Nonlinear Sigma Model, Zakharov-Shabat Method, and New Exact Forms of the Minimal Surfaces in $\mathbb{R}^3$

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General formulas for the construction of exact solutions of the equation of the minimal surface in $\mathbb{R}^3$, which appears in various physical problems, have been derived by the Zakharov-Shabat ”dressing” method. Particular examples are considered.

1. The ”area” functional plays an important role in many topical problems of theoretical and mathematical physics. In particular, according to current concepts [1], a free relativistic string is a continuum of points each moving along its world line in the D-dimensional spacetime. This motion occurs so that the area of the world sheet covered by the string in minimal (informal physical and geometric interpretation of this phenomenon was given in [2]).

We begin with the area functional in the form

$$S[f] = \int d^{D-1}y \sqrt{|\det g^{(0)}|},$$

(1)

where $g^{(0)} = |g^{(0)}_{ij}(y)|$, $g^{(0)}_{ij}(y)$ is induced Riemann metric on a hypersurface, $y = (y^1, y^2, ..., y^{D-1})$, $d^{D-1}y = dy^1 \wedge dy^2 ... \wedge dy^{D-1}$, $(y, y^D) \in \Omega \subset \mathbb{R}^D$, and $y^D = f(x)$ is the smooth hypersurface (it is assumed that the function $f(y)$ is real-valued).

Then, the Euler-Lagrange equation (critical point) for Eq.(1) is written as

$$\text{div} \left( \frac{f_{yi}}{\sqrt{1 + \sum_{i=1}^{D-1} (f_{yi})^2}} \right) = 0.$$  

(2)

This equation is hardly integrable for arbitrary D value. As will be shown below, this equation with $D = 3$ has a Lax representation and, therefore, belongs to the class of completely integrable systems. The aim of this work is to construct its new exact solutions.

Equation (2) for $D = 3$ can be represented in the following more distinct form by setting $y^1 = t$, $y^2 = x$:

$$(1 + f_t^2)f_{xx} - 2f_{xt}f_x f_t + (1 + f_x^2)f_{tt} = 0.$$  

(3)

This two-dimensional nonlinear elliptic equation has long been known in differential geometry. It coincides with the condition of zero average curvature of the surface and is referred to as the ”equation of the minimal surface” (MS) (in $\mathbb{R}^3$). Equation (3) appears, e.g., at some reductions of the $n$-field model [7], in the stationary and dispersionless limit.
of the system of coupled nonlinear Schrödinger equation [8], and some other physical problems. In this sense, it has a certain universal significance. It is worth emphasizing that the so-called integral Enneper-Weierstrass representations [9], as well as linearization of Eq.(3) by means of the Legendre or hodograph transforms [10], generally give only some parametrizations of the solution or their limited set (the plane in $\mathbb{R}^3$, helicoid, catenoid, Scherk surface, and Enneper surface [11] are among the known solutions of Eq. (3)). For this reason, the approach developed below on the basis of direct integration of Eq. (3) seems to be only possible method for solving the formulated problem.

It is also noteworthy that it is geometrically obvious that the image of the minimal surface is invariant under rotation, shift and similarity transformation.

2. Equation (3) can be represented in the form of the compatibility condition for Lax pair [12]

$$\Psi_x = \frac{1}{\lambda^2 + 1} [\lambda (g^{11} I_1 + g^{12} I_2) - I_2] \Psi \equiv U \Psi, \quad (4a)$$

$$\Psi_t = -\frac{1}{\lambda^2 + 1} [\lambda (g^{21} I_1 + g^{22} I_2) + I_1] \Psi \equiv V \Psi, \quad (4b)$$

where

$$g = \begin{pmatrix} u_{tt} & u_{tx} \\ u_{tx} & u_{xx} \end{pmatrix}, \quad \det g = 1, \quad g = g^T, \quad g^2 \neq I, \quad g^{-1} = \{g^{kl}\}, \quad (5)$$

$$u_{xx} = 1 + \frac{f_x^2}{\mathcal{L}}, \quad u_{tt} = \frac{1 + f_t^2}{\mathcal{L}}, \quad u_{xt} = \frac{f_x f_t}{\mathcal{L}}, \quad \mathcal{L} = \sqrt{1 + f_x^2 + f_t^2}, \quad (6)$$

