ON THE CANCELLATION-FREE ANTIPODE FORMULA FOR THE MALVENUTO-REUTENAUER HOPF ALGEBRA

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Abstract. For the Malvenuto-Reutenauer Hopf algebra of permutations, we provide a cancellation-free antipode formula for permutations starting with a decreasing sequence consisting of two positive integers and ending with an increasing one. As a consequence, we confirm two conjectures posed by Carolina Benedetti and Bruce E. Sagan.

1. Introduction

The antipode of a combinatorial Hopf algebra can be computed by the celebrated Takeuchi’s formula [7]. However, this antipode formula usually contains a great number of cancellations in its alternating sum. Recently, significant attention has been devoted to developing cancellation-free antipode formulas for various combinatorial Hopf algebras, including the Hopf algebra of graphs, matroids, simplicial complexes, set operads and posets, among others [3,1,3,3,3,11,14,15]. Notably, Benedetti and Sagan [7] employed the approach of sign-reversing involution to establish cancellation-free antipode formulas for nine different combinatorial Hopf algebras.

The Malvenuto-Reutenauer Hopf algebra \( \mathcal{S} \text{Sym} \) of permutations, introduced by Malvenuto and Reutenauer [12,13], is a connected graded Hopf algebra which has a linear basis consisting of permutations in all symmetric groups. Aguiar and Sottile [2] gave, for the first time, an explicit antipode formula for \( \mathcal{S} \text{Sym} \), but it involves massive cancellations. Then Aguiar and Mahajan [1] provided a cancellation-free expression for the antipode in terms of Hopf monoids. In the seminal paper [1], Benedetti and Sagan recovered the antipode formula for certain permutations using the techniques of sign-reversing involutions and appealing only to the Hopf algebra structure. In the same paper the authors predicted two cancellation-free antipode formulas for some permutations whose image under the Robinson-Schensted map is a column superstandard Young tableau of hook shape (see Conjectures 9.9 and 9.10 of [1] for details).

The purpose of this work is to prove the two conjectures posed by Benedetti and Sagan. More precisely, we give a cancellation-free antipode formula for permutations starting with a decreasing sequence consisting of two positive integers and ending with an increasing one. The two conjectures then follow directly as consequences of our results.

The organization of the paper is as follows. In the next section, after recalling some definitions related to the Malvenuto-Reutenauer Hopf algebra, we state the main result, Theorem 2.3. Then, by adapting sign-reversing involutions, we prove our main result in Section 3, where the two conjectures of Benedetti and Sagan are proved as consequences of Theorem 2.3.

Key words and phrases. Malvenuto-Reutenauer Hopf algebra, permutation, antipode, involution.

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2. Preliminaries

We begin by making clear some notation and definitions concerning the Malvenuto-Reutenauer Hopf algebra, and referring the reader to [10, 13] for more details. For an introduction to Hopf algebras, we refer the reader to [16].

Throughout, let \( \mathbb{N} \) and \( \mathbb{P} \) denote the set of nonnegative integers and positive integers, respectively. Given \( n \in \mathbb{P} \), we let \( \{n\} \) denote the set \( \{1, 2, \ldots, n\} \). Also, \( k \) will usually denote a field of characteristic zero, although it may also be a commutative ring.

For an alphabet \( A \), let \( A^* \) be the set of words on \( A \), including the empty word. Given a word \( w = a_1a_2 \cdots a_n \) on \( A \), for any \( i_1 < i_2 < \cdots < i_k \) in \( \{n\} \), the word \( w_{i_1}w_{i_2} \cdots w_{i_k} \) is called a subword of \( w \). A shuffle of two words \( u \) and \( v \) is a word obtained by “shuffling” together the words \( u \) and \( v \), maintaining the relative order of the letters in each word. The shuffle product of \( u \) and \( v \), denoted by \( u \shuffle v \), is then the sum of all shuffles, with multiplicities. As is well known, the shuffle product is sometimes defined recursively by making the empty word the identity element for each product, and requiring that

\[
(au)\shuffle(bv) = a(ulv) + b(auv)
\]

for all \( a, b \in A \) and \( u, v \in A^* \). It will be convenient to identify the shuffle set with the sum of its elements.

Let \( \mathfrak{S}_n \) be the group of permutations of \( \{n\} \) and \( \mathfrak{S} = \bigcup_{n \geq 0} \mathfrak{S}_n \), where \( \mathfrak{S}_0 \) consists of the empty permutation \( \emptyset \). The Malvenuto-Reutenauer Hopf algebra of permutations [13] is a connected graded Hopf algebra, and referring the reader to [11]. The counit is the projection \( \epsilon : \mathfrak{S} \text{Sym} \to k \mathfrak{S}_0 \). To describe its Hopf structure, we need more notions.

A permutation \( \sigma \in \mathfrak{S}_n \) is usually viewed as a word \( \sigma_1\sigma_2 \cdots \sigma_n \), where \( \sigma_i = \sigma(i) \) for all \( i \in \{n\} \). Let \( w = w_1w_2 \cdots w_n \) be a sequence of positive integers. If \( m \) is a positive integer, then let \( w + m \) be the word obtained by replacing in \( w \) each \( i \) by \( i + m \). The standardization of \( w \), denoted by \( \text{st}(w) \), is the unique permutation \( \sigma \in \mathfrak{S}_n \) such that

\[
\sigma_i < \sigma_j \iff w_i \leq w_j \quad \text{whenever} \quad 1 \leq i < j \leq n.
\]

For example, if \( w = 325 \), then \( w + 3 = 658 \) and \( \text{st}(w) = 213 \). Now if \( \pi \in \mathfrak{S}_m \) and \( \sigma \in \mathfrak{S}_n \), then the product \( \prod \) and coproduct \( \Delta \) are defined by

\[
\prod \sigma = \sum_{\tau \in \mathfrak{S} \prod (\sigma \cdot m)} \tau \quad \text{and} \quad \Delta(\pi) = \sum_{i=0}^{m} \text{st}(\pi_1 \cdots \pi_i) \otimes \text{st}(\pi_{i+1} \cdots \pi_m),
\]

respectively. The product is often called the shifted shuffle product. To illustrate,

\[
21\prod 12 = 2134 + 2314 + 2341 + 3214 + 3241 + 3421
\]

and

\[
\Delta(2314) = \emptyset \otimes 2314 + 1 \otimes 213 + 12 \otimes 12 + 231 \otimes 1 + 2314 \otimes \emptyset.
\]

