A NEW CONVERGENCE ANALYSIS OF TWO-LEVEL HIERARCHICAL BASIS METHODS

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Abstract. This paper is concerned with the convergence analysis of two-level hierarchical basis (TLHB) methods in a general setting, where the decomposition associated with two hierarchical component spaces is not required to be a direct sum. The TLHB scheme can be regarded as a combination of compatible relaxation and coarse-grid correction. Most of the previous works focus on the case of exact coarse solver, and the existing identity for the convergence factor of exact TLHB methods involves a tricky max-min problem. In this work, we present a new and purely algebraic analysis of TLHB methods, which gives a succinct identity for the convergence factor of exact TLHB methods. The new identity can be conveniently utilized to derive an optimal interpolation and analyze the influence of coarse space on the convergence factor. Moreover, we establish two-sided bounds for the convergence factor of TLHB methods with inexact coarse solver, which extend the existing TLHB theory.

1. Introduction

Multigrid is a typical multilevel iterative scheme, which has been proved to be a powerful solver (with linear or near-linear computational complexity) for a large class of linear systems that arise from discretized partial differential equations; see, e.g., [12, 8, 17, 19]. The fundamental module of multigrid is a two-grid scheme, which combines two complementary error-reduction processes: a smoothing (or relaxation) process and a coarse-grid correction process. The smoothing process is typically a simple iterative method such as the Jacobi and Gauss–Seidel iterations. Usually, it is efficient on high-frequency (i.e., oscillatory) error modes, while the low-frequency (i.e., smooth) part cannot be eliminated effectively. One way to capture the low-frequency error is to coarsen the underlying grid so that low-frequency modes on the initial fine-grid appear high-frequency on a coarser-grid. The low-frequency error will be further eliminated by a relaxation method on the coarse-grid. The resulting correction can be interpolated back to the fine-grid by an interpolation operator. Such a process is the so-called coarse-grid correction.

For a given initial guess \( u_0 \in \mathbb{R}^n \), the smoothing iteration for solving \( Au = f \) can be described as

\[
u_{k+1} = u_k + M^{-1}(f - Au_k),
\]

where \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite (SPD) and \( M \in \mathbb{R}^{n \times n} \) is a nonsingular smoother. In the classical two-grid analysis (see, e.g., [10, 11, 19]), \( M \) is assumed to be \( A \)-convergent (i.e., \( \|I - M^{-1}A\|_A < 1 \)), which is equivalent to the positive

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definiteness of $M + M^T - A$; see, e.g., \cite[Proposition 3.8]{19}. This assumption plays a crucial role in the theoretical analysis of two-grid methods.

As mentioned earlier, the global smoothing (1.1) has a little effect on the low-frequency error in general. Alternatively, one can use a local smoother $M_s \in \mathbb{R}^{n_s \times n_s}$ ($n_s < n$) in the smoothing process, which is expected to focus on eliminating the high-frequency error modes. Compared with the global smoother $M$, one has more room to design the local smoother $M_s$ due to its size is relatively small. Inspired by the compatible relaxation iterations in \cite{10} (the idea of compatible relaxation originated with Brandt \cite{6}), we perform the following smoothing iteration:

$$
(1.2) \quad u_{k+1} = u_k + SM_s^{-1}S^T(f - Au_k),
$$

where $S \in \mathbb{R}^{n \times n}$ is of full column rank and $M_s \in \mathbb{R}^{n_s \times n_s}$ is $(S^TAS)$-convergent (noting that $M_s$ is not restricted to $S^TMS$). Clearly, the iteration (1.2) will reduce to (1.1) if $n_s = n$ and $S = I_n$. Such a special case is not our focus here: two-grid theory has been well developed in the literature; see, e.g., \cite{10, 11, 19, 24, 14, 16, 20, 21, 22}. Let $u_k \in \mathbb{R}^n$ be an approximation to the exact solution $u$ (e.g., $u_k$ is generated from (1.2)), and let $P \in \mathbb{R}^{n \times n_c}$ ($n_c < n$) be an interpolation matrix with full column rank. The (exact) coarse-grid correction can then be described as

$$
(1.3) \quad u_{k+1} = u_k + P(P^TAP)^{-1}P^T(f - Au_k).
$$

The two-level hierarchical basis (TLHB) scheme can be obtained by successively performing presmoothing, coarse-grid correction, and postsmoothing iterations (see Algorithm 1). Some pioneering works on TLHB methods can be found in \cite{5, 2, 23, 4, 3, 18}. A basic assumption in the classical TLHB theory is that $(S P)$ is square and nonsingular (which entails that $n_c = n - n_s$). For example,

$$
S = \begin{pmatrix} I_{n_s} \\ 0 \end{pmatrix}, \quad P = \begin{pmatrix} * \\ I_{n-n_s} \end{pmatrix}.
$$

This assumption leads to the positive definiteness of the hierarchical basis matrix

$$
(1.4) \quad \begin{pmatrix} S^T \\ P^T \end{pmatrix} A(S P) = \begin{pmatrix} S^TAS & S^TAP \\ P^TAS & P^TAP \end{pmatrix}.
$$

An important quantity involved in the analysis of multilevel methods is the so-called \textit{Cauchy–Bunyakowski–Schwarz} (C.B.S.) constant \cite{9, 1, 11}. The C.B.S. constant associated with (1.4) is defined as

$$
\gamma := \max_{\substack{v_c \in \mathbb{R}^{n_c} \backslash \{0\} \\ v_s \in \mathbb{R}^{n_s} \backslash \{0\}}} \frac{v_s^TSPAPv_c}{\sqrt{v_s^TSP^TAPv_s} \cdot \sqrt{v_c^TAPv_c}}.
$$

The positive definiteness of (1.4) implies that $\gamma \in [0, 1)$, which can be viewed as the cosine of the abstract angle between the hierarchical components range($S$) and range($P$).

Using a hierarchical expression for the inverse of TLHB preconditioner (see (2.6)), one can easily verify that a \textit{necessary} condition for TLHB convergence is

$$
(1.6) \quad \text{rank}(S P) = n,
$$

which is a foundation of TLHB analysis. In particular, in the case of exact coarse solver, the condition (1.6) is also sufficient for TLHB convergence. Obviously, (1.6) implies that $n_c + n_s \geq n$, i.e., $n_c \geq n - n_s$. If $S$ and $M_s$ are preselected, the classical setting $n_c = n - n_s$ is not the optimal one, at least from the perspective of
convergence (see Theorem 3.7 and Remark 3.8). In addition, if \( n_c > n - n_s \), then \( \gamma \) happens to be 1, which will trivialize some classical TLHB theories (see, e.g., [9, 1, 18, 11]). Under the condition (1.6), Falgout, Vassilevski, and Zikatanov [11, Theorem 4.1] established an identity for the convergence factor of exact TLHB methods. However, the identity (see also (2.15)) involves a tricky max-min problem: it is generally difficult to determine when the ‘min’ is attained, which limits the application of the identity.

In this paper, we derive a new and succinct identity (see (3.1)) for the convergence factor of exact TLHB methods under the condition (1.6). Our proof is not only novel but also much simpler than that in [11]. The new identity provides a straightforward approach to analyze the optimal interpolation (see Theorem 3.4) and the influence of range(\( P \)) on the convergence factor (see Theorem 3.7).

In practice, the Galerkin coarse-grid system is often too costly to solve exactly, especially when its size is still large. Instead, one can solve the system approximately as long as the convergence speed is satisfactory. Compared with the exact case, the convergence analysis of TLHB methods with inexact coarse solver is of more practical significance. Motivated by this observation, we establish two-sided bounds for the convergence factor of inexact TLHB methods, from which one can readily get the identity for the exact case.

