CONNECTIVITY PROPERTIES FOR ACTIONS ON
LOCALLY FINITE TREES

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Abstract. Given an action \( G \acts T \) by a finitely generated group on a locally finite tree, we view points of the visual boundary \( \partial T \) as directions in \( T \) and use \( \rho \) to lift this sense of direction to \( G \). For each point \( E \in \partial T \), this allows us to ask if \( G \) is \((n-1)\)-connected “in the direction of \( E \)”. The invariant \( \Sigma^n(\rho) \subseteq \partial T \) then records the set of directions in which \( G \) is \((n-1)\)-connected. In this paper, we introduce a family of actions for which \( \Sigma^1(\rho) \) can be calculated through analysis of certain quotient maps between trees. We show that for actions of this sort, under reasonable hypotheses, \( \Sigma^1(\rho) \) consists of no more than a single point. By strengthening the hypotheses, we are able to characterize precisely when a given end point lies in \( \Sigma^n(\rho) \) for any \( n \).

1. Introduction

Let \( G \) be a group having type \( F_n \) and let \( M \) be a proper CAT(0) metric space. Let \( \rho : G \to Isom(M) \) be an action by isometries. In [2], Bieri and Geoghegan introduced a collection of geometric “\( \Sigma \)-invariants”, \( \Sigma^n(\rho), n \geq 0 \). These arise naturally from the study of the Bieri-Neumann-Strebel-Renz (BNSR) invariants \( \Sigma^n(G) \), which can then be viewed as a special case. These invariants provide topological insight into \( \rho \) and provide algebraic information about \( G \). In particular, if \( \rho \) has discrete orbits and \( G \) is finitely generated, then

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1By definition, \( G \) has type \( F_n \) iff there exists a \( K(G,1) \)-complex having finite \( n \)-skeleton. This is equivalent to saying that there is an \( n \)-dimensional \((n-1)\)-connected CW-complex on which \( G \) acts freely and cocompactly by permuting cells. All groups have type \( F_0 \), while type \( F_1 \) is equivalent to finitely generated and type \( F_2 \) is equivalent to finitely presented [5] §7.2.

2A CAT(0) space is a geodesic metric space whose geodesic triangles are no fatter than the corresponding “comparison triangles” in the Euclidean plane, and a metric space is proper if every closed ball is compact [7] Ch. II.1.
\[ \Sigma^1(\rho) = \partial M \text{ iff the point stabilizers under } \rho \text{ are finitely generated; more generally, if } G \text{ has type } F_n, \text{ then } \Sigma^n(\rho) = \partial M \text{ iff the point stabilizers under } \rho \text{ have type } F_n. \]

The invariant \( \Sigma^n(\rho) \) depends on a notion of “controlled connectivity”, which we will briefly describe here\(^3\). The action \( \rho \) can be used to impose a sense of direction on \( G \) as follows. The space \( M \) has a CAT(0) boundary \( \partial M \), which is in one to one correspondence with the collection of geodesic rays emanating from any particular point of \( M \). In this way, \( \partial M \) encompasses the set of directions in \( M \) in which one can “go to infinity”. For an end point \( E \in \partial M \) there is a nested sequence of subsets of \( M \) (called horoballs about \( E \)). This nested sequence provides a filtration of \( M \). Because \( G \) has type \( F_n \), there is an \( n \)-dimensional \( (n-1) \)-connected CW-complex \( X \) on which \( G \) acts freely and cocompactly by permuting cells. One can then choose a \( G \)-equivariant “control” map \( h : X \to M \). Fixing an \( E \in \partial M \), \( h \) allows us to lift the sense of direction from \( M \) up to \( X \) (and therefore \( G \) by proxy) by taking the preimages of horoballs about \( E \). If, roughly speaking, the preimages of the horoballs about \( E \) are \( (n-1) \)-connected, the action \( \rho \) is said to be \( \text{controlled } (n-1) \)-connected or \( \text{CC}^{n-1} \) over \( E \).\(^4\) The precise definition ensures that this is independent of choice of \( X \) or \( h \), and is in fact a property of \( \rho \) [2, §3.2].

