THEORIES OF BUNDLES WITH ADDITIONAL HOMOTOPY CONDITIONS

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Abstract. In the present paper we study bundles equipped with extra homotopy conditions, in particular so-called simplicial \( n \)-bundles. It is shown that (under some condition) the classifying space of 1-bundles is the double coset space of some finite dimensional Lie group. We also establish some relation between our bundles and \( C^* \)-algebras.

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Introduction. In the present paper\(^1\) we define a simplicial \( n \)-bundle over a space \( X \) as an object which actually “lives” on the product \( X \times \Delta_n \) of the space by the \( n \)-dimensional simplex whose vertices correspond to some vector bundles \( \xi_i \) over \( X \), \( \dim(\xi_i) = d_i, \ i = 0, \ldots, n \), edges correspond to \( \frac{n(n+1)}{2} \) homotopies between \( \xi_i \otimes [d_j] \) and \( [d_i] \otimes \xi_j, \ i \neq j \) (where \([m]\) denotes a trivial \( \mathbb{C}^m \)-bundle), two-dimensional faces correspond to homotopies between homotopies etc., up to higher cell which corresponds to a homotopy of “\( n \)-th degree”. The corresponding homotopy functor is representative and it is not difficult to give an explicit description of its classifying space. The corresponding definitions are given in Subsection 1.1.

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In Subsection 1.2 of the present paper we study the case \( n = 1 \) more detailed. Assume that positive integers \( k = d_0 \) and \( l = d_1 \) are relatively prime and that the structure groups of related bundles are reducible to the corresponding special linear groups. We show that the classifying space \( B\text{SU}_k \) of the corresponding 1-bundles has the homotopy type of a finite CW-complex. More precisely, we show that the homotopy fibre product

\[
B\text{SU}(k) \times_{B\text{SU}(kl)} B\text{SU}(l)
\]

defined by maps \( B\text{SU}(k) \to B\text{SU}(kl) \) and \( B\text{SU}(l) \to B\text{SU}(kl) \) which are induced by homomorphisms \( \text{SU}(k) \to \text{SU}(kl) \), \( A \mapsto A \otimes E_l \) and \( \text{SU}(l) \to \text{SU}(kl) \), \( B \mapsto E_k \otimes B \) (\( E_m \) denotes the unit \( m \times m \)-matrix, and the symbol “\( \otimes \)” here denotes the Kronecker product of matrices) for \( (k, l) = 1 \) is homotopy equivalent to the double coset space

\[
\text{Gr}_k^l := ((\text{SU}(k) \otimes E_l) \setminus \text{SU}(kl))/(E_k \otimes \text{SU}(l)).
\]

From the other hand this fibre product is precisely \( B\text{SU}_k \). We also give a sketch of the explicit description of classifying spaces for 2-bundles.

In Subsection 1.3 we establish some relation of considered kind of bundles to \( C^* \)-algebras.

Using the previous results, in Subsection 1.4 we propose a triangulated model for \( B\text{PU} \). More precisely, we define it as the geometric realization of the triangulated space related to simplicial \( n \)-bundles.

In Subsection 1.5 we propose the way to simplify the proof that \( B\text{SU}_\otimes \) is an infinite loop space using some \( C^* \)-algebras \( A_k^l \) (defined in Subsection 1.3).

In Subsection 1.6 we briefly discuss some objects which can be obtained by gluing together vector bundles of different dimensions over different elements of an open covering of some manifold, where in order to glue the bundles over \( n \)-fold overlapping we use the structure of a simplicial \( n \)-bundle (cf. [8]). Probably, such objects closely related to the theory of nonabelian bundle gerbes. In place of “usual” principal bundles in the nonabelian setting we should use so-called bibundles, which are simultaneously left and right principal bundles [1]. In the present paper similar objects appear naturally. It seems that the idea of the simplicial \( n \)-bundles fits in the context of so-called “descent data” and the theory of bundle gerbes [6].

In Subsection 2.1 we show that so-called “Matrix Grassmannian” is a classifying space of some topological group which is a group of paths in a Lie group that satisfy some boundary conditions (in the sense that they have origins and endpoints in prescribed subgroups). Using this fact we show in Subsection 2.2 that the existence of an embedding of a given bundle into a trivial one is equivalent to the reducibility of the structure group of the bundle to some “subgroup” (in the homotopic sense).

Finally, in Section 3 we propose some (hypothetical) application of the established relation between spaces and \( C^* \)-algebras related to multiplier algebras.

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1. Simplicial n-bundles

1.1. Main definitions. By k, l, m we shall denote positive integers greater than 1. Sub-
script k in the notation of a vector bundle ξ_k indicates its dimension. We shall consider
only complex vector bundles.

Let us introduce further notation. By (1)

\[ \theta^l_k : BU(k) \to BU(kl), \quad \theta^k_l : BU(l) \to BU(kl) \]

denote the maps of classifying spaces induced by the group homomorphisms U(k) \to U(kl), \ A \mapsto A \otimes E_l, \ A \in U(k) \) and U(l) \to U(kl), \ B \mapsto E_k \otimes B, \ B \in U(l) \) respectively,

where \( E_n \) is the unit \( n \times n \)-matrix, and the symbol \( \otimes \) here denotes the Kronecker product
of matrices.

Some more complicated maps will also be needed for us, for instance (2)

\[ \theta^l_k m : BU(km) \to BU(klm) \]

which is induced by the group homomorphism U(km) \to U(klm) corresponding to the
homomorphism of algebras

(3)

\[ M_{km}(\mathbb{C}) = M_k(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_{klm}(\mathbb{C}), \]

\[ A \otimes B \mapsto A \otimes E_l \otimes B \] for elementary tensors,

where \( A \in M_k(\mathbb{C}), \ B \in M_m(\mathbb{C}) \). Note that the image of this algebra homomorphism is
exactly the centralizer of the subalgebra \( \mathbb{C}E_k \otimes M_m(\mathbb{C}) \) \( \mathbb{C}E_l \subset M_{klm}(\mathbb{C}) \).

Definition 1. A homotopy \( h : \zeta_0 \simeq \zeta_1 \) between two bundles \( \zeta_0, \zeta_1 \) over \( X \) with the same fibre is a bundle \( Z \) over \( X \times I \) (\( I = [0, 1] \)) with the same fibre such that \( Z \mid_{X \times \{i\}} = \zeta_i, \ i = 0, 1 \).

Definition 2. A (simplicial) 0-bundle over \( X \) is a “usual” vector bundle \( \xi_k \). A 1-bundle
over \( X \) is a triple \( \{\xi_k, \xi_l, t_{k,l}\} \) consisting of a couple of vector bundles \( \xi_k, \xi_l \) and a homotopy \( t_{k,l} : \xi_k \otimes [l] \simeq [k] \otimes \xi_l \), i.e. in fact a triple \( \{\psi_k, \psi_l, h_{k,l}\} \) consisting of classifying maps \( \psi_k : X \to BU(k), \psi_l : X \to BU(l) \) for \( \xi_k, \xi_l \) and a homotopy \( h_{k,l} : X \times \Delta_1 \to BU(kl) \) such that

\[ h_{k,l} \mid_{X \times \{0\}} = \theta^l_k \circ \psi_k, \quad h_{k,l} \mid_{X \times \{1\}} = \theta^k_l \circ \psi_l, \]

where by \( \Delta_1 \) we denote a 1-simplex \( 0 \longrightarrow 1 \) with vertices 0, 1 corresponding to \( k, l \) respectively.

Further, two 1-bundles \( \{\xi_k, \xi_l, t_{k,l}\} \) and \( \{\eta_k, \eta_l, t'_{k,l}\} \) are said to be equivalent, if \( \xi_k \cong \eta_k, \xi_l \cong \eta_l \) and \( t_{k,l} \simeq t'_{k,l} \).
Before giving the definition of a 2-bundle let us notice that we can take the space $\Omega^{BU(l)}_{BU(k)}(BU(kl))$ of paths in $BU(kl)$ with origins in the subspace $\theta_k^l(BU(k)) \subset BU(kl)$ and endpoints in the subspace $\theta^k_l(BU(l)) \subset BU(kl)$ as a classifying space $BU(l)^3$ for 1-bundles of the form $\{\xi_k, \xi_l, t_{k,l}\}$. We shall show (see Proposition 8) that if $(k, l) = 1$ then the analogous space $BSU_k^l$ is the double coset space $Gr_{kl}^k = (SU(k) \otimes E_l) \setminus SU(kl)/(E_k \otimes SU(l))$.

**Definition 3.** A 2-bundle over $X$ is a collection of data consisting of bundles $\xi_k, \xi_l, \xi_m$ over $X$, homotopies between bundles

$$t_{k,l}: \xi_k \otimes [l] \simeq [k] \otimes \xi_l, \quad t_{l,m}: \xi_l \otimes [m] \simeq [l] \otimes \xi_m, \quad t_{k,m}: \xi_k \otimes [m] \simeq [k] \otimes \xi_m$$

and one more homotopy between the composition

$$\xi_k \otimes [l] \otimes [m] \xrightarrow{t_{k,l} \otimes \text{id}_m} [k] \otimes \xi_l \otimes [m] \xrightarrow{\text{id}_l \otimes t_{l,m}} [k] \otimes [l] \otimes \xi_m$$

and the composition

$$\xi_k \otimes [l] \otimes [m] \xrightarrow{\text{id}_k \otimes \tau_{l,m}} \xi_k \otimes [m] \otimes [l] \xrightarrow{t_{k,m} \otimes \text{id}_l} [k] \otimes \xi_m \otimes [l] \xrightarrow{\text{id}_k \otimes \tau_{m,l}} [k] \otimes [l] \otimes \xi_m,$$

where $\tau_{l,m}, \tau_{m,l}$ are the canonical isomorphisms induced by interchangings of tensor multipliers.

