Doubly Reflected Backward SDEs Driven by $G$-Brownian Motion—a Monotone Approach

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Abstract

In this paper, we study the doubly reflected backward stochastic differential equations driven by $G$-Brownian motion. We show that the solution can be constructed by a family of penalized reflected $G$-BSDEs with a lower obstacle. The advantage of this construction is that the convergence sequence is monotone, which is helpful to establish the relation between doubly reflected $G$-BSDEs and double obstacle fully nonlinear partial differential equations.

Key words: $G$-expectation, reflected backward SDE, double obstacle PDEs
MSC-classification: 60H10

1 Introduction

In 1997, El Karoui, Kapoudjian, Pardoux, Peng and Quenez [9] first introduced the reflected backward stochastic differential equations (RBSDEs), which means that the first component of the solution is required to be above a given continuous process, called the obstacle. To this end, there is an additional non-decreasing process aiming to push the solution upward, which behaves in a minimal way such that it satisfies the Skorohod condition. This problem is closely related with the optimal stopping problem (see [2]) and the obstacle problem for partial differential equations (PDEs for short, see [1]).

Then, Cvitanic and Karaztas [5] extended the above results to the case where there are two obstacles, that is, the solution $Y$ is forced to remain between two given continuous processes, called lower and upper obstacle. Accordingly, two non-decreasing processes will be added in the doubly RBSDE, whose objective is to push the solution upward and pull the solution downward, respectively, such that the Skorohod conditions are satisfied. They also showed that the solution $Y$ coincides with the value function of a Dynkin game. Due to the importance both in theoretical analysis and in applications, there are amount of works dealing with this problem. We may refer to the papers [4, 8, 10, 11, 12, 26] and the references therein.

Note that the classical theory can only solve the financial problems under drift uncertainty and semi-linear PDEs. Motivated by dealing with financial problems under volatility uncertainty and fully nonlinear PDEs, Peng [23, 24, 25] established a new kind of nonlinear expectation theory, called $G$-expectation theory. A nonlinear Brownian motion, called $G$-Brownian motion, was constructed and the corresponding $G$-Itô’s calculus was introduced.

Based on the $G$-expectation theory, Hu, Ji, Peng and Song [13] investigated the BSDEs driven by $G$-Brownian motion ($G$-BSDEs). Compared with the classical results, there is an additional non-increasing $G$-martingale $K$ in the equation due to nonlinearity. Therefore, the solution is a triple of processes $(Y, Z, K)$ defined universally on the $G$-expectation space. In [13], the well-posedness of $G$-BSDEs is established while the comparison theorem, Feynman-Kac formula and Girsanov transformation can be found in their companion paper [14].

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Recently, Li, Peng and Soumana Hima [16] considered the reflected G-BSDEs with a lower obstacle. Since there will be a non-increasing G-martingale in G-BSDE, the definition is slightly different from the classical case. In fact, they put the non-decreasing process aiming to push up the solution and the non-increasing G-martingale together as a general non-decreasing process, which satisfies a so-called Skorohod condition. Existence is obtained by approximation via penalization and uniqueness can be derived from a priori estimates. However, it is because of the non-increasing G-martingale that the reflected G-BSDEs with an upper obstacle are not equivalent to those with a lower obstacle. The combination of the non-increasing G-martingale and the non-increasing process to pull down the solution ends up with a finite variation process, which makes it difficult to obtain a priori estimates although this finite variation process fulfills the martingale condition. Fortunately, we may show that the solution constructed by the penalization method is the largest one, which can be regarded as the uniqueness. For more details, we may refer to [15].

The reflected G-BSDEs with two obstacles are studied by Li and Song [17]. They proposed a so-called approximate Skorohod condition, which highly related with the construction via penalization method. Roughly speaking, the solution of doubly reflected G-BSDE (Y, Z, A) with terminal value ξ, generator f, lower obstacle L and upper obstacle U is the limit of (Y^n, Z^n, A^n), where (Y^n, Z^n, K^n) is the solution of the following penalized G-BSDEs:

\[ Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s)ds - \int_t^T Z^n_s dB_s - (K^n_T - K^n_t) \]

+ n \int_t^T (Y^n_s - L_s)^- ds - n \int_t^T (Y^n_s - U_s)^+ ds

and A^n_t = n \int_0^t (Y^n_s - L_s)^- ds - n \int_0^t (Y^n_s - U_s)^+ ds - K^n_t. They established the well-posedeness of doubly reflected G-BSDEs when the upper obstacle is a generalized G-Itô process.

It is worth pointing out that the sequence Y^n is not monotone in n. A natural question is that if the solutions of doubly reflected G-BSDEs can be approximated by some monotone sequences. Indeed, consider the following penalized reflected G-BSDEs with a lower obstacle parameterized by n \in \mathbb{N}:

\[
\begin{aligned}
Y^n_t &= \xi + \int_t^T f(s, Y^n_s, Z^n_s)ds - n \int_0^T (\tilde{Y}^n_s - U_s)^+ ds - \int_t^T Z^n_d dB_s + (\tilde{A}^n_t - \tilde{A}^n_t), \\
\tilde{Y}^n_t &\geq L_t, \forall t \in [0, T] \text{ and } \{ - \int_0^t (\tilde{Y}^n_s - L_s)d\tilde{A}^n_s \}_{t \in [0, T]} \text{ is a non-increasing G-martingale. }
\end{aligned}
\]

By the comparison theorem for reflected G-BSDEs, \tilde{Y}^n is non-increasing in n. The function of the penalization term is to pull the solution \tilde{Y}^n downward such that the limit process Y (if exists) stays below U. Therefore, the remaining problem is to prove that the sequence Y^n converges to some process Y, which is the first component of solution to the desired doubly reflected G-BSDE. However, compared with [15] and [17], the main problem is that A^n is no longer a G-martingale. We will encounter some difficulty in proving that (\tilde{Y}^n - U)^+ converges to 0 with the explicit rate 1/n.

Recall that in [16], the solution of reflected G-BSDE with a lower obstacle is constructed by a sequence of penalized G-BSDEs. Now, for each fixed n, m \in \mathbb{N}, consider the following family of G-BSDEs:

\[
Y^{n,m}_t = \xi + \int_t^T f(s, Y^{n,m}_s, Z^{n,m}_s)ds - \int_t^T Z^{n,m}_s dB_s - (K^{n,m}_T - K^{n,m}_t) \\
+ \int_t^T m(Y^{n,m}_s - L_s)^- ds - \int_t^T n(Y^{n,m}_s - U_s)^+ ds.
\]

Set A^{n,m,+}_t = \int_t^T m(Y^{n,m}_s - L_s)^- ds and A^{n,m,-}_t = \int_t^T n(Y^{n,m}_s - U_s)^+ ds. We will show that, as m goes to infinity, (Y^{n,m}, Z^{n,m}, A^{n,m,+} - K^{n,m}) converges to (Y^n, Z^n, A^n), and then, letting n approach infinity,
\((\bar{Y}^n, \bar{Z}^n, \bar{A}^n)\) converges to \((Y, Z, A)\), which is the solution of doubly reflected \(G\)-BSDE. The method for proving the uniform boundedness of \(Y^{n,m}\) is quite different with the one in [17]. Motivated by [20], the uniformly bounded property in this paper is obtained via some comparison results rather than applying \(G\)-Itô’s formula. Recalling the difficulty raised in only considering the penalized reflected \(G\)-BSDEs, the reason why we introduce the penalized \(G\)-BSDEs with parameters \(n, m\) is that it allows us to derive the convergence rate of \((Y^{n,m} - U)^+\) with explicit rate \(\frac{1}{m}\) uniformly in \(m\) (see Lemma 4.4). Hence, the convergence rate remains the same for the limit process \((\bar{Y}^n - U)^+\).

The objective of constructing the solution of doubly reflected \(G\)-BSDE by a monotone sequence is to establish the relation with double obstacle fully nonlinear PDEs. Generally speaking, in a Markovian framework, the solution \(Y\) of the doubly reflected \(G\)-BSDE is the unique viscosity solution of the associated double obstacle PDE, with extends the result in [11] to the fully nonlinear case.

Recall that the \(G\)-expectation theory is closely related with the quasi-sure analysis by Denis and Martini [7] and the second order BSDEs (2BSDEs) proposed by Soner, Touzi and Zhang [28] and Matoussi, Possamaï and Zhou [20] since the \(G\)-expectation is an upper expectation induced by a non-dominated family of probability measures. Therefore, this paper shares many similarities with the doubly reflected 2BSDEs studied by Matoussi, Piozin and Possamaï [19], but the definitions, the assumptions on parameters and the methods of proof are significantly different.

1. The definitions. In the corrigendum [21], the solution to doubly reflected 2BSDE with terminal value \(\xi\), generator \(\hat{F}\) and obstacles \((L, U)\) is a pair \((Y, Z)\) satisfies the following conditions: (1) \(Y_T = \xi\) and \(Y\) stays between \(L\) and \(U\); (2) for any \(P \in \mathcal{P}\), the process \(V^P\) defined below has paths of bounded variation \(P\)-a.s.

\[
V^P_t := Y_0 - Y_t - \int_0^t \hat{F}_s(Y_s, Z_s) \, ds + \int_0^t Z_s \, dB_s,
\]

and can be represented as the difference of two non-decreasing process such that one satisfies the Skorohod condition and \(V^P\) satisfies the minimality condition.

2. The assumptions. The generator \(\hat{F}\) is assumed to be Lipschitz continuous in \((y, z)\), uniformly continuous in \(\omega\) and square integrable. The obstacles are given by two càdlàg processes such that they can be strictly separated, i.e. \(L_t < U_t\) and \(L_{t-} < U_{t-}\) for any \(t \in [0, T]\). Besides, the upper obstacle \(U\) is required to be a semimartingale under every \(P \in \mathcal{P}\).

3. The proofs. Uniqueness can also be obtained using some a priori estimates. In order to derive existence, the first component \(Y\) of the solution is constructed as the supremum of the solutions to doubly reflected BSDEs under a set of non-dominated probability measures defined on the shifted canonical space. The measurability and regularity of the candidate solution can be obtained by some estimates and the second component \(Z\) is derived from a non-linear Doob-Meyer decomposition.

Compared with doubly reflected 2BSDE [19] and [21], we do not need the assumption that the two obstacles are strictly separated and the assumption that the terminal value \(\xi\) is a uniformly continuous and bounded function on \(\Omega\) (Let us mention that the existence for 2BSDEs still holds for square integrable \(\xi\) while some additional results in [22] and [27] are needed). The solution of doubly reflected \(G\)-BSDE is constructed by a penalization method as in the classical case, which makes it possible to obtain some numerical approximations. Besides, the solution \((Y, Z, A)\), especially the finite variation process \(A\) in our framework is defined universally within the \(G\)-expectation space.

