CATEGORY $\mathcal{O}$ FOR THE LIE ALGEBRA OF VECTOR FIELDS ON THE LINE

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Abstract. Let $\mathcal{W}$ be the Lie algebra of vector fields on the line. Via computing extensions between all simple modules in the category $\mathcal{O}$, we give the block decomposition of $\mathcal{O}$, and show that the representation type of each block of $\mathcal{O}$ is wild using the Ext-quiver. Each block of $\mathcal{O}$ has infinite simple objects. This result is very different from that of $\mathcal{O}$ for complex semisimple Lie algebras. To find a connection between $\mathcal{O}$ and the Whittaker category $\Omega_\alpha$, we give an exact functor from $\mathcal{O}$ to $\Omega_\alpha$, which maps simple modules in $\mathcal{O}$ to simple modules in $\Omega_\alpha$ or zero. We also construct new simple $\mathcal{W}$-modules from Weyl modules and modules over the Borel subalgebra $\mathfrak{b}$ of $\mathcal{W}$.

Keywords: Category $\mathcal{O}$, block, Ext-quiver, wild, Whittaker module.

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1. INTRODUCTION

The category $\mathcal{O}$ for complex semisimple Lie algebras was introduced by Joseph Bernstein, Israel Gelfand and Sergei Gelfand in the early 1970s, see [2], and it includes all highest weight modules. This category is very important in the representation theory. For more details on category $\mathcal{O}$, one can see the monograph [12].

The category $\mathcal{O}$ can be defined for any Lie algebra with a triangular decomposition, see the book [14]. For a Lie algebra $\mathfrak{g}$ with a triangular decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$, there always exists an anti-involution $\sigma$ of $\mathfrak{g}$ such that $\sigma(\mathfrak{g}^+) = \mathfrak{g}^-$ and $\sigma|_{\mathfrak{h}} = \text{id}_{\mathfrak{h}}$. For example, the finite dimensional simple Lie algebras, the Virasoro algebra, the affine Kac-Moody algebras, the Heisenberg Lie algebras are all Lie algebras with triangular decompositions, see [14]. For the Kac-Moody algebras and the Virasoro algebra, categories $\mathcal{O}$ were studied in [8, 4] and references therein. For these algebras, the Hom-spaces between Verma modules determine the block decomposition of the category $\mathcal{O}$ to a great extent.

For the Lie algebra $\mathcal{W} = \mathcal{W}^- \oplus \mathfrak{h} \oplus \mathcal{W}^+$ of vector fields on the line, $\mathcal{W}^-$ is one dimensional, $\mathcal{W}^+$ is infinite dimensional. So $\mathcal{W}$ is not a Lie algebra with a triangular decomposition in the sense of [14]. Although we can also define the category $\mathcal{O}$ for $\mathcal{W}$ similar as that of complex semisimple Lie algebras, however several properties for $\mathcal{O}$ in [14] dose not hold for $\mathcal{W}$. For example, the embeddings

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between Verma modules has little impact on the block decomposition of $O$. Our initial motivation of the present paper was to explore the differences between the category $O$ of $\mathfrak{W}$ and the categories $O$ of semi-simple Lie algebras. In this paper, through giving extensions between all simple modules in $O$, we obtain the block decomposition of the category $O$ for $\mathfrak{W}$, and study the representation type of each block of $O$. We also detect the relations between $O$ and the Whittaker category for $\mathfrak{W}$, and construct simple $\mathfrak{W}$-modules from modules over the Weyl algebra and modules over the Borel subalgebra $b$ of $\mathfrak{W}$.

The paper is organized as follows. In Section 2, we introduce the category $O$ and the Whittaker category $\Omega_a$ for the Lie algebra $\mathfrak{W}$, and recall some results on $\Omega_a$ from $[21]$ in case $a \neq 0$. In Section 3, we first study the Verma modules and recall extensions on the $\mathfrak{W}$-modules $F_{\lambda}$ of Feigin and Fuchs defined in $[9]$. Then using these extensions and the duality between $F_{\lambda}$ and the Verma module $\Delta(\lambda)$, we can give all nontrivial extensions between Verma modules in $O$. Consequently, we obtain $\text{Ext}^1_O(M,N)$ for all simple modules $M,N \in O$, see Theorem 3.11. It should be mentioned that extensions between simple modules for the finite dimensional Witt algebra $W(1,1)$ over an algebraically closed field of characteristic $p > 3$ were determined in $[3]$. Furthermore we give the block decomposition $O = \bigoplus_{\lambda \in \mathbb{C}} O[\lambda]$, and show that each block $O[\lambda]$ is wild by studying a sub-quiver of its Ext-quiver, see Theorem 3.14. In subsection 3.5, we construct a functor $\Gamma_a$ from $O$ to $\Omega_a$, where we identify $\Omega_a$ with the category $\Omega'_a$ of finite dimensional $H'_a$-modules. The algebra $H'_a$ is a subalgebra of $U(b)$ which is isomorphic to the endomorphism algebra $H_a$ of the universal Whittaker $\mathfrak{W}$-module $Q_a$ defined in subsection 2.3. When $a \neq 0$, we show that $\Gamma_a$ is an exact functor, and $\Gamma_a$ maps simple modules in $O$ to simple modules in $\Omega_a$ or zero. Therefore the functor $\Gamma_a$ gives a connection between $O$ and the Whittaker category $\Omega_a$. At the end of Section 3, we also conjecture that some non-integral block $O[\lambda]$ may be equivalent to some subcategory of $\Omega_a$. In Section 4, we construct new simple tensor $\mathfrak{W}$-modules $T(P,V)$ from modules $P$ over the Weyl algebra and $b$-modules $V$. The isomorphism criterion for $T(P,V)$ is also given.

2. Preliminaries

In this paper, we denote by $\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$ and $\mathbb{C}$ the sets of integers, positive integers, nonnegative integers and complex numbers, respectively. All vector spaces and Lie algebras are over $\mathbb{C}$. For a Lie algebra $\mathfrak{g}$ we denote by $U(\mathfrak{g})$ its universal enveloping algebra. We write $\otimes$ for $\otimes_{\mathbb{C}}$.

2.1. Witt algebra. Let $A = \mathbb{C}[x]$ be the polynomial algebra and $\mathfrak{W}$ the derivation Lie algebra of $A$, i.e., $\mathfrak{W} = \text{Der}_{\mathbb{C}}A$. The Lie algebra $\mathfrak{W}$ is called the Lie algebra of vector fields on the line, or the Witt algebra of rank one. Denote $\partial = \frac{\partial}{\partial x}$ and $d_i = x^{i+1}\partial$, for any $i \in \mathbb{Z}_{\geq -1}$. Then $\{d_i \mid i \in \mathbb{Z}_{\geq -1}\}$ is a basis of $\mathfrak{W}$.
We can write the Lie bracket in $\mathfrak{W}$ as follows:

$$[d_i, d_j] = (j - i)d_{i+j}, \quad \text{for all } i, j \in \mathbb{Z}_{\geq -1}.$$ 

Note that the subspace $\mathfrak{h} = \mathbb{C}d_0$ is a Cartan subalgebra of $\mathfrak{W}$, i.e., a maximal abelian subalgebra that is diagonalizable on $\mathfrak{W}$ with respect to the adjoint action. Let $\mathfrak{W}^+ = \text{span}\{d_i \mid i \in \mathbb{Z}_{>0}\}$ and $\mathfrak{W}^- = \mathbb{C}d_{-1}$. Then $\mathfrak{W} = \mathfrak{W}^- \oplus \mathfrak{h} \oplus \mathfrak{W}^+$ is a decomposition of $\mathfrak{W}$, and the Lie subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{W}^+$ is called a Borel subalgebra of $\mathfrak{W}$.

