Breakdown of adiabaticity for the order parameter in a low dimensional gapped system.

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We discuss the emergence of non adiabatic behavior in the dynamics of the order parameter in a low dimensional quantum many-body system subject to a linear ramp of one of its parameters. While performing a ramp within a gapped phase seems to be the most favorable situation for adiabaticity, we show that such a change destroys the order, no matter how slowly the ramp is performed. We show this in detail by studying the dynamics of the one dimensional Quantum Ising model subject to linear variation of the transverse magnetic field within the ferromagnetic phase, and then propose a general argument applicable to other systems.

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The nonequilibrium dynamics of isolated quantum many-body systems is one of the most active and interdisciplinary fields that emerged recently.1–3 Indeed, while interest in this area has been considerably spurred by the opportunity to directly access the nonequilibrium dynamics in cold atom gases loaded in optical lattice,4 many of the questions addressed in that context turn out to be of importance in others, such as high energy physics5 and cosmology.6,7 In all intriguing issues addressed in the recent literature, such as the meaning and occurrence of thermalisation in isolated quantum systems, or the quest for “universal” behavior out of equilibrium, a recurring theme has been the characterization of the response of a many-body system to the variation of the Hamiltonian parameters. In particular, the main focus has been on the two extremes of instantaneous changes (quenches) and slow ones (known under the oxymoron “slow quenches”). The latter has been mostly studied for systems driven across a quantum critical point, where a generalization of the classical Kibble-Zurek mechanism led to the prediction of a universal scaling of the excitation density with the speed at which the critical point is crossed,8,9 successively extended also to quenches within gapless phases,10,11 where even full violation of adiabaticity may occur.12 Specifically, universality is expected whenever the scaling dimension of the fidelity susceptibility10 (or its generalization for non linear protocols) is negative, and extends to other quantities besides the excitation density, such as the excess energy. We also mention that spontaneous generation of defects in the non-equilibrium dynamics has been observed experimentally in spinor condensates.13

Intuitive quantum mechanical arguments, rooted ultimately on the adiabatic theorem, suggest that the case of quenches within a gapped phase is much less interesting. Indeed, in this case the scaling dimension of the fidelity susceptibility is always positive, implying that the density of excitations and the excess energy always tends to zero with the square of the switching rate for linear ramps (generalization to generic power-law ramps is straightforward). This also suggests that other thermodynamics quantities share the same property, i.e. corrections with respect to their equilibrium value are quadratic in the rate. However, intuition indicates a different scenario when considering the order parameter in a phase with spontaneous symmetry breaking. Since even when performing a variation of the Hamiltonian within a gapped phase an extensive amount of energy is injected, one expects to be at the end of the protocol in a situation similar to the case of finite temperature. In certain instances, for example in low dimensional systems, the effect of temperature is the complete disruption of long range order, an effect which is very far from being a small correction.

This work addresses this apparent contradiction by studying the dynamics of the order parameter in a one dimensional Quantum Ising chain after a linear variation in time of the transverse field within the ferromagnetic, ordered phase. In particular, we focus on the asymptotic value of the order parameter as a function of the duration $\tau$ of the linear ramp. For every finite $\tau$ we show that the order parameter vanishes, confirming our expectations. The inverse correlation length, however, turns out to depend quadratically on the ramp rate for large $\tau$. These quadratic corrections persist also in the limit of small $\tau$, where the reference value is that of the sudden change of the transverse field, i.e. $g(t) = g_0$ for $0 \leq t < \tau$, $g(t) = g_0 + (g_1 - g_0)t/\tau$ for $0 < t < \tau$...
$g(t) = g_1$ for $t \geq \tau$. We will assume that the system is initially prepared in its ground state for $g = g_0$.

At zero temperature this model exhibits a quantum phase transition at $g_c = 1$ separating a ferromagnetic phase ($g < g_c$), from a paramagnetic one ($g > g_c$), both characterized by a finite gap. At any finite temperature the system is instead paramagnetic. The order parameter is the spontaneous magnetization along the $x$ axis, defined as $m_x = \frac{1}{L} \sum_j \langle \sigma_j^x \rangle$, which is finite in the ferromagnetic phase and zero in the paramagnetic one. As stated above, we are interested in the dynamics within the ordered phase, so we take both $g_0 < 1$ and $g_1 < 1$.

