On the Boundary Point Principle for divergence-type equations *

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September 18, 2018

Abstract

We provide some versions of the Zaremba-Hopf-Oleinik boundary point lemma for general elliptic and parabolic equations in divergence form under the sharp requirements on the coefficients of equations and on the boundaries of domains.

1 Introduction

The Boundary Point Principle, known also as the “normal derivative lemma”, is one of the important tools in qualitative analysis of partial differential equations. This principle states that a supersolution of a partial differential equation with a minimum value at a boundary point, must increase linearly away from its boundary minimum provided the boundary is smooth enough.

The history of this famous principle begins with a pioneering paper of S. Zaremba [Zar10] where the above assertion was established for the Laplace equation in a three-dimensional domain Ω satisfying an interior touching ball condition. Notice that the major part of all known results on the normal derivative lemma concerns equations with nondivergence structure and strong solutions. A key contribution to the investigation of this problem for elliptic equations was made simultaneously and independently by E. Hopf [Hop52] and O.A. Oleinik [Ole52] (by this reason, all the statements of such type are often called the Hopf-Oleinik lemma). The corresponding comprehensive historical review can be found in [AN16].

*AMS Subject Classification: 35J15, 35J67, 35B45
Key words: Hopf-Oleinik lemma, divergence-type equations, Dini continuity, Kato class
The case of the divergence-type elliptic equations
\[ Lu := -D_i (a^{ij}(x) D_j u) + b^i(x) D_i u = 0 \] (1)
is less studied. It is well known that the Boundary Point Principle fails for uniformly elliptic equations in divergence form with bounded and even continuous coefficients \( a^{ij}(x) \) (see, for instance, [Gil60], [GT83, Ch.3], [PS07, Ch.2] and [Naz12]). Thus, the normal derivative lemma requires more smoothness of the leading coefficients.

The sharp requirements on the regularity of the boundary of a domain, providing the validity of the Boundary Point Principle for the Laplace equation, were independently and simultaneously formulated in the papers [VM67] and [Wid67].

The first result for weak solutions of (1) was proved by R. Finn and D. Gilbarg [FG57]. They considered a two-dimensional bounded domain with \( C^{1,\alpha} \)-regular boundary, the Hölder continuous leading coefficients and continuous lower order coefficients. Recently, in [KK18] (see also [SdL15]) the normal derivative lemma was established in \( n \)-dimensional domains (\( n \geq 3 \)) for equations with the lower-order coefficients from the Lebesgue space \( L^q \), \( q > n \), under the same assumptions on the leading coefficients and on the boundary as in [FG57].

The history of the Boundary Point Principle for parabolic equations is much shorter than for elliptic ones and begins with the papers of L. Nirenberg [Nir53] and A. Friedman [Fri58]. For a partial bibliography in the nondivergence case we refer the reader to [Naz12].

As for the divergence-type parabolic equations
\[ Mu := \partial_t u - D_i (a^{ij}(x; t) D_j u) + b^i(x; t) D_i u = 0, \] (2)
we do not know such results. However, the normal derivative lemma for (2) can be extracted from the lower bound estimates of the Green function for the operator \( \mathcal{M} \). These estimates were obtained in [Zha02], [Cho06] and [CKP12] under various assumptions on the coefficients of \( \mathcal{M} \) and on the boundary of a domain. In particular, [CKP12] deals with cylindrical domains with \( C^{1,\alpha} \)-regular lateral surface, Dini-continuous leading coefficients and lower-order coefficient from the so-called parabolic Kato class (see Remark 3 below).

The goal of our paper is to prove the Boundary Point Principle for the general divergence-type elliptic and parabolic equations under strongly weakened assumptions close to the necessary ones.

### 1.1 Notation and conventions

Throughout the paper we use the following notation:
\( x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n) \) is a point in \( \mathbb{R}^n \);
\( (x; t) = (x', x_n; t) = (x_1, \ldots, x_n; t) \) is a point in \( \mathbb{R}^{n+1} \);
\( \mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_n > 0 \} \), \( \mathbb{R}_+^{n+1} = \{ (x; t) \in \mathbb{R}^{n+1} : x_n > 0 \} \);
\( |x|, |x'| \) are the Euclidean norms in corresponding spaces;
\( B_r(x^0) \) is the open ball in \( \mathbb{R}^n \) with center \( x^0 \) and radius \( r \);
\( \mathcal{C}^1_0(\Omega) \) as the space of \( u \in \mathcal{C}(\Omega) \) such that \( Du \in \mathcal{C}(\Omega) \).

**Definition 1.** We say that a function \( \sigma : [0, 1] \to \mathbb{R}_+ \) belongs to the class \( \mathcal{D} \) if
- \( \sigma \) is increasing, and \( \sigma(0) = 0 \);
- \( \sigma(\tau)/\tau \) is summable and decreasing.

**Remark 1.** It should be noted that our assumption about the decay of \( \sigma(\tau)/\tau \) is not restrictive (see [AN16, Remark 1.2] for details). Moreover, we claim that without loss of generality \( \sigma \) can be assumed continuously differentiable on \( (0; 1] \). Indeed, for any function \( \sigma \in \mathcal{D} \), one can define

\[
\hat{\sigma}(r) := 2 \int_{r/2}^{r} \frac{\sigma(\tau)}{\tau} d\tau, \quad r \in (0; 1].
\]

It is easy to see that \( \hat{\sigma} \in \mathcal{C}^1(0; 1] \). Due to monotonicity properties of \( \sigma \) and \( \sigma(\tau)/\tau \), we have

\[
\hat{\sigma}'(r) = \frac{2}{r} (\sigma(r) - \sigma(r/2)) \geq 0,
\]

\[
\left( \frac{\hat{\sigma}(r)}{r} \right)' = \frac{1}{r} \left[ 2\sigma(r) - \sigma(r/2) \frac{r}{(r/2)} - 2 \int_{r/2}^{r} \frac{\sigma(\tau)}{\tau} d\tau \right] \leq 0,
\]

and for all \( r \in (0; 1] \)

\[
\sigma(r) \leq \hat{\sigma}(r) \leq 2\sigma(r/2).
\]

The second inequality in (3) provides \( \hat{\sigma} \in \mathcal{D} \). Finally, the first inequality in (3) allows us to use \( \hat{\sigma} \) instead of \( \sigma \) in all estimates, and the claim follows.
For $\sigma \in \mathcal{D}$ we define the function $J_\sigma$ as

$$J_\sigma(s) := \int_0^s \frac{\sigma(\tau)}{\tau} \, d\tau.$$  

**Definition 2.** Let $\mathcal{E}$ be a bounded domain in $\mathbb{R}^n$. We say that a function $\zeta : \mathcal{E} \to \mathbb{R}$ belongs to the class $C^{0,\mathcal{D}}(\mathcal{E})$, if

- $|\zeta(x) - \zeta(y)| \leq \sigma(|x - y|)$ for all $x, y \in \overline{\mathcal{E}}$, and $\sigma$ belongs to the class $\mathcal{D}$.

Similarly, suppose that $\mathcal{E}$ is a bounded domain in $\mathbb{R}^{n+1}$. A function $\zeta : \mathcal{E} \to \mathbb{R}$ is said to belong to the class $C^{0,\mathcal{D}}_{p}(\mathcal{E})$, if

- $|\zeta(x; t) - \zeta(y; s)| \leq \sigma(\sqrt{|t - s|} + |x - y|^2)$ for all $(x; t), (y; s) \in \overline{\mathcal{E}}$, and $\sigma$ belongs to the class $\mathcal{D}$.

We use the letters $C$ and $N$ (with or without indices) to denote various constants. To indicate that, say, $C$ depends on some parameters, we list them in parentheses: $C(\ldots)$.

## 2 Elliptic case

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, and let $d(x)$ denote the distance between $x$ and $\partial\Omega$.

