Phase shifts in $I=2 \pi\pi$-scattering from two lattice approaches

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We present a lattice QCD study of the phase shift of $I=2 \pi\pi$ scattering on the basis of two different approaches: the standard finite volume approach by Lüscher and the recently introduced HAL QCD potential method. Quenched QCD simulations are performed on a $32^3 \times 128$ lattice with lattice spacing $a \sim 0.115$ fm using a heavy pion mass of $m_\pi \sim 940$ MeV. Results of the phase shift and the scattering length are shown to agree quite well between these two methods. In case of the potential method, the error is dominated by the systematic uncertainty associated with the violation of rotational symmetry due to finite lattice spacing. In Lüscher’s approach, such systematic uncertainty is difficult to be evaluated and thus is not included in this work. In case of the potential method, the phase shift can be calculated for arbitrary energies below the inelastic threshold. In that context, the phase shift obtained from the nonrest-frame extension of Lüscher’s method obtained at a particular center-of-mass momentum lies on top of the curve predicted by the potential method.

Keywords: lattice QCD, hadron-hadron interaction, scattering phase shift, scattering length
I. INTRODUCTION

Quantum chromodynamics, the theory of the strong interaction, is now known for about four decades and its predictive power is undisputed. Today, in the context of present lattice QCD calculations, armed with modern algorithms and peta-flop computing systems, it is becoming possible to put nuclear physics on the ground of QCD. One such approach is based on effective theories in which hadrons are the fundamental degrees of freedom and the appearing low energy coefficients have to be matched to the underlying theories [1, 2]. The standard approach within QCD involves the computation of multi-nucleon correlation functions and thus suffers from the combinatorial growth of the numbers of Wick-contractions which have to be computed [3–6]. Recently, the HAL QCD collaboration proposed an alternative method [7, 8], where interaction kernels (potentials) are first calculated on the lattice and then low-energy observables such as nucleon-nucleon (NN) scattering phase shifts are extracted by solving the Schrödinger equation employing these potentials. This method, which we will call the HAL QCD method or the potential method in this paper, has been widely applied to various hadronic interactions [9–12] (also cf. the recent review [13]).

The aim of this paper is to compare Lüscher’s approach [14, 15] and the potential method applied to the \( I = 2 \) two-pion scattering problem, where more accurate data than in NN system can be obtained. In principle, both methods should give the same results. In practice, however, the systematic uncertainties are different. Using our high precision data, we would like to address these uncertainties in detail in order to make pros and cons of both methods clear. We will study whether, where and how the potential method is comparable to or better than the Lüscher’s method.

II. LÜSCHER’S METHOD

Lüscher’s method allows to extract information on two-particle scattering amplitudes from the energy levels of the specific state in a finite volume. The energy levels, which are discrete on the lattice, are shifted with respect to those of the non-interacting theory. Lüscher found [14, 15] that this deviation could be used to obtain information on the corresponding S-matrix element.

For applying this method, we first have to compute the correlation function of the desired 2-particle state. In our case, this is the \( \pi \pi \) scattering state in the \( I = 2 \) channel and its correlation function is given by

\[
C_{\pi\pi}(t, t_0, r) \equiv \frac{1}{V} \sum_{x} \langle \pi^+(t, x + r) \pi^+(t, x) J_{\pi^-}(t_0) J_{\pi^-}(t_0) \rangle,
\]

where \( V = L^3 \) is the spatial volume of the lattice. We write \( \pi^\pm \) in order to emphasize that we are considering the \( I = 2 \) channel which does not contain disconnected diagrams. The contractions which have to be performed are displayed in Figure 1.

![Figure 1](image1.png)

**FIG. 1.** Contractions necessary to compute the non-singlet double pion correlation function. The different contributions are referred to trace-disconnected (a) and trace connected (b) respectively.