Performing a Jordan-Wigner transformation, we have

$$\sigma_j^z = \prod_{m<j} (1 - 2c_m^\dagger c_m) \left( c_j + c_j^\dagger \right),$$

$$\sigma_j^z = 1 - 2c_j^\dagger c_j,$$

with $\{c_j, c_j^\dagger\} = \delta_{j,0}$ and $\{c_j, c_0\} = 0$, the Hamiltonian can be written as

$$\hat{H}(t) = P^+ \hat{H}^+(t) P^+ + P^- \hat{H}^-(t) P^-,$$

where

$$P^\pm = \frac{1}{2} \left[ 1 \pm \prod_{j=1}^L \sigma_j^\pm \right]$$

are the projectors in the subspace with an even (+) or odd (−) number of fermions and

$$\hat{H}^\pm(t) = -\sum_{i=1}^L \left( c_i^\dagger c_{i+1}^\dagger + c_i^\dagger c_{i+1} + h.c. \right) - 2g(t)c_i^\dagger c_i,$$

with the $c_i$'s obeying antiperiodic boundary conditions $c_{L+1} = -c_1$ in the even sector and periodic boundary conditions $c_{L+1} = c_1$ in the odd one.

For finite chains the ground state is always in the even sector and the order parameter $\sigma_j^z$, which changes the parity of the fermion number, is strictly zero. However, the energy gap between the lowest energy states within each sector, $|\Omega_+\rangle$ and $|\Omega_-\rangle$, vanishes exponentially in the thermodynamic limit and in the ferromagnetic phase, manifestation of spontaneous breaking of the $\mathbb{Z}_2$ symmetry. One can nonetheless recognize spontaneous symmetry breaking even within each separate sector through the long-distance behavior of the correlation function $\rho(t) = \langle \Omega_+ | \sigma_j^z \sigma_{j+r}^z | \Omega_+ \rangle$, which is independent of $j$. Indeed, in the ferromagnetic phase, $\lim_{r \to \infty} \rho_r = m_r^2 > 0$, signaling the established long-range order. We shall hence focus on the even sector, where the finite-size ground state lies, and study the time evolution of

$$R^+_r(t) = \lim_{L \to \infty} \langle \psi(t) | \sigma_j^z \sigma_{j+r}^z | \psi(t) \rangle$$

where $|\psi(t)\rangle = \mathcal{U}(t) |\Omega_+\rangle$, being $\mathcal{U}(t)$ the evolution operator, and $|\Omega_+\rangle$ the initial state assumed to be the ground state at $g = g_0$.

The Hamiltonian can be instantaneously diagonalized performing a Fourier transform $c_j = \frac{e^{i\pi/4}}{\sqrt{L}} \sum_k e^{ikj} \hat{c}_k$, with $k$ odd multiple of $\pi/L$ so to implement the antiperiodic boundary conditions in the even sector, followed by a Bogoliubov transformation,

$$\left( \hat{c}_k \right) = \left( \begin{array}{cc} u_k(t) & -v_k(t) \\ v_k(t) & u_k(t) \end{array} \right) \left( \begin{array}{c} \gamma_k^+ \\ \gamma_k^- \end{array} \right),$$

with coefficients $u_k(t) = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{g(t) - \cos(k)}{\epsilon_k(t)}}$, $v_k(t) = -\frac{1}{\sqrt{2}} \sqrt{1 - \frac{g(t) - \cos(k)}{\epsilon_k(t)}}$, and eigenvalues $\epsilon_k(t) = \sqrt{1 + g^2(t) - 2g(t) \cos(k)}$. The instantaneous ground state is $|\Omega_+\rangle = \sum_k \left( u_k(t) |v_k(t)\right) \hat{c}_k^\dagger \hat{c}_k^\dagger |0\rangle$, with $\epsilon_k(0) = 0 \forall k$, and $|0\rangle = \bigoplus_{k>0} |0\rangle_k$. The dynamics induced by the linear ramp $g(t)$ can be described through the density matrix $\rho(t) = \langle \psi(t) | \psi(t) \rangle$ that, since the $k$-modes are mutually independent, has the form $\rho(t) = \otimes_{k>0} \rho_k(t)$, where $\rho_k(t)$ in the basis $\{|0\rangle_k, c_k^\dagger |0\rangle_k, \hat{c}_k |0\rangle_k, \hat{c}_k^\dagger |0\rangle_k\}$ is given by $\rho_k(t) = \frac{1}{2} \left( |\hat{1} + f_{1,k}(t)\rangle \langle \hat{1} + f_{1,k}(t)| + f_{2,k}(t) \hat{2}|0\rangle \langle 0| + \hat{2}|0\rangle \langle 0| + f_{3,k}(t) |\hat{3}| \langle \hat{3}| + f_{3,k}(t) |\hat{3}| \langle \hat{3}| \right)$, where $f_{X,Y,Z}(t)$ are the Pauli matrices. In vector notation the coefficients $f_k = (f_{1,k}, f_{2,k}, f_{3,k})^T$ satisfy the simple equation

