ON THE ABSOLUTE VALUE OF THE PRODUCT AND THE SUM OF LINEAR OPERATORS

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Abstract. Let $A, B \in B(H)$. In the present paper, we establish simple and interesting facts on when we have $|A||B| = |B||A|$, $|AB| = |A||B|$, $|A \pm B| \leq |A| + |B|$, $||A| - |B|| \leq |A \pm B|$ and $||A| - |B|| \leq ||A \pm B||$, where $|\cdot|$ denotes the absolute value (or modulus) of an operator. The results give some other interesting consequences.

1. Introduction

Let $H$ be a complex Hilbert space and let $A, B \in B(H)$. We say that $A$ is positive, and we write $A \geq 0$, if $< Ax, x > \geq 0$ for all $x \in H$. Since $H$ is a complex Hilbert space, a positive operator is clearly self-adjoint. We say that $A \geq B$ if they are both self-adjoint and $A - B \geq 0$. Recall also that if $A \geq 0$, then there is a unique positive operator $B$ such that $B^2 = A$. We call it the (positive) square root of $A$ and we denote it by $\sqrt{A}$ (or $A^{1/2}$). Next, we gather basic results on square roots of sums and products.

Lemma 1.1. Let $A, B \in B(H)$ be such that $AB = BA$ and $A, B \geq 0$. Then

- $AB \geq 0$.
- $\sqrt{AB} = \sqrt{A}\sqrt{B}$.
- $\sqrt{A + B} \leq \sqrt{A} + \sqrt{B}$.

The unique positive square root of the positive operator $A^*A$ is commonly known as the absolute value (or modulus) of $A$. We denote it by $|A|$, that is, $|A| = \sqrt{A^*A}$. Notice that $\|A\| = \| |A| \|$ always holds.

We usually warn students to be careful with this notation as it may mislead them to think that e.g.

$$|A| = |A^*|, \ |A + B| \leq |A| + |B| \text{ or } |AB| = |A||B|$$

would hold. Counterexamples are easily found in the setting of 2 by 2 matrices. Notice that $A$ is normal if and only if $|A| = |A^*|$. Also, a priori if $A, B$ are arbitrary, then there is no reason why we should expect $|A||B| = |B||A|$ to hold (for instance, just think of positive operators).

2010 Mathematics Subject Classification. Primary 47A63, Secondary 47A62, 47B15, 47B20.

Key words and phrases. Absolute Value. Triangle Inequality. Normal, Hyponormal, Self-adjoint and Positive Operators. Commutativity. Fuglede Theorem.
Even when $AB = BA$, the equality $|A||B| = |B||A|$ need not hold. For example, let 

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

Then as we can easily verify:

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = BA$$

whereas $|A||B| \neq |B||A|$.

Observe that we have purposely avoided normal operators in our counterexample (cf. Proposition 2.1).

The main aim of this paper is to investigate when relations of the types

- $|A||B| = |B||A|$;
- $|A||B| = |A||B|$;
- $|A + B| \leq |A| + |B|$;
- $|A - |B|| \leq |A + B|$;
- $\|A - |B|| \leq \|A + B\|$;

hold. It turns out that normality and sometimes hyponormality plus commutativity are sufficient for these relations to hold. This comes to corroborate the resemblance to complex numbers which is already known to many. Notice also that commutativity is not unnatural as we already have it in $(\mathbb{C}, \times)$.

The idea here is to start from scratch, and use as basic results as possible to make the paper accessible to a wide audience. We note that for example, we have wittingly avoided the use of the spectral theorem of normal operators. Therefore, most of the results here can be taught at elementary courses in Operator Theory.

It is worth noticing that there is a big amount of papers which have dealt with inequalities involving absolute values and/or norms of operators. The literature is so rich that we rather refer readers to books which have gathered most of these results. For example, see [1], [4] and [10].

Finally, we assume the reader is familiar with other basic results on Operator Theory. A well established reference is [2]. We do recall two crucial results though.

**Theorem 1.2.** (Löwner-Heinz Inequality, see [3] for a simple proof) If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.

**Remark.** It is known to readers that $A \geq B \geq 0$ implies that $A^2 \geq B^2$ when $AB = BA$.

Since we will be dealing with sums and products of commuting normal operators, the use of the celebrated Fuglede-Putnam theorem is inevitable. The following lemma will be used below without further notice.

