A symplectic covariant formulation of special Kähler geometry in superconformal calculus.

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Abstract

We present a formulation of the coupling of vector multiplets to $N = 2$ supergravity which is symplectic covariant (and thus is not based on a prepotential) and uses superconformal tensor calculus. We do not start from an action, but from the combination of the generalized Bianchi identities of the vector multiplets in superspace, a symplectic definition of special Kähler geometry, and the supersymmetric partners of the corresponding constraints. These involve the breaking to super-Poincaré symmetry, and lead to on-shell vector multiplets.

This symplectic approach gives the framework to formulate vector multiplet couplings using a weaker defining constraint for special Kähler geometry, which is an extension of older definitions of special Kähler manifolds for some cases with only one vector multiplet.

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1 Introduction

The understanding of the most general coupling of vector multiplets in $N = 2$ supersymmetry or supergravity is important in very different contexts. It is used in the context of the low-energy effective actions of supersymmetric gauge theories and their coupling to hypermultiplets [1]. It is also an essential element in the compactification of string theory on Calabi–Yau manifolds [2]. This general coupling has been studied for the supergravity case in [3, 4] and has been given the name special geometry [5]. The similar coupling in rigid supersymmetry was obtained in [6] and is referred to as ‘rigid special geometry’.

Historically, the coupling of several $N = 2$ matter multiplets to $N = 2$ supergravity in four dimensions was found using superconformal tensor calculus [7, 8, 9]. Superconformal actions are built with representations of a larger algebra, the $N = 2$ superconformal one. The residual symmetries are broken by introducing two compensating multiplets (one vector multiplet to give rise to the graviphoton in the Poincaré gravity multiplet and a hyper-, a linear or a nonlinear multiplet) to end up with an $N = 2$ Poincaré supergravity theory coupled to $N = 2$ matter multiplets. Reviews on superconformal tensor calculus can be found in [10]. The superconformal construction clarifies the origin of many terms in the action. Here we will confine ourselves to the coupling of vector multiplets to supergravity, where after breaking the superconformal symmetry, the complex scalars of the vector multiplets form a special Kähler manifold.

A prepotential, a holomorphic function of second order, was an essential ingredient to construct the theory. In [11, 12], other approaches were used to describe the coupling of vector multiplets to supergravity.

Electric–magnetic duality transformations in four dimensions manifest themselves by symplectic transformations [14]. Symplectic transformations in a special Kähler manifold have been studied in [4, 15]. The duality symmetry is not a symmetry of the complete action, but only of the equations of motion. In particular, the prepotential is not an invariant of the symplectic transformations. On the other hand, for a coordinate-free formulation of special geometry [11], the symplectic symmetry is an essential ingredient. The symplectic set-up also clarified the link to Calabi–Yau manifolds [4]. In [16], vector multiplet couplings to supergravity were constructed for which no prepotential existed, by performing a symplectic transformation of an action based on a prepotential. The resulting action was thus not based on a prepotential. The first and main purpose of this paper is to obtain a symplectic covariant formulation of the coupling of vector multiplets to $N = 2$ supergravity which at the same time uses su-
perconformal tensor calculus. In particular, it should thus contain the coupling of \cite{16}. For obtaining an action in superconformal tensor calculus, one needs a prepotential, and hence to give up the symplectic covariance. The combination of superconformal and symplectic covariance will, however, be possible if we only construct equations of motions without an action.

The various possible actions and geometric formulations were compared in \cite{17}, and one has arrived at a new definition of special geometry (also the definition for the case of rigid supersymmetry is given there). Remarkably, it was also noticed that one part of the definition, expressed by differential constraints, can be formulated in two different ways. These two forms are equivalent when more than one vector multiplet is coupled to supergravity, but inequivalent if only one vector multiplet is coupled. The presentation in \cite{17} contained the constraints such that for one vector multiplet the coupling known previously (e.g. from superconformal tensor calculus) is obtained. However, it was noted that another form of the constraints is possible, which is also symplectic covariant. Obvious physical arguments could not exclude the existence of hitherto unknown couplings of 1 vector multiplet to supergravity which obey the weaker constraint, and not the stronger one. The second aim of this paper is to show that such new couplings are indeed possible.

To be more explicit, let us repeat here one of the formulations of the 3 equivalent definitions of a special Kähler manifold of \cite{17}. The most suitable definition for our discussion is formulated in terms of symplectic products. It is the one which was denoted as definition 3.

\textit{Definition of a special Kähler manifold}

Take a complex manifold $\mathcal{M}$. Suppose we have in every chart a $2(n+1)$-component vector $V(z^\alpha, \bar{z}^{\dot{\alpha}})$ such that on overlap regions there are transition functions of the form

$$e^{i\frac{1}{2}(f(z^\alpha)-\bar{f}(\bar{z}^{\dot{\alpha}}))} S, \quad (1.1)$$

with $f$ a holomorphic function and $S$ a constant $Sp(2(n+1), \mathbb{R})$ matrix. (These transition functions have to satisfy the cocycle condition.) Take a $U(1)$ connection of the form $\kappa_\alpha dz^\alpha + \kappa_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}}$ with

$$\kappa_{\dot{\alpha}} = -\kappa_\alpha, \quad (1.2)$$

The first one is not explicitly symplectic covariant, but we could as well have discussed here definition 2, where the constraint relevant for the discussion below was formulated as $\langle v, \partial_\alpha v \rangle = 0$. The alternative form is then $\langle \partial_\alpha v, \partial_\beta v \rangle = 0$. 

\footnote{The first one is not explicitly symplectic covariant, but we could as well have discussed here definition 2, where the constraint relevant for the discussion below was formulated as $\langle v, \partial_\alpha v \rangle = 0$. The alternative form is then $\langle \partial_\alpha v, \partial_\beta v \rangle = 0$.}
under which $\bar{V}$ has opposite weight as $V$. Denote the covariant derivative by $\mathcal{D}$:

\[
\begin{align*}
U_\alpha &\equiv \mathcal{D}_\alpha V \equiv \partial_\alpha V + \kappa_\alpha V, \\
\bar{U}_\bar{\alpha} &\equiv \mathcal{D}_{\bar{\alpha}} V \equiv \partial_{\bar{\alpha}} V - \kappa_{\bar{\alpha}} \bar{V}, \\
\mathcal{D}_\alpha \bar{V} &\equiv \partial_\alpha \bar{V} - \kappa_\alpha \bar{V}.
\end{align*}
\] (1.3)

We impose the following conditions:

1. $\langle V, \bar{V} \rangle = i$, \hspace{1cm} (1.4)
2. $\mathcal{D}_{\bar{\alpha}} V = 0$, \hspace{1cm} (1.5)
3. $\mathcal{D}_{[\alpha} U_{\beta]} = 0$, \hspace{1cm} (1.6)
4. $\langle V, U_\alpha \rangle = 0$, \hspace{1cm} (1.7)

where $\langle \cdot, \cdot \rangle$ denotes the symplectic inner product, e.g. $\langle V, \bar{V} \rangle = V^T \Omega \bar{V}$, with an antisymmetric matrix $\Omega$, which has as standard form

\[
\Omega_{st} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.
\] (1.8)

Define

\[
g_{\alpha\bar{\beta}} \equiv i \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle,
\] (1.9)

where $\bar{U}_{\bar{\beta}}$ denotes the complex conjugate of $U_\alpha$. If this is a positive-definite metric, $\mathcal{M}$ is called a special Kähler manifold.

It can then be shown that locally a function $K'$ exists such that

\[
\kappa_\alpha = \frac{1}{2} \partial_\alpha K', \hspace{1cm} \kappa_{\bar{\alpha}} = -\overline{\kappa_\alpha} = -\frac{1}{2} \partial_{\bar{\alpha}} \bar{K}'.
\] (1.10)

The real part of $K'$ is the Kähler potential $K$. If there is an imaginary part $\text{Im} \ K'$, then

\[
V' = e^{\frac{i}{2} \text{Im} \ K'/2} V,
\] (1.11)

satisfies the same constraints with $K'$ replaced by the real $K$.

As discussed at the end of section 4.2.2 in \cite{17}, the constraints have a clear physical interpretation, related to the positivity of kinetic terms in the action. However, as suggested there already, the fourth constraint (1.7) could be replaced by

\[
4'. \quad \langle U_\alpha, U_\beta \rangle = 0,
\] (1.12)

without violating the physical arguments. The constraint 4 implies 4’, by taking a covariant derivative and antisymmetrizing, and with 4’ it was shown that 4 follows when $n > 1$. However, for $n = 1$, equation 4’ is empty. Taking 4’ as constraint thus
allows $\langle V, U_z \rangle \neq 0$. Such $N = 2$ models would be new, and this possibility will be investigated in this paper. It will be called ‘the special case’. The case with $n > 1$ or $\langle V, U_\alpha \rangle = 0$ will be called ‘the generic case’.

In appendix C of [17], two $n = 1$ examples are given where the condition (1.7) is not fulfilled. In these examples it was shown that the relaxation of that constraint leads to models not allowed by other definitions of special geometry. Here we will first give further evidence of the non-triviality of this condition. The main result will be that indeed models which violate (1.7), still allow an $N = 2$ supersymmetric formulation.

The scalars of the special Kähler manifolds are contained in the $\theta = 0$ sector of chiral superfields of $N = 2$ supersymmetry. A chiral multiplet is a reducible representation of $N = 2$ supersymmetry. After imposing suitable constraints, it gives a vector multiplet. In rigid supersymmetry these constraints can be written in superspace for a symplectic section of superfields or, in components, as a linear multiplet of constraints of symplectic sections [17, 18]. With a standard symplectic metric (1.8), the rigid special Kähler constraints can be used to write the lower part of the symplectic sections $V$ in terms of the upper one. The reducibility constraints for the lower parts of the sections then give rise to the field equations of the fields in the upper parts. We want to use this approach to construct the field equations of vector multiplets, coupled to $N = 2$ supergravity.

Chiral and vector multiplets can also be defined as representations of the local superconformal algebra. Then the fields of the gauge multiplet of the superconformal gauge invariances, which is the Weyl multiplet, enter in the transformation rules of the multiplets [7, 8]. To describe the coupling of $n$ on-shell vector multiplets to supergravity, we will start from $2n + 2$ chiral multiplets. The linear multiplet of constraints which reduce these chiral multiplets to vector multiplets in supergravity will contain additional terms with fields of the Weyl multiplet [7].

The equations that follow by supersymmetry from this weak definition of special Kähler geometry are derived for the complete set of $2n + 2$ chiral multiplets. The constraints defining special Kähler geometry involve a breaking of dilatations and the $U(1)$ transformations in the superconformal group. We also choose a symplectic fermionic constraint as the gauge choice for $S$-supersymmetry. Special conformal symmetry is broken by a choice for the dilatation gauge field as in previous approaches. So finally, this leads to the breaking of superconformal to super-Poincaré spacetime symmetry with a residual internal $SU(2)$ in a consistent way, without relying on a prepotential or an action. Combining the reducibility constraints with the constraints of special Kähler geometry we find $n$ on-shell vector multiplets, coupled to $24 + 24$ supergravity.
components, remnants of the Weyl multiplet.

These $24 + 24$ components reside in a 'current multiplet', which we identify as a reduced chiral self-dual superfield. The full supergravity equations, however, would rely on a second compensating multiplet, which is independent of the symplectic formulation. For these aspects we refer to the 3 known constructions of auxiliary field formulations [19, 20].

In section 2, the building blocks of the construction, the Weyl multiplet and the chiral multiplet, are given. Their supersymmetry transformation rules and the constraint to make a vector multiplet out of a chiral are recapitulated. In section 3 the special Kähler constraints and the supersymmetric relatives are treated for the most general case. In section 4 we combine the constraints imposed on the chiral multiplets and those found in section 3 to find on-shell vector multiplets. Finally, we comment on the remaining off-shell components of supergravity and their field equations. We recapitulate our results in section 5.

2 The building blocks of the construction

In this section we review the Weyl multiplet, i.e. the gauge multiplet of the $N = 2$ superconformal symmetry, and the superconformal chiral multiplet, coupled to the Weyl multiplet. Most of the material presented here is well known (see e.g., [8, 20]).

