THE TRANSVERSE LATTICE IN 2+1 DIMENSIONS

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INTRODUCTION

Based on an idea due to Bardeen and Pearson, we formulate the light-front Hamiltonian problem for \( SU(N) \) Yang-Mills theory in (2+1)-dimensions using two continuous space-time dimensions with the remaining space dimension discretized on a lattice. We employ analytic and numerical methods to investigate the string tension and the glueball spectrum in the \( N \rightarrow \infty \) limit. Our preliminary results show qualitative agreement with recent Euclidean lattice Monte Carlo simulations. In the following, we attempt to give a more pedagogical introduction to the idea of the transverse lattice; more detail may be found in Ref. [1].

MOTIVATION

There exists a large gap between the quantum field theory of QCD with its many successes in the context of perturbation theory and experimental observables associated with QCD bound states: the hadron mass spectrum, structure functions, form factors, \textit{et cetera}. There has been little progress during the last 20 years in building a bridge between the field theory and these non-perturbative properties of the hadron spectrum.

\( \ast \) Presented by B. van de Sande
Let us contrast two attempts to bridge this gap: the Euclidean lattice Monte Carlo (ELMC) approach and light-front field theory where one uses Hamiltonian techniques on a theory which is quantized on a surface of constant \( x^+ = (x^0 + x^3)/\sqrt{2} \). As far as progress is concerned, ELMC is much further along, benefiting from a large research effort since the mid 1970’s. In contrast, the light-front approach is not so far along; most research effort has occurred since late 1980’s. For ELMC, further progress is limited mainly by the speed of available computers. For light-front, progress is currently limited by conceptual issues, \textit{exempli gratia} renormalization. Even when successful, ELMC is not able to measure many interesting observables, such as structure functions, directly. For light-front field theory, physically interesting observables are quite easily calculated from the bound state wavefunctions. The approach that we will discuss here, the transverse lattice, uses ideas from both the light-front and lattice approaches.

Instead of solving the full theory of QCD, we will focus on a particular model: \( SU(N) \) Yang Mills theory in 2+1 dimensions in the \( N \to \infty \) limit. Why 2+1 dimensions instead of 3+1 dimensions? Aside from the obvious fact that it has fewer degrees of freedom, one should note that the theory is super-renormalizable. In the context of lattice calculations, this means that there is no critical point and the associated ‘critical slowing’ is absent. Consequently, excellent lattice spectra are available for 2+1 dimensions \cite{2} which one can use as a comparison. Why large \( N \)? The \( N \to \infty \) limit allows considerable simplification of our computational problem; in addition, the lattice data indicates that \( 1/N \) corrections to the low energy spectrum are quite small.

THE TRANSVERSE LATTICE

A number of years ago Bardeen and Pearson \cite{3, 4} formulated a light-front Hamiltonian version of lattice gauge theory, which makes use of the fact that two components of the gauge field are unphysical. In this approach two spacetime dimensions are kept continuous \( x^\pm = (x^0 \pm x^2)/\sqrt{2} \) while the remaining transverse spatial dimension \( x^1 \) is discretized on a lattice. Lattice sites are labeled by integer-valued index \( i \) and the lattice constant is \( a \). Following Bardeen and Pearson, we associate longitudinal gauge fields \( A^\pm_i(x^+, x^-) \) with lattice site \( i \) and link variable \( U_i(x^+, x^-) \) with the link between sites \( i \) and \( i+1 \),

\[
A^\pm_i \quad U_i \quad A^\pm_{i+1} \quad U_{i+1}
\]

\[ i \quad \times \quad \times \quad i + 1 \]  \hspace{1cm} (1)

