Lebesgue Currents

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Abstract

In this paper we introduce the notion of Lebesgue currents. They are a special type of currents involving the Lebesgue measure.

Contents

1 Introduction 1
2 Lebesgue currents 2
3 Smoothing operator $R_\varepsilon$ 10
4 Convergence of regularization of Lebesgue currents 15

1 Introduction

Let $\mathcal{X}$ be a real manifold of dimension $m$ (all manifolds in this paper are oriented, but not necessarily compact). In [1], when one of them has a compact support, de Rham defined the intersection

$$T \wedge \omega$$

between a general current $T$ and a current of a $C^\infty$ form $\omega$, which is expressed as

$$\int_{T \wedge \omega} \bullet = \int_T \omega \wedge \bullet,$$
where $\int_{\mathcal{P}}(\bullet)$ denotes the functional of the current $T$. Is there a useful extension of (1.1) to non-smooth $\omega$? G. de Rham answered this question to some extent and showed important applications in topology. The most important tool is the de Rham’s regularization $R_\epsilon T$ where $\epsilon$ is a small positive number. He showed that $R_\epsilon T$ is a smooth form such that $\lim_{\epsilon \to 0} R_\epsilon T = T$ in the space of currents with some topology. Applying the regularization, he showed that the formula (1.2), when $\omega$ is smooth, becomes

$$
\lim_{\epsilon \to 0} \int_T R_\epsilon(\omega) \wedge \bullet.
$$

(1.3)

Hence formula (1.3) provides an access to the extension. But when $\omega$ is singular, the convergence of (1.3) does not follow from the regularization (see [1]), and it is quite elusive. So to achieve the convergence in his case, G. de Rham created a homotopy formula (3.1) to impose conditions which eliminate the dependence of the regularization $R_\epsilon$ and finally obtain a topological invariant. Our recent work also requires an extension of (1.3), but beyond de Rham’s case. In this paper, we present a different method in obtaining the convergence of (1.3) for singular $\omega$. We show that it, indeed, converges to a finite number, but only for a special type of currents which we call Lebesgue currents, and the limit is a regularization-dependent variant.

2 Lebesgue currents

In example 4.6 below, we’ll show that the convergence of the formula (1.3) can’t come from the de Rham’s regularization $R_\epsilon$. This paper is trying to show it comes from the integrability of Lebesgueness of the currents. So let’s define it first.

**Definition 2.1. (Lebesgue current).**

Let $\mathcal{X}$ be a manifold of dimension $m$. Let $T$ be a homogeneous current of degree $p$. Let $q \in \mathcal{X}$ be a point in $\mathcal{X}$ and $U$ be any coordinates neighborhood of $\mathcal{X}$ containing $q$. For a coordinates system, let $\pi : U \to V_p \simeq \mathbb{R}^{m-p}$ be the

\footnote{In the last 40 years, the convergence of other regularizations has also been studied. But we only study the de Rham’s regularization $R_\epsilon$.}
projection to any $m-p$ dimensional coordinates plane $V_p$. Suppose that there is a neighborhood $B$ of $q$ contained in $U$, and let $T_B$ be a current on $U$ which is equal to $\xi T$ for any $C^\infty$ function $\xi$ of $U$ compactly supported in $B$. Then the point $q$ is called a point of Lebesgue type for $T$ if the following conditions are satisfied,

(a) (The Lebesgue distribution) $\pi_*(T_B)$ (which is well-defined) is given by a signed measure absolutely continuous with respect to the Lebesgue measure of the plane. Furthermore its Radon-Nikodym derivative ([4]) is bounded and Riemann integrable\(^2\) on compact sets of $\mathbb{R}^{m-p}$. The absolute continuity is referring to the bounded support of current $\xi$. This condition means that there is a compactly supported, bounded Riemann integrable function $L$ satisfying

\[
\pi_*(T_B)(\phi) = \int_{\mathbb{R}^{m-p}} \phi L d\mu \tag{2.1}
\]

where $\mu$ is the Lebesgue measure and $\phi$ is a test function.

(b) (The polar function) at each point $a \in V_p$, the function

\[
\lim_{\lambda \to 0} L(\lambda x + a), \text{ for } x, a \in V_p, \lambda \in \mathbb{R}^+.
\tag{2.2}
\]

denoted by

\[
\psi_a(x)
\]

exists almost everywhere as a function of $x$. Furthermore $\psi_a(x)$ as a function in $a, x$, which is bounded and unique almost everywhere, is Riemann integrable on any compact sets.

The function $L$, i.e. Radon-Nikodym derivative in (a) depends on the auxiliary function $\xi$ and will be called the Lebesgue distribution. The function $\psi_a$ in (b) will be called the polar function of $T$ or of $L$. If all points of $\mathcal{X}$ are of Lebesgue type of $T$, we say $T$ is Lebesgue. The collection of Lebesgue currents is denoted by $\mathcal{C}^p(\mathcal{X})$. Let $\mathcal{C}(\mathcal{X}) = \sum_p \mathcal{C}^p(\mathcal{X})$.

**Remark**

(1) The auxiliary function $\xi$ can be replaced by a characteristic function. However it requires a longer build-up with the work of L. Schwartz [5] and P. Billingsley [2].

\(^2\) “Riemann integrable” means that the extension function by a constant to a rectangle containing a compact set is Riemann integrable.
The definition seems to be long and technical. However the idea is so simple that it only requires the functionals of the currents are originated from Lebesgue integrals. To make it more precise, we used two technical ingredients: (a) local existence of the Lebesgue integrable function $L$, (b) a stronger condition on $L$.

**Proposition 2.2.** The definition 2.1 is independent of coordinates charts.

**Proof.** Let $T$ be the homogeneous Lebesgue current of degree $p$ and $(U, x)$ is the coordinates chart in the definition 2.1. Let $y = (y_1, \cdots, y_m)$ be another chart for $U$.

$$dy_1 \cdots \wedge dy_{m-p} = \sum_P h_P(x) dx_{i_1} \cdots \wedge dx_{i_{m-p}}$$

(2.3)

where $P$ is the ordered multi-index $i_1 \cdots i_{m-p}$ and $h_P(x)$ are the minors of the Jacobian matrix. The Lebesgue distribution $\pi_*(\xi h_P(x)T)$ to each plane $\mathbb{R}^{m-p}$ of variables $(x_{i_1}, \cdots, x_{i_{m-p}})$ is denoted by $L_P$. Hence the Lebesgue distribution of $\xi T$ to the $\mathbb{R}^{m-p}$ plane of variables $y_1, \cdots, y_{m-p}$ is

$$\sum_P L_P.$$  

(2.4)

It is easy to see the polar function of

$$\sum_P L_P$$

(2.5)

exists because the polar function for each $L_P$ exists.

