Heegaard Floer Homology and Balanced Presentations of Groups

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Abstract

Let $G$ be a group with a finite balanced presentation $P$. We associate a Heegaard Floer homology group $\tilde{HF}_P(G)$ with the pair $(G, P)$ based on some extra choices and technical assumptions. We show that $\tilde{HF}_P(G)$ is independent from these choices and also is invariant under stable Andrews-Curtis transformations on $P$, based on two claims which are not settled in this paper.

1 Introduction

1.1 $\tilde{HF}(Y)$ an invariant associated with $\pi_1(Y)$

Heegaard Floer homology is an invariant associated with a closed oriented three manifold which was introduced by Ozsváth and Szabó in [OS04b]. There are four versions of this invariant: hat, plus, minus, and infinity Heegaard Floer homology groups. In this paper we work with the hat version with coefficients in $\mathbb{Z}_2$. This is, in fact, an invariant associated with the fundamental group of the three manifold. To see this, first note that there is the following Künneth formula for this invariant:

**Proposition 1.1.** (c.f. [OS04a, Theorem 1.5]) Let $Y_1$ and $Y_2$ be a pair of three manifolds, equipped with $\text{Spin}^c$ structures $s_1$ and $s_2$. Then, there is an identification

$$\tilde{HF}_k(Y_1 \sharp Y_2, s_1 \sharp s_2) = \bigoplus_{i+j=k} \tilde{HF}_i(Y_1, s_1) \otimes_{\mathbb{Z}_2} \tilde{HF}_j(Y_2, s_2).$$

On the other hand, the following theorems imply that the fundamental group of a three manifold determines it up to indeterminacy arising from lens spaces.

**Theorem 1.2.** (c.f. [Mil62, Theorem 1]) Every compact 3-manifold $Y$, which is not isomorphic to $S^3$, is isomorphic to a sum $Y_1 \sharp \ldots \sharp Y_k$, of prime manifolds. The summands $Y_i$ are uniquely determined up to order and isomorphism.

**Theorem 1.3.** (c.f. [AFW15, Theorem 2.1.1]) Let $Y$ be a closed, oriented 3-manifold. If $\pi_1(Y) = \Gamma_1 * \Gamma_2$, then there exist closed, oriented 3-manifolds $Y_1$ and $Y_2$ with $\pi_1(Y_i) = \Gamma_i$, for $i = 1, 2$, and $Y = Y_1 \sharp Y_2$.

**Theorem 1.4.** (c.f. [AFW15, Theorem 2.1.2]) Let $Y_1$ and $Y_2$ be two closed, prime 3-manifolds with $\pi_1(Y_1) = \pi_1(Y_2)$. Then either $Y_1$ and $Y_2$ are homeomorphic, or $Y_1$ and $Y_2$ are both lens spaces.

From Proposition 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4, and the fact that the Heegaard Floer homology of a lens space only depends on its fundamental group, we have:
Corollary 1.5. Let $Y_1$ and $Y_2$ be two closed 3-manifolds. If $\pi_1(Y_1) = \pi_1(Y_2)$ then $\widehat{HF}(Y_1) = \widehat{HF}(Y_2)$.

This observation suggests that there must be a way to compute the hat Heegaard Floer homology group of a three manifold from its fundamental group. Corollary 1.5 also states that, for a given presentation of the fundamental group of a three manifold $Y$ which arises from a Heegaard diagram, $\widehat{HF}(Y)$ is invariant under stable Andrews-Curtis transformations. These transformations are extended Nielsen transformations along with a stabilization transformation that act on a group presentation and result in another presentation for the same group (c.f. [AC65, AC66] and Section 2 for definitions).

Motivated by these facts, in this paper we present a plan to associate a Heegaard Floer homology group with a family of finite group presentations and take several steps towards fulfilling this plan. Certain technical parts remain incomplete while we hope to carry out these parts in future.

1.2 Summary of results

In this paper, a Heegaard diagram is a triple $(\Sigma, \alpha, \beta)$ where $\Sigma$ is a surface and $\alpha$ and $\beta$ are collections of disjoint oriented simple closed curves on the surface $\Sigma$. We assume that the number of curves in $\alpha$ is equal to the number of curves in $\beta$, and for each component $\Sigma_i$ of $\Sigma$, $\Sigma_i - \alpha$ and $\Sigma_i - \beta$ are connected. Let $\mathcal{H}$ be a Heegaard diagram for a closed oriented three manifold $Y$. There is a balanced presentation for the fundamental group of $Y$ which arises naturally from $\mathcal{H}$, see Example 3.1.

Let $G$ be a group which has a balanced presentation $P$. Modulo some extra choices and technical assumptions, we associate a Heegaard Floer homology group $\widehat{HF}_P(G)$ with the pair $(G, P)$. We show the independence from some of these extra choices, while two technical steps remain unsettled (see Claim 5.6 and Claim 5.8). Let $[P]$ denote the set containing all presentations $P'$ for $G$ which result from the action of a sequence of stable Andrews-Curtis transformations on $P$. Assuming the aforementioned two claims, we prove:

Claim 1.6. Let $P$ be a balanced presentation for the group $G$. The homology group $\widehat{HF}_P(G)$ is an invariant associated with $G$ and $[P]$. Moreover, if $G$ is the fundamental group of a closed oriented three manifold $Y$ and $P$ is a presentation associated with a Heegaard diagram of $Y$, then we have $\widehat{HF}_P(G) = \widehat{HF}(Y)$.

Claim 1.6 suggests a method to approach the stable Andrews-Curtis conjecture. This conjecture states that:

Stable Andrews-Curtis Conjecture. (c.f. [AC65]) Every balanced group presentation for the trivial group may be changed to the trivial presentation by a finite sequence of stable Andrews-Curtis transformations.

This conjecture has topological interpretations and consequences which would follow from the conjecture and are studied in [AC65, Wri75]. A group-theoretical and a topological survey of the conjecture can be found in [BM93, HAM93]. In [FGMW10, Kir97], the relation between this conjecture and the smooth 4-dimensional Poincaré conjecture is discussed.

It is a general opinion that the stable Andrews-Curtis conjecture is false. There are several potential counterexamples, amongst them one can mention [AK85, MS99, MMS02].

Assuming Claim 1.6, in order to disprove the stable Andrews-Curtis conjecture it suffices to find a group presentation $P$ for the trivial group $I$ such that $\widehat{HF}_P(I) \neq \mathbb{Z}_2$. 

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1.3 Organization

In Section 2, we associate a dual presentation with a given balanced presentation $P$ of a group $G$. We also define a restricted version of stable Andrews-Curtis transformations which act on a presentation $P$ and its dual presentation at the same time. In Section 3, we explain how one can associate a Heegaard diagram with a presentation $P$ and its dual presentation. In Section 4, we describe the changes of this corresponding Heegaard diagram when the presentation $P$ and its dual presentation undergo the transformations defined in Section 2. In Section 5, we associate a Heegaard Floer homology group with the Heegaard diagram constructed in Section 3. Moreover, we present a proof of Claim 1.6 in this section.

2 Dual presentations and AC-moves

First, we recall some elementary concepts from group theory.

**Definition 2.1.** (c.f. [Joh80]) Let $X$ be a set, $F = F(X)$ denote the free group on $X$, and $R$ be a subset of $F$.

- The group $G = \langle F | R \rangle$ is defined as the quotient group $F/N$ where $N$ is the smallest normal subgroup of $F$ which contains $R$. $(X, R)$ is called a free presentation, or simply a presentation of $G$. The elements of $X$ are called the generators and those of $R$ the relators.

- A group $G$ is called finitely presented if it has a presentation with both $X$ and $R$ finite sets.

A finite presentation $(X | R)$ is called a balanced presentation if we have $|X| = |R|$.

**Definition 2.2.** Let $P = \langle a_1, \ldots, a_d | b_1, \ldots, b_d \rangle$ and $P^* = \langle b_1^*, \ldots, b_d^* | a_1^*, \ldots, a_d^* \rangle$ be two balanced presentations. We say $P$ and $P^*$ are dual presentations if, possibly after rearranging the indices, there exist bijections

$$f_{ij} : A_{ij} \rightarrow \overline{A}_{ij} \quad \text{and} \quad \overline{f}_{ij} : \overline{A}_{ij} \rightarrow \overline{A}_{ij}$$

where

$$A_{ij} = \{ k | a_i \text{ is } k^{th} \text{ letter in } b_j, 1 \leq k \leq |b_j| \},$$

$$\overline{A}_{ij} = \{ k | a_i^{-1} \text{ is } k^{th} \text{ letter in } b_j, 1 \leq k \leq |b_j| \},$$

$$A_{ij}^* = \{ k | b_j^* \text{ is } k^{th} \text{ letter in } a_i^*, 1 \leq k \leq |a_i^*| \},$$

$$\overline{A}_{ij}^* = \{ k | b_j^{-1} \text{ is } k^{th} \text{ letter in } a_i^*, 1 \leq k \leq |a_i^*| \},$$

for $1 \leq i, j \leq d$. Here $|b_j|$ denotes the number of letters in the word $b_j$. We denote the dual presentations $P$ and $P^*$ together with the family $\mathcal{F} = \{ f_{ij}, \overline{f}_{ij} \}_{i,j}$ of correspondences by $(P, P^*)_\mathcal{F}$.

