 1 - \frac{\alpha_0^3}{\alpha_0^2 + \alpha_1^2} \right) \right\}. \]

2. **Swapping in XZ basis**

Suppose the measurement is in the XZ basis. Then \( p_m = 1/4 \) and \( \lambda_m = \frac{1}{2} (1 - \sqrt{1 - 16 \alpha_0 \alpha_1}) \) for all \( m \). The average SCP using partial swapping in the XZ basis is then
\[ S_{XZ} = \min \left\{ 1, 2 - \alpha_0 \left( 1 + \sqrt{1 - 16 \alpha_0^2 \alpha_1^2} \right) \right\}. \]

**B. The protocol**

The QEP protocol proceeds as follows. Consider the triangular lattice with each bond consisting of a single, partially entangled pure state. In step 1, we perform partial entanglement swapping on selected bonds as shown in Fig. 3. At the end of step 1 we have a hexagonal lattice with double links. In each pair, one link is the initial state \( |\alpha\rangle \) and one link is one of \( \{|\phi_m\rangle\} \). Step 2 consists of the following. For each double link, if the outcome \( |\phi_m\rangle \) is already a Bell pair, then we do nothing. Otherwise, we attempt to distill a singlet from the two links \( |\phi_m\rangle \) and \( |\alpha\rangle \).

1. **Protocol in ZZ basis**

Suppose we do the partial swap in the ZZ basis. Two outcomes are singlets and two are partially entangled. From (3) we see that we create a singlet on every bond of the hexagonal lattice deterministically if \( \alpha_0 \) is less than the real root \( \alpha_0^* \approx 0.6478 \) of \( \alpha_0^3 - \alpha_0^2 + \alpha_0 - 1/2 = 0 \). Equivalently, the condition is \( \alpha_0^* \approx 0.3522 \). The critical threshold for this protocol is found by using (4) and solving \( S_{ZZ}(\alpha_1) = p_C^Z \), where \( p_C^Z \) is the classical threshold on the hexagonal lattice, with the result \( \alpha_c \approx 0.1988 \).

2. **Protocol in XZ basis**

Suppose we do the partial swap in the XZ basis. The smallest value of \( \alpha_1 \) for which (5) equals 1 is \( \alpha_1^* \approx 0.3246 \). The solution of \( S_{XZ}(\alpha_1) = p_C^X \) is \( \alpha_c \approx 0.2200 \).
FIG. 3. Transformation of triangular to hexagonal lattice.
a) triangular lattice. A partial swap is applied to the dotted (red) lines. In following frames, outcomes of partial swaps are shown as dashed (green) lines. Partial swaps are applied to pairs of links shown as dotted lines. f) portion of hexagonal lattice with double links. One link of each double link is in the state $|\alpha\rangle$, the other link is one of the four outcomes of the partial swap $\{|\phi_m\rangle\}$.

| Protocol      | $\hat{\alpha}_c$ | $\hat{\alpha}_c^*$ |
|---------------|------------------|--------------------|
| CEP           | 0.1736           | 1/2                |
| QEP ZZ        | 0.1988           | 0.3522             |
| QEP XZ        | 0.2200           | 0.3246             |
| QEP optimal   | 0.1961           | 0.3246             |

TABLE II. Percolation thresholds $\hat{\alpha}_c$ and upper thresholds $\hat{\alpha}_c^*$ for entanglement protocols on the triangular lattice.

3. Protocol in other Bell bases

We optimized over all Bell measurements with only two distinct values of $p_m$. Visual inspection showed that the optimum average SCP occurs when the second argument to min in (5) is equal to 1, which occurs for $p_1 = \frac{\alpha_1^2}{\sqrt{1-2\alpha_1}}$. Inserting this into (2), and solving $S_M(\alpha_1) = p_c^\text{opt}$, we find the lower threshold $\hat{\alpha}_c \approx 0.1961$, which is a small improvement over the ZZ basis. Optimizing for the upper threshold $\hat{\alpha}_c^*$, we find $p_m = 1/4$, which is the XZ basis. A numerical search for more general Bell measurements strongly suggests that the optimum Bell protocol has only two distinct values of $p_m$ for all $\alpha_1$.

In summary, we found that the optimal Bell basis has exactly two distinct values of $p_m$, which depend on $\alpha_1$. At the lower threshold, the optimal basis gives only a slight improvement over the ZZ basis. As the upper threshold is approached, the optimal basis approaches the XZ basis. These results are summarized in Fig. 4 and Table II. We did not investigate non-Bell measurements.

IV. DISCUSSION

We have introduced a new tool, partial entanglement swapping, for entanglement percolation via lattice transformation. This adds flexibility in optimally combining the quantum and the geometric aspects of QEPs. We have demonstrated the utility of partial swapping by using it to design a QEP that transforms the triangular lattice to the hexagonal lattice. Partial entanglement swapping allows sufficient concentration of entanglement to overcome lowered connectivity in the transformed lattice. In particular, there is a least initial amount of entanglement above which long-distance entanglement is deterministic. Thus, we have proven that non-decreasing connectivity, as measured by coordination number and percolation threshold, is not required for QEP. However, in the present example, we find that CEP still provides the optimal percolation threshold. It is interesting to note that the only other known QEP for the triangular lattice uses multi-partite entanglement to enhance the connectivity of the lattice by creating a non-planar graph [12]. Thus, the question of whether a transformed lattice with lower connectivity can give a lower critical threshold remains open. Also unknown is whether the critical threshold of the triangular lattice can be lowered via a transformation to a planar graph, or whether the triangular lattice is, in a sense, a maximally connected...
planar graph.

In addition to answering a conjecture on the geometrical constraints on QEP, partial swapping enlarges the toolbox for QEP. It may be combined with other techniques to push the initial entanglement thresholds lower. However, even in this simple example, the search becomes more complicated because we find that the optimal measurement basis for partial swapping depends on the amount of initial entanglement. Still more interesting than each new protocol would be a proof, constructive or otherwise, of the existence of a minimum threshold for a particular lattice or class of lattices.

V. ACKNOWLEDGMENTS

The author thanks Jan Wehr for discussions and for asking a question that led to the present work. This work was supported in part by the Spanish MICINN (TOQATA, FIS2008-00784), by the ERC (QUAGATUA, OSYRIS), and EU projects SIQS, EQUAM, and the Templeton Foundation.

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