Smooth density for the Solution of Scalar SDEs with Locally Lipschitz Coefficients under Hörmander Condition

M. Tahmasebi

Department of Applied Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-134, Tehran, Iran

Abstract

In this paper the existence of a smooth density is proved for the solution of an SDE, with locally Lipschitz coefficients and semi-monotone drift, under Hörmander condition. We prove the non-degeneracy condition for the solution of the SDE, from it an integration by parts formula would result in the Wiener space. To this end we construct a sequence of SDEs with globally Lipschitz coefficients whose solutions converge to the original one and use some Lyapunov functions to show the uniformly boundedness of the p-moments of the solutions and their Malliavin derivatives.

Subject classification: Primary 60H07, Secondary 60H10, 60J05.
Keywords: smoothness of density, stochastic differential equation, semi-monotone drift, Malliavin calculus, Hörmander condition.

1. Introduction

In this paper, we study Malliavin differentiability and the existence of a smooth density for the solution of an SDE. We consider a scalar SDE whose semi-monotone drift and locally Lipschitz coefficients satisfy Hörmander condition. Such equations are considered mostly in finance, biology, and dynamical systems and are more challenging when considered on infinite dimensional spaces. (see e.g. [2; 23; 10]). We prove the existence of a unique, infinitely Malliavin differentiable, strong solution to this SDE satisfying some nondegeneracy condition, and derive both the integration by parts formula in the Wiener space and the existence of a smooth density for this solution.

This subject have been studied by many authors, mostly in the case where the coefficients are globally Lipschitz. In [14] Kusuoka and Stroock have shown that an SDE whose coefficients are $C^\infty$-globally Lipschitz with polynomial growth, has a strong Malliavin differentiable solution of any order. The absolute continuity of the law of the solution of SDEs with respect to the Lebesgue measure and the smoothness of its density under some nondegeneracy condition are shown in [20; 7]. Nualart (2006) shows that the Hörmander condition, posed on the coefficients, condition (H) in this paper, implies this required nondegeneracy condition, if the coefficients are $C^\infty$-globally Lipschitz and have polynomial growth. Assuming the nondegeneracy condition one can derive some integration by parts formula in the Wiener space and also regularity for the distribution of the solution (see e.g. [21]). It is often of interest to investors to derive an option pricing formula and to know its sensitivity with respect to various parameters. The integration by parts formula obtained from Malliavin calculus can transform the derivative of the option price into a weighted integral of random variables. This gives much more accurate and fast converging numerical solutions than obtained by the classical methods ([12; 4]). The interested reader could see ([1; 19]). In recent years, there were attempts to generalize these results to SDEs with non-globally Lipschitz coefficients. For example, in [8] using a Fourier transform argument, some absolute continuity results are obtained for the law of the solution of an SDE with Hölder coefficients. The existence
of densities for a general class of non-Markov Itô processes under some spatial ellipticity condition and that allow the degeneracy of the diffusion coefficient is shown in [3]. Marco [19] has shown that assuming some local properties of coefficients, and uniform ellipticity of diffusion coefficient, the law of the solution of the SDE has smooth density. If the diffusion coefficient is uniformly elliptic, then the Hörmander condition is satisfied. When the noise is a fractional brownian motion or a Levy process the same results are obtained under ellipticity and Hörmander condition as well. For other references on this subject, we refer the reader to [10; 18; 11; 3].

To deal with the SDE with non-globally Lipschitz coefficients, we construct a sequence of SDEs with globally Lipschitz coefficients whose solutions are Malliavin differentiable of any order and satisfy a nondegeneracy condition. In this way we can apply the classical Malliavin calculus to the solutions. We can find also a uniform bound for the moments of the solutions, and all their Malliavin derivatives, by using some Lyapunov functions. Then we will prove the nondegeneracy condition for the original SDE, using the nondegeneracy condition for the sequence of solutions to the constructed SDEs. This result implies the integration by parts formula in the Wiener space and the existence of the smooth density for the solution.

The paper is organized as follows. In section 2, we recall some basic results from Malliavin calculus that will be used in the paper, in particular the integration by parts formula due to [20, Proposition 2.1.4]. In section 3, we state the assumptions and our main results; in section 4, we prove the uniformly boundedness for the moments of Malliavin derivatives of the solution to a sequence of approximating SDEs, as there exist some Lyapunov functions. Section 5 involves the construction of uniformly boundedness for the moments of Malliavin derivatives, by using some Lyapunov functions. Then we will prove the nondegeneracy condition for the original SDE, using the nondegeneracy condition for the sequence of solutions to the constructed SDEs. This result implies the integration by parts formula in the Wiener space and the existence of the smooth density for the solution. Finally, in Appendix we state the detailed proof on selection of approximating processes.

2. Some basic results from Malliavin calculus

In this article, we use the same notations as in [20]. Let $\Omega$ denote the Wiener space $C_0([0,T];\mathbb{R})$ endowed with $\| \cdot \|_\infty$-norm making it a (separable) Banach space. Consider a complete probability space $(\Omega, \mathcal{F}, P)$, in which $\mathcal{F}$ is generated by the open subsets of the Banach space $\Omega$, $W_t$ is a $d$-dimensional Brownian motion, and $\mathcal{F}_t$ is the filtration generated by $W_t$.

Consider the Hilbert space $H := L^2([0,T];\mathbb{R})$. The Malliavin derivartive operator $D$ is closable from $L^p(\Omega)$ to $L^p(\Omega, H)$, for every $p \geq 1$ and the adjoint of the operator $D$ is denoted by $\delta$. We use the notation $\wedge_\mathcal{F} = \|DF\|_H^2$ to show the Malliavin covariance matrix for a random variable $F$, and for every $k \geq 1$, we set $D_{r_1,\ldots,r_k} F = D_{r_k}(D_{r_{k-1},\ldots,r_1} F)$.

Now let $Y_t$ be a solution to the following SDE:

$$dY_t = B(Y_t)dt + A(Y_t)dW_t, \quad Y_0 = x_0, \tag{2.1}$$

where $B : \mathbb{R} \to \mathbb{R}$ is a measurable function and $A : \mathbb{R} \to \mathbb{R}$ is an $C^\infty$ function. Let $Z_t$ be the solution of the following linear SDE:

$$Z_t = 1 + \int_0^t B'(Y_s)Z_s ds + \int_0^t A'(Y_s)Z_s dW_s$$

and

$$C_t := \int_0^t (Z_s^{-1} A(Y_s))^2 ds,$$

and assume the Hörmander’s condition holds as follows:

(H) \quad $A(x_0) \neq 0$ or $A^{(n)}(x_0) B(x_0) \neq 0$ for some $n \geq 1$.

