ON THE RICCI FLOW ON THE ROTATIONALLY SYMMETRIC TWO-BALL

JEAN C. CORTISSOZ

To my advisor and friend José Escobar, in memoriam.

Abstract. In this paper we study a boundary value problem for the Ricci flow in the two-dimensional ball endowed with a rotationally symmetric metric of positive Gaussian curvature. We show short and long time existence results. We construct families of metrics for which the flow uniformizes the curvature along a sequence of times. Finally, we show that if the initial metric has positive Gaussian curvature and the boundary has positive geodesic curvature then the flow uniformizes the curvature along a sequence of times.

1. Introduction

Let $B^2$ be the two dimensional ball. We let $g$ be a time dependent family of metrics. Let $R$ denote its scalar curvature and $k_g$ the geodesic curvature of its boundary $S^1 \approx \partial B^2$ with respect to the outward unit normal. In this paper we begin the study the of following boundary value problem for the Ricci flow in surfaces,

\[
\begin{align*}
\frac{\partial g}{\partial t} &= -R_g \quad \text{in} \quad M \times (0,T), \\
k_g &= \psi \quad \text{on} \quad \partial M \times (0,T), \\
g (\cdot, 0) &= g_0.
\end{align*}
\]

In other words we want to study the boundary value problem consisting of prescribing the geodesic curvature of the boundary.

Flows in surfaces with boundary have been studied by Brendle [11]. In his paper, Brendle gives a short consideration to the homogenous version of the boundary value problem (1). Due to its simplicity and non triviality, and in order to give a strong basis to our general intuition, in this paper we mainly consider rotationally symmetric metrics on $B^2$ (but we should mention that the results of Section 8 are also valid without this restriction), i.e. metrics of the form

\[ds^2 = dr^2 + f(r)^2 d\omega^2,\]

and the following problem boundary value problem,

\[
\begin{align*}
\frac{\partial g}{\partial t} &= -R_g \quad \text{in} \quad M \times (0,T), \\
k_g &= k_0 \quad \text{on} \quad \partial M \times (0,T), \\
g (\cdot, 0) &= g_0.
\end{align*}
\]

where $k_0$ is the geodesic curvature of the boundary with respect to the metric $g_0$. Notice that if $g_0$ is rotationally symmetric, then the solution $g$ will also be rotationally symmetric.

We also consider the normalized “version” of (2). The normalized Ricci flow is obtained from (2) by the following procedure:
Let $\phi(t)$ be such that for $\tilde{g} = \phi g$ we have $Vol(M) = 1$. Then we change the time scale by letting

$$\tilde{t}(t) = \int_0^t \phi(t) \, dt,$$

then $\tilde{g}(\tilde{t})$ satisfies the equation

$$(3) \quad \frac{\partial}{\partial \tilde{t}} \tilde{g} = \frac{2}{n} \tilde{r} \tilde{g} - 2Ric(\tilde{g}).$$

In this paper we prove the following results

**Theorem 1.** Assume $R \geq 0$ at $t = 0$. Then solution to the normalized flow exists for all time.

Then in Section 7 we show some families of examples where the there is a sequence of times $t \to \infty$ such that $g(t) \to g(\infty)$ smoothly and the metric $g(\infty)$ has constant curvature and totally geodesic boundary.

Finally, in Section 8 by using the techniques introduced by Perelman [P1], we show the following result (it is good to say here that the proof of the following theorem do not depend on the rotational symmetry hypothesis in an essential way).

**Theorem 2.** If $k_0 \geq 0$ and $R_0 \geq 0$ then there is a sequence of times $t_k \to T$ so that for the solution of

$$\frac{R_{max}(t_k)}{R_{min}(t_k)} \to 1 \quad \text{as} \quad t_k \to T.$$

The main tool we use to prove Theorem 2 (and any other convergence result in this paper) is blow up analysis. In order to produce blow up limits we need to bound derivatives of the curvature in terms of a bound on the curvature. This is the content of Section 6. Injectivity radius estimates and Fermi inradius estimates (“how big a collar of the boundary can be”) are easy to get in our situation thanks to the symmetries assumed and by means of the Laplacian Comparison Theorem.

Finally, We want to remark that with a little more work we can show that the subsequential convergence is actually uniform convergence. Also, one can show that the convergence of the curvature to a constant is exponential in any $L^p$ norm for $1 \leq p < \infty$.

2. **Short time existence**

From (1) follows that the solution $g$ is in the same conformal class of $g_0$. If we let $\tilde{g} = ug$, for $u > 0$, it is well known that

$$(4) \quad \tilde{R} = \frac{1}{u} (R - \Delta \log u)$$

Let $\frac{\partial}{\partial \eta}$ be the outward unit normal to $\partial M$ with respect to the initial metric $g_0$. Then the transformation law for the geodesic curvature is given by

$$(5) \quad k_{\tilde{g}} = \frac{1}{\sqrt{u}} \left(k_g + \frac{1}{2u} \frac{\partial u}{\partial \eta} \right).$$
Therefore, equation (1) is equivalent to the following Boundary Value Problem

\[
\begin{cases}
u_t = \Delta \log u - R_0 & \text{in } M \times (0, T) \\
u = 2u \left( \psi u_1^2 - k_g \right) & \text{on } \partial M \times (0, T) \\
u(x, 0) = 1.
\end{cases}
\]

From (6), by the theory of linear parabolic equations, an application of the Inverse Function Theorem, gives us the following

**Theorem 2.1** (Short time existence). There is an \( \epsilon > 0 \) such that equation (6) has a solution in \( C^{2, \gamma} (M \times (0, \epsilon)) \).

Also, we can assert the following fact concerning regularity

**Theorem 2.2.** If \( g_0 \), the initial metric, is smooth then the solution to (1) is smooth in \( M \times (0, \epsilon) \).

### 3. Evolution of the curvature

It is important to know how the curvature quantities are evolving if the metric is evolving by the Ricci flow. So, if \( g \) evolves by the Ricci flow in the time interval \( (0, T) \), and \( \frac{\partial}{\partial \nu} \) is the outward unit normal to the boundary with respect to \( g \) (the time varying family of metrics), we have

**Proposition 3.1.** The scalar curvature \( R \) satisfies the following evolution equation.

\[
\begin{cases}rac{\partial R}{\partial t} = \Delta R + R^2 & \text{in } M \times (0, T), \\\frac{\partial R}{\partial \nu} = k_g R - 2k'_g & \text{on } \partial M \times (0, T).
\end{cases}
\]

Here the ' represents differentiation with respect to \( t \).

**Proof.** A straightforward computation. \( \square \)

Since for problem (2) \( k'_g = 0 \), as a consequence of the Maximum Principle we obtain

**Proposition 3.2.** In problem (2), if \( R \geq 0 \) at \( t = 0 \), it remains so.

### 4. Blow Up

The main technique we will use to study the limiting behavior of the Ricci Flow is the analysis of the blow up limits. Therefore it is important to identify situations where the curvature blows up (in the case of the unnormalized flow). This is the content of the next proposition.

**Proposition 4.1.** Assume \( \chi(M) = 1 \) and \( R > 0 \) through the flow. Assume that \( k_g \geq 0 \) and \( k'_g \leq 0 \). Then the scalar curvature blows up in finite time.

