Extreme objects with arbitrary large mass, or density, and arbitrary size

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Abstract

We consider a generalization of the interior Schwarzschild solution that we match to the exterior one to build global $C^1$ models that can have arbitrary large mass, or density, with arbitrary size. This is possible because of a new insight into the problem of localizing the center of symmetry of the models and the use of principal transformations to understand the structure of space.

1 Introduction

We consider in this paper a family of spherically symmetric, static models with bounded sources. To start with we shall consider the following reduced form of the line-element:

$$ds^2 = -A^2 dt^2 + d\hat{s}^2,$$

(1)

with

$$d\hat{s}^2 = B^2 dr^2 + BCr^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2,$$

(2)

$A$, $B$, and $C$ being three functions of $r$. This form of the line-element is fully adapted to the assumption requiring the existence of a global time-like integrable Killing vector field, as well as to the assumption of spherical symmetry. It remains invariant under the adapted time coordinate transformation

$$\bar{t} = Kt + K_0,$$

(3)
where $K$ is an arbitrary positive constant which is usually chosen such that

$$\lim_{r \to \infty} A = 1,$$

(4)

and $K_0$ is a second arbitrary constant that can be completely ignored. It is also invariant under a radial coordinate transformation

$$\bar{r} = \bar{r}(r),$$

(5)

which may serve different purposes: mathematical simplicity or desired interpretation.

Among the coordinate conditions that can be used to choose $r$ one finds:

i) The historical condition

$$A^2 B^4 C^2 r^4 = 1.$$  

(6)

This is equivalent to the coordinate condition used by Schwarzschild to derive his exterior solution.

ii) The curvature condition

$$BC = 1.$$  

(7)

It was first considered for mathematical simplicity by Droste [1], Hilbert [2] and Weyl [3] and it is the most used.

iii) The isotropic condition

$$C = B.$$  

(8)

iv) The harmonic condition

$$B = \frac{1}{2rA} \left( r^2 AC \right)' ,$$

(9)

where the prime means a derivative with respect to $r$. This condition is equivalent to requiring that the three coordinates

$$x^1 = r \sin \theta \cos \varphi, \quad x^2 = r \sin \theta \sin \varphi, \quad x^3 = r \cos \theta$$  

(10)

be harmonic functions in the space-time defined by (1), i.e.,

$$\Delta x^k = \frac{1}{\sqrt{-g}} \partial_i \left( \sqrt{-g} g^{ij} \partial_j x^k \right) = 0, \quad g = \det (g_{\alpha\beta}).$$  

(11)

(Greek indices run from 0 to 3 and Latin indices from 1 to 3.) This condition which leaves open the choice of an arbitrary constant was used, with a
particular value of this constant, by Fock [4] who meant to give a particular meaning to this particular radial coordinate $r$.

v) The Gauss condition:

$$B = 1.$$  \hspace{1cm} (12)

In recent years one of us (Ll. B) has introduced the quo-harmonic condition [5]:

$$B = \frac{1}{2r} \left( r^2 C \right)' .$$  \hspace{1cm} (13)

This is equivalent to requiring that the three functions (10) be harmonic in the 3-dimensional space with line-element (4), i.e.,

$$\dot{x}^k = \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{\hat{g}} g^{ij} \partial_j x^k \right) = 0, \quad \hat{g} = \det \left( g_{ij} \right), \quad \hat{g}^{ij} = g^{ij}.$$  \hspace{1cm} (14)

This new coordinate condition has been thoroughly discussed in [6, 7]. It is intimately connected with the concept of principal transformation of a 3-dimensional Riemannian metric [8]. Although not every such metric possesses a principal transform, many do have it and in particular those with spherical symmetry. In this particular case a principal transform of (2) is a new metric with line-element

$$ds^2 = \Phi^2 B^2 dr^2 + \Psi^2 B C r^2 d\Omega^2 ,$$  \hspace{1cm} (15)

where $\Phi$ and $\Psi$ are two functions of $r$ such that

i) the Riemann tensor of (15) is zero,

$$\bar{R}_{ijkl} = 0 ,$$  \hspace{1cm} (16)

i.e., the metric is flat, and

ii) the transformation from (2) to (15) is harmonic:

$$\left( \hat{\Gamma}^i_{jk} - \bar{\Gamma}^i_{jk} \right) \hat{g}^{jk} = 0 ,$$  \hspace{1cm} (17)

where $\hat{\Gamma}^i_{jk}$ and $\bar{\Gamma}^i_{jk}$ are respectively the second-kind Christoffel symbols of (2) and (13). Since both (16) and (17) are tensor equations as long as only coordinates that are adapted to the main Killing vector are considered, both conditions are intrinsic to the static character of the models.

