CELLULAR RESOLUTIONS OF MONOMIAL IDEALS AND THEIR ARTHINIAN REDUCTIONS

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ABSTRACT. The question we address in this paper is: which monomial ideals have minimal cellular resolutions, that is, minimal resolutions obtained from homogenizing the chain maps of CW-complexes? Velasco gave families of examples of monomial ideals that do not have minimal cellular resolutions, but those examples have large minimal generating sets. In this paper, we show that if a monomial ideal has at most four generators, then the ideal and its (monomial) Artinian reductions have minimal cellular resolutions. When the ideal is generated by two monomials, we can give a precise description of the CW-complex supporting minimal free resolution of the ideal and its Artinian reduction. Also, in this case, we compute the multigraded Betti numbers, Cohen-Macaulay type and determine when the corresponding algebra is a level algebra.

1. INTRODUCTION

The general theme of this paper is to use chain maps of cell complexes to describe free resolutions of monomial ideals in polynomial rings. Let $\mathbb{K}$ be a field and $M$ be an ideal in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ generated by $q$ monomials. Diana Taylor [22] shows that the simplicial chain complex of a simplex on $q$ vertices can be “homogenized” (see [20] for a description of homogenization) into a free resolution of $M$. Taylor’s resolution works for any monomial ideal and as a result it is often far from being minimal. But her insight was further developed by researchers (see [7, 20]) to find smaller topological objects, such as subcomplexes of the simplex or more generally CW-complexes, whose chain complexes can be homogenized to smaller resolutions for specific (classes of) ideals.

While every monomial ideal has a Taylor resolution which comes from the chain complex of a simplex, there are monomial ideals whose minimal resolutions cannot be obtained from any simplicial or even CW-complexes ([23]). So a natural question to ask is: what classes of monomial ideals have minimal cellular resolutions? Or, could one find a cellular resolution for a given class of monomial ideals that is very close to being minimal?

The idea here is to start from the most structured cellular resolution – the Taylor resolution – and systematically reduce the size of the Taylor complex by deleting redundant faces while ensuring that the remaining faces still support a resolution. This method of
pruning extra faces is most effectively carried out by tools from discrete homotopy theory. The specific tool we use in this paper is discrete Morse theory, which encodes the faces of a cell complex in a graph, and uses “acyclic matchings” to prune this graph, and obtain a smaller topological object that is homotopy equivalent to the first one. Discrete Morse theory was developed by Froman [12] as a combinatorial counterpart of Morse theory for manifolds, and was interpreted in terms of matchings in the poset lattice by Chari [8]. Batzies and Welker [4, 5, 19] applied these methods to cellular free resolutions of monomial ideals; see also [3].

The premise of this paper is Artinian monomial ideals. Specifically, let

\[ J = (u_1, \ldots, u_r) \quad \text{and} \quad I = J + (x_1^{e_1}, \ldots, x_n^{e_n}) \]

where \( u_1, \ldots, u_r \) are monomials in \( S \). We show that if \( r \leq 4 \), \( I \) and \( J \) will both have cellular minimal resolutions. More precisely

**Main Theorem** (Theorem 4.5). Let \( J \) be a monomial ideal with at most four monomial generators in the polynomial ring \( S = \mathbb{K}[x_1, \ldots, x_n] \) over a field \( \mathbb{K} \), and

\[ I = J + (x_1^{e_1}, \ldots, x_n^{e_n}) \]

be an Artinian reduction of \( J \), where \( e_1, \ldots, e_n \) are positive integers. Then both of \( I \) and \( J \) have minimal free resolutions supported on a CW-complex.

Our investigations of Artinian monomial ideals were inspired by the work of Alesandroni [2], who looked for characterizations of ideals with Scarf resolutions.

This paper is organized as follows. In Section 2 we briefly review cellular resolutions, Taylor and Scarf resolutions, and multigraded Betti numbers of monomial ideals. In Section 3 we discuss how discrete Morse theory leads to a free resolution of a monomial ideal \( M \) using acyclic matchings on a graph \( G_M \) built from the generators of \( M \) (Theorem 3.1). In Section 4 we analyze the local structure of the lcm lattice of a set of monomials (see Lemma 4.2 and Proposition 4.3) and apply it to prove that when \( J \) has fewer than five generators, both of \( I \) and \( J \) have minimal cellular resolutions.

It is worth highlighting that Proposition 4.3 offers a method to reduce the scope of search for acyclic matchings to a much smaller structure, and can be applied as a tool to find Morse matchings for Artinian reductions of any monomial ideal.

When the ideal \( J \) is generate by two monomials, in Section 5 we describe homogeneous acyclic matchings that produce minimal free resolutions of \( I \) and \( J \) via a concrete algorithm (Theorem 5.4). Finally, we show in Section 6 that \( \beta_{i,n}(I) \in \{0,1\} \) when \( I \) is an Artinian reduction of a monomial ideal with two generators. As a result, we compute the Cohen-Macaulay type of \( S/I \) and determine when \( S/I \) is a level algebra.

2. **Preliminaries**

In this section, we will introduce the tools used later in the paper.

2.1. **Simplicial and cell complexes.** A simplicial complex \( \Delta \) over a set of vertices \( V = \{v_1, \cdots, v_n\} \) is a collection of subsets of \( V \), with the property that \( \{v_i\} \in \Delta \) for all \( i \), and if \( F \in \Delta \) then all subsets of \( F \) are also in \( \Delta \). An element of \( \Delta \) is called a face of \( \Delta \), and the **dimension** of a face \( F \) of \( \Delta \), denoted by \( \dim(F) \), is defined as \( |F| - 1 \), where \( |F| \) is the size of the set \( F \). The faces of dimensions 0 and 1 are called **vertices**
and edges, respectively, and \( \dim(\emptyset) = -1 \). The maximal faces of \( \Delta \) under inclusion are called facets of \( \Delta \). The dimension of the simplicial complex \( \Delta \) is the maximal dimension of its facets.

For a positive integer \( q \), a \( q \)-simplex is a simplicial complex on \( q \) vertices with exactly one facet of dimension \( q - 1 \); in other words, the simplicial complex \( 2[q] \) consisting of all subsets of \([q]\).

A topological space is called a cell of dimension \( d \) if it is homeomorphic to the \( d \)-dimensional open ball

\[
\text{int}(B^d) = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 < 1 \right\}.
\]

**Definition 2.1.** A Hausdorff space \( X \) is a CW-complex, if there exists a collection \( X^* = \{ c_i : i \in I \} \) of cells such that \( X = \bigcup_{i \in I} c_i \), and for every cell \( c \in X^* \) of dimension \( d \), there exists a continuous map

\[
\Phi_c : B^d := \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 \leq 1 \right\} \to X
\]

such that the restriction of \( \Phi_c \) on \( \text{int}(B^d) \) is a homeomorphism

\[
\Phi_c|_{\text{int}(B^d)} : \text{int}(B^d) \xrightarrow{\cong} c.
\]

A subset \( A \subset X \) is closed in \( X \) if and only if \( A \cap \Phi_c(B^d) \) is closed in \( \Phi_c(B^d) \) for all \( c \in X^* \). For a cell \( c \in X^* \), we call the map

\[
\Phi_c : B^d \to X
\]

the characteristic map of \( c \) and \( \Phi_c(B^d) \) the closed cell that belongs to \( c \).

The collection of cells \( X^* \) of \( X \) is a partially ordered set: for cells \( \sigma, \sigma' \in X^* \) we set

\[
\sigma \preceq \sigma' \iff \Phi_\sigma(B^d) \subseteq \Phi_{\sigma'}(B^d).
\]

A cell \( \sigma \) is a facet of \( X \) if \( \sigma \) is maximal with respect to the above partial order on \( X^* \). A CW-complex is also referred to as a cell complex.

### 2.2. Simplicial and cellular resolutions.

Let \( \mathbb{K} \) be a field and \( M \) be a homogeneous ideal in the polynomial ring \( S = \mathbb{K}[x_1, \ldots, x_n] \). A free resolution of \( M \) is an exact sequence of free modules

\[
0 \to S^{c_{p+1}} \to S^{c_p} \to \cdots \to S^{c_2} \to S^{c_1} \to M \to 0,
\]

where each \( S^{c_i} \) denotes a free \( S \)-module of rank \( c_i \). A free resolution with the smallest possible sequence of ranks \( c_0, \ldots, c_p \) (and smallest \( p \)) is called a minimal free resolution of \( M \), and is known to be unique up to isomorphism of complexes. The minimal ranks \( c_1, \ldots, c_p \) are then denoted by \( \beta_1, \ldots, \beta_p \) and are called the Betti numbers of \( M \). The minimal length \( p \) of a free resolution of \( M \) is called the projective dimension of \( M \). For further details see [20].

Now suppose \( M \) is generated by monomials \( m_1, \ldots, m_q \) in \( S \). Taylor [22] shows that the simplicial chain complex of an \( q \)-simplex can be “homogenized” to produce a (most often non-minimal) free resolution of \( M \). This process is done by labeling each vertex
of the $q$-simplex with one of the monomials $m_1, \ldots, m_q$, and then each face is labeled by the lcm of its vertex labels. This labeled $q$-simplex is called the Taylor complex of $M$ and is denoted by $\mathbb{T}(M)$. The monomial labels of $\mathbb{T}(M)$ belong to $\text{LCM}(M)$ – the lcm lattice of $M$ – which is the set of all monomial labels of $\mathbb{T}(M)$ partially ordered by divisibility. The homogenization of the simplicial chain maps is done using the monomial labels of each face. The resulting free resolution is a multigraded resolution, where in each homological degree $i$, the free module $S^{c_i}$ is written as the direct sum of cyclic $S$-modules

$$S(m_1)^{c_{i,m_1}} \oplus \cdots \oplus S(m_n)^{c_{i,m_n}},$$

where $m_1, \ldots, m_n$ are the monomial labels of the $i$-dimensional faces of the Taylor complex, and $c_i = c_{i,m_1} + \cdots + c_{i,m_n}$.

**Example 2.2.** Let $S = \mathbb{K}[x_1, x_2, x_3]$ be a polynomial ring and $M = (x_1x_2, x_1x_3)$. The labeled 2-simplex

```
  x_1 x_2
  |   |
  x_3
```

produces the Taylor resolution of $M$ as follows

$$0 \rightarrow S(x_1x_2x_3) \rightarrow S(x_1x_2) \oplus S(x_1x_3) \rightarrow M \rightarrow 0.$$

Every monomial ideal has a multigraded minimal free resolution contained in the Taylor resolution. More precisely, if $\mathbb{F}$ is a minimal free resolution of $M$, the free module $S^{\beta_i}$ in homological degree $i$ of $\mathbb{F}$ can be refined as a direct sum of multigraded free modules

$$S(m_1)^{\beta_{i,m_1}} \oplus \cdots \oplus S(m_n)^{\beta_{i,m_n}},$$

where $m_1, \ldots, m_n$ are the monomial labels of the $i$-dimensional faces of the Taylor complex, and for $i \geq 0$ and $m \in \text{LCM}(M)$, the number

$$\beta_{i,m}(M) = \text{number of copies of } S(m) \text{ in the } i\text{-th homological degree of } \mathbb{F}$$

is the $i$-th multigraded Betti number of $M$ in multidegree $m$. In particular, the Betti numbers of $M$ are

$$\beta_i(M) = \sum_{m \in \text{LCM}(M)} \beta_{i,m}(M) = \sum_{j=1}^{n} \beta_{i,m_j}(M).$$

For more details on multigraded resolutions, we refer the reader to [18, 20].

