Constructing $N$-soliton solution for the mKdV equation through constrained flows

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Abstract

Based on the factorization of soliton equations into two commuting integrable $x$- and $t$-constrained flows, we derive $N$-soliton solutions for mKdV equation via its $x$- and $t$-constrained flows. It shows that soliton solution for soliton equations can be constructed directly from the constrained flows.

Keywords: soliton solution, constrained flow, mKdV equation, Lax representation

1 Introduction

It is well known that there are several methods to derive the $N$-soliton solution of soliton equations, such as the inverse scattering method, the Hirota method, the dressing method, the Darboux transformation, etc. (see, for example, [1, 2, 3] and references therein). In present paper, we propose a method to construct $N$-soliton solution for mKdV equation directly through two commuting $x$- and $t$-constrained flows obtained from the factorization of mKdV equation. It was shown in [4, 5, 6, 7] that (1+1)-dimensional soliton equation can be factorized by $x$- and $t$-constrained flow which can be transformed into two commuting $x$- and $t$-finite-dimensional integrable Hamiltonian systems. The Lax representation for constrained

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flows can be deduced from the adjoint representation of the auxiliary linear problem for soliton equations [8]. By means of the Lax representation and the standard method in [9, 10, 11] we are able to introduce the separation variables for constrained flows [12]-[16] and to establish their Jacobi inversion problem [14, 15, 16]. Furthermore, the factorization of soliton equations and separability of the constrained flows allow us to find the Jacobi inversion problem for soliton equations [14, 15, 16]. By using the Jacobi inversion technique [17, 18], the $N$-gap solutions in term of Riemann theta functions for soliton equations can be obtained, namely, the constrained flows can be used to derive the $N$-gap solution. The present paper shows that the $x$- and $t$-constrained flows and their Lax representation can also be used to directly construct the $N$-soliton solution for soliton equations. In fact the method proposed in this paper together with that in the previous paper [19] provides a general procedure to derive $N$-soliton solution for soliton equations via their constrained flows.

2 The factorization of the mKdV hierarchy

We first briefly recall the constrained flows of the mKdV hierarchy and their Lax representation. The mKdV hierarchy

$$q_{t_{2n+1}} = D b_{2n+1} = D \frac{\delta H_{2n+1}}{\delta q}, \quad n = 0, 1, \cdots,$$

with

$$H_{2n+1} = \frac{2a_{2n+2}}{2n+1},$$

is associated with the reduced AKNS spectral problem for $r = -q$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad U = \begin{pmatrix} -\lambda & q \\ -q & \lambda \end{pmatrix},$$

and the evolution equation of the eigenfunction

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{t_{2n+1}} = V^{(2n+1)}(q, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where

$$V^{(2n+1)} = \sum_{j=0}^{2n+1} \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \lambda^{2n+1-j}.$$
with
\[
\begin{align*}
a_0 &= -1, \quad b_0 = c_0 = a_1 = 0, \quad b_1 = -c_1 = q, \quad a_2 = -\frac{1}{2}q^2, \quad b_2 = c_2 = -\frac{1}{2}q_x, \quad \ldots,
\end{align*}
\]
and in general
\[
\begin{align*}
b_{2m+1} &= -c_{2m+1} = Lb_{2m-1}, \quad L = \frac{4}{D}D^2 + qD^{-1}qD, \quad D = \frac{d}{dx}, \quad DD^{-1} = D^{-1}D = 1, \\
b_{2m} &= c_{2m} = -\frac{1}{2}Db_{2m-1}, \quad a_{2m+1} = 0, \quad a_{2m} = 2D^{-1}qb_{2m}.
\end{align*}
\] (2.5)

For the well-known mKdV equation
\[
q_t = Db_3 = \frac{1}{4}(q_{xxx} + 6q^2q_x),
\] (2.6)
the \(V^{(3)}\) reads
\[
V^{(3)} = \begin{pmatrix}
-\lambda^3 - \frac{1}{2}q^2\lambda & q\lambda^2 - \frac{1}{2}q_x\lambda + \frac{1}{2}q_{xx} + \frac{1}{2}q^3 \\
-q\lambda^2 - \frac{1}{2}q_x\lambda - \frac{1}{2}q_{xx} - \frac{1}{2}q^3 & \lambda^3 + \frac{1}{2}q^2\lambda
\end{pmatrix}.
\] (2.7)

