Convex synthesis and verification of control-Lyapunov and barrier functions with input constraints

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Abstract—Control Lyapunov functions (CLFs) and control barrier functions (CBFs) are widely used tools for synthesizing controllers subject to stability and safety constraints. Paired with online optimization, they provide stabilizing control actions that satisfy input constraints and avoid unsafe regions of state-space. Designing CLFs and CBFs with rigorous performance guarantees is computationally challenging. To certify existence of control actions, current techniques not only design a CLF/CFB, but also a nominal controller. This can make the synthesis task more expensive, and performance estimation more conservative. In this work, we characterize polynomial CLFs/CBFs using sum-of-squares conditions, which can be directly certified using convex optimization. This yields a CLF and CBF synthesis technique that does not rely on a nominal controller. We then present algorithms for iteratively enlarging estimates of the stabilizable and safe regions. We demonstrate our algorithms on a 2D toy system, a pendulum and a quadrotor.

I. INTRODUCTION

When synthesizing controllers for dynamical systems, it is often paramount to ensure that the closed-loop system always converges to the desired state and always avoids unsafe regions. These goals can be met by using control Lyapunov functions (CLFs) [28] for stability, and control barrier functions (CBFs) [3] for safety. CLFs and CBFs are designed such that their online minimization leads to desired performance goals. Local verification amounts to certifying these minimization problems are feasible on some subset of state-space, which can be a challenging computational task.

CLFs and CBFs have been used extensively in controller design for various applications, including legged locomotion [14], [15], [23], autonomous driving [2], [7], [37], and robot arm manipulation [22]. Recently, they have been used to guide learning-based methods [8], [9], [17]. The CLFs/CBFs are typically synthesized by hand or learned from data [26], [12], [19] without rigorous verification. In this paper, we provide a new synthesis technique with formal guarantees that is conceptually simpler than previous formal methods [16], [20], [1], [33]. In particular, our technique allows one to verify CLFs and CBFs by solving a single convex optimization problem, under the assumption the dynamics are control-affine and the input constraints are polyhedral (for example, robots subject to torque limits for each motor). This in turn leads to synthesis algorithms based on sequential convex optimization.

The basis of our technique is Sum-Of-Squares (SOS) optimization [6], [24], a widely used tool for controller synthesis and verification, including CLF and CBF design [30], [10], [16], [20], [1], [33]. These previous methods either ignore input constraints [30], [10] or rely on the joint synthesis of a nominal control law of polynomial form [16], [20], [1], [33]. Reliance on this polynomial controller is restrictive if the actual stable/safe control policy is not polynomial. It also limits scalability, as the size of the SOS program increases rapidly with the degree of the controller.

In this paper, we derive new necessary and sufficient conditions for CLFs/CBFs for polynomial dynamical systems with input constraints. We formulate these conditions as SOS feasibility problems, and present an iterative algorithm for enlarging an inner approximation of the stabilizable and/or safe region via sequential SOS optimization. We demonstrate our algorithm on different systems, including a 2D toy example, an inverted pendulum and a quadrotor.

II. BACKGROUND

In this section we give a brief introduction to Sum-Of-Squares (SOS) techniques for certifying polynomial non-negativity. To begin, a polynomial \( p(x) \) is a sum-of-squares (sos) iff \( p(x) = \sum_i q_i(x)^2 \) for some polynomials \( q_i(x) \). Clearly \( p(x) \) being sos implies that \( p(x) \geq 0 \) \( \forall x \). If \( p(x) \) has degree \( 2d \), then it is an sos polynomial if and only if

\[
p(x) = m(x)^T S m(x), \quad S \succeq 0,
\]

where \( m(x) \) is a vector consisting of all monomials of degree at most \( d \). Given \( p(x) \) and \( m(x) \), existence of \( S \) can be checked using semidefinite programming [24], [6].

Sum-of-squares optimization can also certify polynomial non-negativity on a semialgebraic set \( K \), namely the feasible set of finitely-many polynomial inequalities of the form
\[ K = \{ x \in \mathbb{R}^n | b_1(x) \geq 0, \ldots, b_m(x) \geq 0 \}. \] The underlying certificates employ the preorder of \( b_i \), defined as
\[
\text{preorder}(b_1(x), \ldots, b_m(x)) = \left\{ \begin{array}{l}
L_0(x) + \sum_{i=1}^{m} L_i(x) b_i(x) + \sum_{i \neq j} L_{ij}(x) b_i(x) b_j(x) \\
+ \sum_{i \neq j \neq k} L_{ijk}(x) b_i(x) b_j(x) b_k(x) \\
+ \ldots | L_0(x), L_i(x), L_{ij}(x), L_{ijk}(x), \ldots \text{ are all sos} \end{array} \right\}
\]
(2)

