Automorphism Groups of Cyclic $p$-gonal Pseudo-real Riemann Surfaces

Emilio Bujalance*
Departamento de Matemáticas Fundamentales
Facultad de Ciencias, UNED
Senda del rey, 9, 28040 Madrid (Spain)
eb@mat.uned.es

Antonio F. Costa*
Departamento de Matemáticas Fundamentales
Facultad de Ciencias, UNED
Senda del rey, 9, 28040 Madrid (Spain)
acosta@mat.uned.es

Abstract. In this article we prove that the full automorphism group of a cyclic $p$-gonal pseudo-real Riemann surface of genus $g$ is either a semidirect product $C_n \rtimes C_p$ or a cyclic group, where $p$ is a prime $> 2$ and $g > (p - 1)^2$. We obtain necessary and sufficient conditions for the existence of a cyclic $p$-gonal pseudo-real Riemann surface with full automorphism group isomorphic to a given finite group. Finally we describe some families of cyclic $p$-gonal pseudo-real Riemann surfaces where the order of the full automorphism group is maximal and show that such families determine some real 2-manifolds embedded in the branch locus of moduli space.

Submitted 7/07/14. This is a revised version 13/03/15.

Mathematics Subject Classification (2000): 30F10, 14H55, 30F50

Keywords: Riemann surface, Automorphism group, $p$-gonal Riemann surface, Moduli space.

1 Introduction

A Riemann surface is called pseudo-real if it admits anticonformal automorphisms but no anticonformal involution. Pseudo-real Riemann surfaces

*Partially supported by MTM2014-55812.
appear in a natural way in the study of the moduli space $\mathcal{M}^K_g$ of Riemann surfaces considered as Klein surfaces. The moduli space $\mathcal{M}_g$ of Riemann surfaces of genus $g$ is a two-fold branch covering of $\mathcal{M}^K_g$ and the preimage of the branch locus consists of the Riemann surfaces admitting anticonformal automorphisms; they are either real Riemann surfaces (admitting anticonformal involutions, see for instance [N]) or pseudo-real Riemann surfaces.

Pseudo-real Riemann surfaces are those Riemann surfaces which are equivalent to their conjugate but the equivalence is not realized by an involution; therefore they admit anticonformal automorphisms of order greater than 2. The square of such automorphisms are not trivial conformal automorphisms, which means that the points in $\mathcal{M}_g$ corresponding to pseudo-real Riemann surfaces are in the singular set (branch locus) of the orbifold $\mathcal{M}_g = T_g/\text{Mod}_g$.

Pseudo-real Riemann surfaces were first studied in [E] and [S]. Hyperelliptic pseudo-real Riemann surfaces were considered in [S3], [BCn], and [BT] and pseudo-real Riemann surfaces with cyclic automorphism groups in [Et]. In [BCC] the existence of pseudo-real Riemann surfaces of every genus $g \geq 2$ is established and this is followed by a study of pseudo-real Riemann surfaces of genus 2 and 3; in [BCS] all the topological types of actions of the automorphisms groups for these two genera and genus 4 are described. In [H1] and [H2] the author finds explicit equations for non-hyperelliptic pseudo-real Riemann surfaces.

The $p$-gonal surfaces are cyclic $p$-fold coverings of the Riemann sphere and there is a great deal of interest in the study of the automorphism groups of these surfaces (see for instance [B], [BCI], [BHS], [Ko], [W]). The groups of (conformal and anticonformal) automorphisms of $p$-gonal real Riemann surfaces are obtained in [BCI], under the assumption that the $p$-gonal morphism is normal; for the conformal automorphisms in the non-normal case see [W].

In the present work, we study the full groups of (conformal and anticonformal) automorphisms of pseudo-real Riemann surfaces of genus $g$ that are cyclic $p$-gonal, where $p$ is a prime $> 2$ and $g > (p-1)^2$. In these conditions we establish that there are only two possible types of full automorphism groups of cyclic $p$-gonal pseudo-real Riemann surfaces: they are either cyclic or semidirect product of cyclic groups $C_n \rtimes C_p$ (Theorem 4). We also obtain necessary and sufficient conditions for the existence of cyclic $p$-gonal pseudo-real Riemann surfaces with given full automorphism group.

In section 4 we obtain the maximal order of the automorphism groups of pseudo-real cyclic $p$-gonal Riemann surfaces of genus $g > (p-1)^2$, when $p-1$ divides $g$. 

2
Finally, in section 5, we apply our results to describe some 2-manifolds embedded in the branch locus of moduli space corresponding to cyclic $p$-gonal pseudo-real Riemann surfaces with maximal symmetry.

**Acknowledgements.** We would like to express our thanks to Marston Conder for advice about metacyclic groups. The authors are indebted to the referee for careful reading of this article, many helpful remarks that improve the quality of the exposition and to point out an error in a preliminary version of Theorem 4.