**Remark 2.3.** For each $i$, $f_{ij}$ and $\overline{f}_{ij}$, $1 \leq j \leq d$ induce a cyclic ordering on all occurrences of the letter $a_i$, independent from its sign, in the relators. In fact, elements of the sets $f_{ij}(A_{ij})$ and $\overline{f}_{ij}(\overline{A}_{ij})$, $j = 1, \ldots, d$, are distinct and show different occurrences of the letters $b_j^*$ and $b_j^{-1}$ in the relation $a_i^*$. Therefore

$$\bigcup_{j=1}^{d} f_{ij}(A_{ij}) \cup \overline{f}_{ij}(\overline{A}_{ij}) = \{ 1, \ldots, |a_i^*| \}.$$
In other words, \( f_{ij} \) and \( T_{ij} \), \( j = 1, \ldots, d \), induce a correspondence between all occurrences of the letter \( a_i \) in the relators \( b_j \) and the elements of \( \{1, \ldots, |a_i^*|\} \). Now the natural cyclic ordering of the elements of \( \{1, \ldots, |a_i^*|\} \) induces the desired cyclic ordering.

**Remark 2.4.** There might be more than one dual presentation for a given presentation and they may present different groups. For the trivial presentation \( T = \langle a \mid a \rangle \) of the trivial group, we have \( T^* = T \).

Andrews-Curtis transformations are defined on a presentation \( P = \langle a_1, \ldots, a_n \mid b_1, \ldots, b_m \rangle \) of a group \( G \) as follows:

1. Replace \( b_i \) with \( b_{i}b_j \) for some \( j \neq i \);
2. Replace \( b_i \) with \( b_i^{-1} \);
3. Replace \( b_i \) with \( b_igg^{-1} \), where \( g \) is one of \( a_j \) or its inverse;

Moreover, we allow the stabilization transformation:

4. Add/remove \( a_{n+1} \) as both a generator and a relator.

These four transformations are called the stable Andrews-Curtis transformations. It is clear that each stable Andrews-Curtis transformation on a presentation \( P \) of the group \( G \) gives another presentation for the group \( G \).

Let \( (P, P^*_F) \) be a pair of dual presentations as in Definition 2.2. Corresponding to each transformation of types 1-4, we associate a dual transformation which acts on the dual presentation \( P^* \) as follows:

1*. Replace all \( b_j^* \)s (resp. \( b_j^{*-1} \)s) in \( a_k^* \), \( k = 1, \ldots, d \), with \( b_j^*b_i^* \) (resp. with \( b_i^{*-1}b_j^{*-1} \)), for \( i \neq j \);
2*. Replace \( b_i^* \) with \( b_i^{*-1} \) in all the relators;
3*. Replace \( a_j^* \) with \( a_j^*b_i^*b_i^{*-1} \);
4*. Add/remove \( b_{d+1}^* \) as both a generator and a relator.

Define the inverse of the third Andrews-Curtis transformation and its dual as follows:

5. Replace a relator \( b_i = b_j^*gg^{-1} \) with \( b_j^* \), where \( g \) is one of \( a_j \) or its inverse;
5*. Remove \( b_i^*b_i^{*-1} \) from the relator \( a_j^* \). If \( g = a_j \), \( b_i^* \) is \( f_{ji}(|b_i|)^{th} \) letter in \( a_j^* \), and if \( g = a_j^{-1} \), \( b_i^* \) is \( f_{ji}(|b_i| - 1)^{th} \) letter in \( a_j^* \).

**Remark 2.5.** Although the transformation 5 is not mentioned in the stable Andrews-Curtis transformations, it is the inverse of 3. Note that corresponding to 5, the transformation 5* is not always possible.

For the dual pair of presentations \( (P, P^*_F) \), we always assume that Andrews-Curtis moves come in pairs. This makes the Andrews-Curtis moves for dual pairs restricted in comparison with the classical Andrews-Curtis moves. Nevertheless, we will see later that there are detours around this extra restriction.

**Definition 2.6.** An AC-move for the pair \( (P, P^*_F) \), is one of the transformations 1-5 along with its corresponding dual transformation.
3 Dual presentations and the associated Heegaard diagram

The following example describes a pair of dual presentations $P$ and $P^*$ for the fundamental group of a closed oriented three manifold.

**Example 3.1.** Let $Y$ be a closed oriented three manifold and

$$H = (\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_g\}, \beta = \{\beta_1, \ldots, \beta_g\})$$

be a Heegaard diagram for $Y$. This diagram gives a balanced presentation

$$P = \langle a_1, \ldots, a_g | b_1, \ldots, b_g \rangle$$

for $\pi_1(Y)$ as follows.

Fix an orientation for each one of the curves $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$. Let $\alpha_i^*, \ldots, \alpha_g^*$ denote oriented simple closed curves in $\Sigma$ based at the point $p \in \Sigma - \alpha - \beta$ such that each $\alpha_i^*$ positively intersects $\alpha_i$ in one point and stays disjoint from the rest of $\alpha$ curves. We call $\alpha_i^*$ a **dual curve** for $\alpha_i$, see Figure 1.

![Figure 1: $\alpha_i^*$ is a dual curve for $\alpha_i$](image)

Attach 2-handles $D_1, \ldots, D_g$ to $\Sigma \times [0, 1]$ along $\alpha_i \times \{0\}$. Now attach a 3-ball to the two sphere boundary of it. This results in a handlebody $H_1$. We have $\pi_1(H_1) \cong \langle a_1, \ldots, a_g \rangle$ where $a_i$ is the homotopy class of $\alpha_i^*$ in $H_1$.

Consider a small cylindrical neighborhood $N_j$ of each $\beta_j$ in $\Sigma$ and let $p_j$ be a point on $\beta_j - \alpha_1 - \cdots - \alpha_g$. Now start from $p_j$ on $\beta_j$ and traverse $\beta_j$ in its direction. In this path, let $\alpha_{k_i}$ be the $i^{th}$ $\alpha$ curve in the neighborhood $N_i$ which intersects $\beta_j$. If the intersection number of $\alpha_{k_i}$ with $\beta_j$ is $\alpha_i$, for $i = 1, \ldots, n$, we obtain a word $b_j = a_{k_1}^{\alpha_1}a_{k_2}^{\alpha_2}\cdots a_{k_n}^{\alpha_n}$ with $\alpha_1, \alpha_2, \ldots, \alpha_n \in \{\pm 1\}$ (see Figure 2). This is called the **relator associated with the curve** $\beta_j$.

![Figure 2: Neighborhood of $\beta_j$ and its associated relator $b_j$.](image)

Consider a corresponding curve $\tilde{\beta}_j = \alpha_{k_1}^{\alpha_1^*}\alpha_{k_2}^{\alpha_2^*}\cdots \alpha_{k_n}^{\alpha_n^*}$ where $\alpha_i^{*^{-1}}$ denotes the curve $\alpha_i^*$ with the reverse orientation. This curve is homotopic to $\beta_j$ in $H_1$. Therefore, the homotopy class of $\beta_j$ in $\pi_1(H_1)$ is $b_j$. 

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Attach 2-handles $\tilde{D}_1, \ldots, \tilde{D}_g$ to $H_1$ along $\beta_j \times \{1\}$ (curves in the boundary $\Sigma \times \{1\}$ of $H_1$) and denote the resulting space with $\tilde{H}_1$. By Van-Kampen Theorem, we have

$$\pi_1(\tilde{H}_1) \cong \langle a_1, \ldots, a_g \rangle / N$$

where $N$ is the normal subgroup of $\pi_1(H_1)$ generated by $\{b_1, \ldots, b_g\}$. Therefore, we have $\pi_1(\tilde{H}_1) \cong \langle a_1, \ldots, a_g | b_1, \ldots, b_g \rangle$. $\tilde{H}_1$ embeds in $Y$ and its complement in $Y$ is an open three-ball. Again by Van-Kampen Theorem, we have

$$\pi_1(Y) \cong \pi_1(\tilde{H}_1) \cong \langle a_1, \ldots, a_g | b_1, \ldots, b_g \rangle.$$

If we use the dual Heegaard diagram for $Y$, i.e.

$$\mathcal{H}^* = (\Sigma, \beta = \{\beta_1, \ldots, \beta_g\}, \alpha = \{\alpha_1, \ldots, \alpha_g\}),$$

another presentation for $\pi_1(Y)$ is obtained, which is denoted by:

$$P^* = \langle b_1^*, \ldots, b_g^* | a_1^*, \ldots, a_g^* \rangle.$$

Here the generators $b_i^*$ are in correspondence with dual curves for $\beta_i$ and the relations $a_j^*$ are obtained from $\alpha_j$ curves with the same method as mentioned above after fixing a point $q_j$ on $\alpha_j - \beta_1 - \cdots - \beta_g$.

With the notation of Definition 2.2, we define $f_{ij} : A_{ij} \to \overline{A}_{ij}$ and $\overline{f}_{ij} : \overline{A}_{ij} \to A_{ij}^*$, $1 \leq i, j \leq g$, as follows. Let $a_i$ be $k$th letter in $b_j$. This means that if we start from $p_j$ on $\beta_j$ and traverse $\beta_j$ in its direction, $\alpha_i$ is $k$th curve which intersects $\beta_j$ and the intersection number of $\alpha_i$ with $\beta_j$ is $+1$. Now start from $q_i$ on $\alpha_i$ and traverse $\alpha_i$ in its direction. Let this intersection of $\beta_j$ with $\alpha_i$, which corresponds to a letter $b_j^{-1}$, be the $l$th letter in $a_i^*$. We define $f_{ij}(k) = l$. $\overline{f}_{ij}$ is defined similarly. If we set $F = \{f_{ij}, \overline{f}_{ij}\}_{i,j}$, $(P, P^*)_F$ is a pair of dual presentations.

We call $P$ and $P^*$ a pair of dual presentations associated with the diagram $\mathcal{H}$. In Example 3.1, we may use a Heegaard diagram $\mathcal{H}$ which does not correspond to a three manifold. In fact, the method of this example can be used to assign such a dual pair $(P, P^*)$ to any Heegaard diagram. The following proposition gives a semi-inverse construction.

