HOMOTOPY TYPE OF THE UNITARY GROUP OF THE UNIFORM ROE ALGEBRA ON \( \mathbb{Z}^n \)

TSUYOSHI KATO, DAI SUKE KISHIMOTO, AND MITSUNOBU TSUTAYA

Abstract. We study the homotopy type of the space of the unitary group \( U_1(C_\ast_u(|\mathbb{Z}^n|)) \) of the uniform Roe algebra \( C_\ast_u(|\mathbb{Z}^n|) \) of \( \mathbb{Z}^n \). We show that the stabilizing map \( U_1(C_\ast_u(|\mathbb{Z}^n|)) \to U_\infty(C_\ast_u(|\mathbb{Z}^n|)) \) is a homotopy equivalence. Moreover, when \( n = 1, 2 \), we determine the homotopy type of \( U_1(C_\ast_u(|\mathbb{Z}^n|)) \), which is the product of the unitary group \( U_1(C_\ast_u(|\mathbb{Z}^n|)) \) (having the homotopy type of \( U_\infty(C_\ast_u(|\mathbb{Z}^n|)) \) or \( \mathbb{Z} \times B U_\infty(C_\ast_u(|\mathbb{Z}^n|)) \) depending on the parity of \( n \)) of the Roe algebra \( C_\ast(|\mathbb{Z}^n|) \) and rational Eilenberg–MacLane spaces.

1. Introduction

For a \( C^\ast \)-algebra \( A \), let \( GL_d(A) \) and \( U_d(A) \) denote the space of the invertible and unitary matrices with entries in \( A \), respectively. It is well-known that they always have the same homotopy type. We will often refer only to \( U_d(A) \) but most statements are valid for \( GL_d(A) \) as well. There have been a lot of works on the homotopy theory of \( U_d(A) \) and some of them have important applications. For finite-dimensional case, the complex-valued unitary matrices \( U_d(\mathbb{C}) \) is just the usual unitary group acting linearly on \( \mathbb{C}^d \). For infinite-dimensional case, Kuiper [Kui65] proved that the space of all unitary operators on an infinite-dimensional Hilbert space is contractible. This result is basic in the Atiyah–Singer index theory. This kind of contractibility result has been extended to \( U_d(A) \) of some other algebras \( A \) while \( A \) is all the bounded operators on an infinite dimensional Hilbert space in the original result. Of course it is not always the case for \( U_d(A) \) of other infinite-dimensional \( C^\ast \)-algebras \( A \). In general, it is hard to determine the homotopy type of \( U_d(A) \).

Let us use the notation

\[ GL_\infty(A) = \lim_{d \to \infty} GL_d(A) \quad \text{and} \quad U_\infty(A) = \lim_{d \to \infty} U_d(A). \]

It is well-known that \( U_\infty(A) \) has the same homotopy type as \( U_1(A \otimes \mathcal{K}) \) where \( \mathcal{K} \) is the space of compact operators. The \( K \)-theory \( K_i(A) \) \((i = 0, 1)\) is a basic homotopy invariant of \( A \), which is characterized as

\[ K_0(A) = \pi_1(U_\infty(A)) \quad \text{and} \quad K_1(A) = \pi_0(U_\infty(A)). \]

Since \( U_d(A) \) is not necessarily homotopy equivalent to \( U_\infty(A) \), \( K_i(A) \) is not a so strong invariant in general. But sometimes the natural map \( U_d(A) \to U_\infty(A) \), which we will call the stabilizing map, becomes a homotopy equivalence. Study on such stability seems to trace back to the work of Bass [Bas64]. There have been a various works on this kind of stability. Rieffel introduced the topological stable rank in [Rie83] and applied it to show the stability of the non-commutative

2010 Mathematics Subject Classification. 55Q52 (Primary), 46L80 (Secondary).

Key words and phrases. uniform Roe algebra, Roe algebra, unitary group, homotopy type, operator \( K \)-theory.

Kato was supported by JSPS KAKENHI 17K18725 and 17H06461. Kishimoto was supported by JSPS KAKENHI 17K05248 and 19K03473. Tsutaya was supported by JSPS KAKENHI 19K14535.
torus in [Rie87], which is a key tool in the present work. It is difficult in general to determine how stable a given $C^*$-algebra is.

In the present paper, we study the stability of the uniform Roe algebra $C_u^*(\mathbb{Z}^n)$ on $\mathbb{Z}^n$ and investigate its homotopy type. The uniform Roe algebra $C_u^*(X)$ of a metric space $X$ introduced by Roe in [Roe88] to establish an index theory on open manifolds, where the index lives in the $K$-theory $K_*(C_u^*(X))$. The algebra $C_u^*(X)$ itself is also important since it encodes a kind of “large scale geometry” of $X$. Studying the homotopy type of $U_d(C_u^*(X))$ will provide more insights from a homotopy theoretic viewpoint, which cannot be obtained only from its $K$-theory. But there are only a few works on the homotopy type of $U_d(C_u^*(X))$ yet. For example, Manuilov and Troitsky [MT21] studied some condition for $U_d(C_u^*(X))$ being contractible. In the present work, we observe the other extreme, that is, $U_d(C_u^*(\mathbb{Z}^n))$ has a highly nontrivial homotopy type.

We give some comment on the relation with our previous work [KKT] on the space $\mathcal{U}$ of finite propagation unitary operators on $\mathbb{Z}$. Note that $U_1(C_u^*(\mathbb{Z}))$ can be viewed as a kind of completion of $\mathcal{U}$. We determined the homotopy type of $\mathcal{U}$ there. But it is not clear whether $\mathcal{U}$ has the same homotopy type as $U_1(C_u^*(\mathbb{Z}))$. Actually, they turn out to have the same homotopy type (Theorem 1.2). Also, the method there does not seem to be extended to $\mathbb{Z}^n$ when $n \geq 2$. We employ rather operator algebraic technique in the present paper. Our method here reduces the problem to determine the homotopy type of $U_1(C_u^*(\mathbb{Z}^n))$ to the one to show the surjectivity of the homomorphism on $K$-theory $K_*(C_u^*(\mathbb{Z}^n)) \to K_*(C^*(\mathbb{Z}^n))$ induced from the inclusion (Proposition 7.4), where $C^*(\mathbb{Z}^n)$ denotes the Roe algebra of $\mathbb{Z}^n$.

For stability, we show the following theorem in Section 3.

