Frobenius-Witt differentials and regularity

Takeshi Saito
November 4, 2021

Abstract

T. Dupuy, E. Katz, J. Rabinoff, D. Zureick-Brown introduced the module of total $p$-differentials for a ring over $\mathbb{Z}/p^2\mathbb{Z}$. We study the same construction for a ring over $\mathbb{Z}_{(p)}$ and prove a regularity criterion. For a local ring, the tensor product with the residue field is constructed in a different way by O. Gabber, L. Ramero.

In another article [11], we use the sheaf of FW-differentials to define the cotangent bundle and the micro-support of an étale sheaf.

Let $p$ be a prime number and $P = (X + Y)^p - X^p - Y^p \in \mathbb{Z}[X, Y]$ be the polynomial appearing in the definition of addition of Witt vectors. For a ring $A$ and an $A$-module $M$, we say a mapping $w : A \to M$ is a Frobenius-Witt derivation (Definition 1.1) or an FW-derivation for short if for any $a, b \in A$, we have

$$w(a + b) = w(a) + w(b) - P(a, b) \cdot w(p),$$
$$w(ab) = b^p \cdot w(a) + a^p \cdot w(b).$$

For rings over $\mathbb{Z}/p^2\mathbb{Z}$, such mappings are studied in [4] and called $p$-total derivation. As we show in Lemma 1.2.3, we have $p \cdot w(a) = 0$ for $a \in A$ if $A$ is a ring over $\mathbb{Z}_{(p)}$ and then we may replace $a^p, b^p$ in (1.3) by $F(\bar{a}), F(\bar{b})$ for the absolute Frobenius morphism $F : A/pA = A_1 \to A_1$. The equalities may be considered as linearized variants of those in the definition of $p$-derivation [3] or equivalently $\delta$-ring [1].

After preparing basic properties of FW-derivations in Section 1, we introduce the module $F_\Omega^1_A$ of FW-differentials for a ring $A$ endowed with a universal FW-derivation $w : A \to F_\Omega^1_A$ in Lemma 2.1. If $A$ is a ring over $\mathbb{Z}_{(p)}$, then $F_\Omega^1_A$ is an $A/pA$-module and the canonical morphism $F_\Omega^1_A \to F_\Omega^1_{A/p^2A}$ is an isomorphism by Corollary 2.4.1. Consequently, the generalization of the definition does not introduce new objects. If $A$ itself is a ring over $\mathbb{F}_p$, then the $A$-module $F_\Omega^1_A$ is canonically identified with the scalar extension $F^*\Omega^1_A$ of $\Omega^1_A$ by the absolute Frobenius $F : A \to A$ by Corollary 2.4.2.

For a local ring $A$ with residue field $k = A/\mathfrak{m}$ of characteristic $p$, we show in Proposition 2.6 that the $k$-vector space $F_\Omega^1_A \otimes_A k$ fits in an exact sequence $0 \to F^*(\mathfrak{m}_A/\mathfrak{m}_A^2) \to F_\Omega^1_A \otimes_A k \to F^*\Omega^1_k \to 0$ where $F^*$ denotes the scalar extension by the absolute Frobenius $F : k \to k$. We deduce from this in Corollary 2.7 that $F_\Omega^1_A \otimes_A k$ is canonically identified with the $k^{1/p}$-vector space $\Omega_A$ defined by Gabber and Ramero in [5, 9.6.12] using an
extension of $A$ involving the ring of Witt vectors $W_2(k)$. They use this module to correct an incomplete proof of a regularity criterion stated in [6, Chapitre 0, Théorème 22.5.4]. In the case where $A$ is a discrete valuation ring, we construct injections from the duals of the graded quotients of the Galois groups of Galois extensions of the fraction field of $A$ by the filtration of ramification groups to twists of $F\Omega^1_A \otimes_A k$ in [10].

The main result is the following regularity criterion. Under a suitable finiteness condition, we prove in Theorem 3.4 that a noetherian local ring $A$ with residue field of characteristic $p$ is regular if and only if the $A/pA$-module $F\Omega^1_A$ is free of the correct rank, using Proposition 2.6.

The construction of $F\Omega^1$ is sheafified and we obtain a sheaf of FW-differentials $F\Omega^1_X$ on a scheme $X$. We will use the sheaf of FW-differentials in [11] to define the cotangent bundle and the micro-support of an étale sheaf in mixed characteristic. In the final section, we study the relation of $F\Omega^1_X$ with $H_1$ of cotangent complexes.

The author thanks Luc Illusie for comments on earlier versions, for discussion on cotangent bundle and on notation and terminology. The author thanks Ofer Gabber for indicating another construction of the module and for the reference to [5] and [4]. The author thanks Alexander Beilinson for suggesting similarity to [3] and [1]. The author thanks Akhil Mathew heartily for pointing out an error in Lemma 1.3 and Corollary 2.4.3 and also for suggesting an argument proving that in Theorem 3.4 the regularity condition (2) implies the flatness of $F\Omega^1_A$ without any finiteness assumption.

The research is partially supported by Grant-in-Aid (B) 19H01780.

1 Frobenius-Witt derivation

We introduce Frobenius-Witt derivations and study basic properties.

**Definition 1.1.** Let $p$ be a prime number.

1. Define a polynomial $P \in \mathbb{Z}[X,Y]$ by

$$P = \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} \cdot X^i Y^{p-i}. \quad (1.1)$$

2. Let $A$ be a ring and $M$ be an $A$-module. We say that a mapping $w: A \to M$ is a Frobenius-Witt derivation or FW-derivation for short if the following condition is satisfied: For any $a, b \in A$, we have

$$w(a + b) = w(a) + w(b) - P(a, b) \cdot w(p), \quad (1.2)$$

$$w(ab) = b^p \cdot w(a) + a^p \cdot w(b). \quad (1.3)$$

For a ring $A$ over $\mathbb{Z}_{(p)}$, Definition 1.1.2 is essentially the same as [11, Definition 2.1.1] since the condition (3) loc. cit. is automatically satisfied by Lemmas 1.2.3 and 1.3.2 below.

**Lemma 1.2.** Let $A$ be a ring and $w: A \to M$ be an FW-derivation.

1. We have $w(1) = 0$. Let $a \in A$ and $n \in \mathbb{Z}$. Then, we have

$$w(na) = n \cdot w(a) + a^p \cdot w(n). \quad (1.4)$$
If $n \geq 0$, we have

\[(1.5) \quad w(a^n) = na^{p(n-1)} \cdot w(a).\]

2. For $n \in \mathbb{Z}$, we have

\[(1.6) \quad w(n) = \frac{n - np}{p} \cdot w(p),\]

In particular, we have $w(0) = 0$.

3. Assume that $A$ is a ring over $\mathbb{Z}_p$. Then, for any $a \in A$, we have $p \cdot w(a) = 0$.

In the most part of this article, $A$ will be a ring over $\mathbb{Z}_p$. Under this assumption, FW-derivations $w: A \to M$ take values in the $p$-torsion part of $M$ by Lemma 1.2.3.

**Proof.** 1. By putting $a = b = 1$ in (1.3), we obtain $w(1) = 0$.

Set $w_a(n) = n \cdot w(a) + a^p \cdot w(n)$. Then, by (1.2) and $P(n, m)a^p = P(na, ma)$, we have $w_a(n + m) = w_a(n) + w_a(m) - P(na, ma) \cdot w(p)$. Since $w_a(1) = w(a)$, we obtain (1.4) by the ascending and the descending inductions on $n$ starting from $n = 1$ by (1.2).

For $n = 0$, we have $w(a^0) = w(1) = 0$. By (1.3) and induction on $n$, we have $w(a^{n+1}) = a^p w(a^n) + a^{pn} w(a) = a^p \cdot na^{p(n-1)} w(a) + a^{pn} w(a) = (n + 1)a^{pn} w(a)$ and (1.5) follows.

2. Set $w_1(n) = \frac{n - np}{p} \cdot w(p)$. Then, by binomial expansion, $w_1$ satisfies (1.2). Hence we obtain (1.6) similarly as in the proof of (1.4). By setting $n = 0$ in (1.6), we obtain $w(0) = 0$.

3. Comparing (1.4) and (1.3), we obtain $(n - np) \cdot w(a) = 0$. Since the $p$-adic valuation $v_p(p - p^r)$ is 1, we obtain $p \cdot w(a) = 0$. \(\square\)

**Lemma 1.3.** Assume that $A$ is flat over $\mathbb{Z}$ and that the Frobenius $F: A/pA \to A/pA$ is an isomorphism.

1. The mapping $w: A \to A/pA$ given by $w(a^0 + pb) \equiv b^p \mod pA$ for $a, b \in A$ is well-defined and is an FW-derivation.

In particular, for $A = \mathbb{Z}_p$, the mapping $w: \mathbb{Z}_p \to F_p$ defined by $w(a) = \frac{a - a^p}{p} \mod p$ is an FW-derivation.

2. Let $\varphi: A \to A$ be an endomorphism satisfying $\varphi(a) \equiv a^p \mod p$ and let $\varphi_1: A \to A$ be the unique mapping satisfying $\varphi(a) = a^p + p\varphi_1(a)$. Let $M$ be any $A$-module and $w: A \to M$ be any FW-derivation. Then, we have

\[w(r) = \varphi_1(r) \cdot w(p)\]

for $r \in A$.

**Proof.** 1. Since $F: A/pA \to A/pA$ is assumed a surjection, any element $r \in A$ may be written as $r = a^p + pb$ for $a, b \in A$. Since $(a + pb)^p \equiv a^p \mod p^2$, the mapping $w$ is well-defined. Since

\[a^p + pb + a^p + pb' = (a + a')^p + p(b + b' - P(a, a'))\]
we have
\[ w(a^p + pb + a^p + pb) = (b + b' - P(a, a'))^p \equiv w(a^p + pb) + w(a^p + pb) - P(a^p + pb, a^p + pb') \mod p \]
and (1.2) is satisfied. Since
\[ (a^p + pb)(a^p + pb') \equiv (aa')^p + p(a^p b + a^p b') \mod p^2, \]
we have
\[ w((a^p + pb)(a^p + pb)) = (a^p b + a^p b')^p \equiv (a^p + pb')^p w(a^p + pb) + (a^p + pb)^p w(a^p + pb') \mod p \]
and (1.3) is satisfied.