The antipode \( S \) for \( \mathfrak{S} \text{Sym} \) can be computed recursively by

\[
S(\sigma) = - \sum_{i=0}^{n-1} S(\text{st}(\sigma_1 \cdots \sigma_i)) \prod \text{st}(\sigma_{i+1} \cdots \sigma_n)
\]
for all permutations $\sigma \in \mathfrak{S}_n$. For notational convenience, for any $\sigma \in \mathfrak{S}_n$ and $j \in [n]$, we write $S(\sigma)_j$ for the sum of the terms of $S(\sigma)$ ending in $j$ with signs, and $S(\sigma)_j^*$ for the sum obtained from $S(\sigma)_j$ by omitting the common suffix $j$. Thus, we have

$$S(\sigma) = \sum_{j=1}^{n} S(\sigma)_j = \sum_{j=1}^{n} S(\sigma)_j^*.$$  

For positive integers $k$ and $l$, let

$$\eta_{k,l} = \begin{cases} k(k+1)\cdots l, & k \leq l, \\ \emptyset, & k = l + 1, \\ 0 & k \geq l + 2, \end{cases}$$  

and

$$\delta_{l,k} = \begin{cases} l(l-1)\cdots k, & k \leq l, \\ \emptyset, & k = l + 1, \\ 0 & k \geq l + 2. \end{cases}$$

Since we only deal with results for $\mathfrak{S}_n\text{Sym}$, the words containing 0 will be considered to be the zero element of $\mathfrak{S}_n\text{Sym}$. For example, when $n = 4$, we have $13112211221122 = 13112211221122$ and $5412121\delta_{a,6} = 54121210$; when $a = 2$, we have $a(a - 1)\eta_{1,a-2}221\delta_{a+2} = 211011\delta_{n,4} = 2111\delta_{n,4}$.

More generally, for a set $A$ of positive integers, let $\eta_A$ and $\delta_A$ denote the increasing and decreasing words whose elements are in $A$, respectively. Given $A \subseteq [n]$, we let $\overline{A} = [n] - A$ be the complement of $A$ in $[n]$, and write $\sigma_{A,n} = \delta_A\eta_{\overline{A}}$. When no confusion will result we simply write $\sigma_A = \sigma_{A,n}$. For example, for $n \geq 5$ and $A = \{2, 5\}$, we have $\delta_A = 52$ and $\eta_{\overline{A}} = 13467 \cdots n$, so that $\sigma_A = 5213467 \cdots n$. More generally, for $A = \{a, b\}$ with $1 \leq b < a \leq n$, we have

$$\sigma_A = ab\eta_{1,b-1}\eta_{b+1,a-1}\eta_{a+1,n}.$$  

The following two conjectures were made in [1], Conjectures 9.9 and 9.10.

**Conjecture 2.1** (See [1], Conjecture 9.9). Let $A = \{a\}$ with $1 < a \leq n$. We have

$$S(\sigma_A) = (-1)^{a-1}(2\Pi\delta_{n,a})31 + (-1)^{a+a}(a - 1)\eta_{1,a-2}2\Pi\delta_{n,a+1}a$$

$$+ \sum_{j=2}^{a-1}(-1)^{a+j}[(j - 1)\eta_{1,j-2}2\Pi\delta_{n,j-1}j] + ((j + 1)\eta_{1,j-1}2\Pi\delta_{n,j+2}j].$$

**Conjecture 2.2** (See [1], Conjecture 9.10). Let $A = \{a, 2\}$ with $2 < a \leq n$. We have

$$S(\sigma_A) = (-1)^{a}[(32\Pi\delta_{n,5})41 + (12\Pi\delta_{n,4})3] + (-1)^{a-1}[(1\Pi(3\Pi\delta_{n,5})4)2]$$

$$+ \sum_{j=3}^{a-1}(-1)^{a+j}[(j + 1)21\eta_{3,j-1}2\Pi\delta_{n,j+2}j] - (j21\eta_{3,j-1}2\Pi\delta_{n,j+2}j).$$

Let $A = \{a, b\}$ with $1 \leq b < a \leq n$. Note that $S(\sigma_{[a]}) = S(\sigma_{[a,1]})$. We will establish the following cancellation-free antipode formula for permutations of the type $\sigma_A$, and thereby verify the two conjectures as simple corollaries.

**Theorem 2.3.** Let $a, b$ and $n$ be positive integers such that $1 \leq b < a \leq n$, and let $A = \{a, b\}$.

(a) For $1 \leq j \leq b - 1$, if $j = 1$, then

$$S(\sigma_A)_j = \begin{cases} (-1)^{a}(32\Pi\delta_{n,5})4, & \text{if } b = 2, \\ (-1)^{a+b+1}[(b + 1)\eta_{2,b-1}2\Pi\delta_{n,b+2}]b + \sum_{k=4}^{b}(-1)^{a+k}(k\eta_{2,k-1}2\Pi\delta_{n,k+1}), & \text{if } b \geq 3. \end{cases}$$
if $2 \leq j \leq b - 1$, then

$$S(\sigma_A)^j = \begin{cases} 
(-1)^{n+1} [111(31\delta_{n,5}) + 43111\delta_{n,5}], & \text{if } j = 2, \\
(-1)^{n+1} [111(j - 1)\eta_{2,j-2} - \delta_{j-1,j-2}\eta_{1,j-3}] \Pi \delta_{n,j+1} \\
+ 111(j + 1)\eta_{2,j-1}11\delta_{n,j+2} + \delta_{j-2,j+1}\eta_{1,j-1}11\delta_{n,j+3}], & \text{if } 3 \leq j \leq b - 1.
\end{cases}$$

(b) For $j = b$, we have

$$S(\sigma_A)^j = \begin{cases} 
(-1)^{n+1} (21\delta_{n,5}), & \text{if } b = 1, \\
(-1)^{n+1} [111(31\delta_{n,5}) + 43111\delta_{n,5}], & \text{if } b = 2, \\
(-1)^{n+b+1} [(111(b - 1)\eta_{2,b-2} - \delta_{b-1,b-2}\eta_{1,b-3}) \Pi \delta_{n,b+1} \\
+ 111(b + 1)\eta_{2,b-1}11\delta_{n,b+2}], & \text{if } b \geq 3.
\end{cases}$$