The rest of this paper is organized as follows. In Section 2, we introduce some properties and convergence results of TLHB methods. In Section 3, we present a new identity for the convergence factor of exact TLHB methods, followed by some discussions on how the new identity can be used to analyze the optimal interpolation and the influence of range(\( P \)) on the convergence factor. In Section 4, we establish a systematic convergence theory for inexact TLHB methods. In Section 5, we give some concluding remarks.

2. Preliminaries

We start with some notation used in the subsequent discussions.

- \( I_n \) denotes the \( n \times n \) identity matrix (or \( I \) when its size is clear from context).
- \( \lambda_{\min}(\cdot), \lambda_{\min}^+(\cdot), \) and \( \lambda_{\max}(\cdot) \) stand for the smallest eigenvalue, the smallest positive eigenvalue, and the largest eigenvalue of a matrix, respectively.
- \( \lambda_i(\cdot) \) denotes the \( i \)-th smallest eigenvalue of a matrix.
- \( \lambda(\cdot) \) denotes the spectrum of a matrix.
- \( \rho(\cdot) \) represents the spectral radius of a matrix.
- \( \| \cdot \|_2 \) denotes the spectral norm of a matrix.
- \( \| \cdot \|_A \) denotes the energy norm induced by an SPD matrix \( A \in \mathbb{R}^{n \times n} \), for any \( v \in \mathbb{R}^n \), \( \| v \|_A = \sqrt{v^T Av} \); for any \( B \in \mathbb{R}^{n \times n} \), \( \| B \|_A = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\| Bv \|_A}{\| v \|_A} \).

Our focus is on TLHB methods for solving the linear system

\[
Au = f,
\]

where \( A \in \mathbb{R}^{n \times n} \) is SPD, \( u \in \mathbb{R}^n \), and \( f \in \mathbb{R}^n \). Some basic assumptions involved in the analysis of TLHB methods are listed below.

- Let \( S \in \mathbb{R}^{n \times n_s} \) and \( P \in \mathbb{R}^{n \times n_c} \) be of full column rank, where
  \[
  \max\{n_s, n_c\} < n \leq n_s + n_c.
  \]
- Assume that \((SP) \in \mathbb{R}^{n \times (n_s + n_c)}\) is of full row rank, or, equivalently, for any \( v \in \mathbb{R}^n \), there exist \( v_s \in \mathbb{R}^{n_s} \) and \( v_c \in \mathbb{R}^{n_c} \) such that \( v = Sv_s + Pv_c \).
• Let $M_s$ be an $n_s \times n_s$ nonsingular matrix such that $M_s + M_s^T - A_s$ is SPD, where $A_s := S^T A S$.
• Let $B_c \in \mathbb{R}^{n_c \times n_c}$ be an SPD approximation to $A_c$, where $A_c := P^T A P$ is the so-called Galerkin coarse-grid matrix.

With the above assumptions, the standard TLHB scheme for solving (2.1) can be described as Algorithm 1 ($u_0 \in \mathbb{R}^n$ is an initial guess). If $B_c = A_c$, then Algorithm 1 is called an exact TLHB method; otherwise, it is called an inexact TLHB method.

**Algorithm 1 TLHB method**

1: Presmoothing: $u_1 \leftarrow u_0 + SM_s^{-1} S^T (f - A u_0)$
2: Restriction: $r_c \leftarrow P^T (f - A u_1)$
3: Coarse-grid correction: $e_c \leftarrow B_c^{-1} r_c$
4: Interpolation: $u_2 \leftarrow u_1 + P e_c$
5: Postsmoothing: $u_{TL} \leftarrow u_2 + SM_s^{-T} S^T (f - A u_2)$

**Remark 2.1.** Due to $n_s < n$, it follows that

$$\|I - SM_s^{-1} S^T A\|_A = 1,$$

which does not satisfy a conventional assumption in two-grid analysis, that is, the smoothing iteration is a contraction in $A$-norm. Moreover, there is no nonsingular matrix $X \in \mathbb{R}^{n \times n}$ such that

$$I - X^{-1} A = (I - SM_s^{-1} S^T A)(I - SM_s^{-T} S^T A).$$

Therefore, the classical two-grid theory is not applicable for Algorithm 1. Compared with the two-grid case, one has more room to design the local smoother $M_s$ instead of limiting it to simple types (e.g., the Jacobi or Gauss–Seidel type). For example, if $M_s$ is taken to be $A_s$, then

$$I - SM_s^{-1} S^T A = I - S A_s^{-1} S^T A,$$

which is an $A$-orthogonal projection along (or parallel to) range($S$) onto null($S^T A$) and hence can remove the error components contained in range($S$). If range($S$) covers most of the high-frequency modes, then the smoothing iteration will eliminate the high-frequency error effectively.

From Algorithm 1, we have

$$u - u_{TL} = \tilde{E}_{TL}(u - u_0)$$

with

$$\tilde{E}_{TL} = (I - SM_s^{-T} S^T A)(I - PB_c^{-1} P^T A)(I - SM_s^{-1} S^T A),$$

which is called the iteration matrix (or error propagation matrix) of Algorithm 1. Define

$$\overline{M}_s := M_s (M_s + M_s^T - A_s)^{-1} M_s^T.$$  

Then, $\tilde{E}_{TL}$ can be expressed as

$$\tilde{E}_{TL} = I - \overline{B}_{TL}^{-1} A,$$
where
\begin{equation}
\tilde{B}_{TL}^{-1} = S\tilde{M}_s^{-1}S^T + (I - SM_s^{-T}S^TA)PB_c^{-1}P^T(I - ASM_s^{-1}S^T).
\end{equation}
Indeed, \( \tilde{B}_{TL}^{-1} \) admits the following hierarchical expression:
\begin{equation}
\tilde{B}_{TL}^{-1} = (S \ P)^\dagger \hat{B}_{TL}^{-1} (S \ P)^T,
\end{equation}
where
\begin{equation}
\hat{B}_{TL} = \begin{pmatrix} I & 0 \\ P^TASM_s^{-1} & I \\ \tilde{M}_s & 0 \\ 0 & B_c \end{pmatrix} \begin{pmatrix} I \ M_s^{-T}S^TAP \ 0 \ I \end{pmatrix}.
\end{equation}
The matrix \( \hat{B}_{TL} \) is referred to as the TLHB preconditioner, whose positive definiteness follows from the positive definiteness of \( \hat{B}_{TL} \) and \( \text{rank}(S \ P) = n \). According to (2.2) and (2.4), we deduce that
\begin{equation}
\|E_{TL}\|_A = \rho(E_{TL}) = \max \{ \lambda_{\max}(\hat{B}_{TL}^{-1}A) - 1, 1 - \lambda_{\min}(\hat{B}_{TL}^{-1}A) \}.
\end{equation}
In particular, if \( B_c = A_c \), then the iteration matrix is denoted by \( E_{TL} \), and
\begin{equation}
E_{TL} = (I - SM_s^{-T}S^T)(I - \Pi_A)(I - SM_s^{-1}S^T),
\end{equation}
where
\begin{equation}
\Pi_A := PA_c^{-1}P^T.
\end{equation}
Similarly, we have
\begin{equation}
E_{TL} = I - B_{TL}^{-1}A.
\end{equation}
where
\begin{equation}
B_{TL}^{-1} = S\tilde{M}_s^{-1}S^T + (I - SM_s^{-T}S^TA)PA_c^{-1}P^T(I - ASM_s^{-1}S^T).
\end{equation}
Note that \( \Pi_A \) is an \( A \)-orthogonal projection. We then have
\begin{equation}
\|E_{TL}\|_A = \lambda_{\max}(E_{TL}) = 1 - \lambda_{\min}(B_{TL}^{-1}A).
\end{equation}
Based on the so-called saddle-point lemma [11, Lemma 3.1], Falgout, Vassilevski, and Zikatanov [11, Theorem 4.1] derived an identity for \( \|E_{TL}\|_A \), as described in the following theorem.

