KoPA: Automated Kronecker Product Approximation

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Abstract

We consider the matrix approximation induced by the Kronecker product decomposition. We propose to approximate a given matrix by the sum of a few Kronecker products, which we refer to as the Kronecker product approximation (KoPA). Because the Kronecker product is an extension of the outer product from vectors to matrices, KoPA extends the low rank approximation, and include the latter as a special case. KoPA also offers a greater flexibility over the low rank approximation, since it allows the user to choose the configuration, which are the dimensions of the two smaller matrices forming the Kronecker product. On the other hand, the configuration to be used is usually unknown, and has to be determined from the data in order to achieve the optimal balance between accuracy and parsimony. We propose to use extended information criteria to select the configuration. Under the paradigm of high dimensional analysis, we show that the proposed procedure is able to select the true configuration with probability tending to one, under suitable conditions on the signal-to-noise ratio. We demonstrate the superiority of KoPA over the low rank approximations through numerical studies, and a benchmark image example.

Keywords: Information Criterion, Kronecker Product, Low Rank Approximation, Matrix Decomposition, Model Selection, Random Matrix
1 Introduction

Observations that are matrix/tensor valued have been commonly seen in various scientific fields and social studies. In recent years, technological advances have made matrix/tensor type data that are of large dimensions possible and more and more prevalent. For example, high resolution images in face recognition and motion detection (Turk and Pentland, 1991; Bruce and Young, 1986; Parkhi et al., 2015), brain images through fMRI (Belliveau et al., 1991; Maldjian et al., 2003), adjacent matrices of social networks of millions of nodes (Goldenberg et al., 2010), the covariance matrix of thousands of stock returns (Ng et al., 1992; Fan et al., 2011), the import/export network among hundreds of countries (Chen et al., 2019), etc. Due to the high dimensionality of the data, it is often useful and preferred to store, compress, represent, or summarize the matrices/tensors through low dimensional structures. In particular, low rank approximations of matrices have been ubiquitous. Finding a low rank approximation of a given matrix is closely related to the singular value decomposition (SVD), see Eckart and Young (1936) for an early paper pointing out the connection. SVD has proven to be extremely useful in matrix completion (Candes and Recht, 2009; Candes and Plan, 2010; Cai et al., 2010), community detection (Le et al., 2016), image denoising (Guo et al., 2015), among many others.

In this paper, we investigate the matrix approximations induced by the Kronecker product. Since the \texttt{\LaTeX} command for the Kronecker product is \texttt{\otimes}, we create the name KoPA (Kronecker otimes Product Approximation) for the proposed method. Kronecker product is an operation on two matrices which generalizes the outer product from vectors to matrices. Specifically, the Kronecker product of a \(m_1 \times n_1\) matrix \(A = (a_{ij})\) and a \(m_2 \times n_2\) matrix \(B = (b_{ij})\), denoted by \(A \otimes B\), is defined as a \((m_1m_2) \times (n_1n_2)\) matrix which takes the form of a block matrix. In \(A \otimes B\), there are \(m_1n_1\) blocks of size \(m_2 \times n_2\): the \((i,j)\)-th block equal to the scalar product \(a_{ij}B\). We refer the readers to Horn and Johnson (1991) and Van Loan and Pitsianis (1993) for overviews of the properties and computations of the Kronecker product. Kronecker product has also found wide applications in signal processing, image restoration and quantum computing, etc. For example, in the statistical model for a multi-input multi-output (MIMO) channel communication system, Werner et al. (2008) modeled the covariance matrix of channel signals as the Kronecker product of the transmit covariance matrix and the receive covariance matrix. In compressed sensing, Duarte and Baraniuk (2012) utilized Kronecker product to provide a sparse basis for high-dimensional signals. In image restoration, Kamm and Nagy (1998) considered the blurring operator as a Kronecker product of two smaller matrices. In quantum computing, Kaye et al. (2007) represented the joint state of quantum bits as a Kronecker product of their individual states.

In SVD, a matrix is represented in the sum of rank one matrices, and each rank one matrix is written as the outer product of the left singular vector, and the corresponding right singular vector (after the transpose). Similarly, we define the Kronecker Product Decomposition (KPD) of a \((m_1m_2) \times (n_1n_2)\) matrix \(C\) as

\[
C = \sum_{k=1}^{d} A_k \otimes B_k,
\]

where \(d = \min\{m_1n_1, m_2n_2\}\), and each \(A_k\) is \(m_1 \times n_1\), and each \(B_k\) is \(m_2 \times n_2\). In defining the KPD, the dimensions of \(A_k\) and \(B_k\) are relevant, and we refer to their dimensions (in this case, \(m_1 \times n_1\) and \(m_2 \times n_2\))
as the *configuration* of the KPD. Apparently further constraints on $A_k$ and $B_k$ are necessary to make the decomposition well defined and unique, but we will defer the exact definition of KPD to Section 2. Since the Kronecker product can be viewed as an extension of the vector outer product, the KPD can be viewed as an extension of the SVD correspondingly. In particular, if $n_1 = 1$, and $m_2 = 1$, then $A_k$ and $B_k$ are column and row vectors, respectively, and the KPD reduces to SVD.

Similarly as in the rank-one approximation, the best matrix approximation given by a Kronecker product is formulated as finding the closest Kronecker product under the Frobenius norm. This was introduced in the matrix computation literature as the nearest Kronecker product (NKP) problem in [Van Loan and Pitsianis (1993)](#), who also demonstrated its equivalence to the best rank one approximation and therefore also to the SVD, after a proper rearrangement of the matrix entries. Such an equivalence is also maintained if one seeks the best approximation of a given matrix by the sum $\sum_{k=1}^{p} A_k \otimes B_k$ of $p$ Kronecker products of the same configuration. Despite of its connection to SVD, finding a best Kronecker approximation also involves a pre-step: determining the configurations of the Kronecker products, i.e., determining the dimensions of $A_k$ and $B_k$. One of our major contributions in this paper is on the selection of the configuration based on the information criterion.

Although the configuration selection poses new challenges, KPD also provides a framework that is more flexible than SVD. Here we use the image of Lenna, a benchmark in image analysis, to illustrate the potential advantage of the KPD over SVD. The left panel in Figure 1 is the $512 \times 512$ pixel portrait of Lenna in gray scale. The middle panel shows the best rank-1 approximation of the original image given by the leading term of SVD. The rank-1 approximation explains 28.14% of the total variation of the original image with 1023 parameters. The right panel in Figure 1 displays the image obtained by the nearest Kronecker product of configuration $(16 \times 32) \otimes (32 \times 16)$. With the same number of parameters as the rank-1 approximation, this nearest Kronecker approximation explains 72.17% of the variance, 2.56 times of the capability of the best rank-1 approximation.

![Figure 1: (Left) Original Lenna image (Mid) SVD approximation (Right) KPD approximation](image-url)
In this paper, we focus on the model
\[ Y = \lambda A \otimes B + \sigma E, \]
where \( E \) is a standard Gaussian ensemble consisting of IID standard normal entries, \( \lambda > 0 \) and \( \sigma > 0 \) indicates the strength of signal and noise respectively. We consider the matrix de-noising problem which aims to recover the Kronecker product \( \lambda A \otimes B \) from the noisy observation \( Y \). Here the configuration of the Kronecker product, i.e. the dimensions of \( A \) and \( B \), are to be determined from the data. We propose to use information criteria (which include AIC and BIC as special cases) to select the configuration, and prove its consistency under mild conditions on the signal-to-noise ratio. The consistency of the configuration selection is established for both random and deterministic \( A \) and \( B \), under the paradigm of high dimensional analysis, where the dimension of \( Y \) diverges to infinity.

The rest of the paper is organized as follows. In Section 2, we provide the definitions of the KPD, and the model under consideration, with a review of some of their basic properties. In Section 3, we introduce the information criteria for selecting the configuration of the Kronecker product. We investigate and establish the consistency of the proposed selection procedure in Section 4. Extension to the two-term Kronecker product models is discussed in Section 5. In Section 6, we carry out extensive simulations to assess the performance of our method, and demonstrate its superiority over the SVD approach. We also present a detailed analysis of the Lenna image.

### Notations:
Throughout this paper, for a vector \( v \), \( \|v\| \) denotes its Euclidean norm. And for a matrix \( M \), \( \|M\|_F = \sqrt{\text{tr}(M'M)} \) and \( \|M\|_S = \max_{\|u\|=1} \|Mu\| \) denote its Frobenius norm and spectral norm respectively. For any two real numbers \( a \) and \( b \), \( a \wedge b \) and \( a \vee b \) stand for \( \min\{a, b\} \) and \( \max\{a, b\} \) respectively. For any number \( x \), \( x_+ \) denotes the positive part \( x \vee 0 = \max\{x, 0\} \).

### 2 Kronecker Product Model

#### 2.1 Kronecker Product Decomposition
We first repeat the definition of the Kronecker product of a \( m_1 \times n_1 \) matrix \( A \) and a \( m_2 \times n_2 \) matrix \( B \), which is given by
\[ A \otimes B = \begin{bmatrix}
  a_{1,1}B & a_{1,2}B & \cdots & a_{1,n_1}B \\
  a_{2,1}B & a_{2,2}B & \cdots & a_{2,n_1}B \\
  \vdots & \vdots & & \vdots \\
  a_{m_1,1}B & a_{m_1,2}B & \cdots & a_{m_1,n_1}B
\end{bmatrix}. \]

Let \( C \) be a \((m_1m_2) \times (n_1n_2)\) real matrix, its Kronecker Product Decomposition (KPD) of configuration \((m_1, n_1, m_2, n_2)\) is defined as
\[ C = \sum_{k=1}^{d} \lambda_k A_k \otimes B_k. \]
where \( d = \min\{m_1n_1, m_2n_2\} \), each \( A_k \) is \( m_1 \times n_1 \) with Frobenius norm \( \|A_k\|_F = 1 \), each \( B_k \) is \( m_2 \times n_2 \) with \( \|B_k\|_F = 1 \), and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0 \). The matrices \( A_k \) are mutually orthogonal in the sense that \( \text{tr}(A_kA_l') = 0 \) for \( 1 \leq k < l \leq d \), and so are the matrices \( B_k \).
The best way to see that the KPD is a valid definition is through its connection with the SVD, after a proper rearrangement of the elements of $C$, as demonstrated in Van Loan and Pitsianis (1993). Denote by vec($\cdot$) the vectorization of a matrix by stacking its rows. If $A = (a_{ij})$ is a $m \times n$ matrix, then

$$\text{vec}(A) := [a_{1,1}, \ldots, a_{1,n_1}, \ldots, a_{m_1,1}, \ldots, a_{m_1,n_1}]'.$$

If $B = (b_{ij})$ is a $m_2 \times n_2$ matrix, then vec($A$)[vec($B$)]' is a $(m_1m_1) \times (m_2m_2)$ matrix containing the same set of elements as the Kronecker product $A \otimes B$, but in different positions. We define the rearrangement operator $R$ to represent this relationship. Write the matrix $C$ as a $m_1 \times n_1$ array of blocks of the same block size $m_2 \times n_2$, and denote by $C_{i,j}^{m_2,n_2}$ the $(i,j)$-th block, where $1 \leq i \leq m_1$, $1 \leq j \leq n_1$. The operator $R$ maps the matrix $C$ to

$$R[C] = [\text{vec}(C_{1,1}^{m_2,n_2}), \ldots, \text{vec}(C_{1,n_1}^{m_2,n_2}), \ldots, \text{vec}(C_{m_1,1}^{m_2,n_2}), \ldots, \text{vec}(C_{m_1,n_1}^{m_2,n_2})]',$$

(2)

When applied to a Kronecker product $A \otimes B$, it holds that

$$R[A \otimes B] = \text{vec}(A)[\text{vec}(B)]'.$$

(3)

In view of (2) and (3), we see that the KPD in (1) corresponds to the SVD of the rearranged matrix $R[C]$, and the conditions imposed on $A_k$ and $B_k$ are derived from the properties of the singular vectors.

### 2.2 Kronecker Product Model

We consider the model where the observed matrix $Y$ is a noisy version of some unknown Kronecker product

$$Y = \lambda A \otimes B + \sigma E,.$$  
(4)

To resolve the obvious unidentifiability regarding $A$ and $B$, we require

$$\|A\|_F = \|B\|_F = 1,$$

(5)

so that $\lambda > 0$ indicates the strength of the signal part. Note that under (5), $A$ and $B$ are identified up to a sign change. We assume that the noise matrix $E$ has IID stand normal entries, and consequently the strength of the noise is controlled by $\sigma > 0$. The dimensions of $A$ and $B$ correspond to the integer factorization of the dimension of $Y$. For convenience, we assume throughout this article that the dimension of the observed matrix $Y$ in (4) is $2^M \times 2^N$ with $M, N \in \mathbb{N}$. As a result, the dimension of $A$ must be of the form $2^{m_0} \times 2^{n_0}$, where $0 \leq m_0 \leq M$ and $0 \leq n_0 \leq N$, and the corresponding dimension of $B$ is $2^{M-m_0} \times 2^{N-n_0}$. Therefore, we can simply use the pair $(m_0, n_0)$ to denote the configuration of the Kronecker product in (4). An implicit advantage of this assumption lies in the fact that if two configurations $(m, n)$ and $(m', n')$ are different, then the number of rows of $A$ under one configurations divides the one under the other, and similarly for the number of columns, and for $B$. For example, if $m \leq m'$, then the number of rows of $A$ under the former configuration, which is $2^m$, divides the number of rows $2^{m'}$ under the latter one. This fact turns out to be crucial for our theoretical analysis in Section 4.

**Remark 1.** For image analysis, assuming the dimension to be powers of 2 seems rather reasonable. On the other hand, for other applications where the dimension of the observed matrix are not powers of 2, one can...
transform the matrix to fulfill the assumption. For example, one can super-sample the matrix to increase
the dimension to the closest powers of 2, or augment the matrix by padding zeros.

We will consider two mechanisms for the signal part $\lambda A \otimes B$.

**Random Scheme.** We assume that

$$\lambda A \otimes B = \lambda_0 \tilde{A} \otimes \tilde{B},$$

(6)

where $\tilde{A}$ and $\tilde{B}$ are independent, and both consisting of IID standard normal entries. In order to fulfill the
identifiability condition (5), we let $A = \tilde{A}/\|\tilde{A}\|_F$, $B = \tilde{B}/\|\tilde{B}\|_F$, and $\lambda = \lambda_0 \cdot \|\tilde{A}\|_F \cdot \|\tilde{B}\|_F$. For the random
scheme, we define the signal-to-noise ratio as

$$\frac{\mathbb{E}\|\lambda A \otimes B\|_F^2}{\mathbb{E}\|\sigma E\|_F^2} = \frac{\lambda^2_0}{\sigma^2}.$$  

(7)

**Deterministic Scheme.** We assume both $A$ and $B$ are deterministic, satisfying (5). In this case the
signal-to-noise ratio is given by

$$\frac{\|\lambda A \otimes B\|_F^2}{\|\sigma E\|_F^2} = \frac{\lambda^2}{\sigma^2 \cdot 2^{M+N}}$$

for the deterministic scheme.

## 2.3 Estimation with a Given Configuration

Suppose we want to estimate $A$ and $B$ based on a given configuration $(m,n)$, i.e. the dimensions of $\hat{A}$ and $\hat{B}$ are $2^m \times 2^n$ and $2^{M-m} \times 2^{N-n}$ respectively. To estimate $A$ and $B$ in (4) from the observed matrix $Y$, we solve the minimization problem

$$\min_{\lambda, A, B} \|Y - \lambda A \otimes B\|_F^2, \text{ subject to } \|A\|_F = \|B\|_F = 1.$$  

(8)

Since we have assumed that the noise matrix contains IID standard normal entries, (8) is also equivalent to
the MLE. This optimization problem has been formulated as the nearest Kronecker product (NKP) problem
in the matrix computation literature [Van Loan and Pitsianis 1993], and solved through the SVD after
rearrangement. According to Section 2.1 after applying the rearrangement operator, the cost function in
(8) is equivalent to

$$\|Y - \lambda A \otimes B\|_F^2 = \|\mathcal{R}[Y] - \text{vec}(A)[\text{vec}(B)]'\|_F^2.$$  

We note that the rearrangement operator $\mathcal{R}$ defined in (2) depends on the configuration of the block matrix,
and in the current case, on the configuration $(m,n)$. We will use the notation $\mathcal{R}_{m,n}[Y]$ if we need to
emphasize this dependence, but will otherwise suppress the subscript for notational simplicity. Let $\mathcal{R}[Y] = \sum_{d=1}^d \lambda_k u_k v_k'$ be the SVD of the rearranged matrix $\mathcal{R}_{m,n}[Y]$, where $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$ are the singular values in decreasing order, $u_k$ and $v_k$ are the corresponding left and right eigenvectors and $d = 2^{m+n} \wedge 2^{M+N} - m - n$.