**Proposition 3.2.** Let $(P, P^*)_F$ be a pair of dual presentations (with the notation of Definition 2.2). There is a unique associated Heegaard diagram $\mathcal{H}_{(P,P^*)}_F = (\Sigma, \alpha, \beta)$ such that all regions in $\Sigma - \alpha - \beta$ are polygons and its associated pair of dual presentations is $(P, P^*)_F$.

**Proof.** Let $\beta_j$ denote an oriented circle with $|b_j|$ marked points on it which are numbered $1, \ldots, |b_j|$. Similarly let $\alpha_i$ denote an oriented circle with $|a_i^*|$ marked points on it which are numbered $1, \ldots, |a_i^*|$. $F$ gives an identification of marked points on $\alpha_i$ with marked points on $\beta_j$. Construct a 4-regular graph from $\alpha_i$ and $\beta_j$ using the correspondence $F$ by identifying these marked points. The vertices are then the intersection points of $\alpha$ curves with $\beta$ curves.

By an $\alpha$-edge, we mean an edge of the graph which is part of an $\alpha$ curve. Similarly, a $\beta$-edge, is an edge of the graph which is part of a $\beta$ curve.

Let $P = A_1B_1A_2B_2 \ldots A_nB_nA_{n+1}$ be a sequence of $\alpha$-edges and $\beta$-edges such that $A_i$ is adjacent to $B_i$ and $B_i$ is adjacent to $A_{i+1}$, for $1 \leq i \leq n$ with $A_{n+1} = A_1$. Let $A_i \cap B_i = \{v_i\}$ and $B_i \cap A_{i+1} = \{w_i\}$, for $i = 1, \ldots, n$. Let $A_i$ be part of $\alpha_j$, and $B_i$ be part of $\beta_k$. Set an orientation on the edge $A_i$ from $w_{i-1}$ to $v_i$, $i = 1, \ldots, n$, with $w_0 = w_n$. If this orientation is the same as the orientation of $\alpha_j$, set $\epsilon_{A_i} = 1$. Otherwise set $\epsilon_{A_i} = -1$. Also, set an orientation on the edge $B_i$ from $v_i$ to $w_i$, $i = 1, \ldots, n$. If this orientation is the same as
the orientation of $\beta_k$, set $\epsilon_{B_k} = 1$. Otherwise set $\epsilon_{B_i} = -1$. Corresponding to $v_i$, there is a term $a_{j_i}^{i}$ in the relator $b_{j_i}$. Define $\epsilon_{v_i} = \epsilon_{A_i} \epsilon_{B_i} \delta_i$. Also, corresponding to $w_i$, there is a term $a_{j_i}^{i}$ in the relator $b_{j_i}$. Define $\epsilon_{w_i} = -\epsilon_{A_i+1} \epsilon_{B_i} \delta_i$. We say $P$ is a good sequence if $\epsilon_{v_i} - \epsilon_{w_i} = \epsilon_{v_{i+1}}, i = 1, \ldots, n$, with $v_{n+1} = v_1$. Each good sequence $P$ determines an oriented polygon. Each pair of successive letters in $a_d^*$ correspond to an $\alpha$-edge which appears in two oriented polygons, and the corresponding polygons may be glued along these edges. Similarly, polygons may be glued along $\beta$-edges. These gluings of polygons give the surface $\Sigma$.

The following corollary is trivial from the above construction.

**Corollary 3.3.** If $(P, P^*)_F$ is a dual pair of presentations associated with a Heegaard diagram $\mathcal{H} = (\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_d\}, \beta = \{\beta_1, \ldots, \beta_d\})$ for which all regions of $\Sigma - \alpha - \beta$ are polygons, then we have $\mathcal{H} = \mathcal{H}(P, P^*)_F$.

### 4 AC-moves on dual pairs of presentations

Let $(P_0, P_0^*)_F$ be a pair of dual presentations as in Definition 2.2.

**Lemma 4.1.** If an AC-move acts on the pair $(P_0, P_0^*)_F$, it results in a pair $(P_1, P_1^*)_F$ of dual presentations where $P_0$ and $P_1$ (resp. $P_0^*$ and $P_1^*$) are presentations of the same group.

**Proof.** Let AC-n denote the $n^\text{th}$ AC-move. First, assume that $n \neq 4$. Let $P_1 = (a_1, \ldots, a_d|b_1, \ldots, b_d), \quad P_1^* = (b_1^*, \ldots, b_d^*|a_1^*, \ldots, a_d^*)$.

For each AC-n, there is a family of correspondences, denoted by $F_1 = \{f_{i,j}, f_{i,j}^*\}_{i,j}$, $f_{i,j}, f_{i,j}^* : A_{i,j} \to \overline{A}_{i,j}^*$ and $f_{i,j}^* : \overline{A}_{i,j} \to A_{i,j}^*$ (for AC-moves on dual pairs of presentations as in Definition 2.2).

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For AC-2, let $b'_i = b_i^{-1}$ and $b'_j = b_j$ for $j \neq i$. Let $a''_k$ be obtained from $a^*_k$ by replacing all $b^*_i$s (resp. $b^*_{i^{-1}}$s) with $b_i^{*^{-1}}$ (resp. with $b_i^*_{-1}$). If we substitute the generator $b'_i$ in $P_i^*$ with $b_i^{*^{-1}}$, then $P_i^*$ is changed to $P_i^*$. Set

$$A'_i = |b_i| - A_i + 1,$$

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$$A'_ij = A_{ij}, \ j \neq i,$$

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For AC-3, let $g = a_j$ (the case for $g = a_j^{-1}$ is similar), $b'_i = b_ia_ja_j^{-1}$, and $b'_k = b_k$ for $k \neq i$. Let $a''_j = a_j^*b_i^{-1}$ and $a''_k = a^*_k$ for $k \neq i$. Set

$$A'_{ij} = A_{ij} \cup \{b_i + 1\},$$

$$A'_{ij} = A_{ij} \cup \{b_i + 2\},$$

$$A'_{ij} = A_{ij} \cup \{a_j + 1\},$$

$$A'_{ik} = A_{ik}, \ k \neq j, \ or \ l \neq i,$$

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For AC-5, let $g = a_j$ (the case for $g = a_j^{-1}$ is similar), $b'_i$ be the relator obtained from $b_i = b_ia_ja_j^{-1}$ by removing $a_ja_j^{-1}$, and $b'_k = b_k$ for $k \neq i$. Let $a''_j$ be a relator obtained from $a^*_j$ by removing $b_i^*b_i^{-1}$ and $a''_k = a^*_k$ for $k \neq i$. Let $A'_{ij} = \{k_1, \ldots, k_m\}$ where the removed $b'_i$ is the $l$th letter in $a'_j$, $A_{ij} = \{l_1, \ldots, l_n\}$ where the removed $b^*_i$ is the $l$th letter in $a'_j$ (note that $k_s = f_j(|b_i|)$ and $l_t = f_j(|b_i| - 1)$ in AC-5). Set

$$A'_{ij} = A_{ij} \setminus \{b_i - 1\},$$

$$A'_{ij} = A_{ij} \setminus \{b_i\},$$

$$A'_{ij} = \{l_1, \ldots, l_{n-1}, l_{n+1} - 2, \ldots, l_m - 2\},$$

$$A'_{ik} = \{k_1, \ldots, k_{s-1}, k_{s+1} - 2, \ldots, l_m - 2\},$$

$$A'_{ik} = \{k_1, \ldots, k_{s-1}, k_{s+1} - 2, \ldots, l_m - 2\},$$

$$A'_{ik} = \{k_1, \ldots, k_{s-1}, k_{s+1} - 2, \ldots, l_m - 2\}.$$ 

For AC-4, let $a_{d+1}$ be added as both a generator and a relator (the removal of a generator and a relator is similar). Let

$$P_1 = \langle a_1, \ldots, a_d, a_{d+1}|b_1, \ldots, b_d, a_{d+1}\rangle, \ P_1^* = \langle b_1^*, \ldots, b_d^*, b_{d+1}^*|a_1^*, \ldots, a_d^*, b_{d+1}^*\rangle.$$

Set

$$A_{d+1d+1}^* = \{1\},$$

$$\overline{A}_{d+1d+1}^* = \{1\},$$

$$A_{kl} = \emptyset, \ k \neq d + 1, l = d + 1 \ (k = d + 1, l \neq d + 1),$$

$$\overline{A}_{kl} = \emptyset, \ k \neq d + 1, l = d + 1 \ (k = d + 1, l \neq d + 1),$$

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Lemma 4.2. If an AC-move acts on the pair $(P_0, P_0^*)_F$ and gives the pair $(P_1, P_1^*)_F$, then $\mathcal{H}(P_1, P_1^*)_{F_1}$ is obtained from $\mathcal{H}(P_0, P_0^*)_F$ by one of the following changes:

- A Heegaard move (i.e. a handleslide or an isotopy);
- Attaching a 1-handle to the Heegaard surface plus a Heegaard move;
- Changing the orientation for a $\beta$ curve;
- Adding/removing a component which is a standard genus one Heegaard diagram for the three sphere;
- A Heegaard move plus removing a 1-handle from the Heegaard surface.

Proof. Let $\mathcal{H}(P,P^*)_F = (\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_d\}, \beta = \{\beta_1, \ldots, \beta_d\})$. We discuss each AC-move separately.

