On a Deformation of 3-Branes

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Abstract

We construct an explicit class of solutions of type IIB supergravity that is a smooth deformation of the 3-brane class of solutions. The solution is nonsupersymmetric and involves nontrivial dilaton and axion fields as well as the standard 5-form field strength. One of the main features of the solution is that for large values of the radius the deformation is small and it asymptotically approaches the undeformed 3-brane solution, signaling a restoration of conformal invariance in the UV for the dual gauge theory. We suggest that the supergravity deformation corresponds to a massive deformation on the dual gauge theory and consequently the deformed theory has the undeformed one as an ultraviolet fixed point. In cases where the original 3-brane solution preserves some amount of supersymmetry we suggest that the gauge theory interpretation is that of soft supersymmetry breaking. We discuss the deformation for D3-branes on the conifold and the generalized conifold explicitly. We show that the semiclassical behavior of the Wilson loop suggests that the corresponding gauge theory duals are confining.
1 Introduction

One of the main virtues of the AdS/CFT correspondence [1] is that it has provided a computationally precise prescription of a gauge/gravity duality. The original statement of the AdS/CFT refers to the duality between IIB supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Super Yang-Mills. A very natural question that arises from phenomenological motivations is the consideration of the gauge/gravity duality in less supersymmetric situations. With QCD as the ultimate motivation, one would naturally be interested in nonsupersymmetric, nonconformal gauge theories. The duals of such gauge theories in the gauge/gravity framework are expected to be nonsupersymmetric solutions that do not contain an $AdS_5$ factor in the near horizon limit.

With this main motivation in mind, in this paper we construct a class of solution of IIB supergravity that is nonsupersymmetric, has running dilaton and axion, and generically does not contain an $AdS_5$ factor. Our departure from the original 3-brane solution is smooth and controlled by one parameter. The deformation we construct joins some examples in the literature where a controlled departure from a supersymmetric solutions has been motivated from gauge/gravity duality considerations. In particular, a desire to resolve the singularities of the KT solution [4] led to the construction of nonextremal solutions [3]. Our class of solutions is closer in spirit to [4–6], and even more so to [7], where nonsupersymmetric generalizations have been found that have an ultraviolet fixed point corresponding to $\mathcal{N} = 4$ supersymmetric Yang-Mills.

The construction is based on the well known fact that the super 3-brane of [8, 9] admits a generalization where the space perpendicular to the brane is replaced by a Ricci-flat noncompact space and the warp factor is then a harmonic function on the perpendicular space. Along these lines, the class of solution presented in this paper generalizes the situation for the case of radial-dependent nontrivial dilaton and axion, the warp factor is no longer a harmonic function in the perpendicular space to the brane but rather satisfies the Laplace equation with a source.

After discussing the general case in section 2, we consider the case of D3-branes on the conifold. Since the gauge theory dual of this situation and its generalizations is well understood [10–12], we use this case as the main explicit example. We suggest that the deformation amounts to adding a nonsupersymmetric mass deformation to the KW theory. For the one-parametric deformation we construct, the soft supersymmetry breaking scale and the confining scale are given by the same parameter. We therefore, also consider the analogous deformation for D3-branes on the generalized conifold [13] which has a mass scale in the undeformed theory and is expected to be dual to a confining gauge theory.

As is common to most of the deformations of p-branes discussed in the gauge/gravity framework, the resulting deformation we construct has naked singularities in the IR. The ultimate resolution of this naked singularity within the full string theory remains an open problem.

