PARAMETER ESTIMATION FOR FRACTIONAL POISSON PROCESSES

Dexter O. Cahoy§  Vladimir V. Uchaikin†  Wojbor A. Woyczynski‡

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Abstract

The paper proposes an estimation procedure for parameters of the fractional Poisson process (fPp) which is based on the method of moments (MoM). The basic tool is the fractional calculus and the link between fractional Poisson process (fPp) and α-stable densities. Based on this result, we establish the asymptotic normality of our estimators for the intensity rate \( \mu \), and the fractional exponent \( \nu \), two parameters appearing in the fractional Poisson stochastic model; its properties are tested using synthetic data.

Keywords: fractional Poisson process, α-stable Lévy densities, fractional calculus, asymptotic normality, method of moments estimators.

1 Introduction

The paper proposes a formal estimation procedure for parameters of the fractional Poisson process (fPp). Such procedures are needed to make the fPp model usable in applied situations. Different versions of fPp have been studied recently by several authors, see, in particular, Repin and Saichev (2000), Wang and Wen (2003), Wang, Wen and Zhang (2006) and Laskin (2003), so we start our exposition from the basic definitions to make it clear which stochastic model we are working with. Some of the preliminary results on this model appeared in Uchaikin, Cahoy and Sibatov (2008) but we restate them in the first couple of sections for the sake of completeness of presentation. The basic idea of fPp, motivated by experimental data with long memory (such as some network traffic, neuronal firings, and other signals generated by complex systems), is to make the standard Poisson model more flexible by permitting non-exponential, heavy-tailed distributions of interarrival times. However, the price one has to pay for such flexibility is loss of the Markov property, a similar

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§ Department of Mathematics and Statistics, Louisiana Tech University, USA
† Dep’t of Statistics, Case Western Reserve University, USA, email: waw@case.edu
‡ Dep’t of Theoretical and Mathematical physics, Ul’yanovsk State University, Russia
situation to that encountered in the case of certain anomalous diffusions, see, e.g., Priyatinska, Saichev and Woyczynski (2005). To partly replace this loss one demands some scaling properties of the interarrival times’ distributions which makes other tools available; in this paper they are the fractional calculus and the link between fractional Poisson process and α-stable Lévy densities. Based on the latter connection we establish the asymptotic normality of our estimators for the two parameters appearing in our fPp model: the intensity rate \( \mu \), and the fractional exponent \( \nu \). This fact permits construction of the corresponding confidence intervals. The properties of the estimators are then tested using synthetic data.

The paper is composed as follows: Section 2 introduces the basic definition of fPp and the fractional calculus tools needed to study it. Section 3 proves the basic structural theorem relating fPp to α-stable Lévy random variables which makes efficient simulation of the former possible. In Section 4, we describe nontrivial scaling limits of the marginal distributions of fPp. Section 5 introduces the concept of the method-of-moments estimators in the fPp context and calculates them. They are proven asymptotically normal in Section 6. Finally, we test our procedures numerically on simulated data in Section 7. The concluding remarks in Section 8 are then followed by two brief appendices, one on \( \alpha^+ \)-stable stable densities, and one on an alternative fPp model.

2 FPP interarrival time

The fractional Poisson process \( N_\nu(t), 0 < \nu \leq 1, t > 0 \), was defined in Repin and Saichev (2000) via the following formula for the Laplace transform of the p.d.f \( \psi_\nu(t) \) of its i.i.d. interarrival times \( T_i, i = 1, 2, \ldots \):

\[
\{L\psi_\nu(t)\}(\lambda) \equiv \tilde{\psi}_\nu(\lambda) \equiv \int_0^\infty e^{-\lambda t} \psi_\nu(t) dt = \frac{\mu}{\mu + \lambda^\nu},
\]

where \( \mu > 0 \) is a parameter. For \( \nu = 1 \), the above transform coincides with the Laplace transform

\[
\tilde{\psi}_1(\lambda) = \frac{\mu}{\mu + \lambda}.
\]

of the exponential interarrival time density of the ordinary Poisson process with parameter \( \mu = \mathbb{E}N_1(1) \).

Using the inverse Laplace transform the above cited authors derived the singular integral equation for \( \psi_\nu(t) \):

\[
\psi_\nu(t) + \frac{\mu}{\Gamma(\nu)} \int_0^t \psi_\nu(\tau) \frac{d\tau}{[\mu(t - \tau)]^{1-\nu}} = \frac{\mu^\nu}{\Gamma(\nu)} \frac{\nu}{\nu-1},
\]

which is equivalent to the fractional differential equation,

\[
0D_+^\nu \psi_\nu(t) + \mu \psi_\nu(t) = \delta(t),
\]
where the Liouville derivative operator $0D^\nu_t = d^\nu/dt^\nu$ (see, e.g., Kilbas, Srivastava and Trujillo (2006)) is defined via the formula

$$0D^\nu_t \psi_\nu(\tau) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t \psi_\nu(\tau) \frac{d\tau}{[\mu(t-\tau)]^{1-\nu}}.$$ 

These characterizations permitted them to obtain the following integral representation for the p.d.f. $\psi_\nu(t)$,

$$\psi_\nu(t) = \frac{1}{t} \int_0^\infty e^{-x} \phi_\nu(\mu t/x) dx,$$  \hspace{1cm} (2)

where

$$\phi_\nu(\xi) = \frac{\sin(\nu \pi)}{\pi [\xi\nu + \xi^{-\nu} + 2 \cos(\nu \pi)]},$$

and demonstrate that the tail probability distribution of the waiting time $T$ is of the form

$$P(T > t) = \int_t^\infty \psi_\nu(\tau) d\tau = E_\nu(-\mu t^\nu),$$ \hspace{1cm} (3)

where

$$E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + 1)}$$ \hspace{1cm} (4)

is the Mittag-Leffler function (see, e.g., Kilbas, Srivastava and Trujillo (2006)).

**Remark 2.1.** Observe that the Mittag-Leffler function is a fractional generalization of the standard exponential function $\exp(z)$; indeed $E_1(z) = \exp(z)$. It has been widely used to describe probability distributions appearing in finance and economics, anomalous diffusion, transport of charge carriers in semiconductors, and light propagation through random media (see, e.g. Piryatinska, Saichev and Woyczynski (2005), and Uchaikin and Zolotarev (1999)).

In view of (3-4), the interarrival time density for the fractional Poisson process can be easily shown to be

$$\psi_\nu(t) = \mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu), \hspace{1cm} t \geq 0,$$ \hspace{1cm} (5)

where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

is the generalized, two-parameter Mittag-Leffler function. Also, the above information automatically gives the p.d.f.

$$f_\nu^{(n)}(t) = \mu^n \nu \frac{t^{\nu n-1}}{(n-1)!} E^{(n)}_\nu(-\mu t^\nu),$$ \hspace{1cm} (6)
of the $n$-the arrival time, $A_n = T - 1 + \cdots + T_n$, because, obviously, its Laplace transform,

$$L\{f_n^\nu(t)\} = \frac{\mu^n}{(\mu + \lambda^\nu)^n}.$$  

As $\nu \to 1$, the above distribution converges to the classical Erlang distribution.

