Self-dual variables, positive semi-definite action, 
and discrete transformations
in four-dimensional quantum gravity

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A positive semi-definite Euclidean action for arbitrary four-topologies can be constructed by adding appropriate Yang-Mills and topological terms to the Samuel-Jacobson-Smolin action of gravity with (anti)self-dual variables. Moreover, on-shell, the (anti)self-dual sector of the new theory corresponds precisely to all Einstein manifolds in four dimensions. The Lorentzian signature action, and its analytic continuations are also considered. A self-contained discussion is given on the effects of discrete transformations C, P and T on the Samuel-Jacobson-Smolin action, and other proposed actions which utilize self- or anti-self-dual variables as fundamental variables in the description of four-dimensional gravity.

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I. Introduction

Self- or anti-self-dual variables which exploit the unique properties of four dimensions have been proposed as fundamental variables in the description of classical and quantum gravity [1, 2, 3]. Not long after the simplification of the constraints of general relativity achieved with these variables was announced [3], the covariant action was found by Jacobson and Smolin [1], and by Samuel [2]. This first order action with independent left-handed (primed or dotted) connection and vierbein as fundamental variables reproduces the Ashtekar variables and constraints [3] naturally. The resultant equations of motion are the same as the Einstein field equations in four dimensions.

Furthermore, all Einstein manifolds in four dimensions are described by Ashtekar connections which are (anti)self-dual with respect to the metrics of the solutions. This is somewhat surprising since not all Einstein manifolds have Riemann-Christoffel curvature tensors which are (anti)self-dual. In four dimensions, the four-index (antisymmetric in pairs) Riemann-Christoffel curvature tensor, $R_{\alpha\beta\mu\nu}$, can be dualized on the left and on the right and it can be viewed as a $6 \times 6$ matrix $\Lambda^\pm \rightarrow \Lambda^\pm$ of the eigenspaces, $\Lambda^\pm_2$, of the Hodge duality operator $[4]$

\[
\begin{pmatrix}
F^+ & C^+
\end{pmatrix}
\begin{pmatrix}
C^- & F^-
\end{pmatrix}
\]

Here $F^+$ ($F^-$) is self-dual (anti-self-dual) with respect to both left and right duality transformations while $C^+$ ($C^-$) is self-dual (anti-self-dual) under left duality and anti-self-dual/self-dual under right duality transformations. A metric is Einstein iff the matrix is block diagonal i.e. iff the $3 \times 3$ blocks $C^\pm$ vanish $[1]$. On-shell, $F^-$, which is doubly (anti)self-dual, is precisely the curvature of the Ashtekar connection $[3]$. Thus, all Einstein manifolds are described by Ashtekar connections which are (anti)self-dual.

This raises the interesting question of whether it is possible to construct a Yang-Mills-like theory based on a left-handed (or (anti)self-dual) connection with the property that, on-shell, the (anti)self-dual sector of the theory corresponds precisely to all Einstein manifolds in four dimensions. We shall show in Sections II and III that this can indeed be achieved. Moreover, unlike the Einstein-Hilbert action or the Samuel-Jacobson-Smolin action, the Euclidean action of this theory is positive semi-definite for arbitrary four-topologies. This positive semi-definite Euclidean action can be constructed

\footnote{These can be thought of as internal and external duality transformations if we consider the Riemann curvature two-form $R_{AB\mu\nu} dx^\mu \wedge dx^\nu$ instead of $R_{\alpha\beta\mu\nu}$.}
by adding to the Samuel-Jacobson-Smolin action appropriate Yang-Mills and topological terms of the (anti)self-dual connection—terms which are naturally associated with actions based upon fundamental gauge connections.

A positive semi-definite Euclidean action may allow for a well-defined path integral formulation of the quantum theory in the same spirit of more familiar Euclideanized Yang-Mills and scalar field quantum theories. This situation is to be contrasted with the proposed Euclidean path integral formulation of quantum gravity based upon the Einstein-Hilbert action. Unlike path integrals in ordinary Euclideanized quantum field theories, here the Einstein-Hilbert action is not bounded from below; although it is possible to achieve convergence of the functional integral by doing a formal conformal rotation, and then performing a suitable contour integral over complex conformal factors. These extra manipulations are however not needed for more familiar theories with positive semi-definite Euclidean actions such as Yang-Mills theories. Moreover, the conformal manipulations achieve formal convergence by starting from a manifestly divergent kinematic formulation which is based on a quantity which is not well-defined—the Euclidean gravitational functional integral of the Einstein-Hilbert action.

After a discussion on the equations of motion in Section III, we consider the Lorentzian action and its analytic continuations in Section IV. We observe that the proposed action and the Samuel-Jacobson-Smolin action contain only projections of the curvature and vierbein combinations which are doubly (anti)self-dual. In spinorial terms, this means that analytic continuations of the action can be phrased solely in terms of continuations from primed spinors to primed spinors in complexified spacetimes.

It is also the purpose of this work to give a self-contained discussion of the effects of discrete transformations C, P and T on the Samuel-Jacobson-Smolin action and others which employ self- or anti-self-dual variables as fundamental variables in the description of four-dimensional gravity. The Einstein-Hilbert action is C, P and T invariant. Although the Samuel-Jacobson-Smolin action gives rise to the same equations of motion, in the first order formulation, the fundamental variables are nevertheless independent (anti)self-dual connection and vierbein fields. Since self- or anti-self-dual combinations are not P-invariant, this raises the possibility that in the first order formulation, gravity described in terms of these variables is off-shell P-non-conserving, despite the fact that the equations of motions are the same as Einstein’s. There are further issues at stake. Naively, CPT is expected to be good since it is well-known by the CPT theorem that any Lorentz-invariant hermitian local action is CPT invariant. However, there
is a subtlety here, due to the fact that in Lorentzian signature spacetimes, self- or anti-self-dual connections are actually complex combinations—for instance, $A_{BC}^{\pm} = \frac{1}{2}(\star A_{BC} \pm iA_{BC})$. Thus it can happen that although local Lorentz transformations remain as symmetries of the theories, actions based on such variables may not be hermitian, and can contain anti-hermitian or pure imaginary Lorentz-invariant local pieces. Such terms are CPT odd. These, and various other issues connected with the discrete transformations C, P and T are discussed in the last section.