Taylor’s homogenization of the chain maps of a simplex can be applied, in the same fashion, to any simplicial or cell complex ([16, 7, 20]), though one does not always get a resolution. When $\Delta$ is a cell complex on $q$ vertices whose cellular chain maps can be homogenized to a (minimal) free resolution of an ideal $M$ generated by $r$ monomials, we say that $M$ has a (minimal) free resolution supported on $\Delta$. If $\Delta$ supports a minimal free resolution of $M$, then

$$\beta_{i,m}(M) = \text{number of } i\text{-faces of } \Delta \text{ labeled with the monomial } m.$$

In Example 2.2 one can verify with the computer algebra software Macaulay2 [14] that the Taylor resolution is indeed a minimal free resolution of $M = (x_1x_2, x_1x_3)$. We therefore have multigraded Betti numbers

$$\beta_{0,x_1x_2}(M) = \beta_{0,x_1x_3}(M) = 1 = \beta_{1,x_1x_2x_3}(M).$$
so that the total Betti numbers are

$$\beta_0(M) = 2, \ \beta_1(M) = 1.$$  

**Example 2.3.** Let \( S = \mathbb{K}[x_1, x_2, x_3] \) be a polynomial ring and

\[ J = (x_1x_2, x_1x_3), \quad \text{and} \quad I = J + (x_1^2, x_2^2, x_3^2) \]

be monomial ideals in \( S \). The Taylor complex of \( I \) is a simplex of dimension 4, and the Taylor resolution of \( I \) is:

\[
0 \to S(x_1^2x_2^2) \oplus S(x_1^2x_3^2) \oplus S(x_1^2x_2x_3) \to S^2(x_1x_2x_3) \to S(x_1x_2x_3) \to S(x_1x_2) \to I \to 0.
\]

The minimal multigraded free resolution of \( I \) is:

\[
0 \to S(x_1x_2^2) \oplus S(x_1^2x_3) \oplus S(x_1x_2x_3) \to S(x_1x_2) \oplus S(x_1x_3) \to I \to 0.
\]

Observe that the Taylor resolution of \( I \) is much larger than its minimal free resolution. A natural question is how eliminate the extra summands from the Taylor resolution to get (close) to the minimal multigraded free resolution of \( I \). Is there a topological object supporting the minimal resolution? It is not difficult to see that there is no simplicial complex supporting a minimal free resolution of \( I \). We will show later in this paper (Example 5.5) that \( I \) has a minimal free resolution that is supported on the cell complex below.

![Cellular Resolution Diagram](attachment:cellular_resolution_diagram.png)

Example 2.3 is the motivating example for this paper. Starting from an (Artinian) monomial ideal \( M \), we asked if it is possible to find a topological object supporting the minimal free resolution of \( M \). The natural place to look is the Taylor resolution: how to eliminate the extra summands in each homological degree? These extra summands correspond, in fact, to faces of the Taylor complex that share a label with a subface. An extreme action would be to delete all faces of the Taylor complex that share a label with any other face. The resulting subcomplex of the Taylor complex is the well known Scarf Complex ([21]).
Definition 2.4. The Scarf complex of a monomial ideal $M$ is a simplicial subcomplex of the Taylor complex of $M$ that is given by the set of those faces $\sigma \in \mathbb{T}(M)$ such that there is no other face $\tau \in \mathbb{T}(M)$ with $\tau \neq \sigma$ and $\text{lcm} \tau = \text{lcm} \sigma$.

Even though the Scarf complex of a monomial ideal is often too small to support a resolution, but its monomial labels appear in any multigraded resolution of the ideal (see [17] for a nice overview of simplicial resolutions). It must also be noted that there are classes of ideals with no cellular minimal resolutions [23].

Upon realizing that an Artinian monomial ideal may not have a simplicial minimal resolution at all, we turned to Discrete Morse Theory: starting from the face poset of the Taylor complex of $M$ and eliminating faces systematically, we could prove that the cell complex in Example 2.3 does in fact support a minimal free resolution of $M$.

The next section is devoted to introducing the main tools of Discrete Morse Theory that are needed for our purposes.

3. HOMOGENEOUSACYCLIC MATCHINGS AND DISCRETE MORSE THEORY

For any integer $q \geq 1$, let $G_q$ be the directed graph with vertex and edge sets

$$
V(G_q) = 2^{|q|},
$$

$$
E(G_q) = \{(T, T') : T, T' \in V(G_q), \ |T| = |T'| + 1 \text{ and } T' \subseteq T\}.
$$

The directed graph $G_q$ can be visualized as a directed hypercube [15, p. 33]. Let $A$ be a matching in $G_q$, that is, a subset of $E(G_q)$ where no two edges in $A$ share a vertex. Let $G_q^A$ be the directed graph on $V(G_q)$ with edge set

$$
E(G_q^A) = (E(G_q) \setminus A) \cup \{(T', T) : (T', T) \in A\}.
$$

An element of $V(G_q) \setminus V(A)$ is called an $A$-critical vertex of $G_q$.

The matching $A$ is acyclic in $G_q$ if $G_q^A$ is an acyclic directed graph, or equivalently, the induced subgraph $G_q^A[V(A)]$ is acyclic. For a multiset $U = \{m_1, \ldots, m_q\}$ of non-trivial monomials in the polynomial ring $S$ over a field $K$, we let $G_U$ denote the directed graph $G_q$ where every vertex $T$ is labeled with the monomial $m_T = \text{lcm}(m_j : j \in T)$. By convention we set $m_\emptyset = 1$.

For a monomial ideal $M$ in $S$, by $G_M$ we mean $G_{G(M)}$ where $G(M)$ is the (unique) minimal monomial generating set of $M$. Note that the vertices of $G_M$ and their monomial labels correspond to the faces of the Taylor complex $\mathbb{T}(M)$ and their monomial labels. A matching $A$ of $G_M$ is called homogeneous if

$$
m_T = m_{T'} \quad \text{for every} \quad (T, T') \in A.
$$

When $A$ is a homogeneous acyclic matching of $G_M$, Batzies and Welker (see Theorem 3.1 below) show that the $A$-critical vertices $T$ of $G_M$ are in one-to-one correspondence with cells $\sigma_T$ of a CW-complex $X_A$ which supports a free resolution of $J$. For any two $A$-critical vertices $T$ and $T'$ of $G_M$ with $|T| = |T'| + 1$ consider the partial order $\leq$ on
\(V(G_M)\) as follows:

\[
\sigma_{T'} \preceq \sigma_T \iff \begin{cases} 
T' \subseteq T \\
\text{or} \\
\text{there exists a directed path from } T'' \text{ to } T' \text{ in } G^A_M \text{ for some } T'' \subseteq T \text{ with } |T''| = |T'|.
\end{cases}
\]

\textbf{Theorem 3.1 (Batzies-Welker [5]).} Let \(M\) be a monomial ideal in the polynomial ring \(S = \mathbb{K}[x_1, \ldots, x_n]\) over a field \(\mathbb{K}\). If \(A\) is a homogeneous acyclic matching on \(G_A\), then there exists a CW-complex \(X_A\) which supports a multigraded free resolution of \(M\). The \(i\)-cells \(\sigma_T\) of \(X_A\) are in one-to-one correspondence with the \(A\)-critical vertices \(T\) of \(G_M\) of cardinality \(i + 1\).

The resolution supported on \(X_A\) is minimal if for any two \(A\)-critical vertices \(T, T'\) such that \(|T'| = |T| - 1\) and \(\sigma_{T'} \preceq \sigma_T\) implies \(m_T \neq m_{T'}\).

\textbf{3.1. The Main Question.} The big general question we are concerned with is the following: given a monomial ideal \(I\), how close to a minimal free resolution of \(I\) can we get using cellular resolutions? For example, it is known that all monomial ideals of projective dimension 1 have minimal cellular resolutions supported on graphs [11], and powers of square-free monomial ideals of projective dimension \(\leq 1\) have minimal cellular resolutions supported on hypercubes [9, 10]. On the other hand, in [23] Velasco presented a family of monomial ideals, including a 23-generated monomial ideal in \(\mathbb{K}[x_1, \ldots, x_{284}]\), none of which have minimal cellular resolutions. What can we say about monomial ideals with fewer generators? In light of Theorem 3.1, to find minimal cellular resolutions of (Artinian reductions of) monomial ideals, we consider the following question:

\textbf{Question 3.2 (BW-matchings).} Let \(u_1, \ldots, u_r\) be monomials in the polynomial ring \(S = \mathbb{K}[x_1, \ldots, x_n]\), where \(\mathbb{K}\) is a field, and

\[I = J + (x_1^{e_1}, \ldots, x_n^{e_n})\]

be an Artinian reduction of \(J = (u_1, \ldots, u_r)\). Under which conditions, we can find (possibly by an algorithm) a homogeneous acyclic matching \(A\) in \(G_J\) (resp. \(G_I\)) such that for any two \(A\)-critical vertices \(T, T' \in V(G^A_J)\) (resp. \(T, T' \in V(G^A_I)\)) \(m_T \neq m_{T'}\) whenever \(|T| = |T'| + 1\) and \(\sigma_{T'} \preceq \sigma_T\)?

We call a matching \(A\) satisfying the requirement of Question 3.2 for an ideal \(J\) a Batizes-Welker matching, or simply a BW-matching of \(G_J\). If a monomial ideal \(M\) has a BW-matching, then Theorem 3.1 implies that \(M\) has a minimal cellular resolution.

In the remainder of this paper, we give a positive answer to Question 3.2 when \(r \leq 4\) (Theorem 4.5). Moreover, when \(r \leq 2\) we are able to find an explicit BW-matching of \(G_I\), which gives us a concrete description of the CW-complex supporting a minimal free resolution of \(I\) (Theorem 5.4). The following lemma is needed in our later discussions.

\textbf{Lemma 3.3 (Cycles arising from a homogeneous matching).} Let \(U\) be a multiset of non-trivial monomials and \(A\) be a homogeneous matching in \(G_I\). Suppose \(C\) is a cycle in \(G^A_{I_U}\). Then

\(i\) \(|U| \geq 3\);
(ii) there exists an integer $t \in \{2, \ldots, |U| - 1\}$ such that for every vertex $T \in V(C)$ either $|T| = t$ or $|T| = t + 1$;
(iii) $C$ must have at least six edges;
(iv) any two vertices $T$ and $T'$ of $C$ have the same monomial label $m_T = m_{T'}$.

Proof. If $|U| = 1$ or $|U| = 2$ then we have no directed cycle $C$ in $G^A_U$, so $|U| \geq 3$. Since $C$ is a directed cycle, if we start from any vertex $T$ of $C$, we can follow a directed path in $C$ bringing us back to $T$:

$$T = T_0 \to T_1 \to \cdots \to T_q = T.$$  

So the only way for $C$ to be a cycle is that it consists of a sequence of alternating “up” arrows (arrows in the matching $A$) and “down” arrows (arrows outside $A$) from $T_0$ to $T_q$. Since $A$ is a matching, no vertex can be part of two consecutive “up” arrows. Moreover, since $A$ is homogeneous, every vertex in $T$ of $A$ satisfies $|T| > 1$; otherwise, $|T| = 1$ and the down arrow from $T$ has the trivial monomial 1 at its end, a contradiction. Therefore $C$ is one of the following two sequences

$$T = T_0 \to T_1 \to \cdots \to T_q = T$$

So $C$ must have an even cardinality of edges, and if we set $t = |T|$ in the first scenario, and $t = |T| - 1$ in the second, then every vertex of $C$ has size $t$ or $t + 1$.