We suppose that $\partial\Omega$ satisfies the **interior $C^{1,\mathcal{D}}$-paraboloid condition**. The latter means that in a local coordinate system $\partial\Omega$ is given by the equation $x_n = F(x')$, where $F$ is a $C^1$-function such that $F(0) = 0$ and the inequality

$$F(x') \leq |x'| \cdot \sigma(|x'|).$$  

(4)

holds true in some neighborhood of the origin. Here $\sigma$ is a $C^1$-function belonging to the class $\mathcal{D}$ (see Remark [1]).

Let an operator $\mathcal{L}$ be defined by the formula (1). Assume that the coefficients of $\mathcal{L}$ satisfy the following conditions:

$$\nu \mathcal{I}_n \leq (a^{ij}(x)) \leq \nu^{-1} \mathcal{I}_n,$$

$$a^{ij} \in C^{0,\mathcal{D}}(\Omega) \quad \text{for all} \quad i, j = 1, \ldots, n,$$

(5)

and

$$\omega(r) := \sup_{x \in \Omega} \int_{B_r(x) \cap \Omega} \frac{|b(y)|}{|x - y|^{n-1}} \cdot \frac{d(y)}{d(y) + |x - y|} \, dy \to 0 \quad \text{as} \quad r \to 0.$$  

(6)
Here $\nu$ is a positive constant, $I_n$ is identity $(n \times n)$-matrix, while $b(y) = (b^1(y), \ldots, b^n(y))$.

**Remark 2.** Notice that condition (6) says that the function $\frac{|b(y)|}{|x - y|^{n-1}} \cdot \frac{d(y)}{d(y) + |x - y|}$ is integrable uniformly with respect to $x$. Moreover, in any strict interior subdomain of $\Omega$ condition (6) means that $b$ is an element of the Kato class $K_{n,1}$. (For the definition of the scale of the Kato classes $K_{n,\alpha}$ with $\alpha < n$ the reader is referred to the paper [DH98].) However, in the whole domain $\Omega$ our condition (6) is weaker then $b \in K_{n,1}$.

The main result of this Section is stated as follows.

**Theorem 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial \Omega$ satisfying the interior $C^1,D$-paraboloid condition, let $L$ be defined by (2), and let assumptions (3)-(4) be fulfilled.

In addition, assume that a nonconstant function $u \in C^1(\overline{\Omega})$ satisfies, in the weak sense, the inequality

$$Lu \geq 0 \quad \text{in} \quad \Omega.$$  

Then, if $u$ attends its minimum at a point $x^0 \in \partial \Omega$, we have

$$\frac{\partial u}{\partial n}(x^0) < 0.$$  

Here $\frac{\partial}{\partial n}$ is the derivative with respect to the exterior normal on $\partial \Omega$.

**Remark 3.** Notice that without restriction, we may assume that $x^0 = 0$ and $\partial \Omega$ is locally a paraboloid $x_n = |x'| \cdot \sigma(|x'|)$ with a smooth function $\sigma \in D$. Further, all the assumptions on $a^{ij}$ and $b$ are invariant under the $C^1,D$-regular change of variables. So, we may consider $\partial \Omega$ locally as a flat boundary $x_n = 0$ and assume, without loss of generality, that $B_R \cap \mathbb{R}^n_+ \subset \Omega$ for some $R > 0$.

Consider for $0 < \rho < R/2$ the point $x^\rho = (0, \ldots, 0, \rho)$ and the annulus $A_\rho := \{x : \rho/2 < |x - x^\rho| < \rho\} \subset \Omega$.

Let $x^\ast$ be an arbitrary point in $\overline{A}_\rho$. Following [FG57] (see also [SdL15]) we define the auxiliary functions $z$ and $\psi_{x^\ast}$ as solutions of the problems

\[
\begin{cases}
L_0 z = 0 & \text{in } A_\rho, \\
z = 1 & \text{on } \partial B_{\rho/2}(x^\rho), \\
z = 0 & \text{on } \partial B_\rho(x^\rho),
\end{cases}
\]

\[
\begin{cases}
L_0^{\ast} \psi_{x^\ast} = 0 & \text{in } A_\rho, \\
\psi_{x^\ast} = 1 & \text{on } \partial B_{\rho/2}(x^\rho), \\
\psi_{x^\ast} = 0 & \text{on } \partial B_\rho(x^\rho),
\end{cases}
\]
where the operators $L_0$ and $L_{0}^{\ast}$ are determined by the formulas

$$L_0 z := -D_i(a^{ij}(x)D_j z)$$ and $$L_{0}^{\ast} \psi := -D_i(a^{ij}(x^{*})D_j \psi),$$

respectively. It is well known that $\psi \in C^\infty(A_\rho)$, and the existence of (unique) weak solution $z$ follows from the general elliptic theory.

**Lemma 2.2.** There exists $C_1 = C_1(n, \nu, \sigma) > 0$ such that the inequality

$$|Dz(x^{*}) - D\psi(x^{*})| \leq C_1 \frac{J_\sigma(2\rho)}{\rho} (11)$$

holds true for all $\rho \leq R/2$.

**Proof.** Setting $w^{(1)} = z - \psi$, we observe that $w^{(1)}$ vanishes on $\partial A_\rho$. Hence, $w^{(1)}$ can be represented in $A_\rho$ as

$$w^{(1)}(x) = \int_{A_\rho} G_{\rho}^{x^{*}}(x, y) L_0^{\ast} w^{(1)}(y) dy \equiv \int_{A_\rho} G_{\rho}^{x^{*}}(x, y) (L_0^{\ast} \psi - L_0 z(y)) dy,$$

where $G_{\rho}^{x^{*}}$ stands for the Green function of the operator $L_0^{\ast}$ in $A_\rho$. The equality $(\ast)$ follows from the relation $L_0^{\ast} \psi = L_0 z = 0$, see (7).

Applying integration by parts we get another version of the representation formula:

$$w^{(1)}(x) = \int_{A_\rho} \nabla x \nabla y G_{\rho}^{x^{*}}(x, y) (a^{ij}(x^{*}) - a^{ij}(y)) \nabla j z(y) dy. (9)$$

Differentiating both sides of equality (9) with respect to $x_k$ we get

$$D_k w^{(1)}(x^{*}) = \int_{A_\rho} D_k D_\rho G_{\rho}^{x^{*}}(x^{*}, y) (a^{ij}(x^{*}) - a^{ij}(y)) \nabla j z(y) dy, (10)$$

where $k = 1, \ldots, n$.

According to Lemma 3.2 [GWS2], $z \in C^1(A_\rho)$, and the following estimate holds for $y \in A_\rho$:

$$|Dz(y)| \leq \frac{N_1}{\rho}, (11)$$

where $N_1$ depends only on $n, \nu, \sigma$. Moreover, due to Theorem 3.3 [GWS2] we have also the estimate for the Green function $G_{\rho}^{x^{*}}(x, y)$:

$$|D_x D_y G_{\rho}^{x^{*}}(x, y)| \leq \frac{N_2}{|x - y|^n} \quad x, y \in A_\rho, (12)$$
where $N_2$ is completely determined by $n$, $\nu$, and $\sigma$.

Finally, combination of (10)-(12) with condition (5) implies
\[
|Dw^{(1)}(x^*)| \leq \frac{N_1 N_2}{\rho} \int_{B_{2\rho}(x^*)} \frac{\sigma(|x^* - y|)}{|x^* - y|^n} \, dy,
\]
and (8) follows.

