Self-Dual Supergravity in Seven Dimensions with Reduced Holonomy $G_2$

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Abstract

We present self-dual $N = 2$ supergravity in superspace for Euclidean seven dimensions with the reduced holonomy $G_2 \subset SO(7)$, including all higher-order terms. As its foundation, we first establish $N = 2$ supergravity without self-duality in Euclidean seven dimensions. We next show how the generalized self-duality in terms of octonion structure constants can be consistently imposed on the superspace constraints. We found two self-dual $N = 2$ supergravity theories possible in 7D, depending on the representations of the two spinor charges of $N = 2$. The first formulation has both of the two spinor charges in the 7 of $G_2$ with 24 + 24 on-shell degrees of freedom. The second formulation has both charges in the 1 of $G_2$ with 16 + 16 on-shell degrees of freedom.

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1. Introduction

It has been well-known that M-theory [1] can produce realistic four-dimensional (4D) theory with chiral fermions upon a particular compactification with extra seven dimensions (7D) with the reduced holonomy $G_2$ instead of the maximal one $SO(7)$. The reduced holonomy $G_2$ is a special case of a series of reduced holonomies, such as 8D with $Spin(7)$ holonomy, or $G_2$, $SU(3)$ and $SU(2)$ holonomies in 7D, 6D and 4D [5][2][3][4]. In the cases of $Spin(7)$ and $G_2$ holonomies, so-called octonion structure constants play a crucial role [6]. This is because for these reduced holonomies, generalized self-duality conditions can be dictated by octonion structure constants [6] that were not present in the case of self-dual supergravities in 4D [7][8]. In particular, the 7D manifold both with $G_2$ holonomy and generalized self-duality [3][4][5] in the compactification of 11D supergravity is compatible with local supersymmetry, as confirmed by Killing spinors as the singlets of $G_2$ [3][4].

These developments indicate the importance of constructing self-dual supergravity theories with reduced holonomies on these manifolds, as the next natural step to take. Actually, in our previous paper [9], we have carried out such a construction of self-dual supergravity in 8D with reduced holonomy $Spin(7) \subset SO(8)$. We have found that self-dual supergravity in 8D has differences from, as well as similarities to self-dual supergravity in 4D [7][8]. The most fundamental difference is the involvement of octonion structure constants [6] making the whole computation non-trivial.

Recently similar but different formulations have been presented, such as using BRST or topological quantum field symmetry as the guiding principle for constructing self-dual supergravity in 8D or 7D with lower-order terms [10]. However, there seems to be no complete self-dual supergravity formulation in 7D with desirable reduced holonomy $G_2$ [5][6][3] before quantization, with all the higher-order interactions in a closed form. For example, in the topological quantum field formulation of self-dual supergravity in [10], only lower-order terms are compared with supergravity theory, due to the complication at fermionic quartic terms. Moreover, topological formulations [10] rely on the BRST symmetry at the quantized level after gauge-fixings, and as such, they are not classically gauge-invariant.

In our present paper, we will formulate complete $N = 2$ self-dual supergravity in Euclidean 7D with the reduced holonomy $G_2 \subset SO(7)$. Even though the existence of such a formulation has been conjectured for some time, the required computations for a complete theory are considerably non-trivial, similarly to the case of self-dual supergravity in 8D [9]. The most important objective of this paper is to complete the self-dual supergravity in 7D with the reduced holonomy $G_2$, including all the higher-order interaction terms in
superspace, in a self-contained and economical fashion.

As we have done in 8D [9], we adopt a very special set of constraints called ‘Beta-Function-Favored Constraints’ (BFFC). This set of constraints had been developed for drastically simplifying \( \beta \)-function computations for Green-Schwarz superstring in 10D [11]. For example, the whole \( \beta \)-function computation is reduced to the evaluation of only one single Feynman graph [11]. As in the corresponding case in 8D [9], we will see the power of BFFC for simplifying our computations, in particular, for the consistent supersymmetrization of self-duality conditions possible in 7D with the reduced holonomy \( G_2 \).

Based on this BFFC constraints, we first formulate self-dual supergravity with ‘restricted’ \( N = 2 \) supersymmetry with both spinor charges in the 7 of \( G_2 \). Such a formulation can be given in terms of extra constraints that are superspace generalization of a bosonic generalized self-duality condition \( R_{ab}{}^{cd} = (1/2)\phi^{cdef} R_{aCDF} \) for a Riemann tensor with the (dual) octonion structure constant \( \phi^{abcd} \). We next give an alternative self-dual supergravity with ‘restricted’ \( N = 2 \) supersymmetry with both spinor charges in the 1 of \( G_2 \). Due to the ‘nilpotent’ character of the spinor charges, the latter supergravity can be also regarded as topological gravity, related to the quantum field theories in [10].

2. \( N = 2 \) Supergravity in Euclidean 7D

Before imposing supersymmetric generalized self-duality conditions, we establish first \( N = 2 \) superspace supergravity in Euclidean 7D with the signature \((++\cdots+)\). This process is analogous to self-dual supergravity in 8D [9]. Namely, we use a particular set of constraints BFFC out of infinitely many possible sets of superspace constraints, linked by super-Weyl rescalings [12]. In other words, we establish a 7D analog of the BFFC in 10D [11] or 8D [9]. The BFFC constraints greatly simplifies the whole computation, such as many fermionic terms considerably simplified, or no dilaton in exponents [11][9].