$I_1 = g^{-1} g_t$, $I_2 = g^{-1} g_x$, $\Psi = \Psi(x,t,\lambda) \in \text{Mat}(2,\mathbb{C})$, $\lambda \in \mathbb{C}$ is the spectral parameter, and $\mathcal{L}$ is the Lagrangian density of Eq. (3). The unimodularity condition for the matrix $g$ means that the function $u = u(x,t)$ satisfies a Monge-Ampere elliptic equation

$$u_{tt} u_{xx} - u_{tx}^2 = 1, \quad (7)$$

so that $\text{Tr} I_1 = \text{Tr} I_2 = 0$ and $I_1$, $I_2 \in \text{sl}(2)$, where $\text{sl}(2)$ is the Lie algebra of the $SL(2)$ group. Then, consistency requirement (4), together with the identity

$$I_2 t - I_1 x - [I_2, I_1] = 0 \quad (8)$$

leads to a nonlinear sigma-model of the form ($\alpha$, $\beta = 1, 2$)

$$\partial_\alpha (g^{\alpha \beta} g^{-1} \partial_\beta g) = 0, \quad (9)$$

which is equivalent to Eq. (3).

Exact solutions of Eq. (3) for the discrete spectrum of associated linear system (4a) are constructed with the use of the Zakharov-Shabat dressing procedure [13] (in the variant used in [14]-[15] and also in [16]).

First, it directly follows from Eq. (4a) that

\footnote{The boundary value problem for nonlinear elliptic equations on a half-plane, the entire plane, and a quarter plane was solved by means of the inverse scattering problem method in [16]-[17], [18], and [19], respectively.}
Then, we know a certain bare (background) solution $f_0 = f_0(x, t)$ of Eq. (3) and this solution, according to Eq. (4), corresponds to the matrix solution $\Psi_0 = \Psi_0(x, t, \lambda)$, and $g = g_0(x, t)$. We set $\Psi = \chi \Psi_0$, where $\chi = \chi(x, t, \lambda) \in \text{Mat}(2, \mathbb{C})$, assuming the canonical normalization $\chi(\infty) = I$ (where $I$ is $2 \times 2$ identity matrix). From the properties of the matrix $U (\sigma_i, i = 1, 3$ are the standard Pauli matrix and the asterisk stands for Hermitian conjugation),

$$U(\lambda) = \bar{U}(\lambda), \quad U(\lambda) = -\sigma_2 U^T(\lambda) \sigma_2, \quad gU(\lambda) = U^*(\lambda)g,$$

the involutions (symmetries) for the function $\Psi$ and $\chi$ follows in the form

$$\bar{\Psi}(\lambda) = \Psi(\lambda) D_1(\lambda), \quad \Psi^T(\lambda) \sigma_2 \Psi(\lambda) = D_2(\lambda), \quad g \Psi(\lambda) D_3(\lambda) = \Psi^{-1T}(-\frac{1}{\lambda}),$$

and

$$\bar{\chi}(\lambda) = \chi(\lambda), \quad \chi^T(\lambda) = \sigma_2 \chi^{-1}(\lambda) \sigma_2, \quad \chi^{-1}(0) \chi(\lambda) \Psi_0(0) = \Psi_0(0) \sigma_2 \chi(-\frac{1}{\lambda}) \sigma_2,$$

where $D_i(\lambda)$ are arbitrary matrices ($i = 1, 3$) such that $D_1(\lambda) \bar{D}_1(\lambda) = I, \quad D_2^T(\lambda) = -D_2(\lambda)$. Taking into account Eqs. (13), we seek $\chi$ and $\chi^{-1}$ in the form