2 Non-Euclidean crystallographic groups and Pseudo-real Riemann surfaces

A non-Euclidean crystallographic group (or NEC group) is a discrete group of isometries of the hyperbolic plane $\mathbb{D}$ (we consider the unit disc model). We shall assume that an NEC group has compact orbit space. If $\Gamma$ is such a group, its algebraic structure is determined by its signature

$$(h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

(1)

The orbit space $\mathbb{D}/\Gamma$ is a surface, possibly with boundary. The number $h$ is called the genus of $\Gamma$ and equals the topological genus of $\mathbb{D}/\Gamma$, while $k$ is the number of its boundary components; the sign is $+$ or $-$ according to whether or not the surface $\mathbb{D}/\Gamma$ is orientable. The integers $m_i \geq 2$ are called the proper periods, and represent the branch indices over interior points of $\mathbb{D}/\Gamma$ in the natural projection $\pi: \mathbb{D} \to \mathbb{D}/\Gamma$. The bracketed expressions $(n_{i1}, \ldots, n_{is_i})$, some or all of them may be empty (with $s_i = 0$), are called period cycles and represent the branchings over the $i$th hole in the surface, and the numbers $n_{ij} \geq 2$ are named link periods.

There exists, associated with each signature, a canonical presentation for the group $\Gamma$, and a formula for the hyperbolic area of its fundamental domain (see [BEGG]). If the signature has sign $+$ then $\Gamma$ has the following generators:

$$x_1, \ldots, x_r$$ (elliptic transformations),
$$c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k}$$ (reflections),
$$e_1, \ldots, e_k$$ (boundary transformations),
$$a_1, b_1, \ldots, a_g, b_g$$ (hyperbolic transformations);

and these generators satisfy the relations

$$x_i^{m_i} = 1 \quad (\text{for } 1 \leq i \leq r),$$
$$c_i^2 = 1, \quad c_i^{-1}c_{ij}c_i = 1 \quad (\text{for } 1 \leq i \leq k, 0 \leq j \leq s_i),$$

$$c_i^{-1}c_{ij}c_i = 1 \quad (\text{for } 1 \leq i \leq k, 0 \leq j \leq s_i).$$

3
If the sign is $-$ then we just replace the hyperbolic generators $a_i, b_i$ by glide reflections $d_1, \ldots, d_h$, and the long relation by $x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2 = 1$.

The hyperbolic area of an arbitrary fundamental region of an NEC group $\Gamma$ with signature $(1)$ is

$$\mu(\Gamma) = 2\pi \left( \varepsilon h - 2 + k + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left( 1 - \frac{1}{n_{ij}} \right) \right)$$

where $\varepsilon = 2$ if the sign is $+$, and $\varepsilon = 1$ if the sign is $-$. Furthermore, any discrete group $\Lambda$ of isometries of $\mathbb{H}$ containing $\Gamma$ as a subgroup of finite index is also an NEC group, and the hyperbolic area of a fundamental region for $\Lambda$ is given by the Riemann-Hurwitz formula:

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda).$$

For any NEC group $\Lambda$, the canonical Fuchsian subgroup $\Lambda^+$ is its subgroup of orientation-preserving elements. If $\Lambda^+ \neq \Lambda$ then $\Lambda^+$ has index 2 in $\Lambda$, and we say that $\Lambda$ is a proper NEC group.

Let $X$ be a compact Riemann surface of genus $g > 1$. Then there is a surface Fuchsian group $\Gamma$, which is an NEC group with signature $(g; +; [-]; \{ - \})$, such that $X = \mathbb{D}/\Gamma$, and the full (conformal and anticonformal) automorphism group $\text{Aut}(X)$ of $X$ is isomorphic to $\Delta/\Gamma$, where $\Delta$ is an NEC group normalizing $\Gamma$. We denote by $\text{Aut}^+(X)$ the group $\Delta^+ / \Gamma$ of all conformal automorphisms of $X$.

**Definition 1** A Riemann surface is called pseudo-real if it admits anticonformal automorphisms but no anticonformal involution.

In [BCC] we have established some basic results on the automorphism groups of pseudo-real Riemann surfaces

**Proposition 2** [BCC] Let $X$ be a pseudo-real Riemann surface, and let $G$ be the full automorphism group of $X$. Then $4$ divides the order of $G$.

**Proposition 3** [BCC] Suppose the pseudo-real surface $X$ is conformally equivalent to $\mathbb{D}/\Gamma$, where $\Gamma$ is a surface Fuchsian group which is normalized by an NEC group $\Delta$ such that $\Delta/\Gamma \cong G = \text{Aut}(X)$. Then the signature of $\Delta$ has the form $(h; +; [m_1, \ldots, m_r])$ and the signature of the canonical Fuchsian subgroup $\Delta^+$ of $\Delta$ is

$$(h-1; +; [m_1, m_1, m_2, m_2, \ldots, m_r, m_r]).$$
3 Automorphism groups of cyclic $p$-gonal pseudo-real Riemann surfaces

In this section we establish our main theorem on the algebraic structure of the full automorphism groups of cyclic $p$-gonal pseudo-real Riemann surfaces:

**Theorem 4** Let $X$ be a cyclic $p$-gonal ($p$ is a prime $>2$) pseudo-real Riemann surface of genus $g > (p-1)^2$. Let $G$ be the full automorphism group of $X$, $H \cong C_p$ be the subgroup of $G$ generated by a $p$-gonality automorphism and $\Delta$ be an NEC group uniformizing $X/G$. Hence the genus $g$ of $X$ must be even and the possible isomorphy types for the group $G$ are:

1. $G \cong C_n \rtimes rH$, where $4$ divides $n$, the first factor is generated by an anticonformal automorphism and

   $C_n \rtimes_r H = \langle x, y : x^p = 1; y^n = 1; y^{-1}xy = x^r \rangle$ (2)

   where $0 < r < p$, $r^n \equiv 1 \mod p$. The NEC group $\Delta$ has signature either:

   $\langle 1; -; [p, \frac{2(g+p-1)}{p-1}, p, n/2] \rangle$ (3)

   or, if in the presentation (2) $r = 1$ (direct product) or $r = p-1$, the signature of $\Delta$ may be also:

   $\langle 1; -; [p, \frac{2g}{p-1}, p, \frac{n}{2}] \rangle$ (4)

2. $G \cong C_{np}$ (where $4$ divides $n$) and $G$ is generated by an anticonformal automorphism. The NEC group $\Delta$ has either signature (3) or (4).