We consider the pseudo-scalar interpolating operator \( \pi^+ = \bar{d}\gamma_5u \) and the source function \( J_{\pi^-} \). In this work, we employ a wall source \( J_{\pi^-} = W_{\pi^-} \), where

\[
W_{\pi^-}(t_0) \equiv \sum_{x, y} \bar{u}(t_0, x) \gamma_5 d(t_0, y),
\]
as well as a Gaussian source $J_{\pi^-} = G_{r,\pi^-,}$, where

$$G_{r,\pi^-}(x_0, t_0) \equiv \sum_{x,y} e^{-\frac{(x-x_0)^2}{r^2}} e^{-\frac{(y-y_0)^2}{r^2}} \bar{u}(t_0, x)\gamma_5 d(t_0, y)$$

with an appropriate radius $r$ (see below). Furthermore, we need the single-pion correlation function given by

$$C_{\pi}(t, t_0) \equiv \sum_x \langle \pi^+(t, x) J_{\pi^-}(t_0) \rangle.$$  

Let us now consider the ratio

$$\tilde{R}(t, t_0) \equiv \sum_r C_{\pi\pi}(t, t_0, r) C^2_{\pi}(t, t_0)$$

and the corresponding effective mass

$$\Delta E(t) \equiv \frac{1}{2} \log \frac{\tilde{R}(t-1, t_0)}{\tilde{R}(t+1, t_0)}.$$  

For $t \to \infty$, $\Delta E(t)$ approaches the (relativistic) energy difference between an interacting two-pion system and a non-interacting one. Thus,

$$\Delta E(t) \xrightarrow{t \to \infty} \sqrt{k^2 + m_\pi^2} - m_\pi,$$

where $k$ is the asymptotic scattering momentum. The scattering phase shift $\delta(k)$ can be obtained by solving a nonlinear equation which involves the toroidal zeta function [14]:

$$k \cot \delta(k) = \frac{1}{\pi L} \sum_{n \in \mathbb{Z}} \frac{1}{|n|^2 - k^2},$$

with $\tilde{k} \equiv kL/(2\pi)$ and box-size $L$. For numerical applications, we use a convergence accelerated version of the r.h.s. of (8) suggested in [14, 15].

The scattering length $a_{\pi\pi}^{l=2}$ is defined by the low-energy limit

$$a_{\pi\pi}^{l=2} \equiv \lim_{k \to 0} \frac{\tan \delta(k)}{k}.$$  

Therefore, one ideally has to compute $\delta(k)$ for many small values of $k$, which requires the use of large lattices (since $k \sim 1/L$). Indeed we take a large lattice in this paper, so that it is suitable to expand the energy level of the lowest lying continuum state in the $A_1$ representation of the cubic group [16] for $L \gg a_{\pi\pi}^{l=2}$ via (cf. [14, 17])

$$\Delta E_0 = -\frac{4\pi a_{\pi\pi}^{l=2}}{m_\pi L^3} \left[ 1 + c_1 \frac{a_{\pi\pi}^{l=2}}{L} + c_2 \left( \frac{a_{\pi\pi}^{l=2}}{L} \right)^2 \right] + \mathcal{O} \left( \frac{1}{L^6} \right),$$

with $c_1 = -2.837297$ and $c_2 = 6.375183$. Thus, for $L \approx 3.7$ fm and $a_{\pi\pi}^{l=2} \approx 0.16$ fm (see below), we expect the truncation error to be negligible.

In order to determine the scattering phase shift $\delta(k)$ at larger asymptotic momenta $k$, we use the moving-frame extension introduced by Rummukainen and Gottlieb [18], which was already applied to the $I=2$ $\pi\pi$-system in [19].

### III. POTENTIAL METHOD

The potential method, which was originally introduced by the HAL QCD collaboration [7, 8], also allows for extracting S-matrix elements from lattice correlation functions. The major difference to Lüsher’s finite volume method is that the interaction kernel (potential) is extracted at short distances in the finite volume, while the scattering phase shifts are calculated by solving the Schrödinger equation in the infinite volume. In practice, the potential method involves an additional expansion in terms of its non-locality with respect to the relative coordinate of the scattered particles. On the long run, the potential method will be used to solve multi-nucleon scattering...
problems and thus only heavy particles such as nucleons appear in the system. For that reason, we mainly focus on heavy pions in this work.

