Topological Matter, Integrable Models and Fusion Rings

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We show how topological $G_k/G_k$ models can be embedded into the topological matter models that are obtained by perturbing the twisted $N = 2$ supersymmetric, hermitian symmetric, coset models. In particular, this leads to an embedding of the fusion ring of $G$ as a sub-ring of the perturbed, chiral primary ring. The perturbation of the twisted $N = 2$ model that leads to the fusion ring is also shown to lead to an integrable $N = 2$ supersymmetric field theory when the untwisted $N = 2$ superconformal field theory is perturbed by the same operator and its hermitian conjugate.

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1. Introduction

The structure of the fusion ring, or Verlinde algebra, for affine Lie algebras has been known for quite some time \[1\] \[2\]. Perhaps one of the simplest characterizations of this fusion ring comes from the “discrete characters” \[2\], whose multiplication table precisely reproduces the fusion rules. More recently, it was shown in \[3\] that the fusion ring of $SU_k(n)$ could be “integrated” to yield a potential $W(x_a; \lambda)$, whose local ring is isomorphic to the fusion ring. That is, the fusion ring could be characterized by a single function $W(x_a; \lambda)$: the variables, $x_a$, are canonically identified with particular representations of $SU_k(n)$, and the algebraic relations implied by the fusion rules consist precisely of those polynomials of $x_a$ that lie in the ideal generated by the partials $\{ \frac{\partial W}{\partial x_a} \}$. At the time, it was unclear whether this remarkable result could be generalized.

The potentials $W(x_a; \lambda)$ also turned up in an apparently unrelated piece of work on the structure of a generalized Zamolodchikov metric \[4\] for the ground states of the perturbed $N = 2$ coset models that have a Landau-Ginzburg structure. It was shown in \[5\] that for a special class of perturbations, this generalized Zamolodchikov metric satisfied some classical Toda equations. This indirectly suggested that the corresponding perturbed conformal models should be integrable. For the $\mathbb{CP}^n$ models, this particular class of perturbations gave rise to an effective Landau-Ginzburg potential, $W(x_a; \lambda)$, that was identical to the potential that characterized the fusion rules of $SU_k(n)$.

Further suggestive results have been obtained in \[6\] and \[7\]. Unfortunately, the first of these two papers had not appeared at the time of writing this paper, but one of the central ideas of \[6\] appears to be that the Verlinde algebra of $G_k$ can be obtained from the correlation functions of a topological coset model based on $G_k/G_k$. The idea in \[7\] is that such a fusion algebra might possibly be naturally identified with a particular topological perturbation of the chiral primary ring \[8\] of a suitably chosen twisted $N = 2$ superconformal field theory. The $SU(2)_k$ example is extensively discussed in \[7\], but a complete understanding of the type $D$ and type $E$ modular invariants appeared elusive. It is however fairly clear from the results in \[8\] that something like this should be true for $SU_k(n)$.

The conjecture in \[6\] about $G/G$ models was recently established in \[9\]. Direct evidence for the connection with topological perturbations of $N = 2$ Landau-Ginzburg models can be found in \[10\], where the Verlinde dimensions for $G_k$ theories are expressed as correlation functions in topological Landau-Ginzburg theories.

Our purpose in this paper is to construct directly the embeddings of $G_k/G_k$ theories into the topological matter theories that are obtained by twisting hermitian symmetric
space, \( N = 2 \) supersymmetric, coset models. Specifically, the correlation functions of \( G_k/G_k \) can all be translated directly into a subset of the topological correlation functions of a twisted \( N = 2 \) supercoset theory. In particular, the fusion algebra of \( G_k \) generally forms a natural sub-ring of the perturbed chiral, primary ring, \( \mathcal{R} \). We describe this embedding in detail in section 3 of this paper. To obtain the \( G_k/G_k \) theory from the \( N = 2 \) theory one perturbs away from the conformal point in a very specific, flat, or geodesic, direction. We identify this direction and the corresponding relevant perturbing operator. Naturally, for \( SU_k(n) \) we recover the results of \([3]\) and \([10]\).

In section 4, we show that if the original \( N = 2 \) superconformal field theory (prior to twisting) is perturbed by the operator considered above, along with the hermitian conjugate perturbation, then the result is an integrable field theory. In particular, we establish that a general Grassmannian model has at least three integrable perturbations.

The key to establishing all of these results is essentially contained in the original paper by Eguchi and Yang \([11]\), where the group \( SU_k(2) \) is considered. The generalization of \([11]\) is relatively straightforward once one has obtained the appropriate description of the \( N = 2 \) supersymmetric, hermitian symmetric space models. For reasons that will soon become obvious, we call this the \textit{paratoda} description, and we will discuss it in section 2.

The paratoda construction is not manifestly supersymmetric. However, for the \( \mathcal{CP}^n \) supercoset models, there is a free superfield realization that can be obtained by hamiltonian reduction \([12]\), \([13]\). In this formulation, the perturbed model is easily seen to be integrable and the top components of the supermultiplets can be seen to provide the quantum integrals of motion. This is discussed in section 5.

2. The Parafermionic-Toda Models

Our purpose in this section is to show how to represent an \( N = 2 \) supersymmetric coset model based upon a hermitian symmetric space (HSS) in terms of an appropriately screened tensor product of generalized parafermions and a Toda theory (or, at least, the free field equivalent of a Toda theory). We will, in fact, obtain such a \textit{paratoda} description for the slightly more general class of coset models:

\[
\mathcal{M}_{k,\ell}(G; H) \equiv \frac{G_k \times H_{\ell}}{H_{k+\ell}},
\]
where $H$ is a subgroup of $G$, with $\text{rank}(H) = \text{rank}(G)$. Let $\alpha_1, \ldots, \alpha_r$ be a system of simple roots for $G$, ordered in such a way that $\alpha_1, \ldots, \alpha_p$ is a system of simple roots for $H$. Let $U = (U(1))^r$ be a torus for $H$, and hence a torus for $G$.

First consider the situation where $H \equiv G$. Such models will be called $G_{k,\ell}$ coset models. These have central charge $c = c_k(G) + c_\ell(G) - c_{k+\ell}(G)$, where $c_m(G)$ is the central charge of $G$ at level $m$. The central charge of the $G_{k,\ell}$ model may be written [14]:

$$c = \left[ \frac{k d_G}{(k + g)} - r \right] + \left[ r - \frac{12 k \rho_G^2}{(\ell + g)(k + \ell + g)} \right], \quad (2.1)$$

where $d_G$ is the dimension of $G$, $g$ is the dual Coxeter number of $G$, and $\rho_G$ is the Weyl vector of $G$. The first term in (2.1) is the central charge of the generalized parafermionic coset $G/U$, and the second term is the central charge of a Feigen-Fuchs, free bosonic field theory, or of its equivalent Toda theory. It has been convincingly argued in [13], [16] [17], [14] that the $G_{k,\ell}$ coset theory can indeed be obtained from a tensor product of such theories. Specifically, one tensors the parafermionic theory $G/U$ with a theory consisting of $r$ free bosons with energy-momentum tensor:

$$T_b(z) = -\frac{1}{2}(\partial \phi)^2 + i \beta_+ \Lambda_+ - \beta_- \Lambda_- \cdot \partial^2 \phi,$$

where

$$\beta_\pm \equiv \frac{1}{\sqrt{k}} \left[ \sqrt{\frac{(k + \ell + g)}{(\ell + g)}} \right]^{\pm 1} \quad (2.2)$$

As has been described in a number of places, such a bosonic free field description can be directly related to a Toda theory (see, for example, [18] [19]).

