Casimir Energy of a Spherical Shell

M. E. Bowers and C. R. Hagen*

Department of Physics and Astronomy
University of Rochester
Rochester, New York 14627

Abstract

The Casimir energy for a conducting spherical shell of radius $a$ is computed using a direct mode summation approach. An essential ingredient is the implementation of a recently proposed method based on Cauchy's theorem for an evaluation of the eigenfrequencies of the system. It is shown, however, that this earlier calculation uses an improper set of modes to describe the waves exterior to the sphere. Upon making the necessary corrections and taking care to ensure that no mathematically ill-defined expressions occur, the technique is shown to leave numerical results unaltered while avoiding a longstanding criticism raised against earlier calculations of the Casimir energy.

I. Introduction

In 1948 Casimir [1] first predicted that two infinite parallel plates in vacuum would attract each other. This quite remarkable result depends only on the universal constants $\hbar$ and $c$ and the geometry of the plate configuration, but is independent of such things as the electric charge. It has its origin in the expression for the energy $E$ of the uncoupled electromagnetic field which is well known to have the form

$$E = \sum_{k,\lambda} \left( \frac{1}{2} + n_{k,\lambda} \right) \hbar \omega_k^{(\lambda)}$$

where $\omega_k^{(\lambda)} = |k|$ and $n_{k,\lambda}$ denotes the photon occupation number in the mode with wave number $k$ and polarization $\lambda$. The sum is to be taken over
all allowed $k$ and $\lambda$. In the absence of radiation this reduces to the zero point vacuum energy, or

$$E = \sum_{k,\lambda} \frac{1}{2} \hbar \omega_k^{(\lambda)}.$$  

(1)

This divergent expression can be rendered finite by use of a cutoff or convergence factor, and (somewhat surprisingly) made to yield measurable effects when boundary surfaces are used to modify the allowed set of modes which are to be included in (1). Thus Casimir found that there is a net attractive force between two parallel conducting plates. This result led, naturally enough, to his subsequent suggestion [2] that such forces could play a significant role in lending stability to the electron. However, the Casimir energy for the sphere was found to be positive, thereby implying a repulsive force rather than the anticipated attractive one. This result was first achieved in a remarkable, albeit intricate, calculation by Boyer [3] and was later verified by a number of authors [4-6].

More recently a direct mode summation approach to the problem of calculating the Casimir energy of a conducting spherical shell has been advanced by Nesterenko and Pirozhenko [7]. Their technique uses Cauchy’s theorem to convert the sum over eigenmodes into an integral and was found to yield the same analytic expression for the Casimir energy as that obtained in refs. 5 and 6. Since the original approach to the parallel plate geometry was also based on a direct mode summation, such a result provides a welcome addition to the literature of this subfield. On the other hand the approach of ref. 7 has some shortcomings which prevent it from being immediately accepted as the natural extension of the Casimir method to the sphere. In particular it is shown in the following section that the standing waves exterior to the sphere have not been found correctly in ref. 7. After finding appropriate expressions for those modes the calculation is put on a more rigorous footing by including a cutoff function to dampen the high frequency contributions (in direct analogy to the Casimir calculation [1]) together with a modification of the contour in order to ensure the convergence of the Cauchy integral expression. It is shown that this eliminates a spurious divergence which Candelas [8] asserts must be added to the result of ref. 6. The approach developed here applies equally well to the Dirichlet and Neumann spheres, but no explicit calculations are carried out for these cases since earlier numerical work
is unaffected. An extension to (2+1) dimensional QED is considered in a subsequent section with the result that there is an intrinsic divergence in the Casimir energy in this case.