For \( n \geq 0 \), the invariant \( \Sigma^n(\rho) \) consists of all those end points over which \( \rho \) is \( \text{CC}^{n-1} \). These form a nested family

\[ \Sigma^0(\rho) \supseteq \Sigma^1(\rho) \supseteq \Sigma^2(\rho) \ldots. \]

The action \( \rho \) induces a topological action by \( G \) on \( \partial M \), under which \( \Sigma^n(\rho) \) is invariant. Those familiar with the BNSR invariant \( \Sigma^n(G) \) may recall that the BNSR invariant is an open subset of the boundary, which in their case is a sphere. It is worth pointing out that the Bieri-Geoghegan invariant \( \Sigma^n(\rho) \) is in general not open in \( \partial M \).

Bieri and Geoghegan have calculated \( \Sigma^n \) for the modular group acting on the hyperbolic plane in [3] and provide information about \( \Sigma^n \) for actions on trees by metabelian groups of finite Pr"ufer rank in [2], Example C in Chapter 10. In his PhD thesis [13], Rehn provides

\(^3\)See Theorem A and the Boundary Criterion in [2], and note that the required condition “almost geodesically complete” is ensured by cocompactness due to Ontaneda [12, Theorem B].

\(^4\)The technical definition of controlled connectivity is provided in [2].

\(^5\)For \( n = 0 \), we take \((-1)\)-connected to mean non-empty.
calculations for the natural action by $SL_n(\mathbb{Z}[\frac{1}{k}])$ on the symmetric space for $SL_n(\mathbb{R})$.

In the case where $M = T$, a locally finite simplicial tree, calculations in [2] led Bieri and Geoghegan to ask whether $\Sigma^1(\rho)$ would always be either empty, a singleton, or the entire boundary of the tree. The “entire boundary” case has been discussed above. In his Frankfurt Diploma Thesis [9], Lehnert gave an example for which this is not the case. However, in this paper we illustrate that there does exist a class of actions for which $\Sigma^n$ is either empty or a singleton.

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1.1. Main Result. All trees are assumed to be simplicial trees viewed as CAT(0) metric spaces by giving each edge a length of 1. All actions under consideration are by simplicial automorphisms and therefore are by isometries. Furthermore, we assume that actions are without inversions — i.e, an edge is stabilized if and only if it is fixed pointwise — since we can simply pass to the barycentric subdivision otherwise. Any tree exhibiting such an action by a group $G$ will be referred to as a $G$-tree. All $G$-trees are assumed to be infinite, and $G$ is always assumed to be finitely generated.

A group action on a tree is minimal if there exists no proper invariant subtree. A cocompact action on an infinite tree is minimal if and only if the tree has no leaves. We define a morphism of trees to mean a map between two trees which sends vertices to vertices, edges to edges, and preserves adjacency. All maps between $G$-trees are assumed to be $G$-equivariant morphisms of trees, and therefore continuous. The star of a vertex is the set of edges adjacent to that vertex, and a morphism is locally surjective (resp. locally injective) if for each vertex of the domain tree, the corresponding map between stars is surjective (resp. injective). See [1] for further discussion. In the context of morphisms of trees (as opposed to graphs), local injectivity is equivalent to injectivity, and local surjectivity implies surjectivity. A tree is locally finite if the star of each vertex is finite; such trees are proper metric spaces.
Figure 1. $G$ admits a normal subgroup $N$, whose action on $\tilde{T}$ collapses $\tilde{T}$ to $T$.

**Theorem 1.1** (Main Theorem). Let $G$ be a finitely generated group, $T$ a locally finite tree, and $G \vartriangleleft \rho T$ a cocompact action by isometries. If there exists a minimal $G$-tree $\tilde{T}$ and a $G$-morphism $q : \tilde{T} \to T$ which is locally surjective, but not locally injective, then $\Sigma^1(\rho)$ consists of at most a single point of $\partial T$.

We do not require $\tilde{T}$ to be locally finite, as it is irrelevant to us whether or not $\tilde{T}$ is proper as a metric space. Also, it is worth noting that the map $q : \tilde{T} \to T$ does not generally extend to a map $\partial \tilde{T} \to \partial T$, as geodesic rays may be collapsed to finite paths by $q$.

As mentioned in the introduction, $\Sigma^1(\rho)$ is a $G$-invariant subset of $\partial T$. Hence, if the conditions of the Main Theorem apply and there does exist a point $E_0 \in \Sigma^1(\rho)$, then $E_0$ is necessarily fixed by $\rho$. In some cases, this allows us to easily determine that $\Sigma^1(\rho)$ is empty, as in the following examples.