**Proof.** Apply the Maximum Principle to the equation for the evolution of the curvature. \( \square \)

It can also be shown that

**Proposition 4.2.** Assume \( \chi(M) = 1 \) and that the geodesic curvature is nonpositive. Then the scalar curvature blows up in finite time.
**Proof.** Let \( A(t) \) be the area of the manifold at time \( t \). Then a computation shows that

\[
A'(t) = -\int_M R \, dV.
\]

This together with the fact \( \chi(M) = 1 \) shows that

\[
A'(t) \leq -\epsilon, \quad \text{for some } \epsilon > 0
\]

Hence, the curvature cannot remain bounded after time \( T = \frac{A(0)}{\epsilon} \) (Otherwise, by the derivative estimates of Section 6, we would be able to continue the solution past \( T \)).

\[\square\]

5. **Existence for all time**

As we have seen, the equation we are studying has a solution for a short time. In general, we do not expect this equation to have a solution for all time. In this section, under certain restrictions, we show longtime existence for the normalized flow (in the rotationally symmetric case). To prove our result, we follow the ideas in [H3]. Let us remark that the results in this section do not depend on derivative estimates on the curvature.

We start with a definition.

**Definition 5.1.** We define the potential function as the solution to the problem

\[
\begin{align*}
\Delta f &= R - r \\
\frac{\partial f}{\partial \nu} &= 0,
\end{align*}
\]

with mean value 0, where \( r \) is the average of the scalar curvature, i.e.,

\[
r = \frac{\int_M R}{A(M)},
\]

where \( A(M) \) represents the area of \( M \).

We compute the evolution equation satisfied by the potential function.

**Lemma 5.2.** The potential function satisfies an evolution equation

\[
\begin{align*}
\frac{\partial f}{\partial t} &= \Delta f + rf + \psi \quad \text{in } M \times (0, T) \\
\frac{\partial f}{\partial \nu} &= 0 \quad \text{on } \partial M \times (0, T),
\end{align*}
\]

where \( \psi \) satisfies

\[
\begin{align*}
\Delta \psi &= -r' \\
\frac{\partial \psi}{\partial \nu} &= -\frac{\partial R}{\partial \nu} = k (r - R) + 2k',
\end{align*}
\]

where \( ' \) denotes derivative with respect to time.

**Proof.** We compute

\[
(R - r) \Delta f + \Delta \left( \frac{\partial f}{\partial t} \right) = \frac{\partial}{\partial t} (\Delta f) = \frac{\partial}{\partial t} (R - r) = \Delta R + R (R - r) - r' = \Delta (\Delta f) + R \Delta f - r'.
\]

From this follows that

\[
\Delta \left( \frac{\partial f}{\partial t} - \Delta f - rf \right) = r'.
\]
On the other hand, if we write $\psi := \frac{\partial f}{\partial t} - \Delta f - rf$, we have,

$$\frac{\partial \psi}{\partial \nu} = -\frac{\partial R}{\partial \nu}.$$  

\[\square\]

Consider the function $h = \Delta f + |\nabla f|^2$.

Then we have that

**Lemma 5.3.** $h$ satisfies an evolution equation

$$\begin{cases}
\frac{\partial h}{\partial t} = \Delta h - 2 |M_{ij}|^2 + rh - r' - 2 \langle \nabla \psi, \nabla f \rangle \\
\frac{h}{t} = R - r \quad \text{on} \quad \partial M \times (0,T),
\end{cases}$$

where $M_{ij} = \nabla_i \nabla_j f - \frac{1}{2} \Delta f g_{ij}$.

**Proof.** We compute

$$\frac{\partial}{\partial t} |\nabla f|^2 = 2g^{ij} (\frac{\partial}{\partial t} g_{ij}) \partial_i f \partial_j f + 2g^{ij} \partial_i (\Delta f + rf + \psi) \partial_j f + (R - r) g^{ij} \partial_i f \partial_j f$$

$$= 2 \langle \nabla \Delta f, \nabla f \rangle + 2r |\nabla f|^2 + \langle \nabla \psi, \nabla f \rangle + (R - r) |\nabla f|^2.$$  

From the Weitzenböck formula we obtain

$$\frac{\partial}{\partial t} |\nabla f|^2 = \Delta |\nabla f|^2 - 2 |Hess f|^2 - R |\nabla f|^2$$

$$+ 2r |\nabla f|^2 + (\nabla \psi, \nabla f) + (R - r) |\nabla f|^2.$$  

To see that $h|_{\partial M} = R - r$ recall that $|\nabla f|^2 = \left|\frac{\partial f}{\partial \nu}\right|^2 = 0$  

\[\square\]

From the formula in the previous Lemma we can deduce the following inequality:

$$\frac{\partial h}{\partial t} \leq \Delta h + rh - r' + 2 |\nabla \psi| |\nabla f|$$

$$\leq \Delta h + rh - r' + 2 |\nabla \psi| (|\nabla f| + R - r + r)$$

(since $R > 0$)

$$= \Delta h + (r + 2 |\nabla \psi|) h - r' + 2\left(r + \frac{1}{2}\right) |\nabla \psi|.$$  

To proceed we will need the following fact.

**Lemma 5.4.** Let $(M_k, g_k)$ be a sequence of Riemannian manifolds with boundary with rotationally symmetric metrics. Let $d_k = \text{diam}_{g_k}(M_k)$ and assume that there is a constant $C > 0$ independent of $k$ such that $d_k < C$, and there exist $\epsilon > 0$ such that $k_{g_k} > -\epsilon$. Then if $l(\partial M) \to 0$ we must have $\text{Vol}(M_k) =: A_k \to 0$.

**Proof.** Denote by $r_k$ the radius of $M_k$. If $r_k \to 0$, we are done. If not, let $(M_k, g_k)$ be a sequence such that $\lim_{k \to \infty} r_k = \alpha$. The hypothesis imply that $\alpha < \infty$. Then, for any $0 < \tau < \alpha$, we have that

$$\frac{f_k'}{f_k} = k_{g_k},$$

and hence,

$$f_k(\alpha) = f_k(\tau) \exp \left( \int_\tau^\alpha k_{g_k} \, d\rho \right).$$
Notice that $k_g$ is decreasing, and then,

$$\begin{align*}
f_k(\alpha) &\geq f_k(\tau) \exp \left( \int_{\alpha}^{\tau} k_g(\rho) \, d\rho \right) \\
&\geq f_k(\tau) \exp \left( -\int_{\alpha}^{\tau} \epsilon \, d\rho \right) \\
&= f_k(\tau) \exp \left( - (\alpha - \tau) \epsilon \right) \\
&\geq f_k(\tau) \min \{1, \exp(-\epsilon \alpha)\}.
\end{align*}$$

Since $l(\partial M_k) \to 0$, $f_k(\alpha) \to 0$ also, and we have that $f_k(\tau) \to 0$ as $k \to \infty$. Therefore, for any $\eta > 0$ we can find a $\tau_0$ such that $f_k(\tau_0) \leq \eta$, and then choose $k$ large enough so that for any $\tau > \tau_0$ $f_k(\tau) < \eta$. This shows that,

$$\begin{align*}
Vol(M_k) &\leq 2\pi \int_0^{\alpha} f_k(\rho) \, d\rho \\
&\leq 2\pi \eta \alpha,
\end{align*}$$

which proves the lemma.

We are ready to prove now,

**Theorem 5.5.** Assume $R \geq 0$ at $t = 0$. Then solution to the normalized flow exists for all time.

**Proof.** The proof is going to be developed in two steps, and is by contradiction. Assume the curvature blows up at some time $T < \infty$.

**Step 1.** We have that

$$\int_0^T |\nabla \psi| \, dt < \infty$$

First we notice that the function $|\nabla \psi|^2$ is subharmonic. Indeed, by the Weitzenböck formula

$$\Delta \left( |\nabla \psi|^2 \right) = 2 |Hess \psi|^2 + 2 \langle \nabla (\Delta \psi), \nabla \psi \rangle + R |\nabla \psi|^2 \geq 0 \quad \text{(notice that } \Delta \psi \text{ is constant in space)},$$

so it attains its maximum at the boundary. Therefore,

$$|\nabla \psi| \leq |k_g(R - r)| + |2k'_g| \leq C |R - r| + C$$

(of course we must show that $k'_g$ is bounded, see Remark 5.6 at the end of this section).

