Cluster Algebras and Scattering Diagrams

Part I
Basics in Cluster Algebras*

Tomoki Nakanishi
Graduate School of Mathematics, Nagoya University

Abstract. This is a first step guide to the theory of cluster algebras. We especially focus on basic notions, techniques, and results concerning seeds, cluster patterns, and cluster algebras.

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0 Introduction to Part I

This is a first step guide to the theory of cluster algebras.

Cluster algebra theory was founded by the series of papers entitled as “Cluster Algebras I–IV” (commonly referred to as CA1–4) by Fomin and Zelevinsky, together with Berenstein [FZ02a, FZ03, BFZ05, FZ07]. These fundamental papers are still the best sources to learn the basic notions, formulations, techniques, important examples, and the perspective on cluster algebras. However, even for someone who seriously wishes to learn cluster algebra theory, reading through all of them, even selectively, at the beginning is not an easy task. This is partly because each of CA1–4 has multiple goals, and some of them are rather advanced and/or specific for the beginners.

Fortunately, several excellent texts/monographs/reviews (including preliminary drafts) are already available to fill the gap. (e.g., [Car06], [Kel10], [GSV10], [Kel13], [Mar13], [Wil14], [GR17], [FZW16], [FZW17], [FZW20], [FZW21]). As another text of the same kind, we especially focus on basic aspects in cluster algebra theory. To be more explicit, we paid attention in the following points while preparing the manuscript.

• Most importantly, the text is aimed for anyone who starts to learn cluster algebras seriously without any preliminary knowledge.
• The text should be read in a self-contained way. Also, it should be reasonably concise.
• Consequently, we focus on (most but not all of) the basic and fundamental aspects of cluster algebra theory, namely, basic notions, techniques, and results concerning seeds, cluster patterns, and cluster algebras in CA1–4.
• We especially employ the formulation of seeds and mutations in CA4 throughout the text, because the structure of mutations is most transparent in our point of view. By the same reason, in some part of proofs, we use the notion of free coefficients (coefficients in a universal semifield) that replaces the role of coefficients of geometric type in CA1–4.
• We try to present elementary manipulations of mutations in the proofs of basic results explicitly, even if many of them are easy or straightforward. This is based on our belief that the essence of cluster algebras is found in these details of mutation mechanism.
• Naturally, we do not try to go deep into a particular topic. In particular, we have to omit the proof of the finite type classification of cluster algebras in CA2, because it requires deep analysis in conjunction with root systems of finite type.
• We add a short section on generalized cluster algebras as extra material beyond CA1–4.
All results presented in this text are well-known, and, except for the last section, they are taken from CA1–4 or easily derived from the results therein unless otherwise mentioned.

Bon voyage!
1 Getting Started

1.1 The very first step: pentagon periodicity

Let us play with a prototypical example of the cluster algebraic structure without explaining the background.

Let $x_1, x_2$ be formal variables. We consider the following recursion, or discrete dynamical system with discrete time $t = 0, 1, 2, \ldots$:

- initial condition: $x_1(0) = x_1, \ x_2(0) = x_2$.
- time development: For even $t$,
  \[
  \begin{align*}
  x_1(t+1) &= \frac{1}{x_1(t)}(x_2(t) + 1), \\
  x_2(t+1) &= x_2(t),
  \end{align*}
  \]

  and, for odd $t$,
  \[
  \begin{align*}
  x_1(t+1) &= x_1(t), \\
  x_2(t+1) &= \frac{1}{x_2(t)}(x_1(t) + 1).
  \end{align*}
  \]

Let us calculate $x_1(t)$ and $x_2(t)$ up to $t = 5$. Anyone who wishes to learn cluster algebras seriously has to work on the following calculations by oneself.

\[
\begin{align*}
  x_1(0) &= x_1, \\
  x_2(0) &= x_2. \\
  x_1(1) &= \frac{x_2 + 1}{x_1}, \\
  x_2(1) &= x_2. \\
  x_1(2) &= \frac{x_2 + 1}{x_1}, \\
  x_2(2) &= \frac{1}{x_2} \frac{x_1 + x_2 + 1}{x_1} = \frac{x_1 + x_2 + 1}{x_1 x_2}.
\end{align*}
\]

So far, nothing special happened. Let us continue.

\[
\begin{align*}
  x_1(3) &= \frac{x_1}{x_2 + 1} \frac{x_1 x_2 + x_1 + x_2 + 1}{x_1 x_2} = \frac{x_1 + 1}{x_2}, \\
  x_2(3) &= \frac{x_1 + x_2 + 1}{x_1 x_2}.
\end{align*}
\]

Now, at the equality with symbol “!” in (1.6), the first nontrivial reduction happens. Let us continue, and we mark the equality with symbol “!” whenever a similar reduction occurs.
After $t = 3$, reductions occur *systematically*, so that at $t = 5$ the result is extremely reduced.

Let us observe the above result more closely.

**Observation 1.1.** (a). *Periodicity.* We have a *half* periodicity at $t = 5$. If we continue calculation, we have a *full* periodicity at $t = 10$.

(b). *Laurent phenomenon.* Thanks to the reduction, each $x_i(t)$ is expressed, not as a general rational function, but as a *Laurent polynomial* in the initial variables $x_1, x_2$ with integer coefficients.

(c). *Laurent positivity.* Moreover, every nonzero coefficient of the above Laurent polynomial is *positive*.

**Remark 1.2.** The property (c) is nontrivial even though the rules (1.1) and (1.2) do not contain any negative coefficients. For example, we have

$$\frac{x^3 + 1}{x + 1} = x^2 - x + 1.$$  \hfill (1.9)

Next, let us consider a similar but different system.

Let $y_1, y_2$ be another formal variables. We consider the following recursion, or discrete dynamical system with discrete time $t = 0, 1, 2, \ldots$:

- **initial condition:** $y_1(0) = y_1, \ y_2(0) = y_2$.
- **time development:** For even $t$,

$$\begin{cases} y_1(t + 1) = y_1(t)^{-1}, \\ y_2(t + 1) = y_2(t)(1 + y_1(t)), \end{cases} \hfill (1.10)$$

and, for odd $t$,

$$\begin{cases} y_1(t + 1) = y_1(t)(1 + y_2(t)), \\ y_2(t + 1) = y_2(t)^{-1}. \end{cases} \hfill (1.11)$$

We calculate up to $t = 5$. As before, we mark the symbol “!” whenever some reduction occurs. Again, it is important to do it oneself.
\[
\begin{align*}
&\{ \begin{array}{l}
y_1(0) = y_1, \\
y_2(0) = y_2,
\end{array} \} \\
&\{ \begin{array}{l}
y_1(1) = y_1^{-1}, \\
y_2(1) = y_2(1 + y_1),
\end{array} \}
\]

(1.12)

\[
\begin{align*}
&\{ \begin{array}{l}
y_1(2) = \frac{1 + y_2 + y_1 y_2}{y_1}, \\
y_2(2) = \frac{1}{y_2(1 + y_1)},
\end{array} \}
\]

(1.13)

\[
\begin{align*}
&\{ \begin{array}{l}
y_1(3) = \frac{y_1}{1 + y_2 + y_1 y_2}, \\
y_2(3) = \frac{\frac{1 + y_1 + y_2}{y_1}}{y_2(1 + y_1)} \overset{!}{=} \frac{1 + y_2}{y_1 y_2},
\end{array} \}
\]

(1.14)

\[
\begin{align*}
&\{ \begin{array}{l}
y_1(4) = \frac{\frac{1 + y_2 + y_1 y_2}{y_1}}{1 + y_2 + y_1 y_2} \overset{!}{=} \frac{1 + y_2}{y_2}, \\
y_2(4) = \frac{y_1 y_2}{1 + y_2},
\end{array} \}
\]

(1.15)

\[
\begin{align*}
&\{ \begin{array}{l}
y_1(5) = y_2, \\
y_2(5) = \frac{y_1 y_2}{1 + y_2} \overset{!}{=} \frac{1 + y_2}{y_2} = y_1.
\end{array} \}
\]

(1.16)

(1.17)

Unlike the previous case, the variables \(y_i(t)\) are not necessarily Laurent polynomials in the initial variables \(y_1, y_2\). Nevertheless, similar reductions systematically occur after \(t = 3\), and the same periodicity is obtained. It is natural to speculate that there is a close relation between two systems.

Indeed this is the simplest example of mutations of cluster variables for \(x_i(t)\) and coefficients for \(y_i(t)\), which we are going to study. The periodicity we observed is the celebrated pentagon periodicity for a cluster algebra of type \(A_2\).

### 1.2 Semifields

We introduce the notion of a semifield. There are some variations of the definition depending on the purpose, and we especially use the one in [FZ03].

**Definition 1.3 (Semifield).** A multiplicative abelian group \(P\) equipped with a binary operation \(\oplus\) is called a **semifield** if the following properties hold:

For any \(a, b, c, \in P\),

\[
\begin{align*}
& a \oplus b = b \oplus a, \\
& (a \oplus b) \oplus c = a \oplus (b \oplus c), \\
& (a \oplus b)c = ac \oplus bc.
\end{align*}
\]

(1.18)

(1.19)

(1.20)
The operation $\oplus$ is called the addition in $P$. Note that there is no subtraction in $P$.

**Example 1.4.** The set of all positive rational numbers $\mathbb{Q}_+$ is a semifield by the usual multiplication and addition. On the other hand, the set of all non-negative rational numbers $\mathbb{Q}_{\geq 0}$ is not a semifield by the usual multiplication and addition, because it is not a multiplicative group due to the presence of 0. Similarly, the set of all positive real numbers $\mathbb{R}_+$ is a semifield by the usual multiplication and addition.

The following three examples are especially important for cluster algebras.

**Example 1.5.** (a). Universal semifield $\mathbb{Q}_{sf}(u)$. Let $u = (u_1, \ldots, u_n)$ be an $n$-tuple of formal variables. Let $Q(u)$ be the rational function field of $u$. We say that a rational function $f(u) \in Q(u)$ has a subtraction-free expression if it is expressed as $f(u) = p(u)/q(u)$, where both $p(u)$ and $q(u)$ are nonzero polynomials in $u$ whose coefficients are nonnegative integers. For example, $f(u) = u_1^2 - u_1 + 1 = (u_1^3 + 1)/(u_1 + 1)$ has a subtraction-free expression. Let $\mathbb{Q}_{sf}(u)$ be the set of all rational functions in $u$ having subtraction-free expressions. Then, $\mathbb{Q}_{sf}(u)$ is a semifield by the usual multiplication and addition in $Q(u)$.

(b). Tropical semifield $\text{Trop}(u)$. Let $u = (u_1, \ldots, u_n)$ be an $n$-tuple of formal variables. Let $\text{Trop}(u)$ be the set of all Laurent monomials in $u$ with coefficient 1, which is a multiplicative abelian group by the usual multiplication. We define the addition $\oplus$ by

$$\prod_{i=1}^{n} u_i^{a_i} \oplus \prod_{i=1}^{n} u_i^{b_i} := \prod_{i=1}^{n} u_i^{\min(a_i, b_i)}.$$  

(1.21)

Then, $\text{Trop}(u)$ becomes a semifield. The addition $\oplus$ is called the tropical sum.

(c). Trivial semifield $1$. Let $1 = \{1\}$ be the trivial multiplicative group. We define the addition by $1 \oplus 1 = 1$. Then, $1$ becomes a semifield.

Let $P$ be any semifield. For any $a \in P$ and any positive integer $m$, we write

$$ma := a \oplus \cdots \oplus a.$$  

(1.22)

Also, any positive integer $m$ is identified with an element of $P$ as $m = m1 \in P$. For example, in the trivial semifield $1$, we have $2 = 1 \oplus 1 = 1$, which is a little confusing. (See Remark 1.12 for another problem of this notation.)

**Definition 1.6** (Semifield homomorphism). For any semifields $P$ and $P'$, a map $\varphi : P \to P'$ is called a semifield homomorphism if it preserves the multiplication and the addition.
1.2. Semifields

**Example 1.7** (Trivial homomorphisms). For any semifield \( \mathbb{P} \), the map

\[
\varphi_{\text{triv}} : \mathbb{P} \to 1, \ a \mapsto 1
\]

(1.23)
is a semifield homomorphism.

The following fact justifies the name “the universal semifield” for \( \mathbb{Q}_{sf}(\mathbf{u}) \).

**Proposition 1.8.** Let \( \mathbb{Q}_{sf}(\mathbf{u}) \) be the universal semifield with \( \mathbf{u} = (u_1, \ldots, u_n) \).

Let \( \mathbb{P} \) be any semifield, and let \( \mathbf{a} = (a_1, \ldots, a_n) \) be any \( n \)-tuple of elements in \( \mathbb{P} \). Then, there is a unique semifield homomorphism \( \pi : \mathbb{Q}_{sf}(\mathbf{u}) \to \mathbb{P} \) such that \( \pi(u_i) = a_i \) for any \( i = 1, \ldots, n \).

**Proof.** First, for any nonzero polynomial \( p(\mathbf{u}) \) with nonnegative integer coefficients, the image \( \pi(p(\mathbf{u})) \) is uniquely determined by replacing \( u_i \) with \( a_i \) and + with \( \oplus \) in \( \mathbb{P} \) in the polynomial \( p(\mathbf{u}) \), where the notation (1.22) is taken into account. The equality \( \pi(p(\mathbf{u})q(\mathbf{u})) = \pi(p(\mathbf{u}))\pi(q(\mathbf{u})) \) is guaranteed by the axiom (1.18)–(1.20). Next, for a given \( f(\mathbf{u}) \in \mathbb{Q}_{sf}(\mathbf{u}) \), take any subtraction-free expression \( f(\mathbf{u}) = p(\mathbf{u})/q(\mathbf{u}) \). Then, the image \( \pi(f(\mathbf{u})) \) is defined by \( \pi(p(\mathbf{u}))/\pi(q(\mathbf{u})) \). To see that it is well-defined, let us take another subtraction-free expression \( f(\mathbf{u}) = p'(\mathbf{u})/q'(\mathbf{u}) \). Then, we have \( p(\mathbf{u})q'(\mathbf{u}) = p'(\mathbf{u})q(\mathbf{u}) \). Thus, \( \pi(p(\mathbf{u}))\pi(q'(\mathbf{u})) = \pi(p'(\mathbf{u}))\pi(q(\mathbf{u})) \). Therefore, we have \( \pi(p(\mathbf{u}))/\pi(q(\mathbf{u})) = \pi(p'(\mathbf{u}))/\pi(q'(\mathbf{u})) \). It is easy to show that \( \pi \) is a semifield homomorphism. \( \square \)

**Definition 1.9** (Specialization). Let \( \mathbb{Q}_{sf}(\mathbf{u}) \), \( \mathbf{a} \in \mathbb{P} \), and \( \pi \) be the ones in Proposition 1.8. For \( f(\mathbf{u}) \in \mathbb{Q}_{sf}(\mathbf{u}) \), the image \( \pi(f(\mathbf{u})) \) is called the **specialization** of \( f(\mathbf{u}) \) at \( \mathbf{a} \) in \( \mathbb{P} \), and denoted by \( f|_{\mathbb{P}}(\mathbf{a}) \).

**Example 1.10** (Tropicalization homomorphism). Let us consider the semifields \( \mathbb{Q}_{sf}(\mathbf{u}) \) and \( \text{Trop}(\mathbf{u}) \) with common formal variables \( \mathbf{u} = (u_1, \ldots, u_n) \).

By applying Proposition 1.8 with \( \mathbb{P} = \text{Trop}(\mathbf{u}) \) and \( \mathbf{a} = \mathbf{u} \), we have a unique semifield homomorphism

\[
\pi_{\text{trop}} : \mathbb{Q}_{sf}(\mathbf{u}) \to \text{Trop}(\mathbf{u})
\]

(1.24)
such that \( \pi_{\text{trop}}(u_i) = u_i \) for any \( i = 1, \ldots, n \). We call it the **tropicalization homomorphism**. For example, for \( \mathbf{u} = (u_1, u_2, u_3) \),

\[
\pi_{\text{trop}} \left( \frac{3u_1u_2^2u_3^2 + 2u_1^2u_2u_3}{3u_2^2 + u_1^2u_2^2 + u_1u_2^3u_3} \right) = \frac{u_1u_2u_3}{u_2^2} = u_1u_2^{-1}u_3.
\]

(1.25)

Roughly speaking, it extracts the “leading Laurent monomial” of a function \( f(\mathbf{u}) \) in \( \mathbb{Q}_{sf}(\mathbf{u}) \).

Since a semifield \( \mathbb{P} \) is a multiplicative abelian group, we can construct the group ring \( \mathbb{Z}\mathbb{P} \) of \( \mathbb{P} \). The addition in \( \mathbb{Z}\mathbb{P} \), denoted by + as usual, should be distinguished from the addition \( \oplus \) in \( \mathbb{P} \).
To construct the field of fractions of the ring \( \mathbb{Z}_P \), the ring \( \mathbb{Z}_P \) should be a domain. This is automatically guaranteed.

**Proposition 1.11.** For any semifield \( P \), the following facts hold.

(a). There is no torsion element in \( P \) other than 1. Namely, for any \( p \in P \), if \( p^n = 1 \) for some positive integer \( n \), then \( p = 1 \).

(b). The group ring \( \mathbb{Z}P \) is a domain. Namely, there is no zero divisor other than 0.

**Proof.** (a). Suppose that \( p^n = 1 \) for some positive integer \( n \). Then, we have

\[
p = \frac{p \oplus p^2 \oplus \cdots \oplus p^n}{1 \oplus p \oplus \cdots \oplus p^{n-1}} = \frac{p \oplus p^2 \oplus \cdots \oplus 1}{1 \oplus p \oplus \cdots \oplus p^{n-1}} = 1. \tag{1.26}
\]

(b). This follows from the known fact that, for any multiplicative abelian group \( G \) with no torsion element other than 1, its group ring \( \mathbb{Z}G \) is a domain (e.g., [May69]). For completeness, let us give a proof. Take any \( a, b \in \mathbb{Z}G \).

They are written as finite sums

\[
a = \sum_i m_i g_i, \quad b = \sum_i m'_i g'_i \quad (g_i, g'_i \in G)
\]

with nonzero coefficients \( m_i, m'_i \in \mathbb{Z} \). Let \( H \) be the subgroup of \( G \) generated by all \( g_i, g'_i \). Then, \( H \) is finitely generated, and we have \( a, b, ab \in \mathbb{Z}H \). By the assumption, \( H \) has no torsion element other than 1. Therefore, thanks to the fundamental theorem of finitely generated abelian groups, we have \( H \cong \mathbb{Z}^n \) for some \( n \). Thus, \( \mathbb{Z}H \) is isomorphic to the Laurent polynomial ring \( \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), which is a domain. Therefore, if \( ab = 0 \), then \( a = 0 \) or \( b = 0 \). \( \square \)

Thanks to Proposition 1.11, the field of fractions of \( \mathbb{Z}_P \) is well-defined, and it is denoted by \( \mathbb{Q}_P \).

**Remark 1.12.** There is some notational conflict between \( P \) and \( \mathbb{Z}_P \) (and also \( \mathbb{Q}_P \)). Namely, for \( a \in P \), \( 2a \), for example, may stand for \( a \oplus a \) in \( P \), or \( a + a \) in \( \mathbb{Z}_P \) depending on the context, and they are distinct. Fortunately, we do not have serious difficulty from this conflict, because usually this notation appears together with \( \oplus \) or \( + \), which clarifies the context. Anyway, we have to be careful.

### 1.3 Matrix and quiver mutations

Below we fix a positive integer \( n \), which is called the *rank* of the forthcoming seeds, cluster patterns, cluster algebras, etc.

**Definition 1.13** (Skew-symmetrizable matrix). An \( n \times n \) integer matrix \( B = (b_{ij})_{i,j=1}^n \) is said to be *skew-symmetrizable* if there is a diagonal matrix \( D = (d_{ij} \delta_{ij})_{i,j=1}^n \) whose diagonal entries \( d_i \) are positive rational numbers such that

\[
B_D = D^{-1}BD = \begin{pmatrix}
d_1 & b_{12} & \cdots & b_{1n} \\
b_{21} & d_2 & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \cdots & d_n
\end{pmatrix}
\]

is skew-symmetric with respect to the diagonal entries in \( D \).


that $DB$ is skew-symmetric, i.e.,

$$d_i b_{ij} = -d_j b_{ji}. \quad (1.27)$$

The matrix $D$ is called a (left) skew-symmetrizer of $B$, which is not unique to $B$. In particular, any skew-symmetric matrix is skew-symmetrizable with a skew-symmetrizer $D = I$.

The condition (1.27) can be rephrased in the matrix notation as

$$DB = -B^T D, \text{ or } DBD^{-1} = -B^T, \quad (1.28)$$

where $B^T$ is the transpose of $B$. By (1.27), we have

$$b_{ii} = 0, \quad (1.29)$$

$$b_{ij} = 0 \iff b_{ji} = 0, \quad (1.30)$$

$$b_{ij} > 0 \iff b_{ji} < 0. \quad (1.31)$$

**Example 1.14.** The following matrices exhaust all $2 \times 2$ skew-symmetrizable matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \pm b \\ \pm a & 0 \end{pmatrix} \quad (a, b \in \mathbb{Z}_{>0}). \quad (1.32)$$

In the former case, any diagonal matrix whose diagonals are positive integers is a skew-symmetrizer. In the latter case, a skew-symmetrizer is given by

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}. \quad (1.33)$$

For any integer $a$, we define

$$[a]_+ := \max(a, 0). \quad (1.34)$$

We have the following useful equalities:

$$a = [a]_+ - [-a]_+, \quad (1.35)$$

$$a[b]_+ + [-a]_+ b = a[-b]_+ + [a]_+ b, \quad (1.36)$$

where (1.36) follows from (1.35).

**Definition 1.15 (Matrix mutation).** For any $n \times n$ skew-symmetrizable matrix $B = (b_{ij})$ and $k \in \{1, \ldots, n\}$, we define a new matrix $B' = \mu_k(B) = (b'_{ij})$ by the following rule:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k, \\ b_{ij} + b_{ik}[b_{kj}]_+ + [-b_{ik}]_+ b_{kj} & i, j \neq k. \end{cases} \quad (1.37)$$

The matrix $\mu_k(B)$ is called the mutation of $B$ in direction $k$. 
Remark 1.16. By the identity \((1.36)\), the second case of \((1.37)\) is also written as

\[
\begin{align*}
b'_{ij} &= b_{ij} + b_{ik}[-b_{kj}]_+ + [b_{ik}]_+ b_{kj} \quad (i, j \neq k).
\end{align*}
\] (1.38)

Remark 1.17. The formula \((1.37)\) is verbally rephrased that \(B'\) is obtained from \(B\) by the following elementary transformations of matrices:

- For each \(j \neq k\), add the \(k\)th column multiplied by \([b_{kj}]_+\) to the \(j\)th column. Also, for each \(i \neq k\), add the \(k\)th row multiplied by \([-b_{ik}]_+\) to the \(i\)th row. (By Remark 1.16, one can simultaneously change the sign of \(b_{kj}\) and \(b_{ik}\) above.)
- Then, multiply \(-1\) to the \(k\)th column and the \(k\)th row. (This is well-defined because \(b_{kk} = 0\).)

Proposition 1.18. Let \(B\) a skew-symmetrizable matrix, and let \(B' = \mu_k(B)\). Then, the following facts hold:

(a). Any skew-symmetrizer of \(D\) is also a skew-symmetrizer of \(B'\). Therefore, \(B'\) is also skew-symmetrizable.

(b). We have \(\det B = \det B'\). In particular, if \(B\) is nonsingular, \(B'\) is also nonsingular.

(c). We have \(B = \mu_k(B')\). Namely, each mutation \(\mu_k\) is involutive.

Proof. (a). For \(i = k\) or \(j = k\), we have

\[
d_i b'_{ij} = -d_i b_{ij} = d_j b_{ji} = -d_j b'_{ji}.
\] (1.39)

For \(i, j \neq k\), we have

\[
d_i b'_{ij} = d_i (b_{ij} + b_{ik}[b_{kj}]_+ + [-b_{ik}]_+ b_{kj})
\]
\[
= -d_j b_{ji} - d_k b_{ki} [b_{kj}]_+ + [d_k b_{ki}]_+ b_{kj}
\]
\[
= -d_j (b_{ji} + b_{ki}[-b_{kj}]_+ + [b_{ki}]_+ b_{kj}) = -d_j b'_{ji}.
\] (1.40)

(b). The verbal version of the mutation in Remark 1.17 preserves the determinant.

(c). Let \(B'' = \mu_k(B')\). For \(i = k\) or \(j = k\), we have

\[
b''_{ij} = -b'_{ij} = b_{ij}.
\] (1.41)

For \(i, j \neq k\),

\[
b''_{ij} = b'_{ij} + b'_{ik}[b_{kj}]_+ + [-b'_{ik}]_+ b_{kj}
\]
\[
= (b_{ij} + b_{ik} [b_{kj}]_+ + [-b_{ik}]_+ b_{kj}) - b_{ik} [-b_{kj}]_+ - [b_{ik}]_+ b_{kj} = b_{ij},
\] (1.42)

where in the last equality we used \((1.35)\) or \((1.36)\).
Example 1.19. Here is an example of a matrix mutation.

\[
B = \begin{pmatrix}
0 & 6 & -3 \\
-12 & 0 & 6 \\
2 & -2 & 0
\end{pmatrix}, \quad B' = \mu_1(B) = \begin{pmatrix}
0 & -6 & 3 \\
12 & 0 & -30 \\
-2 & 10 & 0
\end{pmatrix},
\]

where a common skew-symmetrizer of \( B \) and \( B' \) is given by

\[
D = \begin{pmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 3
\end{pmatrix}.
\]

The matrix mutation is compatible with the matrix decomposition.

Proposition 1.20. Suppose that an \( n \times n \) skew-symmetrizable matrix \( B \) is decomposed as

\[
B = \begin{pmatrix}
B_1 & O \\
O & B_2
\end{pmatrix},
\]

where \( B_1 \) and \( B_2 \) are \( n_1 \times n_1 \) and \( n_2 \times n_2 \) matrices, respectively. Then, we have

\[
\mu_k(B) = \begin{cases}
\begin{pmatrix}
\mu_k(B_1) & O \\
O & B_2
\end{pmatrix} & k = 1, \ldots, n_1, \\
\begin{pmatrix}
B_1 & O \\
O & \mu_k(B_2)
\end{pmatrix} & k = n_1 + 1, \ldots, n.
\end{cases}
\]

Proof. This is clear from the verbal version of the mutation in Remark 1.17. \( \square \)

Any skew-symmetric matrix can be represented by a quiver.

Definition 1.21 (Quiver). A (finite) quiver is a finite directed graph. Namely, it consists of a finite set of vertices and a finite set of arrows between the vertices. The following arrows are called a loop and a 2-cycle, respectively.

Example 1.22. Here is an example of a quiver with a loop and a 2-cycle.
To be more precise, there are two ways to count 2-cycles in this quiver. If we do not distinguish two arrows from the vertex 1 to the vertex 2, there are only one 2-cycle. On the other hand, if we distinguish them, there are two 2-cycles. Here, we employ the former viewpoint. Namely, we only care about the multiplicity of arrows.