The paper is organized as follows. First, we recall some basic results concerning \(G\)-BSDEs and reflected \(G\)-BSDEs with a single obstacle in Section 2. In Section 3, we first formulate the doubly reflected \(G\)-BSDEs in details and establish some a priori estimates. The existence via penalization is obtained in Section 4. Finally, we provide the probabilistic interpretation for solutions of double obstacle PDEs.
2 Preliminaries

We review some basic notions and results of $G$-expectation, $G$-BSDEs and reflected $G$-BSDEs. For simplicity, we only consider the one-dimensional case. The readers may refer to [13], [14], [10], [23], [24], [25] for more details.

2.1 $G$-expectation and $G$-Itô's calculus

Let $\Omega_T = C_0([0, T]; \mathbb{R})$, the space of real-valued continuous functions with $\omega_0 = 0$, be endowed with the supremum norm and let $B$ be the canonical process. Set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \in \mathbb{N}, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b,Lip}(\mathbb{R}^n)\},$$

where $C_{b,Lip}(\mathbb{R}^n)$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^n$.

Let $(\Omega, L_{ip}(\Omega_T), \hat{E})$ be the $G$-expectation space. The function $G : \mathbb{R} \to \mathbb{R}$ is defined by

$$G(a) := \frac{1}{2}\hat{E}[aB_1^2] = \frac{1}{2}(\sigma^2 a^+ - \sigma^2 a^-).$$

In this paper, we only consider the non-degenerate case, i.e., $\sigma^2 > 0$.

Define $||\xi||_{L_p^p} := (\hat{E}[|\xi|^p])^{1/p}$ for $\xi \in L_{ip}(\Omega_T)$ and $p \geq 1$. The completion of $L_{ip}(\Omega_T)$ under this norm is denote by $L_p^p(\Omega)$. For all $t \in [0, T]$, $\hat{E}_t[\cdot]$ is a continuous mapping on $L_{ip}(\Omega_T)$ w.r.t the norm $||\cdot||_{L^p_b}$. Hence, the conditional $G$-expectation $\hat{E}_t[\cdot]$ can be extended continuously to the completion $L_{G}^1(\Omega_T)$. Denis, Hu and Peng [6] prove that the $G$-expectation has the following representation.

**Theorem 2.1** ([6]) *There exists a weakly compact set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega_T))$, such that

$$\hat{E}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in L_{G}^1(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\hat{E}$.***

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{E}$. For this $\mathcal{P}$, we define the capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is called polar if $c(A) = 0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables $X$ and $Y$ if $X = Y$, q.s.

**Definition 2.2** Let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_t)$, $i = 0, 1, 2, \ldots, N - 1$. For each $p \geq 1$ and $\eta \in M_G^0(0, T)$, let $||\eta||_{H^p_G} := (\hat{E}[\int_0^T |\eta_s|^2 ds]^{p/2})^{1/p}$, $||\eta||_{M^p_G} := (\hat{E}[\int_0^T |\eta_s|^p ds]^{1/p}$ and denote by $H^p_G(0, T)$, $M^p_G(0, T)$ the completion of $M^0_G(0, T)$ under the norm $||\cdot||_{H^p_G}$, $||\cdot||_{M^p_G}$, respectively.

We denote by $(B)$ the quadratic variation process of the $G$-Brownian motion $B$. For two processes $\xi \in M^0_G(0, T)$ and $\eta \in M^0_G(0, T)$, the $G$-Itô integrals $\left(\int_0^T \xi_s dB_s\right)_{0 \leq s \leq T}$ and $\left(\int_0^T \eta_s dB_s\right)_{0 \leq s \leq T}$ are well defined, see Li and Peng [13] and Peng [23]. The following proposition can be regarded as the Burkholder–Davis–Gundy inequality under $G$-expectation framework.
Proposition 2.3 ([14]) If \( \eta \in H^\alpha_G(0, T) \) with \( \alpha \geq 1 \) and \( p \in (0, \alpha] \), then we have \( \sup_{u \in [t, T]} \int_t^u \eta_s dB_s |^p \in L^p_G(\Omega_T) \) and

\[
\mathbb{E}^p_c \mathbb{E}_t [\int_t^T |\eta_s|^2 ds]^{p/2} \leq \mathbb{E}_t \left[ \sup_{u \in [t, T]} \int_t^u \eta_s dB_s |^p \right] \leq \sigma^p \mathbb{E}^p_c \mathbb{E}_t [\int_t^T |\eta_s|^2 ds]^{p/2},
\]

where \( 0 < c_p < C_p < \infty \) are constants depending on \( p, T \).

Let \( S^0_G(0, T) = \{ h(t, B_{t_1} \wedge t, \ldots, B_{t_n} \wedge t) : t_1, \ldots, t_n \in [0, T], h \in C_b, L^p(\mathbb{R}^{n+1}) \} \). For \( p \geq 1 \) and \( \eta \in S^0_G(0, T) \), set \( \| \eta \|_{S^p_G} = \left( \mathbb{E} [\sup_{t \in [0, T]} |\eta_t|^p] \right)^{1/p} \). Denote by \( S^p_G(0, T) \) the completion of \( S^0_G(0, T) \) under the norm \( \| \cdot \|_{S^p_G} \). For \( \xi \in L^p(\Omega_T) \), let \( \mathcal{E}(\xi) = \mathbb{E} [\sup_{t \in [0, T]} |\xi_t|] \). For \( p \geq 1 \) and \( \xi \in L^p(\Omega_T) \), define \( \| \xi \|_{p, \mathcal{E}} = \mathbb{E} (|\xi|^p)^{1/p} \) and denote by \( \mathcal{L}^p(\Omega_T) \) the completion of \( L^p(\Omega_T) \) under \( \| \cdot \|_{p, \mathcal{E}} \). Similar to the classical Doob’s maximal inequality, the following theorem holds.

Theorem 2.4 ([29]) For any \( \alpha \geq 1 \) and \( \delta > 0 \), \( L^{\alpha+\delta}_G(\Omega_T) \subset L^p_\mathcal{E}(\Omega_T) \). More precisely, for any \( 1 < \gamma < \beta := (\alpha + \delta) / \alpha, \gamma \leq 2 \), we have

\[
\| \xi \|_{p, \mathcal{E}} \leq \gamma \| \xi \|_{L^{\alpha+\delta}_G} + 14^{1/\gamma} C_{\beta/\gamma} \| \xi \|_{L^{\alpha+\delta}_G}^{(\alpha+\delta)/\gamma}, \quad \forall \xi \in L^p(\Omega_T),
\]

where \( C_{\beta/\gamma} = \sum_{i=1}^\infty i^{-\beta/\gamma}, \beta^* = \gamma / (\gamma - 1) \).

2.2 G-BSDEs

We now introduce some basic results about G-BSDEs. In fact, the solution of G-BSDE with terminal value \( \xi \), generators \( f, g \), is a triple of processes \( (Y, Z, K) \in \mathfrak{G}^\alpha_G(0, T) \), such that

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d(B)_s - \int_t^T Z_s dB_s - (K_T - K_t), \quad (2.1)
\]

where \( \mathfrak{G}^\alpha_G(0, T) \) the collection of processes \( (Y, Z, K) \) such that \( Y \in S^0_G(0, T), Z \in H^\alpha_G(0, T), \) and \( K \) is a non-increasing G-martingale with \( K_0 = 0 \) and \( K_T \in L^\alpha_G(\Omega_T) \). For the well-posedness of G-BSDEs, we need to propose the following assumptions on the generators \( (f, g) \), where

\[
f(t, \omega, y, z), \quad g(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}
\]

satisfy the following properties: there exists a constant \( \beta > 1 \), such that

(H1) for any \( y, z \in \mathbb{R}, f(\cdot, \cdot, y, z), g(\cdot, \cdot, y, z) \in M^\beta_G(0, T); \)

(H2) there exists some \( \kappa > 0 \) such that

\[
|f(t, y, z) - f(t, y', z')| + |g(t, y, z) - g(t, y', z')| \leq \kappa (|y - y'| + |z - z'|).
\]

Hu, Ji, Peng and Song [13] established the existence and uniqueness result for Equation (2.1).

Theorem 2.5 ([13]) Assume that \( \xi \in L^\beta_G(\Omega_T) \) and \( f, g \) satisfy (H1) and (H2) for some \( \beta > 1 \). Then, for any \( 1 < \alpha < \beta \), Equation (2.1) has a unique solution \( (Y, Z, K) \in \mathfrak{G}^\alpha_G(0, T) \). Moreover, we have

\[
|Y_t|^\alpha \leq C \mathbb{E}_t [\|\xi\|^\alpha + \int_t^T |f(s, 0, 0)|^\alpha + |g(s, 0, 0)|^\alpha ds],
\]

where the constant \( C \) depends on \( \alpha, T, \mathcal{E} \) and \( \kappa \).
Below is a generalization of Proposition 3.5 in \[14\].

**Theorem 2.6** (\[17\]) Let \( f, g \) satisfy (H1) and (H2) for some \( \beta > 1 \). Assume

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d(B)_s - \int_t^T Z_sdB_s - (K_T - K_t) + (A_T - A_t),
\]

where \( Y \in S_0^\alpha(0, T), Z \in H_0^\beta(0, T), K, A \) are both non-increasing process with \( A_0 = K_0 = 0 \) and \( A_T, K_T \in L^\alpha_0(\Omega_T) \) for some \( 1 < \alpha < \beta \). Then there exists a constant \( C := C(\alpha, T, \mathcal{E}, \kappa) > 0 \) such that

\[
\mathbb{E}[\int_0^T |Z_s|^2ds]^{\frac{1}{2}} \leq C \left\{ \mathbb{E}[|Y_T^\alpha|^\alpha] + \left( \mathbb{E}[|Y_T^\alpha|^\alpha] \right)^{\frac{1}{2}} \left( \mathbb{E}[\int_0^T f_s^0ds]^{\alpha} \right)^{\frac{1}{2}} + (m_{A,K}^{A,K})^{1/2} \right\},
\]

where \( Y_T^\alpha = \sup_{t \in [0, T]} |Y_t|, f_0^0 = |f(s, 0, 0)| + |g(s, 0, 0)|, m_{A,K}^{A,K} = \min\{\mathbb{E}[|A_T|^\alpha], \mathbb{E}[|K_T|^\alpha]\} \).

Similar with the classical case, the comparison theorem for G-BSDEs still holds.