For a triple of positive integers $k, l, m$ by $\Delta_2$ denote 2-simplex

$$\begin{array}{ccc}
{k} & \to & {l} \\
\downarrow & & \downarrow \\
{l} & \to & {m}
\end{array}$$

From the homotopy point of view a 2-bundle is a collection of data consisting of maps

$$\psi_k: X \to BU(k), \quad \psi_l: X \to BU(l), \quad \psi_m: X \to BU(m)$$

(which are classifying maps for $\xi_k, \xi_l, \xi_m$), maps

$$h_{k,l}: X \times \Delta^{(0)}_1 \to BU(kl), \quad h_{l,m}: X \times \Delta^{(1)}_1 \to BU(lm), \quad h_{k,m}: X \times \Delta^{(2)}_1 \to BU(km)$$

(where $\Delta^{(0)}_1, \Delta^{(1)}_1, \Delta^{(2)}_1$ are the faces of simplex $\Delta_2$ with vertices $(k, l), (l, m), (k, m)$ respectively) such that

$$h_{k,l} |_{X \times \{k\}} = \theta_k^l \circ \psi_k, \quad h_{k,l} |_{X \times \{l\}} = \theta_l^k \circ \psi_l$$

$$h_{l,m} |_{X \times \{l\}} = \theta_l^m \circ \psi_l, \quad h_{l,m} |_{X \times \{m\}} = \theta_l^m \circ \psi_m$$

$$h_{k,m} |_{X \times \{k\}} = \theta_k^m \circ \psi_k, \quad h_{k,m} |_{X \times \{m\}} = \theta_k^m \circ \psi_m$$

and a map $h_{k,l,m}: X \times \Delta_2 \to BU(klm)$ such that

$$h_{k,l,m} |_{X \times \Delta^{(0)}_1} = \theta_{kl}^m \circ h_{k,l}, \quad h_{k,l,m} |_{X \times \Delta^{(1)}_1} = \theta_{kl}^m \circ h_{l,m}, \quad h_{k,l,m} |_{X \times \Delta^{(2)}_1} = \theta_{kl}^m \circ h_{k,m}.$$

\footnote{In fact this space is a classifying space of the group of paths $U_k^l := \Omega^{BU(l)}_{BU(k)}(U(kl))$, whence the notation
In order to describe the classifying space $BU_{k, l, m}$ of 2-bundles corresponding to the triple $k, l, m$ consider the following commutative diagram:

\[
\begin{array}{ccc}
BU(l) & \xrightarrow{\theta^l} & BU(kl) \\
\downarrow{\theta^l} & & \downarrow{\theta^l_m} \\
BU(k) & \xrightarrow{\theta^m} & BU(km)
\end{array}
\]

\[
\begin{array}{ccc}
BU(kl) & \xrightarrow{\theta^l} & BU(klm) \\
\downarrow{\theta^l_m} & & \downarrow{\theta^l_m} \\
BU(k) & \xrightarrow{\theta^m} & BU(km)
\end{array}
\]

The space $BU_{k, l, m}$ can be described as the space (with respect to the compact-open topology) of (continuous) maps

\[\Phi: \Delta_2 \to BU(klm)\]

of simplex (4) to $BU(klm)$ such that for vertices we have $\Phi(\{k\}) \in BU(k)$, $\Phi(\{l\}) \in BU(l)$, $\Phi(\{m\}) \in BU(m)$ and for edges we have $\Phi(\Delta^0_1) \subset BU(kl)$, $\Phi(\Delta^1_1) \subset BU(lm)$, $\Phi(\Delta^2_1) \subset BU(km)$, where $BU(r)$, $BU(rs)$ are identified (with the help of $\theta$'s) with the corresponding subspaces in $BU(klm)$.

The definition of a simplicial $n$-bundle for an arbitrary finite $n$ should be clear now. It seems that $n$-bundles over $X$ define simplicial objects of an appropriate category. For example, the face maps correspond to the arrows in the following commutative diagram (cf. (5))

\[
\begin{array}{ccc}
BU(l) & \xrightarrow{\theta^l} & BU_k \\
\downarrow{\theta^l_m} & & \downarrow{\theta^l_m} \\
BU_k l_m & \xrightarrow{\theta^m} & BU^m
\end{array}
\]

\[
\begin{array}{ccc}
BU^l & \xrightarrow{\theta^l} & BU^l_m \\
\downarrow{\theta^l_m} & & \downarrow{\theta^l_m} \\
BU^l_k & \xrightarrow{\theta^m} & BU(m)
\end{array}
\]

in which every (sub)simplex is a fiber product in the homotopy category. However, we shall not study general properties of $n$-bundles in this paper, instead of this in the next section we shall concentrate mainly on the particular case of 1-bundles.

1.2. Classifying spaces of simplicial 1 and 2-bundles.

**Lemma 4.** Let $G$ be a group, $K, L \subset G$ its subgroups. The left action of $K$ on the homogeneous space $G/L$ is free $\iff K \cap (\cup_{g \in G} gLg^{-1}) = \{e\}$.

**Proof.** For the stabilizer $St$ of a coset $gL$ we have $St(gL) = gLg^{-1}$. □

**Lemma 5.** Take $U(kl)$, $U(k) \otimes E_l$, $E_k \otimes U(l) \subset U(kl)$ in place of $G, K, L$ respectively. Then

\[K \cap (\cup_{g \in G} gLg^{-1}) = \{\lambda E_{kl}\}, \lambda \in \mathbb{C}, \|\lambda\| = 1 \iff (k, l) = 1.\]
Proof. Assume that \((k, l) = 1\), then \(\forall g \in G\) we have \(K \cap gLg^{-1} = \{\lambda E_{kl}\}\). Indeed, every unitary matrix can be diagonalized in some basis, besides every eigenvalue of a matrix \(A \in K\) has multiplicity dividing by \(l\), and every eigenvalue of a matrix \(B \in gLg^{-1}\) has multiplicity dividing by \(k\). Hence every element from the intersection \(K \cap gLg^{-1}\) is actually a scalar matrix. Now the converse assertion is clear. □

Corollary 6. If \((k, l) = 1\), then the left action of \(SU(k) \otimes E_l\) on the left coset space \(SU(kl)/(E_k \otimes SU(l))\) is free, and analogously, the right action of the group \(E_k \otimes SU(l)\) on the right coset space \((SU(k) \otimes E_l) \setminus SU(kl)\) is free.

Below we shall assume that the numbers \(k, l\) are relatively prime unless otherwise stated. Note that this condition has already appeared in the similar situations (e.g. [3], [4]).

Put

\[ Gr_k^l := ((SU(k) \otimes E_l) \setminus SU(kl))/E_k \otimes SU(l) \]

(actually, the arrangement of brackets is not important). Further, by \(\times\) denote the fiber product in the homotopy category.

By analogy with (1), define the maps

\[ \vartheta_k^l : BSU(k) \longrightarrow BSU(kl) \quad \vartheta^k_l : BSU(l) \longrightarrow BSU(kl). \]

Theorem 7. There is a homotopy equivalence \(Gr_k^l \simeq BSU(k) \times_{BSU(kl)} BSU(l)\), i.e. for some maps \(\varphi_k, \varphi_l\) (defined uniquely up to homotopy) the square

\[
\begin{array}{ccc}
Gr_k^l & \xrightarrow{\varphi_k} & BSU(k) \\
\downarrow & & \downarrow \vartheta_k^l \\
BSU(kl) & \xleftarrow{\vartheta^k_l} & BSU(l)
\end{array}
\]

is Cartesian in the homotopy category.

Proof. By \(ESU(n)\) denote the total space of the universal principal \(SU(n)\)-bundle. Consider the Cartesian square

\[
\begin{array}{ccc}
ESU(k) \times_{SU(k)} SU(kl) & \xrightarrow{h} & ESU(kl) \\
\downarrow & & \downarrow \\
BSU(k) & \xleftarrow{\vartheta_k^l} & BSU(kl)
\end{array}
\]

(8)
where $ESU(k) \times_{SU(k)} SU(kl)$ is an $SU(kl)$-bundle associated with the universal $SU(k)$-bundle with respect to the action of $SU(k) = SU(k) \otimes E_l \subset SU(kl)$ on $SU(kl)$ by the left translations. The space $ESU(k) \times_{SU(k)} SU(kl)$ is homotopy equivalent to the right coset space $(SU(k) \otimes E_l) \backslash SU(kl)$. Indeed, the map

$$ESU(k) \times_{SU(k)} SU(kl) \to (SU(k) \otimes E_l) \backslash SU(kl), \quad [e, g] \mapsto [g],$$

where

$$[e, g] \in ESU(k) \times_{SU(k)} SU(kl)$$

denotes the equivalence class

$$\{(e, g) \mid (e, g) \sim (e\alpha^{-1}, \alpha g), \ e \in ESU(k), \ g \in SU(kl), \ \alpha \in SU(k) = SU(k) \otimes E_l \subset SU(kl)\},$$

and $[g]$ denotes the right coset $(SU(k) \otimes E_l) \cdot g$ is well defined and its fibre $ESU(k)$ is contractible. Using Corollary 6 we obtain that the factorization of the upper row of diagram (8) by the free right action of the group $E_k \otimes SU(l)$ gives a diagram which is a Cartesian square equivalent to (7). □

Now let us give a homotopy-theoretic description of the obtained result. In the first place, consider diagram (8). From the viewpoint of homotopy theory a universal principal bundle is nothing but a path fibration. Therefore the space $ESU(kl)$ is the space of paths in $BSU(kl)$ whose endpoints coincide with a base point $\ast \in BSU(kl)$, and the projection sends a path to its origin. The map

$$(9) \quad BSU(k) \xrightarrow{\varphi_{kl}} BSU(kl)$$

can be considered as an embedding. Thus, the total space $ESU(k) \times_{SU(k)} SU(kl)$ of the induced bundle can be considered as the set of pairs consisting of a point in $BSU(k) \subset BSU(kl)$ and a path in $BSU(kl)$ which joins the point and the base point $\ast \in BSU(kl)$. Therefore the map

$$X \to ESU(k) \times_{SU(k)} SU(kl)$$

is a pair consisting of a map $X \to BSU(k)$ and a null-homotopy of its composition with map (9) to the base point. Equivalently, in terms of bundles it is an $SU(k)$-bundle over $X$ together with a trivialization of the $SU(kl)$-bundle obtained from the initial one by the extension of the structure group corresponding to the group homomorphism $SU(k) \to SU(kl), \ A \mapsto A \otimes E_l$.

