NON-ABELIAN GROUP STRUCTURE ON THE URYSOHN UNIVERSAL SPACE

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ABSTRACT. Following the continuing interest in the Urysohn space and, more specifically, the recent problem area of finding and comparing group structures on the Urysohn space we prove that there exists a non-abelian group structure on the Urysohn universal metric space. More precisely, using the Fraïssé theory we construct a free group with countably many generators equipped with a two-sided invariant metric that is isometric to the rational Urysohn space. We provide several open questions and problems related to this subject.

INTRODUCTION

There has been a lot of research recently connected to the Urysohn universal metric space. The space was constructed by P. Urysohn ([13]) in 1920’s but was forgotten for quite a long time. Nowadays, the Urysohn space, as well as the group of all its isometries, are a popular topic of mathematical research. A very interesting result was proved by P. Cameron and A. Vershik in [1] where they proved that there is an abelian (monothetic) group structure on the Urysohn space. Later, P. Niemiec in [10] proved that there is an abelian Boolean metric group that is isometric to the Urysohn space. And recently, Niemiec in [11] rediscovered the Shkarin’s universal abelian Polish group ([12]) and proved that this group is isometric to the Urysohn space as well (it is open though whether it differs from the group structures found by Cameron and Vershik). Niemiec also proved several negative results concerning group structures on the Urysohn space (we again refer to [11]), e.g. he proved there is no abelian metric group of exponent 3 that is isometric to the Urysohn space ([11] Proposition 2.18). Let us also mention our previous work from [2] where we showed an existence of a metrically universal separable abelian metric group (answering an

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open question of Shkarin from [12]) which turned out to be yet another different abelian group isometric to the Urysohn space. Vershik then asked (personal communication) whether there also exists a non-abelian group structure on the Urysohn space. We answer this question affirmatively here. Thus the following is the main result of this paper.

**Theorem 0.1.** There exists a free group $G$ of countably many generators equipped with a two-sided invariant metric that is isometric to the rational Urysohn space. In particular, there is a non-abelian group structure on the Urysohn space: the metric completion of $G$.

The subject of the group structures on the Urysohn space is still far from being finished and there are several open questions provided at the end of the paper. It is worth mentioning here that except the Cameron-Vershik example all other known metric groups, as well as our new example, that are isometric to the Urysohn space are constructed via Fraïssé theory. We provide few questions related to Fraïssé classes of metric groups as well at the end of the paper.

Before proving Theorem 0.1 we give a brief introduction to Fraïssé theory in the next section that is used in the construction. We also introduce some notation there so we do not recommend completely skipping this section. On the other hand, some familiarity with Fraïssé theory is expected from the reader and we refer to [6] and [3] for detailed expositions.

1. Preliminaries

Let $L$ be a countable signature and let $\mathcal{K}$ be some class of finitely generated $L$-structures, resp. isomorphism types of finitely generated $L$-structures - we will not distinguish between these two notions. Moreover, let $\leq$ be the relation of being a substructure for elements of $\mathcal{K}$, i.e. for $A, B \in \mathcal{K}$ we write $A \leq B$ if $A$ is a substructure of $B$. We distinguish a certain subrelation $\sqsubseteq$ of $\leq$ (note that in an ordinary Fraïssé theory the full relation $\leq$ is considered). We then say $\mathcal{K}$ satisfies the $\sqsubseteq$-hereditary property if for every $A \in \mathcal{K}$ and for every $L$-structure $B$ such that $B \sqsubseteq A$ we have that $B \in \mathcal{K}$. We say that $\mathcal{K}$ satisfies the $\sqsubseteq$-joint-embedding property if for every $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ and embeddings $\iota_A : A \hookrightarrow C, \iota_B : B \hookrightarrow C$ such that $\iota_A(A) \sqsubseteq C$ and $\iota_B(B) \sqsubseteq C$. We call it that $A$ and $B$ are $\sqsubseteq$-embeddable into $C$ and $\iota_A, \iota_B$ are $\sqsubseteq$-embeddings. Finally, we say that $\mathcal{K}$ has the $\sqsubseteq$-amalgamation property if whenever we have $A, B, C \in \mathcal{K}$ such that $A$ is $\sqsubseteq$-embeddable into $B$ via some embedding $\phi_B$ and $\sqsubseteq$-embeddable into $C$ via some embedding $\phi_C$, then there exists $D \in \mathcal{K}$ and $\sqsubseteq$-embeddings of $B$ via $\psi_B$ into $D$ and of $C$ via $\psi_C$ into $D$ such that $\psi_C \circ \phi_C = \psi_B \circ \phi_B$. 
If $\mathcal{K}$ with the relation $\sqsubseteq$ is countable (resp. contains countably many isomorphism types), satisfies the $\sqsubseteq$-hereditary property, $\sqsubseteq$-joint embedding property and (one-point) $\sqsubseteq$-amalgamation property, then we call $(\mathcal{K}, \sqsubseteq)$ a $\sqsubseteq$-Fraïssé class.

If $A \in \mathcal{K}$ and $\mathcal{K}$ is an infinite $L$-structure that is the union of some chain $B_1 \sqsubseteq B_2 \sqsubseteq B_3 \sqsubseteq \ldots$, for $B_n \in \mathcal{K}, n \in \mathbb{N}$, then we define $A \sqsubseteq K$ if there is some $m$ such that $A \sqsubseteq B_m$. We then have the following theorem.

