Collapsing and static thin massive charged dust shells in a Reissner-Nordström black hole background in higher dimensions

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Abstract

The problem of a spherically symmetric charged thin shell of dust collapsing gravitationally into a charged Reissner-Nordström black hole in $d$ spacetime dimensions is studied within the theory of general relativity. Static charged shells in such a background are also analyzed. First a derivation of the equation of motion of such a shell in a $d$-dimensional spacetime is given. Then a proof of the cosmic censorship conjecture in a charged collapsing framework is presented, and a useful constraint which leads to an upper bound for the rest mass of a charged shell with an empty interior is derived. It is also proved that a shell with total mass equal to charge, i.e., an extremal shell, in an empty interior, can only stay in neutral equilibrium outside its gravitational radius. This implies that it is not possible to generate a regular extremal black hole by placing an extremal dust thin shell within its own gravitational radius. Moreover, it is shown, for an empty interior, that the rest mass of the shell is limited from above. Then several types of behavior of oscillatory charged shells are studied. In the presence of a horizon, it is shown that an oscillatory shell always enters the horizon and reemerges in a new asymptotically flat region of the extended Reissner-Nordström spacetime. On the other hand, for an overcharged interior, i.e., a shell with no horizons, an example showing that the shell can achieve a stable equilibrium position is presented. The results presented have applications in brane scenarios with extra large dimensions, where the creation of tiny higher dimensional charged black holes in current particle accelerators might be a real possibility, and generalize to higher dimensions previous calculations on the dynamics of charged shells in four dimensions.

Keywords: thin shell, black hole, extra dimensions.
1 Introduction

We propose to study the dynamics, as well as the statics, of a charged dust thin spherical shell collapsing into a Reissner-Nordström black hole in higher \(d\) spacetime dimensions in general relativity. This problem is interesting for two reasons. First, the suggestion that the world is a four-dimensional brane with large extra spatial dimensions [1], equalizing the electroweak and the Planck scales, means that charged Planckian black holes, could be generated in particle accelerator smashers of present technology (see, e.g., [2] and references therein). Since the debris formed in the collision of the energetic particles can be accreted by the higher dimensional black hole in a very first stage of the dynamical process, it is important to study a scenario of higher dimensional charged collapse into a charged black hole background. Second, this problem is also interesting on its own, since it generalizes to higher dimensions results on the dynamics of charged shells in four dimensions, the corroboration of the higher dimensional cosmic censorship hypothesis being a representative case. Higher dimensional spherical charged black holes, the higher dimensional equivalent of the four dimensional Reissner-Nordström black holes, have been studied sometime ago [3], but the analysis of the dynamics of a charged dust shell collapsing in such a higher dimensional background is new. With the help of the well-known junction conditions [4], we render into higher dimensional general relativity coupled to Maxwell’s electromagnetism, the results obtained in four dimensions for a charged dust thin shell collapsing into an existing Reissner-Nordström black hole, see, e.g., [5]-[16] for generic results in general relativity and [17] for results in Lovelock gravity, where, in particular, in [15]-[17] an analysis of the radial equation and of the cosmic censorship in a charged background is performed. In [18] [19] an extended analysis in backgrounds with different charges was discussed. We also will use calculations from static charged shells, see, e.g., [20, 21].

In section 2 we apply the Israel formalism to a \(d\)-dimensional spacetime and find the shell’s equation of motion. In section 3 we show that overcharging a non-extremal black hole by a charged shell is not allowed as long as the rest mass of the shell is positive. Therefore, the cosmic censorship cannot be violated in this process. Then, we point out that the derived radial equation of motion is not equivalent to the original equation of motion. A primary constraint is then derived. We further prove the important new result that a shell with total mass equal to charge, i.e., an extremal shell, in an empty interior, can only stay in neutral equilibrium outside its gravitational radius. Thus, a regular extremal black hole, generated by placing an interior extremal dust thin shell, is ruled out. Moreover, we show explicitly, for an empty interior, that the rest mass is limited from above, with the upper bound given through functions of the total mass and charge of the shell, and with the precise value of the bound depending on the charge to total mass ratio. In section 4 we study in detail the properties of the oscillations of an oscillatory shell. The mass and charge of the
shell that we shall consider are not necessarily small. We find that as long
as the existing black hole is not overcharged, an oscillatory shell must cross
the horizon and reexpand into another asymptotically flat region, in line
with the analysis for test shells of small mass and charge. This result also
indicates that there are no stable stationary solutions for the shell at any
radius. For an extremal black hole, we show that it is possible for the shell
to stay in neutral equilibrium. For an overcharged object, we show that
the shell can be configured to stay in stable equilibrium. We demonstrate
by an example that the trajectory of the shell is determined by the sign
of the normal vector orthogonal to the worldline. The constraint equation
derived in section 3 and the sign of the outward normal studied in section
4 are highlighted in this paper, which were seldom discussed in previous
studies.

2 Thin shell formalism in $d$-dimensional space-time

We apply the Israel formalism [4] for a thin shell to a $d$-dimensional ($d \geq
4$) spacetime. The evolution of the shell then is described by a timelike
hypersurface $\Sigma$. Denote the normal to the shell by $n^a$. Then the associated
extrinsic curvature $K_{ab}$ is given by

$$K_{ab} = h^c \, a \, h^d \, b \, \nabla_c \, n_d,$$

(1)

where $h_{ab}$ is the induced $d-1$ metric on $\Sigma$. Now, let $S_{ab}$ be the surface
stress-energy tensor on $\Sigma$, and $S = h^{ab}S_{ab}$ its trace. $S_{ab}$ is related to the
discontinuity of the extrinsic tensor measured from the outer and inner
sides of $\Sigma$. One can choose units such that the $d$-dimensional Einstein
equation is given by

$$G_{ab} = 8\pi G_d T_{ab},$$

(2)

where $G_d$ is the Newton constant in $d$ dimensions. Then the $d$-dimensional
Lanczos equation takes the same form as the 4-dimensional one [4]

$$\gamma_{ab} - h_{ab} \gamma = 8\pi G_d S_{ab}$$

(3)

where $\gamma_{ab} = [K_{ab}]$ is the jump of the extrinsic curvature across the shell and
$\gamma$ is its trace. By a simple arrangement, we obtain the following equivalent
form where the dimension $d$ appears explicitly

$$[K_{ab}] = -8\pi G_d (S_{ab} - \frac{1}{d-2} S h_{ab}),$$

(4)