**Example 1.1.** Let $G$ be the group given by the presentation

$$G = \langle a, s, t | a^s = a^2, a^t = a^3 \rangle.$$  

As is clear from the given presentation, $G$ can be realized as a fundamental group of a graph of groups, where the graph is a 2-rose (a single vertex with two loops). The Bass-Serre tree $\tilde{T}$ associated with this graph of groups decomposition is a regular 7-valent tree. Let $N$ be the normal closure of $a$. Then $N$ consists of all elements of $G$ which stabilize a vertex in $\tilde{T}$. The quotient group $G/N$ is free on two generators and acts freely on $T = N \backslash \tilde{T}$ with quotient a 2-rose of circles, whereby $T$ is a regular 4-valent tree. Figure 1 demonstrates the collapsing on a neighborhood of a vertex in $\tilde{T}$. (One can take $T$ to be the Cayley graph of $G/N$.) The natural quotient map $\tilde{T} \to T$ satisfies the
conditions of the Main Theorem, and no end point $E \in \partial T$ is fixed by $\rho$. Hence $\Sigma^1(\rho) = \emptyset$.

This example can be generalized to any non-free group with a graph of groups decomposition over a graph containing a single vertex. Such a group always has a free quotient obtained by collapsing the normal closure of the subgroup associated with the vertex, and as above, the Cayley graph of this free group can be viewed as the quotient of the original Bass-Serre tree.

**Example 1.2.** One of Lehnert’s counterexamples to the question of whether $\Sigma^1$ must be either $\emptyset$, a singleton, or $\partial T$ in the case of simplicial trees is closely related to the group $G$ discussed in Example 1.1: Let $H = \mathbb{Z} \left< \frac{1}{6} \right> \rtimes F_2(x, y)$, where $F_2(x, y)$ is a free group generated by the letters $x$ and $y$. One obtains $H$ from $G$ by adding relations corresponding to the commutator subgroup of $N$. The semidirect product structure is given by $t^x = \frac{1}{2}$ and $t^y = \frac{1}{3}$ for $t \in \mathbb{Z} \left< \frac{1}{6} \right>$. This group acts on the same tree $T$, by viewing it as the Cayley graph of its factor $F_2(x, y)$, and one can represent points in $\partial T$ by infinite reduced words in $F_2(x, y)$. Any point represented by an infinite word eventually consisting of only $x$ or only $y$ will not lie in $\Sigma^1$ [9]. This fact is a consequence of the interplay between the actions by $F_2(x, y)$ on $\mathbb{Z} \left< \frac{1}{6} \right>$ and on $T$. The author has a proof of this result in a paper currently in preparation, which is based on the “topological construction of the Bass-Serre tree” [14] [8, Ch.6] and is distinct in flavor from the both the contents of this paper and the proof in [9].

Evidently, for the action $H \curvearrowright T$, there exists no $T$ and $q : T \to T$ as described in Theorem 1.1.

**Example 1.3.** Here is an example where $T$ is not locally finite. Let $K_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the Klein 4-group, and $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ the infinite dihedral group. Consider the quotient map $\pi : D_\infty * D_\infty \to K_4 * K_4$, induced by performing the abelianization map $D_\infty \to K_4$ on each free factor of $D_\infty * D_\infty$. There is an action $\tilde{\rho} : D_\infty * D_\infty \to \text{Aut}(\tilde{T})$, where $\tilde{T}$ (a regular $\infty$-valent tree) is the Bass-Serre tree corresponding to the given free product decomposition. There is also an action $\rho : D_\infty * D_\infty \to \text{Aut}(T)$, where $T$, a regular 4-valent tree, is the Bass-Serre tree for $K_4 * K_4$; this action factors through $\pi$. We can realize $T$ as a quotient of $\tilde{T}$ satisfying the conditions of the Main Theorem. Again, because no point of $\partial T$ is fixed by $\rho$, it follows that $\Sigma^1(\rho)$ is empty. This example is of a kind initially pointed out to the author by Mike Mihalik.
This, too, can be generalized: if $A_1$ and $A_2$ are two finitely generated infinite groups, which admit finite quotients $Q_1$ and $Q_2$, respectively, then $G = A_1 \ast A_2$ admits a quotient map $\pi : G \to Q_1 \ast Q_2$. While $G$ acts on the Bass-Serre tree $\tilde{T}$ corresponding to the decomposition $A_1 \ast A_2$, it also acts on $\ker \pi \setminus \tilde{T}$, which is isomorphic to the Bass-Serre tree corresponding to $Q_1 \ast Q_2$.