**Theorem 1.1** ((generalized) Fraïssé’s theorem; see [3]). Let $(\mathcal{K}, \sqsubseteq)$ be a $\sqsubseteq$-Fraïssé class. Then there exists a unique up to isomorphism countable structure $K$ such that the set $\sqsubseteq \text{−} \text{Age}(K)$ of all finitely generated $\sqsubseteq$-substructures of $K$ is equal to $\mathcal{K}$ and $K$ is $\sqsubseteq$-ultrahomogeneous, i.e. every partial isomorphism between two finitely generated $\sqsubseteq$-substructures of $K$ extends to an automorphism of $K$.

We call such $K$ the Fraïssé limit of $\mathcal{K}$.

Let us also make few notes concerning notations here. If $X$ is a metric space then by $d_X$ we denote the metric. However, in some cases when there is no danger of confusion we may just denote the metric as $d$. Similarly, if $G$ is a group then by $1_G$ we denote the unit, and in some cases we may just denote it as 1.

**2. The construction**

We define a special type of metric and then use it in the definition of our class of groups. The main problem then will be to prove that this class is indeed a Fraïssé class. The tools developed within this proof will also help us check that the Fraïssé limit is isometric to the rational Urysohn space.

**Definition 2.1.** Let $(G, d)$ be a metric group. We say that the metric $d$ is **finitely generated** if there exists a finite set $A_G \subseteq G^2$ (called generating set for $d$) such that for every $a, b \in G$ we have $d(a, b) = \min\{d(a_1, b_1) + \ldots + d(a_n, b_n) : n \in \mathbb{N}, \forall i \leq n((a_i, b_i) \in A_G \land a = a_1 \cdot \ldots \cdot a_n, b = b_1 \cdot \ldots \cdot b_n)\}$. In particular, we assume that for every $a, b \in G$ there exist $(a_1, b_1), \ldots, (a_n, b_n) \in A$ such that $a = a_1 \cdot \ldots \cdot a_n$, $b = b_1 \cdot \ldots \cdot b_n$, thus $G$ must be finitely generated.

Moreover, if $G$ is a finitely generated free group, we say that the metric $d$ is **simply finitely generated** if the elements of the generating
set $A_G$ have the following simple form: if $(a, b) \in A_G$, then either $a$ or $b$ is a multiple of one of the generators of $G$.

For example, if $G$ is freely generated by elements $a, b, c, d$, then $(a \cdot b, c \cdot d)$ is not allowed as an element of $A_G$. On the other hand, $(a^{n_1} \cdot b^{n_2} \cdot c^{n_3}, d^{n_4})$, where $n_1, n_2, n_3, n_4 \in \mathbb{Z}$, can be an element of $A_G$.

Let us state few remarks connected to the previous definition that will be relevant in the text.

Remark 2.2. If $G$ is a free group freely generated by $f_1, \ldots, f_n$ and $d$ is a (simply) finitely generated metric on $G$, we shall always assume that $A_G$ (the generating set for $d$) contains $(f_i, f_i)$ for $i \leq n$, and $d(f_i, f_i) = 0$ of course.

Later in the text, when we define some finitely generated metric on some group by specifying the generating set and the values of the metric on it, we usually omit this fact (that $(f, f)$, where $f$ is a generator, belongs to the generating set), however we always implicitly assume it.

Remark 2.3. If $G$ is a group with finitely generated metric $d$, where $A_G$ is the generating set for $d$, then, by definition, for every $a, b \in G$ there is a decomposition $a = a_1 \cdot \ldots \cdot a_n$ and $b = b_1 \cdot \ldots \cdot b_n$ such that $(a_1, b_1), \ldots, (a_n, b_n) \in A_G$ and $d(a, b) = d(a_1, b_1) + \ldots + d(a_n, b_n)$.

Let us mention here in order to avoid future confusions that later in the text we will often prove the equality $d(a, b) = d(a_1, b_1) + \ldots + d(a_n, b_n)$ for some $a, b, a_1, \ldots, a_n, b_1, \ldots, b_n \in G$ even though not every $(a_i, b_i)$ necessarily belongs to $A_G$. This will be done in order to simplify the notation and the reader can always slightly and straightforwardly modify the proof so that all such $(a_i, b_i)$’s belong to the generating set $A_G$.

Definition 2.4 ($\sqsubseteq$-Fraïssé class of finitely generated free groups equipped with simply finitely generated metric). Elements of the class $\mathcal{G}$ are finite free products of $\mathbb{Z}$, i.e. free groups with finitely many generators equipped with a simply finitely generated rational metric.

For $G, H \in \mathcal{G}$ we say that $G$ is $\sqsubseteq$-embeddable into $H$, $G \sqsubseteq H$, if $G$ is isometrically isomorphic to a free summand of $H$, i.e. there exists some $G', F \in \mathcal{G}$ such that $H = G' * F$ and $G$ is isometrically isomorphic to $G'$.

The following theorem is the key part in proving Theorem 0.1. The fact that the Fraïssé limit of this Fraïssé class is isometric to the rational Urysohn space will follow rather easily.

Theorem 2.5. $(\mathcal{G}, \sqsubseteq)$ is a $\sqsubseteq$-Fraïssé class.
Proof. Because of the demand that metrics attain only rational values and each metric is finitely generated, it is clear that $\mathcal{G}$ is countable. The $\sqsubseteq$-hereditary property is also easy to verify. The joint-$\sqsubseteq$-embedding property is just a special case of the $\sqsubseteq$-amalgamation property which we will check.

