CP-violation parameters from decay rates of $B^{\pm} \rightarrow DK^{\pm}$, $D \rightarrow$ multibody final states

A. Soffer,¹ W. Toki,² and F. Winklmeier³

¹Tel Aviv University, Tel Aviv, 69978, Israel
²Colorado State University, Fort Collins, CO, 80523, USA
³CERN, CH-1211 Geneva 23, Switzerland

(Dated: January 1, 2009)

We describe a method for measuring CP-violation parameters from which the Cabibbo-Kobayashi-Maskawa angle $\gamma$ may be extracted. The method makes use of the total decay rates in $B^{\pm} \rightarrow DK^{\pm}$ decays, where the neutral $D$ meson decays to multibody final states. We analyze the error of the method using experimental CP-violation analysis variables that enable straightforward sensitivity comparison with other methods for extracting $\gamma$, and discuss the use of $B$-factory and charm-factory data to obtain the relevant charm decay information needed for this measurement. Measurement sensitivities are estimated for the currently available $B$-factory data sample, and $D$ decay modes for which use of this method can make a significant contribution toward reducing the total error on $\gamma$ are identified.

PACS numbers: 13.25.Hw, 11.30.Er

I. INTRODUCTION

An important part of the program to study CP violation is the measurement of the angle $\gamma = \arg (-V_{ud}V_{ud}^*/V_{cd}V_{cd}^*)$ of the unitarity triangle related to the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix [1]. Measurement of $\gamma$ performed with tree-level processes defines an experimentally allowed region for the apex of the unitarity triangle. This region should overlap with the region obtained from $B^0 - \bar{B}^0$ and $B_u - \bar{B}_u$ mixing, assuming there are no significant new-physics contributions in the mixing amplitude. With this assumption, current Tevatron measurements [2] of the $B_u - \bar{B}_u$ mixing rate yield an indirect constraint on $\gamma$ that is much tighter than direct measurements [3]. Therefore, precise direct determination of $\gamma$ presents an opportunity to conduct an accurate test of the Standard Model.

The decays $B \rightarrow DK$ can be used to measure $\gamma$ with essentially no hadronic uncertainties, exploiting interference between the $b \rightarrow ucs$ and $b \rightarrow csu$ amplitudes of the decays $B \rightarrow D^0K$ and $B \rightarrow D^0\bar{K}$, respectively [4]. Interference takes place when the $D$ meson [1] is observed in a final state $F$ that is accessible to both $D^0$ and $\bar{D}^0$ decays. Such measurements can be conducted with quite a few $D$ and $B$ decay modes, including those with excited charm and strange mesons, involving different methods for constructing and optimizing CP-violation observables and measuring parameters related to $\gamma$. In fact, there has been a healthy stream of new ideas in this area since the basic method was first proposed in 1991 [4]. The different parameters of the various measurements are then combined statistically, yielding confidence intervals for $\gamma$ [3]. The statistical sensitivity provided by each mode and method is generally poor, mainly due to the strong CKM suppression (and, for most modes [5], color suppression) in the $b \rightarrow ucs$ transition. This necessitates the exploitation of as many modes and methods as possible, in order to achieve a small combined error on $\gamma$.

The most accurate $\gamma$ measurement method to date determines $\gamma$ by analyzing the $D$-decay event distribution in $B^{\pm} \rightarrow DK^{\pm}$ with multibody $D$ decays [6, 7]. This method was initially applied to the Cabibbo-favored decay $D \rightarrow K^0_s\pi^+\pi^-$ [8, 9], and the BABAR Collaboration later used it with $K^0_sK^+K^-$ [10] and the Cabibbo-suppressed decay $D \rightarrow \pi^+\pi^-\pi^0$ [11]. A simulation study has also been conducted for the four-body mode $D \rightarrow K^+K^-\pi^+\pi^-$ [12].

As originally proposed [6], this method extracts the angle $\gamma$ from measurements of $d\hat{\Gamma}_F^\pm(P)/dP$, the differential decay rates of $B^{\pm} \rightarrow DK^{\pm}$ at each phase-space point $P$ of the multibody $D$-decay final state $F$. However, measurements done with the final states $F = K^0_s\pi^+\pi^-$ and $F = K^0_sK^+K^-$ have only made use of the phase-space distributions, given by the relative differential rates

$$\frac{d\hat{\Gamma}_F^\pm(P)}{dP} \equiv \frac{d\Gamma_F^\pm(P)}{dP} \frac{1}{\Gamma_F^\pm},$$

(1)

where

$$\Gamma_F^\pm = \int \frac{d\Gamma_F^\pm(P)}{dP} dP$$

(2)

are the total decay rates. Thus, these measurements were sensitive only to the dependence of the rates on the point $P$, not to their integrated values $\Gamma_F^\pm$. By contrast, the BABAR measurement with $F = \pi^+\pi^-\pi^0$ used both $d\Gamma_F^\pm(P)/dP$ and $\Gamma_F^\pm$. For that mode, the total decay rate $\Gamma_F^\pm$ gave more precise information about the CP-violation parameters than the phase-space distribution $d\Gamma_F^\pm(P)/dP$. While most measurements and sensitivity estimates have focused on use of $d\hat{\Gamma}_F^\pm(P)/dP$ for learning about $\gamma$, it is important to identify and study the

¹ We use the symbol $D$ to indicate any linear combination of a $D^0$ and a $\bar{D}^0$ meson state.
decay modes for which the total decay rate has competitive sensitivity to the CP-violation parameters. This will help ensure that all useful modes are utilized for measuring \( \gamma \), while preventing much effort from being wasted on data analysis of decay modes that are not promising.

