Abstract

We show that the BRST cohomology of the massless sector of the Type IIB superstring on $AdS_5 \times S^5$ can be described as the relative cohomology of an infinite-dimensional Lie superalgebra. We explain how the vertex operators of ghost number 1, which correspond to conserved currents, are described in this language. We also give some algebraic description of the ghost number 2 vertices, which appears to be new. We use this algebraic description to clarify the structure of the zero mode sector of the ghost number two states in flat space, and initiate the study of the vertices of the higher ghost number.
1 Introduction

Pure spinor formalism [Ber00] is a generalization of the BRST formalism with the ghost fields constrained to satisfy a nonlinear (quadratic) equation:

$$\lambda^\alpha \Gamma_{\alpha\beta}^m \lambda^\beta = 0$$  \hspace{1cm} (1)

where $\Gamma_{\alpha\beta}^m$ are the Dirac’s Gamma-matrices. A natural question arises, what kind of nonlinear constraints can ghost fields satisfy in a physical theory? What if we replace (1) by an arbitrary set of equations:

$$\lambda^\alpha C_{\alpha\beta}^i \lambda^\beta = 0, \quad i \in I$$  \hspace{1cm} (2)

Of course, this would generally speaking have nothing to do with the string theory. But the question is, besides coming from superstring theory, what special properties of $C_{\alpha\beta}^m = \Gamma_{\alpha\beta}^m$ are important for physics? This would be useful to know, for example when thinking about possible generalizations of the pure spinor formalism.

It turns out that there is some special property of (1) which plays an important role in the string worldsheet theory. This is the so-called Koszulity — see [GKR06] and references therein. The formalism of Koszul duality was extensively used in the study of the algebraic properties of the supersymmetric Yang-Mills theories in [MS04b, MS04a], and in the classification of the possible deformations of these theories in [MS09].

In this paper we will study the BRST cohomology of the massless sector of the Type IIB superstring in $\text{AdS}_5 \times \text{S}^5$. We will use the formalism of Koszul duality to gain better understanding of the massless BRST cohomology.

The BRST cohomology counts infinitesimal deformations of the background $\text{AdS}_5 \times \text{S}^5$, also called “linearized excitations” or “gravitational waves”. From the point of view of the string worldsheet theory, they are identified with the massless vertex operators. Understanding the properties of these vertex operators is important already because of their role in the scattering theory. Indeed, the correlation function of vertex operators is the main ingredient in the string theory computation of the S-matrix.

Main results

1. We show that the cohomology of the BRST complex of the Type IIB SUGRA on $\text{AdS}_5 \times \text{S}^5$ is equivalent to some relative Lie algebra cohomology.
2. We classify the vertex operators of the ghost number 1, which correspond to the densities of the local conserved charges

3. We give a general Lie-algebraic description of the vertex operators of the ghost number \( \geq 2 \) and use this description to study the properties of the zero momentum states ("discrete states")

**Previous results for ghost number 1** The classification of the vertex operators in the ghost number 1 was done, at least partially, in the Appendix of our previous paper [Mik11b]; the method which we develop here appears more elegant.

**Zero momentum states** In a typical string theory computation one considers the scattering of physical excitations (vertex operators) which depend on the space-time coordinates exponentially:

\[
V(x) \simeq e^{ikx}
\]  

But we find it interesting to also consider vertex operators depending on \( x \) polynomially. We will call them "zero momentum vertices" because their wavefunction in the momentum space is supported at \( k = 0 \). It turns out that this "zero momentum sector" carries one potentially unpleasant surprise: there are some well-defined vertex operators which do not correspond to any physical states [BBMR11, Mik12]. This means that just the requirement of BRST invariance alone does not yet provide a complete characterization of the physically relevant sigma-models. (But the picture becomes complete if one imposes, in addition to the BRST invariance, the condition of the sigma-model being finite at the one-loop level.) In this paper we use the Koszul duality to obtain a dual description of such unphysical states in terms of fields satisfying unusual equations of motion, similar to this one:

\[
\partial_m A_n + \partial_n A_m = 0
\]  

Such equations imply that higher derivatives of \( A \) vanish.

**Plan of the paper** We will start in Sections 2, 3 with the application of Koszul duality to the ten-dimensional supersymmetric Maxwell theory. In Section 4 we apply a similar method to the study of linearized Type IIB SUGRA in \( AdS_5 \times S^5 \). We introduce in Section 4.2 some infinite-dimensional
super-Lie algebra, and show in Section 4.5 that the BRST cohomology is equal to the Lie-algebraic cohomology of some ideal $I$ of this super-algebra. In Section 5 we consider the flat space limit and in particular study the zero momentum states. One unusual finding is the existence of nontrivial cohomology at the ghost number three.

**Note added in the revised version**  The approach developed in this paper is useful for clarifying the construction of integrated vertex [CMV13].

## 2 Pure spinor formulation of the SUSY Maxwell theory

### 2.1 Supersymmetric space-time and basic constraints

Here we will remind the superspace description of the classical supersymmetric Maxwell theory in 10 dimensions. The superspace is formed by 10 bosonic coordinates $x^m$ and 16 fermionic coordinates $\theta^\alpha$. This is the supersymmetric space-time, we will call it $M$:

$$M = \mathbb{R}^{10|16}$$  \hspace{1cm} (5)

The basic superfield is the vector potential $A_\alpha(x, \theta)$. For every $\alpha \in \{1, \ldots, 16\}$, the corresponding $A_\alpha$ is a scalar function:

$$A_\alpha : M \rightarrow \mathbb{R}$$  \hspace{1cm} (6)

The equations of motion of the theory are encoded in the following construction. Let us consider the “covariant derivatives”:

$$\nabla_\alpha = \frac{\partial}{\partial \theta^\alpha} + \Gamma^m_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^m} + A_\alpha(x, \theta)$$  \hspace{1cm} (7)

It turns out [Nil81, Wit86] that the equations of motion of SUSY Maxwell theory are equivalent to the constraint:

- There exists a differential operator $\nabla_m = \frac{\partial}{\partial x^m} + A_m(x, \theta)$ such that:

$$\{\nabla_\alpha, \nabla_\beta\} = \Gamma^m_{\alpha\beta} \nabla_m$$  \hspace{1cm} (8)
The nontrivial requirement of the constraint is that the LHS of (8) is proportional to $\Gamma^m_{\alpha\beta}$, because the most general structure would be:

$$\Gamma^m_{\alpha\beta} \nabla_m + \Gamma^m_{\alpha\beta} m_1 m_2 m_3 m_4 m_5 X_{m_1 m_2 m_3 m_4 m_5}$$

(9)

where $X_{m_1...m_5} = X(x, \theta)_{m_1...m_5}$ some function on the superspace. Equivalently, the constraint (8) can be written:

$$\Gamma^m_{m_1 m_2 m_3 m_4 m_5} \{\nabla_\alpha, \nabla_\beta\} = 0$$

(10)

With the constraint (10) satisfied, we consider (8) as the definition of $\nabla_m$. The pure spinor interpretation of (10) is due to [How91].

2.2 Definition of the Lie superalgebra $\mathcal{L}$.

Now let us forget Eq. (7) and consider the Lie superalgebra $\mathcal{L}$ generated by the letters $\nabla_\alpha$ with the relation (8). This is an infinite-dimensional Lie superalgebra. It turns out that some properties of the SUSY Maxwell theory can be described in terms of this algebra $\mathcal{L}$. In the next Section we will describe an application of the cohomology of $\mathcal{L}$.

3 Lie algebra cohomology and solutions of the SUSY Maxwell theory

3.1 Vacuum solution

Let us consider the vacuum solution $A_\alpha(x, \theta) = 0$. In this case $\nabla_\alpha = \nabla^{(0)}_\alpha = \partial / \partial \theta^\alpha + \Gamma^m_{\alpha\beta} \theta^\beta \partial / \partial x^m$. The vacuum solution is invariant under the supersymmetry algebra $\text{susy}$ generated by the operators $S_\alpha$:

$$S_\alpha = \partial / \partial \theta^\alpha - \Gamma^m_{\alpha\beta} \theta^\beta \partial / \partial x^m$$

(11)

We observe that $\{S_\alpha, \nabla^{(0)}_\alpha\} = 0$, and in this sense the vacuum solution is $\text{susy}$-invariant. It turns out that the operators $\nabla^{(0)}$ themselves generate the same (isomorphic) algebra $\text{susy}$ as do $S_\alpha$. This can be explained using the interpretation of $M$ as the coset space of $\text{susy}$. Let us consider the abstract algebra $\text{susy}$ generated by $t^\text{odd}_\alpha$ and $t^\text{even}_m$ with the commutation relations:

$$\{t^\text{odd}_\alpha, t^\text{odd}_\beta\} = \Gamma^m_{\alpha\beta} t^\text{even}_m$$

(12)
and other commutators all zero. Let us interpret $x^m$ and $\theta^\alpha$ as coordinates on the group manifold of the corresponding Lie group:

$$g = \exp(\theta^\alpha t^\text{odd}_\alpha + x^m t^\text{even}_m)$$  \hspace{1cm} (13)$$

Then $\nabla_\alpha$ acts as the multiplication by $t^\text{odd}_\alpha$ on the left, and $S_\alpha$ as the multiplication by $t^\text{odd}_\alpha$ on the right. We can consider the universal enveloping algebra $U\text{susy}$ as a representation of $\text{susy}$, by the left multiplication. Then the regular representation can be considered as its dual, which will be denoted $(U\text{susy})'$.

**Relation between \mathcal{L} and \text{susy}.
** There is an ideal $I \subset \mathcal{L}$ such that the factoralgebra over this ideal is $\text{susy}$:

$$\mathcal{L}/I = \text{susy}$$  \hspace{1cm} (14)$$

The basic constraint $[\mathcal{L}]$ actually implies the existence of $W^\alpha$ such that:

$$[\nabla_\alpha, \nabla_m] = \Gamma^m_{\alpha\beta} W^\beta$$  \hspace{1cm} (15)$$

This $W^\alpha$ is the element of $I$, because if $\nabla_\alpha$ were the generators of the 10-dimensional supersymmetry algebra, then $W^\alpha$ would be zero.

### 3.2 Deformations of solutions and cohomology

The deformation of the given solution $A_\alpha(x, \theta)$ is:

$$A_\alpha \mapsto A_\alpha + \delta A_\alpha$$  \hspace{1cm} (16)$$

where $\delta A_\alpha$ should satisfy:

$$\{\nabla_\alpha, \delta A_\beta\} = \Gamma^m_{\alpha\beta} \delta A_m$$  \hspace{1cm} (17)$$

The fact that the LHS is proportional to $\Gamma^m_{\alpha\beta}$ is a nontrivial constraint on $\delta A_\beta$, and if it is satisfied then $\delta A_m$ becomes the definition of $\delta A_m$.

Let us introduce *pure spinors* $\lambda^\alpha$ satisfying:

$$\lambda^\alpha \Gamma^m_{\alpha\beta} \lambda^\beta = 0$$  \hspace{1cm} (18)$$

1 A thorough investigation of the consequences of the basic constraint $[\mathcal{L}]$ can be found in [Maf09].
Using these pure spinors, Eq. (17) can be written:

\[ Qv = 0 \] (19)

where \( Q = \lambda^a \nabla_a \) (20)

and \( v = \lambda^a \delta A_a \) (21)

Therefore the problem of classifying the infinitesimal deformations of the vacuum solution is reduced to the computation of the cohomology of \( Q \).

### 3.3 Koszul duality and its application to deformations

Let us consider a representation \( V \) of the Lie algebra \( \text{susy} \), and the following version of the BRST complex:

\[ \ldots \xrightarrow{Q_{\text{BRST}}} V \otimes \mathbb{C} \mathcal{P}^n \xrightarrow{Q_{\text{BRST}}} V \otimes \mathbb{C} \mathcal{P}^{n+1} \xrightarrow{Q_{\text{BRST}}} \ldots \] (22)

where \( \mathcal{P}^n \) is the space of polynomial functions of degree \( n \) on the pure spinors \( \lambda^a \). A representation \( V \) of \( \text{susy} \) is also a representation of \( \mathcal{L} \), because \( \text{susy} = \mathcal{L} / I \).

Koszul duality\(^2\) implies that the cohomology of (22) coincides with the Lie algebra cohomology of \( \mathcal{L} \):

\[ H^n(Q_{\text{BRST}} ; V) = H^n(\mathcal{L} ; V) \] (23)

Notice that \( \bigoplus_{n=0}^{\infty} \mathcal{P}^n \) is a commutative algebra with quadratic relations. This algebra is Koszul dual to the universal enveloping of a Lie algebra \( U\mathcal{L} \).

**Brief review of (23)** The Koszul duality implies that the following sequence:

\[ \ldots \rightarrow \text{Hom}_\mathbb{C}(\mathcal{P}^2, U\mathcal{L}) \rightarrow \text{Hom}_\mathbb{C}(\mathcal{P}^1, U\mathcal{L}) \rightarrow U\mathcal{L} \rightarrow \mathbb{C} \rightarrow 0 \] (24)

is exact, and therefore provides a free resolution of the \( U\mathcal{L} \)-module \( \mathbb{C} \). This fact depends on special properties of the quadratic constraint (1).

