Analysis of error dependencies on NewHope

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Abstract. Among many submissions to the NIST post-quantum cryptography (PQC) project, NewHope is a promising key encapsulation mechanism (KEM) based on the Ring-Learning with errors (Ring-LWE) problem. Since NewHope is an indistinguishability (IND)-chosen ciphertext attack secure KEM by applying the Fujisaki-Okamoto transform to an IND-chosen plaintext attack secure public key encryption, accurate calculation of decryption failure rate (DFR) is required to guarantee resilience against attacks that exploit decryption failures. However, the current upper bound of DFR on NewHope is rather loose because the compression noise, the effect of encoding/decoding of NewHope, and the approximation effect of centered binomial distribution are not fully considered. Furthermore, since NewHope is a Ring-LWE based cryptosystem, there is a problem of error dependency among error coefficients, which makes accurate DFR calculation difficult. In this paper, we derive much tighter upper bound on DFR than the current upper bound using constraint relaxation and union bound. Especially, the above-mentioned factors are all considered in derivation of new upper bound and the centered binomial distribution is not approximated to subgaussian distribution. In addition, since the error dependency is considered, the new upper bound is much closer to the real DFR than the previous upper bound. Furthermore, the new upper bound is parameterized by using Chernoff-Cramer bound in order to facilitate calculation of new upper bound for the parameters of NewHope. Since the new upper bound is much lower than the DFR requirement of PQC, this DFR margin is used to improve the security and bandwidth efficiency of NewHope. As a result, the security level of NewHope is improved by 7.2% or bandwidth efficiency is improved by 5.9%. This improvement in the security and bandwidth efficiency can be easily achieved because there is little change in time/space complexity of NewHope.

Keywords: Bandwidth Efficiency · Chernoff-Cramer Bound · Decryption Failure Rate · Error Dependency · NewHope · NIST · Post-Quantum Cryptography · Relaxation · Security · Union Bound · Upper Bound
1 Introduction

Current public-key algorithms based on integer decomposition, discrete logarithm, and elliptic curve discrete logarithm problems (e.g., RSA and elliptic curve cryptography) have been unlikely to be broken by currently available technology. However, with the advent of quantum computing technology such as Shor’s quantum algorithm for integer factorization, current public-key algorithms can be easily broken. For that reason, in order to avoid such security problem of future systems, new public-key algorithms called post-quantum cryptography (PQC) should be developed to replace the existing public-key algorithms. Therefore, the National Institute of Standards and Technology (NIST) has recently begun a PQC project to identify and evaluate post-quantum public-key algorithms secure against quantum computing [1]. Among the various PQC candidates, lattice-based cryptosystems have become one of the most promising candidate algorithms for post-quantum key exchange. Lattice-based cryptosystems have been developed based on worst-case assumptions about lattice problems that are believed to be resistant to quantum computing.

Among various lattice problems, learning with errors (LWE) problem introduced by Regev in 2005 [2] has been widely analyzed and used. Furthermore, the Ring-LWE problem presented by Lysyanskaya, Peikert, and Regev in 2010 [3], which improves the computational and implementation efficiency of LWE, has also been widely used [4], [5], [6], [7], [8]. NewHope has been proposed by Alkim, Ducas, Pöppelmann, and Schwabe [9], [10], which is one of the various cryptosystems based on Ring-LWE. NewHope has attracted a lot of attention [11], [12], [13] and it was verified in an experiment of Google [14]. The key reasons that NewHope attracts so much attention are the use of simple and practical noise distribution, a centered binomial distribution, and a proper choice of ring parameters for better performance and security.

NewHope is an indistinguishability (IND)-chosen ciphertext attack (CCA) secure key encapsulation mechanism (KEM) that exchanges the shared secret key based on the IND-chosen plaintext attack (CPA) secure public-key encryption (PKE). Note that the IND-CPA secure PKE can be transformed into the IND-CCA secure KEM using Fujisaki-Okamoto (FO) transform [15]. The IND-CCA secure KEM obtained by applying FO transform to IND-CPA secure PKE requires a very low decryption failure rate (DFR) because an attacker can exploit the decryption failure [15], [16]. Therefore, the DFR in NewHope should be lower than $2^{-128}$ to make sure of resilience against attacks that exploit decryption failures. Note that as in Frodo [5] and Kyber [6], this study aims to achieve the DFR lower than $2^{-140}$ to allow enough margin in NewHope. In [1], [9], an upper bound on DFR of NewHope is derived but this upper bound on DFR is rather loose because the compression noise, the effect of encoding/decoding of NewHope, and approximation effect of centered binomial distribution are fully considered. Furthermore, according to [20], [21], accurate calculation of DFR is difficult because there is a problem of error dependency in Ring-LWE based cryptosystems. However, the DFR of IND-CCA secure KEM obtained by apply-
ing FO transform to IND-CPA secure PKE must be calculated as accurately as possible because DFR is closely related to the security.

In this paper, an upper bound on DFR, which is much closer to the real DFR than the previous upper bound on DFR derived in [4], [9], is derived by considering the above-ignored factors. Also, the centered binomial distribution is not approximated to the subgaussian distribution. Especially, the new upper bound on DFR considers the error dependency among error coefficients by using the constraint relaxation, which is an approximation of a difficult problem to a nearby problem that is easier to solve, and union bound. Furthermore, the new upper bound is parameterized by using Chernoff-Cramer (CC) bound in order to facilitate calculation of new upper bound for the parameters of NewHope. Since the new upper bound on DFR is much lower than the DFR requirement of PQC, this DFR margin is used to improve the security and bandwidth efficiency, which is reducing the ciphertext size.

Contributions The contributions of this paper is divided into three categories.

(1) Understanding NewHope as a Digital Communication System
NewHope can be understood as a digital communication system. Bob and Alice are transmitter and receiver, respectively, and the 256-bit shared secret key is a message bit stream. The difference between the encoding output \( v \) and the received signal \( v' \) distorted by many factors can be modeled as a digital communication channel. We analyze all the noise sources of this channel and numerically calculate the noise distribution of NewHope. Also, we analyze the encoding/decoding of Additive threshold encoding (ATE) in NewHope, which is an error-correcting code (ECC) for NewHope.

(2) DFR Analysis of NewHope By Considering Error Dependency
The previous upper bound on DFR of NewHope [4], [9] is loosely derived because the compression noise, effect of encoding/decoding of ATE in NewHope, effect of error dependency among error coefficients, and approximation effect of the centered binomial distribution are not fully considered. However, we derive a much closer upper bound on DFR to the real DFR than the previous upper bound on DFR by considering the above factors ignored in the derivation of previous upper bound [4], [9]. Also, the exact centered binomial distribution is used for deriving the upper bound on DFR without approximating it to the subgaussian distribution. As a result, a new upper bound on DFR is derived, which is less than \( 2^{-418} \) for \( n = 1024 \) and \( 2^{-399} \) for \( n = 512 \). Note that the previous upper bound on DFR is less than \( 2^{-216} \) for \( n = 1024 \) and \( 2^{-213} \) for \( n = 512 \).

(3) Improvement of Security and Bandwidth Efficiency of NewHope
By Using New DFR Margin Since the new upper bound on DFR is much lower than the required \( 2^{-128} \), this DFR margin can be exploited to improve the security level by 7.2% or bandwidth efficiency by 5.9% without changing the procedures of NewHope.
2 NewHope

2.1 Parameters

There are three important parameters in NewHope: \( n \), \( q \), and \( k \).

- \( n \): the dimension \( n = 512 \) or \( 1024 \) for NewHope guarantees the security properties of Ring-LWE and enables efficient number theoretic transform (NTT) [18].
- \( q \): the modulus \( q = 12289 \) is determined to support security and efficient NTT and it is closely related with the bandwidth.
- \( k \): the noise parameter \( k = 8 \) is the parameter of centered binomial distribution, which determines the noise strength and hence directly affects the security and DFR [4].

2.2 Notations

- \( \mathbb{R}_q = \mathbb{Z}_q[x]/(x^n + 1) \): the ring of integer polynomials modulo \( x^n + 1 \) where each coefficient is reduced modulo \( q \).
- \( a \xleftarrow{} \chi \): the sampling of \( a \in \mathbb{R}_q \) following the probability distribution \( \chi \) over \( \mathbb{R}_q \).
- \( \psi_k \): denote the centered binomial distribution with parameter \( k \), which is practically realized by \( \sum_{i=0}^{k-1} (b_i - b'_i) \), where \( b_i \) and \( b'_i \) are uniformly and independently sampled from \{0, 1\}. The variance of \( \psi_k \) is \( k/2 \) [4].
- \( a \circ b \): the coefficient-wise product of polynomials \( a \) and \( b \).

2.3 NewHope Protocol

NewHope is a lattice-based KEM for Alice (Server) and Bob (Client) to share 256-bit secret key with each other. The protocol of NewHope is briefly explained based on Fig. 1 as follows, where the functions are the same ones as defined in [4].

Step 1) \( \text{seed} \xleftarrow{} \{0, 1, \ldots, 255\}^{32} \) denotes a uniform sampling of 32 byte arrays (corresponding to 256 bits) with 32 integer elements selected between 0 and 255 by using a random number generator. Then \( SHAKE256(l, d) \), a strong hash function [19], takes an integer \( l \) that specifies the number of output bytes and a byte array \( d \) as its input. In NewHope, \( z \xleftarrow{} SHAKE256(64, \text{seed}) \) denotes that 32 byte arrays (\text{seed}) are hashed to generate 64 pseudorandom byte arrays (\( z \)) with 64 integer elements uniformly selected between 0 and 255. Then GenA expands 32 pseudorandom byte arrays \( z[0 : 31] \) using \( SHAKE128 \) hash function [19] to generate the polynomial \( \hat{a} \in \mathbb{R}_q \) where \( z[0 : 31] \) is the first 32 byte arrays of \( z \). Since \( \hat{a} \) is generated from the \text{seed} sampled following a uniform distribution, the coefficients of \( \hat{a} \) also follow a uniform distribution on \([0, q - 1]\).

