Subordination and superordination for multivalent functions defined by linear operators

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Abstract

In this paper, certain linear operators defined on \( p \)-valent analytic functions have been unified and for them some subordination and superordination results as well as the corresponding sandwich type results are obtained. A related integral transform is discussed and sufficient conditions for functions in different classes have been obtained.

Keywords: \( p \)-valent function, Linear operator, Starlike function, Strongly starlike function.

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1. Introduction

Let \( \mathcal{H} \) be the class of functions analytic in \( \mathbb{U} := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathcal{H}[a, n] \) be the subclass of \( \mathcal{H} \) consisting of functions of the form \( f(z) = a + anz^p + an_1z^{p+1} + \ldots \). Let \( \mathcal{A}_p \) denote the class of all analytic functions of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \mathbb{U})
\]

and let \( \mathcal{A}_1 := \mathcal{A} \). For two functions \( f(z) \) given by (1.1) and \( g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \), the Hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[
(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z).
\]

For two analytic functions \( f \) and \( g \), we say that \( f \) is \textit{subordinate} to \( g \) or \( g \) \textit{superordinate} to \( f \), if there is a Schwarz function \( w \) with \( |w(z)| \leq |z| \) such that \( f(z) = g(w(z)) \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(\mathbb{U}) \subseteq g(\mathbb{U}) \). The class \( R(\alpha) \) is defined by

\[
R(\alpha) := \left\{ f \in \mathcal{A} : \text{Re} \frac{f(z)}{z} > \alpha, 0 \leq \alpha < 1; z \in \mathbb{U} \right\}
\]

and \( R = R(0) \). The class \( S^*(\alpha) \) of starlike functions of order \( \alpha \) is defined as

\[
S^*(\alpha) := \left\{ f \in \mathcal{A} : \text{Re} \frac{zf'(z)}{f(z)} > \alpha, 0 \leq \alpha < 1; z \in \mathbb{U} \right\}.
\]

Note that \( S^*(0) = S^* \), the class of starlike functions. The class of starlike functions of reciprocal order \( \alpha \) is denoted by \( S^*_\alpha(\alpha) \) and is given by

\[
S^*_\alpha(\alpha) := \left\{ f \in S^* : \text{Re} \frac{zf'(z)}{f(z)} > \alpha, 0 \leq \alpha < 1; z \in \mathbb{U} \right\}.
\]

Note that \( S^*_0(0) = S^* \). For \(-1 \leq B < A \leq 1\), Janowski [14] introduced the class \( S^*[A, B] \) given by

\[
S^*[A, B] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} = \frac{1 + Az}{1 + Bz}, z \in \mathbb{U} \right\}.
\]

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For $A = 1$ and $B = -1$, it reduces to the class $S^*$. A function $f \in A$ is said to be strongly starlike function of order $\eta$ if it satisfies
\[
\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\eta \pi}{2} \quad (0 < \eta \leq 1; z \in U)
\]
or equivalently
\[
\frac{zf'(z)}{f(z)} < \left( \frac{1+z}{1-z} \right)^{\eta} \quad (0 < \eta \leq 1; z \in U).
\]
The class of all such functions is denoted by $SS^*(\eta)$. Obviously, $SS^*(1) = S^*$. The class $\mathcal{L}^r(\eta)$ is defined by
\[
\mathcal{L}^r(\eta) := \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \eta > 0; \ z \in U \right\}
\]
or equivalently
\[
z^r f(z)/f(z) < (1+z)^{\eta} \quad (\eta > 0; \ z \in U).
\]
Note that the class $\mathcal{L} := \mathcal{L}^r(\frac{1}{2})$, was introduced by Sokół and Stankiewicz [34] and studied recently by Rosihan M. Ali et al. [1].

