Universal trade-off relation between coherence and intrinsic concurrence for two-qubit states

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Abstract: Entanglement and coherence are two essential quantum resources for quantum information processing. A natural question arises of whether there are direct link between them. And by thinking about this question, we propose a new measure for quantum state that contains concurrence and is called intrinsic concurrence. Interestingly, we discover that the intrinsic concurrence is always complementary to coherence. Note that the intrinsic concurrence is related to the concurrence of a special pure state ensemble. In order to explain the trade-off relation more intuitively, we apply it in some composite systems composed by a single-qubit state coupling four typical noise channels with the aim at illustrating their mutual transformation relationship between their coherence and intrinsic concurrence. This unified trade-off relation will provide more flexibility in exploiting one resource to perform quantum tasks and also provide credible theoretical basis for the interconversion of the two important quantum resources.

I. INTRODUCTION

Entanglement and coherence are two crucial resources which are widely applied to quantum information processing and computation [1]. For a physical system, commonly used entanglement measures mainly consider correlations between their subsystems, whereas we usually think the physical system as a whole in the research of coherence omitting its structure [2]. Entanglement as one of earlier resource theories is crucial ingredient for various quantum information processing protocols [3], such as remote state preparation [4,5], quantum teleportation [6], super-dense coding [7] and so on. With the development of the entanglement theory, entanglement of formation [8], concurrence [9], relative entropy of entanglement [10], negativity [11] have been proposed. Although entanglement can be measured in a variety of ways, there exist intrinsic relations between them. For instance, a functional relation between the entanglement of formation and concurrence has been put forward by Ref. [9]. As we all know that the negativity of a two-qubit state is invariably less than its concurrence, and its entanglement of formation is always greater its relative entropy of entanglement. On the other hand, coherence is usually a major concern of quantum optics in earlier research [12]. But a different viewpoint that quantum coherence was regarded as one of the key resources, just like entanglement, is proposed by Aberg [13] in 2006. Based on the work of Aberg, the resource theory of quantum coherence has been developed [14–19]. The resource theory is based on the rules that the set of incoherent operations is seen as the free operations and the set of incoherent states is seen as the set of free states. These free operations and free states depend on a reference basis \( \{|n\rangle\}_{n=1,\ldots,d} \) of the d-dimensional Hilbert space. This means that the quantification of quantum coherence intrinsically rests with the reference basis. Now that both entanglement and coherence are characterized by the resource theory, the understanding of common evolution of coherence and entanglement will be crucial. In particular, the research of the intrinsic relations hidden in these quantum resources has been made in recent years [20,21]. Since the chosen types of quantum resources and measure approaches are various, there exist distinct differences for these intrinsic relations among the quantum states. Therefore, the main goal of our research is how to obtain a universal intrinsic relation.

In our research, we work out two puzzles. First, for a generally two-qubit mixed state, we will determine that its intrinsic concurrence can be complementary to its coherence. This reveals that the increase of its coherence causes the decrease of its intrinsic concurrence. Second, we will determine the internal relationship between its intrinsic concurrence and concurrence of a special pure state ensemble which is a special decomposition of a two-qubit state. This will solve the problem where do its coherence and concurrence come from. In Sec. II, we review the quantification of first-order coherence and concurrence. In Sec. III and Sec. IV, we provide detailed proofs of the complementary and the special decomposition respectively. In Sec. V, we illustrate its application for some typical systems.