For a quiver with \(n\) vertices, we assume that its vertices are labeled with 1, \ldots, \(n\) without duplication. For a skew-symmetric matrix \(B = (b_{ij})\), one can associate a quiver \(Q(B)\) without loops and 2-cycles by the following rule:

- For each matrix element \(b_{ij} > 0\), we assign \(b_{ij}\) arrows from the vertex \(i\) to the vertex \(j\).

Since \(b_{ii} = 0\), there are no loops. Also, there are no 2-cycles, because \(b_{ji} = -b_{ij} < 0\) if \(b_{ij} > 0\).

Conversely, one can recover a skew-symmetric matrix \(B\) from a quiver without loops and 2-cycles by applying the above rule in the opposite direction. It is clear that this correspondence is one-to-one.

One can translate the matrix mutation into the quiver mutation as follows.

**Definition 1.23 (Quiver mutation).** For any quiver \(Q\) without loops and 2-cycles and \(k = 1, \ldots, n\), we define a new quiver \(Q' = \mu_k(Q)\) by the following operation:

- For each pair \(i, j\) (\(i \neq j\)) such that \(i, j \neq k\), if there are \(p > 0\) arrows from the vertex \(i\) to the vertex \(k\), and \(q > 0\) arrows from the vertex \(k\) to the vertex \(j\), then add \(pq\) arrows from the vertex \(i\) to the vertex \(j\).
- Remove the resulting 2-cycles as many as possible.
- Invert all arrows into and out of the vertex \(k\).

The quiver \(Q'\) is called the *mutation of \(Q\) at the vertex \(k\).*

To see the equivalence to the matrix mutation \((1.37)\), we rewrite the second case of \((1.37)\) as follows: For \(i, j \neq k\),

\[
 b'_{ij} = \begin{cases} 
 b_{ij} + b_{ik}b_{kj} & \text{if } b_{ij}, b_{kj} > 0, \\
 b_{ij} - b_{ik}b_{kj} & \text{if } b_{ij}, b_{kj} < 0, \\
 b_{ij} & \text{otherwise.}
\end{cases}
\]  

\((1.47)\)

It is easy to see that the first two operations in Definition 1.23 correspond to \((1.47)\), while the last operation corresponds to the first case of \((1.37)\).

**Example 1.24.** Here is an example of a skew-symmetric matrix \(B\) and the corresponding quiver \(Q(B)\), where the number attached to each arrow
represents the multiplicity of the arrows of the same kind:

\[
B = \begin{pmatrix}
0 & 3 & -2 & 2 \\
-3 & 0 & 4 & 0 \\
2 & -4 & 0 & 1 \\
-2 & 0 & -1 & 0
\end{pmatrix}, \quad Q(B) = \begin{pmatrix}
1 & 3 & 2 \\
2 & 2 & 4 \\
4 & 3
\end{pmatrix}.
\]

The mutation of \(Q(B)\) at the vertex 1 is done as follows:

Putting it back to the matrix form, we obtain

\[
\mu_1(B) = \begin{pmatrix}
0 & -3 & 2 & -2 \\
3 & 0 & -2 & 0 \\
-2 & 2 & 0 & 5 \\
2 & 0 & -5 & 0
\end{pmatrix}.
\]  

(1.48)

The above correspondence between skew-symmetric matrices and quivers is the basis of the connection between cluster algebras and quiver representations. See [Pla18] for a concise survey on the subject.
2 Basic Notions

In this section we introduce basic notions for cluster algebras.

2.1 Seeds and mutations

Let us introduce the most fundamental notions in cluster algebra theory, namely, seeds and mutations.

Recall that, for a given semifield \( \mathbb{P} \), \( \mathbb{Q} \mathbb{P} \) is the field of fractions of the group algebra \( \mathbb{Z} \mathbb{P} \) in Section 1.2.

**Definition 2.1 (Seed/Cluster variable/Coefficient).** Let \( n \) be any positive integer. Let \( \mathbb{P} \) be any semifield, and let \( \mathcal{F} \) be a field that is isomorphic to the rational function field of \( n \)-variables with coefficients in the field \( \mathbb{Q} \mathbb{P} \).

- A (labeled) seed with coefficients in \( \mathbb{P} \) (or a seed in \( \mathcal{F} \)) of rank \( n \) is a triplet \( \Sigma = (x, y, B) \) such that \( x = (x_1, \ldots, x_n) \) is an \( n \)-tuple of algebraically independent and generating elements in \( \mathcal{F} \) (i.e., a transcendence basis of \( \mathcal{F} \)), \( y = (y_1, \ldots, y_n) \) is an \( n \)-tuple of any elements in \( \mathbb{P} \), and \( B = (b_{ij})_{i,j=1}^n \) is an \( n \times n \) skew-symmetrizable (integer) matrix.
- We call \( x \), \( y \), and \( B \), respectively, the cluster, the coefficient tuple, and the exchange matrix of \( \Sigma \). The elements \( x_i \) and \( y_i \), respectively, are called the cluster variables and the coefficients.
- We call \( \mathbb{P} \) and \( \mathcal{F} \), respectively, the coefficient semifield and the ambient field of a seed \( \Sigma \), and also, of the forthcoming cluster patterns, cluster algebras, etc.

**Remark 2.2.** Sometimes, cluster variables \( x_i \) and coefficients \( y_i \) are casually called \( x \)-variables and \( y \)-variables, respectively. Or, we may simply say variable \( x_i \), variable \( y_i \), etc. Alternatively, they are also denoted by \( A_i \) and \( X_i \), and called \( A \)-coordinates (\( A \)-variables) and \( X \)-coordinates (\( X \)-variables), respectively, following the convention of another pioneering works on cluster algebra theory by Fock and Goncharov [FG09a, FG09b].

There are some related notions to Definition 2.1.

**Definition 2.3 (Seed without coefficients/Y-seed).**

- When the coefficient semifield \( \mathbb{P} \) is taken to be the trivial semifield \( 1 \), all coefficients \( y_i \) are 1. Then, we can reduce a triplet \( (x, y, B) \) in Definition 2.1 to a pair \( (x, B) \), which is called a seed without coefficients.
- For any coefficient semifield \( \mathbb{P} \), we have an option to ignore cluster variables and concentrate on a pair \( Y = (y, B) \) in Definition 2.1 which is called a \( Y \)-seed in \( \mathbb{P} \).
For any seed Σ = (x, y, B), we attach an n-tuple \( \hat{y} = (\hat{y}_1, \ldots, \hat{y}_n) \) of elements in \( \mathcal{F} \),

\[
\hat{y}_i = y_i \prod_{j=1}^{n} x_j^{b_{ji}}. \tag{2.1}
\]

They play an important role in the cluster algebra theory. We call them \( \hat{y} \)-variables.

**Definition 2.4** (Seed mutation). For any seed \( \Sigma = (x, y, B) \) in \( \mathcal{F} \) and \( k \in \{1, \ldots, n\} \), we define a new seed \( \Sigma' = (x', y', B') \) in \( \mathcal{F} \) by the following rule:

\[
x'_i = \begin{cases} 
x_k^{-1} \left( \prod_{j=1}^{n} x_j^{[b_{jk}]} \right) \frac{1 + \hat{y}_k}{1 + y_k} & i = k, \\
x_i & i \neq k,
\end{cases} \tag{2.2}
\]

\[
y'_i = \begin{cases} 
y_k^{-1} \left( y_i y_k^{[b_{ki}]} \right) (1 + y_k)^{-b_{ki}} & i = k, \\
y_i y_k^{[b_{ki}]} (1 + y_k)^{-b_{ki}} & i \neq k,
\end{cases} \tag{2.3}
\]

\[
b'_{ij} = \begin{cases} 
-b_{ij} & i = k \text{ or } j = k, \\
b_{ij} + b_{ik} [b_{kj}] + [-b_{ik}] + b_{kj} & i, j \neq k,
\end{cases} \tag{2.4}
\]

where \( \hat{y}_k \) in (2.2) is defined by (2.1). The seed \( \Sigma' \) is called the mutation of \( \Sigma \) in direction \( k \), and denoted by \( \mu_k(\Sigma) = \mu_k(x, y, B) \). The mutations of a seed without coefficients \((x, B)\) and a \( Y \)-seed \((x, B)\) are also defined by the same formulas.

We will soon show that \( \Sigma' \) in the above is indeed a seed in \( \mathcal{F} \).

The mutation of \( B \) in (2.4) is the matrix mutation already introduced in Definition 1.15. Thanks to the identity (1.35), the first case of (2.2) is also written as the following more standard form:

\[
x'_k = x_k^{-1} \left( \frac{y_k}{1 + y_k} \prod_{j=1}^{n} x_j^{[b_{jk}]} \right) + \frac{1}{1 + y_k} \prod_{j=1}^{n} x_j^{-[b_{jk}]} \right). \tag{2.5}
\]

Related with the mutation (2.2), we note the following useful identity:

\[
\frac{1 + \hat{y}_k^{-1}}{1 + y_k^{-1}} = \frac{1 + \hat{y}_k}{1 + y_k} \prod_{j=1}^{n} x_j^{-b_{jk}}. \tag{2.6}
\]

The following fact, together with Proposition 1.18 ensures that \((x', y', B')\) is indeed a seed in \( \mathcal{F} \).
Proposition 2.5. (a). The mutation $\mu_k$ is involutive. Namely, $\mu_k(\Sigma') = \Sigma$ for $\Sigma' = \mu_k(\Sigma)$.

(b). The elements in $x'$ are algebraically independent and generating elements in $\mathcal{F}$.

Proof. (a). Let $(x'', y'', B'') = \mu_k(x', y', B')$ for $(x', y', B')$ in Definition 2.4. $B'' = B$ was already shown in Proposition 1.18 (b). To show $x''_k = x_k$, it is enough to show that $x''_k = x_k$. We first note that

$$
\hat{y}_k' = y_k' \prod_{j=1}^n x_j' b_{jk} = y_k^{-1} \prod_{j=1}^n x_j^{-b_{jk}} = \hat{y}_k^{-1}, \tag{2.7}
$$

where we used the fact $b_{kk} = 0$. Then, we have

$$
x_k'' = x_k'^{-1} \left( \prod_{j=1}^n x_j' [-b_{jk}]_+ \right) \frac{1 + \hat{y}_k'}{1 + y_k} = x_k'^{-1} \left( \prod_{j=1}^n x_j' [b_{jk}]_+ \right) \frac{1 + \hat{y}_k^{-1}}{1 + y_k} \tag{2.8}
$$

where we used (2.6) and (1.35). Let us show $y'' = y$. We have $y''_k = y_k^{-1} = y_k$. For $i \neq k$,

$$
y_i'' = y_i' y_k' [b_{ki}]_+ (1 + y_k')^{-b_{ki}} = (y_i y_k' [b_{ki}]_+ (1 + y_k')^{-b_{ki}}) y_k^{-[b_{ki}]_+ (1 + y_k^{-1}) b_{ki}} = y_i. \tag{2.9}
$$

(b). By (a), $x_1, \ldots, x_n$ are expressed as rational functions of $x'_1, \ldots, x'_n$ over $\mathbb{QP}$. It follows that $x'_1, \ldots, x'_n$ generate $\mathcal{F}$; moreover, they are algebraically independent, because, if not, the transcendence degree of $\mathcal{F}$ over $\mathbb{QP}$ becomes less than $n$, which is a contradiction.

The following fact is the first manifestation of the close relationship (duality) between the mutations of $x$ and $y$.

Proposition 2.6. The $\hat{y}$-variables in (2.1) mutate in the ambient field $\mathcal{F}$ as

$$
\hat{y}_i' = \begin{cases} 
\hat{y}_k^{-1} & i = k, \\
\hat{y}_i y_k' [b_{ki}]_+ (1 + \hat{y}_k^{-1})^{-b_{ki}} & i \neq k,
\end{cases} \tag{2.10}
$$

which is the same rule for the coefficients in (2.3).
Proof. Below (and elsewhere) we use the fact $b_{kk} = 0$ effectively. For $i = k$, it was already shown in (2.7). For $i \neq k$,

$$
\hat{y}'_i = y'_i \prod_{j=1}^{n} x'_j b_{ji},
$$

$$
= y_i y_k^{[b_{ki}]_+} (1 + y_k)^{-b_{ki}} \left( \prod_{j=1}^{n} x_j^{b_{ij} + b_{ijk}[b_{ki}]_+ + [-b_{jk}] + b_{ki}} \right)
$$

$$
\times \left( x_k^{-1} \prod_{j=1}^{n} x_j^{-[b_{jk}]_+} \right) \left( 1 + \hat{y}_k \right)^{-b_{ki}}
$$

$$
= \hat{y}_i y_k^{[b_{ki}]_+} (1 + \hat{y}_k)^{-b_{ki}}.
$$

Proof. For the matrix mutation, this was already pointed out in Remark 1.16. The other cases can be also shown by (1.35) as follows:

$$
\hat{y}'_i = \begin{cases} 
  x_k^{-1} \left( \prod_{j=1}^{n} x_j^{-[\varepsilon b_{jk}]_+} \right) \frac{1 + \hat{y}_k^{\varepsilon}}{1 + \hat{y}_k^{\varepsilon}} & i = k, \\
  x_i & i \neq k,
\end{cases}
$$

(2.12)

$$
y'_i = \begin{cases} 
  y_k^{-1} \frac{1 + \hat{y}_k^{\varepsilon} }{1 + \hat{y}_k^{\varepsilon}} & i = k, \\
  y_i y_k^{[\varepsilon b_{ki}]_+} (1 + y_k)^{-b_{ki}} & i \neq k,
\end{cases}
$$

(2.13)

$$
b'_{ij} = \begin{cases} 
  -b_{ij} & i = k \text{ or } j = k, \\
  b_{ij} + b_{ik}[\varepsilon b_{kj}]_+ + [-\varepsilon b_{kj}]_+ b_{kj} & i, j \neq k,
\end{cases}
$$

(2.14)

$$
\hat{y}'_i = \begin{cases} 
  \hat{y}_k^{-1} \frac{1 + \hat{y}_k^{\varepsilon}}{1 + \hat{y}_k^{\varepsilon}} & i = k, \\
  \hat{y}_i \hat{y}_k^{[\varepsilon b_{ki}]_+} (1 + \hat{y}_k^{\varepsilon})^{-b_{ki}} & i \neq k.
\end{cases}
$$

(2.15)

Proof. For the matrix mutation, this was already pointed out in Remark 1.16. The other cases can be also shown by (1.35) as follows:

$$
\left( \prod_{j=1}^{n} x_j^{-[b_{jk}]_+} \right) \frac{1 + \hat{y}_k}{1 + \hat{y}_k} = \left( \prod_{j=1}^{n} x_j^{-b_{jk}} \right) \hat{y}_k y_k^{-1} = 1,
$$

(2.16)

$$
\frac{y_k^{[b_{ki}]_+}}{y_k^{-[b_{ki}]_+}} \frac{(1 + y_k)^{-b_{ki}}}{(1 + y_k^{-1})^{-b_{ki}}} = y_k^{b_{ki}} y_k^{-b_{ki}} = 1.
$$

(2.17)

The case (2.15) is similar.
We continue to give related notions for seeds.

**Definition 2.8 (\(S_n\)-action/Unlabeled seed).**

- For a seed \(\Sigma = (x, y, B)\) in \(\mathcal{F}\) and a permutation \(\sigma\) of \(\{1, \ldots, n\}\), we define the action of \(\sigma\) on \(\Sigma\) by
  \[
  \sigma \Sigma = (\sigma x, \sigma y, \sigma B),
  \]
  where \(\sigma x = x', \sigma y = y', \sigma B = B'\) are defined by
  \[
  x'_i = x_{\sigma^{-1}(i)}, \quad y'_i = y_{\sigma^{-1}(i)}, \quad b'_{ij} = b_{\sigma^{-1}(i)\sigma^{-1}(j)}.
  \]
  Clearly, \(\sigma \Sigma\) is a seed in \(\mathcal{F}\), and this yields a left action of the symmetric group \(S_n\) of degree \(n\) on the set of seeds in \(\mathcal{F}\); namely, we have \(\tau(\sigma \Sigma) = \tau \sigma (\Sigma)\) for \(\sigma, \tau \in S_n\). This also induces the action of \(\sigma\) on \(\hat{y}\)-variables \(\sigma \hat{y} = \hat{y}'\) as
  \[
  \hat{y}'_i := y'_i \prod_{j=1}^n x'_j b'_{ji} = y_{\sigma^{-1}(i)} \prod_{j=1}^n x_{\sigma^{-1}(j)} = \hat{y}_{\sigma^{-1}(i)}.
  \]
- We introduce an equivalence condition for (labeled) seeds in \(\mathcal{F}\),
  \[
  \Sigma' \sim \Sigma
  \]
  if there is some permutation \(\sigma \in S_n\) such that \(\Sigma' = \sigma \Sigma\). Then, each equivalence class \([\Sigma]\) is called an unlabeled seed in \(\mathcal{F}\).

The mutations and the action of \(\sigma\) is compatible in the following sense.

**Proposition 2.9.** The following equality holds:
  \[
  \mu_{\sigma(k)}(\sigma \Sigma) = \sigma(\mu_k(\Sigma)).
  \]

**Proof.** We set the left and right hand sides as \((x', y', B')\) and \((x'', y'', B'')\), respectively. We calculate \((x', y', B')\) below, which turns out to coincide with \((x'', y'', B'')\).

\[
x'_i = \begin{cases} 
  x_k^{-1} \left( \prod_{j=1}^n x_{\sigma^{-1}(j)}^{[-b_{\sigma^{-1}(j)k}]+} \right) \frac{1 + \hat{y}_k}{1 \oplus y_k} & i = \sigma(k), \\
  x_{\sigma^{-1}(i)} & i \neq \sigma(k),
\end{cases}
\]

\[
y'_i = \begin{cases} 
  y_k^{-1} \left( y_{\sigma^{-1}(i)} y_k^{b_{k\sigma^{-1}(i)}+} (1 \oplus y_k)^{b_{k\sigma^{-1}(i)}-}\right) & i = \sigma(k), \\
  y_{\sigma^{-1}(i)} y_k^{b_{k\sigma^{-1}(i)}+} (1 \oplus y_k)^{b_{k\sigma^{-1}(i)}-} & i \neq \sigma(k),
\end{cases}
\]

\[
b'_{ij} = \begin{cases} 
  -b_{\sigma^{-1}(i)\sigma^{-1}(j)} & i = \sigma(k) \text{ or } j = \sigma(k), \\
  b_{\sigma^{-1}(i)\sigma^{-1}(j)} + b_{\sigma^{-1}(i)k} b_{k\sigma^{-1}(j)} & i, j \neq \sigma(k).
\end{cases}
\]
Two mutations are not commutative, in general. However, under some simple condition, they are commutative.

**Proposition 2.10.** For a seed $\Sigma = (x, y, B)$ and a pair $k, \ell (k \neq \ell)$, suppose that

$$b_{k\ell} = b_{\ell k} = 0$$

holds. Then, we have

$$\mu_k \mu_\ell (\Sigma) = \mu_\ell \mu_k (\Sigma),$$

or equivalently,

$$\mu_\ell \mu_k \mu_\ell \mu_k (\Sigma) = \Sigma.$$

**Proof.** We set $\Sigma' = \mu_k (\Sigma)$ and $\Sigma'' = \mu_\ell (\Sigma')$, and we show that $\Sigma''$ is symmetric with respect to $k$ and $\ell$. By (2.26), we have

$$x_i' = \begin{cases} x_k^{-1} \left( \prod_{j=1}^{n} x_j^{[-b_{jk}]+} \right) \frac{1 + \hat{y}_k}{1 + y_k} & i = k, \\ x_i & i \neq k, \end{cases}$$

(2.29)

$$y_i' = \begin{cases} y_k^{-1} & i = k, \\ y_\ell & i = \ell, \\ y_i y_k^{[b_{ki}]+} (1 + y_k)^{-b_{ki}} & i \neq k, \ell, \end{cases}$$

(2.30)

$$b_{ij}' = \begin{cases} -b_{ij} & i = k \text{ or } j = k, \\ b_{ij} & i = \ell \text{ or } j = \ell, \\ b_{ij} + b_{ik} [b_{kj}]+ + [-b_{ik}]_{+} b_{kj} & \text{otherwise}. \end{cases}$$

(2.31)

These also imply that

$$\hat{y}_\ell' = \hat{y}_\ell.$$

(2.32)

Then, we have

$$x_i'' = \begin{cases} x_k^{-1} \left( \prod_{j=1}^{n} x_j^{[-b_{jk}]+} \right) \frac{1 + \hat{y}_k}{1 + y_k} & i = k, \\ x_\ell^{-1} \left( \prod_{j=1}^{n} x_j^{[-b_{j\ell}]+} \right) \frac{1 + \hat{y}_\ell}{1 + y_\ell} & i = \ell, \\ x_i & i \neq k, \ell, \end{cases}$$

(2.33)

$$y_i'' = \begin{cases} y_k^{-1} & i = k, \\ y_\ell^{-1} & i = \ell, \\ y_i y_k^{[b_{ki}]+} y_\ell^{[b_{\ell i}]+} (1 + y_k)^{-b_{ki}} (1 + y_\ell)^{-b_{\ell i}} & i \neq k, \ell, \end{cases}$$

(2.34)
2.2 Cluster patterns and cluster algebras

We introduce cluster patterns and cluster algebras, which are the main objects to be studied in cluster algebra theory.

Definition 2.11 \((n\text{-Regular tree})\).

- Let \(T_n\) denote the \(n\text{-regular tree}\); namely, it is a tree graph such that each vertex has exactly \(n\) edges attached to it. Moreover, the edges are labeled by 1, \ldots, \(n\) so that the edges attached to each vertex are labeled without duplication. By abusing the notation, the set of vertices of \(T_n\) is also denoted by \(T_n\).
- We say that a pair of vertices \(t\) and \(t'\) in \(T_n\) are \(k\text{-adjacent}\), or \(t'\) is \(k\text{-adjacent to} t\), if they are connected with an edge labeled by \(k\).

The graph \(T_n\) is finite for \(n = 1\) and infinite otherwise.

Example 2.12. Here are the \(n\text{-regular trees} T_n\) for \(n = 1, 2, 3\).

\[
\begin{array}{c}
1 \\
n = 1 \\
2 1 2 1 \\
n = 2 \\
\cdots 2 1 2 1 \cdots \\
\end{array}
\quad
\begin{array}{c}
1 2 3 1 2 3 1 \\
n = 3 \\
\cdots 2 1 2 \cdots \\
\end{array}
\]

Definition 2.13 \((\text{Cluster pattern/Y-pattern/B-pattern})\).

- A collection of seeds \(\Sigma = \{\Sigma_t = (x_t, y_t, B_t)\}_{t \in T_n}\) with coefficients in \(\mathbb{P}\) indexed by \(T_n\) is called a \textit{cluster pattern with coefficients in} \(\mathbb{P}\) if, for any pair \(t, t' \in T_n\) that are \(k\text{-adjacent}\), the equality \(\Sigma_{t'} = \mu_k(\Sigma_t)\) holds. A cluster pattern is also called a \textit{seed pattern}.
- We replace the above \(\Sigma\) with a collection of seeds \(\Sigma = \{\Sigma_t = (x_t, B_t)\}_{t \in T_n}\) without coefficients. Then, it is called a \textit{cluster pattern without coefficients}.
- We replace the above \(\Sigma\) with a collection of \(Y\text{-seeds} Y = \{Y_t = (y_t, B_t)\}_{t \in T_n}\) in \(\mathbb{P}\). Then, it is called a \textit{Y-pattern in}\( \mathbb{P}\).
2.2. Cluster patterns and cluster algebras

From the above $\Sigma$, we extract a collection of the exchange matrices $\mathbf{B} = \{B_t\}_{t \in T_n}$. We call it the $B$-pattern of $\Sigma$.

Often, it is convenient to choose arbitrarily a distinguished vertex $t_0 \in T_n$ called the initial vertex. Since any cluster pattern $\Sigma = \{\Sigma_t = (x_t, y_t, B_t) \mid t \in T_n\}$ is uniquely determined from the initial seed $\Sigma_{t_0}$ at $t_0$ by repeating mutations, we may write $\Sigma = \Sigma(\Sigma_{t_0})$.

For a seed $\Sigma_t = (x_t, y_t, B_t)$ in a cluster pattern $\Sigma$, we use the notation

$$x_t = (x_{1;t}, \ldots, x_{n;t}), \quad y_t = (y_{1;t}, \ldots, y_{n;t}), \quad B_t = (b_{ij;t})_{i,j=1}^n. \quad (2.36)$$

Often, we omit the index $t_0$ for the initial seed as $x_{t_0} = x = (x_1, \ldots, x_n)$, $y_{t_0} = y = (y_1, \ldots, y_n)$, $B_{t_0} = B = (b_{ij})_{i,j=1}^n$. \quad (2.37)

We use similar notations for $\hat{y}$-variables as well:

$$\hat{y}_t = (\hat{y}_{1;t}, \ldots, \hat{y}_{n;t}), \quad \hat{y}_{t_0} = \hat{y} = (\hat{y}_1, \ldots, \hat{y}_n). \quad (2.38)$$

By fixing the initial vertex, one may regard each coefficient $y_{i;t}$ as a rational function with a subtraction-free expression in the initial coefficients $y$, since the mutations of coefficients in $(2.3)$ are subtraction-free. Similarly, one may regard each cluster variable $x_{i;t}$ as a rational function with a subtraction-free expression in the initial cluster variables $x$; moreover, its coefficients are rational functions with subtraction-free expressions in $y$.

Now we give the definition of a cluster algebra.

**Definition 2.14** (Cluster algebra). For any cluster pattern $\Sigma$, the cluster algebra $\mathcal{A} = \mathcal{A}(\Sigma)$ associated with $\Sigma$ is the $\mathbb{Z}P$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables $x_{i;t}$ ($i = 1, \ldots, n; t \in T_n$) of $\Sigma$.

If $\Sigma$ has only finitely many distinct cluster variables, $\mathcal{A}(\Sigma)$ is clearly finitely generated. On the other hand, if $\Sigma$ has infinitely many distinct cluster variables, $\mathcal{A}(\Sigma)$ may be finitely generated or not, depending on $\Sigma$.

**Example 2.15** (Type $A_1$). Let $n = 1$. We consider the following arrangement of a cluster pattern $\Sigma$ on $T_1$.

![Diagram of T_1 with two vertices labeled t_0 and t_1 connected by an arrow labeled Sigma_t_0 to Sigma_t_1.]

