Abstract

It is well-known that the $SU(2)$ quantum Racah coefficients or the Wigner 6$j$ symbols have a closed form expression which enables the evaluation of any knot or link polynomials in $SU(2)$ Chern-Simons field theory. Using isotopy equivalence of $SU(N)$ Chern-Simons functional integrals over three balls with one or more $S^2$ boundaries with punctures, we obtain identities to be satisfied by the $SU(N)$ quantum Racah coefficients. This enables evaluation of the coefficients for a class of $SU(N)$ representations. Using these coefficients, we can compute the polynomials for some non-torus knots and two-component links. These results are useful for verifying conjectures in topological string theory.

Keywords: Chern-Simons field theory, Knot polynomials, Ooguri-Vafa conjecture

Contents

1 Introduction 2

2 Chern-Simons Field Theory 3

3 Duality Matrix identities 10
   3.1 Fixing signs of the braiding eigenvalues 11
   3.2 More identities of Racah coefficients from equivalence of states 13
      3.2.1 Six Punctured $S^2$ boundaries 14

4 $SU(N)$ quantum Racah coefficients 17
   4.1 Coefficients when $R_1$ is fundamental 17
   4.2 Coefficients for $R_1$ symmetric or antisymmetric representations 20
5 Discussion and Conclusion

Appendix A Formulae for $U(N)$ link invariants in terms of $SU(N)$ quantum Racah coefficients and braiding eigenvalues 25
   Appendix A.1 Non Torus Knots ................................. 25
   Appendix A.2 Non Torus Links ................................. 27

Appendix B Knot Polynomials 28

Appendix C Link Polynomials 32

Appendix D Reformulated link invariants 34
   Appendix D.1 Reformulated invariant for knots ................ 36
   Appendix D.2 Reformulated invariant for links ................ 38

1. Introduction

Following the seminal work of Witten [1] on Chern-Simons theory as a theory of knots and links, generalised invariants [2, 3] for any knot or link can be directly obtained without going through the recursive procedure. For torus links, which can be wrapped on a two-torus $T^2$, using the torus link operators [4], explicit polynomial form of these invariants could be obtained. However for non-torus links, the generalised invariants [2] in $SU(N)$ Chern-Simons theory involves $SU(N)$ quantum Racah coefficients which are not known in closed form as known for $SU(2)$ [5, 6]. This prevents in writing the polynomial form for the non-torus links.

For simple $SU(N)$ representations placed on knots, whose Young Tableau are $\Box$, $\Box \Box$ and $\overline{\Box}$ we had obtained some Racah coefficients from isotopy equivalence of knots and links which was useful to obtain polynomial for few non-torus knots [2, 3]. Going beyond these simple representations and finding the quantum Racah coefficients appeared to be a formidable task. In fact, determining these coefficients would help in verifying the topological string conjectures for a general non-torus knot or link proposed by Ooguri-Vafa [7] and Labastida-Marino-Vafa [8]. Using the few Racah coefficients data [2], Ooguri-Vafa conjecture for $4_{1,6_1}$ non-torus knots as indicated in Figure 1 were verified [9]. For verifying Labastida-Marinoi-Vafa conjecture for the non-torus two-component links [8], we need to evaluate the non-torus link whose component knots carry different representations.

These non-torus links invariants will also be useful to generalise some of the results of recent papers [10, 11, 12, 13] where torus knots and links are studied. So, it is very important to determine the $SU(N)$ quantum Racah coefficients.
Using the correspondence between Chern-Simons functional integral and the correlator conformal blocks states in the corresponding Wess-Zumino conformal field theory [1], we can derive identities to be obeyed by the $SU(N)$ quantum Racah coefficients. Using the identities and the properties of quantum dimensions for any $N$, we could determine the form of these coefficients for some class of $SU(N)$ representations. These coefficients are needed to obtain polynomial invariants of some non-torus knots and non-torus two-component links.

The plan of the paper is as follows: In section 2, we briefly review Chern-Simons functional integrals and properties of the Racah coefficients. In section 3, we systematically study the equivalence of states and obtain identities to be obeyed by the $SU(N)$ quantum Racah coefficients. In section 4, we tabulate the $SU(N)$ quantum Racah coefficients. In Appendix A we give the generalised Chern-Simons invariant for the non-torus knots and non-torus links in Figures 1 & 2. Then we present the polynomial form of these invariants for few representations in Appendix B and Appendix C. We also verify Ooguri-Vafa conjecture for knots and Labastida-Marino-Vafa conjecture for links in Appendix D. In the concluding section, we summarize and discuss some of the open problems.

2. Chern-Simons Field Theory

Chern-Simons fields on $S^3$ with $U(1) \times SU(N)$ gauge group with levels $k_1, k_2$ respectively is given by by the following action:

$$ S = \frac{k_1}{4\pi} \int_{S^3} B \wedge dB + \frac{k_2}{4\pi} \int_{S^3} Tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), $$

(2.1)

where $B$ is the $U(1)$ gauge connection and $A$ is the $SU(N)$ matrix valued gauge connection. The observables in this theory are Wilson loop operators:

$$ W_{(R_1,n_1),(R_2,n_2),...,(R_r,n_r)}[\mathcal{L}] = \prod_{\beta=1}^{r} Tr_{R_\beta} U^A[K_\beta] Tr_{n_\beta} U^B[K_\beta], $$

(2.2)

where the holonomy of the gauge field $A$ around a component knot $K_\beta$, carrying a representation $R_\beta$, of a $r$-component link is denoted by $U^A[K_\beta] = P[exp \oint_{K_\beta} A]$ and $n_\beta$ is the $U(1)$ charge carried by the component knot $K_\beta$. The expectation value of these Wilson loop operators are the link invariants:

$$ V^{(U(1))}_{(R_1,n_1),...,[\mathcal{L}](q,\lambda)} = \left(W_{(R_1,n_1),...,[\mathcal{L}]}ight) = \frac{\int [DB][DA] e^{iS} W_{(R_1,n_1),...,(R_r,n_r)}[\mathcal{L}]}{\int [DB][DA] e^{iS}} $$

$$ \prod_{\beta=1}^{r} Tr_{R_\beta} U^A[K_\beta] Tr_{n_\beta} U^B[K_\beta]. $$

(2.3)

We make a specific choice of the $U(1)$ charges and the coupling $k_1$ so that the above invariant is a polynomial in two variables $q = exp \left( \frac{2\pi i}{k_1+N} \right)$, $\lambda = q^N$ [14, 15]

$$ n_\beta = \frac{l_\beta}{\sqrt{N}} ; \quad k_1 = k + N, $$

(2.4)
Figure 1: Plat representation for some non-torus knots
Figure 2: Plat representation for some non-torus links
where $l_\beta$ is the total number of boxes in the Young Tableau representation $R_\beta$. The $U(1)$ link invariant for the link with this substitution gives

$$V^{U(1)}_{\Psi_0^{(1)}, \chi_0^{(2)}} [\mathcal{L}] = (-1)^{\sum_l l_\beta p_\beta} \exp \left( \frac{i\pi}{k + N} \sum_{\beta=1}^r l_\beta^2 p_\beta \right) \exp \left( \frac{i\pi}{k + N} \sum_{\alpha \neq \beta} l_\alpha l_\beta l k_{\alpha\beta} \right),$$

(2.5)

where $p_\beta$ is the framing number of the component knot $K_\beta$ and $l k_{\alpha\beta}$ is the linking number between the component knots $K_\alpha$ and $K_\beta$. In order to directly evaluate $SU(N)$ link invariants, we need to use the following two ingredients:

1. The relation between $SU(N)$ Chern-Simons functional integral on the three-dimensional ball to the two-dimensional $SU(N)_k$ Wess-Zumino conformal field theory on the boundary of the three-ball [1].
2. Any knot or link can be drawn as a plating of braids [16].

In Figure 1 and Figure 2 we have drawn some non-torus knots and non-torus links as a plat representation of braids. We have labelled them in the Thistlewaithe notation and written their braid words. We have indicated the orientation and labelled the representation $R_i$ on the component knots in the link. Note that $b_i^{(-)}$ ($\{b_i^{(-)}\}^{-1}$) in the braid word denotes right-handed crossing (left-handed crossing) between $i$-strand and $(i+1)$-th strand which are anti-parallelly oriented. Similarly $b_j^{(+)}$ ($\{b_j^{(+)}\}^{-1}$) denotes right-handed crossing (left-handed crossing) between $j$- and $(j+1)$-th strand which are parallelly oriented. The plat representation of these non-torus knots and non-torus links involves braids with four-strands. Hence we can view these knots and links in $S^3$ as gluing of three-balls with 4-punctured boundary as shown in Figure 3. There are two three-balls $B_1$ and $B_3$ with opposite $S^2$ boundaries. A three-ball denoted as $B_2$ in Figure 3 with two $S^2$ boundaries with a braid $\mathcal{B}$ which can represent any of the braid words corresponding to the non-torus knots and links in Figures 1&2. The gluing of the three-balls are along the oppositely oriented $S^2$ boundary.
$S^2$ boundaries. The functional integral over the ball $B_3$ is given by a state $|\chi^{(2)}\rangle$ where the superscript denotes the label of the $S^2$ boundary. The representation $R_i$ indicate that the lines are going into the $S^2$ boundary of the three-ball and the conjugate representation denotes the lines going out of the $S^2$ boundary. The state corresponding to a functional integral on a three-ball with an oppositely oriented boundary is written in a dual space along with conjugating all the representations as illustrated for the ball $B_1$. The expectation value of the Wilson-loop operator gives the link invariant for a nontorus link $L$:

$$V_{R_1, R_2}^{SU(N)}[\mathcal{L}] = \langle \Psi^{(1)}_0 | B^{(1),(2)}_{\nu} | \chi^{(2)}_0 \rangle.$$  \hspace{1cm} (2.6)

These invariants multiplied with the $U(1)$ invariant (2.5) are polynomials in two variables $q = \exp(2\pi i/(k + N))$ and $\lambda = q^N$. In order to see the polynomial form, we write these states on a four-punctured boundary in a suitable basis of four-point conformal block of the $SU(N)_k$ Wess-Zumino conformal field theory. There are two different four-point conformal block bases as shown in Figure 4 where $t \in (R_1 \otimes R_2) \cap (R_3 \times R_4)$ and $s \in (R_2 \otimes \bar{R}_3) \cap (\bar{R}_1 \times R_4)$. Using these bases, the states corresponding to three balls $B_1, B_2$ and $B_3$ in Figure 3, can be expanded as [2, 3]

$$|\Psi^{(1)}_0\rangle = \sqrt{\text{dim}_q R_1 \text{dim}_q R_2} \langle \phi_0(R_1, R_2, \bar{R}_3, \bar{R}_4) \rangle^{(1)}$$

$$= \sum_{s \in R_1 \otimes R_2} \epsilon_s^{(R_1, R_2)} \sqrt{\text{dim}_q s} \langle \hat{\phi}_s(R_1, R_2, \bar{R}_3, \bar{R}_4) \rangle^{(1)}.$$  \hspace{1cm} (2.7)

$$B^{(1),(2)} = B|\phi_0(R_2, R_2, \bar{R}_3, \bar{R}_4)\rangle^{(1)} \langle \phi_0(R_2, R_2, \bar{R}_3, \bar{R}_4) \rangle^{(2)}$$

$$= \sum_{s \in R_1 \otimes R_2} \epsilon_s^{(R_1, R_2)} \sqrt{\text{dim}_q s} \langle \hat{\phi}_s(R_2, R_2, \bar{R}_3, \bar{R}_4) \rangle^{(2)}.$$  \hspace{1cm} (2.8)

$$|\chi^{(2)}_0\rangle = \sqrt{\text{dim}_q R_1 \text{dim}_q R_2} \langle \phi_0(R_2, R_2, \bar{R}_3, \bar{R}_4) \rangle^{(2)}$$

$$= \sum_{s \in R_1 \otimes R_2} \epsilon_s^{(R_1, R_2)} \sqrt{\text{dim}_q s} \langle \hat{\phi}_s(R_2, R_2, \bar{R}_3, \bar{R}_4) \rangle^{(2)}.$$  \hspace{1cm} (2.9)

where the subscript ‘0’ in the basis state in eqns.(2.7, 2.9) denotes the singlet state. The states are written such that the invariant of a simple circle (also called unknot) carrying representation $R$ is normalised as $\text{dim}_q R$, which is the quantum dimension of
the representation $R$, defined as

$$
dim_q R = \prod_{\alpha > 0} \frac{[\alpha (\rho + \Lambda_R)]}{[\alpha \rho]},
$$

where $\Lambda_R$ denotes the highest weight of the representation $R$, $\alpha$'s are the positive roots and $\rho$ is equal to the sum of the fundamental weights of the Lie group $SU(N)$. The square bracket refers to quantum number defined as

$$
[a] = \frac{q^a - q^{-a}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.  \tag{2.12}
$$

The symbol $\epsilon_s^{(R_1, R_2)} = \pm 1$ which we will fix from equivalence of states in the next section. To operate the braid word $B$ in eqn. (2.8), we need to find the eigenbasis of the braiding generators $b_i^{(\pm)}$'s.

The conformal block $|\phi_s(R_1, R_2, R_3, R_4)\rangle$ is suitable for the braiding operator $b_1^{(\pm)}$ and $b_3^{(\pm)}$. Similarly braiding in the middle two strands involving the operator $b_2^{(\pm)}$ requires the conformal block $|\phi_s(R_1, R_2, R_3, R_4)\rangle$. That is,

$$
\begin{align*}
&b_2^{(\pm)} |\phi_s(R_1, R_2, R_3, R_4)\rangle = \lambda_2^{(\pm)}(R_2, R_3) |\phi_s(R_1, R_3, R_2, R_4)\rangle, \\
&b_1^{(\pm)} |\phi_s(R_1, R_2, R_3, R_4)\rangle = \lambda_1^{(\pm)}(R_1, R_2) |\phi_s(R_2, R_1, R_3, R_4)\rangle, \\
&b_3^{(\pm)} |\phi_s(R_1, R_2, R_3, R_4)\rangle = \lambda_3^{(\pm)}(R_3, R_4) |\phi_s(R_1, R_2, R_4, R_3)\rangle,
\end{align*}
\tag{2.13}
$$

where braiding eigenvalues $\lambda_i^{(\pm)}(R_1, R_2)$ in vertical framing are

$$
\lambda_i^{(\pm)}(R_1, R_2) = \epsilon_i^{(\pm)} q^{cR_1 + cR_2 - cR_i \pm 1}. \tag{2.14}
$$

In this framing, framing number $p_3$ for the component knot is equal to writhe $w$ of that component knot which is the difference between total number of left-handed crossings and total number of right-handed crossing. For example, torus knots $4_1, 5_2$ in Figure 1 have writhe $w$ equal to 0 and 5 respectively. The symbol $\epsilon_i^{(\pm)}$ is a sign which can be fixed by studying equivalence of states or equivalence of links which we shall elaborate for a class of representations in the next section and $C_R$ denotes the quadratic casimir for the representation $R$ given by

$$
C_R = \kappa_R - \frac{l^2}{2N}, \quad \kappa_R = \frac{1}{2}[NL + l + \sum_i (l_i^2 - 2i l_i)],
$$

where $l_i$ is the number of boxes in the $i$-th row of the Young Tableau representation $R$ and $l$ is the total number of boxes. The two bases in Figure 4 are related by a duality matrix $a$ as follows:

$$
|\phi_t(R_1, R_2, R_3, R_4)\rangle = a ts \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} |\phi_s(R_1, R_2, R_3, R_4)\rangle. \tag{2.16}
$$
From the definition of \( t,s \), we can see that the duality matrix \( a \) obeys the following properties:

\[
\alpha_{ts} \begin{bmatrix} R_1 & R_2 \\ \tilde{R}_3 & \tilde{R}_4 \end{bmatrix} = \alpha_{st} \begin{bmatrix} R_3 & R_4 \\ \tilde{R}_1 & \tilde{R}_2 \end{bmatrix} = \alpha_{st} \begin{bmatrix} R_1 & R_4 \\ \tilde{R}_3 & \tilde{R}_2 \end{bmatrix} = \alpha_{ts} \begin{bmatrix} R_3 & R_1 \\ \tilde{R}_1 & \tilde{R}_2 \end{bmatrix}.
\] (2.17)