The estimators for model (4) is given by

$$\hat{\lambda} = \lambda_1 = \|\mathcal{R}[Y]\|_S, \quad \hat{A} = \text{vec}^{-1}(u_1), \quad \hat{B} = \text{vec}^{-1}(v_1), \quad \hat{\sigma}^2 = \frac{\|Y\|_F^2 - \hat{\lambda}^2}{2^{M+N}},$$  

(9)

where vec$^{-1}$ is the inverse operation of vec($\cdot$) that restores a vector back into a matrix.
We look at a few special cases of the configuration \((m, n)\). When \((m, n) = (0, 0)\) or \((m, n) = (M, N)\), the nearest Kronecker product approximation of \(Y\) is always itself. For instance, if \(m = n = 0\), the estimators are

\[
\hat{\lambda} = \|Y\|_F, \quad \hat{A} = 1, \quad \hat{B} = \hat{\lambda}^{-1}Y, \quad \sigma^2 = 0.
\]

These two configurations are obviously over-fitting, and we shall exclude them in the subsequent analysis.

When \(m = 0\), \(n = N\) or \(m = M\), \(n = 0\), the nearest Kronecker product approximation of \(Y\) is the same as the rank-1 approximation of \(Y\) without rearrangement. When the true configuration used to generate \(Y\) is chosen, that is \((m, n) = (m_0, n_0)\), the Kronecker product decomposition approach is equivalent to denoising a perturbed rank-1 matrix, since

\[
\mathcal{R}_{m_0, n_0}[Y] = \lambda \text{vec}(A)\text{vec}(B)' + \sigma \mathcal{R}_{m_0, n_0}[E],
\]

where the rearranged perturbation term \(\mathcal{R}_{m_0, n_0}[E]\) is still a standard Gaussian ensemble. Therefore \(\lambda, A\) and \(B\) can be recovered with negligible error when \(\sigma \|\mathcal{R}_{m_0, n_0}[E]\|_S \ll \lambda\). Details will be discussed in Section 4.

### 3 Configuration Determination through the Information Criteria

Our primary goal is to recover the Kronecker product \(\lambda A \otimes B\) from \(Y\), based on the model (4), and we have shown in Section 2.3 how the NKP problem (8) can be solved through SVD, after a proper rearrangement of the matrix entries. This approach depends on the configuration of the Kronecker product, which is typically unknown. We propose to use the information criteria based procedure to select the configuration.

Recall that we have assumed the dimension of \(Y\) to be \(2^M \times 2^N\). If the dimension of \(A\) is \(2^m \times 2^n\), then the dimension of \(B\) has to be \(2^{M-m} \times 2^{N-n}\). Therefore, the configuration can be indexed by the pair \((m, n)\), which takes value from the Cartesian product set \(\{0, \ldots, M\} \times \{0, \ldots, N\}\).

For any given configuration \((m, n)\), the estimation procedure in Section 2.3 leads to the corresponding estimators \(\hat{\lambda}, \hat{A}\) and \(\hat{B}\). Denote the estimated Kronecker product by \(\hat{Y}(m, n) = \hat{\lambda} \hat{A} \otimes \hat{B}\). Note that all of \(\hat{\lambda}, \hat{A}\) and \(\hat{B}\) depend implicitly on the configuration \((m, n)\) used in estimation, and should be written as \(\hat{\lambda} = \hat{\lambda}(m, n)\) etc. However, we will suppress the configuration index from the notation for simplicity, whenever its meaning is clear in the context. Under the assumption that the noise matrix \(E\) is a standard Gaussian ensemble, we define the information criterion as

\[
\text{IC}_q(m, n) = 2^{M+N} \ln \frac{\|Y - \hat{Y}\|_F^2}{2^{M+N}} + qp,
\]

where \(\hat{Y} = \hat{Y}(m, n)\) is the estimated Kronecker product under the configuration \((m, n)\), \(p = 2^m + n + 2^{M+N-m-n}\) is the number of parameters involved in the Kronecker product of the configuration \((m, n)\), and \(q \geq 0\) controls the penalty on the model complexity. The information criterion (11) can be viewed as another version of the extended BIC introduced by Chen and Chen (2008) and Foygel and Drton (2010) in the linear regression and graphical models setting, respectively. The information criterion (11) reduces to the log mean square error when \(q = 0\), and corresponds to the Akaike information criterion (AIC) (Akaike, 1974) when \(q = 2\), and the Bayesian information criterion (BIC) (Schwarz, 1978) when \(q = \ln 2^{M+N} = (M + N) \ln 2\).
compared with the first term in (11). We conclude that the number of parameters under true configuration is $2\lambda$ the first one. Therefore, under the true configuration, $A$ has a spectral norm of the order $\lambda$ the first term is a rank-1 matrix of spectral norm $2\lambda$. 

Remark 2. Strictly speaking, the number of parameters involved in the Kronecker product $\lambda A \otimes B$ should be $2^{m+n} + 2^{M+N-m-n} - 1$ because of the constraints [3]. Since it does not affect the selection procedure to be introduced in (12), we will use $p = 2^{m+n} + 2^{M+N-m-n}$ for simplicity.

The information criterion (11) can be calculated for all configurations, and the one corresponding to the smallest value of (11) will be selected, based on which the estimation procedure in Section 2.3 proceeds. In other words, the selected configuration $(\hat{m}, \hat{n})$ is obtained through

$$ (\hat{m}, \hat{n}) = \arg \min_{(m,n) \in \mathcal{C}} \text{IC}_q(m,n), $$

where $\mathcal{C}$ is the set of all candidate configurations.

As discussed in Section 2.3 when $m = n = 0$ or $m = M, n = N$, it holds that $\hat{Y} = Y$, and the information criterion (11) will be $-\infty$, no matter what value $q$ takes. Therefore, these two configurations should be excluded in model selection and we use

$$ \mathcal{C} := \{0, \ldots, M\} \times \{0, \ldots, N\} \setminus \{(0,0), (M,N)\}, $$

as the set of candidate configurations in (12). Note that the set $\{0, \ldots, M\} \times \{0, \ldots, N\}$ forms a rectangle lattice in $\mathbb{Z}^2$, and $m = n = 0$ and $m = M, n = N$ are the bottom left and top right corner of the lattice. Therefore, we sometimes refer to these two configurations as the “corner cases” in the sequel, intuitively.

Furthermore, we define $W$ as the set of all wrong configurations

$$ W := \mathcal{C} \setminus \{(m_0, n_0)\}. $$

We now provide a heuristic argument to show how the selection procedure (12) is able to select the true configuration $(m_0, n_0)$, under the random scheme (6). Precise statements of the results and the analysis will be presented in Section 4. Basically, we are reiterating the well known wisdom of balancing the bias and the variance.

According to [3], for a given configuration $(m, n)$, $\mathcal{R}[\hat{Y}]$ equals the first SVD component of $\mathcal{R}[Y]$, and it follows that

$$ \text{tr}\{\mathcal{R}[\hat{Y}](\mathcal{R}[Y - \hat{Y}])'\} = 0. $$

Therefore, $\|Y - \hat{Y}\|_F^2 = \|Y\|_F^2 - \|\hat{Y}\|_F^2 = \|Y\|_F^2 - \lambda^2$, and the information criterion (11) can be rewritten as

$$ \text{IC}_q(m,n) = 2^{M+N} \ln \frac{\|Y\|_F^2 - \lambda^2}{2^{M+N}} + qp. \quad (13) $$

For the true configuration $(m, n) = (m_0, n_0)$, the rearranged matrix $\mathcal{R}[Y]$ takes the form (10), where the first term is a rank-1 matrix of spectral norm $\lambda = \lambda_0\|\hat{A}\|_F\|\hat{B}\|_F \approx \lambda_0 2^{(M+N)/2}$, and the second term has a spectral norm of the order $O(2^{(m_0+n_0)/2} + 2^{M+N-m_0-n_0}/2) \quad (\text{see Vershynin (2010)})$ for an overview of the finite sample bounds of the expected norms of a Gaussian ensemble). If the dimensions of $A$ and $B$ are sufficiently large, and $m_0, n_0, M - m_0, N - n_0 \gg 1$, then the latter spectral norm is negligible compared with the first one. Therefore, under the true configuration, $\lambda, A$ and $B$ can be accurately estimated. Additionally, the number of parameters under true configuration is $2^{m_0+n_0} + 2^{M+N-m_0-n_0} = o(2^{M+N})$, which is negligible compared with the first term in (11). We conclude that

$$ \text{IC}_q(m_0, n_0) \approx 2^{M+N} \ln \frac{\|\lambda A \otimes B + \sigma E\|_F^2 - \lambda^2}{2^{M+N}} \approx 2^{M+N} \ln \frac{\|\sigma E\|_F^2}{2^{M+N}} \approx 2^{M+N} \sigma^2. $$
For the wrong configuration \((m, n) \in W\) that is close to the true one, the rearranged noise matrix \(R_{m,n}[E]\) and the number of parameters \(p\) are negligible as well. However, the estimated coefficient \(\hat{\lambda}\) is much smaller than the one obtained under true configuration:

\[
\hat{\lambda} = \|R_{m,n}[Y]\|_S \approx \|R_{m,n}[\lambda A \otimes B]\|_S < \|R_{m_0,n_0}[\lambda A \otimes B]\|_S = \lambda \approx \lambda_0 2^{(M+N)/2}.
\]

In fact, our detailed analysis in Section 4 will show that \(\hat{\lambda}\) is approximately at most \(\lambda/\sqrt{2}\), which means that the wrong but close-to-truth configuration \((m, n)\) leads to a serious under-fitting. To summarize

\[
IC_q(m, n) \approx 2^{M+N} \ln \left(\frac{\|\lambda A \otimes B + \sigma E\|_F^2 - \hat{\lambda}^2}{2^{M+N}}\right) \approx 2^{M+N} \ln \left(\frac{\|\sigma E\|_F^2 + \lambda^2/2}{2^{M+N}}\right) \\
\approx 2^{M+N} \left(\sigma^2 + \lambda_0^2/2\right) > IC_q(m_0, n_0).
\]

Therefore, the information criterion (11) is in favor of the true configuration over a wrong but close-to-truth one.

On the other hand, if the wrong configuration \((m, n) \in W\) is close to the corner one \((0, 0)\), the number of parameters \(p\) is approximately \(2^{M+N-m-n}\), which is now comparable with the likelihood term in (11). Even though the close-to-corner configurations may result in over-fitting so that \(\hat{\lambda} > \lambda\) (as to be shown in Section 4), the penalty term in information criterion will then dominates and leads to a greater \(IC_q(m, n) > IC_q(m_0, n_0)\), under certain conditions on the signal-to-noise ratio defined in (7).

We have seen that the trade-off between goodness-of-fit and model complexity plays its role here, as expected. Wrong but close-to-truth configurations involve similar numbers of parameters as the true one, but lead to serious under-fitting; on the other hand, close-to-corner configurations may provide a closer fit to the original \(Y\), but require much more parameters. Precise statements and detailed theoretical analysis will be presented in Section 4.

### 4 Theoretical Results

The main purpose of this section is to give a theoretical guarantee of the configuration selection procedure proposed in Section 3. We will consider both the random scheme (6) and the deterministic scheme introduced in Section 2.2. Since the selection procedure is information-criterion based, a classical approach is to establish the asymptotic consistency. For this purpose, we make the following assumption on the size of the component matrices \(A\) and \(B\), which fall in the paradigm of high dimensional analysis. In this section all our discussion will be based on model (4). We use \((m_0, n_0)\) to denote the true configuration, i.e. the matrices \(A\) and \(B\) are of dimensions \(2^{m_0} \times 2^{n_0}\) and \(2^{M-m_0} \times 2^{N-n_0}\) respectively.

**Assumption 1.** Consider the model (4). Assume that the true configuration \((m_0, n_0)\) satisfy

\[
m_0 + n_0 \to \infty, \quad M + N - m_0 - n_0 \to \infty,
\]

and

\[
2^{m_0+n_0} + 2^{M+N-m_0-n_0} = o \left(\frac{2^{M+N}}{\ln(MN)}\right).
\]
The first condition entails that the numbers of entries in $A$ and $B$ will need to diverge to infinity, and so is that of $Y$. The second condition ensures that the true configuration cannot stay arbitrarily close to the corner configurations. On the other hand, we remark that this will be the only condition on the sizes of the involved matrices: we do not require each of $m_0, n_0, M - m_0, N - n_0$ to go to infinity.

The number of parameters involved in the Kronecker product $\lambda A \otimes B$ is

$$p_0 = 2m_0 + n_0 + 2M + N - m_0 - n_0 - 1.$$ 

Comparing with the dimension $2^M \times 2^N$, of the observed matrix $Y$, we see a dimension reduction has been achieved, since $p_0 = o(2^M + N)$, due to Assumption 1.

We also make the following assumption on the error matrix $E$.

**Assumption 2.** Consider the model (4). Assume that $E$ is a standard Gaussian ensemble, i.e. with IID standard normal entries. And also assume that $E$ is independent with $\hat{A}$ and $\hat{B}$ if the random scheme (6) is under consideration.

We first present the convergence rates of the estimators $\hat{\lambda}, \hat{A}, \hat{B}$, given the estimation procedure in Section 2.3 under the true configuration. Since the error matrix $E$ has IID standard normal entries, according to Vershynin (2010), the expectation of the largest singular value of the rearranged error matrix $R_{m_0,n_0}[E]$ is bounded by

$$s_0 = 2^{(m_0 + n_0)/2} + 2^{(M + N - m_0 - n_0)/2}.$$ 

**Theorem 1.** Let $\hat{\lambda}, \hat{A}, \hat{B}$ be the estimators obtained under the true configuration, as given in (9). Suppose Assumption 2 holds, we have for the deterministic scheme of model (4),

$$\frac{\hat{\lambda} - \lambda}{\lambda} = O_p(s_0/\lambda),$$

$$\|\hat{A} - A\|_F^2 = O_p(s_0/\lambda),$$

$$\|\hat{B} - B\|_F^2 = O_p(s_0/\lambda).$$

As an immediate consequence, we see that $\hat{A}$ and $\hat{B}$ are consistent as long as $s_0 = o(\lambda)$. Although Theorem 1 considers only the deterministic scheme, together with a conditioning argument it would also cover the random scheme (6), giving the following corollary.

**Corollary 1.** Assume the same conditions of Theorem 1. If $A$ and $B$ are generated according to the random scheme (6), then

$$\|\hat{A} - A\|_F^2 \xrightarrow{p} 0, \quad \|\hat{B} - B\|_F^2 \xrightarrow{p} 0.$$ 

### 4.1 Random Scheme

If a wrong configuration $(m, n) \in W$ is used for the estimation, $Y$ is rearranged as

$$R_{m,n}[Y] = \lambda R_{m,n}[A \otimes B] + \sigma R_{m,n}[E].$$

The second term on the right hand side of (14) is a $2^{n+m} \times 2^{M+N-m-n}$ standard Gaussian ensemble scaled by $\sigma$. The first term $R_{m,n}[A \otimes B]$ in general has a rank greater than one because of the wrong configuration. Lemma 1 characterizes the spectral norm of the first term as the product of spectral norms of rearranged $A$
and $B$. It is a property of matrices and Kronecker products, so we present it in the general form, without referring to any “true” configuration.