Hence, if $|\nabla \psi|$ is not integrable in $[0, T]$, neither is $R|\partial M$. In such a case, since the conformal factor is given by

$$u(x,t) = \exp \left[ \int_0^t r(\tau) - R(x,\tau) \, d\tau \right],$$

$l(\partial M) \to 0$ as $t \to T$. But $R > 0$ and $k_g \geq 0$, and hence by Lemma 5.4 we must have $A(M) \to 0$, which contradicts the fact that the normalized flow keeps the volume constant.

**Step 2.** If the normalized flow does not exist for all time, we have that for the unnormalized flow the curvature has maximum blow up rate in the boundary.

As we already showed, the following inequality holds

$$\frac{\partial h}{\partial t} \leq \Delta h + (r + 2 |\nabla \psi|) h - r' + 2 \left( r + \frac{1}{4} \right) |\nabla \psi|.$$
Let \( c(t) = \int_0^t (r + 2 |\nabla \psi|) \, dt \), and define
\[
w = \exp(-c(t)) \, h,
\]
then
\[
\frac{\partial w}{\partial t} \leq \Delta w - \exp(-c(t)) \left[ r' + 2 |\nabla \psi| \left( r + \frac{1}{4} \right) \right].
\]
Using the fact that \( c(t) \) is bounded, the inequalities
\[
|r'| \leq \int_{\partial M} |\frac{\partial R}{\partial 
abla \psi}| \, d\sigma \leq C |k_g| |R - r|_{\partial M} + 2 |k'_g|,
\]
and
\[
|\nabla \psi| \leq C |R - r|_{\partial M} + C.
\]
The Maximum Principle shows
\[
h(t) \leq C \hat{R}_{max}(t) + C,
\]
where
\[
\hat{R}_{max}(t) = \max_{(x, \tau) \in \partial M \times [0, t]} R(x, \tau).
\]
This proves the claim since \( h(t) \geq R(t, x) - r(t) \).

The last claim shows that \( \int_0^T R_{max}(\tau) \, d\tau < \infty \), where \([0, T]\) is the maximum interval of existence for the Ricci Flow. Again, we use the fact that the conformal factor \( u \) satisfies the identity,
\[
u(x,t) = \exp \left[ \int_0^t r(\tau) - R(x, \tau) \, d\tau \right].
\]
It follows that there is a constant \( \delta > 0 \) such that \( u \geq \delta \), and hence we can continue the solution past \( T \) (since \( u \) is bounded, by Theorem 1.3 in chapter III of [DB], \( u \) is Hölder continuous. The rest follows from a standard bootstrapping argument).

\[\Box\]

**Remark 5.6.** Assume that the normalized flow becomes singular at time \( T_0 < \infty \) we will show that \( k'_g \) remains bounded. To see this, consider the unnormalized flow. Then \( T_0 \) corresponds to the blow up time \( T < \infty \) for the unnormalized flow. If \( A(t) \) represents the area of the surface at time \( t \), we cannot have \( A(T) = 0 \) (because, by using Gauss-Bonnet and the fact that we keep \( k_g = k_0 \) constant for the unnormalized flow, it can be shown that \( A(t) \leq C(T - t) \), and this would imply that the normalized flow exists for all time). Therefore, there is \( \epsilon > 0 \) such that \( A(t) \geq \epsilon \) for \( 0 \leq t < T \).

On the other hand, if \( t \) and \( \hat{t} \) represent the time parameter for the normalized and unnormalized flow respectively, and \( \phi = 1/A \), we have
\[
\frac{d}{dt} k_g = \frac{d}{dt} \left( \frac{k_0}{\sqrt{\phi(t)}} \right) \frac{d\hat{t}}{dt} = -\frac{k_0}{2\phi^2} \phi' \frac{d\hat{t}}{dt}.
\]
Hence all we must show is that \( \phi' = \left( \frac{1}{4} \right)' \) is bounded. But this is rather easy to see, since
\[
A'(t) = -\int_M R \, dV = -2\pi + \int_{\partial M} k_0 \, d\sigma,
\]
which is clearly bounded.
Let us observe that the long time existence result conveys some interesting information, namely

**Proposition 5.7.** For the normalized flow, \( k_g \to 0 \) exponentially.

### 6. Derivative estimates

In this section, and for the rest of this work, we denote by \( \frac{\partial}{\partial \nu} \) the outward unit normal to the boundary with respect to the evolving metric \( g \), which hence is different at different times (since the conformal class is preserved, this change is only by a rescaling). The purpose is to show that for solutions of (2) a bound on the curvature produces bounds on its derivatives.

#### 6.1. First Order Estimates

The following Theorem and its proof are along the lines of Theorem 7.1 in [H5].

**Theorem 6.1.** There exist constants \( C_1 \) for \( R \geq 1 \) such that if the curvature is bounded \( |R| \leq M \), up to time \( t \) with \( 0 < t \leq \frac{1}{M} \) then the covariant derivative of the curvature is bounded

\[ |DRm| \leq C_1m^{\frac{1}{2}}. \]

**Proof.** First we prove this estimate for the normal derivative. At the boundary we have the estimate

\[ \partial R = k_g R. \]

Define the quantity

\[ F = t|\frac{\partial R}{\partial \nu}|^2 + AR^2, \] where \( A \) is a constant to be chosen. In the interior we have

\[
\frac{\partial F}{\partial t} = (R \nu)^2 + 2t(R \nu)(R \nu + (R \nu)_t) + 2AR \left[ \Delta R + R^2 \right] \\
= (R \nu)^2 + 2t(R \nu) \left[ \Delta R \nu + 2RR \nu + RR \nu \right] + 2AR \left[ \Delta R + R^2 \right] \\
= \Delta \left( t(R \nu)^2 + AR^2 \right) - 2 \left( t|\nabla R \nu|^2 + A|\nabla R|^2 + C_t R(R \nu)^2 + 2AR^3 \right) \\
\leq \Delta F + (CtR - 2(A - 1)) |\nabla R|^2 + 2AR^3.
\]

By choosing \( A \) big enough we get the inequality

\[ \frac{\partial F}{\partial t} \leq \Delta F + 2AR^3. \]

At the boundary we have,

\[ F = tk_g^2 R^2 + AR^2 \leq \tilde{C}tM^3 + \tilde{C}M^2, \]

for some constant \( \tilde{C} \). Then the maximum principle, as long as \( tM \leq 1 \), yields

\[ t|\frac{\partial R}{\partial \nu}|^2 \leq F \leq CM^2, \]

for some constant \( C_1 \).
6.2. Second Order Estimates. We show how to estimate $\frac{\partial^2 R}{\partial \nu^2}$. Higher derivatives follow a similar procedure.

Let

$$\rho(P,t) = d_t(P, \partial M)$$

be the distance from $P$ to the boundary at time $t$, and define

$$F = \exp(k_0 \rho) R$$

where $k_0$ is the geodesic curvature of $\partial M$.

(we want to remark that if $k_0 = 0$ all these preparations become unnecessary).

A simple computation shows that $F_\nu = 0$ in the boundary. $F$ satisfies the equation

$$F_t = \Delta F - 2 \left( \frac{\nabla \exp(k_0 \rho)}{\exp(k_0 \rho)} \right) \cdot \nabla F \quad \text{where } \rho = d_t(P, \partial M)$$

where $\rho'$ denotes differentiation with respect to $t$. To simplify, we rewrite this last expression as

$$\begin{cases} 
F_t = \Delta F + B \cdot \nabla F + CF + EF^2 \\
F_\nu = 0,
\end{cases}$$

where $B, C, E$ have the obvious meanings. Differentiating with respect to $\nu$ and writing $w = F_\nu$ we get,

$$Lw \equiv w_t - \Delta w - B \cdot \nabla w = RF_\nu + B_\nu \cdot \nabla F - C_\nu F + E_\nu F^2 - (C + EF)w.$$

Write

$$G = RF_\nu + B_\nu \cdot \nabla F - C_\nu F + E_\nu F^2 - (C + EF)w,$$

and the boundary condition is $w = 0$.

