Functoriality of
Moduli Spaces of Global $G$-Shtukas

Paul Breutmann

Abstract

Moduli spaces of global $G$-shtukas play a crucial role in the Langlands program for function fields. We analyze their functoriality properties following a change of the curve and a change of the group scheme $G$ under various aspects. In particular, we prove two finiteness results which are of interest in the study of stratifications of these moduli spaces and which potentially allow the formulation of an analog of the André-Oort conjecture for global $G$-shtukas.

Contents

1 Introduction

2 Preliminaries

3 Functoriality of $\nabla^H\mathcal{M}^1(C, G)$
  3.1 The Shtuka Datum
  3.2 Changing the Coefficients
  3.3 Changing the Group $G$

References

1 Introduction

Global $G$-shtukas are the function field analogue of abelian varieties. Their moduli spaces play a crucial role in the Langlands program for function fields. This article is concerned about functoriality properties of these moduli spaces in various aspects. Let us look in more detail. We choose a smooth, projective, geometrically irreducible curve $C$ over a finite field $\mathbb{F}_q$ with $q$ elements. Let $G$ be a smooth, affine group scheme over $C$ and denote by $\sigma$ the $\mathbb{F}_q$-Frobenius of a scheme $S$ over $\mathbb{F}_q$. Then a global $G$-shtuka $G = (G, s_1, \ldots, s_n, \tau_G)$ over $S$ consists of a $G$-torsor $\mathcal{G}$ over $C_S := C \times_{\mathbb{F}_q} S$, $n$ sections $s_i : S \to C$ called legs and an isomorphism $\tau_G : \sigma^* \mathcal{G}|_{C_S \setminus \cup_i \Gamma_{s_i}} \to \mathcal{G}|_{C_S \setminus \cup_i \Gamma_{s_i}}$ outside the union of the graphs $\Gamma_{s_i}$ of the $s_i$. The precise definitions of all the notations used in this article are given in the second section. The stack whose $S$-valued points parametrize the global $G$-shtukas over $S$ with $n$ legs is denoted by $\nabla_n,\mathcal{M}^1(C, G)$. Once we fix $n$ closed points $(v_1, \ldots, v_n) = v$ in $C$ to which we refer as characteristic places we can introduce boundedness conditions $Z_v$ for all $v \in \mathbb{P}^1$ and $H$-level structures. Here a bound $Z_v$ is roughly a $L^*G_v$-invariant closed subscheme of the affine flag variety $\mathcal{F}_{\mathcal{L}_G}$ (see $\S\ 2.6$ for a precise definition) and $H$ is an open, compact subgroup of $G(\mathbb{A}^\infty)$, where $\mathbb{A}^\infty$ denotes the ring of the adeles outside $\mathbb{P}^1$. Then we denote by $\nabla^{\bar{Z}_v, H}_n,\mathcal{M}^1(C, G)$ the stack which parametrize $G$-shtukas $G$ over $S$ bounded by $(\bar{Z}_v)_{v \in \mathbb{P}^1}$ together with an $H$-level structure. At the beginning of the third section we define all the parameters $(C, G, v, Z_v, H)$ as a shtuka datum. This definition results in the natural question of if an appropriate change of this shtuka datum induces a morphism of the corresponding moduli spaces and if so, which properties it has.

In subsection 3.1 we define a morphism of shtuka data and clarify what an appropriate change of the shtuka datum should be. Roughly a morphism from $(C, G, v, Z_v, H)$ to $(C', G', w, Z'_w, H')$ is a pair $(\pi, f)$, where $\pi : C \to C'$ is a finite morphism and $f$ is a morphism of group schemes from the Weil restriction $\pi_*G$ to $G'$ such that $w, Z'_w$ and $H'$ satisfy certain conditions.

In the following subsections we then answer the questions about the functoriality of $\nabla^{\bar{Z}_v, H}_n,\mathcal{M}^1(C, G)$. More precisely, we first consider in subsection $\S\ 3.2$ the case that we basically only change the curve $C$, which yields the following main result of that subsection.
Theorem 1.1 (cf. Theorem 3.14). Let \( C, G, \hat{Z}_w, H \) be a shtuka datum and \( \pi : C \to C' \) a finite morphism of smooth, projective, geometrically irreducible curves over \( \mathbb{F}_q \) with \( w_i = \pi(v_i) \) and \( \underline{w} = (w_1, \ldots, w_n) \). Then the morphism \( (\pi, \id_{\pi, G}) : (C, G, \hat{Z}_w, H) \to (C', \pi, G, \underline{w}, \hat{Z}_w, \pi, H) \) of shtuka data (see Definition 3.9 and Remark 3.10) induces a finite morphism of the moduli stacks
\[
\pi_* : \pi_* \hat{Z}_w^H \mathcal{M}^{1}(C, G) \to \pi_* \hat{Z}_w^H \mathcal{M}^{1}(C', \pi, G, \underline{w}, \hat{Z}_w, \pi, H, G).
\]
The construction of this morphism and the proof of the theorem relies on a lemma in subsection 3.1 that states an equivalence of categories between \( \hat{G} \)-torus over \( C' \) and \( \pi_* \hat{Z}_w, \pi, H \overline{\pi} \). The next subsection 3.2 addresses the fact about the action of function fields in the case that we only change the group scheme \( G \).

For all morphisms \( (\id_C, f) \) of shtuka data, we need to make different assumptions to state different results on the properties of this morphism. Assuming that \( f : G \to G' \) is generically an isomorphism, we get the following first main result of this subsection.

Theorem 1.2 (cf. Theorem 3.20). Let \( w = (w_1, \ldots, w_m) \) be a finite set of closed points in \( C \) and let \( (\id_C, f) : (C, G, \hat{Z}_w, H) \to (C', G', \hat{Z}_w, H) \) be a morphism of shtuka data, where \( f : G \to G' \) is an isomorphism over \( C \). Then the morphism
\[
f_* : \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C, G)} \to \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C', G', \underline{w}, \hat{Z}_w, \pi, H)}
\]
is schematic and quasi-projective. In the case that \( G \) is a parahoric Bruhat-Tits group scheme this morphism is projective. For any morphism \( (\hat{G}', \gamma H) : S \to \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C', G') \underline{w}, \hat{Z}_w, \pi, H} \)
the fiber product
\[
S \times_{\nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C', G') \underline{w}, \hat{Z}_w, \pi, H}} \mathcal{M}^{1}(C, G)
\]
is given by a closed subscheme of
\[
S \times_{\mathbb{F}_q} ((L^w_{v, \hat{G}'(G')}(\gamma H)(G')) \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} ((L^w_{v, \hat{G}'(G')}(\gamma H)(G'))).
\]
If \( \hat{Z}_w \) arises as a base change of \( \hat{Z}_w \) for all \( v \in \underline{w} \), the morphism \( f_* \) is surjective.

This result will again be important for the verification of the axioms on the moduli space \( \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C, G)} \) in [Bre19]. If we assume that \( f : G \to G' \) is a closed immersion instead of a generic isomorphism we get the second main result of this subsection.

Theorem 1.3 (cf. Theorem 3.23). Let \( f : G \to G' \) be a closed immersion of smooth, affine group schemes over \( C \). Then the induced morphism \( f_* : \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C, G)} \to \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C, G', \underline{w}, \hat{Z}_w, \pi, H)} \) is unramified and schematic.

Assuming additionally that \( G \) is a parahoric Bruhat-Tits group scheme, we prove as well:

Theorem 1.4 (cf. Theorem 3.26). Let \( G \) be a parahoric Bruhat-Tits group scheme and \( f : G \to G' \) be a closed immersion of smooth, affine group schemes and \( \underline{w} = (v_1, \ldots, v_n) \) be a set of closed points in \( C \). Then the induced morphism
\[
f_* : \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C, G)} \to \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C, G', \underline{w}, \hat{Z}_w, \pi, H)} \]
is proper and in particular finite.

Apart from the interest of these morphisms in the study of \( \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C, G)} \) in general, there are two other motivations. The first one is that the finiteness results in Theorems 3.14 and 3.20 potentially enable us to formulate an analog of the André-Oort conjecture for moduli spaces of global shtukas, as explained in more detail in Remark 3.28. The second motivation arises from the study of Newton and Kottwitz-Rapoport stratifications of \( \nabla_{\nabla_{\hat{Z}_w}^{H} \mathcal{M}^{1}(C, G)} \) in [Bre10], where the above results are again needed.

We remark that the way to choose a bound \( Z_w \) as we do in this article is quite general. In [Bre19] we will restrict our interest to moduli spaces of global G-shtukas that are bounded by a cocharacter of a maximal torus of \( G \), which can also be formulated using Grassmannians and seems more natural when working with local models.

Acknowledgements. I would like to thank my advisor Urs Hartl, for all his helpful discussions. I am grateful to Timo Richarz and Johannes Anschütz for their comments and remarks. During the work of this project, the author was supported by the SFB 878 “Groups, Geometry & Actions” of the German Science Foundation (DFG), the CNRS and the ERC Advanced Grant 742608 “GeoLocLang”.


2 Preliminaries

Before we start with the functoriality of $\nabla_n \mathcal{H}^1(C, G)$, we introduce the basic objects and notations that we use in this article. Most of the notations introduced in this section can also be found in [AH13] and [AH14].

Let $q$ be a power of some prime number $p$. We start with a smooth, projective, geometrically irreducible curve $C$ over the field $\mathbb{F}_q$ with $q$ elements. We denote by $Q := \mathbb{F}_q(C)$ its function field. For a closed point $v \in C$ we denote by $A_v$ the completion of the stalk $\mathcal{O}_{C,v}$ and by $Q_v$ the fraction field of $A_v$. Furthermore we choose a uniformizer $z_v$ in $A_v$, denote the residue field of $v$ by $\mathbb{F}_v$ and set $deg v = [\mathbb{F}_v : \mathbb{F}_q]$. 

Let $G$ be a smooth, affine group scheme over $C$ and $\mathcal{G} := G \times_C Q$ its generic fiber. We write $G_v := G \times_C Spec A_v$ and $G_s := G \times_C Spec Q_v = G_v \times_{A_v} Spec Q_v$ for the appropriate base changes.

For an $\mathbb{F}_q$-scheme $S$ we denote by $\sigma_S : S \to S$ the absolute $\mathbb{F}_q$-Frobenius, which acts as the $q$-power map on the structure sheaf. Further we define $\sigma$ as the endomorphism $id_C \times \sigma_S$ of $C_S := C \times_{\mathbb{F}_q} S$. For a morphism $s : S \to C$ we denote as usual by $\Gamma_s : S \to C_S$ the graph of $s$, which is a closed immersion.

Let $X$ be a site with a final object $\mathcal{X}$ and $G$ a sheaf of groups on $X$. Then a (right) $G$-torsor is a sheaf $\mathcal{G}$ on $X$ with a right action of $G$ on $\mathcal{G}$ such that $G \times \mathcal{G} \to \mathcal{G}$, $(g, h) \mapsto (hg, h)$ is an isomorphism and $\mathcal{G}(U) \neq \emptyset$ for some covering $U \to X$. When we speak about a torsor, we always mean a right torsor and if nothing else is mentioned we mean a sheaf on the big étale site of a scheme. For any scheme $S$ we write $S_{\text{ét}}$ for the big étale site of this scheme. We denote by $\mathcal{H}^1(C, \mathcal{G})$ the stack fibered over $(\mathbb{F}_q)_{\text{ét}}$, whose fiber category $\mathcal{H}^1(C, \mathcal{G})(S)$ is given by the category of $\mathcal{G}$-torsors over $C_S$.

§ 2.1 Global $\mathcal{G}$-Shtukas:

Let $S$ be an $\mathbb{F}_q$-scheme. A global $\mathcal{G}$-shtuka over $S$ is a tuple $\mathcal{G} = (\mathcal{G}, s_1, \ldots, s_n, \tau_G)$, where

- $\mathcal{G}$ is a $\mathcal{G}$-torsor over $C_S$,
- $s_1, \ldots, s_n$ are morphisms $S \to C$ and
- $\tau_G : \sigma^* G|_{C_S \setminus (\bigcup_{i=1}^n \Gamma_{s_i})} \to G|_{C_S \setminus (\bigcup_{i=1}^n \Gamma_{s_i})}$ is an isomorphism of the $\mathcal{G}$-torsors $\sigma^* G$ and $G$ restricted to $C_S \setminus (\bigcup_{i=1}^n \Gamma_{s_i})$.

We take the notation $\nabla_n \mathcal{H}^1(C, \mathcal{G})$ from [AH14, Definition 2.12] for the stack fibered over $(\mathbb{F}_q)_{\text{ét}}$, whose $S$-valued points for a scheme $S$ are given by $\mathcal{G}$-shtukas $\mathcal{G}$ over $S$. Morphisms from $(\mathcal{G}, s_1, \ldots, s_n, \tau)$ to $(\mathcal{G}', s_1', \ldots, s_n', \tau')$ in the fiber category $\nabla_n \mathcal{H}^1(C, \mathcal{G})(S)$ only exist if $s_i = s_i'$ and are given by morphisms $f : \mathcal{G} \to \mathcal{G}'$ of $\mathcal{G}$-torsors over $C_S$ such that $f \circ \tau = \tau' \circ \sigma^* f$. Given two $\mathcal{G}$-shtukas $\mathcal{G}$ and $\mathcal{G}'$ over $S$ with $s_i = s_i'$, we also define a quasi-isomorphism from $\mathcal{G}$ to $\mathcal{G}'$ to be an isomorphism $f : \mathcal{G}|_{C_S \setminus \Gamma_{s_i}} \to \mathcal{G}'|_{C_S \setminus \Gamma_{s_i}}$ of $\mathcal{G}$-torsors satisfying $f \circ \tau = \tau' \circ \sigma^* f$, where $D$ is some effective divisor on $C$. The moduli space $\nabla_n \mathcal{H}^1(C, \mathcal{G})$ is an ind-algebraic stack that is ind-separated and locally of ind-finite type [AH13, Theorem 3.14].

§ 2.2 Loop Groups:

Let $\mathbb{F}$ be a finite field and $\mathbb{H}$ be a smooth, affine group scheme over $\mathbb{D} := Spec \mathbb{F}[z]$, with generic fiber $\mathbb{H} := \mathbb{H} \times \mathbb{D} \mathbb{D}$ where $\mathbb{D} := Spec \mathbb{F}((z))$. We are mainly interested in the case that $\mathbb{D} = Spec A_v$ and $\mathbb{H} = G_v$ for some closed point $v \in C$.

We recall that the sheaf of groups $L^* \mathbb{H}$ on $\mathbb{F}_{\text{ét}}$, whose $R$-valued points for an $\mathbb{F}$-algebra $R$ are given by

$$L^* \mathbb{H}(R) := \mathbb{H}(R[[z]]) := \mathbb{H}(\mathbb{D}R) := Hom_D(\mathbb{D}R, \mathbb{H}) \quad \text{with } \mathbb{D}R := Spec R[[z]],$$

is an infinite-dimensional affine group scheme over $\mathbb{F}$. It is called the group of positive loops associated with $\mathbb{H}$. The group of loops associated with $\mathbb{H}$ is the sheaf of groups $L \mathbb{H}$ on $\mathbb{F}_{\text{ét}}$, whose $R$-valued points are defined by

$$L \mathbb{H}(R) := \mathbb{H}(R((z))) := \mathbb{H}(\mathcal{D}R) := Hom_D(\mathcal{D}R, \mathbb{H}),$$

where we write $R((z)) = R[[z]]\left[\frac{1}{z}\right]$ and $\mathcal{D}R = Spec R((z))$. The loop group $L \mathbb{H}$ is an ind-scheme of ind-finite type over $\mathbb{F}$. 

3
\[ \text{§ 2.3 Torsors for Loop Groups:} \]
We write \( \mathscr{H}^1(\mathbb{F}, L^*H) \) for the stack fibered over \( (\mathbb{F})_{Et} \) whose fiber category \( \mathscr{H}^1(\mathbb{F}, L^*H)(S) \) is the category of \( L^*H \)-torsors over \( S \). In the same way \( \mathscr{H}^1(\mathbb{F}, LH) \) denotes the stack fibered over \( (\mathbb{F})_{Et} \) whose fiber category \( \mathscr{H}^1(\mathbb{F}, LH)(S) \) is the category of \( LH \)-torsors over \( S \). There is a natural 1-morphism

\[
L : \mathscr{H}^1(\mathbb{F}, L^*H) \to \mathscr{H}^1(\mathbb{F}, LH), \quad L^* \mapsto L
\]

induced by the inclusion of sheaves \( L^*H \subset LH \).

We now consider also the \( z \)-adic completions of \( \mathbb{D} \) and \( \mathbb{H} \) and denote them by \( \hat{\mathbb{D}} \) and \( \hat{\mathbb{H}} \). Later when we pass from global \( G \)-shtukas to local \( G_v \)-shtukas we often need to know that \( L^*H \)-torsors are equivalent to formal \( \hat{\mathbb{H}} \)-torsors. So we recall that for an \( \mathbb{F} \)-scheme \( S \) a \( z \)-adic formal scheme \( \mathcal{H} \) over \( \hat{\mathbb{D}}_S := \hat{\mathbb{D}} \times_{\mathbb{D}} S \) together with an action \( \hat{\mathbb{H}} \times_{\mathbb{H}} \mathcal{H} \to \mathcal{H} \) of \( \mathbb{H} \) is called a formal \( \mathbb{H} \)-torsor if there is an étale covering \( S' \to S \) and an \( \mathbb{H} \)-equivariant isomorphism \( \mathcal{H} \times_{\mathcal{H}} \mathcal{D}_{S'} \to \hat{\mathbb{H}} \times_{\mathbb{H}} \mathcal{D}_{S'} \), where \( \mathbb{H} \) is acting on itself by right multiplication.

We denote by \( \mathscr{H}^1(\hat{\mathbb{D}}, \hat{\mathbb{H}}) \) the category fibered in groupoids over \( (\mathbb{F})_{Et} \) whose fiber category \( \mathscr{H}^1(\hat{\mathbb{D}}, \hat{\mathbb{H}})(S) \) is the groupoid of formal \( \hat{\mathbb{H}} \)-torsors over \( S \). We remark that Arasteh Rad and Hartl proved in \[ \text{AH14}, \text{Proposition 2.4} \] that there is a natural isomorphism of stacks \( \mathscr{H}^1(\hat{\mathbb{D}}, \hat{\mathbb{H}}) \to \mathscr{H}^1(\mathbb{F}, L^*H) \). It sends a formal \( \hat{\mathbb{H}} \)-torsor \( \mathcal{H} \) to the sheaf

\[
L^* : S_{Et} \to \text{Sets}, \quad T \mapsto \text{Hom}_{\mathbb{D}_T}(\hat{\mathbb{D}}_T, \mathcal{H})
\]

which becomes a \( L^*H \)-torsor under the action of \( L^*H(T) = \text{Hom}_{\mathbb{D}}(\hat{\mathbb{D}}_T, \mathcal{H}) \).

\[ \text{§ 2.4 Local } \mathbb{H} \text{-Shtukas:} \]
Let \( S \) be an \( \mathbb{F} \)-scheme and \( \sigma \) its absolute \( \mathbb{F} \)-Frobenius. If \( \mathbb{F} \) equals \( \mathbb{F}_q \) or \( \mathbb{F}_a \), we will write \( \sigma \) and \( \sigma_v \), respectively, instead of \( \sigma \). With the previous notations a local \( \mathbb{H} \)-shtuka over \( S \) is a pair \( (L^*, \tau_L) \) where

- \( L^* \) is a \( L^*H \)-torsor over \( S \) and
- \( \tau_L : \hat{\sigma} \circ \tau_L \to \tau_L \) is an isomorphism of the associated loop group torsors from \( \mathcal{H} \) in \[ \text{§ 2.3} \].

A morphism from \( \mathcal{L} = (L^*, \tau_L) \) to \( \mathcal{L}' = (L'^*, \tau_L') \) of two local \( \mathbb{H} \)-shtukas over \( S \) is a morphism \( f : L^* \to L'^* \) of \( L^*H \)-torsors over \( S \) satisfying \( \tau_L \circ f = \tau_L' \circ \hat{\sigma} \circ f \). A quasi-isogeny from \( \mathcal{L} = (L^*, \tau_L) \) to \( \mathcal{L}' = (L'^*, \tau_L') \) is an isomorphism \( f : L \to L' \) of the associated \( L^*H \)-torsors satisfying \( \tau_L \circ f = \tau_L' \circ \hat{\sigma} \circ f \).

A local \( \mathbb{H} \)-shtuka \( (L^*, \tau_L) \) is called étale if \( \tau_L : \hat{\sigma} \circ \tau_L \to \tau_L \) comes already from an isomorphism \( \tau_L : \hat{\sigma} \circ L^* \to L^* \) of the \( L^*H \)-torsors.

We denote the category of local \( \mathbb{H} \)-shtukas over \( S \) by \( \text{Sh}_{\mathbb{H}}(S) \) and the category of étale local \( \mathbb{H} \)-shtukas over \( S \) by \( \text{EtSh}_{\mathbb{H}}(S) \).

We recall the Corollary \[ \text{AH14}, \text{Corollary 2.9} \] that states that if \( \mathbb{H} \) has a connected special fiber, then any étale local shtuka over an separably closed field \( k \) is already isomorphic to \( (L^*H_k, 1 \cdot \hat{\sigma}^* \circ \hat{\sigma}) \).

Let \( \mathbb{F}[[\zeta]] \) be the power series ring over \( \mathbb{F} \) in a variable \( \zeta \). We denote by \( \text{Nil}_{\mathbb{F}}[[\zeta]] \) the category of schemes over \( \text{Spec} \ \mathbb{F}[[\zeta]] \) on which \( \zeta \) is locally nilpotent in the structure sheaf. Therefore \( \text{Nil}_{\mathbb{F}}[[\zeta]] \) is the full subcategory of formal schemes over \( \text{Spf} \ \mathbb{F}[[\zeta]] \) consisting of ordinary schemes. We will define boundedness conditions only for local shtukas over a scheme \( S \) in \( \text{Nil}_{\mathbb{F}}[[\zeta]] \).

\[ \text{§ 2.5 The Affine Flag Variety:} \]
Let \( \mathbb{H} \) be a smooth, affine group scheme over \( \text{Spec} \ \mathbb{F}[z] \) as before. Then the affine flag variety \( F_{\mathbb{H}} \) is defined as the quotient sheaf \( LH/L^*H \) on \( F_{Et} \), that is the sheaf associated to the pre-sheaf

\[
T \mapsto LH(T)/L^*H(T).
\]

By \[ \text{PR05}, \text{Theorem 1.4} \] \( F_{\mathbb{H}} \) is represented by an ind-scheme which is ind-quasi-projective and in particular ind-separated and of ind-finite type over \( \mathbb{F} \). By \[ \text{Ric16, Theorem A} \] \( F_{\mathbb{H}} \) is ind-projective if and only if \( \mathbb{H} \) is a Bruhat-Tits group scheme over \( \mathbb{F}[z] \) in the sense of \[ \text{BT84}, \text{Definition 5.2.6} \]. We also remark that \( L^*H \) acts from the left on \( F_{\mathbb{H}} \).
§ 2.6 Bounds in $\mathcal{F}_{l|H}$

We fix an algebraic closure $F(\zeta)^a_{l|H}$ of $F(\zeta)$. For a finite extension $R$ of discrete valuation rings $F[\zeta] \subset R \subset F(\zeta)^a_{l|H}$ with residue field $\kappa_R$ we denote similar as before with $N\ell p_{l|H}$ the category of $R$-schemes on which $\zeta$ is locally nilpotent. Furthermore we set $\mathcal{F}_{l|H,R} = \mathcal{F}_{l|H} \times_{\mathcal{F}_{l|H}} Spf R$ as well as $\mathcal{F}_{l|H} := \mathcal{F}_{l|H,F[\zeta]}$.

Now let $R$ and $R'$ be two such finite extensions of discrete valuation rings and let $\bar{Z}_R \subset \mathcal{F}_{l|H,R}$ and $\bar{Z}'_{R'} \subset \mathcal{F}_{l|H,R'}$ be two closed ind-subschemes. We call $\bar{Z}_R$ and $\bar{Z}'_{R'}$ equivalent if there is a finite extension $R$ of discrete valuation rings as above containing $R$ and $R'$ such that $\bar{Z}_R \times_{Spf R} Spf R = \bar{Z}'_{R'} \times_{Spf R} Spf R$ as closed ind-subschemes of $\mathcal{F}_{l|H,R}$.

Now a bound is defined (compare [AH14, Definition 4.8] and [AH13, Definition 4.5]) as an equivalence class $\bar{Z} = [\bar{Z}_R]$ of closed ind-subschemes $\bar{Z}_R \subset \mathcal{F}_{l|H,R}$ satisfying

- firstly, that all subschemes $\bar{Z}_R$ are stable under the left action of $L^*H$ on $\mathcal{F}_{l|H,R}$, and

- secondly, that all the special fibers $Z_R := \bar{Z}_R \times_{Spf R} Spec \kappa_R$ are quasi-compact and connected subschemes of $\mathcal{F}_{l|H,R}$.

We remark that in [AH13] and [AH14] the definition of a bound does not require the special fibers to be connected. We make this assumption because it does not change the theory and simplifies the formulation of certain statements. In fact, if $\bar{Z}$ is the disjoint union of two bounds $\bar{Z}_1 \coprod \bar{Z}_2$ then the moduli space $\nabla_{l|H}^2(H^1(C,G))$ that will be defined in paragraph §2.11 is the disjoint union of the moduli spaces $\nabla_{l|H}^2(H^1(C,G))$ and $\nabla_{l|H}^2(H^1(C,G))$.

§ 2.7 The Reflex Ring:

For an equivalence class $\bar{Z} = [\bar{Z}_R]$ as above we set $G_{\bar{Z}} := \{ \gamma \in Aut_{\bar{F}[\zeta]}(F(\zeta)^a_{l|H}) \mid \gamma(\bar{Z}) = \bar{Z} \}$. The ring $R_{\bar{Z}}$ is defined as the intersection of the fixed field of $G_{\bar{Z}}$ in $F(\zeta)^a_{l|H}$ with all the finite extensions $R$ over which a representative $\bar{Z}_R$ of $\bar{Z}$ exists. In the case that $\bar{Z}$ is a bound, we call $R_{\bar{Z}}$ the reflex ring of $\bar{Z}$. It is not always clear if there exists a representative of $\bar{Z}$ over $R_{\bar{Z}}$. We write $\kappa_{\bar{Z}}$ and $\kappa_R$ for the residue fields of $R_{\bar{Z}}$ and $R$ respectively. Then the special fiber $Z_{\bar{Z}} := \bar{Z}_R \times_{Spf R} Spec \kappa_R$ arises from a unique closed subscheme $Z \subset \mathcal{F}_{l|H} \times_{Spf \kappa_{\bar{Z}}}$. This follows from Galois descent for closed ind-subschemes of $\mathcal{F}_{l|H}$, which is effective. The subscheme $Z$ is called the special fiber of $\bar{Z}$.

§ 2.8 Boundeness of Local $H$-Shtukas:

Let $\bar{Z}$ be a bound with reflex ring $R_{\bar{Z}}$. Furthermore let $L^*$ and $L^*_{\ell}$ be two $L^*H$-torsors over a scheme $S$ in $N\ell p_{l|H}$ and $\delta : L \to L^*$ an isomorphism of the associated $LH$-torsors. We choose a covering $S' \to S$ such that there are trivializations $\alpha : L^* \to \tilde{L}_{S'}$ and $\alpha' : L^*_{\ell} \to \tilde{L}_{S'}$. Then the automorphism $\alpha' \circ \delta \circ \alpha^{-1} : L_{S'} \to L_{S'}$ defines a morphism $S' \to LH$.

For any finite extension $R$ of $R_{\bar{Z}}$ we have an induced morphism

$$ S' \times_{Spf R} R \to LH \times_{Spf R} R \to \mathcal{F}_{l|H,R}. $$

Then $\delta$ is said to be bounded by $\bar{Z}$ if all trivializations $\alpha$ and $\alpha'$ and all finite extensions $R$ of $R_{\bar{Z}}$ with a representative $\bar{Z}_R$, this morphism factors through $R_{\bar{Z}}$. By [AH14, Remark 4.9] $\delta$ is bounded if and only if this condition is satisfied for one trivialization and for one such extension $R$. By definition a local $H$-shtuka $(L^*, \tau_{L^*})$ is bounded by $\bar{Z}$ if $\tau_{L^*}$ is bounded by $\bar{Z}$.

