CAPITAL ASSET PRICING MODEL UNDER DISTRIBUTION UNCERTAINTY

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(Communicated by Yuanguo Zhu)

Abstract. In this paper, we investigate and demonstrate the capital asset pricing model (CAPM) based on distribution uncertainty (or ambiguity, defined as uncertainty about unknown probability).

We first achieve directly capital asset pricing model based on spectral risk measures (abbreviated as SCAPM) in the case of normal distributions; Then we can characterize SCAPM under the condition of uncertain distributions of returns by solving a robust optimal portfolio model based on spectral measures. Specifically, we do it in the following two folds: 1) Completing first the corresponding effective frontier fitting; 2) Getting the valuation of the market portfolio return $r_m$ and the risk parameters of $\beta_\phi$ in use of the kernel density estimation under the distribution uncertainty of returns.

Finally, by selecting 10 stocks from the constituent stocks of the HS300 Index, and comparing the valuation results from the SCAPM formula with the actual yield in the market, we verify the model proposed in the present paper is reasonable and effective.

1. Introduction. With the development of microeconomics in the capital market, there have been more and more discussions on the calculation of returns and risks of assets traded in the market, as well as the correlation between them. Capital asset pricing theory came into being at the historical moment. Following with the heated discussion on risk measurement and capital cost assessment in recent years, the capital asset pricing theory develops rapidly.

Since H. Markowitz [23] put forward mean-variance portfolio theory in 1952, scholars have officialy carried out researches on asset pricing. Soon afterwards, William Sharpe [30] and J. Lintner [22] proposed, respectively, a further theoretical model under strict market hypothesis: Capital asset pricing model (CAPM). CAPM formula is simple and easy to understand, easy to test, it has been rapidly developed in the academic field. In subsequent stages, the academic world continuously put forward various improvements and innovations to the model. The improvement of CAPM is largely dependent on the traceability of the model, also related to the adequacy of the portfolio return distribution [33]. However, in the actual investment process, investors are often exposed to the situation that the distribution of asset...
returns is unknown. In this case, it is difficult to directly and accurately give the asset pricing formula. Even when the distribution parameters of returns on assets are known, the calculating the spectral measure which is in form of high-dimensional integral is also a big challenge.

Dealing with such uncertain decision problems, the maxmin expected utility theory was proposed by Gilboa and Schmidler [12], which has been continuously developed and applied by scholars and extended to solve the portfolio problem with uncertain models. Meanwhile, the robust optimization methods have become increasingly popular in solving portfolio optimization problems with uncertain return distributions [32]. Based on empirical and historical distributions, an optimal “reference” model is assumed, ambiguity parameters are set to define the deviation between the real model and the reference model, and the relative entropy is used to control the deviation. With the help of maxmin expected utility theory, the expected utility maximization problem is solved in the worst situation for investors within the scope of relative entropy control. M. Salahi, F. Mehrdoust and F. Piri [27] extended the model based on the traditional CVaR model, introduced the uncertainty of mean value, and used the mean value form of CVaR to replace the original one. Kang, Li and Li [16] defined the ambiguity set to describe the uncertainty of real income distribution in terms of mean and variance, in addition, used the maximum-minimization expected utility framework to transform the combinatorial optimization problem of distribution uncertainty into a second-order cone programming problem.

Motivated by the researches of previous scholars, this paper investigates and studies asset pricing problems based on distribution uncertainty. The spectral risk measurement is selected as a risk measurement index being due to its excellent properties of consistency measurement and the corresponding descriptions of the subjective risk preferences of different investors. For the basic source of this article, one can refer to [7] for details.

This article is organized as follows. Section 2 introduces roughly the classical portfolio theory and traditional CAPM pricing model, as the main risk measurement tool used in this paper, the spectral measure is widely used in the specific problems in the following chapters. So in this section we also give definition of spectral measures. Section 3 discusses the asset pricing model of spectral measures under the normal distribution. In this section, a portfolio mean-spectrum risk model is constructed when returns follow the normal distribution, and the asset pricing formula SCAPM based on spectral risk measure is derived according to the expression of effective frontier boundary. When returns on risk assets and market portfolios don’t satisfy Sharpe hypothesis, the SCAPM formula is proposed to adjust. In section 4, we consider the SCAPM problem in the view of distribution uncertainty. Robust optimal control theory is adopted to construct a robust optimal portfolio model based on spectral measure, the corresponding effective frontier is fit. Then, the market portfolio \( M(r_m) \) is evaluated using the uncertain distribution of returns, and the estimator form under kernel density estimation method for risk parameter \( \beta_0 \) is given. Finally, empirical tests are carried out to verify the rationality of the model in section 5.

2. Preliminaries.

2.1. Markowitz Portfolio theory and expansion. Modern Asset Portfolio Theory proposed by Markowitz [23], believes that investors love returns and hope to
pursue the maximization of investment returns. They’re also risk averse, and they want to minimize portfolio risk. Therefore, Markowitz believed when making investment decisions, investors usually follow the following two rules: (1) Choose return portfolio with the least risk when expected return rates are the same; (2) Choose return portfolio with the highest expected return rate when investment risks are the same.

Assume that there are \( n \) risk assets available for investment and one risk-free asset in the market, the return rate of risk-free asset is \( r_f \), and the return rate of \( n \) risk assets is \( R = (r_1, \cdots, r_n)' \), expected return rate is \( \mu = (\mu_1, \cdots, \mu_n)' \), covariance matrix is \( \Sigma = (\sigma_{ij})_{n \times n} \), and the investment weight of \( n \) risk assets portfolio is \( x = (x_1, \cdots, x_n)' \), while the investment weight of risk-free asset is \( 1 - x'I \). In addition, it is assumed that rate of return on each asset obeys normal distribution, and the degree of correlation among assets is reflected in covariance.

When the short selling is allowed, we solve the quadratic programming equation

\[
\min_x \frac{1}{2} x' \Sigma x
\]

s.t. \( x' \mu = \mu_P, x'I = 1 \) (1)

The expression of the effective frontier of mean-variance model is obtained:

\[
\frac{\sigma_P}{1/C} - \frac{[\mu_P - A/C]^2}{D/C^2} = 1
\]

For given \( \mu_P \), optimal portfolio proportion is:

\[
x = \frac{C\mu_P - A}{D} \Sigma^{-1} \mu + \frac{B - A\mu_P}{D} \Sigma^{-1} I
\]

Substitute Equation (3) into Equation (1), the expected return and variance of the optimal portfolio can be obtained as follows:

\[
\mu_P = \frac{C\mu_P - A}{D} \mu' \Sigma \mu + \frac{B - A\mu_P}{D} \mu' \Sigma^{-1} I
\]

\[
\sigma_P^2 = \frac{C\mu_P^2 - 2A\mu_P + B}{D}
\]

Where \( A = I' \Sigma^{-1} \mu, B = \mu' \Sigma^{-1} \mu > 0, C = I' \Sigma^{-1} I > 0, D = BC - A^2 \).

When the covariance matrix \( \Sigma \) is a positive definite matrix, we can know that \( \sigma_P^2 \) in Equation (5) is a quadratic function of \( \mu_P \). Standard deviation is used to describe risk, with \( \mu_P \) as the vertical axis and \( \sigma_P \) as the horizontal axis. According to Equation (5), a hyperbola can be obtained, which is called the minimum variance frontier, and the point \((\frac{1}{C}, \frac{A}{C})\) is called the global minimum variance portfolio. The part of the hyperbola that above this point is called effective frontier of the portfolio.

Construct a portfolio for above risk assets considering risk-free asset. The return rate of the new portfolio is \( R_P = (1 - x'I)r_f + x'R \), the expected return rate is \( \mu_P = (1 - x'I)r_f + x'\mu \), and variance is \( \sigma_P^2 = x' \Sigma x \). Substituting the above parametric solution (1), we get the equation of the minimum variance effective frontier containing risk-free asset:

\[
\sigma_P = \frac{\mu_P - r_f}{\sqrt{B - 2r_f A + r_f^2 C}}
\]

The minimum variance effective frontier containing risk-free asset is a straight line starting from point \((0, r_f)\) and intersecting (including tangency) with the effective
frontier of risk assets, which is called capital allocation line (CAL). In addition, The capital allocation line tangent to the effective frontier of risk assets is called the capital market line (CML), and the tangency point is called market portfolio.

**Definition 2.1.** For portfolio $P$, the risk premium per unit risk is:

$$ S = \frac{E(r_p) - R_f}{\sigma_p} $$

$S$ is called Sharpe ratio, which describes the excess risk return of the portfolio $P$.

**Theorem 2.2.** The relationship between Sharpe ratio $S_i$ of assets and Sharpe ratio $S_m$ of the current market portfolio is as follows

$$ S_i = \rho_{im} \cdot S_m. $$

The market portfolio has the largest Sharpe ratio $S_m$.

**Proof.** It is easy to check from [9, 30] that Theorem 2.2 is tenable.

**Theorem 2.3** ([30]). For any single risk asset $i$, whose expected rate of return $E(R_i)$ has the following remarkable CAPM related to the expected rate of return $E(R_m)$ of market portfolios

$$ E(R_i) = R_f + (E(R_m) - R_f)\beta_i, \quad \text{where} \quad \beta_i = \frac{\sigma_{im}}{\sigma_m^2} $$

$\beta_i$ is the systemic risk of a single risk asset, $E(R_m) - R_f$ is called risk premium of the market portfolio, which represents the compensation of return that market portfolio exceeds risks at risk-free rate.

The coefficient $\beta$ in CAPM formula measures the sensitivity of a single asset to market return fluctuations. Take a security as an example, when the security moves exactly in line with the market price, it has a unit $\beta$ value. If the value of $\beta$ is greater than 1, it means that changes of security price are more affected by the market, and the difference of security price changes is greater than the difference of market changes. So investors need to take more risks, in addition, the expected return rate estimated by CAPM formula will be higher.

### 2.2. Spectral risk measure and properties.

**2.2.1. Definition of spectral risk measure.** The concept of spectral measures was proposed by Acerbi [1] in 2002, which is also a coherent measure of risks. Before introducing spectral measure, an important theorem is introduced:

**Theorem 2.4.** [1] Let $\rho_i (i = 1, \cdots, n)$ be a coherent measure of risks, then any convex combination about it $\rho = \sum_i \alpha_i \rho_i$ ($\alpha_i \in \mathbb{R}^+$ and $\sum \alpha_i = 1$) is also a coherent measure of risks.

Similarly, if $\rho_\alpha (\alpha \in [a, b])$ is a single parameter set of coherent measure of risks, then $\forall \alpha \in [a, b]$, for measure $d\mu(\alpha)$, and satisfies $\int_a^b d\mu(\alpha) = 1$, the measure is defined as

$$ \rho = \int_a^b d\mu(\alpha) \rho_\alpha $$

It’s also a coherent measure of risks.
**Definition 2.5.** For a distribution function of portfolio return $F_X(x) = P[X \leq x]$, its inverse function is $F_X^-(p) = \inf\{x | F_X(x) \geq p\} = VaR_p(X)$. Then, at confidence level $(1 - \alpha) \in (0, 1)$, the expected loss ES is defined as:

$$ES_\alpha(X) = -\frac{1}{\alpha} \int_0^\alpha F_X^-(p)dp$$  \hspace{1cm} (11)

It satisfies the definition of coherent measures of risks.