**Proposition 8.** A map $X \to \text{Gr}_k^l$ (see diagram (7)) is a triple $\{\xi_k, \xi_l, t_{k,l}\}$ consisting of vector $C^k$ and $C^l$-bundles $\xi_k, \xi_l$ with structure groups $SU(k)$ and $SU(l)$ respectively and a homotopy $t_{k,l}: \xi_k \otimes [l] \simeq [k] \otimes \xi_l$ (cf. the definition of 1-bundles in Definition 3).
Proof. We apply the previous arguments to the diagram obtained by factorization of the upper row of diagram (5) by the free right action of the group $E_k \otimes SU(l)$. Then we replace it by the equivalent diagram of path fibrations. For subspaces $K, L \subset M$ by $\Omega^l_K(M)$ we denote the space of paths in $M$ with origins in $K$ and endpoints in $L$. It is easy to see that the right column of our diagram

$$\text{ESU}(kl)/(E_k \otimes SU(l)) \to \text{BSU}(kl)$$

is equivalent to the fibration

$$\Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl)) \to \text{BSU}(kl)$$

which sends a path to its origin, with fibre $\Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl))$ homotopy equivalent to the homogeneous space $SU(kl)/(E_k \otimes SU(l))$. Another obvious fibration

$$\Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl)) \to \Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl))$$

sending a path to its endpoint is a homotopy equivalence because its fibre $\Omega^*_l(\text{BSU}(kl))$ is contractible. Therefore the embedding

$$\Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl)) \to \Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl))$$

can be replaced by the homotopy equivalent projection

$$\Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl)) \to \text{BSU}(l)$$

which sends a path to its endpoint.

Thus we obtain an interpretation of the fibration

$$\Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl)) \to \text{BSU}(k)$$

induced from (10) by map (9), and the corresponding Cartesian square

$$\begin{array}{ccc}
\Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl)) & \to & \text{BSU}(l) \\
\downarrow & & \downarrow \sigma^l_k \\
\text{BSU}(k) & \to & \text{BSU}(kl)
\end{array}$$

which is equivalent to (7). In particular, we see that a map

$$X \to \Omega^l_{\text{BSU}(kl)}(\text{BSU}(kl))$$

is a triple consisting of maps

$$X \xrightarrow{\psi_k} \text{BSU}(k), \quad X \xrightarrow{\psi_l} \text{BSU}(l)$$

and a homotopy connecting $\vartheta^k_l \circ \psi_k$ and $\vartheta^l_k \circ \psi_l$. This completes the proof. □

Note that there is the obvious map $\text{Gr}^l_k \to \text{Gr}^l_l$ corresponding to the interchanging of factors of the fibre product (see diagram (7)) or equivalently to the inversion of path’s direction.
Now let us describe an explicit construction of the classifying space for 2-bundles in case when numbers \( k, l \) and \( m \) are pairwise relatively prime. It was asserted after diagram (5) that the space \( \text{BSU}_k^l m \) can be described as the space of continuous maps

\[ \Phi : \Delta_2 \to \text{BSU}(klm) \]

from 2-simplex \( \# \) to \( \text{BSU}(klm) \) such that for vertices we have the “boundary” conditions \( \Phi(\{k\}) \in \text{SU}(k), \ \Phi(\{l\}) \in \text{BU}(l), \ \Phi(\{m\}) \in \text{BSU}(m) \) and for edges the conditions \( \Phi(\Delta^0_1) \subset \text{SU}(kl), \ \Phi(\Delta^{(1)}_1) \subset \text{BSU}(lm), \ \Phi(\Delta^{(2)}_1) \subset \text{BSU}(km) \).

Obviously, \( \text{BSU}_k^l m \) is the classifying space for the topological group \( \text{SU}_k^l m \) consisting of maps from 2-simplex \( \# \) to \( \text{SU}(klm) \) satisfying the analogous “boundary” conditions. Besides, we set

\[ (11) \quad \text{TSU}_k^l m := \left\{ \Psi : \partial \Delta_2 \to \text{SU}(klm) \mid \Psi(\{k\}) \in \text{SU}(k), \ldots, \Psi(\Delta^0_1) \subset \text{SU}(kl), \ldots \right\}, \]

where \( \partial \Delta_2 \) denotes the boundary of 2-simplex \( \# \). We obtain the exact sequence of groups

\[ (12) \quad \Omega^2 \text{SU}(klm) \to \text{SU}_k^l m \to \text{TSU}_k^l m, \]

where \( \Omega^2 \) denotes the twofold loop space (we consider the identity element as a basepoint in \( \text{SU}(klm) \)), and the last map is induced by assigning to a map of \( \Delta_2 \) its restriction to the boundary \( \partial \Delta_2 \). There is a sequence of classifying spaces

\[ \Omega \text{SU}(klm) \to \text{BSU}_k^l m \to \text{BTSU}_k^l m \]

corresponding to the exact sequence of groups.

First of all let us describe the space \( \text{BTSU}_k^l m \). Gluing blocks of the form \( \text{ESU}(k) \times \text{SU}(kl) \times \text{ESU}(l) \), we obtain the following space

\[ (13) \quad \begin{array}{c}
\text{SU}(kl) \\
\times \text{ESU}(k) \\
\times \text{SU}(lm) \\
\times \text{ESU}(l)
\end{array}
\]

\[ \times \begin{array}{c}
\text{ESU}(m) \\
\times \text{SU}(kl) \\
\times \text{SU}(lm) \\
\times \text{ESU}(m)
\end{array}
\]

**Proposition 9.** The space \( \text{BTSU}_k^l m \) is homotopy equivalent to (13).

**Proof.** Note that space (13) is an \( \text{SU}(kl) \times \text{SU}(lm) \times \text{SU}(km) \)-fibration over \( \text{BSU}(k) \times \text{BSU}(l) \times \text{BSU}(m) \). The corresponding projection takes a map \( \Phi : \partial \Delta_2 \to \text{BSU}(klm) \) satisfying conditions as in (11) to the collection of its values in vertices \( \{\Phi(\{k\}), \ \Phi(\{l\}), \ \Phi(\{m\})\} \in \text{BSU}(k) \times \text{BSU}(l) \times \text{BSU}(m) \). The fibre can be identified with the loop space of \( \text{SU}(kl) \times \text{SU}(lm) \times \text{SU}(km) \) i.e. with \( \text{SU}(kl) \times \text{SU}(lm) \times \text{SU}(km) \). □
Remark 10. Let us remark that space (13) is also an $SU(k) \times SU(l) \times SU(m)$-fibration over $Gr^l_k \times Gr^m_l \times Gr^m_k$. The corresponding projection takes a map $\Phi: \partial \Delta_2 \to BSU(klm)$ satisfying conditions as in (11) to the collection of its values on edges \( \{\Phi(\Delta_1^{(0)}), \Phi(\Delta_1^{(1)}), \Phi(\Delta_1^{(2)})\} \subset Gr^l_k \times Gr^m_l \times Gr^m_k \). The corresponding fibre is $SU(k) \times SU(l) \times SU(m)$.

According to Corollary 6 (recall that the numbers $k, l, m$ are assumed to be pairwise relatively prime), space (13) can be replaced by the following finite dimensional quotient space:

\[
SU(mk) \times SU(kl) \times SU(lm).
\]

The required space $BSU^{l_m}_{kl}$ is the total space of some $\Omega(SU(klm))$-fibration over (14) (probably, (14) can be mapped to $SU(klm)$ and $BSU^{l_m}_{kl}$ is induced from the path fibration over $SU(klm)$).

1.3. A relation to $C^*$-algebras. Now we want to establish some relation to $C^*$-algebras. By $C[0,1]$ denote the $C^*$-algebra of continuous complex-valued functions on the segment $[0,1]$. Consider the norm-closed subalgebra $A^l_k$ in $M_{kl}(C[0,1])$ defined as follows:

\[
A^l_k := \{ f \in M_{kl}(C[0,1]) \mid f(0) \in M_k(\mathbb{C}) \otimes CE_l, f(1) \in \mathbb{C}E_k \otimes M_l(\mathbb{C}) \}.
\]

The group $SU^l_k$ acts continuously on $A^l_k$ by conjugations. We have the algebra homomorphisms

\[
\pi_k: A^l_k \to M_k(\mathbb{C}), \quad \pi_l: A^l_k \to M_l(\mathbb{C})
\]

defined as evaluation maps for matrix-valued functions at the points 0 and 1 respectively. By $I^l_k := \ker(\pi_k)$ and $I^l_l := \ker(\pi_l)$ denote their kernels. Bundles classified by the space $BSU^l_k \simeq Gr^l_k$ can naturally be considered as bundles with fibre $A^l_k$ (and the structure group $SU^l_k$). Then, for example, the maps $\varphi_k: Gr^l_k \to BSU(k)$ and $\varphi_l: Gr^l_k \to BSU(l)$ (see square (7)) can be regarded as maps of classifying spaces corresponding to homomorphisms of fibres $\pi_k$ and $\pi_l$, and ideals $I^l_k \subset A^l_k$, $I^l_l \subset A^l_k$ can respectively be regarded as fibres of the corresponding bundles over $Fr_{k,l}$ and $Fr_{l,k}$ (the last spaces are the homotopy fibres of maps $\varphi_k$ and $\varphi_l$, see the next section).

