Abstract. In this paper we investigate a family of Moishezon twistor spaces on the connected sum of 4 complex projective planes, which can be regarded as a direct generalization of the twistor spaces on $3\mathbb{C}P^2$ of double solid type studied by Poon and Kreussler-Kurke. These twistor spaces have a natural structure of double covering over a scroll of 2-planes over a conic. We determine the defining equations of the branch divisors in an explicit form, which are very similar to the case of $3\mathbb{C}P^2$. Using these explicit description we compute the dimension of the moduli spaces of these twistor spaces. Also we observe that similarly to the case of $3\mathbb{C}P^2$, these twistor spaces can also be considered as generic Moishezon twistor spaces on $4\mathbb{C}P^2$. We obtain these results by analyzing the anticanonical map of the twistor spaces in detail, which enables us to give an explicit construction of the twistor spaces, up to small resolutions.

1. Introduction

In their papers, Kreussler-Kurke [12] and Poon [16] investigated algebraic structure of generic twistor spaces on $3\mathbb{C}P^3$, the connected sum of 3 copies of complex projective planes. They showed that if the half-anticanonical system of a twistor space of $3\mathbb{C}P^2$ is base point free, then the morphism associated to the system becomes a generically 2 to 1 covering map whose branch divisor is a quartic surface. Further, they determined defining equation of the quartic surface; for the most generic twistor spaces, with respect to homogeneous coordinates on $\mathbb{C}P^3$, the equation is of the form

\begin{equation}
\label{eq:1}
z_0z_1z_2z_3 = Q(z_0, z_1, z_2, z_3)^2
\end{equation}

where $Q$ is a (homogeneous) quadratic polynomial with real coefficients. From the equation, the intersection of the quadratic surface $Q = 0$ and any of the 4 plane $z_i = 0$ is a double conic, and the intersection points of these 4 conics (consisting of 12 points) are ordinary double points of the quartic surface (1.1). For a generic quadratic polynomial $Q$, these are all singularities of the surface (1.1), but they showed that when (1.1) is actually the branch divisor of the twistor spaces, the surface has one more node, which is necessarily real, so that the branch surface has 13 ordinary nodes in total.

In this paper, we shall find Moishezon twistor spaces on $4\mathbb{C}P^2$ which can be regarded as a direct generalization of these twistor spaces on $3\mathbb{C}P^2$. More concretely, we show the following. There exist twistor spaces on $4\mathbb{C}P^2$ such that (i) the anticanonical system is 4-dimensional as a linear system, and the image of the associated rational map is a scroll $Y$ of 2-planes in $\mathbb{C}P^4$ over a conic, (ii) there is an explicit and simple elimination of the indeterminacy locus of the anticanonical map, whose resulting morphisms is a generically 2 to 1 covering map onto the scroll $Y$, (iii) the branch divisor of the last covering, which will be denoted by $B$ throughout this paper, is an intersection of $Y$ with a quartic hypersurface in $\mathbb{C}P^4$, (iv) if we take homogeneous coordinates on $\mathbb{C}P^4$ such that the scroll $Y$ is defined by $z_0^2 = z_1z_2$, then the quartic hypersurface is defined by the equation

\begin{equation}
\label{eq:2}
z_0z_3z_4f(z_0, z_1, z_2, z_3, z_4) = Q(z_0, z_1, z_2, z_3, z_4)^2,
\end{equation}
where \( f \) and \( Q \) are linear and quadratic polynomials with real coefficients respectively. The double covering structure and similarity of the equations (1.2) with (1.1) would justify to call these twistor spaces a direct generalization of those by Kreussler-Kurke-Poon.

The main tool of the present investigation is the anticanonical system of the twistor spaces. In Section 2 we start by constructing a rational surface \( S \) which will be contained in the twistor space \( Z \) on \( 4\mathbb{CP}^2 \) as a real half-anticanonical divisor, and then clarify the structure of the bi-anticanonical system on \( S \). Next in Section 3 we study the structure of the anticanonical map of the twistor spaces in detail. In Section 3.1 we show that the anticanonical map induces a rational map to \( \mathbb{CP}^4 \), whose image is a scroll \( Y \) of planes over a conic, and give an explicit elimination of the indeterminacy locus, obtaining a degree 2 morphism \( Z_1 \to Y \). Next in Section 3.2 we study the structure of the anticanonical map more in detail, and by applying some explicit blowups and blowdowns we modify the degree 2 morphism \( Z_1 \to Y \) to another morphism \( Z_4 \to Y \) so as to have no divisor to be contracted. Up to contraction of curves, this gives the Stein factorization of the morphism \( Z_1 \to Y \). Although the modifications therein are a little bit complicated, they are rather natural in light of the structure of the anticanonical system, and indispensable for obtaining explicit construction of the twistor spaces. We also obtain a key technical result that the branch divisor of \( Z_1 \to Y \) is a cut of \( Y \) by a quartic hypersurface.

In Section 4, we explicitly determine a defining equation of the branch divisor of the double covering. For this, in Section 4.1 we find 5 hyperplanes in \( \mathbb{CP}^4 \) such that the intersection of the branch divisor with the hyperplanes becomes double curves, i.e. a curve of multiplicity 2. These double curves are analogous to the above 4 conics appeared in the case of \( 3\mathbb{CP}^2 \), but to understand how they intersect each other requires some effort. In addition, in the present case, finding all the double curves is not so easy, since not all of the double curves are obtained as an image of twistor lines as in the case of \( 3\mathbb{CP}^2 \). In Section 4.2 we show that the 5 double curves are contained in a quadratic hypersurface in \( \mathbb{CP}^4 \), and that such a hyperquadric is unique up to the defining equation of the scroll \( Y \). In Section 4.3 we prove the main result which determines the defining equation of the quartic hypersurface (Theorem 4.5). The equation includes not only the quadratic polynomial obtained in Section 4.2 but also a linear polynomial, which might look strange at first sight. We give an account for a geometric meaning of it.

In Section 4.4 we investigate singularities of the branch divisor \( B \) of the double covering. As above \( B \) has 5 double curves, and at most of the intersection points of them, \( B \) has (non-real) ordinary double points. This is totally parallel to the case of \( 3\mathbb{CP}^2 \) explained at the beginning. But in the present case there are exactly 2 special intersection points, at which \( B \) has \( A_3 \)-singularities. Besides these ordinary double points and \( A_3 \)-singular points, we show that \( B \) has other isolated singularities and determine their basic invariants (Theorem 4.10). The result means that in general \( B \) has extra 6 ordinary double points. For obtaining this result, as in Kreussler [10] and Kreussler-Kurke [12] in the case of \( 3\mathbb{CP}^2 \), we compute the Euler numbers of the relevant spaces, especially the branch divisor \( B \). We also note that the concrete modification of the anticanonical map obtained in Section 3.2 is crucial for determining the invariants of the singularities.

In Section 5.1 we compute dimension of the moduli space of the present twistor spaces. We first compute the dimension by counting the number of effective parameters (coefficients) involved in the defining quartic polynomials. Next we show that some cohomology group of the twistor space an be regarded as a tangent space of the moduli space, and see that it coincides with the dimension obtained by counting the number of effective parameters. This
implies a completeness of our description obtained in Section 4. In Section 5.2 we discuss
genericity of our twistor spaces among all Moishezon twistor spaces on $4\mathbb{CP}^2$, indicated
in the title of this paper. The genericity implies a kind of density in the moduli space
of Moishezon twistor spaces on $4\mathbb{CP}^2$. In the course we also give a rough classification of
Moishezon twistor spaces on $4\mathbb{CP}^2$ under some genericity assumption.

In Appendix we show that an inverse of a non-standard contraction map employed in
Section 4.2 can be realized by an embedded blowup with a non-singular center in the
ambient space, and point out in a concrete form that the modification can be regarded as a
singular version of the well-known operation of Hironaka [5] for constructing non-projective
Moishezon threefolds.

We should also mention a relationship between this work and our previous paper [8]. As
explained in the beginning, in the case of $3\mathbb{CP}^2$, the branch quartic divisor of the double
covering becomes of the form $(L,1)$ under a genericity assumption. In non-generic cases, as
showed by Kreussler-Kurke [12], the branch divisor becomes similar but more degenerate
form, and in the most degenerate situation, the branch divisor has a $\mathbb{C}^*$-action. As a
consequence, in that case the twistor spaces have a $\mathbb{C}^*$-action. The twistor spaces studied
in [8] is a generalization of these twistor spaces (with $\mathbb{C}^*$-action) to the case of $n\mathbb{CP}^2$, $n > 3$.
In this respect we remark that the twistor spaces in this paper is obtained as a deformation
of the twistor spaces studied in [8] in the case of $4\mathbb{CP}^2$. It is very natural to expect that
we can obtain a further generalization to the case of $n\mathbb{CP}^2$, $n > 4$. We hope to discuss this
attractive topic in a future paper.

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Notations and Conventions. The natural square root of the anticanonical bundle of
a twistor space is denoted by $F$. If two varieties $X_1$ and $X_2$ are birational under some
blowup or blowdown, and if $Y$ is a subvariety of $X_1$, then we often use the same symbol $Y$
to mean the strict transform or birational image of $Y$ under the blowup or blowdown, as
far as it makes sense. For a line bundle $L$ and a non-zero section $s$ of $L$, $(s)$ means the zero
divisor of $s$. For a linear subspace $V \subset H^0(L)$, we denote by $|V|$ to mean the linear system
$\{s \in H^0(L), s \neq 0\}$. $Bs|V|$ means the base locus of $|V|$. We mean dim $|V| = \dim V - 1$
and $h^i(L) = \dim H^i(L)$.

2. A construction of rational surfaces and their bi-anticanonical system

We are going to investigate twistor spaces on $4\mathbb{CP}^2$ which contain a particular type of
non-singular rational surface $S$ as a real member of $|F|$. In this section we first construct
the surface $S$ as a blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1$, and then study the bi-anticanonical system on
it. For this, we define the line bundles $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as the pullback
of $\mathcal{O}_{\mathbb{CP}^1}(1)$ by the projection to the first and second factors respectively. We simply call
members of the linear system $\mathcal{O}(m,n)$ as $(m,n)$-curves. We define a real structure on
$\mathbb{CP}^1 \times \mathbb{CP}^1$ as a product of the complex conjugation and the antipodal map. Next take any
non-real $(1,0)$-curves $C_1$, any $(0,1)$-curve $C_2$, any distinct 3 points on $C_1 \setminus (C_2 \cup \overline{C_2})$, and 1
point on $C_2 \setminus (C_1 \cup \overline{C_1})$. By taking the images under the real structure, we obtain distinct
8 points in total (see Figure 1). Let $\epsilon : S \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be the blowup at these 8 points.
This surface $S$ has a natural real structure induced from that on $\mathbb{CP}^1 \times \mathbb{CP}^1$, and also has
an anticanonical curve

\[ C := C_1 + C_2 + \overline{C}_1 + \overline{C}_2, \]

where this time \( C_i \) means the strict transform of the original \( C_i \), so that \( C_1^2 = \overline{C}_1^2 = -3 \) and \( C_2^2 = \overline{C}_2^2 = -1 \). The identity component of the holomorphic automorphism group of \( S \) is trivial. In the sequel we investigate the anticanonical and bi-anticanonical systems on \( S \).

**Proposition 2.1.** (i) \( \dim |K_S^{-1}| = 0 \), so that \( C \) is the unique anticanonical curve on \( S \), (ii) \( \dim |2K_S^{-1}| = 2 \), Bs \( |2K_S^{-1}| = C_1 \cup \overline{C}_1 \), and Bs \( |2K_S^{-1} - C_1 - \overline{C}_1| = \emptyset \).

**Proof.** (i) is immediate. For (ii), \( CC_1 = C\overline{C}_1 = -1 \) means the inclusion \( C_1 \cup \overline{C}_1 \subset \text{Bs} \ |2K_S^{-1}| \). Riemann-Roch formula and the rationality of \( S \) imply \( \chi(2K_S^{-1}) = 1 \). These mean \( H^1(K_S^{-1}) = 0 \). Therefore restricting \( 2K_S^{-1} - C_1 - \overline{C}_1 \) to \( C_2 \cup \overline{C}_2 \), we obtain the exact sequence

\[
0 \to H^0(K_S^{-1}) \to H^0(2K_S^{-1} - C_1 - \overline{C}_1) \to H^0(\mathcal{O}_{C_2}) \oplus H^0(\mathcal{O}_{\overline{C}_2}) \to 0.
\]

This implies \( (C_2 \cup \overline{C}_2) \cap \text{Bs} \ |2K_S^{-1} - C_1 - \overline{C}_1| = \emptyset \), and also \( h^0(2K_S^{-1} - C_1 - \overline{C}_1) = 3 \). On the other hand by restricting the same system to \( C_1 \cup \overline{C}_1 \), we obtain the exact sequence

\[
0 \to H^0(\mathcal{O}_S(2C_2 + 2\overline{C}_2)) \to H^0(2K_S^{-1} - C_1 - \overline{C}_1) \xrightarrow{r} H^0(\mathcal{O}_{C_1}(1)) \oplus H^0(\mathcal{O}_{\overline{C}_1}(1)).
\]

As \( h^0(\mathcal{O}_S(2C_2 + 2\overline{C}_2)) = 1 \) clearly, the image of the restriction map \( r \) in (2.2) is 2-dimensional. Projecting this to \( H^0(\mathcal{O}_{C_1}(1)) \) gives either a 1-dimensional subspace or \( H^0(\mathcal{O}_{C_1}(1)) \) itself. In order to prove \( \text{Bs} \ |2K_S^{-1} - C_1 - \overline{C}_1| = \emptyset \), it is enough to exclude the former possibility. Suppose it is the case. Let \( p \in C_1 \) be the zero point of a generator of the 1-dimensional subspace. Then we have \( \text{Bs} \ |2K_S^{-1} - C_1 - \overline{C}_1| = \{p, \overline{p} \} \) and \( p \notin C_2 \cup \overline{C}_2 \) as we have already seen. Let \( S' \to S \) be the blowup at \( p \) and \( \overline{p} \), and \( C'_1 \) and \( \overline{C}'_1 \) the strict transforms of \( C_1 \) and \( \overline{C}_1 \) respectively. As \( (2K_S^{-1} - C_1 - \overline{C}_1)^2 = 2 \), for any member \( D \in |2K_S^{-1} - C_1 - \overline{C}_1| \) with \( D \neq C_1 + 2C_2 + \overline{C}_1 + 2\overline{C}_2 \), we have \( C \cap D = \{p, \overline{p} \} \), and the intersections are transversal. This means that the system \( |C'_1 + 2C_2 + \overline{C}'_1 + 2\overline{C}_2| \) on \( S' \) is base point free. Then as \( C_1'(C_1' + 2C_2 + \overline{C}'_1 + 2\overline{C}_2) = C_1'(C_1' + 2C_2 + \overline{C}'_1 + 2\overline{C}_2) = 0 \), the morphism associated to the system contracts \( C'_1 \) and \( \overline{C}'_1 \) to points. On the other hand, by (2.1), \( C_2 \) and \( \overline{C}_2 \) are
mapped to mutually different points by the same morphism. This is a contradiction because $C_1 \cap C_2 \neq \emptyset$, $C_1 \cap \overline{C}_2 \neq \emptyset$, and $C_1$ is connected.

Let $\phi : S \to \mathbb{CP}^2$ be the morphism associated to the system $|2K_S^{-1}| \simeq |2K_S^{-1} - C_1 - \overline{C}_1|$. Since $\text{Bs} |2K_S^{-1} - C_1 - \overline{C}_1| = \emptyset$ and $(2K_S^{-1} - C_1 - \overline{C}_1)C_2 = (2K_S^{-1} - C_1 - \overline{C}_1)\overline{C}_2 = 0$, $\phi$ factors as $S \to \overline{S} \to \mathbb{CP}^2$, where $S \to \overline{S}$ denotes the blowdown of $C_2$ and $\overline{C}_2$.

**Proposition 2.2.** The morphism $\phi$ is generically 2 to 1, and the branch divisor is a quartic curve. Further, the images $\phi(C_1)$ and $\phi(\overline{C}_1)$ are the same line, and $\phi(C_2)$ and $\phi(\overline{C}_2)$ are different 2 points on the line.