**Example 2.1.** For some values of $\nu$, the p.d.f. of the interarrrival times can be calculated more explicitly. In particular, consider

$$\psi_{1/2}(t) = \mu t^{1/2 - 1} E_{1/2,1/2} (-\mu t^{1/2}), \quad t \geq 0,$$

where

$$E_{1/2,1/2} (-z) = \sum_{n=0}^\infty \frac{(-z)^n}{\Gamma \left( \frac{1}{2} + \frac{1}{2} \right)} = \frac{1}{\sqrt{\pi}} - z E_{1/2,1} (-z). \quad (7)$$

Using the identity,

$$E_{1/2,1} (-z) = e^{z^2} \text{Erfc}(z),$$

where

$$\text{Erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du,$$

is the complementary error function, we obtain,

$$\psi_{1/2}(t) = \mu t^{1/2 - 1/2} \left( \frac{1}{\sqrt{\pi}} - \mu t^{1/2} e^{(\mu t^{1/2})^2} \text{Erfc}(\mu \sqrt{t}) \right)$$

$$= \frac{\mu}{\sqrt{\pi} t} - \mu^2 e^{\mu^2 t} \text{Erfc}(\mu \sqrt{t}), \quad t \geq 0. \quad (8)$$

In another approach to the study of $N^\nu(t)$, Laskin (2003) used the fractional Kolmogorov-Feller-type differential equation system

$$0 D^\nu P^\nu_n(t) = \mu [P^\nu_{n-1}(t) - P^\nu_n(t)] + \delta_{n0} \frac{t^{-\nu}}{\Gamma(1 - \nu)}, \quad n = 1, 2, \ldots, \quad (9)$$

to characterize the 1-D probability distributions $P^\nu_n(t) = \text{P} (N^\nu(t) = n)$. The solutions of the above system of equations (9) can be calculated to be

$$G^\nu(u,t) \equiv \text{E} u^{N^\nu(t)} = E^\nu (\mu t^{\nu} (u - 1)). \quad (10)$$

Hence, expanding $G^\nu(u,t)$ over $u$, and rearranging (10)), we find

$$P^\nu_n(t) = \frac{(-z)^n}{n!} \frac{d^n}{dz^n} E^\nu(z) \bigg|_{z=-\mu^\nu} = \frac{(\mu t^{\nu})^n}{n!} \sum_{k=0}^\infty \frac{(k+n)!}{k!} \frac{(-\mu t^{\nu})^k}{\Gamma(\nu (k + n) + 1)}. \quad (11)$$
Equivalently, one can show (see, Laskin (2003)) that the moment generating function (MGF) of the fractional Poisson process \( N_\nu(t) \) is of the form

\[
M_\nu(s, t) \equiv E e^{-s N_\nu(t)} = \sum_{m=0}^{\infty} \frac{[\mu t^\nu (e^{-s} - 1)]^m}{\Gamma(\nu m + 1)},
\]

which permits calculation (see, Table 1) of the fPp’s moments via the usual formula,

\[
E [N_\nu(t)]^k = (-1)^k \frac{\partial^k}{\partial s^k} M_\nu(s, t) \bigg|_{s=0}.
\]

Table 1: Properties of fPp compared with those of the Poisson process.

|                  | Poisson process \((\nu = 1)\) | Fractional Poisson Process \((\nu < 1)\) |
|------------------|---------------------------------|------------------------------------------|
| \( P_0(t) \)    | \( e^{-\mu t} \)               | \( E_\nu(-\mu t^\nu) \)                |
| \( \psi(t) \)   | \( \mu e^{-\mu t} \)           | \( \mu t^{\nu - 1} E_\nu,\nu(-\mu t^\nu) \) |
| \( P_n(t) \)    | \( \frac{(\mu)^n}{n!} e^{-\mu t} \) | \( \frac{(\mu^\nu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^\nu)^k}{\Gamma(\nu(k+n)+1)} \) |
| \( \mu N(t) \)  | \( \mu t \)                     | \( \frac{\mu^\nu}{\Gamma(\nu + 1)} \left\{ 1 + \frac{\mu^\nu}{\Gamma(\nu + 1)} \left[ \frac{\nu B(\nu, 1/2)}{2^{\nu-1}} - 1 \right] \right\}, \right. \)
| \( \sigma^2_N(t) \) | \( \mu t \)                     | \( B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \) |
| \( \mathbb{E} [N(t)]^k \) | \( \frac{\partial^k}{\partial s^k} \left. \exp \left[ \mu s - 1 \right] \right|_{s=0} \) | \( (-1)^k \frac{\partial^k}{\partial s^k} \sum_{m=0}^{\infty} \frac{[\mu^\nu (e^{-s} - 1)]^m}{\Gamma(m\nu + 1)} \bigg|_{s=0} \) |

More recently, Mainardi et al. (2004, 2005) provided an approach to fPp based on analysis of the survival probability function \( \Theta(t) = P(T > t) \). They have shown that \( \Theta(t) \) satisfies the fractional differential equation

\[
_0D_t^{\nu} \Theta(t) = -\mu \Theta(t), \quad t \geq 0, \quad \Theta(0^+) = 1,
\]

where

\[
_0D_t^{\nu} f(t) = \begin{cases} 
\frac{1}{\Gamma(1-\nu)} \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^\nu} d\tau, & 0 < \nu < 1; \\
\frac{d}{dt} f(t), & \nu = 1.
\end{cases}
\]
is the so-called Caputo derivative. Obviously, for the standard Poisson process with parameter \( \mu \), \( \Theta(t) \) satisfies the ordinary differential equation

\[
\frac{d}{dt} \Theta(t) = -\mu \Theta(t), \quad t \geq 0, \quad \Theta(0^+) = 1.
\]

Some characteristics of the classical and fractional Poisson processes are compared in Table 1, above.

### 3 Simulation of fPp interarrival times \( (T_i) \)

Simulation of the usual Poisson process is very easy and efficient because, given a random variable \( U \), uniformly distributed on \([0, 1]\), the random variable \( |\ln U|/\mu \) has the exponential distribution with parameter \( \mu \). With the interarrival time for fPp exhibiting a more complicated structure described in Section 2 the issue of an efficient simulation of fPp depends on finding a representation for the Mittag-Leffler function which is more computationally convenient than the series (4). Here, the critical observation is that the interarrival times \( T_i = T \), are equidistributed with the random variable,

\[
T' = \frac{|\ln U|^{1/\nu}}{\mu^{1/\nu}} S_\nu,
\]

where \( S_\nu \geq 0 \) is a completely asymmetric \( \nu \)-stable random variable (see, Appendix 1) with the p.d.f. \( g_\nu(s) \) possessing the Laplace transform

\[
\int_0^\infty g_\nu(s)e^{-\lambda s} \, ds = \exp(-\lambda^\nu).
\] (14)