The purpose of this paper is to provide the tools for estimating the CP-parameter sensitivities of measurements of the absolute decay rates \( \Gamma^F_\pm \) for different \( D \) decay modes. We demonstrate that a good estimate of the sensitivities is provided by a single mode-dependent parameter. The impact of each mode on the combined error of \( \gamma \) depends on values of strong phases and decay distributions that in many cases are not well known yet. However, our general analysis of the sensitivities, performed in terms of CP-violation parameters similar to those used in the most accurate experimental analyses to date, provides a good indication as to when using the integrated decay rates is a superposition of the absolute decay rates \( \Gamma \), where \( \delta_B \) is the CP-even phase difference between them. The magnitude \( |A_B| \) is measured [13] from the rate of the process \( B^- \to D^0 K^- \), \( D^0 \to K^- \pi^+ \), where contamination by the interfering decay chain \( B^- \to D_s^0 K^- \), \( D_s^0 \to K^- \pi^+ \) is doubly Cabibbo-suppressed as well as \( r_B \)-suppressed.

We define the magnitudes \( A_F \) and \( R_F A_F \) to be the square roots of the total \( D^0 \) decay rates into \( F \) and \( \bar{F} \),

\[
A_F \equiv \sqrt{\Gamma(D^0 \to F)} = \sqrt{\Gamma(D^0 \to \bar{F})},
R_F \equiv \frac{1}{A_F} \sqrt{\Gamma(D^0 \to \bar{F})} = \frac{1}{A_F} \sqrt{\Gamma(D^0 \to F)}.
\]

The ratio \( R_F \) equals 1 for charge self-conjugate final states \( (F = \bar{F}) \), but can in general have any non-negative value. Eqs. (5) ignore the possible impact of CP-violation in \( D \) decays. In addition, our use below of \( A_F \) and \( R_F \) will also ignore the effect of \( D^0 - D_s^0 \) mixing. It has been demonstrated [14] that these effects can be neglected for the purpose of measuring \( \gamma \), as long as this is done consistently for the \( D \) mesons produced in the \( B \) decay as well as for those used to determine necessary \( D \)-meson quantities, discussed in Section III. Alternatively, previously measured mixing and CP violation in \( D \) decays can be explicitly accounted for in the formalism [15]. For the purpose of the current discussion, it is sufficient to neglect these effects, as we do throughout this paper.

We define the normalized amplitude distribution functions for the \( P \)-dependent charm meson decays,

\[
f_{D^0}^F(P) = \frac{A(D^0 \to F(P))}{A_F},
f_{D_s^0}^F(P) = \frac{A(D_s^0 \to F(P))}{A_F R_F},
f_{D^0}^{-}\bar{F}(P) = \frac{A(D^0 \to \bar{F}(P))}{A_F},
f_{D_s^0}^{-}\bar{F}(P) = \frac{A(D_s^0 \to \bar{F}(P))}{A_F R_F}.
\]

These functions satisfy the relations

\[
f_{D^0}^{-}\bar{F}(P) = f_{D_s^0}^{-}\bar{F}(P), \quad f_{D^0}^{-}\bar{F}(P) = f_{D^0}^F(P)
\]

as a result of CP conservation in the charm meson decays, and are explicitly normalized, such that

\[
\int |f_{D^0}^F(P)|^2 dP = \int |f_{D^0}^{-}\bar{F}(P)|^2 dP = 1.
\]

Accounting for the interference between the \( b \to u\bar{c}s \) and \( b \to c\bar{u}s \) amplitudes in the \( B \) meson decays, the
amplitudes for the four full decay chains are obtained from Eqs. (3), (5), and (6),
\[
\begin{align*}
A(B^- \to F(P)K^-) &= A_0 \left( f_{D^0}^F + R_F f_{D^0}^D(P) z_\pm \right), \\
A(B^+ \to F(P)K^+) &= A_0 \left( R_F f_{D^0}^F + f_{D^0}^D(P) z_+ \right), \\
A(B^- \to \overline{F}(\overline{P})K^-) &= A_0 \left( R_F f_{D^0}^{\overline{F}} + f_{D^0}^{\overline{D}}(\overline{P}) z_\pm \right), \\
A(B^+ \to \overline{F}(\overline{P})K^+) &= A_0 \left( f_{D^0}^{\overline{F}} + R_F f_{D^0}^{\overline{D}}(\overline{P}) z_+ \right),
\end{align*}
\]
(9)

where \( A_0 \equiv |A_B|A_P \). The observable \( P \)-dependent \( B \)-decay rates are the squares of these amplitudes,
\[
\begin{align*}
\frac{d\Gamma_\pm^F(P)}{dP} &= |A(B^\pm \to F(P)K^\pm)|^2, \\
\frac{d\Gamma_\pm^{\overline{F}}(\overline{P})}{d\overline{P}} &= |A(B^\pm \to \overline{F}(\overline{P})K^\pm)|^2.
\end{align*}
\]
(10)

In the case \( F = F \), namely, when the \( D \) decay final state is self-conjugate, only two of the four equations (10) are unique. These are the modes that have been studied experimentally so far [8–11, 16]. As mentioned above, measurements of \( z_\pm \) using \( F = K^{0}\pi^+\pi^- \) and \( F = K^0 K^+ K^- \) have been performed by analyzing only the \( P \)-dependence of the event distributions \( d\Gamma_\pm^F(P)/dP \), disregarding the total decay rates \( \Gamma_\pm^F \). Since fitting \( d\Gamma_\pm^F(P)/dP \) in terms of \( r_B, \gamma \), and \( \delta_B \) leads to an average upward bias in \( r_B \) when \( r_B \) is of order its experimental error, Refs. [8–10, 16] used the CP-violation parameters
\[
x_\pm \equiv \Re \{z_\pm\}, \quad y_\pm \equiv \Im \{z_\pm\},
\]
(11)
which are unbiased for this type of analysis. After these parameters are measured in the analysis of \( d\Gamma_\pm^F(P)/dP \), they are converted into (in general, non-Gaussian) confidence regions in terms of the “physical” parameters \( r_B, \gamma \), and \( \delta_B \).