For general $n$-bundles one can also define the corresponding $C^*$-algebras (such as

\[
A^{l_m}_{kl} := \{ f: \Delta_2 \to M_{klm}(\mathbb{C}) \mid f(k) \in M_k(\mathbb{C}) \otimes CE_{lm}, \ldots; f(\Delta_1^{(0)}) \in M_{kl}(\mathbb{C}) \otimes CE_m, \ldots \}
\]

for 2-bundles, etc.).

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*I am grateful to Ralph Meyer who pointed out this relation in a discussion after my talk in Göttingen.*
THEORIES OF BUNDLES WITH ADDITIONAL HOMOTOPY CONDITIONS

Remark 11. It seems interesting to study the exact sequence of $K$-functors corresponding to the exact coefficient sequence of $C^*$-algebras $I_0 \rightarrow A_k \rightarrow M_k(C)$. It may have relation to the coefficient sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/k\mathbb{Z} \rightarrow 0$ (where $A_k^i$ corresponds to the “left” $\mathbb{Z}$).

Now we want to interpret diagram (6) in terms of the introduced $C^*$-algebras. To this purpose introduce new $C^*$-algebras

$$A_k^{kl} := \{ f \in M_{kl}(C[0,1]) \mid f(0) \in M_k(C) \otimes \mathbb{C} E_i \},$$

$$A_k^I := \{ f \in M_k(C[0,1]) \mid f(1) \in \mathbb{C} E_k \otimes M_l(C) \}.$$

The claimed interpretation follows from the following facts: 1) $A_k^{kl} \simeq M_k(C)$, $A_k^I \simeq M_l(C)$ (and the embeddings $A_k^I \hookrightarrow A_k^{kl}$, $A_k^I \hookrightarrow A_k^I$ correspond to the epimorphisms $\pi_k : A_k^I \rightarrow M_k(C)$, $\pi_l : A_k^I \rightarrow M_l(C)$ under this equivalences) and 2) $A_k^I = A_k^{kl} \cap A_k^I$ (the intersection in $M_{kl}(C[0,1]) \simeq M_k(C)$).

1.4. A triangulated model for $\text{BSU}$. The spaces of the form $\text{BSU}(k_0, \ldots, k_m)$, etc. can be used to construct a “geometric realization” of the corresponding triangulated space (cf. diagram (6)). It allows one to deal with homotopies naturally related to simplicial $n$-bundles.

Let $\Delta_n := \{ x \in \mathbb{R}^{n+1} \mid x_0 + \ldots + x_n = 1, x_i \geq 0 \}$ be the “standard” $n$-simplex, $[n] := \{ 0, \ldots, n \}$, $I \subset [n]$, $\delta_i : \Delta_{|I|} \hookrightarrow \Delta_n$ the natural inclusion of the $i$th face. For any finite ordered set $k_0, \ldots, k_n$ consisting of integers greater than 1 we define the space of maps (=functions) with the corresponding “boundary conditions”:

$$\text{BSU}(k_0, \ldots, k_n) := \{ f : \Delta_n \rightarrow \text{BSU}(k_0 \ldots k_n) \mid (f \circ \delta_i)(\Delta_{|I|}) \subset \text{BSU}(\prod_{i \in I} k_i) \forall I \subset [n] \}$$

(the embeddings $\text{BSU}(\prod_{i \in I} k_i) \subset \text{BSU}(\prod_{i = 0}^n k_i)$ generalize (6) and (2)). For instance, $\text{BSU}(k_0, k_1)$ is the paths space $\Omega^\text{BSU}(k_1)_{\text{BSU}(k_0)}(\text{BSU}(k_0 k_1)) = \text{BSU}_{k_0}^{k_1}$, $\text{BSU}(k_0, k_1, k_2) = \text{BSU}_{k_0 k_1} k_2$, etc.

More specific, one can consider the category $\mathcal{T}_r$ whose objects are simplexes whose vertices are labeled by integers (greater than 1) and whose morphisms are increasing maps preserving labels (i.e. “face maps”). We want to define the contravariant functor (denoted by $\text{BSU}$) from $\mathcal{T}_r$ to the category of topological spaces such that $\text{BSU}(\Delta_n(k_0, \ldots, k_n)) = \text{BSU}(k_0, \ldots, k_n)$. We also have the natural forgetful functor which forgets labels.

For any $i$, $0 \leq i \leq n$ define the $i$th face operator $d_i : \text{BSU}(k_0, \ldots, k_n) \rightarrow \text{BSU}(k_0, \ldots, \hat{k}_i, \ldots, k_n)$ by $f \mapsto f \circ \delta_i$ (we regard $i$ as a one-element subset in $[n]$).

Consider the geometric realization

$$|\text{BSU}| := \coprod_{k_0, \ldots, k_n} \text{BSU}(k_0, \ldots, k_n) \times \Delta_n/\sim,$$
where \( \sim \) denotes the equivalence relation generated by \((d, f, u) \sim (f, \delta_i(u)), f \in \text{BSU}(k_0, \ldots, k_n), u \in \Delta_{n-1}\) labeled by integers \(k_0, \ldots, k_n\) (where \(\hat{k}_i\) means that \(k_i\) is omitted).

**Remark 12.** There are obvious modification of our construction for unitary or projective unitary groups.

**Remark 13.** As follows from the previous results, it is natural to consider the following subspaces in \(|\text{BSU}|\). Fix \(q \geq 1\) and define the full subcategory \(\mathcal{T}r' \subset \mathcal{T}r\) consisting of simplexes \(\Delta_n, n \geq q\) labeled by all sets of integers \(k_0, \ldots, k_n\) that are pairwise relatively prime. The corresponding subspaces in \(|\text{BSU}|\) allows us to avoid the localization.

**Remark 14.** In the similar way one can define the functor from \(\mathcal{T}r\) to the category of unital \(\mathbb{C}\)-algebras which for example takes 2-simplex \(\Delta_2\) labeled by \(k_0, k_1, k_2\) to \(A_{k_0}^{k_1}k_2\), (see (16)) etc. It is a “fiber” of the universal bundle over \(|\text{BPU}|\).

1.5. **A remark about G. Segal’s proof that \(\text{BU}_\otimes\) is an infinite loop space.** Using the concept of \(\Gamma\)-space, G. Segal in [7] proved that various classifying spaces, in particular \(\text{BU}_\otimes\) (it is the space \(\text{BU}\) with the tensor-product composition law), are infinite loop spaces. (To be precise, the paper explicitly dealt with \(\text{BO}_\otimes\) case). But the proof in this case is more complicated than for example for \(\text{BU}_\otimes\). It seems that using the algebras \(A_{k}^l\) and the corresponding groups \(\text{PU}_k^l\) (for \((k, l) = 1\) we can reduce this case to the “common” one (such as \(\text{BU}_\otimes\) etc.) and hence to simplify the proof.

More precisely, fix such a pair \(\{k, l\}\) and put \(G_n := \text{PU}_k^l\) for \(n \in \mathbb{N}\). We see that for each nonnegative integer \(n\) we have the topological group \(G_n\) containing the symmetric group \(\Sigma_n\) (which acts on tensor factors \(A_{k}^l = A_{k}^1 \otimes \ldots \otimes A_{k}^l\) by permutations) and the family of homomorphisms \(G_m \times G_n \to G_{m+n}\) given by the tensor product \(A_{k}^m \otimes A_{k}^n \to A_{k}^{m+n}\).

Hence we can define the \(\Gamma\)-space \(A\) such that

\[
A(1) = \coprod_{n \geq 0} \text{BG}_n, \quad A(2) = \coprod_{m, n \geq 0} (\text{EG}_m \times \text{EG}_n \times \text{EG}_{m+n})/(G_m \times G_n),
\]

and so on (we use the notation from [7]).

**Remark 15.** Note that if we use just \(M_k(\mathbb{C})\) instead of \(A_{k}^n\) we get the localization of the classifying space at \(k\) (in the sense that \(k\) is invertible).

1.6. **A cocycle condition for \(n\)-bundles.** Now we wish to consider briefly some kind of objects which can be constructed by means of \(n\)-bundles.

Suppose \(M,\) say, a manifold, \(\{U_i\}_{i \in I}\) its (locally finite) open covering, \(d_i, i \in I\) a collection of positive integers greater than 1. Assume that for every \(U_i\) we are given a vector bundle \(\xi_i \to U_i,\) \(\dim(\xi_i) = d_i.\) Assume also that for every pairwise overlapping \(U_{ij} := U_i \cap U_j\) there is a homotopy \(t_{i,j} : \xi_i \otimes [d_j] \mid_{U_{ij}} \simeq [d_i] \otimes \xi_j \mid_{U_{ij}},\) i.e. actually a 1-bundle structure over \(U_{ij};\) for every triple overlapping \(U_{ijk}\) there is a 2-bundle structure over it, and so on. Note that if we are given such a structure up to \(n\)-fold overlapping, then its extension to \(n+1\)-fold overlapping can be regarded as a homotopy analog of the \(n\)-cocycle condition (cf. [8]). The relation to simplicial \(n\)-bundles becomes obvious if we consider
the nerve of the open covering. Moreover, if the covering consists of only one open set
$U_i = X \forall i \in I$, then we go back to the initial notion of an $n$-bundle.

