SINGULAR CHAINS ON TOPOLOGICAL STACKS

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Abstract. We extend the functor Sing of singular chains to the category of topological stacks and establish its main properties. We prove that Sing respects weak equivalences and takes a morphism of topological stacks that is both a Serre and a Reedy fibration to a Kan fibration of simplicial sets. When restricted to the category of topological spaces Sing coincides with the usual singular functor.

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1. Introduction

Given a topological stack $\mathcal{X}$ we define the simplicial set $\text{Sing}(\mathcal{X})$ of singular chains on $\mathcal{X}$ and establish its main properties. This generalizes the usual functor $\text{Sing} : \text{Top} \to \mathbf{sSet}$ of singular chains on topological spaces. We address the following questions about $\text{Sing}(\mathcal{X})$:

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functoriality with respect to morphisms of stacks, the homotopy type of Sing($X$), and effect on fibrations of topological stacks.

**Functoriality and the homotopy type of Sing($X$).** There are several ways to define the homotopy type of a topological stack (see for instance [Be, Ha, Mo, No12]). In [No12] the notion of classifying space of a topological stack is introduced to give a better grip on the functoriality of the homotopy type. Nevertheless, the functoriality of the classifying space only makes sense in the homotopy category of topological spaces, that is, the classifying space is a functor CS : topStack $\to$ Ho(Top).

Our construction of singular chains in this paper enhances this by giving us an honest functor $\text{Sing} : \text{topStack} \to \text{sSet}$. When restricted to the subcategory Top, this functor coincides with the usual singular functor on topological spaces. The functor $\text{Sing} : \text{topStack} \to \text{sSet}$ in facts lifts the classifying space functor $\text{CS} : \text{Ho(Top)} \to \text{Ho(sSet)}$.

This is a consequence of one of our main results (Theorem 11.2).

**Theorem 1.1.** Let $f : X \to Y$ be a weak equivalence of Serre stacks. Then, $\text{Sing}(f) : \text{Sing}(X) \to \text{Sing}(Y)$ is a weak equivalence of simplicial sets. In particular, if $X \to X$ is a classifying space for $X$, then the induced map $\text{Sing}(X) \to \text{Sing}(X)$ is a weak equivalence.

In particular, $\text{Sing}(X)$ has the same homotopy type as the classifying space $\text{CS}(X)$. Somewhat surprisingly, the proof of the above theorem is highly nontrivial.

**Effect on fibrations of topological stacks.** It is well known that for a Serre fibration $f : X \to Y$ of topological spaces, the induced map $\text{Sing}(f) : \text{Sing}(X) \to \text{Sing}(Y)$ is a Kan fibration of simplicial sets. The corresponding statement for topological stacks, however, should be formulated more carefully, as there are various notions of fibrations between topological stacks. For example, the above statement would clearly be false if we use the notion of Serre fibration for topological stack as in ([No14], Definition 3.6), because this notion is “intrinsic” (i.e., is invariant under replacing a stack by an equivalent stack – in particular, any equivalence of topological stacks $f : X \to Y$, such as the inclusion of a point into a trivial groupoid, is automatically a Serre fibration).

It turns out, the correct condition on a morphism $f : X \to Y$ to ensure that $\text{Sing}(f) : \text{Sing}(X) \to \text{Sing}(Y)$ is a Kan fibration is that $f$ is a Serre fibration and also a Reedy fibration (Definition 7.8). This is another main result of the paper (Theorem 10.5).

**Theorem 1.2.** Let $p : X \to Y$ be a morphism of Serre topological stacks that is a (weak) Serre fibration and also a Reedy fibration. Then, $\text{Sing}(p) : \text{Sing}(X) \to \text{Sing}(Y)$ is a (weak) Kan fibration.

We point out that the Reedy condition can always be arranged for any morphism of stacks: given $f : X \to Y$ we can replace $X$ by an equivalent stack $\tilde{X}$ such that the corresponding morphism $f' : \tilde{X} \to Y$ is a Reedy fibration (Proposition 7.12). Such a replacement would not affect the property of being a Serre fibration.

The paper is organized as follows. In Section 3 we set up the terminology and review some generalities about stacks and topological stacks. In Section 4 we introduce the tilde construction. This is the left Kan extension of the inclusion $\Delta \to \text{Top}$ and plays a crucial role in the rest of the paper. In Sections 5-6 we review some basic facts about homotopy of maps between morphisms of stacks. We also recall the relevant background on fibrations of
stacks. The only new notion in this section is that of a restricted homotopy which is related to the tilde construction introduced in Section 4.

In Section 7 we look at various model structures on $\text{Gpd}$, $\text{pshGpd}$ and $\text{sGpd}$, and establish some of their properties which are, presumably, well known but we have been unable to locate them in the literature. The notion of Reedy fibration of stacks (Definition 7.8) introduced and studied in this section is central to the paper. It is an adaptation of Reedy fibration of simplicial groupoids.

We introduce the functor $\text{Sing} : \text{topStack} \to \text{sSet}$ in Section 8. Section 9 is the technical heart of the paper where we prove a list of lemmas which play key role in the proofs of our main results. In Section 10 we prove the first main result of the paper, namely, that if $\to$ implies that the (singular simplicial set of the) classifying space of a topological stack is naturally weakly homotopy equivalent to $\text{Sing}(\bullet)$, see Proposition 11.1.

2. Notation and terminology

The category of finite ordinal numbers with order preserving maps between them is denoted by $\Delta$. The simplicial $n$-simplex is denoted by $\Delta^n := \text{Hom}_\Delta(\bullet, [n])$. The topological $n$-simplex is denoted by

$$\{\Delta^n\} = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1, x_i \geq 0\}$$

We denote the cosimplicial object $n \mapsto |\Delta^n|$ in $\text{Top}$ by $|\Delta^n|$. The $k$th horn in $\Delta^n$, namely, the sub-simplicial set of $\Delta^n$ generated by the $i$th faces of the unique nondegenerate $n$-cell in $\Delta^n$, $i \in \{0, 1, \ldots, k, \ldots, n\}$, is denoted by $\Lambda^k_n$. When talking about homotopies between maps we often use the notation $[0, 1]$ instead of $|\Delta^1|$.

The bisimplicial set $\Delta^{m,n}$ is the bisimplicial set $\Delta^{m,n} : \Delta^{op} \times \Delta^{op} \to \text{Set}$ represented by $[m, [n]] \in \Delta \times \Delta$. That is, $\Delta^{m,n} := \text{Hom}_\Delta(\bullet, ([m], [n])) = \Delta^m \otimes \Delta^n$.

We denote the category of small groupoids by $\text{Gpd}$ and the category of compactly generated Hausdorff spaces by $\text{Top}$.

We usually use the notation $[\mathcal{C}, \mathcal{D}]$ for functor categories. The categories of presheaves of groupoids over $\text{Top}$, simplicial sets, simplicial groupoids and bisimplicial sets are denoted by $\text{pshGpd}$, $\text{sSet}$, $\text{sGpd}$ and $\text{bsSet}$, respectively.

For a simplicial set $X \in \text{sSet}$, we use the notation $\tilde{X} \in \text{pshGpd}$ for the left Kan extension of $X$ along $\Delta \hookrightarrow \text{Top}$.

We use functional notation $g \circ f$ for composition of 1-morphisms $f : X \to Y$ and $g : Y \to Z$, and multiplicative notation $\alpha \cdot \beta$ (or simply $\alpha \beta$) for composition of 2-isomorphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$. We use the notation $\alpha \circ h$ for the composition of a 2-isomorphism $\alpha : f \Rightarrow g$ between $f, g : X \to Y$ with a morphism $h : Y \to Z$.

For (presheaves of) groupoids $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ we denote their strict fiber product by $\mathcal{X} \times_Y \mathcal{Z}$ and their 2-fiber product by $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$. The notation $\mathcal{X} \cong \mathcal{Y}$ means an isomorphism of (presheaves of) groupoids, and $\mathcal{X} \sim \mathcal{Y}$ means an equivalence of (presheaves of) groupoids.

3. Topological stacks

Throughout the paper, we will work over the base Grothendieck site $\text{Top}$ of compactly generated Hausdorff topological spaces (with the open-cover topology).

3.1. Presheaves of groupoids. We denote the 2-category of presheaves of groupoids over $\text{Top}$ by $\text{pshGpd}$ = $[\text{Top}^{op}, \text{Gpd}]$. We shall denote the objects of this category with calligraphic letters, i.e., $\mathcal{X} \in \text{pshGpd}$. For $T \in \text{Top}$ we call $\mathcal{X}(T)$ the groupoid of $T$-points of $\mathcal{X}$. 
By an equivalence of presheaves of groupoids we mean a morphism \( f : X \to Y \) such that for every \( U \in \text{Top} \), the induced map \( f(T) : X(T) \to Y(T) \) on the \( T \)-points is an equivalence of groupoids. Two presheaves of groupoids are equivalent if there exists a zigzag of equivalences between them.

### 3.2. The 2-category of stacks

By a stack over \( \text{Top} \) we mean a presheaf of groupoids \( X \in \text{pshGpd} \) which satisfies the descent condition:

\[
\lim_{\text{holim}} \left( \prod X(U_i) \rightarrow \prod X(U_{ij}) \rightarrow \prod X(U_{ijk}) \right)
\]

for every \( T \in \text{Top} \) and every open cover \( \{U_i\} \) of \( T \). Morphisms and 2-isomorphisms of stacks are the ones of the underlying presheaves of groupoids. That is, stacks form a full sub-2-category of \( \text{pshGpd} \).

By a topological stack we mean a stack over \( \text{Top} \) which is equivalent to the quotient stack of a topological groupoid \( X = [R \rightrightarrows X] \), with \( R \) and \( X \) topological spaces. A topological stack is Serre if it has a groupoid presentation such that \( s : R \to X \) is locally (on source and target) a Serre fibration. That is, for every \( y \in R \), \( s \) is a Serre fibration from a neighborhood of \( y \) to a neighborhood of \( s(y) \).

Morphisms and 2-isomorphisms of topological stacks are the ones of the underlying presheaves of groupoids, so topological stacks, as well as Serre topological stacks, form a full sub-2-category of \( \text{pshGpd} \). We denote the 2-category of topological stacks by \( \text{topStack} \).