**Proof.** Let $\Gamma$ be a surface Fuchsian group uniformizing $X$, i.e. $X = \mathbb{D}/\Gamma$. There is an NEC group $\Delta$ and a Fuchsian group $\Lambda$ such that $X/G \cong \mathbb{D}/\Delta$, $X/H \cong \mathbb{D}/\Lambda$ and

   $\Gamma \triangleleft \Lambda < \Delta; \Gamma \triangleleft \Delta; \Lambda/\Gamma \cong H; \Delta/\Gamma \cong G$

   The signature of $\Lambda$ is

   $(0; +; [p, \frac{2g}{p-1}, p])$

   where $q = \frac{2(g+p-1)p}{p-1}$, and by Proposition 3 the signature of $\Delta$ has the form

   $(h; -; [m_1, ..., m_r])$

   Assume that $n = [\Delta : \Lambda]$. If the genus of $X$ satisfies $g > (p-1)^2$, then the $p$-gonal covering $X \to \hat{\mathbb{C}} = X/H$ is unique, [A], hence $\Lambda \triangleleft \Delta$ and there is
\( \varpi : \Delta \to \Delta/\Gamma \cong G \) such that \( \varpi^{-1}(H) = \varpi^{-1}(\Lambda/\Gamma) = \Lambda \). Changing the indices in the canonical generators \( x_1, \ldots, x_r \) in such a way that \( \varpi(x_i^j) \notin \Lambda/\Gamma \), for any \( 0 < j < m_i, i = t + 1, \ldots, r \), but this condition is not satisfied for \( x_1, \ldots, x_t \), we obtain:

\[
[m_1, \ldots, m_r] = [ps_1, \ldots, ps_t, m_{t+1}, \ldots, m_r].
\]

Given the signature of \( \Lambda \), the fact that \( n = [\Delta : \Lambda] \) implies that the integers \( s_1, \ldots, s_t \) divide to \( n \) and \( q = \sum \frac{n}{s_i} \). Applying Riemann-Hurwitz formula we get:

\[
-2 + q\left(1 - \frac{1}{p}\right) = n
\]

(5)

From the above formula and using \( q = \sum \frac{n}{s_i} \) we obtain:

\[
n(h - 2) = -2 + \left(\sum_{i=1}^{t} \frac{n}{s_i}\right)(1 - \frac{1}{p}) - n\sum_{i=1}^{t} \left(1 - \frac{1}{s_ip}\right) - n\sum_{i=t+1}^{r} \left(1 - \frac{1}{m_i}\right) < \left(\sum_{i=1}^{t} \frac{n}{s_i}\right)(1 - \frac{1}{p}) - n\sum_{i=1}^{t} \left(1 - \frac{1}{s_ip}\right) = n\sum_{i=1}^{t} \left(\frac{1}{s_i} - 1\right) \leq 0
\]

Thus \( h = 1 \) (note that \( h > 0 \), since the sign in the signature of \( \Delta \) is \(-\)).

Now formula (5) is equivalent to:

\[
2 - n + n\sum_{i=1}^{t} \left(1 - \frac{1}{s_i}\right) + n\sum_{i=t+1}^{r} \left(1 - \frac{1}{m_i}\right) = 0
\]

where \( s_i \geq 1 \) and \( m_i \geq 2 \). From there it is easy to deduce that there is at most one \( s_i > 1 \). We have two possible cases:

i) \( s_i = 1 \), for all \( i \), in which case we have

\[
\sum_{i=t+1}^{r} \left(1 - \frac{1}{m_i}\right) = 1 - \frac{2}{n},
\]

thus \( r - t = 1 \) and \( m_{t+1} = \frac{n}{2} \).

ii) There is one \( i \) (we may suppose \( i = t \)) such that \( s_t > 1 \), then \( s_t = \frac{n}{2} \), \( s_1 = \ldots = s_{t-1} = 1 \) and \( r = t \).

Therefore the possible signatures for \( \Delta \) are:

\[
i) (1; -; [p, \frac{2(p-1)}{n(p-1)}, p, \frac{n}{2}])
\]

\[
ii) (1; -; [p, \frac{2p}{n(p-1)}, p, \frac{n}{2}p])
\]

(6)
Assume then that $\Delta$ is an NEC group with one of the signatures in (6) and let

$$l = \frac{2(g + p - 1)}{n(p - 1)}$$

in signature i) and

$$l = \frac{2g}{n(p - 1)}$$

in signature ii).

