Shades of hyperbolicity for Hamiltonians

Mário Bessa¹, Jorge Rocha² and Maria Joana Torres³

¹ Departamento de Matemática, Universidade da Beira Interior, Rua Marquês d’Ávila e Bolama, 6201-001 Covilhã, Portugal
² Departamento de Matemática, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
³ CMAT, Departamento de Matemática e Aplicações, Universidade do Minho, Campus de Gualtar, 4700-057 Braga, Portugal

E-mail: bessa@ubi.pt, jrocha@fc.up.pt and jtorres@math.uminho.pt

Received 21 December 2012, in final form 16 August 2013
Published 23 September 2013
Online at stacks.iop.org/Non/26/2851

Recommended by D V Treschev

Abstract
We prove that a Hamiltonian system \( H \in C^2(M, \mathbb{R}) \) is globally hyperbolic if any of the following statements hold: \( H \) is robustly topologically stable; \( H \) is stably shadowable; \( H \) is stably expansive; and \( H \) has the stable weak specification property. Moreover, we prove that, for a \( C^2 \)-generic Hamiltonian \( H \), the union of the partially hyperbolic regular energy hypersurfaces and the closed elliptic orbits, forms a dense subset of \( M \). As a consequence, any robustly transitive regular energy hypersurface of a \( C^2 \)-Hamiltonian is partially hyperbolic. Finally, we prove that stable weakly-shadowable regular energy hypersurfaces are partially hyperbolic.

Mathematics Subject Classification: 37J10, 37D30, 70H05

1. Introduction and a summary of the main results

The set of Hamiltonian functions \( H: M \rightarrow \mathbb{R} \) on a symplectic \( 2n \)-dimensional manifold \( M \) is an important subclass of all continuous-time dynamical systems generated by the \( 2n \) differential equations (Hamilton's equations): \( q_i = \partial_{p_i} H \) and \( p_i = -\partial_{q_i} H \), where \((q_i, p_i) \in M\) and \( i = 1, \ldots, n \). The vast range of their applications throughout several branches of mathematics justify the relevance of these systems. Actually, many problems that appear in the mathematical physics context can be expressed in terms of differential equations, and a well-studied and successful subclass of these differential equations is the one that corresponds to the Hamiltonian equations, whose solutions keep invariant a given symplectic form (see [1, 4]).

A stable property of a dynamical system is a specific property that is satisfied by dynamical systems ‘close’ to the initial one, in other words it is a property that still holds when the initial system is slightly perturbed.
As observable systems always carry slight perturbations of theoretical models, to obtain stable properties is a central problem in describing the dynamics of a given dynamical system. Among them the structurally stable property, introduced in the mid 1930s by Andronov and Pontrjagin, plays a fundamental role. Roughly speaking it means that under small perturbations the dynamics are topologically equivalent. Smale’s program in the early 1960s aimed to prove the (topological) genericity of structurally stable systems. Although Smale’s program was proved to be wrong one decade later, it played a fundamental role in the theory of dynamical systems, in particular it gave rise to the interrelation of the structural stability concept with the uniform hyperbolicity property.

The characterization of structurally stable systems, using topological and geometric dynamical properties, has been one of the main objects of interest in the global qualitative theory of dynamical systems in the last 40 years. In [16] the authors characterized the structurally stable Hamiltonian systems as the Anosov ones. The purpose of one of our main results (see theorem 1) is, thus, to give a characterization of the structurally stable Hamiltonian systems by making use of the notions of topological stability, shadowing, expansiveness and specification. Although these properties seem to be quite different, they all are interconnected with the uniform hyperbolicity.

Anosov systems, and thus, structurally stable Hamiltonians, are topologically stable, expansive and satisfy the shadowing property. But the converse is not true. There are examples of systems, far from the structurally stable ones, that are topologically stable4, that are expansive (see example 3.1) and that satisfy the shadowing property (see [60] and also the symplectic curious example of the existence of shadowing of transition chains of several invariant tori alternating with Birkhoff zones of instability [32]).

Therefore, the problem on the relationship between structural stability and topological and geometric properties of the system is not trivial. The restriction to the $C^2$-interiors of sets of Hamiltonians satisfying a certain geometric property, which is explored in the present work, became one effective approach to the solution of this problem. Let us say that a property stably holds for some system if it holds for any system in some $C^2$-neighbourhood of that system.

Now we explain the geometric and dynamical properties we deal with and state theorem 1.

The shadowing of a given set of points which is an ‘almost orbit’ of some system by a true orbit appears in many branches of dynamical systems and is quite often related with hyperbolicity. Actually, the errors introduced by computational estimates of an orbit are negligible if the pseudo-orbit can be realized by a genuine orbit of the given system, and, in this sense, are mere pixel inaccuracies typical of the computational framework. Despite the fact that shadowable systems can be non-hyperbolic, the stability of shadowing is sufficient to obtain hyperbolicity. We refer the work developed in [8, 31, 38, 47, 52, 57] both for flows and diffeomorphisms and in dissipative and conservative contexts. Here we prove that stable shadowing is equivalent to hyperbolicity.

The notion of topological stability (see the definition in section 3.1), developed in parallel to the theory of structural stability, was first introduced by Walters in [56] when proving that Anosov diffeomorphisms are topologically stable. Then, Nitecki proved that topological stability is a necessary condition to get Axiom A plus strong transversality (see [44]). In the late 1970s [50], Robinson proved that Morse–Smale flows are topologically stable and, in the mid-1980s, Hurley obtained necessary conditions for topological stability (see [35–37]). Here we generalize the results for flows in [18, 41] to the Hamiltonian context (see also [19]) and prove that stable topological stability is equivalent to hyperbolicity.

4 It is immediate that the topological stability is invariant by conjugacy. Moreover, by a result of Gogolev (see [33]) the existence of a conjugacy between two maps where one of which is Anosov is not sufficient to guarantee that the other is Anosov.
A dynamical system is, in brief terms, expansive, whenever two points stay near for future and past iterates, then they must be equal. We can say, in a general scope, that the system has sensitivity to the initial conditions, because two different points must be separated by forward or backward iteration. This notion was first developed in the 1950s (see [58]) and, in the flow context, introduced by the studies of Bowen and Walters (see [28]). In this paper we generalize the recent results in [31, 40, 54] by proving that stable expansiveness is equivalent to hyperbolicity.

The concept of weak specification (see section 3.5), although intricate, is quite well summarized for diffeomorphisms, in simple words in [30, p 193]:

*The weak specification means that whenever there are two pieces of orbits \{f^n(x_1) : a_1 \leq n \leq b_1\} and \{f^n(x_2) : a_2 \leq n \leq b_2\}, they may be approximated up to \(\epsilon\) by one periodic orbit—the orbit of \(x\)—provided that the time for switching from the first piece of orbit to the second (namely \(a_2 - b_1\)) and the time for switching back (namely \(p - (b_2 - a_1)\)) are larger than \(K(\epsilon)\), this number \(K(\epsilon)\) being independent of the pieces of orbit, and in particular independent of their length*,

where \(p\) is any number greater or equal than \(K(\epsilon) + b_2 - a_1\) and, once fixed, \(x\) is some point of period \(p\).

Several authors obtained hyperbolicity as a consequence of the stable specification property (see [6, 53]). We point out that their proofs are based on an argument of change of index in hyperbolic closed orbits. However, in the symplectic setting such situation cannot occur because the index is constant and equal to \(n - 1\). Here, using a new symplectic approach, we obtain similar results for Hamiltonian systems.

Summarizing, in theorem 1, we prove that a Hamiltonian system is globally hyperbolic (Anosov) if any of the following properties stably holds: topological stability, shadowing, expansiveness, specification. Thus, we call these properties shades of hyperbolicity. The proof goes as follows. First let us recall that a Hamiltonian system is a star Hamiltonian if the property of having all periodic orbits of hyperbolic type is stable. In theorem 3 we prove that the stability of any of the shade properties ensures that the Hamiltonian system is a star Hamiltonian. Then, we use the fact, recently proved, that a star Hamiltonian system is Anosov (see [10] for \(n = 2\), [16] for \(n \geq 2\) and also theorem 2).

Dynamical systems exhibiting dense orbits are called transitive. Informally speaking this means that the whole manifold is dynamically indecomposable. Those systems for which this property remains valid for any perturbation are called robustly transitive. Since the pioneering work of Mañé (see [39]) several other results were obtained in order to ensure that robust transitivity, with respect to \(C^1\)-topology, implies a certain form of hyperbolicity (see e.g. [5, 17, 26, 27, 34, 55]). Here, and as a direct consequence of a dichotomy that we discuss in the sequel, we prove that robustly transitive Hamiltonian systems are partially hyperbolic. Moreover, we prove that stably ergodic Hamiltonian systems are partially hyperbolic.

The shadowing property in the weak sense first appears in the paper by Corless and Pilyugin (see [29]) related to the genericity of shadowing among homeomorphisms, with respect to the \(C^0\)-topology. In simple terms weak shadowing allows to approximate ‘almost orbits’ as true orbits, if one considers only the distance between the orbit and the ‘almost orbit’ as two subsets in the manifold, thus forgetting the time parametrization. There exist dynamical systems without the weak shadowing property (see [46, example 2.12]) and dynamical systems satisfying the weak shadowing property but not the shadowing one ([46, example 2.13]). In this paper we generalize a recent result (see [20, 21]) for the setting of Hamiltonians. More precisely, we obtain partial hyperbolicity under the hypothesis of stable weak shadowing (see theorem 6).
A generic property for Hamiltonians is a specific property that is satisfied by a countable intersection of open and dense subsets of these continuous-time systems. Such properties are of great importance because, although they can be not satisfied by an open and dense subset of systems, they give us the typical behaviour, in Baire’s category sense, that one could expect from a wide class of systems (see [10–13, 16, 43, 49]). Indeed it is relevant and useful to know that, given a Hamiltonian system, it can be slightly perturbed in order to obtain a (generic) new one satisfying a certain dynamical property.

Questions concerning the generic behaviour of Hamiltonians were first raised by Robinson (see [49]). One of our main results (see theorem 5) is a generalization of a result stated in [43] by Newhouse and proved in [13] for the four-dimensional context (or a weak 2n-dimensional version): $C^2$-generic Hamiltonians are of hyperbolic type or else exhibit dense 1-elliptic closed orbits. More precisely we prove that, $C^2$-generically, Hamiltonians have only two types of well-differentiated behaviour: partial hyperbolicity (chaotic type, see [25]) or else lots of totally elliptic closed orbits (KAM type, see [59]). We observe that, in the four-dimensional case, 1-elliptic closed orbits are totally elliptic and, moreover, partial hyperbolicity is actually hyperbolicity. It is still an open and quite interesting question to know if these type of results hold for mechanical systems (see [14, section 8]).

With respect to the discrete-time case, Newhouse proved that, in the two-dimensional case, $C^1$-generic symplectomorphisms are Anosov (uniformly hyperbolic) or else the elliptic points are dense [43]. Later, Arnaud (see [2]) proved the four-dimensional version of this result, namely that $C^1$-generic symplectomorphisms are Anosov, partially hyperbolic or have dense elliptic periodic points. Finally, completing the program for the discrete case, Saghin and Xia [51] proved the same result but for any dimension. In this paper, we prove the continuous-time version of Saghin–Xia’s theorem (see theorem 4).

Finally it is worth mentioning that, in this paper, we deal with the $C^2$-topology for Hamiltonians, therefore with the $C^1$-topology for the associated vector field. The main reason for this is that the proofs make use of several perturbation tools only available for this topology (see the perturbation results in section 5).

2. Hamiltonian systems

In this section we present the main definitions in the context of Hamiltonians. Moreover, we introduce the transversal linear Poincaré flow and present some of its relevant properties.

2.1. The Hamiltonian framework

Let $(M, \omega)$ be a symplectic manifold, where $M$ is a $2n$-dimensional ($n \geq 2$), compact, boundaryless, connected and smooth Riemannian manifold, endowed with a symplectic form $\omega$. A Hamiltonian is a real-valued $C^r$ function on $M$, $2 \leq r \leq \infty$. We denote by $C^r(M, \mathbb{R})$ the set of $C^r$-Hamiltonians on $M$. From now on, we shall be restricted to the $C^2$-topology and thus we set $r = 2$. Given a Hamiltonian $H$, the Hamiltonian vector field $X_H$ is defined by $\omega(X_H(p), u) = \nabla H_p(u)$, for all $u \in T_p M$; this vector field generates the Hamiltonian flow $X^t_H$. Observe that $H$ is $C^2$ if and only if $X_H$ is $C^1$. As $H$ is continuous and $M$ is compact, the set of singularities of $X_H$, $\text{Sing}(X_H)$, is non-empty. Let $\mathcal{R}(H) = M \setminus \text{Sing}(X_H)$ stand for the set of regular points.