Example 1.4. More generally, there is a notion of a *morphism of graphs of groups* (essentially, a morphism of graphs together with a collection of homomorphisms of vertex and edge groups that ensure certain squares commute), which lifts to an equivariant morphism between the corresponding Bass-Serre trees (Proposition 2.4 of [1]), and one can determine whether the lift will be locally surjective and not locally injective (Corollary 2.5 of [1]). This can be used to produce maps satisfying the conditions of Theorem 1.1. For example, consider the Baumslag-Solitar groups $BS(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$. There is a projection map $BS(2,4) \twoheadrightarrow BS(1,2)$ obtained by adding the relation $ tat^{-1}a^{-2}$. One can show that this corresponds to a morphism of graphs of groups which lifts to a map between the corresponding Bass-Serre trees and has the desired properties.

Applying Theorems $A$ and $H$ of [2], we have:

**Corollary 1.2.** If $G \acts T$ satisfies the conditions of the Main Theorem, then for any point $z \in T$, the stabilizer $G_z$ of $z$ under the action $\rho$ is not finitely generated. □

1.2. **Collapsing Pairs.** Recall that, in the language of [15], Chapter I.2, each geometric edge of $T$ corresponds to two oriented edges, one pointing in either direction.

**Remark.** We will use the lowercase $e$ to refer to edges of $T$, oriented or not, and the uppercase $E$ to refer to points of $\partial T$.

**Definition 1.1.** Under the hypotheses of the Main Theorem, let $(\tilde{e}_1, \tilde{e}_2)$ be a pair of adjacent distinct oriented edges in $\tilde{T}$ with common initial vertex $\tilde{v}$. If $q(\tilde{e}_1) = q(\tilde{e}_2)$, we call this a *collapsing pair (of edges)* under $q$. Let $e = q(\tilde{e}_1)$ be the resulting oriented edge in $T$. For a vertex $w \in T$ (or end point $E \in \partial T$), we say the pair $(\tilde{e}_1, \tilde{e}_2)$ *faces* $w$ (resp., $E$) if $e$ points toward $w$ (resp., $E$). This is the same as saying the geodesic from $q(\tilde{v})$ to $w$ (resp., $E$) passes through $e$.

The proof of the Main Theorem will follow from two facts: Proposition 3.8 states that because $q$ is not locally injective, all end points of
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$T$ (with the possible exception of a single end point) are faced by a collapsing pair. Proposition 3.4 states that local surjectivity of $q$ forces any end point of $T$ faced by a collapsing pair to lie outside $\Sigma^1(\rho)$.

1.3. The case where stabilizers on $\tilde{T}$ have type $F_n$. If we add the condition that the stabilizers under $\tilde{\rho}$ have type $F_n$, then we can prove that a point $E \in \partial T$ which is not faced by a collapsing pair lies in $\Sigma^1(\rho)$.

Theorem 1.3. Assume the conditions of the Main Theorem. Furthermore, suppose that for $n > 0$, $G$ has type $F_n$ and for each point $\tilde{z}$ of $\tilde{T}$, the stabilizer $G_{\tilde{z}}$ has type $F_n$. Then $E \in \partial T$ lies in $\Sigma^1(\rho)$ if and only if there is no collapsing pair facing $E$.

Corollary 1.4. Let the group $H$ have type $F_n$, and let $\varphi : H \rightarrow H$ be injective, so that $G = \langle H, t \mid a^t = \varphi(a) \forall a \in H \rangle$ is an ascending HNN-extension. If $\chi : G \rightarrow \mathbb{Z}$ maps $t \mapsto 1$ and $\langle \langle H \rangle \rangle \mapsto 0$, then $\chi$ represents a point in $\Sigma^1(G)$.

This corollary is not new [6] [10] [11], but the approach is. For further discussion on this result, see [4].

