STARK-WANNIER LADDERS AND CUBIC EXPONENTIAL SUMS
Alexander Fedotov, Frédéric Klopp

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On $L^2(\mathbb{R})$, we consider the Schrödinger operator
\begin{equation}
H_\epsilon = -\frac{\partial^2}{\partial x^2} + v(x) - \epsilon x,
\end{equation}
where $v$ is a real analytic 1-periodic function and $\epsilon$ is a positive constant. This operator is a model to study a Bloch electron in a constant electric field ([1]). The parameter $\epsilon$ is proportional to the electric field. The operator (1.1) was studied both by physicists (see, e.g., the review [6]) and by mathematicians (see, e.g., [9]). Its spectrum is absolutely continuous and fills the real axis. One of main features of $H_\epsilon$ is the existence of Stark-Wannier ladders. These are $\epsilon$-periodic sequences of resonances, which are poles of the analytic continuation of the resolvent kernel in the lower half plane through the spectrum (see, e.g., [2]). Most of the mathematical work studied the case of small $\epsilon$ (see, e.g., [9, 3] and references therein). When $\epsilon$ is small, there are ladders exponentially close to the real axis. Actually, only the case of finite gap potentials $v$ was relatively well understood. For these potentials, there is only a finite number of ladders exponentially close to the real axis. It was further noticed that the ladders non-trivially “interact” as $\epsilon$ changes, and conjectured that the behavior of the resonances strongly depends on number theoretical properties of $\epsilon$ (see, e.g., [1]).

In the present note, we only consider the periodic potential $v(x) = 2 \cos(2\pi x)$ and study the reflection coefficient $r(E)$ of the Stark-Wannier operator (1.1) in the lower half of the complex plane of the spectral parameter $E$. The resonances are the poles of the reflection coefficient. We show that, as $\text{Im } E \to -\infty$, the function $E \mapsto \frac{1}{r(E)}$ can be asymptotically described in terms of a regularized cubic exponential sum that is a close relative of the cubic exponential sums often encountered in analytic number theory. This explains the dependence of the reflection coefficient on the arithmetic
nature of $\epsilon$. For $\frac{\pi^2}{2x} \in \mathbb{Q}$, we describe the asymptotics of the Stark-Wannier ladders situated far from the real axis.

Let us recall the definition of the reflection coefficient for (1.1) following [2]. Consider the equation
\begin{equation}
-\psi''(x) + (v(x) - \epsilon x)\psi(x) = E\psi(x), \quad x \in \mathbb{C},
\end{equation}
For the sake of simplicity, assume that the potential $v$ is entire. Assume also $\int_0^1 v(x) \, dx = 0$. For any $E \in \mathbb{C}$, there are unique solutions $\psi_{\pm}$ to (1.2) that admit the asymptotic representations
\begin{equation}
\psi_-(x, E) = \frac{1}{\sqrt{-\epsilon x - E}} e^{-\int_{-\epsilon x - E}^{E/\epsilon} \sqrt{-\epsilon t - E} \, dt + o(1)}, \quad x \to -\infty,
\end{equation}
\begin{equation}
\psi_+(x, E) = \frac{1}{\sqrt{\epsilon x + E}} e^{i \int_{-\epsilon x + E}^{E/\epsilon} \sqrt{\epsilon t + E} \, dt + o(1)}, \quad x \to +\infty,
\end{equation}
where the determinations of $\sqrt{}$ and $\sqrt{}$ are analytic in $\mathbb{C} \setminus \mathbb{R}$ and positive along $\mathbb{R}_+$. Consider also the solution $\psi_+(x, E) = \overline{\psi_+(\bar{x}, \bar{E})}$. The solutions $\psi_+$ and $\psi_+^*$ being linearly independent, one has
\begin{equation}
\psi_-(x, E) = w(E)\psi_+(x, E) + w^*(E)\psi_+(x, E), \quad x \in \mathbb{R},
\end{equation}
where the coefficient $w(E)$ is independent of $x$ and the function $E \mapsto w(E)$ is the reflection coefficient. It is an $\epsilon$-periodic meromorphic function of $E$. The reflection coefficient is analytic in $\mathbb{C}_+$, and, for $E \in \mathbb{R}$, one has $|r(E)| = 1$. The poles of $r$ are the resonances of $H_\epsilon$.

