Notes on fast moving strings

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Abstract

We review the recent work on the mechanics of fast moving strings in anti-de Sitter space times a sphere and discuss the role of conserved charges. An interesting relation between the local conserved charges of rigid solutions was found in the earlier work. We propose a generalization of this relation for arbitrary solutions, not necessarily rigid. We conjecture that an infinite combination of local conserved charges is an action variable generating periodic trajectories in the classical string phase space. It corresponds to the length of the operator on the field theory side.

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1 Introduction.

The AdS/CFT correspondence is a strong-weak coupling duality. Weakly coupled Yang-Mills is mapped to the string theory on the highly curved AdS space. When AdS space is highly curved, the string worldsheet theory becomes strongly coupled. Therefore, the weakly coupled Yang-Mills maps to the strongly coupled string worldsheet theory. Nevertheless, in some situations elements of the YM perturbation theory can be reproduced from the string theory side. One of the examples are the “spinning strings”. Spinning strings are a class of solutions of the classical string worldsheet theory. They were first considered in the context of the AdS/CFT correspondence in [1, 2, 3]. These are strings rotating in $S^5$ with a large angular momentum. It was noticed in [1] that the energy of these solutions has an expansion in some small parameter which is similar in form to the perturbative expansion in the field theory on the boundary. Then [4] computed the anomalous dimensions of single trace operators with the generic large R-charge, making the actual comparison possible. In [5] more general solutions were considered, having large compact charges both in $S^5$ and in $AdS_5$. For all these solutions, computations in the classical worldsheet theory lead to the series in the small parameter which on the field theory side is identified with $\lambda/J^2$ where $\lambda$ is the ’tHooft coupling constant and $J$ a large conserved charge. Moreover it was shown in [6] that the quantum corrections to the classical worldsheet theory are suppressed for the solutions with the large conserved charge (see also the recent discussion in [7]). This opened the possibility that the results of the calculations in the classical mechanics of spinning strings, which are valid a priori only in the large $\lambda$ limit, can be in fact extended to weak coupling and therefore compared to the Yang-Mills perturbation theory. It was conjectured that the Yang-Mills perturbation theory in the corresponding sector is reproduced by the classical dynamics of the spinning strings. The following picture is emerging.

Single string states in $AdS_5 \times S^5$ correspond to single-trace operators in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. (We consider the large $N$ limit.) The dynamics of the single-trace operators is described in the perturbation theory by an integrable spin chain. This spin chain has a classical continuous limit [8] which describes a class of operators with the large R-charges. In this limit the spin chain becomes a classical continuous system. We have conjectured in [9] that this classical system is equivalent to the worldsheet theory of the classical string in $AdS_5 \times S^5$. The Yang-Mills
perturbative expansion corresponds to considering the worldsheet of the fast moving string as a perturbation of the null-surface \[8, 9, 10, 11, 12\]. The null-surface perturbation theory was previously considered in a closely related context in \[13\].

In this paper we will try to make the statement of equivalence more precise. We will argue that the string worldsheet theory has a “hidden” \(U(1)\) symmetry which is defined unambiguously by its characteristic properties which we describe. This \(U(1)\) commutes with the group of geometrical symmetries of the target space. It corresponds to the \textit{length of the spin chain} on the field theory side. We conjecture that the phase space of the classical continuous spin chain is equivalent to the Hamiltonian reduction of the phase space of the classical string by the action of this \(U(1)\). The equivalence commutes with the action of geometrical symmetries.

We should stress that the hidden \(U(1)\) symmetry which we discuss in this paper was constructed already in \[12\], but the explicit calculation was carried out only at the first nontrivial order of the null-surface perturbation theory. The main new result of our paper is that we discuss this hidden symmetry from the point of view of the integrability. We conjecture the relation between the \(U(1)\) symmetry and the local conserved charges which if true gives a uniform description of this symmetry at all orders of the perturbation theory.

The classical string on \(AdS_5 \times S^5\) is an integrable system (see \[14, 15, 16, 17, 18\] and references there), and our \(U(1)\) corresponds to an action variable. The existence of the action variables for integrable systems with a finite-dimensional phase space is a consequence of the Liouville theorem \[19\]. The classical string has an infinite-dimensional phase space. We are not aware of the existence of a general theorem which would guarantee that the action variables can be constructed in the infinite-dimensional case. But we will give two arguments for the existence of one action variable for the string in \(AdS_5 \times S^5\), at least in the perturbation theory around the null-surfaces. The first argument gives an explicit procedure to construct the action variable order by order in the perturbation theory (Sections 3, 4.4 and 4.6). The second argument uses the existence of the local conserved charges \[20\] (known as higher Pohlmeyer charges) and the results of the evaluation of these charges on the so-called “rigid solutions” performed in \[21, 22\]. The arguments in Section 4 of our paper together with the results of \[21, 22\] suggest that the action variable is an infinite linear combination of the Pohlmeyer charges and allow in principle to find the coefficients of this
linear combination.

The plan of the paper. In Section 2 we will review the classification of the null-surfaces following mostly \[11, 9\] and stress that the moduli space of the null-surfaces is a $U(1)$-bundle over a loop space. Therefore it has a canonically defined action of $U(1)$. In Section 3 we will explain how to extend the action of $U(1)$ from the null-surfaces to the nearly-degenerate extremal surfaces using the perturbation theory. A large part of Section 3 is a review of \[12\]. In Section 4 we discuss the geometrical meaning of this $U(1)$ as an action variable and argue that it is an infinite sum of the local conserved charges.

Note added in the revised version. The coefficients of the expansion of the action variable in the local conserved charges were fixed to all orders in the first paper of \[29\]. Here we consider only the Pohlmeyer charges for the $S^5$ part of the string sigma-model. The role of the Pohlmeyer charges for $AdS_5$ was discussed in the second paper of \[29\]. In the special case when the motion of the string is restricted to $\mathbb{R} \times S^2 \subset AdS_5 \times S^5$ the action variable discussed here corresponds to the action variable of the sine-Gordon model, see the third paper of \[29\].

2 Null-surfaces.

2.1 The definition.

A two-dimensional surface in a space-time of Lorentzian signature is called a null-surface if it has a degenerate metric and is ruled by the light rays. There is a connection between null-surfaces and extremal surfaces. An extremal surface is a two-dimensional surface with the induced metric of the signature $1 + 1$ which extremizes the area functional. Extremal surfaces are solutions of the string worldsheet equation of motion in the purely geometrical background (no $B$-field). When the string moves very fast, the metric on the worldsheet degenerates and the worldsheet becomes a null-surface. Therefore a null-surface can be considered as a degenerate limit of an extremal surface.

In $AdS_5 \times S^5$ there are two types of the light rays. The light rays of the first type project to points in $S^5$. The light rays of the second type project to the timelike geodesics in $AdS_5$ and the equator of $S^5$. The operators of
the large R-charge correspond to the null-surfaces ruled by the light rays of the second type\(^2\).

2.2 The moduli space of null-surfaces.

It is straightforward to explicitly describe all the null-surfaces of the second type in \(\text{AdS}_5 \times S^5\). We have to first describe the moduli space of the null-geodesics of the second type. An equator of \(S^5\) is specified by a point in the coset space \(g_S \in \frac{SO(6)}{SO(2) \times SO(4)}\). Similarly, a timelike geodesic in \(\text{AdS}_5\) is specified by \(g_A \in \frac{SO(2)}{SO(2) \times SO(4)}\). Given \(g_S\) and \(g_A\), let \(E(g_S) \subset S^5\) and \(T(g_A) \subset \text{AdS}_5\) be the corresponding equator in \(S^5\) and timelike geodesic in \(\text{AdS}_5\), respectively. To specify a light ray in \(\text{AdS}_5 \times S^5\) we have to give also a map \(F : T \rightarrow E\) which pulls back the angular coordinate on \(E\) to the length parameter on \(T\) (see Fig.\(\text{I}\)). Such maps are parametrized by \(S^1\). We see that each light ray is defined by a triple \((T, E, F)\). Therefore, the moduli space of light-rays of the second type in \(\text{AdS}_5 \times S^5\) is geometrically:

\[
\left[ \frac{SO(2, 4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \right] \tilde{\times} S^1
\]  

(1)