II. Mode Summation for a Spherical Shell Using Cauchy’s Theorem

In evaluating the Casimir energy of a sphere it should be noted that each mode is \(2l + 1\) fold degenerate. Thus (1) becomes

\[
E_c = \sum_{l=1}^{\infty} \left( l + \frac{1}{2} \right) \sum_{n=1}^{\infty} \sum_{\lambda=1,2} \omega_{nl}^{(\lambda)}
\]

(2)

where the eigenfrequencies \(\omega_{nl}^{(\lambda)}\) are determined by imposing appropriate boundary conditions on the multipole fields. In particular, for the case of a spherical shell of radius \(a\), the transverse electric (i.e., \(\lambda = 1\)) modes are

\[
j_l(\omega a) = 0
\]

(3)

\[
A_l j_l(\omega a) + B_l n_l(\omega a) = 0
\]

(4)

and the transverse magnetic (i.e., \(\lambda = 2\)) modes are

\[
\frac{d}{dr} \left[ r j_l(\omega r) \right] \bigg|_{r=a} = 0
\]

(5)

\[
\frac{d}{dr} \left[ r [C_l j_l(\omega r) + D_l n_l(\omega r)] \right] \bigg|_{r=a} = 0
\]

(6)

where \(j_l\) and \(n_l\) are the spherical Bessel functions. The constants \(A_l, B_l, C_l,\) and \(D_l\) (more specifically, the ratios \(B_l/A_l, D_l/C_l\)) are determined by prescribing the correct asymptotic behavior at large \(r\). It should be noted that Eqs. (3) and (5) determine the interior \((r < a)\) modes while (4) and (6) specify the exterior \((r > a)\) modes.

The coefficients which appear in the exterior mode equations are determined by enclosing the entire system within a second concentric conducting sphere of radius \(R\). It is straightforward to verify that Eqs. (4) and (6) can then be written as
\[ j_l(\omega a) + \tan \delta_l n_l(\omega a) = 0 \]

and

\[ \left. \frac{d}{dr} \left[ r[j_l(\omega r) - \cot \delta_l n_l(\omega r)] \right] \right|_{r=a} = 0 \]

respectively, where for sufficiently large \( R \)

\[ \delta_l = \omega R - \frac{l\pi}{2} \]  \hspace{1cm} (7)

This is to be contrasted with the exterior mode solutions of ref. 7 where Eqs. (4) and (6) have been replaced by those of Stratton [9], namely

\[ h_l^{(1)}(\omega a) = 0 \]

and

\[ \left. \frac{d}{dr} [r h_l^{(1)}(\omega r)] \right|_{r=a} = 0, \]

where \( h_l^{(1)} \) denotes the Hankel function

\[ h_l^{(1)}(z) = j_l(z) + in_l(z). \]

While such boundary conditions are appropriate to determine the complex frequencies for a radiating sphere, they are not valid to specify the necessarily real frequencies which contribute to the vacuum energy [10].

Having formulated the conditions for the eigenfrequencies of the system it now remains to be shown how Cauchy’s theorem can be applied to the evaluation of the Casimir energy. In analogy to ref. 7, the eigenfrequency equations are defined as

\[ f_l^{(1)}(z) = j_l(z) \]
\[ f_l^{(2)}(z) = j_l(z) + \tan \delta_l(z)n_l(z) \]
\[ f_l^{(3)}(z) = \frac{d}{dz}[z j_l(z)] \]
\[ f_l^{(4)}(z) = \frac{d}{dz} \left[ z[j_l(z) - \cot \delta_l(z)n_l(z)] \right] \]
where \( z = \omega a \) and

\[
\delta_l(z) = z(R/a) - \frac{l\pi}{2}.
\]

Another useful definition is what might be termed the f-product, or

\[
f_l(z) = z^2 \prod_{a} f_l^{(a)},
\]

where a factor of \( z^2 \) has been included for convenience in order to make \( f_l(z) \) finite at \( z = 0 \). Defining \( z \) to be a complex variable, \( f_l(z) \) is seen to be an analytic function of \( z \), the zeros of which correspond to the eigenfrequencies of the system [11].