Further, we introduce the barrier function $v$ defined as the weak solution of the Dirichlet problem
\[
\begin{cases}
\mathcal{L}v = 0 \text{ in } A_\rho, \\
v = 1 \text{ on } \partial B_{\rho/2}(x^*), \\
v = 0 \text{ on } \partial B_\rho(x^*). 
\end{cases}
\tag{13}
\]

**Theorem 2.3.** There exists $\rho_0 > 0$ such that for all $\rho \leq \rho_0$ the problem (13) admits a unique solution $v \in C^1(\overline{A}_\rho)$. Moreover, the inequality
\[
|Dv(x) - Dz(x)| \leq C_2 \frac{\omega(2\rho)}{\rho}
\tag{14}
\]
holds true for any $x \in A_\rho$. Here $C_2 = C_2(n, \nu, \sigma) > 0$, $\rho_0$ is completely defined by $n$, $\nu$, $\sigma$, and $\omega$, while $z \in C^1(\overline{A}_\rho)$ is defined in (7).

**Proof.** Consider in $A_\rho$ the auxiliary function $w^{(2)} = v - z$. We observe that $w^{(2)}$ vanishes on $\partial A_\rho$, and
\[
\mathcal{L}_0 w^{(2)} = \mathcal{L}_0 v = \mathcal{L}v - b^i D_i v = -b^i \left( D_i w^{(2)} + D_i z \right) \text{ in } A_\rho.
\]
Hence, $w^{(2)}$ can be represented in $A_\rho$ via corresponding Green function $G_{0,\rho}(x,y)$ as
\[
w^{(2)}(x) = -\int_{A_\rho} G_{0,\rho}(x,y)b^i(y) \left( D_i w^{(2)}(y) + D_i z(y) \right) \, dy.
\]

Differentiation with respect to $x_k$ gives
\[
D_k w^{(2)}(x) = -\int_{A_\rho} D_{x_k} G_{0,\rho}(x,y)b^i(y) \left( D_i w^{(2)}(y) + D_i z(y) \right) \, dy.
\]

Therefore, we get the relation
\[
(\mathbb{I} + T_1) Dw^{(2)} = -T_1 Dz,
\tag{15}
\]
where $\mathbb{I}$ stands for the identity operator, while $T_1$ denotes the matrix operator whose $(k,i)$ entries are integral operators with kernels $D_{x_k} G_{0,\rho}(x,y)b^i(y)$.

The statement of Theorem follows from the next assertion.
Lemma 2.4. The operator $T_1$ is bounded in $C(\overline{A}_\rho)$, and
\[
\|T_1\|_{C\to C} \leq C_3 \omega(2\rho),
\]
where $C_3$ depends only on $n$, $\nu$, and $\sigma$.

Proof. Theorem 3.3 [GW82] provides the estimate
\[
|D_x G_{0,\rho}(x, y)| \leq N_3 \min \left\{ |x - y|^{1-n}; \text{dist}\{y, \partial A_\rho\}; |x - y|^{\nu}\right\} \tag{16}
\]
for any $x, y \in A_\rho$. Here $N_3$ is the constant depending only on $n$, $\nu$, and $\sigma$.

Since $\text{dist}\{y, \partial A_\rho\} \leq d(y)$ for any $y \in A_\rho$, the combination of estimate (16) with condition (3) gives
\[
\int_{B_\rho(x) \cap A_\rho} |D_x G_{0,\rho}(x, y)| |b(y)| \, dy \leq 2N_3 \omega(r), \quad x \in \overline{A}_\rho, \quad r \leq 2\rho. \tag{17}
\]

For arbitrary vector function $f \in C(\overline{A}_\rho)$ we have
\[
|T_1 f(x)| \leq \|f\|_{C(\overline{x})} \cdot \int_{\overline{A}_\rho} |D_x G_{0,\rho}(x, y)| |b(y)| \, dy \leq 2N_3 \omega(2\rho) \cdot \|f\|_{C(\overline{x})}, \quad x \in \overline{A}_\rho.
\]

It remains to show that $T_1 f \in C(\overline{A}_\rho)$. For $x, \mathring{x} \in \overline{A}_\rho$ and any small $\delta > 0$ we have
\[
(T_1 f)(x) - (T_1 f)(\mathring{x}) = J_1 + J_2
\]
\[= \left( \int_{A_\rho \cap B_\delta(\mathring{x})} + \int_{A_\rho \setminus B_\delta(\mathring{x})} \right) \left( D_x G_{0,\rho}(x, y) - D_x G_{0,\rho}(\mathring{x}, y) \right) |b(y)| \cdot f(y) \, dy.
\]
If $|x - \mathring{x}| \leq \delta/2$ then (17) gives
\[
|J_1| \leq \|f\|_{C(\overline{x})} \cdot \int_{B_\delta(\mathring{x}) \cap \overline{A}_\rho} (|D_x G_{0,\rho}(x, y)| + |D_x G_{0,\rho}(\mathring{x}, y)|) |b(y)| \, dy
\]
\[\leq 2N_3 \omega(3\delta/2) \cdot \|f\|_{C(\overline{x})}.
\]
Thus, given $\varepsilon$ we can choose $\delta$ such that $|J_1| \leq \varepsilon$.

On the other hand, $D_x G_{0,\rho}(x, y)$ is continuous for $x \neq y$. Thus, it is equicontinuous on the compact set
\[
\{(x, y) : x \in \overline{B}_{\delta/2}(\mathring{x}) \cap \overline{A}_\rho, \ y \in \overline{A}_\rho \setminus B_\delta(\mathring{x})\}.
\]
Therefore, for chosen $\delta$ we obtain, as $|x - \mathring{x}| \to 0$,
\[
|J_2| \leq \|f\|_{C(\overline{x})} \cdot \int_{\overline{A}_\rho} |b(y)| \, dy \cdot \max_{y \in \overline{A}_\rho \setminus B_\delta(\mathring{x})} |D_x G_{0,\rho}(x, y) - D_x G_{0,\rho}(\mathring{x}, y)| \to 0,
\]
and the Lemma follows. \qed
We continue the proof of Theorem 2.3. Choose the value of \( \rho_0 \) so small that \( \omega(2\rho_0) \leq (2C_3)^{-1} \), where \( C_3 \) is the constant from Lemma 2.4. Then by the Banach theorem the operator \( I + T_1 \) in (15) is invertible. This gives the existence and uniqueness of \( w^{(2)} \in C^1(\overline{A}_\rho) \), and thus, the unique solvability of the problem (13). Moreover, Lemma 2.4 and inequality (11) provide (14). The proof is complete.

To prove Theorem 2.1 we need the following maximum principle.

**Lemma 2.5.** Let \( \mathcal{L} \) be defined by (1), and let assumptions (5)-(6) be satisfied in a domain \( \mathcal{E} \). Suppose that a function \( w \in C^1(\mathcal{E}) \) satisfies \( \mathcal{L}w \geq 0 \) in \( \mathcal{E} \). If \( w \) attains its minimum in an interior point of \( \mathcal{E} \) then \( w = \text{const} \).

**Proof.** In the paper [Zha96] the Harnack inequality was established for the divergence-type operators with the Hölder continuous coefficients \( a^{ij} \) and \( b^i \) belonging to the Kato class \( K_{n,1} \). However, it is mentioned in [Zha96] that the assumption of the Hölder continuity of leading coefficients is needed only for the pointwise gradient estimate of the Green function for the corresponding parabolic operator \( \mathcal{M}_0 \) without lower order coefficients, see [CKP12] below. Since by Theorem 2.6 [CKP12] this estimate holds for operators with Dini coefficients, and \( b \in K_{n,1} \) in any strict interior subdomain of \( \Omega \) (see Remark 2), the strong maximum principle holds for the operator \( \mathcal{L} \) in \( \Omega \).