For \( a \in A = \mathbb{Z}_p \), we have \( a = a^p + pb \) for \( b \in \mathbb{Z}_p \) and \( w(a) \equiv b^p \equiv b \mod p \). Alternatively, we can also verify directly that the mapping \( w: \mathbb{Z}_p \to F_p \) defined by \( w(a) \equiv (a - a^p)/p \mod p \) satisfies (1.2) and (1.3).

2. Since \( F: A/pA \to A/pA \) is assumed a surjection, we may write \( r = a^p + pb \) for \( a, b \in A \). Since \( \varphi(a) \equiv r \mod p \) implies \( \varphi(a)^p \equiv r^p \mod p^2 \), we have \( \varphi(r) = \varphi(a)^p + p \varphi(b) \equiv r^p + pb^p \mod p^2 \). Further by (1.2), (1.5), (1.3) and by \( p \cdot w(p) = p \cdot w(a) = p \cdot w(b) = 0 \) in Lemma 1.2.3, we have \( w(r) = w(a^p) + w(pb) = b^p \cdot w(p) = \varphi_1(r) \cdot w(p) \).

We give a relation between FW-derivations and Frobenius semi-linear derivations for rings over \( F_p \).

**Lemma 1.4.** Let \( A \) be a ring, \( B \) be a ring over \( F_p \) and \( g: A \to B \) be a morphism of rings. For a \( B \)-module \( M \) and a mapping \( w: A \to M \), the following conditions are equivalent:

1. If we regard \( M \) as an \( A \)-module by \( g: A \to B \), then \( w \) is an FW-derivation and \( w(p) = 0 \).
2. If we regard \( M \) as an \( A \)-module by the composition \( f = F \circ g: A \to B \) with the absolute Frobenius, then \( w \) is a derivation.

**Proof.** (1)\( \Rightarrow \) (2): If \( w \) is an FW-derivation satisfying \( w(p) = 0 \), then \( w \) is additive by (1.2). Further (1.3) means the Leibniz rule with respect to the composition \( f = F \circ g: A \to B \).

(2)\( \Rightarrow \) (1): If \( w \) satisfies the Leibniz rule, then we have \( w(1) = 1 \). Hence the additivity implies \( w(p) = 0 \) and (1.2). The Leibniz rule with respect to the composition \( f = F \circ g \) means (1.3) conversely.

**Lemma 1.5.** Let \( A \) be a ring, \( I \subset A \) be an ideal and let \( M \) be an \( A \)-module. Then an FW-derivation \( w: A \to M \) induces an FW-derivation \( \bar{w}: A/I \to M/(IM + A \cdot w(I)) \).

**Proof.** By (1.2), we have \( w(a + b) \equiv w(a) + w(b) \mod IM \) for \( a \in A \) and \( b \in I \). Hence \( w \) induces a mapping \( \bar{w}: A/I \to M/(IM + A \cdot w(I)) \). Since \( w \) satisfies (1.2) and (1.3), \( \bar{w} \) also satisfies (1.2) and (1.3).

An extension of FW-derivation to the ring of polynomials is uniquely determined by choosing the value at the indeterminate.
Proposition 1.6. Let \( A \) be a ring and \( M \) be an \( A[X] \)-module. Let \( w: A \to M \) be an FW-derivation.

1. Let \( x \in M \) be an element satisfying \( px = 0 \). Then, there exists a unique FW-derivation \( \tilde{w}: A[X] \to M \) extending \( w \) and satisfying \( \tilde{w}(X) = x \).

2. If \( A \) is a ring over \( \mathbb{Z}(p) \), the mapping

\[
Q(f) = \sum_{0 \leq k_0, \ldots, k_n < p, k_0 + \cdots + k_n = p} \frac{(p-1)!}{k_0! k_1! \cdots k_n!} a_0^{k_0} (a_1 X)^{k_1} \cdots (a_n X^n)^{k_n} \in A[X],
\]

\[
w^{(p)}(f) = \sum_{i=0}^n X^{pi} \cdot w(a_i) \in M.
\]

In (1.8), the summation is taken over the integers \( 0 \leq k_0, \ldots, k_n < p \) satisfying \( k_0 + \cdots + k_n = p \).

If \( \tilde{w}: A[X] \to M \) is an FW-derivation extending \( w \) and satisfying \( \tilde{w}(X) = x \), then by (1.2) and (1.3) we have

\[
\tilde{w}(f) = f^p \cdot x + w^{(p)}(f) - Q(f) \cdot w(p)
\]

for \( f \in A[X] \). Hence it suffices to show that \( \tilde{w} \) defined by (1.10) is actually an FW-derivation.

For \( f = \sum_{i=0}^n a_i X^i \), \( g = \sum_{i=0}^n b_i X^i \in A[X] \), set

\[
f^{(p)} = \sum_{i=0}^n a_i^p X^{pi}, \quad R(f, g) = \sum_{i=0}^n P(a_i, b_i) X^{pi}.
\]

Then, we have

\[
(f + g)^{(p)} = f^{(p)} + g^{(p)} + pR(f, g), \quad f^p = f^{(p)} + pQ(f).
\]

From this and \( (f + g)^p = f^p + g^p + pP(f, g) \), by reducing to the universal case where \( A \) is flat over \( \mathbb{Z} \), we deduce

\[
Q(f + g) = Q(f) + Q(g) + P(f, g) - R(f, g).
\]

By (1.12), we have

\[
w^{(p)}(f + g) = w^{(p)}(f) + w^{(p)}(g) - R(f, g) \cdot w(p).
\]

Since \( px = 0 \), we have \( (f + g)^p \cdot x = f^p \cdot x + g^p \cdot x \). This and (1.13) and (1.12) show that the mapping \( \tilde{w} \) satisfies (1.2).
We show that the mapping \( \tilde{w} \) satisfies (1.3). Since \( px = 0 \), we have \((fg)^px = f^p \cdot g^p x + g^p \cdot f^p x\). Hence, we may assume \( x = 0 \). If \( f \) and \( g \) are monomials, we have \( Q(f) = Q(g) = 0 \) and \( w^{(p)}(fg) = f^p \cdot w^{(p)}(g) + g^p \cdot w^{(p)}(f) \) and (1.3) is satisfied in this case. For \( f_1, f_2, g \in A[X] \), we have \( w_0(f_1g + f_2g) - (w_0(f_1g) + w_0(f_2g)) = P(f_1g, f_2g) \cdot w(p) \) and \((f_1 + f_2)^p w_0(g) + g^p w_0(f_1 + f_2) - (f_1^p w_0(g) + g^p w_0(f_1) + f_2^p w_0(g) + g^p w_0(f_2)) = g^p P(f_1, f_2) \cdot w(p)\) by (1.13) and (1.12). Since \( P(f_1g, f_2g) = g^p P(f_1, f_2) \), the equality (1.3) follows by induction on the numbers of non-zero terms in \( f \) and \( g \).

2. If \( \tilde{w}: A[X] \to M \) is an FW-derivation extending \( w \), we have \( \tilde{w}(X) \in M[p] \) by the assumption that \( A \) is a ring over \( Z(p) \) and Lemma 1.2.3. Thus, the assertion follows from 1.

\[ \square \]

2 Frobenius-Witt differentials

We introduce the module of Frobenius-Witt differentials as the target of the universal FW-derivation and study basic properties.

**Lemma 2.1.** Let \( p \) be a prime number and \( A \) be a ring. Then, there exists a universal pair of an \( A \)-module \( F\Omega^1_A \) and an FW-derivation \( w: A \to F\Omega^1_A \).

**Proof.** Let \( A^{(A)} \) be the free \( A \)-module representing the functor sending an \( A \)-module \( M \) to the set \( \text{Map}(A, M) \) and let \([\cdot]: A \to A^{(A)}\) denote the universal mapping. Define an \( A \)-module \( F\Omega^1_A \) to be the submodule generated by \([a + b] - [a] - [b] + P(a, b)[p]\) and \([ab] - a^p[b] - b^p[a]\) for \( a, b \in A \). Then, the pair of \( F\Omega^1_A \) and the composition \( w: A \to F\Omega^1_A \) of \([\cdot]: A \to A^{(A)}\) with the canonical surjection \( A^{(A)} \to F\Omega^1_A \) satisfies the required universal property.

**Definition 2.2.** Let \( A \) be a ring and \( p \) be a prime number. We call the \( A \)-module \( F\Omega^1_A \) and \( w: A \to F\Omega^1_A \) in Lemma 2.1 the module of FW-differentials of \( A \) and the universal FW-derivation. For \( a \in A \), we call \( w(a) \in F\Omega^1_A \) the FW-differential of \( a \).

If \( A \) is a ring over \( Z(p) \), by Lemma 1.2.3, we have \( p \cdot F\Omega^1_A = 0 \). For a morphism \( A \to B \) of rings, the composition \( A \to B \to F\Omega^1_B \) defines a canonical morphism \( F\Omega^1_A \to F\Omega^1_B \) and hence a \( B \)-linear morphism

\[ (2.1) \quad F\Omega^1_A \otimes_A B \to F\Omega^1_B. \]

We study the module of FW-differentials of a quotient ring.

**Proposition 2.3.** Let \( p \) be a prime number and let \( A \) be a ring. Let \( I \subset A \) be an ideal and \( B = A/I \) be the quotient ring.

