Essential tori in link complements: detecting the satellite structure by monotonic simplification

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Abstract. In a recent work “Arc-presentation of links: Monotonic simplification” Ivan Dynnikov showed that each rectangular diagram of the unknot, composite link, or split link can be monotonically simplified into a trivial, composite, or split diagram, respectively. The following natural question arises: Is it always possible to simplify monotonically a rectangular diagram of a satellite knot or link into one where the satellite structure is seen? Here we give a negative answer to that question both for knot and link cases.

Introduction

In [1] Ivan Dynnikov shows that by a sequence of elementary moves that do not increase the complexity of a diagram (number of horizontal or vertical edges) it is possible to transform an arbitrary rectangular diagram of the unknot into a diagram on which it is obviously seen that the knot is trivial (See Fig. 1). He also shows that the composite structure of a knot (See Fig. 2) and split structure of a link (See Fig. 3) can also be found by using elementary moves that do not increase complexity (further we shall call it monotonic simplification of a rectangular diagram). So, the following question naturally arises: Is it always possible to simplify the diagram of a satellite knot monotonically to obtain a diagram on which the satellite structure is seen? Or even more generally: is it always possible to simplify an arbitrary diagram of a link monotonically to obtain a diagram on which the structure of every incompressible torus (See Fig. 4) from the link complement is seen?

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The method that we are going to explore originates from works by J. Birman and W. Menasco published in the beginning of the 90-s in which they developed a technique for studying links and knots presented in the form of a closed braid ([3], [4]). The technique uses foliated surfaces in the complements of links or knots and contains certain tricks which are similar to those introduced earlier by D. Bennequin in [7].

Using and developing the above mentioned technique J. Birman and W. Menasco published in 1994 a paper [5] where they introduced and studied a class of embedded tori in closed braid complements. In particular, they proved that each such torus can be admissibly reduced (by the latter is meant an isotopy of the torus in the complement of a link together with a rearrangement of the link by exchange moves) so as to have a special position in which the foliation of the torus induced by open book fibration admits the so-called standard tiling.

However, the geometric description of tori from this class was incomplete (see [6]). In particular, K. Ng in [2] introduced examples of tori admitting a standard tiling which did not have the form of a thin round tube around a knot, as was stated in [5]. The question of further simplification of such tori into thin tori (we shall use further this term instead of term “type k torus” of [5]) was opened up to this point. In this paper we will give an answer to an analogous question stated in the language of rectangular diagrams instead of the closed braids language.

The idea of studying foliated surfaces in link complements was used not
only in the case of links presented in the form of closed braids. In 1995-1996 P. Cromwell (papers [8], [9]) adopted this idea for studying so-called arc-presentations of links and established some of their basic properties. Further development of that technique and results for rectangular diagrams (which are just another way of thinking about arc-presentations) were obtained 10 years later by I. Dynnikov in [1]. In that paper the author shows that a disc, in the case of a trivial knot, or a sphere in cases of composite and split links can be modified (together with simplifications of links) so as to have specific foliations induced by open book fibration. Analogously, one can show that every torus in the complement of an arbitrary arc-presentation can be modified (together with a simplification of the arc-presentation) to have standard tiling.

So, it is quite natural to restate the question of the further simplification of such tori into thin tori (whose structure can be seen on the rectangular diagram) in the language of rectangular diagrams. Exactly this question is investigated in the current article. Examples of “unsimplifiable” links and an “unsimplifiable” knot rectangular diagrams together with incompressible non-boundary parallel tori in their complements are presented.

1. Preliminaries

We refer the reader to paper [1] for the definitions of rectangular diagram, arc-presentation, and elementary moves.

We use the standard terminology in which a knot is a one-component link in $S^3$. Knots and links are tame and considered up to ambient isotopy.

**Definition.** We shall say that rectangular diagram $K_R$ of a knot $K$ is a satellite of rectangular diagram $L_R$ of link $L$ if the following holds:

1. The x-distance (y-distance) between neighboring vertical (horizontal) edges of $K_R$ is sufficiently large. By the latter we mean that all distances between neighboring edges are greater than some constant Const.

2. There is a set $C$ of $L_R$ components whose vertices are lying sufficiently close to vertices of $K_R$. The vertices of other components are lying sufficiently far from vertices of $K_R$. By the latter we mean that the distance from every vertex of $C$ and the nearest vertex of $K_R$ is not greater than, for example $\text{Const}/10$. At the same time, there is no component of $L_R$ not included in $C$ whose some vertex is lying closer than $C/10$ to the vertex of $K_R$.

3. $K_R$ and $L_R$ represent different links.
Remark. An example of rectangular diagram and its satellite rectangular
diagram can be simply obtained from Fig. 4 by replacing the “fat” shaded
knot with its core.

