DEVIATION OF GEODESICS IN FLRW SPACETIME GEOMETRIES

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Abstract

The geodesic deviation equation ('GDE') provides an elegant tool to investigate the timelike, null and spacelike structure of spacetime geometries. Here we employ the GDE to review these structures within the Friedmann–Lemaître–Robertson–Walker ('FLRW') models, where we assume the sources to be given by a non-interacting mixture of incoherent matter and radiation, and we also take a non-zero cosmological constant into account. For each causal case we present examples of solutions to the GDE and we discuss the interpretation of the related first integrals. The de Sitter spacetime geometry is treated separately.

This paper is dedicated to Engelbert Schücking

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1 Introduction

It has been known for a long while that the geodesic deviation equation (‘GDE’), first obtained by J L Synge [23, 24], provides a very elegant way of understanding features of curved spaces, and, as pointed out by Pirani [14, 15], gives an invariant way of characterising the nature of gravitational forces in spacetime. As such, it is a useful tool to use in examining specific exact solutions of the Einstein field equations (‘EFE’). Indeed, it may be claimed that the GDE is one of the most important equations in relativity, as this is how one measures spacetime curvature\(^1\). This latter aspect has been discussed in some depth by Szekeres [26].

The GDE determines the second rate of change of the deviation vectors for a congruence of geodesics of arbitrary causal character, i.e., their relative acceleration. Consider the normalised tangent vector field \(V^a\) for such a congruence, parametrised by an affine parameter \(v\). Then

\[
V^a := \frac{dx^a(v)}{dv}, \quad V_a V^a := \epsilon, \quad 0 = \frac{\delta V^a}{\delta v} = V^b \nabla_b V^a,
\]

where \(\epsilon = +1, 0, -1\) if the geodesics are spacelike, null, or timelike, respectively, and we define covariant derivative along the geodesics by \(\delta T^a_{\ b}/\delta v := V^c \nabla_c T^a_{\ b}\). A deviation vector \(\eta^a := dx^a(w)/dw\) for the congruence, which can be thought of as linking pairs of neighbouring geodesics in the congruence, commutes\(^2\) with \(V^a\), so

\[
\frac{\delta \eta^a}{\delta v} = \eta^b \nabla_b V^a.
\]

It follows that their scalar product is constant along the geodesics:

\[
\frac{\delta (\eta_a V^a)}{\delta v} = 0 \quad \Leftrightarrow \quad (\eta_a V^a) = \text{const}.
\]

To simplify the relevant equations, we always choose them orthogonal:

\[
\eta_a V^a = 0.
\]

The general GDE takes the form

\[
\frac{\delta^2 \eta^a}{\delta v^2} = -R^a_{\ bcd} V^b \eta^c V^d,
\]

(see, e.g., Synge and Schild [25], Schouten [21], or Wald [29]). The general solution to this second-order differential equation along any geodesic \(\gamma\) will have two arbitrary constants (corresponding to the different congruences of geodesics that might have \(\gamma\) as a member). There is a first integral along any geodesic that relates the connecting vectors for two different congruences which have one central geodesic curve (with affine parameter \(v\)) in common. This is

\[
\eta_1 a \frac{\delta \eta_2^a}{\delta v} - \eta_2 a \frac{\delta \eta_1^a}{\delta v} = \text{const},
\]

and is completely independent of the curvature of the spacetime manifold.

The aim of this paper is to systematically use the GDE to explore the geometry of the standard Friedmann–Lemaître–Robertson–Walker (‘FLRW’) models of relativistic cosmology (see, e.g., Refs. [18, 21, 25, 27]), solving the GDE for timelike, null and spacelike geodesic congruences in these geometries; hence, obtaining the Raychaudhuri equation determining the time evolution of these models [18, 21], the Mattig observational relations [11] underlying the interpretation of cosmological data [19], and determining the nature of their spatial 3-geometry [18, 25, 27]. Also, we identify in each case the first integral for the GDE and comment on its meaning, in the null case leading to the usual reciprocity theorem [1], and in the timelike case obtaining generic solutions of the GDE via this integral. Thus, our purpose is to characterise the major geometrical and physical features of these spacetimes by use of the GDE, hence showing the utility of this equation in obtaining all the essential geometrical and dynamical results of standard cosmology in a unified way. It is a pleasure to dedicate this paper to Engelbert Schücking, who has made a major contribution to obtaining clarity and elegance in understanding many features of relativistic cosmology.

\(^1\)And so is analogous to the Lorentz force law in electrodynamics; cf. Misner, Thorne and Wheeler, Ch.3, Box 3.1 [12].

\(^2\)The Lie derivative of \(\eta^a\) along the integral curves of \(V^a\) is zero; see, e.g., Schouten [4].
1.1 The cosmological context

In the cosmological situation we consider, we assume the sources of the gravitational field to be a non-interacting mixture of incoherent matter and radiation, to each of which the phenomenological fluid description applies (see, e.g., Refs. 2 and 4). For completeness we also include a cosmological constant $\Lambda$.

Notation used is as follows: $u^a$ is the normalised timelike tangent vector field ($u_a u^a = -1$) to the fundamental matter fluid flow, which is geodesic: $0 = u^b \nabla_b u^a := u^a$. The integral curves of $u^a$ are parameterised by the proper time $t$ of comoving fundamental observers. We use standard FLRW comoving coordinates:

$$ds^2 = -dt^2 + a^2(t) f_{\mu\nu}(x^\rho) dx^\mu dx^\nu, \quad f_{\mu\nu} dx^\mu dx^\nu = dr^2 + f^2(r) (d\theta^2 + \sin^2 \theta d\phi^2) ,$$  
$$u^a = (\delta^a_0)^{\alpha} = \delta^0_0 ,$$

where $a(t)$ denotes the time dependent scale factor, and the function $f(r)$ relates to the intrinsic curvature of the spacelike 3-surfaces \{\text{t = const}\} orthogonal to $u^a$. By spatial homogeneity and isotropy, the covariant derivative of $u^a$ reduces to

$$\nabla_a u_b = \frac{1}{a} \Theta h_{ab}, \quad \Theta := D_a u^a = 3 \frac{\dot{a}}{a} .$$  

Here, $h_{ab}$ is the standard orthogonal projection tensor

$$h_{ab} = g_{ab} + u_a u_b \implies h_{\alpha\beta} = g_{\alpha\beta} ,$$

$\Theta$ is the fluid rate of expansion, and the spatial derivative operator (projected orthogonal to $u^a$ on all indices) is denoted by $D_a$ (cf. Ref. 2). It is a well-known consequence of Eq. (1) that FLRW spacetime geometries have vanishing Weyl curvature (cf. Refs. 2 and 4),

$$C_{abcd} = 0 ;$$

the fluid matter flow neither generates tidal gravitational fields nor causes propagation of gravitational waves.

