A New Quantum Operator for Distance and its use in Studying the Weak Equivalence Principle

Daniel Katz
University of Massachusetts Lowell
1 University Ave, Lowell MA
01854
E-mail: daniel_katz@uml.edu

April 2018

Abstract. We introduce a new non-relativistic quantum operator for the distance traveled by a particle in a given interval of time. The purpose of this operator is to investigate the Weak Equivalence Principle (WEP) in the quantum regime. In particular, our operator measures the integrated expected trajectory of the particle. If its expectation value depends on the particle’s mass we can infer that the particle’s motion is also dependent on its mass and thus violates the WEP. In this article the new operator is derived and some of its elementary properties are explored. As a proof of concept we use it to analyze the expected distance traveled by a free Gaussian wavepacket with some initial momentum. It is shown in this case that the distance such a particle travels becomes close to light-like as its mass vanishes and agrees with the classical result for macroscopic masses. This calculation begs a number of questions which are presented as topics for future research.

Keywords: Gravitation, Quantum Mechanics, Equivalence Principle

1. Introduction

It is quite well-known that the simple combination of quantum mechanics/field theory with Einstein’s General Relativity (GR) leads to nonsensical predictions. One reason for this may be the incompatibility of the WEP of GR with the type of statements one is allowed to make about quantum particles. While it does not rigorously justify GR - that honor belongs to the somewhat more restrictive Einstein Equivalence Principle - the WEP is the conceptual foundation on which metric theories of gravity are based. It asserts that gravitational and inertial masses are identical for all objects. This version of the WEP lends itself to conceptually simple, though technically challenging, torsion balance experiments which have historically dominated the experimental WEP
A New Quantum Operator for Distance

Experimental tests of WEP are quantified by measuring the dimensionless Eötvös parameter

$$\eta(A, B) = \frac{(m_g/m_i)_A - (m_g/m_i)_B}{\frac{1}{2}((m_g/m_i)_A - (m_g/m_i)_B)^2},$$

where $m_i$ and $m_g$ are inertial and gravitational mass, respectively, and the subscripts A and B refer to a pair of test masses with those labels. WEP predicts that $\eta$ is identically zero. State-of-the-art torsion balance experiments have managed to determine that if inertial mass and gravitational mass differ from each other they do so at a level below one part in $10^{-13}$ [1]. Other independent methods of testing the WEP at various length and mass scales exist as well. For instance, Baessler et al. [2] used a torsion pendulum to make a $10^{-13}$ level measurement which was also much more sensitive to the self-gravitation of the test masses than other experiments. Careful analysis of time series data from the ongoing Lunar Laser Ranging experiment yields a similar upper bound on the extent of possible WEP violation [3]. Atom interferometry has also been used [4] to validate WEP to a more modest $10^{-8}$ level, but on a much smaller mass scale. Presently, the most stringent upper bound comes from an early data release from the MICROSCOPE [5] mission, which uses a combination of methods aboard an artificial satellite to look for deviations between inertial and gravitational mass. Analysis of their first data release gives an upper bound of about $10^{-14}$ for WEP violation, and future data from the mission are expected to reduce that bound by an order of magnitude or more. In addition, future experiments such as the proposed sounding rocket test by Reasenberg et al. [6] or variants of the canceled QUEST satellite experiment [7] aim to bring the measurement of the Eötvös parameter down to the $10^{-17}$ level.

The equivalence of inertial and gravitational mass is itself equivalent, at least classically, to another formulation of the WEP: if a non-self-gravitating body with no internal structure is subject to no non-gravitational forces, the resulting trajectory of the body is independent of its own properties. This in turn shunts the source of gravity away from Newton’s “spooky action at a distance” and into the structure of spacetime itself. Since the statement doesn’t actually require the body in question to be under any forces (in contrast to, say, the universality of free fall formulation of WEP), it is also relevant to the study of the free particle in section 3. One obvious difficulty in assessing this version of WEP on the quantum scale is the lack of definite trajectories in the quantum description. This can be circumvented by formulating a version of WEP in terms of expectation values; essentially by an intuitive appeal to Ehrenfest’s theorem. This is the approach taken by Greenberger in the appendix of [8] to show that WEP is only an approximate symmetry in quantum theory which becomes exact in the limit of large quantum numbers. He points out that the spread of a localized Gaussian wavefunction evolves in a mass-dependent way, but that contributions to observables from this spread vanish in the classical limit. Put another way, low-lying states overlap significantly with each other while semi-classical states are more or less distinct.