One can see that $\Phi = \{\varepsilon_{-1}, \varepsilon_1, \varepsilon_2, \ldots\}$ is the root system of $\mathfrak{W}$, where $\varepsilon_i \in \mathfrak{h}^*$ such that $\varepsilon_i(d_0) = i$, $i \in \mathbb{Z}_{\geq -1}$. The subalgebra $\mathbb{C}d_{-1} \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1 \cong \mathfrak{sl}_2$, and $z = -d_1d_{-1} + d_0^2 - d_0$ is its Casimir element.

**Definition 2.1.** A $\mathfrak{W}$-module $M$ is called a weight module if $d_0$ acts diagonally on $M$, i.e.,

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda},$$

where $M_{\lambda} := \{v \in M \mid d_0v = \lambda v\}$. For a weight module $M$, denote

$$\text{Supp}(M) := \{\lambda \in \mathbb{C} \mid M_{\lambda} \neq 0\}.$$ 

If $M$ is a simple weight $\mathfrak{W}$-module, then $\text{Supp}(M) \subset \lambda + \mathbb{Z}$ for some $\lambda \in \mathbb{C}$. For a $\lambda \in \text{Supp}(M)$, a nonzero vector $v \in M_{\lambda}$ is called a maximal vector if $\mathfrak{W}^+v = 0$. A weight module is called a highest weight module if it is generated by a maximal weight vector.

We use $U(\mathfrak{W})\text{-Mod}$ to denote the category of all left $U(\mathfrak{W})$-modules.

2.2. **Category $\mathcal{O}$**. Next we introduce the category $\mathcal{O}$ for $\mathfrak{W}$.

**Definition 2.2.** The category $\mathcal{O}$ for $\mathfrak{W}$ is a full subcategory of $U(\mathfrak{W})\text{-Mod}$ whose objects are $\mathfrak{W}$-modules $M$ satisfying the following axioms:

(a) $M$ is a finitely generated $U(\mathfrak{W})$-module;
(b) $M$ is a weight module;
(c) $M$ is locally $\mathfrak{W}^+$-finite: for each $v \in M$, the subspace $U(\mathfrak{W}^+)v$ is finite dimensional.

Let $M$ be a module in $\mathcal{O}$. By (a) and (c) in Definition 2.2, we can assume that $M$ is generated by a finite dimensional $U(\mathfrak{W}^+)$-module $N$. By induction on the dimension of $N$, we can show that $M$ has the following property.

**Lemma 2.3.** Any module $M$ in $\mathcal{O}$ has a finite filtration of submodules as follows:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M,$$

where each factor $M_j/M_{j-1}$ for $1 \leq j \leq m$ is a highest weight module.

So highest weight modules are basic constituents of $\mathcal{O}$.
2.3. **Whittaker category** $\Omega_a$. We also introduce a Whittaker category $\Omega_a$ for $\mathfrak{m}$ which can be related to $\mathcal{O}$.

**Definition 2.4.** [21] For each $a \in \mathbb{C}$, the Whittaker category $\Omega_a$ for $\mathfrak{m}$ is a full subcategory of $U(\mathfrak{m})$-Mod whose objects are $\mathfrak{m}$-modules $M$ satisfying the following axioms:

(a) $d_{-1} - a$ acts locally nilpotently on $M$;
(b) the subspace $\text{wh}_a(M) = \{ v \in M \mid d_{-1}v = av \}$ is finite dimensional.

A module in $\Omega_a$ is called a Whittaker module. An element in $\text{wh}_a(M)$ is called a Whittaker vector. The category $\Omega_a$ was studied in [21].

**Example 2.1.** For any $b \in \mathbb{C}$, the space $A_{a,b} = \mathbb{C}[x]$ is a Whittaker $\mathfrak{m}$-module under the action:

$$d_ix^n = (n + b(i + 1))x^{n+i} + ax^{n+i+1}, \quad i \in \mathbb{Z}_{\geq -1}, n \in \mathbb{Z}_{\geq 0}. $$

We can see that $\text{wh}_a(A_{a,b}) = \mathbb{C}$. By Theorem 3.5 in [13], when $a \neq 0$, $A_{a,b}$ is a simple $\mathfrak{m}$-module for any $b \in \mathbb{C}$.

Let $N_a = \mathbb{C}1_a$ be the one dimensional $\mathbb{C}[d_{-1}]$-module such that $d_{-1}1_a = a1_a$. Let $Q_a = U(\mathfrak{m}) \otimes_{\mathbb{C}[d_{-1}]} N_a$ the induced $\mathfrak{m}$-module, and

$$H_a = \text{End}_\mathfrak{m}(Q_a)^{\text{op}}. $$

Then $Q_a$ is both a left $U(\mathfrak{m})$-module and a right $H_a$-module. Let $H_a$-mod be the category of finite dimensional $H_a$-modules.

**Theorem 2.5.** [21] Suppose that $a \neq 0$. Then

(a) The functors $M \mapsto \text{wh}_a(M)$ and $V \mapsto Q_a \otimes_{H_a} V$ are inverse equivalence between $\Omega_a$ and $H_a$-mod .
(b) Any simple module in $\Omega_a$ is isomorphic to $A_{a,b}$ for some $b \in \mathbb{C}$.

Therefore any finite dimensional simple nonzero $H_a$-module is one dimensional.

3. **Block decomposition of $\mathcal{O}$**

In this section, we study extensions between Verma modules and simple modules in $\mathcal{O}$. Using the Ext-quiver, we show that each block of $\mathcal{O}$ has wild representation type. We also construct an exact functor $\Gamma_a$ from $\mathcal{O}$ to the Whittaker category $\Omega_a$.

3.1. **The Verma modules.** For a $\lambda \in \mathbb{C}$, denote by $\mathbb{C}_\lambda$ the one-dimensional $\mathfrak{b}$-module with the generator $v_\lambda$ and the action given by

$$\mathfrak{m}^+ v_\lambda = 0, \quad d_0v_\lambda = \lambda v_\lambda.$$
The Verma module over $\mathfrak{m}$ is defined as follows:
\[
\Delta(\lambda) := \text{Ind}_{\mathfrak{m}}^W C_\lambda \cong U(\mathfrak{m}) \otimes U(b) C_\lambda.
\]

The module $\Delta(\lambda)$ has the unique simple quotient module $L(\lambda)$. By Lemma 2.3, the modules $L(\lambda)$ for $\lambda \in \mathbb{C}$ provide a complete set of irreducible modules in category $O$.

Lemma 3.1. (1) $zv = \lambda(\lambda + 1)v$ for all $v \in \Delta(\lambda)$.
(2) The module $\Delta(\lambda)$ is simple if and only if $\lambda \neq 0$. So $\Delta(\lambda) = L(\lambda)$ for $\lambda \neq 0$.
(3) The module $\Delta(0)$ is a uniserial module whose structure can be described by the following exact sequence:
\[
(3.1) \quad 0 \to \Delta(-1) \overset{\partial}{\to} \Delta(0) \overset{\partial}{\to} L(0) \to 0.
\]

Proof. (1) The proof follows from $[z, d_{-1}] = 0$, $\Delta(\lambda) = \mathbb{C}[d_{-1}]v_\lambda$ and $zv_\lambda = \lambda(\lambda + 1)v_\lambda$.

(2) For any $i \in \mathbb{Z}_{\geq 0}$, denote $v_{\lambda - i} := d_{-1}^i \cdot v_\lambda$. We can deduce that $d_0 \cdot v_{\lambda - i} = (\lambda - i)v_{\lambda - i}$. Since $\Delta(\lambda)$ is generated by $v_\lambda$, $\Delta(\lambda)$ is reducible if and only if there is an $i \in \mathbb{Z}_{>0}$ such that $d_1 v_{\lambda - i} = d_2 v_{\lambda - i} = 0$.