**Theorem 1.1.** For any integer $n \geq 1$, the stabilizing maps

\[
\begin{align*}
GL_1(C_u^*(\mathbb{Z}^n)) &\to GL_\infty(C_u^*(\mathbb{Z}^n)), \\
U_1(C_u^*(\mathbb{Z}^n)) &\to U_\infty(C_u^*(\mathbb{Z}^n)), \\
GL_1(C^*(\mathbb{Z}^n)) &\to GL_\infty(C^*(\mathbb{Z}^n)), \\
U_1(C^*(\mathbb{Z}^n)) &\to U_\infty(C^*(\mathbb{Z}^n))
\end{align*}
\]

between the spaces of invertible and unitary elements are homotopy equivalences. This implies that these maps induce the following isomorphisms on homotopy groups for all $i \geq 0$:

\[
\pi_i(GL_1(C_u^*(\mathbb{Z}^n))) \cong \pi_i(U_1(C_u^*(\mathbb{Z}^n))) \cong \begin{cases}
K_1(C_u^*(\mathbb{Z}^n)) & i \text{ is even,} \\
K_0(C_u^*(\mathbb{Z}^n)) & i \text{ is odd,}
\end{cases}
\]

\[
\pi_i(GL_1(C^*(\mathbb{Z}^n))) \cong \pi_i(U_1(C^*(\mathbb{Z}^n))) \cong \begin{cases}
K_1(C^*(\mathbb{Z}^n)) & i \text{ is even,} \\
K_0(C^*(\mathbb{Z}^n)) & i \text{ is odd.}
\end{cases}
\]

Let $K(V, i)$ denote the Eilenberg–MacLane space of type $(V, i)$ and $B U_\infty(\mathbb{C})$ denote the classifying space of the unitary group $U_\infty(\mathbb{C})$. Also, for based spaces $X_i$ ($i = 1, 2, \ldots$), define

\[
\bigcap_{i \geq 1} X_i = \lim_{k \to \infty}(X_1 \times X_2 \times \cdots \times X_k).
\]

For the homotopy type of $U_1(C_u^*(\mathbb{Z}))$, we show the following results when $n = 1, 2$.

**Theorem 1.2.** There exist homotopy equivalences of infinite loop spaces

\[
GL_1(C_u^*(\mathbb{Z})) \cong U_1(C_u^*(\mathbb{Z})) \cong \mathbb{Z} \times B U_\infty(\mathbb{C}) \times \prod_{i \geq 1} K(\ell^\infty(\mathbb{Z}, \mathbb{Z})_S, 2i - 1).
\]
Theorem 1.3. There exist homotopy equivalences of infinite loop spaces

$$\text{GL}_1(C^*_u(\mathbb{Z}^2)) \simeq U_1(C^*_u(\mathbb{Z}^2)) \simeq V_1 \times U_\infty(\mathbb{C}) \times \prod_{i \geq 1} (K(V_0, 2i - 1) \times K(V_1, 2i)), $$

where $V_0, V_1$ are the rational vector spaces given by

$V_0 = \ker[K_0(C^*_u(\mathbb{Z}^2)) \to K_0(C^*(\mathbb{Z}^2))], \quad V_1 = K_1(C^*_u(\mathbb{Z}^2))$

and the product factor $V_1$ is a discrete space.

More detailed descriptions of the vector spaces $V_0$ and $V_1$ appear in the proof of Lemma 7.7.

We will see the existence of a homotopy section of the inclusion $U_1(C^*_u(\mathbb{Z}^2)) \to U_1(B^\text{SW})$ in Section 6, where $U_1(B^\text{SW})$ is the Segal–Wilson restricted unitary group having the homotopy type of $\mathbb{Z} \times B U_\infty(\mathbb{C})$. This implies Theorem 1.2. We also show in Section 7 that, for any integer $n \geq 1$, the inclusion $U_1(C^*_u(\mathbb{Z}^2)) \to U_1(C^*(\mathbb{Z}^2))$ admits a homotopy section if and only if the homomorphism $K_r(C^*_u(\mathbb{Z}^2)) \to K_r(C^*(\mathbb{Z}^2))$ is surjective. Since we can see it is surjective when $n = 1, 2$, Theorems 1.2 again and 1.3 follow. If one could show the surjectivity for $n \geq 3$, then a similar homotopy decomposition will immediately follow.

This paper is organized as follows. We fix our notation in Section 2. In Section 3, we recall Rieffel’s results on stability and show Theorem 1.1. In Section 4, we recall the Bott periodicity realized as a $\ast$-homomorphism. In Section 5, we recall the Segal–Wilson restricted unitary group and show its stability. In Section 6, we show Theorem 1.2 using the Segal–Wilson restricted unitary group. In Section 7, we discuss the homotopy type of $U_d(C^*_u(\mathbb{Z}^2))$ for general $n \geq 1$ and show Theorems 1.2 again and 1.3.

2. Notation

The $C^*$-algebra of bounded operators on a Hilbert space $V$ is denoted by $\mathcal{B}(V)$ and the subalgebra of compact operators by $\mathcal{K}(V)$. We write the operator norm of $T \in \mathcal{B}(V)$ as $\|T\|$. The Hilbert space of square summable sequences indexed by a discrete group $\Gamma$ will be written as

$$\ell^2(\Gamma) = \{ (v_g)_g \mid \sum_{g \in \Gamma} |v_g|^2 < \infty \}. $$

We also consider the tensor product Hilbert space $\ell^2(\Gamma) \otimes \mathcal{H}$ with an infinite dimensional separable Hilbert space $\mathcal{H}$.

A bounded operator $T \in \mathcal{B}(\ell^2(\Gamma))$ can be expressed in the matrix form as

$$T = (T_{g,h})_{g,h}, \quad T_{g,h} \in \mathbb{C}. $$

For $T \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathcal{H})$, we also have a similar expression $T = (T_{g,h})_{g,h}$ with $T_{g,h} \in \mathcal{B}(\mathcal{H})$.

Definition 2.1. Let $\Gamma$ be a finitely generated group and $d$ denote the word metric with respect to some finite set of generators. We say that a bounded operator $T \in \mathcal{B}(\ell^2(\Gamma))$ has finite propagation if

$$\text{prop}(T) = \sup \{ d(g, h) \mid T_{g,h} \neq 0 \}$$

is finite. We define finite propagation for $T = (T_{g,h})_{g,h} \in \mathcal{B}(\ell^2(\Gamma) \otimes \mathcal{H})$ similarly.
Example 2.2. The shift \( S_x \in \mathcal{B}(l^2(\Gamma)) \) by \( x \in \Gamma \) is defined by

\[ S_x = ((S_x)_{g,h})_{g,h}, \quad S_{g,h} = \begin{cases} 1 & g^{-1}h = x, \\ 0 & \text{otherwise}. \end{cases} \]

The operator \( S_x \) is a unitary operator with \( \text{prop}(S_x) = d(x, 1) \).

It is easy to see that the definition of having finite propagation is independent of the choice of generators while the value of \( \text{prop}(T) \) depends on the word metric. Since we have

\[ \text{prop}(ST) \leq \text{prop}(S) + \text{prop}(T), \quad \text{prop}(T^*) = \text{prop}(T), \quad \text{prop}(1) = 0 \]

for any finite propagation operators \( S, T \in \mathcal{B}(l^2(\Gamma)) \), the subset of finite propagation operators becomes a unital \(*\)-subalgebra of \( \mathcal{B}(l^2(\Gamma)) \). Similar properties hold for finite propagation operators \( S, T \in \mathcal{B}(l^2(\Gamma) \otimes \mathcal{H}) \) such that the components \( T_{g,h} \) and \( S_{g,h} \) are compact operators.

