Strong edge coloring of subcubic bipartite graphs

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December 9, 2013

Abstract

A strong edge coloring of a graph $G$ is a proper edge coloring in which each color class is an induced matching of $G$. In 1993, Brualdi and Quinn Massey [3] proposed a conjecture that every bipartite graph without 4-cycles and with the maximum degrees of the two partite sets $2$ and $\Delta$ admits a strong edge coloring with at most $\Delta + 2$ colors. We prove that this conjecture holds for such graphs with $\Delta = 3$. We also confirm the conjecture proposed by Faudree et al. [5] for subcubic bipartite graphs.

NOTE: After publishing this paper on ArXiv, we have been notified that an equivalent result has already been proven by Maydanskiy [9] in the scope of incidence colorings. Moreover, the latter result has been verified in a little stronger form in [14] by Wu and Lin.

Keywords: Strong edge coloring, strong chromatic index, subcubic bipartite graph

1 Introduction

A strong edge coloring of a graph $G$ is a proper edge coloring in which each color class is an induced matching of $G$; i.e., any two edges at distance at most two are assigned distinct colors. The minimum number of colors for which a strong edge coloring of $G$ exists is the strong chromatic index of $G$, denoted $\chi'_s(G)$.

In 1985, Erdős and Nešetřil proposed the following conjecture during a seminar in Prague.

Conjecture 1 (Erdős, Nešetřil, 1985). Let $G$ be a graph with maximum degree $\Delta$. Then

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4} \Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{4} (5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

Molloy and Reed [10] established currently the best known upper bound for the strong chromatic index of graphs with sufficiently large maximum degree.

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Theorem 1 (Molloy, Reed, 1997). For every graph $G$ with sufficiently large maximum degree $\Delta$ it holds that
\[ \chi'_s(G) \leq 1.998 \Delta^2. \]

Conjecture 1 was proved for subcubic graphs by Andersen [11] and independently by Horák, He and Trotter [8]. Some work has been done also for graphs with maximum degree four [4, 7], but the conjecture is still open even for them.

In 1990, Faudree et al. [5] proposed a conjecture for bipartite graphs. The complete bipartite graph $K_{\Delta,\Delta}$ shows that the bound, if true, is the best possible.

Conjecture 2 (Faudree et al., 1990). Let $G$ be a bipartite graph. Then $\chi'_s(G) \leq \Delta^2$.

The authors in [5] also considered the graphs with all cycle lengths divisible by four. Such graphs are bipartite and none of their cycles has a chord. Conjecture 2 holds for such graphs, furthermore, the authors conjectured that their strong chromatic index is bounded by a function linear in $\Delta$.

Regarding the small values of maximum degree $\Delta$, the conjectured bound is trivially true for isolated edges, paths and even cycles. In 1993, Steger and Yu [13] showed that Conjecture 2 holds also for subcubic graphs. Note that this conjecture is still open for $\Delta \geq 4$.

Conjecture 2 was generalized by Brualdi and Quinn Massey [3] in 1993 as follows.

Conjecture 3 (Brualdi, Quinn Massey, 1993). Let $H$ be a bipartite graph with bipartition $X$ and $Y$ such that $\Delta(X) = \Delta_1$ and $\Delta(Y) = \Delta_2$. Then $\chi'_s(H) \leq \Delta_1 \Delta_2$.

They proved that Conjecture 3 is true for graphs with all cycle lengths divisible by four. Note that Steger and Yu solved this problem in case when $\Delta_1 = \Delta_2 = 3$. Brualdi and Quinn Massey [3] considered graphs with $\Delta_1 = 2$ and established Conjecture 3 for graphs without any 4-cycles.

Theorem 2 (Brualdi, Quinn Massey, 1993). Let $H$ be a bipartite graph with bipartition $X$ and $Y$ without any 4-cycle such that $\Delta(X) = 2$ and $\Delta(Y) = \Delta$. Then $\chi'_s(H) \leq 2\Delta$.

In fact, they conjectured that $2\Delta$ in above theorem can be replaced by $\Delta + 2$.

Conjecture 4 (Brualdi, Quinn Massey, 1993). Let $H$ be a bipartite graph with bipartition $X$ and $Y$ without any 4-cycle such that $\Delta(X) = 2$ and $\Delta(Y) = \Delta$. Then $\chi'_s(H) \leq \Delta + 2$.