2 The Riemann curvature tensor

In order to determine the explicit form of the GDE (2), we need the Riemann curvature tensor $R_{abcd}$. Because of Eq. (11), $R_{abcd}$ can be expressed purely in terms of the Ricci curvature tensor $R_{ab}$, its trace $R$, and the metric:

$$R_{abcd} = \frac{1}{3} ( R_{ac} g_{bd} - R_{ad} g_{bc} + R_{bd} g_{ac} - R_{bc} g_{ad} ) - \frac{1}{6} R ( g_{ac} g_{bd} - g_{ad} g_{bc} ) .$$  

The EFE algebraically determine $R_{ab}$ from the matter tensor $T_{ab}$:

$$R_{ab} = T_{ab} - \frac{1}{2} T g_{ab} + \Lambda g_{ab} \implies R = - T + 4 \Lambda .$$

When the matter takes a ‘perfect fluid’ form:

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab} \implies T = - (\mu - 3p) ,$$

($\mu$ is the total energy density and $p$ the isotropic pressure), the Ricci tensor expression is

$$R_{ab} = (\mu + p) u_a u_b + \frac{1}{3} (\mu - p + 2\Lambda) g_{ab} \implies R = (\mu - 3p) + 4 \Lambda .$$

Thus, from Eq. (12), the curvature tensor takes the form

$$R_{abcd} = \frac{1}{3} (\mu + \Lambda) ( g_{ac} g_{bd} - g_{ad} g_{bc} ) + \frac{1}{6} (\mu + p) ( g_{ac} u_b u_d - g_{ad} u_b u_c + g_{bd} u_a u_c - g_{bc} u_a u_d ) .$$

$^3$Determined later by use of the 3-D spatial GDE.

$^4$Geometrised units, characterised by $c = 1 = 8\pi G/c^2$, are used throughout.
Then, for any normalised vector field \( V^a \): \( V_a V^a = \epsilon \), by a straightforward contraction one obtains from Eq. (16) the source term in the GDE:

\[
R_{abcd} V^b V^d = \frac{1}{2} (\mu + \Lambda) (\epsilon g_{ac} - V_a V_c) + \frac{1}{2} (\mu + p) \left[ (V_b u^b)^2 g_{ac} - 2 (V_b u^b) u(a V_c) + \epsilon u_a u_c \right].
\]  

(17)

We will also want the GDE in the spacelike 3-surfaces \( \{ t = \text{const} \} \) orthogonal to \( u^a \), which are 3-spaces of maximal symmetry. In the FLRW case, the Gauß embedding equation provides the relation

\[
3 R_{abcd} = (R_{abcd})_\perp - \frac{1}{8} \Theta^2 (h_{ac} h_{bd} - h_{ad} h_{bc})
\]

(18)

for the 3-D Riemann curvature. From Eq. (16), which made use of the EFE, one has

\[
(R_{abcd})_\perp = \frac{1}{8} (\mu + \Lambda) (h_{ac} h_{bd} - h_{ad} h_{bc})
\]

(19)

so that Eq. (18) becomes

\[
3 R_{abcd} = K(t) \left( h_{ac} h_{bd} - h_{ad} h_{bc} \right),
\]

(20)

where the spatial curvature scalar \( K(t) \) is given by

\[
K(t) := \frac{1}{8} 3 R = \frac{1}{8} (\mu - \frac{1}{3} \Theta^2 + \Lambda).
\]

(21)

This factor will determine the 3-D spatial GDE source term:

\[
3 R_{abcd} V^b V^d = K \left( h_{ac} - V_a V_c \right),
\]

(22)

where \( V_a V^a = 1 \) and \( V_a u^a = 0 \).

### 3 The geodesics

Before turning to address the GDE, we need to solve for the geodesic curves along which the GDE will be integrated. Now the fundamental 4-velocity \( u^a = \delta_0^a \) is a geodesic vector field. Any other geodesic can be transformed to have a purely radial spatial part by suitable choice of local coordinates (because the FLRW geometry is isotropic about every point). Hence, w.o.l.g., radial geodesics are considered, with the origin of the local coordinates \( r = 0 \) at the starting point \( v = 0 \), so that in all cases we will have \( x^2 = \theta = \text{const} \), \( x^3 = \phi = \text{const} \Rightarrow 0 = V^2 = V^3 \).

It is convenient to decompose a general geodesic tangent vector field \( V^a \) into parts parallel and orthogonal to \( u^a \):

\[
V^a := E u^a + P e^a,
\]

(23)

where \( e^a = a^{-1} (\partial_r)^a = a^{-1} \delta_0^a \), \( e_a e^a = 1 \), \( e_a u^a = 0 \), such that

\[
V_a V^a = \epsilon = -E^2 + P^2, \quad -(V_a u^a) = E, \quad P = (\epsilon + E^2)^{1/2}.
\]

(24)

As \( e^a \) spans a radial direction, \( P \geq 0 \). By spatial homogeneity and isotropy\(^5\), for a congruence of radial normalised geodesics, starting off isotropically from \( r = 0 \Rightarrow v = 0 \) (so \( E|_{v=0} = \text{const for all of them} \)),

\[
0 = D_a E = D_a P.
\]

(25)

To determine \( E \), note that

\[
- \frac{\delta (V_a u^a)}{\delta v} = -V^b \nabla_b (V_a u^a) = -V^a (\nabla_b u_a) V^b
\]

\[
= -\frac{1}{3} \Theta h_{ab} V^a V^b = -\frac{1}{3} \Theta \left[ \epsilon + (V_a u^a)^2 \right].
\]

(26)

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\(^5\)See section 4.4 below.

\(^6\)Even though \( e^a \) is not invariantly defined, the 1+3 covariant discussion of LRS perfect fluid spacetime geometries given in Ref. 7 still applies. As such, \( e^a \) is the Fermi-transported (along \( u^a \)) unit tangent of a geodesic and shearfree spacelike congruence. Furthermore, in the given context also its spatial rotation vanishes.
3 THE GEODESICS

Thus, we need to solve

$$\frac{dt}{dv} = V^0 = E = -(V^au^a) , \quad \frac{1}{(\epsilon + E^2)} \frac{dE}{dv} = -\frac{1}{a} \frac{da}{dt} ;$$  

(27)

so

$$\frac{E}{(\epsilon + E^2)} \frac{dE}{dv} = -\frac{1}{a} \frac{da}{dt} \quad \Leftrightarrow \quad \frac{1}{a} \frac{da}{dt} \ln \left(\frac{(\epsilon + E^2)}{(\epsilon + E_0^2)}\right) = \frac{d}{dv} \ln \left(\frac{a}{a_0}\right)^{-1} .$$  

(28)

Integrating, we obtain

$$\frac{(\epsilon + E^2)}{(\epsilon + E_0^2)} = \left(\frac{a_0}{a}\right)^2 .$$  

(29)

Now solving for $E$,

$$E^2 = (\epsilon + E_0^2) \left(\frac{a_0}{a}\right)^2 - \epsilon ,$$  

(30)

which implies

$$\frac{dt}{dv} = V^0 = E(a) = \pm \left(\epsilon + E_0^2 \left(\frac{a_0}{a}\right)^2 - \epsilon \right)^{1/2} ,$$  

(31)

with a “+” for future-directed vectors $V^a$ and a “−” for past-directed ones. Also, with Eq. (7),

$$V^ag_{ab}V^b = -(V^0)^2 + a^2 (V^1)^2 = \epsilon \quad \Leftrightarrow \quad V^ah_{ab}V^b = \epsilon + E^2 = a^2 (V^1)^2 ,$$  

(32)

so

$$\frac{dr}{dv} = V^1 = \frac{P(a)}{a} = \left[\epsilon + \frac{E^2(a)}{a^2}\right]^{1/2} ,$$  