When $\alpha = 0$, define $ES_{(0)}(X) = -\text{ess.inf} X$ in the worst case. In the case of continuous distribution of returns, CVaR has the same definition as ES:

$$CVaR_\alpha(X) = TCE_\alpha(X) = -E[X | X \leq F_X^-(\alpha)]$$  \hspace{1cm} (12)

**Definition 2.6.** Let the measure $d\mu_{(\alpha)}$ on $\alpha \in [0, 1]$ satisfy the normalized integral condition $\int_0^1 \alpha d\mu_{(\alpha)} = 1$. The following integral $M_\mu(X)$ is a coherent measure of risks

$$M_\mu(X) = \int_0^1 \alpha ES_{\alpha}(X)d\mu_{(\alpha)} = -\int_0^1 d\mu_{(\alpha)} \int_0^\alpha F_X^-(p)dp$$  \hspace{1cm} (13)

For the above formula, by Fubini-Tonelli integral exchange theorem, then there is

$$M_\mu(X) = -\int_0^1 dpF_X^-(p) \int_0^1 d\mu_{(\alpha)} = -\int_0^1 dpF_X^-(p)\phi(p) = M_\phi(X)$$  \hspace{1cm} (14)

where $\phi(p) = \int_p^1 d\mu_{(\alpha)}$ is called risk spectrum, substituting it into $\int_0^1 \alpha d\mu_{(\alpha)} = 1$, we obtain the normalization condition of $\phi(p)$ as follows:

$$\int_0^1 \phi(p)dp = \int_0^1 dp \int_p^1 d\mu_{(\alpha)} = \int_0^1 d\mu_{(\alpha)} \int_0^\alpha dp = \int_0^1 d\mu_{(\alpha)} \alpha = 1$$  \hspace{1cm} (15)

For any $d\mu(\alpha)$ that satisfies the normalization integral condition, $M_\mu(X)$ defined in Equation (13) can be expressed as the measure $M_\phi(X)$ in Equation (14), in combination with risk spectrum $\phi(p) = \int_p^1 d\mu_{(\alpha)}$.

Define the norm of $\phi(p) \in L^1[0, 1]$ by $\phi(p) = \int_p^1 d\mu_{(\alpha)}$ as:

$$||\phi|| = \int_0^1 |\phi(p)|dp.$$  \hspace{1cm} (16)

According to the properties of monotony and positive definite of elements in the space $L^1[0, 1]$, we have the following definitions.

**Definition 2.7.** [1] $\forall I \subset [a, b]$, if $\int_I \phi(p)dp \geq 0$ is satisfied, the element $\phi \in L^1[0, 1]$ is called “positive definite”; $\forall q \in [a, b]$ and $\forall \varepsilon > 0$, $[q - \varepsilon, q + \varepsilon] \subset [a, b]$, satisfies $\int_{q-\varepsilon}^{q+\varepsilon} \phi(p)dp \geq \int_q^{q+\varepsilon} \phi(p)dp$, the element $\phi \in L^1[0, 1]$ is called decreased.

According to Definition (2.7), the definition of risk spectrum function is given as follows:

**Definition 2.8.** [1] $\phi \in L^1[0, 1]$ is an admissible risk spectrum that satisfies the following three conditions

1. $\phi$ is positive definite
2. $\phi$ is decreased
3. $||\phi|| = 1$

According to Definition (2.8), there are the following
Definition 2.9. Define $M_\phi(X) = -\int_0^1 F_X^+(p)\phi(p)dp$ and $\phi \in L^1[0,1]$. When $\phi$ is the admissible risk spectrum, $M_\phi(X)$ is called spectral risk measure generated by $\phi$.

Definition 2.10. Allowable risk spectrum function $\phi \in L^1[0,1]$ is called “risk aversion function” of risk measure $M_\phi(X) = -\int_0^1 F_X^+(p)\phi(p)dp$. Similarly, the corresponding risk measure $M_\phi$ is called spectral risk measure generated by $\phi$.

So when $M_\phi$ is a coherent measure of risks, the property requirements of risk spectrum $\phi \in L^1[0,1]$ are as follows:

Theorem 2.11. When $\phi \in L^1[0,1]$, if and only if $\phi$ is the allowable risk spectrum, $M_\phi(X)$ is a coherent measure of risks.

Acerbi [1] proposed to use ES expected loss to define $F_X^+(p)$. The advantage of ES is that it takes all expected losses greater than VaR value into account, considering the tail characteristics of return distribution.

Compared with ES, spectral risk measure describes the tail characteristics of return distribution in more detail. In combination with definition 2.9, the risk spectrum function $\phi(p)$ gives different weights to all loss values exceeding VaR$_p$ in the composition of spectral risk measure. In addition, according to decline property of $\phi(p)$, the loss value in the tail is endowed with greater importance of tail loss, which is consistent with the subjective sentiment of rational investors’ risk aversion in the market.

2.2.2. Construction and properties of spectral risk measures. Risk spectrum function $\phi(p)$ plays a decisive role in the composition of spectral risk measures. According to previous researches, any risk spectrum function can be written in the form of truncation function [18]:

$$\phi(p) = f(\alpha, p)I_{0 < p \leq \alpha}, \alpha \in (0,1] \quad (17)$$

Considering that extreme events have a small probability of occurrence but can cause huge losses, they are generally located at the left tail of return distribution, and the confidence level is usually set as $1 - \alpha \geq 95\%$, that is $\alpha \leq 0.1$.

Considering that when risk spectrum $\phi(p)$ is written in the form of the truncation function in Equation (17), in combination with the definition of allowable risk spectrum in 2.8, when spectral risk measure $M_\phi(X)$ is a coherent measure of risk, if and only if $f(\alpha, p)$ meets the following properties:

1. For $\forall I \subset [0,\alpha]$, $f(\alpha, p)$ is positive definite.
2. $\forall q \in [0,\alpha], \varepsilon > 0$ and $[q - \varepsilon, q + \varepsilon] \subset [0,\alpha]$, $f_{q-\varepsilon}^q f(\alpha, p)dp \geq f_{q}^{q+\varepsilon} f(\alpha, p)dp$, that is $f(\alpha, p)$ decreases with respect to $p$.
3. $\int_0^\alpha f(\alpha, p)dp = 1$.

According to Theorem 2.11, there is:

Proposition 1. $f(\alpha, p)$ is separable with respect to $\alpha$ and $p$, that is, $f(\alpha, p) = g(\alpha)h(p)$. If and only if the above three properties of $f(\alpha, p)$ are satisfied, and $h(p) \geq 0, g(\alpha) = 1/\int_0^\alpha h(p)dp$, $\phi(p)$ is the allowable risk spectrum, the corresponding measure $M_\phi(X)$ is spectral risk measure.

Satisfying Proposition 1, $M_\phi(X) = -\int_0^1 F_X^+(p)\phi(p)dp$, abbreviated as $M$, several concrete forms of constructing allowable risk spectrum are given below.
Definition 2.12. (geometric spectral risk function), denote $GM = M$, if
\[
\phi(p) = \frac{1}{\ln(1 + \alpha)} \frac{1}{1 + p} 1_{0 < p \leq \alpha}
\]  
(18)

Definition 2.13. (exponential spectrum risk function), denote $EM = M$, if
\[
\phi(p) = \frac{e^{-p}}{1 - e^{-\alpha}} 1_{0 < p \leq \alpha}
\]  
(19)

GM and EM are all allowable risk spectral functions, and spectral risk measures constituted by them all satisfy properties of coherent measure of risk. As functions of allowable risk spectrum, the decrease property of GM and EM well reflects when tail loss increases, measurement of risk also increases, which is consistent with investors’ subjective risk preference. In addition, GM and EM use weighted average of a single decrease for tail return, it is more in line with the emotion of general investors in the market.

3. Asset pricing based on spectral risk measures. Based on mean-variance model by H. Markowiz, the CAL line and CML line are derived through the effective frontier expression, and the asset pricing formula CAPM, which uses variance to measure risk under normal distribution of returns, is obtained. However, with further research on risk measurement in recent years, it has been found that variance has a major defect as a risk measurement. First, variance measures the risk of bilateral return distribution, which includes both the profit and loss of investment returns in real market. Therefore, using variance will overestimate the risk. Secondly, variance only measures return on the objective market, while it is difficult to effectively measure the subjective emotions that affect investors’ behavior.

Therefore, VaR, CVaR and consistency risk measurement were put forward successively. Value-at-Risk (VaR) was first used widely because of its simple definition. However, it has several limitations, one defect is that it doesn’t satisfy subadditivity. Therefore, consistent risk measures such as conditional value-at-risk (CVaR) and spectral risk measure are derived to overcome the drawbacks. Spectral risk measure introduces spectral function $\phi(p)$ to describe investors’ subjective aversion. Mean-CVaR effective frontier under normal distribution is proposed in [19, 20]; X. L and X. Li [18], Deng [9] continue to put forward the model of portfolio management based on spectral risk measure, in addition, the efficient frontier of variance-spectral risk measure model under normal distribution is given. Considering there are many existing researches on the mean-portfolio problem with spectral risk measure, this section we will do research on the asset pricing problem based on mean-spectral risk measure under normal distribution.

3.1. Mean-spectral risk measure portfolio model. The return on risk assets in the market studied in this section follows normal distribution, and the following parameters are defined as follows: build a portfolio of risk assets with $n$ investable risk assets in the market, the return of the portfolio is $R(x, \xi) = x'\xi$. The investment decision vector is $x = (x_1, x_2, \cdots, x_n)'$, $x_i$ represents the i-th financial asset in the portfolio, and the rate of return vector is $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$, $\xi_i$ represents the rate of investment return of the i-th financial asset in the portfolio.

The feasible set is $X = x \in \mathbb{R}^n : x'I = 1$. Assume that the asset return vector obeys normal distribution, $\xi \sim N(\mu, V)$, then we have the portfolio return $R(x, \xi) = x'\xi \sim N(x'\mu, x'Vx)$. 

For simple, we denote by \( R(x, \xi) = r_x \). For the hypothesis of normal distributions, the spectral risk measure of the portfolio is defined as follows:

\[
M(r_x) = T(\alpha)\sigma(r_x) - E(r_x)
\]

where \( T(\alpha) = -\int_0^\alpha f(\alpha, p)\Phi^{-1}(p)dp \), \( \Phi \) is the distribution function of standard normal distribution \( N(0, 1) \), \( \Phi^{-1}(p) \) is the \( p \) underside quantile of standard normal distribution \( N(0, 1) \), \( \phi \) is admissible spectrum.

We state simply the relationship between \( T(\alpha) \) and \( \alpha \):

**Proposition 2.** [18] In the definition, \( f(\alpha, p) \) is the spectral risk measure in the form of a function defined in Proposition 1, so \( T(\alpha) \) is a minus function of \( \alpha \). And when \( \alpha \to 0^+ \), \( T(\alpha) \to +\infty \).

By Proposition 2, when the portfolio is determined, the corresponding \( E(r_x), \sigma(r_x) \) of portfolio are also determined. By \( M(r_x) = T(\alpha)\sigma(r_x) - E(r_x), \) when \( \alpha \) decreases, \( T(\alpha) \) increases, and \( M(r_x) \) is more affected by \( \sigma(r_x) \).