From (2), it is also clear that a cycle $C$ of two edges is not possible. So $C$ must contain one of the following two sequence of edges

\[
\begin{align*}
T & \to (T \cup \{a\}) \setminus \{b\} \\
T & \to (T \cup \{a\}) \setminus \{b\}
\end{align*}
\]

for distinct elements $a, b, c$. But then $b \in T \setminus \{a\} \subseteq T$ and so

$$T \setminus \{a\} \not\subseteq (T \cup \{c\}) \setminus \{b\} \quad \text{and} \quad T \not\subseteq (T \cup \{a, c\}) \setminus \{b\}.$$  

Accordingly, in both cases, the cycles cannot have length four otherwise we should have the thick edges as follows:

\[
\begin{align*}
T & \to (T \cup \{c\}) \setminus \{b\} \\
T & \to (T \cup \{c\}) \setminus \{b\}
\end{align*}
\]

Thus $C$ must have at least six edges. Therefore $C$ is of the form

$$T_1 \to T_2' \to T_3' \to \cdots \to T_s' \to T_{s+1} = T_1.$$
where for each \( i \in [s] \), \( T_i, T_{i+1} \subseteq T'_i \) and \( m_{T_{i+1}} = m_{T_i} \). It follows that
\[
m_{T_1} = m_{T_{s+1}} = m_{T_s} = \cdots = m_{T_3} = m_{T_2} = m_{T_1},
\]
and hence \( m_{T_1} = m_{T_2} = \cdots = m_{T_s} = m_{T_1} = m_{T_2} = \cdots = m_{T_r} \), as required. \( \square \)

4. (Artinian Reductions of) Monomial Ideals with \( \leq 4 \) Generators

For the rest of the paper, we use the following notation.

**Setup 4.1 (Our Setup).** Let \( S = \mathbb{K}[x_1, \ldots, x_n] \) be a polynomial ring over a field \( \mathbb{K} \).

- If \( \mathcal{U} = \{m_1, \ldots, m_q\} \) is a multiset of monomials in \( S \) and \( m_i \neq 1 \) for all \( i = 1, \ldots, q \), then
  - for \( T \subseteq [q] \), set \( m_T = \text{lcm}(m_j : j \in T) \) and \( m_{\emptyset} = 1 \) as before.
  - for \( T \subseteq [q] \), \( \overline{T} \) denotes the set complement \([q] \setminus T\);
  - \( \text{LCM}(\mathcal{U}) \) denotes the set of least common multiples of any number of elements of \( \mathcal{U} \);
  - if \( u \in \text{LCM}(\mathcal{U}) \), then set
    \[
    \mathcal{U}_u = \{ T \subseteq [q] : m_T = u \}, \quad \overline{\mathcal{U}}_u = \{ T \subseteq [q] : m_T = u \} = \{ \overline{T} : T \in \mathcal{U}_u \}.
    \]
- If \( M \) is an ideal of \( S \) minimally generated by a set of monomials \( \mathcal{U} = \{m_1, \ldots, m_q\} \), then we set
  - \( \text{LCM}(M) = \text{LCM}(\mathcal{U}) \);
  - \( M_u = \mathcal{U}_u \) for \( u \in \text{LCM}(M) \).
- \( J \) stands for an ideal of \( S \) minimally generated by \( r \) monomials \( u_j = \prod_{i \in [n]} x_i^{\alpha_{j,i}} \) for \( j \in [r] \) using all \( n \) variables \( x_1, \ldots, x_n \), and that none of the monomials \( u_1, \ldots, u_r \) are pure powers \( x_j^b \) for some integers \( j \) and \( b \).
- The ideal \( I = J + (x_1^{e_1}, \ldots, x_n^{e_n}) \) is an Artinian reduction of \( J \) with
  \[
e_i > \max \{\alpha_{j,i} : j \in [r]\}
  \]
for all \( i \in [n] \).
- We will always use the following order on the generators of \( I \):
  \[
u_1, \ldots, u_r, x_1^{e_1}, \ldots, x_n^{e_n}.
  \]
In particular, the index set for the generators of \( I \) will be \([r + n]\), where the \( j \)-th generator of \( I \) is
\[
\begin{align*}
u_j, & \quad \text{if } j \leq r, \\
x_j^{e_j-r}, & \quad \text{if } r < j \leq r + n.
\end{align*}
\]
- If \( \mathcal{X}, \mathcal{Y} \) are families of sets, then \( \mathcal{X} \times \mathcal{Y} \) is the set
  \[
  \mathcal{X} \times \mathcal{Y} = \{ X \cup Y : X \in \mathcal{X}, Y \in \mathcal{Y} \}.
  \]

**Lemma 4.2.** With notation as in Setup 4.1 let \( \mathcal{U} = \{m_1, \ldots, m_q\} \) be a multiset of monomials in \( S \), and let \( u \in \text{LCM}(\mathcal{U}) \) and \( V = \{i : m_i \not\in u\} \). Then there exists a simplicial complex \( \Delta \) on vertex set \([q] \setminus V\) such that \( \overline{U}_u = \Delta \star \{V\} \).
Proof. First observe that
\[ T \in \mathcal{U}_u \implies m_T = u \]
\[ \implies T \cap V = \emptyset \]
\[ \implies T \supseteq V. \]
Let \( \Delta = \{ T \setminus V : m_T = u \} \). Then \( \mathcal{U}_u = \Delta \ast \{ V \} \) and \( \Delta \) is a simplicial complex, because if \( F \subseteq T \setminus V \) for some \( T \) with \( m_T = u \), then by letting \( T' = F \cup V \) we have
\[ V \subseteq T' \subseteq T \implies T' \subseteq T \subseteq V \]
\[ \implies u = m_{T'} | m_T = u \]
\[ \implies m_{T'} = u \]
\[ \implies F = T' \setminus V \in \Delta, \]
as required. \( \square \)

Proposition 4.3. With notation as in Setup 4.1, let \( I = (u_1, \ldots, u_r, x_1^{e_1}, \ldots, x_n^{e_n}) \) and \( u = x_1^{b_1} \cdots x_n^{b_n} \in \text{LCM}(I) \). Let
\[ A = \{ i \in [n] : 0 < b_i < e_i \}, \quad B = \{ i \in [n] : b_i = e_i \}, \]
and for \( j \in [r] \) let
\[ u_j' = \prod_{i \in A} x_i^{\alpha_{j,i}}, \quad u' = \prod_{i \in A} x_i^{b_i}, \quad \text{and} \quad U' = \{ u_j' \mid j \in [r] \}. \]
Then
(i) \( u' \in \text{LCM}(U') \);
(ii) \( I_u = U'_u \ast \{ X' \} \) where \( X' = \{ r + i : i \in B \} \);
(iii) There exist a simplicial complex \( \Delta \) on \( \leq r \) vertices and an isomorphism
\[ \phi : G_I[I_u] \longrightarrow G_{U'}[\Delta] \]
of underlying graphs such that \( \phi(T) = ([r] \setminus (T \setminus X')) \setminus V \) where \( T \in V(G_I[I_u]) \), and \( V = \{ i : m_i \mid u \} \);
(iv) If \( T_1, T_2 \) are vertices of \( G_I[I_u] \) such that \( |T_2| = |T_1| + 1 \) then \( |\phi(T_1)| = |\phi(T_2)| + 1 \);
(v) If \( T_1 \subseteq T_2 \) are vertices of \( G_I[I_u] \) then \( \phi(T_2) \subseteq \phi(T_1) \).

Proof. Assume \( u = \text{lcm}(u_{j_1}, \ldots, u_{j_s}, x_1^{e_1}, \ldots, x_n^{e_n}) \) where \( j_1 < \cdots < j_s \). Then
\[ b_i = \max \{ \alpha_{j_1,i}, \ldots, \alpha_{j_s,i} \} \quad \text{for} \quad i \in A \]
and
\[ u' = \text{lcm}(u_{j_1}', \ldots, u_{j_s}') \in \text{LCM}(U'). \]
This settles (i). To show (ii), let \( u'' = \prod_{i \in B} x_i^{b_i} \). Then \( u = u' \cdot u'' \). If \( T \subseteq [r + n] \) is such that \( m_T = u \), put \( T' = T \cap [r] \) and \( T'' = T \cap \{ r + 1, \ldots, r + n \} = X' \). It immediately follows that \( u_j' = \text{lcm}(u_j' \mid j \in T') \) and \( u'' = m_{T''} \). Since \( T = T' \cup T'' \), we have shown that \( T \in U'_u \ast \{ X' \} \). Conversely, if \( T \in U'_u \ast \{ X' \} \), then \( T = T' \cup X' \) where \( T' \subseteq [r] \) and \( \text{lcm}(u_j' \mid j \in T') = u' \). This implies that \( m_T = \text{lcm}(u_j \mid j \in T' \cup X') = \text{lcm}(u', u'') = u \) and so \( T \in I_u \). This ends the proof of (ii). To verify (iii), by (ii) and Lemma 4.2, we have:
\[ I_u = U'_u \ast \{ X' \} \quad \text{and} \quad \mathcal{U}'_u = \Delta \ast \{ V \}, \]
where $\Delta$ is a simplicial complex on vertex set $[r] \setminus V$. Consider the following sequence of graph isomorphisms, where $f_1$ and $f_3$ preserve the directions of the edges, and $f_2$ reverses the directions of the edges:

\[
\begin{align*}
  f_1 : G_{\mathcal{I}}[\Delta] & \to G_{\mathcal{I}}[\mathcal{U}_w], & f_1(T) = T \cup V; \\
  f_2 : G_{\mathcal{I}}[\mathcal{U}_w] & \to G_{\mathcal{I}}[\mathcal{U}_u], & f_2(T) = [r] \setminus T = \overline{T}; \\
  f_3 : G_{\mathcal{I}}[\mathcal{U}_u] & \to G_{\mathcal{I}}[\mathcal{I}_u], & f_3(T) = T \cup X',
\end{align*}
\]

which leads to the following graph isomorphisms

\[
G_{\mathcal{I}}[\Delta] \cong G_{\mathcal{I}}[\mathcal{U}_w] \cong G_{\mathcal{I}}[\mathcal{U}_u] \cong G_{\mathcal{I}}[\mathcal{I}_u] \cong T \leftrightarrow U \cup V \leftrightarrow [r] \setminus (T \cup V) \leftrightarrow ([r] \setminus (T \cup V)) \cup X'.
\]

From the graph isomorphisms above, we have the isomorphism $\phi := f_1^{-1} \circ f_2^{-1} \circ f_3^{-1}$ from $G_{\mathcal{I}}[\mathcal{I}_u]$ to $G_{\mathcal{I}}[\Delta]$.

If $T \in V(G_{\mathcal{I}}[\mathcal{I}_u])$ then $T = T' \cup X'$ where $T' \cap X' = \emptyset$ and

\[
\phi(T) = ([r] \setminus (T \setminus X')) \setminus V = [r] \setminus (T' \cup V).
\]

To verify (iv), let $T_1, T_2 \in V(G_{\mathcal{I}}[\mathcal{I}_u])$ be such that $|T_1| = |T_2| + 1$. Then $|T_1| = |T_1| - |X'| = |T_2| + 1 - |X'| = |T_2| + 1$. Since $V \cap T_2 = V \cap T_1 = \emptyset$, we conclude that $|T_1 \cup V| = |T_1| + |V| = |T_2| + 1 + |V| = |T_2 \cup V| + 1$. Hence $|[r] \setminus (T_1 \cup V)| = |[r] \setminus (T_2 \cup V)| - 1$. Thus $\phi(T_2) = \phi(T_1) + 1$.