\(^2\)A nice review can be found in the introductory part of [GKR06]; cohomology with coefficients in a representation was not considered in [GKR06], but it was discussed in [MS09].
In (24) the action of $U \mathcal{L}$ on $U \mathcal{L}$ is by the left multiplication, and the action of the differential involves the right multiplication by the $\nabla_\alpha$:

\[ d\phi(p) = \phi(\lambda^\alpha p)\nabla_\alpha \]  

(25)

Here on the right hand side we have the product of $\nabla_\alpha \in U \mathcal{L}$ with $\phi(\lambda^\alpha p) \in U \mathcal{L}$. In other words, for $\phi \in \text{Hom}_C(\mathcal{P}^n, U \mathcal{L})$ we have:

\[ d\phi = \mu_{U \mathcal{L}}^{\text{right}}(\nabla_\alpha) \circ \phi \circ \mu_p(\lambda^\alpha) \]  

(26)

where $\mu_p(\lambda^\alpha) : \mathcal{P}^n \to \mathcal{P}^{n+1}$ is a multiplication of a polynomial by $\lambda^\alpha \in \mathcal{P}^1$, and $\mu_{U \mathcal{L}}^{\text{right}}(\nabla_\alpha)$ is the right multiplication by $\nabla_\alpha$ in $U \mathcal{L}$. (The composition $\phi \circ \mu_p(\lambda^\alpha)$ is of the type $\mathcal{P}^n \to U \mathcal{L}$; we then multiply by $\nabla_\alpha \in U \mathcal{L}$.)

Since we have a projective resolution of $C$, we can now use it to compute the Lie algebra cohomology of $\mathcal{L}$ with coefficients in $V$, i.e. $\text{Ext}_{U \mathcal{L}}(C, V)$. It is the cohomology of the following sequence:

\[ 0 \to \text{Hom}_{U \mathcal{L}}(U \mathcal{L}, V) \to \text{Hom}_{U \mathcal{L}}(\text{Hom}_C(\mathcal{P}^1, U \mathcal{L}), V) \to \ldots \]  

\[ \ldots \to \text{Hom}_{U \mathcal{L}}(\text{Hom}_C(\mathcal{P}^n, U \mathcal{L}), V) \to \text{Hom}_{U \mathcal{L}}(\text{Hom}_C(\mathcal{P}^{n+1}, U \mathcal{L}), V) \to \ldots \]  

(27)

where the differential is induced by (26) and acts as follows. For $f \in \text{Hom}_{U \mathcal{L}}(\text{Hom}_C(\mathcal{P}^n, U \mathcal{L}), V)$, the $df \in \text{Hom}_{U \mathcal{L}}(\text{Hom}_C(\mathcal{P}^{n+1}, U \mathcal{L}), V)$ is evaluated on $\phi \in \text{Hom}_C(\mathcal{P}^{n+1}, U \mathcal{L})$ as follows:

\[ (df)(\phi : \mathcal{P}^{n+1} \to U \mathcal{L}) = f(\mu_{U \mathcal{L}}^{\text{right}}(\nabla_\alpha) \circ \phi \circ \mu_p(\lambda^\alpha)) \]  

(28)

There is an isomorphism:

\[ \mathcal{P}^n \otimes_C V \simeq \text{Hom}_{U \mathcal{L}}(\text{Hom}_C(\mathcal{P}^n, U \mathcal{L}), V) \]  

(29)

\[ p \otimes v \mapsto [\phi \mapsto \phi(p)v] \]  

(30)

Here “$\phi(p)v$” means the action of $\phi(p) \in U \mathcal{L}$ on the element $v$ of the representation $V$ of $U \mathcal{L}$. This isomorphism relates (27) to (22).

**Special case** The cohomology problem described in Section 3.2 corresponds to the particular case of $V = (U_{\text{susy}})'$. As we have just explained, this is equivalent to the computation of the Lie algebra cohomology:

\[ H^\bullet(\mathcal{L}, (U_{\text{susy}})') \]  

(31)
Notice that \( \text{susy} = \mathcal{L}/I \) and therefore \( (U_{\text{susy}})' \) is naturally a representation of \( \mathcal{L} \), by the left multiplication. To calculate this cohomology, we notice that the following complex:

\[
\ldots \longrightarrow U\mathcal{L} \otimes_{\mathbb{C}} \Lambda^2 I \longrightarrow U\mathcal{L} \otimes_{\mathbb{C}} I \longrightarrow U\mathcal{L} \longrightarrow U_{\text{susy}} \longrightarrow 0
\]

(32)
is a free resolution of \( U_{\text{susy}} \) as a \( U\mathcal{L} \)-module. This means that:

\[
H^n(\mathcal{L}, (U_{\text{susy}})') = H^n(I, \mathbb{C}) \quad (33)
\]

More specifically, the ghost number one vertex operator \( \lambda^a \delta A_\alpha \) corresponds to the first cohomology:

\[
H^1(I, \mathbb{C}) = \left( \frac{I}{[I, I]} \right)'
\]

(34)

This has the following physical interpretation. The space \( \frac{I}{[I, I]} \) can be identified with the space of field strengths. Then (34) tells us that the classical solutions are linear functionals on the space of field strengths. Indeed, given a classical solution, we can compute the value of the field strength on this classical solution. Therefore, the space of classical solutions is expected to be dual to the space of field strengths, as we indeed observe in (34).

**Explicit description of \( \frac{I}{[I, I]} \)** Elements \( W^\alpha \) of \( I \) were introduced in Eq. (15). Consider the projection of \( W^\alpha \) to \( I/[I, I] \), i.e. \( W^\alpha \mod [I, I] \). We conjecture that all the other elements of \( I/[I, I] \) can be obtained from \( W^\alpha \) by commuting with \( \nabla_{\alpha} \), i.e. acting with \( \text{susy} \). This means that all the gauge invariant operators at the linearized level are \( W^\alpha \) and its derivatives.

4 Type IIB SUGRA in \( AdS_5 \times S^5 \)

**Note in the revised version** The constructions of this paragraph can be illustrated by explicit examples of vertex operator, corresponding to the \( \beta \)-deformation These examples are constructed in [CMV13].

4.1 BRST complex

The BRST complex of Type IIB SUGRA in \( AdS_5 \times S^5 \) [BH02, BC01, Ber05b] is based on the coset space \( G/G_0 \) where \( G \) is the Lie supergroup corresponding
to the Lie superalgebra $g = psu(2, 2|4)$ and $G_0$ is the subgroup corresponding to $g_0 = so(1, 4) \oplus so(5)$. A $\mathbb{Z}_4$-grading of $g$ plays an important role. The generators of $g$ are denoted:

\begin{align*}
&i^3_\alpha \text{ of degree } 3, \quad \alpha \in \{1, \ldots, 16\} \\
&i^1_\dot{\alpha} \text{ of degree } 1, \quad \dot{\alpha} \in \{1, \ldots, 16\} \\
&t^2_n \text{ of degree } 2, \quad n \in \{0, \ldots, 9\} \\
&t^0_{[mn]} \text{ of degree } 0
\end{align*}

The superalgebra $g_0$ is generated by $t^0_{[mn]}$, $g_3$ by $i^3_\alpha$, $g_1$ by $i^1_\dot{\alpha}$, and $g_2$ by $t^2_m$. The index $[mn]$ of $t^0_{[mn]}$ runs over a union of two sets: the set of choices of 2 different elements $m, n$ from $\{0, \ldots, 4\}$, and the set of choices of 2 different elements $m, n$ from $\{5, \ldots, 9\}$. This corresponds to the split of $g_0$ into the direct sum of $so(1, 4)$ and $so(5)$. Both $i^3_\alpha$ and $i^1_\dot{\alpha}$ transform as spinors of both $so(1, 4)$ and $so(5)$ under the adjoint action of $g_0$, and $t^2_m$ transform as vectors.

The BRST complex computing supergravity excitations on the background $AdS_5 \times S^5$ is:

\[ \ldots \xrightarrow{Q_{BRST}} \text{Hom}_{g_0} (U_g, P^n) \xrightarrow{Q_{BRST}} \text{Hom}_{g_0} (U_g, P^{n+1}) \xrightarrow{Q_{BRST}} \ldots \]

where $P^n$ is the space of polynomials of the order $n$ of two independent pure spinors $\lambda_L$ and $\lambda_R$:

\[ \lambda^\alpha_L f_{\alpha\beta} m \lambda^\beta_L = 0, \quad \lambda^{\dot{\alpha}}_R f_{\alpha\dot{\beta}} m \lambda^{\dot{\beta}}_R = 0 \quad \text{for } m \in \{0, \ldots, 9\} \]

where $f_{\bullet\bullet}$ are the structure constants of $g$, and $Q_{BRST}$ is given by:

\[ Q_{BRST} = Q^L_{BRST} + Q^R_{BRST} \]

where \[ Q^L_{BRST} = \lambda^\alpha_L L(t^3_\alpha) \]

and \[ Q^R_{BRST} = \lambda^{\dot{\alpha}}_R L(t^1_{\dot{\alpha}}) \]

Here $L(t)$ is the left multiplication by $t$. We will use the notation $P^{p,q}$ for the space of polynomials of the order $p$ in $\lambda_L$ and $q$ in $\lambda_R$. Therefore $P^n = \bigoplus_{p+q=n} P^{p,q}$.

More generally, we can consider the cohomology with coefficients in an arbitrary representation $V$ of $g$:

\[ \ldots \xrightarrow{Q_{BRST}} V \otimes_{g_0} P^n \xrightarrow{Q_{BRST}} V \otimes_{g_0} P^{n+1} \xrightarrow{Q_{BRST}} \ldots \]
The cohomology of this complex will be denoted $H^n(Q_{BRST} : V)$. With this notation, the cohomology of the “standard” BRST complex (36) is $H^n(Q_{BRST} : (Ug))'$. These complexes were studied in [BC01, Mik11b, Mik11a].

It is useful to consider a filtration $F^p$ on the space of vertex operators, corresponding to the powers of $\lambda_R$. We will consider an element of $\text{Hom}_{g0}(Ug, P^n)$ to be of the order $p$ if it goes like $O(\lambda^p_R)$ when $\lambda_R \to 0$. The space of such operators will be denoted $F^p\text{Hom}_{g0}(Ug, P^n)$. This is a decreasing filtration, i.e. $\ldots \supset F^p \supset F^{p+1} \supset F^{p+2} \supset \ldots$ This allows us to calculate the cohomology of $Q_{BRST}$ using some approximation scheme, starting from the cohomology of $Q_L$ and considering $Q_R$ as a small correction. The first approximation is:

$$E_2^{p,q} = H^p(Q^R_{BRST} ; H^q(Q^L_{BRST} ; V))$$

(42)

### 4.2 Lie algebra formed by the covariant derivatives

Now we will introduce some infinite-dimensional Lie algebra, which we will use later to study the cohomology of the complexes (36) and (41).

**Definition of the Lie algebra $L^{\text{tot}}$.** We will consider the infinite-dimensional super Lie algebra generated by the following letters:

$$\nabla^L_{\alpha}, \nabla^R_{\dot{\alpha}}, t^0_{[mn]}$$

(43)

where the indices $\alpha, \dot{\alpha}$ and $[mn]$ run over the same sets as in (35), and with the following relations:

$$\{\nabla^L_{\alpha}, \nabla^L_{\beta}\} = f^{\alpha\beta}_{m} \nabla^L_{m}$$

(44)

$$\{\nabla^R_{\dot{\alpha}}, \nabla^R_{\dot{\beta}}\} = f^{\dot{\alpha}\dot{\beta}}_{m} \nabla^R_{m}$$

(45)

$$\{\nabla^L_{\alpha}, \nabla^R_{\beta}\} = f^{\alpha\beta}_{[mn]} t^0_{[mn]}$$

(46)

$$[t^0_{[mn]}, \nabla^L_{\alpha}] = f^0_{[mn]\beta} \nabla^L_{\beta}$$

(47)

$$[t^0_{[mn]}, \nabla^R_{\dot{\alpha}}] = f^0_{[mn]\dot{\alpha}} \nabla^R_{\dot{\alpha}}$$

(48)

$$[t^0_{[kl]}, t^0_{[mn]}] = f^0_{[kl][mn]pq} t^0_{[pq]}$$

(49)

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3 Frobenius reciprocity implies a relation between (36) and (41), see [Mik11b].
where Eqs. (44) and (45) are the definitions of $\nabla^L_m$ and $\nabla^R_m$. The coefficients $f_{\bullet\bullet}$ are the structure constants of $psu(2,2|4)$ in the basis (35). We will introduce the following notation for this Lie algebra:

$$\mathcal{L}^\text{tot} = \mathcal{L}^L + \mathcal{L}^R + \mathfrak{g}_0$$

where the sum is as linear spaces. More details are in [Mik13].

**Grading.** We will introduce on $\mathcal{L}^\text{tot}$ a $\mathbb{Z}$-grading as follows:

$$\deg(\nabla^L_{\alpha}) = 1$$
$$\deg(\nabla^R_{\dot{\alpha}}) = -1$$

**Definition of the ideal $I \subset \mathcal{L}^{\text{tot}}$.** There is an ideal $I \subset \mathcal{L}^{\text{tot}}$ such that $\mathcal{L}^{\text{tot}}/I = \mathfrak{g}$. The structure of $\mathfrak{g}$ is explained in Eq. (35). Modulo $I$ the generators $t^0_{[m]}$ become the generators $t^0_{[m]}$ of $\mathfrak{g}_0 \subset \mathfrak{g}$, $\nabla^L_{\dot{\alpha}}$ becomes $t^3_\alpha$, $\nabla^R_{\dot{\alpha}}$ becomes $t^1_\alpha$, and both $\nabla^L_m$ and $\nabla^R_m$ become $t^2_m$. The ideal $I$ is not invariant under the $U(1)$ which defines the $\mathbb{Z}$-grading (51), but only under $\mathbb{Z}_4 \subset U(1)$.