§ 2.9 A Version of the Theorem of Beauville-Laszlo:

Let $\nu \in C$ be a closed point and set $C^\nu := C \setminus \{ \nu \}$ as well as $C^\nu := C^\nu \times_{F} S$. We define $H^1(C^\nu, G)$ as the category fibered in groupoids over $(F_q)_{F[\zeta]}$ whose fiber category $H^1(C^\nu, G)(S)$ consists of those $G$-torsors over $C^\nu$ that can be extended to a $G$-torsor over $C_S$. By restricting a $G$-torsor $G$ over $C_S$ to $C^\nu_S$ we get a morphism $H^1(C^\nu, G) \to H^1(C^\nu, G)$. We further introduce the notation $G_{\bar{v}} = Res_{\bar{v}}(G_{\nu[\bar{z}]})$ and $\bar{G}_{\nu} \subset G_{\bar{v}} \times_{F[\bar{z}]} F_{q}(\bar{z})$. For $G \in H^1(C^\nu, G)$ the base change $G_\nu := G \times_{C_S} (Spf A_\nu \times_{F[q]} S)$ defines a formal $G_\nu$-torsor over $Spf A_{\nu} \times_{F[q]} S$. Its Weil restriction $Res_{\bar{v}}(G_{\nu[\bar{z}]})G_{\nu}$ defines a formal $\bar{G}_{\nu}$-torsor over $Spf F_{q}(\bar{z}) \times_{F[q]} S$. Using the category equivalence in §2.3 it corresponds to an object in $H^1(F_q, L^*\bar{G}_{\nu})(S)$ that we denote by $L^*_{\nu}(G)$ which defines a functor

$$ L^*_{\nu} : H^1(C^\nu, G) \to H^1(F_q, L^*\bar{G}_{\nu}), \quad G \mapsto L^*_{\nu}(G). $$

5
Furthermore we have the functor

\[ L_v : \mathcal{H}^1_c(C^v, \mathbb{G}) \to \mathcal{H}^1_c(\mathbb{F}_q, L\mathcal{G}_v), \quad \mathcal{G}_{|C^v} \mapsto L(L^*_v(\mathcal{G})) = L_v(\mathcal{G}) \]

which is independent of the extension \( \mathcal{G} \) of \( \mathcal{G}_{|C^v} \). Now a version of the theorem of Beauville-Laszlo, that is proven in [AH14, Lemma 5.1], states that the following diagram is cartesian.

\[ \begin{array}{ccc}
\mathcal{H}^1_c(C, \mathcal{G}) & \xrightarrow{\mathcal{G}_{|C^v}} & \mathcal{H}^1_c(C^v, \mathcal{G}) \\
L^*_v & \downarrow & L_v \\
\mathcal{H}^1_c(\mathbb{F}_q, L^*\mathcal{G}_v) & \xrightarrow{\mathcal{G}_{|C^v}} & \mathcal{H}^1_c(\mathbb{F}_q, L\mathcal{G}_v) 
\end{array} \]

§ 2.10 The Global-Local Functor:

Now we fix \( n \) closed points \( v = \{v_1, \ldots, v_n\} \) of \( C \). Then define \( A_{\mathcal{G}} \) as the completion of the local ring \( \mathcal{O}_{C^v} \) at the closed point \( v \). We set

\[ \nabla \mathcal{H}^1_c(C, \mathcal{G})^{\mathcal{G}} := \nabla \mathcal{H}^1_c(C, \mathcal{G}) \times_{\mathbb{G}} \text{Spf} A_{\mathcal{G}}. \]

An \( S \)-valued point of \( \nabla \mathcal{H}^1_c(C, \mathcal{G})^{\mathcal{G}} \) is therefore given by a global \( \mathbb{G} \)-shtuka \( \mathcal{G} = (\mathcal{G}, s_1, \ldots, s_n, \tau_\mathcal{G}) \) such that \( s_i : S \to C \) factors through \( \text{Spf} A_{v_i} \). We now want to associate with such a global \( \mathbb{G} \)-shtuka a local \( \mathbb{G}_{v_i} \)-shtuka for all \( v_i \in \mathcal{G} \). We write \( \mathbb{D}_{v_i} := \text{Spec} A_{v_i} \) and \( \hat{\mathbb{D}}_{v_i} := \text{Spf} A_{v_i} \) as well as \( \hat{\mathbb{D}}_{v_i} : S \to \mathbb{D}_{v_i} \). Then we have:

\[ \hat{\mathbb{D}}_{v_i} \times_{\mathbb{F}_q} S = \bigcup_{v \in \mathbb{D}_{v_i}} V(\mathcal{G}_{v_i,1}) \]

where \( a_{v_i,j} := \langle a \otimes 1 - 1 \otimes s^*(a)^{\vee} \mid a \in \mathbb{F}_q \rangle \) and \( V(\mathcal{G}_{v_i,1}) \) is the closed subscheme given by this ideal. We remark that \( s \) cyclically permutes these components and that the \( S \)-valued Frobenius \( \sigma^e \mathcal{G} \) leaves all these components \( V(\mathcal{G}_{v_i,1}) \) stable. For \( \mathcal{G} \in \nabla \mathcal{H}^1_c(C, \mathcal{G})^{\mathcal{G}} \) define a base change:

\[ \hat{\mathbb{D}}_{v_i} \times_{\mathbb{F}_q} S = \bigcup_{v \in \mathbb{D}_{v_i}} V(\mathcal{G}_{v_i,1}) \]

defines a formal \( \hat{\mathbb{G}}_{v_i} \)-torsor over \( \bigcup_{v \in \mathbb{D}_{v_i}} \hat{\mathbb{D}}_{v_i} \) which is an object in \( \mathbb{H}^1(\hat{\mathbb{D}}_{v_i}, \hat{\mathbb{G}}_{v_i}) \). Each component \( \hat{\mathbb{G}} \times_{\mathbb{G}} V(\mathcal{G}_{v_i,1}) \) defines a formal \( \hat{\mathbb{G}}_{v_i} \)-torsor. Similar to the notation in § 2.3 we denote by \( L^*_0 \) the \( \mathbb{G}_{v_i} \)-torsor associated by [AH14, Proposition 2.4] with the formal \( \hat{\mathbb{G}}_{v_i} \)-torsor. Then \( \mathbb{L}_{v_i,0} \) is a local \( \mathbb{G}_{v_i} \)-shtuka, where \( \sigma^e \mathcal{G} \mathbb{L}_{v_i,0} = \mathbb{L}_{v_i,0} \) is the isomorphism of \( \mathbb{L}_{v_i,0} \)-torsors induced by \( \tau \) (compare also [AH14, Lemma 5.1]). More precisely, it can be written as:

\[ \sigma^e \mathcal{G} \mathbb{L}_{v_i,0} = \tau \sigma \mathbb{L}_{v_i,0} = \cdots \sigma^e \mathcal{G} \mathbb{L}_{v_i,0} \]

This now defines the global-local functor:

\[ \Gamma_{v_i} : \nabla \mathcal{H}^1_c(C, \mathcal{G})^{\mathcal{G}}(S) \to \text{Sh}G_{v_i}(S) \]

\[ \mathcal{G} = (\mathcal{G}, s_1, \ldots, s_n, \tau_\mathcal{G}) \mapsto (\mathcal{L}^*_0, \tau^e \mathcal{G}) = \Gamma_{v_i}(\mathcal{G}) \]

We remark that this functor transforms by [AH14, Definition 5.4] quasi-isogenies into quasi-isogenies. If \( v \neq \mathcal{G} \) the component \( V(\mathcal{G}_{v_i,0}) \) exists only if \( S \) is an \( F_v \)-scheme.

If we do not restrict \( \hat{\mathbb{G}} \times_{\mathbb{G}} V(\mathcal{G}_{v_i,0}) \) to the component \( V(\mathcal{G}_{v_i,0}) \), then we use the Weil restriction \( \mathcal{L}^*_0(\mathcal{G}) \) we get in a similar way a local \( \mathbb{G}_{v_i} \)-shtuka \( \mathbb{L}_{v_i}(\mathcal{G}, \tau_{v_i}) \) for all \( \tau_{v_i} : \mathcal{L}_{v_i}(\mathcal{G}) \to \mathcal{L}^*_0(\mathcal{G}) \)

§ 2.11 Boundness of Global \( \mathbb{G} \)-Shtukas:

Recall that we fixed \( n \) closed points \( \mathcal{G} = \{v_1, \ldots, v_n\} \) in \( C \). If the group scheme \( \mathbb{G} \) is fixed we write for each of these schemes \( \mathbb{F}_{v_i} = \mathbb{F}_{v_i} \mathbb{G} \) for the corresponding affine flag variety over \( \mathbb{F}_v \) and \( \mathbb{F}_{v_i,R} = \mathbb{F}_{v_i} \mathcal{G}_{v_i} \mathbb{G} \) for a finite extension \( A_v \subset R \). In each of these affine flag varieties \( \mathbb{F}_{v_i} = \mathbb{F}_{v_i} A_{v_i} \) we choose a bound \( \hat{Z}_{v_i} = [\hat{Z}_{v_i} R] \) with reflex ring \( R_{\hat{Z}_{v_i}} \) and we write \( \hat{Z}_{v_i} \) for the tuple \( \{\hat{Z}_{v_1}, \ldots, \hat{Z}_{v_n}\} \). Choosing a uniformizer \( \pi_{v_i} \) in \( R_{\hat{Z}_{v_i}} \) and defining \( F_R \) as the compositum of all the residue fields \( R_{\hat{Z}_{v_i}}(\pi_{v_i}) \) we
set \( R_{\mathcal{Z}} := \mathcal{F}_R[\pi_1, \ldots, \pi_n] \). In particular the morphism \( Spf R_{\mathcal{Z}} \to C^n \) factors through \( Spf A_{\mathcal{Z}} \). This means that every point \( g = (s_1, \ldots, s_n, r) \) in \( \nabla_n \mathcal{H}^1(C, G) \times_{C^n} Spf R_{\mathcal{Z}}(S) \) also defines an \( S \)-valued point in \( \nabla_k \mathcal{H}^1(C, G) \mathcal{Z} \) so that we write \( \Gamma_v(g) \) for its associated local \( \mathcal{Z}_{\mathcal{V} \mathcal{V}} \)-shtuka over \( S \). The fact that \( S \in Nilp R_{\mathcal{Z}} \) allows us to ask if \( \Gamma_v(g) \) is bounded by \( \mathcal{Z}_v \).

We define \( \nabla^2 \mathcal{H}^1(C, G) \) to be the stack consisting of these bounded global \( G \)-shtukas. That means the fiber category \( \nabla^2 \mathcal{H}^1(C, G)(S) \) is the full subcategory of \( \nabla \mathcal{H}^1(C, G) \times_{C^n} Spf R_{\mathcal{Z}}(S) \) that consists of those global \( G \)-shtukas \( \mathcal{Z} \) over \( S \) that are bounded by \( \mathcal{Z}_v \). By [AH13] Remark 7.2 the moduli space \( \nabla^2 \mathcal{H}^1(C, G) \) is a closed sub-stack of \( \nabla \mathcal{H}^1(C, G) \times_{C^n} Spf R_{\mathcal{Z}} \). Moreover we denote by \( \nabla^2 \mathcal{H}^1(C, G) \mathcal{Z}_v := \nabla^2 \mathcal{H}^1(C, G) \times_{Spf R_{\mathcal{Z}}} \mathcal{F}_R \) the special fiber of \( \nabla^2 \mathcal{H}^1(C, G) \).

### § 2.12 D-Level Structures:

Let \( D \) be a proper closed subscheme of \( C \) and let \( D_S := D \times_{\mathcal{F}_R} S \) for some \( \mathcal{F}_R \)-scheme \( S \) and \( \mathcal{G} \) a \( G \)-torsor on \( C_S \). By [AH13] Definition 3.1] a D-level structure on \( \mathcal{G} \) is a trivialization \( \Psi : \mathcal{G} \times C_S, D_S \to \mathcal{G} \times C_S D_S \) and \( \mathcal{H}^1(D)(C, G)(S) \) denotes the stack fibered over \( \mathcal{F}_r \) whose fiber category \( \mathcal{H}^1(D)(C, G)(S) \) consists of pairs \( (\mathcal{G}, \Psi) \) where \( \mathcal{G} \in \mathcal{H}^1(C, G)(S) \) and \( \Psi \) is a D-level structure. A morphism from \( (\mathcal{G}, \Psi) \) to \( (\mathcal{G}', \Psi') \) in this fiber category is given by an isomorphism \( f : \mathcal{G} \to \mathcal{G}' \) of \( G \)-torsors such that \( \Psi' = \Psi \circ (f \times id_{D_S}) \). The moduli stack of global \( G \)-shtukas with D-level structure is denoted by \( \nabla \mathcal{H}^1(D)(C, G) \). Its fiber category over \( S \) is given by tuples \( \mathcal{G}, \Psi \) is \( \mathcal{G} \in \nabla \mathcal{H}^1(C, G) \times_{C^n} \mathcal{F}_r \) (i.e. \( s_i : S \to C \) factors through \( C_D \) and \( \Psi \) is a S-level structure on \( \mathcal{G} \) satisfying \( \Psi \circ (\tau \times id_{D_S}) = \sigma'(\Psi) \). A morphism from \( \mathcal{G}, \Psi \) to \( \mathcal{G}', \Psi' \) in this fiber category is a morphism \( f \in \nabla \mathcal{H}^1(C, G)(S) \) (in particular an isomorphism \( f : \mathcal{G} \to \mathcal{G}' \) of \( G \)-torsors) satisfying \( \Psi' = \Psi \circ (f \times id_{D_S}) \). If \( D = \emptyset \) we have \( \nabla \mathcal{H}^1(D)(C, G) = \nabla \mathcal{H}^1(C, G) \). If \( v_1, \ldots, v_n \in D \) and \( \mathcal{Z}_v \) is a bound as before, we use the intuitive notations \( \nabla \mathcal{H}^1(D)(C, G) \mathcal{Z}_v \) for the base change \( \nabla \mathcal{H}^1(D)(C, G) \times_{C^n} Spf A_{\mathcal{Z}} \) and \( \nabla \mathcal{H}^1(D)(C, G) \mathcal{Z}_v \) for the stack of \( G \)-shtukas \( \mathcal{Z}_v \) in \( \nabla \mathcal{H}^1(C, G) \) with a D-level structure.

### § 2.13 Local Shtukas and Local GL_n-Shtukas:

The category of local GL_n-shtukas over an \( \mathcal{F}_R \)-scheme \( S \) can be defined more explicitly. We briefly describe this here since it is useful for the definition of the Tate functors. We denote by \( \mathcal{O}_S[z] \) the sheaf of \( \mathcal{O}_S \)-algebras on \( S \), which associates with every \( S \)-scheme \( Y \) the ring \( \mathcal{O}_S[z](Y) = \Gamma(Y, \mathcal{O}_Y[z]) \). Now every sheaf of \( \mathcal{O}_S[z] \)-modules that is fppf-locally free of rank \( r \) is by [HV11] Prop. 2.3] already Zariski locally free of rank \( r \). We call these locally free sheaves of \( \mathcal{O}_S[z] \)-modules of rank \( r \). For a commutative ring \( R \) we set \( \mathcal{R}(z) := \mathcal{R}[z][\frac{1}{z}] \). This leads to the intuitive notation \( \mathcal{O}_S[z] \mathcal{R}(z) \) for the sheaf on \( S \mathcal{F}_R \) associated to the pre-sheaf \( Y \to \Gamma(Y, \mathcal{O}_Y[z])(z) \). The absolutely \( \mathcal{F} \) Frobenius was denoted \( \sigma \) and we use the same notation for the endomorphism of \( \mathcal{O}_S[z] \) and \( \mathcal{O}(z) \) that acts as \( \sigma \) on sections of \( \mathcal{O}_S \) and as the identity on \( z \). For a sheaf \( M \) of \( \mathcal{O}_S[z] \)-modules we can consider the pullback \( \sigma^* M = M \otimes \mathcal{O}_S[z], \sigma \) \mathcal{O}_S[z] \). Now by [HV11] Definition 4.1] a local shtuka of rank \( r \) over \( S \) is a pair \( (M, \tau_M) \) consisting of a locally free sheaf \( M \) of \( \mathcal{O}_S[z] \)-modules of rank \( r \) and an isomorphism \( \tau_M : \sigma^* M \otimes \mathcal{O}_S[z] \mathcal{O}_S(z) \to M \otimes \mathcal{O}_S(z) \mathcal{O}_S(z) \).

The local shtuka \((M, \tau_M) \) is called étale if \( \tau_M \) arises from an isomorphism \( \sigma^* M \to M \) of \( \mathcal{O}_S[z] \)-modules. A morphism from \((M, \tau_M) \) to \((M', \tau_M') \) between two local shtukas over \( S \) is a morphism \( f : M \to M' \) of \( \mathcal{O}_S[z] \)-modules satisfying \( \tau_M \circ \sigma^* f = f \circ \tau_M \). A quasi-isogeny from \((M, \tau_M) \) to \((M', \tau_M') \) is a morphism \( f : M \otimes \mathcal{O}_S[z] \mathcal{O}_S(z) \to M' \otimes \mathcal{O}_S[z] \mathcal{O}_S(z) \) of \( \mathcal{O}_S[z] \)-modules satisfying \( \tau_M \circ \sigma^* f = f \circ \tau_M \). We denote the category of local shtukas over \( S \) by \( \text{Sh}_{\mathcal{F}_R}(S) \) and the category of étale local shtukas over \( S \) by \( \text{EtSh}_{\mathcal{F}_R}(S) \).

Now there is a category equivalence between local GL_n-shtukas as defined in [2.4] and the category of local shtukas of rank \( r \) over \( S \) with isomorphisms as the only morphisms. It is naturally induced by the category equivalence [HV11] Lemma 4.2] of \( \mathcal{H}^1(\mathcal{F}, L' \mathcal{G}_L)(S) \) and the category of locally free sheaves of \( \mathcal{O}_S[z] \)-modules of rank \( r \) with isomorphisms as morphisms.
§ 2.14 Tate Functors on Local $\mathbb{G}$-Shtukas:

Now let $S$ be a connected $F_\rho$-scheme with geometric base point $\pi \in S$ and algebraic fundamental group $\pi_1(S, \pi)$. We denote by $\mathfrak{M}_{\text{Mod}}[\pi_1(S, \pi)]$ (resp. $\mathfrak{M}_{\text{Mod}}^G[\pi_1(S, \pi)]$) the category of finite and free $F_\rho[z]$-modules (resp. $F(z)$ vector spaces) equipped with a continuous action of $\pi_1(S, \pi)$. Then the dual Tate functor $\check{T}$ on étale local shtukas is defined as

$$\check{T}_S : \text{EtSht}_S(S) \to \mathfrak{M}_{\text{Mod}}^G[\pi_1(S, \pi)]$$

where the superscript $\tau_\mathcal{M}$ denotes the $\mathcal{M}$ invariants. The rational dual Tate functor is defined by

$$\check{V}_S : \text{EtSht}_S(S) \to \mathfrak{M}_{\text{Mod}}^G[\pi_1(S, \pi)]$$

We also need Tate functors for local $\mathbb{H}$-shtukas. To define these we denote by $\text{Rep}_{\mathbb{F}}[\mathbb{H}]$ the category of representations $\rho : \mathbb{H} \to \text{GL}(V)$, where $V$ is a finite-free $\mathbb{F}[z]$-module and $\rho$ a morphism of algebraic groups over $\mathbb{F}[z]$. Any such $\rho$ naturally induces, as described in [AH14, section 3, above Definition 3.5], a functor $\rho_{\mathbb{H}} : \text{EtSht}_S(S) \to \text{EtSht}_S(S)$ that is compatible with quasi-isogenies.

Let $\text{Funct}^\mathbb{G}(\text{Rep}_{\mathbb{F}}[\mathbb{H}], \mathfrak{M}_{\text{Mod}}[\pi_1(S, \pi)])$ and $\text{Funct}^\mathbb{G}(\text{Rep}_{\mathbb{F}}[\mathbb{H}], \mathfrak{M}_{\text{Mod}}[\pi_1(S, \pi)])$ be the categories of the appropriate tensor functors whose morphisms are isomorphisms of functors. Now the dual Tate functor $\check{T}_S$ and the rational dual Tate functor $\check{V}_S$ are defined by

$$\check{T}_S^\mathbb{G}(\mathbb{H}) : \text{EtSht}_S(S) \to \text{Funct}^\mathbb{G}(\text{Rep}_{\mathbb{F}}[\mathbb{H}], \mathfrak{M}_{\text{Mod}}[\pi_1(S, \pi)])$$

and

$$\check{V}_S^\mathbb{G}(\mathbb{H}) : \text{EtSht}_S(S) \to \text{Funct}^\mathbb{G}(\text{Rep}_{\mathbb{F}}[\mathbb{H}], \mathfrak{M}_{\text{Mod}}[\pi_1(S, \pi)])$$

§ 2.15 Tate Functors on Global $\mathbb{G}$-Shtukas:

Now we assume that the tuple $\nu = (v_1, \ldots, v_n)$ is given by $n$ pairwise different places on $C$ and set $\mathcal{O} = C\{v_1, \ldots, v_n\}$. We denote by $\mathcal{O}^{\mathbb{G}} = \prod_{v \in \mathcal{O}} A_v$ the integral adeles of $C$ outside $\nu$ and by $\mathcal{A}^{\mathbb{G}} = \prod_{v \notin \mathcal{O}} Q_v$ the adeles of $C$ outside $\nu$. Let $\text{Rep}_{\mathcal{O}} \mathbb{G}$ be the category of representations $\rho : \mathbb{G} \to \text{Spec} \mathcal{O}^{\mathbb{G}} \to \text{GL}(V)$ where $V$ is a finite free $\mathcal{O}^{\mathbb{G}}$-module and $\rho$ a morphism of group schemes over $\mathcal{O}^{\mathbb{G}}$. Let $S$ be a connected scheme over $\text{Spf} A_\mathcal{O}$ with a fixed geometric base point $\nu$. We denote by $\mathcal{O}^{\mathbb{G}}[\pi_1(S, \mathcal{O})]$ (resp. $\mathcal{A}^{\mathbb{G}}[\pi_1(S, \mathcal{O})]$) the category of $\mathcal{O}^{\mathbb{G}}$-modules (resp. $\mathcal{A}^{\mathbb{G}}$-modules) with a continuous $\pi_1(S, \mathcal{O})$-action. For a finite subsheaf $D \subset C$ we set $D_\nu = D \times_{\mathcal{O}} \nu$ as well as $\mathcal{G}_{\nu} D_\nu = \mathcal{G} \times_{\mathcal{O}} D_\nu$. Then the dual Tate functor $\check{T}_S$ and the rational dual Tate functor $\check{V}_S$ on global $\mathbb{G}$-shtukas are defined by

$$\check{T}_S^\mathbb{G}(\mathbb{H}) : \text{EtSht}_S(S) \to \text{Funct}^\mathbb{G}(\text{Rep}_{\mathcal{O}} \mathbb{G}, \mathcal{O}^{\mathbb{G}}[\pi_1(S, \mathcal{O})])$$

and

$$\check{V}_S^\mathbb{G}(\mathbb{H}) : \text{EtSht}_S(S) \to \text{Funct}^\mathbb{G}(\text{Rep}_{\mathcal{O}} \mathbb{G}, \mathcal{O}^{\mathbb{G}}[\pi_1(S, \mathcal{O})])$$

We remark that the functor $\check{V}$ transforms by [AH13, section 6] quasi-isogenies into isomorphisms. Besides it is useful to know that there is a natural isomorphism $\check{T}_S^\mathbb{G}(\rho D_\nu) \cong \prod_{v \in \mathcal{O}} \check{T}_S^\mathbb{G}(\rho) D_\nu$ writing $\rho = (\rho_v)_{v \in \mathcal{O}}$ with $\rho_v := \rho \times id_{A_v}$. Here $L^\bullet_\nu(\mathcal{G})$ is the étale local $\mathcal{G}_\nu$-shtuka and $\check{T}_S^\mathbb{G}(\rho) = \check{T}_S^\mathbb{G}(\rho \circ_i)$ where $\rho \circ_i$ is the representation of $\mathcal{G}_\nu$ induced from $\rho_v$ by Weil restriction (see [AH14, remark 5.6]).

§ 2.16 $H$-Level Structures:

Let $H$ be an open, compact subgroup of $\mathbb{G}(A_\mathcal{O})$. In this paragraph we define $H$-level structures which are a generalization of the previous $D$-level structures. We denote by

$$\omega^\mathcal{O}_{A_\mathcal{O}} : \text{Rep}_{A_\mathcal{O}} \mathbb{G} \to \mathcal{O}^{\mathbb{G}}, \quad \omega^\mathcal{A}_{A_\mathcal{O}} : \text{Rep}_{A_\mathcal{O}} \mathbb{G} \to \mathcal{A}^{\mathbb{G}}$$

the forgetful functors and by $\text{Isom}^\mathcal{O}(\omega^\mathcal{O}_{A_\mathcal{O}} \mathcal{G}, \check{T}_S^\mathbb{G})$ and $\text{Isom}^\mathcal{A}(\omega^\mathcal{A}_{A_\mathcal{O}} \mathcal{G}, \check{V}_S^\mathbb{G})$ the sets of isomorphisms of tensor functors which are defined for any global $\mathbb{G}$-shtuka $\mathcal{G}$ over $S$, where $S$ is as before a scheme over $\text{Spf} A_\mathcal{O}$ with geometric base point $\nu \in S$. By the definition of the Tate functor $\pi_1(S, \mathcal{O})$ acts on $\check{T}_S^\mathbb{G}$ and $\mathcal{G}(A_\mathcal{O})$ (resp. $\mathcal{G}(A_\mathcal{O})$) acts on $\omega^\mathcal{O}_{A_\mathcal{O}}$ (resp. $\omega^\mathcal{A}_{A_\mathcal{O}}$) since we have $\mathcal{G}(\mathcal{G}) = \text{Aut}^\mathcal{G}(\omega^\mathcal{G}_{A_\mathcal{O}})$ by the generalized tannakian
formalism \cite[corollary 5.20]{Wed04}. This induces an action of $G(\mathcal{O}_{\kappa}) \times \pi_1(S, s)$ on $\text{Isom}^0(\omega_{\kappa, \text{red}}, T_{\kappa})$ and of $G(\mathcal{O}_{\kappa}) \times \pi_1(S, s)$ on $\text{Isom}^0(\omega_{\kappa, \text{red}}, V_{\kappa})$. Now by \cite[Definition 6.3]{AH14} a rational $H$-level structure $\gamma$ on a global $G$-shukta $\mathcal{G}$ in $\nabla_nH^1(C, G)$ is defined as a $\pi_1(S, s)$-invariant $H$-orbit $\gamma = \gamma H$ in $\text{Isom}^0(\omega_{\kappa, \text{red}}, V_{\kappa})$. We denote by $\nabla_n^H \mathcal{H}^1(C, G)$ the category fibered in groupoids over $(\mathbb{F}_q)^{\text{ét}}$ with the following fiber categories. An object in $\nabla_n^H \mathcal{H}^1(C, G)$ is a tuple $(\mathcal{G}, \gamma)$, where $\mathcal{G} \in \nabla_nH^1(C, G)$ and $\gamma$ is a $H$-level structure on $\mathcal{G}$. A morphism from $(\mathcal{G}, \gamma)$ to $(\mathcal{G}', \gamma')$ over $S$ is a quasi-isogeny $f : \mathcal{G} \to \mathcal{G}'$ that is an isomorphism at the characteristic places $v_i$ and that satisfies $V_f \circ \gamma H = \gamma' H$.

In addition we remark that by \cite[section 6]{AH13} for a open, compact subgroup $G$ there is an action of $G$ on the category fibered in groupoids over $\text{Spec} A$ by \cite[Example (3) page 2]{Hei10}. This induces an action of $G$ on the category fibered in groupoids over $\text{Spec} A$.

Now Bruhat-Tits group schemes can be constructed as follows. We start with a reductive group scheme $G$ over $\mathcal{O}_v$ with connected special fiber, with generic fiber equal to $G_0$, and with $H(A_v)$ equal to this parahoric subgroup. Since this group scheme is exactly given by $G_0$, our definition of parahoric Bruhat-Tits group scheme coincides with the one in \cite[Definition 5.2.6]{AH13}.