To estimate derivatives of $w$, we want to use the theory of Fundamental Solutions for linear parabolic equations. In order to do so, we are going to compute the coefficients of the operator $L$ in Fermi coordinates.

Pick a point $P \in \partial M$. Assume that $P$ admits Fermi coordinates $(F_{e,\delta}, \phi)$ in a neighborhood

$$\mathcal{F}_{e,\delta} = \{(x, s) \in \mathbb{R}^2 : -\delta < x < \delta \quad 0 \leq s < \epsilon\}$$

i.e., the $s$-coordinate represents the distance to the boundary, and $x$ is the coordinate on the curves “parallel” to the boundary. By $k(\sigma)$ we will denote the geodesic curvature of

$$\partial M_\sigma = \{P \in M : \pi_2 (\phi^{-1}(P)) = \sigma\},$$

where $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the second coordinate.

In Fermi coordinates the metric can be described explicitly in terms of the relevant geometric quantities. This is the content of the next proposition.

**Proposition 6.2.** In Fermi coordinates,

1. $g = ds^2 + \exp \left(-2 \int_0^s k(\sigma) \, d\sigma \right) dx^2$;
2. $\frac{\partial s}{\partial x} = k^2 + \frac{1}{2}$;
3. $\sqrt{g} = \exp \left(-\int_0^s k(\sigma) \, d\sigma \right)$;

where $s$ is the distance to the boundary.
Proof. Let us prove (1). Just notice that,

\[ \frac{\partial}{\partial s} g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = 2 g \left( \nabla \frac{\partial}{\partial s}, \frac{\partial}{\partial x} \right) = -2 k(s) g \left( \nabla \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right). \]

The result follows by integration and recalling that at \( s = 0 \) \( g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = 1 \). We go for (2). On one hand we have,

\[ k(s) g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = -g \left( \nabla \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right). \]

Now we differentiate the previous equation with respect to \( s \). First the left hand side,

\[ \frac{\partial}{\partial s} \left[ k(s) g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) \right] = k'(s) g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) + 2 k(s) g \left( \nabla \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = k'(s) g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) - 2 [k(s)]^2 g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right). \]

For the right hand side we have

\[ \frac{\partial}{\partial s} g \left( \nabla \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = g \left( \nabla \frac{\partial}{\partial s}, \frac{\partial}{\partial x} \right) + g \left( \nabla \frac{\partial}{\partial x}, \nabla \frac{\partial}{\partial s} \right) = -\frac{R}{2} g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) + [k(s)]^2 g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right). \]

Combining the two previous computations give the result. (3) follows from (1). \( \square \)

As we said before, we are going to study the operator \( L \) in Fermi coordinates. First we study the principal part which corresponds to the Laplacian. Recall that the Laplacian in local coordinates is given by

\[ \Delta g = \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \right). \]

Choose Fermi coordinates at time \( t = 0 \). In this coordinates the Laplacian is given by

\[ \Delta = \frac{\partial^2}{\partial^2 s} + \exp \left( 2 \int_0^s k(\sigma) \, d\sigma \right) \frac{\partial^2}{\partial^2 x} + \exp \left( \int_0^s k(\sigma) \, d\sigma \right) \frac{\partial}{\partial s} \left[ \exp \left( -\int_0^s k(\sigma) \, d\sigma \right) \right] \frac{\partial}{\partial s}. \]

Hence, since \( \Delta_g(t) = \exp \left( \int_0^t R(s, \tau) \, d\tau \right) \Delta \) (that the conformal factor is given by \( u = \exp \left( -\int_0^t R(s, \tau) \, d\tau \right) \)) and in dimension 2, if \( h = \lambda g \) then \( \Delta_h = \lambda^{-1} \Delta_g \), the functions whose Hölder norms we have to estimate are

(1) \( \alpha(s, t) = \exp \left( \int_0^t R(s, \tau) \, d\tau \right) \).
(2) \( \beta(s, t) = \exp \left( \int_0^t R(s, \tau) \, d\tau \right) \exp \left( 2 \int_0^s k(\sigma) \, d\sigma \right). \)
(3) \( \gamma(s, t) = k(s) \exp \left( \int_0^t R(s, \tau) \, d\tau \right). \)

We will check that these functions are differentiable and that their derivatives can be bounded in terms of bounds on the curvature and its first derivative (which we already know how to bound). We do it for (3), since it is the same for the other functions. Notice that by the rotational symmetry of the metric, which is
preserved by the flow, the \( \frac{\partial}{\partial x} \) derivatives are 0. So we only have to compute the \( t \) and \( s \)-derivatives. Let us compute first the \( s \)-derivatives:

\[
\gamma_s = k'(s) \exp \left( \int_0^t R(s, \tau) \, d\tau \right) + k(s) \left( \int_0^t R_s(s, \tau) \, d\tau \right) \exp \left( \int_0^t R(s, \tau) \, d\tau \right).
\]

By taking into account (2) from proposition [6.2], we have an estimate on \( \gamma_s \) in terms of the curvature, its first derivative, and the geodesic curvature. For the time derivative we have,

\[
\gamma_t = R(s, t) k(s) \exp \left( \int_0^t R(s, \tau) \, d\tau \right),
\]

and hence the same conclusion.

Now we work with the lower order terms. We have the following list of explicit formulae

**Proposition 6.3.** The following formulae hold,

1. \( |\nabla_{g(t)} \exp(k\rho)|^2 = k^2 \exp(2k\rho) \),
2. \( \Delta_{g(t)} \exp(k\rho) = \exp \left( \int_0^t R(\sigma, \tau) \, d\tau \right) \left[ k_0^2 \exp(k_0\rho) + k_0 k(s) \exp(k_0\rho) \right] \),
3. \( \rho(s, t) = \int_0^t \exp \left( -\int_0^\sigma R(\sigma, \tau) \, d\tau \right) \, d\sigma \).

With the aid of the previous proposition, we can compute the terms \( B, C, E \), and show that they satisfy a Lipschitz condition, and that their Lipschitz constants depend only on bounds on the curvature. Also, it can be easily checked that \( B, C, E \) are bounded continuous functions and their bounds depend only on bounds on the curvature and its first derivative which we can bound in terms of the curvature.

To take our next step we need the following proposition.

**Proposition 6.4.** In the strip \([-\delta, \delta] \times \left[ \frac{3}{4} \epsilon, \epsilon \right] \) we have uniform bounds on all the covariant derivatives of the curvature, and these bounds depend only on \( \epsilon, t \) and a bound on the curvature.

**Proof.** The proof follows from the local interior estimates (See Theorem 13.1 in [H5]).

Fix a function \( \psi : \mathbb{R}_+ \rightarrow [0, 1] \), such that

\[
\psi(u) = \begin{cases} 
1 & \text{if } u \in [0, 1] \\
0 & \text{if } u \in [2, \infty] 
\end{cases}.
\]

Define,

\[
\chi(s) = \psi \left( \frac{8(\epsilon - s)}{\epsilon} \right),
\]

and let

\[
v = w - \chi w.
\]

A simple computation shows that

\[
v_t - \Delta v - B \cdot \nabla v = \mathcal{G} - \chi \mathcal{G} + w \Delta \chi + 2 \nabla \chi \cdot \nabla w + Bw \nabla \chi - Gv.
\]
From now on we will use the following notation

\[ W = \mathcal{G} - \chi \mathcal{G} + w \Delta \chi + 2 \nabla \chi \cdot \nabla w + B w \nabla \chi - G v. \]

Since for \( s \leq \frac{\epsilon}{4} \) we have \( \chi \equiv 0 \), the left hand side of the previous equation (by Proposition 6.3) is bounded in terms of bounds on the curvature and \( \epsilon \). This shows that the left hand side is a bounded continuous function on the segment \( \{ 0 \leq s < \epsilon \} \).