Under this condition Nualart [20] has shown the following proposition.
Proposition 2.1. For a solution $Y_t$ to an SDE with globally Lipschitz coefficients and polynomial growth for all their derivatives, the Hörmander’s condition (H) implies that for any $p \geq 2$ and any $\varepsilon$ small enough,

$$P\left(C_t \leq \varepsilon\right) \leq e^p$$

(2.2)

and $(\det C_t)^{-1} \in L_p(\Omega)$ for all $p$. Thereby obtaining the nondegeneracy condition for $Y_t$ and thus the integration by parts formula in the Wiener space and an infinitely differentiable density, too.

3. Formulation of main results

In this section we consider the $\mathcal{F}_t$-adapted stochastic process $X_t$ taking values in $\mathbb{R}$, which is a solution to the following stochastic differential equation

$$dX_t = [b(X_t) + f(X_t)]dt + \sigma(X_t)dW_t, \quad X_0 = x_0.$$  

(3.1)

where $b, f : \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable $C^\infty$ function. We denote by $\mathcal{L}$ the second-order differential operator associated to SDE (3.1):

$$\mathcal{L} = \frac{1}{2} \sigma^2(x) \partial^2 + [b(x) + f(x)] \partial$$

Throughout the paper we assume that $b$, $f$ and $\sigma$ satisfy the following Hypothesis.

Hypothesis 3.1. Functions $b$, $f$ and $\sigma$ satisfy the following conditions.

- $b$ and $\sigma$ are $C^\infty$ locally Lipschitz and all of their derivatives have polynomial growth; i.e., for every $j \geq 0$ there exist some constants $\lambda_j$ and $q_j$ such that for every $x \in \mathbb{R}$

$$|b^{(j)}(x)| + |\sigma^{(j)}(x)| \leq \lambda_j (1 + |x|^q_j)$$

(3.2)

Also, set $\xi := \max_{j \geq 1} q_j < \infty$.

- The function $f$ is $C^\infty$, globally Lipschitz with the Lipschitz constant $k_3$ and all of its derivatives are bounded.

Since the coefficients are locally Lipschitz functions, we need further assumptions that the solution of (3.1) does not explode. Also, to obtain the Malliavin differentiability of the solution $X_t$ in all $t \in [0, T]$, we consider the following hypothesis.

Hypothesis 3.2. The function $b$ is a $C^\infty$ uniformly monotone function, i.e., there exists a constant $K$ such that for every $x, y \in \mathbb{R}$,

$$\left( b(y) - b(x) \right)(y - x) \leq -K|y - x|^2.$$  

(3.3)

- For every $p \geq 1$ there exist some constants $\alpha_p$ and $\beta_p$ such that

$$ab(a) + (12p - 1)\sigma^2(a) + (4p - 1)|a\sigma'(a)|^2 \leq \alpha_p + \beta_p |a|^2 \quad \forall a \in \mathbb{R},$$

(3.4)

Remark 1. Notice that for every monotone drift $b$ and every globally Lipschitz diffusion $\sigma$, Hypothesis 3.2 is satisfied. Hypothesis 3.2 is satisfied also for some locally Lipschitz diffusion $\sigma$, for example when $b(x) = -x^3$ and $\sigma(x) = x^2$.

Since the constant $\beta_p$ in (3.4) could be negative, we may use the version of Gronwall’s inequality which is proved in [13, Lemma 1.1].

Now we are going to present the higher order differentiability and the integration by parts formula for the solution of SDE (3.1) in the Wiener space.

Theorem 3.3. Under Hypotheses 3.1 and 3.2 the SDE (3.1) has a unique strong solution in $\mathbb{D}^\infty$.

We also show the nondegeneracy condition for $X_t$ which derive the integration by parts formula in [20, Proposition 2.1.4] and thus an smooth density.

Theorem 3.4. The solution $X_t$ of SDE (3.1) is nondegenerate and has smooth density.
4. Malliavin differentiability and Lyapunov functions

In this section, we consider the SDE \( \frac{d}{dt}X_t = f(X_t) + \sigma(X_t)dW_t \) with Hypothesis 3.1 and introduce some new stochastic systems by using a sequence of \( \{X^n_t\} \) and prove that if there exist some suitable Lyapunov functions for these systems, then the SDE has an infinitely Malliavin differentiable solution. It will be useful for the next section, as we construct desired Lyapunov functions under Hypothesis 3.2.

Consider two sequences of functions \( b_n, \sigma_n \in C^1_{loc} \) and \( G_n \subseteq \mathbb{R} \) such that \( \bigcup G_n = \mathbb{R} \) and for every \( x \in G_n \), \( \sigma_n(x) = \sigma(x) \) and \( b_n(x) = b(x) \). Assume that there exists a Lyapunov function \( F; (i.e., F \) is positive and for some constant \( p \geq 2 \) and every \( x \) we have \( F(x) \geq c_0|\sigma|^p \)) and a sequence of strong solutions \( \{X^n_t\} \subseteq D^{1,p} \) to the SDE’s

\[
X^n_t = x_0 + \int_0^t b_n(X^n_s)ds + \int_0^t \sigma_n(X^n_s)dW_s
\]

such that

\[
\sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[F(X^n_t)] < \infty. \tag{4.1}
\]

Let \( \tau_n \) is the first exit time of \( X_t \) from \( G_n \). Also, assume that the sequence \( X^n_t \) converges almost surely, or for the transition probability measure \( P^{(n)} \) associated with \( X^n_t \),

\[
\lim_{n \to \infty} P^{(n)}(\tau_n \leq t) = 0 \tag{4.2}
\]

Our main result is the following theorem which proves the Malliavin differentiability for the solution of an SDE, using Lyapunov function.