The field content of our \( N = 2 \) supergravity multiplet is \((e_m{}^a, \psi_m{}^\alpha, C_{mn}, A_m{}^i, B_m, \chi_\alpha, \varphi)\) which is formally the same as \( N = 2 \) supergravity in Minkowskian 7D [13]. The component fields \( A_m{}^i \) \((i = 1, 2)\), \( B_m \) and \( C_{mn} \) have the field strengths \( F_{mn}{}^i \), \( G_{mn} \) and \( H_{mnr} \), respectively. Here we use the underlined spinorial indices \( \underline{\alpha}, \underline{\beta}, \ldots \) including \( N = 2 \) indices \( A, B, \ldots = 1, 2 \), so that \( \underline{\alpha} \equiv \alpha A, \underline{\beta} \equiv \beta B, \ldots \), where \( \alpha, \beta, \ldots = 1, 2, \ldots, 8 \) are for the 8 spinors of \( SO(7) \). These indices are also used for fermionic coordinates as usual in superspace [14], while the indices \( m, n, \ldots = 1, 2, \ldots, 7 \) are for curved bosonic coordinates, and \( a, b, \ldots = 1, 2, \ldots, 7 \) for local Lorentz bosonic coordinates. Even though superscript/subscript of these bosonic indices does not matter, we sometimes use them to elucidate their contractions. In the Clifford alge-
bra for Euclidean 7D, we have a symmetric charge conjugation matrix [15], which can be identified with an unit matrix: \( C_{\alpha\beta} = \delta_{\alpha\beta} \). Subsequently, the raising/lowering of spinor indices will not matter, even though we sometimes use their superscripts/subscripts, in order to elucidate contractions. Relevantly, we have the symmetry \( \langle \gamma_{[n]} \rangle_{\alpha\beta} = +(-1)^{n(n+1)/2}\langle \gamma_{[n]} \rangle_{\beta\alpha} \) [15], where the symbol \([n]\) implies the totally antisymmetric bosonic indices in order to save space: \( \gamma_{[n]} \equiv \gamma_{\alpha_1 \cdots \alpha_n} \).

In our superspace, there are superfield strengths \( F_{\alpha\beta} \), \( G_{AB} \), \( H_{ABC} \) together with the supertorsion \( T_{AB}^C \) and supercurvature \( R_{AB}^c \), which satisfy the Bianchi identities

\[
\begin{align*}
\frac{1}{2} \nabla_A [T_{BC}]^D - \frac{1}{2} T_{[AB]} E_{E(C)}^D - \frac{1}{4} F_{[AB]ef} (M_f e)_{(C)}^D & \equiv 0 \ , \\
\frac{1}{6} \nabla_A [H_{BCD}] - \frac{1}{4} T_{[AB]} E_{E(C)}^D - \frac{1}{4} F_{[AB]ef} F_{CD}^i + \frac{1}{4} G_{[AB]GCD} & \equiv 0 \ , \\
\frac{1}{2} \nabla_A [F_{BC}]^i - \frac{1}{2} T_{[AB]} D_{D(C)}^i & \equiv 0 \ , \\
\frac{1}{2} \nabla_A [G_{BC}] - \frac{1}{2} T_{[AB]} D_{D(C)}^i & \equiv 0 \ .
\end{align*}
\]

As has been mentioned, or as in analogous theory in 8D [9], we need to find a BFFC set of constraints. After trials and errors, we found the BFFC set in 7D to be

\[
\begin{align*}
T_{\alpha\beta}^c = + (\gamma c \tau_3)_{\alpha\beta} & \equiv (\gamma c)_{\alpha\beta} (\tau_3)_{AB} \ , \\
T_{\alpha\beta}^c = + i \delta_{\alpha\beta} \chi & - i (\gamma^a \tau_3)_{\alpha\beta} (\gamma_a \tau_3) \chi - i (\tau_3)_{\alpha\beta} (\tau_3) \chi & - i \delta_{\alpha\beta} \chi_{\beta} \ , \\
H_{\alpha\beta} = + \frac{1}{2} (\gamma \tau_3)_{\alpha\beta} & \ , \quad H_{\alpha} = 0 \ , \quad F_{\alpha} = 0 \ , \quad G_{\alpha} = 0 \ , \\
F_{\alpha\beta} = - i \sqrt{2} (\tau_3)_{\alpha\beta} & , \quad G_{\alpha} = - i \sqrt{2} \delta_{\alpha\beta} \ , \quad T_{\alpha}^c = 0 \ , \\
T_{\alpha} = - \frac{1}{2} (\gamma^c \tau_3)_{\alpha\beta} H_{bc} - \frac{i}{\sqrt{2}} (\gamma^c \tau_3)_{\alpha\beta} F_{bc} - \frac{i}{\sqrt{2}} (\gamma^c \tau_3)_{\alpha\beta} G_{bc} & , \\
\nabla_{\alpha} \chi = + \frac{i}{\sqrt{2}} (\gamma^c \tau_3)_{\alpha\beta} \nabla_{c} \varphi - \frac{i}{12} (\gamma [3] \tau_3)_{\alpha\beta} H[3] - \frac{i}{4 \sqrt{2}} (\gamma^c \tau_3)_{\alpha\beta} F_{cd} i & + \frac{i}{4 \sqrt{2}} (\gamma^c \tau_3)_{\alpha\beta} G_{cd} \\
& + \frac{i}{16} (\gamma^a \tau_3)_{\alpha\beta} \chi_a - \frac{i}{32} (\gamma^a \tau_3)_{\alpha\beta} \chi_{ab} & + \frac{i}{16} (\gamma^a \tau_3)_{\alpha\beta} \chi_{ai} \\
& - \frac{i}{32} (\gamma^a \tau_3)_{\alpha\beta} \chi_{abi} - \frac{i}{8} (\tau_3)_{\alpha\beta} \chi_{3} & + \frac{i}{192} (\gamma [3] \tau_3)_{\alpha\beta} \chi_{[3]3} \ , \\
\nabla_{\alpha} \varphi = - \frac{i}{\sqrt{2}} \chi_{\alpha} & , \\
T_{\alpha}^c = + 2 H_{\alpha}^c & , \\
R_{\alpha\beta} = + \sqrt{2} i (\tau_3)_{\alpha\beta} F_{cd} i - \sqrt{2} i \delta_{\alpha\beta} G_{cd} & ,
\end{align*}
\]

