A coupled harmonic oscillator model to describe the near-field radiative heat transfer between nanoparticles and planar surfaces

Karthik Sasihithlu
Department of Energy Science and Engineering, Indian Institute of Technology Bombay, Mumbai 400076, India

(Dated: October 8, 2018)

When two objects made of a material which supports surface modes are brought in close proximity to each other such that the vacuum gap between them is less than the thermal wavelength of radiation, then the coupling between the surface modes provides an important channel for the heat transfer to occur which is different from that mediated by long range propagating electromagnetic waves. Indeed, the heat transfer then exceeds Planck’s blackbody limit by several orders of magnitude, and consequently has been used for several energy applications such as near-field thermophotovoltaic systems. This near-field radiative heat exchange has been traditionally and successfully described using fluctuational electrodynamics principles. Here, we describe an alternate coupled harmonic oscillator model approach which can be used to model the coupling between surface modes and hence the resultant near-field heat transfer. We apply this theory to estimate the near-field heat transfer for the configurations of two metallic nanoparticles and two planar metal surfaces and compare the result with predictions from fluctuational electrodynamics theory.

PACS numbers: 43.35.Pt, 05.45.Xt, 44.40.+a, 41.20.Jb, 73.20.Mf

*ksasihithlu@iitb.ac.in
I. INTRODUCTION

The discovery that a peak in the density of states at certain frequencies where surface modes can be thermally excited will result in a significant enhancement in heat transfer when objects are brought in close proximity to each other has lead to development of applications relevant to energy systems such as near-field thermophotovoltaics [1–3] and thermal rectification [4]. This enhancement in heat transfer which can be several orders higher than that exchanged by objects when separated by large distances ($d \gg \lambda_T$ where $\lambda_T$ is the thermal wavelength of radiation) has been successfully explained using the theory of fluctuational electrodynamics developed by Rytov [5] and first applied for analysing heat transfer between closely spaced bodies by Polder and Van Hove [6]. The theoretical framework of this procedure relies on the introduction of external microscopic thermal fluctuating currents or dipole moments whose correlation functions are related to the dielectric properties of the material via the fluctuation-dissipation theorem [7–9]. While the predictions from this theory has indeed been experimentally verified over the years for several geometries including that for flat surfaces [10], STM tip over substrate [11, 12] and sphere over a flat substrate [13, 14], a main assumption in this theory is the prevalence of local thermodynamic equilibrium in the objects which restricts the cases to be analysed to stationary or quasi-stationary cases only.

Recently, an alternate approach which uses the coupled harmonic oscillator theory to model the coupling between surface polaritons has been proposed which enables us to analyse the near-field heat transfer for both dynamic situations valid for time scales less than the relaxation time scales of surface polariton excitations, as well as in the steady-state [15–17]. In this work we adopt this approach to arrive at expression for the steady-state near-field heat transfer for the configurations of two nanoparticles and also for two planar systems and compare the expressions with that derived using fluctuational electrodynamics. This approach offers a simpler and more general framework for studying heat transfer between surfaces.

The paper is arranged as follows: in Section II the theory of steady state heat transfer between coupled harmonic oscillators is developed and key results are highlighted. In Section III this theory is applied to predict heat exchange between nanostructures and compared with the predictions of fluctuational electrodynamics. In Section IV we apply this theory to predict the near-field heat exchange between planar surfaces whose dielectric properties are given by the Drude model.

II. HEAT TRANSFER BETWEEN COUPLED HARMONIC OSCILLATORS

We model the surface modes between interacting systems as harmonic oscillators which are in contact with two separate heat baths maintained at constant temperatures as shown in Fig. Since the oscillators are in contact with heat bath they are can be modelled as force driven. While the heat flux between the two heat reservoirs for such a system has been well studied [15, 16, 20, 21] here we provide an alternate derivation based on Green’s function theory to arrive at expression for heat transfer for such a system. A similar derivation has been given in Ref. [22] in the context of modelling the heat transfer due to coupling of Rayleigh modes in solids. We begin with the equation of motion for such a system which in the frequency $\omega$ space can be written as:

$$\omega^2 \hat{x}_1(\omega) = \omega_0^2 \hat{x}_1(\omega) - 2\omega \delta \hat{x}_1(\omega) + F(\omega, T)$$