**Proof of Theorem 2.1.** It is well known that the Boundary Point Principle holds true for the operator with constant coefficients. Using this statement for the operator \( \mathcal{L}_0^x \) (see (7)) with \( x^* = 0 \) in the annulus \( A_1 \) and rescaling \( A_1 \) into \( A_\rho \) we get the estimate

\[
D_n \psi(0) \geq \frac{N_4(n, \nu)}{\rho} > 0.
\]

Furthermore, the inequalities (8) and (14) imply for sufficiently small \( \rho \)

\[
D_n v(0) \geq D_n \psi(0) - |Dz(0) - D\psi(0)| - |Dv(0) - Dz(0)|
\geq \frac{N_4}{\rho} - C_1 \frac{J_\sigma(2\rho)}{\rho} - C_2 \frac{\omega(2\rho)}{\rho} \geq \frac{N_4}{2\rho}.
\]

We fix such a \( \rho \). Since \( u \) is nonconstant, Lemma 2.5 ensures \( u - u(0) > 0 \) on \( \partial B_{\rho/2}(x^o) \). Therefore, we have for sufficiently small \( \varepsilon \)

\[
\mathcal{L}(u - u(0) - \varepsilon v) \geq 0 \quad \text{in} \quad A_\rho; \quad u - u(0) - \varepsilon v \geq 0 \quad \text{on} \quad \partial A_\rho.
\]

By Lemma 2.5 the estimate \( u - u(0) \geq \varepsilon v \) holds true in \( A_\rho \), with equality at the origin. This gives

\[
\frac{\partial u}{\partial n}(0) = -D_n u(0) \leq -\varepsilon D_n v(0),
\]
which completes the proof.

**Remark 4.** Notice that the statement of Theorem 2.1 is also valid for weak supersolutions of equation (1). Namely, let \( \Omega \) and the coefficients of \( L \) be the same as in Theorem 2.1, and let a nonconstant function \( u \in W^{1,2}(\Omega) \) with \( |b \cdot Du| \in L^1(\Omega) \) satisfy in \( \Omega \) the inequality \( Lu \geq 0 \) in the weak sense. Then, if \( u \) attains its minimum at a point \( x_0 \in \partial \Omega \), we have
\[
\liminf_{\varepsilon \to 0} \frac{u(x_0 - \varepsilon n(x_0)) - u(x_0)}{\varepsilon} > 0.
\]

**Remark 5.** The assumptions on the lower-order coefficients \( b^i \) \( (i=1, \ldots, n) \) can be also weakened. In fact, one can take as coefficients \( b^i \) the signed measures, satisfying condition (6). Indeed, all our arguments require a convergence of the corresponding integrals only.

### 3 Parabolic case

Let \( Q \) be a bounded domain in \( \mathbb{R}^{n+1} \) with topological boundary \( \partial Q \). We define the parabolic boundary \( \partial' Q \) as the set of all points \((x^0, t^0) \in \partial Q\) such that for any \( \varepsilon > 0 \), we have \( Q_\varepsilon(x^0, t^0) \setminus \overline{Q} \neq \emptyset \). By \( d_p(x; t) \) we denote the parabolic distance between \((x; t)\) and \( \partial' Q \) which is defined as follows:
\[
d_p(x; t) := \sup\{\rho > 0 : Q_\rho(x; t) \cap \partial' Q = \emptyset\}.
\]

Next, we define the lateral surface \( \partial'' Q \) as the set of all points \((x^0, t^0) \in \partial' Q\) such that \( Q_\varepsilon(x^0, t^0) \cap Q \neq \emptyset \) for any \( \varepsilon > 0 \).

We suppose that \( Q \) satisfies the **parabolic interior** \( C^{1,D} \)-paraboloid condition. It means that in a local coordinate system \( \partial'' Q \) is given by the equation \( x_n = F(x'; t) \), where \( F \) is a \( C^1 \)-function such that \( F(0; 0) = 0 \) and the inequality
\[
F(x'; t) \leq \sqrt{|x'|^2 - t} \cdot \sigma(\sqrt{|x'|^2 - t}) \quad \text{for} \quad t \leq 0
\]
holds in some neighborhood of the origin. Here \( \sigma \) is a \( C^1 \)-function belonging to the class \( D \) (see Remark [1]).

Let an operator \( \mathcal{M} \) be defined by the formula (2). Suppose that the coefficients of \( \mathcal{M} \) satisfy the following conditions:
\[
\nu I_n \leq (a^{ij}(x; t)) \leq \nu^{-1} I_n, \quad a^{ij} \in C^{0,D}_p(Q) \quad \text{for all} \quad i, j = 1, \ldots, n, \tag{19}
\]
and
\[
\omega^-_p(r) \to 0 \quad \text{and} \quad \omega^+_p(r) \to 0 \quad \text{as} \quad r \to 0, \tag{20}
\]
where
\[
\omega_p^-(r) := \sup_{(x;t) \in Q} \int_{Q_r(x;t) \cap Q} \frac{|b(y; s)|}{(t - s)^{(n+1)/2}} \cdot \exp \left( -\gamma \frac{|x - y|^2}{t - s} \right) \times \frac{d_p(y; s)}{d_p(y; s) + \sqrt{|x - y|^2 + t - s}} \ dy ds;
\]
\[
\omega_p^+(r) := \sup_{(x;t) \in Q} \int_{Q_r(x;t^2) \cap Q} \frac{|b(y; s)|}{(s - t)^{(n+1)/2}} \cdot \exp \left( -\gamma \frac{|x - y|^2}{s - t} \right) \times \frac{d_p(y; s)}{d_p(y; s) + \sqrt{|x - y|^2 + s - t}} \ dy ds.
\]

Here \( \nu \) and \( \mathcal{I}_n \) are the same as in Section 2, \( b(y; s) = (b^1(y; s), \ldots, b^n(y; s)) \), and \( \gamma \) is a positive constant to be determined later, depending only on \( n, \nu \) and on the moduli of continuity of the coefficients \( a^{ij} \).

**Remark 6.** Similarly to the elliptic case, in any strict interior subdomain of \( Q \setminus \partial^* Q \) condition (20) means that \( b \) is an element of the parabolic Kato class \( K_n \), see [CKP12]. Indeed, in this case (20) can be rewritten as follows:
\[
\sup_{(x;t) \in Q} \int_{(t-r^2,t+r^2) \times B_r(x)} \frac{|b(y; s)|}{(s - t)^{(n+1)/2}} \cdot \exp \left( -\gamma \frac{|x - y|^2}{s - t} \right) \ dy ds \to 0 \quad \text{as} \quad r \to 0.
\]

This condition differs from Definition 3.1 [CKP12] only in that the integration in [CKP12] is over \( (t - r^2, t + r^2) \times \mathbb{R}^n \). However, using the covering of \( \mathbb{R}^n \setminus B_r(x) \) by the balls of radius \( r/3 \) one can check that corresponding suprema converge to zero simultaneously.

In the whole domain \( Q \) our condition (20) is weaker than \( b \in K_n \).

To formulate the parabolic counterpart of Theorem 2.1 we need the following notion.

**Definition 3.** For a point \((x; t) \in \overline{Q}\) we define its dependence set as the set of all points \((y; s) \in Q\) admitting a vector-valued map \( \mathcal{F} : [0, 1] \mapsto \mathbb{R}^{n+1} \) such that the last coordinate function \( \mathcal{F}_{n+1} \) is strictly increasing and
\[
\mathcal{F}(0) = (y; s); \quad \mathcal{F}(1) = (x; t); \quad \mathcal{F}((0, 1)) \subset Q.
\]
If \( Q \) is a right cylinder with generatrix parallel to the \( t \)-axis, then for any \((x; t) \in \overline{Q}\) the dependence set is \( \overline{Q} \cap \{ s < t \} \).
Theorem 3.1. Let $Q$ be a bounded domain in $\mathbb{R}^{n+1}$, let $\partial'Q$ satisfy the interior parabolic $C^{1,\mathcal{D}}$-paraboloid condition, let $M$ be defined by (2), and let assumptions (19)-(20) be satisfied.