**Theorem 4.1.** Assume that there exists some Lyapunov function \( V(.,.) \) and functions \( K(.,.) \) and \( G(.) \) for the linearized system

\[
LS(r) := \begin{cases}
X^n_t = x_0 + \int_0^t \left[b_n(X^n_s) + f(X^n_s)\right]ds + \int_0^t \sigma_n(X^n_s)dW_s \\
Y^n_t = \sigma_n(X^n_t) + \int_0^t \left[b'_n(X^n_s) + f'(X^n_s)\right]Y^n_sds + \int_0^t \sigma'_n(X^n_s)Y^n_sdW_s,
\end{cases}
\]

with infinitesimal operator \( L_{n,r} \) such that for some constant \( c \)

\[
L_{n,r}V(x, y) \leq c \left(V(x, y) + G(x, y) + h(x)\right), \tag{4.3}
\]

\[
\mathbb{E}[G(X^n_t, D_rX^n_t)] = 0, \ \forall t > r, \quad \text{and} \quad \sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[h(X^n_t)] < c, \tag{4.4}
\]

\[
\left(\partial_x V(x, y)\sigma_n(x) + \partial_y V(x, y)\sigma'_n(x)\right)^2 \leq cK(x, y), \tag{4.5}
\]

\[
\sup_{n \geq 1} \mathbb{E}[V(X^n_t, \sigma_n(X^n_t))] \leq c, \quad \text{and} \quad \sup_{r \leq t \leq T} \mathbb{E}[K(X^n_t, D_rX^n_t)] < \infty. \tag{4.6}
\]

Then there exists a random process \( X_t \in D^{1,p} \) which is the solution of SDE \( \frac{d}{dt}X_t = f(X_t) + \sigma(X_t)dW_t \).

\[
X^n_t \to X_t \text{ in } L^p([0, T] \times \Omega), \quad \text{and} \quad DX^n \to DX \text{ in } D^{1,p}.
\]

**Proof.** By Itô’s formula, one can easily show that

\[
\sup_{n \geq 1} \sup_{r \leq t \leq T} \mathbb{E}[V(X^n_t, D_rX^n_t)] < \infty.
\]

Therefore, there exists a subsequence \( (X^{n_k}_t, DX^{n_k}_t) \) convergent weakly in \( L^p(\Omega) \otimes L^p(\Omega; H) \) to some \((X, Y)\). By assumptions, if the sequence \( X^n_t \) converges almost surely, then \( X^{n_k}_t \to X_t \) in \( L^p(\Omega) \), and therefore \( \{X^{n_k}_t\} \) is uniformly integrable. Since the inequality \( \text{(12)} \) holds, by Corollary 10.1.2...
and Theorem 10.1.3 in [22] there exists a unique measure \( P \) solving the the martingale problem for \( f \) and \( \sigma \). Also, because this martingale problem is well-posed, there exists a unique weak solution \( X_t \) for the SDE (3.1) which is the strong solution which we assumed the SDE (3.1) has and \( P_0 = P \). Now, according to Theorem 4.5.2 in [8], we have a sequence \( \{X^n_t\} \) which converges to \( X_t \) in \( L^p \) and \( \sup_{n \geq 1} \mathbb{E}(\|D_t X^n_t\|^p_H) < \infty \). Therefore, by Lemma 1.2.3 in [20], \( X_t \in \mathbb{D}^{1,p} \) and \( DX^{n_k}_t \longrightarrow DX_t \) in \( L^p(\Omega; H) \).

For every \( m \geq 2 \) and \( r = (r_1, \ldots, r_m) \) with \( r_1 < r_2 < \cdots < r_m \), let \( kSm \) be the set of all permutations of \( (r_1, \ldots, r_m) \) of the form \( R = (R_{i_1}, \ldots, R_{i_k}) \) where \( i_0 = 0 \) and \( R_{i_j} := r_{i_{j-1}+1}, \ldots, r_{i_j} \), such that \( i_{j-1} + 1 \leq \cdots \leq i_j \) for every \( 1 \leq j \leq k \). Set \( D_R X^n_t := D_R X^n_t \cdot \cdots \cdot D_{R_{i_k}} X^n_t \) and consider the linearized system \( LS(r_1, \ldots, r_m) \) defined by the form

\[
\begin{align*}
\sum_{k \leq m, 1 \leq k \leq m} LS(r_{i_1}, \ldots, r_{i_k}) & \quad 0 \leq r_1 < \cdots < r_m \leq t \\
& \quad 0 \leq r_1 < \cdots < r_m \leq t \\
& \quad dD_{r_1 \cdots r_m} X^n_t = \sum_{k \leq m, 1 \leq k \leq m} h^{(k)}_n(X^n_t) D_R X^n_t dt + \sum_{k \leq m, 1 \leq k \leq m} \sigma^{(k)}_n(X^n_t) D_R X^n_t dW_t,
\end{align*}
\]

(4.7)

with infinitesimal operator \( L_{n(r_1, \ldots, r_m)}(x, y), y \in \mathbb{R}^{N_m} \), where \( N_m = \sum_{1 \leq k \leq m} \#(kSm) \) and \( \sharp(kSm) \) is the number of elements of \( kSm \). For the sake of simplicity set

\[
A_{m,n}(X^n_t, Y_{t,m-1}^n) := \sum_{k \leq m, 1 \leq k \leq m} \sigma^{(k)}_n(X^n_t) D_R X^n_t.
\]

(4.8)

Here \( Y_{t,m-1}^n \) is a random vector in \( \mathbb{R}^{N_{m-1}} \) which components are the Malliavin derivatives of \( X^n_t \) up to the order \( m - 1 \), appeared in the system \( LS(r_1, \ldots, r_m) \).

The diffusion coefficient in this system will be denoted by

\[
\sigma_{m,n}(X^n_t) = \left( \sigma^0_{m,n}(X^n_t) + A_{m,n}(X^n_t, Y_{t,m-1}^n) \right)
\]

(4.9)

Our next result states that if a Lyapunov function \( V(\cdot) \) is constructed for the system \( LS(r) \), then for every \( m \geq 2 \) another one could be constructed for the linearized system \( LS(r_1, \ldots, r_m) \).

**Theorem 4.2.** Suppose that the assumptions in Theorem 4.1 hold, and for each \( y \in \mathbb{R}^{N_m} \), define the Lyapunov function \( V_m(x, y, z) = V(x, z) \). If for some constant \( c_0 \) the following inequalities hold:

\[
|\partial_x V_m(x, y, z)|^2 + |\partial_y V_m(x, y, z)|^2 + |\partial_{zz} V_m(x, y, z)|^2 \leq c_0 V(x, z) + c_0 F(x) + c_0,
\]

(4.10)

\[
\sup_{n \geq 1} \mathbb{E}\left[V_m(X^n_{r_{m-1}}, Y^n_{r_{m-1}}, \sigma^0_{m,n}(X^n_{r_{m}}))\right] \leq c_m,
\]

(4.11)

\[
\sup_{n \geq 1} \mathbb{E}\left[X^n_t|^q + |Y^n_{t,m-1}|^q\right] < c_0, \quad 1 \leq q \leq 2(m + 1),
\]

(4.12)

then

\[
\sup_{n \geq 1} \mathbb{E}\left[V_m(X^n_t, Y^n_{t,m-1}, D_{r_1 \cdots r_m} X^n_t)\right] < \infty
\]

Proof. According to Theorem 4.1 it is sufficient to show that for every \( m \geq 1 \), there exist some function \( F_m \) and a constant \( c_m \) such that

\[
L_{n, \{r_1, \ldots, r_m\}} V_m(x, y, z) \leq c_m \left( V_m(x, y, z) + G(x) + F_m(x, y) \right),
\]

(4.13)

\[
\sup_{n \geq 1} \mathbb{E}[F_m(X^n_t, Y^n_{t,m-1})] < \infty,
\]

(4.14)
Corollary 4.3.