For each parahoric subgroup in $G_0(Q_v)$ there is a unique smooth, affine group scheme $\mathbb{H}$ over $A_v$ with connected special fiber, with generic fiber equal to $G_0$, and with $H(A_v)$ equal to this parahoric subgroup. Since this group scheme is exactly given by $G_0$, our definition of parahoric Bruhat-Tits group scheme coincides with the one in \cite[Definition 3.11]{AH13}.

Now Bruhat-Tits group schemes can be constructed as follows. We start with a reductive group scheme $G$ over the function field $Q$, which has a reductive model $G$ over an open subscheme $C \smallsetminus \{w_1, \ldots, w_m\}$ of $C$. For each of the pairwise different closed points $w \in \mathcal{V} \setminus \{w_1, \ldots, w_m\}$ we choose furthermore a parahoric subgroup $H_w \subset G(Q_w)$. Then $H_w$ corresponds as explained above to a smooth, affine group scheme $\mathbb{H}$ over $A_v$ with generic fiber $G \times Q_w$. Consequently $(\bigcup_{w \in \mathcal{V}} \text{Spec} A_v) \times C \setminus \{w_1, \ldots, w_m\}$ is a group scheme over $\bigcup_{w \in \mathcal{V}} \text{Spec} A_v \times C \setminus \{w_1, \ldots, w_m\}$. Using the theorem of Beauville-Laszlo \cite[Theorem in section 3]{BL95} the identification $\mathbb{H}_w \times A_v \times_Q Q_w$ allows us to glue this group scheme to a group scheme $G$ over $C$. This group scheme is by \cite[Proposition 2.7.1]{Gro67} smooth and by \cite[Proposition 17.7.1]{Gro67} affine over $C$. Therefore $G$ is by construction a parahoric Bruhat-Tits group scheme satisfying $G_v = \mathbb{H}_v$ and $G \times C = Q$.

Further we remark that if $\pi : \tilde{C} \to C$ is a generically étale covering of $C$ and $G$ is a parahoric Bruhat-Tits group scheme over $\tilde{C}$ then by \cite[Example (3) page 2]{Hei10} the Weil restriction $\pi_* G$ (see also lemma \ref{lem:weil restriction}) of $G$ along $\pi$ is again a parahoric Bruhat-Tits group scheme. In addition we remark that parahoric Bruhat-Tits group schemes give an interesting class of smooth, affine group schemes over $C$ since moduli spaces of global $G$-shtukas for such parahoric Bruhat-Tits group schemes $G$ are used by Lafforgue to establish in \cite{Laf12} and \cite{Laf13} the Langlands-parametrization over the function field $Q$.
3 Functionality of $\nabla_{\mathbb{Z}_{p}^{\infty}}\mathcal{H}^{1}(C, \mathbb{G})$

In this section we establish and analyze morphisms between moduli spaces of global $\mathbb{G}$-shtukas, which are functorial in changing the curve $C$ and the group scheme $\mathbb{G}$. As mentioned in the introduction, apart from the general interest of these morphisms in the study of $\nabla_{\mathbb{Z}_{p}^{\infty}}\mathcal{H}^{1}(C, \mathbb{G})$, there are two other motivations. The first motivation concerns a potential formulation of an Andrée-Oort conjecture for moduli spaces of global $\mathbb{G}$-shtukas using the finiteness results in theorem 3.14 and theorem 3.20. This potential formulation is explained in more detail in remark 3.28. The second motivation arises from the study of stratifications of $\nabla_{\mathbb{Z}_{p}^{\infty}}\mathcal{H}^{1}(C, \mathbb{G})$ in [Bre19], where the results of this third section are needed again.

The third section is divided into three subsections. In the first subsection we define a shtuka datum and morphisms of these. A shtuka datum contains all the necessary parameters to define a moduli space of $\mathbb{G}$-shtukas.

Then, assuming that $\mathbb{G}$ is a bound in the sense of §2.6, and $\mathbb{G}$ is an open, compact subgroup of $\mathbb{G}(\mathbb{A}_{f})$, we prove in theorem 3.20 a projectivity and a surjectivity result. Afterwards, we consider closed immersions of group schemes. In this situation of a closed immersion we prove $\nabla_{\mathbb{Z}_{p}^{\infty}}\mathcal{H}^{1}(C, \mathbb{G})$ to be unramified (theorem 3.23) and even finite if $\mathbb{G}$ is a parahoric Bruhat-Tits group scheme (theorem 3.26).

3.1 The Shtuka Datum

In this subsection, we define the category of Shtuka-data. While we can easily define the objects, we need some further explanations to define the morphisms.

Definition 3.1. A Shtuka-datum is a tuple $(C, \mathbb{G}, \underline{v}, Z_{\mathbb{E}}, H)$ where
- $C$ is a smooth, projective, geometrically irreducible curve over $\mathbb{F}_{q}$,
- $\mathbb{G}$ is a smooth, affine group scheme over $C$,
- $\underline{v} = (v_{1}, \ldots, v_{n})$ is a tuple of $n$ closed points in $C$ (not necessarily disjoint),
- $Z_{\mathbb{E}}$ is a bound in the sense of §2.6,
- $H$ is an open, compact subgroup of $\mathbb{G}(\mathbb{A}_{f})$.

Before we can define morphisms, we need the following lemmas. Let $\pi: X \to Y$ be a morphism of schemes. We recall that for any functor $F: (\text{Sch}/X)^{op} \to \text{Set}$ the push forward $\pi_{*}F: (\text{Sch}/Y)^{op} \to \text{Set}$ with respect to $\pi$ is defined by $(T \to Y) \mapsto F(T \times_{X} Y)$. In the case that $F$ is a scheme (i.e. representable) and $\pi$, $F$ is also representable, we call $\pi_{*}F$ the Weil restriction $\mathcal{R}_{X/Y}(F)$ of $F$. The basic properties and some conditions for the existence of Weil restrictions are discussed and developed in [BLR90, Paragraph 7.6] and [CGP10]. We have the following lemma, where we call a morphism of schemes finite locally free, if it is finite, flat and of finite presentation.

Lemma 3.2. Let $\pi: X \to Y$ be a surjective, finite locally free morphism of schemes, let $\mathbb{G}$ be a smooth, affine group scheme over $X$ and let $\mathbb{G}$ be a $\mathbb{G}$-torsor on the big étale site of $X$, then

1. $\pi_{*}\mathbb{G}$ is a smooth, affine group scheme over $Y$
2. $\pi_{*}\mathbb{G}$ is a $\pi_{*}\mathbb{G}$-torsor on the big étale site of $Y$.

Proof: Since $\pi$ is finite and faithfully flat we can apply theorem 4 in [BLR90, Paragraph 7.6] to see that the Weil restriction $\pi_{*}\mathbb{G}$ exists indeed as a scheme. Let $U, V, W \in \text{Fun}(\text{Sch}/X)^{op}, \text{Set})$ be arbitrary with natural transformations $f_{1}: U \to W$ and $f_{2}: V \to W$, then for $S \in (\text{Sch}/Y)$ we have

$$\pi_{*}((U \times_{W} V)(S)) = Hom_{X}(S \times_{Y} X, U \times_{W} V) = \{(f_{1}, g) \mid f_{1} \in Hom(S \times_{Y} X, U), g \in Hom(S \times_{Y} X, V), f_{1} \circ f = f_{2} \circ g\} = (\pi_{*}U \times_{\pi_{*}W} \pi_{*}V)(S).$$
This shows that $\pi_*$ commutes with fiber products and it follows that $\pi_* G$ becomes a group scheme over $Y$.

Let $U \subset Y$ be an affine open. Then $\pi_*(X \times_Y U) = U$ and the compatibility with the fiber product implies $\pi_*(G \times_X X \times_Y U) = \pi_* G \times_Y U$. Now $\pi_* G \times_Y U$ is affine because $G$ is affine over $X$ and $\pi$ is finite. Since the Weil restriction of an affine scheme is by construction affine we conclude that $\pi_* G$ is affine over $Y$.

Furthermore we know by [BLR90, Chapter 7.6, Proposition 5] that $\pi_* G$ is again of finite type and smooth over $Y$, which proves the first part.

Now let $G$ be a $G$-torsor over $X$. Since $G$ is smooth and affine, $G$ is represented by a smooth, affine scheme over $X$, by faithfully flat descent. [Gro95, Proposition 2.7.1] and [Gro95, Proposition 17.7.1]. [BLR90, paragraph 7.6, Theorem 4] and [BLR90, paragraph 7.6, Proposition 5] tell us again, that $\pi_* G$ is a smooth scheme over $Y$. Using once more the compatibility of the fiber product with the Weil restriction, the action of $G$ on $G$ induces an action of $\pi_* G$ on $\pi_* G$ and additionally the isomorphism $G \times_X G \simeq G \times X G$ yields an isomorphism $\pi_* G \times_Y \pi_* G = \pi_* G \times_Y \pi_* G$. It remains to show that $\pi_* G$ has étale locally on $Y$ a section to $\pi_* G$. Since $\pi_* G \to Y$ is smooth and surjective this is content of proposition [BLR90, paragraph 2.2, Prop. 14].

Now morphisms between $G$-torsors are sent by $\pi_*$ to morphisms of $\pi_* G$-torsors and in fact we have the following lemma.

**Lemma 3.3.** Let $\pi : X \to Y$ be a surjective, finite locally free morphism and $G$ a smooth, affine group scheme over $X$. Then the functor

$$
\pi_* : \{G\text{-torsors on } X\} \longrightarrow \{\pi_* G\text{-torsors on } Y\},
$$

induced by lemma 3.2, is an equivalence of categories. The inverse functor sends some $\pi_* G$-torsor $\tilde{G}$ to $\pi_* G \times \pi_* G$.

**Proof:**

First we prove that $\pi_*$ is fully faithful. So let $G, \tilde{G}$ be two $G$-torsors over $X$ and $f' : \pi_* G \to \pi_* \tilde{G}$ be a morphism of $\pi_* G$-torsors. We choose an étale covering $U' \to Y$ with $\pi_* G(U') \neq \emptyset \neq \pi_* G(U')$. Now this implies automatically that $U := U' \times_Y X \to X$ is an étale covering with $G(U) \neq \emptyset \neq G(U)$. We choose two sections $u \in G(U) = \pi_* G(U')$ and $\bar{u} \in \tilde{G}(U) = \pi_* \tilde{G}(U')$, which determine trivializations

$$
\alpha' : \pi_* G \times_Y U' \to G \times_Y U' \quad \text{and} \quad \alpha : G \times_X U \to G \times_X U
$$

with $\alpha'^{-1}(1) = \alpha^{-1}(1) = u$ and $\tilde{\alpha}^{-1}(1) = \tilde{\alpha}^{-1}(1) = \bar{u}$. Now we consider the following diagrams, where $U_2 := U \times_X U$, $U'_2 := U' \times_Y U'$ with projections $p_1, p_2$, $p'_1$ and $p'_2$ and $h := \tilde{\alpha}^{-1} \circ (f' \times id_{U'}) \circ \alpha'^{-1}$. Note that since $h$ is $\pi_* G$-equivariant $h$ is determined by $h := h(1) \in \pi_* G(U') = G(U)$. This same $h$ defines then a morphism of $h : G \times_X U \to G \times_X U$ of $G$-torsors on $U$ and we set $f \times id_U := \alpha'^{-1} \circ h \circ \alpha : G \times_X U \to G \times_X U$.
are uniquely determined by the preimage of $1 \in \pi_1 \mathbb{G}(U'_2) = \mathbb{G}(U_2)$. But this preimage is in both cases given as $p_i^1(u) = p_i(\alpha^{-1}(1)) = p_i'(\alpha^{-1}(1)) \in \pi_1 \mathbb{G}(U'_2) = \mathbb{G}(U_2)$. Hence the maps $p_1^\alpha$ and $p_1^\alpha'$ coincide and equally $p_1^\alpha h, p_1^\alpha' h, p_2^\alpha, p_2^\alpha'$ coincide with $p_1 h, p_1\alpha, p_2 h$ and $p_2\alpha$ respectively.

We further denote $g' := p_2^\alpha \circ p_1^\alpha \alpha^{-1}(1) \in \pi_1 \mathbb{G}(U'_2)$ and $\overline{g'} := p_2^\alpha \alpha \circ p_1^\alpha \alpha^{-1}(1) \in \pi_1 \mathbb{G}(U'_2)$. So that we have $g' = g := p_2^\alpha \circ p_1^\alpha \alpha^{-1}(1) \in \mathbb{G}(U_2)$ and $\overline{g'} = \overline{g} := p_2^\alpha \circ p_1^\alpha \alpha^{-1}(1) \in \mathbb{G}(U_2)$. With these notations we get the following bijections:

$$\begin{align*}
\text{Hom}(\mathbb{G}, \overline{\mathbb{G}}) & \overset{f \mapsto h := (\overline{\mathbb{G}}(f \times id)) \circ \alpha^{-1}(1)}{\longrightarrow} \{ h \in \mathbb{G}(U) \mid \overline{g} \circ p_1 h = p_2 h \circ g \} \\
\text{Hom}_{\mathbb{G}, \mathbb{G}}(\pi_1 \mathbb{G}, \pi_1 \overline{\mathbb{G}}) & \overset{f' \mapsto h' := (\overline{\mathbb{G}}(f' \times id)) \circ \alpha^{-1}(1)}{\longrightarrow} \{ h' \in \pi_1 \mathbb{G}(U') \mid \overline{g'} \circ p_1 h' = p_2 h' \circ g' \}
\end{align*}$$

Here the horizontal bijections are due to faithfully flat descent [BLR93, paragraph 6.1, Theorem 6] and the fact that the condition $\overline{g} \circ p_1 h = p_2 h \circ g$ is equivalent by definition of $g, \overline{g}$ to $p_1(\alpha^{-1} h \circ \alpha) = p_2(\alpha^{-1} \circ h \circ \alpha)$ and for $\overline{g'} \circ p_1 h' = p_2 h' \circ g'$ respectively. The equality on the right follows from the identifications in the above cubes. To prove the fully faithfulness it remains to show that the bijective dashed arrow is given by $\pi_1$. By definition of $\pi_1 f$ the following diagram commutes

$$\begin{array}{ccc}
\text{Hom}(U', \pi_1 \mathbb{G}) & \overset{\pi_1 f}{\longrightarrow} & \text{Hom}(U', \pi_1 \overline{\mathbb{G}}) \\
\text{Hom}(U' \times Y, X, \pi_1 \mathbb{G}) & \overset{f}{\longrightarrow} & \text{Hom}(U' \times Y, X, \pi_1 \overline{\mathbb{G}})
\end{array}$$

which shows that $f$ and $\pi_1 f$ map to the same $h$ on the right hand side.

It remains to show that $\pi_1$ is essentially surjective. So let $\overline{\mathbb{G}}$ be a $\pi_1 \mathbb{G}$-torsor over $Y$ and choose again an étale covering $U' \to Y$ and a trivialization $\alpha' : \overline{\mathbb{G}} \times Y \to \mathbb{G} \times Y$. Let $U'_2 := \pi_1 \mathbb{G} \times Y$ and $g' := \pi_2^\alpha \circ p_1^\alpha \alpha^{-1}(1) \in \pi_1 \mathbb{G}(U'_2)$, where $p_2^\alpha \circ p_1^\alpha \alpha^{-1} : \pi_1 \mathbb{G} \times Y \to \pi_1 \mathbb{G} \times Y$. So the descent datum of $\overline{\mathbb{G}}$ is isomorphic to $(\pi_1 \mathbb{G} \times Y, g')$. Now $U := U' \to Y \times X$ is an étale covering and we set $\mathbb{G}_U := \mathbb{G} \times X$ as well as $U_2 := \pi_1 \mathbb{G} \times Y \to U_2 \times Y$ with projections $p_1, p_2$. Let $g \in \mathbb{G}(U_2)$ be equal to $g'$ using $\mathbb{G}(U_2) = \pi_1 \mathbb{G}(U'_2)$. Then $(\mathbb{G}_U, g)$ is a descent datum that comes by [BLR93, beginning of paragraph 6.5 and paragraph 6.1, Theorem 6] from a $\mathbb{G}$-torsor $\overline{\mathbb{G}}$ on $X$. Now it is clear that $(\pi_1 \mathbb{G}, \pi_1 g) = (\pi_1 \mathbb{G} \times Y, g')$. Therefore we have $\pi_1 \overline{\mathbb{G}} = \overline{\mathbb{G}}$ which proves that $\pi_1$ is essentially surjective. We only need to prove that for every $\mathbb{G}$-torsor $\overline{\mathbb{G}}$ on $X$ the torsor $\pi_1 \overline{\mathbb{G}} \times \pi_1 \mathbb{G} \subseteq \mathbb{G}$ is isomorphic to $\mathbb{G}$. With the same notation as above $\overline{\mathbb{G}}$ is given by the $\mathbb{G} \times X, (g, x)$ and $\pi_1 \overline{\mathbb{G}}$ is given by the descent datum $(\pi_1 \mathbb{G} \times Y, g')$. Restricting the latter torsor to $X$ we get the descent datum $(\pi_1 \mathbb{G} \times Y, g) \times id_X = (\mathbb{G} \times Y, (x, g \times id_X))$. Using the adjunction $\text{Hom}(\pi_1 \mathbb{G}, \pi_1 \mathbb{G}) = \text{Hom}(\pi_1 \mathbb{G}, \pi_1 \mathbb{G})$ we denote by $\varphi : \pi_1 \mathbb{G} \to \mathbb{G}$ the morphism corresponding to $id_{\pi_1 \mathbb{G}}$. Now applying the functor $\pi_1 \mathbb{G} \times \mathbb{G}$ gives us the descent data $(\pi_1 \overline{\mathbb{G}} \times Y, (g') \times id_X)$, which proves the lemma.

Now let $\pi : C \to C'$ be a finite morphism from $C$ to some other smooth, projective, geometrically irreducible curve $C'$. This morphism is then automatically faithfully flat [Har77, chapter II Prop. 6.8 and chapter III Prop. 9.7]. Let further $(C, \mathbb{G}, \pi, Z, H)$ be a shtuka datum as in definition 3.1. Since $\mathbb{G}$ is a smooth, affine group scheme over $C$, this allows us to apply lemma 3.2 and 3.3 in this situation.

**Remark 3.4.** We denote by $\mathcal{M}(C, \mathbb{G})$ the category fibered in groupoids, whose $S$-valued points for some $\mathbb{G}$-scheme $S$ are given by isomorphism classes of $\mathbb{G}$-torsors over $C_S$. By lemma 3.3, $\pi$, induces an isomorphism $\mathcal{M}(C, \mathbb{G}) \longrightarrow \mathcal{M}(C', \pi_1 \mathbb{G})$.

Let $w_i = (v_i)$ and $w = (w_1, \ldots, w_n)$. Our next goal is to define the bound $\pi, Z_w = (Z_{w_i})$, at the points $w_i$. We need the following lemma and general remark, where $w$ is a closed point in $C'$, $A_w'$ is the completion of the local ring $O_{C', w}$ and $\pi_A = \pi \times id_{\text{Spec} A'} : C \times C' \to \text{Spec} A'$ is the completion of $\mathbb{G}$.

**Remark 3.5.** In the next lemma, we need the following general fact about Weil restrictions. Let $X, Y, S$ be schemes over some base scheme $Z, \pi : X \to Y$ a $Z$-morphism, $M$ an $X$-scheme and $Y_S = Y \times_S S$, $X_S = X \times_S S$ and $M_S = M \times_S S$ the appropriate base changes. Then we have $(\pi \times id_S), (M \times_S S) = \pi, M \times_S S$. 

12
This is easily seen by the equation for $T \in \text{Sch}/\mathcal{Y}_S$:

$$(\pi_* M \times T S)(T) = (\pi_* M \times Y S)(T) = \text{Hom}_Y(T, \pi_* M) = \text{Hom}_X(T \times Y, M)_x = \text{Hom}_X(T \times Y, M) = (\pi \times id_S)(M \times T S)(T).$$

**Lemma 3.6.** We have $(\pi_w)_* (\coprod_{v\in y^{-1}(w)} \mathbb{G}_a) = (\pi_* \mathbb{G})_w$ as a group scheme over $\text{Spec} \mathcal{A}_w$.

**Proof:** This follows formally from remark 3.5 with $M = \mathbb{G}$, $X = C$, $Y = Z = C'$ and $S = \text{Spec} \mathcal{A}_w$ since we have

$$(\pi_w)_* (\coprod_{v\in y^{-1}(w)} \text{Spec} \mathcal{A}_v) = (\pi_w)_* (\mathbb{G} \times_C \text{Spec} \mathcal{A}_v) = (\pi_w)_* (\mathbb{G} \times_C \text{Spec} \mathcal{A}_w) = (\pi_* \mathbb{G})_w.$$

The notation $\mathcal{G}_v$ was introduced in §2.9 and denotes the Weil restriction of the group scheme $\mathcal{G}_v$ along $\text{Spec} \mathcal{A}_v \rightarrow \text{Spec} \mathbb{F}_q[z_v]$. The lemma has the following corollary.

**Corollary 3.7.** We have $\prod_{v\in w} L^* \mathcal{G}_v = L^* (\pi_* \mathbb{G})_w$ as group schemes over $\mathbb{F}_q$.

**Proof:** Let $R$ be a connected $\mathbb{F}_q$-algebra, then we have:

$$L^* (\pi_* \mathbb{G})_w (R) = (\pi_* \mathbb{G})_w (R[z_w]) = \text{Hom}_{\text{Spec} \mathcal{A}_w} (\text{Spec} R[z_w] \otimes \mathbb{F}_q, \mathbb{F}_w, (\pi_* \mathbb{G})_w$$

$$= \text{Hom}_{\text{Spec} \mathcal{A}_w} (\text{Spec} R \otimes \mathbb{F}_q, (\pi_* \mathbb{G})_w)$$

$$= \text{Hom}_{\text{Spec} \mathcal{A}_w} (\text{Spec} R \otimes \mathbb{F}_q, (\pi_* \mathbb{G})_w, (\prod_{v\in w} \mathcal{G}_v))$$

$$= \prod_{v\in w} \text{Hom}_{\text{Spec} \mathcal{A}_v} (\text{Spec} R \otimes \mathbb{F}_q, \mathcal{G}_v) = \prod_{v\in w} \mathcal{G}_v (R[z_v])$$

$$= \prod_{v\in w} L^* \mathcal{G}_v (R).$$

We have the following 2-cartesian diagrams:

$$\begin{array}{ccc}
\prod_{v\in w} \mathcal{F}l_{\mathcal{G}_v} & \xrightarrow{\sim} & \mathcal{F}l_{(\pi_* \mathbb{G})_w} \\
\downarrow & & \downarrow \\
\prod_{v\in w} \mathcal{H}^1 (\mathbb{F}_q, L^* \mathcal{G}_v) & \xrightarrow{\sim} & \mathcal{H}^1 (\mathbb{F}_q, L^* (\pi_* \mathbb{G})_w)
\end{array}$$

By corollary 3.7 the lower stacks in the diagrams are isomorphic, so that we get an isomorphism $\prod_{v\in w} \mathcal{F}l_{\mathcal{G}_v} \xrightarrow{\sim} \mathcal{F}l_{(\pi_* \mathbb{G})_w}$ and by the base change with the compositum $\mathcal{F}$ of the finite fields $\mathbb{F}_v$ for all $v \in w$ we get an isomorphism $\prod_{v\in w} \mathcal{F}l_{\mathcal{G}_v} \times_{\mathbb{F}} \mathcal{F} \simeq \prod_{v\in w} \mathcal{F}l_{(\pi_* \mathbb{G})_w} \times_{\mathbb{F}_w} \mathcal{F}$. Since $\sigma^{deg_w}$ invariant components are mapped to $\sigma^{deg_w}$ invariant components, it restricts to an isomorphism

$$\prod_{v\in w} \prod_{t \in \mathbb{Z}/deg_w} \mathcal{F}l_{\mathcal{G}_v} \times_{\mathbb{F}_w} \mathcal{F} \simeq \mathcal{F}l_{(\pi_* \mathbb{G})_w} \times_{\mathbb{F}_w} \mathcal{F}.$$

Now let $R$ be a DVR with $\mathbb{F} \subset R$ and such that there exists a representative $\hat{Z}_{v,R}$ of $\hat{Z}_v$ for all $v \in w$. Consider the ind-closed subscheme

$$\prod_{v\in w} \prod_{t \in \mathbb{Z}/deg_w} \hat{Z}_{v,t} \subset \prod_{v\in w} \prod_{t \in \mathbb{Z}/deg_w} \mathcal{F}l_{\mathcal{G}_v} \times_{\mathbb{F}_w} \text{Spf } R$$

where $\hat{Z}_{v,t}$ is always the closed stratum $\mathcal{S}(1) \times_{\mathbb{F}_w} \text{Spf } R$ except for $v \in w$ and $t = 0$, where we set $\hat{Z}_{v,0} = \hat{Z}_{v,R}$. Here $\mathcal{S}(1)$ denotes the closed Schubert cell $1 \cdot L^* \mathcal{G}_v$ in $\mathcal{F}l_{\mathcal{G}_v}$. Via the previous isomorphism this defines an
Next we define \( \pi, H \). We recall that \( H \) was an open, compact subgroup of \( G(\mathbb{A}_\infty) \). Since \( v \in \pi^{-1}(w) \subset |C| \) we have a quotient map of topological rings \( \mathbb{A}_w \rightarrow \mathbb{A}_{\pi^{-1}(w)} \). Since this map is open, it induces by [Con12, theorem 3.6] an open continuous group homomorphism \( G(\mathbb{A}_w) \rightarrow G(\mathbb{A}_{\pi^{-1}(w)}) \). We have \( \mathbb{A}_w \times_C C = \mathbb{A}_w \times_C \eta = \mathbb{A}_{\pi^{-1}(w)} \) where \( \eta \) are the generic points of \( C \) and \( C' \). This gives us with the definition of the Weil restriction \( \pi, G(\mathbb{A}_w) = G(\mathbb{A}_{\pi^{-1}(w)}) \), where both groups carry the same topology by [Con12, example 2.4]. Now the image of \( H \) under this morphism gives us an open, compact subgroup in \( \pi, G(\mathbb{A}_w) \) that we denote by \( \pi, H \).

**Remark 3.8.** We have seen in §2.12 that there is the possibility to define level structures using finite closed subschemes \( D \) of \( C \) and in §2.16 we have remarked that \( D \)-level structures of a \( G \)-shtuka correspond bijectively to \( H_D \)-level structures, where \( H_D = \ker(G(\mathbb{O}_w) \rightarrow G(\mathbb{O}_D)) \). Now we can also consider the Weil restriction \( \pi, D \) of \( D \). It is a closed finite subset of \( C' \) consisting of the points \( \{ w \in |C'| \mid w \times_C C \subset D \} \). And with a \( D \)-level structure of some \( G \)-shtuka \( G \) we could associate a \( \pi, D \)-level structure of the corresponding \( \pi, G \)-shtuka \( \pi, G \) (which will be defined in proposition 3.11). But compared to the associated \( \pi, H \)-level structure that we will define in theorem 3.14 we would lose some information at the points \( D \setminus (\pi, D \times_{C'} C) \), which is seen in the following way. Since we have \( \pi, D \times_{C'} C \subset D \) we have \( H_D \subset H_{\pi, D \times_{C'} C} \) and hence \( \pi, H_D \subset \pi, H_{\pi, D \times_{C'} C} \). Now

\[
H_{\pi, D} = \ker(\pi, G(\mathbb{O}_w) \rightarrow \pi, G(\mathbb{O}_D)) = \ker(G(\mathbb{O}^{\pi^{-1}}(w) \rightarrow G(\mathbb{O}_{\pi, D \times_{C'} C})) = \ker(G(\mathbb{O}_{\pi, D \times_{C'} C})) = \pi, H_{\pi, D \times_{C'} C} \circ \pi, H_D
\]

shows that \( \pi, H_D \) is in general a finer level than \( \pi, D \) (or equivalently \( H_{\pi, D} \)) and the previous equation shows that the information is lost exactly at the points \( D \setminus (\pi, D \times_{C'} C) \).

