Exact Solution of an Irreversible One-Dimensional Model with Fully Biased Spin Exchanges

António M. R. Cadilhe and Vladimir Privman

Department of Physics, Clarkson University, Potsdam, New York 13699–5820, USA

ABSTRACT

We introduce a model with conserved dynamics, where nearest neighbor pairs of spins $\uparrow \downarrow$ ($\downarrow \uparrow$) can exchange to assume the configuration $\downarrow \uparrow$ ($\uparrow \downarrow$), with rate $\beta$ ($\alpha$), through energy decreasing moves only. We report exact solution for the case when one of the rates, $\alpha$ or $\beta$, is zero. The irreversibility of such dynamics results in strong dependence on the initial conditions. Domain wall arguments suggest that for more general models with steady states the dynamical critical exponent for the anisotropic spin exchange is different from the isotropic value.
1. INTRODUCTION

Dynamics of one-dimensional systems has received much attention recently.\textsuperscript{1–3} One-dimensional models can be solved exactly in some cases, and they provide convenient test cases for theories of reactions, deposition, ordering. Furthermore, in many instances, such as, for instance, diffusion-limited reactions, low-dimensional models offer examples on non-mean-field fluctuations.\textsuperscript{4} The scope of models studied recently has been extended significantly\textsuperscript{1} from the “classical” Glauber\textsuperscript{5} and Kawasaki\textsuperscript{6} models of the Ising-spin flips and exchanges.

One of the interesting recent developments has been the introduction of anisotropic (spatially biased) dynamical moves in irreversible models of reactions.\textsuperscript{7–12} Anisotropic particle motion also plays key role in hard-core-particle “asymmetric exclusion” models.\textsuperscript{13–14} Steady states of kinetic models with anisotropy have been studied extensively in the field of driven diffusive systems\textsuperscript{15}. Dynamics without detailed balance has also been considered recently in the field of neural networks.\textsuperscript{16} Our aim in this article has been to initiate investigation of the effect of making pair exchanges anisotropic in Kawasaki-type spin exchange models of dynamical behavior.

Most of our results apply in the fully irreversible limit, with only energy-lowering moves allowed. This corresponds to the zero-temperature limit and, in the case of Kawasaki spin exchanges, yields models of freezing. Studies of irreversible low-temperature kinetic Ising models, both spin-flip, Glauber-, and spin-exchange, Kawasaki-type, have attracted attention recently.\textsuperscript{1} The emphasis has been on derivation of exact results for these and certain related reaction models. In particular, models with conserved order parameter, and related systems, were solved by a method which will be also employed here.\textsuperscript{17–20} Various exactly solvable one-dimensional models of freezing have been treated
by other approaches.\textsuperscript{21–23}

Contrary to non-conserved dynamics models of the Glauber type (spin flip), where several exact results are available, conserved dynamics, Kawasaki type models have long resisted any direct analytical solution. Few exact results have become available recently for variants in the zero temperature limit.\textsuperscript{17–18,24} The ordered domain structure in these $T = 0$ models becomes frozen at large times.

Irreversible dynamics of the “freezing” type usually retains infinite memory of the initial conditions. We will use two different initial conditions, namely, the fully alternating lattice corresponding to a concentration of $\frac{1}{2}$ of both spin types, and random initial conditions where each site is either occupied by a $\uparrow$ spin with probability $p$ or by a $\downarrow$ spin with probability $q = 1 - p$.

The outline of the paper is as follows: We start by defining the model in Section 2. Then in Section 3 we present the exact solution for the single rate (fully anisotropic) case. Finally, in Section 4 we discuss the results and consider generally the anisotropic exchange models in the framework of domain-wall arguments which, for the $T > 0$ dynamics, yield information on the dynamical critical exponent.\textsuperscript{21,25–27}
2. THE MODEL

When two spins are exchanged in a ferromagnetic-interaction model, the total energy can be increased, unchanged, or decreased. By ferromagnetic interaction we mean that the energy of a “bond” connecting parallel spins (\(\uparrow\uparrow\) or \(\downarrow\downarrow\)) is lower than the energy of a bond connecting antiparallel spins (\(\uparrow\downarrow\) or \(\downarrow\uparrow\)).

