Fluid kinematics around two circular cylinders moving towards impact

NEKTARIES BAMPALAS *
Department of Aeronautics, Imperial College London, United Kingdom
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Abstract

The scale factors of an arbitrary orthogonal space are a measure of its content of homogeneous orthogonal space. In the present study, it is shown, that their spatial and temporal rates of variation do not contribute to the differential calculus of arbitrary functions in orthogonal space. Based on this, the Navier-Stokes equations are formulated accordingly, to provide a method of studying the kinematics of fluid motion in orthogonal plane space. Employing this formulation and regular perturbation theory, the kinematic physical measures of the flow of an incompressible, viscous fluid around two identical circular cylinders, which move with equal and opposite velocity towards central impact are evaluated. For this case, the space is parametrised according to the bipolar transformation of the cartesian plane.

1 Introduction

The expressions of the basic differential operators, according to an arbitrary orthogonal vector base, are known, see e.g. Spain (1967). The scale factors, which characterise such a base, are, in general, functions of space and time, and they represent a spatial density measure between the two mapping spaces. The present study shows that their spatial rate of variation does not contribute to the spatial rate of variation of arbitrary functions of the orthogonal space.

Based on this result, which affects the differential calculus of functions in orthogonal space, the Navier-Stokes equations are formulated in an arbitrarily parametrised, orthogonal, plane space. An application of this formulation follows, in order to obtain an asymptotic series solution for the kinematic physical measures of the flow of a fluid around two identical circular cylinders, which move with equal and opposite velocity, towards central impact.

Bampalas & Graham (2008) studied the flow, based on a numerical solution of the Navier-Stokes equations in the plane. However, as the boundaries of the cylinders approach, mesh resolution effects affect the accuracy of the numerical solution.

The next section presents the expressions of the basic differential operators in orthogonal space, on which the subsequent analysis is based on. In section 3, the formulation of the Navier-Stokes equations in an arbitrarily parametrised, orthogonal, plane space is presented. In section 4, the bipolar transformation of the cartesian plane is employed, to adapt the formulation to the case of the flow of a fluid around two circular cylinders and the asymptotic series solution is obtained.

*Current email address for correspondence: nektbam@gmail.com
The basic differential operators in an arbitrary orthogonal space

Consider the right-handed, orthogonal vector base \( \mathbf{I}_i, \ i \in \mathbb{N} : i \in [1, 3] \), the spaces \( \mathbf{q}, \mathbf{s} \in \mathbb{R}^3 \) and time \( t \in \mathbb{R} : t \geq 0 \). Implied is the summation convention,

\[
\mathbf{q} := q_i \mathbf{I}_i, \quad s_i(q, t) := s_i(q; t) \mathbf{I}_i;
\]

\( s_i(q; t) \) are independent variables in \( \mathbf{s} \) and \( q_i \) are independent variables in \( \mathbf{q} \).

The units of spatial measure of \( \mathbf{q} \) and \( \mathbf{s} \) are

\[
dq := \{dq_i, \mathbf{I}_i \in \mathbf{q} : dq_i \to 0\} \neq 0, \quad ds := \{ds_i, \mathbf{I}_i \in \mathbf{s} : ds_i \to 0\} \neq 0.
\]

The variation of arbitrary, continuous functions of \( \mathbf{q} \) or \( \mathbf{s} \) along \( dq \) or \( ds \) is assumed to be linear.

Define a mapping

\[
\mathcal{M} := \{s \mapsto \mathbf{q} : ds_i, \mathbf{I}_i = s_i(q; t) dq_i, \mathbf{I}_i, s_i(q; t) \in \mathbb{R} : s_i \neq 0\};
\]

\( \mathcal{M}(\mathbf{q}; t) := s_i \mathbf{I}_i \) represents a spatial density measure of \( \mathbf{q} \) in \( \mathbf{s} \). For the special case \( \mathbf{q}, \mathbf{s} \in \mathbb{R}^2 \) and for the bipolar transformation, \( s_i = s \), \( \mathbf{s} \) is measured as

\[
\mathbf{s} = \int_{0}^{s_i} du \mathbf{I}_i = \int_{0}^{q_i} s_i(\mathbf{q}(u); t)du \mathbf{I}_i.
\]