where $F(\omega, T)$ is the forcing function whose spectral density $S_F(\omega, T)$ is as yet unknown, and $\delta$ is the half-line width. The Green’s function for such a system defined such that $\hat{x}_1(\omega) = G(\omega) F(\omega, T)$ is given as:

$$G(\omega) = \frac{1}{\omega^2 - \omega_0^2 + 2i\omega \delta}$$
Relating the density of states $\rho(\omega)$ to the imaginary part of the Green’s function, and recognizing that it peaks around $\omega \approx \omega_0$, gives:

$$\rho(\omega) = \frac{4}{\pi} \frac{\omega_0^2 \delta}{(\omega^2 - \omega_0^2)^2 + 4\omega_0^2 \delta^2}$$

At equilibrium at temperature $T$ we have the total energy contained in the harmonic oscillator, $E$, to be:

$$E = \int_{0}^{\infty} \rho(\omega) \frac{\hbar \omega}{e^{\hbar \omega / (k_B T)} - 1} d\omega$$

However, the energy contained in the oscillator is also given by:

$$E = \langle \omega_0^2 x^2(t) \rangle = \omega_0^2 \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega = 2\omega_0^2 \int_{0}^{\infty} |G(\omega)|^2 S_F(\omega, T) d\omega$$

where, $\Theta$ is a large time interval and we make use of the definition of spectral density $S_F(\omega, T) = (\pi/\Theta)|F(\omega, T)|^2$ from Ref. [23]. Comparing Eq. 1 and Eq. 2 we obtain:

$$S_F(\omega, T) = \frac{2\delta}{\pi} \frac{\hbar \omega}{e^{\hbar \omega / (k_B T)} - 1}$$

Now consider system of two coupled oscillators with coupling constant $\gamma \omega_0^2$ such that one of the oscillators is in contact with a heat reservoir at temperature $T$ and the other at zero kelvin. The equations of motion can be written as:

$$\begin{align*}
\omega^2 \ddot{x}_1(\omega) &= \omega_0^2 \dot{x}_1(\omega) - 2i\omega \delta \dot{x}_1(\omega) + \gamma \omega_0^2 \dot{x}_2(\omega) + F(\omega, T) \\
\omega^2 \ddot{x}_2(\omega) &= \omega_0^2 \dot{x}_2(\omega) - 2i\omega \delta \dot{x}_2(\omega) + \gamma \omega_0^2 \dot{x}_1(\omega)
\end{align*}$$

The eigenfrequencies of this coupled system in the absence of the forcing function and for $\Gamma/\omega_0 \ll 1$ is given as:

$$\omega_{\pm} = \omega_0^2 (1 \pm \gamma) - i\omega_0 \delta$$

with the corresponding normalized eigenvectors given by: $\begin{pmatrix} +1 \\ +1 \end{pmatrix}$ and $\begin{pmatrix} +1 \\ -1 \end{pmatrix}$. The new Green’s function can be built from the eigenvectors as follows:

$$\tilde{G}(\omega) = \frac{1}{2} \begin{pmatrix} +1 \\ +1 \end{pmatrix} \begin{pmatrix} +1, +1 \end{pmatrix} \frac{1}{\omega^2 - \omega_0^2 (1 + \gamma) + i\omega_0 \delta} + \frac{1}{2} \begin{pmatrix} +1 \\ -1 \end{pmatrix} \begin{pmatrix} +1, -1 \end{pmatrix} \frac{1}{\omega^2 - \omega_0^2 (1 - \gamma) + i\omega_0 \delta}$$

Using this form of the Green’s function we can write the response to be of the form:

$$\begin{align*}
\ddot{x}_1(\omega) &= \frac{F(\omega, T)}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \frac{1}{\omega^2 - \omega_0^2 (1 + \gamma) + i\omega_0 \delta} \\
\ddot{x}_2(\omega) &= \frac{F(\omega, T)}{2} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \frac{1}{\omega^2 - \omega_0^2 (1 - \gamma) + i\omega_0 \delta}
\end{align*}$$

In steady state the rate of energy transfer to the second oscillator via coupling with the first oscillator is equal to the decay in the second oscillator via damping. This should be equal to the heat transfer to the sink. Thus we can write the expression for the heat transfer in the system shown in Fig. as:

$$P = -\omega_0^2 \frac{d}{dt} \langle x_2^2(t) \rangle = 2\delta \omega_0^2 \langle x_2^2(t) \rangle$$

