On the spectral properties of non-selfadjoint discrete Schrödinger operators

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Abstract

Let $H_0$ be a purely absolutely continuous selfadjoint operator acting on some separable infinite-dimensional Hilbert space and $V$ be a compact non-selfadjoint perturbation. We relate the regularity properties of $V$ to various spectral properties of the perturbed operator $H_0 + V$. The structure of the discrete spectrum and the embedded eigenvalues are analysed jointly with the existence of limiting absorption principles in a unified framework. Our results are based on a suitable combination of complex scaling techniques, resonance theory and positive commutators methods. Various results scattered throughout the literature are recovered and extended. For illustrative purposes, the case of the one-dimensional discrete Laplacian is emphasized.

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The spectral theory of non-selfadjoint perturbations of selfadjoint operators has made significant progress in the last decades, partly due to its impact and applications in the physical sciences. The existence of Limiting Absorption Principles (LAP) and the distributional properties of the discrete spectrum have been two of the main issues considered in this field, among the many results obtained so far.

LAP have been evidenced in various contexts, mostly based on the existence of some positive commutators. [30, 31] and [5] have developed respectively non-selfadjoint versions of the regular Mourre theory and the weak Mourre theory. The analysis of the discrete spectrum has been carried out from a qualitative point of view in [27, 28, 15, 16, 7], where the main focus is set on the existence and structure of the set of limit points. Quantitative approaches based on Lieb-Thirring inequalities, and the identification of the distribution law of the discrete spectrum around the limit points, have also been developed, see e.g. [8, 10, 18, 4, 12, 17, 19, 22, 26, 36] and references therein. These results have been established mostly for specific models like the Schrödinger operators and Jacobi matrices. Extensions to non-selfadjoint perturbations of magnetic Hamiltonians, Dirac and fractional Schrödinger operators have also been studied in [32, 33] and [9] respectively. In general, the techniques used depend strongly on the model considered.

The present paper is an attempt to analyze more systematically the issue of the spectral properties of non-selfadjoint compact perturbations of Hamiltonians exhibiting some absolutely continuous spectrum. In particular, we show various relationships between the regularity of the perturbation and, first the properties of the discrete spectrum, second the existence of some LAP. In order to relate our approach with the existing literature, we have focused our discussion on non-selfadjoint compact perturbations of the discrete Schrödinger operator in dimension one. However, most of the strategies developed in this case can be adapted to many standard models in mathematical physics. This is the purpose of a forthcoming work. For a study of non-selfadjoint discrete Schrödinger operator in random regimes, we refer the reader to [13, 14].

We have articulated this paper on the two following axes.

Firstly, for highly regular compact perturbations, we adapt various complex scaling arguments to our non-selfadjoint setting and show that the limit points of the discrete spectrum are necessarily contained in the set of thresholds of the unperturbed Hamiltonian (Theorem 2.2). We also exhibit some LAP on some suitable subintervals of the essential spectrum away from these thresholds (Theorem 2.3). Since for perturbations displaying some exponential decay, the method of characteristic values allows to establish the absence of resonance in some neighbourhood of the thresholds (Theorem 2.4), we conclude about the finiteness of the discrete spectrum in this case (Theorem 2.5). Let us mention that a variation of this result was previously obtained within the more restricted framework of Jacobi matrices with slightly weaker decay assumptions (see [15, 29]).
Theorem 1]). Our approach extends that result in two ways: it applies to perturbations exhibiting a full off-diagonal structure and also proves the existence of a LAP, which seems to be new.

Secondly, for mildly regular compact perturbations, we show that the set of embedded eigenvalues away from the thresholds is finite and exhibits more restricted versions of LAP (Theorems 2.6 and 2.7). It is actually an application of a non-selfadjoint version of the Mourre theory developed under optimal regularity condition (Theorem 5.1 and Corollary 5.2).

The paper is structured as follows. The main concepts and main results are introduced in Section 2 jointly with the model. The complex scaling arguments for non-selfadjoint operators are developed in Section 3 and culminate with the proof of Theorems 2.2 and 2.3. Section 4 is focused on the method of characteristic values and the proof of Theorem 2.4. The development of a Mourre theory for non-selfadjoint operators under optimal regularity conditions is exposed in Section 5, ending with the proofs of Theorem 5.1, Corollary 5.2, Theorems 2.6 and 2.7. It is self-contained and can be read independently. Finally, various results about the concepts of regularity used throughout the text are recalled and illustrated in Section 6.

Notations: Throughout this paper, \( \mathbb{Z}, \mathbb{Z}_+, \) and \( \mathbb{N} \) denote the sets of integral numbers, non-negative and positive integral numbers respectively. For \( \delta \geq 0 \), we define the weighted Hilbert spaces

\[
\ell^2_\delta(\mathbb{Z}) := \{ x \in \mathbb{C}^\mathbb{Z} : \sum_{n \in \mathbb{Z}} e^{\pm\delta|n|} |x(n)|^2 < \infty \}.
\]

In particular, \( \ell^2(\mathbb{Z}) = \ell^2_0(\mathbb{Z}) \). We have the inclusions \( \ell^2_1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^2_\delta(\mathbb{Z}) \). For \( \delta > 0 \), we define the multiplication operators \( W_\delta : \ell^2_\delta(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) by \( (W_\delta x)(n) := e^{(\delta/2)|n|} x(n) \), and \( W_{-\delta} : \ell^2(\mathbb{Z}) \to \ell^2_\delta(\mathbb{Z}) \) by \( (W_{-\delta} x)(n) := e^{-(\delta/2)|n|} x(n) \). We denote by \( (e_n)_{n \in \mathbb{Z}} \) the canonical orthonormal basis of \( \ell^2(\mathbb{Z}) \).

\( \mathcal{H} \) will denote a separable Hilbert space, \( \mathcal{B}(\mathcal{H}) \) and \( \text{GL}(\mathcal{H}) \) the algebras of bounded linear operators and boundedly invertible linear operators acting on \( \mathcal{H} \). \( \mathcal{S}_p(\mathcal{H}) \), \( p \geq 1 \), stand for the subalgebras of compact operators and the Schatten classes. In particular, \( \mathcal{S}_2(\mathcal{H}) \) is the ideal of Hilbert-Schmidt operators acting on \( \mathcal{H} \). For any operator \( H \in \mathcal{B}(\mathcal{H}) \), we denote its numerical range by \( \mathcal{N}(H) := \{ \langle H\psi, \psi \rangle : \psi \in \mathcal{H} : \| \psi \| = 1 \} \), its spectrum by \( \sigma(H) \), its resolvent set by \( \rho(H) \), the set of its eigenvalues by \( \mathcal{E}_p(H) \). We also write

\[
\text{Re}(H) = \frac{1}{2}(H + H^*), \quad \text{Im}(H) := \frac{1}{2i}(H - H^*).
\]

We define its point spectrum as the closure of the set of its eigenvalues and write it \( \sigma_{pp}(H) = \mathcal{E}_p(H) \). Finally, if \( A \) is a self-adjoint operator acting on \( \mathcal{H} \), we write \( \langle A \rangle := \sqrt{A^2 + 1} \).

For two subsets \( \Delta_1 \) and \( \Delta_2 \) of \( \mathbb{R} \), we denote by \( \Delta_1 + i \Delta_2 := \{ z \in \mathbb{C} : \text{Re}(z) \in \Delta_1, \text{Im}(z) \in \Delta_2 \} \). For \( R > 0 \) and \( z_0 \in \mathbb{C} \), we set \( D_R(z_0) := \{ z \in \mathbb{C} : |z - z_0| < R \} \) and \( D_R^+(z_0) := D_R(z_0) \setminus \{ z_0 \} \). In particular, \( \mathcal{D} := D_1(0) \) denotes the open unit disk of the complex plane. For \( \Omega \subseteq \mathbb{C} \) an open domain and \( \mathbb{B} \) a Banach space, \( \text{Hol}(\Omega, \mathbb{B}) \) denotes the set of holomorphic functions from \( \Omega \) with values in \( \mathbb{B} \). We will adopt the following principal determination of the complex square root: \( \sqrt{\cdot} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}^+ := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \). By \( 0 < |k| \ll 1 \), we mean that \( k \in \mathbb{C} \) is sufficiently close to 0.

The discrete Fourier transform \( \mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}) \), where \( \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \), is defined for any \( x \in \ell^2(\mathbb{Z}) \) and \( f \in L^2(\mathbb{T}) \) by

\[
(\mathcal{F}x)(\theta) := (2\pi)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} e^{-in\theta} x(n), \quad (\mathcal{F}^{-1}f)(n) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{T}} e^{in\theta} f(\theta) d\theta.
\]  

(1.1)

The operator \( \mathcal{F} \) is unitary. For any bounded (resp. selfadjoint) operator \( L \) acting on \( \ell^2(\mathbb{Z}) \), we define the bounded (resp. selfadjoint) operator \( \hat{L} \) acting on \( L^2(\mathbb{T}) \) by

\[
\hat{L} := \mathcal{F}L\mathcal{F}^{-1}.
\]

(1.2)
2 Model and Main results

2.1 The model

The unperturbed Hamiltonian. We denote by $H_0$ the one-dimensional Schrödinger operator defined on $\ell^2(\mathbb{Z})$ by
\[
(H_0x)(n) := 2x(n) - x(n+1) - x(n-1).
\] (2.1)
$H_0$ is a bounded selfadjoint operator. We deduce that $\hat{H}_0 = \mathcal{F}H_0\mathcal{F}^{-1}$ is the multiplication operator on $L^2(\mathbb{T})$ by the function $f$ where
\[
f(\vartheta) := 2 - 2\cos \vartheta = \left(4\sin^2 \frac{\vartheta}{2}\right), \quad \vartheta \in [-\pi, \pi].
\] (2.2)
It follows that $\sigma(H_0) = \sigma_{\text{ess}}(H_0) = \sigma_{ac}(H_0) = [0, 4]$, where $\{0, 4\}$ are the thresholds. For $z \in \mathbb{C} \setminus [0, 4]$, we write $R_0(z) = (H_0 - z)^{-1}$. We deduce that for any $x \in \ell^2(\mathbb{Z})$,
\[
(\mathcal{F}R_0(z)x)(\vartheta) = \frac{(Fx)(\vartheta)}{f(\vartheta) - z}, \quad z \in \mathbb{C} \setminus [0, 4].
\]
For $z \in \mathbb{C} \setminus [0, 4]$ small enough, we can introduce the change of variables
\[
z = 4\sin^2 \frac{\phi}{2}, \quad \text{Im}(\phi) > 0.
\]
In this case, the resolvent $R_0(z)$ is represented by the convolution with the function
\[
R_0(z,n) = \frac{ie^{i|n|}}{2\sin \phi} = \frac{e^{i|n|/2}\arcsin \frac{z}{2}}{\sqrt{z}\sqrt{1-z}}, \quad \arcsin \frac{\sqrt{z}}{2} \sim \frac{\sqrt{z}}{2}. \tag{2.3}
\]
The perturbation. For any bounded operator $V$ acting on $\ell^2(\mathbb{Z})$, we define the perturbed operator
\[
H_V := H_0 + V. \tag{2.4}
\]
If $\{V(n,m)\}_{(n,m) \in \mathbb{Z}^2}$ denotes the matrix representation of the operator $V$ in the canonical orthonormal basis of $\ell^2(\mathbb{Z})$, for $x \in \ell^2(\mathbb{Z})$, the sequence $Vx \in \ell^2(\mathbb{Z})$ is given by
\[
(Vx)(n) = \sum_{m \in \mathbb{Z}} V(n,m)x(m) \quad \text{for any } n \in \mathbb{Z}. \tag{2.5}
\]
If $V$ is represented by a diagonal matrix (i.e. $V(n,m) = V(n,m)\delta_{nm}$), we write $V(n) := V(n,n)$ and $V$ is just the multiplicative operator defined by $(Vx)(n) = V(n)x$ for $x \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$.

For further use, let us conclude this paragraph with the following observation (see Subsection 2.3 for more details). Let $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the unitary operator defined by $(Jx)(n) := (-1)^nx(n)$, $x \in \ell^2(\mathbb{Z})$, and define
\[
V_J := JVJ^{-1}. \tag{2.6}
\]
Then, $J^2 = I$ and the matrix representation of the operator $V_J$ in the canonical orthonormal basis of $\ell^2(\mathbb{Z})$ satisfies
\[
V_J(n,m) = (-1)^{n+m}V(n,m), \quad (n,m) \in \mathbb{Z}^2.
\]
In particular, if $V$ is a diagonal matrix, then $V_J = V$. Calculations yield (see e.g. [24, Eq. (A.1)]):
\[
JH_VJ^{-1} = -H_{-V_J} + 4 \quad \text{so that}
\]
\[
J(H_V - z)^{-1}J^{-1} = -(H_{-V_J} - (4 - z))^{-1}. \tag{2.7}
\]
The conjugate operator. Finally, let us define the auxiliary operator $A_0$, conjugate to $H_0$ in the sense of the Mourre theory and acting on $l^2(\mathbb{Z})$ by

$$A_0 := \mathcal{F}^{-1}\hat{A}_0\mathcal{F},$$

where the operator $\hat{A}_0$ is (abusing notation) the unique selfadjoint extension of the symmetric operator

$$\sin(-i\partial_\theta) + (-i\partial_\theta)\sin\theta$$

defined on $C^\infty(\mathbb{T})$.

Remark 2.1 Define the position operator $X$ and shift operator $S$ on span $\{e_n : n \in \mathbb{Z}\}$ by $(Xe_n)(n) = nx(n)$ and $(Se_n)(n) = x(n+1)$. $A_0$ is also the unique selfadjoint extension of the symmetric operator $\text{Im}(S)X + X\text{Im}(S)$ defined on span $\{e_n : n \in \mathbb{Z}\}$.

2.2 Essential and discrete spectra

Let $L$ be a closed operator acting on a Hilbert space $\mathcal{H}$. If $z$ is an isolated point of $\sigma(L)$, let $\gamma$ be a small positively oriented circle centered at $z$ and separating $z$ from the other components of $\sigma(L)$. The point $z$ is said to be a discrete eigenvalue of $L$ if its algebraic multiplicity

$$m(z) := \text{rank}\left(\frac{1}{2i\pi} \int_\gamma (L - \zeta)^{-1}d\zeta\right)$$

is finite. Note that $m(z) \geq \dim(\text{Ker}(L - z))$, the geometric multiplicity of $z$. Equality holds if $L$ is normal (see e.g. [25]). We define the discrete spectrum of $L$ by

$$\sigma_{\text{disc}}(L) := \{z \in \sigma(L) : z \text{ is a discrete eigenvalue of } L\}.$$  

(2.10)

We recall that a closed linear operator is a Fredholm operator if its range is closed, and both its kernel and cokernel are finite-dimensional. We define the essential spectrum of $L$ by

$$\sigma_{\text{ess}}(L) := \{z \in \mathbb{C} : L - z \text{ is not a Fredholm operator}\}.  \quad (2.11)$$

It is a closed subset of $\sigma(L)$.

Remark 2.2 If $L$ is selfadjoint, $\sigma(L)$ can be decomposed always as a disjoint union: $\sigma(L) = \sigma_{\text{ess}}(L) \sqcup \sigma_{\text{disc}}(L)$. If $L$ is not selfadjoint, this property is not necessarily true. Indeed, consider for instance the shift operator $S : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$ defined by $(Sx)(n) := x(n+1)$. We have

$$\sigma(S) = \{z \in \mathbb{C} : |z| \leq 1\}, \quad \sigma_{\text{ess}}(S) = \{z \in \mathbb{C} : |z| = 1\}, \quad \sigma_{\text{disc}}(S) = \emptyset.$$  

However, in the case of the operator $H_V$, we have the following result:

Theorem 2.1 Let $V$ belongs to $S_{\mathcal{C}}(l^2(\mathbb{Z}))$. Then, $\sigma(H_V) = \sigma_{\text{ess}}(H_V) \sqcup \sigma_{\text{disc}}(H_V)$, where $\sigma_{\text{ess}}(H_V) = \sigma_{\text{ess}}(H_0) = [0, 4]$. The possible limit points of $\sigma_{\text{disc}}(H_V)$ are contained in $\sigma_{\text{ess}}(H_V)$.

Proof. It follows from Weyl’s criterion on the invariance of the essential spectrum under compact perturbations and from [21, Theorem 2.1, p. 373].

The reader will note that if $V$ is compact and selfadjoint, then $\sigma_{\text{disc}}(H_V) \subset (-\infty, 0) \cup (4, \infty)$ and the set of limit points of $\sigma_{\text{disc}}(H_V)$ is necessarily contained in $\{0, 4\}$. If $V$ is non-selfadjoint, then $\sigma_{\text{disc}}(H_V)$ may contain non-real numbers and the set of limit points may be considerably bigger, see e.g. [7] for the case of Laplace operators. However, we show in Theorem 2.2 that this cannot be the case if $V$ satisfies some additional regularity conditions.
Definition 2.1 Let $\mathcal{H}$ be a Hilbert space, $R > 0$ and $A$ be a selfadjoint operator defined on $\mathcal{H}$. An operator $B \in \mathcal{B}(\mathcal{H})$ belongs to the class $\mathcal{A}_R(A)$ if the map $\theta \mapsto e^{i\theta A}B e^{-i\theta A}$, defined for $\theta \in \mathbb{R}$, has an extension lying in $\text{Hol}(D_R(0), \mathcal{B}(\mathcal{H}))$. In this case, we write $B \in \mathcal{A}_R(A)$ and $\mathcal{A}(A) := \cup_{R > 0} \mathcal{A}_R(A)$ is the collection of bounded operators for which a complex scaling w.r.t. $A$ can be performed.

Remark 2.3 (a) If $B \in S_\infty(\mathcal{H}) \cap \mathcal{A}_R(A)$ for some selfadjoint operator $A$ and some $R > 0$, then the holomorphic extension of the map $\theta \mapsto e^{i\theta A}B e^{-i\theta A}$ lies actually in $\text{Hol}(D_R(0), S_\infty(\mathcal{H}))$, see e.g. [29, Lemma 5, Section XIII.5].

(b) The main properties of the classes $\mathcal{A}_R(A)$ are recalled in Section 6. In particular, in our case, we have $V \in \mathcal{A}_R(A_0)$ if and only if $\hat{V} \in \mathcal{A}_R(\hat{A}_0)$.

We have:

Theorem 2.2 If $V \in S_\infty(\ell^2(\mathbb{Z})) \cap \mathcal{A}(A_0)$, then the possible limit points of $\sigma_{\text{disc}}(H_V)$ belong to $\{0,4\}$.

The next result provides more details about the essential spectrum $\sigma_{\text{ess}}(H_V) = [0,4]$.

Theorem 2.3 Let $V \in S_\infty(\ell^2(\mathbb{Z})) \cap \mathcal{A}(A_0)$. Then, there exists a discrete subset $\mathcal{D} \subset (0,4)$ whose only possible limits points belong to $\{0,4\}$ and for which the following holds: given any relatively compact interval $\Delta_0$, $\Delta_0 \subset (0,4) \setminus \mathcal{D}$, there exist $\delta_0 > 0$ such that for any analytic vectors $\varphi$ and $\psi$ w.r.t. $A_0$,

$$\sup_{z \in \Delta_0 + i(-\delta_0,0)} |\langle \varphi, (z - H_V)^{-1}\psi \rangle| < \infty,$$

$$\sup_{z \in \Delta_0 + i(0,\delta_0)} |\langle \varphi, (z - H_V)^{-1}\psi \rangle| < \infty.$$

Remark 2.4 If $V$ is selfadjoint, then $H_V = H_V^*$ and $\mathcal{D}$ coincides with the set of eigenvalues of $H_V$ embedded in $(0,4)$ i.e. $\mathcal{D} = \mathcal{E}_p(H_V) \cap (0,4)$. In the non-selfadjoint case $H_V \neq H_V^*$, we expect that $\mathcal{E}_p(H_V) \cap (0,4) \subset \mathcal{D}$.

The proofs of Theorems 2.2 and 2.3 are postponed to Section 3. Among perturbations $V$ which satisfy the hypotheses of Theorem 2.2 and 2.3, we may consider:

- those which satisfy Assumption 2.1 below,
- $V = |\psi\rangle\langle\varphi|$, where $\varphi$ and $\psi$ are analytic vectors for $A_0$.

We refer to Subsection 6.1.2 for more details and further examples.

For $V \in S_\infty(\ell^2(\mathbb{Z})) \cap \mathcal{A}(A_0)$, Theorem 2.2 states that the points of $(0,4)$ cannot be the limit points of any sequence of discrete eigenvalues. In the next section, we consider more specifically the case of the thresholds $\{0,4\}$.

2.3 Resonances

In this section, we show how to control the distribution of resonances, hence the eigenvalues of $H_V$ around the thresholds $\{0,4\}$. We perform this analysis by means of the characteristic values method. First, we formulate a new assumption and recall some basic facts about resonances.

Assumption 2.1 There exist constants $C > 0$ and $\delta > 0$ such that

$$|V(n,m)| \leq C e^{-\delta(|n|+|m|)}, \quad (n,m) \in \mathbb{Z}^2,$$

where $(V(n,m))_{(n,m) \in \mathbb{Z}^2}$ is the matrix representation of the operator $V$ in the canonical orthonormal basis of $\ell^2(\mathbb{Z})$. 

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Remark 2.5 If $V$ satisfies Assumption 2.1, then

(a) $V \in S_1(\ell^2(\mathbb{Z})) \subset S_\infty(\ell^2(\mathbb{Z}))$,

(b) $V \in A(A_0)$ (see Proposition 6.5).

We also note that Assumption 2.1 holds for $V$ if and only if it holds for $V_J$, defined by (2.6). So, for a fixed operator $V$ satisfying Assumption 2.1 and when no confusion arises, we write

$$V = V \text{ or } -V_J. \quad (2.12)$$

As a preliminary, we have the following result, whose proof is contained in Section 4.