To solve the mean-spectrum risk measure model of the portfolio under the confidence level of \( 1 - \alpha \), that is, to solve the following quadratic programming problem.

\[
\begin{align*}
\min_{x \in X} M(r_x) &= T(\alpha)\sigma(r_x) - E(r_x) = T(\alpha)\sqrt{x'Vx} - x'\mu \\
\text{s.t.} E(r_x) &= E(x'\xi) = x'\mu = r, x'I = 1
\end{align*}
\]

(21)

It is easy to show (one can refer [9, 19, 20] for details) that the expression of the effective frontier of mean-variance model is given in (21).

\[
\frac{\sigma^2(r_x)}{1/C} - \frac{[E(r_x) - A/C]^2}{D/C^2} = 1
\]

where \( A = IV^{-1}\mu, B = \mu'V^{-1}\mu > 0, C = IV^{-1}I > 0, D = BC - A^2, \) and \( x = (Cr - A)\sqrt{V^{-1}\mu}/D + (B - Ar)V^{-1}I/D \). According to the given \( r \), feasible solutions satisfying the above model constitute the effective frontier of mean-spectral risk measure, as follows

**Definition 3.1.** At the confidence level \( 1 - \alpha \) \( (0 < \alpha < 1) \), for conditions \( E(r_x) \geq E(r_x^*) \) and \( M(r_x) \leq M(r_x^*) \), if the combination \( x^* \in X \) belongs to the effective frontier of mean-spectral risk measure \( \Leftrightarrow \) there is no combination \( x \in X \), the above inequality can be satisfied at the same time. At least one condition above should be strictly satisfied for \( (E(r_x^*), M(r_x^*)) \) corresponding to the feasible solution \( x^* \), i.e. the points which form the effective frontier.

Definition (3.1) means that the effective frontier of mean-spectral risk measure is inevitable on the optimal portfolio boundary, meanwhile, is the upper part of the boundary, starting from the combination of minimum variance.

According to \( E(r_x) = r, T(\alpha) > 0 \), the mean-variance model and mean-spectrum measure model under normal distribution of returns have the same solution \([9] \), then the following Proposition is true.

**Proposition 3.** Portfolio \( x \) belongs to mean-spectral risk measure boundary \( \Leftrightarrow \) portfolio is also belongs to mean-variance boundary. Substitute Equation (20) into Equation (22), we can obtain the effective frontier boundary of mean-spectral risk measure

\[
\frac{[(M(r_x) + E(r_x))/T(\alpha)]^2}{1/C} - \frac{[E(r_x) - A/C]^2}{D/C^2} = 1
\]

(23)
Proposition 4. [20] For the confidence level as $1 - \alpha (0 < \alpha < 1)$, the minimum spectral risk measure combination $x^*$ exists if and only if $T(\alpha) > \sqrt{D/C}$.

If the condition is satisfied, the minimum spectral risk measure combination $(M(x^*), E(x^*))$, and the decision variable $x^*$ are obtained as follows

$$
x^* = m + nE(x^*), \quad m = \frac{1}{D}[B(V^{-1}I) - A(V^{-1}\mu)], \quad n = \frac{1}{D}[C(V^{-1}\mu) - A(V^{-1}I)]$$

$$
E(r_{x^*}) = \frac{A}{C} + \sqrt{\frac{D}{C}(\frac{T^2(\alpha)}{C \cdot T^2(\alpha) - D} - \frac{1}{C})}
$$

$$
M(r_{x^*}) = T(\alpha) \sqrt{\frac{T^2(\alpha)}{C \cdot T^2(\alpha) - D} - E(r_{x^*})}
$$

At the confidence level $1 - \alpha$, $E(r^*_{x^*}) > E(r^*_{\sigma}) = A/C$, indicating that the portfolio is located above the mean-variance effective frontier with market condition when the existence condition of the minimum spectral risk measure is satisfied.

According to Proposition 3 and Proposition 4, the existence and form of the effective frontier based on spectral risk measure under normal distribution are obtained.

Proposition 5. For the confidence level $1 - \alpha$, the portfolio is at the effective frontier of mean-spectral risk measure if and only if $T(\alpha) > \sqrt{D/C}$, which $\Leftrightarrow$ the portfolio is on the boundary of mean-spectral risk measure, and its expected return level is greater than or equal to that of the minimum spectral risk measure portfolio at the confidence level $1 - \alpha$.

In this case, the mean-spectrum effective frontier equation of risk asset portfolio is:

$$
M = T \sqrt{\frac{1}{C} + \frac{C}{D}(r - A/C)^2 - r} \quad (24)
$$

And $r \geq \frac{A}{C} + \sqrt{\frac{D}{C}(\frac{T^2}{CT^2 - D} - \frac{1}{C})}$, $A = \mu'V^{-1}\mu, B = \mu'V^{-1} > 0, C = \mu'V^{-1}I > 0, D = BC - A^2$.

Proposition 6. At the confidence level $1 - \alpha$, denote the effective set of mean-spectral risk measure and the effective solution set of mean-variance as $X_M$ and $X_\sigma$, respectively, and then there holds the relations between these sets with $X_M \subseteq X_\sigma \subseteq X$.

The above conclusions prove that when spectral risk measure is used as a risk measure, from the perspective of portfolio, it narrates the range of effective solution set compared with original variance measure, which indicates that spectral risk measure is more conservative in risk measurement under normal conditions.

3.2. Investment strategies and market portfolios with risk-free asset. The effective boundary based on spectral risk measure defined by Equation (24) only takes investment in risky assets into consideration. A ray from risk-free assets is tangent to the effective boundary based on spectral risk measure at point $m$, which is called CML. The Sharpe ratio $S_\phi$ based on spectral risk measure at point $m$ is defined as follows

$$
S_\phi = \frac{E(r_m) - r_f}{M(r_m)} \quad (25)
$$

According to Definition 2.1, the tangency point $m$ is the market portfolio, where $S_\phi$ is the maximum, equals to the slope of tangent line CML.
According to the effective frontier of mean-spectral risk measure Equation (24), take the derivative of $r$ on both sides, then we get
\[
\frac{dM}{dr} = \frac{CT(r - A/C)/D}{\sqrt{\frac{1}{C} + \frac{C}{D}(r - A/C)^2}} - 1
\]  
(26)

So:
\[
\frac{dr}{dM} = 1 \frac{CT(r - A/C)/D}{\sqrt{\frac{1}{C} + \frac{C}{D}(r - A/C)^2}} - 1
\]  
(27)

In combination with Equation (25), there is:
\[
\frac{E(r_m) - r_f}{M(r_m)} = \frac{dr}{dM} = 1 \frac{CT(r - A/C)/D}{\sqrt{\frac{1}{C} + \frac{C}{D}(r - A/C)^2}} - 1
\]  
(28)

Solve the following simultaneous equations:
\[
\begin{cases}
M = T \sqrt{\frac{1}{C} + \frac{C}{D}(r - A/C)^2} - r \\
E(r_m) - r_f = 1 \frac{CT(r - A/C)/D}{\sqrt{\frac{1}{C} + \frac{C}{D}(r - A/C)^2}} - 1
\end{cases}
\]  
(29)

The expected return and spectral risk measure of market portfolio at the tangent point $m$ are:
\[
E(r_m) = \frac{B - A r_f}{A - C r_f}
\]
\[
M(r_m) = \frac{T \sqrt{B - 2 A r_f + C r_f^2 + A r_f - B}}{A - C r_f}
\]  
(30)

The investment proportion of market portfolio at tangent point $m$ is
\[
x_m = \frac{V^{-1}(\mu - r f I)}{A - C r_f}
\]  
(31)

3.3. SCAPM model under normal distributions. Suppose there are $n$ kinds of risk assets available for investment, and the return rate of these $n$ kinds of risk assets is $R = (r_1, \cdots, r_n)'$ under normal distribution, the expected return rate is $\mu = (\mu_1, \cdots, \mu_n)'$, the covariance of return is $V$, $I = (1, 1, \cdots, 1)'$ is unit vector, the investment ratio of $n$ kinds of risk assets is $x = (x_1, \cdots, x_n)'$. The risk-free asset is $r_f$. The following portfolio model formula (32) is constructed to make investment on $n$ kinds of risk assets and find the minimum spectral risk measure of portfolio $P$, when the expected return is $r_P$.
\[
\min_x \quad T(\alpha) \sqrt{x'Vx - x'\mu}
\]
\[
s.t. \quad x'\mu = r_P, \quad x'I = 1
\]  
(32)
In the model above, we have $T(\alpha) = -\int_0^\alpha f(\alpha, p)\Phi^{-1}(p)dp$ to get the expected return rate of market portfolio as $E(r_m)$, and the spectral risk measure as $M(r_m)$. For any asset $i$ in the market, there are following theorems.

**Theorem 3.2.** Assume return on any risk asset $i$ in the market satisfies normal distribution, where $M_{ri}$ is spectral risk measure of asset $i$, $\sigma_i$ represents the standard deviation of return on asset $i$, and $E(r_i)$ represents the expected return on asset $i$. When $T(\alpha) > \sqrt{D/C}$, the asset pricing formula based on spectral measure (SCAPM) is:

$$E(r_i) = r_f + (E(r_m) - r_f) \cdot \beta_\phi,$$

$$\beta_\phi = \frac{T \cdot (\sigma_{im} - \sigma_m^2)}{M(r_m)\sigma_m + \sigma_m(E(r_m) - r_f)} + 1$$  \hspace{1cm} (33)

Where $T = T(\alpha) = -\int_0^\alpha f(\alpha, p)\Phi^{-1}(p)dp$.

**Proof.** For any single asset $i$, we construct its portfolio line with the market portfolio $m$. It is known that $m$ is an effective portfolio, so the portfolio line must be tangent to the capital market line at the point $m$, that is, the slope of the point $m$ is $\frac{E(r_m) - r_f}{M(r_m)}$. Assume any point $P$ on the portfolio line is composed of single asset $i$ which weight is $x_i$ and market portfolio weight $1 - x_i$, then we have:

$$E(r_P) = x_i \cdot E(r_i) + (1 - x_i) \cdot E(r_m)$$

$$M(r_P) = T(\alpha) \cdot \sigma_P - E(r_P)$$

where $\sigma_P = \sqrt{x_i^2 \cdot \sigma_i^2 + (1 - x_i)^2 \cdot \sigma_m^2 + 2x_i(1 - x_i)\sigma_{im}}$.

Also

$$\frac{dE(r_P)}{dx_i} = E(r_i) - E(r_m)$$

$$\frac{dM(r_P)}{dx_i} = T \cdot \frac{x_i(\sigma_i^2 + \sigma_m^2 - 2\sigma_{im}) + \sigma_{im} - \sigma_m^2}{\sigma_P} - [E(r_i) - E(r_m)]$$  \hspace{1cm} (35)

So we have

$$\frac{dE(r_P)}{dM(r_P)} = \frac{E(r_i) - E(r_m)}{x_i(\sigma_i^2 + \sigma_m^2 - 2\sigma_{im}) + \sigma_{im} - \sigma_m^2 - [E(r_i) - E(r_m)]}$$  \hspace{1cm} (36)

At point $m$, $x_i = 0$, $\sigma_P = \sigma_m$, substituted into the above formula we get:

$$\left.\frac{dE(r_P)}{dM(r_P)}\right|_{x_i=0} = \frac{E(r_i) - E(r_m)}{T \cdot \frac{\sigma_{im} - \sigma_m^2}{\sigma_m} - [E(r_i) - E(r_m)]} = \frac{E(r_m) - r_f}{M(r_m)}$$  \hspace{1cm} (37)

Sort out:

$$E(r_i) = E(r_m) + \frac{T \cdot (E(r_m) - r_f) \cdot (\sigma_{im} - \sigma_m^2)}{M(r_m)\sigma_m + \sigma_m(E(r_m) - r_f)}$$  \hspace{1cm} (38)

Converted into a form that includes risk parameter $\beta_\phi$ we get:

$$E(r_i) = r_f + (E(r_m) - r_f) \cdot \beta_\phi,$$

$$\beta_\phi = \frac{T \cdot (\sigma_{im} - \sigma_m^2)}{M(r_m)\sigma_m + \sigma_m(E(r_m) - r_f)} + 1$$  \hspace{1cm} (39)
The risk parameter in asset pricing model SCAPM based on spectral measure, is different from that in the traditional CAPM. Risk parameter $\beta_\phi$ contains the function $T(\alpha)$ related to confidence level $1 - \alpha$, and it is known that $T(\alpha)$ is a decreasing function of $\alpha$. The relationship between $\beta_\phi$ and $\alpha$ is discussed as follows.