2.1 The Weyl multiplet

The Weyl multiplet is the gravitational multiplet of $N = 2$ superconformal gravity. It contains the gauge fields $e_\mu^a, \omega_{\mu}^{ab}, b_\mu, f_\mu^a, \Psi_\mu^i, A_\mu, \psi_\mu^i$ and $\phi_\mu^i$. They are, respectively, gauge fields of general coordinate transformations, Lorentz rotations, dilatations, special conformal boosts, chiral $SU(2)$ and $U(1)$, supersymmetry and special supersymmetry. The representation is completed by the Lorentz tensor $T_{ij}^{ab}$, antisymmetric in $[ij]$, the spinor $\chi^i$ and the scalar $D$. Note that $T_{ij}^{ab}$ is the antiself-dual tensor, and its complex conjugate $T_{abij}$ is self-dual. The spin connection and the gauge fields for the special conformal transformations and special supersymmetry are composite gauge fields, given by

$$\omega_{\mu}^{ab} = -2e^{ij[a}\partial_{[\mu}e_{\nu]}^{b]} - e^{ij[a}e^{b\sigma}e_{\mu\nu}\partial_{\sigma}e_{\nu}^{c} - 2e_{\mu}^{[a}e^{b]}\epsilon^i_{\nu}b_\nu$$

\[\text{However, here we use different normalizations, more suited for a manifestly symplectic formulation of the theory. We use the notations of [21]. So the old supersymmetry parameters are } 1/\sqrt{2} \text{ the new ones and the old fermionic fields are } \sqrt{2} \text{ the new ones. Also keep in mind that } \epsilon^{0123} = i.\]
\[ -\frac{1}{2} (2 \bar{\psi}_i^a \gamma^a \psi_i^b + \bar{\psi}^a \gamma_\mu \psi_i^b + \text{h.c.}), \]

\[
\phi^i_\mu = (\sigma^{\rho \sigma} \gamma_\mu - \frac{1}{3} \gamma_\mu \sigma^{\rho \sigma}) \left( \mathcal{D}_\rho \psi_\sigma^i - \frac{i}{8} \sigma \cdot T^{ij} \gamma_\rho \psi_{\sigma j} \right) + \frac{1}{2} \gamma_\mu \chi^i,
\]

\[
f^{\mu a}_i = \frac{1}{2} \mathcal{R}^{\mu a}_i - \frac{1}{2} e_\mu^a f_\nu^\nu - \frac{i}{2} e^{\alpha \nu} \varepsilon_{\mu \nu \rho} R_{\rho a} (U(1)) + \frac{1}{16} T^{ab}_{ij} T^{di}_{kj} - \frac{3}{4} e^{\nu a} D + \left( \bar{\psi}_i^{[a} \sigma^{ab} \phi_{\nu]} + \bar{\psi}_i^{[a} T^{ab}_{\mu j} \psi_{\nu]} - \frac{1}{2} \delta \bar{\psi}_i^{[a} \sigma^{ab} \chi_{\nu]} - \bar{\psi}_i^{[a} \gamma_\nu \hat{R}^{ab} (Q)_i + \text{h.c.} \right) e_{\nu}^\nu.
\]

The following expressions are used in \( f^{\mu a}_i \):

\[
f^{\mu a}_i = \frac{1}{6} \mathcal{R} - D - \left( \frac{1}{6} e^{\rho \mu \nu} \bar{\psi}_i^a \gamma_\rho \mathcal{D}_\nu \psi_{\sigma i} - \frac{1}{6} \bar{\psi}_i^a \gamma_\mu \psi_{\nu} T^{\mu \nu} - \frac{1}{2} \bar{\psi}_i^a \gamma_\mu \chi_i + \text{h.c.} \right),
\]

\[
\hat{R}_{\rho a} (U(1)) = 2 \partial_{[\rho} A_{\nu]} - i \left( 2 \bar{\psi}_i^{[a} \phi_{\nu]} + \frac{3}{2} \bar{\psi}_i^{[a} \gamma_\nu \chi \right) + \text{h.c.},
\]

\[
\hat{R}^{ab} (Q)_i = 2 \mathcal{D}_{[\rho} \psi_{\sigma i]} - \gamma_{[\rho} \phi_{\sigma]} - \frac{1}{2} \sigma \cdot T^{ij} \gamma_{[\rho} \psi_{\sigma j]} + \text{h.c.}.
\]

Also, \( \mathcal{D}_\mu \) is covariant with respect to the linearly realized symmetries: Lorentz transformations, dilatations, \( U(1) \) and \( SU(2) \), i.e.

\[
\mathcal{D}_\mu \psi_\nu^i = \left( \partial_\mu - \frac{i}{2} \omega^{\alpha \beta}_\mu \sigma_{\alpha \beta} + \frac{1}{2} b_\mu + \frac{1}{2} i A_\mu \right) \psi_\nu^i + \frac{3}{2} \gamma_\mu \psi_\nu^j.
\]

Furthermore, \( \mathcal{R} = e_\alpha^\mu e_\nu^\nu R_{\mu \nu}^{ab} \) is the Ricci scalar derived from the Riemann tensor

\[
R_{\mu \nu}^{ab} = 2 \partial_{[\mu} \omega_{\nu]}^{\alpha \beta} - 2 \omega_{[\mu}^{\alpha \beta} e_{\nu]}^\beta
\]

and

\[
R_\mu^a = e_\nu^\nu R_{\mu \nu}^{ab}.
\]

The transformation rules of the independent fields of the Weyl multiplet under supersymmetry, special supersymmetry and special conformal transformations (with parameters \( \epsilon^i \), \( \eta^i \) and \( \Lambda_\alpha^K \) are

\[
\delta e_\mu^a = \epsilon^i \gamma^a \psi_\mu^i + \text{h.c.},
\]

\[
\delta \psi_\mu^i = \mathcal{D}_\mu \epsilon^i - \frac{1}{2} \sigma \cdot T^{ij} \gamma_\mu \epsilon^j - \frac{1}{2} \gamma_\mu \eta^i,
\]

\[
\delta b_\mu = \frac{1}{2} \epsilon^i \phi_\mu^i - \frac{3}{4} \epsilon^i \gamma_\mu \chi_i - \frac{1}{2} \bar{\eta}^i \psi_\mu^i + \text{h.c.} + \Lambda_\alpha^a e_{\mu a},
\]

\[
\delta A_\mu = \frac{1}{2} \epsilon^i \phi_\mu^i + \frac{3}{4} \epsilon^i \gamma_\mu \chi_i + \frac{1}{2} \bar{\eta}^i \psi_\mu^i + \text{h.c.},
\]

\[
\delta \gamma_\mu^i = 2 \epsilon^j \phi_\mu^j - 3 \epsilon^j \gamma_\mu \chi^i + 2 \bar{\eta}^j \psi_\mu^j - (\text{h.c.}; \text{traceless}),
\]

\[
\delta T^{ij}_{ab} = 8 \epsilon^i \hat{R}_{ab}(Q)^j_i,
\]

\[
\delta \chi^i = -\frac{1}{12} \sigma \cdot \mathcal{D} T^{ij} \epsilon^j + \frac{1}{6} \hat{R}(SU(2))^j_i \cdot \sigma \epsilon^j - \frac{1}{3} i \hat{R}(U(1)) \cdot \sigma \epsilon^i \]

\[
+ \frac{1}{2} D \epsilon^i + \frac{1}{12} \sigma \cdot T^{ij} \eta^j,
\]

\[
\delta D = \epsilon^i \mathcal{D} \chi_i + \text{h.c.}.
\]

The expression for the superconformal covariant curvatures \( \hat{R} \) and the other transformation rules (in terms of the old conventions) were given in [7].
2.2 The chiral multiplet

A chiral multiplet is a reducible representation of the superconformal algebra [8]. By imposing a linear multiplet of constraints it becomes a vector multiplet. This is an irreducible representation of the superconformal algebra. The constraints are called the generalized Bianchi identities, because they contain a Bianchi identity for the tensor in the chiral multiplet.

Later we want to couple vector multiplets to conformal supergravity. The scalars of these vector multiplets form a symplectic section. They are the lowest components of a multiplet. Therefore, all the components of the multiplets have to form such a symplectic section. This is the reason to start from a $(2n + 2)$-dimensional section of chiral multiplets:

\[
\tilde{\Phi} = V + \bar{\theta}^i \Omega_i + \frac{1}{4} \bar{\theta}^i \theta^j Y_{ij} + \frac{1}{4} \varepsilon_{ij} \bar{\theta}^i \sigma \cdot \tilde{F} - \theta^j \\
+ \frac{1}{16} \varepsilon_{ij} (\bar{\theta}^i \sigma_{ab} \theta^j) \bar{\theta}^k \sigma_{ab} \bar{\Lambda}_k + \frac{1}{64} (\varepsilon_{ij} \bar{\theta}^i \sigma_{ab} \theta^j)^2 \tilde{C}.
\] (2.7)

The components of the section are denoted by

\[
\tilde{\Phi} = \begin{pmatrix}
\phi^I \\
\phi_{F,I}
\end{pmatrix},
\] (2.8)

with \( I = 0, \ldots, n \), which gives for the components of the chiral superfield

\[
V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \quad \tilde{\Omega}_i = \begin{pmatrix} \Omega^I_i \\ \Omega_{F,i} \end{pmatrix}, \quad \tilde{Y}_{ij} = \begin{pmatrix} Y^I_{ij} \\ Y_{F,ij} \end{pmatrix},
\]

\[
\tilde{F}_{ab}^- = \begin{pmatrix} \tilde{F}_{ab}^- \\ \tilde{G}_{F,ab} \end{pmatrix}, \quad \tilde{\Lambda}_i = \begin{pmatrix} \Lambda^I_i \\ \Lambda_{F,i} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C^I \\ C_{F,I} \end{pmatrix}.
\] (2.9)

This section of multiplets is independent of the existence of a prepotential. The multiplets starting with \( F_I \) are on an equal footing with the ones starting with \( X^I \). As long as we do not impose the constraints to obtain a vector superfield or the special Kähler constraints these are \( 2n + 2 \) independent chiral multiplets. The full superconformal transformation rules are given by

\[
\delta V = e^i \tilde{\Omega}_i + (\Lambda_D - i \Lambda_A) V, \\
\delta \tilde{\Omega}_i = \mathcal{P} V e_i + \frac{1}{2} \tilde{Y}_{ij} e_j + \frac{1}{2} \sigma \cdot \tilde{F} - e_{ij} e^j + V \eta_i + \frac{3}{2} \Lambda_D - \frac{1}{2} i \Lambda_A) \tilde{\Omega}_i + \Lambda_{SU(2)}(i) \tilde{\tilde{\Omega}}_j, \\
\delta \tilde{Y}_{ij} = 2 e^i (i \mathcal{P} \tilde{\Omega}_j) - 2 e^k \tilde{\Lambda}_i (i \varepsilon_{jk}) + 2 \Lambda_D \tilde{Y}_{ij} + 2 \Lambda_{SU(2)}(i) \tilde{Y}_{jk}, \\
\delta \tilde{F}_{ab}^- = e^{ij} e_i \mathcal{P} \sigma_{ab} \tilde{\Omega}_j + e^{ij} \sigma_{ab} \tilde{\Lambda}_i - 2 e^{ij} \bar{\eta}_i \sigma_{ab} \bar{\Omega}_j + 2 \Lambda_D \tilde{F}_{ab}^- , \\
\delta \tilde{\Lambda}_i = -\frac{1}{2} \sigma \cdot \tilde{F} - \tilde{\mathcal{P}} e_i - \frac{1}{2} \mathcal{P} \tilde{Y}_{ij} e_k \varepsilon^jk + \frac{1}{2} \tilde{C} \varepsilon^i \varepsilon_{ij}.
\]
\[
- \frac{1}{8} \delta (V \epsilon^{jk} T_{jk} \cdot \sigma) \epsilon_i - \frac{3}{2} (\bar{\chi}_i \gamma_a \tilde{\Omega}_j) \gamma^a \epsilon_k \epsilon^{jk} \\
- \tilde{Y}_{ij} \epsilon^{jk} \eta_k + \frac{v}{2} \Lambda_\sigma \tilde{\Lambda}_i + \frac{1}{2} \Lambda_\Lambda \tilde{\Lambda}_i + \Lambda_{SU(2)} T_{ij} \tilde{\Lambda}_j,
\]
\[
\delta \tilde{C} = -2 \epsilon^{ij} \epsilon_i \delta \tilde{\Omega}_j - 6 \epsilon_i \chi_j \tilde{Y}_{kl} \epsilon^{ik} \epsilon^{jl} + \frac{1}{2} \epsilon_i \sigma \cdot T_{jk} \delta \tilde{\Omega}_l \epsilon^{ij} \epsilon^{kl} + 2 \epsilon^{ij} \eta_i \tilde{\Lambda}_j \\
+ 3 \Lambda_D \tilde{C} + i \Lambda_\Lambda \tilde{C}.
\]

This superconformal chiral superfield can be reduced to a vector superfield with the constraints

\[
0 = \tilde{Y}_{ij} - \epsilon_{ik} \epsilon_{jl} \tilde{Y}^{kl},
\]
\[
0 = \delta \tilde{\Omega}_i - \epsilon^{ij} \tilde{\Lambda}_j,
\]
\[
0 = D^a (\tilde{F}^+_{ab} - \tilde{F}^-_{ab} + \frac{1}{4} V T_{ab} \epsilon^{ij} - \frac{1}{4} \tilde{\Omega} \epsilon^{ij} - \tilde{C}).
\]

The symplectic vector of chiral multiplets with these constraints define \(2n + 2\) vector multiplets in superconformal gravity. The special Kähler constraints will relate them such that one ends up with \(n + 1\) vectors and \(n\) complex scalars and spinors obeying field equations.

### 3 Gauge choices and special Kähler constraints

To obtain a Poincaré supergravity theory of \(n\) vector multiplets, we start from the assumption that the components in the symplectic sections \(V\) are the lowest components of reduced chiral multiplets, as is the case in previous constructions of matter couplings in \(N = 2\) supergravity. To achieve that, we have to impose the reducibility constraints \((2.11)-(2.14)\) on the chiral multiplets and suitable constraints that impose restrictions on the sections such that the resulting theory contains \(n\) physical vector multiplets and the gravity multiplet. The superfluous symmetries of the superconformal construction need to be broken by suitable gauge choices. The symplectic section \(V\) can be seen as a function of \(n\) scalars \(z^\alpha\) and their complex conjugates \(\bar{z}^{\bar{\alpha}}\) (\(\alpha = 1, \ldots, n\)). These scalars can be interpreted as the coordinates of a special Kähler manifold.