We can compute
\[
d_1 \cdot v_{\lambda - i} = d_1 \cdot d_{-1}^i \cdot v_\lambda = ([d_1, d_{-1}^i] + d_{-1}^i d_1) \cdot v_\lambda = \sum_{t=0}^{i-1} d_{-1}^i [d_1, d_{-1}] d_{-1}^{t-1} \cdot v_\lambda
\]
\[
= -2 \sum_{t=0}^{i-1} d_{-1}^i d_0 d_{-1}^{t-1} \cdot v_\lambda = -2 \sum_{t=0}^{i-1} (\lambda - i + t + 1)v_{\lambda - i + 1}
\]
\[
= i(i - 1 - 2\lambda)v_{\lambda - i + 1}.
\]

Similarly,
\[
d_2 \cdot v_{\lambda - i} = i(i - 1)(3\lambda - i + 2)v_{\lambda - i + 2}.
\]

Consider $d_1 \cdot v_{\lambda - i} = 0$ and $d_2 \cdot v_{\lambda - i} = 0$, $i \in \mathbb{Z}_{>0}$, we have the equations
\[
\begin{cases}
i(i - 1 - 2\lambda) = 0, \\
i(i - 1)(3\lambda - i + 2) = 0.
\end{cases}
\]

The solution of this equation is $\lambda = 0, i = 1$. Moreover when $\lambda = 0$, the submodule generated by $v_{-1}$ is a proper submodule which is isomorphic to $\Delta(-1)$. Thus $\Delta(\lambda)$ is simple if and only if $\lambda \neq 0$. 

(3) By (2), the submodule \(N\) generated by \(v_{-1}\) of \(\Delta(0)\) is simple and \(\Delta(0)/N \cong L(0)\). So \(N\) is the unique nontrivial submodule of \(\Delta(0)\). Hence \(\Delta(0)\) is a uniserial module. \(\square\)

**Remark 3.2.** If we denote \(e_{\lambda-i} := (-1)^i d_{-1}^{-i} \cdot v_{\lambda},\) for any \(i \in \mathbb{Z}_{\geq 0}\), then \(\{e_{\lambda-i} : i \in \mathbb{Z}_{\geq 0}\}\) is also a basis of \(\Delta(\lambda)\) such that the action of \(\mathfrak{m}\) on \(\Delta(\lambda)\) is defined as follows:

\[
d_k e_{\lambda-i} = ((k+1)\lambda + k - i)e_{\lambda-i+k}, \quad \forall k \in \mathbb{Z}_{\geq -1},
\]

where \(e_{\lambda-i+k} = 0\) when \(k - i > 0\).

**Corollary 3.3.** Any module \(M\) in \(\mathcal{O}\) has finite composition length.

**Proof.** By lemma 2.3, \(M\) has a filtration

\[
0 = M_0 \subset M_1 \subset \cdots \subset M_m = M,
\]

such that each factor \(M_j/M_{j-1}\) for \(1 \leq j \leq m\) is a highest weight module. By Lemma 3.1, every Verma module has finite composition length, so does any highest weight module. Consequently the composition length of \(M\) is finite. \(\square\)

**Remark 3.4.** In \([7]\), the authors defined another category \(\mathcal{O}'\) for \(\mathfrak{m}\). Any module \(M\) in \(\mathcal{O}'\) is locally finite over \(\mathbb{C}d_{-1} \oplus \mathbb{C}d_0\) rather than \(\mathfrak{m}^+\). Similar as \(\Delta(\lambda)\), define the \(\mathfrak{m}\)-module:

\[
\Delta'(\lambda) := U(\mathfrak{m}) \otimes_{U(\mathbb{C}d_{-1} \oplus \mathbb{C}d_0)} \mathbb{C}'_{\lambda},
\]

where \(\mathbb{C}'_{\lambda} = \mathbb{C}v'_{\lambda}\) is the \(U(\mathbb{C}d_{-1} \oplus \mathbb{C}d_0)\)-module defined by \(d_{-1}v'_{\lambda} = 0, d_0v'_{\lambda} = \lambda v'_{\lambda}\). Since any simple weight module over \(\mathfrak{m}\) has one dimensional weight spaces, see \([19]\), \(\Delta'(\lambda)\) does not have finite composition length. So \(\mathcal{O}'\) does not satisfy Corollary 3.3.

### 3.2. Extension between Verma modules.

Recall that for \(\mathfrak{m}\)-modules \(M, N \in \mathcal{O}\), the first cohomology space \(\text{Ext}^1_{U(\mathfrak{m})}(M, N)\) classifies the short exact sequences:

\[
0 \to N \xrightarrow{\alpha} K \xrightarrow{\beta} M \to 0,
\]

also called the extension of \(N\) by \(M\). Generally \(K\) may not lie in \(\mathcal{O}\). We are only interested in that \(K \in \mathcal{O}\), i.e., \(K\) needs to be a weight module. Note that \(\mathcal{O}\) is closed under weight module extensions. So \(\text{Ext}^1_{\mathcal{O}}(M, N) \subset \text{Ext}^1_{U(\mathfrak{m})}(M, N)\). In this subsection, we will give all extensions between Verma modules in \(\mathcal{O}\).

**Lemma 3.5.** Let \(\lambda, \mu \in \mathbb{C}\).

1. If \(\lambda - \mu \in \mathbb{Z}_{\geq 0}\) and \(M\) is a highest weight module with the highest weight \(\mu\), then \(\text{Ext}^1_{\mathcal{O}}(\Delta(\lambda), M) = 0\);
2. \(\text{Ext}^1_{\mathcal{O}}(\Delta(\lambda), \Delta(\lambda)) = 0\).
3. Proposition 3.6. \[ \text{Supp}(N) = \text{Supp}(M) \cup \text{Supp}() \] 

Proof. (1) Suppose that $0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} \Delta(\lambda) \rightarrow 0$ is a short exact sequence in $O$, where $N$ is a weight module. As $\lambda - \mu \in \mathbb{Z}_{\geq 0}$, $\text{Supp}(N) = \text{Supp}(M) \cup \text{Supp}()$, so $\lambda$ is a maximum weight in $N$. Recall that $\Delta(\lambda)$ is generated by the highest weight vector $v_\lambda$. Since $\beta$ is surjective, there exists weight vector $0 \neq v \in N_\lambda$, such that $\beta(v) = v_\lambda$. Moreover, $v$ must be a maximal vector. Otherwise, there exists $d_i \in \mathfrak{W}^+, 0 \neq d_i \cdot v \in N_{\lambda+i}$ such that $\lambda + i \in \text{Supp}(N)$, which contradicts to the maximality of $\lambda$. The map such that $v_\lambda \mapsto v$ can be extended to a $U(\mathfrak{W})$-module homomorphism $\beta': \Delta(\lambda) \rightarrow N$, and $\beta\beta' = 1_{\Delta(\lambda)}$. So the exact sequence (3.3) is split and hence $\text{Ext}_0^1(\Delta(\lambda), M) = 0$.

(2) is an immediate corollary of (1). $\square$

Let us recall the $\mathfrak{W}$-modules $F_\lambda$ of Feigin and Fuchs defined in [9], with $\lambda \in \mathbb{C}$. The module $F_\lambda$ has a basis $\{f_j \mid j \in \mathbb{Z}_{\geq 0}\}$ with the $\mathfrak{W}$-action defined by

$$d_i f_j = (j - (i + 1)\lambda)f_{i+j},$$

where $i \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}_{\geq 0}$. These modules are shown to be restricted dualities of Verma modules. For a weight $\mathfrak{W}$-module $V = \oplus_\lambda V_\lambda$, the restricted duality $\mathfrak{W}$-module $V^* = \oplus_\lambda \text{Hom}_\mathbb{C}(V_\lambda, \mathbb{C})$ is defined by the natural action:

$$(d_i \phi)(v) = \phi(-d_i v),$$

for all $i \in \mathbb{Z}_{\geq 0}, \phi \in V^*, v \in V$. By the universal property of $\Delta(\lambda)$, one can check that $F_\lambda^* \cong \Delta(\lambda)$. Feigin and Fuchs gave the classification of the extensions of $F_\mu$ by the modules $F_\lambda$.