Here, however, we wish to focus on and generalize the approach used experimentally in Ref. [11] and first studied theoretically in Ref. [17], by examining the additional information that can be extracted from the total decay rates \( \Gamma_\pm^F \) and \( \Gamma_\pm^{\overline{F}} \). The expressions for these rates are obtained by taking the squared absolute value of Eqs. (9) and integrating over all phase-space points,
\[
\begin{align*}
\Gamma_\pm^F &= A_0^2 \left( 1 + R_F^2 |z_\pm|^2 - 2 R_F \Re \{z_\pm \bar{z}_- \} \right), \\
\Gamma_\pm^{\overline{F}} &= A_0^2 \left( R_F^2 + |z_\pm|^2 - 2 R_F \Re \{z_\pm \bar{z}_+ \} \right), \\
\Gamma^F &= A_0^2 \left( R_F^2 + |z_-|^2 - 2 R_F \Re \{z_- \bar{z}_+ \} \right), \\
\Gamma^{\overline{F}} &= A_0^2 \left( 1 + R_F^2 |z_+|^2 - 2 R_F \Re \{z_+ \bar{z}_- \} \right),
\end{align*}
\]
(12)

where
\[
\begin{align*}
z_\pm &= \mp \int f_{D^0}^F(P) \left( f_{D^0}^{\overline{F}}(P) \right)^* dP \\
&= \mp \int f_{D^0}^{\overline{F}}(P) \left( f_{D^0}^F(P) \right)^* dP
\end{align*}
\]
(13)
is a measure of the interference between the \( D^0 \) and \( \overline{D}^0 \) decay amplitudes into the final state \( F \), averaged over the final-state phase space. The absolute value and argument of \( z_\pm \) are, respectively, the coherence parameter and average strong phase of Ref. [17]. For the purpose of this discussion, it will be more useful to graphically think of \( z_\pm \) as a coordinate-system offset parameter for \( z_\pm \). Methods to measure \( z_\pm \) are outlined in Section III.

The important point for now is that \( z_\pm \) can be measured significantly more precisely than \( z_\pm \) from high-statistics \( D \) decay samples, namely,
\[
|\sigma_{z_\pm}| \ll |\sigma_{z_\pm}^F|.
\]
(14)

It is useful to represent \( z_\pm \) in terms of the parameters
\[
\rho_\pm \equiv \frac{z_\pm - 1}{R_F} z_F, \quad \bar{\rho}_\pm \equiv \frac{z_\pm - R_F z_F^*}{R_F^2},
\]
(15)
We follow Ref. [11] in referring to \( \rho_\pm \) and \( \bar{\rho}_\pm \) as the polar-coordinate parameters. This designation is motivated by the fact that measurement of the absolute decay rates is directly related to the radii \( |\rho_\pm| \) and \( |\bar{\rho}_\pm| \), via the relations
\[
\begin{align*}
\Gamma_\pm^F &= A_0^2 \left( 1 + R_F^2 |\rho_-|^2 - |z_F|^2 \right), \\
\Gamma_\pm^{\overline{F}} &= A_0^2 \left( R_F^2 + |\bar{\rho}_-|^2 - R_F^2 |z_F|^2 \right), \\
\Gamma^F &= A_0^2 \left( R_F^2 + |\bar{\rho}_-|^2 - R_F^2 |z_F|^2 \right), \\
\Gamma^{\overline{F}} &= A_0^2 \left( 1 + R_F^2 |\rho_+|^2 - |z_F|^2 \right).
\end{align*}
\]
(16)

Fig. 1 demonstrates the relationship between the polar coordinates \( \rho_- \), \( \bar{\rho}_- \) and the Cartesian coordinates \( x_- \), \( y_- \) for specific values of \( z_- \) and \( z_F \). The absolute values \( |\rho_\pm| \), \( |\bar{\rho}_\pm| \) extracted from the total decay rates of Eqs. (16) yield two possible values for \( z_- \) and two for \( z_F \) for a solution of \( \gamma \) with a four-fold ambiguity. In that sense, this is identical to measuring \( \gamma \) with two, two-body \( D \) modes, as in the method of Ref. [18], whose discrete ambiguities are further discussed in Ref. [19]. In the case of multibody \( D \) modes, analysis of the distribution of events throughout the \( F \) phase space reduces the ambiguity to two-fold, in addition to improving the total precision [6]. In effect, the event phase-space-distribution analysis measures not only the absolute value but also the phase of \( \rho_\pm \) and \( \bar{\rho}_\pm \) [11]. Combining the phase-space-distribution analysis with the total-rates analysis yields the most precise measurement of \( \gamma \) for a given \( D \) decay mode.

### III. MEASURING \( z_F \)

A general approach for measuring the components of \( z_F \) from decay rates of the \( \psi(3770) \) into neutral-\( D \) final states has been developed in Ref. [17]. Consider the case in which one of the \( \psi(3770) \) decays into \( F(P) \) and the other decays into \( F'(P') \), where the phase-space points \( P \) and \( P' \) do not have to be related. Due to the
The normalized event density in the $PP'$ coordinates for $FIG. 1$: The relationship between the Cartesian and polar coordinates for $x_F = 0.5 - 0.2i$, $z_\pm = -0.1 + 0.1i$, $R_F = 0.7$. The solid (dotted) arrow corresponds to the complex number $\rho_-$ ($\bar{\rho}_-$) of Eq. (15). A measurement of $|\rho_-|$ ($|\bar{\rho}_-|$) implies that the true value of $z_\pm$ may be anywhere on the solid (dotted) circle in the $(x_F, y_F)$ plane. The two crossing points of the circles are the possible solutions of $z_\pm$.

quantum numbers $J^{PC} = 1^{-}$ of the $\psi(3770)$, its two-$D$ decay wave function is antisymmetric under exchange of the daughters, and hence must be $(D^0 \overline{D}^0 - \overline{D}^0 D^0)/\sqrt{2}$. The normalized event density in the $PP'$ phase space is

$$
\frac{2}{B_{D^0 \overline{D}^0}} \frac{d\Gamma_{F,F'}}{dPdP'} = \left| f_{D^0}^F(P) R_F f_{P'}^F(P) - R_F f_{D^0}^F(P) f_{P'}^F(P) \right|^2 ,
$$

where $B_{D^0 \overline{D}^0}$ is the $\psi(3770) \rightarrow D^0 \overline{D}^0$ branching fraction. Integrating this expression over phase space yields the normalized rate,

$$
\Gamma_{F,F'} = \frac{2}{B_{D^0 \overline{D}^0}} A_{F}^2 A_{F'}^2 \Gamma_{F,F'}
= R_F^2 + R_{F'}^2 - 2 R_F R_{F'} \Re \{ z_F z_{F'} \}
= R_F^2 + R_{F'}^2 - 2 R_F R_{F'} \left( x_F x_{F'} + y_F y_{F'} \right)
$$

where Eq. (13) was used, and we have separated $z_F$ and $z_{F'}$ into their real and imaginary parts,

$$
z_F = x_F + iy_F,
$$

$$
z_{F'} = x_{F'} + iy_{F'}.
$$

By measuring the decay rate of Eq. (18) for different final states $F'$, one obtains all the information about $z_F$. We begin with $F' = F$, for which Eq. (18) becomes

$$
\tilde{\Gamma}_{F,F} = 2 R_F^2 \left( 1 - |z_F|^2 \right)
= 2 R_F^2 \left( 1 - x_F^2 - y_F^2 \right) .
$$