The primary fields of the parafermionic theory will be denoted by $A^\Lambda_\lambda$, where $\Lambda$ is a highest weight of $G_k$, and $\lambda$ is a vector of charges under the Cartan subalgebra (CSA), $\mathcal{X}$, of $G$ that generates the torus, $U$. The Toda, or free bosonic, field theory has a natural vertex operator representation for its highest weight states:

$$V_{\Lambda_+,\Lambda_-}(z) = \exp\left[ -i(\beta_+ \Lambda_+ - \beta_- \Lambda_-) \cdot \phi \right]. \quad (2.3)$$

The conformal weight of $A^\Lambda_\lambda$ is:

$$h_\lambda^\Lambda = \frac{\Lambda \cdot (\Lambda + 2 \rho_G)}{2(k + g)} - \frac{\lambda^2}{2k} + \text{integer} \quad \text{1}$$

\footnote{It probably suffices to merely have $H$ be a regularly embedded subgroup of $G$.}
and that of $\mathcal{V}_{\Lambda_+,\Lambda_-}$ is:

$$h_{\Lambda_+,\Lambda_-} = \frac{1}{2}(\beta_+\Lambda_+ - \beta_-\Lambda_-)^2 + (\beta_+ - \beta_-)\rho_G \cdot (\beta_+\Lambda_+ - \beta_-\Lambda_-).$$

One finds that this can be rewritten as:

$$h_{\Lambda_+,\Lambda_-} = \frac{1}{2k}(\Lambda_+ - \Lambda_-)^2 + \frac{\Lambda_+ \cdot (\Lambda_+ + 2\rho_G)}{2(\ell + g)} - \frac{\Lambda_- \cdot (\Lambda_- + 2\rho_G)}{2(k + \ell + g)}. \quad (2.4)$$

From this, and a consideration of the Cartan subalgebra eigenvalues, $\lambda$, it is easy to identify representatives of the primary fields, $\Phi_{\Lambda}^{\Lambda_+\Lambda_-}$, of the $G_{k,\ell}$ coset model. (The labels $(\Lambda, \Lambda_-; \Lambda_-)$ are highest weight labels of affine $G$ at levels $k$, $\ell$ and $k + \ell$ respectively, and correspond to the numerator and denominator factors in $G_{k,\ell}$.) Indeed, we may take

$$\Phi_{\Lambda}^{\Lambda_+\Lambda_-}(z) = A_{\Lambda_+ - \Lambda_-}(z) \mathcal{V}_{\Lambda_+,\Lambda_-}(z). \quad (2.5)$$

Finally, define the operators:

$$S_{+\alpha_i}(z) = \Phi_{0}^{\alpha_i - \alpha_i}(z) = A_{0}^{\alpha_i}(z) \exp [ + i\beta_+ \cdot \phi_i ]$$

$$S_{-\alpha_i}(z) = \Phi_{0}^{-\alpha_i}(z) = A_{0}^{-\alpha_i}(z) \exp [ - i\beta_- \cdot \phi_i ], \quad (2.6)$$

for $i = 1, \ldots, r$. These operators have conformal weight equal to one, and constitute the screening operators for the theory [15][16][17].

While the correspondence of the $G_{k,\ell}$ coset models with the foregoing paratoda description has not been rigorously established, there is a fairly compelling body of evidence. First, the model with $G = SU(2)$ has been extensively discussed in [15][17], where the correspondence was further verified by checking characters. The correspondence for general $G$ has been discussed in [16][14][17] and it is also fairly clear that the arguments in [15] can be extended beyond $SU(2)$. Secondly, for the $k = 1$ models, $G_{1,\ell}$, the free field realization given above reduces to the well known one [20][19][21]. The rigorous proof that this free bosonic realization yields the $G_{1,\ell}$ models may be found in [21]. It is also probable that this proof could be extended to cover the paratoda theories [22]. We will therefore assume that the paratoda theories with the screening currents (2.6) describe the $G_{k,\ell}$ models.

It is elementary to generalize this description to the $\mathcal{M}_{k,\ell}(G; H)$ models. Observe that

$$\mathcal{M}_{k,\ell}(G; H) = \frac{G}{H_k} \times \frac{H_k \times H_\ell}{H_{k+\ell}} = \frac{G}{H_k} \times \frac{H_k}{\mathcal{U}} \times \mathcal{U} = \frac{G}{\mathcal{U}} \times \mathcal{U}, \quad (2.7)$$
where the second equality comes from replacing the $H_{k,\ell}$ theory by its paratoda equivalent. Thus the $\mathcal{M}_{k,\ell}(G;H)$ models can be obtained by tensoring the parafermionic, $\hat{G}$, theory with an $H$-Toda theory. The energy-momentum tensor of the $H$-Toda theory is

$$T_b'(z) = -\frac{1}{2} (\partial \phi)^2 + i \left( \beta'_+ - \beta'_- \right) \rho_H \cdot \partial^2 \phi ,$$  

(2.8)

where $\rho_H$ is the Weyl vector of $H$, and the coupling constants are given by

$$\beta'_\pm \equiv \frac{1}{\sqrt{k}} \left[ \sqrt{\frac{(k + \ell + h)}{(\ell + h)}} \right]^{\pm 1} ,$$  

(2.9)

where $h$ is the dual Coxeter number of $H$. The screening currents are still given by (2.8), but with $i$ restricted to the range $i = 1, \ldots, p$; that is, the $\alpha_i$, are restricted to the simple roots of $H$. Representatives of the highest weight states of the model are still given by (2.5), but with $\beta_\pm$ replaced by $\beta'_\pm$. We will denote these representatives by $\Phi_{\Lambda,\lambda^+,\lambda^-}(z)$, where $\lambda^+$ and $\lambda^-$ are now highest weight labels of affine $H$ at level $\ell$ and level $k + \ell$ respectively. The conformal weight of $\Phi_{\Lambda,\lambda^+,\lambda^-}$ is given by

$$h = \frac{\Lambda \cdot (\Lambda + 2\rho_H)}{2(k + g)} + \frac{\lambda^+ \cdot (\lambda^+ + 2\rho_H)}{2(\ell + h)} - \frac{\lambda^- \cdot (\lambda^- + 2\rho_H)}{2(k + \ell + h)} .$$  

(2.10)

If $H$ has any $U(1)$ factors, then they give rise directly to uncorrupted free $U(1)$ factors in the bosonic torus theory, $\mathcal{U}$, in (2.7). Such $U(1)$ factors in $\mathcal{U}$ are uncorrupted in the sense that they are orthogonal to all screening currents and to the change at infinity. Indeed, if $p(z)$ denotes the vector of currents in the CSA, $\mathcal{X}$, of $H$, and $v \cdot p(z)$ defines some $U(1)$ factor in $H$, then the corresponding $U(1)$ factor in $\mathcal{U}$ is given by $v \cdot \partial \phi$, up to some normalization.

If $G/H$ is a symmetric space, then $H_{g-h}$ can be conformally embedded in $SO(\dim(G/H))$, and so for a special choice of modular invariant for $H_{g-h}$, the $M_{k,\ell=g-h}(G;H)$ model is precisely the super-GKO coset model based on $G/H$. If $G/H$ is a hermitian symmetric space (HSS), then $M_{k,\ell=g-h}(G;H)$ is, of course, an $N=2$ supersymmetric model [23]. We now specialize to these $N=2$, HSS models.

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2 As usual, if $H$ has several factors, the levels, $\ell$ or $k + \ell$, and the dual Coxeter number, $h$, are to be interpreted as vectors with entries corresponding to each such factor. The dual Coxeter number of a $U(1)$ factor is defined to be zero.
By rewriting the construction of [23] in terms of the paratoda theory, one finds that
the $U(1)$ current of the $N = 2$ superalgebra is given by:

$$J(z) = 2i(\beta'_+ - \beta'_-)(\rho_G - \rho_H) \cdot \partial \phi ,$$  \hspace{1cm} (2.11)

and the associated charge, $Q$, of the primary field $\Phi_{\lambda_+\lambda_-}^{\Lambda}$ is:

$$Q = -2(\rho_G - \rho_H) \cdot \left[ \frac{\lambda_+}{g} - \frac{\lambda_-}{k+g} \right].$$  \hspace{1cm} (2.12)

