THETA CORRESPONDENCE
OF AUTOMORPHIC CHARACTERS

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Abstract

This paper describes the lifting of automorphic characters of $O(3)(\mathbb{A})$ to $\widetilde{SL}_2(\mathbb{A})$. It does so by matching the image of this lift with the lift of automorphic characters from $O(1)(\mathbb{A})$ to $\widetilde{SL}_2(\mathbb{A})$. Our matching actually gives a matching of individual automorphic forms, and not just of representation spaces. Let $V$ be a 3–dimensional quadratic vector space and $U$ a certain 1–dimensional quadratic space. To an automorphic form $I_V(\chi, \varphi)$ determined by the Schwartz function $\varphi \in S(V(\mathbb{A}))$ in the lift of the character $\chi$ we match an automorphic form $I_U(\mu, \varphi_0)$ determined by the Schwartz function $\varphi_0 \in S(U(\mathbb{A}))$ in the lift of the character $\mu$.

Our work shows that, the space $U$ is explicitly determined by the character $\chi$. The character $\mu$ is explicitly determined by the space $V$ and the function $\varphi_0$ is given by an orbital integral involving $\varphi$.

Key words: theta lift, automorphic characters

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1 Introduction

The Siegel-Weil formula identifies the integral of a certain theta function as an Eisenstein series on a symplectic group. In modern language, this is an example of a global theta correspondence for the dual pair Sp(W), O(V) in which the trivial representation of O(V)(A) corresponds to an automorphic representation of Sp(W)(A) generated by the values of Eisenstein series. The group O(V)(A) has many other one dimensional automorphic representations (automorphic characters) and it is an open problem to describe where they all go in the correspondence.

This paper deals with the dual pair, $\tilde{SL}_2, O(3)$. The local and global versions of the theta correspondence, in this case, were studied by Waldspurger [14] [15] and he obtained very explicit information about corresponding representations. However he did not consider global one dimensional representations of O(3)(A).
Our main results describes explicitly the lifts of automorphic characters $\chi \neq 1$ of $O(3)$ in terms of theta series coming from various $O(1)$’s, these results provide a complement to the work of Waldspurger and are analogues to the Seigel-Weil Formula.

1.1 Description of the result

Let $F$ be a number field and let $B$ be a quaternion algebra over $F$. For any place $v$ of $F$, let $F_v$ be the completion of $F$, and let $B_v = B \otimes F_v$. The local algebra $B_v$ can be either split, $B_v \simeq M_2(F_v)$ or non-split, so that $B_v$ is a division algebra. The invariant $\text{Inv}(B_v)$ is defined as $1$ when $B_v$ is split and $-1$ when $B_v$ is non-split. $\prod_v \text{Inv}(B_v) = 1$. Let

$$S = S_B := \{v : \text{Inv}(B_v) = -1\}.$$ 

By the product formula $\prod_v \text{Inv}(B_v) = 1$, the cardinality of $S$, is even. The isomorphism class of $B$ is determined uniquely by $S$.

Let $V$ be the trace zero elements of $B$,

$$V := \{x \in B : \text{Tr}(x) = 0\},$$

and equip $V$ with a quadratic form $Q(x) = \nu(x)$, where $\nu : B \to F$ is the reduced norm. The space $V$ is thus a three dimensional quadratic space over $F$ and, up to scaling $Q$, all three dimensional spaces over $F$ are constructed this way. Since the dimension of $V$ is odd, the orthogonal group factors as $O(V) = SO(V) \times \{\pm 1\}$. Moreover, the group $H = B^\times$ acts on $V$ by conjugation and the action gives rise to the exact sequence

$$1 \to \mathbb{Z} \to H \to SO(V) \to 1,$$
where $Z$ is the center of $H$. We can thus identify $H$ with $G\text{Spin}(V)$.

Let $\mathbb{A} = \mathbb{A}_F$ be the adeles of $F$ and $\mathbb{A}^\times$ be the ideles of $F$. Let $\chi$ be a character of $\mathbb{A}^\times / F^\times$ such that $\chi^2 = 1$. Such a character is given by

$$\chi(a) = (a, -\kappa)_{\mathbb{A}} = \prod_v (a_v, -\kappa_{F_v}) = \prod_v \chi_v(a_v)$$

for some $\kappa \in F^\times$ where the coset $\kappa(F^\times)^2$ in $F^\times/(F^\times)^2$ is uniquely determined and where the global Hilbert symbol $(, )_{\mathbb{A}}$ is defined by the local Hilbert symbols $(, )_{F_v}$.

Since $\nu : F^\times \cong Z \to F^\times$ is the map $x \mapsto x^2$ the composition $\chi \circ \nu$ is a quadratic character on $H(\mathbb{A})$ which is trivial on $Z(\mathbb{A})H(F)$. Hence it is an automorphic character of $H(\mathbb{A})$ which descends to $SO(\mathbb{A})$. All automorphic characters of $SO(\mathbb{A})$ are obtained in this way.

Let $\widetilde{\text{SL}}_2(\mathbb{A})$, be the 2-fold metaplectic cover of $\text{SL}(\mathbb{A})$ Fix a nontrivial additive character $\psi : \mathbb{A}/F \to \mathbb{C}^1$. Then $\widetilde{\text{SL}}_2(\mathbb{A})$ acts on the Schwartz space $\mathcal{S}(V(\mathbb{A}))$ by the Weil representation $\omega = \omega_\psi$ determined by $\psi$. The action of $\widetilde{\text{SL}}_2(\mathbb{A})$ commutes with the linear action of $H(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$ given by $h \cdot \varphi(x) = \varphi(h^{-1} \cdot x)$. Using the commuting actions, we define for $h \in H(\mathbb{A}), g \in \widetilde{\text{SL}}_2(\mathbb{A})$ and $\varphi \in \mathcal{S}(V(\mathbb{A}))$ the theta function

$$\theta(g, h, \varphi) = \sum_{x \in V(F)} \omega(g)(h \cdot \varphi)(x) = \sum_{x \in V(F)} \omega(g)\varphi(h^{-1} \cdot x). \quad (1.1)$$

This function is left invariant by $\text{SL}(F) \times H(F)$.

We now assume that $B(\mathbb{A})$ is a division algebra, i.e. $S \neq \emptyset$. In this case, the orthogonal group $SO(V)$ is anisotropic and the quotient $H(F)Z(\mathbb{A})\setminus H(\mathbb{A})$ is
compact. Thus, for a quadratic character $\chi$ as above, the integral

$$I_V(\chi, \varphi)(g) = \int_{H(F)Z(\mathbb{A}) \backslash H(\mathbb{A})} \theta(g, h, \varphi)\chi(\nu(h))dh$$  \hspace{1cm} (1.2)$$

is absolutely convergent and defines an automorphic form on $\widetilde{SL}_2(\mathbb{A})$.

The space

$$\Theta_V(\chi) := \{I_V(\chi, \varphi) : \varphi \in \mathcal{S}(V(\mathbb{A}))\}$$  \hspace{1cm} (1.3)$$
is a subspace of $\mathcal{A}(\widetilde{SL}_2(\mathbb{A}))$ where $\mathcal{A}(\widetilde{SL}_2(\mathbb{A}))$ is the space of genuine automorphic forms on $\widetilde{SL}_2(\mathbb{A})$. Furthermore, the space $\Theta_V(\chi)$ is right invariant under the action of $\widetilde{SL}_2(\mathbb{A})$ by right translation of the functions in $\mathcal{A}(\widetilde{SL}_2(\mathbb{A}))$ \footnote{Technically, only the global Hecke algebra is acting on functions, so we are slightly abusing notation here.}.

Our main result is a description of the space $\Theta_V(\chi)$ for all nontrivial automorphic characters $\chi$. This description will be given in terms of another theta correspondence which we outline next.

Let $U$ be the one dimensional space with a quadratic form $Q(x) = mx^2$. The group $O(U)$ is just $\{\pm 1\}$ and the adelic group $O(U)(\mathbb{A}) = \prod_v O(U)(F_v)$ is the unrestricted direct product. The group $O(U)(\mathbb{A})$ is compact in the usual product topology and $O(U)(F)$ is embedded diagonally in $O(U)(\mathbb{A})$, $-1 \mapsto (-1, -1, \ldots)$

A continuous character $\mu$ of $O(U)(\mathbb{A})$ determines a set of local characters $\mu_v$ of $O(U)(F_v)$ by restriction to the subgroup $O(V)(F_v) \hookrightarrow O(V)(\mathbb{A})$ and continuity of $\mu$ implies that at almost all places $\mu_v = 1_v$, the trivial representation of $O(U)(F_v)$. At the places where $\mu \neq 1_v$ it is the sign character $\text{sgn}_v$. Let

$$S = \{v : \mu_v = \text{sgn}_v\}.$$  \hspace{1cm} (1.4)$$
The automorphy condition on $\mu$ i.e. $\mu(-1) = 1$ implies that $|S|$ is even. Conversely, for any finite set of places $S$ where $|S|$ is even there is a unique automorphic character

$$\mu_S := \bigotimes_{v \in S} \text{sgn}_v \otimes \bigotimes_{v \not\in S} 1_v.$$  

The group $\tilde{\text{SL}}_2(\mathbb{A})$ acts on the Schwartz space $\mathcal{S}(U(\mathbb{A}))$ by the Weil representation $\omega_0 = \omega_{0,\psi}$ as in the case above, the linear action of $O(U)(\mathbb{A})$ given by, $h_0 \cdot \varphi_0(x) = \varphi(h_0^{-1} \cdot x)$ commutes with the action of $\tilde{\text{SL}}_2(\mathbb{A})$ on $\mathcal{S}(U(\mathbb{A}))$. Using the commuting action, we define for $h_0 \in O(U)(\mathbb{A}), \ g \in \tilde{\text{SL}}_2(\mathbb{A}), \ \varphi_0 \in \mathcal{S}(U(\mathbb{A}))$ the theta function

$$\theta(g, h_0, \varphi_0) = \sum_{x \in U(F)} \omega_0(g)(h_0 \cdot \varphi_0)(x) = \sum_{x \in U(F)} \omega_0(g) \varphi_0(h_0^{-1} \cdot x)$$

As above, this function is left invariant by $\text{SL}(F) \times O(U)(F)$ and for an automorphic character $\mu$ of $O(U)(\mathbb{A})$, we define the automorphic form on $\tilde{\text{SL}}_2(\mathbb{A})$

$$I_U(\mu, \varphi_0)(g) = \int_{O(U)(F) \backslash O(U)(\mathbb{A})} \theta(g, h_0, \varphi_0) \mu(h_0) dh_0 \quad (1.5)$$

Note that in this case, $O(U)(\mathbb{A})$ is compact so the integral is absolutely convergent. The space

$$\Theta_U(\mu) = \{I_U(\mu, \varphi_0) : \varphi_0 \in \mathcal{S}(U(\mathbb{A}))\} \quad (1.6)$$

is a right $\tilde{\text{SL}}_2(\mathbb{A})$ invariant subspace of $\mathcal{A}(\tilde{\text{SL}}_2(\mathbb{A}))$. Such spaces for varying $U$ and $\mu$, are colloquially referred to as "coming from $O(1)$". We can now state our main result.

**Main Theorem**  Let $B$ be a quaternion algebra over a totally real field $F$ Let $\chi$ be a quadratic character of $\mathbb{A}^\times/F^\times$ given by $\chi(x) = (x, -\kappa)_\mathbb{A}$, with $\kappa \in F^\times$. Assume that $\chi \neq 1$ and that $S \neq \emptyset$ so that $B$ is a division algebra. Let $\Theta_V(\chi)$ be
the corresponding automorphic representation of $\widetilde{SL}_2(\mathbb{A})$ constructed above. Then $\Theta_V(\chi)$ is cuspidal irreducible and

$$\Theta_V(\chi) = \Theta_U(\mu),$$

where $U$ is the one dimensional space with quadratic form $Q(x) = \kappa x^2$ and $\mu = \mu_S$ is the automorphic representation of $O(U)(\mathbb{A})$ determined by $S$.

**Remark 1** The case of $\chi = 1$ is covered by the classical Siegel-Weil formula. In that case $\Theta_V(1)$ is given by Eisenstein series whereas in our case $\Theta_V(\chi)$ is a space of cusp forms.

One similarity between our result and the Siegel Weil formula is that in the classical case, the theta integral attached to $\varphi$ is identified with an explicit Eisenstein series constructed from $\varphi$. Similarly, we are able to construct a function $\varphi_0 \in \mathcal{S}(U(\mathbb{A}))$ from $\varphi$ and identify the theta lift attached to $\varphi$ with a theta lift attached to $\varphi_0$. This is made precise in the following

**Theorem 2** Suppose that $B, \chi$ and corresponding $U, \mu$ are as in the main theorem. Then there is a unique map $\mathcal{S}(V(\mathbb{A})) \to \mathcal{S}(U(\mathbb{A}))$, $\varphi \mapsto \varphi_0$ such that

$$I_V(\chi, \varphi)(g) = I_U(\mu, \varphi_0)(g)$$

for all $g \in \widetilde{SL}_2(\mathbb{A})$. More explicitly, for factorizable functions $\varphi = \otimes_v \varphi_v$ the map $\varphi \mapsto \varphi_0$ is given by $\varphi_v \mapsto \varphi_{0,v}$ where $\varphi_{0,v}$ is defined on $F_v^\times$ by

$$\varphi_{0,v}(r) = \left| r \right|^{1}_v \chi_v(r) \int_{T_{x_0}(F_v) \backslash H(F_v)} \varphi_v( rh^{-1} \cdot x_0 ) \chi_v(\nu(h)) dh.$$

We will say that the pair of functions $(\varphi, \varphi_0)$ in theorem 2 are matching. In fact theorem 1 follows from theorem 2 and the later is proved via a matching of Whittaker functions.
It turns out that in our case, the automorphic form $I_V(\chi, \varphi)$ is distinguished \cite{4}, i.e., involves only one square class of Fourier coefficients $\alpha$. That square class is the one where $\alpha = \kappa$, and

$$I_V(\chi, \varphi)(n(b)g) = \sum_{t \in F^\times} \psi^{rt^2}(b)W_\kappa(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} g, \varphi).$$

Note that although many authors (e.g. Gelbart and Piatetski Shapiro \cite{4} for $\widetilde{GL}_2(\A)$ and Vignéras, for classical automorphic forms,) proved that “distinguished automorphic forms come from $O(1)$”, their results do not apply in our case.