2. Controlled connectivity

In a CAT(0) space $M$ there is a notion of a (visual) boundary $\partial M$ obtained by taking equivalence classes of geodesic rays [7, Ch. II.8]. This boundary carries a topology, called the cone topology, induced by the topology on $M$. We call points of $\partial M$ end points. CAT(0) spaces are contractible, and the boundary of a proper CAT(0) space is a compact space. Let $\tau$ be a geodesic ray in $M$. Following [2], we define the Busemann function $\beta_\tau : M \rightarrow \mathbb{R}$ by

$$\beta_\tau(p) = \lim_{t \rightarrow \infty} (t - d(\tau(t), p)).$$

For $r \in \mathbb{R}$, the set $HB_\tau(r) = \beta_\tau^{-1}([r, \infty))$ is called a horoball around $E$. Horoballs in CAT(0) spaces are contractible. We can view $HB_\tau(r)$ as the nested union of closed balls $\bigcup_{k \geq \max(0,r)} B_{k-r}(\tau(k))$.

Definition 2.1. Fix $n \in \mathbb{N}$. Let $G$ be a group having type $F_n$ and let $M$ be a proper CAT(0) space admitting an isometric action $G \varnothing M$. Choose an $n$-dimensional $(n-1)$-connected CW-complex $X^n$ on which $X$ acts freely and cocompactly, and choose a continuous $G$-map $h : X^n \rightarrow M$. We call $h$ a control map; one can be found because the
action by $G$ on $X^n$ is free and $M$ is contractible. Fix a geodesic ray $\tau$ representing $E \in \partial M$. For a horoball $HB_r(\tau)$ about $E$, denote the largest subcomplex of $X^n$ contained in $h^{-1}(HB_r(\tau))$ by $X_{(\tau,r)}$. Finally, we need a notion of lag function: any $\lambda(r) > 0$ satisfying $r - \lambda(r) \to \infty$ as $r \to \infty$ is called a lag.

We say $\rho$ is controlled $(n-1)$-connected, or $CC_{n-1}$, over $E$ if for all $r \in \mathbb{R}$ and all $-1 \leq p \leq (n-1)$, there exists a lag $\lambda$ such that every map $f : S^p \to X_{(\tau,r)}$ extends to a map $\tilde{f} : B^{p+1} \to X_{(\tau,r-\lambda(r))}$. 

**Definition 2.2.** The Bieri-Geoghegan invariant $\Sigma^n(\rho)$ is the subset of $\partial M$ consisting of all end points over which $\rho$ is controlled $(n-1)$-connected.

**2.1. Relationship to the BNSR invariant.** If $\rho$ fixes an endpoint $E$, then the pair $(\rho, E)$ determines a homomorphism $\chi_{\rho,E} : G \to \mathbb{R}$, and $E \in \Sigma^1(\rho)$ iff $\chi_{\rho,E}$ represents a point in $\Sigma^1(G)$ [2, §10.6]. In fact, we can obtain the classical BNSR invariant $\Sigma^n(G)$ as the special case where $\rho$ is the action $G \curvearrowright G_{ab} \otimes \mathbb{R}$ [2, Ch. 10, Example A]. This is an action by translations on a finite dimensional real vector space, so every end point is fixed, and $\partial(G_{ab} \otimes \mathbb{R}) \cong \text{Hom}(G, \mathbb{R})$.

The question of finding a single technique for calculating $\Sigma^1$ for arbitrary group actions on trees seems out of reach at this time. To see this, consider an action $G \curvearrowright T$ by translations, where $T$ is a simplicial line. This corresponds to a homomorphism $\chi : G \to \mathbb{Z}$, and calculating $\Sigma^1(\rho)$ determines whether $\chi$ and $-\chi$ represent points of $\Sigma^1(G)$. However, it is known that $\ker \chi$ is finitely generated if and only if both do represent points of $\Sigma^1(G)$ [5, Theorem B1]. Thus a method for calculating $\Sigma^1(\rho)$ even in the special case that the tree is a simplicial line would enable us to determine whether or not the kernel of an arbitrary homomorphism to $\mathbb{Z}$ is finitely generated.

**3. Proof the Main Theorem**

An automorphism $s$ of a tree $T$ having no fixed point is said to be **hyperbolic**. For each such $s$, there is a unique line $A_s$, called the **axis** of $s$, stable under the action of the subgroup $\langle s \rangle$, which acts on $A_s$ by translations. If $e$ is an oriented edge of $T$, then $s$ is said to act **coherently** on $e$ if $e$ and $se$ are consistently oriented (i.e., if they point in the same direction — neither toward each other nor away from each other). For an automorphism $s$, if $e \neq se$, then $s$ acts coherently on

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6By convention $S^{-1} = \emptyset$, and $(−1)$-connected means “non-empty”.
e if and only if $s$ is hyperbolic and both $e$ and $se$ lie on the axis of $s$ \cite[Proposition 25]{[15]}.