**Proof.** The morphism $\phi$ is surjective since $(2K_S^{-1} - C_1 - \overline{C}_1)^2 = 2 > 0$. This also means it is generically 2 to 1. Further, a general member $D$ of $|2K_S^{-1} - C_1 - \overline{C}_1|$, which is an irreducible non-singular curve by Bertini’s theorem, is an elliptic curve because $D(D + K_S) = 0$. Hence the branch curve of $\phi$ must be of degree 4. For the images $\phi(C_1)$ and $\phi(\overline{C}_1)$, as in the proof of Proposition 2.1 we have the exact sequence (2.2), and the image of $r$ projects isomorphically to $H^0(\mathcal{O}(1))$ and $H^0(\mathcal{O}(1))$. Hence $\phi(C_1)$ and $\phi(\overline{C}_1)$ are lines. Further these lines are identical, since they are precisely the 2-dimensional images into the dual space $\mathbb{P}H^0(2K_S^{-1} - C_1 - \overline{C}_1)^* \to \overline{S}$ by the dual map of $r$. Also the last claim for $\phi(C_2)$ and $\phi(\overline{C}_2)$ is clear from the exact sequence (2.1), $C_1 \cap C_2 \neq \emptyset$ and $C_1 \cup \overline{C}_2 \neq \emptyset$. $\square$

The following property of the branch quartic will also be needed later.

**Proposition 2.3.** If we denote the line and the pair of points on it by $l := \phi(C_1) = \phi(\overline{C}_1)$ and $p_2 := \phi(C_2)$ and $\overline{p}_2 = \phi(\overline{C}_2)$ respectively, then $p_2$ and $\overline{p}_2$ are smooth points of the branch quartic curve, and the curve is tangent to $l$ at these 2 points. (Hence $l$ is a bitangent of the branch curve.)

**Proof.** Let $\beta$ be the branch curve of $\phi$. As $C_1 + 2C_2 + \overline{C}_1 + 2\overline{C}_2 \in |2K_S^{-1} - C_1 - \overline{C}_1|$, there is a line $l'$ such that $\phi^{-1}(l') = C_1 + 2C_2 + \overline{C}_1 + 2\overline{C}_2$. But as $\phi(C_1) = l$, we obtain $l' = l$. Since $\phi(C_2) = p_2$ and both $C_2 \cap C_1$ and $\overline{C}_2 \cap \overline{C}_1$ are non-empty, it follows that $p_2 \in \beta$, so that $\overline{p}_2 \in \beta$ by the real structure. Further, after the blowdown $S \to \overline{S}$, the curve $C_1 \cup \overline{C}_1$ is of course locally reducible at the images of $C_2$ and $\overline{C}_2$. Hence we have $\beta|_l = 2p_2 + 2\overline{p}_2$ as divisors. This means that either $\beta$ has double points at $p_2$ and $\overline{p}_2$, or otherwise $\beta$ is smooth at $p_2$ and $\overline{p}_2$ and is tangent to $l$ at these 2 points. But in the former case by smoothness of $S$ there have to be extra exceptional curves over $p_2$ and $\overline{p}_2$, which contradicts $\phi^{-1}(l) = C_1 + 2C_2 + \overline{C}_1 + 2\overline{C}_2$. Hence the claim follows. $\square$

### 3. Analysis of the anticanonical map on the twistor spaces

#### 3.1. The anticanonical map of the twistor spaces.

Let $S$ be the rational surface equipped with the real structure constructed in the previous section, and $C = C_1 + C_2 + \overline{C}_1 + \overline{C}_2$ the unique anticanonical curve on $S$. Let $Z$ be a twistor space on $4\mathbb{CP}^2$ and suppose that $Z$ contains $S$ as a real member of $|F|$. The following property of $|F|$ is immediate to see and we omit a proof.

**Proposition 3.1.** The system $|F|$ satisfies the following: (i) $\dim |F| = 1$, (ii) $\text{Bs} |F| = C$, (iii) the number of reducible members of $|F|$ is two, and both of the members are real. 

We note that it readily follows from (i) and (ii) that a general member $S'$ of the pencil $|F|$ is also obtained from $\mathbb{CP}^1 \times \mathbb{CP}^1$ by blowing up 8 points arranged as in Figure [1], where the
We denote $L_1 := S^+_1 \cap S^-_1$ and $L_2 := S^+_2 \cap S^-_2$, both of which are twistor lines by [16, §1]. Then these divisors and curves form a tetrahedron as illustrated in Figure 2, (a). These will be significant for our analysis of the anticanonical system on the twistor spaces. We show the following basic properties of the anticanonical system. Note that (iv) means that $Z$ is Moishezon.

Proposition 3.2. The anticanonical system $|2F| = |K_Z^{-1}|$ of the twistor space $Z$ satisfies the following: (i) $\dim |2F| = 4$, (ii) $\text{Bs} |2F| = C_1 \cup \overline{C_1}$, (iii) if $\mu_1 : Z_1 \to Z$ denotes the blowup at $C_1 \cup \overline{C_1}$, $E_1 \cup \overline{E_1}$ the exceptional divisor, and $Z_1 := \mu_1^*(2F) - E_1 - \overline{E_1}$, then $\text{Bs} |Z_1| = 0$, (iv) if $\Phi_1$ denotes the morphism associated to $|Z_1|$, then the image $\Phi_1(Z_1)$ is a scroll of 2-planes over a conic, and the morphism $\Phi_1$ is generically 2 to 1 over the scroll.

By the blowup $\mu_1 : Z_1 \to Z$, the tetrahedron in $Z$ is transformed to be a cubic in $Z_1$ as in Figure 2 (b).

Proof of Proposition 3.2 (i) is immediate from Proposition 2.1 (ii), Proposition 3.1 (i), and the exact sequence

$$0 \to H^0(F) \to H^0(2F) \to H^0(2K_S^{-1}) \to 0,$$

where the last zero is a consequence of $h^0(F) = 2$ and the Riemann-Roch formula applied to $F$. The claim (ii) also follows from this exact sequence and Proposition 2.1 (ii).
For (iii), let $\tilde{S}$ be the strict transform of $S$. Then $\tilde{S}$ is biholomorphic to $S$, and $\tilde{S} \in |\mu_1^*F - E_1 - \overline{E}_1|$. Hence we have an exact sequence

$$0 \to \mu_1^*F \to \mu_1^*(2F) - E_1 - \overline{E}_1 \to \mu_1^*(2F) - E_1 - \overline{E}_1|_{\tilde{S}} \to 0.$$  

(3.3)

Since $H^1(\mu_1^*F) \simeq H^1(F) = 0$, we obtain that the restriction map $H^0(\mu_1^*(2F) - E_1 - \overline{E}_1) \to H^0(\mu_1^*(2F) - E_1 - \overline{E}_1|_{\tilde{S}})$ is surjective. Further, as $(\mu_1^*F)|_{\tilde{S}} \simeq F|_{\tilde{S}} \simeq K_{\tilde{S}}^{-1}$ under the biholomorphism $\tilde{S} \simeq S$, we have $\mu_1^*(2F)|_{\tilde{S}} \simeq 2K_{\tilde{S}}^{-1}$. Further, $E_1|_{\tilde{S}} \simeq \sigma_{\tilde{S}}(C_1)$. Hence we obtain an isomorphism $\mu_1^*(2F) - E_1 - \overline{E}_1|_{\tilde{S}} \simeq 2K_{\tilde{S}}^{-1} - C_1 - C_{1}$. Therefore by the third claim of Proposition \[2.1\](ii), we obtain $Bs|\mu_1^*(2F) - E_1 - \overline{E}_1| = \emptyset$.

Let $\Phi : Z \to \mathbb{CP}^4$ be the rational map associated to the anticanonical system $|2F|$, so that $\Phi_1 = \Phi \circ \mu_1$. For (iv) it is enough to show that $\Phi(Z)$ is the 3-dimensional scroll as in the statement, and the rational map $\Phi : Z \to \Phi(Z)$ is generically 2 to 1. Let $S^2H^0(F)$ be the subspace of $H^0(2F)$ generated by all sections of the form $s_1s_2$ where $s_i \in H^0(2F)$. This is a 3-dimensional subspace. Then we have the following left commutative diagram of rational maps:

$$\begin{array}{c|c|c}
 Z & \mathbb{CP}^4 & Y \\
 \Phi & \pi & \Phi \circ \pi \\
 f & \downarrow & f \\
 \mathbb{CP}^2 & \Lambda & \end{array}$$  

(3.4)

where $\pi$ is the linear projection induced by the inclusion $S^2H^0(F) \subset H^0(2F)$ and $f$ is the rational map associated to the subsystem $|S^2H^0(F)|$. Clearly the image $f(Z)$ is a conic, for which we denote by $\Lambda$. Hence writing $Y := \pi^{-1}(\Lambda)$, $Y$ is exactly the scroll as in the statement of (iv), and we obtain the right commutative diagram in (3.4). We have to show that $\Phi : Z \to Y$ is surjective and generically 2 to 1. For these, we note that by the definition of $f$, for any $\lambda \in \Lambda$, $f^{-1}(\lambda)$ belongs to the pencil $|F|$. Then by (3.2) for any non-singular member $S \in |F|$, the restriction $\Phi|_S$ is exactly the rational map associated to the system $|2K_S^{-1}|$, where the target space is the fiber plane $\pi^{-1}(\lambda)$. By Proposition \[2.2\] this means that $\Phi|_{f^{-1}(\lambda)} : f^{-1}(\lambda) \to \pi^{-1}(\lambda)$ is surjective as far as $f^{-1}(\lambda)$ is non-singular. Therefore $\Phi$ itself is also surjective to $Y$. Now the final claim (2 to 1 over $Y$) is immediate from these considerations and Proposition \[2.2\].

Thus the anticanonical map is 2 to 1 over the scroll $Y$. Further, by the above argument and Proposition \[2.2\] the branch locus of the 2 to 1 map has degree 4 on the planes $f^{-1}(\lambda)$, from which one might find similarity with those in the case of $3\mathbb{CP}^2$ [12, 16]. But we are yet far from the goal. In the next subsection we shall investigate structure of the anticanonical map more closely.

3.2. Modification of the anticanonical map. We use the notations $\Phi, \Lambda, Y, f$ and $\pi$ given in the proof of the proposition. Further define $l$ to be the singular locus of the scroll $Y$. $l$ is a line, and is exactly the indeterminacy locus of the projection $\pi$. This line plays an important role throughout this paper. Note that for a hyperplane $H \subset \mathbb{CP}^4$, the intersection $Y \cap H$ splits to planes iff $H$ projects to a line in $\mathbb{CP}^2$ (in which the conic $\Lambda$ is contained), and otherwise $Y \cap H$ is a cone over $\Lambda$ whose vertex is the point $l \cap H$. Further, in the former situation, $Y|_H$ is a double plane (i.e. a non-reduced plane of multiplicity 2) iff the line is tangent to $\Lambda$. Let $\nu : \tilde{Y} \to Y$ be the blowup at $l$, and $\Sigma$ the exceptional divisor.
\(\tilde{Y}\) is biholomorphic to the total space of the \(\mathbb{CP}^2\)-bundle \(\mathbb{P}(\mathcal{O}(2)\oplus\mathcal{O}) \to \Lambda\), and \(\Sigma\) is identified with the subbundle \(\mathbb{P}(\mathcal{O}(2))\), so that it is biholomorphic to \(\mathbb{CP}^1 \times \mathbb{CP}^1\). More invariantly, we have a natural isomorphism \(\Sigma \cong l \times \Lambda\). The composition \(\tilde{Y} \to Y \to \Lambda\) is a morphism which is identified with the bundle projection, for which we denote by \(\tilde{\pi}\).

In order to treat the branch locus of the degree 2 morphism \(\Phi_1\) (see Proposition 3.2) properly, we consider a lifting problem of the morphism \(\Phi_1\) to \(\tilde{Y}\). By definition of \(f\), the composition \(Z_1 \xrightarrow{\mu_1} Z \xrightarrow{\tilde{\pi}} \Lambda \subset \mathbb{CP}^2\) is the rational map associated to the 2-dimensional linear system \(\mu_1^*|S^2H^0(F)|\) on \(Z_1\) (whose members are the total transforms of members of the linear system \(|S^2H^0(F)|\)). This linear system on \(Z_1\) is a subsystem of \(|\mu_1^*(2F)|\). But since the pencil \(|F|\) has \(C_1\) and \(\overline{C}_1\) as components of the base locus, all the above total transforms contain the divisor \(2E_1 + 2\overline{E}_1\). Hence subtracting \(E_1 + \overline{E}_1\) and recalling \(L_1 = \mu_1^*2F - E_1 - \overline{E}_1\), we can regard linear system \(\mu_1^*|S^2H^0(F)|\) as a subsystem of \(|L_1|\). This 2-dimensional subsystem of \(|L_1|\) still has \(E_1 + \overline{E}_1\) as the fixed components, so by subtracting it we obtain a 2-dimensional subsystem of \(|L_1 - E_1 - \overline{E}_1| = |\mu_1^*2F - 2(E_1 + \overline{E}_1)|\), which is readily seen to be coincide with \(|\mu_1^*2F - 2(E_1 + \overline{E}_1)|\) itself. Hence the composition \(Z_1 \to Z \to \Lambda\) can be regarded as the rational map associated to \(|\mu_1^*2F - 2(E_1 + \overline{E}_1)|\).

However, the curves \(C_2\) and \(\overline{C}_2\) are contained in \(\text{Bs}|F|\) (Proposition 3.1 (ii)), and the strict transforms of these curves to \(Z_1\) are exactly the base locus of \(|\mu_1^*2F - 2(E_1 + \overline{E}_1)|\). Thus the composition \(Z_1 \to Z \to \Lambda\) has the strict transforms of \(C_2\) and \(\overline{C}_2\) as its indeterminacy locus. Therefore, the morphism \(\Phi_1 : Z_1 \to \Lambda\) cannot be lifted to \(Z_1 \to \tilde{Y}\) as a morphism, because the composition with \(\tilde{\pi}\) is not a morphism.

So let \(\mu_2 : Z_2 \to Z_1\) be the blowup at \(C_2 \cup \overline{C}_2\), and \(E_2\) and \(\overline{E}_2\) the exceptional divisors. Here we are regarding \(C_2\) and \(\overline{C}_2\) as curves in \(Z_1\). Let \(\Phi_2 := \Phi_1 \circ \mu_2\), and \(L_2 := \mu_2^*L_1\). (This time we do not subtract \(E_2 + \overline{E}_2\) because \(C_2\) and \(\overline{C}_2\) are not base curves of \(|L_1|\).)

Obviously \(\Phi_2\) is the rational map associated to \(|L_2|\), and it is clearly a morphism. Then we have the following:

**Proposition 3.3.** The morphism \(\Phi_2 : Z_2 \to \Lambda\) can be lifted to a morphism \(\tilde{\Phi}_2 : Z_2 \to \tilde{Y}\). Namely there is a morphism \(\tilde{\Phi}_2 : Z_2 \to \tilde{Y}\) such that \(\Phi_2\) factors as \(Z_2 \xrightarrow{\tilde{\Phi}_2} \tilde{Y} \xrightarrow{\nu} \Lambda\).

**Proof.** As in the above explanation, the composition \(Z_2 \to Z_1 \to \Lambda\subset \mathbb{CP}^2\) is the rational map associated to the system \(|\mu_2^*(\mu_1^*2F - 2(E_1 + \overline{E}_1))|\). In the same way for the identification between a conic and a line on a plane by means of the projection from a point, once we fix any non-zero element of \(H^0(\mu_1^*F - E_1 - \overline{E}_1)\), by taking a product with it, members of the system \(|\mu_1^*2F - 2(E_1 + \overline{E}_1)|\) can be identified with those of the pencil \(|\mu_1^*F - (E_1 + \overline{E}_1)|\), and the rational map associated to \(|\mu_1^*2F - 2(E_1 + \overline{E}_1)|\) is identified with the rational map associated to \(|\mu_1^*F - (E_1 + \overline{E}_1)|\). Therefore the composition \(Z_2 \to Z_1 \to \Lambda\subset \mathbb{CP}^2\) can be identified with the rational map associated to the pencil \(|\mu_2^*(\mu_1^*F - E_1 - \overline{E}_1)|\). This pencil has \(E_2 + \overline{E}_2\) as the fixed component, and if we subtract this, the pencil becomes free, since \(|F| = |K^{-1}\|\) and \(|K^{-1}\|\) consists of a single member \(C_1 + \overline{C}_1 + C_2 + \overline{C}_2\) (a reduced curve) by Proposition 2.1 (i). Hence the composition \(Z_2 \to Z_1 \to \Lambda\) has no point of indeterminacy. We write \(f_2\) for this morphism. Thus we are in the following left situation:

![Diagram](3.5)
Then by using the above explicit form of the restriction map, we obtain the strict transform of a member of the pencil \( \pi \) where all maps except \( \pi \) are morphisms, and the 2 triangles are commutative. (The map \( Z_2 \to Y \) is \( \Phi_2 \) and the map \( \tilde{Y} \to \Lambda \) is \( \tilde{\pi} \).) For lifting \( \Phi_2 \) to \( \tilde{Y} \), we need to assign a point of \( \tilde{Y} \) for each point of \( Z_2 \). For this, recall that \( \nu \) is isomorphic outside \( \nu^{-1}(l) = \Sigma \). For \( z \in Z_2 \setminus \Phi_2^{-1}(l) \), of course, we assign the point \( \nu^{-1}(\Phi_2(z)) \). For \( z \in \Phi_2^{-1}(l) \) define \( \lambda := f_2(z) \) and \( y := \Phi_2(z) \in l \). The inverse image \( \nu^{-1}(y) \) is a fiber of the projection \( \Sigma \to l \). In accordance with the natural isomorphism \( \Sigma \cong l \times \Lambda \), this fiber and the fiber \( \tilde{\pi}^{-1}(\lambda) \) intersect at a unique point. Let \( \tilde{y} \in \tilde{Y} \) be this point, and we assign \( \tilde{y} \) to \( z \). Define \( \tilde{\Phi}_2 : Z_2 \to \tilde{Y} \) to be the map thus obtained. Then \( \tilde{\Phi}_2 \) is clearly continuous and is a lift of \( \Phi_2 \). As \( \tilde{\Phi}_2 \) is holomorphic on the complement of the analytic subset \( \Phi_2^{-1}(l) \), Riemann’s extension theorem means that it is automatically holomorphic on the whole of \( Z_2 \). Thus we get the situation right in (3.5) and obtained the desired lift \( \tilde{\Phi}_2 \).