Throughout this paper, the spherically symmetric metrics we are con-
cerned with always have the following form

$$ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \, d\Omega_{d-2}^2,$$

(5)
where
\[
d\Omega_{d-2}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2 + \cdots + \prod_{i=1}^{d-3} \sin^2 \theta_i d\theta_{d-2}^2
\]  \hspace{1cm} (6)
is the metric of the unit \((d-2)\)-sphere. Let \(M_i\) and \(Q_i\) denote the interior charges, i.e., the mass and charge of the central black hole. Then the inside geometry, the geometry for the spacetime interior to the shell, can be described by the Reissner-Nordström metric with
\[
f(r) \equiv f_i(r) = 1 - \frac{8\pi G_d}{(d-2)\Omega_{d-2}} \frac{2M_i}{r^{d-3}} + \frac{\epsilon_d Q_i^2}{r^{2(d-3)}}, \hspace{1cm} (7)
\]
where \(\Omega_n \equiv \frac{2\pi^{n+1}}{n!(d-2)}\) is the unit area of an \(n\) dimensional sphere and \(\epsilon_d\) is a constant proportional to the \(d\)-dimensional vacuum permeability. Similarly, for the outside geometry, the geometry exterior to the shell, one has
\[
f(r) \equiv f_o(r) = 1 - \frac{8\pi G_d}{(d-2)\Omega_{d-2}} \frac{2M_o}{r^{d-3}} + \frac{\epsilon_d Q_o^2}{r^{2(d-3)}}, \hspace{1cm} (8)
\]
where \(M_o\) and \(Q_o\) are the mass and charge measured by an outside observer. Conservation of charge gives the charge \(q\) of the shell:
\[
q = Q_o - Q_i, \hspace{1cm} (9)
\]
and the energy \(E\) of the shell is defined by
\[
E = M_o - M_i. \hspace{1cm} (10)
\]
The zeros of \(f_i\) and \(f_o\), in (7) and (8), respectively, take the form
\[
(r_i^\pm)^{d-3} = \frac{8\pi G_d}{(d-2)\Omega_{d-2}} M_i \pm \sqrt{\left(\frac{8\pi G_d}{(d-2)\Omega_{d-2}}\right)^2 M_i^2 - \epsilon_d^2 Q_i^2}, \hspace{1cm} (11)
\]
\[
(r_o^\pm)^{d-3} = \frac{8\pi G_d}{(d-2)\Omega_{d-2}} M_o \pm \sqrt{\left(\frac{8\pi G_d}{(d-2)\Omega_{d-2}}\right)^2 M_o^2 - \epsilon_d^2 Q_o^2}, \hspace{1cm} (12)
\]
and give the event and Cauchy horizons of the respective metrics, if those exist. As is well-known, the coordinate system we chose above cannot cover the entire spacetime. To make our results as general as possible, the following derivation is not specific to any region (like inside or outside of a horizon). Let \(\tau\) be the proper time of an observer comoving with the shell. Suppose the evolution of the areal radius \(r\) of the shell is given by \(r = r(\tau)\). The \(d-1\)-metric of the shell then takes the form
\[
d s^2 = -d\tau^2 + r^2(\tau) d\Omega_{d-2}^2. \hspace{1cm} (13)
\]
Let \( u^a = \left( \frac{d}{d\tau} \right)^a \) be the four-velocity of the comoving observer on the radial collapsing dust shell. Then \( S_{ab} \) may be written as
\[
S_{ab} = \sigma(\tau) u_a u_b ,
\]
(14)
where \( \sigma \) is the surface mass density. From the conservation law \( (d-1) \nabla_b S^b_a = 0 \), where \( (d-1) \nabla \) is the derivative operator on \( \Sigma \), we obtain
\[
\frac{\dot{\sigma}}{\sigma} + (d-2) \frac{\dot{r}}{r} = 0 ,
\]
(15)
where the overdot denotes the derivative with respect to \( \tau \). The total rest mass of shell is defined as
\[
m = \sigma A_{d-2} r^{d-2} .
\]
(16)
Eq. (15) then implies that \( m \) is constant. We assume throughout that \( m \geq 0 \). The four-velocity \( u^a \) can also be expressed in terms of bulk metric quantities, either inside or outside the shell, as
\[
u^a = \dot{t} \left( \frac{\partial}{\partial t} \right)^a + \dot{r} \left( \frac{\partial}{\partial r} \right)^a,
\]
(17)
where \( \dot{t} = \frac{dt}{d\tau} \). Since the relation \( g_{ab} u^a u^b = -1 \) must be satisfied, we may express \( \dot{t} \) in terms of \( \dot{r} \),
\[
\dot{t} = \pm \sqrt{\frac{f(r) + \dot{r}^2}{f(r)^2}} ,
\]
(18)
where it is of course implicit that all quantities are to be evaluated at \( r = r(\tau) \).