at the mass dimensions \( d \leq 1 \). Here \( \chi_{[n]} \equiv (\overline{\chi} \gamma_{[n]} \chi) \), \( \chi_3 \equiv (\overline{\chi} \tau_3 \chi) \), \( \chi_{[n]} \equiv (\overline{\chi} \gamma_{[n]} \tau_3 \chi) \), and the meaning of the underlined indices is, e.g., \( (\tau_3)_{\alpha\beta} = \delta_{\alpha\beta}(\tau_3)_{AB} \), \( \delta_{\alpha\beta} = \delta_{\alpha\beta} \delta_{\beta\alpha} \), while \( (\gamma_a \tau_3) \chi \equiv (\gamma_a \tau_3) \chi^{\alpha} \chi_{\beta} \), etc. The \( \tau_i \) (\( i = 1, 2 \)) and \( \tau_3 \) are
$2 \times 2$ matrices for the $N = 2$ indices:

$$
\tau_1 = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}.
$$

(2.3)

There are two important features in these constraints which are similar to the self-dual supergravity in 8D [9]. First, the fermionic components $H_{abc}, F_{ab}^i$ and $G_{ab}$ are absent, in contrast to any non-BFFC set, where they contain linear dilatino. Second, no exponential factor with the dilaton appears anywhere in our constraints as in 8D [9] or 10D [11]. With its technical details skipped, we mention that the most frequently-used relationship in these computations is the Fierz identity

$$
(\gamma_a \tau_3)_{(a\beta]} (\gamma^a \tau_3)_{\beta\gamma} + \delta_{(a\beta]} \delta_{\gamma\beta} - (\tau_i)_{(a\beta]} (\tau_i)_{\gamma\beta} \equiv 0.
$$

(2.4)

Other constraints at $d \geq 3/2$ in our BFFC are

$$
\nabla_\alpha F_{bc}^i = -\frac{i}{\sqrt{2}} (\tau^i T_{bc})_\alpha, 
$$

(2.5a)

$$
\nabla_\alpha G_{bc} = -\frac{i}{\sqrt{2}} T_{bec\alpha},
$$

(2.5b)

$$
\nabla_\alpha H_{bcd} = \frac{1}{4} (\gamma_{[b} \tau_3 T_{cd]})_\alpha,
$$

(2.5c)

$$
R_{\alpha \beta c} = -\frac{1}{4} (\gamma^c)^{(cd)} \nabla_\alpha \nabla_\beta (R_{cda} + 2 F_{ab}^i F_{cd}^i - 2 G_{ab} G_{cd})
$$

$$
- \epsilon_{ij} (\tau_3)^{(i)\beta} F_{ac}^i F_{bc}^j - (\tau_3)^{(i)\beta} F_{[a}^i F_{b]c}^i - \frac{1}{\sqrt{2}} (\gamma^c \tau_3 \tau_i)_{(a\beta]} \nabla_\alpha F_{cd}^i + \frac{1}{\sqrt{2}} (\gamma^c \tau_3 \tau_i)_{(a\beta]} \nabla_\alpha G_{cd}
$$

$$
+ i T_{ab\beta\gamma} \nabla^{(c} \tau_3 \tau_i)_{(a\beta]} \nabla_{c\alpha} F_{cd}^i + i T_{ab\beta\gamma} (\gamma^c \tau_3 \tau_i)_{(a\beta]} \nabla_{c\alpha} G_{cd}
$$

$$
+ i (\tau_i T_{ab\gamma}) (\tau_i \chi)_{(a\beta]} - T_{ab\beta\gamma} \nabla^{(c} \tau_3 \tau_i)_{(a\beta]} - i \delta_{(a\beta]} (T_{ab\gamma}) - i \delta_{(a\beta]} (T_{ab\gamma})
$$

(2.5d)

$$
R_{[ab]} = +2 \nabla_\alpha H_{a\beta c},
$$

(2.5e)

$$
R_{a[bcd]} = -4 \nabla_a H_{bcd} - \frac{1}{2} F_{[a}^i F_{cd]}^i + \frac{1}{2} G_{[a} G_{cd]}
$$

(2.5f)

$$
R_{abcd} - R_{cdab} = -2 \nabla_{[a} H_{bcd]} + 4 H_{ab}^c H_{ecd} + 4 H_{c[a}^c H_{e]d} + 2 F_{a}^i F_{cd}^i + 2 F_{[a}^i F_{d]b}^i - 2 G_{ab} G_{cd} - 2 G_{a[c} G_{d]b}.
$$

(2.5g)

These with Bianchi identities at $d = 3/2$ and $d = 2$ lead to the gravitino, graviton, and antisymmetric tensor superfield equations:

$$
(\gamma^b T_{ab})_{(a\beta]} - 2i (\tau_3 \nabla_a \chi)_{(a\beta]} + i (\gamma^b \tau_3 \chi)_{(a\beta]} H_{abc} + \sqrt{2} (\gamma^b \tau_3 \chi)_{(a\beta]} F_{ab}^i + \sqrt{2} (\gamma^b \chi)_{(a\beta]} G_{ab} \equiv 0.
$$

(2.6a)

$$
R_{ab} + 2 (F_{ac}^i F_{bc}^i - G_{ac} G_{bc}^i) + 2 \sqrt{2} \nabla_a \nabla_b \varphi \equiv 0.
$$