**Example 2.3.** Let $c$ be a cell element. Then $c$ is Lebesgue. Furthermore all chains of homogeneous cells are Lebesgue.

To see it, we need to work with one coordinates neighborhood. Assume $c$ is defined as a map

$$\mu : \tilde{\Delta}^{m-p} \to X.$$  

where $\tilde{\Delta}^{m-p}$ is a closed convex polyhedron of dimension $m-p$, and $\mu$ can be smoothly extended to a neighborhood of $\tilde{\Delta}^{m-p}$. Let $U \simeq \mathbb{R}^m$ be a coordinates
neighborhood containing the image $\mu(\Delta^p)$. Then abusing the notations, we denote the chain also by
$$\mu : \Delta^{m-p} \to \mathbb{R}^m,$$
where the image is bounded. Let $V_p$ be any coordinates plane of $\mathbb{R}^m$ and $\pi : \mathbb{R}^m \to V_p$ be the projection.

Next we check the conditions in definition 2.1 for all points.

(a) For a simplicity we may replace $\xi$ by the characteristic function of $c$ (the proof is identical for other smooth $\xi$). For any $C^\infty$ form $\phi$ of $V_p$,
$$\int_{\Delta^{m-p}} \mu^* \circ \pi^*(\phi)$$
is well-defined. Hence $\pi^*(c)$ is well-defined, where $c$ is regarded as $\xi T$ as in the definition 2.1. Let’s find the Lebesgue distribution. The only form $\phi$ that has non-zero $\int_{\Delta^{m-p}} \mu^* \circ \pi^*(\phi)$ is $g(x)\text{vol}(V_p)$ where $g(x)$ is a $C^\infty$ function on $V_p$ and $\text{vol}(V_p)$ is the Euclidean volume form of $V_p$. It is obvious that the positivity of $g$ implies the positivity of $\int_{\Delta^{m-p}} \mu^* \circ \pi^*(\phi)$. This shows $\pi^*(c)$ is a positive distribution on $V_p$. By theorem V in chapter 1, §4, [5], it is a positive measure. Since $\pi^*(c)$ corresponds to a positive measure, it can be extended to a linear functional on the Lebesgue integrable functions of $V_p$. For any characteristic function $\mathcal{X}$ of a set of Lebesgue measure 0, $\pi^*(c)(\mathcal{X}\text{vol}(V_p))$ is well-defined and it is zero. Hence the distribution $\pi^*(c)$ is absolutely continuous with respect to the Lebesgue measure. Applying the Radon-Nikodym theorem there exists the Radon-Nikodym derivative $L$, i.e. the Lebesgue distribution of $c$.

(b) At last we study the property of $L$, i.e. to show the existence of the polar function. Let $S$ be the boundary faces of the polyhedral. The Radon-Nikodym derivative $L$ at almost all points (a.e. or p.p in Lebesgue measure) $x \in V_p$ is defined as the
$$\lim_{r \to 0} \frac{\pi^*(c)B_r}{\mu(B_r)}$$
where $B_r$ is the Euclidean ball in $V_p$ of radius $r$, centered at $q$ and $\mu$ is the Lebesgue measure.

Therefore
$$\psi_x(x) = \lim_{\epsilon \to 0} L(\epsilon x + a) = 0$$
if $a \notin \overline{\pi(\mu(\Delta^p))}$ and
$$\psi_x(x) = \lim_{\epsilon \to 0} L(\epsilon x + a) = 1$$
(2.8)
if \( a \in \pi(\mu(\Delta^p)) \). Most importantly if \( a \in S \), then \( \psi_a(x) \) is a simple function of three rational values (in a suitable coordinates of \( V_p \)) on finitely many either open or closed sets. Therefore \( \psi_a(x) \) is a Riemann integrable function.

Furthermore if \( c \) is a chain of cells, \( c = \sum_i k_i \Delta_i^{m-p} \) where \( \Delta_i^{m-p} \) are convex polyhedrons of dimension \( m-p \) and \( k_i \) are real numbers. Then the Lebesgue distribution of \( c \) is just the sum of the Lebesgue distributions of each \( k_i \Delta_i^{m-p} \). The polar function is also the sum of each polar functions. So we showed the Lebesgueness (definition 2.1) of the currents is linear.

Example 2.4. Let \( \omega \) be a \( C^\infty \) form. Then \( \omega \) is Lebesgue.

Let’s prove it for all points. Assume \( \omega \) has degree \( m-p \). Let \( U \) be a coordinates neighborhood with \( x_1, \cdots, x_p, y_1, \cdots, y_{m-p} \). Assume \( V_p \) has coordinates \( x_1, \cdots, x_p \). For the verification purpose we may assume

\[
\omega|_U = h(x,y)\text{vol}(y)
\]

where \( x, y \) are vectors \( (x_1, \cdots, x_p), (y_1, \cdots, y_{m-p}) \), \( \text{vol}(y) \) is the volume form in the \( y \)-coordinates plane and \( h(x,y) \) is a \( C^\infty \) function. Let \( \xi \) be any function with a compact support. Applying the Fubini’s theorem,

\[
\int_{y \in \mathbb{R}^{m-p}} \xi(x,y)h(x,y)\text{vol}(y)
\]

is a \( C^\infty \) function of \( x \). Let \( \phi = g(x)\text{vol}(x) \) be a form on

\[
V_p = \{ y_1 = \cdots = y_{m-p} = 0 \}
\]

with a compact support. The integral of the product

\[
\int_{(x,y) \in \mathbb{R}^m} \xi(x,y)h(x,y)g(x)\text{vol}(y) \wedge \text{vol}(x)
\]

exists. This shows that

\[
\int_{y \in \mathbb{R}^{m-p}} \xi(x,y)h(x,y)\text{vol}(y)
\]

is the Lebesgue distribution \( L(x) \), which is a \( C^\infty \) function. Then the polar function

\[
\lim_{\epsilon \to 0} L(\epsilon x + a) = \psi_a(x)
\]

is the constant function with constant \( L(a) \).
Example 2.5. There exist currents that are not Lebesgue.