Step 2) Generate polynomials \( (s, s', e, e', e'' \in \mathbb{R}_q) \) whose coefficients are sampled following the centered binomial distribution \( \psi_k \). The polynomials \( (s, s', e) \) are transformed to \( (\hat{s}, \hat{t}, \hat{e}) \), respectively, by applying NTT for efficient
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Fig. 1: NewHope Protocol.

| Step | Alice (Server) | Bob (Client) |
|------|----------------|--------------|
| 1    | $s$ seed $\sim [0,1,\ldots,255]^{22}$ | $s$ seed $\sim [0,1,\ldots,255]^{22}$ |
|      | $z \sim SHAKE256(\{x, seed\})$ | $z \sim SHAKE256(\{x, seed\})$ |
|      | $\hat{a} \sim \text{GenA}(z[0:31])$ | $\hat{a} \sim \text{GenA}(z[0:31])$ |
| 2    | $z, e \sim \mathbb{F}_{256}^e$ | $z', e', e'' \sim \mathbb{F}_{256}^e$ |
|      | $\hat{z} \sim \text{NTT}(z)$ | $\hat{i} \sim \text{NTT}(z')$ |
|      | $\hat{e} \sim \text{NTT}(e)$ | |
|      | $sk \sim \text{EncodePolynomial}(\hat{s})$ | |
| 3    | $\hat{b} \rightarrow \hat{a} \circ \hat{s} + \hat{e}$ | $pk = \text{EncodePK}(\hat{b}, z[0:31])$ |
|      | $\hat{a} \rightarrow \text{GenA}(z[0:31])$ | $(\hat{b}, z[0:31]) = \text{DecodePK}(pk)$ |
| 4    | $\mu \sim [0,1,\ldots,255]^{22}$ | $\hat{a} \rightarrow \text{Encode}(\hat{a})$ |
|      | $v \sim \text{Encode}(\mu)$ | |
| 5    | $(\hat{a}, h) \rightarrow \text{DecodeC}(\hat{c})$ | $\hat{a} \rightarrow \hat{a} + \text{NTT}(\hat{e'})$ |
|      | $\hat{z} \rightarrow \text{DecodePolynomial}(sk)$ | $v' \rightarrow \text{NTT}^{-1}(\hat{b} + \hat{e'}) + e'' + v$ |
|      | $v'_\text{decomp} \rightarrow \text{Decompress}(h)$ | |
|      | $v'' = v'_\text{decomp} - \text{NTT}^{-1}(\hat{b} + \hat{e})$ | |
| 6    | $\hat{c} \rightarrow \text{EncodeC}(\hat{c}, \hat{h})$ | $h \sim \text{Compress}(v'')$ |
|      | $\mu \rightarrow \text{Decode}(\hat{c})$ | |

polynomial multiplication. Then Alice transforms the secret key ($\hat{s}$) into byte arrays using $\text{EncodePolynomial}()$ which converts the polynomial ($\hat{s}$) into 2048 byte arrays.

Step 3) Alice creates a public key ($pk$) by converting $\hat{b} = \hat{a} \circ \hat{s} + \hat{e}$ and $z[0:31]$ into 1824 byte arrays by using $\text{EncodePK}()$, and transmits ($pk$) to Bob. Then Bob transforms the received public key ($pk$) into ($\hat{b}, z[0:31]$) using $\text{DecodePK}()$, and creates ($\hat{a}$) which is the same ($\hat{a}$) generated in Step 1.

Step 4) A 256-bit shared secret key ($\mu$) is created and encoded by ATE encoder to generate a 1024-symbol codewords $v$.

Step 5) Generate a ciphertext ($\hat{u}, v'$) by using the public key components $\hat{b}$, $\hat{a}$, the various errors $\hat{t}, \hat{e}', \hat{e}''$ and $v$.

Step 6) To efficiently reduce bandwidth, compression is performed on the coefficients of $v'$ to generate the polynomial $h$, and then the ciphertext polynomials ($\hat{u}, h$) are transformed into the byte arrays $c$ by using $\text{EncodeC}()$, and $c$ is transmitted to Alice. Alice performs decompression on $\hat{h}$ to restore $v'$. However, this decompressed polynomial $v'_{\text{decomp}}$ is different from $v'$ generated in Step 5, due to the loss from compression and decompression. Alice creates $v''$ by using the received ciphertext $c$ and $sk$ generated in Step 2. Each coefficient of $v''$ is a sum of the corresponding coefficients of $v$ and errors. Note that $v''$ is not a polynomial used in NewHope, but it is added in Fig. [1] for easy explanation of the results in this paper.
Step 7) The 256-bit shared secret key ($\mu$) is recovered (or decrypted) from the coefficients of $v''$ by performing the decoding of ATE.

3 Understanding NewHope as a Digital Communication System

3.1 NewHope as a Digital Communication System

In order to facilitate analysis of DFR of NewHope, it is much more convenient to understand the protocol of NewHope as a digital communication system. For NewHope, the mapping $\mathbb{Z}_{256}^2 \rightarrow \mathbb{Z}_n^2$ and the mapping $\mathbb{Z}_n^2 \rightarrow \mathbb{Z}_{256}^2$ through ATE, $n = 512$ or $n = 1024$, can be regarded as encoding and decoding of ECC, respectively. Also, the mapping $\mathbb{Z}_n^2 \rightarrow \mathbb{R}_q$ and $\mathbb{R}_q \rightarrow \mathbb{Z}_n^2$ through ATE can be regarded as modulation and demodulation, respectively. Then NewHope can be understood as a digital communication system as follows.

Bob and Alice are transmitter and receiver, respectively, and the 256-bit shared secret key ($\mu$) is a message bit stream. Also, the process of transmitting and receiving messages (Steps 4, 5, 6, and 7) can be viewed as a digital communication channel. In more detail, the transmitter (Bob) generates a 256-bit message bit stream, encodes this message into an $n$-bit codeword, modulates each codeword bit to a symbol of $\mathbb{Z}_q$, and transmits the resulting signal (Step 4). At the receiver (Alice), the received signal through the noisy channel is demodulated and decoded (Step 7). For NewHope, a process of adding the compression noise and the difference noise generated in Steps 5 and 6 can be regarded as noisy communication channel. This overall process in Steps 4-7 can be described as a digital communication system shown in Fig. 2.

In Fig. 2 $\mu_{enc}$ is the encoded signal of $\mu$ by applying encoding of ATE, and $n_t$ represents the overall noise generated in Steps 5 and 6, which is called the total noise $n_t$. After interpreting NewHope as a digital communication system, the DFR in NewHope is equivalent to the block error rate $Pr(\mu \neq \mu')$ in a digital communication system. Therefore, in order to calculate tight upper bound on DFR of NewHope, exact analysis of encoding/modulation and decod-
3.2 Analysis of Encoding/Modulation and Decoding/Demodulation of NewHope

Analysis of Encoding/Modulation and Decoding/Demodulation: ATE

In NewHope, ATE is used to encode and modulate a message bit $\mu_i$ and decode and demodulate an erroneous message bit $v''_i$. Note that ATE performs both encoding/decoding as an ECC and modulation/demodulation. The encoding/modulation and decoding/demodulation procedures of ATE with $m$ repetitions are shown in Fig. 3 where $m = 4$ for $n = 1024$ and $m = 4$ for $n = 512$ [17].

The encoding of ATE is performed such that one message bit $\mu_i$ is repeated $m$ times and the modulation of ATE is a mapping of each bit to an element of $\mathbb{Z}_q$ (usually either 0 or $\lfloor \frac{q}{4} \rfloor$) as the coefficients of $v$. Note that the $m$ repetitions is the same operation as the encoding of an $m$-repetition code. The demodulation of ATE is to calculate the absolute value of the difference between the received erroneous symbol $v''_i$ and $\lfloor \frac{q}{2} \rfloor$ over integer domain $\mathbb{Z}_q$. The decoding of ATE is to sum up four absolute values corresponding to the same $\mu'_{\text{enc},i}$, $\mu'_{\text{enc},i+256}$, $\mu'_{\text{enc},i+512}$, $\mu'_{\text{enc},i+768}$ to generate $\mu'_s,i$ and compare it with the threshold $m \cdot q/4$ to determine if the estimate $\mu'_i$ of $\mu_i$ is 0 or 1 as follows.

$$\mu'_s,i \begin{cases} 0 & \text{if } \frac{m \cdot q}{4} \\ 1 & \text{otherwise} \end{cases}$$

Analysis of Difference Noise, Compression Noise, and Total Noise of NewHope

Total noise $n_i$ is defined as the noise contained in the received
signal \(v''\) except the transmitted signal \(v\). The \(i\)th coefficient \(n_{t,i}\) of the total noise polynomial \(n_t\) contained in the polynomial \(v''\) in Step 6 is expressed as follows.

\[
n_{t,i} = (v' - v)_i = (v'_{\text{decomp}} - us - v)_i = (v' + n_c - us - v)_i = (bs' + e'' - ass' - e's)_i + n_{c,i} = (es' - e's + e'')_i + n_{c,i} = n_{d,i} + n_{c,i},
\]

where \((\cdot)_i\) denotes the \(i\)th coefficient of the given polynomial, \(n_c \in \mathcal{R}_q\) is the compression noise polynomial, \(n_{c,i}\) is the \(i\)th coefficient of \(n_c\) contained in \(v''\), \(n_d \in \mathcal{R}_q\) is the difference noise polynomial, and \(n_{d,i}\) is the \(i\)th coefficient of \(n_d\) contained in \(v''\).