In addition, assume that a function $u \in C^{1,0}_{x,t}(\overline{Q})$ satisfies, in the weak sense, the inequality

$$Mu \geq 0 \quad \text{in} \quad Q.$$ 

Then, if $u$ attains its minimum at a point $(x^0; t^0) \in \partial'Q$, and $u$ is nonconstant on the dependence set of $(x^0; t^0)$, we have

$$\frac{\partial u}{\partial n}(x^0; t^0) < 0.$$ 

Here $\frac{\partial}{\partial n}$ denotes the derivative with respect to the spatial exterior normal on $\partial''Q \cap \{t = t^0\}$.

Remark 7. Notice that we do not care of the behavior of $u$ after $t^0$. Thus, without loss of generality we suppose $Q = Q \cap \{t < t_0\}$. Moreover, similarly to the elliptic case, we may assume that $(x^0; t^0) = (0; 0)$, and $\partial''Q$ is locally a paraboloid

$$x_n = P(x'; t) := \sqrt{|x'|^2 - t \cdot \sigma(\sqrt{|x'|^2 - t})},$$

where $\sigma \in \mathcal{D}$ is smooth.

Next, we flatten the boundary of the paraboloid by the coordinate transform

$$\tilde{x}' = x'; \quad \tilde{t} = t; \quad \tilde{x}_n = x_n - P(x'; t). \quad (22)$$

Lemma 3.2. Assumptions (19) and (20) on $a^{ij}$ and $b^i$ remain valid under transform (22).

Proof. It is easy to see that $|D'\mathcal{P}| \in C_0^{0,\mathcal{D}}(Q_R \cap \mathbb{R}^{n+1}_+)$ for some $R > 0$. (Here, we consider $D'\mathcal{P}$ as a function of $(x,t)$-variables, which is independent on $x_n$.) Therefore, the “new” coefficients $\tilde{a}^{ij}$ satisfy (19) in $Q_R \cap \mathbb{R}^{n+1}_+$.

It is also evident that the transformed “old” coefficients $b^i$ satisfy (20).

However, the coordinate change (22) generates an additional term $\tilde{b}_n$ which admits the estimate

$$|\tilde{b}_n(\tilde{x}; t)| \leq C|\partial_t \mathcal{P}(x'; t)| = C\left(\frac{\sigma'(|x'|^2 - t)}{\sqrt{|x'|^2 - t}} + \sigma'(|x'|^2 - t)\right).$$

Estimating the integral entering in the definition of $\omega_p^\pm$, it suffices to assume that $x' = 0$ and $t = 0$. This gives

$$\omega_p^-(r) \leq C \int_{Q_r} \exp\left(-\gamma \frac{|y|^2}{s}\right) \left(\frac{\sigma'(|y|^2 - s)}{\sqrt{|y|^2 - s}} + \sigma'(|y|^2 - s)\right) dy ds.$$
After integration over $y_n$ we make change of variables

$$\varrho = \frac{|y'|}{\sqrt{-s}}; \quad \tau = \sqrt{|y'|^2 - s},$$

and arrive at

$$\omega_p^{-}(r) \leq C \int_0^{r \sqrt{2}} \int_0^\infty \exp(-\gamma \varrho^2) \frac{\varrho^{n-2}}{\varrho^2 + 1} \left( \frac{\sigma(\tau)}{\tau} + \sigma'(\tau) \right) d\varrho d\tau$$

$$\leq C(n, \gamma) \left( J(\sigma(r \sqrt{2}) + \sigma(r \sqrt{2})) \right),$$

and the lemma follows.

Thus, we may consider $\partial'^b Q$ locally as a flat boundary $x_n = 0$ and assume, without loss of generality, that $Q \cap \mathbb{R}_+^{n+1} \subset Q$.

Next, we take for $0 < \rho \leq R/2$ the cylinder $A_\rho = Q_\rho(x^\rho; 0)$ (as in the elliptic case, $x^\rho = (0, \ldots, 0, \rho)$). Define the auxiliary function $\tilde{z}$ as the solution of the initial-boundary value problem

$$\left\{ \begin{array}{l}
\mathcal{M}_0 \tilde{z} := \partial_t \tilde{z} - D_i(a^{ij}(x; t)D_j \tilde{z}) = 0 \quad \text{in} \quad A_\rho, \\
\tilde{z} = 0 \quad \text{on} \quad \partial^b A_\rho, \\
\tilde{z}(x; -\rho^2) = \varphi(x) \quad \text{for} \quad x \in B_\rho(x^\rho),
\end{array} \right. \quad (23)$$

where $\varphi$ is a smooth cut-off function such that

$$\varphi(x) = 1 \quad \text{for} \quad |x| < 1/2; \quad \varphi(x) = 0 \quad \text{for} \quad |x| > 3/4.$$

The existence of (unique) weak solution $\tilde{z}$ follows from the general parabolic theory.

**Theorem 3.3.** The function $\tilde{z}$ belongs to $C^{1,0}_{x,t}(\overline{A}_\rho)$ for sufficiently small $\rho$. Moreover, there exists a positive constant $\tilde{\rho}_0 \leq R/2$ depending only on $n, \nu$ and $\sigma$, such that the inequality

$$|D\tilde{z}(x; t)| \leq \frac{C_4(n, \nu)}{\rho}, \quad (x; t) \in \overline{A}_\rho,$$

holds true for all $\rho \leq \tilde{\rho}_0$.

**Proof.** We partially follow the line of proof of Lemma 2.2. Let $(x^*; t^*)$ be an arbitrary point in $\overline{A}_\rho$. We introduce the auxiliary function $\psi_{x^*, t^*}$ as the solution of the problem

$$\left\{ \begin{array}{l}
\mathcal{M}_0^{x^*,t^*} \psi_{x^*, t^*} = 0 \quad \text{in} \quad A_\rho, \\
\psi_{x^*, t^*} = 0 \quad \text{on} \quad \partial^b A_\rho, \\
\psi_{x^*, t^*}(x; -\rho^2) = \varphi(x) \quad \text{for} \quad x \in B_\rho(x^\rho),
\end{array} \right. \quad (24)$$
where $\mathcal{M}_0^{x^*,t^*} := \partial_t - D_i a^{ij}(x^*,t^*) D_j$ is operator with constant coefficients frozen at the point $(x^*,t^*)$. It is well known that $\psi^{x^*,t^*} \in C^\infty(\overline{\mathcal{A}_\rho})$, and

$$|D\psi^{x^*,t^*}(y,s)| \leq \frac{N_5(n,\nu)}{\rho}, \quad (y,s) \in \overline{\mathcal{A}_\rho}. \quad (25)$$

Setting $w^{(3)} = \tilde{z} - \psi^{x^*,t^*}$ we observe that $w^{(3)}$ vanishes on $\partial' \mathcal{A}_\rho$. Hence, $w^{(3)}$ can be represented in the cylinder $\mathcal{A}_\rho$ as

$$w^{(3)}(x,t) = \int_{\mathcal{A}_\rho \cap \{s \leq t\}} \Gamma^{x^*,t^*}_\rho(x,y,t,s) M^{x^*,t^*}_0 \psi^{x^*,t^*}(y,s) dyds,$$

where $\Gamma^{x^*,t^*}_\rho$ stands for the Green function of the operator $M^{x^*,t^*}_0$ in $\mathcal{A}_\rho$.