In the Euclidean space $\mathbb{R}^m$ of coordinates $x_1, \cdots, x_p, \cdots, x_m$, we let

$$T = \delta_0 dx_1 \wedge \cdots \wedge dx_p$$

with $\delta$-function $\delta_0$ of the origin $O$ of $\mathbb{R}^m$. Notice $T$ already has a compact support. Let $\pi$ be the projection

$$\pi: \mathbb{R}^m \rightarrow V_p = \mathbb{R}^{m-p} = \{(x_{p+1}, \cdots, x_m)\}. \quad (2.10)$$

For a compactly supported $C^\infty$ function $\xi$ on $V_p$ with value 1 at the origin, the projected distribution $\pi_*(\xi T)$ to the coordinates plane $V_p$ is just the $\delta$-function of the origin. So it is a measure on $V_p$ with Borel $\sigma$-algebra. Now we consider the two measures for $V_p$ on the same $\sigma$-algebra. The Lebesgue measurement of the singleton set, the origin $\{0\} \in V_p$ is zero, but the $\pi_*(\xi T)$ measurement of the set $\{0\}$ is 1. Hence $\pi_*(\xi T)$ compactly near $O$ is not absolutely continuous with respect to the Lebesgue measure. Hence the projected distribution $\pi_*(\xi T)$ is not a Lebesgue distribution, i.e. it fails the condition (a) in definition 2.1. Therefore $T$ is not a Lebesgue current.

Proposition 2.6. If $T$ is Lebesgue and $\omega$ is $C^\infty$, then the intersection

$$T \wedge \omega$$

is Lebesgue.

Proof. Let $T$ be homogeneous of degree $p$ and $\deg(\omega) = r$. First we assume all the local data $q, U, B, V_p, \xi$ (those in the definition 2.1) for the current $T \wedge \omega$. Assume $q$ is the origin of coordinates of $U$. Assume $V_p$ has coordinates $x_1, \cdots, x_r, x_{r+1}, \cdots, x_{m-p}$. We may also assume

$$\omega = g dx_1 \wedge \cdots \wedge dx_r. \quad (2.12)$$

Let $\pi_r$ be the projection

$$U \rightarrow \mathbb{R}^{m-p-r} = \{(x_{r+1}, \cdots, x_{m-p})\}. \quad (2.13)$$
and \( \pi_p \) be the projection

\[
U \to V_p \simeq \mathbb{R}^{m-p}. \tag{2.14}
\]

Since \( T \) is Lebesgue, the Radon-Nikodym derivative of the projection currents \( (\pi_p)_*(g\xi T) \), which is called the Lebesgue distribution, exists and is denoted by

\[
\mathcal{L}_p(x_1, \cdots, x_r, x_{r+1}, \cdots, x_{m-p}). \tag{2.15}
\]

By the definition it has a compact support because \( g\xi \) has a compact support, and the Lebesgue distribution \( (\pi_p)_*(\xi[T \wedge \omega]) \) is given by the function

\[
\int_{(x_1, \cdots, x_r) \in \mathbb{R}^r} \mathcal{L}_p(x_1, \cdots, x_r, x_{r+1}, \cdots, x_{m-p}) dx_1 \wedge \cdots \wedge dx_r. \tag{2.16}
\]

Let’s denote it by \( \mathcal{L}_r(x_{r+1}, \cdots, x_{m-p}) \). Since \( \mathcal{L}_p \) is Riemann integrable on a compact set, so it \( \mathcal{L}_r \). Next we need to consider its polar function, i.e. the limit

\[
\lim_{\epsilon \to 0} \mathcal{L}_r(\epsilon(x_{r+1}, \cdots, x_{m-p}) + a), \tag{2.17}
\]

where \( a \in \mathbb{R}^{m-p-r} = \{(x_{r+1}, \cdots, x_{m-p})\} \). By the formula (2.9), it is equal to

\[
\lim_{\epsilon \to 0} \int_{(x_1, \cdots, x_r) \in \mathbb{R}^r} \mathcal{L}_p \left( x_1, \cdots, x_r, \epsilon(x_{r+1}, \cdots, x_{m-p}) + a \right) dx_1 \wedge \cdots \wedge dx_r. \tag{2.18}
\]

Suppose \( \mathcal{L}_p \) has a support in a ball \( B \) and it is bounded by a constant \( C \). Then

\[
\mathcal{L}_p \left( x_1, \cdots, x_r, \epsilon(x_{r+1}, \cdots, x_{m-p}) + a \right)
\]

is bounded by the constant \( C \) which is Lebesgue integrable on \( B \). Therefore by the dominant convergence theorem,

\[
\lim_{\epsilon \to 0} \int_{(x_1, \cdots, x_r) \in \mathbb{R}^r} \mathcal{L}_p \left( x_1, \cdots, x_r, \epsilon(x_{r+1}, \cdots, x_{m-p}) + a \right) dx_1 \wedge \cdots \wedge dx_r
\]

\[
= \int_{(x_1, \cdots, x_r) \in \mathbb{R}^r} \lim_{\epsilon \to 0} \mathcal{L}_p \left( x_1, \cdots, x_r, \epsilon(x_{r+1}, \cdots, x_{m-p}) + a \right) dx_1 \wedge \cdots \wedge dx_r
\]

\[
= \int_{(x_1, \cdots, x_r) \in \mathbb{R}^r} \psi(x_1, \cdots, x_r, x_{r+1}, \cdots, x_{m-p}) (0, x_{r+1}, \cdots, x_{m-p}) dx_1 \wedge \cdots \wedge dx_r \tag{2.19}
\]
where \( \psi_\bullet(\bullet) \) is the polar function for \( T \). We should note the sub plane \( \{(0,x_{r+1},\cdots,x_{m-p})\} \) in \( \mathbb{R}^m \) is general in the sense of algebraic geometry. Then looking at the formula (2.19), since \( \psi_\bullet(\bullet) \) is Riemann integrable, we obtain that the Fubini’s theorem says

\[
\int_{(x_1,\cdots,x_r)\in\mathbb{R}^r} \psi(x_1,\cdots,x_r,a_{r+1},\cdots,a_{m-p})(0,x_{r+1},\cdots,x_{m-p})dx_1 \wedge \cdots \wedge dx_r
\quad (2.20)
\]
is Riemann integrable in variables \( a_{r+1},\cdots,a_{m-p},x_{r+1},\cdots,x_{m-p} \). Hence the polar function of \( L_r \) is Riemann integrable. This completes the proof. \( \square \)

