Faraday waves and droplets in quasi-one-dimensional Bose gases

Lauro Tomio\textsuperscript{a}, A. Gammal\textsuperscript{b}, F. Kh. Abdullaev\textsuperscript{c}, R. K. Kumar\textsuperscript{a,d}

\textsuperscript{a}Instituto de Física Teórica, Universidade Estadual Paulista, 01140-070 São Paulo, SP, Brazil
\textsuperscript{b}Instituto de Física, Universidade de São Paulo, 05508-090 São Paulo, Brazil
\textsuperscript{c}Physical-Technical Institute, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan; and
Physical Department, National University of Uzbekistan, Tashkent, Uzbekistan
\textsuperscript{d}Department of Physics, Centre for Quantum Science, and Dodd-Walls Centre for Photonic and Quantum Technologies, University of Otago, Dunedin 9054, New Zealand

E-mail: lauro.tomio@gmail.com (corresponding author)

Abstract. Faraday waves (FW) are studied in mixtures of Bose gases by taking into account quantum fluctuations beyond the GP mean-field formalism with the Lee-Huang-Yang term, with both inter- and intra-species scattering lengths periodically varying in time. In this case, the effect of periodic modulations on the system is studied, with nonlinear resonances shown to be excited in the oscillation widths of quantum droplets. It was shown that the period of the Faraday patterns is quite sensitive to the estimated value obtained by the beyond mean-field contribution, which can be used to measure quantum fluctuations in the ground state of the quasi-one-dimensional mixture. Variational predictions are confirmed by full numerical simulations. We also add an analysis of quantum droplets and quantum fluctuation contributions when considering three-body interactions.

1. Introduction

In the present contribution, we report our recent studies on the emergence of Faraday waves (FW) [1] in mixtures of Bose gases [2] by taking into account quantum fluctuations beyond the Gross-Pitaevskii (GP) mean-field formalism [3, 4], when the Lee-Huang-Yang (LHY) term [5] is considered with both inter- and intra-species scattering lengths periodically varying in time [6]. Additionally, here we also introduce a short discussion on the possible contribution to quantum fluctuations due to three-body interactions. The effect of periodic modulations on the system is studied in [2], with nonlinear resonances shown to be excited in the oscillation widths of quantum droplets. It was shown that the period of the Faraday patterns is quite sensitive to the estimated value obtained by the beyond mean-field contribution, which can be used to measure quantum fluctuations in the ground state of the quasi-one-dimensional mixture. Ref. [2] extends a previous study on single-species Bose-Einstein condensates (BEC) having two- and three-body interactions periodically varying in time [7], in which two models was considered for the time-dependent three-body interactions; with quadratic and quartic dependence on the two-body atomic scattering length $a_s$. In that work, it was shown that parametric instabilities can lead to the generation of FW resonances with wave-lengths depending on the background scattering length, as well as on the corresponding modulation parameters of $a_s$.

By taking into account quantum fluctuations beyond the GP mean-field formalism with the
LHY term, FW are studied in Ref. [2], considering BEC mixtures of gases assumed trapped in cigar-type geometry, with inter- and intra-species scattering lengths periodically varying in time. The period of the Faraday patterns is shown to be quite sensitive to the estimated value obtained by the beyond mean-field contribution, which can be used to measure ground state quantum fluctuations. It was also shown that nonlinear resonances are excited in the oscillation widths of the quantum droplets. Analytical predictions confirm our numerical simulations.

In the next section 2, we provide some details on the formalism, followed by a modulational instability analysis in section 3. In section 4, we select some of our results obtained in our numerical simulations. The section 5 we present our conclusions with perspectives.

2. Quasi-one-dimensional Bose-gas mixtures with quantum fluctuations

Within an approximation such that the intra-species $s-$wave two-body scattering lengths are repulsive and identical, $a_{11} = a_{22} = a_s$, we consider a quasi-one-dimensional (1D) BEC mixture with two atomic species $i, j = 1, 2$ having the same mass $m$ and the same number of atoms. For the inter-species, we assume attractive interactions, with $a_{12} = -a_s [1 - (\delta a_s)/a_s]$, with $0 < \delta a_s \ll a_s$. Within these assumption, the coupled three-dimensional formalism is reduced to a single 1D GP-type equation for the droplet, given in dimensionless units as [4, 8]

$$\frac{1}{\tau} \frac{\partial \psi}{\partial \tau} = -\frac{\partial^2 \psi}{\partial x^2} + f(\omega t') \delta g |\psi|^2 \psi - \frac{1}{\pi} [f(\omega t') g]^{3/2} |\psi|^2 \psi, \quad (1)$$