Having introduced \(K'\) in \((1.11)\), we have exhausted constraint \((1.6)\). The remaining relevant constraints are then \((1.4), (1.5)\), and we will take the formulation with \((1.12)\). Condition \((1.4)\) gauge fixes the dilatations, choosing the canonical kinetic term for the graviton. Equation \((1.3)\) imposes the holomorphicity of the scalar fields. For the symmetry of the kinetic matrix of the vectors, one needs another constraint, which is \((1.12)\). In all previous papers on special geometry, one imposed instead \((1.7)\), which
is equivalent for \( n > 1 \), but not for \( n = 1 \) as mentioned in the introduction. There is no physical argument known to demand (1.7), but up to now, no physical applications have been found that do not fulfil it.

We have thus seen that we can look upon equations (1.4) and (1.5) in two ways. They are the defining equations of special geometry, as well, they can be considered as gauge choices for the dilatations and chiral \( U(1) \) transformations present in the superconformal algebra. As we will see below a supersymmetric extension of these constraints will include the gauge choice of \( S \)-supersymmetry.

From (1.3) follows

\[
g_{\alpha \bar{\beta}} \equiv \partial_\alpha \partial_{\bar{\beta}} K = i \langle U_\alpha , \bar{U}_{\bar{\beta}} \rangle .
\]  

(3.1)

Furthermore, we impose the ‘physical’ condition (positivity of the kinetic energy terms of the vectors [17]) that

\[
det g_{\alpha \bar{\beta}} > 0 \quad \text{if} \quad \langle V, U_\alpha \rangle = 0 ,
\]

\[
g'_{z \bar{z}} \equiv g_{z \bar{z}} - Z_z \bar{Z}_z > 0 \quad \text{for} \quad Z_z \equiv \langle V , U_z \rangle .
\]  

(3.2)

Using the constraints it can be shown that

\[
\mathcal{W} = (V , U_\alpha , \bar{V} , \bar{U}_{\bar{\alpha}})
\]  

(3.3)

forms for every \( z , \bar{z} \) a basis for symplectic vectors. More information about the expansion coefficients can be found in appendix [A]. This expansion will be used in the derivation of the supersymmetric extension of the special Kähler constraints and of the field equations.

### 3.1 The constraint on the curvature

Covariant derivatives involve the Kähler connection as in (1.3), and after choosing a real Kähler potential one may define a Kähler weight\(^4\) \( p \) for a symplectic section \( W \), such that

\[
\mathcal{D}_\alpha W = \left( \partial_\alpha + \frac{p}{2} (\partial_\alpha K) \right) W , \quad \mathcal{D}_{\bar{\alpha}} W = \left( \partial_{\bar{\alpha}} - \frac{p}{2} (\partial_{\bar{\alpha}} K) \right) W .
\]  

(3.4)

If \( W \) carries indices \( \alpha \) or \( \bar{\alpha} \) there is a further metric connection, defined such that \( \mathcal{D}_\alpha g_{\beta \bar{\gamma}} = 0 \). The curvature of the special Kähler manifold is then defined by

\[
[\mathcal{D}_\alpha , \mathcal{D}_{\bar{\beta}}] X_\gamma = -p g_{\alpha \bar{\beta}} X_\gamma - R_{\alpha \bar{\beta} \gamma \delta} X_\delta ,
\]  

(3.5)

---

\(^3\) Keep in mind that for \( n > 1 \) one always has \( Z_z = 0 \).

\(^4\) \( V \) and \( U_\alpha \) have weight 1, while \( Z_z \) has weight 2, and for their complex conjugates respectively \(-1\) and \(-2\).
where $X_\alpha$ is a generic vector with Kähler weight $p$. Applying this for $X_\alpha$ replaced by $U_\alpha$ and taking the symplectic inner product with $\bar{U}_\delta$ one finds

$$R_{\alpha\beta\gamma\delta} \equiv g_{\delta\bar{\delta}} R_{\alpha\beta\gamma\delta} = -2g_{(\alpha|\bar{\beta}|\gamma\delta)\bar{\delta}} - i\langle \mathcal{D}_\alpha U_{\gamma}, \mathcal{D}_\beta \bar{U}_\delta \rangle .$$

(3.6)

If we introduce a symmetric tensor

$$C_{\alpha\beta\gamma} = \langle U_\alpha, \mathcal{D}_\beta U_{\gamma} \rangle ,$$

(3.7)

and expand the last term of (3.6) in the basis $W$ according to appendix A we obtain the following two cases:

1. The generic case :

$$\mathcal{D}_\alpha U_\beta = iC_{\alpha\beta\gamma} g_{\gamma\bar{\gamma}} \bar{U}_{\bar{\gamma}} ,$$

(3.8)

and the curvature is constrained to

$$R_{\alpha\beta\gamma\delta} \equiv g_{\beta\bar{\beta}} R_{\alpha\gamma\delta} = -2g_{(\alpha|\bar{\beta}|\gamma\delta)\bar{\delta}} + C_{\alpha\gamma\epsilon} g_{\epsilon\bar{\epsilon}} C_{\bar{\beta}\bar{\delta}\bar{\epsilon}} .$$

(3.9)

2. The special case :

In a similar way one finds that in this case

$$\mathcal{D}_z U_z = ig_{zz}^z (C_{zzz} \bar{U}_z - g_{zz} \mathcal{D}_z Z_z \bar{V}') .$$

(3.10)

The curvature becomes

$$R_{zzzz} = -2g_{zz}^2 - g_{zz}^z (\mathcal{D}_z Z_z) g_{zz} (\mathcal{D}_z Z_z) + C_{zzzz} g_{zz}^z C_{zzzz} .$$

(3.11)

### 3.2 An adapted basis and metric for the special case

When $Z_z \neq 0$, one may diagonalize the matrix of symplectic products between $V, \bar{V}, U_z$ and $\bar{U}_z$ by defining

$$U'_z = U_z + iZ_z \bar{V} ; \quad \bar{U}'_z = \bar{U}_z - iZ_z V .$$

(3.12)

We then have symplectic products

$$\langle V, \bar{V} \rangle = i, \quad \langle V, U'_z \rangle = \langle V, \bar{U}'_z \rangle = \langle \bar{V}, U'_z \rangle = \langle \bar{V}, \bar{U}'_z \rangle = 0 ,$$

$$\langle \bar{U}'_z, U'_z \rangle = i(g_{zz} - Z_z \bar{Z}_z) = ig_{zz}^z .$$

(3.13)

In this way we find the Hermitian metric $g'_{zz}$ which is invertible because of (3.2), but is not the second derivative of the Kähler potential $K$, used to define the covariant
derivatives in (3.4). With this definition, covariant derivatives on the above equations lead to
\[ \langle D_z U', V \rangle = \langle D_z U', V \rangle = 0 . \] (3.14)

The defining expressions for \( U_z \) and \( Z_z \) imply
\[ D_z \bar{U}_z = g_{\bar{z}z} \bar{V} , \quad D_z U_z = g_{z\bar{z}} V , \quad D_z \bar{Z}_z = D_z Z_z = 0 , \] (3.15)
which in the new basis give
\[ D'_z \bar{U}'_z = g'_{\bar{z}z} \bar{V} - i \bar{Z}_z \bar{U}'_z , \quad D'_z U'_z = g'_{z\bar{z}} V + i Z_z \bar{U}'_z . \] (3.16)

When we define \( D' \) with metric connection such that \( D'_z g'_{\bar{z}z} = 0 \), all the above relations remain valid for \( D' \), as the non-zero connections are just \( \Gamma_{z\bar{z}}^z \) and \( \Gamma_{\bar{z}z}^\bar{z} \). The new definition now implies
\[ \langle \bar{U}'_z, D'_z U'_z \rangle = 0 . \] (3.17)

The analogue of (3.7) is then the definition
\[ C'_{z\bar{z}z} \equiv \langle U'_z, D'_z U'_z \rangle = C_{z\bar{z}z} . \] (3.18)

This leads again to
\[ D'_z U'_z = i C'_{z\bar{z}z} g'_{\bar{z}z} \bar{U}'_z . \] (3.19)

We define then the curvature based on the metric \( g' \) by
\[ [D'_z, D'_\bar{z}^\bar{z}] X_z = -p g_{z\bar{z}} X_z - R'_{z\bar{z}z} \bar{z} X_z . \] (3.20)

Observe that the first term has \( g \) and not \( g' \) as this is the Kähler curvature. Calculating as before the curvature \( R' \) by replacing \( X_z \) with \( U' \), and an inner product with \( \bar{U}' \), the last terms in (3.16) lead to extra terms such that we find
\[ R'_{z\bar{z}z} \equiv R'_{z\bar{z}z} \bar{z} g'_{\bar{z}z} = -2 g_{z\bar{z}} g'_{\bar{z}z} + C_{z\bar{z}z} g'_{\bar{z}z} \bar{C}_{z\bar{z}z} . \] (3.21)

Rephrasing as much as possible in terms of the metric \( g'_{\bar{z}z} \), we thus recover another geometry as for other special Kähler models. There is an essential difference in the product of metrics in (3.6) and here. We tried to extend our analysis in the basis \( W' = (V, U'_z, \bar{V}, \bar{U}'_z) \), but ran into problems with the transformation rules because we want \( V \) to be the lowest component of a chiral multiplet. So, it is not possible to get rid of \( Z_z \neq 0 \) by choosing another basis while keeping a section of chiral multiplets. The model with \( Z_z \neq 0 \) is really another model compared to those studied in the past.
For some calculations below, it is also useful to introduce another basis with symplectic vectors orthogonal to $U$. That is, we introduce

$$V' = V + iZg^z\bar{U}_z, \quad \bar{V}' = \bar{V} - i\bar{Z}g^zU_z,$$

$$\langle V', \bar{V}' \rangle = i(1 - g^zZ\bar{Z}_z) \quad \langle V', U_z \rangle = \langle V', \bar{U}_z \rangle = \langle \bar{V}', U_z \rangle = \langle \bar{V}', \bar{U}_z \rangle = 0,$$

$$\langle \bar{U}_z, U_z \rangle = ig_{zz}.$$  \hspace{1cm} (3.22)

### 3.3 Supersymmetric extension of special Kähler constraints

It is clear that the constraint (1.4) breaks the superconformal symmetry. The constraints and their supersymmetric partners therefore play the role of gauge conditions for some of the superconformal symmetries. The residual symmetry should then still contain the symmetries of Poincaré supergravity. In this subsection we will derive the supersymmetric partners of the constraints (1.4), (1.5) and (1.12), and compute the decomposition rule for the resulting supergravity, i.e. the rule which gives the remaining symmetry as a linear combination of the original, superconformal, symmetry.

#### 3.3.1 Gauge choices and decomposition rule

Before we go to these constraints, we break the special conformal symmetry by imposing a constraint on $b_\mu$:

$$K\text{-gauge: } b_\mu = 0. \hspace{1cm} (3.23)$$

This does not alter the number of degrees of freedom as $b_\mu$ is pure gauge in the Weyl multiplet (cf table 1).

The decomposition rule for the special conformal symmetry is

$$\Lambda^a_K = -\epsilon^{\mu a} \left( \frac{1}{2} \epsilon^i \phi_{\mu i} - \frac{3}{2} \epsilon^i \gamma_{i} \chi_i - \frac{1}{2} \bar{\eta}^i \psi_{\mu i} + {\text{h.c.}} \right). \hspace{1cm} (3.24)$$

Constraint (1.4) breaks the dilatations. Indeed, the superconformal transformation of (1.4) gives

$$\langle \bar{V}, \bar{\epsilon}^i \check{\Omega}_i \rangle - \langle V, \epsilon^i \check{\Omega}_i \rangle + 2i\Lambda_D = 0, \hspace{1cm} (3.25)$$

and the dilatations are now a combination of other symmetries. We choose as $S$-gauge

$$S\text{-gauge: } \langle \bar{V}, \check{\Omega}_i \rangle = 0 \quad \text{and} \quad \langle V, \check{\Omega}_i \rangle = 0. \hspace{1cm} (3.26)$$

Remark that after this gauge choice the decomposition rule (3.23) simplifies to

$$\Lambda_D = 0, \hspace{1cm} (3.27)$$
The Weyl multiplet (24 + 24)

| fields | d.o.f. | comments |
|--------|--------|----------|
| $e_{\mu}^a$ | 5 | 16 - 4(translation.) - 6(Lorentz) - 1(dilatation) |
| $b_{\mu}$ | 0 | 4 - 4(special conformal.) |
| $A_\mu$ | 3 | 4 - 1(U(1)) |
| $Y_{\mu,j}$ | 9 | 12 - 3(SU(2)) |
| $\psi_{\mu}$ | 16 | 32 - 8(Q-supersymmetry) - 8(S-supersymmetry) |
| $T_{ab}^{ij}$ | 6 | complex antiself-dual |
| $\chi_i$ | 8 | |
| $D$ | 1 | real scalar |

| Symplectic section of chiral multiplets (16(2n + 2) + 16(2n + 2)) |
|-------------------|---------------|
| $V$ | 2(2n+2) |
| $\bar{\Omega}_i$ | 8(2n+2) |
| $\bar{Y}_{ij}$ | 6(2n+2) |
| $\tilde{F}_{ab}$ | 6(2n+2) |
| $\tilde{\Lambda}_i$ | 8(2n+2) |
| $\tilde{C}$ | 2(2n+2) |

Table 1: Degrees of freedom in the model before the constraints.

such that we can forget about the original dilatations completely. Demanding that the sections $V$ depend on $z^\alpha$ and $\bar{z}^\alpha$ in the way described in (1.3)–(1.5), is a *gauge choice for the chiral U(1)-transformations*. In fact, consider the transformation of the first line of (2.10) using these equations:

$$\delta V = U_\alpha \delta z^\alpha - \frac{1}{2} (\partial_\alpha K' \delta z^\alpha - \partial_\dot{\alpha} K' \delta \bar{z}^{\dot{\alpha}}) V .$$

(3.28)

An inner product with $\bar{V}$ gives (using (1.4) and its covariant derivative) a decomposition rule for the $U(1)$-transformations, i.e.

$$\Lambda_A = \text{Im} (\partial_\alpha K' \delta z^\alpha) ,$$

(3.29)

where we have already used (1.10).

