STRONGLY GRADED LEAVITT PATH ALGEBRAS

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Abstract. Let \( R \) be a unital ring, let \( E \) be a directed graph and recall that the Leavitt path algebra \( L_R(E) \) carries a natural \( \mathbb{Z} \)-gradation. We show that \( L_R(E) \) is strongly \( \mathbb{Z} \)-graded if and only if \( E \) is row-finite, has no sink, and satisfies Condition (Y). Our result generalizes a recent result by Clark, Hazrat and Rigby, and the proof is short and self-contained.

1. Introduction

Given an associative unital ring \( R \) and a directed graph \( E \), one may define the Leavitt path algebra \( L_R(E) \) (see Section 2.2). Since their introduction in 2005, Leavitt path algebras have grown into a common theme within modern algebra on the interface between ring theory and operator algebra. For an excellent account of the history of the subject and a review of some of its main developments, we refer the reader to Abrams’ survey article [1].

Every Leavitt path algebra comes equipped with a natural \( \mathbb{Z} \)-gradation (see Remark 2.3) and there are multiple examples of when the graded structure has been utilized in the study of \( L_R(E) \). On the other hand, there are only a few examples of when properties of the \( \mathbb{Z} \)-gradation itself have been studied. Notably, Hazrat has shown that for a finite graph \( E \), the Leavitt path algebra \( L_R(E) \) with coefficients in a unital ring \( R \) is strongly \( \mathbb{Z} \)-graded if and only if \( E \) has no sink (see [7, Theorem 3.15]). The authors of the present article have shown that if \( E \) is a finite graph, then \( L_R(E) \) is always epsilon-strongly \( \mathbb{Z} \)-graded (see [8, Theorem 1.2]).

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In their recent article [6], Clark, Hazrat and Rigby introduced the following condition.

**Definition 1.1** (Clark et al [6]). A directed graph $E$ satisfies *Condition (Y)* if for every $k \in \mathbb{N}$ and every infinite path $p$, there exists an initial subpath $\alpha$ of $p$ and a finite path $\beta$ such that $r(\beta) = r(\alpha)$ and $|\beta| - |\alpha| = k$.

Using results on Steinberg algebras, in the case when $K$ is a commutative unital ring, they showed that $L_K(E)$ is strongly $\mathbb{Z}$-graded if and only if $E$ is row-finite, has no sink, and satisfies Condition (Y) (see [6, Theorem 4.2]).

We now introduce the following seemingly weaker condition, which can actually be shown to be equivalent to Condition (Y).

**Definition 1.2.** A directed graph $E$ satisfies *Condition (Y1)* if for every infinite path $p$, there exists an initial subpath $\alpha$ of $p$ and a finite path $\beta$ such that $r(\beta) = r(\alpha)$ and $|\beta| - |\alpha| = 1$.

Our main result is the following generalization of [6, Theorem 4.2].

**Theorem 1.3.** Let $R$ be a unital ring and let $E$ be a directed graph. Consider the Leavitt path algebra $L_R(E)$ with its canonical $\mathbb{Z}$-gradation. The following three assertions are equivalent:

(i) $L_R(E)$ is strongly $\mathbb{Z}$-graded;

(ii) $E$ is row-finite, has no sink, and satisfies Condition (Y);

(iii) $E$ is row-finite, has no sink, and satisfies Condition (Y1).

In the context of graph $C^*$-algebras, Chirvasitu [5, Theorem 2.14] has shown that the assertions on $E$ made in Theorem 1.3(ii) are equivalent to freeness of the full gauge action on the graph $C^*$-algebra $C^*(E)$.

Here is an outline of this article.

In Section 2 we record definitions and results that will be used in the sequel. In Section 3 we prove Lemma 3.1 which establishes the implication (i)$\Rightarrow$(ii) in Theorem 1.3. Notice that the implication (ii)$\Rightarrow$(iii) is trivial. In Section 4 we prove Proposition 4.3 which establishes the implication (iii)$\Rightarrow$(i) in Theorem 1.3, thereby finishing the proof of Theorem 1.3.

2. **Preliminaries**

In this section we will record definitions and results that will be used in the rest of this article.