To solve for $\Phi$ and $\Psi$ two methods can be used. The first one consist in using directly Eqs. (14) and (17) where these two functions are the unknowns. It has the advantage of allowing to use any coordinate condition that one wishes. The second method is based on the remark that if we consider the problem solved and we use Cartesian coordinates $x^i$ of (13), i.e., such that

$$\bar{\Gamma}^i_{jk} = 0 ,$$  \hspace{1cm} (18)
then from (17) it follows that in this system of coordinates we shall have
\[ \hat{\Gamma}^i_{jk} g^{jk} = 0. \] (19)

These conditions tell us that \( x^i \) is a system of qu-Harmonic coordinates of (2), i.e., solutions of (14). From this remark it follows that if we solve first these equations for the functions
\[ x^1 = R(r) \sin \theta \cos \varphi, \quad x^2 = R(r) \sin \theta \sin \varphi, \quad x^3 = R(r) \cos \theta, \] (20)
that is to say, if we first solve the single equation for the unknown function \( R(r) \)
\[ r^2 R'' + r \left(2 - r B^{-1} B' \right) R' - 2 B^2 R = 0, \] (21)
then to solve Eqs. (16) and (17) we just have to equate the line-element of flat space written in polar coordinates \( R, \theta, \varphi \) with (15):
\[ dR^2 + R^2 d\Omega^2 = \Phi^2 B^2 dr^2 + \Psi^2 B C r^2 d\Omega^2, \] (22)
whence it follows that
\[ \Phi = \frac{R'}{B}, \quad \Psi = \frac{R}{r \sqrt{BC}}. \] (23)

Principal transformations were introduced in [8] as a generalisation to 3-dimensional Riemannian metrics of one of Gauss’s theorems according to which any 2-dimensional metric can be mapped conformally, locally, into an Euclidean space. In this paper they will play also an important role in the interpretation of the models to be presented in Sect. 3.

These models will be required to satisfy the following conditions:

i) A value of the coordinate \( r \) exists, say \( r_1 \), such that for \( r > r_1 \) the space-time model is a solution of the vacuum field equations
\[ S_{\alpha\beta} = 0, \] (24)
where \( S_{\alpha\beta} \) is the Einstein tensor. It is therefore the exterior Schwarzschild solution.

ii) For \( r < r_1 \) the space-time model is an interior solution with a perfect-fluid source
\[ S_{\alpha\beta} = \kappa T_{\alpha\beta}, \quad \kappa = 8\pi, \quad T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + pg_{\alpha\beta}, \] (25)
where \( \rho \) is the energy density, \( p \) the pressure and \( u^\alpha \) the 4-velocity of the fluid, tangent to the main Killing field. The solution that we shall consider
is a rather straightforward, but physically innovating, generalization of the interior Schwarzschild solution. This generalization is crucial to the goal that we pursue in this paper. Namely the possibility of constructing global models with arbitrary mass or arbitrary density, and arbitrary size.

iii) On the border \( r = r_1 \) we shall require the continuity of the three functions \( A, B, C \) as well as the continuity of the three derivatives \( A', B', C' \) thus completing the construction of models of class \( C^1 \). This requirement greatly restricts the choice of a unique global coordinate condition, i.e., being the same on both sides of the border.

## 2 The exterior Schwarzschild solution

The exterior Schwarzschild solution has been described using the coordinate conditions i) to iv) mentioned in the preceding section. We list below the corresponding expressions of the coefficients \( A, B \) and \( C \), and the intervals of the corresponding radial coordinate \( r \) on which the metric is static.

i) Historical condition:

\[
A = \sqrt{1 - \frac{2m}{R}}, \quad B = A^{-1} R', \quad \sqrt{BC} = \frac{R}{r}, \quad r \in [0, \infty[. \tag{26}
\]

where \( R(r) \) is the function

\[
R(r) = \left[ r^3 + (2m)^3 \right]^{1/3}. \tag{27}
\]

ii) Curvature condition:

\[
A = \sqrt{1 - \frac{2m}{r}}, \quad B = A^{-1}, \quad BC = 1, \quad r \in [2m, \infty[. \tag{28}
\]

iii) Isotropic condition:

\[
A = \frac{1 - m/2r}{1 + m/2r}, \quad B = C = \left(1 + \frac{m}{2r}\right)^2, \quad r \in [m/2, \infty[. \tag{29}
\]

iv) Particular harmonic condition:

\[
A = \sqrt{\frac{r - m}{r + m}}, \quad B = A^{-1}, \quad BC = \left(1 + \frac{m}{r}\right)^2, \quad r \in [m, \infty[. \tag{30}
\]
v) Gauss condition:

\[ A = \sqrt{1 - \frac{2m}{R}}, \quad B = 1, \quad C = R, \quad r \in ]D, \infty[ , \quad (31) \]

where \( D \) is the arbitrary constant in the following definition of \( R \):

\[ R \equiv r \sqrt{1 - \frac{2m}{r}} + m \ln \left( -1 + \frac{r}{m} + \frac{r}{m} \sqrt{1 - \frac{2m}{r}} \right) + D. \quad (32) \]

The lowest value of \( r \), say \( r_B \), in the domain of staticity, different for each of the coordinate conditions considered so far, defines a border, say \( \mathcal{B} \), which was used to be called the Schwarzschild singularity and that now is more often called the horizon of the exterior Schwarzschild solution.

In the case of the original Schwarzschild coordinate, but also when using the Gaussian condition and setting \( D = 0 \), one has \( r_B = 0 \), strongly suggesting that Schwarzschild had succeeded, as it was his intention, in identifying the most extreme source of his solution as being a point. This point of view it is still accepted in [10] where the author claims that the non zero values of \( r_B \), suggesting that \( \mathcal{B} \) is in fact a 2-dimensional surface, come from an inadequate choice of the coordinate condition. We want to comment that whether \( \mathcal{B} \) is a point or a surface depends intrinsically on the geometry which is describing the structure of the space in the static reference frame. If the metric of this space is that with line-element (2), i.e., if one accepts that the physical distance \( \hat{d}(x_1, x_2) \) between two points of space is

\[ \hat{d}(x_1, x_2) = \int_{x_1}^{x_2} ds, \quad (33) \]

then \( \mathcal{B} \) is the border of the metric completion of the domain of staticity, and this border is unquestionably a 2-dimensional sphere. No coordinate condition can change that. On the other hand there is no reason whatsoever to take for granted that (33) defines a physical distance between two points of space, instead of, say, an optical distance. And if the distance is changed to a new one \( \bar{d}(x_1, x_2) \) which makes of \( \mathcal{B} \) a point then the problem of choosing an appropriate radial coordinate has to be addressed concomitantly with the new space structure.