The decomposition rule for $\delta_S(\eta_i)$ follows from the variation of the $S$-gauge:

$$\eta_i = -i \langle \bar{V}, D V \rangle \epsilon_i - \frac{1}{2} i \langle \bar{V}, \bar{Y}_{ij} \rangle \epsilon^j - \frac{1}{2} i \langle \bar{V}, \tilde{F}_{ab} \rangle \epsilon_{ij} \sigma^{ab} \epsilon^j - i \langle \bar{\epsilon}_j \bar{\Omega}_i, \bar{\Omega}_i \rangle .$$

(3.30)
From now on, we only request that the constraints are invariant under the resulting Poincaré supersymmetry

$$\delta(\epsilon_i) = \delta_Q(\epsilon_i) + \delta_S(\eta_i) + \delta_A(\Lambda_A) + \delta_K(\Lambda_K),$$

(3.31)

with $\Lambda_K$, $\Lambda_A$ and $\eta_i$ defined in (3.24), (3.29) and (3.30).

Having the symplectic sections as functions of $z$ and $\bar{z}$, we can consider the transformations of the bosonic constraints (1.4)–(1.6) and (1.12). The variation of the first one determined the breaking of dilatations. The constraints (1.5) and (1.6) are used to determine the $z$, $\bar{z}$ dependences of $V$, $U$ and $K$ and their supersymmetry transformations are thus trivial if we compute them in terms of $\delta z$ and $\delta \bar{z}$. The constraint (1.12) is only non-trivial for $n > 1$. Its variation is

$$\delta \langle U_\alpha, U_\beta \rangle = 2 \langle \mathcal{D}_\gamma U_\alpha, U_\beta \rangle \delta z^\gamma,$$

(3.32)

which is 0 due to the symmetry of (3.7). This finishes the supersymmetry variations of the bosonic special Kähler constraints.

### 3.3.2 Physical fermions and fermionic constraints

The first line of (2.10), using (3.27) and (3.29), is in terms of $\delta z$:

$$\bar{\epsilon}^i \Omega_i = U_\alpha \delta z^\alpha.$$

(3.33)

Therefore, the supersymmetry transformation of $z$ is chiral, and we define $\lambda_i^\alpha$ as

$$\bar{\epsilon}^i \lambda_i^\alpha \equiv \delta z^\alpha,$$

(3.34)

leading to

$$\tilde{\Omega}_i = U_\alpha \lambda_i^\alpha,$$

(3.35)

compatible with the $S$-gauge. The relation (3.33) can be inverted to

$$\lambda_i^\alpha = -ig^{\alpha\dot{\alpha}} \langle U_\alpha, \bar{\Omega}_i \rangle.$$

(3.36)

That $\tilde{\Omega}_i$ has only components in the $U$ direction implies the constraints (the primes here and below are irrelevant for $n > 1$ or $Z_z = 0$)

$$\langle V, \tilde{\Omega}_i \rangle = Z_z \lambda_i^\alpha \text{ or } \langle V', \tilde{\Omega}_i \rangle = 0, \quad \langle U_\alpha, \tilde{\Omega}_i \rangle = 0.$$

(3.37)
The transformation rules for $z^\alpha$ and $\lambda_i^\alpha$ are\footnote{In the transformation laws below, there is still the $SU(2)$ transformation which is not gauge fixed and thus independent of the other transformations. We will not indicate these transformations explicitly, as they follow from the position of the $i$ indices.}

\[
\begin{align*}
\delta z^\alpha &= \bar{\epsilon}^i \lambda_i^\alpha, \\
\delta \lambda_i^\alpha &= -\Gamma_{\beta\gamma}^\alpha \lambda_i^\beta \delta z^\gamma + \frac{1}{3} (\partial_\beta K \delta z^\beta - \text{h.c.}) \lambda_i^\alpha \\
&+ \nabla z^\alpha \epsilon_i - \frac{i}{2} g^{\alpha\dot{\alpha}} \langle \bar{U}_\alpha, \bar{Y}_{ij} \rangle \epsilon^j - \frac{1}{2} i g^{\alpha\dot{\alpha}} \langle \bar{U}_\alpha, \bar{F}_{\dot{a}b} \rangle \frac{1}{2} \sum_{\epsilon_1, \epsilon_2} \epsilon \sigma^{ab} e_i,
\end{align*}
\] (3.38)

where

\[
\nabla_\mu z^\alpha = \partial_\mu z^\alpha - \bar{\psi}_i^\lambda \lambda_i^\alpha.
\] (3.39)

### 3.3.3 Further variations of constraints in the generic case

At the first fermionic level we have imposed the gauge choice (3.26), and found furthermore the constraints (3.37), leaving $n$ physical fermions as shown in (3.35) and (3.36). The variation of the $S$-gauge leads to the decomposition rule. Here we will determine the further constraints on the $2(n+1)$ chiral multiplets in the symplectic vector. We first perform this analysis for the generic case where $\langle V, U_\alpha \rangle = 0$, and treat the case $n = 1$ separately afterwards.

The Poincaré transformations on (3.37) give

\[
\begin{align*}
\langle V, \bar{Y}_{ij} \rangle &= 0, \\
\langle U_\alpha, \bar{Y}_{ij} \rangle &= -C_{\alpha\beta\gamma} \bar{\lambda}_i^\beta \lambda_j^\gamma, \\
\langle V, \bar{F}_{\dot{a}b} \rangle &= 0, \\
\langle U_\alpha, \bar{F}_{\dot{a}b} \rangle &= -\frac{1}{2} C_{\alpha\beta\gamma} \varepsilon^{ij} (\bar{\lambda}_i^\beta \sigma_{ab} \lambda_j^\gamma).
\end{align*}
\] (3.40)

To analyse the content of these equations, we make use of lemma B.1 of [17]. This says that the $2(n+1) \times (n+1)$ matrix $(V, U_\alpha)$ has rank $(n+1)$. Thus we can solve (3.40) for half of the components of $\bar{Y}_{ij}$ and $\bar{F}_{\dot{a}b}$.

Straightforward variation of these two equations under Poincaré supersymmetry yields a set of new constraints:

\[
\begin{align*}
\langle V, \bar{A}_i \rangle &= -\frac{1}{2} C_{\alpha\beta\gamma} \varepsilon^{kl} (\bar{\lambda}_i^\beta \sigma_{ab} \lambda_k^\gamma), \\
\langle U_\alpha, \bar{A}_i \rangle &= \frac{1}{2} i C_{\alpha\beta\gamma} \gamma^{\beta\delta} \left( \langle U_\beta, \bar{Y}_{ij} \rangle \varepsilon_{jk} \lambda_k^\gamma + \langle U_\beta, \sigma \cdot \bar{F} \rangle \lambda_i^\gamma \right) \\
&+ \frac{1}{2} D_\alpha C_{\beta\gamma\delta} \cdot \varepsilon^{kl} (\bar{\lambda}_i^\beta \sigma_{ab} \lambda_k^\gamma) \sigma_{ab} \lambda_i^\delta.
\end{align*}
\] (3.41)

(3.42)
Varying constraint (3.41) yields

\[
\langle V, \bar{C} \rangle = \frac{1}{2} i \varepsilon^{ik} \varepsilon^{jl} C_{\alpha \beta \gamma} g_{\alpha \tilde{a}} \langle \bar{U}_{\alpha}, \bar{Y}_{ij} \rangle \bar{\lambda}^\beta_k \lambda^\gamma_l \\
- \frac{1}{2} i C_{\alpha \beta \gamma} g_{\alpha \tilde{a}} \langle \bar{U}_{\alpha}, \bar{F}_{ab} \rangle \varepsilon^{kl} (\bar{\lambda}^\beta_k \sigma^{ab} \lambda^\gamma_l) \\
- \frac{1}{6} D_\alpha C_{\beta \gamma \delta} \cdot \varepsilon^{ij} (\bar{\lambda}^\beta_k \sigma^{ab} \lambda^\gamma_l) \varepsilon^{kl} (\bar{\lambda}^\delta_k \sigma^{ab} \lambda^\gamma_l) .
\]

(3.43)

The variation of (3.42) gives

\[
\langle U_\alpha, \bar{C} \rangle = \frac{1}{4} C_{\alpha \beta \gamma} g^{\beta \bar{a}} g_{\gamma \bar{b}} \varepsilon^{ik} \varepsilon^{jl} \langle \bar{U}_{\bar{\beta}}, \bar{Y}_{ij} \rangle \langle \bar{U}_{\bar{\gamma}}, \bar{Y}_{kl} \rangle \\
- \frac{1}{2} C_{\alpha \beta \gamma} g^{\beta \bar{a}} g_{\gamma \bar{b}} \langle \bar{U}_{\bar{\beta}}, \bar{F}_{ab} \rangle \langle \bar{U}_{\bar{\gamma}}, \bar{F}_{-ab} \rangle \\
- \frac{1}{2} i \varepsilon^{ik} \varepsilon^{jl} [C_{\alpha \beta \gamma} \langle V, Y_{ij} \rangle + D_\alpha C_{\beta \gamma \delta} \cdot g^{\delta \bar{a}} \langle \bar{U}_{\delta}, \bar{Y}_{ij} \rangle \bar{\lambda}^\beta_k \lambda^\gamma_l] \\
+ \frac{1}{2} i C_{\alpha \beta \gamma} \langle V, \bar{F}_{ab} \rangle + D_\alpha C_{\beta \gamma \delta} \cdot g^{\delta \bar{a}} \langle \bar{U}_{\delta}, \bar{F}_{ab} \rangle \varepsilon^{ij} (\bar{\lambda}^\beta_k \sigma^{ab} \lambda^\gamma_l) \\
- 2 i C_{\alpha \beta \gamma} g^{\beta \bar{a}} \varepsilon^{ij} \bar{\lambda}^\beta_k \langle \bar{U}_{\delta \bar{\beta}}, \bar{\Lambda}_{ij} \rangle \\
+ \frac{1}{12} D_\alpha D_\beta C_{\gamma \delta \epsilon} \cdot \varepsilon^{ij} (\bar{\lambda}^\beta_k \sigma^{ab} \lambda^\gamma_l) \varepsilon^{kl} (\bar{\lambda}^\delta_k \sigma^{ab} \lambda^\gamma_l) .
\]

(3.44)

These are all the possible ‘Kähler’ constraints on the sections. Let us review the degrees of freedom. Before imposing the constraints, we have the degrees of freedom as in table [1]. First of all there is the Weyl multiplet with $24 + 24$ degrees of freedom. The gauge invariances have been used to determine the counting. Indeed, the dilation invariance can be seen as removing the trace of the vierbein $e^\mu_\alpha$ and $\gamma^\mu \psi^\mu_\alpha$ is pure gauge under special supersymmetry. Similarly the vectors $A_\mu$ and $V_\mu \psi_i$ lose a degree of freedom because of their gauge transformations. Secondly, we have the symplectic vectors of $2n + 2$ chiral multiplets, which altogether consist of $(2n + 2)16 + (2n + 2)16$ degrees of freedom.

Then we have imposed the constraints (1.14)–(1.16), (1.12) and their supersymmetry partners. The new counting is in table [2]. All the symplectic sections are first reduced to $(n + 1)$ rather than $(2n + 2)$ degrees of freedom, as inner products with $V$ and with $U_\alpha$ are removed by the constraints. The symplectic vector $V$ is further reduced to $n$ complex variables $\varepsilon^a$, by constraints which we have interpreted as gauge choices of dilatations and chiral $U(1)$. These invariances have thus disappeared, and in the upper part of the table we should thus no longer subtract from degrees of freedom of the vierbein and of $A_\mu$. Similarly, the constraint $\langle \bar{V}, \bar{\Omega}_i \rangle = 0$ removed a spinor doublet from the degrees of freedom of $\bar{\Omega}_i$, but this breaks the $S$–symmetry, and thus the gravitino still has 24 degrees of freedom. As a result, the superconformal invariance is reduced to super-Poincaré. The super-Poincaré multiplet contains the graviphoton, which resides in $\langle \bar{V}, \bar{F}_{ab} \rangle$. Similarly the other internal products with $\bar{V}$ can be seen as auxiliary fields.
of the 40 + 40 off-shell super-Poincaré multiplet. In other formulations they are expressed in terms of another compensating multiplet. This compensating multiplet is then also used to gauge fix the $SU(2)$ invariance which we have not broken here.