**Proposition 2.7.** Let \( b \) be the boundary operator of currents. If \( T \) is Lebesgue, so is \( bT \).

**Proof.** We prove it in a neighborhood diffeomorphic to \( \mathbb{R}^m \). Resume all notations in definition 2.1. Let \( \text{dim}(T) = m - p \). Let \( x_1,\cdots,x_{m-p} \) be the coordinates for an \( m - p \) subspace \( V_p \) of \( \mathbb{R}^m \approx U \). Let \( \pi: \mathbb{R}^m \to V_p \) be the projection. Let \( \phi \) be a test function in \( V_p \). Then

\[
\int_{\xi bT} \pi^*(\phi)dx_1 \wedge \cdots \wedge dx_{m-p} = \int_{bT} \xi \pi^*(\phi)dx_1 \wedge \cdots \wedge dx_{m-p}
\quad (2.21)
\]

where \( \frac{\partial \pi^*(\phi)}{\partial x_k} = 0 \) for \( k \not\in \{1,\cdots,m-p\} \). Because \( T \) is Lebesgue, there exist Lebesgue distributions \( L_k(x_0,x_1,\cdots,x_{m-p}) \) of \( \frac{\partial \xi}{\partial x_k} T \) in the subspace

\[
\mathbb{R}^{m-p} \approx \{(x_k,x_1,\cdots,x_{m-p})\},
\]

where we identify \( x_0 = x_k \) for all \( k = m - p + 1,\cdots,m \). Then we obtain the Lebesgue distribution \( \mathcal{L}(x_1,\cdots,x_{m-p}) \) for \( \xi bT \) equals to

\[
\mathcal{L}(x_1,\cdots,x_{m-p}) = \int_{-\infty}^{+\infty} \sum_k \mathcal{L}_k(x_0,x_1,\cdots,x_{m-p})dx_0.
\quad (2.22)
\]

By the dominant convergence theorem and the Fubini’s theorem, since the polar functions for each \( \mathcal{L}_k(x_0,x_1,\cdots,x_{m-p}) \) exists, polar function for

\[
\mathcal{L}(x_1,\cdots,x_{m-p})
\]
also exists. \( \square \)
3 Smoothness operator $R_\epsilon$

We recall the smoothing operators $R_\epsilon$, $A_\epsilon$ defined in chapter III, [1], which involve the Lebesgue measure. Let $\mathcal{X}$ be a manifold of dimension $m$. Denote the space of currents over real numbers by $\mathcal{D}'(\mathcal{X})$, also the subspace of homogeneous currents of degree $i$ (dimension $i$) by $\mathcal{D}^i(\mathcal{X}) (\mathcal{D}'(\mathcal{X}) )$. On a manifold, a test form is a $C^\infty$ form with a compact support in a small neighborhood.

**Definition 3.1. (G. de Rham)**

Let $\epsilon$ be a small positive number. Linear operators $R_\epsilon$ and $A_\epsilon$ on the space of currents $\mathcal{D}'(\mathcal{X})$ are called smoothing operators if they satisfy

1. a homotopy formula

$$R_\epsilon T - T = bA_\epsilon T + A_\epsilon bT.$$  \hspace{1cm} (3.1)

where $b$ is the boundary operator.

2. $\text{supp}(R_\epsilon T), \text{supp}(A_\epsilon T)$ are contained in any given neighborhood of $\text{supp}(T)$ provided $\epsilon$ is sufficiently small, where $\text{supp}$ denotes the support.

3. $R_\epsilon T$ is $C^\infty$;

4. If $T$ is $C^r$, $A_\epsilon T$ is $C^r$.

5. If a smooth differential form $\phi$ varies in a bounded set and $\epsilon$ is bounded above, then $R_\epsilon \phi, A_\epsilon \phi$ are bounded.

6. As $\epsilon \to 0$,

$$R_\epsilon T(\phi) \to T(\phi), A_\epsilon T(\phi) \to 0$$

uniformly on each bounded set $\phi$.

**Theorem 3.2. (G. de Rham)** The operators $R_\epsilon, A_\epsilon$ exist.

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3 de Rham’s regularization has the root in the Schwart’s regularization, §4, chapter VI, [5].
3 SMOOTHING OPERATOR $R_\varepsilon$

Proof. In the following we review the constructions of operators $R_\varepsilon$ and $A_\varepsilon$. The verification of conditions (1)-(6) is contained in [1]. The evaluation of the current $T$ on a test form $\phi$ will be denoted by the symbol

$$\int_T \phi.$$ 

There are three steps in the construction.

Step 1: Local construction. Construction in $X = \mathbb{R}^m$. We denote the operators by $r_\varepsilon$ and $a_\varepsilon$ where $\varepsilon > 0$ is a real number.

Step 2: Preparation for gluing. Convert the operators to the unit ball $B$ in $\mathbb{R}^m$. So operators are denoted by $r^B_\varepsilon, a^B_\varepsilon$ for each $B$.

Step 3: Global gluing. Assume $X$ is covered by open unit balls $B^i$, countable $i$. Regarding $B^i$ as $B$ in step 2, we have operators

$$r^B_i, a^B_i$$

for each $B_i$. Then glue them together to obtain

$$R_\varepsilon, A_\varepsilon$$

on the entire $X$.

Step 1 (construction of $r_\varepsilon$ and $a_\varepsilon$): Let $\mathbb{R}^m$ be the Euclidean space of dimension $m$. Let $x$ be its Euclidean coordinates. Let $T$ be a homogeneous current of degree $p$ on $\mathbb{R}^m$. Let $B$ be the unit ball in $\mathbb{R}^m$. Let $f(x) \geq 0$ be a function of $\mathbb{R}^m$ supported in $B$, satisfying

$$\int_{\mathbb{R}^m} f(x) \text{vol}(x) = 1,$$  \hspace{1cm} (3.3)

and the symmetry,

$$f(-x) \text{vol}(-x) = f(x) \text{vol}(x),$$  \hspace{1cm} (3.4)

where $x$ are the coordinates of $\mathbb{R}^m$ and the volume form

$$\text{vol}(x) = dx_1 \wedge \cdots \wedge dx_m.$$  

In particular $f(x)$ is a bounded function.

