A Method of Classifying All Simply Laced Root Systems

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Abstract

A root system in which all roots have same norm is known as a simply laced root system. We present a simple method of classifying all simply laced root systems.

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This note is motivated by Chapter 12 of [1], a study of the class of all graphs with least eigenvalue \( \geq -2 \); we present a much simpler and shorter method of deriving Theorem 12.7.4 of [1]. Let \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{R} \) be respectively the set of all positive integers, the set of all integers and the set of all reals. Let \( \mathbb{E} \) be the euclidean space of countably infinite dimension; i.e., \( \mathbb{E} \) is the usual innerproduct space defined on \( \{ (r_1, r_2, \ldots) \in \mathbb{R}^\mathbb{N} : \sum_{i=1}^{\infty} r_i^2 < \infty \} \); for any \( x = (r_1, r_2, \ldots) \) and \( y = (s_1, s_2, \ldots) \) which belong to \( \mathbb{E} \), their innerproduct \( \sum_{i=1}^{\infty} r_i s_i \) is denoted by \( \langle x, y \rangle \). We denote the zero-vector of any subspace of \( \mathbb{E} \) by 0 itself. Let \( \mathbb{S} \) be any subset of \( \mathbb{E} \). Then the set \( \{ \alpha_1 v_1 + \cdots + \alpha_n v_n : \text{for each } i \leq n, \alpha_i \in \mathbb{Z} \text{ and } v_i \in \mathbb{S} \} \) is denoted by \( \mathbb{Z}(\mathbb{S}) \); any element (subset) in (of) \( \mathbb{Z}(\mathbb{S}) \) is said to be generated by \( \mathbb{S} \). \( \hat{\mathbb{S}} \) denotes the set \( \{ v \in \mathbb{Z}(\mathbb{S}) : ||v|| = \sqrt{2} \} \). We associate with \( \mathbb{S} \), a graph denoted by \( G[\mathbb{S}] \): its vertex set is \( \mathbb{S} \); two vertices are joined if their innerproduct is nonzero. If \( G[\mathbb{S}] \) is connected, then \( \mathbb{S} \) is called indecomposable; otherwise it is decomposable. Note that \( \mathbb{S} \) is decomposable if and only if it has a proper subset \( T \) such that for all \( x \in T \) and for all \( y \in \mathbb{S} \setminus T, \langle x, y \rangle = 0 \). If \( \mathbb{S} \) is linearly independent and for all distinct \( x, y \in \mathbb{S}, \langle x, y \rangle \leq 0 \), then \( \mathbb{S} \) is called obtuse.

Our object is to produce a method of classifying every non-empty finite set \( \mathbb{X} \) in \( \mathbb{E} \) such that for all \( x, y \in \mathbb{X}, \langle x, x \rangle = 2, \langle x, y \rangle \in \mathbb{Z} \text{ and } x - \langle x, y \rangle y \in \mathbb{X} \). Such a set \( \mathbb{X} \) is known as a simply laced root system in the literature and any element in \( \mathbb{X} \) is called a root of \( \mathbb{X} \). When \( \mathbb{X} \) is indecomposable (decomposable) it is also called irreducible (reducible). Thus note that \( \mathbb{X} \) is a (disjoint) union of mutually orthogonal irreducible simply laced root systems. Henceforth \( \Phi \) denotes a simply laced root system. A subset \( \Delta \) of \( \Phi \) is called a base of \( \Phi \) if \( \Delta \) is obtuse and generates \( \Phi \). (Though this definition appears to be different from that of a base of a general root system (see [2] or [3]), it can be shown that for simply laced root systems, they are equivalent.)

Remark 1. For any \( x \in \Phi, -x = x - 2x = x - \langle x, x \rangle x \in \Phi \). Let \( a, b \in \Phi \); note that \( \langle a, b \rangle = 1 \Rightarrow a - b \in \Phi \) and \( \langle a, b \rangle = -1 \Rightarrow a + b \in \Phi \); since \( ||a|| \leq ||a|| ||b|| \) where\}
The family of all connected graphs with largest eigenvalue $\leq 2$.

equality holds only when one root is a scalar multiple of the other, it follows that if $a \neq \pm b$ then $\langle a, b \rangle \in \{-1, 0, 1\}$.

The following result classifies all simply laced root systems.

**Theorem 2.** If $\Omega$ is an irreducible simply laced root system, then there exists an automorphism $\theta$ of $E$ such that $\theta(\Omega) \in \{A_n : n \in \mathbb{N}\} \cup \{D_n : n \in \mathbb{N} \text{ and } n > 3\} \cup \{E_6, E_7, E_8\}$.

Let $\{e_i : i = 1, 2, \ldots\}$ be an orthonormal basis for $E$. Then

for each $n \in \mathbb{N}$, $A_n = \{\pm(e_i - e_j) : 1 \leq i < j \leq n + 1\}$ and

for each $n \geq 2$, $D_n = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$.