**Theorem 2.7** (\[14\]) Let \((Y_t^l, Z_t^l, K_t^l)_{t \leq T}, l = 1, 2\), be the solutions of the following G-BSDEs:

\[
Y_t^l = \xi^l + \int_t^T f^l(s, Y_s^l, Z_s^l)ds + \int_t^T g^l(s, Y_s^l, Z_s^l)d(B)_s + V_T^l - V_t^l - \int_t^T Z_s^ldB_s - (K_T - K_t^l),
\]

where processes \( \{V_t^l\}_{0 \leq t \leq T} \) are assumed to be right-continuous with left limits (RCLL), quasi-surely, such that \( \mathbb{E}[\sup_{t \in [0, T]} |V_t^l|^{\beta}] < \infty \), \( f^l, g^l \) satisfy (H1) and (H2), \( \xi^l \in L^\beta_G(\Omega_T) \) with \( \beta > 1 \). If \( \xi^1 \geq \xi^2 \), \( f^1 \geq f^2 \), \( g^1 \geq g^2 \) and \( V^1 - V^2 \) is a non-decreasing process, then \( Y_t^1 \geq Y_t^2 \).

It is easy to find that the main difference between G-BSDEs and BSDEs in the classical case is that there is an additional non-increasing G-martingale \( K \) in G-BSDEs, which exhibits the uncertainty of the model and leads to the difficulty for analysis. Song \[30\] proved that the non-increasing G-martingale could not be form of \( \{\int_0^t \eta_sdt\} \) or \( \{\int_0^t \gamma_s d(B)_s\} \), where \( \eta, \gamma \in M^1_G(0, T) \). More precisely, he established the following result.

**Theorem 2.8** (\[30\]) Assume that for \( t \in [0, T], \int_0^t \zeta_s dB_s + \int_0^t \eta_s ds + K_t = L_t \), where \( \zeta \in H^1_G(0, T), \eta \in M^1_G(0, T) \), \( K, L \) are non-increasing G-martingales. Then we have \( \int_0^t \zeta_s dB_s = 0 \), \( \int_0^t \eta_s ds = 0 \) and \( K_t = L_t \).

**Remark 2.9** We call the following process \( u \) a generalized G-Itô process

\[
u_t = u_0 + \int_0^t \eta_s ds + \int_0^t \zeta_s dB_s + K_t,
\]

where \( \eta \in M^1_G(0, T), \zeta \in H^1_G(0, T) \) and \( K \) is a non-increasing G-martingale with \( K_0 = 0 \). By Theorem 2.8, the decomposition for generalized G-Itô processes is unique.

### 2.3 Reflected G-BSDEs with a single obstacle

Now we introduce the reflected G-BSDEs with a lower obstacle studied in \[13\]. Compared with the G-BSDEs, the parameters consist of a terminal value \( \xi \), generators \( f, g \) and an obstacle \( S \), where \( S \) satisfies the following assumption.
for any obstacle $U$ obstacle BSDEs are given by the terminal value $\xi$.

Let us now introduce the reflected $G$-BSDE with a lower obstacle. A triple of processes $(Y, Z, A)$ is called a solution of reflected $G$-BSDE with a lower obstacle with parameters $(\xi, f, g, S)$ if:

(a) $(Y, Z, A) \in S^\beta_G(0, T)$ and $Y_t \geq \xi_t$, $0 \leq t \leq T$;

(b) $Y_t = \xi + \int^t_0 f(s, Y_s, Z_s)ds + \int^T_0 g(s, Y_s, Z_s)d<B>_s - \int^T_0 Z_sdB_s + (A_T - A_t)$;

(c) $\{-\int^T_0 (Y_s - S_s)dA_s\}_{t \in [0, T]}$ is a non-increasing $G$-martingale.

where $S^\beta_G(0, T)$ is the collection of processes $(Y, Z, A)$ such that $Y \in S^\beta_G(0, T)$, $Z \in H^\beta_G(0, T)$, $A$ is a continuous non-decreasing process with $A_0 = 0$ and $A \in S^\beta_G(0, T)$. By the results in [16] and Remark 4.8 in [17], we have the following existence and uniqueness result as well as the comparison theorem for reflected $G$-BSDEs.

**Theorem 2.10** ([16]) Suppose that $f$, $g$ and $S$ satisfy (H1) –(H3) and $\xi$ belongs to $L^\beta_G(\Omega_T)$ such that $\xi \geq S_T$, where $\beta > 2$. Then, the reflected $G$-BSDE with parameters $(\xi, f, g, S)$ has a unique solution $(Y, Z, A)$. Moreover, for any $2 \leq \alpha < \beta$ we have $Y \in S^\beta_G(0, T)$, $Z \in H^\beta_G(0, T)$ and $A \in S^\beta_G(0, T)$.

**Theorem 2.11** ([16]) Let $(\xi^i, f^i, g^i, S^i)$ be two sets of parameters, $i = 1, 2$. Suppose $S^i$, $f^i$ and $g^i$ satisfy (H1) –(H3), $\xi^i$ belong to $L^\beta_G(\Omega_T)$ such that $\xi^i \geq S^i_T$ for $i = 1, 2$, where $\beta > 2$. We furthermore assume the following:

(i) $\xi^1 \leq \xi^2$, q.s.;

(ii) $f^1(t, y, z) \leq f^2(t, y, z)$, $g^1(t, y, z) \leq g^2(t, y, z)$, $\forall (y, z) \in \mathbb{R}^2$;

(iii) $S^1_t \leq S^2_t$, $0 \leq t \leq T$, q.s.

Let $(Y^1, Z^1, A^1)$ be the solutions of the reflected $G$-BSDE with parameters $(\xi^i, f^i, g^i, S^i)$, $i = 1, 2$, respectively. Then

$$Y^1_t \leq Y^2_t, \quad 0 \leq t \leq T \quad q.s.$$ 

3 Doubly reflected $G$-BSDEs and a priori estimates

In this section, we first recall the definition of solutions to doubly reflected $G$-BSDEs introduced in [17] and then provide some a priori estimates, which yield the uniqueness of solutions.

3.1 Formulation of doubly reflected $G$-BSDEs

We first formulate the doubly reflected $G$-BSDEs in details. The parameters of doubly reflected $G$-BSDEs are given by the terminal value $\xi$, the generators $f, g$, the lower obstacle $L$ and the upper obstacle $U$, which satisfy the following assumptions. There exists some $\beta > 2$ such that

(A1) for any $y, z, f(\cdot, y, z), g(\cdot, y, z) \in S^\beta_G(0, T)$;

(A2) $|f(t, \omega, y, z) - f(t, \omega, y', z')| + |g(t, \omega, y, z) - g(t, \omega, y', z')| \leq \kappa(|y - y'| + |z - z'|)$ for some $\kappa > 0$.
Theorem 3.2 Suppose that \( I \in S_G^3(0, T) \). There exists some \( I \in S_G^3(0, T) \) satisfying the following representation

\[
I_t = I_0 + A_t^{I^+} - A_t^{I^-} + \int_0^t \sigma^I(s)dB_s,
\]

where \( A_t^{I^+}, A_t^{I^-} \in S_G^3(0, T) \) are two non-decreasing processes with \( A_0^{I^+} = A_0^{I^-} = 0 \) and \( \sigma^I \in S_G^3(0, T) \) such that \( L_t \leq I_t \leq U_t \) and \( U_t + A_t^{I^+} \) is a generalized G-Itô process of the following form

\[
U_t + A_t^{I^+} = U_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s + K^u_t,
\]

where \( b, \sigma \in S_G^3(0, T) \), \( K^u \in S_G^3(0, T) \) is a non-increasing G-martingale with \( K^u_0 = 0 \);

(A4) \( \xi \in L_G^2(\Omega, T) \) and \( L_T \leq \xi \leq U_T \).

Remark 3.1 Assumption (A3) in this paper is weaker than Assumption (A3) in [17]. Recall that Assumption (A3) in [17] says that the upper obstacle is a generalized G-Itô process of the following form

\[
U_t = U_0 + \int_0^t b^U(s)ds + \int_0^t \sigma^U(s)dB_s + K^U_t,
\]

where \( \{b^U(t)\}_{t \in [0,T]}, \{\sigma^U(t)\}_{t \in [0,T]} \in S_G^3(0, T) \) is a non-increasing G-martingale. We may set \( I = U, \sigma^I = \sigma^U, A_t^{I^+} = \int_0^t (b^U(s))^+ds, A_t^{I^-} = \int_0^t (b^U(s))^+ds - K^U_t \). Clearly, the pair \((I, U)\) satisfies (A3) in the present paper.

Let us recall the definition of solutions to doubly reflected G-BSDEs introduced in [17]. A triple of processes \((Y, Z, A)\) with \( Y, A \in S_G^3(0, T), Z \in H_G^3(0, T) \), for some \( 2 \leq \alpha \leq \beta \), is called a solution to the doubly reflected G-BSDE with the parameters \((\xi, f, g, L, U)\) if the following properties hold:

(S1) \( L_t \leq Y_t \leq U_t, t \in [0, T] \);

(S2) \( Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B \rangle_s - \int_t^T Z_sdB_s + (A_T - A_t) \);

(S3) \((Y, A)\) satisfies Approximate Skorohod Condition with order \( \alpha \) (ASC\(_\alpha\)).

Condition (ASC\(_\alpha\)): We say a pair of processes \((Y, A)\) with \( Y, A \in S_G^3(0, T) \) satisfies the approximate Skorohod condition with order \( \alpha \) (with respect to the obstacles \((L, U)\)) if there exist non-decreasing processes \( \{A_n^{+}\}_{n \in \mathbb{N}}, \{A_n^{-}\}_{n \in \mathbb{N}} \) and non-increasing G-martingales \( \{K_n\}_{n \in \mathbb{N}} \), such that

- \( \mathbb{E}[|A_T^{n^+}|^\alpha + |A_T^{n^{-}}|^{\alpha} + |K_T^n|^{\alpha}] \leq C \), where \( C \) is independent of \( n \);
- \( \mathbb{E}[\sup_{t \in [0,T]}|A_t - (A_t^{n^+} - A_t^{n^{-}} - K_T^n)|^{\alpha}] \to 0 \), as \( n \to \infty \);
- \( \lim_{n \to \infty} \mathbb{E}[\int_0^T (Y_s - L_s)dA_s^{n^+}] = 0 \);
- \( \lim_{n \to \infty} \mathbb{E}[\int_0^T (U_s - Y_s)dA_s^{n^{-}}] = 0 \).

We state the main result of this paper, which extends Theorem 3.2 in [17].

Theorem 3.2 Suppose that \( \xi, f, g, L \) and \( U \) satisfy (A1)-(A4). Then the reflected G-BSDE with data \((\xi, f, g, L, U)\) has a unique solution \((Y, Z, A)\). In fact, for any \( 2 \leq \alpha < \beta \) we have \( Y \in S_G^3(0, T), Z \in H_G^3(0, T) \) and \( A \in S_G^3(0, T) \). Moreover, \( Y \) can be approximated by a non-increasing sequence \( \{Y^n\}_{n \in \mathbb{N}} \).
3.2 Some a priori estimates

In this subsection, we establish some a priori estimates for solutions of doubly reflected G-BSDEs, which generalizes Proposition 3.8 in \[17\] and will be used to obtain the continuity of the value function \( u \) defined in Section 5. For simplicity, we only consider the case that \( g \equiv 0 \) in the following of this paper and the results still hold for the general case. In the sequel, \( C \) always represents a constant depending on \( \alpha, T, \kappa, G \), but not on \( n, m \), which may vary from line to line.