Proposition 2.1 Let Assumption 2.1 hold. Set $z(k) = k^2$. Then, there exists $0 < \varepsilon_0 \leq \delta/8$ small enough such that the resolvent, as an operator-valued function,

$$k \mapsto \left((H_V - z(k))^{-1} : \ell^2_\delta(\mathbb{Z}) \rightarrow \ell^2_\delta(\mathbb{Z})\right)$$

admits a meromorphic extension from $D_{\varepsilon_0}(0) \cap \mathbb{C}^+$ to $D_{\varepsilon_0}(0)$. This extension will be denoted again $R_V(z(k))$.

Now, we define the resonances of the operator $H_V$ near $z = 0$ and $z = 4$. We refer to Section 4 for the details about the next definitions. In particular, the quantity $Ind, \cdot$ denotes the index w.r.t. a contour $\gamma$ and is defined by (4.11).

Definition 2.2 The resonances of the operator $H_V$ near 0 are the poles of the meromorphic extension of the resolvent $R_V(z)$, as introduced in Proposition 2.1. The multiplicity of a resonance $z_0 := z_0(k) = k^2$ is defined by

$$\text{mult}(z_0) := Ind, \gamma \left( I + \mathcal{T}_V \left( z_0(\cdot) \right) \right), \quad (2.13)$$

where $\gamma$ is a circle positively oriented chosen sufficiently small such that $k$ is the only point satisfying $z_0(k)$ is a resonance of $H_V$.

There exists a simple way of defining the resonances of $H_V$ near the threshold $z = 4$. Indeed, relation (2.7) implies that

$$JW_{-\delta} (H_V - z)^{-1} W_{-\delta} J^{-1} = -W_{-\delta} (H_{-V_J} - (4 - z))^{-1} W_{-\delta}. \quad (2.14)$$

Using (2.14) as in Definition 2.2, we define the resonances of the operator $H_V$ near 4, as points $z = u - 4$ with $u$ poles of the meromorphic extension of the resolvent

$$(H_{-V_J} - u)^{-1} : \ell^2_\delta(\mathbb{Z}) \rightarrow \ell^2_\delta(\mathbb{Z}), \quad u := 4 - z, \quad (2.15)$$

near $u = 0$. Therefore, the analysis of the resonances of $H_V$ near the second threshold 4 is reduced to that of the first one 0 (up a sign and a change of variable). More precisely, we have:

Definition 2.3 The resonances of the operator $H_V$ near 4 are the points $z = 4 - u$ where $u$ are the poles of the meromorphic extension of the resolvent $R_{-V_J}(u)$, as introduced in Proposition 2.1. The multiplicity of a resonance $z_4 := z_4(k) = 4 - k^2$ is defined by

$$\text{mult}(z_4) := Ind, \gamma \left( I + \mathcal{T}_{-V_J} \left( 4 - z_4(\cdot) \right) \right), \quad (2.16)$$

where $\gamma$ is a circle positively oriented chosen sufficiently small such that $k$ is the only point satisfying $4 - z_4(k)$ is a pole of $R_{-V_J}(u)$.
We denote by $\text{Res}_\mu (H_V)$ the resonances set of $H_V$ near the threshold $\mu \in \{0, 4\}$. The discrete eigenvalues of the operator $H_V$ near 0 (resp. near 4) are resonances. Moreover, the algebraic multiplicity (2.9) of a discrete eigenvalue coincides with its multiplicity as a resonance near 0 (resp. near 4), defined by (2.13) (resp. (2.16)). Let us justify it briefly in the case of the discrete eigenvalues near 0 (the case concerning those near 4 can be treated similarly). Let $z_0 = z_0(k) \in \mathbb{C} \setminus [0, 4]$ be a discrete eigenvalue of $H_V$ near 0. According to [35, Chap. 9] and since $V = W_{-\delta} \gamma W_{-\delta}$ is of trace class, $\gamma$ being defined by (4.9), this is equivalent to the property $f(z_0) = 0$, where for $z \in \mathbb{C} \setminus [0, 4]$, $f$ is the holomorphic function defined by
\[
f(z) := \det (I + V(H_0 - z)^{-1}) = \det (I + \gamma W_{-\delta}(H_0 - z)^{-1}W_{-\delta}).
\]
Moreover, the algebraic multiplicity (2.9) of $z_0$ is equal to its order as zero of the function $f$. Residue Theorem yields:
\[
m(z_0) = \text{ind}_{\gamma'} f := \frac{1}{2\pi i} \int_{\gamma'} \frac{f'(z)}{f(z)} dz,
\]
$\gamma'$ being a small circle positively oriented containing $z_0$ is the only zero of $f$. The claim follows immediately from the equality
\[
\text{ind}_{\gamma'} f = \text{Ind}_{\gamma} \left( I + T_V \left( z_0(\cdot) \right) \right),
\]
see for instance [3, Eq. (6)] for more details.

**Remark 2.6** (a) The resonances $z_0(k) := k^2$ and $z_4(k) := 4 - k^2$ are defined in some two-sheets Riemann surfaces $\mathcal{M}_\mu$, $\mu \in \{0, 4\}$.

(b) The discrete spectrum and the embedded eigenvalues of $H_V$ near $\mu$ belong to the set of resonances
\[
z_{\mu}(k) \in \mathcal{M}_\mu, \quad \text{Im}(k) \geq 0.
\]

The above considerations lead us to the following result:

**Theorem 2.4** If Assumption 2.1 holds, then for any $0 < r \ll 1$, we have that for $\mu \in \{0, 4\}$,
\[
\# \{ z_{\mu}(k) \in \text{Res}_\mu (H_V) : k \in D_r(0) \} = 0,
\]
the resonances being counted according to their multiplicity defined by (2.13) and (2.16).

According to Theorem 2.4, there are no resonances of $H_V$ in a punctured neighborhood of $\mu$, in the two-sheets Riemann surface $\mathcal{M}_\mu$ (where the resonances are defined). Figure 2.1 gives an illustration of this fact.

Since the discrete eigenvalues of $H_V$ near $\mu$ are part of the set of resonances $z_{\mu}(k) \in \mathcal{M}_\mu$, $\text{Im}(k) \geq 0$, we conclude from Theorems 2.2, 2.4 and Proposition 6.5 that:

**Theorem 2.5** Let Assumption 2.1 hold. Then, $\sigma_{\text{disc}}(H_V)$ has no limit points in $[0, 4]$, hence is finite. There exists also a finite subset $D \subset (0, 4)$ for which the following holds: given any relatively compact interval $\Delta_0$, $\Delta_0 \subset (0, 4) \setminus D$, there exist $\delta_0 > 0$ such that for any vectors $\varphi$ and $\psi$ analytic w.r.t. $A_0$,
\[
\sup_{z \in \Delta_0 + i(0, \delta_0)} |\langle \varphi, (z-H_V)^{-1}\psi \rangle| < \infty,
\]
\[
\sup_{z \in \Delta_0 + i(0, \delta_0)} |\langle \varphi, (z-H_V)^{-1}\psi \rangle| < \infty.
\]

**Remark 2.7** As shown in Lemma 6.2, the linear subspace span $\{e_n ; n \in \mathbb{Z} \}$ is contained in the family of vectors analytic w.r.t. $A_0$. 

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As mentioned in the introduction, the first conclusion of Theorem 2.5 can be produced under weaker decay conditions along the diagonal under the additional assumption that the perturbed operator $H_V$ is still a Jacobi operator (see e.g. [15, Theorem 1]). But, Theorem 2.5 allows to handle perturbations displaying a full off-diagonal structure and proves the existence of some LAP.

If $V$ is selfadjoint and satisfies Assumption 2.1, the sets of eigenvalues and resonances in some neighbourhood of the thresholds $\{0, 4\}$ coincide. It follows from Theorem 2.4, Proposition 2.1 and the usual complex scaling arguments (see e.g. [34]) that:

**Corollary 2.1** Let Assumption 2.1 hold. If the perturbation $V$ is selfadjoint, then:

- $\sigma_{ess}(H_V) = [0, 4]$ and $\sigma_{disc}(H_V)$ is finite.
- There is at most a finite numbers of eigenvalues embedded in $[0, 4]$. Each eigenvalues embedded in $(0, 4)$ has finite multiplicity.
- The singular continuous spectrum $\sigma_{sc}(H_V) = \emptyset$ and the following LAP holds: given any relatively compact interval $\Delta_0 \subset (0, 4) \setminus \mathcal{E}_p(H_V)$, there exist $\delta_0 > 0$ such that for any analytic vectors $\varphi$ and $\psi$ w.r.t. $A_0$,

$$\sup_{z \in \Delta_0 + i(0, \delta_0)} |\langle \varphi, (z - H_V)^{-1}\psi \rangle| < \infty,$$

$$\sup_{z \in \Delta_0 + i(-\delta_0, 0)} |\langle \varphi, (z - H_V)^{-1}\psi \rangle| < \infty.$$

**2.4 Embedded eigenvalues and Limiting Absorption Principles**

For less regular perturbation $V$, we can still take advantage of the existence of some positive commutation relations to control some spectral properties of $H_V$. Let us define the regularity conditions involved in the statement of Theorem 2.6 below.

**Definition 2.4** Let $\mathcal{H}$ be a Hilbert space and $A$ be a selfadjoint operator defined on $\mathcal{H}$. Let $k \in \mathbb{N}$. An operator $B \in \mathcal{B}(\mathcal{H})$ belongs to the class $C^k(A)$, if the map $W_A : \theta \mapsto e^{i\theta A}B e^{-i\theta A}$ is $k$-times strongly continuously differentiable on $\mathbb{R}$. We also denote $C^\infty(A) = \cap_{k \in \mathbb{N}} C^k(A)$.

**Remark 2.8** A bounded operator $B$ belongs to $C^1(A)$ if and only if the sesquilinear form defined on $\mathcal{D}(A) \times \mathcal{D}(A)$ by $(\varphi, \psi) \mapsto \langle A\varphi, B\psi \rangle - \langle \varphi, B A \psi \rangle$, extends continuously to a bounded form on $\mathcal{H} \times \mathcal{H}$. The (unique) bounded linear operator associated to the extension is denoted by $ad_A(B) = [A, B]$ and we have $(\partial_\theta W_A)(0) = iad_A(B)$. 

**Figure 2.1** Resonances near the threshold $\mu \in \{0, 4\}$ in variable $k$. 

Absence of resonances

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**Figure 2.1** Resonances near the threshold $\mu \in \{0, 4\}$ in variable $k$. 

- The physical plane
- The non physical plane

The second sheet of the Riemann surface $\mathcal{M}_\mu$
Following [1], we also consider fractional order regularities:

**Definition 2.5** Let $\mathcal{H}$ be a Hilbert space and $A$ be a selfadjoint operator defined on $\mathcal{H}$. An operator $B \in \mathcal{B}(\mathcal{H})$ belongs to $C_{1,1}(A)$ if

$$
\int_0^1 \| e^{iA\theta} B e^{-iA\theta} + e^{-iA\theta} B e^{iA\theta} - 2B \| \frac{d\theta}{\mathcal{M}} < \infty.
$$

Actually, $C_{1,1}(A)$ is a linear subspace of $\mathcal{B}(\mathcal{H})$, stable under adjunction $\ast$. It is also known that $C^1(A) \subseteq C_{1,1}(A) \subseteq C^0(A)$, see e.g. inclusions (5.2.19) in [1]. Examples of these classes are given in Subsection 6.2 below.

We recall that the operator $A_0$ is defined by (2.8). If we assume that $\text{Im}(V)$ is signed, we obtain Theorem 2.8 below.

**Theorem 2.6** Let $V \in S_{\infty}(\ell^2(\mathbb{Z}))$ and assume $\pm \text{Im}(V) \geq 0$. Fix any open interval $\Delta$ such that $\overline{\Delta} \subset (0,4)$.

1. Assume $V \in C^1(A_0)$ and $\text{ad}_{A_0}(\text{Re}(V))$ also belongs to $S_{\infty}(\ell^2(\mathbb{Z}))$. Then,

$$
\mathcal{E}_p(H_V) \cap \overline{\Delta} \subset \sigma_{pp}(\text{Re}(H_V)) \cap \overline{\Delta}.
$$

The set $\mathcal{E}_p(H_V) \cap \overline{\Delta}$ is finite and its eigenvalues have finite geometric multiplicity.

2. Assume that $V \in C_{1,1}(A_0)$. Given any open interval $\Delta_0$ such that $\overline{\Delta_0} \subset \Delta \setminus \sigma_{pp}(\text{Re}(H))$ and any $s > 1/2$, the following LAP holds:

$$
\sup_{\pm \text{Im}(z) > 0, \text{Re}(z) \in \Delta_0} \| \langle A_0 \rangle^{-s}(z - H_V)^{-1} \langle A_0 \rangle^{-s} \| < \infty.
$$

**Remark 2.9**

(a) If $V \in S_{\infty}(\mathcal{H})$, then $V^\ast$, $\text{Re}(V)$ and $\text{Im}(V)$ also belong to $S_{\infty}(\mathcal{H})$.

(b) If $V$ belongs to $C^1(A)$ (resp. $C_{1,1}(A)$), then, $\text{Re}(V)$ and $\text{Im}(V)$ also belong to $C^1(A)$ (resp. $C_{1,1}(A)$). In particular, if $V$ belongs to $C_{1,1}(A)$ and $\text{Re}(V) \in S_{\infty}(\mathcal{H})$, then $\text{ad}_A(\text{Re}(V)) \in S_{\infty}(\mathcal{H})$, see Remark (ii) in the proof of Theorem 7.2.9 in [1].

Finally, we also have:

**Theorem 2.7** Let $V \in S_{\infty}(\ell^2(\mathbb{Z}))$ and assume that $V \in C_{1,1}(A_0)$.

1. If $\text{Im}(V) > 0$ and $\text{ad}_{A_0}(\text{Re}(V)) + \beta - \text{Im}(V) \geq 0$ for some $\beta \geq 0$, then for any open interval $\Delta$ such that $\overline{\Delta} \subset (0,4)$ and any $s > 1/2$, the following LAP holds:

$$
\sup_{-\text{Im}(z) > 0, \text{Re}(z) \in \Delta} \| \langle A_0 \rangle^{-s}(z - H_V)^{-1} \langle A_0 \rangle^{-s} \| < \infty.
$$

2. If $\text{Im}(V) < 0$ and $\text{ad}_{A_0}(\text{Re}(V)) - \beta + \text{Im}(V) \geq 0$ for some $\beta \geq 0$, then for any open interval $\overline{\Delta}$ such that $\overline{\Delta} \subset (0,4)$ and any $s > 1/2$, the following LAP holds:

$$
\sup_{+\text{Im}(z) > 0, \text{Re}(z) \in \Delta} \| \langle A_0 \rangle^{-s}(z - H_V)^{-1} \langle A_0 \rangle^{-s} \| < \infty.
$$

The proofs of Theorems 2.6 and 2.7 are direct applications of the abstract Mourre theory developed in Section 5. See Section 5.8 for the details.

### 3 Complex scaling

In this section, we use a complex scaling approach to study $\sigma(H_V)$ for compact perturbations $V \in \mathcal{A}(A_0)$. Since the spectral properties of $H_V$ and $\tilde{H}_V$ coincide, we reduce our analysis to those of $\tilde{H}_V$. 

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3.1 Before perturbation

By following the general principles exposed in [34], we first describe the scaling process for the unperturbed operator $\hat{H}_0$, which is the multiplication operator by the function $f$ (see (2.2)). In what follows, we have summarized the main results. When no confusion can arise, the operator $\hat{H}_0$ is identified with the function $f$.

We consider the unitary group $\left\langle e^{i\theta \hat{A}_0}\right\rangle_{\theta \in \mathbb{R}}$, so that for $\psi \in L^2(T)$, we have

$$
\left(e^{i\theta \hat{A}_0}\psi\right)(\vartheta) = \psi(\varphi_\theta(\vartheta)) \sqrt{J(\varphi_\theta)}(\vartheta),
$$

where

- $(\varphi_\theta)_{\theta \in \mathbb{R}}$ is the flow solution of the equation
  $$
  \begin{align*}
  \partial_\theta \varphi_\theta(\vartheta) &= 2 \sin(\varphi_\theta(\vartheta)), \\
  \varphi_0(\vartheta) &= \text{id}_T(\vartheta) = \vartheta \text{ for each } \vartheta \in T,
  \end{align*}
  $$

- $J(\varphi_\theta)(\vartheta)$ denotes the Jacobian of the transformation $\vartheta \mapsto \varphi_\theta(\vartheta)$.

Existence and uniqueness of the solution follow from standard ODE results. Explicitly, $\varphi_\theta(\vartheta) = \pm \arccos \left( \frac{-\text{th}(2\theta) + \cos \vartheta}{1 - \text{th}(2\theta) \cos \vartheta} \right)$ for $\pm \vartheta \in T$.

By using (3.1) and the fact that $\varphi_{\theta_1} \circ \varphi_{\theta_2} = \varphi_{\theta_1 + \theta_2}$ for all $(\theta_1, \theta_2) \in \mathbb{R}^2$, we have for all $\theta \in \mathbb{R}$,

$$
\left(e^{i\theta \hat{A}_0} \hat{H}_0 e^{-i\theta \hat{A}_0} \psi\right)(\vartheta) = f(\varphi_\theta(\vartheta)) \psi(\vartheta).
$$

Let $T : \mathbb{C} \to \mathbb{C}$, $T(z) := 2(1 - z)$. Note that the map $T$ is bijective with $T^{-1}(z) = 1 - \frac{z}{2}$, and maps $[-1, 1]$ onto $[0, 4]$. The points $T(-1) = 4$ and $T(1) = 0$ are the thresholds of $H_0$ and $\hat{H}_0$. Note also that $f = T \circ \cos$. Consider for $\theta \in \mathbb{R}$, the function $G_\theta$ defined on $[0, 4]$ by $G_\theta := T \circ F_\theta \circ T^{-1}$ with

$$
F_\theta(\lambda) := \frac{\lambda - \text{th}(2\theta)}{1 - \lambda \text{th}(2\theta)}, \quad \lambda \in [-1, 1].
$$

Then, for all $\theta \in \mathbb{R}$,

$$
\left(e^{i\theta \hat{A}_0} \hat{H}_0 e^{-i\theta \hat{A}_0} \psi\right)(\vartheta) = G_\theta(\hat{H}_0) \psi(\vartheta),
$$

In other words, for any $\theta \in \mathbb{R}$, $e^{i\theta \hat{A}_0} \hat{H}_0 e^{-i\theta \hat{A}_0}$ is the multiplication operator by the function $(G_\theta \circ f)$, where $G_\theta \circ f = (T \circ F_\theta \circ \cos) = (G_\theta \circ T \circ \cos)$.

**Remark 3.1** In (3.2), the denominator does not vanish since $|\text{th}(2\theta)\lambda| < 1$ for $\lambda \in [-1, 1]$.

We summarise the following properties, whose verification is left to the reader.

**Proposition 3.1** Let $\mathbb{D} := D_1(0)$ denote the open unit disk of the complex plane $\mathbb{C}$. Then,

(a) For any $\lambda \in [-1, 1]$, the map $\theta \mapsto F_\theta(\lambda)$ is holomorphic in $D_{\frac{\pi}{2}}(0)$.

(b) For $\theta \in \mathbb{C}$ such that $|\theta| < \frac{\pi}{2}$, the map $\lambda \mapsto F_\theta(\lambda)$ is a homographic transformation with $F_\theta^{-1} = F_{-\theta}$. In particular, for $\theta \in \mathbb{R}$, $F_\theta(\mathbb{D}) = \mathbb{D}$ and $F_\theta([-1, 1]) = [-1, 1]$.

(c) For $\theta \in \mathbb{C}$ such that $0 < |\theta| < \frac{\pi}{2}$, the unique fixed points of $F_\theta$ are $\pm 1$. 

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(d) For $\theta_1, \theta_2 \in \mathbb{C}$ with $|\theta_1|, |\theta_2| < \frac{\pi}{2}$, we have that: $F_{\theta_1} \circ F_{\theta_2} = F_{\theta_1 + \theta_2}$.

From Proposition 3.1, statements (a) and (b), we deduce:

**Proposition 3.2** The bounded operator valued-function

$$\theta \mapsto e^{i\theta \tilde{A}_0 \tilde{H}_0} e^{-i\theta \tilde{A}_0} \in B(L^2(\mathbb{T})),$$

admits a holomorphic extension from $(-\pi, \pi)$ to $D_{\mathbb{R}}(0)$, with extension operator given for $\theta \in D_{\mathbb{R}}(0)$ by $G_{\theta}(\tilde{H}_0)$, which is the multiplication operator by the function $G_{\theta} \circ f = (T \circ F_\theta \circ \cos) = (G_{\theta} \circ T \circ \cos)$. In the sequel, this extension will be denoted $\tilde{H}_0(\theta)$.

Combining the continuous functional calculus, Proposition 3.2 and unitary equivalence properties, we get for $\theta \in D_{\mathbb{R}}(0)$,

$$\sigma(\tilde{H}_0(\theta)) = \sigma(G_{\theta}(\tilde{H}_0)) = G_{\theta}(\sigma(\tilde{H}_0)) = G_{\theta}([0, 4]).$$

Thus, for $\theta \in D_{\mathbb{R}}(0)$, $\sigma(\tilde{H}_0(\theta))$ is a smooth parametrized curve given by

$$\sigma(\tilde{H}_0(\theta)) = \{T \circ F_\theta(\lambda) = G_{\theta} \circ T(\lambda) : \lambda \in [-1, 1]\}. \quad (3.4)$$

More precisely, we have:

**Proposition 3.3** Consider the family of bounded operators $(\tilde{H}_0(\theta))_{\theta \in D_{\mathbb{R}}(0)}$ defined in Proposition 3.2. Then,

(a) For $\theta_1, \theta_2 \in D_{\mathbb{R}}(0)$ such that $\text{Im}(\theta_1) = \text{Im}(\theta_2)$, we have

$$\sigma(\tilde{H}_0(\theta_1)) = \sigma(\tilde{H}_0(\theta_2)). \quad (3.5)$$

The curve $\sigma(\tilde{H}_0(\theta))$ does not depend on the choice of $\text{Re}(\theta)$.