### Table 2: Degrees of freedom in the model after the special Kähler constraints

| fields          | d.o.f | comments                                    |
|-----------------|-------|---------------------------------------------|
| $e^a_{\mu}$     | 6     | 16 - 4 (translation) - 6 (Lorentz)           |
| $A_{\mu}$       | 4     | gauge vector → vector                        |
| $V^{i}_{\mu \ j}$ | 9     | 12 - 3 ($SU(2)$)                            |
| $\psi^i_{\mu}$  | 24    | 32 - 8 (Q-supersymmetry)                     |
| $T^{ij}_{ab}$    | 6     | complex antiself-dual                       |
| $\chi_i$        | 8     | real scalar                                 |
| $D$             | 1     |                                             |
| $\langle V, \dot{Y}_{ij} \rangle$ | 6     |                                             |
| $\langle V, \dot{F}_{ab} \rangle$ | 6     |                                             |
| $\langle V, \dot{A}_{i} \rangle$ | 8     |                                             |
| $\langle V, \dot{C} \rangle$ | 2     |                                             |

#### 3.3.4 Further variations of constraints in the special case

Now we continue the analysis of the supersymmetry transformations on special Kähler constraints for supergravity theories with $Z_z = \langle V, U_z \rangle \neq 0$. This can only happen for $n = 1$, because that is the only case where this condition is not equivalent with (I.12). Because $n = 1$, $U_\alpha$ and $D_\alpha$ can be replaced by $U_z$ and $D_z$.

The computation of the special Kähler constraint goes along the same track as for the generic case, but extra terms appear because of the weaker constraint. The new
contributions appear for the first time after the supersymmetry variation of \((3.37)\):

\[
\begin{align*}
\langle V', \tilde{Y}_{ij} \rangle &= -D_z Z_z \cdot \tilde{x}^i \lambda_j^z, \\
\langle U'_z, \tilde{Y}_{ij} \rangle &= -C_{zzz} \mathcal{Z} \tilde{x}_{ij} - C_j z \lambda_j^z, \\
\langle V', \tilde{F}_{-ab} \rangle &= -\frac{1}{2} D_z Z_z \cdot \varepsilon_{ij} \left( \tilde{x}_i \sigma_{ab} \lambda_j^z \right), \\
\langle U'_z, \tilde{F}_{-ab} \rangle &= -\frac{1}{2} C_{zzz} \varepsilon_{ij} \left( \tilde{x}_i \sigma_{ab} \lambda_j^z \right).
\end{align*}
\] (3.45)

Define a new vector \(V''\)

\[
V'' \equiv V' - g^{zz} D_z C_{zzz} \cdot U'_z \frac{U'_z}{C_{zzz}} = V' - D_z Z_z \cdot U'_z. \tag{3.46}
\]

In terms of \(V''\) the constraints will have the same form as before:

\[
\langle V'', \tilde{Y}_{ij} \rangle = \langle V'', \tilde{F}_{-ab} \rangle = 0. \tag{3.47}
\]

Then, one finds

\[
\begin{align*}
\langle V', \tilde{\Lambda}_i \rangle &= \frac{1}{2} \varepsilon_{ij} \left( D_z Z_z \right) C_{zzz} \left( \tilde{U}_z, \tilde{Y}_{ij} \right) \varepsilon_{jk} \lambda_j^z + \langle \tilde{U}_z, \sigma \cdot \tilde{F} \rangle \lambda_i^z \\
&\quad - \frac{1}{6} \left( C_{zzz} - D_z D_z Z_z \right) \varepsilon_{ij} \lambda_j^z \sigma_{ab} \lambda_i^z, \tag{3.48}
\end{align*}
\]

\[
\begin{align*}
\langle U'_z, \tilde{\Lambda}_i \rangle &= \frac{1}{2} \varepsilon_{ij} C_{zzz} \left( \tilde{U}_z, \tilde{Y}_{ij} \right) \varepsilon_{kl} \lambda_k^z + \langle \tilde{U}_z, \sigma \cdot \tilde{F} \rangle \lambda_i^z \\
&\quad + \frac{1}{6} D_z C_{zzz} \cdot \varepsilon_{ij} \left( \lambda_k^z \sigma_{ab} \lambda_i^z \right) \sigma_{ab} \lambda_j^z, \tag{3.49}
\end{align*}
\]

where we have used that

\[
\langle V, D_z D_z U_z \rangle = -C_{zzz} + D_z D_z Z_z. \tag{3.50}
\]

Note that these are the analogues of \((3.42)\) and \((3.41)\).

Equation \((3.48)\) can be replaced by

\[
\langle V'', \tilde{\Lambda}_i \rangle = \frac{1}{6} \varepsilon_{ij} \lambda_k^z \sigma_{ab} \lambda_j^z \sigma_{ab} \lambda_i^z \frac{1}{C_{zzz}} \left( \left( -C_{zzz} + D_z D_z Z_z \right) C_{zzz} - D_z Z_z \cdot D_z C_{zzz} \right). \tag{3.51}
\]

Using the notation

\[
O_{zzzz} \equiv g^{zz} \tilde{Z}_z \left[ \left( -C_{zzzz} + D_z D_z Z_z \right) C_{zzzz} - D_z Z_z \cdot D_z C_{zzzz} \right], \tag{3.52}
\]

the variation of \((3.49)\) now gives

\[
\begin{align*}
\langle U'_z, \tilde{C} \rangle &= \frac{1}{4} C_{zzzz} g^{zz} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{ab} \varepsilon_{cd} \tilde{U}_z, \tilde{Y}_{ij} \rangle \langle \tilde{U}_z, \tilde{Y}_{kl} \rangle \varepsilon_{ab} \varepsilon_{cd} \varepsilon_{ef} \varepsilon_{gh} \langle \tilde{U}_z, \tilde{F}_{-ab} \rangle \\
&\quad - \frac{1}{2} C_{zzzz} g^{zz} \varepsilon_{ij} \langle \tilde{U}_z, \tilde{F}_{-ab} \rangle \\
&\quad - \frac{1}{2} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{ab} \langle \tilde{C}_{zzzz} \left( V, \tilde{Y}_{ij} \right) + D_z C_{zzzz} \cdot g^{zz} \langle \tilde{U}_z, \tilde{F}_{-ab} \rangle \lambda_k^z \lambda_l^z \\
&\quad + \frac{1}{2} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{ab} \langle \tilde{C}_{zzzz} \left( V, \tilde{F}_{-ab} \rangle + D_z C_{zzzz} \cdot g^{zz} \langle \tilde{U}_z, \tilde{F}_{-ab} \rangle \varepsilon_{kl} \lambda_k^z \lambda_l^z \\
&\quad - 2 \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{ab} \lambda_j^z \lambda_i^z \langle \tilde{U}_z, \tilde{A}_i \rangle \\
&\quad + \frac{1}{12} \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{ab} \varepsilon_{cd} \langle \tilde{O}_{zzzz} \rangle \varepsilon_{ij} \lambda_k^z \lambda_l^z \rangle \varepsilon_{ij} \lambda_k^z \lambda_l^z \rangle. \tag{3.53}
\end{align*}
\]

\(^6\)Note that \(g_{zz} D_z Z_z = D_z C_{zzzz}\) is not necessarily 0 in this case.
Straightforward variation of constraint (3.48) gives

\[ \langle V', \tilde{C} \rangle = \frac{1}{4} D_z Z_z \cdot g^{zz} g^{z\bar{z}} \varepsilon^{ijl} \langle \bar{U}_{z}, \bar{Y}_{ij} \rangle \langle \bar{U}_{z}, \bar{Y}_{kl} \rangle - \frac{1}{2} D_z Z_z \cdot g^{zz} \langle \bar{U}_{z}, \bar{F}_{\alpha \beta} \rangle \langle \bar{U}_{z}, \bar{F}^{-\alpha \beta} \rangle + \frac{1}{2} \varepsilon^{ijk} \varepsilon^{jl} \left( \left( C_{zzz} - D_z D_z Z_z \right) g^{zz} \langle \bar{U}_{z}, \bar{Y}_{ij} \rangle - D_z Z_z \cdot \langle \bar{V}, \bar{Y}_{ij} \rangle \right) \bar{\lambda}_i^z \bar{\lambda}_i^z - \frac{1}{2} i \left( \left( C_{zzz} - D_z D_z Z_z \right) g^{zz} \langle \bar{U}_{z}, \bar{F}_{\alpha \beta} \rangle - D_z Z_z \cdot \langle \bar{V}, \bar{F}_{\alpha \beta} \rangle \right) \varepsilon^{ijl} \left( \bar{\lambda}_i^z \sigma^{ab} \bar{\lambda}_j^z \right) \\
-2i D_z Z_z \cdot g^{zz} \varepsilon^{ijl} \langle \bar{U}_{z}, \bar{\lambda}_i^z \rangle \bar{\lambda}_i^z - \frac{1}{12} \left( 2 D_z C_{zzz} - D_z D_z D_z Z_z \right) \varepsilon^{ijl} \left( \bar{\lambda}_i^z \sigma^{ab} \bar{\lambda}_j^z \right) \varepsilon^{kl} \left( \bar{\lambda}_k^z \sigma^{ab} \bar{\lambda}_l^z \right). \tag{3.54} \]

This can be rewritten in

\[ \langle V'', \tilde{C} \rangle = \frac{ig^{zz}}{2C_{zzz}} \left[ \left( C_{zzz} - D_z D_z Z_z \right) C_{zzz} + D_z C_{zzz} \cdot D_z Z_z \right] \\
\times \left[ \varepsilon^{ijk} \varepsilon^{jl} \langle \bar{U}_{z}, \bar{Y}_{ij} \rangle \bar{\lambda}_i^z \bar{\lambda}_j^z - \langle \bar{U}_{z}, \bar{F}_{\alpha \beta} \rangle \varepsilon^{ijl} \left( \bar{\lambda}_i^z \sigma^{ab} \bar{\lambda}_j^z \right) \right] \\
-\frac{1}{12} \left[ \left( 2 D_z C_{zzz} - D_z D_z D_z Z_z \right) + \frac{D_z Z_z}{C_{zzz}} \left( D_z D_z C_{zzz} + \frac{1}{2} O_{zzzz} \right) \right] \\
\times \varepsilon^{ijl} \left( \bar{\lambda}_i^z \sigma^{ab} \bar{\lambda}_j^z \right) \varepsilon^{kl} \left( \bar{\lambda}_k^z \sigma^{ab} \bar{\lambda}_l^z \right). \tag{3.55} \]

4 The generalized Bianchi identities combined with the special Kähler constraints

In this section we start by imposing the reduction constraints on the chiral multiplets. Because the constraints on the field strengths are Bianchi identities, this linear multiplet of constraints is called the generalized Bianchi identities. We combine these constraints with the special Kähler constraints of section 3. Together they give the field equations of \( n \) vector multiplets and expressions for the auxiliary fields \( \chi_i \) and \( D \). We derive this for the generic case \( \langle V, U_\alpha \rangle = 0 \). We comment on the supergravity equations of motion in this generic case. Finally, we give the equations for the special case where \( \langle V, U_\alpha \rangle \neq 0 \).

4.1 The field equations for the generic case

To see what follows from equations (2.11)–(2.14), we take the symplectic inner product of these equations with the basis \( \mathcal{W} \). The 4 components of equation (A.6) give 4 equations for each constraint.
From the first identity we learn that the section $\tilde{Y}_{ij}$ is totally constrained, as it should be because it is auxiliary. We have

$$\tilde{Y}_{ij} = -ig^{\alpha\beta} \left[ C_{\alpha\beta\gamma} \lambda^\gamma_i \lambda^\gamma_j \bar{U}_\alpha \right] - C_{\alpha\beta\gamma} \varepsilon_{ik} \varepsilon_{jl} \lambda^{ik} \lambda^{jl} U_\alpha .$$

(4.1)

It is more interesting to take a look at (2.12). Taking the symplectic inner product of this equation with all components of the basis $\mathcal{W}$ (3.3), and using the special Kähler constraints of section 3.3 and (4.1) gives:

$$\langle \tilde{V} , \tilde{\Lambda}_i \rangle = 0 ,$$

$$\langle \tilde{U}_\alpha , \tilde{\Lambda}_i \rangle = -\varepsilon_{ij} C_{\alpha\beta\gamma} \nabla^i \lambda^j ,$$

$$0 = \chi^i - \frac{2}{3} \sigma^{ij} \left( D_{ij} \psi_i \mp \frac{1}{8} \sigma \cdot T^{ij} \psi_{ij} \right) - \frac{1}{2} g_{\alpha\beta} \bar{g} z^\alpha \cdot \lambda^{ij} + \frac{i}{2} \varepsilon^{ij} \gamma^\mu \left( V , \sigma \cdot \bar{F} \right) \psi_{ij} + \frac{1}{12} i C_{\alpha\beta\gamma} \varepsilon^{ijkl} \left( \lambda^{ik} \sigma^{ab} \lambda^j \right) \sigma_{ab} \lambda^\gamma ,$$

$$0 = ig_{\alpha\beta} \left( \gamma^\mu \lambda^{i} + \frac{1}{2} (\sigma \cdot \bar{A}) \lambda^{i} \right) + \frac{1}{2} \varepsilon^{ij} C_{\alpha\beta\gamma} g^{\beta\gamma} (\bar{U}_\alpha , \sigma \cdot \bar{F}) \lambda^\gamma_j + \frac{1}{2} i C_{\alpha\beta\gamma} C_{\alpha\bar{\beta} \bar{\gamma}} g^{\gamma(\lambda_{i}^{a} \lambda_{j}^{\bar{a}})} \lambda^\beta_j + \frac{1}{6} D_{\alpha} C_{\beta \gamma \delta} \varepsilon^{ijkl} \left( \lambda^{(\delta)^{b} \sigma^{ab} \lambda^l) \right) \sigma_{ab} \lambda^\delta ,$$

(4.2)

where

$$\nabla_{\mu} \lambda^{\alpha}_i = \partial_{\mu} \lambda^{\alpha}_i - \frac{1}{2} \omega^{ab}_{\mu} \sigma_{ab} \lambda^{\alpha}_i + \frac{1}{2} \gamma_{\mu} \lambda^{j} \lambda^{\alpha}_j + \Gamma^\alpha_{\beta\gamma} \partial_{\mu} z^\beta \cdot \lambda^\gamma - \frac{1}{2} i Q_{\mu} \lambda^{\alpha}_i$$

$$- \nabla^\alpha \psi_{\mu i} + \frac{1}{2} ig^{\alpha\bar{\alpha}} \left[ (\bar{U}_\alpha , \tilde{V}_{ij}) + \varepsilon_{ij} (\bar{U}_\alpha , \sigma \cdot \bar{F}) \right] \psi_{\mu j} ,$$

and

$$Q_{\mu} = -\frac{1}{2} i (\partial_{\alpha} K \cdot \partial_{\mu} z^\alpha - \text{h.c.})$$

(4.3)

is the Kähler 1-form.