**Theorem 3.3.** The covariance of asset $i$ and market portfolio $m$ is $\sigma_{im}$, when $\sigma_{im} > \sigma_m^2$, $\beta_\phi$ and $\alpha$ change in the same direction; when $\sigma_{im} < \sigma_m^2$, $\beta_\phi$ and $\alpha$ change in the opposite direction.

**Proof.** Take the derivative of $\beta$ with respect to $\alpha$ in Equation (39)

$$
\frac{d\beta_\phi}{d\alpha} = \frac{(\sigma_{im} - \sigma_m^2) \cdot \sigma_m \cdot [(E(r_m) - r_f) + \frac{Ar_f - B}{A - Cr_f}]}{(M(r_m)\sigma_m + \sigma_m (E(r_m) - r_f))} \cdot \frac{d\alpha}{dT}
$$

Substitute Equation (30) into the molecule we obtain

$$
\frac{d\beta_\phi}{d\alpha} = \frac{(\sigma_{im} - \sigma_m^2) \cdot -r_f}{[M(r_m)\sigma_m + \sigma_m (E(r_m) - r_f)]^2} \frac{d\alpha}{dT}
$$

According to Proposition 2, when the return distribution is determined to be normal, there is $\frac{d\alpha}{dT} = \frac{1}{dT} < 0$, then we discuss the positive and negative relationship of $\frac{d\beta_\phi}{d\alpha}$:

1. When $\sigma_{im} > \sigma_m^2$, $\beta_\phi$ increases with increasing $\alpha$. At this time, the volatility of the asset $i$ is greater than the market volatility, that is, it is more sensitive to the market changes, can be called a high-risk asset. With the increase of $\alpha$, investors have a strong aversion to risk, that is, they have a high excess expectation for risk, and return on assets determined by SCAPM formula will be correspondingly higher.

2. When $\sigma_{im} < \sigma_m^2$, $\beta_\phi$ decreases with increasing $\alpha$. At this time, the volatility of the asset $i$ is less than that of the market, indicating that the change of the asset is relatively stable compared with that of the market. This asset may be a medium-low risk asset or related to the necessities of life. When this kind of asset is priced by SCAPM, even if investors are more risk averse, i.e., $\alpha$ increases, the expected return of it will not continue to increase, but will be kept at a reasonable level correspondingly.

When SCAPM formula is used to evaluate risk assets with high market sensitivity, the risk coefficient $\beta_\phi$ can reflect the risk aversion preference of investors better, and return on assets obtained by SCAPM formula is higher.

3.4. Investigation of SCAPM model: $\rho_{im} = 0$ or $\beta = 0$. The classical CAPM model is derived from Sharpe’s basic assumption of asset return. Sharp proposed the basic idea of single-factor model in 1963, believed that there is a linear correlation between risk assets in market and return on market portfolio, in addition, $\beta$ represents the influence degree of market portfolio volatility on return on risk assets.

Due to strict theoretical assumptions of the model, SCAPM problem is discussed when linear correlation hypotheses, between return on assets and market portfolios, are not fully satisfied in the real market.
Firstly, starting from the formula composition, CAPM and SCAPM formulas are studied to observe factors that influence formula measurement results.

CAPM formula proposed based on Markowitz’s mean-variance model is:
\[
E(r_i) = r_f + (E(r_m) - r_f) \cdot \beta, \quad \beta = \frac{\sigma_{im}}{\sigma_m^2} = \rho_{im} \frac{\sigma_i}{\sigma_m}
\] (42)
The expansion of risk coefficient \(\beta\) is shown as above, it contains the correlation coefficient \(\rho_{im}\) between asset \(i\) and market portfolio \(m\).

Similarly, SCAPM formula based on spectral measure and its correlation coefficient \(\beta_\phi\) are as follows:
\[
E(r_i) = r_f + (E(r_m) - r_f) \cdot \beta_\phi, \quad \beta_\phi = \frac{T \cdot (\rho_{im} \sigma_i - \sigma_m)}{M_r + (E(r_m) - r_f)^2} + 1
\] (43)
It can be seen from the formula that the expected return on asset \(i\) is related to the risk coefficient \(\beta\), and Pearson correlation coefficient is included in the composition of \(\beta\) by observing Equation (42) and Equation (43). Furthermore, Pearson correlation coefficient can only measure the degree of linear correlation between variables. However, in the real market, considering that there may be situations that do not fully satisfy Sharpe’s hypotheses, there may be a nonlinear relationship between returns on assets and market portfolios, as follows:
\[
r_i = a_i + b_i r_m^2 + \epsilon, \quad b \neq 0
\] (44)
At this time, there is still a correlation between ROA (returns of assets) and market portfolios. However, since the correlation is nonlinear, Pearson correlation coefficient is used to measure, with \(\rho_{im} = 0\), the correlation between ROA and market portfolio cannot be accurately reflected.

To solve return on assets estimated by SCAPM formula, the key lies in calculating risk coefficient \(\beta_\phi\), meanwhile, Pearson correlation coefficient \(\rho_{im}\) included in the expression of \(\beta_\phi\), measures the linear relationship between return rate \(r_i\) of asset \(i\) and return rate \(r_m\) of market portfolio \(m\). In the case of Sharpe’s linear correlation hypothesis is not fully satisfied, we consider to find a more appropriate measurement coefficient.

Therefore, the generalized correlation coefficient SEV (Sure Explained Variability) is considered to introduce, which can improve Pearson correlation coefficient \(\rho_{im}\) in \(\beta_\phi\).

According to the definition of generalized correlation coefficient SEV[21], we have:

**Definition 3.4.** For asset \(i\), \(r_i\) is the dependent random variable, the return \(r_m\) on market portfolio \(m\) is the explanatory variable, the variance of \(r_i\) is decomposed in terms of \(r_m\) as follows:
\[
var(r_i) = var(E(r_i|r_m)) + E(var(r_i|r_m)),
\] (45)
\(E(r_i|r_m)\) is the conditional expectation of \(r_i\) under a given \(r_m\), and \(var(E(r_i|r_m))\) represents the degree of dispersion of \(r_i\) due to \(r_m\). Define the interpreted variance of \(r_m\) over \(r_i\), which is called as \(SEV(r_i|r_m)\):
\[
SEV(r_i|r_m) = \frac{var(E(r_i|r_m))}{var(r_i)} = 1 - \frac{E[(r_i - E(r_i|r_m))^2]}{var(r_i)}
\] (46)
Assume that \(f(r_m, r_i)\) is the joint density function of random variables \(r_m\) and \(r_i\), while \(f(r_m)\) is the marginal density function of independent variable \(r_m\), when
substituted into Equation (46), the estimator of SEV can be expressed as follows:

\[
SEV(r_i|r_m) = \frac{E[E(r_i|r_m)^2 - (E(r_i))^2]}{\text{var}(r_i)}
\]

\[
= \int (\mu^r_i|r_m)(r_m)^2 f(r_m) dr_m - (E(r_i))^2 \]

(47)

Where

\[
\mu^r_i|r_m(r_m) = E(r_i|r_m) = \frac{\int r_i f(r_m, r_i) dr_i}{f(r_m)}
\]

(48)

Assume that random sample is \{\(\{r_{i,t}, r_{m,t}\}, t = 1, \ldots, n\}\), \(\mu^r_i|r_m\) and \(f(r_m)\) can be estimated. The following is the estimated value obtained by using the kernel density estimation method, where \(K(\cdot)\) is the kernel function and \(h\) is the window width.

\[
\int r_i f(r_m, r_i) dr_i = \frac{1}{nh} \sum_{t=1}^{n} K\left(\frac{r_m - r_{m,t}}{h}\right) r_{i,t}
\]

(49)

\[
f(r_m) = \frac{1}{nh} \sum_{t=1}^{n} K\left(\frac{r_m - r_{m,t}}{h}\right)
\]

By substituting Equation (49) into Equation (48), the estimate value of \(\mu^r_i|r_m\) can be obtained:

\[
\mu^r_i|r_m(r_m) = \begin{cases} \sum_{t=1}^{n} K\left(\frac{r_m - r_{m,t}}{h}\right) r_{i,t}, & \text{when } \sum_{t=1}^{n} K\left(\frac{r_m - r_{m,t}}{h}\right) \neq 0, \\ \frac{\sum_{t=1}^{n} K\left(\frac{r_m - r_{m,t}}{h}\right)}{n}, & \text{when } \sum_{t=1}^{n} K\left(\frac{r_m - r_{m,t}}{h}\right) = 0. \end{cases}
\]

(50)

Substitute it into Equation (47) we finally get generalized correlation coefficient \(S\bar{E}V(r_i|r_m)\) of random variable \(r_m\) and \(r_i\):

\[
S\bar{E}V(r_i|r_m) = \frac{\int (\mu^r_i|r_m(r_m))^2 f(r_m) dr_m - (\mu_i)^2}{\sigma_i^2}
\]

(51)

\(\mu_i\) and \(\sigma_i^2\) are the mean and variance of return rate of asset \(i\).

Use \(SEV(r_i|r_m)\) to replace correlation coefficient in \(\beta_\phi\), we get \(\beta^*_\phi\) and SCAPM modified by SEV.

\[
E(r_i) = r_f + (E(r_m) - r_f) \cdot \beta^*_\phi, \quad \beta^*_\phi = \frac{T \cdot (S\bar{E}V(r_i|r_m)\sigma_i - \sigma_m)}{M_{r_m} + (E(r_m) - r_f)} + 1
\]

(52)

For the relationship between dependent variable \(r_i\) and the independent variable \(r_m\), it is assumed that there is a measurable function \(h(\cdot)\) and \(\text{var}(h(r_m))\). The independent variable \(r_m\) is independent of the random error term \(\epsilon\), and the properties of \(SEV(r_i|r_m)\) are as follows:

1. If the relationship between dependent variable \(r_i\) and independent variable \(r_m\) can be expressed as: \(r_i = h(r_m) + \epsilon\), so \(SEV(r_i|r_m) = \text{var}(h(r_m))(\text{var}(h(r_m)) + \text{var}(\epsilon))\), when \(\epsilon\) does not exist, \(SEV(r_i|r_m) = 1\).

2. If \(h(r_m) = r_m\), the relationship between dependent variable \(r_i\) and independent variable \(r_m\) can be expressed as: \(r_i = a r_m + b + \epsilon\), \(a, b\) are constant and \(a \neq 0\), then \(SEV(r_i|r_m) = \beta^*_m\).
(3) There are: \( \rho_{im} \neq 0 \Rightarrow SEV(r_i|r_m) \neq 0 \) and \( SEV(r_i|r_m) = 0 \Rightarrow \rho_{im} = 0 \).

(4) \( SEV(r_i|r_m) \in [0, 1] \) If there is a completely positive or negative correlation between dependent variable \( r_i \) and independent variable \( r_m \), and \( \rho_{im} = \pm 1 \), then \( SEV(r_i|r_m) = 1 \).