**Lemma 1.** Let $A$ be a $2^m \times 2^n$ matrix and $B$ be a $2^{M-m} \times 2^{N-n}$ matrix. Then for any $m', n' \in \mathbb{Z}$, $0 \leq m' \leq M$, $0 \leq n' \leq N$,

$$||R_{m',n'}[A \otimes B]||_S = ||R_{m\wedge m',n\wedge n'}[A]||_S \cdot ||R_{(m'-m)+,(n'-n)+}[B]||_S$$

(15)

Lemma 1 is critical in proving the main consistency result, so we briefly outline its proof here, while deferring the rigorous one to the Appendix.

The Kronecker product decomposition (1) of $A$ with respect to the configuration $(m \wedge m', n \wedge n', (m - m')_+, (n - n')_+)$ is given by

$$A = \sum_{i=1}^{I} \mu_i C_i \otimes D_i,$$

where $I = 2^{m \wedge m' + n \wedge n'} \wedge 2^{(m-m')_+ + (n-n')_+}$, and $\mu_1 \geq \mu_2 \cdots \geq \mu_I$ are in decreasing order. Similarly, the KPD of $B$ with respect to the configuration $((m' - m)_+, (n' - n)_+, M - m \lor m', N - n \lor n')$ is

$$B = \sum_{j=1}^{J} \nu_j F_j \otimes G_j,$$

where $J = 2^{(m'-m)_+ + (n'-n)_+} \wedge 2^{M + N - m \lor m' - n \lor n'}$ and $\nu_1 \geq \cdots \geq \nu_J$. It follows that

$$A \otimes B = \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_i \nu_j C_i \otimes D_i \otimes F_j \otimes G_j = \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_i \nu_j C_i \otimes F_j \otimes D_i \otimes G_j.$$  

(16)

The last step is critical, where the order of the Kronecker product $D_i \otimes F_j$ is exchanged. Although the Kronecker product does not commutes in general, it does if one of the two components is a scalar, or both are vectors, which is true here, since $(m' - m)_+ (m - m')_+ = (n' - n)_+ (n - n')_+ = 0$. It can then be verified that the equation (16) gives the KPD of $A \otimes B$ with respect to the configuration $(m', n')$, where the leading coefficient is $\mu_1 \nu_1$, same as (15) in Lemma 1.

Applying Lemma 1 to (14) when a wrong configuration is used in the estimation, we see that only the first KPD component of $\lambda A \otimes B$, as shown in (16), can possibly be extracted by the estimation procedure, leading to a serious under-fit. In particular, for the random scheme (6), we have the following bound for the singular value in (15).

**Corollary 2.** Consider the model (4) with random scheme (6), where $A$, $B$ and $E$ are independent, each consisting of IID standard normal entries. Then under Assumption 1, for any $0 \leq m \leq M$ and $0 \leq n \leq N$,

$$||R_{m,n}[\hat{A} \otimes \hat{B}]||_S = 2^{m \wedge n} + 2^{M + N - m \lor m} + 2^{(m-m)_+ + (n-n)_+} + 2^{M + N - |m-m| - |n-n|} + O_p(\tau),$$

(17)

where

$$\tau = 2^{m \wedge n + m \wedge n} + 2^{M + N - m \lor n \lor n} + 2^{(m-m)_+ + (n-n)_+} + 2^{(m-m)_+ + (n-n)_+}.$$
set $\mathcal{W}$ satisfies
$$\max_{(m,n)\in \mathcal{W}} \| \mathcal{R}_{m,n}[\mathbf{A} \otimes \mathbf{B}] \|_S = \left( \frac{1}{\sqrt{2}} + o_p(1) \right),$$
which is attained when $m+n=1$ or $m+n=M+N-1$ or $|m-m_0|+|n-n_0|=1$. On the other hand, it holds that under the true configuration $\| \mathcal{R}_{m_0,n_0}[\mathbf{A} \otimes \mathbf{B}] \|_S = \| \mathbf{A} \|_F \| \mathbf{B} \|_F = 1$. Corollary 2 thus shows that any wrong configuration in $\mathcal{W}$ can possibly recover at most of $1/\sqrt{2}$ part of $\mathbf{A} \otimes \mathbf{B}$, as opposed to 1 with the correct configuration.

Although there is a good separation between the true and wrong configurations at the signal part, the estimator $\hat{\lambda}$ is also affected by the error matrix $\mathbf{E}$. Most severely when $m+n=1$ or $m+n=M+N-1$, the spectral norm of the error matrix is $\| \mathcal{R}_{m,n}[\mathbf{E}] \|_S = 2 \frac{m+n}{2} \left( \frac{1}{\sqrt{2}} + o_p(1) \right)$, which is of the same order as the signal part $\lambda \| \mathcal{R}_{m,n}[\mathbf{A} \otimes \mathbf{B}] \|_S$. Under these close-to-corner configurations, over-fitting may occur. However, they also involve a much larger number of parameters in the Kronecker product than that of the true configuration, and therefore lead to a large penalty term in the information criterion. The following theorem gives sufficient conditions for the information criterion (11) to separate the true and wrong configurations.

**Theorem 2.** Consider the model (4) with random scheme (6), where $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are independent, each consisting of IID standard normal entries. If Assumption 1 and 2 hold, and if

$$q > 2 \ln 2 - 2 \ln \left( 1 + \frac{\lambda_0^2}{\sigma^2} \right) \quad \text{and} \quad q = o \left( \frac{2^{M+N}}{2^{m_0+n_0} + 2^{M+N-m_0-n_0}} \right), \quad (18)$$

then there exists a constant $\alpha > 0$ such that

$$\min_{(m,n)\in \mathcal{W}} \text{IC}_q(m,n) - \text{IC}_q(m_0,n_0) \geq 2^{M+N}(\alpha + o_p(1)),$$

where

$$\alpha = \left[ \ln \left( 1 + \frac{\lambda_0^2}{2\sigma^2} \right) \right] \wedge \left[ \ln \left( \frac{\sigma^2 + \lambda_0^2}{2\sigma^2} \right) + \frac{q}{2} \right].$$

The second condition in (18) guarantees that the penalty term is of a smaller order, compared with likelihood term in the information criterion (11), under the true configuration. The first part of (18) can be equivalently written as the positivity of $\ln[(\sigma^2 + \lambda_0^2)/(2\sigma^2)] + q/2$, which appears in the definition of $\alpha$.

Theorem 2 shows that the information criterion gap between the true and wrong configuration is of the order $2^{M+N}$, scaled by the positive constant $\alpha$. We note that if the signal-to-noise ratio $\lambda_0^2/\sigma^2$, as defined in (7), is larger than 1, then the condition (18) is fulfilled as long as $q \geq 0$. In particular, when $q = 0$, the information criterion (11) is equivalent as using the mean squared error. On the other hand, if AIC (when $q = 2$) or BIC (when $q = (M+N)\ln 2$) is used, the condition $\lambda_0 > 0$ suffices. These observations are summarized in the following corollary.

**Corollary 3.** Consider the model (4) with random scheme (6), where $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are independent, each consisting of IID standard normal entries. Suppose the Assumption 1, 2, and the second half of (18) hold.

(i) If $q = 0$ and $\lambda_0^2 > \sigma^2$, then Theorem 2 holds with

$$\alpha = \ln \left( \frac{1}{2} + \frac{\lambda_0^2}{2\sigma^2} \right).$$
(ii) If $q = 2$ or $q = (M + N) \ln 2$, and if $\lambda_0 > 0$, then Theorem 2 holds with

$$\alpha = \ln \left(1 + \frac{\lambda_0^2}{2\sigma^2}\right).$$

For other values of $q > 0$, we note that if $q > 2 \ln 2$, there does not need to be any condition on the signal-to-noise ratio, and $\lambda_0 > 0$ suffices for (18). On the other hand, a larger signal-to-noise ratio would always increase the gap size $\alpha$.

With the separation of the information criterion provided in Theorem 2 we have the following consistency result regarding the configuration selection procedure.

**Theorem 3.** Assume the same conditions of Theorem 2, then

$$P\left[\text{IC}_q(m_0, n_0) < \min_{(m,n)\in W} \text{IC}_q(m, n)\right] \to 1.$$

As the matrix size goes to infinity as described in Assumption 1, the probability that the true configuration is selected by minimizing the information criterion converges to 1. We also remark that the larger the gap $\alpha$ is, the higher the probability that the true configuration is selected. Although this fact is not directly reflected in the statement of Theorem 3, it can be seen from the the proof in the Appendix.

### 4.2 Deterministic Scheme

Theorem 2 and Theorem 3 give the consistency of configuration selection through information criterion when $A$ and $B$ are generated under the random scheme. Two results directly follow the randomness of $A$ and $B$: (i) Corollary 2, and (ii) $\lambda \approx \lambda_0 2^{(M+N)/2}$. When $A$ and $B$ are deterministic in the model, these two results do not hold automatically, and have to be made as assumptions. Following Lemma 1, define $\epsilon$ by

$$\epsilon = \max_{(m,n)\in W} \|R_{m\wedge m_0, n\wedge n_0}[A]\|_S \cdot \|R_{(m-m_0)_+, (n-n_0)_+}[B]\|_S.$$

Since $\|A\|_F = \|B\|_F = 1$, it always holds that $\epsilon \leq 1$. The quantity $\epsilon$ measures how well the best Kronecker product with wrong configuration can approximate $A \otimes B$. If $\epsilon = 1$, then $A \otimes B$ can be written as a Kronecker product of a different configuration. According to Corollary 2, the chance that $\epsilon$ is close to one is small when $A$ and $B$ are random (in fact it roughly equals $1/\sqrt{2}$). Therefore, it is natural and reasonable to assume that $\epsilon$ is strictly less than 1 in the deterministic case, as detailed in Assumption 3. Note that $\epsilon$ defined earlier depend implicitly on the size of the matrices $A$ and $B$.

**Assumption 3.** There exists a constant $0 < \epsilon_0 < 1$ such that

$$\limsup_{m_0+n_0, M+N-m_0-n_0 \to \infty} \epsilon \leq \epsilon_0.$$

**Assumption 4.** The signal strength $\lambda$ in model (4) satisfies

$$\liminf_{m_0+n_0, M+N-m_0-n_0 \to \infty} \frac{\lambda^2}{2M+N} > 0.$$

We have the following parallel results of Theorem 2 and Theorem 3 for the deterministic scheme.
Theorem 4. Consider the model (4), where $A$ and $B$ are deterministic. Under Assumptions 1, 2, 3 and 4 if
\[ q > -2 \ln \left( \frac{1}{2} + \frac{\lambda^2}{\sigma^2 2^{M+N}} (1-\epsilon_0^2) \right) \quad \text{and} \quad q = o \left( \frac{2^{M+N}}{2^{m_0+n_0} + 2^{M+N-m_0-n_0}} \right), \] (19)
then Theorem 3 holds with the gap size
\[ \alpha = \left[ \ln \left( 1 + \frac{\lambda^2}{\sigma^2 2^{M+N}} (1-\epsilon_0^2) \right) \right] \wedge \left[ \frac{q}{2} + \ln \left( \frac{1}{2} + \frac{\lambda^2}{\sigma^2 2^{M+N}} (1-\epsilon_0^2) \right) \right]. \] (20)
Furthermore, Theorem 3 holds as well.

Assumption 3 and 4 implies that the gap size $\alpha$ satisfies $\lim \inf \alpha > 0$, so that the separation given by the information criterion (11) is still of the order $O(2^{M+N})$, same as the random scheme. On the other hand, if $A$ and $B$ are randomly generated, we have $\lambda^2 2^{-(M+N)} \approx \lambda_0^2$ and $1 - \epsilon^2 \approx 1/2$, and Theorem 4 turns out to be same as those under the random scheme.

The following corollary, regarding special cases of the information criterion (11), is the analogous version of Corollary 3, but for the deterministic scheme.

Corollary 4. Consider the model (4), where $A$ and $B$ are deterministic. Suppose Assumptions 1, 2, 3 and 4, and the second half of (19) hold.

(i) If $q = 0$ and
\[ \lim \inf \frac{\lambda^2}{\sigma^2 2^{M+N}} > \frac{1}{2 (1-\epsilon_0^2)}, \] (21)
then Theorem 4 holds.

(ii) If $q = 2$ or $q = (M+N) \ln 2$, then Theorem 4 holds.

When $q = 0$, the information criterion (11) is equivalent to the mean squared error. For this to consistently select the true configuration, a stronger signal-to-noise ratio is needed, as given in (21). In the condition (21), the signal-to-noise ratio is coupled with the quantity $\epsilon_0$ in Assumption 3: the smaller $\epsilon_0$ is, the weaker the signal-to-noise need to be. For AIC (when $q = 2$) and BIC (when $q = (M+N) \ln 2$), a minimal condition on the signal-to-noise ratio, as given in Assumption 4, suffices to make Theorem 4 valid.

5 Two-term Kronecker Product Models

In this section, we extend the one-term Kronecker product model in (4) to the following two term Kronecker product model.
\[ Y = \lambda_1 A_1 \otimes B_1 + \lambda_2 A_2 \otimes B_2 + \sigma E, \] (22)
where $\lambda_1 \geq \lambda_2 > 0$ and $A_i \in \mathbb{R}^{2^{m_0} \times 2^{n_0}}$, $B_i \in \mathbb{R}^{2^{M-m_0} \times 2^{N-n_0}}$, $i = 1, 2$ satisfy the following orthonormal condition
\[ \text{tr}(A_i A_j^\prime) = \text{tr}(B_i B_j^\prime) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \]
The orthonormal condition implies the identifiability: if $\lambda_1 > \lambda_2 > 0$, then $A_i$ and $B_i$ are identified up to a sign change, see Section 2.1. Note that the two terms in model (22) have the same Kronecker product configuration $(m_0, n_0)$. Therefore, although two terms are present, there is only one configuration to be determined.

It is very natural to consider an extension of the model (4) to the general multi-term case, where the number of terms, called the order of the model, is possibly also unknown. However, a thorough analysis regarding the estimation, configuration selection, and order selection is beyond the scope of this paper. In this section we only study the two-term model (22), and propose to select the configuration by the same procedure introduced in Section 3, and establish its consistency. We remark that similar results can be directly extended to the multi-term model, if the order is known. But we will limit the discussion to the two-term model for ease of presentation.

For the two-term model (22), the SVD solution after rearrangement given in Section 2.3 continues to work: one only needs to take the first two terms from the SVD of $R[Y]$. Therefore, we will focus on the problem of configuration selection in this section.

We propose to use the same configuration selection procedure in Section 3, that is, for any candidate configuration $(m, n) \in \mathcal{C}$, although $Y$ is generated from the two-term model (22), we nonetheless still calculate the information criterion (11) by fitting the one-term Kronecker product model (4) to $Y$. In this case, the estimated $\hat{\lambda}$ used in the information criterion (13) is

$$\hat{\lambda} = \| R_{m,n}[Y] \|_S = \| \lambda_1 R_{m,n} [A_1 \otimes B_1] + \lambda_2 R_{m,n} [A_2 \otimes B_2] + \sigma R_{m,n}[E] \|_S.$$  \hspace{1cm} (23)

Under the true configuration $\hat{\lambda} \approx \lambda_1$. To bound $\hat{\lambda}$ under wrong configurations, we define

$$\hat{\epsilon}_1 = \max_{(m,n) \in \mathcal{W}} \| R_{m,n}^{m_0, n_0} [A_1] \|_S \cdot \| R_{m-n_0, n}^{m_0, n} [B_1] \|_S,$$

$$\hat{\epsilon}_2 = \max_{(m,n) \in \mathcal{W}} \| R_{m,n}^{m_0, n_0} [A_2] \|_S \cdot \| R_{m-n_0, n}^{m_0, n} [B_2] \|_S,$$

and similar to Assumption 3 we assume $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are bounded from 1.