In this article we present a new tool for probing equivalence in the quantum regime: a quantum operator for special relativistic 4-distance. The rest of this work is laid out
as follows. In section 2, we derive the distance operator by canonically quantizing the distance element of Minkowski spacetime. The operator ends up being dependent on the potential of the system it is to act on, so as a proof of concept we begin by analyzing in section 3, free particles, both localized Gaussian wavepackets and delocalized planewaves. We will see that the expected distance traveled by a Gaussian depends on the particle’s mass but that this dependence vanishes in the classical limit. Finally, in section 4, we discuss the distance operator in situations other than infinite free space, as well as some possible generalizations. Natural units ($c = \hbar = 1$) are used throughout.

2. Derivation of the Operator

We begin by considering the distance element in 4-dimensional Minkowski space parameterized by time,

$$ds = \sqrt{1 - \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt}} dt.$$  \hspace{1cm} (2)

The explicit time dependence complicates the evolution of the operator which will result from quantization of this object and we will need the expressions

$$s = \int_0^t dt' \sqrt{1 - \frac{d\vec{x}}{dt'} \cdot \frac{d\vec{x}}{dt'}}$$  \hspace{1cm} (3)

and

$$\frac{\partial s}{\partial t} = \sqrt{1 - \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt}},$$  \hspace{1cm} (4)

which follow immediately from 2. The derivative of $s$ in 4 is written as a partial in order to distinguish it from the total differential which occurs on the left-hand-side of Ehrenfest’s theorem. For simplicity we have chosen $t_0 = 0$ in 3. To quantize $s$ we take the canonical approach of promoting $\vec{x}$ to an operator obeying the commutation relations

$$[x_i, p_j] = i\delta_{ij}, \quad i, j = 1, 2, 3,$$  \hspace{1cm} (5)

where $\delta_{ij}$ is the Kronecker delta, with its conjugate, the momentum operator $\vec{p}$. Now $\vec{x}$ is no longer a curve parameterized by time and representing the trajectory of a particle, but rather a quantum operator whose evolution is determined by Heisenberg’s equation. Although we are dealing with only free particles in this paper, for future purposes it will be useful to know the evolution of our distance operator in the presence of a potential depending on at most $\vec{x}$. The Heisenberg equation for the position operator is then given by

$$\frac{dx_i}{dt} = i[H, x_i]$$

$$= i \left( \left[ \frac{p_i^2}{2m}, x_i \right] + [V, x_i] \right)$$

$$= -\frac{p_i}{m}$$  \hspace{1cm} (6)
where $m$ is the particle’s mass and the last line follows because the potential, $V$, is a function of neither time nor momentum. This condition on $V$ is restrictive, but it still allows for two potentials which are expected to be important to future work involving this operator, namely $V \propto z$ and $V \propto 1/r$. Because of $\vec{x}$'s simple evolution the commutator of $s$ and the Hamiltonian is not nested. We evaluate it by observing that the only sensible interpretation of a non-power function of an operator is the power series representing it, provided that the series actually converges. In light of this we have

$$\left[H, s\right] = \int_0^t dt' \left[H, \sqrt{1 - \left(\frac{p}{m}\right)^2}\right]$$

$$= \int_0^t dt' \left[H, \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left(\frac{ip}{m}\right)^n\right]$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left(\frac{-1}{n}\right) \frac{(1/2)^n}{m^{2n}} \int_0^t dt' [H, p^{2n}]$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left(\frac{-1}{n}\right) \frac{(1/2)^n}{m^{2n}} \int_0^t dt' [V, p^{2n}]$$

(7)

where $\binom{n}{k}$ is a binomial coefficient. We can now put the pieces together and write down an expression for the expectation value of the 4-distance traveled by a particle in time $t$. Ehrenfest’s theorem says of the operator $s$

$$\frac{d}{dt} \langle s \rangle = i\left\langle \left[H, s\right] \right\rangle + \left\langle \frac{\partial s}{\partial t} \right\rangle.$$  

(8)

Integrating both sides, plugging in (4) and (7) and using Cauchy’s formula on the resulting double integral gives our result for the expectation value of the quantum distance operator $s$:

$$\langle s \rangle = \int_0^t dt' \left\langle \sqrt{1 - \left(\frac{p}{m}\right)^2} \right\rangle + i \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left(\frac{-1}{n}\right) \frac{(1/2)^n}{m^{2n}} \int_0^t dt' (t - t') \langle [V, p^{2n}] \rangle.$$  