In [24] the chiral, primary fields were identified by computing the Ramond ground
states. It was shown that there were

$$\mu = \frac{1}{|Z(G)|} N_k(G) \left| \frac{W(G)}{W(H)} \right|$$  \hspace{1cm} (2.13)

such states, where $N_k(G)$ is the number of affine, highest weight labels of $G$ at level $k$,
$Z(G)$ is the center of $G$, and $W(G)$ and $W(H)$ are the (finite) Weyl groups of $G$ and $H$,
respectively. For each highest weight label, $\Lambda$, of $G$, and for each highest weight label, $\lambda$,
of the coset denominator, $H_{k+g-h}$, one obtains a Ramond ground state for the coset if and
only if:

$$\lambda = w(\Lambda + \rho_G) - \rho_H ,$$  \hspace{1cm} (2.14)

where $w$ is a representative of a Weyl-coset element of $W(G)/W(H)$. Moreover, one obtains
exactly one such ground state for each such $\Lambda$ and $\lambda$. The Weyl element $w$ in (2.14) is
chosen so that $w(\Lambda + \rho_G)$ is a dominant weight of $H$. This choice also guarantees that $\lambda$ is
a dominant weight of $H$. The weight $w(\Lambda)$ is manifestly a weight in the $G$ representation
with highest weight $\Lambda$, and it is straightforward to establish that $w(\Lambda)$ is also a highest
weight of affine $H$ at level $k$. Similarly $w(\rho_G) - \rho_H$ is a highest weight label for $H_{g-h}$ and
can be obtained from the spinor ground-state of $SO(\text{dim}(G/H))$. Thus we can construct
representatives of the Ramond vacua directly from the ground states of the $G$ and $H$
factors in the coset model. To obtain representatives of the chiral primary fields from this
description of the Ramond ground states, one merely performs a spectral flow with respect
to the $N = 2 U(1)$ current, $J(z)$. If one simultaneously performs a spectral flow in the CSA
of the denominator, $H$, one can arrange that the net effect of both these spectral flows is
a single spectral flow in the $SO(\text{dim}(G/H))$ factor. This means that one can obtain the
chiral primary fields by taking the affine $G$ and $H$ labels corresponding to the Ramond
ground states, and merely shifting the $H$-labels by $-(\rho_G - \rho_H)$\(^3\). As a consequence, a set of representatives of the chiral primary fields is given by:

$$\Phi_{\lambda_-}^{\Lambda, \lambda_+}, \quad \text{where} \quad \lambda_+ = w(\rho_G) - \rho_G; \quad \lambda_- = w(\Lambda + \rho_G) - \rho_G. \quad (2.15)$$

One can also establish this result directly from our earlier analysis by using (2.12) and (2.10) (with $\ell = g - h$) to write:

$$h - \frac{1}{2}Q = \frac{(|\Lambda + \rho_G|^2 - |\lambda_+ + \rho_G|^2)}{2(k + g)} + \frac{(|\lambda_+ + \rho_G|^2 - |\rho_G|^2)}{2g}. \quad (2.16)$$

It is obvious that the labels in (2.15) satisfy $h - \frac{1}{2}Q = 0$, and the reverse implication may be established by the arguments in the appendix of [24], whose crucial observation is that weights in a $G$-representation have the same length as the highest weight if and only if they are Weyl images of that highest weight.

The field identifications induced by spectral flow in the CSA, $X$, of $H$, map a coset state with weights $(\Lambda, \lambda_+; \lambda_-)$ into another such state according to:

$$\Lambda \rightarrow \Lambda + k \upsilon$$
$$\lambda_+ \rightarrow \lambda_+ + (g - h)v$$
$$\lambda_- \rightarrow \Lambda_- + (k + g - h)v \quad (2.17)$$

where $\upsilon$ is any vector. Spectral flow by an arbitrary vector, $\upsilon$, yields an automorphism of the coset theory provided we use appropriately twisted Kac-Moody currents (see, for example, [25]); that is, we replace the currents of $G$ or $H$ according to:

$$J^\alpha_n \rightarrow J^\alpha_{n+v}\cdot\alpha$$
$$H^i_n \rightarrow H^i_n + kv^i\delta_{n,0}$$

To avoid using such twisted representations one usually restricts $\upsilon$ to be a weight of $G$, and hence a weight of $H$.

If a coset state is constructed from highest weight states of $G_k$, $H_\ell$ and $H_{k+\ell}$, then the spectral flow of this state will not, in general, be represented by highest weight states of the $G$ and $H$ current algebras (but this state will, of course, be a highest weight state of the coset model). The chiral primary fields given in (2.15) are all obtained from primary fields of $G$ and $H$. It turns out (modulo fixed point problems [24][26]) there are still $Z(G)$

\(^3\) The negative sign comes from our reversal of the conventional sign for $J(z)$ used in [23].
inequivalent spectral flows that map the list (2.13) back into itself, and hence each chiral primary field is given \(Z(G)\) times in (2.13). This multiple counting accounts for the factor of \(\frac{1}{Z(G)}\) in (2.13).

An issue that will be important in the next section is to determine which elements of the chiral primary ring, \(\mathcal{R}\), can be represented by \(\Phi^{\Lambda,\lambda_+}_\lambda\) with \(\lambda_+ = 0\), i.e., using only the vacuum in \(H_{g-h}\). Such fields manifestly form a sub-ring, which we will denote by \(\mathcal{F}\), of the chiral primary ring. The elements of \(\mathcal{F}\) are naturally labelled by the highest weights, \(\Lambda\), of \(G\) at level \(k\).

One finds that if \(g-h = 1\) in all the simple factors in \(H\) then one can use spectral flow to set \(\lambda_+\) to zero. This is almost obvious from (2.13) and (2.17), since a flow by \(v = -\frac{1}{(g-h)}[w(\rho_G) - \rho_G]\) would accomplish the desired result. The problem is that the dual Coxeter number, \(h\), vanishes in the \(U(1)\) factor of \(H\), and so such a \(v\) is not necessarily a weight of \(G\), and thus the currents of \(G\) would be twisted by such a flow. However, the only coset models for which one has \(g-h = 1\) in the simple factors of \(H\) are the \(\mathbb{CP}^n\) models, \(\frac{SU(n+1)}{SU(n) \times U(1)}\), and explicit computations \([24][27][3]\) show that one can always find a representative, \(\Phi^{\Lambda,\lambda_+}_\lambda\), of each chiral primary field such that \(\lambda_+ = 0\). Moreover, the set of all such representatives yields all the chiral primary fields exactly once, i.e none of these representatives is equivalent another under spectral flow. As a result, we see that if we only want to consider untwisted representations of \(G\) and \(H\), then \(\mathcal{F} \equiv \mathcal{R}\) only for the \(\mathbb{CP}^n\) models. Otherwise, \(\mathcal{F}\) is a proper sub-ring of \(\mathcal{R}\).

3. Topological Matter Models

To obtain the topological matter model from the \(N = 2\) supersymmetric models discussed above, one first “twists” the energy momentum tensor: \(T(z) \to T(z) + \frac{1}{2} \partial J(z)\), and then uses \(G^+(z)\) as a further screening current \([28][29][11][8]\). From (2.8) and (2.11) one can easily see that, in the paratoda formulation, this corresponds simply to replacing \(T_b'(z)\) of (2.8) by:

\[
T_b^{\text{top}}(z) = -\frac{1}{2}(\partial \phi)^2 + i(\nu_+ - \nu_-)\rho_G \cdot \partial^2 \phi ,
\]

(3.1)

where

\[
\nu_\pm = \beta'_\pm(k, \ell = g-h) = \beta_\pm(k, \ell = 0) = \beta_\pm(k, \ell = 0) = \frac{1}{\sqrt{k}} \left[ \sqrt{\frac{(k+g)}{g}} \right]^{\pm1}.
\]
Thus the twisted paratoda theory has precisely the energy-momentum tensor of a $G_{k,\ell}$ theory but with $\ell = 0$. This generalizes the observation in [30] that when the pure Toda theory is twisted, one obtains a topological model corresponding to $G_{1,0}$.