**Assumption 5.** There exist $0 < \epsilon_1, \epsilon_2 < 1$ such that

$$\limsup_{m_0+n_0 \to \infty, M+N-m_0-n_0 \to \infty} \hat{\epsilon}_1 \leq \epsilon_1,$$

$$\limsup_{m_0+n_0 \to \infty, M+N-m_0-n_0 \to \infty} \hat{\epsilon}_2 \leq \epsilon_2.$$

Even though $\text{vec}(A_1)$ and $\text{vec}(A_2)$ are orthogonal according to the model assumption, the column spaces of $R_{m,n}[A_1 \otimes B_1]$ and $R_{m,n}[A_2 \otimes B_2]$ are not necessarily orthogonal. In the worst case when $R_{m,n}[A_1 \otimes B_1]$ and $R_{m,n}[A_2 \otimes B_2]$ have the same column space and the same row space, then $\hat{\lambda}$ in (23) is close to $\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2$, which may exceed $\lambda_1$. Therefore, we need to bound the distance between the column (and row) spaces of $R_{m,n}[A_1 \otimes B_1]$ and $R_{m,n}[A_2 \otimes B_2]$. For this purpose, we make use of the principal angles between linear subspaces. Specifically, if $M_1$ and $M_2$ are two matrices of the same number of rows, the smallest principal angle between their column spaces, denote by $\Theta(M_1, M_2)$, is defined as

$$\cos \Theta(M_1, M_2) = \sup_{u_1 \neq 0, u_2 \neq 0} \frac{u_1^T M_1^T M_2 u_2}{\| M_1 u_1 \| \| M_2 u_2 \|}.$$
We first discuss the deterministic scheme, where $A_i$ and $B_i$ are non-random. In Assumption 6, $\theta_c$ and $\theta_r$ are lower bounds of the smallest possible principal angles between the column spaces and the row spaces of the two rearranged components, respectively.

**Assumption 6.** There exist $\theta_c, \theta_r \in (0, \pi/2]$ such that

$$\liminf_{m_0+n_0 \to \infty, M+N-m_0-n_0 \to \infty} \min_{(m,n) \in W} \Theta(R_{m,n}[A_1 \otimes B_1], R_{m,n}[A_2 \otimes B_2]) \geq \theta_c,$$

and

$$\liminf_{m_0+n_0 \to \infty, M+N-m_0-n_0 \to \infty} \min_{(m,n) \in W} \Theta([R_{m,n}[A_1 \otimes B_1]]', [R_{m,n}[A_2 \otimes B_2]]') \geq \theta_r.$$

The following upper bound on the spectral norm of a sum involves both the individual spectral norms and the principal angles between the column and row spaces.

**Lemma 2.** Suppose $M_1$ and $M_2$ are two matrices of the same dimension. Let $\|M_1\|_S = \mu$, $\|M_2\|_S = \nu$. Denote the principle angles between the column spaces and the row spaces as $\theta = \Theta(M_1, M_2)$, $\eta = \Theta(M_1', M_2')$, respectively. Then

$$\|M_1 + M_2\|_S^2 \leq \Lambda^2(\mu, \nu, \theta, \eta),$$

where

$$\Lambda^2(\mu, \nu, \theta, \eta) = \frac{1}{2} \left( \mu^2 + \nu^2 + 2\mu\nu \cos \theta \cos \eta + \sqrt{(\mu^2 + \nu^2 + 2\mu\nu \cos \theta \cos \eta)^2 - 4\mu^2\nu^2 \sin^2 \theta \sin^2 \eta} \right).$$

Similar to Assumption 4, we assume the signal strength of the leading term $\lambda_1^2$ is at least of order $2^{M+N}$.

**Assumption 7.** The signal strength $\lambda_1$ in model (22) satisfies

$$\liminf_{m_0+n_0, M+N-m_0-n_0 \to \infty} \frac{\lambda_1^2}{2^{M+N}} > 0.$$

Similar as Theorem 4 we show that for the two-term model, the information criterion obtained by fitting a one-term model can still separate the true and wrong configurations with a gap of the order $2^{M+N}$.

Comparing with Theorem 4 an additional condition (24) is required to guarantee that under the wrong configuration the estimated $\hat{\lambda}$ in (23) does not exceed $\lambda_1$. The first half of (25) is also stronger than that of (19), which makes sure that the penalty term in the information criterion is strong enough to separate the corner configurations, which can potentially over-fit and lead to a large likelihood.

**Theorem 5.** Consider the two-term model (22), where the matrices $A_i$ and $B_i$ are deterministic. Suppose the Assumptions 1, 2, 5, 6 and 7 hold. If $\lambda_1, \lambda_2$ satisfy

$$\lambda_1^2 > \Lambda^2(\lambda_1 \epsilon_1, \lambda_2 \epsilon_2, \theta_c, \theta_r),$$  \hfill (24)  

and $q$ satisfies

$$q > -2 \ln \left( \frac{1}{2} + \frac{\lambda_1^2 + \frac{1}{2} \lambda_2^2 - \Lambda^2(\lambda_1 \epsilon_1, \lambda_2 \epsilon_2, \theta_c, \theta_r)}{\lambda_2^2 + \sigma^2 2^{M+N}} \right) \quad \text{and} \quad q = o \left( \frac{2^{M+N}}{2m_0+n_0 + 2^{M+N-m_0-n_0}} \right),$$  \hfill (25)
then there exists a constant $\alpha > 0$ such that

$$\min_{(m,n)\in W} \text{IC}_q(m,n) - \text{IC}_q(m_0,n_0) \geq 2^{M+N}(\alpha + o_p(1)).$$

Furthermore, Theorem 3 also holds.

There is also a random scheme of the two term model (22) that is similar to (6). For the random scheme we assume

$$\lambda_i A_i \otimes B_i = \lambda_{i0} \tilde{A}_i \otimes \tilde{B}_i, \quad i = 1, 2,$$ (26)

where all of the five matrices $\tilde{A}_i$ and $\tilde{B}_i$ and $E$ are independent, and each consisting of IID standard normal entries. To translate back into the form of (22), we let $A_i = \tilde{A}_i/\|\tilde{A}_i\|_F$, $B_i = \tilde{B}_i/\|\tilde{B}_i\|_F$, and $\lambda_i = \lambda_{i0} \cdot \|\tilde{A}_i\|_F \cdot \|\tilde{A}_i\|_F$.

For the random scheme, under the Assumption (1), the two column spaces in Assumption (6) are asymptotically orthogonal, and so are the two row spaces. Therefore, the stochastic version of the condition (24) is automatically fulfilled, and we have the following corollary regarding the random scheme.

**Corollary 5.** Consider the two-term model (22), where the matrices $A_i$ and $B_i$ are generated as the random scheme (26). Suppose the Assumptions 1, 2, 5 and 7 hold. If $q$ satisfies (25), then Theorem 7 holds.

**Remark 3.** The conditions (24) and (25) are complicated. So we provide some sufficient conditions for them, and leave the proof to the Appendix. First, if

$$\frac{\lambda_1}{\lambda_2} > \epsilon_1 \epsilon_2 \cos \theta_c \cos \theta_r + \sqrt{\epsilon_1^2 \epsilon_2 \cos^2 \theta_c \cos^2 \theta_r + (1 - \epsilon_1^2) \epsilon_2^2},$$

then condition (24) is satisfied. Second, if $q > 2 \ln 2$, then the first part of (25) is satisfied.

### 6 Examples

We illustrate the performance of the estimation and configuration selection procedure through simulation studies in Section 6.1, and an image example in Section 6.2.

#### 6.1 Simulations

We design two simulation studies: the first one on the performance of estimation procedure introduced in Section 2.3, and the second one on the configuration selection proposed in Section 3. It is worthwhile to point out that some implications of the theoretical results in Section 4 are also reflected in the outcome of the numerical studies.

**6.1.1 Estimation with known configuration**

We first consider the performance of the estimators of $\lambda$, $A$ and $B$ given in (9), when the true configuration $(m_0, n_0)$ is known. Throughout this subsection the simulations are based on model (4) with $m_0 = 5$, $n_0 = 5$, $M = 10$, $N = 10$ and $\sigma = 1$. 

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The model (4) after the rearrangement under the true configuration becomes

\[ \mathcal{R}_{m_0,n_0}[Y] = \lambda \text{vec}(A)\text{vec}(B)' + \sigma \mathcal{R}_{m_0,n_0}[E], \]

where \( \text{vec}(A) \in \mathbb{R}^{2m_0+n_0}, \text{vec}(B) \in \mathbb{R}^{2M+N-m_0-n_0} \) are unit vectors. Without loss of generality, set \( \text{vec}(A) = (1, 0, \ldots, 0)' \), \( \text{vec}(B) = (1, 0, \ldots, 0)' \). In this experiment, the noise level is fixed at \( \sigma = 1 \), so the signal-to-noise ratio is controlled by \( \lambda \), which takes values from the set \( \{e_1, e_2, \ldots, e_{16}\} \). For each value of \( \lambda \), we calculate the errors of the corresponding estimators \( \hat{\lambda}, \hat{A} \) and \( \hat{B} \) by

\[ \ln \left( \frac{\hat{\lambda}}{\lambda} - 1 \right)^2 \text{ and } \ln \|\hat{A} - A\|_F^2 + \ln \|\hat{B} - B\|_F^2. \]

The errors based on 20 repetitions are reported in Figure 2.

![Boxplots for errors in \( \hat{\lambda}, \hat{A} \) and \( \hat{B} \) against the signal-to-noise ratio.](image)

Figure 2 displays an interesting linear pattern, that is, as the signal-to-noise ratio increases, \( \ln \left( \frac{\hat{\lambda}}{\lambda} - 1 \right)^2 \) is approximately linear against \( \ln \lambda \) with a slope around \(-2\), and so is the error \( \ln(\|\hat{A} - A\|_F^2, \|\hat{B} - B\|_F^2) \) for the matrix estimators. We note that this pattern is consistent with Theorem 1 which asserts that

\[ \frac{\hat{\lambda}}{\lambda} - 1 = O \left( \frac{1}{\lambda} \right) \text{ and } \|\hat{A} - A\|_F \|\hat{B} - B\|_F = O \left( \frac{1}{\lambda} \right), \]

since \( s_0 \) remains a constant as we vary the signal strength \( \lambda \) in the simulation.

### 6.1.2 Configuration Selection

We now demonstrate the performance of the information criterion based procedure for selecting the configuration. Three criteria will be considered: MSE (when \( q = 0 \)), AIC (when \( q = 2 \)) and BIC (when \( q = (M + N) \ln 2 \)). Since the consistency is established for both the random and deterministic scheme in Section 4, we carry out two experiments correspondingly.

**Experiment 1: Random scheme**

The simulation is based on model (4) with \( \lambda, A, B \) being generated according to the random scheme (6).
for some chosen $\lambda_0 > 0$. Two configurations are considered: (i) $M = N = 10$, $m_0 = 6$, $n_0 = 4$, and (ii) $M = N = 9$, $m_0 = 5$, $n_0 = 4$. Similar as Section 6.1.1, the noise level is fixed at $\sigma = 1$, so the signal-to-noise ratio is controlled by $\lambda_0$. For each value of $\lambda_0$, we report the empirical frequencies of the correct configuration selection made by each criterion in Table 1 out of 100 repetitions.

(a) $M = N = 9$, $(m_0, n_0) = (5, 4)$

| $\lambda_0$ | MSE | AIC | BIC | $\lambda_0$ | MSE | AIC | BIC |
|-------------|-----|-----|-----|-------------|-----|-----|-----|
| 0.01        | 0   | 10  | 10  | 0.01        | 0   | 7   | 7   |
| 0.02        | 0   | 10  | 10  | 0.02        | 0   | 7   | 7   |
| 0.03        | 0   | 11  | 11  | 0.03        | 0   | 14  | 14  |
| 0.04        | 0   | 17  | 17  | 0.04        | 0   | 99  | 99  |
| 0.05        | 0   | 61  | 61  | 0.05        | 0   | 100 | 100 |
| 0.06        | 0   | 97  | 97  | 0.06        | 0   | 100 | 100 |
| 0.10        | 0   | 100 | 100 | 0.10        | 16  | 100 | 100 |
| 0.90        | 0   | 100 | 100 | 0.90        | 0   | 100 | 100 |
| 0.95        | 1   | 100 | 100 | 0.95        | 0   | 100 | 100 |
| 1.00        | 5   | 100 | 100 | 1.00        | 57  | 100 | 100 |
| 1.05        | 39  | 100 | 100 | 1.05        | 95  | 100 | 100 |
| 1.10        | 67  | 100 | 100 | 1.10        | 99  | 100 | 100 |
| 1.15        | 93  | 100 | 100 | 1.15        | 99  | 100 | 100 |
| 1.20        | 99  | 100 | 100 | 1.20        | 100 | 100 | 100 |
| 1.50        | 100 | 100 | 100 | 1.50        | 100 | 100 | 100 |

(b) $M = N = 10$, $(m, n) = (6, 4)$

| $\lambda_0$ | MSE | AIC | BIC | $\lambda_0$ | MSE | AIC | BIC |
|-------------|-----|-----|-----|-------------|-----|-----|-----|
| 0.01        | 0   | 7   | 7   | 0.01        | 0   | 7   | 7   |
| 0.02        | 0   | 7   | 7   | 0.02        | 0   | 7   | 7   |
| 0.03        | 0   | 14  | 14  | 0.03        | 0   | 100 | 100 |
| 0.04        | 0   | 99  | 99  | 0.04        | 0   | 100 | 100 |
| 0.05        | 0   | 100 | 100 | 0.05        | 0   | 100 | 100 |
| 0.06        | 0   | 100 | 100 | 0.06        | 0   | 100 | 100 |
| 0.10        | 16  | 100 | 100 | 0.10        | 10  | 100 | 100 |
| 0.90        | 0   | 100 | 100 | 0.90        | 0   | 100 | 100 |
| 0.95        | 0   | 100 | 100 | 0.95        | 0   | 100 | 100 |
| 1.00        | 10  | 100 | 100 | 1.00        | 16  | 100 | 100 |
| 1.05        | 57  | 100 | 100 | 1.05        | 95  | 100 | 100 |
| 1.10        | 95  | 100 | 100 | 1.10        | 99  | 100 | 100 |
| 1.15        | 99  | 100 | 100 | 1.15        | 99  | 100 | 100 |
| 1.20        | 100 | 100 | 100 | 1.20        | 100 | 100 | 100 |
| 1.50        | 100 | 100 | 100 | 1.50        | 100 | 100 | 100 |

Table 1: The empirical frequencies that the true configuration is selected by the information criteria (MSE, AIC and BIC), out of 100 repetitions.

For extremely weak signal-to-noise ratio $\lambda_0 \leq 0.03$, none of the three criteria is able to select the true configuration with a high chance, for both configurations. We remark this does not contradict with Theorem 3. When the signal is very weak, larger dimensions of the observed matrix $Y$ are required for the consistency. As the signal-to-noise ratio increases from 0.01 to 0.06, the probability that the true configuration is selected increases and eventually gets very close to one for AIC and BIC. This is echoing Corollary 3 (ii), which shows that AIC and BIC only requires a minimal condition $\lambda_0 > 0$ to achieve the consistency. On the other hand, MSE only starts to select the true configuration with a decent probability when the signal-to-noise ratio $\lambda_0 > 1$, which is again in line with the condition $\lambda_0^2 > \sigma^2$ in Corollary 3 (ii).

Comparing the panel (a) and panel (b) of Table 1, we also observe that when the dimension of $Y$ is larger (i.e. panel (b)), it is more likely to select the true configuration, no matter which criterion is under used. On the other hand, when $M = N = 10$ as shown in panel (b), MSE requires a signal-to-noise ratio around 1.15 to make a nearly perfect configuration selection, comparing with the signal-to-noise ratio 1.20 for panel (a), corresponding to a smaller dimension $M = N = 9$. 18
Experiment 2: Deterministic scheme

In the second experiment, we consider the deterministic scheme where \( A \) and \( B \) are non-random. In particular, we control the value of \( \epsilon_0 \) used in Assumption 4, which is critical for Theorem 4. Note that under the random scheme demonstrated in Experiment 1, \( \epsilon_0^2 \) is approximately 0.5. According to Corollary 4, larger \( \epsilon_0^2 \) implies that a larger signal-to-noise ratio is required to achieve the consistency of configuration selection. In this experiment, we consider two choices \( \epsilon_0^2 = 0.7 \) and \( \epsilon_0^2 = 0.9 \), both larger than 0.5. Same configurations as in Experiment 1 are considered: (i) \( M = N = 10, m_0 = 6, n_0 = 4 \), and (ii) \( M = N = 9, m_0 = 5, n_0 = 4 \).