Let us consider the groups $G_0, G_1, G_2 \in \mathcal{G}$ such that $G_0 \sqsubseteq$-embeds into both $G_1$ and $G_2$ by an isometric homomorphism. Denote these $\sqsubseteq$-embeddings as $\iota_1, \iota_2$ respectively and assume for simplicity they are just inclusions, thus we may write $G_1 = G_0 * F_1$ and $G_2 = G_0 * F_2$ for some $F_1, F_2 \in \mathcal{G}$.

We need to find a group $G_3 \in \mathcal{G}$ with $\sqsubseteq$-embeddings $\rho_1 : G_1 \to G_3$ and $\rho_2 : G_2 \to G_3$ such that $\rho_1 \circ \iota_1 = \rho_2 \circ \iota_2$. Algebraically, we just set $G_3 = G_0 * F_1 * F_2$. We denote by $\rho_1$, resp. $\rho_2$ the obvious algebraic $\sqsubseteq$-embeddings which are again in fact inclusions and for which we have $\rho_1 \circ \iota_1 = \rho_2 \circ \iota_2$. We need to define a two-sided invariant metric on $G_3$ that extends those on $G_1$ and $G_2$ so that $\rho_1$ and $\rho_2$ are moreover isometries.

The metric $d_{G_i},$ for $i \in \{1, 2\}$, on $G_i$ is generated by some $A_i \subseteq G_i^2$. We let the metric $d_{G_3}$ on $G_3$ be generated by $A_3 = A_1 \cup A_2$. We have to check that $\rho_1, \rho_2$ are indeed isometries, i.e. for $i \in \{1, 2\}$ and for every $a, b \in G_i \subseteq G_3$ we have $d_{G_i}(a, b) = d_{G_3}(a, b)$.

To simplify the proof, we claim that we may assume that $F_1$ and $F_2$ are just one-generated, i.e. isomorphic to $\mathbb{Z}$. To see how the general case follows, let us say, $F_1$ is generated by $f_1^1, \ldots, f_n^1$ and $F_2$ is generated by $f_1^2, \ldots, f_m^2$, and $n \leq m$. Then we would divide the amalgamation into $n$ steps, successively considering the amalgams $G_0 * \langle f_1^1, f_2^1 \rangle, \ldots, G_0 * \langle f_1^1, \ldots, f_n^1, f_2^2, \ldots, f_m^2 \rangle$ (note that it is not possible to do this for every class of structures in general but it is in this particular case).

Let us denote the generator of $F_i$ as $f_i$ for $i \in \{1, 2\}$. It is sufficient to check that $\rho_1$ is an isometry, the proof for $\rho_2$ is the same. We will accomplish that in a series of lemmas. We note that some of these lemmas will be also useful in proving that the Fraïssé limit is isometric to the rational Urysohn space.

Let us define few notions at first.

**Definition 2.6.** Let $a_1, \ldots, a_n \in G$ be a sequence of elements of $G$, where $G$ is a free group freely generated by some elements $\{g_1, \ldots, g_r, f\}$ and let $G' \leq G$ be the subgroup generated by $\{g_1, \ldots, g_r\}$. Assume there is a sequence $1 = k_1 < \ldots < k_m = n$ such that $\forall i \leq n$ we have $a_i = f^j$, for some $j \in \mathbb{Z} \setminus \{0\}$, iff $i \in \{k_1, \ldots, k_m\}$, otherwise $a_i \in G'$.
Also assume that \( a_1 \cdot \ldots \cdot a_n = 1_G \). Thus in particular \( a_{k_1} \ldots a_{k_m} = 1_G \). Then we call \( a_1, \ldots, a_n \) a cancelling \( f \)-sequence of length \( m \). Let us write \( a_{k_i} \) as \( f^{\varepsilon_i} \), where \( \varepsilon_i \in \mathbb{Z} \setminus \{0\} \), for every \( i \leq m \).

**Definition 2.7.** Let \( S \subseteq \mathbb{N} \) be a finite subset of natural numbers. We say that \( A \subseteq S \) is convex if for every \( a < b < c \) from \( S \), \( a, c \in A \) implies that also \( b \in A \). Also, we call two elements \( a, b \in S \) neighbours in \( S \) if there is no \( c \in S \) such that \( a < c < b \) or \( b < c < a \).

We say that \( \langle A_1, \ldots, A_k \rangle \), where for each \( i \leq k \) \( A_i \subseteq S \), is an exhaustive sequence for \( S \) if the sets \( A_1, \ldots, A_k \) are pairwise disjoint, \( \bigcup_{i \leq k} A_i = S \), \( A_i \) is a convex subset of \( S \) and for each \( 1 < i \leq k \), \( A_i \) is a convex subset of \( S \setminus \bigcup_{j<i} A_j \).

**Lemma 2.8.** Let \( a_1, \ldots, a_n \in G \), where \( G \) is like in Definition 2.6 be a cancelling \( f \)-sequence of length \( m \).

Then there exists an exhaustive sequence for \([1, \ldots, m]\) \( \langle A_1, A_2, \ldots, A_l \rangle \) such that for every \( 1 \leq j \leq l \) we have

\[
\prod_{i \in A_j} a_{k_i} = \prod_{i \in A_j} f^{\varepsilon_i} = 1_G
\]

or equivalently \( \sum_{i \in A_j} \varepsilon_i = 0 \).

