A MODEL OF THE TWISTED $K$-THEORY RELATED TO BUNDLES OF FINITE ORDER

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Abstract. In the present paper we propose a geometric model of the twisted $K$-theory related to elements of finite order in $H^3(X, \mathbb{Z}) \times [X, BBSU]$ for this purpose we consider the monoid of endomorphisms of the direct limit of matrix algebras which acts on the space of Fredholm operators, the representing space of $K$-theory, in such a way that this action corresponds to the multiplication of $K(X)$ by elements of finite order. Being well-pointed and grouplike, this monoid has the classifying space which is the base of the universal Dold fibration. This allows us to define the corresponding twisted $K$-theory as the group of homotopy classes of sections of the associated fibration of Fredholm operators.

Introduction

The complex $K$-theory is a generalized cohomology theory represented by the $\Omega$-spectrum \( \{K_n\}_{n \geq 0} \), where $K_n = \mathbb{Z} \times BU$ if $n$ is even and $K_n = U$ if $n$ is odd. $K_0 = \mathbb{Z} \times BU$ is an $E_\infty$-ring space, and the corresponding space of units $K_\otimes$ (which is an infinite loop space) is $\mathbb{Z}/2\mathbb{Z} \times BU_\otimes$, where $BU_\otimes$ denotes the space $BU$ with the $H$-space structure induced by the tensor product of virtual bundles of virtual dimension 1. Twistings of the $K$-theory over a compact space $X$ are classified by homotopy classes of maps $X \to B(\mathbb{Z}/2\mathbb{Z} \times BU_\otimes) \simeq K(\mathbb{Z}/2\mathbb{Z}, 1) \times BB$, where $B$ denotes the functor of classifying space. The theorem that $BU_\otimes$ is an infinite loop space was proved by G. Segal [18]. Moreover, the spectrum $BU_\otimes$ can be decomposed as follows: $BU_\otimes = K(\mathbb{Z}, 2) \times BSU$. This implies that the twistings in $K$-theory can be classified by homotopy classes of maps $X \to K(\mathbb{Z}/2\mathbb{Z}, 1) \times K(\mathbb{Z}, 3) \times BSU$. In other words, for a compact space $X$ the twistings correspond to elements in $H^1(X, \mathbb{Z}/2\mathbb{Z}) \times H^3(X, \mathbb{Z}) \times [X, BBSU]$, $[X, BBSU] = bsu^1(X)$, where \( \{bsu^n\}_n \) is the generalized cohomology theory corresponding to the infinite loop space $BSU_\otimes$.

Twisted $K$-theory (under the name “$K$-theory with local coefficients”) has its origins in M. Karoubi’s PhD thesis [10] and in paper of P. Donovan and M. Karoubi [3], where the case of a local coefficient system $\alpha \in \mathbb{Z}/2\mathbb{Z} \times H^1(X, \mathbb{Z}/2\mathbb{Z}) \times H^3_{\text{tors}}(X, \mathbb{Z})$ was studied. The case of general (not necessarily of finite order) twistings from $H^3(X, \mathbb{Z})$ was considered by J. Rosenberg in [16]. A modern survey on this subject (including historical remarks) is given in [11]. A very accessible introduction to the subject is also given in [19].

The twisted $K$-theory corresponding to the twistings coming from $H^1(X, \mathbb{Z}/2\mathbb{Z}) \times H^3(X, \mathbb{Z})$ has been intensively studied during the last decade, but not the general case (as far as the author

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knows). It seems that the reason is that there is no known appropriate geometric model for “nonabelian” twistings from \([X, \text{BBSU}_\otimes]\).

In the present paper we make an attempt to give such a model for elements of finite order in
\[H^3(X, \mathbb{Z}) \times [X, \text{BBSU}_\otimes].\]

For this purpose we consider the monoid of endomorphisms of the direct limit of matrix algebras
\[M_{kl}^\infty(\mathbb{C}) := \lim_{m} M_{kl}^m(\mathbb{C})\]
(the limit is taken over unital homomorphisms). More precisely, in the infinite algebra
\[M_{kl}^\infty(\mathbb{C})\] we fix an increasing filtration by unital subalgebras
\[A_k^m \subset A_k^{m+1} \subset \ldots,\]
\[A_k^m \cong M_k^m(\mathbb{C})\] such that \(A_k^{m+1} = M_k(A_k^m)\) and consider endomorphisms of \(M_{kl}^\infty(\mathbb{C})\) that induced by unital homomorphisms of the form \(h_{m,n}: A_k^m \to A_k^{m+n}\) (for some \(m, n\)), i.e. that are of the form \(M_k(\theta_{m,n})\). Such endomorphisms form the topological monoid \(\text{End}(M_{kl}^\infty(\mathbb{C}))\) which is homotopy equivalent to the direct limit \(\text{Fr}_{kl}^{\infty, l^\infty} := \lim_{m,n} \text{Fr}_{kl}^{m,n}, l^m\),

where \(\text{Fr}_{kl}^{m,n} = \text{Hom}_{\text{alg}}(M_k^m(\mathbb{C}), M_{kl}^{m+n}(\mathbb{C}))\) is the space of unital \(*\)-homomorphisms of matrix algebras, and the limit is not contractible for pairs \((k, l)\) such that \((k, l) = 1\). Note that \(\text{Fr}_{k,1} = \text{PU}(k)\), i.e. for \(m = n = 0\) we return to the known case of abelian twistings of finite order (which is described in the next section). Furthermore, the monoid \(\text{End}(M_{kl}^\infty(\mathbb{C}))\) naturally acts on the space of Fredholm operators and this action induces the multiplication of \(K(X)\) by elements of order \(k\). Moreover, the monoid \(\text{End}(M_{kl}^\infty(\mathbb{C}))\) is well-pointed and grouplike and therefore it has the classifying space which is the base of the corresponding universal principal fibration (in the sense of Dold, i.e. with the WCHP).

In fact, “usual” (abelian) twistings of order \(k\) correspond to automorphisms of \(M_k^\infty(\mathbb{C})\) (which form the group \(\lim_{m} \text{PU}(k^m)\) because of \(\text{Fr}_{kl}^{m,n} = \text{PU}(k^m)\) for \(n = 0\) while nonabelian ones correspond to general endomorphisms. Note that these endomorphisms act on the localization of the space of Fredholm operators over \(l\) by homotopy auto-equivalences, i.e. they are invertible in the sense of homotopy.

This paper is organized as follows.

In Section 1 we give a review of standard material about twisted \(K\)-theory related to twistings from \(H^3(X, \mathbb{Z})\). The definition is based on the conjugation action of the projective unitary group \(\text{PU}(\mathcal{H})\) of a separable Hilbert space \(\mathcal{H}\) on the space of Fredholm operators \(\text{Fred}(\mathcal{H})\), the representing space of complex \(K\)-theory. This action induces the action of the Picard group \(\text{Pic}(X)\) on \(K(X)\) by group automorphisms (Theorem 1). We also consider the specialization of this construction to the case of twistings of finite order in \(H^3(X, \mathbb{Z})\) because precisely this particular case we are going to generalize in what follows.

In Section 2 we study the spaces of unital \(*\)-homomorphisms of matrix algebras \(\text{Fr}_{kl}^{m,n}\) which will play in the subsequent consideration the same role as the groups \(\text{PU}(k)\) for twistings of finite order in \(H^3(X, \mathbb{Z})\).

The key result of Section 3 is Theorem 17 which can be regarded as a counterpart of Theorem 1. It states that in terms of the representing space \(\text{Fred}(\mathcal{H})\) the multiplication of the \(K\)-functor by (not necessarily line) bundles of finite order \(k\) can be represented by some maps
\[\gamma_{kl^{m,n}}: \text{Fr}_{kl^{m,n}} \times \text{Fred}(\mathcal{H}) \to \text{Fred}(\mathcal{H}).\]
In order to organize the particular maps $\gamma_{klm,ln}$ for different $m, n$ in a genuine action on $\text{Fred}(H)$ we should take the direct limit $\lim_{m,n} \text{Fr}_{klm,ln}$. It turns out that this limit naturally is a topological monoid, and we give its precise definition in Section 4. In Section 5 we investigate its action on $K$-theory.

Since $\text{Fr}_{kl\infty,ln} := \lim_{m,n} \text{Fr}_{klm,ln}$ is a well-pointed grouplike topological monoid, it has the classifying space $B \text{Fr}_{kl\infty,ln}$ which is the base of the universal principal $\text{Fr}_{kl\infty,ln}$-fibration. This allows us to define the corresponding twisted $K$-theory as the set of homotopy classes of sections of the associated fibration with fiber the space of Fredholm operators. We do this in Section 6.

In Section 7 we sketch an approach via (a homotopy coherent version of) bundle gerbes.

In Section 8 we define maps $B \text{Fr}_{kl\infty,ln} \times B \text{Fr}_{ku\infty,ln} \to B \text{Fr}_{k+l\infty,ln}$ and the corresponding generalization of the (finite) Brauer group.

Sections 9 and 10 contains some results concerning homotopy types of considered spaces, in particular, a calculation of homotopy groups of $\text{End}(M_{kl\infty}(C))$ (which clarifies the origin of the condition $(k, l) = 1$).

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1. Twisted $K$-theory related to twistings from $H^3(X, \mathbb{Z})$

In order to establish the relation with the subsequent construction of more general twistings, we begin with a review of the standard material about twisted $K$-theory with twistings from $H^3(X, \mathbb{Z})$.

Let $X$ be a compact space, $\text{Pic}(X)$ its Picard group consisting of isomorphism classes of line bundles with respect to the tensor product. The Picard group is represented by the $H$-space $\text{BU}(1) \simeq \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ whose multiplication is given by the tensor product of line bundles or (in the appearance of the Eilenberg-MacLane space) by the addition of two-dimensional integer cohomology classes. In particular, the first Chern class $c_1$ defines the group isomorphism $c_1: \text{Pic}(X) \xrightarrow{\simeq} H^2(X, \mathbb{Z})$. The group $\text{Pic}(X)$ is a subgroup of the multiplicative group of the ring $K(X)$ and therefore it acts on $K(X)$ by group automorphisms. This action is functorial on $X$ and therefore it can be described in terms of classifying spaces (see Theorem [1]).

As a representing space for $K$-theory we take $\text{Fred}(H)$, the space of Fredholm operators on the separable Hilbert space $H$. It is known [2] that for a compact space $X$ the action of $\text{Pic}(X)$ on $K(X)$ is induced by the conjugation action

$$\gamma: \text{PU}(H) \times \text{Fred}(H) \to \text{Fred}(H), \quad \gamma(g, T) = gTg^{-1}$$

of the projective unitary group $\text{PU}(H)$ of the Hilbert space $H$ on $\text{Fred}(H)$. The precise statement is given by the following theorem (recall that $\text{PU}(H) \simeq \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$).
Theorem 1. If $f_\xi: X \to \text{Fred}(\mathcal{H})$ and $\varphi_\zeta: X \to \text{PU}(\mathcal{H})$ represent $\xi \in K(X)$ and $\zeta \in \text{Pic}(X)$ respectively, then the composite map

$$X \xrightarrow{\operatorname{diag}} X \times X \xrightarrow{\varphi_\zeta \times f_\xi} \text{PU}(\mathcal{H}) \times \text{Fred}(\mathcal{H}) \xrightarrow{\gamma} \text{Fred}(\mathcal{H})$$

represents $\xi \otimes \zeta \in K(X)$.

Proof see in [2]. □

It is essential for the theorem that the group $\text{PU}(\mathcal{H})$ has the homotopy type of the classifying space for line bundles $\mathbb{C}P^\infty$ and from the other hand its conjugation action on $\text{Fred}(\mathcal{H})$ induces the action of the Picard group on $K(X)$.

In order to define the corresponding version of $K$-theory consider $\text{Fred}(\mathcal{H})$-bundle $\widetilde{\text{Fred}}(\mathcal{H}) \to \text{BPU}(\mathcal{H})$ associated (by means of the action $\gamma$) with the universal principal $\text{PU}(\mathcal{H})$-bundle over the classifying space $\text{BPU}(\mathcal{H}) \simeq K(\mathbb{Z}, 3)$ for $\text{PU}(\mathcal{H})$, i.e. the bundle

$$\begin{array}{ccc}
\text{Fred}(\mathcal{H}) & \xrightarrow{\text{EPU}(\mathcal{H}) \times_{\text{PU}(\mathcal{H})}} & \text{Fred}(\mathcal{H}) \\
\downarrow & & \downarrow \\
\text{BPU}(\mathcal{H}). & & 
\end{array}$$

Then for any map $f: X \to \text{BPU}(\mathcal{H})$ the corresponding twisted $K$-theory $K_f(X)$ is the set (in fact the group) of homotopy classes of sections $[X, f^*\widetilde{\text{Fred}}(\mathcal{H})]'$ (here $[\ldots, \ldots]'$ denotes the set of fibrewise homotopy classes of sections). The group $K_f(X)$ depends up to isomorphism only on the homotopy class $[f]$ of $f$, i.e. in fact on the corresponding cohomology class in $H^3(X, \mathbb{Z})$ called the Dixmier-Douady class.

Remark 2. Although the isomorphism class of the twisted $K$-theory group only depends on the twisting class in $H^3(X, \mathbb{Z})$, it is important to note that this isomorphism is not natural, but that instead one has a natural action of $H^2(X, \mathbb{Z})$ on such isomorphisms$^1$ [5].

In this paper we will consider twistings of finite order, in the abelian case they are related to subgroups $\text{PU}(k) \subset \text{PU}(\mathcal{H})$, $k \in \mathbb{N}$.

Remark 3. It is not true that for every $\alpha \in H^3(X, \mathbb{Z})$ such that $k\alpha = 0$ there exists a $\text{PU}(k)$-bundle with Dixmier-Douady class $\alpha$: in general one has to consider all groups $\text{PU}(k^n)$, $n \in \mathbb{N}$. For example, for $k = 2$ there is no factorization $K(\mathbb{Z}/2\mathbb{Z}, 2) \to \text{BPU}(2) \to K(\mathbb{Z}, 3)$ of the Bockstein map $K(\mathbb{Z}/2\mathbb{Z}, 2) \to K(\mathbb{Z}, 3)$ (otherwise applying the loop functor we obtain the factorization from Remark 4 below which clearly does not exist $^1$).

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on the separable Hilbert space $\mathcal{H}$, $M_k(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^{\otimes k})$ the matrix algebra over $\mathcal{B}(\mathcal{H})$ (of course, it is isomorphic to $\mathcal{B}(\mathcal{H})$), $M_k(\mathbb{C}) \to M_k(\mathcal{B}(\mathcal{H}))$ the inclusion induced by the inclusion of the unit $\mathbb{C} \to \mathcal{B}(\mathcal{H})$, $1 \leftrightarrow \operatorname{Id}$. Thereby $\mathcal{U}(k)$ is a subgroup of the unitary group $\mathcal{U}_k(\mathcal{H})$ of the algebra $M_k(\mathcal{B}(\mathcal{H}))$, and we have the injective homomorphism

$$i_k: \text{PU}(k) \hookrightarrow \text{PU}_k(\mathcal{H}),$$

where $\text{PU}_k(\mathcal{H})$ is the projective unitary group on $\mathcal{H}^{\otimes k}$ (of course, $\text{PU}_k(\mathcal{H}) \cong \text{PU}(\mathcal{H})$).