For a given polynomial \( p(x) \), the Positivstellensatz [29][18, Section 3.6] states that
\[ p(x) \geq 0 \text{ on } K \]
\[ \exists q(x), r(x) \in \text{preorder}(b_1(x), \ldots, b_m(x)), k \in \mathbb{N} \]
\[ \text{s.t. } p(x)q(x) = p(x)^{2k} + r(x) \]  
(3)

In other words, if \( p(x) \) is non-negative on \( K \), then there is a certificate of this fact defined by polynomials \( q(x) \) and \( r(x) \) in the preorder. Further, for a fixed \( k \), finding this certificate can be cast as a semidefinite program [6].

Unfortunately, a generic element of the preorder is the summation of \( 2^m \) different polynomials each scaled by a different sum-of-squares polynomial. This exponential complexity motivates simpler (sufficient) conditions for non-negativity. A common simplification—called the S-procedure [24]—is existence of polynomials \( \tilde{r}_i(x) \), \( i = 0, \ldots, m \) satisfying
\[ (1 + \tilde{r}_0(x))p(x) - \sum_{i=1}^{m} \tilde{r}_i(x)b_i(x) \text{ is sos} \]
(4a)

\[ \tilde{r}_i(x) \text{ is sos, } i = 0, \ldots, m. \]
(4b)

The equation (4a) implies that \( p(x) \geq 0 \) on \( K \) because, by definition, \( \tilde{r}_i(x)b_i(x) \geq 0 \) on \( K \). Frequently, we simplify this condition further by taking \( \tilde{r}_0(x) = 0 \).

### III. Problem Formulation

We consider a control-affine dynamical system of the form
\[ \dot{x} = f(x) + g(x)u, \quad u \in U, \]
(5)

where \( x \in \mathbb{R}^{n_x} \) and \( u \in \mathbb{R}^{n_u} \) denote the state and control, \( f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x} \) and \( g : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_u} \) are polynomial functions of \( x \) and \( U \subset \mathbb{R}^{n_u} \) denotes the set of admissible inputs, which we assume to be a convex polytope. Equivalently, we assume existence of a finite set of points \( u^i \) satisfying
\[ U = \text{ConvexHull}(u^1, \ldots, u^{n_u}). \]
(6)

Without loss of generality, we assume that the goal state is \( x^* = 0 \). Finally, we note that through a change of variables, the dynamics of many robotic systems can be written using polynomials; see, e.g., [25], [27].

### A. Control Lyapunov Functions (CLFs)

A polynomial function \( V(x) \) is a control Lyapunov function (CLF) if it satisfies the following conditions for some positive integer \( \alpha \):
\[ V(x) \geq \epsilon(x^T x)^\alpha \forall x \]  
(7a)

\[ V(0) = 0 \]  
(7b)

if \( V(x) < \rho \) and \( x \neq 0 \) then \( \exists u \in U \)
\[ \text{s.t. } L_f V + L_g V u < -\kappa V, \]
\[ V(x, u) \]
(7c)

where \( L_f V = \frac{\partial V}{\partial x} f(x) \) and \( L_g V = \frac{\partial V}{\partial x} g(x) \) denote Lie-derivatives.

Under these conditions, the sublevel set \( \Omega_\rho = \{ x \in \mathbb{R}^{n_x} | V(x) < \rho \} \) is an inner approximation of the stabilizable region, i.e., the backward reachable set of the goal state \( 0 \). This means that for all initial states in \( \Omega_\rho \), there exist control actions that drive the state to \( 0 \). The condition (7a) guarantees that \( V(x) \) is positive definite and radially unbounded, while the condition (7c) guarantees that \( V(x) \) converges to zero exponentially with a rate larger than \( \kappa V > 0 \). Our goal is to find a polynomial CLF \( V(x) \) and a scalar \( \rho \) that satisfy (7) and maximize (in some sense) the size of the \( \Omega_\rho \). In other words, we seek a CLF that yields a large inner approximation of the stabilizable region. We remark that a previous technique for outer approximation appears in [21].