Let us now state the first of our results. Represent $1/r$ by its Fourier series
\begin{equation}
1/r(E) = \sum_{m \in \mathbb{Z}} e^{2\pi niE/\epsilon} p(m) \quad \text{for } \text{Im} E \leq 0. \quad \text{Let } a(\epsilon) = \sqrt{\frac{2\pi}{e}} e^{i\pi/4}. \quad \text{One has}
\end{equation}

**Theorem 1.** Let $v(x) = 2 \cos(2\pi x)$. Then, as $m \to \infty$,
\begin{equation}
p(m) = a(\epsilon) \sqrt{m} e^{-2\pi i\omega m^3/2 - 2m \log(2\pi m/e) + \delta(m)}, \quad \omega = \left\{ \frac{\pi^2}{3\epsilon} \right\},
\end{equation}
where, for $x$ real, $\{x\}$ denotes the fractional part of $x$, and $\delta(m) = O(\log^2 m/m)$. This estimate is locally uniform in $\epsilon > 0$.

Clearly, the asymptotic behavior of $1/r(E)$ as $\text{Im} E \to -\infty$ is determined by the Fourier series terms with large positive $m$, and so, roughly,
\begin{equation}
\frac{1}{r(E)} \approx a(\epsilon) \mathcal{P}(E/\epsilon), \quad \mathcal{P}(s) = \sum_{m \geq 1} \sqrt{m} e^{-2\pi i\omega m^3/2 - 2m \log(2\pi m/e) + 2\pi is}.
\end{equation}

It is worth to compare the function $\mathcal{P}$ with the cubic exponential sums $\sum_{n=1}^N e^{-2\pi i\omega n^3}$. Such sums were extensively studied in analytic number theory, see, e.g., [4]. They
were proved to depend strongly on the arithmetic nature of $\omega$. This appears to be true in our case too. We have

**Theorem 2.** Let $v(x) = 2 \cos(2\pi x)$. Assume that $\omega \in \mathbb{Q}$ and represent it in the form $\omega = \frac{p}{q}$, where $0 \leq p < q$ are co-prime integers. If $p = 0$, we take $q = 1$. For $\xi \in \mathbb{R}$, we set $I_q(\xi) := \{m \in \mathbb{Z} : |\xi - \frac{m}{q}| \leq 1/2\}$. As $\text{Im } E \to -\infty$, one has

$$r^{-1}(E) = \frac{b(\epsilon) \rho}{q} \sum_{m \in I_q(\xi)} S_q(p, m) e^{2\pi i \frac{\xi m}{q} + \frac{\pi}{4} + \frac{\pi}{4} + \frac{1}{2}},$$

where $b(\epsilon) = \frac{\pi^2}{\sqrt{2\epsilon}}$ and $\xi = \text{Re } E/\epsilon$, $\rho = e^{-\pi \text{Im } E/\epsilon}$, and

$$S_q(p, m) = \sum_{l=0}^{q-1} e^{2\pi i \frac{m^3 - ml}{q}}.$$

The error estimates are locally uniform in $\epsilon > 0$.

Let us discuss this result. First, assume that $\omega = 0$. By Theorem 2,

$$(b(\epsilon) r(E))^{-1} = \sqrt{\tau} e^{\tau E + O(\ln \tau)} + e^{O(\ln \tau)}, \quad \tau = e^{2\pi E/\epsilon},$$

where the determination of $\sqrt{\tau}$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ and positive along $\mathbb{R}_+$. Recall that $1/r$ is $\epsilon$-periodic. Let $B_\epsilon = \{E \in \mathbb{C} : \text{Im } E \leq 0, 0 \leq \text{Re } E \leq \epsilon\}$. Representation (1.8) implies

**Corollary 1.** Assume $\omega = 0$. The resonances located in $B_\epsilon$ have the following properties:

- for sufficiently large $y > 0$, the resonances with $\text{Im } E < -\epsilon y$ are located in the domain $|\text{Re } E - \epsilon/2| \leq C \epsilon^2/|\text{Im } E|$, where $C > 0$ is a constant;
- let $n(y)$ be the number of resonances in the rectangle $[0, \epsilon] - i [0, \epsilon y]$; then, one has

$$n(y) = \frac{1}{\pi} e^{\pi y + o(1)} \quad \text{as } y \to \infty.$$

The first statement immediately follows from Theorem 2; to prove the second one has to use Jensen formula and Levin lower bounds for the absolute values of entire functions, see, e.g., [8].

When $\omega = 0$, it is difficult to obtain the asymptotics of the resonances as, in a neighborhood of the line $\text{Re } E/\epsilon = 1/2 \mod 1$, they are determined by the first Fourier coefficients of $1/r$, i.e., by $p(m)$ with $m = 1, 2, 3, \ldots$. Hence, the problem is not asymptotic in nature.