Next, we take $F'$ to be a CP-even or CP-odd state, namely

$$
F' = D^0_\pm = \frac{1}{\sqrt{2}} (D^0 \pm \overline{D}^0) .
$$

The inverse relations of Eq. (21) yield

$$
z_{D^0_\pm} = \pm 1,
$$

$$
R_{D^0_\pm} = 1,
$$

where the normalization condition Eq. (8) has been taken into account. Then Eq. (18) becomes

$$
\tilde{\Gamma}_{F,D^0_\pm} = 2 \left( 1 \mp x_F \right) .
$$

Eqs. (20) and (23) are sufficient for obtaining $x_F$ and $y_F$, the latter with a sign ambiguity. To resolve this ambiguity, we now take $F'$ to be the 2-body state $K^- \pi^+$. Eq. (18) then gives

$$
\tilde{\Gamma}_{F,K^-\pi^+} = R_F^2 + R_{K^-\pi^+}^2
- 2 R_F R_{K^-\pi^+} (x_F x_{K^-\pi^+} + y_F y_{K^-\pi^+}) .
$$

We have yet to determine $x_{K^-\pi^+}$ and $y_{K^-\pi^+}$. These are obtained from the rates

$$
\tilde{\Gamma}_{K^-\pi^+,K^-\pi^+} = 2 R_{K^-\pi^+}^2 (1 - x_{K^-\pi^+}^2 - y_{K^-\pi^+}^2) ,
$$

$$
\tilde{\Gamma}_{K^-\pi^+,D^0_\pm} = 2 \left( 1 \mp x_{K^-\pi^+} \right) ,
$$

as in Eqs. (20) and (23), respectively. Since $K^-\pi^+$ is a two-body state, $f_{K^-\pi^+}^D$ and $f_{K^-\pi^+}^D$ are numbers rather than functions. It then follows from Eq. (8), that $z_{K^-\pi^+}$ has unit magnitude, and the constraint

$$
x_{K^-\pi^+}^2 + y_{K^-\pi^+}^2 = 1 ,
$$

resolves the ambiguity in $y_{K^-\pi^+}$. Thus, it is possible to measure the real and imaginary parts of $z_F$ with no ambiguities.

Several studies [6, 20, 22] have shown that when obtaining $D$-decay parameters from $\psi(3770)$ decays, the expected error on the CP-violation parameters due to the finite $\psi(3770)$ statistics is relatively small, given current CESR-c and $B$-factory integrated luminosities. A detailed simulation study [22] has shown that in the phase-space-distribution analysis with $F = R_{0}^0 \pi^+ \pi^-$, the error on $\gamma$ due to the finite $\psi(3770)$ statistics is about four times smaller than the error due to the finite $B^+ \rightarrow D K^-$ statistics in the currently available, $\sim 1$ ab$^{-1}$ $B$-factory data sample. Measurements performed by CLEO-c with 818 pb$^{-1}$ of $e^+ e^- \rightarrow \psi(3770)$ data have yielded an estimated $\gamma$ error of $1^\circ - 2^\circ$ due to the measurement of the $D \rightarrow K_S \pi^+ \pi^-$ decay parameters [23]. CLEO-c has also measured $z_F$ for the modes $K^- \pi^+ \pi^0$ and $K^- \pi^+ \pi^- \pi^+$, obtaining the preliminary results [24]

$$
|z_{K^-\pi^+\pi^0}| = 0.79 \pm 0.08 , \arg \{z_{K^-\pi^+\pi^0} \} = \arg \{z_{K^-\pi^+\pi^0} \} = (197^{+02}_{-02})^{\circ} ,
$$

$$
|z_{K^-\pi^+\pi^-\pi^+}| = 0.24^{+0.21}_{-0.17} , \arg \{z_{K^-\pi^+\pi^-\pi^+} \} = (161^{+03}_{-08})^{\circ} .
$$
We note that while these errors are large, their impact on the errors of $|\bar{\rho}_\pm|$, which are the relevant CP-violation parameters for small-$R_F$ modes (see Section IV B) is suppressed by $R_F$, as seen from Eqs. (15).

Furthermore, the newly launched BEPC-II charm factory, with a design luminosity almost twenty times that of CESR-c, will be able to supply the charm data needed to match the large $B$ samples that will be collected at LHCb and possibly at a proposed $e^+e^-$ "super $B$ factory". We conclude that the error on $z_\pm$ in a decay-rate analysis of the type presented in this paper will be dominated by the experimental error on $|\rho_\pm|$ and not by knowledge of $z_F$.