This section discusses the representation of SCAPM formula in the real market where the Sharpe hypothesis is not satisfied. Generalized correlation coefficient \( SEV \) is introduced to replace Pearson correlation coefficient, we use \( SEV(r_i|r_m) \) to measure the correlation between \( r_i \) and \( r_m \). Compared with Pearson correlation coefficient, which only measures the linear correlation between variables, \( SEV \) is not affected by the exact form of the function between two variables. In this section, the kernel density estimator \( \hat{SEV}(r_i|r_m) \) based on samples is presented, and \( \beta_\phi \) is substituted to obtain the improved \( \beta^*_\phi \) and SCAPM model modified by \( SEV \).

### 4. SCAPM problem under distribution uncertainty (Ambiguous)

In this section, we consider SCAPM problem when the distribution of return is uncertain. Referring to [16], we fit the effective frontier corresponding to portfolio model, then estimate the relevant parameters of market portfolio return rate \( r_m \) through numerical simulation. By using kernel density estimation, we get estimator of the risk parameter \( \beta_\phi \), the valuation form of SCAPM is obtained in final.

#### 4.1. Robust mean-spectral measure model under distribution uncertainty

**Definition 4.1.** For the portfolio, define the loss function of decision variable \( x \) as \( l(x, \xi) = -x^T \xi \). Assume that the random return vector \( \xi \) has a continuous probability density function \( p(\xi) \), and the return distribution \( p \) satisfies that the characteristic parameter mean value is \( \mu \), covariance is \( \Sigma \). Given a confidence level of \( 1 - \alpha \in (0, 1) \), there is

\[
VaR_\alpha(x) = \min \left\{ \eta \in \mathbb{R} : \int_{\xi: l(x, \xi) \leq \eta} p(\xi) d\xi \geq 1 - \alpha \right\}. \tag{53}
\]

If distribution of returns is the same, at confidence level of \( 1 - \alpha \), there is

\[
CVaR_\alpha(x, P) = E_P[l(x, \xi)|l(x, \xi) \geq VaR_\alpha(x)]
= \frac{1}{\alpha} \int_{\{\xi: l(x, \xi) \geq VaR_\alpha(x)\}} l(x, \xi)p(\xi) d\xi. \tag{54}
\]

The corresponding spectral risk measure is

\[
\rho_\phi(X) = \int_0^1 F_X^-(\alpha)\phi(\alpha) d\alpha \tag{55}
\]

where \( F_X^-(\alpha) = \inf \{x | F_X(x) \geq 1 - \alpha\} \), \( \phi \) is allowable risk spectrum.

**Lemma 4.2.** [29] For probability space \((\Omega, F, P)\), define risk measure \( \rho : L^2(\Omega, F, P) \to \mathbb{R} \), as shown in Equation \(56\), it is law invariant risk measures.

\[
\rho(X) = \sup_{\phi \in \Phi} \rho_\phi(X) \tag{56}
\]

\( \Phi \subseteq L^1[0, 1] \) represents a set of series of admissible risk spectrum functions \( \phi \). The definition of the worst-case risk measure is given below:
Definition 4.3. Give a set of mean and standard deviation \((\mu, \sigma)\), then define a set of distributions
\[
\Theta(\mu, \sigma) := \left\{ F : (-\infty, \infty) \to \mathbb{R}_{\geq 0} \right\}
\]
where \(\mathbb{R}_{\geq 0}\) represents a set of non-negative real numbers, and the risk measure formula satisfying the distribution set can be obtained from Equation (56)
\[
\rho^\uparrow(\mu, \sigma) := \sup_{F_X \in \Theta(\mu, \sigma)} \rho(F_X).
\] (58)

Substitute Equation (56) into Equation (58) to obtain the worst-case law invariant consistent risk measures (WCLICRM).
\[
\rho^\uparrow(\mu, \sigma) := \sup_{F_X \in \Theta(\mu, \sigma)} \left\{ \sup_{\phi \in \Phi} \rho^\uparrow(\mu, \sigma) \right\}. \] (59)

Definition 4.4. When an admissible spectral risk function \(\phi\) in the set of admissible spectral risk functions \(\Phi\) is determined, as a special form of WCLICRM, the worst-case spectral risk measure (WCSRM) is defined as follows:
\[
\rho^\uparrow(\mu, \sigma) := \sup_{F_X \in \Theta(\mu, \sigma)} \rho^\uparrow(\mu, \sigma). \] (60)

Lemma 4.5. [17] For any WCSRM, if and only if the admissible spectral risk function \(\phi \in L^2([0, 1])\), it can be converted into a return distribution containing a measure, the relationship of parameters \(\mu, \Sigma\) is as follows,
\[
\rho^\uparrow(\mu, \sigma) = \mu + \sigma k, \quad k := \sqrt{\|\phi\|_2^2 - 1}. \] (61)

The inverse function of the corresponding worst-case distribution function \(F\) is
\[
F^{-1}(\alpha) = (\mu - \frac{\sigma}{k} + \frac{\sigma}{k} \phi(\alpha^+)), \quad \alpha \in (0, 1), \] (62)

4.1.2. Robust optimization combination model based on WCSRM.

Definition 4.6. Considering the definition of CVaR[4], here we define spectral risk measure under the worst case in the market. Define the loss function \(l(x, \xi) = -\xi'x, \quad \xi \in \mathbb{R}^n\), and mean of the distribution of return rate \(\xi\) as \(\bar{\mu}\), covariance matrix as \(\Sigma \succ 0\), a set of distributions \(\Theta\) is satisfied. So there is
\[
\Theta = \left\{ P \in M_+ | P(\xi \in \Omega) = 1, E_P(\xi) = \bar{\mu}, \text{Cov}_P(\xi) = \Sigma, \rho^\uparrow(\mu, \sigma) \right\}, \] (65)

When the return distribution of portfolio \(P\) is uncertain, historical return data \(\{\xi[i]\}_{i=1}^S\) is observed, and all samples are independently and identically distributed, we have
\[
\hat{\mu} = \frac{1}{S} \sum_{i=1}^S \xi[i], \quad \hat{\Sigma} = \frac{1}{S-1} \sum_{i=1}^S (\xi[i] - \hat{\mu})(\xi[i] - \hat{\mu})', \] (64)

Definition 4.7. Combined with the way ambiguity set defined by Delaga and Ye[8], the ambiguous parameter set of portfolio \(P\) distribution is as follows
\[
D_\omega(\tau_1, \tau_2) = \left\{ P \in M_+ : \begin{array}{l} P(\xi \in \Omega) = 1, \\
(E_P(\xi) - \hat{\mu})' \hat{\Sigma}^{-1} (E_P(\xi) - \hat{\mu}) \leq \tau_1, \\
||\text{Cov}_P(\xi) - \hat{\Sigma}||_F \leq \tau_2, \text{Cov}_P(\xi) \succ 0 \end{array} \right\}, \] (65)
Ambiguous parameters $\tau_1, \tau_2$ define the difference between the true distribution of returns and the estimated one, meaning the degree of ambiguity, which can be specified externally by self-sampling or based on empirical data. $D_\Theta(0, 0)$ means that the degree of ambiguity is 0, at this time, the historical data of returns accurately estimates the real distribution.

The portfolio problem under the distribution uncertainty is defined as follows:

For the traditional mean-spectral risk measure portfolio problem (MS):

$$(MS): \min_x \rho_\alpha(x, P), \quad s.t. \quad E_P(\xi)'x \geq \nu, \quad x \in \chi,$$

where $\nu$ represents the lower limit of expected target return, $MS$ model is used to find a feasible solution $x$ that minimizes spectral risk measure under the restriction condition that the expected return is greater than or equal to $\nu$.

**Definition 4.8.** Considering robust mean-spectrum risk measure portfolio problem ($RMS$) for arbitrary ambiguous parameter set $D$ under distribution of returns $P$ uncertainty

$$(RMS): \min_x \sup_{P \in D} \rho_\alpha(x, P), \quad s.t. \quad \inf_{P \in D} E_P(\xi)'x \geq \nu, \quad x \in \chi,$$

For $RMS$ problem, we hope to find the feasible solution $x$, which minimizes the spectral risk measure $\rho$ under the constraint that the expected return is greater than or equal to $\nu$, when return distribution $P$ obeies the ambiguous parameter set $D$.

Substitute the ambiguity set $D_\Theta(\tau_1, \tau_2)$ in Equation (65) to replace the parameter $D$ of the generalized ambiguity set in $RMS$, so we can construct a robust spectral risk measure model under distribution uncertainty as follows

$$(RMS - D_\Theta(\tau_1, \tau_2)): \min_x \max_{P \in D_\Theta(\tau_1, \tau_2)} \rho_\alpha(x, P), \quad s.t. \quad \inf_{P \in D_\Theta(\tau_1, \tau_2)} E_P(\xi)'x \geq \nu, \quad x \in \chi,$$

For $RMS$ problem, we hope to find the feasible solution $x$, which minimizes the spectral risk measure $\rho$ under the constraint that the expected return is greater than or equal to $\nu$, when return distribution $P$ obeies the ambiguous parameter set $D$.

**Theorem 4.9.** For loss function $l(x, \xi) = -\xi'x, \ x$ is the decision variable, $x \in \chi$, the solution to $(RMS - D_\Theta(\tau_1, \tau_2))$ problem can be transformed into the solution to the following second-order cone programming (SOCP) problem:

$$\min_x \ k \sqrt{x'(\hat{\Sigma} + \tau_2I_n)x - \hat{\mu}'x + \sqrt{\tau_1} \sqrt{x'\hat{\Sigma}x}},$$

$$s.t. \quad \hat{\mu}'x - \sqrt{\tau_1} \sqrt{x'\hat{\Sigma}x} \geq \nu,$$

$$k = \sqrt{||\phi||_2^2 - 1},$$

where $x \in \mathbb{R}^n$.

**Proof.** Define set:

$$U_{(\mu, \Sigma)_\Theta} = \{(\mu, \Sigma) \in \mathbb{R}^n \times S^n_+ | (\mu - \hat{\mu})'\hat{\Sigma}^{-1}(\mu - \hat{\mu}) \leq \tau_1, ||\Sigma - \hat{\Sigma}||_F \leq \tau_2\}$$
We have
\[
\max_{P \in D_\Theta(\tau_1, \tau_2)} \rho_\alpha(x, P) = \max_{(\mu, \Sigma) \in U_{\hat{\mu}, \hat{\Sigma}}(\Theta)} \max_{P \in \Theta} \rho_\alpha(x, P) = \max_{(\mu, \Sigma) \in U_{\hat{\mu}, \hat{\Sigma}}(\Theta)} \{-\mu' x + \sqrt{\tau_1} \sqrt{x' \Sigma x} \sqrt{||\phi||^2 - 1}\} \quad (71)
\]
\[
= \min_{\mu \in U_{\hat{\mu}}} \mu' x - \sqrt{\tau_1} \sqrt{x' \Sigma x} \quad (72)
\]
Combined with restriction conditions of (RMS - D_\Theta(\tau_1, \tau_2)) optimization problem, the optimization model is transformed into the following (SOCP) problem
\[
\max_{\Sigma \in U_{\hat{\Sigma}}} \sqrt{x' \Sigma x} = \sqrt{x' (\hat{\Sigma} + \tau_2 I_n)x} \quad (74)
\]

\section{Numerical simulation based on WCSRM robust model}

4.1.3 Numerical simulation based on WCSRM robust model. We have already obtained the robust optimization model based on WCSRM in previous section, referring to suggestions of many scholars on SOCP problem, this section we use the interior point method to solve this problem by referring to [32]. We fit the corresponding effective frontier, so as to prepare for finding the optimal portfolio M point.