Note that \( A^\pm_i \) and \( U_i \) are \( N \times N \) color matrices where \( A^\pm_i \) is Hermitian and traceless and \( U_i \in SU(N) \). We can define the color Maxwell tensor

\[
F_{\alpha \beta}^i = \partial^\alpha A^\beta_i - \partial^\beta A^\alpha_i + i[A^\alpha, A^\beta] \]  \hspace{1cm} (2)
where \( \alpha, \beta \in \{+, -\} \) along with a covariant derivative

\[
D^\alpha U_i = \left( \partial^\alpha + iA^\alpha_{i-1} \right) U_i - iU_i A^\alpha_{i+1}.
\]  

(3)

Using this notation, one can write down an action,

\[
A = a \sum_i \int dx^+ dx^- \text{ Tr} \left\{ -\frac{1}{4g^2} F_{\alpha\beta,i} F^{\alpha\beta}_i + \frac{1}{2a^2 g^2} D_{\alpha} U_i D^{\alpha} U_i^\dagger \right\},
\]  

(4)

which has several important properties:

- If we take the naive continuum limit \( a \to 0 \) where \( U_i = \exp(-iaA_i) \), we recover the continuum Yang-Mills action.
- By construction, this action is 100% gauge invariant.
- It has an automatic confinement mechanism. We will say more about this later.

**Linearization**

In an ideal world, Eqn. (4) is the action one would quantize. However, we are faced with a problem. The field theory on each link of this transverse lattice is an \( SU(N) \times SU(N) \) non-linear \( \sigma \)-model. Upon quantizing this theory, we must enforce \( N^2 \)-nonlinear constraints on each link. Although some attempt has been made to do this quantization [5], success has remained elusive. Instead, we perform the linearization suggested by the original authors [3]. In this scheme one replaces the unitary link variables \( U_i \) with \( N \times N \) complex matrices \( M_i \), that is, \( U_i \to \sqrt{2ag^2} M_i \).

The obvious next step would be to include an effective potential to enforce the constraint \( \sqrt{2ag^2} M_i \in SU(N) \) dynamically. For instance the term\[5\]

\[
\lambda_c \text{ Tr} \left\{ \left( M_i^\dagger M_i - 2ag^2 \right)^2 \right\}
\]  

\[
= \lambda_c \text{ Tr} \left\{ M_i^\dagger M_i M_i^\dagger M_i \right\} - 4ag^2 \text{ Tr} \left\{ M_i^\dagger M_i \right\} + \left( 2ag^2 \right)^2
\]  

(5)

is minimized precisely when \( \sqrt{2ag^2} M_i \in U(N) \). One could imagine adding such a term to the Hamiltonian, taking the \( \lambda_c \to \infty \) limit, and recovering the transverse lattice action [6]. Unfortunately, a closer inspection reveals several problems:

- The second term in (6) is a mass term with negative \( (\text{mass})^2 \) coefficient. Quantization of the theory breaks down for negative \( (\text{mass})^2 \) for the same reason that it breaks down in the 't Hooft model [3] and the spectrum becomes unbounded from below. Presumably this negative \( (\text{mass})^2 \) term is a signal for the presence of

\[\text{\textsuperscript{*}}\text{The } O(n) \sigma\text{-models are a good example of where this linearization is known to work.}\]

\[\text{\textsuperscript{†}}\text{For large } N, \text{ we can ignore the distinction between } U(N) \text{ and } SU(N).\]
spontaneous symmetry breaking. One could imagine solving the broken phase of the theory using zero mode techniques [7]. The resulting ‘shifted’ theory would have some complicated effective potential representing the effects of the spontaneous symmetry breaking; this effective potential is undoubtedly different than the above form (5).

- Tadpole contractions associated with four-point interactions, such as the first term in (5), produce a divergent shift in the mass term $\text{Tr}\{M_i^\dagger M_i\}$. Renormalization leaves the mass as a free parameter in the Hamiltonian.

- A successful renormalization group analysis would introduce couplings between neighboring links. For instance, one expects terms associated with the constraint $2ag^2M_iM_{i+1}\in SU(N)$.