The automorphic forms in $\Theta_U(\mu)$ are also distinguished. For any $\varphi_0 \in S(U(\A))$ the Fourier expansion of $I_U(\mu, \varphi_0)$ is

$$I_U(\mu, \varphi_0)(n(b)g) = \sum_{t \in F^\times} \psi^{rt^2}(b)W_\kappa(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} g, \varphi_0),$$

where, in fact, $W_\kappa(g, \varphi_0) = \omega_0(g) \varphi_0(1)$.

To establish theorem 2 for a given $\varphi$ it suffices to construct $\varphi_0$ such that $W_\kappa(g, \varphi_0) = W_\kappa(g, \varphi)$ for all $g \in \widetilde{SL}_2(\A)$. Thus the problem of matching reduces to a problem of matching the Whittaker functions $W_\kappa(g, \varphi) = W_\kappa(g, \varphi_0)$. then, we are reduced to the problem of local matching: given $\varphi_v \in S(V(F_v))$, find $\varphi_{0,v} \in S(U(F_v))$ such that

$$W_\kappa(g_v, \varphi_v) = W_\kappa(g_v, \varphi_{0,v})$$

(1.7)

for all $g_v \in \widetilde{SL}_2(F_v))$. 

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The analysis of the local matching depends on the Bruhat decomposition
\[ \widetilde{\text{SL}}_2(F_v) = \tilde{B} \Pi \tilde{B}w\tilde{B} \]
in the following way. The equality in equation (1.7) for
\[ g \in \tilde{B} \]
determines \( \varphi_{0,v} \) uniquely for \( r \in F_v^\times \) by the formula
\[ \varphi_{0,v}(r) = |r|v \chi_v(r) \int_{T_{\tilde{a}}(F_v)\backslash H(F_v)} \varphi_v(r h^{-1} \cdot x_0) \chi_v(\nu(h)) dh. \quad (1.8) \]

There are then two issues. First we must show that \( \varphi_0 \) defined on \( F_v^\times \) as in
equation (1.8) can be extended to a Schwartz function on \( F_v \). Then, with this
Schwartz function \( \varphi_{0,v} \in \mathcal{S}(U(F_v)) = \mathcal{S}(F_v) \), we must show that the equality
in (1.7) holds on the big cell \( \tilde{B}w\tilde{B} \) as well and hence on all of \( \widetilde{\text{SL}}_2(F_v) \).

The calculation that establishes the local matching on the big cell amounts
to establishing an identity involving orbital integrals, Gauss sums and Weil indexes.

For the global identity we must also check the fundamental lemma which
says that, in the unramified situation, the standard unramified vector \( \varphi_v^0 \in \mathcal{S}(V(F_v)) \) is matched by the standard unramified vector \( \varphi_v^0 \in \mathcal{S}(U(F_v)) \).

One technical point which should be noted is that the usual theta corre-
spondence is between \( \text{O}(V)(\mathbb{A}) \) and \( \widetilde{\text{SL}}_2(\mathbb{A}) \) whereas we have only discussed
automorphic characters of \( \text{SO}(V)(\mathbb{A}) \). However, in our situation, \( \text{O}(V)(\mathbb{A}) = \text{SO}(\mathbb{A}) \times \mathfrak{m}_2(\mathbb{A}) \) where \( \mathfrak{m}_2 = \{ \pm 1 \} \) and this product allows us to extend our
results to \( \text{O}(V)(\mathbb{A}) \). For an automorphic character \( \eta \) of \( \mathfrak{m}_2(\mathbb{A}) \), let
\[ \Theta_V(\chi \otimes \eta) = \{ I_V(\chi, \varphi) : \varphi \in \mathcal{S}(V(\mathbb{A}))^\eta \}, \]
where
\[ \mathcal{S}(V(\mathbb{A}))^\eta = \{ \varphi \in \mathcal{S}(V(\mathbb{A})) : \varphi(\delta x) = \eta(\delta) \varphi(x), \ \forall \ \delta \in \mathfrak{m}_2(\mathbb{A}) \}. \]
The space \( \Theta_V(\chi \otimes \eta) \) is the usual theta lift of the automorphic character \( \chi \otimes \eta \).
of $O(V)(\mathbb{A})$ and as a consequence of our calculation we can determine it.

**Proposition 3** There is a unique character $\eta$ of $\mu_2(\mathbb{A})$ such that $\Theta_V(\chi \otimes \eta) = \Theta_V(\chi)$ and $\Theta_V(\chi \otimes \eta') = 0$ for $\eta' \neq \eta$. More precisely,

$$\eta = \eta_T = (\otimes_{v \in T} \text{sgn}_v) \otimes (\otimes_{v \notin T} 1_v),$$

with

$$T = \{v : \chi_v(-1)\text{Inv}(B_v) = -1\}.$$

A final note about the type of representation we get is that the local representations coming from $O(1)$ which we match are the irreducible even or odd components of the Weil representation for a one dimensional quadratic space. In fact, what we show is that, when $\text{Inv}(B_v) = 1$, then in the local matching, $\varphi_v$ is matched by an even $\varphi_0 \in \mathcal{S}(U(F_v))$ and, when $\text{Inv}(B_v) = -1$, then $\varphi_v$ is matched by an odd $\varphi_0 \in \mathcal{S}(U(F_v))$. This means that the local component $\Theta_U(\mu)_v$ of $\Theta_U(\mu)$ is the image of the irreducible representation of $\widetilde{SL}_2(\mathbb{A})$ on either the space of even Schwartz functions $\mathcal{S}(U(F_v))^+$ or the space of odd Schwartz functions $\mathcal{S}(U(F_v))^{-}$ according to the invariant of $B$. By theorem 2 the local component $\Theta_V(\chi)_v$ of $\Theta_V(\chi)$ coincides with $\Theta_U(\mu)_v$. Therefore the local components $\Theta_V(\chi)_v$ and the global representation $\Theta_V(\chi)$ are exactly the ones coming from $O(1)$.

In [15] Waldspurger considers the space $\mathcal{A}_{00}(\widetilde{SL}_2(\mathbb{A}))$ which is the orthogonal complement in $\mathcal{A}_0(\widetilde{SL}_2(\mathbb{A}))$ to the representation coming from $O(1)$ which he calls the elementary Weil representations. These elementary representations are exactly the ones that correspond to $\Theta_V(\chi)$ in our case and so our result complements the classification carried out by Waldspurger.

Our result can be applied to the question of representations by spinor genera...
to obtain results similar to those of R. Schulze-Pillot in [9]. Details will be included in a future article.

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2 Notation

2.1 Fields and characters

When \( F_v \) is non archimedean we denote the ring of integers by \( \mathcal{O}_v \) and a generator of the maximal subring \( \wp_v \) of \( \mathcal{O}_v \) by \( \wp_v \). Let the modular degree of \( F_v \) over \( \mathbb{Q}_p \) be \( r_v \) so that \( |\wp_v|_v = q = p^{r_v} \).

Fix \( \psi \)-a non trivial additive character of \( A = A_F \) which is trivial on \( F \). The local components of \( \psi \), will be denoted by \( \psi_v \) except when it is clear from context that \( \psi \) means its local component.

The character \( \psi \) is defined in terms of the following parameters. We can define a map \( \lambda \) as the composition of the maps \( F_v \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \leftarrow \mathbb{Q}/\mathbb{Z} \). where
the map from $F_v$ to $Q_p$ is the trace map. We then define characters of $F_v$

$$\psi_{0,v}(x) = \exp(-2\pi i \lambda(x))$$

The product of the local characters $\psi_0 = \prod_v \psi_{0,v}$ is a character of $A/F$ and our fixed character $\psi$ and its local factors are defined as

$$\psi(x) = \psi_0(\epsilon x), \quad \psi_v(x) = \psi_{0,v}(\epsilon x) \quad (2.9)$$

for some $\epsilon \in F^\times$. For $\alpha \in F^\times$, let $\psi^\alpha(x) = \psi(\alpha x)$. Then $\{\psi^\alpha : \alpha \in F^\times\}$ is the space of all non trivial characters on $A/F$.

The conductor of $\psi_{0,v}$ is the smallest power of $\varphi_v$ on which $\psi_{0,v}$ is trivial. It is equal to the inverse different $D^{-1} = \{x \in F_v : \text{Tr}(x) \in \mathbb{Z}_p\}$. The differential exponent is $d$, it is defined such that $D = \varphi^{-d}$. Note that the conductor of $\psi$ is $\epsilon^{-1}D^{-1} = \epsilon^{-1}\varphi^{-d}$.

### 2.2 The quaternions

$B$ is a quaternion algebra over $F$, and $B_v = B \otimes_FF_v$. The Invariant $\text{Inv}(B_v) = +1$ if $B_v \simeq M_2(F_v)$ and $\text{Inv}(B_v) = -1$ if $B_v$ is the (unique up to isomorphism) division quaternion algebra over $F_v$.

The isomorphism class of $B$ is uniquely determined by the set of places $S = S_B := \{v : \text{Inv}(B_v) = -1\}$.

**Definition 4** $V$ is the 3-dimensional $F$-vector space

$$V = \{x \in B : \text{Tr}(x) = 0\}$$

We also define $V_v = V \otimes_FF_v$ and $V(A) = V \otimes_FA$. We will often abuse notation
and write V instead of V_v when the meaning of V is clear from context.

**Definition 5** The group H is the group of invertible quaternions

\[ H = \{ x \in B : \nu(x) \neq 0 \} \]

**Definition 6** The group \( T_x \subset H \) is the centralizer of \( x \)

\[ T_x = \{ y \in H : xy = yx \} \]

The standard definitions of Schwartz spaces are

**Definition 7** When \( V \) is defined over a non archimedean field, the Schwartz space \( S(V) \) is the set of locally constant compactly supported functions on \( V \).

When \( V \) is defined over an archimedean field, the Schwartz space \( S(V) \) is the set of infinitely differentiable rapidly decreasing functions. When \( V \) is split and archimedean, the “restricted Schwartz space” is the subspace of functions of the form

\[
 f \left( \begin{array}{cc}
 u & v \\
 w & -u
 \end{array} \right) = P(u, v, w) \exp(-2\pi |\epsilon| (u^2 + \frac{v^2}{2} + \frac{w^2}{2}))
\]

where \( P \) is a polynomial.

When \( V \) is non split and archimedean, the “restricted Schwartz space” is the space of functions

\[
 f(x) = P(x) \exp(-2\pi |\epsilon| \nu(x))
\]

Where \( P \) is a polynomial and \( x = (u, v, w) \in V \).

This section concludes with some basic geometric properties of B.
Lemma 8 If $B_v = M_2(F_v)$ then $Z(F_v) \backslash H(F_v)$ is not compact and if $B_v = \mathbb{H}(F_v)$ then $Z(F_v) \backslash H(F_v)$ is compact.

Lemma 9 The quotient $Z(\mathbb{A}) \backslash H(\mathbb{A})$ is compact unless $S = \emptyset$.

**PROOF.** This follows from corollary 3.2 in [13]. The product of the reduced norms is isotropic if and only if the reduced norm is isotropic at every $v$.

Lemma 10 For $x \in V$ we have $T_x = (F(x))^\times$

Remark 11 Lemma 10 also holds over all local fields $F_v$ and over the ring of Adeles, $\mathbb{A}$.

Lemma 12 If $B = B_v$ then $\nu(x) \notin -(F^\times)^2$ if and only if $Z \backslash T_x \simeq Z \backslash F(x)$ is compact.

2.3 $\widetilde{\text{SL}}_2(\mathbb{A})$

The group $\widetilde{\text{SL}}_2(\mathbb{A})$ is the unique non trivial 2-fold topological central extension of $\text{SL}_2(\mathbb{A})$

$$1 \to \{\pm\} \to \widetilde{\text{SL}}_2(\mathbb{A}) \to \text{SL}_2(\mathbb{A}) \to 1$$

Group elements are realized as pairs $(g, \epsilon)$ $g \in \text{SL}_2(\mathbb{A})$ $\epsilon \in \{\pm 1\}$. Multiplication is defined by:

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2\beta(g_1, g_2))$$

Where $\beta$ is a 2-cocycle on $\text{SL}_2(\mathbb{A})$. 

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The generators of $\tilde{\text{SL}}_2(\mathbb{A})$ are:

\[
m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1, -1) \quad (2.10)
\]

When calculating the action of group elements by the Weil representation (see below) we will drop the $\epsilon$ from the notation.

2.4 $O(V)(\mathbb{A})$ and $\tilde{\text{SL}}_2(\mathbb{A})$

Recall that we have fixed a division quaternion algebra $B$ with associated quadratic space $V$ and hence identified $H = B^\times = \text{GSpin}(V)$.

The actions of $\tilde{\text{SL}}_2(\mathbb{A})$ and $H(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$ are the following: $H(\mathbb{A})$ acts on elements of $V(\mathbb{A})$ by conjugation and on elements of $\mathcal{S}(V(\mathbb{A}))$ by acting on their arguments i.e. for $\varphi \in \mathcal{S}(V(\mathbb{A}))$, $x \in V(\mathbb{A})$, and $h \in H(\mathbb{A})$, the action is

\[
h \cdot \varphi(x) = \varphi(h^{-1} \cdot x) = \varphi(h^{-1}xh)
\]

The Weil representation of $\tilde{\text{SL}}_2(\mathbb{A})$ as described in Kudla’s notes ([7] p. 32-34) is determined by an additive character $\psi$ of $\mathbb{A}$ which we fixed in chapter 2 and by the quadratic space $(V(\mathbb{A}), \nu)$, determined by the algebra $B(\mathbb{A})$.

The formulas for the actions of the generators of $\tilde{\text{SL}}_2(\mathbb{A})$ on Schwartz functions are:
\[ \omega(n(b)) \varphi(x) = \psi(b \nu(x)) \varphi(x) \quad (2.11) \]
\[ \omega(m(a)) \varphi(x) = |a|^{\frac{3}{2}}(a, -\det(V)) \gamma(a, \psi^{\frac{1}{2}})^{-1} \varphi(ax) \quad (2.12) \]
\[ \omega((w, \epsilon)) \varphi(x) = \epsilon \gamma(\psi \circ V)^{-1} \int_{V(A)} \varphi(y) \psi(-(x, y)) dy. \quad (2.13) \]

Here \( \gamma(a, \psi^{\frac{1}{2}}) \) is the quotient of the Weil indices \( \frac{\gamma(\psi^{\frac{3}{2}})}{\gamma(\psi^{\frac{1}{2}})} \). See the appendix in [11] for the definitions and properties of the Weil index. Also, \( \det(V) \) is the determinant of the matrix of the quadratic form on \( V \) and

\[ \gamma(\psi \circ V) = \gamma(\det(V), \psi^{\frac{3}{2}}) \gamma(\psi^{\frac{1}{2}})^3 \epsilon(V) \]

where \( \epsilon(V) = \pm 1 \) is the Hasse invariant of \( V \), see [12]. The measure \( dy \) in equation (2.13) is the unique Haar measure which makes the transform \( \hat{f}(x) = \int_{V} f(y) \psi(-(x, y)) dy \) self dual. For more details see [7].