Let

$$f^\varepsilon = \varepsilon^{-m} f(\varepsilon^{-1} x).$$
Let
\[ \vartheta_\epsilon(x) = f^\epsilon(x) \text{vol}(x). \] (3.5)
be the \( m \)-form on \( \mathbb{R}^m \). This is the pull-back of \( \vartheta_1(x) \) under the diffeomorphism
\[ \frac{x}{\epsilon} \to x. \]

Next we define two operators on the differential forms of Euclidean space \( \mathbb{R}^m \). Let
\[ s_y(x) \]
be any smooth maps in two variables which is regarded as a family of the maps
\[ \mathbb{R}^m \to \mathbb{R}^m \]
\[ x \to s_y x. \]
parametrized by \( y \in \mathbb{R}^m \). Let \( \phi \) be any test form. For such a two variable map \( s_y(x) \), we denote two operations on the form \( \phi \)
\[ s_y^*(\phi), \quad \text{and} \]
\[ S_y^*(\phi) = \int_{t=0}^1 dt \wedge S_{ty}^*(\phi) \]
where \( S_{ty}^*(\bullet) \) is some homotopy operator associated to the homotopy
\[ [0,1] \times \mathbb{R}^m \to \mathbb{R}^m \]
\[ (t,x) \to s_{ty}(x). \]
See chapter III, \( \S \)14, [1] for the definition.

In the step 1, \( s_y(x) = x+y \). Then operators \( r_\epsilon, a_\epsilon \) have explicit expressions in coordinates. Then we define
\[
\begin{cases}
\int_{r_\epsilon T} \phi = \int_{y \in T} \left( \int_{x \in \mathbb{R}^m} s_x^* \phi(y) \wedge \vartheta_\epsilon(x) \right), \\
\int_{a_\epsilon T} \phi = \int_{y \in T} \left( \int_{x \in \mathbb{R}^m} S_x^* \phi(y) \wedge \vartheta_\epsilon(x) \right).
\end{cases}
\] (3.6)

The differential form
\[ \int_{x \in \mathbb{R}^m} s_x^* \phi(y) \wedge \vartheta_\epsilon(x) \] (3.7)
is denoted by 

\[ r_\epsilon^*(\phi(x)), \quad (3.8) \]

and 

\[ \int_{x \in \mathbb{R}^m} S_\epsilon^* \phi(y) \land \partial_\epsilon(x) \quad (3.9) \]

by 

\[ a_\epsilon^*(\phi(x)). \quad (3.10) \]

Explicitly we use Euclidean coordinates of \( \mathbb{R}^m \) to denote the test form \( \phi = \sum b_{i_1 \ldots i_p} dx_{i_1} \land \cdots \land dx_{i_p} \).

The formulas (3.8) and (3.10) have explicit expressions

\[
 r_\epsilon^*(\phi(x)) = \int_{x \in \mathbb{R}^m} \sum_{i_1 \ldots i_p} b_{i_1 \ldots i_p} (x + y) dy_{i_1} \land \cdots \land dy_{i_p} \land \partial_\epsilon(x) \\
 a_\epsilon^*(\phi(x)) = \int_{x \in \mathbb{R}^m} \sum_{i_1 \ldots i_{p-1}} \int_0^1 x_i b_{i_1 \ldots i_{p-1}} (y + tx) df(\frac{x}{\epsilon}) vol(\frac{x}{\epsilon}) \land dy_{i_1} \land \cdots \land dy_{i_{p-1}}. \quad (3.11)
\]

Let’s elaborate \( r_\epsilon \) more. It was proved on page 65, [1] that the smooth form \( r_\epsilon T \) is calculated as a fibre integral. More precisely let 

\[ \varrho_\epsilon(y, x) = w^m_x \circ A^* \circ s_{-y}(x)^*(\partial_\epsilon) \]

be the double form, where \( w \) is the linearly extended operator on homogeneous forms such that \( w(\phi) = (-1)^{deg(\phi)} \phi \), and \( A^* \) is the converting operator 

\[ single\ form\ on\ a\ product\ of\ two \Rightarrow double\ form.\]

which dependens on the order of the product. Then the current \( r_\epsilon T \) is represented by the smooth form

\[ r_\epsilon T = \int_{y \in T} \varrho_\epsilon(y, x), \quad (3.12) \]

Next we sketch the rest of two steps in the global construction of \( R_\epsilon \).

\textbf{Step 2:} Choose a unit ball \( B \subset \mathbb{R}^m \) diffeomorphic to \( \mathbb{R}^m \). Let \( h \) be a specific diffeomorphism

\[ \mathbb{R}^m \rightarrow B. \]
Let
\begin{align*}
    s_{xy} = \begin{cases}
        hS_xh^{-1}(y) & \text{for } y \in B \\
        y & \text{for } y \notin B
    \end{cases}
\end{align*}

Then we can define the operators $r^B_\epsilon, a^B_\epsilon$ in the same way (with a test form $\phi$):
\begin{equation}
    \begin{cases}
        \int_{r^B_\epsilon T} \phi = \int_{y \in T} T \left( \int_{x \in \mathbb{R}^m} s^*_x \phi(y) \wedge \psi_\epsilon(x) \right), \\
        \int_{a^B_\epsilon T} \phi = \int_{y \in T} T \left( \int_{x \in \mathbb{R}^m} S^*_x \phi(y) \wedge \psi_\epsilon(x) \right).
    \end{cases}
\tag{3.13}
\end{equation}

Then the operators $r^B_\epsilon, a^B_\epsilon$ on $B$ will satisfy
(a) properties (1), (4), (5) and (6) in definition 3.1.
(b) $r^B_\epsilon(T)$ is $C^\infty$ in $B$, $r^B_\epsilon(T) = T$ in the complement of $\bar{B}$;
(c) if $T$ is $C^r$ in a neighborhood of a boundary point of $B$, $r^B_\epsilon(T)$ will have the same regularity in the neighborhood.

Step 3: Cover the $\mathcal{X}$ with countable open sets $B_i$ (locally finite). Now we regard each $B_i$ as $B$ in step 2. Let a neighborhood $U_i$ of $B_i$ deffeomorphic to $\mathbb{R}^m$ as in the step 2. Let $h_i$ be the diffeomorphism to the image
\begin{align*}
    V_i & \to \mathbb{R}^m \\
    B_i & \to B.
\end{align*}

Let $g_i \geq 0$ be a function on $\mathcal{X}$, which is 1 on a compact neighborhood of $B_i$ and zero else where. Let $T' = g_iT$ and $T'' = T - T'$. Then we let
\begin{align*}
    R^i_\epsilon T &= h_i^{-1}r^B_\epsilon h_i T' + T'' \\
    A^i_\epsilon T &= h_i^{-1}a^B_\epsilon h_i T'.
\end{align*}

Finally we glue them together by taking the composition
\begin{align*}
    R^{(h)}_\epsilon &= R^1_\epsilon \cdots R^h_\epsilon, \\
    A^{(h)}_\epsilon &= R^1_\epsilon \cdots R^h_\epsilon A^h_\epsilon.
\end{align*}

Then take the limit as $h \to \infty$ to obtain the global operator $R_\epsilon$ and $A_\epsilon$. \hfill \Box
Lemma 4.1. Let $X = \mathbb{R}^m$ be equipped with de Rham data consisting of single coordinates neighborhood with the bump function $f$. Let $T_1, T_2$ be two Lebesgue homogeneous currents of dimensions $i, j$ with $i + j \geq m$. Let $\phi$ be a test form of degree $i + j - m$. Then

$$\lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge \phi$$

exists.