- From a more practical viewpoint, implementing the $\lambda_c \to \infty$ limit forces half of the dynamical degrees of freedom at each link to decouple from the theory. This is not good news for numerical calculations where the computational difficulty depends critically on the number of dynamical degrees of freedom.

As a consequence of these considerations, a first-principles construction of the correct effective potential $V_i$ is not a simple matter. Instead, we take a more pedestrian approach and include in $V_i$ all operators up to fourth order in $M_i$ containing one color trace

$$V_i = \mu^2 \text{Tr}\{M_i^\dagger M_i\} + \frac{\lambda_1}{aN} \text{Tr}\{M_i^\dagger M_i M_i^\dagger M_i^\dagger\} + \frac{\lambda_2}{aN} \text{Tr}\{M_i^\dagger M_{i+1} M_{i+1}^\dagger M_i^\dagger\}$$

and try to determine the associated coupling constants empirically. Note that the $\lambda_1$ term is local to one link and the $\lambda_2$ term acts on two adjacent links.

**Quantization**

Next, we quantize the theory, write down the Hamiltonian $P^-$, and construct a basis of states. Further details may be found in Ref. [1]. At each lattice site $i$ we have a 1+1 dimensional gauge theory with conserved current

$$J_i^\alpha = i \left( M_i \, \hat{\partial}^\alpha \, M_i^\dagger + M_i^\dagger \, \hat{\partial}^\alpha \, M_i \right).$$

We set $\partial_- A^+_i = 0$ by choice of gauge and throw away the associated dynamical zero mode $\int dx^- A^+_i$. The $A^-_i$ field obeys the equation of motion

$$\left(\partial_-\right)^2 A^-_i = \frac{g^2}{a} J^+_i.$$
As evidenced by the absence of time derivatives $\partial_+$ in Eqn. (9), $A_i^-$ is not a true dynamical degree of freedom but is constrained. We solve the constraint equation (9) for $A_i^-$ and remove it from the theory. At this point, quantization of the theory is straightforward. The momentum conjugate to $M_i(x^-)$ is $\partial_- M_i^\dagger(x^-)$ and we impose the usual equal $x^+$ commutation relations:

$$[M_i(x^-), \partial_- M_j^\dagger(y^-)] = \frac{i}{2} \delta_{ij} \delta(x^- - y^-). \quad (10)$$

The longitudinal momentum operator

$$P^+ = 2 \sum_i \int dx^- \text{Tr}\{\partial_- M_i \partial_- M_i^\dagger\} \quad (11)$$

and Hamiltonian

$$P^- = \sum_i \int dx^- \left( -\frac{g^2}{2a} \text{Tr}\left\{J_i^+ \frac{1}{(\partial_-)^2} J_i^+ \right\} + V_i \right) \quad (12)$$

generate translations in the $x^-$ and $x^+$ directions, respectively.

Finally, we construct a basis of states. The zero mode of the $A_i^-$ constraint equation (8) generates the Gauß law constraint

$$0 = \int dx^- J_i^+ \quad (13)$$

which we must impose on the basis of physical states. That is, physical states must be color singlets at each lattice site. We construct a basis of closed color loops:

$$\begin{align*}
\text{Tr}\{M_i M_i^\dagger\} |0\rangle & \quad \times \times \times \\
\text{Tr}\{M_i M_i^\dagger M_i M_i^\dagger\} |0\rangle & \quad \times \times \\
\text{Tr}\{M_i M_{i+1} M_{i+1}^\dagger M_i^\dagger\} |0\rangle & \quad \times \times \\
\text{Tr}\{M_i M_{i+1} M_{i+1}^\dagger M_{i+1}^\dagger M_i^\dagger M_i^\dagger\} |0\rangle & \quad \times \\
& \text{et cetera.}
\end{align*} \quad (14)$$

where the $x^-$ co-ordinate of each link field remains arbitrary and the number of links $M_i$ must equal the number of anti-links $M_i^\dagger$. Since the loop-loop coupling constant is non-leading in $N$, we do not include states with more than one color trace in the Hilbert space; we deal with a free string theory.