II. PRELIMINARIES

A commonly used entanglement measure is concurrence [8,22]. For a two-qubit pure state \( |\psi\rangle \), its spin-flipped state is defined by \( |\tilde{\psi}\rangle = (\sigma_2 \otimes \sigma_2) |\psi^*\rangle \), where \( |\psi^*\rangle \) is the complex conjugate of \( |\psi\rangle \), and \( \sigma_2 \) is the second one of the Pauli matrices. The concurrence is defined as [9]

\[
C(|\psi\rangle) = \sqrt{\text{tr}(\rho_{12}^2) - \text{tr}(\rho_{12}^4)}.
\]

For a general two-qubit state \( \rho \), its density matrix can be written as the form \( \rho = \sum_{n=1}^4 p_n |\psi_n\rangle \langle \psi_n| \), where \( p_n \) are the eigenvalues, in decreasing order, of the matrix \( \rho \) and \( |\psi_n\rangle \) are the corresponding eigenvectors. Its spin-flipped density matrix \( \tilde{\rho} \) can be expressed as

\[
\tilde{\rho} = (\sigma_2 \otimes \sigma_2) \rho (\sigma_2 \otimes \sigma_2) = \sum_{n=1}^4 p_n |\tilde{\psi}_n\rangle \langle \tilde{\psi}_n|.
\]
The concurrence is defined by the convex-roof \([23, 24]\) as follows

\[
C (\rho) = \min_{\{\varphi_n \mid |\varphi_n\rangle\}} \sum_n q_n C (|\varphi_n\rangle). \tag{3}
\]

The minimization is taken over all possible decompositions \(\rho\) into pure states. An analytic solution of concurrence can be calculated \([9]\)

\[
C (\rho) = \max \left\{ 0, \sqrt{\lambda_1 - \sqrt{\lambda_2} - \sqrt{\lambda_3 - \sqrt{\lambda_4}}} \right\}, \tag{4}
\]

where \(\lambda_n\) are the eigenvalues, in decreasing order, of the non-Hermitian matrix \(\tilde{\rho}\). The definition of concurrence is based on the convex-roof construction, and it is suitable for use in both pure states and mixed states \([25, 27]\).

Besides, a widely used measure of hidden coherence is the first-order coherence \([12]\), which is similar with the degree of polarization coherence \([28]\). Let us consider a two-qubit state \(\rho = \sum_{n=1}^4 p_n |\psi_n\rangle \langle \psi_n|\), composed of subsystems \(A\) and \(B\). This quantum state \(\rho\) can be obtained by applying a unitary operation \(V\) to the non-entanglement state \(\rho_A\), where the unitary operation \(V\) contains the corresponding eigenvectors \(|\psi_n\rangle\) and the state \(\rho_A\) is a diagonal matrix with the eigenvalues \(p_n\). Each subsystem of the state \(\rho\) is characterized by the reduced density matrix \(\rho_A = Tr_B (\rho)\) and \(\rho_B = Tr_A (\rho)\). The degree of first-order coherence of each subsystem is a better methodology for quantifying this coherence, and it can be given by \([12]\)

\[
D_{A,B} = \sqrt{2Tr \left( \rho_{A,B}^2 \right) - 1}. \tag{5}
\]

Therefore, a measure of coherence for both subsystems, when they are considered independently, has the following form \([11]\)

\[
D = \sqrt{D_A^2 + D_B^2}. \tag{6}
\]

III. A TRADE-OFF RELATION BETWEEN INTRINSIC CONCURRENCE AND COHERENCE

For a general two-qubit pure state \(|\psi\rangle\), its concurrence is defined as \([29]\)

\[
C (|\psi\rangle) = \sqrt{2 \left[ 1 - Tr (\rho_B^2) \right]} = \sqrt{2 \left[ 1 - Tr (\rho_A^2) \right]}, \tag{7}
\]

where \(\rho_A\) and \(\rho_B\) are the reduced density matrix of the pure state \(|\psi\rangle\). Combining the definition of the first-order coherence with the formula (7), it is obvious that the trade-off relation of the pure state \(|\psi\rangle\) can be expressed as

\[
C^2 (|\psi\rangle) + D^2 (|\psi\rangle) = 1. \tag{8}
\]

But, for a general two-qubit mixed state, the square sum of these two quantities is not any longer a conserved quantity. It is well-known that if the evolution of the quantum state is unitary, its purity does not change over time. Therefore, the purity can be one of the candidates for conserved quantities. In order to generalize the trade-off relation from the pure state to the mixed state, we try to look for an Hermitian operator whose average value can satisfy the above trade-off relation. Interestingly, we find that the average value of the spin-flipped operator related to a two-qubit mixed state is complementary to its first-order coherence. Here, we introduce some peculiarities about the spin-flipped operator at first.