Now, the surface \( \Sigma \) can be defined by \( \Sigma = r(\tau) = 0 \), which gives the surface’s gradient, \( n_a = N (d) \nabla_a \Sigma \), where \( N \) is some function used to normalize the normal. By imposing \( n_a n^a = 1 \), the normal to the shell \( n^a \) may be determined up to signs. Write \( n^a \) formally as
\[
n^a = n^t \left( \frac{\partial}{\partial t} \right)^a + n^r \left( \frac{\partial}{\partial r} \right)^a .
\]
(19)
Following the definition above, one finds
\[
n^r = \pm \sqrt{f(r) + \dot{r}^2} ,
\]
(20)
The sign of \( n^r \) is associated with the direction of \( n^a \). One can verify, from Eqs. (11) and (5), that the \( \theta_1 - \theta_1 \) component of \( K_{ab} \) is
\[
K_{\theta_1 \theta_1} = r n^r .
\]
(21)
Let \( n^r_1 \) and \( n^r_0 \) represent \( n^r \) inside and outside the shell, respectively. So
\[
n^r_1 = \pm \sqrt{f_1(r) + \dot{r}^2} ,
\]
(22)
\[
n^r_0 = \pm \sqrt{f_0(r) + \dot{r}^2} .
\]
(23)
To make the left-hand side of Eq. (4) explicit, one needs to specify the
direction of \( n^a \). From now on, we choose \( n^a \) to be outward-pointing, i.e.,
pointing from the inside of the shell to the outside. Consistently, \([K_{ab}] = K^o_{ab} - K^i_{ab}\). Thus, the left-hand side of Eq. (4) reads

\[
\text{Left} = r (n^a_r - n^a_i).
\] (24)

(When one chooses \( n^a \) to be inward-pointing, \([K_{ab}] = K^i_{ab} - K^o_{ab}\). It gives the same \([K_{ab}]\) since the signs of \( K^i_{ab} \) and \( K^o_{ab} \) are also reversed.) Its counterpart on the right-hand side is

\[
\text{Right} = -8\pi G_d d^d - 2\sigma r^2 = -8\pi G_d m (d - 2)(\Omega^d - 2)r^{4-d}.
\] (25)

From Eqs. (24) and (25), we have immediately

\[
\frac{8\pi G_d m}{(d - 2)(\Omega^d - 2)} = r^{d-3}(n^r_i - n^r_o).
\] (26)

To be definitive, we consider the asymptotically flat region which
responds to the plus sign in Eq. (20). By substituting the Reissner-Nordström solutions (7) and (8), Eq. (26) yields

\[
\frac{8\pi G_d m}{(d - 2)} = r^{d-3} \left( \sqrt{1 - \frac{2M_i}{r^{d-3}} + \frac{\epsilon_i Q_i^2}{r^{2(d-3)}} + \dot{r}^2} - \sqrt{1 - \frac{2M_o}{r^{d-3}} + \frac{\epsilon_o Q_o^2}{r^{2(d-3)}} + \dot{r}^2} \right). \] (27)

For a given dimension \( d \), we can always choose the units for mass and
charge such that

\[
\frac{8\pi G_d}{(d - 2)(\Omega^d - 2)} = 1 \quad \text{and} \quad \epsilon_d = 1.
\] (28)

Then the zeros of \( f_i \) and \( f_o \), given in (11) and (12), respectively, take the simpler form

\[
(r_{i,\pm})^{d-3} = M_i \pm \sqrt{M_i^2 - Q_i^2},
\] (29)

\[
(r_{o,\pm})^{d-3} = M_o \pm \sqrt{M_o^2 - Q_o^2},
\] (30)

which are the horizons of the respective metrics if they exist, and moreover, Eq. (27) becomes

\[
m = r^{d-3} \left( \sqrt{1 - \frac{2M_i}{r^{d-3}} + \frac{Q_i^2}{r^{2(d-3)}} + \dot{r}^2} - \sqrt{1 - \frac{2M_o}{r^{d-3}} + \frac{Q_o^2}{r^{2(d-3)}} + \dot{r}^2} \right).
\] (31)
In addition, from Eq. (26), the master equation of motion is

\[ m = r^{d-3} (n^r_i - n^r_o) . \]  

(32)

with \( m \) being constant, see Eq. (15). Note that this result applies to any region covered by the coordinate system we have chosen. But the sign of \( n^r_i \) and \( n^r_o \) may vary from region to region. For instance, in the asymptotically flat region, \( n^r \) always points to larger values of \( r \). Therefore, \( n^r > 0 \). But the sign may change inside a horizon. This issue will be discussed later.

3 Cosmic censorship, radial equation and constraints, and application to an empty interior

In this section, we shall discuss some important issues related to the equation of motion (32).

3.1 Proof of the cosmic censorship

The thin shell model has been used to test the cosmic censorship. If an overcharged shell can implode past the horizon of an existing non-extremal black hole, a naked singularity will form and the cosmic censorship will be violated. For an empty interior, Boulware [8] has shown that a naked singularity is not possible for a positive rest mass \( m \). Hubeny [15] made an attempt to prove the cosmic censorship for a shell imploding onto an existing black hole, under the requirement that the energy \( E \) and rest mass \( m \) of the shell obey \( E > m > 0 \). Despite a complicated analysis on the radial equation of motion in [15], a rigorous proof, showing that a shell with overcharged exterior cannot pass through the horizon of the existing black hole, was not given. In [16] a general equation, obtained in Hamiltonian form, for an arbitrary metric function which clearly includes the charged case, was found. This analysis was extended to charged Lovelock gravity in [17]. Here, specializing to the charged case the analysis done in [16], we provide a rather short and straightforward proof from the original equation of motion (32) imposing only the condition \( m > 0 \), instead of using the radial equation derived below (see Eq. (34)).