### 2.5 \( O(U)(A) \) and \( \widetilde{SL}_2(A) \)

For the 1-dimensional quadratic space \( U = F \) with \( Q(x) = mx^2 \), \( m \in F^{\times} \) the Weil representation \( \omega_0 \) of \( \widetilde{SL}_2(A) \), commutes with the action of \( O(U)(A) \). This representation is given by the formulas

\[ \omega_0(n(b)) \varphi_0(x) = \psi(bmx^2) \varphi_0(x) \quad (2.14) \]
\[ \omega_0(m(a)) \varphi_0(x) = |a|^{\frac{3}{2}}(a, 2m) \gamma(a, \psi^{\frac{1}{2}})^{-1} \varphi_0(ax) \quad (2.15) \]
\[ \omega_0((w, \epsilon)) \varphi(x) = \epsilon \gamma(\psi \circ U)^{-1} \int_{U(A)} \varphi_0(y) \psi(-2mxy) dy \quad (2.16) \]

As above \( \gamma(a, \psi^{\frac{1}{2}}) \) is the quotient of the Weil indices \( \frac{\gamma(\psi^{\frac{3}{2}})}{\gamma(\psi^{\frac{1}{2}})} \) and

\[ \gamma(\psi \circ U) = \gamma(\det(U), \psi^{\frac{1}{2}}) \gamma(\psi^{\frac{1}{2}}) \epsilon(U) \]
3 Fourier expansions and Whittaker models

Our matching of the spaces $\Theta_V(\chi)$ and $\Theta_U(\mu)$ will be reduced to matching their Whittaker models so we first realize the two spaces as such.

3.1 Whittaker model of $\Theta_V(\chi)$

We will now calculate the Fourier expansion of functions $I_V(\chi, \varphi) \in \Theta_V(\chi)$. This calculation will show that, like $\Theta_U(\mu)$, the space $\Theta_V(\chi)$ is distinguished.

We start by writing the automorphic forms as sums.

$$I_V(\chi, \varphi)(g) = \int_{H(F)Z(A)\backslash H(A)} \theta(g,h,\varphi)\chi(\nu(h))dh$$

$$= \int_{H(F)Z(A)\backslash H(A)} \sum_{x \in V(F)} \omega(g)\varphi(h^{-1} \cdot x)\chi(\nu(h))dh$$

$$= \int_{H(F)Z(A)\backslash H(A)} \sum_{\delta \in F, \delta \neq -\Box^x} \sum_{x \in V(F)} \omega(g)\varphi(h^{-1} \cdot x)\chi(\nu(h))dh$$

(3.17)

We make the convention that $\Box$ should be read as “a square” and $\Box^x$ be read as “a non zero square”. As mentioned in sections 2.2, we exclude the case of the unramified global algebra $B(\mathbb{A}) = M(2, \mathbb{A})$ and therefore by Lemmas 8 and 12 the quotient $H(F)Z(\mathbb{A})\backslash H(\mathbb{A})$ is compact and the integral in equation (3.17) is absolutely convergent. Also, $V$ is anisotropic so that the norms of $x \in V$ are not equal to a negative of a non zero square. Therefore $-\delta$ is not equal to a non zero square.

Lemma 13 The term $\delta = 0$ in the sum in equation (3.17) is zero.
PROOF. The space V is anisotropic so the only point of norm 0 is 0 and the the \( \delta = 0 \)-term in the sum is the integral of the character \( \chi \circ \nu \) on the compact space \( H(F)Z(\hat{A}) \setminus H(\hat{A}) \) so the term is zero.

The integral in equation (3.17) can be switched with the summation since the domain of integration is compact and the function has compact support. Thus we have

\[
IV(\chi, \varphi)(g) = \sum_{\delta \in F^\times} \int_{H(F)Z(\hat{A}) \setminus H(\hat{A})} \sum_{\gamma \in T_{x_\delta}(F) \setminus H(F)} \omega(g) \varphi(h^{-1} \gamma^{-1} \cdot x_\delta) \chi(\nu(h)) dh,
\]

where \( x_\delta \in V(F) \) is some point such that \( \nu(x_\delta) = \delta \). Notice that the summation over the set \( \{ x \in V(F) : \nu(x) = \delta \} \) is replaced by the sum over \( T_{x_\delta}(F) \setminus H(F) \).

Since B is a division algebra and V is anisotropic H(F) acts transitively on \( \{ x \in F : \nu(x) = \delta \} \) and so the choice of \( x_\delta \) does not matter. Also, since \( \chi \) is trivial on the rational points \( \chi(\nu(\gamma h)) = \chi(\nu(h)) \) for \( \gamma \in H(F) \) so we can include \( \gamma \) in the argument of \( \chi \circ \nu \).

\[
IV(\chi, \varphi)(g) = \int_{H(F)Z(\hat{A}) \setminus H(\hat{A})} \sum_{\gamma \in T_{x_\delta}(F) \setminus H(F)} \omega(g) \varphi(h^{-1} \gamma^{-1} \cdot x_\delta) \chi(\nu(\gamma h)) dh
\]

We now combine the inner sum and integral by a change of variables \( h_0 = \gamma h \).

\[
IV(\chi, \varphi)(g) = \int_{T_{x_\delta}(F)Z(\hat{A}) \setminus H(\hat{A})} \omega(g) \varphi(h_0^{-1} \cdot x_\delta) \chi(\nu(h_0)) dh_0
\]

The next step is to factor \( h_0 = h_1 q_1 \) with \( h_1 \in A_\delta := T_{x_\delta}(F)Z(\hat{A}) \setminus T_{x_\delta}(\hat{A}) \) and, \( q_1 \in B_\delta := T_{x_\delta}(\hat{A}) \setminus H(\hat{A}) \). This gives,

\[
IV(\chi, \varphi)(g) = \sum_{\delta \in F^\times} \int_{B_\delta} \int_{A_\delta} \omega(g) \varphi(q_1^{-1} h_1^{-1} \cdot x_\delta) \chi(\nu(h_1 q_1)) dh_1 \, dq_1 \quad (3.18)
\]

\[
= \sum_{\delta \in F^\times} \int_{B_\delta} \omega(g) \varphi(q_1 \cdot x_\delta) \chi(\nu(q_1^{-1})) dq_1 \int_{A_\delta} \chi(\nu(h_1)) dh_1 \quad (3.19)
\]
To proceed, we need the following description of $T_{x\delta}(A)$.

**Lemma 14** Let $\chi$ is a nontrivial quadratic character of $F\backslash A$ given by the Hilbert symbol $\chi(r) = (r, -\kappa)_A$ and $\chi_v(r) = (r, -\kappa)_{F_v}$. Let $\delta = \nu(x\delta) \neq 0$ with $x\delta \in V(F)$.

1. $T_{x\delta}(F_v) \subset \ker(\chi_v \circ \nu_v)$ if and only if $\delta \in \kappa(F_v)^2$
2. $T_{x\delta}(A) \subset \ker(\chi \circ \nu)$ if and only if $\delta \in \kappa(F^\times)^2$

**Proof.** The first result is contained in Proposition 4.8.2 in [2] when we notice that since $\chi_v(a) = (a, -\kappa)_{F_v}$ the kernel of $\chi_v$ is the set

$$\ker(\chi_v) = \nu(F_v(\sqrt{-\kappa})^\times)$$

and since $T_{x\delta}(F_v) = (F_v(x\delta))^\times = (F_v(\sqrt{\delta})^\times$,

$$\nu(T_{x\delta}(F_v)) = \ker(\chi_\delta)$$

where $\chi_\delta(a) = (a, -\delta)_{F_v}$.

The second statement follows from the fact that if $\mathcal{K}_1$ and $\mathcal{K}_2$ are quadratic extensions of $F$ then they determine characters $\chi_1$ and $\chi_2$, where $\ker(\chi_i) = \nu(\mathcal{K}_i(A)^\times)$, and $\ker(\chi_1) = \ker(\chi_2)$ if and only if $\mathcal{K}_1 \simeq \mathcal{K}_2$.

**Corollary 15** If $\chi_v = 1$ for some place $v \in S$, the set of ramified places for $B$ then $\Theta_V(\chi) = 0$.

**Proof.** If $\chi_v$ is trivial at a ramified place $v$, the square class of $\kappa$ is $-(F_v^\times)^2$. But since $\delta \neq -\square$ and $\delta$ is not a norm of $V(F_v)$, $\delta$ is not in the square class of $\kappa$. Therefore, by Lemma 14 $T_{x\delta}(A) \not\subset \ker(\chi \circ \nu)$ and the integral over $A_\delta$ in equation (3.19) will be zero for any $g$ and $\varphi$. 

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Using Lemma 14 we can see that the inner integral in (3.19) is nonzero exactly when \( \delta \in \kappa(F^\times)^2 \). If we pick \( x_0 \) such that \( \nu(x_0) = \kappa \) then elements with norm \( \alpha^2 \kappa \) are represented by \( \alpha x_0 \), therefore

\[
I_V(\chi, \varphi)(g) = \sum_{\alpha \in F^\times T_{x_0}(A) \backslash H(A)} \int_{T_{x_0}(F) \backslash T(\mathbb{A})} \omega(g) \varphi(q^{-1} \cdot \alpha x_0) \chi(\nu(g)) dq \tag{3.20}
\]

where we normalized the volume such that

\[
\text{Vol}(T_{x_0}(F)Z(A) \backslash T_{x_0}(A)) = \text{Vol}(T_{\delta x_0}(F)Z(A) \backslash T_{\delta x_0}(A)) = 1
\]

When we replace \( g \) in equation (3.20) by \( n(b)g \) then the \( \alpha \) term in the sum is multiplied by \( \psi(\alpha^2 \kappa b) \). Therefore the sum in (3.20) is actually the Fourier expansion of \( I_V(\chi, \varphi) \). We make this explicit in the following definitions:

**Definition 16** The Whittaker function corresponding to the form \( I_V(\chi, \varphi) \) is the function

\[
W_{\varphi}(g) = \int_{T_{x_0}(A) \backslash H(A)} \omega(g) \varphi(q^{-1} \cdot x_0) \chi(\nu(g)) dq.
\]

where \( x_0 \in V_F \) such that \( \nu(x_0) = \kappa \)

**Definition 17** The Whittaker model of the automorphic representation \( \Theta_V(\chi) \) is the space

\[
W_V(\chi) := \{ W_{\varphi}(g) : \varphi \in \mathcal{S}(V(\mathbb{A})) \}
\]

We collect the result of the above discussion in the proposition

**Proposition 18** Let \( \chi \neq 1 \) then

(1) The function \( I_V(\chi, \varphi) \) is a cuspidal automorphic form and therefore

\[
\Theta_V(\chi) \subset \mathcal{A}_0(\widetilde{\text{SL}}_2(\mathbb{A})).
\]
(2) The function $I_V(\chi, \varphi)$ is distinguished in the sense of Gelbart and Piatetski-Shapiro [4] with Fourier coefficients in the square class of $\kappa(\mathbb{F}^\times)^2$.

The information we have so far, allows us to determine some information about the space $U$.

**Remark 19** In order that $\Theta_V(\chi) \subset \Theta_U(\mu)$ it is necessary that the quadratic form on $U$ is defined by $Q(x) = \kappa x^2$.

### 3.2 The Whittaker model of $\Theta_U(\mu)$

In this section we compute the Fourier expansion of the function $I_U(\chi, \varphi_0)$.

First, we write

$$I_U(\chi, \varphi_0) = \int_{O(U)(\mathbb{A})} \sum_{x \in F} \omega_0(g) \varphi_0(h_0^{-1} \cdot x) \mu(h_0) dh_0 = I_U(\mu, \varphi_0)(g) = \frac{1}{2} \sum_{x \in F} \int_{O(U)(\mathbb{A})} \omega_0(g) \varphi_0(u^{-1} \cdot x) \mu(u) du$$

The group $O(U)(\mathbb{A})$ is compact, so the representation on $S(V(\mathbb{A}))$ is a direct sum of isotypic components and it is not hard to see that the inner integral is the volume of $O(U)(\mathbb{A})$ times the projection from $S(U(\mathbb{A}))$ onto the $\mu = \mu_S$ isotypic subspace. Recall that at places $v \in S$ the factor $\mu_v$ is the sign representation and at places $v \notin S$ the factor $\mu_v$ is the trivial character. Therefore the $\mu = \mu_S$ isotypic subspace is the subspace

$$S(U(\mathbb{A}))_\mu = (\otimes_{v \in S} S(U(F_v))^\perp) \otimes (\otimes_{v \notin S} S(U(F_v))^\perp), \quad (3.21)$$

where $S(U(F_v))^\perp$, $S(U(F_v))^\perp$ are the subspaces of odd and even functions respectively.
This means that we can replace \( \varphi_0 \) by its projection to \( \mathcal{S}(U(A))_\mu \) which we also denote by \( \varphi_0 \) and then

\[
I_U(\mu, \varphi_0)(g) = \frac{1}{2} \sum_{x \in F} \omega_0(g) \varphi_0(x).
\]

The space \( \Theta_U(\mu) \) is isomorphic to \( \mathcal{S}(U(A))_\mu \).

To construct the Whittaker model for \( \Theta_U(\mu) \) we consider the Fourier expansion of the forms \( I_U(\mu, \varphi_0) \). Let \( g \in \tilde{\text{SL}}_2(A) \) be any element then for any \( b \in A \)

\[
I_U(\mu, \varphi_0)(n(b)g) = \frac{1}{2} \sum_{x \in F} \omega_0(n(b)g) \varphi_0(x)
= \frac{1}{2} \sum_{x \in F} \psi(bmx^2) \omega_0(g) \varphi_0(x)
\]

The term \( x = 0 \) can be removed from the sum since \( \varphi_0 \in \mathcal{S}(U(A))_\mu \) and \( \mu \neq 1 \), so \( \varphi_0(0) = 0 \).