In [8] it was shown that the Nambu-Bethe-Salpeter (NBS) wave function satisfies the time-independent Schrödinger equation:

\[
\left( \frac{k^2}{m_\pi} - H_0 \right) \psi_k(r) = \int d^3r' U(r, r') \psi_k(r'),
\]

(11)

where \( H_0 \equiv -\nabla^2_r / m_\pi \) is the non-interacting part of the two-pion Hamiltonian. Note that the non-local potential \( U(r, r') \) is independent of \( k \) [8, 20]. Here the equal-time NBS wave function can be written as

\[
\psi_k(r) \equiv \frac{1}{V} \sum_x \langle 0 | \pi^+(x + r) \pi^+(x) | \pi^+(k) \pi^+(-k) \rangle_{in},
\]

(12)

where \( | \pi^+(k) \pi^+(-k) \rangle_{in} \) is a two-pion instate with relative momentum \( k \). We can now relate the two-pion correlation function (1) to the NBS wave function (12) via

\[
C_{\pi\pi}(r, t, t_0) = \sum_k \psi_k(r) a_k e^{-E_k(t-t_0)},
\]

(13)

with Fourier coefficients \( a_k = \langle \pi^+(k) \pi^+(-k) | J_x(-0) J_x(-0) | 0 \rangle \) and \( E_k^2 = 4(k^2 + m_\pi^2) \). We perform a derivative expansion of the non-local potential and only keep the leading-order term. For the two-pion system, we can thus write

\[
U(r, r') = V_C(r) \delta(r - r') + O(\nabla^2_r),
\]

(14)

where \( V_C \) is central potential and only dependent on \( r = |r| \). Combining (14) with (11) and (13) we obtain [8]

\[
V_C(r) = \frac{k^2}{m_\pi} + \frac{1}{m_\pi} \lim_{(t-t_0) \to \infty} \frac{\nabla^2_r C_{\pi\pi}(r, t, t_0)}{C_{\pi\pi}(r, t, t_0)},
\]

(15)

where the Laplacian \( \nabla^2_r \) acts on the spatial components of \( C_{\pi\pi}(r, t, t_0) \).

Both eq. (7) in Lüscher’s method and eq. (15) in the potential method require data for large \( t \) to guarantee the ground state saturation of the correlation function. However, the signal-to-noise ratio becomes worse at large \( t \), particularly for many-nucleon systems. To overcome this difficulty, a generalized potential method has been introduced in [21] on the basis of the time-dependent Schrödinger-like equation: We multiply (11) with \( a_k e^{-E_k(t-t_0)}/C_{\pi}(t, t_0)^2 \equiv \hat{a}_k e^{-\Delta E_k(t-t_0)} \) and sum both sides over \( k \), obtaining

\[
\sum_k \hat{a}_k e^{-\Delta E_k(t-t_0)} \left( \frac{k^2}{m_\pi} - H_0 \right) \psi_k(r) = \int d^3r' U(r, r') R(r', t),
\]

(16)

where we defined, analogous to [5],

\[
R(r, t, t_0) \equiv \frac{C_{\pi\pi}(r, t, t_0)}{C_{\pi}^2(t, t_0)}
\]

(17)

and \( \Delta E_k \equiv E_k - 2m_\pi \). It is straightforward to check that

\[
\frac{k^2}{m_\pi} e^{-\Delta E_k(t-t_0)} = \left( -\frac{\partial}{\partial t} + \frac{1}{4m_\pi} \frac{\partial^2}{\partial t^2} \right) e^{-\Delta E_k(t-t_0)}.
\]

(18)

Hence we find

\[
\left( -\frac{\partial}{\partial t} + \frac{1}{4m_\pi} \frac{\partial^2}{\partial t^2} - H_0 \right) R(r, t, t_0) = \int d^3r' U(r, r') R(r', t, t_0).
\]

(19)