Finally, to show (v), let $T_1, T_2 \in V(G_{\mathcal{I}}[\mathcal{I}_u])$ be such that $T_2 \subseteq T_1$. Since $T_2 \cap X' = T_1 \cap X'$, we have

\[
T_2' = T_2 \setminus X' = T_2 \setminus (T_2 \cap X') \subseteq T_1 \setminus (T_1 \cap X') = T_1'.
\]

So $T_2 \cup V \subseteq T_1 \cup V$ and $[r] \setminus (T_2 \cup V) \subseteq [r] \setminus (T_1 \cup V)$, which implies that $\phi(T_1) \subseteq \phi(T_2)$.

Proposition 4.3 is inductively powerful, as it reduces the search for BW-matching inside a large lattice to a much smaller simplicial complex. Our main theorem (Theorem 4.5) makes full use of Proposition 4.3 to find BW-matches for Artinian reductions of ideals with up to 4 generators. We first do an example.

**Example 4.4.** Let $I = (x_1^2 x_2, x_1 x_3, x_1^3 x_4, x_2^4, x_3^2, x_4^3)$ be a monomial ideal in the polynomial ring $S = K[x_1, x_2, x_3, x_4]$. If $u = x_1^3 x_2 x_3^2 x_4$, then with notation as in Proposition 4.3

\[
A = \{1, 4\}, \quad X' = \{5, 6\}, \quad u' = x_1^3 x_4, \quad \text{and} \quad U' = \{x_1^2, x_1, x_3^3 x_4\}.
\]

So,

\[
\begin{align*}
I_u &= \{\{3, 5, 6\}, \{1, 3, 5, 6\}, \{2, 3, 5, 6\}, \{1, 2, 3, 5, 6\}\} \\
&= \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \ast \{\{5, 6\}\} \\
&= U'_u \ast \{X'\}.
\end{align*}
\]

Hence

\[
\overline{U}_w = \{\{1, 2\}, \{2\}, \{1\}, \{\emptyset\}\} = (12) \ast \{\emptyset\} = \Delta \ast \{V\}.
\]
Therefore, $G_{U'[\Delta]}$ is indeed the graph Figure 1 (left), which leads to the homogeneous acyclic matching $\mathcal{A}_u$ for $G_I[I_u]$ as shown in Figure 1 (right):

![Figure 1](image1)

Also if $u = x_1^3x_2^2x_3^2x_4$, then
$$A = \emptyset, \quad X' = \{4, 5, 6, 7\}, \quad \text{and} \quad u' = 1.$$  

By Proposition 4.3,
$$I_u = \{\{4, 5, 6, 7\}, \{1, 4, 5, 6, 7\}, \{2, 4, 5, 6, 7\}, \{3, 4, 5, 6, 7\}, \{1, 2, 4, 5, 6, 7\}, \{1, 3, 4, 5, 6, 7\}, \{2, 3, 4, 5, 6, 7\}, \{1, 2, 3, 4, 5, 6, 7\}\}$$
$$= \{\emptyset\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \ast \{\{4, 5, 6, 7\}\}$$
$$= U'_{u'} \ast \{X'\}.$$  

Thus
$$\overline{\mathcal{U}}_u = \{\{1, 2, 3\}, \{2, 3\}, \{1, 3\}, \{1, 2\}, \{3\}, \{2\}, \{1\}, \emptyset\}$$
$$= \langle 123 \rangle \ast \emptyset = \Delta \ast \{V\}.$$  

Therefore, $G_{U'[\Delta]}$ is the graph in Figure 2 (left), which leads to the homogeneous acyclic matching $\mathcal{A}_u$ for $G_I[I_u]$ as shown in Figure 2 (right):

![Figure 2](image2)

We are now ready to show that any monomial ideal with at most four generators, and any monomial Artinian reduction of such an ideal, has a minimal free resolution supported on a CW-complex. Now that we have Proposition 4.3, this task is reduced to finding a BW-matching for every single simplicial complex with at most four vertices.
Theorem 4.5 (Main Theorem). Let $J$ be a monomial ideal with at most four monomial generators in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$. Also, let $I = J + (x_1^{e_1}, \ldots, x_n^{e_n})$ be an Artinian reduction of $J$, where $e_1, \ldots, e_n$ are positive integers. Then both $I$ and $J$ have minimal free resolutions supported on a CW-complex.

Proof. Following Setup 4.1, suppose $J$ is minimally generated by monomials $u_1, \ldots, u_r$ and $I$ is an Artinian reduction of $J$ where $r \leq 4$. First we prove the result for $I$ by showing that $G_I[I_u]$ has a BW-matching for every $u \in \text{LCM}(I)$.

Let $u \in \text{LCM}(I)$. By Proposition 4.3, $G_I[I_u] \cong G_{I'}[\Delta]$, where $\Delta$ is a simplicial complex on at most $|U'| \leq r \leq 4$ vertices. According to [1] the only possible simplicial complexes on at most four vertices – and hence candidates for $\Delta$ – are

\[
\langle \rangle, \langle 1 \rangle, \langle 1, 2 \rangle, \langle 1, 2, 3 \rangle, \langle 12 \rangle, \langle 123 \rangle, \langle 123, 23 \rangle, \langle 12, 3, 23 \rangle, \\
\langle 1, 2, 3, 4 \rangle, \langle 12, 34 \rangle, \langle 123, 34 \rangle, \langle 1, 2, 34 \rangle, \langle 12, 3, 34 \rangle, \\
\langle 13, 24, 34 \rangle, \langle 14, 24, 34 \rangle, \langle 12, 34, 23 \rangle, \langle 12, 3, 43 \rangle, \\
\langle 14, 23, 43 \rangle, \langle 13, 43, 23 \rangle, \langle 12, 34, 23 \rangle, \langle 12, 34, 23 \rangle.
\]

(3)

In List 4.6 we highlight an explicit acyclic matching $\mathcal{A}'_u$ for each possible graph $G_{I'}[\Delta]$. For example, in Figure 3 $\Delta = \langle 1, 23 \rangle$. The graph $G_{I'}[\Delta]$ with a matching $\mathcal{A}'_u$ appears on the left, and the corresponding homogeneous acyclic matching $\mathcal{A}_u$ for $G_I[I_u]$ appears on the right with $V$ and $X'$ as in Proposition 4.3. The matchings in the right-hand-side picture are homogeneous as all vertices have the same monomial label $u$.

![Graphs](image-url)

**Figure 3.**

To build a homogeneous matching on $G_I$, we take $\mathcal{A} = \bigcup_{u \in \text{LCM}(I)} \mathcal{A}_u$. We observe that $\mathcal{A}$ is an acyclic homogeneous matching because if $C$ is a cycle in $G_I^A$, then $V(C) \subseteq I_u$ for some $u \in \text{LCM}(I)$ (see Lemma 3.3). This contradicts the fact that $\mathcal{A}_u$ is acyclic.

We show that homogeneous acyclic matching $\mathcal{A}$ is indeed a BW-matching of $G_I$. To this end, for two $\mathcal{A}$-critical vertices $V_1$ and $V_2$ of $G_I$ where $|V| = |V_1| + 1$ and $\sigma_{V_1} \preceq \sigma_{V_2}$, we show $m_{V_1} \neq m_{V_2}$. Suppose on the contrary that $m_{V_1} = m_{V_2} = u$. Then $V_1$ and $V_2$ are $\mathcal{A}_u$-critical vertices of $G_I$. By Proposition 4.3(iv), there exists two vertices...
$T_1, T_2 \in V(G_{\Delta}[\Delta])$ such that $\phi(V_i) = T_i$ for $i = 1, 2$, and $|T_1| = |T_2| + 1$. Since $\sigma_{V_1} \preceq \sigma_{V_2}$, either $V_1 \subseteq V_2$ or there is a directed path from $V_1'$ to $V_1$ in $G_I$ for some $V_1' \subseteq V_2$ satisfying $|V_1'| = |V_1|$. If $V_1 \subseteq V_2$, then by Proposition 4.3(v), we have $T_2 \subseteq T_1$ and hence $\sigma_{T_2} \preceq \sigma_{T_1}$. On the other hand, if there exists a directed path from $V_1'$ to $V_1$ in $G_I$, this path is of the form:

![Diagram](image)

for some $V_1' \subseteq V_2$ with $|V_1'| = |V_1|$, where for each $i \in [s]$, $V_i', V_i' \subseteq V_i$ and

$$m_{i+1} = m_{i+1} \mid m_i.$$

Then

$$m_i = m_{i+1} \mid m_{i+1} \mid m_{i+1} \mid \cdots \mid m_i.$$

Hence $m_i = m_{i+1} = \cdots = m_i = m_{i+1} = \cdots = m_i = u$. Now let

$$T_i = \phi(V_i) \quad \text{for} \quad i \in [s],$$

$$T_j' = \phi(V_j') \quad \text{for} \quad j \in \{3, \ldots, s + 1\}.$$

Then we have the following directed path from $T_s \subseteq T_1$ to $T_2$ showing that $\sigma_{T_2} \preceq \sigma_{T_1}$.

![Diagram](image)

We have seen that both cases lead to $\sigma_{T_2} \preceq \sigma_{T_1}$, but this is not possible because in all cases of the matchings in List 4.6 any two $A_u'$-critical vertices $T_1, T_2$ have the property that $||T_1| - |T_2|| \ne 1$ except when $\Delta = \{1, 23, 24, 34\}$. In this case, we have only the two $A_u'$-critical vertices $T_1 = \{1\}$ and $T_2 = \{24\}$ for which $\sigma_{T_1} \not\preceq \sigma_{T_2}$. Hence homogeneous acyclic matching $A$ is a BW-matching of $G_I$ and $I$ has a minimal free resolution supported on a CW-complex (see Theorem 5.1).

Since $A$ is an acyclic homogeneous matching of $G_I$, $A \cap E(G_J)$ is an acyclic homogeneous matching of $G_J$. For any two $A \cap E(G_J)$-critical vertices $V_1$ and $V_2$ of $G_J$ where $|V_2| = |V_1| + 1$ and $\sigma_{V_1} \preceq \sigma_{V_2}$, we have $V_1, V_2 \in V(G_J^A)$, so that $m_{V_1} \ne m_{V_2}$. Therefore $A \cap E(G_J)$ is a BW-matching of $G_J$, that is $J$ has a minimal free resolution supported on a CW-complex. The proof is complete. \qed

**List 4.6** (Directed graphs corresponding to simplicial complexes with at most four vertices).
\[ \Delta = \langle 14, 24, 34 \rangle \]

\[ \Delta = \langle 1, 23, 24, 34 \rangle \]

\[ \Delta = \langle 12, 13, 24, 34 \rangle \]

\[ \Delta = \langle 14, 23, 24, 34 \rangle \]

\[ \Delta = \langle 13, 14, 23, 24, 34 \rangle \]

\[ \Delta = \langle 12, 13, 14, 23, 24, 34 \rangle \]

\[ \Delta = \langle 12, 13, 14, 234 \rangle \]

\[ \Delta = \langle 14, 23, 24, 34 \rangle \]

\[ \Delta = \langle 14, 234 \rangle \]
Example 4.7. Let $S = \mathbb{K}[x_1, x_2, x_3, x_4]$ be a polynomial ring in four variables and

$$J = (x_1^2 x_2^2, x_1 x_3, x_2^2 x_4), \quad \text{and} \quad I = J + (x_1^4, x_2^4, x_3^2, x_4^2).$$