In order to simplify the notation, we will use the language of particle exchanges from now on. Thus, the \(\uparrow\) (\(\downarrow\)) spins will be replaced by particles \(A\) (\(B\)). We consider only the dynamics in the \(T \to 0\) limit and allow for energy-decreasing particle exchanges. Such dynamics leads to frozen states.\(^{17-18,20}\)

Consider a one-dimensional lattice where each site is occupied by particle \(A\) or \(B\). The only two possible nearest-neighbor particle exchanges that lower the energy locally are

\[
\cdots ABAB \cdots \to \cdots AABB \cdots \quad \text{(rate } \alpha) \quad (1)
\]

\[
\cdots BABA \cdots \to \cdots BBAA \cdots \quad \text{(rate } \beta) \quad (2)
\]

Note that the particle (spin) exchanges are anisotropic. The usual isotropic Kawasaki-model spin exchanges correspond to the equal rates \(\alpha = \beta\). For finite temperatures, exchanges other than those shown in (1)-(2) are allowed, and their relative rates depend on the temperature.\(^6\) In the Ising model notation, exchanges (1)-(2) lower the energy by \(4J\), where \(-J\) is the energy of the \(AA\) and \(BB\) bonds, while \(J > 0\) is the energy of the \(AB\) and \(BA\) bonds.

The model defined by the moves (1)-(2) is conveniently analyzed\(^{18}\) in terms of the probabilities \(P_{C,n}(t)\) that a randomly chosen interval of \(n > 1\) lattice sites is fully alternating, \(ABAB\cdots\) or \(BABA\cdots\), at time \(t\). Here \(C \in \{A, B\}\) denotes the starting site type in the \(n\)-site sequence. For \(n > 3\), the interior of such intervals is “fully reactive,”
i.e., all internal nearest-neighbor exchanges are allowed (reduce energy). In order to be able to claim that the probabilities \( P_{C,n}(t) \) are independent of the position along the one-dimensional lattice, we must assume that the initial conditions are uniform; specific choices will be specified later (see also Section 1). It is important to emphasize that the probabilities only refer to the interior of the interval which might or might not be a part of a larger alternatively-ordered interval. The probabilities are not conditioned on the arrangement of particles outside the selected \( n \)-site sequence.

A closed hierarchy of rate equations can now be written as follows. Consider first the case of even \( n = 2k \), where \( k > 1 \). The rate equation for \( P_{A,2k}(t) \) is obtained in the form

\[
-\dot{P}_{A,2k} = (k-1)\alpha P_{A,2k} + (k-2)\beta P_{A,2k} + \beta P_{A,2k+1} + \beta P_{B,2k+1} + 2\alpha P_{A,2k+2}
\]

Here the dot denotes the time derivative. The first term in (3) represents the rate at which the \( k-1 \) sequences of the type (1), internal to the selected interval of \( 2k \) consecutive sites, undergo the process given in (1) thereby destroying the fully alternatively-ordered state in the selected interval. Similarly, the second term corresponds to the \( k-2 \) different four-site subsequences in the selected interval, which undergo process (2).

The third and the fourth terms correspond to, respectively, the rightmost and the leftmost \( AB \) pairs in the selected interval of \( 2k \) sites, exchanging as parts of four-site sequences of type (2). In such a process the four-site sequence containing the end-pair of the selected interval must in fact be a part of a larger, \((2k+1)\)-site, ordered sequence. Thus, \( P_{C,2k+1} \) is used. Finally, the last term corresponds to processes in which the two end sites of the selected interval exchange with sites external to it. The configuration is of type (1) at both ends, and the alternating order must extend to a larger interval of \( 2k + 2 \) sites, in order for process (1) to occur.
Similarly, we get

\[ -\dot{P}_{B,2k} = (k-1)\beta P_{B,2k} + (k-2)\alpha P_{B,2k} + \alpha P_{B,2k+1} + \alpha P_{A,2k+1} + 2\beta P_{B,2k+2} \quad (4) \]