1And for that purpose, any p-brane.
2 Deformed 3-branes

The field equations of IIB supergravity when only the metric, self-dual 5-form, dilaton and axion field are nontrivial are:

\[
\begin{align*}
\ast d\Phi &= e^{2\Phi} d\chi \wedge \ast d\chi, \\
\ast d(\ast e^{2\Phi} d\chi) &= 0, \\
\ast d\tilde{F}_5 &= dF_5 = 0, \\
R_{MN} &= \frac{1}{2} \partial_M \Phi \partial_N \Phi + \frac{e^{2\Phi}}{2} \partial_M \chi \partial_N \chi + \frac{1}{96} F_{MPQRS} F_{N^{PQRS}}. 
\end{align*}
\] (2.1)

First we consider the dilaton/axion system and show that it effectively decouples from the specific features of the metric, which we take to be of the form

\[
ds^2 = e^{2A(y^m)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(y^m)} g_{mn} dy^m dy^n, \]

(2.2)

where \(g_{mn}\) is the Ricci-flat metric of the noncompact 6-d space perpendicular to the 4-d worldvolume. We assume that the space has one radial coordinate \(r\). The typical example to bear in mind would be a cone over a 5-d compact Einstein space: \(dr^2 + r^2 dX_5^2\). However, the statement is true for more general spaces that are not necessarily cones over \(X_5\); all we need is to be able to extract a volume form for the 5-d compact space. Assume that the dilaton and axion depend only on the radial coordinate in the 6-d perpendicular space: \(\Phi = \Phi(r)\) and \(\chi = \chi(r)\). Then \(d\Phi = \Phi' dr\) and \(d\chi = \chi' dr\) and also

\[
\ast d\chi = -\chi' e^{4A+4B} g^{rr} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge Vol(X^5), \]

(2.3)

where \(Vol(X^5)\) is the volume form of the compact subspace, and we assume it to be independent of the radial coordinate. Introducing a new radial coordinate \(t\) satisfying

\[
\frac{dt}{dr} = (\sqrt{gg_{rr}})^{-1} e^{-4A-4B}, \]

(2.4)

the dilaton and axion equations become:

\[
\begin{align*}
\ddot{\Phi} &= \dot{\chi} e^{2\Phi}, \\
\dot{\chi} e^{2\Phi} &= c_0,
\end{align*}
\] (2.5)

where \(c_0\) is a constant. The dilaton equation becomes \(\ddot{\Phi} = c_0^2 e^{-2\Phi}\), the solution of the dilaton/axion system is:

\[
\begin{align*}
\Phi &= -\ln \frac{c_1}{c_0} + \ln \cosh c_1 t, \\
\chi &= \chi_0 + \frac{c_1}{c_0} \tanh c_1 t.
\end{align*}
\] (2.6)
As was stated above, the decoupling of the dilaton/axion system is very generic, its functional form is therefore determined universally. The dependence on the concrete structure of the metric enters only through \( t \) (see [14] for another example of this universality).

There are two facts needed in considering the Einstein equations. The first one states that for a metric of the form (2.2) the Ricci tensor has the following form

\[
R_{\mu\nu} = -\eta_{\mu\nu}e^{2(A-B)} \left( \nabla^2 A + 4\partial A \partial A + 4\partial A \partial B \right),
\]

\[
R_{mn} = R_{mn}[g] - g_{mn} \left( \nabla^2 B + 4\partial B \partial B + 4\partial A \partial B \right)
\]

\[
- 4 \nabla_m \nabla_n B - 4 \nabla_m \nabla_n A - 4\partial_m A \partial_n A + 4\partial_m B \partial_n B
\]

\[
+ 4(\partial_m A \partial_n B + \partial_n A \partial_m B),
\]

(2.7)

where \( R_{mn}[g] \) is the Ricci tensor in the 6-d space calculated from the metric \( g_{mn} \). Motivated by the super 3-brane picture, from now on we specialize to the case \( A + B = 0 \); in the absence of the deformation this condition is required for supersymmetry [8, 15]. Moreover, this choice is consistent with the equations of motion. The ansatz for the 5-form field strength is

\[
F_5 = \mathcal{F} + *\mathcal{F}, \quad \mathcal{F} = KV ol(X^5).
\]

(2.8)

