Quasinormal modes of Regular Black Hole Charge

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Abstract

The quasinormal modes (QNMs) of a regular black hole with charge are calculated in the eikonal approximation. In the eikonal limit the QNMs of black hole are determined by the parameters of the unstable circular null geodesics. The behaviors of QNMs are compared with QNMs of Reisner-Nordström black hole, it this is by fixed some of the parameters that characterize the black holes and varying another. We observed that the parameter that is related one effective cosmological constant at small distances , determines the behaviors of the QNMs of regular black hole with charge.

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I. INTRODUCTION

A problem of the classical theory of gravity is inevitable existence of singularities. For example the solutions that describing black holes (BH) as Schwarzschild, Reisner-Nordström and Kerr have curvature singularity in their interior, although they are covered by an event horizon, the prediction of Hawking radiation where the negative energy flux of the radiation of black holes, cause that the black hole to shrink until the singularity is reached.

The presence of singularities are generally regarded as indicating the breakdown of the theory, requiring modifications that presumably include quantum theory.

To avoid the black holes singularity problem the construction of regular (i.e., nonsingular) solutions have been proposed, for example the theory of general relativity coupled to nonlinear electrodynamics is a candidate, because in this theory exist the regular magnetic black hole proposed by Bardeen [1] and the regular solution of Bronnikov [2] for example.

Also Hayward [3] proposed a regular space-times are given that describe the formation of a black hole from an initial vacuum region, its quiescence as a static region (behaving as a cosmological constant at small radius.), and its subsequent evaporation to a vacuum region the static region is Bardeen-like. In this context, variations of the Hayward solutions have also been proposed as Rotating Hayward [4] and Hayward with charge [5].

The black holes always interacts with matter and fields around they and as the result of these interactions it takes a perturbed state regardless of whether or not they are regular. Behavior of the perturbations is of special interest at stability as perturbations in linear regime. Perturbed Black Holes are characterized by gravitational waves that are characterized by complex frequency which is called quasinormal modes (QNMs) [6]. The QNMs resonances are crucial to identify the behavior of the parameters that characterizes a black hole, especially the mass and angular momentum of the BH, but as well are going to be important on identifying additional physical parameters arising.

Mashhoon and Ferrari [7] [8] have suggested an analytical technique of calculating the QNMs in the geometric-optics (eikonal) limit. The basic idea is to interpret the BH free oscillations in terms of null particles trapped at the unstable circular orbit and slowly leaking out. In this sense Cardoso [9] showed the relationship among unstable null geodesics, Lyapunov exponents and quasinormal modes in a stationary spherically symmetric space-time.

In [10] applying the ideas of Cardoso, the QNMs frequencies of the regular magnetic BH
model proposed by Bardeen were determined. Also in [11] the QNMs of regular black holes are studied using the sixth order WKB approximation.

The organization of this paper is as follows. First in section II a short summary of the Quasi-normal modes and Lyapunov exponent are given. In Section III the Hayward black hole with charge is described and the effective potentials is shown for null geodesics and then in section IV we analyze the QNMs frequencies in each case the QNMs frequencies are compared with QNMs of the Reissner-Nordström BH. Finally we conclude in Section V with a brief discussion.

II. QUASI-NORMAL MODES AND LYAPUNOV EXPONENT

The connection between the QNMs and bound states of the inverted black hole effective potential was pointed out in [8]. In [9] it is shown that, in the eikonal limit, the QNMs of black holes are determined by the parameters of the circular null geodesics. The real part of the complex QNMs frequencies is determined by the angular velocity at the unstable null geodesics. The imaginary part is related to the instability time scale of the orbit, and therefore related to the Lyapunov exponent that is its inverse. In the case of stationary, spherically symmetric spacetimes it turns out that this exponent can be expressed as the second derivative of the effective potential evaluated at the radius of the unstable circular null orbit. It was also shown the agreement of the so calculated QNMs with the analytic WKB approximation,

\[ \omega_{QNM} = \Omega_c \tilde{l} - i(n + \frac{1}{2})|\lambda| \]  

where \( n \) is the overtone number and \( \tilde{l} \) is the angular momentum of the perturbation. \( \Omega_c \) is the angular velocity at the unstable null geodesic and \( \lambda \) is the Lyapunov exponent, determining the instability time scale of the orbit. From the equations of motion for a test particle in the static spherically symmetric spacetime, \( \dot{r}^2 + V_r = 0 \), where \( V_r \) is the effective potential for radial motion, circular geodesics are determined from the conditions \( V(r_c) = V'(r_c) = 0 \) where \( r_c \) is the radius of the circular orbit. The Lyapunov exponent in terms of the second derivative of the effective potential is given by

\[ \lambda = \sqrt{\frac{-V''(r)}{2l^2}}, \]  