We defend indeed in this paper the point of view that (33) is an optical distance and that the physical distance between two points of space in the static frame of reference is instead

\[ \bar{d}(x_1, x_2) = \int_{x_1}^{x_2} d\bar{s}, \quad (34) \]
where $d\bar{s}$ is the line-element of the principal transform of (33). This leads of course to the conclusion that $\Phi$ and $\Psi$ in (23) are respectively the radial and tangential principal velocities of the speed of light.

To find the principal transform of the exterior Schwarzschild solution we shall start with its line-element written in curvature coordinates (28). The general solution of Eq. (21) is then

$$R(r) = Q_1 f_1(r) + Q_2 f_2(r),$$  \hspace{1cm} (35)

$Q_1$ and $Q_2$ being two arbitrary constants of integration and

$$f_1(r) = r - \frac{3m}{2}, \quad f_2(r) = \sqrt{1 - \frac{2m}{r} \left( r - \frac{m}{2} \right)}.$$  \hspace{1cm} (36)

Requiring

$$\lim_{r \to \infty} R' = 1$$  \hspace{1cm} (37)

leads to

$$Q_2 = 1 - Q_1.$$  \hspace{1cm} (38)

As we shall see in Sect. 4, the remaining constant $Q_1$ will be fixed by matching the exterior Schwarzschild solution to an interior one. Notice however, as it was already pointed out in [7], the remarkable fact that if $Q_1 = 0$ then the domain of staticity of the exterior Schwarzschild solution is in quo-harmonic coordinates the interval $R \in [0, \infty[$ as for Schwarzschild’s historical form. On the other hand quo-harmonic coordinates are intimately related to principal transformations and there is no doubt that in the sense of the l-h-s of (22) $R = 0$ corresponds intrinsically to a point.

The fate of the status of the Schwarzschild singularity remains therefore suspended to whatever we can learn from matching the exterior solution to an interior one using a global system of quo-harmonic coordinates.

### 3 A new perfect fluid, spherically symmetric, static model

We consider now the field equations (25) under the general assumptions of staticity and spherical symmetry which led to the line-element (1), to which we shall add as a simplifying assumption the constancy of the energy density:

$$\rho = \text{constant.}$$  \hspace{1cm} (39)

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\(^2\)A simplified and incomplete version of this point of view was developed in [11] and [12].
Using our notations and curvature coordinates Eqs. (25) can be written as the following system of three first-order differential equations:

$$2B^{-3}B' + r^{-1}(1 - B^{-2}) = \kappa r \rho,$$

$$2B^{-3}(\sigma B)' = \kappa r,$$  

$$2AA' + 2B^{-1}B'A^2 = \kappa r \sigma^{-1}A^2B^2,$$

where

$$\sigma = \frac{1}{\rho + \rho}.$$  

Before writing down solutions of these equations let us remind that we want to obtain solutions that can be matched to the exterior Schwarzschild solution. This demands the existence of a value of \(r\), say \(r_1\), such that

i) on this 2-sphere, say \(R\), the pressure vanishes, \(p_1 = 0\),

ii) and the functions \(A, B\) are continuous across \(R\). From (28) we have to require then, with obvious notations,

$$A_{1}^{-} = A_{1}^{+} = \sqrt{1 - \frac{2m}{r_1}}, \quad A_{1}^{-}B_{1}^{-} = A_{1}^{+}B_{1}^{+} = 1.$$  

We shall discuss the continuity of the derivatives latter on.

The general solution of Eq. (40) is

$$B^{-2} = 1 - \frac{\kappa}{r} \int_{r_0}^{r} r^2 \rho \, dr = 1 - \frac{\kappa \rho}{3r} \left( r^3 - r_0^3 \right), \quad r \geq r_0,$$

where \(r_0\) is an arbitrary constant that we will assume non-negative. We shall refer from now on to \(r_0\) as the center of symmetry of the configuration. That \(r_0\) can be understood as being a point and therefore as a center will be fully justified in Sect. 4.

One has to put \(r_0 = 0\) to obtain the interior Schwarzschild solution. Since this particular solution satisfies the regularity conditions

$$B_0 = 1, \quad B_0' = 0,$$

it has become customary to require the same regularity conditions from any other physically meaningful solution. This is unjustified for several reasons.

i) It is assumed implicitly that the range of the radial variable must be \(r \in [0, \infty[\).  

\(^3\)This assumption and \(r_0 \neq 0\) leads to the singular solution of Volkoff \(13\) and Wyman \(14\).
ii) An intuitive meaning of $r$ is accepted before knowing whether a global $C^1$ model can be completed using curvature coordinates. We shall see that it cannot.

iii) More general regularity conditions can be accepted because they do not contradict any basic mathematical or physical principle.

iv) It ignores that other interpretations for the line-element (2) are available that do not require these regularity conditions.