A null-surface is a one-parameter family of light rays. Therefore it determines a contour in \(\left[ \frac{SO(2, 4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \right] \tilde{\times} S^1\). But we have to also remember that an arbitrary collection of the light rays is not necessarily a null-surface. It is a null-surface only if the induced metric is degenerate. To understand what it means, let us choose a space-like curve belonging to our surface. This space-like curve is a collection of points, one point on each light ray. For the surface to be null, the tangent vector to this curve at each point of the curve should be orthogonal to the light ray to which the point belongs. (This condition does not depend on how we choose a space-like curve.) What kind of a constraint does it impose on the contour? The space \(\left[ \frac{SO(2, 4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \right] \tilde{\times} S^1\) is a \(U(1)\) bundle over \(\frac{SO(2, 4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)}\). The condition of the degeneracy of the metric defines a connection on this bundle. The definition of this connection is: the curve in the total space is considered horizontal, precisely if the corresponding collection of light rays is a degenerate surface. What is the curvature of

\(^2\)The null-surfaces of the first type have a boundary. They describe the shock wave propagating from the cusp of the worldline of a spectator quark in \(\mathbb{R} \times S^3\).
Figure 1: A null-geodesic in $AdS_5 \times S^5$ is specified by the choice of an equator $E$ in $S^5$, a time-like geodesic $T$ in $AdS_5$ and a map $F : T \to E$ which maps the angular parameter $\psi$ on the equator to the time $t$ on the geodesic, up to a constant.

this connection? Both $\frac{SO(2,4)}{SO(2) \times SO(4)}$ and $\frac{SO(6)}{SO(2) \times SO(4)}$ are Kahler manifolds (if we forgive that the metric on the first coset is not positive-definite). Let us denote the Kahler forms $k_A$ and $k_S$. The curvature of our $U(1)$-bundle is $k_A + k_S$. A curve in the base space $\frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)}$ can be lifted to the horizontal curve in the total space if and only if a two-dimensional film ending on this curve has an integer Kahler area (integral of $k_A + k_S$ over this film should be an integer). Moreover, it is lifted as a horizontal curve almost unambiguously, except that there is a “global” action of $U(1)$ shifting $F : T \to E$ on every light ray by the same constant. Therefore, the moduli space of null-surfaces is the $U(1)$ bundle over the space of contours in $\frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)}$ subject to the integrality condition which we described.

To summarize, the moduli space of the null-surfaces of the second type is:

$$\text{Map}_0 \left( S^1, \frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \right) \tilde{\times} S^1$$

Here $\text{Map}(S^1, X)$ means the space of maps from the circle to $X$; for $X$ a
Figure 2: A picture of a null-surface in \( AdS_5 \times S^5 \). A null-surface is a two-dimensional surface with the degenerate metric, ruled by the light rays. We have shown five light rays and a spacial contour with a parameter \( \sigma \). One can visualize the null-surface as the surface swept by the spacial contour as it moves along the light rays.
Kähler manifold $Map_0(S^1, X)$ means the space of maps satisfying the integrality condition. At this point we consider the null-surfaces without a parametrization; therefore we divide by the group $\text{Diff}(S^1)$ of the diffeomorphisms of the circle. Turning on the fermionic degrees of freedom on the worldsheet we get the moduli space of supersymmetric null-surfaces \[9\]:

\[
\text{Map}_0(S^1, Gr(2|2, 4|4)) \sim S^1 \times \text{Diff}(S^1)
\]  

(3)

Here $\text{Map}_0(S^1, Gr(2|2, 4|4))$ is the phase space of the continuous spin chain \[9\]. Therefore the moduli space of null-surfaces is “almost” equivalent to the phase space of the continuous spin chain, except for the fiber $S^1$ and the reparametrizations $\text{Diff}(S^1)$. We have to explain what happens to the fiber and why the null-surface actually comes with the parametrization. Also, we have to explain how the symplectic structure is defined on the moduli space of null surfaces. Let us start with the parametrization.

### 2.3 Parametrized null-surfaces.

The phase space of the classical string has a boundary which consists of strings “moving with the speed of light”. A string moving very fast can be approximated by a null-surface. But one null-surface can approximate many different fast moving strings. The null-surface as we defined it so far “remembers” only the direction of the velocity at each point of the approximated string, but it misses the information about the ratios of the relativistic factors $\sqrt{1 - v^2}$ at different points of the string. Although $\sqrt{1 - v^2} \to 0$ in the null-surface limit, the ratio $\sqrt{1 - v^2(\tau, \sigma_1)}/\sqrt{1 - v^2(\tau, \sigma_2)}$ for two different points on the worldsheet remains finite. Therefore, if we want to think of the moduli space of the null-surfaces as the boundary of the phase space, we have to equip the null-surfaces with an additional structure. This additional structure is the parametrization.

A null-surface is a one-parameter family of the light rays. The parameterization is a particular choice of the parameter. In other words, it is a monotonic function $\sigma$ from the family of light rays forming the null-surface to the circle, defined modulo $\sigma \sim \sigma + \text{const}$. One can also think of it as a density $d\sigma$ on the set of light rays forming the null-surface. This density is roughly speaking proportional to the density of energy on the worldsheet of the fast-moving string, in the limit when it becomes the null-surface. We will now give the definition of $\sigma$. 
Consider the family of string worldsheets \( \Sigma(L) \) converging to the null-surface \( \Sigma_0 = \Sigma(\infty) \). We will introduce a parametrization \( d\sigma \) of \( \Sigma_0 \) in the following way. Consider a Killing vector field \( U \) on \( S^5 \), corresponding to some rotation of the sphere:

\[
U.x^i_S = u^{ij}x^j_S
\]  

(4)

Here \( x^i_S \) parametrizes the \( S^5 \): \( \sum (x^i_S)^2 = 1 \).

When \( L \) is large, \( \Sigma(L) \) is close to \( \Sigma_0 \), the string moves very fast and the conserved charge corresponding to \( U \) is very large. We can approximate this charge by an integral over a spacial contour on the null-surface \( \Sigma_0 \) of \( u^{ij}x^i_{0,S}\partial_\tau x^j_{0,S} \) times some density \( d\sigma \):

\[
Q_U = L \int_{\sigma \in [0,2\pi]} d\sigma \ u^{ij}x^i_{0,S}\partial_\tau x^j_{0,S} + \text{terms vanishing at } L \to \infty
\]  

(5)

Here \( x_{0,S} \) is the \( S^5 \)-part of the null-surface; we choose the \( \tau \) coordinate on the null-surface to be the affine parameter on the light ray normalized by the condition \( x_{0,S}(\tau + 2\pi, \sigma) = x_{0,S}(\tau, \sigma) \). Eq. (4) with the condition \( \int d\sigma = 2\pi \) is the definition of \( d\sigma \), and also the precise definition of the large parameter \( L \), modulo \( O(1/L) \). We choose \( \sigma \) as the parametrization.

We can now say that the moduli space of parametrized null-surfaces is the boundary of the phase space of a classical string. We say that a family \( \Sigma(L) \) of extremal surfaces has a parametrized null-surface \( \Sigma_0 \) as a limit when \( L \to \infty \) if and only if

- \( \Sigma(L) \) has \( \Sigma_0 \) as a limit when \( L \to \infty \), as a continuous family of smooth two-dimensional surfaces in a smooth two-dimensional manifold, and

- the density of \( Q_U \) approaches Eq. (5) in the limit \( L \to \infty \).

This definition of the parametrization does not depend on which particular geometrical symmetry \( U \) we use. An alternative way to define the same parametrization is to use a special choice of the worldsheet coordinates on \( \Sigma \). Let us choose the worldsheet coordinates \( \tau, \sigma' \) so that

\[
\left( \frac{\partial x_S}{\partial \tau} \right)^2 + \left( \frac{\partial x_S}{\partial \sigma'} \right)^2 = -\left( \frac{\partial x_A}{\partial \tau} \right)^2 - \left( \frac{\partial x_A}{\partial \sigma'} \right)^2 = 1
\]

\[
\left( \frac{\partial x_S}{\partial \tau}, \frac{\partial x_S}{\partial \sigma'} \right) = -\left( \frac{\partial x_A}{\partial \tau}, \frac{\partial x_A}{\partial \sigma'} \right) = \text{const}
\]

where \( x_A \) is the projection of the string worldsheet to \( AdS_5 \) and \( x_S \) is the projection to \( S^5 \). Then we define \( \sigma = \sigma'/\int d\sigma' \). In the null-surface limit \( d\sigma \) defines the parametrization of the null-surface.
2.4 The symplectic structure.

