On the conservation of second-order cosmological perturbations in a scalar field dominated universe

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We discuss second-order cosmological perturbations on super-Hubble scales, in a scalar field dominated universe, such as during single field inflation. In this context we show that the gauge-invariant curvature perturbations defined on uniform density and comoving hypersurfaces coincide and that perturbations are adiabatic in the large scale limit. Since it has been recently shown that the uniform density curvature perturbation is conserved on large scales if perturbations are adiabatic, we conclude that both the uniform density and comoving curvature perturbations at second-order, in a scalar field dominated universe, are conserved. Finally, in the light of this result, we comment on the variables recently used in the literature to compute non-Gaussianities.

The study of second-order perturbation theory has recently become important (see e.g., also for earlier work), especially because primordial non-Gaussianities generated by inflation are typically only of second-order level. In the study of cosmological perturbations and non-Gaussianities it is very useful to establish results in terms of quantities that are conserved on large scales, i.e., on super-Hubble scales. At linear order, it is well known that the gauge invariant curvature perturbations defined on uniform density and comoving hypersurfaces – the so called uniform density and comoving curvature perturbations – coincide and are both conserved on large scales for adiabatic perturbations. On the other hand, when the universe is dominated by a single scalar field, such as during inflation, one can easily show that perturbations are adiabatic on large scales. By adiabatic we intend that the entropy perturbation (defined below) vanishes.

At second-order some results have been recently established on the conservation properties of these variables. In it has been shown that it is possible to define a gauge invariant quantity, the second-order generalization of the uniform density curvature perturbation, which is conserved if the entropy perturbation of the content of the universe vanishes. Furthermore, in two conserved (although not gauge invariant) quantities (originally defined in ) have been used to compute the level of non-Gaussianities produced from inflation. Gauge-invariant perturbation theory has also been studied in .

In this note we show, using the Einstein energy and momentum constraint equations at perturbed second-order, that the gauge invariant uniform density and comoving curvature perturbations and , given as the sum of a first and a second-order contribution to be defined below,

\[ \zeta = \zeta_1 + \frac{1}{2} \zeta_2, \quad \mathcal{R} = \mathcal{R}_1 + \frac{1}{2} \mathcal{R}_2, \]

 coincide (up to a sign) on large scales,

\[ \zeta + \mathcal{R} \approx 0. \]

Furthermore, we generalize at second-order the well-known first-order result that the non-adiabatic entropy perturbation of a scalar field on large scales is proportional to the sum of the uniform density and comoving curvature perturbations (see e.g., ), and thus vanishes,

\[ S = \mathcal{H} \left( \frac{\delta P}{P} - \frac{\delta \rho}{\rho} \right) = - \frac{1 - c_s^2}{c_s^2} (\zeta + \mathcal{R}) \approx 0, \]

where and are the density and pressure of the field,

\[ \delta \rho = \delta \rho_1 + \frac{1}{2} \delta \rho_2, \quad \delta P = \delta P_1 + \frac{1}{2} \delta P_2, \]

are their perturbations up to second-order, and

\[ c_s^2 = \frac{\dot{P}}{\dot{\rho}} \]

is the adiabatic speed of sound of the field. Therefore, since the conservation of at second-order has been shown to hold under the condition that the entropy perturbation of the content of the universe vanishes, i.e., when \( S = 0 \), our conclusion implies that both the uniform density and the comoving curvature perturbations are conserved on super-Hubble scales when the universe is dominated by a single minimally coupled scalar field,

\[ \zeta \approx -\mathcal{R}, \quad \dot{\zeta} \approx -\dot{\mathcal{R}} \approx 0. \]

The advantage of using a conserved curvature perturbation to compute non-Gaussianities produced in single field inflation, as it has been done in , is clear: since the comoving curvature perturbation is conserved on super-Hubble scales, non-Gaussianities in this variable can only be generated inside the Hubble radius due to the self-coupling of the inflaton fluctuations in the vacuum, and are later conserved in their super-Hubble evolution . Only the presence of more than one field, which makes perturbations to be non-adiabatic, can source curvature perturbations and generate non-Gaussianities on large scales. Examples of this case have been discussed in using the uniform density curvature perturbation.

We begin with a review of the first-order results. Then we define the gauge invariant second-order uniform density and comoving curvature perturbations for a scalar field and show that the source of their evolution equation, the non-adiabatic pressure perturbation, vanishes.

\[ \zeta = \zeta_1 + \frac{1}{2} \zeta_2, \quad \mathcal{R} = \mathcal{R}_1 + \frac{1}{2} \mathcal{R}_2, \]
on large scales. By virtue of the results of [1], we conclude that these variables are conserved. Finally, in the light of our results, we comment on the different variables used by Maldacena [3] and Acquaviva et al. [2] in computing non-Gaussianities from single field inflation. The equality symbol “=” means here equal on all scales, while “≈” stems for equal only on super-Hubble scales.