(b) For $\pm \text{Im}(\theta) > 0$, $\theta \in D_{\mathbb{R}}(0)$, the curve $\sigma(\tilde{H}_0(\theta))$ lies in $\mathbb{C}_\pm$.

(c) Let $\theta \in D_{\mathbb{R}}(0)$. If $\text{Im}(\theta) \neq 0$, the curve $\sigma(\tilde{H}_0(\theta))$ is an arc of a circle containing the points 0 and 4. If $\text{Im}(\theta) = 0$, $\sigma(\tilde{H}_0(\theta)) = [0, 4]$.

We refer to Figure 3.1 below for a graphic illustration.

**Proof.** We start by justifying statement (a). Let $\theta_1, \theta_2 \in D_{\mathbb{R}}(0)$ with $\text{Im}(\theta_1) = \text{Im}(\theta_2)$. Thanks to (3.4), it suffices to show that

$$\left\{F_{\theta_1}(\lambda) : \lambda \in [-1, 1]\right\} = \left\{F_{\theta_2}(\lambda) : \lambda \in [-1, 1]\right\}, \quad (3.6)$$

to prove (3.5). So, let $z = F_{\theta_1}(\lambda)$, for some $\lambda \in [-1, 1]$. Let us show that there exists $\lambda' \in [-1, 1]$ such that $z = F_{\theta_2}(\lambda')$. By Proposition 3.1, statements (b) and (d), we can write

$$z = F_{\theta_2}\left(F_{\theta_2}^{-1}(F_{\theta_1}(\lambda))\right) = F_{\theta_2}\left(F_{\theta_2}(F_{\theta_1}(\lambda))\right) = F_{\theta_2}(F_{\theta_1 - \theta_2}(\lambda)) = F_{\theta_2}(F_{\text{Re}(\theta_1 - \theta_2)}(\lambda)).$$

Since $\text{th}(2\text{Re}(\theta_1 - \theta_2)) < 1$, then we have $|F_{\text{Re}(\theta_1 - \theta_2)}(\lambda)| \leq 1$ for $\lambda \in [-1, 1]$. Setting $\lambda' = F_{\text{Re}(\theta_1 - \theta_2)}(\lambda)$ yields $z = F_{\theta_2}(\lambda')$ with $\lambda' \in [-1, 1]$. This proves the inclusion

$$\left\{F_{\theta_1}(\lambda) : \lambda \in [-1, 1]\right\} \subset \left\{F_{\theta_2}(\lambda) : \lambda \in [-1, 1]\right\}.$$

The opposite inclusion can be shown similarly by permuting the roles of $\theta_1$ and $\theta_2$. This proves the first claim. Statement (b) follows by direct calculations. Let us prove the last statement.
According to statement (a), \( \theta \) can be chosen with \( \text{Re}(\theta) = 0 \), say \( \theta = iy \) with \( y \in \mathbb{R} \). The case \( \text{Im}(\theta) = 0 \) (i.e. \( y = 0 \)) is immediate since \( \sigma(\tilde{H}_0(\theta)) = \sigma(\tilde{H}_0(0)) = \sigma(\tilde{H}_0) = \sigma(H_0) = [0,4] \). Now, suppose that \( y \neq 0 \). Observe that the map \( T \) is a composition of a translation and a homothety. To prove that \( \sigma(\tilde{H}_0(\theta)) \) is an arc of a circle, it is enough to observe that \( \{ F_{-iy}(\lambda) : \lambda \in [-1,1] \} \) is a continuous parametrised curve and that for \( \lambda \in [-1,1] \), the real and imaginary parts of \( F_{-iy}(\lambda) \) satisfies a circle equation. Indeed, denoting

\[
\text{th}(-2\theta) = \frac{i \sin(-2y)}{\cos 2y} = i \tan(-2y) =: it,
\]

we have

\[
F_{-iy}(\lambda) = \frac{\lambda(1 + t^2)}{1 + \lambda^2 t^2} + i \frac{(1 - \lambda^2) t}{1 + \lambda^2 t^2} =: X + iY,
\]

and

\[
X^2 + \left( Y + \frac{1 - t^2}{2t} \right)^2 = \left( \frac{1 + t^2}{2t} \right)^2.
\]

(3.7)

\[\square\]

**Remark 3.2** For \( \theta \in D_{2R}(0), \text{Im}(\theta) \neq 0 \), Equation (3.7) provides the center \( c_{\theta} \in \mathbb{C} \) and the radius \( R_{\theta} > 0 \) of the circle supporting \( \sigma(\tilde{H}_0(\theta)) \).

### 3.2 After perturbation

Now, we focus on the complex scaling of the perturbation \( \tilde{V} \) together with the perturbed operator \( \tilde{H}_V = \tilde{H}_0 + \tilde{V} \). For \( \tilde{V} \in \mathcal{A}_R(\tilde{A}_0), R > 0 \), and for all \( \theta \in D_R(0) \), we set

\[
\tilde{V}(\theta) := e^{i \theta \tilde{A}_0} \tilde{V} e^{-i \theta \tilde{A}_0}.
\]

The following lemma holds:

**Lemma 3.1** Let \( \tilde{V} \in \mathcal{S}_\infty(L^2(\mathbb{T})) \cap \mathcal{A}_R(\tilde{A}_0), R > 0 \). Then, for all \( \theta \in D_R(0) \), \( \tilde{V}(\theta) \) is compact.

**Proof.** This follows from [29, Lemma 5, Section 5], since \( \tilde{V}(\cdot) \) is the analytic continuation of a bounded operator-valued function with compact values on the real axis. \[\square\]

Now, for all \( \theta \in D_{2R'}(0) \) with \( 2R' := \min \left( R, \frac{\pi}{8} \right) \), we consider

\[
\tilde{H}_V(\theta) := \tilde{H}_0(\theta) + \tilde{V}(\theta), \quad \tilde{V} \in \mathcal{A}_R(\tilde{A}_0).
\]

(3.8)

We obtain:

**Proposition 3.4** Let \( \tilde{V} \in \mathcal{A}_R(\tilde{A}_0), R > 0 \). Then,

(a) \( \tilde{H}_V(\theta) \) is a holomorphic family of bounded operators on \( D_{2R'}(0) \).

(b) For any \( \theta' \in \mathbb{R} \) such that \( |\theta'| < R' \), we have

\[
\tilde{H}_V(\theta + \theta') = e^{i \theta' \tilde{A}_0} \tilde{H}_V(\theta) e^{-i \theta' \tilde{A}_0},
\]

for all \( \theta \in D_{R'}(0) \).

**Proof.** Statement (a) follows from Proposition 3.2. Now, we prove (b). We fix \( \theta' \in \mathbb{R} \) with \( |\theta'| < R' \) and observe that the following maps

\[
\theta \mapsto e^{i \theta' \tilde{A}_0} \tilde{H}_V(\theta) e^{-i \theta' \tilde{A}_0} \quad \text{and} \quad \theta \mapsto \tilde{H}_V(\theta + \theta'),
\]

are bounded and holomorphic on \( D_{R'}(0) \). Moreover, they coincide on \( \mathbb{R} \cap D_{R'}(0) = (-R', R') \). Hence, they also coincide on \( D_{R'}(0) \). \[\square\]

The next proposition gives the key to the proofs of Theorems 2.2 and 2.3.
Proposition 3.5 Let $R > 0$ and $\hat{V} \in S_\infty(L^2(\mathbb{T})) \cap A_R(\hat{A}_0)$, and let $R' > 0$ such that $2R' = \min(R, \frac{\pi}{\theta})$. Then, for any $\theta \in D_{R'}(0)$, we have

(a) $\sigma(\hat{H}_V(\theta))$ depends only on $\text{Im}(\theta)$.

(b) It holds: $\sigma_{\text{ess}}(\hat{H}_V(\theta)) = \sigma_{\text{ess}}(\hat{H}_0(\theta)) = \sigma(\hat{H}_0(\theta))$ and

$$\sigma(\hat{H}_V(\theta)) = \sigma_{\text{disc}}(\hat{H}_V(\theta)) \bigcup \sigma_{\text{ess}}(\hat{H}_0(\theta)),$$

where the possible limit points of $\sigma_{\text{disc}}(\hat{H}_V(\theta))$ lie in $\sigma_{\text{ess}}(\hat{H}_0(\theta))$.

Proof. Statement (a) is a consequence of the unitary equivalence established in statement (b) of Proposition 3.4. Statement (b) follows from Lemma 3.1, the Weyl criterion on the invariance of the essential spectrum and [21, Theorem 2.1, p. 373].

3.3 Proof of Theorem 2.2

The proof of Theorem 2.2 follows from Proposition 3.6 below as an adaptation of the usual complex scaling arguments to our non-selfadjoint setting (see e.g., [29, Theorem XIII.36], [23]).

For any $\theta \in D_{\frac{\pi}{2}}(0)$, $c_0 \in \mathbb{C}$ and $R_0 > 0$ stand respectively for the center and the radius of the circle supporting $\sigma(\hat{H}_0(\theta))$. For $\pm \text{Im}(\theta) \geq 0$, we define the open domains $S^\pm_{\theta} := \mathbb{C} \setminus A^\pm_{\theta}$ where

$$A^\pm_{\theta} := \{z \in \mathbb{C} : \text{Re}(z) \in [0,4], \pm \text{Im}(z) \geq 0, |z - c_0| \geq R_0\}, \quad \pm \text{Im}(\theta) > 0,$$

and $A^0_{\theta} = \{z \in \mathbb{C} : \text{Re}(z) \in [0,4], \pm \text{Im}(z) \geq 0\}$. According to Proposition 3.3, the domains $S^\pm_{\theta}$ depend only on $\text{Im}(\theta)$. In addition, if $0 \leq \text{Im}(\theta') < \text{Im}(\theta)$, $S^-_{\theta'} \subset S^-_{\theta}$ and if $\text{Im}(\theta) < \text{Im}(\theta') \leq 0$, $S^+_{\theta'} \subset S^+_{\theta}$.

Proposition 3.6 Let $R > 0$, $\hat{V} \in A_R(\hat{A}_0)$ and $R' > 0$ such that $2R' = \min(R, \frac{\pi}{\theta})$. Let $(\theta, \theta') \in D_{R'}(0) \times D_{R'}(0)$. Then:

(a) If $0 \leq \text{Im}(\theta') < \text{Im}(\theta)$, we have $\sigma_{\text{disc}}(\hat{H}_V(\theta')) \cap S^-_{\theta'} = \sigma_{\text{disc}}(\hat{H}_V(\theta)) \cap S^-_{\theta'} \subset \sigma_{\text{disc}}(\hat{H}_V(\theta')) \cap S^-_{\theta'}$.

In particular,

$$\sigma_{\text{disc}}(\hat{H}_V(\theta)) \cap S^0_{\theta} = \sigma_{\text{pp}}(\hat{H}_V) \cap S^0_{\theta}.$$

(b) If $\text{Im}(\theta) < \text{Im}(\theta') \leq 0$, we have $\sigma_{\text{disc}}(\hat{H}_V(\theta')) \cap S^+_{\theta'} = \sigma_{\text{disc}}(\hat{H}_V(\theta)) \cap S^+_{\theta'} \subset \sigma_{\text{disc}}(\hat{H}_V(\theta')) \cap S^+_{\theta'}$.

In particular,

$$\sigma_{\text{disc}}(\hat{H}_V(\theta)) \cap S^0_{\theta} = \sigma_{\text{pp}}(\hat{H}_V) \cap S^0_{\theta}.$$

As a consequence, the discrete spectrum of $\hat{H}_V$ (and $H_V$) can only accumulate at 0 and 4.

Proof. We focus our attention on case (a). For a moment, fix $\theta_0 \in D_{R'}(0)$ such that $\text{Im}(\theta_0) > 0$ and suppose that $\lambda \in \sigma_{\text{disc}}(\hat{H}_V(\theta_0))$. Since the map $\theta \mapsto \hat{H}_V(\theta)$ is analytic, there exist open neighborhoods $\mathcal{V}_{\theta_0}$ and $\mathcal{W}_{\lambda}$ of $\theta_0$ and $\lambda$ respectively such that [25, 29]:

- For all $\theta \in \mathcal{V}_{\theta_0}$, the operator $\hat{H}_V(\theta)$ has a finite number of eigenvalues in $\mathcal{W}_{\lambda}$ denoted by $\lambda_j(\theta) \in \{1, \ldots, n\}$, all of finite multiplicity.

- These nearby eigenvalues are given by the branches of a finite number of holomorphic functions in $\mathcal{V}_{\theta_0}$ with at worst algebraic branch point near $\theta_0$.

If $\varphi = \theta - \theta_0 \in \mathbb{R}$ for $\theta \in \mathcal{V}_{\theta_0}$, then $\hat{H}_V(\theta_0 + \varphi)$ and $\hat{H}_V(\theta_0)$ are unitarily equivalent according to Proposition 3.4. So, the only eigenvalue of $\hat{H}_V(\theta_0 + \varphi)$ near $\lambda$ in $\mathcal{W}_{\lambda}$ is $\lambda$. Therefore, $\lambda_j(\theta) = \lambda$ for any $j \in \{1, \ldots, n\}$ and for $\theta \in \mathcal{V}_{\theta_0}$ with $\theta - \theta_0 \in \mathbb{R}$. By analyticity, we deduce that for all $\theta \in \mathcal{V}_{\theta_0}$ and all $j \in \{1, \ldots, n\}$ one has $\lambda_j(\theta) = \lambda$. Finally, we have proved that given $\theta_0 \in D_{R'}(0)$ and $\lambda \in \mathcal{W}_{\lambda}$, the number of nearby eigenvalues is finite and does not depend on $\lambda$. Consequently, the discrete spectrum of $\hat{H}_V(\theta)$ in $\mathcal{V}_{\theta_0}$ is finite and depends only on $\text{Im}(\theta)$. 

...
and applying the above observation yields: $\sigma_{\text{disc}}(\overline{H}_V(\theta_0))$, there exists a neighborhood $V_{\theta_0}$ of $\theta_0$, such that for all $\theta \in V_{\theta_0}$, $\lambda \in \sigma_{\text{disc}}(\overline{H}_V(\theta))$. Now, following [29, Problem 76, Section XIII], if $\gamma \in C^0([0, 1], D_R'(0))$ is a continuous curve and $\lambda \in \sigma_{\text{disc}}(\overline{H}_V(\gamma(0)))$, then either $\lambda \in \sigma_{\text{disc}}(\overline{H}_V(\gamma(1)))$ or $\lambda \in \sigma_{\text{ess}}(\overline{H}_V(\gamma(t)))$ for some $t \in (0, 1]$. We apply this observation twice keeping in mind Proposition 3.5.

Fix $\theta \in D_R'(0)$ with $\text{Im}(\theta) > 0$. Starting with $\lambda \in \sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap S^0_0 = \sigma_{\text{pp}}(\overline{H}_V) \cap S^0_0$, considering the continuous curve $\gamma^+: [0, 1] \to D_R'(0)$

$$\gamma^+(t) = t\theta,$$

and applying the above observation yields: $\sigma_{\text{disc}}(\overline{H}_V) \cap S^-_0 \subset \sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap S^0_0$. Now starting with $\lambda \in \sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap S^-_0$, considering the opposite continuous curve $\gamma^-: [0, 1] \to D_R'(0)$

$$\gamma^-(t) = (1 - t)\theta,$$

and applying the above observation yields the opposite inclusion. So, we have proven that given $\theta \in D_R'(0)$ with $\text{Im}(\theta) > 0$, $\lambda \in S^-_0$, then $\lambda \in \sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap S^-_0$ if and only if $\lambda \in \sigma_{\text{disc}}(\overline{H}_V) \cap S^0_0 = \sigma_{\text{pp}}(\overline{H}_V) \cap S^0_0$.

The proof of the identity $\sigma_{\text{disc}}(\overline{H}_V(\theta')) \cap S^-_0 = \sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap S^-_0$ for any $0 \leq \text{Im}(\theta') < \text{Im}(\theta)$ is similar. The inclusion $\sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap S^-_0 \subset \sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap S^-_0$ follows from the inclusion $S^-_0 \subset S^-_0$, which allows us to conclude on case (a).

The proof of case (b) is analogous. Once proven Statements (a) and (b), we conclude as follows. From Proposition 3.5, we know that the limit points of $\sigma_{\text{disc}}(\overline{H}_V)$ belong necessarily to $\sigma_{\text{ess}}(\overline{H}_V) = [0, 4]$. Pick one of these points, say $\lambda_0$. It is necessarily the limit point of a subsequence of either $\sigma_{\text{disc}}(\overline{H}_V) \cap S^-_0$ or $\sigma_{\text{disc}}(\overline{H}_V) \cap S^+_0$. Without any loss of generality, assume there exists a subsequence of $\sigma_{\text{disc}}(\overline{H}_V) \cap S^-_0$ which converges to $\lambda_0 \in [0, 4]$. Since $\sigma_{\text{disc}}(\overline{H}_V) \cap S^-_0 = \sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap S^-_0$ for any $\text{Im}(\theta) > 0$, $\theta \in D_R'(0)$, $\lambda_0$ also belongs to $\sigma_{\text{ess}}(\overline{H}_V)$. Since $\sigma_{\text{ess}}(\overline{H}_V(\theta)) \cap \sigma_{\text{ess}}(\overline{H}_V) = \{0, 4\}$ for any $\text{Im}(\theta) > 0$, $\lambda_0 \in \{0, 4\}$ and the last part follows.

### 3.4 Proof of Theorem 2.3

Let $R > 0$, $V \in A_0(A_0)$ and $R^* > 0$ such that $2R^* = \min(R, \frac{2}{3})$. Observe first that for any $\theta \in D_{R^*}(0)$ with $\text{Im}(\theta) > 0$, $D_+ := \sigma_{\text{disc}}(\overline{H}_V(\theta)) \cap (0, 4)$ is discrete and its possible accumulation
points belong to $\{0, 4\}$. Let $\varphi$ and $\psi$ be two analytic vectors for $\hat{A}_0$ with convergence radius $R_0 > 0$ and denote by $\varphi : \theta \mapsto \varphi(\theta)$ and $\psi : \theta \mapsto \psi(\theta)$ their analytic extension on $D_{R_0}(0)$. Consider the function $F(z, \theta) = \langle \psi(\theta), (H_V(\theta) - z)^{-1} \varphi(\theta) \rangle$ whenever it exists. For $\theta \in D_{\min(R_0, R')}(0)$, $F(\cdot, \theta)$ is analytic in $\mathbb{C} \setminus \sigma(H_V(\theta))$ and meromorphic in $\mathbb{C} \setminus \sigma_{\text{ess}}(H_V(\theta))$. Now, let us fix $z \in S_0^- \setminus \sigma_{\text{disc}}(H_V)$. Then $F(z, \cdot)$ is analytic on some region $D_{\min(R_0, R')}(0) \cap \{ \theta \in \mathbb{C} : -\epsilon_z < \text{Im}(\theta) \}$, for some $\epsilon_z > 0$. Since for any $\eta \in D_{\min(R_0, R')}(0) \cap \mathbb{R}$, direct calculations yield $F(z, \eta) = F(z, 0)$, we conclude by analyticity that $F(z, \cdot)$ is constant in $D_{\min(R_0, R')}(0) \cap \{ \theta \in \mathbb{C} : -\epsilon_z < \text{Im}(\theta) \}$. In particular, given $\theta_0 \in D_{\min(R_0, R')}(0)$ with $\text{Im}(\theta_0) > 0$, $F(\cdot, \theta_0)$ provides an analytic continuation to the function $F(\cdot, 0)$ from $S_0^- \setminus \sigma_{\text{disc}}(H_V)$ to $S_{\theta_0}^- \setminus \sigma_{\text{disc}}(H_V(\theta_0))$. In particular, for any relatively compact interval $\Delta_0$, $\overline{\Delta_0} \subset (0, 4) \setminus D_+$, the map $\langle \psi, (H_V - z)^{-1} \varphi \rangle$ extends continuously from some region $\Delta_0 - i(0, \delta_0^+)$, for some $\delta_0^+ > 0$, to $\Delta_0 - i[0, \delta_0^-]$. Note that $\delta_0^-$ can be chosen as:

$$
\delta_0^- = \text{dist}(\overline{\Delta_0}, \min\{\text{Im} z; z \in \sigma_{\text{disc}}(H_V) \cap S_0^- \}) > 0.
$$

This proves the first statement.

The proof of the other case is similar. Once proven both cases, we can set $D = D_+ \cup D_-$ and the proposition follows.

### 4 Resonances for exponentially decaying perturbations

#### 4.1 Preliminaries

Throughout this subsection, the perturbation $V$ is assumed to satisfy Assumption 2.1. We define and characterize the resonances of the operator $H_V$ near $z = 0$ and $z = 4$.

Let $R_V(z) := (H_V - z)^{-1}$ denote the resolvent of the operator $H_V$. We have the following lemma:

**Lemma 4.1** Set $z(k) := k^2$. Then, there exists $0 < \epsilon_0 \leq \frac{2}{\pi}$ small enough such that the operator-valued function

$$
k \mapsto W_{-\delta}(H_0 - z(k))^{-1}W_{-\delta},
$$

admits a holomorphic extension from $D_{\epsilon_0}^*(0) \cap \mathbb{C}^+$ to $D_\epsilon^*(0)$, with values in $S_\infty(\ell^2(\mathbb{Z}))$.