$^1$The author is grateful to Thomas Schick who pointed me out to this important fact.
The group $PU(k)$ is the base of the principal $U(1)$-bundle
\begin{equation}
    U(1) \to U(k) \xrightarrow{\chi_k} PU(k).
\end{equation}
Let $\vartheta_{k,1} \to PU(k)$ be the complex line bundle associated with (4) (we introduce the subscripts in $\vartheta_{k,1}$ for unification with the subsequent notation). Analogously, $PU(H)$ is the base of the universal principal $U(1)$-bundle
\begin{equation}
    U(1) \to U(H) \to PU(H).
\end{equation}
Let $[k]$ be the trivial $\mathbb{C}^k$-bundle over $X$.

**Proposition 4.** If a line bundle $\zeta \to X$ satisfies the condition
\begin{equation}
    [k] \otimes \zeta = \zeta \otimes [k] \cong X \times \mathbb{C}^k,
\end{equation}
then its classifying map $\varphi_\zeta : X \to PU_k(H) \cong PU(H)$ can be lifted to a map $\tilde{\varphi}_\zeta : X \to PU(k)$ (see (3)) such that $i_k \circ \tilde{\varphi}_\zeta \simeq \varphi_\zeta$, and vice versa. In particular, $\zeta \cong \tilde{\varphi}_\zeta^*(\vartheta_{k,1})$.

**Proof.** Extend exact sequence (4) to the right to fibration
\begin{equation}
    PU(k) \xrightarrow{\psi_k} BU(1) \xrightarrow{\omega_k} BU(k).
\end{equation}
In particular, $\psi_k : PU(k) \to BU(1) \cong \mathbb{C}P^\infty$ is a classifying map for $U(1)$-bundle $\chi_k$ (4). It is easy to see that the diagram
\begin{equation}
    \begin{array}{ccc}
    PU(k) & \xrightarrow{\psi_k} & BU(1) \\
    & \searrow \downarrow \psi & \\
    & & PU_k(H)
    \end{array}
\end{equation}
commutes, where the vertical arrow is a classifying map for the bundle $U(1) \to U(H) \to PU(H)$.

Let $\zeta \to X$ be a line bundle satisfying condition (5), $ \varphi_\zeta : X \to PU_k(H) \cong PU(H)$ its classifying map. Since $\omega_k$ (see (3)) is induced by taking the direct sum of a line bundle with itself $k$ times (followed by the extension of the structural group to $U(k)$), we see that $\omega_k \circ \varphi_\zeta \simeq \ast$. Now it is easy to see from the exactness of (6) that $\varphi_\zeta : X \to BU(1)$ has a lift $\tilde{\varphi}_\zeta : X \to PU(k)$, and hence the same is true for $\varphi_\zeta$. □

**Remark 5.** Note that the choice of a lift $\tilde{\varphi}_\zeta$ corresponds to the choice of trivialization (5): two choices differ by a map $X \to U(k)$. Thus, a lift is defined up to the action of $[X, U(k)]$ on $[X, PU(k)]$. The subgroup in $Pic(X)$ consisting of (classes of) line bundles satisfying condition (5) is $\text{im}\{\psi_{k*} : [X, PU(k)] \to [X, \mathbb{C}P^\infty]\}$ (or the factor-group $[X, PU(k)]/[X, U(k)]: [X, U(k)]$ is a normal subgroup in $[X, PU(k)]$ because it is the kernel of the group homomorphism $i_{k*} : [X, PU(k)] \to [X, PU_k(H)]$, cf. (3)).

**Remark 6.** We do not claim that every element $\zeta \in Pic(X)$, $\zeta^k = 1$ can be represented by a map $X \to PU(k)$. For example, for $k = 2$ there is no factorization $K(\mathbb{Z}/2\mathbb{Z}, 1) \simeq \mathbb{R}P^\infty \to PU(2) \to \mathbb{C}P^\infty$ of the Bockstein map $K(\mathbb{Z}/2\mathbb{Z}, 1) \simeq \mathbb{R}P^\infty \to \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$. In order to obtain all elements of order $k$ in the sense of the group structure on $Pic(X)$ one has to consider all subgroups $PU(k^n)$, $n \in \mathbb{N}$ (cf. Remark (3)).
Let \( \text{Fred}_k(\mathcal{H}) \) be the subspace of Fredholm operators in \( M_k(\mathcal{B}(\mathcal{H})) \). Clearly, \( \text{Fred}_k(\mathcal{H}) \cong \text{Fred}(\mathcal{H}) \). Being a subgroup in \( \text{PU}_k(\mathcal{H}) \) (see (3)), the group \( \text{PU}(k) \) acts on \( M_k(\mathcal{B}(\mathcal{H})) \). Let

\[
\gamma_{k,1}: \text{PU}(k) \times \text{Fred}_k(\mathcal{H}) \to \text{Fred}_k(\mathcal{H})
\]

be the restriction of this action on \( \text{Fred}_k(\mathcal{H}) \subset M_k(\mathcal{B}(\mathcal{H})) \). Then the diagram

\[
\begin{array}{ccc}
\text{PU}_k(\mathcal{H}) \times \text{Fred}_k(\mathcal{H}) & \xrightarrow{\gamma} & \text{Fred}_k(\mathcal{H}) \\
\downarrow \gamma_{k,1} \downarrow \downarrow \downarrow & \ & \downarrow \\
\text{PU}(k) \times \text{Fred}_k(\mathcal{H}) & \xrightarrow{i_k \times \text{id}} & \text{Fred}_k(\mathcal{H})
\end{array}
\]

commutes and we have the following theorem which is a specialization of Theorem 1.

**Theorem 7.** Let \( f_\xi: X \to \text{Fred}_k(\mathcal{H}) \) be a representing map for some element \( \xi \in K(X) \). Let \( \zeta \) be as in the previous proposition. Then the composite map

\[
X \xrightarrow{\text{diag}} X \times X \xrightarrow{\bar{\zeta} \times f_\xi} \text{PU}(k) \times \text{Fred}_k(\mathcal{H}) \xrightarrow{\gamma_{k,1}} \text{Fred}_k(\mathcal{H})
\]

represents the element \( \xi \otimes \zeta \in K(X) \).

**Remark 8.** Note that the “subgroup” \( U(k) \to \text{PU}(k) \) acts homotopy trivially on \( \text{Fred}_k(\mathcal{H}) \) (and hence trivially on \( K(X) \)), in accordance with Remark 5. Indeed, if \( \varphi_\zeta \) can be lifted to \( U(k) \), then \( \zeta \cong [1] \) is a trivial line bundle over \( X \), from the other hand the action of \( U(k) \) on \( \text{Fred}_k(\mathcal{H}) \) can be extended to the action of the contractible group \( U_k(\mathcal{H}) \cong U(\mathcal{H}) \).

**Remark 9.** It follows from the definition of the inclusion \( i_k \) that the action \( \gamma_{k,1} \) is trivial on elements in \( K(X) \) of the form \( k\xi \). Indeed, a classifying map for \( k\xi \) can be decomposed as \( X \xrightarrow{f_\xi} \text{Fred}(\mathcal{H}) \xrightarrow{\text{diag}} \text{Fred}_k(\mathcal{H}) \). From the other hand, there is the relation \((1 + (\zeta - 1)) \cdot k\xi = k\xi + 0 = k\xi \) in the \( K \)-functor, or, equivalently, \( \zeta \otimes ([k] \otimes \xi) = (\zeta \otimes [k]) \otimes \xi = [k] \otimes \xi \) in terms of bundles.

Note that since inclusion (3) is a group homomorphism, the group structure on \( \text{PU}(k) \) corresponds to the tensor product of line bundles that are classified by this group.

Consider \( \text{Fred}_k(\mathcal{H}) \)-bundle

\[
\begin{array}{ccc}
\text{Fred}_k(\mathcal{H}) & \longrightarrow & \text{EPU}(k) \times \text{Fred}_k(\mathcal{H}) \\
\downarrow & & \downarrow \\
\text{BPU}(k) & & 
\end{array}
\]

associated with the universal principal \( \text{PU}(k) \)-bundle \( \text{EPU}(k) \to \text{BPU}(k) \) by means of the action \( \gamma_{k,1} \). This bundle is the pullback of (2) with respect to \( i_k \). We will denote it by \( \widetilde{\text{Fred}_k(\mathcal{H})} \to \text{BPU}(k) \) for short.

Now the version of the twisted \( K \)-theory related to the conjugation action of \( \text{PU}(k) \) on \( \text{Fred}_k(\mathcal{H}) \), or, equivalently, to the action of the group of (isomorphism classes of) line bundles classified by maps \( X \to \text{PU}(k) \) on \( K(X) \), is defined as follows: for a given map \( f: X \to \text{BPU}(k) \) we define \( K_f(X) \) as the set \([X, f^*(\widetilde{\text{Fred}_k(\mathcal{H})})]\)' of homotopy classes of sections of the induced bundle \( f^*(\widetilde{\text{Fred}_k(\mathcal{H})}) \to X \).
Note that up to (noncanonical) isomorphism the twisted $K$-theory depends only on the cohomology class $\beta = f^*(\alpha) \in H^3(X, \mathbb{Z})$, where $\alpha \in H^3(BPU(k), \mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$ is the generator, therefore the more appropriate notation for it is $K_\beta(X)$.

Note that the considered constructions are well-behaved with respect to the group homomorphisms

$$PU(k^n) \to PU(k^{n+1}), \ T \mapsto T \otimes E_k,$$

i.e. we can take the corresponding direct limits.

There is another way to define the twisted $K$-theory. Namely, let $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators on the separable Hilbert space $\mathcal{H}$. Recall that the group of $*$-automorphisms of the algebra $\mathcal{K}(\mathcal{H})$ is $PU(\mathcal{H})$. For a given $PU(\mathcal{H})$-cocyce on $X$ consider the corresponding $\mathcal{K}(\mathcal{H})$-bundle $A \to X$ and define the corresponding twisted $K$-theory as the algebraic $K$-theory of the Banach algebra $\Gamma(A, X)$ of its continuous sections (we should only remember that the algebra $\mathcal{K}(\mathcal{H})$ is not unital). If the Dixmier-Douady class of $A$ has finite order, then the $\mathcal{K}(\mathcal{H})$-bundle $A \to X$ is of the form $A_k \otimes \mathcal{K}(\mathcal{H})$, where $A_k \to X$ is a matrix algebra bundle with fiber $M_k(\mathbb{C})$ (for some $k$). In this case the twisted $K$-theory can be defined as the $K$-theory of the algebra of sections of $A_k \to X$.

The specific property of the finite-dimensional case is that algebras of sections of nonisomorphic bundles $A_k \to X$ and $A'_m \to X$ can be Morita-equivalent, i.e. they can define the same element in the Brauer group $Br(X)$ (note that if in addition $(k, m) = 1$, then $\Gamma(X, A_k)$ is Morita-equivalent to $C(X)$). This happens precisely when $A_k \otimes \mathcal{K}(\mathcal{H}) \cong A'_m \otimes \mathcal{K}(\mathcal{H})$ as $\mathcal{K}(\mathcal{H})$-bundles (let us notice the relation of this fact to Remarks 5 and 8). In fact, there is the group isomorphism $Br(X) \cong H^3(X, \mathbb{Z})$ defined by the assignment to an algebra bundle its Dixmier-Douady class. The torsion subgroup in $Br(X)$, the so-called “finite Brauer group”, corresponds to (finite dimensional) matrix algebra bundles.

For a fixed $\alpha \in H^3(X, \mathbb{Z})$, $\alpha \neq 0$ the twisted $K$-theory $K_\alpha(X)$ is not a ring, only a $K(X)$-module. However there are maps $K_\alpha(X) \otimes K_\beta(X) \to K_{\alpha+\beta}(X)$ which equip the direct sum $\bigoplus_{\alpha \in Br(X)} K_\alpha(X)$ with the structure of a graded ring.

## 2. Spaces of unital homomorphisms of matrix algebras

In this section we study spaces of unital $*$-homomorphisms of matrix algebras. They can be regarded as analogs of groups of $*$-automorphisms $Aut(M_k(\mathbb{C})) \cong PU(k)$ in the subsequent constructions.

Fix a pair of positive integers $\{k, l\}$, $(k, l) = 1$. Let $Fr_{klm,lm}$ be the space of unital $*$-homomorphisms of matrix algebras $\text{Hom}_{\text{alg}}(M_{klm}(\mathbb{C}), M_{klm+nn}(\mathbb{C}))$. Recall that the group of $*$-automorphisms of the complex matrix algebra $M_n(\mathbb{C})$ is the projective unitary group $PU(n)$, therefore there are the left action of $PU(kl^{m+n})$ and the right action of $PU(kl^m)$ on $Fr_{klm,lm}$. Moreover, $Fr_{klm,lm}$ is a (left) homogeneous space over $PU(kl^{m+n})$:

**Proposition 10.** There is an isomorphism of homogeneous spaces

$$Fr_{klm,lm} \cong PU(kl^{m+n})/(E_{klm} \otimes PU(l^m)),$$

where $E_{klm}$ is the space of unital elements of $M_{klm}(\mathbb{C})$. 

Note that up to (noncanonical) isomorphism the twisted $K$-theory depends only on the cohomology class $\beta = f^*(\alpha) \in H^3(X, \mathbb{Z})$, where $\alpha \in H^3(BPU(k), \mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$ is the generator, therefore the more appropriate notation for it is $K_\beta(X)$.
where $E_n$ and the symbol "$\otimes$" denote the unit matrix and the Kronecker product of matrices respectively.

Proof. It follows from Noether-Skolem's theorem that the group $\text{PU}(kl^{m+n})$ acts transitively on the set of unital $*$-homomorphisms $M_{kl^m}(C) \to M_{kl^{m+n}}(C)$. From the other hand, the stabilizer of such homomorphism $M_{kl^m}(C) \to M_{kl^{m+n}}(C)$, $T \mapsto T \otimes E_l^n$ is the subgroup $E_{kl^m} \otimes \text{PU}(l^n) \subset \text{PU}(k^{l^m+n})$. □

In particular, for $n = 0$ we have $\text{Fr}_{kl^m,1} = \text{PU}(k^m)$.

Proposition 11. A map $\varphi: X \to \text{Fr}_{kl^m,l^n}$ is the same thing as an embedding of trivial bundles $X \times M_{kl^m}(C) \ni (x, T) \mapsto X \times M_{kl^{m+n}}(C)$ whose restriction to a fiber is a unital $*$-homomorphism of matrix algebras.