To analyze the compression noise \(n_{c,i}\), we first need to investigate the coefficient of the polynomial \(v' = ass' + es' + e''\) being compressed, where the coefficients of \(s, s', e,\) and \(e''\) follow the predetermined centered binomial distribution. However, since the coefficients of polynomial \(a\) follow a uniform distribution, the coefficient of the compressed polynomial \(h\) will eventually follow a uniform distribution. A compression to \(v'\) is performed by applying \([v'_i * r/q]\) to the coefficients \(v'_i\) of \(v'\) to generate the coefficient \(h_i\) of \(h\), where \([\cdot]\) is a rounding function that rounds to the closest integer, \(r\) denotes the compression rate on \(v'\), and \(r = 8\) for NewHope. Then the range of the compressed coefficients \(h_i\) of \(h\) is changed from \([0, q - 1]\) to \([0, r - 1]\) so that the number of bits required to store a coefficient is reduced from 14 bits (\(\approx \lceil \log_2 q \rceil\)) for \(v'\) to 3 bits (\(\approx \lceil \log_2 r \rceil\)) for NewHope with \(r = 8\). Note that the smaller the value of \(r\) is, the more compression is performed. A decompression is performed by applying \([h_i * q/r]\) to each of the coefficients of \(h\). Then the coefficient takes the value from \(0, [q/r], [2q/r], \ldots, [(r - 1) \cdot q/r]\). This compression and decompression are illustrated in Fig. 4 where the coefficients \(v'_i\) of \(v'\) from different patterns (or ranges) are mapped to different \(v_{\text{decomp},i}\) values through compression and decompression. In the end, compression and decompression can be seen as a rounding operation. Therefore, the compression noise is inevitably generated with the maximum magnitude \([q/2r]\) and the distribution \(Pr_{n_c}(x)\) of the compression noise is derived as follows:

\[
Pr_{n_c}(x) = \begin{cases} 
q/r, & 0 \leq x \leq \left\lceil \frac{q}{2r} \right\rceil - 1 \\
0, & \text{otherwise} \\
q/r, & q - 2 - \left\lceil \frac{q}{2r} \right\rceil \leq x \leq q - 1.
\end{cases}
\]  

To analyze the difference noise \(n_{d,i} = (es' - e's + e'')_i\), we use the fact that the coefficients of \(e, e', e'', s,\) and \(s'\) are independent and identically distributed (i.i.d.) following the same centered binomial distribution. In order to derive the distribution of coefficient \(n_{d,i}\) of \(n_d\), a number of convolution operations are
required because it is a sum of many i.i.d. random variables, each of which is obtained by multiplying two i.i.d. random variables following the centered binomial distribution. However, since it is difficult to calculate the multiple convolutions of the above distribution in closed form, the distribution of difference noise is numerically calculated [13].

Total noise is a sum of compression noise and difference noise which are independently generated. Thus, the distribution of total noise is obtained by performing convolution of the distributions of compression noise and difference noise as shown in Fig. 5. However, due to the error dependency among total noise coefficients $n_{t,i}$, the distribution of only one total noise coefficient cannot be used to calculate the accurate DFR or derive a better upper bound on DFR [20], [21].

4 DFR Analysis of NewHope By Considering Error Dependency

4.1 New Upper Bound on DFR of NewHope

In this paper, a new upper bound on DFR of NewHope, which is much tighter than the upper bound given in [4], [9], is derived by considering the total noise in section 3 and the centered binomial distribution without doing subgaussian approximation. More importantly, the error dependency is considered in deriving an upper bound on DFR by using constraint relaxation, which is an approximation of a difficult problem to a nearby problem that is easier to solve, and union bound.

A new upper bound on DFR of NewHope is derived by considering two types of error dependency as shown in Fig. 6. The first analysis of BER is performed on the output bit of one ATE decoder to derive an upper bound on the BER $Pr(\mu_i \neq \mu_i')$. In this case, the error dependencies among four input values are considered. Note that analysis of one ATE decoder is good enough because all 256 ATE decoders are statistically identical. The second analysis is performed on 256 output bits $\mu_i'$ of ATE decoders in NewHope to derive an upper bound on DFR $Pr(\mu \neq \mu')$ of NewHope. In this case, the error dependencies among 256 bits $\mu_i'$ are considered.
Propose Upper Bound on BER of NewHope Suppose that \( \Pr(\mu_i = 0) = \Pr(\mu_i = 1) = \frac{1}{2} \), then the BER is average of two conditional probability depending on \( \mu_i \).

\[
\Pr(\mu_i \neq \mu_i') = \Pr(\{\mu_i \neq \mu_i'\} \cap \{\mu_i = 0\}) + \Pr(\{\mu_i \neq \mu_i'\} \cap \{\mu_i = 1\}) = \frac{1}{2} \left( \Pr(\mu_i \neq \mu_i'|\mu_i = 0) + \Pr(\mu_i \neq \mu_i'|\mu_i = 1) \right)
\]

(4)

Since \( \Pr(\mu_i \neq \mu_i'|\mu_i = 0) \) and \( \Pr(\mu_i \neq \mu_i'|\mu_i = 1) \) are statistically identical, we will analysis the BER given \( \mu_i = 1 \). Then the total noise given \( \mu_i = 1 \) is defined by \( n_{l,i}^{\mu_i=1} = (n_{t,i} + \mu_{enc,i}) \mod q \). Note that 0 and 1 of \( \mathbb{Z}_2 \) are mapped into 0 and \( \lfloor \frac{q}{2} \rfloor \) of \( \mathbb{Z}_q \), respectively. The output \( \mu_{s,i}' \) of decoding/demodulation of NewHope, which is defined in section 3.2, is determined by four dependent coefficients of \( v' \) given \( \mu_i = 1 \) as follows:

\[
\mu_{s,i}' = \sum_{l=0}^{3} |n_{l,i+256*l}^{\mu_i}-\lfloor \frac{q}{2} \rfloor|,
\]

\[
= \sum_{l=0}^{3} |(n_{t,i+256*l} + \mu_{i}\lfloor \frac{q}{2} \rfloor) \mod q - \lfloor \frac{q}{2} \rfloor|
\]

(5)

where \( \mu_{s,i}' \in \mathbb{Z} \).
In NewHope, most operations are performed over $\mathcal{R}_q = \mathbb{Z}_q[x]/(X^n + 1)$, but for the convenience of analysis, we consider the two domains $\mathbb{Z}$ and $\mathbb{Z}_q$, and express the polynomials $e, s, e', s', e'', s''$, and $n_c$ in $\mathcal{R}_q = \mathbb{Z}_q[x]/(X^n + 1)$ by the vectors $e, s, e', s', e'', s'' \in \mathbb{Z}^{n \times 1}$ and $n_c \in \mathbb{Z}^{n \times 1}$. Then, it is clear that $e, s, e', s', e'', s'' \in \mathbb{Z}^{n \times 1}$ are the random vectors following the centered binomial distribution with the parameter $k = 8$ and $n_c \in \mathbb{Z}^{n \times 1}$ is the random vector following the uniform distribution over the support $[-\lfloor \frac{q}{2} \rfloor, \lfloor \frac{q}{2} \rfloor]$. To express the product of two polynomials over $\mathcal{R}_q = \mathbb{Z}_q[x]/(X^n + 1)$ as an operation $\circ$ for the corresponding vectors over $\mathbb{Z}^{n \times 1}$, we define a new operation $\odot$, which is called cyclic shift product, as follows:

$$ (e \odot s)_i = (e \odot s)_i = \sum_{j=0}^{n-1} \text{sign}(i-j)e_j s_{(i-j) \mod n}, $$

where $\text{sign}(x) = 1$ when $x \geq 0$, otherwise $\text{sign}(x) = -1$. For examples, if $n = 4$,

$$ (e \odot s)_0 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}^T \begin{pmatrix} +s_0 \\ -s_3 \\ -s_2 \\ -s_1 \end{pmatrix}, (e \odot s)_1 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}^T \begin{pmatrix} +s_1 \\ +s_0 \\ -s_3 \\ -s_2 \end{pmatrix}, $$

$$ (e \odot s)_2 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}^T \begin{pmatrix} +s_2 \\ +s_1 \\ +s_0 \\ -s_3 \end{pmatrix}, (e \odot s)_3 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}^T \begin{pmatrix} +s_3 \\ +s_2 \\ +s_1 \\ +s_0 \end{pmatrix}, $$

Fig. 6: Two types of error dependency in the demodulation and decode of NewHope.
where \((\cdot)^T\) denotes the transpose of vector. Using the newly defined vectors \(e, s, e', s', e'', n, \text{ and operation } \odot\), ATE \(\mu'_{s,t}\) in (5) can be expressed as:

\[
\mu'_{s,t} = \sum_{l=0}^{3} |n'_{l,i+256+l} - q\alpha_{i+256+l}|, 
\]

(7)

where \(n'_{l,i} = (e \odot s') - (e' \odot s) + e'' + n_{c,t}\), and \(\alpha_i\) is an integer making \(n'_{l,i}\) be in \([0, q-1]\) such that, \(n'_{l,i} = n'_{0,i} - \alpha_i q + \lfloor \frac{q}{2} \rfloor\) and \(|\alpha_i| \leq \lfloor (2nk^2 + k + (q-1)/r)/q \rfloor\). For example, if \(|n'_{l,i}| \leq \lfloor \frac{q}{2} \rfloor\), then \(\alpha_i = 0\). Finally, under the condition that an all-one message bit is transmitted, the event of bit error is equivalent to the following inequality.

\[
q - 1 \leq \sum_{l=0}^{3} |n'_{l,i+256+l} - q\alpha_{i+256+l}| \leq 2q - 2, 
\]

(8)

where \(2q - 2\) is a maximum value of \(|n'_{l,i+256+l} - q\alpha_{i+256+l}|\).

In order to find the support satisfying (8), some of sets and vector should be defined. Let \(\Omega = \text{supp}(e, s, e', s', e'', n, c)\) = \{\(e, s, e', s', e'', n_{c,t}\)\} where \(k\) is the centered binomial parameter, \(r\) is the compression rate such that, \(\Pr(\Omega) = 1\) and let the bit error support \(E = \{e \in \Omega | q - 1 \leq \sum_{l=0}^{3} |n'_{l,i+256+l} - q\alpha_{i+256+l}| \leq 2q - 2\}\) such that, \(\Pr(\mu_i \neq \mu'_i) = \Pr(\Omega)\).