We now establish the complete equivalence of the twisted $N = 2$ supersymmetric paratoda theory and the topological $G_{k,0}$ coset models. Recall that the set of screening currents of the $N = 2$ theory is simply a subset of those that we used for the $G_{k,\ell}$ theory. Indeed, the simple roots of $G$ contain one simple root, $\gamma = \alpha_r$, more than the simple roots of $H$. The corresponding “screening” operators $S_{\gamma} = \Phi^{0,-\gamma}_0$ and $S_{-\gamma} = \Phi^{0,0}_{-\gamma}$ are, in fact, primary fields of the $N = 2$ supersymmetric paratoda theory, and have conformal dimensions and $Q$-charges:

\[ h = \frac{3}{2}, \quad Q = 1, \quad Q = \frac{k}{2(k+g)} \]

This suggests that we should identify:

\[ S_{+\gamma}(z) \equiv G^+(z) \quad \text{and} \quad S_{-\gamma}(z) \equiv (G_{-\frac{1}{2}} - \frac{k}{2(k+g)}) \phi(z) \]

where $G^\pm(z)$ are the $N = 2$ supercurrents, and $\phi$ is a chiral primary field. To see that this identification is correct, consider what the operators $\Phi^{0,-\gamma}_0$ and $\Phi^{0,0}_{-\gamma}$ represent in the original $N = 2$ coset model. Let $\lambda_{+\bar{\alpha}}(z)$ be the fermions of $SO(\dim(G/H)) \equiv H_{g-h}$ and let $J_{+\bar{\alpha}}(z)$ be the currents of $G$ that are not currents of $H$. The labels, $\bar{\alpha}$, are the roots of $G$ that are not roots of $H$. (See [23] for details.) The $H_{g-h}$ ground state representation corresponding to $\lambda_+ = -\gamma$ consists of all the fermions $\lambda^{-\bar{\alpha}}(z)$ (and includes $\lambda^{-\gamma}(z)$ itself).

To ensure that $\lambda_- = 0$, i.e., one is in the $H_{k+g-h}$ vacuum, one must pair these fermions with $J_{+\bar{\alpha}}(z)$. This is precisely the construction of the supercurrent in [23]. Similarly, the primary field with $\Lambda = \lambda_+ = 0$ and $\lambda_- = -\gamma$, can be represented by the current $J_{-\bar{\alpha}}(z)$.

Since $G^\pm(z) = \sum_{\bar{\alpha}} J_{+\bar{\alpha}}(z) \lambda_{+\bar{\alpha}}(z)$, one sees that $J_{-\bar{\alpha}}(z)$ is precisely $(G_{-\frac{1}{2}} - \frac{k}{2}) \lambda^{-\bar{\alpha}}(z)$. The fermion $\lambda_{-\bar{\alpha}}(z)$ is a representative of the chiral primary field:

\[ \phi = \Phi^{0,-\gamma}_{-\gamma} \]  

with $h = \frac{k}{2(k+g)}$ and $Q = \frac{k}{(k+g)}$.

Thus, if we wish to pass from the $N = 2$ theory to the $G_{k,\ell=0}$ theory, we must use the two operators $S_{\pm\gamma}$ as screening charges. Using $G^+(z)$ as a screening charge is, of course, already required in going to the topological version of the $N = 2$ theory. One can also easily understand the use of $S_{-\gamma}(z)$ as a screening charge: it is equivalent to perturbing the topological model by inserting

\[ \exp\left[-\lambda \int d^2 z \; G_{-\frac{1}{2}}^\pm G_{-\frac{1}{2}}^\pm \phi\right] \]  

(3.3)
into the topological correlation functions. This follows from the observations of [31, 32], in which one can split the integrals of the form $\int d^2z$, into holomorphic and anti-holomorphic contour integrals, factorize the result and reinterpret the exponential of this integral as precisely the appropriate generalization of the contour prescriptions of [33].

The conclusion of all this is that all the correlation functions of $G_k \times G_0$ are equal to the topological correlations of the $N = 2$ supersymmetric HSS models, perturbed by the top component of the chiral, primarily superfield whose bottom component is the chiral, primary field $\phi$ of (3.2).

For untwisted representations of affine $G$, the only unitary representation of $G_0$ is one dimensional and consists of only the vacuum [34]. Hence the $G_k \times G_0$ theory is precisely the $G_k / G_k$ theory considered in [6, 9]. The three point functions in this theory are precisely the fusion rules of $G_k$. That is, if $\phi_\Lambda$ is a primary field of $G_k$, then

$$\phi_\Lambda \phi_\Lambda' = N_{\Lambda \Lambda' \Lambda''} \phi_\Lambda'' .$$

However, we have established this field theory is isomorphic to the perturbed topological matter model described above, restricted to the sub-ring $F$ of $R$. In particular, the correlation functions of the two models are equal.

We now recall some of the basic properties of the $N = 2$ supersymmetric topological matter models [8]. The general topological correlation function is given by:

$$\langle \phi_{i_1} \ldots \phi_{i_n} \rangle_{\text{top}} \equiv \left\langle \phi_{i_1} \ldots \phi_{i_n} \exp \left[ \sum_\ell t_\ell \int d^2 z G^{-1}_- \tilde{G}^{-1}_- \phi_\ell (z, \bar{z}) \right] \right\rangle ,$$

where the right-hand-side is computed in the twisted conformal $N = 2$ model. The two point function, $\eta_{ij} = \langle \phi_i \phi_j \rangle_{\text{top}}$ is independent of the coupling constants, $t_\ell$, and provides a flat metric for the topological field theory. The three point function, $C_{ijk}(t) \equiv \langle \phi_i \phi_j \phi_k \rangle_{\text{top}}$, is totally symmetric, and satisfies:

$$\partial_\ell C_{ijk} = \partial_i C_{\ell jk} ,$$

where $\partial_\ell = \partial_{t_\ell}$, and the indices are raised by $\eta^{ij} = \eta_{ij}$. The $n$-point functions can be factorized into products of the $C_{ij}$. For $t_\ell = 0$ the three point functions are nothing other than the multiplication table for the chiral primary ring, $R$. This ring can be generated by

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4 The operators $\tilde{G}^{\pm}(z)$ are the anti-holomorphic $N = 2$ supercharges.
a set of fields, which we will denote by \( x_a, a = 1, \ldots, M \), and the ring, \( \mathcal{R} \), is characterized by polynomial vanishing relations \( v_b^{(0)}(x_a) = 0, b = 1, \ldots, M \) in this ring. For \( t_\ell \neq 0 \) the fields \( \phi_i(x_a; t_\ell) \) are polynomials in \( x_a \) that are quasihomogeneous in \( x_a \) and \( t_\ell \). These polynomials satisfy the multiplication rule:

\[
\phi_i(x_a; t) \phi_j(x_a; t) = C_{ij}^k(t) \phi_k(x_a; t) \mod \{ v_b(x_a; t) \}, \tag{3.7}
\]

where the \( v_b(x_a; t) \) are polynomials in \( x_a \), and are the perturbative vanishing relations. For a generic perturbation, there are \( \mu \) isolated points \( x_a^{(\alpha)} \), that satisfy \( v_b(x_a^{(\alpha)}; t) = 0 \). At these points, the equations:

\[
\phi_i(x_a^{(\alpha)}; t) \phi_j(x_a^{(\alpha)}; t) = C_{ij}^k \phi_k(x_a^{(\alpha)}; t)
\]

are satisfied identically.

If the conformal theory has a Landau-Ginzburg potential, \( W_0(x_a) \), then the perturbed topological matter model has an effective Landau-Ginzburg potential \( W(x_a; t) \) such that one has \( \phi_i(x_a; t) = -\frac{\partial W}{\partial t_i} \), and the partials, \( \frac{\partial W}{\partial x_a} \), of \( W \) yield an equivalent set of vanishing relations to those provided by the \( v_b \). The points \( x_a^{(\alpha)} \) are thus the critical points of \( W \).

Introduce a matrix \( D \) such that \( D^2 = \eta \). Since \( \eta \) is symmetric, we can take \( D \) to be symmetric (but not necessarily real). Let \( X_i = DC_iD \), where \( C_i \) is the matrix \( C_{ij}^k \). From (3.6), one sees that the matrices \( X_i \) commute with each other and are totally symmetric, and so they can be simultaneously diagonalized by an orthogonal transformation, \( \tilde{S} \). Let \( S_i^\alpha \) be the elements of the matrix \( D\tilde{S} \), then \( \eta^{ij} S_i^\alpha S_j^\beta = \delta^{\alpha\beta} \), \( S_i^\alpha S_j^\alpha = \eta^{ij} \), and

\[
C_{ijk} S_j^\alpha S_k^\beta = \phi_i^{(\alpha)} \delta^{\alpha\beta}, \tag{3.8}
\]

where \( \phi_i^{(\alpha)} \) are the eigenvalues of \( C_{ij}^k \). From the complete symmetry of \( C_{ijk} \) it follows that \( \phi_i^{(\alpha)} = S_i^\alpha / S_0^\alpha \) and hence:

\[
C_{ijk} = \sum_\alpha \frac{S_i^\alpha S_j^\alpha S_k^\alpha}{S_0^\alpha}. \tag{3.9}
\]