II. Positive semi-definite Euclidean Action

Let us briefly recall some concepts with regard to duality and establish our notations. If a two-form carries a pair of anti-symmetric internal indices AB, with each index taking values from 0 to 3; it is possible to consider the notion of both internal and external self- or anti-self-duality. The curvature two-form, $\frac{1}{2}R_{AB\mu\nu}dx^\mu \wedge dx^\nu$, of the spin connection in four-dimensions is an example. On two-forms, the internal dual transformation is defined to be

$$\star C_{AB\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}\varepsilon_{AB}^{\ CD}C_{CD\mu\nu}dx^\mu \wedge dx^\nu$$

(2)

and the external dual transformation

$$\star C_{AB\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}|e|\varepsilon_{\alpha\beta}C_{AB\mu\nu}dx^\alpha \wedge dx^\beta$$

(3)

The internal (external) indices are raised and lowered by $\eta^{AB}$ ($g^{\mu\nu}$) and $\eta_{AB}$ ($g_{\mu\nu}$) respectively, and $|e|$ is the determinant of the vierbein, $e^{A} = e^{A}_{\ \mu}dx^\mu$.

We shall consider only the case for which the internal indices are Lorentz indices, and the signatures of the internal and external metrics are the same i.e. $(\mp,+,+,+)$. We adopt the convention that upper case Latin indices which run from 0 to 3 denote Lorentz indices, while lower case indices run from 1 to 3 e.g. $A = 0, a ; a = 1,2,3$.

An interesting special case is the two-form $\Sigma_{AB} = e_{A} \wedge e_{B}$. For it, external and internal dual transformations are the same since

$$\star \Sigma_{AB} = \frac{1}{2}|e|\varepsilon_{\alpha\beta}e_{A\mu}e_{B\nu}dx^\alpha \wedge dx^\beta$$

$$= \frac{1}{2}\varepsilon_{AB}^{\ CD}e_{\alpha}e_{D\beta}dx^\alpha \wedge dx^\beta = \star \Sigma_{AB}$$

(4)
Since $*^2 = *^2 = \pm 1$, the eigenvalues are $\pm 1$ and $\pm i$ for Euclidean and Lorentzian signature respectively. Internal self- and anti-self-dual combinations for Lorentzian signature are denoted as

$$G_{AB}^\pm = \frac{1}{2}(\ast G_{AB} \pm iG_{AB})$$  \hspace{1cm} (5)

These satisfy

$$\ast G_{AB}^\pm = \pm iG_{AB}^\pm$$  \hspace{1cm} (6)

Note that $G_{AB}$ does not have to be a two-form for internal self or anti-self-duality to make sense. For instance, one can consider the anti-self-dual combination of the connection one-form $A_{BC}^- = \frac{1}{2}(\ast A_{BC} - iA_{BC})$. The combinations $\frac{1}{2}(\ast C_{AB} \pm iC_{AB})$ are external self- and anti-self-dual two-forms. In the above self and anti-self-dual combinations, the $i$'s should be set to unity for the case of Euclidean signature.

We shall start with the proposed positive semi-definite Euclidean action and consider the Lorentzian case later on. It is a matter of convention to use either self-dual or anti-self-dual variables. We choose to use anti-self-dual variables for all our discussions. Our conventions will then be that anti-self-dual variables are coupled to left-handed fermions fields.

Let $A_{BC} = -A_{CB}$ be a connection one-form, and $A_{BC}^-$ be the anti-self-dual combination $A_{BC}^- = \frac{1}{2}(\ast A_{BC} - A_{BC})$. It can be verified that the curvature of two-form of $A_{BC}^-$,

$$F_{AB}^- = dA_{AB}^- + A_{AC}^-C \wedge A_{CB}^-$$  \hspace{1cm} (7)

satisfies

$$F_{AB}^- = \frac{1}{2}(\ast F_{AB}^- - F_{AB}^-)$$  \hspace{1cm} (8)

where $F_{AB} = dA_{AB} + \frac{1}{2}A_{AC} \wedge A_{CB}^-$. The proposed action is

$$S_E = - \int_M \left[ \frac{1}{2g}(F_{AB}^- - *F_{AB}^-) - \frac{g}{(16\pi G)} \Sigma_{AB}^- \right] \wedge \left[ \frac{1}{2g}(F_{AB}^- - *F_{AB}^-) - \frac{g}{(16\pi G)} \Sigma_{AB}^- \right]$$

$$= - \int_M \left[ \frac{1}{2g^2}(F_{AB}^- \wedge F_{AB}^- - *F_{AB}^- \wedge F_{AB}^-) - \frac{1}{8\pi G} F_{AB}^- \wedge \Sigma_{AB}^- \right] + \frac{g^2}{(16\pi G)^2} \Sigma_{AB}^- \wedge \Sigma_{AB}^-$$  \hspace{1cm} (9)
The combination \[ \frac{1}{2g}(F_{AB}^- - \ast F_{AB}^-) - \frac{g}{(16\pi G)} \Sigma_{AB}^- \] is anti-self-dual under both external and internal duality transformations. If all the variables and couplings are real for Euclidean signature, then the Euclidean action is positive semi-definite for arbitrary topologies because the integrand in (9) is, since the action is also

\[ S_E = \int_M \left[ \frac{1}{2g}(F_{AB}^- - \ast F_{AB}^-) - \frac{g}{(16\pi G)} \Sigma_{AB}^- \right] \wedge \ast \left[ \frac{1}{2g}(F_{AB}^- - \ast F_{AB}^-) - \frac{g}{(16\pi G)} \Sigma_{AB}^- \right] \] (10)

Recall that for Riemannian spacetimes, the inner product for differential forms, \((\alpha, \beta) = \int_M \alpha \wedge \ast \beta\), leads to \((\alpha, \alpha) \geq 0\).

Let us examine the terms in the action \(S_E[e^A, A_{AB}^-]\). The first term within brackets in the second line of (9) is a locally exact topological term which does not contribute to the equations of motion. Locally, it can be written in terms of the Chern-Simons three-form \(C_{\wedge AB}\).

It can also be expressed in the form of topological Euler and signature invariants since

\[ \int_M F_{AB}^- \wedge F_{AB}^- = \int_M \left[ \frac{1}{2} F^{AB} \wedge F_{AB} + \frac{1}{4} \epsilon_{ABCD} F^{AB} \wedge F^{CD} \right] \] (11)

while for compact four-manifolds without boundary, the signature and Euler invariants are respectively

\[ \tau(M) = -\frac{1}{24\pi^2} \int_M R^{AB} \wedge R_{AB} \] (12)

and

\[ \chi(M) = \frac{1}{32\pi^2} \int_M \epsilon_{ABCD} R^{AB} \wedge R^{CD} \] (13)

The remaining term is the Yang-Mills action \(\int_M Tr(F^- \wedge \ast F^-)\) for the Ashtekar fields.