Since (8) is the sum of four absolute values, it can be divided into 16 cases, and the new vector \(y^i \in \{-1, 1\}^{4x1}\), whose \(j\)th element \(y^i_j\) is \(-\text{sign}(i \mod 2^j - 2^{j-1})\) for \(i = 0, 1, \cdots, 15\), is used for dividing into 16 cases. For example, \(y^0 = (1, 1, 1, 1)\) and \(y^1 = (1, -1, -1, -1)\). Then, by using \(y^i \in \{-1, 1\}^{4x1}\), the set \(\Omega_j\) that satisfies each of 16 cases of (8) can be defined as follows:

\[
\Omega_j = \{\omega_j \in \Omega | \forall l \in [0, 3] \text{ s.t. } (n'_{l,i+256+l} - q)y^i_l \geq 0\}, 
\]

(9)

where the details of \(\Omega_j\) is shown in Table II. The \(\Omega_0, \Omega_1, \cdots, \Omega_{15}\) are clearly disjoint set such that \(\Omega = \bigcup_{i=0}^{15} \Omega_i\) and \(\Omega_i \cap \Omega_j = \emptyset\) if \(i \neq j\). If \(\omega_k \in \Omega_k\), then absolute operator in (8) can be replaced with \(y^k\) by \(\Omega_k\) as follows:

\[
\sum_{l=0}^{3} |n'_{l,i+256+l} - q\alpha_{i+256+l}| \leq \sum_{l=0}^{3} (n'_{l,i+256+l} - q\alpha_{i+256+l})y^k_l 
\]

(10)

The bit error support \(E\) can be partitioned into 16 supports \(E_0, E_1, \cdots, E_{15}\) by using the support \(\Omega_k\) as follows:

\[
E_j = \{e_j | e_j \in \Omega_j \cap E\}. 
\]

(11)

It is obvious that \(E_j \subseteq \Omega_j\) for \(j = 0, 1, \cdots, 15\), and \(E_0, E_1, \cdots, E_{15}\) are disjoint such that \(E = \bigcup_{i=0}^{15} E_i\) and \(E_i \cap E_j = \emptyset\) if \(i \neq j\). Also, \(E_k\) is expressed using \(\Omega_k\) as follows:

\[
E_k = \{e_k \in \Omega_k | q - 1 \leq \sum_{l=0}^{3} (n'_{l,i+256+l} - q\alpha_{i+256+l})y^k_l \leq 2q - 2\} 
\]

(12)
Table 1: The details of set $\Omega$

| Set | Set condition |
|-----|----------------|
| $\Omega_0$ | $\{\omega_0 \in \Omega \mid n_{t,i}^* - \alpha_i q \geq 0, n_{t+i+256}^* - \alpha_i + 256q \geq 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_1$ | $\{\omega_1 \in \Omega \mid n_{t,i}^* - \alpha_i q \geq 0, n_{t+i+256}^* - \alpha_i + 256q \geq 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_2$ | $\{\omega_2 \in \Omega \mid n_{t,i}^* - \alpha_i q \geq 0, n_{t+i+256}^* - \alpha_i + 256q \geq 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_3$ | $\{\omega_3 \in \Omega \mid n_{t,i}^* - \alpha_i q \geq 0, n_{t+i+256}^* - \alpha_i + 256q \geq 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_4$ | $\{\omega_4 \in \Omega \mid n_{t,i}^* - \alpha_i q \geq 0, n_{t+i+256}^* - \alpha_i + 256q < 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_5$ | $\{\omega_5 \in \Omega \mid n_{t,i}^* - \alpha_i q \geq 0, n_{t+i+256}^* - \alpha_i + 256q < 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q < 0\}$ |
| $\Omega_6$ | $\{\omega_6 \in \Omega \mid n_{t,i}^* - \alpha_i q \leq 0, n_{t+i+256}^* - \alpha_i + 256q < 0, n_{t,i+512}^* - \alpha_i + 512q < 0, n_{t,i+768}^* - \alpha_i + 768q < 0\}$ |
| $\Omega_7$ | $\{\omega_7 \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q < 0, n_{t,i+512}^* - \alpha_i + 512q < 0, n_{t,i+768}^* - \alpha_i + 768q < 0\}$ |
| $\Omega_8$ | $\{\omega_8 \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q \geq 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_9$ | $\{\omega_9 \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q \geq 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_{10}$ | $\{\omega_{10} \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q \geq 0, n_{t,i+512}^* - \alpha_i + 512q < 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_{11}$ | $\{\omega_{11} \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q \geq 0, n_{t,i+512}^* - \alpha_i + 512q < 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_{12}$ | $\{\omega_{12} \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q < 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q \geq 0\}$ |
| $\Omega_{13}$ | $\{\omega_{13} \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q < 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q < 0\}$ |
| $\Omega_{14}$ | $\{\omega_{14} \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q < 0, n_{t,i+512}^* - \alpha_i + 512q \geq 0, n_{t,i+768}^* - \alpha_i + 768q < 0\}$ |
| $\Omega_{15}$ | $\{\omega_{15} \in \Omega \mid n_{t,i}^* - \alpha_i q < 0, n_{t+i+256}^* - \alpha_i + 256q < 0, n_{t,i+512}^* - \alpha_i + 512q < 0, n_{t,i+768}^* - \alpha_i + 768q < 0\}$ |

For the convenience of explanation, the inequality in [12] is expressed by using the new variable $\beta = -1, 0, \ldots, q - 2$ as follows:

$$q - 1 \leq \sum_{l=0}^{3} (n_{t+i+256+l}^* - q\alpha_i + 256+l)y_l^k \leq 2q - 2$$

$$\Leftrightarrow \sum_{l=0}^{3} (n_{t+i+256+l}^* - q\alpha_i + 256+l)y_l^k = q + \beta.$$  \hspace{1cm}(12)

$$\Leftrightarrow \sum_{l=0}^{3} n_{t+i+256+l}^*y_l^k = q \left( \sum_{l=0}^{3} \alpha_i + 256+l y_l^k + 1 \right) + \beta$$

$$\Leftrightarrow \sum_{l=0}^{3} n_{t+i+256+l}^*y_l^k = q(A_i + 1) + \beta. \hspace{1cm}(13)$$

where $A_i = \sum_{l=0}^{3} \alpha_i + 256+l y_l^k$ and $A_i$ is fully determined by $n_{t,i}^*$, $n_{t+i+256}^*$, $n_{t,i+512}^*$, and $n_{t,i+768}^*$, and $|A_i| < 4\alpha_{max}$ where $\alpha_{max} = \lfloor (2nk^2 + k + (q - 1)/r)/q \rfloor$. There are two constraints in [13] such that $A_i$ is a finite integer, and $\sum_{l=0}^{3} n_{t+i+256+l}^*$ and $\beta$ are congruent modulo $q$. Thus, $E_k$ can be expressed as union of supports.
satisfying two constraint as follows:

\[ E_k = \{ \epsilon_k \in \Omega_k | q - 1 \leq \sum_{l=0}^{3} (n_{t,i+256*l}^* - q\alpha_{i+256*l}) y_l^k \leq 2q - 2 \} \]

\[ = \bigcup_{\beta=-1}^{q-2} \bigcup_{j=A_{\min}}^{A_{\max}} \left( \{ \epsilon_k \in \Omega_k | n_{t,i+256*l}^* y_l^k = q(A_i + 1) + \beta \} \right) \]

\[ = \bigcup_{\beta=-1}^{q-2} \bigcup_{j=A_{\min}}^{A_{\max}} (\{ \epsilon_k \in \Omega_k | n_{t,i+256*l}^* = j\epsilon + \beta \} \cap \{ \epsilon_k \in \Omega_k | A_i = j - 1 \}) \]

where \( A_{\min} = -4\alpha_{\max} \) and \( A_{\max} = 4\alpha_{\max} \).

In order to calculate the BER, the occurring probability \( \Pr(E) \) of the bit error support \( E \) should be calculated. As mentioned above, since the bit error support \( E \) can be disjointly partitioned, \( \Pr(E) = \sum_{i=0}^{15} \Pr(E_i) \). If \( E_0 \) is first considered, it can be expressed as the union of different supports as follows:

\[ E_0 = \bigcup_{j=A_{\min}}^{A_{\max}} \bigcup_{\beta=-1}^{q-2} (\{ \epsilon_k \in \Omega_0 | n_{t,i+256*l}^* = j\epsilon + \beta \} \cap \{ \epsilon_k \in \Omega_0 | A_i = j - 1 \}) \]

\[ = \left( \bigcup_{j:j\neq 1} \bigcup_{\beta=-1}^{q-2} (\{ \epsilon_k \in \Omega_0 | n_{t,i+256*l}^* = j\epsilon + \beta \} \cap \{ \epsilon_k \in \Omega_0 | A_i = j - 1 \}) \right) \]

\[ \bigcup \left( \bigcup_{\beta=-1}^{q-2} (\{ \epsilon_k \in \Omega_0 | n_{t,i+256*l}^* = q + \beta \} \cap \{ \epsilon_k \in \Omega_0 | A_i = 0 \}) \right) \]

\[ = E_{0,j\neq 1} \cup E_{0,j=1} \quad (14) \]

where \( E_{0,j\neq 1} = \bigcup_{j:j\neq 1} \bigcup_{\beta=-1}^{q-2} (\{ \epsilon_k \in \Omega_0 | n_{t,i+256*l}^* = j\epsilon + \beta \} \cap \{ \epsilon_k \in \Omega_0 | A_i = j - 1 \}) \) and \( E_{0,j=1} = \bigcup_{\beta=-1}^{q-2} (\{ \epsilon_k \in \Omega_0 | n_{t,i+256*l}^* = q + \beta \} \cap \{ \epsilon_k \in \Omega_0 | A_i = 0 \}) \). Next, in order to derive the upper bound of the occurring probability of \( E_0 \), the upper bound of occurring probability of each support \( E_{0,j\neq 1} \) and \( E_{0,j=1} \) is derived by using the following Theorems 1 and 2.