To find the solution of Equation (69), the exponential spectrum risk measure EM is selected in this paper, as follows:
\[
EM = \phi(p) = \frac{e^{-p}}{1 - e^{-\alpha}} 1_{0 < p \leq \alpha}
\]
\[
k = \sqrt{||\phi||^2 - 1} = \sqrt{\frac{3e^{-\alpha} - 1}{2(1 - e^{-\alpha})}}, \alpha \in (0, 1) \quad (78)
\]
Substitute the specific form of K into Equation (69), we get
\[
\min_x -\hat{\mu}'x + \sqrt{\tau_1 \sqrt{x'\Sigma x} + \sqrt{\frac{3e^{-\alpha} - 1}{2(1 - e^{-\alpha})} x'(\Sigma + \tau_2 I_n)x}},
\]
\[
\text{s.t.} \quad \hat{\mu}'x - \sqrt{\tau_1 \sqrt{x'\Sigma x}} \geq \nu,
\]
\[
x \in \chi.
\]

Equation (79) is a robust optimization model based on spectral measure, which is the final solution required in this paper. Ambiguous parameters \(\tau_1, \tau_2\) is obtained by self-sampling samples to get the parameter set that measures difference between the new sample and the original one. Then take the quantile under a certain percentage as estimation of ambiguity parameters, the specific steps are described in detail in the empirical analysis.

The steps to fit the corresponding effective frontier based on robust optimization model are briefly described below. For specific ideas, one can refer [16] for details.

**Effective frontier and the steps to define market portfolio:**

1. First, we solve \((RMS - D_\Theta(\tau_1, \tau_2))\) model without constraint conditions, that is, solve the main problem part of Equation (79). The minimum risk portfolio weight \(x_{\min}\) is obtained when the risk measure is minimized, hence, we can further obtain
\[
\nu_{\min} = \mu_w'x_{\min}, \mu_w = \hat{\mu} - \sqrt{\tau_1 \Sigma x} \sqrt{x'\Sigma x}
\]  
(80)

2. Solve Equation of maximizing expected return
\[
\max_{x \in \chi} \{ \min_{P \in D_\Theta(\tau_1, \tau_2)} E_P(\xi)'x \}
\]
(81)

Hence, we obtain portfolio weight \(x_{\max}\) under the condition of maximizing expected return of the portfolio, furthermore, get the optimal expected value \(\nu_{\max}\). In addition, we define the parameter \(\Delta = \nu_{\max} - \nu_{\min}\).

3. Select the step size as \(M\), and determine the number of point sets by step size, which fit the effective frontier. For \(\nu \in \{\nu_{\min}, \nu_{\min} + \frac{\Delta}{M-1}, \nu_{\min} + 2 \cdot \frac{\Delta}{M-1}, \ldots, \nu_{\min} + (M - 1) \cdot \frac{\Delta}{M-1}\}\), we substitute them into the final model of Equation (71) \((RMS - D_\Theta(\tau_1, \tau_2))\) in turn. According to each optimal solution \(x\), the coordinate of the corresponding point in two-dimensional quadrant is determined.

4. Find \((0, r_f)\) point from the vertical axis, then make fellowship with efficient frontier from that point, including tangent line, the largest slope line is tangent with the effective frontier, tangent point is market portfolio \(m\). In order to highlight the following parameters, which are from the process of solving robust mean-spectral risk measure model, we call spectral risk measure of the market portfolio obtained by solving \((RMS - D_\Theta(\tau_1, \tau_2))\) as \(M^*(r_m)\). In addition, we call the expected return rate of the market portfolio as \(E^*(r_m)\), optimal investment ratio as \(x^*_m\), the tangent slope which is Sharpe ratio of the market portfolio as \(SR^*_m = \frac{E^*(r_m) - r_f}{M^*(r_m)}\).

4.2. Robust SCAPM formula.

**Definition 4.10.** Define a group of risk assets \(i\) with a return rate of \(\xi_i, (i = 1, \cdots, m)\), the portfolio is constructed for this group of risk assets according to the
decision variable $x = (x_1, \cdots, x_m)'$, $\sum_{i=1}^{m} x_i = 1$, where $x_i$ represents the weight of investment to risk assets, $M_\phi$ describes spectral risk measure of the portfolio, $E_P(\xi) = \xi^T x_i$ is expected return rate of the portfolio. Mean-spectral risk measure model is as follows:

$$
\min_x \quad M_\phi \\
\text{s.t.} \quad E_P(\xi) \geq \nu
$$

(82)

If Model (82) has corresponding feasible solutions by given different values $\nu$, hence, the corresponding effective frontier curve $EF$ can be determined. The market portfolio $m$ at the tangent point can be determined by CML line, which has the maximum Sharpe ratio $SP_m = \frac{E(r_m) - r_f}{M(r_m)}$ (83)

SCAPM formula for return distribution uncertainty is defined below.

**Theorem 4.11.** Assume that the return distribution of risk assets in the market is uncertain, the market portfolio $m$ is obtained, and the expected return rate is $E(r_m)$, spectral risk measure is $M(r_m)$. For any risk asset $i$, SCAPM formula of expected return rate is:

$$
E(r_i) = r_f + (E(r_m) - r_f) \cdot \beta_\phi, \quad \beta_\phi = \frac{M_\phi'(0, i) + M(r_m)}{M(r_m)}
$$

(84)

**Proof.** For risk asset $i$, a new portfolio $Z$ is constructed with the market portfolio $m$. For $Z$, the return rate is $r_{a,i} = x_{a}' r$, the return rate vector is $r = (r_i, r_m)'$, the decision vector is $x_a = (a, 1-a)'$. Take expected return rate of $r_{a,i}$ as $\mu_{a,i}$, the spectral risk measure of $Z$ as $M_\phi(a, i)$, and value at risk as $VaR(a)$. When take different values of $a$, the portfolio line formed by a series of portfolios is tangent to CML line at the point $m$, so the slope of the tangent point is $SP_m$. Given that

$$
r_{a,i} = x_a' r = ar_i + (1-a)r_m
$$

$$
\mu_{a,i} = a \cdot E(r_i) + (1-a) \cdot E(r_m)
$$

(85)

By definition of spectral risk measure

$$
M_\phi(a, i) = \int_0^\alpha \phi(p) VaR_p(a) dp
$$

(86)

The derivative of $\mu_{a,i}$ and $M_\phi(a, i)$ with respect to weight $a$ is obtained as follows:

$$
\frac{\partial \mu_{a,i}}{\partial a} = E(r_i) - E(r_m)
$$

$$
\frac{\partial M_\phi(a, i)}{\partial a} = \phi(a) VaR_a(a) VaR'_a(a)
$$

(87)

Therefore,

$$
\frac{\partial \mu_{a,i}}{\partial M_\phi(a, i)} = \frac{E(r_i) - E(r_m)}{\phi(a) VaR_a(a) VaR'_a(a)}
$$

(88)

When $a = 0$, portfolio $Z$ is the market portfolio $m$, take $r_{0,i} = r_m, M_\phi(a, i) = M(r_m), VaR_p(a) = VaR_p(r_m)$ into above equation:

$$
\frac{\partial \mu_{a,i}}{\partial M_\phi(a, i)} \bigg|_{a=0} = \frac{E(r_i) - E(r_m)}{M_\phi(0, i)} = \frac{E(r_m) - r_f}{M(r_m)}
$$

(89)

Where $M'_\phi(0, i) = \phi(a) VaR_a(r_M) VaR'_a(0)$.
Hence
\[ E(r_i) - r_f = \frac{M'_\phi(0, i) + M(r_m)}{M(r_m)}(E(r_m) - r_f), i = 1, \ldots, n \] (90)

The spectrum-\( \beta \) is defined as
\[ \beta_\phi = \frac{M'_\phi(0, i) + M(r_m)}{M(r_m)} \] (91)

SCAPM formula under distribution uncertainty is obtained as follows
\[ E(r_i) = r_f + (E(r_m) - r_f) \cdot \beta_\phi, \quad \beta_\phi = \frac{M'_\phi(0, i) + M(r_m)}{M(r_m)} \] (92)

**Definition 4.12.** When we get the market portfolio \( m \), and its spectral risk measure \( M^*(r_m) \), the expected return rate of market portfolio \( E^*(r_m) \), and the optimal investment ratio \( x^*_m \) by using robust mean-spectral measure optimization model, robust SCAPM formula under distribution uncertainty is as follows:
\[ E(r_i) = r_f + (E^*(r_m) - r_f) \cdot \beta_\phi, \quad \beta_\phi = \frac{M'_\phi(0, i) + M^*(r_m)}{M^*(r_m)} \] (93)

**Remark 1.** Define set \( A \) which contains \( n \) risk assets, and their returns within \( t \) period of time are shown as \( r_i = (r_{i,1}, \ldots, r_{i,t}), i = 1, \ldots, n \). We construct a robust mean-spectrum risk model by this set, then the optimal market portfolio \( m \) and spectral risk measure \( M'_A(r_m) \) under the market portfolio can be solved. The asset pricing formula based on set \( A \) is
\[ E(r_i) - r_f = \frac{M'_\phi(0, i) + M_A^*(r_m)}{M_A^*(r_m)}(E_A^*(r_m) - r_f) \] (94)

\( M'_\phi(0, i) \) can also be obtained according to the optimal market portfolio \( M_A^*(r_m) \) under this set.

4.3. **Nuclear density estimation of \( \beta \) under spectral measures.** In real market, sequences of return on assets usually do not satisfy normal hypothesis, easy to have the characteristics of “sharp peak and thick tail”. Considering the nonparametric estimation method to estimate distribution parameters can avoid the error caused by incorrect distribution assumption, and it can directly estimate the form of population density function based on the known sample, when population distribution characteristics of the data are not clear. By substituting the known sample data into the nonparametric estimator, specific nonparametric estimators can be obtained. Nowadays, the widely used kernel density estimation method, proposed by Parzen(1962), can meet the needs of practical applications better because of its advantages in the precision of estimating data.

4.3.1. **The form of nonparametric kernel density estimation.** For the kernel density estimation, we can state first the necessary steps as follows. There are \( m \) stocks under return distribution uncertainty, and each stock takes \( n \) data samples, then the return rate of the \( i \)th stock has samples of \( \xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,n} \). Since the population return distribution is unknown, assuming that they satisfy independent identical
distribution, and the return density function of the \( i \)th stock is defined as \( f(\xi_i) \), the kernel density estimator of \( f(\xi_i) \) can be obtained as follows

\[
\hat{f}(x_i) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h} k(\frac{\xi_i - \xi_{i,n}}{h})
\]  

(95)

In Equation (95), \( k(\cdot) \) is the kernel function and \( h \) is the window width, which play a decisive role in determining the density function.

**Proposition 7.** Kernel function \( k(\cdot) \) is equivalent to weight in the determination of density function, weight is given by the distance \( \xi_i - \xi_{i,n} \) between sample points and estimation parameters.