The verification of the above statement is straightforward in view of (3) and the integral representation for the Mittag-Leffler function implied by (14); cf., e.g., Uchaikin and Zolotarev (1999):

\[
P(T' > t) = P\left(\frac{|\ln U|^{1/\nu}}{\mu^{1/\nu}} S_\nu > t\right)
= \int_0^1 P\left(S_\nu > \frac{t\mu^{1/\nu}}{(-\ln(1-u))^{1/\nu}}\right) \, du
= \int_0^\infty P\left(S_\nu > \frac{t}{\tau^{1/\nu}}\right) \mu e^{-\mu \tau} \, d\tau
= \int_0^\infty \left(\int_{t/\tau^{1/\nu}}^\infty g_\nu(s) \, ds\right) \mu e^{-\mu \tau} \, d\tau
= \int_0^\infty \int_0^{\nu/s^\nu} g_\nu(s) \mu e^{-\mu \tau} \, d\tau \, ds
= \int_0^\infty g_\nu(s) e^{-\mu \tau/s^\nu} \, ds = E_\nu(-\mu t^\nu) = P(T > t).
\]
Utilizing the well known Kanter (-Chambers-Mallows) algorithm (see, e.g., Kanter (1975)) we obtain the following corollary providing an algorithm for simulation of the fPp interarrival times:

**Corollary.** Let $U_1$, $U_2$, and $U_3$, be independent, and uniformly distributed in $[0,1]$. Then the fPp interarrival time

$$T = \frac{\ln U_1^{1/\nu} \sin(\nu \pi U_2) \sin((1 - \nu) \pi U_2)^{1/\nu-1}}{\mu^{1/\nu} \sin(\pi U_2)^{1/\nu} \ln U_3^{1/\nu-1}}. \quad (15)$$

A comparison of sample trajectories for the standard Poisson process and an fPp, with parameter $\nu = 1/2$, can be seen in Figure 1.

![Sample trajectories of: (a) standard Poisson process, (b) fPp with parameter $\nu = 1/2$.](image)

Figure 1: Sample trajectories of: (a) standard Poisson process, (b) fPp with parameter $\nu = 1/2$. 


4 Scaling limit for fractional Poisson distribution

For the standard Poisson process, \( N(t) = N_1(t) \), the central limit theorem and infinite divisibility of the Poisson distribution give us immediately the following Gaussian scaling limit of distributions: as \( \bar{n} = \mu t \to \infty \),

\[
\frac{N(t) - \bar{n}}{\sqrt{\bar{n}}} \Rightarrow N(0, 1).
\]

A more subtle, skew-normal approximation to the Poisson distribution is provided by the following formula: for \( n = 0, 1, 2, \ldots \),

\[
P(N(t) \leq n) \approx \Phi(z) - \frac{1}{6\sqrt{n}}(z^2 - 1)\phi(z),
\]

where \( z = \frac{(n + \frac{1}{2} - \bar{n})/\sqrt{n}}{\bar{n}/\sqrt{n}} \), and \( \Phi \) and \( \phi \) are standard normal c.d.f. and p.d.f., respectively. The above formula, used to calculate the probabilities \( P(m < N(t) \leq n) \) (including \( P(N(t) = n) \)), guarantees, uniformly over \( n, m \), errors not worse than \( 1/(20\bar{n}) \) (as opposed to errors of the order \( 1/\sqrt{\bar{n}} \) if the skewness correction term is dropped), see, e.g., Pitman (1993), p. 225.

Considering the case of the fPp, \( N_\nu(t) \), and introducing the standardized random variable

\[
Z_\nu = \frac{N_\nu(t)}{\bar{n}_\nu}, \quad \text{where} \quad \bar{n}_\nu = EN_\nu(t) = \frac{\mu t^\nu}{\Gamma(\nu + 1)},
\]

and substituting \( u = e^{-\lambda/\bar{n}_\nu} \) in (10), we get the Laplace transform

\[
Ee^{-\lambda Z_\nu} = E_\nu(\bar{n}_\nu \Gamma(\nu + 1)(e^{-\lambda/\bar{n}_\nu} - 1)), \quad \lambda > 0,
\]

which has, for large \( \bar{n}_\nu \) (i.e. large \( t \)) the asymptotics

\[
Ee^{-\lambda Z_\nu} \sim E_\nu(-\lambda'), \quad \lambda' = \lambda \Gamma(\nu + 1).
\]

Since,

\[
E_\nu(-\lambda') = \nu^{-1} \int_0^\infty \exp(-\lambda' x)g_\nu(x^{-1/\nu})x^{-1-1/\nu}dx
\]

\[
= \int_0^\infty e^{-\lambda z} \left\{ \frac{[\Gamma(\nu + 1)]^{1/\nu}}{\nu}g_\nu \left( \frac{z}{\Gamma(\nu + 1)} \right)^{-1/\nu} \right\} z^{-1-1/\nu}dz,
\]

where \( g_\nu(s) \) is the \( \nu \)-stable p.d.f., see Uchaikin and Zolotarev (1999), formula (6.9.8), the random variable \( Z_\nu \) has, for \( \bar{n}_\nu \to \infty \), a non-degenerate limit distribution with the p.d.f.

\[
f_\nu(z) = \left\{ \frac{[\Gamma(\nu + 1)]^{1/\nu}}{\nu}g_\nu \left( \frac{z}{\Gamma(\nu + 1)} \right)^{-1/\nu} \right\} z^{-1-1/\nu}, \quad \text{(16)}
\]
with moments
\[ \langle Z^k \rangle = \frac{[\Gamma(1 + \nu)]^k \Gamma(1 + k)}{\Gamma(1 + k\nu)}, \]
see Uchaikin (1999). Making use of the series expansion for \( g_\nu \), we obtain the series expansion
\[ f_\nu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k! \Gamma(1 - (k + 1)\nu)[\Gamma(\nu + 1)]^{k+1}}. \]
Note that
\[ f_\nu(0) = \frac{1}{\Gamma(1 + \nu)\Gamma(1 - \nu)} = \frac{\sin(\nu\pi)}{\nu\pi}. \]

It is also worth to note, that \( \langle Z^0 \rangle = 1, \langle Z^1 \rangle = 1 \) and \( \langle Z^2 \rangle = 2\nu\text{B}(\nu, 1 + \nu) \), so that the limit relative fluctuations is given by
\[ \delta_\nu \equiv \sigma_{N(t)}/\langle N \rangle = \sqrt{2\nu\text{B}(\nu, 1 + \nu)} - 1 = \begin{cases} 1, & \nu = 0, \\ \sqrt{\pi/2} - 1, & \nu = 1/2 \\ 0, & \nu = 1. \end{cases} \]

For \( \nu = 1/2 \), one can obtain an explicit expression for \( f_\nu(z) \):
\[ f_{1/2}(z) = \frac{2}{\pi} e^{-z^2/\pi}, \quad z \geq 0. \]

The above family of limiting distributions is plotted below.

5 Method of Moments

In this section we derive method-of-moments estimators for parameters \( \nu \), and \( \mu \), based on the first two moments of a transformed random variable \( T \). It is important to emphasize that the Hill (1975), Pickands (1975), and Haan and Resnick (1980) estimators can be used to estimate these parameters as well. However, the above estimators are only using a portion of the information contained in the data making them statistically less efficient. It is this drawback that motivates us to look for estimators that utilize, or even optimize the use, of all the available information in the data.

Recall that
\[ T \overset{d}{=} \frac{[\ln U]^{1/\nu}}{\mu^{1/\nu}} S(\nu), \quad (17) \]
where \( U \) has \( U(0, 1) \) distribution, \( S(\nu) \) is one-sided \( \alpha \)-stable, and the random variables \( U \) and \( S(\nu) \) are statistically independent. Since the first moment doesn’t exist, we consider the log-transformation of the original random variable \( T > 0 \).
Figure 2: Limiting distributions for $\nu = 0.1(0.1)0.9$ and 0.95.