One can ask the natural question: does this construction give us more general objects
than the usual vector bundles? In fact, if all of the higher cocycle conditions are satisfied,
the answer is negative.

Consider a very simple example: bundles over a sphere. Suppose $\xi_{kl} \to S^{2n}$ is an $kl$-
dimensional bundle over the sphere classified by the map $f: S^{2n} \to BSU(kl)$. Assume that
$n < \min\{k, l\}$. The restriction of $f$ to the upper (closed) hemisphere $U$ can be deformed
to a map into $\partial^k(BSU(k)) \subset BSU(kl)$, and the restriction to the down hemisphere $V$ into
$\partial^l(BSU(l)) \subset BSU(kl)$; thus $\xi_{kl}|_U = \xi_k \otimes [l]$, $\xi_{kl}|_V = [k] \otimes \xi_l$ for some (obviously trivial)
bundles $\xi_k$ over $U$ and $\xi_l$ over $V$ (see diagram (7)). It also follows from diagram (7) that
the equator $S^{2n-1} = U \cap V$ goes to $Gr^1_{kl}$. Thus we get some 1-bundle $\{\xi_k|_{U \cap V}, \xi_l|_{U \cap V}, t_k, t_l\}$
(where $t_{k,l}: \xi_k|_{U \cap V} \otimes [l] \simeq [k] \otimes \xi_l|_{U \cap V}$ is the homotopy naturally arising from our con-
struction) over $U \cap V$. By the way, the map of the equator $S^{2n-1} \to Gr^1_{kl}$ can be extended
to the whole sphere $S^{2n}$ (i.e. $f$ can be lifted to a map to $Gr^1_{kl}$) if and only if the homotopy
class of $[f] \in \pi_{2n}(BSU(kl)) \cong \mathbb{Z}$ in the homotopy group is divisible by $kl$ (recall that we
suppose $(k, l) = 1$). Indeed, in this case $\xi_{kl} \to S^{2n}$ can be represented both in the form
$\xi_k \otimes [l] \to S^{2n}$ and in the form $[k] \otimes \xi_l \to S^{2n}$.

Equivalently, we have an $A^l_k$-bundle $A^l_k \to U \cap V \cong S^{2n-1}$ over the equator such that
\[
\pi_k: A^l_k \to \text{End}(\xi_k)|_{U \cap V}, \quad \pi_l: A^l_k \to \text{End}(\xi_l)|_{U \cap V},
\]
where $\pi_k, \pi_l$ are maps of bundles corresponding to (15). Moreover, $A^l_k \to U \cap V$ can be
extended to an $A^l_k$-bundle $A^l_k \to S^{2n}$ over the whole sphere $S^{2n}$ (with conditions which
extend (17) to $S^{2n}$) if and only if $[f] \in \pi_{2n}(BSU(kl)) \cong \mathbb{Z}$ is divisible by $kl$.

### 2. Matrix Grassmannians

In this section we discuss a kind of bundles which is closely connected with 1-bundles.
Using an informal analogy, we can say that the passage from a 1-bundle to a new bundle
is similar to the passage from an $A, B$-bimodule $A M_B$ to a (left) $A \otimes B^n$-module $M$.

We studied this kind of bundles in papers [3] and [4]. In particular, we developed their
stable theory (which can be treated as a noncommutative analog of the Picard group).
The starting point there was the notion of a “Matrix Grassmannian” which is an analog
of the usual Grassmannian for the case of matrix algebras. The idea was to develop
the corresponding theory of bundles together with a natural stable equivalence relation
(which naturally arises under the passage to the direct limit of classifying spaces) starting
with Matrix Grassmannians as classifying spaces. It was noticed that the most interesting
theory corresponds to the case $(k, l) = 1$ (otherwise the localization occurs when we take
the direct limit). In this paper we are mainly interested in a nonstable theory. As an
application, we obtain an interpretation of “floating” bundles (which are actually not just...
matrix bundles but pairs consisting of such a bundle and its embedding into a trivial) as bundles with some structure groups.

2.1. **Matrix Grassmannians as spaces of the type BG.**

**Definition 16.** A *k*-subalgebra in a matrix algebra $M_n(\mathbb{C})$ is a $*$-subalgebra with a unit isomorphic to $M_k(\mathbb{C})$ (obviously, such a subalgebra exists only if $k \mid n$).

By $\text{PU}(k) \times \text{PU}(l)$ denote the subgroup in $\text{PU}(kl)$ which is the image of the embedding

\[(18) \quad \text{PU}(k) \times \text{PU}(l) \to \text{PU}(kl), \ (X, Y) \mapsto X \otimes Y\]

induced by the Kronecker product of matrices. By $\text{Gr}_{k,l}$ denote the homogeneous space

\[(19) \quad \text{PU}(kl)/(\text{PU}(k) \times \text{PU}(l)).\]

**Remark 17.** It follows from Noether-Skolem’s theorem that $\text{Gr}_{k,l}$ parametrizes the set of $k$-subalgebras in $M_{kl}(\mathbb{C})$, whence the title “Matrix Grassmannian”.

**Remark 18.** Note that the space of all (not necessarily $*$-) unital subalgebras in $M_{kl}(\mathbb{C})$ isomorphic to $M_k(\mathbb{C})$ is homotopy equivalent to $\text{Gr}_{k,l}$. Indeed, the projective unitary group $\text{PU}(n)$ is the deformation retract of the corresponding projective general linear group $\text{PGL}_n(\mathbb{C})$. We restrict ourselves to the case of $*$-subalgebras because we want to deal with compact spaces.

**Remark 19.** Note that for every $k$-subalgebra $A_k \subset M_{kl}(\mathbb{C})$ there is a unique corresponding $l$-subalgebra $B_l \subset M_{kl}(\mathbb{C})$ which is the centralizer of $A_k$ in $M_{kl}(\mathbb{C})$, moreover $M_{kl}(\mathbb{C}) = A_k \otimes B_l$. So the space $\text{Gr}_{k,l}$ also parametrizes the set of representations of the algebra $M_{kl}(\mathbb{C})$ in the form of the tensor product $A_k \otimes B_l$ of its $k$ and $l$-subalgebras. Note also the following easy fact: the space $\text{Gr}_{k,l}$ is the homotopy fibre of the map $\text{BPU}(k) \times \text{BPU}(l) \to \text{BPU}(kl)$ induced by the tensor product of bundles (or, equivalently, by the map of classifying spaces induced by the group homomorphism $\text{IS}$).

**Remark 20.** As above, we assume that numbers $k$, $l$ are relatively prime, unless otherwise stated (although some results below are true without this assumption). Homotopy consequences from the condition $(k, l) = 1$ (in particular, related to the passage to the direct limit) were studied in the previous papers (see for example $\text{IS}_1$, $\text{IS}_2$). In general, one can consider the conditions on pairs $\{k, l\}$ of the form $(k, l) = d$, where $d$ is a fixed positive integer, greater than 1 in general. The corresponding (equivalence classes of) bundles form a set equipped with the obvious action of (equivalence classes of) bundles satisfying the condition $(k, l) = 1$.

**Remark 21.** Note that if $(k, l) = 1$, the Matrix Grassmannian $\text{Gr}_{k,l}$ can also be represented as the homogeneous space of the special linear group $\text{SU}(kl)/(\text{SU}(k) \otimes \text{SU}(l))$. The point is that in this case the center $\mu_{kl} \cong \mu_k \times \mu_l$ of the group $\text{SU}(kl)$ is the product of centers of $\text{SU}(k)$ and $\text{SU}(l)$ ($\mu_n$ is the group of $n$th degree roots of unity). The same is true for $(\text{SU}(k) \otimes \mathbb{C}_l) \setminus \text{SU}(kl)/(\mathbb{C}_k \otimes \text{SU}(l))$ and $(\text{PU}(k) \otimes \mathbb{C}_l) \setminus \text{PU}(kl)/(\mathbb{C}_k \otimes \text{PU}(l))$.

Recall that for subspaces $K$, $L \subset M$ by $\Omega^L_K(M)$ we denote the space of paths in $M$ with origins in $K$ and endpoints in $L$. Identify $\text{BPU}(k) \times \text{BPU}(l)$ with the subspace in $\text{BPU}(kl)$ which is the image of the map of classifying spaces induced by $\text{IS}$. Let $\ast \in \text{BPU}(k) \times \text{BPU}(l) \subset \text{BPU}(kl)$ be a base point.
Proposition 22. $\text{Gr}_{k,l} \simeq \Omega^*_{\text{BPU}(k) \times \text{BPU}(l)} \text{BPU}(kl)$.