It is also of interest to write down the rate equation for the case \( n = 2 \). Indeed, the quantities \( P_{C,2} \) yield densities of \( AB \) and \( BA \) interfaces in the system. The interfaces separate ordered domains of \( A \) and \( B \) particles (\( \uparrow \) and \( \downarrow \) spins). We have

\[ -\dot{P}_{A,2} = 2\alpha P_{A,4} + \beta P_{B,4} \quad (5) \]

\[ -\dot{P}_{B,2} = 2\beta P_{B,4} + \alpha P_{A,4} \quad (6) \]

We next turn to the case of odd \( n = 2k + 1 \), with \( k \geq 1 \). The rate equations now read

\[ -\dot{P}_{A,2k+1} = (k-1)\alpha P_{A,2k+1} + (k-1)\beta P_{A,2k+1} + \beta P_{B,2k+2} + \alpha P_{A,2k+2} + (\alpha + \beta) P_{A,2k+3} \quad (7) \]

\[ -\dot{P}_{B,2k+1} = (k-1)\beta P_{B,2k+1} + (k-1)\alpha P_{B,2k+1} + \beta P_{A,2k+2} + \alpha P_{B,2k+2} + (\alpha + \beta) P_{B,2k+3} \quad (8) \]

We must complement the above system of rate equations with initial conditions. For the initially alternating lattice of \( A \) and \( B \) particles, the probability of starting at either site type is 1/2 regardless of the interval size, i.e.,

\[ P_{A,2k}^{\text{alt}}(0) = P_{B,2k}^{\text{alt}}(0) = P_{A,2k+1}^{\text{alt}}(0) = P_{B,2k+1}^{\text{alt}}(0) = \frac{1}{2} \quad (9) \]

For the initially random distribution, placing \( A \) particles with probability \( p \) and \( B \) particles with probability \( q = 1 - p \), at each lattice site, the initial conditions are

\[ P_{A,2k}^{\text{ran}}(0) = P_{B,2k}^{\text{ran}}(0) = (pq)^k \quad (10) \]

\[ P_{A,2k+1}^{\text{ran}}(0) = p(pq)^k \quad (11) \]
\[ P^{\text{ran}}_{B,2k+1}(0) = q(pq)^k \]  

(12)

The above system of rate equations turns out to be exactly solvable in some cases. The case \( \alpha = \beta \) (isotropic exchanges) has been considered in the literature.\textsuperscript{17–18,24} We could not find a solution for the general case of both rates being non-zero and not equal to each other. In the next section we solve the case of \( \alpha > 0 \) and \( \beta = 0 \). We note, however, that the transformation used to eliminate the \( k \)-dependence (see the next section) can be generalized to the case of \( \alpha > 0, \beta > 0 \). Thus, a system of ordinary differential equations can be obtained and analyzed numerically or by approximation methods.
3. THE SINGLE RATE CASE

In this section we consider a solvable case of the fully anisotropic exchanges. Specifically, we take $\beta = 0$, $\alpha > 0$, so that only exchanges of the type (1) are allowed. It is furthermore convenient to redefine the time variable (or alternatively, set $\alpha = 1$) in such a way that

$$t_{\text{new}} = \alpha t_{\text{old}} \tag{13}$$

This eliminates the $\alpha$-dependence from the rate equations.

More importantly, the $k$-dependence can be also eliminated, by a transformation which generalizes the one used in the isotropic case.$^{17-18,24}$ Thus, we try a solution of the form

$$P_{A,2k}(t) = S(t)\gamma^k(t) \tag{14}$$

$$P_{B,2k}(t) = U(t)\gamma^k(t) \tag{15}$$

$$P_{A,2k+1}(t) = V(t)\gamma^k(t) \tag{16}$$

$$P_{B,2k+1}(t) = W(t)\gamma^k(t) \tag{17}$$

Direct substitution in (3)-(4), (7)-(8) yields (with $\alpha = 1$, $\beta = 0$)

$$\dot{\gamma} = -\gamma \tag{18}$$

$$\dot{S} = (1 - 2\gamma)S \tag{19}$$

$$\dot{U} = 2U - (V + W) \tag{20}$$

$$\dot{V} = (1 - \gamma)V - \gamma S \tag{21}$$

$$\dot{W} = (1 - \gamma)W - \gamma S \tag{22}$$
We must also check that the proposed forms (14)-(17) are consistent with the initial conditions. Denoting the initial values by the subscript 0, we identify