### 3.3. Strict and 2-categorical fiber products

The 2-categories of stacks, topological stacks and Serre topological stacks are all closed under 2-fiber products (in fact, all finite limits), and these are computed as presheaves of groupoids. For future reference, we recall the definition of the 2-fiber product of presheaves of groupoids:

\[
(X \times_Z Y)(T) := X(T) \times_{Z(T)} Y(T), \quad \forall T \in \text{Top},
\]

where the latter is the 2-categorical fiber product of groupoids. The reason for using this nonstandard notation is that in this paper we will also be using strict fiber products of (presheaves of) groupoids as well, for which we use the standard notation:

\[
(X \times Y)(T) = X(T) \times Y(T), \quad \forall T \in \text{Top}.
\]

### 3.4. Yoneda

Via the Yoneda functor \( \text{Top} \to \text{pshSet} \) we may identify \( \text{Top} \) with a full subcategory \( \text{pshSet} \) (or \( \text{pshGpd} \), or \( \text{topStack} \)). That is, we identify \( X \in \text{Top} \) with the functor it represents, namely, \( \text{Hom}_{\text{Top}}(-, X) \); note that this is a set-valued functor, but we can regard a set as a groupoid in which the only morphisms are the identity morphisms. We often abuse notation and use the same notation both for \( X \in \text{Top} \) and for the image of \( X \) in \( \text{pshSet} \) (or \( \text{pshGpd} \), or \( \text{topStack} \)) under Yoneda.

The Yoneda embedding preserves fiber products (in fact, all limits), but it seldom preserves colimits. If a stack \( X \) is equivalent to \( \text{Hom}_{\text{Top}}(-, X) \), for some \( X \in \text{Top} \), we often abuse terminology and say that \( X \) is a topological space.

### 3.5. Representable morphisms

A morphism \( f : X \to Y \) of topological stacks is called representable if for every morphism \( Y \to Y \) from a topological space \( Y \), the 2-fiber product \( Y \times_Y X \) is equivalent to a topological space. It turns out that, for every topological stack \( X \), every morphism \( f : X \to X \), with \( X \) a topological space, is representable.

### 3.6. Classifying spaces for topological stacks

The following theorem has been proven in ([No14], Corollary 3.17).
Theorem 3.1. Let $X$ be a topological stack. Then, there exists an atlas $\varphi : X \to X$ that is a weak trivial Serre fibration. This means that, for any map from a topological space $T$, the fiber product

$$X \times_X T \to T$$

has the property that $X \times_X T \to T$ is a weak trivial Serre fibration of topological spaces (in particular, a weak homotopy equivalence).

See Definition 6.3 for the general definition of weak trivial Serre fibration, bearing in mind that the definition simplifies considerably in the case of topological spaces.

We call a map $\varphi : X \to X$ as in Theorem 3.1 a classifying atlas for $X$. Note that in the definition of classifying atlas given in [No12] we only require $\varphi : X \to X$ to be a universal weak equivalence. The definition we are using here is stronger.

The $n^{th}$ homotopy group (set if $n = 0$) of a pointed topological stack $(X, x)$ is defined ([No14], Section 5) to be the group $\pi_n(X, x) = [(S^n, s_0), (X, x)]$ of homotopy classes of pointed maps. Equivalently, it can be defined to be the homotopy group $\pi_n(X, x')$ of a classifying atlas $X$ for $X$ at some lift $x'$ of $x$ to $X$. This definition is independent of the choice of $X$ and $x'$ (up to a natural isomorphism).

A morphism of topological stacks $f : X \to Y$ is called a weak equivalence if it induces isomorphisms $f^* : \pi_n(X, x) \to \pi_n(Y, y)$ for all choices of basepoint and all $n \geq 0$.

4. The tilde construction

Consider the inclusion $\Delta \to \text{Top}$, $[n] \mapsto |\Delta^n|$. Its left Kan extension

$$\text{sSet} \to \text{pshSet} \xrightarrow{\psi} \text{pshGpd}$$

$A \mapsto \tilde{A}$

is uniquely determined by the property that it preserves colimits and sends $\Delta^n$ to $|\Delta^n|$ (rather, the presheaf represented by it). It is left adjoint to the restriction functor

$$-A : \text{pshSet} \to \text{sSet} \xrightarrow{\psi} \text{sGpd}$$

$X \mapsto X_\Delta = \text{Hom}_{\text{pshSet}}(|\Delta^*|, X)$.

More explicitly, $\tilde{A}$ is constructed exactly like the colimit construction of the geometric realization of $A$, except that instead of using the topological simplices $|\Delta^n|$ as building blocks we use the presheaves in $\text{pshSet}$ represented by them.

We have a natural map

$$\psi_A : \tilde{A} \to |A|.$$

This is adjoint to $A \to \text{Sing}(|A|)$, the unit of the adjunction $| - | : \text{Top} \rightleftarrows \text{sSet} : \text{Sing}$. Note that the Yoneda embedding $\text{Top} \to \text{pshSet}$ (or $\text{pshGpd}$) does not necessarily preserve colimits, so $\psi_A$ is often not an isomorphism (but it is when $A = \Delta^n$).

Example 4.1. Write $\Lambda^n_k$ as the coequalizer of

$$\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{i \in \{0, 1, \ldots, n\}, i \neq k} \Delta^{n-1} \to \Lambda^n_k$$

Then, we can write $\tilde{\Lambda^n_k}$ as the coequalizer

$$\coprod_{0 \leq i < j \leq n} |\Delta^{n-2}| \rightrightarrows \coprod_{i \in \{0, 1, \ldots, n\}, i \neq k} |\Delta^{n-1}|$$

in $\text{pshSet}$. The map $\psi_{\Lambda^n_k} : \tilde{\Lambda^n_k} \to |\Lambda^n_k|$ is almost never an isomorphism.
Lemma 4.2. Let $A$ be a simplicial set and $\mathcal{X}$ a presheaf of groupoids. Then, we have an isomorphism (and not just an equivalence) of groupoids

$$\text{Hom}_{\text{pshGpd}}(\tilde{A}, \mathcal{X}) \cong \text{Hom}_{\text{Gpd}}(A, \mathcal{X}_\Delta),$$

$$f \mapsto f_\Delta \circ \iota_A.$$

Here, $\iota_A : A \to \tilde{A}_\Delta$ is the unit of adjunction. In particular, we have the following natural isomorphisms

$$\begin{align*}
\text{Hom}_{\text{pshGpd}}(\Delta^n, \mathcal{X}) & \cong \text{Hom}_{\text{Gpd}}(\Delta^n, \mathcal{X}_\Delta) \\
\mathcal{X}(\Delta^n) & \cong \mathcal{X}(\Delta^n)
\end{align*}$$

Proof. In the case where $\mathcal{X}$ is a presheaf of sets, i.e., $\mathcal{X} \in \text{pshSet}$, this is just the left adjointness of the left Kan extension. For the general case view $\mathcal{X}$ as a groupoid object in $\text{pshSet}$ and apply the above isomorphisms to $\text{Ob}(\mathcal{X})$ and $\text{Mor}(\mathcal{X}) \in \text{pshSet}$. □

4.1. Yoneda and colimits. As we pointed out above, unless $A$ is representable, the natural map $\psi_A : \tilde{A} \to |A|$ is not in general an isomorphism of presheaves of sets. This is due to the fact that the Yoneda functor $\text{Top} \to \text{pshSet}$ (or $\text{Top} \to \text{pshGpd}$) does not preserve colimits.

In certain situations, however, the following lemma comes handy.

Lemma 4.3. Let $\mathcal{X}$ be a Serre topological stack. Let $A \hookrightarrow B$ be a closed embedding of topological spaces which is locally a Serre cofibration, and let $A \to C$ be a finite proper map of topological spaces. Then, the map

$$\text{Hom}_{\text{pshGpd}}(B \vee_A C, \mathcal{X}) \to \text{Hom}_{\text{pshGpd}}(B \vee'_A C, \mathcal{X})$$

induced by the natural map $B \vee'_A C \to B \vee_A C$ is an equivalence of groupoids. Here, $\vee$ stands for colimit in $\text{Top}$ and $\vee'$ stands for colimit in $\text{pshSet}$.

Proof. In ([BeGiNoXu], Proposition 1.3) a similar statement is proved for Hurewicz stacks. The same proof carries over. □

Corollary 4.4. Let $A$ be a finite simplicial set with the property that every nondegenerate cell has nondegenerate faces (e.g., $A=\partial \Delta^n, \Lambda^n_k, \Lambda^n_k \times \Delta^1$). Then, the natural map $\psi_A : \tilde{A} \to |A|$ induces an equivalence of groupoids

$$\text{Hom}_{\text{pshGpd}}(|A|, \mathcal{X}) \cong \text{Hom}_{\text{pshGpd}}(\tilde{A}, \mathcal{X})$$

$$f \mapsto f \circ \psi_A$$

for every Serre topological stack $\mathcal{X}$.

Proof. Use induction on the number of nondegenerate cells and apply Lemma 4.3. □

As we pointed out above, in the case where $A$ is representable, namely $A = \Delta^n$, the above equivalence is indeed an isomorphism of groupoids.

5. Homotopy between morphisms of presheaves of groupoids

We review the notion of homotopy between morphisms of stacks from [No14], and introduce a variant called restricted homotopy.
5.1. Fiberwise homotopy.

**Definition 5.1.** Let \( f, g : A \to X \) and \( p : X \to Y \) be morphisms of presheaves of groupoids, and \( \varphi : p \circ f \Rightarrow p \circ g \) a 2-isomorphism:

\[
\begin{array}{ccc}
A & \xrightarrow{p \circ f} & Y \\
\downarrow{\varphi} & & \downarrow{p \circ g} \\
\xrightarrow{f} & X & \xrightarrow{g} \\
\end{array}
\]

A fiberwise homotopy from \( f \) to \( g \) relative to \( \varphi \) is a quadruple \((H, \epsilon_0, \epsilon_1, \psi)\) where

- \( H : A \times [0,1] \to X \) is a morphism of presheaves of groupoids;
- \( \epsilon_0 : f \Rightarrow H_0 \) and \( \epsilon_1 : H_1 \Rightarrow g \) are 2-isomorphisms;
- \( \psi : p \circ f \circ pr_1 \Rightarrow p \circ H \) is a 2-isomorphism such that \( \psi_0 = p \circ \epsilon_0, \psi_1 \cdot (p \circ \epsilon_1) = \varphi \), and the following diagram 2-commutes:

\[
\begin{array}{ccc}
A \times [0,1] & \xrightarrow{H} & X \\
\downarrow{pr_1} & \nearrow{\psi} & \downarrow{p} \\
A & \xrightarrow{p \circ f} & Y \\
\end{array}
\]

(Notation: \( H_i := H|_{A \times \{i\}}, \psi_i := \psi|_{A \times \{i\}}, \) for \( i = 0,1 \).) A ghost fiberwise homotopy from \( f \) to \( g \) relative to \( \varphi \) is a 2-isomorphism \( \xi : f \Rightarrow g \) such that \( \varphi = p \circ \xi \). A fiberwise homotopy as above is called strict if \( \epsilon_0 \) and \( \epsilon_1 \) are the identity 2-isomorphisms. In the case where \( \varphi \) is the identity 2-isomorphism (so \( p \circ f = p \circ g \)) we say that \( H \) is a homotopy relative to \( Y \).