$$\chi = I + \frac{P_1}{\lambda - \lambda_1} + \frac{\bar{P}_1}{\lambda - \bar{\lambda}_1} + \frac{R_1}{\lambda + \lambda_1} + \frac{\bar{R}_1}{\lambda + \bar{\lambda}_1},$$

$$\chi^{-1} = I + \frac{\sigma_2 P_1^T \sigma_2}{\lambda - \lambda_1} + \frac{\sigma_2 P_1^* \sigma_2}{\lambda - \bar{\lambda}_1} + \frac{\sigma_2 R_1^T \sigma_2}{\lambda + \lambda_1} + \frac{\sigma_2 R_1^* \sigma_2}{\lambda + \bar{\lambda}_1},$$

where $P_1 = P_1(x, t), \quad R_1 = R_1(x, t) \in \text{Mat}(2, \mathbb{C})$ are as yet unknown function and $\lambda_1 \in \mathbb{C}$ is the simple pole of the function $\chi$ such that $\lambda_1 = \lambda_{1R} + \lambda_{1I}, \quad \lambda_{1R} \neq 0$ and $\lambda_{1I} \neq 0$. In terms of $P_1 = a \sigma_2 \sigma_2 \sigma_2$ and $R_1 = b \sigma_2 \sigma_2 \sigma_2$, the condition $\chi \chi^{-1} = I$ after equating combination of residues at the same poles to zero gives the following system of linear equations for the vectors $a >, \quad \bar{a} >, \quad b >$ and $\bar{b} >$:

$$\frac{1}{\lambda_1 - \lambda_1} \bar{a} > \sigma_2 b > < q \sigma_2 > + \frac{1}{\lambda_1 + \lambda_1} \bar{b} > < q \sigma_2 > = p >,$$

$$- \frac{1}{\lambda_1 - \lambda_1} a > < q \sigma_2 > + \frac{1}{\lambda_1 + \lambda_1} \bar{b} > < q \sigma_2 > = - \bar{p} >,$$

$$\frac{1}{\lambda_1 + \lambda_1} a > < q \sigma_2 > + \frac{1}{\lambda_1 - \lambda_1} \bar{a} > < q \sigma_2 > = q >,$$

$$- \frac{1}{\lambda_1 - \lambda_1} a > < \bar{q} \sigma_2 > + \frac{1}{\lambda_1 + \lambda_1} \bar{a} > < \bar{q} \sigma_2 > = - \bar{q} >.$$
The requirement of the absence of doubles poles is reduced to the conditions
\[ < p \sigma_2 p > = < q \sigma_2 q > = 0. \]  
(16)

In addition, it is easy to verify that
\[ < p \sigma_2 \tilde{p} > = - < \tilde{p} \sigma_2 p > = 0, \quad < q \sigma_2 \tilde{q} > = - < \tilde{q} \sigma_2 q > = 0, \]  
(17)
\[ < p \sigma_2 q > = \overline{< q \sigma_2 p >}, \quad < q \sigma_2 \tilde{q} > = - \overline{< p \sigma_2 \tilde{p} >}. \]

Solving system (15) with allowance for Eq. (16) and (17), we obtain the matrices \( P_1 \) and \( R_1 \) in the form
\[ P_1 = \frac{1}{|d_1|^2|l_1|^2 - |d_2|^2|l_2|^2} [\bar{d}_1 l_1 q > - \bar{d}_2 l_2 \tilde{q} > < p], \]  
(18)
\[ R_1 = - \frac{1}{|d_1|^2|l_1|^2 - |d_2|^2|l_2|^2} [\bar{d}_1 l_1 p > < q + d_2 l_2 \tilde{p} > < q]. \]

where
\[ l_1 = < q \sigma_2 p >, \quad l_2 = < \tilde{q} \sigma_2 p >, \quad d_1 = \frac{1}{\lambda_1 + \lambda_1^{-1}}, \quad d_2 = \frac{1}{\lambda_1 + \lambda_1^{-1}}. \]

In this case, \( p > q = (q > p)^*, \tilde{q} > p = (\tilde{p} > q)^T \), and, in view of Eqs. (16) and (17), \( P_1|p >= 0 \) and \( R_1|q >= 0 \), so, that \( |p > \in \ker P_1, |q > \in \ker R_1 \).