### 4.3 Lie algebra cohomology

Let us consider the relative Lie algebra cohomology:

$$H^\ast (\mathcal{L}^{\text{tot}} ; \mathfrak{g}_0 ; V)$$

We claim that this cohomology coincides with the BRST cohomology:

$$H^\ast (\mathcal{L}^{\text{tot}} ; \mathfrak{g}_0 ; V) = H(Q_{\text{BRST}} ; V)$$

We will prove a stronger statement. Let us introduce a decreasing filtration of the Lie algebra cochain complex in the following way. We say that a cochain $c$ belongs to $F^pC^q (\mathcal{L}^{\text{tot}} ; \mathfrak{g}_0 ; V)$ if $c(\xi_1, \ldots, \xi_q)$ is zero whenever there are less than $p$ letters $\nabla^R_{\dot{\alpha}}$ among $\xi_1, \ldots, \xi_q$. For example, for $c \in F^3C^2$ should be true that $c(\nabla^R_{\dot{\alpha}} , \nabla^R_{\dot{\beta}} ) = 0$, but $c(\nabla^R_m , \nabla^R_{\dot{\beta}} )$ does not have to be zero (because $\nabla^R_m$ is defined in (45) as the commutator of two $\nabla^R_{\dot{\alpha}}$, i.e. has degree 2).

In other words, the ghost dual to $\nabla^R_{\dot{\alpha}}$ is considered “small of the order $\varepsilon$”; the ghost dual to $\nabla^R_m$ is considered “small of the order $\varepsilon^2$”, etc. But all
the “left” ghosts are of the order 1. The $F^pC$ consists of cochains which are of the order $\varepsilon^p$ and higher.

Similarly, the BRST complex has a filtration by the powers of $\lambda_R$.

We will construct a filtered quasi-isomorphism between the relative Lie algebra complex $C^\bullet(L_{\text{tot}}; g_0; V)$ and the BRST complex. A filtered quasi-isomorphism of two filtered complexes $C^\bullet_1$ and $C^\bullet_2$ is a map of complexes which is a quasi-isomorphism $\text{gr}^pC^\bullet_1 \to \text{gr}^pC^\bullet_2$ for every $p$. A filtered quasi-isomorphism is a quasi-isomorphism of complexes in the usual sense, if one forgets the grading [Sta, Lemma 05S3]. This can be understood from the point of view of spectral sequences; filtered quasi-isomorphism becomes an isomorphism at $E_1^{\bullet,\bullet}$.

In particular, it follows that the relative Lie algebra cohomology (52) coincides with the BRST cohomology (41).

**Construction of filtered quasi-isomorphism.** Let $C^\bullet(L_{\text{tot}}; g_0; V)$ denote the space of cochains in the relative Lie algebra cohomology complex (52). Let us introduce the operation of restriction from the space of relative cochains to the BRST complex:

$$R : C^\bullet(L_{\text{tot}}; g_0; V) \longrightarrow V \otimes \text{Fun}(\lambda_L, \lambda_R)$$

(54)

which is defined as follows. Given the cochain $c \in C^q(L_{\text{tot}}; g_0; V)$, we have to define $Rc \in V \otimes \text{Fun}(\lambda_L, \lambda_R)$. By definition $c$ is a polylinear function of $q$ elements of $L$:

$$\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_q \mapsto c(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_q)$$

(55)

Elements of the linear space $L_{\text{tot}}/g_0$ are, by definition in Section 4.2, nested commutators of $\nabla^L$s plus nested commutators of $\nabla^R$s. We define $Rc$ as the following function of $\lambda_L$ and $\lambda_R$:

$$Rc(\lambda_L, \lambda_R) = c\left( (\lambda_L^a \nabla^L_a + \lambda_R^a \nabla^R_a)^{\otimes q} \right)$$

for $c \in C^q(L_{\text{tot}}; g_0; V)$

(56)

We used the following notation: $\xi^{\otimes q}$ means $\xi \otimes \xi \otimes \cdots \otimes \xi$. We observe:

$$RQ_{\text{Lie}} = Q_{\text{BRST}}R$$

(57)
**Lemma:** $R$ is a filtered quasi-isomorphism. 

To prove this, we consider the action of $Q_{\text{Lie}}$ on the following space:

$$\text{gr}^p C^{p+q}(L^\text{tot}; g_0 : V) = \frac{F^p C^{p+q}(L^\text{tot}; g_0 : V)}{F^{p+1} C^{p+q}(L^\text{tot}; g_0 : V)} = \bigoplus_{r=0}^{p} C^{q+r}(L^L; V) \otimes_{g_0} \text{gr}^p C^{p-r}(L^R; C) \quad (58)$$

We observe that:

1. The action of $Q_{\text{Lie}}$ on $\text{gr}^p C^{p+q}(L^\text{tot}; g_0 : V)$ coincides with the action of the operator $Q_{\text{Lie}}[H^*(L^L, V)] + Q_{\text{Lie}}[H^*(L^R, C)]$ on $\bigoplus_{r=0}^{p} C^{q+r}(L^L; V) \otimes_{g_0} \text{gr}^p C^{p-r}(L^R; C)$

2. The restriction map $R$ is only nonzero on the $r = 0$ term. It intertwines this complex with the left BRST complex, which has the BRST operator $Q_L = \lambda^0 \alpha^3$. In other words, it is a morphism of complexes:

$$\text{gr}^p C^{p+q}(L^\text{tot}; g_0 : V) \longrightarrow \text{gr}^p C^{p+q}_{\text{BRST}} \quad (59)$$

With these two observations, the Koszul isomorphisms:

$$H^q(L^L; V) \simeq H^q(Q^L_{\text{BRST}}; V) \quad (60)$$

$$H^p(L^R; C) \simeq H^p(Q^R_{\text{BRST}}; C) = \text{Fun}(\lambda^{\otimes p}) \quad (61)$$

imply that $\text{gr}^p R : \text{gr}^p C^{p+q}(L^\text{tot}; g_0 ; V) \longrightarrow \text{gr}^p C^{p+q}_{\text{BRST}}$ is a quasi-isomorphism, i.e. $R$ is a filtered quasi-isomorphism.

#### 4.4 An analogue of the Koszul resolution

In fact, it is possible to glue two Koszul resolutions (one for $L^L$ and another for $L^R$) along $g_0$, as we will now explain.\footnote{Note in the revised version: we are greatful to the referee of [CMV13] for pointing out an error in the original version of this subsection}. Similarly to (24), consider the following BRST-type complex:

$$0 \longrightarrow C \longrightarrow \text{Hom}_{g_0}(U L^\text{tot}, C) \longrightarrow \text{Hom}_{g_0}(U L^\text{tot}, \mathcal{P}^1) \longrightarrow \ldots \quad (62)$$

$$\ldots \longrightarrow \text{Hom}_{g_0}(U L^\text{tot}, \mathcal{P}^n) \longrightarrow \text{Hom}_{g_0}(U L^\text{tot}, \mathcal{P}^{n+1}) \longrightarrow \ldots$$
where the differential acts as follows:

\[ d\phi = \mu_P(\lambda^L_\alpha) \circ \phi \circ \mu_{U_{\mathcal{L}}^{\text{tot}}}^{\text{right}}(\nabla^L_\alpha) + \mu_P(\lambda^R_\alpha) \circ \phi \circ \mu_{U_{\mathcal{L}}^{\text{tot}}}^{\text{right}}(\nabla^R_\alpha) \]  

(notations as in (24)), and Hom\textsubscript{g\textsubscript{0}} means linear maps invariant under the following action of g\textsubscript{0}:

\[(\eta.\phi)(x) = \phi(x\eta) + \eta^{[mn]}t^{\phi}_{[mn]}\phi \]  

We will call the two terms on the right hand side of (72) d\textsubscript{L}\phi and d\textsubscript{R}\phi. We will introduce the abbreviated notation for the terms of (62):

\[0 \to C \to X^0 \to X^1 \to \ldots \]  

There is a bigrading: \[X^n = \bigoplus_{p+q=n} X^{p,q} \] where \[X^{p,q} = \text{Hom}_{g_0}(U_{\mathcal{L}}^{\text{tot}}, \mathcal{P}^{p,q})\]; notice that \[d_L : X^{p,q} \to X^{p+1,q}\] and \[d_R : X^{p,q} \to X^{p,q+1}\].

We will now prove that (62) is a (\(U_{\mathcal{L}}^{\text{tot}}, U_{g_0}\))-injective (\(U_{\mathcal{L}}^{\text{tot}}, U_{g_0}\))-exact resolution of C in the sense of [Hoc56].

**Proof**  Being (\(U_{\mathcal{L}}^{\text{tot}}, U_{g_0}\))-injective follows from Section 1 of [Hoc56] (Lemma 1). Note that every term of (62) is a direct sum of finite-dimensional representations of g\textsubscript{0}. This implies that the kernel and the image of every differential is a direct g\textsubscript{0}-submodule as required in [Hoc56]. It remains to prove the exactness. We will prove the equivalent statement, that the cohomology of the truncated complex:

\[0 \to X^0 \to X^1 \to \ldots \]  

is only nonzero in the zeroth term: \[H^0 = C\]. We will use the spectral sequence of the bicomplex \[d = d_L + d_R\]. Let us first calculate the cohomology of \[d_L\]. We will “normal order” the elements of \(U_{\mathcal{L}}^{\text{tot}}\) by putting elements of \(U_{\mathcal{L}}^{\text{R}}\) to the left and elements of \(U_{\mathcal{L}}^{\text{L}}\) to the right. This gives an isomorphism of linear spaces:

\[\text{Hom}_{g_0}(U_{\mathcal{L}}^{\text{tot}}, \mathcal{P}^{n-p,p}) = \text{Hom}_C(U_{\mathcal{L}}^{\text{L}}, \mathcal{P}^{n-p}) \otimes \text{Hom}_C(U_{\mathcal{L}}^{\text{R}}, \mathcal{P}_R^p)\]  

The differential \[d_L\] only acts on the \(\text{Hom}_C(U_{\mathcal{L}}^{\text{L}}, \mathcal{P}^{n-p})\), while \(\text{Hom}_C(U_{\mathcal{L}}^{\text{R}}, \mathcal{P}_R^p)\) is “inert”. The action of the differential on \(\text{Hom}_C(U_{\mathcal{L}}^{\text{L}}, \mathcal{P}_L^{n-p})\) is the same as in the Koszul complex of \(U_{\mathcal{L}}^{\text{L}}\). Therefore the cohomology of \[d_L\] is \[\text{Hom}_C(U_{\mathcal{L}}^{\text{R}}, \mathcal{P}_R^p)\]. The action of \[d_R\] on the cohomology of \(d_L\) is the same as the action of the differential in the Koszul complex of \(U_{\mathcal{L}}^{\text{R}}\). Therefore \[H(d_R, H(d_L)) = C\], corresponding to constant \(\phi\). This completes the proof.
**Corollary**  This means that for any $U\mathcal{L}_{\text{tot}}$-module $W$, the $\text{Ext}(U\mathcal{L}_{\text{tot}},U\mathfrak{g}_{\bar{0}})(W,C)$ can be computed as the cohomology of the following complex:

$$
\ldots \rightarrow \text{Hom}_{U\mathcal{L}_{\text{tot}}}(W,\text{Hom}_{\mathfrak{g}_{\bar{0}}}(U\mathcal{L}_{\text{tot}},\mathcal{P}^n)) \rightarrow \\
\text{Hom}_{U\mathcal{L}_{\text{tot}}}(W,\text{Hom}_{\mathfrak{g}_{\bar{0}}}(U\mathcal{L}_{\text{tot}},\mathcal{P}^{n+1})) \rightarrow \ldots \tag{68}
$$

As in Section 3.3, there is an isomorphism of complexes (68) and (41):

$$
\text{Hom}_{U\mathcal{L}_{\text{tot}}}(W,\text{Hom}_{\mathfrak{g}_{\bar{0}}}(U\mathcal{L}_{\text{tot}},\mathcal{P}^n)) \simeq \text{Hom}_{\mathfrak{g}_{\bar{0}}}(W,\mathcal{P}^n) \quad (69)
$$

$$
f \mapsto [w \mapsto f(w)(1)] \tag{70}
$$

If $W$ is semisimple as a representation of $\mathfrak{g}_{\bar{0}}$, then this shows that $\text{Ext}(U\mathcal{L}_{\text{tot}},U\mathfrak{g}_{\bar{0}})(W,C)$ can be identified with the cohomology of (41) for $V = W'$.