We may also write this relation as:

\[
\phi_i^{(\alpha)} \phi_j^{(\alpha)} = C_{ij}^k \phi_k^{(\alpha)} \tag{3.9}
\]

(with no sum on \( \alpha \)). Thus the eigenvalues, \( \phi_i^{(\alpha)} \), are therefore precisely the values of the chiral primary fields at the points \( x_a^{(\alpha)} \), i.e. \( \phi_i^{(\alpha)} = \phi_i(x_a^{(\alpha)}; t) \).
In the foregoing analysis the set of \( \phi_i \) formed a basis for the entire chiral primary ring, \( \mathcal{R} \). However, one can consistently truncate to any sub-ring, \( \mathcal{F} \), of \( \mathcal{R} \) provided that one also restricts the perturbations in (3.3) to those that preserve this truncation. This is easily done for the sub-ring, \( \mathcal{F} \), introduced in the last section since the perturbations will preserve the truncation if and only if the perturbing operator, \( G_{-\frac{1}{2}} \tilde{G}_{-\frac{1}{2}} \phi(z, \bar{z}) \), has a representation in the paratoda theory that consists of some operator in the \( G_k \) theory, tensored with the identity in the \( H_g \) theory. In particular, recall that the holomorphic operator \( (G_{-\frac{1}{2}} \phi)(z) \) discussed in the previous section manifestly satisfies this criterion since it was represented by \( \Phi_{0,0}^{\gamma}(\bar{z}) \), which has \( \lambda_\gamma = 0 \). This operator can also be represented by the \( G_k \) current \( J_{-\gamma}^{\gamma}(z) \).

Thus we can consider the topological matter theory restricted to the sub-ring, \( \mathcal{F} \), along with the single perturbation (3.3). For the right value of \( \lambda \), we have \( C_{\Lambda \Lambda', \Lambda''} = N_{\Lambda \Lambda', \Lambda''} \), where the \( N_{\Lambda \Lambda', \Lambda''} \) are the fusion coefficients. Moreover, from the results of [2] we know that the matrix \( S_{\Lambda \Lambda', \Lambda''} \) employed above is precisely the modular inversion matrix for affine \( G \) at level \( k \) [34]:

\[
S_{\Lambda \Lambda'} = (i)^{|\Delta^+|} \frac{\Gamma^*}{(k + g) \Gamma} \left| \frac{1}{\sum_{w \in W(G)} e^{-2\pi i (k + g) \cdot w(\Lambda + \rho_G)} \epsilon(w) e^{-2\pi i (k + g) \cdot w(\Lambda + \rho_G)}} \right|,
\]

where \(|\Delta^+|\) is the number of positive roots of \( G \), \( \Gamma \) is the co-root lattice of \( G \) and \( \Gamma^* \) is its dual. The eigenvalues \( \phi_{\Lambda}^{(\Lambda') \Lambda''} \) of \( C_{\Lambda \Lambda', \Lambda''} \) are therefore given by:

\[
\phi_{\Lambda}^{(\Lambda') \Lambda''} = \frac{S_{\Lambda \Lambda'} \chi_\Lambda}{S_0 \Lambda} \equiv \chi_\Lambda \left( e^{-\frac{2\pi i}{(k + g)} \cdot \Lambda' \cdot p} \right), \tag{3.10}
\]

where \( p \) is the vector of CSA generators of \( G \), and \( \chi_\Lambda \) is the character of the finite \( G \) representation with highest weight \( \Lambda \). The right-hand-side of (3.10) constitute the discrete characters of \( G \), and have long been known to satisfy the fusion rules of [2].

Thus we have established that the values of the \( \phi_{\Lambda}(x_a; \lambda) \) at the solutions of \( v_{\Lambda''}(x_a; \lambda) = 0 \) are given by these discrete characters. In particular, one can easily express the \( x_a^{(\alpha)} \) in terms of these characters.

4. Integrable models

Consider once again the model \( M_{k, \ell}(G; H) \). For simplicity we will specialize to the \( N = 2 \) superconformal theory, with \( \ell = g - h \), but most of the conclusions of this section
will hold for more general $\ell$ and $H$. Define operators

$$
S_{-\psi}(z) = \Phi_0^{0,\psi}(z) \equiv A_{-\psi}^0(z) \exp[-i\beta_+ \psi \cdot \phi]
$$

$$
S_{\psi}(z) = \Phi_{\psi}^{0,0}(z) \equiv A_0^0(z) \exp[i\beta_- \psi \cdot \phi],
$$

(4.1)

where $\psi$ is the highest root of $G$. In section 2 we indentified $S_{+\gamma}$ and $S_{-\gamma}$ with $G^+(z)$ and $(G^{-\frac{1}{2}}\phi)(z)$, and by virtually identical arguments, we can identify $S_{-\psi}$ and $S_{+\psi}$ with the conjugate fields $G^-(z)$ and $(G^{+\frac{1}{2}}\bar{\phi})(z)$. From the properties of hermitean symmetric spaces, there is an automorphism of $G$ that interchanges $\gamma$ and $-\psi$. Indeed if $\alpha_1, \ldots, \alpha_{r-1}, \gamma$ is a system of simple roots of $G$, then so is $\alpha_1, \ldots, \alpha_{r-1}, -\psi$ and hence the two systems are Weyl rotations of each other. One such automorphism corresponds to the $N=2$ $U(1)$ charge conjugation operation of fields in the coset model. More generally there are automorphisms that mix the roots $\alpha_1, \ldots, \alpha_{r-1}$ in non-trivial ways with $\gamma$ and $\psi$.

Consider now the $W$-algebra generators of the $G_{k, \ell}$ model. These can, of course, be represented in the paratoda language, and moreover the $W$-generators all commute with the screening charges obtained from the currents $S_{\pm \alpha_i}, i = 1, \ldots, r$. (Remember that we have: $\alpha_r = \gamma$. ) If one considers the $W$-generators in the $G_{k, \ell=0}$ theories, one can then “untwist” them to obtain the $W$-generators of the corresponding $N=2$ coset model. To give perhaps a better perspective upon this procedure consider the $N=2$ supersymmetric paratoda theory: $G/\mathcal{U} \times \mathcal{U}$. One can think of this theory as a $G_k$ current algebra, but with a rescaled torus whose currents have charges at infinity. One can presumably write down the $W$-generators for this model by using the techniques of [35], provided that one appropriately rescales the torus and adds the proper sub-leading terms. The resulting $W$-generators will commute with the charges obtained from $S_{\pm \gamma}(z)$. Because the $W$-generators are their own conjugates, they are invariant under conjugation, and so they must also commute with the charges obtained from $S_{\pm \psi}(z)$. This means that they commute with $G^\pm_{-\frac{1}{2}}$ and with $\oint (G_{-\frac{1}{2}}^{-1} \phi)(z)dz$ and $\oint G_{+\frac{1}{2}}^+ \bar{\phi}(z)dz$. The former means that these $W$-generators are top components of a superfield, while the latter means that [36] [37] [38], to first order in perturbation theory, the $W$-generators provide integrals of motion for the $N=2$ superconformal model perturbed by

$$
\lambda \int d^2z \ G_{-\frac{1}{2}}^{-1} \bar{G}_{-\frac{1}{2}}^{-1} \phi + \bar{\lambda} \int d^2z \ G_{\frac{1}{2}}^+ \bar{G}_{\frac{1}{2}}^+ \bar{\phi}.
$$

(4.2)
The foregoing argument is somewhat heuristic since it some makes (highly plausible) assumptions about properties of the $W$-algebra of the theory. It also only verifies integrability to first order in perturbation theory. For the $\mathbb{CP}^n$ models one can make a much more complete analysis of the $W$-generators because a simple superfield formulation is available. We will see in the next section that the foregoing argument is born out.

A method for establishing the integrability beyond first order perturbation theory might be provided by generalizing the techniques of [11][37]. That is, one can attempt to construct the integrals of motion as commutators of suitable multiple integrals of the currents $S_-\gamma$ and $S_+\gamma$, the result should then commute with the perturbed Hamiltonian. These techniques appear to work well for $G = SU(2)$ [11].