Since the symplectic form $\omega$ is non-degenerate, given $H \in C^2(M, \mathbb{R})$ and $p \in M$, we know that $\nabla H_p = 0$ is equivalent to $X_H(p) = 0$. Therefore, the extreme values of a Hamiltonian $H$ are exactly the singularities of the associated Hamiltonian vector field $X_H$ or, equivalently, the equilibria of the flow $X^t_H$. 
A scalar $e \in H(M) \subset \mathbb{R}$ is called an energy of $H$ and the pair $(H, e)$ is called the Hamiltonian level. Given an energy $e$, we define the energy level set as $H^{-1}(\{e\})$, an energy hypersurface $\mathcal{E}_{H,e}$ is a connected component of $H^{-1}(\{e\})$ and it is regular if it does not contain singularities. Observe that a regular energy hypersurface is a $X_H$-invariant, compact and $(2n-1)$-dimensional manifold. The energy level set $H^{-1}(\{e\})$ is said to be regular if any energy hypersurface of $H^{-1}(\{e\})$ is regular. We observe that if $H^{-1}(\{e\})$ is regular, then $H^{-1}(\{e\})$ is the union of a finite number of energy hypersurfaces. Finally, a Hamiltonian level $(H, e)$ is said to be regular if the energy level set $H^{-1}(\{e\})$ is regular.

A Hamiltonian system is a triple $(H, e, \mathcal{E}_{H,e})$, where $H$ is a Hamiltonian, $e$ is an energy and $\mathcal{E}_{H,e}$ is a regular connected component of $H^{-1}(\{e\})$.

Fixing a small neighbourhood $W$ of a regular $\mathcal{E}_{H,e}$, there exist a small neighbourhood $U$ of $H$ and $\epsilon > 0$ such that, for all $H \in U$ and $\tilde{e} \in (e - \epsilon, e + \epsilon)$, $H^{-1}(\{\tilde{e}\}) \cap W = \mathcal{E}_{\tilde{H},\tilde{e}}$, where $\mathcal{E}_{\tilde{H},\tilde{e}}$ is an energy hypersurface of $\tilde{H}$. We call $\mathcal{E}_{\tilde{H},\tilde{e}}$ the analytic continuation of $\mathcal{E}_{H,e}$.

In the space of Hamiltonian systems we consider the topology generated by a fundamental system of neighbourhoods. Given a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ we say that $V$ is a neighbourhood of $(H, e, \mathcal{E}_{H,e})$ if there exist a small neighbourhood $U$ of $H$ and $\epsilon > 0$ such that for all $H \in U$ and $\tilde{e} \in (e - \epsilon, e + \epsilon)$ the analytic continuation $\mathcal{E}_{\tilde{H},\tilde{e}}$ of $\mathcal{E}_{H,e}$ is well-defined.

Given a Hamiltonian level $(H, e)$, let $\Omega(H|_{\mathcal{E}_{H,e}})$ be the set of non-wandering points of $H$ on the energy hypersurface $\mathcal{E}_{H,e}$, that is, the points $p \in \mathcal{E}_{H,e}$ such that, for every neighbourhood $U$ of $p$ in $\mathcal{E}_{H,e}$, there is $\tau > 0$ such that $X_H(U) \cap U \neq \emptyset$.

By Liouville’s theorem, the symplectic manifold $(M, \omega)$ is also a volume manifold (see, for example, [1]). This means that the $2n$-form $\omega^n = \omega \wedge \ldots \wedge \omega$ (wedging $n$-times) is a volume form and induces a measure $\mu$ on $M$, which is called the Lebesgue measure associated to $\omega^n$. Notice that the measure $\mu$ on $M$ is invariant by the Hamiltonian flow. So, given a regular Hamiltonian level $(H, e)$, we induce a volume form $\omega_{\mathcal{E}_{H,e}}$ on each energy hypersurface $\mathcal{E}_{H,e} \subset H^{-1}(\{e\})$ as follows: given $p \in \mathcal{E}_{H,e}$, $v \in T_pM$ and $u_k \in T_p\mathcal{E}_{H,e}$, $k \in \{1, \ldots, 2n-1\}$,

$$\omega_{\mathcal{E}_{H,e}}(u_1, u_2, \ldots, u_{2n-1}) \cdot dH(v) = \omega^n(v, u_1, u_2, \ldots, u_{2n-1}).$$

The volume form $\omega_{\mathcal{E}_{H,e}}$ is $X_H$-invariant. Hence, it induces an invariant volume measure $\mu_{\mathcal{E}_{H,e}}$ on $\mathcal{E}_{H,e}$ which is a finite measure, since any energy hypersurface is compact. Observe that, under these conditions, we have that $\mu_{\mathcal{E}_{H,e}}$-a.e. $p \in \mathcal{E}_{H,e}$ is recurrent, by the Poincaré recurrence theorem.

### 2.2. Transversal linear Poincaré flow and hyperbolicity

Let us begin with the definition of the transversal linear Poincaré flow. Later, we state some results using this linear flow. Consider a Hamiltonian vector field $X_H$ and a regular point $p$ in $M$ and let $e = H(p)$. Define $\mathcal{N}_p := N_p \cap T_pH^{-1}(\{e\})$, where $N_p = (\mathbb{R}X_H(p))^{\perp}$ is the normal fibre at $p$, $\mathbb{R}X_H(p)$ stands for the flow direction at $p$ and $T_pH^{-1}(\{e\}) = \text{Ker} \nabla H_p$ is the tangent space to the energy level set. Thus, $\mathcal{N}_p$ is a $(2n - 2)$-dimensional bundle.

**Definition 2.1.** The transversal linear Poincaré flow associated with $H$ is given by

$$\Phi^t_H(p) : \mathcal{N}_p \to \mathcal{N}_{X_H(p)} \quad v \mapsto \Pi_{X_H(p)} \circ DX_H^t(p)(v),$$

where $\Pi_{X_H(p)} : T_{X_H(p)}M \to \mathcal{N}_{X_H(p)}$ denotes the canonical orthogonal projection.

Observe that $\mathcal{N}_p$ is $\Phi^t_H(p)$-invariant. Now, $\mathcal{N}_p$ can be seen as the quotient space $T_p\mathcal{E}_{H,e}/\{X_H\}$ and we can consider the symplectic form $\tilde{\omega}_{\mathcal{E}_{H,e}} : \mathcal{N}_p \times \mathcal{N}_p \to \mathbb{R}$ defined by

$$\tilde{\omega}_{\mathcal{E}_{H,e}}([u], [v]) = \omega(u, v),$$

for any $u, v \in T_p\mathcal{E}_{H,e}$. 

Shades of hyperbolicity for Hamiltonians 2855
Note that this form is well-defined since $\omega_{\mathcal{E}_H,\sigma}(\{X_H\}, \{v\}) = \omega(X_H, v) = dH(v) = 0$, for any $v \in T_p\mathcal{E}_{H,\sigma}$. It is well known (see e.g. [1]) that, given a regular point $p \in \mathcal{E}_{H,\sigma}$, $\Phi_H^t(p)$ is a linear symplectomorphism for $\omega_{\mathcal{E}_H,\sigma}$.

For any symplectomorphism, in particular for $\Phi_H^t(p)$, we have the following result.

**Theorem 2.1 (Symplectic eigenvalue theorem, [1])**. Let $f$ be a symplectomorphism in $M$, $p \in M$ a fixed point of $f$ and $\sigma$ an eigenvalue of $Df_p$ of multiplicity $m$. Then $1/\sigma$ is an eigenvalue of $Df_p$ of multiplicity $m$. If $\sigma$ is non-real, then $\overline{\sigma}$ and $1/\overline{\sigma}$ are also eigenvalues of $Df_p$. Moreover, the multiplicity of the eigenvalues $+1$ and $-1$, if they occur, is even.

Given an $X^s_H$-invariant, regular and compact subset of $M$, $\Lambda$, let $\mathcal{N}_{\Lambda} := \bigcup_{p \in \Lambda} \{p\} \times \mathcal{N}_p$.

The proof of the following result can be found in [12, section 2.3].

**Lemma 2.2**. Take a Hamiltonian $H \in C^2(M, \mathbb{R})$ and let $\Lambda$ be an $X^s_H$-invariant, regular and compact subset of $M$. Then, $\Lambda$ is uniformly hyperbolic for $X^s_H$ if and only if the induced transversal linear Poincaré flow $\Phi_H^t$ is uniformly hyperbolic on $\mathcal{N}_\Lambda$.

So, we can define a uniformly hyperbolic set as follows.

**Definition 2.2**. Let $H \in C^2(M, \mathbb{R})$. An $X^s_H$-invariant, compact and regular set $\Lambda \subset M$ is uniformly hyperbolic if $\mathcal{N}_\Lambda$ admits a $\Phi_H^t$-invariant splitting $\mathcal{N}_\Lambda^c \oplus \mathcal{N}_\Lambda^s$ such that there is $\ell > 0$ satisfying

$$\|\Phi_H^t(p)|\mathcal{N}_\Lambda^c\| \leq \frac{1}{2} \quad \text{and} \quad \|\Phi_H^t(X^s_H(p))|\mathcal{N}_\Lambda^s\| \leq \frac{1}{2},$$

for any $p \in \Lambda$.

We remark that the constant $\frac{1}{2}$ can be replaced by any constant $\theta \in (0, 1)$, possibly with a different constant $\ell$.

Given $p \in \mathcal{R}(H)$, we say that $p$ is a periodic point of the Hamiltonian $H$ if $X^s_H(p) = p$ for some $t$. The smallest $t_0 > 0$ satisfying the condition above is called period of $p$; in this case, we say that the orbit of $p$ is a closed orbit of period $t_0$. According with definition 2.2, a periodic point $p$ is hyperbolic if there exists a splitting of the normal subbundle $\mathcal{N}$ along the orbit of $p$ that satisfies the condition above.

Now, we state the definition of dominated splitting, by using the transversal linear Poincaré flow.

**Definition 2.3**. Take $H \in C^2(M, \mathbb{R})$ and let $\Lambda$ be a compact, $X^s_H$-invariant and regular subset of $M$. Consider a $\Phi_H^t$-invariant splitting $\mathcal{N} = \mathcal{N}^s \oplus \cdots \oplus \mathcal{N}^n$ over $\Lambda$, for $1 \leq k \leq 2n - 2$, such that all the subbundles have constant dimension. This splitting is dominated if there exists $\ell > 0$ such that, for any $1 \leq i < j \leq k$,

$$\|\Phi_H^t(p)|\mathcal{N}_\Lambda^c\| \cdot \|\Phi_H^t(X^s_H(p))|\mathcal{N}_\Lambda^s\| \leq \frac{1}{2}, \quad \forall \ p \in \Lambda.$$

Finally, we state the definition of partial hyperbolicity for the transversal linear Poincaré flow.

**Definition 2.4**. Take $H \in C^2(M, \mathbb{R})$ and let $\Lambda$ be a compact, $X^s_H$-invariant and regular subset of $M$. Consider a $\Phi_H^t$-invariant splitting $\mathcal{N} = \mathcal{N}^a \oplus \mathcal{N}^c \oplus \mathcal{N}^s$ over $\Lambda$ such that all the subbundles have constant dimension and at least two of them are non-trivial. This splitting is partially hyperbolic if there exists $\ell > 0$ such that,

1. $\mathcal{N}^a$ is uniformly hyperbolic and expanding;
2. $\mathcal{N}^c$ is uniformly hyperbolic and contracting and
3. $\mathcal{N}^c$ $\ell$-dominates $\mathcal{N}^c$ and $\mathcal{N}^c$ $\ell$-dominates $\mathcal{N}^s$.
In general, throughout this paper, we consider these three phenomena, i.e. uniform hyperbolicity, dominated splitting and partial hyperbolicity, defined in a set $\Lambda$ that is the whole energy level.

**Remark 2.1.** It was proved in [22] that, in the symplectic world, the existence of a dominated splitting implies partial hyperbolicity. More precisely, if $N^u \oplus N^c$ is a dominated splitting, with $\dim N^u \leq \dim N^c$, then $N^c$ splits invariantly as $N^c = N^c \oplus N^s$, with $\dim N^s = \dim N^u$. Furthermore, the splitting $N^u \oplus N^c \oplus N^s$ is dominated, $N^u$ is uniformly expanding and $N^s$ is uniformly contracting. In conclusion, $N^u \oplus N^c \oplus N^s$ is partially hyperbolic.