Proof. The proof is unrelated to the homotopy formula (3.1). Therefore there is no need for the operator $A_\epsilon$. Thus we concentrate on the construction of $R_\epsilon$ only.

Let $x_1, \ldots, x_m$ be the Euclidean coordinates for $\mathbb{R}^m$ and $y_1, \ldots, y_m$ be the same coordinates for another copy of $\mathbb{R}^m$. Let $K, J$ be ordered multi-indexes such that $K \cup J$ is $\{1, 2, \ldots, m\}$ and the lengths are $m - j, j$ respectively. Let $dX_K, dY_J$ be the oriented volume forms for the planes whose coordinates are indexed by $K, J$. The corresponding vectors of coordinates are denoted by $X_K, Y_J$. Let $I$ be an index of length $i + j - m$. Let

$$\phi = \sum_I g_I dX_I.$$  \hspace{1cm} (4.2)

Applying the formula (3.12), we obtain that

$$\int_{T_1} R_\epsilon T_2 \wedge \phi = \pm \int_{(x,y) \in T_1 \times T_2} \vartheta_\epsilon(x - y) \wedge \phi(x)$$ \hspace{1cm} (4.3)

where we used the product current defined by

$$[T_1 \times T_2](\phi) = T_1 T_2(A^*(\phi)),$$

However the homotopy formula (3.1) plays a crucial role in the topological development discussed elsewhere.
through the tensor product $T_1 T_2$. The product current has the natural product Lebesgue distribution and product polar function. So $[T_1 \times T_2]$ is also a Lebesgue current in $\mathbb{R}^m \times \mathbb{R}^m$.

Next we change the coordinates of $\mathbb{R}^m \times \mathbb{R}^m$,

$$(x, y) \rightarrow (x, z).$$

with $z = x - y$. Then

$$\mathbb{R}^m \times \mathbb{R}^m \simeq \mathbb{R}^m_x \times \mathbb{R}^m_z,$$

where the subscriptions are the corresponding coordinates for the Euclidean space $\mathbb{R}^m$ ($\mathbb{R}^m_z$ is orthogonal to the diagonal). Further the formula (4.3) is equal to

$$\int_{T_1} R_1 T_2 \wedge \phi = \pm \int_{[T_1 \times T_2] \wedge \phi} \vartheta_\epsilon(z)$$

Let $B_\epsilon$ be the ball in $\mathbb{R}^m \times \mathbb{R}^m$ containing the support of $\vartheta_1(z) \wedge \phi$. We let $\xi$ be a $C^\infty$ function that is 1 on $B_\epsilon$ and supported on a bounded neighborhood of $B_\epsilon$. Then

$$\int_{T_1} R_1 T_2 \wedge \phi = \pm \int_{\xi[T_1 \times T_2] \wedge \phi} \vartheta_\epsilon(z)$$

where $[T_1 \times T_2]$ is the product current induced from the de Rham’s tensor product. Because $[T_1 \times T_2]$ is Lebesgue, by proposition 2.6, $[T_1 \times T_2] \wedge \phi$ is Lebesgue. So there is a Riemann integrable function $\mathcal{L}(z)$ on $\mathbb{R}^m_z$ such that

$$\int_{\xi[T_1 \times T_2] \wedge \phi} \vartheta_\epsilon(z) = \int_{\mathbb{R}^m} \mathcal{L}(z) \vartheta_1(\frac{z}{\epsilon})$$

where $\vartheta_\epsilon(z) = \vartheta_1(\frac{z}{\epsilon})$. We change the variable $z' = \frac{z}{\epsilon}$ to obtain that

$$\int_{\xi[T_1 \times T_2] \wedge \phi} \vartheta_\epsilon(z) = \int_{z' \in \mathbb{R}^m} \mathcal{L}(\epsilon z') \vartheta_1(z')$$

By the dominant convergence theorem

$$\lim_{\epsilon \rightarrow 0} \int_{z' \in \mathbb{R}^m} \mathcal{L}(\epsilon z') \vartheta_1(z') = \int_{z' \in \mathbb{R}^m} \psi_0(z') \vartheta_1(z')$$

(4.10)
By the Lebesgueness of \([T_1 \times T_2] \wedge \phi\), \(\psi_0(z')\) exists a.e and is integrable on a compact set. Since \(\vartheta_1(z')\) is \(C^{\infty}\),

\[
\int_{z' \in \mathbb{R}^m} \psi_0(z') \vartheta_1(z')
\]

exists.

\(\square\)

**Theorem 4.2.**  
Let \(\mathcal{X}\) be a manifold equipped with de Rham data. Let \(T_1, T_2\) be two homogeneous Lebesgue currents and one of them has a compact support. Let \(\phi\) be a test form of degree \(i + j - m\). Then

\[
\lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge \phi
\]

exists.

**Proof.** Let \(T_1, T_2\) are homogeneous currents of degrees \(i, j\) respectively. To localize it, let \(p_k(x)\) be a partition of unity for the open covering in the de Rham data. Then \(T_2 = \sum_k p_k T_2\) and each \(p_k T_2\) is a current compactly supported on the open set

\[U_k \simeq \mathbb{R}^m.\]

Since one of \(T_1, T_2\) is compact, there will be finitely many such \(k\) contributing to the integral (4.12). Therefore it suffices to show

\[
\int_{T_1} R_\epsilon (p_k T_2) \wedge \phi.
\]

has a limit for each \(k\). For each fixed \(k\), \(p_k T_2\) is compactly supported on \(U_k\). Therefore we may assume \(X = \mathbb{R}^m\) whose de Rham data only consists of one open set \(\mathbb{R}^m\), also \(T_2\) has a compact support in \(\mathbb{R}^m\). Applying the lemma 4.1, we complete the proof.

\(\square\)
Definition 4.3. Let $T_1, T_2$ be homogeneous Lebesgue currents on a smooth manifold. By the theorem 4.2, we define the intersection

$$[T_1 \wedge T_2]$$

by the formula

$$[T_1 \wedge T_2](\phi) = \lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge \phi$$

for a test form $\phi$. Furthermore we linearly extend the definition to non-homogeneous Lebesgue currents.