\[ \gamma^\text{alt}_0 = 1 \]  
\[ S^\text{alt}_0 = U^\text{alt}_0 = V^\text{alt}_0 = W^\text{alt}_0 = \frac{1}{2} \]  

and

\[ \gamma^\text{ran}_0 = pq \]  
\[ S^\text{ran}_0 = U^\text{ran}_0 = 1 \]  
\[ V^\text{ran}_0 = p \]  
\[ W^\text{ran}_0 = q \]  

These two types of initial conditions were introduced in Section 2.

For both initial conditions the solution can be now obtained straightforwardly. First, we solve (18),

\[ \gamma(t) = \gamma_0 e^{-t} \]  

This result is then used to solve (19),

\[ S(t) = S_0 \exp \left[ 2\gamma_0 e^{-t} - 2\gamma_0 + t \right] \]  

The next step is to solve the equations for \( V \) and \( W \), (21)-(22),

\[ V(t) = (V_0 - S_0) \exp \left[ \gamma_0 e^{-t} - \gamma_0 + t \right] + S_0 \exp \left[ 2\gamma_0 e^{-t} - 2\gamma_0 + t \right] \]  
\[ W(t) = (W_0 - S_0) \exp \left[ \gamma_0 e^{-t} - \gamma_0 + t \right] + S_0 \exp \left[ 2\gamma_0 e^{-t} - 2\gamma_0 + t \right] \]  

Finally, the equation for \( U \) is solved,

\[ U(t) = \left( U_0 - \frac{V_0 + W_0 - S_0}{\gamma_0} \right) e^{2t} + \frac{V_0 + W_0 - 2S_0}{\gamma_0} \exp \left[ \gamma_0 e^{-t} - \gamma_0 + 2t \right] \]
These expressions provide the full, time-dependent solution of the problem. Specifically, the probabilities $P_{C,n}(t)$ are obtained from (30)-(33), with (14)-(17), for all $n \geq 3$. However, for $n = 2$ special expressions apply,

$$P_{A,2}(t) = P_{A,2}(0) + S_0 \gamma_0 \exp[2\gamma_0 e^{-t} - 2\gamma_0] - S_0 \gamma_0$$ \hspace{1cm} (34)

$$P_{B,2}(t) = P_{B,2}(0) + \frac{S_0 \gamma_0}{2} \exp[2\gamma_0 e^{-t} - 2\gamma_0] - \frac{S_0 \gamma_0}{2}$$ \hspace{1cm} (35)

These results were obtained by integrating (5)-(6), with $\alpha = 1$, $\beta = 0$. 

\[ + \frac{S_0}{\gamma_0} \exp \left[ 2\gamma_0 e^{-t} - 2\gamma_0 + 2t \right] \] \hspace{1cm} (33)
4. DISCUSSION

As in the isotropic-exchange case,\textsuperscript{17–18,24} the most profound feature of the dynamics is its strong dependence on the initial conditions. In fact, the initial values are “remembered” for all times, and enter the $t \to \infty$ limiting expressions. The large-time limiting expressions can be obtained by expanding in powers of the quantity $\gamma_0 e^{-t}$ ($\equiv \gamma(t)$) which enters in the exponentials in (31)-(35). In fact, for large times we have $S,V,W \propto e^t$, $U \propto e^{2t}$, where the proportionality constants depend on the initial values. Thus, only the shortest alternatively-ordered intervals survive at large times. Specifically, $P_{A,2}(\infty)$, $P_{B,2}(\infty)$, $P_{A,3}(\infty)$, $P_{B,3}(\infty)$, $P_{B,4}(\infty)$ “survive” (remain nonzero) for the fully anisotropic dynamics. All other probabilities vanish exponentially.