**Lemma 1.3.** Let $A, B \in B(H)$ where $A$ is normal. Then, we have

$$AB = BA \iff A^*B = BA^* \iff AB^* = B^*A \iff A^*B^* = B^*A^*.$$
2. Main Results: Absolute Value and Products

We start with the following:

**Proposition 2.1.** Let \( A, B \in B(H) \) be such that \( AB = BA \). If \( A \) is normal, then \(|A||B| = |B||A|\).

*Remark.* The preceding result was proved in [9] by assuming that both \( A \) and \( B \) are normal.

*Proof.* Since \( AB = BA \) and \( A \) is normal, we have \( A^*B = BA^* \). We then clearly have from the previous two relations:

\[
AB = BA \implies A^*AB = A^*BA = BA^*A.
\]

Hence

\[
|A|B = B|A|.
\]

Since \( |A| \) is self-adjoint, the previous equality gives (by taking adjoints) \(|A|B^* = B^*|A|\). Hence

\[
B^*|A|B = |A|B^*B \implies B^*|A| = |A|B^* \implies |B||A| = |A||B|,
\]

as required. \( \square \)

We have already observed above that in general \(|AB| \neq |A||B|\). The following result is somewhat inspired by a one in [5].

**Theorem 2.2.** Let \( A, B \in B(H) \) be self-adjoint such that \( AB \) is normal. Then

\[
|AB| = |A||B|.
\]

*Remark.* It was noted in [10] that if \( S, T \) are two non-commuting self-adjoint operators, then the inequality \(|ST| \leq |S||T|\) never holds. So, in our result the normality of the product transforms the non valid inequality into a true full equality.

*Remark.* Notice that \( AB \) being a normal product of two self-adjoint operators does not necessarily imply that \( AB \) is self-adjoint, i.e. we do not necessarily have \( AB = BA \). If, however, we impose further that \( A \geq 0 \) (or \( B \geq 0 \)), then \( AB \) becomes self-adjoint. See e.g. [7].

*Proof.* Since \( A \) and \( B \) are self-adjoint, we may write

\[
B(AB) = BAB = (AB)^*B.
\]

Since \( AB \) and \((AB)^*\) are normal, the Fuglede-Putnam theorem gives

\[
B(AB)^* = (AB)^{**}B \text{ or merely } B^2A = AB^2.
\]

Consequently, \( B^2A^2 = AB^2A = A^2B^2 \).

On the other hand, we easily see that

\[
|AB|^2 = (AB)^*AB = AB(AB)^* = AB^2A = A^2B^2
\]
and so 
\[ |AB| = \sqrt{A^2 B^2} = \sqrt{A^2} \sqrt{B^2} = |A||B|, \]
as required. □

Since \(|AB|\) is self-adjoint, we have:

**Corollary 2.3.** Let \(A, B \in B(H)\) be self-adjoint such that \(AB\) is normal. Then \(|A||B|\) is self-adjoint, i.e. \(|A||B| = |B||A|\). Moreover, if we also assume that \(A, B \geq 0\), then \(AB \geq 0\).

The assumptions of the previous theorem cannot just be dropped. We give a counterexample for each hypothesis.

- Let 
\[ A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]
Then each of \(A\) and \(B\) is self-adjoint but \(AB\) is not normal for 
\[ AB = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} \]
We can easily check that 
\[ |AB| = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad |A| = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad |B| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
i.e.
\[ |AB| \neq |A||B|. \]

- Let 
\[ A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}. \]
Then, neither \(A\) nor \(B\) is normal. Their product \(AB\) is, however, self-adjoint (hence normal!) because 
\[ AB = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \]
Next, we have 
\[ |A| = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad |B| = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad |AB| = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \]
Accordingly, 
\[ |AB| = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \neq \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = |A||B|. \]

An akin result to Theorem 2.2 is:

**Theorem 2.4.** Let \(A, B \in B(H)\) be such that \(AB = BA\). If \(A\) is normal, then 
\[ |AB| = |A||B|. \]
Remark. It was also noted in [6] that if $S, T$ are two commuting normal operators, then the inequality $|ST| \leq |S||T|$ holds. So, our result here is stronger.

Proof. Since $AB = BA$ and $A$ is normal, we get $A^*B = BA^*$ or $AB^* = B^*A$. Hence

$$|AB|^2 = (AB)^*AB = B^*A^*AB = A^*B^*AB = A^*AB^*B.$$  

By Proposition 2.1, $A^*AB^*B = B^*BA^*A$. Consequently,

$$|AB| = \sqrt{A^*AB^*B} = \sqrt{A^*A\sqrt{B^*B} = |A||B|},$$

as required.