To generate the matrix \( Y \) in (4), we take \( \lambda = \lambda_0 2^{(M+N)/2} \), generate \( B \) from a normalized standard Gaussian ensemble so that \( \|B\|_F = 1 \), and generate \( A \) as

\[
A = \epsilon_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes D_1 + \sqrt{1 - \epsilon_0^2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes D_2,
\]

where \( D_1, D_2 \in \mathbb{R}^{2^{m-1} \times 2^n} \), and \( \text{vec}(D_1), \text{vec}(D_2) \) are random unit vectors orthogonal to each other. It can be verified that such pair of \( A \) and \( B \) does satisfy Assumption 4 with the prescribed \( \epsilon_0 \). Once \( \lambda, A, B \) are generated, they are held as constants, and 100 repetitions of \( Y \) are generated according to (4) with \( \sigma = 1 \). Various choices of the signal-to-noise ratio \( \lambda_0 \) (since the noise level \( \sigma \) is fixed at 1) are considered. In particular, we note that Corollary 4 requires the condition (21) on the signal-to-noise ratio, for the consistency of the MSE. Under the setup of the simulations in this experiment, this condition translates as whether \( \lambda_0 \) is larger than \( \sqrt{1/(2(1 - \epsilon_0^2))} \), which equals 1.29 for \( \epsilon_0^2 = 0.7 \) and 2.24 for \( \epsilon_0^2 = 0.9 \). We report the empirical frequencies of the correct selection, out of 100 repetitions, in Tables 2 and 3 for the two cases \( \epsilon_0^2 = 0.7 \) and \( \epsilon_0^2 = 0.9 \) respectively.

| \( \lambda_0/\sigma \) | MSE | AIC | BIC | \( \lambda_0/\sigma \) | MSE | AIC | BIC |
|-----------------------|-----|-----|-----|-----------------------|-----|-----|-----|
| 0.03                  | 0   | 9   | 9   | 0.03                  | 0   | 13  | 13  |
| 0.10                  | 0   | 100 | 100 | 0.10                  | 0   | 100 | 100 |
| 1.00                  | 0   | 100 | 100 | 1.00                  | 0   | 100 | 100 |
| 1.27                  | 0   | 100 | 100 | 1.27                  | 0   | 100 | 100 |
| 1.28                  | 6   | 100 | 100 | 1.28                  | 0   | 100 | 100 |
| 1.29                  | 80  | 100 | 100 | 1.29                  | 77  | 100 | 100 |
| 1.30                  | 100 | 100 | 100 | 1.30                  | 100 | 100 | 100 |
| 1.50                  | 100 | 100 | 100 | 1.50                  | 100 | 100 | 100 |
| 2.00                  | 100 | 100 | 100 | 2.00                  | 100 | 100 | 100 |

Table 2: The empirical frequencies that the true configuration is selected, out of 100 repetitions, when \( \epsilon_0^2 = 0.7 \).

For both values of \( \epsilon_0^2 \), AIC and BIC have similar performances as in Experiment 1. They both select the true configuration with a probability close to 1 as long as the signal-to-noise ratio is not too small. The
(a) $M = N = 9$, $(m_0, n_0) = (5, 4)$

(b) $M = N = 10$, $(m_0, n_0) = (6, 4)$

| $\lambda_0/\sigma$ | MSE | AIC | BIC | $\lambda_0/\sigma$ | MSE | AIC | BIC |
|-------------------|-----|-----|-----|-------------------|-----|-----|-----|
| 0.03              | 0   | 14  | 14  | 0.03              | 0   | 14  | 14  |
| 0.10              | 0   | 100 | 100 | 0.10              | 0   | 100 | 100 |
| 2.20              | 0   | 100 | 100 | 2.20              | 0   | 100 | 100 |
| 2.21              | 2   | 100 | 100 | 2.21              | 2   | 100 | 100 |
| 2.22              | 15  | 100 | 100 | 2.22              | 0   | 100 | 100 |
| 2.23              | 77  | 100 | 100 | 2.23              | 28  | 100 | 100 |
| 2.24              | 95  | 100 | 100 | 2.24              | 99  | 100 | 100 |
| 2.25              | 100 | 100 | 100 | 2.25              | 100 | 100 | 100 |
| 2.50              | 100 | 100 | 100 | 2.50              | 100 | 100 | 100 |
| 3.00              | 100 | 100 | 100 | 3.00              | 100 | 100 | 100 |

Table 3: The empirical frequencies that the true configuration is selected, out of 100 repetitions, when $\epsilon_0^2 = 0.9$.

The performance of MSE does have a phase transition at the threshold value 1.29 in Table 2 and at 2.24 in Table 3, which once again confirms the condition (21) in Corollary 4.

6.2 Lenna Image

In this section we revisit and analyze the Lenna image introduced in Section 1. The original image has $512 \times 512$ pixels and three channels (red, green and blue). For simplicity, we first convert the original image in RGB format to the grayscale according to

$$Y_0 = 0.2125R + 0.7154G + 0.0721B,$$

where $R$, $G$ and $B$ are the red, green and blue channels respectively. Each entry of $Y_0$ is a real number between 0 and 1, where 0 codes black and 1 indicates white. The grayscale Lenna image $Y_0$ is displayed in Figure 3.

Our analysis will be based on the de-meaned version $Y$ of the original image $Y_0$. We demonstrate how well the image $Y$ can be approximated by a Kronecker product or the sum of a few Kronecker products, and make comparisons with the low rank approximations given by SVD.

We first consider the configuration selection by MSE, AIC and BIC. Figure 4 plots the heat maps for the information criterion $IC_q(m, n)$ for all candidate configurations in the set

$$\mathcal{C} = \{(m, n) : 0 \leq m, n \leq 9\} \setminus \{(0, 0), (9, 9)\},$$

where the top-left and bottom-right corners are always excluded from the consideration. Since darker cells correspond to smaller values of the information criteria, we see that MSE and AIC select the configuration $(8, 9)$, and BIC selects $(6, 7)$. 

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We also observe an overall pattern in Figure 4: configurations with larger \((m, n)\) values are more preferable than those with smaller \((m, n)\). Note that the Kronecker product does not commute, and \((m, n)\) indicates that it is a \(2^m \times 2^n\) block matrix, with each block of the size \(2^{9-m} \times 2^{9-n}\). Real images usually show the locality of pixels in the sense that nearby pixels tend to have similar colors. Therefore, it can be understood that larger values of \(m\) and \(n\) are preferred, since they are better suited to capture the locality. Actually, for the Lenna image, the configuration \((8, 9)\) accounts for 99.22% of the total variation of \(Y\). The penalty on the number of parameters in AIC is not strong enough to offset the closer approximation given by the configuration \((8, 9)\). With a stronger penalty term, BIC selects a configuration that is closer to the center of the configuration space, involving a much smaller number of parameters.

From the perspective of image compressing, Kronecker product approximation offers a framework that is more flexible than the low rank approximations, because it allows a choice of the configuration, and hence a choice of the compression rate. We compare the performance of KoPA and the low rank approximations in Figure 5. In the graph, each circle corresponds to a configuration \((m, n) \in \mathcal{C}\). Its value on the horizontal axis is the number of parameters involved in the Kronecker product

\[
p = 2^{m+n} + 2^{M+N-m-n},
\]
and its value on the vertical axis indicates the percentage of the variation that is explained by the nearest Kronecker product of the configuration \((m,n)\). On the other hand, each cross stands for a rank-\(k\) approximation of \(Y\), where its value on the horizontal axis is the number of parameters

\[
p = 1 + \sum_{j=1}^{k} (2^M + 2^N - 2j + 1) \quad \text{for } k = 1, \ldots, 2^{M \wedge N},
\]

and its value on the vertical axis has the same interpretation as the circles. According to Figure 5, there always exists a one-term Kronecker product which provides a better approximation of the original Lenna image than the best low rank approximation involving the same number of parameters.

We also consider the blurred image obtained by adding random noise to the original image

\[
Y_\sigma = Y + \sigma E,
\]

where \(E\) is a matrix with IID standard normal entries. We experiment with three levels of blurring strength: \(\sigma = 0.1, 0.3, 0.5\). The blurred images with different \(\sigma\) are shown in Figure 6. For the blurred images, the information criteria \(IC_q(m,n)\) are calculated, and the corresponding heat maps are plotted in Figure 7. With added noise, AIC and BIC tend to select configurations in the middle of the configuration space.

Now we consider the multi-term Kronecker approximation. Following the discussion in Section 5, for each of three blurred images \(Y_\sigma\), we use the configuration selected by BIC in Figure 7. Specifically, configurations \((5,6), (4,6)\) and \((4,5)\) are selected when \(\sigma = 0.1, 0.3\) and \(0.5\), respectively. A two-term Kronecker product

![Figure 5: Percentage of variance explained against number of parameters, for KoPA with all configurations, and for low rank approximations of all ranks.](image)

![Figure 6: Blurred images with different \(\sigma\).](image)

![Figure 7: Heat maps of information criteria.](image)
Figure 6: Blurred Lenna images when (Left) $\sigma = 0.1$ (Mid) $\sigma = 0.3$ (Right) $\sigma = 0.5$

Figure 7: Heat maps for three different information criteria for blurred Lenna image with different noise levels.
Figure 8: The fitted image given by two-term KoPA, and the best rank-k approximation. The number of parameters involved is given on top of each image.

The model is then fitted under the selected configuration. The fitted images are plotted in the left panel of Figure 8. To compare with the low rank approximations, we plot the best rank-k approximation in the right panel, where for each $\sigma$, the rank $k$ is chosen such that the rank-k approximation involves a similar number of parameters as the two-term KoPA. From Figure 8, it is quite evident that the image reconstructed from two-term KoPA can be easily recognized as a human face, while the face is far from being conspicuous in the image given by the low rank approximation.
Finally, we look at the reconstruction error defined by
$$\|Y - \hat{Y}\|^2_F / \|Y\|^2_F.$$ 
For each of the three blurred images, we continue to use the configuration selected by BIC. However, we now keep increasing the terms in the KoPA until $Y_\sigma$ is fully fitted, and plot the corresponding reconstruction error against the number of parameters in Figure 9. It has the familiar “U” shape due to the added noise. A similar curve is given for the low rank approximations exhausting all possible ranks. According to Figure 9, the multi-term KoPA constantly outperforms the low rank approximation at any given number of parameters. Furthermore, the minimum reconstruction error that KoPA can reach is always smaller than that given by the low rank approximation.

![Figure 9: Reconstruction error against the number of parameters for KoPA and low rank approximations. The three panels from left to right correspond to $\sigma = 0.1$, $\sigma = 0.3$ and $\sigma = 0.5$ respectively.](image)

7 Conclusion and Discussions

In this article, we propose the Kronecker product approximation, as an alternative of the low rank approximation of large dimensional matrices. Comparing with the low rank approximation, KoPA is more flexible because any configuration of the Kronecker product can potentially be used, leading to different levels of approximation and compression. To select the configuration, we propose to use the extended information criterion, which includes MSE, AIC and BIC as special cases. Based on the Kronecker product model with either a random or a deterministic Kronecker product, we establish the asymptotic consistency of the configuration selection procedure. Extension to the two-term Kronecker product model is also investigated. Both the simulations and an example on image analysis demonstrate that KoPA can be superior over the low rank approximations in the sense that it can give a closer approximation of the original matrix/image with a higher compression rate.

We conclude with a discussion of future directions. First of all, as mentioned in Section 5, it is of immediate interest to consider an extension to the multi-term model with unknown number of terms. For the multi-term model, the configuration selection and order selection are coupled, and should be studied simultaneously. Another extension is to consider a multi-term model, where each term can have its own configuration. This approach certainly allows a greater flexibility, but also poses new challenges not only on the configuration and order selections, but also on the estimation and algorithms as well. It is of our
utmost interest to develop a natural and interpretable procedure for the estimation, order determination, and configuration selection, with theoretical guarantees.

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Appendix

A Proof of Theorem 1

Noticing that
\[ \hat{\lambda} = \| R_{m_0, n_0} | Y \|_S = \| \lambda \text{vec}(A) \text{vec}(B) \|_S + \sigma \| R_{m_0, n_0} | E \|_S, \]
by triangular inequality, we have
\[ \left| \hat{\lambda} - \| \lambda \text{vec}(A) \text{vec}(B) \|_S \right| \leq \| R_{m_0, n_0} | E \|_S, \]
where \( \| \lambda \text{vec}(A) \text{vec}(B) \|_S = \lambda \). The following bound for \( \| R_{m_0, n_0} | E \|_S \) can be obtained using the concentration inequality from Vershynin (2010),
\[ P(\| R_{m_0, n_0} | E \|_S \geq 2^{(m_0+n_0)/2} + 2^{(M+N-m_0-n_0)/2} + t) \leq e^{-t^2/2}. \]
Therefore, \( \| R_{m_0, n_0} | E \|_S = s_0 + O_p(1) \) and
\[ |\hat{\lambda} - \lambda| \leq s_0 + O_p(1), \]
which yields \( \hat{\lambda} - \lambda = O_p(s_0) \). Let \( u_1 = \text{vec}(\hat{A}) \in \mathbb{R}^{2m_0+n_0} \), \( v_1 = \text{vec}(\hat{B}) \in \mathbb{R}^{2M+N-m_0-n_0} \) be the leading left and right eigenvectors of \( R_{m_0, n_0} | Y \) that satisfy
\[ u_1^T R_{m_0, n_0} | Y | v_1 = \hat{\lambda}. \]
According to (4),
\[ \hat{\lambda} = \lambda [u_1^T \text{vec}(A)] [\text{vec}(B)]' v_1 + \sigma u_1^T R_{m_0, n_0} | E | v_1, \]
where \( u_1^T R_{m_0, n_0} | E | v_1 \) follows standard normal distribution. Therefore,
\[ [u_1^T \text{vec}(A)] [\text{vec}(B)]' v_1 = \frac{\hat{\lambda} + O_p(1)}{\lambda} = 1 - \frac{\hat{\lambda} - \lambda + O_p(1)}{\lambda} = 1 - O_p \left( \frac{s_0}{\lambda} \right). \]
Notice that \( \text{vec}(A), \text{vec}(B), u_1 \) and \( v_1 \) are unit vectors. Hence
\[ 1 \geq u_1^T \text{vec}(A) = \frac{1 - O_p(s_0/\lambda)}{v_1^T \text{vec}(B)} \geq 1 - O_p \left( \frac{s_0}{\lambda} \right), \]
which gives \( \text{tr}[\hat{A}'A] = u_1^T \text{vec}(A) = 1 - O_p(s_0/\lambda) \). Therefore, we have
\[ \| \hat{A} - A \|_F^2 = 2 - \text{tr}[\hat{A}'A] = O_p \left( \frac{s_0}{\lambda} \right). \]
A similar result holds for \( \hat{B} \).
B Proof of Lemma \[1\]

Consider the complete Kronecker product decomposition of \(A\) with respect to the configuration \((m \land m', n \land n', (m - m')_+, (n - n')_+)\):

\[
A = \sum_{i=1}^{I} \mu_i C_i \otimes D_i, \tag{27}
\]

where \(I = 2^{m \land m' + n \land n'} \land 2^{(m - m')_+ + (n - n')_+}, \mu_1 \geq \mu_2 \geq \cdots \geq \mu_I\) are the coefficients in decreasing order. \(C_i\) and \(D_i\) satisfy

\[
\langle C_i, C_j \rangle = \langle D_i, D_j \rangle = \delta_{i,j}, \tag{28}
\]

where \(\delta_{i,j}\) is the Kronecker delta function such that \(\delta_{i,j} = 1\) if and only if \(i = j\) and \(\delta_{i,j} = 0\) otherwise, and \(\langle A, B \rangle := \text{tr}[A' B]\) is the trace inner product. Notice that the decomposition in \((27)\) corresponds to the singular decomposition for \(R \land m \land m', n \land n'[A]\). Therefore, the singular values \(\mu_1, \ldots, \mu_I\) are uniquely identifiable and the components \(C_i, D_i\) are identifiable if the singular values are distinct. In particular,

\[
\mu_1 = \|R \land m \land m', n \land n'[A]\|_S.
\]

Similarly, the KPD of \(B\) with the configuration \(((m' - m)_+, (n' - n)_+, M - m \lor m', N - n \lor n')\) is given by

\[
B = \sum_{j=1}^{J} \nu_j F_j \otimes G_j, \tag{29}
\]

where \(J = 2^{(m' - m)_+ + (n' - n)_+} \land 2^{M - m \lor m' - n \lor n'}\) and \(\nu_1 = \|R_{(m' - m)_+ + (n' - n)_+}[B]\|_S\).