Definition 2.3. The uniform Roe algebra \( C^u(\Gamma) \) of \( \Gamma \) is the norm closure of the algebra of finite propagation operators in \( \mathcal{B}(l^2(\Gamma)) \).

Definition 2.4. The Roe algebra \( C^*(\Gamma) \) of \( \Gamma \) is the norm closure of the algebra of finite propagation operators \( T \in \mathcal{B}(l^2(\Gamma) \otimes \mathcal{H}) \) such that each component \( T_{g,h} \) is a compact operator.

Remark 2.5. We follow the usual notation \( C^*(\Gamma) \) for the Roe algebra of \( \Gamma \) to distinguish it from the group \( C^*\)-algebra of \( \Gamma \) though we do not consider the latter here.

We will consider the uniform Roe algebra \( C^u(\Gamma) \) as a subalgebra of the Roe algebra \( C^*(\Gamma) \) with respect to some inclusion \( \mathbb{C} \subset \mathcal{H} \).

We use the symbol \( l^\infty(\Gamma, \mathbb{C}) \) to express the Banach algebra of \( \mathbb{C} \)-valued bounded sequences indexed by \( \Gamma \) rather than the simpler symbol \( l^\infty(\Gamma) \) since we also consider the abelian group of \( \mathbb{Z} \)-valued bounded sequences \( l^\infty(\Gamma, \mathbb{Z}) \).

The group \( \Gamma \) acts on the algebras \( l^\infty(\Gamma, \mathbb{C}) \) and \( l^\infty(\Gamma, \mathcal{K}(\mathcal{H})) \) by right translation. The action by \( x \in \Gamma \) is compatible with the conjugation by \( S_x \) through the diagonal inclusion \( l^\infty(\Gamma, \mathbb{C}) \to C^u(\Gamma) \) or \( l^\infty(\Gamma, \mathcal{K}(\mathcal{H})) \to C^*(\Gamma) \) given by

\[ (t_g)_g \mapsto (T_{g,h})_{g,h}, \quad T_{g,h} = \begin{cases} t_g & g = h, \\ 0 & \text{otherwise}. \end{cases} \]

Moreover, this inclusion extends to the well-known isomorphisms

\[ l^\infty(\Gamma, \mathbb{C}) \rtimes \Gamma \cong C^u(\Gamma), \quad l^\infty(\Gamma, \mathcal{K}(\mathcal{H})) \rtimes \Gamma \cong C^*(\Gamma) \]

from the reduced crossed products of \( C^*\)-algebras. For example, see [Roe03, Theorem 4.28].

The \( d \times d \)-matrix algebra \( M_d(A) \) of a \( C^*\)-algebra \( A \) is again a \( C^*\)-algebra. The spaces of invertible elements and unitary elements in \( M_d(A) \) will be denoted as \( \text{GL}_d(A) \) and \( U_d(A) \). The stabilizing maps are given as

\[ \text{GL}_1(A) \to \text{GL}_\infty(A) = \lim_{d \to \infty} \text{GL}_d(A), \quad \text{U}_1(A) \to \text{U}_\infty(A) = \lim_{d \to \infty} \text{U}_d(A), \]

where the inductive limits are taken along the inclusions \( \text{GL}_d(A) \subset \text{GL}_{d+1}(A) \) and \( \text{U}_d(A) \subset \text{U}_{d+1}(A) \). The inductive limit spaces \( \text{GL}_\infty(A) \) and \( \text{U}_\infty(A) \) are well-known to be homotopy equivalent to the spaces \( \text{GL}_1(A \otimes \mathcal{K}(\mathcal{H})) \) and \( \text{U}_1(A \otimes \mathcal{K}(\mathcal{H})) \).
3. Stability

The aim of this section is to prove Theorem 1.1. Once the assumption of the following result by Rieffel [Rie87] is verified, the theorem will immediately follow.

**Theorem 3.1** (Rieffel). Let $A$ be a unital $C^*$-algebra. If $A$ is $\text{tsr}$-boundedly divisible, then the stabilizing maps

$$GL_1(A) \to GL_\infty(A) \quad \text{and} \quad U_1(A) \to U_\infty(A)$$

are homotopy equivalences.

**Remark 3.2.** The original statement of Theorem 4.13 in [Rie87] is involved only with homotopy groups. But what is actually proved there is slightly stronger as above.

For the definitions of the topological stable rank $\text{tsr}(A) \in \mathbb{Z}_{\geq 1}$, see [Rie83]. A $C^*$-algebra $A$ is said to be $\text{tsr}$-boundedly divisible [Rie87] if there is a constant $K$ such that for any integer $m$, there exists an integer $d \geq m$ such that $A$ is isomorphic to $M_d(B)$ for some $C^*$-algebra $B$ with $\text{tsr}(B) \leq K$. To verify the assumption, we need to follow the two lemmas.

**Lemma 3.3.** The topological stable ranks of $C^*_u(|\mathbb{Z}^n|)$ and $C^*(|\mathbb{Z}^n|)$ are estimated as

$$\text{tsr}(C^*_u(|\mathbb{Z}^n|)) \leq n + 1 \quad \text{and} \quad \text{tsr}(C^*(|\mathbb{Z}^n|)) \leq n + 1.$$

**Remark 3.4.** We will see that both $C^*_u(|\mathbb{Z}^n|)$ and $C^*(|\mathbb{Z}^n|)$ are $\text{tsr}$-boundedly divisible using this lemma. Thus we will actually obtain the estimates $\text{tsr}(C^*_u(|\mathbb{Z}^n|)) \leq 2$ and $\text{tsr}(C^*(|\mathbb{Z}^n|)) \leq 2$ by [Rie87] Proposition 4.6).

**Proof.** Let $A = \mathbb{C}$ or $\mathcal{K}(\mathcal{H})$. Since the invertible elements in $\ell^\infty(\mathbb{Z}^n, \mathbb{C})$ and $\mathbb{C} \oplus \ell^\infty(\mathbb{Z}^n, \mathcal{K}(\mathcal{H}))$ are dense, we have

$$\text{tsr}(\ell^\infty(\mathbb{Z}^n, \mathbb{C})) = 1$$

by [Rie83] Proposition 3.1]. Considering the restricted action of $\mathbb{Z}^m \subset \mathbb{Z}^n$ on the first $m$ factors of $\mathbb{Z}^n$, we obtain the isomorphism

$$\ell^\infty(\mathbb{Z}^n, \mathbb{C}) \cong (\mathbb{Z}^m) \cong (\mathbb{Z}^m \rtimes \mathbb{Z}^{m+1}).$$

Thus, by [Rie83] Theorem 7.1], we get the desired estimates on $\text{tsr}(C^*_u(|\mathbb{Z}|))$ and $\text{tsr}(C^*(|\mathbb{Z}^n|))$. □

**Lemma 3.5.** For any integer $d \geq 1$, there exist isomorphisms

$$\phi: C^*_u(|\mathbb{Z}^n|) \cong M_d(C^*_u(|\mathbb{Z}^n|)) \quad \text{and} \quad \phi: C^*(|\mathbb{Z}^n|) \cong M_d(C^*(|\mathbb{Z}^n|)).$$