Since all the two-pion scattering states with different \( k \) obey the same non-local potential, the ground state saturation with asymptotically large \( t \) is no longer necessary in this time-dependent method. The only condition for \( t \) is that it should be large enough so that both inelastic contributions in the numerator of \( R \) and the contribution from excited pions in the denominator of \( R \) are negligible. The second condition is essentially the same condition for \( t \) to extract
the pion mass from $C_\pi(t, t_0)$. The milder condition for $t$ is particularly a good news for the two-nucleon system. At lowest order of the derivative expansion, this results in a modified expression for the local potential (cf. [21]):

$$ V_C(r) = \frac{1}{m_\pi} \nabla_2^2 R(r, t, t_0) - \frac{\partial}{\partial t} R(r, t, t_0) + \frac{1}{4m_\pi} \frac{\partial^2}{\partial t^2} R(r, t, t_0). $$

(20)

In this improved method, one can also check the validity of the derivative expansion and improve the potential through the $t$-dependence of the right hand side of (20).

Once the local potential $V_C(r)$ is obtained, it can be fitted by an empirical formula or, alternatively, interpolated. Since we need the potential only as input for the continuum Schrödinger equation, we perform a barycentric rational interpolation [22] to obtain a continuous function on $r$. Furthermore, we set the potential to zero for distances $r > r_{\text{cut}}$, where $r_{\text{cut}}$ is the largest distance available in the considered data set. This cut is far less aggressive than it might appear, since the measured potentials are essentially compatible with zero for distances $r > 2\text{fm}$.

The interpolated potentials are then plugged into the radial part of the Schrödinger equation (11) with eq. (14) for zero orbital angular momentum $L=0$, which can be solved with respect to $\psi_k(r)$ for arbitrary $k$. The scattering phase $\delta(k)$ at a given asymptotic momentum $k$, can be obtained by noting that the asymptotic form of $\psi_k$ is given by a free radial wave [8] [23] [24]:

$$ \psi_k(r) \xrightarrow{r \to \infty} e^{i\delta(k)} (\cos \delta(k) j_0(kr) - \sin \delta(k) n_0(kr)), $$

(21)

with the zeroth-order spherical Bessel and Neumann functions $j_0(r) = \sin r/r$ and $n_0(r) = -\cos r/r$ respectively. Therefore, the scattering phase shift can be obtained by taking the long distance limit of the following ratio:

$$ \tan \delta(k) = \left. \frac{kr j_0(kr) - f_k(r) n'_0(kr)}{kr j_0(kr) - f_k(r) n_0(kr)} \right|_{r \to \infty}, $$

(22)

where

$$ f_k(r) = r \frac{d \ln \psi_k(r)}{dr}. $$

(23)

Due to confinement the pion potential is localized and thus the asymptotic region is reached quickly. This also implies, that finite volume corrections in this method are expected to be very small [8].

In the final step, the scattering length and other phenomenologically relevant parameters such as the effective range can be computed by fitting the scattering phases to the effective range expansion [15]:

$$ \frac{k \cot \delta(k)}{m_\pi} = \frac{1}{m_\pi a_{\pi}^{\pi}} + \frac{1}{2} m_\pi r_e \left( \frac{k^2}{m_\pi^2} \right) + P \langle m_\pi r_e \rangle^3 \left( \frac{k^2}{m_\pi^2} \right)^2 + O \left( \left( \frac{k^2}{m_\pi^2} \right)^3 \right), $$

(24)

where $r_e$ is called the effective range and $P$ the shape parameter. Both quantities parameterize the $k^2$ dependence of the phase shift $\delta(k)$ at low energies.

IV. LATTICE SETUP

We generate approximately 350 statistically independent quenched configurations using the Wilson plaquette action at $\beta=5.8726$ corresponding to $a \sim 0.115 \text{fm}$ and a lattice size of $32^3 \times 128$, which amounts to a spatial extent of $L \sim 3.7 \text{fm}$. In the valence sector, we use 2 HEX smeared, tree-level clover improved Wilson quarks [25] [29] (Note that smeared clover-Wilson actions have small cutoff effects on spectral quantities [30]). Since our ultimate goal is to compute multnucleon scattering, we tune $m_\pi$ to $\sim 940 \text{MeV}$. An exploratory study at smaller pion masses is also given in Appendix A.