For $\alpha_j \in \mathbb{C} \ (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \ (j = 1, 2, \ldots, m)$, the generalized hypergeometric function $F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series
\[
F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n z^n}{(\beta_1)_n \cdots (\beta_m)_n n!} \quad (l \leq m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\})
\]
where $(\alpha)_n$ is the Pochhammer symbol defined by
\[
(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \end{cases}
\]
Corresponding to the function
\[
h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^p F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),
\]
the Dziok-Srivastava operator [12] (see also [33]) $H_p^{[m]}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product
\[
H_p^{[m]}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) := h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \ast f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \cdots (\alpha_l)_{n-p} \cdot a_n z^n}{(\beta_1)_{n-p} \cdots (\beta_m)_{n-p} \cdot (n-p)!}.
\]
For brevity, we write
\[
H_p^{[m]}(\alpha_1) := H_p^{[m]}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)
\]
and we have the following identity:
\[
z(H_p^{[m]}(\alpha_1) f(z))' = \alpha_1 H_p^{[m]}(\alpha_1 + 1) f(z) - (\alpha_1 - p) H_p^{[m]}(\alpha_1) f(z).
\]
Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [13], the Carlson-Shaffer linear operator [27], the Ruscheweyh derivative operator [50], the generalized Bernardi-Libera-Livingston linear integral operator (cf. [4], [17], [20]) and the Srivastava-Owa fractional derivative operators (cf. [24], [27]).

Motivated by the multiplier transformation on $A$, we define the operator $I_p(r, \lambda)$ on $A_p$ by the following infinite series
\[
I_p(r, \lambda)f(z) := z^p + \sum_{n=p+1}^{\infty} \left( \frac{n + \lambda}{p + \lambda} \right)^r a_n z^n \quad (\lambda \in \mathbb{C} \setminus \{-1, -2, \ldots\})
\]
and we have the following identity:
\[
z(I_p(r, \lambda)f(z))' = (p + \lambda)I_p(r + 1, \lambda)f(z) - \lambda I_p(r, \lambda)f(z).
\]
For $\lambda \geq 0$, the operator was introduced and studied by Ravichandran and Sivaprasad Kumar [32] and extensively used by many authors (cf. [2], [3], [33]). The operator $I_p(\lambda)$ is closely related to the Sălăgean derivative operators [31]. The operator $I_p^\lambda := I_p(r, \lambda)$ was studied by Cho and Srivastava [9] and Cho and Kim [10]. The operator $I_r := I_1(r, 1)$ was studied by Uralegaddi and Somanatha [36].

Corresponding to the function $h_p$ defined in [13], Al-Kharasani and Al-Areefi [3] introduced a function $F_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ given by

$$h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \frac{z^p}{(1-z)^{\lambda+p-1}}$$

and defined a new linear operator $J_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$, analogous to $H_p^l m|\alpha_1|$, by

$$J_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) f(z) = F_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \ast f(z)$$

where $\alpha_j \in C \ (j = 1, 2, \ldots, l)$ and $\beta_j \in C \ \{0, -1, -2, \ldots\}$ \ ($j = 1, 2, \ldots, m$), $z \in U$, $\lambda > 0$. For convenience, we write

$$J_k^l m|\alpha_1| := J_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m).$$

They established the following identity:

$$z(J_k^l m|\alpha_1| f(z))' = (\alpha_l - 1)J_k^l m|\alpha_1| - 1 f(z) - (\alpha_1 - p - 1)J_k^l m|\alpha_1| f(z).$$

Special cases of this operator are when $p = 1$, it reduces to the operator defined in [16], when $p = 1, \lambda = 2$ it is the Noor’s integral operator defined in [24]. Now consider the following infinite series:

$$\mathcal{F}_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z) = z^p + \sum_{n=0}^{\infty} \left( \frac{n + \lambda}{p + \lambda} \right)^{\lambda+p-1} z^n \ (z \in C \ \{ -1, -2, \ldots \}),$$

we have

$$I_p(r, \lambda) f(z) = \mathcal{F}_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z) * f(z).$$

Corresponding to the function $F_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ given by

$$z(F_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)) = \frac{z^p}{(1-z)^{\lambda+p-1}} \ (z \in U; \ \lambda > 0),$$

Al-Kharasani and Al-Areefi [3] defined the multiplier transform $T_k(r, \lambda)$ as follows:

$$T_k(r, \lambda) f(z) = F_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \ast f(z) \quad (z \in C \ \{ -1, -2, \ldots \}, \ \lambda > 0; f \in \mathcal{A}_{p}, z \in U)$$

and established the identity

$$z(T_k(r, \lambda) f(z))' = (p + \lambda)T_k(r, \lambda - 1) f(z) - \lambda T_k(r, \lambda) f(z).$$

When $p = 1$, this operator is a generalization of the linear operator defined in [23]. Recently Miller and Mocanu [22] considered certain second order differential superordinations. Using the results of Miller and Mocanu [22], Bulboaca [6] have considered certain classes of first order differential superordinations and Bulboaca [5] considered certain superordination-preserving integral operators. Later many papers in this direction emerged (cf. [2], [3], [12], [32], [33], [35]).