**Proposition 3.6.** [9] Suppose that $\lambda, \mu \in \mathbb{C}$. Then

$$\text{Ext}_0^1(U(\mathfrak{w}), F_\lambda, F_\mu) = \begin{cases} \mathbb{C}, & \text{if } \lambda - \mu = 0, 2, 3, 4; \\ \mathbb{C} \oplus \mathbb{C}, & \text{if } (\lambda, \mu) = (0, -1); \\ \mathbb{C}, & \text{if } (\lambda, \mu) = (0, -5) \text{ or } (4, -1); \\ \mathbb{C}, & \text{if } (\lambda, \mu) = \left(\frac{3+\sqrt{19}}{2}, \frac{-7+\sqrt{19}}{2}\right); \\ 0, & \text{otherwise.} \end{cases}$$

Moreover all nontrivial extensions of $F_\mu$ by $F_\lambda$ were listed in the table 1 on page 207 of [9]. We give these extensions in a slightly different form as follows.

(1) The unique non-split extension $E(F_\lambda, F_\lambda)$ of $F_\lambda$ by itself has a basis $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j - (i + 1)\lambda)f_{i+j},$$

$$d_i f'_j = (j - (i + 1)\lambda)f'_{i+j} + (i + 1)f_{i+j}.$$
There are two non-split extensions: $E(F_0, F_{-1}), E'(F_0, F_{-1})$ of $F_{-1}$ by $F_0$. The module $E(F_0, F_{-1})$ has a basis $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j + i + 1)f_{i+j},
\quad d_i f'_j = j f'_{i+j} + (i + 1) f_{i+j-1}.$$  

The module $E'(F_0, F_{-1})$ has a basis $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j + i + 1)f_{i+j},
\quad d_i f'_j = j f'_{i+j} + (i + 1) f_{i+j-1}.$$  

The unique non-split extension $E(F_\lambda, F_{\lambda-2})$ of $F_{\lambda-2}$ by $F_\lambda$ has a basis $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j - (i + 1)(\lambda - 2))f_{i+j},
\quad d_i f'_j = (j - (i + 1)\lambda) f'_{i+j} + ((i + 1)i(i - 1) + 2(i + 1)ij)f_{i+j-2}.$$  

The unique non-split extension $E(F_\lambda, F_{\lambda-3})$ of $F_{\lambda-3}$ by $F_\lambda$ has a basis $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j - (i + 1)(\lambda - 3))f_{i+j},
\quad d_i f'_j = (j - (i + 1)\lambda) f'_{i+j} + ((i + 1)i(i - 1)j + (i + 1)ij(j - 1))f_{i+j-3}.$$  

The unique non-split extension $E(F_\lambda, F_{\lambda-4})$ of $F_{\lambda-4}$ by $F_\lambda$ has a basis $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j - (i + 1)(\lambda - 4))f_{i+j},
\quad d_i f'_j = (j - (i + 1)\lambda) f'_{i+j} + \left(\frac{(i + 1)!}{(i - 4)!}\lambda + \frac{(i + 1)j}{(i - 3)!}\right)
- 6(i + 1)i(i - 1)j(j - 1) - 4(i + 1)ij(j - 1)(j - 2))f_{i+j-4}.$$  

The unique non-split extension $E(F_0, F_{-5})$ of $F_{-5}$ by $F_0$ has a basis $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j + 5(i + 1))f_{i+j},
\quad d_i f'_j = j f'_{i+j} + \left(2\frac{(i + 1)!j}{(i - 4)!} - 5\frac{(i + 1)!j(j - 1)}{(i - 3)!}\right)
+ 10(i + 1)i(i - 1)j(j - 1)(j - 2) + 5(i + 1)ij(j - 1)(j - 2)(j - 3))f_{i+j-5}.$$  

The unique non-split extension $E(F_4, F_{-1})$ of $F_{-1}$ by $F_4$ has a basis $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j + i + 1)f_{i+j},
\quad d_i f'_j = (j - 4(i + 1))f'_{i+j} + \left(12\frac{(i + 1)!}{(i - 5)!} + 22\frac{(i + 1)!j}{(i - 4)!} + 5\frac{(i + 1)!j(j - 1)}{(i - 3)!}\right)
- 10(i + 1)i(i - 1)j(j - 1)(j - 2) - 5(i + 1)ij(j - 1)(j - 2)(j - 3))f_{i+j-5}.$$
Lemma 3.8. If $(\lambda, \mu) = (\frac{5+\sqrt{19}}{2}, -\frac{7+\sqrt{19}}{2})$, the unique non-split extension $E(F_\lambda, F_\mu)$ of $F_\mu$ by $F_\lambda$ has a basis $\{f_j, f_j' \mid j \in \mathbb{Z}_{\geq 0}\}$ such that

$$d_i f_j = (j - (i + 1) \mu) f_{i+j},$$

$$d_i f_j' = (j - (i + 1) \lambda) f'_{i+j} + \left(\frac{(i + 1)!((22 \pm 5\sqrt{19})}{(i-6)!4} - \frac{(i + 1)!j(31 \pm 7\sqrt{19})}{(i-5)!2}ight)
- \frac{(i + 1)!j(j-1)(25 \pm 7\sqrt{19})}{(i-3)!} - \frac{(i + 1)!j(j-1)(j-2)5}{(i-3)!}
+ 5(i+1)(i-1)(j-1)(j-2)(j-3) + \frac{j!2(i+1)j}{(j-5)!} f_{i+j-6}.$$

In the above formulas, $f_j = f_j' = 0$ if $j < 0$. By the isomorphism $F_\lambda^* \cong \Delta(\lambda)$, we obtain all nontrivial extensions between Verma modules.

Proposition 3.7. Suppose that $\lambda, \mu \in \mathbb{C}$. Then

$$\text{Ext}^1_\mathcal{O}(\Delta(\mu), \Delta(\lambda)) = \begin{cases} 
\mathbb{C}, & \text{if } \lambda - \mu = 2, 3, 4; \\
\mathbb{C}, & \text{if } (\lambda, \mu) = (0, -1), (0, -5) \text{ or } (4, -1); \\
\mathbb{C}, & \text{if } (\lambda, \mu) = (\frac{5\pm\sqrt{19}}{2}, -\frac{7\pm\sqrt{19}}{2}); \\
0, & \text{otherwise.}
\end{cases}$$

Proof. By $\dim \text{Ext}^1_\mathcal{O}(\Delta(\mu), \Delta(\lambda)) \leq \dim \text{Ext}^1_{U(\mathfrak{g})}(\Delta(\mu), \Delta(\lambda))$ and $F_\lambda^* \cong \Delta(\lambda)$, we have

$$\dim \text{Ext}^1_\mathcal{O}(\Delta(\mu), \Delta(\lambda)) \leq \begin{cases} 
1, & \text{if } \lambda - \mu = 0, 2, 3, 4; \\
2, & \text{if } (\lambda, \mu) = (0, -1); \\
1, & \text{if } (\lambda, \mu) = (0, -5) \text{ or } (4, -1); \\
1, & \text{if } (\lambda, \mu) = (\frac{5\pm\sqrt{19}}{2}, -\frac{7\pm\sqrt{19}}{2}).
\end{cases}$$

If $E(F_\lambda, F_\lambda)$ is a weight module, then there are nonzero $a_j \in \mathbb{C}$ such that

$$d_0(f_j' + a_j f_j) = (j - \lambda)(f_j' + a_j f_j)$$

for almost all $j$. However on the other side, $d_0(f_j' + a_j f_j) = (j - \lambda)f_j' + f_j + a_j(j - \lambda)f_j$, which is a contradiction. So $E(F_\lambda, F_\lambda)$ is not a weight module, and hence $\text{Ext}^1_\mathcal{O}(\Delta(\lambda), \Delta(\lambda)) = 0$. In fact, by (2) in Lemma 3.5, we can also see that $\text{Ext}^1_\mathcal{O}(\Delta(\lambda), \Delta(\lambda)) = 0$.