The above formulation (17) implies that

$$\ln(T) \overset{d}{=} \ln \left( \frac{|\ln U|^{1/\nu}}{\mu^{1/\nu}} S(\nu) \right). \quad (18)$$

Simplifying (18), we get the equivalent expression

$$\ln(T) \overset{d}{=} \frac{1}{\nu} \ln \left( \frac{|\ln U|}{\mu} \right) + \ln(S(\nu)). \quad (19)$$

Taking the expectation of (19), we obtain the equality

$$E \ln(T) = \frac{1}{\nu} [E \ln(|\ln U|) - \ln(\mu)] + E \ln(S(\nu)). \quad (20)$$

Our task now is to obtain the first moments of the random variables $\ln(|\ln U|)$ and $\ln(S(\nu))$. We start by finding the distribution of the first. Let $Y = |\ln U| = -\ln U$. The random variable $Y$ has the distribution $e^{-y}$, $y > 0$. After the monotone transformation $X = \ln Y$, one easily shows that $X$ has the probability density function

$$f_X(x) = e^{x-e^x}, \quad x \in \mathbb{R}.$$ 

Thus, the first moment of $\ln(|\ln U|)$ can now be calculated as follows:

$$EX = \int_{\mathbb{R}} xe^{x-e^x} dx = \int_{\mathbb{R}^+} \ln(y)e^{-y} dy = -C. \quad (21)$$
where $C \cong 0.57721566490153286$ is the Euler’s constant, see, e.g., Boros and Moll (2004).

The next step is to find the expectation of $\ln(S(\nu))$. Zolotarev (1986), p. 213-220, shows that

$$E \ln(S(\nu)) = C \left( \frac{1}{\nu} - 1 \right). \tag{22}$$

When (21) and (22) are substituted into (20), the latter equality becomes

$$E \ln(T) = \frac{1}{\nu} \left( -C - \ln(\mu) \right) + C \left( \frac{1}{\nu} - 1 \right) = -\frac{\ln(\mu)}{\nu} - C. \tag{23}$$

From equation (23), we obtain

$$\mu = \exp(-\nu[E \ln(T) + C]). \tag{24}$$

Alternatively, the second moment of the log-transformed random variable $T$ is given by

$$E [\ln(T)]^2 = E \left[ \ln \left( \left( \frac{|\ln U|}{\mu} \right)^{1/\nu} S(\nu) \right) \right]^2 = E \left[ \frac{1}{\nu} \ln \left( \frac{|\ln U|}{\mu} \right) + \ln(S(\nu)) \right]^2. \tag{25}$$

Expanding the right-hand side (RHS) of (25), we obtain the equality

$$E [\ln(T)]^2 = E \left[ \frac{1}{\nu^2} (\ln(|\ln U|) - \ln(\mu))^2 + \frac{2}{\nu} \ln \left( \frac{|\ln U|}{\mu} \right) \ln(S(\nu)) + \ln(S(\nu))^2 \right]$$

$$= E \left[ \frac{1}{\nu^2} (\ln(|\ln U|) - \ln(\mu))^2 + \frac{2}{\nu} \ln(|\ln U|) \ln(S(\nu)) \right. \right.$$

$$\left. \left. - \frac{2}{\nu} \ln(\mu) \ln(S(\nu)) + \ln(S(\nu))^2 \right]\right. \tag{26}$$

$$= E \left( \frac{1}{\nu^2} \left\{ |\ln U| \right\}^2 - 2 \ln(\mu) \ln(|\ln U|) + \ln(\mu)^2 \right)$$

$$+ \frac{2}{\nu} \ln(|\ln U|) \ln(S(\nu)) - \frac{2}{\nu} \ln(\mu) \ln(S(\nu)) + \ln(S(\nu))^2 \right) \right).$$

From another integral formula involving the Euler constant, we can easily obtain

$$E [\ln(|\ln U|)]^2 = \mathbb{E} X^2 = \int_{\mathbb{R}} x^2 e^{-x} \, dx = \int_{\mathbb{R}^+} \ln(y)^2 e^{-y} \, dy = C^2 + \frac{\pi^2}{6}. \tag{27}$$
Note that $\pi^2/6 = \zeta(2)$ is the value of the Riemann zeta function at the point 2. Furthermore, Bening et al. (2004) reveals that
\[
\mathbb{E}[\ln(S(\nu))]^2 = \left(\frac{1}{\nu} - 1\right)^2 C^2 + \frac{\pi^2}{6} \left(\frac{1}{\nu^2} - 1\right). \tag{28}
\]

Using equation (27), equation (28), and the statistical independence of two random variables $U$ and $S(\nu)$, equation (26) becomes
\[
\mathbb{E}[\ln(T)]^2 = \frac{\pi^2}{3\nu^2} + \left(\frac{\ln(\mu)}{\nu^2}\right)^2 + C^2 - \frac{\pi^2}{6} + \frac{2C\ln(\mu)}{\nu}. \tag{29}
\]

From (24),
\[\ln(\mu) = -\nu[\mathbb{E}\ln(T) + C]. \tag{30}\]

Substituting (30) into (29) and simplifying the resulting expression, we get
\[
\mathbb{E}[\ln(T)]^2 - [\mathbb{E}\ln(T)]^2 + \frac{\pi^2}{6} = \frac{\pi^2}{3\nu^2}.
\]

This implies that
\[\nu^2 = \frac{\pi^2}{3(\sigma^2_{\ln T} + \pi^2/6)}.\]

Thus, the method-of-moments estimator for $\nu$ is
\[\hat{\nu} = \frac{\pi}{\sqrt{3(\sigma^2_{\ln T} + \pi^2/6)}} \tag{31}\]

and, similarly, from (24),
\[\hat{\mu} = \exp \left( -\hat{\nu} (\mathbb{E}\ln(T) + C) \right) = \exp \left( -\hat{\nu} (\hat{\mu_{\ln T}} + C) \right) \tag{32}\]

is an estimator for $\mu$.