**Theorem 1** The occurring probability \( \Pr(E_{0,j\neq 1}) \) of \( E_{0,j\neq 1} \) in (14) is at most \( 4\Pr[|n_{t,i}^*| > |\frac{q}{2}|] \).

**Proof.** If \( A_i \neq 0 \), then at least one of \( \alpha_i, \alpha_{i+256}, \alpha_{i+512}, \) and \( \alpha_{i+768} \) is not zero. In the equation \( n_{t,i;\mu_i}=1 = n_{t,i+256+l}^* - \alpha_i q + \frac{q}{2} \), since \( \alpha_i \) makes \( n_{t,i;\mu_i}=1 \) be in \([0, q-1] \) and the message is \( \lfloor \frac{q}{2} \rfloor \), \( \alpha_i = 0 \) if and only if \( \lfloor \frac{q}{2} \rfloor \leq n_{t,i}^* \leq \lfloor \frac{q}{2} \rfloor \). Conversely, \( \alpha_i \neq 0 \) if and only if \( |n_{t,i}^*| > |\frac{q}{2}| \). Therefore, at least one among \( |n_{t,i}^*|, |n_{t,i+256}|, |n_{t,i+512}|, \) and \( |n_{t,i+768}| \) is greater than \( |\frac{q}{2}| \). Then, we can relax the constraint \( \{ \Omega | \sum_{l=0}^{3} n_{t,i+256*l}^* = j\epsilon + \beta \} \) and make the superset whose occurring probability
is greater than or equal to origin set $E_{0,j\neq 1}$ as follows:

$$E_{0,j\neq 1} = \bigcup_{j:j\neq 1} q^{-2} \bigcup_{\beta=-1}^3 \left( \{ \epsilon_0 \in \Omega_0 | \sum_{l=0}^3 n_{t,i+256+l}^* = jq + \beta \} \cap \{ \epsilon_0 \in \Omega_0 | A_i = j - 1 \} \right)$$

$$\subseteq \bigcup_{j:j\neq 1} \{ \epsilon_0 \in \Omega_0 | A_i = j - 1 \}$$

$$\subseteq \bigcup_{l=0}^3 \{ \epsilon_0 \in \Omega_0 | |n_{t,i+256+l}^*| > \frac{q}{2} \}$$

The occurring probability of $E_{0,j\neq 1}$ is bounded by using union bound.

$$\Pr(E_{0,j\neq 1}) \leq \Pr \left( \bigcup_{l=0}^3 \{ |n_{t,i+256+l}^*| > \frac{q}{2} \} \right)$$

$$\leq \sum_{l=0}^3 \Pr \left( |n_{t,i+256+l}^*| > \frac{q}{2} \right)$$

$$\leq 4 \Pr \left( |n_{t,i}^*| > \frac{q}{2} \right).$$

\[ \Box \]

Fig. 7: The distribution of $n_{t,i}^*$ for $n = 1024$ and $n = 512$ where $\sup(n_{t,i}^*)$ denotes the support of $n_{t,i}^*$.

The distribution of $n_{t,i}^*$ can be numerically calculated as shown in Fig. 7. By using this distribution of $n_{t,i}^*$, we can calculate $\Pr(E_{0,j\neq 1}) \leq 2^{-564}$ for $n = 1024$ and $\Pr(E_{0,j\neq 1}) \leq 2^{-908}$ for $n = 512.$
Theorem 2  The occurring probability of $E_{0,j=1}$ is at most $\Pr(q-1 \leq \sum_{l=0}^{3} n_{t,i+256l}^{*} \leq 2q-2)$.

Proof. If $A_i = 0$, then $E_{0,j=1}$ can be relaxed by eliminating the constraints \{$\epsilon_0 \in \Omega_0 | A_i = 0$\} as follows:

$$E_{0,j=1} = \bigcup_{\beta=-1}^{q-2} \left( \{ \epsilon_0 \in \Omega_0 | \sum_{l=0}^{3} n_{t,i+256l}^{*} = q + \beta \} \cap \{ \epsilon_k \in \Omega_k | A_i = 0 \} \right)$$

$$\subseteq \bigcup_{\beta=-1}^{q-2} \left( \{ \epsilon_0 \in \Omega_0 | \sum_{l=0}^{3} n_{t,i+256l}^{*} = q + \beta \} \right).$$

Therefore, $\Pr(E_{0,j=1}) \leq \Pr(q-1 \leq \sum_{l=0}^{3} n_{t,i+256l}^{*} \leq 2q-2)$ by using union bound.

Note that $n_{t,i}^{*}, n_{t,i+256}^{*}, n_{t,i+512}^{*}$, and $n_{t,i+768}^{*}$ are statistically dependent to each other, but the distribution of $\sum_{l=0}^{3} n_{t,i+256l}^{*}$ is numerically computable by using Theorem 3.

Theorem 3 $\sum_{l=0}^{3} n_{t,i+256l}^{*}$ is decomposed into the sum of i.i.d random variables.

Proof. We know that $n_{t,i}^{*} = (e \odot s_i) - (e' \odot s_i) + e''_i + n_{c,i}$. Then,

$$\sum_{l=0}^{3} n_{t,i+256l}^{*} = \sum_{l=0}^{3} (e \odot s_i)_{i+256l} - \sum_{l=0}^{3} (e' \odot s_i)_{i+256l} + \sum_{l=0}^{3} e''_{i+256l} + \sum_{l=0}^{3} n_{c,i+256l}$$

(15)

If $\sum_{l=0}^{3} (e \odot s)_i_{i+256l}$ can be decomposed into i.i.d random vectors, $\sum_{l=0}^{3} n_{t,i+256l}^{*}$ can be also decomposed into i.i.d random vectors. Since the inner product result is not affected by the common index permutation, we can collect every 256th element to split the above inner product into the sum of partial inner products.
as follows.

\[
\sum_{l=0}^{3} (e \odot s)_{i+256+l} = \begin{pmatrix}
  e_0^T \\
  \vdots \\
  e_{256} \\
  e_{512} \\
  e_{768} \\
  \vdots \\
  e_{1023}
\end{pmatrix} \begin{pmatrix}
  +s_0 \\
  \vdots \\
  +s_{256} \\
  +s_{512} \\
  +s_{768} \\
  \vdots \\
  +s_{1023}
\end{pmatrix}
\]

It is clear that each partial inner product has a similar structure and hence, we define new random variable \( W_j \),

\[
W_j = \begin{pmatrix}
  e_j^T \\
  e_{j+256} \\
  e_{j+512} \\
  e_{j+768}
\end{pmatrix} \begin{pmatrix}
  +s_j \\
  -s_{j+256} \\
  -s_{j+512} \\
  -s_{j+768}
\end{pmatrix}
\]

(16)

and \( \sum_{l=0}^{3} (e \odot s)_{i+256+l} = \sum_{j=0}^{255} W_j \). Since the random variables participating in the construction of \( W_j \) are different, \( W_j \)'s are clearly independent to each other. Hence, \( \sum_{l=0}^{3} n_{i,l+256+l} = \sum_{j=0}^{255} W_j \) can be decomposed as

\[
\sum_{l=0}^{3} n_{i,l+256+l} = \sum_{l=0}^{n/2-1} \left[ (e' \odot s)'_{i+256+l} - (e' \odot s)''_{i+256+l} + e''_{i+256+l} + n_{c,i+256+l} \right]
\]

\[
= \sum_{j=0}^{3} W_j + \sum_{l=0}^{n/2-1} \left[ e''_{i+256+l} + n_{c,i+256+l} \right].
\]
Note that both \((e' \circ s)\) and \((e \circ s')\) are decomposed into \(n/4\) i.i.d. random variables \(W_j\), and therefore total \(n/2\) i.i.d. random variables \(W_j\) are obtained to produce \(\sum_{j=0}^{n/2-1} W_j\).

In conclusion, by using Theorems 1 and 2 and union bound, the occurring probability of \(E_0\) is upper bounded as follows:

\[
Pr(E_0) = Pr(E_{0,j \neq 1} \cup E_{0,j = 1}) \\
\leq Pr(E_{0,j \neq 1}) + Pr(E_{0,j = 1}) \\
\leq 4 \Pr\left(\sum_{l=0}^{q-1} n_{t,i+256+l}^* > \frac{q}{2}\right) + \Pr\left(q - 1 \leq \sum_{l=0}^{3} n_{t,i+256+l}^* \leq 2q - 2\right).
\]

Next, in order to calculate the BER, \(Pr(E_1)\), \(Pr(E_2)\), \ldots, and \(Pr(E_{15})\) should be calculated, and they can be calculated by using Theorem 4 as follow.

**Theorem 4** \(\forall j = 0, 1, \ldots, 15\) s.t. \(Pr(E_j) \leq 4 \Pr\left(\sum_{l=0}^{q-1} n_{t,i+256+l}^* \leq 2q - 2\right).