By equation (2.15) \( \omega_0(m(t)g) \varphi_0(1) = \omega_0(g)(t) \) so we can rewrite the Fourier expansion as

\[
I_U(\mu, \varphi_0)(n(b)g) = \frac{1}{2} \sum_{x \in F^x} \psi^{mx^2}(b) \omega_0(m(x)g) \varphi_0(1)
\]

Thus \( I_U(\mu, \varphi_0) \) is cuspidal and the Fourier expansion is completely determined by the function \( \omega_0(g) \varphi_0(1) \). Notice that the character \( \psi^{mx^2} \) appears twice in the sum, at \( \pm x \) and that \( \varphi_0 \in \mathcal{S}(V(A))_\mu \) is even so the coefficient of the character is actually \( \omega_0(m(x)g) \varphi_0(1) \). The above calculation justifies the definitions:

**Definition 20** The Whittaker function corresponding to the automorphic form \( I_U(\mu, \varphi_0) \) is

\[
W_{\varphi_0}(g) = \omega(g) \varphi_0(1)
\]
**Definition 21** The global Whittaker model of $\Theta_U(\mu)$ is the space

$$W_U(\mu) := \{ W_{\varphi_0} : \varphi_0 \in \mathcal{S}(U(\mathbb{A})),_{\mu} \}$$

Recall that $m$ defined the quadratic form on $U$, $Q(x) = mx^2$. We note a result which was evident in the Fourier expansion of $I_U(\mu, \varphi_0)$.

**Lemma 22** The automorphic forms $I_U(\mu, \varphi_0)$ are cuspidal and distinguished in the sense of Gelbart and Piatetski-Shapiro [4]. The characters in the Fourier expansion of $I_U(\mu, \varphi_0)$ are all in the square class of $m$.

### 3.3 Factorizability

If $\varphi \in \mathcal{S}(V(\mathbb{A}))$ is factorizable, we write $\varphi = \otimes_v \varphi_v$ and then

$$W_\varphi(g) = \int_{T_{x_0}(\mathbb{A}) \backslash H(\mathbb{A})} \omega(g) \varphi(q^{-1} \cdot x_0) \chi(q) dq = \prod_v W_{\varphi_v}(g_v)$$

where $W_{\varphi_v}(g_v) = \int_{T_{x_0}(F_v) \backslash H(F_v)} \omega_v(g_v) \varphi(q^{-1} \cdot x_0) \chi_v(q_v) dq_v$ and the $\psi^\kappa$-Whittaker space of all such functions is

$$W_{\psi^\kappa}(V, \chi) := \{ W_{\varphi_v} : \varphi \in \mathcal{S}(V(F_v)) \}.$$
and the $\psi^\kappa$-Whittaker space (by Lemma 19) of all such functions is

$$W_{\psi^\kappa}(U, \mu) := \{W_{\varphi_{0,v}} : \varphi \in S(U(F_v))\}.$$ 

From here on, our calculation will consist of local matching. Therefore we will drop the $v$ subscript from most notation. Unless stated otherwise, all formulas should be understood to be over a completion $F_v$. We will also write $T_0$ instead of $T_{x_0}$.

4 Local Matching

4.1 Structural preliminaries

The various lemmas in this section will be used in the later sections.

Remark 23 By Lemma 15 we can assume that, whenever $\chi = 1$, then $B_v = M_2(F_v)$.

Lemma 24 With notation as above, for every place $v$ there is an element $\xi \in H$ such that $\xi^{-1} \cdot x_0 = -x_0$ and $\chi(-\nu(\xi)) = \text{Inv}(B_v)$.

**Proof.** By equation (1) on page 1 and corollary 2.2 on page 6 in [13] we can realize the quaternion algebra as $F_v(x_0) + F_v(x_0)\xi$ with $\xi$ such that if $m \in F_v(x_0)$ then $\xi m = \bar{m}\xi$. Where $\bar{m}$ is the conjugate of $m$ in $F_v(x_0)$. It follows that $\xi x_0 \xi^{-1} = \bar{x_0} \xi \xi^{-1} = \bar{x_0} = -x_0$. Also, by corollary 2.4 on page 6 in [13] we have $\text{Inv}(B) = 1$ if and only if $-\nu(\xi) \in \nu(F_v(x_0)) = \ker(\chi)$. This means that $\text{Inv}(B) = \chi(-\nu(\xi))$. 

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Definition 25 For $\alpha \in F$, let $h_\alpha \in H$ be any element such that $h_\alpha^{-1}x_0 = \alpha x_0 + x$ with $x_0 \perp x$.

Lemma 26 Let $\chi \neq 1$, then the coset space $T_0 \backslash H$ is equal to the disjoint union of right cosets $h_\alpha T_0$ over the set of $\alpha \in F^\times$ such that

$$\chi(\alpha^2 - 1) = \text{Inv}(B) \text{ or } \alpha = \pm 1.$$  

PROOF. If $h$ and $h'$ are such that for some $\alpha$, $h^{-1}x_0 = \alpha x_0 + x$ and $h'^{-1}x_0 = \alpha x_0 + x'$ with $x$ and $x'$ perpendicular to $x_0$ then $\nu(x) = \nu(x')$ and there exists an element of the orthogonal group of the orthogonal complement to the span of $x_0$ taking $x$ to $x'$. This means that $hT_{x_0} = h'T_{x_0}$ so the decomposition into right cosets indexed by $\alpha$ is proved.

To verify the second part of the claim realize $B_v$ as $F_v(x_0) + F_v(x_0)\xi$ as in the proof of 24.

To carry out the calculation we use the orthogonal basis \{ $x_0, \xi, x_0\xi$ \} for $V$ and write an arbitrary element $x$ in terms of this basis: $x = \alpha x_0 + \beta \xi + \gamma x_0\xi$. In terms of those coordinates, $\nu(x) = \alpha^2\kappa + \beta^2\nu(\xi) + \gamma^2\kappa\nu(\xi)$. If $x$ is in the orbit of $x_0$ then

$$\nu(x) = \alpha^2\kappa + \beta^2\nu(\xi) + \gamma^2\kappa\nu(\xi) = \kappa$$

This means that $(\alpha^2 - 1)\kappa = -\nu(\xi)(\beta^2 + \gamma^2\kappa)$ and then $\chi(\alpha^2 - 1)\chi(\kappa) = \chi(-\nu(\xi))\chi(\beta^2 + \gamma^2\kappa)$. Now, since $\chi(\beta^2 + \gamma^2\kappa) = \chi(\kappa) = 1$ we are done.

The second lemma describes the values of $\chi(h_\alpha)$:

Lemma 27 If $h_\alpha$ is defined as in 25 then

$$\chi(\nu(h_\alpha)) = \chi(2(\alpha + 1)) \quad (4.22)$$
PROOF. we extend the basis defined in the proof of Lemma 26 for V to all of \( B \) and write
\[ h_\alpha = a + bx_0 + c\xi + dx_0\xi \]
then by multiplying out the conjugation action of \( h_\alpha \): and collecting the coefficients of \( x_0 \) we get
\[ \alpha = \frac{a^2 + b^2\kappa - c^2\nu(\xi) - d^2\kappa\nu(\xi)}{a^2 + b^2\kappa + c^2\nu(\xi) + d^2\kappa\nu(\xi)} \]
From that we get that \( \alpha + 1 = \frac{2(a^2 + b^2\kappa)}{\nu(h_\alpha)} \) and since \( \chi(a^2 + b^2\kappa) = 1 \) we get
\[ \chi(2(\alpha + 1)) = \chi(\nu(h_\alpha)) \]

4.2 A formula for \( \varphi_0 \)

We can now derive an explicit formula for the local component \( \varphi_{0,v} \) of \( \varphi_0 \) and obtain some parity results from it. Recall that \( v \) has been fixed and we usually don’t include it in the notation. Also, recall the Bruhat decomposition
\[ \widetilde{SL}_2 = \tilde{B} \sqcup \tilde{B}w\tilde{B} \]

**Lemma 28** In order for the functions \( W_\varphi \) and \( W_{\varphi_0} \) to agree on \( \tilde{B} \) it is necessary and sufficient that
\[
\varphi_0(r) = |r|\chi(r) \int_{T_{x_0}(F_v) \setminus H(F_v)} \varphi(rh^{-1} \cdot x_0) \chi(\nu(h))dh. \quad (4.23)
\]

PROOF. Since the quadratic form on \( U = F_v \) is \( Q(x) = \kappa x^2 \), (see Lemma 19) \( W_\varphi \) and \( W_{\varphi_0} \) are in the same Whittaker space. Therefore, in order for them to agree on \( \tilde{B} \), it is necessary and sufficient that they agree on right \( N \) cosets, i.e. on the set \( \{m(a) : a \in F^\times \} \). So we compare
\[
W_{\varphi}(m(a)) = |a|^{\frac{3}{2}}(a, -\det(V))\gamma(a, \psi^{\frac{1}{2}})^{-1} \int_{T_0 \setminus H} \varphi(h^{-1} \cdot ax_0) \chi(\nu(h))dh
\]

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with

\[ W_{\varphi_0}(m(a)) = |a|^\frac{1}{2}(a, 2\kappa)\gamma(a, \psi^\frac{1}{2})^{-1}\varphi_0(a). \]

Solving for \( \varphi_0(a) \) gives a necessary and sufficient condition for matching.

\[ \varphi_0(a) = |a|(a, -\text{Det}(V))(a, 2\kappa) \int_{\mathcal{P}_0 \setminus \mathcal{H}} \varphi(h^{-1} \cdot ax_0)\chi(\nu(h))dh \]

The quaternion algebra with structure constants \( a, b \) has a reduced norm which can be diagonalized as \((-a, -b, ab)\). This means that the determinant of the bilinear form associated to the reduced norm is \( 8a^2b^2 \) so \((a, -\text{Det}(V))(a, 2\kappa) = (a, -2)(a, 2\kappa) = (a, -\kappa) = \chi(a) \)

In order to define \( \varphi_0 \) according to the formula in Lemma 28 we need to show that the formula can be extended to a Schwartz function of \( F_v \). We start with the definition

**Definition 29** Let \( \varphi \) be a Schwartz function in \( \mathcal{S}(V) \) and \( \chi \) a quadratic character of \( F_v^\times \). Define a function on \( F_v^\times \).

\[ \varphi_0^\times(r) = |r|\chi(r) \int_{\mathcal{P}_0 \setminus \mathcal{H}} \varphi(rh^{-1} \cdot x_0)\chi(\nu(h))dh \quad (4.24) \]

**Proposition 30** If \( F_v \) is non archimedean, and \( \varphi \in \mathcal{S}(V) \) then \( \varphi_0^\times \) defined by equation (4.23) can be extended to a function \( \varphi_0 \in \mathcal{S}(U(F_v)) \).

The next few lemmas will establish Proposition 30. The fact that 4.23 defines a Schwartz function at the archimedean place will follow from an indirect argument in Lemmas 48 and 51.

**Lemma 31** When \( F_v \) is non archimedean then \( \varphi_0^\times \) is locally constant on \( F_v^\times \) and supported inside a compact set of \( F \).
**PROOF.** Since $\varphi$ has compact support there will be some $n$ such that $\nu(supp(\varphi)) \subset \varphi^n$. Therefore for ord($r$) small enough, $\nu(rh^{-1} \cdot x_0) = r^2 \kappa \notin \varphi^n$ for any $h \in T_0 \backslash H$ and so for ord($r$) small enough $\varphi_0^\chi(r) = 0$.

Since $\varphi$ is locally constant and has compact support there is a neighborhood of $1$, $U \subset F_v^\times$ such that for all $e \in U$, for all $x$ in the support of $\varphi$ we have $\varphi(ex) = \varphi(x)$. Thus, for $r \neq 0$ and any $e \in U$,

$$\varphi_0(er) = |er|\chi(r) \int_{T_0 \backslash H} \varphi(erh^{-1} \cdot x_0)\chi(\nu(h))dh$$

$$= |r|\chi(r) \int_{T_0 \backslash H} \varphi(rh^{-1} \cdot x_0)\chi(\nu(h))dh$$

$$= \varphi_0(r)$$

**Lemma 32** When $B_v$ is the division algebra $\mathbb{H}$, $F_v$ is non archimedean, and $\chi$ is non trivial, then $\varphi_0$ vanishes on a neighborhood of $0$.

**PROOF.** Since in the division algebra case $T_0 \backslash H$ is compact, for $r$ small enough $\varphi(rh^{-1} \cdot x_0) = \varphi(0)$ for all $h$. For such $r$

$$\varphi_0^\chi(r) = |r|\chi(r) \int_{T_0 \backslash H} \chi(\nu(h))\varphi(0)dh = 0$$

Since $Z(F_v) \backslash T_{x_0}(F_v)$ is compact. and the integral over $T_0 \backslash H$ is equivalent to an integral over the compact group $Z \backslash H$.

**Lemma 33** When $B_v \simeq M_2(F_v)$, $F_v$ is non archimedean, and $\chi$ is non trivial, then $\varphi_0$ is constant on a neighborhood of $0$. 