The values of $B$ and its derivative at the center $r = r_0$ are now

$$B_0 = 1, \quad B'_0 = \frac{1}{2} \kappa \rho r_0.$$  \hspace{1cm} (48)

A more general statement for the derivative $B'$ is that

$$B' = \frac{\kappa \rho}{6 r^2} B^2 \left(2r^3 + r^3_0\right).$$  \hspace{1cm} (49)

proving that $B$ is a monotonously increasing function.

From (45) and (46) we obtain the mass or the density depending on which parameter we may wish to consider as given:

$$m = \frac{1}{6} \kappa \rho \left(r_1^3 - r_0^3\right), \quad \rho = \frac{6m}{\kappa \left(r_1^3 - r_0^3\right)}. \hspace{1cm} (50)$$

Taking into account (44) the solution of (41) is

$$\sigma B = \frac{1}{\rho} B_1 - \frac{1}{2} \kappa \int_r^{r_1} r B^3 \, dr.$$  \hspace{1cm} (51)

The values of $\sigma$ and $\sigma'$ at the center are

$$\sigma_0 = \frac{1}{\rho} B_1 - \frac{1}{2} \kappa \int_{r_0}^{r_1} r B^3 \, dr, \quad \sigma'_0 = \frac{1}{2} \kappa r_0 \left(1 - \sigma_0 \rho\right).$$  \hspace{1cm} (52)

Taking again into account (45) the solution of (42) is

$$A^2 = B^{-2} \exp \left(-\kappa \int_r^{r_1} r B^2 \sigma^{-1} \, dr\right).$$  \hspace{1cm} (53)

The values of $A$ and $A'$ at the center are

$$A_0 = \exp \left(-\frac{1}{2} \kappa \int_{r_0}^{r_1} r B^2 \sigma^{-1} \, dr\right), \quad A'_0 = \frac{1}{2} \kappa r_0 \rho_0 A_0.$$  \hspace{1cm} (54)

Notice that neither $\sigma'$, nor $A'$ are zero at the origin $r = r_0$. This is again a mild departure from conventional requirements of regularity that does not contradict any mathematical or physical principle we are aware of.
Let us prove that the pressure $p$ is a monotonously decreasing function of $r$. In fact from (51) and (49), using (43)–(44) it follows that at $r_1$ we have $\sigma_1 = \rho - 1$ and

$$\sigma_1' = \frac{\kappa}{6r_1^2}B_1^2 (r_3^3 - r_0^3) > 0,$$

(55)

because $r_1 > r_0$. Since $\sigma$ is increasing at $r_1$, it will be monotonously increasing along the whole interval $[r_0, r_1]$ unless there is a point $r_0 \leq r_2 < r_1$ where $\sigma_2' = 0$ and $\sigma_2 < \sigma_1$, which is impossible because if $\sigma_2' = 0$ from (51) and (49) it follows that

$$\sigma_2 = \frac{3r_2^3}{2r_2^3 + r_0^3} \sigma_1 \geq \sigma_1.$$

(56)

This completes the proof of our statement. We shall see in Sect. 5 that $p_0$ (and, thus, the pressure at any other point) is finite except in the most extreme configurations one can think of. We shall also discuss there when the condition $p < \rho$ is satisfied.

The behaviour of $A$ and $\sigma$ are very simply related. In fact, from (51) and (53) its follows that $A/\sigma = C$ with $C = \rho \sqrt{1 - 2m/r_1}$, the value of this constant being derived from (44) and (45).

Let us write $r_0$ in units of $2m$:

$$r_0 = 2ma, \quad a \geq 0.$$

(57)

Then from (50) it follows that the mass $m > 0$ is the solution of the equation

$$6m + \kappa \rho a^3 (2m)^3 = \kappa \rho r_1^3.$$

(58)

If $0 \leq a < 1$ then, for any given density, the viable values of $r_1 > r_0$ are bounded from above. This is a well-known result when $a = 0$, corresponding to the interior Schwarzschild solution, that follows immediately from (46) but it holds true also when $r_0$ belongs to this more general interval. In fact, if we assume that $r_1 > 2m$, as it is necessary from (45) to guarantee that $B_1^{-2} > 0$, then from (58) it follows that

$$\frac{3}{\kappa \rho} > (1 - a^3)r_1^2,$$

(59)

which proves our assertion.

On the contrary, if $a \geq 1$ all values of $r_1$ are viable whatever the density. In fact, the values of $m > 0$ derived from (58) are such that $2m < r_1$ because otherwise we would have

$$6m < \kappa \rho (1 - a^3)(2m)^3 \leq 0,$$

(60)
in contradiction with the positivity of $m$ derived from (58).

The preceding remarks prove that $r_0 = 2m$ selects a distinguished class of solutions that allow the consideration of models with arbitrary density, or mass, with arbitrary large values of $r_1$.

Consider the sequences of models with different values of $a \geq 1$ that can be constructed with a given mass $m$ and decreasing values of $r_1$. The sequence

$$r_0 = 2m$$

(61)
is the only one that allows to reach the limit where $r$ at the exterior covers the interval $[2m, \infty]$ of the maximum domain of staticity of the exterior Schwarzschild solution. We shall see in the following sections that this corresponds to the most extreme models with point-like sources. And therefore this is the sequence that allows to construct models with arbitrary mass, or density, with arbitrary large or small size.