### B. Control Barrier Functions (CBFs)

Given an unsafe set \( \mathcal{X}_{\text{unsafe}} \), a polynomial function \( h(x) \) is a control barrier function with the safe region \( \{ x \in \mathbb{R}^{n_x} | h(x) > 0 \} \) if it satisfies
\[ h(x) \leq 0 \forall x \in \mathcal{X}_{\text{unsafe}} \]  
(8a)

if \( h(x) > \beta^- \) then \( \exists u \in U \)
\[ \text{s.t. } L_f h + L_g h u < -\kappa_h h, \]
\[ h(x, u) \]
(8b)

where \( \beta^- < 0 \) and \( \kappa_h > 0 \) are given constants. We assume that the unsafe regions (such as obstacles) are given as the union of semialgebraic sets, i.e.,
\[ \mathcal{X}_{\text{unsafe}} = \mathcal{X}_{\text{unsafe}}^1 \cup \ldots \cup \mathcal{X}_{\text{unsafe}}^m \]
(9a)

\[ \mathcal{X}_{\text{unsafe}} = \{ x | p_{i,1}(x) \leq 0, \ldots, p_{i,n_i}(x) \leq 0 \}, \]
(9b)

where \( p_{i,j}(x) \) is a polynomial. Our goal is to find a polynomial CBF \( h(x) \) with a large certified safe region \( \{ x \in \mathbb{R}^{n_x} | h(x) > 0 \} \).

Note that the CLF and CBF conditions are related. Specifically, if \( V(x) \) satisfies (7c) for a given \( \kappa \), then \( h(x) = -V(x) \) satisfies (8b) with the same \( \kappa \) and \( \beta^- = -\rho \). Hence any approach for synthesizing CLFs can be used to synthesize CBFs with slight modification.

Our approach can be equally applied to the scenario where the safe region is given and the goal is to find a CBF to certify this safe region.
IV. APPROACH

In this section, we characterize CLF/CBF functions using SOS conditions, accounting for input constraints. We then show that these conditions can be verified by solving an SOS optimization problem. Finally, we present algorithms for automatic CLF/CBF synthesis that optimize an inner approximation of the stabilizable or safe region by solving a sequence of SOS optimization problems.

A. CLF Certification

For brevity, we first focus on CLFs and later show how these ideas can be modified for CBFs. To begin, observe that direct use of the CLF condition (7) is complicated by equation (7c), which, for each state \( x \in \Omega_p \), requires existence of an admissible control \( u \in \mathcal{U} \) satisfying the Lyapunov inequality (7c). Current CLF design techniques [16], [20] ensure existence of \( u \) by finding an explicit polynomial control policy satisfying (7c). This imposes conservatism, since existence of a CLF does not imply existence of a polynomial controller. It also adds a semidefinite constraint of order \( O((\deg(u))^m) \) to the underlying semidefinite program, limiting scalability.

We next propose an alternative strategy based on the contrapositive statement of condition (7c):

\[
\dot{V}(x, u) \geq -\kappa_V V \quad \forall u \in \mathcal{U},
\]

then \( V(x) \geq \rho \) or \( x = 0 \). \hspace{1cm} (10)

Note by taking contrapositive statement, we have replaced the inconvenient \( \exists \) quantifier with the \( \forall \) quantifier, which is easier to work with in an SOS framework. Indeed, as quoted from [31, Chapter 9], “Our sum-of-squares toolkit is well suited for addressing questions with the \( \forall \) quantifier over indeterminates. Working with the \( \exists \) quantifier is much more difficult”.

The condition \( \dot{V}(x, u) \geq -\kappa_V V \forall u \in \mathcal{U} \) of the contrapositive (10) denotes infinitely many constraints on \( x \) indexed by \( u \in \mathcal{U} \). We next reformulate this finitely many constraints by exploiting the polyhedrality of \( \mathcal{U} \) and the control-affine dynamics. To begin, we note that \( \dot{V}(x, u) = L_f V + L_g V u \) is a linear function of \( u \) under the control-affine assumption. This implies that \( \dot{V}(x, u) \geq -\kappa_V V \) is a linear inequality on \( u \) which, by convexity, holds for all \( u \in \mathcal{U} \) if and only if it holds at each vertex of \( \mathcal{U} \); see Fig. 2 and also Appendix VII-A. Hence, \( \dot{V}(x, u) \geq -\kappa_V V \forall u \in \mathcal{U} \) is equivalent to the finite set of inequalities:

\[
\dot{V}(x, u^i) \geq -\kappa_V V \quad \forall i = 1, \ldots, m. \hspace{1cm} (11)
\]

Substituting (11) into (10), we arrive at the desired reformulation of (10):

\[
\text{if } \dot{V}(x, u^i) \geq -\kappa_V V \forall i = 1, \ldots, m, \quad \text{then } (V(x) - \rho)x^T x \geq 0, \hspace{1cm} (12)
\]

where we have also replaced \( V(x) \geq \rho \) or \( x = 0 \) with the equivalent condition that \( (V(x) - \rho)x^T x \geq 0 \).