All these previous explanations concerned the case that we change the curve in the shtuka datum but we can also change the group scheme in this datum. Let \( f : G \rightarrow G' \) be any morphism of smooth, affine group schemes over \( C \) and \( v \) a closed point in \( C \). Firstly this induces a morphism \( L^*G_v \rightarrow L^*G'_v \) of the positive loop groups as well as a morphism \( LG_v \rightarrow LG'_v \) of the loop groups. Consequently we also get a morphism \( \mathcal{F}G_v \rightarrow \mathcal{F}G'_v \) of the affine flag varieties. Secondly such a morphism induces a morphism \( f_{\mathbb{A}_w} : G(\mathbb{A}_w) \rightarrow G'(\mathbb{A}_w) \) of locally compact Hausdorff spaces by [Con12, Proposition 2.1]. Now we can define morphisms of shtuka data.

**Definition 3.9.** A morphism between two shtuka data \( (C, G, v, \tilde{Z}_w, H) \) and \( (C', G', w, \tilde{Z}'_w, H') \) is a pair \((\pi, f)\) such that:

- \( \pi : C \rightarrow C' \) is a finite morphism with \( \pi(v) = w \),
- \( f : \pi, G \rightarrow G' \) is a morphism of smooth, affine group schemes over \( C' \),
- The morphism \( (\pi, Z_w)_R \rightarrow \bigoplus_{w \in \mathbb{A}_w} \mathcal{F}_l(G_{\mathbb{A}_w} \times R) \rightarrow \bigoplus_{w \in \mathbb{A}_w} \mathcal{F}_l(G'_{\mathbb{A}_w} \times R) \) factors through \( \tilde{Z}'_w \times R \), where \( R \) is a DVR such that there exists representatives \((\pi, Z_w)_R \) and \( \tilde{Z}'_w \times R \) of the corresponding bounds
- \( f_{\mathbb{A}_w}(\pi, H) \subset H' \)

With this definition we have reached the goal of this subsection. In the next two subsections we will prove that such a morphism induces a morphism of the corresponding moduli stacks and determine some of its properties. But before we give some remarks.

**Remark 3.10.**

- Let \( \pi : C \rightarrow C' \) be a finite morphism and \( w = \pi(v) \). With the definition of \( \pi, H \) and \( \pi, Z_w \) on page 13 it is clear that \((\pi, id_{\pi, G}) : (C, G, v, Z_w, H) \rightarrow (C', \pi, G, w, \pi, Z_w, \pi, H) \) defines a morphism of shtuka data
- Every morphism \((\pi, f)\) of shtuka data factorizes as \((id_C, f) \circ (\pi, id_{\pi, G})\).
If \( f : \pi_*G \to G' \) is an isomorphism in the generic fiber we have \( \pi_*G(\Lambda) = G'(\Lambda) \) so that we can naturally choose \( H = H' \).

If \( f : \pi_*G \to G' \) is smooth in the generic fiber, then \( f_\Lambda : \pi_*G(\Lambda) \to G'(\Lambda) \) is an open map by \cite[Theorem 4.5]{Con12} so that we can naturally choose \( H' = f_{\Lambda^n}(\pi, H) \).

If \( f : \pi_*G \to G' \) is proper in the generic fiber, then \( f_\Lambda : \pi_*G(\Lambda) \to G'(\Lambda) \) is a topologically proper map by \cite[Proposition 4.4]{Con12} so that we can naturally choose \( H = f_{\Lambda^n}(H) \).

### 3.2 Changing the Coefficients

In this subsection we prove that a morphism of shtuka data \((\pi, id)\), where we only change the curve, induces a finite morphism of the corresponding moduli stacks. We firstly prove this for the moduli stack \( \nabla_n \mathcal{H}^1(C, G) \), where the characteristic sections are not fixed and no boundedness condition or level structures are imposed. For this purpose we need the following lemma:

**Lemma 3.11.** Let \( S \) be an \( \mathbb{F}_q \)-scheme with \( n \) morphisms \( s_i : S \to C \) for \( i = 1, \ldots, n \). Then the scheme theoretic image of \( \tilde{C}_S := C_S \setminus \bigcup_s \Gamma_{s_1} \) in \( C_S \) equals \( C_S \).

**Proof:** Since \( D := \bigcup_s \Gamma_{s_1} \) is an effective Cartier-Divisor on \( C_S \) over \( S \), we find an affine covering \((U_j)_{j \in J}\) of \( C_S \) with \( U_j := \text{Spec} \, B_j \) such that \( D \) is the vanishing locus of an element \( f_j \in B_j \) that can be written as \( f_j = q_j \) with two regular elements \( a_j, b_j \in B_j \) (see \cite[2.14]{GW10}). Now the ring homomorphism \( B_j \to \Gamma(U_j \setminus \mathcal{D}, \mathcal{O}) \) is injective, which is seen as follows. An element \( x \in B_j \) is sent to 0 if and only if \( f_j^n x = 0 \) for some \( n \in \mathbb{N} \). The latter condition implies \( a_j^n x = 0 \) and since \( a_j \) is a non-zero divisor this means \( x = 0 \) so that \( \text{Spec} \, B_j \setminus \mathcal{D} \) is schematically dense in \( \text{Spec} \, B_j \) (compare also \cite[Lemma 20.2.9]{Gro67}). Now gluing all the \( U_j \) shows that for every affine open \( V \subset C_S \) the ring homomorphism \( \Gamma(V, \mathcal{O}) \to \Gamma(V \setminus \mathcal{D}, \mathcal{O}) \) is injective and we conclude that \( \tilde{C}_S \) is schematically dense in \( C_S \) (see \cite[20.2.1]{Gro67}). \( \square \)

**Proposition 3.12.** Let \( \pi : C \to C' \) be a finite morphism of smooth, projective, geometrically irreducible curves over \( \mathbb{F}_q \), and let \( G \) be a smooth, affine group scheme over \( C \). This induces a finite morphism of the moduli stacks

\[
\pi_* : \nabla_n \mathcal{H}^1(C, G) \to \nabla_n \mathcal{H}^1(C', \pi_* G).
\]

which factors through a closed immersion \( \nabla_n \mathcal{H}^1(C, G) \to \nabla_n \mathcal{H}^1(C', \pi_* G) \times_{C'^n} C^n \).

**Proof:** Let \( S \) be an \( \mathbb{F}_q \)-scheme and \( (G, s_1, \ldots, s_n, \tau_G) \in \nabla_n \mathcal{H}^1(C, G)(S) \). We describe its image \((G', s'_1, \ldots, s'_n, \tau_{G'}) \) to define the morphism. The torsor \( G' \) is given by \((\pi_S)_* G \) and the sections \( s_i : S \to C' \) are mapped to the composition \( s'_i = \pi \circ s_i : S \to C' \). This implies \( \pi_S((\pi_S)_* G) \subset \bigcup_i \Gamma_{s'_i} \subset C'_S \). Let \( \tilde{G}'_S = C'_S \setminus (\bigcup_i \Gamma_{s'_i}) \) and \( \tilde{C}'_S = C'_S \setminus (\bigcup_i \Gamma_{s_1}) \). Then \( U := C \times_{C'} \tilde{C}'_S = C_S \times_{C'_S} \tilde{G}'_S \) is open in \( \tilde{C}'_S \). We denote by \( \pi_U := \pi \times_{id_{C'}} id_{\tilde{C}'_S} : U \to \tilde{C}'_S \) and we have \((\pi_U)_* (G \times_{C'} U) = \pi_S G \times_{C'_S} \tilde{C}'_S \). Now we restrict \( \tau_G : \sigma^* G|_{\tilde{C}'_S} \to \tilde{G}'_S|_{\tilde{C}'_S} \) to \( U_S \) and apply lemma \([3.3]\) to \( \pi_U \). The category equivalence gives us the desired morphism \( \tau_{G'} : (\pi_U)_* (\sigma^* G|_{\tilde{C}'_S} \tilde{C}'_S) = \sigma^* G|_{\tilde{C}'_S} \to (\pi_U)_* (\sigma^* G|_{\tilde{C}'_S} \tilde{C}'_S) = G'|_{\tilde{C}'_S} \). This defines a global \( \pi_* G \)-shtuka \((G', s'_1, \ldots, s'_n, \tau_{G'}) \) over \( S \) and therefore the morphism of the moduli stacks.

We now show that this morphism is representative and finite. Let \( S \) be again an arbitrary scheme over \( \mathbb{F}_q \) and \( \tilde{G}' : S \to \nabla_n \mathcal{H}^1(C', \pi_* G) \) be given by \((G', r', s'_1, \ldots, s'_n) \). Then \( \nabla_n \mathcal{H}^1(C, G) \times_{\nabla_n \mathcal{H}^1(C', \pi_* G)} S \) is the category fibered in groupoids over \( \text{Sch}^{/\mathbb{F}_q} \) whose fiber category over an \( \mathbb{F}_q \)-scheme \( T \) is given by

\[
\{ (G, g : T \to S, \beta) : G = (\pi, \tau_G, s_1, \ldots, s_n) \in \nabla_n \mathcal{H}^1(C, G)(T) \text{ and } \beta : g' G' \to \pi_* G \}. \]

Using the \( n \) sections \( s'_1, \ldots, s'_n : S \to C' \) and the morphism \( \pi : C \to C' \) we set \( \tilde{S} := S \times_{C'^n} C^n \). Since \( S \times_{\mathcal{H}^1(C', \pi_* G)} \mathcal{H}^1(C, G) = S \) by remark \([3.3]\) we know that \( Hom_{\mathcal{G}}(T, S) \) is in bijection with the tuples \((g : T \to S, G, \alpha, g' G' \to \pi_* G)\). Consequently the \( \mathbb{F}_q \)-morphisms \( T \to S \) are in bijection with the tuples \((G, s_1, \ldots, s_n, g, \alpha)\), where \((G, g, \alpha) \in S(T) \) as before and \( s_1, \ldots, s_n : T \to C \) are morphisms making the following diagram commutative for all \( i = 1, \ldots, n \):

\[
\begin{array}{c}
T \xrightarrow{g} S \\
\downarrow s_i \quad \downarrow \pi \\
C \xrightarrow{\pi} C'
\end{array}
\]
We claim that we get a morphism \( \nabla_n \mathcal{H}^1(C, \mathbb{G}) \times_{\nabla_n \mathcal{H}^1(C, \pi, \mathbb{G})} S \rightarrow \tilde{S} \) that is injective on \( T \)-valued points (hence a monomorphism) and satisfies the valuative criterion for properness. Then this implies by [Gro66, Proposition 8.11.5] that \( \nabla_n \mathcal{H}^1(C, \mathbb{G}) \times_{\nabla_n \mathcal{H}^1(C, \pi, \mathbb{G})} S \) is a closed subscheme of \( \tilde{S} \). So first of all a given object \((G, \tau_G, s_1, \ldots, s_n, g, \alpha)\) in \( \nabla_n \mathcal{H}^1(C, \mathbb{G}) \times_{\nabla_n \mathcal{H}^1(C, \pi, \mathbb{G})} S(\mathcal{T}) \) is sent to \((G, s_1, \ldots, s_n, g, \alpha)\). Since \( \alpha : g^* G \rightarrow \pi_*G \) is not only an isomorphism of torsors, but also of \( \pi_*G \)-shtukas the \( n \)-sections \( g \circ s'_i \) of \( g^* G \) and \( s_i \circ \pi \) of \( \pi_*G \) have to coincide, so that \((G, s_1, \ldots, s_n, g, \alpha)\) is a well defined object in \( \tilde{S}(\mathcal{T}) \). This induces the morphism \( \nabla_n \mathcal{H}^1(C, \mathbb{G}) \times_{\nabla_n \mathcal{H}^1(C, \pi, \mathbb{G})} S \rightarrow \tilde{S} \). Further it was claimed, that this morphism is injective on \( T \)-valued points. So given two points \((G, \tau_G, s_1, \ldots, s_n, g, \alpha)\) and \((G, \tau_G', s_1, \ldots, s_n, g, \alpha)\) we need to show that this implies \( \tau_G = \tau_G' \). Since \( \alpha \) is an isomorphism of \( \pi_*G \)-shtukas, we have \( \alpha^{-1} \circ \pi \circ \tau_G = \alpha^{-1} \circ \pi \circ \tau_G' = : \sigma^* (\pi, G)_{C_T} \rightarrow \sigma^* (\pi, G)'_{C_T} \), where we write again \( \tilde{C}_T := C_T \cup \Gamma_{s'_i} \). This implies \( \pi, \tau_G = \pi, \tau_G' \) and using lemma 3.3 applied to \( \pi \times id_{\tilde{C}_T \times \mathcal{O}_C} \) we see \( \tau_G|_{\tilde{C}_T \times \mathcal{O}_C} = \tau_G'|_{\tilde{C}_T \times \mathcal{O}_C} \). We even need to know that \( \tau_G = \tau_G' \). In the following diagram the restriction of \((\tau_G, \tau_G') \) to \( \tilde{C}_T \times \mathcal{O}_C \) factors by the previous observation through the diagonal \( \Delta \).

\[
\begin{array}{ccc}
\sigma^* G|_{\tilde{C}_T} & \xrightarrow{(\tau_G, \tau_G')} & G|_{\tilde{C}_T} \\
\downarrow & & \downarrow \Delta \\
\sigma^* G|_{\tilde{C}_T \times \mathcal{O}_C} & \rightarrow & G|_{\tilde{C}_T} \\
\end{array}
\]

Since \( \mathcal{G} \) is separated over \( C_T \) the diagonal is a closed immersion and \((\tau_G, \tau_G')\) factors already over \( \tilde{C}_T \) through the diagonal if the scheme theoretic image of \( \sigma^* G|_{\tilde{C}_T \times \mathcal{O}_C} \) in \( \sigma^* G|_{\tilde{C}_T} \) equals \( \sigma^* G|_{\tilde{C}_T} \). Since taking the scheme theoretic image is stable under flat base change by [Gro66, Théorème 11.10.5], this is the case if the scheme theoretic image of \( C_T' \times \mathcal{O}_C \) in \( \tilde{C}_T \) equals \( \tilde{C}_T \). By the same argument this follows if the scheme theoretic image of \( \tilde{C}_T' \) in \( \tilde{C}_T \) equals \( \tilde{C}_T \). Now this is content of lemma 3.11 so that we can conclude as desired \( \tau_G = \tau_G' \).

Next we claimed that the morphism satisfies the valuative criterion for properness. So let

\[
\begin{array}{ccc}
\text{Spec} \ K & \xrightarrow{(H, \tau_H, s_1, \ldots, s_n, f, \beta)} & S \\
\downarrow & \text{Spec} \ R & \downarrow \pi \times \text{Spec} \ \mathbb{Z} \\
\text{Spec} \ R & \xrightarrow{(\mathcal{G}, s_1, \ldots, s_n, g, \alpha)} & S \\
\end{array}
\]

be a commutative diagram, where \( R \) is a complete discrete valuation ring with fraction field \( K \), maximal ideal \( m \) and algebraically closed residue field \( K_m = R/m \). Note that \( R \) is a \( k_R \)-algebra. We have to prove that there exists a unique dashed arrow making everything commutative. The commutativity of the square shows \( H = j^* \mathcal{G}, s_1 \circ j = ri : \text{Spec} \ K \rightarrow C \), \( f = g \circ j : \text{Spec} \ K \rightarrow S \) and \( j^* \alpha = \beta \). To define this dashed arrow we have to extend \((G, s_1, \ldots, s_n)\) to a \( \mathbb{G} \)-shtuka \((G, \tau_G, s_1, \ldots, s_n)\) over \( R \) such that \( \alpha \) extends to an isomorphism \( \alpha : g^* G \rightarrow \pi_*G \) and \( j^* \tau_G = \tau_H \). So we define this isomorphism \( \tau_G : \sigma^* G|_{\tilde{C}_T} \rightarrow \tilde{G}^e_{\tilde{C}_T} \).

Since \( \mathcal{H}^1(\tilde{C}_T, \pi, \mathbb{G}) \) and \( \mathcal{H}^1(\tilde{C}_R \times \mathcal{O}_C, \pi, \mathbb{G}) \) are isomorphic, \( \tau_G|_{\tilde{C}_R \times \mathcal{O}_C} \) is defined by \( \alpha \circ g^* \tau_G \circ \sigma^* \alpha^{-1} \).

Furthermore we know that \( \tau_G \) is defined on the generic fiber \( \tilde{C}_K \subset \tilde{C}_R \) by \( \tau_G|_{\tilde{C}_K} = j^* \tau_G = \tau_H \). So let \( p \in \tilde{C}_K \setminus \tilde{C}_K \cap \tilde{C}_R \) is also in \( \Gamma_k \). It remains to show, that \( \tau_G \) is extendable to \( p \). Since \( p \) is closed we choose an open \( U \subset \tilde{C}_R \) with \( \cap \Gamma_k \cap \tilde{C}_R \cap \tilde{C}_R \setminus \tilde{C}_R \cap \Gamma_k \). Then we consider the 2-cartesian diagram of stacks fibered over \( \kappa_R \mathcal{E}_t \) (compare [AH14, Lemma 5.1]):

\[
\begin{array}{ccc}
\mathcal{H}^1(V, \mathbb{G}) & \xrightarrow{L_p} & \mathcal{H}^1_\kappa(V, \mathbb{G}) \\
\downarrow & & \downarrow \mathcal{L}_p \\
\mathcal{H}^1(\kappa_R, L^* \mathbb{G}_p) & \xrightarrow{L_p} & \mathcal{H}^1(\kappa_R, L^* \mathbb{G}_p) \\
\end{array}
\]

Here \( \mathcal{H}^1_\kappa(V, \mathbb{G})(X) \) is the full subcategory of \( \mathcal{H}^1(V, \mathbb{G})(X) \) consisting of those \( \mathbb{G} \)-torsors over \( \tilde{V}_X := \tilde{V} \times_{\kappa_R} X \) that can be extended to a \( \mathbb{G} \)-torsor over \( X \). Now \( \mathcal{G}|_{\tilde{V}_n} \) and \( \mathcal{G}|_{\mathbb{G}_n} \) define 2-R-valued points
in $\mathcal{M}^1(V, G)$ and $\tau_G|_{\mathcal{R}_n}$ is an isomorphism in $\mathcal{M}^1(V, G)(R)$ that is already defined. Since $R$ has algebraically closed residue field, we can choose trivializations $\alpha_1 : L^p_{\rho}(\sigma^* G) \to L^p G$, and $\alpha_2 : L^p_{\rho}(G) \to L^p G$ (see [AH14, Proposition 2.4]). Then $\alpha_2 \circ \tau_G \circ \alpha_1 : LG_{p,R} \to LG_{p,R}$ is an isomorphism in $\mathcal{M}^1(\kappa_R, LG)(R)$ given by an element $h \in LG_{p}(R)$. We know by assumption that the pull back of $h$ to $K$ is given by an element $h_K \in L^{*}\mathcal{G}_P(K)$, since $\tau_G$ is generically already an isomorphism over $V$. But since $L^{*}\mathcal{G}_P$ is closed in $LG_P$, it follows that $h \in L^{*}\mathcal{G}_P(R)$. This implies that the isomorphism $\tau_G|_{\mathcal{R}_n}$ comes from an isomorphism in $\mathcal{M}^1(V, G)(R)$. So $\tau_G$ extends uniquely to $p$ and the valuative criterion is proved. This proves that $\nabla_n \mathcal{M}^1(C, G) \times_{\nabla_n \mathcal{M}^1(C_1, \pi_G)} S$ is a closed subscheme of $\tilde{S}$ and since $\tilde{S}$ is finite over $S$ it proves as well that $\nabla_n \mathcal{M}^1(C, G) \to \nabla_n \mathcal{M}^1(C, \pi_G)$ is a finite morphism. \hfill \square

The next goal is to prove that for any shtuka datum $(C, G, \underline{z}, \underline{\pi}, H)$ the morphism $\pi : C \to C'$ induces also a finite morphism $\nabla_n \mathcal{M}^1(C, G) \to \nabla_n \mathcal{M}^1(C', G)$. For this we need the following lemma that concerns the boundedness condition. Given a global $G$-shtuka $\underline{\pi}$ in $\nabla_n \mathcal{M}^1(C, G)(S)$ over $S$, we recall that we introduced in § 2.10 the global-local functor $\Gamma_{\nu}$ that associates with it a local $G_{\nu}$-shtuka $\Gamma_{\nu}(\underline{\pi})$ over $S$. On the other hand we explained (compare also [AH14, Remark 5.6]) that base change with $Spf A_v \times_{\underline{z}_2} S \cong \prod_{\nu \in \tilde{\mathcal{R}}_{deg v}} V(\alpha_v, \nu)$ gives a global $G_{\nu}$-shtuka $\Gamma_{\nu}(\underline{\pi})$ over $S$. Here $G_{\nu}$ denotes the Well restriction $Res_{A_{\nu}, G}(\underline{z}_2)$. Now let $\Gamma_{\nu}$ be a bound in $\mathcal{F}_{\nu}(\underline{\pi})$, and $R$ an DVR over $A_v = \mathcal{F}_{\nu}[\nu_v]$ with a representative $\nu_{\nu_v} \in \mathcal{F}_{\nu}[\nu_v]$. We have $\mathcal{F}_{\nu}(\underline{z}_2) \times_{\nu_v} \mathcal{F}_{\nu}(\underline{\pi})$ be the closed Schubert variety and $\mathcal{S}(1) = \mathcal{S}(1) \times_{\nu_v} \mathcal{S}(R)$ then $\nu_{\nu_v} \times \mathcal{S}(1) \times \cdots \times \mathcal{S}(1)$ defines a bound in $\mathcal{F}_{\nu}(\underline{\pi})$ that we also denote by $\nu_{\nu_v} \times \mathcal{S}(1) \times \cdots \times \mathcal{S}(1)$. We have the following lemma.

Lemma 3.13. Let $G \in \nabla_n \mathcal{M}^1(C, G)(S)$ as before. The local $G_{\nu}$-shtuka $\Gamma_{\nu}(\underline{\pi})$ is bounded by $\tilde{\mathcal{Z}}_{\nu}$, if and only if the local $G_{\nu}$-shtuka $\Gamma_{\nu}(\underline{\pi})$ is bounded by $\nu_{\nu_v} \times \mathcal{S}(1) \times \cdots \times \mathcal{S}(1)$.

Proof: We choose an étale covering $S'$ of $S$ that trivializes $L^*_\nu(\underline{G})$ as well as $\sigma^* L^*_\nu(\underline{G})$. In particular $S'$ trivializes also $\Gamma_{\nu}(\underline{\pi})$ and $\sigma^* \Gamma_{\nu}(\underline{\pi})$. We fix such trivializations and call them $\tilde{\alpha} : L^*_\nu(\underline{G})_{S'} \to L^*_\nu(\underline{G})_{S'}$, $\tilde{\alpha}' : \sigma^* L^*_\nu(\underline{G})_{S'} \to L^*_\nu(\underline{G})_{S'}$, $\alpha : \Gamma_{\nu}(\underline{\pi})_{S'} \to \underline{G} \times V(\alpha_v, \nu)$ and $\alpha' : \sigma^* \Gamma_{\nu}(\underline{\pi})_{S'} \to \underline{G} \times V(\alpha_v, \nu)$. Denote by $\tau_{\nu}$ the Frobenius morphism $\tau_{\nu}$ restricted to $V(\alpha_v, \nu)$ for $\nu_v = 1, \ldots, d-1$, where $d = deg v_i = [F_v : F_{\nu}]$. So the local shtuka $\Gamma_{\nu}(\underline{\pi})$ is given by $(\underline{G} \times V(\alpha_v, \nu), \tau_{\nu} \circ \tau_{\nu} \circ \cdots \circ \tau_{\nu}(s_{\nu_v}) \circ \nu)$ and $\alpha \circ \tau_{\nu} \circ \alpha' \circ \alpha^{-1} : LG_{\nu,S'} \to LG_{\nu,S'}$ computed in $\mathcal{M}^1(\mathcal{F}_{\nu}, LG_{\nu})(S')$ defines a morphism $S' \to LG_{\nu}$.

Now $\Gamma_{\nu}(\underline{\pi})$ is bounded by $\tilde{\mathcal{Z}}_{\nu_v}$ if and only if the morphism $S' \times_{\mathcal{R}_{\nu_v}} Spf R \to LG_{\nu} \times_{\mathcal{F}_{\nu}} Spf R \to \mathcal{F}_{\nu}(\underline{\pi})$ factors through $\mathcal{R}_{\nu_v}$. Since $\tau$ is an isomorphism outside the graphs of $\nu_{\nu_v}, \tau_{d-1}, \ldots, \tau_1$ are isomorphisms. Hence $\sigma \tau \circ \cdots \circ \tau_{d-1} \circ \tau_{d-1} \circ \tau_{d-1} \circ \cdots \circ \tau_{d-1} \circ \cdots \circ \tau_{d-1} \circ \tau_{d-1}$ is coming from some other trivialization $\tilde{\beta} : \underline{G} \times V(\alpha_v, \nu) \to \sigma \Gamma_{\nu}(\underline{\pi})$. This shows that $\tilde{\alpha} \circ \tau \circ \tilde{\beta} \circ \cdots \circ \tilde{\alpha} \circ \tau \circ \tilde{\beta}$ is bounded by $\tilde{\mathcal{Z}}_{\nu_v}$ if and only if $\nu_{\nu_v} \times \mathcal{S}(1) \times \cdots \times \mathcal{S}(1)$ is bounded by $\nu_{\nu_v} \times \mathcal{S}(1) \times \cdots \times \mathcal{S}(1)$. Note that the morphism in the $j$-th component of $\prod_{\nu \in \tilde{\mathcal{R}}_{deg v}} \mathcal{F}_{\nu}(\underline{z}_2) \times_{\nu_v} \mathcal{S}(R)$ is exactly defined by $\tilde{\alpha} \circ \tau \circ \tilde{\beta} \circ \cdots \circ \tilde{\alpha} \circ \tau \circ \tilde{\beta}$. Since $\tau_{d-1}, \ldots, \tau_1$ are isomorphisms the morphism in the $j$-th component with $j \geq 1$ always factors through $\mathcal{S}(1)$. This implies that $\tau$ is bounded by $\nu_{\nu_v} \times \mathcal{S}(1) \times \cdots \times \mathcal{S}(1)$ if and only if $\nu_{\nu_v}$ is bounded by $\nu_{\nu_v}$.

Now we can prove:

Theorem 3.14. Let $(C, G, \underline{z}, \underline{\pi}, H)$ be a shtuka datum and $\pi : C \to C'$ a finite morphism of smooth, projective, geometrically irreducible curves over $\mathcal{F}_q$ with $\nu_v = \nu(v_i)$ and $w = (w_1, \ldots, w_n)$. Then the morphism $\Gamma_{\nu}(\underline{\pi}, \nu, w) : (C, G, \underline{z}, \underline{\pi}, H) \to (C', \pi, \nu, w, \pi, \underline{z}, \pi, H)$ of shtuka data (see definition § 2.9 and remark § 2.10) induces a finite morphism of the moduli stacks

$$\pi_* : \nabla_n \mathcal{M}^1(C, G) \to \nabla_n \mathcal{M}^1(C', \pi, G).$$

Proof: Let $S$ be an $\mathcal{F}_q$-scheme and $(G, s_1, \ldots, s_n, \tau_G, \gamma) \in \nabla_n \mathcal{M}^1(C, G)(S)$. We describe again its image $(G', s'_1, \ldots, s'_n, \tau_{G'}, \gamma') \in \nabla_n \mathcal{M}^1(C', \pi, G)(S)$ to define the morphism. The $\pi_G$-toral $G' = (G', s'_1, \ldots, s'_n, \tau_{G'})$ is already defined by the morphism in proposition § 2.12 but we have to prove that
it lies indeed in $\mathcal{V}^2_\mathcal{A}_R^1(C, \pi, \mathbb{G})(S)$. We will do this first and then define the $\pi, H$-level structure $\gamma'$. Since the section $s_i : S \to C$ is mapped to $s'_i : \pi \to \mathbb{G}$ and $s_i$ is required to factor through $Spf A_{\mathcal{V}_i}$, we easily see with $\pi(v_i) = w_i$ that $s'_i$ factors through $Spf A_{\mathcal{V}_i}$. Furthermore this shows that $C'_{\mathcal{V}_i} \cup \Gamma_{s'_i} \supset C'' \times_{\mathcal{V}_i} S$ and $C_{\mathcal{V}_i} \cup \Gamma_{s_i} \supset C'' \times_{\mathcal{V}_i} S$ where we use the notation $C'' = C'_{\mathcal{V}_1, \ldots, \mathcal{V}_n}$ and $C'' = C'_{\mathcal{V}_1, \ldots, \mathcal{V}_n}$. It remains to show that $\tau_{\mathcal{V}_i}$ is bounded by $Z_{\mathcal{V}_i}$ to see with $=Z$.