Following the procedure similar to that in Eq. 2, Eq. 9 further reduces to:

$$P = \delta \omega_0^2 \int_{0}^{\infty} S_F(\omega, T) \left( \frac{1}{(\omega^2 - \omega_0^2 (1 + \gamma) + i\omega_0 \delta)} - \frac{1}{(\omega^2 - \omega_0^2 (1 - \gamma) + i\omega_0 \delta)} \right)^2$$

Substituting the expression for $S_F(\omega, T)$ from Eq. 3 and evaluating the integral while recognising that the integral is sharply peaked around $\omega \approx \omega_0$, we get:

$$P = \frac{\hbar \omega_0}{e^{\hbar \omega_0 / (k_B T)} - 1} \frac{\omega_0^2 \gamma^2 \delta}{\left(\omega_0^2 \gamma^2 + \delta^2\right)}$$

This expression matches the form for the steady-state heat transfer for this system derived by other authors using Master equation approach [15] and Langevin theory [10].
III. NANOPARTICLES

FIG. 2. Coupling between the fields due to the surface modes of spherical nanoparticles of diameter $D$ and center-to-center distance $d$ which is analyzed in Section III. Thick arrows indicate the direction of orientation of the dipole moments relative to the separation. The coupling strength between the fields is dependent on whether the dipoles are oriented a) side-by-side or b) head-to-tail.

In this section we derive the parameters of the coupled harmonic oscillator model suitable for describing the coupling of surface plasmons between two nanoparticles, and hence arrive at the expression for the heat transfer exchanged from Eq. 11. We only consider the heat transfer mediated by the coupling of surface modes and neglect contributions due to eddy currents [24], multipoles [25], many-body effects [26]. A similar attempt has been made to arrive at a coupled harmonic oscillator model to analyze the near-field heat transfer between nanoparticles by Biels and Agarwal [15]. The parameters in this model were arrived by comparing the steady-state heat transfer prediction in the coupled harmonic oscillator model with that predicted from Rytov’s fluctuational electrodynamics theory. Here, we will outline an alternate approach based on an analysis of splitting of eigenmodes to arrive at the parameters of the model. The advantage of this approach over that followed by Biels and Agarwal is that it can be used in general to analyse coupling between quasi-particles when the steady state heat flux is not known beforehand.

To find the equivalent natural frequency $\omega_0$ and coupling constant $\gamma$ for the interaction between the surface modes across the vacuum gap we proceed as follows: we use the expression for the eigenmodes of the configuration of two nanoparticles separated by a vacuum gap using Maxwell’s equations and compare this expression with that for two coupled harmonic oscillators. Consider first the case of a single nanoparticle with polarizability $\alpha(\omega)$. The resonant frequency is determined from the poles of the polarizability of the nanoparticle. We will consider the particular case of spherical metallic nanoparticles with diameter $D$ whose polarizability can be expressed as:

$$\alpha(\omega) = \frac{\pi D^3 \varepsilon(\omega) - 1}{2 \varepsilon(\omega) + 2}$$

When the permittivity $\varepsilon(\omega)$ of the particle is given by the Drude form:

$$\varepsilon(\omega) = \varepsilon_\infty \left(1 - \frac{\omega_p^2}{\omega(\omega + i\Gamma)}\right)$$

with $\omega_p$ being the plasma frequency, and $\Gamma$ the damping parameter, the pole of Eq. 12 gives the resonant frequency, in the limit of $\Gamma/\omega_L \ll 1$, to be of the form $\omega = \omega_0 - i\Gamma/2$, where $\omega_0 = \omega_L \sqrt{\varepsilon_\infty/(\varepsilon_\infty + 2)}$.