$$\frac{d}{dt} f_k(t) = \hat{R}_k f_k(t),$$

where the matrix $\hat{R}_k = -4(g(t) - \cos(k)) \hat{X}_1 + 4 \sin(k) \hat{X}_2$, with $\hat{X}_i (i = 1, 2, 3)$ are the 3x3 generators of rotations. Notice that the initial conditions $f_{1,k}(0) = \frac{\epsilon_k(0) - \cos(k)}{\epsilon_k(0)}$, $f_{2,k}(0) = -\frac{\sin(k)}{\epsilon_k(0)}$, $f_{3,k}(0) = 0$.

From the evolution of the density matrix we can calculate $R^+_r(t)$ of Eq. (3). We start by writing $R^+_r(t)$ ad

$$R^+_r(t) = \langle B^r_j A^r_{j+1} B^r_{j+2} \cdots A^r_{j+r} B^r_{j+r+1} | 0 \rangle,$$

where $A^r_j = c_j(t) + c_j^\dagger(t)$, $B^r_j = c_j^\dagger(t) - c_j(t)$, $c_j(t) = \mathcal{U}(t) c_j |\Omega_+(t)\rangle$, and $\langle \cdots | 0 \rangle$ denotes the average over the initial state.

Using Wick’s theorem, Eq. (9) can be expressed in terms of the contractions of the $A^r_j$’s and $B^r_j$’s, which in terms of the function $f_{1,k}$, $f_{2,k}$ and $f_{3,k}$ read

$$\langle A_j(t) A_l(t) \rangle = \delta_{jl} - \frac{1}{L} \sum_k e^{ik(j-l)} f_{3,k}(t),$$

$$\langle B_j(t) B_l(t) \rangle = -\delta_{jl} - \frac{1}{L} \sum_k e^{ik(j-l)} f_{3,k}(t),$$

$$\langle B_j(t) A_l(t) \rangle = -\frac{1}{L} \sum_k e^{ik(j-l)} (f_{1,k}(t) + i f_{2,k}(t))$$

We are interested in the stationary state, that is for $t \to \infty$ after taking the thermodynamic limit, i.e. replacing...
We note that disregarding the oscillatory terms is equivalent to state that the stationary value can be computed in the diagonal ensemble, which is completely determined by the occupation numbers $n_k$. We observe that Eq. (11) is also valid at equilibrium, with the $n_k$’s given by the Fermi-Dirac distribution. The effect of the linear ramp of the transverse field is that of creating a non-equilibrium distribution for such quantities.

As in equilibrium, the correlation $R^x_\tau$ can be expressed as a $r \times r$ Toeplitz determinant,

$$R^x_\tau = \begin{bmatrix} C(0) & C(-1) & \cdots & C(-r + 1) \\ C(1) & C(0) & \cdots & C(-r + 2) \\ \vdots & \vdots & \ddots & \vdots \\ C(r - 1) & C(r - 2) & \cdots & C(0) \end{bmatrix},$$

whose asymptotic behavior in the limit $r \to \infty$ has to be determined. To this end, we first note that $C(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \xi(k)e^{-ikr}$, with

$$\xi(k) = \left( \frac{1 - g_0 e^{ik}}{1 - g_0 e^{-ik}} \right)^{1/2} (1 - 2n_k).$$

In terms of the complex variable $z = e^{ik}$ the function $\xi(z)$ has zero index around the unit circle and is non vanishing, as long as $n_k < 1/2$, $\forall k$, a condition that has been verified numerically and perturbatively, and is equivalent to say that the effective temperature of all the modes is less than infinity. Under this condition we can apply the strong Szegö lemma \textsuperscript{20}, which tells us that $R^x_\tau \sim e^{-r/\xi}$, with the inverse correlation length given by,

$$\xi^{-1} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \log(1 - 2n_k).$$

Therefore, whenever $n_k \neq 0$, the correlation length is finite, implying that $R^x_\tau$ goes to zero exponentially hence that the order parameter is zero. Such a condition is verified for any finite duration of the linear ramp, implying that adiabaticity is broken for the order parameter. From Eq. (15) we observe that tiny deviations of
the occupation numbers with respect to their equilibrium value \((n_k = 0)\) translates into a comparably small inverse correlation length. Nonetheless, such small quantitative corrections lead to a completely different behavior of the correlation function \(R_n\) and of the order parameter.