It follows from Cauchy’s theorem that for two functions \( f_l(z) \) and \( \phi(z) \) analytic within a closed contour \( C \) in which \( f_l(z) \) has isolated zeros at \( x_1, x_2, ... x_n \),

\[
\frac{1}{2\pi i} \oint_C \phi(z) \frac{f_l'(z)}{f_l(z)} \, dz = \sum_i \phi(x_i).
\]

Choosing \( \phi(z) = ze^{-\sigma z} \) where \( \sigma \) is a real positive constant thus leads to

\[
\frac{1}{2\pi i} \oint_C e^{-\sigma z} z \, dz \, \frac{d}{dz} \ln f_l(z) = \sum_i z_i e^{-\sigma z_i}.
\]

(8)

Upon combining Eq.(8) with Eq.(2) the Casimir energy becomes

\[
E_c = \lim_{\sigma \to 0} \frac{1}{2\pi i} \sum_{l=1}^{\infty} (l + \frac{1}{2}) \oint_C dze^{-\sigma z} \frac{d}{dz} \ln f_l(z).
\]

(9)

It should be noted that the factor of \( e^{-\sigma z} \) plays the role of a cutoff function which effectively suppresses the high frequency contributions to the Casimir energy. While such a cutoff function is generally invoked in calculations based on parallel plate geometry, it was not employed in refs. 6 and 7. It will be seen, however, that it plays an important role in ensuring that all integrals encountered in Eq. (9) are well defined.

Before the calculation of the Casimir energy can be completed, it is necessary to specify a contour for the integration. An appropriate contour \( C \) (displayed in Fig.1) can be conveniently broken into three parts. These consist of a circular segment \( C_\Lambda \) and two straight line segments \( \Gamma_1 \) and \( \Gamma_2 \). Since the \( \Gamma \) contours are oriented at a nonzero angle \( \phi \) with respect to the imaginary axis, it follows that the contribution to \( C_\Lambda \) is, by virtue of the cutoff factor, bounded by \( \exp(-\sigma \Lambda \sin \phi) \) where \( \Lambda \) is the radius of the circular arc.
Since the logarithm in the Casimir energy expression grows at most algebraically, it follows that the contribution to $C_{\Lambda}$ vanishes exponentially in the limit of large $\Lambda$ provided that $\phi \neq 0$.

Along $\Gamma_1$ and $\Gamma_2$ the quantity $\tan \delta_l$ becomes $i$ and $-i$ respectively for sufficiently large $R$. Thus on $\Gamma_1$, where $y = |z|$, 

\[
\begin{align*}
    f_{l}^{(2)}(z) & \to h_{l}^{(1)}(iy e^{-i\phi}) \\
    f_{l}^{(4)}(z) & \to \frac{d}{dy} [y h_{l}^{(1)}(iy e^{-i\phi})],
\end{align*}
\]

with the corresponding terms for $\Gamma_2$ obtained by complex conjugation. This allows $f_{l}(z)$ to be written, up to an overall normalization, in terms of the modified Bessel functions as

\[
\left. f_{l}(z) \right|_{\Gamma_1} = I_{\nu}(y e^{-i\phi}) K_{\nu}(y e^{-i\phi})
\]
\[ I\nu(ye^{-i\phi}) + ye^{-i\phi} I'_\nu(ye^{-i\phi}) \]
\[ K\nu(ye^{-i\phi}) + ye^{-i\phi} K'_\nu(ye^{-i\phi}) \]

where \( \nu = l + \frac{1}{2} \). Upon noting that the contributions along \( \Gamma_1 \) and \( \Gamma_2 \) are complex conjugates of each other, the Casimir energy becomes

\[
E_c = -\frac{1}{\pi a} \sum_{l=1}^{\infty} (l + \frac{1}{2}) \text{Re} e^{-i\phi} \int_0^\infty dy e^{\sigma ye^{-i\phi}} y \frac{d}{dy} \ln f_l(ye^{-i\phi}).
\]