Proof. We have the bijection (in obvious notation) $\text{Mor}(X, \text{Hom}_{alg}(M_{kl^m}(C), M_{kl^{m+n}}(C))) \cong \text{Mor}(X \times M_{kl^m}(C), M_{kl^{m+n}}(C)), \ h(x)(T) \mapsto h(x, T), \ x \in X, T \in M_{kl^m}(C)$. But for any map $\lambda: X \times M_{kl^m}(C) \to M_{kl^{m+n}}(C)$ there exists the unique map $\nu: X \times M_{kl^m}(C) \to X \times M_{kl^{m+n}}(C)$, $\nu(x, T) = (x, \lambda(x, T))$ which is the identity on the first factor $X$. □

For an embedding $\mu$ as in the statement of Proposition 11 one can define the subbundle

$$B_{l^n} \subset X \times M_{kl^{m+n}}(C)$$

of centralizers for the image of $\mu$ which is an $M_{l^n}(C)$-bundle such that $M_{kl^m}(C) \otimes B_{l^n} = X \times M_{kl^{m+n}}(C)$.

In particular, applying the previous proposition to $\text{id}: \text{Fr}_{kl^m,l^n} \to \text{Fr}_{kl^m,l^n}$ we obtain the canonical embedding $\widetilde{\mu}: \text{Fr}_{kl^m,l^n} \times M_{kl^m}(C) \hookrightarrow \text{Fr}_{kl^m,l^n} \times M_{kl^{m+n}}(C)$, $(h, T) \mapsto (h, h(T))$ and the corresponding $M_{l^n}(C)$-bundle $B_{kl^m,l^n} \to \text{Fr}_{kl^m,l^n}$. Clearly, we have the canonical isomorphism $M_{kl^m}(C) \otimes B_{kl^m,l^n} \cong \text{Fr}_{kl^m,l^n} \times M_{kl^{m+n}}(C)$ with the trivial bundle, but let us notice that the bundle $B_{kl^m,l^n} \to \text{Fr}_{kl^m,l^n}$ itself is not trivial for $n > 0$, as it follows from the next proposition.

Proposition 12. The $M_{l^n}(C)$-bundle $B_{kl^m,l^n} \to \text{Fr}_{kl^m,l^n}$ is associated with the principal $\text{PU}(l^n)$-bundle

$$(10) \quad \text{PU}(l^n) \to \text{PU}(k^{l^m+n}) \to \text{Fr}_{kl^m,l^n}$$

(see (2)).

Proof. is trivial. □

Note that with respect to the above notation we have $B_{l^n} = \varphi^*(B_{kl^m,l^n})$.

There is the homeomorphism (cf. (3))

$$(11) \quad \text{Fr}_{kl^m,l^n} \cong U(k^{l^m+n})/(E_{kl^m} \otimes U(l^n)),$$
therefore we have the principal $U(l^n)$-bundle (cf. (11))
\[
U(l^n) \rightarrow U(kl^{m+n}) \rightarrow Fr_{kl^{m},l^n}
\]
(12)

over $Fr_{kl^{m},l^n}$. Let $\partial_{kl^{m},l^n} \rightarrow Fr_{kl^{m},l^n}$ be the vector $\mathbb{C}^n$-bundle associated with (12). For example, for $n = 0$ we have the line bundle $\partial_{kl^{m},1} \rightarrow PU(kl^{m})$ associated with $U(1) \rightarrow U(kl^{m}) \rightarrow PU(kl^{m})$. Note that $\text{End}(\partial_{kl^{m},l^n}) = B_{kl^{m},l^n}$.

Let $X$ be a compact topological space. By $[n]$ denote the trivial vector bundle with fiber $\mathbb{C}^n$. Note that there is the canonical trivialization $[kl^{m}] \otimes \partial_{kl^{m},l^n} \cong [kl^{m+n}]$ of the bundle $[kl^{m}] \otimes \partial_{kl^{m},l^n} \rightarrow Fr_{kl^{m},l^n}$.

Proposition 13. (cf. Proposition [4]). For any vector $\mathbb{C}^n$-bundle $\eta_n \rightarrow X$ such that
\[
[kl^{m}] \otimes \eta_n \cong [kl^{m+n}]
\]
(13) there is a map $\varphi = \varphi_{\eta_n} : X \rightarrow Fr_{kl^{m},l^n}$ such that $\varphi^*(\partial_{kl^{m},l^n}) \cong \eta_n$, and vice versa. Note that such $\varphi$ is not unique (even up to homotopy): it also depends on the choice of trivialization (13).

Proof. Consider the fibration (cf. (12))
\[
Fr_{kl^{m},l^n} \xrightarrow{\alpha} BU(l^n) \xrightarrow{\beta} BU(kl^{m+n}),
\]
where $\alpha$ classifies $\partial_{kl^{m},l^n}$ as a $\mathbb{C}^n$-bundle and $\beta$ is induced by the group homomorphism $U(l^n) \rightarrow U(kl^{m+n})$, $T \mapsto E_{kl^{m}} \otimes T$ (the Kronecker product of matrices), hence $\beta$ classifies $[kl^{m}] \otimes \zeta_{kl^{m+n}}^\text{univ}$ as a $\mathbb{C}^{kl^{m+n}}$-bundle (here $\zeta_{kl^{m+n}}^\text{univ}$ is the universal $\mathbb{C}^n$-bundle over $BU(l^n)$).

Vector $\mathbb{C}^n$-bundle $\eta_n$ is represented by a map $\varphi' : X \rightarrow BU(l^n)$, but since its composition with $\beta$ is homotopy trivial (because of (13)), we see that $\varphi'$ has a lift $\varphi : X \rightarrow Fr_{kl^{m},l^n}$ with the required property. □

Note that the previous proposition can be applied to all $\eta_n$ such that $[k] \otimes \eta_n \cong [kl^{m}]$, i.e. those of order $k$. Such bundles are classified (in the sense of Proposition [13]) by maps $X \rightarrow Fr_{k,l^n}$ (it is easy to see from fibration (13) with $m = 0$), and there are inclusions $Fr_{k,l^n} \hookrightarrow Fr_{kl^{m},l^n}$ (for example to a homomorphism $h : M_k(\mathbb{C}) \rightarrow M_{kl^{m}}(\mathbb{C})$ we can associate the homomorphism $M_{l^n}(h) : M_{l^n}(M_k(\mathbb{C})) \rightarrow M_{l^n}(M_{kl^{m}}(\mathbb{C})))$).

Remark 14. We do not assert that every bundle $\eta_n \rightarrow X$ of order $k$ in the sense of the group structure $K_\otimes$ can be classified by a map $X \rightarrow Fr_{kl^{m},l^n}$ (cf. Remark [6]). In order to represent all elements of order $k$ for a compact $X$, one has to consider all spaces $Fr_{k,l^{m+n},l^n}$, $r, m, n \in \mathbb{N}$ (cf. Section 10).

The assignment
\[
\{h : M_{kl^{m}}(\mathbb{C}) \rightarrow M_{kl^{m+n}}(\mathbb{C})\} \leftrightarrow \{M_l(h) : M_{l}(M_{kl^{m}}(\mathbb{C})) \rightarrow M_{l}(M_{kl^{m+n}}(\mathbb{C}))\}
\]
defines the map $\iota_{m+1,n} : Fr_{kl^{m},l^n} \rightarrow Fr_{kl^{m+1},l^n}$ (recall that $M_{m}(M_{n}(\mathbb{C})) = M_{mn}(\mathbb{C})$).

The assignment
\[
\{h : M_{kl^{m}}(\mathbb{C}) \rightarrow M_{kl^{m+n}}(\mathbb{C})\} \leftrightarrow \{M_{kl^{m}}(\mathbb{C}) \xrightarrow{h} M_{kl^{m+n}}(\mathbb{C}) \xrightarrow{i} M_{kl^{m+n+1}}(\mathbb{C})\}
\]
where \( i(T) = T \otimes E_l \), defines the map \( t_{m,n+1}: Fr_{klm,n} \to Fr_{klm,n+1} \).

**Proposition 15.** \( t_{m,n+1}^* (\vartheta_{klm,n+1}, l^n) = \vartheta_{klm,n}, t_{m,n+1}^* (\vartheta_{klm,n+1}) = \vartheta_{klm,n} \otimes [l] \).

**Proof** is trivial. \( \square \)

In particular, for \( t_{m,1}: PU(kl^n) \to Fr_{klm,1} \) we have: \( t_{m,1}^* (\vartheta_{klm,1}) = \vartheta_{klm,1} \otimes [l] \) (recall that \( PU(kl^n) = Fr_{klm,1} \)).

3. Relation to K-theory

Recall (see Section 1) that the group \( PU(k) \) acts on the representing space \( Fred(\mathcal{H}) \) of K-theory and this action induces the action of line bundles of order \( k \) on \( K \)-functor. In this section we will show that the spaces of unital homomorphisms of matrix algebras allow us to describe the analogous “action” of arbitrary (not necessarily line) bundles of finite order in terms of the classifying space \( Fred(\mathcal{H}) \).

Again, let \( B(\mathcal{H}) \) be the algebra of bounded operators on the separable Hilbert space \( \mathcal{H} \), \( M_{klm}(B(\mathcal{H})) \) the matrix algebra over \( B(\mathcal{H}) \) (clearly, it is isomorphic to \( B(\mathcal{H}) \)). One can think of \( M_{klm}(B(\mathcal{H})) \) as the algebra of bounded operators on \( H_{klm}(\mathcal{H}) \). Let \( Fred_{klm}(\mathcal{H}) \) be the subspace of Fredholm operators in \( M_{klm}(B(\mathcal{H})) \). Of course, \( Fred_{klm}(\mathcal{H}) \cong Fred(\mathcal{H}) \).

The evaluation map \( Hom_{alg}(M_{klm}(\mathbb{C}), M_{klm+n}(\mathbb{C})) \times M_{klm}(\mathbb{C}) \to M_{klm+n}(\mathbb{C}) \), i.e.

\[
ev_{klm,n}: Fr_{klm,n} \times M_{klm}(\mathbb{C}) \to M_{klm+n}(\mathbb{C}), \quad ev_{klm,n}(h, T) = h(T)
\]

induces the map (cf. (7))

\[
\gamma_{klm,n}: Fr_{klm,n} \times Fred_{klm}(\mathcal{H}) \to Fred_{klm+n}(\mathcal{H}).
\]

**Remark 16.** Note that map (15) can be decomposed as follows

\[
Fr_{klm,n} \times M_{klm}(\mathbb{C}) \to Fr_{klm,n} \times M_{klm}(\mathbb{C}) \to M_{klm+n}(\mathbb{C}) \text{,}
\]

where the last map is the projection \(\mathcal{A}_{klm,n} \to M_{klm+n}(\mathbb{C})\) of the tautological \( M_{klm}(\mathbb{C})\)-bundle \(\mathcal{A}_{klm,n} \to Gr_{klm,n}\) over the matrix Grassmannian \( Gr_{klm,n} = PU(kl^n)/\{ PU(kl^n) \otimes PU(l^n) \} \) which parameterizes unital \( klm \)-subalgebras in the fixed \( klm+n \)-algebra \( M_{klm+n}(\mathbb{C}) \).

Let \( \eta_{ln} \to X \) be a vector \( \mathbb{C}^n \)-bundle over \( X \) satisfying (13), \( \varphi = \varphi_{\eta_{ln}}: X \to Fr_{klm,n} \) its classifying map (in the sense of Proposition [13]), \( B_{ln} = End(\eta_{ln}) \).

**Theorem 17.** (Cf. Theorem [7]). Assume that \( f_\xi: X \to Fred_{klm}(\mathcal{H}) \) represents an element \( \xi \in K(X) \). Then the composition

\[
X \xrightarrow{\text{diag}} \, X \times X \xrightarrow{\varphi \times f_\xi} Fr_{klm,n} \times Fred_{klm}(\mathcal{H}) \xrightarrow{\gamma_{klm,n}} Fred_{klm+n}(\mathcal{H})
\]

represents the element \( \xi \otimes \eta_{ln} \in K(X) \).

**Proof.** If \( \xi \) is represented by a family of Fredholm operators \( F = \{ F_x \} \) on the Hilbert space \( H_{klm} \), then \( \xi \otimes \eta_{ln} \) is represented by the family of Fredholm operators \( \{ F_x \otimes 1_{B_{ln}} \} \) on the Hilbert bundle \( H_{klm} \otimes \eta_{ln} \). It follows from Proposition [14] that \( \varphi \) defines a trivialization of the last bundle, i.e. finally we obtain a family of Fredholm operators in the fixed space \( Fred_{klm+n}(\mathcal{H}) \). \( \square \)
In particular, for \( n = 0 \) (\( \Rightarrow \text{Fr}_{klm,1} = \text{PU}(klm) \)) we have the action of \( \text{PU}(klm) \) on \( \text{Fred}_{klm} (\mathcal{H}) \) (cf. [17]) which corresponds to the tensor product \( \xi \mapsto \xi \otimes \eta_1 \) by the line bundle \( \eta_1 = \varphi^* (\partial_{klm,1}) \) (see Theorem [7]).

**Remark 18.** The previous theorem can be regarded as a generalization of Theorem [7] which corresponds to the special case \( m = n = 0 \), when the space of homomorphisms \( \text{Fr}_{klm,lm} \) is the group \( \text{PU}(k) \) (see Section 1).

Note that the following Theorem [20] can also be specialized to this case: as we have already noticed, the group structure on \( \text{PU}(k) \) corresponds to the tensor product of line bundles classified by this group.

**Remark 19.** (Cf. Remark[9]). Note that the “action” described in Theorem [17] is trivial on elements of the form \( klm \xi \in K(X) \) which are represented by the subspace \( \text{Fred}(\mathcal{H}) \xrightarrow{\text{diag}} \text{Fred}_{klm} (\mathcal{H}) \). Indeed, the center \( \mathbb{C}E_{klm} \subset M_{klm} (\mathbb{C}) \) is fixed under map [13].

Note that the composition of homomorphisms of matrix algebras defines the map

\[
\kappa: \text{Hom}_\text{alg}(M_{klm+n}(\mathbb{C}), M_{klm+n+r}(\mathbb{C})) \times \text{Hom}_\text{alg}(M_{klm}(\mathbb{C}), M_{klm+n}(\mathbb{C})) \rightarrow \text{Hom}_\text{alg}(M_{klm}(\mathbb{C}), M_{klm+n+r}(\mathbb{C})),
\]

i.e.

\[
(18) \quad \kappa: \text{Fr}_{klm+n}, \text{Fr}_{klm,lm} \rightarrow \text{Fr}_{klm,lm+n+r}.
\]

Clearly, the diagram

\[
\begin{array}{ccc}
\text{Fr}_{klm+n}, l \times \text{Fr}_{klm,lm} \times \text{Fred}_{klm} (\mathcal{H}) & \xrightarrow{\text{id}_{Fr} \times \gamma} & \text{Fr}_{klm+n}, l \times \text{Fred}_{klm+n}(\mathcal{H}) \\
\kappa \times \text{id}_{\text{Fred}} & & \gamma \\
\text{Fr}_{klm,lm+n+r} \times \text{Fred}_{klm} (\mathcal{H}) & \xrightarrow{\gamma} & \text{Fred}_{klm+n+r}(\mathcal{H})
\end{array}
\]

is commutative.

