Stochastic quantization of massive fermions

A.N. Efremov
CPHT, Ecole Polytechnique, CNRS, Université Paris-Saclay,
Route de Saclay, 91128 Palaiseau, France.
alexander@efremov.fr

May 22, 2018

Abstract
We consider a general solution of the Langevin equation describing massive fermions to an appropriate boundary problem. Assuming existence we show that all correlators coincide with the Schwinger functions of corresponding Euclidean Quantum Field Theory.

1 Introduction
In the current work we make an effort to bridge a gap in literature and to establish a mathematically sound connection between the Langevin equation and massive fermionic models in the quantum field theory. A motivation for this analysis is our desire to apply the methods developed recently by A.Kupiainen [1] and M.Hairer [2] to non-abelian gauge theories, Yukawa and Gross–Neveu models [3, 4]. Beside of the subcriticality problem which can be tackled using Wilson’s renormalization group approach [5] proposed by A.Kupianen one inevitably runs into stochastic PDE’s in infinite-dimensional Grassmann algebra [6]. A common way of stochastic quantization as introduced by Parisi and Wu [7] is of course the Fokker–Planck equation. Unfortunately a fermionic field, despite its random or stochastic character, is not a stochastic process. Rather than formally derive the Fokker–Planck equation for fermions, see [8], we follow the functional integral approach proposed by J. Zinn-Justin for the scalar $\phi^4$ model [9]. Although the Fokker–Planck approach suggested in [8] for fermions is simple, intuitive and better suited...
for an initial reading we should keep in mind that a fermionic field is not a stochastic process as we understand it in the probability theory. Thus the whole description remains at a formal level only. On the other hand the functional integral approach makes our construction for fermions mathematically meaningful.

The first important piece is the Euclidian action. Without any loss of generality we consider Dirac spinors in two dimensions. The 2-points Schwinger function for Dirac spinors can be obtained from the corresponding Wightman function on an appropriately chosen subspace by the Wick rotation [10], i.e. by the mapping \( x_1 \mapsto -ix_1 \),

\[
W(x) = \int \frac{d^2p}{(2\pi)^2} \frac{-i\hat{p} + m}{p^2 + m^2 - i\epsilon} e^{ipx},
\]

\[
S(x) = \int \frac{d^2p}{(2\pi)^2} \frac{-i\hat{\psi} + m}{p^2 + m^2} e^{ipx}.
\]

Here \( \hat{\psi} = -p_1\gamma_1 + p_2\gamma_2 \), \( \gamma_1 = -i\gamma_1 \), \( \hat{\psi} = p_k\gamma_k \), \( \{\gamma_i, \gamma_j\} = 2\delta_{ij}, \gamma_i^+ = \gamma_i \). The matrices \( \gamma_i, \gamma_i \) correspond to Minkowski and Euclidean space-time respectively. We make a particular choice of implementation for the matrices \( \Gamma_i \) such that they coincide with the Pauli matrices:

\[
\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Hence \( \gamma_1^T = \gamma_1 \), \( \gamma_3 = -i\gamma_1\gamma_2 \). It is easy to see that the function in (2) is a fundamental solution to a non-homogeneous Dirac equation in Euclidean space-time, i.e.

\[
(\hat{\psi} + m)S(x) = \delta(x).
\]

As an example we coincide the Yukawa model [3]. The model involves a scalar field \( \phi \) and a fermion field \( \psi \),

\[
L = \int d^2x \bar{\psi}(\hat{\phi} + m)\psi + g:\bar{\psi}\psi:\phi - \frac{1}{2}\phi\Delta\phi + \frac{1}{2}M^2\phi^2,
\]

where \( \psi: \mathbb{R}^2 \to \Lambda^2 \), \( \bar{\psi}: \mathbb{R}^2 \to \Lambda^2 \) and \( \phi: \mathbb{R}^2 \to \mathbb{R} \) are respectively independent spinor and scalar fields. Here by \( \Lambda \) we denote an infinite-dimensional Grassmann algebra. Since the expectation \( \langle \bar{\psi}\psi \rangle \) turns out to be divergent we should introduce a regularization to define the Wick product. Thus the Wick product \( \bar{\psi}\psi: \) includes the ordinary product \( \bar{\psi}\psi \) and a counterterm \( c \) which
becomes singular once we remove the regularization, i.e. : $\bar{\psi}\psi = \bar{\psi}\psi - c$. The matrices $\gamma_i$ transform as vectors under the rotation $u\gamma_iu^{-1}$ where $u = e^{i\gamma_3 \frac{\pi}{2}}$. Since $\gamma_3$ is hermitian $u$ is unitary. The quantities $\bar{\psi}\psi$ and $\bar{\psi}\phi\psi$ are scalars, i.e. invariant with respect to the action $\bar{\psi} \rightarrow \bar{\psi}u^+$ and $\psi \rightarrow u\psi$, in appendix B we state other properties of the action.