In 2008 Nakprasit [14] settled Conjecture 3 of Brualdi and Quinn Massey for $\Delta_1 = 2$.

Theorem 3 (Nakprasit, 2008). Let $H$ be a bipartite graph with bipartition $X$ and $Y$ such that $\Delta(X) = 2$ and $\Delta(Y) = \Delta$. Then $\chi'_s(H) \leq 2\Delta$.

This theorem is, together with the result of Steger and Yu, so far the only known partial confirmation of Conjecture 3. Here, let us mention that Nakprasit’s result follows also from the results of Petersen [12] and Hanson, Loten, and Toft [6].

Theorem 4 (Petersen, 1891). Every regular multigraph of positive even degree has a 2-factor.

Theorem 5 (Hanson, Loten, Toft, 1998). For $r, k \in \mathbb{N}$ with $r \geq (3k - 1)/2$, every $(2r + 1)$-regular graph with at most $(2r + 1 - k)/k$ cut-edges has a $2k$-factor.
Short proof of Theorem 3. We proceed in two steps. Firstly, applying an induction on the maximum degree of a graph, and secondly using a contradiction by showing that the minimal counterexample admits the desired coloring.

Let $G$ be a minimal biregular counterexample to the theorem in terms of the number of vertices. Since every bipartite graph is a subgraph of a biregular graph, we consider only biregular graphs. Moreover, bridges in bipartite graphs with maximum degree in one of the partition sets at most 2 are reducible for a strong edge coloring with at most $2\Delta$ colors. Hence, the graph $\hat{G}$, $G$ with all the 2-vertices suppressed, has a 2-factor $\hat{F}$. Let $F$ be the set of edges corresponding to the edges of $\hat{F}$. Then, by induction, $G - F$ admits a strong edge coloring $\varphi_1$ with at most $2\Delta - 4$ colors, furthermore, every cycle in $F$ admits a strong edge coloring with at most 4 colors. Notice that the edges of the two cycles in $F$ are at distance at least 3 in $G$, hence there is a strong edge coloring $\varphi_2$ of $F$ with at most 4 colors. Thus, $\varphi_1$ and $\varphi_2$ induce a strong edge coloring of $G$ with at most $2\Delta$ colors, a contradiction.

There are several further open problems regarding subcubic bipartite graphs. Faudree et al. [5] considered also subcubic bipartite graphs and suggested the problems listed below:

Conjecture 5 (Faudree et al., 1991). Let $G$ be a subcubic bipartite graph. Then

(a) $\chi'_s(G) \leq 6$, if for each edge $uv \in E(G)$, $d(u) + d(v) \leq 5$;

(b) $\chi'_s(G) \leq 7$, if $G$ is $C_4$-free;

(c) $\chi'_s(G) \leq 5$, if the girth of $G$ is large enough.

In this paper we solve Conjecture 4 for the case of subcubic graphs and the case (a) of Conjecture 5.

Throughout the paper, we mostly use the terminology used in [2]. We refer to vertices of degree $k$ as $k$-vertices, and to cycles of length $k$ as $k$-cycles. We say that a pair of vertices of a graph $G$ is 2-adjacent if they have a common neighbor in $G$. A 2-neighbor of a vertex $v$ is any vertex at distance at most 2 from $v$. Similarly, a 2-neighbor of an edge $e$ is any edge at distance at most 2 from $e$. By $B_{2,\Delta}$ we denote the set of all bipartite graphs with maximum degrees of partite sets bounded by 2 and $\Delta$, respectively.

2 Subcubic bipartite graphs without 4-cycles

In this section we consider subcubic bipartite graphs without 4-cycles. In particular, we prove that Conjecture 4 holds if $\Delta = 3$. In the proof of the main theorem of this section we will make use of the following results.

Lemma 1. Let $G$ be a graph without isolated vertices. Then, there exists a set of disjoint stars $S$ whose edges cover all the vertices of $G$, i.e., every vertex in $G$ is a vertex of exactly one star $S$ in $S$.

A cover described in Lemma 1 can be obtained e.g. by recursively removing the middle edge of every path of length 3 in a spanning tree of a graph. Note that each minimal set of edges which covers the vertices of $G$ is of desired type. The following lemma is a corollary of Petersen’s theorem that every cubic graph with at most one bridge has a 1-factor [12].
Lemma 2 (Petersen, 1891). Let $G$ be a simple subcubic graph without bridges and with at most one vertex of degree at most two. Then, there is a matching that covers all 3-vertices.