We have the unique choice of the exchange matrices

$$B_{t_0} = B_{t_1} = (0). \quad (2.39)$$

Let $t_0$ be the initial vertex, and we set $x_{1;t_0} = x_1$ and $y_{1;t_0} = y_1$. We have $\hat{y}_{1;t_0} = \hat{y}_1 = y_1$. Accordingly, we have

$$x_{1;t_1} = x_1^{-1} \frac{1 + y_1}{1 \oplus y_1}, \quad y_{1;t_1} = y_1^{-1}. \quad (2.40)$$
One can directly confirm the involution property $\mu_1(\Sigma_{t_1}) = \Sigma_{t_0}$ as follows:

$$x_{1;1}^{-1} \frac{1 + y_1^{-1}}{1 \oplus y_1^{-1}} = x_1^{-1} \frac{1 + y_1^{-1}}{1 + y_1 \frac{1 + y_1^{-1}}{1 \oplus y_1^{-1}}} = x_1, \quad y_{1;1}^{-1} = y_1.$$  \hspace{1cm} (2.41)

The associated cluster algebra $\mathcal{A}(\Sigma)$ is the $\mathbb{ZP}$-algebra generated by

$$x_1, \quad x_1^{-1} \frac{1 + y_1}{1 \oplus y_1}.$$  \hspace{1cm} (2.42)

This is called a cluster algebra of type $A_1$, which depends on the choice of a semifield $\mathbb{P}$ and also the initial coefficient $y_1$.

**Example 2.16** (Type $A_1 \times A_1$). Let $n = 2$. We consider the following arrangement of a cluster pattern on $\mathbb{T}_2$.

$$\ldots \Sigma_{t-1} \Sigma_{t_0} \Sigma_{t_1} \Sigma_{t_2} \Sigma_{t_3} \ldots$$

$$1 \quad t-1 \quad 2 \quad t_0 \quad 1 \quad t_1 \quad 2 \quad t_2 \quad 1 \quad t_3 \quad 2 \quad \ldots$$

Below we use the simplified notations such as $\Sigma_{t_s} = \Sigma_s$, $B_{t_s} = B_s$, $x_{t_s} = x_s$, $y_{t_s} = y_s$, etc. Let $t_0$ be the initial vertex, and we set $x_0 = x, y_0 = y$. Again, we consider the simplest case

$$B_0 = B = O,$$  \hspace{1cm} (2.43)

so that the initial $\hat{y}$ variables are given by

$$\hat{y}_1 = y_1, \quad \hat{y}_2 = y_2.$$  \hspace{1cm} (2.44)

For any $s \in \mathbb{Z}$, we have

$$B_s = O.$$  \hspace{1cm} (2.45)

Accordingly, we have

\[
\begin{align*}
x_{1;1} &= x_1^{-1} \frac{1 + y_1}{1 \oplus y_1}, & y_{1;1} &= y_1^{-1}, \\
x_{2;1} &= x_2, & y_{2;1} &= y_2, \\
x_{1;2} &= x_1^{-1} \frac{1 + y_1}{1 \oplus y_1}, & y_{1;2} &= y_1^{-1}, \\
x_{1;2} &= x_2^{-1} \frac{1 + y_2}{1 \oplus y_2}, & y_{2;2} &= y_2^{-1}, \\
x_{1;3} &= x_1, & y_{1;3} &= y_1, \\
x_{2;3} &= x_2^{-1} \frac{1 + y_2}{1 \oplus y_2}, & y_{2;3} &= y_2^{-1}, \\
x_{1;4} &= x_1, & y_{1;4} &= y_1, \\
x_{1;4} &= x_2, & y_{2;4} &= y_2.
\end{align*}
\]  \hspace{1cm} (2.46-2.49)
where we did the same calculation as (2.41) in the last two steps. We observe the periodicity with period 4. Namely,
\[ x_{s+4} = x_s, \quad y_{s+4} = y_s, \quad B_{s+4} = B_s \quad (s \in \mathbb{Z}). \] (2.50)

In other words, \( \Sigma_{s+4} = \Sigma_s \). This is regarded as a special case of (2.28). The cluster algebra \( \mathcal{A}(\Sigma) \) is the \( \mathbb{ZP} \)-algebra generated by
\[ x_1, \quad x_2, \quad x_1^{-1} \frac{1 + y_1}{1 + y_1}, \quad x_2^{-1} \frac{1 + y_2}{1 + y_2}. \] (2.51)

This is called a cluster algebra of type \( A_1 \times A_1 \).

Observe that a cluster algebra of type \( A_1 \times A_1 \) obtained above is isomorphic to the tensor product of two cluster algebras of type \( A_1 \) as a \( \mathbb{ZP} \)-algebra. One can extend this result to a more general situation.

**Proposition 2.17.** Let \( \Sigma = \Sigma(\Sigma_{t_0}) \) be a cluster pattern of rank \( n \) with the initial seed \( \Sigma_{t_0} = (x, y, B) \). Suppose that the initial exchange matrix \( B_{t_0} = B \) is decomposed as
\[ B = \begin{pmatrix} B' & O \\ O & B'' \end{pmatrix}, \] (2.52)

where the size of \( B' \) and \( B'' \) are \( n' \) and \( n'' \) (\( n' + n'' = n \)), respectively. Accordingly, consider cluster patterns \( \Sigma' \) and \( \Sigma'' \) of rank \( n' \) and \( n'' \) whose initial seeds are given by
\[ \Sigma'_{t_0} = ((x_i)_{i=1}^{n'}, (y_i)_{i=1}^{n'}, B'), \quad \Sigma''_{t_0} = ((x_i)_{i=n'+1}^{n}, (y_i)_{i=n'+1}^{n}, B''), \] (2.53)

respectively. Then, as \( \mathbb{ZP} \)-algebras, we have
\[ \mathcal{A}(\Sigma) \simeq \mathcal{A}(\Sigma') \otimes_{\mathbb{ZP}} \mathcal{A}(\Sigma''). \] (2.54)

**Proof.** We first note that, by Proposition 1.20 for any \( t \in \mathbb{T}_n \), the exchange matrix \( B_t \) of \( \Sigma \) is decomposed in the same form as
\[ B_t = \begin{pmatrix} B'_t & O \\ O & B''_t \end{pmatrix}. \] (2.55)

For a seed \( \Sigma_t = (x_t, y_t, B_t) \) of \( \Sigma \), let us consider the mutation at \( k \leq n' \). Then, the mutation of \( B_t \) is given by the first case in (1.46). Also, for \( i \leq n' \), \( x_{i;t} \) and \( y_{i;t} \) mutate effectively by the sub-exchange matrix \( B'_t \), while, for \( i > n' \), they are stable. The other case \( k > n' \) is similar. It follows that the set of the cluster variables of \( \Sigma \) is the disjoint union of those of \( \Sigma' \) or \( \Sigma'' \). Also, there is no nontrivial algebraic relation between cluster variables of \( \Sigma' \) and those of \( \Sigma'' \). Therefore, \( \mathcal{A}(\Sigma) \) is naturally isomorphic to \( \mathcal{A}(\Sigma') \otimes_{\mathbb{ZP}} \mathcal{A}(\Sigma'') \) under the identification \( zz' \mapsto z \otimes_{\mathbb{ZP}} z' \) (\( z \in \mathcal{A}(\Sigma) \), \( z' \in \mathcal{A}(\Sigma') \)). \( \square \)
We say that a square matrix $M$ is decomposable (resp. indecomposable) if, after some simultaneous permutation of the column and row indices of $M$, $M$ is a direct sum of two square matrices (resp. otherwise).

Thanks to Proposition 2.17 in many situations it is enough to concentrate on cluster patterns and cluster algebras with indecomposable exchange matrices.

### 2.3 Rank 2 periodicities

We continue to use the parametrization and the notation for cluster patterns of rank 2 in Example 2.16. Let us take the following initial exchange matrix

$$B_0 = B = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}, \quad (a, b > 0).$$

By the mutations (2.4), the exchange matrices are given by

$$B_s = \begin{cases} B & s: \text{even}, \\ -B & s: \text{odd}. \end{cases}$$

Accordingly, we have, for even $s$,

$$\begin{align*}
\hat{y}_{1;s} &= y_{1;s}x_{2;s}^a, \\
\hat{y}_{2;s} &= y_{2;s}x_{1;s}^{-b}, \\
x_{1;s+1} &= x_{1;s}^{-1} + \frac{\hat{y}_{1;s}}{1 \oplus y_{1;s}}, \\
x_{2;s+1} &= x_{2;s},
\end{align*}$$

and, for odd $s$,

$$\begin{align*}
\hat{y}_{1;s} &= y_{1;s}x_{2;s}^a, \\
\hat{y}_{2;s} &= y_{2;s}x_{1;s}^b, \\
x_{1;s+1} &= x_{1;s}, \\
x_{2;s+1} &= x_{2;s}^{-1} + \frac{\hat{y}_{2;s}}{1 \oplus y_{2;s}},
\end{align*}$$

Below we concentrate on the case $ab \leq 3$. Without losing generality we may assume that $b = 1$ and $a = 1, 2, 3$, because the opposite case is obtained by exchanging the indices 1 and 2. We strongly recommend the readers to carry out the following calculations to appreciate the systematic occurrences of reductions ("miracles"). It may require a few hours. So, be prepared!

**Example 2.18 (Type $A_2$).** Consider the case $a = 1$, where

$$B_0 = B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
so that the initial $\hat{y}$-variables are given by
\[
\hat{y}_1 = y_1x_2, \quad \hat{y}_2 = y_2x_1^{-1}.
\] (2.63)

In fact, this is the case already calculated in Section 1.1 where cluster variables are without coefficients. Here, we effectively use Proposition 2.6 so that we do not have to look into the contents of $\hat{y}$-variables during the calculation except for the initial one (2.63). We have the following result:

\[
\begin{align*}
x_{1;1} &= x_1^{-1} \frac{1 + \hat{y}_1}{1 + y_1}, & y_{1;1} &= y_1^{-1}, \\
x_{2;1} &= x_2, & y_{2;1} &= y_2(1 + y_1), \\
x_{1;2} &= x_1^{-1} \frac{1 + \hat{y}_1}{1 + y_1}, & y_{1;2} &= y_1^{-1}(1 + y_2 + y_1y_2), \\
x_{2;2} &= x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 + y_2 + y_1y_2}, & y_{2;2} &= y_1^{-1}(1 + y_1)^{-1}, \\
x_{1;3} &= x_1x_2^{-1} \frac{1 + \hat{y}_2}{1 + y_2}, & y_{1;3} &= y_1(1 + y_2 + y_1y_2)^{-1}, \\
x_{2;3} &= x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 + y_2 + y_1y_2}, & y_{2;3} &= y_1^{-1}y_2^{-1}(1 + y_2), \\
x_{1;4} &= x_1x_2^{-1} \frac{1 + \hat{y}_2}{1 + y_2}, & y_{1;4} &= y_2^{-1}, \\
x_{2;4} &= x_1, & y_{2;4} &= y_1y_2(1 + y_2)^{-1}, \\
x_{1;5} &= x_2, & y_{1;5} &= y_2, \\
x_{2;5} &= x_1, & y_{2;5} &= y_1.
\end{align*}
\] (2.64)

Let us extend Observation 1.1 in more details.

(a). Periodicity/Finiteness. Even with the presence of coefficients for cluster variables, we still have the same pentagon periodicity

\[
\Sigma_{s+5} = \tau_{12}(\Sigma_s),
\] (2.69)

where $\tau_{12}$ is the transpose of 1 and 2, and its action was defined in (2.18). In particular, the cluster variables of $\Sigma$ are exhausted by

\[
\begin{align*}
x_1, \quad x_2, \quad x_1^{-1} \frac{1 + \hat{y}_1}{1 + y_1}, \quad x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 + y_2 + y_1y_2}, \quad x_1x_2^{-1} \frac{1 + \hat{y}_2}{1 + y_2}.
\end{align*}
\] (2.70)

By (2.63), as Laurent polynomials in $x_1$ and $x_2$ with coefficients in $\mathbb{Z}P$, they are written as

\[
\begin{align*}
x_1, \quad x_2, \quad x_1^{-1} \frac{1 + y_1x_2}{1 + y_1}, \quad x_2^{-1}x_1^{-1} \frac{x_1 + y_2 + y_1y_2x_2}{1 + y_2 + y_1y_2}, \quad x_2^{-1}x_1 + y_2.
\end{align*}
\] (2.71)

(b). Laurent phenomenon. In (2.71), each cluster variable $x_{i;t}$ is expressed as a Laurent polynomial in the initial cluster variables $x$ with coefficients in $\mathbb{Z}P$. 
(c). \textit{Laurent positivity.} Every coefficient of the above Laurent polynomial is nonnegative in $\mathbb{Z}^P$.

(d). \textit{F-polynomials.} Let us rephrase the properties (b) and (c) in a more specific way. Each cluster variable $x_{i;t}$ is expressed in a unified way as follows:

$$x_{i;t} = \left( \prod_{j=1}^{n} x_j^{g_{j;i;t}} \right) \frac{F_{i;t}(\hat{y})}{F_{i;t}|_P(y)}, \quad (2.72)$$

where $F_{i;t}(y)$ is a polynomial in formal variables $y = (y_1, y_2)$ with nonnegative integer coefficients, and $F_{i;t}|_P(y)$ is the specialization in $P$ at the initial coefficients $y_{t_0} = y$ in Definition 1.59. (Here, we conveniently abuse the symbol $y$ for two different usages.) Later, $F_{i;t}(y)$ is called an \textit{F-polynomial}.

(e). \textit{Unit constant term.} Every $F$-polynomial $F_{i;t}(y)$ has constant term 1.

(f). \textit{Duality.} As a new observation, we see that the cluster variables and coefficients share some common/parallel structure. In particular, the above $F$-polynomials also appear for coefficients.

\textbf{Example 2.19 (Type $B_2$).} Consider the case $a = 2$, where

$$B_0 = B = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.73)$$

so that the initial $\hat{y}$-variables are given by

$$\hat{y}_1 = y_1 x_2^2, \quad \hat{y}_2 = y_2 x_1^{-1}. \quad (2.74)$$

We have the following result:

\begin{align*}
\begin{cases}
 x_{1;1} &= x_1^{-1} \frac{1 + \hat{y}_1}{1 + y_1}, \\
 x_{2;1} &= x_2,
\end{cases} &
\begin{cases}
 y_{1;1} &= y_1^{-1}, \\
 y_{2;1} &= y_2(1 + y_1),
\end{cases} \quad (2.75) \\
\begin{cases}
 x_{1;2} &= x_1^{-1} \frac{1 + \hat{y}_1}{1 + y_1}, \\
 x_{2;2} &= x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 + y_2 \oplus y_1 y_2},
\end{cases} &
\begin{cases}
 y_{1;2} &= y_1^{-1} (1 + y_2 \oplus y_1 y_2)^2, \\
 y_{2;2} &= y_2^{-1} (1 + y_1)^{-1},
\end{cases} \quad (2.76) \\
\begin{cases}
 x_{1;3} &= x_1 x_2^{-2} \frac{1 + 2 \hat{y}_2 + \hat{y}_2^2 + \hat{y}_1 \hat{y}_2^2}{1 + 2 y_2 \oplus y_2^2 \oplus y_1 y_2^2}, \\
 x_{2;3} &= x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 + y_2 \oplus y_1 y_2},
\end{cases} &
\begin{cases}
 y_{1;3} &= y_1 (1 + y_2 \oplus y_1 y_2)^{-2}, \\
 y_{2;3} &= y_1^{-1} y_2^{-1} (1 + 2 y_2 \oplus y_2^2 \oplus y_1 y_2^2),
\end{cases} \quad (2.77)
\end{align*}
During the calculation we used the following polynomial identities:

\[
\begin{align*}
x_{1:4} &= x_1 x_2^{-2} \frac{1 + 2 \hat{y}_2 + \hat{y}_2^2 + \hat{y}_1 \hat{y}_2}{1 + 2 y_2 \oplus y_2^2 \oplus y_1 y_2^2}, \\
x_{2:4} &= x_1 x_2^{-1} \frac{1 + \hat{y}_2}{1 \oplus y_2}, \\
y_{1:4} &= y_1^{-1} y_2^{-2} (1 \oplus y_2)^2, \\
y_{2:4} &= y_1 y_2 (1 \oplus 2 y_2 \oplus y_2^2 \oplus y_1 y_2^2)^{-1}, \\
x_{1:5} &= x_1, \\
x_{2:5} &= x_1 x_2^{-1} \frac{1 + \hat{y}_2}{1 \oplus y_2}, \\
y_{1:5} &= y_1 y_2^2 (1 \oplus y_2)^{-2}, \\
y_{2:5} &= y_2^{-1}, \\
x_{1:6} &= x_1, \\
x_{2:6} &= x_2.
\end{align*}
\]

(2.78)

During the calculation we used the following polynomial identities:

\[
\begin{align*}
y_1 + (1 + y_2 + y_1 y_2^2)^2 &= (1 + y_1)(1 + 2 y_2 + y_2^2 + y_1 y_2^2), \\
y_1 y_2 + (1 + 2 y_2 + y_2^2 + y_1 y_2^2)^2 &= (1 + y_2 + y_1 y_2)(1 + y_2).
\end{align*}
\]

(2.81)

(2.82)

We observe the periodicity

\[
\Sigma_{s+6} = \Sigma_s.
\]

(2.83)

In particular, the cluster variables of \( \Sigma \) are exhausted by

\[
\begin{align*}
x_1, \\ x_2, \\ x_1^{-1} \frac{1 + \hat{y}_1}{1 \oplus y_1}, \\ x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 \oplus y_2 \oplus y_1 y_2}, \\ x_1 x_2^{-2} \frac{1 + 2 \hat{y}_2 + \hat{y}_2^2 + \hat{y}_1 \hat{y}_2^2}{1 \oplus 2 y_2 \oplus y_2^2 \oplus y_1 y_2^2}, \\ x_1 x_2^{-1} \frac{1 + \hat{y}_2}{1 \oplus y_2},
\end{align*}
\]

(2.84)

As Laurent polynomials in \( x_1 \) and \( x_2 \), they are written as

\[
\begin{align*}
x_1, \\ x_2, \\ x_1^{-1} \frac{1 + y_1 x_2^2}{1 \oplus y_1}, \\ x_2^{-1} x_1^{-1} \frac{x_1 + y_2 + y_1 y_2 x_2^2}{1 \oplus y_2 \oplus y_1 y_2}, \\ x_1^{-1} x_2^{-2} x_1^2 + 2 y_2 x_1 + y_2^2 + y_1 y_2^2 x_2^2}{1 \oplus 2 y_2 \oplus y_2^2 \oplus y_1 y_2^2}, \\ x_2^{-1} x_1 + y_2.
\end{align*}
\]

(2.85)

All other observations for type \( A_2 \) are commonly applied.

**Example 2.20** (Type \( G_2 \)). Consider the case \( a = 3 \), where

\[
B_0 = B = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},
\]

(2.86)

so that the initial \( \hat{y} \)-variables are given by

\[
\hat{y}_1 = y_1 x_2^3, \quad \hat{y}_2 = y_2 x_1^{-1}.
\]

(2.87)
We have the following result:

\[
\begin{align*}
\begin{cases}
  x_{1:1} &= x_1^{-1} + \hat{y}_1 - \frac{1}{1 + y_1}, \\
  x_{2:1} &= x_2,
\end{cases} & \quad \begin{cases}
  y_{1:1} &= y_1^{-1}, \\
  y_{2:1} &= y_2(1 + y_1),
\end{cases} \\
\begin{cases}
  x_{1:2} &= x_1^{-1} + \hat{y}_1 - \frac{1}{1 + y_1}, \\
  x_{2:2} &= x_2^{-1} + \hat{y}_2 + \hat{y}_1 y_2 - \frac{1}{1 + y_2 + y_1 y_2},
\end{cases} & \quad \begin{cases}
  y_{1:2} &= y_1^{-1}(1 + y_2 + y_1 y_2)^{-3}, \\
  y_{2:2} &= y_2^{-1}(1 + y_1)^{-1},
\end{cases} \\
\begin{cases}
  x_{1:3} &= x_1 x_2^{-3} + 3\hat{y}_2 + 3\hat{y}_2^2 + \hat{y}_2^3 + 3\hat{y}_1 \hat{y}_2^2 + 2\hat{y}_1 \hat{y}_2^3 + \hat{y}_1^3 \hat{y}_2^3, \\
  x_{2:3} &= x_2^{-1} + \hat{y}_2 + \hat{y}_1 y_2 - \frac{1}{1 + y_2 + y_1 y_2},
\end{cases} & \quad \begin{cases}
  y_{1:3} &= y_1(1 + y_2 + y_1 y_2)^{-3}, \\
  y_{2:3} &= y_1^{-1} y_2^{-1}(1 + 3 y_2 + 3 y_2^2 + y_2^3 + 3 y_1 y_2^2 + 2 y_1 y_2^3 + y_1^2 y_2^3),
\end{cases} \\
\begin{cases}
  x_{1:4} &= x_1 x_2^{-3} + 3\hat{y}_2 + 3\hat{y}_2^2 + \hat{y}_2^3 + 3\hat{y}_1 \hat{y}_2^2 + 2\hat{y}_1 \hat{y}_2^3 + \hat{y}_1^3 \hat{y}_2^3, \\
  x_{2:4} &= x_2^{-1} + 2\hat{y}_2 + \hat{y}_1 y_2 - \frac{1}{1 + 2 y_2 + y_1 y_2},
\end{cases} & \quad \begin{cases}
  y_{1:4} &= y_1^{-2} y_2^{-3}(1 + 2 y_2 + y_2^2 + y_1 y_2^2)^{-3}, \\
  y_{2:4} &= y_1 y_2(1 + 3 y_2 + 3 y_2^2 + y_2^3 + 3 y_1 y_2^2 + 2 y_1 y_2^3 + y_1^2 y_2^3)^{-1},
\end{cases} \\
\begin{cases}
  x_{1:5} &= x_1 x_2^{-3} + 3\hat{y}_2 + 3\hat{y}_2^2 + \hat{y}_2^3 + 3\hat{y}_1 \hat{y}_2^2, \\
  x_{2:5} &= x_2^{-1} + 2\hat{y}_2 + \hat{y}_1 y_2 - \frac{1}{1 + 2 y_2 + y_1 y_2},
\end{cases} & \quad \begin{cases}
  y_{1:5} &= y_1^3 y_2^3(1 + 2 y_2 + y_2^2 + y_1 y_2^2)^{-3}, \\
  y_{2:5} &= y_1^{-1} y_2^{-2}(1 + 3 y_2 + 3 y_2^2 + y_2^3 + y_1 y_2^3),
\end{cases} \\
\begin{cases}
  x_{1:6} &= x_1 x_2^{-3} + 3\hat{y}_2 + 3\hat{y}_2^2 + \hat{y}_2^3 + \hat{y}_1 \hat{y}_2^2, \\
  x_{2:6} &= x_2^{-1} + \hat{y}_2 - \frac{1}{1 + y_2},
\end{cases} & \quad \begin{cases}
  y_{1:6} &= y_1^{-1} y_2^{-3}(1 + y_2)^{-3}, \\
  y_{2:6} &= y_1 y_2^2(1 + 3 y_2 + 3 y_2^2 + y_2^3 + y_1 y_2^3)^{-1},
\end{cases} \\
\begin{cases}
  x_{1:7} &= x_1, \\
  x_{2:7} &= x_1 x_2^{-1} + \hat{y}_2 - \frac{1}{1 + y_2},
\end{cases} & \quad \begin{cases}
  y_{1:7} &= y_1 y_2^3(1 + y_2)^{-3}, \\
  y_{2:7} &= y_2^{-1},
\end{cases} \\
\begin{cases}
  x_{1:8} &= x_1, \\
  x_{2:8} &= x_2,
\end{cases} & \quad \begin{cases}
  y_{1:8} &= y_1, \\
  y_{2:8} &= y_2.
\end{cases}
\end{align*}
\]
During the calculation we used the following polynomial identities:

\[ y_1 + (1 + y_2 + y_1 y_2)^3 = (1 + y_1)(1 + 3y_2 + 3y_2^2 + y_2^3 + 3y_1 y_2^2 + 2y_1 y_2^3 + y_1^2 y_2^3), \]

\[ y_1 y_2 + (1 + 3y_2 + 3y_2^2 + y_2^3 + 3y_1 y_2^2 + y_1 y_2^3) \]

\[ = (1 + y_2 + y_1 y_2)(1 + 2y_2 + y_2^2 + y_1 y_2^2), \]

\[ y_1^2 y_2^3 + (1 + 2y_2 + y_2^2 + y_1 y_2^2)^3 \]

\[ = (1 + 3y_2 + 3y_2^2 + y_2^3 + 3y_1 y_2^2 + 2y_1 y_2^3 + y_1^2 y_2^3) \]

\[ \times (1 + 3y_2 + 3y_2^2 + y_2^3 + y_1 y_2^3), \]

\[ y_1 y_2^2 + (1 + 3y_2 + 3y_2^2 + y_2^3 + y_1 y_2^3) \]

\[ = (1 + 2y_2 + y_2^2 + y_1 y_2^2)(1 + y_2), \]

where the most complicated identity (2.98) may be quickly checked by computer. We observe the periodicity

\[ \Sigma_{s+8} = \Sigma_s. \]  

(2.100)

In particular, the cluster variables of \( \Sigma \) are exhausted by

\[ x_1, \quad x_2, \quad x_1^{-1} \frac{1 + \hat{y}_1}{1 + y_1}, \quad x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 + y_2 + y_1 y_2}, \]

\[ x_1 x_2^{-3} \frac{1 + 3\hat{y}_2 + 3\hat{y}_2^2 + \hat{y}_2^3 + 3\hat{y}_1 \hat{y}_2^2 + 2\hat{y}_1 \hat{y}_2^3 + \hat{y}_1^2 \hat{y}_2^3}{1 + 3y_2 + 3y_2^2 + y_2^3 + 3y_1 y_2^2 + 2y_1 y_2^3 + y_1^2 y_2^3}, \]

\[ x_1 x_2^{-2} \frac{1 + 2\hat{y}_2 + \hat{y}_2^2 + \hat{y}_1 \hat{y}_2^2}{1 + 2y_2 + y_2^2 + y_1 y_2^2}, \]

\[ x_2^{-2} x_1^{-3} \frac{1 + 3\hat{y}_2 + 3\hat{y}_2^2 + \hat{y}_2^3 + \hat{y}_1 \hat{y}_2^3}{1 + 3y_2 + 3y_2^2 + y_2^3 + y_1 y_2^3}, \]

\[ x_1 x_2^{-1} \frac{1 + \hat{y}_2}{1 + y_2}. \]

(2.101)

As Laurent polynomials in \( x_1 \) and \( x_2 \), they are written as

\[ x_1, \quad x_2, \quad x_1^{-1} \frac{1 + y_1 x_2^3}{1 + y_1}, \quad x_2^{-1} x_1^{-1} \frac{x_1 + y_2 + y_1 y_2 x_2^3}{1 + y_2 + y_1 y_2}, \]

\[ x_1^{-2} x_2^{-3} \frac{x_1^3 + 3y_2 x_1^2 + 3y_2^2 x_1 + y_2^3 + 3y_1 y_2^2 x_1 x_2 + 2y_1 y_2^3 x_2 + y_1 y_2^3 x_2^3}{1 + 3y_2 + 3y_2^2 + y_2^3 + 3y_1 y_2^2 + 2y_1 y_2^3 + y_1^2 y_2^3}, \]

\[ x_1^{-1} x_2^{-2} \frac{x_1^2 + 2y_2 x_1 + y_2^2 + y_1 y_2^3 x_2^2}{1 + 2y_2 + y_2^2 + y_1 y_2^2}, \]

\[ x_1^{-1} x_2^{-3} \frac{x_1^3 + 3y_2 x_1^2 + 3y_2^2 x_1 + y_2^3 + y_1 y_2^3 x_2^3}{1 + 3y_2 + 3y_2^2 + y_2^3 + y_1 y_2^3}, \]

\[ x_2^{-1} x_1 + y_2. \]

(2.102)

All other observations for type \( A_2 \) are commonly applied.