where $t'$ and $x'$ are in units of the inverse transverse trap frequency $\omega_{\perp}^{-1}$ and transverse oscillator length $l_\perp = \sqrt{\hbar/(2m\omega_{\perp})}$, respectively. The wave-function $\psi \equiv \psi(x', t')$ is normalized to the number of atoms $N_0$ = $\int_{-\infty}^{\infty} dx' |\psi|^2$, with the corresponding density given by $n = |\Psi|^2 = l_\perp^{-1} |\psi(x', t')|^2$. In Eq. (1), the parameters $g$ and $\delta g$ are related to the scattering length, such that $g \equiv g_{ii} = 2a_s/l_\perp$, $g_{12} = g(1 - \delta g/g)$, with the corresponding time modulation given by $f(\omega t') \equiv 1 + f_1 \sin(\omega t')$ (here $\omega$ is the frequency of the oscillations, given in units of $\omega_{\perp}$).

2.1. Rescaling approach

By rescaling variables and parameters, such that $|\psi(x', t')| \equiv \xi |u(x, t)|, x \equiv x'/x_0$ and $t \equiv t'/x_0^2$, having $\xi \equiv g^{3/2}/(\pi \delta g)$ and $x_0 \equiv \pi \delta g g^{3/2} (g, \delta g > 0)$, with the modulation frequency $\omega \equiv \Omega/x_0^2$, we obtain the Eq. (1) in a more simplified form [See Refs. [4, 8]]:

$$\frac{1}{\tau} \frac{\partial u}{\partial \tau} = -\frac{\partial^2 u}{\partial x^2} + f(\Omega t)|u|^2 u - [f(\Omega t)]^{3/2} |u|^3 u, \quad (2)$$

where $u \equiv u(x, t) = \psi/\xi$ and $f(\Omega t) \equiv [1 + f_1 \sin(\Omega t)]$. The corresponding scaled Hamiltonian $H$ and the normalization $N$ (with total energy given by $E/N$) are given by

$$H = \int_{-\infty}^{\infty} dx \left[ \frac{1}{\tau} \frac{\partial u}{\partial \tau} + \frac{1}{2} f(\Omega t)|u|^4 - \frac{2}{3} [f(\Omega t)]^{3/2} |u|^3 u \right], \quad N = \int_{-\infty}^{\infty} dx |u|^2. \quad (3)$$

For $f(\Omega t) = 1$ with chemical potential $\mu$, we obtain the analytical solution

$$u(x, t) = \frac{-3e^{-\mu t}}{1 + \sqrt{9\mu/2} e^{-\mu t}} \left[ \frac{1 + \sqrt{9\mu/2}}{\sqrt{1 + \sqrt{9\mu/2}}} - \sqrt{-9\mu/2} \right]. \quad (4)$$

The atom-number normalization, physical density and the original Hamiltonian energy, are expressed in terms of the scaled quantities by

$$N_0 = \frac{N}{\pi} \left( \frac{g}{\delta g} \right)^3, \quad |\Psi|^2 = \frac{g^3}{(\pi \delta g)^2 l_\perp} |u|^2, \quad \text{and} \quad H_0 = \hbar \omega_{\perp} \frac{g^2}{\pi^3} \left( \frac{g}{\delta g} \right)^{5/2} H. \quad (5)$$
So, the non-scaled total energy $E_0 = H_0/N_0$ and chemical potential $\mu_0$ are transformed to the corresponding scaled quantities by $(E_0, \mu_0) = \hbar \omega_1 \left[ g^3/\left(\pi^2 \delta g\right) \right] (E, \mu).

The contribution to the energy of the system from the beyond mean-field term is negative in the region $0 < (\kappa \equiv n/a_s) < 0.15$. Thus, for the existence of droplets, in such 1D system we should consider repulsive two-body interactions. Also, the generation of FW requires to detune $a_s$ to positive (repulsive). The next nonlinear correction to the GP equation in terms of $\kappa$ has the form of an effective three-body interaction; i.e., $\lambda |\psi|^4$, which starts to be relevant for $\kappa > 0.15$.