Definition. Let $L$ be a link in $S^3$ and $T \hookrightarrow S^3 \setminus L$ is an essential non-boundary parallel torus. We shall say that the structure of the torus $T$ is seen on a rectangular diagram $L_R$ of link $L$ if $T$ is parallel (in $S^3 \setminus L$) to the tubular neighborhood of some knot $K$ whose rectangular diagram representation $K_R$ is the satellite of $L_R$.

Definition. Let $L$ be a link in $S^3$ and $T \hookrightarrow S^3 \setminus L$ is an essential non-boundary parallel torus. We shall say that the structure of torus $T$ can be detected on the rectangular diagram $L_R$ of $L$ by monotonic simplification if there exist a sequence of elementary moves not increasing complexity of diagrams which lead from $L_R$ to $L'_R$ where the structure of $T$ is seen.

2. Weak example

Theorem 1. There exist rectangular diagrams $L_R$ and $L'_R$ of the link $L$ and an essential non-boundary parallel torus $T \hookrightarrow S^3 \setminus L$ such that following holds:

1. The structure of $T$ is seen on $L'_R$.
2. The structure of $T$ is not seen on $L_R$.
3. There is no sequence of elementary moves not increasing complexity (destabilizations or exchange moves) converting $L_R$ into $L'_R$.

Proof of theorem 1. An example with diagrams $L_R$ and $L'_R$ is presented on Fig. 5. The sequence of elementary moves (with increase of complexity!) is shown to verify the fact that two diagrams represent the same link. It is also obvious that the structure of torus $T$ which includes two dotted components is seen on $L'_R$ and is not seen on $L_R$. Apart from that, one can easily verify
Remark. Actually, the statement of the theorem is not very surprising because of the following argument. It follows from definition of rectangular diagrams that there can be only finitely many rectangular diagrams of fixed complexity representing some link. But there exist links (for example, link \( L \) from theorem above) whose complements include infinitely many non-parallel to each other and boundary non-parallel incompressible tori. Because of that, it is no wonder that it is impossible to see infinitely many tori on finite number of diagrams.

So, the idea that every incompressible torus from a link complement can be detected by monotonic simplification on every rectangular diagram is wrong. But may be it is possible to detect every incompressible torus in the link complement which belongs to some finite set?

This reasoning suggests the following question:

**Question.** Let \( L \) be a link in \( S^3 \). Then is it true, that every incompressible non-boundary parallel torus from JSJ-decomposition of \( S^3 \setminus L \) can be detected by monotonic simplification on every rectangular diagram \( L_R \) of \( L \)?

As we shall see further the answer on this question is negative.

### 3. Strong example

**Theorem 2.** There exist rectangular diagrams \( L_R \) and \( L'_R \) of a link \( L \) and an essential non-boundary parallel torus \( T \hookrightarrow S^3 \setminus L \) from JSJ-decomposition of \( S^3 \setminus L \) such that following holds:

1. The structure of \( T \) is seen on \( L'_R \).
2. The structure of \( T \) is not seen on \( L_R \).
3. There is no sequence of elementary moves not increasing complexity (destabilizations or exchange moves) converting \( L_R \) into \( L'_R \).

![Fig. 6](image-url)
Proof of theorem 2. An example with diagrams \(L_R\) and \(L'_R\) is presented on Figure 6. One can notice that \(L_R\) and \(L'_R\) differ from corresponding diagrams of theorem 1 only because of red "fat" rings presence. So, the arguments of theorem 1 are absolutely applicable to the current case.

The only difference from theorem 1 case here is that now the torus (represented by a shaded ring on Fig. 6) is a torus from JSJ-decomposition. □

Remark. Theorem 2 shows us that a satellite structure cannot always be found by monotonic simplification when one uses elementary moves presented in the paper \([1]\). But what if one will use more a general class of elementary moves? For example, the so-called flypes of rectangular diagrams, introduced by I. Dynnikov in the paper \([10]\)? After all stabilizations, destabilizations, exchange moves, generalized exchange moves and others are just special cases of flypes. For this purpose let us recall what a flype is.

Definition. Let \(R\) be a rectangular diagram and \(a\) and \(b\) – positive integers such that all vertices of \(R\) are lying in the following set (it is assumed, that we have coordinate axes):

\[
\mathbb{R}^2 \setminus (0, a+b) \times (0, a+b) \cup (0, a) \times (0, b) \cup \{(a+t, t) | t \in (0, b)\} \cup \{(t, b+t) | t \in (0, a)\}.
\]

Let us also assume that there is no point from segments \(\{(a+t, t) | t \in (0, b)\}\) and \(\{(t, b+t) | t \in (0, a)\}\), which is not a vertex of a diagram \(R\), lying on the intersection of two straight lines containing edges of diagram \(R\).