By the proof of Theorem 4.5 and Example 4.4, the matching $A = \bigcup_{u \in \text{LCM}(I)} A_u$ of $G_I$ is as follows:

$$\begin{align*}
(\{123456\}, \{124567\}) & (\{134567\}, \{14567\}) & (\{234567\}, \{24567\}) & (\{123456\}, \{13456\}) \\
(\{123457\}, \{12457\}) & (\{123467\}, \{13457\}) & (\{124567\}, \{13567\}) & (\{12456\}, \{1456\}) \\
(\{34567\}, \{4567\}) & (\{123567\}, \{13457\}) & (\{12567\}, \{1567\}) & (\{234567\}, \{34567\}) \\
(\{23457\}, \{2457\}) & (\{23467\}, \{2467\}) & (\{23567\}, \{3567\}) & (\{12345\}, \{2345\}) \\
(\{12356\}, \{1356\}) & (\{12347\}, \{1347\}) & (\{12346\}, \{1346\}) & (\{12346\}, \{1346\}) \\
(\{12367\}, \{1367\}) & (\{12357\}, \{1357\}) & (\{12456\}, \{456\}) & (\{1245\}, \{245\}) \\
(\{1246\}, \{146\}) & (\{1256\}, \{156\}) & (\{13457\}, \{1457\}) & (\{3467\}, \{467\}) \\
(\{1345\}, \{345\}) & (\{1347\}, \{147\}) & (\{1357\}, \{357\}) & (\{1357\}, \{167\}) \\
(\{2346\}, \{346\}) & (\{2356\}, \{356\}) & (\{2347\}, \{247\}) & (\{2367\}, \{367\}) \\
(\{1235\}, \{235\}) & (\{1236\}, \{136\}) & (\{246\}, \{46\}) & (\{145\}, \{45\}) \\
(\{126\}, \{16\}) & (\{347\}, \{47\}) & (\{135\}, \{35\}) & (\{236\}, \{36\})
\end{align*}$$
By Theorem 4.5, $A$ is a BW-matching of $G_I$ and $A$-critical vertices of $G_I^4$ are

\[
\{2567\}, \{1257\}, \{1234\}, \{1237\}, \{256\}, \{124\}, \{125\}, \{134\}, \{137\}, \{157\}, \{567\}, \{257\}, \{267\}, \{127\}, \{234\}, \{237\}, \{123\}, \{14\}, \{24\}, \{34\}, \{56\}, \{57\}, \{15\}, \{25\}, \{67\}, \{26\}, \{17\}, \{27\}, \{37\}, \{12\}, \{13\}, \{23\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}.
\]

So $I$ has minimal free resolutions supported on a CW-complex. The $i$-cells of this CW-complex are in one-to-one correspondence with the $A$-critical vertices of $G_I$ of cardinality $i + 1$.

Also, $G_J$ is the following directed graph

![Directed Graph](image)

We have $A \cap E(G_J) = \emptyset$, hence $A \cap E(G_J)$ is a BW-matching of $G_J$ from which it follows that $J$ has a minimal free resolution supported on a CW-complex. In fact, every vertex of $G_J$ is critical so that the Taylor’s resolution of $J$ is minimal.

**Example 4.8.** Let $S = \mathbb{K}[x_1, x_2, x_3, x_4]$ be a polynomial ring in 4 variables and

\[ J = (x_1x_2, x_1x_3, x_2x_4), \quad \text{and} \quad I = J + (x_1^2, x_2^2, x_3^2, x_4^2). \]

By the proof of Theorem 4.5 and Example 4.4, the matching $A = \bigcup_{u \in \text{LCM}(I)} A_u$ of $G_I$ is as follows:

\[
\begin{align*}
(\{1234567\}, \{124567\}) & \quad (\{134567\}, \{14567\}) & \quad (\{234567\}, \{24567\}) & \quad (\{123456\}, \{13456\}) \\
(\{123457\}, \{12457\}) & \quad (\{123467\}, \{12467\}) & \quad (\{123567\}, \{124567\}) & \quad (\{12456\}, \{1456\}) \\
(\{13456\}, \{1456\}) & \quad (\{13457\}, \{1457\}) & \quad (\{12567\}, \{1567\}) & \quad (\{23456\}, \{3456\}) \\
(\{12345\}, \{1245\}) & \quad (\{12346\}, \{1246\}) & \quad (\{12356\}, \{1245\}) & \quad (\{12345\}, \{1234\}) \\
(\{1235\}, \{135\}) & \quad (\{1236\}, \{136\}) & \quad (\{1257\}, \{157\}) & \quad (\{126\}, \{16\}) \\
(\{125\}, \{15\}) & \quad (\{1267\}, \{167\}) & \quad (\{123\}, \{13\}) & \quad (\{135\}, \{15\}) \\
(\{135\}, \{135\}) & \quad (\{134\}, \{134\}) & \quad (\{1237\}, \{1237\}) & \quad (\{145\}, \{45\}) \\
(\{126\}, \{16\}) & \quad (\{125\}, \{25\}) & \quad (\{137\}, \{17\}) & \quad (\{123\}, \{23\}) \\
(\{126\}, \{16\}) & \quad (\{134\}, \{34\}) & \quad (\{135\}, \{57\}) & \quad (\{246\}, \{46\})
\end{align*}
\]
By Theorem 4.5, $A$ is a BW-matching of $G_I$ and $A$-critical vertices of $G_I^A$ are

\[
\{2356\} \quad \{2347\} \quad \{2367\} \quad \{356\} \quad \{247\} \\
\{367\} \quad \{347\} \quad \{124\} \quad \{267\} \quad \{127\} \quad \{26\} \quad \{27\} \quad \{37\} \quad \{12\} \\
\{13\} \quad \{1\} \quad \{2\} \quad \{3\} \quad \{4\} \quad \{5\} \quad \{6\} \quad \{7\} \\
\{47\} \quad \{14\} \quad \{24\} \quad \{56\} \quad \{15\} \\
\{67\} \quad \{26\} \quad \{27\} \quad \{37\} \quad \{12\} \\
\{13\} \quad \{1\} \quad \{2\} \quad \{3\} \quad \{4\} \\
\{5\} \quad \{6\} \quad \{7\}
\]

This implies that $I$ has a minimal free resolution supported on a CW-complex. Also, $G_J$ is the following directed graph

![Directed Graph](image)

We have $A \cap E(G_J) = (\{123\}, \{23\})$, and hence $A \cap E(G_J)$ is a BW-matching of $G_J$. It follows that $J$ has a minimal free resolution supported on a CW-complex.

5. (Artinian reductions of) Monomial ideals with 2 generators

In this section, we find an explicit BW-matching of $G_I$ when $I$ is an Artinian reduction of a 2-generated monomial ideal. As a result, we give a description of a CW-complex supporting a minimal free resolution of $I$, and compute the multigraded Betti numbers of $I$.

Using Setup 4.1 with $r = 2$, we somewhat simplify the notation as follows:

\[
I = (u_1, u_2, x_1^{e_1}, \ldots, x_n^{e_n}), \quad u_1 = x_1^{a_1} \cdots x_n^{a_n}, \quad \text{and} \quad u_2 = x_1^{b_1} \cdots x_n^{b_n},
\]

where $a_i + b_i > 0$ for all $i \in [n]$. We first partition the variables $x_1, \ldots, x_n$ into three sets $P_0$, $P_1$, and $P_2$ as follows:

\[
P_0 := \{i + 2: a_i = b_i\}, \\
P_1 := \{i + 2: a_i > b_i\}, \\
P_2 := \{i + 2: a_i < b_i\}.
\]

Put

\[
A := \{i + 2: x_i \mid u_1\} = \{i + 2: a_i > 0\} \supseteq P_0 \cup P_1, \\
B := \{i + 2: x_i \mid u_2\} = \{i + 2: b_i > 0\} \supseteq P_0 \cup P_2.
\]

Example 5.1. Let $S = \mathbb{K}[x_1, x_2, x_3, x_4, x_5]$ be a polynomial ring in five variables and

\[
J = (x_1x_2^2 x_3x_4, x_1x_2x_3^2 x_5) \quad \text{and} \quad I = J + (x_1^2, x_2^3, x_3^3, x_4^2, x_5^2).
\]

Then with notation as above

\[
P_0 = \{3\}, \quad P_1 = \{4, 6\}, \quad P_2 = \{5, 7\}, \quad A = \{3, 4, 5, 6\}, \quad B = \{3, 4, 5, 7\}.
\]
Observe that using the notation above we have $X := \{3, \ldots, n + 2\} = P_0 \cup P_1 \cup P_2$, and for any subset $T \subseteq [n + 2]$ we have:

$$m_T = \begin{cases} 
\prod_{i+2 \in T \cap X} x_i^{e_i}, & \text{if } 1 \notin T, 2 \notin T, \\
\prod_{i+2 \in T \cap X} x_i^{e_i} \cdot \prod_{i \in T \cap X} x_i^{b_i}, & \text{if } 1 \notin T, 2 \in T, \\
\prod_{i \in T \cap X} x_i^{a_i}, & \text{if } 1 \in T, 2 \notin T, \\
\prod_{i \in T \cap X} x_i^{a_i} \cdot \prod_{i \in T \cap X} x_i^{\max\{a_i, b_i\}}, & \text{if } 1 \in T, 2 \in T.
\end{cases} \quad (7)$$

One may easily observe that we can replace $A \setminus T$ instead of $X \setminus T$ in $\prod_{i+2 \in X \cap T} x_i^{a_i}$ and $B \setminus T$ instead of $X \setminus T$ in $\prod_{i \in X \cap T} x_i^{b_i}$ above. As a direct consequence of (7), we have the following result.

**Lemma 5.2.** Let $T \subseteq [n + 2]$.

1. If $1 \in T$, then $m_T = m_{T \setminus \{1\}}$ if and only if
   (i) $P_1 \cup \{1, 2\} \subseteq T$, or
   (ii) $A \cup \{1\} \subseteq T$ and $2 \notin T$.

2. If $2 \in T$, then $m_T = m_{T \setminus \{2\}}$ if and only if
   (i) $P_2 \cup \{1, 2\} \subseteq T$, or
   (ii) $B \cup \{2\} \subseteq T$ and $1 \notin T$.

**Proof.** Assume $1 \in T$ and $T' = T \setminus \{1\}$. Let $T = [n + 2] \setminus T$ and define $T'$ analogously. Then $T \cap X = T' \cap X$ and $T \cap P_i = T' \cap P_i$ for $i = 0, 1, 2$.

(i) If $2 \in T$, then by (7), we have:

$$m_T = m_{T'} \iff \max\{a_i, b_i\} = b_i \text{ for all } i \text{ with } i + 2 \in X \setminus T$$

$$\iff (X \setminus T) \cap P_1 = \emptyset$$

$$\iff P_1 \cup \{1, 2\} \subseteq T.$$ 

(ii) If $2 \notin T$, then by (7), we have:

$$m_T = m_{T'} \iff A \setminus T = T' \cap A = \emptyset \iff A \cup \{1\} \subseteq T.$$ 

This completes the proof of (1). A similar argument as above yields the required result.

Now, let $A$ be the collection of edges of $G_I$ described as below:

$$\begin{align*}
(T, T \setminus \{1\}) : & \ 1 \in T, 2 \in T, P_1 \subseteq T, \\
(T, T \setminus \{2\}) : & \ 1 \in T, 2 \in T, P_1 \not\subseteq T, P_2 \subseteq T, \\
(T, T \setminus \{1\}) : & \ 1 \in T, 2 \notin T, A \subseteq T, \\
(T, T \setminus \{2\}) : & \ 1 \notin T, 2 \in T, B \subseteq T, T \neq [n + 2] \setminus \{1\}. 
\end{align*}$$

\[ \Box \]
In [16] it is shown that $A$ is indeed the output of Algorithm 1 in Appendix A. Algorithm 1 is designed to produce a BW-matching. We also give a direct proof below.