We have
\[
\frac{\delta \lambda}{\delta q} = \psi_1^2 + \psi_2^2, \quad L(\psi_1^2 + \psi_2^2) = \lambda^2(\psi_1^2 + \psi_2^2).
\] (2.8)

The \(x\)-constrained flows of the mKdV hierarchy consist of the equations obtained from the spectral problem (2.2) for \(N\) distinct real numbers \(\lambda_j\) and the restriction of the variational derivatives for the conserved quantities \(H_{2k_0+1}\) (for any fixed \(k_0\)) and \(\lambda_j\) defined by (see, for example, [4]-[7], [20, 21])
\[
\begin{align*}
\psi_{1j,x} &= -\lambda_j\psi_{1j} + q\psi_{2j}, \quad \psi_{2j,x} = -q\psi_{1j} + \lambda_j\psi_{2j}, \quad j = 1, \ldots, N,
\end{align*}
\] (2.9a)
\[
\frac{\delta H_{2k_0+1}}{\delta q} - \frac{1}{2} \sum_{j=1}^N \frac{\delta \lambda_j}{\delta q} \equiv b_{2k_0+1} - \frac{1}{2} \sum_{j=1}^N (\psi_{1j}^2 + \psi_{2j}^2) = 0.
\] (2.9b)

For \(k_0 = 0, (2.9b)\) gives
\[
q = \frac{1}{2}(<\Psi_1, \Psi_1> + <\Psi_2, \Psi_2>),
\] (2.10)
where
\[
\Psi_k = (\psi_{k1}, \ldots, \psi_{kN})^T, \quad k = 1, 2, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N).
\]
By substituting (2.10), (2.9a) becomes a finite-dimensional integrable Hamiltonian system (FDIHS)

$$\Psi_{1x} = -\Lambda \Psi_1 + \frac{1}{2} (\langle \Psi_1, \Psi_1 \rangle + \langle \Psi_2, \Psi_2 \rangle) \Psi_2 = \frac{\partial \overline{H}_0}{\partial \Psi_2},$$

$$\Psi_{2x} = -\frac{1}{2} (\langle \Psi_1, \Psi_1 \rangle + \langle \Psi_2, \Psi_2 \rangle) \Psi_1 + \Lambda \Psi_2 = -\frac{\partial \overline{H}_0}{\partial \Psi_1},$$

with

$$\overline{H}_0 = -\langle \Lambda \Psi_1, \Psi_2 \rangle + \frac{1}{8} (\langle \Psi_1, \Psi_1 \rangle + \langle \Psi_2, \Psi_2 \rangle)^2.$$  

Under the constraint (2.10), the t-constrained flow obtained from (2.3) with $V^{(3)}$ given by (2.7) for $N$ distinct $\lambda_j$ can also be written as a FDIHS

$$\Psi_{1t} = \frac{\partial \overline{H}_1}{\partial \Psi_2}, \quad \Psi_{2t} = -\frac{\partial \overline{H}_1}{\partial \Psi_1},$$

with

$$\overline{H}_1 = -\langle \Lambda^3 \Psi_1, \Psi_2 \rangle - \frac{1}{8} (\langle \Psi_1, \Psi_1 \rangle + \langle \Psi_2, \Psi_2 \rangle)^2 <\Lambda \Psi_1, \Psi_2 >$$

$$+ \frac{1}{4} (\langle \Psi_1, \Psi_1 \rangle + \langle \Psi_2, \Psi_2 \rangle) (\langle \Lambda^2 \Psi_1, \Psi_1 \rangle + \langle \Lambda^2 \Psi_2, \Psi_2 \rangle) - \frac{1}{8} <\Lambda \Psi_1, \Psi_1 >^2$$

$$- \frac{1}{8} <\Lambda \Psi_2, \Psi_2 >^2 + \frac{1}{4} <\Lambda \Psi_1, \Psi_1 > <\Lambda \Psi_2, \Psi_2 > + \frac{1}{128} (\langle \Psi_1, \Psi_1 \rangle + \langle \Psi_2, \Psi_2 \rangle)^4.$$