Furthermore, we employ different combinations of source-types and boundary conditions in time direction. Whereas the spatial boundary conditions are always periodic, we use either anti-periodic or Dirichlet boundary conditions in time-direction. We further use wall [2] and Gaussian [3] sources, where in the latter case the spatial location of one pion is chosen randomly and the second pion was separated by $L/2a \times (1, 1, 1)$ in lattice units. This symmetric setup claims to improve the overlap with the $J=0$ ground state [31]. The size of the sources is set to $r \sim 0.32 \text{fm}$, as it was done in previous studies [32]. To improve statistics, we employ four sources per config separated by 32 time-slices, where the Dirichlet boundary is always separated from the source by $T/2$. 

V. ERROR TREATMENT

The statistical error is always computed by repeating the analysis on 2000 bootstrap samples. In order to estimate the overall systematic uncertainty, we use the histogram method \([20, 20, 32, 33]\). We identified the following sources of systematic uncertainties:

- **Excited states**: Possible contributions are estimated by varying the fitting range of the correlation functions. We use three equally sized ranges within \(t_{\text{min}}/a = 15\) and \(t_{\text{max}}/a = 41\):
  
  \[
  R_1 = \{t/a \in [15, 23]\} \\
  R_2 = \{t/a \in [24, 32]\} \\
  R_3 = \{t/a \in [33, 41]\}. 
  \]

  Below \(t_{\text{min}}\) we could not definitely exclude excited state contributions in Lüscher’s method. We consistently use the same ranges for fitting the effective energy in Lüscher’s method \([6]\) and for extracting the potential \([20]\). The latter was averaged over the corresponding time-slices. This tests both methods with respect to their dependence on the ground-state saturation.

- **Violation of rotational invariance**: Rotational invariance is violated at short distance by the finite lattice cut-off and at long distance by the finite lattice volume. For the potential method, we estimate the resulting systematic uncertainty by extracting the potential for distances \(r\) lying on-axis, on a plane-diagonal or on the cubic-diagonal and performing the full analysis on each of these choices separately. We found that this uncertainty, particularly at short distance, is the dominant source of the systematic error. For Lüscher’s method, a reliable estimate of the systematic uncertainty induced by the violation of rotational invariance is difficult to estimate, since the spatial information is lost. We therefore do not include this specific systematic error in our analysis for Lüscher’s method. However, in both approaches, effects of violation of rotational invariance will be suppressed for smaller lattice spacings and for larger lattice volumes.

- **Interpolation and Fit**: Before the Schrödinger equation is solved, the potential has to be interpolated or fitted by an empirical ansatz; we use a barycentric rational interpolation which is able to sample both the repulsive core and the long-distance tail. For testing the quality of this interpolation, we also performed empirical fits to Yukawa-type models or applied Gaussian process regressions. We have observed a good agreement in \(\delta(k)\) among all three approaches. Thus for simplicity, we only kept the barycentric rational interpolation in our final analysis. The potential is compatible with zero at largest observed distances \(r=R_{\text{max}}\). Therefore, we set \(V(r)=0\) for \(r>R_{\text{max}}\). Assuming an exponentially fast decaying tail instead yields compatible results.

- **Asymptotic regime**: By using \([22]\) for calculating the scattering phase shift, an additional systematic error can appear: it is attributed to the uncertainty of whether the asymptotic region in \(r\) is reached or not. Therefore, we average \([22]\) over five disjoint regions of \(r\) and propagate the differences among these regions into our systematic error.

- **Effective range expansion**: When the scattering length and effective range is computed from the effective range expansion, possible higher order corrections in \(k^2/m^2\pi\) also contribute to the systematic error. This contribution is estimated by applying the effective range expansion fit two times: the first one is made with a conservative fit window in \(E_{\text{CM}}\) and only the first two terms of \([24]\). In the second fit, the window is extended and the third term of \([24]\) is also included.