### 3. Shade properties: topological stability, shadowing, weak shadowing, expansiveness, specification, transitivity and ergodicity

In this section we describe the dynamical properties that we shall deal with in the sequel.

#### 3.1. Topological stability

Let $(H, e, \mathcal{E}_{H,e})$ and $(\tilde{H}, \tilde{e}, \tilde{\mathcal{E}}_{H,\tilde{e}})$ be Hamiltonian systems; we say that $(\tilde{H}, \tilde{e}, \tilde{\mathcal{E}}_{H,\tilde{e}})$ is semiconjugated to $(H, e, \mathcal{E}_{H,e})$ if there exists a continuous and surjective map $h : \tilde{\mathcal{E}}_{H,\tilde{e}} \to \mathcal{E}_{H,e}$ and a continuous real map $\tau : \tilde{\mathcal{E}}_{H,\tilde{e}} \times \mathbb{R} \to \mathbb{R}$ such that:

(a) for any $p \in \mathcal{E}_{H,e}$, $\tau_p : \mathbb{R} \to \mathbb{R}$ is an orientation-preserving homeomorphism where $\tau(p, 0) = 0$ and

(b) for all $p \in \mathcal{E}_{H,e}$ and $t \in \mathbb{R}$ we have $h(X^{\tau(p,t)}_{H}(p)) = X^{\tau(p,t)}_{\tilde{H}}(h(p))$.

We say that the Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is topologically stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any Hamiltonian system $(\tilde{H}, \tilde{e}, \tilde{\mathcal{E}}_{H,\tilde{e}})$ such that $H$ is $\delta$-$C^1$-close to $H$ and $\tilde{e} \in (e - \delta, e + \delta)$, there exists a semiconjugacy from $(\tilde{H}, \tilde{e}, \tilde{\mathcal{E}}_{H,\tilde{e}})$ to $(H, e, \mathcal{E}_{H,e})$, i.e., there exists $h : \tilde{\mathcal{E}}_{H,\tilde{e}} \to \mathcal{E}_{H,e}$ and $\tau : \tilde{\mathcal{E}}_{H,\tilde{e}} \times \mathbb{R} \to \mathbb{R}$ satisfying (a) and (b) above and $d(h(p), p) < \epsilon$ for all $p \in \mathcal{E}_{H,e}$. Observe that the notion of topological stability does not define an equivalence relation. Furthermore, the set of systems semiconjugated to a given Hamiltonian system can be not an open set. This motivates the following definition. We say that $(H, e, \mathcal{E}_{H,e})$ is robustly topologically stable if there exists a neighbourhood $V$ of $(H, e, \mathcal{E}_{H,e})$ such that any $(\tilde{H}, \tilde{e}, \tilde{\mathcal{E}}_{H,\tilde{e}}) \in V$ is topologically stable.

#### 3.2. The shadowing property

Let $(H, e, \mathcal{E}_{H,e})$ be a Hamiltonian system. Let us fix real numbers $\delta, T > 0$. We say that a pair of sequences $(x_i), (t_i)_{i\in \mathbb{Z}}$ $x_i \in \mathcal{E}_{H,e}, t_i \in \mathbb{R}, t_i \geq T$ is a $(\delta, T)$-pseudo-orbit of $H$ if $d(X^t_{H}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. For the sequence $(t_i)_{i\in \mathbb{Z}}$ we write $S(n) = t_0 + t_1 + \ldots + t_{n-1}$ if $n > 0$, $S(n) = -(t_0 + \ldots + t_{-2} + t_{-1})$ if $n < 0$ and $S(0) = 0$. Let $x_0 \star t$ denote a point on a $(\delta, T)$-chain $t$ units time from $x_0$. More precisely, for $t \in \mathbb{R}$,

$$x_0 \star t = X^{t-S(i)}_{H}(x_i) \quad \text{if} \quad S(i) \leq t < S(i+1).$$

Simple examples show that, in the case of a flow, it is unnatural to require in the definition of shadowing the closeness of points of a pseudo-orbit and its exact shadowing orbit corresponding to the same instants of time, as it is posed in the shadowing problem for diffeomorphisms. We need to reparametrize the exact shadowing orbit. By $\text{Rep}$ we denote the set of all increasing homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$, called reparametrizations, satisfying $\alpha(0) = 0$. Fixing $\epsilon > 0$, we define the set $\text{Rep}(\epsilon) = \{\alpha \in \text{Rep} : |\frac{\alpha(t)}{t} - 1| < \epsilon, t \in \mathbb{R}\}$. When we choose a reparametrization $\alpha$ in the previous set, we want $\alpha$ to be taken arbitrarily close to identity. A
$(\delta, T)$-pseudo-orbit $((x_i), (t_i))_{i \in \mathbb{Z}}$ is $\epsilon$-shadowed by some orbit of $H$ if there is $z \in \mathcal{E}_{H,e}$ and a reparametrization $\alpha \in \text{Rep}(\epsilon)$ such that $d(X^\alpha_H(z), x_0 \ast t) < \epsilon$, for every $t \in \mathbb{R}$.

The Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is said to have the shadowing property if, for any $\epsilon > 0$ there exist $\delta, T > 0$ such that any $(\delta, T)$-pseudo-orbit $((x_i), (t_i))_{i \in \mathbb{Z}}$ is $\epsilon$-shadowed by some orbit of $H$.

We say that the Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is stably shadowable if there exists a neighbourhood $\mathcal{V}$ of $(H, e, \mathcal{E}_{H,e})$ such that any $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}}) \in \mathcal{V}$ has the shadowing property.

3.3. The weak shadowing property

We recall the following definition of weakly shadowable systems that was introduced in [29] in connection with the problem of genericity of shadowing. Given a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ and $\delta, T > 0$, a $(\delta, T)$-pseudo-orbit $((x_i), (t_i))_{i \in \mathbb{Z}}$ is weakly $\epsilon$-shadowed by some orbit of $H$ if there exists $z \in \mathcal{E}_{H,e}$ such that $\{x_i\}_{i \in \mathbb{Z}} \subset B_\epsilon(O(z))$.

The Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is said to have the weak shadowing property if, for any $\epsilon > 0$ there exist $\delta, T > 0$ such that any $(\delta, T)$-pseudo-orbit $((x_i), (t_i))_{i \in \mathbb{Z}}$ is weakly $\epsilon$-shadowed by some orbit of $H$.

We say that the Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is stable weakly-shadowable if there exists a neighbourhood $\mathcal{V}$ of $(H, e, \mathcal{E}_{H,e})$ such that any $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}}) \in \mathcal{V}$ has the weak shadowing property.

3.4. The expansiveness property

Let $(H, e, \mathcal{E}_{H,e})$ be a Hamiltonian system. We say that $(H, e, \mathcal{E}_{H,e})$ is expansive if, for any $\epsilon > 0$, there is $\delta > 0$ such that if $x, y \in \mathcal{E}_{H,e}$ satisfy $d(X_H^t(x), X_H^t(y)) \leq \delta$, for any $t \in \mathbb{R}$ and for some continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X^\alpha_H(x)$, where $|s| \leq \epsilon$.

This definition asserts that any two points whose orbits remain indistinguishable, up to any continuous-time displacement, must be in the same orbit.

Observe that the reparametrization $\alpha$ is not assumed to be close to identity and that the expansiveness property does not depend on the choice of the metric on $M$.

We say that the Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is stably expansive if there exists a neighbourhood $\mathcal{V}$ of $(H, e, \mathcal{E}_{H,e})$ such that any $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}}) \in \mathcal{V}$ has the expansiveness property.

The next example, pointed out to us by Pedro Duarte, shows that expansiveness can coexist with elliptic orbits (see the definition of an elliptic orbit in section 4).

**Example 3.1.** Consider the Hamiltonian with one degree of freedom given by $H(x, y) = x^3 - 3xy^2$. The associated Hamiltonian vector field is $X_H(x, y) = (-6xy, 3y^2 - 3x^2)$, for more details see [4, appendix 7]. The origin is a degenerated singularity of the vector field. However, our system exhibits a symmetry; the rotation by $\frac{2\pi}{3}$ centred in $(0, 0)$ keeps the phase portrait invariant (see figure 1).

If we compose the time-one symplectomorphism associated to the flow generated by $X_H$ with the rotation of angle $\frac{2\pi}{3}$ we obtain a symplectomorphism $f$ such that the origin is an elliptic fixed point. Moreover $f$ rotates the initial invariant stable (unstable) directions therefore inheriting the original expansiveness.

3.5. The specification property

Let $(H, e, \mathcal{E}_{H,e})$ be a Hamiltonian system and $\Lambda$ be a $X_H^1$-invariant compact subset of $\mathcal{E}_{H,e}$. A specification $S = (\tau, P)$ consists of a finite collection $\tau = \{I_1, \ldots, I_n\}$ of bounded disjoint
intervals $I_i = [a_i, b_i]$ of the real line and a map $P : \bigcup_{i \in \tau} I_i \to \Lambda$ such that for any $t_1, t_2 \in I_i$ we have

$$X_H^{t_i}(P(t_1)) = X_H^{t_i}(P(t_2)).$$

$S$ is said to be $K$-spaced if $a_{i+1} \geq b_i + K$ for all $i \in \{1, \ldots, m\}$ and the minimal of such $K$ is called the spacing of $S$. If $\tau = \{I_1, I_2\}$ then $S$ is said to be a weak specification. Given $\epsilon > 0$, we say that $S$ is $\epsilon$-shadowed by $x \in \Lambda$ if $d(X_H^t(x), P(t)) < \epsilon$ for all $t \in \bigcup_{I_i \in \tau} I_i$.

We say that $\Lambda$ has the weak specification property if for any $\epsilon > 0$ there exists a $K = K(\epsilon) \in \mathbb{R}$ such that any $K$-spaced weak specification $S$ is $\epsilon$-shadowed by a point of $\Lambda$. In this case the Hamiltonian system $(H, e, \mathcal{E}_H,e)$ is said to have the weak specification property. We say that the Hamiltonian system $(H, e, \mathcal{E}_H,e)$ has the weak specification property if $\mathcal{E}_H,e$ has it.

We say that the Hamiltonian system $(H, e, \mathcal{E}_H,e)$ has the stable weak specification property if there exists a neighbourhood $V$ of $(H, e, \mathcal{E}_H,e)$ such that any $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}}) \in V$ has the weak specification property.

### 3.6. The transitive and ergodic properties

A compact energy hypersurface $\mathcal{E}_H,e$ is topologically transitive if, for any open and non-empty subsets $U$ and $V$ of $\mathcal{E}_H,e$, there is $\tau \in \mathbb{R}$ such that $X_H^\tau(U) \cap V \neq \emptyset$.

We say that the Hamiltonian system $(H, e, \mathcal{E}_H,e)$ is robustly transitive if there exists a neighbourhood $V$ of $(H, e, \mathcal{E}_H,e)$ such that, for any $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}}) \in V$, the regular energy hypersurface $\mathcal{E}_{\tilde{H},\tilde{e}}$ is transitive.

The probability measure $\mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}$ is ergodic if, for any $X_{\tilde{H}}^t$-invariant subset $C$ of $\mathcal{E}_{\tilde{H},\tilde{e}}$, we have that $\mu_{X_{\tilde{H}}^t}(C) = 0$ or $\mu_{X_{\tilde{H}}^t}(C) = 1$. It is well-known that if a Hamiltonian system $(H, e, \mathcal{E}_H,e)$ is such that $\mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}$ is ergodic then $\mathcal{E}_{\tilde{H},\tilde{e}}$ is transitive.

We say that the Hamiltonian system $(H, e, \mathcal{E}_H,e)$ with $H \in C^3(M, \mathbb{R})$, is stably ergodic if there exists a neighbourhood $V$ of $(H, e, \mathcal{E}_H,e)$ such that, for any $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}}) \in V$ with $\tilde{H} \in C^3(M, \mathbb{R})$, the probability measure $\mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}$ is ergodic.
4. Precise statement of the results

A Hamiltonian system \((H, e, \mathcal{E}_{H,e})\) is Anosov if \(\mathcal{E}_{H,e}\) is uniformly hyperbolic for the Hamiltonian flow \(X^t_H\) associated to \(H\).