Jung, Kim and Srivastava introduced the linear operator on $\mathcal{A}$ is defined by

$$Q_\beta^\alpha(f) = \left( \frac{\alpha + \beta}{\beta} \right) \frac{\alpha}{\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt, \quad (\alpha > 0, \beta > -1, f \in \mathcal{A}).$$

Note that

$$Q_\beta^\alpha(f) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta + n)\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha + \beta + n)\Gamma(\beta + 1)} a_n z^n. \quad (1.15)$$

Motivated by the above linear operator introduced by Jung, Kim, Srivastava [15], Liu introduced the following integral operator on $\mathcal{A}_{p}$ [18]:

$$Q_\beta^\alpha_{p}(f) = z^p + \sum_{n=0}^{p} \frac{\Gamma(\beta + n + p)\Gamma(\alpha + \beta + p)}{\Gamma(\alpha + \beta + n + p)\Gamma(\beta + p)} a_n z^n \quad (\alpha > 0, \beta > -1, f \in \mathcal{A}_{p}).$$
Note that if
\[ F^\alpha_p(z) := z^n + \sum_{n=p+1}^{\infty} \frac{\Gamma(\beta + n + p)\Gamma(\alpha + \beta + p)}{\Gamma(\alpha + \beta + n + p)\Gamma(\beta + p)} z^n, \]
then
\[ Q^\alpha_{\beta,p}(f) = F^\alpha_p(z) * f(z). \]

Further it can be shown that
\[ z[Q^\alpha_{\beta,p}(f)]' = (\alpha + \beta + p - 1)Q^\alpha_{\beta,p}^{-1}(f) - (\alpha + \beta - 1)Q^\alpha_{\beta,p}(f). \] (1.16)

Since certain important properties of the classes defined by the above mentioned linear operators essentially depend on the recurrence relation (1.6), (1.8), (1.11), (1.14) and (1.16). We define a class of operators and a corresponding class of functions in the following:

**Definition 1.1.** Let \( O_p \) be the class of all linear operators \( L^a_p \) defined on \( \mathcal{A}_p \) satisfying
\[ z[L^a_p f(z)]' = \alpha_a L^{a+1}_p f(z) - (\alpha_a - p)L^a_p f(z). \]

One can also consider the class of linear operators satisfying
\[ z[L^b_p f(z)]' = \alpha_b L^{b+1}_p f(z) - (\alpha_b - p)L^b_p f(z). \]

However, in this paper, we restrict ourselves to the first case as the results pertaining to the second class of operators are much akin to their counter parts in the first case. We note that if \( L^k_p(f(z)) = \mathcal{L}_k(z) * f(z) \), then \( L^k_p \) unifies the above stated all operators for suitable function \( \mathcal{L}_k(z) \) assumes as follows.

\[ L^a_p = \begin{cases} \quad H^L_{am}[\alpha], \quad \text{for} \quad \mathcal{L}_a(z) = h_p(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m;z), \quad k = a = \alpha_1 \\ \quad I_p(r, \lambda), \quad \text{for} \quad \mathcal{L}_r(z) = z^n + \sum_{n=p+1}^{\infty} \left( \frac{n+p}{p+k} \right) r^n, \quad k = a = r \\ \quad J^L_{k,m}[\alpha], \quad \text{for} \quad \mathcal{L}_a(z) = f_k(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m;z), \quad k = b = \alpha_1 \\ \quad T_k(r, \lambda), \quad \text{for} \quad \mathcal{L}_r(z) = f^L_k(\alpha, z), \quad k = b = r \\ \quad Q^\alpha_{\beta,p}, \quad \text{for} \quad \mathcal{L}_a(z) = F^\alpha_{\beta,p}(z), \quad k = b = \alpha. \end{cases} \]

Thus the operators \( H^L_{am}[\alpha], I_p(r, \lambda), J^L_{k,m}[\alpha], T_k(r, \lambda) \) and \( Q^\alpha_{\beta,p} \) are in the class \( O_p \).