Composition [18] corresponds to the composition \( \mu_2 \circ \mu_1 \) of embeddings \( \mu_1, \mu_2 \) corresponding to maps \( \varphi_1: X \rightarrow \text{Fr}_{klm}, l \), \( \varphi_2: X \rightarrow \text{Fr}_{klm+n}, l \) (cf. Proposition [11]). Note that if \( \mu_1, \mu_2 \) correspond to subbundles \( B_{lm}, B_{lm+n} \) respectively, then \( \mu_2 \circ \mu_1 \) corresponds to the subbundle \( B_{lm} \otimes B_{lm+n} \) in \( X \times M_{klm+n+r}(\mathbb{C}) \).

Moreover, composition [18] corresponds to the tensor product \( \xi \otimes \eta_{lm} \otimes \eta_{lm+n} \):

**Theorem 20.** Let \( \varphi_1 := \varphi_{\eta_{lm}}: X \rightarrow \text{Fr}_{klm}, l, \varphi_2 := \varphi_{\eta_{lm+n}}: X \rightarrow \text{Fr}_{klm+n}, l \) be classifying maps for bundles \( \eta_{lm} \rightarrow X, \eta_{lm+n} \rightarrow X \) respectively. Then the composition

\[
X \xrightarrow{\text{diag}} X \times X \times X \xrightarrow{\varphi_2 \times \varphi_1 \times l} \text{Fr}_{klm+n}, l \times \text{Fr}_{klm,lm} \times \text{Fred}_{klm} (\mathcal{H}) \xrightarrow{\lambda} \text{Fred}_{klm+n+r}(\mathcal{H}),
\]

where \( \lambda = \gamma \circ (\text{id}_{Fr} \times \gamma) = \gamma \circ (\kappa \times \text{id}_{\text{Fred}}) \) (see the above diagram) represents \( \xi \otimes \eta_{lm} \otimes \eta_{lm+n} \in K(X) \) (cf. Theorem [17]).

**Proof** is trivial. \( \square \)

In general for a given bundle \( \eta_{lm} \) there are lot of nonequivalent trivializations [13] (i.e. there are lot of homotopy nonequivalent maps \( \varphi \) classifying \( \eta_{lm} \)). However, different trivializations act on \( K \)-functor trivially. The situation is similar to the one in the case of finite Brauer group which is
the quotient of the monoid of isomorphism classes of (finite dimensional) matrix algebra bundles (with respect to the “\(\otimes\)” operation) by the submonoid of “trivial” bundles of the form \(\text{End}(\xi)\).

Recall (see Proposition 1) that a map \(X \to \text{PU}(k)\) is not just a line bundle \(\zeta \to X\) of order \(k\) but also some choice of a trivialization \([k] \otimes \zeta \cong X \times \mathbb{C}^k\). The point is that the action of \(\text{PU}(k)\) on \(\text{Fred}(\mathcal{H})\) factors through the action of \(\text{PU}(\mathcal{H})\), and the action of \(
\text{U}(k)\) factors through the action of the contractible group \(\text{U}(\mathcal{H})\) respectively, hence the necessity of the factorization (cf. Remark 8).

4. Topological monoid \(\text{Fr}_{kl, l}\)

The space \(\text{Fr}_{kl, l}\) itself does not have any natural algebraic operation, but there is composition which relates such spaces. Using these spaces we construct a topological monoid such that maps \(\text{Fr}_{kl, l}\) give rise to its action on the space of Fredholm operators. More precisely, since maps \(\text{Fr}_{kl, l}\) correspond to the multiplication of \(\mathcal{K}(X)\) by \(l^n\)-dimensional bundles (for \(n \in \mathbb{N}\)), the monoid acts on the localization of the space \(\text{Fred}(\mathcal{H})\) over \(l\). In fact, the theory does not depend (up to homotopy) on the choice of \(l\), \((k, l) = 1\), cf. Proposition 36.

So, consider the direct limit of matrix algebras \(M_{kl, l} := \lim_{\to m} M_{kl} (\mathbb{C})\) (the limit is taken over unital \(*\)-homomorphisms) and fix an increasing filtration by unital \(*\)-subalgebras

\[
A_k \subset A_{kl} \subset \ldots \subset A_{klm} \subset A_{klm+1} \subset \ldots, \quad A_{klm} \cong M_{klm} (\mathbb{C})
\]

in it such that \(A_{klm+1} = M_l (A_{klm})\) (the algebra of \(l \times l\)-matrices with elements from \(A_{klm}\)) for all \(m \geq 0\).

Consider the monoid \(\text{End}(M_{kl, l}(\mathbb{C}))\) of endomorphisms of this direct limit. More precisely, an endomorphism \(h \in \text{End}(M_{kl, l}(\mathbb{C}))\) is induced by a unital \(*\)-homomorphism of the form \(h_{m, n}: A_{klm} \to A_{klm+n}\) (for some \(m, n\)), i.e. has the form \(M_l (h_{m, n})\), \(h_{m, n} \in \text{Fr}_{klm, l^n} = \text{Hom}_{\text{alg}}(A_{klm}, A_{klm+n})\). By \(M_{l^n} (h_{m, n})\) we denote the sequence of homomorphisms

\[
M_l (h_{m, n}): A_{klm+r} = M_l (A_{klm}) \to A_{klm+n+r} = M_l (A_{klm+n}), \quad r \in \mathbb{N}.
\]

In particular, for \(n = 0\) we have an automorphism \(M_{l^n} (h_{m, 0}) \in \text{Aut}(M_{kl, l}(\mathbb{C}))\).

Note that the composition of such endomorphisms is well-defined. For example, we define the composition \(M_{l^n} (h_2) \circ M_{l^n} (h_1)\), where \(h_1: A_k \to A_{kl}\) and \(h_2: A_k \to A_{kl}\) are displayed on the
groups are as follows:  

\[ \lim_{\text{monoid}} \text{End}(\mathbb{M}) \]

as \( M_{l_\infty}(M_{l_2}(h_2) \circ h_1) \). Clearly, the composition of endomorphisms is associative and \( M_{l_\infty}(\text{id}_{A_k}) \), i.e. the sequence \( \{\text{id}_{A_k}, \text{id}_{A_{kl}}, \text{id}_{A_{kl^2}}, \ldots\} \) is its unit. This completes the definition of the topological monoid \( \text{End}(M_{kl_\infty}((\mathbb{C})) \).

By assignment to a homomorphism \( h_{m,n} : A_{kl^m} \to A_{kl^{m+n}} \) the homomorphism \( M_r(h_{m,n}) : A_{kl^{m+r}} \to A_{kl^{m+n+r}}, \ r \in \mathbb{N} \) (cf. (20)) we define the embedding \( \text{Fr}_{kl^m,l^n} \to \text{Fr}_{kl^{m+r},l^n} \). Furthermore, the composition of \( M_r(h_{m,n}) \) with the homomorphism \( A_{kl^{m+n+r}} \to M_{l^n}(A_{kl^{m+n+r}}) = A_{kl^{m+n+r+u}}, \ T \mapsto M_{l^n}(T) \) defines the embedding \( \text{Fr}_{kl^{m+r},l^n} \to \text{Fr}_{kl^{m+r},l^n+u} \). The composition of these two embeddings defines the embedding \( \text{Fr}_{kl^m,l^n} \to \text{Fr}_{kl^{m+r},l^n+u} \). Using these maps we define the direct limit \( \lim_{m,n} \text{Fr}_{kl^m,l^n} \).

**Proposition 21.** The monoid \( \text{End}(M_{kl_\infty}((\mathbb{C})) \) is isomorphic to the direct limit \( \lim_{m,n} \text{Fr}_{kl^m,l^n} := \text{Fr}_{kl^m,l^n} \).

**Proof.** Note that for any pair \( m, n \geq 0 \) there is the obvious embedding \( \text{Fr}_{kl^m,l^n} \to \text{End}(M_{kl_\infty}((\mathbb{C})) \). Now the proposition follows from the universal property of the direct limit. \( \square \)

Because of the previous proposition we will denote the monoid \( \text{End}(M_{kl_\infty}((\mathbb{C})) \) also by \( \text{Fr}_{kl^m,l^n} \) (a particular isomorphism \( \text{End}(M_{kl_\infty}((\mathbb{C})) \cong \text{Fr}_{kl^m,l^n} \) is defined by the particular choice of a filtration \( \{A_{kl^m}\}_{m \in \mathbb{N}} \) in \( M_{kl_\infty}((\mathbb{C})) \) as above).

Since \( \text{Fr}_{kl^m,l^n} = \text{PU}(k^{l^n}) \), we see that the subgroup \( \text{Aut}(M_{kl_\infty}((\mathbb{C})) \subset \text{End}(M_{kl_\infty}((\mathbb{C})) \) of the monoid \( \text{End}(M_{kl_\infty}((\mathbb{C})) = \text{Fr}_{kl^m,l^n} \) consisting of \(-\)automorphisms of \( M_{kl_\infty}((\mathbb{C})) \) is \( \text{PU}(k^{l^n}) \) := \( \lim_{m} \text{PU}(k^{l^n}) \).

Note that under the condition \( (k, l) = 1 \) the monoid \( \text{Fr}_{kl^m,l^n} \) is not contractible: its homotopy groups are as follows: \( \pi_r(\text{Fr}_{kl^m,l^n}) = \mathbb{Z}/k\mathbb{Z} \) for \( r \) odd and 0 for \( r \) even (see Proposition 35). (It is easy to see that the monoid \( \text{Fr}_{kl^m,l^n} \) is contractible if and only if \( p \mid k \Rightarrow p \mid l \) for any prime \( p \)). In particular, \( \pi_0(\text{Fr}_{kl^m,l^n}) = 0 \) and hence the monoid is grouplike. Besides, it is a CW-complex, therefore the embedding of the unit is a cofibration and therefore it is well-pointed.
Remark 22. In place of spaces (11) one can consider spaces \( \widetilde{\text{Fr}}_{kl^n, t^n} := \text{SU}(k^{l^n+n})/(E_{kl^n} \otimes \text{SU}(l^n)) \) which are the universal coverings of the corresponding \( \text{Fr}_{kl^n, t^n} \)'s. The corresponding monoid \( \widetilde{\text{Fr}}_{kl^n, t^n} \) is the universal covering \( \widetilde{\text{Fr}}_{kl^n, t^n} \to \text{Fr}_{kl^n, t^n} \) (with fiber the group \( \rho_k \) of \( k \)'th roots of unity). This monoid gives the “SU”-version of the subsequent constructions. In particular, its action on the space of Fredholm operators (cf. the next section) corresponds to the multiplication of \( K(X) \) by SU-bundles of order \( k \).

The monoid \( \text{Fr}_{kl^n, t^n} \) has the filtration \( \text{PU}(k^{l^n}) = \text{Fr}_{kl^n, t^n, 1} \overset{\epsilon_1}{\rightarrow} \text{Fr}_{kl^n, t^n, 1} \overset{\epsilon_2}{\rightarrow} \text{Fr}_{kl^n, t^n, 2} \overset{\epsilon_3}{\rightarrow} \ldots \) Obviously that the multiplication in \( \text{Fr}_{kl^n, t^n} \) induces maps

\[
\mu_{n, s} : \text{Fr}_{kl^n, t^n} \times \text{Fr}_{kl^n, t^n} \to \text{Fr}_{kl^n, t^n+s} .
\]

Note that \( \text{Fr}_{kl^n, t^n} \) in the base space of the vector \( \mathbb{C}^{l^n} \)-bundle \( \partial_{kl^n, t^n} \) which restricts to \( \partial_{kl^n, t^n} \) under the inclusion \( \text{Fr}_{kl^n, t^n} \subset \text{Fr}_{kl^n, t^n} \) (see Proposition 15). Furthermore,

\[
\mu^*_{n, s} (\partial_{kl^n, t^{n+s}}) = \partial_{kl^n, t^n} \boxtimes \partial_{kl^n, t^n} .
\]

We also have \( \epsilon_{n+1}^* (\partial_{kl^n, t^{n+1}}) = \partial_{kl^n, t^n} \otimes [l] \) (see Proposition 15).

5. AN ACTION OF THE MONOID \( \text{Fr}_{kl^n, t^n} \) ON THE SPACE OF FREDHOLM OPERATORS

\( K(X) \) is a commutative ring, therefore its multiplicative group acts on \( K(X) \) by group automorphisms. Invertible elements in \( K(X) \) are virtual bundles of virtual dimension \( \pm 1 \) (which form the group with respect to the tensor product), while the multiplicative group of the localization \( K(X)[\frac{1}{l}] \) over \( l \) consists of virtual bundles of virtual dimension \( \pm l^n \), \( n \in \mathbb{Z} \) (for a compact \( X \) if \( n \) is a big enough positive integer then a virtual bundle of virtual dimension \( l^n \) can be realized by a geometric bundle \( \eta^n \to X \)).

Because of the functoriality of the mentioned action it can be described in terms of the representing space \( \text{Fred}_{kl^n}(\mathcal{H}) \) of the localized \( K \)-theory. In this section we define the action of the monoid \( \text{Fr}_{kl^n, t^n} \) on \( \text{Fred}_{kl^n}(\mathcal{H}) \) which induces the multiplication of \( K \)-functor by bundles of dimensions \( l^n \) of order \( k \) and coincides with maps from Theorem 17 on its finite subspaces \( \text{Fr}_{kl^n, t^n} \subset \text{Fr}_{kl^n, t^n} \).

Let \( M_{kl^n}(\mathcal{B}(\mathcal{H})) := \lim_{\leftarrow m} M_{kl^m}(\mathcal{B}(\mathcal{H})) \) (the limit is taken over unital \( * \)-homomorphisms of matrix algebras which form filtration (19)), \( \text{Fred}_{kl^n}(\mathcal{H}) \) be the subspace of Fredholm operators in \( M_{kl^n}(\mathcal{B}(\mathcal{H})) \).

The tautological action of \( \text{Fr}_{kl^n, t^n} \) on \( M_{kl^n}(\mathbb{C}) \) (recall that \( \text{Fr}_{kl^n, t^n} = \text{End}(M_{kl^n}(\mathbb{C})) \)) defines the action

\[
\gamma_{kl^n, t^n} : \text{Fr}_{kl^n, t^n} \times \text{Fred}_{kl^n}(\mathcal{H}) \to \text{Fred}_{kl^n}(\mathcal{H})
\]

of the monoid \( \text{Fr}_{kl^n, t^n} \) on the space \( \text{Fred}_{kl^n}(\mathcal{H}) \) whose restrictions to “finite” subspaces of the direct limits coincide with maps (10).
Remark 23. Consider the map
\[ \gamma_{kl^\infty, l^\infty} : \text{Fr}_{kl^\infty, l^\infty} \times \text{Fred}_{kl^\infty}(\mathcal{H}) \to \text{Fred}_{kl^\infty, l^\infty+1}(\mathcal{H}) \]
which is the limit of (15) when \( m \to \infty \). According to Theorem 17 it corresponds to the map \( \xi \mapsto \xi \otimes \varphi^*(\vartheta_{kl^\infty, l^\infty}) \), \( \xi \in K(\mathcal{X}[\mathbb{Z}]) \) for \( \varphi : X \to \text{Fr}_{kl^\infty, l^\infty} \) (see the end of the previous section).