**Proof.** For $k \in D_{\epsilon_0}^*(0) \cap \mathbb{C}^+$, $0 < \epsilon \leq \frac{1}{2}$ small enough, and $x \in \ell^2(\mathbb{Z})$, the operator $W_{-\delta}(H_0 - z(k))^{-1}W_{-\delta}$ satisfies

$$
\left(W_{-\delta}(H_0 - z(k))^{-1}W_{-\delta}x\right)(n) = \sum_{m \in \mathbb{Z}} e^{-\frac{1}{2}|n|} R_0(z(k), n - m) e^{-\frac{1}{2}|m|} x(m),
$$

(4.1)

$R_0(z, \cdot)$ being the function defined by (2.3). Thus, for any $(n, m) \in \mathbb{Z}^2$, we have

$$
e^{-\frac{1}{2}|n|} R_0(z(k), n - m) e^{-\frac{1}{2}|m|} = e^{-\frac{1}{2}|n|} \frac{i e^{\left|\text{Im}(n-m)\right|}}{k^2} e^{-\frac{1}{2}|m|} =: f(k).
$$

(4.2)

Since for $0 < |k| \ll 1$ we have $2 \arcsin \frac{2}{k} = k + o(|k|)$, then, there exists $0 < \epsilon_0 \leq \frac{\pi}{8}$ small enough such that for any $0 < |k| \leq \epsilon_0$,

$$
e^{-\frac{1}{2}|n|} R_0(z(k), n - m) e^{-\frac{1}{2}|m|} \leq e^{-\frac{1}{2}|n|} e^{-\frac{\text{Im}(k)-\frac{1}{2}}{k^2} |n-m|} e^{-\frac{1}{2}|m|}.
$$

(4.3)

It can be checked that the r.h.s. of (4.3) lies in $\ell^2(\mathbb{Z}^2, \mathbb{C})$ whenever $0 < |\text{Im}(k)| < \frac{\pi}{8}$. In this case, the operator $W_{-\delta}(H_0 - z(k))^{-1}W_{-\delta}$ belongs to $S_2(\ell^2(\mathbb{Z}))$. Consequently, the operator-valued function

$$
k \mapsto W_{-\delta}(H_0 - z(k))^{-1}W_{-\delta}
$$

converges as $k \to 0$. 

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can be extended via the kernel (4.2) from $D^*_c(0) \cap \mathbb{C}^+$ to $D^*_c(0)$, with values in $S_\infty(\ell^2(\mathbb{Z}))$. We shall denote this extension $A(k)$, $k \in D^*_c(0)$. It remains to prove that $k \mapsto A(k)$ is holomorphic in $D^*_c(0)$.

For $k \in D^*_c(0)$, introduce $\mathcal{D}(k)$ the operator with kernel given by

$$
e^{-\frac{1}{2}|n|}\partial_k R_0(z(k), n-m) e^{-\frac{1}{2}|m|} = e^{-\frac{1}{2}|n|}e^{i|n-m|2\arcsin \frac{k}{2} k^2 (4 - k^2)} e^{-\frac{1}{2}|m|} \left(2ik|n-m| - \frac{4 - 2k^2}{\sqrt{4 - k^2}} \right).$$

As above, it can be shown that $\mathcal{D}(k) \in S_2(\ell^2(\mathbb{Z}))$. Then, for $k_0 \in D^*_c(0)$, the Hilbert-Schmidt operator $\frac{A(k) - A(k_0)}{k - k_0} - \mathcal{D}(k_0)$ admits the kernel

$$N(k, k_0, n, m) := e^{-\frac{1}{2}|n|} \left(R_0(z(k), n-m) - R_0(z(k_0), n-m) - \partial_k R_0(z(k_0), n-m) \right) e^{-\frac{1}{2}|m|}. \quad (4.4)$$

To conclude the proof of the lemma, it suffices to prove that $\left\| \frac{A(k) - A(k_0)}{k - k_0} - \mathcal{D}(k_0) \right\|_{S_2(\ell^2(\mathbb{Z}))} \to 0$ as $k \to k_0$. Since we have

$$\left\| \frac{A(k) - A(k_0)}{k - k_0} - \mathcal{D}(k_0) \right\|_{S_2(\ell^2(\mathbb{Z}))}^2 \leq \sum_{n,m} |N(k, k_0, n, m)|^2, \quad (4.5)$$

it is then sufficient to prove that the r.h.s. of (4.5) tends to zero as $k \to k_0$. By applying the Taylor-Lagrange formula to the function

$$[0,1] \ni t \mapsto q(t) := f(tk + (1-t)k_0) = R_0(z(tk + (1-t)k_0), n-m),$$

we get that there exists $\theta$ in $(0,1)$ such that

$$R_0(z(k), n-m) = R_0(z(k_0), n-m) + (k - k_0) \partial_k R_0(z(k_0), n-m) + \frac{(k - k_0)^2}{2} k^2 \partial^{(2)} k R_0(z(\theta k + (1-\theta)k_0), n-m). \quad (4.6)$$

Then, it follows from (4.4) and (4.6) that the kernel $N(k, k_0, n, m)$ has the representation

$$N(k, k_0, n, m) = \frac{k - k_0}{2} e^{-\frac{1}{2}|n|} \partial^{(2)} k R_0(z(\theta k + (1-\theta)k_0), n-m) e^{-\frac{1}{2}|m|}, \quad (4.7)$$

for some $\theta \in (0,1)$. Now, by easy but fastidious computations, it can be seen that for any $q \in \mathbb{N}$, there exists a family of functions $G_{j,q}$, $0 \leq j \leq q$, holomorphic on $D^*_c(0)$ such that

$$\partial^{(q)} k R_0(z(k), n-m) = i e^{i |n-m|2\arcsin \frac{k}{2}} \sum_{j=0}^{q} G_{j,q}(k)|n-m|^{q-j}. \quad (4.8)$$

In particular, for $q = 2$, we have

$$\partial^{(2)} k R_0(z(k), n-m) = i e^{i |n-m|2\arcsin \frac{k}{2}} \left(G_{0,2}(k)|n-m|^2 + G_{1,2}(k)|n-m| + G_{2,2}(k) \right).$$

This together with (4.7) imply that

$$\left| N(k, k_0, n, m) \right| \leq \left| \frac{k - k_0}{2} \right| \left| G_{0,2}(\theta k + (1-\theta)k_0) \right| e^{-\frac{1}{2}|n|} e^{i |n-m|2\arcsin \frac{k}{2}} \left| \frac{k - k_0}{2} \right| |n-m|^2$$

$$+ \left| \frac{k - k_0}{2} \right| \left| G_{1,2}(\theta k + (1-\theta)k_0) \right| e^{-\frac{1}{2}|n|} e^{i |n-m|2\arcsin \frac{k}{2}} \left| \frac{k - k_0}{2} \right| |n-m|$$

$$+ \left| \frac{k - k_0}{2} \right| \left| G_{2,2}(\theta k + (1-\theta)k_0) \right| e^{-\frac{1}{2}|n|} e^{i |n-m|2\arcsin \frac{k}{2}} \left| \frac{k - k_0}{2} \right|$$

$$= \sum_{j=0}^{2} |N_j(k, k_0, n, m)|.$$
Since the sequence
\[ \sum_{n,m} |N(k, k_0, n, m)|^2 \]
\[ \leq \text{Const} \left( \sum_{n,m} N_0(k, k_0, n, m)^2 + \sum_{n,m} N_1(k, k_0, n, m)^2 + \sum_{n,m} N_2(k, k_0, n, m)^2 \right), \quad (4.8) \]
In the sequel, we show that \[ \sum_{n,m} N_j(k, k_0, n, m)^2 \to 0 \] as \( k \to k_0 \), \( j \in \{0, 1, 2\} \). Similarly to (4.3), we have for \( k \in D^*_c(0) \)
\[ \left| e^{-\frac{4}{2}|n|} e^{\frac{4}{2}|m/i|} \frac{\pi}{2} e^{-\frac{4}{2}|m|} \right| \leq e^{-\frac{4}{2}|n|} e^{-\left( \text{Im}(k) - \frac{4}{2}\right)|n-m| e^{-\frac{4}{2}|m|} \leq e^{-\frac{4}{2}|n|} e^{\frac{4}{2}|n-m|} e^{-\frac{4}{2}|m|}. \]
Thus, it follows that for \( j \in \{0, 1, 2\} \), we have
\[ N_j(k, k_0, n, m)^2 \leq \frac{|k - k_0|^2}{4} \left| G_{j,2} (\theta k + (1 - \theta) k_0) \right|^2 e^{-\delta |n|} e^{\frac{4}{2}|n-m|} e^{-\delta |m|} |n-m|^{2(2-j)}. \]
Since the sequence \( \left( e^{-\delta |n|} e^{\frac{4}{2}|n-m|} e^{-\delta |m|} |n-m|^{2(2-j)} \right) \) belongs to \( \ell^2(\mathbb{Z}^2, C) \), then we have
\[ \sum_{n,m} N_j(k, k_0, n, m)^2 \leq \text{Const} \frac{|k - k_0|^2}{4} \left| G_{j,2} (\theta k + (1 - \theta) k_0) \right|^2 \to 0, \]
which implies by (4.5) and (4.8) that \( \left\| \frac{A(k) - A(k_0)}{k - k_0} - \mathcal{R}(k) \right\|_{\mathcal{S}_\infty(\ell^2(\mathbb{Z}))} \to 0 \) as \( k \to k_0 \). Consequently, the operator-valued extension \( D^*_c(0) \ni k \mapsto A(k) \) is holomorphic with derivative \( \partial_k A(k) = \mathcal{R}(k) \). This concludes the proof of the lemma. \( \square \)

### 4.2 Resonances as poles: proof of Proposition 2.1

With the aid of the identity
\[ (H_{V} - z)^{-1} (I + V(H_0 - z)^{-1}) = (H_0 - z)^{-1}, \]
we get
\[ W_{-\delta} (H_{V} - z)^{-1} W_{-\delta} = W_{-\delta} (H_0 - z)^{-1} W_{-\delta} (I + W_{-\delta} V(H_0 - z)^{-1} W_{-\delta})^{-1}. \]
Assumption 2.1 on \( V \) together with the Schur lemma imply that there exists a bounded operator \( \mathcal{V} \) on \( \ell^2(\mathbb{Z}) \) such that
\[ W_\delta W_{-\delta} = \mathcal{V}, \iff W_\delta V = \mathcal{V} W_{-\delta} \iff V = W_{-\delta} \mathcal{V} W_{-\delta}. \quad (4.9) \]
By putting this together with Lemma 4.1, it follows that the operator-valued function
\[ k \mapsto \mathcal{T}_V(z(k)) := W_\delta V \left( H_0 - z(k) \right)^{-1} W_{-\delta} = \mathcal{V} W_{-\delta} \left( H_0 - z(k) \right)^{-1} W_{-\delta} \quad (4.10) \]
is holomorphic in \( D^*_c(0) \), with values in \( \mathcal{S}_\infty(\ell^2(\mathbb{Z})) \). Then, by the analytic Fredholm extension theorem,
\[ k \mapsto \left( I + W_\delta V \left( H_0 - z(k) \right)^{-1} W_{-\delta} \right)^{-1} \]
adopts a meromorphic extension from \( D^*_c(0) \cap \mathbb{C}^+ \) to \( D^*_c(0) \). So, we have proved Proposition 2.1.

In the rest of our analysis, the compact operator \( \mathcal{T}_V(z(k)) \) defined by (4.10) will play a main role. For this reason, it is important to note the following:

**Remark 4.1**

(a) In the sequel, when \( V = -V_f \), we will take \( \mathcal{V} = -\mathcal{V}_f \) to avoid confusion.

(b) Of course, the factorization \( V = W_{-\delta} \mathcal{V} W_{-\delta} \) in (4.9) is note unique. For instance, with respect to the polar decomposition of an operator, we have \( V = J|V| = J|V|^{1/2}|V|^{1/2} \). However, sometimes another decompositions are more convenient as in (4.9).
4.3 Resonances as characteristic values

In what follows below, we give a simple and useful characterization of the resonances of $H_V$ near $z = 0$ and $z = 4$.

We recall some basic facts about the concept of characteristic values of a holomorphic operator-valued function. For more details on this subject, we refer for instance to [20] and the book [21, Section 4]. The content of the first part of this section follows [21, Section 4].

**Definition 4.1** Let $U$ be a neighborhood of a fixed point $w \in \mathbb{C}$, and $F : U \setminus \{w\} \rightarrow \mathcal{B}(\mathcal{H})$ be a holomorphic operator-valued function. The function $F$ is said to be finite meromorphic at $w$ if its Laurent expansion at $w$ has the form

$$F(z) = \sum_{n=m}^{+\infty} (z-w)^n F_n, \quad m > -\infty,$$

where (if $m < 0$) the operators $F_m, \ldots, F_{-1}$ are of finite rank. Moreover, if $F_0$ is a Fredholm operator, then, the function $F$ is said to be Fredholm at $w$. In this case, the Fredholm index of $F_0$ is called the Fredholm index of $F$ at $w$.

We have the following proposition:

**Proposition 4.1** [21, Proposition 4.1.4] Let $D \subseteq \mathbb{C}$ be a connected open set, $Z \subseteq D$ be a closed and discrete subset of $D$, and $F : D \rightarrow \mathcal{B}(\mathcal{H})$ be a holomorphic operator-valued function in $D \setminus Z$.

Assume that:

- $F$ is finite meromorphic on $D$ (i.e. it is finite meromorphic near each point of $Z$),
- $F$ is Fredholm at each point of $D$,
- there exists $w_0 \in D \setminus Z$ such that $F(w_0)$ is invertible.

Then, there exists a closed and discrete subset $Z'$ of $D$ such that:

1. $Z \subseteq Z'$,
2. $F(z)$ is invertible for each $z \in D \setminus Z'$,
3. $F^{-1} : D \setminus Z' \rightarrow \text{GL}(\mathcal{H})$ is finite meromorphic and Fredholm at each point of $D$.

In the setting of Proposition 4.1, we define the characteristic values of $F$ and their multiplicities as follows:

**Definition 4.2** The points of $Z'$ where the function $F$ or $F^{-1}$ is not holomorphic are called the characteristic values of $F$. The multiplicity of a characteristic value $w_0$ is defined by

$$\text{mult}(w_0) := \frac{1}{2i\pi} \text{Tr} \int_{|w-w_0|=\rho} F'(z) F(z)^{-1} dz,$$

where $\rho > 0$ is chosen small enough so that $\{w \in \mathbb{C} : |w - w_0| \leq \rho\} \cap Z' = \{w_0\}$.

According to Definition 4.2, if the function $F$ is holomorphic in $D$, then, the characteristic values of $F$ are just the complex numbers $w$ where the operator $F(w)$ is not invertible. Then, results of [20] and [21, Section 4] imply that mult$(w)$ is an integer.

Let $\Omega \subseteq D$ be a connected domain with boundary $\partial \Omega$ not intersecting $Z'$. The sum of the multiplicities of the characteristic values of the function $F$ lying in $\Omega$ is called the index of $F$ with respect to the contour $\partial \Omega$ and is defined by

$$\text{Ind}_{\partial \Omega} F := \frac{1}{2i\pi} \text{Tr} \int_{\partial \Omega} F'(z) F(z)^{-1} dz = \frac{1}{2i\pi} \text{Tr} \int_{\partial \Omega} F(z)^{-1} F'(z) dz.$$  \hspace{1cm} (4.11)

Let us contextualize the previous discussion for our model. We reformulate our characterization of the resonances near $z = 0$ as follows:
Proposition 4.2 For $k \in D_{20}^0(0)$, the following assertions are equivalent:

(a) $z_0(k_1) = k_1^2 \in M_0$ is a resonance of $H_V$,

(b) $z_{0,1} = z_0(k_1)$ is a pole of $R_V(z)$,

(c) $-1$ is an eigenvalue of $T_V(z_0(k_1)) := \mathcal{V}_{-\delta} R_0(z_0(k_1)) W_{-\delta}$,

(d) $k_1$ is a characteristic value of $I + T_V(z_0)$.

Moreover, thanks to (2.13), the multiplicity of the resonance $z_0(k_1)$ coincides with that of the characteristic value $k_1$.

Proof. The equivalence $(a) \iff (b)$ is just Definition 2.2.

The equivalence $(b) \iff (c)$ is an immediate consequence of the identity

$$
\left( I + \mathcal{V}_{-\delta} R_0(z) W_{-\delta} \right) \left( I - \mathcal{V}_{-\delta} R_V(z) W_{-\delta} \right) = I,
$$

which follows from the resolvent equation.

The equivalence $(c) \iff (d)$ follows from Definition 4.2.

In a similar way, the following proposition holds:

Proposition 4.3 For $k \in D_{20}^0(0)$, the following assertions are equivalent:

(a) $z_4(k_0) = 4 - k_0^2 \in M_4$ is a resonance of $H_V$,

(b) $\tilde{z}_{4,0} = \tilde{z}_4(k_0) = 4 - z_4(k_0) = k_0^2$ is a pole of $R_{-V_j}(u)$,

(c) $-1$ is an eigenvalue of $\mathcal{T}_{-V_j}(\tilde{z}_4(k)) := -\mathcal{V}_j W_{-\delta} R_0(\tilde{z}_4(k)) W_{-\delta}$,

(d) $k_0$ is a characteristic value of $I + \mathcal{T}_{-V_j}(\tilde{z}_4)$.

Moreover, thanks to (2.16), the multiplicity of the resonance $z_4(k_0)$ coincides with that of the characteristic value $k_0$.

4.4 Proof of Theorem 2.4

4.4.1 Preliminary results

Our goal in this section is to split near the spectral thresholds $z = 0$ and $z = 4$, the weighted resolvent $T_V(z(k))$, $z(k) = k^2$, into a sum of a singular part at $k = 0$ and a holomorphic part in the whole open disk $D_{20}(0)$, with values in $S_{\infty}(l^2(\mathbb{Z}))$.

From (4.1), we know that the kernel of $W_{-\delta}(H_0 - z(k))^{-1} W_{-\delta}$ is given by

$$
e^{-\frac{j}{2}|n| R_0(z(k), n - m) e^{-\frac{j}{2}|m|},
$$

with

$$
R_0(z(k), n - m) = \frac{ie^{n-m|2\arcsin \frac{z}{2}}}{k\sqrt{4-k^2}} = \frac{i}{k\sqrt{4-k^2}} + \frac{i (e^{i|n-m|2\arcsin \frac{z}{2}} - 1)}{k\sqrt{4-k^2}}
$$

(4.12)

where the functions $\alpha$ and $\beta$ are defined by

$$
\alpha(k) := i \left( \frac{1}{k\sqrt{4-k^2}} - \frac{1}{2k} \right) \quad \text{and} \quad \beta(k) := \frac{i (e^{i|n-m|2\arcsin \frac{z}{2}} - 1)}{k\sqrt{4-k^2}}.
$$
Moreover, it is not difficult to show that the functions $\alpha$ and $\beta$ can be extended to holomorphic functions in $D_{c_0}(0)$. By combining (4.1) and (4.12), we get for $k \in D_{c_0}(0)$

$$
\left(W_{-\delta}(H_0 - z(k))^{-1}W_{-\delta}x\right)(n) = \sum_{m \in \mathbb{Z}} \frac{ie^{-\frac{1}{4}|n|}e^{-\frac{1}{2}|m|}x(m)}{2k} + (A(k)x)(n),
$$

where $A(k)$ is the operator defined by

$$
(A(k)x)(n) := \sum_{m \in \mathbb{Z}} e^{-\frac{1}{4}|n|}\alpha(k)e^{-\frac{1}{2}|m|}x(m) + \sum_{m \in \mathbb{Z}} e^{-\frac{1}{2}|m|}\beta(k)e^{-\frac{1}{2}|m|}x(m).
$$

Let $\Theta : \ell^2(\mathbb{Z}) \rightarrow \mathbb{C}$ be the operator defined by $\Theta(x) := \langle x, e^{-\frac{1}{2}1} \rangle_{\ell^2(\mathbb{Z})}$, so that its adjoint $\Theta^* : \mathbb{C} \rightarrow \ell^2(\mathbb{Z})$ be given by $(\Theta^*(\lambda))(n) := \lambda e^{-\frac{1}{2}|n|}$. Then, this together with (4.13) yields

$$
\left(W_{-\delta}(H_0 - z(k))^{-1}W_{-\delta}x\right)(n) = \frac{i(\Theta^*\Theta x)(n)}{2k} + (A(k)x)(n).
$$

By combining (4.10) and (4.15), we finally obtain

$$
\mathcal{T}_V(z(k)) = \gamma \frac{i\Theta^*\Theta}{2k} + \gamma A(k).
$$

We therefore have proved the following result:

**Proposition 4.4** For $k \in D_{c_0}(0)$, the operator $\mathcal{T}_V(z(k))$ admits the decomposition

$$
\mathcal{T}_V(z(k)) = \frac{i\gamma}{k} \mathcal{M}^* \mathcal{M} + \gamma A(k), \quad \mathcal{M} := \frac{1}{\sqrt{2}} \Theta,
$$

with the operator $A(k)$ given by (4.14). Moreover, $k \mapsto \gamma A(k)$ is holomorphic in the open disk $D_{c_0}(0)$, with values in $S_\infty(\ell^2(\mathbb{Z}))$.

**Remark 4.2** Notice that $\mathcal{M}^* \mathcal{M} : \mathbb{C} \rightarrow \mathbb{C}$ is the rank-one operator given by

$$
\mathcal{M}^* \mathcal{M}(\lambda) = \frac{\lambda}{2}\langle e^{-\frac{1}{2}1}, e^{-\frac{1}{2}1} \rangle_{\ell^2(\mathbb{Z})} = \frac{\lambda}{2} \text{Tr}(e^{-\delta |1|}),
$$

$\text{Tr}(e^{-\delta |1|})$ standing for the trace of the multiplication operator by the function $\mathbb{Z} \ni n \mapsto e^{-\delta |n|}$. Then, so is the operator $\mathcal{M}^* \mathcal{M} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$.

**4.4.2 End of the proof**

Let $\mu \in \{0, 4\}$. Then, from Propositions 4.2 and 4.3 together with Proposition 4.4, it follows that $z_{\mu}(k)$ is a resonance of $H_V$ near $\mu$ if and only if $k$ is a characteristic value of the operator

$$
I + \mathcal{T}_V(z(k)) = I + \frac{i\gamma}{k} \mathcal{M}^* \mathcal{M} + \gamma A(k).
$$

Since the operator $\gamma A(k)$ is holomorphic in the open disk $D_{c_0}(0)$ with values in $S_\infty(\ell^2(\mathbb{Z}))$, while $i\gamma \mathcal{M}^* \mathcal{M}$ is a finite-rank operator, then, Theorem 2.4 follows immediately by applying Proposition 4.1 with

- $\mathcal{D} = D_e(0)$,
- $Z = \{0\}$,
- $F = I + \mathcal{T}_V(z(\cdot))$.