6 Asymptotic Normality of the Estimators $\hat{\nu}$ and $\hat{\mu}$

We will show asymptotic normality of the above estimators for $\nu$ and $\mu$. The discussion in Section 5 implies that
\[
\mathbb{E}\ln(\ln U) = -C, \quad \text{and} \quad \mathbb{E}[\ln(\ln U)]^2 = C^2 + \frac{\pi^2}{6}. \]
A further calculation using Mathematica shows that
\[
\mathbb{E} \left[ \ln(\left| \ln U \right|) \right]^3 = -C^3 - \frac{C\pi^2}{2} - 2\zeta(3)
\]
and
\[
\mathbb{E} \left[ \ln(\left| \ln U \right|) \right]^4 = C^2 (C^2 + \pi^2) + \frac{3\pi^4}{20} + 8C\zeta(3).
\]
Additionally, we have
\[
\mathbb{E} \ln(S(\nu)) = C \left( \frac{1}{\nu} - 1 \right),
\]
and
\[
\mathbb{E} \left[ \ln(S(\nu)) \right]^2 = \left( \frac{1}{\nu} - 1 \right)^2 C^2 + \frac{\pi^2}{6} \left( \frac{1}{\nu^2} - 1 \right).
\]
Reference [Zolotarev(1986)] provides the following formula for higher log-moments of \( S(\nu) \):
\[
\mathbb{E} \left( \ln |S(\nu)| \right)^k = \left( \frac{d^k w_\nu(s)}{ds^k} \right) \bigg|_{s=0},
\]
where
\[
w_\nu(s) = \frac{\Gamma(1-s/\nu)}{\Gamma(1-s)}.
\]
To calculate these moments, we need to find the power series expansion of \( w_\nu(s) \). This turns out to be easier if we first expand
\[
\ln w_\nu(s) = \ln \Gamma(1-s/\nu) - \ln \Gamma(1-s)
\]
into a power series, see, Bening et al. (2004). Using the log-gamma expansion
\[
\ln \Gamma(1-\theta) = \mathbb{C} \theta + \sum_{k=2}^\infty \frac{\zeta(k)}{k} \theta^k,
\]
we get
\[
\ln w_\nu(s) = \mathbb{C} \left( \frac{1}{\nu} - 1 \right) s + \frac{\pi^2}{12} \left( \frac{1}{\nu^2} - 1 \right) s^2 + \frac{1}{3} \zeta(3) \left( \frac{1}{\nu^3} - 1 \right) s^3
\]
\[
+ \frac{1}{4} \zeta(4) \left( \frac{1}{\nu^4} - 1 \right) s^4 + \frac{1}{5} \zeta(5) \left( \frac{1}{\nu^5} - 1 \right) s^5 + O(s^6),
\]
and, hence,
\[
w_\nu(s) = 1 + \mathbb{C} \left( \frac{1}{\nu} - 1 \right) s + \left[ \frac{\pi^2}{12} \left( \frac{1}{\nu^2} - 1 \right) + \frac{1}{2} C^2 \left( \frac{1}{\nu^2} - 1 \right)^2 \right] s^2
\]
\[
+ \left[ \frac{1}{3} \zeta(3) \left( \frac{1}{\nu^3} - 1 \right) + \frac{1}{6} C^3 \left( \frac{1}{\nu} - 1 \right)^3 + \mathbb{C} \left( \frac{1}{\nu} - 1 \right) \left( \frac{1}{\nu^2} - 1 \right) \frac{\pi^2}{12} \right] s^3
\]
\[
+ \frac{1}{1440} \left[ \left( \frac{1}{\nu^3} - \frac{1}{\nu^4} \right) \left( 60C^4(\nu - 1)^3 - 60C^2\pi^2(\nu - 1)^2(1 + \nu) \right.ight.
\]
\[
+ \pi^4(\nu - 3)(1 + \nu)(3 + \nu) + 480C(\nu^3 - 1)\zeta(3)) \right] s^4 + O(s^5).
\]
The $k$th log-moment of $S(\nu)$ is simply the coefficient of the term $s^k/k!$ in the above power series expansion (can also be obtained via $(d^k w_\nu(s)/ds^k)|_{s=0}$). In particular, the third and fourth log-moments can be shown to be

$$\mathbb{E}[\ln(S(\nu))]^3 = \frac{-2(\nu - 1)C^3 + C\pi^2(\nu - 1)^2(1 + \nu) - 4(\nu^3 - 1)\zeta(3)}{2\nu^3},$$

and

$$\mathbb{E}[\ln(S(\nu))]^4 = \frac{1}{60} \left[ \left( \frac{1}{\nu^3} - \frac{1}{\nu^4} \right) \left( 60C^4(\nu - 1)^3 - 60C^2\pi^2(\nu - 1)^2(1 + \nu) + \pi^4(\nu - 3)(1 + \nu)(3 + \nu) + 480C(\nu^3 - 1)\zeta(3) \right) \right],$$

respectively. In addition, the above derivations show that

$$\mu_{\ln T} = -\left( \frac{\ln(\mu)}{\nu} + C \right) \quad \text{and} \quad \sigma_{\ln T}^2 = \frac{\pi^2}{3} \left( \frac{1}{\nu^2} - \frac{1}{2} \right).$$

The second-, third-, and fourth-order moments of $\ln T$ are

$$\mathbb{E}(\ln T)^2 = C^2 - \frac{\pi^2(\nu^2 - 2)}{6\nu^2} + \frac{\ln(\mu)[2C\nu + \ln(\mu)]}{\nu^2},$$

$$\mathbb{E}(\ln T)^3 = -\left[ C\nu + \ln(\mu) \right] \left[ 2C^2\nu^2 - \pi^2(\nu^2 - 2) + 2\ln(\mu)(2C\nu + \ln(\mu)) \right] - 2\zeta(3),$$

and

$$\mathbb{E}(\ln T)^4 = \frac{1}{60\nu^4} \left\{ 60C^4\nu^4 - 60C^2\pi^2(\nu^2 - 2) + \pi^4(28 - 20\nu^2 + \nu^4) \right. \right.$$ 

$$\left. + 60\ln(\mu)[2C\nu + \ln(\mu)] \left( 2C^2\nu^2 - \pi^2(\nu^2 - 2) + 2C\nu\ln(\mu) + [\ln(\mu)]^2 \right) \right.$$ 

$$\left. + 480\nu^3[C\nu + \ln(\mu)]\zeta(3) \right\},$$

respectively. We now calculate higher-order central moments of the random variable $\ln T$. After a tedious algebraic manipulation, we get

$$\mu_3 = \mathbb{E}(\ln T - \mu_{\ln T})^3$$

$$= \mathbb{E} \left\{ \frac{1}{\nu} \ln \left( \frac{|\ln U|}{\mu} \right) + \ln(S(\nu)) - \left[ -\left( \frac{\ln(\mu)}{\nu} + C \right) \right] \right\}^3$$

$$= -2\zeta(3)$$
\[
\mu_4 = \mathbb{E} (\ln T - \mu_{\ln T})^4 = \frac{\pi^4(28 - 20\nu^2 + \nu^4)}{60\nu^4}.
\]

If we let
\[
\ln \frac{T}{\mu_{\ln T}} = \frac{1}{n} \sum_{j=1}^{n} \ln T_j \quad \text{and} \quad \sigma_{\ln T}^2 = \frac{1}{n} \sum_{j=1}^{n} (\ln T_j - \ln \frac{T}{\mu_{\ln T}})^2
\]
then, the standard 2-D Central Limit Theorem implies, as \(n \to \infty\), the following convergence in distribution:
\[
\sqrt{n} \left( \ln \frac{T_n}{\mu_{\ln T}} - \mu_{\ln T} \right) \xrightarrow{d} N\left( \begin{pmatrix} 0 \\ \sigma_{\ln T}^2 - \sigma_{\ln T}^4 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_{\ln T}^4 \\ \mu_3 & \mu_4 - \sigma_{\ln T}^4 \end{pmatrix} \right),
\]
where \(N(\mu, \Sigma)\) represents the 2-D normal distribution with mean \(\mu\), and covariance matrix \(\Sigma\), and \(\mu_3, \mu_4, \text{ and } \sigma_{\ln T}^2\) are defined above.