Notice that since there is always an even number of odd vertices in a graph, an eventual 2-vertex from the above lemma is never covered. Now, we prove a theorem that confirms Conjecture 4 for subcubic graphs.

Theorem 6. Let $G \in B_{2,3}$ be a graph without 4-cycles. Then

$$\chi'_s(G) \leq 5.$$  

The bound in Theorem 6 is tight. Consider for an example the Petersen’s graph with all edges subdivided in Fig. 1. It does not admit a strong edge coloring with at most 4 colors, since it contains a 10-cycle with a pendent edge on every second vertex, for which it is not hard to see that it is the only cycle with pendent edges in $B_{2,3}$ that needs at least 5 colors for a strong edge coloring.

![Figure 1](image.png)

Figure 1: Petersen’s graph with all edges subdivided is bipartite subcubic graph with girth 10 and strong chromatic index equal to 5.

Proof. Suppose, to the contrary, that the theorem is false. Let $G$ be a minimal counterexample to the theorem with the bipartition $(X,Y)$ and $\Delta(X) = 2$. By minimal we mean the graph with the minimum number of vertices in $Y$ and, among such, minimum number of 2-vertices.

Now we consider some structural properties of $G$. Obviously, $G$ is connected. Suppose first that there is a bridge $uv$ in $G$. Let $G_u$ and $G_v$ be the two components of $G - uv$ containing the vertices $u$ and $v$, respectively. By the minimality of $G$, there exist strong edge colorings $\varphi_1$ and $\varphi_2$ of $G_u$ and $G_v$ with at most 5 colors, respectively. It is easy to verify that there exists a permutation of colors in $\varphi_1$ such that $\varphi_1$ and $\varphi_2$ induce a strong edge coloring of $G - uv$ and there is an available color for $uv$. Therefore, $G$ is bridgeless and the minimum degree of $G$ is 2, as the edge incident to a 1-vertex is a bridge.

Denote by $\hat{G}$ the graph obtained from $G$ by removing every 2-vertex from $X$ and connecting its two neighbors by an edge. Since $G$ has no 4-cycles, $\hat{G}$ is a simple subcubic graph. Moreover, since $G$ is bridgeless, so is $\hat{G}$.

Suppose that $\hat{G}$ contains two 2-vertices, $x$ and $y$. Then, $x$ and $y$ are adjacent in $\hat{G}$ for otherwise we may connect them by a subdivided edge in $G$ and so decrease the number of 2-vertices in $G$ (what would contradict the minimality assumption). But in this case, we color the edges of the graph $G \setminus \{x,y\}$ strongly with at most 5 colors and then extend the coloring.
to $G$ (such an extension is trivial and we leave it to the reader). Hence, we may assume that $\hat{G}$ contains at most one 2-vertex.

By Lemma 2 there exists a matching $\hat{M}$ covering all the 3-vertices of $\hat{G}$. By removing all the edges of the matching $\hat{M}$ we obtain a 2-regular graph, i.e., $\hat{C} = \hat{G} \setminus \hat{M}$ is a union of cycles. Let $G^*$ be the graph obtained from $\hat{G}$ by contracting all the cycles in $\hat{C}$, i.e., we identify all the vertices of each cycle and remove the edges of the cycles. Hence, $G^*$ is a connected multigraph with possible loops. By Lemma 1 there exists a set of disjoint stars $S^*$ in $G^*$ whose edges cover all the vertices of $G^*$ (in case when $|V(G^*)| = 1$, $S^* = \emptyset$). Let $\hat{S} \subseteq \hat{M}$ be the set of edges in $\hat{G}$ corresponding to the edges of $S^*$. We call a cycle in $\hat{G}$ associated to a central vertex of a star in $S^*$ a central cycle, and similarly, we refer to a cycle in $\hat{G}$ corresponding to a leaf of a star in $G^*$ as a leaf cycle.