The moduli space of parametrized null-surfaces as a manifold depends only on the conformal structure of the target space. But we can introduce additional structures on this moduli space which use the metric on $AdS_5 \times S^5$.

An important additional structure is the closed 2-form which originates from the symplectic form of the classical string. Strictly speaking a differential form in the bulk of the manifold does not automatically determine a differential form on the boundary. Indeed, suppose that we have a differential form, for example a 2-form $\Omega$ in the bulk. We can try to define the “boundary value” $\omega$ of $\Omega$ on the boundary in the following way. Given two vector fields $v_1, v_2$ on the boundary, we find two vector fields $V_1, V_2$ in the bulk such that $\lim V_1 = v_1$ and $\lim V_2 = v_2$. Then we define $\omega(v_1, v_2) = \lim \Omega(V_1, V_2)$. But the problem is that this definition will depend on the choice of $V_1$ and $V_2$. Intuitively, if $(\tilde{V}_1, \tilde{V}_2)$ is some other choice of a pair of vector fields inducing $(v_1, v_2)$ on the boundary, and the “vertical component” of $\tilde{V}_i - V_i$ is not small enough near the boundary, then $\Omega(V_1, V_2) \neq \Omega(\tilde{V}_1, \tilde{V}_2)$.

Given this difficulty, how do we define the symplectic form on the space of null-surfaces given the symplectic form on the string phase space? When we lift the vector field $v$ on the boundary to the vector field $V$ in the bulk, let us require that $dL(V)$ goes to zero when $L \to \infty$. We define $L$ by Eq. (5); it is only an approximate definition at $L \to \infty$, but this is good enough for the purpose of our definition:

$$\omega(v_1, v_2) = \lim_{L \to \infty} L^{-1} \Omega(V_1, V_2) \quad (6)$$

where $V_1$ and $V_2$ are such that $dL(V_1) = dL(V_2) \approx 0$. One can see that $\omega$ has a kernel, which is precisely the tangent space to the fiber $S^1$ in the numerator of Eq. (3). The moduli space has a symmetry $U(1)$ rotating this fiber; we will discuss this symmetry in the next section; we will call it $U(1)_L$. Therefore $\omega$ is the symplectic form on the moduli space of null-surfaces modulo $U(1)_L$.

Eq. (3) implies that the moduli space of parametrized null-surfaces modulo $U(1)_L$ is the space of parametrized contours in the Grassmanian:

$$\text{Map}_0 \left( S^1, \ Gr(2|2, 4|4) \right) \quad (7)$$

One can see that $\omega$ is equal to the integral of the symplectic form on the super-Grassmanian pointwise on the contour, with the measure $d\sigma$. The symplectic area of the film filling the contour is the generating function of the shift of
the origin of the circle. Therefore the integrality condition guarantees that the symplectic form does not depend on the choice of the origin on $S^1$; the symplectic form is horizontal and invariant with respect to the shifts of the origin of $S^1$.

Our definition of the symplectic form on the space of null-surfaces used the target space metric (just the conformal structure would not be enough) and also the fact that the target space is a product of two manifolds.

3 Nearly-degenerate extremal surfaces and the role of the engineering dimension.

Our discussion in this and the next section will be limited to the classical bosonic string.

3.1 Definition of $U(1)_L$.

The moduli space of null-surfaces is a $U(1)$-bundle. The $U(1)$ symmetry shifting in the fiber $S^1$ plays an important role in the formalism. We will call it $U(1)_L$. On fig. 3 we have shown schematically how $U(1)_L$ acts on the null-surfaces. We conjecture that $U(1)_L$ corresponds to the length of the spin chain. Generally speaking, the length of the spin chain is not conserved in the Yang-Mills perturbation theory [26], but it is probably conserved in the continuous limit (this should be related to the discussion of the “closed sectors” in [27]). It should be conserved modulo the corrections vanishing in the continuous limit. We therefore conjecture that there is a continuation of $U(1)_L$ from the space of null-surfaces to the phase space of the classical string, at least to the region of the phase space corresponding to fast moving strings. We conjecture that this continuation is uniquely defined by the following properties:

1. The action of $U(1)_L$ preserves the symplectic structure.

2. The action of $U(1)_L$ does not change the projection of the worldsheet to $AdS_5$. Moreover, it preserves the projection to $AdS_5$ of the null-directions on the worldsheet.

3. We require that the orbits of $U(1)_L$ are closed (otherwise, we would not have called it $U(1)$).
4. The restriction of $U(1)_L$ to the null-surfaces acts as we described (see fig. 3).

The second property reflects the fact that $U(1)_L$ corresponds to the length of the operator rather than its engineering dimension.

Let $\mathcal{E}$ denote the Hamiltonian of $U(1)_L$. Let $X$ denote the phase space of the classical string, and $X/\!(\mathcal{E} = l)$ denote the Hamiltonian reduction of the phase space on the level set of $\mathcal{E}$. The basic conjecture is:

There is a one-to-one map from the phase space of the spin chain of the length $l$ to the reduced phase space of the classical string $X/\!(\mathcal{E} = l)$ preserving the symplectic structure and commuting with the action of $SO(2, 4) \times SO(6)$.

The reduction by $U(1)_L$ was discussed in [23] but only in a sector [24] in which $U(1)_L$ acts as some element of $SO(6)$. The perturbation theory in this sector was discussed in [25] (see also Section 2 of [11]).
3.2 Action of $U(1)_L$ on nearly-degenerate extremal surfaces.

In this subsection we will explain how to continue the action of $U(1)_L$ from the boundary of the phase space. Most of this section is a partial review\(^3\) of Section 3 of [12].

3.2.1 Particle on a sphere.

Consider the phase space of a particle moving on $S^5$, and restrict to the domain where the velocity of the particle is nonzero. This domain is naturally a bundle over the moduli space of equators of $S^5$; let $\pi$ denote the projection map in this bundle. A point of the phase space, corresponding to the position $x \in S^5$ and the velocity $v \in T_x S^5$, projects by $\pi$ to the equator going through $x$ and tangent to $v$. See the discussion in [11].

The symplectic form on the phase space is expressed in terms of the symplectic form on the base and the connection form $\mathcal{D}\psi$:

$$\omega = df \wedge \mathcal{D}\psi + f \pi^* \Omega$$

(8)

where $\mathcal{D}\psi = \frac{\langle p, dx \rangle}{\sqrt{\langle p, p \rangle}}$, $f = \sqrt{\langle p, p \rangle}$ ($p$ is the momentum of the particle) and $\Omega$ is the symplectic form on the moduli space of equators. The moduli space of equators $SO(6) / SO(2) \times SO(4)$ is a Kahler manifold, the symplectic form $\Omega$ is the Kahler form.

Now it is easy to construct the action of $U(1)$. One takes

$$\mathcal{V} = \frac{\partial}{\partial \psi}$$

(9)

This is a vertical vector field, it does not act on the base. The coordinate $\psi$ is essentially the angle along the equator on which the particle is moving. More explicitly:

$$\frac{\partial}{\partial \psi} x = \frac{1}{\sqrt{\langle \partial_T x, \partial_T x \rangle}} \partial_T x$$

(10)

\(^3\)Section 3 of [12] has more than just a construction of $U(1)_L$. The next step is considering the action of the Killing vector field $\frac{\partial}{\partial T}$ where $T$ is the global time in $AdS_5$ on the invariants of $U(1)_L$ and bringing the result to the form suitable for the comparison with the field theory computation. Here we are discussing only the first step.
It is easy to see that the trajectories of the vector field $\frac{\partial}{\partial \psi}$ on the phase space of a particle on $S^5$ are periodic with the period $2\pi$. One has to remember that this vector field is defined only on the open subset of the phase space, where the velocity of the particle is nonzero. But we consider fast moving strings, and the region of the phase space where the velocity is nearly zero is not important for us.