Now, to show the asymptotic normality of the estimators \(\hat{\nu}\) and \(\hat{\mu}\), we will rely on Cramer’s Theorem (see, e.g., Ferguson (1996), p. 45, which we are stating below without proof.

**Theorem (Cramer).** Let \(g\) be a mapping \(g: \mathbb{R}^d \to \mathbb{R}^k\) such that \(g(x)\) is continuous in a neighborhood of \(\theta \in \mathbb{R}^d\). If \(X_n\) is a sequence of \(d\)-dimensional random vectors such that
\[
\sqrt{n} (X_n - \theta) \xrightarrow{d} X,
\]
then
\[
\sqrt{n} (g(X_n) - g(\theta)) \xrightarrow{d} g(\theta)X.
\]
In particular, if \(\sqrt{n} (X_n - \theta) \xrightarrow{d} N(0, \Sigma)\) where \(\Sigma\) is a \(d \times d\) covariance matrix, then
\[
\sqrt{n} (g(X_n) - g(\theta)) \xrightarrow{d} N(0, g(\theta)\Sigma g(\theta)^T).
\]

Indeed, for \(\sigma_{\ln T}^2 > 0\), Cramer’s Theorem shows that
\[
\sqrt{n} (\hat{\nu} - \nu) \xrightarrow{d} N\left( 0, \frac{18\pi^2}{(6\sigma_{\ln T}^2 + \pi^2)^3} \left( \mu_4 - \sigma_{\ln T}^4 \right) \right) \xrightarrow{d} N\left( 0, \frac{18\pi^2}{(6\sigma_{\ln T}^2 + \pi^2)^3} \left( \mu_4 - \sigma_{\ln T}^4 \right) \right) \xrightarrow{d} N\left( 0, \frac{\pi^6}{5} \left( \pi^4 \frac{32 - 20\nu^2 - \nu^4}{90\nu^4} \right) \right) \xrightarrow{d} N\left( 0, \frac{\pi^6}{5} \left( \pi^4 \frac{32 - 20\nu^2 - \nu^4}{90\nu^4} \right) \right) \xrightarrow{d} N\left( 0, \frac{\nu^2}{40} \left( \pi^4 \frac{32 - 20\nu^2 - \nu^4}{90\nu^4} \right) \right),
\]
where the last line of the preceding simplification is obtained by substituting \(\sigma_{\ln T}^2 = \frac{\pi^2}{3} \left( \frac{1}{\nu^2} - \frac{1}{2} \right)\) .
Similarly, the estimator $\hat{\mu}$ can be rewritten as

$$\hat{\mu} = \exp \left( -\hat{\nu} (\mu_{\text{in}T} + C) \right) = \exp \left( -\frac{\pi}{\sqrt{3(\sigma^2_{\text{in}T} + \pi^2/6)}} (\mu_{\text{in}T} + C) \right).$$

Let

$$g(\mu_{\text{in}T}, \sigma^2_{\text{in}T}) = \exp \left( -\frac{\pi}{\sqrt{3(\sigma^2_{\text{in}T} + \pi^2/6)}} (\mu_{\text{in}T} + C) \right).$$

The gradient then becomes

$$\dot{g}(\mu_{\text{in}T}, \sigma^2_{\text{in}T}) = \alpha \frac{\pi}{\sqrt{\pi^2 + 6\sigma^2_{\text{in}T}}} \exp \left( -\frac{\sqrt{2\pi}(\mu_{\text{in}T} + C)}{\sqrt{\pi^2 + 6\sigma^2_{\text{in}T}}} \right).$$

By Cramer’s theorem,

$$\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} N(0, \sigma^2_{\alpha}),$$

where

$$\sigma^2_{\alpha} = \dot{g}(\mu_{\text{in}T}, \sigma^2_{\text{in}T})^T \begin{pmatrix} \sigma^2_{\text{in}T} \\ \mu_3 \\ \mu_4 - \sigma^4_{\text{in}T} \end{pmatrix} \dot{g}(\mu_{\text{in}T}, \sigma^2_{\text{in}T}) = \frac{\mu^2\left[20\pi^4(2 - \nu^2) - 3\pi^2\nu^4 + 20\nu^2 - 32)(\ln \mu)^2 \right]}{120\pi^2} - \frac{720\nu^3(\ln \mu)\zeta(3)}{120\pi^2}. \quad (33)$$

Therefore, we have shown that our method-of-moments estimators are asymptotically normal (asymptotically unbiased). We can now approximate the $(1 - \varepsilon)100\%$ confidence interval for $\mu$, and $\nu$ as follows:

$$\hat{\mu} \pm z_{\varepsilon/2} \sqrt{\frac{\mu^2\left[20\pi^4(2 - \nu^2) - 3\pi^2(\hat{\nu}^4 + 20\hat{\nu}^2 - 32)(\ln \hat{\mu})^2 - 720\hat{\nu}^3(\ln \hat{\mu})\zeta(3) \right]}{120\pi^2 n}},$$

and

$$\hat{\nu} \pm z_{\varepsilon/2} \sqrt{\frac{\hat{\nu}^2(32 - 20\hat{\nu}^2 - \hat{\nu}^4)}{40n}},$$

where the tail quantile $z_{\varepsilon/2}$ is defined by the equality $P(Z > z_{\varepsilon/2}) = \varepsilon/2$, with $Z \xrightarrow{d} N(0, 1)$. 
7 Testing MoM estimators on simulated data

In this section we computationally compare and test the MoM estimators for $\nu$ and $\mu$ obtained in Section 5 using the mean absolute deviation (MAD) from the true values of our parameters, and the square root of the mean squared error (MSE), as our criteria. Recall our method-of-moment estimators for the fractional order $\nu$ (31) and the intensity rate $\mu$ (32):

$$\hat{\nu}_{\text{mm}} = \frac{\pi}{\sqrt{3\left(\sigma_{\ln T}^2 + \pi^2/6\right)}}$$

and

$$\hat{\mu}_{\text{mm}} = \exp\left(-\hat{\nu} \left(\hat{E}\ln(T) + \mathbb{C}\right)\right) = \exp\left(-\hat{\nu} \left(\hat{\mu}\ln(T) + \mathbb{C}\right)\right).$$

7.1 Simulated fPp

We generate $n = 100$ samples of the fPp jump times with sample sizes $N=100$, 1,000, and 10,000. We then calculate the estimates $\hat{\nu}$ and $\hat{\mu}$ for each of the $n$ samples, and then average them to obtain the means $\overline{\nu}$ and $\overline{\mu}$. These values are shown in the tables below together with their MAD and $\sqrt{\text{MSE}}$. The fPp data were simulated for four different pairs of values of $\mu$’s and $\nu$’s. The tables show that, for the sample sizes $N = 10^4$, the relative fluctuations the estimates of $\nu$, and $\mu$, are all below 5 percent. However, judging this performance one must remember that in many practical applications, such as network traffic data, the typical sample sizes $N$ are of the order of millions, or more.