The \( F_5 \) field equation (which coincides with its Bianchi identity) requires \( K = \text{Const.} \). Note that this is different from the standard 3-brane picture, from now on we specialize to the case \( A + B = 0 \); usually it is, written as \( F_5 = (1+\ast)dh^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \), where \( h = e^{-4A} \). Actually, in our case

\[
*\mathcal{F} = \frac{K}{h^2g^{rr} \sqrt{g}} dr \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.
\]

(2.9)

The last element needed for the Einstein equation is the observation that the energy momentum tensor of the dilaton/axion system contributes a “charge”. Namely, using the solution of the dilaton/axion system given in equation (2.6) one obtains

\[
\frac{1}{2} \Phi'^2 + \frac{1}{2} e^{2\Phi} \chi'^2 = \frac{1}{2} c_1^2 (\frac{dt}{dr})^2.
\]

(2.10)

There are two independent Einstein equations (one of them is the \( rr \) component)

\[
g_{rr} \nabla^2 A - 8A'^2 = -\frac{K^2e^{8A}}{4g(g^{rr})^2} + \frac{1}{2} \frac{c_1^2}{g^{rr}},
\]

\[
\nabla^2 A = \frac{K^2e^{8A}}{4gg^{rr}}.
\]

(2.11)

Combining these two, we obtain an equation for \( A \)

\[
\nabla^2 A - 4g^{rr} A'^2 = \frac{c_1^2}{4gg^{rr}}.
\]

(2.12)
Written in terms of the coordinate \( t \) and the warp factor \( h \), this is simply the equation of the harmonic oscillator

\[
\ddot{h} + c_2^2 h = 0. \tag{2.13}
\]

Note that for trivial dilaton/axion system we recover the typical 3-brane solution \( h = h_0 + h_1 t \). The other independent equation fixes the value of \( K \). Thus, we have shown that the type IIB field equations admit a generalization of the 3-brane solution of the form:

\[
\begin{align*}
\delta \lambda &= -\frac{1}{2\tau_2} \gamma^M \partial_M \tau \epsilon^* = -\frac{1}{2} e^\Phi \gamma^r (\partial_r \chi + i \partial_r e^{-\Phi}) \epsilon^*,
\end{align*}
\]

which can not be made zero\(^2\) for nontrivial values of the \( \Phi \) and \( \chi \).

### 3 Deformed D3-branes on Conifolds

In this section we consider the above construction in the particular case where the 6-d perpendicular space is the conifold. This example will clarify any technical detail left obscure in the general construction of section\(^3\) and, more importantly, will shed light on the physics of the deformed solution. This a phenomenologically very interesting case and it also captures most of the general properties of the solution\(^3\). The metric of the singular conifold is

\[
ds_{10}^2 = h^{-1/2}(r)\eta_{\mu\nu} dx^\mu dx^\nu + h^{1/2}(r)g_{mn} dy^m dy^n,
\]

where \( t \) and \( r \) are related through \( dt/dr = (\sqrt{gg^{rr}})^{-1} \). The limit necessary to recover the standard 3-brane solution involves sending \( c_1 \to 0 \) and \( b_1 \to \infty \) in such a way that \( c_1 b_1 \to \text{const.} \) Note also that in a region where \( t \to 0 \) the solution tends to the standard 3-brane solution. The solution is not supersymmetric. This is easily seen from the supersymmetric variation of the dilatino:

\[
\delta \lambda = -\frac{1}{2\tau_2} \gamma^M \partial_M \tau \epsilon^* = -\frac{1}{2} e^\Phi \gamma^r (\partial_r \chi + i \partial_r e^{-\Phi}) \epsilon^*,
\]

\(\text{where}\)

\[
e_{\theta_i} = d\theta_i, \quad e_{\phi_i} = \sin \theta_i d\phi_i, \quad e_\psi = d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2.
\]

\(^2\)There is, however, a different situation in the Euclidean version as discussed in [7].