(2.6b)

$$
R_{[ab]} = 2 \nabla_a H_{ab}^c \equiv -4 \sqrt{2} H_{ab}^c \nabla_a \varphi + 2i (T_{ab\gamma})
$$

(2.6c)
where expressions such as \( (\gamma^b T_{cd})_\gamma \) involve the gravitino superfield strength \( T_{cd} \), e.g., \( (\gamma^b T_{cd})_\gamma \equiv (\gamma^b)_{\dot{\alpha}} T_{cd} \). The symbol \( \doteq \) stands for a superfield equation of motion. Reflecting the Euclidean nature of our 7D, the \( F^2 \)- and \( G^2 \)-terms in (2.6b) have opposite signs, similarly to the corresponding 8D case [9]. As usual in superspace [11][9], the superfield equations for dilatino and dilaton are obtained by the multiplication of (2.6a) by \( \gamma^\alpha \) and the trace of (2.6b), respectively.

There are several remarks in order. First, note that (2.5d) corresponds to the supersymmetry transformation of the Lorentz connection \( \phi_{bcd} \). In particular, the indices \( cd \) are on the gravitino superfield strength \( T_{cd} \), which is made possible by the particular choice of the bosonic supertorsion component (2.2h), as will be seen in (3.8) and (3.9). Second, similar feature is found in the component \( R_{\alpha \beta cd} \) (2.2i), where the pair of indices \( cd \) appears on the superfield strengths \( F_{cd} \) and \( G_{cd} \). Third, note the particular order of indices \( cdab \) on the Riemann supercurvature \( R_{cdab} \) in (2.4e). To reach this form, we made use of the identity (2.5h). Eq. (2.5h) is verified by (2.5g), while the latter is confirmed by the \( T \)-BI at \( d = 2 \).

The first equality in (2.6c) is the same as (2.5f). As in 8D [9], we need the last pair of indices \( ab \) in \( R_{cdab} \) free in (2.5e), instead of the first pair \( cd \), because we can not impose the self-duality on the first pair of indices of \( R_{cdab} \), but only on the last one, due to the presence of supertorsion \( T_{ab} \).

3. \( N = 2 \) Self-Dual Supergravity in 7D with \( G_2 \) Holonomy

We next present an \( N = 2 \) self-dual supergravity with the reduced holonomy \( G_2 \). The difference from the last section is that now the spinor charges form the 7 of \( G_2 \). Interestingly, we can accomplish both the supersymmetrization of the self-duality condition, and the reduction of the maximal holonomy \( SO(7) \) into \( G_2 \) in 7D.

As a first trial of finding a desirable set of superspace constraints, we can try some dimensional reduction from 8D [9] into 7D. However, we soon find that this will not work as smoothly as we first anticipated. In fact, we had a similar experience for the globally supersymmetric self-dual theories in 7D [16]. The main reason is that the reduced holonomy structure \( G_2 \) complicates such a dimensional reduction. As will be also seen shortly, certain differences in structure of supersymmetries in 7D compared with 8D [9], also cause such complications.

After direct trials and errors within 7D, we have found the following set as the right constraints in addition to the BFFC (2.2) before imposing self-duality:

\[
\nabla_\alpha \doteq N_{\dot{\alpha} \dot{\beta}} \nabla_\dot{\beta} \equiv (N \nabla)_\alpha \quad , \quad N \equiv \frac{7}{8} (I - \frac{1}{7} \Psi) \quad , \quad \Psi \equiv \frac{1}{4!} \phi^{[4]} \gamma^{[4]} \equiv \frac{1}{3!} \psi^{[3]} \gamma^{[3]} \quad , \quad (3.1a)
\]
\[ T_{a_b}^{(\gamma)} = N_{a_b}^{cd} \frac{\gamma}{2} = 0 \quad , \quad N_{a_b}^{cd} = \frac{1}{6} \left( \delta_a^c \delta_b^d - \phi_{a_b}^{cd} \right) \quad , \quad (3.1b) \]
\[ (P \chi)_a^{\hat{\gamma}} = 0 \quad , \quad P \equiv \frac{1}{8} \left( I + \Psi \right) \quad , \quad (3.1c) \]
\[ F_{a_b}^{(-)} = 0 \quad , \quad G_{a_b}^{(-)} = 0 \quad , \quad (3.1d) \]
\[ \nabla_a \varphi \equiv + \frac{1}{\sqrt{2}} \phi_a^b c d H_{b c d} \quad , \quad (3.1e) \]
\[ R_{abcd}^{(-)} \equiv N_{a_b}^{ef} R_{abef} \equiv 0 \quad , \quad (3.1f) \]

These constraints are extra associated with supersymmetric self-duality, in addition to (2.2).

To clarify that these are such extra constraints, we use the symbol \( \overset{*}{=} \). The convention for the octonionic structure constants \( \psi \) is like \( \psi_{123} = \psi_{516} = \psi_{624} = \psi_{435} = \psi_{471} = \psi_{673} = \psi_{572} = +1 \), \( \phi_{abcd} \equiv (1/3!) e_{abcd}^{efg} \psi_{efg} \quad [3][6] \). The matrix \( \mathcal{N} \) projects a spinor 8 into a 7 under \( SO(7) \rightarrow G_2 \). This is complementary to the matrix \( \mathcal{P} \) projecting a 8 into a 1.