Proof. Recall that 16 vectors \(y^0, y^1, \ldots, y^{15}\) determine the signs of four noise elements such as \(\pm n_{t,i}^*\), \(\pm n_{t,i+256}^*, \pm n_{t,i+512}^*\), and \(\pm n_{t,i+768}^*\). Likewise the proof of Theorem 1, the superset of \(E_{k,j \neq 1}\) is found.

\[
E_{k,j \neq 1} = \bigcup_{j:j \neq 1} \bigcup_{\beta=-1}^{q-2} \left\{ \epsilon_k \in \Omega_k \bigg| \sum_{l=0}^{3} n_{t,i+256+l}^* y_i^k = jq + \beta \bigg\} \cap \{ \epsilon_k \in \Omega_k | A_i = j - 1 \}
\]

\[
\subseteq \bigcup_{j:j \neq 1} \{ \epsilon_k \in \Omega_k | A_i = j - 1 \}
\]

\[
= \bigcup_{j:j \neq 1} \{ \epsilon_k \in \Omega_k \big| \sum_{l=0}^{3} \alpha_{i+256+l} y_i^k = j - 1 \}
\]

If \(\sum_{l=0}^{3} \alpha_{i+256+l} y_i^k \neq 0\), then at least one among \(\alpha_{i}, \alpha_{i+256}, \alpha_{i+512}\), and \(\alpha_{i+768}\) is not zero. The fact implies at least one among \(|n_{t,i}^*|\), \(|n_{t,i+256}^*|\), \(|n_{t,i+512}^*|\), and \(|n_{t,i+768}^*|\) is greater than \(\lceil q/2 \rceil\). Clearly, \(Pr(E_{k,j \neq 1})\), \(\forall k = 1, 2, \ldots, 15\) is upper bounded as same as \(Pr(E_{0,j \neq 1})\) by using the union bound as follows:

\[
Pr(E_{k,j \neq 1}) \leq \Pr\left(\bigcup_{l=0}^{3} \{ |n_{t,i+256+l}^*| > \frac{q}{2} \} \right)
\]

\[
\leq \sum_{l=0}^{3} \Pr\left( |n_{t,i+256+l}^*| > \frac{q}{2} \right)
\]

\[
\leq 4 \Pr\left( |n_{t,i}^*| > \frac{q}{2} \right).
\]
Also, Theorem 2 is applied to other $\Pr(E_{k,j}=1)$, $\forall k=1,2,...,15$ as follows:

$$E_{k,j=1} = \bigcup_{\beta=-1}^{q-2} \left( \{ \epsilon_k \in \Omega_k | \sum_{l=0}^{3} n_{t,i+256+l} y_{k}^l = q + \beta \} \cap \{ \epsilon_k \in \Omega_k | A_i = 0 \} \right)$$

$$\subseteq \bigcup_{\beta=-1}^{q-2} \left( \{ \epsilon_k \in \Omega_k | \sum_{l=0}^{3} n_{t,i+256+l} y_{k}^l = q + \beta \} \right).$$

Therefore, we obtain the upper bound on $\Pr(E_{k,j}=1)$ as follows:

$$\Pr(E_{k,j=1}) \leq \Pr \left( q - 1 \leq \sum_{l=0}^{3} n_{t,i+256+l} y_{k}^l \leq 2q - 2 \right), \quad \forall k=0,1,...,15$$

Since expectation of $W_j$ for $j=0,1,\cdots,n/2-1$ in (16) consist of sum of product of i.i.d. e and s whose means are zero, the expectation of $W_j$ is zero. This fact guarantees that for any $\mathbf{y}$, the distributions of $\sum_{l=0}^{3} n_{t,i+256+l} y_{k}^l$ are statistically identical and therefore the upper bounds on $\Pr(E_1)$, $\Pr(E_2)$, $\cdots$, and $\Pr(E_{15})$ are same as $\Pr(E_0)$.

In summary, by using Theorems 1, 2, 3, and 4 the upper bound on $\Pr(E_{k,j=1})$, which is the BER of NewHope, is derived by using union bound as follows:

$$\Pr(E) = \Pr \left( \bigcup_{j=0}^{15} E_j \right)$$

$$\leq \sum_{j=0}^{15} \Pr(E_j)$$

$$\leq 16 \left( 4 \Pr \left( \sum_{l=0}^{3} n_{t,i+256+l} > \frac{q}{2} \right) + \Pr \left( q - 1 \leq \sum_{l=0}^{3} n_{t,i+256+l} \leq 2q - 2 \right) \right).$$

(17)

**Derivation of Upper Bound on DFR of NewHope** By using the $\Pr(E)$ in (17), the DFR can be easily upper bounded by using the union bound.

**Theorem 5** The DFR $\Pr(\mu \neq \mu')$ of NewHope is upper bounded as $\Pr(\mu \neq \mu') \leq \sum_{i=0}^{255} \Pr(\mu_i \neq \mu_i').$

Proof Since the DFR is the union of all bit error events, the DFR is upper bounded by the sum of BERs by using the union bound.

$$DFR = \Pr \left( \bigcup_{i=0}^{255} (\mu_i \neq \mu_i') \right)$$

$$\leq \sum_{i=0}^{255} \Pr(\mu_i \neq \mu_i')$$

$$= 256 \Pr(\mu \neq \mu')$$
Each BER is identical so that the upper bound on the DFR is expressed as:

$$DFR \leq 4096 \left( 4 \Pr \left( |n_{t,i+256+i}^*| > \frac{q}{2} \right) + \Pr(q - 1 \leq \sum_{i=0}^{3} n_{t,i+256+i}^* \leq 2q - 2) \right).$$

(18)

Parametrization of the Proposed Upper Bound on DFR of NewHope

The computational complexity of deriving the distribution of \(\sum_{l=0}^{3} n_{t,i+l}^*\) is \(O(k^8)\). Therefore, as \(k\) increases, the proposed upper bound cannot be easily computed. For this reason, the proposed upper bound on DFR of NewHope is parametrized for easy calculation by using CC bound in spite of losing some tightness.

**Theorem 6 (Chernoff-Cramer bound)** Let \(\Phi\) be a distribution over \(\mathbb{R}\) and let \(\chi_1, \ldots, \chi_n\) be i.i.d. random variable of \(\Phi\), with average \(\mu\). Then, for any \(t\) such that \(M_{\Phi,\chi}(t) = E_{\chi}[\exp(\chi t)] < \infty\) it holds that

$$\Pr \left[ \sum_{i=1}^{n} \chi_i > n\mu + \beta \right] \leq \inf_{t} \exp(\beta t + n \ln[M_{\Phi,\chi}(t)])$$

The proposed upper bound has two occurring probability \(\Pr(|n_{t,i+l}^*| > \frac{q}{2})\) and \(\Pr(q - 1 \leq \sum_{i=0}^{3} n_{t,i+l}^* \leq 2q - 2)\) in (18) and those probabilities can be applied with CC bound respectively. In order to apply CC bound to \(\Pr(|n_{t,i+l}^*| > \frac{q}{2})\), we need to calculate the moment generating function (MGF) of multiplication of two centered binomial random variable. Suppose that \(X\) and \(Y\) follow the binomial distribution with parameter \(2k\), and \(X_c\) and \(Y_c\) follow the centered binomial distribution with parameter \(k\). Then \(X_c = X - k\) and \(Y_c = Y - k\) and the MGF \(M_{\Phi,\chi}(t)\) of \(X_c \cdot Y_c\) is calculated as follows:

$$M_{\Phi,\chi}(t) = E_{X,Y} \left[e^{(x-k)(y-k)t}\right] = E_Y \left[E_X[e^{(x-k)(y-k)t}]\right] = E_Y \left[\sum_{y=0}^{2k} \left(\frac{2k}{x}\right) e^{(x-k)(y-k)t} 2^{-x} 2^{-(2k-x)}\right] = E_Y \left[\sum_{y=0}^{2k} \left(\frac{2k}{x}\right) e^{-kt(y-k)} \left(\frac{1}{2} e^{t(y-k)}\right)^x 2^{-(2k-x)}\right] = E_Y \left[e^{-kt(y-k)} \left(\frac{1}{2} e^{t(y-k)} + 1\right)^{2k}\right] = E_Y \left[\cosh^{2K} \left(\frac{t(y-k)}{2}\right)\right] = E_{Y_c} \left[\cosh^{2K} \left(\frac{tY_c}{2}\right)\right].$$

(19)
Since \( n_{t,i}^* \) is the sum of many products of two independent random variables drawn from the centered binomial distribution, CC bound can be applied as follows:

\[
\text{Pr} \left( |n_{t,i}^*| > \frac{q}{2} \right) = 2 \text{Pr} \left( (e \circ s')_i - (e' \circ s)_i > \frac{q}{2} - (e''_i - n_{c,i}) \right) \\
\leq 2 \text{Pr} \left( \sum_{j=0}^{n-1} \text{sign}(i-j)(e'_j s_{i-j} \mod n - e_j s'_{i-j} \mod n) \right) \\
\leq \frac{q}{2} - \left( k + \frac{q-1}{2r} \right) \\
\leq \inf_t 2 \exp \left( \frac{q}{2} - \left( k + \frac{q-1}{2r} \right) t + 2n \ln \left[ \cosh^{2K} \left( \frac{t y_c}{2} \right) \right] \right).
\]

Although the MGF of \( \sum_{t=0}^{n} n_{t,i}^* + 256 s_i \) is very complicated, Theorem 3 guarantees that \( \sum_{t=0}^{n} n_{t,i}^* + 256 s_i \) can be decomposed into \( i.i.d \) random variables \( W_j \) such as \( \sum_{t=0}^{n} n_{t,i}^* + 256 s_i = \sum_{j=0}^{n/2} W_j + \sum_{t=0}^{n} [e''_i + n_{c,i} + 256 t] \), where \( W_j \) is in (16).