1. Let $p_i \in \beta_i - \bigcup_{k=1}^d \alpha_k$ be such that if we start from $p_i$ and write the associated relator for $\beta_i$ it results in $b_i$, for $i = 1, \ldots, d$. Let us first assume that $p_1$ and $p_2$ are on the edges of one polygon $P$ and let $P'$ be a polygon adjacent to $P$ via the edge containing $p_j$. Depending on the orientations of $\beta_i$ and $\beta_j$, either slide $\beta_i$ over $\beta_j$ through $P$ (see Figure 3-A) or connect $P$ and $P'$ by a 1-handle and slide $\beta_i$ over $\beta_j$ through the handle (see Figure 3-B). Alternatively, if $p_1$ and $p_j$ are on the edges of two distinct polygons $P_1$ and $P_2$, respectively, let $P'$ be a polygon adjacent to $P_2$ via the edge containing $p_j$. Depending on the orientations of $\beta_i$ and $\beta_j$, either connect $P_1$ and $P_2$ (see Figure 3-C) or $P_1$ and $P'_2$ (see Figure 3-D) by a 1-handle and slide $\beta_i$ over $\beta_j$ through the handle. In both cases, slide $\beta_i$ using an edge containing $p_i$ over $\beta_j$ using an edge which contains $p_j$. This gives a diagram $\mathcal{H}$ for which all regions are polygons and its associated pair of dual presentations is $(P_1, P^*_1)_{F_1}$. Therefore, Corollary 3.3 implies that $\mathcal{H} = \mathcal{H}(P_1, P^*_1)_{F_1}$.

2. Associated with the second move, just changes the orientation of $\beta_i$ to obtain $\mathcal{H}(P_1, P^*_1)_{F_1}$.

3. Choose $p_i$ as in part 1, and let $P$, $P'$, $P_1$, $P_2$, $P'_2$ be as before. Depending on the orientation of $\beta_i$ and $\alpha_j$, either isotope $\beta_i$ over $\alpha_j$ through $P$ (see Figure 4-A) or connect $P$ and $P'$ by a 1-handle and isotope $\beta_i$ over $\alpha_j$ through the handle (see Figure 4-B) or connect $P_1$ and $P_2$ (see Figure 4-C) or $P_1$ and $P'_2$ (see Figure 4-D) by a 1-handle and isotope $\beta_i$ over $\alpha_j$ through the handle. In all cases, isotope the edge containing $p_i$ over the edge containing $p_j$. This gives a diagram $\mathcal{H}$ for which all regions are polygons and its associated pair of dual presentations is $(P_1, P^*_1)_{F_1}$. Therefore Corollary 3.3 implies that $\mathcal{H} = \mathcal{H}(P_1, P^*_1)_{F_1}$.

4. The forth move is equivalent to adding or removing a component, which is the standard genus one Heegaard diagram for the three sphere, to or from $\mathcal{H}(P_1, P^*_1)_{F_1}$. Note that the Heegaard surface may be disconnected. The resulting diagram is $\mathcal{H}(P_1, P^*_1)_{F_1}$.
Figure 3: For the first AC-move, $\mathcal{H}_{(P_1, P_1^*)_F}$ is obtained from $\mathcal{H}_{(P, P^*)_F}$ by a handleslide (Part A) or attaching a 1-handle plus a handleslide (Parts B, C, and D). Each 1-handle is considered as $S^1 \times [0, 1]$ where its boundaries are identified with an orientation preserving homeomorphism with the two circles in Parts B, C, or D.

5. Consider the fifth AC-move which removes $a_j a_j^{-1}$ from the relator $b_i$ and removes $b_i^* b_i'^{-1}$ from the relator $a_j^*$. This means that there is a bigon in $\mathcal{H}_{(P, P^*)_F}$ such that its $\beta$-edge is determined by $b_i'^{-1} b_i^*$ and its $\alpha$-edge is determined by $a_j a_j^{-1}$. Let $P_1$ and $P_2$ denote polygons in $\mathcal{H}_{(P, P^*)_F}$ which have vertices but no edges in common with the bigon (see Figure 5). After an isotopy through the bigon, the two intersections between $\beta_i$ and $\alpha_j$ disappear.

If $P_1$ and $P_2$ are two distinct polygons (see Figure 5 in the middle), denote the resulting diagram by $\mathcal{H}$. If $P_1$ and $P_2$ are the same polygon (see Figure 5 on the right), then the isotopy changes $P_1$ and $P_2$ to a cylinder which may be thought of as a 1-handle. Remove this 1-handle from the diagram and denote the resulting diagram by $\mathcal{H}$. All regions of $\mathcal{H}$ are polygons and its associated pair of dual presentations is $(P_1, P_1^*)_F$. Therefore Corollary 3.3 results in $\mathcal{H} = \mathcal{H}_{(P_1, P_1^*)_F_1}$.
Figure 4: For the third AC-move, \( \mathcal{H}(P_1, P_1^*) \) is obtained from \( \mathcal{H}(P, P^*) \) by an isotopy (Part A) or attaching a 1-handle plus an isotopy (Parts B, C, and D).

5 Heegaard Floer homology

We restrict our attention to presentations \( P \) of a group \( G \) which admit a dual presentation \( P^* \) with a correspondence \( F \) such that the Heegaard diagram

\[
\mathcal{H}(P, P^*) = (\Sigma, \alpha, \beta)
\]

has the following properties.

(A) For each component \( \Sigma_i \) of \( \Sigma \), \( \Sigma_i - \alpha \), and \( \Sigma_i - \beta \) are connected;

(B) The diagram has a set of completing curves \( \alpha^c \), as defined in Definition 5.1 below.

From Lemma 4.2, the first property is preserved under AC-moves on the pair \( (P, P^*) \). Assuming Claim 5.8, we show (via Lemma 5.7) that the second property is also preserved under AC-moves.
5.1 Completing curves

Let $P = \langle a_1, \ldots, a_d | b_1, \ldots, b_d \rangle$ and $P^* = \langle b_1^*, \ldots, b_d^* | a_1^*, \ldots, a_d^* \rangle$ be a pair of dual presentations for $G$ with a family of correspondences given by $F$ and

$$\mathcal{H}_{(P,P^*)} = (\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_d\}, \beta = \{\beta_1, \ldots, \beta_d\}).$$

**Definition 5.1.** Let $\mathcal{H}_{(P,P^*)}$ be as above. A set of marked, oriented, disjoint, simple closed curves $\alpha^c = \{\alpha_{d+1}, \ldots, \alpha_g\}$ in $\Sigma$ is called a set of completing curves for $\Sigma$ (or for $\mathcal{H}_{(P,P^*)}$) if

1. $\alpha \cap \alpha^c = \emptyset$;
2. For each component $\Sigma_i$ of $\Sigma$, $\Sigma_i - \{\alpha_1, \ldots, \alpha_g\}$ is a punctured sphere;
3. The relators associated with $\alpha_i$, $i = d + 1, \ldots, g$ are trivial in $\langle b_1^*, \ldots, b_d^* | a_1^*, \ldots, a_d^* \rangle$.

We assume that there is an arc, denoted by $\beta_i$, which only intersects $\alpha_i$ in a single point at the marked point on $\alpha_i$, $i = d + 1, \ldots, g$, and is disjoint from $\alpha$ and $\beta$ curves. We denote the set of these $\beta$ arcs by $\beta^a$.

5.2 Heegaard Floer homology groups for diagrams with $\beta$-arcs

Let $\overline{\mathcal{H}}_{(P,P^*)} = (\Sigma, \alpha \cup \alpha^c = \{\alpha_1, \ldots, \alpha_g\}, \beta \cup \beta^a = \{\beta_1, \ldots, \beta_g\}, z)$, where the marked points $z$ are in $\Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$ and each component of $\Sigma$ contains exactly one marked point. The following proposition is similar to [OS04b, Proposition 7.1].

**Proposition 5.2.** Two Heegaard diagrams

$$\mathcal{H}_{(P,P^*)} = (\Sigma, \alpha \cup \alpha^c = \{\alpha_1, \ldots, \alpha_g\}, \beta \cup \beta^a = \{\beta_1, \ldots, \beta_g\}, z)$$

$$\mathcal{H}_{(P,P^*)} = (\Sigma, \alpha \cup \alpha^c = \{\alpha_1, \ldots, \alpha_g\}, \beta \cup \beta^a = \{\beta_1, \ldots, \beta_g\}, w)$$

completing $\mathcal{H}_{(P,P^*)}$ with different choices of the set of marked points are related by a finite sequence of pointed Heegaard moves (i.e. Heegaard moves supported in the complement of the marked point).
Let $\mathbb{T}_s = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_b = \beta_1 \times \cdots \times \beta_g$ denote the subspaces of the symmetric product $\text{Sym}^d(\Sigma)$ where $\mathbb{T}_s$ is a torus and $\mathbb{T}_b$ is an open subset of a torus. For $x, y \in \mathbb{T}_s \cap \mathbb{T}_b$, let $\mathcal{P}_2(x, y)$ denote the set of homotopy classes of Whitney disks connecting $x$ and $y$. Let $J(T)$ denote a generic path of complex structures on $\Sigma$ such that each boundary component of a fixed neighborhood $\mathcal{N}_i$ of each $\alpha_i, i = d + 1, \ldots, g$ is pinched to a point as $T$ goes to infinity. Let $\mathcal{M}_{J(T)}(\phi)$ denote the moduli space of $J(T)$-holomorphic representatives of the Whitney class $\phi$ (modulo action of $\mathbb{R}$).

**Theorem 5.3.** Let $J(T_k)$ denote a path of almost complex structures associated with $\Sigma$ as above, with $T_k \rightarrow \infty$ as $k \rightarrow \infty$. For every $\phi \in \mathcal{P}_2(x, y)$ the number of solutions in $\mathcal{M}_{J(T_k)}(\phi)$ (counted with sign) becomes stable for sufficiently large values of $k$.