We have
\[ L_{n,(r_1,\ldots,r_m)} V_m(x, y, z) = L_{n,r_1} V(x, z) + \partial_z V(x, z) \left( B_{m,n}(x, y) + A_{m,n}(x, y) \right) \]
\[ + \quad \partial_{xz} V(x, z) A_{m,n}(x, z) + \frac{1}{2} \partial_{zz} V(x, z) \left( A_{m,n}(x, z) \right)^2 \]
\[ \leq c V(x, z) + cG(x) + cF(x) + (\partial_z V(x, z))^2 + (\partial_{zz} V(x, z))^2 \]
\[ + \frac{1}{4} (\partial_{zz} V(x, z))^2 + \left( B_{m,n}(x, y) \right)^2 + \left( A_{m,n}(x, y) \right)^2 + \left( A_{m,n}(x, y) \right)^4 , \]

where in the last inequality we have used (4.3). Let
\[ L \]
\[ X \]
construct Lyapunov functions with polynomial growths thus satisfying (4.11) and conclude that
\[ G \]
\[ X \]

5. Malliavin Differentiability under Hypothesis 3.2

It is well-known that under Hypothesis 3.2 the SDE (3.1) has a strong solution \( \{ X_t \} \) \[ 17 \]. The uniqueness of the solution is obtained by using Itô’s formula and Gronwall’s inequality. In this section, we will show that this solution is in \( D^\infty \). To this end, we first show that \( X_t \in L^p(\Omega) \), does not explode in finite time, and the process \( \sup_{t \leq s \leq t} X_s \) has bounded moments. Then, we construct an almost everywhere convergent sequence of processes \( X^n_t \) whose limit is \( X_t \). For every \( p \geq 2 \), we will find some function \( V_p \) such that conditions (4.1), (4.3) and (4.6) hold, so that \( X_t \in D^{1,p} \). We construct Lyapunov functions with polynomial growths thus satisfying (4.11) and conclude that \( X_t \in D^\infty \). This procedure is followed in the next subsections. In what follows \( G_n \subseteq \mathbb{R} \), is set
\[ G_n = \left\{ x \in \mathbb{R}; \quad |x| \geq n^\xi \right\} \] \[ (5.1) \]
for each \( n \geq 1 \), and \( \tau_n \) is the first exit time of \( X_t \) from \( G_n \).
5.1. Some Properties of the solution

Lemma 5.1. For each $t \in [0, T]$ and $p > 1$, $X_t$ belongs to $L^p(\Omega)$ and does not explode in finite time.

Proof. Using Fatou’s lemma, we first show that $X_t$ is in $L^p(\Omega)$. By definition of the operator $\mathcal{L}$ and inequality [3.4], we have

$$\mathcal{L}X_t^{2p} = 2pX_t^{2p-1} \left( b(X_t) + f(X_t) \right) + p(2p - 1)\sigma^2(X_t)X_t^{2p-2} \leq 2p\beta pX_t^{2p} + 2p\alpha X_t^{2p-2} + 2pX_t^{2p-2}X_t f(X_t) \leq 2p\alpha + 2pX_t^{2p-2}\left( \frac{X_t^{2p}}{2} + \frac{f^2(X_t)}{2} \right) \leq p(2\beta + 2k^2 + 1)X_t^{2p} + 2p(\alpha + |f(0)|)X_t^{2p-2} =: \beta X_t^{2p} + \alpha X_t^{2p-2},$$

where in the last inequality we used the Lipschitz property of $f$. Applying Itô’s formula,

$$\frac{d}{dt}E[X_{t\wedge \tau_n}^{2p}] = E[\mathcal{L}X_{t\wedge \tau_n}^{2p}] \leq \beta X_{t\wedge \tau_n}^{2p} + \alpha X_{t\wedge \tau_n}^{2p-2}.$$ (5.2)

Setting $p = 1$ and using Gronwall’s inequality in [13], we derive

$$E[X_{t\wedge \tau_n}^{2}] \leq \left( |x_0|^2 + \frac{\alpha_1}{\beta_1} \right) exp\{3\beta_1 T\} - \frac{\alpha_1}{\beta_1}$$

By (5.2), we have

$$\left( \frac{n}{2} - 1 \right)^{\frac{p}{2}} P(t \geq \tau_n) \leq \left( |x_0|^2 + \frac{\alpha_1}{\beta_1} \right) exp\{3\beta_1 T\} - \frac{\alpha_1}{\beta_1}$$

Letting $n$ tend to $\infty$, we conclude that $\lim_{n \rightarrow \infty} \tau_n = \infty$ almost surely and thus by Fatou’s lemma

$$E[X_t^{2}] \leq \lim_{n \rightarrow \infty} \inf E[X_{t\wedge \tau_n}^{2}] \leq \lim_{n \rightarrow \infty} \inf E[X_{t\wedge \tau_n}^{2}] \leq \left( x_0^2 + \frac{\alpha_1}{\beta_1} \right) exp\{3\beta_1 T\} - \frac{\alpha_1}{\beta_1}.$$  

Hence from (5.2) and induction on $p$ we get $X_t \in L^{2p}(\Omega)$. Now by the following interpolation inequality

$$E[|X_t|^{2p+1}] \leq (E[X_t^{2p}])^{\frac{1}{p}} \left( E[X_t^{2p+2}] \right)^{\frac{1}{2}}$$

we conclude that $X_t \in L^{p}(\Omega)$ for every $p > 1$. \hfill $\square$

In the proof of the next Lemma we will use the following version of the Young’s inequality. For $r \geq 2$ and for all $a, c$ and $\triangle_1 > 0$, we have:

$$a^{r-2}c^2 \leq \triangle_1^{\frac{r-2}{r}} a^r + \frac{2}{\triangle_1^{\frac{r-2}{r}}} c^r.$$ (5.3)

Lemma 5.2. For every $p \geq 2$, if $\beta p \geq 0$, there exists some constant $c_p'$ such that

$$E\left[ \sup_{0 \leq t \leq T} |X_t|^p \right] < c_p'$$ (5.4)

Proof. We know that for every $p \geq 2$, $E[|X_t|^p] < \infty$. Applying Itô’s formula and the Burkholder-Davis-Gundy inequality in [21], we have
Notice that by Theorem 2.2.1 in Nualart (2006), the SDE (5.11) has a strong solution in M. Exist a constant C for every x.