For the convenience of analysis, the new variable \( W \) is defined as:

\[
W = \begin{pmatrix} c_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}^T \begin{pmatrix} +s_0 & +s_1 & +s_2 & +s_3 \\ -s_3 & +s_0 & +s_1 & +s_2 \\ -s_2 & -s_3 & +s_0 & +s_1 \\ -s_1 & -s_2 & -s_3 & +s_0 \end{pmatrix}.
\]

The MGF \( M_{\phi_W}(t) \) of \( W \) is

\[
M_{\phi_W}(t) = E_{s_0, s_1, s_2, s_3} \left[ E_{e_0} \left[ \exp(c_0(s_0 + s_1 + s_2 + s_3)t) \right] \cdot E_{e_1} \left[ \exp(e_1(s_0 + s_1 + s_2 - s_3)t) \right] \cdot E_{e_2} \left[ \exp(e_2(s_0 + s_1 - s_2 - s_3)t) \right] \cdot E_{e_3} \left[ \exp(e_3(s_0 - s_1 - s_2 - s_3)t) \right] \right]. \tag{20}
\]

By using \( M_{\phi_{X_c Y_c}}(t) = E_{Y_c} [E_{X_c} [\exp(x_c y_c t)]] = E_{Y_c} [\cosh^{2K} \left( \frac{ty_c}{2} \right)] \) in (19),

\[
M_{\phi_W}(t) = E_{s_0, s_1, s_2, s_3} \left[ \cosh^{2k} \left( \frac{t}{2} (s_0 + s_1 + s_2 + s_3) \right) \cdot \cosh^{2k} \left( \frac{t}{2} (s_0 + s_1 + s_2 - s_3) \right) \cdot \cosh^{2k} \left( \frac{t}{2} (s_0 + s_1 - s_2 - s_3) \right) \cdot \cosh^{2k} \left( \frac{t}{2} (s_0 - s_1 - s_2 - s_3) \right) \right]. \tag{21}
\]
Even if the computational complexity of $M_{\Phi^W}$ is $O(k^8)$, by using $\cosh^2 k(t) \leq e^{-kt^2}$ and new random variable $Z = (s_0 + s_1 + s_2 + s_3)^2 + (s_0 + s_1 + s_2 - s_3)^2 + (s_0 + s_1 - s_2 - s_3)^2 + (s_0 - s_1 - s_2 - s_3)^2$, the upper bound of $M_{\Phi^W}$ can be derived, which has the complexity $O(k^4)$ as follows:

$$M_{\Phi^W}(t) \leq E_{s_0, s_1, s_2, s_3} \left[ \exp \left( \frac{kt^2}{4} (s_0 + s_1 + s_2 + s_3)^2 \right) \right. \cdot \exp \left( \frac{kt^2}{4} (s_0 + s_1 + s_2 - s_3)^2 \right) \cdot \exp \left( \frac{kt^2}{4} (s_0 + s_1 - s_2 - s_3)^2 \right) \cdot \exp \left( \frac{kt^2}{4} (s_0 - s_1 - s_2 - s_3)^2 \right) \bigg] \leq E_Z \left[ \exp \left( \frac{zk t^2}{4} \right) \right]. \tag{22}$$

Then, by using CC bound and in (22), $Pr(q - 1 \leq \sum_{l=0}^{3} n_{t,i+256*l} \leq 2q - 2)$ is upper bounded as follows:

$$Pr \left( q - 1 \leq \sum_{l=0}^{3} n_{t,i+256*l} \leq 2q - 2 \right) \leq Pr \left( \sum_{i=0}^{n/2} W_i + \sum_{j=0}^{4} (e_j'' + n_{c,j}) > q - 1 \right) \leq Pr \left( \sum_{i=0}^{n/2} W_i \geq q - 1 - 4 \left( k + \frac{q - 1}{2r} \right) \right) \leq \inf_t \exp \left\{ \left( q - 1 - 4k - \frac{2(q - 1)}{r} \right) t + \frac{n}{2} \ln M_{\Phi^W}(t) \right\} \leq \inf_t \exp \left\{ \left( q - 1 - 4k - \frac{2(q - 1)}{r} \right) t + \frac{n}{2} \ln E_Z \left[ \exp \left( \frac{zk t^2}{4} \right) \right] \right\}.$$

Finally, a simplified upper bound on DFR of NewHope is derived as follows:

$$DFR \leq 4096 \inf_t \exp \left\{ \left( q - 1 - 4k - \frac{2(q - 1)}{r} \right) t + \frac{n}{2} \ln E_Z \left[ \exp \left( \frac{zk t^2}{4} \right) \right] \right\} + 4 \inf_t \exp \left\{ \left( \frac{q}{2} - k - \frac{(q - 1)}{2r} \right) t + 2n \ln E_Y \left[ \cosh^2 \left( \frac{tY}{2r} \right) \right] \right\}. \tag{23}$$

**Verification of the proposed upper bounds on DFR of NewHope** We compare the proposed upper bound in (18) and the simplified upper bound using CC bound in (23) with the current upper bound on DFR of NewHope [4], [9] for various $k$. Note that the current upper bound on DFR of NewHope [4], [9] is only provided when $k = 8$. Additionally, we compare the proposed upper
Fig. 8: Comparison of various upper bounds for various $k$ ($n = 1024$).

bounds with the DFR derived by assuming no error dependency as in [13]. For convenience of expression, we will use "Proposed upper bound" to denote the the upper bound derived in (18), "CC upper bound" to denote the simplified upper bound using CC bound in (23), "Current upper bound" to denote the current upper bound on DFR of NewHope [4], [9], "No error dependency" to denote the DFR values calculated by assuming no error dependency as in [13], and "Monte Carlo" to denote the DFR values obtained by performing Monte Carlo simulation of NewHope protocol. Fig. 8 compares the various upper bounds on DFR of NewHope for various noise parameter $k$ for $n = 1024$. First of all, it is confirmed that the two proposed upper bounds improve the upper bound more than fifty order of magnitude compared to the current upper bound for $k = 8$. If we compare the proposed upper bound and CC upper bound, we can see that CC bound is more loose as expected. Nevertheless, since the computational complexity of the proposed upper bound substantially increases as $k$ increases, the proposed upper bound is difficult to calculate when $k$ is large. However, CC upper bound can be calculated for most $k$ because CC upper bound is parameterized for easy calculation. In Fig. 8 Monte Carlo is the DFR value obtained by performing Monte Carlo simulation of NewHope protocol. Therefore, this DFR value reflects the error dependency, but this simulation is only possible for higher noise case (i.e., larger $k$ values). If we compare the Monte Carlo with no error dependency, it is confirmed that Monte Carlo DFR values are slightly larger than the no error dependency. The reason for this is that NewHope uses error correction codes called ATE [20], and therefore the DFR performance is degraded due to error dependency. Also, according to argument in [20], since NewHope uses ATE as an ECC, no error dependency becomes too positive. Fig.
Fig. 8 shows that as \( k \) increases, the proposed upper bound and no error dependency become almost identical. Therefore, it is guaranteed that the proposed upper bound is a fairly tight upper bound, especially for large \( k \).

Fig. 9 compares the various upper bounds on DFR of NewHope for various noise parameter \( k \) for \( n = 512 \). First of all, it is confirmed that the two proposed upper bounds improve the upper bound more than forty order of magnitude compared to the current upper bound for \( k = 8 \). Unlike the case of \( n = 1024 \), the proposed upper bound can be calculated for most \( k \) when \( n = 512 \). Thus, when \( n = 512 \), we can calculate tight upper bound values for most \( k \).

In conclusion, when \( n = 1024 \) and \( n = 512 \), it is confirmed that the proposed upper bound is fairly tight. Figs 8 and 9 show that when the noise parameter \( k \) is 8, the proposed upper bound on DFR of NewHope is much smaller than the DFR requirement of PQC. Therefore, by utilizing this new DFR margin, the security and bandwidth efficiency of NewHope can be improved, which will be verified in the next section.

5 Improved Security and Bandwidth Efficiency of NewHope Based on New Upper Bound on DFR

5.1 Improved Security

Since there exists a trade-off relation between the security level and the DFR, it is necessary to properly select the noise parameter \( k \) of centered binomial distribution such that the security level and the DFR are appropriately determined to meet the requirements. Since it is confirmed by the new upper bound on DFR
Table 2: Improved security level of NewHope based on new DFR margin (The noise parameter of current NewHope is $k = 8$) and the required DFR is $2^{-140}$.

| $n$ | $k$ | DFR | Cost of primal attack Classical/Quantum [bits] | Cost of dual attack Classical/Quantum [bits] |
|-----|-----|-----|---------------------------------------------|---------------------------------------------|
| 1024| 8   | $2^{-418}$ | 259/235 | 257/233 |
|     | 9   | $2^{-441}$ | 262/238 | 261/237 |
|     | 10  | $2^{-284}$ | 266/241 | 265/240 |
|     | 11  | $2^{-240}$ | 269/244 | 268/243 |
|     | 12  | $2^{-205}$ | 272/247 | 271/246 |
|     | 13  | $2^{-178}$ | 275/249 | 274/248 |
|     | 14  | $2^{-150}$ | 278/252 | 276/250 |
|     | 15  | $2^{-137}$ | 280/254 | 279/253 |
| 512 | 8   | $2^{-399}$ | 112/101 | 112/101 |
|     | 9   | $2^{-325}$ | 114/103 | 113/103 |
|     | 10  | $2^{-270}$ | 115/105 | 115/104 |
|     | 11  | $2^{-228}$ | 117/106 | 117/106 |
|     | 12  | $2^{-195}$ | 119/107 | 118/107 |
|     | 13  | $2^{-169}$ | 120/109 | 119/108 |
|     | 14  | $2^{-147}$ | 121/110 | 121/110 |
|     | 15  | $2^{-130}$ | 122/111 | 122/111 |

that NewHope is designed to have unnecessarily low DFR, the security level can be more improved by using the new DFR margin which is the difference between new upper bound and the required DFR.

Table 2 shows the improved security levels which are calculated as the cost of the primal attack and the cost of dual attack [22] to NewHope. It is possible to improve the security level by 7.2 % ($n = 1024$, $k = 14$) and 8.9 % ($n = 512$, $k = 14$) while guaranteeing the required DFR of $2^{-140}$ compared with the current NewHope. Note that such security level improvement does not require much increase of time/space complexity in NewHope because it only changes the noise parameter $k$ without any additional procedure. Therefore, this improvement of security can be easily applied to NewHope.

### 5.2 Improved Bandwidth Efficiency

The bandwidth efficiency of NewHope can also be improved by utilizing new DFR margin. An improvement of bandwidth efficiency is achieved by reducing (or more compressing) the ciphertext size which, however, increases the compression noise resulting in the DFR degradation. Even with such increased compression noise, both the improvement of bandwidth efficiency and the required DFR of $2^{-140}$ can be achieved by utilizing new DFR margin.