*The factor of 1/2 comes from the Dirac procedure for constrained systems. Also, we drop the constrained zero mode $\int dx^- M_i$.

*In fact it is possible to choose $V_i$ such that one has a theory of free bosonic strings.
is not a color singlet at some site, *exempli gratia*

$$\text{Tr}\{\cdots M_j M_{j+2} \cdots \} |0\rangle,$$  \hspace{1cm} (15)

we find that its energy diverges.

**Confinement**

This theory has built-in linear confinement. Consider two test charges separated in the $x^-$ direction. The first term in Eqn. (12) acts as a linear confining potential.\[\text{Now consider two charges at lattice sites } \text{i and } \text{j. Due to the Gauß' law constraint (13) we must construct a color string of at least } |\text{i} - \text{j}| \text{ link fields between the two charges. If } \mu^2 > 0, \text{ there is some minimum energy associated with each link field and the transverse string tension is nonzero. As we shall see in the next section the resulting potential is, in fact, linear. Similar arguments hold for the original action (4).}\]

**Numerical Techniques**

We solve the spectrum on the computer using DLCQ techniques \[\text{. For the } x^- \text{ coordinate, we impose anti-periodic boundary conditions } \tilde{M}_i(x^-) = -M_i(x^- + L) \text{ and use a momentum space representation. For integer valued cut-off } K = LP^+/2\pi, \text{ momenta are labeled by odd half integers } \kappa_m \in \{1/2, 3/2, \ldots \} \text{ where } \sum_m \kappa_m = K. \text{ This yields a finite basis of states. Also, we employ a truncation in particle number.}\]

The first term of the Hamiltonian (12) dominates the behavior of the theory. Thus it is natural to choose units such that the associated coupling constant $g^2 N/a$ is set equal to 1. This choice of units is assumed in the following.

**STRING TENSION**

Next, we introduce a method for measuring the string tension in the $x^1$ direction. Consider a lattice of $n$ transverse links and periodic boundary conditions. We construct a basis of states that wind once around this lattice and calculate the lowest eigenvalue of the invariant (mass)$^2$ operator $M^2 |\Psi\rangle = 2P^+P^- |\Psi\rangle$. The continuum limit string tension is

$$\lim_{a \to 0} \frac{1}{a} \frac{\Delta M}{\Delta n}, \text{ na fixed.} \hspace{1cm} (16)$$

This definition is be equivalent to the standard definition of string tension using two test charges if the charges are placed sufficiently far apart. If we plot $M^2$ vs. $n$ in Figure 1, we obtain a quadratic corresponding to a linear confining potential.

In fact, we find that the string tension decreases with decreasing $\mu^2$, leading us to

\*Note that $(\partial^-)^2 f(x^-) = \int dy^- |x^- - y^-| f(y^-)/2$ in Eqn. (13).
the conclusion that $\mu^2$ is a measure of the physical lattice spacing. The continuum limit is partially fixed by requiring vanishing $\Delta(M)/\Delta(n)$; see Fig. 2. Above the surface, the string tension is positive corresponding to nonzero lattice spacing.

In addition we have an analytic ansatz for these states that wind once around the lattice that is valid in the $\lambda_2/a < 0$ region of parameter space. It agrees well with the numerical results.

**SPECTRUM**

Before we look at the details of the spectrum, we can make some statements about the allowed region in parameter space. A numerical estimate of the ‘edge’ of this region is plotted in Fig. 3. Comparing Figs. 2 and 3, we see that a continuum limit with a non-tachyonic spectrum occurs only for the “wedge shaped region” $-\lambda_1 \leq \lambda_2 \leq \lambda_1/2$. In this region, the continuum limit occurs for $\mu^2 \approx 0$ to within numerical errors. Finite lattice spacing corresponds to $\mu^2$ slightly above the surface in Fig. 2.