Also, let \( I_1 = \emptyset \) and for every \( 1 < j \leq l \) let \( I_j \subseteq [1, n] \) be the set \( \{i \in [1, n] : \exists u, v \in \bigcup_{k<j} A_k(i \in [k_u, k_v])\} \). Then for every \( j \leq l \) and any two neighbours \( i_1 < i_2 \) in \( A_j \) we have

\[
\prod_{k \in (k_{i_1}, k_{i_2}) \setminus I_j} a_k = 1_G
\]

**Proof of Lemma 2.8.** For every \( i < m \) let \( \bar{b}_i \) denote the element \( a_{k_i+1} \cdot \ldots \cdot a_{k_{i+1}-1} \). We will prove the lemma by induction on the length of the cancelling sequence.

Assume that \( m = 2 \). Then \( a_1 \cdot \ldots \cdot a_n = f^{\varepsilon_1} \cdot \bar{b}_1 \cdot f^{\varepsilon_2} = 1 \), where \( \bar{b}_1 \in G' \) (we adopt the notation from Definition 2.6). Then obviously \( \bar{b}_1 = 1 \) and \( f^{\varepsilon_1} \cdot f^{\varepsilon_2} = 1 \). So we set \( A_1 = \{1, 2\} \) and \( \langle A_1 \rangle \) is the desired exhaustive sequence for \([1, 2]\).

Assume now that \( m = j + 1 \) and the lemma has been proved in case \( m = j \). Then one can easily prove (by induction on \( m \)) that there is some \( i < j + 1 \) such that \( \bar{b}_i = 1 \). There are two possible cases.

- \( f^{\varepsilon_i} \cdot f^{\varepsilon_{i+1}} = 1 \): We reduce the length of the cancelling \( f \)-sequence by two by throwing away \( b_i \) and \( f^{\varepsilon_i} \) and \( f^{\varepsilon_{i+1}} \). Now the cancelling \( f \)-sequence is of length \( j - 1 \) and we use the induction hypothesis. This gives us some exhaustive sequence \( \langle A'_1, \ldots, A'_{j-1} \rangle \) for \([1, \ldots, j - 1]\) which we can interpret as an
exhaustive sequence for \([1, \ldots, i-1, i+2, \ldots, j+1]\). Then the desired exhaustive sequence is \(\langle A_1, \ldots, A_i \rangle\) where \(A_1 = \{i, i+1\}\) and \(A_k = A_{k-1}^l\) for \(1 < k \leq j + 1\).

- **f** if \(f_{i+1} \neq 1\): We reduce the length of the cancelling \(f\)-sequence by one by throwing away \(b_i\) and putting \(f_i\) and \(f_{i+1}\) together (i.e. multiplying them together and obtaining \(f_{i+1}^{\pm} \), for every \(i < j + 1\)).

Now the cancelling \(f\)-sequence is of length \(j\) and we use the induction hypothesis. This gives us some exhaustive sequence \(\langle A'_1, \ldots, A'_i \rangle\) for \([1, \ldots, j]\) which we can interpret as an exhaustive sequence for \([1, \ldots, i, i+2, \ldots, j+1]\). Then the desired exhaustive sequence is \(\langle A_1, \ldots, A_l \rangle\) where for every \(k \leq l\) \(A_k = A'_k\) if \(i \notin A'_k\) and \(A_k = A'_k \cup \{i + 1\}\) if \(i \in A'_k\).

\[\square\] (of Lemma 2.8)

**Lemma 2.9.** Let \(G_0, G_1, G_2, G_3\) be as above. Recall that \(f_3\) is the generator of \(F_2\) where \(G_2 = G_0 \ast F_2\). Let us for simplicity denote it as just \(f\) here. Let \(b \in G_3\) be arbitrary such that \(d_{G_3}(1, b) = d_{G_3}(a_1, b_1) + \ldots + d_{G_3}(a_n, b_n)\), where \(a_1, \ldots, a_n \in G_3\) is a cancelling \(f\)-sequence of length \(m\). Then there exists a sequence \(\tilde{a}_1, \ldots, \tilde{a}_n\) such that \(1 = \tilde{a}_1 \cdot \ldots \cdot \tilde{a}_n\) for every \(i \leq n\) we have \(\tilde{a}_i \in G_1\), and \(d_{G_3}(1, b) = d_{G_3}(\tilde{a}_1, b_1) + \ldots + d_{G_3}(\tilde{a}_n, b_n)\).

**Proof of Lemma 2.9** We apply Lemma 2.8 to obtain an exhaustive sequence for \([1, \ldots, m]\) \(\langle A_1, A_2, \ldots, A_i \rangle\) with the property as in Lemma 2.8. Let us call \(l\) the degree of the cancelling \(f\)-sequence. The rest of the proof below is to show how to reduce the degree of the cancelling \(f\)-sequence. The statement of the lemma then follows by induction on degree of the cancelling \(f\)-sequence.