Congratulations! You have successfully gone through the famous "ordeal" in cluster algebra theory.
Remark 2.21. (a). For the matrices in (2.50) with \( ab \geq 4 \), the periodicity of seeds does not occur. Accordingly, we have infinitely many distinct cluster variables for those cluster patterns.

(b). The periods 5, 6, 8 in the above examples are interpreted as \( h + 2 \), where \( h = 3, 4, 6 \) are the Coxeter numbers of the root systems of type \( A_2 \), \( B_2 \), \( G_2 \), respectively. A more account on the connection to the root systems will be given in Section 3.2.

2.4 Free coefficients

Let us introduce a notion, which is not defined in CA1-4 explicitly.

**Definition 2.22.** We say that a cluster pattern \( \Sigma \) is with free coefficients at \( t_0 \in \mathbb{T}_n \) if the following conditions are satisfied:

- The coefficient semifield of \( \Sigma \) is the universal semifield \( \mathbb{Q}_{sf}(y) \) with generators \( y = (y_1, \ldots, y_n) \), where \( n \) is the rank of \( \Sigma \).
- The coefficient tuple \( y_{t_0} \) at \( t_0 \) coincides with \( y \).

Note that, for each \( t \), \( y_{1;t}, \ldots, y_{n;t} \) are algebraically independent.

**Remark 2.23.** It might be natural to call such coefficients universal coefficients. However, in CA4, a related but different notion is defined and called so. Therefore, we call it differently to avoid confusion.

**Remark 2.24.** Though it is not necessary, we often identify the above base vertex \( t_0 \) with the initial vertex for \( \Sigma \). In that case the notation \( y \) above is compatible with the convention for the initial coefficients (2.37). Otherwise, we may dismiss the notation (2.37) to avoid the conflict.

For any \( t \in \mathbb{T}_n \), the coefficients \( y_{1;t_0}, \ldots, y_{n;t_0} \) in Definition 2.22 are expressed as rational functions in \( y_t \) with subtraction-free expressions by doing mutations from \( t \) to \( t_0 \). They induce a canonical isomorphism between the semifields \( \mathbb{Q}_{sf}(y_{t_0}) \) and \( \mathbb{Q}_{sf}(y_t) \). Therefore, under this identification the base vertex can be shifted arbitrarily, so that the choice of the base point \( t_0 \) is superficial.

Recall the universality of \( \mathbb{Q}_{sf}(y) \) in Proposition 1.8. We are going to extend this homomorphism \( \pi : \mathbb{Q}_{sf}(y) \to \mathbb{P} \) to a map between cluster variables with coefficients in \( \mathbb{Q}_{sf}(y) \) and \( \mathbb{P} \) sharing a common \( B \)-pattern. However, there is a pitfall to avoid. Let \( \varphi : \mathbb{P} \to \mathbb{P}' \) be a semifield homomorphism. Then, it is uniquely extended to a ring homomorphism \( \varphi_1 : \mathbb{ZP} \to \mathbb{ZP}' \). However, it can be extended to a field homomorphism \( \varphi_2 : \mathbb{QP} \to \mathbb{QP}' \) only if \( \varphi \) is injective. Indeed, if \( \varphi \) is not injective, \( \varphi_1 \) is not injective. Then, a fraction of \( \mathbb{ZP} \) whose denominator is in \( \text{Ker} \varphi_1 \) does not have a well-defined image of \( \varphi_2 \).
Below, for \( P = Q_{\text{sf}}(y) \), we write \( QP \) as \( QQ_{\text{sf}}(y) \) according to our convention, though it looks a little cumbersome.

**Proposition 2.25.** Let \( \Sigma \) be a cluster pattern with free coefficients at \( t_0 \). Let \( \Sigma' \) be any cluster pattern with coefficients in any semifield \( P \) sharing the common \( B \)-pattern with \( \Sigma \).

(a) Let \( \pi \) be the semifield homomorphism defined by
\[
\pi : \ Q_{\text{sf}}(y) \rightarrow P,
\]
\[
y_i = y_{i; t_0} \mapsto y'_{i; t_0}.
\]
Then, we have
\[
\pi(y_{i; t}) = y'_{i; t}.
\]

(b) Let \( \mathcal{X}(\Sigma) \) be the set of the cluster variables of \( \Sigma \). Let
\[
\varphi : \mathcal{X}(\Sigma) \rightarrow (Q^{P})(x'_{t_0})
\]
be the map such that, in each element \( x_{i; t} \in (QQ_{\text{sf}}(y))(x_{t_0}) \), \( x_{i; t_0} \) is replaced with \( x'_{i; t_0} \), and \( f(y) \in Q_{\text{sf}}(y) \) is replaced with \( f(y') = \pi(f(y)) \). Then, the map is well defined; moreover, we have
\[
\varphi(x_{i; t}) = x'_{i; t}.
\]

**Proof.** (a) Any coefficient \( y_{i; t} \) is expressed as a rational function \( Y_{i; t}(y) \in Q_{\text{sf}}(y) \). Applying \( \pi \) yields the expression for \( y'_{i; t} \) in \( y'_{i; t_0} \), because both \( y_{i; t} \) and \( y'_{i; t} \) obey formally the same mutation formula (2.3).

(b) Any cluster variable \( x_{i; t} \) is expressed as a rational function \( X_{i; t}(x_{t_0}) \in (QQ_{\text{sf}}(y))(x_{t_0}) \). Since the mutation (2.2) does not involve any subtraction, \( X_{i; t}(x_{t_0}) \) has a subtraction-free expression in \( (QQ_{\text{sf}}(y))(x_{t_0}) \). In particular, for the denominator of any coefficient in \( X_{i; t}(x_{t_0}) \), its image by \( \pi \) does not vanish. Thus, the map \( \varphi \) is well defined. Applying \( \varphi \) yields the expression for \( x'_{i; t} \) in \( x'_{t_0} \), because both \( x_{i; t} \) and \( x'_{i; t} \) obey formally the same mutation formula (2.2). \( \square \)

The result is plainly rephrased that cluster variables with *any specific choice of coefficients* can be obtained from cluster variables with *free coefficients* by *specializing the coefficients*. To see how it works more concretely, let us look at the formulas in Section 2.23 (e.g., (2.64)–(2.68)). They can be viewed as expressions of \( x_{i; t} \) in \( (QQ_{\text{sf}}(y))(x_{t_0}) \) and \( y_{i; t} \) in \( Q_{\text{sf}}(y) \), because we did not use any specific property of a given semifield \( P \) and initial coefficients \( y \) therein. Meanwhile, they can be viewed also as expressions for any specific choice of coefficients. (Viewing so is nothing but the specialization by \( \varphi \) and \( \pi \).)

The following is an immediate consequence of Proposition 2.25.
Proposition 2.26. Let $\Sigma$ be a cluster pattern with free coefficients at $t_0$. Let $\Sigma'$ be a cluster pattern with coefficients in any semifield $\mathbb{P}$ sharing the common $B$-pattern with $\Sigma$. Then, for any $t, t' \in \mathbb{T}_n$ and a permutation $\sigma \in S_n$, the following fact holds:

$$\Sigma_t = \sigma \Sigma_{t'} \implies \Sigma'_t = \sigma \Sigma'_{t'}.$$  \hspace{1cm} (2.107)

In other words, any periodicity of $\Sigma$ implies the same periodicity of $\Sigma'$.

Proof. The equality $\Sigma_t = \sigma \Sigma_{t'}$ reduces to the equality $\Sigma'_t = \sigma \Sigma'_{t'}$ under the specialization in Proposition 2.25. \hfill \Box

Whether the opposite implication of (2.107) holds or not is an important issue posed in CA4. Namely, there is a possibility that additional periodicities of $\Sigma'$ occur under some specialization of coefficients, though it does not happen in the rank 2 examples in Section 2.3. We will discuss more about the problem later in Section 4.4.
3 Fundamental Results

In this section we present some of the most fundamental results in cluster algebra theory.

3.1 Laurent phenomenon

We prove the Laurent phenomenon observed in rank 2 examples. This is the main result in CA1 and also the most fundamental fact on cluster algebras.

Theorem 3.1 (Laurent phenomenon \[FZ02a\]). Let $\Sigma$ be any cluster pattern with coefficients in any semifield $\mathbb{P}$. Let $t_0, t \in T_n$ be any vertices. Then, any cluster variable $x_{i,t}$ is expressed as a Laurent polynomial in $x_{t_0}$ with coefficients in $\mathbb{Z}\mathbb{P}$.

Here we present the “classic proof” in CA1. In view of Proposition 2.25, it is enough to prove Theorem 3.1 for any cluster pattern $\Sigma$ with free coefficients at $t_0$. This is because the Laurent polynomial expression of $x_{i,t}$ in $x_{t_0}$ for $\Sigma$ reduces to the one for any choice of coefficients under the specialization in Proposition 2.25.

Definition 3.2 (Coprime). We say that two Laurent polynomials in $\mathbb{Z}\mathbb{P}[x_{t_0}^{\pm 1}]$ are coprime if there is no common factor except for Laurent monomials in $x_{t_0}$ with coefficients in $\mathbb{Z}\mathbb{P}^\times = \{\pm 1\}\mathbb{P}$.

From now on, we assume that $\Sigma$ has free coefficients at $t_0$, that is, $\mathbb{P} = \mathbb{Q}_{sf}(y)$ and $y_{t_0} = y$. The proof in CA1 relies on the following lemma. (The assumption of coefficients is necessary only for the claim (b).)

Lemma 3.3. Let $t_1, t_2, t_3 \in T_n$ be vertices that are sequentially adjacent to $t_0$ in the following way, where $k \neq \ell$:

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
t_0 & t_1 & t_2 & t_3
\end{array}
\]  

(3.1)

Then, the following facts hold:

(a). The cluster variable $x_{k;t_3}$ can be expressed as a Laurent polynomial in $x_{t_0}$ with coefficients in $\mathbb{Z}\mathbb{P}$.

(b). As elements in $\mathbb{Z}\mathbb{P}[x_{t_0}^{\pm 1}]$, $x_{k;t_1}$ is coprime with $x_{k;t_3}$ and $x_{\ell;t_3}$.

Let us temporarily assume Lemma 3.3 and prove Theorem 3.1.

Let $d(t,t')$ denote the distance between $t$ and $t'$ in $T_n$, that is, the number of edges of $T_n$ between $t$ and $t'$.

Proof of Theorem 3.1. Let $d = d(t_0, t)$. Let $t_1, t_2, t_3$ be the ones in (3.1).
Recall that

\[ x_{i; t_1} = \begin{cases} 
  x_{k; t_0}^{-1} \left( \prod_{j=1}^{n} x_{j; t_0}^{[-b_{jk}; t_0]} \right) \frac{1 + \hat{y}_{k; t_0}}{1 + y_{k; t_0}} & i = k, \\
  x_{i; t_0} & i \neq k.
\] (3.2)

They are certainly Laurent polynomials in \( x_{t_0} \). Thus, the claim holds for \( d = 1 \). The case \( d = 2 \) is similar, where only \( x_{i; t_2} \) is concerned. For \( d = 3 \), the problem arises for \( x_{k; t_3} \), because the term \( x_{k; t_2} = x_{k; t_1} \) in the denominator is no longer a monomial in \( x_{t_0} \). However, this case is covered by Lemma 3.3 (a).

We prove the claim by the induction on \( d = d(t_0, t) \geq 3 \), where we fix \( t \) and vary \( t_0 \). Assume that the claim hold up to \( d \). We consider the situation in the following graph, where \( d(t_1, t) = d(t_3, t) = d \) and \( d(t_0, t) = d + 1 \):

\[ k \rightarrow t_0 \rightarrow t_1 \rightarrow k \rightarrow t_2 \rightarrow \cdots \rightarrow t \] (3.3)

By the induction assumption, \( x_{i; t} \) is expressed as a Laurent polynomial in \( x_{t_1} \). We write

\[ x_{i; t} = x_{k; t_1}^{-a} \tilde{f}(x_{t_1}), \quad \tilde{f}(x_{t_1}) \in \mathbb{Z}[x_{t_1}^{\pm 1}], \] (3.4)

where \( a \geq 0 \) is a sufficiently large integer such that \( \tilde{f}(x_{t_1}) \) does not contain any negative power of \( x_{k; t_1} \). Substituting the expression (3.2) for \( x_{k; t_1} \) in \( \tilde{f}(x_{t_1}) \), we have

\[ x_{i; t} = x_{k; t_1}^{-a} \hat{f}(x_{t_0}), \quad \hat{f}(x_{t_0}) \in \mathbb{Z}[x_{t_0}^{\pm 1}]. \] (3.5)

Meanwhile, \( x_{i; t} \) is also expressed as a Laurent polynomial in \( x_{t_3} \). Similarly, we write

\[ x_{i; t} = x_{k; t_3}^{-b} x_{t_3}^{-c} g(x_{t_3}), \quad g(x_{t_3}) \in \mathbb{Z}[x_{t_3}^{\pm 1}], \] (3.6)

where \( b, c \geq 0 \) are sufficiently large integers such that \( g(x_{t_3}) \) does not contain any negative powers of \( x_{k; t_3} \) and \( x_{t_3} \). Recall that \( x_{k; t_3} \) and \( x_{t_3} \) can be expressed as Laurent polynomials in \( x_{t_0} \). Substituting these expressions for \( x_{k; t_3} \) and \( x_{t_3} \) in \( g(x_{t_3}) \), we have

\[ x_{i; t} = x_{k; t_3}^{-b} x_{t_3}^{-c} \tilde{g}(x_{t_0}), \quad \tilde{g}(x_{t_0}) \in \mathbb{Z}[x_{t_0}^{\pm 1}]. \] (3.7)

Comparing (3.4) and (3.7), we obtain the equality

\[ x_{k; t_3}^{-b} x_{t_3}^{-c} \tilde{f}(x_{t_0}) = x_{k; t_1}^{-a} \tilde{g}(x_{t_0}) \quad \text{in} \ \mathbb{Z}[x_{t_0}^{\pm 1}]. \] (3.8)

Then, by Lemma 3.3 (b), \( \tilde{f}(x_{t_0}) \) is divisible by \( x_{k; t_1}^{-a} \) in \( \mathbb{Z}[x_{t_0}^{\pm 1}] \). Thus, by (3.5), we conclude that \( x_{i; t} \in \mathbb{Z}[x_{t_0}^{\pm 1}] \). \( \square \)
Remark 3.4. The above method is commonly applicable to various systems showing the Laurent phenomenon beyond cluster algebras [FZ02b].

Let us go back and prove Lemma 3.3.

Proof of Lemma 3.3. Below we temporarily view any cluster variable as an element in $\mathbb{Q}P(x_{t_0})$. For $a, b \in \mathbb{Q}P(x_{t_0})$, we write $a \sim b$ if there is a Laurent monomial $m$ in $x_{t_0}$ with coefficients in $\mathbb{Z}P^\times$ such that $a = mb$.

(a). We need to compare two cluster variables

\[ x_{k;1} = x_{k;0}^{-1} \left( \prod_{j=1}^{n} x_{j;0}^{-b_{jk;0}} \right) \frac{1 + \hat{y}_{k;0}}{1 + y_{k;0}}, \]  
\[ x_{k;3} = x_{k;2}^{-1} \left( \prod_{j=1}^{n} x_{j;2}^{-b_{jk;2}} \right) \frac{1 + \hat{y}_{k;2}^\varepsilon}{1 + y_{k;2}^\varepsilon}, \]  

where for $x_{k;3}$ we employ the $\varepsilon$-expression in Proposition 2.7. Recall that $x_{k;2} = x_{k;1}$ and $x_{i;2} = x_{i;0}$ for $i \neq k, \ell$. Thus, we have

\[ x_{k;1} \sim 1 + \hat{y}_{k;0}. \]  

If $b_{k;0} \neq 0$, we set

\[ \varepsilon = \text{sign}(b_{k;0}) = -\text{sign}(b_{k;1}) = \text{sign}(b_{k;2}) \in \{1, -1\}, \]  

where $\text{sign}(a) = 1$ if $a > 0$, and $-1$ if $a < 0$. If $b_{k;0} = 0$, we choose $\varepsilon = \pm 1$ arbitrarily. Then, $[-\varepsilon b_{k;2}] = 0$, so that we have

\[ x_{k;3} \sim x_{k;2}^{-1} \left( 1 + \hat{y}_{k;2}^\varepsilon \right) \frac{1 + \hat{y}_{k;2}^\varepsilon}{1 + \hat{y}_{k;0}}. \]

Meanwhile, by Proposition 2.7 with the same $\varepsilon$ in (3.12), we have

\[ \hat{y}_{\ell;1} = \hat{y}_{\ell;0} y_{\ell;0}^{[b_{k\ell;0}]} (1 + \hat{y}_{\ell;0}^\varepsilon) - b_{k\ell;0} = \hat{y}_{\ell;0} (1 + \hat{y}_{\ell;1}^\varepsilon) - b_{k\ell;1}, \]  
\[ \hat{y}_{k;3} = \hat{y}_{k;2} y_{k;2}^{[b_{k\ell;2}]} (1 + \hat{y}_{k;2}^\varepsilon) - b_{k\ell;2} = \hat{y}_{k;2}^{-1} (1 + \hat{y}_{k;1}^\varepsilon) - b_{k\ell;1}. \]

We note that $-\varepsilon b_{k\ell;0}$ and $-\varepsilon b_{k\ell;1}$ are both nonnegative. Thus, $1 + \hat{y}_{k;2}^\varepsilon \in \mathbb{Z}P[x_{t_0}^{\pm 1}]$. Therefore, it is enough to prove that $1 + \hat{y}_{k;2}^\varepsilon$ is divisible by $1 + \hat{y}_{k;0}$ in $\mathbb{Z}P[x_{t_0}^{\pm 1}]$. If $b_{k;0} = 0$, we have $\hat{y}_{k;2}^\varepsilon = \hat{y}_{k;0}^\varepsilon$ by (3.15). Therefore, the claim holds. Suppose that $b_{k;0} \neq 0$. Let $I$ be the ideal of $\mathbb{Z}P[x_{t_0}^{\pm 1}]$ generated by $1 + \hat{y}_{k;0}^\varepsilon$. By (3.14), we have $\hat{y}_{\ell;1}^\varepsilon \equiv 0 \mod I$. Thus, by (3.15), we have $\hat{y}_{k;2}^\varepsilon \equiv \hat{y}_{k;0}^\varepsilon \mod I$. Therefore, we have

\[ 1 + \hat{y}_{k;2}^\varepsilon \equiv 1 + \hat{y}_{k;0}^\varepsilon = \hat{y}_{k;0}^{-\varepsilon} (1 + \hat{y}_{k;0}^\varepsilon) \equiv 0 \mod I. \]
(b). First we note that
\[ x_{\ell; t_3} = x_{\ell; t_2} \sim x_{k; t_1}^{[-\varepsilon b_{k; t_1}]} (1 + \hat{y}_{\ell; t_1}^\varepsilon) = 1 + \hat{y}_{\ell; t_1}^\varepsilon, \] (3.17)
where \( \varepsilon \) is the same sign in (3.12). Recall that by the assumption of free coefficients, \( y_{\ell; t_0} = y_{\ell} \) and \( y_{k; t_0} = y_k \) are algebraically independent. By (3.14) and (3.15), the following facts hold:

• \( 1 + \hat{y}_{k; t_0} \) is a constant with respect to \( y_{\ell}^\varepsilon \).
• \( 1 + \hat{y}_{k; t_1}^\varepsilon \) is a binomial with respect to \( y_{\ell}^\varepsilon \) whose constant term is 1.
• \( 1 + \hat{y}_{k; t_2}^\varepsilon \) is a polynomial with respect to \( y_{\ell}^\varepsilon \) whose constant term is \( 1 + \hat{y}_{k; t_0} \).

It follows that \( x_{k; t_1} \sim 1 + \hat{y}_{k; t_0} \) and \( x_{\ell; t_3} \sim 1 + \hat{y}_{\ell; t_1}^\varepsilon \) are coprime. Also, it follows that \( (1 + \hat{y}_{k; t_2}^\varepsilon)/(1 + \hat{y}_{k; t_0}) \), which is in \( \mathbb{Z}[x_1^{\pm 1}] \) by (a), is a polynomial with respect to \( y_{\ell}^\varepsilon \) whose constant term is \( \hat{y}_{k; t_0}^{-1} \) if \( \varepsilon = 1 \) and 1 if \( \varepsilon = -1 \). Therefore, \( x_{k; t_3} \sim (1 + \hat{y}_{k; t_2}^\varepsilon)/(1 + \hat{y}_{k; t_0}) \) and \( x_{k; t_1} \sim 1 + \hat{y}_{k; t_0} \) are coprime.

3.2 Finite type classification
We present the finite type classification of cluster algebras and cluster patterns without proofs. This is the main result in CA2.

To state the result, we briefly explain the background in Lie theory.

The following definition is the counterpart of Definition 1.13.

Definition 3.5 (Symmetrizable matrix). An integer square matrix \( A = (a_{ij})_{i,j=1}^n \) is said to be symmetrizable if there is a diagonal matrix \( D = (d_i \delta_{ij})_{i,j=1}^n \) whose diagonal entries \( d_i \) are positive integers such that \( DA \) is symmetric, i.e.,
\[ d_i a_{ij} = d_j a_{ji}. \] (3.18)

The matrix \( D \) is called a (left) symmetrizer of \( A \). In particular, any symmetric matrix is symmetrizable.

Definition 3.6 (Cartan matrix). An \( n \times n \) (integer) square matrix \( A = (a_{ij})_{i,j=1}^n \) is called a (generalized) Cartan matrix if it satisfies the following conditions:

• For any \( i \), we have \( a_{ii} = 2 \).
• For any \( i, j \) (\( i \neq j \)), we have \( a_{ij} \leq 0 \); moreover, \( a_{ij} < 0 \) if and only if \( a_{ji} < 0 \).

For each symmetrizable Cartan matrix \( X \), one can define the associated Kac-Moody algebra, root system, and Weyl group \([\text{Kac90}]\).

There is a natural (many-to-one) correspondence from a skew-symmetrizable matrix to a symmetrizable Cartan matrix.
Definition 3.7 (Cartan counterpart). With any skew-symmetrizable matrix $B = (b_{ij})_{i,j=1}^n$, we associate a symmetrizable Cartan matrix $A = A(B)$ as

$$a_{ij} = \begin{cases} 2 & i = j, \\ -|b_{ij}| & i \neq j. \end{cases}$$

(3.19)

The matrix $A(B)$ is called the Cartan counterpart of $B$.

Example 3.8. For the initial exchange matrices $B_0$ of type $A_2$, $B_2$, $G_2$ in Section 2.3, the associated Cartan matrices $A(B_0)$ are given as follows:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$ (3.20)

They are the Cartan matrices of type $A_2$, $B_2$, $G_2$, respectively, as explained below.

We recall the Dynkin diagrams of finite type, which is well-known in the context of semisimple Lie algebras and crystallographic root systems [Bou68, Hum90].

Definition 3.9. We call the graphs given in the list in Figure 1 the Dynkin diagrams of finite type. Here, $A_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $C_n$ ($n \geq 2$), $D_n$ ($n \geq 4$), and we identify $B_2$ and $C_2$ (up to the relabeling of the vertices).

For each Dynkin diagram of type $X_n$ in Figure 1 we define the $n \times n$ matrix $A = A(X_n)$ such that $a_{ii} = 2$, and nondiagonal entries are given as follows, where we follow the convention in [Kac90], which is the transpose
to the one in [Bou68]:

\[
\begin{align*}
\begin{cases}
   a_{ij} = a_{ji} = 0 & \forall i \neq j \\
   a_{ij} = a_{ji} = -1 & \forall i \neq j \\
   a_{ij} = -1, a_{ji} = -2 & \forall i \neq j \\
   a_{ij} = -1, a_{ji} = -3 & \forall i \neq j 
\end{cases}
\end{align*}
\] (3.21)

The resulting matrix \( A \) is a symmetrizable Cartan matrix. These matrices (up to the simultaneous permutations of row and column indices) are called the Cartan matrices of finite type (of type \( X_n \)).

**Example 3.10.** The Cartan matrices of type \( A_3, B_3, C_3 \) (up to the simultaneous permutations of row and column indices) are given, respectively, by

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix},
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -2 & 2
\end{pmatrix},
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -1 & 2
\end{pmatrix}.
\]

(3.22)

For an indecomposable symmetrizable Cartan matrix \( A \), it is known that the following finiteness conditions are equivalent [Kac90, Prop. 4.9]:

(a). \( A \) is of finite type.

(b). The Kac-Moody algebra associated with \( A \) is finite-dimensional.

(c). The root system associated with \( A \) is a finite set.

(d). The Weyl group associated with \( A \) is a finite group.

Moreover, the isomorphism classes of such Kac-Moody algebras, root systems, and Weyl groups, respectively, are classified by the Dynkin diagrams of finite type. In addition, the above finiteness conditions are equivalent to the following condition [Kac90, Prop. 4.7]:

(e). All principal minors of \( A \) are positive.

Now we consider the counterpart in cluster algebras.

**Definition 3.11** (Finite type). A cluster pattern \( \Sigma \) is said to be of finite type if there are only finitely many distinct seeds of \( \Sigma \). The cluster algebra \( \mathcal{A}(\Sigma) \) associated with a cluster pattern \( \Sigma \) is said to be of finite type if the cluster pattern \( \Sigma \) is of finite type.

**Remark 3.12.** It is known [FZ03] that a cluster pattern \( \Sigma \) is of finite type if and only if there are finitely many distinct clusters (equivalently, finitely many distinct cluster variables) of \( \Sigma \). The only if part holds trivially, while the if part requires a detailed study.
3.2. Finite type classification

**Definition 3.13** (Strongly isomorphic). Let \( \Sigma \) and \( \Sigma' \) be cluster patterns of a common rank \( n \) with a common coefficient semifield \( \mathbb{P} \).

- We say that cluster patterns \( \Sigma \) and \( \Sigma' \) are **isomorphic** if there are some \( t, t' \in \mathbb{T}_n \) and a permutation \( \sigma \in S_n \) such that \( (y'_t, B'_t) = (\sigma(y_t), \sigma(B_t)) \).
- We say that the cluster algebras \( \mathcal{A}(\Sigma) \) and \( \mathcal{A}(\Sigma') \) are **strongly isomorphic** if the underlying cluster patterns \( \Sigma \) and \( \Sigma' \) are isomorphic.

**Proposition 3.14.** If \( \mathcal{A}(\Sigma) \) and \( \mathcal{A}(\Sigma') \) are strongly isomorphic, they are isomorphic as \( \mathbb{ZP} \)-algebras.