We shall study the epimorphism

$$\theta : \Delta \to \Delta/\Lambda$$

Let $d, x_1, ..., x_{l+1}$ be the generators of a canonical presentation of $\Delta$. If $\pi : \Delta/\Gamma \to \Delta/\Lambda$ is the natural projection, then $\pi \circ \varpi = \theta$, and since $\varpi(x_i) \in \Lambda/\Gamma, i = 1, ..., l$, we have that $\theta(x_1) = ... = \theta(x_l) = 1$, so the group $\Delta/\Lambda$ is generated by $\theta(d)$ and $\theta(x_{l+1}) (= \theta(d)^{-2}$ by the long relation of NEC groups). Therefore the group $\Delta/\Lambda$ is cyclic of order $n$ and $G \cong \Delta/\Gamma$ is a normal extension of a cyclic group $C_p \cong \Lambda/\Gamma$ by a cyclic group $C_n \cong \Delta/\Lambda$:

$$0 \to C_p \to G \xrightarrow{\pi} C_n \to 0$$

The covering $X/H \cong \mathbb{D}/\Lambda \to X/G \cong \mathbb{D}/\Delta$ is a cyclic covering with automorphism group $C_n \cong \Delta/\Lambda = \langle \theta(d) \rangle = \langle \tilde{y} \rangle$. The lift $y \in G$ of the automorphism $\tilde{y}$ to $X = \mathbb{D}/\Gamma$ must have order $n$ or $np$. In the first case $G$ is a semidirect product $C_n \rtimes_r H$ with presentation (2) and in the second case $G$ is cyclic $(\cong C_{np})$.

Suppose the signature of $\Delta$ is ii) of (6).

Now $\varpi(x_{l+1})$ is an element of order $q = \frac{n}{2}p$ in $G$ and $\varpi$ has order either $n$ or $np$.

If the order of $\varpi(d)$ is $np$, then $G \cong C_{np}$.

In the case $\varpi(d)$ has order $n$, $G$ is generated by $x, y$ where $\varpi(d) = y$, $\varpi(x_i) = x$, and the following relations hold:

$$x^p = 1; y^n = 1; y^{-1}xy = x^r$$

Since there is an element of order $\frac{n}{2}p$ in $G$ then $\Delta^+/\Gamma$ is cyclic of order $\frac{n}{2}p$. Since $y^2 \in \Delta^+/\Gamma$, we have $y^{-2}xy^2 = x = x^{r^2}$, and then $r = 1$ or $p - 1$.

In all the cases, the surface $X$ being pseudo-real, we know by Proposition 2 that 4 divides $n$. Finally, either $\frac{2(g + p - 1)}{n(p - 1)}$ or $\frac{2g}{n(p - 1)}$ is an integer, thus $g$ is even. ■
Remarks. 1. Condition \( g > (p - 1)^2 \) in the above theorem may be replaced by the assumption that the cyclic group \( H \) is normal in \( G \). The \( n \)-gonal surfaces whose full automorphism group \( G \) is not the normalizer of \( H \) have been studied in [BW]. [W]

2. Theorem [4] remains valid if \( G \) is a group of automorphisms (not necessarily the full automorphism group) containing a \( p \)-gonal automorphism and an anticonformal automorphism of order \( > 2 \).

**Theorem 5** For each prime \( p > 2 \) and each \( n \), such that \( 4 \) divides \( n \), there exists a cyclic \( p \)-gonal pseudo-real Riemann surface of genus \( g \) with full automorphism group isomorphic to \( C_p \rtimes_r C_n \) with presentation \( [2] \) and \( 1 < r < p - 1 \) if and only if \( \frac{2(a + p - 1)}{n(p - 1)} \) is an integer \( > 1 \).

**Proof.** Assume that \( l = \frac{2(a + p - 1)}{n(p - 1)} \) is an integer \( > 1 \). Let \( \Delta \) be a maximal NEC group with signature \((1; -, -; [p, \ldots, p, n/2])\). We define

\[
\varpi : \Delta \to C_n \rtimes_r C_p = \langle x, y : x^p = 1; y^n = 1; y^{-1}xy = x^r, r^n \equiv 1 \pmod{p} \rangle
\]

by

\[
\varpi(x_{2i+1}) = x, \varpi(x_{2i+2}) = x^{-1}, i = 0, \ldots, (l - 2)/2,
\]

\[
\varpi(x_{l+1}) = y^2, \varpi(d) = y^{-1}, \text{ if } l \text{ is even}
\]

and

\[
\varpi(x_{2i+1}) = x, \varpi(x_{2i+2}) = x^{-1}, i = 0, \ldots, (l - 3)/2,
\]

\[
\varpi(x_{l-2}) = x, \varpi(x_{l-1}) = x, \varpi(x_l) = x^{l-2}
\]

\[
\varpi(x_{l+1}) = y^2, \varpi(d) = y^{-1}, \text{ if } l \text{ is odd, } l > 1
\]

Every element of \( C_n \rtimes_r C_p \) can be expressed as \( x^a y^b \), by using the relation \( y^{-1}xy = x^r \). Note that \( \varpi(\Delta^+) = \varpi(\langle x_1, \ldots, x_l, x_{l+1}, d^2 \rangle = \langle x, y^2 \rangle \), 4 divides \( n \) and \( n/2 \) is an even number. Thus the elements \( x^a y^{n/2} \) of order 2 of the group \( C_n \rtimes_r C_p \), that can be written as \( x^a y^{n/2} \), correspond to orientation preserving automorphisms. The epimorphism \( \varpi \) cannot be extended since \( \Delta \) is maximal and \( \mathbb{D}/\ker \varpi \) is the pseudo-real Riemann surface that we are looking for.