**Symmetries**

The theory possesses several discrete symmetries. Charge conjugation induces the symmetry $C : (M_i)_{l,m} \leftrightarrow (M_i^\dagger)_{m,l}$ where $l,m \in \{1, \ldots, N\}$. Parity is the product of two reflections $P_1 : x^1 \to -x^1$ and $P_2 : x^2 \to -x^2$. In light-front quantization, $P_1$ is an exact symmetry $P_1 : M \leftrightarrow M^\dagger$, while $P_2 : x^+ \leftrightarrow x^-$, is complicated. Its explicit operation is known only for free particles [10], which we call “Hornbosch parity.” The latter is nevertheless useful since it is often an approximate quantum number and can be used to estimate $P_2$ [11]. Given $P_2$ and $P_1$ we can determine whether spin $J$ is even or odd using the relation $(-1)^J = P_1 P_2$. If rotational symmetry has been restored in the theory, states of spin $J \neq 0$ should form degenerate $P_1$ doublets $| +J \rangle \pm | -J \rangle$ [2].
Figure 2. Parameters such that the lowest $M^2$ eigenvalues are equal for $n = 4$ and 5 (see Fig. 1), where $K = 10.5$ or 11 ($(n + 4)$-particle truncation). This is an estimate of vanishing string tension. Also shown is a line such that the $M^2$ eigenvalues are approximately degenerate for $n = 3, 4,$ and 5.

Figure 3. Parameters such that the lowest $M^2$ eigenvalue is zero, $K = 10$ to 14 with extrapolation using a fit to $\sum_{m=0}^{4} c_m K^{-m/2}$ (6-particle truncation). Below this surface, the spectrum is tachyonic; above the surface, it is well behaved.
Figure 4. A comparison of our spectrum with SU(3) ELMC data in units of the physical string tension $\Sigma$ for various $|J|_{\pi}^{p_c}$. The parameters $g^2N/a = 3.90$, $\mu^2 = 0.134g^2N/a$, $\lambda_1 = 0.487g^2N$, and $\lambda_2 = 1.108g^2N$ were chosen by a best fit to the lattice data, $\chi^2 = 40$, where $K = 10$ (8-particle truncation). Our error estimates are solely for the purpose of performing the $\chi^2$ fit.

As with the lattice results, we use “spectroscopic notation” $|J|_{\pi}^{p_c}$ to classify states.

One expects the lowest two eigenstates to be approximately two-particle states

$$\sum_i \text{Tr}\left\{M^\dagger_i(x^-)M_i(y^-)\right\}|0\rangle$$

with the lowest state having a symmetric wavefunction corresponding to $0^{++}$ and the first excited state having an antisymmetric wavefunction corresponding to $0^{--}$. Of course, these states also have 6-particle et cetera contributions.

Results

Ideally, we would like to predict the effective potential based on some connection to the continuum theory, restoration of rotational invariance, et cetera. However, as a first step, we use instead a best $\chi^2$ fit to the ELMC results of Teper [2].

An example spectrum is shown in Fig. 4. Similar spectra are found in other regions of coupling constant space above the ‘wedge-shaped region’ (Fig. 3). We label the lowest $2^{--}$ and second $0^{--}$ states based on the expectation value of the number operator and determine $(-1)^J$ based on Hornbostel parity [11]; the exception is the $|J|^++$ sector where Hornbostel parity gave exactly the opposite of the desired results. Beyond this,
$J$ is determined by a best fit to the lattice data.

Since this spectrum is the result of a best fit, it is not very predictive. However, we can use the result to tell us about our model. At first glance, we seem to have a pretty good match. However, we note several problems:

- The energy of the lowest $0^{--}$ state is too low.
- The lowest $2^{++}$ and $2^{--}$ states form a degenerate doublet if rotational symmetry is restored (as indeed happens for the lattice data). In our case, the splitting is large. This discrepancy dominates the error in our $\chi^2$ fitting procedure.