If one of the representation is singlet (denoted by 0), we see that the matrix elements are

\[
\alpha_{ts} \begin{bmatrix} R_1 & 0 \\ \tilde{R}_3 & \tilde{R}_4 \end{bmatrix} = \delta_{tR_1}\delta_{sR_3} \quad \alpha_{ts} \begin{bmatrix} 0 & R_2 \\ \tilde{R}_3 & \tilde{R}_4 \end{bmatrix} = \delta_{tR_2}\delta_{sR_4},
\]

\[
\alpha_{ts} \begin{bmatrix} R_1 & R_2 \\ 0 & \tilde{R}_4 \end{bmatrix} = \delta_{tR_1}\delta_{sR_2} \quad \alpha_{ts} \begin{bmatrix} R_1 & R_2 \\ \tilde{R}_3 & 0 \end{bmatrix} = \delta_{tR_3}\delta_{sR_1}.
\] (2.18)

From the procedure presented in this section, we can write the \( U(N) \) invariants of non-torus knots and links as a product of \( U(1) \) invariant times \( SU(N) \) invariant (2.6). For example, non-torus knot 5\(_2\) with framing \( p = 5 \), the invariant will be

\[
V_{R}^{U(N)}[5_2;p] = (-1)^{\frac{5}{2}} q^{\frac{5^2}{2}} \sum_{s,t,s'} \epsilon_{s}^{R,R} \sqrt{dim_{q} s} \epsilon_{s'}^{R,R} \sqrt{dim_{q} s'} (\lambda_{s}^{(+)}(R,R))^{-1} a_{ts} \begin{bmatrix} R & R \\ \tilde{R} & \tilde{R} \end{bmatrix} \lambda_{ts}^{(-)}(\tilde{R},R))^{-2} a_{ts'} \begin{bmatrix} R & R \\ \tilde{R} & \tilde{R} \end{bmatrix} (\lambda_{ts'}^{(+)}(R,R))^{-2},
\] (2.19)

where \( t \) is the total number of boxes in the Young Tableau representing \( R \) and we have indicated the framing number of the knot. We could add additional framing \( p_1 \) by multiplying these invariants by a framing factor as follows:

\[
V_{R}^{U(N)}[K;(p + p_1)] = (-1)^{p_1} q^{\frac{p_1^2}{2}} (\lambda_{0}^{(-)}(R,\tilde{R}))^{-p_1} V_{R}^{U(N)}[K;p] = (-1)^{p_1} q^{p_1 \kappa_{R}} V_{R}^{U(N)}[K;p].
\] (2.20)

So, to obtain \( 5_2 \) invariant with zero framing, we have to take \( p_1 = -5 \). For all the non-torus knots in Figure 1, we have presented zero framed knot invariants in Appendix A.

Similarly the invariant for link 6\(_2\) with linking number \( lk = 3 \) with framing \( p_1,p_2 \) on the component knots will be

\[
V_{(R_1,R_2)}^{U(N)}[6_2;p_1,p_2] = \prod_{i=1}^{2} \{ (-1)^{l_{R_i}p_i} q^{\frac{p_i^2}{2}} \sum_{s,t,s'} \epsilon_{s}^{R_1,R_2} \sqrt{dim_{q} s} (\lambda_{s}^{(+)}(R_1,R_2))^{-3} a_{ts} \begin{bmatrix} R_2 & R_1 \\ \tilde{R}_2 & \tilde{R}_1 \end{bmatrix} \lambda_{ts}^{(-)}(\tilde{R}_1,R_2))^{-2} a_{ts'} \begin{bmatrix} R_1 & R_2 \\ \tilde{R}_1 & \tilde{R}_2 \end{bmatrix} (\lambda_{ts'}^{(+)}(R_1,R_2))^{-1},
\] (2.21)

where \( l_{R_i} \) is the total number of boxes in the Young diagram representation \( R_i \) placed on the component knots in link 6\(_2\). The invariants for the non-torus links in Figure 2 are presented in Appendix A. However to see the polynomial form of these link invariants,
we have to determine the coefficients of the duality matrices. Unlike the $SU(2)$ duality matrices [5, 6], we do not have a closed form expression for $SU(N)$ duality matrices.

In the following section, we will use equivalence of states to obtain the sign (2.14, 2.7) and also derive identities satisfied by the coefficients of the duality matrix. This enables the evaluation of duality matrix elements for some class of representations.

Once we have these coefficients, we could evaluate the framed link invariants and obtain the reformulated invariants [17]

$$f_{R_1, R_2, \ldots, R_r}(q, \lambda) = \sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{d^m} \sum_{\{\tilde{k}^{(\alpha)}\}} \times \prod_{\alpha=1}^{r} \chi_{R_{\alpha j}} \left( C \left( \sum_{j=1}^{m} \tilde{k}^{(\alpha)} \right) \right) \prod_{j=1}^{m \ell_{\alpha j}} |C(\tilde{k}^{(\alpha)})| ! \times \chi_{R_{\alpha j}}(C(\tilde{k}^{(\alpha)})) V_{R_{11}, R_{21}, \ldots, R_{r1}}[L; \{p_{\alpha}\}](q^{d} \lambda^{d}) ,$$

(2.22)

where $\mu(d)$ is the Moebius function defined as follows: if $d$ has a prime decomposition ($\{p_i\}$), $d = \prod_{i=1}^{r} p_i^{n_i}$, then $\mu(d) = 0$ if any of the $m_i$ is greater than one. If all $m_i = 1$, then $\mu(d) = (-1)^{a}$. The second sum in the above equation runs over all vectors $\tilde{k}^{(\alpha)}$, with $\alpha = 1, \ldots, r$ and $j = 1, \ldots, m$, such that $\sum_{m=1}^{\infty} |\tilde{k}^{(\alpha)}| > 0$ for any $j$ and over representations $R_{\alpha j}$. Further $\tilde{k}$ is defined as follows: $(\tilde{k}_{d})_{d} = k_{d}$ and has zero entries for the other components. Therefore, if $\tilde{k} = (k_1, k_2, \ldots)$, then

$$\tilde{k}_{d} = (0, \ldots, 0, k_1, 0, \ldots, 0, k_2, 0, \ldots),$$

(2.23)

where $k_1$ is in the $d$-the entry, $k_2$ in the $2d$-th entry, and so on. Here $C(\tilde{k})$ denotes the conjugacy class determined by the sequence $(k_1, k_2, \ldots)$ (i.e there are $k_1$ 1-cycles, $k_2$ 2-cycles etc) in the permutation group $S_{\ell}$ ($\ell = \sum_{j} j k_j$). For a Young Tableau representation $R$ with $\ell$ number of boxes, $\chi_{R}(C(\tilde{k}))$ gives the character of the conjugacy class $C(\tilde{k})$ in the representation $R$. The explicit relation of the above expression in terms of Chern-Simons invariants are presented in appendix D for few representations. The reformulated invariants for $r$-component links are expected to obey Labastida-Marino-Vafa conjecture [8]

$$f_{R_1, R_2, \ldots, R_r}(q, \lambda) = (q^{1/2} - q^{-1/2})^{r-2} \sum_{Q,s} N_{(R_1, \ldots, R_r), Q,s} q^{s} \lambda^{Q} ,$$

(2.24)

where $N_{(R_1, \ldots, R_r), Q,s}$ are integers. After determining the identities and some of the matrix elements of the duality matrices in the following two sections, we will obtain the polynomial invariants of the non-torus knots and links in Appendix B and Appendix C.

3. Duality Matrix identities

We had elaborated in the previous section on writing states (2.7,2.8,2.9) corresponding to Chern-Simons functional integral on three balls. We can determine the following
coefficients of the duality matrix by comparing eqn.(2.16) and eqns.(2.7,2.9):

\[ a_{ts} \begin{bmatrix} \bar{R}_1 & R_1 \\ \bar{R}_2 & R_2 \end{bmatrix} = \epsilon_s^{(R_1,R_2)} \frac{\sqrt{\text{dim}_q s}}{\sqrt{\text{dim}_q \text{dim}_q R_1 \text{dim}_q R_2}}. \] (3.1)

This relation along with the property (2.18) suggests that

\[ a_{ts} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_3 & \bar{R}_4 \end{bmatrix} = \epsilon_{R_1;\epsilon_{R_2}}^{\epsilon_{R_3}} \epsilon_{R_4;\epsilon_{R_5}} \sqrt{\text{dim}_q s} \sqrt{\text{dim}_q t} \begin{bmatrix} R_1 & R_2 & t \\ \bar{R}_3 & \bar{R}_4 & s \end{bmatrix}, \] (3.2)

where \( \epsilon_{R_i} = \pm 1 = \epsilon_{R_i} \), and \( \epsilon_0 = 1 \). The term in parenthesis is similar to the \( SU(2) \) quantum Wigner 6j symbol but requires appropriate conjugation of representations under interchange of columns in the following way:

\[ \begin{bmatrix} R_1 & R_2 & t \\ \bar{R}_3 & \bar{R}_4 & s \end{bmatrix} = \begin{bmatrix} t & R_2 & R_1 \\ s & \bar{R}_4 & \bar{R}_3 \end{bmatrix} = \begin{bmatrix} \bar{R}_1 & \bar{R}_2 & \bar{t} \\ \bar{s} & \bar{R}_4 & \bar{R}_3 \end{bmatrix}. \] (3.3)

Using (2.18) and the relation to quantum Wigner symbol with the above properties, the \( SU(N) \) duality matrix can be called as \( SU(N) \) quantum Racah coefficients and hence we propose that the coefficients obey the following property:

\[ a_{ts} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_3 & \bar{R}_4 \end{bmatrix} = \frac{\sqrt{\text{dim}_q \text{dim}_q s}}{\sqrt{\text{dim}_q \text{dim}_q R_1 \text{dim}_q R_2}} \epsilon_{R_1}^{\epsilon_{R_2}} \epsilon_{R_3}^{\epsilon_{R_4}} \epsilon_{s;\epsilon_{t}}^{-1} a_{R_1 R_2} \begin{bmatrix} t & \bar{R}_2 \\ s & \bar{R}_4 \end{bmatrix}. \] (3.4)

Using this property, we can relate the sign in eqn.(3.1) as

\[ \epsilon_s^{(R_1,R_2)} = \epsilon_{R_1;\epsilon_{R_2}}^{\epsilon_{s}} \epsilon_{s}^{-1}. \] (3.5)

Starting from the state \( \nu^{(1),(2)} \) in eqn.(2.8) and the duality relation (2.16), we observe that the Racah coefficients must obey the following identities:

\[ \sum_s a_{ts} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_3 & \bar{R}_4 \end{bmatrix} a_{ts'} \begin{bmatrix} R_1 & \bar{R}_2 \\ \bar{R}_3 & \bar{R}_4 \end{bmatrix} = \delta_{ts'}, \] (3.6)

\[ \sum_t a_{ts} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_3 & \bar{R}_4 \end{bmatrix} a_{ts'} \begin{bmatrix} R_1 & R_2 \\ \bar{R}_3 & \bar{R}_4 \end{bmatrix} = \delta_{ss'}. \] (3.7)

### 3.1. Fixing signs of the braiding eigenvalues

From Figure 5, we can write the invariant for the unknot in two equivalent ways giving the following constraint equation:

\[ \sum_s \text{dim}_q s \lambda_0^{(+)}(R,R) = \lambda_0^{(-)}(R,R) \text{dim}_q R. \] (3.8)

Taking \( \epsilon_{0;R,R}^{(-)} = 1 \), we can determine the signs \( \epsilon_{s;R,R}^{(+)} \) which satisfies the above equation.

We can write a general form for the sign for a class of representations \( R_n = \oplus_{\ell=0}^{n-\ell} R_{n-\ell} \otimes R_{n-2\ell} \) in \( SU(N) \) Wess-Zumino conformal field theory in the large \( k \) limit will be

\[ \begin{array}{c}
R_n \\
\otimes \\
R_n
\end{array} \Rightarrow \oplus_{\ell=0}^{n-\ell} R_{n-\ell} \otimes R_{n-2\ell}. \]
The sign \( \epsilon^{(+)}_{\rho; R_n, R_m} = (-1)^{(n-\ell)} \). Similarly, for antiparallelly oriented strands, the irreducible representations in \( \tilde{\rho}_\ell \in R_n \otimes R_n \) are

\[
\begin{array}{c}
\hline
\hline
\hline
\hline
\hline
\hline
\end{array}
\]

Here boxes with dot represents a column of length \( N - 1 \). We take the sign \( \epsilon^{(-)}_{\rho; R_n, R_n} = (-1)^\ell \) which is +1 for the singlet \( \ell = 0 \). We can generalise these results for tensor product of two different symmetric representations \( \rho_\ell \in R_n \otimes R_m \) and \( \tilde{\rho}_\ell \in R_n \otimes R_m \) as

\[
\epsilon^{(+)}_{\rho_\ell, R_n, R_m} = (-1)^{\frac{n+m}{2} - \ell} ; \quad \epsilon^{(-)}_{\tilde{\rho}_\ell, \tilde{R}_n, \tilde{R}_n} = (-1)^{\frac{n+m}{2} - \ell} ,
\]

where we assume \( n \geq m \) and \( \ell = (n-m)/2, (n-m)/2 + 1, \ldots (n+m)/2 \). Similarly for antisymmetric representations \( \tilde{R}_n \) placed on antiparallelly oriented strands, the irreducible representations \( \tilde{\rho}_\ell \in \tilde{R}_n \otimes \tilde{R}_n \) for parallelly oriented strands carrying antisymmetric representation whose sign will be \( \epsilon^{(+)}_{\tilde{\rho}_\ell, \tilde{R}_n, \tilde{R}_n} = (-1)^{2n-\ell} \).

The signs for antisymmetric representations can be similarly generalised for \( \tilde{\rho}_\ell \in \tilde{R}_n \otimes \tilde{R}_m \)
and \( \tilde{\rho}_\ell \in \tilde{R}_n \otimes \tilde{R}_m \) as
\[
\epsilon_{\tilde{\rho}_\ell; \tilde{R}_n, \tilde{R}_m}^{(+)} = (-1)^{n+m-\ell}, \quad \epsilon_{\tilde{\rho}_\ell; \tilde{R}_n, \tilde{R}_m}^{(-)} = (-1)^{n-m-\ell},
\] (3.10)
where \( n \geq m \& n+m \leq N \) and \( \ell = 0, 1, \ldots, m \) for parallel strands. Similarly for antiparallel strands with \( N - m \geq n, \ \ell = n - m, n - m + 1, \ldots, n \).

It is possible to fix the signs for the mixed representation but cannot be written in the most general form as done for the symmetric and the antisymmetric representations. Some of the mixed representation signs are given in the earlier papers [9, 18]. For simplicity, we will confine to the symmetric or the antisymmetric representations placed on the component knots with the defined signs which will be useful for writing the Racah coefficients. In the following subsection, we will study equivalence of states which is needed to obtain the Racah coefficients.

### 3.2. More identities of Racah coefficients from equivalence of states

We can view the two three balls in Figure 6 as two equivalent states: The three-ball counterpart corresponding to the state \( \hat{\Psi}_0^{(1)} \) can be glued with a similar three-ball with oppositely oriented \( S^2 \) boundary to give two unknots. So, this state can be represented as
\[
|\hat{\Psi}_0^{(1)}\rangle = \epsilon(R_1, R_2) \sqrt{\text{dim}_q R_1 \text{dim}_q R_2} |\hat{\phi}_0(R_2 \tilde{R}_1 R_1 \tilde{R}_2)^{(1)}\rangle
\] (3.11)

\[
= (b_2^{(+)} - 1) b_3^{(-)} |\lambda_0^{(2)}\rangle
\]
\[
= \sum_s a_{0s} \begin{bmatrix} R_2 \\ \tilde{R}_1 \end{bmatrix} \begin{bmatrix} R_2 \\ \tilde{R}_1 \end{bmatrix} (\lambda_s^{(+))(R_1, R_2)})^{-1} a_{ts} \begin{bmatrix} R_2 \\ \tilde{R}_1 \end{bmatrix} \times \lambda_t^{(-)}(R_2, \tilde{R}_1) |\hat{\phi}_t(R_2 \tilde{R}_1 R_1 \tilde{R}_2)^{(1)}\rangle.
\] (3.12)
From eqns.(3.11,3.12) and similar relations for braid word $B = b_2^{(+)}(b_3^{(-)})^{-1}$, we can deduce the following identity

$$
\sum_s \alpha_{0s} \left[ R_2 \bar{R}_3 \lambda_s^{(+)}(R_2) \right] \left( \lambda_s^{(+)}(R_1, R_2) \right)^{-1} a_{ts} \left[ R_2 \bar{R}_3 \lambda_s^{(+)}(R_1, R_2) \right] = \epsilon(R_1, R_2)(\lambda_t^{(-)}(R_2, \bar{R}_1))^{-1} a_{ts} \left[ R_2 \bar{R}_3 \lambda_t^{(-)}(R_2, \bar{R}_1) \right].
$$

(3.13)

Using the data from $SU(2)$ [6], we can fix the signs $\epsilon(R_1, R_2)$ and the signs in the duality matrix for the class of symmetric or antisymmetric representations. Suppose we take symmetric representations for $R_1, R_2$ then the sign $\epsilon(R_n, R_m) = (-1)^{min(n,m)}$ and the signs in the duality matrix is $\epsilon_{R_n} = (-1)^{n/2}$. Similarly, for antisymmetric representations for $R_1, R_2$, $\epsilon(R_n, R_m) = (-1)^{min(2n,2m)}$ and $\epsilon_{R_n} = (-1)^n$. With this sign convention, the above identity enables fixing some of the coefficients of the duality matrix which we tabulate in the next section.