(9)

Our interest from here on will be focused on calculating $\langle s \rangle$ for various scenarios, though in this paper we limit ourselves to the $V = 0$ case for simplicity. We can hazard a guess at the physical meaning of $\langle s \rangle$ by considering its derivation, which began by integrating the 4-distance element in Minkowski space. For a classical object this quantity would be the invariant interval of the object’s motion. We can thus interpret $\langle s \rangle$, at least qualitatively, as the weighted average of the invariant intervals corresponding to well-defined trajectories that a classical object might take. This interpretation as an average over trajectories suggests that there may be some connection between our new operator and the path integral formulation of quantum mechanics, though we admit that any such connection is, at this stage, purely speculative.
3. Quantum Distance Traveled by a Free Particle

For the free particle the second term in (9) vanishes and we have

\[
\langle s \rangle = \int_0^t dt' \sqrt{1 - \left(\frac{p}{m}\right)^2}.
\]  

(10)

Since \([H, s] = 0\) in this case the eigenstates of \(H = p^2/2m\) are also eigenstates of \(s\). Because they have definite momenta one readily shows that plane waves

\[
\langle x | \psi \rangle \propto \exp(-iEt - i\vec{p} \cdot \vec{x})
\]  

(11)

have for their \(s\)-eigenvalues

\[
t\sqrt{1 - \left(\frac{p}{m}\right)^2}
\]  

(12)

in which \(p\) represents a momentum eigenvalue as opposed to the momentum operator. This shows that completely delocalized planewave states travel a continuum of distances which depend, in the classical way, on their momenta. On the other hand, a planewave particle-in-a-box has a discrete spectrum of momentum eigenvalues and thus, according to (12), can only be found to have traveled certain distances within the box. This implies an effective discretization of space or time or both when it comes to the motion of a confined particle. It is important to emphasize that (12) has nothing at all to say about the structure of spacetime, meaning that the discretization phenomenon is limited to the values of \(s\) which may be observed and has no baring on the geometry inside the box.

The case of planewaves was easy to analyze but it doesn’t do much for our goal of testing the equivalence principle in the quantum-into-classical regime since the totally delocalized planewave states lack any classical analog. We must therefore turn our attention to localized wavepackets for which the classical limit corresponds to particles of definite position and momentum. To facilitate the calculation we first notice that computing (10) by power series expansion will involve computing all the even moments of the wavepacket, \(\langle p^{2n} \rangle\). Let \(U\) be the unitary propagator so that

\[
|\psi(t)\rangle = U|\psi(0)\rangle
\]  

(13)

for any initial state ket \(|\psi(0)\rangle\). For \(V = 0\) the propagator depends only on time so that \([U, p] = 0\), allowing us to ignore the time evolution of the initial wavefunction while calculating \(\langle p^{2n} \rangle\):

\[
\langle \psi(t) | p^{2n} | \psi(t) \rangle = \langle \psi(0) | U^\dagger p^{2n} U | \psi(0) \rangle
\]

\[
= \langle \psi(0) | p^{2n} U^\dagger U | \psi(0) \rangle
\]

\[
= \langle \psi(0) | p^{2n} | \psi(0) \rangle.
\]  

(14)

We remark in passing that this simplification will not be possible for particles subject to linear and Coulombic potentials since the corresponding propagators have position dependence and so fail to commute with the momentum operators. The wavepacket we choose saturates the uncertainty bound: a Gaussian initially centered on the origin with
A New Quantum Operator for Distance

initial mean momentum in the negative $z$-direction of $p_0$. The initial wavefunction in momentum space is

$$
\langle p|\psi(0)\rangle = \frac{\sqrt{2} \sigma^{3/2}}{\pi^{1/4}} \exp \left[ -\frac{\sigma^2}{2} (\vec{p} + p_0 \hat{p}_z) \right]
$$