Note that the space \( \text{Fred}_{kl^\infty}(\mathcal{H}) \) is the localization of \( \text{Fred}(\mathcal{H}) \) over \( l \) (in the sense that \( l \) is invertible). It is not surprising because the action of the monoid \( \text{Fr}_{kl^\infty, l^\infty} \) relates to the tensor product by \( l^n \)-dimensional bundles (cf. Theorem 37). In particular, \( \text{Fred}_{kl^\infty}(\mathcal{H}) \) represents \( K \)-theory localized over \( l \), i.e. \( [X, \text{Fred}_{kl^\infty}(\mathcal{H})] = K(\mathcal{X}[\mathbb{Z}]). \)

Since \( \pi_0(\text{Fr}_{kl^\infty, l^\infty}) = 0 \) we see that the monoid acts on \( \text{Fred}_{kl^\infty}(\mathcal{H}) \) by homotopy auto-equivalences that are homotopic to the identity map.

Since the monoid \( \text{Fr}_{kl^\infty, l^\infty} \) is grouplike, we see that the set of homotopy classes \([X, \text{Fr}_{kl^\infty, l^\infty}]\) is a group. Then, using (23) we obtain the representation \([X, \text{Fr}_{kl^\infty, l^\infty}] \to \text{Aut}(K(\mathcal{X}[\mathbb{Z}]))\) which is functorial on \( X \) ("\( \text{Aut} \)" denotes group automorphisms).

The monoid \( \text{Fr}_{kl^\infty, l^\infty} = \text{End}(M_{kl^\infty}(\mathbb{C})) \) contains the subgroup \( \text{PU}(kl^\infty) = \text{Aut}(M_{kl^\infty}(\mathbb{C})) \) which in turn contains the "subgroup" \( \text{U}(kl^\infty) \) (corresponding to the direct limit of the canonical epimorphisms \( \text{U}(kl^m) \to \text{PU}(kl^m) \)). The action of groups \( \text{U}(kl^m) \) on spaces \( \text{Fred}_{kl^m}(\mathcal{H}) \) is homotopy trivial, because it factors through the action of the contractible group \( \text{U}(\mathcal{H}) \) (cf. Remark 8). Analogously, the action of groups \( \text{PU}(kl^m) \) on \( \text{Fred}_{kl^m}(\mathcal{H}) \) factors through the action of \( \text{PU}(\mathcal{H}) \).

Consider the fibration
\[ \text{U}(kl^m) \to \text{U}(kl^{m+n})/(E_{kl^m} \otimes \text{U}(l^n)) \to \text{U}(kl^{m+n})/(\text{U}(kl^m) \otimes \text{U}(l^n)) \]
(cf. (11)) and take the direct limit as \( m, n \to \infty \). Since \( \lim_{m,n} \text{U}(kl^{m+n})/(\text{U}(kl^m) \otimes \text{U}(l^n)) \) is contractible (see Lemma 37) and the group \( \lim_{m,n} \text{U}(l^n) \) acts freely on it, we obtain the homotopy equivalence \( \lim_{m,n} \text{U}(kl^{m+n})/(\text{U}(kl^m) \otimes \text{U}(l^n)) \simeq \text{BU}(l^\infty) \simeq \text{lim}_{n} \text{BU}(l^n) \). Now we see that the homotopy nontrivial part of the action of \( \text{Fr}_{kl^\infty, l^\infty} \) on \( \text{Fred}_{kl^\infty}(\mathcal{H}) \) corresponds to the cokernel \( \text{U}(kl^\infty) \to \text{Fr}_{kl^\infty, l^\infty} \) (more precisely, to the cokernel \([X, \text{U}(kl^\infty)] \to [X, \text{Fr}_{kl^\infty, l^\infty}]\) or, equivalently, to the image of the map \( \text{Fr}_{kl^\infty, l^\infty} \to \text{BU}(l^\infty) \) (cf. (14)) which is a classifying map for the direct limit of bundles \( \vartheta_{kl^m, l^n} \).

Note that the space \( \text{Fr}_{kl^\infty, l^\infty} \) classifies bundles \( \eta_{m,n} \to X \) with the following equivalence relation:
\[ \eta_{m} \sim \eta_{n} \iff \eta_{m} \otimes [l^{-m}] \cong \eta_{n} \otimes [l^{-n}] \text{ for some } t \in \mathbb{N}. \]

In other words, the induced action on the localized \( K \)-theory \( K(\mathcal{X}[\mathbb{Z}]) \) is the action of the multiplicative group of equivalence classes \( [23] \) of bundles of the form \( \eta_{m,n} = \varphi^*(\vartheta_{kl^m, l^n}) \), \( \varphi : X \to \text{Fr}_{kl^m, l^n} \subset \text{Fr}_{kl^\infty, l^\infty} \), i.e. those whose classifying maps can be lifted to \( \text{Fr}_{kl^\infty, l^\infty} \) (cf. (14)), and the group structure is induced by the tensor product of such bundles (cf. Theorem 20 and the end of Section 4). Thus, these automorphisms have the form \( \xi \mapsto \xi \otimes \varphi^*(\vartheta_{kl^m, l^n}) \), cf. Theorems 17 and 20 (note that for a compact \( X \) every map \( X \to \text{Fr}_{kl^\infty, l^\infty} \) can be factorized through \( X \to \text{Fr}_{kl^m, l^n} \subset \text{Fr}_{kl^\infty, l^\infty} \) for some \( m, n \in \mathbb{N} \)).

Thus, we obtain the following main theorem.
Theorem 24. For any compact $X$ action (22) on the representing space $\text{Fred}_{k\ell}^\infty(\mathcal{H})$ of $K[\frac{1}{4}]$-theory induces the action $\xi \mapsto \xi \otimes \varphi^*(\theta_{k\ell m, l^n})$ on $K(X)[\frac{1}{4}]$ of the multiplicative group of equivalence classes (22) of bundles $\eta_n = \varphi^*(\theta_{k\ell m, l^n}) \in K(X)[\frac{1}{4}]$, where $\varphi \in [X, \text{Fr}_{k\ell}^\infty, l^\infty]$.

6. A DEFINITION OF THE CORRESPONDING TWISTED $K$-THEORY

In order to define the twisted $K$-theory for more general twistings by analogy with the definition of the twisted $K$-theory from Section 1 first we should do is to construct the classifying space of the topological monoid $\text{Fr}_{k\ell}^\infty, l^\infty$. Fortunately, a well-pointed grouplike topological monoid has the classifying space given, for example, by May’s geometric bar-construction [12], [17], pp. 210-214.

Recall that in our case $\pi_0(\text{Fr}_{k\ell}^\infty, l^\infty) = 0$, i.e. $\pi_0$ is a group and hence our monoid is grouplike.

Thus, there exists the classifying space $B\text{Fr}_{k\ell}^\infty, l^\infty$ and the universal principal $\text{Fr}_{k\ell}^\infty, l^\infty$-quasifibration $E\text{Fr}_{k\ell}^\infty, l^\infty \to B\text{Fr}_{k\ell}^\infty, l^\infty$ (in particular, the space $E\text{Fr}_{k\ell}^\infty, l^\infty$ is aspherical and even contractible because $\text{Fr}_{k\ell}^\infty, l^\infty$ is a CW-complex). Furthermore, there is the homotopy equivalence $\text{Fr}_{k\ell}^\infty, l^\infty \tilde{\to} \Omega B\text{Fr}_{k\ell}^\infty, l^\infty$ (and hence $\pi_r(\Omega B\text{Fr}_{k\ell}^\infty, l^\infty) = \mathbb{Z}/k\mathbb{Z}$ for $r > 0$ even and 0 for $r$ odd, cf. Proposition 35).

Applying the two-sided geometric bar-construction ([17], ibid.) to the action (23) of $\text{Fr}_{k\ell}^\infty, l^\infty$ on $\text{Fred}_{k\ell}^\infty(\mathcal{H}) := \lim_{\to m} \text{Fred}_{k\ell m}(\mathcal{H})$ we construct the $\text{Fred}_{k\ell}^\infty(\mathcal{H})$-quasifibration

$$\text{Fred}_{k\ell}^\infty(\mathcal{H}) \to B\text{Fr}_{k\ell}^\infty, l^\infty$$

over $B\text{Fr}_{k\ell}^\infty, l^\infty$.

But quasifibration (26) is not appropriate for our purpose: we would like (by analogy with the abelian case, cf. (2)) to define the twisted $K$-theory corresponding to a map $f : X \to B\text{Fr}_{k\ell}^\infty, l^\infty$ as the set of homotopy classes of sections of the “induced quasifibration” $f^*(\text{Fred}_{k\ell}^\infty(\mathcal{H})) \to X$, but the problem is that the pull-back of a quasifibration is not a quasifibration in general.

Fortunately, there are constructions that provide locally homotopy trivial fibrations instead of quasifibrations and therefore allow induced fibrations and classification. One of such constructions is M. Fuch’s modified Dold-Lashof construction [9], the other one [20] given by J. Wirth (note that the homotopy type of the classifying space $B\text{Fr}_{k\ell}^\infty, l^\infty$ does not depend on the choice of a particular construction).

Applying one of these constructions we can assume that (26) is a fibration (in the sense of Dold, i.e. with weak covering homotopy property). We propose this fibration as a model for the twisted $K$-theory for twistings corresponding to the action of bundles of order $k$ on $K(X)$ by the tensor product (cf. Theorem 24). More precisely, for a map $f : X \to B\text{Fr}_{k\ell}^\infty, l^\infty$ we define the corresponding twisted $K$-theory as the set $[X, f^*(\text{Fred}_{k\ell}^\infty(\mathcal{H}))]'$ of homotopy classes of sections of the induced fibration $f^*(\text{Fred}_{k\ell}^\infty(\mathcal{H})) \to X$.

In order to obtain the fibration with fiber the $\Omega$-spectrum $\{K_n\}_{n \geq 0}$, we should verify that the homotopy equivalence $\text{Fred}_{k\ell}^\infty(\mathcal{H}) \to \Omega^2 \text{Fred}_{k\ell}^\infty(\mathcal{H})$ is equivariant with respect to action (23) of the monoid $\text{Fr}_{k\ell}^\infty, l^\infty$. For this purpose we can use the version of Bott periodicity for spaces of Fredholm operators given in [3]. Action (23) consists of the composition of inclusions $\text{Fred}_{k\ell m}(\mathcal{H}) \to \text{Fred}_{k\ell m+n}(\mathcal{H})$ induced by inclusions of filtration [19] and the conjugation action of
PU\((kl^{m+n})\) on \(\text{Fred}_{kl^{m+n}}(\mathcal{H})\). It is easy to see that the homotopy equivalences defined in [3] are equivariant with respect to both mentioned types of maps and therefore can be applied fiberwisely to fibration (26).

7. An approach by means of bundle gerbes

In this section we sketch an approach to twisted \(K\)-theory for “higher” twistings by means of some generalization of bundle gerbes [13], [14]. For this purpose we want to combine the idea of bundle gerbes and bundle gerbe modules from [4] with the idea of homotopy transition cocycles from [20] applying to our monoid \(\text{Fr}_{kl^{m+n}}\) and using the observation that the multiplication (21) in the monoid corresponds to the tensor product of bundles (22).

First, let us recall some facts about “abelian” bundle gerbes with Dixmier-Douady class of finite order [4], [13], [14]. For this purpose we want to combine the idea of some generalization of bundle gerbes [13], [14].

Let \(\hat{g}: \text{PU}(k) \to \text{PU}(k)\) be an open cover of a compact space \(X\), \(Y = Y_U\) the disjoint union of all the elements in the open cover, \(\pi: Y \to X\) the corresponding projection, \(Y^{[2]} = Y \times_{\pi} Y\) the fibre product. For a given \(\text{PU}(k)\) 1-cocycle \(g = \{g_{\alpha\beta}\}_{\alpha,\beta \in A}\) one can associate a bundle gerbe \(L \to Y^{[2]}\) as follows:

\[
L_{\alpha\beta} := g^*_{\alpha\beta}(\hat{g}), \quad g_{\alpha\beta}: U_{\alpha\beta} \to \text{PU}(k)
\]

and the product

\[
\theta_{\alpha\beta\gamma}: L_{\alpha\beta} \otimes L_{\beta\gamma} \xrightarrow{\cong} L_{\alpha\gamma}
\]

over \(U_\alpha \cap U_\beta \cap U_\gamma\) is induced by the group structure on \(\text{U}(k)\) (because \(\mu^*(\hat{g}) = \hat{g} \otimes \hat{g}\) for the group multiplication \(\mu: \text{PU}(k) \times \text{PU}(k) \to \text{PU}(k)\)). The bundle gerbe \((L, Y)\) is nontrivial (equivalently, its Dixmier-Douady class \(d(L, Y) \neq 0 \in H^3(X, \mathbb{Z})\) iff there is no lift of \(g\) to a \(\text{U}(k)\)-cocycle \(\tilde{g}\), i.e. there is no \(\text{U}(k)\) 1-cocycle \(\tilde{g}\) such that \(\chi_k \circ \tilde{g} = g\). In other words, the nontriviality of \((L, Y)\) is an obstruction to the existence of such a lift.

Recall [14] that two bundle gerbes \((L, Y)\) and \((L', Y')\) are called stably isomorphic if there are trivial bundle gerbes \(T_1\) and \(T_2\) such that

\[
L \otimes T_1 \cong L' \otimes T_2
\]

(here “\(\otimes\)” denotes the product of bundle gerbes). Recall also that \((L, Y)\) and \((L', Y')\) are stably isomorphic iff \(d(L, Y) = d(L', Y')\). Any stably equivalence class of bundle gerbes with Dixmier-Douady class of finite order in \(H^3(X, \mathbb{Z})\) contains a representative of the above form (i.e. determined by a projective cocycle \(g\) for \(\text{PU}(k)\) for some \(k \in \mathbb{N}\)). Note also that the product of bundle gerbes \(L \otimes L'\) corresponds to the “tensor product” of groups

\[
\tau: \text{PU}(k_1) \times \text{PU}(k_2) \to \text{PU}(k_1) \otimes \text{PU}(k_2) \subset \text{PU}(k_1k_2)
\]

in the sense that the compositions

\[
(U_\alpha \cap U_\beta) \cap (V_\gamma \cap V_\delta) \xrightarrow{\text{diag}} (U_\alpha \cap U_\beta) \times (V_\gamma \cap V_\delta) \xrightarrow{g_{\alpha\beta} \times g'_{\gamma\delta}} \text{PU}(k_1) \times \text{PU}(k_2) \xrightarrow{\tau} \text{PU}(k_1k_2)
\]
(for all $\alpha, \beta \in A$; $\gamma, \delta \in A'$) form a projective cocycle over $Y \times_{\pi'} Y'$ which determines the product bundle gerbe (where $Y \times_{\pi'} Y'$ is the fibre product).