Now by assumption $\tau_{\mathcal{V}_i}$ is bounded by $Z_{\mathcal{V}_i}$, which means by definition that the local shukas $\Gamma_{v_i}(G)$ are bounded by $Z_{\mathcal{V}_i}$ for all $i$. By lemma 3.13 this is equivalent to the fact that $L_{\mathcal{V}_i}(G)$ is bounded by $Z_{\mathcal{V}_i} \times S(1) \times \cdots \times S(1)$. Now consider the following 2-cartesian diagram (compare [2.9] and [AH14, Lemma 5.1]) where we set $U := C'' \times_{C'} C$.

$$
\begin{array}{c}
\mathcal{H}^1(C, \mathbb{G}) & \longrightarrow & \mathcal{H}^1_\pi(U, \mathbb{G}) \\
\Pi_v L^\pi_w & \downarrow & \Pi_v L^\pi_w \\
\Pi_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L^\pi_{\mathcal{V}_w}) & \longrightarrow & \Pi_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L_{\mathcal{V}_w}) .
\end{array}
$$

Here $\mathcal{H}^1(U, \mathbb{G})(S)$ is the full subcategory of $\mathcal{H}^1(U, \mathbb{G})(S)$ consisting of those $\mathbb{G}$-torsors over $U_S$ that can be extended to a $\mathbb{G}$-torsor over $C_S$. Now the categories $\mathcal{H}^1(C, \mathbb{G})$ and $\mathcal{H}^1_\pi(U, \mathbb{G})$ are by lemma 3.3 equivalent to $\mathcal{H}^1(C', \pi, \mathbb{G})$ and $\mathcal{H}^1_\pi(C'', \pi, \mathbb{G})$. Furthermore the categories

$$
\prod_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L^\pi_{\mathcal{V}_w}) = \prod_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L^\pi_{\mathcal{V}_w}) \quad \text{and}
$$

$$
\prod_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L_{\mathcal{V}_w}) = \prod_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L_{\mathcal{V}_w})
$$

are by corollary 3.4 equivalent to $\prod_{v \in \mathcal{V}_w} \mathcal{H}^1(F_q, L^\pi_{\mathcal{V}_w})$ and $\prod_{v \in \mathcal{V}_w} \mathcal{H}^1(F_q, L_{\mathcal{V}_w})$. Therefore the whole diagram (5) is equivalent to the diagram

$$
\begin{array}{c}
\mathcal{H}^1(C', \pi, \mathbb{G}) & \longrightarrow & \mathcal{H}^1_\pi(C'', \pi, \mathbb{G}) \\
\Pi_v L^\pi_w & \downarrow & \Pi_v L^\pi_w \\
\Pi_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L^\pi_{\mathcal{V}_w}) & \longrightarrow & \Pi_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L_{\mathcal{V}_w}) .
\end{array}
$$

Now we choose some covering $S'$ over $S$ that trivializes $L^\pi_{\mathcal{V}_w}$ for all $v \in \pi^{-1}(w)$ and fix some trivializations $\alpha_v : L^\pi_{\mathcal{V}_w}(G)_{S'} \to L^\pi_{\mathcal{V}_w}(G)_{S'}$, $\alpha'_v : \pi^* L^\pi_{\mathcal{V}_w}(G)_{S'} \to \pi^* L^\pi_{\mathcal{V}_w}(G)_{S'}$. Then $(G, \tau)$ defines a tuple $\Pi_{v \in \pi^{-1}(w)} (\pi_{v|\mathcal{V}_w} L^\pi_{\mathcal{V}_w}, \pi_{v|\mathcal{V}_w} \alpha_v \circ L_v(\tau_{v|\mathcal{V}_w}) \circ \alpha'_v)$. Using the diagrams shows that $\alpha_v$ and $\alpha'_v$ are the trivializations corresponding to $\Pi_{v \in \mathcal{V}_w} \alpha_v$ and $\Pi_{v \in \mathcal{V}_w} \alpha'$. Now choose some finite extension $\mathbb{R} \supset \mathbb{F}_q[\zeta]$ such that there are representatives $Z_{v,R}$ for all $v \in w$. Using the 2-cartesian diagram

$$
\begin{array}{c}
\prod_{v \in \pi^{-1}(w)} \mathcal{F}_{\mathcal{V}_w} & \longrightarrow & \mathcal{F}_q \\
\Pi_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L^\pi_{\mathcal{V}_w}) & \longrightarrow & \Pi_{v \in \pi^{-1}(w)} \mathcal{H}^1(F_q, L_{\mathcal{V}_w})
\end{array}
$$

the tuple $(\Pi_{v|\mathcal{V}_w} L^\pi_{\mathcal{V}_w}, \Pi_{v|\mathcal{V}_w} \alpha_v \circ L_v(\tau_{v|\mathcal{V}_w}) \circ \alpha'_v)$ defines an $\mathcal{S}' \times_{\mathcal{S}} Spf R \rightarrow \mathcal{S}' \times_{\mathcal{S}} Spf R \rightarrow \mathcal{S}' \times_{\mathcal{S}} Spf R$-valued point in $\Pi_{v|\mathcal{V}_w} \mathcal{F}_{\mathcal{V}_w} \times_{\mathcal{S}} Spf R$. By lemma 3.13 the boundedness of $\mathcal{G}$ at all the points $v \in (\pi \cap \pi^{-1}(w))$ by $Z_v$ is equivalent to the boundedness of $L^\pi_{\mathcal{V}_w}(G)$ by $\Pi_{v|\mathcal{V}_w} Z_{v,1}$ with $Z_{v,0} = Z_v$ for all $v \in w$ and $Z_{v,1} = S(1)$ for all $v \notin w$ and $i \neq 0$, which means by definition, that the above $\mathcal{S}' \times_{\mathcal{S}} Spf R$ valued point factors through $\Pi_{v|\mathcal{V}_w} \Pi_{v|\mathcal{V}_w} Z_{v,1}$. The tuple $\Pi_{v|\mathcal{V}_w} L^\pi_{\mathcal{V}_w}$, $\alpha_v \circ L_v(\tau_{v|\mathcal{V}_w}) \circ \alpha'_v$ defines in the same way a morphism $\mathcal{S}' \times_{\mathcal{S}} Spf R \rightarrow \mathcal{S}' \times_{\mathcal{S}} Spf R = \Pi_{v|\mathcal{V}_w} \mathcal{F}_{\mathcal{V}_w} \times_{\mathcal{S}} Spf R$. Composing with the isomorphism $\Pi_{v|\mathcal{V}_w} \mathcal{F}_{\mathcal{V}_w} \cong \mathcal{F}_{\mathcal{V}_w}$, the above morphism factors also through $\Pi_{v|\mathcal{V}_w} \Pi_{v|\mathcal{V}_w} Z_{v,1}$. With lemma 3.13 and the definition of $\pi, Z_v$ on page 14 it follows, that $\pi, Z_v$ is bounded by $\pi, Z_v$. \hfill 18
Next we have to define the $\pi, H$-level structure $\gamma'$. We fix a geometric base point $\mathfrak{s} \in S$ and we choose for all closed points $v \in C_{\mathfrak{s}}$ a trivialization $L_w^*(\mathfrak{G}_w) \simeq (L^*\mathcal{O}_{v_w}, \tau = 1)$, which exists by [AH14, Corollary 2.9]. This provides also trivializations

$$L_w^*(\pi, \mathfrak{G}_w) = \prod_{v \mid w} L_w^* \mathfrak{G}_w \cong \prod_{v \mid w} L_w^* \mathcal{O}_{v_w} = L_w^* \pi, \mathfrak{G}_w(\mathfrak{s}) .$$

We denote by $L_w^*$ and $L_w^*$ the shtukas $L_w^*(\mathfrak{G}_w)$ and $L_w^*(\pi, \mathfrak{G}_w)$. Now these trivializations induce isomorphisms

$$\beta : \omega_{\mathfrak{G}_{\mathfrak{s}}}^o = \prod_{v \in \mathfrak{C}_{\mathfrak{s}}} \mathcal{T}_L^* \mathfrak{G}_w \cong \prod_{v \in \mathfrak{C}_{\mathfrak{s}}} \mathcal{T}_L^* \mathfrak{G}_w = \mathcal{T}^*_\mathfrak{G}$$

$$\pi \circ \beta : \omega_{\mathfrak{G}_{\mathfrak{s}}}^o = \prod_{v \in \mathfrak{C}_{\mathfrak{s}}} \mathcal{T}_L^* \pi^* \mathfrak{G}_w \cong \prod_{v \in \mathfrak{C}_{\mathfrak{s}}} \mathcal{T}_L^* \pi^* \mathfrak{G}_w = \mathcal{T}^*_\pi \mathfrak{G} .$$

We write $\omega_{\mathfrak{G}_{\mathfrak{s}}}^o := \omega_{\mathfrak{G}_{\mathfrak{s}}} \otimes_{\mathfrak{C}_{\mathfrak{s}}} \mathcal{A} \mathcal{M}$ and $\omega_{\mathfrak{G}_{\mathfrak{s}}}^o := w_{\mathfrak{G}_{\mathfrak{s}}} \otimes_{\mathfrak{C}_{\mathfrak{s}}} \mathcal{A} \mathcal{M}$. Now $\beta \circ \gamma \in \text{Aut}^o(\omega_{\mathfrak{G}_{\mathfrak{s}}}^o)$ is given by an element $g \in G(\mathcal{A}_\mathfrak{M})$ and the $H$-orbit of $g$ by $\beta \circ g H$. Now we can use the projection $G(\mathcal{A}_\mathfrak{M}) \to G(\mathfrak{A}^\infty(\omega)) = \pi_! \mathfrak{G}(\mathcal{A}_\mathfrak{M})$ to define $\pi, g \in \pi_!, \mathfrak{G}(\mathcal{A}_\mathfrak{M})$ as the image of $g$. This corresponds to an element in $\text{Aut}(\omega_{\mathfrak{G}_{\mathfrak{s}}}^o)$. Therefore $\pi \circ \beta \circ \pi_!$ defines an element $\gamma'$ and consequently an $\pi, H$-orbit in $\text{Isom}^o(\omega_{\mathfrak{G}_{\mathfrak{s}}}^o, \mathfrak{Y}_{\pi, \mathfrak{G}})$. This orbit is independent of the representative $\gamma$ since $\pi, H$ was defined as the image of $H$ under the above projection. Let $\rho \in \pi_1(S, \mathfrak{s})$ since $\gamma \in \text{Isom}^o(\omega_{\mathfrak{G}_{\mathfrak{s}}}^o, \mathfrak{Y}_{\pi, \mathfrak{G}})$ is $\pi_1(S, \mathfrak{s})$ invariant, we know that there is $h \in H$ such that $\rho_! = \gamma_H$. This defines a group homomorphism $\varphi : \pi_1(S, \mathfrak{s}) \to H$ and we set $\mathfrak{Y} : \pi_1(S, \mathfrak{s}) \to H \to \pi_1(S, \mathfrak{s})$. Let $\rho \in \pi_1(S, \mathfrak{s})$ and $\gamma' \in \text{Isom}^o(\omega_{\mathfrak{G}_{\mathfrak{s}}}^o, \mathfrak{Y}_{\pi, \mathfrak{G}})$ be as above, then $\rho$ operates by $\rho_! = \gamma_\mathfrak{s}(\rho)$ and in particular $\pi_1(S, \mathfrak{s}) = \gamma_\mathfrak{s}$ since $\gamma_\mathfrak{s}$ is an $\pi, H$-equivariant structure and defines a level structure $\gamma'$ of $\mathfrak{G}$.

After constructing this morphism, we now prove that it is representable by a scheme and finite. By property [S.12] it is clear that the morphism $\nabla^\infty_n \mathcal{H}^1(C, \mathfrak{G}) \to \nabla^\infty_n \mathcal{H}^1(C', \pi, \mathfrak{G})$ is finite. Now we find some finite subscheme $D \subset C$ such that $H_D := \ker(G(\mathfrak{C}_D) \to G(\mathfrak{O}_D))$ is a subgroup of finite index in $H$. Then we have by [S.2.16] the following diagram:

$$\nabla^\infty_n \mathcal{H}^1(C, \mathfrak{G}) \overset{\text{finite étale}}{\longrightarrow} \nabla^\infty_n \mathcal{H}^1(C, \mathfrak{G}) \overset{\sim}{\longrightarrow} \nabla^\infty_n \mathcal{H}^1(C', \pi, \mathfrak{G}) \overset{\text{finite}}{\longrightarrow} \nabla^\infty_n \mathcal{H}^1(C', \pi, \mathfrak{G})$$

where the horizontal arrows are finite (and even étale) by [AH13, section 6]. This implies firstly that the morphism $\nabla^\infty_n \mathcal{H}^1(C, \mathfrak{G}) \to \nabla^\infty_n \mathcal{H}^1(C', \pi, \mathfrak{G})$ is finite and consequently that the morphism $\nabla^\infty_n \mathcal{H}^1(C, \mathfrak{G}) \to \nabla^\infty_n \mathcal{H}^1(C', \pi, \mathfrak{G})$ is finite. □

3.3 Changing the Group $\mathbb{G}$

Now let $f : \mathbb{G} \to \mathbb{G}'$ be a morphism of smooth, affine group schemes over $C$. In this subsection we explain how this induces a morphism between the moduli stacks of $\mathbb{G}$-shtukas and $\mathbb{G}'$-shtukas. Further we prove some of its properties, depending on $f$. First of all we recall, that given a sheaf $M$ on $C_{\text{ét}}$, with an action of $\mathbb{G}$, we can define the sheaf $M \times^\mathbb{G} \mathbb{G}'$ whose $R$-valued points are given by the set $\{(a, b) \mid a \in M(R), b \in \mathbb{G}'(R)\}$, where $(a, b) \sim (c, d)$ if and only if $(a, b) = (cg, f(g^{-1})d)$ for some $g \in \mathbb{G}(R)$. Actually this construction works for any sheaf of groups on any site. Now this construction is functorial for $\mathbb{G}$-equivariant morphisms $\varphi : M_1 \to M_2$ and commutes obviously with base change. We also write $f, M = M \times^\mathbb{G} \mathbb{G}'$ and note that if $M$ is a $\mathbb{G}$-torsor then $f, M$ is a $\mathbb{G}'$-torsor. With these facts we see that for a given $\mathbb{G}$-shtuka $(\mathbb{G}, s_1, \ldots, s_n, \tau_\mathbb{G})$ over $S$, the tuple $(f, \mathbb{G}, s_1, \ldots, s_n, f, \tau_\mathbb{G})$ defines a $\mathbb{G}'$-shtuka over $S$. Therefore we get a morphism

$$\nabla_n \mathcal{H}^1(C, \mathfrak{G}) \to \nabla_n \mathcal{H}^1(C, \mathfrak{G}' \to \nabla_n \mathcal{H}^1(C', \pi, \mathfrak{G}) \to \mathfrak{G}, s_1, \ldots, s_n, f, \tau_\mathbb{G}) \to (f, \mathbb{G}, s_1, \ldots, s_n, f, \tau_\mathbb{G}) .$$

Now we want to show that this morphism also induces a morphism of these moduli stacks with additional $H$-level structure. So we fix $n$ closed points $\mathfrak{s} = (v_1, \ldots, v_n)$ in $C$ and let $H \subset G(\mathcal{A}_\mathfrak{M})$ be an open and compact subgroup. Let further $S$ be a connected $\mathbb{F}_q$-scheme with a geometric base point $\mathfrak{s} \in S$ and
\((\mathcal{G}, \gamma) = (G, s_1, \ldots, s_n, \tau_G, \gamma)\) be a \(G\)-shtuka over \(S\) with an \(H\)-level structure \(\gamma H\). We already mentioned that by [Con12, Proposition 2.1] \(f : \mathcal{G} \to \mathcal{G}'\) induces a continuous homomorphism \(f_{\mathcal{L}} : \mathcal{G}(\mathcal{L}_H) \to \mathcal{G}'(\mathcal{L}_H)\) (see also above definition 3.9). For an open, compact subgroup \(H' \subset G'(\mathcal{L}_H)\) satisfying \(f_{\mathcal{L}}(H') \subset H'\) we now construct an \(H'\)-level structure on the shtuka \(f^{-1}\mathcal{G} = (f, \mathcal{G}, s_1, \ldots, s_n, f^{\tau_G})\).

We choose for every \(v \in \mathcal{C} = C_v \mathcal{G}\) a trivialization \(\alpha_v : L_v^* \mathcal{G}(\mathcal{L}_v) \to (L_v^* \mathcal{G}(\mathcal{L}_v) \cdot \sigma^*)\) which exists by [AH14, Proposition 2.9]. Since \(f_v\) commutes with base change this induces trivializations \(f_{\alpha_v} : L_v^* ((f_v \mathcal{G})(\mathcal{L}_v)) \to (L_v^* \mathcal{G}(\mathcal{L}_v) \cdot \sigma^*)\). We denote by \(\omega_{\mathcal{G}} : \text{Rep}_{\text{R}} \mathcal{G} \to \text{Mod}_{\text{C}(\mathcal{G})} (S, s)\) and \(\omega_{\mathcal{G}'} : \text{Rep}_{\text{R}} \mathcal{G}' \to \text{Mod}_{\text{C}(\mathcal{G}')} (S, s)_1\) the forgetful functors and by \(\mathcal{L}_v\) and \(\mathcal{L}'_v\) the local shtukas \(L_v^* \mathcal{G}(\mathcal{L}_v)\) and \(L_v^* ((f_v \mathcal{G})(\mathcal{L}_v))\). Then the previous trivializations provide isomorphisms of tensor functors

\[
\beta : \omega_{\mathcal{G}} = \prod_{v \in \mathcal{C}_v} \mathcal{T}_{L_v^* \mathcal{G}'} \to \prod_{v \in \mathcal{C}_v} \mathcal{T}_{L_v^* \mathcal{G}} = \mathcal{T}_{\mathcal{G}}
\]

\[
f_{\alpha_v} : \beta = \prod_{v \in \mathcal{C}_v} \mathcal{T}_{L_v^* ((f_v \mathcal{G})(\mathcal{L}_v))} \to \prod_{v \in \mathcal{C}_v} \mathcal{T}_{L_v^* \mathcal{G}} = \mathcal{T}_{f_\mathcal{G}}.
\]

It follows that \(\beta^{-1} \circ \gamma \in \text{Aut}_{\text{R}}(\omega_{\mathcal{G}})\) is given by an element \(g \in \mathcal{G}(\mathcal{L}_H)\) and the \(H\)-orbit of \(\gamma\) is given by \(\beta \circ g H\). Now we view the image \(f(g)\) of \(g\) under the map \(f_{\mathcal{L}} : \mathcal{G}(\mathcal{L}_H) \to \mathcal{G}'(\mathcal{L}_H)\) as an automorphism in \(\text{Aut}_{\text{R}}(\omega_{\mathcal{G}})\) and define \(\gamma' := f \circ \gamma \circ f(g)\). Since \(f_{\mathcal{L}}(H) \subset H'\) the \(H'\)-orbit of \(f(g)\) is independent of the chosen representative \(\gamma\) in the orbit \(\gamma H\). Since \(\pi_1(S, s)\) leaves \(\gamma H\) invariant there is for all \(\rho \in \pi_1(S, s)\) an \(h \in H\) such that \(\rho \cdot \gamma = \gamma \cdot h\). This defines a group automorphism \(\varphi : \pi_1(S, s) \to H\) and we set \(\varphi' : \pi_1(S, s) \to H'\). Now \(\rho \circ \pi_1(S, s) \circ \varphi' \circ \rho^{-1}\) operates on \(\gamma' \in \text{Isom}(\omega_\mathcal{G}(\mathcal{L}_H))\) by \(\rho \cdot \gamma' = \gamma' \circ \varphi'(\rho)\). In particular \(\pi_1(S, s) \circ \gamma' \circ \gamma' H'\) so that \(\gamma' H'\) is \(\pi_1(S, s)\) invariant and defines a \(H'\)-level structure on \(f_\mathcal{G}\). A morphism \((\mathcal{G}, \tau)\) to \((\mathcal{F}, \tau)\) induces naturally a morphism \((f, \mathcal{G}, \tau) \to (f, \mathcal{F}, \tau)\) so that we get a morphism of moduli stacks

\[
\nabla^H \mathcal{H}^1(C, \mathcal{G}) \to \nabla^H \mathcal{H}^1(C, \mathcal{G}'), \quad (\mathcal{G}, \tau) \to (f, \mathcal{F}, \tau).
\]

Next we show that this morphism behaves well with respect to boundedness conditions. We note that for all \(v \in \mathcal{C}\) the morphism \(f : \mathcal{G} \to \mathcal{G}'\) induces a morphism \(L_v^* \mathcal{G} \to L_v^* \mathcal{G}'\) as well as a morphism \(L_{G_v} \to L_{G'_v}\) and consequently also a morphism \(\mathcal{F}_{L_{G_v}} \to \mathcal{F}_{L_{G'_v}}\).

**Lemma 3.15.** Let \(Z_v\) be a bound in \(\prod_{v \in \mathcal{C}} \mathcal{F}_{L_{G_v}}\) and \(\mathcal{G}\) a \(G\)-shtuka over \(S\) bounded by \(Z_v\). Let further \(\mathcal{Z}_v\) be a bound in \(\prod_{v \in \mathcal{C}} \mathcal{F}_{L_{G'_v}}\) such that after choosing representatives over some DVR \(R\) the morphism \(Z_{v, R} \to \prod_{v \in \mathcal{C}} \mathcal{F}_{L_{G'_v}}\) factors through \(\mathcal{Z}_{v, R}\). Then \(f^\mathcal{G}\) is bounded by \(\mathcal{Z}_v\).

**Proof.** We have to prove that for \(v \in \mathcal{C}\) the local shtuka \(\Gamma_v(f^\mathcal{G})\) is bounded by \(Z_v\). We choose some covering \(S' \to S\) with \(S'/\text{Spec} R\) that trivializes \(L_v^* \sigma^* \mathcal{G}\) and \(L_v^* \mathcal{G}\) at the same time and fix such trivializations, which we denote by \(\alpha : L_v^* \sigma^* \mathcal{G} \to L_v^* \mathcal{G}_{v, S'}\) and \(\alpha' : L_v^* \mathcal{G} \to L_v^* \mathcal{G}_{v, S'}\). Then \(f, \alpha\) and \(f, \alpha'\) are trivializations of \(L_v^* ((f, \mathcal{G})_{v, S'})\) and \(L_v^* ((f, \mathcal{G}')_{v, S'})\). Now we have the automorphism \(\alpha' \circ \tau_{\mathcal{G}} \circ \alpha^{-1} : L_{G_v, S'} \to L_{G'_v, S'}\) and we let \(\mathcal{G}' \to L_{G'_v, S'}\) be the unit morphism. The composition defines an \(S'\)-valued point \(1_{S'} \circ \alpha' \circ \tau_{\mathcal{G}} \circ \alpha^{-1}\) in \(L_{G_v, S'}\). The composition of this point with the morphism \(L_{G_v, S'} \to L_{G'_v, S'}\) induced by \(f\) defines an \(S'\)-valued point in \(L_{G'_v, S'}\). Since the diagram

\[
\begin{array}{ccc}
S' & \xrightarrow{1_{S'}} & L_{G_v, S'} \\
\downarrow & & \downarrow \\
L_{G'_v, S'} & \xrightarrow{f \circ (\alpha' \circ \tau_{\mathcal{G}} \circ \alpha^{-1})} & L_{G'_v, S'}
\end{array}
\]

commutes, this is exactly the \(S'\)-valued point defined by

\[
f_{\alpha'} f_{\tau_{\mathcal{G}}} f_{\alpha} = f_{\alpha' \circ \tau_{\mathcal{G}} \circ \alpha^{-1}}.
\]

By assumption \(\mathcal{G}\) is bounded by \(Z_v\) and consequently the morphism \(1_{S'} \circ \alpha' \circ \tau_{\mathcal{G}} \circ \alpha^{-1}\) factors after projection to \(\mathcal{F}_{L_{G_v}, S'}\) through \(Z_v\) and maps then into \(Z_v\). This means exactly that \(\Gamma_v(f^\mathcal{G})\) is bounded by \(Z_v\), so that \(f^\mathcal{G}\) is bounded by \(Z_v\).

\[\square\]

This lemma and the previous explanations show.
Corollary 3.16. The morphism \((id, f) : (C, G, w, Z, H) \to (C, G', w, Z', H')\) of shtuka data induces a morphism

\[ f_* : \nabla^Z_n H^1(C, G) \to \nabla^Z_n H^1(C, G') \]

\((G, s_1, \ldots, s_n, \gamma) \mapsto (f_* G, s_1, \ldots, s_n, f_* \gamma).\)

Proof: Follows directly from lemma 3.15 and the morphism \(f\) on page 20. \(\square\)

Now we are interested in some special classes of morphisms \(f : G \to G'\).

Generic Isomorphisms of \(G\)

First of all we want to consider morphisms \(f : G \to G'\) which are generically an isomorphism, that means \(f \times id_G : G \to G'\). In this case \(f : G \to G'\) is already an isomorphism over some open subscheme in \(C\). So we fix such an \(f : G \to G'\) and denote by \(U\) the maximal open subscheme in \(C\) such that \(f \times id_G : G_U \to G'_U\) is an isomorphism and denote by \(w = (w_1, \ldots, w_m)\) the finite set of closed points in the complement \(C \setminus U\).

Before we come to the moduli stacks of the global \(G\)-shtukas, we prove a proposition that describes the morphism \(\mathcal{H}^1(C, G) \to \mathcal{H}^1(C, G')\). For this proposition we need the following lemma.

Lemma 3.17. Let \(L'_i\) be an \(L^* G_w\)-torsor over an \(F_q\)-scheme \(S\). Then the quotient stack \([L'_i/L^* G_w]\) is represented by a scheme \(L'_i/L^* G_w\) over \(S\) that is étale locally on \(S\) isomorphic to \(L^* G_w/L^* G_w\). In the case that \(G_w\) is parahoric \(L'_i/L^* G_w\) is projective.

Proof: Let \(L' := L'_i \times L^* G_w L^* G_w\) be the associated \(L^* G_w\)-torsor of \(L'_i\). By \([AH14, Theorem 4.4]\) the quotient stack \([L'/L^* G_w]\) is represented by an ind-quasi-projective ind-scheme \(L'/L^* G_w\) over \(S\). The closed morphism \(L'_i \to L'\) realizes \([L'_i/L^* G_w]\) as a closed sub-sheaf of \(L'/L^* G_w\). Since \(L'_i\) is affine over \(S\) the quotient \([L'_i/L^* G_w]\) is given by a closed subscheme in \(L'/L^* G_w\). It is clear that after passing to a covering \(S' \to S\) that trivializes \(L'_i\), the scheme \(L'_i/L^* G_w\) becomes isomorphic to \(L^* G_w/L^* G_w \times_{S'} S'\). Since \(L'/L^* G_w\) is by \([AH14, Theorem 4.4]\) ind-projective if \(G_w\) is parahoric, we see that the last statement about the projectivity of \(L'_i/L^* G_w\) follows. \(\square\)

Now we can prove:

Proposition 3.18. Let \(f : G \to G'\) be a morphism of smooth, affine group schemes over \(C\), which is an isomorphism over \(C w\). Then the morphism

\[ f_* : \mathcal{H}^1(C, G) \to \mathcal{H}^1(C, G'), \ G \to f_* G \]

is schematic and quasi-projective. Étale locally it is relatively representable by the morphism

\[(L^* G_w/L^* G_w) \times_{F_q} \cdots \times_{F_q} (L^* G_w/L^* G_w) \to F_q.\]

That means that for any \(F_q\)-morphism \(S \to \mathcal{H}^1(C, G)\) there is an étale covering \(S' \to S\) such that the fiber product \(S' \times_{\mathcal{H}^1(C, G)} \mathcal{H}^1(C, G)\) is given by a tuple \((g, G, \alpha)\) where \(G \in \mathcal{H}^1(C, G)\) and \(\alpha : f_* G \to g' G'\).

