Time dependence of $c$ and its concomitants

Ll. Bel

_Fisika Teorikoa, Euskal Herriko Unibertsitatea,_
_P.K. 644, 48080 Bilbo, Spain_
_e-mail: wtpbedil@lg.ehu.es_

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Abstract

As we showed in a preceding arXiv:gr-qc Einstein equations, conveniently written, provide the more orthodox and simple description of cosmological models with a time dependent speed of light $c$. We derive here the concomitant dependence of the electric permittivity $\epsilon$, the magnetic permeability $\mu$, the unit of charge $e$, Plank’s constant $h$, under the assumption of the constancy of the fine structure constant $\alpha$, and the masses of elementary particles $m$. As a consequence of these concomitant dependences on time they remain constant their ratios $e/m$ as well as their Compton wave length $\lambda_c$ and their classical radius $r_0$.

1 Introduction

To say that light rays are null geodesics of some space-time metric is an intrinsic statement, but any statement about the speed of light is necessarily relative to a frame of reference, as soon as we escape from the semantic tautology that consists in saying that the speed of light is $c$ because the universal constant $c$ is called the speed of light... To say that the speed of propagation of light from a point A to a point B and back is some given value $W = D/\Delta T$ requires that we define the time interval $\Delta T$ as proper time along the trajectory of A, and this does not raise any problem, but it requires also that we define the space distance between the world-lines of A and B and this is subject to debate. More generally this requires to define the time-like congruence of world lines which will be the first ingredient of the frame of reference where the measures of distance are made. To say that
the speed of propagation of light from a point A to a point B is some given value \( V = D / (T_B - T_A) \) further requires to specify a synchronization of time, i.e. the second ingredient of any frame of reference, between the world-line of A and that of B to make sense of the denominator of this formula.

There is a point of view that allows to speak about the speed of light without falling into any fundamental metrology problem. Let us assume that we know of a theory which we know how to test or that has been useful to describe scenarios we are interested in. Let us assume that the formalism of this theory includes a constant, say \( c_0 \), with the dimensions of a velocity and somehow we have some justification to suspect that this constant could in fact be a function, possibly a new unknown of a new theory grafted to the main one. Then it might be legitimate, even before the operational meaning of this function has been clarified, to make the effort to discuss the new theory if we are able to make sense of its new predictions or the new scenarios that follow from it. At least temporarily, because sooner or later we shall need to clarify what we mean by frame of reference, space, time and velocity.

The idea that the speed of light could depend on gravity was one of those that Einstein had in mind since his 1907 paper when he started thinking about a generalization of Special relativity, and later on in his 1911 and 1912 papers. Einstein changed his mind in several occasions and he did it loudly, as when he decided that the field equations had to be covariant after having argued earnestly that covariance was a physical nonsense, or when he regreted to have used a cosmological constant to propose a static model of the Universe. But, to our knowledge, he never made public that he has changed his mind about the speed of light: he just ceased to write about it. To most people, this idea is now anathema and has been abandoned in favor of the hypothesis that the speed of light is the same constant whatever the location and time and whatever the frame of reference, a concept which is often used without care and identified with any system of coordinates.

For us the time and location dependence of the speed of light is an inescapable consequence of a theory of frames of reference that implements the axiom of free mobility. This paper is nevertheless independent from these general considerations and is self-contained.

In Sect. 2 we review the main idea about the time dependence \( c(T) \) of the speed light for Robertson-Walker cosmological models. In the following

\footnotesize

1Our point of view has been described in [1] and references therein

2In our opinion it is still a mistake to confuse covariance with invariance under a dynamical group

3For a recent review on different Variable Speed of Light theories in cosmology read [8] and references therein. These theories compete with inflationary ones to cure problems of standard cosmological models
sections, with an increasing level of speculation, we derive models for the concomitant variation of $\epsilon(T)$, the electronic charge, of $\epsilon(T)$, the electric permittivity of the cosmological medium, of $\mu(T)$, its magnetic permeability, of $h(T)$, Plank’s constant and of $m(T)$, the masses of elementary particles. We assume in the process the constancy of $\alpha$, the fine structure constant, and $G$, Newton’s constant.