Denote by hypothesis 3.2, b X. Then, obviously, A and Gronwall’s inequality completes the proof.

5.2. Approximation of the solution

For every integer n > 0 let choose some smooth functions \( \varphi_n : \mathbb{R} \to \mathbb{R} \) such that \( \varphi_n = 1 \) on \( A_n := \{ x \in \mathbb{R} ; |x| \leq n^2 \} \) and \( \varphi_n = 0 \) outside \( A_{2n^2} \). Also for each multiindex L with \( |L| = l \geq 1 \),

\[
\sup_{n,x} \left( |\partial_L \varphi_n| + |(b, \partial_L \varphi_n)| + |\sigma \partial_L \varphi_n| \right) \leq M_l
\]

for some \( M_l > 0 \). (See Appendix and the proof of Lemma 2.1.1 in [20].) Now, set

\[
b_n(x) := \varphi_n(x) b(x) \quad \text{and} \quad \sigma_n(x) := \varphi_n(x) \sigma(x)
\]

for every \( x \in \mathbb{R}^d \) and \( n > 0 \). Then \( b_n \) would be globally Lipschitz and continuously differentiable. By (3.2) for each \( x \in \mathbb{R}^d \) and each multiindex L with \( |L| = l \geq 1 \), there exist positive constants \( \Gamma_l \) and \( p_l \) such that

\[
|\partial_L b_n(x) + \partial_L \sigma_n(x)|^2 \leq \Gamma_l(1 + |x|^{p_l}).
\]

Then, obviously, \( b_n \) and \( \sigma_n \) are continuously differentiable and therefore globally Lipschitz. By hypothesis 3.2 \( b'(x) \) is negative and since \( ac \leq a^2/2 + c^2/2 \) for all \( a, c \) and \( 0 \leq \phi(.) \leq 1 \), there exists a constant \( C_{0,b} \) independent of \( n \) such that

\[
b'_n(x) \leq C_{0,b}, \quad \text{and} \quad (\sigma'_n(x))^2 \leq 2(\sigma'(x))^2 + C_{0,\sigma}
\]

Notice that by Theorem 2.2.1 in Nualart (2006), the SDE (5.11) has a strong solution in \( \mathbb{D}^\infty \), that is, there exists \( X^n_t \) in \( \mathbb{D}^\infty \) which satisfies

\[
X^n_t = x_0 + \int_0^t b_n(X^n_s)ds + \int_0^t \sigma_n(X^n_s)dW_s.
\]

Denote by \( \mathcal{L}_n \) the infinitesimal operator associated to the latter SDE.
Lemma 5.3. For each $t \in [0, T]$ and $p > 1$, the sequence $\{X^n_t\}$ is uniformly integrable and almost everywhere convergent to $X_t$.

Proof. To prove the convergency, let $X_t^{\tau_n}$ denotes $X$ stopped at $\tau_n$. By the proof of Lemma 5.1, $\tau_n$ increases to infinity as $n$ tends to infinity. By the choice of the function $\phi_n(\cdot)$, it then follows that $X_t^{\tau_n} = X^n_t$ for all $t \leq \tau_n$. Thus, for fix $t \in [0, T]$, letting $n$ tend to infinity, we will have $\lim_{n \to \infty} X^n_t = \lim_{n \to \infty} X_t^{\tau_n} = X_t$ a.s.

Now, we prove that the sequence $\{X^n_t\}$ is uniformly integrable. In fact, we show that for every $p > 1$,

$$\sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[|X_t^n|^p] \leq c_p. \quad (5.12)$$

By definition of the operator $L_n$ and (5.4), since $\phi_n(\cdot) \leq 1$, we have

$$L_n(X_t^n)^{2p} = 2p(X_t^n)^{2p-1} \left( b_n(X_t^n) + f(X_t^n) \right) + p(2p-1)\sigma_n^2(X_t^n)(X_t^n)^{2p-2} \leq 2p\phi_n(X_t^n)\left( \alpha_p + \beta_p X_t^n \right)(X_t^n)^{2p-2} + 2p(X_t^n)^{2p-2}(X_t^n)f(X_t^n)$$

$$\leq 2p\beta_p(X_t^n)^{2p} + 2p\alpha_p(X_t^n)^{2p-2} + 2p(X_t^n)^{2p-2}\left( \frac{(X_t^n)^2}{2} + \frac{f^2(X_t^n)}{2} \right) \leq p(2\beta_p + 2k_1^2)(X_t^n)^{2p} + 2p(\alpha_p + |f(0)|^2)(X_t^n)^{2p-2}.$$  

Using Itô’s formula,

$$\frac{d}{dt}\mathbb{E}[X_t^n]^{2p} = \mathbb{E}[L_n(X_t^n)^{2p}] \leq \beta'_p\mathbb{E}[(X_t^n)^{2p}] + \alpha'_p\mathbb{E}[(X_t^n)^{2p-2}]$$

Now setting $p = 2$ and applying Gronwall’s inequality in (13), for $p = 2$ we derive (5.12). By induction on $p$ and the following interpolation inequality

$$\mathbb{E}[X_t^n]^{2p+1} \leq \left( \mathbb{E}[(X_t^n)^{2p}] \right)^{1/2} \left( \mathbb{E}[(X_t^n)^{2p+2}] \right)^{1/2},$$

we conclude that for every $p \geq 2$ inequality (5.12) holds.

\[ \square \]

Corollary 5.4. The following properties hold

- For every $p \geq 2$, $X_t^n \to X_t$ in $L^p(\Omega)$,
- For every $q \geq q_1$, $b_n(X_t^n) \to b(X_t)$ and $\sigma_n(X_t^n) \to \sigma(X_t)$ in $L^q(\Omega)$.