Using the theorem of Beauville-Laszlo from \([2, 29]\) we write \(G = (G|_{U'}, \Pi_{w \in w} L_w, \varphi)\) with \(U_T := (C|_{w}) \times_{F_q} T, L_w \in \mathcal{H}^1(F_q, L^* G_w)(T)\) and \(g = (g|_{w \in w} \Pi_{w \in w} L_w(G)) \to \Pi_{w \in w} L_w(L_w)\). In the same way we write \(G' = (G'|_{U'}, \Pi_{w \in w} L'_w, \psi)\). In particular \(f_* G\) is given by \((f_* G|_{U'}, f_* \Pi_{w \in w} L'_w, f_* \varphi)\) and the isomorphism \(\alpha\) is determined by \(\alpha_U : f_* G \to g' G'\) and \(\alpha_w : f_* L_w \to L_w(g' G')\).
Since \( f|_U = id \) we have \( f_*(G|_U) = G|_U \) and the point \((G|_U, \alpha_U)\) is equivalent to the point \((g^*G|_U, \alpha_U \circ L_w(\alpha_U))\) by the isomorphism \((\alpha_U^{-1}, \Pi id_{L_w})\). This shows that the category of tuples \((\nu, \alpha)\) as above is equivalent to the category of tuples \((L_w, \alpha)\) over \(w \in w_{\nu} \) where \(L_w \cong \mathcal{H}^1(C, L^*G_w)(T)\) and \(\alpha_w : f_!L_w \to g^*L'_w\). Namely we associate with some arbitrary tuple \((L_w, \alpha)\) over \(\nu\) the tuple \((g^*G|_U, \alpha_g \circ L_w(\alpha_g))\) where \(\alpha_{\nu} = id\) and \(\alpha_w = \alpha_w\) and \(\varphi\) is uniquely determined by the condition \(f_!\varphi = \psi \circ L(\alpha_w^{-1})\). This is unique because \(f \times id_{\mathcal{H}_w} : \mathcal{G}_w \to \mathcal{G}_w\) is an isomorphism.

Now we note that the isomorphisms \(\alpha_w : f_!L_w \to g^*L'_w\) are in bijection with the \(L^*G_w\) equivariant morphisms \(L_w \to g^*L'_w\). This shows that the tuples \((L_w, \alpha)\) parametrize exactly the \(T\)-valued points of the quotient stack \([g^*L'_w/L^*G_w]\) over \(\mathbb{F}_q\). It follows with the lemma [3.17] that the fiber product \(S \times_{\mathcal{H}^1(C, G)} G\) is given by the scheme \(g^*L'_w/L^*G_w \times \cdots \times g^*L'_w/n L^*G_w\). In particular the morphism \(f_!\) is representable and the remaining statements follow directly from the previous lemma. □

Now let us turn to the moduli stacks of global \(G\)-shtukas. Let us firstly assume that \(w \subset \nu\) and that all the closed points \(w\) are \(\mathbb{F}_q\)-rational. In particular the group homomorphism \(f : \mathcal{G} \to \mathcal{G}'\) is an isomorphism outside the fixed characteristic places \(v_1, \ldots, v_n\). Then we have the following theorem.

**Proposition 3.19.** Let \(f : \mathcal{G} \to \mathcal{G}'\) be a morphism of smooth, affine group schemes over \(C\), which is an isomorphism over \(C\) with \(w \subset \nu\) and \(v_1 \in \mathbb{C}(\nu)\) for all \(w \in \mathbb{F}_q\). Let \(H \subset \mathbb{G}(A)=G'(A)\) be an open, compact subgroup, let \(Z_v\) be a bound in \(\mathcal{F}_{\mathcal{G}_v}\) for all \(v\) and let \(Z_v\) be the base change of \(Z_v\) under the map \(\mathcal{F}_{\mathcal{G}_v} \to \mathcal{F}_{\mathcal{G}_v}\). Then the morphism

\[
f_* : \tilde{\mathcal{Z}}_n = \mathcal{H}^1(C, G) \to \tilde{\mathcal{Z}}_n = \mathcal{H}^1(C, G), \quad (G, \gamma H) \mapsto (f_* G, \gamma H)
\]

is schematic and quasi-projective. Étale locally it is relatively representable by the morphism

\[
(L^*G_w/L^*G_w) \times_{\mathcal{F}_q} \cdots \times_{\mathcal{F}_q} (L^*G_w/L^*G_w) \to \mathcal{F}_q.
\]

That means that for any \(\mathbb{F}_q\)-scheme \(S\) there is an étale covering \(S' \to S\) such that the fiber product \(S' \times_{\mathcal{H}^1(C, G)} \mathcal{H}^1(C', \mathcal{G}')\) is given by \(S' \times_{\mathcal{F}_q} (\prod_{w \in \mathbb{F}_q} L^*G_w/L^*G_w)\), where the product is taken over \(S\). This shows that the category of \(\mathcal{G}\)-shtukas is moduli deformation.

**Proof:** Since \(f\) is an isomorphism outside \(w\), for two open subgroups \(\tilde{H} \subset H \subset \mathbb{G}(A)=G'(A)\) the diagram

\[
\begin{array}{ccc}
\tilde{Z}_n^{\tilde{H}} \mathcal{H}^1(C, G) & \to & \tilde{Z}_n^{\tilde{H}} \mathcal{H}^1(C, G) \\
\downarrow f_* & & \downarrow f_* \\
\tilde{Z}_n^{Z_v} \mathcal{H}^1(C, G') & \to & \tilde{Z}_n^{Z_v} \mathcal{H}^1(C, G')
\end{array}
\]

is cartesian. In particular we can assume \(H \subset \mathbb{G}(O)=G'(O)\), because otherwise we can prove the theorem for the compact open subgroup \(\tilde{H} := H \cap \mathbb{G}(O)\). This implies the assertions of the theorem for the vertical arrows on the left and the right in the previous diagram are relatively represented by the same morphism. Now for each \(S\)-valued point \((G, \gamma H)\) in \(\tilde{Z}_n^{\tilde{H}} \mathcal{H}^1(C, G)\) we find an isomorphic point \((G', \gamma' H)\) with \(\gamma' \in Isom^0(\omega_{\mathbb{G}_w}, \mathcal{F}_{\mathcal{G}_w})\). This is due to the fact, that we can pull back global \(G\)-shtukas along quasi-isogenies of local \(\mathbb{G}_w\)-shtukas [AH13 Theorem 5.2] and is explained in the proof of [AH13 Theorem 6.4]. We get a morphism \(\tilde{Z}_n^{\tilde{H}} \mathcal{H}^1(C, G) \to \tilde{Z}_n^{Z_v} \mathcal{H}^1(C, G)\) sending \((G, \gamma H) \to (G', \gamma' H)\) to \(G'\). This is the morphism \(\tilde{Z}_n^{\tilde{H}} \mathcal{H}^1(C, G) \to \tilde{Z}_n^{Z_v} \mathcal{H}^1(C, G)\) from [1] in \(\mathcal{S}^{1.20}\) composed with the morphism \(\mathcal{M} := \mathcal{V}_n^{\tilde{H}} \times_{\mathcal{H}^1(C, G)} \mathcal{H}^1(C, G)\). Now using proposition [3.18] it suffices to prove that \(\tilde{Z}_n^{\tilde{H}} \mathcal{H}^1(C, G)\) is given by the fiber product

\[
\mathcal{M} := \mathcal{V}_n^{\tilde{H}} \times_{\mathcal{H}^1(C, G)} \mathcal{H}^1(C, G).
\]

There is a natural morphism \(p : \tilde{Z}_n^{\tilde{H}} \mathcal{H}^1(C, G) \to \mathcal{M}\) which sends an \(S\)-valued point \((G, \gamma H)\), where we can assume as before \(\gamma \in Isom^0(\omega_{\mathbb{G}_w}, \mathcal{F}_{\mathcal{G}_w})\), to \((f_* G, \gamma H), \mathcal{G}, id_{f_* G}\) which is well defined by [3.19].
need to prove that this morphism induces an equivalence of the fibered categories. First we see that it is fully faithful. Let \((\mathcal{G}_1, \gamma_1H)\) and \((\mathcal{G}_2, \gamma_2H)\) be two \(S\)-valued points in \(\mathcal{V}_n^H.\mathcal{M}^1(C, \mathcal{G})\), where we assume again \(\gamma_1, \gamma_2 \in \text{Isom}^0(\omega_{\mathcal{G}_i}, \mathcal{T}_{\mathcal{G}_i})\). Let \(g \in \text{Hom}(\mathcal{M}(\mathcal{G}_1, \gamma_1H), \mathcal{M}(\mathcal{G}_2, \gamma_2H))\). Since \(V_g \circ \gamma_1 \equiv \gamma_2 \mod H\) we see that \(V_g \circ \gamma_1 \equiv h \circ \gamma_1 \mod h \) for some \(h \in H\), which implies that \(V_g\) already comes from a tensor isomorphism in \(\text{Isom}^0(T_{\mathcal{G}_i}, T_{\mathcal{G}_i})\). By [AH14, Proposition 3.6] it follows that \(g\) is not only a quasi-isogeny but also a morphism of the global \(\mathcal{G}\)-shtukas \(\mathcal{G}_1 \to \mathcal{G}_2\). Therefore \(\text{Hom}(\mathcal{G}_1, \gamma_1H), \mathcal{G}_2, \gamma_2H)\) equals the morphisms of \(\mathcal{G}\)-torsors such that \(g\) is a morphism of the global \(\mathcal{G}\)-shtukas \(\mathcal{G}_1\) and \(\mathcal{G}_2\) compatible with the level structure. Since \(f\) is an isomorphism outside of \(\mathcal{V}\) the latter condition is equivalent to the statement that \(f, g\) is a morphism of \(\mathcal{G}\)-shtukas compatible with the level structure. But this says exactly that

\[
\text{Hom}(\mathcal{G}_1, \gamma_1H), \mathcal{G}_2, \gamma_2H) = \text{Hom}(\mathcal{M}(\mathcal{G}_1, \gamma_1H), \mathcal{M}(\mathcal{G}_2, \gamma_2H), (f, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3), (f, \mathcal{G}_2, \gamma_2H), \mathcal{G}_2, \mathcal{G}_3)).
\]

For the essential surjectivity let \((E, s_1, \ldots, s_n, \tau_E, \gamma_EH), \mathcal{G}, \psi)\) be an \(S\)-valued point in \(\mathcal{M}\), with \(\gamma_E \in \text{Isom}^0(\omega_{\mathcal{G}_i}, \mathcal{T}_{\mathcal{G}_i})\) as before. This is isomorphic to

\[
((f, \mathcal{G}, s_1, \ldots, s_n, \sigma^* \psi \circ \tau_E \circ \psi^{-1}, \mathcal{T}_\psi \circ \gamma_EH), \mathcal{G}, \mathcal{G}_3).
\]

by \((\psi^{-1}, \mathcal{G})\). We need to show that it comes from an element

\[
(\mathcal{G}, \gamma_EH) = (\mathcal{G}, s_1, \ldots, s_n, \tau_E, \gamma_EH) \in \mathcal{V}_n^H.\mathcal{M}^1(C, \mathcal{G}).
\]

Here \(s_1, \ldots, s_n\) and \(\mathcal{G}\) are already uniquely defined. Therefore we need to define the isomorphism \(\tau_E : \mathcal{G}^i_{\mathcal{C}_i} \times_{\mathcal{V}_i} \mathcal{V}_E^i \to \mathcal{G}^i_{\mathcal{C}_i} \times_{\mathcal{V}_i} \mathcal{V}_E^i\). Since all the closed points \(w\) are \(\mathbb{P}\)-rational \((C, \mathcal{G})_w\) is contained in \(C_{\mathcal{S}_i} \cup \Gamma_{\mathcal{S}_i}\). Note that this is not the case if \(w_i\) splits, because in this case \(w_i \times \mathbb{P}\) has degree \(w_i\) components isomorphic to \(S\) and \(\Gamma_{\mathcal{S}_i}\). Just to make sure that \(\tau_{\mathcal{G}}\) is a morphism of \(\mathcal{G}\)-shtukas compatible with the level structure. Since \(f, g\) is a morphism of \(\mathcal{G}\)-shtukas compatible with the level structure. This means that this morphism factors through \(Z_{\mathcal{V}_i, R}\) and since \(Z_{\mathcal{V}_i, R}\) arises from base change it factors by the universal property of the fiber product also through \(Z_{\mathcal{V}_i, R}\). This shows that \(\mathcal{G}\) is bounded by \(Z_{\mathcal{V}_i, R}\) for all \(i \in \mathcal{V}\).

If \((\mathcal{G}, f) : (C, \mathcal{G}, \mathcal{G}_i, \mathcal{G}_i) \to (C, \mathcal{G}^i, \mathcal{G}_i, \mathcal{G}_i, \mathcal{G}_i, \mathcal{G}_i)\) is a morphism of shtuka data, where \(\mathcal{G}_i\) does not arise as a base change of \(\mathcal{G}_i\) or if \(f : \mathcal{G} \to \mathcal{G}^i\) is an isomorphism outside \(\mathcal{G}\) without any conditions relating \(\mathcal{G}\) to the characteristic points \(\mathcal{G}\) or their residue field, the morphism

\[
f_* : \mathcal{V}^H_\mathcal{G}.\mathcal{M}^1(C, \mathcal{G}) \to \mathcal{V}^H_\mathcal{G}.\mathcal{M}^1(C, \mathcal{G}) \text{(} \mathcal{G}, \gamma \mathcal{H}) \to (f, \mathcal{G}, \gamma \mathcal{H})\]

is still representable, but in general not surjective anymore. More precisely, we have the following theorem.

**Theorem 3.20.** Let \(w = (w_1, \ldots, w_m)\) be a finite set of closed points in \(C\) and let

\[(id_{C}, f) : (C, \mathcal{G}, \mathcal{G}_i, \mathcal{G}_i, \mathcal{G}_i, \mathcal{G}_i) \to (C, \mathcal{G}_i, \mathcal{G}_i, \mathcal{G}_i, \mathcal{G}_i, \mathcal{G}_i)\]

be a morphism of shtuka data, where \(f : \mathcal{G} \to \mathcal{G}_i\) is an isomorphism over \(C \backslash \mathcal{W}\). Then the morphism

\[
f_* : \mathcal{V}^H_\mathcal{G}.\mathcal{M}^1(C, \mathcal{G}) \to \mathcal{V}^H_\mathcal{G}.\mathcal{M}^1(C, \mathcal{G}_i) \text{(} \mathcal{G}, \gamma \mathcal{H}) \to (f, \mathcal{G}, \gamma \mathcal{H})\]

is schematic and quasi-projective. In the case that \(\mathcal{G}\) is a parahoric Bruhat-Tits group scheme this morphism is projective. For any morphism \((\mathcal{G}, \gamma \mathcal{H}) : S \to \mathcal{V}^H_\mathcal{G}.\mathcal{M}^1(C, \mathcal{G})\)

the fiber product \(S \times_{\mathcal{V}^H_\mathcal{G}.\mathcal{M}^1(C, \mathcal{G})} \mathcal{V}^H_\mathcal{G}.\mathcal{M}^1(C, \mathcal{G}_i)\) is given by a closed subscheme of
If \( \tilde{Z}_v \) arises as a base change of \( \tilde{Z}_v' \) for all \( v \in \mathcal{V} \), the morphism \( f_* \) is surjective.

Proof: In the case that \( \tilde{Z}_v \) does not arise by base change from \( \tilde{Z}_v \), the immersion \( \tilde{Z}_v \to \tilde{F}_L_{G_\mathcal{V}} \) factors through the base change \( \tilde{Z}_v'' \to \tilde{Z}_v \times_{\tilde{F}_L_{G_\mathcal{V}}} \tilde{F}_L_{G_\mathcal{V}} \). Since \( \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G) \to \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G) \) is a closed substack we may therefore assume that the beginning of \( \tilde{Z}_v \) arises by base change from \( \tilde{Z}_v' \) for all \( v \in \mathcal{V} \).

Furthermore we can as in the previous theorem assume that \( H \subset G(\mathcal{O}_\mathcal{V}) \). Let \( S \to \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G') \) be an \( S \)-valued point given by \( (\mathcal{G}', \gamma' H) = (G', s'_1, \ldots, s'_n, \tau_{G'}, \gamma' H) \), where we can assume as before that \( \gamma' \in \text{Isom}^\circ(\omega^\mathcal{G}_{i\mathcal{V}}, \tilde{T}_G) \). There is a natural morphism

\[
S \times \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G') \to S \times \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G)
\]

sending an \( T \)-valued point \((g, \mathcal{G}, \gamma H, \psi)\) to \((g, \mathcal{G}, \psi)\), where \( g : T \to S \) is a morphism of schemes, \((\mathcal{G}, \gamma H)\) is a \( T \)-valued point in \( \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G) \) and \( \psi : f_*((\mathcal{G}, \gamma H) \to g'(\mathcal{G}', \gamma' H) \) is an isomorphism of global \( G' \)-shtukas. By proposition \( \ref{prop:3.18} \) it is now enough to show that this is a closed immersion.

Given a \( T \)-valued point \((g, \mathcal{G}, \psi)\) in \( S \times \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G) \), there can be at most one \( T \)-valued point \((g, (\mathcal{G}, \gamma H), \psi)\) in \( S \times \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G) \) with \( \mathcal{G} = (G, s_1, \ldots, s_n, \tau_G) \) and \( \psi \in \text{Isom}^\circ(\omega^\mathcal{G}_{i\mathcal{V}}, \tilde{T}_G) \) mapping to \((g, (\mathcal{G}, \gamma H), \psi)\). This is because \( \psi : f_*((\mathcal{G}, \gamma H) \to g'(\mathcal{G}', \gamma' H) \) is an isomorphism of global \( G \)-shtukas. That means namely that \( s_1, \ldots, s_n \) are determined by \( s'_1 \circ g, \ldots, s'_n \circ g \), that \( \gamma H \) equals \( \tilde{T}_G \circ g^*\gamma' H \) and that there is at most one \( \tau_G \) since over the open subset \( X : (C\mathcal{V})^C \cap (G_S^C, \Gamma_{G_S}) \subset C_S \) the isomorphism \( \tau_G \) is determined by \( f_* \tau_G = \sigma^* \psi \circ g^* \tau_G \circ \psi \).

Therefore we have to answer the question if the morphism \( \tau_G|_X : \sigma^* \mathcal{G}|_X \to \mathcal{G}|_X \) can be extended to \( C_S \cup \cap \Gamma_{G_S} \). Note that if this is possible, then the global \( G \)-sshtuka \( \mathcal{G} \) is automatically bounded by \( \tilde{Z}_v \) as we have seen at the end of the proof of the previous proposition \( \ref{prop:3.19} \).

Let \( \mathcal{V} \) be the compositum of all \( \mathcal{V}_i \) with \( v_i \in \mathcal{V} \) and let \( v_i(0) \) be \( C_{\mathcal{V}_i} \) be the closed point lying over \( v_i \) that equals the image of the characteristic morphism \( s_i \). Then the definition \( \mathcal{C}_{\mathcal{V}} := \mathcal{G}|_{\cup v_i(0)} \) satisfies \( \mathcal{C}_{\mathcal{V}} \subset C_S \). Let further

\[
I = \left\{ w \in \mathcal{C}_{\mathcal{V}} \mid \text{for some } w_j \in \mathcal{V}, w \neq v_i(0) \text{ for all } v_i \in \mathcal{V} \right\}.
\]

In other words that means that \( I \) is determined by \( \bigcup_{v_i(0)} \Gamma_{G_S} \subset \bigcup_{v_i(0)} \Gamma_{G_S} \subset C_{\mathcal{V}} \) and \( \Gamma_{G_S} \subset \bigcup_{v_i(0)} \Gamma_{G_S} \subset C_{\mathcal{V}} \). The definition satisfies also the equation \((\mathcal{C}_{\mathcal{V}}|_I) \times_{\mathcal{V}} S = U \). Then by the theorem of Beauville-Laszlo from \( \ref{thm:2.9} \) we have the following cartesian diagram

\[
\begin{array}{c}
\mathcal{H}^1(C, \mathcal{G}_C) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\prod_{w \in I} L_w^* \quad \prod_{w \in I} L_w^* \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^1(C, \mathcal{G}_{\mathcal{V}}) \\
\end{array}
\]

which means that \( \sigma^* \mathcal{G}|_{\mathcal{C}_{\mathcal{V}}} \cdot \Gamma_{G_S} \) and \( \mathcal{G}|_{\mathcal{C}_{\mathcal{V}}} \cdot \Gamma_{G_S} \) are given by tuples \((\sigma^* \mathcal{G}|_{\mathcal{C}_{\mathcal{V}}} \cdot \Gamma_{G_S}), \prod_{w \in I} L_w^* (\sigma^* \mathcal{G}|_{\mathcal{C}_{\mathcal{V}}} \cdot \Gamma_{G_S}) \) and \((\mathcal{G}|_{U}, \prod_{v_i(0)} L_v^* (\mathcal{G}|_{U}) \cdot \prod_{v_i(0)} \Gamma_{G_S} (\mathcal{G}|_{U}) \). The morphism \( \tau_G|_U : \sigma^* \mathcal{G}|_U \to \mathcal{G}|_U \) determines for all \( w \in I \) an isomorphism \( L_w(\tau_G) : L_w (\sigma^* \mathcal{G}|_U) \to L_w \mathcal{G}|_U \). The question if \( \tau_G \) can be extended to \( C_S \cup \cup \Gamma_{G_S} \) is therefore to the question if all the isomorphisms \( L_w(\tau_G) \) in \( \mathcal{H}^1(F, L^* \mathcal{G}_{w}) \) already come from an isomorphism \( L_w^* (\sigma^* \mathcal{G}|_U) \to L_w^* (\mathcal{G}|_U) \) in \( \mathcal{H}^1(F, L^* \mathcal{G}_{w}) \). Since \( L^* \mathcal{G}_{w} \subset L^* \mathcal{G}_{w} \) is a quasi-compact closed subscheme, this is a closed condition on \( T \) which shows that

\[
S \times \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G) \to S \times \nabla_{n(L')^H}^\mathcal{H} \mathcal{H}^1(C, G)
\]
is a closed immersion. It rests to show that under our assumption on \( \hat{\mathcal{Z}} \) the morphism \( f \), is surjective. This is not clear yet, since the closed subscheme

\[
S \times_{\nabla_{n-H}^{\mathcal{H}^{1}(C, G')} \mathbb{P}^{1}} \mathcal{H}^{1}(C, G) \to S \times_{\nabla_{n-H}^{\mathcal{H}^{1}(C, G')} \mathbb{P}^{1}} ((L_{w_{0}}(G')/L^{\mathbb{G}_{w_{0}}}) \times_{\nabla_{n-H}^{\mathcal{H}^{1}(C, G')} \mathbb{P}^{1}} (L_{w_{n}}(G')/L^{\mathbb{G}_{w_{n}}}))
\]

do not necessarily surject to \( S \). For the proof of the surjectivity we show that for any algebraically closed field \( K \) and every global \( G' \)-shtuka \( G' = (G', s_{1}, \ldots, s_{n}, \tau_{G'}) \) in \( \nabla_{n-H}^{\mathcal{H}^{1}(C, G')}(K) \), there is a global \( G \)-shtuka \( G = (G, s_{1}, \ldots, s_{n}, \tau_{G}) \) in \( \nabla_{n-H}^{\mathcal{H}^{1}(C, G)}(K) \) with \( f_{*}(G) = G' \). By proposition 3.18 and the fact that \( K \) is algebraically closed the choice of a \( G \)-torsor \( \mathcal{G} \) over \( C_{K} \) with \( f_{*}(G) = G' \) corresponds to an element in \( \prod_{w \in \mathcal{W}} (L^{\mathbb{G}_{w}}/L^{\mathbb{G}_{w}})(K) \). Now let \( \mathbb{P}^{n} \) be the composition of the fields \( \mathbb{F}_{w} \) for all \( w \in w \). For a closed point \( w \in w \subset C \) there are exactly \( \text{deg } w \) different closed points in \( C_{\mathbb{F}} \) lying above \( w \). We denote them by \( u_{w}^{(0)}, \ldots, u_{w}^{(\text{deg } w-1)} \), where \( u_{w}^{(0)} \) is a randomly chosen one and the others arise by applying successively \( \sigma \) on the residue field. If \( w \in w \) we choose \( u_{w}^{(0)} \) as before to be the image of the characteristic morphism \( s_{i} \). Now once again Beauville and Laszlo help us with the diagram

\[
\mathcal{H}^{1}(C_{\mathbb{P}^{n}}, G_{\mathbb{P}^{n}}) \longrightarrow \mathcal{H}^{1}(C_{\mathbb{P}^{n}}, G_{\mathbb{P}^{n}})
\]

where \( J = \{ v \in C_{\mathbb{P}^{n}} \mid |v|w \) for some \( w \in w \) and \( V := C_{\mathbb{P}^{n}} \setminus J \). It allows us to identify \( G' \) with the tuple \((G'_{|vK}, \prod_{w \in \mathcal{W}} L^{G'_{w_{0}}}, (\epsilon_{w}^{(i)})_{w \in \mathcal{W}}(\epsilon_{w}^{(i)}))_{w \in \mathcal{W}} \)\) where \( (\epsilon_{w}^{(i)} : L^{G'_{w_{0}}}(G') \to L^{G'_{w_{0}}}(G') \) already comes from an isomorphism of \( L^{G'_{w_{0}}} \)-torsors. Consequently \( \sigma^{*}G' \) is identified with \((\sigma^{*}G'_{|vK}, \prod_{w \in \mathcal{W}} L^{G'_{w_{0}}}, (\sigma^{*}\epsilon_{w}^{(i)}_{w(\sigma^{(i)})})_{w \in \mathcal{W}} \)\) where \( \sigma^{*}_{w} : L_{w}(G') \to L_{w}(G') \) coming again from an isomorphism of \( L^{G'_{w_{0}}} \)-torsors.