Let us consider the spin-flipped operator \(F\) corresponding to a Hermitian operator \(F\), whose order is \(2n\), defined as

\[
\tilde{F} = \sigma_2^{\otimes n} F^* \sigma_2^{\otimes n}, \tag{9}
\]

where \(F^*\) is the complex conjugate of \(F\). Obviously, the spin-flipped operator \(\tilde{F}\) satisfies the Hermitian property. If the Hermitian operator \(F\) can be written as the form \(F = F_1 F_2\), then the corresponding spin-flipped operator \(\tilde{F}\) has a similar form

\[
\tilde{F} = \sigma_2^{\otimes n} F_1^* \sigma_2^{\otimes n} F_2^* \sigma_2^{\otimes n} = (\sigma_2^{\otimes n} F_1^* \sigma_2^{\otimes n}) (\sigma_2^{\otimes n} F_2^* \sigma_2^{\otimes n}), \tag{10}
\]

For the Pauli operators, one obtains some special properties

\[
\tilde{\sigma}_i = \sigma_2 \sigma_i^* \sigma_2 = -\sigma_i, \tag{11}
\]

where \(i \in \{1, 2, 3\}\). This indicates that the spin-flipped state \(\tilde{\rho}_A\) is related to the single qubit state \(\rho_A\), is reversed with the state \(\rho_A\) in the Bloch sphere space. For a general mixed state \(\rho = \sum_{n=1}^4 p_n |\psi_n\rangle \langle \psi_n|\), according to the formula (2), one obtains that the average value of the spin-flipped operator \(\tilde{\rho}\) can be expressed as the following form

\[
Tr (\rho \tilde{\rho}) = \sum_{m,n=1}^4 p_m p_n C_{mn} Tr \left( |\psi_m\rangle \langle \tilde{\psi}_n| \right) \tag{12}
\]

where the tilde inner product has the form \(C_{mn} = \langle \psi_m | \tilde{\psi}_n \rangle \). It is evident that the average \(Tr (\rho \tilde{\rho})\) is non-negative. And the tilde inner product \(C_{mn}\) satisfies some properties

\[
\sum_{n=1}^4 |C_{mn}|^2 = \sum_{n=1}^4 \langle \psi_m | \tilde{\psi}_n \rangle \langle \tilde{\psi}_n | \psi_m \rangle \tag{13}
\]

\[
= \langle \psi_m | \left( \sum_{n=1}^4 \tilde{\psi}_n \langle \tilde{\psi}_n | \psi_m \rangle \right) | \psi_m \rangle = \langle \psi_m | \psi_m \rangle = 1,
\]

\[
= \langle \tilde{\psi}_n | \psi_m \rangle = 0.
\]
\[
\sum_{m=1}^{4} |C_{mn}|^2 = \sum_{n=1}^{4} \langle \tilde{\psi}_n | \psi_n \rangle \langle \psi_m | \tilde{\psi}_n \rangle \\
= \langle \tilde{\psi}_n | \sum_{m=1}^{4} |\psi_m \rangle \langle \psi_m | \tilde{\psi}_n \rangle \\
= \langle \tilde{\psi}_n | \tilde{\psi}_n \rangle = 1.
\tag{14}
\]

And then, we give the definition of the intrinsic concurrence. For a general two-qubit state \( \rho \), its intrinsic concurrence is defined as
\[
C_I (\rho) = \sqrt{\text{Tr}(\rho \tilde{\rho})}.
\tag{15}
\]

Here, we obtain two properties about the intrinsic concurrence, which indicate the rough relation between the concurrence and the intrinsic concurrence.