(33)

which, for later reference, can also be cast into the form

$$dt := a \, dr = (\epsilon + E_0^2)^{1/2} \left(\frac{a_0}{a}\right) dv ,$$  

(34)

the definition coming from Eq. (7). Hence,

$$\frac{dt}{dr} = \frac{dt/dv}{dr/dv} = \pm \frac{E(a)}{[a^{-2} (\epsilon + E^2(a))]^{1/2}} = \pm \frac{a^2 E(a)}{a_0 (\epsilon + E_0^2)^{1/2}} ,$$  

(35)

and so

$$\frac{dt}{dr} = \pm a(t) \left[1 - \epsilon \left(\frac{a(t)}{a_0}\right)^2 \right]^{1/2} , \quad a_0 := \pm a_0 (\epsilon + E_0^2)^{1/2} .$$  

(36)

3.1 Timelike

For timelike vector fields, $\epsilon = -1$. If we have $V^a$ initially parallel to $u^a$, then $E_0^2 = 1$, and so $dt/dv = 1$ and $dr/dv = 0$, confirming that $V^a$ then remains parallel to $u^a$ (which is geodesic). Otherwise, for future-directed timelike geodesics $V^a$ that have a non-zero initial hyperbolic angle of tilt with $u^a$ (such that $E_0^2 > 1$), the following relations apply:

$$\frac{dt}{dv} = \left[1 + (E_0^2 - 1) \left(\frac{a_0}{a}\right)^2 \right]^{1/2} , \quad \frac{dr}{dv} = (E_0^2 - 1)^{1/2} \left(\frac{a_0}{a^2}\right) .$$  

(37)

3.2 Spacelike

For spacelike vector fields, $\epsilon = +1$. Setting $E_0 = 0$ means starting off orthogonally, but these geodesics do not remain orthogonal to the flow lines, and so do not remain within the spacelike 3-surfaces $\{t = \text{const} \}$. Indeed, from Eqs. (25) and (31)

$$-(V^au^a)|_P = 0 , \quad \Theta |_P > 0$$  

$$\Rightarrow \quad -(V^au^a) = E < 0 \quad \text{nearby} \quad \Rightarrow \quad \frac{dt}{dv} < 0 ,$$  

(38)
showing that the *geodesic*, nearby spacelike 3-surfaces bend *down* (into the past) relative to the spacelike 3-surfaces \(\{t = \text{const}\}\). In this case \(\alpha_0 = \pm a_0\), and

\[
\frac{dt}{dv} = - \left[ \frac{(a_0)}{a} - 1 \right]^{1/2} .
\]

So, with \(dr/dv = (a_0/a^2)\), we find

\[
\frac{dt}{dr} = - a \left[ 1 - \left( \frac{a}{a_0} \right)^2 \right]^{1/2} .
\] (39)

The geodesic 3-surfaces give the best slicing of a spacetime in order to approximate Newtonian theory in a general spacetime — see the discussion by Ehlers [3] — and have been studied in the FLRW context by Rindler [17], Page [13], and Ellis and Matravers [6].

The simplest dynamical case is the spatially flat Einstein–de Sitter model, which has (pressure-free) incoherent matter as a source, and \(\Lambda = 0\). Here, the length scale factor takes the functional form

\[
a(t) = a_0 \left( \frac{3}{2} H_0 t \right)^{2/3},
\]

where \(H_0\) is the value of the Hubble parameter \(H := (1/a) (da/dt)\) at time \(t_0\). Hence, we obtain (note that \(t \leq t_0\))

\[
r(t, t_0) = - \frac{1}{a_0 (3H_0/2)^{2/3}} \int_{t_0}^{t} dy \left[ 1 - (3H_0/2)^{4/3} y^{4/3} \right]^{1/2} ,
\] (40)

leading to an elliptic integral which gives the value of \(r\) at time \(t\), starting off orthogonally at \(r = 0\) and time \(t_0 = \frac{3}{4} H_0^{-1}\).

### 3.3 Null

In the case of null congruences, \(\epsilon = 0\). Then it follows for the past-directed case that

\[
\frac{dt}{dv} = - \left( E_0^2 \right) \left( \frac{a_0}{a} \right)^2 \right]^{1/2} = - E_0 \left( \frac{a_0}{a} \right) ,
\]

and, as \(dr/dv = |E_0| (a_0/a^2)\),

\[
\frac{dt}{dr} = - a(t) .
\] (41)

Alternatively, we can use the fact that \(\xi_a := a(t) u_a\) is a conformal Killing vector field: \(\nabla_a \xi_b = \dot{a}(t) g_{ab}\). Thus, for any geodesic vector field \(k^a\),

\[
k^b \nabla_b (\xi_a k^a) = \dot{a}(t) g_{ab} k^b = \dot{a}(t) k_a k^a ,
\] (42)

and in the particular case that \(k^a\) is null:

\[
k_a k^a = 0 \Rightarrow \xi_a k^a = a(t) u_a k^a = \text{const}
\]

\[
\Rightarrow (k_a u^a)(t) = \frac{\text{const}}{a(t)} .
\] (43)

Relating this to the redshift, \(z\), defined by

\[
(1 + z) := \frac{(k_a u^a)e}{(k_0 u^0)_0} = \frac{E_e}{E_0} = \left( \frac{a_0}{a(t_e)} \right) ,
\] (44)

for past-directed radial null geodesics it follows that

\[
k^a = \frac{a_0}{a(t)} \left( -1, \frac{1}{a(t)}, 0, 0 \right)^T , \quad \frac{dt}{dv} = k^0 = E = -(1 + z) ,
\] (45)

where we have set \(E_0 = -1\) by choice of the affine parameter \(v\).
4 The geodesic deviation equation

4.1 The deviation vectors

The basic equations relating the geodesic vector $V^a$ and orthogonal deviation vector $\eta^a$ have been given above, see Eqs. (1) - (3). We now restrict the deviation vector further.

4.1.1 The screen space

When $V^a$ is not parallel to $u^a$, the vector $\eta^a$ lies in the screen space of $u^a$ (i.e., the spacelike 2-surface orthogonal to both, $u^a$ and $V^a$) iff, additionally to $(\eta_a V^a) = 0$, $\eta^a$ also lies in the rest 3-space of $u^a$, i.e., $(\eta_a u^a) = 0$. We can choose this to be true initially; will it be maintained along the integral curves of any geodesic vector field $V^a$? With Eq. (2), we have

$$\frac{\delta(\eta_a u^a)}{\delta v} = \frac{1}{3} \Theta_{ab} \eta^a V^b + u_a \eta^b \nabla_b V^a = \frac{2}{3} \Theta_{ab} \eta^a V^b + \eta^b \nabla_b (V_a u^a), \tag{46}$$

so

$$\frac{\delta(\eta_a u^a)}{\delta v} = \eta^b \nabla_b (V_a u^a), \tag{47}$$

which will be zero, if $\eta^b \nabla_b (V_a u^a) = 0$, and this will be true for the congruences we consider (cf. Eq. (25)). Propagation of condition (47) along the integral curves of $u^a$ then confirms its preservation. This can be seen as follows. The fact that $V^a$ and $\eta^a$ commute, Eq. (2), gives rise to the relation

$$0 = u_a \left[ u^c \nabla_c V^b \left( \nabla_b \eta^a \right) + V^b \eta^c \nabla_c \nabla_b \eta^a - u^c \nabla_c \eta^b \left( \nabla_b V^a - \eta^b \nabla_b \nabla_c V^a \right) \right], \tag{48}$$

which is used to eliminate the respective terms in the “dot”-derivative of condition (17). Hence, with Eq. (3),

$$\left[ V^b \nabla_b (\eta_a u^a) - \eta^b \nabla_b (V_a u^a) \right] = \frac{2}{3} h_{ab} \left[ \Theta V^a \right] = 0, \tag{49}$$

which vanishes because $h_{ab}$ is symmetric in its indices. So, the consistent solution to these equations is

$$(\eta_a u^a) = 0, \quad (\eta_a V^a) = 0, \quad D_a (V_b u^b) = 0; \tag{50}$$

i.e., $\eta^a$ starts and remains within the rest 3-spaces of $u^a$, and it also remains orthogonal to $V^a$, which has a constant scalar product with $u^a$ in these rest 3-spaces. From now on we will assume these relations hold.