In an asymptotically flat region outside an event horizon, both \( n^r_i \) and \( n^r_o \) are positive. In this case, substituting Eqs. (22) and (23) into (32) one finds

\[ m = r^{d-3} \left( \sqrt{f_i(r)} + \dot{r}^2 - \sqrt{f_o(r)} + \dot{r}^2 \right) . \]  

(33)

In [16] not only a more general equation of motion has been derived from a Hamiltonian treatment which reduces to Eq. (33) in the spherically symmetric case, but also it was mentioned that \( f_i = 0 \) gives rise to a lower
bound for \( r \) since Eq. (33) obviously breaks down at a certain stage for positive mass \( m \). This simple fact indeed plays an important role in the following proof. Note that Eq. (33) remains valid (\( n_i^r \) and \( n_o^r \) do not change sign) until it reaches a root of \( f_i(r) = 0 \) or \( f_o(r) = 0 \). Suppose \( |Q_i| \leq M_i \), i.e., the interior contains an existing black hole with \( f_i(r) \) vanishing at the horizon \( r = r_{i+} \). In attempt to violate the cosmic censorship, the exterior Reissner-Nordström spacetime must be overcharged, i.e., \( |Q_o| > M_o \). Consequently, \( f_o(r) > 0 \) for any \( r > 0 \) and the negative sign in Eq. (33) will never change. Since \( m > 0 \), it is then evident from Eq. (33) that the shell cannot reach the horizon. Therefore, the cosmic censorship is upheld in this process.

In [18, 19], cosmic censorship was tested by considering U(1) charges different from the one analyzed here. Although the proof above only concerns ordinary electric charges, it certainly can be extended to other type of charges. Note that the modification on a Reissner-Nordström spacetime caused by two charges of other types, \( \epsilon \) and \( q \) say, is direct, one simply replaces \( Q_o^2 \) by \( \epsilon^2 + q^2 \). Therefore, the proof holds when the existing black hole and the thin shell consist of more than one type of U(1) charges.

### 3.2 Radial equation and constraint

Solving Eq. (33) for \( \dot{r}^2 \), we find

\[
\dot{r}^2 = -f_i + \frac{r^{2(d-3)}}{4m^2} \left( f_i - f_o + \frac{m^2}{r^{2(d-3)}} \right)^2 \equiv V(r),
\]

or, equivalently,

\[
\dot{r}^2 = -f_o + \frac{r^{2(d-3)}}{4m^2} \left( f_i - f_o - \frac{m^2}{r^{2(d-3)}} \right)^2.
\]

For \( f_i \) and \( f_o \) given in Eqs. (7) and (8), we have

\[
V(r) = \frac{1}{m^2 r^{2(d-3)}} \left( [(M_o - M_i)^2 - m^2] r^{2(d-3)} \right.
\]

\[
+ \left[ m^2 (M_i + M_o) - (M_i - M_o)(Q_i^2 - Q_o^2) \right] r^{d-3}
\]

\[
+ \left[ m^4 + (Q_i^2 - Q_o^2)^2 - 2m^2 (Q_i^2 + Q_o^2) \right].
\]

The radial behavior of an oscillatory shell can be characterized by its turning points, which are the roots of \( V(r) = 0 \). When \( V(r) \) is negative, the corresponding region is physically forbidden for the shell. When \( V(r) \) is positive, a motion is possible, but not guaranteed. This can be seen by substituting Eqs. (34) and (35) into Eq. (33). Eq. (33) then takes the form

\[
m = \frac{r^{2(d-3)}}{2m} \left[ \sqrt{\left( f_i - f_o + \frac{m^2}{r^{2(d-3)}} \right)^2} - \sqrt{\left( f_i - f_o - \frac{m^2}{r^{2(d-3)}} \right)^2} \right].
\]
This equation holds (i.e., it gives the identity \( m = m \)), if and only if

\[
f_i - f_o - \frac{m^2}{r^{2(d-3)}} \geq 0.
\]

(38)

By substituting Eqs. (7) and (8) and assuming that the energy of the shell given in Eq. (10) is always positive, i.e.,

\[
E = M_o - M_i > 0,
\]

(39)

we obtain the following constraint

\[r \geq r_c.\]

(40)

where \( r_c \) is the constraint radius

\[r_c = \left( \frac{m^2 + Q_o^2 - Q_i^2}{2M_o - 2M_i} \right)^{1/(d-3)}.\]

(41)

Note that the case given by condition (39) is the case we are interested in throughout the paper. Eq. (34) together with Eq. (40) are equivalent to Eq. (83) in the asymptotically flat region (accurately, the region from infinity to the first root of \( f_i(r) = 0 \) or \( f_o(r) = 0 \)). A necessary and sufficient condition for a trajectory existing in this region is that \( V(r) \geq 0 \) (see Eq. (51)) and Eq. (10) holds. As an immediate application of this condition, we give an alternative proof of the cosmic censorship in the last subsection. Note that \( f_i = 0 \) at the horizon radius, \( r_{i+} \), of the black hole. Since \( r_{i+} \) is the only root for \( f_i(r) \) and \( f_o(r) \), Eq. (33) holds for \( r \geq r_{i+} \). This, of course, does not mean a true trajectory can exist anywhere in this region. To see if the shell can reach \( r = r_{i+} \), one needs to check both Eq. (34) and Eq. (40). Since \( V(r) > 0 \) at \( r = r_{i+} \), Eq. (34) appears to allow a violation to the cosmic censorship. However, Eq. (38) (consequently, Eq. (40)) manifestly breaks down at \( r = r_{i+} \) for \( f_i = 0 \) and \( f_o > 0 \), thus saving the cosmic censorship.

### 3.3 Application to an empty interior, \( M_i = 0, Q_i = 0 \)

Now we further explore the constraint, given in Eq. (10), in the case where the interior of the shell is empty. The spacetime is described by the metric of Eq. (5) with

\[f(r) \equiv f_i(r) = 1\]

(42)

for the interior, and

\[f(r) \equiv f_o(r) = 1 - \frac{2M_o}{r^{d-3}} + \frac{Q_o^2}{r^{2(d-3)}}\]

(43)
for the exterior. Since we shall mainly focus on asymptotically flat regions in this sub-section, the relevant equation of motion becomes

\[ m = r^{d-3} \left( \sqrt{1 + r^2} - \sqrt{1 - \frac{2M_o}{r^{d-3}} + \frac{Q_o^2}{r^{2(d-3)}} + r^2} \right). \]  

(44)

In this case condition (40) reduces to \( r \geq r_c \), now with,

\[ r_c \equiv \left( \frac{m^2 + Q_o^2}{2M_o} \right)^{1/(d-3)}. \]  

(45)

By setting \( \dot{r} = 0 \) in Eq. (44), we find that the only turning point for the empty shell is located at

\[ r = r_t = \left( \frac{m^2 - Q_o^2}{2(m - M_o)} \right)^{1/(d-3)}. \]  

(46)

Note also that \( V(r) \) in Eq. (46) takes the form

\[ V(r) = \frac{1}{m^2 r^{2(d-3)}} \left[ (M_o^2 - m^2) r^{2(d-3)} + (m^2 M_o - M_o Q_o^2) r^{d-3} + \frac{m^4}{4} - 2m^2 Q_o^2 + Q_o^4 \right], \]  

(47)

which is essentially a quadratic function of \( r^d \) up to an overall factor \( 1/(m^2 r^{2(d-3)}) \).