The first two equations in (4.2) imply with (3.42) and (3.41) that all components of $\tilde{\Lambda}_i$ are expressed in terms of other fields, and thus they contain no independent degrees of freedom. The third expresses $\chi^i$ in terms of other fields. In a superconformal calculation using a Lagrangian, this expression for $\chi^i$ can be found from the equation of motion of the fermion of the compensating vector multiplet. The fourth equation is the field equation for $n$ fermion doublets $\lambda^\beta_i$.

We now proceed with the analysis of (2.13). We first repeat that (3.40) implies that there are $(n + 1)$ independent antisymmetric tensors in the symplectic vector $\bar{F}_{ab}$. Apart from these there is another antiself-dual tensor in the Weyl multiplet $T^{ij}_{ab}$. A few definitions make (2.13) more transparent. First define the combination in brackets as

$$\bar{F}_{ab} = \tilde{F}_{ab} + \frac{1}{4} VT_{abij} \varepsilon^{ij} + \frac{1}{4} VT_{ab} \varepsilon^{ij} .$$

(4.5)
Then we take out covariantization terms:

\[
\tilde{F}_{ab} = \tilde{\tilde{F}}_{ab} - 2(\tilde{\Omega}^i \gamma_{[a} \psi_{b]}^j \epsilon_{ij} + \tilde{V} \tilde{\psi}_i^j \psi_{b}^j \epsilon_{ij} + \text{h.c.}).
\] (4.6)

This is chosen such that covariant derivatives in (2.13) can be rewritten as ordinary derivatives, and the equation reduces to

\[
\partial_\mu \epsilon^{\mu\nu\rho\sigma} \tilde{F}_{\rho\sigma} = 0.
\] (4.7)

Applying this on the \(n+1\) independent components of \(\tilde{F}_{\mu
u}\), implies that they can be expressed in terms of \(n+1\) vectors. The other \((n+1)\) equations of (4.7) are the equivalent of field equations for these vectors. Here, it is clear how our formulation keeps the symplectic covariance. Only in the interpretation do we distinguish one half of the equations as Bianchi identities and the others as field equations. These could have been interchanged giving the ‘magnetic dual formulation’. Also the fact of whether or not a prepotential exists is hidden here. The difference is seen only when breaking the symplectic formulation in finding an explicit solution of equations (3.40).

If the \((n+1)\times(n+1)\)-matrix, formed by the upper part of \((V, U_\alpha)\) is invertible, then (3.40) expresses the \((n+1)\) lower components of \(F_{ab}^\prime\) in terms of the upper ones. This is the case where there is a prepotential. When this matrix is not invertible\(^7\), then one can still solve (3.40) for other \((n+1)\) components of \(\tilde{F}_{ab}^\prime\).

We thus conclude that we have \(n+1\) on-shell vectors and their field equations also depend on the 6 degrees of freedom of the tensor \(T_{ij}^{ab}\) of the Weyl multiplet.

Now let us have a look at (2.14). It involves the covariant Laplacian,

\[
\Box \tilde{V} \equiv \eta^{mn} D_m D_n \tilde{V} = e^{-1} \partial_\mu (e D^\mu \tilde{V}) + (b^\mu - i A^\mu) D_\mu \tilde{V} + f_\mu^\mu \tilde{V} + 2 \bar{\psi}_i^j \gamma_\mu \psi_{i}^j D_\mu \tilde{V} - \bar{\psi}_i^j D_\mu \tilde{\Omega}^j + \frac{1}{2} \bar{\psi}_i^j \gamma_\mu \tilde{\Omega}^j + \frac{1}{8} \bar{\psi}_i^j \gamma_\mu \sigma \cdot T_{ij} \tilde{\Omega}^j - \frac{3}{2} \bar{\psi}_i^j \gamma_\mu \chi_1 \tilde{V}.
\] (4.8)

To derive this expression, we used a theorem on covariant derivatives in the second reference of [10]. We can again take the symplectic inner product of (2.14) with \(\mathcal{W}\):

\[
\langle \tilde{V}, \tilde{C} \rangle = 0,
\]

\[
\langle \tilde{U}_\alpha, \tilde{C} \rangle = -2 \tilde{C}_{\alpha \beta} \gamma_\mu \tilde{z}^{\beta} \cdot \nabla^{\mu} \tilde{z}^{\beta} + \frac{1}{4} \tilde{C}_{\alpha \beta} \gamma_\mu \tilde{\lambda}^{\beta} \cdot T_{ki} \tilde{\lambda}^{ki},
\]

\[
0 = -2 e^{-1} \partial_\mu (e (Q^\mu - A^\mu)) + 2i g_{\alpha \dot{\alpha}} \partial_\mu z^\alpha \cdot \nabla^{\mu} \tilde{z}^{\dot{\alpha}} - 2i g_{\alpha \dot{\alpha}} \partial_\mu z^\alpha \cdot (\bar{\psi}_i^\mu \chi_1) - 2i (Q^\mu - A^\mu) (Q_\mu - A_\mu) + 2i f_{\mu}^\mu - 4 \bar{\psi}_i^j \gamma_\mu \psi_{i}^j (Q_\nu - A_\nu).
\]

\(^7\) As proven in [17], it is only the matrix \((f_\alpha^\mu X^\mu)\) that is always invertible.
The fourth equation of the expansion in terms of complex scalars. So we find the same structure in the equations as for the fermions: an expression for the constraint using the results of the previous section, we comment on the appearance of equations The real part constrains the divergence of other fields. The real and imaginary part of the third equation have both to be kept terms with an arbitrary power of undifferentiated scalar fields or metric, but only symplectic covariant formulation. With the linearized approximation we mean that we only expose the linear terms. This already shows the essential features of this which gives rise to a 24 + 24 ‘current’ multiplet. The \( \approx \) side of (4.10) would, for example, contain an additional coupling to hypermultiplets.

The first two equations, together with (3.44) and (3.43) (which can be simplified using the second equations in (4.2)) determine that \( \tilde{C} \) is completely determined in terms of other fields. The real and imaginary part of the third equation have both to be 0. The real part constrains the divergence of \( Q_\mu - A_\mu \), and the imaginary part gives an expression for the \( D \)-field of the Weyl multiplet. (\( D \) is hidden in the \( f_\mu^\mu \)-term.) The fourth equation of the expansion in terms of \( W \) gives the field equations for \( n \) complex scalars. So we find the same structure in the equations as for the fermions: \( n + 1 \) equations express \( \tilde{C} \) in terms of other fields, while the \( n + 1 \) other equations give the field equations for \( n \) complex scalars \( z^\alpha \), an expression for \( D \) and a constraint for \( (Q_\mu - A_\mu) \). The degrees of freedom are described in table 3.

4.2 Comments on the supergravity equations

Using the results of the previous section, we find on the appearance of equations of motion for the remaining 24+24 components of table 3 from one symplectic invariant constraint

\[
\langle V, \tilde{F}_\alpha^+ \rangle \approx 0, \tag{4.10}
\]

which gives rise to a 24 + 24 ‘current’ multiplet. The \( \approx \)-sign is used to denote that we only expose the linear terms. This already shows the essential features of this symplectic covariant formulation. With the linearized approximation we mean that we keep terms with an arbitrary power of undifferentiated scalar fields or metric, but only linear in other fields. In a full treatment of \( N = 2 \) supergravity couplings the right-hand side of (4.11) would, for example, contain an additional coupling to hypermultiplets.
To discuss the supersymmetry partners of (4.10), we derive a new \( N = 2 \) multiplet with 24 + 24 components. The multiplet starting with the symplectic expression \( \langle V, \tilde{F}^+_{ab} \rangle \) is a supergravity realization of this multiplet. As shown below the supermultiplet of constraints derived from (4.10) is only equivalent to the supergravity equations of motion, up to integration ‘constants’. These 8 + 8 remaining unknowns can be determined when one of the three possibilities of a second compensating multiplet is introduced as in [19, 7, 9]. In our approach this is the place where the second compensating multiplet, which is also needed for consistency in the Lagrangian formulation, comes into play.

### 4.2.1 A restricted chiral self-dual tensor multiplet

The supermultiplet structure of the ‘current’ multiplet from (4.10) is that of a chiral self-dual tensor multiplet,

\[
W^+_{ab} = A^+_{ab} + \bar{\theta}^i \psi_{abi} + \frac{1}{4} \bar{\theta}^i \theta^j B_{abij} + \frac{1}{4} \epsilon_{ijkl} \bar{\theta}^i \sigma_{cd} \theta^j F_{ab}^{cd} + \frac{1}{6} \epsilon_{ijkl} (\bar{\theta}^i \sigma_{cd} \theta^j) \bar{\theta}^k \sigma_{cd} \chi_{abk} + \frac{1}{48} (\epsilon_{ijkl} \bar{\theta}^i \sigma_{cd} \theta^j)^2 C^+_{ab}. \tag{4.11}
\]

It has the following field content. \( A^+_{ab} \) is a self-dual complex tensor with 6 degrees of freedom. \( \psi_{abi} \) has 24 left-handed fermionic components. The tensor \( B_{abij} \) has 18 components. The tensor \( F_{ab}^{cd} \) is self-dual in its first and antiself-dual in its second pair.
of indices, leading to 9 complex components. It also satisfies the following properties:

\[
F_{ab,cd} + F_{cd,ab} = \frac{1}{2} \varepsilon_{ab} \varepsilon_{cf} \left( F_{ef,cd} - F_{cd,ef} \right),
\]
\[
F_{ab,cd} - F_{cd,ab} = \varepsilon_{abe[d} \left( F_{e]c}^{\epsilon} + F_{e]c}^{\epsilon} \right),
\]
\[
F_{ab,cd} = \delta_{a[c} F_{d]b} - \delta_{b[c} F_{d]a} - \varepsilon_{abe[c} F^{\epsilon}_{d]},
\]
\[
F_{[ab]} = 0,
\]
\[
F_{a}^{a} = 0,
\]

\text{(4.12)}

where

\[
F_{a}^{c} = F_{ab}^{cd} \epsilon_{d}^{b}.
\]

\text{(4.13)}

A general component of this self-dual–antiself-dual tensor \( F_{ab}^{cd} \) can thus be written in terms of the traceless symmetric part \( F_{(ab)} \) with 9 components. The fermion \( \chi_{abi} \) has again 24 left-handed components and \( C_{ab}^{+} \) has 6. So, this is a chiral multiplet with 48 + 48 components.

The transformation rules of this multiplet are the same as for a chiral multiplet with a complex scalar as lowest component, but with the components replaced straightforwardly:

\[
\delta A_{ab}^{+} = \bar{c}^{i} \psi_{abi},
\]
\[
\delta \psi_{abi} = \bar{\theta} A_{ab}^{+} \epsilon_{i} + \frac{1}{2} B_{abij} \epsilon_{j} + \frac{1}{2} \sigma_{cd} F_{ab}^{cd} \varepsilon_{ij} \epsilon_{j},
\]
\[
\delta B_{abij} = 2 \bar{c}_{(i} \bar{\theta} \psi_{abj)} - 2 \bar{c}_{k} \chi_{ab(i} \varepsilon_{j)k},
\]
\[
\delta F_{ab}^{cd} = \varepsilon_{ij} \bar{c}_{i} \bar{\theta} \sigma_{cd} \psi_{abj} + \bar{c}_{i} \sigma_{cd} \chi_{abi},
\]
\[
\delta \chi_{abi} = -\frac{1}{2} \sigma_{cd} F_{ab}^{cd} \bar{\theta} \epsilon_{i} - \frac{1}{2} \bar{\theta} B_{abij} \varepsilon_{jk} \epsilon_{k} + \frac{1}{2} C_{ab}^{+} \varepsilon_{ij} \epsilon_{j},
\]
\[
\delta C_{ab}^{+} = -2 \bar{c}^{ij} \bar{c}_{i} \bar{\theta} \chi_{abj}.
\]

\text{(4.14)}

Since we have broken superconformal symmetry to super-Poincaré and \( SU(2) \), we only need a super-Poincaré version of this multiplet. Note that it cannot be extended to a superconformal one. The commutator of a supersymmetry and a special supersymmetry has to give a Lorentz transformation that can never be realized because of the duality and chirality properties of the spinors. For this reason, it is only possible to construct an antiself-dual chiral tensor multiplet, realizing the superconformal algebra, as given in [22].