**Proof.** By the assumption, there are some \( t_0, t_1 \in \mathbb{T}_n \) and a permutation \( \sigma \in S_n \) such that \( (y'_t, B'_t) = (\sigma(y_{t_0}), \sigma(B_{t_0})) \). Let us consider the cluster pattern \( \sigma \Sigma = \{\sigma \Sigma_t\}_{t \in \mathbb{T}_n} \). Then, the correspondence \( x'_{t_1} \mapsto \sigma x_{t_0} \) induces a \( \mathbb{ZP} \)-algebra isomorphism \( \varphi : \mathcal{A}(\Sigma') \xrightarrow{\sim} \mathcal{A}(\sigma \Sigma) \). Meanwhile, thanks to Proposition 2.9 the set of cluster variables of \( \sigma \Sigma \) coincides with the one of \( \Sigma \). Therefore, \( \mathcal{A}(\sigma \Sigma) = \mathcal{A}(\Sigma) \).

Now we are ready to state the classification of cluster patterns/algebras of finite type. In view of Proposition 2.17 it is enough to concentrate on the cluster patterns whose exchange matrices are indecomposable.

**Theorem 3.15** (Finite type classification [FZ03]). For cluster patterns with indecomposable exchange matrices, the following facts hold.

(a). The cluster pattern \( \Sigma \) is of finite type if and only if there is some \( t \in \mathbb{T}_n \) such that the Cartan matrix \( A(B_t) \) associated with \( B_t \) is of finite type.

(b). The Dynkin type of \( A(B_t) \) does not depend on the choice of such \( t \).

By Theorem 3.15 any isomorphism class of cluster patterns of finite type with indecomposable exchange matrices is uniquely labeled by the Dynkin type of the corresponding Cartan matrix \( A(B_t) \) therein. We have already used this labeling for the rank 2 examples in Section 2.3.

For a cluster pattern of finite type, the root system of the corresponding type naturally parametrizes the cluster variables.

**Definition 3.16** (Denominator vector/Non-initial cluster variable). Let \( \Sigma \) be any cluster pattern with a given initial vertex \( t_0 \).

- For any cluster variable \( x_{i:t} \), we define an integer vector \( d_{i:t} = (d_{j:i:t})_{j=1}^n \) such that \( -d_{j:i:t} \) is the lowest degree of \( x_{j:t_0} \) in the Laurent polynomial expression of \( x_{i:t} \) in \( x_{t_0} \). The vector \( d_{i:t} = (d_{j:i:t})_{j=1}^n \) is called the denominator vector (or \( d \)-vector; for short) of \( x_{i:t} \). In particular, for an initial variable \( x_{i:t_0} \), we have \( d_{i:t_0} = -e_i \).
- We say that a cluster variable \( x_{i:t} \) is **non-initial** if it does not coincide with any initial cluster variables \( x_{1:t_0}, \ldots, x_{n:t_0} \).
Example 3.17. In the examples in Section 2.3, we see the following list of the denominator vectors for the non-initial cluster variables.

\[(A_2) : (1, 0), (1, 1), (0, 1), \quad \text{(3.23)}\]
\[(B_2) : (1, 0), (1, 1), (1, 2), (0, 1), \quad \text{(3.24)}\]
\[(G_2) : (1, 0), (1, 1), (1, 2), (1, 3), (2, 3), (0, 1). \quad \text{(3.25)}\]

They are naturally identified with the positive roots of the corresponding root systems. See Figure 2.

This phenomenon can be fully generalized to any cluster pattern of finite type if we properly choose the initial vertex \( t_0 \).

Theorem 3.18 (FZ03). Let \( \Sigma \) be any cluster pattern of finite type with an initial vertex \( t_0 \) such that \( A(B_{t_0}) \) is a Cartan matrix of finite type. Then, the denominator vector of any non-initial cluster variable is identified with a positive root of the corresponding root system. Moreover, this correspondence is one-to-one.

3.3 Cluster algebras of geometric type

Below we focus on a class of cluster patterns whose coefficient semifields are tropical semifields. They are important in various applications of cluster algebras.

Let \( \text{Trop}(u) \) be the tropical semifield in Example 1.10.

Definition 3.19 (Geometric type). A cluster pattern (resp. cluster algebra) with coefficients in a tropical semifield \( \text{Trop}(u) \) is called a cluster pattern (resp. cluster algebra) of geometric type, where \( u = (u_1, \ldots, u_m) \) and \( m \) is taken independently of the rank \( n \) of a cluster pattern.

Remark 3.20. The terminology originates in the fact that cluster algebras of this type typically arise as the coordinating rings of certain algebraic varieties, which are called the geometric realization of cluster algebras in FZ03.

Let \( \Sigma \) be a cluster pattern of geometric type. Recall that an element of \( \text{Trop}(u) \) is a Laurent monomial in \( u \) with coefficient 1. Therefore, any
3.3. Cluster algebras of geometric type

Coefficient $y_{i;t}$ of $\Sigma$ at $t \in \mathbb{T}_n$ is represented as

$$y_{i;t} = \prod_{j=1}^{m} u_j^{c_{ij;t}}, \quad i = 1, \ldots, n. \quad (3.26)$$

This determines an $m \times n$ integer matrix $C_t = (c_{ij;t})$ for each $t \in \mathbb{T}_n$, and the coefficient tuple $y_t$ and the matrix $C_t$ are identified. Under this identification, the mutation in (2.3) is translated as follows.

**Proposition 3.21.** Let $t, t' \in \mathbb{T}_n$ be $k$-adjacent vertices. Then, we have the following mutation formula of matrices $C_t$:

$$c_{ij;t'} = \begin{cases} -c_{ik;t} & j = k, \\ c_{ij;t} + c_{ik;t} [b_{kj;t}]_+ + [-c_{ik;t}]_+ b_{kj;t} & j \neq k. \end{cases} \quad (3.27)$$

**Proof.** For any integer $a$, the following identity holds:

$$\min(0, a) = -[-a]_+. \quad (3.28)$$

Thus, by the definition of the tropical sum in (1.21), we have

$$1 \oplus y_{k;t} = \prod_{i=1}^{m} u_i^{-[-c_{ik,t}]_+}. \quad (3.29)$$

By replacing the index $i$ in (2.3) with $j$, and comparing the power of $u_i$, we obtain (3.27). \qed

Observe that the formula (3.27) is parallel to the mutation (2.4) of the exchange matrix $B_t$. This motivates us to define the following $(n + m) \times n$ matrix

$$\tilde{B}_t = \begin{pmatrix} B_t \\ C_t \end{pmatrix}, \quad (3.30)$$

which is called an extended exchange matrix of $\Sigma$. Abusing the notation, let us write the $(i, j)$-entry of $\tilde{B}_t$ as $b_{ij}$. Then, we can unify the mutations (2.4) and (3.27) as a single matrix mutation of $\tilde{B}_t$ defined by

$$b_{ij;t'} = \begin{cases} -b_{ij;t} & i = k \text{ or } j = k, \\ b_{ij;t} + b_{ik;t} [b_{kj;t}]_+ + [-b_{ik;t}]_+ b_{kj;t} & i, j \neq k, \end{cases} \quad (3.31)$$

where $i = 1, \ldots, n + m$, and $j, k = 1, \ldots, n$. Next, let us observe that the mutations of cluster variables can be also expressed in terms of the matrix $\tilde{B}_t$. By (3.29), we have

$$\frac{y_{k;t}}{1 \oplus y_{k;t}} = \prod_{j=1}^{m} u_j^{[c_{jk,t}]_+}, \quad \frac{1}{1 \oplus y_{k;t}} = \prod_{j=1}^{m} u_j^{-[c_{jk,t}]_+}. \quad (3.32)$$
Thus, the mutation in the form of Eq. (2.5) is written as
\[
x_{i:t'} = \begin{cases} 
\frac{1}{x_{k:t}} \left( \prod_{j=1}^{m} u_j^{[c_{jk,t}]_+} \prod_{j=1}^{n} x_{j:t}^{[b_{jk,t}]_+} + \prod_{j=1}^{m} u_j^{[-c_{jk,t}]_+} \prod_{j=1}^{n} x_{j:t}^{[-b_{jk,t}]_+} \right) & i = k, \\
x_{i:t} & i \neq k.
\end{cases}
\] (3.33)

To summarize the result so far, the coefficients \( y_t \) of geometric type can be safely replaced with the lower part \( C_t \) of the extended exchange matrices \( \tilde{B}_t \).

To make the picture complete, we introduce an extended cluster,
\[
\tilde{x}_t = (x_{1:t}, \ldots, x_{n+m:t}) := (x_{1:t}, \ldots, x_{n:t}, u_1, \ldots, u_m).
\] (3.34)

Then, together with the extended exchange matrix notation, the mutation (3.33) is written as, for \( k = 1, \ldots, n, \)
\[
x_{i:t'} = \begin{cases} 
\frac{1}{x_{k:t}} \left( \prod_{j=1}^{n+m} x_{j:t}^{[b_{jk,t}]_+} + \prod_{j=1}^{n+m} x_{j:t}^{[-b_{jk,t}]_+} \right) & i = k, \\
x_{i:t} & i \neq k.
\end{cases}
\] (3.35)

This formally coincides with the mutation of cluster variables without coefficients.

Let us formulate this observation more completely in terms of a cluster pattern of rank \( n + m \) without coefficients. We consider the extension of \( \tilde{B}_{t_0} \) to a full \( (n + m) \times (n + m) \) skew-symmetrizable matrix. Such an extension (together with its skew-symmetrizer) is not unique. To give an example, let \( D \) be a skew-symmetrizer of \( B_{t_0} \), and let \( d \) be the least common multiple of the diagonal entries of \( D \). Then, the following matrix gives a skew-symmetrizable extension of \( \tilde{B}_{t_0} \):
\[
\hat{B}_{t_0} = \begin{pmatrix} B_{t_0} & -D^{-1}dC_{t_0}^T \\ C_{t_0} & 0 \end{pmatrix},
\] (3.36)
where a skew-symmetrizer is given by \( \hat{D} = D + dI_m \). (The factor \( d \) ensures that the matrix \( \hat{B}_{t_0} \) is an integer matrix.) For a given initial vertex \( t_0 \) in \( T_{n+m} \), we regard \( T_n \) as a subtree of \( T_{n+m} \) such that \( T_n \) contains \( t_0 \) and all vertices that are sequentially adjacent to \( t_0 \) with edges labeled by \( \{1, \ldots, n\} \) in \( T_{n+m} \). Then, the above cluster pattern \( \Sigma \) of geometric type is equivalent to consider a cluster pattern of rank \( n + m \) without coefficients, restricted to the subtree \( T_n \subset T_{n+m} \), and with the initial seed \( (\tilde{x}_{t_0}, \hat{B}_{t_0}) \). The elements \( x_{n+1:t} = x_{n+1}, \ldots, x_{n+m:t} = x_{n+m} \) are called the frozen variables, because they are not mutated. We note that the right half of \( \hat{B}_t \) does not influence neither (3.31) nor (3.35). Thus, we may safely replace seeds \( (\tilde{x}_t, \hat{B}_t) \) with \( (\tilde{x}_t, \tilde{B}_t) \) without losing information.

Let us summarize the results obtained above as a proposition:
Proposition 3.22. We have the following three equivalent presentations of a cluster pattern of geometric type:

- a cluster pattern consisting seeds \((x_t, y_t, B_t)\) with coefficients in \(\text{Trop}(u)\), by definition.
- a cluster pattern consisting seeds \((x_t, \tilde{B}_t)\) whose coefficients in \(\text{Trop}(u)\) are encoded in the extend exchange matrices \(\tilde{B}_t\).
- a cluster pattern consisting extended seeds \((\tilde{x}_t, \hat{B}_t)\) or \((\tilde{x}_t, \tilde{B}_t)\) without coefficients and with frozen variables \(x_{n+1}, \ldots, x_{n+m}\) that are identified with \(u_1, \ldots, u_m\). (In this case we still say that it is of rank \(n\).)

Even if the above three pictures are equivalent as cluster patterns, there is some discrepancy for the corresponding cluster algebras due to the difference of the coefficient semifields. Note that, for \(\mathbb{P} = \text{Trop}(u)\), the group ring \(\mathbb{Z}\mathbb{P}\) is identified with the Laurent polynomial ring \(\mathbb{Z}[u^{\pm 1}]\). Then, for the first two pictures, the corresponding cluster algebras are given by
\[
\mathcal{A} = \mathbb{Z}[u^{\pm 1}][x_t]_{t \in \mathbb{T}_n} = \mathbb{Z}[x_t, u^{\pm 1}]_{t \in \mathbb{T}_n}, \quad (3.37)
\]
while, for the last one, including the frozen variables as honorary members, the corresponding cluster algebra is given by
\[
\mathcal{A} = \mathbb{Z}[\tilde{x}_t]_{t \in \mathbb{T}_n} = \mathbb{Z}[x_t, u]_{t \in \mathbb{T}_n}. \quad (3.38)
\]
Therefore, one has to be careful in which sense cluster algebra is considered.

For a cluster pattern of geometric type, the Laurent phenomenon in Theorem 3.1 claims
\[
x_{i;t} \in \mathbb{Z}[u^{\pm 1}][x_t]_{t \in \mathbb{T}_n} = \mathbb{Z}[x_t, u^{\pm 1}]. \quad (3.39)
\]

Actually, the following strong version of the Laurent phenomenon holds.

Theorem 3.23 ([FZ03]). For any cluster pattern of geometric type, we have
\[
x_{i;t} \in \mathbb{Z}[x_t^{\pm 1}, u]. \quad (3.40)
\]

Proof. We consider the following claim, which is slightly stronger than the desired result.

Claim. For each \(j\), \(x_{i;t}\) is a polynomial in \(u_j\) whose constant term (with respect to \(u_j\)) is a nonzero polynomial with a subtraction-free rational expression in \(x_{i_0}^{\pm 1}\) and \(u\) other than \(u_j\). (See Definition 1.5 for a subtraction-free expression.)

We prove the claim by the induction on the distance \(d = d(t_0, t)\) in \(\mathbb{T}_n\). For \(d = 0\), we have \(x_{i;t} = x_i\), so that the claim holds. Suppose that the claim holds for \(d = d(t_0, t)\). Let \(t' \in \mathbb{T}_n\) be \(k\)-adjacent to \(t\). It is enough to prove the claim for \(x_{k;t'}\). Let us look at the first case of (3.33). Note
that, for each \( j \), either \([c_{jk}; t]_+\) or \([-c_{jk}; t]_+\) is zero. Thus, by the induction assumption, the numerator in (3.33) is a polynomial in \( u_j \) whose constant term is a nonzero polynomial with a subtraction-free rational expression in \( x_{t_0}^{\pm 1} \) and \( u \) other than \( u_j \). (Here, the condition of having a subtraction-free expression guarantees that, even in the special case \( c_{jk}; t = 0 \), the constant terms from the two products in (3.33) never cancel accidentally.) Also, the denominator \( x_{kt} \) has the same property. Since we already know that \( x_{kt} \) is in \( \mathbb{Z}[x_{t_0}^{\pm 1}, u^{\pm 1}] \), the denominator divides the numerator such that the result is a polynomial in \( u_j \) having the same property. \( \square \)

**Remark 3.24.** The above claim in the proof does not imply that \( x_{kt} \) has a nonzero constant term in \( u \).

### 3.4 Grassmannian \( \text{Gr}(2, 5) \)

Let us present a prototypical example of a cluster algebra (of geometric type) appearing in Lie theory.

First, we briefly recall basic definitions/facts on Grassmannian \( \text{Gr}(2, 5) \). See [Ful97, Section 9] for the background and more information.

**Definition 3.25 (Grassmannian \( \text{Gr}(2, 5) \)).**

- In short, the Grassmannian \( \text{Gr}(2, 5) \) is the complex projective variety consisting of all 2-dimensional subspaces in a 5-dimensional vector space \( V \) over \( \mathbb{C} \).
- An element of \( \text{Gr}(2, 5) \) is identified with its basis, which is represented by a \( 2 \times 5 \) complex matrix \( M = (m_{ij}) \) of rank 2 modulo the left action of \( GL(2, \mathbb{C}) \). This tells us that \( \dim \text{Gr}(2, 5) = 10 - 4 = 6 \).
- For each pair \( 1 \leq i < j \leq 5 \), let
  \[ p_{ij} = p_{ij}(M) := \begin{vmatrix} m_{1i} & m_{1j} \\ m_{2i} & m_{2j} \end{vmatrix}. \]  (3.41)

Then, we define a map \( i : \text{Gr}(2, 5) \rightarrow \mathbb{C}P^9 \) by
  \[ i : M \mapsto [p_{12} : p_{13} : \cdots : p_{45}], \] (3.42)
where \([z_1 : \cdots : z_{10}]\) is the homogeneous coordinates of the complex projective space \( \mathbb{C}P^9 \). The map is injective, and it is called the Plücker embedding, while \( p_{ij} \) are called the Plücker coordinates.

- For \( 1 \leq i < j < k < \ell \leq 5 \), the Plücker coordinates satisfy the relations
  \[ R_{ijkt} : p_{ij}p_{kt} - p_{ik}p_{jt} + p_{it}p_{jk} = 0. \] (3.43)
They are called the Plücker relations. Note that there are five Plücker relations, \( R_{1234}, R_{1235}, R_{1245}, R_{1345}, \) and \( R_{2345} \).
The homogeneous coordinate ring of $\text{Gr}(2, 5)$ is given by
\[\mathbb{C}[\text{Gr}(2, 5)] = \mathbb{C}[\mathbf{P}] / I_R, \quad \mathbf{P} = (p_{ij})_{1 \leq i < j \leq 5},\] (3.44)
where $I_R$ is the homogeneous ideal generated by the Plücker relations; moreover, $I_R$ is prime. Thus, $\text{Gr}(2, 5)$ is a projective variety in $\mathbb{C}P^9$.

Alternatively, we may consider the affine cone $\tilde{\text{Gr}}(2, 5)$ over $\text{Gr}(2, 5)$, that is, the affine variety in the complex affine space $\mathbb{A}^{10}$ defined by the same Plücker relations (3.43). The coordinate ring of $\tilde{\text{Gr}}(2, 5)$ has the same form as (3.44),
\[\mathbb{C}[\tilde{\text{Gr}}(2, 5)] = \mathbb{C}[\mathbf{P}] / I_R.\] (3.45)
However, the ring in (3.44) is regarded as a graded ring, while it is a (not graded) ring here.

We are going to show that $\mathbb{C}[\text{Gr}(2, 5)]$ (or $\mathbb{C}[\tilde{\text{Gr}}(2, 5)]$ by forgetting the grading) has the cluster algebra structure. To do that, we consider a cluster pattern of geometric type, where there are two cluster variables $x_i; t$ ($i = 1, 2$) and five frozen variables $x_a, \ldots, x_e$. The cluster pattern $\Sigma$ we consider is encoded in the following initial extended exchange matrix $\tilde{B}_0$, or equivalently, the corresponding initial extended quiver $\tilde{Q}_0$, where the vertices $a, \ldots, e$ are frozen vertices and we will mutate only at the vertices 1 and 2.

\[
\tilde{B}_0 = \begin{pmatrix}
0 & -1 \\
1 & 0 \\
-1 & 0 \\
1 & 0 \\
-1 & 1 \\
0 & -1 \\
0 & 1 \\
\end{pmatrix}, \quad \tilde{Q}_0 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\begin{array}{c}
b \\
c \\
d \\
a \\
1 \\
2 \\
e \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\] (3.46)

In particular, this is a special case of cluster pattern of type $A_2$.

In addition, we introduce an alternative presentation of the above data by a triangulation of a pentagon, where the frozen vertices of the quiver correspond to the sides of the pentagon, while the unfrozen vertices correspond to two diagonals as follows:

\[
T_0 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\begin{array}{c}
a \\
b \\
c \\
d \\
e \\
1 \\
2 \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\] (3.47)
To explain in more detail, we identify a subquiver in $\tilde{Q}_0$ and a triangle in $T_0$ by the following correspondence:

![Diagram](image)

where we regard that the arrows among frozen vertices (i.e, $a \rightarrow b$ and $d \rightarrow e$) are omitted in $\tilde{Q}_0$. (This is reasonable, because they do not influence the mutations of seeds $(\tilde{x}_t, \tilde{B}_t)$ at all.)

The advantage of the presentation by a triangulation is that one can manage a mutation of a seed pictorially as a flip of a diagonal. For example, the mutation of the quiver $\tilde{Q}_0$ at the vertex 1 gives the following quiver $\tilde{Q}_1$, where again the arrows among frozen vertices are omitted:

![Diagram](image)

The corresponding triangulation $T_1$ is given by the flip of the diagonal 1 inside the quadrilateral with sides $a, b, c, 2$ depicted as below:

![Diagram](image)

Meanwhile, the mutation of the cluster variable $x_1$ is given by

$$x'_1 = \frac{x_b x_2 + x_a x_c}{x_1}.$$  

By rewriting it in the form

$$x_1 x'_1 = x_b x_2 + x_a x_c,$$  

we can recognize it as *Ptolemy’s theorem* for cyclic quadrilaterals.

The mutations of these data are presented up to $t = 5$ in Table [1] where we observe the familiar pentagon periodicity of type $A_2$ as the periodicity of the alternative flips in a pentagon. In fact, this is the reason why we call this periodicity so.

**Remark 3.26.** This is indeed a prototypical example of the *surface realization* of cluster patterns/algebras developed by Fock-Goncharov [FG07]
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| $t$ | $\tilde{B}_t$ | $\tilde{Q}_t$ | $T_t$ | $x_{1:t}, x_{2:t}$ |
|-----|--------------|--------------|------|------------------|
| 0   | $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ | $\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$ | $\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ a \\ b \\ c \\ d \end{array}$ | $x_{13}, x_{14}$ |
| 1   | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$ | $\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ a \\ b \\ c \\ d \end{array}$ | $x_{24}, x_{14}$ |
| 2   | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$ | $\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ a \\ b \\ c \\ d \end{array}$ | $x_{24}, x_{25}$ |
| 3   | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$ | $\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ a \\ b \\ c \\ d \end{array}$ | $x_{35}, x_{25}$ |
| 4   | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$ | $\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ a \\ b \\ c \\ d \end{array}$ | $x_{35}, x_{13}$ |
| 5   | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$ | $\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ a \\ b \\ c \\ d \end{array}$ | $x_{14}, x_{13}$ |

Table 1: Mutations of extended exchange matrices.
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and Fomin-Shapiro-Thurston [FST08]. Only a limited class of cluster patterns/algebras are realized in this way. In [FST08] a complete classification of the class of cluster algebras that admit the surface realization is given.

Let $\mathcal{A}$ be the cluster algebra associated with the cluster pattern $\Sigma$ in the sense of (3.38). In order to match $\mathbb{C}[\text{Gr}(2, 5)]$ in (3.44), we replace the ground ring $\mathbb{Z}$ with $\mathbb{C}$. In view of the result in Example 2.18, it is given by

$$\mathcal{A} = \mathbb{C}[x_{1;0}, x_{2;0}, x_{1;1}, x_{2;2}, x_{1;3}, x_a, \ldots, x_e].$$  

(3.53)

We regard it as a graded ring such that every generator is of degree 1.

We have the following conclusion.

Theorem 3.27 ([FZ02a, FZ03]). The graded ring $\mathbb{C}[\text{Gr}(2, 5)]$ is isomorphic to the cluster algebra $\mathcal{A}$ in (3.53). In particular, $\mathbb{C}[\text{Gr}(2, 5)]$ has a cluster algebra structure.

Proof. Let $[ij]$ be the chord (diagonal or side) whose end points are labeled by $i$ and $j$ in the underlying pentagon of $T_t$. We give the following labeling of cluster/frozen variables of $\mathcal{A}$:

- Each frozen variable $x_i$ ($i = a, \ldots, e$) is labeled as $x_{jk}$ if the side labeled by $i$ is $[jk]$. Explicitly,

$$x_a = x_{12}, \quad x_b = x_{23}, \quad x_c = x_{34}, \quad x_d = x_{45}, \quad x_e = x_{15}.$$  

(3.54)

- Each cluster variable $x_{i;t}$ ($i = 1, 2$) is labeled as $x_{jk}$ if the diagonal labeled by $i$ in $T_t$ is $[jk]$ in Table 1. In particular,

$$x_{1;0} = x_{13}, \quad x_{2;0} = x_{14}, \quad x_{1;1} = x_{24}, \quad x_{2;2} = x_{25}, \quad x_{1;3} = x_{35}.$$  

(3.55)

The cluster algebra $\mathcal{A}$ in (3.53) is now written as

$$\mathcal{A} = \mathbb{C}[\mathbf{X}], \quad \mathbf{X} = (x_{ij})_{1 \leq i < j \leq 5}.$$  

(3.56)

On the other hand, the mutation from $t = 0$ to 1 in (3.52) is identified with the Plücker relation

$$R_{1234} : \quad x_{13}x_{24} = x_{14}x_{23} + x_{12}x_{34}.$$  

(3.57)

Similarly, the mutations from $t = 1$ to 5 are identified with the Plücker relations $R_{1245}, R_{2345}, R_{1325}, R_{1345}$. Therefore, we have a graded ring homomorphism

$$\varphi : \mathbb{C}[\text{Gr}(2, 5)] = \mathbb{C}[\mathbf{P}]/I_R \rightarrow \mathcal{A} = \mathbb{C}[\mathbf{X}]$$  

(3.58)

The map $\varphi$ is clearly surjective. To show that $\varphi$ is injective, we rely on the following facts:
3.4. Grassmannian $\text{Gr}(2, 5)$

- The ideal $I_R$ is prime as stated in Definition 3.25.
- In particular, the ideal $I_R$ contains no monomial in $P$. (If there is a monomial in $I_R$, it should be at least quadratic, which contradict the primeness.)

Suppose that a polynomial $F(P)$ in $P = (p_{ij})$ is in Ker $\varphi$. Namely, $F(X) = 0$ in $A$. By multiplying some monomial $m(X)$ to $F(X)$ and applying the mutation relations (the Plücker relations for $X$), one can eliminate all non-initial cluster variables $x_{24}$, $x_{25}$, $x_{35}$ and obtain a polynomial $G(x_0)$ in the initial cluster variables $x_0$. Namely, we have $G(x_0) = m(X)F(X) = 0$ in $A$. Then, due to the algebraic independence of $x_0$, $G(x_0)$ is the zero polynomial. Applying the same operation to $F(P)$ in $\mathbb{C}[P]$, we have $m(P)F(P) \equiv 0 \text{ mod } I_R$. Since $m(P) \neq 0$, we have $F(P) \equiv 0$ by the primeness of $I_R$.  

Remark 3.28. It was shown by Scott [Sco06] that the homogeneous coordinate ring of a general Grassmannian $\text{Gr}(k, m)$ also has the cluster algebra structure.
4 Separation formulas

In this section we present the separation formulas given in CA4. They are particularly important to study the structure of seeds in a cluster pattern systematically.

4.1 Principal coefficients

We introduce the notion of cluster patterns with principal coefficients. They are a special class of cluster patterns of geometric type and play an important role in studying cluster patterns.

**Definition 4.1.** We say that a cluster pattern $\Sigma$ is with principal coefficients at $t_0 \in T_n$ if the following conditions are satisfied:

- The coefficient semifield of $\Sigma$ is a tropical semifield $\text{Trop}(y)$ with generators $y = (y_1, \ldots, y_n)$, where $n$ is the rank of $\Sigma$.
- The coefficient tuple $y_{t_0}$ at $t_0$ coincides with $y$.