If \( l = 1 \) by Theorem 2.4.7 of [BEGG] the group \( \Delta \) is contained in an NEC group \( \Omega \) with signature \((0; +; [2]; \{(p, n/2)\})\). Let

\[
C_2 \rtimes (C_n \rtimes_r C_p) = \langle s, x, y : s^2 = 1, x^p = 1; y^n = 1; y^{-1}xy = x^r; sx = x^{-1}, sys = y^{-1} \rangle
\]
We may assume that \( \varpi : \Delta \to C_n \rtimes_r C_p \) is given by \( \varpi(d) = y, \varpi(x_1) = x, \varpi(x_2) = y^{-2} \), where \( d, x_1 \) and \( x_2 \) are generators of a canonical presentation of \( \Delta \). The epimorphism \( \varpi \) is the restriction to \( \Delta \) of

\[
\theta^* : \Omega \to C_2 \rtimes (C_n \rtimes C_p)
\]

where \( \alpha \equiv \frac{p+1}{2} r^{n-1} \mod p \). Hence there are anticonformal involutions in \( \text{Aut}(D/\ker \varpi) \) and \( D/\ker \varpi \) is not a pseudo-real Riemann surface. ■

**Theorem 6** For each prime \( p > 2 \) and each \( n \), such that 4 divides \( n \), there exists a cyclic \( p \)-gonal pseudo-real Riemann surface of genus \( g \), with full automorphism group isomorphic to \( C_{np} \) if and only if either \( \frac{2(g+p-1)}{n(p-1)} \) or \( \frac{2g}{n(p-1)} \) are integers \( > 1 \) and \( \gcd(p,n/2) = 1 \).

**Proof.** If either \( \frac{2(g+p-1)}{n(p-1)} \) or \( \frac{2g}{n(p-1)} \) are integers \( > 1 \) with \( \gcd(p,n/2) = 1 \), we may consider maximal NEC groups with signatures either

\[
(1; -; [p, \frac{2(g+p-1)}{n(p-1)}, p, \frac{n}{2}]) \text{ or } (1; -; [p, \frac{2g}{n(p-1)}, p, \frac{n}{2p}])
\]

respectively. The surface Fuchsian groups uniformizing the cyclic \( p \)-gonal pseudo-real Riemann surfaces having cyclic automorphism group are the kernel of the epimorphisms given by Theorem 4 of \([\text{Et}]\).

Assume that there is a cyclic \( p \)-gonal pseudo-real Riemann surface \( X \) of genus \( g \), with full automorphism group \( C_{np} \). By Theorem 4, there is a maximal NEC group \( \Delta \) with signature as given in \([6]\) (then either \( \frac{2(g+p-1)}{n(p-1)} \) or \( \frac{2g}{n(p-1)} \) are integers) and there is an epimorphism \( \varpi : \Delta \to C_{np} \) such that \( \ker \varpi \) uniformizes \( X \). Condition ii) of Theorem 4 of \([\text{Et}]\) implies \( \gcd(p,n/2) = 1 \).

For the first possible signature of \( \Delta \), assume \( \frac{2(g+p-1)}{n(p-1)} = 1 \). The epimorphism \( \varpi : \Delta \to C_{np} = \langle t \rangle \), where \( \Delta \) has signature \((1; -; [p, \frac{2g}{n}])\) is given by \( \varpi(d) = t, \varpi(x_1) = t^{2m}, \varpi(x_2) = t^{-2m-2} \), where \( d, x_1 \) and \( x_2 \) are generators of a canonical presentation of \( \Delta \), and \( 0 < m < p \) (note that \( x_1 \) is an orientation preserving transformation and then \( \varpi(x_1) \) must be an even power of \( t \)). By Theorem 2.4.7 of \([\text{BEGG}]\) the group \( \Delta \) is contained in an NEC group \( \Omega \) with signature \((0; +; [2]; \{(p, \frac{2g}{n})\})\) and the epimorphism \( \varpi \) is the restriction to \( \Delta \) of

\[
\theta^* : \Omega \to D_{np} = \langle s, t : s^2 = t^{np} = 1, sts = t^{-1} \rangle
\]

\[
\theta^*(x_1') = t^{2m+1}, \theta^*(c_0') = t^{2m}s, \theta^*(c_1') = s, \theta^*(c_2') = st^{-2m-2}
\]
where $x'_1, c'_0, c'_1, c'_2$ is a set of generators of a canonical presentation of $\Omega$. Then $\ker \theta^* = \ker \varpi = \Gamma$ and $\text{Aut}^+ (\mathbb{D}/\Gamma)$ contains $D_{np}$ with anticonformal involutions. Hence $\mathbb{D}/\Gamma$ is not pseudo-real. Thus $\frac{2(g+p-1)}{n(p-1)} > 1$.

The argument for the second possible signature of $\Delta$ with $\frac{2g}{n(p-1)} = 1$ is similar. 

**Theorem 7** For each prime $p > 2$ and each $n$, such that 4 divides $n$, there exists a cyclic $p$-gonal pseudo-real Riemann surface of genus $g$ with full automorphism group isomorphic to $C_p \rtimes C_n$ with presentation (2) and $r = 1$ or $p-1$ if and only if either $\frac{2(g+p-1)}{n(p-1)}$ is an integer $> 1$ or $\frac{2g}{n(p-1)}$ is an integer $> 1$ and $\gcd(p, n/2) = 1$.