Then one can construct a new rectangular diagram \(R'\), by replacing each vertex \((x, y) \in (0, a) \times (0, b)\) by vertex \((y+a, x+b)\). Besides, vertices on the segments \(\{(a+t, t) | t \in (0, b)\}\) and \(\{(t, b+t) | t \in (0, a)\}\) are deleting or adding in order to obtain the set of vertices of some rectangular diagram. This can be done uniquely (See Fig. 7 for example).
The transformation $R \mapsto R'$ is called flype. Apart from that every transformation which is conjugated to the described transformation via a turn on $\pi/2$ or $\pi$ is also called flype.

**Claim.** Links $L_R$ and $L'_R$ from Fig. 6 cannot be obtained from each other by complexity preserving flypes.

4. The relationship of our examples and the example of K. Ng

Theorem 2 provides us with an example of link $L$ and an incompressible torus $T \in S^3 \setminus L$ such that $T$ can not be detected by monotonic simplification on rectangular diagram $L_R$ of $L$.

In the language of arc-presentations, this statement means that there exists an arc-presentation $L_A$ of link $L$ and an embedding of the torus $T$ into $S^3 \setminus L_A$ such that $L_A$ cannot be modified without increasing complexity, and the torus $T$ cannot be isotoped in the complement of $L_A$ to be a thin torus (tubular neighborhood of some knot in arc-presentation form). Let us now describe how the embedding of $T$ can be arranged geometrically.

It turns out that $T$ can be realized in $S^3 \setminus L_A$ so as to have the form of the torus from the example constructed by K. Ng (Figures 4a, 4b, 4c of paper [2]). For clarification of that fact we provide here a description of an embedding of $L_A$ and $T \in S^3 \setminus L$ into $S^3$ by showing an $H_\theta$-sequence of their embedding (Fig. 8) and a foliation induced on $T$ by this embedding (Fig. 9). In our pictures we follow the presentation methods used by K. Ng ([2], Figures 4, 7).
Each picture presented in Fig. 8 shows the intersection of $T \cup L$ with corresponding page $H_\theta$. The first picture corresponds to the case $\theta = 0$, the last picture corresponds to the case $\theta = 2\pi$ (and that is why these pictures are similar). Black arcs in Fig. 8 represent intersections of torus $T$ with pages $H_\theta$, red dotted arcs represent arcs of the link $L$. 8 pictures representing critical pages are marked with a legend “saddle n”. Fig. 9 represents standard tiling induced on $T$ by fibration.
5. Example with two components

It was shown that detecting the satellite structure of a link by monotonic simplification of its rectangular diagram is not possible in general. However, the links in the examples above had four and six components lying on both sides of the torus, which itself was unknotted. So it is natural to ask the following question: Can the satellite structure of a satellite link always be detected by monotonic simplification if the link has fewer than 4 components? We shall answer to this question in the following theorem:

**Theorem 3.** There exist rectangular diagrams $L_R$ and $L_R'$ of a link $L$ **consisting of two components** and an essential non-boundary parallel torus $T \hookrightarrow S^3 \setminus L$ such that the following holds:

1. The structure of $T$ is seen on $L_R'$.
2. The structure of $T$ is not seen on $L_R$.
3. There is no sequence of elementary moves not increasing complexity (destabilizations or exchange moves) converting $L_R$ into $L_R'$.

**Proof of theorem 3.** An example with diagrams $L_R$ and $L_R'$ is presented on Fig. 10. □

6. Applying foliated surfaces approach to tori.
It was proved that in case of the link having two or more components lying on both sides of the torus from the link’s complement it is possible to “lock” the torus by the link so as the torus and the link could not be simplified without increase of the complexity. Let us now pose the following question:

**Question.** Let $K$ be a knot in $S^3$. Is it true, that every incompressible non-boundary parallel torus from $S^3 \setminus K$ can be detected by monotonic simplification on every rectangular diagram $L_K$ of $K$?

Using described above standard technique one can show that the torus could be admissibly reduced so as to have standard tiling. The further simplification is possible if the following holds: The foliation of the torus has two neighboring (on the binding circle) vertices $a$ and $b$ such that the arc $ab$ appears when $\theta = t$, exists when $t < \theta < T$ and disappears when $\theta = T$.

**Definition.** Let us call this pattern an *ab-cap* (See Fig. 11). If the segment $ab$ of binding circle contains no vertices of $K$ (it takes place when, for example, the torus is knotted and $ab$ lies outside of the torus) then we shall call this ab-cap *empty*.