**Proof.** Let $V = \mathbb{C}$ or $\mathcal{H}$. According to the decomposition

$$\ell^2(\mathbb{Z}^n) \otimes V = \bigoplus_{(i_1, \ldots , i_n) \in \mathbb{Z}^n} V_{(i_1, \ldots , i_n)}, \quad V_{(i_1, \ldots , i_n)} \cong V,$

we have the matrix expression for $T \in \mathcal{B}(\ell^2(\mathbb{Z}^n) \otimes V)$

$$T = (T_{(i_1, \ldots , i_n)(j_1, \ldots , j_n)}), \quad T_{(i_1, \ldots , i_n)(j_1, \ldots , j_n)}: V_{(j_1, \ldots , j_n)} \to V_{(i_1, \ldots , i_n)}.$$

Consider the map $\phi: \mathcal{B}(\ell^2(\mathbb{Z}^n) \otimes V) \to M_d(\mathcal{B}(\ell^2(\mathbb{Z}^n) \otimes V))$ given by

$$\phi(T)_{(i_1, \ldots , i_n)(j_1, \ldots , j_n)} = \left( \begin{array}{ccc} T_{(d_1, \ldots , d_n)(j_1, \ldots , j_n)} & \cdots & T_{(d_1, \ldots , d_n)(j_1 + d - 1, \ldots , j_n)} \\ \vdots & \ddots & \vdots \\ T_{(d_1 + d - 1, \ldots , d_n)(j_1, \ldots , j_n)} & \cdots & T_{(d_1 + d - 1, \ldots , d_n)(j_1 + d - 1, \ldots , j_n)} \end{array} \right) \in M_d(\mathcal{B}(V)).$$
The restrictions to $C_u^*(\mathbb{Z}^n) \subset \mathcal{B}(\ell^2(\mathbb{Z}^n))$ and $C^*(\mathbb{Z}^n) \subset \mathcal{B}(\ell^2(\mathbb{Z}^n) \otimes \mathcal{H})$ are desired isomorphisms. □

Remark 3.6. When $n = 1$, the map $\phi$ is just taking the block matrix of which each block is a $d \times d$-matrix.

Proof of Theorem 1.1. By Lemmas 3.3 and 3.5 we can apply Theorem 3.1 to $C_u^*(\mathbb{Z}^n)$ and $C^*(\mathbb{Z}^n)$. This completes the proof of the theorem. □

4. Bott periodicity

Let us recall the Bott periodicity of $C^*$-algebras here. Let $A$ be a $C^*$-algebra, which might be non-unital. The direct sum $\mathbb{C} \oplus A$ is considered as the unitization with unit $(1, 0) \in \mathbb{C} \oplus A$. Define the unitary group $U'_d(A)$ by

$$U'_d(A) = \{ U \in U_n(\mathbb{C} \oplus A) \mid U - (I_d, 0) \in M_d(A) \}.$$ 

If $A$ is already unital, we have a canonical isomorphism $U'_d(A) \cong U_d(A)$. So we use the same symbol $U_d(A)$ for $U'_d(A)$ even if $A$ is not unital.

Consider the following space of continuous functions:

$$C_0(\mathbb{R}^m, A) = \{ T : \mathbb{R}^m \to A \mid T \text{ is continuous and } \lim_{|z| \to \infty} T(z) = 0 \}.$$ 

This is a $C^*$-algebra without unit. Notice that $C_0(\mathbb{R}^m, A)$ is isomorphic to the space $\Omega^m A$ of based maps from the $m$-sphere $S^m$ to $A$ where the basepoint $* \in S^m$ is mapped to $0 \in A$.

Set the element

$$p_B(z) = \frac{1}{1 + |z|^2} \begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix} \in M_2(\mathbb{C} \oplus C_0(\mathbb{R}^2, \mathbb{C})) \quad (z \in \mathbb{R}^2),$$

where we identify $\mathbb{R}^2 \cong \mathbb{C}$ in the matrix entries. The Bott map $\beta : A \to M_2(A \oplus C_0(\mathbb{R}^2, A))$ is a $*$-homomorphism defined by

$$\beta(a) = p_B \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} p_B.$$ 

Then we have the commutative square of unital $C^*$-algebras

$$\begin{array}{ccc}
\mathbb{C} \oplus A & \xrightarrow{\epsilon} & \mathbb{C} \\
\beta \downarrow & & \eta \\
M_2(\mathbb{C} \oplus A \oplus C_0(\mathbb{R}^2, A)) & \xrightarrow{\epsilon} & M_2(\mathbb{C} \oplus A)
\end{array}$$

where $\epsilon : \mathbb{C} \oplus A \to \mathbb{C}$ and $\epsilon : M_2(\mathbb{C} \oplus A \oplus C_0(\mathbb{R}^2, A)) \to M_2(\mathbb{C} \oplus A)$ are the projections and $\eta : \mathbb{C} \to M_2(\mathbb{C} \oplus A)$ is the unit map. This square induces the $*$-homomorphism between the kernels of $\epsilon$:

$$\beta : A \to M_2(C_0(\mathbb{R}^2, A)).$$
We call this $\beta$ the Bott map as well. It is natural in the following sense: if $f: A \to B$ is a $\ast$-homomorphism between $C^*$-algebras, then the following square commutes:

$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\beta} & & \downarrow{\beta} \\
M_2(C_0(\mathbb{R}^2, A)) & \xrightarrow{f_*} & M_2(C_0(\mathbb{R}^2, B))
\end{array}$

**Proposition 4.1.** The Bott map $\beta: A \to M_2(C_0(\mathbb{R}^2, A))$ induces an isomorphism on K-theory.

**Remark 4.2.** This can be seen as a formulation of the Bott periodicity. If you wish to deduce this proposition from the results appearing in [Bla86], it follows from the observation 9.2.10 on the generator of $K_0(C_0(\mathbb{R}^2, \mathbb{C}))$ and the Künneth theorem for tensor products (Theorem 23.1.3).

The Bott periodicity provides the natural homotopy equivalence

$U_\infty(A) \xrightarrow{\beta} U_\infty(M_2(C_0(\mathbb{R}^2, A))) \cong \Omega^2 U_\infty(A),$

which is a group homomorphism. Thus we obtain the following proposition on infinite loop structure.

**Proposition 4.3.** The unitary group $U_\infty(A)$ of a $C^*$-algebra $A$ is equipped with a canonical infinite loop space structure such that the map $U_\infty(A) \to U_\infty(B)$ induced from a $\ast$-homomorphism $A \to B$ is an infinite loop map. Moreover, the underlying loop structure of $U_\infty(A)$ coincides with the group structure of $U_\infty(A)$.

**Remark 4.4.** The last sentence in the proposition means that there exists a homotopy equivalence $B U_\infty(A) \cong \Omega U_\infty(A)$ from the classifying space $B U_\infty(A)$ of the topological group $U_\infty(A)$.

5. **Segal–Wilson restricted unitary group**

To study the homotopy type of $U_1(C_0^*(\mathbb{Z}))$, we will relate it with other spaces. One is the Segal–Wilson restricted unitary group $U_1(B_{SW})$ and the other is the unitary group of the Roe algebra $U_1(C_0^*(\mathbb{Z}))$. We recall the former in this section.