Since the lift \( \tilde{\Phi}_2 : Z_2 \to \tilde{Y} \) is a degree 2 morphism between non-singular spaces, we can speak about its branch divisor. Namely we first define the ramification divisor \( R \) on \( Z_2 \) as a zero divisor of a natural section (defined by the Jacobian) of the line bundle \( K_{Z_2} - \Phi_2^* K_{\tilde{Y}} \), and then let the branch divisor \( \tilde{B} \) to be the image \( \tilde{\Phi}_2(R) \), which is necessarily a divisor. We can determine the cohomology class of this divisor as follows:

**Proposition 3.4.** Let \( \tilde{B} \) be the branch divisor of the lift \( \tilde{\Phi}_2 : Z_2 \to \tilde{Y} \) as above. Then \( \tilde{B} \in [\mathcal{O}_Y(4)], \) where \( \mathcal{O}_Y(1) := \nu^* \mathcal{O}_Y(1) = \nu^* \mathcal{O}_{\mathbb{C}P^4}(1)|_Y \).

**Proof.** As before let \( \Sigma \) be the exceptional divisor of the blowup \( \tilde{Y} \to Y \), and let \( \mathfrak{f} \) be the cohomology class of the fiber class \( \tilde{\pi}^* \mathcal{O}_\Lambda(1) \). The cohomology group \( H^2(\tilde{Y}, \mathbb{Z}) \) is a free \( \mathbb{Z} \)-module generated by \( \Sigma \) and \( \mathfrak{f} \). \( \Sigma \) is isomorphic to \( l \times \Lambda \), and the restriction \( \nu^* \Sigma \) can be identified with the projection to \( l \). Define \( (0,1) \) to be the bidegree of a fiber of this projection. Then the normal bundle is \( N_{\Sigma/Y} \cong \mathcal{O}(-2,1) \), while \( \mathfrak{f} \) is restricted to the class \((1,0)\). From these we can readily deduce that the restriction map \( H^2(\tilde{Y}, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z}) \) is isomorphic.

Let \( H \subset \mathbb{C}P^4 \) be any hyperplane containing the line \( l \). Then since \( \Lambda \) is a conic, we have \( \nu^{-1}(H) = \Sigma + 2\mathfrak{f} \), which means

\[
\nu^* \mathcal{O}_Y(1) = \Sigma + 2\mathfrak{f} \quad \text{in} \quad H^2(\tilde{Y}, \mathbb{Z}).
\]

Then by using the above explicit form of the restriction map, we obtain

\[
\mathcal{O}_Y(1)|_{\Sigma} \cong \Sigma + 2\mathfrak{f} |_{\Sigma} = \mathcal{O}(-2,1) + \mathcal{O}(2,0) = \mathcal{O}(0,1). \tag{3.7}
\]

Hence in order to prove \( \tilde{B} \in [\mathcal{O}_Y(4)] \), it suffices to show \( \tilde{B}|_{\Sigma} \in [\mathcal{O}(0,4)]. \)

In order to obtain the restriction \( \tilde{B}|_{\Sigma} \), for each \( \lambda \in \Lambda \) we write \( S_\lambda := f_2^{-1}(\lambda) \), which is the strict transform of a member of the pencil \( |F| \). Then it is not difficult to see that the restriction \( \tilde{\Phi}_2|_{S_\lambda} : S_\lambda \to \tilde{\pi}^{-1}(\lambda) = \mathbb{C}P^2 \) is naturally identified with the restriction of the original restriction \( \Phi|_{S_\lambda} : S_\lambda \to \mathbb{C}P^2 \). If \( S_\lambda \) is non-singular (which is the case for almost all \( \lambda \)), the last restriction \( \Phi|_{S_\lambda} \) is exactly the bi-anticanonical map of \( S_\lambda \). Hence by Propositions 2.2 and 2.3 the branch curve is a quartic curve which is tangent to the line \( l \) at 2 points. The last 2 points are independent of the choice of \( \lambda \), since they are exactly \( p_2 = \Phi(C_2) \) and \( \overline{p}_2 = \Phi(\overline{C}_2) \). Therefore \( \tilde{B}|_{\Sigma} \) contains the fibers of \( \Sigma \to \Lambda \) (the restriction of \( \tilde{Y} \to Y \) to \( \Sigma \)) over the points \( p_2 \) and \( \overline{p}_2 \) by multiplicity 2 respectively. Moreover since the intersection of the line \( l \) with the branch quartic curve of \( S_\lambda \to \mathbb{C}P^2 \) consists of the 2 points \( p_2 \) and \( \overline{p}_2 \), if \( \tilde{B}|_{\Sigma} \) contains an irreducible curve of bidegree \((k,l)\) with \( k > 0 \), then we have \((k,l) = (1,0)\).
From this we deduce that the restriction of the line bundle \((3.8)\) on \(E\) these two fibers are written by 3 bold lines respectively. Different from the projection to \(CP\) and let \(2\) points, both of which consist of three rational curves simultaneously. For this we first consider the divisor \(E\) and \(E_2\) of \(\mu_2 : Z_2 \to Z_1\) are isomorphic to \(CP^1 \times CP^1\) and the normal bundles satisfy

\[
N_{E_2/Z_2} \simeq \mathcal{O}(-1,-1), \quad N_{E_2/Z_2} \simeq \mathcal{O}(-1,-1).
\]

(See Figure 3(c).) Therefore \(E_2\) and \(E_2\) can also be blowdown along the projection different from the original \(E_2 \to C_2\) and \(E_2 \to C_2\). Let \(\mu_2 : Z_2 \to Z_3\) be this blowdown. (See (c) \(\to\) (d) in Figure 3) \(Z_3\) is still non-singular. The birational transformation from \(Z_1\) to \(Z_3\) is exactly Atiyah’s flop at \(C_2\) and \(C_2\). The divisors \(E_1\) and \(E_1\) in \(Z_1\) are also isomorphic to \(CP^1 \times CP^1\), and they are respectively blown up at 2 points through \(\mu_2\). We use the same letters \(E_1\) and \(E_1\) to mean these divisors in \(Z_2\). These divisors are not affected by the blowdown \(\mu_3\), and we still denote by \(E_1\) and \(E_1\) for their images in \(Z_3\), as displayed in Figure 3(d).

Next we show that these 2 divisors \(E_1\) and \(E_1\) in \(Z_3\) can be contracted to non-singular rational curves simultaneously. For this we first consider the divisor \(E_1\) in \(Z_1\), so that \(E_1 \simeq CP^1 \times CP^1\), and take the cohomology class of a fiber of the projection to \(CP^1\) which is different from the projection to \(C_1\). Next pullback the class by the blowup \(\mu_2\) and push it to \(E_1 \subset Z_3\) by \(\mu_3\). (In Figure 3 these cohomology classes are represented by non-dotted lines on \(E_1\).) Thus we obtain a cohomology class on \(E_1 \subset Z_3\) whose self-intersection number is zero. The linear system on this \(E_1\) having this cohomology class is clearly a free pencil, and induces a morphism to \(CP^1\). Let \(g : E_1 \to CP^1\) be this morphism. General fibers of \(g\) are non-singular rational curves, and there exist precisely 2 singular fibers, both of which consist of 2 non-singular rational curves intersecting at a point. The same is true for \(E_1\), and let \(g : E_1 \to CP^1\) the morphism corresponding to \(g\). Now \(g\) and \(g\) naturally fit on the intersection \(E_1 \cap E_1\) and form a morphism \(g \cup g : E_1 \cup E_1 \to CP^1 \cup CP^1\). Here note that these two \(CP^1\)-s are identified at 2 points, and \(g \cup g\) has reducible fibers exactly over these 2 points, both of which consist of three rational curves. In Figure 3(d), these 2 reducible fibers are written by 3 bold lines respectively.

We are going to show that the reducible connected divisor \(E_1 \cup E_1\) on \(Z_3\) can be contracted along \(g \cup \tilde{g}\). For this we need to examine the normal bundle, \([E_1 + E_1]|_{E_1 \cup E_1}\). The restriction of the normal bundle \(N_{E_1/Z_3} = [E_1]|_{E_1}\) is degree \((-2)\) on irreducible fibers of \(g\), and degree \((-1)\) on the 2 irreducible components of the (two) singular fibers. (See Figure 3(d).) From this we deduce that the restriction of the line bundle \([E_1 + E_1]|_{E_1 \cup E_1}\) is \((-2)\) on irreducible fibers of \(g \cup \tilde{g}\), and \((-1)\) on the end components of the reducible fibers, while \((-1) + (-1) = -2\) on the middle component of the reducible fibers. Now by the relative version of Nakai-Moishezon criterion for ampleness, these numerical data imply that the dual line bundle \([E_1 + E_1]|_{E_1 \cup E_1}\) is \((g \cup \tilde{g})\)-ample. Moreover again from the numerical data, for a direct image of the dual bundle, we have

\[
R^1 (g \cup \tilde{g})*([E_1 + E_1]|_{E_1 \cup E_1})^\otimes m = 0 \text{ for any } m > 0.
\]
Figure 3. The transformations from $Z_1$ to $Z_3$. The picture (b) is identical to (b) in Figure II. The present (b) is obtained from the original (b) by just cutting out (just for presentation) along the 2 twistor lines $L_1$ and $L_2$. So in each of (b), (c) and (d), the two $L_1$-s are identified in the direction indicated by the arrows, and the same for $L_2$-s. (e) is obtained from (d) by contracting four $(-1,-1)$-curves, and will be used later.
Therefore by a theorem of Fujiki \[4\] Theorem 2, the divisor \( E_1 \cup \bar{E}_1 \) can be contracted to \( \mathbb{CP}^1 \cup \mathbb{CP}^1 \) along the morphism \( g \cup \bar{g} \). Let \( \mu_4 : Z_3 \to Z_4 \) be the birational morphism obtained this way, and put \( l_4 := \mu_4(E_1) \) and \( \bar{l}_4 := \mu_4(\bar{E}_1) \), so that \( l_4 \cup \bar{l}_4 \) can be naturally identified with the target space of \( g \cup \bar{g} \). As the degree of the restriction of \( N_{E_1/Z_3} \) to irreducible fibers of \( g \) is \((-2)\) as above, we have \( \text{Sing} Z_4 = l_4 \cup \bar{l}_4 \), and possibly outside the 2 points \( l_4 \cap \bar{l}_4 \), \( Z_4 \) has ordinary double points along \( l_4 \cup \bar{l}_4 \). (In Section 3\[4\] we will obtain an explicit defining equation of \( Z_4 \) around the 2 points.) This way by contracting \( E_1 \cup \bar{E}_1 \) in \( Z_3 \) we have obtained a singular variety \( Z_4 \). Then the morphism \( \Phi_2 \) descends to \( Z_4 \):

**Proposition 3.5.** Let \( \Phi_2 : Z_2 \to Y \) be the generically 2 to 1 covering as before. Then \( \Phi_2 \) descends to a morphism \( Z_4 \to Y \). Namely there is a morphism \( \Phi_4 : Z_4 \to Y \) such that \( \Phi_4 \circ \mu_4 \circ \mu_3 = \Phi_2 \).

**Proof.** We first show that \( \Phi_2 \) descends to a morphism \( \Phi_3 : Z_3 \to Y \). Recall that \( \Phi_2 \) is induced by the system \( |L_2| \), where \( L_2 = \mu_3^* L_3 \) and \( L_1 = \mu_4^* 2F - E_1 - \bar{E}_1 \). In accordance with those on \( \Sigma_3 \) in the proof of Proposition 3\[4\] let \((0,1)\) be the fiber class of the projection \( E_1 \to C_1 \). Then we obtain \( L_1|_{E_1} \simeq 2\mu_4^*(F|_{C_1}) - N_{E_1/Z_1} \simeq 2(\mu_4^* K_S^{-1}|_{C_1}) - \mathcal{O}(1,0,2) \simeq 2\mu_1^* \mathcal{O}(1,2) \simeq \mathcal{O}(1,0) \). (See Figure 3\[1\](b) for \( N_{E_1/Z_1} \simeq \mathcal{O}(1,0,2) \).) Further, as the curve \( C_2 \subset Z_1 \) intersects \( E_1 \) transversally at exactly 1 point and the same for \( \bar{E}_1 \) (again see Figure 3\[1\](b)), we have \( L_2|_{C_2} \simeq (\mu_3^* 2F)|_{C_2} = \mathcal{O}(1,2) \simeq \mathcal{O}(2) \). Therefore pulling back to \( Z_2 \) we obtain \( L_2|_{E_1} \simeq \mu_2^* \mathcal{O}(1,0) \), \( L_2|_{E_2} \simeq \mathcal{O}(E_2) \), and analogous result for the restrictions to \( E_1 \) and \( \bar{E}_2 \). These imply that the direct image sheaf \( (\mu_3)_* L_3 =: L_3 \) is still invertible and \( L_3|_{E_1} \simeq \mu_2^* \mathcal{O}(1,0) \). If we use the projection \( g : E_1 \to l_4 \), the last isomorphism can be rewritten as

\[ L_3|_{E_1} \simeq g^* \mathcal{O}(l_4). \]

Then since \( L_2 \simeq \mu_3^* L_3 \), the morphism associated to \( |L_2| \) factors through the morphism associated to \( |L_3| \). Letting \( \Phi_3 \) be the last morphism, this means \( \Phi_2 = \Phi_3 \circ \mu_3 \) as claimed.

In a similar way we next show that \( \Phi_3 \) descends to a morphism \( \Phi_4 : Z_4 \to Y \). From \( (\mu_3)_* L_3 =: L_3 \) is still an invertible sheaf (on \( Z_4 \)), whose restriction to \( l_4 \) is of degree 1. Then by the natural isomorphisms \( H^0(Z_2, L_2) \simeq H^0(Z_3, L_3) \simeq H^0(Z_4, L_4) \) the map associated to \( |L_3| \) factors through the map induced by \( |L_4| \). Therefore if we define \( \Phi_4 \) to be the map associated to \( |L_4| \), we have \( \Phi_3 = \Phi_4 \circ \mu_4 \), as claimed. Hence we have obtained \( \Phi_2 = \Phi_3 \circ \mu_3 = \Phi_4 \circ \mu_4 \circ \mu_3 \).