**Proposition 5.3 (A is a homogeneous matching of $G_1$).** Let $I = (u_1, u_2, x_1^{e_1}, \ldots, x_n^{e_n})$ be an ideal of $S = \mathbb{K}[x_1, \ldots, x_n]$ as in (4), and let $A$ be the set of edges of $G_1$ defined above. Then $A$ is a homogeneous matching of $G_1$ with $A$-critical vertices $T \subset [n + 2]$ such that

$$
1 \in T, \ 2 \in T, \ P_1 \not\subseteq T, \ P_2 \not\subseteq T \quad \text{or} \\
1 \in T, \ 2 \not\in T, \ P_1 \cup P_2 \subseteq T, \ P_0 \not\subseteq T, \quad \text{or} \\
1 \in T, \ 2 \not\in T, \ P_2 \not\subseteq T, \ A \not\subseteq T, \quad \text{or} \\
1 \not\in T, \ 2 \in T, \ P_1 \not\subseteq T, \ B \not\subseteq T, \quad \text{or} \\
1 \not\in T, \ 2 \not\in T, \ A \not\subseteq T, \ B \not\subseteq T.
$$

**Proof.** To see that $A$ is a matching, we consider a few scenarios where a vertex of $A$ might appear in two different edges.

- $(T, T \setminus \{1\}), (T, T \setminus \{2\}) \in A$. In this case, $1 \in T$, $2 \in T$, and the edges are of types (8) and (9), respectively. Thus $P_1 \subseteq T$ and $P_1 \not\subseteq T$, which is a contradiction.
- $(T', T), (T, T \setminus \{1\}) \in A$. In this case $1 \in T$ and $2 \in T'$. Then $(T', T)$ is of type (2) so that $P_1 \not\subseteq T$. Also, $(T, T \setminus \{1\})$ is of type (8) so that $P_1 \subseteq T$, a contradiction.
- $(T', T), (T, T \setminus \{2\}) \in A$. In this case $2 \in T$ and $1 \in T'$. Then $(T', T)$ is of type (2) so that $P_1 \subseteq T$. Also, $(T, T \setminus \{2\})$ is of type (11) so that $B \subseteq T$. It follows that $T = [n + 2] \setminus \{1\}$, which contradicts (11).
- $(T', T), (T'', T) \in A$. Assume without loss of generality, that $T = T'' \setminus \{1\} = T'' \setminus \{2\}$. It follows that $1 \not\in T''$ and $2 \not\in T'$ so that $(T', T)$ and $(T'', T)$ are of types (10) and (11), respectively. Therefore, $A \cup B \subseteq T$. Thus

$$
T' = [n + 2] \setminus \{2\}, \quad T'' = [n + 2] \setminus \{1\}, \quad \text{and} \quad T = [n + 2] \setminus \{1, 2\},
$$

which is a contradiction because the choices of $T, T'$, and $T''$ yield $(T', T) \in A$ and $(T'', T) \not\in A$.

The above discussion shows that $A$ is a matching in $G_1$. Observe that $A$ is homogeneous by Lemma 5.2.

Now we focus on critical vertices. Let $T \subset [n + 2]$ be an $A$-critical vertex of $G_1^A$. Then by (8–11), we have the following cases:

- If $1, 2 \in T$, then $T \not\subseteq V(A)$ if and only if $P_1 \not\subseteq T$ and $P_2 \not\subseteq T$.
- If $1 \in T$, $2 \not\in T$, then $T \not\subseteq V(A)$ if and only if

$$
(P_1 \subseteq T \ \text{or} \ P_2 \not\subseteq T) \ \text{and} \ A \not\subseteq T
$$

$$
\iff (P_1 \subseteq T \ \text{and} \ A \not\subseteq T) \ \text{or} \ (P_2 \not\subseteq T \ \text{and} \ A \not\subseteq T)
$$

$$
\iff (P_1 \cup P_2 \subseteq T \ \text{and} \ A \not\subseteq T) \ \text{or} \ (P_2 \not\subseteq T \ \text{and} \ A \not\subseteq T)
$$

$$
\iff (P_1 \cup P_2 \subseteq T \ \text{and} \ P_0 \not\subseteq T) \ \text{or} \ (P_2 \not\subseteq T \ \text{and} \ A \not\subseteq T)
$$

- If $1 \not\in T, 2 \in T$, then $T \not\subseteq V(A)$ if and only if

$$
B \not\subseteq T \ \text{and} \ P_1 \not\subseteq T.
$$
- If $1 \notin T$, $2 \notin T$, then $T \notin V(A)$ if and only if
  
  $$A \subseteq T \quad \text{and} \quad B \subseteq T.$$  

The above discussion shows that the $A$-critical vertices are exactly the same as in the statement of the theorem. \hfill \Box

The following theorem shows that $A$ is indeed a BW-matching of $G_I$.

**Theorem 5.4** ($A$ is a BW-matching). Let $I = (u_1, u_2, x_1^{e_1}, \ldots, x_n^{e_n})$ be an ideal of $S = \mathbb{K}[x_1, \ldots, x_n]$ as in (4), and $A$ be the homogeneous matching of $G_I$ defined as above. Then

(i) $G_I^A[V(A)]$ is acyclic; 
(ii) for any two $A$-critical vertices $T, T' \in V(G_I^A)$ we have $m_T \neq m_{T'}$. 

In particular, $A$ is a BW-matching on $G_I$.

**Proof.** To see (i), suppose on the contrary that $G_I^A[V(A)]$ has a directed cycle, say $C$. Then $C$ contains a path of length five shown in Figure 4 (left) (see Lemma 3.3), where $j_1, j_2, j_5 \in \{1, 2\}$, $j_2 \notin \{j_1, j_2\}$, $j_4 \notin \{j_3, j_5\}$, $j_3 \notin \{j_1, j_5\}$, and $T \neq [n + 2]$.

![Figure 4.](image)

It follows that $j_5 = j_1$ and $j_3 \notin T$. So $j_3 \in \overline{T} = [n + 2] \setminus T$.

Assume that $j_1 = j_5 = 1$ and $j_3 = 2$. Then $1 \in T$ and $2 \notin T$ (see Figure 4 (right)). By Lemma 5.2 and (3)–(11), we have:

$$((T \cup \{1\}) \setminus \{j_2\}, T \setminus \{j_2\}) \in A \quad \Rightarrow \quad \begin{cases} P_2 \subseteq (T \cup \{2\}) \setminus \{j_2\}, \\ P_1 \subseteq (T \cup \{2\}) \setminus \{j_2\}. \end{cases} \quad (12)$$

Thus

$$P_1 \setminus \{j_2\} \subseteq T \setminus \{j_2\} \quad \text{and} \quad P_1 \subseteq T \setminus \{j_2\},$$

which implies that $j_2 \in P_1$.

Remind that $P_0 \cup P_1 \cup P_2 = X = \{3, \ldots, n + 2\}$, and that $P_0$, $P_1$, and $P_2$ are pairwise disjoint. So, using (12) and considering that $P_0 \cup P_1 \subseteq A \subseteq T$, we must have

$$\overline{T} \setminus \{2\} \subseteq X \setminus (P_0 \cup P_1) \subseteq P_2 \quad \text{and} \quad (\overline{T} \cup \{j_2\}) \setminus \{2\} \subseteq X \setminus P_2 \subseteq P_0 \cup P_1.$$

Therefore

$$(\overline{T} \cup \{j_2\}) \setminus \{2\} \subseteq (P_2 \cup \{j_2\}) \cap (P_0 \cup P_1) = (P_0 \cup P_1) \cap \{j_2\} = \{j_2\}.$$
Example 5.5. Let \( (T \cup \{j_2\}) \setminus \{2\} = \{j_2\} \), hence \( T = \{2\} \). Therefore, \( T = [n+2] \setminus \{2\} \).

By Lemma 5.2, \(((T \cup \{1,2\}) \setminus \{j_2, j_4\}, (T \cup \{2\}) \setminus \{j_2, j_4\}) \in A\). Hence \( P_1 \subseteq (T \cup \{1,2\}) \setminus \{j_2,j_4\} \), which implies that

\[
(T \cup \{j_2,j_4\}) \setminus \{1,2\} \subseteq X \setminus P_1 \subseteq P_0 \cup P_2.
\]

But \( T = \{2\} \), so \( \{j_2,j_4\} \subseteq P_0 \cup P_2 \), that is \( j_2 \in P_0 \cup P_2 \), a contradiction since \( j_2 \in P_1 \).

If \( j_1 = 2 \), an analogous argument shows that \( T = \{1\} \). Thus \( T = [n + 2] \setminus \{1\} \), which is a contradiction (because by (8)–(11) we have \( ([n + 2] \setminus \{1\}, [n + 2] \setminus \{1,2\}) \notin A \)). This settles (i).

(ii) Let \( T, T' \) be two distinct \( A \)-critical vertices of \( G^A_T \) with \( m_T = m_{T'} \). Since \( X = \{3, \ldots, n+2\} \), we observe by (7) that \( T \cap X = T' \cap X \) and hence \( T \) and \( T' \) may only differ in their intersections with \( \{1,2\} \). We check all of the possible scenarios.

- If \( T \cap \{1,2\} = \emptyset \), then by (7) we must have \( X \setminus T = X \setminus T' = \emptyset \), and so \( X \subseteq T \cap T' \). Thus \( T = [n+2] \) is a vertex of \( A \), a contradiction.
- If \( T \cap \{1,2\} = \{1\} \) and \( T' \cap \{1,2\} = \{2\} \), then by (7) for every \( i \) with \( i + 2 \in X \setminus T = X \setminus T' \) we have \( a_i = b_i \), and hence \( X \setminus T = X \setminus T' \subseteq P_0 \). Therefore, \( P_1 \cup P_2 \subseteq T \cap T' \), and so \( P_1 \subseteq T' \), which by (8)–(11) implies that \( T' \) is a vertex of \( A \), a contradiction.
- If \( T \cap \{1,2\} = \{1,2\} \) and \( T' \cap \{1,2\} = \{1\} \), then by (7) for every \( i \) with \( i + 2 \in X \setminus T = X \setminus T' \) we have \( \max\{a_i, b_i\} = a_i \), which means \( X \setminus T = X \setminus T' \subseteq P_0 \cup P_1 \). This implies that \( P_2 \cup \{1,2\} \subseteq T \). Hence by (8)–(11), \( T \) is a vertex of \( A \), a contradiction.

\[\Box\]

**Example 5.5.** Let \( I = (x_1x_2, x_1x_3, x_1^2, x_2^2, x_3^2) \) be as in Example 2.3 and \( A \) be the BW-matching obtained from (8)–(11) as it is shown in Figure 5.
that I

... minimal free resolution of \( \{ \sigma \} \).

Notice that the following two directed paths starting from \( T \) lead to \( T' \). 

Therefore, by Theorem 5.4, there is a CW-complex with exactly five vertices, six 1-dimensional and two 2-dimensional cells which supports a minimal free resolution of \( I \).

To describe this complex, we use Equation (1) and Theorem 5.4. Recall that for subsets \( T, T' \subseteq [n + 2] \) with \( |T| = |T'| + 1 \), we have:

\[
\sigma_{T'} \preceq \sigma_T \iff \begin{cases} 
T' \subseteq T \\
\text{there exists a directed path from } T'' \text{ to } T' \text{ in } G^A_I \\
\text{for some } T'' \subseteq T \text{ with } |T''| = |T'|.
\end{cases}
\]

For example, if we consider \( \sigma_{\{145\}} \), then we have \( \sigma_{T'} \preceq \sigma_T \) for all \( \mathcal{A} \)-critical \( T' \subseteq \{145\} \) with \( |T'| = 2 \). Therefore, \( \sigma_{\{14\}} \preceq \sigma_{\{145\}} \) and \( \sigma_{\{45\}} \preceq \sigma_{\{145\}} \). Moreover, from the following two directed paths starting from \( T'' = \{15\} \subseteq \{145\} \), it follows that \( \sigma_{\{25\}} \preceq \sigma_{\{145\}} \) and \( \sigma_{\{12\}} \preceq \sigma_{\{145\}} \).