$E_8 = D_8 \cup \left\{ \frac{1}{2} \sum_{i=1}^{8} \epsilon_i e_i : \epsilon_i = \pm 1 \text{ for } i \leq 8 \text{ and } \prod_{i=1}^{8} \epsilon_i = 1 \right\}$,

$E_7 = \{v \in E_8 : \langle v, a \rangle = 0\}$ and

$E_6 = \{v \in E_7 : \langle v, b \rangle = 0\}$

where $a$ and $b$ are two vectors in $E_8$ such that $\langle a, b \rangle = 1$.

Let us denote the largest eigenvalue and the least of a graph $G$ by $\Lambda(G)$ and $\lambda(G)$ respectively. Derivation of Theorem 2 can be done in two steps: (1) For each irreducible root system, finding a base $\Delta$ such that $\Lambda(G[\Delta]) < 2$. (2) Associating with each connected graph whose largest eigenvalue is less than 2, a base of one of the root systems defined above. The second part is routine; the reader is referred to [2] and [3].
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for this. We focus only on the first one; our method involves heavily the family of all
connected graphs with largest eigenvalue \( \leq 2 \) displayed in the figure—for each graph
\( G \) on the left (right), \( \Lambda(G) = 2 \) (\( \Lambda(G) < 2 \)). (For computation of this family, see [4].)

Proposition 3. If \( X \) is a base of \( \Phi \), then \( \tilde{X} = \Phi \).

Proof. For any \( w = t_1x_1 + \cdots + t_mx_m \in \tilde{X} \), where \( x_1, \ldots, x_m \) are distinct vectors in \( X \),
let \( \rho(w) = |t_1| + \cdots + |t_m| \). It is enough to show that \( \tilde{X} \subseteq \Phi \). Let \( u = \alpha_1v_1 + \cdots + \alpha_nv_n \)
where \( v_1, \ldots, v_n \) are distinct vectors in \( X \) be an element in \( \tilde{X} \); we can assume that each
\( x \in \tilde{X} \) such that \( \rho(x) < \rho(u) \) belongs to \( \Phi \) and \( x \not\in X \). Now for some \( k \in \{1, \ldots, n\} \),
\( \alpha_k \neq 0 \); we can assume that for each \( j \leq k \), \( \alpha_j > 0 \) and for each \( j \in \{ i \in \mathbb{N} : k < i \leq n \} \),
\( \alpha_j < 0 \). Then for some \( \ell \in \{1, \ldots, k\} \), \( \langle u, v_\ell \rangle > 0 \). Since \( u \neq v_\ell \), it can be verified that
\( \langle u, v_\ell \rangle = 1 \). Therefore \( |u - v_\ell|^2 = 2 \); since \( \rho(u - v_\ell) = \rho(u) - 1 \), \( u - v_\ell \in \Phi \). Now
\( \langle u - v_\ell, v_\ell \rangle = -1 \Rightarrow (u - v_\ell) + v_\ell \in \Phi \); i.e., \( u \in \Phi \).

Proposition 4. \( \Phi \) has a base.

Proof. We can assume that \( \Phi \) is irreducible. Let \( S \) be an indecomposable obtuse
subset of \( \Phi \) such that \( Z(S) \cap \Phi \) is as large as possible. Suppose that the latter is a
proper subset of \( \Phi \). Since \( \Phi \) is irreducible, we can find some \( p \in \Phi - Z(S) \) and \( a \in S \)
such that \( \langle p, a \rangle < 0 \). Let \( T \) be the set of all roots \( r \) such that the following holds: for
some \( x \in S \), \( \langle r, x \rangle < 0 \) and \( r \) can be expressed in the form \( p - \sum_{x \in S} \alpha_xx \) where for each
\( x \in S \), \( \alpha_x \in \mathbb{N} \cup \{0\} \). For any such \( r \), let \( \sum_{x \in S} \alpha_xx \)—it is easy to verify that this sum is
independent of the form for \( r \)—be denoted by \( \rho(r) \). Note that \( T \) is non-empty because
\( p \in T \). Choose a root \( q \) in \( T \) so that \( \rho(q) \) is as large as possible. If \( q \) can be easily verified
that \( \langle q, x \rangle \in \{-1, 0\} \) for each \( x \in S \). Now assuming \( S \cup \{q\} \subseteq \mathbb{R}^n \) where \( n = |S| + 1 \), let
\( B \) be the \( n \times n \) matrix whose rows are the vectors in \( S \cup \{q\} \). Since \( S \cup \{q\} \) is not obtuse,
it is linearly dependent. Therefore \( B \) is singular. Then the relation \( \lambda(\hat{B}^T \hat{B}) \geq 0 \) where
\( \hat{B}^T \) is the transpose of \( B \) becomes an equality. Since \( \hat{B}^T = 2I - A \) where \( A \) is the
adjacency matrix of \( G[S \cup \{q\}] \), it follows that \( \lambda(2I - A) = 0 \); therefore \( \Lambda(A) = 2 \).
Then \( G[S \cup \{q\}] \) is one of the graphs on the left side of the figure. It can be verified
that \( \alpha A = 2\alpha \) where \( \alpha \) is the vector formed by the labels assigned to the vertices of
\( G[S \cup \{q\}] \). Now \( (\alpha B)(\alpha B)^T = \alpha B B^T \alpha^T = \alpha(2I - A)\alpha^T = 0 \alpha^T = 0 \). Therefore
\( \alpha B = 0 \). Since one of the coordinates of \( \alpha \) is 1, a vector in \( S \cup \{q\} \), say \( u \), belongs to
\( Z(X) \) where \( X = (S \cup \{q\}) \setminus \{u\} \). Since \( u \) is a pendent vertex of \( G[S \cup \{q\}] \), \( G[X] \) is
connected; therefore \( X \) is indecomposable. Since \( S \subseteq Z(X) \), \( X \) is linearly independent.
Note also that \( Z(S) \not\subseteq Z(X) \) because \( S \cup \{p\} \subseteq Z(X) \). Thus the presence of \( X \)
contradicts the choice of \( S \). Therefore \( S \) is a base of \( \Phi \). □