Thus we arrived at the following situation:

\[ \begin{array}{ccc}
Z_2 & \xrightarrow{\mu_4 \circ \mu_3} & Z_4 \\
\mu_2 & \xrightarrow{\Phi_2} & \Phi_4 \\
Z_1 & \xrightarrow{\Phi_1} & Y
\end{array} \]

where all maps are morphisms and the 2 triangles are commutative. Since \( \mu_4 \circ \mu_3 \) is birational, \( \Phi_4 \) is still a degree 2 morphism branching at the divisor \( B \). But in contrast with \( \Phi_2 \), it does not contract divisors anymore:

**Proposition 3.6.** The morphism \( \Phi_4 \) does not contract any divisor to a point or a curve.

**Proof.** It is enough to show that the morphism \( \Phi_1 : Z_1 \to Y \) does not contract any irreducible divisor other than \( E_1 \) and \( \bar{E}_1 \). Let \( D \) be such a divisor. If \( D \) is real, then
$D \in |\mu_1^* (kF) - lE_1 - \overline{E}_1|$ for some $k \geq 1$ and $l \geq 0$, and $(\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot D = 0$ by the contractedness property. For computing this intersection number, we notice, as $\mu_1^* F|_{E_1} \simeq \mu_1^*(K_{S^{-1}}|_{C_1}) \simeq \mathcal{O}_{E_1}(0,-1)$, that we have $(\mu_1^* F)^2 \cdot E_1 = (\mu_1^* F|_{E_1})^2 = 0$. Hence we have
\[
(\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot E_1 = 4 \mu_1^* F^2 \cdot E_1 - 4 \mu_1^* F \cdot (E_1 + \overline{E}_1) \cdot E_1 + (E_1 + \overline{E}_1)^2 \cdot E_1
\]
\[
= 4 \cdot 0 - 4 \mu_1^* F \cdot E_1^2 + E_1^3
\]
\[
= -4 \mathcal{O}_{E_1}(0, -1) \cdot \mathcal{O}_{E_1}(-1, -2) + \mathcal{O}_{E_1}(-1, -2)^2
\]
\[
= -4 + 4 = 0,
\]
and the same for $\overline{E}_1$. From these, recalling $F^3 = 0$ (as we are over $4\mathbb{CP}^2$), we obtain
\[
(\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot D = (\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot (\mu_1^* (kF) - lE_1 - \overline{E}_1)
\]
\[
= (\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot \mu_1^* (kF)
\]
\[
= E_1^2 \cdot \mu_1^* (kF) + \overline{E}_1 \cdot \mu_1^* (kF) - 4k \mu_1^* F^2 \cdot (E_1 + \overline{E}_1)
\]
\[
= 2N_{E_1/Z_5} : \mu_1^* (kF)|_{E_1} + 2k \mathcal{O}_{E_1}(-1, -2) \cdot \mathcal{O}_{E_1}(0, -1) = 2k.
\]
Therefore we have $(\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot D > 0$. Hence $D$ is not contracted to a curve or a point by $\Phi_1$. When $D$ is not real, by applying the above computations for $D + \overline{D}$ instead of $D$, we obtain $(\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot (D + \overline{D}) > 0$. Hence, since $(\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot D = (\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot \overline{D}$, we again conclude $(\mu_1^* 2F - E_1 - \overline{E}_1)^2 \cdot D > 0$. Hence in the non-real case too, $D$ cannot be contracted to a point or a curve by $\Phi_1$, as claimed. □

As a consequence, we obtain the following

**Proposition 3.7.** The branch divisor $B$ has only isolated singularities.

**Proof.** Let $Z_4 \xrightarrow{\mu_5} Z_5 \xrightarrow{\Phi_5} Y$ be the Stein factorization of the morphism $\Phi_4$. $\mu_5$ is necessarily birational. Then $\Phi_5$ is just a double covering with branch $B$. Hence if $B$ has singularities along a curve, so is $Z_5$. Since $B$ does not contain $l$ and $\text{Sing} \ Z_4 = \mathcal{I}_4 \cup \overline{\mathcal{I}_4}$, this means that the birational morphism $\mu_5$ resolves the singularities along the curve. Hence $\mu_5$ contracts a divisor. This contradicts Proposition 3.6 □

We note that the proof of Proposition 3.6 means that the original anticanonical map $\Phi : Z \to Y$ does not contract any divisor. We also note that the morphism $\mu_5$ in the proof of Proposition 3.7 contracts (at worst) finitely many curves, and all these curves are over singular points of $B$. We will investigate these singularities in detail in Section 4.4

**4. Defining equation of the branch quartic hypersurface**

In the last section we analyzed the anticanonical system on the twistor space in detail and obtained the space $Z_4$ and a degree 2 morphism $\Phi_4 : Z_4 \to Y$ which is explicitly birational to the original anticanonical map, and which does not contract any divisor. We further showed that the branch divisor $B$ of $\Phi_4$ is a cut of $Y$ by a quartic hypersurface in $4\mathbb{CP}^4$. In this section we shall determine defining equation of this quartic hypersurface. We also determine the number of singularities of the branch divisor.
4.1. Finding double curves on $B$. Our way for obtaining the equation includes finding hyperplanes $H \subset \mathbb{CP}^4$ such that, regarding the intersection $H \cap Y$ (which is either a plane or a cone as in the beginning of Section 3.2) as a reduced divisor on $Y$, the restriction $B|_{H}\cap Y$ is a double curve (i.e. a non-reduced curve of multiplicity 2). So it is similar to the method of Poon [14] (for the case of $3\mathbb{CP}^2$), but the origin of some of the double curves is different from the case of $3\mathbb{CP}^2$.

We keep the notations from the last section. We are going to show the existence of five double curves, two of which are easy to find as we see now. First we recall there are diagrams in (3.4); $S$ is a conic in $\mathbb{CP}^2$ and for each $\lambda \in \Lambda$, $S_\lambda := f^{-1}(\lambda)$ is a member of the pencil $[F]$, i.e. $\Lambda$ is a parameter space of $|F|$. For any $\lambda \in \Lambda$, $\pi^{-1}(\lambda)$ is a plane containing the line $l$. Let $S_1 = S_1^+ + S_1^-$ and $S_2 = S_2^+ + S_2^-$ be the reducible members as in (3.1), and let 0 and $\infty$ be the points of $\Lambda$ such that $f^{-1}(0) = S_1$ and $f^{-1}(\infty) = S_2$ hold. Let $H_1$ and $H_2$ be the hyperplanes in $\mathbb{CP}^4$ which are the inverse images of the tangent line of $\Lambda$ at the points 0 and $\infty$ respectively under the projection $\pi$. Then the restrictions $H_1|_{Y}$ and $H_2|_{Y}$ are double planes, and we have $\Phi^{-1}(H_1) = 2S_1$ and $\Phi^{-1}(H_2) = 2S_2$. Letting $L_1 = S_1^+ \cap S_1^-$ and $L_2 = S_2^+ \cap S_2^-$ be the twistor lines as before, we define

\[(4.1) \quad \mathcal{C}_1 := \Phi(L_1) \quad \text{and} \quad \mathcal{C}_2 := \Phi(L_2).\]

Then since $\Phi$ is a real map and 0 and $\infty$ are real points, and since $\Phi$ does not contract any divisor by Proposition 3.6, $S_1^+$ and $S_1^-$ are mapped birationally to the plane $\pi^{-1}(0)$ and $S_2^+$ and $S_2^-$ are mapped birationally to the plane $\pi^{-1}(\infty)$. Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are contained in the branch divisor $B$ in such a way that the restrictions $B|_{H_i\cap Y}$ and $B|_{H_2\cap Y}$ respectively contain $\mathcal{C}_1$ and $\mathcal{C}_2$ by multiplicity 2. But as we know that $B$ is a cut of $Y$ by a hyperquartic surface by Proposition 3.4, $\mathcal{C}_1$ and $\mathcal{C}_2$ must be conics. So we call these two double conics.

These double conics are analogous to the 4 conics contained in the coordinates tetrahedron in $\mathbb{CP}^3$ used in [14] in the case of $3\mathbb{CP}^2$, but in the present case there are only 2 since there are only 2 reducible members of $|F|$. Next we find other 3 double curves. For this recall that our twistor space $Z$ contains the surface $S$ constructed in Section 2 as a real member of $|F|$. Take up the blowup $\epsilon : S \to \mathbb{CP}^1 \times \mathbb{CP}^1$ which was concretely given when constructing $S$, and let $e_i$ and $\overline{e}_i$, $1 \leq i \leq 4$, be the exceptional curves of the blowup. Here we take indices such that $e_i \cdot C_1 = 1$ for $i = 1, 2, 3$ and $e_4 \cdot C_2 = 1$. Next let $\{e_i \mid 1 \leq i \leq 4\}$ be an orthonormal basis of $H^2(4\mathbb{CP}^2, \mathbb{Z})$ determined by $\{e_i\}$; namely letting $t : Z \to 4\mathbb{CP}^2$ be the twistor fibration, $t^*\alpha_i|_S = e_i - \overline{e}_i$ in $H^2(S, \mathbb{Z})$. Then we have the following proposition which is technically significant for our purpose:

**Proposition 4.1.** For each $i$ with $1 \leq i \leq 3$ (not 4), the linear systems $|F + t^*\alpha_i|$ and $|F - t^*\alpha_i|$ consist of a single element. Moreover, all these 6 divisors are irreducible.

**Proof.** In this proof for simplicity we write $\alpha_i$ for $t^*\alpha_i$. It is enough to prove the claim for the system $|F + \alpha_i|$. Fix any $i$ with $1 \leq i \leq 3$. We have $(F + \alpha_i)|_S = K_S^{-1} + (e_i - \overline{e}_i) = e^*\mathcal{O}(2, 2) - \sum_{j=1}^{4}(e_j + \overline{e}_j) + (e_i - \overline{e}_i)$, which can be rewritten as

\[
\left(e^*\mathcal{O}(1, 0) - \sum_{1 \leq j \leq 3} \overline{e}_j\right) + \left(e^*\mathcal{O}(1, 2) - \overline{e}_i - e_4 - \overline{e}_4 - \sum_{1 \leq j \leq 3, j \neq i} e_j\right).
\]

The intersection number between $\overline{e}_1 \sim e^*\mathcal{O}(1, 0) - \sum_{j=1}^{3} \overline{e}_j$ and the above $(F + \alpha_i)|_S$ is easily computed to be $-2$. Hence $\overline{e}_1$ is a fixed component of $|(F + \alpha_i)|_S|$. Further counting dimension, the remaining system $e^*\mathcal{O}(1, 2) - \overline{e}_i - e_4 - \overline{e}_4 - \sum_{j=1, j \neq i}^{3} \overline{e}_j$ consists of a single member, which is the strict transform of a $(1, 2)$-curve passing through the 5 points
The unique anticanonical curve (i.e. the cycle of 4 rational curves). On the other hand we have the half of Chern classes of these divisors are given by irreducibility. (On the other hand, by (4.2), the systems $S_z, Z, F$ of reducible members of $|F|$ are degree 1 on $Z$. Moreover, it is not difficult to show that the Chern classes of these divisors are given by the half of the following classes:

$$F - \sum_{1 \leq j \leq 4} \alpha_j, \quad F + \sum_{1 \leq j \leq 4} \alpha_j, \quad F + \alpha_4 - \sum_{1 \leq j \leq 3} \alpha_j, \quad F - \alpha_4 + \sum_{1 \leq j \leq 3} \alpha_j.$$ 

Then it is a easy to check that a sum of any two of these 4 classes (allowing to choose the same one) are not equal to $2(F + \alpha_i)$, for any $1 \leq i \leq 3$. This implies the desired irreducibility. (On the other hand, by (4.2), the systems $|F \pm \alpha_i|$ are also non-empty, but both of them consist of a single reducible member.)

In the following for $1 \leq i \leq 3$ we denote by $X_i$ for the unique member of $|F + t^i \alpha_i|$. Then $X_i \in |F - t^i \alpha_i|$, and $X_i + \bar{X}_i \in |2F|$. Thus we obtained 3 reducible real members of the anticanonical system on $Z$. We remark that from the proof of Proposition 4.1 these 3 members originally come from the choice of 3 points on $C_1$ in the construction of $S$ at the beginning of Section 2. By using these we obtain a special basis of $H^0(2F) \simeq \mathbb{C}^5$ as follows:

**Proposition 4.2.** For any $i$ with $1 \leq i \leq 3$, let $\xi_i \in H^0(Z, 2F)$ be an element such that $(\xi_i) = X_i + \bar{X}_i$. Then $S^2H^0(Z, F) \simeq \mathbb{C}^5$ and any two among $\{\xi_i | 1 \leq i \leq 3\}$ generate $H^0(2F) \simeq \mathbb{C}^5$.

**Proof.** Let $S \in |F|$ be any real irreducible member and take $s_0 \in H^0(F)$ with $(s_0) = S$. Let $s_1 \in H^0(F)$ be any element satisfying $s_1 \not\in \mathbb{C}s_0$. Then $\{s_0, s_1\}$ is a basis of $H^0(F)$ and $\{s_0^2, s_0s_1, s_1^2\}$ is a basis of $S^2H^0(F)$. We consider the exact sequence

$$0 \rightarrow H^0(F) \otimes s_0 \rightarrow H^0(2F) \rightarrow H^0(2K_S^{-1}) \rightarrow 0$$

appeared in the proof of Proposition 3.2. For proving the claim of the proposition, it suffices to show that for any subset $\{i, j\} \subset \{1, 2, 3\}$, the images of $s_1^2, \xi_i, \xi_j$ by the restriction map to $S$ generate $H^0(2K_S^{-1})$. For this, the divisor $(s_1^2|_S)$ is exactly $2C$, where $C$ is the unique anticanonical curve (i.e. the cycle of 4 rational curves). On the other hand we have $(\xi_i|_S) = X_i|_S + \bar{X}_i|_S$, and from the proof of Proposition 1.1 we know the curves $X_i|_S$ and $\bar{X}_i|_S$ in concrete forms, and it is not difficult to verify that the 3 bi-anticanonical curves $2C, X_i|_S + \bar{X}_i|_S, X_j|_S + \bar{X}_j|_S$ are linearly independent. Hence the 3 images generate $H^0(2K_S^{-1})$.

In the sequel for obtaining nice coordinates, we choose a slightly different basis $\{u_1, u_2\}$ of $H^0(Z, F)$ as follows. Namely respecting the reducible members, we choose those satisfying $(u_1) = S_1$ and $(u_2) = S_2$. By Proposition 4.2 the collection $\{u_1u_2, u_1^2, u_2^2, \xi_1, \xi_2\}$ is a basis of $H^0(Z, 2F)$. The target space of the anticanonical map $\Phi : Z \rightarrow \mathbb{C}P^4$ is nothing but the dual projective space $\mathbb{P}H^0(Z, 2F)^* \simeq \mathbb{C}P^4$, and if we put

$$z_0 := u_1u_2, \quad z_1 := u_1^2, \quad z_2 := u_2^2, \quad z_3 := \xi_1, \quad z_4 := \xi_2,$$
then \((z_0, z_1, z_2, z_3, z_4)\) can be used as homogeneous coordinates on it. (Here we remark that there is no special reason to choose \(\xi_1\) and \(\xi_2\). Any choice of two among \(\{\xi_1, \xi_2, \xi_3\}\) leads to the same description below.) As \(\mathbb{CP}^4 = \mathbb{P}H^0(2F)^*\) as above, \(\mathbb{CP}^4\) is equipped with a real structure and by (4.4) it is just the complex conjugation with respect to the above coordinates. In these coordinates the scroll \(Y = \Phi(Z)\) is explicitly defined by the equation

\[
z_0^2 = z_1 z_2,
\]

and the ridge \(l\) of \(Y\) is given by \(z_0 = z_1 = z_2 = 0\). For each \(0 \leq i \leq 4\) we define a hyperplane by \(H_i := \{z_i = 0\}\). Obviously \(l \subset H_i\) for \(i = 0, 1, 2\), and \(l \not\subset H_i\) for \(i = 3, 4\). In particular \(H_1|_Y\) and \(H_2|_Y\) are double planes, \(H_0|_Y\) is the sum of these 2 planes, and \(H_3|_Y\) and \(H_4|_Y\) are cones whose vertices are the points \(H_3 \cap l\) and \(H_4 \cap l\) respectively.

Let \(z_5 \in H^0(2F, Z)\) be an element such that \((z_5) = X_3 + X_3\). Then by Proposition 4.2 we can write

\[
z_5 = a_0 z_0 + a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4
\]

for some \(a_i \in \mathbb{R}\). Let \(H_5 := \{z_5 = 0\}\). Since \(z_5 \not\in S^2 H^0(Z, F)\) clearly, \(H_5|_Y\) is also a cone.

Now the following proposition provides the promised 3 double curves on \(B\):

**Proposition 4.3.** For \(i = 3, 4, 5\), the intersection of the branch divisor \(B\) with the cone \(H_i \cap Y\) is a double curve of \(B\).