**Theorem 2.2.** Define
\begin{equation}
\tilde{M}_s := M_s^T(M_s + M_s^T - A_s)^{-1}M_s.
\end{equation}
The convergence factor of Algorithm 1 with \( B_c = A_c \) can be characterized as
\begin{equation}
\|E_{TL}\|_A = 1 - \frac{1}{K_{TL}},
\end{equation}
where
\begin{equation}
K_{TL} = \max \limits_{v \in \text{range}(I - \Pi_A)} \min \limits_{v : v^T(I - \Pi_A)sv_x \neq 0} \frac{v^T\tilde{M}_sv_x}{v^TAv_x}.
\end{equation}
The identity (2.15) is valid as long as \( \text{rank}(S \ P) = n \). In particular, if \( (S \ P) \) is square and nonsingular, Falgout, Vassilevski, and Zikatanov [11, Corollary 4.1 and Theorem 4.2] further proved the following results.
Theorem 2.3. If \((S \ P)\) is an \(n \times n\) nonsingular matrix, then

\[
K_{TL} = \max_{v_s \in \mathbb{R}^n \setminus \{0\}} \frac{v_s^T \tilde{M}_s v_s}{v_s^T \tilde{M}_s v_s}
\]

and

\[
K_{TL} \leq \frac{\lambda_{\max}(A_c^{-1} \tilde{M}_s)}{1 - \gamma^2},
\]

where \(\gamma\) is defined by (1.5). In the case of inexact coarse solver, if

\[
\lambda(B_c^{-1} A_c) \subset [\frac{1}{1 + \delta}, 1]
\]

with \(\delta > 0\), then

\[
\|\tilde{E}_{TL}\|_A \leq 1 - \frac{1}{K_{TL} + \frac{\delta}{1 - \gamma^2}}.
\]

3. Convergence analysis of exact TLHB methods

In this section, we present a new convergence analysis of Algorithm 1 with exact coarse solver (under the condition \(\text{rank}(S \ P) = n\)), which gives a succinct identity for the convergence factor \(\|E_{TL}\|_A\). The new identity can be conveniently used to analyze the optimal interpolation and the influence of \(\text{range}(P)\) on \(\|E_{TL}\|_A\).

Observe that the identity (2.15) involves a tricky max-min problem. In general, it is difficult to determine when the ‘min’ is attained, which limits the application of (2.15). Furthermore, the proof of (2.15) provided in [11] is not very direct.

The following theorem gives a new and succinct identity for \(\|E_{TL}\|_A\), whose proof is straightforward.

Theorem 3.1. The convergence factor of Algorithm 1 with \(B_c = A_c\) can be characterized as

\[
\|E_{TL}\|_A = 1 - \sigma_{TL},
\]

where

\[
\sigma_{TL} = \lambda_{\min}^+(S \tilde{M}_s^{-1} S^T A(I - \Pi_A)) = \lambda_{\min}^+(\tilde{M}_s^{-1} S^T A(I - \Pi_A)S).
\]

Proof. By (2.9) and (2.11), we have

\[
B_{TL}^{-1} A = I - (I - S \tilde{M}_s^{-1} S^T A)(I - \Pi_A)(I - S \tilde{M}_s^{-1} S^T A).
\]

Then

\[
\lambda(B_{TL}^{-1} A) = \lambda(I - (I - S \tilde{M}_s^{-1} S^T A)(I - S \tilde{M}_s^{-1} S^T A)(I - \Pi_A))
\]

\[
= \lambda(I - (I - S \tilde{M}_s^{-1} S^T A)(I - \Pi_A))
\]

\[
= \lambda(S \tilde{M}_s^{-1} S^T A(I - \Pi_A) + \Pi_A).
\]

Since \(\Pi^2_A = \Pi_A\) and \(\text{rank}(\Pi_A) = n_c\), there exists a nonsingular matrix \(Y \in \mathbb{R}^{n \times n}\) such that

\[
\Pi_A = Y^{-1} \left( \begin{array}{cc} I_{n_c} & 0 \\ 0 & 0 \end{array} \right) Y.
\]

Let

\[
S \tilde{M}_s^{-1} S^T A = Y^{-1} \left( \begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array} \right) Y,
\]

\[
\Pi_A = Y^{-1} \left( \begin{array}{cc} I_{n_c} & 0 \\ 0 & 0 \end{array} \right) Y.
\]

Let

\[
S \tilde{M}_s^{-1} S^T A = Y^{-1} \left( \begin{array}{cc} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{array} \right) Y,
\]
where $Z_{ij} \in \mathbb{R}^{n_i \times n_j}$ with $n_1 = n_c$ and $n_2 = n - n_c$. Direct computations yield

\begin{equation}
S\tilde{M}_s^{-1}S^TA(I - \Pi_A) = Y^{-1}\begin{pmatrix} 0 & Z_{12} \\ 0 & Z_{22} \end{pmatrix}Y, \tag{3.5}
\end{equation}

\begin{equation}
S\tilde{M}_s^{-1}S^TA(I - \Pi_A) + \Pi_A = Y^{-1}\begin{pmatrix} I_n & Z_{12} \\ 0 & Z_{22} \end{pmatrix}Y. \tag{3.6}
\end{equation}

Note that $A^{\frac{\gamma}{2}}E_{TL}A^{-\frac{\gamma}{2}}$ is symmetric positive semidefinite (SPSD) and $B_{TL}$ is SPD. We then have

$$\lambda(S\tilde{M}_s^{-1}S^TA(I - \Pi_A) + \Pi_A) = \lambda(B_{TL}^{-1}A) \subset (0,1],$$

which, together with (3.6), leads to

$$\lambda(Z_{22}) \subset (0,1].$$

Hence,

$$\lambda_{\text{min}}(B_{TL}^{-1}A) = \lambda_{\text{min}}(S\tilde{M}_s^{-1}S^TA(I - \Pi_A) + \Pi_A) = \lambda_{\text{min}}(Z_{22}) = \lambda_{\text{min}}^{+}(S\tilde{M}_s^{-1}S^TA(I - \Pi_A)), $$

where we have used the expressions (3.5) and (3.6). The identity (3.1) then follows immediately from (2.13).

**Remark 3.2.** According to the proof of Theorem 3.1, we deduce that

\begin{equation}
\lambda(S\tilde{M}_s^{-1}S^TA(I - \Pi_A)) = \{ 0, \ldots, 0, \nu_1, \ldots, \nu_{n-n_c} \}, \tag{3.7}
\end{equation}

where $0 < \nu_i \leq 1$ for all $i = 1, \ldots, n - n_c$. Then

\begin{equation}
\lambda(S\tilde{M}_s^{-1}S^TA(I - \Pi_A)S) = \{ 0, \ldots, 0, \nu_1, \ldots, \nu_{n-n_c} \}. \tag{3.8}
\end{equation}

In particular, if $(S\ P)$ is an $n \times n$ nonsingular matrix, then

$$\sigma_{\text{TL}}^{-1} = \lambda_{\text{min}}^{-1}(M_s^{-1}S^TA(I - \Pi_A)S)$$

\begin{equation}
= \left( \min_{v_s \in \mathbb{R}^{n_s} \setminus \{0\}} \frac{v_s^T S^T A(I - \Pi_A) S v_s}{v_s^T M_s v_s} \right)^{-1}
\end{equation}

\begin{equation}
= \max_{v_s \in \mathbb{R}^{n_s} \setminus \{0\}} \frac{v_s^T \tilde{M}_s v_s}{v_s^T S^T A(I - \Pi_A) S v_s},
\end{equation}

which gives the expression (2.17). If $M_s$ is further taken to be $A_s$, then

$$\|E_{\text{TL}}\|_A = 1 - \lambda_{\text{min}}(A_s^{-1}S^TA(I - \Pi_A)S)$$

\begin{equation}
= 1 - \lambda_{\text{min}}(I - A_s^{-1}S^TA \gamma^{-1} P T A S)$$

\begin{equation}
= \lambda_{\max}(A_s^{-1}S^TA \gamma^{-1} P T A S) \tag{2.17}
\end{equation}

$$\|E_{\text{TL}}\|_2 = \|A_s^{-\frac{\gamma}{2}} S^T A \gamma^{-\frac{\gamma}{2}} \|^2 = \gamma^2,$$

where $\gamma$ is defined by (1.5).