2.1. **Strongly $\mathbb{Z}$-graded rings.** Recall that an associative ring $S$ is said to be *$\mathbb{Z}$-graded* if, for each $n \in \mathbb{Z}$, there is an additive subgroup $S_n$ of $S$ such that $S = \oplus_{n \in \mathbb{Z}} S_n$ and for all $n, m \in \mathbb{Z}$ the inclusion $S_n S_m \subseteq S_{n+m}$ holds. If, in addition, for all $n, m \in \mathbb{Z}$, the equality $S_n S_m = S_{n+m}$ holds, then $S$ is said to be *strongly $\mathbb{Z}$-graded*.

Throughout the rest of this subsection $S$ denotes a, not necessarily unital, $\mathbb{Z}$-graded ring.
Proposition 2.1. The ring $S$ is strongly $\mathbb{Z}$-graded if and only if for every $n \in \mathbb{Z}$ the $S_0$-bimodule $S_n$ is unital, and the equalities $S_1S_{-1} = S_{-1}S_1 = S_0$ hold.

Proof. The "only if" statement is immediate. Now we show the "if" statement. Take positive integers $m$ and $n$. First we show by induction that $S_m = (S_1)^m$. The base case $m = 1$ is clear. Next suppose that $S_m = (S_1)^m$. Then we get that $S_{m+1} = S_0S_{m+1} = S_1S_{-1}S_{m+1} \subseteq S_1S_{-1}S_m = S_1(S_1)^m = (S_1)^{m+1} \subseteq S_{m+1}$. Next we show by induction that $S_{n-S} = (S_1)^n$. The base case $n = 1$ is clear. Next suppose that $S_{n-S} = (S_1)^n$. Then we get that $S_{n-S-1} = S_{n-S-1}S_0 = S_{n-S-1}S_{-1} \subseteq S_{n-S-1}S_{-1} = S_{n-S-1} = (S_1)^{n-S-1}S_1 = (S_1)^{n-S} \subseteq S_{n-S}$.

Case 1: $S_mS_n = (S_1)^m(S_1^n) = (S_1)^{m+n} = S_{m+n}$.

Case 2: $S_{m-n}S_n = (S_1)^m(S_1^{-n}) = (S_1)^{m+n} = S_{m-n}$.

Case 3: Now we show that $S_mS_n = S_{m-n}$. We get that $S_mS_n = (S_1)^m(S_1^{-n}) = (S_1)^{m+n} = S_{m-n}$, if $m \geq n$, or $(S_1)^m(S_1^{-n}) = (S_1)^{n-m} = S_{m-n}$, otherwise.

Case 4: $S_{m-n}S_n = S_{n-m}$. This is shown in a similar fashion to Case 3, using the equality $S_{-1}S_1 = S_0$, and is therefore left to the reader. □

2.2. Leavitt path algebras. Let $R$ be an associative unital ring and let $E = (E^0, E^1, r, s)$ be a directed graph. Recall that $r$ (range) and $s$ (source) are maps $E^1 \to E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ are called edges. A vertex $v$ for which $s^{-1}(v)$ is empty is called a sink. A vertex $v$ for which $r^{-1}(v)$ is empty is called a source. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then $E$ is called row-finite. If $s^{-1}(v)$ is an infinite set, then $v \in E^0$ is called an infinite emitter. If both $E^0$ and $E^1$ are finite sets, then we say that $E$ is finite. A path $\mu$ in $E$ is a sequence of edges $\mu = \mu_1 \ldots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i \in \{1, \ldots, n-1\}$. In such a case, $s(\mu) := s(\mu_1)$ is the source of $\mu$, $r(\mu) := r(\mu_n)$ is the range of $\mu$, and $|\mu| := n$ is the length of $\mu$. For any vertex $v \in E^0$ we put $s(v) := v$ and $r(v) := v$. The elements of $E^1$ are called real edges, while for $f \in E^1$ we call $f^*$ a ghost edge. The set $\{f^* \mid f \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(f^*)$ denote $s(f)$, and we let $s(f^*)$ denote $r(f)$. For $n \geq 2$, we define $E^n$ to be the set of paths of length $n$, and $E^* = \cup_{n \geq 0} E^n$ is the set of all finite paths. If $\mu = \mu_1 \mu_2 \mu_3 \ldots$, where $\mu_i \in E^1$, for all $i \in \mathbb{N}$, and $r(\mu_i) = s(\mu_{i+1})$ for all $i \in \mathbb{N}$, then $\mu$ is said to be an infinite path. The set of all infinite paths is denoted by $E^{\infty}$. If $p \in E^* \cup E^{\infty}$ and some $\alpha \in E^*$, $p' \in E^* \cup E^{\infty}$ satisfy $p = \alpha p'$, then $\alpha$ is said to be an initial subpath of $p$.