**Property 1.** For a two-qubit pure state \( |\psi \rangle \), there is an equivalence relation about its concurrence and intrinsic concurrence. The formula can be expressed as
\[
C_I (|\psi \rangle) = C (|\psi \rangle).
\tag{16}
\]

**Proof.** According to the definition (15), one obtain
\[
C_I (|\psi \rangle) = \sqrt{\text{Tr}(\rho \tilde{\rho})} = \sqrt{\langle \psi | \tilde{\psi} \rangle \text{Tr}(\rho \tilde{\rho}) \langle \tilde{\psi} | \psi \rangle} = \sqrt{\text{Tr}(\rho \tilde{\rho})} = C (|\psi \rangle).
\tag{17}
\]

**Property 2.** For a general two-qubit state \( \rho \), there is a lower bound of its intrinsic concurrence. And the lower bound is its concurrence. The inequality about its concurrence and intrinsic concurrence can be written as
\[
C_I (\rho) \geq C (\rho).
\tag{18}
\]

**Proof.** Combining the analytic solution (4) with the definition (15), one obtain
\[
C_I (\rho) = \sqrt{\text{Tr}(\rho \tilde{\rho})} = \sqrt{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} \geq \sqrt{\lambda_1} \\
\geq \max \left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\} = C (\rho).
\tag{19}
\]

According to the property 2, one obtain an inference that the necessary and sufficient condition of \( C_I (\rho) = C (\rho) \) is \( 0 \leq R(\rho \tilde{\rho}) \leq 1 \), where \( R(\rho \tilde{\rho}) \) is the rank of the non-Hermitian matrix \( \rho \tilde{\rho} \). The inference reveals that if a rank-2 mixed state satisfies \( C_I (\rho) = C (\rho) \) is \( 0 \leq R(\rho \tilde{\rho}) \leq 1 \), the square sum of its first-order coherence and concurrence is a conserved quantity. We will illustrate this for an example in Sec. V.

Finally, we introduce the trade-off relation between the intrinsic concurrence and the first-order coherence. For a general two-qubit state \( \rho \), there is a complementary relation
\[
C_I^2 (\rho) + D^2 (\rho) = \text{Tr} (\rho^2).
\tag{20}
\]

**Proof.** In general, a two-qubit state \( \rho \) is denoted as
\[
\rho = \frac{1}{4} \left[ I \otimes I + (\tilde{A} \cdot \tilde{\sigma}) \otimes I + I \otimes (\tilde{B} \cdot \tilde{\sigma}) + \sum_{m,n=1}^{3} T_{mn} \rho_m \otimes \rho_n \right],
\tag{21}
\]
where \( I \) stands for identity operator of single qubit, \( \rho_n \) stand for three Pauli operators, \( \tilde{A} = (a_1, a_2, a_3) \) and \( \tilde{B} = (b_1, b_2, b_3) \) are vectors in \( R^3 \) and \( \tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \). Just to make it easy to calculate, let us rewrite the state \( \rho \) as
\[
\rho = \rho_A \otimes \rho_B + \frac{1}{4} \sum_{m,n=1}^{3} (T_{mn} - a_m b_n) \sigma_m \otimes \sigma_n,
\tag{22}
\]
where \( \rho_A = \text{Tr}_B (\rho) = \frac{1}{2} (I + \tilde{A} \cdot \tilde{\sigma}) \) and \( \rho_B = \text{Tr}_A (\rho) = \frac{1}{2} (I + \tilde{B} \cdot \tilde{\sigma}) \). According to the defining (5), we obtain that the coherence of each subsystem has the following form
\[
D_K = \sqrt{2 \text{Tr}(\rho_K^2) - 1} = |\tilde{K}|,
\tag{23}
\]
where \( K \in \{A, B\} \). Therefore, combining the defining (6) with the formula (23), one obtains that its first-order coherence can be given by the following formula
\[
D (\rho) = \sqrt{\frac{|A|^2 + |B|^2}{2}}.
\tag{24}
\]