A. Self-Conjugate Modes

So far, all the multibody D-decay modes studied experimentally within the context of $B^\pm \to DR^\pm$ have been charge self conjugate, i.e., $F = \overline{F}$. From Eq. (5), one sees that self-conjugate modes satisfy $R_F = 1$. In addition, Eq. (7), together with the condition $F = \overline{F}$, implies

$$f_F^E(P) = f_{\overline{F}}(P).$$

As a result, such states satisfy

$$y_F = 0,$$

(28)

as we demonstrate by dividing the phase space of $F$ into two equal-volume regions $V$ and $\overline{V}$, such that every point $P \in V$ is related to a point $\overline{P} \in \overline{V}$ by the CP transformation. For example, in a three-body decay of the type $D \to a^+a^-b_0^-$, the division is along the line $(p_+ + p_0)^2 = (p_-- + p_0)^2$, where $p_j$ is the four-momentum of particle $j$. Such a division can be performed for any multibody final state that is self conjugate, regardless of its particle multiplicity. Then

$$y_F = \Re \left\{ \int_V f_{D^0}^E(P) \left( f_{\overline{D}^0}^E(P) \right)^* dP \right\} + \Re \left( \int_{\overline{V}} f_{D^0}^{\overline{E}}(\overline{P}) \left( f_{\overline{D}^0}^{\overline{E}}(\overline{P}) \right)^* d\overline{P} \right).$$

(29)

Using Eq. (27), the second integral in Eq. (29) can be written as

$$\Re \left[ \int_{\overline{V}} f_{D^0}^{\overline{E}}(\overline{P}) \left( f_{\overline{D}^0}^{\overline{E}}(\overline{P}) \right)^* d\overline{P} \right].$$

(30)

The integrand in Eq. (30) is the complex conjugate of the integrand of the first term in Eq. (29). Therefore, their imaginary parts cancel in the sum, yielding $y_F = 0$.

In the $\gamma$-related measurements performed so far with $B^\pm \to DR^\pm$ and $D \to F$ decays into a multibody, self-conjugate state, a particular model was assumed for the functional form of $f_F^E(P)$. The parameters of the model were obtained by fitting the phase-space distribution of $D^0 \to F$ decays, where the flavor of the $D^0$ was tagged by its production in the decay $D^{\pm} \to D^0\pi^{\pm}$. In this case, Eq. (27) guarantees that $z_F$ can be fully determined by inserting the model $f_F^E(P)$ into Eq. (13), as done in Ref. [11]. The same cannot be done for modes that are not self conjugate, where one must resort to the use of $\psi(3770)$ decays.

IV. EXPERIMENTAL SENSITIVITIES

Due to the linear relationship (16) between the experimentally observable rates and $|\rho_\pm|^2$, $|\bar{\rho}_\pm|^2$, these squared radii are the unbiased CP-violation parameters of choice for the rates analysis, given that decay rates can almost always be obtained from reasonably unbiased estimators. If the errors on $|\rho_\pm|^2$ and $|\bar{\rho}_\pm|^2$ are significantly smaller than the values of these parameters, then their roots $|\rho_\pm|$ and $|\bar{\rho}_\pm|$ are also unbiased parameters.

In terms of the errors on the rates, the errors on $|\rho_\pm|$ and $|\bar{\rho}_\pm|$ are

$$\sigma_{|\rho_-|} = \frac{\sigma_{\Gamma_F}}{2|\rho_-|A_0 F^2},$$

$$\sigma_{|\rho_+|} = \frac{\sigma_{\Gamma_F}}{2|\rho_+|A_0 F^2},$$

$$\sigma_{|\bar{\rho}_-|} = \frac{\sigma_{\Gamma_F}}{2|\bar{\rho}_-|A_0 F^2},$$

$$\sigma_{|\bar{\rho}_+|} = \frac{\sigma_{\Gamma_F}}{2|\bar{\rho}_+|A_0 F^2},$$

(31)

where we have used Eq. (14) to neglect the error on $|z_F|$. As a result of Eqs. (14) and (15), the errors on $|\rho_\pm|$ and $|\bar{\rho}_\pm|$ are similar in magnitude to the errors on $x_\pm$ and $y_\pm$. Therefore, studying Eqs. (31) provides a simple means to compare the $\gamma$ sensitivity of a rates analysis using any final state $F$ to the sensitivity of the current-best measurement, namely, that of $x_\pm$ and $y_\pm$ from the phase-space-distribution analysis of $F = K_S^0\pi^+\pi^-$. In what follows, we make quantitative estimates of the errors on $|\rho_\pm|$ and $|\bar{\rho}_\pm|$.

A. Self-conjugate modes

As a result of Eq. (28), Eq. (15) simplifies to

$$\bar{\rho}_\pm = \rho_\pm = z_\pm - x_F$$

(32)

for self-conjugate modes. Therefore, the two circles of Fig. 1 collapse onto each other, and the rates measurement of the radii $|\rho_\pm|$ is no longer sufficient for fully determining $z_\pm$. This is hardly a problem, for two reasons. First, the phases of $\rho_\pm$ may be determined from the event-distribution analysis, as was done in Ref. [11], yielding a measurement of $z_\pm$ whose precision is enhanced
due to the use of all available experimental information. Second, as stated in the introduction, precise knowledge of $\gamma$ can in any case be obtained only by combining many measurements of parameters related to $\gamma$. Therefore, measurement of $|\rho_\pm|$ helps reduce the overall error on $\gamma$, even if it is not sufficient for extracting $\gamma$ without information obtained from other $\gamma$-related measurements.

Since $x_F$ is well known, it is useful to estimate the errors on $|\rho_\pm|$ for relevant $D$ decay modes, as they will correspond closely to the errors on $z_\pm$. Since we are dealing with the case $R_F = 1$, Eqs. (31) become

$$
\sigma_{|\rho_\pm|} = \frac{\sigma_{\Gamma_\pm}}{2|\rho_\pm|A_0^2} = \frac{1 + |\rho_\pm|^2 - x_F^2}{2|\rho_\pm|}, \quad (33)
$$

where we have used $\Gamma_\pm \equiv \Gamma_F^\pm = \Gamma_K^\pm$, and the second equality of Eq. (33), obtained from Eq. (16), conveniently relates the $|\rho_\pm|$ errors to the relative errors on the signal branching fractions. We rely on previous “reference” experimental studies of the relevant decay modes to obtain these relative errors for any hypothetical value of $|\rho_\pm|$. Suppose that in a reference measurement performed with $B$-factory data of integrated luminosity $L$, one observed $N_\pm$ signal $B^\pm \to FK^\pm\pi^\pm\pi^\mp$ events, from which the rates $\Gamma_F^\pm$ were determined and the CP-violation parameter values $|\rho_\pm|^2$ were calculated. Let $N \equiv N_+ + N_-$. Then the numbers of signal events that would be observed in an experimentally identical, hypothetical measurement of luminosity $L$ given hypothetical values $|\rho_\pm|^2$ for the CP-violation parameters, are

$$
N_\pm = \bar{N} \bar{r}_\pm \frac{L}{L}, \quad (34)
$$

where

$$
\bar{r}_\pm = \frac{\Gamma_F^\pm}{\Gamma_F^+ + \Gamma_F^-} = \frac{1 + |\rho_\pm|^2 - x_F^2}{2 + |\rho_-|^2 + |\rho_+|^2 - 2x_F^2}. \quad (35)
$$

is the ratio between the value of the rate $\Gamma_F^\pm$ given the hypothetical parameter values $\rho_\pm$ and the sum of the rates $\Gamma_F^+ + \Gamma_F^-$ measured in the reference measurement. The second equality in Eq. (35) arises from Eqs. (16).