Note that the index \( i \) is computed in \( \mathbb{Z}/\text{deg } w \) so that \( -1 = \text{deg } w - 1 \). We use again the intuitive notation \( \tau'_{w} := L_{w}(\tau_{G'}) : L_{w}(\sigma^{*}G'_{0}) \to L_{w}(G') \) and define for all \( w \in J \) the element \( \epsilon_{w}^{(i)} := \epsilon_{w}^{(i)} \circ \tau'_{w(\sigma^{(i)})} \circ \sigma^{*}_{w}((\epsilon_{w}^{(i-1)}))^{-1} \) in \( L^{G'_{w_{0}}}(K) \). The fact that \( \tau_{G'} \) is an isomorphism over \( C_{K} \setminus \bigcup_{k=1}^{n} \Gamma_{s_{k}} \) implies that \( \epsilon_{w}^{(i)} \) is an element in \( L^{G'_{w_{0}}}(K) \) for all \( w \in J \). Equivalently we have

\[
b_{w}^{(i)} := 1 \quad \text{and } \quad b_{w}^{(i)} := \sigma^{*}b_{w}^{(i-1)}, (\epsilon_{w}^{(i)})^{-1} \in L^{G_{w_{0}}}(K)
\]

for all \( i = 1, \ldots, \text{deg } w - 1 \). Now if \( w \in w \) we write \( \tau'_{w} := \tau'_{w}^{(i)} \sigma^{*}((\epsilon_{w}^{\text{deg } w-1})_{w} \cdot \sigma^{*}(\epsilon_{w}^{\text{deg } w-2})_{w} \cdot \ldots \cdot \sigma^{*}(\epsilon_{w}^{\text{deg } w-1})_{w} \cdot \epsilon_{w}^{(i-1)})_{w} \in L^{G'_{w_{0}}}(K) \). We can choose by \([AH14] Corollary 2.9\) an element \( b_{w}^{(0)} \in L^{G'_{w_{0}}}(K) \) satisfying \( b_{w}^{(0)} \circ \tau'_{w} \sigma^{*}(b_{w}^{(0)})^{-1} = 1 \). Additionally we define \( b_{w}^{(i)} := \sigma^{*}b_{w}^{(i-1)} \cdot (\epsilon_{w}^{(i-1)})^{-1} \in L^{G'_{w_{0}}}(K) \) for all \( i = 1, \ldots, \text{deg } w - 1 \), so that we have \( b_{w}^{(i)} \cdot (\epsilon_{w}^{(i)} \cdot \sigma_{w}^{*}(b_{w}^{(i-1)}) = 1 \). Further more we see

\[
b_{w}^{(0)} \cdot (\epsilon_{w}^{(i)} \cdot \sigma_{w}^{*}(b_{w}^{(i-1)}) \cdot (\epsilon_{w}^{(i)} \cdot \sigma_{w}^{*}(b_{w}^{(i-1)}))^{-1} = 1
\]

Now if \( w \in w \) we define \( \mathcal{G} := (G'_{|vK}, \prod_{w \in \mathcal{W}} L^{G'_{w_{0}}}, (\epsilon_{w}^{(i)})_{w \in \mathcal{W}} \)\) is given, as described in proposition 3.18 by

\[
\mathcal{G} = (G'_{|vK}, \prod_{w \in \mathcal{W}} L^{G'_{w_{0}}}, (\epsilon_{w}^{(i)})_{w \in \mathcal{W}})
\]
It lies indeed in the pre-image of $\mathcal{G}'$ since $f, \hat{G} = (\mathcal{G}'|_U, \prod_{w \in S}^{d_{\deg w} - 1} L^+ \mathcal{G}'(w), (b_w \circ \varphi^{(w)}_{U(w)}))$ is isomorphic to $\mathcal{G}'$ by $\{id_{\mathcal{G}'}|_U, \{\hat{b}^{(w)}_{U(w)}\}_{w \in S}\}$. Now, we show that there is $\tau_\mathcal{G} : \sigma^* \mathcal{G}|_{C^1_{\mathcal{K}' \setminus \Gamma_{S_{\mathcal{K}}}}} \to \mathcal{G}|_{C^1_{\mathcal{K}' \setminus \Gamma_{S_{\mathcal{K}}}}}$ with $f, \tau_\mathcal{G} = \tau_{\mathcal{G}}$. We set $\tau_\mathcal{G} = \tau_{\mathcal{G}}$ and need to convince ourselves that it extends to $C^1_{\mathcal{K}' \setminus \Gamma_{S_{\mathcal{K}}}}$. This is the case if and only if for all $w^{(i)} \in J_0$ the vertical right hand side morphism $b_{w}^{(i)} \circ \epsilon_{w}^{(i)} \circ L\mathcal{G}(w) \circ b_{w}^{(i)} = 1$ for all $w^{(i)} \in J_0$. This proves $f, \hat{G} = \mathcal{G}'$ and finally the theorem.

Closed Subgroups of $\mathcal{G}'$

Secondly, we take a closer look to the case that $f : \mathcal{G} \to \mathcal{G}'$ is a closed immersion of group schemes over $C$. We start with the following lemma that we mainly need for theorem 3.22.

**Lemma 3.21.** Let $f : \mathcal{G} \to \mathcal{G}'$ be a closed immersion of smooth, affine group schemes over the curve $C$. Then the diagonal morphism $\Delta : \mathcal{H}(C, \mathcal{G}) \to \mathcal{H}(C, \mathcal{G}) \times \mathcal{H}(C, \mathcal{G}') \mathcal{H}(C, \mathcal{G})$ of the induced morphism $f, : \mathcal{H}(C, \mathcal{G}) \to \mathcal{H}(C, \mathcal{G}')$ is a monomorphism.

The same is true for the diagonal morphism $\Delta : \mathcal{H}(C, \mathcal{G}) \to \mathcal{H}(C, \mathcal{G}) \times \mathcal{H}(C, \mathcal{G}') \mathcal{H}(C, \mathcal{G})$ of the induced morphism $f, : \mathcal{H}(C, \mathcal{G}) \to \mathcal{H}(C, \mathcal{G}')$.

**Proof:** For the first diagonal morphism we have to prove that for any $F_{\mathcal{G}}$-scheme $S$ the functor $\Delta_S : \mathcal{H}(C, \mathcal{G})(S) \to \mathcal{H}(C, \mathcal{G}) \times \mathcal{H}(C, \mathcal{G}') \mathcal{H}(C, \mathcal{G})$ is fully faithful. Let $\mathcal{G} \in \mathcal{H}(C, \mathcal{G})(S)$, then this functor is clearly always faithful since $\varphi \in Aut(\mathcal{G})$ is sent to $(\varphi, \varphi) \in Aut(\Delta(\mathcal{G}))$, where $\Delta(\mathcal{G}) = (\mathcal{G}, \mathcal{G}, id_{\mathcal{G}})$. Note that it suffices to consider $\varphi \in Aut(\mathcal{G})$ since all morphisms in $\mathcal{H}(C, \mathcal{G})$ are isomorphisms. To show that $\Delta_S$ is full, let $(\varphi, \psi) \in Aut(\Delta(\mathcal{G}))$ which means by definition that

$$f, \varphi \circ f, \psi = f, \varphi \circ f, \psi = f, \psi \circ f, \varphi.$$  

commutes. Therefore we have $f, \varphi = f, \psi$ and since $f : \mathcal{G} \to \mathcal{G}'$ is a closed immersion this implies $\varphi = \psi$ and hence that $\Delta_S$ is full.

More precisely, to see this, one chooses a covering $U \to C_S$ that trivializes $\mathcal{G}$ so that $\varphi$ and $\psi$ correspond to morphisms $\varphi, \psi : U \to C_S$ satisfying the corresponding cocycle condition. The morphisms $f, \varphi$ and $f, \psi$ correspond to the compositions $U \xrightarrow{\varphi} \mathcal{G} \xrightarrow{f} \mathcal{G}'$ and the equality $f, \varphi = f, \psi$ means $f \circ \varphi = f \circ \psi$. Since $f$ is a closed immersion this implies $\varphi = \psi$, which proves that the first diagonal morphism is a monomorphism. The proof for the second diagonal morphism $\Delta : \mathcal{H}(C, \mathcal{G}) \to \mathcal{H}(C, \mathcal{G}) \times \mathcal{H}(C, \mathcal{G}') \mathcal{H}(C, \mathcal{G})$ works literally in the same way.

**Corollary 3.22.** The morphism $f, : \mathcal{H}(C, \mathcal{G}) \to \mathcal{H}(C, \mathcal{G}')$ is representable by an algebraic space. In particular for every $F_{\mathcal{G}}$-morphism $\mathcal{G}' : S \to \mathcal{H}(C, \mathcal{G}')$ and the natural projection $\pi : C_S \to S$, the Weil restriction $\pi^*_{\mathcal{G}'}(\mathcal{G}'/\mathcal{G}_{\mathcal{S}})$ is an algebraic space, that equals the fiber product $S \times_{\mathcal{H}(C, \mathcal{G}') \mathcal{H}(C, \mathcal{G}))}$

**Proof:** Since the diagonal morphism in lemma 3.21 is a monomorphism it follows by [LMB06, Corollary 8.1.2] that $f, : \mathcal{H}(C, \mathcal{G}) \to \mathcal{H}(C, \mathcal{G}')$ is representable by an algebraic space. By definition this means that the fiber product $S \times_{\mathcal{H}(C, \mathcal{G}') \mathcal{H}(C, \mathcal{G}))}$ is an algebraic space and in particular given by a functor.
Theorem 3.33. Let $f : G \to G'$ be a closed immersion of smooth, affine group schemes over $C$. Then the induced morphism $f : \nabla_n \mathcal{H}^1(C, G) \to \nabla_n \mathcal{H}^1(C, G')$ is unramified and schematic. 

Proof: We first show that $f_*$ is unramified and then conclude that it is representable by a scheme. Let $B$ be any ring and $I \subseteq B$ an ideal with $I^2 = 0$ and $p : \text{Spec } B := \text{Spec } B/I \to \text{Spec } B$ the natural projection arising in a diagram of the form

$$
\begin{array}{ccc}
\text{Spec } B/I & \xrightarrow{g} & \nabla_n \mathcal{H}^1(C, G) \\
p \downarrow & & \downarrow f_* \\
\text{Spec } B & \xrightarrow{g_1} & \nabla_n \mathcal{H}^1(C, G')
\end{array}
$$

To prove that $f_* : \nabla_n \mathcal{H}^1(C, G) \to \nabla_n \mathcal{H}^1(C, G')$ is unramified, we need to show that for any diagram of this kind there exists at most one dashed arrow making the diagram commutative, that means $g_1 = g_2$. This suffices since $\nabla_n \mathcal{H}^1(C, G)$ and $\nabla_n \mathcal{H}^1(C, G')$ are locally of ind-finite type over the noetherian scheme $C^n$. The morphism $g : \text{Spec } B \to \nabla_n \mathcal{H}^1(C, G)$ corresponds to a global $G$-shtuka $\mathcal{G} := (\mathcal{G}, \tau_0, \tau_1, \ldots, \tau_n)$ over $\text{Spec } B$, where $g_1$ and $g_2$ correspond to global $G$-shtukas $\mathcal{G}_1 = (\mathcal{G}_1, s_1, \ldots, s_n, \tau_{G_1})$ and $\mathcal{G}_2 = (\mathcal{G}_2, s'_1, \ldots, s''_n, \tau_{G_2})$ over $\text{Spec } B$. The commutativity of the upper triangle means that there are isomorphisms $\beta_1 : p^* \mathcal{G}_1 \to \mathcal{G}$ and $\beta_2 : p^* \mathcal{G}_2 \to \mathcal{G}$ of global $G$-shtukas over $\text{Spec } B$. Therefore we have to prove that the isomorphism $\beta_2 \circ \beta_1$ arises already from an isomorphism $\mathcal{G}_1 \to \mathcal{G}_2$ of global $G$-shtukas over $\text{Spec } B$. Furthermore we denote by $\mathcal{G}' := (\mathcal{G}', s_1, \ldots, s_n, \tau_{G'})$ the global $G'$-shtuka over $\text{Spec } B$ corresponding to $g' : \text{Spec } B \to \nabla_n \mathcal{H}^1(C, G')$. The commutativity of the lower triangle gives us isomorphisms $\alpha_1 : f_1 \mathcal{G}_1 \to \mathcal{G}'$ and $\alpha_2 : f_1 \mathcal{G}_2 \to \mathcal{G}'$ of global $G'$-shtukas over $\text{Spec } B$ satisfying $\gamma = p^* \alpha_2 \circ f_1 \beta_2 = p^* \alpha_1 \circ f_1 \beta_1$ where $\gamma : f_1 \mathcal{G} \to \mathcal{G}'$ is the isomorphism of global $G'$-shtukas over $\text{Spec } B$ coming from the commutativity of the square. Now these isomorphisms imply directly that the paws $s_i, s'_i$ and $s''_i$ coincide for all $i$ with $1 \leq i \leq n$. Although $f : G \to G'$ is a closed immersion it is by the following remark a priori not so clear that the torsors $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic, but we now prove this as follows. The $G$-torsors $\mathcal{G}_1$ and $\mathcal{G}_2$ over $C_B$ come with $G$-equivariant maps to $G'$ which are induced by $\alpha_1 : f_1 \mathcal{G}_1 \to G'$ and $\alpha_2 : f_1 \mathcal{G}_2 \to G'$. Therefore they define two $C_B$-valued points $h_1, h_2 : C_B \to G'/G_B$. In other words one can describe them as follows. Since

$$
\begin{array}{ccc}
\text{Spec } B & \xrightarrow{\mathcal{G}_1} & \mathcal{H}^1(C, G) \\
\downarrow f & & \downarrow f_1 \\
\text{Spec } B & \xrightarrow{\mathcal{G}_2} & \mathcal{H}^1(C, G')
\end{array}
$$

commute, that means $f_* \mathcal{G}_1 = f_* \mathcal{G}_2$, the $G$-torsors $\mathcal{G}_1$ and $\mathcal{G}_2$ induce morphisms $h_1, h_2$ from $\text{Spec } B$ to the fiber product $\text{Spec } B \times_{\text{Spec } C, \mathcal{H}^1(C, G)} \mathcal{H}^1(C, G)$. In corollary this fiber product was seen to be $p_B.(G'/G_B)$. Therefore we have $\text{Hom}_{C_B}(C_B, (G'/G_B)) = \text{Hom}_B(\text{Spec } B, p_B,(G'/G_B))$ by definition of the Weil restriction, so that $h_1$ and $h_2$ correspond consequently to morphisms $C_B \to G'/G_B$. First we show that they coincide on $C_B := C_B \backslash \bigcup_i \Gamma_i$. The $F_q$-Frobenius induces a morphism $j : B/I \to B$, $b \mapsto b^q$ which is well defined, because $I^2 = 0$ and in particular $I^q = 0$. We get the following commutative diagram:

$$
\begin{array}{ccc}
\text{Spec } B & \xrightarrow{j} & \text{Spec } B/I \\
\downarrow \sigma_B & & \downarrow g_1 \\
\text{Spec } B & \xrightarrow{g_2} & \mathcal{H}^1(C, G)
\end{array}
$$

$\square$
which implies

\[ \sigma^* G_1 = j^* p^* G_1 \overset{\sim}{\longrightarrow} j^* \beta_1^* \overset{\sim}{\longrightarrow} j^* p^* G_2 = \sigma^* G_2. \]

By restricting this isomorphism to \( \mathcal{C}_B \) and composing with \( \tau_{G_1} \) and \( \tau_{G_2} \) we get

\[ \delta_0 : G_1|_{\mathcal{C}_B} \overset{\sim}{\longrightarrow} \sigma^* G_1|_{\mathcal{C}_B} \overset{\sim}{\longrightarrow} j^* \beta_1|_{\mathcal{C}_B} \overset{\sim}{\longrightarrow} \sigma^* G_2|_{\mathcal{C}_B} \overset{\sim}{\longrightarrow} G_2|_{\mathcal{C}_B}, \]

an isomorphism \( \delta_0 := \tau_{G_2} \circ j^* \beta_2^{-1} \circ j^* \beta_1 \circ \tau_{G_1}^{-1} \) of \( G \)-torsors over \( \mathcal{C}_B \). It satisfies

\[
\begin{align*}
\alpha_2|_{\mathcal{C}_B} \circ f_\delta \circ \alpha_1^{-1}|_{\mathcal{C}_B} &= \alpha_2 \circ f \circ \tau_{G_2} \circ f \circ j^* \beta_2^{-1} \circ f \circ j^* \beta_1 \circ \sigma \circ \alpha_1^{-1} \circ \tau_{G_1}^{-1} \\
= &\tau_{G_1} \circ \sigma \circ \alpha_1^{-1} \\
= &j^* (f \circ \beta_1 \circ \alpha_1^{-1}) \\
= &\tau_{G_1} \circ \sigma \circ \alpha_2 \circ j^* f \circ \beta_2^{-1} \circ j^* (f \circ \beta_2 \circ p \circ \alpha_2^{-1}) \circ \tau_{G_1}^{-1} = \text{id}_{\mathcal{C}_B}.
\end{align*}
\]

In other words \( \delta_0 \) is an isomorphism from \((G_1|_{\mathcal{C}_B}, \alpha_1|_{\mathcal{C}_B})\) to \((G_2|_{\mathcal{C}_B}, \alpha_2|_{\mathcal{C}_B})\) of \( \mathcal{C}_B \)-valued points in \( G'/G_B \).

Therefore the restriction of \((h_1, h_2) : C_B \to G'/G_B \times_{C_B} G'/G_B\) to the open subscheme \( \mathcal{C}_B \) in \( C_B \) factors through the diagonal in the following diagram

\[
\begin{tikzcd}
C_B \arrow{rr}{(h_1, h_2)} \arrow{dr}[swap]{\Delta} & & G'/G_B \times_{C_B} G'/G_B \\
& \mathcal{C}_B \arrow{ur} & \to G'/G_B.
\end{tikzcd}
\] (9)

To see that \( G_1 \simeq G_2 \) over \( C_B \) we have to show that the morphism \((h_1, h_2)\) factors through the diagonal \( \Delta \) as well. Now since \( f \) is a closed immersion, the quotient \( G'/G_B \) exists as a scheme by [Ana73, Theorem 4.C] and it is smooth and separated by [SGA70, VI_B, Proposition 9.2(xii) and (x)]. In particular the diagonal \( \Delta \) is a closed immersion. Therefore \( C_B \) factors through the diagonal if the scheme theoretic image of \( \mathcal{C}_B \) in \( C_B \) equals \( C_B \). This was proven in lemma \[\text{LIM}\text{.11}\]. As a result of this, we conclude that \( \delta_0 \) extends to an isomorphism \( \delta : G_1 \simeq G_2 \) of \( G \)-torsors over \( C_B \). The computation

\[
\begin{align*}
\delta_0^{-1} \circ \tau_{G_2} \circ \sigma \circ \delta_0 &= \tau_{G_1} \circ j^* \beta_1^{-1} \circ j^* \beta_2 \circ \tau_{G_2} \circ \sigma \circ \tau_{G_2} \circ \sigma \circ j^* \beta_1 \circ \sigma \circ \tau_{G_1}^{-1} \\
= &\tau_{G_1} \circ j^* \beta_1^{-1} \circ j^* \beta_2 \circ \sigma \circ j^* \beta_1 \circ \sigma \circ \tau_{G_1}^{-1}
\end{align*}
\]

shows that \( \delta : G_1 \simeq G_2 \) is an isomorphism of \( G \)-shtukas over \( B \), which finishes the proof that the morphism \( f_* : \nabla_n \mathcal{H}^1(C, G) \to \nabla_n \mathcal{H}^1(C, G') \) is unramified.

It rests to show that this morphism is schematic. We have proven in lemma \[\text{LIM}\text{.21}\] that the diagonal \( \Delta_f \) of \( f_* \) is a monomorphism, which implies together with [LMB00, Corollary 8.1.2] that the morphism \( f_* \) is representable by an algebraic space. It is clear that \( f_* \) is a separated morphism, since the moduli spaces of global \( G \)-shtukas are separated. Furthermore we have proven that \( f_* \) is unramified and in particular locally quasi-finite [Gro67, Corollaire 17.4.3]. All together this allows us to apply [LMB00, Theorem A.2] which states that a separated, locally quasi-finite morphism of algebraic stacks that is representable by an algebraic space is already schematic. This finishes the proof of the theorem. \( \square \)

**Remark 3.24.** Note that this is a particular property of the morphism of shtukas. Even if \( f : G \to G' \) is a closed immersion, it is not true that \( \mathcal{H}^1(C, G) \to \mathcal{H}^1(C, G') \) is an unramified morphism.

**Corollary 3.25.** Let \((\text{id}_C, f) : (C, G) \to (C, G')\) be a morphism of shtuka data, where \( f : C \to C' \) is a closed immersion of smooth, affine group schemes over \( C \). Then the induced morphism

\[
f_* : \nabla_n \mathcal{H}^1(C, G) \to \nabla_n \mathcal{H}^1(C, G')
\]

is unramified and schematic.
**Proof:** We first consider the induced morphism \( f_* : \mathfrak{H}_n^{2, H}(C, \mathbb{G}) \to \mathfrak{H}_n^{2, H'}(C, \mathbb{G}') \) of the moduli spaces of global \( \mathbb{G} \)-shtukas without level structures. We have the following commutative diagram

\[
\begin{array}{c}
\mathfrak{H}_n^{2, H}(C, \mathbb{G}) \\
\downarrow f_* \\
\mathfrak{H}_n^{2, H'}(C, \mathbb{G}') \\
\end{array}
\]

The vertical arrow on the right is an unramified morphism by theorem \( \text{[3.23]} \) where the horizontal arrows are closed immersions and in particular also unramified. As a consequence the vertical arrow on the left is unramified as well. To prove the statement for the morphism \( f_* \) of moduli spaces of global \( \mathbb{G} \)-shtukas with level \( H \)-structure, we choose similar to \( \text{[3.14]} \) some finite subscheme \( D \subset C \) such that \( H_D := \ker(\mathbb{G}(\mathbb{O}_D) \to \mathbb{G}(\mathbb{O}_D)) \) and \( H'_D := \ker(\mathbb{G}'(\mathbb{O}_D) \to \mathbb{G}'(\mathbb{O}_D)) \) are subgroups of finite index in \( H \) (resp. \( H' \)). Then we have by \( \text{[2.16]} \) the following commutative diagram

\[
\begin{array}{c}
\mathfrak{H}_n^{2, H}(C, \mathbb{G}) \\
\downarrow f_* \\
\mathfrak{H}_n^{2, H'}(C, \mathbb{G}') \\
\end{array}
\]

All the horizontal arrows are etale and in particular unramified. Furthermore we have seen that the vertical arrow on the right is unramified. It follows that \( \mathfrak{H}_n^{2, H}(C, \mathbb{G}) \to \mathfrak{H}_n^{2, H'}(C, \mathbb{G}') \) is unramified.

**Theorem 3.26.** Let \( \mathbb{G} \) be a parahoric Bruhat-Tits group scheme and \( f : \mathbb{G} \to \mathbb{G}' \) be a closed immersion of smooth, affine group schemes and \( v = (v_1, \ldots, v_n) \) be a set of closed points in \( C \). Then the induced morphism

\[
f_* : \mathfrak{H}^1_n(C, \mathbb{G}) \to \mathfrak{H}^1_n(C, \mathbb{G}')
\]

is proper and in particular finite.

**Proof:** We know by theorem \( \text{[3.23]} \) that this morphism is unramified and schematic and in particular locally quasi-finite. Moreover the morphism is quasi-compact. Since \( \mathfrak{H}^1_n(C, \mathbb{G}) \to \mathfrak{H}^1_n(C, \mathbb{G}') \) is of ind-finite type, this follows from \( \text{[AH13]} \) Theorem 2.5] after choosing a representation \( \rho : \mathbb{G}' \to \text{GL}(V_0) \) for some vector bundle \( V_0 \) such that the quotient \( \text{GL}(V_0)/\mathbb{G} \) is quasi-affine (see \( \text{[AH13]} \) Proposition 2.2]). Therefore it suffices to prove that \( f_* \) satisfies the valuative criterion for properness to see that this morphism is proper and consequently also finite, due to the quasi-finiteness. Thus let \( R \) be a complete discrete valuation ring with uniformizer \( \pi \) such that its residue field \( \kappa_R = R/\pi \) is algebraically closed and let \( K = \text{Frac}(R) \) be the fraction field of \( R \). Let us further denote by \( K^{alg} \) an algebraic closure of \( K \) and by \( R^{alg} \) the integral closure of \( R \) in \( K^{alg} \). We need to prove that in every diagram of the form

\[
\begin{array}{c}
\text{Spec } K^{alg} \\
\downarrow i_k \downarrow g_1 \\
\text{Spec } K \\
\downarrow f_* \\
\text{Spec } K^{alg} \\
\end{array}
\]

there exists a unique dashed arrow making the diagram commutative.

Here \( g_1, g_2, i_K, i_R, j \) and \( \overline{j} \) are defined by the diagram. Choosing the closed embedding \( \rho : \mathbb{G}' \to \text{GL}(V_0) \) it suffices, due to the separateness of the moduli spaces, to prove the valuative criterion for the composition \( \rho_* \circ f_* : \mathfrak{H}^1_n(C, \mathbb{G}) \to \mathfrak{H}^1_n(C, \text{GL}(V_0)) \). Therefore we may assume that \( \mathbb{G}' \) equals \( \text{GL}(V_0) \). We denote by \( \mathbb{G} = (\mathbb{G}, s_1, \ldots, s_n, \tau) \) the global \( \mathbb{G} \)-shtuka over \( K \) corresponding to \( g_1 \) and by \( \mathbb{G}' = (\mathbb{G}', s'_1, \ldots, s'_n, \tau' \mathbb{G}) \) the global \( \mathbb{G}' \)-shtuka over \( K \) corresponding to \( g_2 \). Furthermore the commutativity of the square gives an isomorphism \( \alpha : f_* \mathbb{G} \to f_* \mathbb{G}' \) of global \( \mathbb{G}' \)-shtukas over \( K \).

Let \( S := \{ v \in C \mid \mathbb{G} \times_C \mathbb{F}_v \text{ is not reductive} \} \) \( \cup \mathbb{G} \). Then \( \mathbb{G}'/\mathbb{G} \times_C (C \setminus S) \) is by \( \text{[Alp14]} \) Theorem 9.4.1 and Corollary 9.7.7] an affine scheme over \( C \setminus S \) and in particular \( \mathbb{G}'/\mathbb{G} \times_C (C \setminus S) \) is an affine scheme over
Now the $G$-torsor $\mathcal{G}|_{(C\setminus S)_R}$ with its $G$ equivariant morphism $\mathcal{G} \rightarrow \mathcal{G}'$ induced by $\alpha$ defines an $(C\setminus S)_K$ valued point of the quotient $\mathcal{G}'/\mathcal{G}_R$.

\[
\begin{array}{c}
(C\setminus S)_R \\
\rightarrow \mathcal{G}'/\mathcal{G}_R \\
\downarrow \\
(C\setminus S)_R
\end{array}
\]

(11)

Now the proof consists of several steps. In a first step we want to show that $s$ factors through $(C\setminus S)_R$ which means that it gives a section $s_R : (C\setminus S)_R \rightarrow \mathcal{G}'/\mathcal{G}_R \times_{C_R} (C\setminus S)_R$ of the vertical morphism in diagram (11). This morphism $s_R$ corresponds to a unique $G$-torsor $\tilde{E}$ over $(C\setminus S)_R$ together with an isomorphism $\sigma_R : f_\ast \tilde{E} \rightarrow \mathcal{G}'|_{(C\setminus S)_R}$ satisfying $j^\ast \tilde{E} = \mathcal{G}'|_{(C\setminus S)_K}$.

In the second step of the proof we then show that the base change of $\tilde{E}$ to $R^{alg}$ extends uniquely to a $G$-torsor over the whole relative curve $C_{R^{alg}}$. More precisely, we show that there is a $G$-torsor $\tilde{E}$ over $C_{R^{alg}}$ such that firstly the restriction $\tilde{E}|_{(C\setminus S)_R}$ is isomorphic to the $G$-torsor $i_R^\ast \tilde{E}$ over $(C\setminus S)_R$ and secondly $f_\ast \tilde{E} \cong i_R^\ast \tilde{E}$. Then we show in the third step that this $G$-torsor $\tilde{E}$ over $C_{R^{alg}}$ gives rise to a unique $G$-shutuka $(E, \tau_1, \ldots, \tau_n, \tau_E)$ in $\mathcal{V}_n(H^1(C, G)(R^{alg}))$ making the diagram (10) commutative. This will then finish the proof.

(Step 1) We can assume that $Spec A = C\setminus S$ is affine by enlarging $S$ if necessary. Since we have seen that $\mathcal{G}'/\mathcal{G}_R \times_{C_R} (C\setminus S)_R$ is affine over $(C\setminus S)_R = Spec A_R = Spec (A \times_{\mathcal{G}_R} C_R)$ we can set $\mathcal{G}'/\mathcal{G}_R \times_{C_R} Spec A_R = Spec B$ for some ring $B$. Therefore, to prove the assertion of the first step, namely that $s$ in diagram (11) factors through $(C\setminus S)_R$ it is enough to show that the ring morphism $s^\ast : B \rightarrow A \otimes_{\mathcal{G}_R} K = A_K$ factors through $A_R$. We write $L := Frac(A_R)$ for the function field of $C_R$ and $\mathcal{O} := (A_R)(\pi) \subset L$ for the localisation of $A_R$ at the prime ideal $(\pi) := Ker(A_R \rightarrow A_K)$. The fact that $A_R$ is normal due to the smoothness of $C_R$ over $R$ and the fact that the prime ideal $(\pi) \subset A_R$ corresponding to the generic point of $Spec A_K$ is of height 1 in $A_R$, implies that $\mathcal{O}$ is a discrete valuation ring with uniformizer $\pi$. The normality of $A_R$ allows us also by [Har77, chapter II, 6.3.1] to write $A_R = \bigcap_{p \subset A_K \text{ max ideal}} A_{R,p}$. For all prime ideals $p \subset A_R$ of height 1 we have either $p = (\pi)$ or $p \not\subset (\pi)$. In the second case $p$ comes from a closed point in $Spec A_K$ which means $A_{R,p} = A_K$. Since $A_K = \bigcap_{q \subset A_{K, max ideal}} A_{K,q}$ we conclude

\[
A_R = \mathcal{O} \cap A_K \subset L.
\]

Due to this equation it is enough to show that the composition $s^\ast_L : Spec L \rightarrow Spec A_K$ factors through $Spec \mathcal{O}$.