The Lax representation for the constrained flows (2.11) and (2.12), which can be obtained from the adjoint representation of the Lax representation for mKdV hierarchy [3, 8], is given by

$$M_x = [\tilde{U}, M], \quad M_t = [\tilde{V}^{(3)}, M]$$

where $\tilde{U}$ and $\tilde{V}^{(3)}$ are obtained from $U$ and $V^{(3)}$ by inserting (2.10) and the Lax matrix $M$ is of the form

$$M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \quad A(\lambda) = -\lambda - \sum_{j=1}^{N} \frac{\lambda \lambda_j \psi_{1j} \psi_{2j}}{\lambda^2 - \lambda_j^2},$$

$$B(\lambda) = \frac{1}{2} (\langle \Psi_1, \Psi_1 \rangle + \langle \Psi_2, \Psi_2 \rangle) + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda_j}{\lambda^2 - \lambda_j^2} [(\lambda + \lambda_j) \psi_{1j}^2 - (\lambda - \lambda_j) \psi_{2j}^2],$$

$$C(\lambda) = -\frac{1}{2} (\langle \Psi_1, \Psi_1 \rangle + \langle \Psi_2, \Psi_2 \rangle) + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda_j}{\lambda^2 - \lambda_j^2} [(\lambda - \lambda_j) \psi_{1j}^2 - (\lambda + \lambda_j) \psi_{2j}^2].$$

The compatibility of (2.2), (2.3) and (2.1) ensures that if $\Psi_1, \Psi_2$ satisfies two commuting FDIHSs (2.11) and (2.12), simultaneously, then $q$ given by (2.10) is a solution of mKdV equation (2.6), namely, the mKdV equation (2.6) is factorized by the $x$-constrained flow (2.11) and $t$-constrained flow (2.12).
3 Constructing the $N$-soliton solution for the mKdV equation

Hereafter we assume that $q(x,t), \psi_{1j}, \psi_{2j}$ be real functions. For soliton solution we have $q(x,t) \rightarrow 0, \psi_{1j} \rightarrow 0, \psi_{2j} \rightarrow 0$, when $|x| \rightarrow \infty$. In order to obtain convenient formulas to construct $N$-soliton solution, we need to rewrite all the formulas by using the complex version instead of the vector version. We denote

$$\Phi = \Psi_1 + i\Psi_2, \quad \phi_j = \psi_{1j} + i\psi_{2j}.$$ 

Then (2.11) and (2.12) become

$$\Phi_x = -\Lambda \Phi^* - \frac{i}{2} \Phi^T \Phi \Phi,$$  \hspace{1cm} (3.1)

$$\Phi_t = -\Lambda^3 \Phi^* - \frac{i}{2} \Phi^T \Phi^* \Lambda^2 \Phi + \frac{i}{2} \Lambda \Phi^* \Phi^T \Lambda \Phi - \frac{i}{2} \Phi \Phi^T \Lambda^2 \Phi^*,$$  \hspace{1cm} (3.2)

where we have used $\overline{H}_0 = 0$.

The generating function of integrals of motion for the system (3.1) and (3.2),

$$\frac{1}{2} Tr M^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda),$$

gives rise to

$$A^2(\lambda) + B(\lambda)C(\lambda) = \lambda^2 - 2\overline{H}_0 + \sum_{j=1}^N \frac{F_j}{\lambda^2 - \lambda_j^2},$$

where $F_j, j = 1, \ldots, N,$ are $N$ independent integrals of motion for the systems (3.1) and (3.2)

$$F_j = 2\lambda_j^3\psi_{1j}\psi_{2j} - \frac{1}{2} \Phi^T \Phi^* \lambda_j^2(\psi_{1j}^2 + \psi_{2j}^2) + \frac{1}{4} \lambda_j^2(\psi_{1j}^2 + \psi_{2j}^2)^2 + \frac{1}{2} \sum_{k \neq j} \frac{\lambda_j^2}{\lambda_j^2 - \lambda_k^2} P_{kj},$$

$$P_{kj} = \lambda_j \lambda_k(4\psi_{1j}\psi_{2j}\psi_{1k}\psi_{2k} + \psi_{1j}^2\psi_{1k}^2 + \psi_{2j}^2\psi_{2k}^2 - \psi_{1j}^2\psi_{2k}^2 - \psi_{2j}^2\psi_{1k}^2)$$

$$-\lambda_k^2(\psi_{1j}^2\psi_{1k}^2 + \psi_{2j}^2\psi_{2k}^2 + \psi_{1j}^2\psi_{2k}^2 + \psi_{2j}^2\psi_{1k}^2), \quad j = 1, \ldots, N.$$
\[
\lambda_k \phi_j \phi_k^* - \lambda_j \phi_k \phi_j^* = (\lambda_j^2 - \lambda_k^2) \partial_x^{-1}(\phi_j \phi_k). \tag{3.3}
\]