We work with conformal time and use the perturbed Friedmann-Lemaître-Robertson-Walker metric in the so called generalized longitudinal gauge [2],

\[ ds^2 = a^2 \left\{ -(1 + 2\phi)dt^2 + \left[ (1 - 2\psi)\delta_{ij} + \chi_{ij} \right] dx^i dx^j \right\} , \]

where we can expand the (spatially dependent) metric perturbation variables at first and second-order, \( \phi = \phi_1 + \frac{1}{2}\phi_2 \) and \( \psi = \psi_1 + \frac{1}{2}\psi_2 \), while \( \chi_{ij} = \frac{1}{2} \chi_{2ij} \) is only second-order and divergence-free, \( \partial_i \chi_{ij} = 0 \). A dot stems for a derivative with respect to conformal time and \( \dot{H} = \dot{a}/a \). The universe is dominated by a perturbed scalar field with background value \( \varphi = \varphi(\eta) \), energy density \( \rho = \varphi^2/(2a^2) + a^2V \), and pressure \( P = \varphi^2/(2a^2) - a^2V \). The scalar field perturbation can be expanded into first and second-order contributions, \( \delta \varphi = \delta \varphi_1 + \frac{1}{2} \delta \varphi_2 \).

The Friedmann equation is

\[ 3H^2 = \kappa^2 \left( \frac{1}{2} \dot{\varphi}^2 + a^2V \right) , \]

where \( \kappa^2 = 8\pi G \). We shall use the background scalar field evolution equation, \( \dot{\rho} = -3H \varphi^2/a^2 \), or

\[ \ddot{\varphi} + 2H \dot{\varphi} + V' = 0 , \]

where \( V' \) is the derivative of the scalar field potential with respect to the field. We shall also repeatedly make use of the relation \( \kappa^2 \varphi^2/2 = H^2 - \dot{H} \), which can be derived from the Friedmann equation. The adiabatic speed of sound of the scalar field defined in Eq. [3] is

\[ c_s^2 = 1 - 2V'\dot{\varphi}/\dot{\rho} . \]

**First-order perturbations.** At first-order, the perturbed energy constraint reads (see e.g., [14])

\[ 6H(\dot{\psi}_1 + \dot{\varphi}(\phi)) - 2\Delta \psi_1 = -\kappa^2 \delta \rho_1 a^2 , \]

with

\[ \delta \rho_1 = \frac{1}{a^2}(\dot{\varphi}\delta \varphi_1 - \dot{\varphi}_1 \varphi^2) + V' \delta \varphi_1 , \]

\( \Delta = \partial^i \partial_i \), and the momentum constraint is

\[ 2(\dot{\psi}_1 + H\phi_1) = \kappa^2 \dot{\varphi} \delta \varphi_1 . \]

The uniform density curvature perturbation and the comoving curvature perturbation are defined as [12] [13]

\[ \zeta_1 = -\psi_1 - H \frac{\delta \rho_1}{\rho} , \quad \mathcal{R}_1 = \psi_1 + H \frac{\delta \varphi_1}{\dot{\varphi}} . \]

By definition, both these variables reduce to the curvature perturbation \( \psi_1 \) (up to a sign) on setting \( \delta \rho_1 = 0 \) and \( \delta \varphi_1 = 0 \), respectively. Indeed, for a scalar field, the comoving hypersurfaces are the hypersurfaces of uniform field.

On using the energy and momentum constraints, Eqs. [11] and [13], at first-order one can show that uniform density and uniform field (or comoving) hypersurfaces coincide on large scales. Indeed we have,

\[ \frac{\delta \rho_1}{\rho} - \frac{\delta \varphi_1}{\dot{\varphi}} = \frac{\Delta \psi_1}{3H(H^2 - \dot{H})} \approx 0 . \]

Thus, uniform density and comoving curvature perturbations coincide on large scales, up to a sign,

\[ \zeta_1 \approx -\mathcal{R}_1 . \]

Equation [15], together with [12], also yields

\[ \delta \varphi_1 \approx \phi_1 \dot{\varphi} + (\dot{\varphi}/\dot{\varphi}) \delta \varphi_1 - \dot{H} \delta \varphi_1 . \]

For a general fluid and in absence of anisotropic stress, both curvature perturbation variables are sourced by the non-adiabatic pressure perturbation [21],

\[ \delta P_{1,\text{nad}} = \delta P_1 - c_s^2 \delta \rho_1 . \]