(2)
where $t$ is the time coordinate. The dot denotes the derivative with respect to an affine parameter and the prime stands for the derivative with respect to $r$. The orbital angular velocity is given by

$$\Omega_c = \frac{d\varphi}{dt} = \frac{\dot{\varphi}}{t}. \quad (3)$$

For our purpose both expressions should be evaluated at $r_c$, the radius of the unstable null circular orbit. For a static spherically symmetric background

$$ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2d\Omega^2, \quad (4)$$

for equatorial orbits ($\theta = \pi/2$) the Lagrangian is;

$$L = -f(r)\dot{t}^2 + \frac{1}{g(r)}\dot{r}^2 + r^2\dot{\phi}^2 = \delta_1. \quad (5)$$

Here $\delta_1 = 1, 0$ for spacelike and null geodesics, respectively, the energy $E (\dot{t} = \frac{E}{g_{tt}})$ and the angular momentum $L (\dot{\phi} = \frac{L}{g_{\phi\phi}})$ of a test particle are conserved quantities then for the case of a static spacetime in (4), and the effective potential is given by;

$$V_{\epsilon f} = g(r) \left[ -\frac{E^2}{f(r)} + \frac{L^2}{r^2} - \delta_1 \right] \quad (6)$$

If we consider the null geodesics ($\delta_1 = 0$), we obtain;

$$V_{\epsilon f}'' = \frac{L^2 g(r)}{r^4 f(r)} \left[ r^2 f''(r) - 2f(r) \right] \quad (7)$$

while the orbital angular velocity, which is proportional to the real part of the QNM frequencies, is given by

$$\Omega = \sqrt{\frac{f(r)}{2r^2}}. \quad (8)$$

III. HAYWARD BLACK HOLE WITH CHARGE

The Hayward black hole with charge (HBH (see [5])) is described by the static spherically symmetric space-time (4), where $f(r) = g(r) = \left(1 - \frac{(2mr-q^2)r^2}{r^4+(2mr+q^2)l^2}\right)$, with $m$ corresponding to mass of the black hole, $q$ is the electric charge and $l$ is a convenient encoding of the central energy density $3/8\pi l^2$ assumed positive and $l$ also is related an effective cosmological
constant at small distances. The metric function the HBH becomes as Reissner-Nordström (RN) black hole when $l \to 0$. The Hayward black hole with charge is non-singular (regular) at center of the metric where $r \to 0$ also HBH is flat for $m = q = 0$. The analysis of $f(r)$ for zeros show a critical mass $m_\ast = \left( \frac{16(l^2+1)^3}{27(l^2-1)^4}q^6 \right)^{\frac{1}{4}}$. Then if $m_\ast = 0$ the spacetime is flat, when $m_\ast = m$ the solution HBH has a horizon (extreme case $r_\ast$), $m_\ast < m$ has two horizons (the exterior $r_+$ and inner $r_-$ horizons) finally if $m_\ast > m$ $f(r)$ has no zero. This is shown in Fig. (1).

![FIG. 1: Behavior of $f(r)$ for fixed values of parameters $l = 0.5$, $q = 1$ and different values of $m$ with $m_\ast$ the critical mass](image)

In Fig (2) the comparison between the Hayward black hole with charge and Reissner-Nordström effective potentials is shown for null geodesics. The presence of maximum and minimum in the effective potential indicates that there exist circular orbits, both stable and unstable for both black holes then the method of Cardoso [9] can be easily applied. Figure (2) shows that the effective potential of Hayward black hole is the greatest compared to Reissner-Nordström black hole that has the least effective potential. Also is possible to observe that there is a difference between the radius of the circular orbits $r_c$ of the black holes.