To study the field equations of the fields of table 3, we need a multiplet with 24 + 24 components. A suitable multiplet of constraints is:

\[
0 = \partial^{a} \left( B_{abij} + \varepsilon_{ik} \varepsilon_{jl} \tilde{B}_{ab}^{kl} \right),
\]

\text{(4.15)}
0 = \partial^a (\chi^i_{ab} - \varepsilon^{ij} \partial \psi_{abj}) , \quad (4.16) \\
0 = \partial^a (C_{ab}^c - \Box A_{ab}^c) , \quad (4.17) \\
0 = \partial^a \partial_c (F_{ab}^{\, cd} + \tilde{F}_{ab}^{\, cd}) . \quad (4.18)

These are the analogues of the constraints (5.4) in [22]. This set contains \((9 + 6 + 9) + 24\) equations. The constraint for \(F_{abcd}^{\, cd}\) splits up in a part symmetric in \((bd)\) (6 independent equations) and an antisymmetric part in \([bd]\) (3 independent equations), which correspond to the real and imaginary part of \(F_{ac}^{\, c}\):

\[
0 = -\partial_c \left( \partial_b (F_{djc} + \tilde{F}_{djc}) \right) + \frac{1}{2} \delta_{bd} \partial^e \partial^c (F_{ac} + \tilde{F}_{ac}) + \frac{1}{2} \Box (F_{bd} + \tilde{F}_{bd}) \\
+ \frac{1}{2} \varepsilon_{bde} \partial^e (F_{c}^{\, e} - \tilde{F}_{c}^{\, e}) . \quad (4.19)
\]

As far as we know, this reduced multiplet is a new representation of the rigid \(N = 2\) algebra.

An explicit supergravity realization of this reduced multiplet is given by

\[
A_{ab}^+ = \langle V, \tilde{F}_{ab}^+ \rangle , \\
\psi_{abi} \approx -i \varepsilon^{ij} \gamma^a \sigma_{ab} \phi^j , \\
B_{abik} \approx 2i \varepsilon_{ij} R_{SU(2)}^+ \, ^{+^j}_{\, ^k} , \\
F_{ab}^{\, cd} \approx 2 \delta^{[c}_{[d} \left( \partial_{|b|} (Q_{d]}^{\, |a|} - A_{d]}^{\, |a|}) + (\partial^{[d} (Q_{b]}^{\, |a|} - A_{b]}^{\, |a|}) - 2i R_{b]\, ^l_{|f|} + \frac{1}{2} i \delta^{|l|}_{d]b]} R \right) \\
- \varepsilon^{\, cd}_{e f} \delta^{[e}_{[a} \left( \partial_{|b|} (Q^{f]} - A^{f]} \right) + (\partial^{[f} (Q_{b]} - A_{b]})) - 2i R_{b}\, ^f_{|l|} + \frac{1}{2} i \delta^{|l|}_{b]} R \right) \quad . (4.20)
\]

In deriving this multiplet we used the constraints of sections [3] and [4]. The expression for \(B_{abij}\) satisfies constraint (1.17), which is a Bianchi identity that expresses the existence of \(SU(2)\)-vectors. The expression for \(F_{ab}^{\, cd}\) fulfils (1.12). It also satisfies (1.18) when the third equation of (1.9) for \((Q_{\mu}^{\, |a|} - A_{\mu}^{\, |a|})\) is used. Therefore, the multiplet derived from \(\langle V, \tilde{F}_{ab}^+ \rangle\) has 24 + 24 components.

### 4.2.2 Some comments on the multiplet of equations from \(\langle V, \tilde{F}_{ab}^+ \rangle \approx 0\)

Putting the ‘current’ multiplet (4.20) to zero, will give rise to some supergravity field equations. These are 24 + 24 equations for the 24 + 24 remaining degrees of freedom of table [3]. The counting in this table subtracts the gauge degrees of freedom. The multiplet here is a multiplet of curvatures and the counting is equivalent if we take into account the Bianchi identities.

However, our equations are not equivalent to the complete supergravity equations of motion. They differ modulo ‘integration constants’. These can be determined when
a second compensating multiplet is coupled \[7, 19\]. Since this step is independent of the symplectic formulation of the coupling of vector multiplets to supergravity, we do not treat it here.

Let us give a brief discussion of the content of the equations following from (4.10). Equation (4.10) reduces 6 degrees of freedom. It expresses the ‘graviphoton’ field strength \( T_{abij} \) as a combination of the \( n + 1 \) on-shell vectors obtained above. It is the symplectic expression for the algebraic equation of motion that one finds in the Lagrangian approach, (4.11) in \[9\].

Using (4.2) in (2.1) with

\[
R_{\mu i} \equiv e^{-1}e^{\mu \nu \rho \sigma} \gamma_5 \gamma_\nu \left( D_{\rho} \psi^j - \frac{1}{8} \sigma \cdot T^{ij} \gamma_\rho \psi_{\sigma j} \right)
\]

(4.21)
in the second component of the current multiplet gives that

\[
\phi^i_\rho = R^i_\rho - \frac{1}{4} \gamma_\rho \gamma \cdot R^i_\rho \approx 0 ,
\]

(4.22)
the traceless part of the field equation of the gravitini. Therefore, this equation cannot determine the trace-part \( \gamma \cdot R^i \). However, combining (4.22) with the Bianchi identity for the gravitino field strength \( \partial^\mu R^i_\mu \approx 0 \), yields

\[
\partial \gamma \cdot R^i \approx 0 ,
\]

(4.23)
which determines \( \gamma \cdot R^i \) in terms of 8 ‘integration constants’.

The \( B_{abij} \) component yields

\[
R_{SU(2)ab}^i \approx 0 .
\]

(4.24)
Together with the Bianchi identity for the \( SU(2) \) curvature it states that the gauge fields \( \psi_{\mu}^{ij} \) are pure gauge, i.e.

\[
\psi_{\mu}^{i j} = (\varphi^{-1} \partial_\mu \varphi)^{i j} ,
\]

(4.25)
where \( \varphi \) is a group element of \( SU(2) \). The three local parameters defining \( \varphi \) are left undetermined.

\( F_{ab}^{cd} \) has its components in the traceless part of \( F_{(ac)} \). From \( F_{ab}^{cd} \approx 0 \) follows

\[
F_{ac} = 2 \partial_{(a} (Q_{c)} - A_{c}) - 2 i R_{ac} + \frac{1}{2} i g_{ac} \mathcal{R} \approx 0 .
\]

(4.26)
The imaginary part is the traceless part of the Einstein equation. Again we cannot determine the scalar curvature \( \mathcal{R} \) from this equation. However, combining this equation with the Bianchi identity for the Einstein tensor

\[
\partial^a (\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R}) = 0 ,
\]

(4.27)
gives
\[ \partial^a R \approx 0 \]  
(4.28)
and again $R$ is determined up to a constant. The real part of the $F$-component gives that $A_\mu \approx Q_\mu$ up to a constant vector. Also in the Lagrangian approach [3], one finds
\[ A_\mu \approx Q_\mu. \]  
(4.29)

The additional $8 + 8$ remaining unknowns can be determined through the field equations of a second compensating multiplet. This concludes the short discussion of the supergravity equations of motion.

4.3 The field equations for the special case

In this subsection, the expressions of section 4.1 are generalized for the case $n = 1$ where further $Z_z = \langle V, U_z \rangle \neq 0$. This is the case that was excluded by the former definitions and where our less restrictive definitions becomes relevant. The equations are found by expanding the constraints in terms of the basis of symplectic vectors using the methods mentioned at the end of the appendix.

The section $\tilde{Y}_{ij}$ remains totally constrained:

\[ \tilde{Y}_{ij} = g'_{ij} \left( -i\varepsilon_{ik}\varepsilon_{jl}g_{zz}\nabla\tilde{z} \cdot \tilde{\lambda}_k \tilde{\lambda}_l \tilde{V} + ig_{zz}\nabla z Z_z \cdot \tilde{\lambda}_i \lambda_j \tilde{V} + i\varepsilon_{ik}\varepsilon_{jl} \tilde{C}_{zzz} \tilde{\lambda}_k \lambda^l U_z - i\tilde{C}_{zzz} \tilde{\lambda}_i \lambda^j \tilde{U}_z \right). \]  
(4.30)

This equation reduces to the former equation (4.11) when $\langle V, U_z \rangle = 0$.

The equations that can be derived from the constraints for the fermions are the following ones:

\[ \langle \tilde{V}', \tilde{A}_i \rangle = -\nabla z \tilde{Z}_z \cdot \varepsilon_{ij} \nabla \tilde{z} \cdot \lambda^{ij}, \]
\[ \langle \tilde{U}_z', \tilde{A}_i \rangle = -\varepsilon_{ij} \tilde{C}_{zzz} \nabla \tilde{z} \cdot \lambda^{ij} + \frac{1}{2}\gamma^\mu \varepsilon_{ij} \left( \langle \tilde{U}_z', \tilde{Y}^{jk} + \sigma \cdot \tilde{F}^+ \varepsilon^{jk} \rangle \right) \psi_{\mu j}, \]
\[ 0 = \chi^i - \frac{2}{3} \sigma^{\mu \nu} (D_\mu \psi^j_\nu - \frac{1}{8} \sigma \cdot T^{ij} \gamma_\mu \psi_{ij}) - \frac{1}{2} \left( g_{zz} \bar{\theta} \cdot \lambda^{ij} - i\gamma^\mu (\bar{Q} - \bar{A}) \right) \psi^i_j - g_{zz} \frac{1}{3} \tilde{Z}_z \tilde{C}_{zzz} \bar{\theta} \cdot \lambda^{ij} + \frac{1}{4}\gamma^\mu (g_{zz} \langle \tilde{V}', \tilde{Y}^{ij} + \sigma \cdot \tilde{F}^+ \varepsilon^{ij} \rangle) \psi_{\mu j} - \frac{1}{4} \varepsilon^{ij} \nabla z Z_z \cdot \langle \tilde{U}_z, \tilde{Y}_{jk} \rangle \varepsilon^{kl} \lambda^i_k + \langle \tilde{U}_z, \sigma \cdot \tilde{F}^- \rangle \lambda^i_j \]
\[ -\frac{1}{12} \varepsilon^{ij} \langle \tilde{U}_z, \tilde{Y}_{jk} \rangle \varepsilon^{kl} \lambda^i_k \sigma_{ab} \lambda^l_j \]  
(4.31)
\[ 0 = ig'_{zz} \left( \nabla \tilde{z} \lambda^{ij} + \frac{1}{2} (\bar{Q} - \bar{A}) \lambda^{ij} \right) \]
\[ + \frac{1}{2} i\varepsilon^{ij} \tilde{C}_{zzz} g_{zzz} \left( \langle \tilde{U}_z, \tilde{Y}_{jk} \rangle \varepsilon^{kl} \lambda^i_k + \langle \tilde{U}_z, \sigma \cdot \tilde{F}^- \rangle \lambda^i_j \right) \]
\[ + \frac{1}{6} \tilde{D}_z \tilde{C}_{zzz} \cdot \varepsilon^{ij} \varepsilon^{kl} \langle \tilde{\lambda}_k \sigma_{ab} \lambda^l_j \rangle \sigma_{ab} \lambda^i_j - iZ_z \tilde{D}_z \tilde{Z}_z \cdot \nabla \tilde{z} \cdot \lambda^{ij}. \]  
(4.31)
Also here, the equations reduce to those that we have found for the generic case where \( \langle V, U_\z \rangle = 0 \). The same comments as in section 4.1 are valid here.

Repeating the analysis for the equations of the vectors, it appears that there is no information used about \( Z_\z \). This means that the analysis of the equations for the vectors of section 4.1 remains valid. This is no surprise because the equations for the vectors are a symplectic section of equations. All the other equations are singlets for the symplectic group and can therefore be written as symplectic invariant equations.