(15)

where $\sigma$ is the initial spread of the wavefunction in position space. The even moments are then

$$
\langle p^{2n} \rangle = \frac{2\sigma^3}{\sqrt{\pi}} \exp(-\sigma^2 p_0^2) \int_0^\infty dp \: p^{2n+2} \exp(-\sigma^2 p^2) \int_{-1}^1 d(cos \theta) \exp(-2\sigma^2 p p_0 \cos \theta)
$$

$$
= -\frac{2\sigma^3}{\sqrt{\pi} p_0} \exp(-\sigma^2 p_0^2) \int_0^\infty dp \: p^{2n+1} \exp(-\sigma^2 p^2) \sinh(2\sigma^2 p_0).
$$

(16)

To evaluate this last integral we make frequent use of identities and formulas from Buchholtz’ compendium on confluent hypergeometric functions [9] and begin with a change of variables $u = (\sigma p)^2$. Then we write the hyperbolic sine as a $0 F_1$ generalized hypergeometric function so that

$$
\int_0^\infty dp \: p^{2n+1} \exp(-\sigma^2 p^2) \sinh(2\sigma^2 p_0)
$$

$$
= \frac{\sqrt{\pi} p_0}{2\Gamma(3/2)\sigma^{2n+1}} \int_0^\infty du \: e^{-u} u^{n+1/2} {}_0F_1( , 3/2, \sigma^2 p_0^2 u)
$$

(17)

where $\Gamma(z)$ is the gamma function. This integral is the special case of the integral representation

$$
\Gamma(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty dt \: e^{-t} t^{a-1} {}_0F_1( , b, zt)
$$

which is valid so long as the real part of $a$ is positive, with $a = n + 3/2$, $b = 3/2$ and $z = \sigma^2 p_0^2$. In this way the integral in $\langle p^{2n} \rangle$ can be expressed as a $1 F_1$ hypergeometric function which, after an application of Kummer’s transformation, reduces to a generalized Laguerre polynomial. Thus, the even moments of a free Gaussian wavefunction with average momentum $-p_0 \hat{p}_z$ are

$$
\langle p^{2n} \rangle = -\frac{1}{2\sqrt{\pi}} \left( \frac{-1}{2\sigma} \right)^{2n} \Gamma(-n - 1/2) \Gamma(2n + 2) L_n^{1/2}(-\sigma^2 p_0^2)
$$

(19)

where $L_n^{\alpha}(z)$ is a generalized Laguerre polynomial. To evaluate we expand the radical in a power series, plug in the moments and simplify the resulting combination of gamma functions. Since the expectation values are in this case time-independent the time integration in is trivial and we have

$$
\langle s \rangle = -\frac{t}{2\sqrt{\pi}} \sum_{n=0}^\infty \Gamma(n - 1/2) x^n L_n^{1/2}(-\beta^2/x)
$$

(20)

where we have introduced the parameter $x \equiv 1/(m\sigma)^2$ for convenience (note that it is proportional to $\hbar^2$) and $\beta$ is the usual relative velocity of the particle, $\beta = v_0/c$. The expectation value in is clearly dependent on the mass of the particle whose motion it describes, implying that WEP does not hold on the quantum scale. To validate these
results we take the classical $\hbar \to 0$ limit of Eq. 20. This is achieved by replacing the Laguerre polynomials with the first term in their asymptotic expansions,

$$L_\alpha^\gamma(z) \sim \frac{(-z)^n}{\Gamma(n+1)}, \quad (21)$$

and summing the resulting series:

$$\lim_{\hbar \to 0} \langle s \rangle = -\frac{t}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n-1/2)}{\Gamma(n+1)} \beta^{2n} \sim t \sqrt{1 - \beta^2}. \quad (22)$$

This is the classical result for the 4-interval traversed by a free particle with mass $m$ and constant momentum $p_0$. We consider equations 20 and 22 to be proof that the distance operator introduced in this paper is interesting and worthy of study. In particular, we see that the interval’s expectation value depends on the particle’s mass on the scale set by $\hbar$ but in the classical limit the particle moves independent of its own properties. We take this to indicate that WEP is only an approximate feature of the quantum world.