This concludes the proof of Theorem 2.4.
5 On positive commutators

In this section, we extend the validity of the Mourre theory in a non-selfadjoint setting under optimal regularity assumptions. In what follows, $\mathcal{H}$ is a fixed Hilbert space and $A$ is a selfadjoint operator densely defined on $\mathcal{H}$. We recall that the classes $C^k(A)$ and $C^{\alpha,1}(A)$, $k \in \mathbb{N}$, have been introduced in Definitions 2.4 and 2.5. Throughout this section, $H$ belongs to $B(\mathcal{H})$ and we denote $\sigma_+: = \max \sigma(\text{Im}(H))$ and $\sigma_-: = \min \sigma(\text{Im}(H))$.

As a general rule, a statement involving the symbol $\pm$ has to be understood as two independent statements.

5.1 Abstract results

The abstract results we are about to introduce associate the existence of positive commutation relations with the control of some spectral properties of the operator $H$. We start by describing these positivity conditions. Let $\Delta \subset \mathbb{R}$ be a Borel subset:

**Assumptions**

(M+) $\Re(H) \in C^1(A)$ and there exist $a_+, b_+ > 0$ and $\beta_+ \geq 0$ such that

$$\text{iad}_A(\Re(H)) + \beta_+ (\sigma_- - \text{Im}H) \geq a_+ E_{\text{Re}(H)}(\Delta) - b_+ E_{\text{Re}(H)}^\perp(\Delta).$$

(M−) $\Re(H) \in C^1(A)$ and there exist $a_-, b_- > 0$ and $\beta_- \geq 0$ such that

$$\text{iad}_A(\Re(H)) + \beta_- (\text{Im}H - \sigma_-) \geq a_- E_{\text{Re}(H)}(\Delta) - b_- E_{\text{Re}(H)}^\perp(\Delta).$$

(M) $\Re(H) \in C^1(A)$ and there exist $c_\Delta > 0$, $K \in S_\infty(\mathcal{H})$, such that

$$E_{\text{Re}(H)}(\Delta) i(\text{ad}_A(\Re(H)) E_{\text{Re}(H)}(\Delta) \geq c_\Delta E_{\text{Re}(H)}(\Delta) + K, \quad (5.1)$$

**Remark 5.1** Assumptions (M±) stated with $\beta_\pm = 0$ are equivalent to Assumption (M) stated with $K = 0$. Indeed, if $B \in B(\mathcal{H})$ is symmetric and $E$ is an orthogonal projection acting on $\mathcal{H}$, then the following statements are equivalent:

(a) There exists $c > 0$ such that $EBE \geq cE$.

(b) There exist $a > 0$, $b > 0$, such that $B \geq aE - bE^\perp$, with $E^\perp = 1 - E$.

(c) There exist $a > 0$, $b > 0$, such that $B \geq a - (a+b)E^\perp$, with $E^\perp = 1 - E$,

where inequalities are understood in the sense of quadratic forms.

We start by a reformulation of [31, Proposition 2.4] and refer to Section 5.2 for its proof.

**Lemma 5.1** Let $H \in B(\mathcal{H})$ and $\Delta \subset \mathbb{R}$ be a Borel subset.

- Assume (M) holds on $\Delta$. Then, $\mathcal{E}_p(H) \cap \{\Delta + i\{\sigma_\pm\}\} \subset (\sigma_{pp}(\Re(H)) \cap \Delta) + i\{\sigma_\pm\}$. In particular, the set $\mathcal{E}_p(H) \cap \{\Delta + i\{\sigma_\pm\}\}$ is finite and each of these eigenvalues has finite geometric multiplicity. If (M) holds with $K = 0$, then $\mathcal{E}_p(H) \cap \{\Delta + i\{\sigma_\pm\}\} = \emptyset$.

- Assume that $\Re(H) \in C^1(A)$ and that $\text{iad}_A(\Re(H)) > 0$ (i.e. positive and injective). Then, $\mathcal{E}_p(H) \cap \{\mathbb{R} + i\{\sigma_\pm\}\} = \emptyset$.

**Remark 5.2** If $H \in C^1(A)$, then, $\Re(H)$ and $\text{Im}(H)$ belong to $C^1(A)$ and,

$$\text{ad}_A(\Re(H)) = i \text{Im}(\text{ad}_A(H)), \quad \text{ad}_A(\text{Im}(H)) = -i \text{Re}(\text{ad}_A(H)).$$

Indeed, $H^* \in C^1(A)$ and $\text{ad}_A(H^*) = -(\text{ad}_A(H))^*$, see e.g. [1, Proposition 5.1.7].
We recall that if $H \in C^{1,1}(A)$, then $H \in C^{1}(A)$.

**Theorem 5.1** Let $H \in C^{1,1}(A)$, $\Delta \subset \mathbb{R}$ be an open interval and $s > 1/2$.

- Assume (M$^+$) holds on $\Delta$. For any relatively compact interval $\Delta_0$, $\Delta_0 \subset \Delta$, we have
  \[ \sup_{z \in \Delta_0 + i[\sigma_+, \infty)} \left\| (A)^{-s}(z - H)^{-1}(A)^{-s} \right\| < \infty. \]

- Assume (M$^-$) holds on $\Delta$. For any relatively compact interval $\Delta_0$, $\Delta_0 \subset \Delta$, we have
  \[ \sup_{z \in \Delta_0 + i(-\infty, \sigma_-)} \left\| (A)^{-s}(z - H)^{-1}(A)^{-s} \right\| < \infty. \]

The proof of Theorem 5.1 is carried out through Sections 5.3–5.5.

**Remark 5.3** Once observed that for any $s > 0$ and for any $z \in \Delta_0 + i((-\infty, \sigma_- - 1] \cup [\sigma_+ + 1, \infty))$, we have
\[ \left\| (z - H)^{-1} \right\| \leq 1/\text{dist}(z, \mathcal{N}(H)) \leq 1, \]
the proof of Theorem 5.1 reduces to bound the weighted resolvent uniformly for $z \in \Delta_0 + i[\sigma_+, \sigma_+ + 1]$ and $z \in \Delta_0 + i[\sigma_- - 1, \sigma_-)$ respectively.

**Remark 5.4** Theorem 5.1 can be formulated equivalently with weights of the form $h(A)$, $\tilde{h}(A)$ instead of $(A)^{-s}$, where $h : \mathbb{R} \to \mathbb{C}$ is such that: $0 < c_1 \leq h(x)(x)^s \leq c_2$ for some $c_1 > 0$, $c_2 > 0$ and $s > 1/2$. For technical reasons, the proof of Theorem 5.1 is actually developed with weights of the form $h(A)$ where $h(x) = (|x| + 1)^{-s}$.

Interpolation spaces can also be introduced within this framework in order to obtain an optimal version on the Besov scales.

Theorem 5.1 can be reformulated as follows:

**Corollary 5.1** Let $H \in C^{1,1}(A)$, $\Delta \subset \mathbb{R}$ be an open interval and $s > 1/2$. Let $\Delta_1$ be an open bounded interval such that $\Delta_0 \subset \Delta_1 \subset \Delta$. Let $\chi \in C_0^\infty(\mathbb{R})$ be supported on $\Delta_1$ and taking value 1 on $\Delta_0$.

- Assume (M$^+$) holds on $\Delta$. Then, we have
  \[ \sup_{z \in \mathbb{R} + i[\sigma_+, \infty)} \left\| (A)^{-s}\chi(\text{Re}(H))(z - H)^{-1}\chi(\text{Re}(H))(A)^{-s} \right\| < \infty. \]

- Assume (M$^-$) holds on $\Delta$. Then, we have
  \[ \sup_{z \in \mathbb{R} + i(-\infty, \sigma_-)} \left\| (A)^{-s}\chi(\text{Re}(H))(z - H)^{-1}\chi(\text{Re}(H))(A)^{-s} \right\| < \infty. \]

Synthesizing Lemma 5.1 and Theorem 5.1, we obtain:

**Corollary 5.2** Let $H \in C^{1,1}(A)$. Assume (M) holds on some open interval $\Delta \subset \mathbb{R}$. Then,

- $\mathcal{E}_p(H) \cap (\Delta + i[\sigma_\pm]) \subset (\sigma_{pp}(\text{Re}(H)) \cap \Delta) + i[\sigma_\pm]$. In particular, the set $\mathcal{E}_p(H) \cap (\Delta + i[\sigma_\pm])$ is finite and each of these eigenvalues has finite geometric multiplicity.

- Let $s > 1/2$. For any relatively compact interval $\Delta_0 \subset \Delta$ such that $\overline{\Delta_0} \subset \Delta \setminus \mathcal{E}_p(\text{Re}(H))$,
  \[ \sup_{z \in \Delta_0 + i(\sigma_+, \infty)} \left\| (A)^{-s}(z - H)^{-1}(A)^{-s} \right\| < \infty. \]

For any relatively compact interval $\Delta_0 \subset \Delta$ such that $\overline{\Delta_0} \subset \Delta \setminus \mathcal{E}_p(\text{Re}(H))$,
\[ \sup_{z \in \Delta_0 + i(-\infty, \sigma_-)} \left\| (A)^{-s}(z - H)^{-1}(A)^{-s} \right\| < \infty. \]
The proofs of Corollaries 5.1 and 5.2 are postponed to Sections 5.6 and 5.7 respectively.

We conclude this section by the following remark. If $H$ is selfadjoint in Theorem 5.1, that is $H = \text{Re}(H)$, $\text{Im}(H) = 0$ and $\sigma_{\pm} = 0$, we recover [1, Theorem 7.3.1]. Namely,

**Theorem 5.2** Let $H \in C^{1,1}(A)$ be selfadjoint, $\Delta \subset \mathbb{R}$ be an open interval and $s > 1/2$. Assume \((M \pm)\) hold on $\Delta$ with $\beta_{\pm} = 0$, then for any relatively compact interval $\Delta_0, \overline{\Delta}_0 \subset \Delta$, we have

$$\sup_{z \in \Delta_0 + i\mathbb{R}} \| (A)^{-s} (z - H)^{-1} (A)^{-s} \| < \infty. \quad (5.2)$$

### 5.2 Proof of Lemma 5.1

We start with the following lemma:

**Lemma 5.2** Let $H \in B(\mathcal{H})$. Assume that there exist $\varphi \in \mathcal{H}$ and $\lambda \in \mathbb{R}$ such that $H \varphi = (\lambda + i\sigma_{\pm}) \varphi$. Then, $\text{Re}(H) \varphi = \lambda \varphi$. In particular,

$$\mathcal{E}_p(H) \cap (\mathbb{R} + i\{\sigma_{\pm}\}) \subset (\mathcal{E}_p(\text{Re}(H)) \cap \mathbb{R}) + i\{\sigma_{\pm}\}. \quad (5.3)$$

**Proof.** We have

$$\langle \varphi, \text{Im}(H) \varphi \rangle = \text{Im} \left( \langle \varphi, H \varphi \rangle \right) = \sigma_{\pm} \| \varphi \|^2,$$

$$\sigma_- \| \varphi \|^2 \leq \langle \varphi, \text{Im}(H) \varphi \rangle \leq \sigma_+ \| \varphi \|^2. \quad (5.4)$$

To fix ideas, let us consider the $+$ case. We have $\| \sqrt{\sigma_+ - \text{Im}(H) \varphi} \|^2 = \langle \varphi, (\sigma_+ - \text{Im}(H)) \varphi \rangle = 0$, so $\sqrt{\sigma_+ - \text{Im}(H) \varphi} = 0$, and hence $\text{Im}(H) \varphi = \sigma_+ \varphi$ and $\text{Re}(H) \varphi = \lambda \varphi$. □

Let us prove now Lemma 5.1. Assume $(M)$ holds. As a consequence of the Virial Theorem, see e.g. [1, Corollary 7.2.11], $\sigma_{pp}(\text{Re}(H)) \cap \Delta$ is necessarily finite with finite multiplicity. Moreover, it is empty if $K = 0$. In view of Lemma 5.2, this proves the first statement of Lemma 5.1.

If there exists $\varphi \in \mathcal{H}$ such that $H \varphi = (\lambda + i\sigma_-) \varphi$ for some $\lambda \in \mathbb{R}$, then $\text{Re}(H) \varphi = \lambda \varphi$, due to Lemma 5.2. Assuming that $\text{Re}(H) \in C^1(A)$, we can apply the Virial Theorem (see e.g. [1, Proposition 7.2.10]) to deduce that $\langle \varphi, \text{ad}_A(\text{Re}(H)) \varphi \rangle = 0$. If in addition $\text{ad}_A(\text{Re}(H)) > 0$, then necessarily $\varphi = 0$. This means that $\mathcal{E}_p(H) \cap (\mathbb{R} + i\{\sigma_-\}) = \emptyset$. Similarly, we have that $\mathcal{E}_p(H) \cap (\mathbb{R} + i\{\sigma_+\}) = \emptyset$, which proves the second statement of Lemma 5.1.

### 5.3 Deformed resolvents and first estimates

The proof of Theorem 5.1 is based on Mourre’s differential inequality strategy. Our presentation interpolates between [1] and [30]. Throughout this section, $\Delta \subset \mathbb{R}$ denotes an open interval and $H \in C^1(A)$. In particular $\text{Re}(H) \in C^1(A)$ (see Remark 5.2). For further use, we formulate various additional hypotheses, that will be related to the condition $H \in C^{1,1}(A)$ in Section 5.5.

We assume there exist $0 < \epsilon_0 \leq 1$ and two maps

$$S : (0, \epsilon_0) \rightarrow B(\mathcal{H}),$$

$$B : (0, \epsilon_0) \rightarrow B(\mathcal{H}),$$

such that

**Assumptions**

(A1) there exists $C > 0$ so that for any $\epsilon \in (0, \epsilon_0)$, $\| S(\epsilon) - H \| \leq C \epsilon$,

(A2) $\lim_{\epsilon \to 0} \| B(\epsilon) - i \text{ad}_A(H) \| = 0$,

(A3) $\lim_{\epsilon \to 0} \epsilon^{-1} \| S(\epsilon) - H \| = 0$. 

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Assumptions (A1)–(A4) allow to extend continuously the functions $S$, $B$ and $\partial S$ on $[0, \epsilon_0)$, by setting $S(0) = H$, $B(0) = i \text{ad}_A(H)$ and $(\partial S)(0) = 0$.

For $\epsilon \in (0, \epsilon_0)$, $\partial S$ and $\partial \partial S$ are continuously differentiable on $[0, \epsilon_0)$ w.r.t. the operator norm topology.

Due to Assumptions (A1)–(A4), the maps $\epsilon \mapsto Q^\pm_\epsilon$ and $\epsilon \mapsto R^\pm_\epsilon$ are continuous on $[0, \epsilon_0)$ w.r.t. the operator norm topology. Using (A1), (A3) and (A5), we also have

$$C_R := \sup_{\epsilon \in [0, \epsilon_0)} \| R^\pm_\epsilon \| < \infty.$$  

Thus, it follows that:

**Lemma 5.3** Assume that (A1)–(A5) hold. Then, for any $z \in \mathbb{C}$, any $\epsilon \in [0, \epsilon_0)$, we have

$$\| T^\pm_\epsilon(z) - T_0(z) \| \leq C_R \epsilon. \tag{5.7}$$

**Lemma 5.4** Assume that (A1)–(A4) hold. Then, for any $z \in \mathbb{C}$ such that $\text{Im}(z) \in [\sigma_+, \sigma_+ + 1] \cup [\sigma_- - 1, \sigma_-]$, any $\epsilon \in (0, \epsilon_0]$, any $p > 0$ and any $\psi \in \mathcal{H}$, we have

$$\langle \psi, \text{Im}(T_0(z)) \psi \rangle \leq \frac{1}{2p} \| T^\pm_\epsilon(z) \psi \|^2 + \left( C_R \epsilon + \frac{p}{2} \right) \| \psi \|^2, \tag{5.8}$$

$$\| \text{Im}(T_0(z)) \| \leq \frac{1}{2p} \| T^\pm_\epsilon(z) \|^2 + \left( C_R \epsilon + \frac{p}{2} \right) \| \psi \|^2.$$  

In particular, for any $z \in \mathbb{C}$ such that $\text{Im}(z) \in [\sigma_+, \sigma_+ + 1] \cup [\sigma_- - 1, \sigma_-]$, any $\epsilon \in (0, \epsilon_0]$, any $p > 0$ and any $\psi \in \mathcal{H}$, we also have

$$\| \text{Im}(T_0(z)) \psi \|^2 \leq \frac{\sigma_0}{2p} \| T^\pm_\epsilon(z) \psi \|^2 + \sigma_0 \left( C_R \epsilon + \frac{p}{2} \right) \| \psi \|^2, \tag{5.9}$$

$$\| \text{Im}(T_0(z)) \|^2 \leq \frac{\sigma_0}{2p} \| T^\pm_\epsilon(z) \| ^2 + \sigma_0 \left( C_R \epsilon + \frac{p}{2} \right) \| \psi \|^2,$$

where $\sigma_0 := \sup_{z \in \mathbb{C}, \text{Im}(z) \in [\sigma_+, \sigma_+ + 1] \cup [\sigma_- - 1, \sigma_-]} \| \text{Im}(T_0(z)) \| = \sigma_+ - \sigma_- + 1$.

**Proof.** Due to Lemma 5.3, for any $z \in \mathbb{C}$, any $\psi \in \mathcal{H}$ and any $\epsilon \in (0, \epsilon_0)$,

$$\| \psi, \text{Im}(T_0(z)) \psi \| \leq \| \psi, T_0(z) \psi \| \leq \| \psi, T^\pm_\epsilon(z) \psi \| + C_R \epsilon \| \psi \|^2 \leq \| \psi \| \| T^\pm_\epsilon(z) \psi \| + C_R \epsilon \| \psi \|^2,$$

while for any $p > 0$,

$$\| \psi \| \| T^\pm_\epsilon(z) \psi \| \leq \frac{1}{2p} \| T^\pm_\epsilon(z) \psi \|^2 + \frac{p}{2} \| \psi \|^2.$$
Once observed that \(\text{Im}(T_0(z)^*) = -\text{Im}(T_0(z))\), some analog inequalities hold with \(T_0(z)^*\) and \(T_0^*(z)^*\) instead of \(T_0(z)\) and \(T_0^*(z)\). This allows us to conclude about the first estimates. Now, note that if \(\text{Im}(z) \in [\sigma_+, \sigma_+ + 1]\), \(\text{Im}(T_0(z)) \geq 0\), and if \(\text{Im}(z) \in [\sigma_- - 1, \sigma_-]\), \(\text{Im}(T_0(z)) \leq 0\). So, for any \(\psi \in \mathcal{H}\) and any \(z \in \mathbb{C}\) such that \(\text{Im}(z) \in [\sigma_+, \sigma_+ + 1] \cup [\sigma_- - 1, \sigma_-]\),

\[
\|\text{Im}(T_0(z))\psi\|^2 \leq \|\text{Im}(T_0(z))\|\|\psi, \text{Im}(T_0(z))\psi\| \leq \sigma_0 \langle \psi, \text{Im}(T_0(z))\psi \rangle. \tag{5.10}
\]

The last estimates follow immediately from the first ones.

In order to shorten our notations, we denote for any Borel set \(\Delta_0 \subset \mathbb{R}\),

\[
\Delta_0^\pm := \Delta_0 + i[\sigma_+, \sigma_+ + 1], \\
\Delta_0^- := \Delta_0 + i[\sigma_- - 1, \sigma_-],
\]

and deduce from the triangular inequality that

\[
\text{dist} \big[0, \Delta_0^\pm \big] \leq 3 \delta_0^2 \left(1 + \frac{\sigma_0}{2p}\right) \|T_0^*(z)\| \leq \sigma_0 \langle \psi, \text{Im}(T_0(z))\psi \rangle.
\]

The conclusion follows now from Lemma 5.4.

**Lemma 5.5** Assume that (A1)–(A5) hold. For any \(z \in \Delta_0^\pm\), any \(\epsilon \in [0, \epsilon_0]\), any \(p > 0\) and any \(\varphi \in \mathcal{H}\), we have

\[
\|E_{\text{Re}(H)}^\pm(\Delta)\varphi\|^2 \leq 3 \delta_0^2 \left(1 + \frac{\sigma_0}{2p}\right) \|T_0^*(z)\| \leq \sigma_0 \langle \psi, \text{Im}(T_0(z))\psi \rangle.
\]

Proof. For for \(\text{Re}(z) \in \Delta_0\) we have \(\delta_0 \|\text{Re}(T_0(z))^{-1} E_{\text{Re}(H)}^\pm(\Delta)\| \leq 1\). So, if \(z \in \Delta_0^\pm\), we can write

\[
E_{\text{Re}(H)}^\pm(\Delta) = \left(\text{Re}(T_0(z))^{-1} E_{\text{Re}(H)}^\pm(\Delta) \left(T_0^*(z) - i \text{Im}(T_0(z)) + \epsilon R_\epsilon^\pm\right)\right),
\]

and deduce from the triangular inequality that

\[
\|E_{\text{Re}(H)}^\pm(\Delta)\varphi\|^2 \leq 3 \delta_0^2 \left(1+ \|T_0^*(z)\| \leq \sigma_0 \langle \psi, \text{Im}(T_0(z))\psi \rangle \right).
\]

The conclusion follows now from Lemma 5.4.