We are looking for the conditions that allow the shell to have a trajectory in the asymptotically flat region. This is equivalent to checking if \( V(r) \) has positive solutions under the constraint (45). We discuss the following possibilities.

(i) \( |Q_o| > M_o \): This describes an overcharged shell. There are three subcases:

   (ia) \( m < M_o \). Eq. (47) shows that \( V(r) \) is positive for sufficiently large values of \( r \) and constraint (45) is satisfied automatically. Therefore a trajectory exists in this case.

   (ib) \( m = M_o \). Now \( V(r) > 0 \) for \( r < \tilde{r}_t \), where \( \tilde{r}_t \equiv \left( \frac{Q_o^2 - m^2}{4m} \right)^{1/(d-3)} \). But \( \tilde{r}_t < r_c \). Hence, there is no solution.

   (ic) \( m > M_o \). Then \( V(r) > 0 \) for \( r < r_t \). But \( r_t < r_c \). Again, there is no solution for this case.

(ii) \( |Q_o| = M_o \): The shell consists of extremal charged dust. There are three subcases:

   (ia) \( m < M_o \). By an argument parallel to (ia), we see that a trajectory in the asymptotically flat region is guaranteed.
(iib) \( m = M_\text{o} \). Now \( V(r) \) vanishes identically, meaning the shell can stay in neutral equilibrium at any radius bigger than the horizon radius \( r^{d-3} = M_\text{o} \). Can this result be extended to the region inside the horizon \( r^{d-3} = M_\text{o} \)? To answer this question, we need first write down the equation of motion for \( r^{d-3} < M_\text{o} \). According to the discussion in section 2, the equation of motion (22) applies to all regions and \( n^r \) is determined by Eqs. (22) and (23) up to signs. Recall the normal \( n^a \) points from the inside to outside by our convention. Thus, for the extremal Reissner-Nordström solution, \( n^r \) takes the plus sign for both \( r^{d-3} > M_\text{o} \) and \( r^{d-3} < M_\text{o} \). Therefore, Eq. (41) holds for \( r^{d-3} < M_\text{o} \). By substituting \( m = M_\text{o} = Q_\text{o} \) into Eq. (44), we see immediately that \( \dot{R} = 0 \) is no longer a solution for \( r^{d-3} < m \).

Thus, we have proved the following new result which can be stated as a theorem: one cannot form an extremal Reissner-Nordström black hole by placing an extremal charged dust shell somewhere within its event horizon. For an analysis of static extremal charged dust shells, also called Bonnor shells, made of Majumdar-Papapetrou matter see [20].

(iic) \( m > M_\text{o} \). Then \( V(r) \geq 0 \) for \( r \leq r_\text{t} \). However, it is easy to find \( r_c > r_\text{t} \), which means no solution exists for this case.

(iii) \(|Q_\text{o}| < M_\text{o} \): This describes an undercharged shell. There are three subcases:

(iii) m < M_\text{o}. Again, there exists a solution.

(iii) m = M_\text{o}. From Eq. (17), one sees immediately that a solution can be found for sufficiently large values of \( r \).

(iii) m > M_\text{o}. Now we only need to check condition (40), i.e., a trajectory can exist outside the horizon if and only if \( r_t \geq r_c \). A simple calculation shows that this is equivalent to

\[
m^2 - 2mM_\text{o} + Q_\text{o}^2 \leq 0.
\]

(48)

The inequality holds if and only if \( (r_{o-})^{d-3} \leq m \leq (r_{o+})^{d-3} \), where \( (r_{o\pm})^{d-3} = M_\text{o} \pm \sqrt{M_\text{o}^2 - Q_\text{o}^2} \). Since \( (r_{o-})^{d-3} \leq M_\text{o} \leq (r_{o+})^{d-3} \), the constraint reduces to

\[
m \leq (r_{o+})^{d-3} = M_\text{o} + \sqrt{M_\text{o}^2 - Q_\text{o}^2}.
\]

(49)

Note that Eq. (49) states that the proper mass \( m \) is smaller than the outer horizon radius \( r_{o+} \), or some power of it. This case is noteworthy and so we dwell upon it. Eq. (49) can be inverted to yield

\[
M_\text{o} \geq \bar{m}.
\]

(50)
with
\[ \bar{m} \equiv \frac{m}{2} + \frac{Q_o^2}{2m}, \]  
(51)

Equation (50) is valid in \( d \) dimensions, it was derived for four dimensions in [13], from a different perspective. The lowest possible value for \( \bar{m} \) in (51) is when \( m = |Q_o| \), so that \( \bar{m} = m = |Q_o| \). It also follows then that \( M_o \geq |Q_o| \). Now, from (46) one finds that \( r_t \) obeys \( r_t^{d-3} \geq m \), the inequality being saturated precisely when \( m = |Q_o| \). Since \( r_t \) is to be interpreted as the maximum stationary value of \( r \) a given shell can have, this inequality is to be expected, any charged shell can never be permanently immersed inside its horizon. This lower bound for the radius of the shell forces then the finite, non-zero, value, for the minimal energy, \( M_o = |Q_o| = m \). A shell with radius and mass given by \( r = M_o = |Q_o| \) is about to form an extremal black hole. However, as shown in [20, 21], instead it forms a quasi-extremal black hole, called more simply, a quasi-black hole. The binding energy of a generic shell \( E_{\text{bind}} = m - M_o \) thus obeys
\[ E_{\text{bind}} \leq \frac{m}{2} - \frac{Q_o^2}{2m}. \]  
(52)

The binding energy is zero when \( |Q_o| = m \), and is maximum when the charge \( |Q_o| \) vanishes. Equation (52) gives thus a constraint on the binding energy of these \( d \) dimensional shells. Shells with stronger binding would be placed in the opposite sector of a Carter-Penrose diagram, with a pair of horizons separating it from asymptotic infinity, but that is another subject.