Notice that \( \sigma_{\{23\}} \not\preceq \sigma_{\{145\}} \) since \( \{23\} \) is not in a directed path starting from a subset of \( \{145\} \).

Following the same argument, we see that the only CW-complex that supports the minimal free resolution of \( I \) is the one in Figure 6. For being minimal free resolution, observe that \( m_{T'} \neq m_T \) when \( \sigma_{T'} \preceq \sigma_T \) (see Figure 6(right)).

6. Some Algebraic Invariants

We now use the results in Section 5 to study algebraic invariant of the monomial ideal

\[
I = (u_1, u_2, x_1^{e_1}, \ldots, x_n^{e_n}) \text{ where } u_1 = x_1^{a_1} \cdots x_n^{a_n}, \quad u_2 = x_1^{b_1} \cdots x_n^{b_n}.
\]
In Corollary 6.1 we compute the multigraded Betti numbers and the Cohen-Macaulay type of $S/I$. Also, we show, in Theorem 6.3 that if $u_1$ and $u_2$ have disjoints support, then $I$ has a minimal free resolution supported on its Scarf complex. Finally, in Corollary 6.6 we characterize the conditions under which $S/I$ is a level algebra.

Recall that for a Cohen-Macaulay ideal $M$ of projective dimension $\rho$, the number $\beta_\rho(S/M)$ is called the Cohen-Macaulay type of $S/M$.

**Corollary 6.1.** With the notation as in (4) and (5)-(6), let $I = (u_1, u_2, x_1^{e_1}, \ldots, x_n^{e_n})$ and $A$ be the homogeneous matching obtained from (8)-(11). Then

$$\beta_{i,u}(I) = \begin{cases} 1 & \text{if } u \in \{m_T : T \text{ is an $A$-critical vertex of } G^A_T \text{ with } |T| = i + 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the Cohen-Macaulay type of $S/I$ is $|A \cap B| + |P_1||P_2|$.

**Proof.** First of all note that $\dim(S/I) = 0$ so $\text{depth}(S/I) = 0$. Thus $S/I$ is Cohen-Macaulay. Let $\Delta$ be the CW-complex supporting the minimal free resolution of $I$. Then $\beta_{i,u}(I)$ is the number of $i$-faces of $\Delta$ labeled with the monomial $u$.

By definition of $\beta_{i,u}(I)$, we have $\beta_{i,u}(I) = t$ if and only if there exist $T_1, \ldots, T_t$ of $A$-critical vertices of $G^A_T$ with $|T_j| = i + 1$ for $j \in \{1, \ldots, t\}$, and $m_{T_1} = \cdots = m_{T_t} = u$. Theorem 5.4 implies that $m_T \neq m_{T'}$ for any two critical vertices $T \neq T'$. Thus

$$\beta_{i,u}(I) = 1 \text{ for all } u \in \{m_T : T \text{ is an $A$-critical vertex of } G^A_T \text{ with } |T| = i + 1\}.$$  

To obtain the Cohen-Macaulay type of $S/I$, we note that $\text{projdim}(S/I) = n$, so in order to compute $\beta_n(S/I)$, we have to count the number of $A$-critical vertices $T$ with $|T| = n$. By Proposition 5.3 $A$-critical vertices $T \subset \{n + 2\}$ are of the following forms:

- $1 \in T$, $2 \in T$, $P_1 \not\subset T$, $P_2 \not\subset T$ or
- $1 \in T$, $2 \not\in T$, $P_1 \cup P_2 \subseteq T$, $P_0 \not\subset T$, or
- $1 \not\in T$, $2 \not\in T$, $P_2 \not\subset T$, $A \not\subset T$, or
- $1 \not\in T$, $2 \in T$, $P_1 \not\subset T$, $B \not\subset T$, or
- $1 \not\in T$, $2 \not\in T$, $A \not\subset T$, $B \not\subset T$.

So, $A$-critical vertices of size $n$ are exactly one of the following three forms:

- For any pair $(a, b) \in P_1 \times P_2$, $T = \{n + 2\} \setminus \{a, b\}$ is an $A$-critical vertex.
- For any $a \in P_1 \cap B$, $T = \{n + 2\} \setminus \{a, 1\}$ is an $A$-critical vertex.
- For any $a \in (P_2 \cap A) \cup P_0$, $T = \{n + 2\} \setminus \{a, 2\}$ is an $A$-critical vertex.

The above shows that

$$\beta_n(S/I) = |P_1||P_2| + |P_1 \cap B| + |(P_2 \cap A) \cup P_0|$$

$$= |P_1||P_2| + |(P_1 \cap B) \cup (P_2 \cap A) \cup P_0|$$

$$= |P_1||P_2| + |A \cap B|. \quad \square$$

**Remark 6.2.** Corollary 6.1 is not necessarily true when $I$ is an Artinian reduction of a monomial ideal with more than two generators. For example, let

$$I = (x_1^2 x_3, x_1 x_2 x_3 x_4, x_1^2 x_2 x_4, x_1^3, x_2^2, x_3^2, x_4^2)$$
and \( A \) be any acyclic homogeneous matching. Since 
\[
\mathbf{m}_{\{1235\}} = \mathbf{m}_{\{125\}} = \mathbf{m}_{\{135\}} = \mathbf{m}_{\{235\}} = x_1^2 x_2 x_3 x_4,
\]
there are distinct vertices \( u, v \in \{\{125\}, \{135\}, \{235\}\} \) such that \( u, v \notin V(A) \). Hence \( u \) and \( v \) are \( A \)-critical vertices of \( G^A_I \). It follows that \( \beta_{2, x_1^2 x_2 x_3 x_4}(I) > 1 \) (see Figure 7).

It is known from [20] that any multigraded free resolution of \( I \) contains the Scarf multigraded resolution of \( I \), but the Scarf complex is too small to support a resolution itself in general. In the next theorem we show if \( A \cap B = \emptyset \) then Scarf\((I)\) supports the minimal free resolution of \( I \). In this case, in order to explain Scarf\((I)\), we introduce \( d \)-skeleton.

The \( d \)-skeleton \( \Delta^d \) of a simplicial complex \( \Delta \) is the subcomplex of \( \Delta \) consisting of those faces of \( \Delta \) having dimension at most \( d \). In other words, the \( d \)-skeleton of \( \Delta \) is the subcomplex 
\[
\Delta^d = \{ \sigma \in \Delta : |\sigma| \leq d + 1 \}.
\]

**Theorem 6.3.** Let \( I = (u_1, u_2, x_1^{e_1}, \ldots, x_n^{e_n}) \) be as in (4) and (5)–(6), and let \( A \) be as in (8)–(11). If \( A \cap B = \emptyset \), then the set of \( A \)-critical vertices of \( G_I^A \) forms a simplicial complex. In this case, 
\[
\text{Scarf}(I) = \langle P_1 \rangle^{\frac{1}{|P_1|}} \ast \langle P_2 \rangle^{\frac{1}{|P_2|}} \ast \langle \{1, 2\} \rangle
\]
supports the minimal free resolution of \( I \). In particular,
\[
\beta_i(I) = \sum_{a+b+c=i+1} \left( \begin{array}{c} |P_1| \\ a \end{array} \right) \left( \begin{array}{c} |P_2| \\ b \end{array} \right) \left( \begin{array}{c} 2 \\ c \end{array} \right).
\]

**Proof.** Since \( A \cap B = \emptyset \) we have \( P_0 = \emptyset \). That means no variable appears with the same nonzero exponent in both \( u_1, u_2 \). In this case, Scarf\((I)\) supports a minimal free resolution of \( I \) by [2] Theorem 5.6. Utilizing Theorem 5.4 for any two \( A \)-critical vertices \( T, T' \in V(G^A_I) \) we have \( \mathbf{m}_T \neq \mathbf{m}_{T'} \), so the set of \( A \)-critical vertices of \( G^A_I \) coincides with Scarf\((I)\) by the definition of Scarf\((I)\). It turns out that 
\[
\text{Scarf}(I) = \langle T \subset [n + 2] : T \text{ is an } A \text{-critical vertex of } G^A_I \rangle.
\]

Let \( T \) be a facet of the simplicial complex on the right hand side of (13). By Proposition 5.3 we have \( |T| = n \) and \( \{1, 2\} \subseteq T \). So (13) can be read as 
\[
\text{Scarf}(I) = \langle [n + 2] \setminus \{a, b\} : a \in P_1, b \in P_2 \rangle.
\]
Now we just need to prove that
\[ \langle [n+2] \setminus \{a, b\} : a \in P_1, b \in P_2 \rangle = \langle P_1 \rangle_{|P_1| - 2} \ast \langle P_2 \rangle_{|P_2| - 2} \ast \langle 12 \rangle. \]
Let \( U \in \langle P_1 \rangle_{|P_1| - 2}, V \in \langle P_2 \rangle_{|P_2| - 2}, \) and \( S \in \langle 12 \rangle. \) Then there exist \( a, b \) such that \( a \in P_1 \setminus U \) and \( b \in P_2 \setminus V. \) So
\[ S \cup U \cup V \subseteq \langle [n+2] \setminus \{a, b\} : a \in P_1, b \in P_2 \rangle. \]
Conversely, let \( a \in P_1, b \in P_2, \) and \( T' = [n+2] \setminus \{a, b\}. \) Since \( P_1 \cup P_2 \cup \{1, 2\} = [n+2], \) it is evident that \( T' \in \langle P_1 \rangle_{|P_1| - 2} \ast \langle P_2 \rangle_{|P_2| - 2} \ast \langle 12 \rangle. \)
For the last part of the theorem, we know that
\[ \beta_i(I) = \sum_u \beta_{i,u} \]
where \( u \) ranges over all monomials \( m_T \) such that \( T \) is an \( A \)-critical vertex of \( G_I^A \) with \( |T| = i + 1. \) So, to compute \( \beta_i(I), \) we have to count the number of \( A \)-critical vertices of size \( i + 1 \) by Corollary 6.1. Thus,
\[ \beta_i(I) = \sum_{a+b+c=i+1 \atop a < |P_1|, b < |P_2|} \binom{|P_1|}{a} \binom{|P_2|}{b} \binom{2}{c}, \]
as required. \( \square \)

**Example 6.4.** Let \( I = (x_1x_2, x_3x_4, x_1^2, x_2^3, x_3^3, x_4^3). \) Then with notation as in (5)–(6), we have
\[ P_0 = \{ \emptyset \}, \quad P_1 = \{3, 4\}, \quad P_2 = \{5, 6\}. \]
Let \( A \) be the BW-matching obtained from (5)–(11). Then, by Proposition 5.3 the \( A \)-critical vertices of \( G_I^A \) are
\[
\begin{align*}
\{1235\} & \quad \{1236\} & \quad \{1245\} & \quad \{1246\} \\
\{123\} & \quad \{124\} & \quad \{125\} & \quad \{126\} & \quad \{135\} & \quad \{136\} \\
\{145\} & \quad \{146\} & \quad \{235\} & \quad \{236\} & \quad \{245\} & \quad \{246\} \\
\{12\} & \quad \{13\} & \quad \{14\} & \quad \{15\} & \quad \{16\} & \quad \{23\} \\
\{24\} & \quad \{25\} & \quad \{26\} & \quad \{35\} & \quad \{36\} & \quad \{45\} & \quad \{46\} \\
\{1\} & \quad \{2\} & \quad \{3\} & \quad \{4\} & \quad \{5\} & \quad \{6\}
\end{align*}
\]
By Theorem 5.4, there is a CW-complex with exactly six vertices, thirteen 1-dimensional, twelve 2-dimensional and four 3-dimensional cells which support a minimal free resolution of \( I. \) Clearly, this CW-complex is Scarf and
\[ \text{Scarf}(I) = \langle \{3\}, \{4\} \rangle \ast \langle \{5\}, \{6\} \rangle \ast \langle \{1, 2\} \rangle. \]

**Remark 6.5.** The converse of Theorem 6.3 is not true. Indeed, if \( I = (x_1^2x_2, x_1x_2^2, x_1^3, x_2^3), \) one can easily check that
\[ \text{Scarf}(I) = \langle \{12\}, \{13\}, \{24\} \rangle \]
supports a minimal free resolution of \( I \) but \( A \cap B \neq \emptyset. \)
Let $R$ be an Artinian standard graded algebra. Then $R$ is called a level algebra if there is exactly one shift in the last free module in a minimal free resolution of $R$. It turns out that if $I$ is a monomial ideal, then $S/I$ is a level algebra if the last module in the minimal free resolution of $S/I$ is of the form $S^\alpha(m)$ where $\alpha$ is a positive integer and $S^\alpha(m)$ is the free $R$-module with generators in multidegree $m$ where $m$ is a monomial label of the faces of the Taylor complex \((13)\).  