Similarly to (9), we integrate by parts and obtain

$$w^{(3)}(x,t) = \int_{\mathcal{A}_\rho \cap \{s \leq t\}} D_y i \Gamma^{x^*,t^*}_\rho(x,y,t,s) (a^{ij}(x^*,t^*) - a^{ij}(y,s)) D_j \tilde{z}(y,s) dyds.$$

Differentiating both sides with respect to $x_k$, $k = 1, \ldots, n$, we get the system of equations

$$D_k \tilde{z}(x,t) - \int_{\mathcal{A}_\rho \cap \{s \leq t\}} D_{x_k} D_y i \Gamma^{x^*,t^*}_\rho(x,y,t,s) \times$$

$$\times (a^{ij}(x^*,t^*) - a^{ij}(y,s)) D_j \tilde{z}(y,s) dyds = D_k \psi^{x^*,t^*}(x,t). \quad (26)$$

Now we put $(x^*,t^*) = (x,t)$ and get the relation

$$(I - T_2) D \tilde{z} = \Psi, \quad (27)$$

where

$$\Psi = D \psi^{x^*,t^*}(x,t) \big|_{(x^*,t^*)=(x,t)}$$

while $T_2$ denotes the matrix integral operator whose kernel is matrix $T_2$ with entries

$$T_2^{ij}(x,y,t,s) = D_{x_k} D_y i \Gamma^{x^*,t^*}_\rho(x,y,t,s) \big|_{(x^*,t^*)=(x,t)} \chi_{\{s \leq t\}} \big( a^{ij}(x,t) - a^{ij}(y,s) \big).$$

It is easy to see that $\Psi \in C(\overline{\mathcal{A}_\rho})$. Therefore, the statement of Theorem follows from the next assertion.

**Lemma 3.4.** The operator $T_2$ is bounded in $C(\overline{\mathcal{A}_\rho})$, and

$$\|T_2\|_{C \to C} \leq C_5 J_n(2\sqrt{2}\rho),$$

where $C_5$ depends only on $n$ and $\nu$. 14
Proof. The following estimate for the Green function $\Gamma_{\rho}^{x*,t^*}(x,y; t, s)$ is well known:

$$
|D_xD_y\Gamma_{\rho}^{x*,t^*}(x,y; t, s)| \leq \frac{N_6}{(t-s)^{(n+2)/2}} \exp \left(-N_7 \frac{|x-y|^2}{t-s}\right), \quad (28)
$$

where $N_6$ and $N_7$ are completely determined by $n$ and $\nu$.

Combination of (28) with condition (19) gives for $r \leq 2\rho$ and $(x; t) \in \overline{A}_\rho$

$$
\int_{Q_r(x;t) \cap A_\rho} |T_2(x,y; t, s)| \, dy \, ds
\leq \int_{t-r^2 B_r(x)} \int_{t-s}^t \frac{N_6 \sigma(\sqrt{t-s} + |x-y|^2)}{(t-s)^{(n+2)/2}} \exp \left(-N_7 \frac{|x-y|^2}{t-s}\right) \, dy \, ds.
$$

Change of variables $\varrho = |x-y|/\sqrt{t-s}$, $\tau = \sqrt{t-s + |x-y|^2}$ gives

$$
\int_{Q_r(x;t) \cap A_\rho} |T_2(x,y; t, s)| \, dy \, ds
\leq \int_0^{r \sqrt{2}} \int_{0}^{\varrho \infty} N_8 \exp (-N_7 \varrho^2) \varrho^{n-1} \sigma(\tau) \, d\varrho \, d\tau \leq C_5 J_\sigma(\varrho \sqrt{2})
$$

($N_8$ and $C_5$ depend only on $n$ and $\nu$).

For a vector function $f \in C(\overline{A}_\rho)$ and for all $(x; t) \in \overline{A}_\rho$ we have

$$
|T_2 f(x; t)| \leq \|f\|_{C(\overline{A}_\rho)} \cdot \int_{A_\rho} |T_2(x,y; t, s)| \, dy \, ds \leq C_5 J_\sigma(2\sqrt{2}\rho) \cdot \|f\|_{C(\overline{A}_\rho)}.
$$

It remains to show that $T_2 f \in C(\overline{A}_\rho)$. For $(x; t), (\hat{x}; \hat{t}) \in \overline{A}_\rho$ and any small $\delta > 0$ we have

$$
(T_2 f)(x; t) - (T_2 f)(\hat{x}; \hat{t}) = \tilde{J}_1 + \tilde{J}_2
$$

$$
:= \left( \int_{A_\rho \cap Q_\delta(\hat{x}; \hat{t})} + \int_{A_\rho \setminus Q_\delta(\hat{x}; \hat{t})} \right) (T_2(x,y; t, s) - T_2(\hat{x}, y; \hat{t}, s)) f(y; s) \, dy \, ds.
$$

Similarly to the proof of Theorem 2.3 if $\sqrt{|t-s| + |x-y|^2} \leq \delta/2$ then (29) gives

$$
|\tilde{J}_1| \leq 2C_5 J_\sigma(3\sqrt{2}\rho/2) \cdot \|f\|_{C(\overline{A}_\rho)}.
$$
Thus, given ε we can choose δ such that |J_1| ≤ ε.

Next, D_x D_y \Gamma_{p,\rho}^x (x, y; t, s) is continuous w.r.t. (x; t) and w.r.t. (x^*; t^*) for (x; t) ≠ (y; s). Therefore, T_2 (x, y; t, s) is continuous w.r.t. (x; t) for (x; t) ≠ (y; s). Similarly to the proof of Theorem 2.3 for chosen δ we obtain, as (x; t) → (\tilde{x}; t),

|J_2| ≤ \|f\|_{C(\overline{A}_\rho)} \cdot \max_{(y; s) \in A_\rho \setminus Q_\delta(\tilde{x}; t)} \left| T_2 (x, y; t, s) - T_2 (\tilde{x}, y; t, s) \right| → 0,

and the Lemma follows.

We continue the proof of Theorem 3.3. Choose the value of \tilde{\rho}_0 so small that \mathcal{J}_\phi (2\sqrt{2} \tilde{\rho}_0) ≤ (2C_5)^{-1}, where C_5 is the constant from Lemma 3.4. Then by the Banach theorem the operator I - T_2 in (27) is invertible. This gives \tilde{z} \in C_{x,t}^{1,0} (\overline{A}_\rho). Moreover, Lemma 2.5 and inequality (25) provide (24). The proof is complete.

For \rho ≤ \tilde{\rho}_0 we introduce the Green function \Gamma_{0,\rho}(x; y; t, s) of the operator \mathcal{M}_0 in the cylinder A_\rho. By Theorem 2.6 [CKP12], D_x \Gamma_{0,\rho}(x; y; t, s) is continuous for (x; t) ≠ (y; s), and the estimate

|D_x \Gamma_{0,\rho}(x; y; t, s)| ≤ N_8 \min \left\{ \frac{1}{(t - s)^{(n+1)/2}}, \frac{\text{dist} (y, \partial \Gamma_{0,\rho}(x^p))}{(t - s)^{(n+2)/2}} \right\} \times \exp \left( -N_9 \frac{|x - y|^2}{t - s} \right)

(30)

holds for any (x; t), (y; s) ∈ A_\rho, s < t. Here N_8 and N_9 are the constants depending only on n, \nu, and \sigma.

Further, we introduce the barrier function \tilde{v} defined as the weak solution of the initial-boundary value problem

\begin{align*}
\begin{cases}
\mathcal{M} \tilde{v} &= 0 \quad \text{in } A_\rho, \\
\tilde{v} &= 0 \quad \text{on } \partial'' A_\rho, \\
\tilde{v}(x; -\rho^2) &= \varphi (\frac{x - x^p}{\rho}) \quad \text{for } x \in B_\rho(x^p),
\end{cases}
\end{align*}

(31)

where \varphi is the same as in (24).

**Theorem 3.5.** Let b satisfy the first relation in (20) with \gamma = N_9 (n, \nu, \sigma) (here N_9 is the constant in (34)). Then there exists a positive \hat{\rho}_0 ≤ \tilde{\rho}_0 such that for all \rho ≤ \hat{\rho}_0 the problem (31) admits a unique solution \tilde{v} ∈ C_{x,t}^{1,0} (\overline{A}_\rho).