Now, consider $G$ again. Every edge of $\hat{M}$, $\hat{C}$, and $\hat{S}$ is subdivided once in $G$; denote by $M$, $C$, and $S$ the corresponding sets of edges in $G$. In the sequel we describe how a strong edge coloring of $G$ with at most 5 colors can be constructed. Firstly, we color almost all edges of the cycles in $C$ and all edges in $S$ with the colors 1, 2, and 3 and show that the remaining noncolored edges in $G$ can be colored by at most two additional colors. In particular, after coloring the edges with the colors 1, 2, and 3, we consider the graph $R(G)$ whose vertices are yet noncolored edges of $G$, and two vertices are connected if the corresponding edges are at distance at most 2 in $G$. We refer to $R(G)$ as a conflict graph.

Obviously, a proper coloring of the vertices of $R(G)$ is a strong edge coloring of the corresponding edges in $G$. Our aim is to show that the graph $R(G)$ is bipartite and hence 2-colorable.

Constructing the coloring, we consider two cases regarding the number of the cycles in $C$:

(a) The set $C$ contains exactly one cycle $C_0$. Let $v_1, v_2, \ldots, v_k$ be the consecutive vertices of $C_0$ from the partition $Y$, i.e., the 3-vertices of $G$ with an eventual 2-vertex. Since $G$ is bipartite, the length of $C_0$ is $2k$. Choose the natural orientation of $C_0$ induced by the labeling of the vertices. We label the incoming and outgoing edge incident to a vertex $v_t$, $1 \leq t \leq k$, as $e^+_t$ and $e^-_t$, respectively.

Since $G$ is a minimal counterexample with respect to the number of 2-vertices, $C_0$ has at least one subdivided chord in $M$, represented by the edges $f_i$ and $f_j$, where $f_i$ is incident to $v_i$, for some $i, 1 \leq i \leq k - 2$, and $f_j$ is incident to $v_j$, for some $j, i + 2 \leq j \leq k$. By a subdivided chord we mean the two edges in $G$ corresponding to a chord of the cycle in $\hat{G}$. If possible, we always choose a subdivided chord which does not divide $C_0$ into two cycles, where one is of length 6.

Next, color the edges of $C_0$ repetitively by the colors 1, 2, 3 starting with the edge $e^+_{i+1}$ and continue with the edge $e^-_{i+1}$. If $2k$ is divisible by 3, all the edges of $C_0$ are strongly colored. Notice that the conflict graph is a union of $\lfloor k/2 \rfloor$ copies of $K_2$, and hence bipartite.

So, suppose that $2k$ is not divisible by 3. Uncolor the edges $e^+_i, e^-_i, e^+_j$ and $e^-_j$. The edge $f_i$ has at most two colored 2-neighbors, hence we can color it with some color from $\{1, 2, 3\}$. Now we color $f_j$, which has three colored 2-neighbors, $f_i, e^-_{j-1}$, and $e^+_{j+1}$. However, the edges $e^-_{j-1}$ and $e^+_{j+1}$ are assigned the same color, therefore we can color $f_j$ with some color from $\{1, 2, 3\}$ also.

Now consider the conflict graph of $G$. We show that it contains no cycle of odd length and hence its vertices are properly colorable with two colors. In case when $C_0$ is an
8-cycle with two subdivided chords, then \( R(G) \) is comprised of two 4-cycles with a common edge, and hence it is bipartite.

In case when \( C_0 \) is of length at least 10 and every its subdivided chord divides \( C_0 \) into a 6-cycle and a cycle of higher length, then \( R(G) \) is comprised of a 4-cycle and a path with a common vertex, and a disjoint union of \( \left\lfloor \frac{k-4}{2} \right\rfloor \) copies of \( K_2 \). Otherwise, if no subdivided chord divides \( C_0 \) into a 6-cycle, \( R(G) \) is comprised an 8-cycle or a path on 10 vertices, and some copies of \( K_2 \), hence, it is again bipartite.

![Figure 2: An example of a strong edge coloring in case when \( G \) is comprised of only one cycle with some diagonals.](image)

(b) The set \( C \) contains at least two cycles. Consider an arbitrary star in \( S^* \) with a central vertex \( c \) of degree \( p \). Let \( C_c \) be the central cycle in \( G \) corresponding to \( c \). As in the previous case, let the length of \( C_c \) be \( 2k \) and let \( v_1, v_2, \ldots, v_k \) be its vertices from the partition \( Y \). There are \( p \) leaf cycles around \( C_c \), for \( 1 \leq p \leq k \). A leaf cycle of \( C_c \) is called \( C_i \) if one of the two edges of \( S_i \), connecting \( C_c \) and \( C_i \), is incident to \( v_i \). We say that the leaf cycles \( C_i \) and \( C_{i+1} \) are adjacent. Furthermore, let \( f_i \) and \( g_i \) be the two edges of \( S \) connecting the cycle \( C_c \) and \( C_i \) such that \( f_i \) is incident to \( v_i \) and consecutively adjacent to \( e_i^+ \) and \( e_i^- \), and \( g_i \) is adjacent to two edges of \( C_i \).