With the two KPD of \(A\) and \(B\), we can rewrite \(A \otimes B\) by

\[
A \otimes B = \left( \sum_{i=1}^{I} \mu_i C_i \otimes D_i \right) \otimes \left( \sum_{j=1}^{J} \nu_j F_j \otimes G_j \right) = \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_i \nu_j C_i \otimes D_i \otimes F_j \otimes G_j.
\]

Notice that the Kronecker product satisfies distributive law and associative law. The matrix \(D_i\) is \(2^{(m - m')_+} \times 2^{(n - n')_+}\) and the matrix \(F_j\) is \(2^{(m' - m)_+} \times 2^{(n' - n)_+}\). For all possible values of \(m, m', n, n'\), either one of \(D_i\) and \(F_j\) is a scalar, or they are both vectors; and for both cases \(D_i \otimes F_j = F_j \otimes D_i\). Therefore,

\[
A \otimes B = \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_i \nu_j C_i \otimes F_j \otimes D_i \otimes G_j = \sum_{i=1}^{I} \sum_{j=1}^{J} \mu_i \nu_j P_{ij} \otimes Q_{ij}, \tag{29}
\]

where

\[
P_{ij} := C_i \otimes F_j, \quad Q_{ij} := D_i \otimes G_j.
\]

Notice that \(P_{ij}\) is a \(2^{m'} \times 2^{n'}\) matrix and \(Q_{ij}\) is a \(2^{M - m'} \times 2^{N - n'}\) matrix. Therefore, \((29)\) is a KPD of \(A \otimes B\) indexed by \((i, j)\) with respect to the Kronecker configuration \((m', n', M - m', N - n')\) as long as \(P_{ij}\)
and $Q_{ij}$ satisfy the orthonormal condition in (28). One can verify
\[
\langle P_{ij}, P_{kl} \rangle = \text{tr}[P_{ij}' P_{kl}]
\]
\[
= \text{tr}[(C_i \otimes F_j)'(D_k \otimes G_l)]
\]
\[
= \text{tr}[(C_i'D_k) \otimes (F_j'G_l)]
\]
\[
= \text{tr}[C_i'D_k]\text{tr}[F_j'G_l]
\]
\[
= \delta_{i,j}\delta_{k,l},
\]
and similar results hold for $Q_{ij}$. It follows that
\[
\|R_{m',n'}[A \otimes B]\|_S = \max_{i,j} \mu_i \nu_j = \mu_1 \nu_1 = \|R_{m'=m,n'=n}(A)\|_S \|R_{m'-m,n'-n}(B)\|_S,
\]
and the proof is complete.

C Proof of Lemma 2

We first prove the following technical lemma.

Lemma 3. Let $U$, $V$ be two vector subspaces of $\mathbb{R}^n$ with $\Theta(U,V) = \theta \in [0, \pi/2]$, where $\Theta(U,V)$ denotes the smallest principal angle between $U$ and $V$. Suppose $w \in \mathbb{R}^n$ is a unit vector and
\[
\|P_Uw\| = \cos \alpha,
\]
for some $\alpha \in [0, \pi/2]$, where $P_U$ denotes the orthogonal projection to the space $U$. Then it holds that
\[
\|P_Vw\| \leq \begin{cases} 
\cos(\theta - \alpha) & \text{if } \alpha \leq \theta, \\
1 & \text{if } \alpha > \theta.
\end{cases}
\]

Proof. Let
\[
u = \frac{P_Uw}{\|P_Uw\|}
\]
such that $\|\nu\| = 1$ and $\nu \in U$. Let $\{u_1, u_2, \ldots, u_n\}$ be an orthogonal basis of $\mathbb{R}^n$ such that $u_1 = \nu$. For any vector $v \in V$, we have
\[
v'w = v'\left(\sum_{i=1}^n u_i u_i'\right)w
\]
\[
= v'u_1u_1'w + \sum_{i=2}^n v'u_iu_i'w
\]
\[
\leq v'u_1u_1'w + \sqrt{\sum_{i=2}^n v'u_i} \sqrt{\sum_{i=2}^n u_i'w}
\]
\[
= \cos \eta \cos \alpha + \sin \eta \sin \alpha
\]
\[
= \cos(\eta - \alpha),
\]
where $v'u_1 = \cos \eta$. Given $\cos \eta = v'u_1 \leq \cos \theta$, the Lemma is proved.
Now we prove Lemma 2.

Proof of Lemma 2. Suppose $M_1$ and $M_2$ are $m \times n$ matrices. To avoid confusion, the notation $m, n$ is only used for this proof and is not related to the configuration $(m, n)$ used in the main text.

Consider maximizing $\| (M_1 + M_2)u \|^2$ over all unit vectors $u$ such that $\| u \| = 1$. Note that

$$
\| (M_1 + M_2)u \|^2 = \| M_1u + M_2u \|^2 \\
= \| M_1P_{M_1^\perp}u + M_2P_{M_1^\perp}u \|^2 \\
= \| M_1P_{M_1^\perp}u \|^2 + \| M_2P_{M_1^\perp}u \|^2 + 2(M_1P_{M_1^\perp}u)^\top M_2P_{M_1^\perp}u,
$$

where $P_M$ denotes the projection matrix to the column space of $M$. Since $\| M_1 \|_S = \mu$ and $\| M_2 \|_S = \nu$, we have

$$
\| M_1P_{M_1^\perp}u \|^2 \leq \mu^2\| P_{M_1^\perp}u \|^2 \quad \text{and} \quad \| M_2P_{M_1^\perp}u \|^2 \leq \nu^2\| P_{M_1^\perp}u \|^2.
$$

Since $M_1P_{M_1^\perp}u \in \text{span}(M_1)$ and $M_2P_{M_1^\perp}u \in \text{span}(M_2)$, it holds that

$$
(M_1P_{M_1^\perp}u)^\top M_2P_{M_1^\perp}u \leq \cos \theta \mu \nu \| P_{M_1^\perp}u \| \| P_{M_2^\perp}u \|.
$$

It follows that

$$
\| (M_1 + M_2)u \|^2 \leq \mu^2\| P_{M_1^\perp}u \|^2 + \nu^2\| P_{M_2^\perp}u \|^2 + 2\mu \nu\| P_{M_1^\perp}u \| \| P_{M_2^\perp}u \| \cos \theta.
$$

Suppose $\| P_{M_1^\perp}u \| = \cos \alpha$ for some $\alpha \in [0, \pi/2]$. If $\alpha > \eta$, then $\| P_{M_1^\perp}u \| \leq 1$. The right hand side of the preceding inequality attains its maximum when $\| P_{M_1^\perp}u \| = \cos \eta$ and $\| P_{M_2^\perp}u \| = 1$. Hence, we only consider the case $\alpha \leq \eta$, which implies that $\| P_{M_2^\perp}u \| \leq \cos(\eta - \alpha)$, and

$$
\| (M_1 + M_2)u \|^2 \leq \mu^2 \cos^2 \alpha + \nu^2 \cos^2(\eta - \alpha) + 2\mu \nu \cos \theta \cos \alpha \cos(\eta - \alpha).
$$

Therefore,

$$
\mu^2 \cos^2 \alpha + \nu^2 \cos^2(\eta - \alpha) + 2\mu \nu \cos \theta \cos \alpha \cos(\eta - \alpha) \\
= \frac{1}{2} \mu^2 (1 + \cos 2\alpha) + \frac{1}{2} \nu^2 (1 + \cos 2(\eta - 2\alpha)) + \mu \nu \cos \theta \cos \eta + \cos(\eta - 2\alpha) \\
= \frac{1}{2} (\mu^2 + \nu^2 + 2\mu \nu \cos \theta \cos \eta) \\
+ \left( \frac{1}{2} \mu^2 + \frac{1}{2} \nu^2 \cos 2\eta + \mu \nu \cos \theta \cos \eta \right) \cos 2\alpha + \left( \frac{1}{2} \nu^2 \sin 2\eta + \mu \nu \cos \theta \sin \eta \right) \sin 2\alpha \\
\leq \frac{1}{2} (\mu^2 + \nu^2 + 2\mu \nu \cos \theta \cos \eta) \\
+ \sqrt{\left( \frac{1}{2} \mu^2 + \frac{1}{2} \nu^2 \cos 2\eta + \mu \nu \cos \theta \cos \eta \right)^2 + \left( \frac{1}{2} \nu^2 \sin 2\eta + \mu \nu \cos \theta \sin \eta \right)^2} \\
= \frac{1}{2} \left( \mu^2 + \nu^2 + 2\mu \nu \cos \theta \cos \eta + \sqrt{\left( \mu^2 + \nu^2 + 2\mu \nu \cos \theta \cos \eta \right)^2 - 4\mu^2 \nu^2 \sin^2 \theta \sin^2 \eta} \right).
$$

The proof is complete.

Here we also show two useful corollaries and their proofs.
Corollary 6. If $X$ is a $m \times n$ matrix with $n = o(\sqrt{m})$ and $E$ is a matrix of i.i.d. standard Gaussian entries, then for $\sigma > 0$,

$$\|X + \sigma E\|_S^2 \leq \|X\|_S^2 + \sigma^2 \|E\|_S^2 + 2\sigma \|X\|_S \|E\|_S \cos \Theta(X,E) \leq \|X\|_S^2 + \sigma^2 (\sqrt{m} + \sqrt{n})^2 + R,$$

where $R = O_p(n\|X\|_S + \sqrt{m})$.

**Proof of Corollary 6** By setting $\cos \Theta(X',E') = 0$ in Lemma 2, we have

$$\|X + \sigma E\|_S^2 \leq \|X\|_S^2 + \sigma^2 \|E\|_S^2 + 2\sigma \|X\|_S \|E\|_S \cos \Theta(X,E).$$

Let $V_x$ be a $m \times n$ matrix whose columns are the left orthonormal eigenvectors of $X$ and $V_e = [v_1, \ldots, v_n]$ be a $m \times n$ matrix whose columns are the left orthonormal eigenvectors of $E$. Then

$$\cos \Theta(X,E) = \|V_x' V_e\|_S.$$

Since for any $i = 1, \ldots, n$, $\|V_x' v_i\|^2$ follows Beta($n/2$, $(m-n)/2$) distribution, we have

$$\|V_x' v_i\|^2 = \frac{n}{m} \left(1 + O_p \left(\frac{1}{\sqrt{n}}\right)\right).$$

Therefore,

$$\|V_x' V_e\|_S \leq \|V_x' V_e\|_F = \sqrt{\sum_{i=1}^n \|V_x' v_i\|^2} = \frac{n}{\sqrt{m}} + O_p \left(\sqrt{\frac{n}{m}}\right).$$

By plugging the values of $\|E\|_S$ and $\cos \Theta(X,E)$, we have

$$\|X + \sigma E\|_S^2 \leq \|X\|_S^2 + \sigma^2 (\sqrt{m} + \sqrt{n})^2 + R,$$

where

$$R = O_p\left(n\|X\|_S + \sqrt{m}\right).$$

$R$ is a higher order term compared with $\|X\|_S^2 + \sigma^2 (\sqrt{m} + \sqrt{n})^2$.

Corollary 7. If $X$ is a $m \times n$ matrix with $n = o(\sqrt{m})$ and $E$ is a matrix of i.i.d. standard Gaussian entries, then for $\sigma > 0$, when $m \to \infty$

$$P(\|X + \sigma E\|_S \geq U + \sigma t) \leq e^{-t^2/2},$$

where

$$U^2 = \|X\|_S^2 + \sigma^2 (\sqrt{m} + \sqrt{n})^2 + R,$$

and

$$R = \sqrt{2\sigma^2 (\sqrt{m} + \sqrt{n}) + 2\sigma^2 + 2n\sigma \|X\|_S} \left(1 + \sqrt{\frac{n}{m}}\right) = O_p\left(n\|X\|_S + \sqrt{m}\right).$$

**Proof of Corollary 7** Since for any $t > 0$

$$P(\|E\|_S > \sqrt{m + \sqrt{n}} + t) \leq e^{-t^2/2},$$

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we have
\[ \mathbb{E}[\|E\|_S^2] = \int_{t=0}^{\infty} P(\|E\|_S > t^2)2tdt = \int_{t=0}^{\sqrt{m}+\sqrt{n}} 2tdt + \int_{t=0}^{\sqrt{m}+\sqrt{n}+t} 2tdt \leq (\sqrt{m} + \sqrt{n})^2 + 2\pi(\sqrt{m} + \sqrt{n}) + 2. \]

Therefore, given \( \mathbb{E}\|E\|_S \leq \sqrt{m} + \sqrt{n} \) and \( \mathbb{E}\cos^2 \Theta(X,E) \leq n^2/m \), we have
\[ \mathbb{E}\|X + \sigma E\|_S^2 \leq \|X\|_S^2 + \sigma^2 \left( (\sqrt{m} + \sqrt{n})^2 + 2\pi(\sqrt{m} + \sqrt{n}) + 2 \right) + 2\sigma\|X\|_S(\sqrt{m} + \sqrt{n}) \frac{n}{\sqrt{m}} = U^2, \]
and \( \mathbb{E}\|X + \sigma E\|_S \leq \sqrt{\mathbb{E}\|X\|_S^2} + \sigma^2 \|X\|_S^2 \leq U \).

Noticing that \( \|X + \sigma E\|_S \) is a function of \( E \) with Lipschitz norm \( \sigma \), by concentration inequality, we have
\[ P(\|X + \sigma E\|_S \geq U + \sigma t) \leq e^{-t^2/2}. \]

**D Proof of Theorem 2**

*Proof of Theorem 2.* Note that, in this proof, some minor terms of less interest are denoted with the letter \( D \). In some simple cases, the formula of \( D \) will be given. But in some complicated cases, we only provide the order or the stochastic order of the minor term.

We begin with the following expansion:
\[ \|Y\|_F^2 = \lambda_0^2\|\hat{A}\|_F^2\|\hat{B}\|_F^2 + \sigma^2\|E\|_F^2 + 2\sigma\lambda_0 \text{tr}[E'(\hat{A} \otimes \hat{B})]. \] (30)

When \( \hat{A} \) and \( \hat{B} \) are realizations of Gaussian random matrices, by central limit theorem, we have
\[ \|\hat{A}\|_F^2 = 2^{m_0+n_0} + 2^{(m_0+n_0)/2}Z_a, \quad \|\hat{B}\|_F^2 = 2^{M+N-m_0-n_0} + 2^{(M+N-m_0-n_0)/2}Z_b, \]
where \( Z_a \) and \( Z_b \) converge in law to \( \mathcal{N}(0,2) \).

Similarly, we have
\[ \|E\|_F^2 = 2^{M+N} + 2^{(M+N)/2}Z_e, \]
where \( Z_e \) converges in law to \( \mathcal{N}(0,2) \) as well.