**Proposition 3.3** Let \( (\xi^i, f^i, L^i, U^i) \), \( i = 1, 2 \) be two sets of data, each one satisfying all the assumptions (A1)-(A4). Let \( (Y^i, Z^i, A^i) \) be a solution of the reflected G-BSDE with data \( (\xi^i, f^i, L^i, U^i) \) and let \( \{A^{i,n,+}\}_{n \in \mathbb{N}}, \{A^{i,n,-}\}_{n \in \mathbb{N}}, \{K^{i,n}\}_{n \in \mathbb{N}} \) be the approximation sequences for \( A^i, i = 1, 2 \) respectively. Set \( \hat{Y}_t = Y^1_t - Y^2_t, \hat{\xi} = \xi^1 - \xi^2, \hat{L}_t = L^1_t - L^2_t \) and \( \hat{U}_t = U^1_t - U^2_t \). Then there exists a constant \( C := C(\alpha, T, \kappa, \sigma) > 0 \) such that

\[
|\hat{Y}_t|^\alpha \leq \liminf_{n \to \infty} C \left( \sum_{i=1}^{2} \mathbb{E}_t \sup_{s \in [t,T]} |Y^i_s|^\alpha \right)^{\frac{\alpha-2}{\alpha}} \times \left( \sum_{i=1}^{2} \mathbb{E}_t \sup_{s \in [t,T]} |A^{i,n,+}|^\alpha + \mathbb{E}_t \sup_{s \in [t,T]} |A^{i,n,-}|^\alpha \right)^{\frac{1}{\alpha}}
\]

\[
\times \left( \mathbb{E}_t \sup_{s \in [t,T]} |\hat{L}_s|^\alpha + \mathbb{E}_t \sup_{s \in [t,T]} |\hat{U}_s|^\alpha \right) + C \mathbb{E}_t |\hat{\xi}|^\alpha + \int_t^T |\hat{\lambda}_s|^\alpha ds,
\]

where \( \hat{\lambda}_s = |f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)| \).

**Proof.** Set \( \hat{Z}_t = Z^1_t - Z^2_t, \hat{A}_t = A^1_t - A^2_t \). For any \( r > 0 \), applying G-Itô’s formula to \( H^\alpha_t e^{r_t} = (|\hat{Y}_t|^\alpha e^{r_t})^\alpha \), we have

\[
H^\alpha_t e^{r_t} + \int_t^T r e^{r_s} H^\alpha_s e^{r_t} ds + \int_t^T \frac{\alpha}{2} e^{r_s} H^{\alpha/2 - 1}_s d(B)_s
\]

\[
= |\hat{\xi}|^\alpha e^{r_T} + \alpha(1 - \frac{\alpha}{2}) \int_t^T e^{r_s} H^{\alpha/2 - 2}_s (\hat{Z}_s)^2 d(B)_s - \int_t^T \alpha e^{r_s} H^{\alpha/2 - 1}_s \hat{Y}_s \hat{Z}_s dB_s
\]

\[
+ \int_t^T \alpha e^{r_s} H^{\alpha/2 - 1}_s (\hat{Y}_s^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)) ds + \int_t^T \alpha e^{r_s} H^{\alpha/2 - 1}_s \hat{Y}_s d\hat{A}_s.
\]

By the assumption of \( f^1 \) and the Young inequality, similar analysis as the proof of Proposition 3.8 in \[17\] implies that

\[
\int_t^T \alpha e^{r_s} H^{\alpha/2 - 1}_s (\hat{Y}_s^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)) ds
\]

\[
\leq \tilde{r} \int_t^T e^{r_s} H^{\alpha/2}_s ds + \frac{\alpha(\alpha - 1)}{4} \int_t^T e^{r_s} H^{\alpha/2 - 1}_s (\hat{Z}_s)^2 d(B)_s + \int_t^T e^{r_s} |\hat{\lambda}_s|^\alpha ds,
\]

where \( \tilde{r} = \alpha - 1 + \alpha \kappa + \frac{\alpha \kappa}{2(\alpha - 1)} \). Set \( A^{i,n,+} = A^{i,n,+} - A^{i,n,-} - K^{i,n}, i = 1, 2, \hat{Y}^L_t = (Y^1_t - L^1_t) - (Y^2_t - L^2_t) \) and \( \hat{Y}^U_t = (U^1_t - Y^1_t) - (U^2_t - Y^2_t) \). Noting that \( \hat{Y}^L_t \leq Y^1_t - L^1_t, \hat{Y}^U_t \leq U^1_t - Y^1_t \) and \( A^{1,n,+}, A^{1,n,-} \) are
non-decreasing processes, it is easy to check that
\[
\int_t^T \alpha e^{rs} H_s^{\alpha/2-1} \hat{Y}_s dA_s^1 \\
= \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} \hat{Y}_s d(A_s^1 - A_s^{1,n}) + \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} \hat{Y}_s dA_s^{1,n} \\
\leq \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} (Y_s^1 - L_s^1) dA_s^{1,n,+} + \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} |\hat{L}_s| dA_s^{1,n,+} \\
+ \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} (U_s^1 - Y_s^1) dA_s^{1,n,-} + \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} |\hat{U}_s| dA_s^{1,n,-} \\
+ \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} \hat{Y}_s d(A_s^1 - A_s^{1,n})| - \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} (\hat{Y}_s) + dK_s^{1,n}.
\]

By Lemma 3.7 in [17], we have for any \( t \in [0, T] \)
\[
\lim_{n \to \infty} \hat{E}[\| \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} \hat{Y}_s d(A_s^1 - A_s^{1,n}) \|] = 0.
\]

It is easy to check that
\[
\hat{E}[\int_t^T \alpha e^{rs} H_s^{\alpha/2-1} (U_s^1 - Y_s^1) dA_s^{1,n,-}] \\
\leq C \hat{E} \left[ \sup_{t \in [0, T]} (|Y_t^1| + |Y_t^2|)^{\alpha - 2} \int_t^T (U_s^1 - Y_s^1) dA_s^{1,n,-} \right] \\
\leq C \hat{E} \left[ \sup_{t \in [0, T]} (|Y_t^1| + |Y_t^2|)^{\alpha - 2} \hat{E}[\| \int_t^T (U_s^1 - Y_s^1) dA_s^{1,n,-} \| \hat{F}] \right]^{\frac{1}{2}}.
\]

It follows from the approximate Skorohod condition that
\[
\lim_{n \to \infty} \hat{E}[\| \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} (U_s^1 - Y_s^1) dA_s^{1,n,-} \|] = 0.
\]

Similar analysis as above yields that
\[
\lim_{n \to \infty} \hat{E}[\| \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} (Y_s^1 - L_s^1) dA_s^{1,n,+} \|] = 0,
\]
\[
\lim_{n \to \infty} \hat{E}[\| \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} (Y_s^2 - L_s^2) dA_s^{2,n,+} \|] = 0,
\]
\[
\lim_{n \to \infty} \hat{E}[\| \int_t^T \alpha e^{rs} H_s^{\alpha/2-1} (U_s^2 - Y_s^2) dA_s^{2,n,-} \|] = 0.
\]

By simple calculation, we obtain that
\[
\hat{E}_s[\int_t^T \alpha e^{rs} H_s^{\alpha/2-1} |\hat{L}_s| dA_s^{1,n,+}] \\
\leq C \hat{E}_s \left[ \sup_{s \in [t, T]} (|Y_s^1| + |Y_s^2|)^{\alpha - 2} \sup_{s \in [t, T]} |\hat{L}_s| \| A_T^{1,n,+} \| \right] \\
\leq C I_1^2(Y) \left( \hat{E}_s[\| \sup_{s \in [t, T]} |\hat{L}_s|^\alpha \|] \right)^{\frac{1}{2}} \left( \hat{E}_s[\| A_T^{1,n,+} |^\alpha \|] \right)^{\frac{1}{2}},
\]

\[10\]
where \( I^a_n(Y) = (\sum_{i=1}^2 \hat{E}_t[|Y_s^i|[\sup_{s \in [t,T]} |Y_s^i|]^{\frac{2}{1-a}}]^{1-a}) \). Similarly, we have

\[
\hat{E}_t[\int_t^T \alpha e^{rs} H_s^{a/2-1} |\tilde{U}_s| dA_s^{1,n,-}] \leq CI^a_t(Y) \left( \hat{E}_t[\sup_{s \in [t,T]} |\tilde{U}_s|^{a}] \right)^{\frac{1}{2-a}} \left( \hat{E}_t[|A_T^{1,n,-}|^{a}] \right)^{\frac{1}{a}},
\]

\[
\hat{E}_t[\int_t^T \alpha e^{rs} H_s^{a/2-1} |\bar{L}_s| dA_s^{2,n,+}] \leq CI^a_t(Y) \left( \hat{E}_t[\sup_{s \in [t,T]} |\bar{L}_s|^{a}] \right)^{\frac{1}{2-a}} \left( \hat{E}_t[|A_T^{2,n,+}|^{a}] \right)^{\frac{1}{a}},
\]

\[
\hat{E}_t[\int_t^T \alpha e^{rs} H_s^{a/2-1} |\tilde{U}_s| dA_s^{2,n,-}] \leq CI^a_t(Y) \left( \hat{E}_t[\sup_{s \in [t,T]} |\tilde{U}_s|^{a}] \right)^{\frac{1}{2-a}} \left( \hat{E}_t[|A_T^{2,n,-}|^{a}] \right)^{\frac{1}{a}}.
\]

Set \( M^n_t = \int_0^t \alpha e^{rs} H_s^{a/2-1} (\bar{Y}_s \bar{Z}_s dB_s + (\bar{Y}_s)^+ dK_s^{1,n} + (\bar{Y}_s)^- dK_s^{2,n}) \). By Lemma 3.4 in \([13]\), \( M^n \) is a G-martingale. Let \( r = \beta + 1 \). Combining the above inequalities, we get

\[
H^{n/2}_t e^{rt} + (M^n_T - M^n_t) \leq \hat{\xi} e^{T} + \int_t^T e^{rs} |\hat{\lambda}_a| ds + \sum_{i=1}^2 \left( \int_t^T \alpha e^{rs} H_s^{a/2-1} \hat{Y}_s d(A_s^i - A_s^{i,n}) \right) \\
+ \int_t^T \alpha e^{rs} H_s^{a/2-1} |\bar{L}_s| d(A_s^{1,n,+} + A_s^{2,n,+}) + \int_t^T \alpha e^{rs} H_s^{a/2-1} |\tilde{U}_s| d(A_s^{1,n,-} + A_s^{2,n,-}) \\
+ \sum_{i=1}^2 \int_t^T \alpha e^{rs} H_s^{a/2-1} (U_s^i - Y_s^i) dA_s^{i,n,-} + \sum_{i=1}^2 \int_t^T \alpha e^{rs} H_s^{a/2-1} (Y_s^i - L_s^i) dA_s^{i,n,+}.
\]