Consider an arbitrary, continuous scalar function \( f(\mathbf{s}, t) \in \mathbb{R} \), with continuous derivatives. The spatial rate of variation of \( f \) along \( \mathbf{I}_i \) is described by the directional derivative \( f_{s_i} \) along \( \mathbf{I}_i \) (see e.g. [Kreyszig, 1999]), which is defined as

\[
f_{s_i}[s(\mathbf{q}; t), t] = \frac{df}{ds_i} := \lim_{ds_i \to 0} \frac{f[s(\mathbf{q}; t) + ds_i(\mathbf{q}; t) \mathbf{I}_i, t] - f[s(\mathbf{q}; t), t]}{ds_i} =
\]

\[
= \lim_{s_i(q, t) dq_i \to 0} \frac{f[s(\mathbf{q}; t) + s_i(q; t) dq_i \mathbf{I}_i, t] - f[s(\mathbf{q}; t), t]}{s_i(q; t) dq_i} =
\]

\[
= s^{-1}_i(q; t) \lim_{dq_i \to 0} \frac{f[q + dq_i \mathbf{I}_i, t; \mathcal{M}(\mathbf{q}; t)] - f[\mathbf{q}, t; \mathcal{M}(\mathbf{q}; t)]}{dq_i}.
\]

Therefore,

\[
f_{s_i}[s(\mathbf{q}; t), t] = f_{s_i}[\mathbf{q}, t; \mathcal{M}(\mathbf{q}; t)] = s^{-1}_i(q; t) f_{q_i}[\mathbf{q}, t; \mathcal{M}(\mathbf{q}; t)].
\]

From the first fundamental theorem of calculus (see e.g. [Apostol, 1967]), the indefinite integral of \( f_{s_i} \) along \( \mathbf{I}_i \), is

\[
f[s(\mathbf{q}; t), t] = f[\mathbf{q}, t; \mathcal{M}(\mathbf{q}; t)] = \int_{s_i}^{s_i} f_{s_i}[s(u), t] du = \int_{q_i}^{q_i} f_{q_i}[\mathbf{q}(u), t; \mathcal{M}(\mathbf{q}; t)] du,
\]

where \( \xi \in \mathbf{s} \cdot \mathbf{I}_i, \zeta \in \mathbf{q} \cdot \mathbf{I}_i \).

Space and time are considered to be independent sets. Therefore, the temporal rate of variation of \( f[s(\mathbf{q}; t), t] \) is
\[ f_t[s(q; t), t] = \frac{df[s(q; t), t]}{dt} := \lim_{dt \to 0} \frac{f[s(q; t), t + dt] - f[s(q; t), t]}{dt} = f_t[q, t; S(q; t)] . \]

The indefinite integral of \( f_t \) along time is

\[ \int_t^\theta f_t[q, u; S(q; t)] du , \quad \theta \in [0, t] . \]

Starting from the fact that \( s_i(q; t) \) are independent variables in \( s \) and following the way usually employed in vector analysis, see e.g. Spain (1967), the expressions for the basic differential operators in \( s \) (see the appendix) are

\[
\begin{align*}
\nabla &:= \partial_{s_i} I_i , \quad \nabla^i := \iota \partial_{s_i} , \quad \nabla \times := [m(i+2) \partial_{s_{m(i+1)}} - m(i+1) \partial_{s_{m(i+2)}}] I_i , \\
\nabla^2 &:= \partial_{s_i^2} , \quad \nabla^2 := \iota^2 I_i , \quad \nabla^4 := \partial_{s_i^2} + \partial_{s_{i3}^2 s_{m(i+1)}} + \partial_{s_{i3}^2 s_{m(i+2)}} , \quad \nabla^4 := \iota^4 I_i ;
\end{align*}
\]

\( \partial \) denotes partial differentiation, a left superscript of an operator indicates the vector component on which it operates and

\[
m(i) := \begin{cases} 
    i & \text{if } i \in [1, 3] \\
    i \mapsto i - 3 & \text{if } i \in [4, 5].
\end{cases}
\]

The differential operators in \( q \) are obtained by applying \( M \) on the corresponding operators in \( s \). The operators in \( s, q \in \mathbb{R}^2 \) are obtained by those in \( s, q \in \mathbb{R}^3 \) by setting \( \partial_{s_3} = 0, \partial_{q_3} = 0. \)