PROOF. Since $\chi$ is nontrivial, by Lemma 12 the set $Z(F_v) \setminus T_{x_0}(F_v)$ is compact and we can define the function

$$\bar{\varphi}(x) = \int_{Z(F_v) \setminus T_{x_0}(F_v)} \varphi(t^{-1}x)dt.$$ 

It is not hard to see that $\bar{\varphi}$ depends only on the norm of $x$ and on $\alpha$ as defined in 25. Next, by Lemma 26 we can see that the measure $dh$ on $Z(F_v) \setminus H(F_v)$ is a positive scalar multiple of the product measure $d\alpha dt$ with $t \in Z(F_v) \setminus T_0(F_v)$ and $\alpha$ ranging over the set of $\alpha$ such that $\chi(\alpha^2 - 1) = 1$. Also, by Lemma 27 $\chi(h_\alpha) = \chi(2(\alpha + 1))$. This means that for the purpose of showing that $\varphi_0$ is constant near 0 we can replace the integral over $Z(F_v) \setminus H(F_v)$ by the integral

$$\varphi_0(r) = C\chi(r)|r| \int_{\{\alpha : \chi(\alpha^2 - 1) = 1\}} \bar{\varphi}(h_\alpha^{-1}rx_0)\chi(2(\alpha + 1))d\alpha$$

for some constant $C$. With an abuse of notation we can write $\bar{\varphi}$ as a function of two variables

$$\bar{\varphi}(\alpha, b) = \int_{Z(F_v) \setminus T_0(F_v)} \varphi(t^{-1}(\alpha x_0 + y))dt.$$ 

where $y \perp x_0$ and $\nu(\alpha x_0 + y) = b$. Therefore we can write $\bar{\varphi}(h_\alpha^{-1}rx_0) = \bar{\varphi}(\alpha r, r^2\kappa)$. Furthermore, the characteristic function of the set $\{\alpha : \chi(\alpha^2 - 1) = 1\}$ is the function $\frac{1}{2}[\chi(\alpha^2 - 1) + 1]$. When we multiply by the characteristic function we can extend the integral to an integral over all of $F_v$.

$$\varphi_0(r) = C\chi(r)|r| \int_{F_v} \bar{\varphi}(\alpha r, r^2\kappa)\frac{1}{2}[\chi(\alpha^2 - 1) + 1]\chi(2(\alpha + 1))d\alpha$$ 

$$= \frac{\chi(2)C}{2} \chi(r)|r| \int_{F_v} \bar{\varphi}(\alpha r, r^2\kappa)[\chi(\alpha + 1) + \chi(\alpha - 1)]d\alpha$$

We change variables to $\alpha' = r\alpha$ and we get:

$$\varphi_0(r) = \frac{\chi(2)C}{2} \int_{F_v} \bar{\varphi}(\alpha', r^2\kappa)[\chi(\alpha' + r) + \chi(\alpha' - r)]d\alpha'$$
To continue, we will need to extend $x_0$ to a basis for $V$ with $x_0, x_1, x_2$ such that $\nu(x_0) = \kappa = -\nu(x_1)$ and $\nu(x_2) = -1$ and we also need the following lemma.

**Lemma 34** For some $m > 0$, if $r \in \wp^m$ and $\chi(\alpha' + r) + \chi(\alpha' - r) \neq 0$ then

$$\tilde{\varphi}(\alpha', r^2\kappa) = \int_{\mathbb{Z}(F_v) \setminus T_0(F_v)} \varphi(t^{-1}(\alpha'x_0 + \alpha'x_1))dt = \tilde{\varphi}(\alpha', 0)$$

**PROOF.** The condition that $\chi(\alpha' + r) + \chi(\alpha' - r) \neq 0$ means that $\chi(\alpha^2 - r^2) = 1$ so the square class of $\alpha^2 - r^2$ can be either $\kappa$ or 1.

If $\alpha^2 - r^2 = \square$ i.e. the square class of $\alpha^2 - r^2$ is 1, let $\epsilon_1 = -\alpha' + \sqrt{\alpha^2 - r^2}$, it is not hard to see that $|\epsilon_1| \leq \frac{|r|}{2}$ regardless of $\alpha'$. Let $x = \alpha'x_0 + \alpha'x_1 + \epsilon_1x_1$, then $\nu(x) = r^2\kappa$ and the $x_0$ coordinate of $x$ is $\alpha'$ so

$$\tilde{\varphi}(\alpha', r^2\kappa) = \int_{\mathbb{Z}(F_v) \setminus T_0(F_v)} \varphi(t^{-1}(\alpha'x_0 + \alpha'x_1 + \epsilon_1x_1))dt$$

Since $\varphi$ is locally constant and $\mathbb{Z}(F_v) \setminus T_0(F_v)$ is compact, for $r$ small enough

$$\tilde{\varphi}(\alpha', r^2\kappa) = \int_{\mathbb{Z}(F_v) \setminus T_0(F_v)} \varphi(t^{-1}(\alpha'x_0 + \alpha'x_1))dt = \tilde{\varphi}(\alpha', 0).$$

Similarly when $\alpha^2 - r^2 = \kappa\square$ since $\chi$ is not the trivial character, $\kappa \neq -\square$ so it is not possible that $|r| > q^g|\alpha'|$ where $g$ is the index of the coset of squares in $1 + \wp$, also since in this case $\alpha^2 - r^2 \neq \square$, it is not possible that $|\alpha| > q^g|r|$.

Let $\epsilon_1 = -\alpha'$ and $\epsilon_2 = \sqrt{(\alpha^2 - r^2)\kappa}$. As above, $|\epsilon_1| \leq q^g|r|$ and if we let $x = \alpha'x_0 + \alpha'x_1 + \epsilon_1x_1 + \epsilon_2x_2$ we get that for small enough $r$

$$\tilde{\varphi}(\alpha', r^2\kappa) = \int_{\mathbb{Z}(F_v) \setminus T_0(F_v)} \varphi(t^{-1}(\alpha'x_0 + \alpha'x_1 + \epsilon_1x_1 + \epsilon_2x_2))dt$$

$$= \int_{\mathbb{Z}(F_v) \setminus T_0(F_v)} \varphi(t^{-1}(\alpha'x_0 + \alpha'x_1))dt$$

$$= \tilde{\varphi}(\alpha', 0).$$

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To continue with the proof of Lemma 33 notice that for $\alpha'$ in some neighborhood of 0 we have

$$\int_{Z(F_v) \setminus T_0(F_v)} \varphi(t^{-1}(\alpha' x_0 + \alpha' x_1)) dt = \varphi(0)$$

since we normalized the volume of $Z(F_v) \setminus T_0(F_v)$ to be 1.

To show that for $r$ small enough $\varphi_0(r)$ is independent of $r$ let $r$ be small enough that $\bar{\varphi}(\alpha', r^2 \kappa) = \bar{\varphi}(\alpha', 0)$ for any $\alpha'$ such that $\chi(\alpha' + r) + \chi(\alpha' - r) \neq 0$. This means that

$$\varphi_0(r) = \frac{\chi(2)C}{2} \int_{F_v} \bar{\varphi}(\alpha', 0)[\chi(\alpha' + r) + \chi(\alpha' - r)] d\alpha'.$$

If $\alpha' \not\in r\mathcal{g}$ then $\chi(\alpha' - r) = \chi(\alpha' + r) = \chi(\alpha')$ so we can split the integral into two parts

$$\varphi_0(r) = \chi(2)C \int_{\alpha' \not\in r\mathcal{g}} \bar{\varphi}(\alpha', 0)\chi(\alpha') d\alpha' + \frac{\chi(2)C}{2} \int_{r\mathcal{g}} \bar{\varphi}(\alpha', 0)[\chi(\alpha' + r) + \chi(\alpha' - r)] d\alpha'.$$

For $r$ small enough the integral over $r\mathcal{g}$ becomes

$$\frac{\chi(2)C}{2} \int_{r\mathcal{g}} \varphi(0)[\chi(\alpha' + r) + \chi(\alpha' - r)] d\alpha'$$

which splits into the sum of the integrals

$$\frac{\chi(2)C}{2} \varphi(0) \left( \int_{r\mathcal{g}} \chi(\alpha' + r) d\alpha' \int_{r\mathcal{g}} + \chi(\alpha' - r) d\alpha' \right)$$

and when we change variables in the two integrals we get

$$\chi(2)C \varphi(0) \int_{r\mathcal{g}} \chi(\alpha') d\alpha'$$

so the sum of the two integrals is

$$\varphi_0(r) = \chi(2)C \int_{F_v} \bar{\varphi}(\alpha', 0)\chi(\alpha') d\alpha'$$
We have now established proposition 30.

The following proposition is needed in order to define the global representation.

**Proposition 35 (fundamental Lemma)** If $F_v$ is nonarchimedean with $q$ odd, $\chi$ is unramified, $\text{Inv}(B_v) = 1$, $\nu(x_0) \in \mathcal{O}_v^\times$ and $\varphi = \text{Char}_{V \cap M_2(\mathcal{O}_v)}$ then $\varphi_0 = \text{Char}_{\mathcal{O}_v}$.

**PROOF.** From Lemma 28 we have that when $\chi = 1$, $\varphi_0$ must be

$$
\varphi_0(r) = |r| \int_{T_0(F_v) \backslash H(F_v)} \varphi(rh^{-1} \cdot x_0) dh
$$

(4.25)

If we normalize the measure on $T_{x_0}(F_v) \backslash H(F_v)$ such that $\varphi_0(1) = 1$ then the volume of the set $\{h \in T_0(F_v) \backslash H(F_v) : h^{-1} \cdot x_0 \in M_2(\mathcal{O}_v)\}$ is one.

Notice that for $\text{ord}(r) = n$ the set $\{h \in T_0(F_v) \backslash H(F_v) : h^{-1} \cdot rx_0 \in M_2(\mathcal{O}_v)\}$ is the same as the set $\{h \in T_0(F_v) \backslash H(F_v) : h^{-1} \cdot x_0 \in \varpi^{-n} M_2(\mathcal{O}_v)\}$ and so by Lemma 36 its volume is $q^n$. This means that for $r \in \mathcal{O}_v$ we have $\varphi_0(r) = 1$.

Also for $r \notin \mathcal{O}_v$, $\nu(h \cdot rx_0) \notin \mathcal{O}_v$ so $h \cdot rx_0 \notin M_2(\mathcal{O}_v)$ and $\varphi_0(r) = 0$.

When $\chi$ is unramified non trivial we normalize the measure such that $\text{Vol}_{T_0 \backslash H(H(\mathcal{O}_v))} = 1$ then by Lemma 37 $\varphi_0 = \text{Char}_{\mathcal{O}_v}$.

**Lemma 36** When $\chi_v = 1$ (and hence $B_v \simeq M_2$), $F_v$ is nonarchimedean and $L \subset V(F_v)$ is any lattice then if

$$
\text{Vol}(\{h \in T_0(F_v) \backslash H(F_v) : h^{-1} \cdot x_0 \in L\}) := \text{Vol}_H(L) \neq 0
$$

then $\text{Vol}_H(\varpi^{-n}L) = q^n \text{Vol}_H(L)$. 

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PROOF. A few things are clear: It is enough to show the result in the case of \( n = 1 \), and that the volume is independent of the choice of \( x_0 \) of norm \( \kappa \). It is also not hard to see that we can assume that \( x_0 \in L \).

To carry out the calculation, pick an isomorphism of \( B \) with \( M_2(F_v) \) such that \( L \) is the trace zero matrices with integer coefficients and

\[
x_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Let \( K = \text{GL}_2(\mathcal{O}_v) \). It turns out that the double coset \( T_0 \backslash \text{GL}_2(F_v)/K \) is represented by the set \( \{ q_i : i = 0 \ldots \infty \} \) with

\[
q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_i = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \quad i > 0.
\]

This means that the coset space \( T_{x_0} \backslash H \) is the disjoint union of the cosets \( q_iK \).

It is clear that \( K \cdot L = L \) and \( K \cdot \varpi^n L = \varpi^n L \) so \( (q_iK)^{-1} \cdot x_0 \subset L \) if and only if \( q_i^{-1} \cdot x_0 \in L \) and also \( (q_iK)^{-1} \cdot x_0 \subset \varpi^{-1} L \) if and only if \( q_i^{-1} \cdot x_0 \in \varpi^{-1} L \).

The only \( q_i \) such that \( q_i \cdot x_0 \in L \) is \( q_0 \), so that \( \text{Vol}_H(L) \) is equal to the volume of the set \( K \subset T_0 \backslash H \).

Similarly \( q_1 \) and \( q_0 \) are the only \( q_i \) such that \( q_i \cdot x_0 \in \varpi^{-1} L \) and so \( \text{Vol}_H(L) \) is the sum of the volumes of \( K \) and \( q_1 K \). Comparison of the volumes of the two lattices now follows from a simple group calculation.

**Lemma 37** If \( q \) is odd, \( \chi \) is unramified non trivial, \( \varphi = \text{Char}(M_2(\mathcal{O}_v)) \) and \( \text{Vol}_{T_0 \backslash H}(M_2(\mathcal{O}_v)) = 1 \) then \( \varphi_0 \) as defined by equation 4.23 is equal to \( \text{Char}(\mathcal{O}_v) \).
PROOF. Pick

\[ x_0 = \begin{pmatrix} 0 & -\gamma \\ 1 & 0 \end{pmatrix} \]

with \( \gamma \neq -\Box, \; \gamma \in \mathcal{O}_v^\times \). By Lemma 10 we can see that

\[ T_0 = \left\{ \begin{pmatrix} a & -\gamma b \\ b & a \end{pmatrix} : a, b \in F_v \right\} \]

Notice that the function \( \varphi \) is invariant by the action of \( K = H^\times(\mathcal{O}_v) = \text{GL}_2(\mathcal{O}_v) \) so to calculate

\[ \varphi_0(r) = \chi(r) |r| \int_{T_0 \backslash H} \varphi(h^{-1} \cdot rx_0) \chi(\nu(h)) dh \]

The integral can be replaced by a sum over coset representative of

\[ T_0 \backslash H/K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \omega^n \end{pmatrix} := q_n : n \geq 0 \right\} \]

To actually calculate the integral we need to calculate the volume of the cosets \( \text{Vol}_{T_0 \backslash H}(q_iK) \). Since we set \( \text{Vol}_{T_0 \backslash H}(K) = 1 \) we obtain the volume of the other cosets by comparing them to the volume of \( K \). And in fact, for \( i > 0 \), \( \text{Vol}_{T_0 \backslash H}(q_iK) = (q + 1)q^{i-1} \). We can now calculate \( \varphi_0 \). First notice that if \( r \notin \mathcal{O}_v \) then \( \nu(rx_0) \notin \mathcal{O}_v \) and \( \varphi_0(r) = 0 \). When \( r \in \mathcal{O}_v \) then

\[ \varphi_0(r) = |r| \chi(r) \left( \varphi(rx_0) + \sum_{i=1}^{\infty} (q + 1)q^{i-1} \varphi(q_i^{-1} \cdot x_0) \chi(\nu(q_i)) \right) \]
now, \( q_i^{-1} \cdot r x_0 = \begin{pmatrix} 0 & -\varpi^{-i} r \gamma \\ r \varpi^i & 0 \end{pmatrix} \) so for \( i < \text{ord}(r) \), \( q_i^{-1} \cdot r x_0 \) is not in the support of \( \varphi \) so

\[
\varphi_0(r) = q^{-\text{ord}(r)} (-1)^{\text{ord}(r)} \left( 1 + \sum_{i=1}^{\text{ord}(r)} (q + 1) q^{i-1} (-1)^i \right) = 1
\]

We can now obtain some parity properties for \( \varphi \) and \( \varphi_0 \).

**Corollary 38** The parity of \( \varphi_0 \) is even when \( \text{Inv}(B_v) = 1 \) and odd if \( \text{Inv}(B_v) = -1 \).