**Variation**  Similarly, we can consider the following projective resolution:

$$
\ldots \rightarrow (\mathcal{P}^{n+1})' \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}} \rightarrow (\mathcal{P}^n)' \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}} \rightarrow \ldots \tag{71}
$$

$$
\ldots \rightarrow (\mathcal{P})' \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}} \rightarrow C \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}} \rightarrow C \rightarrow 0
$$

where the differential acts as follows:

$$
\partial(s \otimes \xi) = (s \circ \mu_{\mathcal{P}'}(\lambda_L^\alpha)) \otimes \xi \nabla^L_{\alpha} + (s \circ \mu_{\mathcal{P}'}(\lambda_R^\alpha)) \otimes \xi \nabla^R_{\dot{\alpha}} \tag{72}
$$

This means that $\text{Ext}(U\mathcal{L}_{\text{tot}},U\mathfrak{g}_{\bar{0}})(C,V)$ can be computed as the cohomology of the following complex:

$$
\ldots \rightarrow \text{Hom}_{U\mathcal{L}_{\text{tot}}}(\mathcal{P}^n)' \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}}, V) \rightarrow \\
\text{Hom}_{U\mathcal{L}_{\text{tot}}}(\mathcal{P}^{n+1})' \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}}, V) \rightarrow \ldots \tag{73}
$$

As in Section 3.3, there is an isomorphism of complexes (73) and (41):

$$
\text{Hom}_{U\mathcal{L}_{\text{tot}}}(\mathcal{P}^n)' \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}}, V) \simeq \mathcal{P}^n \otimes_{\mathfrak{g}_{\bar{0}}} V \quad (74)
$$

$$
f \mapsto [\lambda \mapsto f(\lambda \otimes 1)] \tag{75}
$$

The expression $[\lambda \mapsto f(\lambda \otimes 1)]$ on the right hand side of (75) denotes an element of $\mathcal{P}^n \otimes_{\mathfrak{g}_{\bar{0}}} V$, understood as a $\mathfrak{g}_{\bar{0}}$-invariant polynomial function of pure spinors of the order $n$, whose value on a pair of pure spinors $\lambda = (\lambda_L, \lambda_R)$ is defined as follows. Since $\lambda$ can be interpreted as an element of $(\mathcal{P}^n)'$, we can consider $\lambda \otimes 1$ an element of $(\mathcal{P}^n)' \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}}$; then we can act on it by $f \in \text{Hom}_{U\mathcal{L}_{\text{tot}}}(\mathcal{P}^n)' \otimes_{\mathfrak{g}_{\bar{0}}} U\mathcal{L}_{\text{tot}}, V)$.  

Eq. (74) is another proof of (53).
4.5 Reduction to the cohomology of the ideal $I \subset \mathcal{L}_{\text{tot}}$

The following construction works for an arbitrary completely reducible representation $A$ of $\mathfrak{g}_0$. Given such an $A$, let us consider $H^n(Q_{\text{BRST}} ; V)$ in the special case:

$$V = \text{Hom}_C (Ug \otimes_{\mathfrak{g}_0} A , C)$$  \hspace{1cm} (76)

According to Section 4.3 $H^n(Q_{\text{BRST}} ; V)$ is equivalent to $H^n(L_{\text{tot}} ; \mathfrak{g}_0 ; V)$, which in the case (76) is the same as $\text{Ext}^n(U L_{\text{tot}}, U g \otimes_{\mathfrak{g}_0} A ; C)$ [Hoc56]. Consider the following complex of $U L_{\text{tot}}$-modules:

$$\ldots \rightarrow U L_{\text{tot}} \otimes_{\mathfrak{g}_0} (\Lambda^2 I \otimes_C A) \rightarrow U L_{\text{tot}} \otimes_{\mathfrak{g}_0} (I \otimes_C A) \rightarrow$$

$$\rightarrow U L_{\text{tot}} \otimes_{\mathfrak{g}_0} A \rightarrow Ug \otimes_{\mathfrak{g}_0} A \rightarrow 0$$  \hspace{1cm} (77)

Here the action of $\mathfrak{g}_0$ on $\Lambda^p I \otimes_C A$ is the sum of the adjoint action on $I$ and the action on $A$. The complex (77) is a $(U L_{\text{tot}}, U g \otimes_{\mathfrak{g}_0} A)$-projective and $(U L_{\text{tot}}, U g \otimes_{\mathfrak{g}_0} A)$-exact resolution of $U g \otimes_{\mathfrak{g}_0} A$ as a $U L_{\text{tot}}$-module, in the sense of [Hoc56]; see Appendix A. Therefore:

$$H^n (L_{\text{tot}} ; \mathfrak{g}_0 ; \text{Hom}_C (U g \otimes_{\mathfrak{g}_0} A , C)) = \text{Hom}_{\mathfrak{g}_0} (A , H^n(I))$$  \hspace{1cm} (78)

Geometrical interpretation  Consider the case when $A$ is a finite-dimensional representation. With $V$ defined by (76) the BRST complex of (41) is:

$$\text{Hom}_{\mathfrak{g}_0} (U g \otimes_{\mathfrak{g}_0} A , P^*)$$  \hspace{1cm} (79)

Geometrically, this is the space of $A'$-valued functions $f_a(g, \lambda_3, \lambda_1)$ where the index $a$ enumerates a basis of $A'$, such that for $h \in G_0$:

$$f_a(hg, h\lambda_3 h^{-1}, h\lambda_1 h^{-1}) = f_a(g, \lambda_3, \lambda_1)$$  \hspace{1cm} (80)

$$f_a(gh, \lambda_3, \lambda_1) = f_b(g, \lambda_3, \lambda_1) \rho^b_a(h)$$  \hspace{1cm} (81)

More precisely, this is the space of Taylor series of sections of the pure spinor bundle over $AdS_5 \times S^5$; the universal enveloping algebra is the space of finite linear combinations, i.e. we do not care about the convergence of the Taylor series $f$. Equation (80) says that $f$ is a section of a bundle over the homogeneous space. On the other hand, Eq. (81) requires that $f$ transform
in a fixed representation $A'$ under the group $G_0$ of *global rotations* around $g = 1$.

The space of Taylor series, as a representation of the global rotations $G_0$, is the direct sum of infinitely many finite-dimensional representations:

$$\text{Hom}_{g_0} (U g, P^*) = \bigoplus_A A \otimes \text{Hom}_{g_0} (U g \otimes g_0 A, P^*)$$

Therefore (78) implies that:

$$H^n (Q_{\text{BRST}}, (U g)') = H^n (I)$$

**Action of the global symmetries** Notice that $g$ naturally acts on $H^m (I)$. This corresponds to the right action of $g$ on the BRST complex (36), i.e. to the global symmetries of the $AdS_5 \times S^5$ sigma-model.

### 4.6 Ghost number 1: global symmetry currents

The elements of $H^1(Q_{\text{BRST}}; (U g)') = H^1(I)$ correspond to the global symmetry currents of the $\sigma$-model [Ber05a, Ber05b, BBMR11]. There are finitely many global symmetries. We have:

$$H^1(I) = \left( \frac{I}{[I, I]} \right)'$$

We will now show that $\frac{I}{[I, I]}$ is a *finite-dimensional* representation of $g$, actually the adjoint representation of $g$.

**Special notations for summation over repeating indices.** As already introduced in (35), the index $m$ enumerates the basis of the vector representation of $g_0 = so(1,4) \oplus so(5)$, and runs from 0 to 9; more precisely, $m \in \{0, \ldots, 4\}$ enumerates vectors of $so(1,4)$, and $m \in \{5, \ldots, 9\}$ vectors of $so(5)$. For a vector $v^m$ we denote:

$$v^m = \begin{cases} v^m & \text{if } m \in \{0, \ldots, 1\} \\ -v^m & \text{if } m \in \{5, \ldots, 9\} \end{cases}$$
For two vectors \(v^m\) and \(w^m\) we denote:

\[
v^m w^m = v^0 w^0 - \sum_{i=1}^{9} v^i w^i
\]

\[
v^m w^m = v^0 w^0 - \sum_{i=1}^{4} v^i w^i + \sum_{i=5}^{9} v^i w^i
\]  \(\text{(86)}\)

**Proposition.** As a representation of \(\mathfrak{g}\), \(I/I\) is generated by the following objects\(\text{[6]}\):

\[
T_m^2 = \nabla^L_m - \nabla^R_m
\]  \(\text{(87)}\)

\[
T_{[mn]}^0 = [\nabla^L_m, \nabla^L_n] - [\nabla^R_m, \nabla^R_n]
\]  \(\text{(88)}\)

\[
Z^L_{\alpha} = \nabla^L_{\alpha} - \frac{1}{10} [\nabla^L_{\alpha}, [\nabla^L_m, \nabla^L_]]
\]  \(\text{(89)}\)

\[
Z^R_{\alpha} = \nabla^R_{\alpha} - \frac{1}{10} [\nabla^R_{\alpha}, [\nabla^R_m, \nabla^R_{\alpha}]]
\]  \(\text{(90)}\)

Notice that \([\nabla^L_m - \nabla^R_m, (\nabla^L_n - \nabla^R_n)] \in [I, I]\) implies that:

\[
[\nabla^L_m, \nabla^L_n] + [\nabla^R_m, \nabla^R_n] - 2\eta^0_{[mn]} = 0 \mod [I, I]
\]  \(\text{(91)}\)

Similarly, \([\nabla^L_m - \nabla^R_m], [\nabla^L_{\alpha} - \nabla^R_{\alpha}]] \in [I, I]\) implies that:

\[
\nabla^L_{\alpha} - \frac{1}{10} f_{\alpha}^{m\alpha}[\nabla^R_m, \nabla^R_{\alpha}] = -Z^L_{\alpha} \mod [I, I]
\]  \(\text{(92)}\)

We will write “\(\equiv 0\)” instead of “\(= 0 \mod [I, I]\)”.

The \((30|32)\)-dimensional linear space generated by \(T_m^2, T_{[mn]}^0, Z^L_{\alpha}, Z^R_{\alpha}\) is closed under the action of \(\mathfrak{g}\). It must be the adjoint representation of \(\mathfrak{g}\).

For example, let us consider \(\{\nabla^L_{\alpha}, Z^L_{\beta}\}\). Modulo \([I, I]\) this is same as \(\{[\nabla^R_m, \nabla^R_{\alpha}], Z^L_{\beta}\}\), and using \((46), (47)\) and \((48)\) this is proportional to \(T_m\).

**Proof of the proposition.** Let \(J\) denote the subspace of \(I/[I, I]\) generated by the action of \(\mathfrak{g}\) on \([87], (88), (89)\) and \((90)\). We have to prove that \(J = I\).

---

\(\text{\footnote{The coefficient }\frac{1}{10}\text{ depends on the choice of normalization for }\nabla_{\alpha}; \text{ in our conventions }f_{\alpha\beta}^m = \Gamma^m_{\alpha\beta},\text{ and the projection pr}(\nabla_m)\text{ of }\nabla_m\text{ to }\mathfrak{g}\text{ satisfies: }\text{ad}_{\text{pr}(\nabla_m)}^2|_{\mathfrak{g}_3} = 1 --- \text{ no summation over }m.}\)
Let us consider some linear combination of commutators of $\nabla^L_\alpha$, for example:

$$\sum_{\vec{\alpha}} C^{\alpha_1 \ldots \alpha_q} [\nabla^L_{\alpha_1}, \{\nabla^L_{\alpha_2}, \ldots \{\nabla^L_{\alpha_{q-2}}, \{\nabla^L_{\alpha_{q-1}}, \nabla^L_{\alpha_q}\}\}\ldots]\}$$  \hspace{1cm} (93)

Suppose that the coefficients $C$ are such that this expression belongs to $I$. We will prove that it also belongs to $J$, using the induction in $q$ — the number of commutators. Suppose that for $q < n$, all such expressions lie in $J$. We will prove that for $q = n$, (93) is also in $J$.

Notice that:

$$\sum_{\vec{\alpha}} C^{\alpha_1 \ldots \alpha_5} [\nabla^L_{\alpha_1}, \{\nabla^L_{\alpha_2}, \ldots \{\nabla^L_{\alpha_{q-1}}, f^{\alpha_5}_{\alpha q} [\nabla^R_m, \nabla^R_\beta]\}\ldots}\} \in J$$  \hspace{1cm} (94)

because $\nabla^L_{\alpha} = \frac{1}{10} f^{\alpha}_a \bar{m} \beta [\nabla^R_m, \nabla^R_\beta] \in J$. Therefore, it remains to prove that the following expression belongs to $J$:

$$\sum_{\vec{\alpha}} C^{\alpha_1 \ldots \alpha_5} [\nabla^L_{\alpha_1}, \{\nabla^L_{\alpha_2}, \ldots \{\nabla^L_{\alpha_{q-1}}, f^{\alpha_5}_{\alpha q} [\nabla^R_m, \nabla^R_\beta]\}\ldots}\} \in J$$  \hspace{1cm} (95)

(notice that it automatically belongs to $J$). When we commute $\nabla^R$ with $\nabla^L$, the number of commutators drops and we are left with $q - 4$ commutators. This provides the step of the induction.