2. $c$ is a function of time

In a recent preprint, [4], we considered the general Robertson-Walker line-element written as follows:

$$dS^2 = -dT^2 + \frac{1}{c^2(T)} \left( \frac{dR^2}{1 - kR^2} + R^2 d\Omega^2 \right), \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$$ (1)

with $c(T)$ having the dimensions of velocity and $k$, the space constant curvature, those of inverse square length. The idea that this rewriting makes explicit is that $c(T)$ is a varying speed of light and that, what is usually called the scale factor is:

$$F(T) = c_0/c(T)$$ (2)

where $c_0$ is the measured speed of light at any reference time $T_0$, and thus $F(T)$ can be interpreted as the corresponding varying refractive index of the cosmological medium with $F(T_0) = 1$ at the time when $c(T_0) = c_0$.

Notice that from this point of view $c_0$ ceases to be a universal cosmological constant to become a local constant to be measured at any time that one wishes to describe local physics.

The field equations are:

$$S_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi GT_{\alpha\beta}$$ (3)

where $S_{\alpha\beta}$ is the Einstein tensor of the line-element [11], $\Lambda$ is the proper global cosmological constant of the model with dimensions $T^{-2}$ and $T_{\alpha\beta}$ is:

$$T_{\alpha\beta} = \rho(T)u_\alpha u_\beta + \frac{P(T)}{c(T)^2}(g_{\alpha\beta} + u_\alpha u_\beta)$$ (4)

with $u_0 = -1$ and $u_i = 0$. $G$ is Newton’s constant and it is assumed to be indeed constant. $\rho(T)$ is mass density and $P(T)$ is pressure. Eqs. (3) are

4Observational constraints on the variation of $\alpha$ and $G$ are discussed in [3].
then the usual equations with a small but important difference: the function $c(T)$ in the r-h-s of (4) is a replacement for a constant $c_0$, considered to be a universal constant. This is the single presence of the speed of light in the r-h-s. In other words, there is no need for Einstein’s constant.

The explicit Einstein’s equations (3) reduce to the following two:

\[
3 \frac{k c(T)^4 + \dot{c}(T)^2}{c(T)^2} = 8\pi G \rho(T) + \Lambda
\]

(5)

\[
\frac{k c(T)^4 + 5 \dot{c}(T)^2 - 2c(T)\ddot{c}(T)}{c(T)^2} = -8\pi G \frac{P(T)}{c(T)^2} + \Lambda
\]

(6)

where a dot means a derivative of a function with respect to $T$.

Eqs. (3) remain covariant keeping in mind that the metric coefficients $g_{\alpha\beta}$ in both sides transform as a covariant tensor, that $u_\alpha$ transform as a covariant vector, and that $\rho(T)$, $p(T)$ and $c(T)$ in the r-h-s must be transformed as scalars. From the usual point of view, i.e. writing in the r-h-s $c_0$ instead of $c(T)$ the Eqs. (3) are also obviously manifestly covariant but the physics that they describe is local in time and different from the physics that we consider to describe the cosmological model as a whole.

3 $\epsilon$, $\mu$ and the unit of charge $e$ are functions of time

Assuming, as we do, that the cosmological medium behaves as a linear dielectric, and that the speed of light is a function $c(T)$, means equivalently to assume that either the electric permittivity $\epsilon(T)$ or the magnetic permeability $\mu(T)$, or both are also functions of time, the relationship between these three quantities being:

\[
c(T) = 1/\sqrt{\epsilon(T)\mu(T)} \quad \text{or} \quad F(T) = c_0\sqrt{\epsilon(T)\mu(T)}
\]

(7)

We know, (5) and (6), that under these circumstances the space-time trajectories of light rays are not the null geodesics of (ref(1.1)) with metric $g_{\alpha\beta}$ but those of the metric:

\[
g_{\alpha\beta} = g_{\alpha\beta} + (1 - F(T)^{-2})u_\alpha u_\beta
\]

(8)

We know also that under the same circumstances the Maxwell equations are, if no charges nor currents are present:

\[
\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0
\]