To prove this theorem, we use the cylindrical reformulation of Heegaard Floer homology (c.f. [Lip06]).

**Proof.** We claim that there is some $N$ such that for any $k \geq N$, the part of boundary of any holomorphic disk in $\mathcal{M}_{J(T_k)}(\phi)$ which uses $\beta_i$ does not leave the stretched neighborhood $\mathcal{N}_i$.

If this is not true, then there is some $d + 1 \leq i_0 \leq g$, and a sequence of holomorphic disks $u_k \in \mathcal{M}_{J(T_k)}(\phi)$ such that some part of the boundary of $u_k$ which uses $\beta_{i_0}$ leaves the neighborhood $\mathcal{N}_{i_0}$. Corresponding to each $u_k$, there is a surface $S_k$ (with boundary) which is constructed from $\mathcal{D}(u_k)$, the domain associated with $u_k$, and $u_k$ is a holomorphic map from $S_k$ to $\Sigma \times \mathbb{D}$. Here $\mathbb{D}$ is the unit disk and the almost complex structure on $\Sigma$ is $J(T_k)$. Let $u_k^\Sigma : S_k \rightarrow \Sigma$ and $\tilde{u}_k^\Sigma : S_k \rightarrow \mathbb{D}$ denote the projections of this map to $\Sigma$ and $\mathbb{D}$ respectively. There are parts of the boundary components of $S_k$ which are mapped to $\alpha_i$ and $\beta_i, i = d + 1, \ldots, g, \tilde{u}_k^\Sigma$.

Since $\{u_k\}_k$ is a sequence of holomorphic curves with bounded energy, it has a weak limit. Let $\tilde{u} : S \rightarrow \Sigma \times \mathbb{D}$ denote a component of this weak limit such that at least one boundary component of $S$ is mapped to $\beta_{i_0} \setminus \mathcal{N}_{i_0}$ by $\tilde{u}^\Sigma$, the projection of $\tilde{u}$ to $\Sigma$. Therefore, there is a boundary component of $S$ which is mapped to $\beta_{i_0}$ and no part of it is mapped to $\alpha_{i_0}$. Since $\beta_{i_0}$ only intersects $\alpha_{i_0}$, the whole boundary component of $S$ is mapped to $\beta_{i_0}$. If this boundary component is projected by $\tilde{u}^\mathbb{D}$ to the whole boundary of $\mathbb{D}$, then all the boundary components of $S$ are projected to $\beta$ curves and $\beta$ arcs by $\tilde{u}^\Sigma$. On the other hand, if this boundary component is projected by $\tilde{u}^\mathbb{D}$ to a single point (i.e. a point with negative real coordinate on the boundary of $\mathbb{D}$) then the maximum principle implies that $S$ is mapped to this single point by $\tilde{u}^\mathbb{D}$. Once again, this means that all the boundary components of $S$ are mapped to $\beta$ curves and $\beta$ arcs by $\tilde{u}^\Sigma$. In both cases, we conclude that $\mathcal{D}(\tilde{u})$ is a periodic domain which crosses the marked point. This is in contradiction with the assumption $n_z(\phi) = 0$. 

The path $J(T)$, for $T$ sufficiently large so that the condition of Theorem 5.3 is true, will be called sufficiently pinched near $\alpha_{d+1}, \alpha_{d+2}, \ldots, \alpha_g$.

Let $\widehat{CF}(\overline{\mathcal{H}}_{(P, P^*), x})$ be a free abelian group over $\mathbb{Z}/2$ generated by the $g$-tuples $x = \{x_1, \ldots, x_g\} \in \mathbb{T}_s \cap \mathbb{T}_b$ such that $x_i$ is an intersection point of $\alpha_i$ with some $\beta_{\sigma(i)}$, where $\sigma$ is a permutation on $g$ letters. For sufficiently large values of $T_i$, as stated in Theorem 5.3, let

$$
\partial_{J(T_i)} : \widehat{CF}(\overline{\mathcal{H}}_{(P, P^*), x}) \rightarrow \widehat{CF}(\overline{\mathcal{H}}_{(P, P^*), x})
$$

be the map defined by

$$
\partial_{J(T_i)} x = \sum_{y \in \mathbb{T}_s \cap \mathbb{T}_b} \sum_{\phi \in \mathcal{P}_2(x, y) | \mu(\phi) = 1, \ n_z(\phi) = 0} \# \mathcal{M}_{J(T_i)}(\phi) y.
$$

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A small modification of standard arguments in Heegaard Floer theory implies that
\[
(CF(\mathcal{H}_{(P,P^*)}), \partial_{J(T_i)})
\]
is a chain complex. We define the Floer homology group \(\hat{H}F(\mathcal{H}_{(P,P^*)})\) to be the homology group associated with the chain complex \((CF(\mathcal{H}_{(P,P^*)}), \partial_{J(T_i)})\).

Similar to [OS04b, Theorem 7.3 and Theorem 9.5], we can prove the following proposition.

**Proposition 5.4.** Let \(\mathcal{H}_1 = (\Sigma, \alpha \cup \alpha^c = \{\alpha_1, \ldots, \alpha_g\}, \beta \cup \beta^a = \{\beta_1, \ldots, \beta_g\}, z)\) and suppose that \(\alpha \cup \alpha^c\) (resp. \(\beta\)) is obtained from \(\alpha \cup \alpha^c\) (resp. \(\beta\)) by a sequence of handleslides and isotopies, and let \(\mathcal{H}_2 = (\Sigma, \beta \cup \beta^c = \{\beta_1, \ldots, \beta_g\}, w)\). Then we have \(\hat{H}F(\mathcal{H}_1) \cong \hat{H}F(\mathcal{H}_2)\).

**5.3 Attaching a one handle**

Let \(\mathcal{H} = (\Sigma, \alpha = \{\alpha_1, \ldots, \alpha_g\}, \beta = \{\beta_1, \ldots, \beta_g\}, z)\) be a Heegaard diagram, possibly with \(\beta\)-arcs. In a component of \(\Sigma\), we connect two regions of this diagram by a 1-handle and denote the new surface by \(\Sigma_1\). Let \(\alpha_{g+1}\) be the meridian of this 1-handle and \(\beta_{g+1}\) be an arc which intersects only \(\alpha_{g+1}\) in a single point.

Let \(\mathcal{H}_1 = (\Sigma_1, \alpha \cup \{\alpha_{g+1}\}, \beta \cup \{\beta_{g+1}\}, z)\). \(\hat{CF}(\mathcal{H}_1)\) is then a free abelian group generated by \(g + 1\)-tuples \(x = \{x_1, \ldots, x_{g+1}\}\) such that \(x_i\) is an intersection point of \(\alpha_i\) with a \(\beta_{\sigma(i)}\), where \(\sigma\) is a permutation on \(g + 1\) letters. Clearly \(x_{g+1}\) is the unique intersection point between \(\alpha_{g+1}\) curve and \(\beta_{g+1}\) arc. Corresponding to each generator \(x\), we consider a \(g\)-tuple \(\bar{x} = \{x_1, \ldots, x_g\}\) for the diagram \(\mathcal{H}\).

Let \(D_1, \ldots, D_m\) denote domains of \(\mathcal{H}_1\), i.e. closure of components of \(\Sigma - \alpha \cup \{\alpha_{g+1}\} - \beta\), such that \(D_1\) and \(D_2\) contain \(\alpha_{g+1}\) in their boundaries. Let \(\bar{D}_1\) and \(\bar{D}_2\) be domains obtained from \(D_1\) and \(D_2\) by attaching disks to their \(\alpha_{g+1}\) boundaries. Therefore, the domains of \(\mathcal{H}\) are \(\bar{D}_1, \bar{D}_2, D_3, \ldots, D_m\).

Let \(J\) be a path of complex structure on \(\Sigma\) and \(J(T)\) denote a path of complex structure on \(\Sigma_1\) which is sufficiently pinched near \(\alpha_{g+1}\) such that as \(T\) goes to infinity, each boundary of a tubular neighborhood of \(\alpha_{g+1}\) is pinched to a point.

Similar to [AE, Proposition 5.1] one can prove:

**Proposition 5.5.** Let \(J(T_i)\) denote a path of complex structures associated with \(\Sigma_1\) such that \(\alpha_{g+1}\) is pinched as \(T_i \to \infty\). Choose \(\phi_1 \in \pi_2(x, y)\) with \(D(\phi_1) = \sum_{i=1}^m a_i D_i\) and let \(\phi\) be a corresponding Whitney disk in \(\mathcal{H}\) connecting corresponding generators \(\bar{x}\) and \(\bar{y}\) with \(D(\phi) = a_1 D_1 + a_2 D_2 + \sum_{i=3}^m a_i D_i\). We have \(\mu(\phi_1) = \mu(\phi)\) and if \(\mathcal{M}(\phi_1)\) is nonempty for the sequence \(J(T)\) of almost complex structures, then \(\mathcal{M}(\phi)\) is also nonempty. Moreover, if \(\mu(\phi_1) = 1\), then we have \(\tilde{\mathcal{M}}(\phi_1) \cong \tilde{\mathcal{M}}(\phi)\) for sufficiently large \(T_i\).