We assume that the error on the number of events $\bar{N}$ in the reference measurement can be written as the sum in quadrature of a Poisson signal part and a background part, namely,

$$
\sigma_{\bar{N}}^2 = \bar{N} + \sigma_{\bar{N},bgd}^2. \quad (36)
$$

Using this relation and the published reference-measurement quantities $\bar{N}$ and $\sigma_{\bar{N}}$, we obtain the background contribution to the error, which we assume to be CP symmetric. Then the errors on the numbers of events in the hypothetical measurement, in which $N_\pm$ will be observed, are

$$
\sigma_{N_\pm} = \sqrt{\frac{1}{2} \left( \sigma_{\bar{N}}^2 - \bar{N} \right)} + \bar{N} \bar{r}_\pm \sqrt{\frac{L}{L}}, \quad (37)
$$

where the statistical assumption leading to Eq. (36) was again used. From Eqs. (34) and (37), we obtain the relative branching-fraction errors for the hypothetical measurement,

$$
\sigma_{\Gamma_\pm} = \frac{\sigma_{N_\pm}}{N_\pm} = \sqrt{\frac{1}{2} \left( \sigma_{\bar{N}}^2 - \bar{N} \right)} + \bar{N} \bar{r}_\pm \sqrt{\frac{L}{L}}. \quad (38)
$$

In Table I we report $x_F$, $|\rho_\pm|$, and $\sigma_{|\rho_\pm|}$ for several three-body $D$ decay modes, assuming a data sample of $10^9 e^+e^- \to BB$ events, similar to the currently available $B$-factory sample. We obtain the values of $x_F$ from Eq. (13), using the Dalitz-plot distributions $f_{D^0}(P)$, whose parameterizations are reported in Refs. [11], [8], [10], and [25] for the $D$-decay final states $\pi^+\pi^-\pi^0$, $K^0_S\pi^+\pi^-$, $K^0_SK^+K^-$, and $K^+K^0\pi^0$, respectively. We also obtain $\bar{N}, \sigma_{\bar{N}}$, and $L$ from these references, except for $K^+K^-\pi^0$, where we estimate $\bar{N}$ and $\sigma_{\bar{N}}$ from their values in $\pi^+\pi^-\pi^0$ [11], taking into account the ratio of branching fractions $B(D^0 \to K^+K^-\pi^0)/B(D^0 \to \pi^+\pi^-\pi^0)$ [26] and an assessment that the background yield in $K^+K^-\pi^0$ will be 20% of that in $\pi^+\pi^-\pi^0$. Since extraction of $|\rho_\pm|^2$ from the total rates has been reported only for the $F = \pi^+\pi^-\pi^0$ mode [11], we take $|\rho_\pm| = |\rho_\pm|$ when evaluating $\bar{r}_\pm$ for all other modes, for lack of a better value. We take the hypothetical CP-violation parameter values $|\rho_\pm|$ from the averages of the values of $x_\pm$ and $y_\pm$ reported in Refs. [10] and [16],

$$
x_- = 0.097 \pm 0.034, \\
y_- = 0.054 \pm 0.058, \\
x_+ = -0.087 \pm 0.031, \\
y_+ = -0.038 \pm 0.042. \quad (39)
$$

The errors of Eq. (39) reflect the sensitivity of a measurement conducted with $1.04 \times 10^9 e^+e^- \to BB$ events, comparable to the value used to produce Table I.

One can see from Eq. (33), that the error $\sigma_{|\rho_\pm|}$ is small when $|\rho_\pm|$ is large. Large $|\rho_\pm|$ requires $x_F$ to be large, by virtue of Eq. (32) and the smallness of $z_\pm$, demonstrated in Eq.(39). We note that some insight into the value of $x_F$ for a particular mode can be obtained by studying the distribution of events in the $D^0$-decay Dalitz plot, since generally, high level of apparent symmetry under

| Mode | $x_F$ | $|\rho_\pm|^2$ | $|\rho_\pm|$ | $\sigma_{|\rho_\pm|}$ |
|------|-------|-------------|-----------|----------------|
| $\pi^+\pi^0\pi^-$ | 0.85 | 0.13 | 0.10 | 0.75 | 0.94 | 0.07 | 0.06 |
| $K^0_S\pi^+\pi^-$ | -0.02 | 0.05 | 0.03 | 0.13 | 0.08 | 0.19 | 0.32 |
| $K^0_SK^+K^-$ | -0.31 | 0.09 | 0.10 | 0.41 | 0.23 | 0.12 | 0.20 |
| $K^+K^-\pi^0$ | 0.20 | 0.11 | 0.12 | 0.29 | 0.48 | 0.19 |
the exchange of the two charged particles leads to a high value of $x_F$.

It is evident from Table I that of the three-body modes studied here, only $x_F^{\rho_5-\pi^+\pi^0}$ is large enough for Eq. (33) to yield $|\rho_\pm|$ errors that are competitive with the errors of Eq. (39). In particular, the high-statistics, low-background mode $K^0\pi^+\pi^-$ ends up having large $|\rho_\pm|$ errors due to the very small value of $x_F^{K^0\pi^+\pi^-}$. On the other hand, we expect that the methods for suppression of the significant background in $\pi^+\pi^-\pi^0$, which were first developed in Ref. [31], will improve in upcoming analyses. That should reduce $\sigma_{|\rho_\pm|}$ for this mode below the simple extrapolation shown in Table I.