The well-known braiding identity relates the two three-balls with two $S^2$ boundaries as pictorially shown in Figure 7. Operating these braiding operators on the two $S^2$

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{braiding_identity.png}
\end{array} \]

Figure 7: Braiding Identity

boundary states, we can obtain the following identity for the duality matrix:

$$
\sum_{s,t} \lambda_s^{(+)}(R_1, R_2) a_{st} \left[ R_2 \bar{R}_3 \lambda_t^{(-)}(R_1, R_2) \right]^{-1} a_{st} \left[ R_2 \bar{R}_3 \lambda_t^{(-)}(R_1, R_2) \right] = \sum_{s,t,s',t'} a_{st} \left[ R_2 \bar{R}_3 \lambda_t^{(-)}(R_2, \bar{R}_3) \right]^{-1} a_{s't'} \left[ R_2 \bar{R}_3 \lambda_t^{(-)}(R_2, \bar{R}_3) \right]^{-1}
$$

$$
\times a_{s't'} \left[ R_2 \bar{R}_3 \lambda_t^{(+)}(R_1, R_2) a_{s't'} \right] \left[ R_2 \bar{R}_3 \lambda_t^{(+)}(R_1, R_2) a_{s't'} \right].
$$

(3.14)

3.2.1. Six Punctured $S^2$ Boundaries

We have obtained these identities by studying equivalence of three-balls with four-punctured $S^2$ boundaries. We shall now look at a generalisation of Figure 6 for three-balls with six-punctured $S^2$ boundaries as depicted in Figure 8 where the braiding operator $B = g_6 = \{ (b_4^{(+)})^{-1}b_3^{(-)} \} \{ (b_5^{(+)})^{-1}b_4^{(-)} \} \{ (b_3^{(+)})^{-1}b_2^{(-)} \}$. The Chern-Simons functional integral on these three-balls corresponds to a state in a space of six-point correlator conformal blocks in $SU(N)_k$ Wess-Zumino conformal field
Figure 8: Six-punctured boundary

Figure 9: Six point conformal block bases
theory. They can be expanded in a convenient six point conformal block bases. Two such basis states are drawn in Figure 9. In terms of these six-point conformal block bases, we can relate the state $|\Psi\rangle$ and the state $|\xi\rangle$ as follows:

$$|\Psi\rangle = \epsilon(R_1, R_2, R_3) \prod_{i=1}^{3} \sqrt{\dim R_i} \phi_{0,0,0}(R_1, R_2, R_3, \bar{R}_2, \bar{R}_3, \bar{R}_1) = B \prod_{i=1}^{3} \sqrt{\dim R_i} \phi_{0,0,0}(R_1, \bar{R}_1, R_2, \bar{R}_2, R_3, \bar{R}_3),$$

(3.15)

where $\epsilon(R_1, R_2, R_3) = \pm 1 = \epsilon(R_1, R_2)\epsilon(R_1, R_3)\epsilon(R_2, R_3)$ for symmetric and antisymmetric representations. Applying the braiding operator $B = g_6$ on the six-point conformal block in the above equation, we obtain the following relation:

$$\epsilon(R_1, R_2, R_3) \phi_{0,0,0}(R_1, R_2, R_3, \bar{R}_2, \bar{R}_3, \bar{R}_1) = \sum_{\lambda_{q_1}} \lambda_{q_1}^{-\lambda}(\bar{R}_1, R_2) \{\lambda_{p_1}^{(\lambda)}(\bar{R}_1, R_2)^{-1}

\times \lambda_{p_2}^{-\lambda}(\bar{R}_1, R_3) \{\lambda_{R_3}^{(\lambda)}(\bar{R}_1, R_3)^{-1}

\times \lambda_{p_3}^{-\lambda}(\bar{R}_1, R_3) \{\lambda_{R_3}^{(\lambda)}(\bar{R}_1, R_3)^{-1}

× \lambda_{p_4}^{-\lambda}(\bar{R}_1, R_3) \{\lambda_{R_3}^{(\lambda)}(\bar{R}_1, R_3)^{-1}

× \lambda_{p_5}^{-\lambda}(\bar{R}_1, R_3) \{\lambda_{R_3}^{(\lambda)}(\bar{R}_1, R_3)^{-1}

× \lambda_{p_6}^{-\lambda}(\bar{R}_1, R_3) \{\lambda_{R_3}^{(\lambda)}(\bar{R}_1, R_3)^{-1}

\times \lambda_{p_7}^{-\lambda}(\bar{R}_1, R_3) \{\lambda_{R_3}^{(\lambda)}(\bar{R}_1, R_3)^{-1}

\times \lambda_{p_8}^{-\lambda}(\bar{R}_1, R_3) \{\lambda_{R_3}^{(\lambda)}(\bar{R}_1, R_3)^{-1} \}

(3.16)

We can simplify the RHS of the above expression using the property (2.18). Further, the summation over index $q_1, \mu_1$ can be done using the identity (3.13). The simplified equation suggests another identity for the Racah coefficients:

$$\sum_s a_{ts} \left[ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \right] p_{R_1}^{a_{ts}} \left[ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \right] = (\epsilon_t \epsilon_s)^{-1} \left[ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \right] a_{ts} \left[ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \right].$$

(3.17)

This is a generalisation of identity (3.13). Using this identity, we can do the summation over index $p_2$ and $\mu_2$ and further simplify the RHS of the expression (3.16). The close similarity of these $SU(N)$ coefficients to the $SU(2)$ Racah coefficient identities suggests that all the identities of $SU(2)$ quantum coefficients must be generalisable to the $SU(N)$ coefficients and hence we postulate the following identity

$$\sum_{l_1} a_{r_1 l_1} \left[ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \right] a_{r_1 l_2} \left[ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \right] = \left[ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \right] a_{r_1 l_3} \left[ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \right],$$

(3.18)

where we have appropriately chosen conjugate representations which are consistent with the definition of the Racah matrix. Again, using the above identity with $l_3 = 0$, we can
do the summation over $\nu_1$ index in the simplified RHS of eqn.(3.16). Finally, we see the eqn.(3.16) reducing to

$$\langle \hat{\varphi}_{0,0,0}(R_1,R_2,\overline{R}_3,\overline{R}_2,\overline{R}_1) \rangle = a_{p_1,0} R_1 R_2 \overline{R}_2 \overline{R}_1 \langle \varphi_{p_1,0,p_1}(R_1,R_2,\overline{R}_3,\overline{R}_2,\overline{R}_1) \rangle .$$

(3.19)

Using the properties of the duality matrix, RHS can be seen to be LHS. This elaborate exercise on the equivalence of two states corresponding to six-punctured boundary confirms that the correctness of the identity (3.18). Armed with these identities, we try to determine the Racah coefficients for some representations which we present in the next section.

4. $SU(N)$ quantum Racah coefficients

We shall use the properties and identities derived in the previous sections to obtain the duality matrix coefficients which will be useful for computing the non-torus knot and non-torus links.

For knots, all the strands carry same representation. So for obtaining non-torus knot invariants, we have to evaluate two types of Racah matrices—namely,

$$a_{ij} \begin{bmatrix} R & \hat{R} \\ \hat{R} & \hat{R} \end{bmatrix}, a_{li} \begin{bmatrix} R & R \\ \hat{R} & \hat{R} \end{bmatrix},$$

where first type can be shown to be a symmetric matrix from the properties of the Racah matrix. We could evaluate the symmetric first type Racah coefficients for $R = (0,0,0)$ using eqns.(3.6,3.7) [2, 3]. However, for the second type Racah matrix we could only evaluate the coefficients for the fundamental representation ($R = (0,0)$).

Similarly for the two-component links, we can place two representations $R_1, R_2$ on the component knots. In this case, we can have three types of Racah matrices as follows:

$$a_{ij} \begin{bmatrix} R_1 & \hat{R}_1 \\ \hat{R}_2 & \hat{R}_2 \end{bmatrix}, a_{li} \begin{bmatrix} R_1 & R_2 \\ \hat{R}_2 & \hat{R}_1 \end{bmatrix}, a_{li} \begin{bmatrix} R_1 & R_2 \\ \hat{R}_1 & \hat{R}_2 \end{bmatrix} .$$

(4.2)

Now, we will present the Racah coefficients for some representations which will be useful to compute the non-torus knot and link polynomials in $U(N)$ Chern-Simons theory.

4.1. Coefficients when $R_1$ is fundamental

1. For the simplest fundamental representation $R = (0,0)$, the two types of Racah coefficient matrices are [2, 3]:

$$a_{ts} \begin{bmatrix} R & \hat{R} \\ \hat{R} & \hat{R} \end{bmatrix} = \frac{1}{2\sin R} \begin{bmatrix} s = 0 & t = 0 \\ \sqrt{N-1}[N+1] & -1 \end{bmatrix} \begin{bmatrix} \bar{R} = (0,0,0,0) \\ \sqrt{N-1}[N+1] \end{bmatrix} .$$

17
\[ a_{ts} \left[ \begin{array}{cc} R = \square & R = \square \\ \bar{R} = \square & \bar{R} = \square \end{array} \right] = \frac{1}{\dim_q R} \begin{pmatrix} t = 3 \quad \begin{cases} s = 0 \quad \frac{n}{[n]} \\ s \quad \frac{n+1}{[n]} \end{cases} \\ t \end{pmatrix} \begin{pmatrix} \begin{cases} \sqrt{\frac{N}{[N+1]}} \quad \frac{n}{[n]} \\ \sqrt{\frac{N}{[N+1]}} \quad \frac{n+1}{[n+1]} \end{cases} \\ \begin{cases} \sqrt{\frac{N}{[N+1]}} \quad \frac{n}{[n]} \\ \sqrt{\frac{N}{[N+1]}} \quad \frac{n+1}{[n+1]} \end{cases} \end{pmatrix}, \]

where \( \dim_q(R = \square) = [N] \).

2. Next, we look at Racah coefficient matrices where \( R_1 = \square \neq R_2 \). This will be useful for the computation of two-component links.

\[ a_{ts} \left[ \begin{array}{cc} R_1 = \square & R_2 = \square \\ R_2 = \square & R_2 = \square \end{array} \right] = \frac{1}{K} \begin{pmatrix} t = 0 \quad \begin{cases} s = \square \quad \frac{n}{[n]} \\ s \quad \frac{n+1}{[n]} \end{cases} \\ t \end{pmatrix} \begin{pmatrix} \begin{cases} \sqrt{\frac{N}{[N+1]}} \quad \frac{n}{[n]} \\ \sqrt{\frac{N}{[N+1]}} \quad \frac{n+1}{[n+1]} \end{cases} \\ \begin{cases} \sqrt{\frac{N}{[N+1]}} \quad \frac{n}{[n]} \\ \sqrt{\frac{N}{[N+1]}} \quad \frac{n+1}{[n+1]} \end{cases} \end{pmatrix}, \]

where \( K = \sqrt{\dim_q R_1 \dim_q R_2} \). Similarly, the second and third type Racah matrix coefficients for \( R_1 = \square, R_2 = \square \) are

\[ a_{ts} \left[ \begin{array}{cc} R_1 = \square & R_2 = \square \\ R_1 = \square & R_2 = \square \end{array} \right] = \frac{1}{K} \begin{pmatrix} t = 0 \quad \begin{cases} s = \square \quad \frac{n}{[n]} \\ s \quad \frac{n+1}{[n]} \end{cases} \\ t \end{pmatrix} \begin{pmatrix} \begin{cases} \sqrt{\frac{N}{[N+1]}} \quad \frac{n}{[n]} \\ \sqrt{\frac{N}{[N+1]}} \quad \frac{n+1}{[n+1]} \end{cases} \\ \begin{cases} \sqrt{\frac{N}{[N+1]}} \quad \frac{n}{[n]} \\ \sqrt{\frac{N}{[N+1]}} \quad \frac{n+1}{[n+1]} \end{cases} \end{pmatrix}, \]

3. Interestingly, we could find the coefficients for \( R_1 = \square, R_2 = \square \) using the identities. The corresponding conjugate representations are \( \bar{R}_1 = \square, \bar{R}_2 = \square \). The three types of Racah coefficients for this class of representations are

\[ a_{ts} \left[ \begin{array}{cc} R_1 = \square & R_2 = \square \\ R_2 = \square & R_1 = \square \end{array} \right] = \frac{1}{K} \begin{pmatrix} t = 0 \quad \begin{cases} s = \square \quad \frac{n}{[n]} \\ s \quad \frac{n+1}{[n]} \end{cases} \\ t \end{pmatrix} \begin{pmatrix} \begin{cases} \sqrt{\frac{N}{[N+1]}} \quad \frac{n}{[n]} \\ \sqrt{\frac{N}{[N+1]}} \quad \frac{n+1}{[n+1]} \end{cases} \\ \begin{cases} \sqrt{\frac{N}{[N+1]}} \quad \frac{n}{[n]} \\ \sqrt{\frac{N}{[N+1]}} \quad \frac{n+1}{[n+1]} \end{cases} \end{pmatrix}, \]
\[
a_{ts} \begin{bmatrix} R_1 & R_2 \\ R_1 & R_2 \end{bmatrix} = \begin{pmatrix} t = \begin{pmatrix} & 1 & 0 & 1 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} s = \begin{pmatrix} & n+1 \\ & n+1 \\ & n+1 \\ & n+1 \end{pmatrix} \\ -\sqrt{\frac{[N-1][N+n]}{[n+1][N+n-1]} - \frac{[n][N+n]}{[n+1][N+n-1]} & \sqrt{\frac{N-1}{[3][N+2]}} \end{pmatrix} \end{pmatrix},
\]

where \( K = \sqrt{\text{dim}_q R_1 \text{dim}_q R_2} \).