Another striking difference between the two cases $a = 0$ and $a > 0$ comes from the comparison of the geometry described by the 3-dimensional metric (4). Using the orthonormal co-basis

$$\theta^1 = dr, \quad \theta^2 = r \, d\theta, \quad \theta^3 = r \sin \theta \, d\varphi,$$

(62)

the non-zero components of the Ricci tensor are

$$R_{11} = \frac{\kappa \rho}{3r^3}(2r^3 + r_0^3), \quad R_{22} = R_{33} = \frac{\kappa \rho}{6r^3}(4r^3 - r_0^3).$$

(63)

It follows from these formulas the well-known result that for $r_0 = 0$ the three non-zero components are identical, meaning that the metric (4) has constant curvature. This is not the case if $r_0 > 0$ whatever this value might be. This means that our solution is essentially different from the Schwarzschild one.

We postpone the discussion of the physical acceptability of these solutions as well as the discussion about the geometrical nature of their center.

## 4 Matching interior solutions to the exterior one

We have already implemented the continuity of the functions $A$ and $B$ on $r = r_1$. It so happens that there remains no freedom to add new conditions on the derivatives when using curvature coordinates both in the interior and the exterior solutions. In fact, the derivative of $B$ is inescapably discontinuous, and the derivative of $A$ is already continuous without the necessity of requiring it. This is of course the same that happens when trying to match
the exterior and interior Schwarzschild solutions but let us remind how this comes about.

We shall use again, when necessary, super-indexes $\pm$ to refer to quantities that belong to the exterior or interior solutions, and the sub-index 1 to indicate that the corresponding quantity has been evaluated on the border of the object. From (40) and (45) we have

$$2B_1^{-3}B_1^{-\prime} + \frac{2m}{r_1^2} = \kappa \rho r_1,$$

and from (28) we have

$$2B_1^{-3}B_1^{+\prime} + \frac{2m}{r_1^2} = 0,$$

and therefore inescapably

$$2B_1^{-3} \left( B_1^{+\prime} - B_1^{-\prime} \right) = -\kappa \rho r_1 \neq 0.$$  

Also from (12)-(15) it follows that

$$2A_1 A_1^{-\prime} + 2B_1^{-3}B_1^{-\prime} = \kappa \rho r_1,$$

while from (28) we have

$$2A_1 A_1^{+\prime} + \frac{2m}{r_1^2} = 0.$$  

Subtracting (67) from (64) and using the last result proves as stated that

$$2A_1 \left( A_1^{+\prime} - A_1^{-\prime} \right) = 0.$$  

Because of (66) the model thus constructed is not of class $C^1$ across the border of the object. Most authors of papers and books when dealing with Schwarzschild’s solutions feel satisfied with an identical situation. This is a mistake because it perverts the nice axiomatization of General Relativity proposed by Lichnerowicz [17] and introduces an ambiguity in the theory if one wants to have an interpretation for the metric (4), and the coefficients (23), as we have for instance for the function $A$. Besides, it is well known\(^4\) that when (44) holds then an interior static spherical solution can always be matched to the exterior Schwarzschild metric. Below we prove it for our particular interior solution.\(^5\)

\(^4\)See for example [18] or [19].
\(^5\)Another example can be seen in [20].
Let us consider the differential equation (21) with \( B = B^- \) given by Eq. (16). The expansion of \( B \) around \( r = r_0 \) is

\[
B = 1 + \frac{1}{2} \kappa pr_0 (r - r_0) + O \left[ (r - r_0)^2 \right],
\]

(70)

and therefore we have to distinguish two cases.

i) If \( r_0 = 0 \), i.e., when the metric is the interior Schwarzschild solution then \( r = r_0 = 0 \) is a regular singular point and the solutions that are finite at this origin are all proportional to the following one:

\[
R = \frac{3}{2qr^2} \left[ \arcsin(\sqrt{qr}) - \sqrt{qr} \sqrt{1 - qr^2} \right], \quad q \equiv \frac{1}{3} \kappa \rho.
\]

(71)

These finite solutions are such that

\[
R^- (0) = R_0^-, \quad R^-'(0) = R_0^-' ,
\]

(72)

the derivative at the origin \( R_0^-' \) remaining arbitrary at this point.

ii) If \( r = r_0 > 0 \) then the origin is an ordinary point of the differential equation. Therefore we can choose arbitrarily the values of \( R \) and its derivative at this point:

\[
R^- (r_0) = R_0^-, \quad R^-'(r_0) = R_0^-' .
\]

(73)

Each solution \( R(r) \) of the differential equation defines new coordinates (20) which are quo-harmonic for the interior solution. But we do not have, and we shall not use, their explicit analytic expression.

These initial conditions can be chosen such that if \( R^+ \) is the solution of the same equation (21), with \( B = B^+ \) given by Eq. (28), corresponding to initial conditions on the border \( r = r_1 \),

\[
R^+_1 = R^-_1, \quad R^+_1' = R^-_1' ,
\]

(74)

then the global model defined by the coordinate transformation \( R = R^+(r) \) at the exterior and \( R = R^- (r) \) in the interior is of class \( C^1 \) across the border.