The anti-self-dual projector \( N_{a_b}^{cd} \) reduces an adjoint representation 21 into a 7 under \( SO(7) \rightarrow G_2 \). This is complementary to the self-dual component projector

\[ P_{a_b}^{cd} \equiv \frac{1}{3} \left( \delta_a^c \delta_b^d + \frac{1}{2} \phi_{a_b}^{cd} \right) \quad . \quad (3.2) \]

projecting a 21 into the adjoint representation 14 of \( G_2 \). The superscript \((\cdot)\) in (3.1) is the anti-self-dual component for the pair of indices \( a_b \), projected by the operator \( N_{a_b}^{cd} \). Even though we do not write explicitly, other constraints, such as \( R_{abcd}^{(-)} \equiv N_{a_b}^{ef} R_{abef} \overset{*}{=} 0 \) or \( R_{a\beta cd}^{(-)} \overset{*}{=} 0 \) will follow, as the necessary conditions of (3.1).

The consistency check of these constraints is performed by applying a spinorial derivative \( \nabla_a \) on each of the extra constraints (3.1b) through (3.1f), with the aid of identities, such as

\[ \psi_{abc} \phi_{a_b}^{cd} \equiv -4 \psi_c^{a_b} \quad [6] \].

First, the spinorial derivative acting on (3.1b) is shown to vanish:

\[ 0 \overset{\nabla_a}{=} \nabla_a T_{a_b}^{(-)} \overset{\gamma}{=} - \epsilon_{i j} N_{a_b}^{cd} (\mathcal{N} \tau_3) \frac{\delta}{\delta g} F_{c e}^i F_d^j - 2 N_{a_b}^{cd} (\mathcal{N} \tau_5) \frac{\delta}{\delta e} G_d^c \overset{\gamma}{=} 0 \quad , \quad (3.3) \]
due to the identity \([6][3]\)

\[ N_{a_b}^{cd} P_{c e}^f P_d^h k \overset{\gamma}{=} 0 \quad . \quad (3.4) \]

Second, the case of (3.1c) is also straightforward, owing to the useful identities:

\[ (\mathcal{N} \gamma_a \mathcal{P})_{a_b} \equiv + \delta_{aa} \delta_{b_8} \quad , \quad (\mathcal{N} \gamma_a \mathcal{P})_{a_b} \equiv + \psi_{a_b} \delta_{b_8} \quad , \quad (3.5) \]

\[ (\mathcal{N} \gamma_{a_b})_{a_{\beta}} \equiv + i \delta_{aa} \delta_{b_8} \quad , \quad (\mathcal{N} \gamma_{a_b})_{a_{\beta}} \equiv + \psi_{a_{b_{\beta}}} \delta_{b_8} \quad , \quad (3.5) \]

\[ (\mathcal{N} \gamma_{a_{\beta}})_{a_{\beta}} \equiv + i \psi_{a_{b_{\beta}}} \quad , \quad (\mathcal{N} \gamma_{a_{\beta}})_{a_{\beta}} \equiv + \phi_{a_{b_{\beta}}} + \delta_{a_{b_{\beta}}} \quad , \quad (3.5) \]

\[ (\mathcal{N} \gamma_{a_{b_{\beta}}})_{a_{\beta}} \equiv - \frac{1}{2} \delta_{[a_{b_{\beta}}} \psi_{c_b]} + \frac{1}{2} \psi_{a_{b_{\beta}}} \delta_{[a_{b_{\beta}}} \psi_{c_b]} + i \delta_{a_{b_{\beta}}} \delta_{b_8} \psi_{a_{b_{\beta}}} - \delta_{a_{b_{\beta}}} \phi_{a_{b_{\beta}}} \quad , \quad (3.5) \]

\[ (\mathcal{N} \gamma_{a_{b_{\beta}}})_{a_{\beta}} \equiv \psi_{a_{b_{\beta}}} \delta_{b_8} \quad , \quad P_{c e}^f \psi_{e_{f_g}} \equiv 0 \quad , \quad \psi_{a_{b_{\beta}}} \phi_{a_{b_{\beta}}} \equiv -4 \psi_c^{a_{b_{\beta}}} \quad . \quad (3.5) \]

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Third, the case of (3.1d) is very simple, due to the self-duality $T^{(-)}_{ab} \equiv 0$. Fourth, the spinorial derivative of (3.1e) needs a special care, because it needs the gravitino field equation (2.6a):

$$0 \equiv \nabla_\alpha \left( \nabla_a \varphi - \frac{1}{6\sqrt{2}} \phi_a^\alpha \phi_{bcd} H_{bcd} \right) \equiv - \frac{1}{2\sqrt{2}} \tau_3 \left[ + \gamma^b T_{ab} - 2i\tau_3 \nabla_a \chi + i(\gamma^{bc} \tau_3 \chi) H_{abc} + \sqrt{2}(\gamma^b \tau^i \chi) F_{ab} + \sqrt{2}(\gamma^b \chi) G_{ab} \right]_\alpha \equiv 0 \ . \quad (3.6)$$

Finally, the case of (3.1f) is

$$0 \equiv \nabla_\alpha R^{(-)}_{bcde} \equiv + \nabla_{[b} R^{(-)}_{\alpha [c]de} + T_{\alpha [b [f} R^{(-)}_{f |c]de} + T_{[c} \nabla_{[b] R^{(-)}_{\alpha f]de} + T]_b \nabla_{[c] R^{(-)}_{\alpha fde]} \equiv 0 \ . \quad (3.7)$$

Here use is made of the $R$-Bianchi identity starting with $\nabla_\alpha R_{bcde} + \cdots \equiv 0$, as well as other facts, such as $R^{(-)}_{bcde} \equiv 0$ and $R^{(-)}_{bedc} \equiv 0$.