4.1.2 The force term

The “force term” (cf., e.g., Pirani [14]) for the general GDE (3) for geodesic congruences of either timelike, null or spacelike causal character, specialised to the FLRW case, can now be evaluated from Eqs. (17) and (50) to yield

$$R_{abcd} V^b \eta^c V^d = \left[ \epsilon \{ (\mu + \Lambda) + \frac{1}{2} (\mu + p) E^2 \} \right] \eta^a \tag{51}$$

where, as before, $-(V_a u^a) = E$. Note that this force term is proportional to $\eta^a$ itself, i.e., according to the GDE (3) only the magnitude $\eta$ will change along a geodesic, while its spatial orientation will remain fixed. Consequently, the GDE (3) reduces to give just a single differential relation for the scalar quantity $\eta$. This reflects the spatial isotropy of the Riemann curvature tensor about every point in the present situation; anisotropic effects as induced, e.g., by non-zero electric Weyl curvature, $E_{ab}$, or shear viscosity, $\pi_{ab}$, are not involved.

We deal, now, with three cases: the GDE for a fundamental observer, for past-directed geodesic null congruences, and for other families of geodesics.

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7 Also, Eq. (25) has no component proportional to $u^a$, confirming the consistency of the above screen space analysis.
4.2 Geodesic deviation for a fundamental observer

Case 1: $V^a = u^a$ for the central geodesic. In this case the affine parameter coincides with the proper time of the central fundamental observer, i.e., $v = t$. From Eq. ($51$), with $\epsilon = -1$ and $E = 1$,

$$R_{abcd} u^b u^c = \frac{1}{3} (\mu + 3p) \eta_a - \frac{1}{3} \Lambda \eta_a .$$

Let the deviation vector be $\eta^a = \ell e^a, \ e_a e^a = 1, \ e_a u^a = 0$, such that it connects neighbouring flow lines in the radial direction. Then $\delta\varepsilon^a / \delta t = u^b \nabla_b e^a = 0$ (as there is no shear or vorticity!), i.e., a basis is used which is parallelly propagated along $u^a$, and Eq. ($51$) gives

$$\frac{d^2 \ell}{dt^2} = - \frac{1}{6} (\mu + 3p) \ell + \frac{1}{3} \Lambda \ell ,$$

which is the Raychaudhuri equation ($16$). However, this equation applies to both comoving matter of active gravitational mass density $(\mu + 3p)$, and to test matter that is not comoving. On the basis of this relation, it is clear that for positive active gravitational mass density and non-negative cosmological constant all families of past- and future-directed timelike geodesics will experience focusing, provided $(\mu + 3p) > 2 \Lambda$, and so gives rise to the standard singularity theorems (see, e.g., Refs. ($16$, $2$, $8$, $4$)).

4.2.1 Comoving matter

For comoving matter, $V^a = u^a \Rightarrow |E_0| = 1 \Rightarrow |E| = 1$ for the whole family of geodesics. Then, set $\ell = a$ and multiply by $da/dt$ to get

$$0 = \frac{da}{dt} \frac{d^2 a}{dt^2} + \frac{1}{6} (\mu + 3p) a \frac{da}{dt} - \frac{1}{3} \Lambda a \frac{da}{dt} .$$

Using the conservation equation for comoving matter,

$$\frac{d\mu}{dt} = - \frac{3}{a} \frac{da}{dt} (\mu + p) \Rightarrow \frac{d(\mu a^2)}{dt} = - (\mu + 3p) a \frac{da}{dt} ,$$

one finds the familiar Friedmann equation

$$\left( \frac{1}{a} \frac{da}{dt} \right)^2 - \frac{1}{3} \mu a^2 - \frac{1}{3} \Lambda a^2 = - k , \quad k = \text{const} ,$$

giving the usual time evolution of $a(t)$ for a given equation of state. In terms of invariants,

$$\left( \frac{1}{a} \frac{da}{dt} \right)^2 - \frac{1}{3} \mu - \frac{1}{3} \Lambda = - \frac{k}{a^2} ,$$

which is just the trace of the Gauß equation, Eq. ($21$), if we identify

$$K = \frac{k}{a^2}$$

as the constant curvature of the spacelike 3-surfaces $\{ t = \text{const} \}$. Hence, we recover the standard dynamical equations for the FLRW models from the GDE. As usual, whenever $K$ is non-zero, by rescaling $a(t)$ by a constant the dimensionless quantity $k$ can be normalised to $\pm 1$, which is then the curvature of the 3-spaces of maximal symmetry with metric $f_{\mu\nu} dx^\mu dx^\nu$ (cf. Eq. ($1$)).

If one considers a non-interacting mixture of both incoherent matter and radiation, one has

$$\mu = 3 H_0^2 \Omega_m \left( \frac{a_0}{a} \right)^3 + 3 H_0^2 \Omega_r \left( \frac{a_0}{a} \right)^4 , \quad p = H_0^2 \Omega_r \left( \frac{a_0}{a} \right)^4 .$$

Then, evaluating Eq. ($57$) at $t = t_0$ shows that

$$H_0^2 - \frac{1}{3} (\mu_{m_0} + \mu_{r_0}) - \frac{1}{3} \Lambda = - \frac{k}{a_0^2} \quad \Rightarrow \quad H_0^2 (\Omega_{m_0} + \Omega_{r_0} + \Omega_{\Lambda_0} - 1) = K_0 ,$$

If $\Lambda < 0$, for $(\mu + 3p) > 0$ there will be focusing anyway.
where
\[ K_0 := \frac{k}{a_0^3} \]  
(61)
and, as familiar, \( \Omega_{i0} \) denotes dimensionless cosmological density parameters \( \Omega_i := \mu_i/(3H^2) \) at \( t = t_0 \); \( \Omega_L := \Lambda/(3H^2) \) defines an analogous quantity for the cosmological constant. Similarly, evaluating the Raychaudhuri equation (53) at \( t = t_0 \) gives
\[ q_0 = -\frac{1}{3H_0^2} \left( \frac{\dddot{a}}{a} \right) \bigg|_{t_0} = \frac{1}{3} (\Omega_m 0 + 2\Omega_{r0} - 2\Omega_{\Lambda0}) \simeq \frac{1}{3} \Omega_m 0 - \Omega_{\Lambda0} , \]  
(62)
the \( t_0 \) value of the dimensionless cosmological deceleration parameter \( q := -(a d^2a/dt^2)/(da/dt)^2 \). These results will be useful in deriving the observational relations for null data (see section 4.3).