In fact let the new expression of the metric (2) be

\[
d\tilde{s}^2 = \tilde{B}^2 dR^2 + \tilde{D}^2 R^2 d\Omega^2 ,
\]

(75)

where

\[
\tilde{B} = \frac{B}{R^+}, \quad \tilde{D} = \frac{r}{R}.
\]

(76)

\(^6\)This case has been considered by P. Teyssandier [14].
\( \tilde{B}, \tilde{D} \) and \( \tilde{D}' \) are continuous because \( R \) and \( R' \) have been chosen to be continuous. From the equation (21) that satisfy both \( R^\pm \) we can derive that

\[ R_1^{++} - R_1^{--} - 2B_1^{-1} \left( B_1^{++'} - B_1^{--'} \right) R_1^{'} = 0. \quad (77) \]

From (76) we have

\[ \tilde{B}' = B'R'^{-1} - BR'^{-2}R''. \quad (78) \]

Therefore,

\[ \frac{d\tilde{B}}{dR} = \tilde{B}'R'^{-1} = R'^{-2}B' - R'^{-3}BR''. \quad (79) \]

and

\[ \frac{d\tilde{B}_1^+}{dR} - \frac{d\tilde{B}_1^-}{dR} = R_1^{--} \left( B_1^{++'} - B_1^{--'} \right) - R_1^{--}B_1 \left( R_1^{++} - R_1^{--} \right). \quad (80) \]

Using (77) in this equation we finally obtain

\[ \frac{d\tilde{B}_1^+}{dR} - \frac{d\tilde{B}_1^-}{dR} = 0, \quad (81) \]

which proves that the global quo-harmonic model is of class \( C^1 \).

Once \( R_1^- \) and \( R_1'^- \) have been found integrating (21) with initial conditions (73) the constants \( Q_1 \) and \( Q_2 \) of (35) can be found solving the system of linear equations

\[ R_1^- = Q_1f_1(r_1) + Q_2f_2(r_1), \quad R_1'^- = Q_1f_1'(r_1) + Q_2f_2'(r_1). \quad (82) \]

Since the two functions \( f_1 \) and \( f_2 \) are independent this system has always a solution.

In case i) above the supplementary condition (38) determines the remaining arbitrary constant \( R_0'^- \), while in case ii) this condition gives only a relation between the two initial conditions (73). To obtain a second relation and make our model completely determine let us consider the principal transform of the interior metric of space. By definition we shall have (22) with \( R = R^- \) and appropriate functions \( \Phi, \Psi, B^- \) and \( C^- \). If we accept, as we have done, that the structure of space in the static frame of reference of the source is given by the l-h-s of this equation and we want to have a center for our configuration and no hole in the space we are forced to choose our function \( R \) with the initial condition

\[ R^-(r_0) = R_0^- = 0. \quad (83) \]

Since the functions \( R(r) \) and \( r(R) \) are not explicitly known when \( r_0 > 0 \), neither is known explicitly the space-time metric (1) using the radial coordinate \( R \). Therefore it is not possible using \( R \) to discuss directly the behavior.
of the causal geodesics that go through the center of the configuration. But this can be done looking first at this problem using the original curvature coordinate $r$. Let us consider a causal geodesic reaching $P_{\text{in}}$ with $r = r_0$ at some time $\tilde{t}$. Since the space trajectory is contained in a plane, say $\Pi$, of the auxiliary Euclidean space with line-element $dr^2 + r^2 d\Omega^2$, we can use polar coordinates such that $\Pi$ is the plane $\theta = \pi/2$. Let $\tilde{\varphi}$ be the value of the azimuth angle at $P_{\text{in}}$; and let $\dot{r} = \tilde{v}_r$ and $\dot{\varphi} = \tilde{v}_{\varphi}$ be the two corresponding derivatives. Let us consider now the causal geodesic starting at time $\tilde{t}$ from the point $P_{\text{out}}$ defined by $r = r_0$ and $\varphi = \varphi + \pi$ with derivatives given by $\dot{r} = -\tilde{v}_r$ and $\dot{\varphi} = -\tilde{v}_{\varphi}$. The space-time trajectory of this geodesic branch is contained also on $\Pi$ and is the symmetric image of the incident branch with respect to the diameter joining $P_{\text{in}}$ and $P_{\text{out}}$. We see therefore that the process of contraction of the sphere $r = r_0$ to the point $R = 0$ described by (20) joins the end-point of the incoming branch of the geodesic with the origin of the outgoing one. And in the process the matched geodesic is as smooth as they were the two branches. Notice also that, since starting from any point there are always a bunch of geodesics reaching $r = r_0$ with different values of $\theta$ and $\varphi$, our construction of the center of the new models endows this center with a focalising property.

The quo-harmonic class of coordinates derived from the radial coordinate $R$ are natural coordinates in the sense that they have the following properties: i) they exhibit the center as a point in the sense of (14) and ii) they make the space-time metric smooth across the border of the spherical source. Another system of natural coordinates sharing these two properties are the Gauss coordinates derived from the radial coordinate:

$$R_G = \int_{r_0}^{r} B \, dr.$$  \hspace{1cm} (84)

Instead for large values of $r$ the quo-harmonic $R$ behaves as

$$R = r - \frac{3}{2}m + O(1/r^2),$$  \hspace{1cm} (85)

while the Gaussian $R_G$ behaves as

$$R = r - m \left[ 1 + \ln \left( \frac{m}{2r} \right) \right] + O(1/r),$$  \hspace{1cm} (86)

which makes the asymptotic behaviour of the metric to depart a little bit more from the Newtonian intuition.