Taking conditional expectations on both sides and letting \( n \to \infty \), we finally get the desired result. \( \blacksquare \)

### 4 Existence via penalization

In this section, we prove the existence of solutions to doubly reflected G-BSDEs by a penalization method. More importantly, we show that the first component \( Y \) can be approximated by a monotone sequence. For this purpose, for each fixed \( n \in \mathbb{N} \), consider the following family of reflected G-BSDEs with lower obstacle \( L \):

\[
\begin{align*}
\bar{Y}_n^t &= \xi + \int_t^T f(s, \bar{Y}_n^s, \bar{Z}_n^s) ds - n \int_t^T (\bar{Y}_n^s - U_s)^+ ds - \int_t^T \bar{Z}_n^s dB_s + (\bar{A}_n^T - \bar{A}_n^t), \\
\bar{Y}_n^t &\geq L_t, \forall t \in [0,T], \{ - \int_0^t (\bar{Y}_n^s - L_s) dA_s^a \}_{a \in [0,T]} \text{ is a non-increasing G-martingale.}
\end{align*}
\]

(4.1)

Suppose that the lower obstacle \( L \) satisfies Assumption (H3) and \( f \) satisfies (H1), (H2) in Section 2, \( U \in M^\beta(0,T) \) with \( \beta > 2 \). Then, by Theorem 2.10, the reflected G-BSDE (4.1) admits a unique solution \( (\bar{Y}_n^*, \bar{Z}_n^*, \bar{A}_n^*) \) for any \( n \in \mathbb{N} \).

The main objective in this paper is to show that, the solution of doubly reflected G-BSDE can be approximated by the solutions of penalized reflected G-BSDEs (1.1). The advantage of this construction is that \( \bar{Y}_n^* \) is non-increasing in \( n \). Besides, since the assumptions on the parameters are weaker, the existence result extends the one in \([17]\).

To this end, let us furthermore consider the following penalized G-BSDEs parameterized by \( n, m \in \mathbb{N} \):

\[
\begin{align*}
Y_n^{t,m} &= \xi + \int_t^T f(s, Y_n^{t,m}, Z_n^{t,m}) ds - \int_t^T Z_n^{t,m} dB_s - (K_n^{t,m} - K_t^{n,m}) \\
&\quad + \int_t^T m(Y_n^{t,m} - L_s^-) ds - \int_t^T n(Y_n^{t,m} - U_s)^+ ds.
\end{align*}
\]

(4.2)
Set $A^{n,m,+}_t = \int_t^T m(Y^{n,m} - L_s) - ds$ and $A^{n,m,-}_t = \int_t^0 n(Y^{n,m} - U_s) + ds$. Clearly, $A^{n,m,+}$ and $A^{n,m,-}$ are non-decreasing processes and Equation (4.2) can be written as:

$$Y^{n,m}_t = \xi + \int_t^T f(s, Y^{n,m}_s, Z^{n,m}_s)ds - \int_t^T Z^{n,m}_s dB_s - (K^{n,m}_T - K^{n,m}_t) + (A^{n,m,+}_t - A^{n,m,+}_0) - (A^{n,m,-}_t - A^{n,m,-}_0).$$

(4.3)

Remark 4.1 Especially, if $m = n$, set $Y^n = Y^{n,m}$, $Z^n = Z^{n,m}$, $K^n = K^{n,n}$. Then the penalized G-BSDEs (4.2) reduce to Equation (4.1) studied in [17].

In the following, we show that the sequence $(Y^{m,n}, Z^{m,n}, A^{m,n,+} - K^{n,m})$ converges to $(\hat{Y}^n, \hat{Z}^n, \hat{A}^n)$ as $m$ goes to infinity under Assumptions (A1)-(A4). The first step is to prove that $Y^{n,m}$ is uniformly bounded under the norm $\| \cdot \|_{\mathcal{G}_t}$. 

Lemma 4.2 For $2 \leq \alpha < \beta$, there exists a constant $C$ independent of $n, m$, such that

$$\mathbb{E}[\sup_{t \in [0,T]} |Y^{n,m}_t|^{\alpha}] \leq C.$$ 

Proof. Set $Y^*_t = I_t$, $Z^*_t = \sigma^*_t$. It is easy to check that

$$Y^*_t = I_T - \int_t^T Z^*_s dB_s + (A^{1,+}_T - A^{1,+}_t) = I_T + \int_t^T f(s, Y^*_s, Z^*_s)ds - \int_t^T Z^*_s dB_s + (A^{1,+}_T - A^{1,+}_t) - (A^{1,-}_t - A^{1,-}_t),$$

(4.4)

where $A^{1,+}_t = A^{1,+}_t + \int_t^T f^-(s, Y^*_s, Z^*_s)ds$ and $A^{1,-}_t = A^{1,-}_t + \int_t^T f^+(s, Y^*_s, Z^*_s)ds$. Clearly, $A^{1,+}, A^{1,-} \in S^0_{\mathcal{G}}(0, T)$. Consider the following two G-BSDEs:

$$Y^+_t = U_T + \int_t^T f(s, Y^+_s, Z^+_s)ds + (A^{*,+}_T - A^{*,+}_t) - \int_t^T Z^+_s dB_s - (K^+_T - K^+_t),$$

(4.5)

$$Y^-_t = L_T + \int_t^T f(s, Y^-_s, Z^-_s)ds - (A^{*,-}_T - A^{*,-}_t) - \int_t^T Z^-_s dB_s - (K^-_T - K^-_t).$$

(4.6)

By Theorem 2.7, we have for any $t \in [0, T]$, $Y^-_t \leq Y^*_t \leq Y^+_t$, which implies that $Y^+_t \geq L_t$ and $Y^-_t \leq U_t$. Therefore, we may add the terms $\int_t^T m(Y^+_s - L_s) - ds$ and $\int_t^0 n(Y^-_s - U^-_s) + ds$ into Equation (4.5) and (4.6), respectively. By Theorem 2.7 again, for any $t \in [0, T]$ and $n, m$, we have $Y^+_t \leq Y^{n,m}_t \leq Y^+_t$. By the estimates for G-BSDEs (see Theorem 2.3), we get that

$$\mathbb{E}[\sup_{t \in [0,T]} |Y^{*,+}_t|^{\alpha}] \leq C\mathbb{E}[\sup_{t \in [0,T]} \mathbb{E}_t[|U_T + A^{*,+}_T|^{\alpha} + \int_t^T (|f(s, 0, 0)|^{\alpha} + |A^{*,+}_t|^{\alpha})ds]],$$

$$\mathbb{E}[\sup_{t \in [0,T]} |Y^{*,-}_t|^{\alpha}] \leq C\mathbb{E}[\sup_{t \in [0,T]} \mathbb{E}_t[|L_T - A^{*,-}_T|^{\alpha} + \int_t^T (|f(s, 0, 0)|^{\alpha} + |A^{*,-}_t|^{\alpha})ds]].$$

Consequently, there exists a constant $C$ independent of $n, m$ such that

$$\mathbb{E}[\sup_{t \in [0,T]} |Y^{n,m}_t|^{\alpha}] \leq C.$$
Remark 4.3 In fact, according to the proof, Lemma 4.2 still holds if $f$ satisfies (H1) with $\beta > 2$ and Assumption (A3) without (3.1) in it.

The following lemma gives us the explicit convergence rate of $(Y^{n,m} - U)^+$, which is helpful to get the convergence rate of $(Y^n - U)^+$. The latter one is difficult to obtain if we only consider the penalization sequence (4.1) since $\bar{A}^n$ is not a non-increasing $G$-martingale. That is the reason why we introduce the penalization sequence (4.2) with two parameters $n, m$. Assumption (A1) and Equation (3.1) are important for the proof.

Lemma 4.4 For $2 \leq \alpha < \beta$, there exists a constant $C$ independent of $n, m$, such that

$$\hat{E}[ \sup_{t \in [0,T]} |(Y^{n,m}_t - U_t)^+|^\alpha] \leq \frac{C}{n^\alpha}.$$  

Proof. Consider the following G-BSDE:

$$\hat{Y}^{n}_t = U_T + \int_t^T f(s, \hat{Y}^{n}_s, \hat{Z}^{n}_s)ds - \int_t^T n(\hat{Y}^{n}_s - U_s)^+ ds + (A^{n,+}_t - A^{n}_t) - \int_t^T \hat{Z}^{n}_s dB_s - (\hat{K}_T^{n} - \hat{K}_t^{n}).$$  

Noting that $Y^*_t = I_t \leq U_t$, we may add the $- \int_t^T n(Y^*_s - U_s)^+ ds$ term into Equation (4.3). By Theorem 2.7 we have $\hat{Y}^n_t \geq Y^*_t$ and hence $\hat{Y}^n_t \geq L_t$ for any $n \in \mathbb{N}$ and $t \in [0, T]$. Therefore, we may add the $+ \int_t^T m(Y^*_s - L_s)^- ds$ term into Equation (4.3). Applying Theorem 2.4 again implies $\hat{X}^n_t \geq Y^{n,m}_t$. It suffices to prove that for any $2 \leq \alpha < \beta$, there exists a constant $C$ independent of $n, m$, such that

$$\hat{E}[ \sup_{t \in [0,T]} |(\hat{Y}^{n}_t - U_t)^+|^\alpha] \leq \frac{C}{n^\alpha}.$$  

Set $\hat{Y}^{n}_t = \hat{Y}^n_t + A^{n,+}_t$, $\hat{\xi} = U_T + A^{n,+}_T$ and $\hat{U}_t = U_t + A^{n,+}_t$. Equation (4.7) can be written as

$$\hat{Y}^{n}_t = \hat{\xi} + \int_t^T \hat{f}(s, \hat{Y}^{n}_s, \hat{Z}^{n}_s)ds - \int_t^T n(\hat{Y}^{n}_s - U_s)^+ ds - \int_t^T \hat{Z}^{n}_s dB_s - (\hat{K}_T^{n} - \hat{K}_t^{n}),$$  

where $\hat{f}(s, y, z) = f(s, y - A^{n,+}_s, z)$. By Lemma 4.5 in [17] or Lemma 4.5 in [15], we have

$$\hat{E}[ \sup_{t \in [0,T]} |(\hat{Y}^{n}_t - \hat{U}_t)^+|^\alpha] \leq \frac{C}{n^\alpha},$$  

which is the desired result. ■

Then, we show that the sequences $A^{n,m,+}$, $A^{n,m,-}$, $K^{n,m}$ and $Z^{n,m}$ are uniformly bounded.