Interesting to note, that along the lines sketched in [13], and developed in [14] in four dimensions, one can show that in \( d \) dimensions the minimum value for \( M_o \), i.e., \( M_o = |Q_o| \), follows in Newtonian gravitation if one replaces \( m \) by \( M_o \), as required by the strong equivalence principle, and takes the limit \( r \to 0 \). Indeed, if from special relativity, one uses the equivalence from the inertial mass with total energy \( M_o \), and from general relativity, the equivalence of inertial and gravitational masses, so that gravitation also sees \( M_o \), one has in \( d \) dimensions, in the units we are using, \( M_o = m + Q_o^2 - \frac{M_o^2}{r} \), i.e., the total energy of the \( d \) dimensional shell is equal to the rest mass energy plus the Coulomb energy plus the gravitational energy. Obviously, the kinetic energy is missing in this “total energy” expression. However, if the shell is held still at the radius \( r \), \( M_o \) is indeed the total energy. After the shell is released at \( r \), \( M_o \) changes with the radius and is no longer the total energy. Solving for \( M_o \) yields, \( M_o(r) = \frac{1}{2} \left[ (r^2 (d-3) + 4 m r^{d-3} + 4 Q_o^2)^{1/2} - r \right] \). Now,
\[ \frac{dM_o}{dt} = \frac{(d-3)(M_o^2 - Q_o^2)}{2M_o r^{d-3} + 2} \]. The shell collapses, when the kinetic energy increases, i.e., when \( M'_o(r) \) decreases at \( r \), so that \( M_o \geq |Q_o| \) for collapse. Another conclusion one can draw is that as the radius of the shell shrinks, down to \( r = 0 \), upon collapse or otherwise, one obtains \( M_o = |Q_o| \), and also \( M_o = m \). Thus, through these mixed Newtonian and relativistic arguments, one recovers a lower limit for the total mass, \( M_o \geq |Q_o| \), as we have determined using full general relativity. The difference is, whereas in upgraded Newtonian gravitation, one predicts a point particle with \( M_o = |Q_o| \), in general relativity one expects an extremal black hole, or more correctly, a quasi-extremal black hole [20, 21].

By summarizing all the cases above, we find the upper bound of \( m \) for given \( M_o \) and \( Q_o \):

\[
\begin{align*}
    m &< M_o & \text{when } |Q_o| > M_o, \\
    m &\leq M_o + \sqrt{M_o^2 - Q_o^2} & \text{when } |Q_o| \leq M_o.
\end{align*}
\]

(53)

This bound is valid in any \( d \) dimensional spacetime, with \( d \geq 4 \).

4 Oscillatory shells

4.1 Properties of the oscillations

To study the properties of the oscillations of the charged shell in a \( d \) dimensional spacetime we divide the interior solution in its three distinct phases, a generic black hole solution (with two horizons), an extremal black hole solution (with one horizon), and an overcharged solution (a naked singularity with no horizons).

(i) Generic interior black hole solution, \(|Q_i| < M_i|\): Since, after skipping the overall factor \( 1/r^{2(d-3)} \), Eq. (34) is a quadratic equation, there are at most two roots, i.e., two turning points. Cruz and Israel [5] showed that a test shell, i.e., a shell with mass and charge vanishingly small when compared to those of the central black hole, oscillates between a maximum radius, larger than the event horizon radius, \( M_i + \sqrt{M_i^2 - Q_i^2} \), and a minimum radius, smaller than Cauchy horizon radius, \( M_i - \sqrt{M_i^2 - Q_i^2} \).

Thus, it is important to show that this property of oscillatory motion can be extended to higher dimensions, and to any charged shell, not only test shells. This we do now. Denote by \( r_{i+} \) and \( r_{i-} \) the two roots of \( f_i(r) = 0 \), which satisfy, \( r_{i+}^{d-3} = M_i + \sqrt{M_i^2 - Q_i^2} \) and \( r_{i-}^{d-3} = M_i - \sqrt{M_i^2 - Q_i^2} \), see Eq. (29). Also, \( r_o+ \) and \( r_o- \) are defined similarly, see Eq. (30). Let \( r_1 \) and \( r_2 \) be the two roots of \( V(r) = 0 \) in Eq. (34). From Eq. (34), we see immediately that neither of the two
roots can lie between \( r_{i+} \) and \( r_{i+} \) and \( V(r) \) must be non-negative in the region with \( r_{i+} < r < r_{i+} \). Since we are interested in oscillations, the system must be bound, \( E < m \). To study the distribution of the two roots \( r_1 \) and \( r_2 \), we first assume that the two roots of \( V(r) \) satisfy \( r_{i+} < r_1 < r_2 \) and the oscillation condition means that \( V(r) \) is positive for \( r_1 < r < r_2 \). But it contradicts the fact that \( V(r) \geq 0 \) in the region \( r_{i+} < r < r_{i+} \), for \( V(r) \) is a quadratic function of \( r^{d-3} \).