\(^3\)The study of this example motivated the general construction presented in section\(^3\). I thank A.A. Tseytlin for originally suggesting this example for consideration.
All the geometrical statements made in this section apply as well to any 6-d space that is a cone over a 5-d compact Einstein manifold. The explicit form for the selfdual 5-form is

\[ F_5 = \mathcal{F} + *\mathcal{F}, \quad \mathcal{F} = K(\rho) e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2}. \]

(3.3)

At the expense of being repetitive but with the hope of completely clarifying the structure of the Einstein equation we consider them explicitly in this case. The two independent Einstein equations are

\[ A'' + \frac{5A'}{r} = -\frac{108K^2 e^{8A}}{r^{10}}, \]
\[ A^2 = -\frac{27K^2 e^{8A}}{r^{10}} - \frac{1}{16}(\Phi'^2 + e^{2\Phi} \chi'^2). \]

(3.4)

The equation for \( A \) becomes

\[ A'' + \frac{5A'}{r} - 4A^2 = \frac{c_1^2}{4r^{10}}. \]

(3.5)

To solve this equation take \( h = e^{-4A} \) and note that

\[ (r^5(e^{-4A}))' = -4r^5 e^{-4A}(A'' + \frac{5A'}{r} - 4A^2), \]

which linearizes it into

\[ h'' + \frac{5h'}{r} + \frac{c_1^2 h}{r^{10}} = 0. \]

(3.6)

In the \( t \) coordinate \( dt/dr = r^{-5} \) this equation becomes

\[ \ddot{h} + c_1^2 h = 0. \]

(3.7)

Note that in the absence of the deformation, \( \ddot{h} = 0 \) implies that \( h \) has the form of a harmonic function: \( h = h_0 + h_1 t = h_0 + Q_1/r^4 \). The solution of the whole system, setting \( c_1 = 4L^4 \), is

\[ e^\Phi = g_s \cosh \frac{L^4}{r^4}, \]
\[ \chi = -g_s^{-1} \tanh \frac{L^4}{r^4}, \]
\[ h = b_1 \sin \frac{L^4}{r^4} + b_2 \cos \frac{L^4}{r^4}, \]
\[ K = \frac{L^4}{27} \sqrt{b_1^2 + b_2^2}. \]

(3.8)

Taking \( b_2 = 1 \) in the above expressions gives asymptotically flat space far away from the brane. The other interesting asymptotic to explore corresponds to removing the
flat region, i.e. taking \( b_2 = 0 \). The limit of D3 without deformation can be recovered by sending \( L^4 \to 0 \) and \( b_1 \to \infty \) in such a way that \( L^4 b_1 \to \text{const} \). The trigonometric behavior of the metric signals the existence of a very pathological array of naked singularities. We thus, consider the solution reliable away from this region. Namely, to avoid the highly oscillatory \( r \sim 0 \) region we assume \( r \geq r^* \), where \( r^* \) is the largest value of \( r \) at which \( h \) has a maximum. This pathological behavior is captured by the scalar curvature which equals

\[
R = \frac{8L^8}{h^{1/2}r^{10}}.
\]  

(3.9)

Showing that zeros of \( h \) are points at which \( R \) blows up. In the limit of no deformation we have that, as expected for 3-branes, the scalar curvature is identically zero. This behavior of \( R \) is then solely an effect of the dilaton/axion system being nontrivial.