Lemma 4.5 For $2 \leq \alpha < \beta$, there exists a constant $C$ independent of $n, m$, such that

$$\hat{E}[|A_T^{n,m,+}|^\alpha] \leq C, \; \hat{E}[|A_T^{n,m,-}|^\alpha] \leq C, \; \hat{E}[|K_T^{n,m}|^\alpha] \leq C, \; \hat{E}[\int_0^T |Z_s^{n,m}|^2 ds]^\alpha/2 \leq C.$$  

Proof. By Lemma 4.4 it is easy to check that $\hat{E}[|A_T^{n,m,-}|^\alpha] \leq C$. By Theorem 2.6 we have

$$\hat{E}[\int_0^T |Z_s^{n,m}|^2 ds]^\alpha/2 \leq C \left\{ \hat{E}[ \sup_{t \in [0,T]} |Y^{n,m}_t|^\alpha] + \left( \hat{E}[ \sup_{t \in [0,T]} |Y^{n,m}_t|^\alpha] \right)^{1/2} \times \left( \hat{E}[\int_0^T |f^{n,m}_s ds|^{\alpha/2}] + (\hat{E}[|A_T^{n,m,-}|^\alpha])^{1/2} \right) \right\},$$  

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where \( f_s^0 = f(s, 0, 0) \). By Lemma 4.2 we get that \( \hat{\mathbb{E}}[(\int_0^T |Z_s^n|^2 ds)^{\alpha/2}] \leq C \). Note that

\[
A_T^{n,+}_m - K_T^{n,m} = Y_0^{n,m} - \xi - \int_0^T f(s, Y_s^{n,m}, Z_s^{n,m}) ds + \int_0^T Z_s^{n,m} dB_s + A_T^{n,-}_m.
\]

By simple calculation, we obtain that

\[
\hat{\mathbb{E}}[|A_T^{n,+}_m - K_T^{n,m}|^\alpha] \leq C \left\{ \hat{\mathbb{E}} \left[ \sup_{t \in [0,T]} |Y_t^{n,m}|^\alpha \right] + \hat{\mathbb{E}} \left[ \int_0^T |Z_s^{n,m}|^2 ds \right]^{\alpha/2} \right. \nonumber
\]

\[
+ \hat{\mathbb{E}}[|A_T^{n,-}_m|^\alpha] + \hat{\mathbb{E}} \left[ \int_0^T |f_s^0|^\alpha ds \right].
\]

Since \( A_T^{n,+}_m \) and \(-K_T^{n,m}\) are non-negative, we get the desired result.

By a similar analysis as the proof of Lemma 4.3, Lemma 4.4 and Theorem 5.1 in [16], for any fixed \( n \) and \( 2 \leq \alpha < \beta \), we have

(a) \( \lim_{m \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |(Y_t^{n,m} - L_t)^-|^{\alpha}] = 0 \);

(b) letting \( m \) go to infinity, \((Y_t^{n,m}, Z_t^{n,m}, A_t^{n,m,+} - K_t^{n,m})\) converges to \((\tilde{Y}_t^n, \tilde{Z}_t^n, A_t^n)\), which is the solution of Equation (4.1). More precisely,

\[
\lim_{m \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |\tilde{Y}_t^n - Y_t^{n,m}|^\alpha] = 0,
\]

\[
\lim_{m \to \infty} \hat{\mathbb{E}}[\int_0^T |\tilde{Z}_t^n - Z_t^{n,m}|^2 dt]^{\alpha/2} = 0,
\]

\[
\lim_{m \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |A_t^n - (A_t^{n,+} - K_t^{n,m})|^\alpha] = 0.
\]

Building upon Lemma 4.2, Lemma 4.3, Lemma 4.5 and the statements (a), (b), we have the following result.

**Lemma 4.6** For any \( 2 \leq \alpha < \beta \), there exists a constant \( C \) independent of \( n \), such that

\[
\hat{\mathbb{E}}[\sup_{t \in [0,T]} |\tilde{Y}_t^n|^\alpha] \leq C, \quad \hat{\mathbb{E}}[(\int_0^T |\tilde{Z}_t^n|^2 dt)^{\alpha/2}] \leq C,
\]

\[
\hat{\mathbb{E}}[\sup_{t \in [0,T]} |\tilde{A}_t^n|^\alpha] \leq C, \quad \hat{\mathbb{E}}[\sup_{t \in [0,T]} |(\tilde{Y}_t^n - U_t)^+|^\alpha] \leq \frac{C}{n^\alpha}.
\]

**Proof of Theorem 3.2**

We first prove the uniqueness. The uniqueness for \( Y \) is a direct consequence of Proposition 5.3.

For the uniqueness of \( Z \) and \( A \), we may refer to the proof of Theorem 3.2 in [17].

We then prove the existence. If \( m = n \), Lemma 4.2, Lemma 4.3, Lemma 4.5 still hold. We employ the notations in Remark 4.1. Set \( A^n = A^{n,-} - K^n - A^{n,+} \), where \( A^{n,-} = A^{n,n,-} \) and \( A^{n,+} = A^{n,n,+} \). By a similar analysis as the proof of Lemma 4.7 in [17], we have

\[
\lim_{n,m \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |Y_t^n - Y_t^m|^\alpha] = 0,
\]

\[
\lim_{n,m \to \infty} \hat{\mathbb{E}}[(\int_0^T |Z_s^n - Z_s^m|^2 ds)^{\alpha/2}] = 0,
\]

\[
\lim_{n,m \to \infty} \hat{\mathbb{E}}[\sup_{t \in [0,T]} |A_t^n - A_t^m|^\alpha] = 0.
\]
Denote by $(Y, Z, A)$ the limit of $(Y^n, Z^n, A^n)$. By the proof of Theorem 3.2 in [17], $(Y, Z, A)$ is a solution to doubly reflected $G$-BSDE with parameters $(\xi, f, L, U)$.

Now it remains to prove that $\hat{Y}^n$ converges to $Y$ decreasingly. By Theorem 2.11 for any $n_1 \leq n_2$ and $t \in [0, T]$, we have $\hat{Y}^{n_1} \geq \hat{Y}^{n_2}$. It is sufficient to prove that for any $2 \leq \alpha < 2$,

$$\lim_{n \to \infty} \mathbb{E}[\sup_{t \in [0, T]} |Y_t^n - \hat{Y}_t^n|^\alpha] = 0,$$

(4.8)

Noting that $\hat{Y}_t^n \geq L_t$ for any $n \in \mathbb{N}$ and any $t \in [0, T]$, $(\hat{Y}^n, \bar{Z}^n, \bar{A}^n)$ satisfy the following equation

$$\hat{Y}_t^n = \xi + \int_t^T f(s, \hat{Y}_s^n, \bar{Z}_s^n) ds - \int_t^T \bar{Z}_s^n dB_s + (\bar{A}_t^n - \bar{A}_t^0) - \int_t^T n(\bar{Y}_s^n - U_s)^+ ds + \int_t^T n(\bar{Y}_s^n - L_s)^- ds.$$

By Theorem 2.7 for any $n \in \mathbb{N}$ and $t \in [0, T]$, $\hat{Y}_t^n \geq 0$, where $\hat{Y}_t^n = \hat{Y}_t^n - Y_t^n$. For any constant $r$, applying G-Itô’s formula to $e^{rt}(H^n_t)^{\frac{\alpha}{2}}$, where $H^n_t = |\hat{Y}^n_t|^2$, we have

$$|H_t^n|^{\alpha/2}e^{rt} + \int_t^T e^{rs} |H_s^n|^{\alpha/2} ds + \int_t^T \frac{\alpha}{2} e^{rs} |H_s^n|^{\alpha/2-1} (\hat{Z}_s^n)^2 d[B]_s$$

$$= \alpha(1 - \frac{\alpha}{2}) \int_t^T e^{rs} |H_s^n|^{\alpha/2-1} (\hat{Y}_s^n)^2 d[B]_s - \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} \hat{Y}_s n(Y_s^n - L_s)^- ds$$

$$+ \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} \hat{Y}_s f_s^n ds - \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} (\hat{Y}_s^n \hat{Z}_s^n dB_s - \hat{Y}_s^n dK_s^n - \hat{Y}_s^n d\bar{A}_s^n)$$

$$- \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} \hat{Y}_s n(|\hat{Y}_s^n - U_s|^+ - (Y_s^n - U_s)^+) ds,$$

(4.9)

where $\hat{Z}_t^n = \hat{Z}_t^n - Z_t^n$ and $f_s^n = f(s, \hat{Y}_s^n, \bar{Z}_s^n) - f(s, Y_s^n, Z_s^n)$. Applying the Hölder inequality, we have

$$\int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} \hat{Y}_s f_s^n ds \leq (\alpha + \frac{\alpha^2}{2^2(\alpha - 1)}) \int_t^T e^{rs} |H_s^n|^{\alpha/2} ds$$

$$+ \frac{\alpha(\alpha - 1)}{4} \int_t^T e^{rs} |H_s^n|^{\alpha/2-1} (\hat{Z}_s^n)^2 d[B]_s.$$

Noting that $\hat{Y}_t^n \geq 0$, it is easy to check that

$$+ \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} \hat{Y}_s d\bar{A}_s^n \leq \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} [(\hat{Y}_s^n - L_s) + (Y_s^n - L_s)^-] d\bar{A}_s^n,$$

$$- \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} \hat{Y}_s n(|\hat{Y}_s^n - U_s|^+ - (Y_s^n - U_s)^+) ds \leq 0,$$

$$- \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} \hat{Y}_s n(Y_s^n - L_s)^- ds \leq 0,$$

$$+ \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} \hat{Y}_s dK_s^n \leq 0.$$

(4.10)

Set $M_t^n = \int_0^t \alpha e^{rs} |H_s^n|^{\alpha/2-1} (\hat{Y}_s^n \hat{Z}_s^n dB_s - (\hat{Y}_s^n - L_s) d\bar{A}_s^n)$, which is a $G$-martingale. Letting $r = 1 + \alpha \kappa + \frac{\alpha^2}{4(\alpha - 1)}$, all the above analysis indicates that

$$e^{rt} |\hat{Y}_t^n|^\alpha + (M_t^n - M_0^n) \leq \int_t^T \alpha e^{rs} |H_s^n|^{\alpha/2-1} (Y_s^n - L_s)^- d\bar{A}_s^n.$$
Taking conditional expectations on both sides, we have
\[
|\hat{Y}^n_t|^\alpha \leq C\hat{E}_t[(\int_T^t |H^n_s|^\alpha/2-1(Y^n_s - L_s)^-d\bar{A}^n_s)].
\]
By Theorem 2.4, it is sufficient to prove that there exists some \( \gamma > 0 \), such that
\[
\lim_{n \to \infty} \hat{E}[(\int_0^T |H^n_s|^\alpha/2-1(Y^n_s - L_s)^-d\bar{A}^n_s)^\gamma] = 0.
\]
Indeed, for any \( 1 < \gamma < \beta/\alpha \), we have
\[
\hat{E}[(\int_0^T |H^n_s|^\alpha/2-1(Y^n_s - L_s)^-d\bar{A}^n_s)^\gamma] \
\leq \hat{E} \sup_{s \in [0,T]} |\hat{Y}^n_s|^{(\alpha-2)\gamma} \sup_{s \in [0,T]} ((Y^n_s - L_s)^-)^{\gamma} (\bar{A}^n_T)^\gamma \
\leq \left( \hat{E} \sup_{s \in [0,T]} |\hat{Y}^n_s|^{\alpha\gamma} \right) \frac{\hat{E} \sup_{s \in [0,T]} ((Y^n_s - L_s)^-)^{\alpha\gamma}}{\hat{E} (\bar{A}^n_T)^{\alpha\gamma}} \frac{\hat{E} (\bar{A}^n_T)^{\alpha\gamma}}{\hat{E} (\bar{A}^n_T)^{\alpha\gamma}},
\]
which converges to 0 as \( n \) goes to infinity by Lemma 4.3 and Lemma 4.1, Lemma 4.4 in [17].