**5.4 Invariance**

Let \(\alpha_0^c\) be a set of completing curves for \(\mathcal{H}_{(P,P^*)}\). If \(\alpha_0^c\) is obtained from \(\alpha^c\) by a sequence of isotopies and handleslides over curves in \(\alpha \cup \alpha^c\), it clearly determines a set of completing curves and from Proposition 5.4, \(\hat{H}F(\mathcal{H}_{(P,P^*)})\) is invariant under these changes of completing curves. The set of completing curves is thus partitioned into equivalence classes, where the sets of completing curves in each equivalence class are related to each other by isotopies and handleslides. \(\hat{H}F(\mathcal{H}_{(P,P^*)})\) remains invariant on each equivalence class. However, it is not clear that different equivalence classes give the same Heegaard Floer homology group. This is the main unknown part of the proof of Claim 1.6.
Claim 5.6. $\tilde{\mathcal{HF}}(\tilde{\mathcal{H}}_{(P,P^*)_F})$ remains invariant under different choices of completing curves for $\alpha$.

Assuming this claim is valid, we denote $\tilde{\mathcal{HF}}(\tilde{\mathcal{H}}_{(P,P^*)_F})$ by $\tilde{\mathcal{HF}}(\mathcal{H}_{(P,P^*)_F})$. The above claim is used in the arguments of this section.

Lemma 5.7. Let $(P_1, P_1^*)_F$ be a pair of dual presentations obtained from $(P, P^*)_F$ by an AC-$i$ move, $1 \leq i \leq 4$. If there is a set of completing curves for $\mathcal{H}_{(P,P^*)_F}$, then $\mathcal{H}_{(P_1,P_1^*)_F}$ also has a set of completing curves and

$$\tilde{\mathcal{HF}}(\mathcal{H}_{(P,P^*)_F}) \cong \tilde{\mathcal{HF}}(\mathcal{H}_{(P_1,P_1^*)_F}).$$

Proof. According to the proof of Lemma 4.2, $\mathcal{H}_{(P_1,P_1^*)_F}$ is obtained from $\mathcal{H}_{(P,P^*)_F}$ by a Heegaard move, or by attaching a 1-handle plus a Heegaard move, or by changing the orientation for a $\beta$ curve, or by adding/removing a component which is the standard genus one Heegaard diagram for $S^3$. For the third and forth case, it is clear that $\mathcal{H}_{(P,P^*)_F}$ and $\mathcal{H}_{(P_1,P_1^*)_F}$ have the same set of completing curves and $\tilde{\mathcal{HF}}(\mathcal{H}_{(P,P^*)_F}) \cong \tilde{\mathcal{HF}}(\mathcal{H}_{(P_1,P_1^*)_F})$.

Let $\mathcal{H}_{(P_1,P_1^*)_F}$ be obtained from $\mathcal{H}_{(P,P^*)_F}$ by a Heegaard move. The proof of Lemma 4.2 implies that this Heegaard move is a handleslide of a $\beta_i$ curve over a $\beta_j$ curve or an isotopy of a $\beta_i$ curve over an $\alpha_j$ curve. Therefore, $\tilde{\mathcal{H}}_{(P,P^*)_F}$ is obtained from $\tilde{\mathcal{H}}_{(P_1,P_1^*)_F}$ by Heegaard moves and lemma is proved in this case, by standard arguments in Heegaard Floer theory.

Let $\mathcal{H}_{(P_1,P_1^*)_F}$ be obtained from $\mathcal{H}_{(P,P^*)_F}$ by inserting a handle to $\Sigma$ and doing an isotopy of $\beta_i$ on $\alpha_j$ or doing a handleslide of $\beta_i$ over $\beta_j$, $i, j \in \{1, \ldots, d\}$ through this handle. Let $\Sigma_1$ denote the new surface. First suppose that the new handle connects two different components of $\Sigma$ and two different components of $\Sigma - \beta$. In this case, we may assume that the polygons containing the legs of the 1-handle contain marked points from $z$. In fact, each component of $\Sigma$ contains a marked point and according to Proposition 5.2, there is a finite sequence of Heegaard moves which relates different choices of marked points in each component of $\Sigma$ which (by Proposition 5.4) result in the same Heegaard Floer homology group. Then, we have

$$\tilde{\mathcal{H}}_{(P_1,P_1^*)_F} = (\Sigma_1, \alpha \cup \alpha^c, \beta_1 \cup \beta^a, z_1),$$

where $\beta_1$ is obtained from $\beta$ by doing a Heegaard move and $z_1$ is the same set as $z$ except that the two marked points next to the legs of the 1-handle are identified. Let $\mathcal{H} = (\Sigma_1, \alpha \cup \alpha^c, \beta \cup \beta^a, z_1)$. We then have $\tilde{\mathcal{HF}}(\tilde{\mathcal{H}}_{(P_1,P_1^*)_F}) \cong \tilde{\mathcal{HF}}(\mathcal{H})$. The two diagrams $\mathcal{H}$ and $\tilde{\mathcal{H}}_{(P,P^*)_F}$ have the same set of generators and since the Whitney disks in $\mathcal{H}$ do not use the handle, it follows that the Whitney disks in $\mathcal{H}$ and the corresponding moduli spaces are in correspondence with the Whitney disks in $\tilde{\mathcal{H}}_{(P,P^*)_F}$ and the corresponding moduli spaces. From here, $\tilde{\mathcal{HF}}(\tilde{\mathcal{H}}_{(P,P^*)_F}) \cong \tilde{\mathcal{HF}}(\mathcal{H})$ and the lemma follows in this case from Claim 5.6.

Suppose now that the new handle is attached to a single component of $\Sigma$. Let $\alpha_{g+1}$ be the meridian of this 1-handle and $\beta_{g+1}$ be an arc which only intersects $\alpha_{g+1}$ in a single point. We may assume that one of the polygons containing the legs of the 1-handle contains a marked point $z \in z$. In fact, according to Proposition 5.2, there is a finite sequence of moves which relates different choices of marked points which (by Proposition 5.4) result in the same Heegaard Floer homology group. Let

$$\mathcal{H} = (\Sigma_1, \alpha \cup \alpha^c \cup \{\alpha_{g+1}\}, \beta \cup \beta^a \cup \{\beta_{g+1}\}, z).$$

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There is a correspondence between the generators of $H$ and $H(P, P^*)_F$. In fact, if $x_{g+1}$ denotes the intersection point of $\alpha_{g+1}$ and $\beta_{g+1}$, then each generator of $H$ is of the form $\mathbf{x} \cup \{x_{g+1}\}$ where $\mathbf{x}$ is a generator of $H(P, P^*)_x$. Let $D_1, D_2, D_3, \ldots, D_m$ denote the domains of $H$, i.e. closure of components of

$$\Sigma_1 - \alpha \cup \alpha^c \cup \{\alpha_{g+1}\} - \beta \cup \beta^c,$$

such that $D_1$ and $D_2$ contain $\alpha_{g+1}$ in their boundaries and $D_1$ contains the marked point $z$. Let $D_1$ and $D_2$ be domains obtained from $D_1$ and $D_2$ by attaching disks to their $\alpha_{g+1}$ boundaries. Therefore, domains of $H(P, P^*)_x$ are $\bar{D_1}, \bar{D_2}, \bar{D_3}, \ldots, \bar{D_m}$. Let $\phi \in \pi_0^g(\mathbf{x}, \mathbf{y})$, where $\pi_0^g(\mathbf{x}, \mathbf{y})$ denotes the space of homotopy classes of Whitney disks in $H(P, P^*)_x$, connecting $\mathbf{x}$ and $\mathbf{y}$, which do not cross the marked point. Further assume that $\mathcal{D}(\phi) = c_2\bar{D_2} + \sum_{i=3}^m c_i\bar{D_i}$. Now a corresponding disk $\tilde{\phi} \in \pi_2(\{x_{g+1}\} \cup \mathbf{x}, \{x_{g+1}\} \cup \mathbf{y})$ is determined by the domain $\sum_{i=2}^m c_i\bar{D_i}$. From Proposition 5.5, we have

$$\widehat{HF}(\overline{H}(P, P^*)_x) \cong \widehat{HF}(H)$$

and lemma is proved in this case.

After doing the isotopy or handleslide through the handle (corresponding to the AC-move), the relator associated with $\alpha_{g+1}$ is $b_i^{-1}b_i^* - 1$ or $b_i^{-1}b_i^*$, therefore curves in $\alpha^c \cup \{\alpha_{g+1}\}$ satisfy condition 3 in Definition 5.1. Clearly the curves in $\alpha^c \cup \{\alpha_{g+1}\}$ satisfy conditions 1-2 and 4 of Definition 5.1. Let

$$\overline{H}(P, P^*)_{F_1} = (\Sigma_1, \alpha \cup \alpha^c \cup \{\alpha_{g+1}\}, \beta_1 \cup \beta^a \cup \{\beta_{g+1}\}, z),$$

where $\beta_1$ is obtained by $\beta$ after doing the Heegaard move. From Proposition 5.4, we have

$$\widehat{HF}(\overline{H}(P, P^*)_{F_1}) \cong \widehat{HF}(H),$$

and therefore $\widehat{HF}(\overline{H}(P, P^*)_{F_1}) \cong \widehat{HF}(\overline{H}(P, P^*)_F)$. \hfill $\square$

The second technical step which remains unsettled in this paper is the following.

**Claim 5.8.** Let $(P_1, P_1^*)_{F_1}$ be a pair of dual presentations obtained from $(P, P^*)_F$ by the fifth AC-move. If there is a set of completing curves for $H(P, P^*)_F$, then $H(P_1, P_1^*)_{F_1}$ also has a set of completing curves and

$$\widehat{HF}(H(P, P^*)_F) \cong \widehat{HF}(H(P_1, P_1^*)_{F_1}).$$

Assuming Claim 5.6 and Claim 5.8, the invariance under several other choices involved in the construction of the groups $\widehat{HF}(H(P, P^*)_F)$ may be proved.
Lemma 5.9. Let \((P, P^*)_\mathcal{F}\) and \((P, P^*)_\mathcal{G}\) be two pairs of dual presentations. Then we have 

\[ \overline{HF}(\mathcal{H}(P, P^*_\mathcal{F})) \cong \overline{HF}(\mathcal{H}(P, P^*_\mathcal{G})) \]

Proof. First consider the simplest case where the two families \(\mathcal{F} = \{ f_{ij}, \overline{f}_{ij}\}_{i,j}\) and \(\mathcal{G} = \{ g_{ij}, \overline{g}_{ij}\}_{i,j}\) are the same except that \(f_{11}\) is obtained from \(g_{11}\) by composing with a transposition (the case where \(\overline{f}_{11}\) is obtained from \(\overline{g}_{11}\) by composing with a transposition is similar).