3 A normalisation of the Navier-Stokes equations in orthogonal plane space

Consider the mechanics of fluid of density \( \rho \) and kinematic viscosity \( \nu \), in plane space \( s \in \mathbb{R}^2 \) and in time. Applying a mapping \( M : \{ s \mapsto q \} \), such as that obtained by a conformal mapping of the cartesian plane in \( q \), it is

\[ S(q; t) = s_i(q; t) I_i = s(q; t) I_i . \]

The fundamental physical measures, of fluid mechanics in the plane, are the streamfunction \( \Psi = \Psi I_3 \), the velocity \( U = U_i I_i \), \( i = 1, 2 \), the vorticity \( \omega = \omega I_3 \) and the pressure \( P \). The Navier-Stokes equations in the plane are

\[
\begin{align*}
D_t U &= -\rho^{-1} \nabla P + \nu \nabla^2 U , \quad \nabla \cdot U = 0 , \quad \text{or} \\
D_t \omega &= \nu \nabla^2 \omega , \quad \nabla^2 \Psi = -\omega ;
\end{align*}
\]

\( D_t \) signifies the material derivative

\[ D_t := \partial_t + U \cdot \nabla . \]

The kinematic relation between the velocity and the streamfunction for plane motion is

\[ U = \nabla \times \Psi . \]
The physical measures are functions of the independent variables and the parameters as

\[
\Psi[s(q; t), t; v; \rho] = \Psi[q, t]; s(q; t); v; \rho], \quad U_s[s(q; t), t; v; \rho] = U_s[q, t]; s(q; t); v; \rho], \
\omega[s(q; t), t; v; \rho] = \omega[q, t]; s(q; t); v; \rho], \quad P[s(q; t), t; v; \rho] = P[q, t]; s(q; t); v; \rho].
\]

Consider the boundary \( \partial q \) of \( q \) to be rectangular and symmetrical with respect to \( q = 0 \). Set the lines \( q_i = \pm l_i \), to confine \( q \) and define \( l = l_i l_i \). Introduce a characteristic speed \( U_s \), a characteristic, cartesian length scale \( l \) and for \( s(q; t) \) with units of length, set \( s(\mathbf{q}; t) \) to scale length in \( s \) and \( s(\mathbf{q}; t)U_s^{-1} \) time. The characteristic length and time scales, are functions of space and time, but based on the analysis presented in section 2, they are parameters in the differential calculus of the physical measures of fluid motion in \( q \). The scaled variables, parameters and measures are

\[
\tilde{q}_i := q_i l_i^{-1}, \quad \tilde{s}(\tilde{\mathbf{q}}; \tilde{t}) := \tilde{s} l^{-1}, \quad \tilde{s}_i(\tilde{\mathbf{q}}; \tilde{t}) := s_i(\mathbf{q}; t) \hat{l} \tilde{l}^{-1}, \
\tilde{l} := t U_s l^{-1}, \quad Re_s := U_s s \nu^{-1}, \quad Re_l := U_s \hat{l} \nu^{-1}, \
\tilde{\Psi} := \Psi(U_s s)^{-1}, \quad \tilde{U} := U U_s^{-1}, \quad \tilde{\omega} := \omega \nu s^{-1}, \
\tilde{P} := P(\rho U_s^2)^{-1}, \quad \tilde{\Psi} := \Psi \tilde{s}, \quad \tilde{\omega} := \tilde{s} \tilde{\omega}^{-1}.
\]

Introduce the scaled differential operators

\[
\tilde{\nabla} := s \nabla, \quad \tilde{\nabla} := \nabla, \quad \tilde{\nabla} := s \nabla, \quad \tilde{\nabla} := \nabla, \
\nabla^2 := s^2 \nabla^2, \quad D_l := \partial_l + \tilde{U} \cdot \tilde{\nabla}, \quad \tilde{\nabla} := \nabla, \quad \tilde{\nabla} := \nabla,
\]

and the scaled Navier-Stokes equations in the plane are

\[
D_l \tilde{U} = -\tilde{\nabla} \tilde{P} + Re_s^{-1} \tilde{\nabla}^2 \tilde{U}, \quad \tilde{\nabla} \cdot \tilde{U} = 0, \
D_l \tilde{\omega} = Re_s^{-1} \tilde{\nabla}^2 \tilde{\omega}, \quad \tilde{\nabla}^2 \tilde{\Psi} = -\tilde{\omega}, \
\tilde{U} = \tilde{\nabla} \times \tilde{\Psi}.
\]