In the present investigation, we unify certain linear operators defined on \( p \)-valent functions and for them some key results with subordination and superordination leading to some sandwich results are obtained. A related integral transform is also discussed. Further sufficient conditions for functions belonging to the classes \( R, S^*, S'_*, SS^* \) and \( SL^* \) have been obtained using our key results. Hence most of the earlier results in this direction becomes special cases to our results, for instance, the results of Al-Kharsani and Al-Arefi [3] become special case to our main results when \( \mu = 1 \) and \( \nu = 0 \).

2. Preliminaries

In our present investigation, we need the following:

**Definition 2.1.** [22, Definition 2, p.817] Denote by \( \mathcal{D} \), the set of all functions \( f(z) \) that are analytic and injective on \( \mathbb{U} - E(f) \), where
\[ E(f) = \{ \zeta \in \partial \mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \}, \]
and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial \mathbb{U} - E(f) \).

**Lemma 2.1** (cf. Miller and Mocanu[21, Theorem 3.4h, p.132]). Let \( \psi(z) \) be univalent in the unit disk \( \mathbb{U} \) and let \( \varphi \) and \( \psi \) be analytic in a domain \( D \supset \psi(\mathbb{U}) \) with \( \varphi(w) \neq 0 \), when \( w \in \psi(\mathbb{U}) \). Set
\[ Q(z) := z\psi(z)\varphi(\psi(z)), \quad h(z) := \varphi(\psi(z)) + Q(z). \]

Suppose that
1. \( Q(z) \) is starlike univalent in \( \mathbb{U} \) and
2. \( \Re_{\frac{h(z)}{Q(z)}} > 0 \) for \( z \in \mathbb{U} \).
If \( q(z) \) is analytic in \( \mathbb{U} \), with \( q(0) = \psi(0) \), \( q(\mathbb{U}) \subset D \) and

\[
\vartheta(q(z)) + zq'(z)\varphi(q(z)) < \vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)),
\]
(2.1)

then \( q(z) \prec \psi(z) \) and \( \psi(z) \) is the best dominant.

**Lemma 2.2.** [4, Corollary 3.2, p.289] Let \( \psi(z) \) be univalent in the unit disk \( \mathbb{U} \) and \( \vartheta \) and \( \varphi \) be analytic in a domain \( D \) containing \( \psi(\mathbb{U}) \). Suppose that

1. \( \mathbb{R} \left[ \vartheta'(\psi(z)) / \varphi(\psi(z)) \right] > 0 \) for \( z \in \mathbb{U} \),
2. \( \Omega(z) := z\psi'(z)\varphi(\psi(z)) \) is starlike univalent in \( \mathbb{U} \).

If \( q(z) \in \mathbb{H}[\psi(0), 1] \cap \mathcal{D} \), with \( q(\mathbb{U}) \subseteq D \), and \( \vartheta(q(z)) + zq'(z)\varphi(q(z)) \) is univalent in \( \mathbb{U} \), then

\[
\vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)) < \vartheta(q(z)) + zq'(z)\varphi(q(z)),
\]
(2.2)

implies \( \psi(z) \prec q(z) \) and \( \psi(z) \) is the best subordinant.

**Definition 2.2.** Let \( f \in \mathcal{A}_p \), we define the function \( \Omega_{L, \mu, \nu}^p \) by

\[
\Omega_{L, \mu, \nu}^p(f(z)) = \left( \frac{I_p f(z)}{z^p} \right)^\mu \left( \frac{z^p}{I_p f(z)} \right)^\nu
\]

where the powers are principal one, \( \mu \) and \( \nu \) are real numbers such that they do not assume the value zero simultaneously. For the sake of convenience, let us denote

\[
\Omega_{L, \mu, \nu}^p(f(z), F(z)) := \frac{\Omega_{L, \mu, \nu}^p(f(z))}{\Omega_{L, \mu, \nu}^p(F(z))}.
\]

3. Sandwich Results

We begin with the following theorem.