4. Similar exercise could be done for \( R_1 = \bigcirc \) and \( R_2 = \bigcirc \) and their conjugate representations are \( \overline{R}_1 = \bigcirc \) and \( \overline{R}_2 = \overline{\bigcirc} \). These representations give

\[
a_{ts} \begin{bmatrix} R_1 & R_2 \\ R_2 & R_1 \end{bmatrix} = \frac{1}{K} \begin{pmatrix} t = \begin{pmatrix} & 1 & 0 & 1 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} s = \begin{pmatrix} & n+1 \\ & n+1 \\ & n+1 \\ & n+1 \end{pmatrix} \\ -\sqrt{\frac{[N-1][N+n]}{[n+1][N+n-1]} - \frac{[n][N+n]}{[n+1][N+n-1]} & \sqrt{\frac{N-1}{[3][N+2]}} \end{pmatrix} \end{pmatrix},
\]

5. These results can be generalised for \( R_1 = \bigcirc \) and \( R_2 = \bigcirc \) and \( R_1 = \bigcirc \) and \( R_2 = \bigcirc \), whose conjugate representations are \( \overline{R}_1 = \bigcirc \) and \( \overline{R}_2 = \overline{\bigcirc} \), as follows:

19
4.2. Coefficients for $R_1$ symmetric or antisymmetric representations

1. For $R = \square$, $\bar{R} = \square$

\[
\alpha_{ts} \left[ \begin{array}{c} R_1 \\ R_2 \end{array} \right] = \frac{1}{K} \left( \begin{array}{c} s = \frac{n}{2} - 1 \\ t = 0 \end{array} \right) \left( \begin{array}{c} \sqrt{\frac{N}{2}(N - 1)(N - n + 2)(n - 1)} \\ \sqrt{\frac{N + 1}{2}(N - n)} \end{array} \right),
\]

\[
\alpha_{ts} \left[ \begin{array}{c} R_1 \\ R_1 \end{array} \right] = \left( \begin{array}{c} s = \frac{n}{2} - 1 \\ t = 0 \end{array} \right) \left( \begin{array}{c} \sqrt{\frac{N + 1}{2}(N - n)} \\ \sqrt{\frac{n}{2}(N - n + 2)(n + 1)} \end{array} \right),
\]

\[
\alpha_{ts} \left[ \begin{array}{c} R_1 \\ R_2 \end{array} \right] = \frac{(-1)^n}{K} \left( \begin{array}{c} s = 0 \\ t = \frac{n}{2} - 1 \end{array} \right) \left( \begin{array}{c} \sqrt{\frac{N}{2}(N - 1)(N - n + 2)(n - 1)} \\ \sqrt{\frac{N + 1}{2}(N - n)} \end{array} \right),
\]

where $K = \sqrt{\text{dim}_q R_1 \text{dim}_q R_2}$.

2. For $R = \square$, $\bar{R} = \square$

\[
\alpha_{ts} \left[ \begin{array}{c} R \\ R \end{array} \right] = \frac{1}{K} \left( \begin{array}{c} s = \tilde{\rho}_0 \\ t = \tilde{\rho}_0 \end{array} \right) \left( \begin{array}{c} \sqrt{\text{dim}_q \tilde{\rho}_0} \\ \sqrt{\text{dim}_q \tilde{\rho}_1} \end{array} \right),
\]

\[
\alpha_{ts} \left[ \begin{array}{c} R \\ R \end{array} \right] = \frac{1}{K} \left( \begin{array}{c} s = \tilde{\rho}_1 \\ t = \tilde{\rho}_0 \end{array} \right) \left( \begin{array}{c} \frac{\sqrt{\text{dim}_q \tilde{\rho}_1} \text{dim}_q \tilde{\rho}_2}{\text{dim}_q R - 1} - 1 \\ \sqrt{\text{dim}_q \tilde{\rho}_1} \text{dim}_q \tilde{\rho}_2 \end{array} \right),
\]

where

\[
K = \text{dim}_q R = \frac{[N][N + 1] + [N][N - 1]}{2}, \quad \text{dim}_q \tilde{\rho}_0 = 1, \quad \text{dim}_q \tilde{\rho}_1 = [N + 1][N - 1],
\]

\[
\text{dim}_q \tilde{\rho}_2 = \frac{[N - 1][N][N + 3]}{2^2}. \tag{4.3}
\]

The second type Racah matrix coefficients are
where the quantum dimensions of the representations are

\[
\dim_q \rho_1 = \frac{[N - 1][N]^2[N + 1]}{[2][3]}, \quad \dim_q \rho_2 = \frac{[N - 1][N][N + 1][N + 2]}{[4][2]},
\]

\[
\dim_q \rho_3 = \frac{[N][N + 1][N + 2][N + 3]}{[4][3][2]}.
\]

The variables \( x, y, u, v \) are

\[
x = \frac{[N + 3][N]}{[N + 2]} - [N - 1], \quad y = \frac{[N]}{[N + 2]}, \quad u = \frac{[2][N + 1]}{[N + 2]}, \quad v = \frac{[N + 1][2]}{[N + 2]} - 1.
\]

2. For \( R = \), \( \tilde{R} = \)

\[
\left( \begin{array}{c}
\tilde{t} = \tilde{\rho}_0 \\
\tilde{\rho}_1 \\
\tilde{\rho}_2
\end{array} \right),
\]

where

\[
K = \dim_q R = \frac{[N][N - 1]}{[2]}, \quad \dim_q \tilde{\rho}_0 = 1, \quad \dim_q \tilde{\rho}_1 = [N + 1][N - 1],
\]

\[
\dim_q \tilde{\rho}_2 = \frac{[N + 1][N]^2[N - 3]}{[2]^2}.
\]

The second type Racah matrix coefficients are
where
\[
\dim\rho_1 = \frac{[N-1][N]^2[N+1]}{2} [2][3],
\]
\[
\dim\rho_2 = \frac{[N+1][N][N-1][N-2]}{4} [2],
\]
\[
\dim\rho_3 = \frac{[N][N-1][N-2][N-3]}{4}[3][2],
\]
and the variables are
\[
x = \frac{[N+1]}{[N-2]} - \frac{[N-3][N]}{[N-2]}, \quad y = \frac{[N]}{[N-2]},
\]
\[
u = \frac{[2][N-1]}{[N-2]}, \quad v = \frac{[N-1][2]}{[N-2]} - 1.
\tag{4.6}
\]

3. For \( R_1 = \neq R_2 = \neq \) which is will be useful for the computation of links, the second type Racah coefficients are
\[
\alpha_{ts} = \left[ \begin{array}{ccc}
R_1 & R_2 & R_1 \\
R_1 & R_1 & R_1 \\
R_2 & R_2 & R_2 \\
\end{array} \right] = \left( \begin{array}{cccc}
t & s = 0 & \hat{\rho}_0 & \hat{\rho}_1 \\
\rho_0 & \hat{\rho}_0 & \hat{\rho}_0 & \hat{\rho}_0 \\
\rho_1 & \hat{\rho}_1 & \hat{\rho}_1 & \hat{\rho}_1 \\
\rho_2 & \hat{\rho}_2 & \hat{\rho}_2 & \hat{\rho}_2 \\
\end{array} \right),
\tag{4.7}
\]
where \( \rho_0 = \neq, \quad \rho_1 = \neq, \quad \rho_2 = \neq \) and the variables \( z_i \)'s are
\[
z_1 = \frac{[2]}{[N+1][N+2]}, \quad z_2 = z_1 \left( \frac{[N+3][N+4] - [N][N-1]}{[3]} + [N+4] \right),
\]
\[
z_3 = \frac{[2]}{[N+2][N+3]}, \quad z_4 = \frac{[2]}{[N+1][N+3]}.
\]
The Racah coefficients of third type is given by
\[
\alpha_{ts} = \left[ \begin{array}{ccc}
R_1 & R_2 & R_1 \\
R_2 & R_2 & R_2 \\
R_1 & R_1 & R_1 \\
\end{array} \right] = \frac{1}{\sqrt{K}} \left( \begin{array}{cccc}
t & s = 0 & \hat{\rho}_0 & \hat{\rho}_1 \\
\rho_0 & \hat{\rho}_0 & \hat{\rho}_0 & \hat{\rho}_0 \\
\rho_1 & \hat{\rho}_1 & \hat{\rho}_1 & \hat{\rho}_1 \\
\rho_2 & \hat{\rho}_2 & \hat{\rho}_2 & \hat{\rho}_2 \\
\end{array} \right),
\tag{4.7}
\]

22
The quantum dimensions of the representations in terms of the \( q \)-numbers are

\[
\sqrt{K} = \sqrt{\text{dim}_q R_1 \text{dim}_q R_2} = \frac{[N][N + 1]}{2} \sqrt{\frac{[N + 2]}{3}}, \quad \text{dim}_q \rho_0 = \frac{[N - 1][N][N + 1][N + 2]}{4[3][2]},
\]
\[
\text{dim}_q \rho_1 = \frac{[N - 1][N][N + 1][N + 2][N + 3]}{2[3][5]}, \quad \text{dim}_q \rho_2 = \frac{[N][N + 1][N + 2][N + 3][N + 4]}{2[3][4][5]}
\]

Using the identities, it should be possible to generalise these Racah matrices for \( R_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \), which are again \( 3 \times 3 \) matrices.

4. Equivalently, we could write the Racah matrix coefficients when \( R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \neq R_2 \)

where \( R_2 \) is totally antisymmetric \( n \)-th rank tensor (represented by \( n \)-vertical box).

For \( R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) and \( R_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) the second type Racah coefficient matrix is

\[
\begin{pmatrix}
\begin{array}{c|ccc}
 & \rho_0 & \rho_1 & \rho_2 \\
\hline
\rho_0 & \sqrt{\frac{[N + 1][N][N - 1][N - 2]}{[3][5][2][2]}} & -z_1 & z_3 \\
\rho_1 & -z_1 & \sqrt{\frac{[N + 1][N][N - 2][N - 3]}{[5][3][5][2]}} & -z_4 \\
\rho_2 & z_3 & -z_4 & \sqrt{\frac{[N + 1][N][N - 3][N - 4]}{[5][3][5][2]}} \\
\end{array}
\end{pmatrix}
\]

where \( \rho_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \), \( \rho_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \), \( \rho_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) and the variables are

\[
z_1 = \frac{[2]}{[N - 1][N - 2]}, \quad z_2 = z_1 \left( -\frac{[N - 3][N - 4][N + 1]}{[N - 1][N - 2]} \right), \quad z_3 = \frac{[2]}{[N - 2][N - 3]}, \quad z_4 = \frac{[2]}{[N - 1][N - 3]}
\]

Finally, with these data available, we evaluate the polynomial form for the non-torus knots and links in Figures 1\&2 in Appendix B and Appendix C using the formula in Appendix A. From these invariants, the reformulated invariants \( f_{R}[\mathcal{K}], f_{R_1R_2}[\mathcal{L}] \) (2.22) are obtained and shown to obey eqn.(2.24).

5. Discussion and Conclusion

In this paper, we have attempted a challenging problem of obtaining matrix elements of the duality matrix which has properties and identities similar to the quantum Racah coefficients. Particularly, we derived these identities and properties by studying the equivalence of states in the space of correlator conformal blocks in the \( SU(N)_k \) Wess-Zumino conformal field theory.

We have tabulated the Racah coefficients for some class of representations which will be useful to compute non-torus knots and non-torus two component links. We have
presented the polynomial form for all the non-torus knots and non-torus links in Figures 1&2 (see Appendix B and Appendix C) and obtained their reformulated invariants. These invariants obey the conjectured form (2.24) [7, 8] confirming the correctness of our Racah coefficients in section 4.

We believe that there must be a systematic way of writing a closed form expression similar to the expression obtained for $SU(2)$ quantum Racah coefficients [5, 6]. There are papers in the literature addressing classical Racah and quantum Racah coefficients. Unfortunately, we do not see such explicit coefficients in section 4 to compare. We hope to study those papers which may help us to obtain a closed form expression for $SU(N)$ quantum Racah coefficients.

There are interesting recent developments relating torus knots to spectral curve in the $B$-model topological strings [11], Poincare polynomial computation from refined Chern-Simons theory, Khovanov homology, fivebranes [10, 12], and the polynomial invariants from counting of solutions in four-dimensional theories [13]. We hope to extend these recent works to non-torus links and report in future.

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Appendix A. Formulae for $U(N)$ link invariants in terms of $SU(N)$ quantum Racah coefficients and braiding eigenvalues

In this appendix we give the expression of $U(N)$ link invariant for all the non torus knots and links in Figure 1 and Figure 2 in terms of $SU(N)$ quantum Racah coefficients and braiding eigenvalues.

Appendix A.1. Non Torus Knots

$$V_{R}^{U(N)}[41; 0] = \sum_{s,t,s'} \epsilon_{s}^{R,R} \sqrt{\text{dim}_{q} s} \epsilon_{s'}^{R,R} \sqrt{\text{dim}_{q}s'} (\lambda_{s'}^{(-)}(R, R))^{2} a_{ts} \begin{bmatrix} R & R & \bar{R} \\ \bar{R} & \bar{R} & R \end{bmatrix} (\lambda_{s}^{(-)}(R, R))^{-1} a_{ts'} (\lambda_{s'}^{(+)}(R, R))^{-1}. \quad (A.1)$$

$$V_{R}^{U(N)}[52; 0] = q^{(-5s + \frac{5}{2})} \sum_{s,t,s'} \epsilon_{s}^{R,R} \sqrt{\text{dim}_{q} s} \epsilon_{s'}^{R,R} \sqrt{\text{dim}_{q}s'} (\lambda_{s}^{(-)}(R, R))^{-2} a_{ts} \begin{bmatrix} \bar{R} & R & R \\ R & \bar{R} & \bar{R} \end{bmatrix} (\lambda_{s'}^{(+)}(R, R))^{-1}. \quad (A.2)$$

$$V_{R}^{U(N)}[61; 0] = q^{(-5s + \frac{5}{2})} \sum_{s,t,s'} \epsilon_{s}^{R,R} \sqrt{\text{dim}_{q} s} \epsilon_{s'}^{R,R} \sqrt{\text{dim}_{q}s'} (\lambda_{s}^{(-)}(R, R))^{2} a_{ts} \begin{bmatrix} R & R & \bar{R} \\ \bar{R} & \bar{R} & R \end{bmatrix} (\lambda_{s'}^{(+)}(R, R))^{-1}. \quad (A.3)$$

$$V_{R}^{U(N)}[62; 0] = q^{(-5s + \frac{5}{2})} \sum_{s,t,s',u,v} \epsilon_{s}^{R,R} \sqrt{\text{dim}_{q} s} \epsilon_{s'}^{R,R} \sqrt{\text{dim}_{q}s'} \lambda_{s}^{(+)}(R, R) a_{ts} \begin{bmatrix} \bar{R} & R & R \\ R & \bar{R} & \bar{R} \end{bmatrix} \lambda_{s'}^{(-)}(R, \bar{R}) a_{ts'} \begin{bmatrix} \bar{R} & R & R \\ R & \bar{R} & \bar{R} \end{bmatrix} \lambda_{u}^{(-)}(R, \bar{R}) a_{uv} \begin{bmatrix} \bar{R} & R & R \\ R & \bar{R} & \bar{R} \end{bmatrix} (\lambda_{u}^{(-)}(R, \bar{R}))^{-1}. \quad (A.4)$$

$$V_{R}^{U(N)}[63; 0] = \sum_{s,t,s',u,v} \epsilon_{s}^{R,R} \sqrt{\text{dim}_{q} s} \epsilon_{s'}^{R,R} \sqrt{\text{dim}_{q}s'} \lambda_{s}^{(+)}(R, R) a_{ts} \begin{bmatrix} \bar{R} & R & R \\ R & \bar{R} & \bar{R} \end{bmatrix} (\lambda_{s'}^{(-)}(R, R))^{-1} a_{ts'} \begin{bmatrix} \bar{R} & R & R \\ R & \bar{R} & \bar{R} \end{bmatrix} \lambda_{u}^{(-)}(R, R) a_{uv} \begin{bmatrix} \bar{R} & R & R \\ R & \bar{R} & \bar{R} \end{bmatrix} (\lambda_{u}^{(-)}(R, R)). \quad (A.5)$$
\[
V^{(U(N))}_R [T_2; 0] = q^{(7\kappa_R - \frac{77}{2})} \sum_{s, t, s'} \epsilon^r_s \sqrt{\dim q} \epsilon^r_s \sqrt{\dim q} (\lambda^{(s)}(R, R))^{-2} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix}
\]
\[
(\lambda^{(-)}(R, \tilde{R}))^{-4} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix} (\lambda^{(s)}(R, R))^{-1}.
\] (A.6)

\[
V^{(U(N))}_R [T_3; 0] = q^{(7\kappa_R - \frac{77}{2})} \sum_{s, t, s'} \epsilon^r_s \sqrt{\dim q} \epsilon^r_s \sqrt{\dim q} (\lambda^{(s)}(R, R))^{3} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix}
\]
\[
(\lambda^{(+)}(R, \tilde{R}))^{3} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix} \lambda^{(-)}(R, R).
\] (A.7)

\[
V^{(U(N))}_R [T_4; 0] = q^{(7\kappa_R - \frac{77}{2})} \sum_{s, t, s', u, v} \epsilon^r_s \sqrt{\dim q} \epsilon^r_u \sqrt{\dim q} \lambda^{(+)}(R, R) a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix}
\]
\[
(\lambda^{(-)}(R, \tilde{R}))^{2} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix} (\lambda^{(s)}(\tilde{R}, R))^{-2} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix}
\]
\[
(\lambda^{(+)}(R, R))^{2} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix} \lambda^{(-)}(R, R).
\] (A.8)