### 3.2.2 String on a sphere.

In some sense, a string is a continuous collection of particles. Therefore, it is natural to apply a similar construction to the string. Treating the string as a continuous collection of particles requires the choice of the coordinates on the worldsheet. We will therefore introduce the conformal gauge:

$$
(\partial_\tau x)^2 + (\partial_\sigma x)^2 = 0 \\
(\partial_\tau x, \partial_\sigma x) = 0
$$

In this gauge the symplectic form is:

$$
\omega = \oint d\sigma (\delta_1 x, \overset{\rightarrow}{D}_\tau \delta_2 x)
$$

In the Hamiltonian formalism, we introduce $p_A = \partial_\tau x_A \in T(AdS_5)$ — the $AdS_5$-component of the momentum, and $p_S = \partial_\tau x_S \in T(S^5)$ — the $S^5$-component of the momentum. Now we will interpret the string as a collection of particles parametrized by $\sigma$. We are tempted to interpret the vector field $(9),(10)$ acting pointwise in $\sigma$ as the required $U(1)_L$ symmetry. The generator of this symmetry would be $\int d\sigma |p_S|$. But this would be wrong. This field preserves the symplectic structure, does have periodic trajectories and acts correctly on the null surfaces. But unfortunately it does not preserve the gauge (11). It only commutes with the second constraint, $(p, \partial_\sigma x) = 0$. But it does not commute with the first one, $(p, p) + (\partial_\tau x, \partial_\sigma x) = 0$. Indeed, it commutes with $(\partial_\tau x_S)^2 = (p_S, p_S)$ but not with $(\partial_\sigma x_S)^2$. Therefore we should modify this vector field so that it still has periodic trajectories, but also commutes with the constraint. There is a systematic procedure to do this, order by order in $\frac{1}{(p_S,p_S)}$, developed in [12].

Let us summarize this procedure, or perhaps a variation of it. To make sure that the modified vector field is Hamiltonian (preserves the symplectic structure) we construct it as a conjugation of $\frac{\partial}{\partial \psi}$ with some canonical
transformation, which we denote $F$:

$$
\mathcal{V}.x = F^{-1} \left[ \frac{\partial}{\partial \psi} F[x] \right]
$$

(13)

or schematically $\mathcal{V} = F^{-1} \circ \frac{\partial}{\partial \psi} \circ F$. Since $F$ is a canonical transformation, $\mathcal{V}$ is automatically a Hamiltonian vector field. Since $F$ is single-valued, $\mathcal{V}$ generates periodic trajectories. It remains to construct $F$ such that $\mathcal{V}$ commutes with the constraint $(p, p) + (\partial_\sigma x)^2$. But to require that $F^{-1} \circ \frac{\partial}{\partial \psi} \circ F^{-1}$ commutes with $(p, p) + (\partial_\sigma x)^2$ is the same as to require that $\frac{\partial}{\partial \psi}$ commutes with $F^*[(p_S, p_S) + (\partial_\sigma x_S)^2]$ — the pullback of $(p_S, p_S) + (\partial_\sigma x_S)^2$ by $F$. Therefore we have to find such a canonical transformation $F$ that the pullback of $(p_S, p_S) + (\partial_\sigma x_S)^2$ with $F$ is annihilated by the vector field $\frac{\partial}{\partial \psi}$. In other words, we have to find a canonical transformation which removes $\psi$ from $(p_S, p_S) + (\partial_\sigma x_S)^2$; after this canonical transformation $|p_S|^2 + (\partial_\sigma x_S)^2$ becomes $|p_S|^2 + \phi_0 + \phi_1 + \ldots$ where all the $\phi_k$ for $k \geq 0$ are in involution with $\int d\sigma' |p_S| d\sigma$ and $\phi_k$ is of the order $1/|p_S|^{2k}$. This was done in Section 3 of [12]. The canonical transformation can be expanded in $1/(p_S, p_S)$; the corresponding generating function is expanded in the odd powers of $1/|p_S|$. The authors of [12] gave the explicit expression for $F$ to the first order in $1/|p_S|$, but they also give a straightforward algorithm for constructing the higher orders. (We will reconsider the higher orders from a slightly different point of view in Section 4.4, perhaps making this algorithm more precise.)

At the first order we need to find $h^{(1)}$ such that the canonically transformed constraint, which is a function of $\sigma$:

$$(p_S, p_S)(\sigma) + (\partial_\sigma x_S, \partial_\sigma x_S)(\sigma) + \{h^{(1)}, (p_S, p_S)(\sigma) + (\partial_\sigma x_S, \partial_\sigma x_S)(\sigma)\}$$

has zero Poisson bracket with $\int d\sigma'|p_S|(\sigma')$ up to the terms of the order $1/|p_S|^3$, for every $\sigma$. And $h^{(1)}$ should be of the order $1/|p_S|$. In other words, we should have:

$$
\left\{ \int d\sigma |p_S|(\sigma), \left( (\partial_\sigma x_S)^2(\sigma') + \{h^{(1)}, |p_S|^2(\sigma')\} \right) \right\} = 0 \quad (14)
$$

One can see that

$$
h^{(1)} = -\frac{1}{4} \int \frac{d\sigma}{|p_S|} \left( \partial_\sigma x_S, \frac{p_S}{|p_S|} \right)(\sigma)
$$

(15)
works. Notice also that this $h^{(1)}$ is reparametrization invariant (where $|p_S|$ transforms as a density of weight one). Therefore it commutes also with the second constraint $(p_S, \partial_\sigma x_S) = \text{const}$. Therefore, to the first order in $1/|p_S|$ the canonical transformation we are looking for is generated by this $h^{(1)}$. Then the generator of the $U(1)_L$ is, up to the terms of the order $1/|p_S|^3$:

$$E = \int d\sigma |p_S| - \{h^{(1)}, \int d\sigma |p_S|(\sigma)\} + \ldots =$$

$$= \int d\sigma \left[ |p_S| + \frac{1}{4|p_S|} \left( (\partial_\sigma x_S)^2 - \left( D_\sigma \frac{p_S}{|p_S|}, D_\sigma \frac{p_S}{|p_S|} \right) - \frac{(p_S, \partial_\sigma x_S)^2}{(p_S, p_S)} \right) + \ldots \right]$$

One can see immediately that the trajectories of this charge are closed up to the terms of the order subleading to $1/|p_S|$. Indeed, we have explained in Section 3.2.1 why the leading term gives periodic trajectories. And the second term (which as we have seen is needed to make the charge commuting with the Virasoro constraints) averages to zero on the periodic trajectories of the first term. Therefore (see for example Section 3 of [11]) the trajectories of $E$ do not drift at this order.

We will discuss the higher orders in Section 4.4.

4 **Length of the operator and local conserved charges.**

We have seen that the null-surface perturbation theory has a “hidden” symmetry $U(1)_L$. The existence of $U(1)$ symmetries acting on the phase space is typical for integrable systems, at least for those which have a finite-dimensional phase space. Corresponding conserved charges are called action variables [19]. Classical string in $AdS_5 \times S^5$ is an integrable system. Therefore, we should not be surprised to find such an action variable$^4$.

$^4$Strictly speaking, the integrability is not necessary for the construction of the action variable in *perturbation theory*. A typical example is a particle on a sphere $S^2$ in an arbitrary (polynomial) potential. When the particle moves very fast, it does not feel the potential. All the trajectories are periodic in the limit of an infinite velocity. Therefore on the boundary of the phase space, when the velocity is infinite, we have an action variable $|p|$ — the absolute value of the momentum. It is well known that the perturbation theory in $1/|p|$ allows us to extend this action variable from the boundary inside the phase space, but only in the perturbation theory. For an arbitrary potential, the perturbation
The local conserved charges in involution for the classical string in $AdS_5 \times S^5$ are explicitly known. Therefore, instead of constructing $U(1)_L$ in the perturbation theory, we can try to build it as some linear combination of the already known conserved charges. In this section, we will argue that the coefficients of this linear combination are actually fixed by the calculation of $[21, 22]$.