**Theorem 3.1.** Let \( \psi \) be convex univalent in \( \mathbb{U} \) with \( \psi(0) = 1 \). Let \( \mathbb{R} \left[ \alpha_{n+1} \mu - \alpha_n \nu \right] \geq 0 \), \( \alpha_{n+1} \neq 0 \) and \( f \in \mathcal{A}_p \). Assume that \( \chi \) and \( \Phi \) are respectively defined by

\[
\chi(z) := \frac{1}{\alpha_{n+1}} \left[ (\alpha_{n+1} \mu - \alpha_n \nu) \psi(z) + z\psi'(z) \right]
\]
(3.1)

and

\[
\Phi(z) := \Omega_{L, \mu, \nu}^p(f(z)) \Upsilon_L(z),
\]
(3.2)

where

\[
\Upsilon_L(z) := \mu \Omega_{L, 1, 1}^{n+1}(f(z)) - \frac{\alpha_n \nu}{\alpha_{n+1}} \Omega_{L, 1, 1}^p(f(z)).
\]

1. If \( \Phi(z) \prec \chi(z) \), then

\[
\Omega_{L, \mu, \nu}^p(f(z)) \prec \psi(z)
\]

and \( \psi(z) \) is the best dominant.

2. If \( \chi(z) \prec \Phi(z) \),

\[
0 \neq \Omega_{L, \mu, \nu}^p(f(z)) \in \mathbb{H}[1, 1] \cap \mathcal{D} \quad \text{and} \quad \Phi(z) \text{ is univalent in } \mathbb{U},
\]
(3.3)

then

\[
\psi(z) \prec \Omega_{L, \mu, \nu}^p(f(z))
\]

and \( \psi(z) \) is the best subordinant.
Proof. Define the function \( q(z) \) by

\[
q(z) := \Omega_{L,\mu,v}^+(f(z)),
\]

(3.4)

where the branch of \( q(z) \) is so chosen such that \( q(0) = 1 \). Then \( q(z) \) is analytic in \( U \). By a simple computation, we find from (3.4) that

\[
\frac{z q'(z)}{q(z)} = \frac{z^{(n-1)}(f(z))'}{\Omega_{L,\mu,v}^+(f(z))} - \frac{\nu}{\nu+1} - \frac{\mu}{\mu+1} + p(v - \mu).
\]

(3.5)

By making use of the identity

\[
z(L_p^+ f(z))' = \alpha_\nu L_p^+ f(z) - (\alpha_\nu - p)L_p^+ f(z),
\]

(3.6)

in (3.5), we have

\[
\Omega_{L,\mu,v}^+(f(z)) \left( \mu \Omega_{L,\nu,1}^{(n)}(f(z)) - \frac{\alpha_\nu}{\alpha_{r+1}} \Omega_{L,\nu,1}^{(n)}(f(z)) \right) = \frac{1}{\alpha_{r+1}} [(\alpha_{r+1} - \alpha_\nu) q(z) + z q'(z)].
\]

(3.7)

In view of (3.7), the subordination \( \Phi(z) \prec \chi(z) \) becomes

\[
(\alpha_{r+1} - \alpha_\nu) q(z) + z q'(z) \prec (\alpha_{r+1} - \alpha_\nu) \psi(z) + z \psi'(z)
\]

and this can be written as (2.7). By defining

\[
\psi(w) := (\alpha_{r+1} - \alpha_\nu) w \quad \text{and} \quad \varphi(w) := 1.
\]

Note that \( \varphi(w) \neq 0 \) and \( \varphi(w), \varphi(w) \) are analytic in \( \mathbb{C} \setminus \{0\} \). Set

\[
Q(z) := z \psi'(z)
\]

\[
h(z) := \varphi(\psi(z)) + Q(z) = (\alpha_{r+1} - \alpha_\nu) \psi(z) + z \psi'(z).
\]

In light of the hypothesis of our Theorem 3.1, we see that \( Q(z) \) is starlike and

\[
\operatorname{Re} \left( \frac{h'(z)}{Q(z)} \right) = \operatorname{Re} \left( \frac{\alpha_{r+1} - \alpha_\nu + 1 + z \psi'(z)}{\psi'(z)} \right) > 0.
\]

By an application of Lemma 2.1 we obtain that \( q(z) \prec \psi(z) \) or

\[
\Omega_{L,\mu,v}^+(f(z)) \prec \psi(z).
\]

The second half of Theorem 3.1 follows by a similar application of Lemma 2.2.

Using Theorem 3.1 we obtain the following “sandwich result”.