Proposition 4.4. If $T_1, T_2$ are Lebesgue, so is

$$[T_1 \wedge T_2].$$

Remark The proposition extends the proposition 2.6.

Proof. We may assume $T_i$ are homogeneous of dimensions $i, j$ respectively. Recall just as in the proof of theorem 4.2, it suffices to consider the current $[T_1, T_2]$ in a coordinates chart $\mathbb{R}^m$. Then we’ll adapt all notations in lemma 4.1. As in the local calculation of it, let

$$X_I = (x_{m-j+1}, \cdots, x_i).$$

(4.17)

Let $\mathbb{R}^{i+j-m}_{X_I}$ be the plane of variables $X_I$. Hence

$$\{0\} \times \mathbb{R}^{i+j-m}_{X_I}$$

is a subspace of $\mathbb{R}^m \times \mathbb{R}^m$. Let $\tilde{R}^{i+j-m}$ be a general subspace of dimension $i + j - m$ inside of

$$\mathbb{R}^m \times (\{0\} \times \mathbb{R}^{i+j-m}_{X_I}).$$

(4.18)

Consider the projections diagram

$$\begin{array}{c}
\mathbb{R}^m \times \mathbb{R}^m \\
\downarrow_{\text{Proj}_1} \\
\tilde{R}^{i+j-m} \quad \Downarrow \\
\downarrow_{\text{Proj}_2} \\
\{0\} \times \mathbb{R}^{i+j-m}_{X_I}.
\end{array}$$

(4.19)
As in lemma 4.1, let $\xi$ be a compactly supported $C^\infty$ function of $\mathbb{R}^m \times \mathbb{R}^m$, whose value at the compact set $\text{supp}(T_1 \times T_2)$ is 1. As we proved in lemma 4.1, the current

$$\xi[T_1 \times T_2] \wedge \vartheta$$

as $\epsilon \to 0$ converges weakly to a current, whose local projection to $\mathbb{R}^m$ is $[T_1 \wedge T_2]$. Let’s denote the limit by $T_3$. Then inside the diagram (4.18), we have the diagram of the projections of currents,

$$T_3 \xrightarrow{\text{Proj}_1 \wedge} \xrightarrow{\text{Proj}_2} (\text{Proj}_1)_*T_3 \to (\text{Proj}_2)_*(T_3),$$

Notice $(\text{Proj}_2)_*(T_3)$ is the projection of $[T_1 \wedge T_2]$. Thus it suffices to show $(\text{Proj}_2)_*(T_3)$ satisfies condition (a), (b) in the definition 2.1. Since $l$ is a diffeomorphism, it suffices to prove $(\text{Proj}_1)_*T_3$ is so. Notice $\text{Proj}_1$ is factored through

$$\mathbb{R}^m_z \times \tilde{\mathbb{R}}^{i+j-m}.$$

We denote the coordinates of

$$\mathbb{R}^m_z \times \tilde{\mathbb{R}}^{i+j-m}$$

by $\tilde{z}, \tilde{x}$. The current $T_1 \times T_2$ is Lebesgue. The Lebesgue distribution of it in

$$\mathbb{R}^m_z \times \tilde{\mathbb{R}}^{i+j-m}$$

is denoted by $L_1(\tilde{z}, \tilde{x})$, and its polar function by $\psi^1_{(a,b)}(\tilde{z}, \tilde{x})$, where $(a, b), (\tilde{z}, \tilde{x})$ are two points in

$$\mathbb{R}^m_z \times \tilde{\mathbb{R}}^{i+j-m}.$$

We should note that in the coordinates of

$$\mathbb{R}^m_z \times (\{0\} \times \tilde{\mathbb{R}}^{i+j-m})$$

the Lebesgue distribution $L_1(z, x)$ will be changed by a Jacobian multiple. Notice by the proof in lemma 4.1, we obtain that the Lebesgue distribution of $T_3$, projected to the plane $\tilde{\mathbb{R}}^{i+j-m}$ is

$$L_2(\tilde{x}) = \int_{\mathbb{R}^m} \psi^1_{(0, \tilde{x})}(\tilde{z}, 0) \vartheta_1(\tilde{z}).$$

(4.23)
Since \( \psi^1_{(a,b)}(\tilde{z}, \tilde{x}) \) is Riemann integrable in all variables, so is \( \mathcal{L}_2(\tilde{x}) \) by Fubini’s theorem. This verifies the condition (a) of definition 2.1. Next we concentrate on the polar function to show the limit of

\[
\mathcal{L}_2(\epsilon \tilde{x} + c) = \int_{\tilde{z} \in \mathbb{R}^m} \psi^1_{(0, \epsilon \tilde{z} + c)}(\tilde{z}, 0) \vartheta_1(\tilde{z})
\]

as \( \epsilon \to 0 \) is Riemann integrable. Applying the Riemann-Lebesgue theorem ([4]) in higher dimensions, we obtain that because the Lebesgue distribution \( \mathcal{L}_1(z, x) \) is Riemann integrable, the discontinuity set \( E \) has the Lebesgue measure 0. Since the subspace \( \tilde{\mathbb{R}}^{i+j-m} \) is general (in the sense of algebraic geometry), we obtain that the discontinuity set

\[
E_{\tilde{z}} = E \cap \tilde{\mathbb{R}}^{i+j-m}
\]

has measure 0 in the \( i+j-m \) dimensional Lebesgue measure. Therefore for almost all \( \tilde{x} \in \mathbb{R}^{i+j-m}, (0, \tilde{x}) \) are continuous points of the function \( \mathcal{L}_1(\tilde{z}, \tilde{x}) \).

\[
\lim_{\epsilon \to 0} \mathcal{L}_1(\epsilon \tilde{z}, \tilde{x}) = \mathcal{L}_1(0, \tilde{x}).
\]

We just calculated that for almost all \( \tilde{x} \in \mathbb{R}^{i+j-m},

\[
\mathcal{L}_2(\tilde{x}) = \lim_{\epsilon \to 0} \int_{\tilde{z} \in \mathbb{R}^m} \mathcal{L}_1(\epsilon \tilde{z}, \tilde{x}) \vartheta_1(\tilde{z}) = \int_{\tilde{z} \in \mathbb{R}^m} \mathcal{L}_1(0, \tilde{x}) \vartheta_1(\tilde{z}) = \mathcal{L}_1(0, \tilde{x})
\]

Then at any point \( c \in \mathbb{R}^{i+j-m}, \) the polar function of \( \mathcal{L}_2(\tilde{x}) \) at almost all \( \tilde{x} \in \mathbb{R}^{i+j-m} \) can be calculated using the Lebesgue distribution of the \( T_1 \times T_2 \) as

\[
\lim_{\epsilon \to 0} \mathcal{L}_2(\epsilon \tilde{x} + c) = \lim_{\epsilon \to 0} \mathcal{L}_1(0, \epsilon \tilde{x} + c).
\]

Since the polar function of \( \mathcal{L}_1 \) is Riemann integrable. Then by the Fubini’s theorem, it is Riemann integrable over a general plane of dimension \( i+j-m \). the polar function of \( \mathcal{L}_2(\tilde{x}) \) at \( c \) which equals to

\[
\psi^1_{(0, c)}(0, \tilde{x})
\]

is Riemann integrable because \( \psi^1_{(\cdot)}(\cdot) \) is Riemann integrable in all variables and 0 corresponds to a general subspace. We complete the proof.
Remark
(1) The definition 4.3 is based on the de Rham’s treatment of Kronecker index in [1].