The system becomes unstable for $\kappa \approx 3$ when the time-perturbed correction is of the order $f$. When $\kappa > 0.15$, the negative contribution of the beyond of mean-field term is changed to a repulsive one. Within our present setup, the possibility to enter into a collapsing regime with the increasing of the density does not occur, since the character of the correction in quasi-1D limit is changing from an attractive to a repulsive one. Another problem could be to consider the LHY term as local [8]. This is possible for disturbances which, by scaling, are larger than the healing length. Such scaling modulations of the system was studied in Ref. [2], with some details provided in the next section.

3. Modulational instability
To study the modulational instability (MI), we consider the evolution of the nonlinear plane-wave solution for Eq.(2), given by

$$u(x, t) = \left(1 + \frac{\delta u}{A}\right) w_0(t) \equiv [A + \delta u] e^{-iF(t)A^2+iG(t)A},$$

$$\frac{dF(t)}{dt} = f(\Omega t), \quad \frac{dG(t)}{dt} = \frac{f^3}{2}(\Omega t),$$

where $\delta u \equiv \delta u(x,t) \ll A$. So, the corresponding equation for the correction $\delta u$ is

$$\left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right) \delta u - Af(\Omega t) left[2A - \sqrt{f(\Omega t)} right] \frac{\delta u + \delta u^*}{2} = 0,$$

with the real-part component of the Fourier transform of $\delta u$ given by

$$\frac{\partial^2 V}{\partial t^2} = \sqrt{\delta u^2} \left[ k^2 + Af(\Omega t) left(2A - \sqrt{f(\Omega t)} right) \right] = 0,$$

where $V \equiv V(k,t)$ (where $k$ is the wave number). For the unperturbed case ($f(\Omega t) = 1$),

$$\partial^2 V_0/\partial t^2 = -2 \left[ k^2 + A(2A - 1) \right] V_0 \equiv -k^2 \left[ k^2 + B \right] V_0 \equiv -\omega_0^2 V_0,$$

where $B \equiv A(2A - 1)$ and $\omega_0^2 \equiv k^2(B + 1)$. It follows that the plane wave is stable with respect to modulations for $A > 1/2$ ($B > 0$), becoming unstable for $B < 0$. For $f_1 \ll 1$, with $\sqrt{1 + f_1 \sin(\Omega t)} \approx 1 + (f_1/2) \sin(\Omega t)$, the corresponding equation is given by

$$\frac{\partial^2 V}{\partial t^2} + \sqrt{\delta u^2} \left[ k^2 + 2A f_1 \sin(\Omega t) + 2h_1 \cos(2\Omega t) \right] = 0,$$

where $B_1 \equiv A(2A - 1) - 3Af_f^2/16$, $\omega_1^2 \equiv k^2(k^2 + B_1)$, $h_1 \equiv \frac{f_1 A(2A - 3/2)}{k^2 + B_1}$, $h_2 \equiv \frac{3Af_f^2}{16(k^2 + B_1)}$.

The system becomes unstable for $A < (1 + 3f_f^2/16)/2$, when $B_1 < 0$. By keeping the expansion only till the first order in $f_1$, we have $h_2 \approx 0$, $B_1 \approx A \left(2A - 1 - \frac{3}{16} f_f^2 \right) \approx B$ and $\omega_1^2 \equiv k^2(k^2 + B_1) \approx \omega_0^2$. The next order term, in $f_f^2$, can be relevant for the case that $A \approx 3/4$, when the time-perturbed correction is of the order $f_f^3$. We assume small corrections, $B_1 \approx B$, in the next. In the case that $B < 0$, when $k < \sqrt{|B|}$, we have $\omega_0^2 < 0$, with the MI gain $p_c = k\sqrt{|B| - k^2}$. The maximal MI gain is achieved when $k = \pm \sqrt{|B|/2}$, with $p_c = p_m = |B|/2$. 

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Figure 1. The time evolutions of the central densities are shown for initial amplitudes given by $A = 0.4$ and $A = 0.6$. The profiles are scaled by a power of 10, as indicated inside the frame. As expected, the solution is unstable for $A = 0.6$. Results extracted from Ref. [2].