Now consider the two-particle system shown in Fig. , with their centers being separated by a gap $d$. The presence of a second particle induces a change in the net polarizability of the two-particle system to the form [27, 28]:

$$\alpha'(\omega) = \frac{\alpha(\omega)}{1 - \frac{\kappa \alpha(\omega)}{4\pi \varepsilon_0 d^3}}$$
where \( \kappa \) is an alignment factor which depends on the possible polarization modes such that \( \kappa = -1 \) for dipoles aligned side-by-side (perpendicular polarization) as shown in Fig. 2a and \( \kappa = 2 \) for dipoles aligned head-to-tail (parallel polarization) as shown in Fig. 2b. In deriving Eq. 14 it is assumed that the gap \( d \) between the nanoparticles is such that \( 1/d^3 \) component of the electric field dominates compared to the other terms which vary as \( 1/d^2 \) and \( 1/d \). The eigenmodes for this two-particle system, with the higher energy mode corresponding to the case with \( \kappa = -1 \) and the lower energy mode corresponding to the case with \( \kappa = 2 \), can be solved for by substituting the expression for \( \alpha \) in Eq. 12 into Eq. 14 and solving for the poles of Eq. 14. In the limit of dipole approximation (\( d \gg D \)) and under small losses (\( \Gamma \ll \omega_L \)) we get the eigenmodes to be of the form:

\[
\omega_{\pm} = \omega_{np}^0 \sqrt{1 \pm \gamma_{np}^2 - \frac{i \Gamma}{2}}
\]

where,

\[
\omega_{np}^0 = \omega_L \sqrt{\frac{\varepsilon_{\infty}}{\varepsilon_{\infty} + 2}}
\]

and

\[
\gamma_{np}(d) = \frac{9}{16} \frac{D^3}{d^3} \frac{1}{\varepsilon_{\infty} + 2}
\]

For large gaps \( d \gg D \), \( \omega_{np}^0 \approx \omega_0 \) as expected. Comparing Eq. 15 with Eq. 5 we can directly relate the parameters of the coupled harmonic oscillator model with that of the coupled nanoparticles as follows:

\[
\omega_0 \rightarrow \omega_{np}^0; \quad \gamma \rightarrow \gamma_{np}; \quad \delta \rightarrow \frac{\Gamma}{2}
\]

Substituting these parameters from Eq. 18 into Eq. 11 we arrive at the the expression for heat transfer between two dipolar nanoparticles as:

\[
P_{np}(d) = \frac{\hbar \omega_{np}^0}{e^{\hbar \omega/(k_B T)} - 1} \left( \frac{\omega_{np}^0 \gamma^2}{(\omega_{np}^0)^2 \gamma^2 + \Gamma^2/4} \right) \frac{\Gamma}{2}
\]

For analytical simplicity if we consider the case where \( k_B T \gg \hbar \omega_0 \), the expression in Eq. 19 can be shown to reduce to:

\[
P_{np}(d) = \frac{81}{128} k_B T \left( \frac{D}{d} \right)^6 \frac{\varepsilon_{\infty}}{(\varepsilon_{\infty} + 2)^3} \frac{\omega_L^2}{\Gamma}
\]

This expression can be compared with the expression for heat transfer between two dipolar nanoparticles derived using fluctuational electrodynamics principles \([24, 25, 29–31]\). This, for small gaps \( d/\lambda \ll 1 \) reads \([24, 30]\):

\[
P_{FE}(d) = \frac{3}{4 \pi^3 \hbar^6} \int_0^\infty \frac{\hbar \omega}{e^{\hbar \omega/(k_B T)} - 1} \text{Im} [\alpha(\omega)]^2 d\omega
\]

Considering the limit \( k_B T \gg \hbar \omega \) and using the expression for \( \alpha(\omega) \) from Eq. 12 to evaluate the integral in Eq. 21 we get:

\[
P_{FE}(d) = \frac{27}{128} k_B T \left( \frac{D}{d} \right)^6 \frac{\varepsilon_{\infty}}{(\varepsilon_{\infty} + 2)^3} \frac{\omega_L^2}{\Gamma}
\]

The expression for the heat transfer in Eq. 20 derived using the coupled harmonic oscillator model agrees with that derived using fluctuational electrodynamics principles in Eq. 22 except for a numerical factor. There is currently no consensus in literature regarding the constant numerical factor in Eq. 22 - it being 27/128 in Ref. \([24, 30]\), 27/(32\pi) in Ref. \([29]\), 27/256 in Ref. \([25]\) and 27/8 in Ref. \([31]\).