Note that now, in contrast to the standard D3-brane, the 10-d metric does not factorize into a product metric after removing the asymptotically flat region. Removing the asymptotically flat region (taking \( b_2 = 0 \)) for the deformed 3-brane leaves the \( \sin(L^4/r^4) \) factor in the angular part, preventing the metric from becoming a product space. One, however, recovers the \( AdS_5 \times T^{1,1} \) space in the limit of no deformation discussed above. Also, very far from the brane \( r \gg L \) the deformation becomes asymptotically small. Thus, in terms of the AdS asymptotics, our solution tends to \( AdS_5 \times T^{1,1} \) for large values of the radius (which corresponds to the UV region on the gauge theory side). As one approaches small \( r \) (flow to the IR in gauge theory) the metric is no longer a direct product. Moreover, in order to avoid a very erratic behavior (the warp factor going to zero and even changing signs) we should consider our solution being valid only in the region \( r \geq r^* \) where \( r^* \) is the largest value of \( r \) at which \( h \) has a maximum. We suggest that around \( r \approx r^* \) the supergravity description starts to break down. The asymptotic behavior is very different from the KT solution in the sense that KT is not asymptotically \( AdS_5 \times T^{1,1} \), there is the characteristic logarithmic running of the warp factor [2] which is also present for all of its generalizations [12, 13, 16]. Perhaps the most important difference with the KT solution is the fact that for our solution \( K \) is constant, i.e. no running 5-form flux (or number of colors). This allows us to conclude that the deformation does not alter the gauge group structure of the KW solution. In other words, the deformation amounts to a perturbation that dies in the UV but has a very strong influence in the IR. The situation described above is the trademark of a massive deformation. This situation is somehow analogous to that considered for holomorphic \( \tau = \chi + ie^{-\Phi} \) [17, 18]. One difference is, however, supersymmetry. The solution presented above is not supersymmetric, as was shown above from the supersymmetric transformation of the dilatino. In the absence of nontrivial 3-form field strengths the complex dilaton \( \tau \) has to be a holomorphic or antiholomorphic function (see for example [14]). Thus, our deformation amounts, in the gauge theory dual, to a nonsupersymmetric mass deformation. Since supersymmetry is restored at high energy we can conclude that this is a situation where supersymmetry is softly broken. Recalling that the \( \mathcal{N} = 1 \) superconformal field theory (the limit with the mass deformation going to zero) is known [10, 12], we suggest that the KW gauge theory

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6
is a UV fixed point of the deformed theory; and also that the deformation is given by
adding dimension two operators of the form $m^2 \text{Tr}(a^\dagger a + b^\dagger b + ab)$, where $a$ and $b$ are
the components of the chiral superfields $A$ and $B$ transforming as $(N, \bar{N})$ and $(\bar{N}, N)$
respectively under the gauge group $U(N) \times U(N)$. For the gauge couplings one has:

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} \sim e^{-\Phi} = \frac{1}{g_s \cosh \frac{L^4}{r^4}}. \quad (3.10)$$

The difference of the squares of the inverses of the gauge couplings is proportional to $B_2$
which is constant for this solution. The behavior of the gauge couplings is moderate, the
above sum ranges from $1/g_s$ to $1/(g_s \cosh L^4/r^4)$.

The dependence of the dilaton on the radius is very mild in the region where the solution is reliable, even after applying a $SL(2, R)$ transformation to the above background. Namely:

$$\tilde{\tau} = \frac{a \tau + b}{c \tau + d}; \quad \tau = \chi + ic^\Phi, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, R), \quad (3.11)$$

results in the following dilaton

$$e^\Phi = g_s \left[ \frac{c^2}{g_s^2} + d^2 - \frac{2cd}{g_s} \tanh \frac{L^4}{r^4} \right] \cosh \frac{L^4}{r^4}. \quad (3.12)$$

It is worth noting that for $r \gg L$ we basically recover the strong/weak relation. If the
original string coupling $g_s$ was weak ($g_s \ll 1$), then after the $SL(2, R)$ transformation we get $\tilde{g}_s \approx c^2/g_s \gg 1.$

### 3.1 Deformed D3-branes on the generalized conifold

In the previous subsection we considered a deformation of the background of D3-branes on
the singular conifold. One feature of the solution was that there was only one scale which
determined, on the gauge theory side, the confining (see appendix) and the supersymmetry
breaking scale. In this subsection we consider adding a deformation to the background
of D3 on the generalized conifold [13]. The generalized conifold is not a cone over a 5-d
Einstein space. Nevertheless, the general discussion of section [2] directly applies since we
are able to define a volume form for the compact 5-d space that is independent of the
radial coordinate, basically the same one as in the case of the singular conifold[14]. D3-branes on the generalized conifold provide supersymmetric vacua of IIB [19], and they
are dual to a confining theory [13]. Thus, D3-branes on the generalized conifold provide a
supersymmetric massive deformation of KW. By considering the axion/dilaton deformation on
the background of D3 on the generalized conifold we believe, in principle, to be able to
separate the confining and susy breaking scales in the gauge theory by adjusting the
two corresponding parameters in the sugra solution.