Moreover, the inequality

|D \tilde{v}(x; t) - D \tilde{z}(x; t)| ≤ C_6 \frac{\omega_p (2\rho)}{\rho}

(32)
holds true for any \((x; t) \in \mathcal{A}_\rho\). Here \(C_6 = C_6(n, \nu, \sigma) > 0\), \(\tilde{\rho}_0\) is completely defined by \(n, \nu, \sigma,\) and \(\omega\), while \(\tilde{z} \in C_{1, t}(\overline{\mathcal{A}}_\rho)\) is defined in (23).

**Proof.** We follow the line of proof of Theorem 2.3. Consider in \(\mathcal{A}_\rho\) the auxiliary function \(w^{(4)} = \tilde{v} - \tilde{z}\). We observe that \(w^{(4)}\) vanishes on \(\partial' \mathcal{A}_\rho\), and

\[
M_0 w^{(4)} = -b^i \left( D_i w^{(4)} + D_i \tilde{z} \right) \quad \text{in} \quad \mathcal{A}_\rho.
\]

Similarly to the proof of Theorem 2.3, \(D_k w^{(4)}\) can be represented in \(\mathcal{A}_\rho\) as

\[
D_k w^{(4)}(x; t) = -\int_{\mathcal{A}_\rho \cap \{s \leq t\}} D_x \Gamma_{0, \rho}(x, y; t, s) \times b^i(y; s) \left( D_i w^{(4)}(y; s) + D_i \tilde{z}(y; s) \right) dy ds.
\]

Therefore, we get the relation

\[
(I + T_3) D w^{(4)} = -T_3 D \tilde{z},
\]

where \(T_3\) denotes the matrix operator whose \((k, i)\) entries are integral operators with kernels \(D_x \Gamma_{0, \rho}(x, y; t, s)\). The statement of Theorem follows from the next assertion.

**Lemma 3.6.** The operator \(T_3\) is bounded in \(C(\overline{\mathcal{A}}_\rho)\), and

\[
\|T_3\|_{C \to C} \leq C_7 \omega^-_p(2\rho),
\]

where \(C_7\) depends only on \(n, \nu,\) and \(\sigma\).

**Proof.** Recall that \(\rho \leq R/2\) and \(Q_R \cap \mathbb{R}^{n+1}_+ \subset Q\). Thus \(\text{dist}\{y, \partial B_\rho(x^p)\} \leq d_\rho(y; s)\) for any \((y; s) \in \mathcal{A}_\rho\), and the combination of estimate (30) with the first relation in (21) gives for \(r \leq 2\rho\)

\[
\int_{Q_r(x; t) \cap \mathcal{A}_\rho} |D_x \Gamma_{0, \rho}(x, y; t, s)| |b(y; s)| dy ds \leq N_{10}(n) N_8 \omega^-_p(r), \quad x \in \overline{\mathcal{A}}_\rho, \quad (34)
\]

(here \(N_8\) is the constant in (30)).

The rest of the proof repeats literally the proof of Lemma 2.4.

We continue the proof of Theorem 3.5. Choose the value of \(\hat{\rho}_0\) so small that \(\omega(2\hat{\rho}_0) \leq (2C_7)^{-1}\), where \(C_7\) is the constant from Lemma 3.6. Then by the Banach theorem the operator \(I + T_3\) in (33) is invertible. This gives the existence and uniqueness of \(w^{(4)} \in C_{1, t}(\overline{\mathcal{A}}_\rho)\), and thus, the unique solvability of the problem (31). Moreover, Lemma 3.6 and inequality (24) provide (32). The proof is complete.

\[\square\]
To prove Theorem 3.1 we need the following maximum principle.

**Lemma 3.7.** Let $M$ be defined by (2), and let assumptions (19)-(20) be satisfied in a domain $E \subset \mathbb{R}^{n+1}$. Let a function $w \in C^{1,0}_x, \partial_t(E)$ satisfy $Mw \geq 0$ in $E$. If $w$ attains its minimum in a point $(x^0; t^0) \in \bar{E} \setminus \partial' E$ then $w = \text{const}$ on the closure of the dependence set of $(x^0; t^0)$.

**Proof.** The Harnack inequality for parabolic divergence-type operators was established in [Zha96] under the assumptions that the leading coefficients $a^{ij}$ are Hölder continuous and $b$ satisfy (21) with arbitrary $\gamma > 0$ (and integration over $(t - r^2, t + r^2) \times \mathbb{R}^n$ that is inessential, see Remark 6).

As it was mentioned in the proof of Lemma 2.5, the first assumption can be replaced by the Dini continuity. Further, in fact only (21) with a certain $\gamma$ occurring in the estimate of $D\Gamma_{0,\rho}$ is used in [Zha96]. The latter coincides with the assumption $b \in K_n$.

Since our assumption (20) implies $b \in K_n$ in any strict interior subdomain of $\bar{Q} \setminus \partial'Q$ (see Remark 6), the strong maximum principle holds for the operator $M$.

**Remark 8.** This Lemma is the only point where we need the second relation in (20). If we could prove at least weak maximum principle for the operator $M$ using only the quantity $\omega^-_p$, we did not need $\omega^+_p$ at all. Unfortunately, we cannot do it, and the question whether the second relation in (20) is necessary for the Boundary Point Principle remains open.

**Proof of Theorem 3.1.** It is well known that the Boundary Point Principle holds true for the operator with constant coefficients. Using this statement for the operator $M^{x*,t*}_0$ with $x^* = 0$, $t^* = 0$ in the cylinder $A_1$ and rescaling $A_1$ into $A_\rho$ we get the estimate

$$D_n\psi_{0,0}(0;0) \geq \frac{N_{11}(n,\nu)}{\rho}.$$ 

Next, the relation (26), Lemma 3.1 and inequality (24) imply for sufficiently small $\rho$

$$D_n\tilde{z}(0;0) \geq D_n\psi_{0,0}(0;0) - \|T_2D\tilde{z}\|_{C(A_\rho)} \geq \frac{N_{11}}{\rho} - C_4 C_5 \frac{\mathcal{J}_{0}(2\sqrt{2}\rho)}{\rho} \geq \frac{N_{11}}{2\rho}.$$ 

The relation (32) gives for sufficiently small $\rho$

$$D_n\tilde{v}(0;0) \geq D_n\tilde{z}(0;0) - |D\tilde{v}(0;0) - D\tilde{z}(0;0)| \geq \frac{N_{11}}{2\rho} - C_6 \frac{\omega^-_p(2\rho)}{\rho} \geq \frac{N_{11}}{4\rho}.$$ 

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We fix such a \( \rho \). Since \( u \) is nonconstant on the dependence set of \((0; 0)\), Lemma 3.7 ensures
\[
u(x; -\rho^2) - u(0; 0) > 0 \quad \text{for} \quad x \in B_{3\rho/4}(x^\rho).
\]
Therefore, we have for sufficiently small \( \varepsilon \)
\[
\mathcal{M}(u - u(0; 0) - \varepsilon \tilde{v}) \geq 0 \quad \text{in} \quad A_\rho; \quad u - u(0; 0) - \varepsilon \tilde{v} \geq 0 \quad \text{on} \quad \partial' A_\rho.
\]
By Lemma 3.7 the estimate \( u - u(0; 0) \geq \varepsilon \tilde{v} \) holds true in \( A_\rho \), with equality at the origin. This gives
\[
\frac{\partial u}{\partial n}(0; 0) = -D_n u(0; 0) \leq -\varepsilon D_n \tilde{v}(0; 0),
\]
which completes the proof. \( \square \)

Remark 9. As in elliptic case, the statement of Theorem 3.1 is also valid for weak supersolutions of equation (2). Namely, let \( Q \) and the coefficients of \( \mathcal{M} \) be the same as in Theorem 3.1. Suppose that for a function \( u \in L^2(Q) \) with \( Du \in L^2(Q) \) the assumptions
\[
\sup_t \| u(\cdot; t) \|_{L^2} < \infty, \quad \text{and} \quad |b \cdot Du| \in L^1(Q)
\]
are fulfilled. Finally, let \( u \) satisfy the inequality \( \mathcal{M} u \geq 0 \) in the weak sense.