The differential equation for \( \tilde{\Psi} \) is

\[
D_l (\tilde{\nabla}^2 \tilde{\Psi}) = Re_s^{-1} \tilde{\nabla}^4 \tilde{\Psi}
\]

and in explicit form

\[
\tilde{\Psi}_{\tilde{x}_2} + 2 \tilde{\Psi}_{\tilde{x}_1 \tilde{x}_2} + \tilde{\Psi}_{\tilde{x}_2} = \left( \tilde{\Psi}_{\tilde{x}_1} + \tilde{\Psi}_{\tilde{x}_1} + \tilde{\Psi}_{\tilde{x}_2} \tilde{\Psi}_{\tilde{x}_1} + \tilde{\Psi}_{\tilde{x}_2} \tilde{\Psi}_{\tilde{x}_1} + \tilde{\Psi}_{\tilde{x}_2} \tilde{\Psi}_{\tilde{x}_1} + \tilde{\Psi}_{\tilde{x}_2} \tilde{\Psi}_{\tilde{x}_1} + \tilde{\Psi}_{\tilde{x}_2} \tilde{\Psi}_{\tilde{x}_1} \right) Re_s.
\]

The functional dependence of the scaled physical measures on the scaled variables and the parameters is

\[
\tilde{\Psi}(\tilde{s}; \tilde{t}; \tilde{s}; Re_s) = \tilde{\Psi}(\tilde{q}; \tilde{t}; \tilde{s}; Re_s; 1), \quad \tilde{U}(\tilde{s}; \tilde{t}; \tilde{s}; Re_s) = \tilde{U}(\tilde{q}; \tilde{t}; \tilde{s}; Re_s; 1), \
\tilde{\omega}(\tilde{s}; \tilde{t}; \tilde{s}; Re_s) = \tilde{\omega}(\tilde{q}; \tilde{t}; \tilde{s}; Re_s; 1), \quad \tilde{P}(\tilde{s}; \tilde{t}; \tilde{s}; Re_s) = \tilde{P}(\tilde{q}; \tilde{t}; \tilde{s}; Re_s; 1).
\]

4 The kinematics of the flow of a fluid around two circular cylinders moving towards central impact

Consider two identical, circular cylinders of radius \( \hat{l} = R \), immersed in incompressible fluid. The fluid is set into motion due to the motion of the cylinders, with constant and equal in
where $V$ denotes the initial conditions for the streamfunction are

$$\Psi(\sigma, \pm 1; \tilde{t}; \tilde{s}; R_{sa}; \tau_0) = \mp V(\tilde{\sigma}; \tilde{t}; \tau_0)$$

and the boundary conditions for the streamfunction are

$$\Psi(0, 0; \tilde{t}; \tilde{s}; R_{sa}; \tau_0) = 0$$

where $V(\tilde{\sigma}; \tilde{t}; \tau_0) = \tilde{y}(\tilde{\sigma}, \tau_0)\tilde{\sigma}^{-1} = \tilde{y}'(\tilde{\sigma}; \tau_0)\tilde{\sigma}^{-1}$ and $\tilde{\psi}(\tilde{\sigma}, \tau_0)$ denotes the initial flow field.

Figure 1: Description of the plane space between two circular cylinders of zero roughness surfaces according to the bipolar transformation (see also [Milne-Thomson, 1960]).
Remark on the no-slip boundary condition: The motion of the cylinders is along the $i_1$-direction. Therefore, the no-slip boundary condition is
\[
\tilde{\Psi}_2(\sigma, \pm 1, \tilde{t}, \tilde{\theta}; Re \sigma; \tau'_0) = 0 .
\]
But, $\sigma = \sigma(\tilde{y}'(\tau'_0))$ and $\tau'_{0 \tilde{t}} = 0$, therefore,
\[
\tilde{\Psi}_2 \left[ \sigma(\tilde{y}'(\tau'_0)), \tilde{t}, \tilde{\theta}; Re \sigma; \tau'_0 \right] = \tilde{\Psi}_2 \sigma \tilde{y}' \tilde{y}_x + \tilde{\Psi}_2 \tilde{t}.
\]
For arbitrary values of $\tilde{t}$, $\tilde{y}_x' \neq 0$. However, on the boundaries of the cylinders, it is
\[
\frac{d\tilde{y}'(\sigma; \tau'_0)}{d\tilde{x}(\sigma, \pm 1; \tau'_0)} = \frac{d\tilde{y}'(\sigma, \pm 1; \tau'_0)}{d\tilde{x}(\sigma, \pm 1; \tau'_0)} = 0 .
\]
Therefore, the no-slip boundary condition is imposed by the requirement, that
\[
\tilde{\Psi}_2(\sigma, \pm 1, \tilde{t}, \tilde{\theta}; Re \sigma; \tau'_0) = 0 .
\]