\[
V^{(U(N))}_R [T_5; 0] = q^{(7\kappa_R - \frac{77}{2})} \sum_{s, t, s', u, v} \epsilon^r_s \sqrt{\dim q} \epsilon^r_u \sqrt{\dim q} (\lambda^{(s)}(R, R))^{-1} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix}
\]
\[
(\lambda^{(+)}(R, R))^{-1} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix} (\lambda^{(-)}(\tilde{R}, R))^{-2} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix}
\]
\[
(\lambda^{(+)}(R, R))^{-2} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix} (\lambda^{(-)}(\tilde{R}, R))^{-1}.
\] (A.9)

\[
V^{(U(N))}_R [T_6; 0] = q^{(7\kappa_R - \frac{77}{2})} \sum_{s, t, s', u, v} \epsilon^r_s \sqrt{\dim q} \epsilon^r_u \sqrt{\dim q} (\lambda^{(s)}(R, R))^{-2} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix}
\]
\[
(\lambda^{(-)}(R, \tilde{R}))^{2} a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix} (\lambda^{(s)}(\tilde{R}, R))^{-1} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix}
\]
\[
(\lambda^{(+)}(R, R))^{-1} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix} (\lambda^{(-)}(\tilde{R}, R))^{-1}.
\] (A.10)

\[
V^{(U(N))}_R [T_7; 0] = q^{(7\kappa_R - \frac{77}{2})} \sum_{s, t, s', u, v, w, x} \epsilon^r_s \sqrt{\dim q} \epsilon^r_u \sqrt{\dim q} (\lambda^{(s)}(R, R)) a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix}
\]
\[
(\lambda^{(-)}(R, \tilde{R})) a_{ts'} \begin{bmatrix} \tilde{R}, \tilde{R}, \tilde{R} \end{bmatrix} (\lambda^{(s)}(\tilde{R}, R))^{-1} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix}
\]
\[
(\lambda^{(+)}(R, \tilde{R}))^{-1} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix} (\lambda^{(-)}(\tilde{R}, R))^{-1} a_{uv} \begin{bmatrix} R, R, R \end{bmatrix}
\]
\[
\lambda^{(-)}(R, \tilde{R}) a_{uv} \begin{bmatrix} R, R, R \end{bmatrix} (\lambda^{(s)}(R, R)).
\] (A.11)
The invariants are hence called multicolored links.

Appendix A.2. Non Torus Links

In the context of links, we can place different representations on the component knots.

For framed knots $K$ with framing number $p$, the invariants will be related to the zero-framed knot invariants as

$$V_R^{[(U(N))]}[K; p] = q^{p \cdot \kappa} V_R^{[(U(N))]}[K; 0].$$ (A.15)

Appendix A.2. Non Torus Links

In the context of links, we can place different representations on the component knots. The invariants are hence called multicolored links.

$$V_{(R_1, R_2)}^{[(U(N))]}[0; 0] = q^{- 4 \cdot \kappa_+ \cdot \kappa_0} \sum_{s, t, s'} \varepsilon_s^{R_1, R_2} \sqrt{\dim_q s} \varepsilon_{s'}^{R_1, R_2} \sqrt{\dim_q s'} (\lambda_s^+(R_1, R_2))^2 a_{ts'} \begin{bmatrix} R_1 & R_2 & \bar{R}_1 \\ R_2 & \bar{R}_2 & \bar{R}_1 \end{bmatrix}$$ (A.12)

$$V_{(R_1, R_2)}^{[(U(N))]}[0; 0] = q^{- 7 \cdot \kappa_1 \cdot \kappa_0 + \frac{27}{2} \kappa_0} \sum_{s, t, s'} \varepsilon_s^{R_1, R_2} \sqrt{\dim_q s} \varepsilon_{s'}^{R_1, R_2} \sqrt{\dim_q s'} (\lambda_s^+(R, R))^2 a_{ts'} \begin{bmatrix} R & R & \bar{R} \\ R & \bar{R} & \bar{R} \end{bmatrix}$$ (A.13)

$$V_{(R_1, R_2)}^{[(U(N))]}[10; 0] = q^{- 6 \cdot \kappa_1 \cdot \kappa_0 + \frac{27}{2} \kappa_0} \sum_{s, t, s'} \varepsilon_s^{R_1, R_2} \sqrt{\dim_q s} \varepsilon_{s'}^{R_1, R_2} \sqrt{\dim_q s'} (\lambda_s^+(R, R))^2 a_{ts'} \begin{bmatrix} R & \bar{R} & \bar{R} \\ R & \bar{R} & \bar{R} \end{bmatrix}$$ (A.14)

$$V_{(R_1, R_2)}^{[(U(N))]}[0; 0] = q^{2 \cdot \kappa_1 \cdot \kappa_0 + \frac{27}{4} \kappa_1 \cdot \kappa_0} \sum_{s, t, s', \mu, v} \varepsilon_s^{R_1, R_2} \varepsilon_\mu \varepsilon_v \sqrt{\dim_q s} \varepsilon_\mu \varepsilon_v \sqrt{\dim_q s'} (\lambda_s^+(R_1, R_2))^3 a_{ts'} \begin{bmatrix} R_1 & R_1 & \bar{R}_1 \\ R_2 & R_2 & \bar{R}_2 \end{bmatrix}$$ (A.16)

$$V_{(R_1, R_2)}^{[(U(N))]}[0; 0] = q^{- 2 \cdot \kappa_1 \cdot \kappa_0 + \frac{27}{4} \kappa_1 \cdot \kappa_0} \sum_{s, t, s', \mu, v} \varepsilon_s^{R_1, R_2} \varepsilon_\mu \varepsilon_v \sqrt{\dim_q s} \varepsilon_\mu \varepsilon_v \sqrt{\dim_q s'} (\lambda_s^+(R_1, R_2))^3 a_{ts'} \begin{bmatrix} R_1 & R_1 & \bar{R}_1 \\ R_2 & R_2 & \bar{R}_2 \end{bmatrix}$$ (A.17)

$$V_{(R_1, R_2)}^{[(U(N))]}[0; 0] = q^{- 2 \cdot \kappa_0 \cdot \kappa_0 + \frac{27}{4} \kappa_0 \cdot \kappa_0} \sum_{s, t, s', \mu, v} \varepsilon_s^{R_1, R_2} \varepsilon_\mu \varepsilon_v \sqrt{\dim_q s} \varepsilon_\mu \varepsilon_v \sqrt{\dim_q s'} (\lambda_s^+(R_1, R_2))^3 a_{ts'} \begin{bmatrix} R_1 & R_1 & \bar{R}_1 \\ R_2 & R_2 & \bar{R}_2 \end{bmatrix}$$ (A.18)
\[ V_{(R_1, \bar{R}_2)}^{\{U(N)\}}[\bar{r}_2; 0, 0] = q^{-\kappa_2} \left( \frac{\bar{r}_2}{\bar{r}_2} \right) \sum_{s.t.s',u,v} \epsilon_{s.t.s',u,v}^{R_1,R_2} \sqrt{\text{dim}_q} \epsilon_{R_1,R_2}^{R_1} \sqrt{\text{dim}_q} (\lambda_u^{(+)}(R_1, \bar{R}_2))^{-2} \]

\[ a_s \begin{bmatrix} \bar{R}_1 & R_1 \\ \bar{R}_2 & R_2 \end{bmatrix} \lambda_1^{(-)}(R_2, \bar{R}_2) \lambda_u^{(+)}(R_1, \bar{R}_2) \]

\[ a_{u.s'} \begin{bmatrix} \bar{R}_1 & R_1 \\ \bar{R}_2 & R_2 \end{bmatrix} \lambda_0^{(+)}(R_1, \bar{R}_2)^2 a_{uv} \begin{bmatrix} \bar{R}_2 & R_1 \\ \bar{R}_1 & R_1 \end{bmatrix} \lambda_v^{(-)}(\bar{R}_1, R_2). \]

(A.19)

\[ V_{(R_1, \bar{R}_2)}^{\{U(N)\}}[\bar{r}_3; 0, 0] = q^{-3\kappa_2} \sum_{s.t.s',u,v} \epsilon_{s.t.s',u,v}^{R_1,R_2} \sqrt{\text{dim}_q} \epsilon_{R_1,R_2}^{R_1} \sqrt{\text{dim}_q} \lambda_1^{(-)}(R_1, \bar{R}_2) \lambda_0^{(+)}(R_1, \bar{R}_2)^2 a_{uv} \begin{bmatrix} \bar{R}_2 & R_1 \\ \bar{R}_1 & R_1 \end{bmatrix} \lambda_v^{(-)}(\bar{R}_1, R_2). \]

(A.20)

Including the framing numbers \( p_1, p_2 \) on the component knots of these two-component torus links \( \mathcal{L} \), the framed multicolored invariant will be

\[ V_{(R_1, R_2)}^{\{U(N)\}}[\mathcal{L}; p_1, p_2] = q^{p_1\kappa_1 + p_2\kappa_2} V_{(R_1, R_2)}^{\{U(N)\}}[\mathcal{L}; 0, 0]. \]

(A.21)

**Appendix B. Knot Polynomials**

In this appendix we present the polynomial form of the \( U(N) \) link invariant for non torus knots in Figure 1 for representation whose Young tableau diagrams are \( \Box \) and \( \Box \). The polynomial corresponding to representation \( \Box \) is proportional to HOMFLY-PT polynomial \( P(\lambda, t)[K] \) \([19, 20]\) upto unknot \( U \) normalisation:

\[ P(\lambda, q)[K] = \frac{V_{\Box}^{\{U(N)\}}[K; 0]}{V_{\Box}^{\{U(N)\}}[U]} = \frac{(q^{1/2} - q^{-1/2})}{(\lambda^{1/2} - \lambda^{-1/2})} V_{\Box}^{\{U(N)\}}[K; 0]. \]

We list them so that we can directly use them in the computation of reformulated invariants in Appendix D.

1. For fundamental representation \( R = \Box \) placed on the knot, the \( U(N) \) knot polynomials are

\[ V_{\Box}^{\{U(N)\}}[\lambda_1] = \frac{(-1)}{\sqrt{(q-1)^2}} \left[ -\lambda - \lambda q^2 + (\lambda^2 + \lambda + 1) q \right] \]

\[ V_{\Box}^{\{U(N)\}}[\lambda_2] = \frac{1}{\sqrt{1+q}} \left[ q - q\lambda^3 + \lambda(-1 + \lambda^2) + q^2\lambda(-1 + \lambda^2) \right] \]

\[ V_{\Box}^{\{U(N)\}}[\lambda_3] = \frac{1}{\sqrt{(q-1)^2}} \left[ -\lambda^3 + \lambda + (\lambda - \lambda^3)q^2 + (\lambda^4 + \lambda^3 - \lambda^2 - 1)q \right] \]
\[ V^{L(N)}_{62} = \frac{(-1+\lambda)}{(1+\lambda)q^{7/2}q^{\lambda^2}} [-\lambda - q^4 \lambda - q^3(1 + 2\lambda) + q(1 + \lambda + \lambda^2) + q^3(1 + \lambda + \lambda^2)] \]

\[ V^{L(N)}_{63} = \frac{(-1+\lambda)}{(1+\lambda)q^{1/2}q^{\lambda^2}} [-\lambda - q^4 \lambda + q(1 + \lambda + \lambda^2) + q^3(1 + \lambda + \lambda^2) - q^2(1 + 3\lambda + \lambda^2)] \]

\[ V^{L(N)}_{72} = \frac{1}{(1+\lambda)\sqrt{q}q^{\lambda^2}} \left[ \lambda(1 + \lambda^3) + q^2\lambda(1 + \lambda^3) - q(-1 - \lambda^2 + \lambda^3 + \lambda^4) \right] \]

\[ V^{L(N)}_{73} = \frac{\lambda^3}{(1+\lambda)\sqrt{q}q^{\lambda^2}} \left[ -1 + q^2 - q^3\lambda + q^4(-1 + 1\lambda^2) - q^3(1 - 1\lambda - 2\lambda^2) - q^2(-1 + 1\lambda^3) \right] \]

\[ V^{L(N)}_{74} = \frac{(-1+\lambda)\lambda^{1/2}/2}{(1+\lambda)q^{1/2}q^{\lambda^2}} [(1 + \lambda)^2 + q^2(1 + \lambda)^2 - q(2 + 2\lambda + 2\lambda^2 + \lambda^3)] \]

\[ V^{L(N)}_{75} = \frac{(-1+\lambda)}{(1+\lambda)q^{1/2}q^{\lambda^2}} \left[ \lambda(1 + \lambda) + q^4\lambda(1 + \lambda) - q(1 + \lambda)^2 - q^3(1 + \lambda^2) + q^2(1 + 2\lambda + 2\lambda^2 + \lambda^3) \right] \]

\[ V^{L(N)}_{76} = \frac{-1+\lambda}{(-1+\lambda)q^{1/2}q^{\lambda^2}} \left[ \lambda^2 + q^2\lambda^2 - q\lambda(2 + 2\lambda + \lambda^2) - q^3\lambda(2 + 2\lambda + \lambda^2) + q^2(2 + 4\lambda + 2\lambda^2 + \lambda^3) \right] \]

\[ V^{L(N)}_{77} = \frac{(-1+\lambda)}{(1+\lambda)q^{1/2}q^{\lambda^2}} \left[ \lambda + q^4\lambda - q(1 + 2\lambda + 2\lambda^2) - q^4(1 + 2\lambda + 2\lambda^2) + q^2(2 + 4\lambda + 2\lambda^2 + \lambda^3) \right] \]

\[ V^{L(N)}_{81} = \frac{1}{(q-1)\sqrt{q}q^{\lambda^2}} \left[ -\lambda^4 + \lambda + q^2(\lambda - \lambda^4) + q(\lambda^5 + \lambda^4 - \lambda^2 - 1) \right] \]

\[ V^{L(N)}_{91} = \frac{1}{(q-1)\sqrt{q}q^{\lambda^2}} \left[ \lambda(\lambda^4 - 1)q^2 - (\lambda^5 + \lambda^4 - \lambda^2 - 1)q + \lambda(\lambda^4 - 1) \right] \]

\[ V^{L(N)}_{101} = \frac{1}{(q-1)\sqrt{q}q^{\lambda^2}} \left[ -\lambda^5 + \lambda + q^2(\lambda - \lambda^5) + q(\lambda^6 + \lambda^5 - \lambda^2 - 1) \right] \]