4.2.2 Non-comoving matter

For isotropically distributed test matter moving with other 4-velocities about the fundamental observers, i.e., \( V^a = v^a \Rightarrow |E_0| > 1 \), except for the central curve of the congruence \( v^a \) which coincides with \( u^a \), we need to obtain other solutions to the GDE for timelike curves, evaluated along this central fundamental world line (where again proper time \( t \) is the same as the preferred affine parameter \( v \), and also here the deviation vectors have radial orientation). There are two ways to do this.

One way is to fully specify the matter source in the equations of the previous discussion on the comoving matter case, solve these equations to obtain the source term in the GDE (53), and then solve the GDE to obtain its general solution (with two arbitrary constants). In the case of the de Sitter universe, we have \( 0 = \mu = p, \Lambda \neq 0 \) so Eq. (53) becomes
\[ 0 = \frac{d^2\ell}{dt^2} - \frac{1}{3} \Lambda \ell , \]  
(63)
and the solution is
\[ \ell(t) = \begin{cases} C_1 \cosh(\alpha t) + C_2 \sinh(\alpha t) & \Lambda > 0 \\ C_1 \cos(\alpha t) + C_2 \sin(\alpha t) & \Lambda < 0 \end{cases} , \]  
(64)
with \( \alpha := (1/3|\Lambda|)^{1/2} \) and \( C_1 \) and \( C_2 \) integration constants carrying the dimension of \( \ell(t) \). This shows the deviation for arbitrary (i.e., independent of \( |E_0| \geq 1 \)) timelike geodesics in the de Sitter (\( \Lambda > 0 \)) and anti-de Sitter (\( \Lambda < 0 \)) cases.

When dynamical matter is present, life is more complex. Defining a dimensionless conformal time variable \( \tau \) by \( dt/d\tau := a \Rightarrow d^2t/d\tau^2 = da/d\tau \), for a matter source according to Eq. (53) the Friedmann equation (67) yields
\[ \frac{da}{d\tau} = \left[ \frac{1}{3} \Lambda a^4 - k a^2 + a_0^3 H_0^2 \Omega_m 0 a + a_0^4 H_0^2 \Omega_{r0} \right]^{1/2} . \]  
(65)
This can easily be solved when \( \Lambda = 0 \), for given value of the spatial curvature parameter \( k \). It follows that the GDE for timelike congruences, Eq. (53), can be rewritten as
\[ 0 = \frac{d^2\ell}{d\tau^2} - \frac{1}{a} \frac{da}{d\tau} \frac{d\ell}{d\tau} + \frac{1}{3} a_0^3 H_0^2 \left[ \Omega_m \left( \frac{a_0}{a} \right) + 2\Omega_{r0} \left( \frac{a_0}{a} \right)^2 \right] \ell - \frac{1}{3} \Lambda a^2 \ell , \]  
(66)
where \( a = a(\tau) \), and \( da/d\tau \) is determined through Eq. (65). Unfortunately, this linear homogeneous second-order ordinary differential equation is very complicated, except for the de Sitter universe (where \( 0 = \Omega_m = \Omega_{r0}, \Lambda \neq 0 \)), which we already considered.

To provide a simple example with dynamical matter, we fall back onto the Einstein–de Sitter model, where \( \Lambda = 0, k = 0, \Omega_{r0} = 0 \Rightarrow \Omega_m 0 = 1 \). In dimensionless conformal time, the length scale factor is \( a(\tau) = \frac{1}{4} a_0^3 H_0^2 \tau^2 \), and so the solution to Eq. (66), which then reduces to
\[ 0 = \frac{d^2\ell}{d\tau^2} - \frac{2}{\tau} \frac{d\ell}{d\tau} + 2 \frac{d^2\ell}{d\tau^2} , \]  
(67)
\[ \text{Or, equivalently, } (\mu + p) = 0 \Rightarrow (\mu + 3p) = -2\mu = \text{const}, \] giving an effective cosmological constant.
is given by
\[ \ell (\tau) = C_1 \tau + C_2 \tau^2 ; \] (68)
again, the integration constants \( C_1 \) and \( C_2 \) carrying the dimension of \( \ell (\tau) \). Fixing initial conditions such as to describe a set of test particles isotropically emanating from the central reference geodesic at \( \eta = \eta_0 \), one has \( C_2 = -\frac{1}{2} C_1 / \eta_0 \).

Another way to obtain solutions to the timelike GDE \( (73) \) is to use the first integral which relates different solutions to the GDE along a central reference geodesic \( \gamma_0 \) (which is common to both congruences, and on which the affine parameters coincide and are equal to the preferred time coordinate, i.e., \( v|_{\gamma_0} = t \)). Let \( \eta_1 \) relate to the fundamental family of world lines and \( \eta_2 \) to another family. Then \( \eta_1 = a(t) \), and as \( dt/dv = -(v_0 w^0) = E \) takes the value \( E = 1 \) on the central reference geodesic, \( (1/\eta_1) (d\eta_1/dv) = H = \frac{1}{2} \Theta \). Considering parallel (radial) deviation vectors for the two families, we obtain for their magnitudes
\[
\eta_1 \frac{d\eta_2}{dt} - \eta_2 \frac{d\eta_1}{dt} = \text{const} \quad \Rightarrow \quad \frac{d\eta_2}{dt} - \eta_2 H(t) = \frac{\text{const}}{a(t)}. \] (69)
In terms of initial data at time \( t = t_0 \),
\[
\left. \frac{d\eta_2}{dt} \right|_{t_0} - \eta_2 |_{t_0} H_0 = \frac{\text{const}}{a_0}, \] (70)
which leads to
\[
\frac{d\eta_2}{dt} - \eta_2 \frac{1}{a(t)} \frac{da(t)}{dt} = \frac{a_0}{a(t)} \left[ \left. \frac{d\eta_2}{dt} \right|_{t_0} - \eta_2 |_{t_0} H_0 \right], \] (71)
and so
\[
\left. \frac{d}{dt} \left( \frac{\eta_2}{a(t)} \right) \right|_{t_0} = \frac{a_0}{a^2(t)} \left[ \left. \frac{d\eta_2}{dt} \right|_{t_0} - \eta_2 |_{t_0} H_0 \right]. \] (72)

Then, integration yields
\[
\eta_2(t) = \eta_2 |_{t_0} \left( \frac{a(t)}{a_0} \right) + \left[ \left. \frac{d\eta_2}{dt} \right|_{t_0} - \eta_2 |_{t_0} H_0 \right] a(t) \int_{t_0}^{t} \frac{a_0}{a^2(y)} dy. \] (73)

For the Einstein–de Sitter example, which we referred to before, \( a(t) = a_0 \left( t/t_0 \right)^{2/3} \) (as \( H_0 = \frac{2}{3} t_0^{-1} \)), and we find
\[
\eta_2(t) = \eta_2 |_{t_0} \left( \frac{t}{t_0} \right)^{2/3} + 3 \left[ \left. \frac{d\eta_2}{dt} \right|_{t_0} - \frac{2}{3} \eta_2 |_{t_0} t_0^{-1} \right] t_0^{2/3} t^{2/3} \left( t_0^{-1/3} - t^{-1/3} \right). \] (74)

Special cases:

**A:** Suppose \( \eta_2 = 0 \) at \( t = t_0 \) (matter flowing out isotropically from the central line at that instant), then
\[
\eta_2(t) = 3 \left. \frac{d\eta_2}{dt} \right|_{t_0} t_0^{2/3} t^{2/3} \left( t_0^{-1/3} - t^{-1/3} \right), \] (75)
giving the radial motion of free particles relative to the fundamental observers, that start off by diverging from them. The graph of Eq. \( (73) \) was plotted in Fig. 1.