Proof. By inequality (5.12) and uniform integrability of sequence $\{X_t^n\}$, we derive uniform integrability of sequences $\{b'_n(X_t^n)\}$ and $\{\sigma'_n(X_t^n)\}$. Also, we know that $t X_t^n$ converges to $X_t$ almost surely. Thus $X_t^n \to X_t$ in $L^p(\Omega)$ and $b'(X_t^n) \to b'(X_t)$ in $L^p(\Omega)$. On the other hand, for every $\epsilon \geq 0$ and every integer $p \geq 2$

$$P(|b'_n(X_t^n) - b(X_t^n)| > \epsilon) \leq P(|X_t^n| > n) \leq \sup_n \frac{\mathbb{E}[|X_t^n|^p]}{n^p} \leq \frac{c_p}{n^p}.$$  

So that, $b'_n(X_t^n) - b'(X_t^n) \to 0$ in probability and thus in $L^p(\Omega)$ by uniform integrability of $\{b'_n(X_t^n) - b'(X_t^n)\}$. The triangle inequality completes the proof.

\[ \square \]

5.3. Weak differentiability of the solution

In this subsection, we prove the weak differentiability of $X_t$, using Lemma 1.2.3 in [20] and Theorem 4.1. Then, this fact and Theorem 4.2 results Theorem 5.3.
Lemma 5.5. Assuming Hypothesis 3.2 for every \( p > 1 \) the unique strong solution of SDE (3.1) is in \( D^{1,p} \). In addition, for \( r > t \), \( D_r X_t = 0 \) and for \( r \leq t \)

\[
D_r X_t = \sigma(X_r) + \int_r^t \left( b'(X_s) + f'(X_s) \right) D_r X_s ds + \int_r^t \sigma'(X_s) D_r X_s dW_s.
\]

(5.13)

Proof. By Theorem 2.2.1 in [20] we know that for every \( r > t \), \( D_r X_t^n = 0 \) and for every \( r \leq t \)

\[
D_r X_t^n = \sigma_n(X^n_r) + \int_r^t \left( b'_n(X^n_s) + f'(X^n_s) \right) D_r X^n_s ds + \int_r^t \sigma'_n(X^n_s) D_r X^n_s dW_s.
\]

By Theorem 4.1 it is sufficient to find a Lyapunov function satisfying conditions (4.3) and (4.6), which implies that for some constant \( c_q \),

\[
\sup_{n \geq 1} \sup_{0 \leq t \leq T} E \left[ |DX_t^n|^p \right] \leq c_q,
\]

(5.14)

Then we will show that \( DX_t \) is the solution of SDE (5.13). To this end, we proceed through the following three steps.

**Step 1.** In this step, we introduce some Lyapunov functions \( V_q(.) \) for every \( q \geq 1 \) and obtain some upper bounds for \( L_{n,r} V_q \) in terms of \( V_q(.) \) and \( \sigma'(.) \).

For every \( q \geq 1 \) and every \( M > 0 \) large enough, choose the following Lyapunov function

\[
V_q(x, y) := x^{2q} + x^{2q} y^{2q} + y^{2q} + M,
\]

then by definition of \( V_q \), equation (5.12) and Theorem 2.2.1 in [20], the conditions (4.5) and (4.6) hold. To prove (4.3), note that by definition of \( L_{n,r} \), it holds

\[
L_{n,r} V_q(x, y) = 4q x^{2q-2} \left[ x \left( b_n(x) + f(x) \right) + \frac{4q - 1}{2} \sigma_n^2(x) \right]
+ 2q y^{2q} x^{2q-2} \left[ x \left( b_n(x) + f(x) \right) + \frac{2q - 1}{2} \sigma_n^2(x) \right]
+ 2q y^{2q} \left[ 2q x^{2q} \sigma_n^2(x) + \frac{2q - 1}{2} \sigma_n'(x)^2 \left( x^{2q} + 1 \right) \right]
+ 2q y^{2q} \left[ \left( b'_n(x) + f'(x) \right) \left( x^{2q} + 1 \right) \right] =: (I_1 + I_2 + I_3 + I_4)(x, y).
\]

(5.15)

Since \( f' \) is bounded, using (5.10) we get

\[
I_1(x, y) \leq 2q(k_1 + C_{0,b}) V_q(x, y).
\]

(5.16)

Applying (5.10)

\[
I_3(x, y) \leq 2q y^{2q} \left[ 2q x^{2q-2} \left( \frac{(x \sigma_n'(x))^2}{2} + \frac{\sigma_n^2}{2} \right) + \frac{2q - 1}{2} \sigma_n'(x)^2 \left( x^{2q} + 1 \right) \right]
\leq 2q y^{2q} x^{2q-2} \left[ \frac{2q}{2} \sigma^2(x) + (4q - 1) x^2 \sigma^2(x) \right]
+ 2q (2q - 1) y^{2q} \sigma^2(x) + q(4q - 1) C_{0,s} x^{2q} y^{2q} + q(2q - 1) C_{0,s} y^{2q}.
\]

Now by (3.4), we have

\[
(I_1 + I_2 + I_3)(x, y) \leq 4q x^{2q-2} \left[ \alpha_q + \beta_q \left| x \right|^2 \right] + 4q x^{2q-2} \left[ \frac{x^2}{2} + \frac{f^2(x)}{2} \right]
+ 2q y^{2q} x^{2q-2} \left[ \alpha_q + \beta_q \left| x \right|^2 \right] + 2q y^{2q} x^{2q-2} \left[ \frac{x^2}{2} + \frac{f^2(x)}{2} \right]
+ 2q (2q - 1) y^{2q} (\sigma'(x))^2 + q(4q - 1) C_{0,s} x^{2q} y^{2q} + q(2q - 1) C_{0,s} y^{2q}
\]

(5.17)
To show that the terms in the right hand side of the latter inequality are bounded, we need the Young’s inequality. Using (5.3) for \( r = 4q \) and again for \( r = 2q \) with \( c = \Delta_1 = 1 \), we have

\[
x^{4q-2} \leq \frac{2q - 1}{2q} x^{4q} + \frac{1}{2q}, \quad \text{and} \quad x^{2q-2} \leq \frac{q - 1}{q} x^{2q} + \frac{1}{q}.
\]  

(5.18)

Applying (5.18) in (5.17), we find some constant \( c_q \) such that

\[
I_1(x, y) + I_2(x, y) + I_3(x, y) \leq c_q V_q(x, y) + 2q(2q - 1)y^{2q}(\sigma'(x))^2.
\]  

(5.19)

Substituting (5.10) and (5.19) in (5.15), we conclude that

\[
L_{n,r} V_q(x, y) \leq (c_q + 2q(k_1 + C_{0, b}) V_q(x, y) + 2q(2q - 1)y^{2q}(\sigma'(x))^2
\]  

(5.20)

**Step 2.** Here, we show the inequality (4.3) for every \( q \geq 1 \). First, let \( q = q_1 \). By (3.2) for some \( c_q \) independent of \( n \), we have

\[
2q(2q - 1)y^{2q}(\sigma'(x))^2 \leq \gamma_1 4q(2q - 1)y^{2q} + \gamma_1 4q(2q - 1)y^{2q}|x|^{2q_1} \leq c_{q_1} V_q(x, y).
\]

Substitute this bound in (5.20) and use Theorem 4.1 to derive (5.14) for \( p = q_1 \), from which and interpolation inequality we derive the result for each \( p < q_1 \).