Next is

\[ \frac{1}{8\pi G} \int_M F_{AB}^- \wedge \Sigma_{AB}^- = \frac{1}{16\pi G} \int_M \left[ F_{AB}^- \wedge e^A \wedge e^B - \ast F_{AB}^- \wedge e^A \wedge e^B \right] \] (14)

This is the Samuel-Jacobson-Smolin action \([\ast \Sigma = \ast \Sigma\) for the Ashtekar variables. The second term on the RHS of the above is the Einstein-Hilbert-Palatini action.

\[ \text{Note that } \ast \Sigma = \ast \Sigma \text{ implies that } \ast \Sigma^- = \frac{i}{4} (\ast \Sigma - \Sigma) = -\Sigma^- . \]

\[ \text{The Ashtekar variables can be assumed to be just } A^a_{\dot{a} b} \text{ since } A^a_{\dot{a} c} \text{ and } A^a_{\dot{a} a} \text{ are not independent, but are related by } A^a_{\dot{a} c} = i e^{\dot{a} a}_{b c} A^b_{\dot{a} a}. \text{ Again, for Euclidean signature, the } i \text{ here should be set to unity.} \]
The last entry in (9) is just the cosmological term since
\[\Sigma^{-AB} \wedge \Sigma^{-AB} = -3! e^0 \wedge e^1 \wedge e^2 \wedge e^3 = -6(\ast 1)\] (15)

By comparing the coupling constants, the (positive) cosmological constant is related to \(g\) and the gravitational constant, \(G\), by
\[g^2 = \frac{16\pi \lambda G}{3}\] (16)

Putting everything together, the total Lagrangian in (9) corresponds to adding a topological term as well as a non-topological Yang-Mills \(Tr(F^{-} \wedge \ast F^{-})\) Lagrangian four-form to the Samuel-Jacobson-Smolin Lagrangian with cosmological constant.

A related action without the Yang-Mills term was proposed recently by Nieto et al [9] in the context of an (anti)self-dual version of the SO(3,2) MacDowell-Mansouri action [10] for gravity. Their action can also be written as
\[S = -\int \left[ \frac{-1}{g} F^{-AB} - \frac{g}{(16\pi G)} \Sigma^{-AB} \right] \wedge \left[ \frac{1}{g} F^{-AB} - \frac{g}{(16\pi G)} \Sigma^{-AB} \right]\] (17)

This particular action leads to exactly the same equations of motion as Einstein’s since it differs from the Samuel-Jacobson-Smolin action with cosmological constant by only a topological invariant. However, since \(F^{-}\) is not externally anti-self-dual off-shell, the Euclidean action is not positive semi-definite, unlike the proposed action (9). In (9) only doubly (anti)self-dual fluctuations of the curvature and \(\Sigma\) contribute to the action. It is intriguing to observe that this is also true for the Samuel-Jacobson-Smolin action which can be rewritten as \[\frac{1}{16\pi G} \int_M (F^{-AB} - \ast F^{-AB}) \wedge \Sigma^{-AB}\] because of the anti-self-dual nature of \(\Sigma^{-}\) with respect to \(\ast\). However, this action is again not positive definite. In this respect, it is quite natural to view the proposed action (9) (or (10)) as the natural positive definite extension which preserves the condition that only doubly (anti)self-dual fluctuations of the curvature and \(\Sigma\) contribute. As we shall discuss later in Section IV, the fact that the action contains only doubly anti-self-dual projections means that, in spinorial terms, when spacetimes are complexified, analytic continuations of the action can be phrased solely in terms of continuation from primed to primed spinors.

An alternative to the action (10) which also satisfies the criterion of positive semi-definite action is
\[S = -\int \left[ \frac{-1}{g} F^{-AB} + \frac{g}{(16\pi G)} \Sigma^{-AB} \right] \wedge \ast \left[ \frac{-1}{g} F^{-AB} + \frac{g}{(16\pi G)} \Sigma^{-AB} \right]\] (18)
Although this latter action contains both (externally) self and (anti)self-dual projections of the curvature $F^-$, the equations of motion obtain from (10) and (18) are the same because they differ only by a topological term proportional to $\int_M Tr(F^- \wedge F^-)$.

**III. (Anti)self-duality and Einstein Manifolds**

We turn next to the equations of motion of the proposed actions. We shall show that all Einstein manifolds are solutions to the equations of motion of the new actions. Furthermore, $F^-$ is (anti)self-dual (with respect to $\star$) iff the solution is an Einstein manifold. Thus, the on-shell (anti)self-dual sector of the theory corresponds precisely to all Einstein manifolds in four dimensions.

Varying the first order action $S_E[e^A, A_{AB}^-]$ with respect to $A^-\mu$ yields the equation of motion

$$-\frac{1}{g^2}D_{A^-} \star F^- + \frac{1}{16\pi G} D_{A^-} \Sigma^- = 0$$

(19)

The additional metric dependent Yang-Mills term contributes to the energy-momentum tensor. So varying with respect to $e_A^\mu$ produces an extra contribution. These equations of motion come from

$$\delta S_{E,A^-} = \int_M -\frac{1}{8\pi G} (F^-_{AB} \wedge e^B \wedge dx^\mu + \frac{\lambda}{3!} \epsilon^{ABCD} e^B \wedge e^C \wedge e^D \wedge dx^\mu) \delta e^A_{\mu}$$

$$+ \frac{2}{g^2} \int_M T_{YM}^{\mu\nu} \delta e^A_{\mu} \star 1 = 0$$

(20)

Here $T_{YM}^{\mu\nu}$ is the energy-momentum tensor from the Yang-Mills Lagrangian. It takes the form

$$T_{YM}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^-_{AB\alpha\beta} F^{-AB\alpha\beta} - F^-_{AB} \mu^\alpha F^{-AB\nu\alpha}$$

$$= -\frac{1}{2} (F^-_{AB} \mu^\alpha F^{-AB\nu\alpha} \mp \star F^-_{AB} \mu^\alpha \star F^{-AB\nu\alpha})$$

(21)

In arriving at the last equality we have used the identity

$$\frac{1}{2} g^{\mu\nu} F^-_{AB\alpha\beta} F^{-AB\alpha\beta} - F^-_{AB} \mu^\alpha F^{-AB\nu\alpha} = \pm \star F^-_{AB} \mu^\alpha \star F^{-AB\nu\alpha}$$

(22)

### Footnotes

4 Usually, self- or anti-self-duality of Yang-Mills gauge fields refers to self- or anti-self-duality of the curvature with respect to $\star$. This is what is being discussed here. The difference here is that we are also using variables which are also internally (anti)self-dual.
where the upper(lower) sign is for Euclidean(Lorentzian) signature. From (21) it is clear that the energy-momentum tensor vanishes if the curvature is self- or anti-self-dual.\footnote{For Lorentzian signature, the eigenvalues of $*$ are $\pm i$. So the result that the energy-momentum from the Yang-Mills action vanishes for self- or anti-self-dual curvatures $\mathcal{F}$ which satisfy $\mathcal{F} = \pm i \mathcal{F}$, also holds for Lorentzian signature.} This will be used to show that all Einstein manifolds are solutions to the equations of motion of the proposed actions.