Pick any \( \theta > 0 \), and set \( f = v|_{t=\theta} \). Let \( \Gamma(s; t; \zeta, \tau) \) be the Green’s function of the operator \( L \) in the layer \( \{ 0 < s < \epsilon \} \times [\theta, T] \). Since \( v = 0 \) at \( s = 0 \) and \( s = \epsilon \), we can represent \( v \) as follows

\[
(7) \quad v(s, t) = \int_{\theta}^{t} \int_{0}^{\epsilon} \Gamma(s; \zeta; t, \tau) W(\zeta, \tau) d\zeta d\tau + \int_{0}^{\epsilon} \Gamma(s; \zeta; t, \theta) f(\zeta) d\zeta.
\]

It is well known that we can differentiate \( v \) with respect to \( s \), and the derivative is given by direct differentiation under the integral sign. That is to say,

\[
\frac{\partial v}{\partial s} = \int_{\theta}^{t} \int_{0}^{\epsilon} \frac{\partial}{\partial s} \Gamma(s; \zeta; t, \tau) W(\zeta, \tau) d\zeta d\tau + \int_{0}^{\epsilon} \frac{\partial}{\partial s} \Gamma(s; \zeta; t, \theta) f(\zeta) d\zeta.
\]

We are ready to produce bounds on \( \frac{\partial v}{\partial s} \). First of all, it is well known that

\[
\Gamma(s; t; \xi, \tau) \leq C(t - \tau)^{-\frac{1}{2}} \exp \left[ -C \frac{|s - \xi|^2}{t - \tau} \right]
\]

and

\[
\frac{\partial}{\partial s} \Gamma(s; t; \xi, \tau) \leq C(t - \tau)^{-\frac{3}{2}} \exp \left[ -C \frac{|s - \xi|^2}{t - \tau} \right],
\]

where \( C \) depends only on the Lipschitz bounds on the coefficients of \( L \). Therefore, if

\[
|f(\xi)| \leq M_0 \quad \text{and} \quad |W(\xi, \tau)| \leq M_1,
\]

writing

\[
\int_{\theta}^{t} \int_{0}^{\epsilon} C(t - \tau)^{-\frac{1}{2}} \exp \left[ -C \frac{|s - \xi|^2}{t - \tau} \right] d\zeta d\tau = K_1
\]

and

\[
\int_{0}^{\epsilon} C(t - \tau)^{-\frac{3}{2}} \exp \left[ -C \frac{|s - \xi|^2}{t - \tau} \right] d\xi = K_0,
\]

we find that

\[
\left| \frac{\partial v}{\partial s} \right| \leq K_1 M_1 + \frac{M_0 K_0}{(t - \theta)^{\frac{1}{2}}}.
\]

Finally, notice that \( M_0, M_1, K_0, K_1 \) depend only on bounds on the curvature and its first derivative. Since

\[
\frac{\partial}{\partial s} (x, t) = \frac{\partial^2 w}{\partial s \partial w}(x, t) \quad \text{recalling that} \quad \frac{\partial}{\partial w} = g \left( \frac{\partial}{\partial s} - \frac{\partial}{\partial x} \right)^{-1} \frac{\partial}{\partial s}
\]

\[
= \exp \left( -\frac{1}{2} \int_{t=0}^{t} R(x, \tau) d\tau \right) \frac{\partial^2 w}{\partial s \partial w}(x, t)
\]

We have proved the following

**Theorem 6.5.** Second partial derivatives of \( R \) (in a Fermi coordinate neighborhood) can be bounded up to the boundary, and the bounds only depend on bounds on \( R \), the geodesic curvature of the boundary, the time \( t > 0 \) up to where we are bounding, and \( \epsilon > 0 \), the “size” of the Fermi coordinate neighborhood at \( t = 0 \).
7. Subsequential Convergence: Some Families of Examples

Our purpose now is to show a family of examples where the curvature is uniformized by the (unnormalized) flow in the sense that if \([0,T)\) is the maximum interval of existence of the solution for the unnormalized flow, then along a sequence of times \(t_k \to T\)

\[
\lim_{k \to \infty} \frac{R_{\text{max}}(t_k)}{R_{\text{min}}(t_k)} = 1,
\]

where \(R_{\text{max}}(t) = \max_{x \in M} R(x, t)\), and \(R_{\text{min}}(t)\) is defined analogously.

To accomplish our goal we will use blow up analysis. First we describe how to form a blow up limit. Let \((0,T)\) be the maximum time of existence of the Ricci flow. Then we can find a sequence of times \(t_i \to T\) and points \(p_i \in M\) such that

\[
\lambda_i := R(p_i, t_i) = \max_{M \times [0,t_i]} R(x,t),
\]

and we have \(\lambda_i \to \infty\). Then we define the dilations

\[
g_i(t) := \lambda_i g \left( t_i + \frac{t - \lambda_i t_i}{\lambda_i} \right) - \lambda_i t_i \leq t < \lambda_i (T - t_i).
\]

Then, using the derivative estimates (plus injectivity radius and Fermi inradius assumptions, easily seen to be fulfilled in the rotationally symmetric case via the Laplacian Comparison Theorem), we can find a subsequence of times, which we denote again by \(t_i \to T\), such that \((M, g_i(t), p_i)\) converges smoothly to a solution to the Ricci flow \((M_\infty, g_\infty(t), p_\infty)\).

The following theorem is the key to our approach.

**Theorem 7.1** (See Theorems 26.1 and 26.3 in [H5]). There are two possible blow up limits for solutions of \(E\) with \(R > 0\). If the blow up limit is compact, then it is the round hemisphere \(S^2_+\) with totally geodesic boundary. If the blow up limit is noncompact, then it (or its double) is isometric to the cigar metric in \(\mathbb{R}^2\).

Recall that the cigar metric in \(\mathbb{R}^2\) is given by

\[
ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}
\]

**Proof.** The main observation here is that the geodesic curvature of a rescaled metric \(\lambda^2 g\) is given by

\[
k_{\lambda^2 g} = \frac{k_g}{\lambda}.
\]

Therefore the blow up limits (as long as the geodesic curvature is kept uniformly bounded throughout the flow) have totally geodesic boundary. Once this has been established, it is not difficult to show (by following the corresponding arguments in the boundaryless case) that the conclusions of the Theorem hold.

\[\square\]

Having Theorem 7.1 in our toolbox, one way to proceed is as follows: we study a scaling invariant geometric property of the initial condition that is preserved by the Ricci flow, and show that this property does not hold for the cigar. To see how this argument works, let us show an example.

Consider a rotationally symmetric metric

\[
ds^2 = dr^2 + f(r)^2 d\omega
\]
on $B^2$ that satisfies the following properties:

(P1) $R > 0$ and $R$ is radially decreasing;
(P2) $\frac{\partial R}{\partial r} = k \rho R$ in $\partial B^2$.

For this type of metrics we are going to show that the Ricci flow uniformizes curvature in the sense described above.