Our first main result states that the stability of any of the properties: topological stability, shadowing, expansiveness and specification guarantees global hyperbolicity:

**Theorem 1.** Let \((H, e, \mathcal{E}_{H,e})\) be a Hamiltonian system. If any of the following statements hold:

1. \((H, e, \mathcal{E}_{H,e})\) is robustly topologically stable;
2. \((H, e, \mathcal{E}_{H,e})\) is stably shadowable;
3. \((H, e, \mathcal{E}_{H,e})\) is stably expansive;
4. \((H, e, \mathcal{E}_{H,e})\) has the stable weak specification property,

then \((H, e, \mathcal{E}_{H,e})\) is Anosov.

It is well known from classical hyperbolic dynamics that Anosov implies shadowing, expansiveness and topological stability. With respect to weak specification the issue is more subtle. For instance, mixing Anosov flows satisfy the specification property. Furthermore it is known that, for volume-preserving flows, the specification property implies the mixing property, see [6, lemma 3.1]. Thus, the following example provides an Anosov flow without the specification property.

**Example 4.1 (Non-mixing Anosov suspension flow).** Let be given an Anosov diffeomorphism in a surface \(\Sigma_f: \Sigma \to \Sigma\), and a ceiling function \(h: \Sigma \to \mathbb{R}^+\) satisfying \(h(x) \geq \beta > 0\) for all \(x \in \Sigma\). We consider the space \(M_h \subseteq \Sigma \times \mathbb{R}^+\) defined by

\[
M_h = \{(x, t) \in \Sigma \times \mathbb{R}^+: 0 \leq t \leq h(x)\}
\]

with the identification between the pairs \((x, h(x))\) and \((f(x), 0)\). The flow defined on \(M_h\) by \(S_t(x, r) = (f^n(x), r + s - \sum_{i=0}^{n-1} h(f^i(x)))\) is an Anosov suspension flow, where \(n \in \mathbb{N}_0\), is uniquely defined by the condition

\[
\sum_{i=0}^{n-1} h(f^i(x)) \leq r + s < \sum_{i=0}^{n} h(f^i(x)).
\]

If we choose \(h(x) = 1\), then the suspension flow cannot be topologically mixing. To see this just observe that the integer iterates of \(\Sigma \times (0, 1/2)\) are disjoint from \(\Sigma \times (1/2, 1)\).

A Hamiltonian system \((H, e, \mathcal{E}_{H,e})\) is a Hamiltonian star system if there exists a neighbourhood \(V\) of \((H, e, \mathcal{E}_{H,e})\) such that, for any \((\tilde{H}, \tilde{e}, \tilde{\mathcal{E}}_{H,e})\) \(\in V\), the correspondent regular energy hypersurface \(\tilde{\mathcal{E}}_{H,e}\) has all hyperbolic closed orbits.

The next result was proved in [10] for \(n = 2\) and recently generalized by the authors in [16], for \(n \geq 2\).

**Theorem 2.** If \((H, e, \mathcal{E}_{H,e})\) is a Hamiltonian star system, then \((H, e, \mathcal{E}_{H,e})\) is Anosov.

Thus, theorem 1 is a direct consequence of theorems 2 and 3.

**Theorem 3.** Let \((H, e, \mathcal{E}_{H,e})\) be a Hamiltonian system. If any of the following statements hold:

a. \((H, e, \mathcal{E}_{H,e})\) is robustly topologically stable;
b. \((H, e, \mathcal{E}_{H,e})\) is stably shadowable;
c. \((H, e, \mathcal{E}_{H,e})\) is stably expansive;
d. \((H, e, \mathcal{E}_{H,e})\) has the stable weak specification property,

then \((H, e, \mathcal{E}_{H,e})\) is a Hamiltonian star system.
It is interesting to note that the specification property implies the topological mixing property (see lemma 6.2). Moreover, we also note that recently it was proved (see [11]) that $C^2$-generically Hamiltonian systems are topologically mixing. Clearly, topological mixing implies transitivity, thus $C^2$-stable weak specification implies $C^2$-stable transitivity.

Given $1 \leq k \leq n - 1$, we recall that a $k$-elliptic closed orbit has $2k$ simple non-real eigenvalues of the transversal linear Poincaré flow (see definition 2.1) at the period of norm one, and its remaining eigenvalues with norm different from one. In particular, when $k = n - 1$, (totally) elliptic closed orbits have all eigenvalues at the period of norm one, simple and non-real.

Partial hyperbolicity guarantees the decomposition of the normal bundle at energy levels into three invariant subbundles such that, the dynamics is uniformly expanding in one direction, uniformly contracting in the other direction and central in the remaining direction (see definition 2.4). If the central subbundle is trivial the system is Anosov.

A regular energy hypersurface is far from partially hyperbolic if it is not in the closure (w.r.t. the $C^2$-topology) of partially hyperbolic surfaces. Notice that, by structural stability, the union of partially hyperbolic energy level sets is open [10]. Moreover, partially hyperbolic hypersurfaces do not contain elliptic closed orbits. Now, we state the following main result:

**Theorem 4.** Given an open subset $U \subset M$, if a $C^2$-Hamiltonian has a regular energy hypersurface, far from partially hyperbolic, intersecting $U$, then it can be $C^2$-approximated by a $C^\infty$-Hamiltonian having a closed elliptic orbit intersecting $U$.

The previous result generalizes the result stated in [43, theorem 6.2] and proved in [13]. As an almost direct consequence, we arrive at the Newhouse–Arnaud–Saghin–Xia dichotomy for $2n$-dimensional ($n \geq 2$) Hamiltonians [2, 43, 51]. Recall that, for a $C^2$-generic Hamiltonian, all but finitely many points are regular.

**Theorem 5.** For a $C^2$-generic Hamiltonian $H \in C^2(M, \mathbb{R})$ the union of the partially hyperbolic regular energy hypersurfaces and the closed elliptic orbits, forms a dense subset of $M$.

At this point it is worth to recall the four-dimensional result that motivated the proof of the Hamiltonian version of Franks’ lemma (see section 5 and theorem 5.2). Thérèse Vivier showed in [55] that any robustly transitive regular energy surface of a $C^2$-Hamiltonian is Anosov. See also [34] for the symplectomorphism case. It is easy to see that our results imply the multidimensional version of Vivier’s theorem. In fact, if a regular energy hypersurface $E_{H,e}$ of a $C^2$-Hamiltonian $H$ is far from partial hyperbolicity, then, by theorem 4, there exists a $C^2$-close $C^\infty$-Hamiltonian with an elliptic closed orbit on a nearby regular energy hypersurface. This invalidates the chance of robust transitivity for $H$ according to a KAM-type criterium (see [55, corollary 9]). The same argument shows that the presence of a regular energy hypersurface $E_{H,e}$ of a $C^2$-Hamiltonian $H$ which is far from partial hyperbolicity invalidates the chance of stable ergodicity.

**Corollary 1.** Let $(H, e, E_{H,e})$ be a robustly transitive Hamiltonian system. Then $E_{H,e}$ is partially hyperbolic.

**Corollary 2.** Let $(H, e, E_{H,e})$ be a stably ergodic Hamiltonian system. Then $E_{H,e}$ is partially hyperbolic.

Finally, we obtain the following result that states that any robust weakly-shadowable regular energy hypersurface of a $C^2$-Hamiltonian is partially hyperbolic.

**Theorem 6.** Let $(H, e, E_{H,e})$ be a stable weakly-shadowable Hamiltonian system. Then $E_{H,e}$ is partially hyperbolic.
5. Perturbation lemmas

In this section we present three key perturbation results for Hamiltonians that we shall use in the sequel. The first one (theorem 5.1) is a version of the $C^1$-pasting lemma (see [5, theorem 3.1]) for Hamiltonians. Actually, in the Hamiltonian setting, the proof of this result is much more simple (see [11]). The second perturbation result (theorem 5.2), due to Vivier, is a version of Franks’s lemma for Hamiltonians (see [55, theorem 1]). Roughly speaking, it says that we can realize a Hamiltonian corresponding to a given perturbation of the transversal linear Poincaré flow. The last perturbation result (theorem 5.3) is a Hamiltonian suspension theorem (see [15, theorem 3]), especially useful for the conversion of perturbative results between symplectomorphisms and Hamiltonian flows. Indeed, if we perturb the Poincaré map of a periodic orbit, there is a nearby Hamiltonian realizing the new map.

**Theorem 5.1 (Pasting lemma for Hamiltonians).** Fix $H \in C^r(M, \mathbb{R}), 2 \leq r \leq \infty$, and let $K$ be a compact subset of $M$ and $U$ a small neighbourhood of $K$. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $H_\delta \in C^l(M, \mathbb{R})$ for $2 \leq l \leq \infty$, is $\delta$-$C^{\min(r, l)}$-close to $H$ on $U$ then there exist $H_0 \in C^\ell(M, \mathbb{R})$ and a closed set $V$ such that:

- $K \subset V \subset U$;
- $H_0 = H_\delta$ on $V$;
- $H_0 = H$ on $U^c$;
- $H_0$ is $\epsilon$-$C^{\min(r, l)}$-close to $H$.

Let $p \in M$ be a regular point of a Hamiltonian $H$ and define the arc $X_H^{[\tau_1, \tau_2]}(p) = \{X_H^{\tau}(p), \tau \in [\tau_1, \tau_2]\}$. Given a transversal section $\Sigma$ to the flow at $p$, a flowbox associated to $\Sigma$ is defined by $\mathcal{F}(p) = X_H^{[\tau_1, \tau_2]}(\Sigma)$, where $\tau_1, \tau_2$ are chosen small enough such that $\mathcal{F}(p)$ is a neighbourhood of $p$ foliated by regular orbits.

**Theorem 5.2 (Franks’ lemma for Hamiltonians).** Take $H \in C^r(M, \mathbb{R}), 2 \leq r \leq \infty$, $\epsilon > 0$, $\tau > 0$ and $p \in M$. Then, there exists $\delta > 0$ such that for any flowbox $\mathcal{F}(p)$ of an injective arc of orbit $X_H^{[0, \tau]}(p)$, with $\tau \geq \tau$, and a transversal symplectic $\delta$-perturbation $\Psi$ of $\Phi_H(p)$, there is $H_0 \in C^\ell(M, \mathbb{R})$ with $\ell = \max\{2, k - 1\}$ satisfying:

- $H_0$ is $\epsilon$-$C^2$-close to $H$;
- $\Phi_H(p) = \Psi$;
- $H = H_0$ on $X_H^{[0, \tau]}(p) \cup (M \setminus \mathcal{F}(p))$.

Consider a Hamiltonian system $(H, \epsilon, \mathcal{E}_{H, \epsilon})$ and a periodic point $p \in \mathcal{E}_{H, \epsilon}$ with period $\pi$. At $p$ consider a transversal $\Sigma \subset M$ to the flow, i.e. a local $(2n - 1)$-submanifold for which $X_H$ is not tangential anywhere. Define the $2n - 2$ symplectic submanifold

$$\Sigma = \Sigma \cap \mathcal{E}_{H, \epsilon}.$$

Thus, for any $p \in \Sigma$

$$T_p \mathcal{E}_{H, \epsilon} = T_p \Sigma \oplus \mathbb{R}X_H(p).$$

Let $U \subset M$ be some open neighbourhood of $p$ and $V = U \cap \Sigma$. The Poincaré (section) map $f : V \to \Sigma$ is the return map of $X_H^\tau$ to $\Sigma$. It is given by

$$f(p) = X_H^\tau(p), \quad p \in V,$$

where $\tau$ is the return time to $\Sigma$ defined implicitly by the relation $X_H^\tau(p) \in \Sigma$ and satisfying $\tau(p) = \pi$. In addition, $p$ is a fixed point of $f$. Notice that one needs to assume that $U$ is a small neighbourhood of $p$. Thus, $f$ is a $C^1$-symplectomorphism between $V$ and its image. Moreover, any two Poincaré section maps of the same closed orbit are conjugate by a symplectomorphism.
Theorem 5.3 (Hamiltonian suspension). Let $H \in C^\infty(M, \mathbb{R})$ with Poincaré map $f$ at a periodic point $p$. Then, for any $\epsilon > 0$ there is $\delta > 0$ such that for any symplectomorphism $f_0$ being $\delta$-$C^1$-close to $f$, there is a Hamiltonian $H_0 \in C^2$-close with Poincaré map $f_0$.

6. Hyperbolicity versus stable shades (proof of theorem 3)

We shall start with a key lemma that states that the presence of a non-hyperbolic periodic point $p$ for a Hamiltonian $H$ ensures the existence of a Hamiltonian $H_1$, arbitrarily $C^2$-close to $H$, exhibiting a continuum of periodic points close to $p$.

Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ and a periodic point $p \in \mathcal{E}_{H,e}$ with period $\pi$. Let $\Sigma_p^c$ denote a submanifold of $\Sigma$, associated to $p$, such that $T_p \Sigma_p^c \oplus \mathbb{R}X_H(p) = N_p^c \oplus \mathbb{R}X_H(p)$, where $N_p^c$ denotes the subspace of $N_p$ associated with norm-one eigenvalues of $\Phi_H^\pi(p)$.

Lemma 6.1 (Explosion of periodic orbits). Let $(H, e, \mathcal{E}_{H,e})$ be a Hamiltonian system and let $p \in \mathcal{E}_{H,e}$ be a non-hyperbolic periodic point. Then, there exists a Hamiltonian system $(H_1, e_1, \mathcal{E}_{H_1,e_1})$, arbitrarily close to $(H, e, \mathcal{E}_{H,e})$, such that $H_1$ has a non-hyperbolic periodic point $q \in \mathcal{E}_{H_1,e_1}$, close to $p$, and such that every point in a small neighbourhood of $q$, in $\Sigma_q^c$, is a periodic point of $H_1$.

Proof. Let $(H, e, \mathcal{E}_{H,e})$ be a Hamiltonian system and let $p \in \mathcal{E}_{H,e}$ be a non-hyperbolic periodic point of period $\pi$. If $H \in C^\infty(M, \mathbb{R})$, take $H_0 = H$; otherwise we use the pasting lemma for Hamiltonians (theorem 5.1) to obtain a Hamiltonian $\overline{H} \in C^\infty(M, \mathbb{R})$ such that $\overline{H}$ is arbitrarily $C^2$-close to $H$ and that $\overline{H}$ has a periodic point $\overline{p}$, close to $p$, with period $\pi$ close to $\pi$. We observe that $\overline{p}$ may not be the analytic continuation of $p$ and this is precisely the case when 1 is an eigenvalue of $\Phi_{\overline{H}}^\pi(p)$. If $\overline{p}$ is not hyperbolic, take $H_0 = \overline{H}$. If $\overline{p}$ is hyperbolic then, since $\overline{H}$ is arbitrarily $C^2$-close to $H$, the distance between the spectrum of $\Phi_{\overline{H}}^\pi(p)$ and $\mathbb{S}^1$ can be taken arbitrarily close to zero (weak hyperbolicity). Now, we are in a position to apply Franks’ lemma for Hamiltonians (theorems 5.2) to obtain a new Hamiltonian $H_0 \in C^\infty(M, \mathbb{R})$, $C^2$-close to $\overline{H}$ and having a non-hyperbolic periodic point $q$ close to $\overline{p}$.

Clearly, the Poincaré map $f_0$ at $q$, associated to the Hamiltonian flow $X_{H_0}$, is a $C^\infty$ local symplectomorphism. In order to go on with the argument and obtain the Hamiltonian $H_1$, the first step is to use the weak pasting lemma for symplectomorphisms [5, lemma 3.9] to change the Poincaré map $f_0$ by its derivative. In this way we get a symplectomorphism $f_1$, arbitrarily $C^\infty$-close to $f_0$, such that (in the local canonical coordinates mentioned in [42] and given by Darboux’s theorem) $f_1$ is linear and equal to $Df_0$ in a neighbourhood of the periodic non-hyperbolic periodic point $q$.

Next, we use the Hamiltonian suspension theorem (theorems 5.3) to realize $f_1$, i.e. in order to obtain a Hamiltonian $H_1 \in C^\infty(M, \mathbb{R})$ such that $f_1$ is linear and equal to $Df_0$ in a neighbourhood of the non-hyperbolic periodic point $q$.

Moreover, the existence of an eigenvalue, $\lambda$, with modulus equal to one is associated to a symplectic invariant two-dimensional subspace $E$ contained in the subspace $N_p^c$, associated to norm-one eigenvalues. Furthermore, up to a perturbation using again the Hamiltonian suspension theorem, $\lambda$ can be taken to be rational, thus creating periodic points related with $E$. This argument can be repeated for each norm-one eigenvalue, if necessary (see the proof of ([16, theorem 2]) for the details). This ensures the existence of a Hamiltonian system $(H_1, e_1, \mathcal{E}_{H_1,e_1})$ arbitrarily close to $(H, e, \mathcal{E}_{H,e})$ such that $q \in \mathcal{E}_{H_1,e_1}$, and of a small neighbourhood of $q$ in $\Sigma_q^c$ composed of periodic points. □
Remark 6.1. Let \((H_1, e_1, \mathcal{E}_{H_1,e_1})\) and \(q \in \mathcal{E}_{H_1,e_1}\) be given by lemma 6.1. We observe that the proof of lemma 6.1 guarantees that \((H_1, e_1, \mathcal{E}_{H_1,e_1})\) is such that the Poincaré map \(f_1\) at \(q\) associated to \(X'_{H_1}\) is a linear map (in the local canonical coordinates mentioned above). This fact will be implicitly used in the proofs of (a), (b) and (c) of theorem 3 and in the proof of proposition 8.1.

Now we are in position to prove item (a) of theorem 3.

Proof. Given a robustly topologically stable Hamiltonian system \((H, e, \mathcal{E}_{H,e})\), we prove that all its periodic orbits in \(\mathcal{E}_{H,e}\) are hyperbolic; from this it follows that \((H, e, \mathcal{E}_{H,e})\) is a Hamiltonian star system.

By contradiction, let us assume that there exists a robustly topologically stable Hamiltonian system \((H, e, \mathcal{E}_{H,e})\) having a non-hyperbolic periodic point \(p \in \mathcal{E}_{H,e}\). It follows from lemma 6.1 that there exists a robustly topologically stable Hamiltonian system \((H_1, e_1, \mathcal{E}_{H_1,e_1})\), arbitrarily close to \((H, e, \mathcal{E}_{H,e})\), and there exists a non-hyperbolic periodic point \(q \in \mathcal{E}_{H_1,e_1}\) of \(H_1\) for which every point in a small neighbourhood of \(q\), in \(\Sigma_{q_1}\), is a periodic point of \(H_1\).

Finally, we approximate \((H_1, e_1, \mathcal{E}_{H_1,e_1})\) by \((H_2, e_2, \mathcal{E}_{H_2,e_2})\), also robustly topologically stable, such that \(q\) is an hyperbolic periodic point or an isolated \(k\)-elliptic periodic point (for \(H_2\)). This is a contradiction because \((H_2, e_2, \mathcal{E}_{H_2,e_2})\) is semiconjugated to \((H_1, e_1, \mathcal{E}_{H_1,e_1})\), although there is an \(H_1\)-orbit (different from \(q\)) contained in a small neighbourhood of \(q\) and the same cannot occur for \(H_2\) because \(q\) is a hyperbolic periodic point or an isolated \(k\)-elliptic periodic point of \(H_2\).

Let us now prove item (b) of theorem 3.

Proof. Given a stably shadowable Hamiltonian system \((H, e, \mathcal{E}_{H,e})\), we prove that all its periodic orbits in \(\mathcal{E}_{H,e}\) are hyperbolic; from this it follows that \((H, e, \mathcal{E}_{H,e})\) is a Hamiltonian star system.

By contradiction, let us assume that there exists a stably shadowable Hamiltonian system \((H, e, \mathcal{E}_{H,e})\) having a non-hyperbolic periodic point \(p \in \mathcal{E}_{H,e}\). It follows from lemma 6.1 that there exists a stably shadowable Hamiltonian system \((H_1, e_1, \mathcal{E}_{H_1,e_1})\), arbitrarily close to \((H, e, \mathcal{E}_{H,e})\), and there exists a non-hyperbolic periodic point \(q \in \mathcal{E}_{H_1,e_1}\) of \(H_1\) for which every point, say in a \(\xi\)-neighbourhood of \(q\), in \(\Sigma_{q_1}\), is a periodic point of \(H_1\).

Fix any \(\epsilon \in (0, 1/2)\). Since \((H_1, e_1, \mathcal{E}_{H_1,e_1})\) has the shadowing property there exist \(\delta \in (0, \epsilon)\) and \(T > 0\) such that every \((\delta, T)\)-pseudo-orbit \(((x_i), (t_i))_{i \in \mathbb{Z}}\) is \(\epsilon\)-shadowed by some orbit of \(H_1\).

Let \(x_0 = q\). Take \(y \in \Sigma_{q_1}^\circ\) such that \(d(q, y) = \frac{\pi}{2} > 2\epsilon\) and fix \(\delta \in (0, \epsilon)\) sufficiently small. We construct a bi-infinite sequence of points \(((x_i), (t_i))_{i \in \mathbb{Z}}\) with \(x_i \in \Sigma_{q_1}^\circ\) such that \(((x_i), (t_i))_{i \in \mathbb{Z}}\) is a \((\delta, T)\)-pseudo orbit for some \(T > 0\). If \(k \in \mathbb{N}\) is sufficiently large then there exist \(x_i \in \Sigma_{q_1}^\circ, i \in \mathbb{Z}\), such that:

- \(x_i = x_0\) and \(t_i = \pi\) for \(i \leq 0\);
- \(d(x_i, x_{i-1}) < \delta\) and \(t_i = \pi\) for \(1 \leq i \leq k\);
- \(x_i = y\) and \(t_i = \pi\) for \(i \geq k\).

Observe that we are considering that the return time at the transversal section is the same and equal to \(\pi\). Clearly, it is not exactly equal to \(\pi\), however it is as close to \(\pi\) as we want by just decreasing the \(\xi\)-neighbourhood. Therefore, \(((x_i), (\pi))_{i \in \mathbb{Z}}\) is a \((\delta, T)\)-pseudo-orbit for some \(T > 0\) such that \(\pi \geq T\).

By the shadowing property, there is a point \(z \in \mathcal{E}_{H,e}\) and a reparametrization \(\alpha \in \text{Rep}(\epsilon)\) such that \(d(X_{H_1}^\alpha(z), x_0 \ast t) < \epsilon\), for every \(t \in \mathbb{R}\). Hence, there is no forward/backward
expansion and so, \( z \in \Sigma_q^c \). However, since \((H_1, e_1, \mathcal{E}_{H_1,e_1})\) has the shadowing property and \( x_0 \ast k \pi = x_k \), we have that
\[
d(q, y) \leq d(q, X_{H_1}^{n(k\pi)}(z)) + d(X_{H_1}^{n(k\pi)}(z), x_k) < 2\epsilon,
\]
which is a contradiction. \( \square \)

Now we prove item (c) of theorem 3.

**Proof.** Given a stably expansive Hamiltonian system \((H, e, \mathcal{E}_{H,e})\), we prove that all its periodic orbits are hyperbolic; from this it follows that \((H, e, \mathcal{E}_{H,e})\) is a Hamiltonian star system.

By contradiction, let us assume that there exists a stably expansive Hamiltonian system \((H, e, \mathcal{E}_{H,e})\) having a non-hyperbolic periodic point \( p \in \mathcal{E}_{H,e} \). If follows from lemma 6.1 that there exists a stably expansive Hamiltonian system \((H_1, e_1, \mathcal{E}_{H_1,e_1})\), arbitrarily close to \((H, e, \mathcal{E}_{H,e})\), and there exists a non-hyperbolic periodic point \( q \in \mathcal{E}_{H_1,e_1} \) of period \( \pi \) of \( H_1 \) for which every point in a small neighbourhood of \( q \), in \( \Sigma_q^c \), is a periodic point of \( H_1 \) with period close to \( \pi \).

Finally, we just have to pick two points \( x, y \in \Sigma_q^c \) sufficiently close in order to obtain
\[
d(X_t^{H_1}(x), X_t^{H_1}(y)) < \delta \quad \text{for all} \quad t \in \mathbb{R}.
\]
It is clear that \((H_1, e_1, \mathcal{E}_{H_1,e_1})\) cannot be expansive which is a contradiction and theorem 3(c) is proved. \( \square \)

To prove theorem 3(d), we shall start by deducing some consequences of the weak specification property. Let us first recall that a compact energy hypersurface \( \mathcal{E}_{H,e} \) of a Hamiltonian \((H, e, \mathcal{E}_{H,e})\) is topologically mixing if, for any open and non-empty subsets of \( \mathcal{E}_{H,e} \), say \( U \) and \( V \), there is \( \tau \in \mathbb{R} \) such that \( X_t^H(U) \cap V \neq \emptyset \), for any \( t \geq \tau \). The first lemma is a particular case of [6, lemma 3.1].