Similarly $E(F_0, F_{-1})$ is not a weight module. By the action of $d_0$ on $f_j'$, we can see that $E'(F_0, F_{-1})$, $E(F_\lambda, F_{\lambda-2})$, $E(F_\lambda, F_{\lambda-3})$, $E(F_\lambda, F_{\lambda-4})$, $E(F_0, F_{-5})$, $E(F_4, F_{-1})$, and $E(F_{\frac{5+\sqrt{19}}{2}}, F_{\frac{7+\sqrt{19}}{2}})$ are weight modules. So these modules are also no-split extensions between Verma modules in $\mathcal{O}$. Then we can complete the proof.

3.3. Extensions between simple modules. In this subsection, we compute $\text{Ext}^1_\mathcal{O}(M, N)$ for all simple modules $M, N \in \mathcal{O}$.

Lemma 3.8. If $\lambda - \mu \notin \mathbb{Z}$, then $\text{Ext}^1_\mathcal{O}(L(\lambda), L(\mu)) = 0$.
Proof. If $M \in \mathcal{O}$ such that $L(\mu) \subset M$ and $M/L(\mu) \cong L(\lambda)$, then $\text{Supp}(M) = \text{Supp}(L(\mu)) \cup \text{Supp}(L(\lambda))$. Since $\lambda - \mu \not\in \mathbb{Z}$, $\text{Supp}(L(\mu)) \cap \text{Supp}(L(\lambda)) = \emptyset$. So $M \cong L(\mu) \oplus L(\lambda)$.

Lemma 3.9. (1) For all $\lambda \in \mathbb{C}$, $\dim \text{Ext}^1_{\mathcal{O}}(L(\lambda), L(\lambda)) = 0$.

(2) We have $\dim \text{Ext}^1_{\mathcal{O}}(L(0), L(-1)) = 1$. That is, if

$$0 \to \Delta(-1) \to M \to L(0) \to 0,$$

is a non-split exact sequence of $\mathcal{M}$-modules in $\mathcal{O}$, then $M \cong \Delta(0)$.

(3) $\dim \text{Ext}^1_{\mathcal{O}}(L(0), L(\lambda)) = 0$ for all $\lambda \in \mathbb{C} \setminus \{-1\}$.

Proof. (1) When $\lambda \neq 0$, thanks to (1) in Lemma 3.5 and $L(\lambda) = \Delta(\lambda)$, we obtain $\text{Ext}^1(L(\lambda), L(\lambda)) = 0$. It is enough to prove that $\text{Ext}^1_{\mathcal{O}}(L(0), L(0)) = 0$.

Consider the short exact sequence

$$0 \to \Delta(-1) \to \Delta(0) \to L(0) \to 0,$$

where $\Delta(-1)$ is the unique nonzero submodule of $\Delta(0)$. We can get a long exact sequence by using the functor $\text{Hom}_{\mathcal{O}}(-, L(0))$:

$$\cdots \to \text{Hom}_{\mathcal{O}}(\Delta(-1), L(0)) \to \text{Ext}^1_{\mathcal{O}}(L(0), L(0)) \to \text{Ext}^1_{\mathcal{O}}(\Delta(0), L(0)) \to \cdots.$$

According to (1) in Lemma 3.5, and the fact that $L(0)$ is not a composition factor of $\Delta(-1)$, we have $\text{Ext}^1_{\mathcal{O}}(\Delta(0), L(0)) = 0$, $\text{Hom}_{\mathcal{O}}(\Delta(-1), L(0)) = 0$, whence $\text{Ext}^1_{\mathcal{O}}(L(0), L(0)) = 0$. Thus $\text{Ext}^1_{\mathcal{O}}(L(\lambda), L(\lambda)) = 0$ for any $\lambda \in \mathbb{C}$.

(2) Consider the short exact sequence

$$0 \to \Delta(-1) \to \Delta(0) \to L(0) \to 0,$$

where $\Delta(-1)$ is the maximal submodule of codimension 1 of $\Delta(0)$. According to Lemma 3.5, we have $\text{Ext}^1_{\mathcal{O}}(\Delta(0), L(-1)) = 0$. Applying $\text{Hom}_{\mathcal{O}}(-, L(-1))$ to (3.5), from $\text{Hom}_{\mathcal{O}}(\Delta(0), L(-1)) = 0$, we can get

$$0 \to \text{Hom}_{\mathcal{O}}(\Delta(-1), L(-1)) \to \text{Ext}^1_{\mathcal{O}}(L(0), L(-1)) \to \text{Ext}^1_{\mathcal{O}}(\Delta(0), L(-1)) \to \cdots.$$

Thus $\text{Ext}^1_{\mathcal{O}}(L(0), L(-1)) \cong \text{Hom}_{\mathcal{O}}(\Delta(-1), L(-1)) \cong \mathbb{C}$.

(3) By (1) and (2), we can assume that $\lambda \neq 0, -1$. Let $M$ be a non-split extension of $L(\lambda)$ by $L(0)$ in $\mathcal{O}$. We can suppose that $L(\lambda) \subset M$ and $M/L(\lambda) = L(0)$. Then $1 \leq \dim M_0 \leq 2$. There must exists $e_0' \in M_0 \setminus L(\lambda)_0$ such that $d_1e_0' = 0$. Thus $d_1d_-1e_0' = 0$. As $zd_-1e_0' = (\lambda + 1)d_-1e_0' = d_-1ze_0' = 0$ and $\lambda \neq 0, -1$, we deduce that $d_-1e_0' = 0$. Since $M$ is indecomposable, $d_2e_0'$ is a nonzero element in $L(\lambda)$. So $[d_-1, d_2]e_0' = 0$ implies that $d_-1d_2e_0' = 0$, contradicting with that $d_-1$ acts injectively on $L(\lambda)$.

$\square$
Next we use the relative Lie algebra cohomology to compute \( \dim \text{Ext}^1_\mathcal{O}(L(\lambda), L(0)) \).

For two weight \( \mathfrak{m} \)-modules \( M, N \),
\[
\text{Ext}^1_{\mathfrak{m}, \mathfrak{n}}(M, N) \cong H^1(\mathfrak{m}, \mathfrak{n}; \text{Hom}_\mathcal{O}(M, N))
\cong C^1(\mathfrak{m}, \mathfrak{n}; \text{Hom}_\mathcal{O}(M, N))/B^1(\mathfrak{m}, \mathfrak{n}; \text{Hom}_\mathcal{O}(M, N)),
\]
where the set of 1-cocycles \( C^1(\mathfrak{m}, \mathfrak{n}; \text{Hom}_\mathcal{O}(M, N)) \) is the subspace of all \( c \in \text{Hom}_\mathfrak{m}(\mathfrak{m}, \mathfrak{n}; \text{Hom}_\mathcal{O}(M, N)) \) such that
\[
(3.6) \quad c(\mathfrak{h}) = 0, \quad c([g_1, g_2]) = [g_1, c(g_2)] - [g_2, c(g_1)];
\]
for all \( g_1, g_2 \in \mathfrak{m} \), where \( [g, \psi] \in \text{Hom}_\mathcal{O}(M, N) \) such that
\[
[g, \psi](v) = g\psi(v) - \psi(gv),
\]
for \( g \in \mathfrak{m} \), \( \psi \in \text{Hom}_\mathcal{O}(M, N), v \in M \). A 1-cocycle \( c \) is a coboundary if there is a \( \psi \in \text{Hom}_\mathfrak{m}(M, N) \) such that \( c(g) = [g, \psi] \) for any \( g \in \mathfrak{m} \).

**Lemma 3.10.** \( \dim \text{Ext}^1_\mathcal{O}(L(\lambda), L(0)) = \begin{cases} 1, & \text{if } \lambda = -1, -2; \\ 0, & \text{if } \lambda \neq -1, -2. \end{cases} \)

**Proof.** By (1) in Lemma 3.9, we can assume \( \lambda \neq 0 \). So \( L(\lambda) = \Delta(\lambda) = U(\mathfrak{m})v_\lambda = \mathbb{C}[d_{-1}]v_\lambda \). Suppose that \( L(0) = \mathbb{C}v_0 \).