In summary, we can certify the CLF condition (7c) by checking if the polynomial \( (V(x) - \rho)x^T x \) is always non-negative on the semialgebraic set \( \{x | \dot{V}(x, u^i) \geq -\kappa_V V \} \).

The constraint (14a) is bilinear in \( V(x) \) and \( \lambda(x) \). Hence, we can apply the standard bilinear alternation technique to fix one and search for the other in an alternating fashion using SOS optimization [35], [34], [30].
To enlarge the sublevel set $\Omega_\rho = \{ x | V(x) < \rho \}$, we borrow an idea from [30], which measures the size of the sublevel set $\Omega_\rho$ with an inner ellipsoid $E_d = \{ x | (x - x_\mathcal{E}(x - x_\mathcal{E})) \leq d \} \subset \text{Cl}(\Omega_\rho)$, where $x_\mathcal{E}$ and $\mathcal{E}$ are given, and $\text{Cl}(\Omega_\rho)$ is the closure of the set $\Omega_\rho$. The goal is to expand the ellipsoid $E_d$ by maximizing $d$.

When $V(x)$ and $\rho$ are fixed, we can find a large inner ellipsoid $E_d \subset \text{Cl}(\Omega_\rho)$ by solving the following SOS program.

$$\max_{d, s_1(x)} \quad d$$
$$\text{s.t.} \quad (x - x_\mathcal{E})^T S_\mathcal{E}(x - x_\mathcal{E}) - d - s_1(x)(V(x) - \rho) \text{ is sos}$$
$$s_1(x) \text{ is sos.}$$

Constraints (15b)-(15c) are obtained by applying the S-procedure to the statement “$x \notin E_d$ if $x \notin \text{Cl}(\Omega_\rho)$”, which is the contrapositive of $\text{Cl}(\Omega_\rho) \subseteq E_d$. Keeping $V(x)$ and $\rho$ fixed, we then find the polynomial $\lambda(x)$ in (14) through the following SOS program:

$$\text{find } \lambda(x)$$
$$\text{subject to constraint (14).}$$

Finally, we update the CLF $V(x)$, aiming to enlarge the sublevel set $\Omega_\rho$ while containing the ellipsoid $E_d$. To do this, we increase the margin between the sublevel set $\Omega_\rho$ and the ellipsoid $E_d$ by minimizing the maximal value of $V(x)$ on $E_d$. Formally, we solve the following optimization

$$\min_{V(x), s_2(x), t} \quad t$$
$$\text{s.t.} \quad t - V(x) - s_2(x)(d - (x - x_\mathcal{E})^T S_\mathcal{E}(x - x_\mathcal{E})) \text{ is sos}$$
$$s_2(x) \text{ is sos}$$
$$\text{Constraints (7a), (7b), (14a),}$$

where $\lambda(x)$ is fixed to the solution in the program (16). Constraints (17b)-(17c) guarantee that $t \geq \max_{x \in E_d} V(x)$.

We present our algorithm in Algorithm 1, and visualize it pictorially in Fig. 3.

**Algorithm 1** Finding CLF with inscribed ellipsoids through bilinear alternation

Start with $V(x^{(0)}), i = 0, \text{converged}=\text{False}$

while not converged do

Solve SOS program (15) to find the ellipsoid $E_d(i)$.

if $d(i) \leq d(i-1) \text{ then} \text{ converged}=\text{True}$

else

Solve SOS program (16) to find the polynomials $\lambda(i)(x)$.

Solve SOS program (17) to find the CLF $V(i+1)$,

$i = i + 1$.

end if

end while

The goal of Algorithm 1 is to maximize the size of $\Omega_\rho$. An alternative goal is to stabilize a specified set of initial conditions. To this end, we ensure that $\Omega_\rho$ contains a set of specified states $x^{(j)}, j = 1, \ldots, n_{\text{sample}}$ by minimizing the maximal of $V(x^{(j)})$ on this set:

$$\min_{V(x), \lambda(x)} \quad \max_{j=1,\ldots,n_{\text{sample}}} V(x^{(j)})$$
subject to constraint (14).

This optimization program (18) has the bilinear product between $\lambda(x)$ and $V(x)$. We solve it with bilinear alternations using Algorithm 2.

**Algorithm 2** Find CLF by minimizing sample values with bilinear alternation

Start with $V(x^{(0)}), i = 0, \text{converged}=\text{False}$

while not converged do

Fix $V(i)(x)$, find $\lambda(i)(x)$ through solving the SOS program (18).