**Calculation of $\{\nabla^L_\alpha, Z^R_\alpha\}$ and $\{\nabla^R_\alpha, Z^L_\alpha\}$**. Here we will prove that both $\{\nabla^L_\alpha, Z^R_\alpha\}$ and $\{\nabla^R_\alpha, Z^L_\alpha\}$ are proportional to $f^{[mn]}_{\alpha \bar{m} \bar{a}, T^0_{[mn]}}$, and $[\nabla^R_m, T^2_{n}]$ is proportional to $f^{[pq]}_{mn, T^0_{[pq]}}$. Let us define $\nabla^R_\alpha$ and $\nabla^L_\alpha$ so that:

$$[\nabla^L_m, \nabla^L_\alpha] = f^{\alpha}_{m \bar{m} \bar{a}} \nabla^L_{\bar{a}}$$  \hspace{1cm} (96)

$$[\nabla^R_m, \nabla^R_\alpha] = f^{\alpha}_{m \bar{m} \bar{a}} \nabla^R_{\bar{a}}$$  \hspace{1cm} (97)

That the RHS of (96) is proportional to $f^{\alpha}_{m \bar{m} \bar{a}}$ and the RHS of (97) is proportional to $f^{\alpha}_{m \bar{m} \bar{a}}$ follows from (44) and (45).

To calculate $\{\nabla^L_\alpha, Z^R_\alpha\}$, $\{\nabla^R_\alpha, Z^L_\alpha\}$ and $[\nabla^R_m, T^2_{n}]$ we start with the following observation:

$$\{\nabla^L_\alpha, Z^R_\alpha\} + \{\nabla^R_\alpha, Z^L_\alpha\} \equiv 0$$  \hspace{1cm} (98)

This follows from:

$$0 \equiv \{\nabla^L_\alpha - \nabla^R_\alpha, \nabla^L_\alpha - \nabla^R_\alpha\} = \{\nabla^L_\alpha, \nabla^L_\alpha\} + \{\nabla^R_\alpha, \nabla^R_\alpha\} - 2t^0_{\alpha \bar{a}}$$  \hspace{1cm} (99)
Also notice:
\[
\{[\nabla_m^L, \nabla^L_\beta], \nabla^R - \nabla^L_\bar{\alpha}\} \equiv f_{m\beta} \{\nabla^L_\beta, \nabla^R_\bar{\alpha} - \nabla^L_\bar{\alpha}\} = \\
= [\nabla_m^L, \{\nabla^L_\beta, \nabla^R_\bar{\alpha} - \nabla^L_\bar{\alpha}\}] - \{\nabla^L_\bar{\beta}, [\nabla_m^L, \nabla^R - \nabla^L_\bar{\alpha}]\} \equiv \\
\equiv - f_{\bar{\alpha}\beta} n [\nabla_m^L, \nabla^L_\beta - \nabla^R_n] - f_{\bar{m}\gamma} \{\nabla^L_\beta, \nabla^R_\gamma - \nabla^L_\gamma\}.
\] (100)

This implies:
\[
f_{m\beta} \{\nabla^L_\beta, Z^R_\bar{\alpha}\} - f_{\bar{m}\gamma} \{\nabla^R_\beta, Z^L_\gamma\} = - f_{\bar{m}\beta} n [\nabla_m^L, \nabla^L_n - \nabla^R_n] \] (101)

Similarly:
\[
f_{m\beta} \{\nabla^R_\beta, Z^L_\bar{\alpha}\} - f_{\bar{m}\gamma} \{\nabla^L_\beta, Z^R_\gamma\} = - f_{\bar{m}\beta} n [\nabla_m^R, \nabla^R_n - \nabla^L_n] \] (102)

Taking into account (98), we get the following system of equations for \(X_{\alpha\dot{\alpha}} = \{\nabla^L_\alpha, Z^R_\bar{\alpha}\}\) and \(X_{mn} = [\nabla^L_m, \nabla^L_n - \nabla^R_n]\):
\[
2 f_{m(\dot{\alpha}) \gamma} X^{\gamma}_{\beta\dot{\alpha}} + f_{\dot{\alpha}\beta} n X^{n}_{mn} = 0 \] (103)
\[
2 f_{m(\dot{\alpha}) \gamma} X^{\gamma}_{\beta\dot{\alpha}} + f_{\dot{\alpha}\beta} n X^{n}_{mn} = 0 \] (104)

This system of equations has the following solution, which defines \(T^0_{[pq]}\):
\[
X_{\alpha\dot{\alpha}} = f_{\alpha\dot{\alpha}} [pq] T^0_{[pq]} \] (105)
\[
X_{mn} = - f_{mn} [pq] T^0_{[pq]} \] (106)

We have to prove that there are no other solutions. Let us use the identity:
\[
f_{m\alpha\beta} f_{\alpha\beta} n = 16 \delta^n_m \] (107)

Contracting (103) and (104) with \(f_{\dot{\alpha}\beta} \) and \(f_{\alpha\beta} \) we get:
\[
2 f_{m(\dot{\alpha}) \gamma} f_{\dot{\alpha}\beta} X^{\gamma}_{\beta\dot{\alpha}} + 16 X^{n}_{mk} = 0 \] (108)
\[
2 f_{m(\dot{\alpha}) \gamma} f_{\dot{\alpha}\beta} X^{\gamma}_{\beta\dot{\alpha}} + 16 X^{n}_{mk} = 0 \] (109)

This implies:
\[
f_{m(\dot{\alpha}) \gamma} f_{\dot{\alpha}\beta} X^{\gamma}_{\beta\dot{\alpha}} + f_{\dot{\alpha}\beta} f_{\dot{\alpha}\beta} X^{\gamma}_{\beta\dot{\alpha}} = 0 \] (110)

Let us assume that the pair \((m, k)\) is such that:
- either \(m \in \{0, \ldots, 4\}\) and \(k \in \{5, \ldots, 9\}\)
- or \(m \in \{5, \ldots, 9\}\) and \(k \in \{0, \ldots, 4\}\);
then \((110)\) implies that for such pairs \((m,k)\) the expression \(f_{m\dot{\alpha}}\gamma f_{k\dot{\beta}}^\delta \gamma X_{\gamma\dot{\beta}}\) is symmetric under the exchange \(m \leftrightarrow k\). But \(X_{mk}\) is always antisymmetric under such an exchange. Therefore Eq. \((108)\) implies that \(X_{mk}\) is only nonzero when either both \(m\) and \(k\) belong to \(\{0, \ldots, 4\}\), or both \(m\) and \(k\) belong to \(\{5, \ldots, 9\}\). This means that \(X_{mk}\) is proportional to \(f_{mk}^\bullet\), and we can define \(T^0_{[pq]}\) from \((105)\). Then \((103)\) gives:

\[
2f_{m(\dot{\alpha})}^\gamma \left( X_{\gamma|\dot{\beta}} - f_{\gamma|\dot{\beta}}^\bullet [pq] Y_{[pq]} \right) = 0
\]

which implies that \(X_{\alpha\dot{\alpha}} = f_{\alpha\dot{\alpha}}^\bullet [pq] Y_{[pq]}\).

To summarize, \(\frac{f^0_{[pq]}}{[pq]}\) is a finite-dimensional space, the adjoint representation of \(g\).

### 4.7 Ghost number 2: vertex operators

The cohomology group \(H^n(I)\) is a linear space dual\(^7\) to the homology \(H_n(I)\). The vertex operators correspond to \(H^2(I) = (H_2(I))'\). The linear space \(H_2(I)\) consists of the expressions of the form:

\[
a = \sum_i x_i \wedge y_i
\]

\[
\sum_i [x_i, y_i] = 0
\]

where \(x_i\) and \(y_i\) are elements of \(I\), with the equivalence relations:

\[
a \simeq a + [x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y
\]

We do not have the complete analysis at the ghost number two. It must be true that \(H_2(I)\) correspond to the space of gauge-invariant\(^8\) operators at a marked point in \(AdS_5 \times S^5\). This is an infinite-dimensional representation of \(g\). The simplest element of \(H_2(I)\) is:

\[
\mathcal{O} = C^\alpha_{\dot{\alpha}} (\nabla^L_\alpha - W^R_\alpha) \wedge (\nabla^R_\dot{\alpha} - W^L_\dot{\alpha})
\]

---

\(^7\)This is the Poincaré duality, Section VI.3 of [Kna88].

\(^8\)Gauge invariance means is the diffeomorphism invariance plus various gauge symmetries of the Type IIB SUGRA.
This probably corresponds to the value of the dilaton\(^9\). It should be possible to obtain other fields by acting on (115) with \(\nabla_\alpha^L\) and \(\nabla_\dot{\alpha}^R\).

5 Flat space limit

In this section we will study the cohomology of the BRST operator in flat space.

In flat space \(\mathcal{L}^{\text{tot}} = \mathcal{L}^L \oplus \mathcal{L}^R\). The limit of the BRST complex (36) is:

\[ Q_{\text{SUGRA}} = \lambda_L^\alpha \left( \frac{\partial}{\partial \theta_L^\alpha} + \Gamma_m^{\alpha_\beta} \theta_L^\beta \frac{\partial}{\partial x^m} \right) + \lambda_R^{\dot{\alpha}} \left( \frac{\partial}{\partial \theta_R^{\dot{\alpha}}} + \Gamma_m^{\dot{\alpha}_\beta} \theta_R^{\beta} \frac{\partial}{\partial x^m} \right) \] (116)

acting on functions of \(\theta_L, \theta_R, x, \lambda_L, \lambda_R\).

5.1 Ghost number 1.

The space \([\mathcal{I}, \mathcal{I}]\) is generated by \(\nabla^L_m - \nabla^R_m, [\nabla^L_m, \nabla^L_n], W_\alpha^L\) and \(W^{\dot{\alpha}}_R\). We observe:

\[ [\nabla^L_m, \nabla^L_n] = - [\nabla^R_m, \nabla^R_n] \mod [\mathcal{I}, \mathcal{I}] \] (117)

As a representation of susy, this space should be the dual to \(\text{susy} + \text{Lorentz}\).

We observe:

\[ \{\nabla_{(\alpha}, \Gamma^m_{\beta)} W^\gamma_L \} \equiv \frac{1}{2} \Gamma^n_{\alpha\beta} [\nabla^L_n, \nabla^L_m] \] (118)

As explained in [Maf09], Eq. (118) implies that \(\nabla_\alpha W^\gamma_L\) is proportional to \((\Gamma_{mn})^\gamma_\alpha [\nabla^L_n, \nabla^L_m]\).

5.2 Ghost number 2.

We do not have the complete analysis at the ghost number two. The RR should correspond to \(W_\alpha^L \wedge W^{\dot{\alpha}}_R\). The NSNS 3-form field strength \(H = dB\) should correspond to:

\[ H_{klm} = (\nabla^L_k - \nabla^R_k) \wedge [\nabla^L_l, \nabla^L_m] \] (119)

---

\(^9\)We did not prove that (115) is not exact. One can compute its value on some vertex operator and show that it it nonzero; but this is technically a nontrivial computation, and we did not do it.
The following expression
\[
R'_{klmn} = [\nabla^L_k, \nabla^L_l] \wedge [\nabla^R_m, \nabla^R_n] + [\nabla^L_m, \nabla^L_n] \wedge [\nabla^R_k, \nabla^R_l] \quad (120)
\]
should correspond to a linear combination of the curvature tensor \( R_{klmn} \) and the second derivatives of the dilaton — see Eq. (131). It satisfies the relations:
\[
\begin{align*}
R'_{klmn} &= R'_{mnkl} = -R'_{lkmn} \quad (121) \\
R'_{[klmn]} &= 0 \quad (122) \\
\nabla_j R'_{klmn} &= 0 \quad (123)
\end{align*}
\]
Notice that \( R'_{k[lmn]} = 0 \) follows from (121) and (122). Eq. (121) follows immediately from (120). Here is the proof of (122):
\[
\begin{align*}
R'_{klmn} &= [\nabla^L_k, \nabla^L_l] \wedge [\nabla^R_m, \nabla^R_n] + [\nabla^L_m, \nabla^L_n] \wedge [\nabla^R_k, \nabla^R_l] \\
&= 4 \left[ [\nabla^L_k, \nabla^L_l] \wedge [\nabla^R_m, \nabla^R_n] - [\nabla^L_m, \nabla^L_n] \wedge [\nabla^R_k, \nabla^R_l] \right] = 0 \quad (124)
\end{align*}
\]
— here \( [[\nabla^L_k, \nabla^L_l], \nabla^L_m] \) = 0 because of the Jacobi identity. To prove (123) we observe that when calculating \( \nabla_j \phi \) for any element \( \phi \) of \( H_2(I) \), we can use either \( \nabla_j^L \phi \) or \( \nabla_j^R \phi \). Since both terms on the right hand side of (120) are in \( H_2(I) \), we are free to use \( \nabla_j^L \) when calculating \( \nabla_j([\nabla^L_k, \nabla^L_l] \wedge [\nabla^R_m, \nabla^R_n]) \) and \( \nabla_j^R \) when calculating \( \nabla_j([\nabla^R_m, \nabla^R_n] \wedge [\nabla^L_k, \nabla^L_l]) \). Those are both zero because of the Jacobi identity.