Straightforward manipulation [5] of the argument of the logarithm and rescaling of the integration variable allows this to be recast in the form

\[
E_c = -\frac{1}{\pi a} \sum_{l=1}^{\infty} (l + \frac{1}{2})^2 \text{Re} e^{-i\phi} \int_0^\infty dy e^{\sigma ye^{-i\phi}} y \frac{d}{dy} \ln \left[ 1 - \left( \lambda_l(ye^{-i\phi}) \right)^2 \right]
\]

where

\[
\lambda_l(y) = \frac{d}{dy} [y I_\nu(y) K_\nu(y)].
\]

Upon using the uniform expansions of \( I_\nu(\nu y) \) and \( K_\nu(\nu y) \) for large \( \nu \) [12] it is found that the integral over \( y \) is finite for \( \sigma = 0 \). Adding and subtracting the asymptotic form of the integrand for large \( \nu \), the Casimir energy assumes the form

\[
E_c = E_{\text{fin}} + E_{\sigma},
\]

where the finite part is

\[
E_{\text{fin}} = \frac{1}{\pi a} \sum_{l=1}^{\infty} \int_0^\infty dy \left[ (l + \frac{1}{2})^2 \ln[1 - (\lambda(y))^2] + \frac{1}{4} (1 + y^2)^{-3} \right]
\]

and the cutoff dependent part is

\[
E_{\sigma} = \frac{1}{4\pi a} \sum_{l=1}^{\infty} \text{Re} e^{-i\phi} \int_0^\infty dy e^{\sigma ye^{-i\phi}} y \frac{d}{dy} (1 + y^2 e^{-2i\phi})^{-3}.
\]
In arriving at the $\phi$ independent form for $E_{fin}$ one has in succession taken the limit $\sigma \to 0$ and replaced $y$ with $ye^{i\phi}$. The resulting integral over the interval $0 < y < e^{-i\phi}\infty$ is then trivially seen to be equivalent to the corresponding integral over the real interval $0 < y < \infty$. The quantity $E_{fin}$ has been evaluated numerically in ref. [6] and consequently no attempt is made here to obtain that result independently.

Although the integral which appears in $E_{\sigma}$ is finite even for $\sigma = 0$, the limit cannot be taken at this point since it would yield the divergent result

$$E_{\sigma}\big|_{\sigma=0} = -\frac{3}{64} \sum_{l=1}^{\infty} (l + \frac{1}{2})^0.$$ 

Consequently, $E_{\sigma}$ is to be evaluated for finite $\sigma$ before taking the limit of vanishing cutoff. Since this is clearly a delicate limiting process, some care is warranted. It should first of all be noted that the cutoff factor used here was initially introduced as a real exponential prior to its emergence in $E_{\sigma}$ as the term $\exp(-i\nu \sigma ye^{-i\phi})$. Thus there is a damping term of the form $\exp(-\nu \sigma y \sin \phi)$. This may be contrasted with ref. [6] which objects to using an explicit cutoff but finds, by using a temporal point separation, that an oscillatory factor of the form $e^{i\nu y}$ can appear in $E_{\sigma}$. It is significant that such a term (unlike that found here) has no residual exponential damping effect. This has led to a criticism of the ref. 6 result by Candelas [8] who asserts that because the waves of different $l$ all contribute an amount of equal absolute value to the Casimir energy, an explicit cutoff must be introduced in the sum over $l$. His result is that such an insertion adds a divergent term to the Casimir energy that is proportional to $\int_{0}^{\infty} d\nu$ (where $\nu$ is a frequency). Since, however, the calculation presented here has included a cutoff ab initio there should be no need for the inclusion of an additional cutoff of the type used in [8]. This can now be verified by explicitly carrying out the summation over $l$ to give

$$E_{\sigma} = -\frac{3}{2\pi a} \text{Re}e^{-i\phi} \int_{0}^{\infty} dy \frac{y^2 \exp(-i\sigma ye^{-i\phi}/2)}{(1 + y^2 e^{-2i\phi})^4} \frac{e^{-2i\phi}}{\exp(i\sigma ye^{-i\phi}) - 1}.$$ 