(c) For $j = b + 1$, we have

$$S(\sigma_A)^j = \begin{cases} 
(-1)^n (111\delta_{n,3}), & \text{if } b = 1, a = 2, \\
(-1)^n (111\delta_{n,3} + 311\delta_{n,4}), & \text{if } b = 1, a \geq 3, \\
(-1)^{n+b} [(111b\eta_{b-1} - \delta_{b-1,b-2}\eta_{1,b-2}) \Pi \delta_{n,b+2}], & \text{if } b \geq 2, a = b + 1, \\
(-1)^{n+b} [(111b\eta_{b-1} - \delta_{b-1,b-2}\eta_{1,b-2}) \Pi \delta_{n,b+2} \\
- (b + 2)b\eta_{b-1}11\delta_{n,b+3}], & \text{if } b \geq 2, a \geq b + 2.
\end{cases}$$

(d) For $b + 2 \leq j \leq n$, we have

$$S(\sigma_A)^j = \begin{cases} 
(-1)^{n+j} [(j - 1)b\eta_{b-1} - \delta_{b-1,b-2}\eta_{1,b-2} \Pi \delta_{n,j+1} \\
+ (j + 1)b\eta_{b-1} - \delta_{b-1,b-2}\eta_{1,b-2} \Pi \delta_{n,j+2}], & \text{if } b + 2 \leq j \leq a - 1, \\
(-1)^{n+j} [(j - 1)b\eta_{b-1} - \delta_{b-1,b-2}\eta_{1,b-2} \Pi \delta_{n,j+1}], & \text{if } j = a, \\
0, & \text{if } a + 1 \leq j \leq n.
\end{cases}$$

In particular, $S(\sigma_{[b,a],n})^j$ is independent of $a$ for any $j$ with $1 \leq j \leq b$.

For better understanding, we first compute $S(\sigma_{A,n})$ directly for $n \in \{2, 3, 4\}$.

**Example 2.4.** A simple computation yields that $S(21) = 12$, $S(321) = -123$, and

$$S(213) = 231 - (1113)2, \\
S(312) = 231 - (1113 + 31)2 + 213, \\
S(2134) = -(2114)3 + (11143)2, \\
S(3124) = -(2114)3 + (11143 + 3114)2 - (21114)3, \\
S(3214) = 3241 - (11134)2 + (12143)2, \\
S(4123) = -(2114)3 + (11143 + 3114)2 - (21114 + 412)3 + 3124, \\
S(4213) = 3241 - (11134)2 + [(1112 - 21)114 - 421]3 + 3214, \\
S(4312) = 4231 - (11134 + 431)2 + [(1112 - 21)114 + 11142]3 - (11132 - 321)4.$$

3. **Proof of the main theorem**

In this section, we prove Theorem 2.3. In particular, we verify [7], Conjectures 9.9 and 9.10. We begin with some useful lemmas where the first one was proved in [7].
Lemma 3.1. [7] Propositions 9.3 and 9.5] We have

\[ S(\eta_{1,n}) = (-1)^n \delta_{n,1} \quad \text{and} \quad S(\delta_{n,1}) = (-1)^n \eta_{1,n}. \]

Lemma 3.2. For any \( k \in \mathbb{N} \) and \( n \in \mathbb{P} \) with \( k \leq n \), we have

\[ \sum_{i=k}^{n} (-1)^i \delta_{i,k+1} \Pi i \eta_{i+1,n} = 0. \]

Proof. Assume that \( w \) is a term appearing in the sum on the left-hand side of the desired identity. Let \( j, k \leq j \leq n \), be the largest integer such that \( \delta_{j,k+1} \) is a subword of \( w \). Then \( w \) appears only in \((−1)^j \delta_{j,k+1} \Pi i \eta_{i+1,n}\) and, by the maximality of \( j \), in \((−1)^{j+1} \delta_{j-1,k+1} \Pi i \eta_{i+1,n}\), from which we see that the coefficient of \( w \) is 0, so the proof follows.

Lemma 3.3. For any positive integer \( j, k \) and \( n \) with \( j \leq k \leq n - 2 \), we have

\[ (-1)^j \delta_{j,k} \eta_{k+1,n} + \sum_{i=k}^{n} (-1)^i (\delta_{k,j} \Pi i \delta_{i,k+2})(k+1) \Pi i \eta_{i+1,n} = 0. \]

Proof. Let \( w \) be any word contained in the left hand-side shuffle set and let \( i, k+1 \leq i \leq n \), be the largest integer such that \( \delta_{i,k+1} \) is a subword of \( w \). If \( i = k+1 \), then \( k+1 \) appears to the left of \( k+2 \) in \( w \), so that \( w \) only appears in \((−1)^k \delta_{k,j} \eta_{k+1,n}\) and

\[ \sum_{i=k+1}^{n} (-1)^i (\delta_{k,j} \Pi i \delta_{i,k+2})(k+1) \Pi i \eta_{i+1,n} = 0. \]

Thus, the two occurrences of \( w \) have opposite signs and hence will cancel each other. If \( i \geq k + 2 \), then \( \delta_{i,k+1}(i+1) \) is a subword of \( w \). Consequently, \( w \) appears in \((−1)^j (\delta_{k,j} \Pi i \delta_{i,k+2})(k+1) \Pi i \eta_{i+1,n}\) and, by maximality of \( i \), in \((−1)^{j+1} (\delta_{k,j} \Pi i \delta_{i-1,k+2})(k+1) \Pi i \eta_{i+1,n}\), yielding that the coefficient of \( w \) is 0, and the proof follows.

Another identity of shuffle sets, whose proof we omit, is the following.

Lemma 3.4. For any positive integer \( k \) and \( n \) with \( 3 \leq k \leq n \), we have

\[ (-1)^k (k+1) \eta_{2,k} \eta_{k+2,n} + \sum_{i=k+1}^{n} (-1)^i [(k+1) \eta_{2,k-1} \Pi i \delta_{i,k+2}] k \Pi i \eta_{i+1,n} = 0. \]

Proof. The proof is analogous to that of Lemma 3.3; the details are omitted.

Lemma 3.5. Let \( w \in \mathcal{W}_n \), \( n \geq 2 \). If there exists \( i, 2 \leq i \leq n \), such that \( w_i \cdots w_n = \eta_{i,n} \), then \( S(w)_j = 0 \) for all \( j \) with \( i \leq j \leq n \).