**Proof.** If either $\frac{2(g+p-1)}{n(p-1)}$ or $\frac{2g}{n(p-1)}$ are integers $> 1$, we may consider maximal NEC groups with signatures either $(1; -; [p, \frac{2g}{n(p-1)}], p, \frac{n}{2})$ or $(1; -; [p, \frac{2g}{n(p-1)}], p, \frac{n}{2}p)$ respectively. For the first case considering the epimorphism defined in Theorem 5 we obtain the surface that we are looking for. If the signature is $(1; -; [p, \frac{2g}{(p-1)}], p, \frac{n}{2})$, with $l = \frac{2g}{n(p-1)}$, and $\gcd(p, n/2) = 1$, then we define:

$$\varpi : \Delta \to C_n \rtimes r C_p = \langle x, y : x^p = 1; y^n = 1; y^{-1}xy = x^r \rangle$$

by

$$\varpi(x_{2i+1}) = x, \varpi(x_{2i+2}) = x^{-1}, i = 0, \ldots, (l - 3)/2,$$

$$\varpi(x_l) = x, \varpi(x_{l+1}) = x^{-1}y^2, \varpi(d) = y^{-1}, \text{if } l \text{ is odd}$$

and

$$\varpi(x_{2i+1}) = x, \varpi(x_{2i+2}) = x^{-1}, i = 0, \ldots, (l - 4)/2,$$

$$\varpi(x_{l-1}) = x, \varpi(x_l) = x$$

$$\varpi(x_{l+1}) = x^{-2}y^2, \varpi(d) = y^{-1}, \text{if } l \text{ is even}$$

Note that $x^{-1}y^2$ and $x^{-2}y^2$ have order $\frac{2}{2}p$ as consequence of the condition $\gcd(p, n/2) = 1$.

For the first possible signature of $\Delta$, if we assume $\frac{2(g+p-1)}{n(p-1)} = 1$, the argument in Theorem 5 shows that the surfaces in these conditions are not pseudoreal. Assume now that $\Delta$ has the second possible signature with
We may assume that the epimorphism \( \varpi : \Delta \to C_p \rtimes C_n \), where \( \Delta \) has signature \((1; -; [p, n^2p])\) is given by \( \varpi(d) = y, \varpi(x_1) = x, \varpi(x_2) = x^{-1}y^{-2} \), where \( d, x_1 \) and \( x_2 \) are generators of a canonical presentation of \( \Delta \). By Theorem 2.4.7 of \cite{BEGG} the group \( \Delta \) is contained in an NEC group \( \Omega \) with signature \((0; +; [2]; \{(p, n^2p)\})\). If \( r = 1 \), the epimorphism \( \varpi \) is the restriction to \( \Delta \) of \( \theta^* : \Omega \to C_2 \rtimes (C_n \rtimes C_p) \)

\[
\theta^*(x'_1) = sy^{-1}, \quad \theta^*(c'_0) = s, \quad \theta^*(c'_1) = sx, \quad \theta^*(c'_2) = sy^{-2}, \quad \theta^*(e') = sy^{-1}
\]

where

\[
C_2 \rtimes (C_n \rtimes C_p) = \langle s, x, y : s^2 = 1, x^p = 1; y^n = 1; y^{-1}xy = x^r; sx = x^{-1}, sys = y^{-1} \rangle,
\]

\( r = 1, p - 1 \)

Hence the surface uniformized by \( \ker \varpi \) has anticonformal involutions.

4 Cyclic \( p \)-gonal pseudo-real Riemann surfaces with automorphism group of maximal order for a fixed genus.

For each type of automorphism groups of pseudo-real \( p \)-gonal Riemann surfaces of genus \( g \), next proposition determines its maximal order:

**Proposition 8** Let \( p \) be a prime \( > 2 \) and \( g > (p - 1)^2 \). Assume that \( p - 1 \) divides \( g \).

1. If \( \frac{g}{p-1} \equiv 3 \mod 4 \) there are pseudo-real cyclic \( p \)-gonal Riemann surfaces of genus \( g \) with full automorphism group of order \( \frac{g(p^{g+p-1})}{p-1} \) and this order is maximal for pseudo-real cyclic \( p \)-gonal Riemann surfaces of genus \( g \). The full automorphism group is either isomorphic to a semidirect product of cyclic groups or, if \( \gcd(p, \frac{g+p-1}{2(p-1)}) = 1 \), may be isomorphic to a cyclic group.

2. If \( \frac{g}{p-1} \equiv 0 \mod 4 \) and \( \gcd(p, \frac{g+p-1}{2(p-1)}) = 1 \) there are pseudo-real cyclic \( p \)-gonal Riemann surfaces of genus \( g \) with full automorphism group of order \( \frac{pg}{p-1} \) and this order is maximal for pseudo-real cyclic \( p \)-gonal Riemann surfaces of genus \( g \). In this case the full automorphism group is either cyclic or \( C_p \rtimes C_n \) with \( r = 1 \) or \( p - 1 \).
Proof. Assume that $X$ is a pseudo-real cyclic $p$-gonal Riemann surface of genus $g$ with $X/\text{Aut}(X)$ uniformized by an NEC group where signature $i)$ of (6) in theorem 4 and then

$$2 \leq \frac{2(g + p - 1)}{n(p-1)}$$

or equivalently:

$$n \leq \frac{g}{(p-1)} + 1$$

When $p - 1$ divides $g$ we may have $n = \frac{g}{(p-1)} + 1$. Hence there are surfaces with automorphism groups of order $np = \frac{p(g+p-1)}{p-1}$ and this order is maximal. Note that $\frac{g}{p-1} \equiv 3 \mod 4$, since 4 must divide $n$. By theorem 4 the automorphism group is either isomorphic to $C_p \times C_n$ or, if $\gcd(p, \frac{p(g+p-1)}{2(p-1)}) = 1$, may be isomorphic to $C_{np}$.