**Remark 4.7** Furthermore, for any \( 2 \leq \alpha < \beta \), we have
\[
\lim_{n \to \infty} \hat{E}[(\int_0^T |\hat{Z}^n_s - Z^n_s|^2 ds)^{\beta/2}] = 0, \quad \lim_{n \to \infty} \hat{E} \sup_{t \in [0,T]} |\bar{A}^n_t - A^n_t|^\alpha = 0,
\]
where \( \bar{A}^n_t = \bar{A}^n_t - L^n_t \) and \( L^n_t = \int_0^t n(\bar{Y}^n_s - U_s)ds \). That is, the solution \((Y, Z, A)\) of doubly reflected G-BSDE can be constructed by the penalized reflected G-BSDEs (4.1). We give a short proof here.

Letting \( r = 0, \alpha = 2 \) in Equation (4.9) and applying (4.10), we have
\[
\int_0^T (\hat{Z}^n_s)^2 dB_s \leq \int_0^T 2\hat{Y}^n_s \hat{f}^n_s ds - \int_0^T 2\hat{Y}^n_s \hat{Z}^n_s dB_s + \int_0^T 2\hat{Y}^n_s d\bar{A}^n_s \\
\leq \kappa_\varepsilon \int_0^T (\hat{Y}^n_s)^2 ds + \varepsilon \int_0^T (\hat{Z}^n_s)^2 ds + 2 \sup_{t \in [0,T]} |\hat{Y}^n_t| |\bar{A}^n_t| - \int_0^T 2\hat{Y}^n_s \hat{Z}^n_s dB_s,
\]
where \( \varepsilon > 0 \) and \( \kappa_\varepsilon = 2k + \frac{\alpha^2}{\varepsilon} \). By Proposition 2.3, for any \( \varepsilon' > 0 \), we obtain
\[
\hat{E}[(\int_0^T Y^n_s \hat{Z}^n_s dB_s)^{\beta/2}] \leq C\hat{E}[(\int_0^T |Y^n_s|^2 (\hat{Z}^n_s)^2 ds)^{\beta/2}] \\
\leq C(\hat{E} \sup_{t \in [0,T]} |Y^n_t|^\alpha)^1/2 (\hat{E}[(\int_0^T |\hat{Z}^n_s|^2 ds)^{\beta/2})^{1/2} \\
\leq C\frac{4e^\varepsilon}{\varepsilon'} \hat{E}[(\int_0^T |Y^n_t|^\alpha] + C\varepsilon' \hat{E}[(\int_0^T |\hat{Z}^n_s|^2 ds)^{\beta/2}],
\]
Choosing \( \varepsilon \) and \( \varepsilon' \) small enough, it is easy to check that
\[
\hat{E}[(\int_0^T (\hat{Z}^n_s)^2 ds)^{\beta/2}] \leq C(\hat{E} \sup_{t \in [0,T]} |Y^n_t|^\alpha] + (\hat{E} \sup_{t \in [0,T]} |\hat{Y}^n_t|^\alpha)^1/2 (\hat{E}[(\int_0^T |\bar{A}^n_t|^\alpha]^{1/2}.
\]
It follows from Lemma 4.6 and Equation (4.8) that \( \lim_{n \to \infty} \mathcal{E}[\sup_{t \in [0,T]} |\tilde{A}_t^n - A_t^n|^\alpha] = 0 \). By simple calculation, we have

\[
\mathcal{E}[\sup_{t \in [0,T]} |\tilde{A}_t^n - A_t^n|^\alpha] \leq C \mathcal{E}[\sup_{t \in [0,T]} |\tilde{Y}_t^n|^\alpha + (\int_0^T|\tilde{f}_s^n|ds)^{\alpha} + \sup_{t \in [0,T]} |\int_0^t \tilde{Z}_s^n dB_s|^\alpha]
\leq C[\mathcal{E}[\sup_{t \in [0,T]} |\tilde{Y}_t^n|^\alpha] + \mathcal{E}[\int_0^T |\tilde{Z}_s^n|^2ds]^{\alpha/2}] \to 0, \text{ as } n \to \infty.
\]

The proof is complete.

5 Relation with double obstacle fully nonlinear PDEs

In this section, we establish the relation between double obstacle fully nonlinear PDEs and doubly reflected G-BSDEs studied in the previous sections. To this end, we consider the doubly reflected G-BSDEs in a nonlinear Markovian framework. For simplicity, we only consider the doubly reflected BSDEs driven by 1-dimensional G-Brownian motion with generator \( g \) corresponding to the \( \langle B \rangle \) term equals to 0. Similar results still holds for the other cases.

For each \( 0 \leq t \leq T \) and \( \xi \in L_G^p(\Omega_t), p \geq 2 \), let \( \{X_s^{t,\xi}, t \leq s \leq T\} \) be the solution of the following G-SDE:

\[
X_t^{t,\xi} = x + \int_t^s \tilde{b}(r, X_r^{t,\xi})dr + \int_t^s \tilde{l}(r, X_r^{t,\xi})dB_r + \int_t^s \sigma(r, X_r^{t,\xi})dB_r.
\]  

(5.1)

For any \( (t, x) \in [0,T] \times \mathbb{R} \), the parameters \( (\xi^{t,x}, f^{t,x}, L^{t,x}, U^{t,x}) \) of doubly reflected G-BSDEs take the following form:

\[
\xi^{t,x} = \phi(X_t^{t,x}), \quad f^{t,x}(s, y, z) = f(s, X_s^{t,x}, y, z),
\]

\[
L^{t,x} = h(s, X_s^{t,x}), \quad U^{t,x} = h'(s, X_s^{t,x}),
\]

where \( b, l, \sigma, h, h' : [0,T] \times \mathbb{R} \to \mathbb{R} \), \( \phi : \mathbb{R} \to \mathbb{R} \) and \( f : [0,T] \times \mathbb{R}^3 \to \mathbb{R} \) are deterministic functions and satisfy the following conditions:

(Ai) \( b, l, \sigma, h, h' \) are continuous in \( t \);

(Aii) There exist a positive integer \( k \) and a constant \( \kappa \) such that

\[
|b(t, x) - b(t, x')| + |l(t, x) - l(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq \kappa|x - x'|,
\]

\[
|\phi(x) - \phi(x')| \leq \kappa(1 + |x|^k + |x'|^k)|x - x'|, \quad |h(t, x) - h(t, x')| \leq \kappa|x - x'|,
\]

\[
|f(t, x, y, z) - f(t, x', y', z')| \leq \kappa(1 + |x|^k + |x'|^k)|x - x'| + |y - y'| + |z - z'|;
\]

(Aiii) \( h' \) belongs to the space \( C_{lip}^{1,2}([0,T] \times \mathbb{R}) \), \( h(t, x) \leq h'(t, x) \) and \( h(T, x) \leq \phi(x) \leq h'(T, x) \) for any \( x \in \mathbb{R} \) and \( t \in [0,T] \), where \( C_{lip}^{1,2}([0,T] \times \mathbb{R}) \) is the space of all functions of class \( C^{1,2}([0,T] \times \mathbb{R}) \) whose partial derivatives of order less than or equal to 2 and itself are continuous in \( t \) and Lipschitz continuous with respect to \( x \).

We have the following estimates of G-SDEs, which come from Chapter V of Peng [25].

Proposition 5.1 [25] Let \( \xi, \xi' \in L_G^p(\Omega_t) \) and \( p \geq 2 \). Then we have, for each \( \delta \in [0,T-t] \),

\[
\mathcal{E}_t[\sup_{s \in [t,t+\delta]} |X_s^{t,\xi} - X_s^{t,\xi'}|^p] \leq C|\xi - \xi'|^p,
\]

\[
\mathcal{E}_t[|X_t^{t,\xi}|^p] \leq C(1 + |\xi|^p),
\]

\[
\mathcal{E}_t[\sup_{s \in [t,t+\delta]} |X_s^{t,\xi} - \xi|^p] \leq C(1 + |\xi|^p)\delta^{p/2},
\]

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where the constant $C$ depends on $\kappa, G, p$ and $T$.

We now define

$$u(t, x) := Y_t^{t, x}, \quad (t, x) \in [0, T] \times \mathbb{R}, \tag{5.2}$$

where $Y_t^{t, x}$ is the first component of the solution to doubly reflected $G$-BSDE with parameters $(\ell_t^{t, x}, f_t^{t, x}, L_t^{t, x}, U_t^{t, x})$. Our first observation is that $u$ is a deterministic and continuous function.