We have another matrix expression for $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$ as

$T = \begin{pmatrix} T_{--} & T_{+-} \\ T_{+--} & T_{++} \end{pmatrix},$

where

$T_{--}: \ell^2(\mathbb{Z}_{<0}) \to \ell^2(\mathbb{Z}_{<0}), \quad T_{+-}: \ell^2(\mathbb{Z}_{\geq 0}) \to \ell^2(\mathbb{Z}_{<0}),$

$T_{+--}: \ell^2(\mathbb{Z}_{<0}) \to \ell^2(\mathbb{Z}_{\geq 0}), \quad T_{++}: \ell^2(\mathbb{Z}_{\geq 0}) \to \ell^2(\mathbb{Z}_{\geq 0}).$

**Definition 5.1.** We define the $C^*$-algebra $B_{SW}$ by

$B_{SW} := \{ T \in \mathcal{B}(\mathcal{H}) \mid T_{--}, T_{+-} \text{ are compact} \}.$

The symbol “SW” stands for Segal–Wilson. The unitary group $U_1(B_{SW})$ is called the restricted unitary group in the work of Segal and Wilson [SW85]. They used it as a model of the infinite Grassmannian.
Lemma 5.2 (Segal–Wilson). The space $U_1(B^{SW})$ has the homotopy type of $\mathbb{Z} \times B U_\infty(\mathbb{C})$. Moreover, the map

$$\pi_0(U_1(B^{SW})) \to \mathbb{Z}, \quad [U] \mapsto \text{ind}(U_{++}),$$

is bijective, where ind$(U_{++})$ denotes the Fredholm index of the Fredholm operator $U_{++}$.

Let $S = S_{+1} \in B^{SW}$ the shift operator as in Example 2.2. We have ind $S^n = n$.

The goal of this section is to see the following.

Proposition 5.3. The stabilizing maps

$$GL_1(B^{SW}) \to GL_\infty(B^{SW}) \quad \text{and} \quad U_1(B^{SW}) \to U_\infty(B^{SW})$$

are homotopy equivalences.

To show this, we do not use a kind of stability as in Section 3.

Lemma 5.4. For any integer $d \geq 1$, the inclusion

$$U_1(B^{SW}) \to U_d(B^{SW})$$

induces an isomorphism on $\pi_0$.

Proof. Consider the composite of the inclusion and the isomorphism $\phi: B^{SW} \to M_d(B^{SW})$ similar to the one in the proof of Lemma 3.5.

$$U_1(B^{SW}) \to U_d(B^{SW}) \xrightarrow{\phi^{-1}} U_1(B^{SW}).$$

It is easy to see that the image of the shift $S \in U_1(B^{SW})$ under this composite again has index 1. This implies the lemma. \hfill \Box

Lemma 5.5. The K-theory of $B^{SW}$ is computed as

$$K_i(B^{SW}) \cong \begin{cases} 0 & i = 0, \\ \mathbb{Z} & i = 1, \end{cases}$$

where $K_1(B^{SW})$ is generated by the shift $S \in U_1(B^{SW})$.

Proof. This follows from the isomorphisms

$$K_0(B^{SW}) \cong \lim_{d \to \infty} \pi_1(U_d(B^{SW})) \quad \text{and} \quad K_1(B^{SW}) \cong \lim_{d \to \infty} \pi_0(U_d(B^{SW}))$$

and Lemmas 5.2 and 5.4. \hfill \Box

Lemma 5.6. For any $i \geq 0$, there exists an integer $m \geq 1$ such that the iterated Bott map

$$\beta^i: U_d(B^{SW}) \to U_d(M_2(C_0(\mathbb{R}^{2i}, B^{SW})))$$

induces an isomorphism on $\pi_0$ if $d \geq m$.

Proof. From the isomorphisms

$$\pi_0(U_d(M_2(C_0(\mathbb{R}^{2i}, B^{SW})))) \cong \pi_2(U_2d(B^{SW})) \cong \pi_2(U_1(B^{SW})) \cong \mathbb{Z}$$

and

$$K_1(M_2(C_0(\mathbb{R}^{2i}, B^{SW}))) \cong \lim_{d \to \infty} \pi_0(U_d(M_2(C_0(\mathbb{R}^{2i}, B^{SW})))) \cong K_1(B^{SW}) \cong \mathbb{Z},$$
we can find an integer \( m \geq 1 \) such that the stabilizing map
\[
\pi_0(U_d(M_2(C_0(\mathbb{R}^{2i}, B^{SW})))) \to K_1(C_0(M_2(\mathbb{R}^{2i}, B^{SW})))
\]
is an isomorphism if \( d \geq m \). Consider the commutative diagram
\[
\begin{array}{ccc}
\pi_0(U_d(B^{SW})) & \xrightarrow{=} & K_1(B^{SW}) \\
\downarrow \beta^i & & \downarrow \beta^i \\
\pi_0(U_d(M_2(C_0([\mathbb{R}^{2i}, B^{SW}])))) & \xrightarrow{=} & K_1(M_2(C_0([\mathbb{R}^{2i}, B^{SW}])))
\end{array}
\]
where the top arrow is an isomorphism by Lemma 5.4 and the right Bott map \( \beta^i \) is an isomorphism by Proposition 4.1. Then the lemma follows. \( \square \)

Proof of Proposition 5.3 Take an integer \( i \geq 0 \). We can find an integer \( m \geq 1 \) as in Proposition 5.6 and
\[
\pi_{2i}(U_d(B^{SW})) \to \pi_{2i}(U_\infty(B^{SW})) \cong \mathbb{Z}
\]
is an isomorphism if \( d \geq m \). Consider the following commutative diagram:
\[
\begin{array}{ccc}
U_1(C_0(\mathbb{R}^{2i}, B^{SW})) & \xrightarrow{=} & U_1(M_2(C_0(\mathbb{R}^{2i}, B^{SW}))) \\
\downarrow & & \downarrow \beta^i \\
U_d(C_0(\mathbb{R}^{2i}, B^{SW})) & \xrightarrow{=} & U_d(M_2(C_0(\mathbb{R}^{2i}, B^{SW})))
\end{array}
\]
where the left horizontal arrows are the isomorphisms similar to the one in Lemma 3.5 and the vertical arrows are the inclusions. Since the composite
\[
U_1(B^{SW}) \to U_d(B^{SW}) \xrightarrow{\beta^i} U_d(M_2(C_0(\mathbb{R}^{2i}, B^{SW})))
\]
induces an isomorphism on \( \pi_0 \), the middle vertical arrow
\[
U_1(M_2(C_0(\mathbb{R}^{2i}, B^{SW}))) \to U_d(M_2(C_0(\mathbb{R}^{2i}, B^{SW})))
\]
induces a surjection on \( \pi_0 \). But it is indeed an isomorphism as their \( \pi_0 \) are isomorphic to \( \mathbb{Z} \). Then the map
\[
U_1(C_0(\mathbb{R}^{2i}, B^{SW})) \to U_d(C_0(\mathbb{R}^{2n}, B^{SW}))
\]
induces an isomorphism on \( \pi_0 \). This implies that the map
\[
U_1(B^{SW}) \to U_d(B^{SW})
\]
induces an isomorphism on \( \pi_{2i} \). Thus the map
\[
U_1(B^{SW}) \to U_\infty(B^{SW})
\]
induces an isomorphism on \( \pi_{2i} \). This completes the proof. \( \square \)
6. Homotopy type of $U_1(C^*_{\ell}(\mathbb{Z}))$

The goal of this section is to prove Theorem 1.2. The components $T_{-+}$ and $T_{+-}$ of a finite propagation operator $T \in \mathcal{B}(H)$ are finite rank operators. This implies the inclusion

$C^*_{\ell}(\mathbb{Z}) \subset B^{SW}.$

This map is a key to the proof of Theorem 1.2.