Therefore, the first term in (30) is
\[ \lambda_0^2\|\hat{A}\|_F^2\|\hat{B}\|_F^2 = 2^{M+N}\lambda_0^2 + 2^{(M+N)/2}\lambda_0^2D_1, \]
where \( D_1 = 2^{(M+N-m_0-n_0)/2}Z_a + 2^{(m_0+n_0)/2}Z_b + Z_aZ_b = O_p(s_0) \). Since
\[ \text{tr}[E'(\hat{A} \otimes \hat{B})] = \text{tr}[\mathcal{R}_{m_0,n_0}[E][\text{vec}(\hat{A})][\text{vec}(\hat{B})]] = \text{tr}[\text{vec}(\hat{B})'\mathcal{R}_{m_0,n_0}[E][\text{vec}(\hat{A})]] \sim \mathcal{N}(0,\|\hat{A}\|_F^2\|\hat{B}\|_F^2), \]
the last term in (30) is
\[ \text{tr}[E'(\hat{A} \otimes \hat{B})] = 2^{(M+N)/2}D_2, \]
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where $D_2 = O_p(1)$. Therefore, (30) can be rewritten as

$$\frac{\|Y\|^2_F}{2M+N} = \lambda_0^2 + \sigma^2 + D_3,$$

(31)

where

$$D_3 = 2^{-(M+N)/2}\lambda_0^2 D_1 + 2^{-(M+N)/2}2\lambda_0 \sigma D_2 + 2^{-(M+N)/2}\sigma^2 Z_e = O_p(r_0)$$

and

$$r_0 = \frac{s_0}{2^{(M+N)/2}} = 2^{-(m_0+n_0)/2} + 2^{-(M+N-m_0-n_0)/2} > 2^{-(M+N)/2}.$$

For the true configuration $(m_0, n_0)$, from Theorem 1 we have

$$\hat{\lambda}(m_0, n_0) = \lambda + Z_\lambda,$$

where $Z_\lambda = O_p(s_0)$, which gives

$$\frac{\hat{\lambda}^2}{2M+N} = \lambda_0^2 + D_4,$$

(32)

where $D_4 = 2^{-(M+N)/2}2\lambda_0 Z_\lambda + 2^{-(M+N)}Z_\lambda^2 = O_p(r_0)$.

With (31) and (32), the information criterion (11) for the true configuration is given by

$$IC_q(m_0, n_0) = 2^{M+N} \ln(\sigma^2 + D_3 - D_4) + q_0 = 2^{M+N} \ln \sigma^2 + 2^{M+N} D_4 + 2 q_0 2^{(M+N)/2},$$

(33)

where $p_0$ is the number of parameters for true model and

$$D_5 = \ln \frac{\sigma^2 + D_3 - D_4}{\sigma^2} + q_0^2 = O_p(r_0 + q_0^2).$$

Consider a wrong configuration $(m, n) \in W$. Then

$$\hat{\lambda}(m, n) = \|R_{m,n}[Y]\|_S = \|\lambda_0 R_{m,n}[\hat{A} \otimes \hat{B}] + \sigma R_{m,n}[E]\|_S.$$  

(34)

The spectral norm of the two terms in (34) can be calculated with Lemma 1 and the concentration inequality for the spectral norm of a Gaussian ensemble (Vershynin, 2010). Specifically, we have

$$\|R_{m_0,n_0}[\hat{A}]\|_S \leq 2^{(m_0-m)/2} + 2^{(n_0-n)/2} + Z_{ra},$$

$$\|R_{(m-m_0)+,(n-n_0)+}[\hat{B}]\|_S \leq 2^{(m-m_0)+(n-n_0)/2} + 2^{(M+N-m\cdot m_0-n\cdot n_0)/2} + Z_{rb},$$

for some $O_p(1)$ random variables $Z_{ra}$ and $Z_{rb}$. Furthermore,

$$\|R_{m,n}[\hat{A} \otimes \hat{B}]\|_S = \|R_{m_0,n_0}[\hat{A}]\|_S \|R_{(m-m_0)+(n-n_0)+}[\hat{B}]\|_S \leq 2^{(m+n)/2} + 2^{(M+N-m-n)/2} + 2^{(m_0-m)+(n_0-n)/2} + 2^{(M+N-m_0-m-n_0-n)/2} + D_6 \leq 2^{(M+N-1)/2} + \sqrt{s_0} + \sqrt{\sigma^2} s_0 + D_6,$$

(35)

where

$$D_6 = \left(2^{(m-m_0)+(n-n_0)/2} + 2^{(M+N-m\cdot m_0-n\cdot n_0)/2}\right) Z_{ra} + \left(2^{(m_0-m)/2} + 2^{(m_0-m)+(n_0-n)/2}\right) Z_{rb} + Z_{ra} Z_{rb} = O_p(s_0).$$

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Similarly, we have
\[
\| R_{m,n}[E] \|_S \leq 2^{(m+n)/2} + 2^{(M+N-m-n)/2} + Z_{re},
\]
(36)
for some random variable \( Z_{re} = O_p(1) \).

We partition the configuration space \( W \) into three disjoint subsets:
\[
W = W_s \cup W_c \cup W_l,
\]
where \( W_s = \{(m,n) \in W : m + n < (M + N)/4\} \) is the subset with matrix \( A \) much smaller than \( B \),
\( W_c = \{(m,n) \in W : (M + N)/4 \leq m + n \leq 3(M + N)/4\} \) is the subset with \( A \) and \( B \) having similar sizes
and \( W_l = \{(m,n) \in W : m + n > 3(M + N)/4\} \) is the subset with matrix \( A \) much larger than \( B \).

For \((m,n)\) in \(W_c\), \( \hat{\lambda}(m,n)^2 \) can be bounded above by (35) and (36) through triangular inequality such that
\[
\frac{\hat{\lambda}(m,n)^2}{2^{M+N}} \leq 2^{-(M+N)} \left( \lambda_0 \| R_{m,n}[\hat{A} \otimes \hat{B}] \|_S + \sigma \| R_{m,n}[E] \|_S \right)^2 \leq \frac{\lambda_0^2}{2} + D_7,
\]
(37)
where
\[
D_7 = 2^{-(M+N)/2} \left( \frac{\lambda_0^2}{2} + s_0 + \frac{D_6}{\sqrt{2}} + \frac{\lambda_0}{\sqrt{2}} (s_{m,n} + Z_{re}) \right) + 2^{-(M+N)} \left( \frac{\lambda_0}{\sqrt{2}} + \sqrt{2} s_0 + D_6 + s_{m,n} + Z_{re} \right)^2
\]
\[
= O_p \left( r_0 + 2^{-(M+N)/8} \right),
\]
with \( s_{m,n} = 2^{(m+n)/2} + 2^{(M+N-m-n)/2} \).

Note that \( s_{m,n} \) is related to the number of parameters under \((m,n)\) through
\[
s_{m,n}^2 = 2^{(m+n)} + 2^{(M+N-m-n)} + 2 \cdot 2^{(M+N)/2} = p + 2 \cdot 2^{(M+N)/2}.
\]
(38)
The equality in (37) is attained when \( |m_0 - m| + |n_0 - n| = 1 \). The upper bound for IC\(_q\)(\(m,n\)) can therefore be established with (31) and (37) such that
\[
\text{IC}_q(m,n) \geq 2^{M+N} \left[ \ln \left( \frac{\lambda_0^2 + 2\sigma^2}{2} + D_3 - D_7 \right) + qr_{m,n}^2 - 2 \cdot 2^{-(M+N)/2} \right]
\]
\[
= 2^{M+N} \left[ \ln \frac{\lambda_0^2 + 2\sigma^2}{2} + O_p \left( r_0 + 2^{-(M+N)/8} \right) \right],
\]
(39)
where \( r_{m,n} = s_{m,n}/2^{(M+N)/2} \).

For \((m,n)\) in \(W_s \cup W_l\), we bound \( \hat{\lambda}(m,n) \) in (34) with Corollary 6 such that
\[
\frac{\hat{\lambda}(m,n)^2}{2^{M+N}} \leq \frac{\lambda_0^2 \| R_{m,n} [\hat{A} \otimes \hat{B}] \|_S^2 + \sigma^2 s_{m,n}^2 + R_1}{2^{M+N}} \leq \frac{\lambda_0^2}{2} + \sigma^2 r_{m,n}^2 + D_8,
\]
(40)
where \( R_1 = 2^{(M+N)/2} O_p (2^{m+n} \land 2^{M+N-m-n}) \) is the residual term from Corollary 6 and
\[
D_8 = 2^{-(M+N)} (\sqrt{2} + \sqrt{2} s_0 + D_6)^2 + 2^{-(M+N)/2} (1 + s_0 + D_6/\sqrt{2}) + 2^{-(M+N)} R_1 = O_p(r_0).
\]

Based on (31), (40) and (38), we have
\[
\text{IC}_q(m,n) \geq 2^{M+N} \left[ \ln \left( \frac{\lambda_0^2}{2} + \sigma^2 (1 - r_{m,n}^2) + D_3 - D_8 \right) + qr_{m,n}^2 - 2 \cdot 2^{-(M+N)/2} \right].
\]
(41)
Note that in \([41]\), IC\(_q(m, n)\) is a uni-modal function of \(r_{m,n}^2\), which has a support
\[
r_{m,n}^2 \in \left(2^{-3(M+N)/8} + 2^{-3(M+N)/8}, 2^{-1/2} + 2^{-(M+N-1)/2} \right).
\]
When \(r_{m,n}^2 = 2^{-3(M+N)/8} + 2^{-3(M+N)/8}\), the right hand side of \(41\) is at least
\[
2^{M+N} \left[ \ln \left( \frac{\lambda_0^2}{2} + \sigma^2 \right) + O_p \left( r_0 + 2^{-3(M+N)/8} \right) \right].
\]
When \(r_{m,n}^2 = 2^{-1/2} + 2^{-(M+N-1)/2}\), the right hand size of \(41\) is
\[
2^{M+N} \left[ \ln \left( \frac{\lambda_0^2}{2} + \sigma^2 \right) + \frac{q}{2} + O_p(r_0) \right].
\]

Therefore, when \((m, n) \in \mathcal{W}_\alpha \cup \mathcal{W}_\beta\), IC\(_q(m, n)\) is bounded from below by the smaller one of \(42\) and \(43\), and hence it holds that
\[
IC_q(m, n) \geq 2^{M+N} \left[ \ln \left( \frac{\lambda_0^2}{2} + \sigma^2 \right) + o_p(1) \right] \land 2^{M+N} \left[ \ln \left( \frac{\lambda_0^2}{2} + \sigma^2 \right) + \frac{q}{2} + o_p(1) \right].
\] \(44\)

By combining \(33\), \(39\) and \(44\), we have for any \((m, n) \in \mathcal{W}\),
\[
IC_q(m, n) - IC_q(m_0, n_0) \geq 2^{M+N} (\alpha + o_p(1)),
\]
where
\[
\alpha = \left[ \ln \left( 1 + \frac{\lambda_0^2}{2\sigma^2} \right) \right] \land \left[ \ln \left( \frac{\lambda_0^2}{2\sigma^2} + \frac{q}{2} \right) \right].
\] \(45\)
The condition
\[
q > 2 \ln 2 - 2 \ln \left( \frac{\lambda_0^2}{2\sigma^2} \right)
\]
ensures that \(\alpha\) in \(45\) is strictly positive. The proof is complete.

E Proof of Theorem 3

We calculate the tail bounds for \(\|Y\|_{P}^2\) and \(\hat{\lambda}^2\), which appear in the information criterion formula \(41\). According to the tail bounds for \(\chi^2\) random variable given in \cite{Laurent and Massart, 2000}, we have the following tail bounds for \(\|\hat{A}\|_{P}^2\), \(\|\hat{B}\|_{P}^2\) and \(\|E\|_{P}^2\). For any \(t > 0\),
\[
P \left[ 2^{-(m_0+n_0)} \|\hat{A}\|_{P}^2 > 1 + \frac{\sqrt{2t}}{2(m_0+n_0)/2} + \frac{t^2}{2m_0+n_0} \right] \leq e^{-t^2/2},
\]
\(46\)
\[
P \left[ 2^{-(m_0+n_0)} \|\hat{A}\|_{P}^2 < 1 - \frac{\sqrt{2t}}{2(m_0+n_0)/2} \right] \leq e^{-t^2/2},
\]
\(47\)
\[
P \left[ 2^{-(M+N-m_0-n_0)} \|\hat{B}\|_{P}^2 > 1 + \frac{\sqrt{2t}}{2(M+N-m_0-n_0)/2} + \frac{t^2}{2M+N-m_0-n_0} \right] \leq e^{-t^2/2},
\]
\(48\)
\[
P \left[ 2^{-(M+N-m_0-n_0)} \|\hat{B}\|_{P}^2 < 1 - \frac{\sqrt{2t}}{2(M+N-m_0-n_0)/2} \right] \leq e^{-t^2/2},
\]
\(49\)
\[
P \left[ 2^{-(M+N)} \|E\|_{P}^2 > 1 + \frac{\sqrt{2t}}{2(M+N)/2} + \frac{t^2}{2M+N} \right] \leq e^{-t^2/2},
\]
\(50\)
where

\[ P \left[ 2^{-(M+N)} \| \mathbf{E} \|_F^2 < 1 - \frac{\sqrt{2} t}{2(M+N)^{1/2}} \right] \leq e^{-t^2/2}, \]

Moreover, the tail bounds of \( \| \mathbf{A} \|_F \) are

\[
P \left[ 2^{-(m_0+n_0)/2} \| \mathbf{A} \|_F > 1 + \frac{t}{2(m_0+n_0)^{1/2}} \right] = P \left[ 2^{-(m_0+n_0)} \| \mathbf{A} \|_F^2 > 1 + 2t + \frac{t^2}{2(m_0+n_0)} \right] \leq e^{-t^2/2},
\]

\[
P \left[ 2^{-(m_0+n_0)/2} \| \mathbf{A} \|_F < 1 - \frac{\sqrt{2} t}{2(m_0+n_0)^{1/2}} \right] = P \left[ 2^{-(m_0+n_0)} \| \mathbf{A} \|_F^2 < \left(1 - \frac{\sqrt{2} t}{2(m_0+n_0)^{1/2}}\right)^2 \right] \leq e^{-t^2/2},
\]

and similarly for \( \| \mathbf{B} \|_F \).

\[
P \left[ 2^{-(M+N-m_0-n_0)/2} \| \mathbf{B} \|_F > 1 + \frac{t}{2(M+N-m_0-n_0)^{1/2}} \right] \leq e^{-t^2/2},
\]

\[
P \left[ 2^{-(M+N-m_0-n_0)/2} \| \mathbf{B} \|_F < 1 - \frac{\sqrt{2} t}{2(M+N-m_0-n_0)^{1/2}} \right] \leq e^{-t^2/2}.
\]

Combining (46), (47), (48) and (49), we have

\[
P \left[ 2^{-(M+N)} \| \mathbf{A} \|_F^2 \| \mathbf{B} \|_F^2 > 1 + \sqrt{2} r_0 t + r_0 t^2 + \sqrt{2} : 2^{-(M+N)/2} r_0 t^3 + 2^{-(M+N)/2} t^4 \right] \leq 2 e^{-t^2/2},
\]

\[
P \left[ 2^{-(M+N)} \| \mathbf{A} \|_F^2 \| \mathbf{B} \|_F^2 < 1 - \sqrt{2} r_0 t + 2 \cdot 2^{-(M+N)/2} t^2 \right] \leq e^{-t^2/2}.
\]

Similarly, by combining (52), (54) with

\[ P[Z > t] \leq e^{-t^2/2}, \]

where \( Z \sim \mathcal{N}(0, 1) \), we have

\[
P \left[ 2^{-(M+N)/2} \text{vec}(\mathbf{A})' \text{vec}(\mathbf{B}) > t + r_0 t^2 + 2^{-(M+N)/2} t^3 \right] \leq 3 e^{-t^2/2},
\]

\[
P \left[ 2^{-(M+N)/2} \text{vec}(\mathbf{A})' \text{vec}(\mathbf{B}) < - \left( t + r_0 t^2 + 2^{-(M+N)/2} t^3 \right) \right] \leq 3 e^{-t^2/2}.
\]