Following Hazrat [7] we make the following definition.

Definition 2.2. The Leavitt path algebra of $E$ with coefficients in $R$, denoted by $L_R(E)$, is the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{f \mid f \in E^1\}$ and $\{f^* \mid f \in E^1\}$ with the coefficients in $R$, subject to the relations:

1. $uv = \delta_{u,v}v$ for all $u, v \in E^0$;
Then for all \( m, n \) the desired properties. Now we show the claim.

\[ \text{Lemma 2.4.} \]

The Leavitt path algebra \( L_R(E) \) carries a natural \( \mathbb{Z} \)-gradation. Indeed, put \( \deg(v) = 0 \) for each \( v \in E^0 \). For each \( f \in E^1 \) we put \( \deg(f) = 1 \) and \( \deg(f^*) = -1 \). By assigning degrees to the generators in this way, we obtain a \( \mathbb{Z} \)-gradation on the free algebra \( F_R(E) = R \langle v, f, f^* \mid v \in E^0, f \in E^1 \rangle \). Moreover, the ideal coming from relations (1)–(4) in Definition 2.2 is homogeneous. Using this it is easy to see that the natural \( \mathbb{Z} \)-gradation on \( F_R(E) \) carries over to a \( \mathbb{Z} \)-gradation on the quotient algebra \( L_R(E) \).

\[ \text{Remark 2.3.} \]

The following lemma follows from a general result concerning non-emptiness of inverse limits in [4] Chapter III § 7.4. For the convenience of the reader, we give a short direct proof adapted to the situation at hand.

\[ \text{Lemma 2.4.} \]

If \( (X_n)_{n \in \mathbb{N}} \) is a sequence of finite non-empty sets and for all \( n \in \mathbb{N} \), \( g_n \) is a function \( X_{n+1} \to X_n \), then there exists an element \( (x_1, x_2, x_3, \ldots) \in \prod_{n \in \mathbb{N}} X_n \) such that for each \( n \in \mathbb{N} \) the equality \( g_n(x_{n+1}) = x_n \) holds.

\[ \text{Proof.} \]

We claim that there exists a sequence of sets \( (Z_n)_{n \in \mathbb{N}} \) such that for all \( n \in \mathbb{N} \), \( Z_n \) is a non-empty subset of \( X_n \) and \( g_n(Z_{n+1}) = Z_n \). Let us assume for a moment that the claim holds. Define an element \( (x_1, x_2, x_3, \ldots) \in \prod_{n \in \mathbb{N}} Z_n \) inductively in the following way. Let \( x_1 \) be any element in \( Z_1 \). Take \( m \in \mathbb{N} \). Suppose that we have defined \( x_n \in Z_n \) for all \( n \leq m \). Then let \( x_{m+1} \) be any element in \( g_m^{-1}(x_m) \cap Z_{m+1} \). It is clear that the element \( (x_1, x_2, x_3, \ldots) \) has the desired properties. Now we show the claim. For all \( m, n \in \mathbb{N} \) put

\[ Y^m_n = (g_n \circ g_{n+1} \circ \cdots \circ g_{m+n-1})(X_{m+n}). \]

Then for all \( m, n \in \mathbb{N} \), the set \( Y^m_n \) is a finite and non-empty subset of \( X_n \), and the relations

\[ Y^{m+1}_n \subseteq Y^m_n \quad (1) \]

and

\[ g_n(Y^m_{n+1}) = Y^{m+1}_n \quad (2) \]

hold. For all \( n \in \mathbb{N} \) put \( Z_n = \bigcap_{m \in \mathbb{N}} Y^m_n \). From (1) it follows that every \( Z_n \) is a finite and non-empty subset of \( X_n \). In fact, for all \( n \in \mathbb{N} \), there is \( p(n) \in \mathbb{N} \) with the property that for all \( k \geq p(n) \), the equalities \( Y^k_n = Y^{p(n)}_n = Z_n \) hold. Take \( n \in \mathbb{N} \) and \( k = \max(p(n+1), p(n)) \). Then, from (2), we get that

\[ g_n(Z_{n+1}) = g_n(Y^{p(n+1)}_{n+1}) = g_n(Y^k_{n+1}) = Y^{k+1}_n = Z_n \]

which shows the claim. \( \square \)
3. Necessary conditions

In this section we will prove Lemma 3.1 which establishes the implication (i) \(\Rightarrow\) (ii) in Theorem 1.3.