It is known that the Einstein field equations in four-dimensions can be obtained from the Samuel-Jacobson-Smolin action with cosmological constant. In terms of $A^-$ and $e$, they read

\[ D_{A^-} \Sigma^- = 0 \quad (23) \]

and

\[ F_{AB}^- \wedge e^B \wedge + \frac{\lambda}{6} \epsilon_{ABCDEF} e^B \wedge e^C \wedge e^D = 0 \quad (24) \]

Given an invertible vierbein, the unique solution to (23) is that $A^-$ is the (anti)self-dual part of the torsionless spin connection i.e.

\[ A^-_{AB} = \frac{1}{2} (\ast \omega_{AB}(e) - \omega_{AB}(e)) \quad (25) \]

which makes the curvature $F_{AB}^- = \frac{1}{2} (\ast R_{AB} - R_{AB}) = R_{AB}^-(e)$. Solutions that satisfy this and (24) are Einstein manifolds. For vierbeins of Einstein manifolds, the Ashtekar connections $A^-_{\text{Einstein}}$ as in (25) are (externally) (anti)self-dual as well i.e. $F^- = - \ast F^-$.\footnote{Recall that the Riemann-Christofel curvature tensor for Einstein manifolds in four dimensions, $R_{ABCD}$, obey the condition that the left and right dual are equal. So for $R_{AB} = \frac{1}{2} R_{ABCD} e^C \wedge e^D$, $\ast R_{AB} = \ast R_{AB}$. Consequently, $\ast R_{AB} = - R_{AB}$.} As noted previously, the extra energy-momentum tensor contribution from the Yang-Mills term in (20) vanishes for self or anti-self-dual curvatures. So Ashtekar connections and vierbeins for Einstein manifolds also obey the set of equations (19) and (20).

Conversely, if the connection $A^-_{AB}$ is such that $F_{AB}^- = - \ast F_{AB}^-$, then the set (19) and (20) reduces to the set (23) and (24) due to the Bianchi identity $D_{A^-} F^- = 0$, and the vanishing of the Yang-Mills energy-momentum tensor. Therefore we can conclude that, on-shell, the (anti)self-dual sector of the new actions (10) and (18) corresponds precisely to all Einstein manifolds in four dimensions.

Expression (10) tells us that the positive semi-definite Euclidean action $S_E$ is minimized by configurations which obey

\[ \frac{1}{2} (F^- - \ast F^-) = \frac{g^2}{(16\pi G)} \Sigma^- = \frac{\lambda}{3} \Sigma^- \quad (26) \]
For Einstein manifolds, the Weyl two-form is given by
\[ W_{AB} = R_{AB} - \frac{\lambda}{3} e_A \wedge e_B \] (27)
and \( *R_{AB} = *R_{AB} \). The latter leads to
\[ F_{AB}^- (A_{Einstein}^-) = -*F_{AB}^- (A_{Einstein}^-). \]
The anti-self-dual part of the Weyl two-form is therefore
\[
W_{AB}^- = \frac{1}{2} (\star W_{AB} - W_{AB}) \\
= \frac{1}{2} (F_{AB}^- - *F_{AB}^-) - \frac{g^2}{16\pi G} \Sigma_{AB}^- \] (28)
Thus, the lower bound of zero action is attained for conformally self-dual \((W_{AB}^- = 0)\) Einstein manifolds. Configurations which obey (26) may correspond to the ground state of the theory.

IV. The Lorentzian action and analytic continuation

What is the Lorentzian signature action, \( S_L \), which corresponds to the positive semi-definite action \( S_E \) in (10)? We would like the continuation from Lorentzian to Euclidean signature to have certain properties. In particular, the actions should have the property that \( exp(iS_L) = exp(-S_E) \). The continuation should also preserve the (anti)self-dual nature of the fields \( A^- \), \( F^- \) and \( \Sigma^- \) with respect to \( \star \) as well as the (anti)self-duality of the combination \( \frac{1}{2} (F^- - (i)F^-) \) and \( \Sigma^- \) with respect to \( * \). We shall first show explicitly, that it is possible to achieve these and continue from Lorentzian to Euclidean signature and vice versa by a Wick rotation before giving a more general analytic continuation prescription.

The Lorentzian action is
\[
S_L = - \int_M \left[ \frac{1}{2g} (F_{AB}^- + i * F_{AB}^-) - \frac{g}{(16\pi G)} \Sigma_{AB}^- \right] \wedge * \left[ \frac{1}{2g} (F^{-AB} + i * F^{-AB}) - \frac{g}{(16\pi G)} \Sigma^{-AB} \right] \\
= i \int_M \left[ \frac{1}{2g^2} (F_{AB}^- \wedge F^{-AB} + i * F_{AB}^- \wedge F^{-AB}) - \frac{1}{8\pi G} F_{AB}^- \wedge \Sigma^{-AB} \right] \\
+ \frac{g^2}{(16\pi G)^2} \Sigma_{AB}^- \wedge \Sigma^{-AB} \] (29)

In this section and henceforth, unless stated otherwise, all variables are Lorentzian. To be clear, these may be denoted with \( L \) subscripts. Euclidean variables will be denoted by \( E \) subscripts.

\(^7\) Our convention for Lorentzian signature is \( \epsilon_{0123} = -\epsilon^{0123} = 1 \).
A Wick rotation with \( (e_L)_0 = -i(e_E)_0 \) and \( (e_L)_a = (e_E)_a \) will result in the metric having Euclidean signature \((+,+,+,+)\). The corresponding change induced in \( \Sigma^- \) is

\[
(\Sigma_L^-)_{0a} \mapsto (\Sigma_E^-)_{0a} = \frac{1}{2}(\frac{1}{2} \epsilon_{0a}^{\ bc}(\Sigma_E)_{bc} - (\Sigma_E)_{0a})
\]

Thus

\[
\Sigma_L^{-0a} = -(\Sigma_L^-)_{0a} \mapsto -(\Sigma_E^-)_{0a} = -\Sigma_E^{-0a}
\]

We also have \( \text{det}(e^A_{\mu})_L \mapsto \text{idet}(e^A_{\mu})_E \). With

\[
(A_L)_{0a} \mapsto -i(A_E)_{0a}
\]

\[
(A_L)_{bc} \mapsto (A_E)_{bc}
\]

we obtain

\[
(A_L^-)_{0a} \mapsto (A_E^-)_{0a} = \frac{1}{2}(\frac{1}{2} \epsilon_{0a}^{\ bc}(A_E)_{bc} - (A_E)_{0a})
\]