**Proof.** Let \(i\) be any one of 3, 4, 5. By definition of the hyperplane \(H_i\), we have \(\Phi^{-1}(H_i) = X_{i-2} + X_{i-2}\). Since \(\Phi\) does not contract any divisor by Proposition 3.6 this implies \(\Phi(X_{i-2}) = \Phi(X_{i-2}) = H_i \cap Y\). As \(\Phi\) is degree 2, this means that the restrictions \(\Phi|_{X_{i-2}}\) and \(\Phi|_{X_{i-2}}\) are birational over the cone \(H_i|_Y\). (In particular, the non-real degree 2 divisors \(X_{i-2}\) and \(X_{i-2}\) are birational to a cone.) Therefore the curve \(X_{i-2} \cap X_{i-2}\) is the ramification divisor of the restriction \(\Phi|_{\Phi^{-1}(H_i)} : \Phi^{-1}(H_i) \to H_i \cap Y\), and hence

\[
\mathcal{C}_3 := \Phi(X_1 \cap X_1), \quad \mathcal{C}_4 := \Phi(X_2 \cap X_2) \quad \text{and} \quad \mathcal{C}_5 := \Phi(X_3 \cap X_3)
\]

are branch divisors when restricted to \(\Phi^{-1}(H_i)\). This implies the claim of the proposition. \(\square\)

As in the proof, we use the letters \(\mathcal{C}_3, \mathcal{C}_4\) and \(\mathcal{C}_5\) to mean the 3 double curves in the proposition. Then because we know that \(B\) is a cut of \(Y\) by a quartic hypersurface, we have \(\mathcal{C}_i \in |\mathcal{O}_Y|_H(2)|\), where \(\mathcal{O}_{Y|H}(2) := \mathcal{O}_{\mathbb{CP}^4}(2)|_Y|_H\). Namely, \(\mathcal{C}_3, \mathcal{C}_4\) and \(\mathcal{C}_5\) are intersection of the cone \(Y \cap H_i\) with a quadratic in \(H_i = \mathbb{CP}^3\). From this it follows that these 3 curves are of degree 4 in \(\mathbb{CP}^4\). So in the following we call these double curves double quartic curves.

From our choice of the coordinates, any double curves can be written as , as sets,

\[
(4.7) \quad \mathcal{C}_i = B \cap H_i, \quad 1 \leq i \leq 5.
\]

Since \(B\) and \(H_i\) are real, all \(\mathcal{C}_i\)-s are real curves.

**4.2. Quadratic hypersurfaces containing the double curves.** In this section we show that there exists a quadratic hypersurface in \(\mathbb{CP}^4\) which contains the double conics \(\mathcal{C}_1\), \(\mathcal{C}_2\) and the double quartic curves \(\mathcal{C}_3, \mathcal{C}_4\) and \(\mathcal{C}_5\), and also show that such hyperquadric is unique up to the defining equation of the scroll \(Y\).

First we make it clear how the 5 double curves of \(B\) intersect each other. For this for each \(0 \leq i \leq 4\) we denote by \(e_i \in \mathbb{CP}^4\) for the point whose coordinates are zero except \(z_i\)-component, and define some lines as follows: for each pair \((i, j)\) with \(i = 1, 2\)
and \( j = 3, 4, 5 \), define \( l_{ij} := H_i \cap Y \cap H_j \). Since \( H_i \cap Y \) is a (double) plane for \( i = 1, 2 \), this is a line. Thus we get 6 lines. If \( j \neq 5 \), these are coordinate lines and in Figure 4, \( l_{14} = e_2 e_3 \), \( l_{13} = e_2 e_4 \), \( l_{24} = e_1 e_3 \), \( l_{23} = e_1 e_4 \). (We do not write pictures of \( l_{15} \) and \( l_{25} \) because these are not coordinate lines. But this is just a matter of a choice of coordinates and these two play the same role as other 4 lines.) Also, for the ridge \( l \) we have \( l = e_3 e_4 \) (the bold line on the left picture in Figure 4).

Then since the intersection of the branch divisor \( B \) with the plane \( H_i \cap Y = \{ z_0 = z_i = 0 \} \) (\( i = 1, 2 \)) is the double conics \( C_i \) and since the line \( l_{ij} \) (\( 3 \leq j \leq 5 \)) is contained in this plane, the intersection \( B \cap l_{ij} \) consists of (not 4 but) 2 points, and \( B \cap l_{ij} = C_i \cap l_{ij} \). Moreover, as \( C_i = \Phi(L_i) \), these 2 points cannot be identical. Thus for each of the 6 lines \( l_{ij} \), \( B \cap l_{ij} \) consists of 2 points. (In Figure 4 these points are represented by numbered points \( 1, 2, \ldots, 7, 8 \).) On the other hand, we have \( B \cap H_j = C_j \). Hence as \( l_{ij} \subset H_j \), we obtain \( B \cap l_{ij} \subset C_j \cap l_{ij} \) for \( i = 1, 2 \) and \( j = 3, 4, 5 \). But since \( C_j \) is an intersection of \( Y \cap H_j \) with a quadric surface in \( H_j \), \( C_i \cap l_{ij} \) consists at most 2 points. Hence we have the coincidence \( B \cap l_{ij} = C_j \cap l_{ij} \) for these \( i \) and \( j \). Therefore we have \( B \cap l_{ij} = C_i \cap C_j \) for \( i = 1, 2 \) and \( j = 3, 4, 5 \). By a similar reason, the intersection \( B \cap l \) also consists of 2 points, which are exactly \( C_1 \cap C_2 \).

In Figure 4 these points are displayed as \( p_2 \) and \( \overline{p}_2 \). On the other hand, for each pair \((j, k)\) with \( 3 \leq j < k \leq 5 \) we define a plane \( P_{jk} \) by \( P_{jk} = H_j \cap H_k \). Then since \( B \cap H_j \) is contained in a quadric surface, and \( Y \cap P_{jk} \) is a conic, \( B \cap P_{jk} \) (\( 3 \leq j < k \leq 5 \)) consists of 4 points, and it coincides with \( C_j \cap C_k \). In Figure 4 for the case \((j, k) = (3, 4)\), these are represented by numbered points \( 9, 10, 11, 12 \). (For avoiding confusion we do not write a picture for \( C_3 \cap C_5 \) and \( C_1 \cap C_5 \). The way how these curves intersect is completely analogous to that of \( C_3 \) and \( C_4 \).)

We list all these intersections:

- 2 points \( C_1 \cap C_2 \), which are exactly \( p_2 \) and \( \overline{p}_2 \).
• 12 points \( C_i \cap C_j \) with \( i = 1, 2 \) and \( j = 3, 4, 5 \).
• 12 points \( C_3 \cap C_4, \ C_3 \cap C_5 \) and \( C_4 \cap C_5 \).

Collecting these, we obtain 26 points in total. Since all the double curves are the image of curves in \( Z \) by a map which is degree 1 on these curves, these 26 points form 13 conjugate pairs. Among these 26 points, the 2 points \( C_1 \cap C_2 \) are on the singular locus \( l \) of \( Y \), and other 24 points are ordinary double points of \( B \). (In some sense these 24 points are analogous to the 12 ordinary double points of the branch quartic surface appeared in \([16]\) and \([12]\).) In Section \([4.4]\) we will show that the 2 points \( p_2 \) and \( \overline{p}_2 \) are \( A_3 \) singular points of \( B \).

With these situation in hand, we next show the existence of a hyperquadric which contains all the double conics:

**Proposition 4.4.** There exists a real quadratic hypersurface in \( \mathbb{C}P^4 \) which contains all the 5 double curves \( C_i, 1 \leq i \leq 5 \), and which is different from the scroll \( Y \). Moreover, such a hyperquadric is unique in the following sense: if \( Q \) and \( Q' \) are defining quadratic polynomials of two such hyperquadrics, then there exists \((c, c') \in \mathbb{R}^2 \) with \((c, c') \neq (0, 0)\) such that \( cQ - c'Q' \in (z_0^2 - z_1z_2) \). (Note that since the scroll \( Y \) contains all the double curves, presence of this ambiguity is obvious from the beginning.)

**Proof.** As we have already seen, for \( i = 3, 4 \) the intersection \( H_i \cap Y \) is a quadratic cone in \( H_i = \mathbb{C}P^3 \), and the double quartic curve \( C_i \) belongs to \( |\mathcal{O}_{H_i \cap Y}(2)| \). In the above homogeneous coordinates the intersection \( H_3 \cap H_4 \) is a plane defined by \( z_3 = z_4 = 0 \), and \( H_3 \cap H_4 \cap Y \) is a conic defined by \( z_0^2 = z_1z_2 \), on which the 4 points \( C_3 \cap C_4 \) lie. Conics on the plane passing through these 4 points form a pencil, which is invariant under the real structure. Choose any real one of such conics, and let \( q(z_0, z_1, z_2) \) be its defining equation with real coefficients, which is of course uniquely determined up to rescaling. Among the above 26 points there are exactly 8 points lying on \( H_3 \) (which are the points 3, 4 and 7 to 12 in Figure \([4]\), four of which are the above 4 points on \( H_3 \cap H_4 \) (the points 9, 10, 11, 12 in Figure \([4]\). Any quadratic polynomial on \( H_3 \) whose restriction to \( H_3 \cap H_4 \) equals \( q \) is of the form

\[
Q_3 = q(z_0, z_1, z_2) + a_0z_0z_3 + a_1z_1z_3 + a_2z_2z_3 + a_3z_3^2.
\]

Imposing that the quadric \((Q_3)\) passes the remaining 4 points \((3, 4, 7, 8)\), the coefficients \(a_0, a_1, a_2\) and \(a_3\) are uniquely determined (without an ambiguity of rescaling), and they are real since the set of 4 points \((3, 4, 7, 8)\) is real. Then since elements of \( |\mathcal{O}_{H_3 \cap Y}(2)| \) which go through the 8 points is unique by dimension counting, it follows that the quadratic surface \((Q_3)\) automatically contains \( C_3 \). The situation is the same for \( H_4 \), and let

\[
Q_4 = q(z_0, z_1, z_2) + b_0z_0z_4 + b_1z_1z_4 + b_2z_2z_4 + b_3z_4^2.
\]

be the quadratic polynomial on \( H_4 \) with real coefficients, which is uniquely determined by \( q \) and the 8 points \((1, 2, 5, 6 \text{ and } 9 \text{ to } 12)\) on \( H_4 \). Then \((Q_4) \supset C_4 \).

Since \( Q_3|_{H_3 \cap H_4} = Q_4|_{H_3 \cap H_4} (= q) \), the pair \((Q_3, Q_4)\) is naturally regarded as a section of a line bundle \( \mathcal{O}_{H_3 \cup H_4}(2) \). By the exact sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{C}P^4} \otimes z_3^3 \otimes \mathcal{O}_{\mathbb{C}P^4}(2) \rightarrow \mathcal{O}_{H_3 \cup H_4}(2) \rightarrow 0,
\]

we obtain that the section \((Q_3, Q_4)\) can be extended to a quadratic polynomial on \( \mathbb{C}P^4 \), and that such polynomial is unique up to adding a constant multiple of \( z_3z_4 \). Explicitly such an extension has to be of the form

\[
Q(z_0, z_1, z_2, z_3, z_4) := q(z_0, z_1, z_2) + \sum_{0 \leq i \leq 3} a_i z_i z_3 + \sum_{0 \leq i \leq 3} b_i z_i z_4 + cz_3 z_4,
\]
where \( c \) is an arbitrary constant. Of course, the hyperquadric satisfies \((Q) \supset \mathcal{C}_3 \cup \mathcal{C}_4\). Then if we further impose that \((Q)\) contains the point \( p_2 \in \mathcal{C}_1 \cap \mathcal{C}_2 \subset l = \{z_0 = z_1 = z_2 = 0\}\), then from (4.10) a linear equation for \( a_3, b_3, c \) is obtained, from which \( c \) is uniquely determined. \( c \) is real since \( a_3 \) and \( b_3 \) are real. Summarizing up, we have obtained that once we fix a real quadratic polynomial \( q(z_0, z_1, z_2) \), then there exists a unique real quadratic polynomial \( Q(z_0, z_1, z_2, z_3, z_4) \) whose restriction to \( H_3 \cap H_4 \) equals \( q \) and which goes through the 9 points (1 to 8 and \( p_2 \)). In particular, if \( q = z_0^2 - z_1 z_2 \), then \( Q = z_0^2 - z_1 z_2 \).

Next we show that this polynomial \( Q \) (which is uniquely determined from \( q \)) always contains the double curves \( \mathcal{C}_i \) for any \( 1 \leq i \leq 5 \). It remains to show \( \mathcal{C}_i \subset (Q) \) for \( i = 1, 2, 5 \). For \( i = 1, 2 \) this is immediate since \((Q)\) already goes through 5 points on the conic \( \mathcal{C}_i \), which means that it goes through the remaining 1 point \( p_2 \). \( \mathcal{C}_5 \subset (Q) \) is also immediate if we notice that as \( \mathcal{C}_i \subset (Q) \) for \( 1 \leq i \leq 4 \), \((Q)\) already goes through all the 12 points \( \mathcal{C}_i \cap \mathcal{C}_5 \) for \( 1 \leq i \leq 4 \) on \( H_5 \) among the 26 points obtained in Section 4.2 and that since \( h^{\mathcal{C}_5}(\mathcal{C}_H \cap Y(2)) = 9 \), eight points already and uniquely determine the quadric. Thus we have proved the existence of a quadratic polynomial \( Q \) satisfying \( \mathcal{C}_i \subset (Q) \) for any \( 1 \leq i \leq 5 \).

For the uniqueness in the sense of the proposition, let \( Q \) and \( Q' \) be as stated in the proposition. Then since both \( Q|_{H_5 \cap H_4} \) and \( Q'|_{H_5 \cap H_4} \) are real and belong to the pencil (determined from the 4 points \( \mathcal{C}_3 \cap \mathcal{C}_4 \)), there exists \((c, c') \in \mathbb{R}^2 \) with \((c, c') \neq (0, 0)\) such that \( cQ - c'Q'|_{H_5 \cap H_4} \in (z_0^2 - z_1 z_2) \). Further, the hyperquadric \((cQ - c'Q')\) goes through the points 1 to 8 and \( p_2 \) at least, and therefore must belong to the ideal \((z_0^2 - z_1 z_2)\) by the uniqueness which was already proved. Thus we get the required uniqueness.

4.3. **Defining equation of the branch divisor.** With the results in the previous 2 subsections, we are ready to provide the main result in this paper:

**Theorem 4.5.** Let \( Z \) be any twistor space on \( 4\mathbb{CP}^2 \) containing the surface \( S \) (constructed in Section 2) as a real member of \(|F|\). Let \( \Phi_4 : Z_4 \to Y \) be the generically 2 to 1 morphism canonically obtained from the explicit birational transformations in Section 3, and \( B \) the branch divisor of \( \Phi_4 \). Let \( Q(z_0, z_1, z_2, z_3, z_4) \) be a defining equation of the hyperquadric containing all the 5 double curves, obtained in Proposition 4.4. Then \( B \) is an intersection of the scroll \( Y = \{z_0^2 = z_1 z_2\} \) with the quartic hypersurface defined by the equation of the form

\[
(4.11) \quad z_0 z_3 z_4 f(z_0, z_1, z_2, z_3, z_4) = Q(z_0, z_1, z_2, z_3, z_4)^2
\]

where \( f(z_0, z_1, z_2, z_3, z_4) \) is a linear polynomial with real coefficients.

**Proof.** By Proposition 4.4 there exists a real hyperquadric such that the intersection with \( Y \) is the branch divisor \( B \). Let \( \mathcal{B} \subset \mathbb{CP}^4 \) be any one of such hyperquadrics and \( F = F(z_0, z_1, z_2, z_3, z_4) \) a defining equation of \( \mathcal{B} \). We note that \( \mathcal{B} \) is not unique in the sense that \( F \) is determined only up to quartic polynomials in the ideal \((z_0^2 - z_1 z_2)\). Then for \( i = 1, 2 \), the restriction of \( \mathcal{B} \) to a plane \( H_i \cap Y = \{z_0 = z_i = 0\} \) is the twice of the double conic \( \mathcal{C}_i \) (see (4.7)). Also, by the choice of \( Q \), the restriction of the hyperquadric \((Q)\) to the same plane is \( \mathcal{C}_i \). These two mean that there exists a constant \( c_i \in \mathbb{C} \) such that \( F - c_i Q^2 \) belongs to the ideal \((z_0, z_i)\). Namely there exist cubic polynomials \( f_i \) and \( g_i \) (in \( z_0, z_1, z_2, z_3, z_4 \)) satisfying

\[
(4.12) \quad F - c_i Q^2 = z_0 f_i + z_i g_i \quad (i = 1, 2).
\]

Taking the difference, we obtain

\[
(4.13) \quad (c_1 - c_2) Q^2 = z_0 (f_1 - f_2) + z_1 g_1 - z_2 g_2.
\]
If $c_1 \neq c_2$, substituting $z_0 = z_1 = 0$, the hyperquadric $(Q)$ restricted to the plane $\{z_0 = z_1 = 0\}$ is defined by $z_2g_2 = 0$. This means that $\mathcal{C}_1$ is reducible, which cannot happen since it is the image of the twistor line $L_1$ (see (4.11)). Hence we obtain $c_1 = c_2$. Similarly, for $i = 3, 4$, considering the restrictions of $F$ and $Q^2$ to the cone $H_i \cap Y = \{z_i = z_0^2 - z_1z_2 = 0\}$, again by coincidence, there exist a constant $c_i \in \mathbb{C}$, a cubic polynomial $f_i$ and a quadratic polynomial $g_i$ satisfying

\[
(4.18) \quad F - c_iQ^2 = z_if_i + (z_0^2 - z_1z_2)g_i \quad (i = 3, 4).
\]

By (4.12) with $i = 1$ and (4.14) we obtain

\[
(4.15) \quad (c_3 - c_1)Q^2 = z_0f_1 + z_1g_1 - z_if_i - (z_0^2 - z_1z_2)g_i \quad (i = 3, 4).
\]

From this we again obtain that the hyperquadric $\{Q = 0\}$ restricted to the plane $\{z_0 = z_1 = 0\}$ is given by $\{z_if_i = 0\}$, contradicting the irreducibility of $\mathcal{C}_1$. Hence we obtain $c_3 = c_1$ for $i = 3, 4$. Thus we get $c_1 = c_2 = c_3 = c_4$. If $c_1 = 0$, by (4.12), we have $F = z_0f_1 + z_1g_1$. But this cannot happen since this means $\mathcal{B} \supset \{z_0 = z_1 = 0\}$, contradicting $\mathcal{B} \cap \{z_0 = z_1 = 0\} = \mathcal{C}_1$. Hence $c_1 \neq 0$. So replacing $Q$ with $Q/\sqrt{c_1}$, we may assume that all the four $c_i$-s in (4.12) and (4.14) are one.