The second important piece is the Langevin equation:

$$\partial_t \phi = -\frac{\delta L}{\delta \phi} + \xi = \Delta \phi - M^2 \phi - g\bar{\psi}\psi + \xi,$$  \hspace{1cm} (6)

$$\partial_t \psi = -\frac{\delta L}{\delta \bar{\psi}} + \eta = -(\dot{\phi} + m)\psi - g\bar{\psi}\phi + \eta,$$  \hspace{1cm} (7)

$$\partial_t \bar{\psi} = \frac{\delta L}{\delta \psi} + \bar{\eta} = (\dot{\phi}^T - m)\bar{\psi} - g\bar{\psi}\phi + \bar{\eta},$$  \hspace{1cm} (8)

where $\xi$, $\eta$, $\bar{\eta}$ denote the corresponding Gaussian white noise. Having a solution of these equations one can calculate different correlators. Below we show that such correlators coincide with the Schwinger functions which we obtain using the corresponding generating functional.

## 2 Stochastic quantization of massive fermions

First let us write the fundamental solution of the linear equations corresponding to the retarded Green functions

$$(\partial_t + \dot{\phi} + m)G = \delta(t, x),$$  \hspace{1cm} (9)

$$(\partial_t - \dot{\phi}^T + m)\bar{G} = \delta(t, x),$$  \hspace{1cm} (10)

$$(\partial_t - \Delta + M^2)P = \delta(t, x).$$  \hspace{1cm} (11)

These functions vanish $\forall t < 0$, i.e. $P(t, x) = 0$, $G(t, x) = 0$ and $\bar{G}(t, x) = 0$.

$$G(t, x) = \frac{1}{i} \int \frac{dwd^2p}{(2\pi)^3} \frac{w - im - \dot{\phi}}{(w - im)^2 - p^2} e^{iw(t,x)},$$  \hspace{1cm} (12)

$$\bar{G}(t, x) = \frac{1}{i} \int \frac{dwd^2p}{(2\pi)^3} \frac{w - im + \phi^T}{(w - im)^2 - p^2} e^{iw(t,x)},$$  \hspace{1cm} (13)

$$P(t, x) = \frac{1}{i} \int \frac{dwd^2p}{(2\pi)^3} \frac{1}{w - i(p^2 + M^2)} e^{iw(t,x)}.$$  \hspace{1cm} (14)
Here $\tilde{G} = \gamma_2 G\gamma_2$. We can write equations (6)-(8) in the following form

$$\partial_t \Psi + \tilde{D} \Psi + V(\Psi) - \Xi = 0,$$

(15)

where

$$\Xi = \begin{pmatrix} \eta \\ \bar{\eta} \\ \xi \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi \\ \bar{\psi} \\ \phi \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} \partial + m \\ -\partial^T + m \\ -\Delta + M^2 \end{pmatrix}, \quad V(\Psi) = g \begin{pmatrix} \psi \phi \\ \bar{\psi} \phi \\ \bar{\psi} \psi \end{pmatrix}. \quad (16)$$

Let $t_0 \in (a, b) \subset \mathbb{R}$. A distribution $\Psi(t, x)$ in $(a, b) \times \mathbb{R}^2$ is a solution to an initial value problem $\Psi(t_0, x) = \Psi_0(x)$ for equation (15) if and only if

$$\Psi = G_{[\infty, t_0]}(-V + \Xi) + e^{-\tilde{D}(t-t_0)} \Psi_0, \quad G = \begin{pmatrix} G & 0 & 0 \\ 0 & \tilde{G} & 0 \\ 0 & 0 & P \end{pmatrix} \quad (17)$$

$$(G_{[\infty, t_0]}f)(t, x) = \int_{t_0}^{\infty} ds \int d^2 z \, G(t-s, x-z) f(s, z). \quad (18)$$

This result follows immediately from the definition of the retarded Green function $G$, i.e. $(\partial_t + \tilde{D}) G = \delta(t, x)$. Let assume that a solution for the initial value problem exists for an infinite time interval, $t_0 = -\infty$. Since $m > 0$ it follows from (17) that $\Psi$ should solve a fix point problem

$$\Psi = G(-V + \Xi), \quad (19)$$

where $Gf = G_{[+\infty, -\infty]}f$. Under the above assumption and an appropriate regularity of the potential $V$ a particular choice of the model, the Yukawa model in two dimensions in our case, is not important in the coming lemmas. Since in a general situation the potential $V$ contains all required counterterms which are fine-tuned for a particular model to cancel singularities we still use in appendix A the whole action $L$.