Furthermore, Tables 2-5 strongly suggest that our method-of-moments estimators are asymptotically unbiased; they did fairly well in our simulations and could be regarded as reasonable starting values for other iterative estimation procedures.

| Table 2: Mean estimates of and dispersions from the true parameter for a simulated fPp data with $(\nu, \mu) = (0.9, 10)$. |
|---|---|---|---|---|---|---|---|
| & $N=100$ & $N=1,000$ & $N=10,000$ |
| | Mean & MAD & $\sqrt{\text{MSE}}$ & Mean & MAD & $\sqrt{\text{MSE}}$ & Mean & MAD & $\sqrt{\text{MSE}}$ |
| $\hat{\nu}_{\text{mm}}$ | .9027 | .0449 | .0556 | .9008 | .0128 | .0141 | .9012 | .0045 | .0056 |
| $\hat{\mu}_{\text{mm}}$ | 10.06 | 1.289 | 1.649 | 10.05 | 1.4130 | 1.5130 | 10.05 | 1.386 | 1.683 |
Table 3: Mean estimates of and dispersions from the true parameter for a simulated fPp data with \((\nu, \mu) = (0.3, 1)\).

|        | N=100  |          | N=1,000 |          | N= 10,000 |          |
|--------|--------|----------|---------|----------|-----------|----------|
|        | Mean   | MAD      | \(\sqrt{\text{MSE}}\) | Mean   | MAD      | \(\sqrt{\text{MSE}}\) | Mean   | MAD      | \(\sqrt{\text{MSE}}\) |
| \(\hat{\nu}_{mm}\) | .3048  | .0233    | .0279   | .3001  | .0059    | .0073   | .3004  | .0021    | .0025   |
| \(\hat{\mu}_{mm}\) | 1.025  | .1403    | .1789   | 1.009  | .0473    | .0616   | .9998  | .0137    | .0179   |

Table 4: Mean estimates of and dispersions from the true parameter for a simulated fPp data with \((\nu, \mu) = (0.2, 100)\).

|        | N=100  |          | N=1,000 |          | N= 10,000 |          |
|--------|--------|----------|---------|----------|-----------|----------|
|        | Mean   | MAD      | \(\sqrt{\text{MSE}}\) | Mean   | MAD      | \(\sqrt{\text{MSE}}\) | Mean   | MAD      | \(\sqrt{\text{MSE}}\) |
| \(\hat{\nu}_{mm}\) | .2062  | .0159    | .0197   | .2008  | .0041    | .0054   | .1999  | .0013    | .0017   |
| \(\hat{\mu}_{mm}\) | 127.9  | 47.87    | 70.94   | 102.3  | 10.13    | 13.42   | 100.2  | 3.599    | 4.519   |

Table 5: Mean estimates of and dispersions from the true parameter for a simulated fPp data with \((\nu, \mu) = (0.6, 1000)\).

|        | N=100  |          | N=1,000 |          | N= 10,000 |          |
|--------|--------|----------|---------|----------|-----------|----------|
|        | Mean   | MAD      | \(\sqrt{\text{MSE}}\) | Mean   | MAD      | \(\sqrt{\text{MSE}}\) | Mean   | MAD      | \(\sqrt{\text{MSE}}\) |
| \(\hat{\nu}_{mm}\) | .6023  | .0378    | .0462   | .5999  | .0119    | .0141   | .5998  | .0034    | .0042   |
| \(\hat{\mu}_{mm}\) | 1226  | 531.7    | 758.8   | 1019  | 143.0    | 189.0   | 997.4  | 38.68    | 48.56   |
In Section 6 we have also derived the asymptotic probability distributions of our estimators which give the following 95%-confidence intervals for different values of the parameters to be estimated. Not surprisingly, they turned out to be much tighter than the bootstrap ones which were calculated using the built-in function in R, see, e.g. DiCiccio and Efron (1996). They were also better centered around the true values of the parameters. For the “Average” column of the tables shown below we simulated 100 sets of sample size \( N \) and averaged the lower and upper 95% confidence bounds calculated from the expressions obtained in Section 6. For the “Bootstrap” column we simulated 100 bootstrap replicates using the basic nonparametric bootstrap CI procedure. To see the asymptotic behavior of the confidence intervals for larger \( N \), our tables are given for \( N=10,000, 100,000, \) and \( 1,000,000 \).

Table 6: 95% CI’s for a simulated fPp data with \((\nu, \mu) = (0.9, 10)\).

| \( N \) | Average | Bootstrap |
|---------|---------|-----------|
| 10,000  | (.8896, .9113) (.8824, .9107) | (.8967, .92036) (.8926, .9004) | (.8988, .9010) (.8987, .9008) |
| 100,000 | (9.668, 10.31) (9.563, 10.19) | (9.900, 10.11) (9.849, 10.059) | (9.965, 10.03) (9.948, 10.01) |
| 1,000,000 | | | |

Table 7: 95% CI’s for a simulated fPp data with \((\nu, \mu) = (0.3, 1)\).

| \( N \) | Average | Bootstrap |
|---------|---------|-----------|
| 10,000  | (.2947, .3049) (.2945, .3042) | (.2985, .3017) (.2988, .3020) | (.2994, .3004) (.2994, .3004) |
| 100,000 | (.9657, 1.035) (.9880, 1.061) | (.9886, 1.010) (.9970, 1.022) | (.9964, 1.003) (.9970, 1.005) |
| 1,000,000 | | | |

Table 8: 95% CI’s for a simulated fPp data with \((\nu, \mu) = (0.2, 100)\).

| \( N \) | Average | Bootstrap |
|---------|---------|-----------|
| 10,000  | (.1966, .2035) (.1995, .2052) | (.1988, .2010) (.1984, .2006) | (.1997, .2003) (.1998, .2006) |
| 100,000 | (91.48, 108.9) (94.60, 112.9) | (97.11, 102.6) (96.79, 102.2) | (99.15, 100.9) (99.6, 101.3) |
| 1,000,000 | | | |

Table 9: 95% CI’s for a simulated fPp data with \((\nu, \mu) = (0.6, 1000)\).

| \( N \) | Average | Bootstrap |
|---------|---------|-----------|
| 10,000  | (.5906, .6091) (.5847, .6016) | (.5968, .6026) (.5985, .6031) | (.5990, .6008) (.5985, .6003) |
| 100,000 | (892.2, 1111) (840.2, 1028) | (962.9, 1031) (960, 1036) | (988.4, 1010) (982.5, 1002) |
| 1,000,000 | | | |
8 Concluding remarks

Our analysis shows that, in comparison to the standard Poisson process, the fractional Poisson process offers more modeling flexibility and the ability to accommodate some clumping (burstiness) in the set of the jump points of their sample path, see Fig. 1. We have also succeeded in computing the limiting distributions of the scaled $n$th arrival time for the fPp as well as the limiting distribution of $Z = N(t)/E[N(t)]$ for fPp. Lastly, we were able to find asymptotically normal estimators of the parameters of the fractional Poisson process. The role of $\alpha$-stable densities turned out to be critical in analyzing the theoretical and numerical properties of fPp.

A number of interesting issues remain to be investigated including an extension of our model to the fractional order $1 < \nu < 2$, and to fractional Poisson fields. The nonstationary fPp models permitting nonconstant intensity rates would be also of obvious interest. To the best of our knowledge, the multiscaling property and long-range dependence of fPp has not been investigated either. Application of the above theory to model real physical phenomena, such as network traffic, particle streams, economic “events”, is in progress.