1. The canonical morphism \( F\Omega^1_A \otimes_A B \to F\Omega^1_B \) (2.1) induces an isomorphism

\[ (2.2) \quad (F\Omega^1_A \otimes_A B)/(B \cdot w(I)) \to F\Omega^1_B. \]

In particular, if the ideal \( I \) is generated by \( a_1, \ldots, a_n \in A \), we have an isomorphism

\[ (2.3) \quad F\Omega^1_A/(I \cdot F\Omega^1_A + \sum_{i=1}^n A \cdot w(a_i)) \to F\Omega^1_B. \]
2. Let $B \rightarrow B'$ be a morphism of rings to a ring $B'$ over $\mathbb{F}_p$, and let $F^*(I/I^2 \otimes_B B')$ denote the tensor product with respect to the absolute Frobenius $F: B' \rightarrow B'$. Then the isomorphism \((2.2)\) defines an exact sequence

\[(2.4) \quad F^*(I/I^2 \otimes_B B') \rightarrow F\Omega^1_A \otimes_A B' \rightarrow F\Omega^1_B \otimes_B B' \rightarrow 0\]

of $B'$-modules.

Proof. 1. By Lemma \[1.3\] the universal FW-derivation $w: A \rightarrow F\Omega^1_A$ induces an FW-derivation $\tilde{w}: B \rightarrow M = (F\Omega^1_A \otimes_A B)/(B \cdot w(I))$. This defines a $B$-linear mapping $F\Omega^1_B \rightarrow M$ in the opposite direction. Since the composition $F\Omega^1_A \rightarrow F\Omega^1_B \rightarrow M$ with the morphism induced by $A \rightarrow B$ is the canonical surjection, the composition $M \rightarrow F\Omega^1_B \rightarrow M$ with \((2.2)\) is the identity of $M$. Since the other composition $F\Omega^1_B \rightarrow M \rightarrow F\Omega^1_B$ is also the identity, \((2.2)\) is an isomorphism.

If $I$ is generated by $a_1, \ldots, a_n \in A$, the image of $w: I \otimes \mathbb{Z} B \rightarrow F\Omega^1_A \otimes_A B$ is generated by $w(a_1), \ldots, w(a_n)$ as a $B$-module by \[1.2\] and \[1.3\].

2. The additive mapping $w: I \rightarrow F\Omega^1_A \otimes_A B'$ is compatible with the composition $A \rightarrow B'$ with the Frobenius $F: B' \rightarrow B'$ by \[1.3\]. Hence $w$ induces a $B'$-linear mapping $F^*(I/I^2 \otimes_B B') \rightarrow F\Omega^1_A \otimes_A B'$. Since its image is $B' \cdot w(I)$, the sequence \((2.4)\) is exact by the isomorphism \((2.2)\). \hfill \Box

Corollary 2.4. Let $A$ be a ring over $\mathbb{Z}_{(p)}$ and set $B = A/pA$ and $B_2 = A/p^2A$. For a $B$-module $M$, let $F^*M$ denote the tensor product $M \otimes_B B$ with respect to the absolute Frobenius $F: B \rightarrow B$.

1. The $A$-module $F\Omega^1_A$ is a $B$-module. The morphism $F\Omega^1_A \rightarrow F\Omega^1_{B_2}$ induced by the surjection $A \rightarrow B_2 = A/p^2A$ is an isomorphism.

2. The derivation $d: A \rightarrow F^*\Omega^1_B$ is an FW-derivation and defines an isomorphism

\[(2.5) \quad F\Omega^1_A/(A \cdot w(p)) \rightarrow F^*\Omega^1_B\]

of $B$-modules. In particular, if $p = 0$ in $A = B$, the isomorphism \((2.5)\) gives an isomorphism

\[(2.6) \quad F\Omega^1_B \rightarrow F^*\Omega^1_B.\]

3. Assume that $A$ is faithfully flat over $\mathbb{Z}_{(p)}$ and that the Frobenius $F: A/pA \rightarrow A/pA$ is an isomorphism. Then, $F\Omega^1_A$ is a non-zero $A/pA$-module generated by $w(p)$.

In particular, if $A$ is a discrete valuation ring with perfect residue field $k$ such that $p$ is a uniformizer, then $F\Omega^1_A$ is a $k$-vector space of dimension 1 generated by $w(p)$.

4. Assume that $A$ is noetherian and that the quotient $A/\sqrt{pA}$ by the radical of the principal ideal $pA$ is of finite type over a field $k$ with finite $p$-basis. Then, the $A$-module $F\Omega^1_A$ is of finite type.

By Lemma \[1.2\] and Corollary \[2.4\], if $A$ is a ring over $\mathbb{Z}_{(p)}$, an FW-derivation $w: A \rightarrow M$ is always induced by an FW-derivation $A/p^2A \rightarrow M[p]$ to the $p$-torsion part. Examples after the proof show that we cannot relax the assumption in 4. in essential ways.
Proof. 1. The $A$-module $F\Omega^1_A$ is a $B$-module by Lemma 2.3. Since $p \cdot F\Omega^1_A = 0$, we have $w(p^2) = 2p^2 \cdot w(p) = 0$. Hence the isomorphism $F\Omega^1_A/(p^2 \cdot F\Omega^1_A + B_2 \cdot w(p^2)) \to F\Omega^1_{B_2}$ for $I = p^2A$ gives the required isomorphism $F\Omega^1_A \to F\Omega^1_{B_2}$.

2. Let $M$ be a $B$-module. By the universality of $F\Omega^1_A$, $A$-linear morphisms $F\Omega^1_A/(A \cdot w(p)) \to M$ correspond bijectively to FW-derivations $w: A \to M$ satisfying $w(p) = 0$. By the universality of $F^*\Omega^1_{B_1}$, $B$-linear morphisms $F^*\Omega^1_{B_1} \to M$ correspond bijectively to usual derivations $B \to M$ with respect to the Frobenius $B \to B$. Since $B = A/pA$, usual derivations $B \to M$ further correspond bijectively to derivations $A \to M$ with respect to the composition $A \to B$ with the Frobenius. Hence the assertion follows from Lemma 1.4.

3. Since $F: A/pA \to A/pA$ is assumed a surjection, we have $\Omega^1_{A/pA} = 0$. Hence by the isomorphism (2.5), $F\Omega^1_A$ is an $A/pA$-module generated by one element $w(p)$. Let $w: A \to A/pA$ be the FW-derivation in Lemma 1.3.1 defined by $w(a^p + pb) \equiv b^p \mod pA$ for $a, b \in A$. If $A/pA \neq 0$, then we have $w(p) = 1 \neq 0$ and $F\Omega^1_A \neq 0$.

4. A field $k$ is formally smooth over $F_p$ by [3] Chapitre 0, Théorème (19.6.1). Since the ideal $\sqrt{pA}/pA \subset A/pA = B$ is a nilpotent ideal of finite type, the morphism $k \to A/\sqrt{pA}$ is lifted to a morphism $k \to A/pA = B$ of finite type. Since $k$ is of finite $p$-basis, the $k$-vector space $\Omega^1_k$ is of finite dimension and the $B$-module $\Omega^1_B$ is of finite type by the exact sequence $\Omega^1_k \otimes B \to \Omega^1_B \to \Omega^1_{B/k} \to 0$. Thus, the assertion follows from the isomorphism (2.8) of $B$-modules.

Example 1. Let $A = k$ be a field of characteristic $p > 0$. Then, the $k$-vector space $F\Omega^1_k = F^*\Omega^1_k$ is finitely generated if and only if $k$ has a finite $p$-basis.

2. Let $k$ be a perfect field of characteristic $p > 0$ and let $K \subset k((t))$ be a subextension of finite type of transcendental degree $n \geq 1$ over $k$ as in [3] Proposition 11.6. Then, $A = k[[t]]/K \subset k((t))$ is a discrete valuation ring with residue field $k$ and dim$_k F\Omega^1_A \otimes_A k \leq 1$ by (2.4). Since the surjection $A \to A/m^2_A = k[t]/(t^2)$ induces a surjection $F\Omega^1_A \to F\Omega^1_{A/m^2_A} \neq 0$, we have dim$_k F\Omega^1_A \otimes_A k = 1$. On the other hand, we have dim$_K F\Omega^1_A \otimes_A K = \dim_K F^*\Omega^1_K = n$. Hence if $n > 1$, the $A$-module $F\Omega^1_A$ is not finitely generated.

Let $A \to B$ be a surjection of rings over $\mathbb{Z}_p$ with kernel $I \subset A$. Set $A_1 = A/pA$ and $B_1 = B/pB$ and let $I_1 \subset A_1$ be the image of $I$. Then the exact sequence (2.4), the isomorphism (2.8) for $A$ and $B$ and the Frobenius pull-back of the exact sequence $I_1/I^2_1 \to \Omega^1_{A_1} \otimes_A B_1 \to \Omega^1_{B_1} \to 0$ define a commutative diagram

$$
\begin{array}{cccccc}
F^*(I/I^2 \otimes_B B_1) & \to & F\Omega^1_A \otimes_A B_1 & \to & F\Omega^1_B \otimes_B B_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
F^*(I_1/I^2_1) & \to & F^*\Omega^1_{A_1} \otimes_{A_1} B_1 & \to & F^*\Omega^1_{B_1} & \to & 0
\end{array}
$$

(2.7)

of exact sequences. The morphism $w: F^*(I/I^2 \otimes_B B_1) \to F\Omega^1_A \otimes_A B_1$ is induced by the restriction of the universal FW-derivation $w: A \to F\Omega^1_A$ and the vertical arrows are the canonical surjections. By the isomorphism (2.8), the bottom terms on the middle and right are the quotients of the top terms by the $B_1$-submodules generated by $w(p)$.

Proposition 2.5. Let $p$ be a prime number and let $A$ be a ring.
1. If \( A = \lim_{\lambda \in \Lambda} A_\lambda \) is a filtered inductive limit, the canonical morphism \( \lim_{\lambda \in \Lambda} F\Omega^1_{A_\lambda} \to F\Omega^1_A \) is an isomorphism.

2. Let \( S \subset A \) be a multiplicative subset. Then, the canonical morphism

\[
(2.8) \quad S^{-1}F\Omega^1_A \to F\Omega^1_{S^{-1}A}
\]

is an isomorphism.

3. Assume that \( A \) is a ring over \( \mathbb{Z}(p) \) and let \( B = A[X] \) be a polynomial ring. Then, \( F\Omega^1_B \) is the direct sum of \( F\Omega^1_{A_\lambda} \otimes_A B \) with a free \( B/pB \)-module of rank 1 generated by \( w(X) \).