Proof. We proceed by induction on \( n \), the base case \( n = 2 \) is trivial by Lemma 3.1. Assume for positive integers less than \( n \). It then follows from Eq. (1) that the only shuffle sets containing \( S(w)_j \), \( 1 \leq j \leq n \), must be

\[ - \sum_{k=j}^{n-1} S(w_1 \cdots w_{i-1} \eta_{i,k}) \Pi i \eta_{k+1,n}. \]

So \( S(w)_j \) is only contained in the shuffle sets \( S(w_1 \cdots w_{i-1} \eta_{i,k}) \Pi i \eta_{k+1,n} \), where \( j \leq k \leq n - 1 \). Thus, by the induction hypothesis, \( S(w_1 \cdots w_{i-1} \eta_{i,k})_j = 0 \) for all \( k \) with \( j \leq k \leq n - 1 \), and the proof follows.

We are now in a position to prove Theorem 2.3.
Proof of Theorem 2.3. The proof is by induction on \( n \); for \( n \in \{2, 3, 4\} \), the assertion is true by Example 2.4. Now assume for any positive integer less than \( n \) where \( n \geq 5 \). It follows from Eq. (1) and Lemma 5.1 that

\[
S(\sigma_A) = -ab\eta_{1, b-1}\eta_{b+1, a-1}\eta_{a+1, n} + 131(b + 1)\eta_{2, b}\eta_{b+2, n} - 123\eta_{3, n}
\]

Now we show that the terms \( S(\sigma_A)^j \), \( 1 \leq j \leq n \), coincide with the desired identities.

We need two simple but useful observations. First note that for any positive integers \( i \) and \( j \) with \( 1 \leq j \leq b \) and \( a \leq i \leq n - 1 \), the induction hypothesis guarantees that \( S(\sigma_{(b, a), i})^j \) is independent of \( a \). So we can substitute \( i \) for \( a \) in \( S(\sigma_{(b, a), i})^j \) without changing its value, that is, we have

\[
S(\sigma_{(b, a), i})^j = S(\sigma_{(b, i), j}), \quad \text{where} \quad 1 \leq j \leq b.
\]

Second, if \( b \geq 2, i = 3 \), then the fourth term in Eq. (3) is nonempty and we have

\[
S(\sigma_{(b, a), i})^j = S(\sigma_{(b, i), j})^j = S(\sigma_{(b, i), j})^j = S(\sigma_{(b, i), j})^j = S(\sigma_{(b, i), j})^j = S(\sigma_{(b, i), j})^j.
\]

Suppose \( 1 \leq j \leq b - 1 \). Then \( b \geq j + 1 \geq 2 \).

Case 1. \( j = 1 \). Applying the induction hypothesis and Eqs. (3) and (4) to Eq. (2) yields that

\[
S(\sigma_A)^j = (b + 1)\eta_{2, b}\eta_{b+2, n} - \sum_{i=4}^{b+1} S(\sigma_{(i-1, i), j})^j \eta_{i+1, n} - \sum_{i=b+2}^{n-1} S(\sigma_{(b, i), j})^j \eta_{i+1, n}.
\]

Thus, if \( b = 2 \), then the second sum in Eq. (5) is empty, so that

\[
S(\sigma_A)^j = 32\eta_{4, n} - \sum_{i=4}^{n-1} S(\sigma_{(i-1, i), j})^j \eta_{i+1, n} = 32\eta_{4, n} + \sum_{i=4}^{n-1} (-1)^{i+1}(321i\eta_{4, n})\eta_{i+1, n}.
\]

According to Lemma 3.3, where put \( j = 2 \) and \( k = 3 \), we obtain that

\[
S(\sigma_A)^j = (-1)^{n}(321i\eta_{4, n})\eta_{i+1, n}.
\]

If \( b \geq 3 \), then, by the induction hypothesis (i.e., Item (2)) for \( 4 \leq i \leq b + 1 \), we have

\[
S(\sigma_{(i-1, i), j})^j = (i\eta_{2, i-2}i\eta_{i+1}) (i - 1) + \sum_{k=4}^{i-1} (-1)^{i+k}(k\eta_{2, k-1}i\eta_{i+1}) = i(-1)^{i+k}(k\eta_{2, k-1}i\eta_{i+1}),
\]

and for \( b + 2 \leq i \leq n - 1 \), we have

\[
S(\sigma_{(b, i), j})^j = (-1)^{i+b+1}[(b + 1)\eta_{2, b-1}i\eta_{b+2}b + \sum_{k=4}^{b+1} (-1)^{i+k}(k\eta_{2, k-1}i\eta_{i+1})].
\]

Thus, Eq. (5) becomes

\[
S(\sigma_A)^j = (b + 1)\eta_{2, b}\eta_{b+2, n} + \sum_{i=4}^{b+1} (-1)^{i+k+1}(k\eta_{2, k-1}i\eta_{i+1})\eta_{i+1, n}
\]

\[
+ \sum_{i=b+2}^{n-1} \sum_{k=4}^{n-1} (-1)^{i+k+1}(k\eta_{2, k-1}i\eta_{i+1})\eta_{i+1, n} + \sum_{i=b+2}^{n-1} (-1)^{i+b} [(b + 1)\eta_{2, b-1}i\eta_{b+2}b \eta_{i+1, n}.
\]
\[(b + 1)\eta_{2,b}\eta_{b+2,n} = (b + 1)\eta_{2,b}III\eta_{b+2,n} + \sum_{k=4}^{b+1} \sum_{i=k}^{b+1} (-1)^{i+k+1}(k\eta_{2,k-1}III\delta_{i,k+1})III\eta_{i+1,n} + \sum_{i=b+2}^{n-1} (-1)^{i+b} [(b + 1)\eta_{2,b-1}III\delta_{i,b+2}] bIII\eta_{i+1,n}.
\]

Combining the second and fifth terms, and the third and fourth terms, respectively, we obtain that

\[S(\sigma_A)_j^* = (b + 1)\eta_{2,b}\eta_{b+2,n} + \sum_{k=4}^{b+1} (-1)^{k+1}k\eta_{2,k-1}III\left(\sum_{i=k}^{n-1} (-1)^{i+k+1}\delta_{i,k+1}III\eta_{i+1,n}\right)
+ \sum_{i=b+1}^{n-1} (-1)^{i+b} [(b + 1)\eta_{2,b-1}III\delta_{i,b+2}] bIII\eta_{i+1,n}.
\]