(9)
\[ \partial_{\alpha}(\sqrt{-g} G^{\alpha \beta}) = 0, \quad \sqrt{-g} = \frac{F(T)^3 R^2 \sin \theta}{c(T)^3 \sqrt{1 - k R^2}} \]  
(10)

where:

\[ G^{\alpha \beta} = \frac{1}{\mu} \bar{g}^{\alpha \rho} \bar{g}^{\beta \sigma} F_{\rho \sigma} \]  
(11)

with:

\[ \bar{g}^{\alpha \rho} \bar{g}_{\rho \beta} = \delta^\alpha_\beta \]  
(12)

If the line-element were Minkowski’s metric, i.e. if \(c(T) = c_0\) were constant and \(k\) were zero, the electromagnetic field \(F_{\alpha \beta}\) of a point monopole charge, say \(e_0\), would be:

\[ F_{10}(T, R) = \frac{e_0}{4\pi \epsilon_0 R^2}, \quad F_{20} = F_{30} = 0, \quad F_{ij} = 0 \]  
(13)

where \(i, j, \ldots = 1, 2, 3\).

We want to find the corresponding solution of Maxwell equations under the assumption that \(\epsilon(T), \mu(T)\) as well as \(e(T)\) may be functions of \(T\), and that \(k \neq 0\) may change the \(R\) dependence. More precisely we look for a solution of the following form:

\[ F_{10}(T, R) = \frac{e(T)}{4\pi \epsilon(T) D(R)^2}, \quad F_{20} = F_{30} = 0, \quad F_{ij} = 0 \]  
(14)

This field already satisfies Eqs. (9). From (8) it follows that the non zero components of \(\bar{g}_{\alpha \beta}\) are:

\[ \bar{g}_{00} = -F(T)^{-2}, \quad \bar{g}_{11} = c(T)^{-2}(1-kR^2)^{-1}, \quad \bar{g}_{22} = \bar{g}_{33} = \sin^{-2} \theta = c(T)^{-2} R^2 \]  
(15)

and from (7), (11) and (12) it follows that the non zero component of \(G^{\alpha \beta}\), which reduces to \(-e_0/(4\pi \epsilon_0 R^2)\) in Minkowski’s space-time, is:

\[ G^{10} = -(1-kR^2) \frac{e_0^4 e(T)}{4\pi F(T)^2 D(R)^2} \]  
(16)

Giving to \(\beta\) the value 0, Eq. (10) gives:

\[ D(R)^2 = R^2 \sqrt{1 - k R^2} \]  
(17)

and giving to \(\beta\) the value 1, Eq. (10) gives:
\[ e(T) = e_0 F(T)^{-1} \]  

(18)
e_0 \text{ being the value of } e(T) \text{ at the time of reference when } c(T) = c_0.

From (15) it follows that:

\[ \sqrt{-\bar{g}} = F^{-1} \sqrt{-g}, \quad \bar{g} = \det(\bar{g}_{\alpha\beta}), \]  

(19)
therefore, using (11), Eq. (10) can be written:

\[ \partial_{\alpha}(\sqrt{-g}(F/\mu) \bar{F}^{\alpha\beta}) = 0, \quad \bar{F}^{\alpha\beta} = g^{\alpha\rho} g^{\beta\sigma} F_{\rho\sigma} \]  

(20)
Although not compelling it is attractive to make the further assumption:

\[ \mu(T) = \mu_0 F(T) \]  

(21)
\(\mu_0\) being the value of \(\mu(T)\) when \(c(T) = c_0\). This amounts to make an assumption on the cosmological medium, namely that \((F_{\rho\sigma}, \bar{F}^{\alpha\beta})\) will behave as it does the electromagnetic field in a perfect vacuum in a space-time with metric (8). This allows, mutatis mutandis to refer, if necessary, to electromagnetic physics as usual.