Mismatch. It turns out that the linearized SUGRA equations of motion are not satisfied, because \( \nabla^k H_{klm} \neq 0 \). Using the identities from Appendix B of [Mafl09], we derive using (119):
\[
\nabla^k H_{klm} = -\frac{2}{3} [\nabla^L_k, \nabla^L_l] \wedge [\nabla^L_m, \nabla^L_n] + \frac{1}{3} (\nabla^L_k - \nabla^R_k) \wedge [\nabla^L_m, \nabla^L_n] + \frac{1}{3} (\nabla^L_l - \nabla^R_l) \wedge \Gamma_{m|n|\alpha} \{ W^\alpha_L, W^\beta_L \} \quad (125)
\]
However, the derivatives of \( \nabla^k H_{klm} \) are all zero\(^\text{10}\):
\[
\nabla_n \nabla^k H_{klm} = 0 \quad (126)
\]
\(^\text{10}\) Since the homology of \( I \) is \( I \)-invariant, we can calculate either \( \nabla^L_n \nabla^k H_{klm} \) or \( \nabla^R_n \nabla^k H_{klm} \); it is easier to calculate \( \nabla^R_n \nabla^k H_{klm} \).
therefore this is a “zero mode effect”. Moreover, we have:
\[

\nabla^k H_{klm} = \nabla [l A_m^L] = \nabla [l A_m^R] \tag{127}
\]

where
\[

A_m^L = \frac{2}{3} (\nabla_n^L - \nabla_R^L) \wedge [\nabla_n^L, \nabla_m^L] + \frac{1}{3} \Gamma_{\alpha \beta m} W_{L \alpha}^\beta \wedge W_{L \alpha}^\beta
\]
\[

A_m^R = \frac{2}{3} (\nabla_n^R - \nabla_R^L) \wedge [\nabla_n^R, \nabla_m^R] + \frac{1}{3} \Gamma_{\alpha \beta m} W_{R \alpha}^\beta \wedge W_{R \alpha}^\beta \tag{128}
\]

Notice that $A_m^L$ and $A_m^R$ are both in $H_2(I)$.

The dilaton The difference $A_m^L - A_m^R$ should be identified with the first derivative of the dilaton $\partial_m \phi$. Notice that:
\[

\nabla_n (A_m^L - A_m^R) = \frac{4}{3} [\nabla_k^L, \nabla_m^L] \wedge [\nabla_n^R, \nabla_k^R] \tag{129}
\]

This is in agreement with the statement that (120) is a linear combination of the Riemann-Christoffel tensor $R_{klmn}$ and the derivatives of the dilaton $\partial [g_k][m] \partial [\n] \phi$. Indeed, we have:
\[

g^{lm} \left( [\nabla_k^L, \nabla_l^L] \wedge [\nabla_m^R, \nabla_n^R] + [\nabla_m^L, \nabla_n^L] \wedge [\nabla_k^R, \nabla_l^R] \right) - \frac{3}{4} \nabla_n (A_k^L - A_k^R) = 0 \tag{130}
\]

which is the Einstein’s equation $R_{kn} = 0$ for the Ricci tensor $R_{kn} = g^{lm} R_{klmn}$, if we identify:
\[

[\nabla_k^L, \nabla_l^L] \wedge [\nabla_m^R, \nabla_n^R] + [\nabla_m^L, \nabla_n^L] \wedge [\nabla_k^R, \nabla_l^R] =
\]
\[

= R_{klmn} + \partial [\n] g_k[\n] \partial [\n] = 0 \tag{131}
\]

where $R_{klmn}$ is the Riemann-Christoffel tensor in the Einstein frame, and $\partial_n \phi = \frac{3}{8} (A_m^L - A_m^R)$. Also observe that $\nabla_n (A_m^L - A_m^R) = 0$ — the Klein-Gordon equation for the dilaton. Indeed:
\[

[\nabla_k^L, \nabla_l^L] \wedge [\nabla_k^R, \nabla_l^R] =
\]
\[

= [(\nabla_k^L - \nabla_R^L), (\nabla_l^L - \nabla_R^L)] \wedge [\nabla_k^R, \nabla_l^R] \simeq
\]
\[

\simeq 2(\nabla_k^L - \nabla_R^L) \wedge [\nabla_l^R, \nabla_k^R] \simeq -(\nabla_k^L - \nabla_R^L) \wedge \Gamma_{k \alpha \beta} [W_{R \alpha}^\alpha, W_{R \alpha}^\beta] \simeq
\]
\[

\simeq \Gamma_{k \alpha \beta} [W_{R \alpha}^\alpha, W_{R \alpha}^\beta] \wedge W_{R \alpha}^\beta = 0 \tag{132}
\]

(We used the Dirac equation $\Gamma_{k \alpha \beta} [\nabla_{k \alpha}^L, W_{R \alpha}^\alpha] = 0$.)
Unphysical operator  We have seen that the difference $A_m^L - A_m^R$ corresponds to the derivative of the dilaton: $\partial_m \phi$. But the sum $A_m^L + A_m^R$ presents a problem. Observe that:

$$\nabla_l(A_m^L + A_m^R) = \nabla_p(A^L_m + A^R_m) \quad (133)$$

$$\nabla_k \nabla_l(A^L_m + A^R_m) = 0 \quad (134)$$

This means that the first derivative of $(A_m^L + A_m^R)$ is a constant.

Relation to the results of [BBMR11, Mik12] This mismatch is not surprising. We know from [BBMR11] that the zero momentum states are not correctly reproduced as the cohomology of the “naive” BRST complex (36). Therefore we do expect a mismatch in the zero mode sector of the space of local operators.

A state on which $A_m^L + A_m^R$ is nonzero is described in [Mik12]. It is obtained as the flat space limit of the nonphysical AdS vertex of [BBMR11] with the internal commutator taking values in $g_2$ (using the notations of Section 4.1). In this case $A_m^L + A_m^R$ is constant — the gradient of the “asymmetric dilaton”.

Besides being constant, $A_m^L + A_m^R$ can also be depending on $x$ linearly. To obtain the state with $A_m^L + A_m^R$ depending linearly on $x$, we have to consider the flat space limit of the nonphysical vertex $B_{ab} j^a \wedge j^b$ with the internal commutator $f^{ab} B_{ab}$ taking values in $g_0$ [BBMR11, Mik12]. It depends on a constant antisymmetric tensor $B_{mn}$. The leading term in the flat space limit is a trivial constant NSNS $B$-field $B_{mn} dx^m \wedge dx^n$, which can be gauged away. Discarding the terms with $\theta$'s, the leading nontrivial term is:

$$B_{mn} dx^m \wedge \left( x^n \sum_{k=0}^4 (dx_k x^k) - dx^n \sum_{k=0}^4 (x_k x^k) \right) \quad (135)$$

This does not solve the SUGRA equations $\partial^n H_{nm} = 0$, instead $\partial^n H_{nm}$ is proportional to $B_{mn} dx^m \wedge dx^n$ — a constant 2-form.

In terms of the unintegrated vertex, the observable $A_m^L + A_m^R$ should be identified as follows. It is proportional to $\partial^n B_{mn}$ in the gauge where the vertex has ghost number $(1,1)$, i.e. only $\lambda_L \lambda_R$ terms, no $\lambda_L \lambda_L$ and $\lambda_R \lambda_R$ terms.\footnote{If we try to change the gauge $B_{mn} \rightarrow B_{mn} + \partial_m [\Lambda_n]$ to get rid of $\partial^n B_{mn}$, this would generate some $\lambda_L \lambda_L$ and $\lambda_R \lambda_R$ terms [BBMR11].}
Nonphysical operator: summary

Let us denote:

\[ \nabla L_k, \nabla L_l \land \nabla R_m, \nabla R_n + \nabla L_m, \nabla L_n \land \nabla R_k, \nabla R_l = \mathcal{R}_{klmn} \]

\[ (\nabla^L_k - \nabla^R_k) \land [\nabla^L_i, \nabla^L_j] = H_{klm} = \partial_k B_{lm} \]

\[ A^\pm_m = A^L_m \pm A^R_m \]

We get the following equations of motion:

\[ g^{lm} \mathcal{R}_{klmn} = \frac{3}{4} \nabla (k A^\pm_n) \]

\[ 0 = \nabla [k A^\pm_n] \]

\[ \nabla^k H_{klm} = \nabla [l A^+_m] \]

\[ 0 = \nabla (l A^+_m) \]

The gradient of the dilaton corresponds to \( A^-_n \), while \( A^+_n \) does not have a clear interpretation in the Type IIB supergravity. The “observable” \( A^+_n \) is dual to the unphysical vertex of [Mik12]. The unphysical vertex is not BRST trivial. However, as we explained in [Mik12], it should be thrown away because it leads to a quantum anomaly in the worldsheet sigma-model at the 1-loop level.

Generic element of \( H_2(I) \)

The “generic” element is:

\[ \mathcal{O} = x_L \land x_R \]

where \( x_L \in I \cap \mathcal{L}^L \) and \( x_R \in I \cap \mathcal{L}^R \). Notice that the following expression:

\[ (\nabla_m x_L) \land x_R - x_L \land (\nabla_m x_R) \]

is zero in homology, i.e. exact:

\[ (\nabla_m x_L) \land x_R - x_L \land (\nabla_m x_R) = \delta ((\nabla^L_m - \nabla^R_m) \land x_L \land x_R) \]

Indeed, the generic gauge-invariant SUGRA operator can be understood as the product of two gauge-invariant Maxwell operators \( \mathcal{O}_L \) and \( \mathcal{O}_R \), with the condition that \( \mathcal{O}_L \frac{\partial}{\partial x_m} \mathcal{O}_R = 0 \). The zero momentum special operators of the form \([127]\) are not of this form.
5.3 Higher ghost numbers

This section was added in the revised version of the paper. We have previously claimed that the cohomology at the ghost number higher than 2 vanishes. We are greateful to the referee for insisting that we present a proof of this statement. Upon careful examination, it turns out that the statement is wrong. There is some nontrivial cohomology at least at the ghost number 3. Here we will only do a preliminary analysis:

- We prove that the cohomology at the ghost number $> 4$ vanishes.
- We give an example of the nontrivial cohomology class at the ghost number 3.

We suspect that the cohomology at the ghost numbers 3 and 4 is a finite-dimensional space, and is in some way related to the unphysical states of [BBMR11, Mik12].

We will start by proving the vanishing theorem for the super-Maxwell cohomology at the ghost number higher than 1. We will then point out that the SUGRA BRST complex is *almost* the tensor product of two super-Maxwell complexes (the “left sector” and the “right sector”). If it were, literally, the tensor product, that would indeed imply the vanishing theorem at the ghost number $> 2$. But in fact, even in flat space there is some “interaction” between the left and the right sector, and this leads to a nontrivial cohomology at least at the ghost number 3.

5.3.1 Super-Maxwell BRST complex

The cohomology of the super-Maxwell BRST complex:

$$Q_{\text{SM}axw} = \lambda^\alpha \left( \frac{\partial}{\partial \theta^\alpha} + \Gamma^m_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^m} \right)$$

is only nontrivial at the ghost numbers 0 and 1.

**Sketch of the proof** This fact is well-known in the pure spinor formalism. At the ghost number 0, the cohomology is formed by the constants (no dependence on $\lambda$, $x$ and $\theta$). At the ghost number 1, the cohomology is the solutions of the free Maxwell equation and the free Dirac equation. The vanishing of the cohomology at the ghost number 2 is equivalent to the following
two statements: 1) for any current $j_m$ such that $\partial_m j_m = 0$ always exists the
gauge field $F_{mn}$ satisfying $\partial_k F_{lm} = 0$ and $\partial_m F_{mn} = j_n$ and 2) for any spinor
$\psi$ exists a spinor $\phi$ such that $\Gamma^m \partial_m \phi = \psi$. The vanishing of the cohomology
at the ghost number 3 is equivalent to the statement that for any $\rho$ exists
$j_m$ such that $\partial_m j_m = \rho$. All these facts are proven in any graduate course of
classical electrodynamics.

5.3.2 Type IIB BRST complex

The BRST complex of Type IIB in flat space is *almost* the tensor product of
two SMaxwell complexes:

$$Q_{S\text{Maxw} \otimes \text{SMaxw}} = \lambda^\alpha_L \left( \frac{\partial}{\partial \theta^\alpha_L} + \Gamma^m_{\alpha \beta} \theta^\beta_L \frac{\partial}{\partial x^m_L} \right) + \lambda^\alpha_R \left( \frac{\partial}{\partial \theta^\alpha_R} + \Gamma^m_{\alpha \beta} \theta^\beta_R \frac{\partial}{\partial x^m_R} \right)$$  \hspace{1cm} (147)

The cohomology of (147) is the tensor product of the cohomologies of two
super-Maxwell complexes. Therefore it is only nontrivial at the ghost num-
bers 0,1 and 2. However, in the Type IIB BRST complex there is no sepa-
ration of $x$ into $x_L$ and $x_R$. The actual BRST complex is therefore different
from (147):

$$Q_{\text{SUGRA}} = \lambda^\alpha_L \left( \frac{\partial}{\partial \theta^\alpha_L} + \Gamma^m_{\alpha \beta} \theta^\beta_L \frac{\partial}{\partial x^m} \right) + \lambda^\alpha_R \left( \frac{\partial}{\partial \theta^\alpha_R} + \Gamma^m_{\alpha \beta} \theta^\beta_R \frac{\partial}{\partial x^m} \right)$$ \hspace{1cm} (148)

The difference is that the left and the right sector have a common $x$ instead
of separate $x_L$ and $x_R$. We also write:

$$Q_{\text{SUGRA}} = Q_L + Q_R$$  \hspace{1cm} (149)

where $Q_L$ and $Q_R$ are the first and second terms on the right hand side of
(148).