The next step in the dressing method is the determination of the dependence of the vectors \( |p > \) and \( |q > \) on the variables \( x \) and \( t \). To this end, system (4) is rewritten in terms of the solutions \( \chi \) and \( \chi^{-1} \); since the functions \( U(\lambda) \) and \( V(\lambda) \) have poles only at the points \( \pm i \), we impose the conditions \( (U_0 = \Psi_0 \chi^{-1}, V_0 = \Psi_0 \chi^{-1}) \):
\[ \text{Res}_{\lambda = \lambda_1} \{ \chi (-\partial_x + U_0(\lambda)) \chi^{-1} \} = 0, \quad \text{Res}_{\lambda = \lambda_1} \{ \chi (-\partial_t + V_0(\lambda)) \chi^{-1} \} = 0. \]  
(19)

According to Eq. (14), near the point \( \lambda_1 \)
\[ \chi(\lambda) = \frac{P_1}{\lambda_1^2 - \lambda}, \quad \chi^{-1}(\lambda) = \frac{\sigma_2 P_1^T \sigma_2}{\lambda - \lambda_1} + B_0(\lambda), \]  
(20)
where \( A_0(\lambda) \) and \( B_0(\lambda) \) are the functions analytic at the point \( \lambda_1 \), such that \( B_0(\lambda) = \sigma_2 A_0^T(\lambda) \sigma_2 \). The substitution of the Eq. (20) into Eq. (19) gives
\[ A_0(\lambda_1)(\partial_x - U_0)B_0 = 0, \]  
(21)
and
\[ A_0(\lambda_1)(\partial_x - U_0)\sigma_2 P_1^T \sigma_2 + P_1(\partial_x - U_0)\sigma_2 A_0^T(\lambda_1)\sigma_2 = 0, \]  
(22)
\[ A_0(\lambda_1)(\partial_t - V_0)\sigma_2 P_1^T \sigma_2 + P_1(\partial_t - V_0)\sigma_2 A_0^T(\lambda_1)\sigma_2 = 0. \]
These equalities are obviously satisfied if \( (\partial_x - U_0)\sigma_2 P_1^T = (\partial_t - V_0)\sigma_2 P_1^T = 0 \). Comparing them with Eqs. (4a) and (4b), we obtain \(|\alpha_1\rangle\) is a constant complex vector

\[
p > = \Psi_0(x, t, \lambda_1) |\alpha_1\rangle. \tag{23}
\]

The same result can be obtained by repeating the above calculations near the point \( \bar{\lambda}_1 \). The dependence of the vector \( q > \) is obtained similarly by applying the same reasons near the point \(-1/\lambda_1\) or \(-1/\bar{\lambda}_1\). Then \(|\alpha_2\rangle\) is a constant complex vector,

\[
q > = \Psi_0(x, t, -\frac{1}{\lambda_1}) |\alpha_2\rangle. \tag{24}
\]

To close the above procedure, it is necessary to obtain a formula for the reconstruction of the "potential" \( f \), and to construct the solution \( \Psi_0(x, t, \lambda) \). To this end, it is easy to derive the following equation for the Lagrangian density of the "dressed" solution from Bianchi relations (6) and formula (10):

\[
L^2 - (Tr \Psi(0))L + 1 = 0. \tag{25}
\]

Therefore (under the assumption that \(|Tr \Psi(0)| \geq 2\),

\[
|\nabla f|^2 = \frac{1}{2} Tr \Psi(0) \left[ Tr \Psi(0) \pm \sqrt{Tr^2 \Psi(0) - 4} \right] - 2. \tag{26}
\]