**Proof.** 1. For any \( A \)-module \( M \), \( \text{FW-derivations} \ A \to M \) are in bijection with projective systems of \( \text{FW-derivations} \ A_\lambda \to M \). \( A \)-linear mappings \( \lim_{\lambda \in \Lambda} F\Omega^1_{A_\lambda} \to M \) are also in bijection with projective systems of \( A_\lambda \)-linear mappings \( F\Omega^1_{A_\lambda} \to M \). Hence the assertion follows from the universality of \( F\Omega^1 \).

2. By (1.3), the mapping \( w: S^{-1}A \to S^{-1}F\Omega^1_A \) given by \( w(a/s) = 1/s^p \cdot w(a) - (a/s^p)^p \cdot w(s) \) is well-defined. Since this is an \( \text{FW-derivation} \), we obtain a morphism \( F\Omega^1_{S^{-1}A} \to S^{-1}F\Omega^1_A \). The composition \( F\Omega^1_A \to F\Omega^1_{S^{-1}A} \to S^{-1}F\Omega^1_A \) is the canonical morphism and the composition \( F\Omega^1_{S^{-1}A} \to S^{-1}F\Omega^1_A \to F\Omega^1_{S^{-1}A} \) is the identity. Hence the morphism (2.8) has an inverse and is an isomorphism.

3. Let \( M \) be a \( B \)-module. Then, by Proposition 1.6 and by the universality of \( F\Omega^1 \), \( B \)-linear morphisms \( F\Omega^1_B \to M \) corresponds bijectively to pairs of \( A \)-linear morphisms \( F\Omega^1_A \otimes_A B \oplus (B/pB) \to M \), the assertion follows.

We give a description as an extension of the fiber of the module of \( \text{FW-differentials} \) of a local ring at the closed point.

**Proposition 2.6.** Let \( A \) be a local ring such that the residue field \( k = A/m_A \) is of characteristic \( p \). For a \( k \)-vector space \( M \), let \( F^*M \) denote the tensor product \( M \otimes_k k \) with respect to the Frobenius \( F: k \to k \). Let \( w: F^*(m_A/m_A^2) \to F\Omega^1_A \otimes_A k = F\Omega^1_A/m_A F\Omega^1_A \) be the morphism induced by the universal \( \text{FW-derivation} \) \( w: A \to F\Omega^1_A \). Then, the sequence

\[
(2.9) \quad 0 \longrightarrow F^*(m_A/m_A^2) \xrightarrow{w} F\Omega^1_A \otimes_A k \longrightarrow F^*\Omega^1_k \longrightarrow 0
\]

(2.9) of \( k \)-vector spaces is exact.

**Proof.** The exactness except the injectivity of \( w \) follows from (2.8). First, we show the case where \( A \) is the localization at a prime ideal of a polynomial ring \( A_0 = W_2(k)[T_1, \ldots, T_n] \) over the ring \( W_2(k_0) \) of Witt vectors of length 2 for a perfect field \( k_0 \) and an integer \( n \). Then, by Proposition 2.3 and 2.5 and Corollary 2.4 and 2.4, the \( A_0 \)-module \( F\Omega^1_{A_0} \) is free of rank \( n+1 \). Hence by Proposition 2.5, the \( k \)-vector space \( F\Omega^1_A \otimes_A k = F\Omega^1_{A_0} \otimes_{A_0} k \) is of dimension \( n+1 \).

Let \( d \) be the transcendence degree of \( k \) over \( k_0 \). Then, we have \( \dim \Omega^1_k = d \). The localization \( B \) at the inverse image of \( m_A \) by the composition \( W(k)[T_1, \ldots, T_n] \to W_2(k)[T_1, \ldots, T_n] \to A \) is a regular local ring of dimension \( n+1-d \) and the canonical morphism \( m_B/m_B^2 \to m_A/m_A^2 \) is an isomorphism. Hence we have \( \dim m_B/m_B^2 = n+1-d \). Since (2.9)
is exact except possibly at $F^*(m_A/m_A^2)$ by Proposition 2.3.2, it follows that (2.9) is exact everywhere.

We show the general case. By taking the limit, we may assume that $A$ is a localization of a ring $A_0$ of finite type over $\mathbb{Z}$. By Corollary 2.4.1, we may assume that $A_0$ is of finite type over $\mathbb{Z}/p^2 \mathbb{Z} = \mathbb{W}(k_0)$ for $k_0 = F_p$. We take a surjection $B_0 = W_2(k)[T_0, \ldots, T_n] \to A_0$. Let $B$ be the localization of $B_0$ at the inverse image of $m_A$ by the composition $B_0 \to A_0 \to A$ and let $I$ be the kernel of the surjection $B \to A$. Then, by Proposition 2.3.2, we have a commutative diagram

\[
\begin{array}{c}
F^*(I \otimes_B k) \longrightarrow F^*(I \otimes_B k) \\
\downarrow \quad \downarrow \\
0 \longrightarrow F^*(m_B/m_B^2_B) \longrightarrow F\Omega^1_B \otimes_B k \longrightarrow F^*\Omega^1_k \longrightarrow 0 \\
\downarrow \quad \downarrow \\
F^*(m_A/m_A^2_A) \longrightarrow F\Omega^1_A \otimes_A k \longrightarrow F^*\Omega^1_k \longrightarrow 0 \\
\downarrow \quad \downarrow \\
0 \quad 0
\end{array}
\]

of exact sequences. Hence the assertion follows. 

We prove a relation with $\Omega_A$ defined by Gabber-Ramero. For the definition of $\Omega_A$, we refer to [5, 9.6.12].

**Corollary 2.7.** Let $A$ be a local ring such that the residue field $k = A/m_A$ is of characteristic $p$. Let $\Omega_A$ be the $k^{1/p}$-vector space defined in [5, 9.6.12] and regard $d_A: A \to \Omega_A$ as an FW-derivation by identifying the inclusion $k \to k^{1/p}$ with the Frobenius $F: k \to k$. Then, the morphism $F\Omega^1_A \otimes_A k \to \Omega_A$ induced by $d_A$ is an isomorphism.

**Proof.** For a $k$-vector space $V$, we identify $V \otimes_k k^{1/p}$ with $F^*V$ by identifying the inclusion $k \to k^{1/p}$ with the Frobenius $F: k \to k$. We consider the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & F^*(m_A/m_A^2) & \longrightarrow & F\Omega^1_A \otimes_A k & \longrightarrow & F^*\Omega^1_{k/F_p} & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \longrightarrow & m_A/m_A^2 \otimes k^{1/p} & \longrightarrow & \Omega_A & \longrightarrow & \Omega^1_{k/F_p} \otimes k^{1/p} & \longrightarrow & 0.
\end{array}
\]

The upper line is exact by Proposition 2.6 and the lower exact sequence is defined in [5, Proposition 9.6.14]. The middle vertical arrow is induced by the FW-derivation $d_A: A \to \Omega_A$ and the diagram is commutative. Hence the assertion follows. 

We give a criterion of regularity which will be used in the proof of the main theorem in the next section.

**Corollary 2.8.** Let $A$ be a regular local ring such that the residue field $k = A/m_A$ is of characteristic $p$. Let $B = A/I$ be the quotient by an ideal $I \subset m_A$. We set $A_1 = A/pA$, $B_1 = B/pB$, and for a $B_1$-module $M$, let $F^*M = M \otimes_{B_1} B_1$ denote the tensor product

\[
\begin{array}{c}
\]
with respect to the Frobenius $F: B_1 \to B_1$. Let $w: F^*(I \otimes_A B_1) \to F\Omega^1_A \otimes_A B_1$ be the morphism induced by the universal FW-derivation $w: A \to F\Omega^1_A$.

We consider the following conditions:

1. The sequence
   \[ 0 \to F^*(I \otimes_A B_1) \xrightarrow{w} F\Omega^1_A \otimes_A B_1 \to F \Omega^1_B \to 0 \]
   of $B_1$-modules is a split exact sequence.

2. $B$ is regular.
   1. We always have $(1) \Rightarrow (2)$.
   2. Assume that $F \Omega^1_A$ is a free $A$-module of finite rank. Then, we have $(2) \Rightarrow (1)$ and $F \Omega^1_B$ is a free $B_1$-module of finite rank.

Proof. First, we show that the condition $(2)$ is equivalent to the following condition:

$(2')$ The sequence $0 \to I \otimes_A k \to m_A/m_A^2 \to m_B/m_B^2 \to 0$ is exact.

$(2) \Rightarrow (2')$: The condition $(2)$ means that $I$ is generated by a part of regular system of parameters of $A$ by [6, Chapitre 0, Corollaire (17.1.9)]. This condition means that the images of a minimal system of generators of $I$ form a basis of the kernel of $m_A/m_A^2 \to m_B/m_B^2$. Hence the condition $(2)$ implies $(2')$.

$(2') \Rightarrow (2)$: Conversely, a lifting of the basis of $I \otimes_A k$ is a part of regular system of parameters of $A$ and is a system of generators of $I$ by Nakayama’s lemma.

By Proposition 2.6 for $A$ and $B$, $(2')$ is equivalent to the following:

$(1')$ The sequence
   \[ 0 \to F^*(I \otimes_A k) \xrightarrow{w} F\Omega^1_A \otimes_A k \to F \Omega^1_B \otimes_B k \to 0 \]
   induced by $(2.11)$ is exact.

1. The condition $(1)$ obviously implies $(1')$.

2. Since $F^*(I \otimes_A B_1)$ and $F\Omega^1_A \otimes_A B_1$ are free $B_1$-modules of finite rank, the condition $(1')$ conversely implies $(1)$ and that $F \Omega^1_B$ is a free $B_1$-module of finite rank.