Altogether we perform \(3\cdot3\cdot5\cdot2=90\) analyses for the potential method and 3 different analyses for Lüscher’s method in our final analysis. Additionally, we also use 2 different boundary conditions and 2 different source types.

The results obtained from the different analyses are collected in a histogram and weighted by their quality of fit \(Q\). For computing the overall systematic uncertainty of the scattering phase \(\delta(k)\) we use a unit weight whereas in case of the scattering length we apply the fit-quality obtained by the effective range expansion fit (we have checked that choosing a unit weight instead yields compatible results, i.e. the central value changes by less than 0.1\% and the systematic error increases by less than 1\%). In all cases, the median gives our central value and the central 68\% the systematic error.

VI. RESULTS

We are going to discuss the results of our analysis in detail now, focusing on a few possible sources of systematic uncertainties at a time.
FIG. 2. Constant fits to effective mass plateau (6) for the wall/Dirichlet data. Bands indicate the corresponding errors from the fits. Note that the systematic uncertainty attributed to the choice of the fitting range ($R_1, R_2, R_3$) is small.

FIG. 3. Two-pion potential for the different ranges $R_i$ computed on the cubic diagonal (points). The curves represent the results from the interpolation.
FIG. 4. Two-pion potential for $R_2$ and evaluated on axis (green squares), plane-diagonals (red circles) and cubic-diagonal (purple diamonds) and their corresponding interpolations.

FIG. 5. Source dependence of potential for $R_2$. The potential obtained from Gaussian sources (circles) is consistent with that obtained by employing wall sources (squares).
A. Source- and boundary-dependence

For the first test, we compare the Gaussian and wall sources with Dirichlet boundary conditions at $m_{\pi} \sim 940$ MeV for both methods. The wall source is commonly used in conjunction with this type of boundary conditions [8, 10–12, 21]. The pion masses are extracted from the single-pion correlation functions by applying exponential fits. The energy shift $\Delta E$ is obtained by fitting the corresponding effective mass plateau [6] to a constant as shown for the
FIG. 7. Dependence of the scattering phase shift on the temporal boundary conditions. The plot shows $\delta(k)$ vs. $k$ for the wall-source-data with anti-periodic or Dirichlet boundary conditions.

FIG. 8. Scattering lengths obtained by the potential or Lüscher’s method in dependence on the temporal range used for extracting the potential or $k^2$.

wall source in Figure 2. From that value and $m_\pi$, the asymptotic momentum $k$ can be computed and plugged into (8), yielding $\delta(k)$. We obtain the lattice potential by computing (20), where the spatial and temporal derivatives in the right hand side have been evaluated numerically by symmetric five-point formulas. Figure 3 depicts the $r$ dependence of the potential obtained from the wall/Dirichlet data lying on the cubic diagonals. The plot includes all time slices ranging from $t_{\text{min}}$ to $t_{\text{max}}$ along with the interpolations. A good agreement among potentials from (20) suggests that contributions from higher order terms in the derivative expansion are small. The situation is different for the effect of rotational invariance violation: Figure 4 shows the potential on $R_2$ for on-axis, plane-diagonal and cubic-diagonal data points. Here we observe stronger deviations from the on-axis data compared to the other two. This uncertainty induced by rotational invariance violation is the major contribution to the systematic uncertainties of the potential method. Figure 5 depicts the source dependence of the potential by comparing the potential obtained from the two sources. The largest differences can be found in the tail, but everything is well compatible within errors. Therefore, observables sensitive to the asymptotic behavior of the potential - such as the scattering length - are only mildly affected by the choice of the source. This suggests that contributions form higher order terms in the derivative expansion are small. The fitted potential can be used to solve the Schrödinger equation (11) to obtain the two-pion
FIG. 9. Results for the scattering lengths obtained using different sources and methods. The upper two results are obtained by fitting the data to the effective range expansion (24), the lower two by solving the equation (10). In all cases, the statistical and overall uncertainties are displayed with solid and dotted error bars, respectively. In the upper panel the statistical error bars are smaller than the symbols. For Lüüscher’s method, the systematic error is underestimated (cf. section V).