In the next section we shall discuss the numerical solutions and in the final section we shall discuss the physical relevance of them.
5 Numerical analysis of the model

The numerical study of the model is greatly simplified by using the following dimensionless variables:

\[ x \equiv \frac{r}{2m}, \quad a \equiv x_0 = \frac{r_0}{2m}, \quad b \equiv x_1 = \frac{r_1}{2m}. \]  

(87)

Now, using (50) the expression (46) reduces to

\[ B^{-2} = 1 - \frac{1}{x} \frac{x^3 - a^3}{b^3 - a^3}. \]  

(88)

5.1 The pressure

For the interior solution corresponding to \( x \in [a, b] \), Eq. (51) reduces to

\[ \mu \equiv \frac{\sigma}{4km^2} = B^{-1} \left[ \frac{1}{3} \sqrt{\frac{b}{b-1}} \left( b^3 - a^3 \right) - \frac{1}{2} \int_x^b xB^3 \, dx \right], \]  

(89)

which can be written in terms of elliptic functions, but is far more easily computed numerically.

Since the pressure decreases from the center \( r = r_0 \) to the border \( r = r_1 \), to make sure that it does not go to infinity it is enough to check that it does not diverge at \( r = r_0 \), i.e., that \( \mu_0 \equiv \mu(a) \) does not become zero (or negative). This can be easily checked by numerical quadrature. In Fig. 1 we see that, in the case \( a = 1 \) (\( r_0 = 2m \)), \( \mu_0 \) only vanishes for \( b = a \), which corresponds to the limit case in which the interior reduces to a point. The same happens for \( a > 1 \), but for \( 0 \leq a < 1 \) the pressure becomes infinite at some interior point unless the matching radius \( r_1 = 2mb \) is greater than a given value \( r_{1\text{min}} \), which is displayed in Fig. 2. In the case \( r_0 = 0 \) corresponding to the interior Schwarzschild solution, the well-known minimum value is \( r_{1\text{min}} = \frac{9}{4}m \).

One may also check numerically that the dominant energy condition \( p < \rho \) —i.e., \( \mu \geq \mu_0 > \frac{1}{2} \mu_1 \equiv \frac{2}{3} \mu(b) \) — is satisfied always if \( r_0 \geq 2m \) and \( r_1 > r_0 \). For \( 0 \leq r_0 < 2m \), however, it is satisfied only for values of \( r_1 \) greater than a minimum value \( \tilde{r}_{1\text{min}} \) which happens to be higher than the \( r_{1\text{min}} \) discussed above, as displayed in Fig. 3. In the case of the interior Schwarzschild solution \((r_0 = 0)\) one has \( \tilde{r}_{1\text{min}} = \frac{8}{3} m \).

Thus, we conclude that \( 2m \) is the smallest value of \( r_0 \) for which the physical conditions on the pressure do not put any limit on the matching radius \( r_1 > r_0 \).
5.2 Quo-harmonic coordinates

To compute the quo-harmonic coordinates (24) we have to solve the differential equation (21), which in the dimensionless variables of this section reduces to

\[ 2xH \ddot{S} + \left( 4H - 2x^3 - a^3 \right) \dot{S} - 4 \left( b^3 - a^3 \right) S = 0, \]  
\[ (90) \]

where

\[ S \equiv \frac{R}{2m}, \]  
\[ (91) \]
a prime indicates derivative with respect to \( x \) and we have defined

\[ H \equiv x \left( b^3 - a^3 \right) B^2 - 2 = x \left( b^3 - a^3 \right) - \left( x^3 - a^3 \right) > 0. \]  
\[ (92) \]

In the case \( a = 0 \) (corresponding to the interior Schwarzschild solution), the origin is regular singular, but the solution satisfying \( \tilde{S}(a) = 0, \tilde{S}'(a) = 1 \) exists in \([0, b]\) and can be explicitly written \[14\] as \( (71) \), which reduces to

\[ \tilde{S} = \frac{3b^3}{2x^2} \left[ b^{3/2} \arcsin \left( b^{-3/2} x \right) - x \sqrt{1 - b^{-3} x^2} \right]. \]  
\[ (93) \]

In the remaining cases \((a > 0)\), the differential equation is linear, the coefficients are continuous, and \( H \) does not vanish for \( x \in [a, b] \) (we are excluding the limit case \( b = a \)), so that there exists a unique solution defined for \( a \leq x \leq b \) that satisfies the initial conditions \( S(a) = 0, S'(a) = \beta \), where \( \beta \) is a constant to be computed later. In fact, \( S = \beta \tilde{S} \), if \( \tilde{S} \) is the solution of Eq. \( (90) \) satisfying

\[ \tilde{S}(a) = 0, \quad \tilde{S}'(a) = 1. \]  
\[ (94) \]

Now the numerical method to compute \( R \) is straightforward. After selecting the parameter \( a \), we compute \( \tilde{S} \) by solving \( (90) \) with \( (94) \). Then the auxiliary condition \( (38) \) and the matching conditions \( (82) \) read

\[ \beta \tilde{S}(b) = Q_1 g_1(b) + (1 - Q_1) g_2(b), \quad \beta \tilde{S}'(b) = Q_1 g_1'(b) + (1 - Q_1) g_2'(b), \]  
\[ (95) \]

where functions \( (36) \) are written as

\[ g_1(x) \equiv \frac{f_1}{2m} = x - \frac{3}{4}, \quad g_2(x) \equiv \frac{f_2}{2m} = \sqrt{1 - \frac{1}{x} \left( x - \frac{1}{4} \right)}. \]  
\[ (96) \]

Since Eqs. \( (95) \) are readily solved for \( \beta \) and \( Q_1 \), \( R \) is given by

\[ S = \frac{R}{2m} = \begin{cases} \beta \tilde{S}(x), & \text{for } a \leq x \leq b, \\ Q_1 g_1(x) + (1 - Q_1) g_2(x), & \text{for } x \geq b. \end{cases} \]  
\[ (97) \]
Two particular cases, for \( a = 0, 1 \) and \( b = 2 \) (i.e., for \( r_0 = 0, 2m \) and \( r_1 = 4m \)) are displayed in Fig. 3. Notice the different definition ranges of \( R \) in curvature coordinates.