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4Similarly, one can define such volume form for the resolved and deformed conifolds.
The metric of the generalized conifold is:

\[ ds^2 = \kappa^{-1} dr^2 + \kappa \frac{r^2}{9} e_\phi^2 + \frac{r^2}{6} (e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2), \]  

(3.13)

with

\[ \kappa = 1 - \frac{b^6}{r^6}. \]  

(3.14)

The radial coordinate \( t \) is given by

\[ \frac{dt}{dr} = \frac{1}{\kappa r^5}. \]  

(3.15)

The solution is

\[ e^\Phi = g_s \cosh c_1 t, \quad \chi = g_s^{-1} \tanh c_1 t. \]  

(3.16)

and

\[ h = b_1 \sin c_1 t + b_2 \cos c_1 t, \quad \text{and} \quad K = \frac{c_1}{108} \sqrt{b_1^2 + b_2^2}. \]  

(3.17)

The relation between the radial coordinate \( t \) and the original \( r \) is

\[ t = \frac{1}{2b^4} \left[ \frac{1}{6} \ln \left( \frac{\bar{r}^2 - 1}{\bar{r}^2} \right) + \frac{1}{\sqrt{3}} \left( \frac{\pi}{2} - \arctan \frac{2\bar{r}^2 + 1}{\sqrt{3}} \right) \right], \]  

(3.18)

where \( \bar{r} = r/b \). In the \( r \gg b \) limit we basically recover the previous singular conifold situation \( t \to -1/(4r^4) \). The generalized conifold metric requires \( r \geq b \). In the \( r \to b \) limit we have

\[ t \to \frac{1}{6b^4} \ln \left( \frac{r - 1}{b} \right). \]

This behavior means that the restriction \( r \geq b \) is not good enough since by the time we reach \( r \approx b \) the metric has experienced some naked singularities \( h = 0 \). We need, therefore, to restrict the radial domain further. As in the case of the conifold, we take \( r \geq r_* \) where \( r_* \) is the largest values of \( r \) for which \( h \) has a maximum.

**Acknowledgments**

I am especially thankful to A.A. Tseytlin for originally suggesting the deformation of the conifold for consideration, as well as many other comments and suggestions that considerably improved this paper. I also benefited from discussions with D. Chung, M. Einhorn and J. Liu. I would like to acknowledge the Office of the Provost at the University of Michigan and the High Energy Physics Division of the Department of Energy for support.
Appendix A  Wilson Loop behavior

Let us now investigate, following [20, 21], the behavior of the Wilson loop corresponding to the “quark-antiquark” potential in the dual gauge theory. It is given by the exponential of the classical fundamental string action in this background evaluated for a static configuration of open string ending on the probe D3-brane placed at the “boundary” $r = \infty$.

We will show that generically it is possible to obtain an area law (confining) behavior for the deformed D3-brane background. The existence of an area law reinforces the interpretation of the deformation as a massive deformation, since otherwise dimensional analysis will require the potential to be Coulombic. This is different from what is found in the standard conifold case [10] where the potential is Coulombic as in [20, 21] in any regime. We will also show that, as expected, in the limit when the deformation is turned off we recover the Coulombic potential characteristic of D3-branes.