Consider the quantity $\inf \frac{L^2}{A}$ over balls centered at the pole in a rotationally symmetric manifold, where by $A$ we denote the area of such ball and by $L$ the length of its boundary. Then we have

**Lemma 7.2.** For $R \geq 0$ we have that

$$\inf \frac{L^2}{A} = \frac{L (\partial M)^2}{A (M)}.$$ 

**Proof.** Since

$$\frac{L^2 (\partial B_\rho)}{A (B_\rho)} = \frac{4\pi^2 f^2 (\rho)}{2\pi \int_0^\rho f (r) \, dr},$$

all we must show is that for $\rho > 0$ the expression

$$f (\rho) f' (\rho) = f' (\rho)$$

is nonincreasing. But this is not difficult to see since $f'' = -K f$, where $K$ is the Gaussian curvature of $M$. 

Condition (P2) guarantees that the solution to (6) is at least $C^3$, and hence we can use the Maximum Principle to show the following,

**Proposition 7.3.** For metrics satisfying properties (P1) and (P2), under the Ricci flow the scalar curvature $R$ remains radially nonincreasing.

**Proof.** The solution metric can be written as

$$ds^2 = h (r, t)^2 \, dr^2 + f (r, t)^2 d\omega^2.$$ 

Let $\frac{\partial}{\partial \rho}$ be the unit vector in the direction of $\frac{\partial}{\partial r}$, i.e., $\frac{\partial}{\partial \rho} = \frac{1}{h} \frac{\partial}{\partial r}$. Define $w = f R_\rho$.

Then, from the evolution equation for the curvature, we have for $\rho > 0$

$$R_t = \frac{1}{f} w_\rho + R^2.$$ 

Differentiate with respect to $\rho$ to get,

$$R_{t \rho} = -\frac{f_\rho}{f^2} w_\rho + w_{\rho \rho} + 2 RR_\rho.$$ 

On the other hand

$$R_{\rho t} = -\frac{1}{2} RR_\rho + R_{t \rho},$$

which in turn implies,

$$f \left( \frac{1}{2} RR_\rho + R_{\rho t} \right) = -\frac{f_\rho}{f} w_\rho + w_{\rho \rho} + 2 Rw.$$ 

Also,

$$(f R_\rho)_\rho = f' R_\rho + f R_{\rho t}$$
From the Ricci flow equation follows that,

\[ f' = -\frac{1}{2}Rf. \]

This yields,

\[ fR_{\rho t} = (fR_{\rho})_t + \frac{1}{2}RfR_{\rho}, \]

from where we obtain,

\[ \frac{1}{2}Rw + (fR_{\rho})_t + \frac{1}{2}Rw = -\frac{f_{\rho}}{f}w_{\rho} + w_{\rho\rho} + 2Rw. \]

This last equation simplifies to

\[ w_t = w_{\rho\rho} - \frac{f_{\rho}}{f}w_{\rho} + Rw, \]

and the boundary conditions are

\[
\begin{align*}
    w &= 0 \quad \text{at} \quad \rho = 0 \\
    w &< 0 \quad \text{at} \quad \rho = \text{radius of the surface} \\
    w|_{t=0} &\leq 0.
\end{align*}
\]

The result then follows from the Maximum Principle.

Consider the normalized flow now. From the previous proposition follows that

\[ r \geq R_{\min}(t) = R|_{\partial M \times \{t\}}, \]

and hence,

\[ L_t(\partial M) = \exp \left[ \int_0^t (r - R) \, d\tau \right] L_0(\partial M) \geq L_0(\partial M). \]

This last fact, together with Lemma 7.2, shows that

\[ \inf \frac{L^2}{A} = L_t(\partial M) \geq L_0(M), \]

i.e., the isoperimetric ratio remains bounded away from 0 throughout the flow, and for the cigar we have that this isoperimetric ratio is equal to 0. This shows that we are in alternative (A) of Theorem 7.1, and hence the flow is uniformizing the curvature along a subsequence of times.

7.1. A family of examples. Now we want to construct explicitly a family of metrics satisfying conditions (P1) and (P2). Consider the family of metrics

\[ ds^2 = dr^2 + f_\epsilon(r)^2 d\omega^2, \]

where

\[ f_\epsilon(r) = (1 - \epsilon) \sin r + \epsilon r. \]

Then, an elementary calculation yields

\[ f' = (1 - \epsilon) \cos r + \epsilon \]

\[ f'' = - (1 - \epsilon) \sin r. \]

The Gaussian curvature is given by

\[ -\frac{f''}{f} = \frac{(1 - \epsilon) \sin r}{(1 - \epsilon) \sin r + \epsilon r}. \]
which can be rewritten as

\[ K = \frac{1}{1 + \left( \frac{\epsilon}{1 - \epsilon} \right) \frac{\sin r}{r}}. \]

This expression shows that \( K \) is decreasing in \( r \) (in fact, \( \frac{\sin r}{r} \) is increasing). On the other hand, the expression for the mean curvature of the geodesic spheres is given by

\[ f' = \frac{(1 - \epsilon) \cos r + \epsilon}{(1 - \epsilon) \sin r + \epsilon r}. \]

The equation we want to be satisfied for certain value of \( r \) is

\[ f'' = 2 \frac{f'' f'}{f}, \]

that is to say,

\[ (1 - \epsilon) \cos r = \frac{2 (1 - \epsilon) \sin r [(1 - \epsilon) \cos r + \epsilon]}{(1 - \epsilon) \sin r + \epsilon r}, \]

which can be manipulated to become,

\[ \epsilon \cos r + (1 - \epsilon) \cos r \sin r = 2 (1 - \epsilon) \cos r \sin r + 2 \epsilon \sin r, \]

and finally simplifies to

\[ (10) \quad \epsilon \cos r - 2 \epsilon \sin r - (1 - \epsilon) \cos r \sin r = 0. \]

This is the equation to be solved. First of all notice that the expression in the left hand side is always negative at \( \frac{\pi}{2} \). At \( r = \frac{3\pi}{4} \), the left hand side evaluates to

\[ \epsilon \left( 3\sqrt{2\pi} + 1 - \sqrt{2} + \frac{1}{2} \right), \]

which is positive for any \( \epsilon > 0 \). Therefore, there is \( r_0 \in \left( \frac{\pi}{2}, \frac{3\pi}{4} \right) \) solving equation (10). Notice that \( f'(r_0) < 0 \): this follows readily from the fact that \( \frac{\partial R}{\partial r} < 0 \), \( R > 0 \) and \( \frac{\partial R}{\partial r} = kR \).

Let us remark that we can also construct metrics satisfying

(P1’) \( R > 0 \) and \( R \) is radially increasing ;

(P2) \( \frac{\partial R}{\partial r} = k_0 R \) in \( \partial \mathcal{B}^2 \).

For these metrics we can also show subsequential convergence of the Ricci flow in a similar way. However, for the case \( k_0 \geq 0 \), we prove a more general result in the following section.

8. Subsequential Convergence: the case of convex boundary \( (k_0 \geq 0) \).

As we have seen, blow up techniques can be used in certain specific examples to show “convergence” to a metric of constant curvature. When we try to prove the general case, we always stumble against the same obstacle: we need to show that among all the possibilities, the only possible blow up limit is \( S^2 \). Since we already know that the only other possibility is the soliton metric in \( \mathbb{R}^2 \) (by Theorem 7.1), all we have to do is rule out this case, and here is where Perelman’s work (see \([P1]\)) comes handy. We will follow his ideas and will apply them in a less general framework.

Following Perelman, we define the following functional,

\[ (11) \quad \mathcal{F}(g, f) = \int_M \left( R + |\nabla f|^2 \right) \exp(-f) \, dV - 2 \int_{\partial M} k_0 \exp(-f) \, d\sigma, \]
and compute its first variation.