An asymptotic series solution for the streamfunction, valid for $0 < \tau'_0 \ll 1$ and $0 < \tau'_0^2 Re R \ll 1$ is obtained by expressing the streamfunction and its boundary and initial conditions as
\[
\dot{\Psi}(\sigma, \tilde{t}; Re \sigma; \tau'_0) = \frac{1}{i} \dot{\Psi}(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) \tau'_0^{2j} , \quad i, j \in \mathbb{N}_0 ,
\]
\[
\psi(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) = \frac{1}{i} \psi(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) \tau'_0^{2j} , \quad i, j \in \mathbb{N}_0 ,
\]
\[
\frac{1}{i} \dot{\Psi}(\sigma, \pm 1, \tilde{t}; \tilde{\theta}) = \mp V(\sigma; \tau'_0; \tilde{\theta}) \delta_{ij} , \quad \frac{1}{i} \dot{\Psi}(\sigma, \pm 1, \tilde{t}; \tilde{\theta}) = 0 ,
\]
\[
\dot{\Psi}(0, \tilde{t}; \tilde{\theta}) = 0 , \quad \frac{1}{i} \dot{\Psi}(\sigma, \tilde{t}; \tilde{\theta}) = \frac{1}{i} \psi(\sigma, \tilde{t}; \tilde{\theta}) ;
\]
$\delta$ denotes the Kronecker delta function.

The solution for the streamfunction is obtained by substituting the assumed asymptotic series in its differential equation and solving according to regular perturbation theory. The velocity and the vorticity are obtained similarly by the kinematic relations between the velocity or the vorticity and the streamfunction.