Note that a projective cocycle $g$ with values in $\text{PU}(k)$ not just determines a bundle gerbe but contains some additional information. More precisely, it gives rise to a module over the bundle gerbe $L = g^*(\vartheta_{k,1})$. Its construction is based on the following proposition.

**Proposition 25.** A map $\varphi : X \to \text{PU}(k)$ is nothing but an isomorphism

$$\hat{\varphi} : \varphi^*(\vartheta_{k,1}) \otimes \mathbb{C}^k \to X \times \mathbb{C}^k.$$ 

**Proof.** By definition, the total space $\vartheta_{k,1}$ is the set of equivalence classes $[g, l]$ of pairs $(g, l)$, $(g, l) \sim (gu, u^{-1}l)$, where $g \in U(k)$, $u \in U(1)$, $l \in \mathbb{C}$. Then for $\varphi = \text{id}$, $X = \text{PU}(k)$ isomorphism (31) is defined as follows:

$$[g, l] \otimes w \mapsto (\bar{g}, g(l \otimes w)),$$

where $w \in \mathbb{C}^k$, $\bar{g} = \chi_k(g) \in \text{PU}(k)$. □

Applying this proposition to the projective cocycle $g = \{g_{\alpha\beta}\}$, we obtain isomorphisms

$$\hat{g}_{\alpha\beta} : L_{\alpha\beta} \otimes \mathbb{C}^k \cong U_{\alpha\beta} \times \mathbb{C}^k$$

(recall that $L_{\alpha\beta} = g_{\alpha\beta}^*(\vartheta_{k,1})$). Let $E_{\alpha} \to U_{\alpha}$ be trivial bundles $U_{\alpha} \times \mathbb{C}^k$. Thus we have isomorphisms $\hat{g}_{\alpha\beta} : L_{\alpha\beta} \otimes E_{\beta} \cong E_{\alpha}$ over $U_{\alpha} \cap U_{\beta}$. The cocycle condition $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ gives rise to the “associativity” condition

$$\begin{array}{ccc}
L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes E_{\gamma} & \xrightarrow{\text{id} \times \hat{g}_{\beta\gamma}} & L_{\alpha\beta} \otimes E_{\beta} \\
\theta_{\alpha\beta\gamma} \times \text{id} & & \hat{g}_{\alpha\gamma} \\
L_{\alpha\gamma} \otimes E_{\gamma} & \xrightarrow{\hat{g}_{\alpha\gamma}} & E_{\alpha}
\end{array}$$

over $U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$.

Following [4], denote the set (in fact, the semi-group) of isomorphism classes of bundle gerbe modules over $L = (L, Y)$ by $\text{Mod}(L)$. The corresponding Grothendieck group $K(L)$ is the twisted $K$-theory group $K_{d(L)}(X)$.

For example, if $d(L) = 0$, we have an isomorphism $\text{Mod}(L) \cong \text{Bun}(X)$ with the semi-group $\text{Bun}(X)$ of all isomorphism classes of vector bundles over $X$. Note that this isomorphism is not canonical, but depends on the choice of a trivialization of $L$, i.e. on isomorphisms $L_{\alpha\beta} \cong L_{\alpha}^* \otimes L_{\beta}$ for line bundles $L_{\alpha} \to U_{\alpha}$. Hence even in case of trivial $L$ we can not canonically identify $L$-modules and vector bundles over $X$.

How one can describe $\text{Mod}(L)$? The above discussion shows that there is a close relation between projective bundles and bundle gerbe modules. The precise statement is that

$$\text{Mod}(L)/\text{Pic}(X) \cong \text{Pro}(X, d(L)),$$

the quotient set of $\text{Mod}(L)$ by the (obvious) action of $\text{Pic}(X)$ is $\text{Pro}(X, d(L))$, the set of all isomorphism classes of projective bundles over $X$ with class $d(L)$ (see [4], Proposition 4.4.).
The outlined results concerning abelian bundle gerbes and their modules will serve as a guideline for our generalization.

As above, let \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in A} \) be an open cover of a compact space \( X \), \( Y = \bigcup_{\mathcal{U}} \) the disjoint union of all the elements in the open cover, \( \pi: Y \to X \) the corresponding projection, \( Y^{[2]} = Y \times_\pi Y \) the fibre product. In our generalization the monoid \( Fr_{kl} \subset \infty \) will play the same role as the projective group \( PU(k) \) in the just described abelian case. According to [20], the local description of a fibration with a structural monoid can be given by a homotopy transition cocycle \( g \).

Let us introduce further notation for specific maps between spaces \( Fr_{kl,m,n} \). By \( \iota \) denote maps

\[
Fr_{kl,m,n} \to Fr_{kl,m,n+1}, \; h \mapsto i \circ h,
\]

where \( h \in \text{Hom}_{alg}(A_{kl,m}, A_{kl,m+n}) = Fr_{kl,m,n} \) (see [19]) and \( i: A_{kl,m+n} \hookrightarrow M_{l}(A_{kl,m+n}) = A_{kl,m+n+1} \) is the inclusion in filtration (19). In the matrix form we have

\[
(33) \; Fr_{kl,m,n} \subset Fr_{kl,m,n+1} \setminus Fr_{kl,m,n+1} = \text{the inclusion in filtration (19)}. \quad \text{In the matrix form we have } i(a) = \text{diag}(a), \text{ where } a \in A_{kl,m+n}.
\]

So, firstly, we have a collection of functions \( g_{\alpha \beta}: U_\alpha \cap U_\beta \to Fr_{kl,m,n} \). For simplicity we assume that these functions take values in the subspace \( Fr_{k,l} \subset Fr_{kl,m,n} \), i.e. in fact \( g_{\alpha \beta}: U_\alpha \cap U_\beta \to Fr_{k,l} \).

Note that below we make the analogous assumption for homotopies \( g_{\alpha \beta \gamma}, \) etc.

Denote \( U_{a_0} \cap \ldots \cap U_{a_n} \) by \( U_{a_0 \ldots a_n} \) for short. So, for any ordered pair \( \{ \alpha, \beta \} \in A^2 \) we have a map \( g_{\alpha \beta}: U_{a_0 \ldots a_n} \to Fr_{k,l} \). On triple intersection \( U_{a_0 a_1 a_2} \) we have the composition

\[
M_1(g_{\alpha \beta}) \circ g_{\beta \gamma}: U_{a_0 a_1 a_2} \to Fr_{k,l},
\]

where \( M_1(g_{\alpha \beta}): U_{a_0 a_1 a_2} \to Fr_{k,l}, \) \( M_1(g_{\alpha \beta})(x) = M_1(g_{\alpha \beta}(x)), \) \( x \in U_{a_0 a_1 a_2} \) and “\( \circ \)” here is induced by the composition \( \mu: Fr_{kl,m} \times Fr_{k,l} \to Fr_{kl,m} \times Fr_{k,l} \) of homomorphisms, i.e. \( M_1(g_{\alpha \beta}) \circ g_{\beta \gamma} = \mu(M_1(g_{\alpha \beta}) \times g_{\beta \gamma}) \).

We also have the composition \( \iota \circ g_{\alpha \gamma}: U_{a_0 a_1 a_2} \to Fr_{k,l} \), where \( \iota: Fr_{k,l} \to Fr_{k,l} \) as above.

Under our assumption there is a homotopy

\[
g_{\alpha \beta \gamma}: U_{a_0 a_1 a_2} \times I \to Fr_{k,l}, \; g_{\alpha \beta \gamma}|_{U_{a_0 a_1 a_2} \times \{ 0 \}} = M_1(g_{\alpha \beta}) \circ g_{\beta \gamma}, \; g_{\alpha \beta \gamma}|_{U_{a_0 a_1 a_2} \times \{ 1 \}} = \iota \circ g_{\alpha \gamma}|_{U_{a_0 a_1 a_2} \times \{ 1 \}}.
\]

On 4-fold intersections \( U_{a_0 a_1 a_2 a_3} \) we have the diagram of homotopies:

\[
\begin{align*}
M_2(g_{\alpha \beta}) \circ M_1(g_{\beta \gamma}) & \circ g_{\gamma \delta} \quad \frac{M_2(g_{\alpha \beta}) \circ M_1(g_{\beta \gamma}) \circ g_{\gamma \delta}}{M_2(g_{\alpha \beta}) \circ g_{\gamma \delta}} \quad \frac{M_1(\iota \circ g_{\alpha \gamma}) \circ g_{\gamma \delta}}{M_1(\iota \circ g_{\alpha \gamma}) \circ g_{\gamma \delta}} \\
M_2(g_{\alpha \beta}) \circ g_{\gamma \delta} & = \frac{\iota \circ M_1(g_{\beta \gamma}) \circ g_{\gamma \delta}}{\iota \circ M_1(g_{\beta \gamma}) \circ g_{\gamma \delta}} \quad \frac{\iota \circ \iota \circ g_{\alpha \delta}}{\iota \circ \iota \circ g_{\alpha \delta}} = M_1(\iota \circ \iota \circ g_{\alpha \delta}).
\end{align*}
\]

The equality in the low left corner of the diagram follows from the equality \( M_1(h) \circ i = i \circ h \) (cf. (33)). Note that \( M_1(\iota) \neq \iota \) but \( \iota \circ \iota = M_1(\iota) \circ \iota \) hence the equality in the low right corner and therefore two compositions of homotopies depicted on the above diagram are homotopies between maps

\[
M_2(g_{\alpha \beta}) \circ M_1(g_{\beta \gamma}) \circ g_{\gamma \delta} \quad \text{and} \quad \iota \circ \iota \circ g_{\alpha \delta}: U_{a_0 a_1 a_2 a_3} \to Fr_{k,l}.
\]

We assume that there is a homotopy

\[
g_{\alpha \beta \gamma \delta}: U_{a_0 a_1 a_2 a_3} \times I^2 \to Fr_{k,l},
\]

such that

\[
g_{\alpha \beta \gamma \delta}|_{U_{a_0 a_1 a_2 a_3} \times \{ 0 \}} = M_1(g_{\alpha \beta \gamma}) \circ g_{\gamma \delta}|_{U_{a_0 a_1 a_2 a_3} \times \{ 0 \}}
\]
In particular, we can regard (cf. the above diagram), and so on.

Further, we have $M$ transition cocycles (for projective bundles). Recall (see Proposition 12) that there is a canonical $g$ which are compatible with $M$ are in the same relation to bundle gerbes as homotopy transition cocycles to usual transition cocycles (for projective bundles). Let $B$ be the pullback of $B, Y$ (see (35) in the next section) for two homotopy bundle gerbes $(\alpha, Y)$, $(\beta, Y')$ such that $(\alpha, Y) \sim (\beta, Y')$. Now we can define the stable equivalence relation on the set of homotopy bundle gerbes by analogy with (29).

The general pattern now should be clear. We should consider a collection of “higher” homotopies $g_{a_0 \cdots a_n} \colon U_{a_0 \cdots a_n} \times I^{n-1} \to \text{Fr}_{k,l^n}$ which are compatible with $g_{a_0 \cdots \tilde{a_k} \cdots a_n}$ in the obvious way.

Now we are ready to define a homotopic analog of bundle gerbes. One can say that homotopy bundle gerbes are in the same relation to bundle gerbes as homotopy transition cocycles to usual transition cocycles (for projective bundles). Recall (see Proposition 12) that there is a canonical $M_I(C)$-bundle $B_{klm,I^n} \to \text{Fr}_{klm,I^n}$. Let $B_{a_0 \cdots a_n} \to U_{a_0 \cdots a_n} \times I^{n-1}$ be the pullback of $B_{klm,I^n} \to \text{Fr}_{klm,I^n}$ via $g_{a_0 \cdots a_n} : U_{a_0 \cdots a_n} \times I^{n-1} \to \text{Fr}_{k,l^n}$, i.e. $B_{a_0 \cdots a_n} := g_{a_0 \cdots a_n}(B_{klm,I^n})$. So $B_{a_0 \cdots a_n}$ is an $M_I(C)$-bundle over $U_{a_0 \cdots a_n} \times I^{n-1}$.

For example, we have $M_I(C)$-bundles $B_{\alpha \beta} \to U_{\alpha \beta}$, $B_{\alpha \beta} = g_{\alpha \beta}^*(B_{klm,I^n})$ over double intersections $U_{\alpha} \cap U_{\beta}$ (cf. (27)). We also have $M_I(\mathbb{C})$-bundles $B_{\alpha \beta \gamma} \to U_{\alpha \beta \gamma} \times I$, $B_{\alpha \beta \gamma} = g_{\alpha \beta \gamma}^*(B_{klm,I^n})$ such that $B_{\alpha \beta \gamma} \mid U_{\alpha \beta \gamma} \times \{0\} = B_{\alpha \beta} \otimes B_{\beta \gamma} \mid U_{\alpha \beta \gamma}$ (cf. (21) and (22)) and $B_{\alpha \beta \gamma} \mid U_{\alpha \beta \gamma} \times \{1\} = M_I(B_{\alpha \gamma}) \mid U_{\alpha \beta \gamma}$.

Further, we have $M_I(\mathbb{C})$-bundles $B_{\alpha \beta \gamma \delta} \to U_{\alpha \beta \gamma \delta} \times I^2$, $B_{\alpha \beta \gamma \delta} = g_{\alpha \beta \gamma \delta}^*(B_{klm,I^n})$ such that $B_{\alpha \beta \gamma \delta} \mid U_{\alpha \beta \gamma \delta} \times \{0\} = B_{\alpha \beta \gamma} \otimes B_{\gamma \delta}$, $B_{\alpha \beta \gamma \delta} \mid U_{\alpha \beta \gamma \delta} \times \{1\} = M_I(B_{\alpha \beta \gamma \delta})$, $B_{\alpha \beta \gamma \delta} \mid U_{\alpha \beta \gamma \delta} \times \{0\} \times I = B_{\alpha \beta} \otimes B_{\beta \gamma \delta}$, $B_{\alpha \beta \gamma \delta} \mid U_{\alpha \beta \gamma \delta} \times \{1\} \times I = M_I(B_{\alpha \beta \gamma \delta})$ (cf. the above diagram), and so on.

We call such collection of bundles that are compatible to each other as described above a homotopy bundle gerbe. In particular, we can regard $B_{\alpha \beta \gamma}$ as an analog of bundle gerbe product from $B_{\alpha \beta} \otimes B_{\beta \gamma}$ to $M_I(B_{\alpha \gamma})$ (cf. (23)). Bundles $B_{\alpha \beta \gamma \delta}$ express (the first of infinite number of) associativity conditions.