Next in the expression (4.12) we take $f_i$ and $g_i$ in such a way that $g_i$ does not contain $z_0$. Then since the right hand side of (4.13) is zero, $z_1g_1 - z_2g_2 = 0$ follows. Hence $g_1 \in \langle z_2 \rangle$, and we can write $g_1 = z_2h_1$ by a quadratic polynomial $h_1$ which does not contain $z_0$. Similarly, in the expression (4.14) we take $f_i$ and $g_i$ in such a way that $f_3$ and $f_4$ do not belong to the ideal $(z_0^2 - z_1z_2)$. Then this time from (4.14) for the case $i = 3$ and $i = 4$, we obtain

\[
(4.16) \quad (z_3f_3 - z_4f_4) + (z_2^3 - z_1z_2)(g_3 - g_4) = 0.
\]

From the choice of $f_3$ and $f_4$, it follows $f_3 \in \{z_3\}$ and $f_4 \in \{z_3\}$. Hence we can put $f_3 = z_4f_5$ for some quadratic polynomial $f_5$. From these, we obtain

\[
(4.17) \quad F - Q^2 = z_0f_1 + z_1z_2h_1 = z_3z_4f_5 + (z_0^2 - z_1z_2)g_3.
\]

Then since $h_1$ does not contain $z_0$, from the latter equality we can readily deduce that if we write $f_5 = z_0f_6 + f_7$ in a way that $f_7$ does not contain $z_0$, then $f_7$ is a multiple of $z_1z_2$, so that $f_5 = z_0f_6 + cz_1z_2$ for some $c \in \mathbb{C}$. Hence by (4.17) we obtain

\[
(4.18) \quad F - Q^2 = z_3z_4(z_0f_6 + cz_1z_2) + (z_0^2 - z_1z_2)g_3.
\]

Defining a linear polynomial $f_8$ by $f_6 = -cz_0 + f_8$ (so that $f_8$ may contain $z_0$) and substituting into (4.18), we finally get

\[
(4.19) \quad F - Q^2 = z_0z_3z_4f_8 + (z_0^2 - z_1z_2)(g_3 - cz_3z_4).
\]

Thus we obtain $F = Q^2 + z_0z_3z_4f_8 + (z_0^2 - z_1z_2)(g_3 - cz_3z_4)$. Hence modulo quartic polynomials in the ideal $(z_0^2 - z_1z_2)$, $\mathcal{B}$ is defined by the equation of the form (4.11). This completes a proof of the theorem. \qed

From the quartic equation (4.11) it is immediate to see that the intersection of the hyperquadric $(Q)$ and the hyperplanes $H_3$ and $H_4$ are double quadric surfaces, and this is of course consistent with the fact that the restrictions $B|_{H_3 \cap Y}$ and $B|_{H_4 \cap Y}$ are double curves. On the other hand, for $i = 1, 2$, in order to see that $B|_{H_i \cap Y}$ are also double curves from the equation, we just need to notice that, on the scroll $Y$, $z_i = 0$ means $z_0 = 0$.

In comparison with the case of $3\mathbb{CP}^2$, appearance of the linear polynomial $f$ in our defining equation (4.11) might look strange at first sight. As the following proposition shows, $f$ comes from the fifth double curve $\mathcal{C}_5$, which does not exist in the case of $3\mathbb{CP}^2$. 


Proposition 4.6. Up to non-zero constants, the linear polynomial \( f \) in (4.11) is exactly \( z_0 \) we have defined in (4.6). In other words, for the third double quartic curve, we have \( \mathcal{C}_3 = \{ f = Q = z_0^2 - z_1 z_2 = 0 \} \).

Proof. We will find all hyperplanes in \( \mathbb{CP}^4 \) (which is the target space of the anticanonical map \( \Phi \)), which correspond to reducible members of the anticanonical system \( |2F| \). First as above for any hyperplane \( H \) defined by the equation of the form \( a_0 z_0 + a_1 z_1 + a_2 z_2 = 0 \), the corresponding member \( \Phi^{-1}(H) \in |2F| \) is clearly reducible. (We are including a non-reduced case.) Also, \( \Phi^{-1}(H_3) \) and \( \Phi^{-1}(H_4) \) are reducible since \( B \cap H_3 \) and \( B \cap H_4 \) are double curves. By the same reason, if \( H_f = \{ f = 0 \} \), the divisor \( \Phi^{-1}(H_f) \) is reducible. Then recalling that the double quartic curve \( \mathcal{C}_3 \) is obtained as an image of the third reducible member of \( |2F| \) obtained in Proposition 4.1, in order to prove the claim of the proposition, it suffices to show that there exists no other hyperplane \( H \) such that \( \Phi^{-1}(H) \) is reducible.

If \( H \) is such a hyperplane, then either \( Y \cap H \) is reducible, or \( Y \cap H \) is irreducible (i.e. a cone) and \( B|_{Y \cap H} \) is a double curve. The former occurs exactly when \( H \) is defined by the equation of the form \( a_0 z_0 + a_1 z_1 + a_2 z_2 = 0 \). So suppose the latter happens. Then recalling \( B = \mathcal{R} \cap Y \) and \( H|_Y \) is reduced by the assumption, \( B|_{Y \cap H} \) can be a double curve only when \( \mathcal{R}|_H \) is a double surface. We show by algebraic mean that this happens only when \( H \) is defined by one of the 4 factors of the left-hand side of (4.11).

If \( \mathcal{R}|_H \) is a double surface, there exists a quadratic polynomial \( q \) on \( H \) such that \((z_0 z_3 z_4 f - Q^2)|_H = q^2\); namely
\[
(4.20) \quad z_0 z_3 z_4 f |_H = (Q|_H)^2 + q^2.
\]
If \( H \) is defined by the equation of the form \( z_i = b_0 z_0 + b_2 z_2 + b_3 z_3 + b_4 z_4 \), then even after substitution the left-hand side of (4.20) does not have monomial of the form \( z_i^3 z_j \) for any \( 1 \leq i, j \leq 4 \). Therefore the coefficient of \( z_i^3 z_j \) of the right-hand side of (4.20) must be zero for any \( 1 \leq i, j \leq 4 \). By an elementary argument, it is possible to show this can happen only when \( (Q|_H)^2 + q^2 = 0 \). Hence by (4.20), \( H \) is equal to one of \( H_0, H_3, H_4 \) and \( H_f \). By symmetry of the equation, if \( H \) is defined by a equation of the form \( z_2 = b_0 z_0 + b_1 z_1 + b_3 z_3 + b_4 z_4 \), then (4.20) is possible only when \( H \) is one of \( H_0, H_3, H_4 \) and \( H_f \). If \( H \) is of the form \( z_0 = b_3 z_3 + b_4 z_4 \), then the left-hand side of (4.20) cannot contain monomials of the form \( z_i^3 z_j \), \( z_i^2 z_i \), \( z_i z_i^2 \) (for any \( i \)), and \( z_i z_j z_i z_i \), \( z_i z_i z_j z_i \), \( z_i^3 z_i z_i \). Therefore the coefficients of these monomials of the right-hand side of (4.20) must vanish. From these, again by an elementary argument it is possible to show that \( (Q|_H)^2 + q^2 = 0 \). Hence again \( H \) has to be one of \( H_0, H_3, H_4 \) and \( H_f \). The remaining 2 cases immediately follow from symmetry of the equation. Thus we have shown that \( \mathcal{R}|_H \) is a double surface only when \( H \) is one of \( H_0, H_3, H_4 \) and \( H_f \). \( \square \)

We again emphasize that the role of the 3 double quartic curves is symmetric, and any choice of two leads to the equation of the form of (4.11).

Remark 4.7. One may wonder whether the linear polynomial \( f \) can be taken as one of the homogenous coordinates on \( \mathbb{CP}^4 \). At least in generic situation this is possible, but if we do so, we lose simplicity of the defining equation of the scroll \( Y \), and it makes more difficult the counting the number of effective parameters in defining equations of \( \mathcal{R} \cap Y \) which will be done in Section 5.1.

4.4. The number of singularities of the branch divisor. In this subsection by using the quartic equation obtained in the previous subsection we determine the number of singularities of the branch divisor of the double covering. Similarly to the method by Kreussler
and Kreussler-Kurke [12], we resort to topology; more precisely we compute the Euler number of the relevant spaces to determine the number of singularities. Though we require much more complicated computation than the case of $3\mathbb{CP}^2$, we do it since this result is crucial for determining the dimension of the moduli space of the present twistor spaces.

As in the proof of Proposition 3.7 let $Z_4 \overset{\Phi_4}{\to} Z_5 \overset{\Phi_5}{\to} Y$ be the Stein factorization of the degree 2 morphism $\Phi_4 : Z_4 \to Y$. We already know that $\mu_5$ contracts finitely many curves, whose images are singular points of the branch divisor $B$. If we put $l_5 = \mu_5(l_4)$ and $\overline{l}_5 = \mu_5(\overline{l}_4)$, $Z_5$ also has ordinary double points along $l_5 \cup \overline{l}_5$. All other singularities of $Z_5$ are lying on singularities of the branch divisor $B$. Among these singularities we already know that there are 26 singularities listed in Section 4.2 and 24 points among them are ordinary double points. The 2 points excluded here are exactly the points $\mathcal{C}_1 \cap \mathcal{C}_2$, for which we still denote by $p_2$ and $\overline{p}_2$ (see Figure 4 again). We begin with determining the type of singularities of these 2 points:

**Proposition 4.8.** At the 2 points $p_2$ and $\overline{p}_2$, the branch divisor $B$ has $A_3$-singularities.

**Proof.** Recall that $l = \{z_0 = z_1 = z_2 = 0\}$, and $\mathcal{C}_1 \cap \mathcal{C}_2 = B \cap l = \{Q = 0\} \cap l$. As above let $p_2$ be any one of the 2 points and we work in a neighborhood of $p_2$. We put $x := z_0/z_4, y := z_1/z_4, z := z_2/z_4$ and $u := Q$. Then by transversality for the intersection of $Q$ and $l$, we can use $(x, y, z, u)$ as coordinates in a neighborhood of $p_2$ in $\mathbb{CP}^4$, and noticing $z_3z_4f \neq 0$ at $p_2$, we may suppose that the hyperquartic (4.11) is defined by a very simple equation, $x = u^2$. Since $Y$ is defined by $x^2 = yz$, we deduce that $p_2$ is an $A_3$-singular point of the surface $B$. By reality, $\overline{p}_2$ is also an $A_3$-singular point. $\square$

Next, as the transformation from $Z$ to $Z_4$ is explicit, it is easy to show the following:

**Proposition 4.9.** For the variety $Z_4$ we have $e(Z_4) = 10$.

**Proof.** Since $Z$ is a twistor space on $4\mathbb{CP}^2$, we have $e(Z) = 2 + 2(4\mathbb{CP}^2 + 1) = 12$. Because the blowup $\mu_1$ replaces two disjoint $\mathbb{CP}^1$-s by two $\mathbb{CP}^1 \times \mathbb{CP}^1$-s, we have $e(Z_1) = 12 + 4 = 16$. Then since a flop does not change the Euler number we obtain $e(Z_3) = 16$. Finally looking Figure 3 (d), the exceptional divisor $E_1 \cup E_1$ of the contraction $\mu_4 : Z_3 \to Z_4$ has Euler number 8, and the image $l_4 \cup \overline{l}_4$ of the exceptional divisor has Euler number 2. Hence we obtain $e(Z_4) = 16 - (8 - 2) = 10$. $\square$

The next result means that the 26 points that we have already found are not all singularities of the branch divisor $B$, but in the generic situation $B$ has extra 6 ordinary double points:

**Theorem 4.10.** Let $\{b_1, \cdots, b_k\}$ be the set of all singular points of $B$ which are different from the 26 singular points listed in Section 4.2. Let $\mu_i$ be the Milnor number of the singular point $b_i$, and put $\beta_i := \mu_i^{-1}(b_i)$ for the exceptional curve of $\mu_i$ over the point $b_i$. (Of course we do not assume irreducibility of $\beta_i$.) Then we have the relation

$$
\sum_{i=1}^{k} \{e(\beta_i) + \mu_i - 1\} = 12.
$$

In particular, if all the singularities are ordinary double points, we have $k = 6$. 

Proof. First since \( \mu_5 : Z_4 \to Z_5 \) replaces each of the 24 ordinary double points by smooth \( \mathbb{CP}^1 \) and also replaces the singular point \( b_i \) by the curve \( \beta_i \) for \( 1 \leq i \leq k \), we obtain

\[
e(Z_4) = e(Z_5) + 24 + \sum_{1 \leq i \leq k} \{e(\beta_i) - 1\}.
\]

On the other hand by the double covering \( Z_5 \to Y \) we have \( e(Z_5) = 2e(Y) - e(B) \). Further as \( Y \) is obtained from the \( \mathbb{CP}^2 \)-bundle \( \tilde{Y} \) over \( \mathbb{CP}^1 \), and it replaces \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) by \( l \simeq \mathbb{CP}^1 \), we have \( e(Y) = e(\tilde{Y}) - (4 - 2) = 6 - 2 = 4 \). Hence we have \( e(Z_5) = 8 - e(B) \), giving

\[
e(Z_4) = 32 - e(B) + \sum_{1 \leq i \leq k} \{e(\beta_i) - 1\}.
\]

Next for computing \( e(B) \) let \( D \) be a general member of the system \( |\mathcal{O}_Y(4)| \). Then since the scroll \( Y \) has ordinary double points along the line \( l \), the divisor \( D \) has ordinary double points at the 4 points \( D \cap l \). As before let \( \nu : \tilde{Y} \to Y \) be the blowup at \( l \), and let \( \tilde{D} \) be the strict transform of \( D \). By Bertini’s theorem we may suppose that \( \tilde{D} \) is non-singular. We shall compute \( e(\tilde{D}) \).

As before write \( f := \tilde{\pi}^*\mathcal{O}_A(1) \in H^2(\tilde{Y}, \mathbb{Z}) \). From the standard relationship of the total Chern class \( c(T_{\tilde{Y}}|\tilde{D}) = c(T_{\tilde{D}}) \cdot c(N_{\tilde{D}/\tilde{Y}}) \) and adjunction formula, we readily obtain

\[
e(\tilde{D}) = c_2(T_{\tilde{D}}) = c_2(T_{\tilde{Y}}) \cdot \tilde{D} + (K_{\tilde{Y}} + \tilde{D}) \cdot \tilde{D},
\]

where the dot means the product in \( H^*(\tilde{Y}, \mathbb{Z}) \). As in the proof of Proposition 3.4 let \( \mathcal{O}(0, 1) := (\nu^*\mathcal{O}_1)|_{\Sigma} \), so that \( \mathcal{O}(1, 0) = f|_{\Sigma} \). Then by using \( N_{\Sigma/\tilde{Y}} \simeq \mathcal{O}(-2, 1) \) and the adjunction formula applied to a fiber of \( \tilde{\pi} \), we readily obtain \( K_{\tilde{Y}} \sim -3\Sigma - 6f \). Further, in the cohomology ring of \( \tilde{Y} \), we have \( \Sigma^3 = N_{\Sigma/\tilde{Y}} \cdot N_{\Sigma/\tilde{Y}} = -4, \Sigma^2 \cdot f = N_{\Sigma/\tilde{Y}} \cdot f = \mathcal{O}(-2, 1) \cdot \mathcal{O}(1, 0) = 1, \Sigma \cdot f^2 = 0 \) and \( f^3 = 0 \). Furthermore by recalling \( \nu^*\mathcal{O}(1) \sim \Sigma + 2f \) (see \( \mathcal{O}(3, 6) \)), we obtain \( K_{\tilde{Y}} + \tilde{D} \sim (-3\Sigma - 6f) + 4(\Sigma + 2f) = \Sigma + 2f \). From these we readily get \( (K_{\tilde{Y}} + \tilde{D}) \cdot \tilde{D} = 32 \).