Now, let \( q > q_1 \). Using Hölder inequality, for some constant \( c'_{q_1} \) independent of \( n \) we have

\[
y^{2q} x^{2q_1} = y^{2q-2q_1} x^{2q_1} y^{2q_1} \leq \frac{q - q_1}{q} y^{2q-2q_1} \left( y^{2q-2q_1} \right)^{\frac{q}{q_1}} + \frac{q_1}{q} x^{2q_1} y^{2q_1} \leq \frac{q - q_1}{q} y^{2q} + \frac{q_1}{q} x^{2q_1} y^{2q} \leq c'_{q_1} V_q(x, y)
\]

Again, substitute this bound in (5.20) and use Theorem 4.1 to derive (5.14) for \( p > q_1 \).

**Step 3.** Now, we show that \( DX_t \) is the solution of SDE (5.13). For every \( \epsilon > 0 \), by Lemma 5.2 we have

\[
P(|DX^n_t| > \epsilon) \leq P(t \geq \tau_n) \leq P(\sup_{0 \leq s \leq \tau_n} |X^n_s| > \epsilon) \leq \frac{E \left( \sup_{0 \leq s \leq \tau_n} |X^n_s|^p \right)}{n^p} \leq c'_{p} \frac{c'_{p}}{n^p}
\]

Therefore, \( DX^n_t \rightarrow DX_t \) in probability. Since the sequence \( |DX^n_t|^p \) is uniformly integrable, this convergence still hold in \( L^p(\Omega) \) for every \( p \geq 2 \). From Corollary 5.4 \( DX_t \) is the solution to SDE (5.13) and the proof is completed.

Now, Since the Lyapunov function \( V_q \) and the functions \( b \) and \( \sigma \) have polynomial growth, by induction on \( m \) in Theorem 4.2 we can derive inequalities (4.11) and (4.12). Condition (4.10) is also obviously true, and Theorem 3.3 follows as a result.

6. The nondegeneracy condition

In this section, we will show how the regularity of the distribution of \( X(t) \) could be derived from the nondegeneracy condition of it.

Denote the Malliavin covariance matrix of \( X^n_t \) and \( X_t \) by \( \Lambda_{X^n}(t) \) and \( \Lambda_X(t) \), for each \( 0 \leq t \leq T \), respectively. Let \( Z^n_t \) be the solution of the following linear SDE;

\[
Z^n_t = 1 + \int_0^t [b''_n(X^n_s) + f'(X^n_s)] Z^n_s ds + \int_0^t \sigma'_n(X^n_s) Z^n_s dW_s
\]

and

\[
C^n_t := \int_0^t \exp \left\{ -2 \int_0^r [b''_n(X^n_s) - \frac{1}{2}(\sigma''_n)^{2}(X^n_s)] ds + 2 \int_0^r \sigma'_n(X^n_s) dW_s \right\} \sigma^2(X^n_r) dr.
\]
Then \( \Lambda_{X^n}(t) = C^n_t(Z^n_t)^{-2} \). Also, by the proof of Theorem 5.5 one can easily show that for every \( p \geq 2 \), there exist some constant \( l_p \) such that
\[
\sup_{n \geq 1} \mathbb{E} \left[ |Z^n_t|^p \right] \leq l_p.
\]

**Lemma 6.1.** The nondegeneracy condition is satisfied for \( X_t \), and for every \( p \geq 2 \) and \( \epsilon < \epsilon_0(p) \),
\[
P \left( \Lambda_X(t) \leq \epsilon \right) \leq e^p
\]

**Proof.** By definition of \( b_n \) and \( \sigma_n \), it is easy to derive condition \( (H) \) for these coefficients. So, the nondegeneracy condition is satisfied for \( X^n_t \) for every \( n \geq 1 \). From (6.1) and Proposition 2.1 for every \( n \geq 1 \) and small enough \( \epsilon \),
\[
P \left( \Lambda_{X^n}(t) \leq \epsilon \right) \leq P \left( C^n_t \leq \epsilon \right) + P \left( (Z^n_t)^{-2} < \epsilon \right) \leq e^p \left( 1 + \mathbb{E} \left[ (Z^n_t)^p \right] \right).
\]

Also,
\[
\{ \Lambda_X(t) \leq \epsilon \} \subseteq \{ \Lambda_{X^n}(t) \leq 2\epsilon \} \cup \{ \Lambda_{X^n}(t) > 2\epsilon \} \quad \text{and} \quad \Lambda_X(t) \leq \epsilon
\]
\[
\subseteq \{ \Lambda_{X^n}(t) \leq 2\epsilon \} \cup \{ \Lambda_{X^n}(t) \leq 2\epsilon \} \cup \{ |\Lambda_{X^n}(t) - \Lambda_X(t)| > \epsilon \}
\]

Now, by Step3 in the proof of Theorem 5.5 we can choose \( N \geq 1 \) such that for every \( n \geq N \),
\[
\mathbb{E} \left[ |D_r X^n_t - D_r X_t|^2 \right] < e^{p+1}.
\]

Then,
\[
P \left( \Lambda_X(t) \leq \epsilon \right) \leq P \left( \Lambda_{X^n}(t) \leq 2\epsilon \right) + P \left( |\Lambda_{X^n}(t) - \Lambda_X(t)| > \epsilon \right)
\]
\[
\leq (1 + l_p)e^p + \frac{1}{\epsilon} \mathbb{E} \left[ |\Lambda_{X^n}(t) - \Lambda_X(t)|^2 \right]
\]
\[
\leq (1 + l_p)e^p + (t - r)e^p,
\]
and the nondegeneracy condition holds for \( X_t \).

Therefore, we have a nondegenerate solution \( X_t \) to SDE \((6.1)\) which has a \( C^\infty \)-density and thus Theorem 3.4 hold.