From now on, we fix

\[
p_\pm = \frac{a_\pm\delta_0^2}{4\beta_\pm d_\pm^2 + 6(a_\pm + b_\pm)\sigma_0}.
\]

In view of Hypotheses (A2), (A3) and (M), we can also assume without any restriction that \(\epsilon_0 > 0\) is chosen such that:

**Assumption (A7)**

\[
\frac{3(a_\pm + b_\pm)}{\delta_0^2} \left(\|C_\epsilon\epsilon_0^2 + \sigma_\epsilon C_\epsilon\epsilon_0\| + \beta_\pm C_\epsilon\epsilon_0 + \sup_{\epsilon \in [0, \epsilon_0]} \|\text{Im}(Q_\epsilon^\pm)\| \right) \leq \frac{a_\pm}{4}.
\]

Assumption (A7) combined with (5.14) ensures that

\[
\frac{3(a_\pm + b_\pm)}{\delta_0^2} \sup_{\epsilon \in [0, \epsilon_0]} \left(\|C_\epsilon\epsilon_0^2 + \sigma_\epsilon C_\epsilon\epsilon_0 + \sigma_\epsilon p_\epsilon^\pm \right) + \beta_\pm \sup_{\epsilon \in [0, \epsilon_0]} \left(C_\epsilon\epsilon_0 + \frac{p_\pm}{2}\right) + \sup_{\epsilon \in [0, \epsilon_0]} \|\text{Im}(Q_\epsilon^\pm)\| \leq \frac{a_\pm}{2}.
\]

(5.15)
For $z \in \mathbb{C}$, we have
\[
\text{Im} \left( T^\pm_e(z) \right) = \text{Im}(z) - \text{Im} \left( S(\epsilon) \right) \pm \epsilon \text{Re} \left( B(\epsilon) \right) = \text{Im}(z - H) \pm \epsilon \text{adi}_A \left( \text{Re}(H) \right) - \epsilon \text{Im}(Q^\pm_e). \tag{5.16}
\]
Assuming (M+), we get for $z \in \Delta + i[\sigma_+ , \infty)$,
\[
\text{Im} \left( T^+_e(z) \right) = - \text{Im} \left( T^+_e(z)^* \right) \\
\geq \left( \text{Im}(z) - \sigma_+ \right) + a_+ \epsilon - (a_+ + b_+) \epsilon E^+_\text{Re}(H)(\Delta) - \epsilon \beta_+ (\sigma_+ - H - \epsilon \text{Im}(Q^e_+)) \\
\geq \left( \text{Im}(z) - \sigma_+ \right) + a_+ \epsilon - (a_+ + b_+) \epsilon E^+_\text{Re}(H)(\Delta) - \epsilon \beta_+ (\text{Im}(T_0(z)) - \epsilon \text{Im}(Q^e_+)). \tag{5.17}
\]
Assuming (M−), we get for $z \in \Delta + i(-\infty, \sigma_-]$,
\[
- \text{Im} \left( T^-_e(z) \right) = \text{Im} \left( T^-_e(z)^* \right) \\
\geq (\sigma_- - \text{Im}(z)) + a_- \epsilon - (a_- + b_-) \epsilon E^-_\text{Re}(H)(\Delta) - \epsilon \beta_- (\text{Im} H - \sigma_-) + \epsilon \text{Im}(Q^-_e) \\
\geq (\sigma_- - \text{Im}(z)) + a_- \epsilon - (a_- + b_-) \epsilon E^-_\text{Re}(H)(\Delta) + \epsilon \beta_- (\text{Im}(T_0(z)) + \epsilon \text{Im}(Q^-_e)). \tag{5.18}
\]
This leads us to:

**Proposition 5.1** Assume that (A1)–(A5) and (A7) hold.

- If (M+) holds, then there exists $C^+_1 > 0$ such that for any $\epsilon \in [0, \epsilon_0)$, any $\varphi \in \mathcal{H}$ and any $z \in \Delta^+_0$,
\[
\text{Im} \left( (\varphi, T^+_e(z)\varphi) \right) + C^+_1 \epsilon \| T^+_e(z)\varphi \|^2 \geq d_+ (\text{Im}(z), \epsilon) \| \varphi \|^2, \tag{5.19}
\]
\[
- \text{Im} \left( (\varphi, (T^+_e(z))^*\varphi) \right) + C^+_1 \epsilon \| (T^+_e(z))^*\varphi \|^2 \geq d_+ (\text{Im}(z), \epsilon) \| \varphi \|^2, \tag{5.20}
\]
with $d_+ (\text{Im}(z), \epsilon) := (\text{Im}(z) - \sigma_+) + \epsilon a_+ + \beta_+ = a_+ / 2$.

- If (M−) holds, then there exists $C^-_1 > 0$ such that for any $\epsilon \in [0, \epsilon_0)$, any $\varphi \in \mathcal{H}$ and any $z \in \Delta^-_0$,
\[
- \text{Im} \left( (\varphi, T^-_e(z)\varphi) \right) + C^-_1 \epsilon \| T^-_e(z)\varphi \|^2 \geq d_- (\text{Im}(z), \epsilon) \| \varphi \|^2, \tag{5.21}
\]
\[
\text{Im} \left( (\varphi, (T^-_e(z))^*\varphi) \right) + C^-_1 \epsilon \| (T^-_e(z))^*\varphi \|^2 \geq d_- (\text{Im}(z), \epsilon) \| \varphi \|^2, \tag{5.22}
\]
with $d_- (\text{Im}(z), \epsilon) := -(\text{Im}(z) - \sigma_-) + \epsilon a_- + \beta_- = a_- / 2$.

In particular, $T^+_e(z)$ is boundedly invertible as soon as $\epsilon \in [0, \epsilon_0)$, $z \in \Delta^+_0$ and $d_+ (\text{Im}(z), \epsilon) > 0$.

**Remark 5.5** The constants $C^\pm_1$ can be explicitly chosen as:
\[
C^\pm_1 := \frac{3(a_\pm + b_\pm)}{\delta^2_\pm} \left( 1 + \frac{\sigma_\pm}{2p_\pm} \right)^2 + \frac{\beta_\pm}{2p_\pm}. \tag{5.23}
\]

*Proof.* Inequalities (5.19) and (5.20) follow from (5.17), Lemmata 5.4 and 5.5, while inequalities (5.21) and (5.22) follows from (5.18), Lemmata 5.4 and 5.5.

Fix $\epsilon \in [0, \epsilon_0)$, $z \in \Delta^+_0$ with $d_+ (\text{Im}(z), \epsilon) > 0$. Inequalities (5.19) and (5.20) show that $T^+_e(z)$ and $(T^+_e(z))^*$ are injective from $\mathcal{H}$ into itself and have closed ranges. Since
\[
\text{Ran}(T^+_e(z)) = (\text{Ker}(T^+_e(z))^*)^\perp,
\]
we deduce that $T^+_e(z)$ and $(T^+_e(z))^*$ are actually bijective and boundedly invertible. Similarly, fix $\epsilon \in [0, \epsilon_0)$, $z \in \Delta^-_0$ with $d_- (\text{Im}(z), \epsilon) > 0$. Inequalities (5.21) and (5.22) show $T^-_e(z)$ and $(T^-_e(z))^*$ are injective from $\mathcal{H}$ into itself and have closed ranges. Since
\[
\text{Ran}(T^-_e(z)) = (\text{Ker}(T^-_e(z))^*)^\perp,
\]
we deduce that $T_+^*(z)$ and $(T_+^*)^*$ are actually bijective and boundedly invertible.

In view of Proposition 5.1, we define for any $\epsilon \in [0, \epsilon_0)$, $z \in \Delta_0^\pm$ with $d_\pm(\Im(z), \epsilon) > 0$,

$$G_+^\pm(z) := (T_+^\pm(z))^{-1}.$$  \hspace{1cm} (5.24)

As a direct consequence of Proposition 5.1, we obtain

**Proposition 5.2** Assume that (A1)–(A5), (M\pm) and (A7) hold. First, we have

$$C_0^\pm := \sup_{(\epsilon, z) \in [0, \epsilon_0) \times \Delta_0^\pm} d_\pm(\Im(z), \epsilon) \|G_+^\pm(z)\| < \infty.$$  \hspace{1cm} (5.25)

In addition, for $C_1^+$ defined in Proposition 5.1, we have:

- For any $\epsilon \in (0, \epsilon_0)$, $z \in \Delta_0^+$,

  $$\|G_+^+(z)\varphi\| \leq \frac{2C^+_1}{a_+} \|\varphi\| + \sqrt{\frac{2}{a_+} \Im((\varphi, G_+^+(z)\varphi))}.$$  \hspace{1cm} (5.26)

- For any $\epsilon \in (0, \epsilon_0)$, $z \in \Delta_0^-$,

  $$\|G_+^-(z)\varphi\| \leq \frac{2C^-_1}{a_-} \|\varphi\| + \sqrt{\frac{2}{a_-} \Im((\varphi, G_+^-(z)\varphi))}.$$  \hspace{1cm} (5.27)

- For any $\epsilon \in (0, \epsilon_0)$, $z \in \Delta_0^\pm$,

  $$\|G_+^-(z)\varphi\| \leq \frac{2C^-_1}{a_-} \|\varphi\| + \sqrt{\frac{2}{a_-} \Im((\varphi, G_+^-(z)\varphi))}.$$  \hspace{1cm} (5.28)

  $$\|G_+^+(z)\varphi\| \leq \frac{2C^+_1}{a_+} \|\varphi\| + \sqrt{\frac{2}{a_+} \Im((\varphi, G_+^+(z)\varphi))}.$$  \hspace{1cm} (5.29)

**Proof.** Observe first that for any $\epsilon \in [0, \epsilon_0)$, $z \in \Delta_0^\pm$ with $d_\pm(\Im(z), \epsilon) > 0$,

$$G_+^\pm(z) \Im(T_+^\pm(z)) G_+^\pm(z) = - \Im(G_+^\pm(z)).$$  \hspace{1cm} (5.30)

Once replaced $\varphi$ by $G_+^+(z)\varphi$ in (5.19), resp. by $G_+^-(z)\varphi$ in (5.21), Proposition 5.1 reads: for any $\epsilon \in [0, \epsilon_0)$, $z \in \Delta_0^\pm$, $d_\pm(\Im(z), \epsilon) > 0$,

$$\Im((G_+^\pm(z)\varphi, \varphi)) + C_1^\pm \epsilon \|\varphi\|^2 \geq d_\pm(\Im(z), \epsilon) \|G_+^\pm(z)\varphi\|^2.$$  \hspace{1cm} (5.31)

But, for all $\epsilon \in [0, \epsilon_0)$, $z \in \Delta_0^\pm$ with $d_\pm(\Im(z), \epsilon) > 0$, we have

$$\Im((G_+^\pm(z)\varphi, \varphi)) \leq \|G_+^\pm(z)\varphi\| \leq \|\varphi\| \|G_+^\pm(z)\varphi\|.$$  \hspace{1cm} (5.31)

Multiplying both sides by $d_\pm(\Im(z), \epsilon)$ and using the fact that $d_\pm(\Im(z), \epsilon) \leq 1 + \epsilon_0 a_\pm$ if $z \in \Delta_0^\pm$, we deduce that

$$d_\pm(\Im(z), \epsilon) \|G_+^\pm(z)\varphi\| \|\varphi\| + C_1^\pm (1 + \epsilon_0 a_\pm) \epsilon_0 \|\varphi\|^2 \geq d_\pm(\Im(z), \epsilon) \epsilon_0 \|G_+^\pm(z)\varphi\|^2.$$  \hspace{1cm} (5.32)

Taking first supremum over the vectors $\varphi$ of norm 1, we deduce that $C_0^\pm < \infty$.

For any $\epsilon \in (0, \epsilon_0)$, $z \in \Delta_0^\pm$, inequality (5.3) implies that

$$\Im((G_+^\pm(z)\varphi, \varphi)) + C_1^\pm \epsilon \|\varphi\|^2 \geq \epsilon a_\pm \|G_+^\pm(z)\varphi\|^2,$$

which implies estimates (5.26) and (5.28).
Note that $\|G^\pm_\epsilon(z)^*\| = \|G^\pm_\epsilon(z)\|$ whenever it exists. Once replaced $\varphi$ by $G^\pm_\epsilon(z)^*\varphi$ in (5.20), resp. by $G^\pm_\epsilon(z)\varphi$ in (5.22), Proposition 5.1 reads: for any $\epsilon \in [0, \epsilon_0)$, $z \in \Delta_0^\pm$, $d_\pm(\text{Im}(z), \epsilon) > 0$,

$$\left| \text{Im} \left( (G^\pm_\epsilon(z)^*\varphi, \varphi) \right) \right| + C^\pm_1 \epsilon\|\varphi\|^2 \geq \epsilon a_\pm\|G^\pm_\epsilon(z)^*\varphi\|^2.$$ 

Estimates (5.27) and (5.29) follow at once. \Box

We recall that for $z \in \mathbb{C} \setminus \mathcal{N}(H)$, we have $\|z - H\|^{-1} \leq 1/\text{dist}(z, \mathcal{N}(H))$. In particular, if $z \in \Delta_0^\pm$, $\|z - H\|^{-1} \leq |\text{Im}(z) - \sigma_\pm|^{-1}$. The following estimates follows directly from Proposition 5.2 and Lemma 5.3:

**Corollary 5.3** Assume that (A1)--(A5), (M±) and (A7) hold. For any $\epsilon \in [0, \epsilon_0)$ and any $z \in \Delta_0^\pm$, we have

$$\|G^\pm_\epsilon(z) - (z - H)^{-1}\| \leq \min \left( \frac{C_R C^\pm_0 \epsilon}{\text{Im}(z) - \sigma_\pm}, \frac{2C_R C^\pm_0}{a|\text{Im}(z) - \sigma_\pm|^2} \right).$$

**Proof.** For any $\epsilon \in [0, \epsilon_0)$ and any $z \in \Delta_0^\pm$, $G^\pm_\epsilon(z) - (z - H)^{-1} = G^\pm_\epsilon(z)(T_0(z) - T^\epsilon_\epsilon(z))(z - H)^{-1}$. We obtain the first estimate by using Lemma 5.3 and Proposition 5.2. The proof of the second estimate is analogous. \Box

Let $1/2 < s \leq 1$. For any $\epsilon \in (0, \epsilon_0)$, $z \in \Delta_0^\pm$, we define

$$F^\pm_{s,\epsilon}(z) := W_s(\epsilon)G^\pm_\epsilon(z)W_s(\epsilon),$$

where $W_s(\epsilon) := (|A| + 1)^{-s}(\epsilon|A| + 1)^{-s-1}$.

Note that for any $s \in (1/2, 1]$, $(W_s(\epsilon))_{\epsilon \in [0, \epsilon_0)}$ is a family of bounded selfadjoint operators. In particular, $\sup_{\epsilon \in [0, 1]}\|W_s(\epsilon)\| \leq 1$. Proposition 5.2 entails immediately:

**Corollary 5.4** Assume that (A1)--(A5), (M±) and (A7) hold. We have

$$\sup_{\epsilon \in (0, \epsilon_0), z \in \Delta_0^\pm} \epsilon\|F^\pm_{s,\epsilon}(z)\| \leq \frac{2C^\pm_0}{a_\pm} < \infty.$$ 

In addition, for any $\epsilon \in (0, \epsilon_0)$, $z \in \Delta_0^\pm$, we have

$$\|G^\pm_\epsilon(z)W_s(\epsilon)\| \leq \sqrt{\frac{2C^\pm_1}{a_\pm} + \frac{2\epsilon\|F^\pm_{s,\epsilon}(z)\|}{a_\pm}},$$

$$\|W_s(\epsilon)G^\pm_\epsilon(z)\| = \|G^\pm_\epsilon(z)^*W_s(\epsilon)\| \leq \sqrt{\frac{2C^\pm_1}{a_\pm} + \frac{2\epsilon\|F^\pm_{s,\epsilon}(z)\|}{a_\pm}}.$$ 

### 5.4 Differential Inequalities

Next, we derive a system of differential inequalities for the weighted deformed resolvents $F^\pm_{s,\epsilon}(z)$.

We recall the following basic facts and refer to [1, Lemma 7.3.4] for proofs.

**Lemma 5.6** Assume that (A1)--(A5), (M±) and (A7) hold. Then, for any fixed $z \in \Delta_0^\pm$, the map $\epsilon \mapsto G^\pm_\epsilon(z)$ is continuous on the interval $[0, \epsilon_0)$ w.r.t. the operator norm topology, continuously differentiable on $(0, \epsilon_0)$ w.r.t. the operator norm topology, with

$$\partial_\epsilon G^\pm_\epsilon(z) = G^\pm_\epsilon(z)(\partial_\epsilon S(\epsilon) - i\epsilon\partial_\epsilon B(\epsilon) - iB(\epsilon))G^\pm_\epsilon(z),$$

$$\partial_\epsilon G^\pm_\epsilon(z) = G^\pm_\epsilon(z)(\partial_\epsilon S(\epsilon) + i\epsilon\partial_\epsilon B(\epsilon) + iB(\epsilon))G^\pm_\epsilon(z).$$
Lemma 5.7 Assume that (A1)–(A7), (M±) hold. Then, for any \( \epsilon \in [0, \epsilon_0) \), \( z \in \Delta_0^\pm \), we have \( G^\pm_\epsilon(z) \in C^1(A) \), with

\[
\text{ad}_A G^+_\epsilon(z) = G^+_\epsilon(z) \left( \text{ad}_A(S(\epsilon)) - i\text{ad}_A(B(\epsilon)) \right) G^+_\epsilon(z),
\]
\[
\text{ad}_A G^-_\epsilon(z) = G^-_\epsilon(z) \left( \text{ad}_A(S(\epsilon)) + i\text{ad}_A(B(\epsilon)) \right) G^-_\epsilon(z).
\]

Summing up, we obtain:

**Proposition 5.3** Assume that (A1)–(A7), (M±) hold. Then, for any \( \epsilon \in (0, \epsilon_0) \), \( z \in \Delta_0^\pm \), the map \( \epsilon \mapsto G^\pm_\epsilon(z) \) is continuously differentiable on \((0, \epsilon_0)\) w.r.t. the operator norm topology with

\[
\partial_\epsilon G^+_\epsilon(z) = \text{ad}_A(G^+_\epsilon(z)) + G^+_\epsilon(z)Q^+(\epsilon)G^+_\epsilon(z),
\]
\[
\partial_\epsilon G^-_\epsilon(z) = -\text{ad}_A(G^-_\epsilon(z)) + G^-_\epsilon(z)Q^-(\epsilon)G^-_\epsilon(z),
\]

where

\[
Q^+ = \partial_s S - i\partial_w B - iB - \text{ad}_A S + i\text{ad}_A B,
\]
\[
Q^- = \partial_S S + i\partial_w B + iB + \text{ad}_A S + i\text{ad}_A B.
\]

This is the first key to state some differential inequalities. Now, for any fixed \( 1/2 < s < 1 \), the map \( \epsilon \mapsto W_s(\epsilon) \) is strongly continuous on \([0, \epsilon_0)\) and converges strongly to \( (|A| + 1)^{-s} \) as \( \epsilon \) tends to zero. Let us introduce for any \( \epsilon \in (0, \epsilon_0) \),

\[
q^\pm(\epsilon) := \epsilon^{-1} \|Q^\pm(\epsilon)\|. \tag{5.33}
\]

We have:

**Proposition 5.4** Assume that (A1)–(A7), (M±) hold. Let \( 1/2 < s < 1 \). For any fixed \( z \in \Delta_0^\pm \), the map \( \epsilon \mapsto F_{s,\epsilon}^\pm(z) \) is weakly continuously differentiable on \((0, \epsilon_0)\) and for any \( \varphi \in \mathcal{H} \), any \( \epsilon \in (0, \epsilon_0) \),

\[
\left| \langle \varphi, \partial_\epsilon F_{s,\epsilon}^\pm(z)\varphi \rangle \right| \leq \frac{2C^\pm_1}{a^\pm_\epsilon} \|\varphi\|^2 \left( \delta_0 \omega_s(\epsilon) + 2q^\pm(\epsilon) \right)
\]
\[
+ 2 \sqrt{\frac{2|\langle \varphi, F_{s,\epsilon}^\pm(z)\varphi \rangle|}{a^\pm_\epsilon}} \|\varphi\| \omega_s(\epsilon) + \frac{4}{a^\pm_\epsilon} q^\pm(\epsilon) |\langle \varphi, F_{s,\epsilon}^\pm(z)\varphi \rangle|. \tag{5.34}
\]

where the function \( \omega_s \) is defined for \( s > 0 \) and \( \epsilon \in (0, \infty) \) by \( \omega_s(\epsilon) := 1 + (1 - s)\epsilon^{s-1} \).

**Proof.** First, note that the map \( \epsilon \mapsto W_s(\epsilon) \) is strongly continuously differentiable on the interval \((0, \epsilon_0)\) and that for any \( \epsilon \in (0, \epsilon_0) \) and any \( \varphi \in \mathcal{H} \), we have

\[
\|\partial_\epsilon W_s(\epsilon)\varphi\| \leq (1 - s)\epsilon^{s-1} \|\varphi\|. \tag{5.35}
\]

Due to Proposition 5.3, for any fixed \( z \in \Delta_0^\pm \), the map \( \epsilon \mapsto F_{s,\epsilon}^\pm(z) \) is weakly continuously differentiable on \((0, \epsilon_0)\). Fix \( z \in \Delta_0^\pm \). We have that for any \( \epsilon \in (0, \epsilon_0) \),

\[
|\langle \varphi, \partial_\epsilon F_{s,\epsilon}^\pm(z)\varphi \rangle| \leq |\langle \partial_\epsilon W_s(\epsilon)\varphi, G^\pm_\epsilon(z)W_s(\epsilon)\varphi \rangle + (W_s(\epsilon)\varphi, G^\pm_\epsilon(z)\partial_\epsilon W_s(\epsilon)\varphi) |
\]
\[
+ |\langle W_s(\epsilon)\varphi, \partial_\epsilon (G^\pm_\epsilon(z))W_s(\epsilon)\varphi \rangle|. \tag{5.36}
\]

Using consecutively Cauchy Schwarz inequality, Proposition 5.2 and estimate (5.35), the first term on the r.h.s. of (5.36) is bounded by

\[
\|\partial_\epsilon W_s(\epsilon)\varphi\| \left( \|G^\pm_\epsilon(z)^*W_s(\epsilon)\varphi\| + \|G^\pm_\epsilon(z)W_s(\epsilon)\varphi\| \right)
\]
\[
\leq 2(1 - s)\epsilon^{s-1} \left( \sqrt{\frac{2C^\pm_1}{a^\pm_\epsilon}} \|\varphi\| + \sqrt{\frac{2|\langle \varphi, F_{s,\epsilon}^\pm(z)\varphi \rangle|}{a^\pm_\epsilon}} \|\varphi\| \right).
\]
Proposition 5.6

Assume that (A1)–(A7), (Minequality, the r.h.s. of (5.36) is bounded by
\[ |\langle W_s(e)\varphi, \text{ad}_A(G_s^\pm(z))W_s(e)\varphi\rangle| + |\langle G_s^\pm(z)^*W_s(e)\varphi, Q_s^\pm(e)G_s^\pm(z)W_s(e)\varphi\rangle| \leq ||\varphi|| (||G_s^\pm(z)W_s(e)\varphi|| + ||G_s^\pm(z)^*W_s(e)\varphi|| + eq^\pm(e)||G_s^\pm(z)^*W_s(e)\varphi||G_s^\pm(z)W_s(e)\varphi||. \]

In view of Proposition 5.2, the r.h.s. of this last inequality is bounded itself by
\[ 2\|\varphi\| \left( \sqrt{\frac{2C_1^\pm}{a_\pm}} \|\varphi\| + \frac{2\|\varphi, F_s^\pm(z)\varphi\|}{ca_\pm} \right) + 4eq^\pm(e) \left( \frac{C_1^\pm}{a_\pm} \|\varphi\|^2 + \frac{||\varphi, F_s^\pm(z)\varphi||}{ca_\pm} \right). \]

From (5.23), we deduce that
\[ \frac{a_\pm}{C_1^\pm} \leq \frac{\epsilon_0^2}{3}, \]
which allows to obtain (5.34).