Indeed, their evolution is given by [18]

\[ \dot{\zeta}_1 = \frac{3H^2}{\rho} \delta P_{1,\text{nad}} + \frac{1}{3} \Delta \dot{v}_1 \approx \frac{3H^2}{\rho} \delta P_{1,\text{nad}} , \]

\[ \dot{\mathcal{R}}_1 = -\frac{3H^2}{\rho} \delta P_{1,\text{nad}} + 3Hc_s^2(\zeta_1 + \mathcal{R}_1) \approx -\frac{3H^2}{\rho} \delta P_{1,\text{nad}} , \]

where \( v_1 \) is the scalar component of the three-velocity of the fluid (for a scalar field \( v_1 = \delta \varphi_1/\dot{\varphi} \)). However, for a scalar field

\[ \delta P_1 = \frac{1}{a^2}(\varphi \delta \varphi_1 - \dot{\psi}_1 \varphi^2) - V' \delta \varphi_1 = \delta \rho_1 - 2V' \delta \varphi_1 , \]

and the definition [18] with Eq. [10] yields [14]

\[ \dot{H} \frac{\delta P_{1,\text{nad}}}{\rho} = -(1 - c_s^2)(\zeta_1 + \mathcal{R}_1) \approx 0 , \]

where we have used Eq. [10] for the second equality. This implies that both uniform density and comoving curvature perturbations are conserved on large scales

\[ \zeta_1 \approx -\mathcal{R}_1 \approx 0 . \]

**Second-order perturbations.** Now we generalize these results to second-order. For simplicity we use that \( \psi_1 = \phi_1 \) in a scalar field dominated universe, which yields from the traceless \( ij \) part of the Einstein equations, and from
the fact that the scalar field anisotropic stress vanishes at first-order. It is not a necessary condition for our results but it considerably simplifies their proof. Note, however, that at second-order, in the generalized longitudinal gauge, we have $\psi_2 \neq \phi_2$, which makes second-order calculations quite involved.

We shall consider the perturbed energy and momentum constraints only on large scales. At second-order they read

$$6\mathcal{H}(\dot{\psi}_2 + H\phi_2) - 24\mathcal{H}^2\psi_1^2 - 6\dot{\psi}_1^2 \approx -\kappa^2\delta\rho_2 a^2,$$

(24)

and

$$\partial^i(\dot{\psi}_2 + H\phi_2) + 2\partial^i(\dot{\psi}_1 + 6\psi_1 \partial^i \dot{\psi}_1) \approx \kappa^2 \left[ \frac{1}{2}\dot{\varphi}\partial^i \delta\varphi_2 + (\delta\varphi_1 + 2\varphi \delta\varphi_1) \partial^i \delta\varphi_1 \right].$$

(25)

[The large scale definition of $\delta\rho_2$ is given in Eq. 15 below.]

We can use the first-order momentum constraint to write the last term in the first line of Eq. 24 as

$$6\psi_1 \partial^i \dot{\psi}_1 \approx 3\kappa^2 \dot{\varphi} \partial^i \delta\varphi_1 - 3H \partial^i(\dot{\psi}_1^2),$$

(26)

and we can replace $\delta\varphi_1$ in the second line of Eq. 25 by the expression 17. By this replacement we find a much simpler form for the perturbed momentum constraint at second-order, that holds only on large scales, and does not involve spatial derivatives,

$$\dot{\psi}_2 + H\phi_2 + 2\psi_1 \dot{\psi}_1 - 3H\dot{\psi}_1^2 \approx \kappa^2 \frac{1}{2} \left[ \dot{\varphi}\delta\varphi_2 + \left( \frac{\dot{\varphi}}{\varphi} - H \right) \delta\varphi_1^2 \right].$$

(27)

We can combine this equation and the second-order energy constraint, Eq. 24, to eliminate the term $\dot{\psi}_2 + H\phi_2$ obtaining

$$\frac{\delta\rho_2}{\dot{\rho}} - \frac{\delta\varphi_2}{\dot{\varphi}} \approx \left( \frac{\dot{\varphi}}{\rho} - \frac{\dot{\varphi}}{\varphi} \right) \frac{\delta\varphi_1^2}{\varphi^2},$$

(28)

where we have made use of Eq. 13. This relation between the energy density and the field perturbations is the second-order analog of Eq. 15 and we shall use it below to show the equivalence between the two curvature perturbations.