Ghost homotopies are precisely those quadruples \((H, \epsilon_0, \epsilon_1, \psi)\) for which \( H \) and \( \psi \) remain constant along \([0,1]\), that is, they factor through \( pr_1 \). In this case, \( \xi := \epsilon_0 \cdot \epsilon_1 \) is a ghost fiberwise homotopy from \( f \) to \( g \) relative to \( \varphi \).

5.2. Restricted fiberwise homotopy. The notion of restricted homotopy we introduce below only applies to morphisms of the form \( \tilde{A} \to X \), where \( \tilde{A} \) is a simplicial set and \( X \) is a presheaf of groupoids.

**Definition 5.2.** Let \( A \) be a simplicial set. Let \( f, g : \tilde{A} \to X \) and \( p : X \to Y \) be morphisms of presheaves of groupoids, and \( \varphi : p \circ f \Rightarrow p \circ g \) a 2-isomorphism:

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{p \circ f} & Y \\
\downarrow{\varphi} & & \downarrow{p \circ g} \\
\xrightarrow{f} & X & \xrightarrow{g} \\
\end{array}
\]

A restricted fiberwise homotopy from \( f \) to \( g \) relative to \( \varphi \) is a quadruple \((H, \epsilon_0, \epsilon_1, \psi)\) where

- \( H : \tilde{A} \times \Delta^1 \to X \) is a morphism of presheaves of groupoids;
- \( \epsilon_0 : f \Rightarrow H_0 \) and \( \epsilon_1 : H_1 \Rightarrow g \) are 2-isomorphisms;
• $\psi : p \circ f \circ \bar{\text{pr}}_1 \Rightarrow p \circ H$ is a 2-isomorphism such that $\psi_0 = p \circ \epsilon_0, \psi_1 \cdot (p \circ \epsilon_1) = \varphi$, and the following diagram 2-commutes

\[ \begin{array}{ccc}
A \times \Delta^1 & \xrightarrow{H} & X \\
\downarrow \text{pr}_1 & & \downarrow p \\
A & \xrightarrow{p \circ f} & Y
\end{array} \]

(Notation: $H_0 := H \circ i$, where $i : A \to A \times \Delta^1$ is the time 0 map.) A restricted fiberwise homotopy as above is called strict if $\epsilon_0$ and $\epsilon_1$ are the identity 2-isomorphisms. In the case where $\varphi$ is the identity 2-isomorphism (so $p \circ f = p \circ g$) we say that $H$ is a restricted homotopy relative to $Y$.

Remark 5.3. In view of the adjunction of Lemma 4.2, we can replace the diagrams above with their corresponding diagram in the category of simplicial groupoids. For example,

\[ \begin{array}{ccc}
X_{\Delta} & \xrightarrow{f} & Y_{\Delta} \\
\downarrow p' \circ f' & & \downarrow p' \circ g' \\
A & \xrightarrow{p' \circ f'} & Y_{\Delta}
\end{array} \]

Thus, we can regard a restricted homotopy as a homotopy in the category of simplicial groupoids.

An ordinary homotopy gives rise to a restricted homotopy.

Lemma 5.4. Let $A$ be a simplicial set and let $\bar{A} := \tilde{A}$. Notation being as in Definition 5.1, suppose that we are given a fiberwise homotopy $(H, \epsilon_0, \epsilon_1, \psi)$ from $f$ to $g$ relative to $\varphi$. Then, precomposing with the natural map $\bar{A} \times \Delta^1 \to \bar{A} \times [0,1]$ gives rise to a restricted fiberwise homotopy from $f$ to $g$ relative to $\varphi$.

Proof. Straightforward. $\square$

6. Lifting conditions

We shall review some of the material from [No14] and recall the notion of (weak) Serre fibration between stacks. For a full account see Sections 2 and 3 of [No14]. Before we start, it is worthwhile to emphasize the difference between the notion of fibration in this section and the standard ones in well known model category structures on the category of presheaves of groupoids: our notion is more geometric, in the sense that it does not distinguish between equivalent presheaves; in particular, any equivalence of presheaves of groupoids is a fibration in our sense.

In what follows, by a Serre cofibration we mean a map of topological spaces that is a retract of relative cell complexes. These are precisely the cofibrations in the model structure on topological spaces in which weak equivalences are weak homotopy equivalences and fibrations are Serre fibrations.
Definition 6.1. Let \( i : \mathcal{A} \rightarrow \mathcal{B} \) and \( p : \mathcal{X} \rightarrow \mathcal{Y} \) be morphisms of presheaves of groupoids. Then, \( i \) has the \textit{weak left lifting property (WLLP)} with respect to \( p \) if given a commuting diagram:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{X} \\
\downarrow{i} & \swarrow{\alpha} & \downarrow{p} \\
\mathcal{B} & \xrightarrow{g} & \mathcal{Y}
\end{array}
\]

there is a morphism \( h : \mathcal{B} \rightarrow \mathcal{X} \), a 2-isomorphism \( \gamma : p \circ h \Rightarrow g \) and a fiberwise homotopy \( H \) from \( f \) to \( h \circ i \) relative to \( \alpha \cdot (\gamma \circ i)^{-1} \):}

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{X} \\
\downarrow{i} & \swarrow{H} & \downarrow{p} \\
\mathcal{B} & \xrightarrow{g} & \mathcal{Y}
\end{array}
\]

We say that \( i \) has the \textit{left lifting property (LLP)} with respect to \( p \) if \( H \) can be taken to be a ghost homotopy. In other words, there are 2-isomorphisms \( \beta : f \Rightarrow h \circ i \) and \( \gamma : p \circ h \Rightarrow g \) such that the following diagram commutes (\( \alpha \) is not shown in the diagram):

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{X} \\
\downarrow{i} & \searrow{\beta} & \downarrow{p} \\
\mathcal{B} & \xrightarrow{g} & \mathcal{Y}
\end{array}
\]

Remark 6.2. The usage of the term ‘weak’ (which means, ‘up to homotopy’) in the above definition is in conflict with our usual usage of the term weak (which means, ‘up to 2-isomorphism’, as opposed to ‘strict’). But since the above definition is quite standard in the homotopy theory literature, we deemed it inappropriate to change it. We apologize for the confusion this may cause.

Definition 6.3. A morphism of presheaves of groupoids \( p : \mathcal{X} \rightarrow \mathcal{Y} \) is a \textit{(weak) Serre fibration} if every trivial Serre cofibration \( i : A \rightarrowtail B \) of topological spaces has the (W)LLP with respect to \( p \). It is a \textit{(weak) trivial Serre fibration} if every Serre cofibration \( i : A \rightarrowtail B \) has the (W)LLP with respect to \( p \).

Remark 6.4. As opposed to the notion of Reedy fibration that we will introduce in Definition 7.8, the notion of (weak) Serre fibration is “intrinsic” (or “geometric”) in the sense that if \( p : \mathcal{X} \rightarrow \mathcal{Y} \) is a (weak) Serre fibration and \( p' : \mathcal{X}' \rightarrow \mathcal{Y} \) is a morphism equivalent to it, then \( p' \) is also a (weak) Serre fibration.

Proposition 6.5. Let \( p : \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of topological stacks, and assume that \( \mathcal{X} \) is Serre. Then, \( p \) is a trivial Serre fibration if and only if it is a Serre fibration and a weak equivalence.

Proof. By ([No14], Lemma 2.4), every morphism \( p : \mathcal{X} \rightarrow \mathcal{Y} \) of topological stacks with \( \mathcal{X} \) a Serre stack is a Serre morphism (in the sense of [No14], Definition 2.2). The result now follows from ([No14], Proposition 3.20). □
7. Reedy fibrations of stacks

In this section, we introduce Reedy fibrations between presheaves of groupoids (Definition 7.8) and establish some of their basic properties.

7.1. Model structure on $\text{Gpd}$.

Definition 7.1. Let $p : G \to H$ be a morphism in $\text{Gpd}$. We say that $p$ is a fibration if for any $x \in G$ and any isomorphism $\varphi : y \to p(x)$ in $H$, there exists an isomorphism $\psi : z \to x$ in $G$ such that $p(\psi) = \varphi$.

There is a model category structure on the category $\text{Gpd}$ of groupoids where

- weak equivalences are equivalences of groupoids;
- cofibrations are maps that are injective on the set of objects;
- fibrations are as in Definition 7.1.

We refer the reader to ([Ho], Theorem 2.1) for more details and further references.

Lemma 7.2. A morphism $i : G \to H$ in $\text{Gpd}$ is a trivial cofibration if and only if it is essentially surjective and induces an isomorphism of groupoids between $H$ and a full subcategory of $G$. When this is the case, $G \times K \to H \times K$ is a trivial cofibration for every groupoid $K$.

Proof. Straightforward. □

Proposition 7.3. The above model structure on $\text{Gpd}$ is left proper, simplicial, cofibrantly generated, combinatorial and monoidal (with respect to cartesian product).

Proof. The properties left proper, simplicial and cofibrantly generated are proved in ([Ho], Theorem 2.1). Since $\text{Gpd}$ is cofibrantly generated and locally presentable, it is, by definition, combinatorial.

To check that the model structure is monoidal we need to verify conditions (i)-(iii) of ([Lu], Definition A.3.1.2). Conditions (ii) and (iii) are obvious. To check (i) we have to show that the cartesian product $\times : \text{Gpd} \times \text{Gpd} \to \text{Gpd}$ is a left Quillen bifunctor. That is, the following two conditions are satisfied:

(a) Let $i : A \to A'$ and $j : B \to B'$ be cofibrations in $\text{Gpd}$. Then, the induced map

$$i \land j : (A' \times B') \coprod_{A \times B} (A \times B') \to A' \times B'$$

is a cofibration in $\text{Gpd}$. Moreover, if either $i$ or $j$ is a trivial cofibration, then $i \land j$ is also a trivial cofibration.

(b) The cartesian product preserves small colimits separately in each variable.