Appendix A. $\alpha^+$ stable densities

The $\alpha^+$-density, or one-sided alpha-stable distribution, denoted by $g^{(\alpha)}(t)$ is determined by its Laplace transform as follows, see, e.g. Samorodnitsky and Taqqu (1994), and Uchaikin and Zolotariev (1999):

$$\{Lg^{(\alpha)}(t)\}(\lambda) \equiv \tilde{g}^{(\alpha)}(\lambda) \equiv \int_0^\infty g^{(\alpha)}(t)e^{-\lambda t}dt = e^{-\lambda^\alpha}. \quad (A.1)$$

It is equal to 0 on the negative halfline, including the origin, positive on the positive halfline, and satisfies the normalization condition

$$\int_0^\infty g^{(\alpha)}(t)dt = 1.$$

The term “stable” means that these densities belong to the class of the L’evy stable laws: the convolution of two $\alpha^+$-densities is again the $\alpha^+$-density (up to a scale factor):

$$\int_0^t g^{(\alpha)}(t-t')g^{(\alpha)}(t')dt' = 2^{1/\alpha}g^{(\alpha)}(2^{-1/\alpha}t).$$

This is easily seen in terms of Laplace transforms:

$$\tilde{g}^{(\alpha)}(\lambda)\tilde{g}^{(\alpha)}(\lambda) = \tilde{g}^{(\alpha)}(2^{1/\alpha}\lambda).$$
Their role in the non-Gaussian central limit theorem is crucial: if $T_1, T_2, \ldots, T_n$ are independent and identically distributed random variables with with slowly decaying tail probabilities $P(T_j > t) \sim at^{-\alpha}$, $t \to \infty$, then the probability density of their sum is, asymptotically, as $n \to \infty$,

$$f_{\sum T_j}(t) \sim [a\Gamma(1 - \alpha)]^{1/\alpha} g^{(\alpha)} \left([a\Gamma(1 - \alpha)]^{1/\alpha} t\right)$$

A few additional important properties of these densities are worth mentioning:

(i) If $\alpha \to 1$, then $g^{(\alpha)}(t) \to \delta(t - 1)$;

(ii) Moments of the $\alpha^+$ densities can be explicitly calculated:

$$\int_0^\infty g^{(\alpha)}(t)t^\nu dt = \begin{cases} \Gamma(1 - \nu/\alpha)/\Gamma(1 - \nu), & -\infty < \nu < \alpha; \\ \infty, & \nu \geq \alpha, \end{cases} \quad (A.2)$$

(iii) For $\alpha = 1/2$ the density can be written out explicitly,

$$g^{(1/2)}(t) = \frac{1}{2\sqrt{\pi}}t^{-3/2}\exp[-1/(4t)], \quad t > 0, \quad (A.3)$$

(iv) For numerical calculations, the following integral formula is convenient:

$$g^{(\alpha)}(t) = \frac{\alpha t^{1/(\alpha - 1)}}{\pi(1 - \alpha)} \int_{-\pi/2}^{\pi/2} \exp \left\{ -t^{\alpha/(\alpha - 1)}U(\phi; \alpha) \right\} U(\phi; \alpha) d\phi, \quad (A.4)$$
where

\[ U(\phi; \alpha) = \left[ \frac{\sin(\alpha(\phi + \pi/2))}{\cos \phi} \right]^{\alpha/(\alpha-1)} \frac{\cos ((\alpha-1)\phi + \alpha \pi/2)}{\cos \phi}; \]

(v) The following asymptotic approximation may be obtained by the saddle-point method:

\[ g^{(\alpha)}(t) \sim \frac{1}{\sqrt{2\pi(1-\alpha)\alpha}} (t/\alpha)^{(\alpha-2)/(2-2\alpha)} \exp[-(1-\alpha)(t/\alpha)^{-\alpha/(1-\alpha)}], \quad t \to 0. \]

Results of numerical calculations, using (A.3), for \( \alpha = 1/2 \), and (A.6), for all other values of \( \alpha \), are shown in Fig. 3. For a complete discussion of \( \alpha \)-stable distributions, see the two monographs cited at the beginning of this Appendix.

Appendix B. Alternative fPp

It is worth mentioning that there exists another fractional generalization of the Poisson process based on the analogy with the fractional Brownian motion. Instead of the stochastic differential equation

\[ \frac{d^\nu B_\nu}{dt^\nu} = W(t), \]

where \( W(t) \) is a Gaussian white noise, we can consider the equation

\[ \frac{d^\nu Y_\nu}{dt^\nu} = X(t), \tag{A.5} \]

where the random function \( X(t) \) denotes the standard Poisson flow

\[ X(t) = \sum_{j=1}^{\infty} \delta(t - T^{(j)}), \]

with \( T^{(j)} = T_1 + T_2 + \ldots + T_j \), and \( T_1, T_2, \ldots, T_j \) being independent random variables with common density

\[ \psi(t) = \mu e^{-\mu t}, \quad t \geq 0, \; \mu > 0. \]

Integrating the stochastic fractional differential equation (A.5) yields, see, e.g., Kilbas, Srivastava and Trujillo (2006),

\[ Y_\nu(t) = \frac{1}{\Gamma(\nu)} \int_0^t X(\tau)d\tau = \frac{1}{\Gamma(\nu)} \sum_{j=1}^{N(t)} \int_0^t \delta(\tau - T^{(j)})d\tau = \frac{1}{\Gamma(\nu)} \frac{1}{(t - T^{(j)})^{1-\nu}}. \]

It is easy to see that, for \( \nu = 1 \), the process becomes the standard Poisson process. The stochastic process \( Y_\nu \) can be interpreted as a signal generated by the Poisson flow of pulses, each of which giving the contribution

\[ A(t - T^{(j)}) = \frac{1}{\Gamma(\nu)(t - T^{(j)})^{1-\nu}}. \tag{A.6} \]
It is also well known that, conditional on \( N(t) = n \), the unordered random times \( T^{(1)}, T^{(2)}, \ldots, T^{(n)} \) at which events occur, are distributed independently and uniformly in the interval \((0, t)\). Therefore,

\[
Y_\nu(t) |_{N(t)=n} = \sum_{j=1}^{n} A_j,
\]

where \( A_j \) is determined by equation (A.6). Now,

\[
P(A_j > y) = P \left( \Gamma(\nu)(t - T^{(j)})^{1-\nu} < y^{-1} \right)
= P \left( t - T^{(j)} < \left[ \Gamma(\nu)y \right]^{-1/(1-\nu)} \right)
= P \left( T^{(j)} > t - \left[ \Gamma(\nu)y \right]^{-1/(1-\nu)} \right)
= P \left( T^{(j)} < \left[ \Gamma(\nu)y \right]^{-1/(1-\nu)} \right)
= \frac{1}{t \left[ \Gamma(\nu)y \right]^{1/(1-\nu)}}.
\]

Because \( \nu > 0 \), the expectation of \( A_j \) exists, and according to the law of large numbers, in this model the limit distribution of the scaled random variable \( Z \) (defined as in Section 4) has the degenerate limit distribution \( f_\nu(z) = \delta(z - 1) \). We will discuss statistical estimation procedures for this model in another paper.

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