4.1 Local conserved charges.

Consider a string in the target space which is a product of two manifolds $A$ and $S$. We assume that the metric on $A$ has the Lorentzian signature, and the metric on $S$ has the Euclidean signature. We will need $A = AdS_5$ and $S = S^5$, but let us first consider the general $A \times S$. The string worldsheet will be denoted $\Sigma$. The classical trajectory of the string is an embedding $x : \Sigma \rightarrow A \times S$. We are going to use the fact that the target space is a direct product. A point of $A \times S$ is obviously a pair $(x_A, x_S)$ where $x_A$ is a point of $A$ and $x_S$ is a point of $S$. Therefore for each point $\zeta \in \Sigma$ we have $x(\zeta) = (x_A(\zeta), x_S(\zeta))$, where $x_A \in A$ and $x_S \in S$. Consider the 1-forms $dx_A$ and $dx_S$ on the string worldvolume, $dx_A$ taking values in $T_xA$ and $dx_S$ in $T_xS$. In other words, $\left[ \begin{array}{c} dx_A \\ dx_S \end{array} \right]$ is a differential of $x$.

The metric on $A \times S$ has the Lorentzian signature, and we consider the string worldsheets which have the induced metric with the Lorentzian signature. Pick two vector fields $\xi_+$ and $\xi_-$ on $\Sigma$, which are both lightlike but have a nonzero scalar product:

\[
\begin{align*}
(\xi_+, \xi_+) &= 0 \\
(\xi_-, \xi_-) &= 0 \\
(\xi_+, \xi_-) &\neq 0
\end{align*}
\]

These vector fields have a simple geometrical meaning. Since the worldsheet is two-dimensional, at each point we have two different lightlike directions. The vector $\xi_+$ points along one lightlike direction, and $\xi_-$ points along another. Pick a spacial contour $C$ on $\Sigma$, and a 1-form $\nu$ on $\Sigma$ such that $\nu(\xi_-) = 0$ series must diverge, because in fact there is no additional conserved quantity besides the energy. Therefore the $U(1)$ will be actually broken by effects which are not visible in the perturbation theory, unless if the potential is such that the system is integrable. We want to thank V. Kaloshin and A. Starinets for discussions of this subject.
and $\nu(\xi_+) \neq 0$. Consider the following functional:

$$Q^{[1]}[x] = \oint_C \nu \frac{\sqrt{(dx_S(\xi_+))^2}}{\nu(\xi_+)}$$  \hspace{1cm} (17)

We will prove that this functional does not depend on a particular choice of $\xi_+$, $\xi_-$, $\nu$ and $C$. This is therefore a correctly defined functional on the phase space of the string. Indeed, the only ambiguity in the choice of $\xi_+$ is $\widetilde{\xi}_+ = f(\zeta)\xi_+$ where $f$ is some function on the worldsheet. But this function cancels in (17). The ambiguity in the choice of $\xi_-$ and $\nu$ is also in rescaling which does not change (17). It remains to prove that (17) does not depend on the choice of the integration contour $C$. To prove that (17) is independent of $C$, let us choose coordinates $(\tau^+, \tau^-)$ on the worldsheet in such a way that the induced metric is $ds^2 = \rho(\tau^+, \tau^-)d\tau^+d\tau^-$. Then $\xi_+$ is proportional to $\frac{\partial}{\partial \tau^+}$ and $\xi_-$ is proportional to $\frac{\partial}{\partial \tau^-}$. In these coordinates

$$Q^{[1]} = \oint_C d\tau^+ \sqrt{(\partial_+x_S)^2}$$  \hspace{1cm} (18)

The variation of $Q^{[1]}$ under the variation of the contour is measured by the differential of the form:

$$d \left( d\tau^+ \sqrt{(\partial_+x_S)^2} \right) =$$

$$= -d\tau^+ \wedge d\tau^- \left( \partial_+x_S, D_+\partial_+x_S \right) \frac{1}{\sqrt{(\partial_+x_S)^2}}$$

But on the equations of motion $D_+\partial_+x_S = 0$. Therefore the integral does not depend on the choice of the contour.

Let us explain why on the equations of motion we have $D_+\partial_+x_S = 0$. Let $N$ be the second quadratic form of the surface, $N : S^2(T\Sigma) \to N\Sigma$ (here $N\Sigma = T(A \times S)/T\Sigma$ is the normal bundle to $\Sigma$ in $A \times S$). The second quadratic form is defined in the following way: suppose that the particle moves on $\Sigma$ with the velocity $v$, then the acceleration of the particle is $N(v)$ modulo a vector parallel to $T\Sigma$. For the surface to be extremal, the trace of $N$ should be zero. The trace of $N$ is the contraction of $N$ with the induced metric on $\Sigma$; it is a section of $N\Sigma$. The trace of $N$ is proportional to $D_+\partial_-x$, therefore we should have:

$$D_+\partial_-x = f^+(\tau^+, \tau^-)\partial_+x + f^-(\tau^+, \tau^-)\partial_-x$$
But notice that \((D_+ \partial_-, \partial_+) = (D_- \partial_+, \partial_+) = 0\) therefore \(f^+ = f^- = 0\).

Another conserved charge is:

\[
\tilde{Q}^{[1]} = \oint_C d\tau^\pm \sqrt{\partial_\pm x_S}^2
\]  

(19)

Are there charges containing higher derivatives of \(x_S\)? Let us consider the following expression:

\[
J^{[2]}(\tau_+, \tau_-) = \frac{1}{|\partial_+ x_S|} \left( D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|}, D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|} \right)
\]

(20)

Even though \(D_+ \partial_+ x_S = 0\) it is not true that \(\partial_- J^{[2]}_\pm\) is zero. The covariant derivatives \(D_+\) and \(D_-\) do not commute, therefore \(D_- D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|} \neq 0\). In fact, for any function \(w : \Sigma \rightarrow T(A \times S)\) we have

\[
[D_+, D_-]w = R(\partial_+ x, \partial_- x).w
\]

(21)

where \(R\) is the Riemann tensor of \(A \times S\). Now we have to start using that \(S\) is a sphere. For \(S = S^5\), the Riemann tensor is constructed from the metric tensor, and

\[
[D_+, D_-]w = \partial_+ x(\partial_- x, w) - \partial_- x(\partial_+ x, w)
\]

(22)

Now consider the following differential form:

\[
\lambda = 2 \frac{d\tau^-}{|\partial_+ x_S|} (\partial_- x_S, \partial_+ x_S) + \frac{d\tau^+}{|\partial_+ x_S|} \left( D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|}, D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|} \right)
\]

(23)

Using (22) we can show that \(d\lambda = 0\), therefore \(\oint \lambda\) is a local conservation law. We use the formula \(D_- D_+ \partial_+ x_S = (\partial_+ x_S)^2 \partial_- x_S - (\partial_+ x_S, \partial_- x_S) \partial_+ x_S\) which is special for \(S^5\). We will denote this charge \(Q^{[2]}\). There is also a charge \(\tilde{Q}^{[2]}\) which is obtained from (23) by replacing \(\tau^+\) with \(\tau^-\) and \(\partial_+\) or \(D_+\) with \(\partial_-\) or \(D_-\).

These charges are just the first examples of an infinite family of charges, which are all in involution. This infinite family was constructed in [20].

A particularly important linear combination is

\[
\mathcal{E}_2 = \frac{1}{2} (Q^{[1]} - \tilde{Q}^{[1]})
\]

(24)

The construction of this charge requires only that the target space is a direct product of two manifolds.
4.2 Local conserved charges are invariant under $U(1)_L$.