According to (3.1.2) in [17], we have
\[
\text{Ext}^1_\mathcal{O}(L(\lambda), L(0)) \cong \text{Ext}^1_{(\mathfrak{m}, \mathfrak{n})}(\Delta(\lambda), L(0))
\cong \text{Ext}^1_{(\mathfrak{m}, \mathfrak{n})}(U(\mathfrak{m}) \otimes U(\mathfrak{n}) \mathbb{C}_\lambda, L(0))
\cong \text{Ext}^1_{(\mathfrak{h}, \mathfrak{n})}(\mathbb{C}_\lambda, L(0))
\cong H^1(\mathfrak{h}, \mathfrak{n}, \text{Hom}_\mathcal{O}(\mathbb{C}_\lambda, L(0))),
\]
where \( \mathbb{C}_\lambda = \mathbb{C}v_\lambda \) is the one dimensional \( \mathfrak{h} \)-module.

For \( \omega \in C^1(\mathfrak{h}, \mathfrak{n}; \text{Hom}_\mathcal{O}(\mathbb{C}_\lambda, L(0))), k, j \in \mathbb{Z}_{\geq 0}, \) we have
\[
(3.7) \quad (j - k)\omega(d_{k+j}) = [d_k, \omega(d_j)] - [d_j, \omega(d_k)].
\]

Taking \( k = 0 \), by \( \omega(d_0) = 0 \), we have \( j\omega(d_j) = [d_0, \omega(d_j)] \). After multiplying a suitable scalar, we can assume that \( \omega(d_j)(v_\lambda) = \delta_{\lambda+j,0}v_0 \). If \( \lambda \in \mathbb{Z}_{\leq -3} \), then \( \omega(d_1) = \omega(d_2) = 0 \), hence \( \omega = 0 \) and \( \dim \text{Ext}^1_\mathcal{O}(L(\lambda), L(0)) = 0 \). If \( \lambda = -2 \), then \( \omega(d_2)(v_{-2}) = v_0, \omega(d_j) = 0 \) for any \( j \neq 2 \). So from \( B^1(\mathfrak{h}, \mathfrak{n}; \text{Hom}_\mathcal{O}(\mathbb{C}_\lambda, L(0))) = 0 \), we have \( \dim \text{Ext}^1_\mathcal{O}(L(-2), L(0)) = 1 \). Similarly \( \dim \text{Ext}^1_\mathcal{O}(L(-1), L(0)) = 1 \).

We can summarize the results on extensions of simple modules as follows:

**Theorem 3.11.** Suppose that \( \lambda, \mu \in \mathbb{C} \). Then
\[
\text{Ext}^1_\mathcal{O}(L(\mu), L(\lambda)) = \begin{cases} 
\mathbb{C}, & \text{if } \lambda - \mu = 2, 3, 4, \lambda \mu \neq 0; \\
\mathbb{C}, & \text{if } (\lambda, \mu) = (0, -1), (0, -2), (-1, 0) \text{ or } (4, -1); \\
\mathbb{C}, & \text{if } (\lambda, \mu) = \left(\frac{5 \pm \sqrt{13}}{2}, \frac{7 \pm \sqrt{13}}{2}\right); \\
0, & \text{otherwise.}
\end{cases}
\]
It should be mentioned that extensions between simple modules for the finite dimensional Witt algebra $W(1, 1)$ over an algebraically closed field of characteristic $p > 3$ were determined in [3].

3.4. Block decomposition of $\mathcal{O}$. We first recall the notion of blocks of an abelian category $\mathcal{C}$. We assume that any object of $\mathcal{C}$ has finite composition length. We introduce an equivalence relation on the set of isomorphism classes of simple objects of $\mathcal{C}$ as follows: two simple objects $V, V'$ are equivalent if there exists a sequence $V = V_1, V_2, \ldots, V_r = V'$ of simple objects satisfying $\text{Ext}^1_{\mathcal{C}}(V_i, V_{i+1}) \neq 0$ or $\text{Ext}^1_{\mathcal{C}}(V_{i+1}, V_i) \neq 0$ for all $i$. Then for each equivalence class $\chi$, we denote by $\mathcal{C}_\chi$ the full subcategory of $\mathcal{C}$ consisting of objects whose all composition factors belong to $\chi$. Each $\mathcal{C}_\chi$ is called a block of $\mathcal{C}$ and

\begin{equation}
\mathcal{C} = \bigoplus_{\chi} \mathcal{C}_\chi.
\end{equation}

Moreover, each $\mathcal{C}_\chi$ cannot be decomposed into a direct sum of two nontrivial abelian full subcategories. The decomposition in (3.8) is called the block decomposition of the category $\mathcal{C}$.

For any $\lambda \in \mathbb{Z}$, let $\mathcal{O}_{[\lambda]}$ be the full subcategory of $\mathcal{O}$ consisting of modules $M$ such that $\text{Supp}M \subset \lambda + \mathbb{Z}$.

**Proposition 3.12.** We have the block decomposition $\mathcal{O} = \bigoplus_{\lambda \in \mathbb{C}/\mathbb{Z}} \mathcal{O}_{[\lambda]}$, each $\mathcal{O}_{[\lambda]}$ is indecomposable. The set $\{L(\lambda + n) \mid n \in \mathbb{Z}\}$ is the set of all simple modules in $\mathcal{O}_{[\lambda]}$.

**Proof.** This result follows from Theorem 3.11. □

Let $\mathbb{C}\langle x_1, x_2 \rangle$ be the free associative algebra over $\mathbb{C}$ in two variables $x_1, x_2$. Recall that an abelian category $\mathcal{C}$ is wild if there exists an exact functor from the category of finite dimensional representations of the algebra $\mathbb{C}\langle x_1, x_2 \rangle$ to $\mathcal{C}$ which preserves indecomposability and takes non-isomorphic modules to non-isomorphic ones, see Definition 2 in [18]. The following Lemma is useful for the study of representations of infinite dimensional algebras. For its proof, one can see Proposition 2.1 in [11]

**Lemma 3.13.** Let $\mathcal{C}$ be an abelian category. If the Ext-quiver of $\mathcal{C}$ contains a finite subquiver $Q$ whose underlying unoriented graph is neither a Dynkin nor an affine diagram such that two arrows in $Q$ can not be concatenated, then $\mathcal{C}$ is wild.

**Theorem 3.14.** For any $\lambda \in \mathbb{C}$, the block $\mathcal{O}_{[\lambda]}$ is wild.
Proof. By Theorem 3.11, the Ext-quiver of every block $O_{[\lambda]}$ contains the following subquiver:

$$
\begin{array}{c}
L(\mu) \\
\downarrow \\
L(\mu - 3) \\
\downarrow \\
L(\mu - 4) \longrightarrow L(\mu - 2) \\
\downarrow \\
L(\mu - 1),
\end{array}
$$

where $\mu \in \lambda + \mathbb{Z}$ such that $\mu(\mu - 1)(\mu - 2)(\mu - 3)(\mu - 4) \neq 0$. This subquiver is neither a Dynkin nor an affine diagram. So the block $O_{[\lambda]}$ is wild by Lemma 3.13. 

Remark 3.15. We can compare the category $O$ of $\mathfrak{W}$ with the category $O_{\mathfrak{sl}_2}$ of $\mathfrak{sl}_2$. Each non-regular block of $O_{\mathfrak{sl}_2}$ is semi-simple. Every regular block of $O_{\mathfrak{sl}_2}$ is equivalent to the category of finite dimensional representations over $\mathbb{C}$ of the following quiver

$$
\begin{array}{c}
\bullet \\
\Gamma \\
\bullet
\end{array}
$$

$ab = 0.$

So every block of $O_{\mathfrak{sl}_2}$ is not wild. It should be mentioned that representation types of all blocks of $O$ for complex simple Lie algebras were independently obtained in [6] and [10].