Proof. In the homotopy category consider the Cartesian square
\[
\begin{array}{ccc}
\Omega^*_{\text{BPU}(k) \times \text{BPU}(l)} \text{BPU}(kl) & \to & \ast \\
\downarrow & & \downarrow \\
\text{BPU}(k) \times \text{BPU}(l) & \subseteq & \text{BPU}(kl),
\end{array}
\]
so the space $\Omega^*_{\text{BPU}(k) \times \text{BPU}(l)} \text{BPU}(kl)$ is the homotopy fibre of the inclusion $\text{BPU}(k) \times \text{BPU}(l) \to \text{BPU}(kl)$. From the other hand, it is easy to see from the representation (19) of the Matrix Grassmannian as a homogeneous space that it is also the homotopy fibre of this map. □

Remark 23. Note that the Cartesian square (20) can be replaced by the equivalent square
\[
\begin{array}{ccc}
(\text{EPU}(k) \times \text{EPU}(l))_{\text{PU}(k) \otimes \text{PU}(l)} & \times & \text{PU}(kl) \\
\downarrow & & \downarrow \\
\text{BPU}(k) \times \text{BPU}(l) & \subseteq & \text{BPU}(kl)
\end{array}
\]
in which the vertical arrows are fibrations; it is easy to see that the total space
\[
(\text{EPU}(k) \times \text{EPU}(l))_{\text{PU}(k) \otimes \text{PU}(l)} \times \text{PU}(kl)
\]
is homotopy equivalent to $\text{Gr}_{k,l}$ (cf. the proof of Theorem 7).

Put $SU_{k,l} := \Omega^*_{\text{SU}(k) \otimes \text{SU}(l)} \text{SU}(kl)$, where $e \in \text{SU}(kl)$ is the identity element of the group (considered also as a base point). This is a topological group with respect to the pointwise multiplication of paths in the group $\text{SU}(kl)$.

Theorem 24. There is a homotopy equivalence $\text{Gr}_{k,l} \simeq \text{BSU}_{k,l}$.

Proof. Applying Milnor’s construction of a classifying space to the group of paths $\text{SU}_{k,l}$ we obtain the space $\Omega^*_{\text{BSU}(k) \times \text{BSU}(l)} \text{BSU}(kl)$ which according to Proposition 22 is homotopy equivalent to $\text{Gr}_{k,l}$. □

Corollary 25. The group $\text{SU}_{k,l}$ is equivalent to the loop space of the Matrix Grassmannian $\text{Gr}_{k,l}$ (as a group in the homotopy category).

Remark 26. The fact that the loop space $\Omega^*_{\text{BSU}(k) \times \text{BSU}(l)} \text{BSU}(kl)$ is $SU_{k,l} := \Omega^*_{\text{SU}(k) \otimes \text{SU}(l)} \text{SU}(kl)$ can be proved more directly. The base point in $\Omega^*_{\text{BSU}(k) \times \text{BSU}(l)} \text{BSU}(kl)$ is the constant path with origin and endpoint in the basepoint $* \in \text{BSU}(k) \times \text{BSU}(l) \subseteq \text{BSU}(kl)$. Due to the homotopy equivalence $\Omega \text{BSU}(n) \simeq \text{SU}(n)$ we see that a loop in $\Omega^*_{\text{BSU}(k) \times \text{BSU}(l)} \text{BSU}(kl)$ with origin and endpoint in the base point is the same thing as a path in $\text{SU}(kl)$ with origin in the subgroup $\text{SU}(k) \otimes \text{SU}(l)$ and endpoint in the identity element $e \in \text{SU}(kl)$. 

Now we define a tautological $M_k(\mathbb{C})$-bundle $\mathcal{A}_{k,l}$ over $\text{Gr}_{k,l}$ as follows. It is a subbundle of the product bundle $\text{Gr}_{k,l} \times M_{kl}(\mathbb{C})$ whose fibre $(\mathcal{A}_{k,l})_x$ over a point $x \in \text{Gr}_{k,l}$ is the $k$-subalgebra in $M_{kl}(\mathbb{C})$ corresponding to the point (see Remark 17). There is also an $M_l(\mathbb{C})$-bundle $\mathcal{B}_{k,l}$ over $\text{Gr}_{k,l}$ whose fibre over $x \in \text{Gr}_{k,l}$ is the centralizer of $k$-subalgebra $(\mathcal{A}_{k,l})_x \subset M_{kl}(\mathbb{C})$ (cf. Remark 19 above). Since $\forall x \in \text{Gr}_{k,l}$ fibres $(\mathcal{A}_{k,l})_x$ and $(\mathcal{B}_{k,l})_x$ are identified with the corresponding subalgebras in $M_{kl}(\mathbb{C})$, it follows that there is the canonical trivialization $\mathcal{A}_{k,l} \otimes \mathcal{B}_{k,l} \cong \text{Gr}_{k,l} \times M_{kl}(\mathbb{C})$. The trivialization can be regarded as a homotopy (see Definition 1)

\[(21) \quad H_{k,l}: \mathcal{A}_{k,l} \otimes \mathcal{B}_{k,l} \simeq \text{Gr}_{k,l} \times M_{kl}(\mathbb{C})\]

from the tensor product of bundles to the trivial bundle.

Consider a pair of $\text{SU}(k)$ and $\text{SU}(l)$-bundles $\mathcal{A}_k$, $\mathcal{B}_l$ (with fibres $M_k(\mathbb{C})$ and $M_l(\mathbb{C})$ respectively) over $X$ such that their tensor product bundle is trivial, together with a homotopy $h_{k,l}: \mathcal{A}_k \otimes \mathcal{B}_l \simeq X \times M_{kl}(\mathbb{C})$. Such collections $(\mathcal{A}_k, \mathcal{B}_l, h_{k,l})$ and $(\mathcal{A}_k', \mathcal{B}_l', h_{k,l}')$ are said to be equivalent if $A_k \cong A_k'$, $B_l \cong B_l'$ and $h_{k,l}$ is homotopic to $h_{k,l}'$.

From the previous results (see Proposition 22) one can easily deduce the following corollary.

**Corollary 27.** There is a natural bijection between the set of homotopy classes of maps $[X, \text{Gr}_{k,l}]$ and just introduced equivalence classes of collections $(\mathcal{A}_k, \mathcal{B}_l, h_{k,l})$, moreover, the triple $(\mathcal{A}_{k,l}, \mathcal{B}_{k,l}, H_{k,l})$ is a universal triple (for fixed $k, l$).

**Remark 28.** A map

$$\varphi: X \to \Omega^*_{\text{BSU}(k) \times \text{BSU}(l)} \text{BSU}(kl)$$

is the same thing as a map

$$\tilde{\varphi}: CX \to \text{BSU}(kl),$$

such that

$$\tilde{\varphi} |_{X \times \{0\}} \subset \text{BSU}(k) \times \text{BSU}(l) \subset \text{BSU}(kl), \quad \tilde{\varphi}(*) = *,$$

where $CX := (X \times [0, 1])/(X \times \{1\})$ is the cone of $X$. By $[x, t]$ denote a point of the cone $CX$ corresponding to $x \in X$, $t \in [0, 1]$. Then the explicit form of the mentioned correspondence is given by the formula $\tilde{\varphi}([x, t]) = \varphi(x)(t), \quad x \in X, \ t \in [0, 1]$. Moreover, there is a one-to-one correspondence between homotopy classes of maps $\varphi$ as above and homotopy classes of maps $\tilde{\varphi}$, where in the last case we consider homotopies preserving base points and in addition such that the image of the subspace $X \times \{0\} \subset CX$ remains inside the subspace $\text{BSU}(k) \times \text{BSU}(l) \subset \text{BSU}(kl)$ during a homotopy.

**Remark 29.** Clearly, the exact sequence of groups

$$\Omega^*_{\text{SU}(k) \otimes \text{SU}(l)} \text{SU}(kl) \to \Omega^*_{\text{SU}(k) \otimes \text{SU}(l)} \text{SU}(kl) \to \text{SU}(kl)$$

(where the second homomorphism is defined by the assignment $\gamma \mapsto \gamma(1) \in \text{SU}(kl)$) corresponds to the fibration

$$\text{Gr}_{k,l} \to \text{BSU}(k) \times \text{BSU}(l) \overset{\otimes}{\to} \text{BSU}(kl).$$
We propose the following interpretation of the considered topological constructions from the viewpoint of $C^*$-algebras. Recall that the (minimal) unitization of $C^*$-algebra $C_0[0,1]$ consisting of functions vanishing at $1 \in [0,1]$ is the $C^*$-algebra $C[0,1]$ which contains $C_0[0,1]$ as an essential ideal. Thus, $C[0,1] \cong C_0[0,1] \oplus \mathbb{C}$, $f \mapsto (f - f(1), f(1))$ as vector spaces. For the matrix algebra $M_n(C[0,1])$ we have the analogous decomposition
\begin{equation}
M_n(C[0,1]) = M_n(C_0[0,1]) \oplus M_n(\mathbb{C}).
\end{equation}

In order to make the analogy with the above considered case of $1$-bundles more transparent, denote $M_{kl}(C_0[0,1])$ by $A_{k,l}$ (it is just the cone over $M_{kl}(\mathbb{C})$). Clearly, the group $SU_{k,l}$ acts continuously on $A_{k,l}$ by conjugations such that for a matrix-valued function $f \in A_{k,l}$ the condition $f(0) \in M_k(\mathbb{C}) \otimes \mathbb{C}E_l \subset M_{kl}(\mathbb{C})$ implies the condition $(gf)(0) \in M_k(\mathbb{C}) \otimes \mathbb{C}E_l \subset M_{kl}(\mathbb{C}) \quad \forall g \in SU_{k,l}$ and the same for $\mathbb{C}E_k \otimes M_l(\mathbb{C})$ in place of $M_k(\mathbb{C}) \otimes \mathbb{C}E_l$. Moreover, this gives us an embedding $PU_{k,l} \hookrightarrow Aut(A_{k,l})$.