Consider a local conserved charge $Q$ acting trivially on the $AdS$ part of the worldsheet. In the conformal gauge, this means that $Q$ is constructed as a contour integral of some combination of $x_S$ and $p_S$. Let us decompose $Q$ in the inverse powers of $|p_S|$: 

$$Q = Q_m + Q_{m+1} + Q_{m+2} + \ldots$$

where $m$ is a non-negative integer, the “order” of the charge; $Q_m$ is of the order $1/|p_S|^{2m-1}$, $Q_{m+1}$ is of the order $1/|p_S|^{2m+1}$ etc. We have to require that $Q$ is in involution with the Virasoro constraints. In particular, it should be in involution with $|p_S(\sigma)|^2 + (\partial_\sigma x_S(\sigma))^2$ for an arbitrary $\sigma$. (Here we used that $Q$ is trivial in AdS-part.) Let us now apply the canonical transformation $F$ which we described in Section 3.2.2. After this canonical transformation $|p_S|^2 + (\partial_\sigma x_S)^2$ becomes $|p_S|^2 + \phi_0 + \phi_1 + \ldots$ where all the $\phi_k$ for $k \geq 0$ are in involution with $|p_S(\sigma)|^2 + (\partial_\sigma x_S(\sigma))^2$ for an arbitrary $\sigma$. And $Q = Q_m + Q_{m+1} + \ldots$ becomes $Q' = Q'_m + Q'_{m+1} + \ldots$, where $Q'$ is the canonically transformed $Q$. We should have:

$$\{ |p_S(\sigma)|^2 + \phi_0(\sigma) + \phi_1(\sigma) + \ldots, Q'_m + Q'_{m+1} + \ldots \} = 0$$

for an arbitrary $\sigma$. At the leading order in $|p_S|$ this implies that $\int d\sigma |p_S(\sigma)|$ is in involution with $Q'_m$. At the next order, it follows that for all values of $\sigma$ the expression $\{|p_S(\sigma)|^2, Q'_{m+1}\}$ is in involution with $\int d\sigma' |p_S(\sigma')|$. This implies that:

$$\left\{ \int d\sigma' |p_S(\sigma')|, \left\{ \int d\sigma |p_S(\sigma)|, Q'_{m+1} \right\} \right\} = 0$$

Since the vector field generated by $\int d\sigma |p_S(\sigma)|$ is periodic, this equation implies that $\int d\sigma |p_S(\sigma)|$ is in involution with $Q'_m$. An analogous argument at higher orders shows that all the $Q'_{m+j}$ commute with $\int d\sigma |p_S(\sigma)|$. Therefore $Q'$ is in involution with the expression $\int d\sigma |p_S(\sigma)|$ which is the generator of $U(1)_L$. The conserved charges of [20] do have an expansion of the form (25) therefore they should commute with $U(1)_L$. This reinforces our conjecture that $U(1)_L$ should be a combination of the local conserved charges.

4.3 A geometrical meaning of $U(1)_L$.

We can try to make more transparent the geometrical meaning of $U(1)_L$ by drawing an analogy with the Liouville theorem for finite-dimensional inte-
grable systems. A mechanical system with $2n$-dimensional phase space is integrable if there are $n$ functions $F_1, \ldots, F_n$ in involution with each other, and the Hamiltonian is a function of $F_1, \ldots, F_n$. Then, there are $n$ action variables $I_1, \ldots, I_n$, each of them being some combination of $F_1, \ldots, F_n$:

$$I_j = I_j(F_1, \ldots, F_n)$$

such that each $I_j$ generates $U(1)$ (has periodic orbits). In this paper we are dealing with an infinite-dimensional system, a classical string in $AdS_5 \times S^5$. We can take the first Pohlmeyer charge $Q^{[1]} - \tilde{Q}^{[1]}$ as a Hamiltonian. This Hamiltonian is presumably integrable, because there is an infinite family of higher charges commuting with it. On the other hand, it does not have any special periodicity properties (we do not see any reason why it would). This means that the closure of the orbit of $Q^{[1]} - \tilde{Q}^{[1]}$ is an invariant torus. Our $U(1)_L$ commutes with $Q^{[1]} - \tilde{Q}^{[1]}$ (This fact is seen immediately, because $Q^{[1]}$ can be rewritten as $\int d\tau \sqrt{-(\partial_+ x_A, \partial_+ x_A)}$ and by definition $U(1)_L$ does not act on the AdS-part of the worldsheet.) Therefore $U(1)_L$ should be a shift of one of the angles parametrizing the invariant torus of $Q^{[1]} - \tilde{Q}^{[1]}$. The angles parametrizing the invariant torus are in correspondence with its one-dimensional cycles. Which cycle corresponds to $U(1)_L$? Every invariant torus can be connected by a one-parameter family of invariant tori to a torus on the boundary of the phase space (or the one very close to the boundary). This means that every 1-cycle is connected to some 1-cycle on a torus on the boundary — the space of null-surfaces. We should take that 1-cycle which is connected to the orbit of $U(1)_L$ on the null-surfaces, described in Section 3.1. The corresponding action variable is $E$ — the generator of $U(1)_L$. These arguments show the uniqueness of $U(1)_L$.

The first Pohlmeyer charge $Q^{[1]} - \tilde{Q}^{[1]}$ has a special property: it actually generates $U(1)_L$ on the boundary. Therefore the difference between $Q^{[1]} - \tilde{Q}^{[1]}$ and $E$ should be a combination of charges vanishing at the boundary. We expect that this is an infinite and linear combination. Indeed, the construction of [12] tells us that the charge we are looking for is local at each order in $1/|p_S|$. A nonlinear combination of the charges would be non-local (a product of integrals).

---

5 It is a natural Hamiltonian on the phase space of a classical string in any case when the target space is a direct product of two manifolds.
4.4 A different point of view on the perturbation theory; higher orders.

In Section 3 we constructed \( U(1)_L \) as \( F^{-1} \circ \frac{\partial}{\partial \psi} \circ F \) where \( \frac{\partial}{\partial \psi} \) is generated by \( \int d\sigma |p_S(\sigma)| \) and \( F \) is the canonical transformation such that \( F^{-1} \circ \frac{\partial}{\partial \psi} \circ F \) commutes with \( |p_S(\sigma)|^2 + |\partial_\sigma x_S(\sigma)|^2 \). This canonical transformation is constructed in the perturbation theory, order by order in \( \frac{1}{|p_S|^2} \).

A disadvantage of this procedure is that at each order we have to require that our \( U(1)_L \) commutes with \( |p_S(\sigma)|^2 + |\partial_\sigma x_S(\sigma)|^2 \) for any \( \sigma \). Since there are infinitely many values of \( \sigma \) we have to impose infinitely many conditions on \( F \) at each order. At the first order, we have seen in Section 3.2.2 that these conditions are not really independent; one generating function \( h^{(1)} \) takes care of all of them — see Eq. (14). At the higher orders, this is not immediately obvious. Therefore, we would like to propose a slightly different way of constructing \( F \). Let us forget for a moment about the Virasoro constraint; instead of the phase space of the string consider the space of harmonic maps \( x(\tau, \sigma) \). Instead of requiring that \( U(1)_L \) commutes with \( |p_S(\sigma)|^2 + |\partial_\sigma x_S(\sigma)|^2 \), let us require that \( U(1)_L \) commutes with \( Q^{[1]} = \int d\sigma |\partial_\tau x_S(\sigma)| \). We will see that the requirement that \( U(1)_L \) commutes with \( Q^{[1]} \) already fixes \( U(1)_L \) in the perturbation theory, and the resulting \( U(1)_L \) will automatically commute with the Virasoro constraints.

As in Section 3, we look for the generator of \( U(1)_L \) as a pullback by a canonical transformation of \( \int |p_S(\sigma)| d\sigma \). In other words, let us look for such a canonical transformation \( F \) that \( \int d\sigma |p_S(\sigma)| \) commutes with \( F^* Q^{[1]} \) (the pullback of \( Q^{[1]} \) by \( F \)). We can construct such a canonical transformation order by order in the perturbation theory. Let us denote \( K = \int d\sigma |p_S(\sigma)| \). We have:

\[
Q^{[1]} = K + q_1 + q_2 + \ldots
\] (28)

Under the rescaling \( p_S \to tp_S \): \( K \to tK, q_1 \to t^{-1}q_1, q_2 \to t^{-3}q_2, q_m \to t^{1-2m}q_m \). The symplectic structure is of the degree 1: \( \omega \to t\omega \), therefore the Poisson brackets are of the degree -1: \( \{,\} \to t^{-1}\{,\} \). We can construct \( F \) order by order in this grading. We have:

\[
F^*(Q^{[1]}) = K + q'_1 + q'_2 + \ldots + q'_m + \ldots
\] (29)

Suppose that we have already found \( F \) such that \( q'_1, \ldots, q'_{m-1} \) commute with \( K \). At the order \( m \), we want to modify \( F \) by the canonical transformation
with the generating function $f_m$ of the order $|p_S|^{1-2m}$ so that $q'_m + \{ f_m, K \}$ commutes with $K$. Since $K$ is periodic, we can decompose

$$ q'_m = q'_{m,0} + \sum_{k \neq 0} q'_{m,k} \tag{30} $$

where $\{ K, q'_{m,k} \} = ikq'_{m,k}$. Then we should take

$$ f_m = \sum_{k \neq 0} \frac{1}{ik} q'_{m,k} \tag{31} $$

Repeating this procedure at higher orders, we end up with the function $F$ such that $\{ K, F^*(Q^{[1]}) \} = 0$.