**IV. PLANAR SURFACES**

In this section we extend the procedure outlined in Sec. III for deriving the expression for near-field heat transfer between two nanoparticles to that between two planar surfaces. We follow the derivation shown in Ref. \([16, 17]\) to
FIG. 3. Coupling between two surface modes located at the interface between a planar surface and vacuum which is analysed in Section [IV]. The dielectric properties is taken to vary only with the frequency $\omega$ and the coupling constant $\gamma\omega_0^2$ between the surface modes is derived using Maxwell’s equations.

derive the equivalent parameters of the coupling oscillator, analyze the heat transfer between closely spaced metallic surfaces whose dielectric properties are given by the Drude form shown in Eq. [13] and contrast it with the parameters derived for coupling between nanoparticles in Sec. [III]. Consider two half-spaces separated by a gap of width $d$ occupying the regions $z < -d/2$ and $z > d/2$. The in-plane two dimensional component $k_x$ and the $z$ component $k_z$ of the wavevector of a planar wave of frequency $\omega$ in the vacuum gap of this system are related as $k_x^2 + k_z^2 = (\omega/c)^2$ with $c$ being the velocity of light. From the dispersion relation for surface polaritons $\omega(k_x)$ [32] and close to the surface polariton frequency the surface mode is seen to acquire an electrostatic character with $k_x \gg \omega/c$ (where group velocity $d\omega/dk_x \to 0$) and hence $k_z \approx ik_x$. In this electrostatic limit the expression for the symmetric and antisymmetric forms of the potential in the vacuum gap due to the surface mode can be written as [33, 34]:

$$\phi_{\pm} = \Phi \exp (ik_x x - i\omega t) \begin{cases} e^{-k_x z} + e^{k_x z} \\ e^{-k_x z} - e^{k_x z} \end{cases}$$

and for $z > d/2$:

$$\phi_{\pm} = \Phi \exp (ik_x x - i\omega t) \begin{cases} (e^{-k_x d/2} + e^{k_x d/2}) & \exp[-k_x (z - d/2)] \\ (e^{-k_x d/2} - e^{k_x d/2}) & \exp[-k_x (z - d/2)] \end{cases}$$

where $\Phi$ is an arbitrary constant. Such a form of the potential, which oscillates harmonically in time with frequency $\omega$, ensures continuity of potential at the interfaces. Satisfying continuity of perpendicular component of displacement field gives the condition for surface modes as:

$$\varepsilon(\omega_{\pm}) = \begin{cases} (1 - e^{k_x d}) / (1 + e^{k_x d}) \\ (1 + e^{k_x d}) / (1 - e^{k_x d}) \end{cases}$$

(23)

where $\varepsilon(\omega)$ is the dielectric function of the half-space. Using the Drude model for the dielectric function shown in Eq. [13] and solving for $\omega$ in Eq. [23] in the low-loss limit $\Gamma \ll \omega_L$ we obtain the eigenfrequencies of the coupled surface modes to be of the form:

$$\omega_{\pm} = \omega_{0}^{pl}(k_x, d) \sqrt{1 \pm \gamma^{pl}(k_x, d)} - i \frac{\Gamma}{2}$$

(24)

where,

$$\omega_{0}^{pl}(k_x, d) = \sqrt{\varepsilon_{\infty}} \omega_L \sqrt{\frac{(\varepsilon_{\infty} + 1) - (\varepsilon_{\infty} - 1)e^{-2k_x d}}{(\varepsilon_{\infty} + 1)^2 - (\varepsilon_{\infty} - 1)^2 e^{-2k_x d}}}$$

(25)

and

$$\gamma^{pl}(k_x, d) = \frac{2e^{-k_x d}}{(\varepsilon_{\infty} + 1) - (\varepsilon_{\infty} - 1)e^{-2k_x d}}$$

(26)
In the limit of large gaps i.e., $k_x d \to \infty$ we have $\omega_0^0(k_x d) \to \omega_L \sqrt{\varepsilon_\infty/(\varepsilon_\infty + 1)}$ which is the surface plasmon frequency for a single half-space, and $\gamma^{pl}(k_x d) \to 0$ as expected. From Eq. [5] and [24] we have for planar surfaces:

$$\omega_0 \rightarrow \omega_0^0(k_x d); \quad \gamma \rightarrow \gamma^{pl}(k_x d); \quad \delta \rightarrow \frac{\Gamma}{2}$$  \hspace{1cm} (27)