The same remark in Remark 2.24 is applicable for the notation.

In other words, a cluster pattern with principal coefficients at $t_0$ is a cluster pattern of geometric type such that the extended exchange matrix in (3.30) at $t_0$ is given by

$$\tilde{B}_{t_0} = \begin{pmatrix} B_{t_0} \\ I \end{pmatrix}. \quad (4.1)$$

All properties of cluster patterns of geometric type hold to cluster patterns with principal coefficients.

**Remark 4.2.** In contrast to a cluster pattern of free coefficients in Definition 2.22, the notion of principal coefficients crucially depends on the choice of the base vertex $t_0$. To explain it in more detail, let $y_t$ and $y'_t$ ($t \in T_n$) are principal coefficients at base vertices $t_0$ and $t'_0$, respectively. Then, the correspondence $\varphi : y_{i;t} \mapsto y'_{i;t}$ cannot be extended to a semifield homomorphism from $\text{Trop}(y_{t_0})$ to $\text{Trop}(y'_{t'_0})$, in general.

Let $\Sigma$ be a cluster pattern with principal coefficients at $t_0$. One may regard the coefficients of $\Sigma$ as the tropicalization of free coefficients at $t_0$. Namely, let $\Sigma'$ be a cluster pattern with free coefficients at $t_0$ such that it shares the common $B$-pattern with $\Sigma$. Then, as a special case of Proposition 2.25, $y'_t$ and $y_t$ are related by the tropicalization homomorphism in (1.24) by

$$\pi_{\text{trop}} : Q_{sf}(y) \rightarrow \text{Trop}(y)$$

$$y'_{i;t} \mapsto y_{i;t}.$$  \quad (4.2)
4.2 C- and G-matrices, and F-polynomials

For a given cluster pattern $\Sigma$ with principal coefficients at $t_0$, one can define important quantities called C-matrices, G-matrices, and F-polynomials. They are the building blocks of seeds of a cluster pattern with coefficients in any semifield $\mathbb{P}$, and together with the separation formula in Theorem 4.16, they clarify the structure of seeds as well as the relation between cluster variables and coefficients.

(a). C-matrices. Let us start with C-matrices. They are the matrices defined in (3.26) specialized for a cluster pattern $\Sigma$ with principal coefficients at $t_0$.

**Definition 4.3** (C-matrix/c-vector). For a given cluster pattern $\Sigma$ with principal coefficients at $t_0$, the C-matrix $C_t = (c_{ij};t)$ of $y_t$ is the $n \times n$ integer matrix defined by

$$y_{i;t} = \prod_{j=1}^{n} y_{j}^{c_{ji};t}. \quad (4.3)$$

Equivalently, they are defined as the lower half of the $2n \times n$ extended exchange matrix $\tilde{B}_t$ of (3.30) with the initial condition (4.1). The $i$th column vector $c_{i;t} = (c_{ji})_{j=1}^{n}$ of $C_t$ is called the c-vector of $y_{i;t}$. We call the collection of the C-matrices $C_{t_0} = \{C_t\}_{t \in T_n}$ the C-pattern of $\Sigma$.

The C-matrices are uniquely determined by the underlying $B$-pattern $B$ of $\Sigma$ and $t_0$, thus, eventually by $t_0$ and $B_{t_0}$ only.

**Proposition 4.4.** The C-pattern $C_{t_0}$ of $\Sigma$ is uniquely determined by the following initial condition and the mutation formula:

$$C_{t_0} = I, \quad (4.4)$$

$$c_{ij;t'} = \begin{cases} -c_{ik;t} & j = k; \\ c_{ij;t} + c_{ik;t}[b_{kj};t]_+ + [-c_{ik;t}]_+ b_{kj;t} & j \neq k, \end{cases} \quad (4.5)$$

where $t$ and $t'$ are $k$-adjacent.

**Proof.** The initial condition (4.4) holds by (4.1). The mutation (4.5) was established in (3.27). \qed

For the second case of (4.5), we have an alternative expression, which is parallel to the $\varepsilon$-expressions in Proposition 2.7.

**Proposition 4.5.** The following expression does not depend on the choice of $\varepsilon \in \{1, -1\}$:

$$c_{ij;t} + c_{ik;t}[\varepsilon b_{kj};t]_+ + [-\varepsilon c_{ik;t}]_+ b_{kj;t}. \quad (4.6)$$

Proof. This follows from (1.36). Alternatively, one can derive each expression from (2.13) in the same way as (3.27).

(b). G-matrices. Next, we define G-matrices. By Theorem 3.23 we have

\[ x_{i;t} \in \mathbb{Z}[x^{\pm 1}, y], \]  

(4.7)

where we set \( x_{t_0} = x \). We introduce a certain degree vector in \( \mathbb{Z}^n \) for each cluster variables \( x_{i;t} \).

Definition 4.6 (Principal \( \mathbb{Z}^n \)-grading). For each monomial in \( \mathbb{Z}[x^{\pm 1}, y] \), we define its \( \mathbb{Z}^n \)-grading by

\[ \deg(x_i) = e_i, \quad \deg(y_i) = -b_{i;t_0}, \]  

(4.8)

where \( e_i \) is the \( i \)th unit vector and \( b_{i;t_0} \) is the \( i \)th column vector of \( B_{t_0} \). We call it the principal \( \mathbb{Z}^n \)-grading.

A seemingly artificial degree of \( y_i \) is designed to ensures the following property:

\[ \deg(\hat{y}_i) = \deg \left( y_i \prod_{j=1}^{n} x_j^{b_{j;i;t_0}} \right) = -b_{i;t_0} + b_{i;t_0} = 0. \]  

(4.9)

Lemma 4.7. Every \( x_{i;t} \in \mathbb{Z}[x^{\pm 1}, y] \) is homogenous with respect to the principal \( \mathbb{Z}^n \)-grading.

Proof. We first note that, thanks to Proposition 2.6 any \( \hat{y} \)-variable \( \hat{y}_{i;t} \) is written as a rational function of the initial \( \hat{y} \)-variables \( \hat{y} \). Then, we can effectively treat it as \( \deg(\hat{y}_{i;t}) = 0 \) in the following calculation, even though \( \hat{y}_{i;t} \) does not belong to \( \mathbb{Z}[x^{\pm 1}, y] \). We prove the claim by the induction on \( d = d(t_0, t) \). The claim is trivial for the initial cluster variables \( x_i \). Suppose that claim is true up to \( d = d(t_0, t) \). Let \( t' \in \mathbb{T}_n \) be \( k \)-adjacent to \( t \). We look at the mutation formula in the form (2.2),

\[ x_{k;t'} = x_{k;t}^{-1} \left( \prod_{j=1}^{n} x_j^{-[b_{j;k;t}]} \right) \frac{1 + \hat{y}_{k;t}}{1 \oplus y_{k;t}}. \]  

(4.10)

Then, the binomial \( 1 + \hat{y}_{k;t} \) is homogeneous with degree \( 0 \). Also, \( 1 \oplus y_{k;t} \) is actually a monomial in \( y \) because it belongs to \( \text{Trop}(y) \). Thus, the right hand side of (4.10) is homogeneous thanks to the induction assumption.

Based on Lemma 4.7 we define G-matrices as follows.
Definition 4.8 (G-matrix/g-vector). For a given cluster pattern Σ with principal coefficients at \(t_0\), we define a \(\mathbb{Z}^n\)-vector

\[ g_{i;t} = \deg(x_{i;t}) \in \mathbb{Z}^n, \tag{4.11} \]

where \(\deg\) is the principal \(\mathbb{Z}^n\)-grading in Definition 4.6. We call it the \(g\)-vector of \(x_{i;t}\). For each \(t \in T_n\), we define the \(G\)-matrix \(G_t = (g_{ij;t})\) of \(x_t\) such that its \(i\)th column vector is \(g_{i;t} = (g_{ji;t})_{j=1}^n\). We call the collection of the \(G\)-matrices \(\{G_t\}_{t \in T_n}\) the \(G\)-pattern of Σ.

Again, the \(G\)-pattern \(G_{t_0}\) is uniquely determined by the underlying \(B\)-pattern \(B\) and \(t_0\) (together with the \(C\)-pattern \(C_{t_0}\) determined from them).

Proposition 4.9. The \(G\)-pattern \(G_{t_0}\) of Σ is uniquely determined by the following initial condition and the mutation formula:

\[ G_{t_0} = I, \tag{4.12} \]

\[ g_{ij;t'} = \begin{cases} -g_{ik;t} + \sum_{\ell=1}^n g_{i\ell;t}[-b_{\ell k;t}]_+ - \sum_{\ell=1}^n b_{i\ell;t_0}[-c_{\ell k;t}]_+ & j = k, \\ g_{ij;t} & j \neq k, \end{cases} \tag{4.13} \]

where \(t\) and \(t'\) are \(k\)-adjacent.

Proof. The initial condition (4.12) follows from the fact \(\deg x_i = e_i\). The mutation (4.13) follows from (4.10) and the formula

\[ \deg \left( \frac{1}{1 + y_{k,t}} \right) = \deg \left( \prod_{j=1}^n y_j^{[-c_{jk;t}]_+} \right) = -\sum_{j=1}^n [-c_{jk;t}]_+ b_{j:t_0}, \tag{4.14} \]

where we used (3.32) in the first equality.

We have the following duality relation between \(C\)- and \(G\)-matrices.

Proposition 4.10. The following equality holds:

\[ G_t B_t = B_{t_0} C_t. \tag{4.15} \]

Proof. This follows from the formula

\[ \deg(\hat{y}_{i;t}) = \deg \left( y_{i;t} \prod_{j=1}^n x_{j;t}^{b_{j;i:t}} \right) = \deg \left( \prod_{j=1}^n y_j^{c_{j;i;t}} \prod_{j=1}^n x_{j;t}^{b_{j;i:t}} \right) = -B_{t_0} c_{i;t} + G_t b_{i;t}. \tag{4.16} \]

and the fact \(\deg(\hat{y}_{i;t}) = 0\).

For the first case of (4.13), we have an alternative expression, which is parallel to the \(\varepsilon\)-expressions in Proposition 2.7.
Proposition 4.11. The following expression does not depend on the choice of \( \varepsilon \in \{1, -1\} \):

\[
-g_{ik,t} + \sum_{\ell=1}^{n} g_{i\ell,t} [-\varepsilon b_{\ell k,t}]_+ - \sum_{\ell=1}^{n} b_{i\ell,t_0} [-\varepsilon c_{\ell k,t}]_+.
\]  

(4.17)

**Proof.** We take the difference of two expressions. Then, it vanishes thanks to (1.35) and (4.15). \( \square \)

(c). \( F \)-polynomials.

The following definition relies on the fact (4.7).

**Definition 4.12** (\( F \)-polynomial). For a given cluster pattern \( \Sigma \) with principal coefficients at \( t_0 \), we define polynomials \( F_i; t(y) \in \mathbb{Z}[y] \) in formal variables \( y = (y_1, \ldots, y_n) \) by specializing Laurent polynomials \( x_i; t(x, y) \in \mathbb{Z}[x^{\pm 1}, y] \) with \( x_1 = \cdots = x_n = 1 \). We call \( F_i; t(y) \) the \( F \)-polynomial of \( x_i; t \). Let \( F_t = \{F_1; t(y), \ldots, F_n; t(y)\} \). We call the collection of the \( F \)-polynomials \( F_t \) the \( F \)-pattern of \( \Sigma \).

Again, the \( F \)-pattern \( F_t \) is uniquely determined by the underlying \( B \)-pattern \( B \) and \( t_0 \) (together with the \( C \)-pattern \( C_t \) determined from them).

**Proposition 4.13.** The \( F \)-pattern \( F_t \) of \( \Sigma \) is uniquely determined by the following initial condition and the mutation formula:

\[
F_{i; t_0}(y) = 1, \tag{4.18}
\]

\[
F_{i; t'}(y) = \begin{cases} 
M_{k; t}(y) & i = k, \\
\frac{F_{k; t}(y)}{F_{i; t}(y)} & i \neq k,
\end{cases} \tag{4.19}
\]

where \( t \) and \( t' \) are \( k \)-adjacent, and

\[
M_{k; t}(y) = \prod_{j=1}^{n} y_j^{c_{jk; t}} \prod_{j=1}^{n} F_{j; t}(y)^{b_{jk; t}} + \prod_{j=1}^{n} y_j^{-c_{jk; t}} + \prod_{j=1}^{n} F_{j; t}(y)^{-b_{jk; t}}.
\]  

(4.20)

\[
= \left( \prod_{j=1}^{n} y_j^{-c_{jk; t}} \prod_{j=1}^{n} F_{j; t}(y)^{-b_{jk; t}} \right) \left( 1 + \prod_{j=1}^{n} y_j^{c_{jk; t}} \prod_{j=1}^{n} F_{j; t}(y)^{b_{jk; t}} \right). \tag{4.21}
\]

**Proof.** This is obtained by specializing the mutation of \( x_t \) in the form (3.33) with \( x_1 = \cdots = x_n = 1 \). \( \square \)

We may regard \( F \)-polynomials also as elements in \( \mathbb{Q}_{sf}(y) \) because the mutation in (4.19) is a subtraction-free operation. Then, we may apply the tropicalization homomorphism \( \pi_{trop}: \mathbb{Q}_{sf}(y) \to \text{Trop}(y) \) in (4.2).
Proposition 4.14. The following fact holds:

\[ \pi_{\text{trop}}(F_{i:t}(y)) = 1. \]  

(4.22)

Proof. We prove it by the induction on the distance \( d = d(t_0, t) \). For \( t = t_0 \), the claim holds by (4.18). Assume that the claim holds for \( d = d(t_0, t) \). Let \( t' \) be \( k \)-adjacent to \( t \). Note that, for each \( j \), either \([c_{jk};t]_+ \) or \([-c_{jk};t]_+ \) is zero. Then, \( \pi_{\text{trop}}(M_{k:t}(y)) = 1 \) for \( M_{k:t}(y) \) in (4.20). So, the claim also holds for \( t' \) by (4.19).

Remark 4.15. The fact (4.22) does not imply that \( F_{i:t}(y) \) has a nonzero constant term as a polynomial in \( y \). For example, \( \pi_{\text{trop}}(y_1 + y_2) = 1 \).

In summary, even if the patterns \( C^{t_0}, G^{t_0}, F^{t_0} \) are originally extracted from a cluster pattern \( \Sigma \) with principal coefficients at \( t_0 \), it turned out that they are uniquely and directly defined from the underlying \( B \)-pattern \( B \) and the base vertex \( t_0 \). Therefore, one can associate these patterns \( C^{t_0}, G^{t_0}, F^{t_0} \) to any cluster pattern \( \Sigma \) with coefficients in any semifield \( \mathbb{P} \).

4.3 Separation formulas

Let \( \Sigma \) be any cluster pattern with coefficients in any semifield \( \mathbb{P} \). Let \( t_0 \) be a given initial vertex \( t_0 \), and let \( C^{t_0}, G^{t_0}, F^{t_0} \) be the associated \( C \)-, \( G \)-, \( F \)-polynomials, respectively. As already mentioned, each \( F \)-polynomial \( F_{i:t}(y) \) belongs to \( \mathbb{Q}_{\text{sf}}(y) \). Following Definition 1.9 let \( F_{i:t}|_{P}(y) \) be the specialization of \( F_{i:t}(y) \) in \( \mathbb{P} \) at the initial coefficient \( y_{t_0} = y \) of \( \Sigma \), where the abuse of the symbol \( y \) does not cause a serious problem.

The cluster variables and coefficients of any cluster pattern are expressed by the initial cluster variables and coefficients together with \( C \)- and \( G \)-matrices, and \( F \)-polynomials. This is one of the most fundamental and useful properties of cluster patterns.

Theorem 4.16 (Separation Formulas \([FZ07]\)). Let \( \Sigma \) be any cluster pattern with coefficients in any semifield \( \mathbb{P} \) and with a given initial vertex \( t_0 \). Let

\[ x_{t_0} = x, \quad y_{t_0} = y, \quad \dot{y}_{t_0} = \dot{y} \]  

(4.23)

be the initial cluster variables, coefficients, and \( \dot{y} \)-variables. Then, the following formulas hold.

\[ x_{i:t} = \left( \prod_{j=1}^{n} x_j^{g_{ji};t} \right) \frac{F_{i:t}(\dot{y})}{F_{i:t}|_{P}(y)}, \]  

(4.24)

\[ y_{i:t} = \left( \prod_{j=1}^{n} y_j^{c_{ji};t} \right) \prod_{j=1}^{n} F_{j:t}|_{P}(y)^{b_{ji};t}, \]  

(4.25)
I.4. Separation formulas

\[ \hat{y}_{i:t} = \left( \prod_{j=1}^{n} \hat{y}_{j}^{c_{j:i:t}} \right) \prod_{j=1}^{n} F_{j:t}(\hat{y})^{b_{j:i:t}}. \]  

(4.26)

**Proof.** The formulas hold for \( t = t_{0} \) due to the initial condition of \( C- \) and \( G- \)matrices and \( F- \)polynomials in Propositions 4.4, 4.9, and 4.13. Therefore, it is enough to show that their right hand sides mutate in the same way as \( x_{i:t}, y_{i:t}, \hat{y}_{i:t} \) based on the mutations of \( C_{t}, G_{t}, \) and \( F_{i:t} \).

We first treat the formula (4.25). Let \( y_{i:t} \) temporarily denote the one in the right hand side of (4.25). Let \( t' \in \mathbb{T}_{n} \) be \( k \)-adjacent to \( t \). For \( i = k \),

\[ y_{k:t'} = \left( \prod_{j=1}^{n} y_{j}^{c_{j:k:t}} \right) \prod_{j=1}^{n} F_{j:t|\mathbb{P}(\mathbf{y})}^{-b_{j:k:t}} = y_{k:t}, \]  

(4.27)

where in the first equality we used the fact \( b_{kk:t} = 0 \). (Below we do not repeat this remark.) For \( i \neq k \),

\[ y_{i:t'} = \left( \prod_{j=1}^{n} y_{j}^{c_{j:i:t} + b_{j:k:t}} \right) \prod_{j=1}^{n} F_{j:t|\mathbb{P}(\mathbf{y})}^{b_{j:k:t}} \]  

times \( \prod_{j \neq k} \left( \prod_{j=1}^{n} y_{j}^{c_{j:k:t}} \prod_{j=1}^{n} F_{j:t|\mathbb{P}(\mathbf{y})}^{-b_{j:k:t}} \right) \) \times \left( 1 + \prod_{j=1}^{n} y_{j}^{c_{j:k:t}} \prod_{j=1}^{n} F_{j:t|\mathbb{P}(\mathbf{y})}^{-b_{j:k:t}} \right) \]  

\[ = y_{i:t} y_{k:t}^{b_{k:i:t}} (1 + y_{k:t} y_{i:k:t}^{b_{k:i:t}}). \]  

(4.28)

The formula (4.26) is proved in the exactly same manner as above.

Now we treat (4.24). Let \( x_{i:t} \) temporarily denote the one in the right hand side of (4.24). For \( i \neq k \), \( x_{i:t} = x_{i:t} \) holds. For \( x_{k:t'} \), we have

\[ x_{k:t'} = \left( \prod_{j=1}^{n} x_{j}^{g_{j:k:t} + \sum_{t=1}^{n} g_{j:t}^{[b_{j:k:t}]}} \right) \]  

\[ \times \frac{F_{k:t}(\hat{y})^{-1} \prod_{j=1}^{n} y_{j}^{c_{j:k:t}} \prod_{j=1}^{n} F_{j:t}(\hat{y})^{-b_{j:k:t}}}{F_{k:t|\mathbb{P}(\mathbf{y})}^{-1} \prod_{j=1}^{n} y_{j}^{c_{j:k:t}} \prod_{j=1}^{n} F_{j:t|\mathbb{P}(\mathbf{y})}^{-b_{j:k:t}}}. \]
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\[ 1 + \prod_{j=1}^{n} \hat{y}_{j}^{c_{j}} \prod_{j=1}^{n} F_{j; t} (\hat{y})^{b_{j}} \]
\[ \times \frac{n \prod_{j=1}^{n} y_{j}^{c_{j}} \prod_{j=1}^{n} F_{j; t} (y)^{b_{j}}}{1 \oplus n \prod_{j=1}^{n} y_{j}^{c_{j}} \prod_{j=1}^{n} F_{j; t} (y)^{b_{j}}} \]
\[ = x_{k; t}^{-1} \left( \prod_{j=1}^{n} x_{j; t}^{-\left[b_{j} + b_{j} \right]} \right) \frac{1 + \hat{y}_{k; t}}{1 \oplus y_{k; t}}, \]

where we used (4.25) and (4.26) in the last equality.

\[ \square \]

Remark 4.17. Originally in [FZ07], the formula (4.24) was referred to as the separation formula, where the additions $\oplus/+$ in $\mathbb{F}/\mathbb{F}$ are separated in the denominator and the numerator. Here we also included the formulas (4.25) and (4.26) in the family, because they together exhibit a duality relation between cluster variables and coefficients/\hat{y}-variables.

Example 4.18 (Type $A_2$). Let us consider the cluster pattern of type $A_2$ in Example 2.18. By comparing the result therein with the separation formulas (4.24) and (4.25), one can easily read off $C$- and $G$-matrices and $F$-polynomials as follows:

\[
C_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} F_{1;0}(y) = 1, \\ F_{2;0}(y) = 1, \end{cases} \tag{4.29}
\]

\[
C_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} F_{1;1}(y) = 1 + y_1, \\ F_{2;1}(y) = 1, \end{cases} \tag{4.30}
\]

\[
C_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} F_{1;2}(y) = 1 + y_1, \\ F_{2;2}(y) = 1 + y_2 + y_1y_2, \end{cases} \tag{4.31}
\]

\[
C_3 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \begin{cases} F_{1;3}(y) = 1 + y_2, \\ F_{2;3}(y) = 1 + y_2 + y_1y_2, \end{cases} \tag{4.32}
\]

\[
C_4 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{cases} F_{1;4}(y) = 1 + y_2, \\ F_{2;4}(y) = 1, \end{cases} \tag{4.33}
\]

\[
C_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{cases} F_{1;5}(y) = 1, \\ F_{2;5}(y) = 1. \end{cases} \tag{4.34}
\]

We have the following observations.

(a). Duality. The following relation between $C$- and $G$-matrices holds:

\[ G_t^T C_t = I, \quad (4.35) \]
where $G_t^T$ is the transpose of $G_t$.

(b). $G$-fan. For each $G$-matrix $G_t$, let
\[
s(\sigma):= \mathbb{R}_{\geq 0}g_{1}\tau + \mathbb{R}_{\geq 0}g_{2}\tau
\]
be the cone in $\mathbb{R}^2$ spanned by its $g$-vectors, which we call a $G$-cone. $G$-cones intersect only in their boundaries, thereby forming a fan (the $G$-fan, or the $g$-vector fan of $\Sigma$). See Figure 3.

In view of the definitions of $C$- and $G$-matrices and $F$-polynomials, we define the action of a permutation $\sigma\in S_n$ on them as follows:
\[
\begin{align*}
\sigma C_t &= C'_t, \quad c'_{ij} := c_{i\sigma^{-1}(j);t}, \\
\sigma G_t &= G'_t, \quad g'_{ij} := g_{i\sigma^{-1}(j);t}, \\
\sigma F_t &= F'_t, \quad F'_i(y) := F_{\sigma^{-1}(i);t}(y).
\end{align*}
\]
(4.37)
(4.38)

The following is a consequence of the separation formulas.

**Proposition 4.19.** Let $\Sigma$ be a cluster pattern with free coefficients at $t_0$. Let $\Sigma'$ be a cluster pattern with principal coefficients at $t_0$ sharing the common $B$-pattern with $\Sigma$. Then, for any $t, t'\in \mathbb{T}_n$ and a permutation $\sigma\in S_n$, the following fact holds:
\[
\Sigma_t = \sigma \Sigma'_t \iff \Sigma'_t = \sigma \Sigma_t.
\]
(4.39)

In other words, the periodicities of $\Sigma$ and $\Sigma'$ coincide.

**Proof.** The implication $\implies$ was given in Proposition 2.26. Let us show the implication $\iff$. Assume the equality $\Sigma'_t = \sigma \Sigma_t$. Then, it implies the same periodicity for $C$- and $G$-matrices and $F$-polynomials,
\[
C_t = \sigma C'_t, \quad G_t = \sigma G'_t, \quad F_t = \sigma F'_t.
\]
(4.40)

Also, we have $B_t = \sigma B'_t$. Then, applying the separation formulas (4.24) and (4.25) to $\Sigma$, we have
\[
x_t = \sigma x'_t, \quad y_t = \sigma y'_t.
\]
(4.41)
In summary, through the separation formulas, a cluster pattern of principal coefficients controls a cluster pattern of any coefficients (including free coefficients) sharing the common $B$-pattern. This is the reason why it is called "principal" in CA4.

4.4 Further results

In the rest of the section, we present without proof some advanced results on $C$- and $G$-matrices and $F$-polynomials, which are fundamentally important in cluster algebra theory.

Definition 4.20. We say that a vector $v \in \mathbb{Z}^n$ is positive (resp. negative) if it is a nonzero vector, and all nonzero components are positive (resp. negative). We say that a square matrix is column sign-coherent if each column vector is either positive or negative.

The following theorems were conjectured by Fomin-Zelevinsky [FZ02a, FZ07] and proved by Gross-Hacking-Keel-Kontsevich [GHKK18] by the scattering diagram method.

Theorem 4.21 (Sign-coherence of $C$-matrices [GHKK18]). Every $c$-vector $c_{i,t}$ is either positive or negative. In other words, every $C$-matrix $C_t$ is column sign-coherent.

Theorem 4.22 (Unit constant property [GHKK18]). Every $F$-polynomial $F_{i,t}(y)$ has constant term 1.

Theorem 4.23 (Laurent positivity [GHKK18]). Every $F$-polynomial $F_{i,t}(y)$ has no negative coefficients.

It was shown in [FZ07] that Theorems 4.21 and 4.22 are equivalent to each other.

Proof of equivalence between Theorems 4.21 and 4.22. Let $t, t' \in T_n$ be $k$-adjacent. Assume that the constant term of $F_{i,t}(y)$ is 1. Then, by (4.19), the following two conditions are equivalent:

(a). The constant term of $F_{k,t'}(y)$ is 1.

(b). The $c$-vector $c_{k,t'}$ is either positive or negative.

Therefore, Theorem 4.21 follows from Theorem 4.22. Conversely, Theorem 4.22 can be shown from Theorem 4.21 and the fact $F_{i,t_0}(y) = 1$ by the induction on $d = d(t_0, t)$.

There are several important consequences of Theorems 4.21, 4.22, 4.23 together with the separation formulas.