**Lemma 3.1.** Let $T$ be a cocompact $G$-tree, and let $E \in \partial T$. Then for any geodesic ray $\tau$ representing $E$, any $r \in \mathbb{R}$, and any oriented edge $e$ of $T$ oriented toward $E$, there exists an element of the $G$-orbit of $e$ which is oriented toward $E$ and does not lie in $HB_r(\tau)$.

**Proof.** The ray of oriented edges beginning at $e$ and representing $E$, with all edges pointing toward $E$, contains infinitely many edges. Because the action is cocompact, the pigeon-hole principle ensures that there must be edges $e_1$ and $e_2$ from this ray in the same $G$-orbit. Hence, there is an $h \in G$ with $he_1 = e_2$. Because $e_1$ and $e_2$ are consistently oriented, $h$ is hyperbolic. Let $v_1$ be the terminus of $e_1$ (the vertex of $e_1$ where $\beta_\tau$ is maximized). By choosing $k \in \mathbb{Z}$ such that (i) $|k| > \beta_\tau(v_1) - r$ and (ii) $h^k$ moves $e_1$ away from $E$, we ensure that $h^k e_1$ is oriented toward $E$ and does not lie in $HB_r(\tau)$. Thus $h^k e$ is the edge we seek. \hfill $\square$

**Observation 3.2.** For trees $\hat{T}$ and $T$, let $q : \hat{T} \to T$ be locally surjective. If $\tau = (e_0, e_1, \ldots)$ is a geodesic edge ray in $T$ and $\hat{e}_0$ is an edge of $\hat{T}$ satisfying $q(\hat{e}_0) = e_0$, then there exists a lift $\hat{\tau}$ of $\tau$ to $\hat{T}$ having initial edge $\hat{e}_0$ and which is also a geodesic edge ray. \hfill $\square$

**Observation 3.3.** Given a nonempty connected $G$-graph $\Gamma$ and a minimal $G$-tree $T$, any $G$-morphism $h : \Gamma \to T$ is surjective.

**Proposition 3.4.** Let $T$ be a cocompact $G$-tree and let $\hat{T}$ be a minimal $G$-tree. Suppose $q : \hat{T} \to T$ is a $G$-morphism which is locally surjective. If $E \in \partial T$ is such that there exists a collapsing pair facing $E$, then $E$ does not lie in $\Sigma^1(\rho)$.

**Proof.** Let $\Gamma$ be a free cocompact $G$-graph, and choose any $G$-morphism $h : \Gamma \to \hat{T}$. Then the composition $q \circ h$ is a suitable control map for determining controlled connectivity over $E$.

Let $\tau : [0, \infty) \to T$ be a geodesic edge ray representing $E$. We will show that for any lag $\lambda > 0$, there exist points in the subgraph $\Gamma_{(\tau, 0)}$ that cannot be connected via a path in $\Gamma_{(\tau, -\lambda)}$.

By Lemma 3.1, we can choose a collapsing pair $(\hat{e}_1, \hat{e}_2)$ facing $E$ but whose image in $\hat{T}$ does not lie in $HB_{-\lambda}(\tau)$. Let $\hat{v}$ be the vertex shared by $\hat{e}_1$ and $\hat{e}_2$, and let $v$ be its image in $T$. Let $\gamma$ be the geodesic ray representing $E$ and emanating from $v$. By Observation 3.2, there exist two distinct lifts $\hat{\gamma}_i$, $i = 1, 2$, of $\gamma$ to $\hat{T}$, with $\hat{\gamma}_i$ having initial edge $\hat{e}_i$. 

Because $\gamma$ and $\tau$ both represent $E$, they eventually merge, so that $\gamma$ intersects $HB_r(\tau)$ nontrivially for all $r \in \mathbb{R}$. Hence, both $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ intersect $q^{-1}(HB_r(\tau))$ for all $r$.