For the curvature,

\[
(F^-_L)_{0a} \mapsto (F^-_E)_{0a} = \frac{1}{2}(\frac{1}{2} \epsilon_{0a}^{\ bc}(F_E)_{bc} - (F_E)_{0a})
\]

To render the continuation explicit, we note that \( \Sigma_{bc}^- = (i)\epsilon_{bc}^{\ 0a}\Sigma_{0a}^- \) and \( F_{bc}^- = (i)\epsilon_{bc}^{\ 0a}F_{0a}^- \). The action is therefore

\[
S_L = i \int_M \left[ \frac{1}{2g^2} (F^-_{AB} \wedge F^-{AB} + i \ast F^-_{AB} \wedge F^-{AB}) - \frac{1}{8\pi G} F^-_{AB} \wedge \Sigma^-{AB} \right.
\]

\[
+ \frac{g^2}{(16\pi G)^2} \Sigma^-_{AB} \wedge \Sigma^-{AB} \bigg] \right.
\]

\[
= i \int_M \left[ \frac{4}{2g^2} (F^+_{0a} \wedge F^-{0a} + i \ast F^+_{0a} \wedge F^-{0a}) - \frac{4}{8\pi G} F^+_{0a} \wedge \Sigma^-{0a} \right.
\]

\[
+ \frac{4g^2}{(16\pi G)^2} \Sigma^-_{0a} \wedge \Sigma^-{0a} \bigg] \right.
\]

Thus

\[
iS_L = \int_M \left[ \frac{4}{2g^2} (F^+_{0a} \wedge (-F^-{0a}) + i \ast F^+_{0a} \wedge (-F^-{0a})) - \frac{4}{8\pi G} F^+_{0a} \wedge (-\Sigma^-{0a}) \right.
\]

\[
+ \frac{4g^2}{(16\pi G)^2} \Sigma^-_{0a} \wedge (-\Sigma^-{0a}) \bigg] \right.
\]
is continued to
\[
\int_{M} \left[ \frac{4}{2g^2} ((F_E^+)^{0a} \wedge F_{E}^{-0a} - \ast E(F_E^+)^{0a} \wedge F_{E}^{-0a}) - \frac{4}{8\pi G} (F_E^+)^{0a} \wedge \Sigma^{-0a} \right. \\
+ \frac{4g^2}{(16\pi G)^2} (\Sigma^{-0a})^{0a} \wedge \Sigma^{-0a} \\
\left. = \int_{M} \left[ \frac{1}{2g^2} ((F_E^{-})_{AB} \wedge F_{E}^{-AB} - \ast E(F_E^{-})_{AB} \wedge F_{E}^{-AB}) - \frac{1}{8\pi G} (F_E^{-})_{AB} \wedge \Sigma^{-AB} \right. \\
+ \frac{g^2}{(16\pi G)^2} (\Sigma^{-AB})_{AB} \wedge \Sigma^{-AB} \right]
\right]
= -S_E
\] (37)

where \(S_E\) is precisely as in expression (9). So we indeed have a continuation of \(\exp(iS_L)\) to \(\exp(-S_E)\) with positive semi-definite Euclidean action \(S_E\).

Although the Euclidean action (9) is positive semi-definite for arbitrary topologies, it remains to be seen whether this can provide all the necessary convergence properties for a well-defined Euclidean path integral approach to quantum gravity. The actions (9), (17) and (18) also contain a dimensionless coupling constant, \(g = \sqrt{16\pi G/3}\), but the perturbative renormalizability (or non-renormalizability) of these theories has not yet been studied.

The action can be thought of as \(S_L[e^A, A_{0a}]\). In the explicit example of continuation from Lorentzian to Euclidean signature, we Wick rotated only the imaginary part of \(A^-\). In continuing from Euclidean to Lorentzian signature however, we have to be careful since \(A^-\) is real in Euclidean signature spacetimes. How then can we distinguish which part of \(A^-\) to Wick rotate which to leave invariant if we insists on using solely (anti)self-dual variables? In general, what we are actually seeking is a continuation which preserves the (anti)self-dual nature of the fields. Precisely, in spinorial terms, the relevant analytic continuations that we seek are continuations from primed spinors to primed spinors.

In complexifying spacetimes, quantities which are complex on Lorentzian sections must be treated as independent of their complex conjugates. This is the case for fermions and scalar fields in usual Euclideanized quantum field theories. Thus \(A^-\) and \(A^+\), which are complex conjugates of each other in Lorentzian signature spacetimes, have to be analytically continued

\(^8\)A common concept for continuing between Lorentzian and Euclidean signature and vice versa is that we should Wick rotate the parity odd part of \(A^-\) and leave the parity even part of \(A^-\) unchanged. We shall have more to say on the properties of \(A^-\) under parity in the next section.
to different independent fields in complex spacetimes. When continued to
the Euclidean section these are again independent.

In spinorial terms, $A^\alpha_\mathcal{A} \epsilon_{\mathcal{B}'} = A_{\bar{0}a} \frac{1}{\sqrt{2}} (\tau^a)_{\mathcal{A}'} \epsilon_{\mathcal{B}'}$ (the scripted spinorial indices take values 0 and 1; and $\tau^a$ are Pauli matrices), is a primed (left-handed or dotted) spinor. Note that here $A_{\bar{0}a}$ is a one-form, with components $A_{\bar{0}a\mu}$ which can also be expressed in primed and unprimed spinorial indices by contracting with curved-space spinors $\sigma^\mu_{\mathcal{A}\mathcal{A}'}$. This contraction can be done for each external or spacetime index of any tensor. In particular, it can be done for the components of the externally (anti)self-dual objects, $\Sigma^\mu$ and $\frac{1}{2}(\ast F^\mu - (i)\ast F^\mu)$. However, we may note that combinations such as $\Sigma^\mu_{\mu\nu}$ and $\frac{1}{2}(\ast F_{\mu\nu} - (i)\ast F_{\mu\nu})$ are also internally (anti)self-dual; and therefore in spinorial terms, contain only primed projections due to their doubly (anti)self-dual nature. The action consists of only these projections, and can thus be written exclusively in terms of primed spinors. As a result, the Lorentzian-Euclidean continuation can be phrased in the more general and rigorous context of analytic continuations of primed spinors to primed spinors in complex spacetimes.

V. Discrete transformations C, P and T; and (anti)self-dual variables

In this section we discuss the charge conjugation (C), parity (P) and time reversal (T) transformations and their combinations; and examine the effects of these discrete transformations on the theory.