So suppose that \(l = 1\), i.e. \(A_1 = \{1, \ldots, m\}\). We have that \(\prod_{i \leq m} a_k = 1\) and for every \(i \leq m\), \(b_k \in G_0\). In particular, \(\bar{b} = \prod_{i \leq m} b_k \in G_0\). We have

\[
d_{G_3}(a_{k_1}, b_{k_1}) + \ldots + d_{G_3}(a_{k_m}, b_{k_m}) = d_{G_2}(a_{k_1}, b_{k_1}) + \ldots + d_{G_2}(a_{k_m}, b_{k_m})
\]

However, since \(\bar{b} = \prod_{i \leq m} b_k \in G_0\) and by assumption \(\iota_2 : G_0 \to G_2\) is an isometry, there exists a decomposition \((a'_1, b'_1), \ldots, (a'_k, b'_k)\) such that \(a'_1 \cdot \ldots \cdot a'_k = 1, b'_1 \cdot \ldots \cdot b'_k = \bar{b}\), for every \(i \leq k, (a'_i, b'_i) \in A_0\) and

\[
d_{G_0}(1, b) = d_{G_0}(a'_1, b'_1) + \ldots + d_{G_0}(a'_k, b'_k) = d_{G_2}(1, \bar{b}) = d_{G_3}(a_{k_1}, b_{k_1}) + \ldots + d_{G_3}(a_{k_m}, b_{k_m})
\]

In order to simplify the notation and without loss of generality (see Remark 2.3) we shall assume that \(k = m\), i.e. the decomposition above is \((a'_1, b'_1), \ldots, (a'_m, b'_m)\). If it were longer, i.e. \(k > m\), we would
Notice that, by the definition of $c\in\mathbb{G}$ the definition of $\hat{1}$ (2.2) we get
\[
\sum a_i \cdot \rho_i = \hat{1}
\]
Finally, we get back to the original sequence $(a_1, b_1), \ldots, (a_n, b_n)$. For every $i \leq m$ we replace $a_k$ by $\hat{a}_i$. Denote the resulting sequence as $\hat{a}_1, \ldots, \hat{a}_n$. Thus we have

- $\hat{a}_k = \hat{a}_i$ for $i \leq m$
- $\hat{a}_i = a_i$ for $i \notin \{k_1, \ldots, k_m\}$

We claim this is the desired sequence from the statement of the lemma. It is easy to check that for each $i \leq n$, $\hat{a}_i \in G_1$. If $i = k_j$ for some $j \leq m$, then $\hat{a}_i = \hat{a}_j \in G_0 \subseteq G_1$. If $i \notin \{k_1, \ldots, k_m\}$ then $\hat{a}_i = a_i \in G_1$ by assumption.

We now prove that $\hat{a}_1 \cdot \ldots \cdot \hat{a}_n = 1$. Recall that for each $i < m$

$$\sum_{k_i < i < k_{i+1}} a_j = \sum_{k_i < i < k_{i+1}} \hat{a}_j = 1.$$

So together with the facts that $\hat{a}_{k_1} = \hat{a}_1 = a'_1 \cdot c_1$, $\hat{a}_{k_i} = \hat{a}_i = c_i^{-1} \cdot a'_i \cdot c_i$ for $1 < i < m$ and $\hat{a}_{k_m} = \hat{a}_m = c_m^{-1} \cdot a'_m$ we obtain that

$$\hat{a}_1 \cdot \ldots \cdot \hat{a}_n = a'_1 \cdot \ldots \cdot a'_m = 1$$

Finally, since for every $i \leq m$ we have $(\hat{a}_{k_i}, b_{k_i}) = (\hat{a}_i, b_i)$ and because of [2.1] and [2.2] we get

$$\sum_{i \in \{k_1, \ldots, k_m\}} d_{G_3}(a_i, b_i) = \sum_{i \in \{k_1, \ldots, k_m\}} d_{G_3}(\hat{a}_i, b_i),$$

and thus we have

$$d_{G_3}(\hat{a}_1, b_1) + \ldots + d_{G_3}(\hat{a}_n, b_n) = \sum_{i \in \{k_1, \ldots, k_m\}} d_{G_3}(a_i, b_i) + \sum_{i \notin \{k_1, \ldots, k_m\}} d_{G_3}(a_i, b_i)$$

$$= \sum_{i \in \{k_1, \ldots, k_m\}} d_{G_3}(a_i, b_i) + \sum_{i \notin \{k_1, \ldots, k_m\}} d_{G_3}(a_i, b_i) = d_{G_3}(1, b)$$

and we are done.
from the definition of $i \in k$.

We shall again denote the generator of $\mathbb{G}$ as $\hat{G}$. Then since $f \notin G_1$ there must exist some $i_S < i \leq n$ such that $a_i = f^l$, for some $l \in \mathbb{Z}$, and $\prod_{i_S \leq j \leq n} a_j \in G_1$. Let $i_F$ be the least such index. However, $a_{i_S}, \ldots, a_{i_F}$ is then a cancelling $f$-sequence. We can apply Lemma 2.9 to this sequence and $b_{i_S} \cdot \ldots \cdot b_{i_F}$ to obtain a sequence of pairs $(\hat{a}_{i_S}, b_{i_S}), \ldots, (\hat{a}_{i_F}, b_{i_F})$ such that for every $i \in [i_S, i_F] \hat{a}_i \in G_1$, $\prod_{i \in [i_S, i_F]} \hat{a}_i = 1$, and moreover $d_{G_3}(a_{i_S}, b_{i_S}) + \ldots + d_{G_3}(a_{i_F}, b_{i_F}) = d_{G_3}(\hat{a}_{i_S}, b_{i_S}) + \ldots + d_{G_3}(\hat{a}_{i_F}, b_{i_F})$. So we can replace $a_{i_S}, \ldots, a_{i_F}$ by $\hat{a}_{i_S}, \ldots, \hat{a}_{i_F}$ in the decomposition of $a$.