From (7) and our new assumption (21) we have then:

\[ \epsilon(T) = \epsilon_0 F(T) \]  

(22)
\(\epsilon_0\) being the value of \(\epsilon(T)\) when \(c(T) = c_0\). This completes the determination of the electromagnetic concomitants to a varying speed of light in cosmology.\(^5\)

4 \(h \text{ is a function of time and } \alpha \text{ it is not}\)

Although both the initial interpretation of Sect. 2 which led to (7) and the assumption in Sect. 3 that led to (22) are speculative we feel that they are sufficiently justified to deserve a calm evaluation. The considerations that follow on the contrary are at this time very loosely motivated and are given as possible candidates to a time variation of some other fundamental quantities, according to our personal ingenuity.

Let us consider the fine structure constant:

\[ \alpha(T) = \frac{e(T)^2}{2\epsilon(T) h(T)c(T)^2} \]  

(23)
\(^5\text{An Ad hoc field theoretical theory describing a space-time dependence } e(x^\alpha) \text{ was proposed in [7].}\)
where we have assumed that Plank’s constant $h$ can be a function of time. From the preceding section we get:

$$\frac{\dot{\alpha}(T)}{\alpha(T)} = -2H(T) - \frac{\dot{h}(T)}{h(T)}$$

(24)

where $H(T) = \dot{F}(T)/F(T)$ is the Hubble’s constant. If we assume that $h$ does not depend on $T$ and we consider that $H_0$, the Hubble’s constant now is of the order of 70 Km/s/Mpc we obtain:

$$\frac{\dot{\alpha}_0}{\alpha_0} = 7.1 \times 10^{-11} \text{ yr}^{-1}$$

(25)

which is a ridiculous large figure compared with present estimates (9):

$$\frac{\dot{\alpha}_0}{\alpha_0} \approx 2.2 \times 10^{-16} \text{ yr}^{-1}$$

(26)

Therefore we can accept either that the Plank’s constant is indeed constant and then our model of varying speed of light is grossly inconsistent with observations or that (26) supports, for the time being, a variation of the form:

$$h(T) = h_0 F(T)^{-2}$$

(27)

i.e.: that the fine structure constant is indeed constant, with a precision of $\approx 1$ part in $10^5$.

5 The masses of elementary particles

Our next step of speculation concerns the time dependence of the masses of elementary particles. In (4) we proposed a model to describe the time-dependent local dynamics of an otherwise isolated gravitational system influenced by the global or local cosmology. This, in the particular case of a spherically symmetric compact mass $m_0$, led to the consideration, at the lowest non trivial approximation, to describe this dynamics by the Lagrangian:

$$L = \frac{1}{2} F(T)^2(\dot{R}^2 + R^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)) - \frac{Gm_0}{R} F(T)^{-1}$$

(28)

In (4) we mentioned that the last term in this formula suggested that $G$ was a time dependent constant. This is actually in contradiction with the fact that the constancy of $G$ is essential to give an unambiguous meaning to

\footnote{In [8] a functional dependence $h(c)$ was discussed}
Therefore we claim now that a better interpretation is to propose the following time-variation of the mass $m(T)$ of elementary particles:

$$m(T) = m_0 F(T)^{-1}$$  \hfill (29)

It is rather reassuring the fact that this implies that $e(T)/m(T)$ remains a true constant as well as the Compton wave-length and the classical radius of elementary particles:

$$\lambda_c = \frac{h(T)}{m(T)c(T)} = \lambda_c(T_0), \quad r_0 = \frac{e(T)^2}{4\pi\epsilon(T)m(T)c(T)^2} = r_0(T_0)$$  \hfill (30)

We point out below another nice consequence of this choice \textsuperscript{29}. Let us consider the Klein-Gordon equation for a massive scalar field:

$$\left(\square - \frac{m(T)^2c(T)^4}{h(T)^2}\right)\Phi = 0$$  \hfill (31)

Using the line-element \textsuperscript{11} this equation becomes:

$$\left(-\partial^2_T - 3H(T)\partial_T + c(T)^2\Delta - \frac{m(T)^2c(T)^4}{h(T)^2}\right)\Phi = 0$$  \hfill (32)

$\Delta$ being the Laplacian of the space metric:

$$d\bar{S}^2 = \frac{dR^2}{1 - kR^2} + R^2d\Omega^2$$  \hfill (33)