Following the great success in the construction of the massive scalar model using methods of functional integral [12] K.Osterwalder and R.Schrader constructed Euclidean theory of free fermions [11]. To put it simply, given an action $L$ which involves fermions one defines a generating functional $Z_{\mathcal{F}t}$ for connected Schwinger functions using an integral over infinite dimensional
Grassmann algebra \([6]\) along with a usual functional integral for bosons.

\[
Z_{fi}(\tilde{K}) = \int D\tilde{\Psi} e^{-L+\tilde{\Psi}\tilde{Q}\tilde{K}}, \quad Q = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \tilde{K} = \begin{pmatrix}
\tilde{k} \\
\tilde{j}
\end{pmatrix}.
\]

(20)

To shorten the notation we use \(\tilde{\Psi}Q\tilde{K}\) instead of \(\int d^2x \tilde{\Psi}(x)Q\tilde{K}(x)\). The composite field \(\tilde{\Psi}\) does not depend on the stochastic time. Given a fixed time \(T\) we could define \(\tilde{\Psi}(x) = \Psi(T,x)\) but it is not necessary here. We use tilde to show that a variable or an operator is explicitly independent of the stochastic time. For convenience we define

\[
\tilde{E}(\Psi) = \tilde{D}\Psi + V(\Psi).
\]

(21)

Since the elements of Grassmann algebra are anti-commuting we should distinguish the left \(\delta_l\) and right derivatives \(\delta_r\).

**Lemma 1** The partition function \(Z_{fi}\) for the corresponding QFT satisfies the following equation

\[
\left(\tilde{E}(Q^T \frac{\delta}{\delta_l K}) - \tilde{K}\right) Z_{fi}(\tilde{K}) = 0.
\]

(22)

**Proof**

\[
\tilde{K} Z_{fi}(\tilde{K}) = \int D\Psi e^{-L} \left(Q^T \frac{\delta e^{\Psi Q\tilde{K}}}{\delta_l \Psi}\right) = \int D\Psi \left(-Q^T \frac{\delta e^{-L}}{\delta_l \Psi}\right) e^{\Psi \tilde{K}}
\]

(23)

\[
= Q^T \frac{\delta L}{\delta_l \Psi} \bigg|_{\Psi = Q^T \delta_l \tilde{K}} Z_{fi}(\tilde{K}) = \tilde{E}(Q^T \delta_l \tilde{K}) Z_{fi}(\tilde{K}).
\]

(24)

Let \(\Psi\) be a solution of (19) and define a generating functional \(Z\) for correlators

\[
Z(K) = \int D\Xi e^{-\frac{1}{2}\Xi\Xi + K\Psi}, \quad K = \begin{pmatrix}
\tilde{k} \\
\tilde{j}
\end{pmatrix}.
\]

(25)

We want to show that at some finite time \(t = T\) the generating functional \(Z(K)\) is a solution of equation (22). Since \(Z(0) = Z_{fi}(0) = 1\) it will imply that \(Z(K) = Z_{fi}(\tilde{K})\), i.e. the correlators obtained from (25) at the same time \(T\), i.e. \(\langle \Psi(T,x_1)...\Psi(T,x_n)\rangle\), coincide with the Schwinger functions of (20).
Lemma 2 Denote \( \tilde{Z}(\tilde{K}) = Z(K) \) where \( K(t, x) = \delta(t - T) \tilde{K}(x) \). The partition function \( \tilde{Z} \) for stochastic process \( \Psi \) corresponding to the Langevin equation \( \partial_t \Psi + \tilde{E}(\Psi) - \Xi = 0 \) satisfies the equation

\[
\left( \tilde{E}(Q^T \frac{\delta}{\delta_t K}) - \tilde{K} \right) \tilde{Z}(\tilde{K}) = 0.
\]  

(26)

Proof Using (25) we calculate the expectation of the noise

\[
\langle \Xi \rangle_K = Q^T \frac{\delta \Psi^T}{\delta \Xi} Q K Z(K), \quad \frac{\delta \Psi^T}{\delta \Xi} = G^T \left( 1 + \frac{\delta \nu^T}{\delta \psi} G^T \right)^{-1}.
\]  