It then follows from Lemmas 5.2 and 5.3 that

\[S(\sigma_A)_1^* = (-1)^{n+b+1} [(b + 1)\eta_{2,b-1}III\delta_{n,b+2}] b + \sum_{k=4}^{b+1} (-1)^{n+k}(k\eta_{2,k-1}III\delta_{n,k+1}).
\]

**Case 2.** \[j = 2.\] In this case we have \(b \geq 3.\) Since \(n \geq 5,\) it follows from Eqs. (4), (5) and (6) that

\[S(\sigma_A)_2^* = -III\eta_{3,n} - \sum_{i=4}^{b+1} S(\sigma_{[i-1,i,i]})_2^*III\eta_{i+1,n} - \sum_{i=b+2}^{n-1} S(\sigma_{[i,b,i]})_2^*III\eta_{i+1,n}.
\]

By the induction hypothesis (i.e., Item (a)),

\[S(\sigma_A)_2^* = -III\eta_{3,n} + \sum_{i=4}^{n-1} (-1)^i [III(3III\delta_{i,5})4 + 431III\delta_{i,5}] III\eta_{i+1,n}
= III\left(-\eta_{3,n} + \sum_{i=4}^{n-1} (-1)^i(3III\delta_{i,5})4III\eta_{i+1,n}\right) + 431III\left(\sum_{i=4}^{n-1} (-1)^i\delta_{i,5}III\eta_{i+1,n}\right).
\]

Putting \(k = j = 3\) in Lemma 5.5 and \(k = 4\) in Lemma 5.2, respectively, there follows

\[S(\sigma_A)_2^* = (-1)^{n+1} [III(3III\delta_{n,5})4 + 431III\delta_{n,5}].
\]

**Case 3.** \(3 \leq j \leq b - 1.\) Then \(b \geq 4.\) If \(j = n - 2,\) then \(b = n - 1, a = n\) so that

\[ab\eta_{1,b-1}\eta_{b+1,a-1}\eta_{a+1,n} = \delta_{n,n-1}\eta_{1,n-2},
\]

and hence we deduce from Eq. (4) that

\[S(\sigma_A)_j^* = S(\sigma_A)_{n-2}^* = -\delta_{n,n-1}\eta_{1,n-3} - \sum_{i=3}^{n-1} S(\sigma_{[i-1,i,i]}_{n-2}^*III\eta_{i+1,n})
= -\delta_{n,n-1}\eta_{1,n-3} - \sum_{i=2}^{n-1} S(\sigma_{[i-1,i,i]}_{n-2}^*III\eta_{i+1,n}.
\]

By the induction hypothesis (i.e., Items (b) and (c)),

\[S(\sigma_A)_j^* = -\delta_{n,n-1}\eta_{1,n-3} + [III(n - 3)\eta_{2,n-4} - \delta_{n-3,n-4}\eta_{1,n-5}] III\eta_{n-1,n}
- [III(n - 3)\eta_{2,n-4}III(n - 1) - \delta_{n-3,n-4}\eta_{1,n-3}III(n - 1) + 1III(n - 1)\eta_{2,n-3}] III\eta_{n-1,n}
= -\delta_{n,n-1}\eta_{1,n-3} - 1III(n - 3)\eta_{2,n-4}III\delta_{n,n-1} + \delta_{n-3,n-4}\eta_{1,n-5}III\delta_{n,n-1} - 1III(n - 1)\eta_{2,n-3}III\eta_{n-1,n}.
\]
which coincides with the formula for $S(\sigma_A^*)_{n-2}$ given by Item (2).

If $j \leq n - 3$, then from Eqs. (2) and (3) it follows that

$$
S(\sigma_A)_j^* = -\sum_{i=3}^{b+1} S(\sigma_{[i-1,i],i})_j^* \Pi \eta_{i,1,n} - \sum_{i=b+2}^{a} S(\sigma_{[b,i],i})_j^* \Pi \eta_{i+1,n} - \sum_{i=a+1}^{n-1} S(\sigma_{[a,i],i})_j^* \Pi \eta_{i+1,n},
$$

$$
= -\sum_{i=j}^{b+1} S(\sigma_{[i-1,i],i})_j^* \Pi \eta_{i,1,n} - \sum_{i=b+2}^{n-1} S(\sigma_{[b,i],i})_j^* \Pi \eta_{i+1,n}.
$$

Since $3 \leq j \leq b - 1$, by the induction hypothesis (i.e., Item (2)), $S(\sigma_{[b,i],i})_j^*$ is independent of $b$, so we can write $S(\sigma_{[b,i],i})_j^* = S(\sigma_{[i-1,i],i})_j^*$. Thus,

$$
S(\sigma_A)_j^* = -\sum_{i=j}^{n-1} S(\sigma_{[i-1,i],i})_j^* \Pi \eta_{i,1,n}.
$$

Now observe that when $i = j$ and $j + 1$, $S(\sigma_{[i-1,i],i})_j^*$ are given by Items (1) and (2), respectively, which coincide with the formula of $S(\sigma_{[i-1,i],i})_j^*$ with $3 \leq j \leq b - 1$ given by Item (2). Thus,

$$
S(\sigma_A)_j^* = \sum_{i=j}^{n-1} (-1)^{i+j} \left[ (1\Pi(j - 1)\eta_{2,j-2} - \delta_{j-1,j-2}\eta_{1,j-3})\Pi \delta_{i,j,j+1}
+ 1\Pi(j + 1)\eta_{2,j-2}\Pi \delta_{i,j,j+2} + \delta_{j+1,j+1}\eta_{1,j-1}\Pi \delta_{i,j,j+3} \right] \Pi \eta_{i,1,n}
$$

$$
= (-1)^j \left[ (1\Pi(j - 1)\eta_{2,j-2} - \delta_{j-1,j-2}\eta_{1,j-3})\Pi \delta_{i,j,j+1}
+ 1\Pi(j + 1)\eta_{2,j-2}\Pi \delta_{i,j+1,j+2} + \delta_{j+2,j+2+1}\eta_{1,j-1}\Pi \delta_{i,j,j+3} \right] \Pi \eta_{i,1,n}
$$

$$
+ (-1)^j \Pi \delta_{i,j+1,j+1} \eta_{1,j-1}\Pi \delta_{i,j+2,j+3} \Pi \eta_{i,1,n}.
$$

Applying Lemma (2), we see that

$$
S(\sigma_A)_j^* = (-1)^{n+j+1} \left[ (1\Pi(j - 1)\eta_{2,j-2} - \delta_{j-1,j-2}\eta_{1,j-3})\Pi \delta_{n,j,1+1}
+ 1\Pi(j + 1)\eta_{2,j-2}\Pi \delta_{n,j+2} + \delta_{j+2,j+2+1}\eta_{1,j-1}\Pi \delta_{n,j+3} \right],
$$

so the proof of Item (2) follows.