The relevant metric is the string metric rather than the Einstein metric of equation (3.8),

$$ds^2_{\text{string}} = \frac{e^{\Phi/2}}{g_s} ds^2_{\text{Einstein}},$$

which in our background is

$$ds^2_{\text{st}} = \left( \cosh \left( \frac{L^4}{r^4} \right) \right)^{1/2} \left[ h^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + h^{1/2} ds^2_6 \right].$$

(A.1)

where $ds^2_6$ is the conifold metric (3.1) and $h = b_1 \sin \left( \frac{L^4}{r^4} \right)$ in order to remove the asymptotically flat region. The Nambu-Goto string action which determines the expression for the Wilson loop depends on this 10-d string metric $G_{MN}$ as

$$S = T \int dx \sqrt{G_{00} G_{xx} + G_{00} G_{rr} (\partial_x r)^2} = T \int dx e^{\Phi/2} \sqrt{h^{-1} + (\partial_x r)^2}.$$  

(A.2)

Since the Lagrangian of this “mechanical system” does not depend explicitly on the “time” $x$, we have a conserved quantity $\frac{e^{\Phi/2} h^{-1}}{\sqrt{h^{-1} + (\partial_x r)^2}} = e_0$. The natural way to parametrize $e_0$ is by interpreting it as the total energy of the “mechanical system” and therefore relating it to the turning point $r_0$. Then

$$e_0^2 = \frac{\cosh \left( \frac{L^4}{r^4_{\text{cr}}} \right)}{b_1 \sin \frac{L^4}{r^4_{\text{cr}}}}.$$  

(A.3)

From the above expressions we find

$$dx = \frac{h^{1/2} dr}{\sqrt{e^\Phi / (e_0^2 h) - 1}} = \frac{b_1^{1/2} \sin^{1/2} \left( \frac{L^4}{r^4} \right)}{\sqrt{\cosh \left( \frac{L^4}{r^4} / (e_0^2 b_1 \sin \left( \frac{L^4}{r^4} \right)) \right) - 1}} dr.$$  

(A.4)

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5I thank M. Einhorn for useful discussions on this appendix.

6$T$ is the time interval and the string tension is set equal to 1.
The energy of a static string configuration is thus

\[ E = \frac{S}{T} = \int dxe^{\Phi/2} \sqrt{h^{-1} + (\partial_x r)^2} = \int \frac{e^{\Phi/2}}{e_0 h^{1/2}} \frac{dr}{\sqrt{e^{\Phi/(e_0^2 h) - 1}}}. \]  

(A.5)

Following [20, 21], the question about confinement is then reduced to finding the dependence of the energy \( E \) on the distance \( \ell \) between the string end-points (between the “quark” and the “antiquark”). More precisely, it has been discussed extensively (see for example [22]), that the behavior of the Wilson Loop is essentially determined by the properties around the turning point \( r_0 \) of the probe string. Following that prescription we obtain

\[ \ell \approx \frac{2b_1^{1/2} \sin^{1/2}(L^4/r_0^4) r_0^3}{L^2 \sqrt{\cot(L^4/r_0^4) - \tanh(L^4/r_0^4)}}, \quad E \approx \frac{r_0^3}{L^2 \sqrt{\cot(L^4/r_0^4) - \tanh(L^4/r_0^4)}}. \]  

(A.6)

It is worth noting that in the limit of zero deformation we reproduce the Coulombic behavior of [20, 21]. Namely, taking \( L^4 \to 0 \), \( b_1 \to \infty \) with \( b_1 L^4 = R^4 \) we obtain

\[ \ell \approx \frac{2R^2}{r_0}, \quad E \approx r_0, \]  

(A.7)

which is precisely the Coulombic behavior \( E \approx 2R^2/\ell \). Recalling that taking the zero deformation limit and considering the region far away from the brane are formally the same we see that if \( r_0 \) is very large (that is the string does not probe deep into the bulk) we also obtain a Coulombic potential. More generally, however, we have

\[ E \approx \frac{1}{2b_1^{1/2} \sin^{1/2}(L^4/r_0^4)} \ell. \]  

(A.8)

In the above expression \( r_0(\ell) \) but we find almost linear confinement whenever the sin is not small, which is everywhere except in the previously discussed limit. In particular in the region \( r_0 \sim L \sim \ell \) we obtain that \( E \approx \ell \).

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