Let \(b_1 = a_1 \cdot b_2 \cdot a_1^{-1} \cdot b_2^{-1}\) letter in the word \(a_1^k a_2^l\) and \(a_1^k a_2^l\) letter in the word \(b_1\). Let \(f_{11}\) send \(k_1\) to \(l_1\) and \(k_2\) to \(l_2\) and \(g_{11}\) send \(k_1\) to \(l_2\) and \(k_2\) to \(l_1\). Consider a Heegaard diagram \(\mathcal{H}(P, P^*_\mathcal{F}) = (\Sigma, \alpha = \{ \alpha_1, \ldots, \alpha_d \}, \beta = \{ \beta_1, \ldots, \beta_l \})\). Figure 7 (on the top) shows part of this diagram. Let \(\mathcal{H}' = (\Sigma_1, \alpha, \beta)\) where \(\Sigma_1\) is a surface obtained from \(\Sigma\) by adding two 1-handles as in Figure 7 on the bottom. Here two parts of \(\beta_1\) are connected by two 1-handles. One may imagine each handle as an \(S^1 \times [0, 1]\) such that its boundaries are identified with the two oriented circles with the same color, with an orientation preserving homeomorphisms. Let us first assume that all regions in \(\mathcal{H}'\) are polygons. Then, by Corollary 3.3, \(\mathcal{H}' = \mathcal{H}(P, P^*_\mathcal{G})\). Let \(\alpha^c = \{ \alpha_{d+1}, \ldots, \alpha_g \}\) denote the completing curves for \(\mathcal{H}(P, P^*_\mathcal{F})\). If \(\alpha_i\), for \(i = g + 1, g + 2\), denote the two simple closed curves as illustrated in Figure 8, then clearly \(\alpha^c \cup \{ \alpha_{g+1}, \alpha_{g+2} \}\) are completing curves for \(\mathcal{H}(P, P^*_\mathcal{G})\).

![Figure 7](image-url)

Figure 7: The figure on the top is part of a diagram determined by \((P, P^*_\mathcal{F})\) and the figure on the bottom is part of a diagram determined by \((P, P^*_\mathcal{G})\). In these figures, the boxes \(A, B, C\) shows other possible \(\alpha\) curves which intersect \(\beta_1\).

Let \(\beta_i\), for \(i = g + 1, g + 2\), denote two arcs as illustrated on the top of Figure 8 and \(\beta_i'\), \(i = g + 1, g + 2\), denote two simple closed curves as illustrated on the bottom of Figure 8. Let

\[ \mathcal{H} = (\Sigma_1, \alpha \cup \alpha^c, \beta \cup \{ \beta_{g+1}', \beta_{g+2}' \}) \]

For each diagram in Figure 8, starting from the part of \(\alpha_1\) which intersects \(\beta_1\) between \(\alpha\) curves in the boxes \(A\) and \(B\), handleslide of \(\alpha_1\) over \(\alpha_{g+1}\), then handleslide the \(\alpha\) curves in the box \(B\), first over \(\alpha_{g+2}\) and then over \(\alpha_{g+1}\). Finally, starting from the part of \(\alpha_1\) which intersects \(\beta_1\) between \(\alpha\) curves in the boxes \(B\) and \(C\), handleslide \(\alpha_1\) over \(\alpha_{g+2}\). After doing these handleslides, we obtain the two diagrams on the top of Figure 9. Now, isotope \(\beta_1\) in the first diagram of Figure 9, and isotope \(\beta_g, \beta_{g+2}'\) and \(\beta_1\) in the second diagram of Figure 9, to obtain the two diagrams on the bottom of Figure 9 respectively.
Figure 8: $\alpha_{g+1}$ and $\alpha_{g+2}$ along with curves in $\alpha^c$ are completing curves for the diagram $\mathcal{H}_{(P, P^*)g}$.

Let $x_i$, $i = g + 1, g + 2$, denote the unique intersection point of $\alpha_i$ with the arc $\beta_i$ for the third diagram in Figure 9 and

$$x = \{x_1, \ldots, x_g, x_{g+1}, x_{g+2}\}, \quad y = \{y_1, \ldots, y_g, x_{g+1}, x_{g+2}\}$$

be two generators for this diagram where $x_i, y_i \in \alpha_i \cap \beta_{\sigma(i)}$, $i = 1, \ldots, g$ and $\sigma$ is a permutation on $g$ letters. Let $x'_i$, $i = g + 1, g + 2$, denote the unique intersection point of $\alpha_i$ with $\beta'_i$, $x_0$ (resp. $x'_0$) be the intersection point of $\alpha_{g+1}$ (resp. $\alpha_{g+2}$) with $\beta_1$ as denoted in Figure 9 on the bottom and

$$x' = \{x_1, \ldots, x_g, x'_{g+1}, x'_{g+2}\}, \quad y' = \{y_1, \ldots, y_g, x'_{g+1}, x'_{g+2}\}$$

be the two corresponding generators for this diagram. Consider the class of a Whitney disk $\phi \in \pi_2(x', y')$ in this diagram and let $m_i$ and $k_i$, $i = 1, 2$ denote local coefficients of $\phi$ on the two sides of $\alpha_{g+1}$ and $\alpha_{g+2}$, as denoted in Figure 9 on the bottom. A computation of coefficients for the disk $\phi$ around $x_0$ and $x'_0$ shows that

$$m_1 + k_1 = m_2 + k_1 \quad \text{and} \quad m_1 + k_1 = m_1 + k_2.$$ 

This means that for each disk $\phi$ as above, the coefficients of $\phi$ around $x_{g+1}$ (resp. $x_{g+2}$) are the same. As a result, for the two diagrams in Figure 9 on the bottom, the generators and the Whitney disks are in correspondence. An argument similar to the proof of Theorem 5.3 proves that the moduli spaces of holomorphic disks in the two diagrams are also in correspondence. Therefore, from Proposition 5.4, $\hat{HF}(\mathcal{H}_{(P, P^*)g}) \cong \hat{HF}(\mathcal{H})$.

On the other hand, for the diagram $\mathcal{H}$, after doing a handleslide of $\beta_1$ over $\beta_{g+1}$ (see diagram B in Figure 10) and then doing isotopies on $\beta_1$ (see diagram C in Figure 10), no $\beta$ curve intersects $\alpha_{g+2}$ and we can remove a 1-handle which has $\alpha_{g+1}$ as its meridian (see diagram D in Figure 10). Now, doing a second handleslide of $\beta_1$ over $\beta_{g+1}$ (see diagram E in Figure 10) and then doing some isotopies of $\beta_1$ (see diagram F on the right in Figure 10), no $\beta$ curve intersects $\alpha_{g+1}$ and we can remove the 1-handle which has $\alpha_{g+1}$ as its meridian (see diagram G in Figure 10). In this diagram, the associated relator for $\beta_1$ is $b_1$ and the
Figure 9: Doing a sequence of handleslides of some curves $\alpha \in \alpha \cup \alpha^c$ over $\alpha_{g+1}$ and $\alpha_{g+2}$, we obtain the two diagrams on the top. Doing isotopies of $\beta_{g+1}$, $\beta_{g+2}$, and $\beta_1$, we obtain the two diagrams on the bottom.

correspondences are given by $\mathcal{F}$. Therefore from Lemma 5.7, we have $\tilde{HF}(\mathcal{H}_{(P,P^*)_F}) \cong \tilde{HF}(\mathcal{H})$, which proves
$$\tilde{HF}(\mathcal{H}_{(P,P^*)_F}) \cong \tilde{HF}(\mathcal{H}_{(P,P^*)_G}).$$

Now let us consider the case where some regions for the diagram $\mathcal{H}'$ are not polygons. By doing isotopies in the diagram $\mathcal{H}_{(P,P^*)_F}$, which are equivalent to the first AC-move on the pair $(P,P^*)_F$, we can assume that $P_i$ and $P'_i$, $i = 1, \ldots, 4$, are disjoint polygons (see Figure 7). Let $(P_1,P^*_{1})_{\mathcal{F}}$ be obtained from $(P,P^*)_{\mathcal{F}}$ by these AC-moves. Also let $(P_1,P^*_{1})_{\mathcal{G}}$ be obtained from $(P,P^*)_{\mathcal{G}}$ by the corresponding AC-moves. Since $P_i$ and $P'_i$, $i = 1, \ldots, 4$, are disjoint polygons in the diagram $\mathcal{H}_{(P_1,P^*_{1})_{\mathcal{F}}}$, the diagram $\mathcal{H}_{(P_1,P^*_{1})_{\mathcal{G}}}$ is obtained as above by adding two one-handle to $\mathcal{H}_{(P_1,P^*_{1})_{\mathcal{F}}}$ and two completing curves. Therefore, from the above discussion we have
$$\tilde{HF}(\mathcal{H}_{(P_1,P^*_{1})_{\mathcal{F}}}) \cong \tilde{HF}(\mathcal{H}_{(P_1,P^*_{1})_{\mathcal{G}}}).$$
Figure 10: The figure shows how a sequence of handleslides and isotopies removes the two handles which are determined by the meridians $\alpha_{g+1}$ and $\alpha_{g+2}$. 
Also from Lemma 5.7 and Claim 5.8, we have
\[ \overline{HF}(\mathcal{H}(P,P^*)) \cong \overline{HF}(\mathcal{H}(P_1,P_1')) \]
which proves the lemma in this case.