As in the previous case, we color all the edges of \( C_c \) by a repetitive sequence of colors 1, 2, and 3. If there exists at least one pair of adjacent leaf cycles \( C_i \) and \( C_{i+1} \), \( 1 \leq i < k \), we start to color the edges of \( C_c \) with the edge \( e_{i+1}^- \). Then, we uncolor the edge \( e_i^+ \) and every edge \( e_j^+ \), for every \( j \neq i \) for which there is a leaf cycle \( C_j \). In case when there is no pair of adjacent leaf cycles around \( C_c \), we start to color \( C_c \) with the edge \( e_{i+1}^- \), for an arbitrary \( i \), and uncolor every edge \( e_j^+ \) for which there is a leaf cycle \( C_j \); if \( 2k \) is not divisible by 3, we uncolor also the edge \( e_i^- \).

Now we color the edges \( f_j \) and \( g_j \) for every leaf cycle \( C_j \) of \( C_c \). There is at least one available color in \( \{1, 2, 3\} \) for the edge \( f_j \), since we can use the color of the adjacent uncolored edge, so we color \( f_j \). The edge \( g_j \) has at most two colored 2-neighbors, hence we can color it also by one of the colors 1, 2, or 3. Finally, we color the edges of the leaf cycles of \( C_c \), except for the two edges adjacent to the edge \( g_j \), alternately with the colors 1, 2, and 3. Since \( g_j \) is already colored, we may have to permute the colors in order to obtain a partial strong edge coloring.

We repeat the described procedure for every central vertex in \( S^* \) and obtain a partial strong edge coloring of \( G \) with 3 colors. It remains to show that the conflict graph \( R(G) \) of the noncolored edges of \( G \) is bipartite. It is easy to see that every component in the graph \( N(G) \) induced by the noncolored edges of \( G \) is an edge (the uncolored edges on
the central cycles) or a 2-path (the edges of $M \setminus S$, the pairs of edges incident to the edges $g_i$, and the pairs of edges incident to the edges $f_i$ and $f_{i+1}$ in cases of adjacent leaf cycles, or the pairs of edges incident to $f_i$ otherwise).

Notice that vertices in $R(G)$ corresponding to single edge components of $N(G)$ do not lie on any cycle in $R(G)$. Moreover, the maximum degree of $R(G)$ is at most 2, therefore we have that $R(G)$ contains no odd cycle, and so it is bipartite. Thus, we can color the noncolored edges of $G$ by two additional colors. By that, we showed that there exists a strong edge coloring of $G$ with at most 5 colors, and so establish the theorem.

\[ \square \]

3 Subcubic bipartite graphs with bounded edge weights

Using Theorem 6, we are able to solve the case (a) of Conjecture 5 by Faudree et al..

**Theorem 7.** Let $G$ be a bipartite subcubic graph with $d(u) + d(v) \leq 5$ for every edge $uv$. Then $\chi'_s(G) \leq 6$.

Note that the bound is tight and is attained e.g. by the complete bipartite graph $K_{2,3}$.

![Figure 3: The strong chromatic index of $K_{2,3}$ is 6.](image)

**Proof.** Suppose, to the contrary, that $G$ is a counterexample to the theorem with the minimum number of vertices. We firstly discuss some structural properties of $G$. Notice that every 3-vertex in $G$ has three neighbors of degree at most 2. The proof will be an easy consequence of the following three claims:

(a) the minimum degree $\delta(G) = 2$; and

(b) there are no adjacent 2-vertices in $G$; and

(c) there are no 4-cycles in $G$.

We prove each of the claims separately.

(a) This property simply follows from the minimality of $G$.

(b) Suppose first that $u$ and $v$ are adjacent 2-vertices. By the minimality of $G$, the graph $G \setminus \{u, v\}$ admits a strong edge coloring $\varphi$ with at most 6 colors. Let $x$ and $y$ be the second neighbors of $u$ and $v$, respectively. We will show that $\varphi$ can always be extended to the edges $ux$, $uv$, and $vy$ and hence to $G$. 