The first part of (a) is easy as it only concerns the object sets of the groupoids in question, and the corresponding statement is true in the category of sets. To prove the second part of (a), assume that $i : A \to A'$ is a trivial cofibration. The claim follows from Lemma 7.2 and two-out-of-three applied to

$$A \times B' \to (A' \times B') \coprod_{A \times B} (A \times B') \to A' \times B'.$$

Condition (b) follow from the following characterization of cartesian product $G \times H$ of groupoids $G$ and $H$: for a groupoid $K$, to give a morphism $G \times H \to K$ is equivalent to giving a pair of groupoid morphisms

$$\phi : G \times (\text{Ob} H) \to K \quad \text{and} \quad \psi : (\text{Ob} G) \times H \to K$$

such that

i) $\phi$ and $\psi$ agree on $\text{Ob} G \times \text{Ob} H$;
ii) the images of arrows of $G$ and arrows of $H$ under $\phi$ and $\psi$, respectively, commute in $K$.

We leave the details to the reader. \hfill \Box

**Proposition 7.4.** The model structure on $\text{Gpd}$ is excellent in the sense of ([Lu], Definition A.3.2.16).

**Proof.** Axioms (A1)-(A4) of [ibid.] are straightforward to check. Axiom (A5), the Invertibility Hypothesis, follows from ([Lu], Lemma A.3.2.20) applied to the fundamental groupoid functor $\Pi_1 : s\text{Set} \to \text{Gpd}$. \hfill \Box

**7.2. Injective model structure on $[\text{C}^{\text{op}}, \text{Gpd}]$.** Let $\text{C}$ be a small category. Since the model structure on $\text{Gpd}$ is combinatorial (Proposition 7.3), by ([Lu], Proposition A.2.8.2) there is a model structure on the category $[\text{C}^{\text{op}}, \text{Gpd}]$ of presheaves of groupoids, called the injective model structure, where

- weak equivalences are the objectwise weak equivalences as in Section 7.1;
- cofibrations are the objectwise cofibrations as in Section 7.1;
- fibrations have the right lifting property with respect to the trivial cofibrations.

We refer the reader to ([Lu], A.2.8) for more details on the injective model structure.

**Proposition 7.5.** The injective model structure on $[\text{C}^{\text{op}}, \text{Gpd}]$ is $\text{Gpd}$-enriched in the sense of ([Lu], Definition A.3.1.5).

**Proof.** This follows from ([Lu], Remark A.3.3.4). \hfill \Box

We are particularly interested in the cases $\text{C} = \text{Top}$ and $\text{C} = \Delta$. In the case $\text{C} = \Delta$, we have an explicit description of fibrations thanks to Proposition 7.7 below.

**7.3. Reedy model structure on $s\text{Gpd}$.** The Reedy model structure on the category of simplicial groupoids $s\text{Gpd} = [\Delta^{\text{op}}, \text{Gpd}]$ is defined as follows:

- weak equivalences are the objectwise weak equivalences;
- cofibrations are morphisms $X \to Y$ such that for every $n$ the map
  $$L_nY \coprod_{L_nX} X_n \to Y_n$$
  is a cofibration of groupoids (as in Section 7.1);
- fibrations are morphisms $X \to Y$ such that for every $n$ the map
  $$X_n \to M_nX \times_{M_nY} Y_n$$
  is a fibration of groupoids (as in Section 7.1).

Here, $L_nX$ stands for the latching object

$$L_nX := \text{colim}_{[k] \not\sim [n]} X_k,$$

and $M_nX$ stands for the matching object

$$M_nX := \text{lim}_{[n] \not\sim [k]} X_k.$$

Let us unravel the above definitions. First of all, recall that the matching object $M_nX$ can be alternatively described by

$$M_nX = \text{Hom}_{\text{Gpd}}(\partial\Delta^n, X),$$

where we regard the simplicial set $\partial\Delta^n$ as a simplicial groupoid. The map $X_n \to M_nX$ is the one induced by the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.

The Reedy fibration condition can now be restated as saying that

$$\text{Hom}_{\text{Gpd}}(\Delta^n, X) \to \text{Hom}_{\text{Gpd}}(\partial\Delta^n, X) \times_{\text{Hom}_{\text{Gpd}}(\partial\Delta^n, Y)} \text{Hom}_{\text{Gpd}}(\Delta^n, Y)$$

is a fibration of groupoids.
Lemma 7.6. Let $X$ and $Y$ be simplicial sets, regarded as objects in $\mathsf{sGpd}$. Then, any morphism $p : X \to Y$ is a Reedy fibration.

Proof. This follows from the definition of a Reedy fibration and the fact that every map of sets, regarded as objects in $\mathsf{Gpd}$, is a fibration of groupoids.

The Reedy cofibrations turn out to coincide with the objectwise cofibrations. That is, a morphism $p : X \to Y$ of simplicial groupoids is a Reedy cofibration if and only if $X_n \to Y_n$ is a cofibration of groupoids (in the sense of Section 7.1) for all $n$. This is a consequence of the following proposition.

Proposition 7.7. The Reedy model structure and the injective model structure on $\mathsf{sGpd} = [\Delta^{op}, \mathsf{Gpd}]$ coincide.

Proof. We know that, by definition, the two model structures have the same weak equivalences. It remains to show that they have the same fibrations. Let $N : \mathsf{sGpd} = [\Delta^{op}, \mathsf{Gpd}] \to [\Delta^{op}, \mathsf{sSet}]$ be the objectwise nerve functor, and let $\Pi_1$ be its left adjoint, the objectwise fundamental groupoid functor. We show that the following are equivalent:

1. $p : X \to Y$ is a Reedy fibration in $\mathsf{sGpd}$.
2. For all $n$, $N(X_n) \to M_n(N(X)) \times_{M_n(N(Y))} N(Y_n)$ is a Kan fibration of simplicial sets.
3. $N(p) : N(X) \to N(Y)$ is a Reedy fibration in $[\Delta^{op}, \mathsf{sSet}]$.
4. $N(p) : N(X) \to N(Y)$ is an injective fibration in $[\Delta^{op}, \mathsf{sSet}]$.
5. $p : X \to Y$ is an injective fibration in $[\Delta^{op}, \mathsf{Gpd}] = \mathsf{sGpd}$.

(1) $\iff$ (2) is true since the nerve functor preserves fiber products and $G \to H$ is a fibration of groupoids if and only if $N(G) \to N(H)$ is a Kan fibration. (2) $\iff$ (3) is true by definition. (3) $\iff$ (4) follows from the fact that injective model structure on $[\Delta^{op}, \mathsf{sSet}]$ is the same as the Reedy model structure (Lemma 7.3, Example A.2.9.8).

The implication (5) $\Rightarrow$ (4) follows from the fact that $\Pi_1$ takes a trivial cofibration of (presheaves of) simplicial sets to a trivial cofibration of (presheaves of) groupoids.

Finally, to prove (4) $\Rightarrow$ (5) we use the fact that $N$ preserves trivial cofibrations (for this use Lemma 7.3) and that $\Pi_1 \circ N = \text{id}_{\mathsf{sGpd}}$. More precisely, to solve a lifting problem in $[\Delta^{op}, \mathsf{Gpd}]$, we can first apply $N$, solve the lifting problem in $[\Delta^{op}, \mathsf{sSet}]$, and then apply $\Pi_1$ to obtain a solution to the original lifting problem.

7.4. Reedy fibrations in $\mathsf{pshGpd}$. From now on, $C = \mathsf{Top}$. We will use the notation $\mathsf{pshGpd}$ instead of $[C^{op}, \mathsf{Gpd}]$. We begin with our main definition.

Definition 7.8. We say that a map of presheaves of groupoids $p : \mathcal{X} \to \mathcal{Y}$ is a Reedy fibration if $p_\Delta : \mathcal{X}_\Delta \to \mathcal{Y}_\Delta$ is a Reedy fibration in $\mathsf{sGpd}$ (see Section 7.3).

Lemma 7.9. Let $X$ and $Y$ be presheaves of simplicial sets, regarded as objects in $\mathsf{pshGpd}$. Then, any morphism $p : X \to Y$ is a Reedy fibration.

Proof. This follows from Lemma 7.6.

Proposition 7.10. If $p : \mathcal{X} \to \mathcal{Y}$ is an injective fibration of presheaves of groupoids, then $p$ is a Reedy fibration.

Proof. We have to show that, for every $n$, the map

$$
\text{Hom}_{\mathsf{pshGpd}}(\Delta^n, \mathcal{X}_\Delta) \to \text{Hom}_{\mathsf{pshGpd}}(\partial \Delta^n, \mathcal{X}_\Delta) \times_{\text{Hom}_{\mathsf{pshGpd}}(\partial \Delta^n, \mathcal{Y}_\Delta)} \text{Hom}_{\mathsf{pshGpd}}(\Delta^n, \mathcal{Y}_\Delta)
$$

is a fibration of groupoids. Via the tilde construction, the above map is isomorphic to

$$
\text{Hom}_{\mathsf{pshGpd}}(\tilde{\Delta}^n, \mathcal{X}) \to \text{Hom}_{\mathsf{pshGpd}}(\tilde{\partial} \Delta^n, \mathcal{X}) \times_{\text{Hom}_{\mathsf{pshGpd}}(\tilde{\partial} \Delta^n, \mathcal{Y})} \text{Hom}_{\mathsf{pshGpd}}(\tilde{\Delta}^n, \mathcal{Y}).
$$

This map is a fibration of groupoids because $p : \mathcal{X} \to \mathcal{Y}$ is a fibration and $\tilde{\partial} \Delta^n \to \tilde{\Delta}^n = \Delta^n$ is a cofibration in the injective model structure on $\mathsf{pshGpd}$ (to see the latter, write $\tilde{\partial} \Delta^n$ as...
the colimit of its faces and use the fact that the tilde construction preserves colimits). The
claim now follows from Proposition 7.7 (also see [Lu], Remark A.3.1.6(2')). □

**Proposition 7.11.** Let \( p : X \to Y \) be a Reedy fibration of presheaves of groupoids, and let \( A \to B \) be a monomorphism of simplicial sets. Then, the map

\[
\Hom_{\text{Gpd}}(B, X_\Delta) \to \Hom_{\text{Gpd}}(A, X_\Delta) \times_{\Hom_{\text{Gpd}}(A, Y_\Delta)} \Hom_{\text{Gpd}}(B, Y_\Delta)
\]

and, equivalently (see Lemma 4.2), the map

\[
\Hom_{\text{pshGpd}}(\tilde{B}, X) \to \Hom_{\text{pshGpd}}(\tilde{A}, X) \times_{\Hom_{\text{pshGpd}}(\tilde{A}, Y)} \Hom_{\text{pshGpd}}(\tilde{B}, Y)
\]

are fibrations of groupoids.

