ON THE EXISTENCE OF RATIO LIMITS OF WEIGHTED N-GENERALIZED FIBONACCI SEQUENCES WITH ARBITRARY INITIAL CONDITIONS

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Abstract. We study ratio limits of the consecutive terms of weighted $n$-generalized Fibonacci sequences generated from arbitrary complex initial conditions by linear recurrences with arbitrary complex weights. We prove that if the characteristic polynomial of such a linear recurrence is asymptotically simple, then the ratio limit exists for any sequence generated from arbitrary nontrivial initial conditions and it is equal to the unique zero of the characteristic polynomial.

1. Introduction

Sequences generated by linear recurrences of an arbitrary order $n \geq 2$ and the ratio limits of their consecutive terms have been studied for over a half a century. The unweighted $n$-generalized Fibonacci sequences* with weights $(1, \ldots, 1)$ and initial conditions $(0, \ldots, 0, 1)$ were introduced and investigated in 1960 by Miles [7]. In 1967, Fielder introduced and studied the unweighted $n$-generalized Lucas sequences that are generated from the initial conditions $(-1, \ldots, -1, n)$ [3].

Also in 1967, Byrd [1] and, independently, Flores [4] showed that when $n$ goes to infinity, the ratio limits of the unweighted $n$-generalized Fibonacci sequences converge to 2. We recently extended this result [10] to the weighted $n$-generalized Fibonacci sequences with weights $(p, \ldots, p)$, $p > 0$, and initial conditions $(0, \ldots, 0, 1)$ by proving that, when $n$ goes to infinity, the ratio limits of these sequences converge to $p + 1$. In a subsequent paper [11], we further showed that for any given $p$ and arbitrary $n$, these ratio limits can be represented geometrically using dilations of a collection of convex compact sets with rising dimensions $n$, such as $n$-balls, $n$-pyramids, $n$-cones, or $n$-simplexes.

An extensive study of ratio limits of weighted $n$-generalized Fibonacci sequences with complex weights and arbitrary complex initial conditions was conducted in 1997 by Dubéau et al. [2]. They proved among the other things that if the characteristic polynomial of a given linear recurrence is asymptotically simple, i.e., among the polynomial’s zeros of maximal modulus there is a unique zero $\lambda_0$ of maximal multiplicity $\nu$, then the ratio limit of the sequence $(F_k^n)_{k=−n+1}^\infty$, generated by this linear recurrence from the initial conditions $(0, \ldots, 0, 1)$, exists and is equal $\lambda_0$. The authors also showed that if a sequence $(F_k^n)_{k=−n+1}^\infty$ generated from arbitrary complex initial conditions $a = (a_{−n+1}, \ldots, a_0)$ by a linear recurrence with an asymptotically simple characteristic polynomial satisfies

\[
\lim_{k \to \infty} \left( F_k^n / k^{\nu−1} \lambda_0^{\nu} \right) \neq 0,
\]

then the sequence’s ratio limit exists and equals $\lambda_0$ as well.

In this paper, we show that condition (1.1) is redundant, i.e., we prove that if the characteristic polynomial of a given linear recurrence with complex weights is asymptotically simple, then the ratio limit exist and is equal to $\lambda_0$ for any sequence generated by this linear recurrence from arbitrary complex initial conditions.

2. Main result

Given a linear recurrence of an order $n \geq 2$ with the characteristic polynomial

\[
\lambda^n - b_1 \lambda^{n-1} - \cdots - b_n
\]

with complex weights $b_i$, $i = 1, \ldots, n$, such that $b_n \neq 0$. We study sequences $(F_k^n)_{k=−n+1}^\infty$ generated by this linear recurrence from arbitrary nontrivial complex initial conditions $a = (a_{−n+1}, \ldots, a_0)$, i.e.,

\[
F_k^n = b_1 F_{k−1}^n + \cdots + b_n F_{k−n}^n, \quad k > 0, \quad \text{and} \quad F_k^n = a_k, \quad k = −n + 1, \ldots, 0.
\]