It is elementary to calculate the perturbative corrections to the superalgebra and establish a Bogomolny bound for the mass of a soliton in the perturbed theory [39]. One finds,

$$m \geq |\lambda (\omega - 1) \Delta \phi|,$$

where $\lambda$ is the coupling constant, $\omega = \frac{k}{k+g}$ is the $N = 2$ $U(1)$ charge of $\phi$, and $\Delta \phi = (\phi(\sigma = +\infty) - \phi(\sigma = -\infty))$ is the difference of the spatial asymptotic values of $\phi$. This bound is saturated by chiral solitons. If the theory has a Landau-Ginzburg formulation then $W(x_a; \lambda)$ is quasihomogeneous, where $x_a$ has weight $Q_a$, its $N = 2$, $U(1)$ charge, and $\lambda$ has weight $1 - \omega$. Hence

$$W(x_a; \lambda) = \left( \sum_a Q_a x_a \frac{\partial W}{\partial x_a} \right) + (1 - \omega) \lambda \frac{\partial W}{\partial \lambda}.$$  

(4.4)

Since $\lambda$ is the flat coordinate of the perturbation, we have $\frac{\partial W}{\partial \lambda} = -\phi$. Moreover, at critical points, one has $\frac{\partial W}{\partial x_a} = 0$, and hence

$$\Delta W = -(1 - \omega) \lambda (\Delta \phi).$$  

(4.5)

Thus we recover the semi-classical result [10][38].

$$m \geq |\Delta W|.$$  

(4.6)

For the $\mathbb{CP}^n$ models, the perturbing field, $\phi$, is equal to some $\phi_\Lambda$ in $\mathcal{F}$. We can thus express $\Delta \phi$ and the Bogomolny bound in terms of the character $\chi_\Lambda$. One should also note that the effective Landau-Ginzburg potentials, $W(x_a; \lambda)$, for the integrable models are not necessarily first order in $\lambda$. Indeed, for $G_k = SU_k(2)$, the potential is a Chebyshev polynomial [8][7]. The general form of this effective potential can be obtained by checking that it also reproduces the fusion algebra, and has also been computed in [3][10].
5. Examples

The $N = 2$ supersymmetric Grassmanian models:

$$G_{m,n,k} = \frac{SU_k(m + n) \times SO_1(2mn)}{SU_{k+n}(m) \times SU_{k+m}(n) \times U(1)}$$

have the property that they are symmetric in $m, n$ and $k$. It is an immediate consequence that they all have three integrable perturbations (provided that $m, n$ and $k$ are distinct). For $m = 1$ and $n = 1$, one obtains the type $A$ minimal model and finds that the integrable perturbation, $\phi$, is the least relevant chiral primary field. For $k = 1$ and $m = 1$ one also obtains the type $A$ minimal models, but this time the integrable perturbation, $\phi$, is the most relevant, chiral primary field [41]. (It should be noted that this is not an exhaustive list of the perturbations that lead to integrable models since it is also known that perturbing the type $A$ models by the next-to-most relevant field also leads to an integrable theory [38].)

The first non-trivial example of a Grassmanian for which the analysis of the preceding section implies that there are three distinct integrable perturbations is provided by the model with $m = 3$, $n = 2$ and $k = 1$ (or some permutation thereof). This has a Landau-Ginzburg potential:

$$W_0(x_1, x_2) = \frac{1}{3}x_2^3 + 2x_1^2x_2^2 - x_1^4x_2.$$

Let $W(x_1, x_2; \lambda)$ be the effective potential after the appropriate perturbation has been added. One then finds the following expressions for $W(x_1, x_2; \lambda) - W_0(x_1, x_2)$:

$$8\lambda x_1; \quad -4\lambda(x_2 - x_1^2); \quad 8\lambda(x_1x_2 + 1),$$

corresponding to writing the coset model in terms of $G_k = SU_1(5), SU_2(4)$ or $SU_3(3)$. (Note that the perturbations have a charge equal to the level of $G$ divided by $k + g = 6$.)

The fusion algebras are obtained by taking $\lambda = 1$. The perturbation $-4(x_2 - x_1^2)$ provides exactly the fusion rules of $SU_2(4)$, provided that one identifies $x_1$ with the 4 of $SU(4)$ and $\frac{1}{2}(x_2 + x_1^2)$ with the 6. The perturbation $8(x_1x_2 + 1)$ of the potential $W_0$ yields the fusion rules of $SU_3(3)$ provided that one identifies $x_1$ with the 3 and $\frac{1}{2}(x_2 + x_1^2)$ with the 6. Finally, the fusion ring of $SU_1(5)$ is a sub-ring of dimension 5, embedded in the chiral ring $\mathcal{R}$ whose dimension is 10. The 5 of $SU(5)$ is identified with $y \equiv \frac{1}{2}(x_1^2 - x_2)$, while all the other antisymmetric tensors of $SU(5)$ map onto powers of $y$. One also finds that $y^5 = 1$, as one expects.
Another example is provided by the type $D$ modular invariant, minimal models. These can be described by taking $G = SO_1(2n + 2)$ and $H = SO_3(2n) \times SO(2)$. The dimension of $\mathcal{R}$ is $2(n + 1)$. The perturbed (integrable) Landau-Ginzburg potential is:

$$W(x_1, x_2) = \frac{1}{2n+1}x_1^{2n+1} - x_1x_2^2 - \lambda x_1.$$  

The fusion ring, $\mathcal{F}$, of $SO_3(2n + 2)$ is only four dimensional, and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for $n$ odd and $\mathbb{Z}_4$ for $n$ even. To obtain this ring, one takes $\lambda = 1$ and identifies the singlet of $SO(2n + 2)$ with $1$ and the vector of $SO(2n + 2)$ with $(x_1^{2n} + x_2^2)$. The two spinors of $SO(2n + 2)$ can be identified with $(x_1^n + \alpha x_2)$, where $\alpha = 1$ for $n$ odd and $\alpha = i$ for $n$ even.

6. $\mathbb{C}P^{n-1}$-models and superfields

In this section we study the superfield formulation of supersymmetric $\mathbb{C}P^{n-1}$ models with $G = SU(n)$ and $H = SU(n-1)$. We find it convenient to use $N = 1$ superfields rather than $N = 2$ superfields. The free superfield formulation of the $\mathbb{C}P^{n-1}$ models can be obtained from the Lie superalgebra $A(n,n-1)$ through a Hamiltonian reduction. Before we study the $\mathbb{C}P^{n-1}$ models we first review some basic properties of the super Lie algebra $A(n,n-1)$ that are relevant for our analysis [12].

The super Lie algebra $A(n,n-1)$ has a $\mathbb{Z}_2$-grading, and therefore the roots are either even or odd. If we denote the simple roots by $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}, \alpha_{2n}$, then the even roots are:

$$\alpha_i + \alpha_{i+1} + \ldots + \alpha_{i+2k-2} + \alpha_{i+2k-1} \quad k = 1, 2, \ldots, \left[\frac{2n+1-i}{2}\right] \quad (6.1)$$

and the odd roots are

$$\alpha_i + \alpha_{i+1} + \ldots + \alpha_{i+2k-1} + \alpha_{i+2k} \quad k = 0, 1, \ldots, \left[\frac{2n-i}{2}\right]. \quad (6.2)$$

The simple roots of $A(n,n-1)$ satisfy the following relations

$$\alpha_{2i-1} \cdot \alpha_{2i} = 1$$
$$\alpha_{2i+1} \cdot \alpha_{2i} = -1 \quad (6.3)$$

with all other inner products being zero (including $\alpha_i \cdot \alpha_i$). Next we introduce the fundamental weights $\lambda_1, \ldots, \lambda_{2n}$ defined by

$$\alpha_i \cdot \lambda_j = \delta_{ij} \quad (6.4)$$

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It is easy to see from (6.3), that in terms of the simple roots, the fundamental weights are given by

\[ \lambda_{2i} = \alpha_1 + \alpha_3 + \ldots + \alpha_{2i-3} + \alpha_{2i-1} \]

\[ \lambda_{2i-1} = \alpha_{2i} + \alpha_{2i+2} + \ldots + \alpha_{2n-2} + \alpha_{2n} . \]

(6.5)

The super Lie algebra \( A(n, n-1) \) contains the even subalgebras \( A_n \) and \( A_{n-1} \). The simple roots of the \( A_n \) subalgebra have the form

\[ \alpha_{2i-1} + \alpha_{2i} \quad i = 1, \ldots, n \]

(6.6)

and for \( A_{n-1} \) the simple roots are given by

\[ \alpha_{2i} + \alpha_{2i+1} \quad i = 1, \ldots, n-1 . \]

(6.7)

From (6.3) we see that the root system for \( A_n \) has a positive definite metric, whereas for \( A_{n-1} \), the metric is negative definite.