We begin with computing the $K$-theory.

**Proposition 6.1.** The following isomorphism holds:

$$K_i(C^*_{\ell}(\mathbb{Z})) \cong \begin{cases} \ell^\infty(\mathbb{Z}, \mathbb{Z})_S & i = 0, \\ \mathbb{Z} & i = 1. \end{cases}$$

where

$$\ell^\infty(\mathbb{Z}, \mathbb{Z})_S = \ell^\infty(\mathbb{Z}, \mathbb{Z})/\{a - S(a) \mid a \in \ell^\infty(\mathbb{Z}, \mathbb{Z})\}$$

is the coinvariant by the shift $S : \ell^\infty(\mathbb{Z}, \mathbb{Z}) \to \ell^\infty(\mathbb{Z}, \mathbb{Z}).$

**Proof.** Applying the Pimsner–Voiculescu exact sequence [PV80] to the crossed product

$C^*_{\ell}(\mathbb{Z}) \cong \ell^\infty(\mathbb{Z}, \mathbb{C}) \rtimes \mathbb{Z},$

we get the six-term cyclic exact sequence:

$$
\begin{array}{ccccccc}
K_0(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \xrightarrow{1 - S} & K_0(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \to & K_0(C^*_{\ell}(\mathbb{Z})) \\
K_1(C^*_{\ell}(\mathbb{Z})) & \to & K_1(\ell^\infty(\mathbb{Z}, \mathbb{C})) & \xrightarrow{1 - S} & K_1(\ell^\infty(\mathbb{Z}, \mathbb{C}))
\end{array}
$$

As is well-known, we have

$$K_i(\ell^\infty(\mathbb{Z}, \mathbb{C})) \cong \begin{cases} \ell^\infty(\mathbb{Z}, \mathbb{Z}) & i = 0, \\ 0 & i = 1, \end{cases}$$

where the induced homomorphism $S : \ell^\infty(\mathbb{Z}, \mathbb{Z}) \to \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is the shift as well. Thus we can compute $K_i(C^*_{\ell}(\mathbb{Z}))$ by the previous exact sequence. \qed

We saw the homotopy stabilities as in Theorem 1.1 and Proposition 5.3. Then it is sufficient to investigate the inclusion $U_\infty(C^*_{\ell}(\mathbb{Z})) \to U_\infty(B^{SW}).$

**Lemma 6.2.** The inclusion $U_\infty(C^*_{\ell}(\mathbb{Z})) \to U_\infty(B^{SW})$ induces isomorphisms on $\pi_{2i}$ for $i \geq 0.$

**Proof.** By Lemma 5.5, $K_1(B^{SW})$ is isomorphic to $\mathbb{Z}$ and generated by the shift $S \in B^{SW}.$ Since $S \in C^*_{\ell}(\mathbb{Z})$ and $K_1(C^*_{\ell}(\mathbb{Z})) \cong \mathbb{Z},$ the map $K_1(C^*_{\ell}(\mathbb{Z})) \to K_1(B^{SW})$ is an isomorphism. Thus the map $\pi_{2i}(U_\infty(C^*_{\ell}(\mathbb{Z}))) \to \pi_{2i}(U_\infty(B^{SW}))$ is also an isomorphism. \qed

Let $F_1$ be the homotopy fiber of the inclusion $U_\infty(C^*_{\ell}(\mathbb{Z})) \to U_\infty(B^{SW}).$

**Proposition 6.3.** The space $F_1$ has the homotopy type of the product of Eilenberg–MacLane spaces

$$\prod_{i \geq 1} \ker(\ell^\infty(\mathbb{Z}, \mathbb{Z})_S, 2i - 1),$$

where $\ell^\infty(\mathbb{Z}, \mathbb{Z})_S$ is a rational vector space of uncountable dimension.
We can find a projection.

Define a

Lemma 6.4. Let $A$ be a $C^*$-algebra, where we do not require the existence of unit. For any element $u \in K_0(A)$, there exists a (non-unital in general) $\ast$-homomorphism $f: \mathbb{C} \to M_d(A)$ such that $f_*([p_1]) \in K_0(M_d(A)) \cong K_0(A)$ equals to $u$.

Proof. Consider the inclusion of based loop spaces $U_{\infty}(C_0(\mathbb{R}, C^*_u(\mathbb{Z}))) \to U_{\infty}(C_0(\mathbb{R}, B^{SW}))$. By Proposition 4.3, Lemma 6.4 and $K_0(C_0(\mathbb{R}, C^*_u(\mathbb{Z}))) \cong \mathbb{Z}$, there exists an infinite loop map $f: U_{\infty}(\mathbb{C}) \to U_{\infty}(C_0(\mathbb{R}, C^*_u(\mathbb{Z})))$ which induces an isomorphism on $\pi_{2i-1}$ for any $i \geq 1$. It follows from this and Lemma 6.2 that the composite

$$U_{\infty}(\mathbb{C}) \to U_{\infty}(C_0(\mathbb{R}, C^*_u(\mathbb{Z}))) \to U_{\infty}(C_0(\mathbb{R}, B^{SW}))$$

is a homotopy equivalence. Then the inclusion of based loop spaces $U_{\infty}(C_0(\mathbb{R}, C^*_u(\mathbb{Z}))) \to U_{\infty}(C_0(\mathbb{R}, B^{SW}))$ admits a homotopy section. This implies that the inclusion of the double loop space $U_{\infty}(C_0(\mathbb{R}^2, C^*_u(\mathbb{Z}))) \to U_{\infty}(C_0(\mathbb{R}^2, B^{SW}))$ also admits a homotopy section. Thus the inclusion $U_{\infty}(C^*_u(\mathbb{Z})) \to U_{\infty}(B^{SW})$ admits a homotopy section by Bott periodicity, which is again an infinite loop map.

Proof of Theorem 1.2. By Proposition 6.5 we have a homotopy equivalence

$$U_{\infty}(C^*_u(\mathbb{Z})) \cong U_{\infty}(B^{SW}) \times F_1$$

as infinite loop spaces. The homotopy types of the spaces $U_{\infty}(B^{SW})$ and $F_1$ are determined in Lemma 5.2 and Proposition 6.3, respectively. Together with the homotopy stability in Theorem 1.1, this completes the proof of the theorem.