The second-order uniform density curvature perturbation has been defined in 1. On large scales, for a general fluid, it yields

$$\zeta_2 \approx \frac{3\mathcal{H}^2}{\dot{\rho}} \left( \delta P_{2,\text{nad}} - 2\frac{\delta\rho_1}{\rho} \delta P_{1,\text{nad}} \right) + \left( \frac{6\mathcal{H}}{\dot{\rho}} \delta P_{1,\text{nad}} + 4\zeta_1 \right) \dot{\zeta}_1,$$

(32)

where the non-adiabatic second-order pressure perturbation is defined as

$$\delta P_{2,\text{nad}} = \delta P^{(2)} - c_s^2 \delta\rho^{(2)} - c_s^2 \frac{\delta\rho_1^2}{\rho}.$$  

(33)

Equation 31 implies

$$\dot{\mathcal{R}}_2 \approx -\frac{3\mathcal{H}^2}{\dot{\rho}} \left( \delta P_{2,\text{nad}} - 2\frac{\delta\rho_1}{\rho} \delta P_{1,\text{nad}} \right) + \left( \frac{6\mathcal{H}}{\dot{\rho}} \delta P_{1,\text{nad}} - 4\mathcal{R}_1 \right) \mathcal{R}_1.$$  

(34)

For a scalar field $\delta P_{1,\text{nad}} \approx 0$, from Eq. 22. Thus $\delta P_{2,\text{nad}}$ is the only possible non-vanishing contribution on the right hand side of Eqs. (32) and (34). Now we show that $\delta P_{2,\text{nad}}$ vanishes as well.

From the large scale definition of $\delta\rho_2$ and $\delta P_2$ 2,

$$\delta P_2 \approx \frac{1}{a^2} \left[ \varphi \delta\varphi_2 - \varphi_2 \varphi^2 + \delta\varphi_1^2 + 4\psi_1 \varphi(\psi_1 \varphi - \delta\varphi_1) \right]\left( \frac{\dot{\varphi}}{\varphi} + V' \delta\varphi_2 - V'' \varphi \delta\varphi_1 \right),$$

(35)

$$\delta P_2 \approx \frac{1}{a^2} \left[ \varphi \delta\varphi_2 - \varphi_2 \varphi^2 + \delta\varphi_1^2 + 4\psi_1 \varphi(\psi_1 \varphi - \delta\varphi_1) \right]\left( \frac{\dot{\varphi}}{\varphi} - V' \delta\varphi_2 + V'' \varphi \delta\varphi_1 \right),$$

(36)

we find $\delta P_2 \approx 2V' a^2 \delta\varphi_2 - 2V'' a^2 \delta\varphi_1^2$. This yields

$$\frac{\mathcal{H}\delta P_{2,\text{nad}}}{\dot{\rho}} = (1 - c_s^2) \mathcal{H} \left( \frac{\delta\rho_2}{\rho} - \frac{\delta\varphi_2}{\varphi} \right) - c_s^2 \mathcal{H} \frac{\delta\rho_1^2}{\rho^2}$$

$$+ \mathcal{H} \left( 1 - c_s^2 \right) \left( \frac{\varphi}{\varphi} - \frac{\dot{\varphi}}{\rho} \right) + c_s^2 \frac{\delta\varphi_1^2}{\varphi^2} \approx -(1 - c_s^2)(\zeta_2 + \mathcal{R}_2) \approx 0,$$

(37)
by virtue of Eq. (31). The second-order non-adiabatic pressure perturbation of a dominating scalar field vanishes on large scales, \( \delta P_{2,\text{nad}} \approx 0 \), i.e., single scalar field perturbations are adiabatic on large scales to second-order. Equations (31) and (37) are our main results. As a consequence, \( \zeta_2 \) and \( R_2 \) as defined by Eqs. (28) and (30) are both conserved,

\[
\dot{\zeta}_2 \approx -\dot{R}_2 \approx 0.
\]

A final remark concerning the literature on non-Gaussianities is in order here. The second-order comoving curvature perturbation definition of Eq. (30) differs from the one given by Acquaviva et al. in Ref. [2] and used in [3, 10]. Our variable is related to their variable from the one given by Acquaviva et al. in Ref. [2] and Gaussianities is in order here. The second-order comov-

Indeed by virtue of Eq. (31). The second-order non-adiabatic perturbation is proportional to the \( \delta P \) of a dominating scalar field van-

Indeed, one can check that the second-order variables used by Maldacena in [8] are related to our variable by \( \nu^M = -R - R^2 \) in the flat gauge, where \( \psi = 0 \), and by \( \nu^M = R + R^2 \) in the uniform field (or comoving) gauge, where \( \delta \psi = 0 \). These variables are thus perfectly conserved for adiabatic perturbations.

Note added: While writing this note, a proof of the conservation of the uniform density and comoving curvature perturbations on large scales without perturbative expansion has been given in [19]. A geometric and fully non-linear generalization of the uniform density and comoving curvature perturbations will be presented in [20].

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