Using the product of monoids $\text{Fr}_{k\ell \infty, \ell \infty} \times \text{Fr}_{k'\ell \infty, \ell \infty} \to \text{Fr}_{k \ell + u \infty, \ell \infty}$ (see (35) in the next section) for two homotopy bundle gerbes ($B, Y$), ($B', Y'$) one can define their product ($B \otimes B', Y_\pi \times \pi' Y'$) (cf. (30)). We call a homotopy bundle gerbe $(T, Y)$ trivial if the corresponding homotopy transition $\text{Fr}_{k\ell \infty, \ell \infty}$-cocycle can be lifted to the total space of the bundle $PU(k\ell \infty) \to \text{Fr}_{k\ell \infty, \ell \infty}$ (which is the direct limit of principal bundles $PU(k\ell \infty) \to \text{Fr}_{k\ell \infty, \ell \infty}$ with fibers $PU(l\ell)$). Now we can define the stable equivalence relation on the set of homotopy bundle gerbes by analogy with (29).
It was shown in [11] that there is an “analysis-free” definition of twisted K-theory by means of bundle gerbe modules. We have already seen above that such modules can be constructed by projective cocycles. In our situation we can assume that there is the similar relation to the appropriate notion of a “homotopy bundle gerbe modules”. Rather than give a general definition we consider a simple example of (a candidate for) such object below. We start with the following observation (cf. Proposition 25).

**Proposition 26.** A map \( \varphi : X \to \text{Fr}_{k,l} \) is nothing but an isomorphism

\[
(34) \quad \hat{\varphi} : B \otimes M_k(\mathbb{C}) \cong X \times M_{kl}(\mathbb{C}),
\]

where \( B \xrightarrow{M_i(\mathbb{C})} X \) is the pullback \( \varphi^*(B_{k,l}) \).

**Proof.** Recall that \( B_{k,l} = \text{PU}(kl) \times M_i(\mathbb{C}) \), i.e. elements of \( B_{k,l} \) are equivalence classes of pairs \((g, a)\), where \((g, a) \sim (gu, u^{-1}a)\), \( g \in \text{PU}(kl) \), \( u \in \text{PU}(l) = E_k \otimes \text{PU}(l) \subset \text{PU}(kl) \), \( a \in M_i(\mathbb{C}) \).

By \([g, a] \in B_{k,l}\) we denote the corresponding equivalence class. Then isomorphism (34) for \( \varphi = \text{id} \), \( X = \text{Fr}_{k,l} \) is defined by

\[
[g, a] \otimes b \mapsto (\tilde{g}, g(a \otimes b)),
\]

where \( b \in M_k(\mathbb{C}) \) and \( \tilde{g} \in \text{Fr}_{k,l} \) is the coset \( \{gu \mid u \in \text{PU}(l) = E_k \otimes \text{PU}(l) \subset \text{PU}(kl)\} \). \( \square \)

Note that a trivialization of \( B \) is equivalent to a lift of \( \varphi \) to \( X \to \text{PU}(kl) \) in the fibration \( \text{PU}(l) \to \text{PU}(kl) \to \text{Fr}_{k,l} \).

Let \( U = \{U_\alpha\}_{\alpha \in A} \) be an open cover of a compact space \( X \). Suppose that there are trivial \( M_k(\mathbb{C})\)-bundles \( A_\alpha \to U_\alpha \) with given trivialization. Applying the previous proposition, we see that the homotopy transition cocycle \( g \) defines isomorphisms

\[
\tilde{g}_{\alpha\beta} : B_{\alpha\beta} \otimes A_\beta \cong M_i(A_\alpha)
\]

(cf. the discussion after Proposition 25), where the trivialization \( M_i(A_\alpha) \cong U_\alpha \times M_{kl}(\mathbb{C}) \) is defined by the trivialization of \( A_\alpha \). Note that the map

\[
g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \times I \to \text{Fr}_{k,l}^2, \quad g_{\alpha\beta\gamma}|_{U_{\alpha\beta\gamma} \times \{0\}} = M_i(g_{\alpha\beta}) \circ g_{\beta\gamma}, \quad g_{\alpha\beta\gamma}|_{U_{\alpha\beta\gamma} \times \{1\}} = \ell \circ g_{\alpha\gamma}|_{U_{\alpha\beta\gamma}},
\]

defines the map

\[
\hat{g}_{\alpha\beta\gamma} : B_{\alpha\beta\gamma} \otimes A_\gamma \to M_{\ell}(A_\alpha)
\]

which is a homotopy (through isomorphisms) between the composition

\[
B_{\alpha\beta} \otimes B_{\beta\gamma} \otimes A_\gamma \xrightarrow{1 \otimes \hat{g}_{\beta\gamma}} B_{\alpha\beta} \otimes M_i(A_\beta) \cong M_i(B_{\alpha\beta} \otimes A_\beta) \xrightarrow{M_i(\tilde{g}_{\alpha\beta})} M_{\ell}(A_\alpha)
\]

and

\[
M_i(B_{\alpha\gamma}) \otimes A_\gamma \xrightarrow{M_i(\tilde{g}_{\alpha\gamma})} M_{\ell}(A_\alpha).
\]

On four-fold intersections \( U_{\alpha\beta\gamma\delta} \) we have a homotopy between homotopies, etc. This collection of data can be regarded as an analog of a bundle gerbe module over the homotopy bundle gerbe \( B := \{B_{\alpha\beta\gamma\delta}\} \). One can define the notion of isomorphism on such objects, form their direct
sum with the diagonal “action” of the bundle gerbe and therefore define the corresponding semi-
group (whose Grothendieck group is a candidate to the role of the corresponding twisted $K$-theory
localized over $l$), etc.

Let $\text{AB}_l(X)$ be the group of equivalence classes of matrix algebra bundles with fibers $M_n(\mathbb{C})$, $n \in \mathbb{N}$ (it is classified by the $H$-space $\text{BPU}(l^\infty)_0$). It can be regarded as a “noncommutative
analog” of the Picard group $\text{Pic}(X)$ and it acts on the set of homotopy bundle gerbe modules.
Then the counterpart of (32) should be the following: $\text{Mod}(B Fr_k m, n, r, s)$

Taking the direct limits as

$$
\pi(38)
$$

(35) $Fr_k m, n \to Fr_k m, n$ by vector bundles $\vartheta_k m, n \to Fr_k m, n$

(37) $\text{HTC}(X, d(B)) := \text{End}(M_{k l^\infty}(\mathbb{C})) \to \text{End}(M_{k l^\infty}(\mathbb{C}))$.

It is easy to see that maps (36) define the structure of an $H$-space on the direct limit

$$
\pi_r(\text{B Fr}_k l^\infty, l^\infty) = \lim_\pi \mathbb{Z}/k^r \mathbb{Z} = \mathbb{Z}[\frac{1}{k}]/\mathbb{Z} \text{ for } r > 0 \text{ even and } 0 \text{ for } r \text{ odd}.
$$

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Remark 27. For the sake of clarity we have considered the “projective” version of homotopy bundle
gerbes and modules with matrix algebras as fibers. But in order to define twisted $K$-theory one
should consider “linear” version replacing $B_k m, n, r, s$ (see (12)), etc.

Remark 28. In fact, the assignment to the homotopy transition cocycle $g$ the stable equivalence
class of the corresponding homotopy bundle gerbe $B$ corresponds to the projection in the fibration

$$
\text{B}(k l^\infty) \to \text{B Fr}_{k l^\infty, l^\infty} \to \text{B}(\text{BPU}(l^\infty)_0),
$$

i.e. $d(B) \in H^3(X, \mathbb{Z}) \times \text{bsu}_1^1[\frac{1}{k}]$, cf. Remark 29 (and moreover, $d(B)$ has finite order).

8. A GENERALIZATION OF THE BRAUER GROUP

Note that the tensor product of matrix algebras induces the maps

$$
\text{Fr}_{k l^\infty, l^\infty} \times \text{Fr}_{k l^\infty, l^\infty} \to \text{Fr}_{k l^\infty, l^\infty}
$$

Taking the direct limits as $m, n, r, s \to \infty$ we obtain the monoid homomorphism

$$
\text{Fr}_{k l^\infty, l^\infty} \times \text{Fr}_{k l^\infty, l^\infty} \to \text{Fr}_{k l^\infty, l^\infty}
$$

and due to the functoriality of the classifying space constructions the corresponding map of class-
sifying spaces

$$
\text{B Fr}_{k l^\infty, l^\infty} \times \text{B Fr}_{k l^\infty, l^\infty} \to \text{B Fr}_{k l^\infty, l^\infty}.
$$

Note that homomorphisms (35) are defined by the tensor product of the direct limits of matrix
algebras

$$
(37)
$$

\begin{align*}
M_{k l^\infty}(\mathbb{C}) \times M_{k l^\infty}(\mathbb{C}) & \to M_{k l^\infty}(\mathbb{C}) \otimes M_{k l^\infty}(\mathbb{C}) \\
& \cong M_{k l^\infty}(\mathbb{C}).
\end{align*}

It is easy to see that maps (36) define the structure of an $H$-space on the direct limit

$\text{B Fr}_{k l^\infty, l^\infty} := \lim_\pi \text{B Fr}_{k l^\infty, l^\infty}$ (the direct limit is induced by the monoid homomorphisms

$\text{End}(M_{k l^\infty}(\mathbb{C})) \to \text{End}(M_{k l^\infty}(\mathbb{C}))$).

Using Proposition 35 one can compute the homotopy groups of the space $\text{B Fr}_{k l^\infty, l^\infty}$:

$$
\pi_r(\text{B Fr}_{k l^\infty, l^\infty}) = \lim_\pi \mathbb{Z}/k^r \mathbb{Z} = \mathbb{Z}[\frac{1}{k}]/\mathbb{Z} \text{ for } r > 0 \text{ even and } 0 \text{ for } r \text{ odd}.
$$
As we have already mentioned, the monoids \( Fr_{k|t^∞, l^∞} \) play in our case the same role as groups \( PU(k^n) \) in the “usual” twisted \( K \)-theory, therefore the space \( B Fr_{k|t^∞, l^∞} \) can naturally be considered as an analog of the \( H \)-space \( BPU(k^n) := \lim_{n \to \infty} BPU(k^n) \). We consider \( BPU(k^n) \) as an \( H \)-space with respect to the product induced by maps \( BPU(k^m) \times BPU(k^n) \to BPU(k^{m+n}) \) corresponding to the tensor product of matrix algebras, while \( (36) \) are also induced by tensor product \( (37) \).

Recall that the \( k \)-primary component \( Br_k(X) \) of the “finite” Brauer group is \( \ker\{[X, BU(k^∞)] \to [X, \text{BPU}(k^∞)]\} \), where \( \chi: U(k^∞) \to PU(k^∞) \) is induced by the canonical group epimorphisms \( \chi_{k^n}: U(k^n) \to PU(k^n) \), see \( (4) \). Alternatively, it can be defined as \( \text{im}\{[X, BPU(k^∞)] \to [X, K(Z, 3)]\} \) (cf. \( (11) \)), whence it is just \( H^2_{k-\text{tors}}(X, \mathbb{Z}) \). It can also be interpreted as the group of obstructions for the lift (= the reduction of the structural group) of \( PU(k^m) \)-bundles to \( U(k^m) \)-bundles.

Note that there is the \( H \)-space homomorphism \( BU(k^∞l^∞) \to B Fr_{k|t^∞, l^∞} \) induced by the composition of homomorphisms \( U(k^tl^∞) \to PU(k^tl^∞) \) with inclusions \( PU(k^tl^∞) \to Fr_{k|t^∞, l^∞} \) of the subgroups of automorphisms of \( M_{k|t^∞}(\mathbb{C}) \) to the monoids of endomorphisms. Thus it is natural to define the \( k \)-primary component of the generalized Brauer group as \( GBr_k(X) := \ker\{[X, BU(k^∞l^∞)] \to [X, B Fr_{k|t^∞, l^∞}]\} \). The new part of the generalized Brauer group comparing with the “classical” one consists of those (classes of) \( M_{k|t^∞}(\mathbb{C}) \)-fibrations whose structural monoid \( \text{End}(M_{k|t^∞}(\mathbb{C})) \) cannot be reduced to the group \( \text{Aut}(M_{k|t^∞}(\mathbb{C})) \subset \text{End}(M_{k|t^∞}(\mathbb{C})) \).

As a justification of our definition let us note that the fibration induced from \( \text{Fred}_{k|t^∞}(\mathcal{H}) \to B Fr_{k|t^∞, l^∞} \) (see \( (26) \)) by the map \( BU(k^tl^∞) \to B Fr_{k|t^∞, l^∞} \) is trivial (cf. the discussion at the end of Section 5). It seems that like the “classical” Brauer group, the generalized one parameterizes twisted \( K \)-theories (cf. the end of Section 1). However in contrast with “classical” it does not admit a simple cohomological description.

From the purely homotopy point of view the generalized Brauer group is the extension of the “classical” one by 2-periodicity, as the homotopy groups \( (35) \) show. While the unique obstruction (to reduction of the structural group from \( PU(k^m) \) to \( U(k^m) \)) in the case of the “classical” Brauer group is the three-dimensional cohomology class in \( H^3_{\text{tors}}(X, \mathbb{Z}) \), in case of \( GBr_k \) there are obstructions in all odd dimensions (cf. \( (35) \)). In this connection note that the homotopy fiber of the map

\[
BPU(k^tl^∞) \to B Fr_{k|t^∞, l^∞}
\]

induced by inclusion of the subgroup \( PU(k^tl^∞) \subset Fr_{k|t^∞, l^∞} \), \( PU(k^tl^∞) = \text{Aut}(M_{k|t^∞}(\mathbb{C})) \) is the space \( Gr_{k|t^∞, l^∞} := \lim_{m, n} Gr_{kl|m, l^n} \), where \( Gr_{kl|m, l^n} := PU(k^tl^{m+n})/(PU(k^tl^m) \otimes PU(l^n)) \) is the so-called “matrix Grassmannian” \( (7) \).

**Remark 29.** The fibration \( Gr_{kl,t^∞} \to \text{BSU}(kl^∞) \to B \tilde{Fr}_{kl,t^∞} \)

relates to the part

\[
bsu^0_{\otimes, l^∞} \to bsu^0_{\otimes, l^∞} \to bsu^0_{\otimes}(\mathbb{Z}/k\mathbb{Z})
\]
of the exact sequence for the generalized cohomology theory \( \{bsu^n_\otimes \}_n \) (see the Introduction) corresponding to the coefficient sequence

\[
0 \to \mathbb{Z}[\frac{1}{l}] \xrightarrow{k} \mathbb{Z}[\frac{1}{l}] \to \mathbb{Z}/k\mathbb{Z} \to 0.
\]

In fact, our new twistings correspond to the coboundary map \( \delta : bsu^0_\otimes (\mathbb{Z}/k\mathbb{Z}) \to bsu^1_\otimes [\frac{1}{l}] \) (while “classical” ones of finite order \( k \) correspond to the coboundary map \( H^2(X, \mathbb{Z}/k\mathbb{Z}) \to H^3(X, \mathbb{Z}) \)).