While the result of taking the classical limit of the expectation value of the distance operator $s$ for a localized wavepacket, Eq. 22, is compelling there is a complication. Having a gamma function in the numerator of the terms in Eq. 20 without one or more such functions in the denominator does not bode well for the series’s convergence. Indeed, since

$$\lim_{n \to \infty} \frac{L_{n+1}}{L_n} = 1 \quad (23)$$

(see for example [10]), a quick check with the ratio test shows that the series diverges everywhere except in the classical limit. What are we to make of this? Retracing our steps we find that this divergence is the result of combining the relativistic interval 2 with the non-relativistic Heisenberg equation. Undoing the time integral in Eq. 10 and expressing $\langle s \rangle$ in momentum space gives

$$\frac{d}{dt} \langle s \rangle = \int d^3p |\langle \psi |p \rangle|^2 \sqrt{1 - (p/m)^2}. \quad (24)$$

When the integration variable exceeds $m$ the integrand becomes imaginary. Essentially, this is a consequence of integrating a non-relativistic integrand over relativistic momenta. We can manually enforce special relativity and make numerical evaluation of the integral 24 easier in one stroke. Recall that $\langle p|\psi \rangle$ is a Gaussian centered on $p_0$. If we assume that $p_0 \ll m$ and that $\sigma$ is not too large then the integrand is negligibly small in the relativistic $p \gtrsim m$ regime. Thus, it is reasonable to approximate the infinite integral in 24 by a truncated finite integral. Carrying out the integral over solid angle in momentum space exactly, separating the resulting hyperbolic sine function into exponentials, changing variables to $\rho = \sigma p$ and applying the truncation gives

$$\frac{d}{dt} \langle s \rangle \approx \frac{1}{\beta} \sqrt{\frac{x}{\pi}} \left\{ \int_{\xi_1}^{\xi_2} d\rho \exp[-(\rho - \rho_0)^2] \sqrt{1 - x\rho^2} \rho \right. \nonumber$$

$$\left. - \int_0^{\xi_1} d\rho \exp[-(\rho + \rho_0)^2] \sqrt{1 - x\rho^2} \rho \right\}, \quad (25)$$
A New Quantum Operator for Distance

\[ \xi_1 = \min(\rho_0/\beta, \max(0, \rho_0 - 3/\sqrt{2})) , \]

\[ \xi_2 = \min(\rho_0/\beta, \rho_0 + 3/\sqrt{2}) , \]

\[ \xi_3 = \min(\rho_0/\beta, \max(0, -\rho_0 + 3/\sqrt{2})) \]

in which \( \rho_0 = m\sigma v_0 = \beta/\sqrt{x} . \) The finite limits of integration are chosen so as to keep the integration variable within three standard deviations of each Gaussian as well as respecting relativity. Figure 1 shows the result of numerical integration of 25 for a range of \( x \)-values and initial particle mean velocities. The graph is set up so that the

**Figure 1.** Numerical integration of the truncated integral 25. The trivial time integral has been carried out and, as it only serves to set the overall scale of the graph, \( t \) is set to unity. Each curve corresponds to a different value of \( \beta \), with the \( \beta = 0.01 \) and \( \beta = 0.1 \) curves nearly coinciding. For reference the classical interval values, \( \sqrt{1 - \beta^2} \), are displayed as horizontal asymptotes of the curves.

parameter \( x \) decreases to the right which means that side of the chart shows the classical limit. We see that the curves, each representing the expected interval of a particle with relative velocity \( \beta \), are indeed approaching their classical values when \( x \) gets small, as 22 insists upon. On the large-\( x \) side of the graph we see that all of the intervals become light-like. This makes sense when one recalls that \( x \) goes like \( 1/m^2 \): massless particles in relativity must traverse light-like intervals. While most of the curves climb from nearly zero to their asymptotic values monotonically, the curves corresponding to the fastest particles considered here (\( \beta = 0.990 \) and \( \beta = 0.999 \)) achieve maxima which are actually larger than the classical values. Since the operator 9 is at least partly non-relativistic and we have had to enforce relativity in the integral 25 manually, the validity of the high-\( \beta \) curves is questionable. As such, we decline to attempt physically interpreting the separation of supremum and asymptote for these curves.
The decision to truncate the infinite integrals in (25) at three standard deviations is arbitrary, but numerical investigation shows that it makes little difference if one extends the region of integration further, provided that \( p \) gets no greater than \( m \). Figure 2 shows an example for \( \beta = 0.1 \).