Lemma 2.9. Let $f: A \to B$ be a morphism of rings over $\mathbb{Z}(p)$ and set $A_1 = A/pA$ and $B_1 = B/pB$. Then, the isomorphism $(2.5)$ induces an isomorphism

$(2.13)$ \[ \text{Coker}(F \Omega^1_A \otimes_A B \to F \Omega^1_B) \to F^* \Omega^1_{B_1/A_1}. \]

Proof. By the isomorphism $(2.5)$ for $A$ and $B$ and its functoriality, we have a commutative diagram

\[
\begin{array}{cccccc}
B_1 & \xrightarrow{-w(p)} & F \Omega^1_A \otimes_A B_1 & \to & F^*(\Omega^1_A \otimes_A B_1) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
B_1 & \xrightarrow{-w(p)} & F \Omega^1_B & \to & F^* \Omega^1_{B_1} & \to 0
\end{array}
\]

of exact sequences and the assertion follows.

We give a criterion for the smoothness.
Proposition 2.10. Let \( f : A \to B \) be a morphism of finite presentation of rings over \( \mathbb{Z}_{(p)} \) and set \( A_1 = A/pA \) and \( B_1 = B/pB \). We consider the sequence

\[
\begin{array}{ccccccc}
0 & \to & F\Omega_A^1 \otimes_A B & \xrightarrow{2.1} & F\Omega_B^1 & \to & F^*\Omega_{B_1/A_1}^1 & \to & 0 \\
\end{array}
\]

of \( B_1 \)-modules

1. Assume that \( f \) is smooth. Then, the sequence (2.14) is a split exact sequence and (2.13) is an isomorphism of projective \( B_1 \)-modules of finite rank.

2. Let \( q \) be a prime ideal of \( B \) such that the residue field \( k = B_q/qB_q \) is of characteristic \( p \) and let \( p \subset A \) be the inverse image of \( q \). Assume that \( A_p \) and \( B_q \) are regular and that (2.14) is a split exact sequence after \( \otimes_B B_q \). Then \( f : A \to B \) is smooth at \( q \).

Proof. 1. Since \( f \) is smooth, the \( B_1 \)-module \( F^*\Omega_{B_1/A_1}^1 = \text{Coker}(F\Omega_A^1 \otimes_A B \to F\Omega_B^1) \) is projective of finite rank.

Since \( A \to B \) is étale, after a localization, there exists a monic polynomial \( f \in A[T] \) such that \( \text{Spec} A \) is isomorphic to an open subscheme of \( \text{Spec} A[T]/(f)[1/f] \) by [6, Théorème (18.4.6)]. Hence we may further assume \( B = A[T]/(f)[1/f] \) for a monic polynomial \( f \in A[T] \). Then, by Proposition 2.5.3 and 2.5.2 and Proposition 2.3.1, the \( B/pB \)-module \( F\Omega_B^1 \) is the quotient of \( (F\Omega_A^1 \otimes_A B) \oplus (B/pB \cdot w(T)) \) by the submodule generated by \( \tilde{w}(f) = f^{(p)}(T^p) \cdot w(T) + w^{(p)}(f) + Q(f) \cdot w(p) \) in the notation of the proof of Proposition 1.6. Since \( f^{(p)}(T^p) \equiv f^p \mod pB \) is invertible in \( B/pB \) and \( w^{(p)}(f) + Q(f) \cdot w(p) \in F\Omega_A^1 \otimes_A B \), the morphism \( F\Omega_A^1 \otimes_A B \to ((F\Omega_A^1 \otimes_A B) \oplus (B/pB \cdot w(T))/B \cdot \tilde{w}(f) \) is an isomorphism as required.

2. Since the assertion is local by Proposition 2.5.2, we may assume that \( A = A_p \). We take a surjection \( C = A[T_1, \ldots, T_n] \to B \) and let \( C_t \) be the localization at the inverse image \( t \) of \( q \). Then, we have a split exact sequence

\[
\begin{array}{ccccccc}
0 & \to & F\Omega_A^1 \otimes_A C & \to & F\Omega_C^1 & \to & F^*(\Omega_{C/A}^1 \otimes_C C/pC) & \to & 0 \\
\end{array}
\]

by Proposition 2.5.3.

By Proposition 2.6 for \( C_t \) and \( B_q \), we have a commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & F^*(tC_t \otimes t^2C_t) & \to & F\Omega_C^1 \otimes_C k & \to & F^*\Omega_k^1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & F^*(qB_q \otimes q^2B_q) & \to & F\Omega_B^1 \otimes_B k & \to & F^*\Omega_k^1 & \to & 0 \\
\end{array}
\]

of exact sequences. The vertical arrows are surjections. Since the kernel \( I \) of the surjection \( C_t \to B_q \) of regular local rings is generated by a part of a regular system of local parameters, the sequence \( 0 \to I \otimes_{C_t} k \to tC_t/t^2C_t \to qB_q/q^2B_q \to 0 \) is exact. Hence we obtain an exact sequence

\[
\begin{array}{ccccccc}
0 & \to & F^*(I \otimes_{C_t} k) & \to & F\Omega_C^1 \otimes_C k & \to & F\Omega_B^1 \otimes_B k & \to & 0. \\
\end{array}
\]
If $F \Omega^1_A \otimes_A B_q \to F \Omega^1_{B_q}$ is a split injection, by (2.15) and (2.16), the induced morphism $F^*(I \otimes_C k) \to F^*(\Omega^1_{C/A} \otimes_C k)$ is an injection. This means that the morphism $I/I^2 \to \Omega^1_{C/A} \otimes_C B_q$ of free $B_q$-modules is a split injection. Since $A \to C$ is smooth, $A \to B$ is also smooth at $q$.

3 Regularity criterion

We recall some facts from commutative algebra and field theory in positive characteristic used in the proof of the main theorem. Let $W$ be a noetherian complete local ring and assume that the characteristic of the residue field is a prime number $p$. Then, $W$ is said to be a Cohen ring [6, Chapitre 0, D"efinition (19.8.4)] if $W$ is flat over $\mathbb{Z}_p$ and $W/pW$ is a field, or equivalently if $W$ is an absolutely unramified discrete valuation ring.

**Theorem 3.1.** Let $p$ be a prime number.

1. ([6 Chapitre 0, Théorème (19.8.2) (i)]) Let $W$ be a Cohen ring such that the residue field is a field of characteristic $p$. Then $W$ is formally smooth over $\mathbb{Z}_p$.

2. ([6 Chapitre 0, Théorème (19.8.6) (ii)]) If $k$ is a field of characteristic $p$, there exists a Cohen ring $W$ such that the residue field is isomorphic to $k$.

A local noetherian ring $A$ is said to be of complete intersection if its completion $\hat{A}$ is isomorphic to the quotient of a regular complete local noetherian ring $B$ by the ideal generated by a regular sequence of $B$ [6 Chapitre IV, D"efinition (19.3.1)]. Let $f: X \to S$ be a flat morphism of finite type of noetherian schemes and $x \in X, s = f(x) \in S$. We say that $X$ is locally of complete intersection relatively to $S$ at $x$ if the local ring $O_{X_s,x}$ of the fiber $X_s = X \times_S s$ is of complete intersection [6, Chapitre IV, D"efinition (19.3.6)]. Let $i: X \to Y$ be a closed immersion of schemes of finite type over a noetherian schemes $S$ and $x \in X$. We say that $i$ is transversally regular relatively to $S$ at $x$ if on a neighborhood $V \subset Y$ of $x$ there exists a regular sequence $(f_i; 1 \leq i \leq n)$ generating the ideal $I_X \subset O_Y$ defining $X$ such that $O_Y/(f_i; 1 \leq i \leq j)$ are flat over $S$ for $1 \leq j \leq n$ [6 Chapitre IV, D"efinition (19.2.2)].

**Proposition 3.2.** 1. ([6 Chapitre IV, Proposition (19.3.2)]) Let $A = B/I$ be a quotient ring of a regular local noetherian ring $B$. Then, $A$ is of complete intersection if and only if $I$ is generated by a regular sequence of $B$.

2. ([6 Chapitre IV, Proposition (19.3.7)]) Let $i: X \to Y$ be a closed immersion of flat schemes of finite type over a noetherian scheme $S$ and $x \in X$. Assume that $Y$ is smooth over $S$. Then, the immersion $i$ is transversally regular relatively to $S$ at $x$ if and only if $X$ is locally of complete intersection relatively to $S$ at $x$.

**Theorem 3.3.** Let $k$ be a field of characteristic $p > 0$.

1. ([2 Section 13, No. 2, Théorème 2 c)]) If $[k : k^p] = n$ is finite, $\dim_k \Omega^1_{k/F_p} = n$.

2. ([2 Section 16, No. 6, Corollaire 3]) Let $k_1$ be a subfield such that $k$ is finitely generated over $k_1$ of transcendental degree $d$ and that $[k_1 : k_1^p]$ is finite. Then $[k : k^p] = p^d \cdot [k_1 : k_1^p]$.
We say that a local ring $A$ is essentially of finite type over a field $k$ if $A$ is isomorphic to the localization at a prime ideal of a ring of finite type over $k$. We state and prove the regularity criterion.

**Theorem 3.4.** Let $A$ be a noetherian local ring with residue field $k = A/m_A$ of characteristic $p$. Assume that $k$ has a finite $p$-basis and set $d = \dim A$, $[k : k^p] = p^r$ and $A_1 = A/pA$. We consider the following conditions:

1. The $A_1$-module $F\Omega^1_{A_1}$ is free of rank $d + r$.
2. $A$ is regular.
3. Assume that the quotient $A/\sqrt{pA}$ by the radical of the principal ideal $pA$ is essentially of finite type over a field $k_1$ with finite $p$-basis and that either of the following conditions is satisfied:
   - (a) $A$ is flat over $\mathbb{Z}_{(p)}$.
   - (b) $A$ is a ring over $\mathbb{F}_p$.

Then the 3 conditions are equivalent.

Let $A$ be the discrete valuation ring in Example 2 after Corollary 2.4. Then $A$ satisfies (2) and (1′) for $d = 1$, $r = 0$ but not (1) unless $n = 1$.