### B. Non-self-conjugate modes

We proceed to estimate the errors on $\rho_\pm$ and $\bar{\rho}_\pm$ in $D$-decay final states that are not self-conjugate, i.e., $F \neq \overline{F}$. As in the procedure leading up to Eq. (33), we replace $A_0^F$ in Eqs. (31) using Eq. (16):

$$\sigma_{|\rho_-|} = \frac{\nu_\rho_-}{2|\rho_-|R_F^0} \frac{\sigma_{\rho_-}}{\Gamma_F^0},$$

$$\sigma_{|\rho_+|} = \frac{\nu_\rho_+}{2|\rho_+|R_F^0} \frac{\sigma_{\rho_+}}{\Gamma_F^0},$$

$$\sigma_{|\bar{\rho}_-|} = \frac{\bar{\nu}_\rho_-}{2|\bar{\rho}_-|R_F^0} \frac{\sigma_{\rho_-}}{\Gamma_F^0},$$

$$\sigma_{|\bar{\rho}_+|} = \frac{\bar{\nu}_\rho_+}{2|\bar{\rho}_+|R_F^0} \frac{\sigma_{\rho_+}}{\Gamma_F^0},$$

(40)

where

$$\nu_\rho \equiv 1 + R_F^0|\rho_\pm|^2 - |z_F|^2,$$

$$\bar{\nu}_\rho \equiv R_F^0|\rho_\pm|^2 - R_F^0|z_F|^2.$$  

(41)

As in Eq. (38), the relative errors in Eqs. (40) are obtained from the number of signal events $N_F^F$, $N_F^{\overline{F}}$ and their errors, observed in existing reference measurements,

$$\frac{\sigma_{\rho_\pm}}{\Gamma_F^0} = \frac{\sqrt{\frac{1}{2} \left( \sigma_{N_F^F}^2 - N_F^F \Gamma_F^0 \right) + \frac{\sigma_{N_F^{\overline{F}}}^2}{2} \sqrt{L}}} {\sqrt{L},}$$

(42)

with an analogous expression for $\overline{F}$, where by analogy with Eq. (35),

$$\tilde{\nu}_F = \frac{\nu_\rho_-}{\nu_\rho_- + \bar{\nu}_\rho_+},$$

$$\tilde{\nu}_\overline{F} = \frac{\nu_\rho_+}{\nu_\rho_- + \bar{\nu}_\rho_+},$$

$$\tilde{\bar{\nu}}_F = \frac{\bar{\nu}_\rho_-}{\nu_\rho_- + \bar{\nu}_\rho_+},$$

$$\tilde{\bar{\nu}}_\overline{F} = \frac{\bar{\nu}_\rho_+}{\nu_\rho_- + \bar{\nu}_\rho_+},$$

(43)

As in Section IV A, the symbols $\tilde{\rho}_\pm$, $\tilde{\bar{\rho}}_\pm$ in Eq. (43) refer to the CP-violation parameters extracted from the reference measurements $N_F$ and $N_{\overline{F}}$. If the total rates were not used to extract CP-violation parameters, one can naively take $\tilde{\rho}_\pm$ and $\tilde{\bar{\rho}}_\pm$ from Eq. (39) for the purpose of performing this error estimate.

Let us consider this error estimate in the case of the non-self-conjugate, three-body final state $F = K_0^{\pm}K^-\pi^+$. With as little as 5% of their currently available data sample, the $\mathrm{BaBar}$ collaboration has performed a preliminary analysis of this mode's Dalitz-plot amplitude-distribution functions $f_{D_0}^F(P)$ and $f_{D_0}^\overline{F}(\overline{P})$ [27], from which we compute $|z_F| = 0.47$. The ratio $R_{K_0^{\pm}K^-\pi^+} = 0.68$ is easily extracted from the results reported in Ref. [27]. With $R_{K_0^{\pm}K^-\pi^+}$ being different from 1 yet of order 1, this mode is in a class of Cabibbo-suppressed decays expected to exhibit large interference between the $b \to m\nu\overline{m}$ and $b \to c\pi\nu$ decays [28]. Unfortunately, as we show below, the combination of a small branching fraction and a medium-sized $|z_F|$, render $K_0^{\pm}K^-\pi^+$ unattractive for extracting $\gamma$ via the total-rate method.

In addition to $R_{K_0^{\pm}K^-\pi^+}$ and $|z_{K_0^{\pm}K^-\pi^+}|$, calculation of all four errors of Eq. (40) also requires knowledge of $\arg\{z_{K_0^{\pm}K^-\pi^+}\}$, which has not been measured. However, a rough estimate of the CP-parameter errors shows them to be comparable to those of the $K_0^{\pm}K^+K^-$ mode, due to the following two observations. First, the combined branching fraction $B(D^0 \to K_0^{\pm}K^-\pi^+)$ + $B(D^0 \to K_0^{\pm}K^+\pi^-)$ is approximately 85% of $B(K_0^{\pm}K^+K^-)$. One therefore expects the relative error on $N_{K_0^{\pm}K^+K^-}$ to be somewhat larger than that on $N_{K_0^{\pm}K^-\pi^+}$. Experimental details, such as kaon vs. pion multiplicities and combinatoric background under the larger $K_0^{\pm}K^+K^-$ Dalitz plot, slightly increase our expectation for the ratio between the relative errors on $N_{K_0^{\pm}K^+K^-}$ and $N_{K_0^{\pm}K^-\pi^+}$. The second observation is that $|z_{K_0^{\pm}K^-\pi^+}|$ is about 50% larger than $x_{K_0^{\pm}K^+K^-}$. Combining these two competing effects, we conclude that the errors on components of the CP-violation parameters obtained from $K_0^{\pm}K^+K^-$ and $K_0^{\pm}K^-\pi^+$ should be of similar magnitudes. As seen in Table I, this implies error values that are too large to be of practical interest.