Arguing as above, any occurence of a pair $(a_j, b_j) \in A_2 \setminus A_1$ in the decomposition of $a$ and $b$ implies an occurence of a cancelling $f$-sequence. Thus applying Lemma 2.9 finitely many times we can get rid of all pairs from $A_2 \setminus A_1$ in the decomposition and obtain a new one $(\hat{a}_1, \hat{b}_1), \ldots, (\hat{a}_n, \hat{b}_n)$ such that for every $i \leq n (\hat{a}_i, \hat{b}_i) \in \hat{G}^2$, $a = \hat{a}_1 \cdot \ldots \cdot \hat{a}_i$, $b = \hat{b}_1 \cdot \ldots \cdot \hat{b}_i$ and $d_{G_3}(a_1, b_1) + \ldots + d_{G_3}(a_n, b_n) = d_{G_3}(\hat{a}_1, \hat{b}_1) + \ldots + d_{G_3}(\hat{a}_n, \hat{b}_n)$. This proves that $d_{G_3}(a, b) \leq d_{\hat{G}_3}(a, b)$ and thus $\rho_1$ is an isometry.

This also finishes the proof of Theorem 2.5.

Lemma 2.10. The inclusion embedding $\rho_1$ is an isometry.

Proof of Lemma 2.10 Let $a, b \in G_1$ be arbitrary. It is immediate from the definition of $d_{G_3}$ that $d_{G_1}(a, b) \geq d_{G_3}(a, b)$.

We shall prove the other inequality $d_{G_3}(a, b) \leq d_{G_1}(a, b)$. Let $a_1 \cdot \ldots \cdot a_n = a$, $b_1 \cdot \ldots \cdot b_n = b$ with $(a_1, b_1), \ldots, (a_n, b_n) \in A_3 = A_1 \cup A_2$ be the decomposition of $a$, resp. $b$, such that $d_{G_3}(a, b) = d_{G_3}(a_1, b_1) + \ldots + d_{G_3}(a_n, b_n)$.

If for every $i \leq n (a_i, b_i) \in A_1$, then there is nothing to prove and we are done. So suppose there is some $i \leq n$ such that $(a_i, b_i) \in A_2 \setminus A_1$. We shall again denote the generator of $F_2$ simply as $f$ here. Realize that either $a_i = f^k$ and $b_i \in G_0$ or $a_i \in G_0$ and $b_i = f^k$, where $k \in \mathbb{Z}$. Let $i_S \leq n$ be the least such index. Let us say that $a_{i_S} = f^k$, for some $k \in \mathbb{Z}$, and $b_{i_S} \in G_0$. Then since $f \notin G_1$ and $\prod_{i_S \leq j \leq n} a_j \in G_1$ there must exist some $i_S < i \leq n$ such that $a_i = f^l$, for some $l \in \mathbb{Z}$, and $\prod_{i_S \leq j \leq i} a_j = 1$. Let $i_F$ be the least such index. However, $a_{i_S}, \ldots, a_{i_F}$ is then a cancelling $f$-sequence. We can apply Lemma 2.9 to this sequence and $b_{i_S} \cdot \ldots \cdot b_{i_F}$ to obtain a sequence of pairs $(\hat{a}_{i_S}, b_{i_S}), \ldots, (\hat{a}_{i_F}, b_{i_F})$ such that for every $i \in [i_S, i_F] \hat{a}_i \in G_1$, $\prod_{i \in [i_S, i_F]} \hat{a}_i = 1$, and moreover $d_{G_3}(a_{i_S}, b_{i_S}) + \ldots + d_{G_3}(a_{i_F}, b_{i_F}) = d_{G_3}(\hat{a}_{i_S}, b_{i_S}) + \ldots + d_{G_3}(\hat{a}_{i_F}, b_{i_F})$. So we can replace $a_{i_S}, \ldots, a_{i_F}$ by $\hat{a}_{i_S}, \ldots, \hat{a}_{i_F}$ in the decomposition of $a$.

Since we just checked that $G$ is a Fraïssé class it has a Fraïssé limit which we denote as $G$. It is a free group with countably many generators carrying a two-sided invariant metric. It follows that the group operations on $G$ are continuous with respect to the topology induced by the metric. To see this, just observe that by invariance for any $g, h \in G$ we have

\begin{equation}
d(g, h) = d(g^{-1}, h^{-1})
\end{equation}
so the operation inverse is continuous (an isometry), and for any \( g_1, g_2, h_1, h_2 \in G \) we have

\[
(2.4) \quad d(g_1 \cdot h_1, g_2 \cdot h_2) = d(g_1 \cdot g_2^{-1}, h_2 \cdot h_1^{-1}) \leq d(g_1, g_2) + d(h_1, h_2)
\]

by invariance and triangle inequality.

As a consequence of the Fraïssé’s theorem we immediately get the following fact generally called a finite extension property of the Fraïssé limit which characterizes this group up to an isometric isomorphism.

**Fact 2.11.** Let \( F \in G \) be such that \( F \subseteq G \). Let \( F \ast C \in G \), where \( C \in G \) and the inclusion embedding \( F \hookrightarrow F \ast C \) is an isometry. Then there exists an isometric homomorphism \( \iota : F \ast C \hookrightarrow G \) such that \( \iota \upharpoonright F = \text{id}_F \).

Consider now the metric completion, denoted \( \overline{G} \), of \( G \). It is a separable complete metric space and the group operations extend to the completion. Indeed, the inverse operation extends because it is an isometry \((2.3)\) and the group multiplication extends because if \((g_n)_n, (h_n)_n \subseteq G\) are two Cauchy sequences then \((g_n \cdot h_n)_n\) is a Cauchy sequence as well \((2.4)\). It follows that \( G \) is a Polish group equipped with a two-sided invariant metric. We refer the reader to [4] for an exposition on Polish (metric) groups.