Consider the radial geodesic when $L = 0$. Then, $V_{ef}$ is given by $V_{ef} = -E^2$ then this shows that particle will behave like a free particle for $E = 0$. In [12] the study of geodesic structure of regular Hayward black hole is devoted.
FIG. 2: The effective potential $V_{ef}$, for null geodesics in RN and in BI is compared. The constants are with $m = 1.3$, $q = 1$, $L = 1$ y $E = 1$ and for HBH $l = 0.5$

IV. THE QNMS OF HAYWARD BLACK HOLE WITH CHARGE

In this section we analyzed the quasinormal modes (QNMs) of regular black hole with charged in the case of null geodesics and those are then compared with the QNMs of RN black hole.

A. The real part of the QNMs frequencies

The angular velocity for massless particles of the Hayward black hole is given by;

$$
\Omega = \sqrt{1 - \frac{(2mr-q^2)r^2}{r^4+(2mr+q^2)l}}
$$

The expression (9) should be evaluated at $r_c$, the radius of the unstable null circular orbit. Hereafter we shall consider $\omega_r/\tilde{l} \mapsto \omega_r$ for our analysis. In Fig. 3 and 4 the behaviors of the QNMs frequencies $\omega_r$ of the HBH for different values of the parameter $l$ are shown and compared with RN black hole. Both frequencies approach the RN limit as $l$ decreases. When we vary $q$ frequency $\omega_r$ increases when $q$ increases (see Fig (4)). In the other case when we fixed $q$ the QNMs frequencies $\omega_r$ decrease when the mass $m$ increases (see Fig (3)).

The period for the circular orbits includes the length of time it takes for the particle to bypass the circular orbit. For the coordinate time $T_t$ is proportional to $1/\omega_r$ then when $q$ is
FIG. 3: QNMs frequencies $\omega_r$ of the HBH and RN are shown as functions of the charge $m$; the other parameters fixed to $q = 1$, $l = 0.7$.

FIG. 4: QNMs frequencies $\omega_r$ of the HBH and RN are shown as functions of the charge $q$; the other parameters fixed to $m = 1$, $l = 0.7, 0.3$

fixed the $T_t$ increases as $m$ increases, the opposite occurs when we vary $q$.

In nonlinear electromagnetic the null geodesics are modified by an effective metric then the QNMs also are modified as can be consulted in [13]. In the table [1] illustrates the discrepancy of $w_r$ between two regular black holes in nonlinear electromagnetic theory and Hayward with charge. The Bardeen black hole can be interpreted as the solution to a nonlinear magnetic monopole with mass and charge and the Bronnikov black hole is a regular solutions with a nonzero magnetic charge. Nevertheless the behavior is similar $\omega_r$. 
increases when the charge $q$ is increased.

| Charge | Hayward BH | Bardeen BH | Bronnikov BH |
|--------|------------|------------|--------------|
| 0.1    | 0.402871   | 0.47158    | 0.385544     |
| 0.2    | 0.405635   | 0.47304    | 0.387518     |
| 0.3    | 0.410495   | 0.472984   | 0.390938     |
| 0.4    | 0.41793    | 0.47422    | 0.39603      |
| 0.5    | 0.428936   | 0.475814   | 0.403162     |
| 0.6    | 0.446173   | 0.477752   | -            |

TABLE I: The $w_r$ for the Hayward BH with $\hat{l} = 2$, $m = 1$ and $l = 0.7$ are compared with the $w_r$ of the effective metric of two regular black holes in nonlinear electromagnetic

B. The imaginary part of the QNMs frequencies

We calculate the the imaginary part of QNMs frequencies of HBH solution corresponding to different values of parameter $l$ a comparison with RN black hole is established as well. Considering the equation (7) we obtained:

$$V''_{ef} = -\left[\frac{2L^2(l^4(q^2 + 2mr)^2 + r^6(2q^2 + r(-3m + r)) + 2l^2r^3(q^2r + m(q^2 + 2r^2)))}{r^3(r^4 + l^2(q^2 + 2mr)^2)}\right]_{r_c} \quad (10)$$

Then in the eikonal or geometric-optics limit, the QNMs frequency $\omega_i$ is given by (1) and (2).