Before generalizing the previous result, we give some direct consequences. The first one is a funny application.

Corollary 2.5. Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ and $B$ are normal, then

$$|AB| = |A^*B| = |AB^*| = |A^*B^*| = |B^*A^*| = |B^*A| = |BA|.$$  

Proof. Since $A$ and $B$ are normal, $|A| = |A^*|$ and $|B| = |B^*|$. As $AB = BA$, then $|A||B| = |B||A|$ for $A$ (or $B$!) is normal. Now, apply Theorem 2.4 to each of the eight products.

Corollary 2.6. Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ is normal and $B$ is invertible, then

$$|AB^{-1}| = |A||B^{-1}|.$$  

Proof. Since $AB = BA$ and $B$ is invertible, we have $AB^{-1} = B^{-1}A$. Theorem 2.4 does the remaining job.

Corollary 2.7. Let $A \in B(H)$ be normal and invertible. Then

$$|A^{-1}| = |A|^{-1}.$$  

Proof. It is clear that

$$I = |AA^{-1}| = |A||A^{-1}|.$$  

So, the self-adjoint $|A|$ is right invertible and so it is invertible (cf. [3]) and:

$$|A|^{-1} = |A^{-1}|.$$  

Theorem 2.4 may be generalized as follows:

Proposition 2.8. Let $(A_i)_{i=1, \ldots, n}$ be a family of pairwise commuting elements of $B(H)$. If all $(A_i)_{i=1, \ldots, n}$ but one are normal, then

$$|A_1A_2 \cdots A_{n-1}A_n| = |A_1||A_2| \cdots |A_{n-1}||A_n|.$$
Proof. If \( A_1, A_2, \cdots, A_{n-1} \) are normal, then just apply the preceding theorem by using a proof by induction. Otherwise, just use commutativity to push the non normal factor to the right as many times as possible until it will be the last factor on the right of the product \( \prod_{i=1}^{n} A_i \). Then proceed as just indicated three lines above.

It is simple to see that \( |A^2| = |A|^2 \) does not hold in general. For instance, let

\[
A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.
\]

Then

\[
|A^2| = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \neq |A|^2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.
\]

But for normal \( A \), things are better.

**Corollary 2.9.** Let \( A \in B(H) \) be normal and invertible. Let \( n \in \mathbb{Z} \). Then

\[
|A^n| = |A|^n.
\]

**Proof.** The case \( n \geq 0 \) follows from Proposition 2.8. The case \( n < 0 \) follows from Proposition 2.8 and Corollary 2.7.

3. **Main Results: Absolute Value and Sums**

We now turn to the triangle inequality w.r.t. \( | \cdot | \). We have two different versions.

Before all else, we state a result (perhaps known to many) which will be called on later. Its proof relies on the following yet simpler result.

**Lemma 3.1.** If \( A \in B(H) \) is anti-symmetric, i.e. \( A^* = -A \), then \( A^2 \leq 0 \).

**Lemma 3.2.** Let \( T \in B(H) \) be hyponormal (i.e. \( TT^* \leq T^*T \), that is, \( \|T^*x\| \leq \|Tx\| \) for all \( x \in H \)). Then

\[
\text{Re} T = \frac{T + T^*}{2} \leq \sqrt{T^*T} = |T|.
\]

**Proof.** It is clear that \( T - T^* \) is anti-symmetric and so by Lemma 3.1

\[
(T - T^*)^2 \leq 0 \iff T^2 + T^* - TT^* - T^*T \leq 0.
\]

Hence

\[
(T - T^*)^2 \leq 0 \iff T^2 + T^* - TT^* - T^*T \leq 0.
\]

But, \( T \) is hyponormal and so \(-TT^* - T^*T \geq -2T^*T \). So,

\[
T^2 + T^* - 2T^*T \leq 0
\]

or

\[
T^2 + T^* + TT^* + T^*T \leq T^2 + T^* + 2T^*T \leq 4T^*T.
\]

Therefore,

\[
(T + T^*)^2 \leq 4T^*T \text{ or } |T + T^*| \leq 2|T|
\]

by Theorem 1.2.
Remembering that \( S^- = \frac{1}{2}(|S| - S) \geq 0 \) whenever \( S \) is self-adjoint, we conclude that
\[
T + T^* \leq |T + T^*| \leq 2|T| = 2\sqrt{T^*T},
\]
as required. \( \square \)

The following fairly simple result is also useful to us.

**Lemma 3.3.** Let \( A, B \in B(H) \) such that \( A \) is normal and \( B \) is hyponormal. If \( AB = BA \), then \( A^*B \) is hyponormal.