The rest of this section is devoted to finishing the proof of Theorem 0.1. We shall prove that \( G \leq \overline{G} \) is isometric to the rational Urysohn space.

Let us recall that a function \( f : X \to \mathbb{R}^+_0 \) is called Katětov, where \((X, d)\) is some metric space, if for every \( x, y \in X \) we have \(|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)\). One should think about the Katětov function \( f \) as about a function that prescribes distances from some, potentially new, point.

The following well known fact characterizes the rational Urysohn space.

**Fact 2.12.** Let \((X, d)\) be a countable metric space with rational metric. Then it is isometric to the rational Urysohn space iff for every finite subset \( A \subseteq X \) and for every rational Katětov function \( f : A \to \mathbb{Q}^+ \) there exists \( x \in X \) such that \( \forall a \in A(d(a, x) = f(a)) \).

We shall prove the following proposition which will thus also be a proof of Theorem 0.1.

**Proposition 2.13.** \((G, d)\) satisfies the condition from Fact 2.12.
Proof. Let $f : B \to \mathbb{Q}^+$, where $B$ is a finite subset of $G$, be a Katětov rational function. We may suppose that $B$ is symmetric (otherwise, we would extend $f$). Let $H \in \mathcal{G}$ be a finitely generated free group such that $H \subseteq G$ and $B \subseteq H$. We shall prove that there exists $H \ast F \in \mathcal{G}$ such that $F$ is generated by a single element $f$ (i.e., $F$ is algebraically isomorphic to $\mathbb{Z}$), the inclusion embedding $H \hookrightarrow H \ast F$ is an isometry, and for every $b \in B$ we have $d(f, b) = f(b)$. Then we will be done by Fact 2.11.

Let $A_H$ be the generating set for the metric $d_H$ on $H$. Let $F$ be a group algebraically isomorphic to the integers generated by an element $f$. We define a simply finitely generated rational metric on $H \ast F$. The metric $d$ on $H \ast F$ is generated by $A = A_H \cup \{(f^\varepsilon, b) : \varepsilon \in \{-1, 1\}, b \in B\}$, and for any $(a, b) \in A_H$ we set $d(a, b) = d_H(a, b)$ and for any $b \in B$ we set $d(f, b) = f(b)$ and $d(f^{-1}, b) = f(b^{-1})$.

We claim this works. We need to prove two things: that for every $a, b \in H$ we have $d_H(a, b) = d(a, b)$ (in other words, that the inclusion $H \hookrightarrow H \ast F$ is an isometry) and that for every $b \in B$ and every $b_1, \ldots, b_n, c_1, \ldots, c_n \in H \ast F$ such that $b = b_1 \cdot \ldots \cdot b_n$ and $f = f_1 \cdot \ldots \cdot f_n$ we have $d(b_1, f_1) + \ldots + d(b_n, f_n) \geq f(b)$.

- Let us start with the former. The proof is the same as the proof of Lemma 2.10. Just instead of applying Lemma 2.9 in that proof, we apply the following claim.

**Claim** Let $a_1, \ldots, a_n$ be a cancelling $f$-sequence (of some length) and let $b_1, \ldots, b_n \in H \ast F$ be arbitrary such that $d(a_1 \cdot \ldots \cdot a_n = 1, b) = d(a_1, b_1) + \ldots + d(a_n, b_n)$, where $b$ is the product $b_1 \cdot \ldots \cdot b_n$. Analogously as in Lemma 2.9 we claim that there exists a sequence $\hat{a}_1, \ldots, \hat{a}_n \in H$ such that $\hat{a}_1 \cdot \ldots \cdot \hat{a}_n = 1$ and $d(1, b) = d(\hat{a}_1, b_1) + \ldots + d(\hat{a}_n, b_n)$.

Let us prove the claim. As in the proof of Lemma 2.9, we apply Lemma 2.8 to the cancelling $f$-sequence (say of length $m$) to obtain an exhaustive sequence for $\{1, \ldots, m\} \langle A_1, \ldots, A_n \rangle$. The proof then proceeds by induction on the degree $l$. We will show the case $l = 1$. The general inductive step is then completely straightforward and analogous as in the proof of Lemma 2.9.

Thus suppose that $l = 1$. Observe that $A$ contains elements of the form $(f^\varepsilon, a)$, where $a \in H$ and $\varepsilon \in \mathbb{Z} \setminus \{0\}$, only when $\varepsilon \in \{-1, 1\}$. It follows that $a_1 = f^\varepsilon$, $a_n = f^{-\varepsilon}$ and $a_i \in H$ for $1 < i < n$. Let us say $\varepsilon = 1$. Then since $f : B \to \mathbb{Q}^+$ is a Katětov function, we have $d(b_1, b_n^{-1}) \leq f(b_1) + f(b_n^{-1}) =
\[ d(f, a_1) + d(f^{-1}, b_n) \]. It follows that we may put \( \hat{a}_1 = b_{n}^{-1}, \hat{a}_n = b_{n} \) and \( \hat{a}_i = a_i \) for \( 1 < i < n \). It is easy to check this is the desired sequence.