Thus, over the space $BSU_{k,l}$ (which according to Theorem 22 is homotopy equivalent to $Gr_{k,l}$) we have an $A_{k,l}$-bundle associated with the universal $SU_{k,l}$-bundle, represented as the tensor product of some $M_k(\mathbb{C})$ and $M_l(\mathbb{C})$-bundles over $0 \in [0,1]$ (these bundles correspond to the tautological bundle $A_{k,l}$ and its centralizer $B_{k,l}$ respectively) and extended to the trivial bundle (with a fixed trivialization) over $1$. This extension can be regarded as an analog of the one-point compactification (or as an analog of the unitization, if we prefer terminology of algebras). Indeed, if $\Gamma(\mathfrak{A}_{k,l})$ is the algebra of continuous sections of some $A_{k,l}$-bundle $\mathfrak{A}_{k,l}$ over $X$ classified by the map $X \to BSU_{k,l}$, then (cf. (22)) $\mathfrak{A}_{k,l} \oplus M_{kl}(\mathbb{C})$ is the algebra of sections of the corresponding (see Remark 28) $M_{kl}(\mathbb{C})$-bundle over $CX$.

Now we want to describe spaces corresponding to above defined ideals $I^1_0$, $I^0_k$. These spaces actually are fibres of bundles $\varphi_k$: $Gr^l_k \to BSU(k)$, $\varphi_l$: $Gr^l_k \to BSU(l)$ (it follows from the representations of $Gr^l_k$ and $Gr_{k,l}$ in the form of homogeneous spaces that they are also fibres of maps $Gr_{k,l} \to BSU(k)$, $Gr_{k,l} \to BSU(l)$ which are classifying maps for $M_k(\mathbb{C})$ and $M_l(\mathbb{C})$-bundles $A_{k,l} \to Gr_{k,l}$ and $B_{k,l} \to Gr_{k,l}$ respectively).

**Definition 30.** A (unitary) $k$-frame in a matrix algebra $M_{kl}(\mathbb{C})$ is an ordered collection of $k^2$ linearly independent matrices
\[
\alpha := \{ \alpha_{i,j} \in M_{kl}(\mathbb{C}) \mid 1 \leq i, j \leq k \}
\]
such that
\[
\alpha_{i,j} \alpha_{r,s} = \delta_{jr} \alpha_{i,s}, \quad 1 \leq i, j, r, s \leq k
\]
(here $\delta_{jr}$ is the Kronecker symbol),
\[
\sum_{i=1}^{k} \alpha_{i,i} = E_{kl} \quad \text{and} \quad \langle \alpha_{i,j}, \alpha_{r,s} \rangle = \delta_{ir} \delta_{js},
\]
where $\langle \ , \ \rangle$ is the hermitian inner product
\[
(X, Y) := \frac{1}{i} \text{tr}(XY^*)
\]
in $M_{kl}(\mathbb{C})$. Clearly that $\alpha$ is a unitary base in some $k$-subalgebra in $M_{kl}(\mathbb{C})$.

It is not difficult to show that the space $\text{Fr}_{k,l}$ of all $k$-frames in $M_{kl}(\mathbb{C})$ is the homogeneous space $\text{PU}(kl)/(\text{E}_k \otimes \text{PU}(l))$ over $\text{PU}(kl)$.

Note that the tautological $M_k(\mathbb{C})$-bundle $\mathcal{A}_{k,l}$ over $\text{Gr}_{k,l}$ is associated with the principal $\text{PU}(k)$-bundle

$$\rho_{k,l}: \text{PU}(kl)/(\text{E}_k \otimes \text{PU}(l)) \to \text{Gr}_{k,l}$$

which to a frame $\alpha$ assigns the corresponding $k$-subalgebra.

Consider the group of paths $\Omega^e_{\text{E}_k \otimes \text{PU}(l)} \text{PU}(kl)$. Define the group homomorphism

$$\varepsilon_k: \text{PU}_{k,l} \to \text{PU}(k), \ g(t) \mapsto \text{Pr}_k(g(0)),$$

where

$$\text{Pr}_k: \text{PU}(k) \otimes \text{PU}(l) \to \text{PU}(k), \ (g_k, g_l) \mapsto g_k.$$  

Clearly, $\Omega^e_{\text{E}_k \otimes \text{PU}(l)} \text{PU}(kl) = \ker(\varepsilon_k)$. Moreover, the following fact takes place.

**Proposition 31.** The space $\text{Fr}_{k,l}$ is homotopy equivalent to the classifying space $\text{B} \Omega^e_{\text{E}_k \otimes \text{PU}(l)} \text{PU}(kl)$.

**Proof.** Note that the exact sequence of groups

$$\Omega^e_{\text{E}_k \otimes \text{PU}(l)} \text{PU}(kl) \to \Omega^e_{\text{PU}(k) \otimes \text{PU}(l)} \text{PU}(kl) \xrightarrow{\varepsilon_k} \text{PU}(k)$$

corresponds to the exact sequence

$$\text{Fr}_{k,l} \xrightarrow{\rho_{k,l}} \text{Gr}_{k,l} \to \text{BPU}(k)$$

of classifying spaces and therefore $\text{Fr}_{k,l} \simeq \Omega^e_{\text{BPU}(l)} \text{BPU}(kl) \simeq \text{B} \Omega^e_{\text{E}_k \otimes \text{PU}(l)} \text{PU}(kl)$. □

Clearly, the homomorphism $\varepsilon_k$ defines the functor which assigns to an $A_{k,l}$-bundle (with the structure group $\text{PU}_{k,l}$) an $M_k(\mathbb{C})$-bundle. Moreover, it takes $A_{k,l}$ bundles whose structure group can be reduced to $\Omega^e_{\text{E}_k \otimes \text{PU}(l)} \text{PU}(kl) \subset \text{PU}_{k,l}$, to a trivial bundle. Note that it is natural to consider bundles with the structure group $\Omega^e_{\text{PU}(k) \otimes \text{E}_l} \text{PU}(kl)$ as $I^0_{k,l}$-bundles (see page 10). More precisely, define the subalgebra $I^0_{k,l} \subset A_{k,l}$ as follows: $I^0_{k,l} := \{f \in A_{k,l} \mid f(0) \in \mathbb{C}E_k \otimes M_l(\mathbb{C})\}$. Then the sequence of fibres $I^0_{k,l} \to A_{k,l} \to M_k(\mathbb{C})$ corresponds to sequence of classifying spaces (23).

**Remark 32.** There is a relation between 1-bundles $\{\xi_k, \xi_l, t_{k,l}\}$ and bundles classified by Matrix Grassmannian $\text{Gr}_{k,l}$ (at least in case $(k, l) = 1$). In particular, one can show that both cases give equivalent “stable” theories (i.e. $\lim \text{Gr}_{k,j}^{I^0_{k,j}}$ and $\lim \text{Gr}_{k,j, l}^{I^0_{k,j}}$ are isomorphic as $H$-spaces). moreover, they do not depend on the choice of a sequence of pairs $\{k_j, l_j\}$ satisfying the conditions: $k_j, l_j \to \infty$ if $j \to \infty$; $k_j \mid k_{j+1}, l_j \mid l_{j+1}; \ (k_j, l_j) = 1 \forall j \in \mathbb{N}$. One can ask the following question: are the spaces $\text{Gr}_{k}^{I^0_{k,l}} = (\text{SU}(k) \otimes E_l)/\text{SU}(k) \otimes \text{SU}(l))$ and $\text{Gr}_{k,l}$ homeomorphic to each other? It seems that the answer is negative, but they are “close” in some sense, according to the following result.

---

4 moreover, if a sequence of pairs $\{k_j, l_j\}$ satisfies the formulated below conditions, then $\lim \text{Gr}_{k_j, l_j}$ as an $H$-space with respect to the operation, induced by the tensor product of bundles, is isomorphic to $\text{BSU}_{\otimes}$ [4, 3].
Proposition 33. There exists a homeomorphism \( \varphi : SU_k \to SU_{k,l} \).

Proof. Define \( \varphi \) by the formula \((\varphi(\gamma))(t) = \gamma(t)\gamma(1)^{-1}, \gamma \in SU_k, \ t \in [0, 1] \). Then \((\varphi(\gamma))(0) \in SU(k) \otimes SU(l) \subset SU(kl)\), \((\varphi(\gamma))(1) = e\), i.e. indeed \( \varphi(\gamma) \in SU_{k,l} \). Now define the map \( \psi : SU_{k,l} \to SU_k \) which is inverse for \( \varphi \). Suppose \( \kappa \in SU_{k,l} \), then by definition \( \kappa(0) \in SU(k) \otimes SU(l) \subset SU(kl) \). Put \( \kappa(0) = (\kappa_k, \kappa_l) \in SU(k) \otimes SU(l) \). Then \((\psi(\kappa))(t) = \kappa(t)\kappa_l^{-1} \in E_k \otimes SU(l) \subset SU(kl)\), i.e. \( \psi(\kappa) \in SU_k \). In the second place \( ((\psi \circ \varphi)(\gamma))(t) = \psi((\varphi(\gamma))(t)) = \psi(\gamma(t)\gamma(1)^{-1}) = \gamma(t), ((\varphi \circ \psi)(\kappa))(t) = \varphi((\psi(\kappa))(t)) = \varphi(\kappa(t)\kappa_l^{-1}) = \kappa(t) \). \( \Box \)

Note that just defined homeomorphism \( \varphi \) is not a group homomorphism, therefore its existence does not imply that the classifying spaces \( Gr^k \) and \( Gr_{k,l} \) are homotopy equivalent.