Now assume that $X$ is a pseudo-real cyclic $p$-gonal Riemann surface of genus $g$ with $X/\text{Aut}(X)$ uniformized by an NEC group with signature $ii)$ of (6).

Since 4 must divide $n$, if $\frac{g}{p-1} \equiv 0 \mod 4$ and $\gcd(p, \frac{g}{2(p-1)}) = 1$, there are surfaces with full automorphism group isomorphic to a maximal order group (the order is $\frac{p(g+p-1)}{p-1}$). Note that when $\frac{g}{p-1} \equiv 0 \mod 4$, 4 does not divide $\frac{2(g+p-1)}{p-1}$ and then case 1 of theorem 4 is not possible. By theorem 4 case 2, the automorphism group is isomorphic to $C_{np}$ or $C_p \times C_n$ with $r = 1, p - 1$.

5 Cyclic $p$-gonal pseudo-real Riemann surfaces in the moduli space.

In this section we apply the previous results to the study of branch locus of moduli spaces.

First we define:

$$\mathcal{M}_g^{PR}(n, p) = \{ X : X \text{ is a cyclic } p\text{-gonal pseudo-real Riemann surface of genus } g \text{ and with full automorphism group of order } np \}.$$  

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g$, $\mathcal{T}_g$ be the corresponding Teichmüller space and

$$\pi : \mathcal{T}_g \to \mathcal{T}_g/\text{Mod}_g = \mathcal{M}_g$$

be the natural projection.
Proposition 9 Let $p$ be a prime $> 2$ and $g \geq 2$. Assume that $p - 1$ divides $g$.

1. If $\frac{g}{p-1} \equiv 3 \mod 4$ then the set $\mathcal{M}^{PR}(\frac{2g+p-1}{p-1}, p)$ is a 2-manifold in the branch locus of the orbifold $\mathcal{M}_g$.

2. If $\frac{g}{p-1} \equiv 0 \mod 4$ and $\gcd(p, \frac{g+p-1}{2(p-1)}) = 1$ then the set $\mathcal{M}^{PR}(\frac{g}{p-1}, p)$ is a 2-manifold in the branch locus of the orbifold $\mathcal{M}_g$.

Proof. First we consider the case $\frac{g}{p-1} \equiv 3 \mod 4$ and we write $n = \frac{g}{p-1} + 1$.

Let $T_{(1;\ldots;[p,p,n/2])}$ be the Teichmüller space for NEC groups with signature $(1;\ldots;[p,p,n/2])$ and $M$ be the subset of maximal groups of $T_{(1;\ldots;[p,p,n/2])}$.

Consider the set $S = \{ (\sigma, i) : \sigma$ is a signature of NEC groups and $i$ is an inclusion of a group of signature $(1;\ldots;[p,p,n/2])$ in a group of signature $\sigma \}$. Using Riemann-Hurwitz and canonical presentations of NEC groups it is possible to show that the set $S$ is finite. Each element $(\sigma, i) \in S$ produces an embedding $i^* : T_{\sigma} \rightarrow T_{(1;\ldots;[p,p,n/2])}$ and $i^* (T_{\sigma})$ is closed (see [MS]). Then $M = T_{(1;\ldots;[p,p,n/2])} - \bigcup_{(\sigma, i) \in S} i^* (T_{\sigma})$ is an open set.

Let $G$ be $C_p \times_r C_n$, $D$ be an abstract group with presentation

$$\left\langle d, x_1, x_2, x_3 : d^2 x_1 x_2 x_3 = 1, x_1^p = x_2^p = x_3^{n/2} = 1 \right\rangle$$

and $\varpi : D \rightarrow G$ be an epimorphism as considered in the proof of theorem 4. If $\Delta$ has signature $(1;\ldots;[p,p,n/2])$ and $\eta : \Delta \rightarrow D$ is the isomorphism given by the canonical presentation then $\ker \varpi \circ \eta$ uniformizes a pseudo-real cyclic $p$-gonal Riemann surface of genus $g$. The epimorphism $\varpi$ induces an embedding $T_{\Delta} \xrightarrow{\varpi^*} T_g$ defined by $[\Delta] \mapsto [\ker \varpi \circ \eta]$ (see [MS]). For two embeddings $\varpi_1^*$ and $\varpi_2^*$ defined by two epimorphisms $\varpi_1$ and $\varpi_2$, we have that $\varpi_1^*(M) \cap \varpi_2^*(M) = \emptyset$, since a possible point in the intersection would admit two different actions of maximal order which is absurd. Finally the action of the modular group $\text{Mod}_g$ is fixed point free on $\pi^{-1}(\mathcal{M}^{PR}(n, p))$, which implies the projection

$$\pi : \pi^{-1}(\mathcal{M}^{PR}(n, p)) = \bigsqcup_{\varpi} \varpi^*(M) \rightarrow \mathcal{M}^{PR}(n, p)$$

is a local homeomorphism. Hence $\mathcal{M}^{PR}(n, p)$ is a manifold of (real) dimension 2 in $\mathcal{M}_g$.