7. Generalization

In this section, we study the relation between the homotopy type of $U_1(C^*_u(\mathbb{Z}^n))$ and the inclusion $U_1(C^*_u(\mathbb{Z}^n)) \subset U_1(C^*(\mathbb{Z}^n))$ for general $n \geq 2$. In view of Theorem 1.2, we propose the following question.

Question 7.1. Does the inclusion $U_d(C^*_\Gamma(\Gamma)) \to U_d(C^*(\Gamma))$ admits a homotopy section? Are the homotopy groups of its homotopy fiber are rational vector spaces?
Let us see the case when $\Gamma = \mathbb{Z}^n$ in view of this question.

**Lemma 7.2.** The $K$-theory of the Roe algebra $C^*([\mathbb{Z}^n])$ is computed as

$$K_i(C^*([\mathbb{Z}^n])) \cong \begin{cases} \mathbb{Z} & i \equiv n \mod 2, \\ 0 & i \not\equiv n \mod 2. \end{cases}$$

**Proof.** Let

$$A_m = \ell^\infty(\mathbb{Z}^m, \mathcal{H}) \rtimes \mathbb{Z}^m$$

with respect to the action of $\mathbb{Z}^m$ ($m \leq n$) on the first $m$ factors of $\mathbb{Z}^n$. Let $S_j$ denote the shift on the $j$-th factor. Then by the Pimsner–Voiculescu exact sequence

$$\xymatrix{ K_0(A_{m-1}) \ar[r]^{1-S_m} & K_0(A_{m-1}) \ar[r] & K_0(A_m) \\ K_1(A_m) \ar[r] & K_1(A_{m-1}) \ar[r]^{1-S_m} & K_1(A_{m-1}) }$$

for $A_m = A_{m-1} \rtimes_{S_m} \mathbb{Z}$, we obtain the short exact sequence

$$0 \to K_i(A_{m-1})_{S_m} \to K_i(A_m) \to K_{i-1}(A_{m-1})^{S_m} \to 0$$

for $i = 0, 1$, where $K_i(A_{m-1})_{S_m}$ and $K_i(A_{m-1})^{S_m}$ denote the coinvariant and the invariant by $S_m$, respectively. Since $A_0 = \ell^\infty(\mathbb{Z}^n, \mathcal{H})$ and we have the well-known isomorphism

$$K_i(\ell^\infty(\mathbb{Z}^n, \mathcal{H})) \cong \begin{cases} \mathbb{Z}^{\mathbb{Z}^n} & i = 0, \\ 0 & i = 1, \end{cases}$$

where $\mathbb{Z}^{\mathbb{Z}^n}$ is the group of all $\mathbb{Z}$-valued sequences over $\mathbb{Z}^n$, we obtain

$$K_i(A_m) \cong \begin{cases} \mathbb{Z}^{\mathbb{Z}^n-m} & i \equiv m \mod 2, \\ 0 & i \not\equiv m \mod 2, \end{cases}$$

by induction on $m$. The lemma is just the case when $m = n$. \hfill $\Box$

Together with the previous lemma, the homotopy type of $U_\infty(C^*([\mathbb{Z}^n]))$ is determined by the following lemma.

**Lemma 7.3.** Let $A$ be a $C^*$-algebra, where we do not require the existence of unit. Consider the following two conditions on $K$-theory:

(i) $K_i(A) \cong \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i = 1, \end{cases}$

(ii) $K_i(A) \cong \begin{cases} 0 & i = 0, \\ \mathbb{Z} & i = 1. \end{cases}$

If (i) holds, then $U_\infty(A)$ has the homotopy type of $U_\infty(\mathbb{C})$ as an infinite loop space. If (ii) holds, then $U_\infty(A)$ has the homotopy type of $\mathbb{Z} \times B U_\infty(\mathbb{C})$ as an infinite loop space.

**Proof.** Suppose the condition (i). By Lemma 6.4 there exists a homotopy equivalence $f: U_\infty(\mathbb{C}) \to U_\infty(A)$, which is an infinite loop map. When the condition (ii) holds, apply the result for the condition (i) to the algebra $C_0(\mathbb{R}, A)$. This implies that $U_\infty(C_0(\mathbb{R}, A))$ is homotopy equivalent to $U_\infty(\mathbb{C})$. By the Bott periodicity, $U_\infty(A)$ is homotopy equivalent to $\Omega U_\infty(\mathbb{C}) \simeq \mathbb{Z} \times B U_\infty(\mathbb{C})$. \hfill $\Box$

**Proposition 7.4.** The inclusion $U_\infty(C^*(([\mathbb{Z}^n])) \to U_\infty(C^*([\mathbb{Z}^n]))$ admits a homotopy section as an infinite loop map if and only if the homomorphism $K_*(C^*([\mathbb{Z}^n])) \to K_*(C^*([\mathbb{Z}^n]))$ is surjective.
Proof. The only if part is obvious. For the if part, when $n$ is odd, this follows from Lemma 7.2 and the same argument as in the proof of Proposition 6.5. When $n$ is even, apply the same argument to the map on the based loop spaces $U_\infty(C_0(\mathbb{R}, C^*_u([\mathbb{Z}^n]))) \to U_\infty(C_0(\mathbb{R}, C^*([\mathbb{Z}^n])))$. Then the proposition follows from the existence of the homotopy section of the map on the double loop spaces $U_\infty(C_0(\mathbb{R}^2, C^*_u([\mathbb{Z}^n]))) \to U_\infty(C_0(\mathbb{R}^2, C^*([\mathbb{Z}^n])))$ and the Bott periodicity. \(\square\)

Now all we have to do is to see that the homomorphism $K_s(C^*_u([\mathbb{Z}^n])) \to K_s(C^*([\mathbb{Z}^n]))$ is surjective. Let

$$B_m = \ell^\infty(\mathbb{Z}^n, \mathbb{C}) \rtimes \mathbb{Z}^m$$

with respect to the action $\mathbb{Z}^m$ ($m \leq n$) on the first $m$ factors of $\mathbb{Z}^n$ and $S_j$ denote the shift on the $j$-th factor. We obtain the short exact sequences similar to (1).

(2) \[ 0 \to K_i(B_{m-1})_{S_m} \to K_i(B_m) \to K_{i-1}(B_{m-1})^{S_m} \to 0 \]

for $i = 0, 1$. For $n = 1, 2$, we can see the surjectivity as follows.