Comparing the coupling parameters $\omega_0^0$ and $\gamma^{pl}$ derived in Eqs. [25] and Eq. [26] for planar surfaces with those derived for nanoparticles $\omega_0^{np}$ and $\gamma^{np}$ in Eqs. [16] and Eq. [17] it is observed that in the former case the coupling parameters are a function of the in-plane wavevector modes $k_x$. We thus have a spectrum of oscillators (labelled by the modes $k_x$) and since only modes of the same in-plane wavevector components interact across the vacuum gap we can consider each of the oscillators to contribute independently to the heat transfer. We can thus write the expression for the near-field heat transfer between two planar surfaces from Eq. [11] and Eq. [27] to be of the form:

$$P^{pl}(d) = \frac{1}{4\pi^2} \int \frac{h\omega_0^0(k_x d)}{e^{h\omega_0^0(k_x d)/(k_B T)} - 1} \left( \frac{[\omega_0^0(k_x d)^2\gamma^{pl}(k_x d)]}{[\omega_0^0(k_x d)]^2\gamma^{pl}(k_x d) + \Gamma^2/4} \right) \frac{\Gamma}{2} d^2 k_x$$ \hspace{1cm} (28)

For analytical simplicity considering the case when $k_B T \gg h\omega_0^0$ and substituting the expressions for the coupling parameters from Eqs. [25] and [26] Eq. [28] reduces to:

$$P^{pl}(d) = \frac{k_B T}{2\pi d^2} \int_0^\infty e^{-2x\Gamma(\varepsilon_\infty - 1)^3 + \Gamma e^{2x}(\varepsilon_\infty + 1)^3 - 2\Gamma \varepsilon_\infty(\varepsilon_\infty^2 - 1) + 16\varepsilon_\infty^2\Gamma} x dx$$ \hspace{1cm} (29)

This expression can be compared with the expression of near-field heat transfer between two planar surfaces derived from fluctuational electrodynamics principles [17] which, in the limit $k_x d \to 0$ and $k_B T \gg h\omega$, reads:

$$P^{pl}_{FE}(d) = \frac{k_B T}{\pi^2 d^2} \int_0^{\infty} d\omega \int_0^{\infty} \left( \frac{\text{Im}\varepsilon(\omega)}{[\varepsilon(\omega) + 1]^2 - (\varepsilon(\omega) - 1)^2 e^{-x}|^2} \right)$$ \hspace{1cm} (30)

Substituting the expression for the dielectric function $\varepsilon(\omega)$ from Eq. [13] and expanding, Eq. [30] reduces to the form:

$$P^{pl}_{FE}(d) = \frac{k_B T}{\pi^2 d^2} \int_0^{\infty} x e^{-x} dx \int_0^{\infty} \frac{M(\omega)}{N(\omega)} d\omega$$ \hspace{1cm} (31)

where $M(\omega)$ and $N(\omega)$ are complicated polynomial even functions of $\omega$. By carrying out the integral over $\omega$ in the complex plane using Cauchy’s residue theorem the expression in Eq. [31] reduces to that in Eq. [29].

To conclude, we have arrived at an expression for heat transfer between two heat baths maintained by constant temperatures mediated by a coupled harmonic oscillator system using Green’s function theory and shown its equivalence to those derived in literature using Master equation and Langevin dynamics. We use this expression to find surface plasmon mediated heat transfer between two closely spaced nanoparticles, and also for two closely spaced planar surfaces whose dielectric properties are of the Drude form. In order to establish an equivalence between the coupled harmonic oscillator system and the coupled nanoparticles/planar surfaces configuration we have compared the splitting in the eigenmodes of the two systems to arrive at the equivalent coupling parameters. The expression of heat transfer thus obtained for both these configurations is shown to be consistent with that obtained using the established theory of fluctuational electrodynamics.

**ACKNOWLEDGEMENTS**

We acknowledge useful discussions with Prof. Girish Sarin Agarwal and with Dr. Svend-Age Biehs.

**REFERENCES**

[1] A. Narayanaswamy and G. Chen, “Surface modes for near field thermophotovoltaics,” Appl. Phys. Lett., vol. 82, pp. 3544–3546, 2003.
[34] N. Van Kampen, B. Nijboer, and K. Schram, “On the macroscopic theory of van der waals forces,” *Physics letters A*, vol. 26, no. 7, pp. 307–308, 1968.