After these basic definitions of the Lie super-algebra we are ready to write down a free superfield description of the \( \mathbb{CP}^{n-1} \) models. We use an \( N = 1 \) superfield formulation and therefore introduce a single anti-commuting coordinate \( \theta \), with \( \theta^2 = 0 \). The superderivative, \( D \), is the square root of the ordinary derivative:

\[ D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial \theta} ; \quad D^2 = \frac{\partial}{\partial z} . \]

(6.8)

For the \( \mathbb{CP}^{n-1} \) model we need \( 2n \) (real) superfields

\[ \Phi^i(z, \theta) = \phi^i(z) + \theta \psi^i(z) , \]

(6.9)

where \( \phi^i(z) \) is a free bosonic field and \( \psi^i(z) \) is a free, real fermion. The superfields satisfy the following operator product expansion

\[ \Phi^i(z_1, \theta_1) \Phi^j(z_2, \theta_2) = -\delta^{ij} \log z_{12} , \]

(6.10)

where

\[ z_{12} = z_1 - z_2 - \theta_1 \theta_2 . \]

(6.11)

The stress tensor is obtained from the the Lax operator \([12][13]\),

\[ L = \prod_{i=1}^{2n+1} [\alpha_0 D - (-1)^i (\lambda_i - \lambda_{i-1}) \cdot D\Phi] , \]

(6.12)
where
\[ \lambda_0 \equiv \lambda_{2n+1} \equiv 0 \quad . \] (6.13)

The parameter \( \alpha_0 \) is background charge of Feigin-Fuchs representation. In order to reproduce the \( CP_{n-1} \) models, whose central charge is:
\[ c = \frac{3k(n-1)}{k+n} \quad , \] (6.14)
we must set
\[ \alpha_0 = \frac{i}{\sqrt{k+n}} \quad . \] (6.15)

In the \( N = 1 \) superfield formulation the stress tensor \( T(z) \) is the top component of an \( N = 1 \) superfield \( T(z, \theta) \) with conformal dimension \( 3/2 \),
\[ T(z, \theta) = \frac{1}{2} \left( G^+(z) + G^-(z) \right) + \theta \; T(z) \quad . \] (6.16)

The fields \( G^\pm(z) \) in (6.16) are the two supersymmetry generators of the \( N = 2 \) supersymmetry algebra. The \( U(1) \) current, \( J(z) \), of the \( N = 2 \) algebra is the lowest component of the superfield \( J(z, \theta) \)
\[ J(z, \theta) = J(z) + \theta \frac{1}{2} \left( G^+(z) - G^-(z) \right) \quad . \] (6.17)

From the Lax operator we obtain \([12][13]\) the currents of the \( N = 2 \) superalgebra for \( CP_{n-1} \) model:
\[ T(z, \theta) = -\frac{1}{2} \sum_{i=1}^{2n} \lambda_{2i} \cdot D\Phi^i \alpha_{2i} \cdot \partial \Phi^i - \frac{1}{2} \sum_{i=1}^{2n} \alpha_{2i} \cdot D\Phi^i \lambda_{2i} \cdot \partial \Phi^i \] 
\[ - \frac{i}{2\sqrt{k+n}} \sum_{i=1}^{2n} \lambda_i \cdot D^3 \Phi \quad . \] (6.18)

and
\[ J(z, \theta) = \sum_{i=1}^{n} \left( \lambda_{2i} \cdot D\Phi \right) \left( \alpha_{2i} \cdot D\Phi \right) - \frac{i}{\sqrt{k+n}} \sum_{i=1}^{n} \left( \lambda_{2i} - \lambda_{2i-1} \right) \cdot \partial \Phi \quad . \] (6.19)

To fully define the conformal model we need the screening operators. These are in one to one correspondence with the roots of the Lie superalgebra \( A(n, n-1) \) and its even subalgebras \( A_n \) and \( A_{n-1} \). Furthermore, the screening operators commute with the Lax operator \([6.12][13]\) and hence they commute with all extended symmetries of the \( CP_{n-1} \)
model. From sections 3 and 4, we know that the screening operators that correspond to the subalgebra $A_{n-1}$ are the ones that play an important role in determining the integrable perturbation. (The other screening operators merely serve to define the parafermionic subsector.) For completeness we will write down all the screening operators. First, the screening operators corresponding to the roots of $A_n$ have the form:

$$Q_{\alpha_{2i-1} + \alpha_{2i}} = \int dz d\theta \left( \alpha_{2i} - \alpha_{2i-1} \right) \cdot D\Phi e^{\frac{i}{\sqrt{k+n}} (\alpha_{2i-1} + \alpha_{2i}) \cdot \Phi). \quad (6.20)$$

The screening operators that are in one to one correspondence with the roots of the even subalgebra $A_{n-1}$ have the form

$$Q_{\alpha_{2i} + \alpha_{2i+1}} = \int dz d\theta \left( \alpha_{2i} - \alpha_{2i+1} \right) \cdot D\Phi e^{\frac{i}{\sqrt{k+n}} (\alpha_{2i} + \alpha_{2i+1}) \cdot \Phi}. \quad (6.21)$$

Finally, associated with every simple root of the Lie subalgebra $A(n,n-1)$, we have a screening operator

$$Q_{\alpha_i} = \int dz d\theta e^{i\sqrt{k+n} \alpha_i \cdot \Phi}. \quad (6.22)$$

There is a natural $\mathbb{Z}_{n+1}$ symmetry of the roots of $A(n,n-1)$ that map our system of simple roots into a new system. Under this symmetry the superfields $\Phi^i$ transform as follows,

$$\alpha_{2i} \cdot \Phi \rightarrow \alpha_{2i+2} \cdot \Phi \quad i = 1, \ldots, n-1$$
$$\alpha_{2n} \cdot \Phi \rightarrow -\lambda_1 \cdot \Phi \quad (6.23)$$
and

$$\alpha_{2i-1} \cdot \Phi \rightarrow \alpha_{2i+1} \cdot \Phi \quad i = 1, \ldots, n-1$$
$$\alpha_{2n-1} \cdot \Phi \rightarrow -\lambda_2 \cdot \Phi \quad (6.24)$$

The screening operators (6.21) transform under the $\mathbb{Z}_{n+1}$ according to:

$$Q_{\alpha_{2i} + \alpha_{2i+1}} \rightarrow Q_{\alpha_{2i+2} + \alpha_{2i+3}} \quad i = 1, \ldots, n-2$$
$$Q_{\alpha_{2n-2} + \alpha_{2n-1}} \rightarrow Q_{\alpha_{2n} - \lambda_{2n}} \equiv \int dz d\theta \left( \alpha_{2n} + \lambda_{2n} \right) \cdot D\Phi e^{\frac{i}{\sqrt{k+n}} (\alpha_{2n} - \lambda_{2n}) \cdot \Phi} \quad (6.25)$$
$$Q_{\alpha_{2n} - \lambda_{2n}} \rightarrow Q_{\alpha_{1} - \lambda_{1}} \equiv -\int dz d\theta \left( \lambda_{1} + \alpha_{1} \right) \cdot D\Phi e^{\frac{i}{\sqrt{k+n}} (\alpha_{1} - \lambda_{1}) \cdot \Phi}$$
$$Q_{\alpha_{1} - \lambda_{1}} \rightarrow Q_{\alpha_{2} + \alpha_{3}}.$$

The roots, $\alpha_{2i-1} + \alpha_{2i}, \ i = 1, \ldots, n-1$, $\alpha_{2n} - \lambda_{2n}$ and $\lambda_{1} - \alpha_{1}$, which appear in the foregoing transformations, in fact define a system of simple roots for $A_n$. 