For computing \( c_2(T_{\tilde{Y}}) \in H^4(\tilde{Y}, \mathbb{Z}) \), as generators of \( H^4(\tilde{Y}, \mathbb{Z}) \) we take any element \( \zeta \in |\mathcal{O}_\Sigma(1, 0)| \) and \( \eta \in |\mathcal{O}_\Sigma(0, 1)| \), viewed as submanifolds in \( \tilde{Y} \), and put \( c_2(T_{\tilde{Y}}) = a\zeta + b\eta \). From the exact sequence associated to the inclusion \( \Sigma \subset \tilde{Y} \), we immediately obtain \( c_2(T_{\tilde{Y}})|_{\Sigma} = c_1(\Sigma) \cdot c_1(N_{\Sigma/\tilde{Y}}) + c_2(\Sigma) \). Then since \( c_1(\Sigma) = \mathcal{O}(2, 2), c_1(N_{\Sigma/\tilde{Y}}) = (-2, 1) \) and \( c_2(\Sigma) = e(\Sigma) = 4 \), we obtain \( c_2(T_{\tilde{Y}})|_{\Sigma} = 2 \). On the other hand, from the inclusion \( f \subset \tilde{Y} \) we readily obtain \( c_2(T_{\tilde{Y}})|_{f} = 3 \). Further, in the cohomology ring of \( \tilde{Y} \) we have \( \zeta \cdot \Sigma = \mathcal{O}(1, 0) \cdot \mathcal{O}(-2, 1) = 1, \eta \cdot \Sigma = \mathcal{O}(1, 0) \cdot \mathcal{O}(-2, 1) = -2, \zeta \cdot f = \mathcal{O}(1, 0) \cdot \mathcal{O}(1, 0) = 0 \) and \( \eta \cdot f = \mathcal{O}(0, 1) \cdot \mathcal{O}(1, 0) = 1 \). Therefore by restricting to \( \Sigma \) and \( f \) respectively, we get \( a - 2b = 2 \) and \( b = 3 \). Hence \( a = 8 \).

Therefore we obtain \( c_2(\tilde{Y}) \cdot \tilde{D} = (8\zeta + 3\eta) \cdot 4(\Sigma + 2f) \), which is readily computed to be 32. Hence from (4.24) we obtain \( e(\tilde{D}) = 32 + 32 = 64 \).

As \( \tilde{D} \to D \) contracts four \( \mathbb{CP}^1 \)-s to 4 points, we have \( e(D) = 60 \). Then \( D \) is obtained from the actual branch divisor \( B \) by (a) smoothing the 24 nodes, (b) smoothing \( k \) singular points \( b_1, \ldots, b_k \), and (c) deforming each of the two \( A_3 \)-singularities (which is exactly \( \mathcal{G}_1 \cap \mathcal{G}_2 = \{p_2, \overline{p_2}\} \)) to two \( A_1 \) singularities. Adding the Milnor number of the singularities for the cases (a) and (b), and also taking the difference of the Milnor number of \( A_3 \)-singularity and two \( A_1 \)-singularities into account, we obtain

\[
e(B) = e(D) - 26 - \sum_{1 \leq i \leq k} \mu_i,
\]
and hence \( e(B) = 34 - \sum_{1 \leq i \leq k} \mu_i \). Substituting this into (4.23) and using Proposition 4.9 we obtain (4.21).  

Remark 4.11. If the blown-up 8 points on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) are arranged as in Figure [1] is in a general position (in certain precise sense), then \( C_2 \) and \( C_2' \) are all curves that are contracted to points by the bi-anticanonical map. But if the 8 points are in a special position (in certain precise sense), then the map contracts extra curves. It is not difficult to classify all positions which yield this situation. The appearance of this kind of curves is exactly the reason why the anticanonical map of the twistor spaces contracts some rational curves which cannot be found from the equation of the branch divisor.

5. Moduli space and genericity of the twistor spaces

5.1. Dimension of the moduli space. In this subsection we compute the dimension of the moduli space of our twistor spaces by counting the number of effective parameters, and verify that it agrees with the dimension of the cohomology group which is relevant to the present case.

For the former purpose, we recall from Section 3 that \( Z \) canonically determines a birational model \( Z_4 \) and the degree 2 morphism \( \Phi_4 : Z_4 \to Y \), and from Section 4.3 the branch divisor of \( \Phi_4 \) is an intersection of \( Y \) with the quartic surface defined by

\[
(5.1) \quad z_0 z_3 z_4 f(z_0, z_1, z_2, z_3, z_4) = Q(z_0, z_1, z_2, z_3, z_4)^2,
\]

where \( f \) and \( Q \) are linear and quadratic polynomials with real coefficients respectively. In this subsection we denote this quartic hypersurface by \( \mathcal{B}(f, Q) \). Since the quartic hypersurface \( \mathcal{B}(f, Q) \) uniquely determines the double cover via the natural quadratic map \( O_Y(2) \to O_Y(4) \) (which takes squares), up to small resolutions, \( Z \) is uniquely determined by the quartic hypersurface. Of course, \( f \) has 5 coefficients and \( Q \) has 15 coefficients, so the equation (5.1) contains 20 parameters. Further it is elementary to see that two pairs \( (f, Q) \) and \( (f', Q') \) of linear and quadratic polynomials (over \( \mathbb{R} \)) determine the same hyperquartic surface if and only if \( (f', Q') = (c^2 f, c Q) \) for some \( c \in \mathbb{R} \). This decreases the number of parameters by one. On the other hand, projective transformations which preserve \( Y \) and which preserves the form of the equation (5.1) have to be \( (z_0, z_1, z_2, z_3, z_4) \mapsto (abz_0, az_1, bz_2, cz_3, dz_4) \) for some \( a, b, c, d \in \mathbb{R}^\times \). (Here we are only considering transformations which are homotopic to the identity.) If a pair \( (f', Q') \) is obtained from a pair \( (f, Q) \) by one of these transformations, then the intersections \( Y \cap \mathcal{B}(f, Q) \) and \( Y \cap \mathcal{B}(f', Q') \) are mutually biholomorphic, so that they define mutually isomorphic double cover. But taking an effect of the above equivalence \( (c^2 f, c Q) = (f, Q) \) into account, we can suppose \( ab = 1 \) and therefore these projective transformations decrease the number of parameters by 3. Thus up to now the number of parameters is \( 20 - (1 + 3) = 16 \). However what we have to consider is not the hyperquartics (5.1) themselves but the intersection with \( Y \); namely if \( Q' = Q + c(z_0^2 - z_1 z_2) \) for some \( c \in \mathbb{R} \), then we have the coincidence \( Y \cap \mathcal{B}(f, Q) = Y \cap \mathcal{B}(f, Q') \subset \mathbb{CP}^4 \). Clearly these transformations are not included in the above projective transformations, so they drop the dimension by one. Thus we have obtained that the number of effective parameters in the quadratic hypersurface (5.1) is 15. Finally, by Theorem 4.10 the pair \( (f, Q) \) must satisfy the constraint that \( \mathcal{B}(f, Q) \cap Y \) has extra 6 ordinary double points in general, which decreases the number of parameters by 6. Therefore we conclude that the space of isomorphic classes of the divisors of the form \( \mathcal{B}(f, Q) \cap Y \), which can be the branch divisor for the twistor spaces under consideration, is 15-dimensional.
Next we compute the dimension of the moduli space of our twistor spaces by determining the dimension of the first cohomology group of an appropriate subsheaf of the tangent sheaf. We begin with a computation for the full moduli space.

**Proposition 5.1.** Let $Z$ be a twistor space on $4\mathbb{CP}^2$ which contains the surface $S$ constructed in Section 2 as a real member of $|F|$. Then we have $H^i(Z, \Theta_Z) = 0$ for $i \neq 1$ and $h^1(Z, \Theta_Z) = 13$.

**Proof.** As computed in [14], for any twistor space on $n\mathbb{CP}^2$, by the Riemann-Roch formula, we have $\chi(\Theta_Z) = 15 - 7n$. Also, since $Z$ is Moishezon and $|F|$ has an irreducible member, we have $H^2(\Theta_Z) = 0$ by [2]. Further we always have $H^3(\Theta_Z) = 0$. Hence it suffices to show $H^0(\Theta_Z) = 0$. Let $\text{Aut}_0Z$ be the identity component of the holomorphic automorphism group of $Z$. Then the real part $(\text{Aut}_0Z)^\sigma$ is naturally identified with the identity component of conformal automorphism group of the self-dual structure. Also, since $|F|$ has just 2 irreducible components, $(\text{Aut}_0Z)^\sigma$ acts on $S^+_1 \cup S^-_1$. Hence as the twistor projections $S^+_1 \to 4\mathbb{CP}^2$ and $S^-_1 \to 4\mathbb{CP}^2$ are of degree 1, $(\text{Aut}_0Z)^\sigma$ acts effectively on $S^+_1 \cup S^-_1$. Furthermore, the degree 1 divisor $S^+_1$ and $S^-_1$ are obtained from $\mathbb{CP}^2$ by blowing-up 4 points, exactly 3 of which are collinear. From this it readily follows that the subgroup of $\text{Aut} S^+_1$ which consists of automorphism preserving the twistor line $L_1$ is 0-dimensional. Hence so is $\text{Aut}(S^+_1 \cup S^-_1)$. Thus $\text{Aut}_0Z$ cannot be of positive dimension. Therefore $H^0(\Theta_Z) = 0$. □

From the proposition, the real part of the Kuranishi family of our twistor space $Z$ is 13-dimensional. Of course, generic twistor spaces on $4\mathbb{CP}^2$ is algebraic dimension 1 and generic members of the Kuranishi family have the same property. In order to restrict to the Moishezon twistor spaces under consideration, we show the following.

**Proposition 5.2.** Let $Z$ and $S$ be as in Proposition 5.1 and $C_1$ and $\overline{C}_1$ the $(-3)$-curves on $S$. Then deformation theory of the pair $(Z, C_1 \cup \overline{C}_1)$ is unobstructed and its Kuranishi family is 9-dimensional. Further, for all sufficiently small deformations preserving the real structure in the Kuranishi family, the twistor spaces contain a non-singular surface constructed in Section 2 as a real member of $|F|$.

Of course, the last property means that the deformed spaces are still the Moishezon twistor spaces under consideration.

**Proof of Proposition 5.2.** Let $\Theta_{Z, C_1 + \overline{C}_1}$ be the subsheaf of $\Theta_Z$ whose germs are vector fields that are tangents to $C_1$ and $\overline{C}_1$. For the former claim on the Kuranishi family, it suffices to show that $H^2(Z, \Theta_{Z, C_1 + \overline{C}_1}) = 0$ and $h^1(Z, \Theta_{Z, C_1 + \overline{C}_1}) = 9$. Recalling $N_{C_1/Z} \simeq N_{\overline{C}_1/Z} \simeq \mathcal{O}(-2)^{\oplus 2}$, we obtain the standard exact sequence

$$0 \to \Theta_{Z, C_1 + \overline{C}_1} \to \Theta_Z \to \mathcal{O}_{C_1}(-2)^{\oplus 2} \oplus \mathcal{O}_{\overline{C}_1}(-2)^{\oplus 2} \to 0,$$

which induces an exact sequence

$$0 \to H^1(\Theta_{Z, C_1 + \overline{C}_1}) \to H^1(\Theta_Z) \to \mathbb{C}^4 \to H^2(\Theta_{Z, C_1 + \overline{C}_1}) \to 0. \quad (5.2)$$

Hence with the aid of Proposition 5.1 we have only to show $H^2(\Theta_{Z, C_1 + \overline{C}_1}) = 0$. For this we first deduce from duality and rationality that $H^2(\Theta_S(-C_1 - \overline{C}_1)) = 0$, which implies, from the exact sequence $0 \to \Theta_S(-C_1 - \overline{C}_1) \to \Theta_{S, C_1 + \overline{C}_1} \to \Theta_{C_1 \cup \overline{C}_1} \to 0$, that $H^2(\Theta_{S, C_1 + \overline{C}_1}) = 0$. Moreover, noting $N_{S/Z} \simeq F|_S \simeq -K_S$, we have an exact sequence

$$0 \to \Theta_{S, C_1 + \overline{C}_1} \to \Theta_{Z, C_1 + \overline{C}_1} \mid S \to -K_S \otimes \mathcal{O}_S(-C_1 - \overline{C}_1) \to 0.$$
For the last term we have \(-K_S \otimes \mathcal{O}_S(-C_1 - \overline{C_1}) \simeq \mathcal{O}_S(C_2 + \overline{C_2})\), and it is easy to see \(H^2(\mathcal{O}_S(C_2 + \overline{C_2})) = 0\). Hence for the middle term we obtain \(H^2(\Theta_{Z,C_1+\overline{C_1}}|s) = 0\). Then by the exact sequence \(0 \rightarrow \Theta_Z(-S) \rightarrow \Theta_{Z,C_1+\overline{C_1}} \rightarrow \Theta_{Z,C_1+\overline{C_1}}|s \rightarrow 0\) and \(H^2(\Theta_Z(-S)) = 0\) \([2]\), we finally obtain \(H^2(\Theta_{Z,C_1+\overline{C_1}}) = 0\), as claimed.

For the latter claim about the existence of the surface in the deformed space, let \(Z_t\) be any one of the deformed twistor space which is sufficiently close to the original \(Z\), and \(C_{1t}, \overline{C}_{1t} \subset Z_t\) the curves corresponding to the original curves \(C_1\) and \(\overline{C_1}\). Let \(F_t\) be the fundamental line bundle on \(Z_t\). Then as \(\dim |F| = 1\), we may suppose \(\dim |F_t| = 1\) by upper-semicontinuity of dimensions of cohomology groups under deformations and the Riemann-Roch formula \(\chi(F_t) = 2\). We also have an invariance \(F_t \cdot C_{1t} = F \cdot C_1\), and the latter is equal to \(K_S^{-1} \cdot C_1 = -1\), and therefore \(F_t \cdot C_{1t} = -1\). This means that \(C_{1t}\) and \(\overline{C}_{1t}\) are base curves of the pencil \(|F_t|\). Let \(S_t \in |F_t|\) be any real irreducible member. Through the Kuranishi family, this surface can be regarded as a small deformation of some real irreducible \(S \in |F|\), which means that \(S_t\) is obtained from \(\mathbb{C}P^1 \times \mathbb{C}P^1\) by moving the blowup 8 points from the original positions (indicated as in Figure 1). But since \(S_t\) contains the curves \(C_{1t}\) and \(\overline{C}_{1t}\) as \((-3)\)-curves, the property that 3 points belong to a \((1,0)\)-curve must be preserved. This means that the structure of \(S_t\) is the same as that of the original \(S\), and we are done. □

5.2. Genericity of the twistor spaces. In this subsection, by using a theorem of Pedersen-Poon about structure of real irreducible members of \(|F|\), we show that the present twistor spaces are in a sense generic among all Moishezon twistor spaces on \(4\mathbb{C}P^2\). We first recall the theorem of Pedersen-Poon \([15]\) in a precise form:

**Proposition 5.3.** Let \(Z\) be a twistor space on \(n\mathbb{C}P^2\) and \(S \in |F|\) a real irreducible member. Then \(S\) is non-singular with \(K_S^2 = 8 - 2n\), and the set of twistor lines lying on \(S\) is exactly the real part of a real pencil whose self-intersection number is zero. Moreover there is a birational morphism \(\epsilon : S \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1\) preserving the real structure, such that the twistor lines are mapped to \((1,0)\)-curves.

Thus \(S\) is always obtained from \(\mathbb{C}P^1 \times \mathbb{C}P^1\) by blowing up 2\(n\) points, where some of the points might be infinitely near in general. As is well-known the position of the blowing up points has a strong effect on algebraic structure of twistor spaces. For example, if a twistor space \(Z\) contains \(S\) that is obtained from the 2\(n\) points lying on an irreducible \((1,2)\)-curve, then it follows \(\dim |F| = 2\), and detailed structure of such twistor spaces is investigated by Campana-Kreussler \([3]\). Then in terms of the configuration of the blowing up points, the genericity of our twistor spaces refers the following property:

**Proposition 5.4.** Let \(Z\) be a Moishezon twistor space on \(4\mathbb{C}P^2\) which is not of Campana-Kreussler type. Suppose that there exists a real irreducible member \(S \in |F|\) such that the images of the 8 exceptional curves of the blowing-down \(\epsilon : S \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1\) in Proposition 5.3 can be taken as distinct points. Then the configuration of the 8 points falls into exactly one of Figure 5.