To give a coordinate independent description, it is convenient to use differential forms. P and T are improper Lorentz transformations of determinant $-1$ and act on the Lorentz indices. In particular under P

$$e^0 \mapsto e^0; \quad e^a \mapsto -e^a \quad (38)$$

while under T

$$e^0 \mapsto -e^0; \quad e^a \mapsto e^a \quad (39)$$

Both P and T are orientation-reversing ($M \rightarrow \overline{M}$) operations but unlike P, which is to be implemented unitarily, T is to be implemented by anti-unitary transformations. So under T, c-numbers are complex conjugated.

$^9\sigma^\mu_{\mathcal{A}\mathcal{A}'}$ are “soldering” spinors which satisfy $\bar{g}_{\mu\nu} \sigma^\mu_{\mathcal{A}\mathcal{A}'} \sigma^\nu_{\mathcal{B}\mathcal{B}'} = \epsilon_{\mathcal{A}\mathcal{B}'} \epsilon_{\mathcal{A}\mathcal{B}'}$.

$^{10}$Again, the $(i)$ should be set to unity for Euclidean signature.

$^{11}$As we have mentioned before, this is also a property of the Samuel-Jacobson-Smolin action.
C acts trivially on the vierbein. On the spin connections, $\omega$, the induced transformations can be deduced from the torsionless condition,

$$de^A + \omega^A{}_{B} \wedge e^B = 0$$

(40)

and they are such that under both P and T,

$$\omega_0^a \mapsto -\omega_0^a; \quad \omega_{bc} \mapsto \omega_{bc}$$

(41)

The induced transformations on the curvature, $R_{AB} = d\omega_{AB} + \omega_A{}^C \wedge \omega_{CB}$, are that

$$R_0^a \mapsto -R_0^a; \quad R_{bc} \mapsto R_{bc}$$

(42)

For Lorentzian signature, the anti-self-dual and self-dual combinations $\omega_0^a \equiv \frac{1}{2}(\pm i\omega_0^a + \frac{1}{2}\epsilon_0{}^{abc}\omega_{bc})$ behave as

$$\omega_0^a \leftrightarrow \omega_0^+$$

under P, while under T (which is anti-unitary)

$$\omega_0^a \leftrightarrow \omega_0^-$$

(43)

(44)

$\omega^\pm$ are trivially invariant under C.

All the terms in the action are invariant under local $SO(3, C)$ gauge transformations and diffeomorphisms. $F^-$ and $\Sigma^-$ transform covariantly while $A^-$ is an (anti)self-dual (left-handed) connection, and a singlet under right-handed Lorentz transformations. With the Samuel-Jacobson-Smolin action, the Ashtekar connection is indeed the (anti)self-dual combination of the spin connection in the second order formulation. So it is reasonable to assume that the behaviour of the Ashtekar connections under the discrete transformations is the same as the behaviour of the (anti)self-dual part of the spin connection. This is in agreement with the fact that the connection carries Lorentz indices according to $A_{AB}^- = \frac{1}{2}(\frac{1}{2}\epsilon_{AB}{}^{CD}A_{CD} - iA_{AB})$.

In order to verify that these properties of $A^\pm$ under discrete transformations are indeed correct, and are compatible with the usual notions of P and T; we can couple fermions to the theory and consider the invariance of the Dirac Lagrangian or the Dirac equation.\footnote{Since the Ashtekar-Sen connections are either self- or anti-self-dual, fermions of only one chirality can be coupled to either of these connections. However, it is possible to write the bispinor Dirac equation and Dirac Lagrangian in terms of a pair of left-handed Weyl fermions by substituting $\phi_R = -i\tau^2(\chi_L)^*$\footnote{For a discussion of anomaly-free fermion couplings to Ashtekar-Sen connection, the effects of discrete transformations, and whether such couplings can produce the phenomenology of the Standard Model and Beyond, see Ref.\footnote{\cite{14}}.}}
The massless Dirac equation is

\[ \gamma^A E_A D \Psi = 0 \]  

(45)

Here \( E_A \) are the inverse vierbein vector fields \( E_A^\mu \partial_\mu \), and the contractions are such that \( E_A [e^B = \delta_A^B \), \( E_A D = E_A^\mu D_\mu \). \( \Psi \) is a four-component Dirac bispinor; and in the chiral representation with

\[ \gamma^5 = \left( \begin{array}{cc} I_2 & 0 \\ 0 & -I_2 \end{array} \right) \]  

(46)

the covariant derivative is

\[ D \Psi = \left[ dI_4 - i \left( \begin{array}{cc} A_0^+ \tau^a/2 & 0 \\ 0 & A_0^- \tau^a/2 \end{array} \right) \right] \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \]  

(47)

where \( \phi_{R,L} \) are two-component right and left-handed Weyl spinors. \( A_0^\pm = \pm iA_{0a} - \frac{1}{2} \epsilon_{0a}^{\ \ bc} A_{bc} = -2A_{0a}^{-} \). In terms of Pauli matrices, \( \tau^a = -\tau^a \), and \( \tau^0 = \tau^0 = -I_2 \), the Dirac matrices in the chiral representation are

\[ \gamma^A = \left( \begin{array}{cc} 0 & i\tau^A \\ i\tau^A & 0 \end{array} \right) \]  

(48)

Note that

\[ \frac{1}{4} A_0^+ \tau^B \tau^C = (iA_{0a} + \frac{1}{2} \epsilon_{0a}^{\ \ bc} A_{bc}) \frac{\tau^a}{2} = -A_0^+ \frac{\tau^a}{2} \]  

(49)

and

\[ \frac{1}{4} A_0^- \tau^B \tau^C = (iA_{0a} + \frac{1}{2} \epsilon_{0a}^{\ \ bc} A_{bc}) \frac{\tau^a}{2} = -A_0^- \frac{\tau^a}{2} \]  

(50)

So (47) is analogous to the usual fermion coupling to spin connections for which the covariant derivative is

\[ D_\omega \Psi = \left[ dI_4 + i \left( \begin{array}{cc} \frac{1}{2} \omega^+_{BC} \tau^B \tau^C & 0 \\ 0 & \frac{1}{2} \omega^-_{BC} \tau^B \tau^C \end{array} \right) \right] \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \]  

(51)

Under P, we have \( \Psi \mapsto i\gamma^0 \Psi \), so that \( \phi_L \mapsto \phi_R \). Under T (which is anti-unitary), \( \Psi \mapsto -i\tau^2 \Psi \); while under C, \( \Psi \mapsto C \Psi^T \) where \( C = -i\gamma^0 \gamma^2 \). It is then straightforward to check that for the Dirac equation to hold under

\[ \text{We shall leave out all the intrinsic phases since these complications will not come into play in our discussion of the transformation properties of } A^- \]
C, P and T; together with (38) and (39), \( A^\pm \) have to transform according to
given by
\[ A^-_a \leftrightarrow A^+_a \quad (52) \]
under P, and
\[ A^\mp_a \leftrightarrow A^\mp_a \quad (53) \]
under T (which is anti-unitary), and be invariant under C.