(3) Suppose $j = b$. Then $1 \leq j \leq n - 1$. Applying the induction hypothesis to Eq. (2), together with Eqs. (3) and (4) and Lemmas (3.2 and 3.3) we see that when $b = 1$,

$$
S(\sigma_A)_j^* = \eta_{2,n} - \sum_{i=3}^{n-1} S(\sigma_{[i,i],i})_j^* \Pi \eta_{i,1,n} = \eta_{2,n} + \sum_{i=3}^{n-1} (-1)^i (2\Pi \delta_{i,4}) \Pi \eta_{i+1,n} = (-1)^n + (2\Pi \delta_{n,4})3;
$$

when $b = 2$,

$$
S(\sigma_A)_j^* = -\Pi \eta_{3,n} - \sum_{i=4}^{n-1} S(\sigma_{[i,i],i})_j^* \Pi \eta_{i,1,n} = -\Pi \eta_{3,n} - \sum_{i=4}^{n-1} (-1)^i [1\Pi (3\Pi \delta_{i,5})4] \Pi \eta_{i+1,n}.
$$
\[
= \Pi_n \left( -\eta_{3,n} + \sum_{i=4}^{n-1} (-1)^i (3 \Pi_{i,5}) \Pi_i \eta_{i+1,n} \right) = (-1)^{n+1} [1 \Pi_3 (3 \Pi_{n,5}) 4];
\]

when \(3 \leq b < n - 2\),

\[
S (\sigma_A)^j = - \sum_{i=b}^{b+1} S (\sigma_{(i-1,i),i})^*_b \eta_{i+1,n} - \sum_{i=b+2}^{n-1} S (\sigma_{(b,i),i})^*_b \eta_{i+1,n}
\]

\[
= \sum_{i=b}^{n-1} (-1)^{i+b} \left[ (1 \Pi_i (b - 1) \eta_{2,b-2} - \delta_{2,b-2,\eta_{1,b-3}}) \Pi_i + 1 \Pi_i (b + 1) \eta_{2,b-1} \Pi_i \delta_{b+2} \right] \eta_{i+1,n}
\]

\[
= (-1)^b [1 \Pi_i (b - 1) \eta_{2,b-2} - \delta_{2,b-2,\eta_{1,b-3}}] \Pi_i \left( \sum_{i=b}^{n-1} (-1)^i \delta_{i+1,1} \eta_{i+1,n} \right)
\]

\[
+ (-1)^b 1 \Pi_i (b + 1) \eta_{2,b-1} \Pi_i \left( \sum_{i=b+1}^{n-1} (-1)^i \delta_{i+1,2} \eta_{i+1,n} \right)
\]

\[
= (-1)^{n+b+1} \left[ (1 \Pi_i (b - 1) \eta_{2,b-2} - \delta_{2,b-2,\eta_{1,b-3}}) \Pi_i \delta_{n,b+1} + 1 \Pi_i (b + 1) \eta_{2,b-1} \Pi_i \delta_{n,b+2} \right];
\]

when \(b = n - 1\), we have \(a = n\), and hence

\[
S (\sigma_A)^j = 1 \Pi_n \eta_{2,n-2} - \sum_{i=n-1}^{n-1} S (\sigma_{(i-1,i),i})_{n-1} \Pi_i \eta_{i+1,n}
\]

\[
= 1 \Pi_n \eta_{2,n-2} + [1 \Pi_i (n - 2) \eta_{2,n-3} - \delta_{n-2,n-3,\eta_{1,n-4}}] \Pi_i n,
\]

which is a special case of the formula given by Item 2 when \(b \geq 3\). This completes the proof of Item 2.

(1) Assume that \(j = b + 1\). First consider the case \(a = b + 1\). Note that \(1 \leq b < n - 1\). Applying the induction hypothesis to Eq. 2, together with Lemmas 3.2 and 3.3, we see that if \(b = 1\), then \(j = 2\) and

\[
S (\sigma_A)^j = - 1 \Pi_n \eta_{2,n-2} - \sum_{i=2}^{n-1} S (\sigma_{(i-1,i),i})^*_2 \Pi_i \eta_{i+1,n}
\]

\[
= - 1 \Pi \left( \sum_{i=2}^{n-1} (-1)^i \delta_{i+1,3} \Pi_i \eta_{i+1,n} \right) = (-1)^n (1 \Pi \delta_{n,3});
\]

if \(2 \leq b \leq n - 2\), then

\[
S (\sigma_A)^j = - \sum_{i=b+1}^{b+1} S (\sigma_{(i-1,i),i})^*_b \Pi_i \eta_{i+1,n} - \sum_{i=b+2}^{n-1} S (\sigma_{(b,i),i})^*_b \Pi_i \eta_{i+1,n}
\]

\[
= \sum_{i=b+1}^{n-1} (-1)^{i+b+1} \left[ (1 \Pi_i b \eta_{2,b-1} - \delta_{2,b-1,\eta_{1,b-2}}) \Pi_i \eta_{i+1,n} \right]
\]

\[
= (-1)^{b+1} \left( 1 \Pi_{i,b-1} - \delta_{b,b-1,\eta_{1,b-2}} \right) \Pi_i \left( \sum_{i=b+1}^{n-1} (-1)^i \delta_{i+1,2} \eta_{i+1,n} \right)
\]

\[
= (-1)^{n+b} \left( 1 \Pi_{b,b-1} - \delta_{b,b-1,\eta_{1,b-2}} \right) \Pi \delta_{n,b+2};
\]
if \( b = n - 1 \), then \( j = a = n \), and Eq. (2) becomes

\[
S(\sigma_A) = -\delta_{n,n-1}\eta_{1,n-2} + 11\Pi n\eta_{2,n-1} - 12\Pi\eta_{3,n} - \sum_{i=3}^{n-1} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n},
\]

so that

\[
S(\sigma_A)^j = -11\Pi\eta_{3,n-1} - \sum_{i=3}^{n-1} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n-1}
\quad \text{(7)}
\]

\[
\quad = -11\Pi\eta_{3,n-1} - \sum_{i=3}^{n-2} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n-1} - S(\sigma_{(n-2,n-1),n-1}).
\]