Fix $\lambda(i)(x)$, find $V(i+1)(x)$ through solving the SOS program (18). Denote the objective value as $o(i+1)$

if $o(i) - o(i+1) < \text{tol} \text{ then} \text{ converged}=\text{True}$

end if

end while

After we find $V(x)$ through either Algorithm 1 or 2, we can then further enlarge the sublevel set $\Omega_\rho$ using bisection on $\rho$ to satisfy the necessary and sufficient condition (13).

**C. Extension to CBFs**

Similar to CLFs, we can synthesize CBFs by solving SOS programs using arguments identical to those from IV-A. We sketch the details here and defer details to the appendix. To begin, we note that the contrapositive statement of (8b) is

$$\hat{h}(x,u) \leq -\kappa_h \forall u \in \mathcal{U}, \text{ then } h(x) \leq \beta^-,$$

which, same as for CLFs, can be rewritten using finitely many inequalities. Specifically, it is equivalent to

$$\hat{h}(x,u^i) \leq -\kappa_h \forall i = 1, \ldots, m, \text{ then } h(x) \leq \beta^-,$$

where $\lambda(x)$ is fixed to the solution in the program (16). Constraints (17b)-(17c) guarantee that $t \geq \max_{x \in E_d} V(x)$. We present our algorithm in Algorithm 1, and visualize it pictorially in Fig. 3.
a consequence of the control-affine dynamics and polyhedrality of $\mathcal{U}$ as described in IV-A.

The CBF condition requires non-negativity of $\beta^i - h(x)$ on the semialgebraic set $\{x| h(x, u^i) \leq -\kappa_i h, \ i = 1, \ldots, m\}$, which, by the Positivstellensatz can be reformulated as a SOS program, as done for CLFs in IV-A. To search for a CBF while expanding the certified safe region, we can formulate the sufficient condition as an SOS program and solve a sequence of SOS programs, similar to the algorithms for CLFs in IV-B. We present the detailed mathematical formulation and algorithms for certifying and searching CBFs in Appendix VII-B and VII-C².

V. RESULTS

We show our results on a 2D toy system, an inverted pendulum and a 3D quadrotor. We use Mosek [4] to solve the SOS optimization problems arising in our synthesis procedures.

A. 2D toy system

Our first example is a 2D toy system from [30], given by

$$
\dot{x}_1 = u, \\
\dot{x}_2 = -x_1 + \frac{1}{6}x_1^3 - u, 
$$

with input constraint $-0.4 \leq u \leq 0.4$. Using Algorithm 1, we certify a stabilizable region $\Omega_\rho = \{x|V(x) \leq \rho\}$ using a CLF $V(x)$ of degree 8, initializing with $V(x) = x_1^2 + x_2^2$ and $\rho = 0.3$. We maximize the final $\rho$ using bisection and Theorem 4.1. The initial and final sublevel sets are plotted as the inner-most green curve and outer-most red curve in Fig. 4. As illustrated, our algorithm greatly increases the size of the certified region. The computation time for each SOS program is less than 0.01s, and we run 1000 iterations of bilinear alternation in Algorithm 1.

An alternative approach to certify $\Omega_\rho$ proceeds by explicitly searching for a polynomial controller $u(x)$ satisfying $\dot{V}(x, u(x)) \leq -\kappa V$ and the input limits. Using the formulation proposed in [16] (and also explained in the Appendix VII-D), we plot the certified regions obtained for controllers of increasing degrees in Fig. 4. As shown, for the same $V(x)$, our CLF approach certifies a larger $\rho$, as it does not restrict to polynomial control laws. Rather, it allows any stabilizing control action within the input limits. This demonstrates the advantage of our approach versus jointly searching for a polynomial Lyapunov function and a polynomial controller.

B. Inverted pendulum

For the inverted pendulum (Fig. 5) described in [31] (with mass $m = 1$ kg, length $l = 0.5$ m and damping $b = 0.1$ N/m), we aim to stabilize the upright equilibrium state $(\theta, \dot{\theta}) = (0, 0)$ with the certified sublevel set $\Omega_\rho$. For a 4-th degree $V(x)$, we plot the final $\Omega_\rho$ as the red contour in Fig. 6a. We simulate the pendulum using a QP controller derived from the CLF. Specifically, we minimize $u^2$ subject to the constraint $\dot{V}(x, u) \leq -\kappa V$, $-4.6 \leq u \leq 4.6$. In Fig. 6b we plot the torque values from this QP-based controller on this simulated trajectory. The computation time for each SOS program in Algorithm 2 is about 0.15s, and we solve 10 SOS programs.