Note that if \( s = 1 \), stronger conclusions hold:

**Proposition 5.5**

Assume that (A1)–(A7), (Mz) hold. For any fixed \( z \in \Delta_0^\pm \), the map \( \epsilon \mapsto F_{1,\epsilon}^\pm(z) \) is continuously differentiable on \((0, \epsilon_0)\) w.r.t. the operator norm topology, and for any \( \epsilon \in (0, \epsilon_0) \),
\[ \|\partial_\epsilon F_{1,\epsilon}^\pm(z)\| \leq 2\frac{C_1^\pm}{a_\pm} \left( \delta_0 + 2eq^\pm(e) \right) + 2\sqrt{\frac{2\|F_{1,\epsilon}^\pm(z)\|}{ca_\pm}} + \frac{4}{a_\pm} q^\pm(e) \|F_{1,\epsilon}^\pm(z)\|. \]

**Proof.** Due to Proposition 5.3, for any fixed \( z \in \Delta_0^\pm \), the map \( \epsilon \mapsto F_{1,\epsilon}^\pm(z) \) is continuously differentiable on \((0, \epsilon_0)\) w.r.t. the operator norm topology. Fix \( z \in \Delta_0^\pm \). We have that for any \( \epsilon \in (0, \epsilon_0) \),
\[ \|\partial_\epsilon F_{1,\epsilon}^\pm(z)\| = \|(\|A| + 1)^{-1} (\partial_\epsilon G_s^\pm(z))(\|A| + 1)^{-1}\| \]

Note that \( \|A(\|A| + 1)^{-1}\| = \|(\|A| + 1)^{-1}A\| \leq 1 \). Using again Proposition 5.3, we deduce that
\[ \|\partial_\epsilon F_{1,\epsilon}^\pm(z)\| \leq \|(\|A| + 1)^{-1}\text{ad}_A(G_s^\pm(z))(\|A| + 1)^{-1}\| + \|(\|A| + 1)^{-1}G_s^\pm(z)Q_s^\pm(e)G_s^\pm(z)(\|A| + 1)^{-1}\| \]
\[ \leq ||G_s^\pm(z)(\|A| + 1)^{-1}\| + ||(\|A| + 1)^{-1}G_s^\pm(z)|| + eq^\pm(e)||G_s^\pm(z)(\|A| + 1)^{-1}\||G_s^\pm(z)||G_s^\pm(z)(\|A| + 1)^{-1}||. \]

In view of Corollary 5.4 (with \( s = 1 \)), the r.h.s. of the last inequality can be bounded by
\[ 2 \left( \sqrt{\frac{2C_1^\pm}{a_\pm}} + \sqrt{\frac{2\|F_{1,\epsilon}^\pm(z)\|}{ca_\pm}} \right) + 4eq^\pm(e) \left( \frac{C_1^\pm}{a_\pm} + \frac{\|F_{1,\epsilon}^\pm(z)\|}{ca_\pm} \right). \]

From (5.38), we deduce (5.39).

The next result follows from Propositions 5.4, 5.5 and Gronwall lemma:

**Proposition 5.6**

Assume that (A1)–(A7), (Mz) hold. Fix \( s \in (1/2, 1) \). If the function \( q^\pm \) belongs to \( L^1(0, \epsilon_0) \), then for any \( 0 < \epsilon_1 < \epsilon_0 \),
\[ \sup_{\epsilon \in (0, \epsilon_1), z \in \Delta_0^\pm} \|F_{s,\epsilon}^\pm(z)\| < \infty. \]
Proof. Assume \( q^\pm \in L^1(0,\epsilon_0) \). If \( s = 1 \), we deduce directly (5.6) from Proposition 5.5 and Gronwall Lemma, as stated in e.g. [11] or [1] Lemma 7.A.1, i.e.

\[
\sup_{\epsilon \in (0,\epsilon_1), z \in \Delta_0^\pm} \| F^\pm_{1,\epsilon}(z) \| \leq \frac{4}{a^\pm} \left( C_0^\pm \epsilon_1^{-1} + C_1^\pm (\delta_0 + 2C_q) + \left( \int_0^1 \frac{d\mu}{\sqrt{\mu}} \right)^2 \right)^{\frac{4C_q}{\epsilon^{1/2}}} e^{\frac{4C_q}{\epsilon^{1/2}}} < \infty,
\]

where \( C_q = \| q^\pm \|_{L^1(0,\epsilon_0)} \). If \( 1/2 < s < 1 \), an extra step is required. For any \( \epsilon \in (0, \epsilon_1) \), we have

\[
|\langle \varphi, F^\pm_{s,\epsilon}(z) \varphi \rangle| \leq \| \varphi \|_2^2 + 2 \int_\epsilon^1 \omega_s(\mu) \sqrt{\delta_0 + 2C_q} \| \varphi \|_2 \left( \int_0^1 \frac{\omega_s(\mu)}{\sqrt{\mu}} \, d\mu \right) d\mu.
\]

Using again Gronwall Lemma, we deduce that

\[
\sup_{\epsilon \in (0,\epsilon_1), z \in \Delta_0^\pm} |\langle \varphi, F^\pm_{s,\epsilon}(z) \varphi \rangle| \leq \frac{4}{a^\pm} \left( C_0^\pm \epsilon_1^{-1} + C_1^\pm (\delta_0 + 2C_q) + \left( \int_0^1 \frac{\omega_s(\mu)}{\sqrt{\mu}} \, d\mu \right)^2 \right)^{\frac{4C_q}{\epsilon^{1/2}}} e^{\frac{4C_q}{\epsilon^{1/2}}} < \infty,
\]

with \( \int_0^1 \omega_s(\mu) \frac{d\mu}{\sqrt{\mu}} = \frac{2s}{2s-1} \).

Since \( \| F^\pm_{s,\epsilon}(z) \| = \sup_{\| \varphi \| = \| \psi \| = 1} |\langle \varphi, F^\pm_{s,\epsilon}(z) \psi \rangle| \), Proposition 5.6 follows by polarisation. \( \square \)

**Proposition 5.7** Assume that (A1)–(A7), (M\pm) hold. Fix \( s \in (1/2, 1) \). If the function \( q^\pm \) belongs to \( L^1(0,\epsilon_0) \), then

\[
\sup_{z \in \Delta_0^\pm} \| (|A| + 1)^{-s}(z - H)^{-1}(|A| + 1)^{-1} \| < \infty.
\]

**Proof.** Assume first that \( s = 1 \) and fix \( z \in \Delta_0^\pm \). Due to Corollary 5.3, for any \( \epsilon \in (0, \epsilon_0) \),

\[
\| (|A| + 1)^{-1}(z - H)^{-1}(|A| + 1)^{-1} - F^\pm_{1,\epsilon}(z) \| \leq \frac{C_RC_0^\pm \epsilon}{\text{Im}(z) - \sigma^\pm}.
\]

In other words, for any fixed \( z \in \Delta_0^\pm \), \( \lim_{\epsilon \to 0^+} F^\pm_{1,\epsilon}(z) = (|A| + 1)^{-1}(z - H)^{-1}(|A| + 1)^{-1} \) in the operator norm topology. By considering Proposition 5.6 and taking the limit when \( \epsilon \) tends to zero, we deduce that

\[
\sup_{z \in \Delta_0^\pm} \| (|A| + 1)^{-1}(z - H)^{-1}(|A| + 1)^{-1} \| < \infty.
\]

Assume now that \( 1/2 < s < 1 \). Pick two vectors \( \varphi, \psi \) in \( \mathcal{H} \) and fix again \( z \in \Delta_0^\pm \). For any \( \epsilon \in (0, \epsilon_0) \),

\[
\langle \varphi, (|A| + 1)^{-s}(z - H)^{-1}(|A| + 1)^{-s} \psi \rangle - \langle \varphi, F^\pm_{1,\epsilon}(z) \psi \rangle = \langle W_s(\epsilon)\varphi, ((z - H)^{-1} - G^\pm(z))W_s(\epsilon)\psi \rangle + \langle W_s(0)\varphi, (z - H)^{-1}(W_s(0) - W_s(\epsilon))\psi \rangle + \langle (W_s(0) - W_s(\epsilon))\varphi, (z - H)^{-1}W_s(\epsilon)\psi \rangle.
\]

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Using the facts that \( \|W_s(\epsilon)\| \leq 1 \) for any \( \epsilon \in [0,1] \), \( \|(z - H)^{-1}\| \leq 1/\text{dist}(z, \mathcal{N}(H)) \leq |\text{Im}(z) - \sigma_{\pm}|^{-1} \) and Corollary 5.3, we deduce that

\[
\langle \varphi, (|A| + 1)^{-s}(z - H)^{-1}(|A| + 1)^{-s} \psi \rangle - \langle \varphi, F_{1,\epsilon}^{\pm}(z) \psi \rangle \leq \frac{C_B C_0^+ \epsilon}{|\text{Im}(z) - \sigma_{\pm}|^2} \|\varphi\|\|\psi\| + \frac{1}{|\text{Im}(z) - \sigma_{\pm}|} \|(W_s(0) - W_s(\epsilon))\varphi\|\|\psi\| + \frac{1}{|\text{Im}(z) - \sigma_{\pm}|} \|\varphi\||(W_s(0) - W_s(\epsilon))\psi||.
\]

In other words, for any fixed \( z \in \Delta_0^\pm \), \( w = \lim_{\epsilon \to 0^+} F_{1,\epsilon}^{\pm}(z) = (|A| + 1)^{-s}(z - H)^{-1}(|A| + 1)^{-s} \). Considering Proposition 5.6, taking the limit when \( \epsilon \) vanishes yields and finally taking supremum over vectors \( \varphi \) and \( \psi \) of norm one, gives

\[
\sup_{z \in \epsilon \Delta_0^\pm} \|(|A| + 1)^{-s}(z - H)^{-1}(|A| + 1)^{-s}\| < \infty.
\]

\[\square\]

In order to conclude about the proof of Proposition 5.1, it remains to show that if \( H \in C^{1,1}(A) \), then it satisfies the hypotheses (A1)–(A6) (hence (A7)) and the integrability of the functions \( q^\pm \). This is the purpose of the next section.

### 5.5 Last step to the proof of Proposition 5.1

The next result is quoted from [1, Lemma 7.3.6].

**Lemma 5.8** Let \( S \in \mathcal{B}(\mathcal{H}) \) be a bounded selfadjoint operator of class \( C^{1,1}(A) \). Then, there exists a family \( (S(\epsilon))_{\epsilon \in (0,1)} \) of bounded operators with the following properties:

- The map \( \epsilon \mapsto S(\epsilon) \) with values in \( \mathcal{B}(\mathcal{H}) \) is of class \( C^\infty \), \( \|S(\epsilon) - S\| \leq C\epsilon \) for some \( C > 0 \) and

\[
\int_0^1 \frac{\|\partial_\epsilon S(\epsilon)\|}{\epsilon} \, d\epsilon < \infty.
\]

- For any \( \epsilon \in (0,1) \), \( S(\epsilon) \in C^\infty(A) \) and

\[
\int_0^1 \|\text{ad}_{A}^2 S(\epsilon)\| \, d\epsilon < \infty.
\]

- \( S \in C^1(A) \), \( \lim_{\epsilon \to 0^+} \|\text{ad}_{A}(S(\epsilon)) - \text{ad}_{A}(S)\| = 0 \), the map \( \epsilon \mapsto \text{ad}_{A}(S(\epsilon)) \) with values in \( \mathcal{B}(\mathcal{H}) \) is of class \( C^\infty \) and

\[
\int_0^1 \|\partial_{\epsilon}\text{ad}_{A} S(\epsilon)\| \, d\epsilon < \infty.
\]

If \( H \in C^{1,1}(A) \), its adjoint \( H^* \), hence \( \text{Re}(H) \) and \( \text{Im}(H) \) also belong to \( C^{1,1}(A) \), see e.g. Lemma 5.2. Applying Lemma 5.8 to both \( \text{Re}(H) \) and \( \text{Im}(H) \) shows that the conclusions of Lemma 5.8 are still valid if \( H \) is not selfadjoint.

Given such an operator \( H \), choose the function \( S \) as given by Lemma 5.8 \((\epsilon_0 = 1)\) and define the function \( B \) by \( B = \text{id}_{A}(S) \). The associated functions \( Q^\pm \) are defined on \((0,1)\) by \( Q^\pm = \partial_{\epsilon}S \mp i\partial_{\epsilon}B + i\text{id}_{A}(B) \). By Lemma 5.8, Assumptions (A1)–(A6) are verified and \( q^\pm \) belong to \( L^1((0,1)) \).

Note that if \( s \geq 1 \),

\[
\sup_{z \in \epsilon \Delta_0^\pm} \|(|A| + 1)^{-s}(z - H)^{-1}(|A| + 1)^{-s}\| \leq \sup_{z \in \epsilon \Delta_0^\pm} \|(|A| + 1)^{-1}(z - H)^{-1}(|A| + 1)^{-1}\|.
\]

The proof of Theorem 5.1 follows now directly from Remarks 5.3, 5.4 and Proposition 5.7.
5.6 Proof of Corollary 5.1

**Lemma 5.9** Let $H \in B(\mathcal{H})$, $\chi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ and $A$ be a selfadjoint operator acting on $\mathcal{H}$. If $\text{Re}(H) \in C^1(A)$, then, for all $s \in [-1, 1]$, the operator $(A)^s\chi(\text{Re}(H))(A)^{-s}$ extends as bounded operator on $\mathcal{H}$.

See e.g. [31, Lemma 6.4] for a proof.

Let us prove Corollary 5.1. Without any loss of generality, we suppose that $1/2 < s \leq 1$. First, note that for any $z \in \mathbb{R} + i((-\infty, \sigma_-, 1) \cup (\sigma_+ + 1, \infty))$, $\|(z-H)^{-1}\| \leq 1/\text{dist}(z, \mathcal{N}(H)) \leq 1$. Hence,

$$\sup_{z \in \mathbb{R} + i((-\infty, \sigma_-, 1) \cup (\sigma_+ + 1, \infty))} \|(A)^{-s}\chi(\text{Re}(H))(z-H)^{-1}\chi(\text{Re}(H))(A)^{-s}\| < \infty.$$ 

So, by combining Theorem 5.1 and Lemma 5.9, we get

$$\sup_{z \in \mathbb{R} \setminus \Delta_1 + i(\sigma_+ \cup (\sigma_+ + 1))} \|(A)^{-s}\chi(\text{Re}(H))(z-H)^{-1}\chi(\text{Re}(H))(A)^{-s}\| < \infty,$$

Finally, for $z \in (\mathbb{R} \setminus \Delta_1) + i[(-\infty, \sigma_-) \cup (\sigma_+, \sigma_+ + 1)]$ and any $(\varphi, \psi) \in \mathcal{H} \times \mathcal{H}$, the resolvent identity yields

$$\|(\chi(\text{Re}(H))(A)^{-s}\varphi, (z-H)^{-1}\chi(\text{Re}(H))(A)^{-s}\psi)\| \leq \|(\chi(\text{Re}(H))(A)^{-s}\varphi, (\text{Re}(A_0(z)))^{-1}\chi(\text{Re}(H))(A)^{-s}\psi)\| + \|(\text{Re}(A_0(z))^{-1}\chi(\text{Re}(H))(A)^{-s}\varphi)\| \|(\text{Im}(A_0(z))(z-H)^{-1}\chi(\text{Re}(H))(A)^{-s}\psi)\|,$$

(5.41)

where $C_\chi = \sup_{\lambda \in \mathbb{R} \setminus \Delta_1} \|(\lambda - \text{Re}(H))^{-1}\chi(\text{Re}(H))\| < \infty$. Due to (5.10), for any $\psi \in \mathcal{H}$ and any $z \in \mathbb{C}$ such that $\text{Im}(z) \in [\sigma_- - 1, \sigma_-) \cup (\sigma_+ + 1, \sigma_+ + 1]$, we have

$$\|\text{Im}(A_0(z))\| \leq \sigma_0(\langle \psi, A_0(z) \psi \rangle),$$

whence

$$\|\text{Im}(A_0(z))(z-H)^{-1}\chi(\text{Re}(H))(A)^{-s}\psi\| \leq \sigma_0(\langle A \rangle^{s} \chi(\text{Re}(H))(z - H)^{-1}\chi(\text{Re}(H))(A)^{-s}\psi, \psi \rangle).$$

From inequality (5.41), we deduce that

$$\|(\chi(\text{Re}(H))(A)^{-s}\varphi, (z-H)^{-1}\chi(\text{Re}(H))(A)^{-s}\psi)\| \leq C_\chi \|\chi\|_\infty \|\varphi\| \|\psi\| + C_\chi \|\varphi\| \sqrt{\sigma_0} \sqrt{(\langle A \rangle^{s} \chi(\text{Re}(H))(z - H)^{-1}\chi(\text{Re}(H))(A)^{-s}\psi, \psi \rangle)}.$$

This entails that

$$\sup_{z \in (\mathbb{R} \setminus \Delta_1) + i[\sigma_-, \sigma_-]} \|(A)^{-s}\chi(\text{Re}(H))(z-H)^{-1}\chi(\text{Re}(H))(A)^{-s}\| < \infty,$$

$$\sup_{z \in (\mathbb{R} \setminus \Delta_1) + i[\sigma_+, \sigma_+ + 1]} \|(A)^{-s}\chi(\text{Re}(H))(z-H)^{-1}\chi(\text{Re}(H))(A)^{-s}\| < \infty,$$

which concludes the proof.

5.7 Proof of Corollary 5.2

Since $H \in C^1(A)$, $\text{Re}(H) \in C^1(A)$. The first statement paraphrases Lemma 5.1. Given any relatively compact interval $\Delta_0$, $\overline{\Delta_0} \subset \Delta \setminus \mathcal{E}_p(\text{Re}(H))$, $(M \pm) \text{ hold both on } \overline{\Delta_0}$, with $\beta_\pm = 0$. We refer to e.g. [1, Paragraph 7.2.2] for details. The second statement follows then from Theorem 5.1.
5.8 Proof of Theorems 2.6 and 2.7

Following [2, Section 5], \( H_0 = L \in C^\infty(A_0) \). Actually,
\[
\text{iad}_{A_0}(H_0) = 4H_0 - H_0^2.
\]

\textbf{Proof of Theorem 2.6.} If \( V \) (hence \( \text{Re}(V) \)) belongs to \( C^1(A_0) \) (resp. \( C^{1,1}(A_0) \)), then \( H_V \) (hence \( \text{Re}(H_V) \)) belongs to \( C^1(A_0) \) (resp. \( C^{1,1}(A_0) \)). In addition, if \( V \) (hence \( \text{Re}(V) \)) and \( \text{iad}_{A_0}(\text{Re}(V)) \) belong to \( S_\infty(\ell^2(\mathbb{Z})) \), then (M) holds with \( c_\Delta > 0 \) if \( \overline{\Delta} \subset (0,4) \). The conclusion follows as a particular case of Corollary 5.2.

\textbf{Proof of Theorem 2.7.} If \( V \) belongs to \( C^{1,1}(A_0) \), then \( H_V \) belongs to \( C^{1,1}(A_0) \). The conclusion follows directly from Theorem 5.1.

6 Regularity classes

In this section, \( \mathcal{H} \) is a Hilbert space and \( A \) denotes a selfadjoint operator acting on \( \mathcal{H} \), with domain \( D(A) \). We sum up the main properties of the regularity classes considered in this paper. We also propose some explicit criteria in the case \( \mathcal{H} = \ell^2(\mathbb{Z}) \) and \( A = A_0 \) defined by (2.8).

6.1 The \( \mathcal{A}(A_0) \) class

We refer to [25, Chapter III] for general considerations on bounded operator-valued analytic maps.

6.1.1 General considerations

The classes \( \mathcal{A}(A) \) and \( \mathcal{A}_R(A), R > 0 \), have been introduced in Definition 2.1. For \( L \in \mathcal{A}_R(A) \), we denote the corresponding analytic map by
\[
a[L] : D_\Delta(0) \to \mathcal{B}(\mathcal{H})
\]
\[
\theta \mapsto e^{i\theta A} B e^{-i\theta A}.
\]

The next propositions follow from direct calculations:

\textbf{Proposition 6.1} Let \( R > 0 \), \( (L_1, L_2) \in \mathcal{A}_R(A) \times \mathcal{A}_R(A) \) and \( \alpha \in \mathbb{C} \). Then,
\begin{itemize}
  \item \( \alpha L_1 + L_2 \in \mathcal{A}_R(A) \) and \( a[\alpha L_1 + L_2] = \alpha a[L_1] + a[L_2] \),
  \item \( L_1 L_2 \in \mathcal{A}_R(A) \) and \( a[L_1 L_2] = a[L_1] a[L_2] \),
  \item \( I \in \mathcal{A}_R(A) \) and \( a[I] \equiv I \),
  \item \( L_1^* \in \mathcal{A}_R(A) \) and \( a[L_1^*](\theta) = (a[L_1](\bar{\theta}))^* \), for any \( \theta \in D_\Delta(0) \).
\end{itemize}

In particular, \( \mathcal{A}(A) \) is a sub-algebra of \( \mathcal{B}(\mathcal{H}) \).