**PROOF.** Be Lemma 24 there is an element \( \xi \in \mathbb{H}(F_v) \) such that \( \xi^{-1} \cdot x_0 = -x_0 \) and \( \chi(-\nu(\xi)) = \text{Inv}(B_v) \). With that we can calculate

\[
\varphi_0(-r) = |r| \chi(-r) \int_{T_{x_0} \setminus H} \varphi(r(\xi h)^{-1}) \cdot x_0 \chi(\nu(h)) dh
\]

\[
= |r| \chi(-r) \int_{T_{x_0} \setminus H} \varphi(r h'^{-1} \cdot x_0) \chi(\nu(\xi^{-1} h')) dh'
\]

with \( h' = \xi h \)

\[
= |r| \chi(-r) \chi(\nu(\xi)) \int_{T_{x_0} \setminus H} \varphi(r h'^{-1} \cdot x_0) \chi(\nu(h')) dh'
\]

\[
= \chi(-\nu(\xi)) |r| \chi(r) \int_{T_{x_0} \setminus H} \varphi(r h'^{-1} \cdot x_0) \chi(\nu(h')) dh'
\]

\[
= \text{Inv}(B) \varphi_0(r)
\]

**Corollary 39** Let \( \varphi \in \mathcal{S}(V(F_v)) \) \( \varphi = \varphi^+ + \varphi^- \) with \( \varphi^+ \) an even function and \( \varphi^- \) an odd function. If \( \chi(-1) \text{Inv}(B_v) = 1 \) then \( \varphi_0^- = 0 \) and if \( \chi(-1) \text{Inv}(B_v) = -1 \) then \( \varphi_0^+ = 0 \).
PROOF. Consider $\varphi^+$. By corollary 38 $\varphi_0(-r) = \text{Inv}(B_v)\varphi_0(r)$ so

\[
\varphi_0^+(r) = \text{Inv}(B_v)\varphi_0^+(-r)
\]

\[
= \text{Inv}(B_v)|r|\chi(-r) \int_{T_0 \backslash H} \varphi^+(-rh^{-1} \cdot x_0)\chi(\nu(h))dh
\]

\[
= \text{Inv}(B_v)\chi(-1)|r|\chi(r) \int_{T_0 \backslash H} \varphi^+(rh^{-1} \cdot x_0)\chi(\nu(h))dh
\]

\[
= \text{Inv}(B_v)\chi(-1)\varphi_0^+(r)
\]

and we see that if $\text{Inv}(B_v)\chi(-1) \neq 1$ then $\varphi_0^+(r) = -\varphi_0^+(r)$. The case of $\varphi_0^-$ is similar.

Recall that the choice of parity at places of $F$ determines the character $\mu$. We can restate corollary 38 in terms of $\mu$ and obtain the remaining part of the local data defining $W(U, \mu)$.

**Lemma 40** In order that $W_V(\chi) \subset W_U(\mu)$ it is necessary that the character $\mu = \otimes_v \mu_v$ is defined by

\[
\mu_v = \begin{cases} 
1 & \text{if } \text{Inv}(B_v) = 1 \\
\text{Sgn} & \text{if } \text{Inv}(B_v) = -1
\end{cases}
\]

5 Nonarchimedean local matching

5.1 An equivariant property

For the following, we will need to recall the Bruhat decomposition. From Lemma 28 we know that if $\varphi_0$ is defined according to equation (4.23) then $W_\varphi$ and $W_{\varphi_0}$ agree on $\bar{B}$. It remains to show that with the same $\varphi_0$, the Whittaker
functions $W_\varphi$ and $W_{\varphi_0}$ will also agree on $\tilde{B} w \tilde{B}$. The verification of this result is simplified by the following lemma.

**Lemma 41** Let the map $i : S(V) \to S(U)$ be defined by $i(\varphi) = \varphi_0$ where $\varphi_0$ is given by equation (4.23), then for any $g \in \tilde{B}$

$$i(\omega(g)(\varphi)) = \omega_0(g)(i(\varphi)).$$

**PROOF.** Since $\tilde{B} = NA = AN$, it will suffice to show that for any $a \in F_v^\times$,

$$i(\omega(m(a))(\varphi)) = \omega_0(m(a))(i(\varphi))$$

and for any $b \in F_v$

$$i(\omega(n(b))(\varphi)) = \omega_0(n(b))(i(\varphi)).$$

To establish equation (5.26) we compute $i(\omega(m(a))(\varphi))(r)$

$$= |r| \chi(r) \int_{T_0 \setminus H} \omega(m(a)) \varphi(r h^{-1} \cdot x_0) \chi(\nu(h)) dh$$

$$= |r| \chi(r) |a|^{\frac{3}{2}} (a, - \det(V)) \gamma(a, \psi^\frac{1}{2})^{-1} \int_{T_0 \setminus H} \varphi(ar h^{-1} \cdot x_0) \chi(\nu(h)) dh.$$ 

On the other hand $\omega_0(m(a))(i(\varphi))(r)$

$$= |a|^{\frac{3}{2}} (a, 2\kappa) \gamma(a, \psi^\frac{1}{2})^{-1} |ar| \chi(ar) \int_{T_0 \setminus H} \varphi(ar h^{-1} \cdot x_0) \chi(\nu(h)) dh$$

$$= |r| \chi(r) |a|^{\frac{3}{2}} \chi(a, 2\kappa) \gamma(a, \psi^\frac{1}{2})^{-1} \int_{T_0 \setminus H} \varphi(ar h^{-1} \cdot x_0) \chi(\nu(h)) dh.$$ 

Comparing the two sides of equation (5.26), we see that they are equal if

$$\chi(a)(a, 2\kappa) = (a, - \det(V)).$$ 

Recall from proof of lemma 28 that $\det(V)$ is in the square class of 2 for both division and matrix algebra so it is true that

$$\chi(a)(a, 2\kappa) = (a, - \kappa)(a, 2\kappa) = (a, - \det(V)).$$

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To establish equation (5.27) we calculate \( i(\omega(n(b))(\varphi))(r) \)

\[
= |r| \chi(r) \int_{T_0 \backslash H} \omega(n(b))\varphi(rh^{-1} \cdot x_0)\chi(\nu(h))dh \\
= \psi(b \kappa)|r| \chi(r) \int_{T_0 \backslash H} \varphi(rh^{-1} \cdot x_0)\chi(\nu(h))dh \\
= \psi(b \kappa)i(\varphi)(r) \\
= \omega_0(n(b))i(\varphi)(r)
\]

**Lemma 42** If for all \( \varphi \in S(V) \), \( W_\varphi(w) = CW_{\varphi_0}(w) \) for some constant \( C \), then for all \( \varphi \in S(V) \) and for all \( g \in BwB \), \( W_\varphi(g) = CW_{\varphi_0}(g) \).

**PROOF.** The subgroup \( \tilde{B} \) can be factored as \( NA \). Since \( m(a)w = wm(a^{-1}) \)

The cell \( BwB \) can be written as \( NwB \). Consider any \( wg \in wB \), by assumption, for any \( \varphi \in S(V) \) we have \( W_\varphi(w) = CW_{i(\varphi)}(w) \) so, using Lemma 41

\[
W_\varphi(wg) = W_{\omega(g)\varphi}(w) \\
= CW_{i(\omega(g)\varphi)}(w) \\
= CW_{\omega_0(g)i(\varphi)}(w) \\
= CW_{i(\varphi)}(wg)
\]

So \( W_\varphi \) and \( CW_{i(\varphi)} \) coincide on all of \( wB \). Finally, since both \( W_\varphi(w) \) and \( CW_{i(\varphi)}(w) \) are in the same Whittaker space by construction, they coincide on all of \( NwB = BwB \).

We will use two different arguments to show that \( W_\varphi \) matches \( W_{\varphi_0} \). In some cases we will match a single \( \varphi \) and use general information about the Weil representation to conclude that in fact, all other functions \( \varphi \) are also matched by \( \varphi_0 \). In other cases we will show directly that for any \( \varphi \in S(V) \), \( W_\varphi(w) = CW_{\varphi_0}(w) \) for some constant \( C \) and then use Lemma 42 to conclude that the
$$W_\varphi = CW_\varphi_0$$ on all of $\tilde{B}w\tilde{B}$ for some constant $C$. Finally, we will show that $C = 1$ and deduce the actual equality on all of $\tilde{S}L_2$ from that relation.

5.2 $\chi \neq 1$.

From equation (2.16) and the definition of $W_\varphi_0$, (definition 20.) we have that

$$W_\varphi_0(w) = \gamma(\psi \circ U)^{-1} \int_U \varphi_0(y)\psi(-2\kappa y)dy$$

Notice that we set $m$ in (2.16) equal to $\kappa$ according to Lemma 19.

The measure $dy$ is self dual with respect to the Fourier transform $\hat{f}(x) = \int_{U(F_v)} f(y)\psi(-2\kappa xy)dy$. Since the conductor of $\psi$ is $\epsilon^{-1} \infty^{-d} \mathcal{O}_v$ it follows by a standard argument that $dy = |\infty^d\epsilon 2\kappa|^\frac{1}{2} du$ where $du$ gives $\mathcal{O}_v$ volume 1.

Since $\det(U) = 2\kappa$ and $\epsilon(U) = 1$ we can simplify the expression for $\gamma(\psi \circ U) = \gamma(\det(U), \psi^{\frac{1}{2}})\gamma(\psi^{\frac{1}{2}})\epsilon(U)$ to $\gamma(\psi^{\kappa})$. And substituting the explicit expression for $\varphi_0$ in terms of $\varphi$, we have

$$W_\varphi_0(w) = |\infty^d\epsilon 2\kappa|^{\frac{1}{2}} \gamma(\psi^{\kappa})^{-1} \int_U \varphi_0(u)\psi(-2\kappa u)du$$

$$= |\infty^d\epsilon 2\kappa|^{\frac{1}{2}} \gamma(\psi^{\kappa})^{-1} \int_U |u|\chi(u) \int_{T_0 \backslash H} \varphi(h^{-1} \cdot u x_0)$$

$$\times \chi(\nu(h))dh \psi(-2\kappa u)du$$

(5.28)

On the other hand, by the definition of the action of $\omega(w)$ (equation (2.13)) and the definition of $W_\varphi$ we have

$$W_\varphi(w) = \gamma(\psi \circ V)^{-1} \int_{T_0 \backslash H} \int_V \varphi(y)\psi(-(h^{-1} x_0, y))dy \chi(\nu(h))dh$$

(5.29)

Remark 43 As mentioned above, we need to verify the equality of (5.28) and (5.29). Notice that both equations involve integrals over $V(F_v)$ and $T_0 \backslash H$ so
verification of the equality is independent of the realization of the local algebra $B_v$ as long as the measures $dy$ and $du$ are self dual with respect to the action of $w$ and the measure on $T_0 \setminus H$ is the same in both equations.

We continue with the calculation of the right hand side of (5.29). Since $\chi(r) = (r, -\kappa)$ is nontrivial, and $\kappa = \nu(x_0)$ we know from Lemma 12 that $Z \setminus T_0$ is compact. It is also clear that the integrand in (5.29) is constant on left $T_0$-cosets so we will replace the integral over $T_0 \setminus H$ with an integral over $Z \setminus H$. To do that, Let $M_\kappa = \text{Vol}(Z \setminus T_0)$ and we have:

$$W_\varphi(w) = \gamma(\psi \circ V)^{-1} \int_{T_0 \setminus H} \int_V \varphi(y) \psi(-(h^{-1}x_0, y)) dy \chi(\nu(h)) dh$$

$$= M_\kappa^{-1} \gamma(\psi \circ V)^{-1} \int_{Z \setminus H} \int_V \varphi(y) \psi(-(h^{-1}x_0, y)) dy \chi(\nu(h)) dh.$$

We now change variables in the inner integral from $y$ to $y' = h \cdot y$

$$W_\varphi(w) = \gamma(\psi \circ V)^{-1} M_\kappa^{-1} \int_{Z \setminus H} \int_V \varphi(h^{-1}y') \psi(-(x_0, y')) dy' \chi(\nu(h)) dh.$$

Next we let $C_L$ be the characteristic function of a compact set $L \subset V(F_v)$ and we note that for any increasing sequence of compact sets

$$W_\varphi(w)$$

$$= \lim_L \gamma(\psi \circ V)^{-1} M_\kappa^{-1} \int_{Z \setminus H} \varphi(h^{-1}y) \psi(-(x_0, y)) C_L(y) dy \chi(\nu(h)) dh$$

$$= \lim_L \gamma(\psi \circ V)^{-1} M_\kappa^{-1} \int_V \psi(-(x_0, y)) \int_{Z \setminus H} \varphi(h^{-1}y) \chi(\nu(h)) dh C_L(y) dy$$

$$= \lim_L \gamma(\psi \circ V)^{-1} M_\kappa^{-1} \int_{\Omega^x} \psi(-(x_0, y)) \int_{Z \setminus H} \varphi(h^{-1}y) \chi(\nu(h)) dh C_L(y) dy,$$

where in the last step we replaced the integral over $V(F_v)$ by the integral over $\Omega^x = \{ v \in V : \nu(v) \neq 0 \}$. This is justified by the fact that the set of elements with 0 norm have measure 0 in $V$. 

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Now, from Lemma 14 we know that if \( \nu(y) \) is not in the square class of \( \kappa \) then there is an element \( h_y \) in its stabilizer \( T_y(F_v) \) such that \( \chi(\nu(h_y)) = -1 \). In that case the inner integral is zero. So we can assume that the support of \( \varphi \) is contained in the set

\[
\Omega_\kappa^x = \{ v \in \Omega^x : \nu(v) \in \kappa(F_v) \}
\]

We state this as a lemma for reference.

**Lemma 44** Let \( \chi(a) = (a, -\kappa) \) be a nontrivial character and let \( \varphi_\kappa \) be the restriction of \( \varphi \) to \( \Omega_\kappa^x \) then \( W_\varphi(w) = W_{\varphi_\kappa}(w) \).

To continue, we will parameterize \( \Omega_\kappa^x \) by using the two fold covering

\[
F^x \times T_0 \backslash H \to \Omega_\kappa^x \quad (c, h) \to ch^{-1}x_0.
\]

The measure on \( \Omega^x \) is the measure \( dy \) which makes the Fourier transform \( \hat{f}(x) = \int_{V(F_v)} \hat{\psi}(-(x, y))f(y)dy \) self dual. This measure is invariant under the action of \( H \) and transforms by \( |c|^3 \) under multiplication by a scalar \( c \).