**Lemma 5.2** For any fixed $t \in [0, T]$, $u$ is a continuous function in $x$.

**Proof.** By Proposition 3.3 and Proposition 5.1 it is easy to check that, for any $t \in [0, T]$ and any $x, x' \in \mathbb{R}$, there exists a constant $C$ depending on $T, k, \kappa, \sigma$ such that

$$|u(t, x) - u(t, x')|^2 \leq C(|x - x'|^2 + |x - x'|).$$

The proof is complete. □

**Lemma 5.3** For any fixed $x \in \mathbb{R}$, $u$ is continuous in $t$.

**Proof.** For any fixed $t \in [0, T]$, we define $X_s^{t, x} := x, Y_s^{t, x} := Y_t^{t, x}, Z_s^{t, x} := 0, A_s^{t, x} := 0, U_s^{t, x} := h'(t, x)$ and $L_s^{t, x} := h(t, x)$ for $0 \leq s \leq t$. Obviously, $(Y_s^{t, x}, Z_s^{t, x}, A_s^{t, x})_{s \in [0, T]}$ is the solution to the doubly reflected $G$-BSDE with parameters $(\phi(X_s^{t, x}), f_t^{t, x}, L_t^{t, x}, U_t^{t, x})$, where $f_t^{t, x}(s, y, z) = \mathbf{1}_{[t, T]}(s)f(s, X_s^{t, x}, y, z)$. For each fixed $x \in \mathbb{R}$, suppose that $0 \leq t_1 \leq t_2 \leq T$, by Proposition 3.3 there exists a constant $C$ depending on $T, k, \kappa, \sigma, x$, such that

$$|u(t_1, x) - u(t_2, x)|^2 = |Y_0^{t_1, x} - Y_0^{t_2, x}|^2$$

$$\leq C\left( \mathbb{E}\left[ \sup_{t \in [0, T]} |L_t^{t_1, x} - L_t^{t_2, x}|^2 \right] + \mathbb{E}\left[ \sup_{t \in [0, T]} |U_t^{t_1, x} - U_t^{t_2, x}|^2 \right] \right)^{\frac{1}{2}} + C\mathbb{E}[|\phi(X_T^{t_1, x}) - \phi(X_T^{t_2, x})|^2]$$

Note that

$$\sup_{t \in [0, T]} |L_t^{t_1, x} - L_t^{t_2, x}|$$

$$\leq |h(t_1, x) - h(t_2, x)| + \sup_{t \in [t_1, t_2]} |h(t, X_t^{t_1, x}) - h(t, X_t^{t_2, x})| + \sup_{t \in [t, T]} |h(t, X_t^{t_1, x}) - h(t, X_t^{t_2, x})|$$

$$\leq 2 \sup_{t \in [t_1, t_2]} |h(t, x) - h(t_2, x)| + \sup_{t \in [t_1, t_2]} |h(t, X_t^{t_1, x}) - h(t, x)| + \sup_{t \in [t_2, T]} k|X_t^{t_1, x} - X_t^{t_2, x}|$$

$$\leq 2 \sup_{t \in [t_1, t_2]} |h(t, x) - h(t_2, x)| + \sup_{t \in [t_1, t_2]} k|X_t^{t_1, x} - x| + \sup_{t \in [t_2, T]} k|X_t^{t_2, x} - X_t^{t_1, x}|.$$

Letting $\delta = t_2 - t_1$, by Proposition 5.1 we have

$$\lim_{\delta \to 0} \mathbb{E}\left[ \sup_{t \in [0, T]} |L_t^{t_1, x} - L_t^{t_2, x}|^2 \right] = 0.$$

A similar analysis yields that

$$\lim_{\delta \to 0} \mathbb{E}\left[ \sup_{t \in [0, T]} |U_t^{t_1, x} - U_t^{t_2, x}|^2 \right] = 0,$$

$$\lim_{\delta \to 0} \mathbb{E}[|\phi(X_T^{t_1, x}) - \phi(X_T^{t_2, x})|^2] = 0.$$
By simple calculation, we obtain that

\[\int_0^T \left( f(t_1, x, X_{t_2}, Y_{t_2}, Z_{t_2}) - f(t_2, x, X_{t_2}, Y_{t_2}, Z_{t_2}) \right)^2 ds \leq C \int_0^T \left( \left( f(t_1, 0, 0) \right)^2 + |X_{t_1}|^2 + |Y_{t_1}|^2 + |Z_{t_1}|^2 \right) ds \]

Similarly, we define a "parabolic superjet" of \( u \) by \( \mathcal{P}^{2}+(u(t, x)) \) the set of triples \((p, q, X) \in \mathbb{R}^3 \) satisfying

\[ u(s, y) \leq u(t, x) + p(s - t) + q(y - x) + \frac{1}{2} X(y - x)^2 + o(|s - t| + |y - x|^2). \]

Similarly, we define \( \mathcal{P}^{2}-(u(t, x)) \) the "parabolic subjet" of \( u \) by \( \mathcal{P}^{2}-(u(t, x)) := -\mathcal{P}^{2,+}(-u(t, x)). \)

**Definition 5.4** Let \( u \in C((0, T) \times \mathbb{R}) \) and \((t, x) \in (0, T) \times \mathbb{R} \). We denote by \( \mathcal{P}^{2}+(u(t, x)) \) (the "parabolic superjet" of \( u \) at \((t, x)\)) the set of triples \((p, q, X) \in \mathbb{R}^3 \) satisfying

\[ u(s, y) \leq u(t, x) + p(s - t) + q(y - x) + \frac{1}{2} X(y - x)^2 + o(|s - t| + |y - x|^2). \]

Similarly, we define \( \mathcal{P}^{2}-(u(t, x)) \) (the "parabolic subjet" of \( u \) at \((t, x)\)) by \( \mathcal{P}^{2}-(u(t, x)) := -\mathcal{P}^{2,+}(-u(t, x)). \)

**Definition 5.5** Let \( u \) be a continuous function defined on \([0, T] \times \mathbb{R} \). It is called a viscosity:

(i) sub-solution of \( \mathcal{P}^{2}+(u(t, x)) \) if \( u(T, x) \leq \phi(x), \ x \in \mathbb{R}, \) and at any point \((t, x) \in (0, T) \times \mathbb{R}, \) for any \((p, q, X) \in \mathcal{P}^{2}+(u(t, x)), \)

\[ \max \left( u(t, x) - h'(t, x), \ min \left( u(t, x) - h(t, x), -p - F(X, q, u(t, x), x, t) \right) \right) \leq 0; \]

(ii) super-solution of \( \mathcal{P}^{2}-(u(t, x)) \) if \( u(T, x) \geq \phi(x), \ x \in \mathbb{R}, \) and at any point \((t, x) \in (0, T) \times \mathbb{R}, \) for any \((p, q, X) \in \mathcal{P}^{2}-(u(t, x)), \)

\[ \max \left( u(t, x) - h'(t, x), \ min \left( u(t, x) - h(t, x), -p - F(X, q, u(t, x), x, t) \right) \right) \geq 0; \]

(iii) solution of \( \mathcal{P}^{2} \) if it is both a viscosity sub-solution and super-solution.
Denote by \( \{(Y_{s,t,x}^n, Z_{s,t,x}^n, A_{s,t,x}^n)\}_{s \in [t,T]} \) the solution of the following penalized reflected G-BSDEs:

\[
\begin{cases}
Y_{s,t,x}^n = \phi (X_{T,t,x}^s) + \int_s^T f(r, X_{r,t,x}^n, Y_{r,t,x}^n, Z_{r,t,x}^n) \, dr - n \int_s^T (Y_{r,t,x}^n - h'(r, X_{r,t,x}^n)) \, dr \\
- \int_s^T Z_{r,t,x}^n \, dB_r + (A_{T,t,x}^n - A_{s,t,x}^n), \quad s \in [t,T],
\end{cases}
\]

By the results of the previous section, \( Y_{t,x}^n \) is the limit of \( Y_{t,x}^n \). We define

\[ u_n(t, x) := Y_{t,x}^n, \quad (t, x) \in [0, T] \times \mathbb{R}. \]

By Theorem 6.7 in [16], \( u_n \) is the viscosity solution of the following parabolic PDE

\[
\begin{cases}
\min \left( u_n(t, x) - h(t, x), -\partial_t u_n - F_n(D_x^2 u_n, D_x u_n, u_n, x, t) \right) = 0, \quad (t, x) \in [0, T] \times \mathbb{R} \\
u_n(T, x) = \phi (x),
\end{cases}
\]

where

\[ F_n(D_x^2 u, D_x u, u, x, t) = F(D_x^2 u, D_x u, u, x, t) - n(u(t, x) - h'(t, x))^+. \]

**Theorem 5.6** The function \( u \) defined by (5.2) is the unique viscosity solution of the double obstacle problem (5.3).

**Proof.** By the previous results, for each \((t, x) \in [0, T] \times \mathbb{R}\), we have

\[ u_n(t, x) \downarrow u(t, x), \quad \text{as } n \to \infty. \]

Note that \( u_n \) is continuous by Lemma 6.4-6.6 in [16]. Since \( u \) is also continuous, applying Dini’s theorem yields that the sequence \( u^n \) uniformly converges to \( u \) on compact sets. The proof will be divided into the following parts.

**Step 1: subsolution.** For each fixed \((t, x) \in (0, T) \times \mathbb{R}\), let \((p, q, X) \in P^{2,+} u(t, x)\). Suppose that \( u(t, x) = h(t, x) \). Noting that \( u(t, x) \leq h(t, x) \), it is easy to check that

\[
\max \left( u(t, x) - h'(t, x), \min (u(t, x) - h(t, x), -p - F(X, q, u(t, x), x, t)) \right) \leq 0.
\]

Now assume that \( u(t, x) > h(t, x) \). It remains to prove that

\[-p - F(X, q, u(t, x), x, t) \leq 0.\]

By Lemma 6.1 in [9], there exist sequences

\[ n_j \to \infty, \quad (t_j, x_j) \to (t, x), \quad (p_j, q_j, X_j) \in P^{2,+} u_{n_j}(t_j, x_j), \]

such that \((p_j, q_j, X_j) \to (p, q, X)\). Recalling that \( u_n \) is the viscosity solution to equation (5.4), hence a subsolution, we have, for any \( j \),

\[-p_j - F(X_j, q_j, u_{n_j}(t_j, x_j), x_j, t_j) \leq -n_j(u_{n_j}(t_j, x_j) - h'(t_j, x_j))^+ \leq 0.\]

Letting \( j \) go to infinity in the above inequality yields the desired result. Therefore, \( u \) is a subsolution of (5.3).
Step 2: supersolution. For each fixed \((t, x) \in (0, T) \times \mathbb{R}\), and \((p, q, X) \in \mathcal{P}^- u(t, x)\). It is sufficient to show that when \(u(t, x) < h'(t, x)\),

\[-p - F(X, q, u(t, x), x, t) \geq 0.\]

Applying Lemma 6.1 in [3] again, there exist sequences

\[n_j \to \infty, \quad (t_j, x_j) \to (t, x), \quad (p_j, q_j, X_j) \in \mathcal{P}^- u_{n_j}(t_j, x_j),\]

such that \((p_j, q_j, X_j) \to (p, q, X)\). Since \(u_n\) is the viscosity solution to equation (5.4), hence a supersolution, we derive that for any \(j\),

\[-p_j - F_{n_j}(X_j, q_j, u_{n_j}(t_j, x_j), x_j, t_j) \geq 0.\]

Noting that \(u_n\) converges uniformly on compact sets, for \(j\) large enough, \(u_{n_j}(t_j, x_j) < h'(t_j, x_j)\) under the assumption that \(u(t, x) < h'(t, x)\). Therefore, letting \(j\) approach infinity in the above inequality implies that

\[-p - F(X, q, u(t, x), x, t) \geq 0,\]

which is the desired result. Thus, \(u\) is a viscosity solution of (5.3).

By a similar analysis as the proof of Theorem 6.3 in [11], we may obtain that the viscosity solution to (5.3) satisfying polynomial growth is unique. The proof is complete.

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