In a similar way as we did in [19], in order to constructing \(N\)-soliton solution, we have to set \(F_j = 0\). By using (3.1) and (3.3) \(F_j\) can be rewritten as

\[
F_j = \frac{i}{2} \lambda_j^2 \phi_j [-\phi_{jx} + \frac{i}{2} \sum_{k=1}^{N} \lambda_k \phi_k \partial_x^{-1}(\phi_j \phi_k)] - \frac{i}{2} \lambda_j^2 \phi_j [-\phi_{jx}^* - \frac{i}{2} \sum_{k=1}^{N} \lambda_k \phi_k \partial_x^{-1}(\phi_j^* \phi_k^*)] = 0,
\]

which leads to

\[
\phi_{jx} = -\gamma_j \phi_j + \frac{i}{2} \sum_{k=1}^{N} \lambda_k \phi_k \partial_x^{-1}(\phi_j \phi_k), \quad j = 1, ..., N,
\]

or equivalently

\[
\Phi_x = -\Gamma \Phi + \frac{i}{2} \partial_x^{-1}(\Phi \Phi^T) \Lambda \Phi = -\Gamma \Phi + R \Phi, \tag{3.4}
\]

where \(\Gamma = \text{diag}(\gamma_1, ..., \gamma_N)\), \(\gamma_j\) are undetermined real numbers and

\[
R = \frac{i}{2} \partial_x^{-1}(\Phi \Phi^T) \Lambda. \tag{3.5}
\]

Notice that

\[
\frac{i}{2} \Phi \Phi^T = R_x \Lambda^{-1}, \quad \Lambda R = R^T \Lambda, \tag{3.6}
\]

it follows from (3.4) and (3.5) that

\[
R_x = \frac{i}{2} \partial_x^{-1}(\Phi_x \Phi^T + \Phi \Phi^T_x) \Lambda
\]

\[
= \partial_x^{-1}(-\Gamma R_x + RR_x - R_x \Gamma + R_x R) = -\Gamma R - \Gamma R + R^2. \tag{3.7}
\]

We now show that \(\Gamma = \Lambda\). In fact, it is found from (3.4) and (3.7) that

\[
\Phi_{xx} = -\Gamma \Phi_x + R \Phi_x + R_x \Phi = -\Gamma (-\Gamma \Phi + R \Phi) + R(-\Gamma \Phi + R \Phi)
\]

\[+ (-\Gamma R - \Gamma R + R^2) \Phi = \Gamma^2 \Phi + 2 R_x \Phi = \Gamma^2 \Phi + i \Phi \Phi^T \Lambda \Phi.
\]

On the other hand (3.1) yields

\[
\Phi_{xx} = \Lambda^2 \Phi + i \Phi \Phi^T \Lambda \Phi,
\]

which implies \(\Gamma = \Lambda\). Therefore we have

\[
\Phi_x = -\Lambda \Phi + R \Phi, \tag{3.8}
\]
$$R_x = \frac{i}{2} \Phi \Phi^T \Lambda = -\Lambda R - R \Lambda + R^2. \quad (3.9)$$

To solve (3.8), we first consider the linear system

$$\Psi_x = -\Lambda \Psi.$$ 

It is easy to see that

$$\Psi = (\alpha_1(t)e^{-\lambda_1 x}, ..., \alpha_N(t)e^{-\lambda_N x})^T.$$ 

Take the solution of (3.8) to be of the form

$$\Phi = (I - M) \Psi, \quad (3.10)$$

then $M$ has to satisfy

$$M_x = M \Lambda - \Lambda M - R + R M. \quad (3.11)$$

Comparing (3.11) with (3.9), one finds

$$M = \frac{1}{2} R \Lambda^{-1} = \frac{i}{4} \partial_x^{-1}(\Phi \Phi^T). \quad (3.12)$$