If $\omega \neq 0$, then the description of the resonances is determined by the values of $S_q(p, m)$ for $m = 1, 2, \ldots q - 1$ (the map $m \to S_q(p, m)$ is $q$-periodic). The $S_q(p, m)$ are cubic complete rational exponential sums, see, e.g., [7]. One easily checks

**Lemma 1.** For any $q \in \mathbb{N}$, $\sum_{m=0}^{q-1} |S_q(p, m)|^2 = q^2$. 

This implies that, for $q \geq 1$, there is at least one integer $0 \leq m_0 < q - 1$ such that $S_q(p, m_0) \neq 0$.

If $S_q(p, m)$ is non zero for only one $0 \leq m < q$ (this happens, for example, for $q = 2, 3, 6$), then one can characterize the resonances as when $\omega = 0$. Now, they live near the lines $\{\text{Re} E/\epsilon = m_0/q + 1/2 + n\}$, $n \in \mathbb{Z}$.

For large $q$, there are actually many non-zero values $S_q(p, m)$:

**Lemma 2.** There exists a constant $C > 0$ such that, for any co-prime $q > p > 0$, one has $\#\{0 \leq m < q : S_q(p, m) \neq 0\} \geq Cq^{2/3}$.

This statement follows from Lemma 1 and the well-known upper bound for general complete rational exponential sums of Hua ([7]).

In general, the behavior of $m \mapsto S_q(p, m)$ is nontrivial; it is known to depend strongly on the prime factorization of $q$. Computer calculations lead to the following conjecture: if $q$ is prime, $0 < p < q$, and $0 < m < q$, then $S_q(p, m) \neq 0$.

If $S_q(p, m)$ is non zero for at least two values of $m$ such that $0 \leq m < q$, then, using (1.5), one can describe asymptotically all the resonances with sufficiently negative imaginary part. One has

**Corollary 2.** Assume that, for some integers $m_1 < m_2$ such that $m_2 - m_1 < q$, one has $S_q(p, m_1) \neq 0$, $S_q(p, m_2) \neq 0$, and $S_q(p, m) = 0$ for all $m_1 < m < m_2$. Then, for sufficiently large $y > 0$, in the vertical half-strip

$$\left\{ E \in \mathbb{C} : -\text{Im} E \geq \epsilon y, \quad \frac{m_1}{q} \leq \frac{\text{Re} E}{\epsilon} \leq \frac{m_2}{q} \right\},$$

there are resonances, and they are described by the asymptotic formulas:

$$E/\epsilon = -i \left( \frac{\ln(\pi k)}{\pi} - \ln \sin \frac{\pi (m_2 - m_1)}{q} \right) + \frac{m_2 + m_1}{q} + o(1), \quad k \in \mathbb{N},$$

where $o(1) \to 0$.

This statement easily follows from Theorem 2.

Finally, let us describe very briefly the ideas leading to Theorems 1 and 2. Buslaev’s solutions $\psi_{\pm}$ used to define the reflection coefficient (see (1.3)) are entire functions of $x$ and $E$; they satisfy the relations $\psi_{\pm}(x + 1, E) = \psi_{\pm}(x, E + \epsilon)$. It appears that the analytic properties of such solutions can naturally be described in terms of a system of two first order difference equations on the complex plane (see, for example, [5]). To get the asymptotics of the Fourier coefficients of the reflection coefficient, we study the solutions of this system far from the origin. The idea leading from Theorem 1 to Theorem 2 is analogous to one used to study the behavior of the exponential sums $\sum_{n=1}^{N} e^{-2\pi i \omega n^3}$ with $\omega \in \mathbb{Q}$ for large $N$, see [4]. However, to use it successfully, one has to carry out a non trivial analysis of properties of the error term in (1.5).
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(Alexander Fedotov) St. Petersburg State University, 7/9 Universitetskaya nab., St.Petersburg, 199034, Russia
E-mail address: a.fedotov@spbu.ru

(Frédéric Klopp)
Sorbonne Universités, UPMC Univ. Paris 06, UMR 7586, IMJ-PRG, F-75005, Paris, France
Univ. Paris Diderot, Sorbonne Paris Cité, UMR 7586, IMJ-PRG, F-75205 Paris, France
CNRS, UMR 7586, IMJ-PRG, F-75005, Paris, France
E-mail address: frederic.klopp@imj-prg.fr