The scalar functions \( Q_1 = Q_1(x, t, \lambda_1) \), \( Q_2 = Q_2(x, t, \lambda_1) \) are introduced as

\[
Q_1 = \Psi_{11}(0) = \frac{(1 + f_{0x}^2)\chi_{11}(0) - f_{0x}f_{0t}\chi_{12}(0)}{\mathcal{L}_0} = \frac{E(0)\chi_{11}(0) - F(0)\chi_{12}(0)}{\sqrt{E(0)G(0) - F(0)^2}},
\]

\[
Q_2 = \Psi_{22}(0) = \frac{(1 + f_{0t}^2)\chi_{22}(0) - f_{0x}f_{0t}\chi_{21}(0)}{\mathcal{L}_0} = \frac{G(0)\chi_{22}(0) - F(0)\chi_{21}(0)}{\sqrt{E(0)G(0) - F(0)^2}}, \tag{27}
\]

where \( \Psi(0) = \{\Psi_{ij}(0)\} \), \( \chi(0) = \{\chi_{ij}(0)\} \), \( \mathcal{L}_0^2 = 1 + f_{0x}^2 + f_{0t}^2 = E(0)G(0) - F(0)^2 \), and \( E(0) \), \( F(0) \), and \( G(0) \) are the coefficients of the first quadratic form of the "initial" ("bare") surface. If \( \Psi_0(0) \) is real-valued, \( Q_1 \) and \( Q_2 \) are also real-valued, because \( \chi(0) = \bar{\chi}(0) \) according to Eq. (14). Taking into account Eqs. (10) and (26), we obtain

\[
f_x^2 = \frac{Q_1}{2} [Q_1 + Q_2 \pm \sqrt{(Q_1 + Q_2)^2 - 4}] - 1 \equiv 2Q_1e^{Q_3} - 1 \equiv \Delta_1^2,
\]

\[
f_t^2 = \frac{Q_2}{2} [Q_1 + Q_2 \pm \sqrt{(Q_1 + Q_2)^2 - 4}] - 1 \equiv 2Q_2e^{Q_3} - 1 \equiv \Delta_2^2,
\]

where \( Q_1 + Q_2 = 4 \cosh Q_3 \). Direct calculation shows that \( \Delta_{1t} = \Delta_{2x} \). This means that the differential 1-form \( \omega = \Delta_1 dx + \Delta_2 dt \) is exact. Consequently, the curvilinear integral

\[
f(x, t) = \pm \int_{(x_0, t_0)}^{(x, t)} \Delta_1(x', t) dx' + \Delta_2(x, t') dt' = \pm \int_{(x_0, t_0)}^{(x, t)} \sqrt{2Q_1e^{Q_3} - 1} dx + \sqrt{2Q_2e^{Q_3} - 1} dt, \tag{29}
\]
where \((x_0, t_0) \in \Omega\) is a certain point, is independent of integration path and the presence of both signs corresponds to the invariance of Eq. (3) under the change \(f \to -f\). Expressions (27) and (28), together with Eq. (29), specify a "single-soliton" solution of Eq. 3 against the background of the solution \(f_0(x, t)\). In this case, from the geometric point of view, the dressing procedure is reduced to a nonlinear transformation of the coefficients of the first quadratic form of the initial surface.

To determine \(\Psi_0(x, t, \lambda)\) we note that the system (4) after some algebra can be rewritten in the form

\[
\Psi_{0x} = \frac{1}{\lambda^2 + 1}[-\lambda G_{0x} - G_0^{-1} G_{0x}] \Psi_0, \quad \Psi_{0t} = \frac{1}{\lambda^2 + 1}[-\lambda G_{0t} - G_0^{-1} G_{0t}] \Psi_0, \tag{30}
\]

where \(G_0 = i\sigma_2 g_0\), so that \(G_0^2 = -I\) (in view of Eq. (7)). This system has the solution