For nonzero $\phi$ the sum is well defined and thus requires no additional cutoff factor. Upon expanding the $\sigma$ dependent terms and dropping those which
go to zero as some positive power of $\sigma$ the above expression reduces to

$$E_\sigma = \frac{3}{2\pi a} \text{Re} \int_0^\infty dy \frac{y^2 e^{-3i\phi}}{(1 + y^2 e^{-2i\phi})^4} [1 + \frac{i}{\sigma ye^{-i\phi}}].$$

As before, let $y \to e^{i\phi}y$ and perform a rotation of the resulting contour. This allows the term singular in $\sigma$ to be dropped (it is purely imaginary) and leaves the finite result

$$E_\sigma \bigg|_{\sigma=0} = \frac{3}{64a}.$$

Upon combining this with the numerical result of ref. [6] for $E_{fin}$, the expression for the Casimir energy is obtained, namely

$$E_c = 0.09235/2a.$$

As a final comment concerning the conducting sphere, it should be noted that the method presented here can equally well be applied to the Dirichlet and Neumann spheres. The results for these cases can be rigorously obtained using the technique of mode summation in conjunction with Cauchy’s theorem. As it would consist merely of a rewriting of ref. [7] using the approach developed here, this is left as an exercise for the interested reader.

### III. Casimir Effect of the Electromagnetic Circle

A natural question which arises subsequent to a consideration of the electromagnetic Casimir effect for a sphere is whether similar results can be obtained in higher and lower dimensions. As will be shown here this question can readily be answered at least in the latter case. Although the electromagnetic field is trivial (i.e., there is no photon) in one spatial dimension, for the two dimensional case a procedure analogous to that of the preceding section can be carried out. This requires the extraction of appropriate boundary conditions from the field equations. The latter are contained in the covariant equations

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (11)$$

$$\partial_\nu F^{\mu\nu} = J^\mu \quad (12)$$
where $J^\mu = (J_i, \rho)$ with $i = 1, 2$. From (11) it follows that

$$B = \nabla \times A$$

and

$$E = -\partial_0 A - \nabla A^0$$

where $B$ is the (scalar) magnetic field and $E$ is the (two-vector) electric field. Correspondingly, Eq.(12) yields

$$\nabla \cdot E = \rho$$

and

$$-\partial_0 E + \nabla B = J$$

where $(\nabla)_i = \epsilon_{ij} \nabla_j$.

In the case that all fields have time dependence $e^{-i\omega t}$ these equations imply that in current free regions

$$(\nabla^2 + \omega^2)B = 0.$$ 

Thus the $B$ field can be written in the form

$$B(r, \phi) = \sum_{m=-\infty}^{\infty} \left[ a_m J_m(kr) + b_m N_m(kr) \right] e^{im\phi}$$

with the corresponding $E$ field given by

$$-i\omega E = \nabla B.$$ 

Since

$$\nabla \times E = -\partial_0 B,$$

it follows that for a conducting circle of radius $a$ the appropriate boundary condition is $r \times E|_{r=a} = 0$, or equivalently, $\frac{d}{dr}B|_{r=a} = 0$. In other words, the Casimir effect in (2+1) dimensional QED is equivalent to that of a scalar field satisfying Neumann boundary conditions.