In order to discuss the effects of the discrete transformations in a clear and concise manner, and also to compare with conventional actions, it is better to refer to the transformation of the variables \( A_{BC} = -A_{CB} \) and its curvature components. Recall that
\[ A_0^a = i(A^-_{0a} - A^+_{0a}); \quad A_{bc} = \epsilon_0^{a}{}_{bc}(A^-_{a} + A^+_{a}) \quad (54) \]
The curvature \( F_{BC} = dA_{BC} + A_{B}{}^{D} \wedge A_{DC} \) then has components
\[ F_{0a} = i(F^-_{0a} - F^+_{0a}); \quad F_{bc} = \epsilon_0^{a}{}_{bc}(F^-_{0a} + F^+_{0a}) \quad (55) \]
Thus
\[ A_{0a} \mapsto -A_{0a}; \quad A_{bc} \mapsto A_{bc} \]
\[ F_{0a} \mapsto -F_{0a}; \quad F_{bc} \mapsto F_{bc} \quad (56) \]
under P and T and are trivially invariant under C. The torsion is defined to be
\[ T^A = de^A + A^A{}_{B} \wedge e^B \quad (57) \]

It can then be shown that the Samuel-Jacobson-Smolin action with cosmological constant is
\[ S_{SJS} = - \frac{1}{(16\pi G)} \int_M [e^A \wedge e^B \wedge *F_{AB} - 2\lambda(*)] \\
- \frac{i}{(16\pi G)} \int_M [d(e^A \wedge T_A) - T^A \wedge T_A] \quad (58) \]
and the total action (9) is
\[ S_L = S_{SJS} + \frac{1}{4g^2} \int_M [-iF_{AB} \wedge F^{AB} + \frac{1}{2} \epsilon_{ABCD} F^{AB} \wedge F^{CD}] \\
+ \frac{i}{4g^2} \int_M [*F_{AB} \wedge F^{AB} + i \epsilon_{ABCD} *F^{AB} \wedge F^{CD}] \quad (59) \]
We take the opportunity here to clarify and emphasize some salient features which are not usually mentioned in the literature with regard to the Samuel-Jacobson-Smolin action.

The RHS of the first line of (58) is the Einstein-Hilbert-Palatini action with cosmological constant term. It can be deduced from the identity (58) that the Samuel-Jacobson-Smolin action reproduces the same equations of motion as Einstein’s theory. The equation of motion that is obtained by varying with respect to $A^-$ is $D_A^- \Sigma^- = 0$. This has the unique solution that $A^-$ is the anti-self-dual part of the spin connection, which makes $A^+ = (\text{the complex conjugate of } A^-)$ the self-dual part of the spin connection and $F_{AB}^\pm = R_{AB}^\pm(\omega(e))$. The torsion then vanishes off shell. Therefore the term quadratic in the torsion, which is not a total divergence, cannot give rise to the any extra equations of motion due to its quadratic dependence on the torsion. So, modulo the equation of motion $D_A^- \Sigma^- = 0$, varying with respect to the vierbein then reproduces Einstein’s equations from the RHS of the first line of (58). Remarkably, the first order action $S_{SJS}$ gives the same equations of motion as Einstein’s theory although, without further conditions on the independent variables, it cannot even be regarded as being (complex) canonically related to the Einstein-Hilbert-Palatini action due to the presence of a torsion-squared term which is not a total divergence.

In the quantum theory, off-shell fluctuations can be expected to contribute. So it is not clear that the quantum theory from $S_{SJS}$ is the same as that from Einstein’s theory. This is not a bad thing by itself given the difficulties with quantizing Einstein’s theory. However, if one wishes to be faithful to the latter and still use $S_{SJS}[A^-, e]$ and (anti)self-dual variables, one possibility is to strictly impose a condition equivalent to the vanishing of the torsion off-shell. This is the analog in the path integral approach of the reality conditions that have to be imposed on the conjugate variables in the canonical quantization program. Otherwise, the torsion terms lead to an action which contains imaginary terms in Lorentzian signature spacetimes, and can cause off-shell CPT violation since purely imaginary (real) local Lorentz-invariant action terms are CPT odd (even). However, both theories are equivalent on passing to the second order formulation since in

14When fermion couplings are included, the torsion-free condition has to be supplemented by fermionic contributions, but an analogous condition can be imposed. However, there can still be subtle discrete symmetry violations due to the presence of instantons and the Adler-Bell-Jackiw anomaly.

15The behaviour of various terms in the action under C, P and T can be checked using the explicit transformation properties of the basic variables $A^\pm$ and $e$ discussed previously.
this latter case, $A^-$ is eliminated in terms of the vierbein, $F_{AB}$ is replaced by $R_{AB}$, and the non-hermitian torsion terms are identically set to zero.

It must be said that even if imaginary Lorentz-invariant terms are not killed by imposing additional conditions in the first order formulation, it remains to be seen whether the Samuel-Jacobson-Smolin action (or its extensions which are Euclidean positive semi-definite) can actually give rise to a successful quantum theory of gravity. Given the difficulties with a quantum theory based upon the Einstein-Hilbert action, these actions which contain all classical solutions of the Einstein-Hilbert action may be worth exploring as alternatives. In this respect, universal P, T and CPT violations or conservation checks and experimental signatures \[16\] of these, can be useful in the evaluation of the various options.