It is obvious that the spin-flipped operator \( \tilde{\rho} \) corresponding to the state \( \rho \) can be expressed as
\[
\tilde{\rho} = \tilde{\rho}_A \otimes \tilde{\rho}_B + \frac{1}{4} \sum_{m,n=1}^{3} (T_{mn} - a_m b_n) \sigma_m \otimes \sigma_n,
\tag{25}
\]
where \( \tilde{\rho}_A = \sigma_2 \rho_A \sigma_2 = \frac{1}{2} (I - \tilde{A} \cdot \tilde{\sigma}) \) and \( \tilde{\rho}_B = \sigma_2 \rho_B \sigma_2 = \frac{1}{2} (I - \tilde{B} \cdot \tilde{\sigma}) \). Therefore, for each subsystem, the definition of the first-order coherence can be rewritten as
\[
D_K = |\tilde{K}| = \sqrt{\text{Tr}[\rho_K (\tilde{\sigma} \otimes I)]} \\
= \sqrt{\text{Tr}[\rho_K (\rho_K - \tilde{\rho}_K)]}.
\tag{26}
\]

Similarly, for the whole system, the definition of the first-order coherence can be rewritten as
\[
D (\rho) = \sqrt{\frac{|A|^2 + |B|^2}{2}} \\
= \sqrt{\text{Tr}[\rho (\tilde{\sigma} \otimes I) + \text{Tr}[\rho (I \otimes \tilde{\sigma})] - 2 \text{Tr}[\rho (\rho_A \otimes \rho_B - \tilde{\rho}_A \otimes \tilde{\rho}_B)]}.
\tag{27}
\]
process can be described as follows
\[ \vec{A} \]

In the same way, we can always apply a unitary operation

And the state \( \rho \) is a pair of complementary quantities.

And then, combining the formula (32) with (33), we obtain

So, one obtains that the eigenvalue-eigenvector equation of the non-Hermitian matrix \( \rho \) has the form

Theorem 2. For a general two-qubit state \( \rho \), if the eigenvalue-eigenvector equation of the non-Hermitian matrix \( \rho \) has the form \( \rho \langle \varphi_n | \varphi_n \rangle = \lambda_n \langle \varphi_n | \varphi_n \rangle \), these eigenvectors \( \langle \varphi_n | \varphi_n \rangle \) will satisfy the tilde orthogonal relation \( \langle \varphi_m | \tilde{\varphi}_n \rangle = \delta_{mn} \langle \varphi_n | \tilde{\varphi}_n \rangle \)

\[ 4 \]

where these pure states \( |\varphi_n \rangle \) satisfy the tilde orthogonal relation

And then, combining the formula (32) with (33), we obtain that the non-Hermitian matrix \( \rho \) can be expressed as

\[ 4 \]

So, one obtains that the eigenvalue-eigenvector equation of the non-Hermitian matrix \( \rho \) has the form

\[ 4 \]

Theorem 1. If a two-qubit state \( \rho \) has a pure state decomposition \( \rho = \sum_{n=1}^{4} q_n |\varphi_n \rangle \langle \varphi_n | \), and these pure states \( |\varphi_n \rangle \) satisfy the tilde orthogonal relation \( \langle \varphi_m | \tilde{\varphi}_n \rangle = \delta_{mn} \langle \varphi_n | \tilde{\varphi}_n \rangle \), then the eigenvectors of the non-Hermitian matrix \( \rho \) will be \( |\varphi_n \rangle \) and the corresponding eigenvalues will be expressed as \( \lambda_n = q_n^2 C^2 (|\varphi_n \rangle) \).