The proof of Theorem 3.1 also yields a characterization for the spectrum of $E_{\text{TL}}$, as described in the following corollary.
Corollary 3.3. The spectrum of $E_{TL}$ is given by
\begin{equation}
\lambda(E_{TL}) = \{0, \ldots, 0, 1 - \nu_1, \ldots, 1 - \nu_{n_c}\},
\end{equation}

where $\{\nu_i\}_{i=1}^{n_c}$ are the positive eigenvalues of $\tilde{M}_s^{-1}S^T A(I - \Pi_A)S$.

Compared with (2.15), the identity (3.1) is more convenient for TLHB analysis. Of particular interest is an interpolation $P$ that minimizes the convergence factor $\|E_{TL}\|_A$, provided that $S$ and $M_s$ are preselected.

Using (3.1), we can derive the following optimal interpolation theory.

Theorem 3.4. Let $\{(\mu_i, v_i)\}_{i=1}^n$ be the eigenpairs of $S\tilde{M}_s^{-1}S^T A$, where
\[\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \quad \text{and} \quad v_i^T A v_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}\]

Then
\begin{equation}
\|E_{TL}\|_A \geq 1 - \mu_{n_c},
\end{equation}

and the equality holds if $\text{range}(P) = \text{span}\{v_1, \ldots, v_{n_c}\}$.

Proof. Due to $\bar{M}_s - A_s$ is SPSD and $S\tilde{M}_s^{-1}S^T A$ has the same nonzero eigenvalues as $\tilde{M}_s^{-1}A_s$, it follows that
\[0 = \mu_1 = \cdots = \mu_{n-n_s} < \mu_{n-n_s+1} \leq \cdots \leq \mu_n \leq 1.\]

Let
\[V = (v_1, \ldots, v_n) \quad \text{and} \quad U_1 = V^{-1}P(P^T V^{-T} V^{-1} P)^{-\frac{1}{2}}.\]

It is easy to check that $V^T A V = I$ and $U_1$ is an $n \times n_c$ matrix with orthonormal columns (i.e., $U_1^T U_1 = I_{n_c}$). Let $U_2$ be an $n \times (n - n_c)$ matrix such that $(U_1 U_2)$ is orthogonal. Then
\[S\tilde{M}_s^{-1}S^T A(I - \Pi_A) = S\tilde{M}_s^{-1}S^T A(I - PA_c^{-1}P^T A) = S\tilde{M}_s^{-1}S^T A(I - V U_1 U_1^T V^{-1}) = S\tilde{M}_s^{-1}S^T AV(I - U_2 U_2^T V^{-1}) = V \Sigma U_2 U_2^T V^{-1},\]

where $\Sigma = \text{diag}(0, \ldots, 0, \mu_{n-n_s+1}, \ldots, \mu_n)$.

According to (3.7) and
\[\Sigma U_2 U_2^T = V^{-1} S\tilde{M}_s^{-1}S^T A(I - \Pi_A) V,\]

we deduce that $\Sigma U_2 U_2^T$ has $n - n_c$ positive eigenvalues. Since
\[\begin{bmatrix} U_1 \ U_2 \end{bmatrix}\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \Sigma U_2 U_2^T (U_1 \ U_2) = \begin{bmatrix} 0 \\ U_2^T \Sigma U_2 \end{bmatrix},\]

it follows that $U_2^T \Sigma U_2$ is positive definite. Hence,
\[\sigma_{TL} = \lambda_{\min}^+(S\tilde{M}_s^{-1}S^T A(I - \Pi_A)) = \lambda_{\min}^+(\Sigma U_2 U_2^T) = \lambda_{\min}(U_2^T \Sigma U_2).
\]

Using the Poincaré separation theorem (see, e.g., [13, Corollary 4.3.37]), we obtain
\[\lambda_{\min}(U_2^T \Sigma U_2) = \lambda_1(U_2^T \Sigma U_2) \leq \lambda_{n_c+1}(\Sigma) = \mu_{n_c+1}.\]
Consequently, 
\[ \|E_{TL}\|_A = 1 - \lambda_{\min}(U_2^T \Sigma U_2) \geq 1 - \mu_{n+1}. \]

In particular, if range\((P) = \text{span}\{v_1, \ldots, v_{n_c}\}\), then there exists a nonsingular matrix \(P_1 \in \mathbb{R}^{n_c \times n_c}\) such that
\[ P = V \begin{pmatrix} P_1 \\ 0 \end{pmatrix}. \]

In this case,
\[ U_1 = \begin{pmatrix} P_1 \\ 0 \end{pmatrix} (P_1^T P_1)^{-\frac{1}{2}}. \]

Then
\[ U_2^T = I - U_1 U_1^T = I - \begin{pmatrix} P_1 \\ 0 \end{pmatrix} (P_1^T P_1)^{-1} \begin{pmatrix} P_1^T \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & I_{n-n_c} \end{pmatrix}. \]

Hence,
\[ \|E_{TL}\|_A = 1 - \lambda_{\min}(\Sigma U_2^T U_2^T) = 1 - \mu_{n+1}. \]

This completes the proof. \(\square\)

**Remark 3.5.** As mentioned in [11, Page 483], \(A_s\) happens to be well-conditioned in the classical TLHB methods, so it is not that impractical to take \(M_s = A_s\). In such a case, the eigenvalues of \(\tilde{S}^{-1} M_s^{-1} S^T A\) are
\[ 0 = \mu_1 = \cdots = \mu_{n-n_s} < \mu_{n-n_s+1} = \cdots = \mu_{n} = 1. \]

Then, \(\mu_{n+1} = 1\) (since \(n_c \geq n-n_s\)), which gives the optimal convergence factor 0.

**Remark 3.6.** Unlike the optimal interpolation theory for two-grid methods [20, 7], \(\tilde{S}^{-1} M_s^{-1} S^T\) here is a singular matrix. That is, \((\tilde{S}^{-1} M_s^{-1} S^T)^{-1}\) is not well-defined and hence cannot induce an inner product in \(\mathbb{R}^n\).

Besides the optimal interpolation analysis, the identity (3.1) is also convenient for analyzing the influence of range\((P)\) on \(\|E_{TL}\|_A\).

**Theorem 3.7.** Let \(\tilde{P} \in \mathbb{R}^{n \times \tilde{n}_c}\) be of full column rank (with \(n_c \leq \tilde{n}_c < n\)), and let range\((S \tilde{P}) = n\). If range\((P) \subseteq\) range\((\tilde{P})\), then
\[ \sigma_{TL} \leq \tilde{\sigma}_{TL}, \]
where \(\sigma_{TL}\) is given by (3.2) and
\[ \tilde{\sigma}_{TL} = \lambda_{\min}^+(\tilde{S}\tilde{M}_s^{-1} S^T A(I - \hat{\tilde{P}}(\hat{\tilde{P}}^T A\hat{\tilde{P}})^{-1}\hat{\tilde{P}}^T A)). \]

**Proof.** Since range\((P) \subseteq\) range\((\tilde{P})\), there exists an \(\tilde{n}_c \times n_c\) matrix \(W\) such that \(P = \tilde{P} W\).