By design, $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = \tilde{v}$, and $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$ is a line. By Observation 3.3, $h$ is onto, so that $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$ lies in the image of $h$. For $i = 1, 2$, choose a vertex $\tilde{y}_i \in \tilde{\gamma}_i \cap q^{-1}(HB_0(\tau))$, and choose $x_i \in h^{-1}(\tilde{y}_i)$. Then both $x_i$ lie in $\Gamma(\tau, 0)$, but any path through $\Gamma(\tau, -\lambda)$ joining $x_1$ to $x_2$ would be mapped to a path in $q^{-1}(HB_{-\lambda}(\tau))$ joining $\tilde{y}_1$ to $\tilde{y}_2$. Since $\tilde{T}$ is a tree, no such path exists.

Lemma 3.5. Let $T$ be a minimal $G$-tree and let $E$ be a nonempty $G$-invariant set of oriented edges. Then there is no vertex $v$ in $T$ such that all edges of $E$ are oriented away from $v$.

Proof. The full subtree of $T$ on the vertex subset

$$\{v \mid \text{each edge of } E \text{ is oriented away from from } v\}$$

is a proper $G$-invariant subtree. By minimality, this set must be empty.

Corollary 3.6. Let $T$ be a cocompact $G$-tree and let $\tilde{T}$ be a minimal $G$-tree. Suppose $q : \tilde{T} \to T$ is a $G$-morphism which is surjective but not locally injective. Then every vertex of $T$ is faced by a collapsing pair.

Proof. Let $\tilde{E}$ be the set of oriented edges of $\tilde{T}$ that are part of a collapsing pair. This is a $G$-invariant set, and it is nonempty because $q$ is not locally injective. By Lemma 3.5, each vertex $\tilde{v}$ of $\tilde{T}$ must therefore have an edge $\tilde{e}$ in $\tilde{E}$ oriented toward $\tilde{v}$. Set $v = q(\tilde{v})$. Then if $q(\tilde{e})$ is not oriented toward $v$, the image of the path from $\tilde{e}$ to $\tilde{v}$ must contain points of backtracking. The point of backtracking closest to $v$ gives rise to a collapsing pair facing $v$. Because $q$ is surjective, all vertices of $T$ are of this form.

Observation 3.7. If a cocompact $G$-tree $T$ has a nonempty $G$-invariant subtree $T'$, then $T$ is a Hausdorff neighborhood of $T'$. Hence, $T$ and $T'$ have the same set of end points.

Proposition 3.8. Let $T$ be a cocompact $G$-tree and let $\tilde{T}$ be a minimal $G$-tree. Suppose $q : \tilde{T} \to T$ is a $G$-morphism which is not locally injective. Then there exists at most one point $E_0 \in \partial T$ such that no collapsing pairs face $E_0$. 

Proof. By Observation 3.7, the ends of $T$ and the ends of $q(\tilde{T})$ are the same, so we may assume $q$ is surjective. By Corollary 3.6, each vertex of $T$ is faced by a collapsing pair in $\tilde{T}$. If two points of $\partial T$ were not faced by a collapsing pair, then no vertex on the line between them would be faced by a collapsing pair. Hence, there can be at most one point of $\partial T$ not faced by a collapsing pair.

This proposition has an interesting consequence. If such an end $E_0$ exists, it must clearly be fixed by $\rho$. Yet points of the boundary which are fixed by $\rho$ correspond to homomorphisms $G \to \mathbb{R}$, and such an end point lies in $\Sigma^a(\rho)$ if and only if the corresponding homomorphism lies in the BNSR invariant $\Sigma^a(G)$, as discussed in §2. Since we only consider simplicial trees, such points in fact correspond to homomorphisms $G \to \mathbb{Z}$. This leads to the following corollary:

**Corollary 3.9.** Under the conditions of Proposition 3.8, if an end point $E_0 \in \partial T$ is faced by no collapsing pair in $\tilde{T}$, then there exists a canonically associated discrete character $\chi : G \to \mathbb{Z}$ such that $E_0 \in \Sigma^a(\rho)$ if and only if $[\chi] \in \Sigma^a(G)$, the BNSR invariant.