\[
\Psi_0(x, t, \lambda) = ic_0(\lambda)[G_0^{-1} + (\lambda - \lambda^3) I + \lambda^2 G_0] \sigma_2, \tag{31}
\]

where \(c_0(\lambda)\) is the factor, determined by the requirement \(\det \Psi_0(\lambda) = 1\). Then, in view of the identity \(\sigma_2 g_0 \sigma_2 = g_0^{-1}\), Eq. (31) can be represented in terms of the derivations of the function \(u_0\) (initial solution of Eq. (7)):

\[
\Psi_0(\lambda) = \{\Psi_0_{ij}(\lambda)\} = \frac{1}{\sqrt{1 + \lambda^2}} \left( \begin{array}{cc} u_{0xx} & \lambda - u_{0tx} \\ -\lambda + u_{0tx} & u_{0tt} \end{array} \right), \quad \Psi_0(0) = \bar{\Psi}_0(0). \tag{32}
\]

3. We define the vectors \(|\alpha_i\rangle, i = 1, 2\), entering into Eqs. (23) and (24) as \(|\alpha_i\rangle = (\mu_i, \nu_i)^T\), where \(\mu_i, \nu_i \in \mathbb{C}\) are arbitrary numbers. In order to simplify the formulas, it is convenient to set \(|\alpha_1\rangle = \mu_10(1 + i\mu_0)(1, -i)^T\) and \(|\alpha_2\rangle = \mu_20(1 + i\mu_0)(1, i)^T\), where \(\mu_0, \mu_0 \in \mathbb{R}\) are the parameters. Furthermore, the parameter \(\lambda_1\) is taken on a unit circle, i.e., \(\lambda_1 = e^{i\theta_1}, \theta_1 \neq 0, \pi/2, \pi, 3\pi/2, 2\pi\), which corresponds to "kink" excitations in spectral problem (4a) [17]. Then, in the general situation, i.e., for any MS \(f_0 = f_0(x, t)\), according to Eqs. (23), (24), and (32) we obtain \((d_1 = 1/(2 \cos \theta_1), d_2 = (1/2)e^{-i\theta_1})\)

\[
|p\rangle = (p_1, p_2)^T = \frac{\mu_10(1 + i\mu_0)e^{i\theta_1}}{L_0 \sqrt{2} |\cos \theta_1|} \left( \begin{array}{c} L_0[1 + e^{-i\theta_1}(1 - i)] + i(e^{-i\theta_1}f_{0x}f_{0t} - f_{0x}^2 - f_{0t}^2) \\ e^{-i\theta_1}[L_0(2 \cos \theta_1 - e^{-i\theta_1}) - f_{0x}f_{0t}] - 1 - f_{0t}^2 \end{array} \right),
\]

\[
|q\rangle = (q_1, q_2)^T = \frac{\lambda_1 \mu_10(1 + i\mu_0)e^{i\theta_1}}{L_0 \sqrt{2} |\cos \theta_1|} \left( \begin{array}{c} L_0[1 + e^{-2i\theta_1}(1 - i)] + i(e^{-i\theta_1}f_{0x}f_{0t} - 1 - f_{0t}^2) \\ e^{-i\theta_1}[L_0(2 \cos \theta_1 - e^{-i\theta_1}) - f_{0x}f_{0t}] - f_{0x}f_{0t} - 1 \end{array} \right),
\]

\[
l_1 = -l_{10} \frac{1 + L_0^2 - 2L_0 \sin \theta_1}{L_0 |\cos \theta_1|}, \quad l_2 = l_{10} \frac{f_{0x}^2 - f_{0t}^2 + 2if_{0x}f_{0t}}{L_0}, \quad l_{10} = \mu_10 \mu_20(1 + \mu_0^2), \tag{33}
\]

\[
\Delta = \Delta(L_0, \theta_1) = |d_1|^2|l_1|^2 - |d_2|^2|l_2|^2 = \frac{l_{10}^2}{4L_0^2}(1 + L_0^2 - 2L_0 \sin \theta_1)^2 - (1 - L_0^2)^2 \cos^4 \theta_1 \neq 0,
\]