**Appendix A. constructing the approximating functions for the drift**

In this Appendix we present how we choose the functions \( b_n \). This construction is motivated by Berhanu in [K. Theorem 2.9.] Assume that \( U \subset V \) be two open sets in \( \mathbb{R}^d \) with distance \( a \). For \( 0 \leq \epsilon \leq a \), define \( U_\epsilon = \{ x; d(x, U) < \epsilon \} \). Then \( U_\epsilon = \bigcup_{x \in U} B_\epsilon(x) \) and \( U \subseteq U_\epsilon \subseteq V \). Fix \( \epsilon \) such that \( 0 < 2\epsilon \leq a \) and let \( h^\epsilon(x) \) be the characteristic function of \( U_\epsilon \). Let \( \psi \in C^\infty_0(\mathbb{R}^d) \) with \( \text{supp}\psi \subseteq B_1(0) \) and \( \int \psi(x)dx = 1 \). Set \( \psi_\epsilon(x) = \frac{1}{\epsilon^d} \psi \left( \frac{x}{\epsilon} \right) \). Consider now the convolution function \( \psi_\epsilon * h^\epsilon \) for \( 0 < 2\epsilon < a \). Since \( \text{supp}\psi_\epsilon \subseteq B_1(0) \), \( \psi_\epsilon * h^\epsilon = 1 \) on \( U \) and \( \psi_\epsilon * h^\epsilon = 0 \) outside \( U_{2\epsilon} \). Note that for each multiindex \( \alpha \),
\[
\partial_\alpha (\psi_\epsilon * h^\epsilon)(x) = \int \partial_\alpha (\psi_\epsilon(y))h^\epsilon(x - y)dy = \frac{1}{\epsilon^d + |\alpha|} \int (\partial_\alpha \psi)(\frac{y}{\epsilon})h^\epsilon(x - y)dy
\]
\[
= \frac{1}{\epsilon^{|\alpha|}} \int (\partial_\alpha \psi)(z)h^\epsilon(x - \epsilon z)dz \leq \| \psi \|_\infty \frac{1}{\epsilon^{|\alpha|}}. \tag{A.1}
\]

Now, for every \( n \geq 1 \) consider \( U = B_{n\epsilon}(0) \), \( V = B_{2n\epsilon}(0) \) and \( \epsilon = n^\xi \). Then there exist the functions \( \phi_n(x) := \psi_\epsilon * h^\epsilon \) such that \( \phi_n(x) = 1 \) on \( U \) and \( \phi_n(x) = 0 \) outside \( V \). Since \( \text{supp}\phi_n(x) \subseteq B_{2n\epsilon}(0) \),
by (A.1) and (3.2) for each multiindex \( \alpha \) with \( |\alpha| = c \geq 1 \), we have

\[
|b(x)\partial_\alpha \phi_n(x)| \leq |b(x)\chi_{|x| \leq 2n\xi}| \| \psi \|_\infty \frac{1}{n^{\xi|\alpha|}} \\
\leq \gamma_c (1 + 2^\xi n^{\xi}) \| \psi \|_\infty \frac{1}{n^{\xi|\alpha|}} \leq 2^{\xi+1} \gamma_c \| \psi \|_\infty.
\]

In the same way, it hold true when we replace \( b \) by \( \sigma \). Also,

\[
|\partial_\alpha \phi_n(x)| \leq \| \psi \|_\infty.
\]

References

[1] Alòs, E., and Ewald, C. O. 2008. Malliavin differentiability of the Heston volatility and applications to option pricing, Adv. in Appl. Probab. 40(1): 144–162.

[2] Bahlali, K. 1999. Flows of homeomorphisms of stochastic differential equations with measurable drifts, Stochastics and Stochastics Reports 67: 53–82.

[3] Baudoin, F., Ouyang, C., and Tindel, S., Upper bounds for the density of solutions of stochastic differential equations driven by fractional Brownian motions, Submitted on 19 Apr 2011

[4] Bavouzet, M. P., and Messaoud, M. 2006. Computation of Greeks using Malliavin’s calculus in jump type market models, Electronic Journal of Probability 11(10): 276–300.

[5] Bell, D. R., 2004, Stochastic differential equations and hypoelliptic operators, Real and Stochastic Analysis, Trends in Mathematics 9–42.

[6] Berhanu, S. Approximation by smooth functions and distributions, http://www.math.temple.edu/~berhanu (2001).

[7] Bichteler, K., Gravereaux, J-B., and Jacod, J. 1987. Malliavin Calculus for Processes with Jumps, Vol 2, Gordon and Breach Science Publishers, Amesterdam.

[8] Chung, K. L. 2001, A Course In probability Theory, Third edition, Academic Press.

[9] Fournier, N., and Printems, J. 2010. Absuletly continuous for some one-dimensional processes, Bernoulli 16(2): 343–360.

[10] Gyöngy, I., and Millet, A. 2007. Rate of convergence of implicit approximations for stochastic evolution equations, Stochastic Differential Equations: Theory and Applications, A volume in honor of professor Boris L. Rosovskii, Interdisciplinary Mathematical Sciences, 2, World Scientific 281–310.

[11] Hiraba, S. 1992. Existence and smoothness of transition density for jump-type Markov processes: Applications of Malliavin calculus, Kodai. Math. J. 15(1): 28–49.

[12] Kohatsu-Higa, A., and Montero, M. 2004. Malliavin Calculus in Finance, Handbook of computational and numerical methods in finance, Birkhäuser Boston, Boston, MA, 111–174.

[13] Khas’minskii, R.Z. 2012. Stochastic Stability of Differential Equations, second ed., Springer-Verlag, Berlin.

[14] Kusuoka, S., and Stroock, D. 1984. Applications of the Malliavin calculus, Part I, Stochastic Analysis, (Kyoto/Katata, 1982), North-Holland Math. Library 32: 271–306.
[15] Kusuoka, S., and Stroock, D. 1985. Applications of Malliavin calculus, Part II, *J. Fac. Sci. Uni. Tokyo*, Sect. IA. MAth. 32: 1–76.

[16] Kusuoka, S. 2010. Existence of densities of solutions of stochastic differential equations by Malliavin calculus, *J. Functional Analysis* 258: 758–784.

[17] Mao, X. 1997, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Limited, England.

[18] Marco, S. D. 2010, *On probability distributions of diffusions and financial models with nonglobally smooth coefficients*, PhD. Thesis, [http://cermics.enpc.fr/ de-marcs/home.html](http://cermics.enpc.fr/ de-marcs/home.html)

[19] Marco, S. D. 2011. Smoothness and asymptotic estimates of densities for SDEs with locally smooth coefficients and applications to square-type diffusions, *Annals of Applied Probability* 21(4): 1282–1321.

[20] Nualart, D. 2006. *The Malliavin Calculus and Related Topics*, second ed., Springer Verlag, Berlin.

[21] Ren Y-F., 2008. On the Burkholder-Davis-Gundy inequalities for continuous martingales, *Statistics and Probability Letters* 78: 3034–3039.

[22] Stroock, W., and Varadhan, S. R. S. 1979. *Multidimensional Diffusion Processes*, Springer Verlag, Berlin.

[23] Zangeneh, B. Z. 1995. Semilinear stochastic evolution equations with monotone nonlinearities, *Stochastics and Stochastics Reports* 3: 129–174.