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In the transformations of the screening operators, we introduced two new operators:
\[
\int d\theta \left( \alpha_{2n} + \lambda_{2n} \right) \cdot D\Phi \ e^{-\frac{i}{\sqrt{k+n}}(\alpha_{2n} - \lambda_{2n}) \cdot \Phi} \\
\int d\theta \left( \lambda_1 + \alpha_1 \right) \cdot D\Phi \ e^{-\frac{i}{\sqrt{k+n}}(\lambda_1 + \alpha_1) \cdot \Phi}.
\]
(6.26)

From the operator product expansion of these fields with stress tensor we find that the conformal dimension of both of these fields is \((2k + n)/(2k + n)\). The operator product of these operators with the \(U(1)\) current of the \(N = 2\) algebra shows that these fields have opposite \(U(1)\) charge. These fields correspond to the operators \(S_{-\gamma}\) and \(S_{\psi}\) that were discussed in section 4.

One can use the explicit realization of the super \(W\)-generators provided by (6.12) to verify that the top component of the \(N = 2\) supermultiplet are invariant, up to total derivatives, under the \(\mathbb{Z}_{n+1}\) symmetry. As a result the charges associated with the generators commute with the perturbation. Hence, to first order in this perturbation, the top components of the \(N = 2\) super \(W\)-algebra can be extended to integrals of motion for the perturbed model. (This argument is a straightforward generalization of the one that was used in [37] and [41] to establish similar results for purely bosonic formulations of \(W\) and super-\(W\) algebras.)

It is relatively easy to identify the affine super-Toda action that underlies the integrable model. Motivated by the purely bosonic theory, one pairs a holomorphic screening charge (or a relevant perturbation) with its anti-holomorphic counterpart to obtain the following potential term of a super-Toda action:
\[
\sum_{i=1}^{n-1} \int d^2z d\theta d\bar{\theta} (\alpha_{2i} - \alpha_{2i+1}) \cdot D\Phi (\alpha_{2i} - \alpha_{2i+1}) \cdot \bar{D}\Phi e^{-\frac{i}{\sqrt{k+n}}(\alpha_{2i} + \alpha_{2i+1}) \cdot \Phi} + \\
\int d^2z d\theta d\bar{\theta} (\alpha_{2n} + \lambda_{2n}) \cdot D\Phi (\alpha_{2n} + \lambda_{2n}) \cdot \bar{D}\Phi e^{-\frac{i}{\sqrt{k+n}}(\alpha_{2n} - \lambda_{2n}) \cdot \Phi} + \\
\int d^2z d\theta d\bar{\theta} (\alpha_1 + \lambda_1) \cdot D\Phi (\alpha_1 + \lambda_1) \cdot \bar{D}\Phi e^{-\frac{i}{\sqrt{k+n}}(\alpha_1 - \lambda_1) \cdot \Phi},
\]
(6.27)

where \(\Phi = \Phi(z, \theta, \bar{z}, \bar{\theta})\). One can verify, using the techniques of [31] [32], that when employed in a perturbation expansion, the foregoing Toda potential generates the appropriate screening for the conformal sector, along with the proper insertions of the relevant perturbations.
To conclude this section we examine the action of $\mathbb{Z}_{n+1}$ symmetry of the other screening operators. For the screening operators (6.20) we have the following transformation law

\[
Q_{\alpha_{2i-1}+\alpha_{2i}} \rightarrow Q_{\alpha_{2i+1}+\alpha_{2i+2}} \quad i = 1, 2, \ldots, n-1
\]
\[
Q_{\alpha_{2n-1}+\alpha_{2n}} \rightarrow \int dz d\theta (\lambda_{2n} - \lambda_1) \cdot D\Phi e^{-\frac{ix}{\sqrt{k+n}}(\lambda_1+\lambda_{2n})} \equiv Q_{\lambda_1+\lambda_{2n}}
\]
(6.28)
\[
Q_{\lambda_1+\lambda_{2n}} \rightarrow Q_{\alpha_1+\alpha_2}
\]

(It is interesting to note that sum of the weights $\lambda_1 + \lambda_{2n}$ appearing in (6.28) is just the sum of the simple roots of the super Lie algebra $A(n,n-1)$. From (6.28) we see that $\mathbb{Z}_{n+1}$ transformation generates a new operator

\[
\int d\theta (\lambda_{2n} - \lambda_1) \cdot D\Phi e^{-\frac{ix}{\sqrt{k+n}}(\lambda_1+\lambda_{2n})}.
\]
(6.29)

From the operator product of this operator with the stress tensor $T(z)$ and the $U(1)$ current we find that the conformal dimension of the field (6.29) is \(\frac{k+n}{k+n}\) and the $U(1)$ charge is zero. This operator is irrelevant and therefore does not give rise to an interesting perturbed conformal field theory.

The $\mathbb{Z}_{n+1}$ symmetry acts on the operators (6.20) according to:

\[
Q_{\alpha_{2i-1}} \rightarrow Q_{\alpha_{2i+1}} \quad i = 1, 2, \ldots, n-1
\]
\[
Q_{\alpha_{2n-1}} \rightarrow \int dz d\theta e^{-i\sqrt{k+n}\lambda_{2n}} \cdot \Phi \equiv Q_{\lambda_{2n}}
\]
(6.30)
\[
Q_{\lambda_{2n}} \rightarrow Q_{\alpha_1}
\]

and

\[
Q_{\alpha_{2i}} \rightarrow Q_{\alpha_{2i+2}} \quad i = 1, 2, \ldots, n-1
\]
\[
Q_{\alpha_{2n}} \rightarrow \int dz d\theta e^{-i\sqrt{k+n}\lambda_{1}} \cdot \Phi \equiv Q_{\lambda_{1}}
\]
(6.31)
\[
Q_{\lambda_{1}} \rightarrow Q_{\alpha_2}
\]

From equations (6.30) and (6.31) we see that the $\mathbb{Z}_{n+1}$ symmetry generates two new operators,

\[
\int d\theta e^{-i\sqrt{k+n}\lambda_{2n}} \cdot \Phi,
\]
(6.32)
\[
\int d\theta e^{-i\sqrt{k+n}\lambda_{1}} \cdot \Phi.
\]

As before the operator product with the stress tensor shows that these also correspond to irrelevant operators and therefore do not give rise to any interesting perturbations.
7. Conclusions

Using the paratoda description of $N = 2$ supersymmetric models we have established the connection between the coset models $G_k/G_k$ and twisted $N = 2$ superconformal models. To obtain the equivalence of these models we saw that we have to perturb the topological $N = 2$ model by the top component of a chiral primary superfield, whose bottom component is a very particular chiral primary field (defined in (3.2)) with conformal dimension $\frac{k}{2(k+g)}$. Once this perturbation is made, the fields of the $G_k/G_k$ theory can be identified as a sub-ring, $\mathcal{F}$, of the chiral ring, $\mathcal{R}$, of the perturbed $N = 2$ theory. The correlation functions of the $G/G$ theory are equal to the topological correlators of the fields in the sub-ring, $\mathcal{F}$, of $\mathcal{R}$ in the $N = 2$ theory, and, in particular, the fusion coefficients of $G$ can be identified with the three point functions of the perturbed $N = 2$ theory.

We have also shown that the chiral primary fields, $\Phi_\Lambda(x_a; \lambda)$, when evaluated at the points $x^{(\alpha)}_a$ that solve the vanishing relations, are nothing other than the discrete characters of $G_k$.

As has been conjectured elsewhere [3], we also find that the perturbation that leads to the $G_k/G_k$ theory is also precisely the one that leads to integrable models. Once again, the paratoda formulation leads to a simple understanding of the integrals of motion of these perturbed conformal field theories.

There remain several interesting questions. We have, for simplicity, taken the diagonal modular invariant for $G$. The result that we have established here is valid for any choice of modular invariant for $G_k$. Thus one can take the attitude that this paper defines $G_k/G_k$ theories for special modular invariants by relating them to perturbed, twisted $N = 2$, superconformal theories. It would be interesting to understand whether there is some fusion algebra interpretation of $G_k/G_k$ theory with a special modular invariant. In view of the results in the last section of [44], the modular invariants for $SU_k(2)$ will almost certainly provide some interesting ring structures.

Finally, we have seen that the fusion ring is naturally a sub-ring of $\mathcal{R}$. However, all the fields in $\mathcal{R}$ can be interpreted in terms of $\frac{G_k \times G_0}{G_k}$ provided one twists $G_0$. It might be interesting to study such extensions of the $G_k/G_k$ theory.

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