(27)

Furthermore

\[
\partial_t \Psi = \left( 1 + G \frac{\delta \nu}{\delta_t \psi} \right)^{-1} \partial_t G \Xi.
\]

(28)

Since \( \Psi \) satisfies the Langevin equation we have

\[
\left[ \left( 1 + G \frac{\delta \nu}{\delta_t \psi} \right)^{-1} \partial_t G - 1 \right] \Xi + \tilde{E}(\Psi) = 0.
\]

(29)

Taking the expectation of (29) and using (27) we get

\[
\left( \tilde{E} + \left[ \left( 1 + G \frac{\delta \nu}{\delta \psi} \right)^{-1} \partial_t G - 1 \right] Q^T G^T \left( 1 + \frac{\delta \nu^T}{\delta \psi} G^T \right)^{-1} Q K \right) Z(K) = 0.
\]

(30)

We restrict our interest to the correlators at \( t = T \), i.e. \( K = \delta(t - T) \tilde{K}(x) \). Since \( G(t, x) = 0 \) for \( t < 0 \) we have

\[
\lim_{t \to 0} \left[ G^T \left( 1 + \frac{\delta \nu^T}{\delta \psi} G^T \right)^{-1} \right]_{t, x} = \lim_{t \to 0} G^T(t, x) = \delta(x).
\]

(31)

Defining

\[
U_r = \frac{\delta \nu}{\delta_t \psi}, \quad \tilde{R} = \left( 1 + GU_r \right)^{-1} \partial_t G Q T G^T \left( 1 + U_t^T G^T \right)^{-1},
\]

(32)

we rewrite equation (30) in the form

\[
(\tilde{E} - \tilde{K} + \tilde{R} Q \tilde{K}) Z(K) = 0
\]

(33)
One can show that for the Green functions the following holds

\[ \partial_t G_{t_1-t_2,x-y} = \frac{Q^T G_{t_1-t_2,x-y}^T - G_{t_1-t_2,x-y} Q^T}{2}. \] (34)

This quantity vanishes whenever \( t_1 = t_2 = T \). Expanding the inverse operators appearing in (32) in power series over \( U \) we obtain for a fixed order \( m \) the following expression

\[ \sum_{n=0}^{m} (GU_r)^n (Q^T G^T - GQ^T)(U_l^T G^T)^{m-n} = \sum_{n=1}^{m} (GU_r)^n Q^T G^T (U_l^T G^T)^{m-n} \]

\[ - \sum_{n=0}^{m-1} (GU_r)^n GQ^T (U_l^T G^T)^{m-n} + Q^T G^T (U_l^T G^T)^{m} - (GU_r)^m GQ^T. \] (35)

Here the last two terms vanish if the time argument on the both ends is the same. Thus for \( m > 0 \) this expression becomes

\[ - \sum_{n=0}^{m-1} (GU_r)^n G[Q^T U_l^T - U_r Q^T] G^T (U_l^T G^T)^{m-1-n}. \] (36)

After summing up all orders \( m \) we obtain the final expression for \( \tilde{R} \) at equal time on the both ends

\[ \tilde{R} = \left( 1 + GU_r \right)^{-1} G[Q^T U_l^T - U_r Q^T] G^T \left( 1 + U_l^T G^T \right)^{-1}. \] (37)

The quantity \( Q^T U_l^T = U_r Q^T \) vanishes for a general action, see appendix A. Consequently equation (33) yields (26).

3 Acknowledgments

I thank the Institute for Theoretical Physics at the University of Leipzig, Germany for the financial support.
A  \( \mathcal{U}_r \) and \( \mathcal{U}_T \)

Below all derivatives are left derivatives.

\[
\mathcal{U}_r = \begin{pmatrix}
-\frac{\delta^2 L}{\delta \psi_i \delta \psi_j} & -\frac{\delta^2 L}{\delta \psi_i \delta \phi_j} & -\frac{\delta^2 L}{\delta \psi_i \delta \bar{\psi}_j} \\
\frac{\delta^2 L}{\delta \psi_i \delta \psi_j} & -\frac{\delta^2 L}{\delta \psi_i \delta \phi_j} & -\frac{\delta^2 L}{\delta \psi_i \delta \bar{\psi}_j} \\
-\frac{\delta^2 L}{\delta \phi_i \delta \psi_j} & -\frac{\delta^2 L}{\delta \phi_i \delta \phi_j} & -\frac{\delta^2 L}{\delta \phi_i \delta \bar{\psi}_j}
\end{pmatrix}
= \begin{pmatrix}
u_{11} & u_{12} & u_{13} \\
u_{21} & u_{22} & u_{23} \\
u_{31} & u_{32} & u_{33}
\end{pmatrix}
\tag{38}
\]