The on-shell degrees of freedom in this system are matched under supersymmetry in the following way: First, each of the three vectors $A_m^i$ and $B_m$ have only three degrees of freedom, so in total $3 \times 3 = 9$ degrees of freedom, due to their self-dualities (3.1d). The siebenbein $e_m^a$ has $(3 \times 4)/2 - 1 = 5$ degrees of freedom, where 3 is like that for the index $m$ of a self-dual vector $B_m$, and $(3 \times 4)/2 = 6$ is for its symmetry, while the subtraction of unity is due to the traceless-ness. The relationship (3.1e) says simply that one degree of freedom by $\varphi$ is completely determined by the field strength $H_{abc}$. Namely, out of the sum $(5 \times 4)/2 + 1 = 11$ of freedoms of $\varphi$ and $C_{mn}^{rr}$, only $11 - 1 = 10$ degrees of freedom remain. Therefore, all the bosonic fields have $9 + 5 + 10 = 24$ degrees of freedom. Now due to the self-duality (3.1b), the gravitino has only $(4 \times 4 \times 2)/2 = 16$ degrees of freedom, where the number $2^{[7/2]-1} = 4$ is for a Majorana spinor in 7D, the additional doubling is due to $N = 2$ indices, and $7 - 3 = 4$ is for the index $m$ for $\psi_m$, while the division by 2 is due to the self-duality (3.1b). Similarly, $\chi$ has $4 \times 2 = 8$ degrees of freedom. In total, there are $16 + 8 = 24$ degrees of freedom, matching those for the bosonic fields, as $24 + 24$.

As has been mentioned, the absence of the $(-)$ components implies the absence of the 7 in $21 = 14 + 7$ under $SO(7) \rightarrow G_2$, namely, these indices are reduced into the adjoint representation 14 of the reduced holonomy $G_2$. This is also reflected into the supersymmetry transformation of the Lorentz spinor connection $\omega_{bcde}$ constructed [14] from (2.5d) as

$$\delta_Q \omega_{bcde} = + i(\tau_3 \tau_3 T_{cd}) \ , \quad (3.8)$$

agreeing with the self-duality of the last two indices on $\omega_{bcde}$:

$$\omega_{bcde} \equiv + \frac{1}{2} \phi_{ef}^\alpha \omega_{bcde} \ , \quad (3.9)$$
thanks to the self-duality $T_{cd}^\delta = + (1/2) \phi_{e}^{ef} T_{ef}^\delta$. As in the corresponding case in 8D [9],
our BFFC has been chosen to be compatible with such a requirement. Exactly as in 8D [9],
any other constraint set away from the BFFC will cause some problem in the component
transformation rule $\delta Q \omega_{abc}$, not consistent with the self-duality in the last two indices. Since
this aspect is just parallel to the 8D case [9], we skip the details here.

The gravitational superfield equation (2.6b) is also consistent with the self-duality of the
Riemann supercurvature. To be more specific, we have [9]

$$R_{ac} = + \delta^{bd} R_{abcd} + \frac{1}{2} \phi_{e}^{def} R_{a[def]} = + \frac{1}{12} \phi_{e}^{def} R_{a[def]}$$

$$= + \frac{1}{12} \phi_{e}^{def} \left( - 4 \nabla_{a} H_{def} - \frac{1}{2} F_{[ad} i F_{ef]} i + \frac{1}{2} G_{[ad} G_{ef]} \right) \equiv - 2 \sqrt{2} \nabla_{a} \nabla b \phi . \ (3.10)$$

As in the analogous case in 8D [9], this is a modification of the usual Ricci flatness condition
in the torsion-full space with the $G_2$ holonomy [3].

Compared with self-dual supergravity in 8D with reduced holonomy $Spin(7)$ [9], there
are similarities and differences. The similarities are such as the self-dual Riemann tensor, or
the maximal holonomy $SO(7)$ reduced to $G_2$, as our first desirable goals. Other technical
similarities are such as the Ricci tensor with the second derivative of the dilaton as in (3.10),
with a structure similar to 8D [9]. One difference is that the asymmetry based on chirality
played an important role for the supersymmetric self-duality in 8D. For example, in 8D the
spinor charge with positive chirality $\nabla_{\alpha}$ was constrained to $1$ as $8 \rightarrow 7 + 1$ under
$SO(8) \rightarrow Spin(7)$, while the gravitino for negative chirality was truncated $T_{ab}^\gamma \neq 0$ [9].
In this sense, the self-dual supergravity had $N = (1,0)$ supersymmetry in 8D. In 7D, we
do not have such ‘asymmetry’ depending on the spinor charges $\nabla_{\alpha 1}$ and $\nabla_{\alpha 2}$, but the condition $\nabla_{\alpha}^* = (\mathcal{P} \nabla)^{\alpha}$ is common to both charges. In 7D, since both spinor charges are
present, and in that sense we still have $N = 2$ supersymmetry, but it is a ‘restricted’ one.

4. Topological $N = 2$ Self-Dual Supergravity in 7D with Reduced Holonomy $G_2$

As we have promised in the Introduction, we next present an alternative ‘topological’
self-dual supergravity in 7D. The difference from the last section is that both of the spinor
charges are now in the $1$ of $G_2$, instead of $7$.

After trials and errors, we found such a set of constraints for supersymmetric generalized
self-duality, as

$$\nabla_{\underline{\alpha}} = (\mathcal{P} \nabla)^{\underline{\alpha}} , \quad (4.1a)$$

$$T_{ab}^{(-)\gamma} = N_{ab}^{cd} T_{cd}^{\gamma} = 0 , \quad (4.1b)$$
\[
\chi_\alpha \overset{*}{=} 0 \ , \\
F_{ab}^{(-)} \overset{*}{=} 0 \ , \quad G_{ab}^{(-)} \overset{*}{=} 0 \ , \\
\varphi \overset{*}{=} 0 \ , \\
\phi_{a}^{	ext{bcd}} H_{bcd}^{	ext{*}} \overset{*}{=} 0 \ , \quad \psi_{abc}^{	ext{*}} H_{abc} \overset{*}{=} 0 \ , \\
R_{abcd}^{(-)} \equiv N_{cd}^{ef} R_{abef} \overset{*}{=} 0 \ ,
\]

(4.1c) - (4.1g)

These constraints are extra, in addition to the original BFFC set (2.2). In eq. (4.1g), the anti-self-duality symbol refers only to the last pair \( cd \), but \textit{not} the first one.