Proof. Assuming that the two-qubit state \( \rho \) has a pure state decomposition
\[ 4 \]

\[ 2 \]

where these pure states \( |\varphi_n \rangle \) satisfy the tilde orthogonal relation

\[ 2 \]

And then, combining the formula (32) with (33), we obtain that the non-Hermitian matrix \( \rho \) can be expressed as

\[ 2 \]

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Theorem 2. For a general two-qubit state \( \rho \), if the eigenvalue-eigenvector equation of the non-Hermitian matrix \( \rho \) has the form \( \rho \langle \varphi_n | \varphi_n \rangle = \lambda_n \langle \varphi_n | \varphi_n \rangle \), these eigenvectors \( \langle \varphi_n | \varphi_n \rangle \) will satisfy the tilde orthogonal relation \( \langle \varphi_m | \tilde{\varphi}_n \rangle = \delta_{mn} \langle \varphi_n | \tilde{\varphi}_n \rangle \)

\[ 2 \]

Proof. The eigenvalue-eigenvector equation of the non-Hermitian matrix \( \rho \) about the state \( \rho \) can be expressed as

\[ 2 \]
The relationship (41) reveals that the intrinsic concurrence $C_I (\rho)$ is inseparable from the concurrence $C (|\varphi_n\rangle)$ of the special pure state ensemble about the state $\rho$.

TABLE I: The unitary evolution operators $U_n$ and the evolution states $\rho_n$ by these channels respectively.

| Channels | AD | BF |
|----------|----|----|
| Unitary evolution operators $U_n$ | $U_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & \sqrt{p} \\ 0 & \sqrt{p} & 0 & \sqrt{1-p} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $U_1 = \begin{pmatrix} \sqrt{1-p} & 0 & 0 & -\sqrt{p} \\ 0 & \sqrt{1-p} & \sqrt{p} & 0 \\ 0 & \sqrt{p} & 0 & \sqrt{1-p} \\ 0 & 0 & 0 & \sqrt{1-p} \end{pmatrix}$ |
| Evolution states $\rho_n$ | $\rho_0 = U_0 \rho U_0^\dagger$ | $\rho_1 = U_1 \rho U_1^\dagger$ |

TABLE II: The ranks $R_n$ and ratios $S_n$ of the evolution states $\rho_n$ by these channels respectively.

| Channels | AD | BF |
|----------|----|----|
| Ranks $R_n$ | $R_0 = 1$ | $R_1 = 2$ |
| Ratios $S_n$ | $S_0 = 1$ | $S_1 = \sqrt{\frac{1+|A|^2-2a^2}{2(|A|^2-a^2)}}$ |

V. THE COMPLEMENTARY RELATION OF QUANTUM STATE IN OPEN SYSTEM

In fact, a quantum system is inevitably coupled with the surrounding environment, and quantum resources are constantly exchanged between the quantum system and the environment. In general, in order to explore the evolution of a single qubit state $\rho_A = \frac{1}{2} (I + \vec{A} \cdot \vec{\sigma})$ in open system, one can be simplified into a closed composite system by Markovian approximation. We assume that the initial state of the environment is $|0\rangle$, $|0\rangle$. Then the state of the composite system at the initial time (IT) can be described as $\rho_{IT} = \rho_A \otimes |0\rangle \langle 0|$. And its evolution state $\rho$ can be described by unitary evolution operator $U$, i.e., $\rho = U \rho_{IT} U^\dagger$.

We will discuss only a few common channels, namely amplitude damped (AD) channel, bit flip (BF) channel, bit-phase flip (BPF) channel, and phase flip (PF) channel. The unitary evolution operators $U_n$ and the evolution states $\rho_n$, which formed by the coupling of the single qubit state $\rho_A$ and these channels respectively, are given in the TABLE I. For these channels, we obtain that the ratios $S_n = \frac{C_I (\rho_A)}{C (|\varphi_n\rangle)}$ are independent of the parameter $p$ of the channels. The ranks $R_n$ of the non-Hermitian matrix $\rho_n \rho_n^\dagger$ and the ratios $S_n$ are given in the TABLE II.