**Proof.** 1. The implication $1 \Rightarrow 1′$ is obvious. We show $1′ \Rightarrow 2$. By Proposition 2.6 we have $\dim_k m_A/m_A^2 = \dim_k F\Omega^1_{A_1} \otimes_A k - \dim_k \Omega^1_{k_1} = (d + r) - r = d = \dim A$. Hence $A$ is regular.

2. It suffices to show (2) $\Rightarrow 1$. First, we show the case (a). Assume that $A$ is flat over $\mathbb{Z}_{(p)}$. Let $W$ be a Cohen ring with residue field $k_1$. Then, since $W_2 = W/p^2W$ is formally smooth over $\mathbb{Z}/p^2\mathbb{Z}$ by Theorem 3.1.2 and the ideal $\sqrt{pA}/p^2A \subset A_2 = A/p^2A$ is nilpotent, the morphism $k_1 \to A/\sqrt{pA}$ is lifted to a morphism $W_2 \to A_2$. By the exact sequence $0 \to A/pA \to A/p^2A \to A/pA \to 0$, we have $\text{Tor}_1^{W_2}(A_2, k_1) = 0$ and the ring $A_2$ is flat over $W_2$.

Since the ideal $\sqrt{pA}/p^2A \subset A_2$ is finitely generated, there exists a morphism $C_2 = W_2[T_1, \ldots, T_N] \to A_2$ over $W_2$ for an integer $N \geq 0$ such that for the localization $B_2$ of $C_2$ at the inverse image of $m_{A_2}$, the induced morphism $B_2 \to A/\sqrt{pA}$ is a surjection and that the image $C_2 \to A_2$ contains a system of generators of $\sqrt{pA}/p^2A \subset A_2$. Then, since $\sqrt{pA}/p^2A$ is nilpotent, the local morphism $B_2 \to A_2$ is a surjection.

Set $B_1 = B_2/pB_2$, $C_1 = C_2/pC_2$ and $n = d + \text{tr. deg}_{k_1} k$. Since $B_1$ is the local ring of $k_1[T_1, \ldots, T_N]$ at a prime ideal with the residue field $k$, we have $\dim B_1 = N - \text{tr. deg}_{k_1} k$. Since $A$ is regular and $p \in A$ is a non-zero divisor, the quotient $A_1 = A/pA$ is of complete intersection. Since $B_1$ is regular, the kernel $I_1$ of the surjection $B_1 \to A_1$ is generated by a regular sequence of length $\dim B_1 - (\dim A - 1) = (N - \text{tr. deg}_{k_1} k) - (d - 1) = N - n + 1$.

Let $X \subset \mathbb{A}^N_{W_2} = \text{Spec} W_2[T_1, \ldots, T_N]$ be a closed subscheme such that $A_2$ is isomorphic to the local ring at a point $x \in X$. Since $A_2$ is flat over $W_2$ and $A_1$ of complete intersection, the closed immersion $X \to \mathbb{A}^N_{W_2}$ is transversally regular relatively to $W_2$ at $x$ by Proposition 3.2.2. Hence the kernel $I_2$ of the surjection $B_2 \to A_2$ is also generated by
a regular sequence of length $N - n + 1$ and the canonical surjection $I_2/I_2 \otimes_{A_2} A_1 \to I_1/I_1$ is an isomorphism of free $A_1$-modules of rank $N - n + 1$.

The canonical morphism $F \Omega^1_A \to F \Omega^1_{A_2}$ is an isomorphism of $A_1$-modules by Corollary \[\text{2.4}.1\]. Hence, we obtain an exact sequence

\[ (3.1) \quad F^*(I_2/I_2 \otimes_{A_2} A_1) \to F \Omega^1_{C_2} \otimes_{C_1} A_1 \to F \Omega^1_A \to 0 \]

of $A_1$-modules by Proposition \[\text{2.3}.2\] and $F^*(I_2/I_2 \otimes_{A_2} A_1) = F^*(I_1/I_1^2)$ is a free $A_1$-module of rank $N - n + 1$.

Set $[k_1 : k_1^p] = p^{n_1}$. We have $\dim_k \Omega^1_{k_1} = r_1$ by Theorem \[\text{3.3}.1\]. The $W_2$-module $F \Omega^1_{W_2}$ is a $k_1$-vector space by Corollary \[\text{2.4}.1\] and is of dimension $r_1 + 1$ by Proposition \[\text{2.6}\]. Hence by Proposition \[\text{2.5}.3\], the $C_2$-module $F \Omega^1_{C_2}$ is a free $C_1$-module of rank $N + r_1 + 1$.

We have $\dim_k \Omega^1_k = \dim_k \Omega^1_{k_1} + \deg_{k_1} k$ by Theorem \[\text{3.3}\]. Since $A$ is regular, by Proposition \[\text{2.6}\], the $k$-vector space $F \Omega^1_{\mathcal{A}} \otimes_A k$ is of dimension $d + r = d + \deg_{k_1} k + r_1 = n + r_1$.

Since $N + r_1 + 1 = (N - n + 1) + (n + r_1)$, the exact sequence \[3.1\] induces an exact sequence $0 \to F^*(I_1/I_1^2) \otimes_{A_1} k \to F \Omega^1_{C_2} \otimes_{C_1} k \to F \Omega^1_A \otimes_{A_1} k \to 0$. Consequently the morphism $F^*(I_1/I_1^2) \to F \Omega^1_{C_2} \otimes_{C_1} A_1$ of free $A_1$-modules of finite rank is a split injection and $F \Omega^1_A$ is a free $A_1$-module of rank $d + r$.

The proof in the case (b) is similar and easier. Since $k$ is formally smooth over $\mathbb{F}_p$, we may assume that $A$ is the localization at a prime ideal of a ring $B$ of finite type over $k_1$ and take a surjection $C = k_1[T_1, \ldots, T_N] \to B$. By Corollary \[\text{2.4}.2\], $F \Omega^1_{C}$ is isomorphic to the free $C$-module $F^*\Omega^1_{C}$ of rank $N + r_1$. Hence it suffices to apply Corollary \[\text{2.5}.2\] to the localization of $C \to A$.

\begin{corollary}
Let $A \to A/I = B$ be a surjection of regular local rings. Assume that the quotient $A/\sqrt{pA}$ by the radical of the principal ideal $pA$ is essentially of finite type over a field $k_1$ with finite $p$-basis. Then for $B_1 = B/pB$, the sequence

\[ (3.2) \quad 0 \to F^*(I/(I^2 + pI)) \xrightarrow{w} F \Omega^1_{A} \otimes_A B_1 \to F \Omega^1_{B} \to 0 \]

of $B_1$-modules is a split exact sequence.
\end{corollary}

\begin{proof}
Since the $A/pA$-module $F \Omega^1_A$ is free of finite rank by Theorem \[\text{3.4}.2\], the assertion follows from Corollary \[\text{2.8}.2\].
\end{proof}

\begin{corollary}
Let $A$ be a regular local ring faithfully flat over $\mathbb{Z}_{(p)}$ and set $A_1 = A/pA$. We consider the following conditions:

(1) The morphism $A_1 \to F \Omega^1_A$ of $A_1$-modules sending 1 to $w(p) \in F \Omega^1_A$ is a split injection.

(2) $A_1$ is regular.

1. We have always (1) $\Rightarrow$ (2).

2. Assume that the quotient $A/\sqrt{pA}$ by the radical of the principal ideal $pA$ is essentially of finite type over a field $k_1$ with finite $p$-basis. Then we have (2) $\Rightarrow$ (1).
\end{corollary}

\begin{proof}
It suffices to apply Corollary \[\text{2.8}.1\] and Corollary \[\text{3.5}\] to $B = A/pA$ respectively.
\end{proof}
4 Relation with cotangent complex

By Proposition 2.5.2, we may sheafify the construction of $F\Omega^1_X$ on a scheme $X$. We call $F\Omega^1_X$ the sheaf of FW-differentials on $X$. In this section, we study the relation of $F\Omega^1_X$ with cotangent complex. Before starting, we prepare basic properties of sheaves of FW-differentials.

Lemma 4.1. Let $X$ be a scheme over $\mathbb{Z}_p$. Let $X_{F_p}$ and $F: X_{F_p} \to X_{F_p}$ denote the closed subscheme $X \times_{\text{Spec} \mathbb{Z}} \text{Spec} F_p \subset X$ and the absolute Frobenius morphism.

1. The $O_X$-module $F\Omega^1_X$ is a quasi-coherent $O_{X_{F_p}}$-module. The canonical isomorphism (2.5) defines an isomorphism

\[ F\Omega^1_X / (O_X \cdot w(p)) \to F^* \Omega^1_{X_{F_p}}. \]

2. Assume that $X$ is noetherian and that the reduced part $X_{F_p,\text{red}}$ is a scheme of finite type over a field $k$ with finite $p$-basis. Then, the $O_X$-module $F\Omega^1_X$ is a coherent $O_{X_{F_p}}$-module. Further if $X$ is regular of dimension $n$, then $F\Omega^1_X$ is a locally free $O_{X_{F_p}}$-module of rank $n$.

Proof. 1. If $X = \text{Spec} A$, the $O_X$-module $F\Omega^1_X$ is defined by the $A$-module $F\Omega^1_A$. Hence the $O_X$-module $F\Omega^1_X$ is quasi-coherent. The $O_X$-module $F\Omega^1_X$ is an $O_{X_{F_p}}$-module by Corollary 2.4.1. The isomorphism (4.1) is clear from (2.5).

2. This follows from Corollary 2.4.4 and Theorem 3.4.2.

A morphism $f: X \to Y$ of schemes defines a canonical morphism

\[ f^* F\Omega^1_Y \to F\Omega^1_X \]

of $O_X$-modules.

We recall some of basic properties on cotangent complexes from [7, Chapitres II, III]. For a morphism of schemes $X \to S$, the cotangent complex $L_{X/S}$ is defined [7, Chapitre II, 1.2.3] as a chain complex of flat $O_X$-modules, whose cohomology sheaves are quasi-coherent. There is a canonical isomorphism $H_0(L_{X/S}) \to \Omega^1_{X/S}$ [7, Chapitre II, Proposition 1.2.4.2]. This induces a canonical morphism $L_{X/S} \to \Omega^1_{X/S}[0]$.