**Lemma 6.2.** If a Hamiltonian system \((H, e, \mathcal{E}_{H,e})\) has the weak specification property, then \( \mathcal{E}_{H,e} \) is topologically mixing.

Let \((H, e, \mathcal{E}_{H,e})\) be a Hamiltonian system and let \( p \in \mathcal{E}_{H,e} \) be a periodic point of period \( \pi \) such that the spectrum of \( \Phi^H_{\tau}(p) \) outside the unit circle is a non-empty set. Let \( S^0(H, p) \) be the part of the spectrum that does not lie on the unit circle. Observe that this set contains both eigenvalues with modulus greater than one and smaller than one.

We define the strong stable and stable manifolds of \( p \) as
\[
W_{ss}(p) = \{ y \in \mathcal{E}_{H,e} : \lim_{t \to +\infty} d(X_t^{H}(y), X_t^{H}(p)) = 0 \}
\]
and
\[
W^s(O(p)) = \bigcup_{t \in \mathbb{R}} W^s(X_t^{H}(p)),
\]
where \( O(p) \) stands for the orbit of \( p \). For small \( \epsilon > 0 \), the local strong stable manifold is defined as
\[
W_{ss}^{1}(p) = \{ y \in \mathcal{E}_{H,e} : d(X_t^{H}(y), X_t^{H}(p)) < \epsilon \quad \text{if} \quad t \geq 0 \}.
\]
By the stable manifold theorem, there exists an \( \epsilon = \epsilon(p) > 0 \) such that
\[
W^{ss}(p) = \bigcup_{t \geq 0} X^{-t}_{H}(W_{ss}^{1}(X_{H}(p))).
\]
Analogous definitions hold for unstable manifolds.

The next result is an adaptation of [6, theorem 3.3].

**Lemma 6.3.** If a Hamiltonian system \((H, e, \mathcal{E}_{H,e})\) has the weak specification property, then for all distinct periodic points \( p, q \in \mathcal{E}_{H,e} \) such that \( S^0(H, p) \neq \emptyset \) and \( S^0(H, q) \neq \emptyset \), we have that \( W^s(O(p)) \cap W^s(O(q)) = \emptyset \).
Proof. We denote by $\epsilon(p)$ the size of the local strong unstable manifold of $p$ and by $\epsilon(q)$ the size of the local strong stable manifold of $q$. Let $\epsilon = \min(\epsilon(p), \epsilon(q))$ and let $K = K(\epsilon)$ be given by the weak specification property. If $t > 0$ then take $I_1 = [0, t]$ and $I_2 = [K + t, K + 2t]$. Now define $P(s) = X_H^{s-t}(p)$ if $s \in I_1$ and $P(s) = X_H^{s-t-k-l}(q)$ if $s \in I_2$. Note that this is a $K$-spaced weak specification.

So, there exists $\epsilon(p)$ which shadows this weak specification:

$$d(X_H^s(x_i), P(s)) \leq \epsilon$$ if $s \in I_1 \cup I_2$.

Using the change of variable $u = t - s$, for every $u \in [0, t]$ we have:

$$d(X_H^s(X_H^u(x_i)), X_H^u(p)) = d(X_H^{s-u}(x_i), X_H^u(p)) \leq \epsilon$$

and using $u = s - K - t$, for every $u \in [0, t]$ we have

$$d(X_H^u(X_H^{K+u}(x_i)), X_H^u(q)) \leq \epsilon.$$

If $y_t = X_H(x_i)$ then we can assume that $y_t \rightarrow y$. Taking limits in the previous inequalities we obtain

$$d(X_H^{-u}(y), X_H^{-u}(p)) \leq \epsilon$$ for every $u \geq 0$ and

$$d(X_H^{u}(X_H^{K+u}(y)), X_H^{u}(q)) \leq \epsilon$$ for every $u \geq 0$.

The first says that $y \in W^{ss}(p) \subseteq W^{ss}(O(p))$ and the second says that $X_H^K(y) \in W^{uu}(q)$, hence $y \in W^s(O(p))$. □

Proposition 6.4. If $(H, e, E_{H,e})$ satisfies the stable weak specification property, then $E_{H,e}$ is partially hyperbolic. In particular, due to remark 2.1, if $n = 2$, then $E_{H,e}$ is hyperbolic.

Proof. The proof is by contradiction; let us assume there exists a Hamiltonian system $(H, e, E_{H,e})$ that has the stable weak specification property and such that $E_{H,e}$ is not partially hyperbolic. Then, by theorem 4, there exists a $C^2$-close $C^\infty$-Hamiltonian $H_0$ with an elliptic closed orbit on a nearby regular energy hypersurface $E_{H_0,e_0}$. This invalidates the chance of mixing for $E_{H_0,e_0}$ according to a KAM-type criterium (see [55, corollary 9]), which contradicts lemma 6.2.

We recall that a Hamiltonian system $(H, e, E_{H,e})$ is a Kupka–Smale Hamiltonian system if (see [49]):

1. the union of the hyperbolic and $k$-elliptic closed orbits ($1 \leq k \leq n - 1$) in $E_{H,e}$ is dense in $E_{H,e}$;
2. the intersection of $W^s(O(p))$ with $W^s(O(q))$ is transversal, for any closed orbits $O(p)$ and $O(q)$.

Lemma 6.5. Let $(H, e, E_{H,e})$ be a Hamiltonian system satisfying the stable weak specification property and let $V$ be a neighbourhood of $(H, e, E_{H,e})$ such that any $(H_0, e_0, E_{H_0,e_0}) \in V$ has the weak specification property. Then, every Kupka–Smale Hamiltonian system in $V$ has hyperbolic periodic points.

Proof. Let $V$ be a neighbourhood of $(H, e, E_{H,e})$ as in the hypothesis of the lemma. Let $p, q \in E_{H_0,e_0}$ be two periodic points of a Kupka–Smale Hamiltonian system $(H_0, e_0, E_{H_0,e_0}) \in V$ and suppose, by contradiction, that $p$ is a non-hyperbolic periodic point. Then, $\dim W^u(O(p)) < (2n - 2)/2$ and, therefore, $\dim W^u(O(p)) + \dim W^s(O(q)) < 2n - 2$. Since, the stable/unstable manifolds intersect in a transversal way, we must have $W^u(O(p)) \cap W^s(O(q)) = \emptyset$. However this contradicts lemma 6.3. □
Remark 6.2. Fix some Hamiltonian system \((H, e, E_{H,e})\) such that \(E_{H,e}\) has a \(k\)-elliptic closed orbit, \(1 \leq k \leq n - 1\). We get that the analytic continuation of \(E_{H,e}, E_{\hat{H},\hat{e}}\), has still a \(k\)-elliptic closed orbit (its analytic continuation). Therefore, the set of Hamiltonians exhibiting \(k\)-elliptic \((1 \leq k \leq n - 1)\) closed orbits is open in \(C^2(M, \mathbb{R})\) (see e.g. [49]).

Lemma 6.6. Let \((H, e, E_{H,e})\) be a Hamiltonian system and let \(p \in E_{H,e}\) be a non-hyperbolic periodic point. Then, there exists \((H_0, e_0, E_{H_0,e_0})\), arbitrarily close to \((H, e, E_{H,e})\), such that \((H_0, e_0, E_{H_0,e_0})\) is a Kupka–Smale Hamiltonian system exhibiting 1-elliptic periodic points.

Proof. Let \((H, e, E_{H,e})\) be a Hamiltonian system with a non-hyperbolic periodic point \(p \in E_{H,e}\). As the boundary of the Anosov Hamiltonian systems has no isolated points (see [16, corollary 1]), it follows from the Newhouse dichotomy for Hamiltonians [43, 13] that \(H\) can be \(C^2\)-approximated by a Hamiltonian exhibiting 1-elliptic periodic points. Since, by remark 6.2, 1-elliptic periodic points are stable, it follows from Robinson’s version of the Kupka–Smale theorem (see [49]) that there exists a Kupka–Smale Hamiltonian system \((H_0, e_0, E_{H_0,e_0})\), arbitrarily close to \((H, e, E_{H,e})\), such that \(H_0\) has a 1-elliptic periodic point in \(E_{H_0,e_0}\).

Now we are in position to prove item (d) of theorem 3.

Proof. Given a Hamiltonian system \((H, e, E_{H,e})\) satisfying the stable weak specification property, we prove that all its periodic orbits are hyperbolic; from this it follows that \((H, e, E_{H,e})\) is a Hamiltonian star system.

By contradiction, let us assume that there exists a Hamiltonian system \((H, e, E_{H,e})\) satisfying the stable weak specification property and having a non-hyperbolic periodic point \(p \in E_{H,e}\). Let \(V\) be a neighbourhood of \((H, e, E_{H,e})\) such that the weak specification property holds here. By lemma 6.6 there exists a Kupka–Smale Hamiltonian \((H_0, e_0, E_{H_0,e_0})\) \(\in V\) such that \(H_0\) has a non-hyperbolic periodic point which contradicts lemma 6.5.

7. Partial hyperbolicity versus dense elliptic orbits (proof of Theorems 4 and 5)

7.1. Proof of theorem 5

A Hamiltonian system \((H, e, E_{H,e})\) is partially hyperbolic if \(E_{H,e}\) is partially hyperbolic. Let \(\mathcal{P}H^2_{\mathbb{R}}(M) \subset C^2(M, \mathbb{R})\) denote the subset of partially hyperbolic Hamiltonians\(^5\). Fix \(H \in \mathcal{P}H^2_{\mathbb{R}}(M)\) and let \(e \in H(M)\) be an energy such that the subset \(H^{-1}(\{e\})\) has a partial hyperbolic component \(E_{H,e}\). For any \(H\) arbitrarily \(C^2\)-close to \(H\) and \(e\) arbitrarily close to \(e\), we get that the analytic continuation of \(E_{H,e}, E_{\hat{H},\hat{e}}\), is still partially hyperbolic. Thus, in other words, partial hyperbolicity is an open property. The proof is similar to the openness of the hyperbolicity done in [10] and mainly uses cone field arguments.

The proof of theorem 5 is a consequence of theorem 4.

Proof. Consider the set

\[ \mathcal{G} = C^2(M, \mathbb{R}) \times M \]

endowed with the product topology associated to the \(C^2\)-topology in \(C^2(M, \mathbb{R})\) and with the topology inherited by the Riemannian structure in \(M\). Given \(p \in M\), let \(E_{H,e}\) be the energy surface passing through \(p\). As we have mentioned before the subset \(\mathcal{P}H\) defined as

\[ \{(H, p) \in \mathcal{G}; E_{H,e} \text{ is a partially hyperbolic regular energy hypersurface and } E_{H,e} \ni p\} \]

is open. Let \(\mathcal{P}H\) be its closure (w.r.t. the \(C^2\)-topology) with complement \(\mathcal{N} = \mathcal{G} \setminus \mathcal{P}H\).

\(^5\) Observe that, due to remark 2.1, if \(n = 2\), then \(\mathcal{P}H^2_{\mathbb{R}}(M)\) is equal to the Anosov Hamiltonian systems.
Given $\epsilon > 0$ and an open set $\mathcal{U} \subset \mathcal{N}$, we can define the subset $\mathcal{O}(\mathcal{U}, \epsilon)$ of pairs $(H, p) \in \mathcal{U}$ for which $H$ has a closed elliptic orbit intersecting the $(2n - 1)$-dim ball $B(p, \epsilon) \cap \mathcal{E}_{H,e}$. This is possible due to theorem 4. It follows from theorem 4 and the fact that (totally)-elliptic orbits are stable (remark 6.2), that $\mathcal{O}(\mathcal{U}, \epsilon)$ is dense and open in $\mathcal{U}$.

Let $(\epsilon_k)_{k \in \mathbb{N}}$ be a positive sequence such that $\epsilon_k \to 0$ as $k \to \infty$. Then, define the sequence of dense and open sets $\mathcal{U}_k := \mathcal{O}(\mathcal{N}, \epsilon_{k-1})$, $k \in \mathbb{N}$. Notice that $\bigcap_{k \in \mathbb{N}} \mathcal{U}_k$ is the set of pairs $(H, p)$ yielding the property that $p$ is accumulated by closed elliptic orbits.