Therefore, the complementary relation of the composite system $\rho_n$, which is formed by the single qubit state $\rho_A$ and its coupling channel, can be rewritten as

$$D^2 (\rho_n) + S_n^2 C^2 (|\varphi_n\rangle) = \frac{1 + |A|^2}{2},$$

where $n \in \{0, 1, 2, 3\}$. The above equation (42) expresses the following two meanings: one is that when a single qubit state $\rho_A$ is coupled with the AD channel, the relation between

And then let’s apply spin-flip to both sides of the equation (36), one can obtain

$$\hat{\rho} \rho |\varphi_n\rangle = \lambda_n |\varphi_n\rangle.$$

It is obvious that the non-Hermitian matrix $\hat{\rho} \rho$ is the Hermitian conjugate of the non-Hermitian matrix $\rho \hat{\rho}$. Therefore, we obtain that the eigenvalue spectral decomposition of the non-Hermitian matrix $\hat{\rho} \rho$ can be expressed as

$$\hat{\rho} \rho = \sum_{n=1}^{4} \lambda_n \langle \varphi_n | \varphi_n \rangle |\varphi_n\rangle,$$

where $\lambda_n$ and $|\varphi_n\rangle$ satisfy the tilde orthogonal relation $\langle \varphi_m | \varphi_n \rangle = \delta_{mn} \langle \varphi_n | \varphi_n \rangle$. And there is a special decomposition of the state $\rho$

$$\rho = \sum_{n=1}^{4} q_n |\varphi_n\rangle \langle \varphi_n |,$$

where $q_n$ satisfy the relation

$$\lambda_n = q_n^2 C^2 (|\varphi_n\rangle).$$
the concurrence and first-order coherence can be revealed by using a circular curve in Fig.1 and the concurrence of the composite system will be increased completely from the decrease of its first-order coherence; the other is that when the state $\rho_A$ is coupled with the BF, BPF and PF channels respectively, the relationship between the concurrence and first-order coherence can be represented by an elliptic curve in Fig.1 and the decrease of the first-order coherence can be converted to the concurrence with a conversion efficiency $\frac{1}{2\sqrt{2}}$.

VI. CONCLUSION

In this paper, we have solved three tasks about the trade-off relation between intrinsic concurrence and first-order coherence. First of all, we put forward the definition of intrinsic concurrence for a general two-qubit state and establish a universal relation that its intrinsic concurrence can be complementary to its first-order coherence. Then, we provide a special decomposition method for a two-qubit state and derive the relation between its concurrence and intrinsic concurrence. Finally, as an application, we give out the unified complementary relation of single-qubit state under four different noise channels. Our results provide a deep physical meaning about the relation between quantum coherence and entanglement which can be widely applied in various contexts. Quantum entanglement of the compound system will be increased completely from the decrease of quantum coherence through an amplitude damped channel. But if quantum systems pass through others channels, the decrease of quantum coherence can be converted to quantum entanglement with a conversion efficiency. We hope that these results will find interesting applications in controlling interconversion of quantum coherence and entanglement for a two-qubit system and sending a single-qubit system under noisy channels that tend to enhance quantum coherence or entanglement.