2. For symmetric second rank representation \( R = \square \), the knot polynomials are

\[ V^{L(N)}_{41} = \frac{(-1+\lambda)(-1+\lambda)}{(1+\lambda)q^{3/4}(1+q)^{3/2}q^{\lambda^2}} \left[ (-1 + \lambda)\lambda + 3q^2\lambda^2 - q^3(1 + \lambda)\lambda^2 + q^4\lambda(-1 + \lambda^2 + \lambda^3) - q^5\lambda^2(-1 - \lambda + \lambda^2) - q(1 - \lambda^2 + \lambda^3) + q^2(2 + \lambda - \lambda^2) \right] \]

\[ V^{L(N)}_{52} = \frac{1}{(-1+\lambda)\sqrt{q}q^{1/2}(1+q)^{3/2}q^{\lambda^2}} \left[ q(-1 + \lambda)(-1 + \lambda)\lambda + q^2\lambda^2(-1 + \lambda^2 + \lambda^3) - q^4\lambda^2(-1 - \lambda + \lambda^2 + \lambda^3) - q^5\lambda^2(-1 - \lambda + \lambda^2 + \lambda^3) + q^6(-1 + \lambda + \lambda^2 + \lambda^3) \right] \]

\[ V^{L(N)}_{61} = \frac{(-1+\lambda)(-1+\lambda)}{(1+\lambda)q^{3/4}(1+q)^{3/2}q^{\lambda^2}} \left[ (-1 + \lambda)\lambda + q(-1 + \lambda)(-1 + \lambda)^2(1 + \lambda) + q^2(-1 + \lambda)^2\lambda(1 + \lambda) + q^3\lambda^2(-3 + \lambda^2) + q^4(-1 + \lambda^2)(-1 + \lambda + \lambda^2) + q^5\lambda^2(-1 + 2\lambda + 4\lambda^2) - q^4\lambda(1 - \lambda - 3\lambda^2 + 2\lambda^3 + \lambda^4) + q^6(\lambda^3 - \lambda^5) \right] \]
\[
V_{\text{N}}^{g_2} = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)\varphi(1+q\lambda)} [g + (-1 + \lambda)\lambda - q\lambda^2 - q^3(-4 + \lambda)\lambda^2 - q^2(-1 + \lambda)\lambda^2 - q^a(1 + \lambda)^2(1 + \lambda) + q^b(1 + \lambda)^2(-3 + 2\lambda) - 2q^c(-1 + \lambda^2) + q^d(1 - 3\lambda - 3\lambda^2 + \lambda^3) + q^e(1 - 2\lambda - 3\lambda^2 + \lambda^4) + q^f(1 + 3\lambda + 2\lambda^2 - 2\lambda^3 + \lambda^4)]
\]

\[
V_{\text{N}}^{g_3} = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)\varphi(1+q\lambda)^2} [(-1 + \lambda)\lambda + q^a(-1 + \lambda)^2\lambda^2 + q^b(1 + \lambda - 4\lambda^2) + q^c(-1 + \lambda^2) + q^d(\lambda - 2\lambda^2 + \lambda^3) + q^e(1 - 2\lambda - \lambda^2 + \lambda^3) + q^f(1 - 3\lambda - 3\lambda^2 + 2\lambda^3) + q^g(\lambda - \lambda^2 + 4\lambda^4)]
\]

\[
V_{\text{N}}^{g_7} = \frac{1}{(-1+q)^2\varphi(1+q\lambda)^2} [g(-1 + \lambda)\lambda - q^a(-1 + \lambda)\lambda^2 + q^b(1 - 3\lambda^2 + 2\lambda^3) + q^c(1 - 2\lambda + \lambda^2 + \lambda^3) + q^d(1 + \lambda - 4\lambda^2 + \lambda^3) + q^e(1 - 2\lambda - \lambda^2 + \lambda^3) + q^f(1 - 3\lambda - 3\lambda^2 + 2\lambda^3) + q^g(\lambda - \lambda^2 + 4\lambda^4)]
\]

\[
V_{\text{N}}^{g_9} = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+q)\varphi(1+q\lambda)^2} \left[ \lambda + q(-1 + \lambda) - q^a(-1 + \lambda)^2\lambda^2 + q^b(1 + \lambda - 4\lambda^2) + q^c(-2 - \lambda + 3\lambda^2) + q^d(1 - 2\lambda^2 + 2\lambda^3) + q^e(1 - 3\lambda - 2\lambda^2 + \lambda^3) + q^f(2 - 2\lambda - \lambda^2 + 3\lambda^3) + q^g(1 - 3\lambda^2 - 2\lambda^3 + \lambda^4) \right]
\]

\[
V_{\text{N}}^{g_4} = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+\varphi)(1+q\lambda)^2} \left[ -q^a(1 + \lambda)\lambda^2 + 2q(-1 + \lambda^2) + q^b(-1 - 6\lambda - \lambda^2 + \lambda^3) + q^c\lambda(2 - 2\lambda - \lambda^2 + 3\lambda^3) - q^d(-4 - 2\lambda + 5\lambda^2 + \lambda^3) + q^e(3 - 2\lambda - 3\lambda^2 + 2\lambda^3) + q^f(-2 - 4\lambda + 6\lambda^2 + 3\lambda^3 - 3\lambda^4) + q^g(2 - 2\lambda - 4\lambda^2 + 3\lambda^3 + \lambda^4) - q^h(1 - 6\lambda - 4\lambda^2 + 4\lambda^3 + \lambda^4) + q^i(1 - 2\lambda - 4\lambda^2 + 5\lambda^3 + \lambda^4 - \lambda^5) \right]
\]

\[
V_{\text{N}}^{g_5} = \frac{(-1+\lambda)(-1+q\lambda)}{(-1+\varphi)(1+q\lambda)^2} \left[ (-1 + \lambda)\lambda - q^a(-1 + \lambda)^3 + q^b(1 + \lambda)^3 + q^c(1 + \lambda)\lambda^2 + q^d\lambda(3 - 4\lambda - \lambda^3) + q^e(1 - 2\lambda^2 + \lambda^3) + q^f(3 - 5\lambda - 5\lambda^2 + 3\lambda^3) + q^g(5 + 5\lambda^2 - 3\lambda^3 - 3\lambda^4) + q^h(2 - 2\lambda - 5\lambda^2 + 3\lambda^3 + \lambda^4) + q^i(1 - 3\lambda - 4\lambda^2 + 6\lambda^3 + \lambda^4 - \lambda^5) \right]
\]
\[ V_{U(1)}^{(N)}[7_6] = \frac{(-1 + \lambda)(-1 + q \lambda)}{(-1 + q)^2 q (1 + q) \lambda^4} \left[ (-1 + \lambda)^2 \lambda^2 - q^{12}(-1 + \lambda)\lambda^2 \right. \\
\left. + q^{11} \lambda^2 (-2 + \lambda^2) + q\lambda (-2 + 3\lambda + \lambda^2 - 2\lambda^3) - q^{10} \lambda^3 (2 - 3\lambda^2 + \lambda^3) - q^9 \lambda^3 (-1 - 7\lambda + 2\lambda^2 + \lambda^3) + q^7 \lambda^2 (3 - 7\lambda - 7\lambda^2 + 4\lambda^3) + q^6 \lambda^2 (-4 - 6\lambda + 10\lambda^2 + \lambda^3 - \lambda^4) + q^4 \lambda (-1 + 8\lambda - \lambda^2 - 7\lambda^3 + \lambda^4) + q^3 \lambda (2 - 8\lambda^2 + 4\lambda^3 + \lambda^4) + q^2 \lambda^2 (1 + 5\lambda - 4\lambda^2 - 4\lambda^3 + 2\lambda^4) + \\
q^2 \lambda (-2 - 2\lambda + 10\lambda^2 + 2\lambda^3 - 4\lambda^4 + \lambda^5) - q^2 (-1 + \lambda + 4\lambda^2 - 3\lambda^3 - 2\lambda^4 + \lambda^5) \right] \]

\[ V_{U(1)}^{(N)}[7_7] = \frac{(-1 + q \lambda)}{(-1 + q)^2 q (1 + q) \lambda^2 \lambda^4} \left[ (-1 + \lambda)^2 \lambda + q^{12}(-1 + \lambda)^3 \lambda^2 - q(-1 + \lambda)^2 (1 + 2\lambda) - \\
2q^{11}(-1 + \lambda)^2 \lambda^2 (-1 - \lambda + \lambda^2) + q^3 (-1 + \lambda)^2 (1 + 7\lambda + 2\lambda^2) + q^2 (-1 + \lambda)^2 (1 - 9\lambda - 8\lambda^2 + 2\lambda^3) + q^2 (-1 + \lambda)^2 (1 - \lambda + 8\lambda - \lambda^2 + 2\lambda^3) + q^2 (2 - 5\lambda + \lambda^2 + 4\lambda^3 - 2\lambda^4) - q^2 (-1 + \lambda)^2 (1 - 5\lambda - 11\lambda^2 + 2\lambda^4) + q^8 \lambda (-3 + 11\lambda - 16\lambda^2 + 8\lambda^3) + q^6 (2 - 18\lambda^2 + 15\lambda^3 + 5\lambda^4 - 4\lambda^5) + q^4 (-4 + 6\lambda + 8\lambda^2 - 12\lambda^3 + \lambda^4 + \lambda^5) + q^10 \lambda (1 - 2\lambda - 4\lambda^2 + 8\lambda^3 - \lambda^4 - 3\lambda^5 + \lambda^6) \right] \]

\[ V_{U(1)}^{(N)}[8_1] = \frac{1}{(-1 + q)^2 q (1 + q) \lambda^2 \lambda^4} \left[ q(-1 + \lambda)^2 \lambda - q^2 (-1 + \lambda)^2 \lambda^2 + q^3 (-1 + \lambda)^2 \lambda^2 + q^5 \lambda (-1 + \lambda)^2 + q^6 \lambda^2 (-1 + \lambda + \lambda^2 - \lambda^3 + \lambda^4) + q^7 \lambda^2 (1 - \lambda - 2\lambda^2 + \lambda^3) + q^8 \lambda^2 (1 - 1 + \lambda + \lambda^2 - \lambda^3 - \lambda^4 - \lambda^5 + \lambda^6) + q^9 (\lambda^4 + \lambda^6 - \lambda^8 - \lambda^9) \right] \]

\[ V_{U(1)}^{(N)}[9_2] = \frac{1}{(-1 + q)^2 q (1 + q) \lambda^2 \lambda^4} \left[ q(-1 + \lambda)^2 \lambda - q^2 (-1 + \lambda)^2 \lambda + q^3 (-1 + \lambda)^2 \lambda^2 + q^4 (-1 + \lambda)^2 \lambda^2 + q^5 \lambda (-1 + \lambda + \lambda^2 + \lambda^3) + q^6 \lambda^2 (1 + 8\lambda + 16\lambda^2 + 3\lambda^3) + q^7 \lambda^2 (1 - \lambda - 2\lambda^2 + \lambda^3 - \lambda^4 + \lambda^5) - q^8 \lambda^4 (1 - 2\lambda + 2\lambda^2 - \lambda^3 - \lambda^4 + \lambda^5) + q^9 (\lambda^5 + \lambda^6 - \lambda^7 - \lambda^8 - \lambda^9) \right] \]

\[ V_{U(1)}^{(N)}[10_2] = \frac{1}{(-1 + q)^2 q (1 + q) \lambda^2 \lambda^4} \left[ q(-1 + \lambda)^2 \lambda - q^2 (-1 + \lambda)^2 \lambda + q^3 (-1 + \lambda)^2 \lambda^2 + q^4 \lambda (-1 + \lambda)^2 + q^5 \lambda^2 (-1 + \lambda + \lambda^2 + \lambda^3) + q^6 \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^7 \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^8 \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^9 \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^{10} \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^{11} \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^{12} \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^{13} \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^{14} \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) + q^{15} \lambda^2 (1 + 1 + \lambda + \lambda^2 + \lambda^3) \right] \]

There seems to be a symmetry transformation on the polynomial variables which gives the \( U(1) \) invariants of knots carrying antisymmetric second rank tensor representation \( R = \mathbb{R}^2 \). The symmetry relation for these non-torus knots (see also eqn.(7) in [21])

\[ V_{U(1)}^{(N)}[\mathbb{R}^2 q^{-1}, \lambda] = V_{U(1)}^{(N)}[\mathbb{R}, \lambda]. \quad (B.2) \]
We checked our polynomials for knots $6_2, 6_3, 7_2, 7_3$ with the results obtained in Ref.[22] using character expansion approach. Before we use these polynomial invariants in verifying Ooguri-Vafa conjecture, we shall enumerate the multi-colored link polynomials in the following appendix.

Appendix C. Link Polynomials

In this appendix we list the $U(N)$ link invariant for non torus links given in Figure 2 for representations $R_1, R_2 \in \{\square \square \square \}$. 

1. For $R_1 = \square, R_2 = \square:$

$$V^{U(N)}_{\square \square \square}[6_2] = \frac{(-1 + \lambda)}{(-1 + \lambda^q)q^{\lambda^q}} [\lambda(-1 + \lambda^2) + q^4 \lambda(-1 + \lambda^2) + q(1 + \lambda - 2\lambda^3) + q^3(1 + \lambda - 2\lambda^3) + q^2(-1 - 2\lambda + \lambda^2 + 2\lambda^3)]$$

$$V^{U(N)}_{\square \square \square}[6_3] = \frac{(-1 + \lambda)}{(-1 + \lambda^q)q^{\lambda^q}} [\lambda + q^4 \lambda - q(1 + 3\lambda + 2\lambda^2) - q^3(1 + 3\lambda + 2\lambda^2) + q^2(2 + 4\lambda + 3\lambda^2 + \lambda^3)]$$

$$V^{U(N)}_{\square \square \square}[7_1] = \frac{(-1 + \lambda)}{(-1 + \lambda^q)q^{\lambda^q}} [\lambda - q^6 \lambda + q(1 + \lambda)^2 + q^5(1 + \lambda)^2 - q^2(2 + 3\lambda + \lambda^2) + q^3(2 + 3\lambda + \lambda^2) - q^4(2 + 3\lambda + \lambda^2)]$$

$$V^{U(N)}_{\square \square \square}[7_2] = \frac{(-1 + \lambda)}{(-1 + \lambda^q)q^{\lambda^q}} [-\lambda - q^4 \lambda + q(1 + \lambda)^2 - 2q^2(1 + \lambda)^2 - 2q^4(1 + \lambda)^2 + q^5(1 + \lambda)^2 + q^6(2 + 5\lambda + 3\lambda^2)]$$

$$V^{U(N)}_{\square \square \square}[7_3] = \frac{(-1 + \lambda)}{(-1 + \lambda^q)q^{\lambda^q}} [-\lambda + q^3(1 + \lambda) + q(1 + \lambda)^3 + q^3(1 + \lambda)^3 - q^2(2 + 4\lambda + 5\lambda^2 + \lambda^3)]$$

2. For $R_1 = \square, R_2 = \square \square$:

$$V^{U(N)}_{\square \square \square}[6_2] = \frac{(-1 + \lambda)}{(-1 + \lambda^q)q^{\lambda^q} \lambda^q} [g - \lambda - q^7 \lambda^2 + q^8 \lambda^3 + q^2(1 + \lambda + \lambda^2) - q^6 \lambda(1 + \lambda + \lambda^2) + q^5 \lambda(1 + 2\lambda + 2\lambda^2) - q^4(-1 + \lambda^3) - q^3(1 + 2\lambda + 2\lambda^2 + \lambda^3)]$$

$$V^{U(N)}_{\square \square \square}[6_3] = \frac{(-1 + \lambda)}{(-1 + \lambda^q)q^{\lambda^q} \lambda^q} [-\lambda + q^7 \lambda^2 - q^2(1 + \lambda) + q^5 \lambda^3(1 + \lambda) + q(1 + \lambda)^2 - q^6 \lambda(1 + \lambda + 2\lambda^2) + q^4(1 + 3\lambda + 2\lambda^2 + \lambda^3) - q^3(1 + 2\lambda + 2\lambda^2 + 2\lambda^3)]$$

$$V^{U(N)}_{\square \square \square}[7_1] = \frac{(-1 + \lambda)}{(-1 + \lambda^q)q^{\lambda^q} \lambda^q} [q + q^2(-1 + \lambda) + (-1 + \lambda)\lambda - q^10(-1 + \lambda)\lambda^2 - q\lambda^3 - q^8(-1 + \lambda)\lambda^3 + q^7(1 + \lambda - 2\lambda^3) - 2q^4(-1 + \lambda^3) + q^9\lambda(-1 + \lambda^3) + q^7(-1 + \lambda + \lambda^3 - \lambda^4) + q^9(-1 - \lambda + \lambda^2 + \lambda^4) + q^6(-1 - \lambda + \lambda^3 + \lambda^4)]$$
For presenation and replacing the second rank symmetric representation \( R \) in the two components of the link, gives the same polynomial:

\[
V^{U(N)}_{\text{[\square\square]\square}}(q, \lambda) = V^{U(N)}_{\text{[\square\square]\square}}(q, \lambda) = \lambda^3 - q - \lambda^2 + \lambda + 1.
\]

(C.1)

and replacing the second rank symmetric representation \( \boxdot \) by antisymmetric reprenteation \( \boxdot \) the link polynomials are related as follows:

\[
V^{U(N)}_{\text{[\square\square]\square}}(q^{-1}, \lambda) = -V^{U(N)}_{\text{[\square\square]\square}}(q, \lambda)
\]

(C.2)