Now we vary the charge $q$. The QNMs frequencies $\omega_i$ (see Fig (5)), for RN case, increases as $q$ augment and presents a maximum then decreasing; for RN the value of $q$ cannot exceed $q = 1$ that corresponds to the extreme BH, $q = M$. HBH does not have this behavior for large values of $l$ and the behavior of the $\omega_i$ for HBH is similar to RN for small values of $l$.

When vary the mass $m$, $\omega_i$ decrease when the mass $m$ is increases as shown in the Fig (6) in both cases.

The instability time scale of the circular null geodesics of HBH is the greatest compared to Reissner-Nordström black hole regardless of the value of the mass (see Fig (6)).
FIG. 5: QNM frequencies $\omega_i$ of the HBH and RN are shown as functions of the charge $q$; the other parameters fixed to $m = 1, l = 0.7, 0.3$

FIG. 6: QNM frequencies $\omega_i$ of the HBH and RN are shown as functions of the charge $m$; the other parameters fixed to $q = 1, l = 0.5, 0.3$

opposite occurs when vary the charge $q$ the instability time scale is suppressed as compared with RN black hole (see Fig (5)). However in both cases the instability time tends to zero.

Table II shows some comparative values to get insight on the discrepancy between the respective frequencies $w_i$ for two regular black holes in nonlinear electromagnetic and Hayward with charge. The behavior is similar for Bardeen BH and HBH ( when $l$ is increased) in both cases $w_i$ decrease when the charge $q$ increases. This behavior is in contrast with RN black hole and Bronnikov, in the case of Bardeen BH one possible explanation may be the
nature of the solution because charge and mass parameters are not independent, and in fact when charge is turned off, so does the mass whose origin is purely electromagnetic (see [13]) but in the case of Hayward with charge the parameter $l$ that is a convenient encoding of the central energy density is the cause of such behavior because when $l \to 0$ the HBH becomes as Reissner-Nordström (RN) black hole.

| Charge | Hayward BH | Bardeen BH | Bronnikov BH |
|--------|------------|------------|--------------|
| 0.1    | 0.091711   | 0.0960758  | 0.0962784    |
| 0.2    | 0.09145    | 0.0956186  | 0.0964377    |
| 0.3    | 0.0909885  | 0.0948243  | 0.0966995    |
| 0.4    | 0.0901247  | 0.0936395  | 0.0970555    |
| 0.5    | 0.088398   | 0.091981   | 0.0974872    |
| 0.6    | 0.0845255  | 0.091981   | -            |

TABLE II: The $w_i$ for the Hayward black hole with $n = 0$, $m = 1$ and $l = 0.5$ are compared with the $w_i$ of the effective metric of two regular black holes in nonlinear electromagnetic

V. CONCLUSIONS

We have studied the QNMs frequencies of Hayward black hole with charge through the Lyapunov exponent in the optical approximation. QNMs frequencies were calculated from the unstable null geodesics when the charge or mass are varied and in all cases comparison is done with the QNMs frequencies of the RN back hole.

When we keeping fixed the charge the imaginary and real part of the QNMs decreases when the mass increases for both cases approaches zero. In the case of fixed the mass the imaginary part of QNMs frequencies for HBH, increases as the charge augments and presents a maximum then decreasing is more notorious for small values of $l$ and approach the RN limit. In the case of real part increases when the charge increases but is limited by the value of the mass and if the parameter $l$ increases.

The $w_r$ for HBH are greater than for RN black hole in the case of varying the charge but they stop growing before the $w_r$ of RN black hole as the parameter $l$ increases. In the case of the imaginary part the values of $w_i$ are less than for RN black hole when the parameter
l also increases. The effective potential of Hayward black hole is the greatest compared to Reissner-Nordström black hole.

Finally the analysis of the QNMs show as that in the case of the consider a regular black hole modifications the the stability and the period for the circular orbits. Also is shows some comparative values to get insight on the discrepancy between the respective frequencies for two regular black holes in nonlinear electromagnetic and Hayward with charge.

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