**B:** Suppose \( d\eta_2/dt = 0 \) at \( t = t_0 \) (matter released from rest at that instant, hence, not comoving with the expanding fundamental matter), then
\[
\eta_2(t) = \eta_2 |_{t_0} \left( \frac{t}{t_0} \right)^{1/3} \left[ 2 - \left( \frac{t}{t_0} \right)^{1/3} \right], \] (76)
gives their radial motion relative to the fundamental observers. The graph of Eq. \( (76) \) was plotted in Fig. 2.
Figure 1: Plot of the deviation vector magnitude $\eta_2(t)$ according to Eq. (75). The parameter values chosen are $H_0 = 60 \text{ km/s/Mpc}$, i.e., $t_0 = 0.01 (\text{Mpc/km})$, and $d\eta_2/dt|_{t_0} = 0.1$.

Figure 2: Plot of the deviation vector magnitude $\eta_2(t)$ according to Eq. (76). The parameter values chosen are $H_0 = 60 \text{ km/s/Mpc}$, i.e., $t_0 = 0.01 (\text{Mpc/km})$, and $\eta_2|_{t_0} = 1$ unit length.

C: Suppose $d\eta_2/dt = \eta_2|_{t_0} H_0$ at $t = t_0$ (matter initially comoving with the expanding fundamental matter), then the matter continues to move as the fundamental observers, i.e., $\eta_2(t) = \eta_2|_{t_0} (t/t_0)^{2/3}$.

Generically, the first integral (6), applied to this timelike case, relates the out-going and in-coming geodesics that link two (timelike separated) points $O$ and $P$, on fixing boundary conditions for the first integral: namely it relates the positions and velocities of each congruence at $O$ to those at $P$. Apart from the cases just considered, the other one that arises naturally is if particles 1 are at rest at $O$ and coincide at $P$, whereas for particles 2 the situation is the converse: they are at rest at $P$ but coincide at $O$. Then

$$\eta_1|_O \frac{d\eta_2}{dt}|_O = \eta_2|_P \frac{d\eta_1}{dt}|_P.$$  \hspace{1cm} (77)

This relates the positions and velocities at $O$ and $P$, showing that if both distances are the same (in absolute, not comoving terms), then the velocities will be the same.
4.3 Past directed null vector fields

**Case 2:** $V^a = k^a$, $k_a k^a = 0$, $k^0 < 0$. Equation (51) now gives

$$R_{abcd} k^b \eta^c k^d = \frac{1}{2} (\mu + p) E^2 \eta_a ,$$

so writing $\eta^a = \eta e^a$, $e_a e^a = 1$, $0 = e_a u^a = e_a k^a$, and using a parallelly propagated and aligned basis, $\delta e^a/\delta v = k^b \nabla_b e^a = 0$, we find from (5),

$$d^2 \eta dv^2 = - \frac{1}{2} (\mu + p) E^2 \eta .$$

Again, in line with the timelike case of Eq. (53), all families of past-directed (and future-directed) null geodesics experience focusing, provided $(\mu + p) > 0$ (while the sign of $\Lambda$ has no influence). Equation (79) is easily solved in the case of the de Sitter universe, where $(\mu + p) = 0$, and the solution is $\eta(v) = C_1 v + C_2$, equivalent to the (flat) Minkowski spacetime case. For null rays diverging from the origin, $C_2 = 0$, and we have the same angular size-distance relation as in flat space (provided we measure distance in terms of the affine parameter $v$).

When dynamical matter is present, we need to express the quantities contained in Eq. (79) in terms of the (non-affine parameter) redshift $z$, defined in Eq. (44). A standard collection of mathematical formulae [1] gives for the derivative operator of Eq. (79) the expression

$$\frac{d^2 \eta}{dv^2} = \left( \frac{dv}{dz} \right)^{-2} \left[ \frac{d^2 \eta}{dz^2} - \left( \frac{dv}{dz} \right)^{-1} \frac{d \eta}{dz} \frac{d}{dz} \frac{d}{dz} \right] .$$

From Eq. (44) we know that

$$(1 + z) = \frac{a_0}{a} = \frac{E}{E_0} \Rightarrow \frac{dz}{(1 + z)} = - \frac{da}{a} = \frac{dE}{E} ,$$

hence, (in the past-directed case),

$$dz = (1 + z) \frac{1}{a} \frac{da}{dv} dv = (1 + z) \frac{1}{a} \frac{da}{dt} E dv = E_0 H (1 + z)^2 dv ,$$

which leads to

$$\frac{dv}{dz} = \frac{1}{E_0 H (1 + z)^2} .$$

The Hubble parameter is to be determined via the Friedmann equation, Eq. (57), from which one obtains

$$H^2 = \frac{1}{3} \mu + \frac{1}{3} \Lambda + H_0^2 (1 - \Omega_m - \Omega_r) (1 + z)^2 .$$

By use of the Raychaudhuri equation, Eq. (53), one finds, furthermore,

$$\frac{d^2 v}{dz^2} = - \frac{3}{E_0 H (1 + z)^2} \left[ 1 + \frac{1}{18 H^2} (\mu + 3p) - \frac{1}{9 H^2} \Lambda \right] .$$

So, altogether, the null GDE, Eq. (79), can be expressed in the new form

$$0 = \frac{d^2 \eta}{dz^2} + \frac{3}{(1 + z)} \left[ 1 + \frac{1}{18 H^2} (\mu + 3p) - \frac{1}{9 H^2} \Lambda \right] \frac{d \eta}{dz} + \frac{1}{2(1 + z)^2} \frac{1}{H^2} (\mu + p) \eta .$$

If we consider again the non-interacting mixture of incoherent matter and radiation, we have

$$\mu = 3 H_0^2 \Omega_m (1 + z)^3 + 3 H_0^2 \Omega_r (1 + z)^4 , \quad p = H_0^2 \Omega_r (1 + z)^4 .$$