**Corollary 6.6.** Let $I = (u_1, u_2, x_1^{e_1}, \ldots, x_r^{e_n})$ be as in (4) and (5)–(6), and let $A$ be as in (8)–(11). The algebra $S/I$ is a level algebra if and only if there exist constants $\alpha, \beta, \gamma$ such that

(i) $a_i - e_i = \alpha$, for all $i + 2 \in P_1$,
(ii) $b_i - e_i = \beta$, for all $i + 2 \in P_2$,
(iii) $a_i - e_i = b_j - e_j = \gamma$, for all $i + 2 \in (P_2 \cap A) \cup P_0$ and $j + 2 \in P_1 \cap B$,
(iv) $\gamma = \alpha + \beta$.

**Proof.** Let $\mathcal{A}$ be the BW-matching obtained from (8)–(11) and $T \subset [n+2]$ be an $\mathcal{A}$-critical vertex of $G^A_I$. Then $|T| = n$ if and only if $T$ is of one of the following forms:

- For any pair $(i + 2, j + 2) \in P_1 \times P_2$, the set $T = [n + 2] \setminus \{i + 2, j + 2\}$ is an $\mathcal{A}$-critical vertex. In this case, we have
  $$\deg m_T = \sum_{k=1}^{n} e_k + (a_i - e_i) + (b_j - e_j).$$

- For any $i + 2 \in P_1 \cap B$, the set $T = [n + 2] \setminus \{i + 2, 1\}$ is an $\mathcal{A}$-critical vertex. In this case, we have
  $$\deg m_T = \sum_{k=1}^{n} e_k + (b_i - e_i).$$

- For any $i + 2 \in (P_2 \cap A) \cup P_0$, the set $T = [n + 2] \setminus \{i + 2, 2\}$ is an $\mathcal{A}$-critical vertex. In this case, we have
  $$\deg m_T = \sum_{k=1}^{n} e_k + (a_i - e_i).$$

If $S/I$ is a level algebra, then there exist constants $\alpha, \beta, \gamma$, and $\gamma'$ such that

$$a_i - e_i = \alpha \quad \text{for all } i + 2 \in P_1,$$
$$b_i - e_i = \beta \quad \text{for all } i + 2 \in P_2,$$
$$b_i - e_i = \gamma \quad \text{for all } i + 2 \in P_1 \cap B,$$
$$a_i - e_i = \gamma' \quad \text{for all } i + 2 \in (P_2 \cap A) \cup P_0.$$

Note that $P_2 \cap ((P_2 \cap A) \cup P_0) \neq \emptyset$ and $P_1 \cap (P_1 \cap B) \neq \emptyset$. Hence $\gamma' = \alpha + \beta = \gamma$. Conversely, if (i)–(iv) holds for some constants $\alpha, \beta, \gamma$, then all $\mathcal{A}$-critical vertices of $G^A_I$ with size $n$ have the same degrees by the above discussion. Thus $S/I$ is a level algebra. This completes the proof. \qed
References

[1] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, 2020. [https://oeis.org]
[2] Alesandroni, G., Minimal resolutions of dominant and semidominant ideals, *J. Pure Appl. Algebra* **221** (4) (2017), 780–798.
[3] Álvarez Montaner, J., Fernández-Ramos, O. and Gimenez, P., Pruned cellular free resolutions of monomial ideals, *J. Algebra* **541** (2020), 126–145.
[4] Batzies, E., *Discrete Morse theory for cellular resolutions*, Ph.D. Thesis, University of Marburg, (2002).
[5] Batzies, E., Welker, V., Discrete Morse theory for cellular resolutions, *J. Reine Angew. Math.* **543** (2002), 147–168.
[6] Bayer, D., Peeva, I., Sturmfels, B., Monomial resolutions, *Math. Res. Lett.* 5, no. 1-2 (1998) 31–46.
[7] Bayer, D., Sturmfels, B., Cellular resolutions of monomial modules, *J. Reine Angew. Math.* **503** (1998), 123–140.
[8] Chari, M., On discrete Morse functions and combinatorial decomposition, *Discrete Math.* **217** (2000), no. 1-3, 101 - 113.
[9] Cooper, S. M., El Khoury, S., Faridi, S., Mayes-Tang, S., Morey, S., Šega, L. M., Spiroff, S., Morse resolutions of powers of square-free monomial ideals of projective dimension one, *J. Algebraic Comb.* **55** (2022) 1085–1122.
[10] Cooper, S. M., El Khoury, S., Faridi, S., Mayes-Tang, S., Morey, S., Šega, L. M., Spiroff, S., Powers of graphs and applications to resolutions of monomial ideals, *Research in the Mathematical Sciences*, **9** (2022), no. 2, Paper No. 31.
[11] Faridi, S., Hersey, B., Resolutions of monomial ideals of projective dimension 1, *Comm. Algebra* **45** (12) (2017), 5453–5464.
[12] Forman, R., Morse theory for cell complexes, *Adv. Math.* **134** (1) (1998), 90–145.
[13] Geramita. A. V., Level algebras-some remarks, *Rend. Sem. Mat. Univ. Pol. Torino* - Vol. **62**,3 (2004).
[14] Grayson, D.R., Stillman, M.E., *Macaulay2, a software system for research in algebraic geometry*, Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/)
[15] Godsil, C., Royle, G., *Algebraic graph theory*. Graduate Texts in Mathematics, 207. Springer-Verlag, New York, (2001)
[16] Ghorbani, R., *Minimal free resolutions that are supported on a simplicial-complex*, in preparation.
[17] Mermin, J., *Three simplicial resolutions*, Progress in commutative algebra 1, 127–141, de Gruyter, Berlin, (2012).
[18] Miller, E., Sturmfels, B., *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics 227. Springer Science+Business Media, Inc., New York, 2005.
[19] Orlik, P., Welker, V., *Algebraic combinatorics*. Lectures from the Summer School held in Nordfjordeid, June 2003, Universitext. Springer, Berlin, (2007).
[20] Peeva, I., *Graded syzygies*, Algebra and Applications, 14. Springer-Verlag London, Ltd., London, (2011).
[21] Peeva, I., Sturmfels, B., Generic lattice ideals, *J. Amer. Math. Soc.* **11** (2) (1998), 363–373.
[22] Taylor, D., *Ideals generated by monomials in an R-sequence*, Ph.D. Thesis, University of Chicago (1966).
[23] Velasco, M., Minimal free resolutions that are not supported by a CW-complex, *J. Algebra* **319**(1) (2008), 102–114.

Appendix A. Finding a Homogeneous Matching $A$ in $G_I$

In the following, we present an algorithm which enables us to describe $i$-faces of a CW-complex $\Delta$ supporting $I$, where $I$ is an Artinian reduction of a monomial ideal with two generators. Indeed, this algorithm yields a matching $A$ such that the $A$-critical vertices of $G_I$ of size $i$ are in one-to-one correspondence with $i$-faces of $\Delta$. For more information
about this algorithm the reader is referred to [16]. To state our algorithm, we use the following definition.

**Definition A.1.** A subset $T \subseteq [n + 2]$ is admissible set if $T \cap \{1, 2\} \neq \emptyset$ and

(i) if $1 \notin T$ then $P_0 \cup P_2 \cup \{2\} \subseteq T$;
(ii) if $2 \notin T$ then $P_0 \cup P_1 \cup \{1\} \subseteq T$.

Let $\mathcal{A}$ be a homogeneous matching of $G_I$. If $(T, T \setminus \{j\}) \in \mathcal{A}$, then since $m_T = m_{T \setminus \{j\}}$, we must have $j \in \{1, 2\} \cap T$. In the following lemma, we show that the possible candidates for $T$ such that $(T, T \setminus \{j\}) \in \mathcal{A}$ are admissible subsets of $[n + 2]$.

**Lemma A.2.** Let $T \subseteq [n + 2]$ and $j \in T \cap \{1, 2\}$. If $m_T = m_{T \setminus \{j\}}$, then $T$ is admissible.

**Proof.** If $\{1, 2\} \subseteq T$, then $T$ is admissible by definition. If $1 \notin T$, then we must have $2 \in T$, and similarly if $2 \notin T$, we must have $1 \in T$. The rest now follows directly from Lemma 5.2. □
Algorithm 1 Finding a homogeneous matching $\mathcal{M}$ in $G_I$

Require: Monomial ideal $I$

Ensure: A homogeneous matching $\mathcal{A}$ in $G_I$

1: $\mathcal{A}' \leftarrow \{(n + 2), [n + 2 \setminus \{1\}]\}$.
2: $m'_1 \leftarrow \prod_{i+2 \in P_1} x_i^{a_i}, m'_2 \leftarrow \prod_{i+2 \in P_2} x_i^{b_i}$.
3: $T \leftarrow 2^{[n+2]} \setminus \{[n + 2]\}$.
4: $k \leftarrow n + 1$.
5: while $k \geq 3$ do
6: $\mathcal{T}_k \leftarrow \{T \in \mathcal{T} : |T| = k\} \setminus V(\mathcal{A}')$.
7: while $\mathcal{T}_k \neq \emptyset$ do
8: Choose $T \in \mathcal{T}_k$.
9: if $T$ is admissible set then
10: $T^* \leftarrow T \cap \{1, 2\}$.
11: if $T^* = \emptyset$ then
12: if $\gcd(m_T, m'_1) = 1$ then
13: $\mathcal{A}' \leftarrow \mathcal{A}' \cup \{(T, T \setminus \{1\})\}$.
14: else
15: if $\gcd(m_T, m'_2) = 1$ then
16: $\mathcal{A}' \leftarrow \mathcal{A}' \cup \{(T, T \setminus \{2\})\}$.
17: end if
18: end if
19: else
20: if $T^* = \{1\}$ then
21: $\mathcal{A}' \leftarrow \mathcal{A}' \cup \{(T, T \setminus \{2\})\}$.
22: end if
23: if $T^* = \{2\}$ then
24: $\mathcal{A}' \leftarrow \mathcal{A}' \cup \{(T, T \setminus \{1\})\}$.
25: end if
26: end if
27: end if
28: $\mathcal{T}_k \leftarrow \mathcal{T}_k \setminus \{T\}$.
29: end while
30: $k \leftarrow k - 1$.
31: end while
32: return $\mathcal{A} := \{(T, T^*) : ([n + 2] \setminus T, [n + 2] \setminus T^*) \in \mathcal{A}'\}$