Proof of Main Theorem. Because $q$ is not locally injective, Proposition 3.8 ensures there is at most one end point faced by a collapsing pair. Because $q$ is locally surjective, Proposition 3.4 ensures that every end point faced by a collapsing pair lies outside $\Sigma^1(\rho)$. □

3.1. **The case where stabilizers under $\tilde{\rho}$ have type $F_n$.** Recall the “topological construction of the Bass-Serre tree”, discussed in §6.2 of [8], and in [14]: the action $\tilde{\rho}$ corresponds to a graph of groups decomposition of $G$. From this we can build a $K(G, 1) \tilde{X}$ admitting the quotient $G \backslash \tilde{T}$ as a retract. Let $p : \tilde{X} \to X$ be the universal covering projection. There is a natural $G$-map $h : \tilde{X} \to \tilde{T}$, and it is clear from the construction of $h$ that for any connected subset $A \subseteq \tilde{T}$, $h^{-1}(A) \subseteq \tilde{X}$ is contractible. If for an integer $n \geq 1$ all point stabilizers under $\tilde{\rho}$ have type $F_n$, then we can take $X$ to have compact $n$-skeleton. Hence, letting $\Gamma$ be the $n$-skeleton of $\tilde{X}$, the composition $\tilde{h} = q \circ h|_{\Gamma} : \Gamma \to T$ is an appropriate control map for $\rho$.

**Definition 3.1.** While the map $q$ does not induce a map $\partial \tilde{T} \to \partial T$, each geodesic ray in $T$ can be lifted to one or more geodesic rays in $\tilde{T}$ (see Observation 3.2) as long as $q$ is locally surjective. Hence, given $E \in \partial T$, we can consider the set $q^{-1}(E) \subseteq \partial \tilde{T}$ of end points represented by lifts of rays representing $E$. 
Lemma 3.10. If $q$ is locally surjective, then $q^{-1}(E)$ is a singleton if and only if there are no collapsing pairs facing $E$.

Proof. Suppose that $q^{-1}(E)$ is not a singleton. Then for $\tau$ representing $E$, there exist two distinct lifts $\tilde{\tau}_1$ and $\tilde{\tau}_2$, representing distinct points $\tilde{E}_1$ and $\tilde{E}_2$ of $\partial \tilde{T}$. If these lifts are not disjoint, then where they split (as they must, eventually) there is a collapsing pair facing $E$. If they are disjoint, consider the geodesic path $P$ through $\tilde{T}$ connecting them. The image of $P$ in $T$ is a finite subtree of $T$. Choose any vertex $v \neq \tau(0)$ which is a leaf of this subtree. This leaf and the corresponding edge lie under a collapsing pair of edges of $P$ facing $E$.

Now suppose there is a collapsing pair $\tilde{(e_1, e_2)}$ of edges of $\tilde{T}$ facing $E$. Let $e$ be their common image in $T$, and let $\zeta$ be the geodesic ray in $T$ representing $E$ and beginning with the edge $e$. Then there are distinct lifts $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ of $\zeta$, each representing a distinct end point of $\tilde{T}$. Hence $q^{-1}(E)$ is not a singleton.

Proof of Theorem 1.3. If there is a collapsing pair facing $E$, then by Proposition 3.4, $E \not\in \Sigma^1(\rho)$.

If there is no collapsing pair facing $E$, we take the control map $\tilde{h}$ described above. By construction of $\tilde{h}$, we need only show that for any horoball $H B_r(\tau)$ about $E$, $q^{-1}(H B_r(\tau))$ is connected.

For $i = 1, 2$, let $\tilde{z}_i$ be a point in $q^{-1}(H B_r(\tau))$, and let $z_i$ be its image in $T$. We will find a path between $\tilde{z}_1$ and $\tilde{z}_2$ lying in $q^{-1}(H B_r(\tau))$.

For $i = 1, 2$, there exists a unique geodesic ray $\zeta_i$ in $T$ which emanates from $z_i$ and represents $E$. Let $\tilde{\zeta}_i$ be the lift of $\zeta_i$ to $\tilde{T}$ emanating from $\tilde{z}_i$. Since $\zeta_i$ lies in $H B_r(\tau)$, $\tilde{\zeta}_i$ lies in $q^{-1}(H B_r(\tau))$. Moreover, since $q^{-1}(E)$ is a singleton, $\tilde{\zeta}_1(\infty) = \tilde{\zeta}_2(\infty)$. Hence, $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ must eventually merge. The closure of $\text{im } \tilde{\zeta}_1 \cup \text{im } \tilde{\zeta}_2$ is the geodesic connecting $\tilde{z}_1$ to $\tilde{z}_2$.

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