Therefore, we have ruled out the possibility that the shell can oscillate entirely outside the horizon \( r = r_{i+} \). Similarly, \( r_1 \) and \( r_2 \) cannot be both smaller than \( r_{i+} \). Therefore, a massive oscillatory shell in a \( d \) dimensional spacetime (\( d \geq 4 \)) obeys the following pattern: the two turning points are separated by \( r_{i+} \) and \( r_{i+} \). To explore the oscillatory shell further, we consider all the three cases, \( |Q_o| < M_o \), \( |Q_o| = M_o \), and \( |Q_o| > M_o \), where the first two can be studied together.

(iia) For \( |Q_o| \leq M_o \), \( f_o(r) \) has two roots \( r_{o+} \) and \( r_{o-} \). By a parallel argument, the bounds on \( r_1 \) and \( r_2 \) can be made tighter:

\[
\begin{align*}
   r_1 & \leq \min \{r_{i+}, r_{o-} \} \\
   r_2 & \geq \max \{r_{i+}, r_{o+} \} \tag{54}
\end{align*}
\]

Since the two roots must be separated, it is then obvious that the shell can never achieve a stable equilibrium position.

(ii) Extremal interior black hole, \( |Q| = M_i \): The above arguments are based on the hypothesis that the existing black hole is non-extremal. We now consider the case where the existing black hole is extremal, i.e. \( |Q| = M_i \), and thus \( r_{i+} = r_{i-} \). There are three cases to consider, \( |Q_o| < M_o \), \( |Q_o| = M_o \), and \( |Q_o| > M_o \).

(iia) For \( |Q_o| < M_o \), Eq. \( 54 \) remains valid, with \( r_{i+} = r_{i-} \).

(iib) For \( |Q_o| = M_o \), both the inside and outside of the shell are extremal. The radial equation \( 54 \) reduces to

\[
   r^2 = \frac{[m^2 - (M_o - M_i)^2]}{4m^2r^{2(d-3)}} \times (m + M_i + M_o - 2r^{d-3}) (m - M_i - M_o + 2r^{d-3}) \tag{55}
\]

For oscillating solutions, \( E < m \), i.e. \( M_o - M_i < m \), there are two roots \( r_1 \) and \( r_2 \) satisfying \( r_1^{d-3} = \frac{1}{2}(M_o + M_i - m) \) and \( r_2^{d-3} = \frac{1}{2}(M_o + M_i + m) \). Using \( E < m \), it is easy to check that \( r_1^{d-3} < M_i \) and \( r_2^{d-3} > M_o \). Therefore, the behavior is similar
to the non-extremal case. When \( m = M_o - M_i \), \( \dot{R} = 0 \) for all values of \( r \). By an argument similar to that in the case (iia) in section 3.3, this means the shell can stay in neutral equilibrium only for \( r^{d-3} > M_i \).

(iic) For \( |Q_o| > M_o \), still keeping \( |Q_i| = M_i \), the roots for Eq. (34) are

\[
r_{1}^{d-3} = \frac{m^2 + 2mM_i - M_i^2 - Q_i^2}{2(m + M_i - M_o)} \quad \text{and} \quad r_{2}^{d-3} = \frac{-m^2 + 2mM_i - M_i^2 + Q_i^2}{2(m + M_i - M_o)}.
\]

Assuming the oscillation condition, \( E = M_o - M_i < m \), one can check that the sign of \( r_{1}^{d-3} - M_i \) is opposite to that of \( r_{2}^{d-3} - M_i \). Therefore, the two roots are distributed on different sides of the interior extremal horizon \( r_i^{d-3} = M_i \), and the shell cannot oscillate entirely in the outside region. It is possible that the two roots coincide at \( r_{1}^{d-3} = M_i \) provided \( m = \sqrt{M_i^2 - 2M_iM_o + Q_o^2} \). This indicates that \( r_{1}^{d-3} = M_i \) could be a stable equilibrium position for the shell. However, we have shown in section 3 that the horizon cannot be reached by a shell with an overcharged exterior. Thus, a stable equilibrium configuration is not possible for the extremal interior.

(iii) Overcharged interior solution \( |Q_i| > M_i \): There are again three cases to consider, \( |Q_o| < M_o \), \( |Q_o| = M_o \), and \( |Q_o| > M_o \), where the first two can be studied together.

(iiiia) For \( |Q_o| \leq M_o \), i.e., the exterior is not overcharged, the oscillation shares the same feature as we have discussed, i.e., the shell cannot be confined in a single asymptotically flat region.

(iiiib) For \( |Q_o| = M_o \), the discussion has just been done in (iiiia).

(iiiic) For \( |Q_o| > M_o \), we are left with the situation that both the interior and exterior of the shell are overcharged. This case seems less interesting since no horizon can possibly appear. However, the following example shows that a stable equilibrium position can be found for some overcharged configurations (stable equilibrium has been ruled out in (i) and(iii)). We choose \( d = 4 \), \( Q_i = 30 \), \( M_i = 25 \), \( M_o = 40 \), \( m = 17 \) and \( Q_o = 40.9105 \) (\( Q_o \) is numerically solved such that the discriminant of the quadratic function in Eq. (36) vanishes). These choices guarantee that \( \dot{r}^2 = 0 \) is a maximum at \( r = 56.093 \), i.e. the shell possesses a stable equilibrium position.

In summary, we have therefore the following conclusions: (1) As long as the interior contains a black hole (non-extremal or extremal), the shell cannot oscillate in any single asymptotically flat region. If an oscillation occurs, the shell must enter a horizon and re-emerge in a new region of the extended Reissner-Nordström spacetime. (2) Only when both the interior and exterior of the shell are overcharged Reissner-Nordström solutions, can the shell achieve a stable equilibrium position for certain configurations.
4.2 Trajectory of the shell

To determine the trajectory of the shell more specifically, our analysis below will follow Boulware’s outline [8], but extend his arguments from an empty interior to a black hole interior. In this subsection, our discussion shall be confined to non-extremal cases. The analysis holds for all dimensions with \( d \geq 4 \).