\textbf{Proposition 6.2} Let \( R > 0 \) and \( \mathcal{H} \) be an auxiliary Hilbert space. Let \( U : \mathcal{H} \to \mathcal{H} \) be a unitary operator. Then, \( L \in \mathcal{A}_R(A) \) if and only if \( U L U^* \in \mathcal{A}_R(UAU^*) \). In addition, for all \( \theta \in D_\Delta(0) \),
\[
(U(a[L]) U^*)(\theta) = (a(U L U^*))(\theta).
\]

Next, we turn to some characterizations of the classes \( \mathcal{A}_R(A) \). Let us recall the following lemma:

\textbf{Lemma 6.1} Let \( \Omega \subseteq \mathbb{C} \) be an open subset and \( (F_n)_{n \in \mathbb{N}} \subset \text{Hol}(\Omega, \mathcal{B}(\mathcal{H})) \). Assume that \( (F_n) \) converges uniformly to a function \( F_\infty \) in any compact subset included in \( \Omega \). Then, \( F_\infty \in \text{Hol}(\Omega, \mathcal{B}(\mathcal{H})) \).
Proof. Let $D$ be any open disk such $\overline{D} \subset \Omega$, and $T$ be a triangle in $D$. Then, Goursat’s theorem implies that for any $n \in \mathbb{N}$, we have

$$\int_T F_n(z)dz = 0.$$ 

Since $(F_n)$ converges uniformly to $F_\infty$ in $\overline{D}$, then $F_\infty$ is continuous in $\overline{D}$ together with

$$\int_T F_n(z)dz \rightarrow \int_T F_\infty(z)dz = 0.$$ 

Thus, Morera’s theorem implies that $F_\infty \in \text{Hol}(D, \mathcal{B}(\mathcal{H}))$, and the claim follows since $D$ is arbitrary.

Proposition 6.3 Let $\Sigma$ be a countable set and $(L_\nu)_{\nu \in \Sigma} \subset A_R(A)$ for some $R > 0$. Assume that there exist $0 < R' < R$, and a family of finite subsets $(\Sigma_n)_{n \in \mathbb{N}}$ with

$$\Sigma_0 \subset \cdots \subset \Sigma_n \subset \Sigma_{n+1} \subset \cdots \subset \Sigma, \quad \bigcup_{n \in \mathbb{N}} \Sigma_n = \Sigma,$$

such that

$$\sum_{n \in \mathbb{N}} \sum_{\nu \in \Sigma_n \setminus \Sigma_{n-1}} \sup_{\theta \in D_R'(0)} \|e^{i\theta A} L_\nu e^{-i\theta A}\| < \infty. \quad (6.1)$$

Then, the operator $\sum_{\nu \in \Sigma} L_\nu \in A_R(A)$.

Proof. By hypotheses, for any $\nu \in \Sigma$, the map $a[L_\nu] : D_{R'}(0) \rightarrow \mathcal{B}(\mathcal{H})$ belongs to $\text{Hol}(D_{R'}(0), \mathcal{B}(\mathcal{H}))$.

Then, so are the maps

$$a \left[ \sum_{\nu \in \Sigma_n} L_\nu \right] : D_{R'}(0) \rightarrow \mathcal{B}(\mathcal{H}), \quad \theta \mapsto \sum_{\nu \in \Sigma_n} e^{i\theta A} L_\nu e^{-i\theta A},$$

for any $n \in \mathbb{N}$. Now, hypothesis (6.1) implies that $(a[\sum_{\nu \in \Sigma_n} L_\nu])_{n \in \mathbb{N}}$ converges uniformly to $a[\sum_{\nu \in \Sigma} L_\nu]$ on $D_{R'}(0)$, hence on any compact subset included in $D_{R'}(0)$. The claim follows from Lemma 6.1.

A first application of Proposition 6.3 is the following result:

Theorem 6.1 Let $\Sigma = \mathbb{N}$ or $\mathbb{Z}$. Let $(\varphi_n)_{n \in \Sigma}$ and $(\psi_n)_{n \in \Sigma}$ be two sequences of analytic vectors for $A$ such that for any $n \in \Sigma$ the series

$$\sum_{k=0}^\infty \frac{|\theta|^k}{k!} \|A^k \varphi_n\|, \quad \sum_{k=0}^\infty \frac{|\theta|^k}{k!} \|A^k \psi_n\|, \quad n \in \Sigma,$$

converge on $D_R(0)$ for some $R > 0$. Let $(\alpha_n)_{n \in \Sigma}$ be a complex sequence.

1. Assume that there exists $0 < R' < R$ such that

$$\sum_{n \in \Sigma} |\alpha_n| \sup_{\theta \in D_{R'}(0)} \|e^{i\theta A} \varphi_n\| \|e^{i\theta A} \psi_n\| < \infty. \quad (6.2)$$

Then, the operator $\sum_{n \in \Sigma} \alpha_n |\varphi_n\rangle \langle \psi_n|$ belongs to $A_R'(A)$ with extension given by

$$a \left[ \sum_{n \in \Sigma} \alpha_n |\varphi_n\rangle \langle \psi_n| \right](\theta) = \sum_{n \in \Sigma} \alpha_n e^{i\theta A} |\varphi_n\rangle \langle \psi_n| e^{-i\theta A},$$

for all $\theta \in D_{R'}(\theta)$. 36
2. Assume that there exists $0 < R' < R$ such that
\[
\sum_{n \in \Sigma} |\alpha_n| \sup_{\theta \in D_{R'}(0)} \|e^{i\theta A} \psi_n\| < \infty. \tag{6.3}
\]

Then, for any fixed $m \in \Sigma$, the operator $\sum_{n \in \Sigma} \alpha_n \langle \varphi_m, \psi_n \rangle$ belongs to $A_{R'}(A)$, with extension given by
\[
a \left[ \sum_{n \in \Sigma} \alpha_n \langle \varphi_m, \psi_n \rangle \right](\theta) = \sum_{n \in \Sigma} \alpha_n e^{i\theta A} \langle \varphi_m, \psi_n \rangle e^{-i\theta A},
\]
for all $\theta \in D_{R'}(0)$.

Proof. The analyticity properties of the sequences $(\varphi_n)$ and $(\psi_n)$ reads: for any $(n, m) \in \Sigma \times \Sigma$, the operator $|\varphi_n\rangle \langle \psi_m| \in A_R(A)$ and the associated holomorphic map satisfies
\[
a[|\varphi_n\rangle \langle \psi_m|](\theta) = e^{i\theta A} |\varphi_n\rangle \langle \psi_m| e^{-i\theta A},
\]
for any $\theta \in D_{R}(0)$. In the sequel, we only give the proof of the first statement. Since
\[
\|e^{i\theta A} |\varphi_n\rangle \langle \psi_m| e^{-i\theta A}\| = \|e^{i\theta A} |\varphi_n\rangle \langle e^{i\theta A} \psi_m|\| = \|e^{i\theta A} \varphi_n\| \|e^{i\theta A} \psi_m\|,
\]
for all $\theta \in D_{R}(0)$, it follows from (6.2) that
\[
\sum_{n \in \Sigma} |\alpha_n| \sup_{\theta \in D_{R'}(0)} \|e^{i\theta A} |\varphi_n\rangle \langle \psi_m| e^{-i\theta A}\| < \infty.
\]

Now, let us introduce the family of finite subsets $(\Sigma_n)_{n \in \mathbb{N}}$ defined by $\Sigma_n = \{ k \in \Sigma : |k| \leq n \}$. Thus, we have
\[
\sum_{n \in \Sigma} |\alpha_n| \sup_{\theta \in D_{R'}(0)} \|e^{i\theta A} |\varphi_n\rangle \langle \psi_m| e^{-i\theta A}\| = \sum_{n \in \mathbb{N}} \sum_{k \in \Sigma_n} |\alpha_k| \sup_{\theta \in D_{R'}(0)} \|e^{i\theta A} |\varphi_k\rangle \langle \psi_m| e^{-i\theta A}\|,
\]
and the claim follows from Proposition 6.3. \hfill \Box

\subsection{6.1.2 Applications}

In this paragraph, we consider the case $\mathcal{H} = \ell^2(\mathbb{Z})$ and $A = A_0$. We recall that $(e_n)_{n \in \mathbb{Z}}$ denotes the canonical orthonormal basis of $\ell^2(\mathbb{Z})$.

First, we state the following result:

Lemma 6.2 For any $n \in \mathbb{Z}$, the series $\sum_{k=0}^{\infty} |\frac{k^{|n|}}{k!}| A_k^0 e_n$ converge on $D_\frac{1}{2}(0)$. In particular, $e_n$, $n \in \mathbb{Z}$, is an analytic vector for $A_0$.

Proof. First, note that for any $n \in \mathbb{Z}$, $A_0 e_n = -i n (e_{n+1} - e_{n-1})$. We deduce that for any $k \in \mathbb{Z}_+$, $n \in \mathbb{Z}^*$, we have
\[
\|A_k^0 e_n\| \leq 2^k \frac{(|n| + k)!}{(|n| - 1)!},
\]
The conclusion follows. \hfill \Box

In the sequel, let $\varphi_\theta$ (resp. $J(\varphi_\theta)$) denote the analytic extension w.r.t. $\theta \in D_{R_0}(0)$, $R_0 \leq \frac{1}{2}$, of the flow $\varphi_\theta$ (resp. the Jacobian $J(\varphi_\theta)$) defined by (3.1). Note that by [34, Lemma 3.3], such extensions exist and furthermore $R_0$ can be chosen such that $\sqrt{J(\varphi_\theta)}$ is analytic in $D_{R_0}(0)$.

Lemma 6.3 Let $0 < R < R_0$. Then, there exists a constant $C > 0$ such that for any $n \in \mathbb{Z}$,
\[
\sup_{\theta \in D_{R}(0)} \|e^{i\theta A_0} e_n\| \leq C e^{\frac{|n|}{\sup_{(a, V) \in D_{R_0}(0) \setminus D_{R}(0)} |\text{Im}(\varphi_\theta)|}}. \tag{6.4}
\]
show that there exists

We only prove the first point. According to Lemma 6.2 and Theorem 6.1, it suffices to

Proof.

n

Let

then the claim follows.

Moreover, since for \( \theta \in D_R(0) \) fixed we have

\[
\| e^{i\theta A_0} e^{in} \| = \| e^{i\theta A_0} e^{in} \| \leq \sup_{\vartheta \in \Gamma} \sqrt{|J(\varphi_{\theta})(\vartheta)|} \| e^{in} \| \sup_{\vartheta \in \Gamma} |\text{Im}(\varphi_{\theta}(\vartheta))|,
\]

then the claim follows. \( \square \)

**Proposition 6.4** Let \( \Sigma = \mathbb{N} \) or \( \mathbb{Z} \), and \( (\alpha_n)_{n \in \Sigma} \) be a complex sequence satisfying \( |\alpha_n| \leq \gamma e^{-\delta|n|} \), \( n \in \Sigma \), for some constants \( \gamma, \delta > 0 \). Then, there exists \( R_\delta > 0 \) such that:

(a) The operator \( \sum_{n \in \Sigma} \alpha_n |e_n| e_n \) belongs to \( \mathcal{A}_{R_\delta}(A_0) \) with extension given by

\[
\theta \mapsto \sum_{n \in \Sigma} \alpha_n e^{i\theta A_0} |e_n| e^{i\theta A_0}.
\]

(b) For any \( m \in \Sigma \) fixed, the operator \( \sum_{n \in \Sigma} \alpha_n |e_m| e_n \) belongs to \( \mathcal{A}_{R_\delta}(A_0) \) with extension given by

\[
\theta \mapsto \sum_{n \in \Sigma} \alpha_n e^{i\theta A_0} |e_m| e^{i\theta A_0}.
\]

**Proof.** We only prove the first point. According to Lemma 6.2 and Theorem 6.1, it suffices to show that there exists \( 0 < R_\delta < \frac{1}{2} \) such that

\[
\sum_{n \in \Sigma} |\alpha_n| \sup_{\theta \in D_{R_\delta}(0)} \| e^{i\theta A_0} e_n \| \| e^{i\theta A_0} e_n \| < \infty. \tag{6.5}
\]

Let \( R_0 \) be a constant as in Lemma 6.3, and note that for any constant \( \Gamma > 0 \), there exists \( 0 < R_\Gamma < R_0 \) such that we have

\[
\sup_{(\theta, \vartheta) \in D_{R_\gamma}(0) \times \Gamma} |\text{Im}(\varphi_{\theta}(\vartheta))| < \Gamma. \tag{6.6}
\]

In particular, by taking \( \Gamma = \frac{\delta}{4} \) in (6.6) and by using Lemma 6.3, it follows that there exists \( 0 < R_\delta < R_0 \) such that

\[
\sup_{\theta \in D_{R_\delta}(0)} \| e^{i\theta A_0} e_n \| \leq C e^{\frac{\delta}{4}|n|}, \quad C > 0. \tag{6.7}
\]

This together with \( |\alpha_n| \leq \gamma e^{-\delta|n|} \), \( n \in \Sigma \), \( \gamma > 0 \), imply (6.5), and the claim follows. \( \square \)

**Proposition 6.5** Let \( V \) satisfy Assumption 2.1. Then, there exists \( R_\delta > 0 \) such that \( V \) belongs to \( \mathcal{A}_{R_\delta}(A_0) \) with extension given by

\[
\theta \mapsto \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} V(n, m) e^{i\theta A_0} |e_m| e_n |e^{i\theta A_0}.
\]

**Proof.** Canonically, the operator \( V \) can be written as

\[
V = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} V(n, m) |e_m| e_n = \sum_{m \in \mathbb{Z}} L_m,
\]

38
where \( L_m := \sum_{n \in \mathbb{Z}} V(n, m) |e_m \rangle \langle e_n| \). Since for any \( m \in \mathbb{Z} \) fixed \( |V(n, m)| \leq \text{Const.} e^{-\delta|m|} e^{-\delta|n|} \), then, it follows from Proposition 6.4 (ii) that \( L_m \in \mathcal{A}_{R}\delta(A_0) \) for some \( R\delta > 0 \), with extension given by

\[
\theta \mapsto \sum_{m \in \mathbb{Z}} V(n, m) e^{i\theta A_0} |e_m \rangle \langle e_n| e^{-i\theta A_0}.
\]

Now, let \( \Sigma_n, n \in \mathbb{N} \), be the finite subsets defined by \( \Sigma_n = \{ k \in \mathbb{Z} : |k| \leq n \} \). We have

\[
\sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{Z} \setminus \Sigma_{n-1}} \sup_{\theta \in D_{R\delta}(0)} \| e^{i\theta A_0} L_m e^{-i\theta A_0} \| = \sum_{m \in \mathbb{Z}} \sup_{\theta \in D_{R\delta}(0)} \| e^{i\theta A_0} L_m e^{-i\theta A_0} \|
\]

\[
\leq \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |V(n, m)| \sup_{\theta \in D_{R\delta}(0)} \| e^{i\theta A_0} |e_m \rangle \langle e_n| e^{-i\theta A_0} \|
\]

\[
= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |V(n, m)| \sup_{\theta \in D_{R\delta}(0)} \| e^{i\theta A_0} e_m \| \| e^{-i\theta A_0} e_n \| < \infty,
\]

since \( R\delta \) is the constant appearing in the proof of Proposition 6.4, so that (6.7) holds. Thus, the claim follows from Proposition 6.3.

\[\Box\]

### 6.2 The \( C^k(A) \) and \( C^{s,p}(A) \) classes

The classes \( C^k(A) \) and \( C^{1,1}(A) \), \( k \in \mathbb{N} \), have been introduced in Definitions 2.4 and 2.5. Let us consider also the following definition:

**Definition 6.1** Let \( \mathcal{H} \) be a Hilbert space and \( A \) be a selfadjoint operator defined on \( \mathcal{H} \). An operator \( B \in \mathcal{B}(\mathcal{H}) \) belongs to \( C^{0,1}(A) \) if

\[
\int_0^1 \| e^{i\theta A} B e^{-i\theta A} - B \| \frac{d\theta}{\theta} < \infty.
\]

We recall that these classes are linear subspaces of \( \mathcal{B}(\mathcal{H}) \), stable under adjunction, according to [1, Chapter 5]. The results exposed in this paragraph illustrate Theorem 2.6.

We start by introducing the sequence subspaces:

\[
Q_k(\mathbb{Z}) := \left\{ x \in \mathbb{C}^\mathbb{Z} : \sum_{j=0}^k q_j(x) < \infty \right\}, \quad k \in \{0, 1, 2\},
\]

\[
c_0(\mathbb{Z}) := \left\{ x \in \mathbb{C}^\mathbb{Z} : \lim_{|n| \to \infty} x(n) = 0 \right\},
\]

where the maps \( q_k : \mathbb{C}^\mathbb{Z} \to [0, \infty], k \in \{0, 1, 2\} \) are defined by \( q_0(x) := \sup_{n \in \mathbb{Z}} |x(n)| \), \( q_1(x) := \sup_{n \in \mathbb{Z}} |n(x(n+1) - x(n))| \) and \( q_2(x) := \sup_{n \in \mathbb{Z}} |n^2(x(n+2) - 2x(n+1) + x(n))| \). Direct calculations allow to show:

**Lemma 6.4** Consider the linear operator \( V \) defined on the canonical orthonormal basis of \( l^2(\mathbb{Z}) \) by \( V e_n = v_n e_n, n \in \mathbb{Z} \). Then, denoting \( v = (v_n)_{n \in \mathbb{Z}} \), we have:

(a) if \( v \in Q_0(\mathbb{Z}) \), then \( V \) is bounded,

(b) if \( v \in c_0(\mathbb{Z}) \), then \( V \) is compact,

(c) if \( v \in Q_k(\mathbb{Z}) \) for some \( k \in \{1, 2\} \), then \( V \in C^k(A_0) \).

**Lemma 6.5** Let \( N \in \mathbb{N} \) and consider two finite families of vectors \( (\varphi_k)_{k=1}^N \) and \( (\psi_k)_{k=1}^N \) whose elements belong to \( \mathcal{D}(A_0) \) for some \( k \in \mathbb{N} \). Then, for any \( (\beta_1, \ldots, \beta_N) \in \mathbb{C}^N \),

\[
V = \sum_{k=1}^N \beta_k |\psi_k \rangle \langle \varphi_k| \in C^k(A_0).
\]
In order to capture fractional order regularities, let us denote for any $0 < a < b$

$$S_{a,b}(Z) := \left\{ \gamma \in C^2 : \int_1^{\infty} \sup_{ar \leq n \leq br} |x(n)| \, dr < \infty \right\},$$  \hspace{1cm} (6.9) $$M_{a,b}(Z) := \left\{ \gamma \in C^2 : \int_1^{\infty} \sup_{ar \leq n \leq br} |x(n + 1) - x(n)| \, dr < \infty \right\},$$

and define $S(Z) = \cup_{0 < a < b} S_{a,b}(Z)$ and $M = \cup_{0 < a < b} M_{a,b}(Z)$.

**Proposition 6.6** Consider the linear operator $V$ defined on the canonical orthonormal basis of $\ell^2(Z)$ by $V e_n = v_n e_n, n \in Z$. Then, denoting $v = (v_n)_{n \in Z}$, we have:

(a) if $v \in S(Z)$, then $V \in C^{1,1}(A_0)$,

(b) if $v \in M(Z) \cap Q_1(Z)$, then $V \in C^1(A_0)$ and $\text{ad}_{A_0}(V) \in C^{0,1}(A_0)$. In particular, $V \in C^{1,1}(A_0)$.

**Proof.** The first statement follows from [1, Theorem 7.5.8]. The second statement, which is based on the inclusions 5.2.19 in [1], was proved in [6].

Let us also consider the subset of vectors $D_{a,b}, 0 < a < b < \infty$,

$$D_{a,b} := \left\{ \xi \in \ell^2(Z) : \int_1^{\infty} \left( \sum_{n \in \mathbb{N} \cap [ar,br]} |\xi_n|^2 \right)^{1/2} \, dr < \infty \right\},$$  \hspace{1cm} (6.10) $$\text{and define } D = \cup_{0 < a < b} D_{a,b}. \text{ We have the following proposition:}$$

**Proposition 6.7** Let $N \in \mathbb{N}$ and consider two finite families of vectors $(\varphi_k)_{k=1}^N$ and $(\psi_k)_{k=1}^N$ whose elements belong to $D$. Then, for any $(\beta_1, \ldots, \beta_N) \in C^N$,\n
$$V = \sum_{k=1}^N \beta_k |\psi_k\rangle \langle \varphi_k| \in C^{1,1}(A_0).$$  \hspace{1cm} (6.11)

We refer to [2, Lemmata 3.13 and 3.14] for the proof.

**Remark 6.1** Since $C^{1,1}(A_0)$ is a linear subspace of $B(\ell^2(Z))$, stable under adjunction, which also contains $C^2(A_0)$, it also possible to combine Lemmata 6.4, 6.5 with Propositions 6.6, 6.7 to obtain more elaborated examples. To name a few, we have:

1. if $v \in S(Z) + (M(Z) \cap Q_1(Z)) + Q_2(Z)$, then the linear operator $V$ defined on the canonical orthonormal basis of $\ell^2(Z)$ by $V e_n = v_n e_n, n \in Z$ belongs to $C^{1,1}(A_0)$,

2. if $\varphi \in D(A_0^0)$ and $\psi \in D$ (or vice-versa), then for any $\beta \in C$, $V = \beta |\psi\rangle \langle \varphi|$ belongs to $C^{1,1}(A_0)$.

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