- We now prove the latter. Without loss of generality, we may assume that \( f_1, \ldots, f_n \) does not contain any cancelling \( f \)-sequence. Otherwise, we would use the previous item to get rid of it. Thus we may suppose that there is \( 1 \leq i \leq n \) such that \( f_i = f \) and for every \( 1 \leq j \leq n \), \( j \neq i \), we have \( f_j \in H \). Let us write \( \overline{b}_1 = b_1 \cdot \ldots \cdot b_{i-1} \) and \( \overline{b}_2 = b_{i+1} \cdot \ldots \cdot b_n \). It follows that \( f_1 \cdot \ldots \cdot f_{i-1} = f_{i+1} \cdot \ldots \cdot f_n = 1, b = \overline{b}_1 \cdot b_i \cdot \overline{b}_2 \). We have to prove that \( f(b) \leq d(\overline{b}_1, 1) + d(b_i, f_i)(= f(b_i)) + d(\overline{b}_2, 1) \). Since the function \( f \) is Katetov, and \( b, b_i \in \text{dom}(f) \), we have \( f(b) \leq f(b_i) + d(b_i, b) \leq f(b_i) + d(1, b_1) + d(1, \overline{b}_2) \). To see the last inequality, notice that \( d(b_i, b) = d(1 \cdot b_i \cdot 1, b_1 \cdot b_i \cdot \overline{b}_2) \leq d(1, b_1) + d(b_i, b) + d(1, \overline{b}_2) \).

This finishes the proof of the proposition and thus also of Theorem 0.1. \( \square \)

3. Open questions and problems

3.1. Groups isometric to the Urysohn space. To summarize, there are now five known group structures on the Urysohn space\(^1\), the groups from papers \([1], [10], [11] \) (and \([12] \), \([2] \) and the present paper. Four of them are known to be different, it is open whether Shkarin/Niemiec’s group belongs to the Cameron-Vershik’s class. We provide some open questions from this area.

Let us start with the groups of finite exponent. We already mentioned in the introduction that Niemiec in \([10] \) proved that there is an abelian metric group of exponent 2 isometric to the Urysohn space and that he proved in \([11] \) that there is no abelian metric group of exponent 3 isometric to the Urysohn space. Moreover, consider the Fraïssé class of all finite abelian groups of exponent \( n \), where \( n > 3 \), equipped with invariant rational metric. He showed (Theorem 5.5 in \([11] \) ) that, surprisingly, the corresponding Fraïssé limit is not isometric to the rational Urysohn space. However, the following problem is still open.

**Question 3.1** (Niemiec). Does there exist an abelian metric group of finite exponent other than 2 and 3 that is isometric to the Urysohn space?

\(^1\)One should rather talk about classes of group structures since Cameron-Vershik’s example is a class of continuum many different monothetic group structures on the Urysohn space
Since all known metric groups isometric to the Urysohn space have an invariant metric and a countable dense subgroup isometric to the rational Urysohn space, it is probably worthy to work on the following problem.

**Problem 3.2.** Characterize countable groups that admit a *two-sided* invariant metric with which they are isometric to the rational Urysohn space.

The reason why we stressed that the metric should be two-sided invariant is because in such a case the group operations are automatically continuous and the operations extend to the metric completion. The following question is thus natural in this context.

**Question 3.3.** Does there exist a metric group that is isometric to the (rational) Urysohn space such that its metric is not two-sided invariant?

### 3.2. Fraïssé classes of metric groups.

The natural class of all finite abelian groups equipped with invariant rational metric is rather easily checked to be a Fraïssé class and the metric completion of the corresponding Fraïssé limit is the universal Polish abelian group from papers [12] and [11]. However, the analogous problem for the non-abelian case is open.  

**Question 3.4.** Does the class of all finite groups equipped with two-sided invariant rational metric have the amalgamation property?

Let us note that the class of all finite groups does have the amalgamation property ([9]) and the Fraïssé limit is the Hall’s universal locally finite group ([5]). It is not hard to check that if the class from Question 3.4 were Fraïssé, then the Fraïssé limit would be algebraically isomorphic to the Hall’s group. It is not clear though whether it would be isometric to the rational Urysohn space.

### 3.3. Universality properties.

The classical results are that there is a universal compact Polish group (i.e. contains every other compact Polish group as a closed subgroup) and that there is no universal locally compact Polish group in the same sense. V. Uspenskij then proved that there are universal Polish groups ([14], [15]). Then comes the mentioned result of Shkarin that there is a universal abelian Polish group. Quite recently, M. Malicki proved that there is no universal Polish group admitting compatible complete left-invariant metric ([8]). The following question is to the best of our knowledge still open.

**Question 3.5.** Does there exist a universal Polish group admitting a *two-sided* invariant metric?
Let us make a conjecture here immediately. In our previous work in [2] we constructed a metrically universal abelian separable group equipped with invariant metric (i.e. any abelian separable group equipped with invariant metric embedds via isometric monomorphism). The Fraïssé class used there is the class of all finite direct sums of \( \mathbb{Z} \) equipped with finitely generated (not necessarily simply finitely generated) rational metric.

**Conjecture 3.6.** The metric completion of the Fraïssé limit of the class of all free groups with finitely many generators equipped with finitely generated rational metric is a (topologically or even metrically) universal Polish group admitting two-sided invariant metric.

Let us remark that the class from the previous conjecture is larger than the class considered in the main construction of this paper. We restricted to simply finitely generated metrics instead of arbitrary finitely generated metric since the amalgamation is easier to prove in the former case and it is sufficient for proving the main result.

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