Now we have obtained the tail bounds for every term in (30). Therefore, with (56), (57) and (58), we have

\[
P \left[ 2^{-(M+N)} \| \mathbf{Y} \|_F^2 > \lambda_0^2 + \sigma^2 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \right] \leq 5 e^{-t^2/2},
\]

where

\[
a_1 = \sqrt{2} \lambda_0^2 r_0 + (\sqrt{2} \sigma^2 + 2 \lambda_0 \sigma) 2^{-(M+N)/2},
\]

\[
a_2 = \lambda_0^2 r_0^2 + \sigma^2 2^{-(M+N)} + 2 \lambda_0 \sigma r_0 2^{-(M+N)/2},
\]

\[
a_3 = \sqrt{2} \lambda_0^2 r_0^2 2^{-(M+N)/2} + 2 \lambda_0 \sigma 2^{-(M+N)},
\]

\[
a_4 = \lambda_0^2 2^{-(M+N)}.
\]
The lower tail bound can be obtained using (57), (51) and (59)

\[ P \left[ 2^{-(M+N)} \| Y \|_F^2 < \lambda_0^2 + \sigma^2 - b_1 t - b_2 t^2 - b_3 t^3 \right] \leq 5e^{-t^2/2}, \]  

where

\[ b_1 = \sqrt{2}\lambda_0^2 r_0 + (\sqrt{2}\sigma^2 + 2\lambda_0\sigma)2^{-(M+N)/2}, \]
\[ b_2 = (2\lambda_0^2 + 2\lambda_0\sigma r_0)2^{-(M+N)/2}, \]
\[ b_3 = 2\lambda_0\sigma 2^{-(M+N)}. \]

Now we derive the tail bound for \( \hat{\lambda} \). First, according to Rudelson and Vershynin (2010), for any configuration \((m,n)\), it holds that

\[ P \left[ \| \mathcal{R}_{m,n}[E] \|_S \geq 2^{(m+n)/2} + 2^{(M+N-m-n)/2} + t \right] \leq e^{-t^2/2}. \]  

Under true configuration \((m_0,n_0)\), \( \lambda = \lambda_0 \| \hat{A} \|_F \| \hat{B} \|_F \). Since

\[ \hat{\lambda}(m_0,n_0) \geq \lambda - \sigma \| \mathcal{R}_{m_0,n_0}[E] \|_S, \]

the lower tail bound on \( \hat{\lambda} \) can be obtained from (53), (55) and (62):

\[ P \left[ \hat{\lambda}(m_0,n_0) < \lambda_0 2^{(M+N)/2} - \sigma s_0 - (\sqrt{2}\lambda_0 s_0 + \sigma) t + 2\lambda_0 t^2 \right] \leq 3e^{-t^2/2}. \]  

We now consider the wrong configurations. For a wrong configuration \((m,n) \in \mathcal{W}\), denote \( \mu_1 \) and \( \nu_1 \) be the leading coefficients in (29). Since \( \hat{A} \) and \( \hat{B} \) are realizations of the standard Gaussian ensemble, according to Rudelson and Vershynin (2010), we have the upper tail bounds for \( \mu_1 \) and \( \nu_1 \):

\[ P[\mu_1 > \mu_1^* + t] \leq e^{-t^2/2}, \]
\[ P[\nu_1 > \nu_1^* + t] \leq e^{-t^2/2}, \]

where

\[ \mu_1^* = 2^{(m_0 \wedge m + n_0 \wedge n)/2} + 2^{(m_0-m)_+ + (n_0-n)_+)/2}, \]
\[ \nu_1^* = 2^{(m-m_0)_+ + (n-n_0)_+)/2} + 2^{(M+N-m_0 \vee m-n_0 \vee n)/2}. \]

By combining (64) with (65), we have

\[ P[\lambda_0 \mu_1 \nu_1 > \lambda_0 \mu_1^* \nu_1^* + \lambda_0 (\mu_1^* + \nu_1^*) t + \lambda_0 t^2] \leq 2e^{-t^2/2}. \]  

Therefore, for \((m,n) \in \mathcal{W}_c\), noticing that

\[ \hat{\lambda}(m,n) \leq \lambda_0 \mu_1 \nu_1 + \sigma \| \mathcal{R}_{m,n}[E] \|_S, \]

we have

\[ P[\hat{\lambda}(m,n) > \lambda_0 \mu_1^* \nu_1^* + \sigma s_{m,n} + (\lambda_0 \mu_1^* + \lambda_0 \nu_1^* + \sigma) t + \lambda_0 t^2] \leq 3e^{-t^2/2}. \]  

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For \((m, n) \in W_s\), we use Lemma \ref{lemma} and rewrite the tail bound in Corollary \ref{corollary} as
\[ P \left[ \hat{\lambda}_{m,n}^2 \geq (\lambda_0 \mu_1 \nu_1)^2 + \sigma^2 \hat{s}_{m,n}^2 + R_0 + R_1 t + \sigma^2 t^2 \right] \leq e^{-t^2/2}, \tag{68} \]
where \(R_0 = O(2^{m+n} \mu_1 \nu_1 + 2^{M+N-m-n}/2)\) is the residual term from Corollary \ref{corollary} and
\[ R_1 = 2\sigma \sqrt{(\lambda_0 \mu_1 \nu_1)^2 + \sigma^2 \hat{s}_{m,n}^2 + R_0} \leq O_p(2^{(M+N)/2}). \]

Now we are ready to establish the consistency in Theorem \ref{theorem}. Let \(\alpha\) be the positive constant in Theorem \ref{lemma}. We have
\[ P \left[ IC_q(m_0, n_0) > 2^{M+N} \left( \ln \sigma^2 + qp_0 + \alpha/2 \right) \right] \]
\[ = P \left[ 2^{-M-N} \left( \|Y\|_{f^2}^2 - \hat{\lambda}^2 \right) > \sigma^2 e^{\alpha/2} \right] \]
\[ \leq P \left[ 2^{-M-N} \left( \|Y\|_{f}^2 > \lambda_0^2 + \sigma^2 + g \right) \right] + P \left[ 2^{-M-N} \hat{\lambda}^2 < \lambda_0^2 - \sigma^2 g \right] \leq 5 \exp \left\{ - \frac{c_1^2}{2r_0^2} + o_p(r_0^{-2}) \right\} + 3 \exp \left\{ - \frac{c_2^2}{2r_0^2} + o_p(r_0^{-2}) \right\}, \tag{69} \]
where
\[ g = \frac{e^{\alpha/2} - 1}{2}, \quad c_1 = \frac{1}{2} \sqrt{2 + \frac{4q}{\lambda_0^2} - \sqrt{2}}, \quad c_2 = \frac{1}{2} \sqrt{2 - \frac{2q}{\lambda_0^2}} \vee 0. \]

To go from \eqref{69} to \eqref{70}, equations \eqref{60} and \eqref{63} are used. Similarly, for \((m, n) \in W_c\), we have
\[ P \left[ IC_q(m, n) < 2^{M+N} \left( \ln \sigma^2 + qp_0 + \alpha/2 \right) \right] \]
\[ \leq P \left[ IC_q(m, n) < 2^{M+N} \left( \ln(\lambda_0^2 + \sigma^2 - 2^{-M-N}((\lambda_0 \mu_1 \nu_1)^2 + qp - \alpha/2) \right) \right] \]
\[ \leq P \left[ 2^{-(M+N)} \left[ \|Y\|_{f}^2 - \hat{\lambda}^2 \right] < \lambda_0^2 + \sigma^2 - 2^{-(M+N)}((\lambda_0 \mu_1 \nu_1)^2 - 1 - e^{-\alpha/2} \frac{\lambda_0^2 + 2\sigma^2}{2}) \right] \]
\[ \leq P \left[ 2^{-(M+N)} \left[ \|Y\|_{f}^2 < \lambda_0^2 + \sigma^2 - h \right] \right] + P \left[ 2^{-(M+N)} \hat{\lambda}^2 > 2^{-(M+N)}((\lambda_0 \mu_1 \nu_1)^2 + h) \right] \]
\[ \leq 5 \exp \left\{ - \frac{c_3^2}{2r_0^2} + o_p(r_0^{-2}) \right\} + 3 \exp \left\{ - \frac{c_4^2}{2r_0^2} + o_p(r_0^{-2}) \right\}, \tag{71} \]
where \eqref{61} and \eqref{67} are used to get \eqref{71}.

where \(h = [1 - e^{-\alpha/2} \frac{\lambda_0^2 + 2\sigma^2}{4}], \quad c_3 = \sqrt{2 + \frac{2h}{\lambda_0^2} - \sqrt{2}}, \quad c_4\) is the solution of
\[ 2\lambda_0(\lambda_0 + \sigma)c_4 + \lambda_0(2\lambda_0 + \sigma)c_4^2 + 2\lambda_0^2 c_4^2 + \lambda_0^2 c_4^2 = h. \]

Finally, for \((m, n) \in W_s\), it holds that
\[ P \left[ IC_q(m, n) < 2^{M+N} \left( \ln \sigma^2 + qp_0 + \alpha/2 \right) \right] \]
\[ \leq P \left[ IC_q(m, n) < 2^{M+N} \left( \ln(\lambda_0^2 + \sigma^2 - 2^{-M-N}((\lambda_0 \mu_1 \nu_1)^2 + \sigma^2 s_{m,n}^2)) + qp - \alpha/2 \right) \right] \]
\[ \leq P \left[ 2^{-(M+N)} \left[ \|Y\|_{f}^2 - \hat{\lambda}^2 \right] < \lambda_0^2 + \sigma^2 - 2^{-(M+N)}((\lambda_0 \mu_1 \nu_1)^2 + \sigma^2 s_{m,n}^2) - [1 - e^{-\alpha/2} \frac{\lambda_0^2 + 2\sigma^2}{2}) \right] \]
\[ \leq P \left[ 2^{-(M+N)} \left[ \|Y\|_{f}^2 < \lambda_0^2 + \sigma^2 - w \right] \right] + P \left[ 2^{-(M+N)} \hat{\lambda}^2 > 2^{-(M+N)}((\lambda_0 \mu_1 \nu_1)^2 + \sigma^2 s_{m,n}^2) + w \right] \]
\[ \leq 5 \exp \left\{ - \frac{c_3^2}{2r_0^2} + o_p(r_0^{-2}) \right\} + \exp \left\{ -2^{(M+N)} \left( \frac{c_4^2}{2} + o(1) \right) \right\} + 2 \exp \left\{ - \frac{c_4^2}{2r_0^2} + o_p(r_0^{-2}) \right\}, \tag{72} \]
where we use (61), (68) and (66) to obtain (72), and
\[
w = [1 - e^{-\alpha/2}] \frac{\lambda_0^2 + \sigma^2}{4}, \quad c_5 = \sqrt{2 + \frac{2w}{\lambda_0^2} - \sqrt{2}},
\]
c_6 is the solution to
\[
2\sigma \sqrt{\lambda_0^4 + \sigma^2 c_6 + \sigma^2 c_6^2} = \frac{w}{2},
\]
and c_7 is the solution to
\[
2c_7 + 3c_7^2 + c_7^3 + c_7^4 = \frac{w^2}{2\lambda_0^2}.
\]

With a similar result for \((m, n) \in W_l\), we conclude that
\[
P \left[ IC_q(m_0, n_0) \geq \min_{(m, n) \in W} IC_q(m, n) \right]
\leq \sum_{(m, n) \in W} P[IC_q(m_0, n_0) \geq IC_q(m, n)]
\leq (M + 1)(N + 1) \left[ (70) + \min\{(71), (72)\} \right]
\rightarrow 0.
\]
The convergence is a consequent of the second condition in Assumption 1. The proof is complete.

F Proof of Theorem 4

The proof is similar to that of Theorem 2, with the assumption that \(\lambda_0 \mu_1^* \nu_1^*\) is bounded by \(\epsilon_0\). We will omit the details.

G Proofs of Theorem 5, Corollary 5

The proof of Theorem 5 is similar to the proofs of Theorem 2, Theorem 3 and Theorem 4, so we only point out the main steps here. The information criterion for the true configuration is
\[
IC_q(m_0, n_0) = 2^{M+N} \ln \left( \frac{\|Y\|_F^2 - \hat{\lambda}^2}{2^{M+N}} \right) + qp,
\]
where \(2^{-(M+N)}\|Y\|_F^2 = 2^{-(M+N)}(\lambda_1^2 + \lambda_2^2) + \sigma^2 + O_p(s_0)\) and \(\hat{\lambda} = \lambda_1 + O_p(s_0)\). Therefore
\[
IC_q(m_0, n_0) = 2^{M+N} \left[ \ln \left( 2^{-(M+N)} \lambda_1^2 + \sigma^2 \right) + o_p(1) \right].
\]

For \((m, n) \in W\) and \((M + N)/4 \leq m + n \leq 3(M + N)/4\), we have
\[
IC_q(m, n) - IC_q(m_0, n_0) = 2^{M+N} \left[ \ln \left( \frac{\lambda_1^2 + \lambda_2^2 + \sigma^{2M+N}}{\lambda_2^2 + \sigma^{2M+N}} - \Lambda^2(\lambda_1 \epsilon_1, \lambda_2 \epsilon_2, \theta_c, \theta_r) \right) + o_p(1) \right],
\]
which is greater than 0 since \(\lambda_1^2 > \Lambda^2(\lambda_1\epsilon_1, \lambda_2\epsilon_2, \theta_c, \theta_r)\). For \((m, n) < (M + N)/4\), the difference \(IC_q(m, n) - IC_q(m_0, n_0)\) is bounded from below by

\[
\geq 2^{M+N} \left[ \ln \left( \frac{\lambda_1^2 + \lambda_2^2 + \sigma^2 2^{M+N} - \Lambda^2(\lambda_1\epsilon_1, \lambda_2\epsilon_2, \theta_c, \theta_r) - \frac{1}{2} \sigma^2 2^{M+N}}{\lambda_2^2 + \sigma^2 2^{M+N}} \right) + \frac{1}{2} q + o_p(1) \right].
\]

This lower bound is always positive, and it is only attained when \(m + n = 1\). A similar inequality holds for \(m + n > 3(M + N)/4\). As a conclusion we know that \(IC_q(m, n) - IC_q(m_0, n_0)\) is at least \(2^{M+N}\alpha\), where

\[
\alpha = \left[ \ln \left( 1 + \frac{\lambda_1^2 - \Lambda^2(\lambda_1\epsilon_1, \lambda_2\epsilon_2, \theta_c, \theta_r)}{\lambda_2^2 + \sigma^2 2^{M+N}} \right) \right] \wedge \left[ \frac{q}{2} + \ln \left( \frac{1}{2} + \frac{\lambda_1^2 - \Lambda^2(\lambda_1\epsilon_1, \lambda_2\epsilon_2, \theta_c, \theta_r)}{\lambda_2^2 + \sigma^2 2^{M+N}} \right) \right].
\]

When \(A_1, B_1, A_2\) and \(B_2\) are generated according to random scheme, we have \(\theta_c = \theta_r = \pi/2 + o_p(1)\) and \(\epsilon_1 = \epsilon_2 = 1/\sqrt{2} + o_p(1)\).

For the random scheme, Corollary 5 follows immediately, since \(\Lambda^2(\lambda_1\epsilon_1, \lambda_2\epsilon_2, \theta_c, \theta_r) = \frac{\lambda_2^2}{2} + o_p(1)\). The proof is complete.

At the end of this section, we note that since \(\Lambda^2(\lambda_1\epsilon_1, \lambda_2\epsilon_2, \theta_c, \theta_r) \leq \lambda_1^2\epsilon_1^2 + \lambda_2^2\epsilon_2^2 + 2\lambda_1\lambda_2\epsilon_1\epsilon_2 \cos \theta_c \cos \theta_r\), a sufficient condition of (24) is

\[
\frac{\lambda_1}{\lambda_2} > \epsilon_1\epsilon_2 \cos \theta_c \cos \theta_r + \sqrt{\epsilon_1^2\epsilon_2^2 \cos^2 \theta_c \cos^2 \theta_r + (1 - \epsilon_1^2)\epsilon_2^2},
\]

as pointed out in Remark 3.