The duality between $C$- and $G$-matrices observed in Example 4.18 is a consequence of Theorem 4.21.
Theorem 4.24 (Duality [NZ12]). Let $D$ be a common skew-symmetrizer of $B$-pattern $B$ of $\Sigma$. Then, the following equality holds:

$$D^{-1}G_t^T DC_t = I. \quad (4.42)$$

**Proof.** Thanks to Theorem 4.21 for each $t \in T_n$ and $k \in \{1, \ldots, n\}$, the sign $\varepsilon_{k;t} \in \{1, -1\}$ of the $c$-vector $c_{k;t}$ is ambiguously determined. We set $\varepsilon = \varepsilon_{k;t}$ in (4.6) and (1.17). Note that $[-\varepsilon_{k;t}b_{\ell;k}]_+ = 0$ for any $\ell = 1, \ldots, n$ due to the sign-coherence. Then, the mutation formulas in (4.5) and (4.13) are simplified as follows:

$$c_{ij; t'} = \begin{cases} -c_{ik;t} & j = k, \\ c_{ij;t} + c_{ik;t}[\varepsilon_{k;t}b_{kj;t}]_+ & j \neq k, \end{cases} \quad (4.43)$$

$$g_{ij; t'} = \begin{cases} -g_{ik;t} + \sum_{\ell=1}^{n} g_{i\ell;t}[-\varepsilon_{k;t}b_{\ell;k}]_+ & j = k, \\ g_{ij;t} & j \neq k. \end{cases} \quad (4.44)$$

They are written in the matrix form

$$C_t' = C_t P, \quad G_t' = G_t Q, \quad (4.45)$$

$$P = \begin{pmatrix} \varepsilon_{k;t}b_{1;k;t} & \ldots & \varepsilon_{k;t}b_{k-1;k;t} & -1 & \varepsilon_{k;t}b_{k,k+1;t} & \ldots & \varepsilon_{k;t}b_{kn,t} \\ 1 & \ldots & 1 & \ldots & \ldots & \ldots & 1 \end{pmatrix}, \quad (4.46)$$

$$Q = \begin{pmatrix} 1 & [-\varepsilon_{k;t}b_{1,k;t}]_+ & \ldots & \vdots & \vdots & \varepsilon_{k;t}b_{k-1,k,t} & -1 & \varepsilon_{k;t}b_{k,k+1,t} & \ldots & \varepsilon_{k;t}b_{kn,k,t} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & [-\varepsilon_{k;t}b_{n,k,t}]_+ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \quad (4.47)$$

By (1.27), we have

$$DP = Q^T D. \quad (4.48)$$

Also, it is easy to see that

$$P^2 = I. \quad (4.49)$$
Now, we prove (4.42) by the induction on the distance \( d = d(t_0, t) \). For \( t = t_0 \), the claim holds by (4.31) and (4.12). Assume that the claim holds for \( d = d(t_0, t) \). Let \( t' \) be \( k \)-adjacent to \( t \). Then, we have

\[
D^{-1}G^T_t DC_t' = D^{-1}(Q^T G^T_t)D(C_t P) = P(D^{-1}G^T_t DC_t)P = I. \tag{4.50}
\]

We have the following immediate consequences of Theorem 4.2.

**Proposition 4.25** ([NZ12]). (a). (Unimodularity)

\[
det C_t = det G_t \in \{1, -1\}. \tag{4.51}
\]

(b). The matrix \( B_t \) is determined by \( B_{t_0} \) and \( C_t \) as follows:

\[
DB_t = C^T_t (DB_{t_0})C_t. \tag{4.52}
\]

**Proof.** (a). Since \( C_t \) and \( G_t \) are integer matrices, the equality (4.42) implies the claim (4.51). (This can be also shown more directly by (4.42) and the fact \( det P = det Q = -1 \).) (b). By (4.15) and (4.42), we have

\[
C^T_t DB_{t_0} C_t = C^T_t DG_t B_t = DB_t. \tag{4.53}
\]

The following result was shown by Cao-Huang-Li [CHL20] using Theorems 4.21, 4.22, 4.23, 4.24 and the separation formulas all together.

**Theorem 4.26** (Detroptalization [CHL20]). Let \( \Sigma \) be any cluster pattern with a given initial vertex \( t_0 \). For any \( t, t' \in T_n \), the following implications hold:

\[
G_t = G_{t'} \implies x_t = x_{t'}, \tag{4.54}
\]

\[
C_t = C_{t'} \implies y_t = y_{t'}. \tag{4.55}
\]

In other words, the cluster variables and coefficients are uniquely determined by their tropical parts, respectively.

The above theorem is refined and completed as the following equivalences of periodicities.

**Theorem 4.27** (Synchronicity [Nak21]). Let \( \Sigma \) be any cluster pattern with a given initial vertex \( t_0 \). For any \( t, t' \in T_n \) and any permutation \( \sigma \in S_n \), the following four conditions are equivalent:

(a). \( G_t = \sigma G_{t'} \).

(b). \( C_t = \sigma C_{t'} \).

(c). \( x_t = \sigma x_{t'} \).
(d). $\Sigma_t = \sigma \Sigma_{t'}$.

Moreover, if $\Sigma$ has free or principal coefficients at some $t'_0$, then the above conditions are also equivalent to the following condition:

(e). $y_t = \sigma y_{t'}$.

As a corollary of Theorem 4.27, we obtain the following result, which answer the problem in CA4.

**Corollary 4.28** ([Nak21]). In Proposition 2.26, the opposite implication in (2.107) holds for any cluster pattern $\Sigma'$.

**Proof.** By Theorem 4.27 the both conditions in (2.107) is equivalent to the condition $G_t = \sigma G_{t'}$ for their common $G$-matrices.

\qed
5 Upper cluster algebras

So far, we mainly consider the properties of seeds and cluster patterns. In this section we study the structure of cluster algebras via the notion of upper cluster algebras. All results are taken from CA3.

5.1 Upper cluster algebras

Upper cluster algebras was introduced in CA3. The motivation was two-fold as follows:

- They are useful to study the structure of cluster algebras.
- The coordinate rings of some algebraic varieties (e.g., double Bruhat cells) are isomorphic to some upper cluster algebras, not cluster algebras. In other words, cluster algebras are too small in some situation.

Let $\Sigma$ be any cluster pattern with coefficients in any semifield $P$, and let $A = A(\Sigma)$ be the associated cluster algebra in the ambient filed $F$.

**Definition 5.1 (Upper cluster algebra).** The upper cluster algebra $\overline{A} = \overline{A}(\Sigma)$ is a $\mathbb{Z}P$-subalgebra of $F$ defined by

$$\overline{A} = \bigcap_{t \in T_n} \mathbb{Z}P[x_t^{\pm1}].$$

(5.1)

Thanks to the Laurent phenomenon in Theorem 3.1, any cluster variable $x_{i,t}$ belong to $\overline{A}$. Therefore,

$$A \subset \overline{A}.$$  

(5.2)

In some cases, $A = \overline{A}$ occurs, but, in general, $A \neq \overline{A}$.

Let us introduce some related notions.

**Definition 5.2.** For each $t \in T_n$, let $t_i$ ($i = 1, \ldots, n$) be the vertex that is $i$-adjacent to $t$. We define $\mathbb{Z}P$-subalgebras of $F$,

$$U_t = \mathbb{Z}P[x_t^{\pm1}] \cap \bigcap_{i=1}^n \mathbb{Z}P[x_i^{\pm1}],$$

(5.3)

$$L_t = \mathbb{Z}P[x_t, x_{t_1}, \ldots, x_{t_n}].$$

(5.4)

We call them the upper bound and lower bound of $A$ at $t$, respectively.

Clearly, we have

$$L_t \subset A \subset \overline{A} \subset U_t,$$

(5.5)

which explains their names.

We introduce the following technical condition.
Definition 5.3. For a given seed \( \Sigma_t \) and \( k = 1, \ldots, n \), let

\[
P_{k;t} := \frac{1}{1 \oplus y_{k;t}} \left( y_{k;t} \prod_{j=1}^{n} x_{j;t}^{[b_{j,k}]} + \prod_{j=1}^{n} x_{j;t}^{-[b_{j,k}]} \right) \in \mathbb{Z}\mathbb{P}[x_t]
\]  

be the polynomial appearing in the mutation formula (2.5). We say that a seed \( \Sigma_t \) is coprime if \( P_{1;t}, \ldots, P_{n;t} \) are pairwise coprime in \( \mathbb{Z}\mathbb{P}[x_t] \), namely, any common factor of \( P_{i;t} \) and \( P_{j;t} \) \((i \neq j)\) belongs to \( \mathbb{Z}\mathbb{P}^\times = \{\pm 1\}\mathbb{P} \), where \( \mathbb{Z}\mathbb{P}^\times \) is the unit group of the ring \( \mathbb{Z}\mathbb{P} \).

For our purpose, the following sufficient condition for coprimeness is useful.

Lemma 5.4. Let \( \Sigma \) be any cluster pattern with free coefficients at \( t_0 \). Then, \( \Sigma_t \) is coprime for any \( t \in \mathbb{T}_n \).

Proof. We see in (5.6) that, up to a multiplicative factor \( 1 \oplus y_{k;t} \) in \( \mathbb{Z}\mathbb{P}[x_t] \), \( P_{k;t} \) is a binomial of degree 1 in \( y_{k;t} \). Thus, for \( P = Q_{sf}(y_t) \simeq Q_{sf}(y_{t_0}) \), \( P_{k;t} \) is clearly irreducible. Therefore, \( P_{1;t}, \ldots, P_{n;t} \) are pairwise coprime.

Remark 5.5. In [BFZ05], instead of free coefficients, the following condition is used to guarantee the coprimeness of \( \Sigma_{t_0} \):

- A cluster pattern \( \Sigma \) is of geometric type and its initial extended exchange matrix \( \tilde{B}_{t_0} \) has full rank.

5.2 Rank 2 case

Let us concentrate on a cluster pattern \( \Sigma \) of rank 2 with a given initial vertex \( t_0 \). Let \( t_i \) \((i = 1, 2)\) be the vertex that is \( i \)-adjacent to \( t_0 \). Let

\[
x_{t_0} = x = (x_1, x_2), \quad x'_{1} = x_{1;t_1}, \quad x'_{2} = x_{2;t_2},
\]

so that

\[
U_{t_0} = \mathbb{Z}\mathbb{P}[x_{1}^{\pm 1}, x_{2}^{\pm 1}] \cap \mathbb{Z}\mathbb{P}[x_{1}'^{\pm 1}, x_{2}'^{\pm 1}] \cap \mathbb{Z}\mathbb{P}[x_{1}^{\pm 1}, x_{2}'^{\pm 1}],
\]

\[
L_{t_0} = \mathbb{Z}\mathbb{P}[x_{1}, x_{2}, x_{1}', x_{2}'],
\]

We take the initial exchange matrix \( B_{t_0} \) as in (2.56), where we include the case \( a = b = 0 \) therein. Then, the mutations at \( t_0 \) in directions 1 and 2 are written explicitly as

\[
x_1 x'_1 = P_{1;t_0} = p_1^+ x_2^+ + p_1^-, \quad x_2 x'_2 = P_{2;t_0} = p_2^+ + p_2^- x_1^b,
\]

\[
p_{k}^+ = \frac{y_k}{1 \oplus y_k}, \quad p_{k}^- = \frac{1}{1 \oplus y_k},
\]

where we set \( y_{t_0} = y \). For \( a, b > 0 \), it is clear that \( \Sigma_{t_0} \) is coprime. For \( a = b = 0 \), there are cases that \( \Sigma_{t_0} \) is not coprime.
Example 5.6. Suppose that $a = b = 0$ and $y_2 = y_1^3$ in $\mathbb{P}$. Then,

$$P_{1; t_0} = \frac{1 + y_1}{1 + y_1}, \quad P_{2; t_0} = \frac{1 + y_1^3}{1 + y_1^3} \quad (5.12)$$

have a non-monomial common factor $1 + y_1$. Thus, $\Sigma_{t_0}$ is not coprime.

Proposition 5.7 ([BFZ05]). Let $\Sigma$ be any cluster pattern of rank 2 such that the initial seed $\Sigma_{t_0}$ is coprime. Then, the following equality holds:

$$L_{t_0} = U_{t_0}. \quad (5.13)$$

This further implies that

$$L_{t_0} = A = \overline{A} = U_{t_0}. \quad (5.14)$$

It is clear that (5.13) implies (5.14) thanks to (5.5).

First we present a consequence of Proposition 5.7.

Theorem 5.8 ([BFZ05]). Let $\Sigma$ be any cluster pattern of rank 2 with coefficients in any semifield $\mathbb{P}$. Then, the cluster algebra $A$ is generated by $x_1, x_2, x'_1, x'_2$. In particular, it is finitely generated.

Proof. First we assume that $\Sigma$ has free coefficients at $t_0$. Then, by Lemma 5.4, $\Sigma_{t_0}$ is coprime. Thus, by Proposition 5.7, $A = L_{t_0} = Z\mathbb{P}[x_1, x_2, x'_1, x'_2]$. Therefore, the claim holds. This means that any cluster variable $x_i; t$ is expressed as a polynomial in $x_1, x_2, x'_1, x'_2$. This expression holds under any specialization of coefficients including the non-coprime case. (Alternatively, one can prove the claim for the case $a = b = 0$, where the non-coprime case happens, directly from the result in Example 2.16. See also Proposition 5.9.)

Now we prove Proposition 5.7.

Proof of Proposition 5.7. The goal is to show that (5.8) reduces to (5.9).

First, we show that

$$Z\mathbb{P}[x_1^{\pm 1}, x_2^{\pm 1}] \cap Z\mathbb{P}[x'_1^{\pm 1}, x_2^{\pm 1}] = Z\mathbb{P}[x_1, x'_1, x'_2^{\pm 1}]. \quad (5.15)$$

The inclusion $\supset$ is clear by (5.10). Let us show the inclusion $\subset$. Let $L \in Z\mathbb{P}[x_1^{\pm 1}, x_2^{\pm 1}]$. Then, we have

$$L = \sum_{m \in \mathbb{Z}} x_1^m Q_m(x_2) \quad (Q_m(x_2) \in Z\mathbb{P}[x_2^{\pm 1}]) \quad (5.16)$$

$$= \sum_{m \in \mathbb{Z}} x'_1^{-m}(p_1^+ x_2^a + p_1^-)^m Q_m(x_2),$$
where the sum is finite. Imposing that $L$ also belongs to $\mathbb{Z}P[x_1^\pm, x_2^\pm]$, we have, for any $m < 0$,

$$ (p_1^+ x_2^a + p_1^-)^m Q_m(x_2) \in \mathbb{Z}P[x_2^\pm]. \quad (5.17) $$

Then,

$$ L = \sum_{m \geq 0} x_2^m Q_m(x_2) + \sum_{m < 0} x_2^{-m} (p_1^+ x_2^a + p_1^-)^m Q_m(x_2) \in \mathbb{Z}P[x_1, x'_1, x_2^\pm], \quad (5.18) $$

which proves (5.15). By (5.15) and the one obtained by interchanging $x_1$ and $x_2$, we obtain the first step reduction

$$ U_{t_0} = \mathbb{Z}P[x_1, x'_1, x_2^\pm] \cap \mathbb{Z}P[x_2, x'_2, x_1^\pm]. \quad (5.19) $$

Now our goal is to prove the following claim.

**Claim.** The following equality holds:

$$ \mathbb{Z}P[x_1, x'_1, x_2^\pm] \cap \mathbb{Z}P[x_2, x'_2, x_1^\pm] = \mathbb{Z}P[x_1, x_2, x'_1, x'_2]. \quad (5.20) $$

The inclusion $\supset$ is clear. We separate the proof of the inclusion $\subset$ into two cases.

**Case 1:** $a, b > 0$. First we prove the following equality.

$$ \mathbb{Z}P[x_1, x'_1, x_2^\pm] = \mathbb{Z}P[x_1, x_2, x'_1, x'_2] + \mathbb{Z}P[x_1, x_2^\pm]. \quad (5.21) $$

The inclusion $\supset$ is clear. Let us show the inclusion $\subset$. It is enough to prove that, for $k, \ell > 0$,

$$ x_1^k x_2^{-\ell} \in \mathbb{Z}P[x_1, x'_1, x_2^\pm] + \mathbb{Z}P[x_1, x_2^\pm]. \quad (5.22) $$

Let $q := -p_2/p_2^+$. Then, by (5.10), we have

$$ x_2^{-1} \equiv q x_1^b x_2^{-1} \mod \mathbb{Z}P[x_2]. \quad (5.23) $$

Applying it repeatedly, we have, for any $k > 0$,

$$ x_2^{-1} \equiv q x_1^b x_2^{-1} \equiv q^2 x_1^{2b} x_2^{-1} \equiv \cdots \equiv q^{k} x_1^{kb} x_2^{-1} \mod \mathbb{Z}P[x_1, x'_2]. \quad (5.24) $$

Therefore,

$$ x_2^{-\ell} \in \mathbb{Z}P[x_1, x'_2] + x_1^k \mathbb{Z}P[x_1, x_2^\pm], \quad (5.25) $$

where we used the assumption $b > 0$. Noticing that $x'_1 x_1 \in \mathbb{Z}P[x_2]$, we obtain (5.22). Now, applying the equality (5.21) to the left hand side of (5.20), we have

$$ \mathbb{Z}P[x_1, x'_1, x_2^\pm] \cap \mathbb{Z}P[x_2, x'_2, x_1^\pm] \cap \mathbb{Z}P[x_2, x'_2, x_1^\pm] \quad (5.26) $$

$$ = (\mathbb{Z}P[x_1, x_2, x'_1, x'_2] + \mathbb{Z}P[x_1, x_2^\pm]) \cap \mathbb{Z}P[x_2, x'_2, x_1^\pm] \quad (5.26) $$

$$ = \mathbb{Z}P[x_1, x_2, x'_1, x'_2] + (\mathbb{Z}P[x_1, x_2^\pm] \cap \mathbb{Z}P[x_2, x'_2, x_1^\pm]). \quad (5.26) $$
Then, the claim (5.20) follows from the following equality:

$$\left( \mathbb{Z}P[x_1, x_2^{\pm 1}] \cap \mathbb{Z}P[x_2', x_2^{\pm 1}] \right) = \mathbb{Z}P[x_1, x_2, x_2']. \quad (5.27)$$

The inclusion $\supset$ is clear. We show the inclusion $\subset$. Let $L \in \mathbb{Z}P[x_2, x_2', x_2^{\pm 1}]$. We have

$$L = \sum_{m,k,l} c_{m,k,l} x_1^m x_2^k x_2'^\ell = \sum_{m,k,l} c_{m,k,l} x_1^m x_2^k x_2'^\ell (p_2^+ + p_2^-)^{\ell}, \quad (m \in \mathbb{Z}, k, \ell \geq 0, \, c_{m,k,l} \in \mathbb{Z}P).$$

(5.28)

Imposing that $L$ also belong to $\mathbb{Z}P[x_1, x_2^{\pm 1}]$, we see that the powers $m$ in the above should be nonnegative. This is the desired result.

**Case 2:** $a = b = 0$. In this case, one can directly show (5.20). We only need to show the inclusion $\subset$. Let $L \in \mathbb{Z}P[x_2, x_2', x_2^{\pm 1}]$. Then, we have

$$L = \sum_{m,k,l} c_{m,k,l} x_1^m x_2^k x_2'^\ell = \sum_{m,k,l} c_{m,k,l} x_1^m x_2^k x_2'^\ell (p_2^+ + p_2^-)^{\ell}, \quad (c_{m,k,l} \in \mathbb{Z}P, \, m \in \mathbb{Z}, k, \ell \geq 0).$$

(5.29)

A similar expression holds for $L' \in \mathbb{Z}P[x_1, x_1', x_2^{\pm 1}]$. It follows that

$$L'' = \sum_{m,k} c_{m,k} x_1^m x_2^k \in \mathbb{Z}P[x_1^{\pm 1}, x_2^{\pm 1}], \quad (c_{m,k} \in \mathbb{Z}P) \quad (5.30)$$

belongs to the left hand side of (5.20) if the following condition is satisfied:

(i). $c_{m,k}$ is divisible by $(p_1^+ + p_1^-)^{-m}$ if $m < 0$,

(ii). $c_{m,k}$ is divisible by $(p_2^+ + p_2^-)^{-k}$ if $k < 0$.

Here, we recall that $P_{1,t_0} = p_1^+ + p_1^-$ and $P_{2,t_0} = p_2^+ + p_2^-$ in (5.10) are coprime in $\mathbb{Z}P$ by assumption. (This is the only place where the coprimeness condition is concerned.) Then, the conditions (i) and (ii) ensure the following condition,

(iii). $c_{m,k}$ is divisible by $(p_1^+ + p_1^-)^{-m}(p_2^+ + p_2^-)^{-k}$ if $m, k < 0$.

The conditions (i)–(iii) guarantee that $L''$ belongs to $\mathbb{Z}P[x_1, x_2, x_1', x_2']$ after replacing negative powers of $x_i$ with positive power of $x_i/(p_i^+ + p_i^-)$.

This completes the proof of Proposition 5.7.

In the non-coprime case, one can directly verify the following fact, using the result in Example 2.16.

**Proposition 5.9.** Let $\Sigma$ be any cluster pattern of rank 2 such that the initial seed $\Sigma_{t_0}$ is not coprime. Then, we have

$$\mathcal{L}_{t_0} = \mathcal{A} = \mathcal{A} \subseteq \mathcal{U}_{t_0}. \quad (5.31)$$
Proof. Since $\Sigma$ is not coprime, we have $a = b = 0$. Then, as shown in Example 2.16 there are four clusters $(x_1, x_2), (x'_1, x_2), (x_1, x'_2), (x'_1, x'_2)$, where

$$x_1x'_1 = P_{1:t_0} = p_1^+ + p_1^-, \quad x_2x'_2 = P_{2:t_0} = p_2^+ + p_2^-.$$ (5.32)

Therefore, $L_{t_0} = \mathbb{Z}[x_1, x_2, x'_1, x'_2] = \mathcal{A}$ holds. By assumption, there is some common factor $R \in \mathbb{Z}$ of $P_{1:t_0}$ and $P_{2:t_0}$ with $R \not\in \mathbb{Z}^\times$ such that

$$P_{1:t_0} = Q_1 R, \quad P_{2:t_0} = Q_2 R \quad (Q_1, Q_2 \in \mathbb{Z}).$$ (5.33)

We have

$$x^{-1}_1x_2^{-1}Q_1Q_2R = x'_1x'_2^{-1}Q_2 = x^{-1}_1x'_2Q_1 \in \mathcal{U}_{t_0}.$$ (5.34)

On the other hand, this element does not belong to $\mathcal{A}$, because one cannot eliminate the negative powers of $x_1$ and $x_2$ simultaneously. Therefore, $\mathcal{A} \not\subseteq \mathcal{U}_{t_0}$. On the other hand, any element $L$ of $\mathcal{A}$ is explicitly written as ($c_{mn} \in \mathbb{Z}$)

$$L = \sum_{m,n \geq 0} c_{mn}x_1^m x_2^n + \sum_{m \geq 0, n < 0} c_{mn}P_{2:t_0}^{-n} x_1^m x_2^n$$

$$+ \sum_{m < 0, n \geq 0} c_{mn}P_{1:t_0}^{-m} x_1^m x_2^n + \sum_{m,n < 0} c_{mn}P_{1:t_0}^{-m} P_{2:t_0}^{-n} x_1^m x_2^n$$

$$= \sum_{m,n \geq 0} c_{mn}x_1^m x_2^n + \sum_{m \geq 0, n < 0} c_{mn}x_1^m x_2^{-n}$$

$$+ \sum_{m < 0, n \geq 0} c_{mn}x'_1^{-m} x_2^n + \sum_{m,n < 0} c_{mn}x'_1^{-m} x'_2^{-n} \in \mathcal{A}.$$ (5.35)

Therefore, $\mathcal{A} = \overline{\mathcal{A}}$. \hfill \Box

5.3 Alternative proof of Laurent phenomenon

As another application of Proposition 5.7 (and the results in the proof), we present alternative proof of the Laurent phenomenon in Section 3.1.

First, we prove the following fact.

Proposition 5.10. Let $\Sigma$ be any cluster pattern of rank 2 such that any seed $\Sigma_t$ is coprime. Then, the following equality holds for any $t \in \mathbb{T}_2$:

$$\mathcal{U}_t = \mathcal{L}_t = \mathcal{U}_{t_0} = \mathcal{L}_{t_0}.$$ (5.36)

In particular, the upper bound $\mathcal{U}_t$ is independent of $t$.

Proof. Since any seed $\Sigma_t$ is coprime by assumption, Proposition 5.7 is applicable to any $t$, so that we have $\mathcal{L}_t = \mathcal{U}_t$. Therefore, it is enough to prove

$$\mathcal{L}_{t_0} = \mathcal{L}_{t_1}.$$ (5.37)
5.3. Alternative proof of Laurent phenomenon

where

\[ \mathcal{L}_{t_1} = \mathbb{Z}[x_1, x_2, x_1', x_2'], \quad (5.38) \]

and \( x''_2 \) is the mutation of \( x_2 = x_{2:t_1} \) at \( t_1 \) in direction 2. Moreover, by the symmetry of \( x_2' \) and \( x_2'' \), it is enough to prove that

\[ x''_2 \in \mathcal{L}_{t_0} = \mathbb{Z}[x_1, x_2, x_1', x_2']. \quad (5.39) \]

We apply (2.61) with \( s = 1 \) therein to obtain

\[ x_2 x''_2 = q^+_2 x_1^b + q^-_2, \quad (5.40) \]

\[ q^+_2 = \frac{y'_2}{1 \oplus y'_2}, \quad q^-_2 = \frac{1}{1 \oplus y'_2}, \quad y'_2 = y_2 (1 \oplus y_1)^b. \quad (5.41) \]

The following relations hold:

\[ \frac{q^+_2}{q^-_2} = y'_2 = \left( p_2^+ / p_2^- \right) (p_1^-)^{-b}, \quad (5.42) \]

\[ x_1 x'_1 = p_1^+ x_2^a + p_1^-; \quad (5.43) \]

\[ x_2 x'_2 = p_2^+ + p_2 x_1^b. \quad (5.44) \]

It follows that

\[ x''_2 = \frac{q^+_2 x_1^b + q^-_2}{x_2} = \frac{q^+_2 x_1^b (x_2 x'_2 - p_2^- x_1^b)}{p_2^+ x_2} + \frac{q^-_2}{x_2} \]

\[ = \frac{q^+_2 x_1^b x'_2}{p_2^+ x_2} - \frac{q^-_2 p_2^-}{p_2^+ x_2} \left( x_1^b x_1 - \frac{q^-_2}{q^+_2} p_2^+ \right) \]

\[ = \frac{q^+_2 x_1^b x'_2}{p_2^+ x_2} - \frac{q^-_2 p_2^-}{p_2^+ x_2} \left( x_1^b x_2 + p_1^- \right)^b - \left( p_1^+ \right)^b. \quad (5.45) \]

The numerator of the second term in the last expression is divisible by \( x_2 \). Thus, we have the property \( (5.39) \).

We once again prove the Laurent phenomenon in Theorem 3.1 based on Proposition 5.10.