Note that \(W \in \mathbb{R}^{\tilde{n}_c \times n_c}\) is of full column rank. One can find an \(\tilde{n}_c \times n_c\) nonsingular matrix \(\tilde{W}\) such that
\[ W = \tilde{W} \begin{pmatrix} I_{n_c} \\ 0 \end{pmatrix}, \]
which yields
\[ P = (\tilde{P} \tilde{W}) \begin{pmatrix} I_{n_c} \\ 0 \end{pmatrix}. \]

Hence,
\[ \tilde{P} = (P C) \tilde{W}^{-1} \]
for some $C \in \mathbb{R}^{n \times (\tilde{n}_c - n_c)}$. From (3.12), we have

$$
\sigma_{TL} = \lambda_{\min}^+ \left( S \tilde{M}_s^{-1/2} S^T A (I - \hat{P}_0 (\hat{P}_0^T A \hat{P}_0)^{-1} \hat{P}_0^T A) \right),
$$

where $\hat{P}_0 = (P^T C)$.

In light of (3.2), (3.8), and (3.13), we have

$$
\sigma_{TL} = \lambda_{\min}^+ \left( \tilde{M}_s^{-1/2} S^T A (I - \Pi_A) S \tilde{M}_s^{-1/2} \right) = \lambda_{n_c + n_c - n + 1} \left( \tilde{M}_s^{-1/2} S^T A (I - \Pi_A) S \tilde{M}_s^{-1/2} \right),
$$

Then

$$
D = \tilde{M}_s^{-1/2} S^T A (I - \Pi_A) S \tilde{M}_s^{-1/2} - \tilde{M}_s^{-1/2} S^T A (I - \hat{P}_0 (\hat{P}_0^T A \hat{P}_0)^{-1} \hat{P}_0^T A) S \tilde{M}_s^{-1/2}.
$$

Let

$$
D = \tilde{M}_s^{-1/2} S^T A (I - \Pi_A) S \tilde{M}_s^{-1/2} - \tilde{M}_s^{-1/2} S^T A (I - \hat{P}_0 (\hat{P}_0^T A \hat{P}_0)^{-1} \hat{P}_0^T A) S \tilde{M}_s^{-1/2}.
$$

Then

$$
D = \tilde{M}_s^{-1/2} S^T A \left[ \hat{P}_0 (\hat{P}_0^T A \hat{P}_0)^{-1} \hat{P}_0^T A - P (P^T A P)^{-1} P^T A \right] S \tilde{M}_s^{-1/2}
$$

$$
= \tilde{M}_s^{-1/2} S^T A \left[ \hat{P}_0 (\hat{P}_0^T A \hat{P}_0)^{-1} \hat{P}_0^T A - \hat{P}_0 \left( \frac{(P^T A P)^{-1}}{0} \right) \hat{P}_0^T A \right] S \tilde{M}_s^{-1/2}
$$

$$
= \tilde{M}_s^{-1/2} S^T A \hat{P}_0 \left[ (\hat{P}_0^T A \hat{P}_0)^{-1} - \left( \frac{(P^T A P)^{-1}}{0} \right) \hat{P}_0^T A \right] S \tilde{M}_s^{-1/2}.
$$

It is easy to verify that

$$
(\hat{P}_0^T A \hat{P}_0)^{-1} - \left( \frac{(P^T A P)^{-1}}{0} \right)
$$

is an SPSD matrix of rank $\tilde{n}_c - n_c$ [15, Lemma 2.7]. Accordingly, $D$ is an SPSD matrix of rank at most $\tilde{n}_c - n_c$. Using [13, Corollary 4.3.5], we obtain

$$
\sigma_{TL} = \lambda_{n_c + n_c - n + 1} \left( \tilde{M}_s^{-1/2} S^T A (I - \Pi_A) S \tilde{M}_s^{-1/2} \right) = \lambda_{n_c + n_c - n + 1} \left( \tilde{M}_s^{-1/2} S^T A (I - \hat{P}_0 (\hat{P}_0^T A \hat{P}_0)^{-1} \hat{P}_0^T A) S \tilde{M}_s^{-1/2} + D \right)
$$

$$
\leq \lambda_{n_c + n_c - n + 1 + \text{rank}(D)} \left( \tilde{M}_s^{-1/2} S^T A (I - \hat{P}_0 (\hat{P}_0^T A \hat{P}_0)^{-1} \hat{P}_0^T A) S \tilde{M}_s^{-1/2} \right)
$$

$$
\leq \lambda_{n_c + \tilde{n}_c - n + 1} \left( \tilde{M}_s^{-1/2} S^T A (I - \hat{P}_0 (\hat{P}_0^T A \hat{P}_0)^{-1} \hat{P}_0^T A) S \tilde{M}_s^{-1/2} \right) = \sigma_{TL}.
$$

This completes the proof.

Remark 3.8. According to (3.1) and (3.11), we deduce that $\|E_{TL}\|_A$ decreases when increasing the number of columns in $P$ (i.e., $n_c$). In other words, $n_c$ cannot be very small in order to achieve a satisfactory convergence.

4. Convergence analysis of inexact TLHB methods

In practice, the Galerkin coarse-grid system is often too costly to solve exactly. Without essential loss of convergence speed, it is advisable to solve the problem approximately (one way is to apply Algorithm 1 recursively). In this section, we establish a new convergence theory for Algorithm 1 with inexact coarse solver under the condition rank($S^T P$) = $n$. 


We remark that the estimate (2.19) is only applicable for \( n_c = n - n_s \). In fact, if \( n_c > n - n_s \), then
\[
\begin{pmatrix} S^T \\ P^T \end{pmatrix} A \begin{pmatrix} S & P \\ P^T A S & A_c \end{pmatrix}
\]
is an SPSD matrix. Hence, the Schur complement \( A_s - S^T APA_c^{-1}P^T AS \) is SPSD, which leads to the positive semidefiniteness of \( I - A_s^{-\frac{1}{2}}S^T APA_c^{-1}P^T ASA_s^{-\frac{1}{2}} \). Then
\[
\gamma^2 = \lambda_{\max} \left( A_s^{-\frac{1}{2}}S^T APA_c^{-1}P^T ASA_s^{-\frac{1}{2}} \right) = 1.
\]
Obviously, the upper bound in (2.19) is always 1, no matter how close \( B_c \) is to \( A_c \). That is, the estimate (2.19) will be trivial if \( n_c > n - n_s \).

In what follows, we establish two-sided bounds for the convergence factor \( \|E_{TL}\|_A \) under the general condition \( n_c \geq n - n_s \).

The following identities on the extreme eigenvalues of \((I - \tilde{S}M^{-1}_s S^T A)(I - \Pi_A)\) and \((I - \tilde{S}M^{-1}_s S^T A)\Pi_A\) will be frequently used in the subsequent analysis.

**Lemma 4.1.** The following eigenvalue identities hold:

\[
\begin{align*}
(4.1a) \quad & \lambda_{\min}\left((I - \tilde{S}M^{-1}_s S^T A)(I - \Pi_A)\right) = 0, \\
(4.1b) \quad & \lambda_{\max}\left((I - \tilde{S}M^{-1}_s S^T A)(I - \Pi_A)\right) = 1 - \sigma_{TL}, \\
(4.1c) \quad & \lambda_{\min}\left((I - \tilde{S}M^{-1}_s S^T A)\Pi_A\right) = 0, \\
(4.1d) \quad & \lambda_{\max}\left((I - \tilde{S}M^{-1}_s S^T A)\Pi_A\right) = \begin{cases} 1 - \lambda_{\min}^+\left(\tilde{S}M^{-1}_s S^T A\Pi_A\right), & \text{if } r = n_c, \\
1, & \text{if } r < n_c, \end{cases}
\end{align*}
\]
where \( r = \text{rank}(S^T A) \).