*They are also referred to as $n$-step Fibonacci sequences [6].
If the ratio limit of the consecutive terms of a sequence \((F^n_k)_{k=-n+1}^\infty\) exists, we denote it as
\[
\Phi^a = \lim_{k_0 < k \to \infty} (F^n_{k+1}/F^n_k), \quad \text{where } F^n_k \neq 0 \text{ for } k > k_0.
\]

The assumptions that the initial conditions are nontrivial and that the weight \(b_n \neq 0\) imply that no more than \(n-1\) consecutive elements of the sequence \((F^n_k)_{k=-n+1}^\infty\) are equal to zero. Let \(F^n_k\) be a nonzero element. Obviously, finding the ratio limit \(\Phi^a\) of the sequence \((F^n_k)_{k=-n+1}^\infty\) is equivalent to finding the ratio limit \(\Phi^a'\) of the sequence \((F^n_k)_{k=-n+1}^\infty\) with initial conditions \(a' = (a'_{-n+1}, \ldots, a'_0) = (F^n_{-n+1}, \ldots, F^n_{0})\), i.e.,
\[
\text{where } a'_{-n+1} \neq 0.
\]

**Theorem 2.1.** Given a linear recurrence with complex weights \((b_1, \ldots, b_n)\), let \((F^n_k)_{k=-n+1}^\infty\) be a sequence generated by this linear recurrence from arbitrary complex initial conditions \(a = (a_{-n+1}, \ldots, a_{-n+1})\), \(a_{-n+1} \neq 0\), and let \((F^0_k)_{k=-n+1}^\infty\) be a sequence generated from the initial conditions \((0, \ldots, 0, 1)\).

(i) If there exists \(k_0\) such that \(F^n_k \neq 0\) for \(k > k_0\), then \(F^a_k \neq 0\) for \(k > k_0 + n - 1\).

(ii) If the characteristic polynomial of the linear recurrence is asymptotically simple, then the ratio limit \(\Phi^a\) exists for any nontrivial initial conditions \(a\) and it coincides with the characteristic polynomial’s unique zero \(\lambda_0\).

**Proof.** (i) Dubeau et al. [2] showed that elements of the sequence \((F^n_k)_{k=0}^\infty\) can be expressed by elements of the sequence \((F^0_k)_{k=-n+1}^\infty\) in the following way:
\[
F^a_k = a_0 F^0_k + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F^0_{k-j}.
\]

Let us assume that \(F^a_{k_1} = 0\) for some \(k_1 > k_0 + n - 1\). Due to our assumptions, the sum in the RHS of equation (2.4) with \(k = k_1\) includes the nonzero term \(a_{-n+1} b_0 F^0_{k_1-1}\). Thus, the sum can equal zero only if it contains at least one more nonzero term so that all the terms add up to zero.

Consequently, the following linear relation between the sequence elements \((F^0_l)\), \(l = k_1 - n + 1, \ldots, k_1\), is implied by our assumption that \(F^a_{k_1} = 0\) for \(k_1 > k_0 + n - 1\):
\[
F^0_{k_1-n+1} = c_1 F^0_{k_1} + \cdots + c_{n-2} F^0_{k_1-n+2},
\]
\text{where coefficients } c_i \text{ are determined by weights } (b_2, \ldots, b_n) \text{ and initial conditions } (a_{-n+1}, \ldots, a_0), \text{ and at least one of the } c_i \text{s is not equal to zero.}

Inserting the relation (2.5) in formula (2.2) with \(k = k_1 + 1\) and initial conditions \((0, \ldots, 0, 1)\) allows us to express the sequence element \(F^a_{k_1+1}\) by its predecessors without using the sequence element \(F^0_{k_1-n+1}\), i.e., relying on a new set of weights \((b'_1, \ldots, b'_{n-1}, 0)\). We conclude by induction that the latter is also true for all elements of the sequence \((F^n_k)_{k=k_1+1}^\infty\). But this contradicts our assumption that characteristic polynomial (2.1) includes \(b_n \neq 0\).