**Proof.** In fact, the first map is a fibration of groupoids for any Reedy fibration \( X \to Y \) in \( \text{sGpd} \) (in our case \( X = X_\Delta \) and \( Y = Y_\Delta \)). In view of Proposition 7.7 this follows from Proposition 7.5 with \( C = \Delta \) (also see [Lu], Remark A.3.1.6(2')).

Alternatively, use ([Du], Lemma 4.5), with \( M = \text{Gpd}, K = A, L = B, X = X_\Delta \) and \( Y = Y_\Delta \). □

**Proposition 7.12.** For any morphism of presheaves of groupoids \( p : X \to Y \), there exists a strictly commutative diagram

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow^g & \nearrow^p' & \\
X' & \rightarrow & \ \\
\end{array}
\]

where \( p' \) is an injective (hence, also Reedy) fibration and \( g : X \to X' \) is an equivalence of presheaves of groupoids.

**Proof.** Take the usual fibrant replacement in the injective model structure on \( \text{sGpd} \) and use Proposition 7.10. □

8. **Singular functor for stacks**

We shall define the functor \( \text{Sing} : \text{pshGpd} \to \text{sSet} \) of singular chains and establish some of its basic properties. This functor will be the focus of the rest of the paper.

8.1. **The functors** \( B, \text{Diag}, \text{and Sing} \). We extend the restriction functor \(-_\Delta\) defined in Section 4 to \( \text{pshGpd} \):

\[
-\Delta : \text{pshGpd} \to \text{sGpd},
\]

\[
X \mapsto X_\Delta = \Hom_{\text{pshGpd}}(|\Delta^*|, X).
\]

In addition to this, we define two more functors.

**Definition 8.1.** Let

\[
B : \text{pshGpd} \to \text{bsSet},
\]

\[
X \mapsto N(X_\Delta),
\]

\[
\text{Sing} : \text{pshGpd} \to \text{sSet},
\]

\[
X \mapsto \text{Diag}(N(X_\Delta)).
\]

Here, \( \text{Diag} : \text{bsSet} \to \text{sSet} \) refers to taking the diagonal of a bisimplicial set.

In the above definition, \(|\Delta^*|\) stands for the cosimplicial object \( n \mapsto |\Delta^n| \) in \( \text{Top} \), and \( N : \text{Gpd} \to \text{bsSet} \) is the levelwise nerve functor obtained from \( N : \text{Gpd} \to \text{sSet} \). Lemma 4.2.
Remark 8.2. When restricted to \( \text{Top} \), the functor \( \text{Sing} \) coincides with the usual singular chains functor \( \text{Sing} : \text{Top} \to s\text{Set} \). More precisely, the following diagram commutes:

\[
\begin{array}{ccc}
\text{Top} & \to & \text{pshGpd} \\
\downarrow & & \downarrow \\
\text{Sing} & \to & \text{Sing} \\
\downarrow & & \downarrow \\
s\text{Set} & \to & s\text{Set}
\end{array}
\]

The top arrow in this diagram is the Yoneda embedding.

Lemma 8.3. Let \( f : X \to Y \) be a morphism of simplicial groupoids that induces equivalences of groupoids \( X_n \to Y_n \) for all \( n \). Then, the induced map \( \text{Diag}(NX) \to \text{Diag}(NY) \) is a weak equivalence of simplicial sets.

Proof. This follows from ([GoJa], Chapter IV, Proposition 1.7). \( \square \)

Corollary 8.4. Let \( f : X \to Y \) be an equivalence of presheaves of groupoids. Then, \( \text{Sing}(f) : \text{Sing}(X) \to \text{Sing}(Y) \) is a weak equivalence of simplicial sets.

We will need the following definition from ([GoJa], Chapter IV, Section 3.3).

Definition 8.5. Define the functor \( d^* : s\text{Set} \to b\text{sSet} \) to be the one uniquely determined by the following two properties:

- \( d^*(\Delta^n) = \Delta^n, n \) (see Section 5 for notation);
- \( d^* \) preserves colimits.

Proposition 8.6. The functors \( \text{Diag} : b\text{sSet} \to s\text{Set} \) and \( N : s\text{Gpd} \to b\text{sSet} \) have left adjoints:

\[
\begin{array}{ccc}
s\text{Gpd} & \xrightarrow{N} & b\text{sSet} \\
\downarrow & \xRightarrow{\Pi_1} & \downarrow \\
\text{bsSet} & \xleftarrow{\text{Diag}} & s\text{Set}.
\end{array}
\]

Here, \( \Pi_1 \) denotes the fundamental groupoid functor, and \( d^* \) is as in (Definition 8.5). Therefore, \( \text{Sing} = \text{Diag} \circ N \) also has \( \Pi_1 \circ d^* \) as left adjoint. In particular, the functors \( N, \text{Diag} \) and \( \text{Sing} \) preserve limits.

Proof. For the first adjunction see ([Ho], Corollary 2.3). The second adjunction is discussed in ([GoJa], Chapter IV, Section 3.3). \( \square \)

Lemma 8.7. Let \( f, g : \mathcal{X} \to \mathcal{Y} \) be morphisms of presheaves of groupoids.

(i) If \( \alpha : f \Rightarrow g \) is a 2-isomorphism, then we have an induced homotopy \( \hat{\alpha} \) from \( \text{Sing}(f) \) to \( \text{Sing}(g) \).

(ii) If \( h \) is a strict homotopy from \( f \) to \( g \) (see Definition 5.1), then we have an induced homotopy \( \hat{h} \) from \( \text{Sing}(f) \) to \( \text{Sing}(g) \).

Proof. Part (ii) follows from the fact that \( \text{Sing} \) commutes with products (Proposition 8.6). To prove part (i), let \( I \) be the constant presheaf of categories \( I : T \mapsto \{0 \to 1\} \), where \( \{0 \to 1\} \) is the ordinal category (also denoted \([1]\)). A 2-isomorphism \( \alpha \) as above is the same thing as a morphism

\[
\Phi_\alpha : \mathcal{X} \times I \to \mathcal{Y}
\]

whose restrictions to \( \{0\} \) and \( \{1\} \) are \( f \) and \( g \), respectively. It is easy to see that \( \text{Sing}(I) = \Delta^1 \). (Note that we have only defined \( \text{Sing} \) for presheaves of groupoids, but clearly the same definition makes sense for presheaves of categories as well.) By Proposition 8.6, we obtain a map of simplicial sets

\[
\hat{\alpha} := \text{Sing}(\Phi_\alpha) : \text{Sing}(\mathcal{X}) \times \Delta^1 \to \text{Sing}(\mathcal{Y})
\]

This is the desired homotopy. \( \square \)
Remark 8.8.

(1) The operation $\alpha \mapsto \hat{\alpha}$ respects composition of 2-isomorphisms in the sense that $\hat{\alpha \cdot \beta}$ is canonically homotopic to the “composition” of $\hat{\alpha}$ and $\hat{\beta}$. More precisely, $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\alpha \cdot \beta}$ are the three faces of a canonical map

$$\text{Sing}(\mathcal{X}) \times \Delta^2 \to \text{Sing}(\mathcal{Y}).$$

We also have higher coherences. That is, every string of $k$ composable 2-isomorphisms defines a canonical map

$$\text{Sing}(\mathcal{X}) \times \Delta^k \to \text{Sing}(\mathcal{Y})$$

whose restriction to various faces represent different ways of composing (a subset) of homotopies associated to these 2-isomorphisms.

(2) In the statement of Lemma 8.7(ii) we could use a general homotopy $h = (H, \epsilon_0, \epsilon_1)$ from $f$ to $g$ (see Definition 5.1), but in this case instead of a homotopy from $f$ to $g$ we obtain a sequence of three composable homotopies $\hat{\epsilon}_0$, $\hat{\epsilon}_1$ and $\hat{h}$.

Example 8.9. In Lemma 8.10 below we will discuss the effect of $\text{Sing}$ on 2-fiber products of presheaves of groupoids. To motivate the assumptions made there, we look at the following examples.

(1) The functor $\text{Sing}$ does not respect 2-fiber products. For examples, let $\mathcal{Z}$ be (the stack associated to) the constant presheaf on $\text{Top}$ with value $J$, where $J = \{0 \leftrightarrow 1\}$ is the interval groupoid, and let $\mathcal{X} = \mathcal{Y} = *$ be singletons mapping to the points 0 and 1 in $\mathcal{Z}$, respectively. Then,

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} = * \times *$$

is equivalent to a point, while

$$\text{Sing}(\mathcal{X}) \times_{\text{Sing}(\mathcal{Z})} \text{Sing}(\mathcal{Y}) = * \times *$$

is the empty set.

(2) It is not reasonable to expect that $\text{Sing}$ takes 2-fiber products to homotopy fiber products either. For example, let $\mathcal{Z} = \mathcal{X} = \mathcal{Y} = [0, 1]$ be the unit interval, and let $\mathcal{X} = \mathcal{Y} = *$ be singletons mapping to the points 0 and 1 in $\mathcal{Z}$, respectively. Then,

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} = * \times *$$

is the empty set, while

$$\text{Sing}(\mathcal{X}) \times_{\text{Sing}(\mathcal{Z})} \text{Sing}(\mathcal{Y}) = * \times [0, 1]$$

is non-empty (in fact, homotopy equivalent to a point).

Lemma 8.10. Consider the following diagram in $\text{pshGpd}$:

$$\begin{array}{ccc}
\mathcal{X} \\
\downarrow^p \\
\mathcal{Y} \quad \xrightarrow{\mathcal{Z}} \\
\end{array}$$

Suppose that $p$ is a Reedy fibration (by Lemma 7.9 this is automatic if $\mathcal{X}$ and $\mathcal{Z}$ are presheaves of sets). Then, there is a natural weak equivalence of simplicial sets

$$\text{Sing}(\mathcal{X}) \times_{\text{Sing}(\mathcal{Z})} \text{Sing}(\mathcal{Y}) \xrightarrow{\sim} \text{Sing}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}).$$
\textbf{Proof.} Since $p$ is a Reedy fibration (hence objectwise fibration when restricted to $\Delta$), the natural map

$$X \times_Z Y \to \mathbf{X \times_Z Y}$$

is an objectwise weak equivalence when restricted to $\Delta$. It follows from Lemma \ref{lem:weak_equivalence} that the induced map

$$\text{Sing}(X \times_Z Y) \xrightarrow{\sim} \text{Sing}(\mathbf{X \times_Z Y})$$

is a weak equivalence of simplicial sets. Precomposing with the isomorphism of Proposition \ref{prop:isomorphism} we obtain the desired weak equivalence

$$\text{Sing}(X) \times_{\text{Sing}(Z)} \text{Sing}(Y) \xrightarrow{\sim} \text{Sing}(X \times_Z Y) \xrightarrow{\sim} \text{Sing}(\mathbf{X \times_Z Y}).$$

\hfill $\square$

### 8.2. Explicit description of the bisimplicial set $B\mathcal{X}$

For $X \in \mathbf{pshGpd}$, we give an explicit description of the elements of the bisimplicial set $B(X)$ and the simplicial set $\text{Sing}(X)$. This description will not be used anywhere else in the paper.