Note that

\[
S(\sigma_{(n-2,n-1),n-1}) = -\delta_{n-1,n-2}\eta_{1,n-3} + 11\Pi(n-1)\eta_{2,n-2} - 12\Pi\eta_{3,n-1} - \sum_{i=3}^{n-2} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n-1}.
\]

Substituting it into Eq. (7), we obtain that

\[
S(\sigma_A)^j = -11\Pi(n-1)\eta_{2,n-2} + \delta_{n-1,n-2}\eta_{1,n-3},
\]

as desired.

Now consider the case \( a \geq b + 2 \). Then \( 1 \leq b \leq n - 2 \). Applying the induction hypothesis to Eq. (2) and using Eq. (5) and Lemmas 5.2 and 5.3, we see that if \( b = 1 \), then \( j = 2 \) and \( a \geq 3 \), so that

\[
S(\sigma_A)^j = -11\Pi\eta_{3,n} - \sum_{i=3}^{a-1} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n} - \sum_{i=a+1}^{n-1} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n}
\]

\[
\quad = -11\Pi\eta_{3,n} - \sum_{i=3}^{n-1} (-1)^i(11\Pi\delta_{i,3} + 31\Pi\delta_{i,4})\Pi\eta_{i+1,n}
\]

\[
\quad = -11\Pi\left(\sum_{i=2}^{n-1} (-1)^i\delta_{i,3}\Pi\eta_{i+1,n}\right) - 31\Pi\left(\sum_{i=3}^{n-1} (-1)^i\delta_{i,4}\Pi\eta_{i+1,n}\right)
\]

\[
\quad = (-1)^n(11\Pi\delta_{n,3} + 31\Pi\delta_{n,4});
\]

if \( 2 \leq b \leq n - 3 \), then

\[
S(\sigma_A)^j = -\sum_{i=b+1}^{b+1} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n} - \sum_{i=b+2}^{a} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n} - \sum_{i=a+1}^{n-1} S(\sigma_{(i-1,i,i)})\Pi\eta_{i+1,n}
\]

\[
\quad = \sum_{i=b+1}^{n-1} (-1)^{i+b+1} (11\Pi b\eta_{2,b-1} - \delta_{b,b-1}\eta_{1,b-2})\Pi\delta_{i,b+2}\Pi\eta_{i+1,n} + \sum_{i=b+2}^{n-1} (-1)^{i+b}(b+2)b\eta_{1,b-1}\Pi\delta_{i,b+3}\Pi\eta_{i+1,n}
\]

\[
\quad = (-1)^{b+1} (11\Pi b\eta_{2,b-1} - \delta_{b,b-1}\eta_{1,b-2})\Pi\left(\sum_{i=b+1}^{n-1} (-1)^i\delta_{i,b+2}\Pi\eta_{i+1,n}\right)
\]
\[ + (-1)^b(b + 2)b\eta_{1,b-1}\Pi\left(\sum_{i=b+2}^{n-1} (-1)^i\delta_{i,b+1}\Pi\eta_{i+1,n}\right) \]

\[ = (-1)^{n+b}\left[ (1\Pi b\eta_{2,b-1} - \delta_{b,b-1}\eta_{1,b-2})\Pi\delta_{n,b+2} - (b + 2)b\eta_{1,b-1}\Pi\delta_{n,b+3}\right]; \]

if \( b = n - 2 \), then \( a = n \) and we have

\[ S(\sigma_A)_j^\ast = -n(n-2)\eta_{1,n-3} - S(\sigma_{[n-2,n-1],n-1})_n^\ast \Pi n \]

\[ = -n(n-2)\eta_{1,n-3} + (1\Pi(n-2)\eta_{2,n-3} - \delta_{n-2,n-3}\eta_{1,n-4})\Pi n, \]

as required.

(3) Suppose \( b + 2 \leq j \leq n \). By Lemma 3.2, for any positive integer \( j \) with \( a + 1 \leq j \leq n \), we have \( S(\sigma_A)_j^\ast = 0 \). It remains to prove Item (1) for all \( j \) with \( b + 2 \leq j \leq a \). First consider the case \( a = n \). If \( j = n \), then from Eq. (3) we see that

\[ S(\sigma_A)_j^\ast = 1\Pi(b + 1)\eta_{2,0}b\eta_{b+2,n-1} - 12\Pi\eta_{3,n-1} - \sum_{i=3}^{b+1} S(\sigma_{[i-1,i],i})\Pi\eta_{i+1,n-1} \]

\[ - \sum_{i=b+2}^{n-2} S(\sigma_{[b,i],i})\Pi\eta_{i+1,n-1} - S(\sigma_{[b,n-1],n-1}). \]

On the other hand, by Eq. (3),

\[ S(\sigma_{[b,n-1],n-1}) = - (n - 1)b\eta_{1,b-1}\eta_{b+1,n-2} + 1\Pi(b + 1)\eta_{2,0}b\eta_{b+2,n-1} - 12\Pi\eta_{3,n-1} \]

\[ - \sum_{i=3}^{b+1} S(\sigma_{[i-1,i],i})\Pi\eta_{i+1,n-1} - \sum_{i=b+2}^{n-2} S(\sigma_{[b,i],i})\Pi\eta_{i+1,n-1}, \]

so that

\[ S(\sigma_A)_j^\ast = (n - 1)b\eta_{1,b-1}\eta_{b+1,n-2}. \]