Then, if \( u \) attains its minimum at a point \((x^0; t^0) \in \partial'' Q \), and \( u \) is non-constant on the dependence set of \((x^0; t^0)\), we have
\[
\liminf_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} \frac{u(x^0 - \varepsilon n(x^0); t^0) - u(x^0; t^0)}{\varepsilon} > 0.
\]

Remark 10. Similarly to Remark 5, one can take as coefficients \( b^i \) the signed measures, satisfying condition (20).

4 Some sufficient conditions for the validity of (6) and (20)

In this section we list several simple sufficient conditions on lower-order coefficients providing the validity of assumptions (6) and (20) for elliptic and parabolic operators, respectively. These conditions are close to ones imposed in [Naz12], where equations in non-divergence form were studied.

Throughout this section we will denote various constants depending on \( n \) only by the letter \( N \) without indices, and the constants depending on \( n \) and
γ only by the letter \( \hat{N} \) without indices. To simplify the notation, we assume that \( \mathbf{b} \) is extended by zero outside of \( \Omega \) (of \( Q \)).

In the elliptic case we consider two types of restrictions: distributed drift

\[
\mathbf{b} \in L^n(\Omega); \quad \sup_{x \in \Omega} \| \mathbf{b} \|_{n, B_\rho(x)} \leq C \sigma(\rho), \tag{35}
\]

and near-boundary drift

\[
| \mathbf{b}(y) | \leq C \frac{\sigma(d(y))}{d(y)} \tag{36}
\]

(recall that \( \sigma \in \mathcal{D} \)).

**Lemma 4.1.** Any of the restrictions \( (35) \) and \( (36) \) implies the validity of condition \( (\Omega) \).

**Proof.** Let \( (35) \) hold. We set \( B^k = B_r/2^k \) and estimate

\[
\omega(r) \leq \sup_{x \in \Omega} \sum_{k=0}^\infty \int_{B^k \setminus B^{k+1}} \frac{|\mathbf{b}(x-y)|}{|y|^{n-1}} \, dy \\
\leq \sup_{x \in \Omega} \sum_{k=0}^\infty \left( \int_{B^k \setminus B^{k+1}} |\mathbf{b}(x-y)|^n \, dy \right)^{\frac{1}{n}} \cdot \left( \int_{B^k \setminus B^{k+1}} \frac{dy}{|y|^n} \right)^{\frac{n-1}{n}} \\
\leq NC \sum_{k=0}^\infty \sigma(r/2^k) \leq NC \mathcal{J}_\sigma(2r),
\]

and \( (\Omega) \) follows.

Now let \( (36) \) hold. We use the decay of \( \sigma(\tau)/\tau \) to estimate

\[
\frac{\sigma(d(y))}{d(y) + |x-y|} \leq \frac{\sigma(d(y) + |x-y|)}{d(y) + |x-y|} \leq \frac{\sigma(|x-y|)}{|x-y|}.
\]

Therefore,

\[
\omega(r) \leq C \int_{B_r(x)} \frac{\sigma(|x-y|)}{|x-y|^n} \leq NC \mathcal{J}_\sigma(r),
\]

and \( (\Omega) \) again follows. \( \square \)

In the parabolic case we consider the following analogue of \( (35) \):

\[
\mathbf{b} \in L^{n+1}(Q); \quad \sup_{(x,t) \in Q} \| \mathbf{b} \|_{n+1, Q_\rho(x;t)} \leq C \sigma(\rho)^{1/(n+1)}, \tag{37}
\]

as well as the analog of \( (36) \):

\[
| \mathbf{b}(y; s) | \leq C \frac{\sigma(d_p(y; s))}{d_p(y; s)}. \tag{38}
\]

20
Lemma 4.2. Any of the restrictions (37) and (38) implies the validity of condition (20) with arbitrary $\gamma > 0$.

Proof. We estimate the quantity $\omega_p^-(r)$; the case of $\omega_p^+(r)$ is considered along the same lines.

Let (37) hold. We set $Q^k = Q_{r/2^k}$ and obtain

$$\omega_p^-(r) \leq \sup_{(x,t) \in Q} \sum_{k=0}^{\infty} \int_{Q_k \setminus Q_{k+1}} \frac{|b(x - y; t + s)|}{(-s)^{(n+1)/2}} \exp\left(-\gamma \frac{|y|^2}{-s}\right) dyds$$

$$\leq \sup_{(x,t) \in Q} \sum_{k=0}^{\infty} \left( \int_{Q_k \setminus Q_{k+1}} |b(x - y; t + s)|^{n+1} dyds \right)^{1/(n+1)} \Phi_k^{n/(n+1)}$$

$$\leq C \sum_{k=0}^{\infty} \sigma(r/2^k) \cdot (r/2^k)^{1/(n+1)} \cdot \Phi_k^{n/(n+1)},$$

where

$$\Phi_k = \int_{Q_k \setminus Q_{k+1}} \exp\left(-\gamma \frac{n+1}{n} \frac{|y|^2}{-s}\right) \cdot \frac{dyds}{(-s)^{(n+1)/2}}.$$

Change of variables $\varrho = |y|/\sqrt{-s}$, $\tau = \sqrt{|y|^2 - s}$ gives

$$\Phi_k \leq \int_{r/2^{k+1}}^{r\sqrt[3]{2}/2^k} \int_0^{\infty} \exp\left(-\gamma \frac{n+1}{n} \frac{\varrho^2}{\tau}\right) \cdot \frac{2\varrho^{n-1}(\varrho^2 + 1)^{1/2n}}{\tau^{1+1/n}} d\varrho d\tau \leq \frac{\tilde{N}}{(r/2^k)^{1/n}}.$$

Thus,

$$\omega_p^-(r) \leq \tilde{N}C \sum_{k=0}^{\infty} \sigma(r/2^k) \leq \tilde{N}C \mathcal{J}_\sigma(2r),$$

and the first relation in (20) follows.

Now let (38) hold. As in Lemma 4.1, we use the decay of $\sigma(\tau)/\tau$ to estimate

$$\frac{\sigma(d_p(y; s))}{d_p(y; s) + \sqrt{|x - y|^2 + t - s}} \leq \frac{\sigma(\sqrt{|x - y|^2 + t - s})}{\sqrt{|x - y|^2 + t - s}}.$$

Therefore,

$$\omega_p^-(r) \leq C \int_{Q_r(x,t)} \exp\left(-\gamma \frac{|x - y|^2}{t - s}\right) \cdot \frac{\sigma(\sqrt{|x - y|^2 + t - s})}{\sqrt{|x - y|^2 + t - s}} \frac{dyds}{(t - s)^{(n+1)/2}}.$$
Change of variables \( \varrho = |x - y|/\sqrt{t - s} \), \( \tau = \sqrt{t - s + |x - y|^2} \) gives

\[
\omega_p(r) \leq C \int_0^{r\sqrt{2}} \int_0^{\infty} \exp\left(-\gamma \varrho^2\right) \cdot \frac{\varrho^q - 1}{\sqrt{\varrho^2 + 1}} \frac{\sigma(\tau)}{\tau} \, d\varrho \, d\tau \leq \hat{N} C J_\sigma(r\sqrt{2}),
\]

and the first relation in (20) again follows.

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