The reparametrization invariance is manifestly preserved at each order, therefore the resulting charge $F^{-1} \circ \frac{\partial}{\partial \phi} \circ F$ will commute with $(p_S, \partial_\sigma x_S)(\sigma)$ for any $\sigma$. Also, the fact that $Q^{[1]}$ is reparametrization-invariant and the arguments analogous to the discussion at the end of Section 4.2 show that $F^{-1} \circ \frac{\partial}{\partial \phi} \circ F$ will automatically commute with $|p_S(\sigma)|^2 + |\partial_\sigma x_S(\sigma)|^2$, as well as with the higher Pohlmeyer charges. Indeed, we know that $F^* Q^{[1]} = K + q'_1 + q'_2 + \ldots$ commutes with $F^* (|p_S|^2(\sigma) + |\partial_\sigma x_S|^2(\sigma)) = |p_S|^2 + \phi_0 + \phi_1 + \ldots$; therefore

$$ \{ K, \phi_0 \} = \{|p_S|^2(\sigma), q'_1 \} \Rightarrow \{ K, \{ K, \phi_0 \} \} = 0 \Rightarrow \{ K, \phi_0 \} = 0 $$

$$ \{ K, \phi_1 \} + \{ q'_1, \phi_0 \} = \{|p_S|^2(\sigma), q'_2 \} \Rightarrow \{ K, \{ K, \phi_1 \} \} = 0 \Rightarrow \{ K, \phi_1 \} = 0 $$

etc.

We used the periodicity of the trajectories of $K$ when we claimed that $\{ K, \{ K, \phi \} \} = 0$ implies $\{ K, \phi \} = 0$. Indeed, for any functional $\phi$ on the phase space, if $\{ K, \{ K, \phi \} \} = 0$ then $\{ K, \phi \}$ is constant on the trajectories of $K$. But if this constant were nonzero, then the change of $\phi$ along the trajectory of $K$ would accumulate over the period of $K$, which would contradict the single-valuedness of $\phi$ on the phase space.

### 4.5 An infinite combination of local conserved charges.

Expanding (23) in the conformal gauge in the powers of $\frac{1}{|p_S|}$ we get:

$$ \frac{1}{2} \left( Q^{[2]} - \tilde{Q}^{[2]} \right) = \int d\sigma \left[ -2|p_S| + \frac{3}{|p_S|}(\partial_\sigma x_S)^2 - \frac{1}{2} \right] \tag{32} $$

22
\[-\frac{3}{|p_S|^3}(p_S, \partial_\sigma x_S)^2 + \frac{4}{|p_S|} \left( D_\sigma \frac{p_S}{|p_S|}, D_\sigma \frac{p_S}{|p_S|} \right) + \ldots \]

And for \(Q^{[1]}\) we get:

\[Q^{[1]} - \tilde{Q}^{[1]} = \int d\sigma \left[ 2|p_S| + \frac{1}{|p_S|}(\partial_\sigma x_S)^2 - \frac{1}{|p_S|^3}(p_S, \partial_\sigma x_S)^2 + \ldots \right] \quad (33)\]

We have:

\[\frac{1}{16} \left[ 7(Q^{[1]} - \tilde{Q}^{[1]}) - \frac{1}{2}(Q^{[2]} - \tilde{Q}^{[2]}) \right] = \int d\sigma \left[ |p_S| + \frac{1}{4|p_S|} \left( (\partial_\sigma x_S)^2 - \left( D_\sigma \frac{p_S}{|p_S|}, D_\sigma \frac{p_S}{|p_S|} \right) - \frac{(p_S, \partial_\sigma x_S)^2}{(p_S, p_S)} \right) + \ldots \right]\]

This coincides with the result (16) for \(E\) which we know from the perturbation theory. We see that up to the terms of the order \(\frac{1}{|p_S|^3}\) the Hamiltonian of \(U(1)_L\) can be represented as a sum of the first two commuting local charges. We conjecture that \(U(1)_L\) is in fact an infinite combination of the local conserved charges. The perturbation theory construction suggests that it should be a worldsheet parity-invariant combination.

The coefficients of this linear combination can be found from considering the conserved charges of particular solutions. There is a special class of fast moving strings, the so-called “rigid” strings. For these “rigid” strings, the corresponding field theory operators are known a priori. These operators provide local extrema of the anomalous dimension in the sector with the given charges. These “rigid” solutions were classified in [10, 28]. They are related to the solutions of the Neumann integrable system. The local conserved charges of some rigid strings were computed in [21, 22].

In [21], the local conserved charges are denoted \(\mathcal{E}_k\). (This agrees with our notation \(E\) for the Hamiltonian of \(U(1)_L\).) The precise definition of \(\mathcal{E}_k\) is given in Section 3 of [21]. The relation to our notations is: \(Q^{[1]} - \tilde{Q}^{[1]} = 2\mathcal{E}_2, Q^{[2]} + 2Q^{[1]} - \tilde{Q}^{[2]} - 2\tilde{Q}^{[1]} = -4\mathcal{E}_4\). The conserved charges have the following structure:

\[\mathcal{E}_n = \delta_{2n\mathcal{J}} + \epsilon_n^{(1)} \mathcal{J} + \epsilon_n^{(2)} \mathcal{F}_3 + \epsilon_n^{(3)} \mathcal{F}_5 + \ldots \quad (34)\]

where \(\mathcal{J}^2 = \lambda/J^2\), and \(J\) is a particular combination of the \(SO(6)\) momenta. The coefficients \(\epsilon_n^{(m)}\) depend on what kind of a rigid string is considered (the ratio of spins). But the authors of [21] noticed that the coefficients \(\epsilon_n^{(m)}\)
for different values of $n$ are not independent. For all the solutions they considered, they find that:

$$E_{10} + \frac{74}{7} E_8 + \frac{1898}{35} E_6 + \frac{6922}{35} E_4 + \frac{32768}{35} (E_2 - J) \sim \frac{1}{J^9}$$

(35)

This means that up to the terms of the order $1/|p_S|^9$ we should have:

$$J = E_2 + \frac{6922}{32768} E_4 + \frac{1898}{32768} E_6 + \frac{370}{32768} E_8 + \frac{35}{32768} E_{10} + \ldots$$

(36)

At first this formula looks rather strange, because it seems to imply that a certain combination of Pohlmeyer charges (which all commute with $SO(6)$) is equal to some component of the angular momentum (which transforms in the adjoint of $SO(6)$). We propose the following resolution of this puzzle. The right hand side of (36) is actually the action variable, which for a particular class of the solutions considered in [21, 22] happens to be equal to the $SO(6)$ charge $J$ (because these particular solutions correspond to the chiral operators on the field theory side; see Section 2 of [11]). In other words, for this particular class of solutions the angular momentum $J$ should be equal to our action variable $E$. The general formula should have on the left hand side $E$, the generator of $U(1)_L$, instead of $J$:

$$E = E_2 + \frac{6922}{32768} E_4 + \frac{1898}{32768} E_6 + \frac{370}{32768} E_8 + \frac{35}{32768} E_{10} + \ldots$$

(37)

This gives the expansion of the generating function of $U(1)_L$ to the order $1/|p_S|^9$. It would be interesting to check explicitly, beyond the order $1/|p_S|$, that this Hamiltonian generates periodic trajectories.