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[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information. Cambridge University Press, Cambridge, England (2003).
[2] M.-L. Hu, X. Y. Hu, J. C. Wang, Y. Peng, Y.-R. Zhang, H. Fan. Quantum coherence and geometric quantum discord. Phys. Rep. 762-764, 1-100 (2018).
[3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. Rev. Mod. Phys. 81, 865 (2009).
[4] A. K. Pati. Minimum classical bit for remote preparation and measurement of a qubit. Phys. Rev. A 63, 014302 (2000).
[5] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters. Remote state preparation. Phys. Rev. Lett. 87, 077902 (2001).
[6] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Phys. Rev. Lett. 70, 1895 (1993).
[7] C. H. Bennett and S. J. Wiesner. Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states. Phys. Rev. Lett. 69, 2881 (1992).
[8] J. Eisert, M. Plenio. A comparison of entanglement measures. J. Mod. Opt. 46, 145 (1999).
[9] W. K. Wootters. Entanglement of Formation of an Arbitrary State of Two Qubits. Phys. Rev. Lett. 80, 2245 (1998).
[10] M. Piani. Relative Entropy of Entanglement and Restricted Measurements. Phys. Rev. Lett. 103, 160504 (2009).
[11] A. Miranowicz, A. Grudka. Ordering two-qubit states with concurrence and negativity. Phys. Rev. A. 70, 032326 (2004).
[12] L. Mandel and E. Wolf. Optical Coherence and Quantum Optics. Cambridge University Press, Cambridge (1995).
[13] Aberg J. Quantifying superposition. arXiv:quant-ph/0612146 (2006).
[14] A. Winter and D. Yang. Operational resource theory of coherence. Phys. Rev. Lett. 116, 120404 (2016).
[15] D. Girolami. Observable measure of quantum coherence in finite dimensional systems. Phys. Rev. Lett. 113, 170401 (2014).
[16] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso. Measuring quantum coherence with entanglement. Phys. Rev. Lett. 115, 020403 (2015).
[17] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio. Converting nonclassicality into entanglement. Phys. Rev. Lett. 116, 080402 (2016).
[18] E. Chitambar, A. Streltsov, S. Rana, M. N. Bera, G. Adesso, and M. Lewenstein. Assisted distillation of quantum coherence. Phys. Rev. Lett. 116, 070402 (2016).
[19] E. Chitambar and G. Gour. Critical examination of incoherent operations and a physically consistent resource theory of quantum coherence. Phys. Rev. Lett. 117, 030401 (2016).
[20] W.-Y. Sun, D. Wang, B.-L. Fang, Z.-Y. Ding, H. Yang, F. Ming, L. Ye. Intrinsic relations of bipartite quantum resources in tripartite systems. Ann. Phys. (Berlin) 521, 1800358 (2019).
[21] D. E. Bruschi, C. Sabín, G. S. Paraoanu. Entanglement, coherence, and redistribution of quantum resources in double spontaneous down-conversion processes. Phys. Rev. A. 95, 062324 (2017).
[22] O. Ghne and G. Th. Entanglement detection. Phys. Rep. 474, 1 (2009).
[23] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters. Mixed-state entanglement and quantum error correction. Phys. Rev. A. 54, 3824 (1996).
[24] H. Barnum and N. Linden. Monotones and Invariants for Multi-partite Quantum States. J. Phys. A. 34, 6787 (2001).
[25] K. Audenaert, F. Verstraete, and B. De Moor. Variational characterizations of separability and entanglement of formation. Phys. Rev. A. 64, 052304 (2001).
[26] P. Rungta, V. Buzek, C. M. Caves, M. Hillery, and G. J. Mil
burn. Universal state inversion and concurrence in arbitrary dimensions. Phys. Rev. A 64, 042315 (2001).

[27] P. Badziag, P. Deuar, M. Horodecki, P. Horodecki, and R. Horodecki. Concurrence in arbitrary dimensions. J. Mod. Opt. 49, 1289 (2002).

[28] X.-F. Qian, T. Malhotra, A. N. Vamivakas, J. H. Eberly. Coherence Constraints and the Last Hidden Optical Coherence. Phys. Rev. Lett. 117, 153901 (2016).

[29] Y. Dai, W.-L. You, Y.-L. Dong, and C.-J. Zhang. Triangle inequalities in coherence measures and entanglement concurrence. Phys. Rev. A 96, 062308 (2017).

[30] Meyer C D. Matrix Analysis and Applied Linear Algebra. SIAM: 603 (2000).