For periodic time-dependent frequency, the above limiting equation for $f_1 \ll 1$ is known as the Mathieu equation, having parametric resonant solutions. The first is given for $\Omega = 2\omega_0$:

$$\Omega^2 = 4\omega_0^2 = 4k^2(k^2 + B) \rightarrow 2k^2 = -B + \sqrt{B^2 + \Omega^2}. \quad (10)$$

The growing of this mode leads to the generation of periodic modulations of the condensate along the cigar-type trap (Faraday wave), with period $L_F = 2\pi k = 2\pi \sqrt{2 \left( \sqrt{B^2 + \Omega^2} + B \right)}$. The corresponding gain at the parametric resonance is $p_c = \omega_0 h_1 = k \left[ f_1 A(2A - 3/2) / \left( \sqrt{k^2 + B} \right) \right]$. The frequency region for the first parametric resonance, with $\delta$ detuned from the resonance, is

$$-2f_1|B| \left( -B + \sqrt{B^2 + \Omega^2} \right) < \delta < \frac{2f_1|B|}{\Omega} \left( -B + \sqrt{B^2 + \Omega^2} \right). \quad (11)$$

3.1. Numerical simulations

Numerical simulations are detailed in Ref. [2], from where we present in Fig. 1 the time evolution of the density $A^2 - |u(x, t)|^2$, by considering the central position ($x = 0$) of the unperturbed case, $f_1 = 0$. The results are shown for two amplitudes: $A = 0.6$ ($B = 0.12$), which is stable; and $A = 0.4$ ($B = -0.08$), which is unstable. These are expected, as the solutions become unstable for $A < 0.5$.

For the amplitude $A = 5$, with $\Omega = 20$ and $f_1 = 0.2$, we show in Fig. 2 that the first parametric resonance occurs at $k = 1.46$. In the left frames, we show time-evolution for $x = 0$, considering 3 values of $k$, enhancing the resonance at $k = 1.46$ in the middle frame. In the right frame, the profiles are for two instants, $t = 0$ and 30, when $k = 1.46$, showing the density oscillations of the profiles.

4. Quantum droplet oscillations

The dynamics of localized quantum-droplet state under temporal oscillations of the transverse trap frequency was analyzed in Ref. [2], using a time-dependent variational approach, with a Gaussian wave function parametrized by the width $\sigma \equiv \sigma(t)$, chirp $b \equiv b(t)$ and phase $\phi \equiv \phi(t)$, $u(x, t) = A(t) \exp \left[ -x^2/(2\sigma^2) + ibx^2 + i\phi \right]$, normalized to the scaled number of atoms.
\( N = \sqrt{\pi} A^2 \sigma \). The characteristic number of atoms is \( N_0 = (N/\pi) (g/\delta g)^{3/2} \approx (1000/\pi)N \), when taking \( \delta g/g = 0.01 \). In such 1D-model, the non-linear cubic and LHY terms are, respectively, repulsive and attractive, with the LHY changing to repulsive as the density increases.

From the Lagrangian density for Eq. (2), with \( f = 1 + f_1 \sin(\Omega t) \), the corresponding averaged Lagrangian, \( \bar{L} = \int_{-\infty}^{\infty} L \, dx \), is such that

\[
\bar{L} = \frac{1}{N} \frac{\sigma^2}{2} \frac{db}{dt} - \frac{d\phi}{dt} - \frac{1}{2\sigma} - 2b^2 \sigma^2 - f \frac{N}{2\sqrt{2\pi} \sigma} + \left( \frac{2}{3} f \right)^{3/2} \frac{\sqrt{N}}{\sigma \sqrt{\pi}},
\]

from which the droplet width equation is obtained; defining the effective potential \( U(\sigma) \):

\[
\frac{d^2 \sigma}{dt^2} = \frac{4}{\sigma^2} + \left( f \sqrt{\frac{2N}{\pi} \sigma} - \frac{4f^{3/2}}{3\sigma^{3/2}} \sqrt{\frac{2N}{3\sqrt{\pi}}} \right) = -\frac{dU(\sigma)}{d\sigma},
\]

\[
U(\sigma) = \frac{2}{\sigma^2} + \sqrt{2f} \left( \frac{N}{\sqrt{\pi} \sigma} - \frac{8}{3} \frac{f N}{3 \sqrt{\pi} \sigma} \right).
\]

As shown in Ref. [2], where \( \sigma \) is presented as a function of \( N \), a critical \( N_c = 1.15 \) will correspond to the minimum of \( \sigma = 4.37 \). From Eq. (13), the stationary state, for \( f = 1 \), is given by

\[
N\sigma = (4\sqrt{\pi}/27) \left( \sigma \pm \sqrt{\sigma^2 - 27/\sqrt{2}} \right)^2. \]

As \( N\sigma \geq 0 \) and real, stationary states are obtained for \( \sigma > \sigma_{min} = \sqrt{27/\sqrt{2}} = 4.37 \), where the droplet is more compact, which will correspond to \( N_c\sigma_{min} = \sqrt{8\pi} \approx 5.01 \) \((N_c = 1.15)\). By increasing \( \sigma \), we obtain two stationary solutions, \( N_+ \).