For the proof, we first show the following.

**Proposition 5.5.** If \(Z\) is a twistor space on \(4\mathbb{C}P^2\) which satisfies \(\dim |F| = 2\), then \(Z\) is either a Campana-Kreussler twistor space, or otherwise non-Moishezon.

**Proof.** Let \(S \in |F|\) be a real irreducible member, which is necessarily non-singular as above. By the assumption, \(S\) satisfies \(\dim |K_S^{-1}| = 1\). Let \(\epsilon : S \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1\) be the birational
morphism fulfilling the properties of Proposition 5.3. (The images of the exceptional curves of $\epsilon$ can be infinitely near.) Since $\epsilon$ is a composition of blowups, by the canonical bundle formula for blowups, the image of the pencil $|K_S^{-1}|$ by $\epsilon$ necessarily has to be a pencil of anticanonical curves on $\mathbb{CP}^1 \times \mathbb{CP}^1$; namely a pencil of $(2,2)$-curves. Let $\mathcal{P}$ be this pencil on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then again by the canonical bundle formula all the images of the exceptional curves of $\epsilon$ must be contained in the base locus of $\mathcal{P}$.

Suppose that general members of the pencil $\mathcal{P}$ are irreducible. Then the pencil $|K_S^{-1}|$ does not have a fixed component. If this pencil has a base point, by taking a sequence of blowups $\tilde{S} \to S$, we obtain a morphism $\tilde{S} \to \mathbb{CP}^1$, which is, again by the canonical bundle formula for blowups, the anticanonical map on $\tilde{S}$. Therefore, the morphism $\tilde{S} \to \mathbb{CP}^1$ must be an elliptic fibration. But then by the canonical bundle formula for elliptic surfaces, we obtain $c_2(\tilde{S}) = 0$. Since $c_2(S) = 0$, this means that $\tilde{S}$ and $S$ are biholomorphic. Hence the pencil $|K_S^{-1}|$ is base point free, and the anticanonical map induces an elliptic fibration $S \to \mathbb{CP}^1$. This implies the anti-Kodaira dimension of $S$ is one, which means that $\mathcal{Z}$ is non-Moishezon.

So in the sequel we suppose that general members of the pencil $\mathcal{P}$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are reducible. Then if $\mathcal{P}$ does not have a fixed component, we have $\dim |K_S^{-1}| \geq 2$, which contradicts our assumption. Hence $\mathcal{P}$ has a fixed component. Let $C_0$ be any one of its irreducible components. Then among the image points of the exceptional curves of $\epsilon$, there exists at least 1 point on $C_0$ because otherwise $C_0$ is not a fixed component. Suppose that $C_0 \in |\mathcal{O}(0,1)|$. Then $C_0' \neq C_0$ by the induced real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$, and $C_0'$ is also a fixed component of $\mathcal{P}$. Hence the movable part of $\mathcal{P}$ must be a free 1-dimensional subsystem of $|\mathcal{O}(2,0)|$, or the system $|\mathcal{O}(1,0)|$ itself with another fixed $(1,0)$-curve $C_0''$. But the former cannot occur because general members of the movable part of the 1-dimensional subsystem would be reducible by freeness and both components actually move, so that all the images of the exceptional curves of $\epsilon$ have to be contained in $C_0 \cup C_0'$, which means $\dim |K_S^{-1}| = 2$. So suppose the latter is the case. Then the fixed $(1,0)$-curve $C_0'$ cannot be real, since if so, we would have $C_0' = \epsilon(L)$ for some twistor line $L \subset S$ by the property of $\epsilon$, whereas on $C_0''$ there is at least one point among the images of the exceptional curves of $\epsilon$, which means $L^2 < 0$ on $S$. Hence $C_0' \neq C_0''$. But this is impossible since $C_0 + C_0' + C_0'' + C_0''$, which is clearly $(2,2)$-curves, would be fixed components of the pencil $\mathcal{P}$ of $(2,2)$-curves. Thus we obtained $C_0 \notin |\mathcal{O}(0,1)|$; namely $\mathcal{P}$ does not have a $(0,1)$-curve as a fixed component.

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**Figure 5.** Possible configurations of distinct 8 points on $\mathbb{CP}^1 \times \mathbb{CP}^1$ for Moishezon twistor spaces, except for the Campana-Kreussle's case.
Also if a fixed component $C_0$ is a $(1,0)$-curve, then it cannot be real by the same reason. Hence the movable part of $\mathcal{P}$ is a free 1-dimensional subsystem of $|\mathcal{O}(0,2)|$, or the system $|\mathcal{O}(0,1)|$ with another fixed component $C'_0 \in |\mathcal{O}(0,1)|$. But the former implies $\dim |K_S^{-1}| = 2$ by the same argument as above, and the latter cannot occur since this time there is no real $(0,1)$-curve. Thus the fixed component $C'_0$ of $\mathcal{P}$ cannot be a $(1,0)$-curve. Further we have $C' \notin |\mathcal{O}(1,1)|$, since any $(1,1)$-curve is not real and hence $C_0 \neq C'_0$ has to be also a base curve, which contradicts that $\mathcal{P}$ is a pencil. Similarly we have $C_0 \notin |\mathcal{O}(2,1)|$ by the real structure. Hence $C_0$ must be in the remaining possibility, $C_0 \in |\mathcal{O}(1,2)|$. In this case $\dim |K_S^{-1}| = 1$ means that all the images of the exceptional curves of $\epsilon$ belong to $C_0$. This implies that the structure of $S$ is exactly as in the case of Campana-Kreussler, and we are done.

\begin{proof}[Proof of Proposition 5.4] First by a result by Kreussler [11, Theorem 6.5], on $n\mathbb{CP}^2$ with $n \geq 3$ we always have $\dim |F| \leq 3$ and the equality holds iff $Z$ is a LeBrun twistor space [13]. Suppose that $Z$ is a LeBrun twistor space. Then it is well-known that a configuration of $2n$ points for generic real irreducible member $S \in |F|$ is as in (i) of Figure 5.

So suppose that $Z$ is a Moishezon twistor space on $4\mathbb{CP}^2$ which is different from LeBrun’s nor Campana-Kreussler’s, and let $S$ be a real irreducible member of $|F|$ such that the $2n$ points on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are distinct. By Proposition 5.3 we have $\dim |F| = 1$. This means $\dim |K_S^{-1}| = 0$. Let $C$ be the unique anticanonical curve, $\epsilon : S \to \mathbb{CP}^1 \times \mathbb{CP}^1$ the birational morphism as in Proposition 5.3 whose images of the exceptional curves are distinct. We put $C_0 := \epsilon(C)$, which is necessarily a real $(2,2)$-curve. All the 8 points are on $C_0$. If $C_0$ is irreducible, $C_0$ must be a non-singular elliptic curve by using the real structure, and from this we readily see that $h^0(mK_S^{-1}) \leq m$ for all $m > 0$, which means that $Z$ is non-Moishezon. Hence $C_0$ is reducible. Taking the form of the induced real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ into account, we can easily show that the decomposition of $C_0$ into irreducible components is one of the following 3 types: (a) $(1,0) + (0,1) + (1,0) + (0,1)$, (b) $(1,1) + (1,1)$, both components being irreducible, or (c) $(1,2) + (1,0)$, both components being irreducible. We note that since $h^0(K_S^{-1}) = 1$, on any of these components, there exists at least 1 point among the 8 points. Repeating an argument in the last part of the proof of Proposition 5.5 we deduce that (c) cannot happen under our assumption. If $C_0$ has a multiple component, since there exists no real $(0,1)$-curve, it must be a real $(1,0)$-curve. But this cannot happen since as remarked above among the 8 points there is at least one point on any irreducible component of $C_0$, contradicting the family of twistor lines on $S$. Hence in both cases (a) and (b) $C_0$ has no multiple component.

Next we show that in the case (a) there is an irreducible component of $C_0$ on which precisely 3 points among the 8 points lie. If not, then because we are excluding LeBrun twistor spaces, on each of the 4 irreducible components exactly 2 points are lying among the 8 points. In this case, the restriction $K_S^{-1}|C \simeq [C]|C$ belongs to $\text{Pic}^0C \simeq \mathbb{C}^*$, which again implies $h^0(-mK_S) \leq m$ for any $m > 0$ as in the above case. This implies that $Z$ is not Moishezon. Hence the component actually exists.

Next we prove that if $C_0$ is in the case (b), there exists another birational morphism $\epsilon' : S \to \mathbb{CP}^1 \times \mathbb{CP}^1$ preserving the real structure such that $C'_0 = \epsilon'(C)$ falls into the case (a), and such that the images of twistor lines in $S$ are $(1,0)$-curves. For this we write $C_0 = C_1 + C_{1}'$ with $C_1$ and $C_1'$ being irreducible $(1,1)$-curves. If exactly 4 points belong to $C_0$, then the remaining 4 points belong to $C_1$, and also no point coincides with the 2 points $C_1 \cap C_{1}'$. Then by a similar argument for the case (a), this implies that $Z$ is not
Moishezon. So the 2 points $C_1 \cap \overline{C}_1$ are included in the 8 points. Therefore $\epsilon$ factors as $S \to S_1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ where the latter arrow is the blowup at $C_1 \cap \overline{C}_1$. We obtain 6 points on $S_1$ as the images of the exceptional curves of $S \to S_1$. These 6 points are not on the exceptional curves of $S_1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ by the assumption that the 8 points are distinct. This implies that the unique anticanonical curve $C$ on $S$ is a cycle of 4 rational curves, whose self-intersection numbers are $(-3), (-1), (-3), (-1)$. Then for another blowdown $\epsilon'$ in the proposition, it is enough to choose a blowingdown $S_1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ which does not contract the 2 exceptional curves of the original $S_1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$, and composite it with the morphism $S \to S_1$. Thus we obtained another $\epsilon'$ as claimed, and we can neglect the case (b).

Hence $C_0$ can be supposed to be in the case (a) and that there is at least one component on which exactly 3 points among 8 points lie. This directly means the 8 points have to be put on $C_0$ arranged as in (ii), (iii) or (iv), as claimed. □

Needless to say, the case (ii) of Proposition 5.4 is exactly the situation we have investigated in this paper. Since it is clear that all other 3 cases ((i), (iii) and (iv)) can be obtained as small deformations of the case (ii), it would be reasonable to say that among the surface $S$ obtained from the 8 points arranged as in (i)–(iv), the case (ii) is most generic. By deformation theory including a co-stability theorem of Horikawa [9], the same is true for the twistor spaces containing these surfaces. Namely any twistor spaces on $4\mathbb{C}P^2$ which has $S$ obtained from (i), (iii) and (iv) as a real member of $|F|$ can be obtained as a limit of the twistor spaces investigated in this paper. In particular, the present twistor spaces can be obtained as a small deformation of a LeBrun twistor space, and this proves the existence of our twistor spaces. We also remark that by using the Horikawa’s theorem, it is possible to show that the present twistor spaces can also be obtained as a small deformation of the twistor spaces studied in [8] (on $4\mathbb{C}P^2$, of course). These are the reason why we call the present twistor spaces to be generic.

Finally we remark that a converse of Proposition 5.4 also holds. Namely if a twistor space $Z$ on $4\mathbb{C}P^2$ has real irreducible $S \in |F|$ which is obtained from the 8 points in the case (ii), (iii), or (iv), then $\dim |F| = 1$ and $Z$ is Moishezon. Concerning structure of these twistor spaces, the case (iii) can be regarded as a mild degeneration of the present twistor spaces, in the sense that the twistor space still has a double covering structure over the scroll $Y$ by the anticanonical system. These twistor spaces (having $S$ obtained from the configuration (iii)) are analogous to a double solid twistor spaces on $3\mathbb{C}P^2$ of a degenerate form found by Kreussler-Kurke [12, p. 50, Case (b)]. Here we mention that there is one more another type of twistor spaces on $4\mathbb{C}P^2$ of a degenerate form which do not admit $\mathbb{C}^*$-action. We will study these 2 kinds of double solid twistor spaces on $4\mathbb{C}P^2$ in a separate paper. On the other hand, although Moishezon, it turns out that the remaining case (iv) does not have structure of double solids, because the anticanonical map of the twistor spaces becomes birational. So they are rather similar to the twistor spaces of Joyce metrics on $4\mathbb{C}P^2$ of non-LeBrun type [7]. But contrary to the Joyce’s case, explicit realization of the anticanonical model seems difficult.

6. Appendix: Inverting the contraction map $Z_3 \to Z_4$ by a blowup

We recall from Section 3.2 that the singular variety $Z_4$ is obtained from the twistor space $Z$ via non-singular spaces $Z_1, Z_2$ and $Z_3$, and the transformations therein are standard until getting $Z_3$. On the other hand, the map $\mu_4 : Z_3 \to Z_4$ contracts the reducible connected
divisor $E_1 \cup E_1$ to a reducible connected curve $l_4 \cup \overline{l}_4$, and there we used Fujiki’s contraction theorem. One would wish to find such a birational morphism through the usual procedure of blowups with non-singular center. In this subsection, we explicitly see that an embedded blowup at and a small resolution provide the desired situation and point out that the process is in a sense a singular version of the Hironaka’s construction of non-projective Moishezon 3-folds.

For this we consider the double covering of $Y$ with branch $B$ which is the intersection with the quartic hypersurface \((4.11)\). Recall that $l = \{z_0 = z_1 = z_2 = 0\}$, and $B \cap l$ consists of two points $\{Q = 0\} \cap l$. As before let $p_2$ be any one of the two points and we work in a neighborhood of $p_2$ as the situation around $\overline{p}_2$ can see by just taking the image under the real structure. As in the proof of Proposition \([4.8]\) putting $x = z_0/z_4, y = z_1/z_4, z = z_2/z_4$ and $u = Q$, we can use $(x, y, z, u)$ as coordinates in a neighborhood of $p_2$ in $\mathbb{CP}^4$, and we may suppose that the hyperquartic \((4.11)\) is defined by $x = u^2$, while $Y$ is defined by $x^2 = yz$. Next for studying the structure of the double covering, we introduce another coordinate $w$ over the neighborhood of $p_2$, so that the double cover is defined by

\[(6.1)\]

\[x^2 = yz, \quad w^2 = u^2 - x \text{ in } \mathbb{C}^5 \text{ with coordinates } (x, y, z, u, w).\]

(This is the equation of $Z_4$ around the points $l_4 \cap \overline{l}_4$ we promised in Section \([3.2]\).) Let $W$ be this double covering and $\varpi : W \to Y$ the projection. (Of course this is valid only in a neighborhood of $p_2$.) The singular locus of $W$ is

\[(6.2)\]

\[\varpi^{-1}(l) = \{x = y = z = u - w = 0\} \cup \{x = y = z = u + w = 0\},\]

which is a union of 2 lines, and $W$ has $A_1$-singularities along these lines minus the origin. We note that substituting $x = u^2 - w^2$ to $x^2 = yz$, we obtain that $W$ contains the following distinguished 4 surfaces

\[(6.3)\]

\[\{x = y = w - u = 0\}, \quad \{x = z = w - u = 0\},\]

\[(6.4)\]

\[\{x = y = w + u = 0\}, \quad \{x = z = w + u = 0\}.\]

Then obviously we have $W \cap \{x = y = z = 0\} = \varpi^{-1}(l)$. So if we let $\tilde{W}^5 \to \mathbb{C}^5$ to be the blowup at the plane $\{x = y = z = 0\}$ and $\tilde{W}$ to mean the strict transform of $W$, then $\tilde{W} \to W$ is an embedded blowup at $\text{Sing} W$. Then by concrete computations using coordinates it is not difficult to see that the exceptional locus of $\tilde{W} \to W$ consists of 2 irreducible divisors which are over the 2 lines \((6.2)\), respectively, that the inverse image of the origin is a non-singular rational curve, and that the singularities of $\tilde{W}$ consists of 2 points lying on this rational curve, both of which are ordinary double points. Further, by the effect of the blowup, the pair of planes \((6.3)\), both of which contain the same line $\{x = y = z = w - u = 0\}$ are separated by one of the exceptional divisors, and the same for another pair \((6.4)\). This way we get the situation of Figure \([3]\) (e). Then an appropriate small resolution (which is obvious from the figure) gives the desired space $Z_3$.

As above the center of the blowup is a reducible curve whose fundamental group is $\mathbb{Z}$. Thus, together with an inspection of the choice of the small resolution displayed in Figure \([3]\), it would be possible to say that the transformation from $Z_4$ to $Z_3$ is a singular version of Hironaka’s well-known example of non-projective Moishezon 3-folds \([5]\) in the sense that the center of the blowup in the present situation is a singular locus of the 3-fold.
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