For \( b + 2 \leq j \leq n - 1 \), applying the induction hypothesis to Eq. (3) and using Lemmas 3.2 and 3.3, we see that if \( j = n - 1 \), then

\[ S(\sigma_A)_j^\ast = -nb\eta_{1,b-1}\eta_{b+1,n-2} - S(\sigma_{[b,n-1],n-1})_n^\ast \Pi n \]

\[ = -nb\eta_{1,b-1}\eta_{b+1,n-2} - (n - 2)b\eta_{1,b-1}\eta_{b+1,n-3}\Pi n; \]

if \( b + 2 \leq j \leq n - 2 \), then it follows from Eq. (3) and Lemma 3.2 that

\[ S(\sigma_A)_j^\ast = -\sum_{i=j}^{n-1} S(\sigma_{[b,i],i})\Pi\eta_{i+1,n} \]

\[ \sum_{i=j+1}^{n-1} (-1)^{i+j+1}(j + 1)b\eta_{1,b-1}\eta_{b+1,j-1}\Pi\delta_{i,j+1}\Pi\eta_{i+1,n} \]

\[ =((-1)^{i+j}\left[(j - 1)b\eta_{1,b-1}\eta_{b+1,j-2}\Pi\delta_{n,j+1} + (j + 1)b\eta_{1,b-1}\eta_{b+1,j-1}\Pi\delta_{n,j+2}\right]. \]
Now consider the case $a \leq n - 1$. Then we have $3 \leq b + 2 \leq j \leq a \leq n - 1$. Applying the induction hypothesis to Eq. (8), which together with Lemmas 3.2 and 3.3 yields that if $j = a$, then

$$S(\sigma_A)_j = -\sum_{i=a}^{a} S(\sigma_{[b,i]},i)_a \mathcal{P}_i \eta_{i+1,n} - \sum_{i=a+1}^{n-1} S(\sigma_{[b,a],i})_a \mathcal{P}_i \eta_{i+1,n}$$

$$= \sum_{i=a}^{n-1} (-1)^{i+a+1}(a-1)b\eta_{1,b-1}\eta_{b+1,a-2} \mathcal{P}_i \eta_{i+1,n}$$

$$=(-1)^{n+a}(a-1)b\eta_{1,b-1}\eta_{b+1,a-2} \mathcal{P}_n \eta_{n+1,n};$$

if $b + 2 \leq j \leq a - 1$, then

$$S(\sigma_A)_j = -\sum_{i=j}^{a} S(\sigma_{[b,i],i})_j \mathcal{P}_j \eta_{j+1,n} - \sum_{i=j+1}^{n-1} S(\sigma_{[b,a],j})_j \mathcal{P}_j \eta_{j+1,n}$$

$$= \sum_{i=j}^{n-1} (-1)^{i+j+1}((j-1)b\eta_{1,b-1}\eta_{b+1,j-2} \mathcal{P}_i \eta_{i+1,n}$$

$$+ \sum_{i=j+1}^{n-1} (-1)^{i+j+1}((j+1)b\eta_{1,b-1}\eta_{b+1,j-1} \mathcal{P}_i \eta_{i+1,n}$$

$$=(-1)^{n+j}[(j-1)b\eta_{1,b-1}\eta_{b+1,j-2} \mathcal{P}_n \eta_{n+1,n} + (j+1)b\eta_{1,b-1}\eta_{b+1,j-1} \mathcal{P}_n \eta_{n+1,n}]$$

as desired. The proof of Theorem 3.3 is completed.

The Conjectures 2.1 and 2.2 now follows easily from Theorem 2.3.

**Corollary 3.6.** Conjectures 2.1 and 2.2 are true.

**Proof.** For $A = \{a\}$ where $1 < a \leq n$, we have $\sigma_A = \sigma_{[1,a]}$. Putting $b = 1$ in Theorem 2.3 yields that

$$(8) \quad S(\sigma_A) = \sum_{j=1}^{a} S(\sigma_A)_j,$$

where

$$S(\sigma_A)_1 = (-1)^{a-1} (2a \mathcal{P}_a) \eta_{a+1,n}, \quad S(\sigma_A)_a = (-1)^{n+a} \left[ (a-1)\eta_{1,a-2} \mathcal{P}_a \eta_{a+1,n} \right] a,$$

and

$$S(\sigma_A)_j = \left[ (j-1)\eta_{1,j} \mathcal{P}_j \eta_{j+1,n} \right] a + \left[ (j+1)\eta_{1,j+1} \mathcal{P}_j \eta_{j+1,n} \right] a$$

for $2 \leq j \leq a - 1$,

so the proof of Conjecture 2.1 follows.

Now let $A = \{a, 2\}$ with $2 < a \leq n$. Putting $b = 2$ in Theorem 2.3, then Eq. (8) still holds, but now we have

$$S(\sigma_A)_1 = (-1)^a (32a \mathcal{P}_a) \eta_{a+1,n}, \quad S(\sigma_A)_2 = (-1)^{n-1} \left[ 32a (2a \mathcal{P}_a) \eta_{a+1,n} \right] 2.$$

If $a = 3$, then, by Theorem 2.3[3],

$$S(\sigma_A)_3 = (-1)^a \left[ (12a \eta_{2,1} - \delta_{2,1} \eta_{1,0}) \mathcal{P}_3 \right] 3$$

$$= (-1)^a \left[ (12a - 2a) \mathcal{P}_4 \right] 3 = (-1)^a (12a \mathcal{P}_4) 3.$$

Hence in this case Conjecture 2.2 is true.
If $a \geq 4$, then it follows from Theorem 2.3(c) that
\[
S(\sigma_A) = (-1)^n \left[ (1\Pi_2\eta_{2,1} - \delta_{2,1}\eta_{1,0})\Pi_3\delta_{n,4} - 42\eta_{1,1}\Pi_3\delta_{n,5} \right] 3
\]
\[
= (-1)^n \left[ 12\Pi_3\delta_{n,4} - 421\Pi_3\delta_{n,5} \right] 3.
\]
According to Theorem 2.3(d), for any $j \geq 4$, we have
\[
S(\sigma_A)_j = \begin{cases} 
(-1)^{n+j+1}[(j+1)21\eta_{3,j-1}\Pi_3\delta_{n,j+2} + (j-1)21\eta_{3,j-2}\Pi_3\delta_{n,j+1}]j, & 4 \leq j \leq a - 1, \\
(-1)^{n+a}[(a-1)21\eta_{3,a-2}\Pi_3\delta_{n,a+1}]a, & j = a. 
\end{cases}
\]
So the proof follows for $a \geq 4$. Therefore, Conjecture 2.2 is true. \qed

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