3.5. Relation between $O$ and $\Omega_a$. In this subsection, we always assume $a$ is a nonzero complex number. Let

$$H'_a = \{ u \in U(\mathfrak{b}) \mid u(d_{-1} - a) \subset (d_{-1} - a)U(\mathfrak{W}) \}$$

which is a subalgebra of $U(\mathfrak{b})$.

Lemma 3.16. The algebra $H'_a$ is isomorphic to the algebra $H_a$ defined in (2.1).

Proof. For any $u \in H'_a$, we define a $\psi_u \in H_a = \text{End}_{\mathfrak{W}}(Q_a)^{\text{op}}$ such that $\psi_u(1_a) = u1_a$. Then one can check that the linear map

$$\psi : H'_a \rightarrow H_a, u \mapsto \psi_u,$$

is an algebra isomorphism. 

In view of Lemma 3.16, by Theorem 2.5, any finite dimensional nonzero simple $H'_a$ is one dimensional.

We define a functor $\Gamma_a$ (called the Whittaker coinvariants functor) from $O$ to the category of finite dimensional $H'_a$-modules. For any module $M \in O_{[\lambda]}$, let

$$\Gamma_a(M) = M/(d_{-1} - a)M.$$ 

Clearly $\Gamma_a(M)$ is an $H'_a$-module. The following result is immediate.
Lemma 3.17. \( \dim \Gamma_a(L(\lambda)) = \begin{cases} 1, & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases} \)

So \( \Gamma_a \) maps simple modules in \( \mathcal{O} \) to simple \( H_a' \)-modules or zero. This property is similar as the functor defined by Backelin, see [1]. Let \( \Omega'_a \) be the category of finite dimensional \( H_a' \)-modules. By Theorem 2.5 and Lemma 3.16, \( \Omega'_a \) is equivalent to the Whittaker category \( \Omega_a \).

Theorem 3.18. The functor \( \Gamma_a : \mathcal{O} \to \Omega'_a \) is exact.

Proof. Suppose that

\[
0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0,
\]

is an exact sequence in \( \mathcal{O} \). We will show that

\[
0 \to \Gamma_a(N) \xrightarrow{\Gamma_a(\alpha)} \Gamma_a(M) \xrightarrow{\Gamma_a(\beta)} \Gamma_a(L) \to 0,
\]

is exact.

From the surjectivity of \( \beta \) and \( \Gamma_a(\beta)\Gamma_a(\alpha) = 0 \), we see that \( \Gamma_a(\beta) \) is surjective and \( \text{Im} \Gamma_a(\alpha) \subset \text{Ker} \Gamma_a(\beta) \). We need to show that \( \Gamma_a(\alpha) \) is injective and \( \text{Ker} \Gamma_a(\beta) \subset \text{Im} \Gamma_a(\alpha) \).

For \( n + (d_{-1} - a)N \in \text{Ker} \Gamma_a(\alpha) \), there exists some \( m \in M \) such that \( \alpha(n) = (d_{-1} - a)m \). Then \( (d_{-1} - a)\beta(m) = \beta \alpha(n) = 0 \). Since \( a \neq 0 \) and \( L \) is a weight module, \( d_{-1} - a \) acts injectively on \( L \). Thus \( \beta(m) = 0 \) and there is an \( n_1 \in N \) such that \( m = \alpha(n_1) \). Consequently, \( \alpha(n) = (d_{-1} - a)\alpha(n_1) \). Then the injectivity of \( \alpha \) implies that \( n = (d_{-1} - a)n_1 \), i.e., \( n \in (d_{-1} - a)N \). So \( \Gamma_a(\alpha) \) is injective.

For \( m + (d_{-1} - a)M \in \text{Ker} \Gamma_a(\beta) \), we have that \( \beta(m) = (d_{-1} - a)m \) for some \( l \in L \). As \( \beta \) is surjective, \( \beta(m) = (d_{-1} - a)\beta(m') \) for some \( m' \in M \), i.e., \( m - (d_{-1} - a)m' \in \text{Ker} \beta = \text{Im} \alpha \). So \( m - (d_{-1} - a)m' = \alpha(n) \) for some \( n \in N \), hence \( \text{Ker} \Gamma_a(\beta) \subset \text{Im} \Gamma_a(\alpha) \). Therefore \( \Gamma_a : \mathcal{O} \to \Omega'_a \) is exact.

If we identify \( \Omega'_a \) with \( \Omega_a \), then \( \Gamma_a \) is actually an exact functor from \( \mathcal{O} \) to the Whittaker category \( \Omega_a \). By Lemma 3.17 and the exactness of \( \Gamma_a \), we obtain the following property of \( \Gamma_a \).

Corollary 3.19. If \( \lambda \notin \mathbb{Z} \), then for any \( M \in \mathcal{O}_{[\lambda]} \), \( \dim \Gamma_a(M) \) is equal to the composition length of \( M \).

Denote the restriction of \( \Gamma_a \) to \( \mathcal{O}_{[\lambda]} \) by \( \Gamma_a^{[\lambda]} \), and by \( \Omega'_{a,[\lambda]} \) the subcategory of \( \Omega'_a \) consisting of the \( H_a' \)-modules isomorphic to \( \Gamma_a^{[\lambda]}(M) \) for \( M \in \mathcal{O}_{[\lambda]} \). Finally we give a conjecture on \( \mathcal{O}_{[\lambda]} \).

Conjecture 3.1. There is a \( \lambda \notin \mathbb{Z} \) such that \( \mathcal{O}_{[\lambda]} \) is equivalent to \( \Omega'_{a,[\lambda]} \).
4. Tensor \( \mathfrak{W} \)-modules from \( \mathfrak{b} \)-modules and Weyl modules

Let \( \mathfrak{D} \) be the Weyl algebra of rank one, that is, \( \mathfrak{D} \) is the associative algebra over \( \mathbb{C} \) generated by \( x, \partial \) subject to the relation \( [\partial, x] = 1 \). In this section, we construct simple \( \mathfrak{W} \)-modules from \( \mathfrak{D} \)-modules and \( \mathfrak{b} \)-modules.

4.1. Tensor module \( T(P, V) \). The following interesting algebra homomorphism was given in [20] and [5] independently, which plays an important role in the classification of simple weight modules for \( \mathfrak{W}_n \), see [20, 16].

**Lemma 4.1.** There is an algebra monomorphism \( \phi \) from \( \mathfrak{U}(\mathfrak{W}) \) to \( \mathfrak{D} \otimes \mathfrak{U}(\mathfrak{b}) \) such that

\[
\begin{align*}
    d_{-1} &\mapsto \partial \otimes 1, \\
    d_m &\mapsto x^{m+1} \partial \otimes 1 + \sum_{r=0}^{m} \binom{m+1}{r+1} x^{m-r} \otimes d_r, \quad m \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

By Lemma 4.1, for any \( \mathfrak{D} \)-module \( P \), any \( \mathfrak{b} \)-module \( V \), the tensor product \( P \otimes V \) can be made to be a \( \mathfrak{W} \)-module denoted by \( T(P, V) \).

**Remark 4.2.** Let \( S = \mathbb{C}[x^{\pm 1}] / \mathbb{C}[x] \) which is a simple \( \mathfrak{D} \)-module. Then we have a functor \( T(S, -) \) from the category \( \mathfrak{b} \)-mod of finite dimensional \( \mathfrak{b} \)-modules to the category \( \mathcal{O} \). We originally intended to use the functor \( T(S, -) \) to study \( \mathcal{O} \). However, in view of extensions between simple modules in \( \mathcal{O} \), \( T(S, V) \) may be decomposable for some indecomposable \( \mathfrak{b} \)-module \( V \). Nevertheless, we can use the bifunctor \( T(-, -) \) to construct simple \( \mathfrak{W} \)-modules.

In the following theorem, we will give the simplicity of \( T(P, V) \) under some natural conditions.