**Alternative proof of Theorem 3.1** Let \( \Sigma \) be any cluster pattern with rank \( n \). As in the previous proof in Section 3.1, it is enough to prove Theorem 3.1 assuming that \( \Sigma \) has free coefficients at \( t_0 \). Then, by Lemma 5.4, any seed \( \Sigma \) is coprime. In particular, the equality (5.36) is applicable for any rank 2 restriction of \( \Sigma \), where the frozen variables are regarded as a part of coefficients. For example, when we consider a cluster pattern restricted in direction 1 and 2, we replace \( \mathbb{Z}[x_2, \ldots, x_n] \) with \( \mathbb{Z}[x_3^1, \ldots, x_n^\pm 1] \). Then,
in the same notation in the proof of Proposition 5.10 we have the following equality corresponding to (5.37):

\[
(Z\mathbb{P}[x_3^{\pm 1}, \ldots, x_n^{\pm 1}])[x_1, x_2, x'_1, x'_2] = (Z\mathbb{P}[x_3^{\pm 1}, \ldots, x_n^{\pm 1}])[x_1, x_2, x'_1, x''_2],
\]

which is also written as

\[
Z\mathbb{P}[x_1, x_2, x'_1, x'_2, x_3^{\pm 1}, \ldots, x_n^{\pm 1}] = Z\mathbb{P}[x_1, x_2, x'_1, x''_2, x_3^{\pm 1}, \ldots, x_n^{\pm 1}]. \tag{5.46}
\]

Let us prove that the upper bound \( U_t \) of \( \Sigma \) is independent of \( t \). Let \( t_i \) \((i = 1, \ldots, n)\) be the vertex that is \( i \)-adjacent to \( t_0 \). Let \( x'_i = x_{i,t_i} \). It is enough to prove \( U_{t_0} = U_{t_1} \). Indeed, we have

\[
U_{t_0} = Z\mathbb{P}[x_{t_1}^{\pm 1}] \cap \bigcap_{i=1}^{n} Z\mathbb{P}[x_{t_i}^{\pm 1}]
\]

\[
= \bigcap_{i=1}^{n} (Z\mathbb{P}[x_{t_1}^{\pm 1}] \cap Z\mathbb{P}[x_{t_i}^{\pm 1}])
\]

\[
= \bigcap_{i=1}^{n} Z\mathbb{P}[x_1^{\pm 1}, \ldots, x_{i-1}^{\pm 1}, x_i, x'_i, x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1}] \quad \text{(by (5.45))}
\]

\[
= \bigcap_{i=2}^{n} (Z\mathbb{P}[x_1, x'_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \cap Z\mathbb{P}[x_1^{\pm 1}, \ldots, x_{i-1}^{\pm 1}, x_i, x'_i, x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1}])
\]

\[
= \bigcap_{i=2}^{n} Z\mathbb{P}[x_1, x'_1, x_2^{\pm 1}, \ldots, x_{i-1}^{\pm 1}, x_i, x'_i, x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1}]
\]

\[
= \bigcap_{i=2}^{n} Z\mathbb{P}[x_1, x'_1, x_2^{\pm 1}, \ldots, x_{i-1}^{\pm 1}, x_i, x''_2, x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1}] \quad \text{(by (5.47))}
\]

\[
= U_{t_1}, \tag{5.48}
\]

where the last equality is obtained by reversing the above procedure. Then, the upper cluster algebra \( \overline{\mathcal{A}} \) is given by

\[
\overline{\mathcal{A}} = \bigcap_{t \in T_n} Z\mathbb{P}[x_t^{\pm 1}] = \bigcap_{t \in T_n} U_t = U_{t_0}. \tag{5.49}
\]

On the other hand, for the cluster algebra \( \mathcal{A} \), we have

\[
\mathcal{A} = Z\mathbb{P}[x_t \mid t \in T_n] \subset \bigcup_{t \in T_n} U_t = U_{t_0}. \tag{5.50}
\]

Therefore, we obtain \( \mathcal{A} \subset \overline{\mathcal{A}} \).
5.4 Further results

Let us present generalizations of Proposition 5.7 and Theorem 5.8, which are part of the main result in CA3, without proof.

**Definition 5.11.** An $n \times n$ skew-symmetrizable matrix $B$ is acyclic if the following condition holds:

- For any $3 \leq m \leq n$, there is no cyclic sequence $i_1, i_2, \ldots, i_m, i_{m+1} = i_1$ in $\{1, \ldots, n\}$ such that $b_{i_s, i_{s+1}} > 0$ holds for any $s = 1, \ldots, m$.

**Example 5.12.** (a) Any $2 \times 2$ skew-symmetrizable matrix $B$ is acyclic.

(b) Let $B$ be any skew-symmetrizable matrix such that the associated Cartan matrix $A(B)$ is of finite type given in Definition 3.9. Then, $B$ is acyclic because the corresponding Dynkin diagram is a tree.

By a similar technique using upper and lower bounds, a parallel result to Proposition 5.7 and Theorem 5.8 hold for any cluster pattern with an acyclic initial exchange matrix.

**Theorem 5.13 (BFZ05).** Let $\Sigma$ be a cluster pattern with coefficients in any semifield $\mathbb{P}$ with a given initial vertex $t_0$. Suppose that the initial exchange matrix $B_{t_0}$ is acyclic. Then, the following facts hold.

(a). We have

$$\mathcal{L}_{t_0} = A.$$  \hspace{1cm} (5.51)

In particular, the cluster algebra $A$ is finitely generated.

(b). If the initial seed $\Sigma_{t_0}$ is coprime, we have

$$\mathcal{L}_{t_0} = A = \overline{A} = U_{t_0}.$$  \hspace{1cm} (5.52)
6 Generalized cluster algebras

To conclude this introductory guide, we present a generalization of cluster algebras introduced by Chekhov and Shapiro as an extra material beyond basics. Essentially all nice properties of cluster algebras are inherited to this generalization. Therefore, they extensively widen the perspective of cluster algebra theory.

6.1 Generalized cluster algebras

Chekhov and Shapiro [CS14] introduced a generalization of cluster algebras called generalized cluster algebras (GCA, for short), motivated by some examples naturally appeared in the study of Teichmüller space for the Riemann surface with orbifold points. Moreover, it turned out that they are indeed a natural generalization such that all essential properties of cluster algebras presented in this text are shown or conjectured to hold [CS14, Nak15].

Let us explain the idea shortly. The mutations of ordinary cluster algebras are defined by binomials $1 + \hat{y}_k$ and $1 \oplus y_k$ in (2.2) and (2.3). We may ask how/why this binomial condition is essential. It turns out that one can replace it with fairly general polynomials, and they are still as good as the ordinary one, if we adequately adjust the other parts of mutations.

**Definition 6.1** (Mutation data). Let $\mathbb{P}$ be a given semifield. We fix the following mutation data $(\mathbf{r}, \mathbf{z})$ in $\mathbb{P}$:

- an $n$-tuple of positive integers $\mathbf{r} = (r_1, \ldots, r_n)$, which are called mutation degrees.
- a collection $\mathbf{z} = (z_{i,s})_{i=1,\ldots,n; s=0,\ldots,r_i}$ of elements in $\mathbb{P}$ satisfying the following conditions:
  
  (unit constant term/monic property) $z_{i,0} = z_{i,r_i} = 1$, \hspace{1cm} (6.1)
  
  (reciprocity) $z_{i,s} = z_{i,r_i-s}$. \hspace{1cm} (6.2)

They are equivalent to give an $n$-tuple of monic and reciprocal polynomials in a single formal variable $y$ with coefficients in $\mathbb{P}$,

$$P_{i,\mathbf{z}}(y) = \sum_{s=0}^{r_i} z_{i,s} y^s, \hspace{1cm} (i = 1, \ldots, n).$$ \hspace{1cm} (6.3)

In the simplest case $\mathbf{r} = (1, \ldots, 1)$, they reduce to $P_{i,\mathbf{z}}(y) = 1 + y$, which is the binomial for the ordinary mutations of seeds mentioned above.

**Definition 6.2** (Seed mutation for GCA). Let $(\mathbf{r}, \mathbf{z})$ be a mutation data in $\mathbb{P}$. Let $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ be an (ordinary) seed with coefficients in $\mathbb{P}$ in Definition 2.1. For the polynomials in (6.3), let

$$P_{k,\mathbf{z}}(\hat{y}_k) = \sum_{s=0}^{r_k} z_{k,s} \hat{y}_k^s, \hspace{1cm} \left. P_{k,\mathbf{z}}(y_k) = \bigoplus_{s=0}^{r_k} z_{k,s} y_k^s \right|_{P}.$$ \hspace{1cm} (6.4)
6.1. Generalized cluster algebras

where \( \hat{y}_k \) is the ordinary \( \hat{y} \)-variable for \( \Sigma \). Then, the \((r, z)\)-mutation of \( \Sigma \) in direction \( k \), \( \mu_k(\Sigma) = (x', y', B') \), is defined as follows:

\[
x_i' = \begin{cases} 
x_k^{-1} \left( \prod_{j=1}^{n} x_j^{-b_{jk} r_k} \right) \frac{P_{k, z}(\hat{y}_k)}{P_{k, z}[\hat{y}_k]} & i = k, \\
x_i & i \neq k,
\end{cases}
\]

\[
y_i' = \begin{cases} 
y_k^{-1} & i = k, \\
y_i y_k [r_k b_k]_+ P_{k, z}[\hat{y}_k]^{-b_k} & i \neq k,
\end{cases}
\]

\[
b_{ij}' = \begin{cases} 
-b_{ij} & i = k \text{ or } j = k, \\
b_{ij} + b_{ik} [r_k b_k]_+ + [-b_{ik} r_k]_+ b_{kj} & i, j \neq k.
\end{cases}
\]

It is easy to check the following properties.

**Proposition 6.3.** (a). Any skew-symmetrizer \( D \) of \( B \) is also a skew-symmetrizer of \( B' \).

(b). The mutation \( \mu_k \) is an involution. In particular, \((x', y', B')\) is a seed.

(c). The \( \hat{y} \)-variables mutate as

\[
\hat{y}_i' = \begin{cases} 
\hat{y}_k^{-1} & i = k, \\
\hat{y}_i y_k [r_k b_k]_+ P_{k, z}(\hat{y}_k)^{-b_k} & i \neq k.
\end{cases}
\]

(d). The \( \epsilon \)-expressions, which are parallel to the ones in Proposition 2.7, hold, where we put \( \epsilon \) in every \([ \cdot ]_+\) and also replace \( P_{k, z}(\hat{y}_k) \) and \( P_{k, z}[\hat{y}_k] \) with \( P_{k, z}(\hat{y}_k^\epsilon) \) and \( P_{k, z}[\hat{y}_k^\epsilon] \) in (6.5)–(6.7).

**Proof.** (a). This is proved in the same way as Proposition 1.18 (a).

(b). By the reciprocity condition (6.2), we have a parallel formula to (2.6),

\[
\frac{P_{k, z}(\hat{y}_k^{-1})}{P_{k, z}[\hat{y}_k^{-1}]} = \frac{P_{k, z}(\hat{y}_k)}{P_{k, z}[\hat{y}_k]} \prod_{j=1}^{n} x_j^{-b_{jk} r_k}. 
\]

Then, one can repeat the proofs of Propositions 1.18 (c) and 2.5 (a).

The properties (c) and (d) can be proved in a similar way as Propositions 2.6 and 2.7.

**Remark 6.4.** One can lift the reciprocity condition (6.2) by putting the data \( z \) in a seed and introducing its mutation [NR16].

**Definition 6.5.** For a given mutation data \((r, z)\) in \( \mathbb{P}\), a generalized cluster pattern \( \Sigma \) with coefficients in \( \mathbb{P}\) are defined in the same way as the ordinary one by replacing mutations with \((r, z)\)-mutations. The generalized cluster algebra \( A \) associated with \( \Sigma \) is defined in the same way as the ordinary one.
The first serious test for this generalization is the Laurent phenomenon, which is the raison d’être of cluster algebras. This indeed holds literally in the same way as the ordinary ones.

**Theorem 6.6 (Laurent phenomenon [CS14]).** Let $\Sigma$ be any generalized cluster pattern with coefficients in any semifield $\mathbb{P}$. Let $t_0, t \in \mathbb{T}_n$ be any vertices. Then, any cluster variable $x_{i,t}$ is expressed as a Laurent polynomial in $x_{t_0}$ with coefficients in $\mathbb{Z}\mathbb{P}$.

**Proof.** The proof is also the same as the proof of Theorem 3.1, just by replacing the binomial $1 + \hat{y}_k$ therein with the polynomial $P_{k,z}(\hat{y}_k)$. □

**Example 6.7.** Let us consider a rank 2 example. As the simplest nontrivial mutation data $(r, z)$, we take $r = (2, 1)$, where the only nontrivial data in $z$ is $z_{1,1}$. We choose $z_{1,1} = z$ to be arbitrary in $\mathbb{P}$. We also choose the simplest nontrivial initial exchange matrix

$$B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (6.10)$$

Accordingly,

$$\hat{y}_1 = y_1x_2, \quad \hat{y}_2 = y_2x_1^{-1}. \quad (6.11)$$

We note that, for the diagonal matrix $R = \text{diag}(r_1, r_2)$,

$$B_0R = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \quad (6.12)$$

which is the exchange matrix for an ordinary cluster algebra of type $B_2$ we studied in Section 2.3. We repeat a similar calculation as in Section 2.3. The result is given below up to $t = 6$.

\[
\begin{align*}
\{ x_{1;0} &= x_1, \quad y_{1;0} = y_1, \\
x_{2;0} &= x_2, \quad y_{2;0} = y_2, \}
\end{align*}
\]

\[
\begin{align*}
\{ x_{1;1} &= x_1^{-1} \frac{1 + z\hat{y}_1 + \hat{y}_1^2}{1 + zy_1 + y_1^2}, \quad y_{1;1} = y_1^{-1}, \\
x_{2;1} &= x_2, \quad y_{2;1} = y_2(1 + zy_1 + y_1^2), \}
\end{align*}
\]

\[
\begin{align*}
\{ x_{1;2} &= x_1^{-1} \frac{1 + z\hat{y}_1 + \hat{y}_1^2}{1 + zy_1 + y_1^2}, \\
x_{2;2} &= x_2^{-1} \frac{1 + \hat{y}_2 + z\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2}{1 + y_2 + zy_1y_2 + y_1^2y_2}, \}
\end{align*}
\]

\[
\begin{align*}
\{ y_{1;2} &= y_1^{-1}(1 + y_2 + zy_1y_2 + y_1^2y_2), \\
y_{2;2} &= y_2^{-1}(1 + zy_1 + y_1^2)^{-1} \}
\end{align*}
\]
6.2 Separation formulas for GCA

As one more confirmation of the rightness of GCA, we present the separation formulas for generalized cluster patterns, which are parallel to the ordinary ones.

**Definition 6.8.** We say that a generalized cluster pattern $\Sigma$ with mutation data $(r, z)$ is *with principal coefficients at* $t_0 \in \mathbb{T}_n$ if the following conditions are satisfied:

- The coefficient semifield of $\Sigma$ is a tropical semifield $\text{Trop}(y, z)$ with generators $y = (y_1, \ldots, y_n)$, $z = (z_{i,s})_{i=1,\ldots,n; s=1,\ldots,r_i-1}$ with $z_{i,s} = z_{i,r_i-s}$ as formal variables.
- The coefficient tuple $y_{t_0}$ at $t_0$ coincides with $y$.
- The mutation data $z$ coincides with $z$.

We observe that all characteristic feature of ordinary cluster patterns are preserved. In particular, we see the same periodicity of the ordinary cluster pattern of type $B_2$ in Section 2.3. In fact, if we formally set $z = 0$ in the above (though such a specialization is prohibited in $\mathbb{F}$) we recover the result for the ordinary cluster pattern of type $B_2$ therein. This is an example of the general fact that any generalized cluster pattern reduces to some (ordinary) cluster pattern called the *right companion cluster pattern* under such a specialization [NR16].

\[
\begin{align*}
\begin{cases}
x_{1;3} = x_1 x_2^{-1} \frac{1 + 2 \hat{y}_2 + \hat{y}_2^2 + z \hat{y}_1 \hat{y}_2 + z \hat{y}_1 \hat{y}_2^2 + \hat{y}_1^2 \hat{y}_2^2}{1 \oplus 2 y_2 \oplus y_2^2 \oplus z y_1 y_2 \oplus z y_1 \hat{y}_2 \oplus \hat{y}_1 y_2 \oplus y_1 \hat{y}_2}, \\
x_{2;3} = x_2^{-1} \frac{1 + \hat{y}_2 + z \hat{y}_1 \hat{y}_2 + \hat{y}_1^2 \hat{y}_2^2}{1 \oplus y_2 \oplus z y_1 y_2 \oplus \hat{y}_1^2 y_2^2}, \\
y_{1;3} = y_1 (1 \oplus y_2 \oplus z y_1 y_2 \oplus y_1^2 y_2)^{-1}, \\
y_{2;3} = y_1^2 y_2^{-1} (1 \oplus 2 y_2 \oplus y_2^2 \oplus z y_1 y_2 \oplus z y_1 \hat{y}_2 \oplus \hat{y}_1 y_2 \oplus y_1 \hat{y}_2), \\
x_{1;4} = x_1 x_2^{-1} \frac{1 + 2 \hat{y}_2 + \hat{y}_2^2 + z \hat{y}_1 \hat{y}_2 + z \hat{y}_1 \hat{y}_2^2 + \hat{y}_1^2 \hat{y}_2^2}{1 \oplus 2 y_2 \oplus y_2^2 \oplus z y_1 y_2 \oplus z y_1 \hat{y}_2 \oplus \hat{y}_1 y_2 \oplus y_1 \hat{y}_2}, \\
x_{2;4} = x_1 x_2^{-1} \frac{1 + \hat{y}_2}{1 \oplus y_2}, \\
y_{1;4} = y_1^{-1} y_2^{-1} (1 \oplus y_2), \\
y_{2;4} = y_1^2 y_2 (1 \oplus 2 y_2 \oplus y_2^2 \oplus z y_1 y_2 \oplus z y_1 \hat{y}_2 \oplus \hat{y}_1 y_2 \oplus y_1 \hat{y}_2)^{-1}, \\
x_{1;5} = x_1, \\
x_{1;5} = x_1 x_2^{-1} \frac{1 + \hat{y}_2}{1 \oplus y_2}, \\
y_{1;5} = y_1 y_2 (1 \oplus y_2)^{-1}, \\
y_{1;5} = y_2^{-1}, \\
y_{1;6} = y_1, \\
x_{1;6} = x_1, \\
x_{2;6} = x_2, \\
y_{2;6} = y_2.
\end{cases}
\]

(6.16) (6.17) (6.18) (6.19)
We need a stronger version of the Laurent phenomenon, which is parallel to Theorem 3.23.

**Theorem 6.9** ([Nak15]). For any generalized cluster pattern with principal coefficients at \( t_0 \), we have

\[ x_{i; t} \in \mathbb{Z}[x_{t_0}^{\pm 1}, y, z]. \]  

**(Proof.)** This is proved by showing a similar claim in the proof of Theorem 3.23 for both variables \( y \) and \( z \), where the property (6.1) of \( z \) is essential. \( \square \)

For a generalized cluster pattern one can define its (generalized) \( C \)- and \( G \)-matrices and \( F \)-polynomials in two ways:

- Define them through a generalized cluster pattern with principal coefficients at \( t_0 \)
- Define them by the underlying (generalized) \( B \)-pattern \( B \) and \( t_0 \).

Below we skip the first definition, which is parallel to the ordinary case, and only give the second one. One notable difference to the ordinary case is that the \( F \)-polynomials are now polynomials in both \( y \) and \( z \), where \( z \) are the formal variables \( z \) corresponding to the mutation data \( z \) (under our usual abuse of notations).

**Definition 6.10.** For a given generalized cluster pattern \( \Sigma \) with a given mutation data \( (r, z) \) and a given initial vertex \( t_0 \), the (generalized) \( C \)-, \( G \)-, \( F \)-patterns \( C_{t_0}, G_{t_0}, F_{t_0} \) of \( \Sigma \) are uniquely determined by the following initial conditions and the mutation formulas, where \( t \) and \( t' \) are \( k \)-adjacent:

\[
C_{t_0} = I, \tag{6.21}
\]

\[
c_{ij; t'} = \begin{cases} 
-c_{ik; t} & j = k, \\
c_{ij; t} - c_{ik; t} [r_k b_{kj; t}] + [-c_{ik; t} r_k] + b_{kj; t} & j \neq k, 
\end{cases} \tag{6.22}
\]

\[
G_{t_0} = I, \tag{6.23}
\]

\[
g_{ij; t'} = \begin{cases} 
-g_{ik; t} + \sum_{\ell=1}^{n} g_{i\ell; t} [-b_{\ell k; t} r_k] + \sum_{\ell=1}^{n} b_{i\ell; t_0} [-c_{\ell k; t} r_k] & j = k, \\
g_{ij; t} & j \neq k, 
\end{cases} \tag{6.24}
\]

\[
F_{i; t_0}(y, z) = 1, \tag{6.25}
\]

\[
F_{i; t'}(y, z) = \begin{cases} 
M_{k; t}(y, z) & i = k, \\
F_{k; t}(y, z) & i \neq k, 
\end{cases} \tag{6.26}
\]
6.2. Separation formulas for GCA

where

\[
M_{k,t}(y) = \left( \prod_{j=1}^{n} y_{j}^{[-c_{jk,t}r_{k}]} \right) \prod_{j=1}^{n} F_{j,t}(y,z)^{[-b_{jk,t}r_{k}]} \\
\times \sum_{s=0}^{r_{k}} z_{k,s} \left( \prod_{j=1}^{n} y_{j}^{c_{jk,t}} \prod_{j=1}^{n} F_{j,t}(y,z)^{b_{jk,t}} \right)^{s}.
\] (6.27)

In the first definition of \( F \)-polynomials, \( F_{i,t}(y,z) \) is defined from the cluster variable \( x_{i,t} \) with principal coefficients in \( t_{0} \) by the specialization \( x_{1} = \cdots = x_{n} = 1 \). Therefore, thanks to Theorem 6.9, \( F_{i,t}(y,z) \) is a polynomial in \( y \) and \( z \).

As expected, the separation formulas for generalized cluster patterns are given exactly in the same form as the ordinary ones in Theorem 6.11 just by adding the variables \( z \) for \( F \)-polynomials.

**Theorem 6.11** (Separation Formulas \[^{[Nak15]}\]). Let \( \Sigma \) be any generalized cluster pattern with coefficients in \( \mathbb{P} \), mutation data \( (r,z) \), and a given initial vertex \( t_{0} \). Let

\[
x_{t_{0}} = x, \quad y_{t_{0}} = y, \quad \hat{y}_{t_{0}} = \hat{y}
\] (6.28)

be the initial cluster variables, coefficients, and \( \hat{y} \)-variables. Then, the following formulas hold.

\[
x_{i,t} = \left( \prod_{j=1}^{n} x_{j}^{a_{ji,t}} \right) \frac{F_{i,t}(\hat{y},z)}{F_{i,t}|_{\mathbb{P}}(y,z)},
\] (6.29)

\[
y_{i,t} = \left( \prod_{j=1}^{n} y_{j}^{c_{ji,t}} \right) \prod_{j=1}^{n} F_{j,t}|_{\mathbb{P}}(y,z)^{b_{ji,t}},
\] (6.30)

\[
\hat{y}_{i,t} = \left( \prod_{j=1}^{n} \hat{y}_{j}^{c_{ji,t}} \right) \prod_{j=1}^{n} F_{j,t}(\hat{y},z)^{b_{ji,t}}.
\] (6.31)

**Proof.** One can repeat the proof of Theorem 4.16 taking care of the modification by the mutation degree \( r \).

There is one notable feature of GCA compared with the ordinary one. For any generalized cluster pattern with mutation data \( (r,z) \), let \( B = \{B_{t}\}_{t\in T_{n}} \) be the (generalized) \( B \)-pattern of \( \Sigma \). Also, let \( C_{t_{0}} = \{C_{t}\}_{t\in T_{n}} \) and \( G_{t_{0}} = \{G_{t}\}_{t\in T_{n}} \) be the (generalized) \( C \)- and \( G \)-patterns of \( \Sigma \).

Let \( R = (r_{ij}\delta_{ij})_{i,j=1}^{n} \) be the diagonal matrix whose diagonal entries are given by the mutation degrees. Let \( RB = \{RB_{t}\}_{t\in T_{n}} \) and \( BR = \{Br_{t}\}_{t\in T_{n}} \). Then, one can verify from (6.7) by inspection that both \( RB \) and \( BR \) are \( B \)-pattern by the ordinary matrix mutations.
Let $L_{C_{t_0}} = \{L_{C_t}\}_{t \in \mathbb{T}_n}$ and $L_{G_{t_0}} = \{L_{G_t}\}_{t \in \mathbb{T}_n}$ be the (ordinary) $C$- and $G$-patterns associated with the (ordinary) $B$-pattern $RB$. Similarly, let $R_{C_{t_0}} = \{R_{C_t}\}_{t \in \mathbb{T}_n}$ and $R_{G_{t_0}} = \{R_{G_t}\}_{t \in \mathbb{T}_n}$ be the (ordinary) $C$- and $G$-patterns associated with the (ordinary) $B$-pattern $BR$.

Proposition 6.12 ([Nak15]). The following relations holds:

\begin{align*}
C_t &= L_{C_t} = R(R_{C_t})R^{-1}, \quad (6.32) \\
G_t &= R_{G_t} = R^{-1}(L_{G_t})R. \quad (6.33)
\end{align*}

**Proof.** Let us prove (6.32). The first equality follows from the inspection of (6.22). To obtain the second equality, multiply $r_{i}^{-1}r_{j}$ to (6.22). It yields the equality $R^{-1}C_{t}R = R_{C_{t}}$. The proof of (6.33) is similar, where we also use (6.32).

Finally, we present a parallel result to Theorem 4.24.

Theorem 6.13 (Duality [Nak15]). (a). For a common skew-symmetrizer $D$ of the (ordinary) $B$-pattern $RB$, we have the equality

\begin{equation}
D^{-1}R^{-1}G_{t}^{T}R_{D}C_{t} = I. \quad (6.34)
\end{equation}

(b). For a common skew-symmetrizer $D$ of the (ordinary) $B$-pattern $BR$, we have the equality

\begin{equation}
D^{-1}RG_{t}^{T}R^{-1}DC_{t} = I. \quad (6.35)
\end{equation}

(c). For a common skew-symmetrizer $D$ of the generalized $B$-pattern $B$, we have the equality

\begin{equation}
D^{-1}G_{t}^{T}DC_{t} = I. \quad (6.36)
\end{equation}

**Proof.** (a). By applying Theorem 4.24 to $L_{C_t}$ and $L_{G_t}$, we obtain the equality

\begin{equation}
D^{-1}(L_{G_t})^{T}D(L_{C_t}) = I. \quad (6.37)
\end{equation}

Then, by (6.32) and (6.33), it is written as (6.34).

(b). This is obtained in the same way as (a). Alternatively, note that, if $D$ is a common skew-symmetrizer of the $B$-pattern $RB$, $DR^2$ is a common skew-symmetrizer of the $B$-pattern $BR$. Then, (6.35) follows from (6.34).

(c). If $D$ is a common skew-symmetrizer of the $B$-pattern $RB$, $DR$ is a common skew-symmetrizer of the generalized $B$-pattern $B$. Then, (6.36) follows from (6.34).

More results on generalized cluster patterns are found in [Nak21].
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