Corollary 3.1. Let \( \psi_j (j = 1, 2) \) be convex univalent in \( U \) with \( \psi_j(0) = 1 \). Assume that \( \operatorname{Re} [\alpha_{r+1} - \alpha_\nu] \geq 0 \) and \( \Phi \) be as defined in (3.2). Further assume that

\[
\chi_j(z) := \frac{1}{\alpha_{r+1}} \left[ (\alpha_{r+1} - \alpha_\nu) \psi_j(z) + z \psi_j'(z) \right].
\]

If (3.3) holds and \( \chi_1(z) \prec \Phi(z) \prec \chi_2(z) \), then

\[
\psi_1(z) \prec \Omega_{L,\mu,v}^+(f(z)) \prec \psi_2(z).
\]

Theorem 3.2. Let \( \psi \) be convex univalent in \( U \) with \( \psi(0) = 1 \) and \( \alpha_\nu \) be a complex number. Assume that \( \operatorname{Re}(\mu \alpha_{r+1} - v \alpha_\nu) \geq 0 \) and \( f \in \mathcal{S}_p \). Define the functions \( F, \chi \) and \( \Psi \) respectively by

\[
F(z) := \frac{\alpha_\nu}{(\alpha_{r+1} - v \alpha_\nu)} \int_0^z t^{\alpha_{r+1} - v} f(t) dt,
\]

(3.8)

\[
\chi(z) := (\mu \alpha_{r+1} - v \alpha_\nu) \psi(z) + z \psi'(z)
\]

(3.9)

and

\[
\Psi(z) := \Omega_{L,\mu,v}^+(F(z)) \left( \mu \alpha_{r+1} \Omega_{L,\nu,1}^+(f(z), F(z)) - v \alpha_\nu \Omega_{L,\nu,-1}^+(f(z), F(z)) \right).
\]

(3.10)
1. If \( \Psi(z) \prec \chi(z) \), then
\[
\Omega^2_{L,\mu,\nu}(F(z)) < \psi(z)
\]
and \( \psi(z) \) is the best dominant.

2. If \( \chi(z) < \Psi(z) \),
\[
0 \neq \Omega^2_{L,\mu,\nu}(F(z)) \in \mathcal{H}[1,1] \cap \mathcal{D} \text{ and } \Psi(z) \text{ is univalent in } \mathbb{U}, \quad (3.11)
\]
then
\[
\psi(z) < \Omega^2_{L,\mu,\nu}(F(z))
\]
and \( \psi(z) \) is the best subordinant.

**Proof.** From the definition of \( F \), we obtain that
\[
\alpha_d L_d^+(f(z)) = (\alpha_d - p)L_d^+(F(z)) + z(L_d^+(F(z)))'. \quad (3.12)
\]
Define the function \( q \) by
\[
q(z) := \Omega^2_{L,\mu,\nu}(F(z)), \quad (3.13)
\]
where the branch of \( q(z) \) is so chosen that \( q(0) = 1 \). Clearly \( q(z) \) is analytic in \( \mathbb{U} \). Using \((3.12)\) and \((3.13)\), we have
\[
\Omega^2_{L,\mu,\nu}(F(z)) (\mu \alpha_{q+1} + \Omega^2_{L,1,0}(f(z), F(z)) - v \alpha_d \Omega^2_{L,0,-1}(f(z), F(z))) = (\mu \alpha_{q+1} - v \alpha_d)q(z) + zq'(z). \quad (3.14)
\]
Using \((3.14)\), the subordination \( \Psi(z) \prec \chi(z) \) becomes
\[
(\mu \alpha_{q+1} - v \alpha_d)q(z) + zq'(z) \prec (\mu \alpha_{q+1} - v \alpha_d)\psi(z) + z\psi'(z)
\]
and this can be written as \((3.15)\), by defining
\[
\vartheta(w) := (\mu \alpha_{q+1} - v \alpha_d)\psi(z) \text{ and } \varphi(w) := 1.
\]
Note that \( \varphi(w) \neq 0 \) and \( \vartheta(w) \) are analytic in \( \mathbb{C} \setminus \{0\} \). Set
\[
Q(z) := z\psi'(z)
\]
\[
h(z) := \vartheta(\psi(z)) + Q(z) = (\mu \alpha_{q+1} - v \alpha_d)\psi(z) + z\psi'(z). \quad (3.16)
\]
In light of the assumption of our Theorem \(3.2\), we see that \( Q(z) \) is starlike and
\[
\text{Re} \left( \frac{z\psi'(z)}{Q(z)} \right) = \text{Re} \left( \frac{\mu \alpha_{q+1} - v \alpha_d + z\psi'(z)}{\psi'(z)} \right) > 0.
\]
An application of Lemma \(2.1\) gives \( q(z) \prec \psi(z) \) or
\[
\Omega^2_{L,\mu,\nu}(F(z)) < \psi(z).
\]
By an application of Lemma \(2.2\), the proof of the second half of Theorem \(3.2\) follows at once.