The final configuration is therefore incompletely ordered. Some interfaces remain at $t = \infty$. These are pairs $AB$ and $BA$, the probability of which is given by $P_{A,2}$ and $P_{B,2}$, respectively. Some of the interfaces survive as nearby pairs, represented by the nonzero values of $P_{A,3}$ and $P_{B,3}$. Furthermore, some triplets of interfaces remain unreacted, due to $\beta = 0$, corresponding to configuration (2). The probability of the latter is given by $P_{B,4}$. For illustration, let us calculate the total density of interfaces per site, $I(t)$, in the system, which was also obtained in the isotropic-model studies.\textsuperscript{17–18,24} We get

$$I(t) = P_{A,2}(t) + P_{B,2}(t)$$ \hspace{1cm} (36)

$$I^{\text{alt}}(t) = \frac{1}{4} \{1 + 3 \exp [2 \left( e^{-t} - 1 \right)] \}$$ \hspace{1cm} (37)

$$I^{\text{ran}}(t) = \frac{pq}{2} \{1 + 3 \exp [2pq \left( e^{-t} - 1 \right)] \}$$ \hspace{1cm} (38)

We note that both the large-time values themselves, and the coefficients of the exponential approach to these values, depend on the initial conditions.
For the initially fully alternating lattice, it is interesting to notice that

\[ P_{A,2}^{\text{alt}}(t) = P_{A,3}^{\text{alt}}(t) = P_{B,3}^{\text{alt}}(t) = P_{B,4}^{\text{alt}}(t) = \frac{1}{2} \exp \left[ 2 \left( e^{-t} - 1 \right) \right] \]  

(39)

For this particular case, the lattice can be viewed as initially fully covered by \( AB \) “objects.” Each nearest-neighbor pair of such objects can “react away” via the process (1). At the end of the process, all the original unreacted \( AB \) pairs remain in the centers of configurations \( BABA \). This process is therefore identical to Random Sequential Adsorption of dimers on the linear lattice, and in fact one can derive the expression for \( P_{B,4}(t) \) from the exact expression available for the adsorption of dimers. However, other quantities in (39) are less straightforward; we were not able to propose a “dynamical” argument for the equalities (39).

We turn next to domain wall arguments as applied to more general one-dimensional models with anisotropic spin exchanges. We will not specify the model but only assume that the \( \uparrow \) spins favor to move to the left, while the \( \downarrow \) spins are favorably moved to the right. In the particle notation, \( BA \rightarrow AB \) exchanges are favored over \( AB \rightarrow BA \).

For very low but finite temperatures, the configuration mainly consists of single isolated interfaces, separating large ordered domains, at the distance of the correlation length, \( \xi \). If we denote by \( \Delta E = 2J > 0 \) the energy cost to create an interface by “breaking” a ferromagnetic bond, then for equilibrium states entropy arguments suggest that

\[ \xi \sim \exp \left[ \Delta E/k_B T \right] = \exp \left[ 2J/k_B T \right] \]  

(40)

for low temperatures. Of course, with anisotropic spin exchanges, the system may not have an equilibrium state but rather reach a steady state at large times. The identification (40) then must be checked by exact or approximate methods.

The usual domain wall argument for isotropic conserved order parameter models at low temperatures, can be phrased as follows. The slowest relaxation mode
in the system will be a locally correlated, diffusional motion of isolated interfaces, on
the time scale proportional to the inverse of the rate of diffusion and to the separation
squared, i.e., to $\xi^2$. However, the rate of diffusion is rather small. An isolated interface
cannot move on its own by spin exchanges. It must first be turned into a triplet of inter-
faces by a spin exchange of energy cost $4J$. Thus, the diffusion rate will be proportional
to the inverse Boltzmann factor, $\exp[-4J/k_B T]$.

In addition, a pair of interfaces will “unbind” and diffuse away at the rate of order
1 (because spin exchanges that lead to motion of a pair of interfaces do not cost energy).
This pair must reach a neighboring isolated interface at a distance of order $\xi$ for the
process to be complete. If it instead returns to the interface from which it originated,
then the configuration will be restored to the original one. The probability of not
returning back is $\sim 1/\xi$; this yields another factor of $\xi$ in the relaxation time.$^{21,25}$
The resulting expression for the relaxation time $\tau$ is written as

$$\tau_{\text{isotropic}} \sim \xi^5$$

where we used (40). This yields the value $z = 5$ for the dynamical critical exponent in
$\tau \propto \xi^z$.