C. Quadrotor

We apply our approach to a 3D quadrotor to demonstrate its scalability. The system has 13 states (with the orientation represented by a unit quaternion $z$), whose dynamics can be written as polynomial functions of states [13] with the additional algebraic constraint $z^T z = 1$ on the unit quaternion.

Fig. 4: We draw the certified inner approximation of the stabilizable region for the 2D dynamical system (Eq.(21)). Starting from the certified region with the innermost circle, our algorithm expands the sublevel set $\Omega_\rho$ to the outermost curve. We also compare with the certified inner approximation of the stabilizable region by searching over a polynomial controller, and draw the region for each controller degree.

Fig. 5: Pendulum state-vector $x = [s, c + 1, \dot{\theta}]$ where $s = \sin \theta$, $c = \cos \theta$. Specifically, we take

$$
\dot{x} = \begin{bmatrix}
x_2 - 1
x_1 + 1
\end{bmatrix} + \begin{bmatrix} 0 & 0 \\
0 & \frac{1}{m^2} 
\end{bmatrix} u,
$$

and impose the additional algebraic constraint that $x_2^2 + (x_2 - 1)^2 = 1$ (since $\sin^2 \theta + \cos^2 \theta = 1$). We incorporate this constraint into the SOS programs using the S-procedure [27], [25].

Initializing the CLF to the LQR quadratic cost-to-go function (green contour in Fig. 6a), we apply Algorithm 2 to cover the downright equilibrium state $(\theta, \dot{\theta}) = (0, 0)$ with the certified sublevel set $\Omega_\rho$. For a 4-th degree $V(x)$, we plot the final $\Omega_\rho$ as the red contour in Fig. 6a. We simulate the pendulum using a QP controller derived from the CLF. Specifically, we minimize $u^2$ subject to the constraint $\dot{V}(x, u) \leq -\kappa V$, $-4.6 \leq u \leq 4.6$. In Fig. 6b we plot the torque values from this QP-based controller on this simulated trajectory. The computation time for each SOS program in Algorithm 2 is about 0.15s, and we solve 10 SOS programs.

²Contemporarily, [38] derived a similar SOS-based formulation for verifying another safety certificate called safety index, and solved the SOS program through nonlinear optimization instead of convex optimization.
We impose the input limit constraint as $0 \leq u \leq 0.75mg$ for each rotor thrust.

To find a CLF, we initialize Algorithm 1 with the quadratic LQR cost-to-go function. For a quadrotor with body length $= 0.15m$, our certified stabilizable region covers distant initial states including $p_{xyz} = (10, 0, 0)$, RollPitchYaw = $(100^\circ, 0, 0)$, vel $= 0$, and the state $p_{xyz} = (5, 5, 0)$, RollPitchYaw = $(140^\circ, 0, 0)$, vel $= 0$ (shown in Fig. 1). Each SOS program in Algorithm 1 takes about 615 seconds, and we solve 20 SOS programs. We compare with the computation time of jointly searching for a Lyapunov function and a polynomial controller (as explained in Appendix VII-D). Finding a linear controller takes 11 seconds, and a cubic controller 1540 seconds. Synthesis of a 5-th degree controller fails due to insufficient memory on our 128 GB machine. Computation time grows rapidly with the controller degree given that the SOS program has an underlying semidefinite constraint of order $O((\text{degree}(u))^{13})$ for a 13-state robot. In contrast, our approach avoids this scalability issue, since it does not require explicit construction of a polynomial control law.

For this example, we also synthesize a controller that respects a minimum height constraint. Specifically, we synthesize a CBF for the unsafe set $\mathcal{X}_{\text{unsafe}} = \{x | p_z \leq -0.15\}$. We start with the initial CBF as $0.0001 - V(x)$ and expand the certified safe region using Algorithms 3 and 4. We simulate the quadrotor with the CBF-CLF-QP controller [1] which considers both CBF and CLF constraints, and compare with the CLF-QP controller that does not take the unsafe region into consideration. We visualize the simulated trajectory (starting from hovering at $p_z = 1m, p_g = p_z = 0m$) in Fig. 7a. We compare the simulation result with the CBF-CLF-QP controller versus with the CLF-QP controller in Fig. 7b. Without the CBF constraint the quadrotor drops below $p_z = -0.15m$ into the unsafe set. With the CBF constraint it always stays within the safe region.

VI. CONCLUSION AND DISCUSSION

In this paper we characterized polynomial CLF/CBF functions for input-constrained, polynomial dynamical-systems using SOS conditions. We then posed CLF/CBF synthesis as an SOS program whose solution provides an inner approximation of the stabilizable/safe region. We presented iterative algorithms to expand the inner approximation of the stabilizable/safe region using sequential SOS optimization. Finally, we showcased our approach on different dynamical systems and compared with explicitly searching for polynomial controllers using Lyapunov functions.