\[
\mathcal{U}_T = \begin{pmatrix}
\frac{\delta^2 L}{\delta \psi_i \delta \psi_j} & \frac{\delta^2 L}{\delta \psi_i \delta \phi_j} & \frac{\delta^2 L}{\delta \psi_i \delta \bar{\psi}_j} \\
\frac{\delta^2 L}{\delta \phi_i \delta \psi_j} & \frac{\delta^2 L}{\delta \phi_i \delta \phi_j} & \frac{\delta^2 L}{\delta \phi_i \delta \bar{\psi}_j} \\
\frac{\delta^2 L}{\delta \bar{\psi}_i \delta \psi_j} & \frac{\delta^2 L}{\delta \bar{\psi}_i \delta \phi_j} & \frac{\delta^2 L}{\delta \bar{\psi}_i \delta \bar{\psi}_j}
\end{pmatrix}
= \begin{pmatrix}
u_{22} & -u_{21} & -u_{T}^{T} \\
u_{12} & u_{11} & -u_{32} \\
u_{T}^{T} & u_{23} & u_{33}
\end{pmatrix}
\tag{39}
\]

The equation \( Q^T \mathcal{U}_T = \mathcal{U}_r Q^T \) implies \( u_{31} = u_{T}^{T} \) and \( u_{13} = -u_{32} \).

B  Symmetries of the action

We define as usual the time and parity reversal operators

\[
\mathcal{T} : (x_1, x_2) \mapsto (-x_1, x_2), \quad \mathcal{P} : (x_1, x_2) \mapsto (x_1, -x_2),
\tag{40}
\]

and corresponding transformations for the spinor \( \psi \)

\[
P : \psi(x) \mapsto \gamma_1 \psi(\mathcal{P} x), \quad T : \psi(x) \mapsto \gamma_3 \psi(\mathcal{T} x), \quad C : \psi(x) \mapsto \gamma_1 \bar{\psi}(x),
\tag{41}
\]

where \( T \) is anti-linear, i.e. \( T \alpha T^{-1} = \alpha^{*} T\psi T^{-1} \). The action \( L \), see (5), is invariant under \( P \). Invariance under \( CT \) can be obtained if one simultaneous makes inversion of the sign of \( m \)

\[
\psi(x) \mapsto \gamma_2 \bar{\psi}(\mathcal{T} x), \quad \bar{\psi}(x) \mapsto \psi(\mathcal{T} x) \gamma_2, \quad m \mapsto -m.
\tag{42}
\]

References

[1] A. Kupiainen. Renormalization group and stochastic pde’s. \textit{Annales Henri Poincaré}, 17, 2014.

[2] M. Hairer. Introduction to regularity structures. \textit{Braz. J. Probab. Stat.}, 29:175–210, 2015.
[3] J. Glimm. The yukawa coupling of quantum fields in two dimensions. Communications in Mathematical Physics, 6(1):61–76, Mar 1967.

[4] D. J. Gross and A. Neveu. Dynamical symmetry breaking in asymptotically free field theories. Phys. Rev. D, 10:3235–3253, Nov 1974.

[5] K.G. Wilson. Renormalization group and critical phenomena. Phys. Rev. B, 4(9):3174–3183, 1971.

[6] F.A. Berezin. Introduction to Superanalysis. Springer, 1987.

[7] G. Parisi and Y. Wu. Perturbation theory without gauge fixing. Scientia Sinica, 24:483, 1981.

[8] P. H. Damgaard and H. H"uffel. Stochastic quantization. Physics Reports, 152(5):227 – 398, 1987.

[9] J. Zinn-Justin. Renormalization and Stochastic Quantization. Nucl. Phys., B275:135–159, 1986.

[10] K. Osterwalder. Euclidean fermi fields, pages 326–331. Springer Berlin Heidelberg, Berlin, Heidelberg, 1973.

[11] K. Osterwalder and R. Schrader. Feynman-kac formula for euclidean fermi and bose fields. Phys. Rev. Lett., 29:1423–1425, Nov 1972.

[12] J. Glimm and A. Jaffe. Quantum physics. Springer, 1987.