Equation (3.10) implies that

$$\Psi = \sum_{n=0}^{\infty} M^n \Phi. \quad (3.13)$$

By using (3.12) and (3.13), it is found from that

$$\frac{i}{4} \partial_x^{-1}(\Psi \Psi^T) = \frac{i}{4} \partial_x^{-1} \sum_{n=0}^{\infty} \sum_{l=0}^{n} M^l \Phi \Phi^T M^{n-l}$$

$$= \partial_x^{-1} \sum_{n=0}^{\infty} \sum_{l=0}^{n} M^l M_x M^{n-l} = \sum_{n=1}^{\infty} M^n.$$

Set

$$V = (V_{kj}) = \frac{i}{4} \partial_x^{-1}(\Psi \Psi^T), \quad V_{kj} = -\frac{i}{4} \frac{\alpha_k(t) \alpha_j(t)}{\lambda_k + \lambda_j} e^{-(\lambda_k + \lambda_j) x},$$

one obtain

$$(I + V) \Phi = \Psi, \quad \text{or} \quad \Phi = (I - M) \Psi = (I + V)^{-1} \Psi. \quad (3.14)$$
Notice that (3.1) and (3.8) gives rise to
\[ \Lambda \Phi^* = (\Lambda - R - \frac{i}{2}q)\Phi. \] (3.15)

By inserting (3.9) and (3.15), (3.2) reduces to
\[ \Phi_t = [\Lambda^2(\Lambda - R - \frac{i}{2}q) - \frac{i}{2}q\Lambda^2 + (\Lambda - R - \frac{i}{2}q)(-\Lambda R - RA + R^2) \\
-(-\Lambda R - RA + R^2)(\Lambda - R - \frac{i}{2}q)]\Phi = -\Lambda^3\Phi + RA^2\Phi. \] (3.16)

Let \( \Psi \) satisfy the linear system
\[ \Psi_t = -\Lambda^3\Psi, \] (3.17)
then
\[ \Psi = (\alpha_1(t)e^{-\lambda_1 x}, \ldots, \alpha_N(t)e^{-\lambda_N x})^T, \quad \alpha_i(t) = \beta_j e^{-\lambda_j t}, \quad j = 1, \ldots, N. \] (3.18)

We now show that \( \Phi \) determined by (3.14) and (3.18) satisfy (3.16). In fact, we have
\[ \Phi_t = -(1 - M)\Lambda^3V + VA^3\Phi - (1 - M)\Lambda^3(1 + V)\Phi = -\Lambda^3\Phi + (I - M)V\Lambda^3\Phi + MA^3\Phi = -\Lambda^3\Phi + 2MA^3\Phi = -\Lambda^3\Phi + RA^2\Phi. \]

Therefore \( \Phi \) given by (3.14) and (3.18) satisfy (3.1) and (3.2), simultaneously, and \( q = \Phi^T\Phi^* \) is the solution of mKdV equation (2.6). Notice that
\[ \partial_x(\Psi^T\Phi) = -\Psi^T\Lambda\Phi + \Psi^T(-\Lambda + R)\Phi \]
\[ = \Psi^T(-2I + 2M)\Lambda\Phi = -2\Phi^T\Lambda\Phi, \]
\[ q_x = \frac{1}{2}(\Phi_x^T\Phi^* + \Phi^T\Phi_x^*) = \frac{1}{2}[(-\Phi_x^T\Lambda - \frac{i}{2}q\Phi^T)\Phi^* + \Phi^T(-\Lambda\Phi + \frac{i}{2}q\Phi^*)] \]
\[ = -\frac{1}{2}(\Phi_x^T\Lambda\Phi^* + \Phi^T\Lambda\Phi) = -\text{Re}(\Phi^T\Lambda\Phi). \]

So we have
\[ q = \frac{1}{2}\text{Re}(\Psi^T\Phi) = \frac{1}{2}\text{Re}\sum_{k=1}^{N}\alpha_k(t)e^{-\lambda_k x}\phi_k. \] (3.19)

Finally, as pointed out in [1], formulas (3.14) and (3.19) gives rise to the well-known \( N \)-soliton solution of mKdV equation (2.6)
\[ u = 2\partial_x\text{ImLn(det}(I + V)). \]
4 Conclusion

We first factorize the mKdV equation into two commuting integrable $x$- and $t$-constrained flows, then use them and their Lax representation to directly derive the $N$-soliton solution for mKdV equation. The method proposed in present paper and previous paper [19] provides a general procedure to construct $N$-soliton solution for soliton equations via their $x$- and $t$-constrained flows and can be applied to other soliton equations.

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