wave functions in infinite volume. When combined with (22), we are able to compute the scattering phase shift $\delta(k)$ for arbitrary $k$. The corresponding curves can be compared to the value computed by using Lüüscher’s method. This is done in Figure 6 and we observe that the results of both agree very well (upper panel). The error bar of the data point at $E_{CM} \approx 30$ MeV obtained from Lüüscher’s method is rather large since it has been extracted in a boosted system. The lower panel of Figure 6 compares the phase shifts obtained using Gaussian and wall sources. A good agreement indicates that the source dependence is very mild, thus concluding that contributions form higher order terms in the derivative expansion are insignificant. Errors in the potential method are mainly due to violation of rotational invariance, whereas this systematic uncertainty has not been accounted for in Lüüscher’s method (cf. section V). We observe that the potential method not only gives a value consistent with the Lüüscher’s method at particular $E_{CM}$ but also provides a powerful tool for mapping out the dependence of the scattering phase $\delta(k)$ on the wide range of $E_{CM}$ (Note that the data points on that curve are highly correlated, since they are all obtained deterministically by solving the Schroedinger-like equation (16) for various input parameters $|k|$ using the potential (20) computed from the same four point correlation function. Figure 7 demonstrates that the dependence of the phase shift on the temporal boundary conditions is also negligible. This result is important when dynamical fermions are used, since anti-periodic boundary conditions are the canonical choice in that case.

B. Ground state saturation

The chosen $t_{min}/a = 15$ roughly corresponds to $t_{min} \approx 1.7$ fm which might also be reachable in nucleon-nucleon systems. The scattering phases obtained from Lüüscher’s method show tiny deviations for different $t_{min}$, whereas those from the potential method are barely affected since the ground state saturation is not required here. Indeed, the scattering phase at low energy from the potential method is stable for even smaller $t$ as long as $t/a \geq t_{inelastic}/a \approx 11$ (cf. Figure 8). A variation of the scattering phase at $t < t_{inelastic}$ is a sign of inelastic contributions. Although not shown here, we observe, on the other hand, a weak $t$ dependence of the scattering phases at larger energy. This can be regarded as truncation errors of the derivative expansion.
TABLE I. Results for scattering lengths $a^{I=2}_{\pi\pi}$ obtained from either method at $m_\pi \sim 940$ MeV, including a full error budget. The breakups include the effect of excited states (column 5), the violation of rotational invariance (column 6, not estimated for Lüscher’s method, cf. section V), the asymptotic behavior of the wave-function (column 7, applicable only for the potential method) and different orders in the effective range expansion (column 8). The latter is not estimated for Lüscher’s method, too, since this would require to repeat the calculation for different lattice sizes. Nevertheless, we expect that it is of the same size as in the potential method and thus subdominant. Due to correlations, the errors do not sum up to 100% when added in square.

|        | value | stat. | sys. total | exc. states | rot.-inv. | asympt. | ERE |
|--------|-------|-------|------------|-------------|------------|---------|-----|
| Potential | -0.1571 | 0.0005 | 0.0063 | 0.0006 | 0.0062 | 0.0 | 0.0004 |
| Lüscher | -0.1597 | 0.0025 | 0.0024 + ? | 0.0024 | ? | – | only one V |

C. Low energy parameters

The scattering length is a useful quantity in phenomenological applications, since many interesting phenomena in low-energy scattering theory can be described by this single number. Therefore, we compare the scattering lengths obtained from the two approaches: in the case of the potential method, we make use of the effective range expansion (24), whereas for the Lüscher method we make use of the (non-relativistic) large volume expansion given in (10) (we expect the higher orders in $1/L$ to be negligible). The resulting scattering lengths are displayed in Figure 9 which shows an excellent agreement. Table I displays the error budgets for the scattering lengths obtained from either method. As seen from the Table, the overall uncertainty in the potential method is dominated by systematics associated with the violation of rotational symmetry.