For a commutative diagram

\[ \begin{array}{ccc} X' & \longrightarrow & S' \\ \downarrow f & & \downarrow \\ X & \longrightarrow & S, \end{array} \]

a canonical morphism $L f^* L_{X/S} \to L_{X'/S'}$ is defined [7, Chapitre II, (1.2.3.2)']. For a morphism $f: X \to Y$ of schemes over a scheme $S$, a distinguished triangle

\[ L f^* L_{Y/S} \to L_{X/S} \to L_{X/Y} \to \]

is defined [7, Chapitre II, Proposition 2.1.2].
The cohomology sheaf $\mathcal{H}_1(L_{X/S})$ is studied as the module of imperfection in [6, Chapitre 0, Section 20.6]. If $X \to S$ is a closed immersion defined by the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_S$ and if $N_{X/S} = \mathcal{I}_X/\mathcal{I}_X^2$ denotes the conormal sheaf, there exists a canonical isomorphism

\[(4.5) \quad \mathcal{H}_1(L_{X/S}) \to N_{X/S}\]

The canonical morphism $\Omega^1$ for a smooth scheme $E$ follows from Lemma 4.2.1. This induces a canonical morphism $L_{X/S}$ in the function field of a scheme $S$ follows from Lemma 4.2.2.

Lemma 4.3. Let $E$ be a scheme smooth over a field $k$ of characteristic $p > 0$.

1. The canonical morphism $L_{E/F_p} \to \Omega^1_{E/F_p}[0]$ is a quasi-isomorphism and the $\mathcal{O}_E$-module $\Omega^1_{E/F_p}$ is flat.

2. Let $E'$ be a scheme smooth over a field $k'$ of characteristic $p > 0$ and $E' \to E$ be a morphism of schemes. Then, we have an exact sequence

\[(4.6) \quad 0 \to \mathcal{H}_1(L_{E'/E}) \to \Omega^1_{E/F_p} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'} \to \Omega^1_{E'/F_p} \to \mathcal{H}_0(L_{E'/E}) \to 0\]

and $\mathcal{H}_q(L_{E'/E}) = 0$ for $q > 1$.

3. ([8, Theorem (7.2)]) Let $F : E \to E$ denote the absolute Frobenius morphism. Then, the sequence $0 \to \mathcal{O}_E \to F_*\mathcal{O}_E \xrightarrow{d} F_*\Omega^1_{E/F_p}$ is exact.

Proof. 1. By the distinguished triangle $L_{E/F_p} \otimes_k \mathcal{O}_E \to L_{E/F_p} \to L_{E/k}$ and Lemma 4.2.2, the assertion is reduced to the case where $E = \text{Spec } k$. Since the formation of cotangent complexes commutes with limits, we may assume $k$ is of finite type over $F_p$. Hence, we may assume that $k$ is the function field of a smooth scheme $E$ over $F_p$. Thus the assertion follows from Lemma 4.2.2.

2. By the distinguished triangle $L_{E/F_p} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'} \to L_{E'/F_p} \to L_{E'/E} \to$, the assertion follows from 1 for $E$ and $E'$.

3. We may assume that $k$ is finitely generated over $F_p$. Then $k$ is isomorphic to the function field of a scheme $S$ smooth over $F_p$. We may assume that $E$ is the generic fiber of a smooth scheme $E_S$ over $S$. Thus, it is reduced to the case where $k = F_p$ is perfect. Then, the canonical morphism $\Omega^1_{E/F_p} \to \Omega^1_{E/k}$ is an isomorphism and the assertion follows from the Cartier isomorphism $[8, \text{Theorem (7.2)}]$. \qed
Lemma 4.4. Let $X$ be a scheme. Let $p$ be a prime number and $E$ be a scheme over $\mathbb{F}_p$.

Let $f : E \to X$ be a morphism of schemes.

1. We consider the following conditions:

   (1) The morphism $f : E \to X$ factors through the absolute Frobenius morphism $F : E \to E$.

   (2) The canonical surjection

   $\Omega_{E/F_p}^1 = \Omega_{E/\mathbb{Z}}^1 \to \Omega_{E/X}^1$

   is an isomorphism.

   We have (1) $\Rightarrow$ (2). If $E$ is a smooth scheme over a field $k$, we have (2) $\Rightarrow$ (1).

2. Assume that $X$ is a regular noetherian scheme, that $E$ is smooth over a field and that $f$ is of finite type and satisfies the equivalent conditions in 1. Then the $\mathcal{O}_E$-module $\mathcal{H}_1(L_{E/X})$ is locally free of finite rank.

Proof. 1. (1) $\Rightarrow$ (2): Suppose $f : E \to X$ factors through $F : E \to E'$. Then since the surjection $\Omega_{E/F_p}^1 \to \Omega_{E/E'}^1$ is an isomorphism, the surjections $\Omega_{E/F_p}^1 \to \Omega_{E/E'}^1 \to \Omega_{E/E'}^1$ are isomorphisms.

   (2) $\Rightarrow$ (1): The condition (2) means that the composition of $f^{-1}\mathcal{O}_X \to \mathcal{O}_E$ and $d : \mathcal{O}_E \to \Omega_{E/F_p}^1$ is the 0-morphism. Since $F : E \to E$ is a homeomorphism on the underlying topological spaces, the continuous mapping $f : E \to X$ is the composition of $F : E \to E$ with a unique continuous mapping $g : E \to X$. Thus, the condition (2) is equivalent to the condition that the composition $g^{-1}\mathcal{O}_X \to F_*\mathcal{O}_E \to F_*\Omega_{E/F_p}^1$ is the 0-morphism.

   By Lemma 4.3.3, the sequence $0 \to \mathcal{O}_E \to F_*\mathcal{O}_E \xrightarrow{d} F_*\Omega_{E/F_p}^1$ is exact. Thus, the condition (2) is further equivalent to the condition that the morphism $g^{-1}\mathcal{O}_X \to F_*\mathcal{O}_E$ factors through $g^{-1}\mathcal{O}_X \to \mathcal{O}_E$. Since $F : E \to E$ is affine, this defines a morphism $g : E \to X$ of schemes and the condition (2) is equivalent to (1).

2. Since the assertion is local on $E$, we may assume that $E$ and $X$ are affine and there exists a closed immersion $E \to P = \mathbb{A}_X^n$ for some $n$. Since $E$ and $X$ are affine and there exists a closed immersion $E \to P$ is a regular immersion. Then, the distinguished triangle

   $L_{P/X} \otimes_{\mathcal{O}_P} \mathcal{O}_E \to L_{E/X} \to L_{E/P} \to \mathcal{H}_1(L_{E/X}) \to \mathcal{H}_1(L_{P/X} \otimes_{\mathcal{O}_P} \mathcal{O}_E)$

   defines an exact sequence $0 \to \mathcal{H}_1(L_{E/X}) \to \mathcal{H}_1(L_{P/X}) \otimes_{\mathcal{O}_{\mathbb{A}_X^n}} \mathcal{O}_E$.

   By Lemma 4.4.2 for $P \to X$ and Lemma 4.4.3 for $E \to P$. The $\mathcal{O}_E$-modules in the exact sequence other than $\mathcal{H}_1(L_{E/X})$ are locally free of finite rank by the isomorphism $\mathcal{L}$. Hence $\mathcal{H}_1(L_{E/X})$ is also locally free of finite rank. $\square$

We give a construction yielding an FW-derivation.

Lemma 4.5. Let $X$ be a scheme and set $\mathbb{A}_X^1 = X \times_{\text{Spec} \mathbb{Z}} \text{Spec} \mathbb{Z}[T]$.

1. Let $E$ be a scheme over $\mathbb{F}_p$ and let $E \to \mathbb{A}_X^1$ be a morphism of schemes. Then, the distinguished triangle

   $L_{\mathbb{A}_X^1/X} \otimes_{\mathcal{O}_{\mathbb{A}_X^1}} \mathcal{O}_E \to L_{E/X} \to L_{E/\mathbb{A}_X^1} \to \mathcal{H}_1(L_{E/X}) \to \mathcal{H}_1(L_{E/\mathbb{A}_X^1}) \to \Omega_{\mathbb{A}_X^1/X}^1 \otimes_{\mathcal{O}_{\mathbb{A}_X^1}} \mathcal{O}_E$

   defines an exact sequence.

2. Let $u \in \Gamma(X, \mathcal{O}_X)$. Define a closed subscheme $W \subset \mathbb{A}_X^1$ by the ideal $(u - T^p, p)$ and identify $\mathcal{H}_1(L_{W/\mathbb{A}_X^1})$ with the conormal sheaf $N_{W/\mathbb{A}_X^1}$ by the canonical isomorphism $\mathcal{L}$. 
Then, the section \( u - T^p \) of the conormal sheaf \( N_{W/A_X^1} \) lies in the image of the injection
\[
\Gamma(W, \mathcal{H}_1(L_{W/X})) \rightarrow \Gamma(W, \mathcal{H}_1(L_{W/A_X^1})) = \Gamma(W, N_{W/A_X^1})
\]
defined by (4.8) for \( E = W \). In other words, there exists a unique section
\[
\omega \in \Gamma(W, \mathcal{H}_1(L_{W/X}))
\]
such that the image in \( \Gamma(W, N_{W/A_X^1}) \) equals \( u - T^p \).

**Proof.**
1. Since the \( \mathcal{O}_{A_X^1} \)-module \( \Omega^1_{A_X^1/X} \) is flat, the assertion follows from the canonical isomorphism \( L_{A_X^1/X} \rightarrow \Omega^1_{A_X^1/X}[0] \) in Lemma 4.22.