Currently, our approach applies to systems with continuous dynamics. Future work could extend it to hybrid systems with rigid contacts by leveraging ideas from [25].

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VII. APPENDIX

A. Alternative derivation for Eq. (12)

We present an alternative approach to derive (12) from (7c). Note that (7c) can be written as

$$\text{if } V(x) < \rho \text{ and } x \neq 0 \text{ then } \min_{u \in \mathcal{U}} \dot{V}(x, u) < -\kappa_V V$$

Since $\dot{V}(x, u)$ is linear in $u$, the minimization of this linear function over the polytope $\mathcal{U}$ has to occur at one of the polytope vertices. Hence condition (23) is equivalent to

$$\text{if } V(x) < \rho \text{ and } x \neq 0 \text{ then } \min_{i=1, \ldots, m} \dot{V}(x, u^i) < -\kappa_V V$$

(24)

The contrapositive statement of (24) is

$$\text{if } \min_{i=1, \ldots, m} \dot{V}(x, u^i) \geq -\kappa_V V \text{ then } V(x) \geq \rho \text{ or } x = 0,$$

(25)

which is equivalent to (12).

B. CBF certification

Similar to CLFs in section IV-A, we apply Positivstellensatz on equation (20) to certify the CBF condition. Given a polynomial $h(x)$ satisfying $h(x) \leq 0 \forall x \in \mathcal{X}_{safe}$, the necessary and sufficient condition for $h(x)$ being a CBF is

$$3q_h(x), r_h(x) \in \text{preorder}(-h(x, u^i) - \kappa_h h, i = 1, \ldots, m)$$

$h_h \in \mathbb{N}, \text{s.t } q_h(x)(\beta_p - h) = (\beta_p - h)^{2k} + r_h(x).$

(26)

$h(x)$ is a valid CBF if the SOS program (26) is feasible.

C. Search certifiable CBFs

Similar to the CLF case, this necessary and sufficient condition is applicable when we certify whether a given $h(x)$ is a valid CBF, but not suitable to search for $h(x)$ due to the nonlinear product of $h(x)$ in the preorder. Instead we present the following sufficient condition of (20) which is linear in $h(x)$.

**Lemma 7.1:** A sufficient condition for the CBF condition (20) and (8b) is the existence of polynomials $\mu_i(x), i = 0, \ldots, m,$ such that

$$\begin{align*}
(1 + \mu_0(x))(\beta_p - h(x)) + \sum_{i=1}^{m} \mu_i(x)(\dot{h}(x, u^i) + \kappa_h h) & \text{ is sos} \\
\mu_i(x) & \text{ is sos, } i = 0, \ldots, m.
\end{align*}$$

(27a)

(27b)
A sufficient condition for \( h(x) \leq 0 \forall x \in \mathcal{X}_{\text{unsafe}} \) (Equation (8a)) is the existence of polynomials \( \phi(x) \) such that
\[
-(1 + \phi_{i,0}(x))h(x) + \sum_{j=1}^{s_i} \phi_{i,j} p_{i,j}(x) \text{ is sos}
\]
(28a)
\[
\phi_{i,j} \text{ is sos} \quad i = 1, \ldots, n_{\text{unsafe}}, j = 0, \ldots, s_i,
\]
(28b)
where the polynomials \( p_{i,j}(x) \) define the unsafe region \( \mathcal{X}_{\text{unsafe}} \) as unions of semialgebraic sets (Equation (9)).

In order to find a CBF \( h(x) \) that certifies a large safe region \( \Phi = \{x|h(x) > 0\} \), we again measure the size of the superlevel set \( \Phi \) with the inscribed ellipsoid \( \mathcal{E}_d = \{x|(x-x_E)^T S_E(x-x_E) \leq d\} \) given \( x_E \) and \( S_E \). We find a large \( d \) such that \( \mathcal{E}_d \subset \Phi \) (through the following SOS program
\[
\max_d \quad d \quad \text{s.t.} \\
(x-x_E)^T S_E(x-x_E) - d + \psi(x)h(x) \text{ is sos}
\]
(29a)
\[
\psi(x) \text{ is sos.}
\]
(29b)

With a given \( h(x) \), we can certify if it is a valid CBF by searching for \( \mu(x), \phi(x) \) through the following SOS program
\[
\text{find } \mu(x), \phi(x) \quad \text{s.t. constraint (27) and (28)},
\]
(30a)
\[
\max_{t, \mu(x), \phi(x)} \quad t \quad \text{s.t.} \\
h(x) - t - \nu(x)(d - (x-x_E)^T S_E(x-x_E)) \text{ is sos}
\]
(31a)
\[
\nu(x) \text{ is sos}
\]
(31b)
\[
\text{constraint (27a) and (28a)},
\]
(31c)
\[
h(x_{\text{anchor}}) \leq 1,
\]
(31d)
where \( x_{\text{anchor}} \) is a given safe state. We impose constraint (31e) to prevent scaling \( h(x) \) with an infinitely large factor.