For general families of correspondences \( F = \{ f_{ij}, f'_{ij} \}_{i,j} \) and \( G = \{ g_{ij}, g'_{ij} \}_{i,j} \), note that each map \( f_{ij} \) (resp. \( f'_{ij} \)) is a composition of \( g_{ij} \) (resp. \( g'_{ij} \)) with some transpositions. This proves the lemma in the general case.

**Lemma 5.10.** Let \( (P_1, P^*_1) \) and \( (P_2, P^*_2) \) be two pairs of dual presentations with the same dual presentation \( P^* \) for the two presentations \( P_1 \) and \( P_2 \). Then we have
\[ \overline{HF}(\mathcal{H}(P_1,P^*_1)) \cong \overline{HF}(\mathcal{H}(P_2,P^*_2)). \]

**Proof.** Let \( P_1 = \langle a_1, \ldots, a_d| b_1, \ldots, b_d \rangle \) and \( P_2 = \langle a_1, \ldots, a_d'| b_1', \ldots, b_d' \rangle \). Let
\[ A_{ij} = \{ k | a_i \text{ is } k^{th} \text{ letter in } b_j, 1 \leq k \leq |b_j| \}, \]
\[ A'_{ij} = \{ k | a_i \text{ is } k^{th} \text{ letter in } b'_j, 1 \leq k \leq |b'_j| \}, \]
and
\[ A_{ij} = \{ k | a_i^{-1} \text{ is } k^{th} \text{ letter in } b_j, 1 \leq k \leq |b_j| \}, \]
\[ A'_{ij} = \{ k | a_i^{-1} \text{ is } k^{th} \text{ letter in } b'_j, 1 \leq k \leq |b'_j| \}, \]
as in Defintion 2.2. Since \( P_1 \) and \( P_2 \) have the same dual presentation, from Definition 2.2 we have \( d = d' \), \( |A_{ij}| = |A'_{ij}| \), and \( |A_{ij}| = |A'_{ij}| \), for each \( 1 \leq i, j \leq d \). This means that for each \( j \), \( j = 1, 2, \ldots, d \), using a permutation on the letters of the relation \( b_j \), we can obtain the relation \( b'_j \).

Let us consider the simplest case where \( b_j = b'_j \) for \( j = 2, \ldots, d \), and \( b'_1 \) is obtained from \( b_1 \) by a transposition which permutes two letters, say \( a_1 \) and \( a_2 \). Figure 11 on the top shows part of the diagram for \( \mathcal{H}(P,P^*_1) = \langle \Sigma, \alpha, \beta \rangle \) where the curves \( \alpha \) and \( \beta \) correspond to \( a_1 \) and \( a_2 \). If we connect two parts of \( \beta_1 \) with two 1-handles, as we did in the proof of Lemma 5.9, we obtain a Heegaard diagram \( \mathcal{H} = \langle \Sigma_1, \alpha, \beta \rangle \) where \( \Sigma_1 \) is obtained from \( \Sigma \) by adding two 1-handles, see Figure 11 on the bottom. Similar to the discussion in Section 2, we can associate the pair of dual presentations \( (P_2, P^*_2)' \), with this diagram, for some correspondence \( \mathcal{G}' \). If all the regions in the right diagram of Figure 11 are polygons, then from Corollary 3.3, we have \( \mathcal{H} = \mathcal{H}(P_2, P^*_2)' \). An argument similar to the proof for Lemma 5.9 shows that
\[ \overline{HF}(\mathcal{H}(P_2, P^*_2)) \cong \overline{HF}(\mathcal{H}(P_2, P^*_2)'). \]
From Lemma 5.9, we have \( \overline{HF}(\mathcal{H}(P_2, P^*_2)) \cong \overline{HF}(\mathcal{H}(P_2, P^*_2)'). \) which proves the lemma in this case.

Otherwise, if connecting two parts of \( \beta_1 \) in the diagram \( \mathcal{H}(P,P^*_1) \) makes some regions non-polygons, we follow an strategy similar to the proof of Lemma 5.9. We do some isotopies in the diagram \( \mathcal{H}(P,P^*_1) \), which are equivalent to the first AC-move on the pair \( (P, P^*)_F \) and do the corresponding isotopies for the diagram \( \mathcal{H} \) to obtain diagrams \( \mathcal{H}(P', P^*)_F \) and \( \mathcal{H}' \), respectively. Let \( (P', P^*)_F \) be obtained from \( (P, P^*)_F \) by these AC-moves and suppose that the associated pair of dual presentations of \( \mathcal{H}' \) is \( (P'_2, P^*_2)' \), which is obtained from \( (P_2, P^*_2)' \) by the corresponding AC-moves. \( \mathcal{H}' \) has the property that all regions are polygons. Therefore, from Corollary 3.3, we have \( \mathcal{H}' = \mathcal{H}(P'_2, P^*_2)' \) and from the proof of Lemma 5.9, we have \( \overline{HF}(\mathcal{H}(P'_2, P^*_2)) \cong \overline{HF}(\mathcal{H}(P'_2, P^*_2)'). \) From Lemma 5.7,
\[ \overline{HF}(\mathcal{H}(P_1, P^*_1)) \cong \overline{HF}(\mathcal{H}(P'_1, P^*_1)), \]
\[ \overline{HF}(\mathcal{H}(P_2, P^*_2)) \cong \overline{HF}(\mathcal{H}(P'_2, P^*_2)). \]
From Lemma 5.9, \( \widehat{HF}(\mathcal{H}_{(P, P^*)}) \cong \widehat{HF}(\mathcal{H}_{(P_2, P^*)}) \). This proves the lemma in this case.

For the general presentations, note that each \( b'_j \) is obtained from \( b_j \) by composition with some transpositions. \( \square \)

**Remark 5.11.** Note that if \( (P, P^*)_F \) is a pair of dual presentations with \( F = \{ f_{ij}, \overline{f}_{ij} \}_{i,j} \), then clearly \( (P^*, P)_F \) with \( F = \{ f_{ij}^{-1}, \overline{f}_{ij}^{-1} \}_{i,j} \) is also a pair of dual presentations.

Remark 5.11 and Lemma 5.10 state that \( \widehat{HF}(\mathcal{H}) \) is independent from the choice of dual presentation \( P^* \) and from Lemma 5.9, \( \widehat{HF}(\mathcal{H}) \) is independent from the choice of the correspondence \( F \). We may thus denote \( \widehat{HF}(\mathcal{H}) \) by \( \widehat{HF}_P(G) \).

**Proof of Claim 1.6 based on Claim 5.6 and Claim 5.8.** Let \( P^* \) be a dual presentation for \( P \) with a correspondence \( F \) which satisfies conditions A and B at the beginning of Section 5.

If the Andrews-Curtis moves 1-4 act on \( P \), the corresponding AC-moves 1-4 act on the pair \( (P, P^*)_F \). From Lemma 5.7, \( \widehat{HF}(P, P^*)_F \) is invariant under AC-moves. Therefore \( \widehat{HF}_P(G) \) is invariant under Andrews-Curtis moves 1-4.

Let the Andrews-Curtis move 5, which is the inverse of the Andrews-Curtis move 3, act on \( P \). Change \( P^* \) such that \( b'_j \) and \( b_j^{*-1} \) are consecutive letters in the relation \( a_i^j \) (note that from Lemma 5.10 and Remark 5.11, we are allowed to change \( P^* \) as described). Also from Lemma 5.9, one can change the correspondence \( F \) such that these two letters \( b'_j \) and \( b_j^{*-1} \) are consecutive in the cyclic ordering determined by \( F \) (see Remark 2.3). Now, the fifth AC-move can be used on this new pair \( (P, P^*)_F \) of dual presentations. Based on Claim 5.8, \( \widehat{HF}(P, P^*)_F \) is invariant under the fifth AC-move. Therefore \( \widehat{HF}_P(G) \) is invariant under the inverse of third Andrews-Curtis move.

To prove the second part of the claim, let \( (P, P^*) \) be a pair of dual presentations associated with a Heegaard diagram \( \mathcal{H} = (\Sigma, \alpha, \beta) \) of \( Y \) (see Example 3.1). \( P \) is a presentation for \( G = \pi_1(Y) \). Since \( \Sigma \setminus \alpha \) is a punctured sphere, we have \( \alpha^c = \emptyset \) and by doing isotopies in the diagram \( \mathcal{H} \), which are equivalent to the first AC-move on the pair \( (P, P^*)_F \), we can obtain a Heegaard diagram \( \mathcal{H}_1 \) in which all regions are polygons. Let \( (P_1, P_1^*)_F \) be a pair obtained from \( (P, P^*)_F \) by these AC-moves. \( (P_1, P_1^*)_F \) is a pair of dual presentations associated with
the diagram $\mathcal{H}_1$, therefore, from Corollary 3.3, we have $\mathcal{H}_1 = \mathcal{H}(P_1, P_1^\ast)$. $\hat{HF}_1(G)$ is defined as the Heegaard Floer homology group associated with the chain complex $(\widehat{CF}(\mathcal{H}_1), \partial_{J(T_i)})$ which is the same as $\widehat{HF}(Y)$. From the first part of the claim, $\hat{HF}_1(G) \cong \hat{HF}_1(P_1, P_1^\ast)$. This completes the proof.

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