VII. SUMMARY

We have performed an $I=2$ $\pi\pi$ scattering study in quenched QCD with heavy pions of $m_\pi \sim 940$ MeV. In the determination of scattering phase shifts on the lattice, two different approaches have been employed with a particular emphasis on the examination of systematic uncertainties in each method: Lüscher’s finite volume approach and the HAL QCD potential method. The results of the phase shift and the scattering length have been found to agree well between the two methods. We have observed that the largest systematic uncertainty in the potential method stems from the violation of rotational invariance, while such a systematic uncertainty is difficult to estimate in Lüscher’s method and thus has been neglected in this study. While Lüscher’s method is sensitive to excited state contaminations, the time-dependent potential method can compensate a gross of these effects. This is especially important when multi-baryon systems are considered. Furthermore, the potential approach allows for extracting the scattering phase at arbitrary momenta, as long as the energy of the system is below the inelastic threshold.

ACKNOWLEDGMENTS

All computations were carried out on FZ Jülich and University of Wuppertal computing resources. This work is supported in part by the Japanese Grant-in-Aid for Scientific Research (No. 24740146), the Grant-in-Aid for Scientific Research on Innovative Areas (No. 2004: 20105001, 20105003), SPIRE (Strategic Program for Innovative Research) and the German DFG SFB TR 55 (Sonderforschungsbereich Transregio 55). Special thanks go to Zoltan Fodor for useful discussions and continuous support. We would also like to thank Kalman Szabo, Stefan Krieg, Stephan Dürr, Christian Höfling, Szabolcs Borsanyi and Balint Toth for their support at different stages of the project.

Appendix A: Quark Mass dependence

Systematic uncertainties in Lüscher’s approach and in the potential method could depend on pion masses. We make a supplemental investigation of such a dependence by lowering the pion mass from $m_\pi \sim 940$ MeV down to $m_\pi \sim 330$ MeV and comparing the scattering lengths obtained from the two methods. We keep our physical volume fixed, so that finite volume effects increase when the pion mass is decreased. Both methods are expected to be barely affected by finite volume effects, as long as the volume remains much larger than the interaction range of the pions. In Figure 10 we plot the scattering lengths as a function of $m^2_\pi$. We see that even for pion masses as low as $m_\pi \sim 330$ MeV, the results of the two methods agree very well. As in the case of $m_\pi \sim 940$ MeV, the overall uncertainty in the potential
FIG. 10. Results for the scattering lengths as a function of pion mass squared. Our results from two methods (note that the results are slightly displaced in order to improve readability) are compared with results in Ref. \cite{34}. The black square denoted the result obtained from continuum $\chi$PT \cite{35,36,37}. Note that the behavior $a_{\pi\pi} \sim 1/m_{\pi}$, predicted by $q\chi$PT \cite{38} is not visible, neither in our nor in the CP-PACS data. This is probably attributed to lattice artifacts induced by the large lattice spacing (cf. \cite{34}) and use of the unimproved plaquette action.

\[ \begin{align*}
   & a_{\pi\pi} = 2/m_{\pi}^2 \ [\text{GeV}^2] \\
   & m_{\pi}^2 \ [\text{GeV}^2]
\end{align*} \]

-2.5 -2 -1.5 -1 -0.5 0 0.2 0.4 0.6 0.8 1

method is dominated by systematics due to the violation of rotational invariance. The results are also compared to a previous quenched calculation by CP-PACS \cite{31}, in which the lattice spacing, volume and pion masses are similar to the parameters in this work. Figure 10 also shows the physical (continuum) value for $a_{\pi\pi}^2/m_{\pi}$ estimated from $\chi$PT \cite{35,36,37}.

\[ a_{\pi\pi}^2/m_{\pi} \]

Note that the behavior $a_{\pi\pi}^2 \sim 1/m_{\pi}$, predicted by $q\chi$PT \cite{38} is not visible, neither in our nor in the CP-PACS data. This is probably attributed to lattice artifacts induced by the large lattice spacing (cf. \cite{34}) and use of the unimproved plaquette action.

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