**Theorem 4.3.** Suppose that \( P \) is a simple \( \mathfrak{D} \)-module, and \( V \) is a simple \( \mathfrak{b} \)-module such that there is an \( l \in \mathbb{Z}_{>0} \) satisfying

1. \( d_l \) acts injectively on \( V \);
2. \( d_i V = 0 \) for any \( i > l \),

then \( T(P, V) \) is a simple \( \mathfrak{W} \)-module.

**Proof.** Let \( N \) be a nonzero submodule of \( T(P, V) \). Suppose that \( u = \sum_{n=0}^{q} p_n \otimes v_n \) is a nonzero element in \( N \), where \( p_0, \ldots, p_q \) are linearly independent.

**Claim 1.** For any \( X \in \mathfrak{D} \), we have that \( \sum_{n=0}^{q} X p_n \otimes d_l^2 v_n \in N \).
For any $k$ with $k \geq 2l$ and any $m$ with $-1 \leq m \leq 2l + 1$, we can compute that
\[
d_{k-m}d_m(p \otimes v) = d_{k-m}(d_mp \otimes v + \sum_{r=0}^{m} \binom{m+1}{r+1} x^{m-r} p \otimes d_r v)
\]
\[
= (d_{k-m}d_mp) \otimes v + \sum_{r=0}^{m} \binom{m+1}{r+1} (d_{k-m}x^{m-r}p) \otimes d_r v
\]
\[
+ \sum_{s=0}^{k-m} \binom{k-m+1}{s+1} x^{k-m-s}d_mp \otimes d_sv
\]
\[
+ \sum_{s=0}^{k-m} \sum_{r=0}^{m} \binom{k-m+1}{s+1} \binom{m+1}{r+1} x^{k-r-s}p \otimes d_sd_r v
\]
\[
= m^{2l+2}x^k x^{-2l} p \otimes \frac{d_i^2 v}{((l+1)!)^2} + g(m),
\]
where $g(m)$ is the term with degree of $m$ smaller than $2l + 2$. Consider the coefficient of $m^{2l+2}$ in $d_{k-m}d_mu$. By letting $m = -1, 0, 1, \ldots, 2l + 1$, using the Vandermonde matrix, we deduce that $\sum_{n=0}^{q} x^i p_n \otimes d_i^2 v_n \in N$, for all $i \in \mathbb{Z}_{\geq 0}$. From the action of $d_{-1}$ on $N$, we see that $\sum_{n=0}^{q} \partial^i x^j p_n \otimes d_i^2 v_n \in N$ for all $i, j \in \mathbb{Z}_{\geq 0}$. So $\sum_{n=0}^{q} Xp_n \otimes d_i^2 v_n \in N$ for any $X \in \mathfrak{D}$. Then Claim 1 follows.

**Claim 2.** $P \otimes d_i^2 v_n \subset N$, for any $0 \leq n \leq q$.

Since $P$ is a simple $\mathfrak{D}$-module, by the Jacobson density theorem, for any $p \in P$, there is a $X_n$ such that
\[
X_n p_i = \delta_{n,i} p, \quad i = 0, \ldots, q.
\]
By Claim 1, we obtain that $P \otimes d_i^2 v_n \subset N$, for any $0 \leq n \leq q$. Claim 2 follows.

**Claim 3.** $N = T(P, V)$. Hence $T(P, V)$ is simple.

Let $V_1 = \{v \in V \mid P \otimes v \subset N\}$. By Claim 2 and that $d_l$ acts injectively on $V$, $V_1 \neq 0$. For any $v \in V_1, p \in P$, taking $m = 0, 1, \ldots, l$ in
\[
d_m(p \otimes v) = d_mp \otimes v + \sum_{r=0}^{m} \binom{m+1}{r+1} x^{m-r} p \otimes d_r v,
\]
we can see that $p \otimes d_0 v, p \otimes d_1 v, \ldots, p \otimes d_l v \in N$. So $V_1$ is a nonzero $\mathfrak{b}$-submodule of $V$. The simplicity of $V$ forces that $V_1 = V$. Then Claim 3 follows.

\[\square\]

4.2. Isomorphism criterion for $T(P, V)$. Next we give the following isomorphism criterion for $T(P, V)$.

**Theorem 4.4.** Suppose that $P, P'$ are simple $\mathfrak{D}$-modules, $V, V'$ are simple $\mathfrak{b}$-modules such that there are $l, s \in \mathbb{Z}_{\geq 0}$ satisfying $d_l$ (resp. $d_s$) acts injectively on
\( V \) (resp. \( V' \)) and \( d_i V = 0 \) (resp. \( d_i V' = 0 \)) for any \( i > 1 \) (resp. \( i > s \)). Then \( T(P, V) \cong T(P', V') \) if and only if \( P \cong P', l = s \) and \( V \cong V' \).

**Proof.** The sufficiency is obvious. Now suppose that

\[
\psi : T(P, V) \to T(P', V')
\]

is an isomorphism of \( \mathfrak{W} \)-modules. Let \( p \otimes v \) be a nonzero element in \( T(P, V) \). Write

\[
\psi(p \otimes v) = \sum_{n=0}^{q} p'_n \otimes v'_n \in T(P', V'),
\]

where \( p'_0, \ldots, p'_q \) are linearly independent. Similar to the proof of Claim 1 of Theorem 4.3, comparing the the highest degree of \( m \) on both sides of

\[
\psi(d_{k-m}d_m(p \otimes v)) = d_{k-m}d_m \psi(p \otimes v),
\]

we have that \( l = s \) and

\[
\psi(Xp \otimes v) = \sum_{n=0}^{q} Xp_n \otimes d_l^2 v_n, \quad \forall X \in \mathfrak{D}.
\]

By the Jacobson density theorem, there exists \( Y \in \mathfrak{D} \) such that \( Yp_i = \delta_{i0} p_0 \). Then

\[
\psi(Yp \otimes v) = p_0 \otimes d_l^2 v_n.
\]

Replacing \( Yp \) by \( p, p_0 \) by \( p' \) and \( d_l^2 v_n \) by \( v' \), we have

\[
\psi(Xp \otimes v) = Xp' \otimes v', \quad \forall X \in \mathfrak{D}.
\]

Consequently \( \psi_1 : P \to P' \) satisfying \( \psi_1(Xp) = Xp' \) is a well-defined map. Since \( P, P' \) are simple \( \mathfrak{D} \)-modules,

\[
\text{Ann}_{\mathfrak{D}}(p) = \text{Ann}_{\mathfrak{D}}(p'), \quad \text{and} \quad P \cong \mathfrak{D}/\text{Ann}_{\mathfrak{D}}(p) \cong P'.
\]

Thus \( \psi_1 \) is a \( \mathfrak{D} \)-module isomorphism and

\[
\psi(p \otimes v) = \psi_1(p) \otimes v', \quad \forall p \in P.
\]

Then from \( \psi(d_m(p \otimes v)) = d_m \psi(p \otimes v) \), we obtain that

\[
\psi(p \otimes d_r v) = \psi_1(p) \otimes d_r v', \quad \forall p \in P, \quad r \in \mathbb{Z}_{\geq 0}.
\]

So

\[
\psi(p \otimes yv) = \psi_1(p) \otimes yv', \quad \forall p \in P, \forall y \in U(b).
\]

Therefore we have \( \text{Ann}_{U(b)}(v) = \text{Ann}_{U(b)}(v') \). The simplicity of \( V \) and \( V' \) implies that \( V \cong U(b)/\text{Ann}_{U(b)}(v) \cong V' \).
Remark 4.5. For each $r > 0$, denote the quotient algebra $b/(d_{r+i} : i > 0)$ by $a_r$. From Theorem 4.3, we know that, to obtain new simple $\mathfrak{g}$-modules $T(P, V)$, it is enough to construct infinite dimensional simple modules $V$ over $a_r$ for $r > 0$ such that the action of $d_r$ on $V$ is injective. Simple modules over $a_1$ and $a_2$ were classified in [15].

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