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The number of colored 2-neighbors of the edges $ux$, $uv$, and $vy$ is always at most 4, hence there are two available colors for each of them. In case when the union of the available colors of all three edges is at least three, $\varphi$ is easily extended. Thus, we may assume that the same two colors, say 5 and 6, are available for all three edges. Notice that in this case $x$ and $y$ are 3-vertices and the union of the colors of the colored edges incident to them is of size 4. Consider the labeling of the vertices and the assignment of the colors as in Figure 4.

![Figure 4](image_url)

**Figure 4:** The labeling of the vertices together with a coloring of the edges in the neighborhoods of $u$ and $v$.

Observe that there is another possible nonisomorphic coloring of the edges where the colors of $y_1y_2$ and $y_3y_4$ are interchanged, however the argument is analogous. Notice also that the edges incident to the vertex $x_2$ (resp. $x_4$, $y_2$, $y_4$) distinct from $x_1x_2$ (resp. $x_3x_4$, $y_1y_2$, $y_3y_4$) are colored by 5 and 6 for otherwise we color $xx_1$ (resp. $xx_3$, $yy_1$, $yy_3$) with 5 or 6 inferring that the three lists of available colors for $ux$, $uv$, and $vy$ are not equal. Observe that it immediately follows that $x_2$, $x_4$, $y_2$, and $y_4$ are distinct vertices.

Next, the two edges incident to $x_5$ and $x_6$ that are not incident to $x_2$, are colored by the colors 1 and 2, respectively. Otherwise we interchange the color of the edges $x_1x_2$ and $xx_1$ or $xx_2$, respectively, again introducing nonequal sets of available colors for the three noncolored edges. However, in this setting, we are able to color $x_1x_2$ with the color 4, and color the edges $ux$, $uv$, $vy$ with the colors 3, 5, and 6, respectively. Hence, we extended $\varphi$ to $G$, a contradiction.

(c) Suppose that $C = uvwz$ is a 4-cycle in $G$. By (a) and (b), precisely two vertices, say $u$ and $w$, are of degree 3, while $v$ and $z$ are 2-vertices. By the minimality, there exists a strong edge coloring $\varphi$ of $G \setminus \{v, z\}$ with at most 6 colors. It remains to color the four edges of $C$. Let $x$ and $y$ be the third neighbors of $u$ and $w$, respectively. In case when $x = y$, $G = K_{2,3}$, hence it is strongly edge colorable with at most 6 colors. Otherwise, each of the noncolored edges has three colored 2-neighbors and there are three available colors for each of them. In case when the sets of available colors are not equal, $\varphi$ is easily extended. Hence, we consider the case when the same colors are forbidden for the edges of $C$.

Without loss of generality, we may assume that the color of $ux$ is 1, and the second edge incident to $x$ is colored by 2. Similarly, let $wy$ be colored with 3 and the second edge $yy_1$ incident to $y$ must be colored with 2. Our aim is to change the color of $yy_1$ in order to obtain nonequal sets of available colors of the edges of $C$. Since there are at
most five 2-neighbors of $yy_1$, each of them must be colored by different color, otherwise, there is another color available for $yy_1$. But in this case, we recolor $wy$ with 2 and $yy_1$ with 3. Then we can color $uw, vw, wz, uz$ with 3, 4, 5, and 6, respectively. Hence, we extended $\varphi$ to $G$, a contradiction.

By the above claims, $G$ is a subcubic bipartite graph without 4-cycles. Moreover, since there is no pair of adjacent 2-vertices in $G$, it follows that the vertices of one partition are all of degree 3, and the vertices in the second partition are all 2-vertices. Such graphs, by Theorem 6, admit a strong edge coloring with at most 5-colors, a contradiction.

Acknowledgement. The authors where supported in part by bilateral project SK-SI-0005-10 between Slovakia and Slovenia, by Science and Technology Assistance Agency under the contract No. APVV-0023-10 (R. Soták), by Slovak VEGA Grant No. 1/0652/12 (M. Mockovčiaková, R. Soták), and VVGS UPJŠ No. 59/12-13 (M. Mockovčiaková), and by ARRS Program P1-0383 and Creative Core FISNM-3330-13-500033 (B. Lužar, R. Soták, R. Škrekovski).

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