If an embedding of a torus has an empty ab-cap then the foliation of the torus induced by open book fibration will be looking locally as it is presented on Fig. 12 on the top left picture. One can now see that there is no obstruction to remove the intersection of the ab-cap with the binding circle (see Fig. 12). After this removal is made the complexity of the torus
embedding is decreased and one can use the standard simplification algorithm leading to new standardly tiled embedding.

**Theorem 4.** If the torus $T$ embedded into the knot complement has the complexity less than 22 then it could be admissibly reduced to thin torus.

**Proof of theorem 4.** Without loss of generality one can suppose that the torus is standardly tiled. It turns out (this is the result of the computer program written by the author) that if the standardly tiled torus embedded into $S^3 \setminus K$ has the complexity less than 22, then ab-caps exist from both sides of the torus. The torus is embedded into $S^3 \setminus K$, so it has no knot points inside or outside of it. So, there exists an empty ab-cap, which can be removed. Therefore the complexity of the embedding will be decreased and one can perform standard simplification algorithm and receive eventually standardly tiled embedding of smaller complexity. Performing this procedure consecutively one will eventually receive a thin torus. □

As we shall see in the next section, there exists an example of the torus of complexity 22 which doesn’t posses an empty ab-cap.

7. **An example of a torus and a knot which could not be simplified**

The conventional way to describe how a torus is embedded into the sphere considers consecutive intersections of regular and singular pages with the torus. The resulting picture is called an $H_\theta$-sequence (See. Fig. 8 for example). The same embedding of the torus as in Figure 8 can be represented by the following picture:

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**Fig. 13**

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This example gives us a new way (introduced by I. Dynnikov) of tori embedding representation. Each rectangle on the picture above represents the “life” of some arc. X-coordinates of horizontal edges of each rectangle correspond to coordinates of arc endpoints (on the binding circle). Y-coordinates of vertical rectangle edges represent values of $\theta$ when this arc appears and disappears. Two rectangles can overlap but no rectangle contains any vertex of another rectangle (this property comes from the fact that two interleaving arcs cannot exist simultaneously). The figure above contains two sets of overlapping rectangles (dark grey rectangles of size $1 \times 3$ and light gray of size $3 \times 1$) corresponding to the arcs with odd left point coordinate and even left point coordinate respectively. As usual, the picture is cyclical. Using this way of representation we can prove the following theorem.

**Theorem 5.** There exist a knot $K$ and a standardly tiled torus $T \hookrightarrow S^3 \setminus K$ such that there is no empty ab-caps on one side of it.

**Proof of theorem 5.** An example of such torus is presented on the figure below:
One can verify that there are no ab-caps where a is even and b is odd. One also can notice that the figure determines standardly tiled embedding, because each vertex participates in precisely 4 saddles. It follows from the fact that there are precisely 4 rectangles adjacent to each vertical line – two from one side and two from another. Apart from that, each saddle rearranges precisely two arcs into other two arcs (two rectangles are adjacent under and
two rectangles are adjacent over each horizontal line).

In order to explain how the torus is embedded into $S^3$ let us look at the following figures.

Fig. 15 presents a staircase tiling pattern describing the torus in a canonical way. On Fig. 16 the intersection of the torus with two half-planes corresponding to $\theta = 0$ and $\theta = \pi$ is pictured. □

So, we have found a torus $T$, which has no empty ab-caps from one side. It turns out that there is a way to place a rigid knot into the complement of this torus such that the pair $(T, K)$ could not be simplified anymore. An example of the knot together with the torus is presented on the following figure:
The following theorem shows that the knot presented above really gives an answer to the question stated in paragraph 6.

**Theorem 6.** There exist rectangular diagrams $L_R$ and $L'_R$ of the knot $K$ and an essential non-boundary parallel torus $T \hookrightarrow S^3 \setminus L$ such that following holds:

1. The structure of $T$ is seen on $L'_R$.
2. The structure of $T$ is not seen on $L_R$.
3. There is no sequence of elementary moves not increasing complexity (destabilizations or exchange moves) converting $L_R$ into $L'_R$.

**Proof of theorem 6.** An example of diagrams $L_R$ and $L'_R$ is presented on Fig. 18.
Remark 1. There are four canonical ways to receive a braid from the rectangular diagram presented on Fig. 18 (depending on the orientation of the knot and the 90°-turn). It is not clear whether one of them gives a counterexample to the initial statement of J. Birman and W. Menasco about braids ([5], theorem 1) or not.

Remark 2. It is possible to make 531 different flypes without increase of the complexity with rectangular diagram presented on Fig. 18 (it is the result of the computer program written by the author). However, none of them changes the combinatorial type of the diagram.

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