3. For \( R_1 = \text{[\square\square]} \) \( R_2 = \text{[\square\square]} \):

\[
V^{U(N)}_{\text{[\square\square]\square}}[6_2] = \left(1 + \frac{1}{1+q}\lambda\right) \frac{1}{1+q} \left[ - q^2(1 + \lambda)^2 - (1 + \lambda)^2 + q(1 + \lambda)^2(1 + \lambda) + q^{15} \lambda^4(-1 + \lambda^2) - q^{13} \lambda^4(-2 + \lambda^2 + \lambda^2) + q^3(1 + \lambda)^2 \lambda(-2 + \lambda + \lambda^2 + \lambda^3) + q^{12} \lambda^3(-2 - \lambda - 2 \lambda^2 + 5 \lambda^3) + q^{12} \lambda^3(-1 + \lambda - 2 \lambda^2 + 4 \lambda^3 - 2 \lambda^4) + q^{10} \lambda(1 + \lambda) + 2 \lambda + \lambda^2 + 3 \lambda^3 - 5 \lambda^5) + q^8 \lambda(1 - 4 \lambda + \lambda^2 - 3 \lambda^3 + 4 \lambda^4 + \lambda^5) + q^5(-1 + \lambda - 4 \lambda^2 + 2 \lambda^5) + q^5 \lambda(1 + 2 \lambda - 3 \lambda^2 - 5 \lambda^4 + 5 \lambda^5) \right] + q^7(1 - 4 \lambda + \lambda^2 + 2 \lambda^3 + 3 \lambda^4 + \lambda^5 - 4 \lambda^6) + q^4(\lambda^3 + \lambda^5 - 2 \lambda^6) + 2q^4(1 - 2 \lambda + \lambda^3 + \lambda^5 - \lambda^6) + q^6(-1 + \lambda + 5 \lambda^2 - 4 \lambda^3 + \lambda^4 - 5 \lambda^5 + 3 \lambda^6)
\]

\[
V^{U(N)}_{\text{[\square\square]\square}}[6_3] = \left(1 + \frac{1}{1+q}\lambda\right) \frac{1}{1+q} \left[ - q^2(1 + \lambda)^2 - (1 + \lambda)^2 + q(1 + \lambda)^2(1 + \lambda) + q^{12} \lambda^2(-1 + \lambda) + q^{12} \lambda^2(-2 + \lambda^2 + \lambda^2) + q^3(1 + \lambda)^2 \lambda(-2 + \lambda + \lambda^2 + \lambda^3) + q^{12} \lambda^3(-2 - \lambda - 2 \lambda^2 + 5 \lambda^3) + q^{11} \lambda^2(-1 + \lambda - 2 \lambda^2 + 4 \lambda^3 - 2 \lambda^4) + q^{10} \lambda(1 + \lambda) + 2 \lambda + \lambda^2 + 3 \lambda^3 - 5 \lambda^5) + q^8 \lambda(1 - 4 \lambda + \lambda^2 - 3 \lambda^3 + 4 \lambda^4 + \lambda^5) + q^5(-1 + \lambda - 4 \lambda^2 + 2 \lambda^5) + q^5 \lambda(1 + 2 \lambda - 3 \lambda^2 - 5 \lambda^4 + 5 \lambda^5) \right] + q^7(1 - 4 \lambda + \lambda^2 + 2 \lambda^3 + 3 \lambda^4 + \lambda^5 - 4 \lambda^6) + q^4(\lambda^3 + \lambda^5 - 2 \lambda^6) + 2q^4(1 - 2 \lambda + \lambda^3 + \lambda^5 - \lambda^6) + q^6(-1 + \lambda + 5 \lambda^2 - 4 \lambda^3 + \lambda^4 - 5 \lambda^5 + 3 \lambda^6)
\]
we find the following relation between the link polynomials:

\[
V_{(\otimes \otimes)}^{L(N)}[\tau_1] = \frac{(-1+\lambda)}{(1+q)q^2(1+q^2)^3} \left[ (-1+\lambda) \lambda - q^{12} \lambda^3(4-4\lambda-3\lambda^2+\lambda^3) + q(1+\lambda-3\lambda^2+\lambda^3) + q^{13} \lambda^3(1-3\lambda+\lambda^3) + q^2(-2+4\lambda-3\lambda^2+\lambda^3) - q^{11} \lambda^2(2-8\lambda-8\lambda^2+5\lambda^3+\lambda^4) - q^3(1+5\lambda-9\lambda^2+3\lambda^3) + q^9 \lambda(2+3\lambda-20\lambda^2-4\lambda^3+7\lambda^4) + q^{14} \lambda^3(-5-2\lambda-2\lambda^2-6\lambda^3+\lambda^4) + q^8(1+13\lambda-17\lambda^2-12\lambda^3+5\lambda^4) \right]
\]

\[
V_{(\otimes \otimes)}^{L(N)}[\tau_2] = \frac{(-1+\lambda)}{(1+q)q^2(1+q^2)^3} \left[ (-1+\lambda) \lambda - q^{18} (-1+\lambda)\lambda^2 + q(1+\lambda-2\lambda^2+\lambda^3) + q^{17} \lambda^2(-2+\lambda^2) + q^4 \lambda(-4+3\lambda+\lambda^2) + q^2(2+2\lambda+\lambda^2-\lambda^3) + q^7(5+5\lambda-7\lambda^2+\lambda^3) + q^3(3+2\lambda-5\lambda^2+\lambda^3) - q^{15} \lambda(-1-6\lambda+2\lambda^2+\lambda^3) + q^{14} \lambda(3-5\lambda-4\lambda^2+2\lambda^3) + q^{13}(1-5\lambda-7\lambda^2+5\lambda^3) + q^{12}(-2-3\lambda+11\lambda^2+2\lambda^3-2\lambda^4) + q^6(18A+3\lambda^2+\lambda^3-\lambda^4) + q^{11}(1+10\lambda+2\lambda^2-6\lambda^3+\lambda^4) - q^{16}(\lambda-3\lambda^3+\lambda^4) - q^9(2+11\lambda-5\lambda^2-3\lambda^3+\lambda^4) + q^5(-4+3\lambda+2\lambda^2-3\lambda^3+\lambda^4) + q^8(-5+7\lambda+6\lambda^2-3\lambda^3+\lambda^4) + q^{10}(5-\lambda-12\lambda^2+\lambda^3+\lambda^4) \right]
\]

\[
V_{(\otimes \otimes)}^{L(N)}[\tau_3] = \frac{(-1+\lambda)}{(1+q)q^2(1+q^2)^3} \left[ (-1+\lambda) \lambda - q^{18} (-1+\lambda)\lambda^2 + q(1+\lambda-2\lambda^2+\lambda^3) + q^{17} \lambda^2(-2+\lambda^2) + q^4 \lambda(-4+3\lambda+\lambda^2) + q^2(2+2\lambda+\lambda^2-\lambda^3) + q^7(5+5\lambda-7\lambda^2+\lambda^3) + q^3(3+2\lambda-5\lambda^2+\lambda^3) - q^{15} \lambda(-1-6\lambda+2\lambda^2+\lambda^3) + q^{14} \lambda(3-5\lambda-4\lambda^2+2\lambda^3) + q^{13}(1-5\lambda-7\lambda^2+5\lambda^3) + q^{12}(-2-3\lambda+11\lambda^2+2\lambda^3-2\lambda^4) + q^6(18A+3\lambda^2+\lambda^3-\lambda^4) + q^{11}(1+10\lambda+2\lambda^2-6\lambda^3+\lambda^4) - q^{16}(\lambda-3\lambda^3+\lambda^4) - q^9(2+11\lambda-5\lambda^2-3\lambda^3+\lambda^4) + q^5(-4+3\lambda+2\lambda^2-3\lambda^3+\lambda^4) + q^8(-5+7\lambda+6\lambda^2-3\lambda^3+\lambda^4) + q^{10}(5-\lambda-12\lambda^2+\lambda^3+\lambda^4) \right]
\]

Changing both the rank two symmetric representation \(\otimes\otimes\) by antisymmetric representation \(\otimes\bar{\otimes}\), we find the following relation between the link polynomials:

\[
V_{(\otimes \bar{\otimes})}^{L(N)}[\mathcal{L}](q, \lambda) = V_{(\otimes \otimes)}^{L(N)}[\mathcal{L}](q^{-1}, \lambda).
\]

With these polynomial invariants available for the non-torus knots and links in Figures 1, 2, we are in a position to verify Ooguri-Vafa [7] and Labastida-Marino-Vafa [8] conjectures.

**Appendix D. Reformulated link invariants**

In this appendix we explicitly write the reformulated link invariant for the non torus knots and links in Figure 1 and Figure 2. Rewriting the most general form of reformulated
The reformulated invariants in terms of two-component link invariants has the following form for \( R \in \{ \Box, \Diamond, \Box \Diamond, \Box \Box \} \):

\[
\begin{align*}
  f_{(\Box \Box)}[\mathcal{L}] &= V_{(\Box \Box)}[\mathcal{L}] - V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] \quad \text{(D.5)} \\
  f_{(\Box \Diamond)}[\mathcal{L}] &= V_{(\Box \Diamond)}[\mathcal{L}] - V_{(\Box \Diamond)}[\mathcal{L}]V_{(\Box \Diamond)}[\mathcal{L}] - V_{(\Box \Diamond)}[\mathcal{L}]V_{(\Box \Diamond)}[\mathcal{L}] \\
  &\quad + V_{(\Box \Diamond)}[\mathcal{L}]V_{(\Box \Diamond)}[\mathcal{L}]^2 \quad \text{(D.6)} \\
  f_{(\Box \Box)}[\mathcal{L}] &= V_{(\Box \Box)}[\mathcal{L}] - V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] - V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] \\
  &\quad + V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}]^2 \quad \text{(D.7)} \\
  f_{(\Box \Diamond)}[\mathcal{L}] &= V_{(\Box \Diamond)}[\mathcal{L}] - V_{(\Box \Diamond)}[\mathcal{L}]V_{(\Box \Diamond)}[\mathcal{L}] - V_{(\Box \Diamond)}[\mathcal{L}]V_{(\Box \Diamond)}[\mathcal{L}] \\
  &\quad + V_{(\Box \Diamond)}[\mathcal{L}]V_{(\Box \Diamond)}[\mathcal{L}]^2 \quad \text{(D.8)} \\
  f_{(\Box \Box)}[\mathcal{L}] &= V_{(\Box \Box)}[\mathcal{L}] - V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] - V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] \\
  &\quad + V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}]^2 \quad \text{(D.9)} \\
  f_{(\Box \Box)}[\mathcal{L}] &= V_{(\Box \Box)}[\mathcal{L}] - V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] - V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] \\
  &\quad - V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] + \frac{1}{2} V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}]^2 + 2 V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}] \\
  &\quad + V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}]^2 V_{(\Box \Box)}[\mathcal{L}] - \frac{3}{2} V_{(\Box \Box)}[\mathcal{L}]V_{(\Box \Box)}[\mathcal{L}]^2 V_{(\Box \Box)}[\mathcal{L}] \\
  &\quad - \frac{1}{2} V_{(\Box \Box)}[\mathcal{L}] + \frac{1}{2} V_{(\Box \Box)}[\mathcal{L}]^2 V_{(\Box \Box)}[\mathcal{L}] \quad \text{(D.10)}
\end{align*}
\]
\[ f_{\{BB\}}[L] = V_{\{BB\}}[L] - V_{\{B_1\}}[K_1]V_{\{B_2\}}[K_2] - V_{\{B_o\}}[L]V_{\{K_2\}}[K_2] \]

\[ = -V_{\{B_o\}}[L]V_{\{K_1\}}[K_1] + \frac{1}{2}V_{\{B_o\}}[L][E]^2 + 2V_{\{B_o\}}[L]V_{\{B_1\}}[K_1]V_{\{K_2\}}[K_2] \]

\[ + V_{\{B_1\}}[K_1]^2V_{\{B_2\}}[K_2] + V_{\{B_2\}}[K_1]V_{\{K_2\}}[K_2] - \frac{3}{2}V_{\{B_o\}}[K_1]^2V_{\{K_2\}}[K_2]^2 \]

\[ = \frac{1}{2}V_{\{B_o\}}[L] + \frac{1}{2}V_{\{B_1\}}[K_1]V_{\{B_2\}}[K_2] \tag{D.11} \]

Here the components knots \( K_1 \) and \( K_2 \) are unknots for the non-torus links in Figure 2. The generalisation of Ooguri-Vafa conjecture for links was proposed in [8] which states that reformulated invariants for \( r \)-component link should have the following structure

\[ f_{(R_1,R_2,...,R_r)}(q,\lambda) = (q^{1/2} - q^{-1/2})^{r-2} \sum_{Q,s} N_{(R_1,...,R_r),Q,s} q^Q \lambda^s, \tag{D.12} \]

where \( N_{(R_1,...,R_r),Q,s} \) are integer and \( Q \) and \( s \) are half integers.

We can see below that all the reformulated invariants we calculate indeed satisfy the conjecture.

**Appendix D.1. Reformulated invariant for knots**

We have already seen in appendix Appendix B, \( V_{\{K\}} \) has the Ooguri-Vafa form given in eqn.(D.4).

For the symmetric second rank tensor \( R = \begin{array}{c}
\end{array} \) placed on the knot, \( f_{\{K\}}[K] \) are:

\[ f_{\{K\}}[41] = \frac{(-1+\lambda)^2}{(-1+q^2)\lambda^2} \left[ -q + \lambda - q^2 \lambda^3 + q^4 \lambda^4 + q^3 \lambda(1 + \lambda) - q^2 \lambda^2(1 + \lambda) \right] \]

\[ f_{\{K\}}[52] = \frac{(-1+\lambda)^2}{(-1+q^2)\lambda^2} \left[ q(-1 + \lambda) + q^2(-1 + \lambda) + \lambda + q^4(1 + \lambda + \lambda^2) - q^3(1 + \lambda^2 + \lambda^3 + \lambda^4) \right] \]

\[ f_{\{K\}}[61] = \frac{(-1+\lambda)^2}{(-1+q^2)\lambda^2} \left[ q - \lambda - q^2 \lambda + q^7 \lambda^4(1 + \lambda) + q^8(1 + \lambda^2) - q^5 \lambda(1 + \lambda + \lambda^2)^2 + q^4 \lambda(-1 + \lambda + 2 \lambda^2 + 2 \lambda^3 + \lambda^4) + q^6(\lambda^2 + \lambda^3 + \lambda^4 - \lambda^5 - \lambda^6) \right] \]

\[ f_{\{K\}}[62] = \frac{(-1+\lambda)^2}{(-1+q^2)\lambda^2} \left[ -\lambda - q^5 \lambda + q^8(-1 + \lambda) + q^9 \lambda^3 + q^6 \lambda^2(1 + \lambda) + q(1 + \lambda^2) - q^2 \lambda(2 + \lambda^2) + q^4(1 + 2 \lambda^2) - q^3 \lambda(1 + \lambda + \lambda^3) \right] \]

\[ f_{\{K\}}[63] = \frac{(-1+\lambda)^2}{(-1+q^2)\lambda^2} \left[ -\lambda + q^2 \lambda^2 - q^7 \lambda^2 + q^8 \lambda^3 + q(1 + \lambda^2) + q^4(-1 + \lambda + \lambda^2) + q^6 \lambda^2(-1 + \lambda + \lambda^2) - q^4 \lambda(2 + 2 \lambda + 2 \lambda^2 + \lambda^3) + q^5(1 + 2 \lambda + 2 \lambda^2 + 2 \lambda^3) - q^8(\lambda^2 + \lambda^4) \right] \]