\[
\dot{\Psi}(\sigma, \tilde{t}; Re R; \tau'_0) = \left( \frac{1}{2} \tilde{t}^3 - \frac{3}{2} \tilde{t} + \frac{3}{5!} \tilde{t}^7 - 3 \tilde{t}^3 + 2 \tilde{t} \right) \tilde{y}' y'_x (\tau'_0 Re R) + O(\tau'_0^2 Re^2 R) + O(\tau'_0) + O(\tau'_0^4) ;
\]
\[
\dot{U}(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) = \pi^{-1} \left\{ \frac{3}{2} \tilde{t}^2 - 1 \right\} \tilde{y}' \tilde{y}'^{-1} + \frac{3}{5!} \tilde{t}^5 - 9 \tilde{t}^3 + 2 \tilde{t} \right\} \tilde{y}' \tilde{y}'^{-1} \tilde{b}^{-1} (\tau'_0 Re R) + O(\tau'_0^2 Re R) + O(\tau'_0^4) ;
\]
\[
\omega(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) = -\pi^{-2} \left\{ \frac{3}{2} \tilde{t}^2 - 1 \right\} \tilde{y}' \tilde{y}'^{-1} + \frac{3}{5!} \tilde{t}^5 - 9 \tilde{t}^3 + 2 \tilde{t} \right\} \tilde{y}' \tilde{y}'^{-1} \tilde{b}^{-1} \pi^{-1} (\tau'_0 Re R) + O(\tau'_0^2 Re R) + O(\tau'_0^4) ;
\]
\[
\omega(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) = -\pi^{-2} \left\{ \frac{3}{2} \tilde{t}^2 - 1 \right\} \tilde{y}' \tilde{y}'^{-1} + \frac{3}{5!} \tilde{t}^5 - 9 \tilde{t}^3 + 2 \tilde{t} \right\} \tilde{y}' \tilde{y}'^{-1} \tilde{b}^{-2} (\tau'_0 Re R) + O(\tau'_0^2 Re R) + O(\tau'_0^4) ;
\]
\[
\omega(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) = -\pi^{-2} \left\{ \frac{3}{2} \tilde{t}^2 - 1 \right\} \tilde{y}' \tilde{y}'^{-1} + \frac{3}{5!} \tilde{t}^5 - 9 \tilde{t}^3 + 2 \tilde{t} \right\} \tilde{y}' \tilde{y}'^{-1} \tilde{b}^{-2} \pi^{-1} (\tau'_0 Re R) + O(\tau'_0^2 Re R) + O(\tau'_0^4) ;
\]
\[
\omega(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) = -\pi^{-2} \left\{ \frac{3}{2} \tilde{t}^2 - 1 \right\} \tilde{y}' \tilde{y}'^{-1} + \frac{3}{5!} \tilde{t}^5 - 9 \tilde{t}^3 + 2 \tilde{t} \right\} \tilde{y}' \tilde{y}'^{-1} \tilde{b}^{-2} \pi^{-1} (\tau'_0 Re R) + O(\tau'_0^2 Re R) + O(\tau'_0^4) ;
\]
\[
\omega(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) = -\pi^{-2} \left\{ \frac{3}{2} \tilde{t}^2 - 1 \right\} \tilde{y}' \tilde{y}'^{-1} + \frac{3}{5!} \tilde{t}^5 - 9 \tilde{t}^3 + 2 \tilde{t} \right\} \tilde{y}' \tilde{y}'^{-1} \tilde{b}^{-2} \pi^{-1} (\tau'_0 Re R) + O(\tau'_0^2 Re R) + O(\tau'_0^4) ;
\]
\[
\omega(\sigma, \tilde{t}; \tilde{\theta}; Re \sigma; \tau'_0) = -\pi^{-2} \left\{ \frac{3}{2} \tilde{t}^2 - 1 \right\} \tilde{y}' \tilde{y}'^{-1} + \frac{3}{5!} \tilde{t}^5 - 9 \tilde{t}^3 + 2 \tilde{t} \right\} \tilde{y}' \tilde{y}'^{-1} \tilde{b}^{-2} \pi^{-1} (\tau'_0 Re R) + O(\tau'_0^2 Re R) + O(\tau'_0^4) ;
\]
For $0 < \tau'_0^2 \ll 1$ and $0 < \tau'^2_0 Re \ll 1$, when the asymptotic series solution is valid, the solution is independent of the initial condition and unique.

For comparison, consider the case of irrotational fluid motion. For this case, the equation and the boundary conditions for the streamfunction $\tilde{\Psi}(\tilde{\sigma}, \tilde{\tau}; \tilde{\tau}'_0)$, are

$$\tilde{\tau}'_0^2 \tilde{\nabla}^2 \tilde{\Psi} = \frac{\tilde{\Psi}_x}{x^2} + \tilde{\tau}'_0^2 \tilde{\Psi}_\theta = 0 \quad ,$$

$$\tilde{\Psi}(\tilde{\sigma}, \pm 1; \tilde{\tau}'_0) = \mp V(\tilde{\sigma}; \tilde{\tau}'_0) \quad , \quad \tilde{\Psi}(0, 0; \tilde{\tau}'_0) = 0 .$$

An asymptotic series solution for the streamfunction, for irrotational flow, valid for $0 < \tau'_0^2 \ll 1$, is obtained by expressing the streamfunction and its boundary conditions as

$$\tilde{\Psi}(\tilde{\sigma}, \tilde{\tau}; \tilde{\tau}'_0) = j \tilde{\Psi}(\tilde{\sigma}, \tilde{\tau}; \tilde{\tau}'_0), \quad j \in \mathbb{N}_0 ,$$

$$j \tilde{\Psi}(\tilde{\sigma}, \pm 1; \tilde{\tau}) = \mp V(\tilde{\sigma}; \tilde{\tau}'_0) \delta_{j0} , \quad j \tilde{\Psi}(0, 0; \tilde{\tau}) = 0 .$$