The measure \( |u|^2dhdu \) on the cover \( F^x \times T_{x_0} \backslash H \) is also invariant under the action of \( H \) and also transforms by \( |c|^3 \) under multiplication by \( c \). It follows that the two measures are related by a positive constant \( C \). We defer the calculation of \( C \) since we will soon change variables again and will normalize the measure that results from the two changes of variables.

The relation between the two measures allows us to write \( W_\varphi(w) \) in terms of
integrals over $F^\times \times T_0 \backslash H$

$$W_\varphi(w) = \lim_{L} \gamma(\psi \circ V)^{-1} CM_\kappa^{-1} \int_{F^\times \times T_0 \backslash H} \psi(-(x_0, h_1^{-1}ux_0)) \times \int_{Z \backslash H} \varphi(h^{-1}h_1^{-1}ux_0)\chi(\nu(h))dh C_L(h_1^{-1}ux_0)|u|^2 dh_1 du$$

We let $h_2 = h_1 h$ and we get

$$W_\varphi(w) = \lim_{L} \gamma(\psi \circ V)^{-1} CM_\kappa^{-1} \int_{F^\times \times T_0 \backslash H} \psi(-(x_0, h_1^{-1}ux_0))\chi(\nu(h_1))|u|^2 \times \int_{Z \backslash H} \varphi(h_2^{-1}ux_0)\chi(\nu(h_2))dh_2 C_L(h_1^{-1}ux_0)dh_1 du$$

Notice that the integrand in the integral over $Z \backslash H$ is left invariant under $Z \backslash T_0$ and recall that the volume of $Z \backslash T_0$ is $M_\kappa$ and so the inner integral can be changed to an integral over $T_0 \backslash H$.

$$W_\varphi(w) = \lim_{L} \gamma(\psi \circ V)^{-1} C \int_{F^\times \times T_0 \backslash H} \psi(-(x_0, h_1^{-1}ux_0))\chi(\nu(h_1))|u|^2 C_L(h_1^{-1}ux_0)dh_1 \times \left( \int_{T_0 \backslash H} \varphi(h_2^{-1}ux_0)\chi(\nu(h_2))dh_2 \right) du.$$

Comparing with equation (4.23) we see that the third integral is $\varphi_0(u)|u|^{-1}\chi(u)$ so we get

$$W_\varphi(w) = \lim_{L} \gamma(\psi \circ V)^{-1} C \int_{F^\times \times T_0 \backslash H} \psi(-(x_0, h_1^{-1}ux_0))\chi(\nu(h_1))C_L(h_1^{-1}ux_0)dh_1 \times |u|\chi(u)\varphi_0(u)du.$$

The integrand in $W_\varphi(w)$ depends only on cosets of the form $h_\alpha T_0$ so using
Lemmas 26 25 we can change the integral in $W_\varphi(w)$ over $T_0 \setminus H$ to an integral over the product of the set of $\alpha$ such that $\chi(\alpha^2 - 1) = \text{Inv}(B_v)$ and $Z(F_v) \setminus T_{x_0}(F_v)$.

We have now changed measures from $dy$ to the product of the measures $d\alpha du dt$, with $\alpha$ as above, $t \in Z \setminus T_0$ and $u \in F_v^\times$. At this point in the calculation we note that the constant relating the two measures will depend on the type of algebra $B_v$ and on the norm of $x_0$ defining the torus $T_0$. We denote the constant relating the measures $dy$ and $|u^2|d\alpha du dt$ by $C_{\kappa,v}$.

So far the formula for $W_\varphi(w)$ looks like:

$$C_{\kappa,v} \lim_{L^\gamma} \frac{1}{\varphi_0(u)} \int |u| \chi(u) \varphi_0(u) \int \psi(-(x_0, t^{-1}h_\alpha^{-1}ux_0)) \chi(\nu(th_\alpha))$$

$$C_L(t^{-1}h_\alpha^{-1}ux_0)dt \ d\alpha du,$$

where $\alpha$ ranges over elements such that $\chi(\alpha^2 - 1) = \text{Inv}(B)$. Also recall that $h_\alpha^{-1}x_0 = \alpha x_0 + x$ with $x \perp x_0$ so

$$(x_0, h_\alpha ux_0) = \alpha u(x_0, x_0) = \alpha u \nu(x_0) = 2\alpha u \kappa$$

With the above observation and with Lemmas 26 and 27 the two inner integrals become

$$C_{\kappa,v} \lim_{L^\gamma} \int \int \psi(-2u\alpha \kappa) \chi(2(\alpha + 1)))C_L(t^{-1}h_\alpha^{-1}ux_0)dt \ d\alpha.$$

Since we are taking the limit over all increasing sequences of lattices $L$ the limit would not change if we considered only lattices which are invariant under the action of the compact group $Z \setminus T_0$. In that case, the integrand does not
depend on $t$. Let $V_{x_0} = \text{Vol}(\mathbb{Z} \setminus T_0)$ and the inner integrals become

$$V_{x_0} \lim_L \int_{\alpha} \psi(-2u\alpha \kappa) \chi(2(\alpha + 1)) C_{L}(h_{\alpha}^{-1}ux_0)d\alpha.$$ 

The characteristic function of the set of $\alpha \neq \pm 1$ such that $\chi(\alpha^2 - 1) = \text{Inv}(B)$ is the function

$$\frac{1}{2}(1 + \text{Inv}(B) \chi(\alpha^2 - 1))$$

so we can extend the integral over $\alpha$ to an integral over $F_v - \{\pm 1\}$ by multiplying the integrand by the characteristic function. The integral over $F_v$ is only defined as a limit and should be understood that when we write $\int_{F_v} \psi(-\alpha') \chi(\alpha')d\alpha$ we will mean $\lim_n \int_{\varphi^{-n}} \psi(-\alpha') \chi(\alpha')d\alpha$.

We split the integral into two integrals and change variables to $\alpha + 1$ and to $\alpha - 1$ respectively. We also define $\chi(0) = 0$ and extend the integral to all of $F_v$.

$$= V_{x_0} \chi(2) \int_{F_v - \{\pm 1\}} \psi(-2u\alpha \kappa) \frac{1}{2}(\chi(\alpha + 1) + \text{Inv}(B) \chi(\alpha - 1))d\alpha.$$ 

We use another change of variable $\alpha' = 2u\alpha \kappa$ and we get

$$= V_{x_0} \chi(u) |2u\kappa|^{-\frac{1}{2}} \psi(2\kappa u) + \text{Inv}(B) \psi(-2\kappa u) \int_{\tilde{F}_v} \psi(-\alpha') \chi(\alpha')d\alpha'.$$

We give the above integral a name

**Definition 45** The 'gauss integral' is the quantity

$$g_v(\psi, \chi) = \int_{F_v} \psi(-\alpha) \chi(\alpha)d\alpha = \lim_n \int_{\varphi^{-n}} \psi(-\alpha) \chi(\alpha)d\alpha.$$
So far the inner integrals in equation 5.30 are

\[ g_v(\psi, \chi)V_{x_0}(u)|2u\kappa|^{-\frac{1}{2}}(\psi(2\kappa u) + \text{Inv}(B)\psi(-2\kappa u)) \]  

(5.31)

putting equation 5.31 together with equation (5.30) we get

\[ W_{\phi}(w) = C_{\kappa,v,\gamma}(\psi \circ V)^{-1}|2\kappa|^{-1}g_v(\psi, \chi)V_{x_0} \int_{F_v^\times} \varphi_0(u)\frac{1}{2}(\psi(2\kappa u) + \text{Inv}(B)\psi(-2\kappa u))du \]

Now by Lemma 38 \( \varphi_0(-u) = \text{Inv}(B_v)\varphi_0(u) \) and so

\[ W_{\phi}(w) = C_{\kappa,v} \text{Inv}(B_v)\gamma(\psi \circ V)^{-1}|2\kappa|^{-1}|2\kappa|^{-\frac{1}{2}}2\kappa|^{-\frac{1}{2}}\gamma(\psi^{\kappa})W_{\phi_0}(w) \]

What we showed so far is that on the "big cell", the two Whittaker functions \( W_{\phi} \) and \( W_{\phi_0} \) are related by a constant which depends on \( \chi, F_v, B_v \) but does not depend on the argument of the function. I.e.

\[ W_{\phi}(g) = C_{\kappa,v,B_v}W_{\phi_0}(g) \]

with

\[ C_{\kappa,v,B_v} = C_{\kappa,v}V_{x_0}g_v(\psi, \chi)|2\kappa|^{-\frac{1}{2}}2\kappa|^{-\frac{1}{2}}\text{Inv}(B_v)\gamma(\psi \circ V)^{-1}\gamma(\psi^{\kappa}) \]

If we showed that \( C_{\kappa,v,B_v} = 1 \) we would be done.

**Proposition 46** When \( F_v \) is non archimedean and \( \chi \neq 1 \) then for any \( \phi \in \mathcal{S}(V) \)

\[ W_{\phi}(w) = W_{\phi_0}(w). \]
Furthermore, $\Theta(H, \chi) \simeq \Theta(O(U), \mu)$ and for any $\varphi \in S(V)$ we have $W_\varphi = W_{\varphi_0}$.

**PROOF.** The discussion above shows that for any $\varphi$, and for $\varphi_0$ defined by equation (4.23) $W_\varphi(w) = C_{\kappa,v,B_v} W_{\varphi_0}(w)$. By Lemma 42 we have that $W_\varphi(g) = C_{\kappa,v,B_v} W_{\varphi_0}(g)$ for all $g \in \tilde{B} w \tilde{B}$. Since the Weil representation of $\widetilde{SL}_2$ is smooth, the stabilizers of $\varphi$ and $\varphi_0$ are both open subgroups and so the intersection of the stabilizers $G$ is an open neighborhood of $I \in \widetilde{SL}_2$. It is not hard to see that

$$\forall g \in G, \quad W_\varphi(g) = W_{\omega(g)\varphi}(I) = W_\varphi(I)$$

and

$$\forall g \in G, \quad W_{\varphi_0}(g) = W_{\omega_0(g)\varphi_0}(I) = W_{\varphi_0}(I)$$

We can find $g' \in G$ such that $g' \in \tilde{B} w \tilde{B}$. If we pick $\varphi$ such that $W_\varphi(I) \neq 0$ then

$$W_\varphi(g') = C_{\kappa,v,B_v} W_{\varphi_0}(g')$$

$$= C_{\kappa,v,B_v} W_{\varphi_0}(I)$$

and

$$W_\varphi(g') = W_\varphi(I)$$

$$= W_{\varphi_0}(I)$$

since $I \in \tilde{B}$. Therefore $C_{\kappa,v,B_v} = 1$ and we are done.
5.3 \( \chi = 1 \).

The argument for matching in this case is different from the argument in the case of a nontrivial character. In this case, the equality of the spaces \( W_{\psi^\chi}(V, \chi) \) and \( W_{\psi^\chi}(U, \mu) \) follows from results of Waldspurger and Rallis. The explicit matching of the functions \( W_\varphi \) and \( W_{\varphi_0} \) will then follow from Lemma 28.

**Proposition 47** For \( \chi = 1 \) and \( F_v \) non archimedean

\[
W_{\psi^\chi}(V, \chi) = W_{\psi^\chi}(U, \mu).
\]

Furthermore, for every \( \varphi \in S(V) \) and \( \varphi_0 \) defined by equation (4.23) we have

\[
W_\varphi = W_{\varphi_0}
\]

**PROOF.** We define the map \( j : S(V) \to W_{\psi^\chi}(V, \chi) \) by \( j(\varphi) = W_\varphi \). The map \( j \) is \( \tilde{\text{SL}}_2(F_v) \) equivariant and it is easy to see that it factors through the space \( S(V)_H \) of \( H \) coinvariant.

Since we know from Lemma 39 that \( W_\varphi = W_{\varphi^*} \), the map \( j \) actually factors through the \( \text{O}(V) \) coinvariants \( S(V)_{\text{O}(V)} \).

When \( \chi = 1 \) the formula for \( W_\varphi \) is

\[
W_\varphi(g) = \int_{T \backslash H} \omega(g) \varphi(h^{-1} \cdot x_0) dh
\]

If we let \( \varphi \) be the characteristic function of some neighborhood containing elements of norm \( \kappa \) then it is easy to see that \( W_\varphi \neq 0 \). Therefore \( j \neq 0 \).
On the other hand, let \( I(\eta) = \text{Ind}_B^{\widetilde{\text{SL}}_2}(\eta) \) where \( \eta \) is the character

\[
\begin{pmatrix}
a & b \\
0 & a^{-1}
\end{pmatrix} \mapsto |a|^2 \gamma(a, \psi^{-1}).
\]

There is an \( \widetilde{\text{SL}}_2(F_v) \) equivariant map \( T : \mathcal{S}(V) \to I(\eta) \) given by \( T(\varphi) = (g \mapsto \omega(g)\varphi(0)) \) which also factors through \( \mathcal{S}(V)_{O(V)} \). Let \( R(V) \) be the image of \( T \). A result of Rallis ([10] thm II.1.1) says that the map \( T \) induces an isomorphism \( \mathcal{S}(V)_{O(V)} \simeq R(V) \). It is also known (e.g. Gelbart [5]) that since \( V \) is a split space, then in fact \( R(V) \simeq I(\eta) \). Furthermore, the composition series of \( I(\eta) \) has length two, it has a unique irreducible submodule and a unique irreducible quotient.

In [15], Waldspurger analyzed the two factors in the composition series. His result is that the unique quotient is \( \Theta_U(1) \): the local theta correspondence of the trivial character from \( U \) (which Waldspurger denotes by \( r_{\psi,q}^{+} \)). This unique quotient has a \( \psi^K \) Whittaker model whereas the unique irreducible submodule of \( I(\eta) \) (which Waldspurger denotes by \( \tilde{\sigma}(\cdot | \cdot |_2) \)) has Whittaker models for all square classes of characters \( \psi^\alpha \) except for \( \psi^K \) (Lemma 3 on p 227 in [15]).

Since \( j \) factors through \( \mathcal{S}(V)_{O(V)} \simeq I(\eta) \) there is a map \( \tilde{j} : I(\eta) \to W_{\psi^K}(V, \chi) \) and by considering the Whittaker models we see that \( \ker(\tilde{j}) = \tilde{\sigma}(\cdot | \cdot |_2) \) and so \( j(\mathcal{S}(V)) = W_{\psi^K}(V, \chi) \). I.e. \( j(\mathcal{S}(V)) \) is the unique irreducible quotient of \( I(\eta) \) which is isomorphic to \( \Theta_U(1) \), that is \( j(\mathcal{S}(V)) = W_{\psi^K}(U, \mu) \).
6 Archimedean Local matching

The argument splits into three cases handled separately in the next subsections.