**Proof.** The positive semidefiniteness of \( \tilde{M}_s - A_s \) implies that \( I - A_s^{\frac{1}{2}}\tilde{S}M^{-1}_s S^T A_s^{\frac{1}{2}} \) is SPSD, which yields the positive semidefiniteness of \( A^{-1} - \tilde{S}M^{-1}_s S^T \). Then
\[
\lambda\left((I - \tilde{S}M^{-1}_s S^T A)(I - \Pi_A)\right) = \lambda\left((A^{-1} - \tilde{S}M^{-1}_s S^T)A(I - \Pi_A)\right)
= \lambda\left((A^{-1} - \tilde{S}M^{-1}_s S^T)^{\frac{1}{2}}A(I - \Pi_A)(A^{-1} - \tilde{S}M^{-1}_s S^T)^{\frac{1}{2}}\right)
\subset [0, +\infty).
\]
Similarly,
\[
\lambda\left((I - \tilde{S}M^{-1}_s S^T A)\Pi_A\right) \subset [0, +\infty).
\]
The identities (4.1a) and (4.1c) then follow immediately from the facts
\[
\det\left((I - \tilde{S}M^{-1}_s S^T A)(I - \Pi_A)\right) = 0 \quad \text{and} \quad \det\left((I - \tilde{S}M^{-1}_s S^T A)\Pi_A\right) = 0.
\]
Here, \( \det(\cdot) \) denotes the determinant of a matrix.

According to the proof of Theorem 3.1, it holds that
\[
\lambda_{\max}\left((I - \tilde{S}M^{-1}_s S^T A)(I - \Pi_A)\right) = 1 - \lambda_{\min}\left(B_{TL}^{-1}A\right) = 1 - \sigma_{TL},
\]
which is exactly the identity (4.1b). By (3.3) and (3.4), we have
\[
\tilde{S}M^{-1}_s S^T A\Pi_A = Y^{-1} \begin{pmatrix} Z_{11} & 0 \\ Z_{21} & 0 \end{pmatrix} Y,
\]
\[
(I - \tilde{S}M^{-1}_s S^T A)\Pi_A = Y^{-1} \begin{pmatrix} I_{n_c} - Z_{11} & 0 \\ -Z_{21} & 0 \end{pmatrix} Y.
\]
Then
\[ \lambda_{\max}\left((I - \tilde{S}\tilde{M}_{s}^{-1}ST)A\right) = 1 - \lambda_{\min}(Z_{11}). \]

- If \( r = n_c \), then \( \text{rank}\left(\tilde{M}_{s}^{1/2}STAP\tilde{A}^{1/2}\right) = n_c \), which leads to the positive definiteness of \( A_{c}^{1/2}PT AS\tilde{M}_{s}^{-1}STAP\tilde{A}^{1/2} \). This implies that \( \tilde{S}\tilde{M}_{s}^{-1}ST A\Pi_{A} \) has \( n_c \) positive eigenvalues. Consequently,
\[ \lambda_{\min}(Z_{11}) = \lambda_{\min}^{+}(\tilde{S}\tilde{M}_{s}^{-1}ST A\Pi_{A}). \]

- If \( r < n_c \), we deduce from the above argument that \( \lambda_{\min}(Z_{11}) = 0 \).

Thus, the identity (4.1d) is proved. \( \square \)

To analyze the convergence of Algorithm 1, we need an important tool for eigenvalue analysis, i.e., the well-known Weyl's theorem (see, e.g., [13, Theorem 4.3.1]).

**Lemma 4.2.** Let \( H_1, H_2 \in \mathbb{C}^{n \times n} \) be Hermitian. Assume that the spectra of \( H_1, H_2, \) and \( H_1 + H_2 \) are \( \{\lambda_i(H_1)\}_{i=1}^{n}, \{\lambda_i(H_2)\}_{i=1}^{n}, \) and \( \{\lambda_i(H_1 + H_2)\}_{i=1}^{n} \), respectively. Then, for each \( k = 1, \ldots, n \), it holds that
\[ \lambda_{k-j+1}(H_1) + \lambda_j(H_2) \leq \lambda_k(H_1 + H_2) \leq \lambda_{k+r}(H_1) + \lambda_{n-r}(H_2) \]
for all \( j = 1, \ldots, k \) and \( r = 0, \ldots, n-k \). In particular, one has
\begin{align}
(4.2a) & \quad \lambda_{\min}(H_1 + H_2) \geq \lambda_{\min}(H_1) + \lambda_{\min}(H_2), \\
(4.2b) & \quad \lambda_{\min}(H_1 + H_2) \leq \min \{\lambda_{\min}(H_1) + \lambda_{\max}(H_2), \lambda_{\max}(H_1) + \lambda_{\min}(H_2)\}, \\
(4.2c) & \quad \lambda_{\max}(H_1 + H_2) \geq \max \{\lambda_{\max}(H_1) + \lambda_{\min}(H_2), \lambda_{\min}(H_1) + \lambda_{\max}(H_2)\}, \\
(4.2d) & \quad \lambda_{\max}(H_1 + H_2) \leq \lambda_{\max}(H_1) + \lambda_{\max}(H_2). 
\end{align}

**Remark 4.3.** Certainly, the Weyl's theorem is applicable for real symmetric matrices. It is worth noting that this theorem can also be applied to the nonsymmetric matrix \( (I - \tilde{S}\tilde{M}_{s}^{-1}ST)A(I - t\Pi_{A}) \) with a parameter \( t \), which is based on the fact
\[ \lambda((I - \tilde{S}\tilde{M}_{s}^{-1}ST)A(I - t\Pi_{A})) = \lambda((A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}A(I - t\Pi_{A})(A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}). \]
One can first apply the Weyl's theorem to the symmetric matrix
\[ (A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}A(I - t\Pi_{A})(A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}, \]
and then transform the result into a form related to
\[ I - \tilde{S}\tilde{M}_{s}^{-1}ST A, \quad (I - \tilde{S}\tilde{M}_{s}^{-1}ST)A\Pi_{A}, \quad \text{or} \quad (I - \tilde{S}\tilde{M}_{s}^{-1}ST A)(I - \Pi_{A}). \]
For example, if \( t \geq 0 \), using (4.2a), we obtain
\[ \lambda_{\min}\left((I - \tilde{S}\tilde{M}_{s}^{-1}ST)A(I - t\Pi_{A})\right) \]
\[ = \lambda_{\min}\left((A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}A(I - t\Pi_{A})(A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}\right) \]
\[ \geq \lambda_{\min}\left((A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}A(I - t\Pi_{A})(A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}\right) \]
\[ - t\lambda_{\max}\left((A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}A\Pi_{A}(A^{-1} - \tilde{S}\tilde{M}_{s}^{-1}ST)^{1/2}\right) \]
\[ = \lambda_{\min}(I - \tilde{S}\tilde{M}_{s}^{-1}ST A) - t\lambda_{\max}(I - \tilde{S}\tilde{M}_{s}^{-1}ST A)\Pi_{A}. \]
For the sake of brevity, such a trick will be implicitly used in the proof of the next theorem.

Now, we are ready to present the new convergence theory for Algorithm 1 with inexact coarse solver.
Theorem 4.4. Let

\begin{equation}
\alpha = \lambda_{\min}(B_c^{-1}A_c) \quad \text{and} \quad \beta = \lambda_{\max}(B_c^{-1}A_c).
\end{equation}

(i) If rank\((S^TAP) = n_c\), then

\begin{equation}
\mathcal{L}_1 \leq \|\tilde{E}_{TL}\|_A \leq \mathcal{V}_1,
\end{equation}

where

\begin{align*}
\mathcal{L}_1 &= \begin{cases}
1 - \min\{\beta - \beta\lambda_{\min}^+(S\tilde{M}_s^{-1}S^TAP), \sigma_{TL}\}, & \text{if } \beta \leq 1, \\
1 - \min\{\lambda_{\max}(\tilde{M}_s^{-1}A_s), \beta\sigma_{TL}\}, & \text{if } \alpha \leq 1 < \beta, \\
\max\{\alpha - 1 - \alpha\lambda_{\min}^+(S\tilde{M}_s^{-1}S^TAP), 1 - \lambda_{\max}(\tilde{M}_s^{-1}A_s), \\
(\alpha - 1)(1 - \lambda_{\max}(\tilde{M}_s^{-1}A_s)), 1 - \beta\sigma_{TL}\}, & \text{if } 1 < \alpha,
\end{cases} \\
\mathcal{V}_1 &= \begin{cases}
1 - \alpha\sigma_{TL}, & \text{if } \beta \leq 1, \\
\max\{1 - \alpha\sigma_{TL}, (\beta - 1)(1 - \lambda_{\min}^+(S\tilde{M}_s^{-1}S^TAP))\}, & \text{if } \alpha \leq 1 < \beta, \\
\max\{1 - \sigma_{TL}, (\beta - 1)(1 - \lambda_{\min}^+(S\tilde{M}_s^{-1}S^TAP))\}, & \text{if } 1 < \alpha.
\end{cases}
\end{align*}