\( k(\cdot) \) satisfies following properties:

1. **Nonnegative:** \( \int_{-\infty}^{+\infty} k(x)dx = 1, \ k(x) \geq 0; \)
2. **Symmetry:** \( k(-x) = k(x); \)
3. **Bounded:** \( \sup |k(x)| \leq A \leq \infty \)

\[
\int_{-\infty}^{+\infty} x^i k(x)dx = 0, (i = 1, 2, \cdots, s-1, s);
\]

\[
\int_{-\infty}^{+\infty} x^s k(x)dx \neq 0; \quad \int_{-\infty}^{+\infty} x^s |k(x)|dx < 0.
\]

One dimensional kernel functions commonly used in practical applications include Gaussian kernel, uniform kernel, exponential kernel, etc. Gaussian kernel is selected in this paper:

\[
k(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\]  

(96)

The influence of window width \( h \) on the result of kernel function is even greater than that of kernel function. It is found that the result of nonparametric kernel density estimation is very sensitive to the choice of window width \( h \). The size of the window width \( h \) determines the distribution length of the sample data and has an impact on the estimated distribution smoothness. If the window width is too large, it will be too smooth and characteristic information will be lost. While if the window width is too small, the smoothness will be insufficient and estimated result will be close to sample data, thus losing its significance.

Optimal theoretical window width can be obtained by integrating MISE minimization standard of the kernel density estimation function, but it is not easy to use because the parameters are too many and difficult to be estimated. In practical application, the least square cross validation method is proposed to make the kernel density estimator function reach the minimum integral square variance ISE, it can obtain the best actual window width. In addition, there are two relatively simple methods in practical application:

1. \( h = \left( \frac{4}{3n} \right)^{\frac{1}{2}} \sigma \) determined by Silverman’s thumb rule;
2. \( h = 3 \frac{R(K)}{3n} \sigma \) determined by the principle of maximum smoothness, where \( n \) is the sample size of return rate, \( \sigma \) is the sample standard deviation, and \( k \) is the kernel function.

4.3.2. **Kernel density estimation form of spectral-\( \beta \).** Observe Equation \( (84) \), which is the asset pricing formula under generalized spectral measure, and decompose spectral-\( \beta \):

\[
\beta_\phi = \frac{M'(r_m) + M(r_m)}{M(r_m)}
\]  

(97)

\[
M'(r_m) = \phi(\alpha) \cdot VaR_\alpha(r_m) \cdot VaR'_{\alpha}(r_m)
\]  

(98)
According to Theorem (4.11), let $VaR_\alpha(r_m) = VaR_{\alpha,i}$, by definition of VaR

$$Pr(r_{a,i} > VaR_{a,i}) = \alpha$$  \hspace{1cm} (99)

Take $r_t = (r_{i,t} - r_{m,t})'(t = 1, \ldots, T)$ as $T$ samples of return vectors, then by Equation (95)

$$\frac{1}{T} \sum_{t=1}^{T} \Phi\left(\frac{-x^{'a}r_t - VaR_{a,i}}{h}\right) = \alpha$$  \hspace{1cm} (100)

$\Phi(\cdot)$ is a standard normal distribution function with the distribution of $N(0, 1)$, and $h$ is a positive window width, differentiate $\alpha$ on both sides we get

$$\frac{1}{Th} \sum_{t=1}^{T} \varphi\left(\frac{-x^{'a}r_t - VaR_{a,i}}{h}\right)\left(\langle r_i - r_m \rangle - \frac{\partial VaR_{a,i}}{\partial a}\right) = 0$$  \hspace{1cm} (101)

When $a = 0$, there is $x^{'a}r_t = r_{m,t}, VaR_{a,i} = VaR(r_m)$, substitute it into the above equation to get the expression when $a = 0$:

$$VaR^{'0}_{0,i} = \sum_{t=1}^{T} \frac{\varphi\left(\frac{-r_{m,t} - VaR(r_m)}{h}\right)}{\varphi\left(\frac{-r_{m,t} - VaR(r_m)}{h}\right)} - \langle r_{i,t} - r_{m,t} \rangle$$  \hspace{1cm} (102)

Take the above equation into Equation (97), the nonparametric kernel density estimator expression of spectral-$\beta$ can be obtained.

$$\beta_\phi = \frac{1}{M(r_m)}\left[M(r_m) + \phi(\alpha) \cdot VaR_\alpha(r_m) \cdot \left(\sum_{t=1}^{T} \frac{\varphi\left(\frac{-r_{m,t} - VaR(r_m)}{h}\right)}{\varphi\left(\frac{-r_{m,t} - VaR(r_m)}{h}\right)} - \langle r_{i,t} - r_{m,t} \rangle\right)\right]$$  \hspace{1cm} (103)

5. Empirical analysis. In this section, we take empirical data to solve the parameters, then obtain an estimate of ROA by using SCAPM formula.

(1) The effective frontier curve, composed of optimal solutions, satisfying the model conditions, is obtained by fitting and solving mean-spectrum robust optimization model, and the relevant parameter $M(r_m)$ of the optimal market portfolio under the model conditions is obtained by tangent line based on the determined risk-free interest rate;

(2) For the risk coefficient which describes risk assets, we estimate it by using kernel density estimation;

(3) Finally, ROA estimated by SCAPM formula is compared with the actual stock market return rate to demonstrate the effectiveness of the model. The empirical stock price data is from NetEase Finance.

5.1. Choice of data. The data selected in this numerical simulation process is based on Shanghai and Shenzhen stock markets, and the standard of portfolio investment is to spread risk. Ten stocks from different industries with high representativeness and low correlation of return rate, from CSI 300 index are selected. They are Kweichow Moutai (600519), China International Monetary Fund (601888) and China Merchants Bank(600036), Wuliangye (000858), CITIC Securities (600030), Hengrui Pharmaceutical (600276), Midea Group (000333), Yili Group (600887), Industrial Bank (601166), Ping An of China (601318). Sample time length is 1 year (from January 1, 2019 to December 31, 2019), according to statistics of actual trading days in 2019 stock market, we take 244 effective closing price data of each stock respectively (for there are a few trading days that stocks were suspended, without
price information, as a result of the company’s major issues, we replenish empty data by taking the method of average growth rate.

As there is a large price difference between Kweichow Moutai (600519) and the other nine stocks, Figure 1 and Figure 2 show the changes of their daily closing price respectively.

![Figure 1. Closing price chart of Kweichow Moutai (600519) in 2019](image1)

![Figure 2. Closing price chart of other nine stocks in 2019](image2)

As we can see from the charts, every stock has shown an upward trend in 2019, with a few have notable returns. The variable data mainly discussed in asset pricing process in this paper is the return rate of assets, so according to Definition (5.1), we do conversion of stock price to return rate.

**Definition 5.1.** Define the closing price of the $i$th company stock on the $t$th day as $P_{i,t}$, ($i = 1, 2, \cdots, m$; $t = 1, 2, \cdots, n$), in this empirical analysis, there is $m = 10, n = 244$. Return rate of the day is $\xi_{i,t}$. The relationship between the two is as follows:

$$
\xi_{i,t} = \ln \frac{P_{i,t}}{P_{i,t-1}}
$$

(104)

Let $\bar{\mu}_i = \frac{1}{n} \sum_{t=1}^{n} \xi_{i,t}$ be mean of return on stock $i$ in the sample, $\hat{\Sigma}$ be the covariance of stocks $i$ and $j$ in the sample. The whole sample stock data has mean vector $\bar{\mu} = (\bar{\mu}_1, \cdots, \bar{\mu}_m)'$, and covariance matrix is $\hat{\Sigma}$. Table 1 shows mean value and covariance data of the stocks corresponding to the whole sample in detail.
Figure 3 shows the fluctuation of each stock’s daily return rate in 2019, it is observed that the basic fluctuation trend of all stocks is similar, with small fluctuations in a period of time and large fluctuations after the middle of the year (see left side of the figure). In addition, large fluctuations and small fluctuations tend always appear in the accompanying state.

The purpose of portfolio is to “avoid putting all your eggs in one basket”, diversification reduces risk while maintaining returns, so the individual asset that make up the portfolio should maintain a low correlation. The specific data of annual return rate and covariance matrix of the 10 stocks are shown in Table 1, and the return correlation matrix of each stock is shown in Figure 4. From it we can see, the return correlation of 10 stocks is relatively low, which satisfies the basic requirements of portfolio construction.

Figure 3. Fluctuation trend on daily returns of 10 stocks in 2019

Figure 4. Correlation matrix figure of return on 10 stocks

5.2. Numerical simulation of robust mean-spectral measure model.

Definition 5.2. Based on Definition 5.1, let $x_i$ be stock investment weight of stock $i$, so $x = (x_1, x_2, \cdots, x_m)'(\sum_{i=1}^{m} x_i = 1)$ is the decision vector of portfolio $p$ constructed this time. Meanwhile, for the new portfolio $P$, its expected real return $E_P(\xi)$ is defined, and the covariance matrix of return rate is $Cov_P(\xi)$. The ambiguity parameter $\tau$ describes the deviation level between return rate distribution
Table 1. Annualized return rate and covariance matrix of 10 stocks

| Stock code  | 600519 | 601888 | 600036 | 600858 | 600030 | 600276 | 000333 | 600887 | 601166 | 601318 |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\mu_i$    | 0.0887 | 0.0462 | 0.0391 | 0.0905 | 0.0473 | 0.0481 | 0.0441 | 0.0411 | 0.0337 | 0.0518 |
| $\hat{\sigma}_{ij}$ |       |        |        |        |        |        |        |        |        |        |
| 600519   | 0.0887 | 0.0462 | 0.0391 | 0.0905 | 0.0473 | 0.0481 | 0.0441 | 0.0411 | 0.0337 | 0.0518 |
| 601888   | 0.0462 | 0.0929 | 0.0274 | 0.0621 | 0.0344 | 0.0429 | 0.0380 | 0.0355 | 0.0227 | 0.0373 |
| 600036   | 0.0391 | 0.0274 | 0.0578 | 0.0417 | 0.0475 | 0.0298 | 0.0387 | 0.0284 | 0.0442 | 0.0490 |
| 600858   | 0.0905 | 0.0621 | 0.0417 | 0.1541 | 0.0685 | 0.0574 | 0.0661 | 0.0643 | 0.0364 | 0.0632 |
| 600030   | 0.0473 | 0.0429 | 0.0298 | 0.0574 | 0.1390 | 0.0416 | 0.0547 | 0.0416 | 0.0497 | 0.0633 |
| 600276   | 0.0481 | 0.0429 | 0.0298 | 0.0574 | 0.0416 | 0.1192 | 0.0444 | 0.0390 | 0.0236 | 0.0410 |
| 000333   | 0.0441 | 0.0344 | 0.0475 | 0.0685 | 0.1390 | 0.0416 | 0.0557 | 0.0416 | 0.0497 | 0.0633 |
| 600887   | 0.0411 | 0.0355 | 0.0284 | 0.0627 | 0.0416 | 0.0497 | 0.0436 | 0.0772 | 0.0241 | 0.0363 |
| 601166   | 0.0337 | 0.0227 | 0.0442 | 0.0364 | 0.0497 | 0.0236 | 0.0397 | 0.0241 | 0.0617 | 0.0458 |
| 601318   | 0.0518 | 0.0373 | 0.0490 | 0.0632 | 0.0633 | 0.0410 | 0.0560 | 0.0363 | 0.0458 | 0.0682 |

parameters estimated through the sample and real portfolio $P$, satisfying the following relationship

$$D_{\Theta} (\tau_1, \tau_2) = \left\{ P \in M_+: \begin{array}{l} P(\xi \in \Omega) = 1, \\
(E_P(\xi) - \hat{\mu})' \hat{\Sigma}^{-1} (E_P(\xi) - \hat{\mu}) \leq \tau_1, \\
||\text{Cov}_P(\xi) - \hat{\Sigma}||_F \leq \tau_2, \text{Cov}_P(\xi) \succ 0 \end{array} \right\}$$

(105)

It is considered that the true distribution of return rates on portfolio $P$ belongs to a distribution set $M_+$, which satisfies the above relation conditions.