**Remark 30.** Note that \( Fr_{k^m, l^n} \) is the total space of the principal \( PU(k^m) \)-bundle \( PU(k^m) \to Fr_{k^m, l^n} \to Gr_{k^m, l^n} \). There is the commutative diagram (cf. (18))

(40)

\[
\begin{array}{ccc}
Fr_{k^{m+n}, l^r} \times Fr_{k^m, l^n} & \longrightarrow & Fr_{k^{m+n+r}} \\
\downarrow & & \downarrow \\
Fr_{k^{m+n}, l^r} \times Gr_{k^m, l^n} & \longrightarrow & Gr_{k^{m+n+r}}
\end{array}
\]

which defines the action of the monoid \( Fr_{k^\infty, l^\infty} \) on \( Gr_{k^\infty, l^\infty} \) and there is the equivalence

\[
E Fr_{k^\infty, l^\infty} \times Fr_{k^\infty, l^\infty} \simeq BPU(k^\infty)
\]

of \( Gr_{k^\infty, l^\infty} \)-fibrations.

**Remark 31.** In this remark we establish a relation to constructions from paper \([8]\). Let \( A^{univ}_{k^m} \to BPU(k^m) \) be the universal \( M_{k^m}(\mathbb{C}) \)-bundle. Applying the functor \( \text{Hom}_{alg}(\ldots, M_{k^m+n}(\mathbb{C})) \) to it fiberwisely we obtain the \( Fr_{k^m, l^n} \)-bundle

(41)

\[
\begin{array}{ccc}
Fr_{k^m, l^n} & \longrightarrow & H_{k^m, l^n}(A^{univ}_{k^m}) \\
\downarrow & & \downarrow \\
& & BPU(k^m)
\end{array}
\]

Its total space \( H_{k^m, l^n}(A^{univ}_{k^m}) \) is homotopy equivalent to \( Gr_{k^m, l^n} \) \([8]\). Moreover, (homotopy classes of) lifts in (41) of a map \( f : X \to BPU(k^m) \) correspond to (homotopy classes of) bundle embeddings

(42)

\[
\begin{array}{ccc}
f^*(A^{univ}_{k^m}) & \longrightarrow & X \times M_{k^{m+n}}(\mathbb{C}) \\
\downarrow & & \downarrow \\
X & & \end{array}
\]

(note that not every map \( f \) has such a lift, see \([8]\)). Applying composition map (18) to (41) fiberwisely, we obtain
A MODEL OF THE TWISTED $K$-THEORY RELATED TO BUNDLES OF FINITE ORDER

which is equivalent (under $H_{klm,ln}(A_{klm}^{univ}) \simeq Gr_{klm,ln}$, $H_{klm,ln+r}(A_{klm}^{univ}) \simeq Gr_{klm,ln+r}$) on total spaces to the bottom arrow in (10). Given a map $\varphi: X \to Fr_{klm+ln,ln}$ and a lift $\tilde{f}$ of $f$ in (11) we obtain some new bundle embedding $f^*(A_{klm}^{univ}) \to X \times M_{klm+ln+r}(C)$ corresponding to the composition

$$X \xrightarrow{\text{diag}} X \times X \xrightarrow{\varphi \times \tilde{f}} Fr_{klm+ln,ln} \times H_{klm,ln}(A_{klm}^{univ}) \xrightarrow{\lambda} H_{klm,ln+r}(A_{klm}^{univ}).$$

Such maps (after taking the direct limit) define the action of the monoid $Fr_{k\infty,l\infty}$ on (classes of) embeddings (12). This gives us an interpretation of the principal $Fr_{k\infty,l\infty}$-fibration induced from the universal one by map (39) (with $t = 1$).

Note that the existence of an embedding $A_k \hookrightarrow X \times M_{kl}(C)$ for $A_k M_{k}(C) \to Fr_{klm,ln}$ implies the triviality of the corresponding $\text{End}(M_{kl}(C))$-fibration $H_{kl\infty,l\infty}(A_k) \to X$.

In order to define a new cohomological obstruction consider the monoid $\tilde{Fr}_{k\infty,l\infty}$ from Remark [22]. An easy calculation shows that $H^5(B \tilde{Fr}_{k\infty,l\infty}, Z) \cong Z/k't'Z$ and this class is the first obstruction to the reduction of the structural monoid to the group $SU(kl\infty)$.

Note that the important feature of the classical Brauer group is its relation to the Morita-equivalence of $C^*$-algebras [15]. More precisely, there is another (equivalent, see Definition 3.4 in [1]) definition of the “usual” twisted $K$-theory as the $K$-theory of continuous-trace algebras of sections of locally trivial algebra bundles with fibers $K(H) \subset B(H)$ (whose local triviality follows from Fell’s condition) with the structural group $PU(H)$. In our case we have bundles of algebras with fibers $M_{kl}(C)$ with the structural monoid $Fr_{k\infty,l\infty}$ which are locally homotopy trivial. It seems to be an interesting task to investigate the relation of the generalized Brauer group to the Morita-equivalence of such bundles.

9. APPENDIX 1: HOMOTOPY GROUPS, ETC.

Lemma 32. The homotopy groups of the space $Fr_{klm,ln}$ up to dimension $\sim 2l^n$ are as follows: $\pi_r(Fr_{klm,ln}) = \mathbb{Z}/kl^m\mathbb{Z}$ for $r$ odd and $0$ for $r$ even.

Proof follows from the homotopy sequence of principal fibration [12] together with the Bott periodicity for unitary groups. □

Note that the Bott periodicity allows us to compute homotopy groups in the previous Lemma only up to dimension $\sim 2l^n$. In what follows such homotopy groups will be called “stable”.

Unital homomorphisms of matrix algebras induce maps $Fr_{klm,ln} \hookrightarrow Fr_{k'l',l''}$ for all $l \geq m$, $u \geq n$. We want to obtain some information about the direct limit $\lim_{m,n} Fr_{klm,ln}$.

Lemma 33. The maps $Fr_{klm,ln} \to Fr_{k+l,ln}$ induce the injective homomorphisms of stable homotopy groups.
Proof. Consider the morphism of homotopy sequences of principal fibrations
\[ U(l^n) \longrightarrow U(kl^{m+n}) \longrightarrow Fr_{kl^m, l^n} \]
which in stable odd dimensions gives the commutative diagram
\[ 0 \longrightarrow \mathbb{Z} \xrightarrow{k^m} \mathbb{Z} \longrightarrow \mathbb{Z}/kl^m\mathbb{Z} \longrightarrow 0 \]
whence we get the injective homomorphisms
\[ \pi_r(Fr_{kl^m, l^n}) \longrightarrow \pi_r(Fr_{kl^m, l^{n+1}}), \mathbb{Z}/kl^m\mathbb{Z} \longrightarrow \mathbb{Z}/kl^{m+1}\mathbb{Z}, \alpha (\mod kl^m) \mapsto l\alpha (\mod kl^{m+1}) \]
in odd stable dimensions. □

Lemma 34. The maps \( Fr_{kl^m, l^n} \rightarrow Fr_{kl^m, l^{n+1}} \) induce the following homomorphisms of stable homotopy groups in odd dimensions:
\[ \pi_r(Fr_{kl^m, l^n}) \rightarrow \pi_r(Fr_{kl^m, l^{n+1}}), \mathbb{Z}/kl^m\mathbb{Z} \rightarrow \mathbb{Z}/kl^{m+1}\mathbb{Z}, \alpha (\mod kl^m) \mapsto l\alpha (\mod kl^m). \]
Hence such a homomorphism has the kernel \( \cong \mathbb{Z}/k\mathbb{Z} \).

Proof. Again, consider the morphism of homotopy sequences of principal fibrations
\[ U(l^n) \longrightarrow U(kl^{m+n}) \longrightarrow Fr_{kl^m, l^n} \]
which in odd stable dimensions turns into the commutative diagram
\[ 0 \longrightarrow \mathbb{Z} \xrightarrow{k^m} \mathbb{Z} \longrightarrow \mathbb{Z}/kl^m\mathbb{Z} \longrightarrow 0 \]
gives us homomorphisms \( \pi_r(Fr_{kl^m, l^n}) \rightarrow \pi_r(Fr_{kl^m, l^{n+1}}) \) as in the statement of the lemma. □

Proposition 35. The homotopy groups of the space \( \lim_{m, n} Fr_{kl^m, l^n} \) are as follows: \( \mathbb{Z}/k\mathbb{Z} \) in all odd dimensions and 0 in all even dimensions.
Proof follows from the previous lemmas. More precisely, we consider the direct limit of cyclic groups with respect to the homomorphisms

\[
\cdots \\
\mathbb{Z}/k l^{m+1} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/k l^{m+1} \mathbb{Z} \xrightarrow{1} \cdots \\
\downarrow \downarrow \\
\mathbb{Z}/k l^m \mathbb{Z} \xrightarrow{1} \mathbb{Z}/k l^m \mathbb{Z} \xrightarrow{1} \cdots ,
\]

where the horizontal arrows have nonzero kernels. Therefore the \( l \)-primary component vanishes in the direct limit (recall that \((k, l) = 1\)). □

Note that the previous proposition shows the reason of the assumption \((k, l) = 1\). This guarantees the homotopy nontriviality of the space \( \lim_{m, n} \text{Fr}_{k l^m, l^n} \).

**Proposition 36.** The inclusion \( \lim_{n} \text{Fr}_{k, l^n} \to \lim_{m, n} \text{Fr}_{k l^m, l^n} \) is a homotopy equivalence. Moreover, the homotopy type of \( \lim_{m, n} \text{Fr}_{k, l^n} \) does not depend on the choice of \( l \) such that \((k, l) = 1\).

**Proof.** Clearly, the considered spaces are CW-complexes, therefore it is sufficient to prove their weak homotopy equivalence. It can be done in analogy with the proofs of the previous lemmas.

More precisely, consider the diagram

\[
\text{Fr}_{k, l^n} \longrightarrow \text{Fr}_{k l^m, l^n} \\
\downarrow \downarrow \\
\text{Fr}_{k, l^{n+1}} \longrightarrow \text{Fr}_{k l^m, l^{n+1}}
\]

and the corresponding diagram of the homotopy sequences in odd stable dimensions (cf. Lemma 32):

\[
\mathbb{Z}/k \mathbb{Z} \xrightarrow{c} \mathbb{Z}/k l^m \mathbb{Z} \cong \mathbb{Z}/k \mathbb{Z} \oplus \mathbb{Z}/l^m \mathbb{Z} \\
\downarrow \downarrow \\
\mathbb{Z}/k \mathbb{Z} \xrightarrow{c} \mathbb{Z}/k l^m \mathbb{Z} \cong \mathbb{Z}/k \mathbb{Z} \oplus \mathbb{Z}/l^m \mathbb{Z},
\]

where the horizontal arrows are injective according to Lemma 33 and the vertical ones are nilpotent on the \( l \)-primary component by Lemma 34.

To prove the second part first suppose that \((l, l') = 1\), then \( \text{Fr}_{k, l^n} \xrightarrow{\sim} \text{Fr}_{k, l^n} \xrightarrow{\sim} \text{Fr}_{k, l^n} \) are homotopy equivalences. In the case \((l, l') = d > 1\) we take \( l'' \) such that \((l, l'') = 1 = (l', l'')\). Then

\[
\text{Fr}_{k, l^n} \xrightarrow{\sim} \text{Fr}_{k, l^n} \xrightarrow{\sim} \text{Fr}_{k, l^n} \xrightarrow{\sim} \text{Fr}_{k, l^n} .
\]

**Lemma 37.** The space \( \lim_{m, n} U(k l^m+n)/(U(k l^m) \otimes E_{l^n}) \) is contractible.

**Proof.** Since this space is a CW-complex, it is sufficient to prove that it is weakly homotopy equivalent to a point. But this is obvious because the only nontrivial stable homotopy groups in
odd dimensions map under
\[ U(kl^{2n})/(U(kl^n) \otimes E_l^n) \to U(kl^{2n+2})/(U(kl^{n+1}) \otimes E_l^{n+1}) \]
as follows: \[ \mathbb{Z}/l^n \mathbb{Z} \to \mathbb{Z}/l^{n+1} \mathbb{Z}, \alpha \pmod{l^n} \mapsto l^2 \alpha \pmod{l^{n+1}}. \]

10. APPENDIX 2: \( \text{Fr}_{k^\infty, l^\infty} \) AS A CLASSIFYING SPACE

Let us show that any bundle of order \( k^n \) in \( K_\otimes \) can be represented by a map \( X \to \text{Fr}_{k^\infty, l^\infty} \), and vice versa.

Consider the fibration
\[
\tilde{\text{Fr}}_{k^m, l^n} \to \text{Gr}_{k^m, l^n} \beta_{m,n} \to \text{BSU}(k^m),
\]
where \( \tilde{\text{Fr}}_{k^m, l^n} := \text{SU}(k^m l^n)/(E_{k^m} \otimes \text{SU}(l^n)) \), and the map \( \beta_{m,n} \) is a classifying map for the tautological \( M_{k^m}(\mathbb{C}) \)-bundle over the matrix Grassmannian \( \text{Gr}_{k^m, l^n} := \text{SU}(k^m l^n)/(\text{SU}(k^m) \otimes \text{SU}(l^n)) \) [7]. Now taking the limit in (43) as \( m, n \to \infty \) with respect to maps induced by the tensor product and using the \( H \)-space isomorphism \( \lim_{m,n} \text{Gr}_{k^m, l^n} \cong \text{BSU}_\otimes \) (where the \( H \)-space structure on \( \lim_{m,n} \text{Gr}_{k^m, l^n} \) is defined by the maps \( \text{Gr}_{k^m, l^n} \times \text{Gr}_{k^m, l^n} \to \text{Gr}_{k^m, l^n} \)) induced by the tensor product of matrix algebras) [7] we see that \( \tilde{\text{Fr}}_{k^\infty, l^\infty} := \lim_{m,n} \tilde{\text{Fr}}_{k^m, l^n} \) is the homotopy fiber of the localization map \( \lim_{m,n} \beta_{m,n} : \text{BSU}_\otimes \to \text{BSU}_{\otimes \mathbb{C}[\frac{1}{k}]} \). In particular, for any SU-bundle over \( X \) of order \( k^n \), \( n \in \mathbb{N} \) a classifying map has a lift to \( \tilde{\text{Fr}}_{k^\infty, l^\infty} \).

The general case (recall that \( \text{BSU}_\otimes \cong K(\mathbb{Z}, 2) \times \text{BSU}_\otimes \)) corresponds to the fibration \( \text{Fr}_{k^\infty, l^\infty} \to \text{BU}_\otimes \to \text{BSU}_{\otimes \mathbb{C}[\frac{1}{k}]} \), and \( \text{Fr}_{k^\infty, l^\infty} \) itself is the fiber of the fibration \( \text{Fr}_{k^\infty, l^\infty} \to \text{BU}_{\otimes \mathbb{C}[\frac{1}{k}]} \to \text{BU}_{\otimes \mathbb{C}[\frac{1}{k}]} \) (cf. [14] and Proposition 36).

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