Using \textsuperscript{2}, \textsuperscript{27} and \textsuperscript{29}, \textsuperscript{29}, \textsuperscript{32} becomes:

$$\left(-\partial^2_T - 3H(T)\partial_T + c(T)^2(\Delta - \frac{m_0^2c_0^2}{h_0^2})\right)\Phi = 0.$$  \hfill (34)

$n_I$ being the collection of eigen-values of the Laplacian operator $\Delta$ and $J_I(X^i)$ being the corresponding eigen-functions\textsuperscript{7}:

$$\triangle J_I(x^i) = n_I J_I(x^i)$$  \hfill (35)

the modes of the scalar field $\Phi(x^\alpha)$ that define the quantum vacuum are a complete set of solutions of this equation of the following form:

$$\varphi_I(T,x^i) = u_I^+(T) J_I(x^i),$$  \hfill (36)

\textsuperscript{7} for details see \textsuperscript{10}, \textsuperscript{11} and references therein
which is the union of two sets of positive and negative energy.\(^8\)

Eq. (34) has to be compared with the most commonly used:

\[
\left(-\partial_T^2 - 3H(T)\partial_T + F(T)^{-2}c_0^2\Delta - \frac{m_0^2 c_0^2}{\hbar_0^2}\right) \Phi = 0 \quad (37)
\]

where one assumes that \(c_0, h_0\) and \(m_0\) are universal constants. The comparison shows that using Eq. (34) a non zero mass \(m_0\) just shifts by a constant the eigen-values \(n_I\) but keeps invariant the functional dependence of the modes with respect to the space-time variables. In other words once the modes for \(m_0 = 0\) are known the modes with \(m_0 \neq 0\) are trivially deduced from them.

## 6 Conclusion

Assuming that a fundamental constant, say \(c\), is in fact a function of time requires a close examination of all the other constants, here called concomitants, that with \(c\) enter in expressions that have a physical meaning. This paper has been an attempt to present a coherent scheme to fulfil such requirement. The points that we want to remind or emphasize are the following:

1.- General relativity and Robertson-Walker models, conveniently interpreted, describe a time-dependence of the speed of light. There is no need to graft any new field theoretical theory to have a varying speed of light.

Our approach is a simple example of a more general one, based on a full fledged theory of frames of reference, that considers the speed of light in a round trip to be in general anisotropic and space-time dependent.

2.- The time dependence of \(e(T)\) follows from a particular application of the equivalence principle for short intervals of time. It guarantees that at this level of approximation, as far as electromagnetism is concerned, the cosmological medium behaves as vacuum behaves in local physics.

3.- We have assumed that \(\alpha\) is a true constant. This is as much conservative as an assumption as it is reckless to take for granted that it is a function of time. It must be understood that what we claim is that in our general framework, where a few constants \(x\) have indeed a time dependence of the form \(x(T) = x_0 F(T)^n\), in particular \(n = 0\) for \(\alpha\), but also for \(r_0\) and \(\lambda_c\).\(^9\) But this is, we claim, the ground level behaviour and we do not exclude that \(\alpha\) as any other of the quantities that we have considered can have slightly different behaviours if perturbations of the background cosmological models are necessary to describe more realistic ones.

\(^8\)See also \(12\)

\(^9\)or the Bohr radius of an hydrogen atom \(a_0 = \epsilon(T)h(T)^2/(\pi m(T)e(T)^2)\)
Notice also that we could have assumed that \( r_0 \) and \( \lambda_c \) were true constants and derive from this the time dependence of \( h(T) \) and the true constancy of \( \alpha \). This would emphasize the primacy of the dimension length and the dependence of the dimension velocity as being derived from measures of length and time.

4.-The time dependence of \( m(T) \) follows from more indirect considerations, and is actually a substitute for a frequent claim about a time dependence of \( G \), which follows from other theories but would be contradictory with our approach.

The fact that the time dependence of \( c(T) \) and the derived dependences of \( h(T) \) and \( m(T) \) leads to a different relationship between the massless and massive modes of quantized fields in a cosmological context provides an example where, without entering into deep metrology problems, accepting time dependence of fundamental constants can lead to interesting new scenarios.

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