### 4.6 More on the perturbation theory.

Here we want to present a slightly different and perhaps simpler way of thinking about the continuation of $U(1)_L$ in the perturbation theory. Consider the Hamiltonian vector field $\xi_{E_2}$ corresponding to the first Pohlmeyer charge $E_2$. Consider the canonical transformation

$$F = e^{2\pi \xi_{E_2}}$$

(38)

This canonical transformation is the Hamiltonian flow generated by $E_2$ by the time $2\pi$. The trajectories of $E_2$ are almost periodic in the null-surface limit, therefore we can write

$$F = e^{v_1}$$

(39)
where \( v_1 \) is a vector field of the order \( 1/|p_S|^2 \). This vector field can be constructed in the following way. Let us choose the conformal gauge on the worldsheet. We know from (33) that \( \mathcal{E}_2 = \int \sigma |p_S| + f = K + f \) where \( f \) is of the order \( 1/|p_S| \). Taking into account that \( e^{2\pi \xi K} = 1 \) we get

\[
F = 1 + \int_0^{2\pi} ds e^{-s \xi K} \xi_f e^{s \xi K} + \int_{s_1 < s_2} ds_2 ds_1 e^{-s_2 \xi K} \xi_f e^{s_2 \xi K} e^{-s_1 \xi K} \xi_f e^{s_1 \xi K} + \ldots = \exp \left\{ \int_0^{2\pi} ds e^{-s \xi K} \xi_f e^{s \xi K} + \frac{1}{2} \int_{s_1 < s_2} ds_1 ds_2 [e^{-s_2 \xi K} \xi_f e^{s_2 \xi K}, e^{-s_1 \xi K} \xi_f e^{s_1 \xi K}] + \ldots \right\}
\]

This defines \( v_1 = \int_0^{2\pi} ds e^{-s \xi K} \xi_f e^{s \xi K} + \frac{1}{2} \int_{s_1 < s_2} ds_1 ds_2 [e^{-s_2 \xi K} \xi_f e^{s_2 \xi K}, e^{-s_1 \xi K} \xi_f e^{s_1 \xi K}] + \ldots \) in the perturbation theory. (Notice that \( f \) can be decomposed in the Fourier series \( f = \sum f_k \) so that \( \{K, f_k\} = ik f_k \) and then the leading term of \( v \) is the zero mode \( \xi_{f_0} \); this is the “averaging” procedure of [11].) The vector field \( v_1 \) defines the vector field on the moduli space of null-surfaces as a limit \( \lim_{|p_S| \to \infty} (\mathcal{E}_2^2 v_1) = \lim_{|p_S| \to \infty} (L^2 v_1) \) (where \( L \) was defined in Section 2.3).

This vector field determines the slow evolution of the null-surface; it is the Hamiltonian vector field of the Landau-Lifshitz model on the moduli space of the null-surfaces modulo \( U(1) \). As in [21] we can consider the improved currents \( \mathcal{E}_{2n}' \). By definition \( \mathcal{E}_{2n}' \) is a linear combination of \( \mathcal{E}_2, \ldots, \mathcal{E}_{2n} \) such that \( \mathcal{E}_{2n}' = O(|p_S|^{-2n+3}) \). The Hamiltonian of the Landau-Lifshitz model is the null-surface limit of \( \mathcal{E}_4' \), more precisely \( \lim_{|p_S| \to \infty} (\mathcal{E}_2^2 \mathcal{E}_4') \). Given that \( \mathcal{E}_2 \) and \( \mathcal{E}_4' \) are in involution, this implies that for some \( a_1 \) we have

\[
F_1 = e^{2\pi (\xi_2 + a_1 \xi_{\mathcal{E}_4})} = e^{v_2}
\]

where \( v_2 \) is of the order \( 1/|p_S|^4 \). Again, \( \lim_{|p_S| \to \infty} (\mathcal{E}_4^2 v_2) \) determines a vector field on the moduli space of null-surfaces. (It can be also defined as \( \lim_{|p_S| \to \infty} (L^4 v_2) \).) This vector field commutes with the time evolution of the Landau-Lifshitz model. We conjecture that this vector field is generated
by the second conservation law of the Landau-Lifshitz model, which is proportional to the null-surface limit\(^6\) of \(E'_6\), more precisely \(\lim_{|p_S| \to \infty} (E'^3_2 E'_6)\) or \(\lim_{|p_S| \to \infty} (L^3 E'_6)\). Repeating this procedure we get
\[
e^{2\pi(\xi E_2 + a_1 E'_4 + a_2 E'_6 + \ldots)} = 1
\]
in the perturbation theory. These arguments lead us to the following conclusion. First, we see once again that there is a linear combination \(E_2 + a_1 E'_4 + a_2 E'_6 + \ldots\) generating periodic trajectories. Second, the moduli space of null-surfaces in \(AdS_5 \times S^5\) modulo \(U(1)_L\) is naturally equipped with the infinite tower of Hamiltonians in involution which are the null-surface limit of the Pohlmeyer charges. This is the generalized Landau-Lifshitz model.

## 5 Conclusion.

Given a manifold with the metric of the Lorentzian signature it is possible to construct the extremal surfaces in this manifold as perturbations of the null-surfaces. In the special case when the manifold is \(AdS_5 \times S^5\) the AdS/CFT correspondence predicts that the extremal surfaces (which are the same as classical string worldsheets) correspond to the states of the large R-charge in the \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory. From this point of view considering the extremal surface as a perturbation around the null-surface corresponds to considering the state of the interacting Yang-Mills theory as a perturbation of the state of the free Yang-Mills theory. This correspondence has the following important features:

1. Locality. In the planar limit (the limit of infinitely many colors) the Yang-Mills perturbation theory is local in the following sense: the Feynman diagrams involve only interactions of those elementary field operators which stand next to each other in the product under the trace. We expect that the correspondence between the parton chains and the string worldsheets is local in each order of the perturbation theory, and therefore the locality of the planar Yang-Mills perturbation theory should correspond to the locality of the string worksheet theory.

\(^6\)Notice that the null-surface limit of \(E'_{2n}\) is invariant under the \(U(1)_L\) symmetry of the null-surfaces, because the \(U(1)_L\) symmetry of the null-surfaces is generated by the conserved quantity \(E_2\) which commutes with \(E'_{2n}\).
2. Integrability. The classical string worldsheet theory in $AdS_5 \times S^5$ is an integrable system.

Because of the integrability, there is an infinite family of local conserved charges in involution. In this paper we have argued that an infinite linear combination of these local charges generates periodic trajectories on the string phase space. This statement can be verified order by order in the null-surface perturbation theory, and it is local at each order. This means that the "slow evolution" of nearly-degenerate extremal surfaces \[11\] is essentially controlled by the Pohlmeyer charges (we will further discuss the slow evolution and how it is related to the Pohlmeyer charges in the second paper of \[29\]).

It would be interesting to further study the null-surface perturbation theory from the point of view of the integrability. It would be especially interesting to study those manifestations of the integrability which are local. The Bäcklund transformations \[20\] is one example. They allow us to construct a new extremal surface from a given extremal surface, and in the null-surface perturbation theory these transformations are well-defined and local at each order. The Bäcklund transformations are closely related to the local conserved charges, and in fact the hidden symmetry $U(1)_L$ can be considered as a consequence of the special properties of these transformations. We will discuss the relation between $U(1)_L$ and the Bäcklund transformations in the third paper of \[29\].

The general problem is, to study those aspects of the integrability which are local in the null-surface perturbation theory. (Without a reference to the null-surface perturbation theory, we would define the locality as some sort of an independence of the choice of the boundary conditions.) This problem arises also on the field theory side. The Feynman diagrams in the planar limit are local, but we usually compute the anomalous dimension of the single-trace operators which requires summing over the whole parton chain. The spectrum of single-trace operators at large $N$ is certainly an invariant of the theory, but it is non-local. If it is true that the planar $\mathcal{N} = 4$ Yang-Mills theory is integrable, it would be important to understand the integrability as much as possible in terms of the local properties of the parton chain (perhaps on the level of the individual Feynman diagrams).

We have defined the $U(1)_L$ strictly speaking in the perturbation theory, but it should be actually well-defined in the domain of the string phase space where the velocity of the string is large enough. In other words, the series
defining the $U(1)_L$ in fact converges if the string moves fast enough. It would be interesting to study the global properties of $U(1)_L$.

An important question is what happens to $U(1)_L$ after the quantization. To answer this question we should first include fermions. Important steps in this direction were made recently in [30, 31, 32].

It would be interesting to understand better why the "length" is conserved on the field theory side. (Why is there a quantum number $L$ with a well-defined classical limit?) To which extent the conservation of $L$ is related to the integrability of the planar Yang-Mills theory? What happens to $L$ when we turn on the fermions?

Null-surfaces are obviously an important ingredient in our construction. The correspondence between the null-surfaces and the "engineering" operators in the free field theory is rather straightforward; the null-surfaces in $AdS_5 \times S^5$ appear very naturally in the description of the coherent states of the free theory [31, 32]. Is there any way to see directly on the field theory side, that turning on the Yang-Mills interaction corresponds to the deformation of the null-surface into the extremal surface?

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