We shall next turn our attention to the other terms in the action (59). The topological instanton term\[16\] is

$$\frac{i}{2g^2} \int_M F_{AB}^- \wedge F^{-AB} = \frac{1}{4g^2} \int_M [-iF_{AB} \wedge F^{AB} + \frac{1}{2} \epsilon_{ABCD} F^{AB} \wedge F^{CD}]$$  (60)

The first term is the analog of the signature invariant. However, due to the presence of $i$; it is P, CP and CPT odd but T even (recall that T is anti-unitary). To check that this is indeed to be expected, we note that in the second order formulation, the Samuel-Jacobson-Smolin action with an additional topological term (60) reduces to

$$S_{\text{second order}}[e^A] = - \frac{1}{(16\pi G)} \int_M [e^A \wedge e^B \wedge \ast R_{AB} - \lambda(\ast 1)]$$
$$+ \frac{1}{4g^2} \int_M [-iR^{AB} \wedge R_{AB} + \frac{1}{2} \epsilon_{ABCD} R^{AB} \wedge R^{CD}]$$  (61)

Thus, the second order action differs from the Einstein-Hilbert-Palatini form by precisely two topological invariants which correspond to $\tau(M)$ and $\chi(M)$ (see equations (12) and (13) in Section I). The metric is real but observe that in the second order action, the term associated with $R^{AB} \wedge R_{AB}$ is pure imaginary. It can be checked from our earlier analysis in this section that $\int_M (R^{AB} \wedge R_{AB})$ is P, T odd hence PT even, while $\int_M (\epsilon_{ABCD} R^{AB} \wedge R^{CD})$ is P, T even. Since T is anti-unitary, this means that, due to the $i$, the action (61) is not P, CP and CPT conserving iff $\int_M (R^{AB} \wedge R_{AB})$ does not vanish, although the equations of motion are identical to Einstein’s. This

\[16\] It also occurs in the action (17) of Nieto et al, and the $i$ is present in the Lorentzian case \[9\].
means that in this case gravitational instantons with non-vanishing signature invariant, $\tau$, should give rise to $P$, $CP$ as well as CPT violations. ($T$ is however not violated in this manner due to its anti-unitary nature). These violations cannot be cured by making $g^2$ imaginary since the term associated with $\chi$ would then be $T$ and CPT odd, while the term associated with $\tau$ would be $P$, $T$ odd. The root of the strange behaviour and violations under the discrete transformations lies in the anti-self-dual nature of $A^-$.

The complex connection $A^-$, which carries Lorentz indices according to $A^-_{AB} = \frac{1}{2} \varepsilon^{CD} A_{CD} - i A_{AB}$, is neither even nor odd under $P$, but has even and odd parts due to its (anti)self-dual nature. In terms of spinors, $A^-$ is a left-handed (primed or dotted) connection. We have $A^- \leftrightarrow A^+$ under $P$.

As a consequence, the combination $\int_M Tr((F^- \wedge F^-)$ is neither even nor odd under $P$, and is not real even after reality conditions are taken into account—for instance by going to the second order formulation.

It should be emphasized that the usual $\theta$-angle action in non-abelian gauge theories with hermitian gauge curvature $F$ is of the form $\theta \int_M Tr(F \wedge F)$ for Lorentzian signature and is continued to the Euclidean action $-i\theta \int_M Tr(F \wedge F)$. Similarly, the usual $\theta$-angle term for gravity is proportional to $-i\theta \tau$ (for Euclidean signature) and is $-24\pi^2 \theta \int_M (R^{AB} \wedge R_{AB})$ for Lorentzian signature spacetimes. Note that this latter form has the normal instanton $\theta$-term behaviour of being $P$, $T$, $CP$ odd, and $PT$, $CPT$ even; in contradistinction with the previous form in (60). To be clear about the $i$'s, under a chiral rotation $\Psi \rightarrow exp(i\alpha\gamma^5)\Psi$, instantons contribute to the change of $exp(-i\alpha\tau/4)$ (note the $i$ associated with $\tau$ for Euclidean signature) in the Euclidean bispinor fermion measure, $D\Psi D\bar{\Psi}$. This results in the usual Adler-Bell-Jackiw [13] anomaly equation

$$\nabla_{\mu} j^{\mu\bar{5}} = -\frac{i}{192\pi^2} R_{\mu\nu\alpha\beta} * R^{\mu\nu\alpha\beta}$$

(62)

for Euclidean signature spacetimes.

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17 $g$ is needed to be real on the Euclidean section for the Euclidean action to be positive semi-definite.

18 In usual Yang-Mills theories with real connections, it is the self- and anti-self-dual curvature combinations $\pm iE + B$ which are neither even nor odd under $P$, but has even and odd parts.

19 $\theta$ is real and note the absence of $i$. Furthermore, $\int_M Tr(F \wedge F)$ is odd under both $P$ and $T$.

20 See also Ref. [14] for anomaly computations in the context of couplings to Ashtekar-Sen connections.
It has been proposed \cite{15} that there will be an analogous Yang-Mills instanton $\theta$-angle term in the first order formulation because of large $SO(3, C)$ gauge transformations of $A^{-}$. If this were equivalent to an addition of

$$\frac{\theta}{16\pi^2} \int_M F_{AB}^{-} \wedge F^{-AB}$$

to the Lorentzian action as has been suggested \cite{15}, the additional $\theta$-term is therefore

$$\frac{\theta}{16\pi^2} \int_M F_{AB}^{-} \wedge F^{-AB} = -\frac{\theta}{32\pi^2} \int_M [R^{AB} \wedge R_{AB} + \frac{i}{2} \epsilon_{ABCD} R^{AB} \wedge R^{CD}] \quad (63)$$

if the condition that the Ashtekar connection is the (anti)self-dual part of the spin connection is enforced. The first signature invariant $\tau(M)$ contribution has the normal behaviour of a Yang-Mills $\theta$-term of being $P$, $T$ odd and $CPT$ even. However, due to the presence of the $i$, this time the second term (which is an Euler number $\chi(M)$\cite{22} contribution in Euclidean signature) is $P$ even, $T$ odd, and $CPT$ odd.

The remaining terms in (59) are

$$\frac{1}{4g^2} \int_M [\ast F_{AB} \wedge F^{AB} + \frac{i}{2} \epsilon_{ABCD} \ast F^{AB} \wedge F^{CD}] \quad (64)$$

The first term is the usual Yang-Mills action and it is $C$, $P$ and $T$ even. The second is $P$ odd, and $T$ even due to the $i$; and therefore $CPT$ odd.

It may be that the inner product in the canonical version of the theory, or the path integral measure in path integral approach, for the yet unavailable quantum theory can restore $CPT$ invariance and other discrete symmetries despite the apparent non-conservation of these in the actions. If so, the observations made here may serve to narrow down the search for the “correct” measure or inner product. On the other hand, it may turn out that these violations and their ramifications are genuine features which emerge from the use of self- or anti-self-dual fundamental variables and cannot be compensated for. In either case, it is hoped that this work may serve to draw attention to the peculiar behaviour of theories with self- or anti-self-dual variables under discrete transformations, the possible non-hermiticity of the actions, and some of the physical implications of utilizing these variables in the description of classical and quantum gravity in four-dimensions.

\footnote{More precisely, in the canonical formulation, large gauge transformations can come only from the local pure rotations of the Lorentz group $SO(3, C)$.}

\footnote{Note that the Euclidean Schwarzschild solution has $\chi = 2, \tau = 0$.}
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