36
\[
\begin{align*}
\text{f}[72] &= \frac{-(-1+\lambda)^2}{(1+q+\lambda^2)} [g - \lambda - q^2 \lambda - 2q^4 \lambda + q^2 (1 + \lambda^2) - q^8 \lambda (1 + \lambda + 2\lambda^2 + 2\lambda^3 + \lambda^4) - q^6 \lambda (2 + \lambda + 3\lambda^2 + 3\lambda^3 + 3\lambda^4 + \lambda^5) + q^5 (1 + \lambda^2 + \lambda^4 + \lambda^5 + \lambda^6) + q^7 (1 + \lambda + 3\lambda^2 + 3\lambda^3 + 4\lambda^4 + 3\lambda^5 + \lambda^6)] \\
\text{f}[73] &= \frac{(-1+\lambda)^2\lambda^3}{(1+q+\lambda^2)} [-2q^{10}\lambda^3 - q^{12}\lambda^3 + q^{11}\lambda^4 - \lambda(1 + \lambda + \lambda^3) + q^6\lambda(-1 - \lambda - 5\lambda^2 + \lambda^3) + q^4\lambda(-1 - \lambda - 4\lambda^2 + \lambda^3) - 2q^2(1 + 2\lambda + 2\lambda^2 + 2\lambda^3) + q^3\lambda(1 + 2\lambda - \lambda^2 + 3\lambda^3) + q^8(-1 - \lambda - 2\lambda^2 - 3\lambda^3 + \lambda^4) + q(1 + 3\lambda + 3\lambda^2 + 3\lambda^3 + \lambda^4) + q^3(2 + 3\lambda + 4\lambda^2 + 2\lambda^4) + q^9(1 + \lambda + \lambda^2 - \lambda^3 + 2\lambda^4) + q^7(1 + 2\lambda + 2\lambda^2 - \lambda^3 + 2\lambda^4)] \\
\text{f}[74] &= \frac{(-1+\lambda)^2\lambda^3}{(1+q+\lambda^2)} [-q^9\lambda^5 + \lambda^2(1 + \lambda) - q^7\lambda^3(1 + \lambda) + q^8\lambda^4(-1 + \lambda^2) + q^6\lambda^2(1 + \lambda + \lambda^2 + \lambda^3) - q\lambda(2 + 4\lambda + 4\lambda^2 + 3\lambda^3 + \lambda^4) + q^5(\lambda + \lambda^2 - 2\lambda^4 - \lambda^5) - q^3(2 + 5\lambda + 5\lambda^2 + 5\lambda^3 + 4\lambda^4 + \lambda^5) + q^4(1 + \lambda + 2\lambda^2 + \lambda^3 - \lambda^4 + \lambda^5 + \lambda^6) + q^2(1 + 5\lambda + 8\lambda^2 + 7\lambda^3 + 3\lambda^4 + 2\lambda^5 + \lambda^6)] \\
\text{f}[75] &= \frac{(1-q^2+q^3)(-1+\lambda)^2}{(1+q+\lambda^2)} [-q + \lambda - q^7(-1 + \lambda)\lambda^3 + q^8\lambda(1 + \lambda^3) + q^{10}\lambda(1 + 2\lambda + 2\lambda^2) + q^4(-1 + \lambda + \lambda^2 + 2\lambda^3) + q^6(\lambda^3 - 2\lambda^4) + q^3(\lambda - \lambda^4) - q^5(1 + \lambda + \lambda^2 + \lambda^4) - \sqrt{9(1 + 2\lambda + 3\lambda^2 + 3\lambda^3 + 2\lambda^4)] \\
\text{f}[76] &= \frac{-(-1+\lambda)^2}{(1+q+\lambda^2)} [-(-1 + \lambda)\lambda^2 + q^11\lambda^5 - q^{10}\lambda^5(1 + \lambda) + q^9\lambda^3(-2 - 2\lambda - \lambda^2 + \lambda^3) + q\lambda(-2 + \lambda + \lambda^2 + \lambda^3) - q^3\lambda(1 - 3\lambda + \lambda^2 + \lambda^4) - q^2(-1 + \lambda^2 + 3\lambda^3 + \lambda^4) - q^8\lambda^2(2 + 4\lambda + 5\lambda^2 + 4\lambda^3 + \lambda^4) - q^6\lambda(3 + 4\lambda + 8\lambda^2 + 6\lambda^3 + 2\lambda^4) - q^7\lambda(2 + 2\lambda + 7\lambda^2 + 6\lambda^3 + 4\lambda^4 + 2\lambda^5) + q^4(1 - \lambda + 4\lambda^2 + 3\lambda^3 + 5\lambda^4 + 2\lambda^5) + q^6(1 + \lambda + 6\lambda^2 + 6\lambda^3 + 7\lambda^4 + 2\lambda^5 + \lambda^6)] \\
\text{f}[77] &= \frac{-(-1+\lambda)^2}{(1+q+\lambda^2)} [\lambda + 2q^4 \lambda + q^11(-1 + \lambda)\lambda^3 - q(1 + \lambda)^2 + q^2(2 + \lambda) - q^8\lambda^2(-1 + \lambda - 2\lambda^2 + \lambda^3) + q^9\lambda^2(-1 - 2\lambda - \lambda^2 - 3\lambda^3 + \lambda^4) - q^6\lambda(7 + 12\lambda + 12\lambda^2 + 5\lambda^3 + 2\lambda^4) + q^4(-2 + 5\lambda^2 + 5\lambda^3 + 4\lambda^4) + q^{10}(\lambda^2 + \lambda^3 + 2\lambda^4 - 2\lambda^5) + q^6(2 + 8\lambda + 11\lambda^2 + 9\lambda^3 + 4\lambda^4 - \lambda^5) + q^7(-1 - 3\lambda - 4\lambda^2 - 3\lambda^3 + 3\lambda^4 - \lambda^5 + \lambda^6)] \\
\text{f}[81] &= \frac{-(-1+\lambda)^2}{(1+q+\lambda^2)} [q - \lambda - q^2\lambda - 2q^4\lambda + q^3(1 + \lambda^2) + q^5(1 + \lambda^2) + q^6\lambda^5(1 + \lambda + \lambda^2) - q^8\lambda^2(-1 - 2\lambda - 2\lambda^2 - 2\lambda^3 + \lambda^5 + \lambda^6) + q^6\lambda(-1 + \lambda + 2\lambda^2 + 3\lambda^3 + 3\lambda^4 + 2\lambda^5 + \lambda^6) - q^7\lambda(1 + 2\lambda + 4\lambda^2 + 5\lambda^3 + 4\lambda^4 + 3\lambda^5 + \lambda^6)]
\end{align*}
\]
Following relation between the reformulated invariants:

\[ f^{[0]}_2 = -\frac{(\frac{n-1}{n})^2}{q\lambda} \left[ q - \lambda - q^2 \lambda - 2q^3 \lambda - 2q^4 \lambda + q^5 (1 + \lambda^2) - q^6 \lambda (1 + \lambda + 2\lambda^2 + 3\lambda^3 + 3\lambda^4 + 2\lambda^5 + \lambda^6) - q^8 \lambda (2 + \lambda + 2\lambda^2 + 4\lambda^3 + 4\lambda^4 + 4\lambda^5 + 3\lambda^6 + \lambda^7) + q^9 (1 + \lambda^2 + 5\lambda^3 + 5\lambda^4 + 3\lambda^5 + 6\lambda^6 + 3\lambda^7 + \lambda^8) \right] \]

\[ f^{[10]}_1 = -\frac{(\frac{n-1}{n})^2}{q\lambda} \left[ q - \lambda - q^2 \lambda - 2q^3 \lambda - 2q^4 \lambda + q^5 (1 + \lambda^2) + q^7 (1 + \lambda^2) + q^{11} \lambda^6 + q^{10} \lambda^2 (1 + 2\lambda + 3\lambda^2 + 3\lambda^3 + 3\lambda^4 + \lambda^5 - \lambda^7 - \lambda^8) + q^9 \lambda (1 + 2\lambda + 3\lambda^2 + 3\lambda^3 + 4\lambda^4 + 4\lambda^5 + 3\lambda^6 + 2\lambda^7 + \lambda^8) - q^9 \lambda (1 + 2\lambda + 4\lambda^2 + 6\lambda^3 + 7\lambda^4 + 6\lambda^5 + 5\lambda^6 + 3\lambda^7 + \lambda^8) \right] \]

Changing the symmetric representation by antisymmetry representation, we find the following relation between the reformulated invariants:

\[ f^{[K]}(q^{-1}, \lambda) = f^{[\Box]}(q, \lambda). \]  

**Appendix D.2. Reformulated invariant for links**

1. For \( R_1 = \Box, R_2 = \Box \):

\[ f^{[\Box]}_2 = -\frac{(\frac{n-1}{n})^2}{q\lambda} \left[ q + \lambda + q^2 \lambda + q^3 \lambda^2 \right] \]

\[ f^{[\Box]}_3 = -\frac{(\frac{n-1}{n})^2}{q\lambda} \left[ \lambda + q^2 \lambda - q (1 + 2\lambda^2) \right] \]

\[ f^{[\Box]}_7_1 = -\frac{(\frac{n-1}{n})^2}{q^{2}\lambda^3} \left[ -\lambda - q^4 \lambda + q^2 (-2 + \lambda) \lambda + q(1 + \lambda^2) + q^3 (1 + \lambda^2) \right] \]

\[ f^{[\Box]}_7_2 = -\frac{(\frac{n-1}{n})^2}{q^{2}\lambda^3} \left[ -\lambda + 3q^2 \lambda - q^4 \lambda + q(1 + \lambda^2) + q^3 (1 + \lambda^2) \right] \]

\[ f^{[\Box]}_7_3 = -\frac{(\frac{n-1}{n})^2}{q^{2}\lambda^3} \left[ q - \lambda - q^2 \lambda + q^2 \lambda \right] \]

2. For \( R_1 = \Box, R_2 = \Box \):

\[ f^{[\Box]}_2 = -\frac{(\frac{n-1}{n})^2}{q^{1+\lambda} \sqrt{q}\lambda} \left[ \lambda^2 + q\lambda^2 + q^3 \lambda^2 + q^4 \lambda^2 + q^2 (-1 - \lambda + \lambda^2) \right] \]

\[ f^{[\Box]}_3 = -\frac{1}{q^{\sqrt{2}\lambda} \gamma} \left[ 1 + q \right] \left[ \lambda + q^2 \lambda - q (1 + \lambda^2) \right] \]

\[ f^{[\Box]}_7_1 = \frac{(\frac{n-1}{n})^2}{q^{\sqrt{2}\lambda} \gamma} \left[ q - \lambda - q^6 \lambda^2 + q^3 (-2 + \lambda) \lambda^2 + q^2 \lambda^3 + q^4 \lambda (1 - \lambda + \lambda^2) + q^5 \lambda (1 - \lambda + \lambda^2) \right] \]
\[ f_{\bigotimes \bigotimes}[7_2] = \frac{(-1+\lambda)}{q^{2+\lambda^2}} [(q^4 - \lambda - 3q^3\lambda - q^2\lambda^2 + q^5\lambda^3 + q(1 - \lambda + \lambda^2) + q^2(1 - \lambda + \lambda^2)] \]

\[ f_{\bigotimes \bigotimes}[7_3] = \frac{(-1+\lambda)}{q^{2+\lambda^2}} [q - q^3\lambda^2 + q^2\lambda^3] \]

We have checked that \( f_{\bigotimes \bigotimes}[L] = f_{\bigotimes \bigotimes}[L] \) for these links. We also have the symmetry relation

\[ f_{\bigotimes \bigotimes}[\mathcal{L}](-q^{-1}, \lambda) = -f_{\bigotimes \bigotimes}[\mathcal{L}](q, \lambda). \quad \text{(D.14)} \]

3. For \( R_1 = \bigotimes \bigotimes \), \( R_2 = \bigotimes \):

\[ f_{\bigotimes \bigotimes}[6_2] = \frac{1}{q^{2+\lambda}} [q(-1 + \lambda)^2\lambda + q^3(-1 + \lambda)^2\lambda^2 + \lambda^3 - \lambda^5 + q^7(-1 + \lambda)^2\lambda^2(3 + \lambda) + q^{10}\lambda^3(-1 + \lambda^2) + q^8\lambda^2(2 - 3\lambda + \lambda^2) + q^4(-1 + \lambda)^2(-1 + 2\lambda + 2\lambda^2) + q^3(-1 + \lambda)^2(1 + 2\lambda^2) + q^4(-1 + \lambda + 4\lambda^2 - 5\lambda^3 + \lambda^4) + q^6\lambda(-1 + 5\lambda - 8\lambda^2 + 3\lambda^3 + \lambda^4) + q^2(-1 + \lambda^2 + \lambda^4 - \lambda^5)] \]

\[ f_{\bigotimes \bigotimes}[6_3] = \frac{1}{q^{2+\lambda}} [-(1 + \lambda)^2\lambda + q^2(1 + \lambda)^2\lambda^2 - q^3(1 + \lambda)^2\lambda^2(2 + \lambda^2) - q^4(-1 + \lambda)^2\lambda(1 - \lambda + 2\lambda^2) + q^8(-1 + \lambda)^2\lambda(1 + \lambda + 2\lambda^2) + q^3\lambda^2(1 - 2\lambda + 2\lambda^2) + q^4(-1 + \lambda)^2(1 + 3\lambda^2 + 2\lambda^2 + 2\lambda) + q(1 - 2\lambda + \lambda^2 - \lambda^3 + \lambda^4) - q^7(1 + 2\lambda^2 - 2\lambda^3 - 2\lambda^4 + \lambda^6)] \]

\[ f_{\bigotimes \bigotimes}[7_1] = \frac{1}{q^{2+\lambda}} [-(1 + \lambda)^2\lambda + q^{11}(1 + \lambda)^2\lambda^2 - q^{13}(1 + \lambda)^2\lambda^3 + q(-1 + \lambda)^2(1 + \lambda^2) + 3q^9(-1 + \lambda)^2\lambda^2(1 - \lambda + \lambda^2) + q^3(-1 + \lambda^2)(1 - \lambda + 2\lambda^2) + q^2\lambda(-2 + 4\lambda - 3\lambda^2 + \lambda^3) + q^{12}\lambda^2(1 - 2\lambda + 3\lambda^2 - 3\lambda^3 + \lambda^4) - q^6(1 + \lambda - \lambda^2 - 2\lambda^3 + \lambda^4) + q^7(-1 + \lambda)^2(1 + 2\lambda^2 - \lambda^3 + \lambda^4) + q^8\lambda^2(2 - 4\lambda + 7\lambda^2 - 7\lambda^3 + 2\lambda^4) - q^5\lambda(-1 + \lambda + 2\lambda^2 - 3\lambda^4 + \lambda^5) + q^{10}\lambda(-1 + 2\lambda - 7\lambda^2 + 11\lambda^3 - 7\lambda^4 + 2\lambda^5) - q^2(-1 + 3\lambda - 4\lambda^2 + \lambda^3 + \lambda^4 - \lambda^5 + \lambda^6)] \]

\[ f_{\bigotimes \bigotimes}[7_2] = -\frac{1}{q^{2+\lambda}} [-(1 + \lambda)^2\lambda + q^{13}(-1 + \lambda)^2\lambda^3 - q(-1 + \lambda)^2(1 + \lambda^2) + q^7(-1 + \lambda)^2(3 - 2\lambda + \lambda^2) + q^{11}(-1 + \lambda)^2\lambda(-1 - \lambda + \lambda^2) + q^5(-1 + \lambda)^2(-1 + 2\lambda - 5\lambda^2 + \lambda^3) - q^2\lambda(-1 + 4\lambda - 4\lambda^2 + \lambda^3) - q^{12}\lambda^3(-1 + 2\lambda - 2\lambda^2 + \lambda^3) - q^7(-1 + \lambda)^2(1 + \lambda + 5\lambda^2 + \lambda^3) + q^{10}\lambda^2(1 - 2\lambda - 2\lambda^2 + 3\lambda^3) + q^6\lambda(3 - 10\lambda + 9\lambda^2 - 5\lambda^3 + 3\lambda^4) + q^4(-1 + 6\lambda - 11\lambda^2 + 12\lambda^3 - 7\lambda^4 + \lambda^5) - q^9\lambda(3 - 3\lambda - 2\lambda^2 + \lambda^4 + \lambda^5) + q^8(1 + \lambda + 2\lambda^2 - 5\lambda^3 + \lambda^4 - \lambda^5 + \lambda^6)] \]
\[ f_{\text{\textcircled{2}}} (q, \lambda) = \frac{(-1+\lambda)^2}{q \lambda^6} \left[ \left(-q + \frac{\lambda^3 + q^2 \lambda^4 (1+\lambda) + q^4 \lambda^2 (1+\lambda)^2 + q^6 \lambda^2 (1+\lambda)^2 - q^3 \lambda^2 (1+\lambda + \lambda^2) - q^7 (1+2\lambda + 3\lambda^2 + 3\lambda^3 + 3\lambda^4 + \lambda^5) + q^8 (\lambda^3 + \lambda^3 + 2\lambda^4 - \lambda^6) \right) \right] \]

Changing both the rank two symmetric representation \( \text{\textcircled{2}} \) by antisymmetric representation \( \text{\textcircled{1}} \), we find the following relation between the reformulated invariants:

\[ f_{\text{\textcircled{1}}} (\mathcal{L} | q, \lambda) = f_{\text{\textcircled{2}}} (\mathcal{L} | q^{-1}, \lambda). \]
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