We note that Eqs. (40) also hold for Cabibbo-allowed final states involving a single charged kaon, such as $F = K^-\pi^-\pi^0$, for which $R_{K^-\pi^-\pi^0} \approx 0.05$ [29] (where we have ignored the effect of $D^0 - \overline{D^0}$ mixing [14]). Eqs. (16) show that in this case, the sensitivity of $\Gamma^F$ and $\Gamma_{\overline{F}}^F$ to the CP-violation parameters is suppressed by $R^2_{K^-\pi^-\pi^0}$, making these rates useful for obtaining $A_B$, as mentioned in Section II for the $D^0 \to K^-\pi^+$ decay. However, the absolute rates $\Gamma^F$ and $\Gamma_{\overline{F}}$ do provide a good measurement of $|\rho_\pm|$. Searching for these decays in a data sample of $226 \times 10^6 e^+e^- \to BB$ events, $\mathrm{BaBar}$ [30] has put an
upper limit on the ratio
\[
R_{ADS} \equiv \frac{\Gamma_F^+ + \Gamma_F^-}{\Gamma_F^+ + \Gamma_F^-} = \frac{\bar{\nu}_{\rho^-} + \nu_{\rho^+}}{\nu_{\rho^-} + \bar{\nu}_{\rho^+}},
\]
for which the central value obtained was \( R_{ADS} = 0.013^{+0.010}_{-0.004} \). The rates that appear in the numerator of Eq. (44), to which we refer as the ADS rates \([18]\), are suppressed by factors of second order in the small parameters \( \tau_B, R_{K^-\pi^+\pi^0} \) relative to the rates in the denominator. The error on \( R_{ADS} \) is dominated by the statistical errors on the ADS rates. To properly account for this when calculating the relative errors on the ADS rates, we evaluate Eq. (42) with
\[
\rho_{\pm} = \frac{\bar{\nu}_{\rho_{\pm}}}{\nu_{\rho_{\pm}} + \bar{\nu}_{\rho_{\pm}}}
\]
instead of the expressions in Eq. (43), and take \( N_{K^+\pi^0} \) to be the number of ADS events detected in Ref. [30], namely, 19±10, where the 10-event error is obtained from the naive average of the positive and negative errors on \( R_{ADS} \).

The resulting errors on \( |\rho_{\pm}| \) are shown in Fig. 2, calculated with Eqs. (40) for different values of \( z_F \). As in the case of Table I, we have assumed a data sample of \( 10^3 e^+e^- \to B\bar{B} \) events and the CP-violation parameter values of Eq. (39). The errors reach values as low as 0.016 and as high as 0.035 (0.045) for \( |\rho_-| \) (|\rho_+|). We see that at least one of the errors is smaller than about 0.025 for any value of \( z_F \). For the CLEO-c central values of \( z_{K^-\pi^+\pi^0} \) [24], we find \( \sigma_{|\rho_{\pm}|} \approx 0.02 \).

These results suggest that one can expect measurement of the CP-violation parameters with \( F = K^-\pi^+\pi^0 \) to yield errors that are very competitive with the current-best measurement, Eq. (39), once the luminosity is high enough for observation of the ADS decays.

V. DISCUSSION

Of the self-conjugate final states studied quantitatively here, the errors obtained from \( \pi^+\pi^-\pi^0 \) are the smallest, due to the large value of \( x_F = \Re\{z_F\} = 0.85 \) in this mode. The errors are expected to decrease beyond the estimate shown in Table I, as background suppression improves in subsequent analyses of this mode. By contrast, the final state \( K^0\pi^+\pi^- \), which thanks in part to its large branching fraction and high purity has yielded the most precise phase-space-distribution measurements of \( \gamma \) to date, has a very small \( x_F \), rendering its absolute decay rates poor measures of the CP-violation parameters.

Our calculations show that measuring \( |\rho_{\pm}| \) with the final state \( K^-\pi^+\pi^0 \) can yield very small errors, smaller than or of similar magnitude to the errors from the phase-space-distribution analysis of \( K^0\pi^+\pi^- \). We note that similar precision may be obtained with the two- and four-body final states \( K^-\pi^+ \) and \( K^-\pi^-\pi^+ \), whose study is outside the scope of this paper.

The results presented here cover the major three-body \( D \)-decay final states with known and significant branching fractions. It is possible that the absolute decay rates into some of the higher-multiplicity states will also turn out to yield competitive errors on \( \gamma \). Among the Cabibbo-suppressed modes, this includes the final state \( K^+K^-\pi^+\pi^- \), whose phase-space-distribution analysis has been studied in simulation [12], and \( 2\pi^+2\pi^- \). The Cabibbo-favored mode \( D^0 \to K^0\pi^+\pi^-\pi^0 \) has a large branching fraction, (5.3 ± 0.6)% [29], and may therefore be attractive for both phase-space-distribution and absolute-decay-rate analyses. Since almost half the rate is due to the resonant contribution \( K^0\pi^+(892)\rho^+ \), the phase-space distribution is highly asymmetric under exchange of the two charged pions. Therefore, it is unlikely that \( x_F \) is large for this mode. Nonetheless, given that the large branching fraction, even \( x_F \) as small as 0.1 could make this mode attractive for studying \( \gamma \).
We have studied the use of the absolute $B^\pm \to DK^\pm$ decay rates, where the $D$ decays to a multibody final state, for obtaining information with which to improve the overall knowledge of the CKM unitarity-triangle phase $\gamma$. This information is complementary to that obtained from other $\gamma$-related measurements, including analysis of the $D^0 - \bar{D}^0$ interference pattern seen in the phase-space distributions of the $D$ decay products. We have developed a formalism for estimating the error on the CP-violating parameters $|\rho_\pm|$ and $|\tilde{\rho}_\pm|$. The parameter that most strongly affects the errors is $z_F$ of Eq. (13). We have evaluated $z_F$ for three-body $D$ final states for which the necessary input information is available, and have estimated the errors on the CP-violation parameters for these self-conjugate modes and for the modes $K^0_{\bar{s}}K^-\pi^+$ and $K^-\pi^+\pi^0$.

Acknowledgments

This research was supported in part by grant number 2006219 from the United States-Israel Binational Science Foundation (BSF), Jerusalem, Israel. The authors thank Werner Sun, Jure Zupan, and Jim Libby for useful suggestions.