![Figure 2. Expectation values of \( s \) approximated by (25) modified to have a variable truncation range. The curves each represent integrals whose limits of integration are \( \pm n \) standard deviations away from the center of the Gaussian in the integrand. The curves with \( n = 3 \) and \( n = 4 \) coincide, indicating that we achieve sufficient numerical accuracy with \( n = 3 \).](image)

### 4. Conclusions & Future Work

We have presented here a new quantum mechanical operator based on the 4-interval element in Minkowski space. Using Heisenberg’s equation for the time evolution of operators we deduced its expectation value and found that it depends in a complicated way on the potential the particle is exposed to. The free particle \((V = 0)\) case was then analyzed for two types of states: completely delocalized planewaves and maximally localized Gaussian wavefunctions. The classical limit of \( \langle s \rangle \) for the latter states was then shown to agree with the standard classical result for a particle moving at constant speed \( v_0 \). Since it seems likely that \( \langle s \rangle \) is, in some sense, an average over the possible paths the particle could take, it provides us with information on whether and how a quantum particle’s mass influences its mean trajectory. The truncated integral (25) and the classical limit (22) then imply that WEP fails to hold for quantum particles but is restored for classical masses and momenta. It has thus been demonstrated that the \( s \) operator is a useful tool for studying WEP in quantum mechanics.
A New Quantum Operator for Distance

There are a number of possible extensions to this work which may prove insightful. Some of them are, in no particular order:

- Effects of confinement. We have already addressed this for particle-in-a-box eigenstates, but what about a localized particle-in-a-box? While the $s$ operator itself is the same in this case as it is for the free particle in infinite space, the computation of its expectation values is complicated by the requirement that the wavefunction vanish at the box walls.

- Since the WEP is intimately connected to gravity, we wonder aloud about the expectation value of the $s$ operator for a localized particle in either linear or Newton/Coulomb inverse potentials.

- What effect does bestowing a particle with orbital and/or intrinsic angular momentum have on the 4-interval it traverses in a given time?

- The $s$ operator as defined in this work is an amalgamation of relativistic and non-relativistic parts. Can this be remedied by replacing the use of Heisenberg’s equation with an analogous one based on the Dirac or Klein-Gordon equations?

- How do the interactions of several particles effect the distances they travel? What sort of interval is covered by an entangled pair?

The investigation of these and other topics shall be the subject of future work by the author.

References

[1] Todd A Wagner, S Schlamminger, JH Gundlach, and EG Adelberger. Torsion-balance tests of the weak equivalence principle. Classical and Quantum Gravity, 29(18):184002, 2012.

[2] Stefan Baeßler, BR Heckel, EG Adelberger, JH Gundlach, U Schmidt, and HE Swanson. Improved test of the equivalence principle for gravitational self-energy. Physical Review Letters, 83(18):3585, 1999.

[3] James G Williams, Slava G Turyshev, and Dale H Boggs. Lunar laser ranging tests of the equivalence principle. Classical and Quantum Gravity, 29(18):184004, 2012.

[4] Lin Zhou, Shitong Long, Biao Tang, Xi Chen, Fen Gao, Wencui Peng, Weitao Duan, Jiaqi Zhong, Zongyuan Xiong, Jin Wang, et al. Test of equivalence principle at 10-8 level by a dual-species double-diffraction raman atom interferometer. Physical review letters, 115(1):013004, 2015.

[5] Pierre Touboul, Gilles Métris, Manuel Rodrigues, Yves André, Quentin Baghi, Joël Bergé, Damien Boulanger, Stefanie Bremer, Patrice Carle, Ratana Chhun, et al. Microscope mission: First results of a space test of the equivalence principle. Physical review letters, 119(23):231101, 2017.

[6] Robert D Reasenberg, Biju R Patla, James D Phillips, and Rajesh Thapa. Design and characteristics of a wep test in a sounding-rocket payload. Classical and Quantum Gravity, 29(18):184013, 2012.

[7] Brett Altschul, Quentin G Bailey, Luc Blanchet, Kai Bongs, Philippe Bouyer, Luigi Cacciapuoti, Salvatore Capozziello, Naceur Gaaloul, Domenico Giulini, Jonas Hartwig, et al. Quantum tests of the einstein equivalence principle with the ste–quest space mission. Advances in Space Research, 55(1):501–524, 2015.

[8] Daniel M Greenberger. The neutron interferometer as a device for illustrating the strange behavior of quantum systems. Reviews of Modern Physics, 55(4):875, 1983.
[9] H. Buchholtz. *The Confluent Hypergeometric Function*. Springer-Verlag, 1969.

[10] A. Deaño, E.J. Huertas, and F. Marcellán. Strong and ratio asymptotics for laguerre polynomials revisited. *Journal of Mathematical Analysis and Applications*, 403:477, 2013.