**Proposition 8.1.** Let $\delta g_{ij} = v_{ij}, \delta f = h, g^{ij} v_{ij} = v$. Then we have,

\[
\delta F = \int_M \exp(-f) \left[ -v_{ij} (R_{ij} + \nabla_i \nabla_j f) + \left( \frac{g_{ij}}{2} - h \right) \left( 2\Delta f - |\nabla f|^2 + R \right) \right] \]

\[
- \int_{\partial M} \left( \frac{\partial v}{\partial \nu} + v \frac{\partial f}{\partial \nu} \right) \exp(-f) \, d\sigma - 2 \int_{\partial M} k_0 (v_\partial/2 - h) \exp(-f) \, d\sigma + \int_{\partial M} \nabla_i v_{ij} \nu^j \, d\sigma.
\]

Here, $\frac{\partial }{\partial \nu} = \{\nu^i\}$ is the outward unit normal to $\partial M$ with respect to $g$ (notice that $\frac{\partial }{\partial \nu}$ depends on time), $\nabla$ represents covariant differentiation with respect to the metric $g$, and $v_\partial/2$ represents the variation of the volume element of $\partial M$ induced by $v_{ij}$.

**Proof.** Equation (12) follows from the following formulas

\[
\delta R = -\Delta v + \nabla_i \nabla_j v_{ij} - R_{ij} v_{ij}
\]

\[
\delta |\nabla f|^2 = -v^{ij} \nabla_i f \nabla_j f + \langle \nabla f, \nabla h \rangle
\]

\[
\delta (\exp(-f) \, dV) = \left( \frac{g_{ij}}{2} - h \right) \exp(-f) \, dV,
\]

and integration by parts ($\Delta$ is the Laplacian operator of the metric).

Consider the evolution equations

\[
\begin{aligned}
(g_{ij})_t &= -2 (R_{ij} + \nabla_i \nabla_j f) \quad \text{in} \quad M \times (0, T) \\
k_\eta &= k_0 \quad \text{on} \quad \partial M \times (0, T) \\
f_t &= -R - \Delta f \quad \text{in} \quad M \times (0, T) \\
\frac{\partial f}{\partial \eta} &= 0 \quad \text{on} \quad \partial M \times (0, T).
\end{aligned}
\]

Let $\frac{\partial }{\partial \eta}$ be the outward normal vector to $\partial M$ with respect to the initial metric. We will make the following assumptions:

$\frac{\partial }{\partial \eta}$ remains normal to $\partial M$ through the flow, and $R_{ij} = 0$ for $\partial_i \in T \partial M$.

Notice that this is always true in dimension 2 (and also for weakly umbilic boundaries, but we do not know if this monotonicity formula is valid or will be useful in such context). Then we have the following computation.

**Proposition 8.2.**

\[
F_t = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 \exp(-f) \, dV.
\]

**Proof.** First of all notice that under the assumption $\nabla \nabla f = 0$ we have that $\nabla \Delta f = \Delta \nabla f$ at the boundary. Indeed, using all our previous assumptions,

\[
f_{ii\nu} = f_{i\nu i} + R_{i\nu} f_i = f_{ii\nu}.
\]

In order to show (13) we will use the following formula. In Fermi coordinates (here $s$ represents the distance to the boundary)

\[
\Delta f = \frac{\partial^2 f}{\partial s^2} + \Delta_s f - k \frac{\partial f}{\partial s},
\]

where $k(s)$ is the mean curvature of the hypersurface at distance $s$ from the boundary, and $\Delta_s$ represents the Laplacian of the induced metrics.
Since we have \( f_t = -\Delta f - R \) and \( \nabla_\nu f = 0 \) at the boundary, we have
\[
0 = -\nabla_\nu R - \nabla_\nu \Delta f.
\]
In Fermi coordinates, \( \nabla^k_\nu = (-1)^k \frac{\partial^k}{\partial s^k} \), and hence
\[
\nabla_\nu \Delta f = -\frac{\partial^3 f}{\partial s^3} + \frac{\partial k}{\partial s} \frac{\partial f}{\partial s} + k \frac{\partial^2 f}{\partial s^2}.
\]
From the previous equations we obtain
\[
\nabla^3_\nu f = -\nabla_\nu R - \frac{\partial k}{\partial s} \frac{\partial f}{\partial s} - k \nabla^2_\nu f.
\]
On the other hand we have,
\[
(\nabla_\nu)^2 = -R - \nabla_\nu f + \Delta f + R = \nabla^2_\nu f + R.
\]
This shows that the term
\[
-2 \int_{\partial M} k_0 \left( \frac{v_0}{2} - h \right) \exp(-f) \, d\sigma
\]
cancels the term
\[
\int_{\partial M} \exp(-f) \nabla_i v_{ij} \nu^j \, d\sigma.
\]
Since \( v = -R - \Delta f = f_t \) and \( \nabla_\nu f = 0 \) at the boundary, it follows that \( \frac{\partial v}{\partial M} = 0 \).
It is not difficult to show that
\[
\int_{\partial M} \nabla_j \exp(-f) v_{ij} \nu^j \, d\sigma = 0.
\]
Putting all this information together we get our monotonicity formula.

We will need the following more sophisticated version of \( F \),
\[
W(g, f, \tau) = \int_M \left[ \tau \left( \left| \nabla f \right|^2 + R \right) + f - 2 \right] (4\pi \tau)^{-1} \exp(-f) \, dV
-2 \int_{\partial M} k_0 \exp(-f) (4\pi)^{-1} \, d\sigma
\]
restricted to \( f \) satisfying
\[
\int_M (4\pi \tau)^{-\frac{1}{2}} \exp(-f) \, dV = 1.
\]
As pointed out by Kleiner and Lott, by making the substitution \( \Phi = \exp\left( -\frac{f}{2} \right) \), we get the functional
\[
W(g, \Phi, \tau) = (4\pi \tau)^{-1} \int_M \left[ 4\pi \left| \nabla \Phi \right|^2 + (\tau R - 2 \log \Phi - 2) \Phi^2 \right] dV
-\int_{\partial M} k_0 \Phi^2 (4\pi)^{-1} \, d\sigma,
\]
which has been extensively studied by Rothaus (See [R1], [R2]) in a domain with boundary \( \Omega \) under the further restriction that the infimum is taken over smooth...
functions vanishing in the boundary. Using his methods is not difficult to show that for our modified functional there is a minimizer.

Just as before, given a time dependent family of metrics evolving by the Ricci flow, i.e., satisfying
\[ \begin{align*} \frac{\partial g_t}{\partial t} &= -R_{g_t} \quad \text{in} \quad M \times (0, T) \\ k_g &= k_0 \quad \text{on} \quad \partial M \times (0, T) \end{align*} \]
and \( f(\cdot, t) \) satisfying the backward heat equation
\[ f_t = -\Delta f + |\nabla f|^2 - R + \frac{1}{\tau}, \quad \tau_t = -1, \]
plus the boundary condition
\[ \frac{\partial f}{\partial \nu} = 0. \]

We have the following monotonicity formula. Its proof is an adaptation of the calculations in the proof of Proposition 8.2.

**Theorem 3.**
\[ \frac{dW}{dt} = \int_M 2\tau |R_{ij} + \nabla_i \nabla_j f - \frac{1}{4\tau} g_{ij}|^2 (4\pi \tau)^{-1} \exp(-f) \ dV \]
\[ + 2 \int_{\partial M} k_0 \exp(-f) (4\pi \tau)^{-1} \ d\sigma. \]

It follows immediately that the quantity \( \mu(g, \tau) = \inf_{f \in \mathcal{C}} \mathcal{W}(g_{ij}, f, \tau) \) is increasing in \( \tau \) along the flow (of course under the assumption that \( k_0 \geq 0 \)).