Also the last constraint can be decomposed with respect to the symplectic basis. Then the equations become:

\[
\langle \vec V', \vec C' \rangle = -2\bar \psi^\mu_i \langle \vec V, \vec Y^{ij} + \sigma \cdot \vec F^+ \epsilon^{ij} \rangle \psi_{\mu j} - 2\partial_{\mu \z} \cdot D_\z Z_\z \cdot \left( \nabla^\mu \z - \bar \psi_i^\mu \lambda^i \z \right) + \frac{1}{4} D_\z \bar Z_\z \lambda^i \z \sigma \cdot T_{ij} \lambda^j \z ,
\]

\[
\langle \vec U', \vec C' \rangle = -2\bar C_{\z \z \z \z} \nabla^\mu \z \cdot \nabla^\mu \z + \frac{1}{4} \bar C_{\z \z \z \z} \left( \lambda^k \z \sigma \cdot T_{kl} \lambda^l \z \right),
\]

\[
0 = g^{\z \z} g'^{\z \z} \left( 2e^{-1} \partial_{\mu} \left( e (Q^\mu - A^\mu) \right) + 2i (Q^\mu - A^\mu)(Q_{\mu} - A_{\mu}) - 2i f_{\mu} + 3i \bar \psi_{\mu} \gamma^\mu \chi_i \frac{1}{2} e^{-1} \partial_{\mu} \left( e (Q^\mu - A^\mu) \right) + 2i (Q^\mu - A^\mu)(Q_{\mu} - A_{\mu}) - 2i f_{\mu} + 3i \bar \psi_{\mu} \gamma^\mu \chi_i \right) + 4 \bar \psi^\mu_i \gamma_{\mu} \psi^j_i(Q_{\nu} - A_{\nu}) + 2i \bar \psi^\mu_i (Q - A_i) \psi_{\mu j} - 2i \bar \psi_{\mu} \phi_{\mu j}
\]

\[
+ \bar \psi_{\mu j} \bar \psi_{\mu j} \bar \epsilon^{ik} \bar \epsilon^{jl} D_\z Z_\z \cdot \bar \lambda_i^k \lambda_i^l \z - \bar \psi_{\mu j} \bar \psi_{\mu j} (V', \sigma \cdot \vec F^+) \epsilon^{ij} \psi_{\mu j} \z - \frac{1}{4} g^{\z \z} g'^{\z \z} \langle \vec V', \bar F_{\mu}^+ \rangle T_{ij} \varepsilon^{ij}
\]

\[
+ 2i g^{\z \z} Z_\z \left( iD_\z Z_\z \cdot \partial_{\mu} \z \cdot (Q_{\mu} - A_{\mu}) + \bar \psi_{\mu} C_{\z \z \z \z} \partial_{\mu} \z \cdot \lambda^i \z \right)
\]

\[
+ \frac{1}{2} g^{\z \z} D_\z Z_\z \cdot g^{\z \z} \left( \frac{1}{2} \bar \epsilon^{ik} \bar \epsilon^{jl} \langle U_\z, \bar Y_{ij} \rangle \langle U_\z, \bar Y_{kl} \rangle - \langle U_\z, \bar F^- \rangle \langle U_\z, \bar F_{-ab} \rangle \right)
\]

\[
- \frac{1}{4} \bar \epsilon^{ik} \bar \epsilon^{jl} \left( D_\z Z_\z \cdot (V, \bar Y_{ij}) + (-C_{\z \z \z \z} + D_\z D_\z Z_\z) g^{\z \z} \langle U_\z, \bar Y_{ij} \rangle \bar \lambda_i^k \lambda_i^l \z
\]

\[- \frac{1}{2} i g^{\z \z} D_\z Z_\z \cdot \bar \epsilon^{ij} \bar \lambda_i^k \langle U_\z, \bar \lambda_j \rangle
\]

\[- \frac{1}{2} i D_\z Z_\z \cdot (V, \bar F_{-ab}) + (-C_{\z \z \z \z} + D_\z D_\z Z_\z) g^{\z \z} \langle U_\z, \bar F_{-ab} \rangle \varepsilon^{kl} \langle \bar \lambda_i^k \sigma_{ab} \lambda_i^l \z \rangle
\]

\[
0 = -2i g^{\z \z} \left[ \varepsilon^{-1} \partial_{\mu} \left( e \nabla^\mu \z \right) + 2i (Q_{\mu} - A_{\mu}) \nabla^\mu \z \right]
\]

\[
+ \frac{1}{2} i (Q_{\mu} - A_{\mu}) \bar \psi_{\mu} \gamma_{\mu} \psi_{\mu} \nabla_{\mu} \z
\]

\[
+ 3 \bar \psi_{\mu} \lambda^i \z - \frac{1}{2} \bar \lambda^i \z \gamma_{\mu} \phi_{\mu} + \Gamma_{\z \z \z \z} \partial_{\mu} \z \cdot \nabla^\mu \z + \frac{1}{4} \bar \psi_{\mu} \gamma_{\mu} \sigma \cdot T_{ij} \lambda^i \z
\]

\[
- \bar \psi_{\mu} \left( \nabla_{\mu} \lambda^i \z + \frac{1}{2} i (Q_{\mu} - A_{\mu}) \lambda^i \z + \frac{1}{2} g^{\z \z} \langle U_\z, \bar Y_{ij} \rangle + \sigma \cdot \bar \psi_{\mu} \gamma_{\mu} \chi_i \varepsilon^{ij} \langle \psi_{\mu j} \rangle \right)
\]

\[
+ 2i Z_\z \left( \partial_{\mu} \z \cdot D_\z Z_\z \cdot \nabla_{\mu} \z - \bar \psi_{\mu} \lambda^i \z \right) + \bar \psi_{\mu} \langle V, \bar Y_{ij} \rangle + \sigma \cdot \bar \psi_{\mu} \gamma_{\mu} \chi_i \varepsilon^{ij} \langle \psi_{\mu j} \rangle
\]

\[
+ \frac{1}{4} C_{\z \z \z \z} g^{\z \z} g'^{\z \z} \bar \epsilon^{ik} \bar \epsilon^{jl} \langle U_\z, \bar Y_{ij} \rangle \langle U_\z, \bar Y_{kl} \rangle - \frac{1}{2} C_{\z \z \z \z} g^{\z \z} g'^{\z \z} \langle U_\z, \bar F_{-ab} \rangle \langle U_\z, \bar F_{-ab} \rangle
\]

\[
- \frac{1}{2} i \bar \epsilon^{ik} \bar \epsilon^{jl} \left( C_{\z \z \z \z} \langle V, \bar Y_{ij} \rangle + D_\z C_{\z \z \z \z} \cdot g^{\z \z} \langle U_\z, \bar Y_{ij} \rangle \right) \bar \lambda_i^k \lambda_i^l \z + \frac{1}{4} \langle U_{ij}, \bar F_{-ab} \rangle T_{ij} \varepsilon^{ij}
\]

\[
+ \frac{1}{2} i \left( C_{\z \z \z \z} \langle V, \bar F_{-ab} \rangle + D_\z C_{\z \z \z \z} \cdot g^{\z \z} \langle U_\z, \bar F_{-ab} \rangle \right) \varepsilon^{ij} \langle \bar \lambda_i^k \sigma_{ab} \lambda_i^l \z \rangle
\]
\[ + \frac{1}{12} \left( D_x D_y C_{zzz} + \frac{1}{2} \mathcal{O}_{zzzzz} \right) \varepsilon^{ij}(\bar{\lambda}_i^z \sigma_{ab} \lambda_j^z) \varepsilon^{kl}(\bar{\lambda}_k^z \sigma_{ab} \lambda_l^z) \]
\[ - 2i C_{zzz} g^{zz} \varepsilon^{ij} \bar{\lambda}_i^z (\bar{U}_z, \bar{\Lambda}_j) \, . \]  
(4.32)

The metric in front of the kinetic term of the scalar in the fourth equation is positive because of the physical condition (3.2). Again, all these equations reduce to the equations of section 4.1 if \( Z_z = 0 \) and the same conclusions can be drawn as in section 4.1. Therefore, we conclude at this point that the ‘special case’ is a valid alternative for a theory with \( N = 2 \) supergravity and one vector multiplet.

### 5 Conclusions

We have presented a fully symplectic invariant formulation of the coupling of an arbitrary number \( n \) of vector multiplets to \( N = 2 \) supergravity in 4 dimensions by using superconformal tensor calculus. This approach does not start from a prepotential, but rather from a \( 2(n+1) \) symplectic vector of chiral superconformal multiplets. We imposed the reducibility constraints (2.11)–(2.14) of chiral multiplets in supergravity to end up with vector multiplets in a superconformal background. Furthermore, we imposed the symplectic covariant defining equations of special Kähler geometry, and supersymmetric partners thereof. The bosonic defining equations include a breaking of dilatations and \( U(1) \)-transformations, the special conformal symmetry being broken as usual by imposing a constraint (3.23) on the dilatational gauge field. In the fermionic sector, one of the constraints, (3.26), breaks special supersymmetry. This results in unbroken Poincaré supersymmetry and \( SU(2) \) gauge symmetry. The other constraints are determined by demanding the preservation of the Poincaré supersymmetry.

The combination of all the special Kähler constraints (1.4)–(1.7) and their supersymmetric partners with the generalized Bianchi identities of the chiral multiplets gave rise to a full set of field equations for the vector multiplet fields.

Furthermore, we also discussed part of the equations of motion for the gravity sector. This could be done by imposing a new symplectic constraint (4.10) and its supersymmetric partners. The full analysis would need a second compensating multiplet.

Finally, we did the same analysis for a weaker definition of the special Kähler constraints where the constraint (1.7) is replaced by (1.12). This is a weaker constraint for the case of one physical vector multiplet. In appendix C of [17] two examples were given where (1.7) is not satisfied. They are not suitable for illustrating non-trivial aspects of our construction. The first example does not fulfil the positivity condition (3.2). The second example does agree with our definition, but is trivial in the sense
that there $\mathcal{D}_z U_z = 0$. Therefore, the extra terms that appear in this paper are absent for this model. A nontrivial realization of our new models can e.g. be obtained by taking

$$V = \begin{pmatrix} 1 \\ az \\ -\bar{z}\bar{z}^2 \\ 3z^2 \end{pmatrix} e^{K/2}. \tag{5.1}$$

For $a = 1$ it is the well-known $SU(1,1)/U(1)$ symmetric space with positive metric in the complex upper half plane. Deviating from this value gives a non-zero value to

$$\langle V, U_z \rangle = \frac{3i(a - 1)z^2}{z^3 - 3az^2\bar{z} + 3az\bar{z}^2 - \bar{z}^3}. \tag{5.2}$$

Then the new metric

$$g'_{zz} = \frac{-3a(z - \bar{z})^2}{(z^2 + (1 - 3a)z\bar{z} + \bar{z}^2)^2} \tag{5.3}$$

has a well-defined positivity domain, and

$$C_{zzz} = \frac{6ia(z - \bar{z})}{(z^2 + (1 - 3a)z\bar{z} + \bar{z}^2)^2} \tag{5.4}$$

is not covariantly constant.

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A basis for symplectic vectors

In this appendix we show that

$$\mathcal{W} = (V, U\alpha, \bar{V}, \bar{U}\bar{\alpha})$$  \hspace{1cm} (A.1)

is a basis for symplectic vectors. Since we are dealing with a $2(n + 1)$-dimensional vector space we only have to show that these vectors are independent.

Proof: Suppose

$$\lambda^0 V + \lambda^0 \bar{V} + \lambda^\alpha U\alpha + \lambda^{\bar{\alpha}} \bar{U}\bar{\alpha} = 0,$$  \hspace{1cm} (A.2)

then it follows that all $\lambda^i = 0$ if and only if the determinant obtained by left symplectic inner products with, respectively, $\bar{V}$, $V$, $\bar{U}\bar{\beta}$ and $U\beta$, is non–zero:

$$\det \begin{pmatrix} -i & 0 & 0 & \langle \bar{V}, \bar{U}\bar{\alpha} \rangle \\ 0 & i & \langle V, U\alpha \rangle & 0 \\ 0 & \langle \bar{U}\bar{\beta}, \bar{V} \rangle & ig_{\alpha\bar{\beta}} & 0 \\ \langle U\beta, V \rangle & 0 & 0 & -ig_{\alpha\bar{\beta}} \end{pmatrix} \neq 0.$$  \hspace{1cm} (A.3)

We can split this up in two cases:

1. The generic case:
   Then $\langle V, U\alpha \rangle = 0$, and (A.3) is
   $$(\det g_{\alpha\bar{\beta}})^2 > 0,$$  \hspace{1cm} (A.4)

   which is satisfied by the metric.

2. The special case:
   Then we define $Z = \langle V, U\alpha \rangle$ and the determinant equation leads to
   $$(g_{\alpha\bar{\beta}} - Z Z_{\bar{\beta}})^2 \neq 0.$$  \hspace{1cm} (A.5)

However, this follows from the 'physical' condition on the sections that leads to the right signs for the kinetic energy of the scalars and the vectors, cf (3.2).

Now that we have a basis, we can expand every symplectic vector in this basis. Take a generic symplectic vector $X_A$, where the index $A$ denotes a generic index. It is again useful to separate two cases.

1. The generic case:
   This leads to
   $$X_A = i\langle \bar{V}, X_A \rangle V - i\langle V, X_A \rangle \bar{V} + ig^{\alpha\bar{\alpha}} \left( \langle U\alpha, X_A \rangle \bar{U}\bar{\alpha} - \langle \bar{U}\bar{\alpha}, X_A \rangle U\alpha \right).$$  \hspace{1cm} (A.6)
2. The special case:  
In the basis $\mathcal{W}$, the expansion becomes

\[
X_A = -ig'_{z\bar{z}} \left( (-g_{zz} \langle \bar{V}, X_A \rangle + i\bar{Z}_z (U_z, X_A))V 
+ (g_{zz} \langle V, X_A \rangle + iZ_z (\bar{U}_z, X_A))\bar{V} 
+ (-i\bar{Z}_z \langle V, X_A \rangle + \langle \bar{U}_z, X_A \rangle)U_z 
- (iZ_z \langle \bar{V}, X_A \rangle + \langle U_z, X_A \rangle)\bar{U}_z \right). \tag{A.7}
\]

In this case we better use the basis

\[
\mathcal{W}' = (V, U'_z, \bar{V}, \bar{U}'_z). \tag{A.8}
\]

The same formulae hold as above, when replacing $g_{\alpha\beta}$ with $g'_{z\bar{z}}$.

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