(ii) If rank\((S^TAP) < n_c\), then

\begin{equation}
\mathcal{L}_2 \leq \|\tilde{E}_{TL}\|_A \leq \mathcal{V}_2,
\end{equation}

where

\begin{align*}
\mathcal{L}_2 &= \begin{cases}
1 - \min\{\beta, \sigma_{TL}\}, & \text{if } \beta \leq 1, \\
1 - \min\{\lambda_{\max}(\tilde{M}_s^{-1}A_s), \beta\sigma_{TL}\}, & \text{if } \alpha \leq 1 < \beta, \\
\max\{\alpha - 1, 1 - \lambda_{\max}(\tilde{M}_s^{-1}A_s), 1 - \beta\sigma_{TL}\}, & \text{if } 1 < \alpha,
\end{cases} \\
\mathcal{V}_2 &= \begin{cases}
1 - \alpha\sigma_{TL}, & \text{if } \beta \leq 1, \\
\max\{1 - \alpha\sigma_{TL}, \beta - 1\}, & \text{if } \alpha \leq 1 < \beta, \\
\max\{1 - \sigma_{TL}, \beta - 1\}, & \text{if } 1 < \alpha.
\end{cases}
\end{align*}

Proof. By (2.2) and (2.4), we have

\[B_{TL}^{-1}A = I - (I - SM_s^{-T}S^TA)(I - PB_c^{-1}PTA)(I - SM_s^{-1}S^T),\]

which yields

\[\lambda(B_{TL}^{-1}A) = \lambda(I - (I - SM_s^{-1}S^TA)(I - SM_s^{-T}S^TA)(I - PB_c^{-1}PTA))\]

\[= \lambda(I - (I - SM_s^{-1}S^T)(I - PB_c^{-1}PTA)).\]

Then

\[\lambda_{\max}(B_{TL}^{-1}A) = 1 - \lambda_{\min}((I - SM_s^{-1}S^T)(I - PB_c^{-1}PTA)),\]

\[\lambda_{\min}(B_{TL}^{-1}A) = 1 - \lambda_{\max}((I - SM_s^{-1}S^T)(I - PB_c^{-1}PTA)).\]

Note that \((I - SM_s^{-1}S^TA)(I - PB_c^{-1}PTA)\) has the same spectrum as the symmetric matrix \((A^{-1} - SM_s^{-1}S^T)^*A(I - PB_c^{-1}PTA)(A^{-1} - SM_s^{-1}S^T)^*\). In view of (4.3), we have

\begin{align}
-\alpha_1 &\leq \lambda_{\max}(B_{TL}^{-1}A) - 1 \leq -\alpha_2, \\
 b_2 &\leq 1 - \lambda_{\min}(B_{TL}^{-1}A) \leq b_1,
\end{align}
where
\[ a_1 = \lambda_{\min}((I - S\tilde{M}_s^{-1}S^T A)(I - \alpha I_A)), \]
\[ a_2 = \lambda_{\min}((I - S\tilde{M}_s^{-1}S^T A)(I - \beta I_A)), \]
\[ b_1 = \lambda_{\max}((I - S\tilde{M}_s^{-1}S^T A)(I - \alpha I_A)), \]
\[ b_2 = \lambda_{\max}((I - S\tilde{M}_s^{-1}S^T A)(I - \beta I_A)). \]

According to (2.8), (4.6a), and (4.6b), we deduce that
\[ \max\{-a_1, b_2\} \leq \|\tilde{E}_{TL}\|_A \leq \max\{-a_2, b_1\}. \]

The remaining task is to establish the upper bounds for \( a_1 \) and \( b_1 \), as well as the lower bounds for \( a_2 \) and \( b_2 \).

**Case 1:** \( \text{rank}(S^T A) = n_c \).

**Subcase 1.1:** \( \beta \leq 1 \). By (4.2b), we have that
\[ a_1 = \lambda_{\min}((I - S\tilde{M}_s^{-1}S^T A)(I - \alpha I_A)) \]
\[ \leq \lambda_{\min}(I - S\tilde{M}_s^{-1}S^T A) - \alpha \lambda_{\min}(I - S\tilde{M}_s^{-1}S^T A)\Pi_A) \]
\[ = 1 - \lambda_{\max}(\tilde{M}_s^{-1}A_s) \]

and
\[ a_1 = \lambda_{\min}((I - S\tilde{M}_s^{-1}S^T A)(I - I_A + (1 - \alpha)\Pi_A)) \]
\[ \leq \lambda_{\min}(I - S\tilde{M}_s^{-1}S^T A) - \alpha \lambda_{\max}(I - S\tilde{M}_s^{-1}S^T A)\Pi_A) \]
\[ = 1 - \alpha - (1 - \alpha)\lambda_{\min}(S\tilde{M}_s^{-1}S^T A\Pi_A), \]

where we have used the identities (4.1a), (4.1c), and (4.1d). We then have
\[ a_1 \leq 1 - \max\{\lambda_{\max}(\tilde{M}_s^{-1}A_s), \alpha + (1 - \alpha)\lambda_{\min}(S\tilde{M}_s^{-1}S^T A\Pi_A)\}. \]

Using (4.2a), we obtain
\[ a_2 = \lambda_{\min}((I - S\tilde{M}_s^{-1}S^T A)((1 - \beta)I + \beta(I - I_A))) \]
\[ \geq (1 - \beta)\lambda_{\min}(I - S\tilde{M}_s^{-1}S^T A) + \beta\lambda_{\min}((I - S\tilde{M}_s^{-1}S^T A)(I - I_A)), \]

which, together with (4.1a), yields
\[ a_2 \geq (1 - \beta)(1 - \lambda_{\max}(\tilde{M}_s^{-1}A_s))^\lambda_{\max}(S\tilde{M}_s^{-1}S^T A\Pi_A). \]

By (4.2d), we have
\[ b_1 = \lambda_{\max}((I - S\tilde{M}_s^{-1}S^T A)((1 - \alpha)I + \alpha(I - I_A))) \]
\[ \leq (1 - \alpha)\lambda_{\max}(I - S\tilde{M}_s^{-1}S^T A) + \alpha\lambda_{\max}((I - S\tilde{M}_s^{-1}S^T A)(I - I_A)). \]

The above inequality, together with (4.1b), gives
\[ b_1 \leq 1 - \alpha\sigma_{TL}. \]

In light of (4.2c), we have that
\[ b_2 = \lambda_{\max}((I - S\tilde{M}_s^{-1}S^T A)(I - I_A)) \]
\[ \geq \lambda_{\max}(I - S\tilde{M}_s^{-1}S^T A) - \beta\lambda_{\max}((I - S\tilde{M}_s^{-1}S^T A)\Pi_A) \]
\[ = 1 - \beta + \beta\lambda_{\min}(S\tilde{M}_s^{-1}S^T A\Pi_A) \]
and
\[ b_2 = \lambda_{\max}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A + (1 - \beta)\Pi_A)) \]
\[ \geq \lambda_{\max}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A)) + (1 - \beta)\lambda_{\min}((I - S\tilde{M}_s^{-1}S^TA)\Pi_A) \]
\[ = 1 - \sigma_{\text{TL}}, \]
where we have used the identities (4.1b)–(4.1d). Then
\[ b_2 \geq 1 - \min \{ \beta - \beta\lambda_{\min}^+(S\tilde{M}_s^{-1}S^TA\Pi_A), \sigma_{\text{TL}} \}. \tag{4.11} \]