**Proof.** Let \( x \in H \). As \( AB = BA \), by the normality of \( A \) and the hyponormality of \( B \) we have
\[
\|(A^*B)^*x\| = \|B^*Ax\| \leq \|BAx\| = \|ABx\| = \|A^*Bx\|,
\]
establishing the hyponormality of \( A^*B \). \( \square \)

Here is the first version of the triangle inequality.

**Theorem 3.4.** Let \( A, B \in B(H) \) be such that \( AB = BA \). If \( A \) is normal and \( B \) is hyponormal, then the following triangle inequality holds:
\[
|A + B| \leq |A| + |B|.
\]

**Proof.** Since \( A \) is normal and \( AB = BA \), we know from Proposition 2.1 that \( |A||B| = |B||A| \). Hence
\[
|A + B|^2 \leq (|A| + |B|)^2 \iff (A + B)^* (A + B) \leq A^*A + B^*B + 2\sqrt{A^*A\sqrt{B^*B}}
\]
\[
\iff A^*B + B^*A \leq 2\sqrt{A^*A\sqrt{B^*B}}.
\]

We already know from above that \( \sqrt{A^*A\sqrt{B^*B}} = \sqrt{A^*AB^*B} \). So, to prove the desired triangle inequality, we are only required to prove
\[
A^*B + B^*A \leq 2\sqrt{A^*AB^*B}.
\]

But
\[
A^*AB^*B = AA^*B^*B = AB^*A^*B = B^*AA^*B.
\]

If we set \( T = A^*B \), then are done with the proof if we come to show that the following holds:
\[
T + T^* \leq 2\sqrt{T^*T}.
\]

But this is just Lemma 3.2 once we show that \( A^*B \) is hyponormal. This is in effect the case as \( A^*B \) is hyponormal by Lemma 3.3.

Therefore, under the assumptions of our theorem we have shown that
\[
|A + B|^2 \leq (|A| + |B|)^2.
\]

Hence, by Theorem 1.2 we have ended up with
\[
|A + B| \leq |A| + |B|,
\]
and this is precisely what we wanted to prove. \( \square \)
Remark. The foregoing result need not hold if commutativity is dropped even if $A$ and $B$ are self-adjoint. The reader may check this easily via the following example:

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

**Corollary 3.5.** Let $T \in B(H)$ be normal. Then

$$|T| \leq |\text{Re}T| + |\text{Im}T|.$$ 

**Proof.** Write $T = \text{Re}T + i\text{Im}T$ where $\text{Re}T$ and $\text{Im}T$ are commuting self-adjoint operators. Then apply Theorem 3.4. \qed

We have another simple consequence of Theorem 3.4.

**Corollary 3.6.** Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ is normal and $B$ is hyponormal, then the following triangle inequality holds:

$$|A - B| \leq |A| + |B|.$$ 

**Proof.** Since $AB = BA$, we know that $A(-B) = (-B)A$. Also $-B$ is hyponormal. Then, apply Theorem 3.4. \qed

Theorem 3.4 may be generalized to a finite sum of operators. Before, recall that the sum of two commuting normal operators remains normal. This too may be generalized (the proof by induction is omitted).

**Proposition 3.7.** Let $(A_i)_{i=1,\ldots,n}$ be a family of normal pairwise commuting elements of $B(H)$. Then $A_1 + A_2 + \cdots + A_n$ is normal.

We are ready for the promised generalization of Theorem 3.4 whose proof is again a proof by induction.

**Corollary 3.8.** Let $(A_i)_{i=1,\ldots,n}$ be a family of pairwise commuting elements of $B(H)$. If all $(A_i)_{i=1,\ldots,n}$ are normal except one which is assumed to be hyponormal, then

$$|A_1 + A_2 + \cdots + A_n| \leq |A_1| + |A_2| + \cdots + |A_n|.$$ 

It is known that an inequality of the type $|||A| - |B||| \leq ||A \pm B||$ is not true in general even if $A$ and $B$ are self-adjoint.

**Proposition 3.9.** Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ and $B$ are normal, then the following inequality holds:

$$|||A| - |B||| \leq ||A + B||.$$ 

Probably the following lemma has been noted elsewhere but we state it here anyway with a proof.