**Lemma 3.1.** Let \(R\) be a unital ring and let \(E\) be a directed graph. Consider the Leavitt path algebra \(L_R(E)\) with its canonical \(\mathbb{Z}\)-gradation. If \(L_R(E)\) is strongly \(\mathbb{Z}\)-graded, then the following three assertions hold:

(i) \(E\) has no sink;
(ii) \(E\) is row-finite;
(iii) \(E\) satisfies Condition (Y).

**Proof.** Suppose that \(S = L_R(E)\) is strongly \(\mathbb{Z}\)-graded.

(i) Seeking a contradiction, suppose that there is a sink \(v\) in \(E\). Then \(v \in S_0 = S_1S_{-1}\). Using that \(v\) is a sink, we get that \(v = v^2 \in vS_1S_{-1} = \{0\}\). This is a contradiction.

(ii) Let \(v \in E^0\) be an arbitrary vertex. From the strong gradation we get that \(v \in S_0 = S_1S_{-1}\), i.e., \(v = \sum_{i=1}^{n} \alpha_i \beta_i^* \gamma_i \delta_i^*\) where \(\alpha_i \beta_i^* \in S_1\) and \(\gamma_i \delta_i^* \in S_{-1}\). Notice that \(|\delta_i| > 0\) for each \(i\). Seeking a contradiction, suppose that \(v\) is an infinite emitter. Then \(f = vf = \sum_{i=1}^{n} \alpha_i \beta_i^* \gamma_i \delta_i^* f\) for infinitely many \(f\)’s. But that is not possible since \(n < \infty\). This is a contradiction. We conclude that \(E\) is row-finite.

(iii) Let \(p\) be an infinite path and let \(k > 0\) be an arbitrary integer. Put \(v = s(p)\). Using that \(S_0 = S_{-k}S_k\) we may write

\[ v = \sum_{i=1}^{n} \alpha_i \beta_i^* \gamma_i \delta_i^* \]

where \(\alpha_i \beta_i^* \in S_{-k}\) and \(\gamma_i \delta_i^* \in S_k\). Let \(p'\) be an initial subpath of \(p\) such that \(|p'| > |\delta_i|\) for each \(i\). Clearly, \(vp' = s(p')p' = p'\). Hence there must be some \(m\) such that \(\delta_m\) is an initial subpath of \(p'\) (and thus also of \(p\)), for otherwise we would have ended up with \(vp' = 0\). Using the notation of Definition 1.1, put \(\alpha := \delta_m\) and \(\beta := \gamma_m\) and notice that \(r(\delta_m) = r(\gamma_m)\) and \(|\gamma_m| - |\delta_m| = k\).

This shows that \(E\) satisfies Condition (Y).

\[ \square \]

4. Sufficient conditions

In this section we will prove Proposition 4.3 which establishes the implication (iii) \(\Rightarrow\) (i) in Theorem 1.3.

**Lemma 4.1.** Let \(R\) be a unital ring and let \(E\) be a directed graph. Consider the Leavitt path algebra \(S = L_R(E)\) with its canonical \(\mathbb{Z}\)-gradation. If \(E\) is row-finite and has no sink, then \(S_1S_{-1} = S_0\).

**Proof.** It suffices to show that \(E^0 \subseteq S_1S_{-1}\). Take \(v \in E^0\). Then \(v = \sum_{s(f)=v} ff^* \in S_1S_{-1}\).

\[ \square \]

**Definition 4.2.** If \(\alpha \in E^n\), for some \(n \in \mathbb{N}\), then we say that \(r(\alpha)\) is a turning node for \(\alpha\) if there exists \(\beta \in E^{n+1}\) with \(r(\alpha) = r(\beta)\).
Proposition 4.3. Let $R$ be a unital ring and let $E$ be a directed graph. Consider the Leavitt path algebra $L_R(E)$ with its canonical $\mathbb{Z}$-gradation. If $E$ is row-finite, has no sink, and satisfies Condition (Y1), then $L_R(E)$ is strongly $\mathbb{Z}$-graded.