Then, from Eq. (81), for $\Lambda = 0$ the Hubble parameter evaluates to

$$H^2 = H_0^2 \left( 1 + \Omega_m z + \Omega_r (2 + z) \right) (1 + z)^2 ,$$
considerable astronomical importance (see, e.g., Refs. [19] and [4]). Using \( \eta \) observer area distance, we have at our disposal, we are now, in a position to easily infer an expression for the

rays diverging from the source \( S \) and arriving at the observer \( O \), with deviation vector \( \eta \), originally derived by Mattig [11] for the dust case (\( \Omega_0 = 0 \)), which is of considerable astronomical importance (see, e.g., Refs. [19] and [4]). Using \( d/d\ell = E_0^{-1} (1 + z)^{-1} d/dv = H (1 + z) d/dz \) (cf. Eqs. [31] and [33]) and choosing the integration constants in Eq. (90) such that \( \eta(z = 0) = 0 \), its definition

\[
\eta(z) = \frac{1}{(1 + z)^2} \left[ C_1 \left( 2 - \Omega_0 - 2 \Omega_r + \Omega_m z \right) + C_2 \left( 1 + \Omega_0 z + \Omega_r z(2 + z) \right)^{1/2} \right], \tag{90}
\]

which we obtained with support from some computer algebra packages. The integration constants \( C_1 \) and \( C_2 \) carry the dimension of \( \eta(z) \). With this explicit form for the deviation vector of a (past-directed) geodesic null congruence at our hands, we are, now, in a position to easily infer an expression for the observer area distance, \( r_0(z) \), originally derived by Mattig [11] for the dust case (\( \Omega_0 = 0 \)), which is of considerable astronomical importance (see, e.g., Refs. [19] and [4]). Using \( d/d\ell = E_0^{-1} (1 + z)^{-1} d/dv = H (1 + z) d/dz \) (cf. Eqs. [31] and [33]) and choosing the integration constants in Eq. (90) such that \( \eta(z = 0) = 0 \), its definition

\[
r_0(z) := \sqrt{\left| \frac{dA_0(z)}{d\Omega_0} \right|} = \left| \frac{\eta(z') \big|_{z' = z}}{\frac{d\eta(z')}{d\ell} \big|_{z' = 0}} \right|, \tag{90}
\]

yields

\[
r_0(z) = H_0^{-1} \left[ 2 \Omega_m - (2 - \Omega_0 - 2 \Omega_r + \Omega_m z) \right]^{-1} \frac{2}{(1 + z)^2} \left[ \left( 2 - \Omega_0 - 2 \Omega_r + \Omega_m z \right) - (2 - \Omega_0 - 2 \Omega_r + \Omega_m z) (1 + \Omega_m z + \Omega_r z(2 + z))^{1/2} \right], \tag{91}
\]
giving the observer area distance as a function of the redshift \( z \) in units of the present-day Hubble radius \( H_0^{-1} \) for an arbitrary non-interacting mixture of matter and radiation (and containing as a special case the Mattig formula when \( \Omega_r = 0 \)). The graph of Eq. (91) was plotted in Fig. 3.

The formula (91) is equivalent to the one stated earlier by Matravers and Aziz [13], but — unlike the usual calculations — is obtained in a uniform way from the null GDE (irrespective of the intrinsic curvature of the spacelike 3-surfaces \( t = \text{const} \)). In the usual approach, three separate calculations are needed (one for each value of \( k \)), and it is a matter of some amazement that they all fit the same formula in the end. In the present approach, one integration is needed, leading to one formula — a considerable increase in clarity.

The first integral relation can be investigated analogously to the timelike case above. Consider null rays diverging from the observer at \( O \) and arriving at the source \( S \), with deviation vector \( \eta_1 \), and null rays diverging from the source \( S \) and arriving at the observer \( O \), with deviation vector \( \eta_2 \). The first integral is the same as before, but now we need to convert (for past-directed null rays) from the affine parameter \( v \) to \( \ell \) according to Eq. (33), with \( a_0/a = (1 + z) \). One obtains

\[
\eta_2 \bigg|_O \frac{d\eta_1}{d\ell} \bigg|_O = \eta_1 \bigg|_S \frac{d\eta_2}{d\ell} \bigg|_S (1 + z), \tag{92}
\]

where the terms \( d\eta/d\ell \) are the angles subtended by the pairs of null rays corresponding to the deviation vectors. Expressed in terms of angular diameter distances, \( r_O \) and \( r_S \), defined by

\[
\eta_1 \bigg|_S := r_O \frac{d\eta_1}{d\ell} \bigg|_O, \quad \eta_2 \bigg|_O := r_S \frac{d\eta_2}{d\ell} \bigg|_S, \tag{93}
\]

\( d\Omega_0 \) here denotes an infinitesimal solid angle rather than a change in density parameter.
(which, for FLRW geometry, are the same as area distances), we find the familiar null reciprocity theorem for FLRW models \[30, 5\]:

\[ r_S = r_0 (1 + z) \]  \hspace{1cm} (94)

This underlies the equivalence (up to redshift factors) of area distance and luminosity distance, and the fact that measured radiation intensity is independent of area distance, depending only on redshift (see Ref. \[4\] for a more detailed discussion). These features are fundamental in analysing observations of distant sources and measurements of the cosmic microwave background radiation.

### 4.4 Generic geodesic vector fields

**Case 3:** \( V^a = \epsilon, \) not parallel to \( u^a, \) nor null. The force term in the generic case is provided by Eq. (51). Writing \( \eta^a = \ell \epsilon^a, e_a \epsilon^a = 1, 0 = e_a V^a = e_a u^a, \) and employing a parallelly propagated and aligned basis, \( \delta e^a / \delta v = V^b \nabla_b e^a = 0, \) we find from Eq. (5),

\[ \frac{d^2 \ell}{dv^2} = - \epsilon \left( \frac{1}{3} (\mu + \Lambda) \ell - \frac{1}{2} (\mu + p) E^2 \ell \right), \]  \hspace{1cm} (95)

giving the spatial orthogonal separation of these geodesics within the 2-D screen space as they spread out in spacetime.

#### 4.4.1 Orthogonal spacelike geodesics

A particular case is the spatial geodesics that start off orthogonal to \( u^a \) (so \( E_0 = 0, \) which implies that the corresponding geodesics are indeed spacelike), but then bend down towards the past thereafter (see the discussion in section 3.2). The above equation applies with \( \epsilon = 1. \) The simplest case is a de Sitter universe where \( 0 = \mu = p, \) and then the solution for all \( |E_0| \geq 0 \), i.e., all spacelike geodesics is

\[ \ell(v) = \begin{cases} 
C_1 \cos(\alpha v) + C_2 \sin(\alpha v) & \Lambda > 0 \\
C_1 \cosh(\alpha v) + C_2 \sinh(\alpha v) & \Lambda < 0
\end{cases} \]  \hspace{1cm} (96)

with \( \alpha := (\frac{1}{3} |\Lambda|)^{1/2} \) (note this is just the exact converse to the timelike case of Eq. (64) above).

In the case of non-zero dynamical matter, however, \( \mu \) and \( p \) are not constants along the initially orthogonal geodesics, as these geodesics do not remain within a spacelike 3-surface \( \{ t = \text{const} \}; \) we have to find \( \mu(t(v)) \) or \( \mu(t(r)) \) from the geodesic equation. However, near the starting point \( v = 0 \) at \( t_0 \) we have (for \( \Lambda = 0 \))

\[ 0 = \frac{d^2 \ell}{dv^2} \bigg|_{t_0} + \frac{1}{3} \mu_0 \ell, \]  \hspace{1cm} (97)
giving the solution near this origin, on carrying out a first-order expansion, by
\[ \ell(v) = \ell_0 \cos(\omega_0 v) + \frac{dl}{dv} \bigg|_{t_0} \sin(\omega_0 v), \quad \omega_0 := \left(\frac{1}{2} \mu_0\right)^{1/2}. \] (98)

This is always convergent for normal matter, irrespective of the intrinsic curvature of the particular spacelike 3-surface \( \{ t_0 = \text{const} \} \) considered. However, as soon as the distance is appreciable, the geodesics will have bent down and lie below the initial 3-surface \( \{ t_0 = \text{const} \} \), where the density of matter will be higher and the curvature greater. Thus, the geodesics will tend to converge even more strongly.