Figure 1 shows the correlation length as a function of \(\tau\) for different ramps computed by numerically solving Eqs. \((8)\) and evaluating Eq. \((15)\). We can see that for long durations the correlation length grows quadratically, while for small \(\tau\) of order one it displays oscillations. The inset of the figure shows that also for small \(\tau\) the growth of \(\xi\) above the sudden-quench value is quadratic. The two limiting cases of slow and sudden quenches can be captured by two different perturbative expansions (more details can be found in the Supplementary Material).

For small \(\tau\) the result of the perturbative expansion of Eqs. \((8)\) at the leading order is

\[
\xi(\tau) = -\frac{1}{\log \left[ \frac{1+2g_0 - \sqrt{1-g_0^4}}{2} \right]} + \tau^2 \frac{2(g_1 - g_0)^2 \left( 1 + g_0g_1 - \sqrt{(1-g_1^2)(1-g_0^2)} \right)}{3(g_0 + g_1)^2 \log^2 \left[ \frac{1+2g_0 - \sqrt{1-g_0^4}}{2} \right]} + O(\tau^4),
\]

where the first term is the result for a sudden quench (\(\xi_{\text{sd}}\)). Higher order can be straightforwardly computed. In particular we notice that only even powers of \(\tau\) are present in the expansion, and all computed corrections are even under \(g_0 \leftrightarrow g_1\), i.e. inversion of the ramp. Figure 2 shows a comparison between the perturbative and the numerical results, and we can see that the agreement is excellent up to \(\tau \simeq 1\) provided corrections up to eighth order are taken into account.

For large \(\tau\), instead, one can use the adiabatic perturbation theory described in Ref. \(21\) which predicts that the occupation numbers \(n_k\) for large \(\tau\) vanish as \(1/\tau^2\) in an oscillating fashion. This is actually the source of oscillations observed in \(\xi\). Indeed, by applying the adiabatic perturbation theory one obtains,

\[
\xi(\tau) = \frac{64(1-g_0^2)^3(1-g_1^2)^3}{(g_1-g_0)^2[(1-g_0^2)^3+(1-g_1^2)^3]} \tau^2 + f(\tau) \sqrt{\tau} + \Lambda + O(\tau^{-1/2}),
\]

where \(f(\tau)\) is an oscillating function and \(\Lambda\) is a constant, see the supplementary material. Thus, the relative oscillations of the correlation length goes to zero as \(\tau^{-3/2}\). Also in this case all the corrections are invariant under the transformation \(g_0 \leftrightarrow g_1\). Figure 3 shows a comparison between this adiabatic perturbative expansion and the numerical data. We see that by including correction up to \(O(1)\) there is quite a good agreement for \(\tau \gtrsim 10\).

In conclusion, we have shown that the order parameter of the one dimensional Quantum Ising model does not evolve adiabatically in the ferromagnetic phase, however small the switching rate of the transverse field is. This occurs in spite of the fact that the Hamiltonian is gapped, which in principle is the most favorable situation for an adiabatic evolution. Such a behavior of the order parameter has to be expected whenever the system has a phase transition only at zero temperature and it is driven within the ordered phase. Indeed a finite density of defects \(n_{df} \sim 1/\tau^4\) will always be generated and in this situation will always be sufficient to destroy order. From this, one can estimate also the behavior of the correlation length, which, following the same reasoning as the Kibble-Zurek argument, will be \(\xi \sim 1/n_{df}^4 \sim \tau^{-2/d}\), with \(d\) being the dimension of the system. A natural question that comes up is what happens instead in an analogous system where the transition survives at finite temperature. One possibility is that there is a transition in the value of the order parameter as a function of \(\tau\), namely for sufficiently slow ramp its asymptotic value is expected to be finite, while it should go to zero for fast ramps. If this is really the case, and in the affirmative case if the value of the order parameter is vanishing or not are interesting question to consider in following studies.

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