\[
A_1 = q > p = (\bar{p}_1|q >, \bar{p}_2|q >), \quad A_2 = q > p = (\bar{p}_1|q >, \bar{p}_2|q >), \quad A_1, A_2 \in \text{Mat}(2, \mathbb{C}),
\]

\[
\chi(0) = I - \frac{2}{\Delta} \text{Re}\{[e^{-i\theta_1}(d_1 l_1 A_1 - d_2 l_2 A_2) - e^{i\theta_1}(d_1 l_1 A_1^T + d_2 l_2 A_2^T)]\sigma_2\}.
\]

As a result, solution (29) is very lengthy and, for this reason, is not given herein a more explicit form. It is nonsingular and depends on four real parameters \(\mu_0, \mu_{10}, \mu_{20}, \theta_1\).
Two particular examples of implementations of the above relations are as follows.

i). Let $f_0(x,t) = 0$; i.e., the procedure of dressing of the trivial solution ("horizontal" plane) is considered. In this case, $\mathcal{L} = \mathcal{L}_0 = 1$, $u_{0xx} = u_{0tt} = 1$, $u_{tx} = 0$, $g_0 = I$, $I_1 = I_2 = 0$, and $\Psi_0(\lambda_1) = (1/\sqrt{2|\cos \theta_1|})e^{-i\theta_1/2}[I + i(\cos \theta_1 \sigma_2 + \sin \theta_1 \sigma_1)]$. The matrix $\chi(0)$ and, therefore, the functions $Q_1$ and $Q_2$ are constants. According to Eq. (29), this means that the solution $f(x,t)$ is a linear function of $x$, i.e., an "inclined" plane, which can be transformed to the initial position by rotation and shift transformations. This result corresponds to the classical Bernshtein result [20]: the image of the minimal surface over the entire plane is the plane itself. Thus, we have a certain "exclusion principle" for new solutions over the entire plane.

The situation for the bare solution in the form of an inclined plane is similar, i.e., $f_0(x,t) = a_0x + b_0t + c_0$, where $a_0$, $b_0$, $c_0 \in \mathbb{R}$ are constants.

ii). Let $f_0(x,t)$ be the MS different from a plane, e.g., a helicoid $f_0 = x \tan t$, $x \in \mathbb{R}$, $t \in \mathbb{R} \setminus (\pi/2)(2k + 1)$, $k = 0, \pm 1, \pm 2, \ldots$. In this situation, $f_{0x} = \tan t$, $f_{0t} = x/\cos^2 t$, $\mathcal{L}_0 = (\sqrt{x^2 + \cos^2 t})/\cos^2 t$, and evolution in $x$ and $t$ is nontrivial (at least for $f_{2x}$ and $f_{2t}$) and is determined according to the first two formulas in Eqs. (33). In this case,

$$
\begin{align*}
I_1 &= -l_{10} \frac{x^2 + \cos^2 t(1 + \cos^2 t) + 2 \sin \theta_1 \cos^2 t \sqrt{x^2 + \cos^2 t}}{\cos \theta_1 \sqrt{x^2 + \cos^2 t}} , \\
I_2 &= l_{10} \frac{\sin^2 t - x^2 + 2ix \tan t}{\sqrt{\cos^2 t + x^2}} , \\
\Delta &= \frac{l_{10}^2}{\cos^6 t \cos^4 \theta_1} \frac{1}{\cos^4 \theta_1} (\cos^4 t + \cos^2 t + x^2 - 2 \sin \theta_1 \sqrt{\cos^2 t + x^2})^2 - (\cos^4 t - \cos^2 t - x^2)^2 \cos^4 \theta_1 .
\end{align*}
$$

To conclude, it is noteworthy that the procedure used above makes it possible to simultaneously solve three equations: equation of MS (3), sigma-model (9), and Monge-Ampere equation (7).

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