The Frobenius pullback $\mathcal{G}' = (f_\ast \mathcal{G}) \rightarrow \mathcal{G}'$ gives an $L$-valued point of the quotient $\mathcal{G}'/\mathcal{G}_R$ $\mathcal{G}'$ over $\mathcal{G}_R$. As before this quotient is affine over $A_R$ and given exactly by $\mathcal{G}'/\mathcal{G}_R \times_{C_R} Spec A_R = Spec (B \otimes_{A_R} A_R) = Spec A_R$, where the $A_R$-algebra structure of $B \otimes_{A_R} A_R$ is given by multiplication in the second component. This means that the $L$-valued point $(\mathcal{G}'_L, \sigma_L)$ is given by an $A_R$-morphism $Spec L \rightarrow Spec (B \otimes_{A_R} A_R)$. In other words we can describe this morphism as follows. The Frobenius $\sigma := id_A \otimes \sigma_R : A_R \rightarrow A_R$ induces of course a morphism of the fraction field $L$ which we denote again by $\sigma : L \rightarrow L$, $\frac{a}{b} \mapsto \frac{\sigma(a)}{\sigma(b)}$ for $a, b \in A_R$. It is not the absolute Frobenius. Now the composition $\sigma \circ s^\ast_L : B \rightarrow L$ is not an $A_R$-linear morphism, but it induces a unique $A_R$-linear morphism $\sigma \circ s^\ast_L : B \otimes_{A_R} A_R \rightarrow L$ making the following diagram commutative.

\[
\begin{array}{c}
b \\
\downarrow \\
b \otimes 1
\end{array}
\begin{array}{c}
B \\
\downarrow \\
B \otimes_{A_R, \sigma} A_R
\end{array}
\begin{array}{c}
s^\ast_L \\
\downarrow \\
s^\ast_L
\end{array}
\begin{array}{c}
L \\
\downarrow \\
L
\end{array}
\]

This morphism $\sigma \circ s^\ast_L$ is the one coming from the tuple $(\mathcal{G}'_L, \sigma L)$. The $\mathcal{G}'$-shutuka $\mathcal{G}'$ is defined over $R$. In particular the restriction of $\mathcal{G}'$ to $Spec A_R$ is an isomorphism $\mathcal{G}'|_{A_R} \rightarrow \mathcal{G}'|_{A_R}$ that induces an isomorphism $\tau_{\mathcal{G}'}$ of $A_R$-algebras

\[
\tau_{\mathcal{G}'} : Spec (B \otimes_{A_R, \sigma} A_R) \rightarrow Spec B.
\]
It sends a $T$-valued point $(\mathcal{E}_0, \delta)$ with $\delta : f_0, \mathcal{E}_0 \rightarrow \sigma^* \mathcal{G}'$ to $(\mathcal{E}_0, \tau_{\mathcal{G}} \circ \delta)$. We then would like to know, that the following diagram

$$
\begin{array}{ccc}
B & \xrightarrow{s'} & B \otimes_{A_R, \sigma} A_R \\
\downarrow & & \downarrow \\
L & \xrightarrow{\tau} & L
\end{array}
$$

of $A_R$-morphisms commutes, which can be seen as follows. By assumption (see diagram (10)) the diagram

$$
\begin{array}{ccc}
f_* \sigma^* \mathcal{G}_L & \xrightarrow{\sigma^* \alpha_L} & \sigma^* \mathcal{G}'_L \\
\downarrow \tau_{\mathcal{G}} & & \downarrow \\
f_* \mathcal{G}_L & \xrightarrow{\alpha_L} & \mathcal{G}'_L
\end{array}
$$

is a commutative diagram of isomorphisms of $\mathcal{G}'$-torsors over $L$. (Actually the whole diagram is already defined over $A_K$ and the vertical arrow on the right is even defined over $A_R$.) Now $\sigma^* s_L$ was corresponding to $(\sigma^* \mathcal{G}_L, \sigma^* \alpha_L)$, so that by the description of the morphism (12) the composition $\tau_{\mathcal{G}} \circ \sigma^* \alpha_L$ corresponds to the $L$-valued point $(\sigma^* \mathcal{G}_L, \tau_{\mathcal{G}} \circ \sigma^* \alpha_L)$ of $\text{Spec } B$. This point in the fiber category $(\text{Spec } B)(L)$ is by $\tau_{\mathcal{G}}^{-1}$ isomorphic to $(\mathcal{G}_L, \tau_{\mathcal{G}} \circ \sigma^* \alpha_L \circ f_1, \tau_{\mathcal{G}}^{-1})$, which is by diagram (14) equal to $(\mathcal{G}_L, \alpha_L)$. Since $(\mathcal{G}_L, \alpha_L)$ is exactly the $L$-valued point $s_L$ the commutativity of diagram (13) follows.

Now we choose a closed point $v \in C \setminus S$. Then we can consider the associated étale local $\overline{G}$-shtuka $L_v(\mathcal{G}') = (L^v_v(\mathcal{G}'), \tau^v_v := L_v(\tau_{\mathcal{G}}))$ over $R$, which arises from the formal $\overline{G}$-torsor $\mathcal{G} \times_{C_R} \text{Spf } A_v, R$ as described in [2.10]. Since $R$ is strictly henselian the $L_v^\mathcal{G}$-torsor $L^v_v(\mathcal{G}')$ is trivial so that we choose a trivialization $\beta : L^v_v(\mathcal{G}') \xrightarrow{\approx} L_v^\mathcal{G}_v$. In particular the composition $\beta \circ \tau^v_v \circ \sigma^* \beta^{-1} : L^v_v(\mathcal{G}') \xrightarrow{\approx} L_v^\mathcal{G}_v$ is given by an element $b \in L_v^\mathcal{G}_v(R)$ so that $\beta : L_v(\mathcal{G}') \xrightarrow{\approx} (L_v^\mathcal{G}_v, b)$.

We define $R_i := R/\pi^i$ and $b_i \in L_v^\mathcal{G}_v(R_i)$ as the image of the projection of $b$ under the map $L^\mathcal{G}_v(R_i) \rightarrow L_v^\mathcal{G}_v(R_i), b \mapsto b_i$. Since $R_0 = \kappa_R$ is algebraically closed, there exists by [AH14, Corollary 2.9] a $c_0 \in L_v^\mathcal{G}_v$ with $b_0 = b \circ \sigma \circ c_0$. Note that $\sigma^* c_0 \in L_v^\mathcal{G}_v(R_i)$. We set inductively $c_i := b_i \circ \sigma \circ c_{i-1}$ for $i > 1$ and $c := \lim_{i \rightarrow \infty} c_i = b \circ \sigma \circ c$. Replacing the trivialization $\beta$ by $c^{-1} \circ \beta$ gives therefore an isomorphism of local $\overline{G}$-shtuka $c^{-1} : L_v(\mathcal{G}') \xrightarrow{\approx} (L_v^\mathcal{G}_v, id)$ as becomes clear from the diagram

$$
\begin{array}{ccc}
s^* L^v_v(\mathcal{G}') & \xrightarrow{\sigma^* \beta} & L^v_v(\mathcal{G}) \\
\downarrow & & \downarrow \\
\tau^v_v & \xrightarrow{\beta} & \tau^v_v
\end{array}
$$

Let $A_{v,R} := A_v \otimes_{\mathcal{O}_S} R$ and $\Gamma(A, \mathcal{G}'/\mathcal{G})$ the ring of sections of $\mathcal{G}'/\mathcal{G}$ over $\text{Spec } A$. The trivializations $c^{-1} \circ \beta : L^v_v(\mathcal{G}') \rightarrow L_v^\mathcal{G}_v$ and $\sigma^* (c^{-1} \circ \beta) : \sigma^* L^v_v(\mathcal{G}') \rightarrow L_v^\mathcal{G}_v$ and the isomorphism $\tau_{\mathcal{G}}$ induces after passing to the $v$-adic completion morphisms $c^{-1} \beta$, $\sigma^* (c^{-1} \beta)$ and $\tau_{\mathcal{G}}$ as in the following diagram

$$
\begin{array}{ccc}
\Gamma(A, \mathcal{G}'/\mathcal{G}) \otimes_A A_{v,R} & \xrightarrow{c^{-1} \beta} & B \otimes_{A_R} A_{v,R} \\
\downarrow & & \downarrow \\
\Gamma(A, \mathcal{G}'/\mathcal{G}) \otimes_A A_{v,R} & \xrightarrow{\sigma^* (c^{-1} \beta)} & B \otimes_{A_R, \sigma} A_{v,R}
\end{array}
$$

The right hand side of the diagram arises as the $v$-adic completion of the diagram (13), where $s^v_L$ and $\sigma^* s^v_L$ denote the induced morphism of the completion. Since $\text{ord}_v(\sigma(x)) = q \cdot \text{ord}_v(x)$ for all $x \in L_v^v$ the diagram (15) implies

$$
\text{ord}_v(s^v_L \circ c^{-1} \beta(y)) = \text{ord}_v(\sigma^* (s^v_L \circ c^{-1} \beta)(y)) = q \cdot \text{ord}_v(s^v_L \circ c^{-1} \beta(y))
$$

31
for \( g \in \Gamma(A, \mathcal{G}/\mathcal{G}) \). This means that \( \text{ord}_e(s_L^* \circ \varphi^{-1}(\beta)(y)) \) equals 0 or \( \infty \). In particular we have

\[
s_L^* \circ \varphi^{-1}(\beta) : \Gamma(A, \mathcal{G}/\mathcal{G}) \otimes_A A_{v,R} \to \{ x \in \mathcal{L}^v | \text{ord}_a(x) \geq 0 \}
\]

which implies

\[
B \otimes_A A_{v,R} \xrightarrow{s_L^*} \{ x \in \mathcal{L}^v | \text{ord}_a(x) \geq 0 \}
\]

This finishes the first step.

**Step 2** As we have described above, the proof of the first step gives us a \( \mathcal{G} \)-torsor \( \mathcal{E} \) over \( (C \setminus S)_R \) with \( j^* \mathcal{E} = \mathcal{G}(\mathcal{C}(S)) \) and an isomorphism \( \alpha_R : f_! \mathcal{E} \xrightarrow{\sim} \mathcal{G}(\mathcal{C}(S)) \). We now show that \( \mathcal{E} \times_{\mathcal{G}(\mathcal{C}(S))} \mathcal{G}(\mathcal{C}(S)) \) extends to a \( \mathcal{G} \)-torsor \( \mathcal{E} \) over \( C_{\text{alg}} \) with \( \mathcal{E} \times_{\mathcal{C}_{\text{alg}}} \mathcal{G}(\mathcal{C}(S)) \). Now the field \( \tilde{L} \) is a torsor \( \mathcal{E} \) over \( L \). Therefore we can choose a finite extension \( K'/K \) and a trivialization \( \tilde{\tau} : \mathcal{E} \rightarrow \mathcal{G}(L) \), where \( L' \) is the integral closure in \( K' \). We recall that we denoted by \( \tau \) a uniformizer of \( C \) at \( v \), so that \( A_v \otimes R' = \mathcal{O}(z_\tau) \) (do not confuse \( A_v \) with \( A \)).

**Step 3** We need to show that the \( \mathcal{G} \)-torsor \( \mathcal{E} \) over \( C_{\text{alg}} \) is a part of a global \( \mathcal{G} \)-shtuka \( \mathcal{E} = (E, r_1, \ldots, r_n, \tau_E) \) defining the dashed arrow in diagram (10). The condition that \( \alpha_{R_{\text{alg}}} : f_! \mathcal{E} \xrightarrow{\sim} \mathcal{G}(\mathcal{C}(S)) \) needs to be an isomorphism of global \( \mathcal{G} \)-shtukas defines \( r_i \) by \( r_i = s'_i \circ r_i \) for all \( 1 \leq i \leq n \). So we have to construct \( \tau_E \).

From the proof of the first step we get the commutative diagram

\[
\begin{array}{ccc}
(C \setminus S)_R & \xrightarrow{\sigma} & \mathcal{G}(\mathcal{C}(S))_R \\
\downarrow{s_L} & & \downarrow{s_R} \\
\sigma^* \mathcal{G}'/\mathcal{G}_R \times_{\mathcal{C}_R} (C \setminus S)_R & \xrightarrow{\tau} & \mathcal{G}'/\mathcal{G}_R \times_{\mathcal{C}_R} (C \setminus S)_R
\end{array}
\]

(17)
We defined \((\mathcal{E}, \mathcal{O}_R)\) with \(\mathcal{O}_R : f, \mathcal{E} \to G'\) to be the \((C|S)_R\)-valued point in \(G'/G_R\) corresponding to \(s_R\). Hence \((\sigma^* \mathcal{E}, \sigma^* \mathcal{O}_R)\) corresponds to \(\sigma^* s_R\) and the composition \(\tau_{G'} \circ \sigma^* s_R\) corresponds to the tuple \((\sigma^* \mathcal{E}, \sigma^* \mathcal{O}_R)\). The commutativity of (17) means that \((\sigma^* \mathcal{E}, \sigma^* \mathcal{O}_R)\) and \((\mathcal{E}, \mathcal{O}_R)\) are isomorphic as \((C|S)_R\)-valued points in \(G'/G_R\). This gives us therefore an isomorphism \(\tau_G : \sigma^* \mathcal{E} \to \mathcal{E}\) of \(G\)-torsors over \((C|S)_R\) satisfying \(\tau_G \circ \sigma^* \mathcal{O}_R = \mathcal{O}_R \circ f, \tau_G\). This defines the isomorphism \(\tau_G\) restricted to \((C|S)_R^{alg}\) by \(\tau_G|_{(C|S)_R^{alg}} = \tau_G \times \text{id}_{R^{alg}}\) and we have to extend it to \(C_{R^{alg}}\setminus \Gamma_{r_i}\). We know additionally by \(\mathcal{E}_{C_{R^{alg}}} = G\) and \(\alpha : f, \mathcal{G} \rightarrow f, \mathcal{G}_{C_{R^{alg}}}\) that \(\tau_G\) extends to \(C_{R^{alg}}\setminus \Gamma_{r_i}\). Therefore we only have to extend \(\tau_G\) at finitely many closed points of \(C_{R^{alg}}\setminus \Gamma_{r_i}\). This works similar as at the end of the proof of proposition 3.27. So for \(p \in C_{R^{alg}}\setminus \Gamma_{r_i}\) we choose an open neighborhood \(V \subset C_{R^{alg}}\) with \((V \times_{R^{alg}} R^{alg}) \cap ((\Gamma_{r_i}) \cup (C|S)_{R^{alg}}) = p\). We write \(\tilde{V} := V\setminus p\) so that \(\tau_G\) is defined on \(\tilde{V}_{R^{alg}}\) and need to be extended to \(V_{R^{alg}}\). Moreover the \(G\)-torsors \(\sigma^* \mathcal{E}|_{V_{R^{alg}}}\) and \(\mathcal{E}|_{V_{R^{alg}}}\) are \(R^{alg}\)-valued points in \(\mathcal{H}^1(V, G)\). \(\mathcal{H}^1(V, G)\) is an isomorphism in \(\mathcal{H}^1(\tilde{V}, G)\). Thanks again to Beauville and Laszlo (§ 2.9) the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{H}^1(V, G) & \longrightarrow & \mathcal{H}^1(\tilde{V}, G) \\
\downarrow L^p & & \downarrow L^p \\
\mathcal{H}^1(\kappa R, L^+ \mathcal{G}_p) & \longrightarrow & \mathcal{H}^1(\kappa R, L^+ \tilde{\mathcal{G}}_p)
\end{array}
\]

makes it sufficient to show that the isomorphism \(L^p(\tau_G) : L^p(\sigma^* \mathcal{E}) \to L^p(\mathcal{E})\) in \(\mathcal{H}^1(\kappa R, L^+ \mathcal{G}_p)\) comes from an isomorphism in \(\mathcal{H}^1(\kappa R, L^+ \tilde{\mathcal{G}}_p)\). After trivializing \(L^p(\mathcal{E})\) the morphism \(L^p(\tau_G)\) is given by an element \(h \in L^+ \mathcal{G}_p(R^{alg})\). Since \(\tau_G\) is already defined on \(V_{R^{alg}}\) the pullback of \(h\) by \(h_K \in L^+ \mathcal{G}_p(K^{alg})\) is already given by an element in \(L^+ \tilde{\mathcal{G}}_p(K^{alg})\). Since \(L^+ \tilde{\mathcal{G}}_p \subset L^+ \mathcal{G}_p\) is a closed subgroup we conclude that \(h\) is already an element in \(L^+ \tilde{\mathcal{G}}_p(R^{alg})\). This shows that \(\tau_G\) extends uniquely to \(V_{R^{alg}}\) and hence to \(C_{R^{alg}}\setminus \Gamma_{r_i}\). Hereby we found the \(G\)-shtuka \(\mathcal{E}\) over \(R^{alg}\) defining a unique dashed arrow in the diagram (10), which ends the proof of the theorem.

**Corollary 3.27.** Let \(G\) be a parahoric Bruhat-Tits group scheme and \((id_C,f) : (C,G,H) \to (C,G',w,Z,H')\) be a morphism of shtuka data, where \(f : G \to G'\) is a closed immersion of smooth, affine group schemes over \(C\). Then the induced morphism

\[
f_* : \nabla^{Z,H}_n \mathcal{H}^1(C, G) \to \nabla^{Z',H'}_n \mathcal{H}^1(C, G')
\]

is finite.

**Proof:** The proof of this corollary works literally in the same way as the proof of corollary 3.24 with replacing unramified by finite.

**Remark 3.28.** The results of this section can potentially be used to formulate and prove an analog of the Andr\'e-Oort conjecture for global \(G\)-shtukas. To formulate such a conjecture one needs the notion of special points and special subvarieties. In the case of Drinfeld modular curves an analog of the Andr\'e-Oort conjecture has been formulated and proved in [Bre05]. Later the notion of special subvarieties and the formulation of the Andr\'e Oort conjecture was generalized in [Bre12] to the higher dimensional Drinfeld modular varieties. In the same paper this Andr\'e-Oort conjecture was proven in some special cases. These results were extended in [Hub13]. To define Drinfeld modular varieties, one fixes a point \(\infty \in C\) so that \(C \setminus \infty = Spec A\) is affine and \(M^{\lambda}_A\) is the moduli space for Drinfeld \(A\)-modules of rank \(r\). Now for certain finite extensions \(A' \subset A\) coming from a morphism \(C \to C'\) of curves, Breuer shows that there is a proper morphism \(M^{\lambda}_A \to M^{\lambda}_{A'}\) of moduli spaces which is also finite by [HH06]. Then Breuer uses the image of this morphism to define special subvarieties. Drinfelds modular variety \(M^{\lambda}_A\) can be embedded into \(\nabla^{Z}_2 \mathcal{H}^1(C, GL_r)\) for \(n = 2\) and some specific choose bound \(Z_w\). The morphism \(M^{\lambda}_A \to M^{\lambda}_{A'}\) then corresponds to a morphism \(\nabla^{Z}_2 \mathcal{H}^1(C, GL_r) \to \nabla^{Z'}_2 \mathcal{H}^1(C', GL_r|_{C'C'})\) coming from a morphism of shtuka data \((C, GL_r, Z_w) \to (C', GL_r|_{C'C'}, Z_w)\). So extending the coefficients for Drinfeld modules generalizes to changing the curve for global \(G\)-shtukas.
as in subsection 3.2 since we are not restricted to choose $n = 2$, $G = \text{GL}_r$ or some specific bound. Moreover we have seen that additionally to changing the curve, we can also change the group scheme $\mathcal{G}$ as in subsection 3.3. Although we do not know if this is precisely the correct definition it is conceivable to define a special subvariety of $\mathcal{H}^n_{\mathcal{G}}(C', G')$ to be the image of the morphism

$$f_\ast \circ \pi_\ast : \mathcal{H}^n_{\mathcal{G}}(C, \mathcal{G}) \to \mathcal{H}^n_{\mathcal{G}}(C', G')$$

arising from a morphism $(\pi, f)$ of shtuka data, where $f : \pi : \mathcal{G} \to G'$ is a closed immersion of (Bruhat-Tits) group schemes. Special points in $\mathcal{H}^n_{\mathcal{G}}(C', G')$ would then be defined to be those points which arise in the image of a morphism $\tilde{f}_\ast : \mathcal{H}^n_{\mathcal{G}}(C', T) \to \mathcal{H}^n_{\mathcal{G}}(C', G')$, where $\tilde{f} : T \to G'$ is a closed (Bruhat-Tits) group scheme that is generically a torus in $G'$.

Following this, an André-Oort conjecture for global $G$-shtukas would then say that given a set $S$ of special points, the Zariski closure of these points is a finite union of special subvarieties. Again, this is not a precise formulation but should give an impression of the flavor of a possible statement.

References

[AH13] Esmail Arasteh Rad and Urs Hartl. “Uniformizing The Moduli Stacks of Global $\mathcal{G}$-Shtukas”. In: ArXiv e-prints (Feb. 2013). arXiv:1302.6351 [math.NT].

[AH14] Esmail Arasteh Rad and Urs Hartl. “Local $P$-shtukas and their relation to global $\mathcal{G}$-shtukas”. In: Münster J. Math. 7.2 (2014), pp. 623–670. ISSN: 1867-5778.

[Alp14] Jarod Alper. “Adequate moduli spaces and geometrically reductive group schemes”. In: Algebra. Geom. 1.4 (2014), pp. 489–531. ISSN: 2214-2584. DOI: 10.14231/AG-2014-022. URL: https://doi.org/10.14231/AG-2014-022

[Ans18] J. Anschütz. “Extending torsors on the punctured Spec($A_{inf}$)”. In: ArXiv e-prints (Apr. 2018). arXiv:1804.06356 [math.NT].

[BL95] Arnaud Beauville and Yves Laszlo. “Un lemme de descente”. In: C. R. Acad. Sci. Paris Sér. I Math. 320.3 (1995), pp. 335–340. ISSN: 0764-4442.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Néron models. Vol. 21. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990, pp. x+325. ISBN: 3-540-50587-3. URL: https://doi.org/10.1007/978-3-642-51438-8

[Bre05] Florian Breuer. “The André-Oort conjecture for products of Drinfeld modular curves”. In: J. Reine Angew. Math. 579 (2005), pp. 115–144. ISSN: 0075-4102. DOI: 10.1515/crll.2005.2005.579.115. URL: https://doi.org/10.1515/crll.2005.2005.579.115.

[Bre12] Florian Breuer. “Special subvarieties of Drinfeld modular varieties”. In: J. Reine Angew. Math. 668 (2012), pp. 35–57. ISSN: 0075-4102.

[Bre19] Paul Breutmann. “Stratifications of Moduli Spaces of Global $G$-Shtukas”. In: in preparation (2019).

[BS68] A. Borel and T. A. Springer. “Rationality properties of linear algebraic groups. II”. In: Tôhoku Math. J. (2) 20 (1968), pp. 443–497. ISSN: 0040-8735. URL: https://doi.org/10.2748/tmj/1178243073

[BT84] F. Bruhat and J. Tits. “Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une domaîne radicelle valuée”. In: Inst. Hautes Études Sci. Publ. Math. 60 (1984), pp. 197–376. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1984__60__5_0

[CGP10] Brian Conrad, Ofer Gabber, and Gopal Prasad. Pseudo-reductive groups. Vol. 17. New Mathematical Monographs. Cambridge University Press, Cambridge, 2010, pp. xx+533. ISBN: 978-0-521-19560-7. URL: https://doi.org/10.1017/CBO9780511661143

[Con12] Brian Conrad. “Weil and Grothendieck approaches to adelic points”. In: Enseign. Math. (2) 58.1-2 (2012), pp. 61–97. ISSN: 0013-8584. URL: https://doi.org/10.4171/LEM/58-1-3
A. Grothendieck. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II”. In: *Inst. Hautes Études Sci. Publ. Math.* 24 (1965), p. 231. issn: 0073-8301. url: [http://www.numdam.org/item?id=PMIHES_1965__24__231_0](http://www.numdam.org/item?id=PMIHES_1965__24__231_0).

A. Grothendieck. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III”. In: *Inst. Hautes Études Sci. Publ. Math.* 28 (1966), p. 255. issn: 0073-8301. url: [http://www.numdam.org/item?id=PMIHES_1966__28__255_0](http://www.numdam.org/item?id=PMIHES_1966__28__255_0).

A. Grothendieck. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV”. In: *Inst. Hautes Études Sci. Publ. Math.* 32 (1967), p. 361. issn: 0073-8301. url: [http://www.numdam.org/item?id=PMIHES_1967__32__361_0](http://www.numdam.org/item?id=PMIHES_1967__32__361_0).

Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I: Schemes with examples and exercises*. Vieweg + Teubner, Wiesbaden, 2010, pp. viii+615. isbn: 978-3-8348-0676-5. doi: 10.1007/978-3-8348-9722-0. url: [https://doi.org/10.1007/978-3-8348-9722-0](https://doi.org/10.1007/978-3-8348-9722-0).

Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. isbn: 0-387-90244-9.

Jochen Heinloth. “Uniformization of $G$-bundles”. In: *Math. Ann.* 347.3 (2010), pp. 499–528. issn: 0025-5831. doi: 10.1007/s00208-009-0443-4. url: [https://doi.org/10.1007/s00208-009-0443-4](https://doi.org/10.1007/s00208-009-0443-4).

Urs Hartl and Markus Hendler. “Change of Coefficients for Drinfeld Modules, Shtuka, and Abelian Sheaves”. In: *arXiv Mathematics e-prints*, math/0608256 (2006), math/0608256. arXiv: math/0608256 [math.NT].

Patrik Hubschmid. “The André-Oort conjecture for Drinfeld modular varieties”. In: *Compos. Math.* 149.4 (2013), pp. 507–567. issn: 0010-437X. doi: 10.1112/S0010437X12000681. url: [https://doi.org/10.1112/S0010437X12000681](https://doi.org/10.1112/S0010437X12000681).

Urs Hartl and Eva Viehmann. “The Newton stratification on deformations of local $G$-shtukas”. In: *J. Reine Angew. Math.* 656 (2011), pp. 87–129. issn: 0075-4102. doi: 10.1515/CRELLE.2011.044. url: [https://doi.org/10.1515/CRELLE.2011.044](https://doi.org/10.1515/CRELLE.2011.044).

V. Lafforgue. “Chotoucas pour les groupes réductifs et paramétrisation de Langlands globale”. In: *ArXiv e-prints* (Sept. 2012). arXiv: 1209.5352 [math.AG].

V. Lafforgue. “Introduction aux chotoucas pour les groupes réductifs et à la paramétrisation de Langlands globale”. In: *ArXiv e-prints* (Apr. 2014). arXiv: 1404.3998 [math.AG].

Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000, pp. xii+208. isbn: 3-540-65761-4.

G. Pappas and M. Rapoport. “Twisted loop groups and their affine flag varieties”. In: *Adv. Math.* 219.1 (2008). With an appendix by T. Haines and Rapoport, pp. 118–198. issn: 0001-8708. doi: 10.1016/j.aim.2008.04.006. url: [https://doi.org/10.1016/j.aim.2008.04.006](https://doi.org/10.1016/j.aim.2008.04.006).

Timo Richarz. “Affine Grassmannians and geometric Satake equivalences”. In: *Int. Math. Res. Not. IMRN* 12 (2016), pp. 3717–3767. issn: 1073-7928. doi: 10.1093/imrn/rnv226. url: [https://doi.org/10.1093/imrn/rnv226](https://doi.org/10.1093/imrn/rnv226).

Jean-Pierre Serre. *Cohomologie galoisienne*. Fifth. Vol. 5. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1994, pp. x+181. isbn: 3-540-58002-6. doi: 10.1007/BFb0108758. url: [https://doi.org/10.1007/BFb0108758](https://doi.org/10.1007/BFb0108758).

SGA3. *Schémas en groupes. I: Propriétés générales des schémas en groupes*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151. Springer-Verlag, Berlin-New York, 1970, pp. xv+564.

Toro Wedhorn. “On Tannakian duality over valuation rings”. In: *J. Algebra* 282.2 (2004), pp. 575–609. isbn: 0021-8693. doi: 10.1016/j.jalgebra.2004.07.024. url: [https://doi.org/10.1016/j.jalgebra.2004.07.024](https://doi.org/10.1016/j.jalgebra.2004.07.024).