The consistency of this set of constraints for supersymmetric generalized self-duality can be confirmed, by applying fermionic derivatives on the constraints (4.1b) - (4.1g). After applying these derivatives, we can use these constraints again, in order to see whether they vanish consistently. A typical example is on (4.1c):

\[
0 \overset{?}{=} \nabla_{\alpha_{1}} \chi_{\beta_{1}} \overset{*}{=}, (\mathcal{P} \nabla)_{\alpha_{1}} \chi_{\beta_{1}} \\
\overset{*}{=} - \frac{i}{4\sqrt{2}} (\mathcal{P} \gamma_{cd}^{(-)})_{\alpha_{1}} \beta \left( F_{cd}^{2} - G_{cd} \right) + i \frac{1}{8} (\mathcal{P} \gamma^{c})_{\alpha_{1}} \beta \left( \nabla_{1} \gamma_{c} \chi_{1} \right) - \frac{i}{16} (\mathcal{P} \gamma^{cd})_{\alpha_{1}} \beta \left( \nabla_{1} \gamma_{cd} \chi_{1} \right) \overset{*}{=} 0 \ ,
\]

(4.2)

where we have used (4.1c), (4.1d) and the identity \( \mathcal{P} \gamma^{cd} \equiv \mathcal{P} \gamma^{cd}_{(-)} \). The subtlety arises with the fermionic derivative on the constraint (4.1b), because we need a peculiar lemma (3.4) for the \( FG \)-term \( N_{ab}^{cd} F_{ce}^{(+)} G_{de}^{(+)} \). As for the constraint (4.1g), we use the \( R \)-Bianchi identity \( \nabla_{\underline{a}} R_{bcde} + \cdots \equiv 0 \), as in (3.7).

Since our fermionic derivatives \( \nabla_{\underline{a}} \) have only the singlet component \( 1 \) in \( 8 = 7 + 1 \) under \( SO(7) \rightarrow G_{2} \), the usual commutator \( \{ \nabla_{\underline{a}}, \nabla_{\underline{b}} \} \) vanishes, as the identity \( \mathcal{P} \gamma^{c} \mathcal{P} \equiv 0 \) also shows. This implies that this system of self-dual supergravity is ‘topological’, like nilpotent BRST symmetry. As a matter of fact, this is also consistent with the result in topological quantum field theory of self-dual (super)gravity [10]. The advantage of our formulation, however, is the usage of BFFC that drastically simplified the whole computation, compared with component formulation [10], where higher-order terms are considerably involved. We also see that the non-self-dual supergravity can be recovered by releasing the condition \( \nabla_{\underline{a}} = (\mathcal{P} \nabla)_{\underline{a}} \), much like the link between the BRST formulation and supergravity discussed in [10].

Due to \( R_{abcd}^{(-)} \overset{*}{=} 0 \), the self-duality \( \omega_{bcd}^{(-)} \overset{*}{=} 0 \) is also consistent with supersymmetry in this system, as in (3.8). Similarly, the Ricci-flatness of this system resembles (3.10):

\[
R_{ac} = \delta^{bd} R_{abcd} \overset{*}{=} + \frac{1}{2} \phi_{c}^{	ext{def}} R_{a[def]} = + \frac{1}{12} \phi_{c}^{	ext{def}} R_{a[def]} \\
= + \frac{1}{12} \phi_{c}^{	ext{def}} \left( -4 \nabla_{a} H_{def} - \frac{1}{2} F_{[ad}^{i} F_{ef]}^{+} + \frac{1}{2} G_{[ad} G_{ef]} \right) \overset{*}{=} 0 \ .
\]

(4.3)
Compared with (3.10), or with the analogous case in 8D [9], the Ricci-flatness of this system with the $G_2$ holonomy [3] is not modified. These results imply that the manifold we are dealing with in this formulation has definitely non-trivial $G_2$ holonomy, but supersymmetry is ‘topological’ [10], instead of the usual one generating translations.

The on-shell degrees of freedom are matched under supersymmetry as follows. First, the three self-dual vectors $A_m^i$ and $B_m$ have $3 \times 3 = 9$ degrees of freedom. Next, $C_{mn}$ has $(5 \times 4)/2 - 1 = 2$ degrees of freedom due to the $7 + 1$ conditions (4.1f). The siebenbein has $(3 \times 4)/2 - 1 = 5$ degrees of freedom as in the last section, while the dilaton has no freedom. The gravitino has $(4 \times 8)/2 = 16$ degrees of freedom, due to the self-duality (4.1b). Finally, $\chi_\alpha$ has zero degree of freedom. In total, the bosons and fermions have the same 16 degrees of freedom, as $16 + 16$ in this system.

Compared with the first formulation in the last section, there are similarities and differences. The similarities are such as the self-dual Riemann tensor and the reduced holonomy $G_2 \subset SO(7)$. The difference is that the dilaton and dilatino superfields are constrained to vanish, while the superfield strength $H_{abc}$ is subject to the constraint (4.1f) without the dilaton. Moreover, the spinorial derivative $\nabla_\alpha$ is subject to (4.1a). This is consistent with the Killing spinor condition for the singlet spinor $1$ under the reduction of the holonomy $SO(7) \to G_2$ for the compactification of M-theory into 4D with chiral fermions [3][4]. It is this fundamental aspect that is reflected into the conditions, such as the vanishing of the dilatino (4.1c) and dilaton (4.1e), or the condition on $H_{bcd}$ (4.1f) without $\varphi$. In other words, this self-duality in 7D is more restrictive than the corresponding 8D case or the first formulation in the last section, with spinor charges realized only as singlets under the holonomy $G_2$.