4.4.2 Geodesics in the orthogonal spacelike 3-surfaces

This is to be contrasted with geodesic congruences within the spacelike 3-surfaces \( \{ t = \text{const} \} \) orthogonal to \( u^a \), which are 3-spaces of maximal symmetry. In contrast to Eq. (1), these geodesics satisfy the 3-D equations
\[ V^a := \frac{dx^a(v)}{dv}, \quad V_a V^a = 1, \quad V_a u^a = 0, \quad 0 = V^b D_b V^a. \] (99)

From Eq. (22), the force term for the resulting 3-D spatial GDE takes the form
\[ 3 R_{abcd} V^b \eta^c \eta^d = \frac{1}{2} \left( \mu - \frac{1}{2} \omega^2 + \Lambda \right) \eta_a = K \eta^a, \] (100)

where \( K(t) \) is the curvature of these 3-spaces (cf. Eq. (21)). Consequently, whether geodesics in these spacelike sections converge or diverge depends on the sign of \( K \). Setting \( \eta^a = e^a \) where \( e_a e^a = 1 \) and \( e_a u^a = 0 \), as before we choose a congruence of vectors such that \( \delta e^a/\delta V = V^b D_b e^a = 0 \) and the 3-D spatial GDE becomes
\[ \frac{d^2 \eta}{dv^2} = -K \eta. \] (101)

\( K = K(t) \) is indeed constant along these spatial geodesics (because they lie within the 3-surfaces \( \{ t = \text{const} \} \)). If \( K > 0 \), one deals again with the familiar oscillator equation, i.e., two neighbouring spatial geodesics will harmonically converge to and diverge from each other as \( v \) increases. If \( K < 0 \), they will exponentially diverge, and if \( K = 0 \), they diverge linearly.

Focusing on radial spatial geodesics, the local FLRW coordinates of the spacelike 3-surfaces \( \{ t = \text{const} \} \) arise as follows. We consider a 3-space with metric \( f_{\mu \nu} dx^\mu dx^\nu \), and constant dimensionless scalar curvature, if non-zero, normalised to \( k = \pm 1 \), (cf. Eqs. (5) and (58)). Note that the full 3-space metric \( h_{\mu \nu}(t) \) at arbitrary time \( t \) is just given by \( h_{\mu \nu}(t) = a^2(t) f_{\mu \nu} \). Choosing an affine parameter \( v = r \), \( V^a = (\partial_r)^a = \delta_1^a \) is the geodesic unit normal to the 2-surfaces \( \{ r = \text{const} \} \), which are 2-spheres of area \( 4\pi f^2(r) \). Thus, it is tangent to the orthogonal coordinate curves \( x^2 = \text{const} \), \( x^3 = \text{const} \). A basis of deviation vectors in the 2-D screen space is given by \( \eta_1^a = \delta_2^a \) and \( \eta_2^a = \delta_3^a \) (these commute with the geodesic vector \( V^a = \delta_1^a \), because each of these is a coordinate basis vector). Employing an orthonormal basis with components \( (e_1)^a = \delta_1^a \), \( (e_2)^a = f^{-1}(r) \delta_2^a \), \( (e_3)^a = f^{-1}(r) (\sin \theta) \delta_3^a \), parallelly propagated along the radial geodesics \( V^a \), Eq. (101) yields
\[ 0 = \frac{d^2 \eta}{dv^2} + k \eta \quad \Rightarrow \quad 0 = \frac{d^2 f}{dr^2} + k f, \] (102)

the second relation following because relative to the orthonormal basis, \( \eta_1^a = f(r) \delta_2^a \) and \( \eta_2^a = f(r) \sin \theta \delta_3^a \) (apply the first equation to either vector to get the second). Then the solution we want corresponds to that solution for which \( \eta(r = 0) = 0 \); we find
\[ f(r) = \begin{cases} \sin r & k = +1 \\ r & k = 0 \\ \sinh r & k = -1 \end{cases}, \] (103)

\footnote{Determined by Eqs. (3), (4) and (9).}

\footnote{That is, the 3-D version of Eq. (1) that applies in these 3-spaces.}

\footnote{When \( a(t) \) is of unit magnitude, say at time \( t = \bar{t} \), then \( f_{\mu \nu} \) is equal to the metric \( h_{\mu \nu}(\bar{t}) \) on the 3-surface \( \{ \bar{t} = \text{const} \} \), except for a dimensional unit factor, and similarly for \( k \) and \( K(\bar{t}) \).}
showing how the GDE within the spacelike 3-surfaces \( \{ t = \text{const} \} \) determines the function \( f(r) \) in Eq. (7). The corresponding solutions with \( d\eta/dr = 0 \) at \( r = 0 \) exhibit precisely how Euclid’s parallel postulate breaks down for these curved 3-space sections, according to the spatial curvature.

In this context it is of interest to remark that the Lorentz-invariant de Sitter spacetime geometry, which is the case \( 0 = \mu = p, \Lambda > 0 \), can be sliced by spacelike 3-surfaces \( \{ t = \text{const} \} \) of either constant positive, zero, or negative intrinsic curvature (cf. Ref. [22]), depending on the sign of the sum \( 3K = -\frac{1}{3} \Theta^2 + \Lambda \) (see Eq. (21)). For anti-de Sitter \( (\Lambda < 0) \) only the negative curvature case applies. The different FLRW forms of the de Sitter spacetime metric follow from arguments essentially identical to that just given for the 3-space metric, because it is a 4-space of constant curvature, i.e., maximal symmetry (and the argument applies also to the 2-sphere, leading to the form of the terms in the last bracket in Eq. (7)). In each case, the GDE, together with the constant curvature condition (20), leads to the harmonic equation (102).

Similarly to the null case, the 3-D geometrical reciprocity theorem can be stated as

\[
\eta_1 \bigg|_O \frac{d\eta_2}{dr} \bigg|_O = \eta_2 \bigg|_P \frac{d\eta_1}{dr} \bigg|_P ,
\]

showing how geodesics diverging about a central geodesic from \( P \) to \( O \) at an angle \( \alpha_0 \) reach a separation \( d \) at \( O \), and geodesics diverging from \( O \) at the same angle will reach the same distance apart at \( P \) (irrespective of the spatial curvature which is constant). Corresponding statements hold for the families of geodesics that diverge from \( P \) and \( O \), and end up parallel at \( O \) and \( P \), respectively.

5 Conclusion

One way of solving the EFE is to treat them as algebraic equations relating \( R_{abcd} \) to \( R_{ab} \) and \( C_{abcd} \), then solving the GDE (which characterises relative acceleration due to spacetime curvature) to determine both the spacetime geometry and its properties. In the case of a FLRW model, this can be carried out explicitly, as shown above: integrating the GDE (cf. Eqs. (53), (86) and (95)) allows complete characterisation of all interesting geometrical features of the exact FLRW geometry in an elegant manner — determining the timelike evolution, spacelike geometry, and null ray properties, which in turn determine the basic observational properties. The Newtonian analogue of some of this has been given by Tipler [27, 28].

An interesting project is to extend this calculation to perturbed FLRW models in order to work out the effects of linear anisotropies on the present results as regards all three causal cases (timelike, spacelike, null). This would allow investigation of both dynamical and observational features of such models, for example examining aspects of gravitational lensing theory [20].

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14 That is, \( a^n \) is not uniquely defined.
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