For the inside modes of the system it follows that the eigenfrequencies are determined by the condition

$$\left. \frac{d}{dr} J_m(\omega r) \right|_{r=a} = 0.$$
The outside modes require that in analogy to the three dimensional case the system be enclosed in a large circle of radius $R$ which is subsequently taken to infinity. Thus the boundary condition in the exterior region takes the form

$$\frac{d}{dr} [J_m(\omega r) + \tan \delta_l N_m(\omega r)] \bigg|_{r=a} = 0$$

where $\delta_l$ is given by Eq.(7). Upon repeating the procedure of the preceding section it is found that the Casimir energy of the circle has the form

$$E_c = -\frac{1}{2\pi a} \sum_{m=-\infty}^{\infty} \text{Re} e^{-i\phi} \int_0^\infty dy y e^{-i\phi} y \frac{d}{dy} \ln[y J'_m(ye^{-i\phi}) K'_m(ye^{-i\phi})].$$

The question which needs to be addressed is whether upon performing the sum over $m$ a finite result for the Casimir energy can be obtained in the $\sigma \to 0$ limit. Using the uniform expansions for the modified Bessel functions for large $|m|$ and considering only those $m$ values with absolute value greater than some large integer $M$, it is found that

$$E_c \to \frac{1}{2a} \sum_{m=M}^{\infty} me^{-m\sigma}.$$

This yields the divergent result

$$E_c \to \frac{1}{2a} \frac{1}{\sigma^2}$$

in the limit of vanishing $\sigma$, a result at variance with the claim in ref. [13] of a finite Casimir energy. On the other hand, it is not unexpected in view of the divergence found by Bender and Milton [14] for the two dimensional Dirichlet Casimir effect.
IV. Conclusion

Since the inception of this subfield, there have been a number of approaches developed for calculating the Casimir energy of a conducting sphere in vacuum. The first successful evaluation consisted of analytically summing the individual mode contributions. This method, closely modelled after the much simpler parallel plate problem, was the most obvious at the time but included tedious algebraic manipulations. More advanced methods followed which rendered the involved mode summation approach obsolete prior to its re-examination in ref. 7. The introduction of Cauchy’s theorem (replacing the mode summation with an integration in the complex frequency plane) opened the door to a revitalization of this method. However, the formalism was based upon an improper treatment of the external modes which led to their being complex and finite in number rather than real and infinite as required. A correct set of equations for the exterior modes was obtained upon placing the entire system in a second conducting sphere whose radius $R$ was eventually allowed to go to infinity.

A second problem associated with the Cauchy integral method consisted in the fact that the neglect of the semicircular part of the contour integral was problematical. This difficulty was eliminated by invoking a cutoff function (as in the original parallel plate calculation), and by introducing simultaneously a kink into the integration contour. This allowed the integral to be evaluated along the imaginary axis. It also led to a mathematically well defined summation over partial waves which eliminated the need for a cutoff of the type proposed by Candelas. Thus an unobservable but divergent contribution to the Casimir energy which he claimed to be essential was found not to be relevant. It was noted that the method developed here applies equally well to the Dirichlet and Neumann spheres, although no explicit calculations have been carried out. Finally, an extension to the case of $(2+1)$ dimensional QED was noted with divergent results being obtained in the limit of vanishing cutoff.

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* Electronic address: hagen@urhep.pas.rochester.edu
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10. It is also of interest to note that the boundary conditions of ref. 7 imply [9] that there are only $2l + 1$ exterior modes for each $l$ even though Eqs. (3) and (5) allow an infinite number of interior modes. The approach presented here, however, preserves the symmetry between the interior and exterior regions (i.e., there are an infinite number of real eigenfrequencies in each case).

11. The fact that there are no other zeros of $f_l(z)$ for $Imz \neq 0$ is perhaps most easily seen from the fact that the eigenfrequencies of this system correspond to those of a free quantum mechanical particle in three dimensions which is confined to the interior of a sphere of radius $a$ or to the region between two concentric spheres of radii $a$ and $R$, subject to either Dirichlet or Neumann boundary conditions. Since all the energy eigenvalues of such a system are known to be real, it follows that $f_l(z)$ has zeros only on the real axis.
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