Finally, the above implies that, for each $k \in \mathbb{N}$, $\mathcal{P} \mathcal{H} \cup \mathcal{U}_k$ is open and dense in $\mathcal{G}$ and $\mathfrak{F} := \bigcap_{k \in \mathbb{N}} (\mathcal{P} \mathcal{H} \cup \mathcal{U}_k)$ is residual. By [24, proposition A.7], we write $\mathfrak{F} = \bigcup_{H \in \mathcal{R}} \{H\} \times \mathcal{M}_H$, where $\mathcal{R}$ is $C^2$-residual in $C^2(M, \mathbb{R})$ and, for each $H \in \mathcal{R}$, $\mathcal{M}_H$ is a residual subset of $M$, having the following property: if $H \in \mathcal{R}$ and $p \in \mathcal{M}_H$, then $\mathcal{E}_{H,e}$ is partially hyperbolic or $p$ is accumulated by closed elliptic orbits.

\section{7.2. Proof of theorem 4}

We begin by considering the following result which is a kind of closing lemma of strong type.

\textbf{Lemma 7.1.} For any homoclinic point $z$ associated to the periodic hyperbolic point $x$ of $H \in C^\infty(M, \mathbb{R})$, there exists an arbitrarily small $C^3$ perturbation of $H$, supported in a small neighbourhood of $x$ such that $z$ becomes a periodic point.

\textbf{Proof.} By [51, lemma 10], for any homoclinic point $z$ associated to the periodic hyperbolic point $x$ of $f \in \text{Diff}_\omega^3(M^{2n-2})$, there exists an arbitrarily small $C^3$ perturbation of $f$, $\tilde{f} \in \text{Diff}_\omega^3(M^{2n-2})$, supported in a small neighbourhood of $x$ such that $z$ becomes a periodic point.

Since periodic points are dense in the homoclinic class, we can choose a periodic point $p$ close to $x$. We consider the Poincaré map of $H$ in a small transversal section at $x$ and define it as the symplectic map $f$ obtained in [51, Lemma 10]. Finally, the Hamiltonian suspension theorem (theorem 5.3), gives the perturbation required in the statement of the lemma. \qed

Take $H \in C^2(M, \mathbb{R})$. Since the time-1 map of any tangent flow derived from a Hamiltonian vector field is measure preserving, we obtain a version of Oseledets’ theorem [45] for Hamiltonian systems. Namely, there exists a decomposition $\mathcal{N}_p = \mathcal{N}_p^1 \oplus \mathcal{N}_p^2 \oplus \ldots \oplus \mathcal{N}_p^{k(p)}$ called Oseledets splitting and, for $1 \leq i \leq k(p) \leq 2n$, there are well-defined real numbers $\lambda_i(H, p) = \lim_{t \to \pm \infty} \frac{1}{t} \log \|\Phi_H^t(p) \cdot v_i\|$, $\forall v_i \in \mathcal{N}_p^i \setminus \{0\}$, called the Lyapunov exponents associated to $H$ and $p$. Since we are dealing with Hamiltonian systems (which imply the volume-preserving property), we obtain that $\sum_{i=1}^{k(p)} \lambda_i(H, p) = 0$. \hfill (7.1)

Notice that the spectrum of the symplectic linear map $\Phi_H^t$ is symmetric with respect to the $x$-axis and to the unit circle. In fact, if $\sigma \in \mathbb{C}$ is an eigenvalue with multiplicity $m$ so is $\sigma^{-1}$.
σ and σ⁻¹ keeping the same multiplicity (see theorem 2.1). Consequently, in the Hamiltonian context the Lyapunov exponents come in pairs and, for all \(i \in \{1, \ldots, n - 1\}\), we have

\[
\lambda_i(H, p) = -\lambda_{2n-i-1}(H, p) := -\lambda_i(H, p).
\]  

(7.2)

Therefore, not counting the multiplicity and abbreviating \(\lambda(H, p) = \lambda(p)\), we have the decreasing set of real numbers,

\[
\lambda_1(p) \geq \lambda_2(p) \geq \cdots \geq \lambda_{n-1}(p) \geq 0 \geq -\lambda_{n-1}(p) \geq \cdots \geq -\lambda_2(p) \geq -\lambda_1(p),
\]

or simply

\[
\lambda_1(p) \geq \lambda_2(p) \geq \cdots \geq \lambda_{n-1}(p) \geq 0 \geq \lambda_{n-1}(p) \geq \cdots \geq \lambda_2(p) \geq \lambda_1(p).
\]

Associated to the Lyapunov exponents we have the Oseledets decomposition

\[
T_pM = N^1_p \oplus N^2_p \oplus \cdots \oplus N^{n-1}_p \oplus N^{n-2}_p \oplus \cdots \oplus N^2_p \oplus N^1_p.
\]  

(7.3)

When all the Lyapunov exponents are equal to zero, we say that Oseledets splitting is trivial. The vector field direction \(\mathbb{R}X_H(p)\) is trivially an Oseledets direction with zero Lyapunov exponent and its ‘symplectic conjugate’ is the direction transversal to the energy level.

**Remark 7.1.** Let \(p \in M\) be a closed orbit for \(X_H^\prime\) of period \(\pi\). Then, \(\lambda_i = \pi^{-1}\log|\sigma_i|\) are the Lyapunov exponents, where \(\sigma_i\) are the eigenvalues of \(\Phi_H^\prime(p)\). Moreover, Oseledets decomposition is defined by the eigendirections. Observe that eigenvalues can be complex and Lyapunov exponents are real numbers.

Define \(\Lambda_i(p) = \lambda_1(p) + \lambda_2(p) + \cdots + \lambda_i(p)\) which represents the top exponential growth of the \(i\)-dimensional volume corresponding to the evolving of \(\Phi_H^\prime(p)\) (for details see [3, section 3.2.3]).

The main principle that makes the argument for the proof of our results possible is the following one due to Mañé:

**The Mañé principle.** In the absence of a dominated splitting some perturbation of \(\Phi_H^\prime\), by rotating its solutions, can be done in order to lower the Lyapunov exponents associated to the splitting without domination.

Actually, the ideas presented here are based on Mañé’s now well-known seminal ideas of mixing different Oseledets directions in order to lower expansion rates and was deeply explored in [7, 9, 12, 23, 51, 55]. This is the content of the following two lemmas. We observe that our notation with respect to the order of the Lyapunov exponents is inverted when compared to the one used in [51], however, the proofs follows equally. We recall that a splitting \(E \oplus F\) has index \(k\) if \(\dim(E) = k\). In our case the index is the dimension of the Oseledets subspace associated with the exponents \(\lambda_1, \ldots, \lambda_{i-1}\).

**Lemma 7.2.** Let \(H \in C^2(M, \mathbb{R})\), \(p \in E_{H,e}\) a hyperbolic periodic point for \(X^\prime_H\) and \(\lambda_{i-1}(H, p) - \lambda_i(H, p) > \delta\) where \(\delta > 0\). Moreover, we assume that \(H\) does not have an \(\ell\)-dominated splitting of index \(i - 1\). Then, there exist \(H_0\), such that \(\|H - H_0\| < \epsilon(\ell)\) and \(y \in E_{H_0,e_0}\) a hyperbolic periodic point of \(H_0\), arbitrarily close to \(p\), such that:

\[
\Lambda_{i-1}(H_0, y) < \Lambda_{i-1}(H, p) - \frac{\delta}{2}.
\]  

(7.4)

**Proof.** The proof follows the same lines of the one in [51, proposition 9]. Let us recall the main steps: First, by [11, corollary 3.9], we know that there is a residual set \(\mathcal{R}\) in \(C^2(M, \mathbb{R})\) such that, for any \(H \in \mathcal{R}\), there is an open and dense set \(S(H)\) in \(H(M)\) such that if \(e \in S(H)\) then any energy hypersurface of \(H^{-1}(\{e\})\) is a homoclinic class. Actually, we can make a
small perturbation on the Hamiltonian and on the energy in order to obtain that, given any hyperbolic periodic point \( p \) of \( H \), the set of its homoclinic related points, \( H_p \), is dense on \( \mathcal{E}_{H,e} \).

Moreover, we can make these perturbations arbitrarily small to guarantee that we still do not have \( \ell \)-dominated splitting of index \( i - 1 \) for the analytic continuation of \( p \).

Second, using the spectral gap hypothesis on \( p \), i.e., \( \lambda_{i-1}(H, p) - \lambda_i(H, p) > \delta \), we can spread this property to \( H_p \) by defining subbundles \( E \) and \( F \) of \( NH_p \), where \( E \) is associated with Lyapunov exponents greater than or equal to \( \lambda_{i-1} \) and \( F \) is associated to Lyapunov exponents less than or equal to \( \lambda_i \). Since the dominated splitting can be extended to the closure (see [25]), if \( E \) \( \ell \)-dominates \( F \) in \( H_p \), then it can be extended to the whole energy hypersurface which is a contradiction.

Third, we use the lack of dominated splitting on \( E \oplus F \) (say \( E \) does not \( \ell \)-dominate \( F \)) to send directions in \( E \) into directions in \( F \) by small \( C^2 \) local perturbations along the segment of the orbit of a homoclinic point \( z \). To put into operation Mañé’s principle we must use theorem 5.2 several times. This will imply the desired inequalities (7.4) for the homoclinic point \( z \).

Finally, we just have to use lemma 7.1 to obtain a small perturbation that makes \( z \) periodic. □

As an almost immediate consequence of lemma 7.2, we obtain (see [51, corollary 11]):

**Corollary 7.3.** Let \( H \in C^2(M, \mathbb{R}) \) and \( \mathcal{E}_{H,e} \) be an energy hypersurface without a dominated splitting of index \( i - 1 \). Then, there exists \( H_0 \) arbitrarily close to \( H \) such that \( H_0 \) has a closed orbit \( p \) with \( \lambda_i(p) = \lambda_{i-1}(p) \).

Now, since we follow closely [51, section 7], we give the highlights of the proof of theorem 4.

**Proof.** Let be given an open subset \( U \subset M \) and let \( H \) be a \( C^2 \)-Hamiltonian with a far from partially hyperbolic regular energy hypersurface intersecting \( U \). We will prove that \( H \) can be \( C^2 \)-approximated by a \( C^\infty \)-Hamiltonian \( H_0 \) having a closed elliptic orbit through \( U \).

By remark 2.1, the existence of a dominated splitting implies partial hyperbolicity. Thus, if some energy hypersurface \( \mathcal{E}_{H,e} \) intersects \( U \) and is not partially hyperbolic, then \( \mathcal{E}_{H,e} \) does not have a dominated splitting at any fibre decomposition of the normal subbundle \( N \) that we consider.

Observe that \( \epsilon \)-close to \( H \) we have that all systems have energy hypersurfaces far from being \( \ell \)-dominated. By contradiction, we assume that the system is ‘far’ from having elliptic closed orbits, i.e., arbitrarily close to \( H \) there are no elliptic closed orbits inside the intersection of a regular energy hypersurface and \( U \). Thus, all closed orbits have some positive Lyapunov exponent \( \lambda \).

Then, lemma 7.2 is used several times to create a sequence of Hamiltonians \( C^2 \)-converging to \( H \) with a Lyapunov exponent at the closed orbits passing throughout \( U \) less than \( r \lambda \), where \( r \in (0, 1) \) but close to 1 which is a contradiction. □

**8. Weak shadowing (proof of theorem 6)**

The next result says, in brief terms, that if a Hamiltonian system can be perturbed in order to create elliptic points, then for small perturbations an iterate of the Poincaré map associated to the elliptic point is the identity. This prevents the weak shadowing property. Its proof is very similar to the one of item (b) of theorem 3.
Proposition 8.1. Let \((H, e, \mathcal{E}_{H,e})\) be a stable weakly-shadowable Hamiltonian system. Then, there exists a neighbourhood \(\mathcal{V}\) of \((H, e, \mathcal{E}_{H,e})\) such that any \((H_0, e_0, \mathcal{E}_{H_0,e_0}) \in \mathcal{V}\) does not have elliptic points in \(\mathcal{E}_{H_0,e_0}\).