The solution for the streamfunction, for irrotational flow, is obtained by substituting the assumed asymptotic series in the Laplace equation for the streamfunction and solving according to regular perturbation theory to obtain

$$\tilde{\Psi}(\tilde{\sigma}, \tilde{\tau}; \tilde{\tau}'_0) = (-1)^{j+1} G(\tilde{\tau}; 2j) V_{\tilde{\tau}'_0} \tilde{\tau}'_0^{2j} , \quad j \in \mathbb{N}_0 ,$$

where $G(\tilde{\tau}; 2j)$ denote the polynomial functions of $\tilde{\tau}$, expressed recursively as

$$G(\tilde{\tau}; 0) := \tilde{\tau} ; \quad G(\tilde{\tau}; 2j - 1) := \frac{\tilde{\tau}^{2j+1} - \tilde{\tau}}{(2j + 1)!} , \forall j \in \mathbb{N}_1 \quad ; \quad G(\tilde{\tau}; 2) := G(\tilde{\tau}; 1) ;$$

$$G(\tilde{\tau}; 2j) := G(\tilde{\tau}; 2j - 1) - \frac{G(\tilde{\tau}; 2k + 2)}{2(j - k) - 1} , \quad k \in \mathbb{N}_0 : k \in [0, j - 2] ,$$

$$\forall j \in \mathbb{N} : j \in [2, +\infty) .$$

The velocity vector, for irrotational flow is then

$$\tilde{U}(\tilde{\sigma}, \tilde{\tau}; \tilde{\tau}'_0) = (-1)^{j+1} G(\tilde{\tau}; 2j) V_{\tilde{\tau}'_0} \tilde{\tau}'_0^{2j-1} I_1 + (-1)^{j} G(\tilde{\tau}; 2j) V_{\tilde{\tau}'_0+1} \tilde{\tau}'_0^{2j} I_2 .$$

**Remark on the unsteady fluid motion:** The flow field, induced by the motion of the circular cylinders, is time dependent. The present formulation separates the effects of the unsteady fluid motion into two parts: a) fluid motion induced solely by and which is synchronised with the motion of the boundaries of the flow field and b) time-dependent fluid motion, which is independent of the motion of the boundaries. For irrotational flow, the second kind of fluid motion can be induced only by imposing time-dependent kinematic boundary conditions. For rotational flow, however, inertial effects, which are described by the $\partial_t$ derivative, can also induce unsteady fluid motion of the second kind.

For the present case of fluid motion, induced by the motion of the circular cylinders with constant velocity towards impact, the unsteady fluid motion is only of the first kind at the asymptotic limit of approach of the cylinders. For irrotational flow, this is only because the kinematic boundary conditions are independent of time. For rotational flow, this is because of the steady boundary conditions and the fact that the $\partial_t$ derivative does not appear in the differential equation for the leading order term of the asymptotic series for the streamfunction. The implication of this, is that the initial condition becomes redundant at the asymptotic limit considered and that the asymptotic series solution is independent of any initial flow field and thus, unique.
5 An interpretation of the asymptotic series solution

Consider the asymptotic series of the streamfunction for rotational and irrotational flow, respectively,

\[ \hat{\Psi}(\tilde{\sigma}, \tilde{\tau}; R; \tau_0') = i \hat{\Psi}(\tilde{\sigma}, \tilde{\tau})(\tau_0' R)^{1/2} \quad \text{and} \quad \hat{\Psi}(\tilde{\sigma}, \tilde{\tau}; \tau_0') = i \hat{\Psi}(\tilde{\sigma}, \tilde{\tau}) \tau_0'^{1/2} . \]

The streamfunction for irrotational flow, is decomposed into infinite kinematic scales of volume flow rates of order \( O[(U_s R)^{1/2}] \). The streamfunction for rotational flow is also decomposed, into infinite kinematic scales of volume flow rates of order \( O[(U_s R)^{1/2}] \). However, for this case, every kinematic scale is decomposed further, into infinite dynamic scales according to the dynamic scaling \( (\tau_0' R)^{1/2} \). The largest scale of the flow field of order \( O(U_s R) \) is described by \( i \hat{\Psi}(\tilde{\sigma}, \tilde{\tau}). \) The lower order terms, \( i \hat{\Psi}(\tilde{\sigma}, \tilde{\tau})(1 - \delta_{ij}) \), of the asymptotic series solution, are present and non-zero for all \( \tau_0' \), but, as \( \tau_0' \to 0 \), they represent asymptotically vanishing physical flow scales of order \( O[(U_s R)^{1/2}] \). \( i \hat{\Psi}(\tilde{\sigma}, \tilde{\tau}) \) are inter-dependent according to the differential equation

\[ \mathcal{D} (i \hat{\Psi}(\tilde{\sigma}, \tilde{\tau})(\tau_0' R)^{1/2}) = 0 , \]

where \( \mathcal{D} = (D_t \tilde{\nabla}^2 - Re^{-1} \tilde{\nabla}^4) \).