Exponential spectral measure (EM): $\phi(p) = \frac{e^{-p}}{1 - e^{-\alpha}} 1_{0 < p \leq \alpha}$ is used to solve the optimization model, and $k_{EM}$ value under EM is obtained from Equation (78)

$$k_{EM} = \sqrt{||\phi||_2^2 - 1} = \sqrt{\int_0^\alpha \left( \frac{\exp^{-p}}{1 - e^{-\alpha}} \right)^2 dp} - 1 = \sqrt{\frac{3e^{-\alpha} - 1}{2(1 - e^{-\alpha})}}.$$  

(106)

Substitute above defined parameters into Equation (69) to obtain the empirical solution of robust mean-spectral measure optimization model.

$$\min x \quad -\hat{\mu}' x + \sqrt{\tau_1} \sqrt{x' \hat{\Sigma} x} + \sqrt{\frac{3e^{-\alpha} - 1}{2(1 - e^{-\alpha})}} \sqrt{x' (\hat{\Sigma} + \tau_2 I_n) x},$$

$$s.t. \quad \hat{\mu}' x - \sqrt{\tau_1} \sqrt{x' \hat{\Sigma} x} \geq \nu, \quad x \in \chi,$$

(107)

Determine the effective frontier curve

(1) **Determine the ambiguity parameter $\tau$** Random sampling with replacement of the sample set of return rate on each stock $\Xi_i = \{\xi_{i,t} \in \Xi_i; t = 1, 2, \cdots, n\}$, $i = 1, 2, \cdots, m$, the sampling capacity of each stock is also $t = 244$. All 10 stocks are drawn out in turn for a total of $B = 10000$ times, so a new sample set $\{Y_1, Y_2, \cdots, Y_B\}$ is obtained. There is a set of corresponding distribution parameters

$$\Omega = \left\{ (\hat{\mu}_b, \hat{\Sigma}_b) : b = 1, \cdots, B \right\}.$$  

(108)
Estimate ambiguity parameters based on parameters of the new sample and the original one:

\[ \Omega_{\tau_1} = \left\{ \tau_{1b} : \tau_{1b} = (\hat{\mu}_b - \hat{\mu})' \hat{\Sigma}_n^{-1} (\hat{\mu}_b - \hat{\mu}), b = 1, \cdots, B \right\}. \]

\[ \Omega_{\tau_2} = \left\{ \tau_{2b} : \tau_{2b} = ||\hat{\Sigma}_b - \hat{\Sigma}||_F, b = 1, \cdots, B \right\}, \]

\[ \hat{\tau}_1 = q(\Omega_{\tau_1}), \]

\[ \hat{\tau}_2 = q(\Omega_{\tau_2}). \]

\[ q_\zeta(.) \] represents taking the upper \( \zeta \) quantile on the action number set, here we take \( \zeta = 95\% \), and the solution is \( \tau_1 = 0.1682, \tau_2 = 0.2351 \).

(2) Expected return range According to the effective frontier determination steps in Section 4.2, substitute ambiguous parameters \( \tau_1, \tau_2 \) estimated in the previous step, to solve the main formula without constraint conditions in Equation (107) and the maximized expected return formula respectively:

\[ \max_{x \in X} \min_{P \in D_\Omega(\tau_1, \tau_2)} E_P'(\xi)'x = \hat{\mu}'x - \sqrt{\tau_1} \sqrt{x'\hat{\Sigma}x} \]

The confidence level is \( 1 - \alpha = 0.95 \), and the final solution is \( \nu_{\min} = 0.5081, \nu_{\max} = 0.7107 \).

(3) Draw the efficient frontier We select step as \( M = 50 \) based on previous result, then for evenly divide interval \( (\nu_{\min}, \nu_{\max}) \), we get interval point \( \nu_k = \nu_{\min} + k \cdot \frac{\nu_{\max} - \nu_{\min}}{M - 1}, k = 0, \cdots, (M - 1) \), and take all \( \nu_k \) into Equation (107), the optimal solution \( x_{\text{opt}} \) satisfying the model is obtained successively, and a point on the two-dimensional coordinate system \( (M_{\phi, \text{opt}}, E_{p, \text{opt}}) \) is determined.

\[ M_{\phi, \text{opt}} = -\hat{\mu}'x_{\text{opt}} + \sqrt{\tau_1} \sqrt{x_{\text{opt}'}} \hat{\Sigma}x_{\text{opt}} + \sqrt{\frac{3e^{-\alpha} - 1}{2(1 - e^{-\alpha})}} \sqrt{x_{\text{opt}'}}(\hat{\Sigma} + \tau_2 I_n)x_{\text{opt}} \]

\[ E_{p, \text{opt}} = \hat{\mu}'x_{\text{opt}} - \sqrt{\tau_1} \sqrt{x_{\text{opt}'}} \hat{\Sigma}x_{\text{opt}} \]

Connect all the above points we can obtain a initial solution curve satisfying the model, and the effective frontier curve is obtained after smoothing the initial curve, as shown in Figure 5.
(4) **Determine the market portfolio** Risk-free rate for one-year in this empirical analysis is taken by one-year Treasury bond yield in 2019, that is $r_f = 0.024$. A tangent that crosses $(0, r_f)$ for the effective frontier is shown in Figure 6, then we determine the tangent point $m(M(r_m), E(r_m))$, and get related parameters of the optimal market portfolio $M(r_m) = 0.82594$, $E(r_m) = 0.51945$, maximum sharpe ratio $SP_m = 0.5383$ and the optimal investment ratio vector $x_{r_m} = (x_1, x_2, \cdots, x_{10})'$ at the market portfolio point as shown in Table 2.

![Figure 6. Market portfolio of \((RMS - D_\Theta(\tau_1, \tau_2))\) model](image)

**Table 2.** Optimal investment ratio of market portfolio in model \((RMS - D_\Theta(\tau_1, \tau_2))\)

| Stock code | 600519 | 601888 | 600036 | 000858 | 600030 | 600276 | 600333 | 600887 | 601166 | 601318 |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $x_i$    | 0.1788 | 0.0656 | 0.1144 | 0.2543 | 0.0364 | 0.1042 | 0.0970 | 0.0342 | 0.0505 | 0.0638 |

5.3. **The kernel density estimate of spectrum-\(\beta\).** According to Equation (97), Equation (98) and Equation (102), the kernel density estimation formula of spectrum-\(\beta\) is as follows

$$
\beta_{\phi} = \frac{M'(r_m) + M(r_m)}{M(r_m)}
$$

$$
M'(r_m) = \phi(\alpha) \cdot VaR_{\alpha}r_m \cdot VaR'_\alpha(r_m)
$$

$$
VaR'_\alpha(r_m) = VaR'_{0,t} = \sum_{t=1}^{T} \sum_{t=1}^{T} \frac{\varphi(-r_{m,t} - VaR_{t}r_m)}{\varphi(-r_{m,t} - VaR_{t}r_m)} - ((r_{i,t} - r_{m,t}))
$$

(112)

According to the optimal investment vector $x_{r_m}$ of the market portfolio obtained in Table 2, we get return rate set $R_{m,t} = \{r_{m,t}; r_{m,t} = \xi^t x_{r_m}, t = 1, \cdots, n\}$ at each time point within one year to construct market portfolio, and:

$$
VaR'_\alpha(r_m) = q_{\zeta}(R_{m,t}) \zeta = 0.95t = 1
$$

$$
\phi(\alpha) = \exp(-\alpha)/(1 - \exp(-\alpha)) = 19.5
$$

(113)
When the confidence level is $1 - \alpha = 0.95$, $VaR'_\alpha(r_m) = 0.0226$ is obtained through program fitting, substitute it into Equation (112) $\beta_{\phi,i}$ corresponding to each stock is finally obtained as shown in Table 3.

| Stock code | 600519 | 601888 | 600036 | 600858 | 600030 | 600276 | 600333 | 600887 | 601166 | 601318 |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\beta_{\phi,i}$ | 1.0004 | 0.9990 | 0.9974 | 1.0026 | 0.9987 | 1.0010 | 0.9991 | 0.9968 | 0.9971 | 0.9989 |

5.4. Result analysis.

**Definition 5.3.** Define $hp_i$ to describe the real return rate of the $i$th stock within one year, $\delta_i$ represents the relative difference between estimated value $E(r_i)$ and real return rate $hp_i$, $\Delta_i$ is absolute difference. According to the following equation

$$E(r_i) = r_f + [E(r_m) - r_f] \cdot \beta_{\phi,i}$$

$$hp_i = \ln(\xi_{i,t}^{(t)})$$

$$\Delta_i = E(r_i) - hp_i$$

$$\delta_i = (E(r_i) - hp_i)/hp_i, \quad i = 1, \cdots, 10$$

Equation (114)

$\Delta_i, \delta_i$ evaluate effectiveness of the pricing formula SCAPM in this paper respectively, from the perspective of absolute difference and relative difference. Since the traditional effective frontier is made up by several envelope, and effected by the selection of risk assets quantity, category, time, length, etc. Therefore, it's normal with differences. If we forecast assets accurately, $\Delta_i, \delta_i$ is small, so model evaluation is effective. We get the estimated differences of each stock $\Delta_i, \delta_i$ through Equation (114), as shown in Table 4 and Table 5.

| Stock code | 600519 | 601888 | 600036 | 600858 | 600030 | 600276 | 600333 | 600887 | 601166 | 601318 |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\Delta_i$ | -21.18% | 7.11% | 4.25% | -50.98% | 3.23% | -3.31% | -0.09% | 14.77% | 17.22% | 3.07% |

| Stock code | 600519 | 601888 | 600036 | 600858 | 600030 | 600276 | 600333 | 600887 | 601166 | 601318 |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\delta_i$ | -31.13% | 17.90% | 10.00% | -52.05% | 7.41% | -6.60% | -0.19% | 46.25% | 58.34% | 7.01% |

Compare the estimated results and actual one in Table 4 and Table 5 comprehensively, we find that the absolute difference of stock code 00033 is 0.09%, and the relative difference is only 0.19%, which indicates the difference between the predicted results using SCAPM formula and actual one is very small.

In addition, comprehensive conclusion is that the absolute difference of stock codes 60030, 600276 and 601318 is controlled within 5%, and relative difference
is controlled within 10%. Theoretically, it can be considered that the deviation of estimation results is not large. In actual investment, if the investment scale is small, the loss caused by estimation error will have only a little influence on investment returns. However, if the scale of actual investment is large, the loss level caused by estimation error should be considered.

In general, the empirical analysis shows that SCAPM formula under distribution uncertainty proposed in this paper is effective to some extent, and it can be continuously updated and improved in practical application and research.

6. **Conclusion.** In this paper, we do research on asset pricing problem based on uncertainty of return distribution, and use spectral risk measure as a index of risk measurement. We derive the asset pricing formula SCAPM based on spectral risk measure, under normal distribution and distribution uncertainty. The research logic follows the derivation process of the traditional pricing formula, starting from the portfolio process under deterministic normal distribution, we define the optimization model form. Then the effective frontier curve is obtained and the optimal market portfolio form is determined, furthermore, we consider asset pricing model under distribution uncertainty. Finally, return data of 10 constituent stocks of CSI 300 Index are used to compare the valuation results with actual one, which indicates that SCAPM formula proposed in this paper is reasonable and effective in the case of distribution uncertainty.

**Acknowledgments.** The authors gratefully thank the reviewers and the editor for their valuable suggestions and constructive feedbacks. This research was funded by NNSF of China (no.11871275).

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Received March 2021; revised April 2021.

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