We mention the possible ‘twisted’ version of $N = 2$ supersymmetries, i.e., the hybrid of two supergravity theories in this paper, such as the twisting $\nabla_{\alpha_1} \equiv (\mathcal{P}\nabla)_{\alpha_1}$ and $\nabla_{\alpha_2} \equiv (\mathcal{N}\nabla)_{\alpha_2}$. We have tried to formulate this version, so far with no success. There are several technical obstructions according to our trials. First, we can no longer put both $\chi_{\alpha_1}$ and $\chi_{\alpha_2}$ to zero, because of non-vanishing term $\psi_{(\alpha}^d H_{\beta)de}$ arising in $\nabla_{\alpha_2} \chi_{\beta_1}$. This is also related to the non-vanishing of the combination $(\mathcal{N}\gamma_{cde})_{\alpha_3} H_{cde}$. Second, even if we allow only $\chi_{\alpha_2}$ to be zero, the condition $\nabla_a \varphi \equiv - (1/6\sqrt{2}) \phi_a^{bcd} H_{bcd}$ is not consistent with its spinorial derivative, which requires the opposite sign between these two terms. There does not seem to be any way out to avoid these problems. Even though the result for global supersymmetry [16] indicate more naturally a twisted $N = 2$ supergravity in 7D, we do

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3The non-vanishing torsion $T_{ab}^c \equiv +2H_{ab}^c$ does not eventually affect the Ricci-flatness in this system, due to $\varphi \equiv 0$. 

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not know any conceptual reason for this obstruction at the present time.

We repeat that our formulation is based on the combination of the peculiar feature of the octonionic structure constants $\psi_{abc}$ leading to generalized self-duality, and the usage of BFFC constraints, all closely related to each other consistently in superspace.

5. Concluding Remarks

In this paper, we have presented two formulations of self-dual supergravity in Euclidean 7D with the reduced holonomy $G_2$. The first formulation has both of the $N = 2$ spinor charges as the 7 of $G_2$, while the second one has both spinor charges as in 1 of $G_2$. The second formulation is more closely related to compactifications of 11D M-theory into 4D, with singlet Killing spinors [2][3][4]. These formulations have certain differences from as well as similarities to the corresponding self-dual supergravity in 8D [9].

The similarities are such as the crucial role played by octonion structure constants, the importance of the special set BFFC for superspace constraints, or the self-dual Riemann tensor with the reduced holonomy $G_2 \subset SO(7)$. Or the simplification by superspace formulation itself is already the common feature in these dimensions, because the component formulations [10] will get more involved for fixing higher-order terms involving fermions. The most important difference is that the surviving supersymmetries invariant under the reduced holonomy $G_2$ required from compactifications [3][4], impose rather strong conditions on the fermionic derivatives, such as $\nabla_{\sigma} \hat* = (P \nabla)_{\sigma}$ in the second formulation.

The analysis of globally supersymmetric self-dual theories in diverse dimensions [17] indicates that $N = 2$ supersymmetry is minimally needed in 7D. Our result in this paper is also consistent with this general conclusion that at least $N = 2$ supersymmetry is needed to maintain the desired self-duality. Our recent component formulation for self-dual supersymmetric Yang-Mills theory in 7D [16] is also consistent with this conclusion.

Another interesting feature of self-dual supergravity in 7D we have found is as follows: In our previous paper [9], we have mentioned the possible usage of dimensional reduction of self-dual supergravity in 8D into lower dimensions, including 7D. However, there is one caveat for this statement. As we have seen in this paper, such a dimensional reduction does not work in a simple way, but needs a special care. This is because of the different aspects of even vs. odd dimensions associated with fermions [17]. For example, we saw in [9] that the $N = (1,0)$ chiral supersymmetry was crucial for the compatibility between self-duality and supersymmetry in 8D, while extended $N = 2$ supersymmetry is minimally required for self-duality in 7D. Due to such different features in 8D and 7D, a simple dimensional
reduction does not work smoothly for getting self-dual supergravity in 7D. As a matter of fact, we have seen a similar situation already in global supersymmetry between 8D and 7D [16][17]. In fact, when we need supersymmetric self-duality in 7D, a naïve simple dimensional reduction from 8D does not give a clue for the necessity of $N = 2$ supersymmetry in 7D.

To our knowledge, our present paper gives the first complete formulations of self-dual supergravity in 7D including all higher-order terms, before quantization. We have presented two different versions of $N = 2$ self-dual supergravity, which will be of great importance for future studies of such self-dualities in 7D, in particular, explicitly formulated in superspace. Our second superspace formulation for topological supergravity in 7D also gives an important ‘bridge’ between topological (super)gravities [10] and conventional superspace formulation which was originally constructed for the conventional local supersymmetry generating translation. As a by-product, we have found it possible to formulate the $N = 2$ self-dual supergravity in 7D, with both spinor charges in the $7$ of $G_2$, that was not predicted by M-theory compactifications [1][3][4].

In our supergravity formulations, the maximal holonomy $SO(7)$ in 7D is reduced into $G_2$ consistently with local supersymmetry. The compatibility of supergravity with reduced holonomies is analogous to certain supergravity formulations with no manifest Lorentz covariances in higher dimensions $D \geq 12$ [18]. To put it differently, there is accumulating evidence that reduced or non-manifest holonomies become more and more important in higher-dimensional supergravity theories in $D \geq 4$.

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