An element of the set $B(X)_{m,n}$ is given by a chain

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n,$$

where $\eta^i : |\Delta^n| \to X$ are objects and $\alpha^i$ are morphisms in the groupoid $\text{Hom}_{\mathbf{pshGpd}}(|\Delta^n|, X)$.

The vertical face and degeneracy maps of $B(X)$ are ‘nerve-wise’, e.g.,

$$d^V_i : B(X)_{m,n} \to B(X)_{m,n+1},$$

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n \mapsto \eta^0 \xrightarrow{\cdots} \eta^{i-1} \xrightarrow{\alpha^{i+1}} \eta^i \xrightarrow{\cdots} \eta^n,$$

for $i \neq 0, n$. For $i = 0$ we have

$$d^V_0 : B(X)_{m,n} \to B(X)_{m,n-1},$$

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n \mapsto \eta^0 \xrightarrow{\cdots} \eta^{i-1} \xrightarrow{\alpha^i} \eta^i \xrightarrow{\cdots} \eta^n.$$

For $i = n$ we have

$$d^V_n : B(X)_{m,n} \to B(X)_{m,n-1},$$

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n \mapsto \eta^0 \xrightarrow{\cdots} \eta^{i-1} \xrightarrow{\alpha^n} \eta^i \xrightarrow{\cdots} \eta^n.$$

The horizontal face and degeneracy maps of $B(X)$ are ‘geometric’, e.g.,

$$d^H_i : B(X)_{m,n} \to B(X)_{m-1,n},$$

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n \mapsto \eta^0 \xrightarrow{\cdots} \eta^{i-1} \xrightarrow{\alpha^i} \eta^i \xrightarrow{\cdots} \eta^n.$$

is induced from $d^i : |\Delta^{m-1}| \to |\Delta^m|$. That is, $\eta^i_j = \eta^i \circ d^i : |\Delta^{m-1}| \to X$ and $\alpha^i_j = \alpha^i \circ d^i$.

An element of $\text{Sing}(X)_n = B(X)_{n,n}$, can be described as a chain

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n$$

with $\eta^i : |\Delta^n| \to X$. For $i \neq 0, n$ the effect of the face map $d_i := d^H_i \circ d^V_i : \text{Sing}(X)_n \to \text{Sing}(X)_{n-1}$ is given by

$$d_i : \text{Sing}(X)_n = B(X)_{n,n} \to B(X)_{n-1,n-1} = \text{Sing}(X)_{n-1},$$

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n \mapsto \eta^0 \xrightarrow{\cdots} \eta^{i-1} \xrightarrow{\alpha^{i+1}} \eta^i \xrightarrow{\alpha^{i+1}} \eta^{i+1} \xrightarrow{\cdots} \eta^n \xrightarrow{\alpha^n} \eta^n.$$

For $i = 0$ we have

$$d_0 : \text{Sing}(X)_n \to \text{Sing}(X)_{n-1},$$

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n \mapsto \eta^0 \xrightarrow{\cdots} \eta^{n-1} \xrightarrow{\alpha^n} \eta^{n-1} \xrightarrow{\alpha^n} \eta^n.$$
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For $i = n$ we have

$$d_n : \text{Sing}(X)_n \rightarrow \text{Sing}(X)_{n-1},$$

$$\eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} \eta^n \Rightarrow \eta^0 \xrightarrow{\alpha^1} \eta^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^{n-1}} \eta^{n-1}.$$

### 9. Lifting Lemmas

In this section we prove some lifting lemmas which will be used in the subsequent sections in the proofs of our main results. We invite the reader to consult Remark 6.2 before reading this section to prevent possible confusion caused by our usage of the term ‘weak’ in what follows.

The following lemma is useful when we want to replace a lax solution to a strict lifting problem with a strict solution.

**Lemma 9.1.** Consider the following strictly commutative diagram, where $p$ is a Reedy fibration of presheaves of groupoids (Definition 7.8) and $i$ is a monomorphism of simplicial sets:

$$\begin{array}{ccc}
\tilde{A} & \xrightarrow{f} & \tilde{X} \\
\downarrow{i} & & \downarrow{p} \\
\tilde{B} & \xrightarrow{g} & \tilde{Y}
\end{array}$$

Suppose that there exists a lift $h$ and 2-isomorphisms $\beta$ and $\gamma$ making the following diagram 2-commutative:

$$\begin{array}{ccc}
\tilde{A} & \xrightarrow{f} & \tilde{X} \\
\downarrow{i} & \xleftarrow{\beta} & \downarrow{p} \\
\tilde{B} & \xleftarrow{h} & \tilde{Y} \\
\downarrow{\gamma} & & \downarrow{\gamma}
\end{array}$$

Then, we can replace $h$ by a 2-isomorphic morphism $h'$ so that $\beta$ and $\gamma$ become the identity 2-isomorphisms. More precisely, $h' \circ \tilde{i} = f$, $p \circ h' = g$, and there is $\alpha : h' \Rightarrow h$ such that $\alpha \circ \tilde{i} = \beta$ and $p \circ \alpha^{-1} = \gamma$.

**Proof.** By the adjunction of Lemma 4.2, we can replace the top diagram with the following strictly commutative diagram in simplicial groupoids:

$$\begin{array}{ccc}
A & \xrightarrow{f'} & X_\Delta \\
\downarrow{i} & & \downarrow{p_\Delta} \\
B & \xrightarrow{g'} & Y_\Delta
\end{array}$$

We can do the similar thing with the bottom diagram. The result now follows from Proposition 7.11. □

In most of our applications of the above lemma, we will have $B = \Delta^n$, in which case $\tilde{B} = |\Delta^n|$. 
Corollary 9.2. Let \( p : X \to Y \) be a Reedy fibration of presheaves of groupoids, \( A \) a simplicial set, and \( H : \tilde{A} \times \Delta^1 \to X \) a restricted homotopy relative to \( Y \) starting at \( H_0 := H|_{\tilde{A} \times \{0\}} : \tilde{A} \to X \). Then, for every 2-isomorphism \( \beta : f \Rightarrow H_0 \) relative to \( Y \), there exists a restricted homotopy \( H' : \tilde{A} \times \Delta^1 \to X \) relative to \( Y \) and a 2-isomorphism \( \alpha : H' \Rightarrow H \) relative to \( Y \) such that \( f = H_0' := H'|_{\tilde{A} \times \{0\}} \), \( \beta = \alpha_0 := \alpha|_{\tilde{A} \times \{0\}} \).

Proof. With the notation of Lemma 9.1, let \( B = A \times \Delta^1 \), \( i : A \to A \times \Delta^1 \) the inclusion at time 0, \( f = f' \), \( g = p \circ H \), \( h = H \), \( \beta = \beta \) and \( \gamma = \text{id} \), as in the diagram

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{f} & X \\
\downarrow \alpha & \searrow H & \\
B & \xrightarrow{g} & Y \\
\end{array}
\]

The fact that \( \beta \) is relative to \( Y \) guarantees that the outer square strictly commutes. The result now follows from Lemma 9.1. \( \square \)

Lemma 9.3. Let \( p : X \to Y \) be a morphism of Serre topological stacks and \( i : A \to B \) a monomorphism of simplicial sets. If \( p \) is a (weak) Serre fibration and either \( p \) or \( i \) is a weak equivalence, then \( \tilde{i} : \tilde{A} \to \tilde{B} \) has (weak) LLP with respect to \( p \) (see Definition 6.1).

Proof. Consider the lifting problem

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{f} & X \\
\downarrow \tilde{i} & \searrow \alpha & \\
B & \xrightarrow{g} & Y \\
\end{array}
\]

First note that to solve it we are allowed to replace each of \( f \) and \( g \) with a 2-isomorphic morphism (and adjust \( \alpha \) accordingly). So, we may assume, by Corollary 4.4, that there are maps \( f' : |A| \to X \) and \( g' : |B| \to Y \) such that \( f = f' \circ \psi_A \) and \( g = g' \circ \psi_B \). Here, \( \psi_A : \tilde{A} \to |A| \) is as in Section 4. Thus, our lifting problem translates to

\[
\begin{array}{ccc}
|A| & \xrightarrow{f'} & X \\
\downarrow |\tilde{i}| & \searrow \alpha' & \\
|B| & \xrightarrow{g'} & Y \\
\end{array}
\]

(The existence of the unique \( \alpha' \) is guaranteed by Corollary 4.4.) This problem can now be solved under the given assumptions. Precomposing with the \( \psi \) maps, we obtain a solution to the original lifting problem. (Also see Proposition 6.5.) \( \square \)

Lemma 9.4. Let \( p : X \to Y \) be a morphism of Serre topological stacks and \( i : A \to B \) a monomorphism of simplicial sets. If \( p \) is a Serre fibration and also a Reedy fibration, and either \( p \) or \( i \) is a weak equivalence, then \( \tilde{i} : \tilde{A} \to \tilde{B} \) has strict LLP with respect to \( p \). That
is, if in the diagram

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{f} & X \\
\downarrow{\tilde{i}} & & \downarrow{p} \\
\tilde{B} & \xrightarrow{g} & Y
\end{array}
\]

the outer square is strictly commutative, then there exists a lift \(h\) making both triangles strictly commutative.

**Proof.** First use Lemma 9.3 to find a solution \(h\) which makes the two triangles commutative up to 2-isomorphism. Then use Lemma 9.1 to rectify \(h\) to make the triangles strictly commutative. \(\square\)

**Corollary 9.5.** Assumptions being as in Lemma 9.4, the map

\[
\text{Hom}_{\mathbf{pshGpd}}(\tilde{B}, X) \to \text{Hom}_{\mathbf{pshGpd}}(\tilde{A}, X) \times \text{Hom}_{\mathbf{pshGpd}}(\tilde{A}, Y) \text{Hom}_{\mathbf{pshGpd}}(\tilde{B}, Y)
\]

and, equivalently (see Lemma 9.4), the map

\[
\text{Hom}_{\mathbf{Gpd}}(B, X_\Delta) \to \text{Hom}_{\mathbf{Gpd}}(A, X_\Delta) \times \text{Hom}_{\mathbf{Gpd}}(A, Y_\Delta) \text{Hom}_{\mathbf{Gpd}}(B, Y_\Delta)
\]

are fibrations of groupoids that are surjective on objects (hence, also on morphisms).