To understand the rationale, suppose first a test shell, still in a black hole interior geometry. Suppose that the shell has two turning points and starts moving in region \( I_+ \) as illustrated in the Carter-Penrose diagram of Fig. 1. According to our discussion, the world line of the shell will pass through region \( II_+ \) and will reach a minimum radius \( r_1 < r_- \), where we have dropped the subscript \( i \) for the interior horizon, i.e., \( r_{i-} = r_- \), since we are dealing with a test shell. However, there are two possible ways to reach the minimum: entering region \( III_+ \) or entering region \( III_- \), see Fig. 1. Similarly, after passing through region \( II_- \), we need to choose if the shell will enter \( I_+ \) or \( I_- \).

![Carter-Penrose diagram](image)

Figure 1: Carter-Penrose diagram of the extended Reissner Nordström spacetime with trajectories of a test shell shown. There are two possible paths for the oscillating shell to choose.

In fact, due to the test character of the shell, Fig. 1 is a simplified diagram for illustration. In the real case, for a massive shell, the left side and the right side of the shell are described by two Reissner-Nordström solutions with different parameters. We need to decide for each side of the
Figure 2: Trajectory of the shell. The shell that starts from I+ falls into III− in both its inside and outside because both \( n_i \) and \( n_o \) are negative at the minimum \( r = r_1 \).

Carter-Penrose diagram the shell goes. The answer relies on the sign of the outward normal \( n^r \). Recall we put \( n_i \) and \( n_o \) as representing \( n^r \) inside and outside the shell, respectively (see Eqs. (22) and (23)). By our convention, \( n^a \) is pointing from inside to outside, as depicted in Fig. 1. Consequently, its component \( n^r \) takes different signs in different regions. The normal \( n^r \) is positive in I+ (\( r > r_{o+} \) or \( r > r_{i+} \)), i.e., both the right-hand side of Eqs. (22) and (23) are positive in I+. It is also positive in region III+, where \( n^a \) points toward larger values of \( r \), opposite to that in region III−. Therefore, the shell will fall into III+ if \( n^r > 0 \) at \( r = r_1 \) and into III− if \( n^r < 0 \). For a flat interior, the sign of \( n_o \), as the shell evolves, can be solved directly from the given initial parameters [8]. On the other hand, for a black hole interior, we have to determine the signs of \( n_i^r \) and \( n_o^r \) simultaneously. The strategy is as follows. We first substitute the minimum turning point \( r = r_1 \) into Eqs. (22) and (23), using the fact that \( \dot{R} \) vanishes at \( r = r_1 \). Therefore,
we obtain the values of $n^r_i$ and $n^r_o$ up to signs. However, given a set of parameters for the shell and spacetime, there is only one set of signs that makes Eq. (32) hold. Thus, the signs can be uniquely determined and consequently, we can decide which regions (inside and outside the shell) the shell will pass. After the shell leaves III$_+$ or III$_-$, it will inevitably fall into II$_-$. To choose between I$_+$ and I$_-$ after II$_-$, we note that $n^r > 0$ in I$_+$ and $n^r < 0$ in I$_-$. Therefore, for a shell that starts originally in I$_+$, it must re-enter I$_+$ in the future.

Now we demonstrate how it works by an explicit example where, the shell is not a test shell, but has mass comparable to the black hole mass, so that the left side and the right side of the shell are described by two Reissner-Nordström solutions with different parameters, see the Carter-Penrose diagram in Fig. 2. We choose the parameter set to be \{\(d = 4\), \(M_o = 110\), \(Q_o = 45\), \(M_i = 100\), \(Q_i = 40\), \(m = 11\)\}. Note that we have chosen \(m > E\), with the energy of the shell being \(E = M_o - M_i\), to guarantee an oscillatory solution. We can solve \(V(r) = 0\) in Eq. (34) to find the two turning points and calculate the characteristic radii of the spacetime. The results are listed in the table below. Note that the data in the table agree with the bounds we derived in section 4.1, i.e., \(r_1 < \min\{r_{i-}, r_{o-}\}\) and \(r_2 > \max\{r_{i+}, r_{o+}\}\). By substituting \(r = r_1\) into Eqs. (22) and (23) and using the fact that \(\dot{r}\) vanishes at \(r = r_1\), we have

\[
\begin{align*}
n^r_i &= \pm 0.79, \\
n^r_o &= \pm 2.15.
\end{align*}
\] (56)

It then follows that the only way to make Eq. (32) hold is both \(n_i\) and \(n_o\) take the negative sign. Therefore, we find that the shell will pass through the III$_-$ regions on both inside and outside. The spacetime diagram of the shell is shown in Fig. 2, where the relation \(r_{o+} > r_{i+} > r_{o-} > r_{i-}\) has been displayed.

5 Conclusions

We have analyzed the interesting case of higher dimensional collapsing and static thin massive charged dust shells in a Reissner-Nordström black hole background. We have derived the equation of motion in a \(d\)-dimensional spacetime of such a thin shell and proved that the cosmic censorship conjecture for the collapsing shell holds. We have also derived a constraint equation from which an upper bound for the rest mass of a shell with empty interior is obtained. Moreover, for a black hole interior, an oscillatory shell always crosses the horizon and reemerges in another asymptotically flat region. For an extremal black hole, a shell with an extremal exterior and a
certain proper mass can stay in neutral equilibrium only outside its horizon, ruling out the existence of a regular extremal black hole, generated by placing an interior extremal dust thin shell. A stable equilibrium is possible only when both the interior and exterior are overcharged. Finally, we have shown how to use the sign of $n^r$ to determine the shell’s trajectory. Presently, it is of real interest to generalize from four to $d$ dimensions. Now, following our results, spherical gravitational collapse in $d$ dimensions is not qualitatively different from four dimensions, but the quantitative correct analysis is worth doing for scenarios with large extra dimensions. Although the collapse in those scenarios may not be spherically symmetric, the assumption of spherically symmetry can be considered a first approximation.

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