2. By 1 applied to \( E = W \), to show that \( u - T^p \) lies in the image of (4.9), it suffices to show that this vanishes in \( \Gamma(W, \Omega^1_{A_X^1/X} \otimes \mathcal{O}_{A_X^1} \mathcal{O}_W) \). By Lemma 4.21, the last arrow in (4.8) for \( E = W \) is \( -d: N_{W/A_X^1} \rightarrow \Omega^1_{A_X^1/X} \otimes \mathcal{O}_{A_X^1} \mathcal{O}_W \). Since \( d(u - T^p) = -pT^{p-1}dT = 0 \) on \( W \), the assertion follows. \( \square \)

**Definition 4.6** (cf. [10, Definition 1.1.6] or [11, Definition 1.1.6 in v1]). Let \( X \) be a scheme and \( u \in \Gamma(X, \mathcal{O}_X) \) be a section. Let \( E \) be a scheme over \( \mathbb{F}_p \) and let \( f : E \rightarrow X \) be a morphism of schemes. Let \( v \in \Gamma(E, \mathcal{O}_E) \) be a section such that \( u \mid_E = f^*u \in \Gamma(E, \mathcal{O}_E) \) is the \( p \)-th power of \( v \). Let \( W \subset A_X^1 \) be the closed subscheme as in Lemma 4.5 and define a morphism \( E \rightarrow W \) over \( X \) by sending \( T \) to \( v \in \Gamma(E, \mathcal{O}_E) \). We define a section
\[
w(u, v) \in \Gamma(E, \mathcal{H}_1(L_{E/X}))
\]
to be the image of \( \omega \) in (4.10) by the morphism \( \Gamma(W, \mathcal{H}_1(L_{W/X})) \rightarrow \Gamma(E, \mathcal{H}_1(L_{E/X})) \) defined by \( E \rightarrow W \).

**Proposition 4.7** (cf. [10, Lemma 1.1.4] or [11, Proposition 1.1.5 in v1]). Let \( X \) be a scheme and \( u \in \Gamma(X, \mathcal{O}_X) \). Let \( f : E \rightarrow X \) be a morphism of schemes and assume that \( E \) is a scheme over \( \mathbb{F}_p \). Let \( v \in \Gamma(E, \mathcal{O}_E) \) be a section satisfying \( u \mid_E = f^*u \in \Gamma(E, \mathcal{O}_E) \) is the \( p \)-th power of \( v \).

1. Assume \( u \mid_E = 0 \) and let \( E \rightarrow Z \subset X \) be the morphism to the closed subscheme defined by \( u \). Then \( w(u, 0) \in \Gamma(E, \mathcal{H}_1(L_{E/X})) \) is the image of \( u \in \Gamma(Z, N_{Z/X}) \) by the morphism \( \Gamma(Z, N_{Z/X}) \rightarrow \Gamma(E, \mathcal{H}_1(L_{E/X})) \) defined by \( L_{Z/X} \otimes \mathcal{O}_Z \mathcal{O}_E \rightarrow L_{E/X} \).

2. Let \( u' \in \Gamma(X, \mathcal{O}_X) \) and \( v' \in \Gamma(E, \mathcal{O}_E) \) be another pair of sections satisfying \( u' \mid_E = v'^p \). Then, we have
\[
w(u + u', v + v') = w(u, v) + w(u', v') - P(v, v') \cdot w(p, 0),
\]
\[
w(uu', vv') = uu' \cdot w(u, v) + u \cdot w(u', v').
\]

3. Let \( X \rightarrow S \) be a morphism of schemes. Then, the minus of the boundary mapping \( -\partial : \mathcal{H}_1(L_{E/X}) \rightarrow \Omega^1_{X/S} \otimes \mathcal{O}_X \mathcal{O}_E \) of the distinguished triangle \( L_{X/S} \otimes \mathcal{O}_X \mathcal{O}_E \rightarrow L_{E/S} \rightarrow L_{E/X} \) sends \( w(u, v) \in \Gamma(E, \mathcal{H}_1(L_{E/X})) \) to \( du \in \Gamma(E, \Omega^1_{X/S} \otimes \mathcal{O}_X \mathcal{O}_E) \).
Proof. 1. Since the morphism \( E \to W \subset \mathbb{A}^1_X \) factors through the 0-section \( Z \subset \mathbb{A}^1_X \), the assertion follows from \( T^p = 0 \) in \( \Gamma(Z, N_{Z/\mathbb{A}^1_X}) \).

2. By 1, \( w(p,0) \in \Gamma(E, \mathcal{H}_1(L_{E/X})) \) is the image of \( p \in N_{\mathbb{F}_p/Z} \). Let \( W' \) be the closed subscheme of \( \mathbb{A}^2_X \) defined by the ideal \( (T^p - u, T'^p - u') \) and define \( E \to W' \) by \( T \mapsto v, T' \mapsto v' \). Then, (4.12) follows from the binomial expansion

\[
(u + u') - (T + T')^p = (u - T^p) + (u' - T'^p) - P(T, T') \cdot p
\]

Similarly, (4.13) follows from

\[
(uu') - (TT')^p = u'(u - T^p) + u(u' - T'^p) - (u - T^p)(u' - T'^p).
\]

3. The morphisms \( E \to W \to \mathbb{A}^1_X \to X \to S \) define a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_1(L_{E/X}) & \longrightarrow & \mathcal{H}_1(L_{E/\mathbb{A}^1_X}) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
\]

\[
\begin{array}{ccc}
\Omega^1_{X/S} \otimes_{O_X} O_E & \longrightarrow & \Omega^1_{\mathbb{A}^1_X/S} \otimes_{O_{\mathbb{A}^1_X}} O_E \\
\downarrow \phi & & \downarrow \phi \\
\Omega^1_{\mathbb{A}^1_X/S} \otimes_{O_{\mathbb{A}^1_X}} O_E & \longrightarrow & \Omega^1_{\mathbb{A}^1_X/S} \otimes_{O_{\mathbb{A}^1_X}} O_E
\end{array}
\]

by Lemma 4.2.1. Since \( d(u - T^p) = du \in \Gamma(E, \Omega^1_{\mathbb{A}^1_X/S} \otimes_{O_{\mathbb{A}^1_X}} O_E) \) and since the lower left horizontal arrow is an injection, the assertion follows.

\[\square\]

Corollary 4.8. Let \( X \) be a scheme and let \( E \) be a scheme over \( \mathbb{F}_p \). Let \( g: E \to X \) be a morphism of schemes and let \( L_{E/X} \) denote the cotangent complex for the composition \( f = g \circ F: E \to X \) with the absolute Frobenius \( F: E \to E \). Then, the mapping

\[
(4.14) \quad w: \Gamma(X, O_X) \to \Gamma(E, \mathcal{H}_1(L_{E/X}))
\]

sending \( u \in \Gamma(X, O_X) \) to \( w(u, v) \) for \( v = g^*u \in \Gamma(E, O_E) \) is an FW-derivation.

\[\square\]

Proof. The assertion follows from Proposition 4.1.2.

The construction of the FW-derivation \( w \) (4.14) is functorial in \( X \) and \( E \).

Definition 4.9. Let \( X \) be a scheme and let \( E \) be a scheme over \( \mathbb{F}_p \). Let \( g: E \to X \) be a morphism of schemes and let \( L_{E/X} \) denote the cotangent complex for the composition \( f = g \circ F: E \to X \) with the absolute Frobenius \( F: E \to E \). By sheafifying the morphism (4.14), we define an FW-derivation \( w: g^{-1}O_X \to \mathcal{H}_1(L_{E/X}) \) and the morphism

\[
(4.15) \quad g^*F\Omega^1_X \to \mathcal{H}_1(L_{E/X})
\]

defined by the universality of \( F\Omega^1_X \).

We study condition for the morphism (4.15) to be an isomorphism.

Lemma 4.10. Let \( g: E \to Z \) be a morphism of schemes over \( \mathbb{F}_p \) and and let \( L_{E/Z} \) denote the cotangent complex for the composition \( f = g \circ F: E \to Z \) with the absolute Frobenius \( F: E \to E \).

1. The morphism \( g^*F\Omega^1_Z \to \mathcal{H}_1(L_{E/Z}) \) (4.15) is a split injection.

2. The split injection (4.15) is an isomorphism if \( \mathcal{H}_1(L_{E/\mathbb{F}_p}) = 0 \). The condition \( \mathcal{H}_1(L_{E/\mathbb{F}_p}) = 0 \) is satisfied if \( E \) is smooth over a field.
Corollary 4.12. Let $g : Z \to X = \text{Spec} A$.

Proof. It suffices to apply Proposition 4.11 to $g : Z = \text{Spec} k \to X = \text{Spec} A$. □
References

[1] B. Bhatt, P. Scholze, *Prisms and prismatic cohomology*, arXiv:1905.08229

[2] N. Bourbaki, *Algèbre*, Chapitre V, Springer, Réimpression, 2006.

[3] A. Buium, *Arithmetic analogues of derivations*, J. Algebra 198 (1997), no. 1, 290-299.

[4] T. Dupuy, E. Katz, J. Rabinoff, D. Zureick-Brown, *Total p-differential on schemes over $\mathbb{Z}/p^2$*, Journal of Algebra 524, 110-123 (2019).

[5] O. Gabber, L. Ramero, *Foundations for almost ring theory – Release 7.5*, https://arxiv.org/abs/math/0409584.

[6] A. Grothendieck, *Éléments de géométrie algébrique IV*, Étude locale des schémas et des morphismes de schémas, Publ. Math. IHES 20, 24, 28, 32 (1964-67).

[7] L. Illusie, *Complexe cotangent et déformations I*, Springer Lecture Notes in Math., 239, Springer-Verlag, Berlin, Heidelberg, New York 1971.

[8] N. Katz, *Nilpotent connections and the monodromy theorem*, Publ. Math. IHÉS, vol. 39,1970, p. 175–232.

[9] M. Raynaud rédigé par Y. Laszlo, *Anneaux excellents*, Travaux de Gabber sur l’uniformisation locale et la cohomologie étale des schémas quasi-excellents, Astérisque 363-364, (2014).

[10] T. Saito, *Graded quotients of ramification groups of local fields with imperfect residue fields*, arXiv:2004.03768

[11] —–, *Cotangent bundle and micro-supports in mixed characteristic case*, arXiv: 2006.00448