According to the present interpretation, the initial condition \( \hat{\psi}(\tilde{\sigma}, \tilde{\tau}; R; \tau_0') = i \hat{\psi}(\tilde{\sigma}, \tilde{\tau})(\tau_0' R)^{1/2} \), for the rotational flow field, in section 4, represents a multi-scale initial flow field. The magnitude of these scales, however, is adjusted according to the change of the \( \tau_0' \) parameter, although the flow field at every scale can be arbitrary.

At the limit \( Re \to 0 \), the asymptotic series solution, for rotational fluid motion, becomes an asymptotic series solution for \( \tilde{\nabla}^4 \hat{\Psi} = 0 \), as expected.

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The gradient $\nabla$ of a scalar $f(s)$ is

$$\nabla f(s) = f_i(s) I_i.$$  

$s_i$ are independent variables; therefore,

$$I_i = I_{m(i+1)} \times I_{m(i+2)} = \nabla s_i = \nabla s_{m(i+1)} \times \nabla s_{m(i+2)}.$$  

The divergence $\nabla \cdot$ of a vector field $f(s)$ is

$$\nabla \cdot f(s) = f_i(s) I_i.$$  

The local rotation of an arbitrary vector field $\nabla \times f(s)$ is obtained as

$$\nabla \times [f_i(s) I_i] = \nabla \times \{f_i(s) \cdot [\nabla s_{m(i+1)} \times \nabla s_{m(i+2)}]\} =$$

$$= f_i(s) \nabla \cdot [\nabla s_{m(i+1)} \times \nabla s_{m(i+2)}] + [\nabla s_{m(i+1)} \times \nabla s_{m(i+2)}] \cdot \nabla f_i(s) =$$

$$= f_i(s) \nabla \cdot [\nabla s_{m(i+1)} \times \nabla s_{m(i+2)}] - (\nabla \times \nabla s_{m(i+2)}) \cdot \nabla f_i(s) + [\nabla s_{m(i+1)} \times \nabla s_{m(i+2)}] \cdot \nabla f_i(s) = I_i \cdot \nabla f_i(s) = f_i(s).$$

Therefore,

$$\nabla \cdot f(s) = f_i(s).$$

The Laplace operator $\nabla^2$ of a scalar $f(s)$ and the operator $\nabla^2$ of a vector $f(s)$ satisfies the identities

$$\nabla^2 f(s) = \nabla \cdot [\nabla f(s)] \quad , \quad \nabla^2 f(s) = \nabla[\nabla \cdot f(s)] - \nabla \times [\nabla \times f(s)].$$

Therefore,

$$\nabla^2 f(s) = f_i(s) I_i.$$  

The biharmonic operator of a scalar $f(s)$ and of a vector $f(s)$, respectively, is

$$\nabla^4 f(s) = \nabla^2[\nabla^2 f(s)] = f_i(s) + f_i s_{m(i+1)}^2 (s) + f_i s_{m(i+2)}^2 (s),$$

$$\nabla^4 f(s) = \nabla^4 f_i(s) I_i.$$  

Therefore, in operational form,

$$\nabla := \partial_s I_i \quad , \quad \nabla \cdot := i \partial_{s_i} \quad , \quad \nabla \times := \left[ m^{(i+2)} \partial_{s_{m(i+1)}} - m^{(i+1)} \partial_{s_{m(i+2)}} \right] I_i,$$

$$\nabla^2 := \partial_{s_i}^2 \quad , \quad \nabla^2 := i \nabla^2 I_i \quad , \quad \nabla^4 := \partial_{s_i}^4 + \partial_{s_{m(i+1)}}^2 + \partial_{s_{m(i+2)}}^2 \quad , \quad \nabla^4 := i \nabla^4 I_i.$$