Proof. Put $S = L_R(E)$. Notice that, by Lemma 4.1, $S_0 = S_1S_{-1}$. Using that $S$ has a set of local units which is contained in $S_0$, it is clear that $S_n$ is a unital $S_0$-bimodule for every $n \in \mathbb{Z}$. In view of Proposition 2.4, it remains to show that $S_0 = S_{-1}S_1$. Clearly, any vertex $v \in E^0$ which is not a source belongs to $S_{-1}S_1$, since $v = r(f) = f^*f$ for some $f \in E^1$. Thus, in order to show that $L_R(E)$ is strongly $\mathbb{Z}$-graded it remains to show that $v \in S_{-1}S_1$ for every source $v$.

Suppose that $v$ is a source. Using $v$ we will now inductively define a sequence $(X_n)_{n \in N}$ of sets in the following way. Put
\[
X_1 = \{ f \in E^1 \mid s(f) = v, \text{and } r(f) \text{ is not a turning node for } f \}
\]
and notice that it is a finite set since $E$ is row-finite. Suppose that we have defined the finite set $X_n \subseteq E^n$ for some $n \in \mathbb{N}$. Put
\[
X_{n+1} = \{ \alpha f \in E^{n+1} \mid \alpha \in X_n, f \in E^1, s(f) = r(\alpha), \text{and } r(f) \text{ is not a turning node for } \alpha f \}.\]
By finiteness of $X_n$ and row-finiteness of $E$ we conclude that $X_{n+1}$ is finite. Seeking a contradiction, suppose that $X_n$ is non-empty for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define a function $g_n : X_{n+1} \to X_n$ by putting $g_n(\alpha f) = \alpha$, for $\alpha f \in X_{n+1}$. By Lemma 2.4, there exists an element $(x_1, x_2, x_3, \ldots) \in \prod_{n \in \mathbb{N}} X_n$ such that for all $n \in \mathbb{N}$ the equality $g_n(x_{n+1}) = x_n$ holds. In other words, $x_1x_2x_3 \cdots$ is an infinite path in $E$ such that $s(x_1) = v$ and with the property that for all $n \in \mathbb{N}$, $r(x_n)$ is not a turning node for $x_1x_2 \cdots x_n$. This contradicts condition (Y1). Therefore, for some $n \in \mathbb{N}$, the set $X_n$ is empty.

Put $k = \min \{ n \in \mathbb{N} \mid X_n = \emptyset \}$.

We claim that $v = \sum_{i=1}^m \alpha_i \alpha_i^* \in E^*$ for some $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in E^*$ such that, for each $i \in \{1, \ldots, m\}$, $r(\alpha_i)$ is a turning node for $\alpha_i$. If we assume that the claim holds, then for each $i \in \{1, \ldots, m\}$ there is some $\beta_i \in E^*$ such that $r(\alpha_i) = r(\beta_i)$ and $|\beta_i| - |\alpha_i| = 1$. Thus,
\[
v = \sum_{i=1}^m \alpha_i \alpha_i^* = \sum_{i=1}^m \alpha_i r(\alpha_i) \alpha_i^* = \sum_{i=1}^m \alpha_i \beta_i \beta_i^* \alpha_i^* \in S_{-1}S_1
\]
as desired.

Now we show the claim. Using that $E$ is row-finite and that $v$ is not a sink, we may write
\[
v = \sum_{i=1}^{m'} f_i f_i^*
\]
with $\{f_1, \ldots, f_{m'}\} = s^{-1}(v)$. If, for some $i$, $r(f_i)$ is not a turning node for $f_i$, then we may replace $f_i f_i^* = f_i r(f_i) f_i^* = f_i \sum_{h \in s^{-1}(r(f_i))} hh^* f_i^*$ in
Equation (3). By repeating this procedure (if necessary) it is clear that we, in a finite number of steps, will be able to identify \( m \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_m \in E^* \) such that \( |\alpha_i| \leq k \), and with the properties that \( v = \sum_{i=1}^{m} \alpha_i \alpha_i^* \) and for each \( i \in \{1, \ldots, m\} \), \( r(\alpha_i) \) is a turning node for \( \alpha_i \). \( \square \)

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