Proof. Let \(\mathcal{V}\) be a neighbourhood of \((H, e, \mathcal{E}_{H,e})\) such that any Hamiltonian system in \(\mathcal{V}\) is weakly shadowable. By contradiction, let us assume that \((H_0, e_0, \mathcal{E}_{H_0,e_0}) \in \mathcal{V}\) has an elliptic point \(q \in \mathcal{E}_{H_0,e_0}\) of period \(\pi\). It follows from lemma 6.1 and from the stability of elliptic points (see remark 6.2) that there exists a Hamiltonian system \((H_1, e_1, \mathcal{E}_{H_1,e_1}) \in \mathcal{V}\) such that every point in a \(\xi\)-neighbourhood of \(q\), in \(\Sigma_q\), is a periodic point. But, since in the current setting, \(q\) is elliptic, we have that \(\Sigma_q^0 = \Sigma_q\) and, therefore, as \(f_1\) is linear there exists \(m > 0\) such that \(f_1^m\) is the identity map in a \(\xi\)-neighbourhood of \(q\). In order to simplify our arguments, let us suppose that \(m = 1\). Fix any \(\epsilon \in (0, \frac{\xi}{2})\). Since \(H_1\) has the weak shadowing property there exist \(\delta \in (0, \epsilon)\) and \(T > 0\) such that every \((\delta, T)\)-pseudo-orbit \(((x_i), (t_i))_{i \in \mathbb{Z}}\) is weakly \(\epsilon\)-shadowed by a trajectory \(\mathcal{O}(z)\).

Let \(x_0 = q\). Take \(y \in \Sigma_q\) such that \(d(q, y) = \frac{3\epsilon}{4} > 2\epsilon\) and fix \(\delta \in (0, \epsilon)\) to be sufficiently small. We construct a bi-infinite sequence of points \(((x_i), (t_i))_{i \in \mathbb{Z}}\) with \(x_i \in \Sigma_q\) such that \(((x_i), (t_i))_{i \in \mathbb{Z}}\) is a \((\delta, T)\)-pseudo-orbit for some \(T > 0\). If \(k \in \mathbb{N}\) is sufficiently large then there exist \(x_1 \in \Sigma_q\), \(i \in \mathbb{Z}\) such that:

- \(x_i = x_0\) and \(t_i = \pi\) for \(i \leq 0\);
- \(d(x_i, x_{i-1}) < \delta\) and \(t_i = \pi\) for \(1 \leq i \leq k\);
- \(x_i = y\) and \(t_i = \pi\) for \(i \geq k\).

Observe that we are considering that the return time at the transversal section is the same and equal to \(\pi\). Clearly, it is not exactly equal to \(\pi\), however, we can make it as close to \(\pi\) as we want by just decreasing the \(\xi\)-neighbourhood. Therefore, \(((x_i), (t_i))_{i \in \mathbb{Z}}\) is a \((\delta, T)\)-pseudo-orbit for some \(T > 0\) such that \(\pi \geq T\).

By the weakly shadowing property, there is a point \(z \in E_{H,e}\) such that \(\{x_i\}_{i \in \mathbb{Z}} \subset B_{\delta}(\mathcal{O}(z))\). Without loss of generality, we may assume that \(z \in B(x_0, \epsilon)\). Since \(H_1\) is weakly shadowable, we have that for some \(T = n\pi\),

\[
d(x_0, x_k) \leq d(x_0, z) + d(z, x_k) = d(x_0, z) + d(X^T_{H_1}(z), x_k) < 2\epsilon,
\]

which is a contradiction. \(\square\)

Finally, the proof of theorem 6 is a consequence of theorem 4 and proposition 8.1.

Proof. Let \((H, e, \mathcal{E}_{H,e})\) be a stable weakly-shadowable Hamiltonian system and suppose, by contradiction, that \(\mathcal{E}_{H,e}\) is not partially hyperbolic. Then, by theorem 4, there exists a \(C^2\)-close \(C^\infty\)-Hamiltonian \(H_0\) with an elliptic closed orbit on a nearby regular energy hypersurface \(\mathcal{E}_{H_0,e_0}\) and this contradicts proposition 8.1. \(\square\)

Acknowledgments

The authors would like to thank the referee for the very useful comments, detailed corrections and suggestions that significantly improved this paper. MB was partially funded by the Portuguese Government through the FCT under the project PEst-OE/MAT/UI0121/2011. JR was partially funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT under the project PEst-C/MAT/UI0144/2011 and MJT was partially financed by FEDER Funds through ‘Programa Operacional Factores de Competitividade—COMPETE’ and by Portuguese
Funds through FCT—‘Fundação para a Ciência e a Tecnologia’, within the Project PEst-C/MAT/UI0013/2011. JR was also partially supported by the FCT—‘Fundação para a Ciência e a Tecnologia’, project PTDC/MAT/099493/2008.

References

[1] Abraham R and Marsden J E 1980 Foundations of Mechanics 2nd edn (New York: Benjamin-Cummings)
[2] Arnaud M-C 2002 The generic symplectic $C^1$-diffeomorphisms of four-dimensional symplectic manifolds are hyperbolic, partially hyperbolic or have a completely elliptic periodic point Ergod. Theory Dyn. Syst. 22 1621–39
[3] Arnold L 1998 Random Dynamical Systems (New York: Springer)
[4] Arnold V I 1989 Mathematical Methods of Classical Mechanics 2nd edn (Berlin: Springer)
[5] Arbieto A and Matheus C 2007 A pasting lemma and some applications for conservative systems Ergod. Theory Dyn. Syst. 27 1399–417
[6] Arbieto A, Senos L and Sodero T 2012 The specification property for flows from the robust and generic view point J. Diff. Eqs 253 1893–909
[7] Bessa M 2007 The Lyapunov exponents of generic zero divergence-free three-dimensional vector fields Ergod. Theory Dyn. Syst. 27 1445–72
[8] Bessa M 2013 $C^1$-stably shadowable conservative diffeomorphisms are Anosov Bull. Korean Math. Soc. at press
[9] Bessa M and Rocha J 2012 Contributions to the geometric and ergodic theory of conservative flows Ergod. Theory Dyn. Syst. at press
[10] Bessa M, Ferreira C and Rocha J 2010 On the stability of the set of hyperbolic closed orbits of a Hamiltonian Math. Proc. Camb. Phil. Soc. 149 373–83
[11] Bessa M, Ferreira C and Rocha J 2012 Generic Hamiltonian Dynamics, arXiv: 1203.3849
[12] Bessa M and Lopes Dias J 2008 Generic dynamics of 4-dimensional $C^2$ Hamiltonian systems Commun. Math. Phys. 281 597–619
[13] Bessa M and Lopes Dias J 2009 Hamiltonian elliptic dynamics on symplectic 4-manifolds Proc. Am. Math. Soc. 137 585–92
[14] Bessa M and Lopes Dias J 2011 Generic Hamiltonian dynamical systems: an overview Dynamics, Games and Science (Proceedings in Mathematics) vol 1 (Berlin: Springer) pp 123–138
[15] Bessa M and Lopes Dias J 2013 Hamiltonian suspension of perturbed Poincaré sections and an application Math. Proc. Camb. Phil. Soc. at press
[16] Bessa M, Rocha J and Torres M J 2013 Hyperbolicity and stability for Hamiltonian flows J. Diff. Eqs 254 309–22
[17] Bessa M and Rocha J 2008 On $C^1$-robust transitivity of volume-preserving flows J. Diff. Eqs 245 3127–43
[18] Bessa M and Rocha J 2011 Topological stability for conservative systems J. Diff. Eqs 250 3960–6
[19] Bessa M and Rocha J 2012 A remark on the topological stability of symplectomorphisms Appl. Math. Lett. 25 163–5
[20] Bessa M, Lee M and Vaz S 2013 Stable weakly shadowable volume-preserving systems are volume-hyperbolic Acta. Math. Sin. at press (arXiv:1207.5546v1)
[21] Bessa M and Vaz S 2012 Stably weakly shadowing symplectic maps are partially hyperbolic, arXiv:1203.5139
[22] Bochi J and Viana M 2004 Lyapunov exponents: how frequently are dynamical systems hyperbolic? Modern Dynamical Systems and Applications (Cambridge: Cambridge University Press) pp 271–97
[23] Bochi J and Viana M 2005 The Lyapunov exponents of generic volume-preserving and symplectic maps Ann. Math. 161 1423–85
[24] Bonatti C, Díaz L and Pujals E 2003 A $C^1$-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources Ann. Math. 158 355–418
[25] Bonatti C, Gourmelon N and Vivier T 2006 Perturbations of the derivative along periodic orbits Ergod. Theory Dyn. Syst. 26 1307–37
[26] Bowen R and Walters P 1972 Expansive one-parameter flows J. Diff. Eqs 12 180–93
[27] Corless R M and Pilyugin S Yu 1995 Approximate and real trajectories for generic dynamical systems J. Math. Anal. Appl. 189 409–23
[28] Denker M, Grillenberger C and Sigmund K 1976 Ergodic Theory on Compact Spaces (Lecture Notes in Mathematics vol 527) (Berlin: Springer)
[29] Ferreira C 2010 Shadowing, Expansiveness and Stability of Divergence-Free Vector Fields (arXiv:1011.3546)
Shades of hyperbolicity for Hamiltonians

[32] Gidea M and Robinson C 2007 Shadowing orbits for transition chains of invariant tori alternating with Birkhoff zones of instability Nonlinearity 20 1115–43

[33] Gogolev A 2010 Diffeomorphisms Hölder conjugate to Anosov diffeomorphisms Ergod. Theory Dyn. Syst. 30 441–56

[34] Horita V and Tahzibi A 2006 Partial hyperbolicity for symplectic diffeomorphisms Ann. Inst. Henri Poincaré 23 641–61

[35] Hurley M 1986 Fixed points of topologically stable flows Trans. Am. Math. Soc. 294 625–33

[36] Hurley M 1984 Consequences of topological stability J. Diff. Eqns 54 60–72

[37] Hurley M 1984 Combined structural and topological stability are equivalent to Axiom A and the strong transversality condition Ergod. Theory Dyn. Syst. 4 81–8

[38] Lee K and Sakai K 2007 Structural stability of vector fields with shadowing J. Diff. Eqns 232 303–13

[39] Mather R 1982 An ergodic closing lemma Ann. Math. 116 503–40

[40] Moriyasu K, Sakai K and Sun W 2005 C1-stably expansive flows J. Diff. Eqns 213 352–67

[41] Moriyasu K, Sakai K and Sumi N 2001 Vector fields with topological stability Trans. Am. Math. Soc. 353 3391–408

[42] Moser J and Zehnder E 2005 Notes on Dynamical Systems (Courant Lectures Notes in Mathematics vol 12) (Providence, RI: American Mathematical Society)

[43] Newhouse S 1977 Quasi-elliptic periodic points in conservative dynamical systems Am. J. Math. 99 1061–87

[44] Nitecki Z 1971 On semi-stability for diffeomorphisms Invent. Math. 14 83–122

[45] Oseledets V I 1968 A multiplicative ergodic theorem. Characteristic Ljapunov exponents of dynamical systems Tr. Moskov. Mat. Obsc. 19 179–210

[46] Pilyugin S Yu 1999 Shadowing in Dynamical Systems (Lecture Notes in Mathematics vol 1706) (Berlin: Springer)

[47] Pilyugin S Yu 1997 Shadowing in structurally stable flows J. Diff. Eqns 140 283–65

[48] Pilyugin S Yu and Tikhomirov S B 2010 Lipschitz shadowing implies structural stability Nonlinearity 23 2509–15

[49] Robinson C 1970 Generic properties of conservative systems I and II Am. J. Math. 92 562–603

[50] Robinson C 1977 Stability theorems and hyperbolicity in dynamical systems Rocky Mountain J. Math. 7 425–37

[51] Saghin R and Xia Z 2006 Partial Hyperbolicity or dense elliptical periodic points for C1-generic symplectic diffeomorphisms Trans. Am. Math. Soc. 358 5119–38

[52] Sakai K 2008 C1-stably shadowable chain components Ergod. Theory Dyn. Syst. 28 987–1029

[53] Sakai K, Sumi N and Yamamoto K 2010 Diffeomorphisms satisfying the specification property Proc. Am. Math. Soc. 138 315–21

[54] Senos L 2012 Generic Bowen-expansive flows Bull. Braz. Math. Soc. 43 59–71

[55] Vivier T 2005 Robustly Transitive 3-Dimensional Regular Energy Surfaces are Anosov (Dijon: Institut de Mathématiques de Bourgogne) (http://math.u-bourgogne.fr/topo/prepub/pre05.html)

[56] Walters P 1970 Anosov diffeomorphisms are topologically stable Topology 9 71–8

[57] Wen X, Gan S and Wen L 2009 C1-stably shadowable chain components are hyperbolic J. Diff. Eqns 246 340–57

[58] Williams R F 1955 A note on unstable homeomorphisms Proc. Am. Math. Soc. 6 308–9

[59] Yoccoz J-C 1992 Travaux de Herman sur les tores invariants Astérisque 754 311–44

[60] Yuan C and Yorke J A 2000 An open set of maps for which every point is absolutely nonshadowable Proc. Am. Math. Soc. 128 909–18