For anisotropic exchange, the unbinding interface pairs will be immediately re-
turned back for “attracting” interfaces, but they will move away by biased diffusion
(with asymptotically negligible probability of return) for “repelling” interfaces. (The
two types of interfaces alternate in the system.) Therefore, the probability factor of not
returning back, the inverse of $1/\xi$, will not be present. We expect

$$\tau_{\text{anisotropic}} \sim \xi^2 \exp[4J/k_B T]$$

If (40) applies for the specific model’s steady state, then we identify $z = 4$, a different
value as compared to the isotropic case.

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In summary, we reported new exact results for the anisotropic exchange model with only the most energy-favorable moves allowed. This restriction allows exact solvability in the fully anisotropic case. The resulting dynamics is that of freezing. Domain wall arguments suggest that for more realistic, $T > 0$ models with similar dynamical rules, there will be new interesting effects such as the new value of the dynamical critical exponent.
REFERENCES

1. V. Privman, editor, Nonequilibrium Statistical Mechanics in One Dimension (Cambridge University Press, Cambridge, 1996), in print.

2. J. Stat. Phys. 65, nos. 5/6 (1991), Proceedings of Models of Non-Classical Reaction Rates, NIH (March 25-27, 1991).

3. V. Privman, Dynamics of Nonequilibrium Processes: Surface Adsorption, Reaction-Diffusion Kinetics, Ordering and Phase Separation, in Trends in Statistical Physics, Vol. 1, p. 89 (Council for Scientific Information, Trivandrum, India, 1994).

4. Review: S. Redner, Ch. 1 in Ref. 1.

5. R. Glauber, J. Math. Phys. 4, 294 (1963).

6. K. Kawasaki, Phys. Rev. 145, 224 (1966).

7. V. Privman, J. Stat. Phys. 72, 845 (1993).

8. S. A. Janowsky, Phys. Rev. E51, 1858 (1995).

9. S. A. Janowsky, Spatial Organization in the Reaction A + B → inert for Particles with a Drift, preprint.

10. I. Ispolatov, P. L. Krapivsky, and S. Redner, Kinetics of A + B → 0 with Driven Diffusive Motion, preprint.

11. V. Privman, E. Burgos, and M. Grynberg, Phys. Rev. E52, 1866 (1995).

12. V. Privman, A. M. R. Cadilhe, and M. L. Glasser, J. Stat. Phys. 81, 881 (1996).

13. Review: S.A. Janowsky and J.L. Lebowitz, Ch. 13 in Ref. 1.

14. Review: B. Derrida and M.R. Evans, Ch. 14 in Ref. 1.
15. B. Schmittmann and R.K.P. Zia, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz, Vol. 17 (Academic, London, 1995).

16. N. Mousseau and D. Sherrington, J. Phys. A28, 6557 (1995).

17. Review: V. Privman, Mod. Phys. Lett. B8, 143 (1994).

18. V. Privman, Phys. Rev. Lett. 69, 3686 (1992).

19. S.N. Majumdar and V. Privman, J. Phys. A26, L743 (1993).

20. S. N. Majumdar and C. Sire, Phys. Rev. Lett. 70, 4022 (1993).

21. Review: S.J. Cornell, Ch. 6 in Ref. 1.

22. S.J. Cornell, K. Kaski and R.B. Stinchcombe, Phys. Rev. B44, 12263 (1991).

23. R. Schilling, J. Stat. Phys. 53, 1227 (1988).

24. J.-C. Lin and P.L. Taylor, Phys. Rev. E48, 4305 (1993).

25. R. Cordery, S. Sarker and J. Tobochnik, Phys. Rev. B24, 5402 (1981).

26. W. Zwerger, Phys. Lett. 84A, 269 (1981).

27. U. Decker and F. Haake, Z. Physik B35, 281 (1979).

28. Review: J.W. Evans, Ch. 10 in Ref. 1.

29. V. Privman and M.E. Fisher, J. Stat. Phys. 33, 385 (1983).

30. M.E. Fisher, IBM J. Res. Dev. 32, 76 (1988).