**Vanishing theorem:** $H^n_{Q_{\text{SUGRA}}} = 0$ for $n > 4$. Let us consider, for example,
a vertex of the ghost number 5.

**Lemma** Given a vertex at the ghost number 5, we can always modify it
by adding $Q$-exact terms so that the new vertex has only terms of the type
$\lambda^5_L \lambda^4_R$.

We have to prove that the terms with $\lambda^5_R$, $\lambda^2_L \lambda^3_R$, $\lambda^3_L \lambda^2_R$, $\lambda^4_L \lambda_R$ and $\lambda^5_L$ can
be gauged away. The term with $\lambda^5_R$ is $Q_R$-closed. Suppose that the term with
the lowest power of $\theta_R$ is proportional to $\lambda_R^5 \theta_R^p$. We observe that this term is closed under $\lambda_R^p \partial/\partial \theta_R$ and therefore is equal to $\lambda_R^p \partial/\partial \theta_R$ of some expression proportional to $\lambda^4 \theta_R^{p+1}$. This means that we can add $Q$-exact terms so that the new vertex has terms of the order $\lambda_R^5$ starting with $\lambda_R^5 \theta_R^{p+2}$. An induction by $p$ implies that the terms containing $\lambda_R^5$ can be all gauged away. Similarly, we can gauge away terms proportional to $\lambda^5_L \lambda_R^3$, then terms proportional to $\lambda^4_L \lambda^2_R$, then $\lambda^3_L \lambda^2_R$, then $\lambda^2_L \lambda^3_R$. This proves the Lemma.

Now we are left with the terms proportional to $\lambda^1_L \lambda^4_R$. In this gauge the vertex operator is both $Q_R$-closed and $Q_L$-closed. Let us look at the expansion in powers of $\theta_R$. Schematically:

$$V = \lambda_R^5 \left( \theta_R^k \phi_k (\lambda_L, \theta_L, x) + \theta_R^{k+1} \phi_{k+1} (\lambda_L, \theta_L, x) + \ldots \right)$$ (150)

were every $\phi_j$ is linear in $\lambda_L$. We observe that all these $\phi_j$s are annihilated by $Q_L$ (because $Q_L V = 0$ and $Q_L$ does not act on $\theta_R$):

$$Q_L \phi_j = 0$$ (151)

We also observe that in the leading term, the coefficient of $\phi_k$ is annihilated by $\lambda_R^5 \partial/\partial \theta_R$. This implies:

$$V = Q_{\text{SUGRA}} \left( \lambda^3_R \theta^k \phi_k (\lambda_L, \theta_L, x) \right) +$$

$$+ \lambda^4_R \left( \theta^{k+1} \phi_{k+1} (\lambda_L, \theta_L, x) + \theta^{k+2} \phi_{k+2} (\lambda_L, \theta_L, x) + \ldots \right)$$ (152)

This means that we are able to increase the order of the leading term by adding a $Q_{\text{SUGRA}}$-exact expression. The induction in $k$ proves the Theorem.

But is it true that $H^n_{\text{SUGRA}} = 0$ for $n = 3$ and $n = 4$? It turns out that at least for $n = 3$ the cohomology is nontrivial. The fact that the cohomology at the ghost number higher than 2 is nontrivial is (for us) unexpected. We will leave this for future research, giving here only an example.

**Example of a vertex at the ghost number 3** For any constant 5-form $F$, let us denote $\tilde{F} = F_{klnmp} \Gamma^{klnmp}$. Consider the following coboundary of $Q_{\text{SMaxw@SMaxw}}$:

$$\Phi [F] = Q_{\text{SMaxw@SMaxw}} \Psi [F]$$ (153)
where

\[ \Psi[F] = \left( \theta_L \Gamma^p \lambda_L \right) \left( \theta_L \Gamma_p \left( x_L^m \Gamma_m x_R^n \Gamma_n + 5 ||x_L||^2 \right) \hat{F} \Gamma_q \theta_R \right) \left( \lambda_R \Gamma^q \theta_R \right) + \]

\[ + \left( \theta_L \Gamma^p \lambda_L \right) \left( \theta_L \Gamma_p x_L^m \Gamma_m f[\lambda_R \theta_R^4] \right) + \left( g_n[\lambda_L \theta_L^4] x_R^n \hat{F} \Gamma_q \theta_R \right) \left( \lambda_R \Gamma^q \theta_R \right) \]

(154)

where \( f[\lambda_R \theta_R^4] \) is chosen so that:

\[ \left( \lambda_R \frac{\partial}{\partial \theta_R} \right) \left( x_R^n \Gamma_n \hat{F} \Gamma_q \theta_R (\lambda_R \Gamma^q \theta_R) + f[\lambda_R \theta_R^4] \right) = 0 \] (155)

and \( g[\lambda_L \theta_L^4] \) is chosen so that:

\[ \left( \lambda_L \frac{\partial}{\partial \theta_L} \right) \left( (\theta_L \Gamma^p \lambda_L) \theta_L \Gamma_p \left( x_L^m \Gamma_m \Gamma_n - 10 x_L^n \right) + g^n[\lambda_L \theta_L^4] \right) = 0 \] (156)

Such \( f[\lambda_R \theta_R^4] \) and \( g^n[\lambda_L \theta_L^4] \) exist because the expression \( x_R^n \Gamma_n \hat{F} \) satisfies the “right” Dirac equation:

\[ \frac{\partial}{\partial x_R^k} \left( x_R^n \Gamma_n \hat{F} \right) \Gamma_k = 0 \] (157)

and the expression \( (x_L^m \Gamma_m \Gamma_n - 10 x_L^n) \) satisfies the “left” Dirac equation:

\[ \frac{\partial}{\partial x_L^k} \Gamma_k (x_L^m \Gamma_m \Gamma_n - 10 x_L^n) = 0 \] (158)

We will now prove that \( \Phi[F] \) depends on \( x_L \) and \( x_R \) only in the combination \( x_L + x_R \). Indeed, for a constant \( c^m \) let us introduce \( \Xi[c, F] \) as follows:

\[ \Xi[c, F] = c^m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) \Psi[F] = \]

\[ = \left( \theta_L \Gamma^p \lambda_L \right) \left( \theta_L \Gamma_p c^m \Gamma_m x_R^n \Gamma_n \hat{F} \Gamma_q \theta_R \right) \left( \lambda_R \Gamma^q \theta_R \right) + \]

\[ + \left( \theta_L \Gamma^p \lambda_L \right) \left( \theta_L \Gamma_p c^m \Gamma_m f[\lambda_R \theta_R^4] \right) - \]

\[ - \left( \theta_L \Gamma^p \lambda_L \right) \left( \theta_L \Gamma_p (x_L^m \Gamma_m c^n \Gamma_n - 10 (x_L c)) \hat{F} \Gamma_q \theta_R \right) \left( \lambda_R \Gamma^q \theta_R \right) - \]

\[ - \left( g^n[\lambda_L \theta_L^4] c^n \hat{F} \Gamma_q \theta_R \right) \left( \lambda_R \Gamma^q \theta_R \right) \] (159)
and we observe that:

\[ Q_{\text{Maxw} \otimes \text{Maxw}} \ c^m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) \Psi[F] = 0 \]  

Since \( Q_{\text{Maxw} \otimes \text{Maxw}} \) commutes with \( c_m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) \), Eq. (160) implies that \( \Phi[F] \) depends on \( x_L \) and \( x_R \) only in the combination \( x_L + x_R \), and is therefore a cocycle of \( Q_{\text{SUGRA}} \). We will now prove that \( \Phi[F] \) is not a coboundary of \( Q_{\text{SUGRA}} \). We know that \( \Phi[F] \) is a coboundary of \( Q_{\text{Maxw} \otimes \text{Maxw}} \), i.e. once we introduce separate \( x_L \) and \( x_R \) we have (153). The question is:

\[ \text{can we modify } \Psi[F], \text{ by adding to it something } Q_{\text{Maxw} \otimes \text{Maxw}} \text{-closed, so that the modified } \Psi[F] \text{ is annihilated by } \frac{\partial}{\partial x_L} - \frac{\partial}{\partial x_R} ? \]  

In order to answer this question, it is useful to consider \( c \) as a ghost and interpret \( \Xi[c, F] \) as a cocycle of the nilpotent operator \( c^m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) \) acting on the cohomology of \( Q_{\text{Maxw} \otimes \text{Maxw}} \). The answer to the question (161) is positive only if \( \Xi[c, F] \) is a coboundary in this complex. The cohomology of \( Q_{\text{Maxw} \otimes \text{Maxw}} \) is the tensor product of two super-Maxwell solutions. We will now prove that \( \Xi[c, F] \) represents a nonzero element of:

\[ H^1 \left( c^m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right), \text{Maxw}(x_L) \otimes \text{Maxw}(x_R) \right) \]  

Remember that super-Maxwell is a direct sum of a solution of the free Maxwell equations and a solution of the free Dirac equation. Looking at (159), the corresponding cocycle corresponds to the tensor product of two solutions of the free Dirac equation. Such an element of \( \text{Maxw}(x_L) \otimes \text{Maxw}(x_R) \) can be represented as a bispinor field \( \psi^{\alpha\dot{\beta}}(x_L, x_R) \) satisfying:

\[ \Gamma^m_{\alpha\alpha'} \frac{\partial}{\partial x_L^m} \psi^{\alpha'\dot{\beta}}(x_L, x_R) = 0 \]  
\[ \frac{\partial}{\partial x_R^m} \psi^{\alpha\dot{\beta'}}(x_L, x_R) \Gamma^m_{\dot{\beta'}\dot{\beta}} = 0 \]  

The element of (162) corresponding to \( \Xi[c, F] \) is:

\[ \psi(c; x_L, x_R)^{\alpha\dot{\beta}} = \left( \hat{c} x_R \hat{F} - (\hat{x}_L \hat{c} - 10(x_L \cdot c)) \hat{F} \right)^{\alpha\dot{\beta}} \]
where hat over letter stands for the contraction with the gamma-matrices, e.g. \( \hat{x}_R = \Gamma^m x^m_R \). Let us analyze the possibility of (165) being in the image of \( c^n \left( \frac{\partial}{\partial x^n_L} - \frac{\partial}{\partial x^n_R} \right) \):

\[
\left( \hat{c} \hat{x}_R \hat{F} - (\hat{x}_L \hat{c} - 10 (x_L \cdot e) \hat{F}) \right)^{\alpha \beta}_{\gamma} \nonumber = c^n \left( \frac{\partial}{\partial x^m_L} - \frac{\partial}{\partial x^m_R} \right) \left( \phi^{\alpha\beta}_{mn} x^m_L x^m_R + \chi^{\alpha\beta}_{mn} x^m_L x^m_R + \sigma^{\alpha\beta}_{mn} x^m_R x^n_R \right) \tag{166}
\]

with all three \( \phi^{\alpha\beta}_{mn} x^m_L x^m_R \), \( \chi^{\alpha\beta}_{mn} x^m_L x^m_R \) and \( \sigma^{\alpha\beta}_{mn} x^m_R x^n_R \) satisfying both (163) and (164). Looking at the part linear in \( x_R \), this implies:

\[
\left( \Gamma^m \hat{x}_R \hat{F} \right)^{\alpha \beta}_{\gamma} = -2 \sigma^{\alpha\beta}_{mn} x^m_R + \chi^{\alpha\beta}_{mn} x^n_R \tag{167}
\]

The left Dirac equation on \( \chi \) implies \( \Gamma^m_\alpha \chi^\alpha_{mn} = 0 \), therefore:

\[
10 \left( \hat{x}_R \hat{F} \right)^{\beta}_{\alpha} = -2 \Gamma^m_\alpha \sigma^{\alpha\beta}_{mn} x^n_R \tag{168}
\]

This implies that \( \sigma \) is of the form:

\[
\sigma^{\alpha\beta}_{mn} = -5 \delta^{\alpha\beta}_{mn} \hat{F} + s^{\alpha\beta}_{mn} \tag{169}
\]

where \( \Gamma^m_\alpha s^{\alpha\beta}_{mn} = 0 \tag{170} \)

for some \( s^{\alpha\beta}_{mn} \) symmetric in \( m \leftrightarrow n \). As we have already mentioned, \( \sigma \) should satisfy the right Dirac equation:

\[
\sigma^{\alpha\beta}_{mn} \Gamma^{n}_{\beta \gamma} = 0 \tag{171}
\]

Equations (170) and (171) imply that the traces of \( \sigma \) and \( s \) are zero:

\[
\sigma^{\alpha\beta}_{mm} = s^{\alpha\beta}_{mm} = 0 \tag{172}
\]

but this contradicts (169) because the trace of \( \delta^{\alpha\beta}_{mn} \hat{F} \) is not zero. This shows that (165) is not in the image of \( c^n \left( \frac{\partial}{\partial x^n_L} - \frac{\partial}{\partial x^n_R} \right) \), and therefore it represents a nonzero element of the cohomology group (162). This implies that \( \Phi[F] \) is a BRST-nontrivial vertex operator at the ghost number three.
**Generalization**  The cohomology of $Q_{\text{SMaxw} \otimes \text{SMaxw}}$ at the ghost number 3 is trivial, i.e. any cocycle with three $\lambda$’s can be represented as $Q_{\text{SMaxw} \otimes \text{SMaxw}} \Psi$. But sometimes $\Psi$ cannot be chosen to depend on $x_L$ and $x_R$ through $x_L + x_R$ only. The obstacle for that is in $H^1(R^{10}, \text{SMaxw} \otimes \text{SMaxw})$ where $R^{10}$ is the abelian group of translations, the Lie cohomology differential is $Q_{\text{Lie}} = c^m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right)$. Notice that $\text{SMaxw} \otimes \text{SMaxw}$ splits into components:

$$\text{SMaxw} \otimes \text{SMaxw} = (\text{Maxw} \otimes \text{Maxw}) \oplus (\text{Maxw} \otimes \text{Dirac}) \oplus (\text{Dirac} \otimes \text{Maxw}) \oplus (\text{Dirac} \otimes \text{Dirac})$$