Now we can prove the main result: By Proposition 3, \( \Omega \) has a base \( X \). Taking \( X \subseteq \mathbb{R}^{|X|} \),
let \( B \) be the \(|X| \times |X| \) matrix whose rows are the elements of \( X \). Then \( B B^T = (2I - A) \)
where \( A \) is the adjacency matrix of \( G[X] \). Since \( B \) is non-singular, so is \( B B^T \); therefore
\( \lambda(B B^T) > 0 \); i.e., \( \lambda(2I - A) > 0 \). Hence \( \Lambda(A) < 2 \). Since \( \Omega \) is irreducible, \( X \)
is indecomposable; therefore \( G[X] \) is one of the graphs on the right side of the figure.
Now as mentioned in the discussion before Proposition 3 there is a root system $\mathcal{R} \in \{A_n : n \in \mathbb{N}\} \cup \{D_n : n \in \mathbb{N} \text{ and } n \geq 4\} \cup \{E_6, E_7, E_8\}$ having a base $Y$ such that $G[Y] = G[X]$. Therefore there is a bijection $f : X \mapsto Y$ such that for all $x, y \in X$, $\langle f(x), f(y) \rangle = \langle x, y \rangle$. Now the map $\theta^*$ from the linear span of $X$ to that of $Y$ defined by $\theta^*(\sum_{x \in X} \alpha_x x) = \sum \alpha_x f(x)$ is an isomorphism. Since these subspaces are finite dimensional, $\theta^*$ can be extended to an automorphism $\theta$ of $\mathbb{E}$. Now by Proposition 3, $\theta(\Omega) = \theta(\hat{X}) = \hat{\theta}(X) = \hat{Y} = \mathcal{R}$. □

Remark 5. Let $X = \{v_1, \ldots, v_n\}$ be a subset of $\mathbb{E}$ such that for all $i, j \in \{1, \ldots, n\}$, $\|v_i\| = \sqrt{2}$ and $\langle v_i, v_j \rangle \in \mathbb{Z}$. For each $x \in \hat{X}$, define $\eta(x) = (\langle x, v_1 \rangle, \ldots, \langle x, v_n \rangle)$. Then $|\hat{X}| = |\{\eta(x) : x \in X\}|$ because for all $a, b \in \hat{X}$, $\eta(a) = \eta(b) \Rightarrow a = b$. Therefore $\hat{X}$ is finite. Note for all $a, b \in \hat{X}$, $a - \langle a, b \rangle b \in \hat{X}$ because $\|a - \langle a, b \rangle b\|^2 = \|a\|^2 - 2 \langle a, b \rangle^2 + \langle a, b \rangle^2 \|b\|^2 = 2$. Thus it follows that $\hat{X}$ is a simple laced root system.

Let $A$ be the adjacency matrix of a signed graph whose least eigenvalue $\geq -2$. (By terming each edge of a graph as positive or negative, we get a signed graph; from the adjacency matrix of the former, that of the latter is obtained by replacing each entry which corresponds to a negative edge by $-1$.) Then $\lambda(A + 2I) \geq 0$. Therefore for some real matrix $B$, $A + 2I = BB^\top$. Thus there is a subset $X = \{v_1, \ldots, v_n\}$ of $\mathbb{E}$ such that for all $i, j \in \{1, \ldots, n\}$, $\|v_i\| = \sqrt{2}$ and $\langle v_i, v_j \rangle \in \mathbb{Z}$ and $([v_i, v_j])_{i,j=1}^n$—known as the Gram matrix of $X$—equals $A + 2I$. By Remark 3, $X$ is a subset of a simply laced root system. Therefore by Theorem 2, we have the following.

Theorem 6. If $A$ is the adjacency matrix of a connected signed graph such that its least eigenvalue is at least $-2$, then $A + 2I$ is the Gram matrix of a subset of a root system which is either $D_n$ for some $n \in \mathbb{N}$ or $E_8$.

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