8.1. **The Argument.** First we show an interesting and well known property of the soliton metric, but before we start, let us set some notation

**Notation.** The annulus in the manifold \((M, g)\) of inner radius \( r_1 > 0 \) and outer radius \( r_2 > r_1 \) (in the metric \( g \)) will be denoted by \( A(r_1, r_2) \), i.e.,
\[ A(r_1, r_2) := B(0, r_2) \setminus B(0, r_1). \]

Sometimes (and it will be clear from the context), the area of this annulus will be also denoted \( A(r_1, r_2) \).

**Lemma 8.3.** Fix \( k > 1 \). Then the area of the annulus \( A(r, kr) \) in the cigar soliton satisfies
\[ A(r, kr) \sim Ckr. \]

**Proof.** The soliton metric is given by
\[ ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}. \]
Then, if \( P = (x, y) \) we have (we denote by \( d \) the distance function in the manifold)
\[ d(0, P) = \int_0^\alpha \frac{1}{\sqrt{1 + \rho^2}} \ d\rho \quad \text{where} \quad \alpha = \sqrt{x^2 + y^2} \]
\[ \sim \log \alpha. \]

Therefore, if \( r = \log \alpha \),
\[ A(\log \alpha, k \log \alpha) \sim 2\pi \int_\alpha^{\alpha^k} \frac{r^{1/r^2}}{1 + r^2} \ dr \]
\[ = \pi \log(1 + \alpha^k) - \pi \log(1 + \alpha^2) \]
\[ \sim \frac{k}{2} \pi \log \alpha. \]

This shows the claim. \( \square \)
We proceed now with the argument as suggested by Perelman in [P1]. Given \( r > 0 \), define the following function

\[
\phi = \begin{cases} 
1 & \text{if } p \in A(2r, 3r) \\
\frac{1}{r} & \text{if } p \in B(0, r) \\
\epsilon \exp(-d(0, p)) & \text{if } p \not\in B(0, 4r),
\end{cases}
\]

where \( \epsilon > 0 \) is a very small number, and

\[
|\nabla \phi| \leq \frac{1}{r}.
\]

Finally, define

\[
f = -\log \phi + c,
\]

where \( c \) is a constant such that

\[
\frac{1}{4\pi r^2} \int_M \phi \exp(-c) \, dV = 1.
\]

In the case of the soliton metric, for \( \phi \) defined as in (15) we have

\[
\frac{1}{4\pi r^2} \int_M \phi \exp(-c) \, dV \sim \int_{A(r, 4r)} \exp(-c) \, dV \sim \frac{1}{4\pi r^2} \exp(-c) \quad \text{by Lemma 8.3}
\]

which implies

\[
c \sim -\log(r).
\]

Also, we have the following well known estimate on the decay of the curvature on the soliton metric

**Proposition 8.4.** The curvature satisfies the following estimate

\[
R(P) \sim \exp(-d(0, P)).
\]

**Proof.** By the transformation law of the curvature under conformal change of metric, we have for a point \( P = (x, y) \in \mathbb{R}^2 \),

\[
R(P) = \frac{4}{1 + x^2 + y^2}.
\]

But, as we computed in Proposition 8.3,

\[
\sqrt{x^2 + y^2} \sim \exp(d(0, P))
\]

and the statement of the Proposition follows.

We let \( \tau = r^2 \) in (15). Then,

\[
\int_M \tau |\nabla f|^2 \exp(-f) \cdot \frac{1}{4\pi r^2} \, dV = \int_M 4r^2 |\nabla \exp\left(-\frac{f}{r}\right)|^2 \exp(-c) \cdot \frac{1}{4\pi r^2} \, dV \\
\leq \int_{A(r, 4r)} 4r^2 \frac{1}{r^2} \cdot \frac{\exp(-c)}{4\pi r^2} \, dV,
\]

which remains bounded as \( r \to \infty \).

Since \( R \leq \exp(-r) \), the integral

\[
\int_M R \cdot r^2 \frac{1}{4\pi r^2} \exp(-f) \, dV
\]

remains bounded.

It is clear also that the expression

\[
\int_M n \frac{1}{4\pi r^2} \exp(-f) \, dV
\]
remains bounded.

Finally,
\[ \int_M f \exp(-f) \frac{1}{4\pi r^2} \, dV = \int_M \phi \log \phi \frac{1}{4\pi r^2} \exp(-c) \, dV - \int_M c \exp(-f) \frac{1}{4\pi r^2} \, dV. \]

The first integral in the righthand side is bounded. Indeed,
\[ \int_M \phi \log \phi \frac{1}{4\pi r^2} \exp(-c) \, dV \leq C \int_{A(t, 4r)} \frac{1}{4\pi r^2} \exp(-c) \, dV \leq C \frac{1}{r^2} \cdot r \cdot r. \]

The second integral goes to \(-\infty\) as \(r \to \infty\).

Replacing all these estimates into (11), this choice of \(\phi\) shows that, if \(g(t)\) is a solution to the Ricci flow, and if there is a blow up sequence \((p_l, t_l)\) converging to the soliton metric, the there is a sequence of radii \(r_l\) such that
\[ \mu_l = \mu(g(t_l), r_l^2) \to -\infty \quad \text{as} \quad l \to \infty. \]

On the other hand, let \(\tau = t_l - t + r_l^2\), and \(\tilde{f}(\cdot, t)\) be the solution in the interval \([0, t_l]\) to
\[
\begin{cases}
  \tilde{f}_t = -\Delta f + |\nabla f|^2 - R + \frac{1}{4\pi} \\
  \frac{\partial}{\partial \nu} \tilde{f} (\cdot, t_l) = f_l,
\end{cases}
\]
where \(f_l\) is chosen so that \(\mu(g(t_l), f_l, r_l^2) \leq \mu_l + 1\).

Then, by the monotonicity formula, for \(t = 0\)
\[ \mu(g(0), t_l + r_l^2) \leq \mathcal{W}(g(0), \tilde{f}(\cdot, 0), t_l + r_l^2) \leq \mu_l + 1 \]
if \(\lim_{l \to \infty} t_l = T\), it is not difficult to show that (see [R1], [R2]), \(\mu(g(0), t_l) \to \mu(g(0), T)\), and this gives a contradiction. This shows the following

**Theorem.** If \(R_0 > 0\), and \(k_{20} \geq 0\), then the Ricci flow as considered in this paper uniformizes the curvature along a sequence of times.

**References**

[B1] S. Brendle, *Curvature flows on surfaces with boundary*, Math. Ann. 324 (2002), 491–519
[B2] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag (1993).
[H1] R.S. Hamilton, *Harmonic maps of manifolds with boundary*, Lecture Notes in Mathematics 471. Springer, 1975
[H2] R.S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), 255–306.
[H3] R.S. Hamilton, *The Ricci flow on surfaces*, Mathematics and General Relativity, Contemporary Mathematics 71 (1988), 237–261.
[H4] R.S. Hamilton, *A compactness property for solutions of the Ricci flow*, Amer. J. Math. 117 (1995), 545–572.
[H5] R.S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry 2 (1995), 7–136.
[I] T. Ivey, *The Ricci flow on radially symmetric \(\mathbb{R}^3\)*, Comm. Partial Differential Equations 19 (1994) no 9-10, 1481-1500.
[P1] G. Perelman, *The entropy formula for the Ricci flow and geometric applications*, arXiv:math.DG/0211159 (2002).
[R1] O. Rothaus, *Logarithmic Sobolev inequalities and the spectrum of Sturm-Liouville operators*, J. Funct. Anal. 39 (1980), 42–56.
[R2] O. Rothaus, *Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators*, J. Funct. Anal. 42 (1981), 110–120.
[She] Y. Shen, *On Ricci Deformation of a Riemannian Metric on Manifolds with Boundary*, Pacific J. Math. 173 No 1 (1996), 203–221.