**Proof.** Surjectivity on objects is simply a restatement of Lemma 9.4. They are fibrations by Proposition 7.11. \(\square\)

**Lemma 9.6.** Let \(p : \mathcal{X} \to \mathcal{Y}\) be a morphism of Serre topological stacks and \(i : A \to B\) a monomorphism of simplicial sets. If \(p\) is a weak Serre fibration and also a Reedy fibration, and either \(p\) or \(i\) is a weak equivalence, then \(\tilde{i} : \tilde{A} \to \tilde{B}\) has strict WLLP with respect to \(p\) in the following sense. If in the diagram

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{f} & X \\
\downarrow{\tilde{i}} & & \downarrow{p} \\
\tilde{B} & \xrightarrow{g} & Y
\end{array}
\]

the outer square is strictly commutative, then there exists a lift \(h\) and a morphism \(H : \tilde{A} \times \Delta^1 \to \mathcal{X}\) such that

i) the lower triangle is strictly commutative, and

ii) \(H\) is a strict restricted fiberwise homotopy from \(f\) to \(h \circ \tilde{i}\) relative to \(\mathcal{Y}\) (see Section 5.4), where strictness means that \(H_0 = f\) and \(H_1 = h \circ \tilde{i}\).

**Proof.** First use Lemma 9.3 to find a solution \(h\) which makes the lower triangle commutative up to a 2-isomorphism and the upper triangle commutative up to fiberwise homotopy \(H' : A \times [0, 1] \to \mathcal{X}\), as in the diagram

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{f} & X \\
\downarrow{\tilde{i}} & & \downarrow{p} \\
\tilde{B} & \xrightarrow{g} & Y
\end{array}
\]
First, we rectify $H'$ using Corollary 9.2. Note that Corollary 9.2 only applies to restricted homotopy and not ordinary homotopy. So, we need to replace $H'$ by the corresponding restricted homotopy $H : A \times \Delta^1 \to \mathcal{X}$ (see Lemma 5.4). Now we can use Corollary 9.2 to rectify $H$ so that $H_0 = f$; note that doing this will alter $h$.

There are two more things to do now: ensure that the 2-isomorphism $\epsilon_1 : H_1 \Rightarrow h \circ \tilde{i}$ becomes an equality, and that $\gamma = \text{id}$. This is achieved by applying Lemma 9.1 to the diagram

$$
\begin{array}{ccc}
\tilde{A} & \xrightarrow{H_1} & \mathcal{X} \\
\downarrow{\tilde{i}} & \searrow{h} & \downarrow{p} \\
\tilde{B} & \xrightarrow{g} & \mathcal{Y}
\end{array}
$$

to adjust $h$ so that $\epsilon_1$ and $\gamma$ become the identity 2-isomorphisms. □

Let $p : \mathcal{X} \to \mathcal{Y}$ be a map of presheaves of groupoids and $i : A \to B$ a map of simplicial sets. Define $L$ to be the groupoid

$$L := \text{Hom}_{\text{pshGpd}}(\text{Cyl}(i), \mathcal{X})_p \cong \text{Hom}_{\text{Gpd}}(\text{Cyl}(i), \mathcal{X}_\Delta)_p,$$

where

$$\text{Cyl}(i) := (A \times \Delta^1) \cup_A B$$

is the mapping cylinder of $i : A \to B$. The subscript $p$ means that we only allow fiberwise maps $H : \text{Cyl}(i) \to \mathcal{X}$. That is, $p \circ H$ is required to strictly factor through the projection $\tilde{p} : \text{Cyl}(i) \to \tilde{B}$. In other words,

$$L \cong \text{Hom}_{\text{pshGpd}}(\text{Cyl}(i), \mathcal{X}) \times_{\text{Hom}_{\text{pshGpd}}(\text{Cyl}(i), \mathcal{Y})} \text{Hom}_{\text{pshGpd}}(\tilde{B}, \mathcal{Y})$$

$$\cong \text{Hom}_{\text{Gpd}}(\text{Cyl}(i), \mathcal{X}_\Delta) \times_{\text{Hom}_{\text{Gpd}}(\text{Cyl}(i), \mathcal{Y}_\Delta)} \text{Hom}_{\text{Gpd}}(\tilde{B}, \mathcal{Y}_\Delta)$$

More explicitly, $L$ is isomorphic to the groupoid

$$\text{Hom}_{\text{pshGpd}}(A \times \Delta^1, \mathcal{X})_p \times_{\text{Hom}_{\text{pshGpd}}(A, \mathcal{X})} \text{Hom}_{\text{pshGpd}}(\tilde{B}, \mathcal{X})$$

of pairs $(H, h)$, where $h : \tilde{B} \to \mathcal{X}$ is a morphism and $H : A \times \Delta^1 \to \mathcal{X}$ is a fiberwise homotopy relative to $\mathcal{Y}$ such that $H_1 = h \circ \tilde{i}$. (Here $H_1 : \tilde{A} \to \mathcal{X}$ stands for the precomposition of $H$ with the time 1 inclusion map $\tilde{A} \to A \times \Delta^1$.)

**Corollary 9.7.** Notation being as above and assumptions being as in Lemma 9.6, the map

$$\text{Hom}_{\text{pshGpd}}(\text{Cyl}(i), \mathcal{X})_p \to \text{Hom}_{\text{pshGpd}}(\tilde{A}, \mathcal{X}) \times_{\text{Hom}_{\text{pshGpd}}(\tilde{A}, \mathcal{Y})} \text{Hom}_{\text{pshGpd}}(\tilde{B}, \mathcal{Y})$$

$$(H, h) \mapsto (H_0, p \circ h)$$

and, equivalently (see Lemma 4.4), the map

$$\text{Hom}_{\text{Gpd}}(\text{Cyl}(i), \mathcal{X}_\Delta)_p \to \text{Hom}_{\text{Gpd}}(A, \mathcal{X}_\Delta) \times_{\text{Hom}_{\text{Gpd}}(A, \mathcal{Y}_\Delta)} \text{Hom}_{\text{Gpd}}(B, \mathcal{Y}_\Delta)$$

are fibrations of groupoids that are surjective on objects (hence, also on morphisms).

**Proof.** Surjectivity on objects is simply a restatement of Lemma 9.6. To prove that they are fibrations use Corollary 9.2 and Proposition 7.11 applied to the last square appearing in the proof of Lemma 9.6. □
10. Sing preserves fibrations

In this section we study the effect of the functor Sing on fibrations of stacks. We begin with a simple example to show why the Reedy condition is necessary in the statement of our main result (Theorem 10.5).

Example 10.1. Let $X$ be a trivial groupoid, namely one that is equivalent to a point. Let $\mathcal{X}$ be the stack associated to the constant presheaf with value $X$. Pick a point in $X$ and consider the map $* \to \mathcal{X}$. This map is an equivalence of stacks, hence is a Serre fibration. However, the induced map of simplicial sets

$$\text{Sing}(*) = * \to N(X) = \text{Sing}(\mathcal{X})$$

is definitely not a Kan fibration (unless $X$ is a point).

Definition 10.2. A map of simplicial sets $p : X \to Y$ is defined to be a weak Kan fibration if for every $n$ and $k$, and every lifting problem

\[
\begin{array}{ccc}
\Lambda^n_k & \overset{f}{\to} & X \\
i & \swarrow h & \searrow p \\
\Delta^n & \to & Y \\
\end{array}
\]

there exists $h : \Delta^n \to X$ such that the bottom triangle commutes and $f : \Lambda^n_k \to X$ is fiberwise homotopic to $h \circ i : \Lambda^n_k \to X$ relative to $Y$.

Recall that in the above definition a fiberwise homotopy relative to $Y$ means a map of simplicial sets $H : \Lambda^n_k \times \Delta^1 \to X$ such that $p \circ H$ is the trivial homotopy from $p \circ f$ to itself.

Lemma 10.3. Let $p : \mathcal{X} \to \mathcal{Y}$ be a morphism of Serre topological stacks that is a (weak) Serre fibration and a Reedy fibration. Then, the induced maps

\[
\begin{align*}
\text{Ob}(\mathcal{X}_\Delta) & \to \text{Ob}(\mathcal{Y}_\Delta) \\
\text{Mor}(\mathcal{X}_\Delta) & \to \text{Mor}(\mathcal{Y}_\Delta)
\end{align*}
\]

are (weak) Kan fibrations of simplicial sets.

Proof. This is simply a restatement of Corollary 9.5 (respectively, Corollary 9.7), with $i$ being the inclusion $\Lambda^n_k \to \Delta^n$, for arbitrary $k$ and $n$. \qed

To prove our first main result we need a generalization of Lemma 4.8 of (GoJa, Chapter IV) which we now prove. For this proof we will need the notion of exterior product of simplicial sets which we briefly recall.

Given simplicial sets $X$ and $Y$, their exterior product is the bisimplicial set $X \boxtimes Y$ defined by

$$(X \boxtimes Y)_{m,n} := X_m \times Y_n.$$  

We have $\text{Diag}(X \boxtimes Y) = X \times Y$. The exterior product has the property that the functor

$$\text{sSet} \to \text{bsSet},$$

$$A \mapsto A \boxtimes \Delta^n,$$

is left adjoint to

$$\text{bsSet} \to \text{sSet},$$

$$X \mapsto X_{*,n}. $$
Lemma 10.4. Let \( f : X \to Y \) be a Reedy fibration of bisimplicial sets. (Here, we are regarding \( X \) as the simplicial object \([m] \mapsto X_{m,*} \) in \( sSet \).) Suppose further that \( f \) is a horizontal pointwise (weak) Kan fibration, in the sense that the maps \( f_{*,n} : X_{*,n} \to Y_{*,n} \) are (weak) Kan fibration for all \( n \). Then, \( f \) is a diagonal (weak) Kan fibration. That is, \( \text{Diag}(f) : \text{Diag}(X) \to \text{Diag}(Y) \) is a (weak) Kan fibration.

Proof. We need to show that the inclusion \( \Lambda^a_k \to \Delta^n \) has (W)LLP with respect to \( \text{Diag}(f) \). By adjunction, this is equivalent to solving the lifting problem

\[
\begin{array}{ccc}
\Lambda^a_k & \xrightarrow{i} & \Lambda^a_k \otimes \Delta^n \\
\downarrow{d^*} & & \downarrow{f} \\
\Delta^n & \xrightarrow{u} & Y
\end{array}
\]

in bisimplicial sets, with the caveat that, in the ‘weak’ setting, instead of a fiberwise homotopy in the upper triangle of (\*) we should be asking for a map \( d^* (\Lambda^a_k \times \Delta^1) \to X \) (with the obvious properties).