(2) The limiting process in the definition 4.3 is in real analysis, but it has another origin in algebraic geometry as “deformation to the normal cone” ([3]).

Proposition 4.5. (Graded commutativity) Resume all notations in definition 4.3.

\[ [T_1 \wedge T_2] = (-1)^{ij}[T_2 \wedge T_1] \quad (4.29) \]

Proof. We need to prove that

\[
\lim_{\epsilon \to 0} \int_{T_1} R_\epsilon(T_2) \wedge \phi = \lim_{\epsilon \to 0} \int_{T_2} R_\epsilon(T_1) \wedge \phi. \quad (4.30)
\]

Let’s consider the problem in a coordinates neighborhood of de Rham data. Adapt all notations in the proof of the lemma 4.1. We know that in the neighborhood,

\[
\int_{T_1} R_\epsilon T_2 \wedge \phi = \int_{\xi[T_1T_2] \wedge \phi(x)} \varrho_\epsilon(y, x) \quad (4.31)
\]

where the evaluation occurs as a double currents \(T_1 T_2\) in the \(\mathbb{R}^m \times \mathbb{R}^m\), \(x, y\) are coordinates of the first and the second copies. Let

\[
\kappa : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m \quad (x, y) \to (y, x) \quad (4.32)
\]

be the diffeomorphism. Then

\[
\int_{\kappa(\xi[T_1T_2] \wedge \phi(x))} \varrho_\epsilon(x, y) = \int_{\xi[T_1T_2] \wedge \phi(x)} \varrho_\epsilon(y, x). \quad (4.33)
\]

Notice

\[
\int_{\kappa(\xi[T_1T_2] \wedge \phi(x))} \varrho_\epsilon(x, y) = (-1)^{ij} \int_{\xi[T_2T_1] \wedge \phi(y)} \varrho_\epsilon(x, y) \quad (4.34)
\]

(This is due to the order in the double form. See p 51, [1]), So

\[
\int_{\xi[T_1T_2] \wedge \phi(x)} \varrho_\epsilon(y, x) = (-1)^{ij} \int_{\xi[T_2T_1] \wedge \phi(y)} \varrho_\epsilon(x, y) \quad (4.35)
\]
Applying the calculation in lemma 4.1, we take the limits of both sides as $\epsilon \to 0$ to obtain that the left side of (4.35) is

$$\lim_{\epsilon \to 0} \int_{T_1} R_\epsilon(T_2) \wedge \phi;$$

(4.36)

the right side is

$$\lim_{\epsilon \to 0} \int_{T_2} R_\epsilon(T_1) \wedge \phi.$$

(4.37)

This completes the proof.

Example 4.6. In this example we show the necessity of the Lebesgue-ness in the “intersection of currents”, i.e. the formula (1.3) could diverge if there was no assumption of Lebesgue-ness.

Let $\mathcal{X} = \mathbb{R}^m$ be equipped with de Rham data consisting of single open set. Assume it has coordinates $x_1, \ldots, x_m$. Let

$$T_1 = \delta_0 dx_1 \wedge \cdots \wedge dx_p, \quad 0 < p < m$$

with the $\delta$-function $\delta_0$ at the origin 0 of $\mathbb{R}^n$. Let $T_2$ be the $p$ dimensional plane $\{x_{p+1} = \cdots = x_m = 0\}$. Now we consider the integral

$$I_\epsilon = \int_{T_1} R_\epsilon T_2.$$

(4.38)

By the formula (3.12),

$$I_\epsilon = \pm \int_{x \in T_1} \int_{y \in T_2 = \mathbb{R}^p} \frac{1}{\epsilon^m} f\left(\frac{x - y}{\epsilon}\right) dx_{p+1} \wedge \cdots \wedge dx_m \wedge dy_1 \wedge \cdots \wedge dy_p.$$

Now let $T_1$ evaluate at the differential form

$$\frac{1}{\epsilon^m} f\left(\frac{x - y}{\epsilon}\right) dx_{p+1} \wedge \cdots \wedge dx_m$$

to obtain that

$$I_\epsilon = \pm \int_{y \in \mathbb{R}^p} \frac{1}{\epsilon^m} f\left(\frac{-y}{\epsilon}, \ldots, \frac{-y}{\epsilon}, 0, \ldots, 0\right) dy_1 \wedge \cdots \wedge dy_p.$$
Since
\[
\int_{y \in \mathbb{R}^p} \oint_{\epsilon} f\left(\frac{-y_1}{\epsilon}, \cdots, \frac{-y_p}{\epsilon}, 0, \cdots, 0\right) dy_1 \wedge \cdots \wedge dy_p = \pm \int_{y \in \mathbb{R}^p} f(y_1, \cdots, y_p, 0, \cdots, 0) dy_1 \wedge \cdots \wedge dy_p
\] (4.40)
converges to a non-zero number as \( \epsilon \to 0 \), \( I_\epsilon \) diverges to infinity.

Notice that \( T_1 \) as shown in example 2.5 is not Lebesgue. So the non-Lebesgueness is the reason for the divergence.

References

[1] G. de Rham, *Differential manifold*, English translation of “Variétés différentiables”, Springer-Verlag (1980).

[2] P. Billingsley, *Convergence of probability measures*, John Wiley & Sons (1999).

[3] W. Fulton, *Intersection theory*, Springer-Verlag (1980).

[4] H. L. Royden and P. M. Fitzpatrick, *Real Analysis*, 4th Edition(2010), Pearson Education Asia Limited and China Machine Press.

[5] L. Schwartz, *Théorie des distributions*, Hermann, Nouveau tirage (1978).