**Lemma 7.5.** The homomorphism $K_1(C^*_u([\mathbb{Z}])) \to K_1(C^*([\mathbb{Z}]))$ is an isomorphism.

**Proof.** Consider the commutative square

$$
\begin{array}{ccc}
K_1(C^*_u([\mathbb{Z}])) & \cong & \ell^\infty(\mathbb{Z}, \mathbb{Z})^S \\
\downarrow & & \downarrow \\
K_1(C^*([\mathbb{Z}])) & \cong & (\mathbb{Z}^2)^S
\end{array}
$$

obtained from the exact sequences (1) and (2). Thus the lemma follows. \(\square\)

**Lemma 7.6.** The homomorphism $K_0(C^*_u([\mathbb{Z}^2])) \to K_0(C^*([\mathbb{Z}^2]))$ is surjective.

**Proof.** When $n = 2$, we can compute $K_1(B_1)$ by the exact sequence (2) as follows:

$$K_1(B_1) \cong \begin{cases} 
\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1} & i = 0, \\
\ell^\infty(\mathbb{Z}^2, \mathbb{Z})^{S_1} & i = 1.
\end{cases}$$

Again by the exact sequences (1) and (2) for $m = 2$, we have the commutative diagram

$$
\begin{array}{ccccc}
0 & \to & \ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1S_2} & \to & K_0(C^*_u([\mathbb{Z}^2])) & \to & \ell^\infty(\mathbb{Z}^2, \mathbb{Z})^{S_1S_2} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & K_0(C^*([\mathbb{Z}^2])) & \cong & (\mathbb{Z}^2)^{S_1S_2} & \to & 0
\end{array}
$$

Thus the homomorphism $K_0(C^*_u([\mathbb{Z}^2])) \to K_0(C^*([\mathbb{Z}^2]))$ is surjective by the right square. \(\square\)

To determine the homotopy type of $C^*_u([\mathbb{Z}^2])$, we also need its $K$-theory.

**Lemma 7.7.** The $K$-theory $K_1(C^*_u([\mathbb{Z}^2]))$ and the kernel of the homomorphism $K_0(C^*_u([\mathbb{Z}^2])) \to K_0(C^*([\mathbb{Z}^2]))$ are rational vector spaces of uncountable dimension.

**Proof.** As seen in the proof of Lemma 7.6, the latter group is isomorphic to $\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1S_2}$. The coinvariant $\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1}$ can be seen to be a rational vector space of uncountable dimension by the same argument as in [KKT], Section 5. Then, since $S_2: \ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1} \to \ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1}$ is a
linear map on a rational vector space, the coinvariant \( \ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1, S_2} \) is a rational vector space of uncountable dimension. For \( K_1(C^*_u([\mathbb{Z}^2])) \), we obtain the exact sequence

\[
0 \to (\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1})_{S_2} \to K_1(C^*_u([\mathbb{Z}^2])) \to (\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1})_{S_2} \to 0
\]

from \([2]\). Since \((\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1})_{S_2} \cong \ell^\infty(\mathbb{Z}, \mathbb{Z})_{S_2}\) and \(\ell^\infty(\mathbb{Z}^2, \mathbb{Z})_{S_1}\) are rational vector spaces, \(K_1(C^*_u([\mathbb{Z}^2]))\) is also a rational vector space of uncountable dimension.

**Proof of Theorem 1.3.** By Proposition 7.4 and Lemma 7.6, the inclusion \(U_\infty(C^*_u([\mathbb{Z}^2])) \to U_\infty(C^*([\mathbb{Z}^2]))\) admits a homotopy section as an infinite loop map. Let \(F\) be the homotopy fiber of the inclusion. Then we have a homotopy equivalence

\[
U_\infty(C^*_u([\mathbb{Z}^2])) \cong U_\infty(C^*([\mathbb{Z}^2])) \times F
\]

as infinite loop spaces. By Lemmas 7.2 and 7.3, \(U_\infty(C^*([\mathbb{Z}^2]))\) is homotopy equivalent to \(U_\infty(\mathbb{C})\) as an infinite loop space. By the naturality of the Bott maps

\[
\beta: U_\infty(C^*_u([\mathbb{Z}^2])) \overset{\cong}{\to} U_\infty(C_0(\mathbb{R}^2, C^*_u([\mathbb{Z}^2]))) \quad \text{and} \quad \beta: U_\infty(C^*([\mathbb{Z}^2])) \overset{\cong}{\to} U_\infty(C_0(\mathbb{R}^2, C^*([\mathbb{Z}^2])))
\]

we have the homotopy equivalence

\[
\tilde{\beta}: F \overset{\cong}{\to} \Omega^2 F
\]

as well. The homotopy group of \(F\) can be computed by Lemma 7.6

\[
\pi_i(F) \cong \begin{cases} V_1 & i \text{ is even}, \\ V_0 & i \text{ is odd}, \end{cases}
\]

where

\[
V_0 = \ker[K_0(C^*_u([\mathbb{Z}^2])) \to K_0(C^*([\mathbb{Z}^2]))], \quad V_1 = K_1(C^*_u([\mathbb{Z}^2]))
\]

are rational vector spaces by Lemma 7.7. Again as in the proof of [KKT] Lemma 5.4, we can find maps

\[
\prod_{i \geq 1} K(V_0, 2i - 1) \to F \quad \text{and} \quad \prod_{i \geq 1} K(V_1, 2i - 1) \to \Omega F
\]

inducing isomorphisms on the odd degree homotopy groups. Then, using the homotopy equivalence \(\tilde{\beta}\), we obtain the homotopy equivalence

\[
\prod_{i \geq 1} (K(V_0, 2i - 1) \times K(V_1, 2i)) \to F.
\]

This completes the proof of the theorem. \(\square\)

Moreover, Lemma 7.5 provides another proof of Theorem 1.2 in a similar manner.

**References**

[Bas64] H. Bass. *K*-theory and stable algebra. *Inst. Hautes Études Sci. Publ. Math.*, (22):5–60, 1964.

[Bla86] Bruce Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1986.

[KKT] T. Kato, D. Kishimoto, and M. Tsutaya. Homotopy type of the space of finite propagation unitary operators on \(\mathbb{Z}\). *preprint*, arxiv: 2007.06787.

[Kui65] Nicolaas H. Kuiper. The homotopy type of the unitary group of Hilbert space. *Topology*, 3:19–30, 1965.
[MT21] Vladimir Manuilov and Evgenij Troitsky. On Kuiper type theorems for uniform Roe algebras. *Linear Algebra Appl.*, 608:387–398, 2021.

[PV80] M. Pimsner and D. Voiculescu. Exact sequences for $K$-groups and Ext-groups of certain cross-product $C^*$-algebras. *J. Operator Theory*, 4(1):93–118, 1980.

[Rie83] Marc A. Rieffel. Dimension and stable rank in the $K$-theory of $C^*$-algebras. *Proc. London Math. Soc. (3)*, 46(2):301–333, 1983.

[Rie87] Marc A. Rieffel. The homotopy groups of the unitary groups of noncommutative tori. *J. Operator Theory*, 17(2):237–254, 1987.

[Roe88] John Roe. An index theorem on open manifolds. I, II. *J. Differential Geom.*, 27(1):87–113, 115–136, 1988.

[Roe03] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[SW85] Graeme Segal and George Wilson. Loop groups and equations of KdV type. *Inst. Hautes Études Sci. Publ. Math.*, (61):5–65, 1985.

**Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan**

*Email address:* tkato@math.kyoto-u.ac.jp

**Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan**

*Email address:* kishi@math.kyoto-u.ac.jp

**Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan**

*Email address:* tsutaya@math.kyushu-u.ac.jp