**Subcase 1.2:** \( \alpha \leq 1 < \beta \). In this subcase, the inequalities (4.8) and (4.10) still hold. We next focus on the lower bounds for \( a_2 \) and \( b_2 \). Using (4.2a), we obtain
\[ a_2 = \lambda_{\min}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A + (1 - \beta)\Pi_A)) \]
\[ \geq \lambda_{\min}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A)) + (1 - \beta)\lambda_{\max}((I - S\tilde{M}_s^{-1}S^TA)\Pi_A), \]
which, together with (4.1a) and (4.1d), leads to
\[ a_2 \geq (1 - \beta)(1 - \lambda_{\min}^+(S\tilde{M}_s^{-1}S^TA\Pi_A)). \tag{4.12} \]

By (4.1b), (4.1c), and (4.2c), we have that
\[ b_2 = \lambda_{\max}((I - S\tilde{M}_s^{-1}S^TA)(I - \beta\Pi_A)) \]
\[ \geq \lambda_{\min}(I - S\tilde{M}_s^{-1}S^TA) - \beta\lambda_{\min}((I - S\tilde{M}_s^{-1}S^TA)\Pi_A) \]
\[ = 1 - \lambda_{\max}(\tilde{M}_s^{-1}A_s) \]
and
\[ b_2 = \lambda_{\max}((I - S\tilde{M}_s^{-1}S^TA)((1 - \beta)I + (1 - \beta)(I - \Pi_A))) \]
\[ \geq (1 - \beta)\lambda_{\max}(I - S\tilde{M}_s^{-1}S^TA) + \beta\lambda_{\max}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A)) \]
\[ = 1 - \lambda_{\max}(\tilde{M}_s^{-1}A_s) \]
Accordingly,
\[ b_2 \geq 1 - \min \{ \lambda_{\max}(\tilde{M}_s^{-1}A_s), \beta\sigma_{\text{TL}} \}. \tag{4.13} \]

**Subcase 1.3:** \( 1 < \alpha \). In this subcase, the estimates (4.12) and (4.13) are still valid. We then consider the upper bounds for \( a_1 \) and \( b_1 \). In light of (4.1a), (4.1d), and (4.2b), we have that
\[ a_1 = \lambda_{\min}((I - S\tilde{M}_s^{-1}S^TA)(I - \alpha\Pi_A)) \]
\[ \leq \lambda_{\max}(I - S\tilde{M}_s^{-1}S^TA) - \alpha\lambda_{\min}((I - S\tilde{M}_s^{-1}S^TA)\Pi_A) \]
\[ = 1 - \alpha + \alpha\lambda_{\min}^+(S\tilde{M}_s^{-1}S^TA\Pi_A) \]
and
\[ a_1 = \lambda_{\min}((I - S\tilde{M}_s^{-1}S^TA)((1 - \alpha)I + \alpha(I - \Pi_A))) \]
\[ \leq (1 - \alpha)\lambda_{\min}(I - S\tilde{M}_s^{-1}S^TA) + \alpha\lambda_{\min}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A)) \]
\[ = 1 - \alpha + (\alpha - 1)\lambda_{\max}(-\tilde{M}_s^{-1}A_s). \]
Hence,
\[ a_1 \leq 1 - \alpha + \min \{ \alpha\lambda_{\min}^+(S\tilde{M}_s^{-1}S^TA\Pi_A), (\alpha - 1)\lambda_{\max}(\tilde{M}_s^{-1}A_s) \}. \tag{4.14} \]
Using (4.2d), we obtain
\[ b_1 = \lambda_{\text{max}}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A + (1 - \alpha)\Pi_A)) \]
\[ \leq \lambda_{\text{max}}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A)) + (1 - \alpha)\lambda_{\text{min}}((I - S\tilde{M}_s^{-1}S^TA)\Pi_A), \]
which, together with (4.1b) and (4.1c), yields
\[ b_1 \leq 1 - \sigma_{\text{TL}}. \tag{4.15} \]

Combining (4.7)–(4.15), we can arrive at the estimate (4.4) immediately.

**Case 2:** \( \text{rank}(S^TAP) < n_c \). The detailed proof of Case 2 is omitted for the sake of conciseness. With the identities (4.1a)–(4.1c) and
\[ \lambda_{\text{max}}((I - S\tilde{M}_s^{-1}S^TA)\Pi_A) = 1, \]
one can prove the following inequalities similarly.

**Subcase 2.1:** \( \beta \leq 1 \). It is easy to show that
\[ a_1 \leq 1 - \max\{\lambda_{\text{max}}(\tilde{M}_s^{-1}A), \alpha\}, \tag{4.16} \]
\[ a_2 \geq (1 - \beta)(1 - \lambda_{\text{max}}(\tilde{M}_s^{-1}A)), \tag{4.17} \]
\[ b_1 \leq 1 - \alpha\sigma_{\text{TL}}, \tag{4.18} \]
\[ b_2 \geq 1 - \min\{\beta, \sigma_{\text{TL}}\}. \tag{4.19} \]

**Subcase 2.2:** \( \alpha \leq 1 < \beta \). In this subcase, the inequalities (4.16) and (4.18) still hold. In addition,
\[ a_2 \geq 1 - \beta, \tag{4.20} \]
\[ b_2 \geq 1 - \min\{\lambda_{\text{max}}(\tilde{M}_s^{-1}A), \beta\sigma_{\text{TL}}\}. \tag{4.21} \]

**Subcase 2.3:** \( 1 < \alpha \). In such a case, the inequalities (4.20) and (4.21) still hold. Furthermore, one has
\[ a_1 \leq 1 - \alpha, \tag{4.22} \]
\[ b_1 \leq 1 - \sigma_{\text{TL}}. \tag{4.23} \]

The estimate (4.5) then follows directly from (4.7) and (4.16)–(4.23). This completes the proof. \( \square \)

**Remark 4.5.** If \( \text{rank}(S^TAP) = n_c \), we get from (4.1b) and (4.1d) that
\[ 2 - \sigma_{\text{TL}} - \lambda_{\text{min}}^+(S\tilde{M}_s^{-1}S^TA\Pi_A) = \lambda_{\text{max}}((I - S\tilde{M}_s^{-1}S^TA)(I - \Pi_A)) \]
\[ + \lambda_{\text{max}}((I - S\tilde{M}_s^{-1}S^TA)\Pi_A) \]
\[ \geq \lambda_{\text{max}}(I - S\tilde{M}_s^{-1}S^TA) = 1, \]
which yields
\[ 1 - \lambda_{\text{min}}^+(S\tilde{M}_s^{-1}S^TA\Pi_A) \geq \sigma_{\text{TL}}. \tag{4.24} \]

With the relation (4.24), one can easily check that the estimate (4.4) reduces to (3.1) when \( B_c = A_c \). Likewise, the estimate (4.5) will reduce to (3.1) if \( B_c = A_c \).
5. Conclusions

In this paper, we present a purely algebraic convergence analysis of TLHB methods, provided that $(S \ P)$ is of full row rank ($S$ and $P$ correspond to two hierarchical components). A new and succinct identity for the convergence factor of exact TLHB methods is derived, which can be conveniently used to analyze the optimal interpolation and the influence of range($P$) on the convergence factor. Two-sided bounds for the convergence factor of inexact TLHB methods are also established, which provide a theoretical framework for the convergence analysis of multilevel methods (a multilevel method can be treated as an inexact two-level scheme). An interesting question involved in the inexact TLHB theory is how to design an efficient coarse solver, which serves as a motivation for developing new multilevel algorithms.

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