6.1 \( \chi \circ \nu \) is trivial on \( O(V)(\mathbb{R}) \).

Throughout this subsection, \( F_\nu \) will be isomorphic to \( \mathbb{R} \) and we will pick a particular realizations of the matrix algebra and division algebra such that \( \kappa = 1 \) when \( \chi \) is nontrivial and \( \kappa = -1 \) when \( \chi \) is trivial. We will not mention these conditions in the statements of the lemmas, but they should be understood to be stated with respect to these choices of \( \kappa \). Since the equality we will establish is defined independently of a particular realization there is no loss of generality.

The cases included in this subsection are the case of \( \chi = 1 \) (which by lemma 15, can only exist when \( B \) is the matrix algebra) and the case \( \chi \neq 1 \) and \( B \) is the division algebra (in which case \( \nu(H) \subset \mathbb{R}^+ \) so \( \chi \circ \nu = 1 \)). Note that in both cases, once we know that we can match all the forms coming from the representation of \( SO(V)(\mathbb{R}) \), by Lemma 39, the parity of all \( \varphi \in \mathcal{S}(V(\mathbb{R})) \) can be assumed to be even. Hence only the trivial extension of \( \chi \circ \nu \) to \( O(V)(\mathbb{R}) \) will have a nonzero lift. This is the reason for describing both cases in this subsection as the case of \( \chi \circ \nu \) trivial on \( O(V)(\mathbb{R}) \).

The matching in this case will be established by the matching of the Whittaker functions corresponding to the Gaussians and using the fact that the space \( \Theta(H, \chi) \) is generated by the image of the Gaussians. The matching of the
Gaussians will depend on calculating their $K$-types and using the Iwasawa decomposition to conclude that the two Whittaker functions agree on all of $\tilde{\text{SL}}_2$. The main argument is contained in the following theorem.

**Theorem 48** Let $\chi$ be a the trivial character when $B$ is a matrix algebra or the nontrivial character when $B$ is the division algebra. Then $W_{\psi^x}(V, \chi) = W_{\psi}(U, \mu)$ and for any $\varphi \in \mathcal{S}(V)$ we have and $W_{\varphi_0} = W_{\varphi}$.

**PROOF.** As explained above, in this case the characters extend to the trivial character on $O(V)$. Therefore, As mentioned in [8], it follows from a result of Howe [6] that $W_{\psi^x}(V, \chi)$ is generated by the image of the Gaussian $W_{\varphi}$. The calculation that we carry out in this proof will show that if $\varphi$ is the Gaussian then $W_{\varphi_0} = W_{\varphi}$ and therefore $W_{\psi^x}(V, \chi) \subset W_{\psi^x}(U, \mu)$. Therefore, for any $\varphi$, there is some $f \in \mathcal{S}(U)$ such that $W_f = W_{\varphi}$ and by Lemma 28 we know that $f$ must be $\varphi_0$. This would also show that the formula for $\varphi_0$ defines a Schwartz function in the restricted Schwartz space.

The second result is a consequence of Lemma 39, which implies that the space $W_{\psi^x}(U, \mu)$ is the image of the irreducible representation on even functions $\mathcal{S}(U)^+$ so it is also irreducible and $W_{\psi^x}(V, \chi) = W_{\psi^x}(U, \mu)$.

By the above argument, the calculation that remains is to show that when $\varphi$ is the Gaussian we have $W_{\varphi_0} = W_{\varphi}$. By Lemma 50 we know that for all elements of

$$K = \left\{ k_\alpha = \begin{pmatrix} \cos(\alpha) - \sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \right\}$$

we have $\omega(k_\alpha)\varphi = \exp(2\pi i m \alpha)\varphi$ and $\omega(k_\alpha)\varphi_0 = \exp(2\pi i m \alpha)\varphi_0$ for some $m$. 

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Using the Iwasawa decomposition $\tilde{SL}_2 = \tilde{B}K$ we write an arbitrary element of $\tilde{SL}_2$ as $bk_\alpha$ for some $\alpha$ and then

$$W_{\varphi_0}(bk_\alpha) = W_{\omega_0(k_\alpha)\varphi_0}(b)$$

$$= \exp(2\pi i m \alpha)W_{\varphi_0}(b)$$

$$= \exp(2\pi i m \alpha)W_{\varphi}(b)$$

$$= W_{\omega(k_\alpha)\varphi}(b)$$

$$= W_{\varphi}(bk_\alpha)$$

Where we have used the fact that by construction $W_{\varphi}(b) = W_{\varphi_0}(b)$.

**Remark 49** As we mentioned above we picked a realization of the algebras with $\nu(x_0) = \kappa = 1, -1$ when $\chi$ is nontrivial or trivial, respectively. By Lemma 19, this means that we must choose the quadratic form on $U$ to be $x \rightarrow -x^2$ and $x \rightarrow x^2$ respectively.

**Lemma 50** Let $\chi$ be trivial on $\nu(H)$ and let $\varphi \in S(V)$ be the Gaussian, then for some $m$ $\omega(k_\alpha)\varphi = \exp(2\pi i m \alpha)\varphi$ and $\omega_0(k_\alpha)\varphi_0 = \exp(2\pi i m \alpha)\varphi_0$

**PROOF.** The Lie group $K$ is connected so its representation is determined by the action of the Lie algebra. This Lie algebra is generated by the element

$$\zeta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so it will be enough to calculate the actions of $\zeta$ to show that the actions of the whole algebra are the same.

We denote the Lie algebra action of $\omega$ and $\omega_0$ by $\bar{\omega}$ and $\bar{\omega}_0$, respectively. Instead
of calculating the action of $\zeta$ directly, we write $\zeta = e_- - e_+$ with

$$e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and calculate the following actions

$$\tilde{\omega}_0(\zeta) = \left. \frac{d}{dt} \right|_{t=0} \omega_0(\exp(te_-)) + \left. \frac{d}{dt} \right|_{t=0} \omega_0(\exp(-te_+))$$

and

$$\tilde{\omega}(\zeta) = \left. \frac{d}{dt} \right|_{t=0} \omega(\exp(te_-)) + \left. \frac{d}{dt} \right|_{t=0} \omega(\exp(-te_+))$$

on $\varphi_0$ and $\varphi$ respectively. It is not hard to see that

$$\exp(-te_+) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = n(-t)$$

and

$$\exp(te_-) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = w \exp(-te_+) w^{-1}$$

The rest of the proof is a lengthy but standard calculus computation. Just as an illustration of the type of calculation involved we calculate the action on $\varphi$ when $\chi$ is trivial. In this case the Gaussian is the function

$$\varphi\left(\begin{pmatrix} u & v \\ w & -u \end{pmatrix}\right) = \exp(-2\pi|\epsilon|(u^2 + \frac{v^2}{2} + \frac{w^2}{2}))$$
and we calculate the first term in the action of $\zeta$.

$$\bar{\omega}(-e_+)\varphi(x) = \frac{d}{dt} \bigg|_{t=0} \omega(\exp(-te_+))\varphi(x)$$

$$= \frac{d}{dt} \bigg|_{t=0} \omega(n(-t))\varphi(x)$$

$$= \frac{d}{dt} \bigg|_{t=0} \exp(-t2\pi\epsilon\nu(x))\varphi(x)$$

$$= -2\pi\epsilon\nu(x)\varphi(x)$$  (6.32)

We now calculate the second term in the action of $\zeta$.

$$\bar{\omega}(e_-)\varphi(x) = \frac{d}{dt} \bigg|_{t=0} \omega(\exp(te_-))\varphi(x)$$

$$= \frac{d}{dt} \bigg|_{t=0} \omega(wn(-t)w^{-1})\varphi(x)$$

$$= \frac{d}{dt} \bigg|_{t=0} \gamma(\psi \circ V)\omega(wn(-t)\varphi(x))$$

The last step uses a Lemma (which is also omitted) which says that the Gaussian is an eigenfunction of $\omega(w)$ with eigenvalue $\gamma(\psi \circ V)^{-1}$.

$$\bar{\omega}(e_-)\varphi(x) = \frac{d}{dt} \bigg|_{t=0} \int_V \omega(n(-t))\varphi(y)\psi(-(x,y))dy$$

To continue the calculation we let

$$x = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$ \quad and \quad $$y = \begin{pmatrix} u & v \\ w & -u \end{pmatrix}$$

and then

$$\bar{\omega}(e_-)\varphi(x) = \frac{d}{dt} \bigg|_{t=0} \int \int \int \exp(2\pi \epsilon i t (u^2 + vw))$$

$$\exp(-2\pi \epsilon |t| (u^2 + \frac{v^2}{2} + \frac{w^2}{2}))$$

$$\exp(2\pi \epsilon i (2u\alpha + \gamma v + \beta w)) du dv dw$$
Using standard formulas from calculus and integrating with respect to the self dual measure

\[ \bar{\omega}(e_-)\varphi(x) = \frac{\text{sgn}(\epsilon) \exp(-2\pi|\epsilon|((\alpha^2 + \frac{\beta^2}{2} + \frac{\gamma^2}{2})))}{2} \]

\[ + 2\pi\epsilon i(-\alpha^2 - \beta\gamma) \exp(-2\pi|\epsilon|((\alpha^2 + \frac{\beta^2}{2} + \frac{\gamma^2}{2})) \]

so that

\[ \bar{\omega}(e_-)\varphi(x) = \frac{\text{sgn}(\epsilon)}{2} \varphi(x) + 2\pi i\epsilon \nu(x) \varphi(x) \]  \hspace{1cm} (6.33)

Adding up equation (6.33) and (6.32) we get the action of \( \zeta \) on the Gaussian

\[ \bar{\omega}(\zeta)\varphi(x) = \bar{\omega}(e_-)\varphi(x) + \bar{\omega}(-e_+)\varphi(x) \]  \hspace{1cm} (6.34)

\[ = \frac{\text{sgn}(\epsilon)}{2} \varphi(x) \]  \hspace{1cm} (6.35)

6.2 \( \chi \circ \nu \) is nontrivial on \( H(\mathbb{R}) \).

The remaining archimedean case is the case of a nontrivial \( \chi \) and \( B \) is a matrix algebra. The argument that will show matching is similar to the one for the other real cases. We will use the fact that the representation is known to be irreducible and the matching of one function to conclude that the Whittaker spaces of \( W_{\psi^*}(V, \chi) \) and \( W_{\psi^*}(U, \mu) \) coincide.

Throughout this section \( F_v \simeq \mathbb{R}, \chi \) is nontrivial and \( B \simeq M_2 \).

**Theorem 51** Let \( \chi \) be the nontrivial character on \( \mathbb{R} \) then

\[ W_{\psi^*}(V, \chi) = W_{\psi^*}(U, \mu) \]

and for any \( \varphi \in \mathcal{S}(V) \) \( W_\varphi = W_{\varphi_0} \).

**Proof.** Since in this case the quadratic space \( V \) is split with signature (2,1)
we can use a theorem of Zhu, (1.2 in [3]) which says that in this case, when 
\( \chi = \text{sgn} \), 
\( W_{\psi^*}(V, \text{sgn}) = W_{\psi^*}(V, 1) \). By Lemma 48 we know that 
\( W_{\psi^*}(V, 1) \) is irreducible and therefore so is 
\( W_{\psi^*}(V, \chi) \).

Since \( W_{\psi^*}(V, \chi) \) is irreducible and since we also know that 
\( W_{\psi^*}(U, \mu) \) is irreducible it will be enough to show that their Whittaker spaces intersect to show that they coincide.

We will show that when \( \varphi \) is the gaussian then 
\( W_{\varphi} = W_{\varphi_0} \), this would mean that the Whittaker spaces are equal and by Lemma 28 it follows that for any 
\( \varphi \in \mathcal{S}(V) \) we have, 
\( W_{\varphi} = W_{\varphi_0} \). It will also follow that \( \varphi_0 \) is actually a function in the restricted Schwartz space of \( \mathcal{S}(\mathbb{R}) \). As in the proof of theorem 48 it will be enough to show that when \( \varphi \) is the gaussian then \( \varphi \) and \( \varphi_0 \) are of the same K-type. This is done in Lemma 52 and completes the argument.

**Lemma 52** Let \( \chi \) be a nontrivial character, let \( B \) be the matrix algebra and let 
\( \varphi \in \mathcal{S}(V) \) be the gaussian character, then for some \( m \), 
\( \omega(k_\alpha) \varphi = \exp(2\pi i m \alpha) \varphi \) 
and 
\( \omega(k_\alpha) \varphi_0 = \exp(2\pi i m \alpha) \varphi_0 \).

**PROOF.** omitted

6.3 The case of \( O(V)(\mathbb{C}) \)

The local data in the complex case is as simple as possible. Since every complex number is a square, the quadratic character \( \chi = \chi_v \) is trivial. For the same reason, the quaternion algebra is ramified i.e. \( \text{Inv}(B_v) = 1 \).

The correspondence in this case can be deduced from known facts without any
additional calculation. In [1] (Proposition 2.1) Adams and Barbasch show that
the theta lift of the trivial character from $O(V)(\mathbb{C})$ to $SL_2(\mathbb{C})$ is isomorphic to
the irreducible representation $\omega_+$ which is the Weil representation of $SL_2(\mathbb{C})$
on $\mathcal{S}(U)(U)^+$- the space of even Schwartz functions.

If $I'_V(\chi \otimes 1, \varphi)$ is the automorphic form in the theta lift of the character $\chi \circ \nu$
extended trivially from $SO(V)(\mathbb{C})$ to $O(V)(\mathbb{C})$ defined similarly to $I_V(\chi, \varphi)$
then $I'_V(\chi \otimes 1, \varphi) = 2I_V(\chi, \varphi)$. Therefore, $\Theta_V(\chi) \simeq \omega^+ \simeq \Theta_U(\mu)$ and to con-
clude the matching of functions we need to show that the spaces of functions
$\Theta_V(\chi)$ and $\Theta_U(\mu)$ actually coincide. This follows from the multiplicity one
of spherical representations of $SL_2(\mathbb{C})$. The fact that the representations are
spherical is verified by the existence of the gaussian functions in both rep-
resentations. Therefore, the matching for the actual function is given by the
formula for $\varphi_0$ (4.23).

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