Table 3 shows the improved bandwidth efficiency of NewHope achieved by additional ciphertext compression. It is possible to improve the bandwidth efficiency by 5.9 % by changing the compression rate on $v'$ from 8 (3 bits per
Table 3: Improved bandwidth efficiency of NewHope based on new DFR margin (The noise parameter and compression rate of current NewHope are $k = 8$ and $r = 8$, respectively and the required DFR is $2^{-140}$).

| $n$  | $r$ | $k$ (Current NewHope) | DFR          |
|------|-----|------------------------|--------------|
| 1024 | 8   | 0                      | $\leq 2^{-418}$|
| 512  | 8   | 0                      | $\leq 2^{-399}$|

coefficient) to 4 (2 bits per coefficient) and the security level by 2.5% by changing the noise parameter from 8 to 10 for $n = 1024$. Similarly, it is possible to improve the bandwidth efficiency by 5.9% and the security level by 1.9% by changing the noise parameter from 8 to 9 for $n = 512$. The improvement of the security and bandwidth efficiency requires little change in the protocol of NewHope, so that this improvement can be easily applied to NewHope.

5.3 Closeness of Centered Binomial Distribution and the Corresponding Rounded Gaussian Distribution for Various $k$

The properties of rounded Gaussian distribution $\xi$ are key factor to the worst-case to average-case reduction for Ring-LWE. However, since a very high-precision and high-complexity sampling is required for the rounded Gaussian distribution, NewHope uses the centered binomial distribution $\psi_k$ for practical sampling without having rigorous security proof. It is generally accepted that as the centered binomial distribution and the rounded Gaussian distribution are closer to each other, NewHope is regarded as more secure. The closeness of two distribution can be measured through many methods. Among them, Rényi divergence is a well-known method, which is parameterized by a real $a > 1$ and defined for two distributions $P$ and $Q$ as follows [23], [24].

$$R_a(P||Q) = \left( \sum_{x \in \text{sup}(P)} \frac{P(x)^a}{Q(x)^{a-1}} \right)^{\frac{1}{a-1}}$$ (24)

where $\text{sup}(P)$ represents the support of $P$ and $Q(x) \neq 0$ for $x \in \text{sup}(P)$.

We define $\xi_k$ to be the rounded Gaussian distribution with the variance $\sigma^2 = k/2$, which is the distribution of $\lfloor \sqrt{k/2} \cdot x \rfloor$ where $x$ follows the standard normal distribution.

Fig. 10 shows that the Rényi divergence ($a = 9$ is used as in [4]) of the centered binomial distribution $\psi_k$ and the rounded Gaussian distribution $\xi_k$ with the same variance $k/2$. It is clear that the Rényi divergence decreases as
For NewHope with $k = 8$

Fig. 10: Rényi divergence of the centered binomial distribution $\psi_k$ and the rounded Gaussian distribution $\xi_k$ with the same variance $k/2$ according to $k$ ($a = 9$).

$k$ increases. Therefore, an increase in the noise parameter $k$ can quantitatively and qualitatively improve the security of NewHope although the time complexity increases a little bit due to the complexity increase of calculating $\sum_{i=0}^{k-1} (b_i - b'_i)$.

6 Conclusions

Since NewHope is an IND-CCA secure KEM by applying the FO transform to an IND-CPA secure PKE, accurate DFR calculation is required to guarantee resilience against attacks that exploit decryption failures. However, the upper bound on DFR of NewHope derived in [4], [9] is rather loose because the compression noise and effect of encoding/decoding of ATE in NewHope are not fully considered. Also, the centered binomial distribution is approximated by subgaussian distribution. Furthermore, since NewHope is a Ring-LWE based cryptosystem, there is a problem of error dependency among error coefficients, which makes accurate DFR calculation difficult.

In this paper, an upper bound on DFR, which is much closer to the real DFR than previous upper bound on DFR derived in [4], [9], is derived by considering the above-ignored factors. Also, the centered binomial distribution is not approximated by the subgaussian distribution. Especially, the new upper bound on DFR considers the error dependency among error coefficients by using the constraint relaxation and union bound. Furthermore, the new upper bound on DFR is parameterized by using CC bound in order to facilitate calculation of new upper bound on DFR for the parameters of NewHope.
According to the new upper bound on DFR of NewHope, since it is much lower than the DFR requirement of PQC, this DFR margin can be used to improve the security and bandwidth efficiency. As a result, the security level of NewHope is improved by 7.2%, or the bandwidth efficiency is improved by 5.9%. This improvement in the security and bandwidth efficiency can be easily achieved in NewHope because there is little change in time/space complexity of NewHope.

References

1. Lily, C., Stephen, J., Yi-Kai, L., Rene, P., Ray, P., and Daniel, S-T.: Report on post-quantum cryptography. In National Institute of Standards and Technology, 8105 NIST Interagency/ Internal Report (NISTIR), Gaithersburg. MD (2016)
2. Lindner, R. and Peikert, C.: Better key (and attacks) for LWE-Based encryption. In CT-RSA, vol. 6558, pp. 319-339. Springer (2011)
3. Regev, O.: On lattices, learning with errors, random linear codes, and cryptography. In ACM Symposium on Theory of Computing, pp. 84-93. Baltimore. MD (2005)
4. Pöppelmann, T., Alkim, E., Avanzi, R., Bos, J., Ducas, L., Piedra, A. D., Schwabe, P., Stebila, D., Albrecht, M. R., Orsini, E., Osheter, V., Paterson, K. G., Peer, G., and Smart, N. P.: NewHope, Technical report, https://csrc.nist.gov/projects/post-quantum-cryptography/round-2-submissions
5. Naehrig, M., Alkim, E., Bos, J., Ducas, L., Easterbrook, K., LaMacchia, B., Longa, P., Mironov, I., Nikolaenko, V., Peikert, C., Raghunathan, A., and Stebila, D.: FrodoKEM, Technical report, https://frodokem.org/files/FrodoKEM-specification-20190330.pdf
6. Schwabe, P., Avanzi, R., Bos, J., Ducas, L., Kiltz, E., Lepoint, T., Lyubashevsky, V., Schanck, J. M., Seiler, G., and Stehle, D.: CRYSTALS-KYBER, Technical report, https://pq-cryptals.org/kyber/data/kyber-specification-round2.pdf
7. Lu, X., Liu, Y., Jia, D., Xue, H., He, J., Zhang, Z., Liu, Z., Yang, H., Li, B., and Wang, K.: LAC, Technical report, https://csrc.nist.gov/projects/post-quantum-cryptography/round-2-submissions
8. Saarinen, M.O.: HILA5: On reliability, reconciliation, and error correction for ring-LWE encryption. In Selected Areas in Cryptography 2017, LNCS, vol. 10719, pp. 192–212. Springer (2018)
9. Alkim, E., Ducas, L., Pöppelmann, T., and Schwabe, P.: Post-quantum key exchange - a New Hope. In 25th USENIX Security Symposium, pp. 327–343. USENIX Association, Austin. TX (2016)
10. Alkim, E., Ducas, L., Pöppelmann, T., and Schwabe, P.: Newhope without reconciliation. In IACR Cryptology ePrint Archive, Report 2016/1157 (2016)
11. Deneuville, J., Gaborit, P., Guo, Q., and Johansson, T.: Ouroboros-E: An efficient lattice-based key-exchange protocol. In 2018 IEEE International Symposium on Information Theory (ISIT), Vail, CO, pp. 1450–1454 (2018)
12. Streit, S.and Santis, F.D.: Post-quantum key exchange on ARMv8-A: A new hope for NEON made simple. In IEEE Transactions on Computers, vol. 67, no. 11, pp. 1651–1662 (2018)
13. Fritzmann, T., Pöppelmann, T., and Sepulveda, J.: Analysis of error-correction codes for lattice-based key exchange. In Selected Areas in Cryptography 2018, LNCS, vol. 11349, pp. 369–390. Springer (2018)
14. Brown, J.: Bringing HSTS to www.google.com. In: Security Blog, Google, https://security.googleblog.com/2016/07/
15. Targhi, E. E. and Unruh, D.: Post-quantum security of the Fujisaki-Okamoto and OAEP transforms. In Selected Areas in Cryptography 2016, LNCS, vol. 9986, pp. 192–216. Springer (2016)
16. Fluhrer, S.: Cryptanalysis of ring-LWE based key exchange with key share reuse. In IACR Cryptology ePrint Archive, Report 2016/085 (2016)
17. Pöppelmann, T. and Güneysu, T.: Towards practical lattice-based public-key encryption on reconfigurable hardware. In Selected Areas in Cryptography 2013, LNCS, vol. 8282, pp. 68-85. Springer (2014)
18. Pollard, J. M.: The fast Fourier transform in a finite field. In Mathematics of Computation, vol. 25, no. 114, pp. 365–374 (1971)
19. Nguyen, P. Q. and Vallée, B.: The LLL algorithm: survey and applications. 1st edition. Springer Publishing Company, New York (2005)
20. D’Anvers, J. P., Vercauteren, F., and Verbauwhede, I.: The impact of error dependencies on ring/mod-LWE/LWR based schemes. In IACR Cryptology ePrint Archive, Report 2018/1172 (2018)
21. D’Anvers, J. P., Vercauteren, F., and Verbauwhede, I.: On the impact of decryption failures on the security of LWE/LWR based schemes. In IACR Cryptology ePrint Archive, Report 2018/1089 (2018)
22. Albrecht, M. R., Player R., and Scott. S.: On the concrete hardness of learning with errors. In Journal of Mathematical Cryptology, vol. 9, no. 3, pp. 169-203. (2015)
23. Rényi, A.: On measures of entropy and information. In Fourth Berkeley symposium on mathematical statistics and probability, vol. 1, pp. 547-561. (1961)
24. Bai, S., Langlois, A., Lepoint, T., Stehlé, D. and Steinfeld, R.: Improved security proofs in lattice-based cryptography: Using the Rényi divergence rather than the statistical distance, In Advances in Cryptology-ASIACRYPT 2015, LNCS, vol. 9452, pp. 3-24. Springer (2015)