**Lemma 3.10.** Let $S, T \in B(H)$ be self-adjoint where $S \geq 0$. If $-S \leq T \leq S$, then $||T|| \leq ||S||$. 

Proof. By assumption, for all $x \in H$

$- < Sx, x > \leq < Tx, x > \leq < Sx, x >$ or merely $| < Tx, x > | \leq < Sx, x >$

Therefore,

$$\|T\| = \sup_{\|x\|=1} |< Tx, x >| \leq \sup_{\|x\|=1} < Sx, x > = \|S\|,$$

as desired. □

Let us prove Proposition 3.9.

Proof. Since $A$ and $B$ are commuting normal operators, we know that $A + B$ too is normal. Since $A + B$ commutes with $B$, by Corollary 3.6 we have

$$|A| = |A + B - B| \leq |A + B| + |B| \Rightarrow |A| - |B| \leq |A + B|.$$

Similarly, as $A + B$ commutes with $A$, we get

$$|B| - |A| \leq |A + B|.$$

Whence

$$-|A + B| \leq |A| - |B| \leq |A + B|.$$

By Lemma 3.10 (and remembering that $\|T\| = \|T\|$ for $T \in B(H)$), we obtain

$$\||A| - |B|| \leq \|A + B\| = \|A + B\|,$$

as required. □

Remark. In the previous proposition, if $B$ is only hyponormal, then at the moment we are only sure that:

$$|B| - |A| \leq |A + B|$$

because we can only prove that $A + B$ is hyponormal. We will remedy this little problem shortly.

Corollary 3.11. Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ and $B$ are normal, then the following inequality holds:

$$\||A| - |B|| \leq \|A - B\|.$$

Proposition 3.9 can be improved as it is a particular case of the following remarkable result:

Proposition 3.12. Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ is normal and $B$ is hyponormal, then the following inequality holds:

$$\||A| - |B|| \leq \|A - B\|.$$

Proof. We easily see as $|A||B| = |B||A|$ that

$$||A| - |B||^2 \leq |A - B|^2 \iff |A|^2 + |B|^2 - 2|A||B| \leq |A|^2 + |B|^2 - A^*B - B^*A$$

$$\iff A^*B + B^*A \leq 2\sqrt{A^*A}\sqrt{B^*B}$$

$$\iff A^*B + B^*A \leq 2\sqrt{B^*AA^*B}. $$
But, this is always true in virtue of Lemma 3.2 as $A^*B$ is hyponormal. Therefore, we have shown
\[ ||A| - |B||^2 \leq |A - B|^2. \]

A glance at Theorem 1.2 finally gives
\[ ||A| - |B|| \leq |A - B|. \]

\[ \square \]

**Corollary 3.13.** Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ is normal and $B$ is hyponormal, then the following inequality holds:
\[ ||A| - |B|| \leq |A + B|. \]

**Proof.** Since $B$ is hyponormal, so is $-B$. The rest is obvious. \[ \square \]

Here is the improvement of Proposition 3.9:

**Corollary 3.14.** Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ is normal and $B$ is hyponormal, then the following inequality holds:
\[ ||A| - |B|| \leq ||A \pm B||. \]

**Proof.** By Proposition 3.9 and Corollary 3.13 we know that
\[ ||A| - |B|| \leq |A \pm B|. \]

Then, calling on Lemma 3.10 yields
\[ ||A| - |B|| = |||A| - |B|| | \leq || A \pm B || = || A \pm B ||. \]

\[ \square \]

If we want to drop commutativity in Theorem 3.4, then this is at the cost of adding an extra condition. Also, we only have to assume that one of the two operators is normal.

**Theorem 3.15.** Let $A, B \in B(H)$ be such that $AB = BA$. If $A$ is normal and $A^*B + B^*A \leq 0$, then
\[ |A + B| \leq |A| + |B|. \]

**Proof.** Clearly,
\[ (A + B)^*(A + B) = A^*A + A^*B + B^*A + B^*B. \]

As $A^*B + B^*A \leq 0$, then
\[ A^*A + A^*B + B^*A + B^*B \leq A^*A + B^*B. \]

By Theorem 1.2 we have
\[ |A + B| = \sqrt{A^*A + A^*B + B^*A + B^*B} \leq \sqrt{A^*A + B^*B}. \]

Since $AB = BA$ and $A$ is normal, Proposition 2.4 implies that $|A||B| = |B||A|$ or $|A|^2|B|^2 = |B|^2|A|^2$. Finally, Lemma 1.1 does the remaining job, i.e. it gives us
\[ |A + B| \leq |A| + |B| \]

and this completes the proof. \[ \square \]
ACKNOWLEDGEMENT

The author wishes to thank Mr. S. Dehimi for a discussion which led to a slight improvement of the result of Theorem 3.3.

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