In Fig. 3 the coefficient \( Q_1 \) is displayed for \( a = 1 \) and different values of \( b \). One may see there that in the limit \( b \rightarrow a = 1 \), i.e., when the interior collapses to the point \( R = 0 \) in quo-harmonic coordinates and to the center \( r_0 = 2m \) in curvature coordinates, \( Q_1 \) vanishes, so that the radial coordinate \( R \) associated to quo-harmonic coordinates is given everywhere by \( R = f_2(r) \), as pointed in Sect. 3.

### 5.3 Metric coefficients and principal transform

The metric coefficient \( B \) is given, in the dimensionless curvature coordinates, by \( (88) \) for \( a \leq x \leq b \) and by \( B = (1 - 1/x)^{-1/2} \) for \( x \geq b \). Similarly, the other metric component in curvature coordinates is given by Eq. \((53)\), which reduces to

\[
A^2 = B^{-2} \exp \left( - \int_x^b x B^2 \mu^{-1} \, dx \right),
\]

for \( a \leq x \leq b \) and is \( A = B^{-1} \) for the exterior \( x \geq b \). Since we can compute numerically the quo-harmonic coordinates, they can be used to display the metric of different models in the same physical coordinates.

The functions in Eqs. \((23)\) now are written as

\[
\Phi = \frac{S'}{B}, \quad \Psi = \frac{S}{x},
\]

and can be readily computed.

For instance, in Fig. 3 we show the metric coefficients \( A \) and \( \tilde{B} \) for two models with \( a = 0, 1 \) (i.e., \( r_0 = 0, 2m \)). In both cases the matching of the interior and exterior metrics happens at the same location in quo-harmonic coordinates: at the spherical surface of radius \( S = 2 \) \( (R = 4m) \), which in curvature coordinates has radius \( x \approx 2.7134 \) when \( a = 0 \) and \( x \approx 2.7916 \) if \( a = 1 \). The continuity of the metric coefficients and their first derivatives is apparent in the figure. The functions \( \Phi \) and \( \Psi \) corresponding to the same special cases are plotted in Fig. 3.

### 6 Concluding remarks

Among all possible values of \( r_0 \) two of them are distinctly distinguished:
i) the value \(r_0 = 0\) because it leads to the most well-behaved models, although only in a restrictive range of the density \(\rho\) and the radius of the interior configuration \(r_1\) or \(R_1\), and

ii) the value \(r_0 = 2m\) because this leads to the closest sequence to Schwarzschild’s one, allows unrestricted values of the parameters \(\rho\) and \(r_1\) or \(R_1\) and contains the emblematic most extreme configuration we can think of as a limit, namely that with a point-like source.

Whether or not new considerations or difficulties will suggest selecting or excluding particular values of \(r_0\) remains to be seen. But if only one new value remains then be pretty sure that this value will be \(r_0 = 2m\).

There exists a range of parameters \(\rho\) and \(r_1\) for which Schwarzschild’s interior solution and ours coexist. Nevertheless the two solutions are quite different, whatever these parameters might be. This is so because the geometry described by (2) is homogeneous and isotropic in one case and inhomogeneous and anisotropic in the other.

We have assumed for simplicity that the energy-density was constant. It might be objected that this is not a realistic “equation of state” to describe the properties of our objects, and in particular the extreme ones, i.e., those without analog based on the interior Schwarzschild solution. But no equation of state will be realistic, and would be un-realistic to guess one, as long as new physics is not available for them. Not to mention that to assume the existence of an equation of state is also a simplifying assumption, as it is to use a perfect fluid description. The main properties of our models do not depend crucially, we believe, on these simplifying assumptions. They depend instead crucially on the identification of \(r = 2m\) with the origin and the center of of symmetry of the model and also on the re-interpretation of the metric (2) based on the consideration of principal transformations.

We do not claim that such objects will be found in nature, but we do claim that if they are found General Relativity is a good theory to understand its geometrical properties and gravitational field. We also claim that it is a sound attitude to look for them. Everybody knows where there is a good chance to discover them.

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Figure 1: Function $\mu_0$ for $a = 1$ and small values of $b$. 
Figure 2: Values of $r_1$ for which the pressure becomes infinite ($r_{1\text{min}}$) or equal to $\rho(\tilde{r}_{1\text{min}})$ at the origin $r_0$. 
Figure 3: Function $R$ and its derivative $R'$ for $r_0 = 0, 2m$ and $r_1 = 4m$. 
Figure 4: Coefficient $Q_1$ for $r_0 = 2m$ and different values of $r_1$. 
Figure 5: Metric coefficients $A$ and $\tilde{B}$ when the matching radius is $R_1 = 4m$ and $r_0 = 0.2m$. The same $R$ coordinate associated to quo-harmonic coordinates is used in both cases.
Figure 6: Functions $\Phi$ and $\Psi$ when the matching radius is $R_1 = 4m$ and $r_0 = 0, 2m$. The same $R$ coordinate associated to quo-harmonic coordinates is used in both cases.