For \( \sigma \gg \sigma_{min} \), asymptotically, we have \( N_+ \rightarrow (16/27)\sqrt{\pi}\sigma \) with \( N_- \) going to zero.

Next, in Ref. [2], it was considered small oscillations around a fixed point \( \sigma_c \), represented by \( \sigma(t) = \sigma_c + \sigma_1(t) \), with \( \sigma_1(t) \ll \sigma_c \). For \( \sigma_1(t) \), we obtain

\[
\frac{d^2 \sigma_1}{dt^2} + \omega_0^2 \sigma_1 = (\epsilon_1 + \epsilon_2 \sigma_1) f_1 \sin(\Omega t), \quad \text{with} \quad \omega_0 = \frac{1}{\sigma_c^2} \left( 6 + \frac{N\sigma_c}{\sqrt{2\pi}} \right), \quad \epsilon_1 = -\omega_0^2 \sigma_c, \quad \epsilon_2 = \frac{6}{\sigma_c^2} + \frac{\omega_0^2}{2}.
\]

At the minimum for the width, we have \( N\sigma_c = N_c\sigma_{min} = 2\sqrt{2\pi} \), such that \( \omega_0 = \sqrt{8}/\sigma_{min} = 4/27 = 0.148, \epsilon_1 = -\left( 8/\sigma_{min}^2 \right), \) and \( \epsilon_2 = (10/\sigma_{min}^2) \). This will correspond to \( N = N_c = 1.15 \), as shown above. For \( N = 1 \), we have \( \sigma_c = 4.379 \), with \( \omega_0 = 0.145 \); and for \( N = 0.1 \), \( \sigma_c = 6.754 \), with \( \omega_0 = 0.055 \). For the periodic variations of \( f(\Omega t) \) we can expect the resonances in the quasi-1D droplet oscillations, with the strongest one being at \( \Omega = \omega_0 \).

The results for the droplet width are shown in four frames (left two columns) presented in Fig. 3 for \( N = 0.1 \) and 1, by assuming two different amplitudes for the oscillations: \( f_1 = 0.2 \) and \( f_1 = 0.1 \). In both the cases it was assumed the same value \( \Omega = 0.055 \) for the oscillation frequency. The main resonance position for \( N = 0.1 \) is expected for this frequency. The variational results are compared with the corresponding numerical simulations \( \sqrt{2(x^2)} \), obtained by using the modified GP equation. The corresponding time evolutions of the total energies are shown in the other four frames (right two columns) of Fig. 3. In the case of resonant value of frequency with \( N = 0.1 \), we can verify that the mean value of the droplet energies are increasing (becoming less negative). With \( N = 1 \), we are out of the resonance, with the corresponding panels indicating that the energy oscillates near a constant mean value.

From the number of atoms \( N_0 \), associated to \( N \) by Eq. (5), a good agreement is verified between variational and numerical results in the initial stage of the evolution, with the deviations occurring at large times being connected with the well-known limitations of the variational procedure. One should also note that for large values of \( N \) the profile will deviate from the usual Gaussian ansatz, such that the description can be improved by using the super-Gaussian ansatz.
Figure 2. The occurrence of the resonance at $k = 1.46$ is shown in the above two panels for $A = 5$, $\Omega = 20$ and $f_1 = 0.2$, extracted from Ref. [2]. In the top frame, we have $A^2 - |u(x,t)|^2$ as a function of $x$ for $t = 0$ and $t = 10$. In the lower frames, we show the time-evolution of the central density, $A^2 - |u(0,t)|^2$, by considering three values of $k$, evidencing the resonance at $k = 1.46$. All quantities are in dimensionless units.

Figure 3. For $\Omega = 0.055$, with $N$ and $f_1$ as indicated, we show the time evolution of the variational widths $\sigma$ (solid) with the corresponding full-numerical ones (dashed) $\sqrt{\langle x^2 \rangle}$, in the left two columns with four panels. In the right two columns, we show the corresponding results for the evolution of the total scaled energies $E$, for the variational (solid) and full-numerical (dashed) solutions. This results are extracted from Ref. [2].
Figure 4. For $\Omega = 0.145$, with $N = 1$ and $f_1 = 0.1$ and 0.2, we show the time evolution of the total scaled energies $E$, for the variational (solid) and full-numerical (dashed) solutions.