Consider the cohomology in the sector $\text{Dirac} \otimes \text{Dirac}$, and more specifically those elements of it which have linear $x$-dependence. It turns out that this cohomology is identified with the quadratic in $x$ solutions $f$ of the “double Dirac equation” modulo solutions presentable as a sum of a solution of the left Dirac equation and a solution of the right Dirac equation:

$$\frac{\partial}{\partial x^m} \Gamma_{\alpha' \alpha} \frac{\partial}{\partial x^n} \Gamma_{\hat{\alpha}' \hat{\alpha}} f^{\alpha' \hat{\alpha}'}(x) = 0$$

but $\hat{\alpha}' \hat{\alpha}$'s and $\sigma$ such that: $f^{\alpha \hat{\alpha}} = s^{\alpha \hat{\alpha}} + \sigma^{\alpha \hat{\alpha}}$ \hspace{1cm} (174)

$$\frac{\partial}{\partial x^m} \Gamma_{\alpha' \alpha} s^{\alpha \hat{\alpha}} = 0 \quad \text{and} \quad \frac{\partial}{\partial x^n} \sigma^{\alpha \hat{\alpha}} \Gamma_{\hat{\alpha}' \hat{\alpha}} = 0$$

Indeed, given such an $f^{\alpha \hat{\alpha}}$ with the quadratic $x$-dependence, we construct $\psi(c)$ in the following way:

$$\psi(c) = \tilde{c} \Gamma^m \frac{\partial}{\partial x_R^m} f(x_R) + \xi(x_L, c)$$ \hspace{1cm} (175)

where $\xi$ is some solution of the left Dirac equation, chosen so that $Q_{\text{Lie}} \psi = 0$; such a solution always exists because $H^2(R^{10}, \text{Dirac}) = 0$. Suppose that $\psi$ is in the image of $Q_{\text{Lie}}$ acting on the quadratic (in $x_L|x_R$) elements of $\text{Dirac} \otimes \text{Dirac}$, i.e.:

$$\psi(c) \overset{?}{=} c^m \left( \frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) (\sigma \langle x_R \otimes x_R \rangle + \chi \langle x_R \otimes x_L \rangle + \phi \langle x_L \otimes x_L \rangle)$$ \hspace{1cm} (176)

The part of $\psi(c)$ linear in $x_R$ would be:

$$-c^m \frac{\partial}{\partial x_R^m} \sigma \langle x_R^2 \rangle + c^m \frac{\partial}{\partial x_L^m} \chi \langle x_R \otimes x_L \rangle$$ \hspace{1cm} (177)
This implies:

\[
\Gamma^m \frac{\partial}{\partial c^m} \psi(c) \langle x_R \rangle = 10 \Gamma^n \frac{\partial}{\partial x_R^n} f(x_R) = -\Gamma^m \frac{\partial}{\partial x_R^m} \sigma \langle x_R^{\otimes 2} \rangle
\]  

(178)

in other words \( f = s + \sigma \) where \( \sigma \) satisfies the right Dirac equation and \( s \) the left Dirac equation. This contradicts (174).

Eq. (169) has \( f^{\alpha\dot{\alpha}} = ||x||^2 \hat{F}^{\alpha\dot{\alpha}} \) with a 5-form \( \hat{F} \); there are also solutions corresponding to a 3-form or 7-form \( \hat{G} \):

\[
f = \hat{G} ||x||^2 - \frac{1}{52} \hat{\Gamma}_p \hat{G} \hat{\Gamma}_p \hat{x}
\]  

(179)

and a 1-form or 9-form \( \hat{A} \):

\[
f = \hat{A} ||x||^2 - \frac{1}{28} \hat{\Gamma}_p \hat{\Gamma} \hat{x}
\]  

(180)

This means that the cohomology at the ghost number 3 at least includes states with the quantum number of a bispinor.

5.3.3 Dual picture

We conjecture that the dual element of \( H_3(I) \) is of the form:

\[
\mathcal{O}^{\alpha\beta} = \left[ [\nabla_m^L, W_\alpha^L] \wedge W_\beta^R \wedge (\nabla_m^L - \nabla_m^R) - \right.
\]

\[
- W_\alpha^L \wedge [\nabla_m^R, W_\beta^R] \wedge (\nabla_m^L - \nabla_m^R) +
\]

\[
+ \frac{1}{2} W_\alpha^L \wedge W_\beta^R (\Gamma_m^{mn})^\beta^\gamma \wedge [\nabla_m^R, \nabla_n^R] +
\]

\[
+ \frac{1}{2} W_\alpha^L (\Gamma_m^{mn})^\alpha^\gamma \wedge W_\beta^R \wedge [\nabla_m^L, \nabla_n^L]
\]  

(181)

5.3.4 Conjecture about the vertices at the ghost number 3

Generally speaking, the physical interpretation of vertex operators is:

- Ghost number 1: global symmetries of the space-time
- Ghost number 2: infinitesimal deformations of the space-time
- Ghost number 3: obstructions to continuing the infinitesimal deformations of the space-time to the second order in the deformation parameter
It is natural to conjecture that the vertices at the ghost number 3 obstruct those and only those infinitesimal deformations which are unphysical in the sense of \[\text{Mik12}\].

The cohomology at the ghost numbers 3 and 4 deserves systematic investigation. We hope to return to this subject in the future work.

6 Conclusion

In this paper we presented a relation between the cohomology of the pure spinor BRST complex in AdS space and the relative Lie algebra cohomology.

We used this relation to develop a “dual” point of view on the vertex operators in Type IIB. In this approach, instead of looking at the vertex operators, we look at the dual linear space which is identified with the gauge-invariant local operators of the Type IIB SUGRA. This works both in flat space and in AdS. We observe that some elements of the BRST cohomology do not correspond to any physical states, \(e.g.\) the \(A^+\) of \([138]\). It turns out that there are also vertex operators at the ghost number three. They correspond to the obstructions for nonlinear deformations in the actions. Physically, these obstructions should not be present.

Such “unphysical” elements should go away if we restrict the BRST complex to the operators annihilated by the Virasoro constraints. We do not know what this restriction means from the point of view of the Lie algebra cohomology.

We conclude that the BRST complex \([36]\) in \(AdS_5 \times S^5\) and its flat space limit \([116]\) both have rich mathematical structure. But at the same time the cohomology does not give a complete description of the supergravity excitations. The difference is in some unphysical states. These unphysical states have polynomial \(x\)-dependence, as opposed to the usually considered exponential \(x\)-dependence. This polynomial (or “zero-momentum”) sector could be important in the calculation of the scattering amplitude, because the momentum conservation implies that the product of the scattering vertices has zero total momentum.
A  Exactness of (77)

This is similar to the proof of the exactness of the standard Koszul resolution
of the Lie algebra in [Kna88]. For any Lie algebra $L$, the universal enveloping
$UL$ is filtered so that $\text{gr}^p UL = F^p UL/F^{p-1}UL = S^p L$. The differential in
our complex acts in such a way, that we can consistently define:

$$
\ldots \rightarrow F^{p-2}UL^\text{tot} \otimes_{g_0} (\Lambda^2 I \otimes_{C} A) \rightarrow F^{p-1}UL^\text{tot} \otimes_{g_0} (I \otimes_{C} A) \rightarrow \\
 \rightarrow F^{p}UL^\text{tot} \otimes_{g_0} A \rightarrow F^{p}Ug \otimes_{g_0} A \rightarrow 0
$$

This defines a series of complexes $d : X^p_n \rightarrow X^p_{n-1}$ parametrized by an integer
$p$, where $X^p_{n-1} = F^p Ug \otimes_{g_0} A$, $X^p_n = F^p U\mathcal{L}^\text{tot} \otimes_{g_0} A$, and $X^p_n = F^{p-n}U\mathcal{L}^\text{tot} \otimes_{g_0} (\Lambda^n I \otimes_{C} A)$ for $n > 0$. At $p = 0$ we get the exact sequence:

$$
0 \rightarrow A \rightarrow A \rightarrow 0
$$

(183)

On the other hand, the factor-complex $X^p/X^{p-1}$ is:

$$
\ldots \rightarrow S^{p-2} \left( \mathcal{L}^\text{tot}/g_0 \right) \otimes_{C} \Lambda^2 I \otimes_{C} A \rightarrow S^{p-1} \left( \mathcal{L}^\text{tot}/g_0 \right) \otimes_{C} I \otimes_{C} A \rightarrow \\
 \rightarrow S^p \left( \mathcal{L}^\text{tot}/g_0 \right) \otimes_{C} A \rightarrow S^p \left( g/g_0 \right) \otimes_{C} A \rightarrow 0
$$

(184)

This is exact, being the de Rham complex of the linear space $I$ times functions
of additional “inert” variables corresponding to a complement to $g_0 + I$ in
$\mathcal{L}^\text{tot}$. By induction, the complexes $X^p$ are exact for all values of $p$, and
therefore the complex (77) is exact.

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References

[BBMR11] Oscar A. Bedoya, I.Ibiapina Bevilaqua, Andrei Mikhailov, and Victor O. Rivelles, Notes on beta-deformations of the pure spinor superstring in $\text{AdS}(5) \times \text{S}(5)$, Nucl.Phys. B848 (2011), 155–215, arXiv/1005.0049.

[BC01] Nathan Berkovits and Osvaldo Chandia, Superstring vertex operators in an $\text{ads}(5) \times \text{s}(5)$ background, Nucl. Phys. B596 (2001), 185–196, hep-th/0009168.

[Ber00] Nathan Berkovits, Super-Poincare covariant quantization of the superstring, JHEP 04 (2000), 018, hep-th/0001035.

[Ber05a] _____, BRST cohomology and nonlocal conserved charges, JHEP 02 (2005), 060, hep-th/0409159.

[Ber05b] _____, Quantum consistency of the superstring in $\text{AdS}(5) \times \text{S}(5)$ background, JHEP 03 (2005), 041, hep-th/0411170.

[BH02] Nathan Berkovits and Paul S. Howe, Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring, Nucl. Phys. B635 (2002), 75–105, hep-th/0112160.

[CMV13] Osvaldo Chandia, Andrei Mikhailov, and Brenno C. Vallilo, A construction of integrated vertex operator in the pure spinor sigma-model in $\text{AdS}5 \times \text{S}5$, arXiv/1306.0145.

[FF88] B. L. Feigin and D. B. Fuchs, Cohomology of lie groups and algebras (in russian), VINITI t. 21, 1988.

[GKR06] Alexey L. Gorodentsev, A. S. Khoroshkin, and Alexei N. Rudakov, On syzygies of highest weight orbits, arXiv/math/0602316.

[Hoc56] G. Hochschild, Relative homological algebra, Trans. Amer. Math. Soc. 82 (1956), 246–269. MR 0080654 (18,278a)

[How91] Paul S. Howe, Pure spinors lines in superspace and ten-dimensional supersymmetric theories, Phys.Lett. B258 (1991), 141–144.
[Kna88] Anthony W. Knapp, *Lie groups, lie algebras, and cohomology*, Princeton University Press, 1988.

[Maf09] Carlos R. Mafra, *Superstring Scattering Amplitudes with the Pure Spinor Formalism*, arXiv/arXiv:0902.1552.

[Mik11a] Andrei Mikhailov, *Finite dimensional vertex*, JHEP 1112 (2011), 5, arXiv/1105.2231.

[Mik11b] , *Symmetries of massless vertex operators in AdS(5) x S**5*, Journal of Geometry and Physics (2011), arXiv/0903.5022.

[Mik12] , *Cornering the unphysical vertex*, JHEP 082 (2012), arXiv/1203.0677.

[Mik13] , *A generalization of the Lax pair for the pure spinor superstring in AdS5 x S5*, arXiv/1303.2090.

[MS04a] M. Movshev and Albert S. Schwarz, *Algebraic structure of Yang-Mills theory*, arXiv/hep-th/0404183.

[MS04b] , *On maximally supersymmetric Yang-Mills theories*, Nucl.Phys. B681 (2004), 324–350, arXiv/hep-th/0311132.

[MS09] , *Supersymmetric Deformations of Maximally Supersymmetric Gauge Theories*, arXiv/0910.0620.

[Nil81] Bengt E.W. Nilsson, *SIMPLE TEN-DIMENSIONAL SUPERGRAVITY IN SUPERSPACE*, Nucl.Phys. B188 (1981), 176.

[Sta] The Stacks Project Authors, *Stacks Project*, http://math.columbia.edu/algebraic_geometry/stacks-git

[Wit86] Edward Witten, *Twistor - Like Transform in Ten-Dimensions*, Nucl.Phys. B266 (1986), 245.