We solve (\*) in two steps, by writing the left vertical map

\[
d^* (\Lambda^a_k) \to d^* (\Delta^n) = \Delta^{n,n} = \Delta^n \otimes \Delta^n
\]

as composition of two inclusions

\[
d^* (\Lambda^a_k) \xrightarrow{i} \Lambda^a_k \otimes \Delta^n \xrightarrow{j} \Delta^n \otimes \Delta^n.
\]

Here, the map \( i \) is left adjoint to the diagonal inclusion

\[
d : \Lambda^a_k \to \text{Diag}(\Lambda^a_k \otimes \Delta^n) = \Lambda^a_k \times \Delta^n.
\]

**Step 1.** We begin by solving the lifting problem

\[
\begin{array}{ccc}
\Lambda^a_k \otimes \Delta^n & \xrightarrow{i} & X \\
\downarrow{d^*} & & \downarrow{f} \\
\Lambda^a_k & \xrightarrow{u} & \text{Diag}(X)
\end{array}
\]

We show that the map \( i \) is a pointwise trivial cofibration. That is, for every \( m \), the map

\[
i_m : (d^* \Lambda^a_k)_{m,*} \to (\Lambda^a_k \otimes \Delta^n)_{m,*}
\]

is a trivial cofibration of simplicial sets. We have

\[
(d^* \Lambda^a_k)_{m,*} = \coprod_{\alpha \in (\Lambda^a_k)_m} C_\alpha \quad \text{and} \quad (\Lambda^a_k \otimes \Delta^n)_{m,*} = \coprod_{\alpha \in (\Lambda^a_k)_m} \Delta^n,
\]

where \( C_\alpha \) is the subcomplex of \( \Lambda^a_k \) generated by all faces of \( \Lambda^a_k \) containing \( \alpha \). (The second equality is obvious from the definition of exterior product. For the first equality see [GoJa], top of the page 221, just before Lemma 3.12.)

The map \( i_m \) is then the disjoint union of the inclusions \( C_\alpha \hookrightarrow \Lambda^a_k \hookrightarrow \Delta^n \), one for each \( \alpha \in (\Lambda^a_k)_m \). It is easy to show that each \( C_\alpha \) is contractible (see [GoJa], Chapter IV, proof of Lemma 3.12), so \( i_m \) is a trivial cofibration. Hence, \( i \) is a pointwise trivial cofibration.

Since \( i \) is a pointwise trivial cofibration, it has strict LLP with respect to \( f \), as \( f \) is a Reedy fibration (see [GoJa], Chapter IV, Lemma 3.3(1)). So our lifting problem has indeed a strict solution.
Consider the adjoint lifting problem

\[ \Lambda^n_k \otimes \Delta^n \xrightarrow{h} X \]
\[ \Delta^n \otimes \Delta^n \xrightarrow{v} Y \]
\[ j \]
\[ l' \]
\[ f \]
\[ \Lambda^n_k \rightarrow X_{*,n} \]
\[ \Delta^n \rightarrow Y_{*,n} \]

Consider the adjoint lifting problem

\[ \Lambda^n_k \rightarrow X_{*,n} \]
\[ \Delta^n \rightarrow Y_{*,n} \]
\[ \delta \]
\[ \star \]
\[ l' \]
\[ f_{*,n} \]

If \( f_{*,n} : X_{*,n} \rightarrow Y_{*,n} \) is a Kan fibration, this problem has a strict solution. Hence, our original problem (*) also has a strict solution, and we are done.

If \( f_{*,n} : X_{*,n} \rightarrow Y_{*,n} \) is a weak Kan fibration, a lift \( l : \Delta^n \rightarrow X_{*,n} \) exists, but the upper triangle commutes only up to a fiberwise homotopy \( H : \Lambda^n_k \times \Delta^1 \rightarrow X_{*,n} \) (relative to \( Y_{*,n} \)). By adjunction, this gives rise to a lift

\[ l' : \Delta^n \otimes \Delta^n \rightarrow X \]
in (***). The upper triangle in (***), however, is not, strictly speaking, homotopy commutative. Rather, instead of a homotopy we have a map \( H' : (\Lambda^n_k \times \Delta^1) \otimes \Delta^n \rightarrow X \), the adjoint of \( H \). Let \( H'' \) be the composition

\[ H'' : d^*(\Lambda^n_k \times \Delta^1) \rightarrow (\Lambda^n_k \times \Delta^1) \otimes \Delta^n \xrightarrow{H'} X. \]

Here, the first map is left adjoint to

\[ \delta \times \text{id}_{\Delta^1} : \Lambda^n_k \times \Delta^1 \rightarrow \text{Diag}((\Lambda^n_k \times \Delta^1) \otimes \Delta^n) = (\Lambda^n_k \times \Delta^1) \times \Delta^n = \Lambda^n_k \times \Delta^n \times \Delta^1, \]

where \( \delta : \Lambda^n_k \rightarrow \Lambda^n_k \times \Delta^n \) is the diagonal inclusion. It follows that the pair

\[ l' : \Delta^n \otimes \Delta^n \rightarrow X, \quad H'' : d^*(\Lambda^n_k \times \Delta^1) \rightarrow X \]
is the desired solution to (*). \qed

We are finally ready to prove our first main result.

**Theorem 10.5.** Let \( p : X \xrightarrow{Y} \) be a morphism of Serre topological stacks that is a (weak) Serre fibration and also a Reedy fibration. Then, \( \text{Sing}(p) : \text{Sing}(X) \rightarrow \text{Sing}(Y) \) is a (weak) Kan fibration.

**Proof.** Let the simplicial set \( R_m(X) := B(X)_{*,m} \) be the \( m \)th row of the bisimplicial set \( B(X) \). Note that we have

\[ R_0(X) = \text{Ob}(X_\Delta), \quad R_1(X) = \text{Mor}(X_\Delta), \quad R_m(X) = R_1(X) \times_{R_0(X)} \cdots \times_{R_0(X)} R_1(X). \]

It follows from Lemma 10.3 that, for every \( m \), \( R_m(X) \rightarrow R_m(Y) \) is a (weak) Kan fibration. That is, the map \( B(p) : B(X) \rightarrow B(Y) \) is a horizontal pointwise (weak) Kan fibration (in the sense of Lemma 10.4). Furthermore, \( B(p) \) is a Reedy fibration of bisimplicial sets because, by assumption, \( p_\Delta \) is a Reedy fibration of simplicial groupoids, and the nerve functor \( N : \text{Gpd} \rightarrow \text{sSet} \) preserves fibrations and limits. It follows now from Lemma 10.4 that \( B(p) \) is a diagonal (weak) Kan fibration. In other words, \( \text{Sing}(p) : \text{Sing}(X) \rightarrow \text{Sing}(Y) \) is a (weak) Kan fibration. \qed

**Corollary 10.6.** Let \( X \) be a Reedy fibrant Serre topological stack. Then, \( \text{Sing}(X) \) is a Kan complex.
Corollary 10.7. For every (weak) Serre fibration of Serre stacks $p : X \to Y$ there exists a strictly commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & Y \\
\sim & \downarrow{g} & \sim \\
X' & \xrightarrow{p'} & Y
\end{array}
\]

where $p'$ is a (weak) Serre fibration as well as an injective (hence, also Reedy) fibration, and $g : X \sim X'$ is an equivalence of Serre stack. In particular, $\text{Sing}(p') : \text{Sing}(X') \to \text{Sing}(Y)$ is a (weak) Kan fibration.

Proof. This follows from Proposition 7.12 and Theorem 10.5. (Also see Remark 6.4.) □

Corollary 10.8. For every Serre stack $X$ there exists a Serre stack $X' \sim X$ equivalent to it such that $\text{Sing}(X')$ is a Kan complex.

11. $\text{Sing}$ preserves weak equivalences

In this section, we prove that the singular functor has the correct homotopy type by showing that it takes a weak equivalence of topological stacks to a weak equivalence of simplicial sets (Theorem 11.2). We begin with a special case.

Proposition 11.1. Let $X$ be a Serre stack, and let $\varphi : X \to X$ be a trivial weak Serre fibration with $X$ (equivalent to) a topological space (i.e., $X$ is a classifying space for $X$ in the sense of Theorem 3.1). Then, $\text{Sing} (\varphi) : \text{Sing}(X) \to \text{Sing}(X)$ is a weak equivalence of simplicial sets.

Proof. We may assume that $X$ is connected. Using Corollary 10.7 and Corollary 8.4, we may assume that $\varphi : X \to X$ is a trivial weak Serre fibration as well as a Reedy fibration. (Note that we are not insisting on $X$ being isomorphic to a topological space but only equivalent.)

For an arbitrary point in $X$, we have, by Lemma 8.10, a weak equivalence of simplicial sets

\[ \text{Sing}(X) \times_{\text{Sing}(X)}^\sim \text{Sing}(X_x \times_x^\sim). \]

The latter is contractible by Theorem 3.1 and Corollary 8.4 (i.e., all its homotopy groups vanish), hence so is $\text{Sing}(X) \times_{\text{Sing}(X)}^\sim$. It follows that $\text{Sing}(\varphi)$ is a weak Kan fibration (Theorem 10.2) with contractible fibers. To see that it is a weak equivalence note that $|\text{Sing}(\varphi)|$ is a weak Serre fibration with contractible fiber, so the fibration homotopy exact sequence implies that it is a weak equivalence. □

Theorem 11.2. Let $f : X \to Y$ be a weak equivalence of Serre stacks. Then, $\text{Sing}(f) : \text{Sing}(X) \to \text{Sing}(Y)$ is a weak equivalence of simplicial sets.

Proof. By ([No12], Theorem 1.2) we can choose classifying atlases $\varphi : X \to X$ and $\psi : Y \to Y$ (in the sense of Theorem 3.1) fitting in a 2-commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\varphi & \downarrow{\sim} & \sim \\
X & \xrightarrow{f'} & Y
\end{array}
\]

By the two-out-of-three property, $f'$ is a weak equivalence. Applying $\text{Sing}$, we find a homotopy commutative diagram in simplicial sets where $\text{Sing}(f')$, $\text{Sing}(\varphi)$ and $\text{Sing}(\psi)$ are weak equivalences of simplicial sets (by Proposition 11.1, also see Remark 8.2). Therefore, $\text{Sing}(f)$ is also a weak equivalence by the two-out-of-three property. □
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