ansatz, which will require a separate investigation. For the numerical simulation, by considering $N = 0.1$ and 1.0, the other parameters are chosen as follows: The widths are $\sigma(N = 0.1) = 6.76$ and $\sigma(N = 1) = 4.33$; the amplitudes $A(N = 1) = 0.361$ and $A(N = 0.1) = 0.091$; and the oscillations frequencies of the width, $\omega_0(N = 0.1) = 0.055$ and $\omega_0(N = 1) = 0.145$. In addition to the results shown in Fig. 3 (for the widths and energies, by considering $\Omega = 0.055$), in this contribution we are also presenting the results obtained for the total scaled energies, in Fig. 4, by considering $\Omega = 0.145$, which corresponds to the resonant behavior for $N = 1$. In the case of $^{39}$K mixture, the estimated resonant frequency is near $\Omega = \omega_0(N = 1) = 2\pi \times 1.5$Hz.

In the model which is considered in this paper we assume that all scattering lengths are varying in time simultaneously. It can be achieved by using the Feshbach resonance management. According to this technique, the external magnetic fields are varying near the resonant value periodically in time, which lead to the periodic variations of the atomic scattering lengths [10]. It will be of interest to consider different modulations in the time of the scattering lengths in the framework of two-field description, which can be more accessible from the experimental point of view. One should also note that the frequency of modulations can induce heating in the condensate for long-time evolutions, with observed temperature effects on the pattern formations (on this regard, see Ref. [11]). These are kind of problems that will require separate investigations.

As observed from the results given in Fig. 3, the droplet energies keep oscillating near a mean value for non-resonant modulations of the scattering length, increasing when close to the resonant $\Omega$. The loss of atoms with energy decreasing in droplets has been pointed out recently by using 3D setup, within an investigation of collective oscillations in binary mixtures, where the transition from solitonic regime to the droplet one was verified [12]. In the droplet side of the crossover, the emission of atoms has been found. In the intermediate value region of the atomic scattering length this effect is diminished. In the quasi-1D geometry, in the region of parameters considered here, the losses are suppressed with the droplet being more stable. In view of actual experimental possibilities, interesting enough is to study a wider region of parameters in the crossover between 1D and 3D setups, when the emission of correlated jets of atoms can occur by modulating the scattering lengths.

4.1. Quantum droplets with quintic, without cubic nonlinear contributions

A particular case to be further explored is the one in which we have $\delta g = 0 \ (g_{12} = -g_{11} = -g_{22})$, such that no cubic term appears in the nonlinear contributions of Eq. (1). Instead, we have a
non-zero contribution from three-body interactions, with a quintic term appearing in Eq. (1). In this case, the contributions coming from quantum fluctuations and three-body interactions can be of the same order, leading to the emergence of quantum droplets. By making the assumption that the additional nonlinear term describing the three-body interactions is given by $-g|u|^4u$ in addition to the quantum fluctuations, we can derive the corresponding GP modified formalism. By using a variational approach with the Gaussian ansatz $u(x,t) = \langle \frac{gN^2}{3\sqrt{3\pi}a} - \left(\frac{2}{3}\right)^{3/2} \sqrt{\frac{N}{a\sqrt{\pi}}} \rangle$. (16)

Then, for the equilibrium value of the droplet width, one gets $\sigma_c = g^{2/3}N/(2\sqrt{\pi})$. This approach will be followed in more detail in a future investigation.

5. Conclusion and Perspectives
The generation of Faraday patterns was studied in a mixture with two species of Bosonic gases, by considering that the intra-species contact interactions are identical and repulsive, with the inter-species being attractive but with absolute value very close to the intra-species one. All the interactions are assumed varying in time with the same frequency. So, the original coupled GP formalism is reduced to a quasi-1D system, where the quantum fluctuations are taken into account with the inclusion of a beyond mean-field term. The period of FW is found and shown that can serve for measuring the strength of the quantum fluctuations. The dynamics of quantum droplet is also studied, by verifying that the nonlinear resonance and bi-stability can occur in the droplet oscillations. As for future investigations, it is quite interesting to perform studies on the emergence of FW and quasi-1D droplets under periodic modulations of the scattering lengths by considering the binary mixtures with both components in the framework of a two-field description. Richer structures are expected of the FW and resonance response for the coupled droplets. Another quite interesting possibility discussed briefly in section 4.1 is the one in which the cubic nonlinear term of the GP equation is suppressed, with the contributions from three-body interactions being taken into account for the formations of quantum droplets.

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