(ii) If the characteristic polynomial of the linear recurrence is asymptotically simple, the ratio limit \(\Phi^0 = \lim_{k_0 < k \to \infty} (F^n_{k+1}/F^n_k)\), where \(F^n_k \neq 0\) for \(k > k_0\), of the sequence \((F^n_k)_{k=k_0+1}^\infty\) exists and equals \(\lambda_0\). Moreover, part (i) of the theorem implies that for \(k > k_0 + n - 1\), we have \(F^a_k \neq 0\). Thus, it follows from formula (2.4) that
\[
\lim_{k_0 < k \to \infty} (F^n_{k+n}/F^n_{k+n-1}) = \lim_{k_0 < k \to \infty} \frac{a_0 F^n_{k+n} + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F^n_{k+n-j}}{a_0 F^n_{k+n-1} + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F^n_{k+n-1-j}} = \frac{a_0 F^0_{k+n}/F^n_{k-n+1} + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F^0_{k-n+1-j}/F^n_{k-n+1}}{a_0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F^0_{k-n+1-j}/F^n_{k-n+1}}
\]
\[
= \lim_{k_0 < k \to \infty} \frac{a_0 F^0_{k+n}/F^n_{k+n-1} + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F^0_{k+n-1-j}/F^n_{k+n-1}}{a_0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} (F^0)^{-i}} = \Phi^0,
\]
i.e., the limit \(\Phi^a\) exists and equals \(\lambda_0\).
3. Conclusions

A criterion proven in 1966 by Ostrowski [8, Theorem 12.2] states that a linear recurrence with weights 
\( b_i \geq 0, \ i = 1, \ldots, n \), has an asymptotically simple polynomial with unique dominant zero \( \lambda_0 \) of multiplicity \( \nu = 1 \) if the gcd of indices \( j \) corresponding to positive weights \( b_j \) equals 1. Thus, in the case of a linear recurrence with nonnegative weights, our theorem allows establishing whether all sequences generated by such a linear recurrence from arbitrary nontrivial initial conditions have the same ratio limit \( \lambda_0 \) by simply finding the gcd of the positive weights.

In particular, our theorem implies that for a given \( n \), all sequences with weights \( (1, \ldots, 1) \) and arbitrary initial conditions, e.g., the \( n \)-generalized Lucas sequence, have the same ratio limit as the \( n \)-generalized Fibonacci sequence. Also, it follows from our theorem that over 340 integer sequences with signatures \( (m, \ldots, m), \ m \in \mathbb{N} \), and a variety of initial conditions, cataloged in the Sloane’s Online Encyclopedia of Integer Sequences [9], have their ratio limits equal to the dominant zero of the corresponding characteristic polynomial. Moreover, according to our theorem, the results proven by us for the weighted \( n \)-generalized Fibonacci sequences with weights \( (p, \ldots, p), \ p > 0 \), and initial conditions \( (0,0,\ldots,1) \) [10, 11], are, in fact, applicable for all sequences with such weights and arbitrary nontrivial initial conditions.

The determination whether all sequences generated by a linear recurrences with complex weights have the same ratio limit equal to the unique zero \( \lambda_0 \) of its characteristic polynomial can be achieved by using a criterion introduced by Dubeau et al. [2] that, as the authors showed, is valid in numerous cases, cf. [2, Theorem 15]. According to this criterion, if for any characteristic polynomial’s zero \( \lambda \) it holds

\[
\sum_{j=1}^{n-1} \left| \sum_{i=j}^{n-1} \frac{b_{i+j}}{\lambda^{i+j}} \right| < 1,
\]

then the characteristic polynomial is asymptotically simple with the dominant zero \( \lambda = \lambda_0 \) of multiplicity \( \nu = 1 \).

Of course, our theorem implies that all integer sequences that are generated by a linear recurrence that satisfies either of the criteria described above have the same ratio limit equal to the dominant zero \( \lambda_0 \) of the linear recurrence’s characteristic polynomial. The OEIS [9] and Khovanova’s website Recursive Sequences [5] catalogs and describes applications of thousands of such integer sequences.

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