2.2. Topological obstructions for embedding of a bundle into a trivial one. In paper 5 topological obstructions for lifting in bundle (23) were considered. Let us discuss this problem more detailed. First note that the space \( Fr_{k,l} \) can also be interpreted as the space \( \text{Hom}_{alg}(M_k(C), M_{kl}(C)) \) of \(*\)-homomorphisms of unital algebras. Indeed, if we fix a \( k \)-frame \( \alpha \) in \( M_k(C) \), then a \(*\)-homomorphism is uniquely determined by a \( k \)-frame in \( M_k(C) \) which is the image of \( \alpha \) under the map induced on frames by the homomorphism.

Let \( A_k^{univ} \to BPU(k) \) be the universal \( M_k(C) \)-bundle. The fibrewise application of the functor \( \text{Hom}_{alg}(\ldots, M_{kl}(C)) \) to it gives us some \( \text{Hom}_{alg}(M_k(C), M_{kl}(C)) \)-bundle \( p_{k,l} : H_{k,l}(A_k^{univ}) \to BPU(k) \). Note that the lifted \( M_k(C) \)-bundle \( p_{k,l}^*(A_k^{univ}) \to H_{k,l}(A_k^{univ}) \) is equipped with the canonical embedding into the product bundle \( H_{k,l}(A_k^{univ}) \times M_{kl}(C) \) defined by the formula:

\[ \{a, h\} \mapsto \{h, h(a)\}, \quad a \in (A_k^{univ})_x, \ h \in H_{k,l}(A_k^{univ}), \ p_{k,l}(h) = x \in BPU(k). \]

Remark 34. The last bundle can be constructed as a \( \text{Hom}_{alg}(M_k(C), M_{kl}(C)) \)-bundle associated with the universal principal \( PU(k) \)-bundle \( EPU(k) \) using the following action of \( PU(k) \) on \( \text{Hom}_{alg}(M_k(C), M_{kl}(C)) \):

\[ (g, \varphi) \mapsto \varphi \circ g^{-1}, \ g \in PU(k), \ \varphi \in \text{Hom}_{alg}(M_k(C), M_{kl}(C)). \]

Now we can completely understand the geometric sense of the homotopy equivalence \( Gr_{k,l} \simeq H_{k,l}(A_k^{univ}) \). Thus, we can substitute the fibration

\[
\begin{array}{ccc}
Fr_{k,l} & \longrightarrow & EPU(k) \times_{PU(k)} Fr_{k,l} \\
\downarrow & & \downarrow p_{k,l} \\
& & BPU(k),
\end{array}
\]

(24)

where \( EPU(k) \times_{PU(k)} Fr_{k,l} \simeq Gr_{k,l} \), for sequence (23).

A map \( f : X \to BPU(k) \) is actually an \( M_k(C) \)-bundle (up to isomorphism), and its lift \( \tilde{f} : X \to H_{k,l}(A_k^{univ}) \) can be treated as the choice of an embedding of the bundle \( f^*(A_k^{univ}) \to X \) into the trivial \( X \times M_{kl}(C) \) such that every fibre is embedded as a central subalgebra. It is not difficult to calculate that for \( (k, l) = 1 \) in stable dimensions (in the
sense of Bott periodicity for unitary groups) \( \pi_{2r-1}(\Fr_{k,l}) = \mathbb{Z}/k\mathbb{Z} \) and even-dimensional homotopy groups of \( \Fr_{k,l} \) are equal to 0. For instance, the first obstruction for the embedding belongs to \( H^2(\text{BPU}(k), \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z} \).

**Definition 35.** We say that an \( M_k(\mathbb{C}) \)-bundle \( A_k \to X \) is embeddable if there exists a fiberwise embedding \( A_k \to X \times M_{kl}(\mathbb{C}) \) for some \( l, (k, l) = 1 \).

**Remark 36.** One can easily show that if \( A_k \) is an embeddable bundle then it can be embedded into a trivial \( X \times M_{km}(\mathbb{C}) \) for every large enough \( m \).

In analogy with the Brauer group [2] we define the following homotopy functor taking values in the category of abelian groups. Two algebra bundles \( A_k \) and \( B_l \) over \( X \) are said to be equivalent if there exist embeddable bundles \( C_m, D_n \) over \( X \) such that \( A_k \otimes C_m \cong B_l \otimes D_n \) (in particular, this implies \( km = ln \)).

Passing to the direct limit in (24) over all pairs \( \{k, l\} \) of relatively prime numbers, we obtain the fibration

\[
\begin{align*}
\text{Gr} & \rightarrow K(\mathbb{Q}/\mathbb{Z}, 2) \times \prod_{q > 1} K(\mathbb{Q}, 2q), \\
\end{align*}
\]

where \( \Fr := \lim_{(k, l) \to 1} \Fr_{k,l}, \quad \Fr_{k,l} := \text{SU}(kl)/(E_k \otimes \text{SU}(l)), \quad \text{Gr} := \lim_{(k, l) \to 1} \text{Gr}_{k,l} \simeq \text{BSU} \) (see Remark [32]). Every direct limit above is taken over maps induced by the tensor product with trivial bundles. In particular, \( \text{Gr} \simeq \text{BSU} \), \( \pi_{2r+1}(\Fr) = \mathbb{Q}/\mathbb{Z} \) for \( r \geq 1 \) and \( \pi_n(\Fr) = 0 \) for others \( n \). It is easy to see from the last fibration that the first obstruction for the embedding is in fact the obstruction for the reduction of the structure group from the projective PU to the special SU and that every class \( \alpha \in H^2(X, \mathbb{Q}/\mathbb{Z}) \) is an obstruction for such an embedding. But in contrast with the Brauer group (which is isomorphic to \( H^3_{\text{tors}}(X, \mathbb{Z}) [2] \)) there are lot of higher obstructions in our lifting problem.

In order to determine the next obstruction, consider the following diagram:

\[
\begin{align*}
\Fr_{k,l} & \rightarrow \text{EPU}(k) \times \Fr_{k,l} \rightarrow \text{BPU}(k), \\
\Fr_{k,l} & \rightarrow \text{ESU}(k) \times \Fr_{k,l} \rightarrow \text{BPU}(k), \\
& \rightarrow \text{BSU}(k).
\end{align*}
\]
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Note that the existence of a homotopy equivalence $\text{ESU}(k) \times \tilde{F}_{k,l} \simeq \text{Gr}_{k,l}$ can easily be deduced from Remark 21. Note also that there is the covering

\[ \mu_k \to \tilde{F}_{k,l} \to F_{k,l}, \]

where $\mu_k$ is the group of $k$th degree roots of unity. Hence $\pi_n(\tilde{F}_{k,l}) = \pi_n(F_{k,l})$ for $n \geq 2$ and $\pi_1(\tilde{F}_{k,l}) = 0$ (while $\pi_1(F_{k,l}) = \mathbb{Z}/k\mathbb{Z}$).

Let us return to the classifying map $f : X \to \text{BPU}(k)$ for some $M_k(\mathbb{C})$-bundle. We have already seen that if the first obstruction vanishes then $f$ can be lifted to $\tilde{f} : X \to \text{BSU}(k)$. Now it can be noticed from the diagram (25) that the next obstruction belongs to the group $H^4(X, \mathbb{Z}/k\mathbb{Z})$. Clearly, it is just the second Chern class $c_2$ reduced modulo $k$. Note that the obstructions are stable in the sense that they do not vanish when we take the direct limits over pairs $(k, l)$ satisfying the condition $(k, l) = 1$ as in Remark 32.

After the previous section the whole lifting procedure can be interpreted as the reduction of the structure group from $\text{PU}(k)$ to $\text{SU}_{k,l}$ (or to $\text{PU}_{k,l}$).

3. SOME SPECULATIONS

In this section we propose a hypothetical way to extend fibration (26) to the right, using the relation between the fibration of classifying spaces

\[ F_{k,l} \to \text{Gr}_k^l \to \text{BPU}(k) \]

and the exact sequence of $C^*$-algebras

\[ 0 \to I_0^l \xrightarrow{j} A_k^l \xrightarrow{i} M_k(\mathbb{C}) \to 0. \]

Recall that the relation between (26) and (27) is based on the fact that (26) is the sequence of classifying spaces for the exact sequence

\[ \Omega_{E_k \otimes \text{PU}(l)} \to \Omega_{E_k \otimes \text{PU}(l)} \to \text{PU}(kl) \to \text{PU}(k) \]

of inner automorphisms groups of (27), where the last homomorphism is the evaluation at 0.

The reason why we are interested in such an extension is that it may provide a generalization of the Brauer group. Indeed, the space $\lim_{(k, l) \to 1} \text{Gr}_k^l \simeq \text{BSU}_{(k, l)}$ is a noncommutative analog of $\mathbb{C}P^\infty$, because it represents the group of equivalence classes of virtual bundles of virtual dimension 1 (while $\mathbb{C}P^\infty$ represents the group of geometric line bundles, i.e. the Picard group).

The theory of $C^*$-algebras provides us with some tool which might help us to solve the problem, namely the concept of multiplier algebra. More precisely, let $M(I_0^l)$ be the
multiplier algebra of the ideal $I_0^l$. We have the morphism of exact sequences of $C^*$-algebras:

$$
0 \longrightarrow I_0^l \xrightarrow{\varphi} \mathcal{M}(I_0^l) \xrightarrow{\psi} Q(I_0^l) \longrightarrow 0
$$

where $Q(I_0^l)$ is the “corona algebra” or “Calkin algebra” (i.e. the factor-algebra $\mathcal{M}(I_0^l)/I_0^l$), and the homomorphism $\nu$ is defined by the commutativity of the diagram.

Notice that the homomorphism $\mu$ is injective because $I_0^l$ is an essential ideal in $A_k^l$.

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