The argument for the second case is similar.
References

[A] Accola, R.D.M., Strongly branched coverings of closed Riemann surfaces, Proc. Amer. Math. Soc. 26 (1970) 315-322.

[B] Broughton, S. A., Superelliptic surfaces as $p$-gonal surfaces. To appear in Riemann and Klein Surfaces, Automorphisms, Symmetries and Moduli Spaces, Contemporary Mathematics.

[BCn] Bujalance, E.; Conder, M., On cyclic groups of automorphisms of Riemann surfaces, J. London Math. Soc. (2) 59 (1999), no. 2, 573-584.

[BCs] Bujalance, E.; Costa, A. F. Automorphism groups of pseudo-real Riemann surfaces of low genus. Acta Math. Sin. (Engl. Ser.) 30 (2014), no. 1, 11-22.

[BHS] Beshaj, L.; Hoxha, V.; Shaska, T. On superelliptic curves of level $n$ and their quotients, I. Albanian J. Math. 5 (2011), no. 3, 115-137.

[BT] Bujalance, E.; Turbek, P. Asymmetric and pseudo-symmetric hyperelliptic surfaces. Manuscripta Math. 108 (2002), no. 1, 1-11.

[BCC] Bujalance, E.; Conder, M. D. E.; Costa, A. F. Pseudo-real Riemann surfaces and chiral regular maps. Trans. Amer. Math. Soc. 362 (2010), no. 7, 3365-3376.

[BCGG] Bujalance, E.; Cirre, J.; Gamboa, J. M.; Gromadzki, G. Symmetries of compact Riemann surfaces, Lecture Notes in Mathematics nº 2007, Springer-Verlag, Berlin.

[BEGG] Bujalance, E.; Etayo, J. J.; Gamboa, J. M.; Gromadzki, G. Automorphism groups of compact bordered Klein surfaces. A combinatorial approach. Lecture Notes in Mathematics, 1439. Springer-Verlag, Berlin, 1990. xiv+201 pp.

[BCI] Bartolini, G.; Costa, A. F.; Izquierdo, M. On automorphisms groups of cyclic $p$-gonal Riemann surfaces. J. Symbolic Comput. 57 (2013), 61-69.

[BI] Bartolini, G.; Izquierdo, M. On the connectedness of the branch locus of the moduli space of Riemann surfaces of low genus. Proc. Amer. Math. Soc. 140 (2012), no. 1, 35-45.
[BW] Broughton, S. A.; Wootton. Exceptional automorphisms of (generalized) super elliptic surfaces. To appear in Riemann and Klein Surfaces, Automorphisms, Symmetries and Moduli Spaces, Contemporary Mathematics.

[CI1] Costa, A. F.; Izquierdo, M. Equisymmetric strata of the singular locus of the moduli space of Riemann surfaces of genus 4. Geometry of Riemann surfaces, 120-138, London Math. Soc. Lecture Note Ser., 368, Cambridge Univ. Press, Cambridge, 2010.

[CI2] Costa, A. F.; Izquierdo, M. On the connectedness of the branch locus of the moduli space of Riemann surfaces of genus 4. Glasg. Math. J. 52 (2010), no. 2, 401-408.

[E] Earle, C. J., On the moduli of closed Riemann surfaces with symmetries, Advances in the Theory of Riemann surfaces, pp. 119-130. Ed. L. V. Ahlfors et al., Princeton Univ. Press, Princeton 1971.

[Et] Etayo Gordejuela, J. J. Nonorientable automorphisms of Riemann surfaces, Arch. Math. (Basel) 45 (1985), no. 4, 374-384.

[H1] Hidalgo, R. A., Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals, Arch. Math. 93 (2009) 219-224.

[H2] Hidalgo, R. A., Erratum to: Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals, Arch. Math. 98 (2012) 449-451.

[K] Kimura, H. Classification of automorphism groups, up to topological equivalence, of compact Riemann surfaces of genus 4. J. Algebra 264 (2003), no. 1, 26-54.

[Ko] Kontogeorgis, A. The group of automorphisms of cyclic extensions of rational function fields, J. Algebra 216 (1999), no. 2, 665-706.

[MS] Macbeath, A. M.; Singerman, D. Spaces of subgroups and Teichmüller space. Proc. London Math. Soc. (3) 31 (1975), no. 2, 211-256.

[N] Natanzon, S. Moduli of Riemann surfaces real algebraic curves and their superanalogs, Translations of Mathematical Monographs, 225, American Mathematical Society, Providence 2004.
[S] Shimura, G., On the field of rationality for an Abelian variety, *Nagoya Math. J.* **45** (1971) 167-178.

[S1] Singerman, D. Finitely maximal Fuchsian groups, *J. London Math. Soc.* (2) **6** (1972), 29-38.

[S2] Singerman, D. Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* **210** (1974), 17-32.

[S3] Singerman, D. Symmetries and pseudo-symmetries of hyperelliptic surfaces, *Glasgow Math. J.* (1) **21** (1980), 39-49.

[W] Wootton, A. The Full Automorphism Group of a Cyclic $p$-gonal Surface, *Journal of Algebra*, 213 No. 1 (2007) 377-396