Let us review the possible sources of error in our calculation.

**Large $N$**. We compare $N \to \infty$ spectra to $SU(3)$ lattice results. However, based on lattice calculations for $SU(2)$, $SU(3)$, and $SU(4)$, $1/N$ corrections to the low energy spectrum are small [2].

**Finite $K$**. Our discretization of the longitudinal momentum introduces some error. However, we have generated spectra for $K = 10, 11, 12, 13, 14$, extrapolated to large $K$ (6-particle truncation), and compared to large $N$ extrapolated ELMC spectra. We saw no real improvement in our results.

**Particle Number**. We also impose a truncation in the number of particles. We have examined spectra for 4-, 6-, and 8-particle truncations ($K = 10$), extrapolated to large number of particles, and compared to large $N$ extrapolated ELMC spectra. We saw no real improvement in our results.

**Hamiltonian**. The effective potential $V_i$ that we chose (7) did not contain any 6-point or higher interactions. In addition, we did not include any operators containing multiple traces, for instance $(\text{Tr}\{M_i M_i^{\dagger}\})^2$.

**CONCLUSIONS**

We have investigated the transverse lattice model of Bardeen and Pearson [3] for $(2 + 1)$-dimensions in the large-$N$ limit using linearized link variables and an empirical effective potential $V_i$. We identified a choice for $V_i$ corresponding to vanishing string tension. The glueball spectrum in the vicinity agreed qualitatively with that coming from the presumably reliable Euclidean lattice Monte Carlo results [2]. Most importantly however, we did not see significant signs of rotational invariance which could lead one to conclude that the transverse gauge dynamics were correctly accounted for by $V_i$. We believe that our choice (7) is probably too simple and that higher order terms are necessary to see improvement in our spectrum.

*Although this term is generally not leading order in $N$, it does act on the two particle subspace of the theory.*
Future work includes the addition of more operators in the effective potential. Also, we can use our method of measuring string tension to measure the physical lattice spacing. This issue needs further investigation. Most importantly, we need a more concrete connection between our model and the continuum theory. This would allow us to better predict the correct effective potential.

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REFERENCES

[1] S. Dalley and B. van de Sande, preprint No. DAMTP-96-21 and hep-ph/9602291.
[2] M. Teper, Phys. Lett. 289B, 115 (1992); Phys. Lett. 311B, 223 (1993); Lecture at Rutherford-Appleton Laboratory, U. K. (December 1995).
[3] W. A. Bardeen and R. B. Pearson, Phys. Rev. D 14, 547 (1976).
[4] W. A. Bardeen, R. B. Pearson, and E. Rabinovici, Phys. Rev. D 21, 1037 (1980).
[5] P. Griffin, Nucl. Phys. B372, 270 (1992).
[6] G. ’t Hooft, Nucl. Phys. B72, 461 (1974); M. Einhorn, Phys. Rev. D 14, 3451 (1976).
[7] D. Robertson, Phys. Rev. D 47, 2549 (1993); S. S. Pinsky and B. van de Sande, Phys. Rev. D 49, 2001 (1994); S. S. Pinsky, B. van de Sande, J. Hiller, Phys. Rev. D 51, 726 (1995).
[8] I. R. Klebanov and L. Susskind, Nucl. Phys. B309, 175 (1988).
[9] H.-C. Pauli and S. Brodsky, Phys. Rev. D 32, 1993 and 2001 (1985).
[10] K. Hornbostel, Ph. D Thesis, SLAC-PUB No. 333 (1988).
[11] B. van de Sande and M. Burkardt, preprint No. MPI H-V 1995 and hep-th/9510104, to appear in Phys. Rev. D.
[12] S. Dalley and T. R. Morris, Int. Journal Mod. Phys. A5, 3929 (1990).