We present our algorithm to iterative search for CBF \( h(x) \) while enlarging the safe region in Algorithm 3.

Similar to the CLF case, we can also search for \( h(x) \) with the aim of certifying some given states \( x^{(k)}, k = 1, \ldots, n_{\text{sample}} \). The certified safe region \( \Phi = \{x|h(x) > 0\} \) covers the sampled state if \( h(x^{(k)}) > 0 \). So our goal is to maximize the minimal value of \( h(x^{(k)}), k = 1, \ldots, n_{\text{sample}} \) so as to bring them all positive through the following program
\[
\max_{h(x), \mu(x), \phi(x)} \min_{k=1, \ldots, n_{\text{sample}}} h(x^{(k)}) \quad \text{s.t. constraint (27)(28)(31e)}.
\]
(32a)

Problem (32) has the bilinear product of \( h(x) \) and \( \mu(x), \phi(x) \), we present our bilinear alternation algorithm in Algorithm 4.

**Algorithm 3** Search CBFs through bilinear alternation with inner ellipsoid

Start with \( h^{(0)}(x), i = 0 \) and converged=False

**while** not converged do

Fix \( h^{(i)}(x) \), solve SOS program (29) to find \( d^{(i)} \).

\[ \text{if } d^{(i)} - d^{(i-1)} < \text{tol then} \]

\[ \text{converged = True} \]

**else**

Fix \( h^{(i)}(x) \), solve SOS program (30) to find \( \mu^{(i)}(x), \phi^{(i)}(x) \).

Find \( \mu^{(i)}(x), \phi^{(i)}(x) \), solve SOS program (31) to find \( h^{(i+1)}(x), i = i + 1, \)

**end if**

**end while**

**Algorithm 4** Search CBFs through bilinear alternation with sampled states

Given \( h^{(0)}(x), i = 0 \) and converged=False

**while** not converged do

Fix \( h^{(i)}(x) \), search \( \mu^{(i)}(x), \phi^{(i)}(x) \) satisfying constraints (32b).

Find \( \mu^{(i)}(x), \phi^{(i)}(x) \), search \( h^{(i+1)}(x) \) through the SOS program (32). Denote the objective value as \( o^{(i+1)}(x) \).

**if** \( o^{(i+1)} - o^{(i)} < \text{tol then} \)

\[ \text{converged= True} \]

**end if**

**end while**

**D. Formulation for searching a polynomial controller and \( \rho \)**

Given a polynomial function \( V(x) \) that satisfies \( V(x) \geq \epsilon(x^T x)^\alpha, V(0) = 0 \), to verify that this \( V(x) \) is a valid Lyapunov function with polynomial control policy \( u(x) \) and inner approximation of region-of-attraction \( \{x|V(x) \leq \rho\} \), we impose the following constraint
\[
L_f V + L_g V u(x) \leq -\kappa V \text{ if } V(x) \leq \rho \quad (33a)
\]
\[
-\kappa V \leq u_{\text{lo}} - u(x) \leq u_{\text{up}} \text{ if } V(x) \leq \rho. \quad (33b)
\]

Using S-procedure, a sufficient condition for (33) is that the following SOS program is feasible.
\[
\text{Find } u_{\text{lo}}(x), \gamma(x), \eta_{\text{lo}}(x), \eta_{\text{up}}(x) \quad (34a)
\]
\[
-\kappa V \leq L_f V - L_g V u(x) \leq \gamma(x)(\rho - V(x)) \text{ is sos } \quad (34b)
\]
\[
u_{\text{up}} - u(x) - \eta_{\text{up}}(x) \rho - V(x) \text{ is sos } \quad (34c)
\]
\[
u_{\text{up}} - u_{\text{lo}} - \eta_{\text{lo}}(x) \rho - V(x) \text{ is sos } \quad (34d)
\]
\[
\gamma(x), \eta_{\text{lo}}(x), \eta_{\text{up}}(x) \text{ are sos. } \quad (34e)
\]

We can then find the maximal \( \rho \) through bisection with SOS program (34).