As a consequence of Theorem \(3.3\), we obtain the following “sandwich result”.

**Corollary 3.2.** Let \( \psi_j \) \((j = 1, 2)\) be convex univalent in \( \mathbb{U} \) with \( \psi_j(0) = 1 \) and \( \alpha_d \) be a complex number. Further assume that \( \text{Re}(\mu \alpha_{q+1} - v \alpha_d) \geq 0 \) and \( \Psi \) be as defined in \((3.17)\). If \((3.17)\) holds and \( \chi_1(z) \prec \Psi(z) \prec \chi_2(z), \) then
\[
\psi_j(z) \prec \Omega^2_{L,\mu,\nu}(F(z)) \prec \psi_2(z),
\]
where
\[
\chi_j(z) := (\mu \alpha_{q+1} - v \alpha_d)\psi_j(z) + z\psi_j'(z) \quad (j = 1, 2)
\]
and \( F \) is defined by \((3.8)\).

**Theorem 3.3.** Let \( \phi \) be analytic in \( \mathbb{U} \) with \( \phi(0) = 1 \) and \( \alpha_d \) is independent of \( a \). If \( f \in \mathcal{A}_p \), then
\[
\Omega^{2+1}_{L,\mu,\nu}(f(z)) < \phi(z) \iff \Omega^2_{L,\mu,\nu}(F(z)) \prec \phi(z).
\]
Further
\[
\phi(z) \prec \Omega^2_{L,\mu,\nu}(f(z)) \iff \phi(z) \prec \Omega^{2+1}_{L,\mu,\nu}(F(z)),
\]
where \( F \) is defined by \((3.8)\).
Proof. From the definition of \( F \), we obtain
\[
\alpha_a f(z) = (\alpha_a - p)F(z) + zF'(z).
\] (3.15)
By convoluting (3.13) with \( \mathcal{L}_k(z) \) and using the fact that \( z(f \ast g)'(z) = f(z) \ast zg'(z) \), we obtain
\[
\alpha_a L_p^\alpha(f(z)) = (\alpha_a - p)L_p^\alpha(F(z)) + z(L_p^\alpha(F(z)))'
\] and by using the identity
\[
z[L_p^\alpha(f(z))]' = \alpha_a L_p^{\alpha+1}(f(z)) - (\alpha_a - p)L_p^\alpha(f(z)),
\] (3.16)
we get
\[
L_p^\alpha(f(z)) = L_p^{\alpha+1}(F(z)).
\] (3.17)
Since \( \alpha_a \) is independent of \( a \), \( \alpha_{a+1} = \alpha_a \), we have
\[
\alpha_a L_p^{\alpha+1}(f(z)) = z(L_p^\alpha(f(z)))' + (\alpha_a - p)L_p^\alpha(f(z)) = z(L_p^{\alpha+1}(F(z)))' + (\alpha_a - p)L_p^{\alpha+1}(F(z)) = \alpha_{a+1} L_p^{\alpha+2}(F(z)).
\] (3.18)
Therefore, from (3.17) and (3.18), we have
\[
\Omega_{L,\mu,v}^{\alpha+1}(F(z)) = \Omega_{L,\mu,v}^\alpha(f(z))
\]
and hence the result follows at once.

Now we will use Theorem 3.3 to state the following “sandwich result”.

Corollary 3.3. Let \( f \in \mathcal{A}_p \) and \( \alpha_a \) is independent of \( a \). Let \( \phi_i \ (i = 1, 2) \) be analytic in \( \mathbb{U} \) with \( \phi_i(0) = 1 \) and \( F \) is defined by (3.3). Then
\[
\phi_1(z) \prec \Omega_{L,\mu,v}^\alpha(f(z)) \prec \phi_2(z)
\]
if and only if
\[
\phi_1(z) \prec \Omega_{L,\mu,v}^{\alpha+1}(F(z)) \prec \phi_2(z).
\]

4. Applications

We begin with some interesting applications of subordination part of Theorem 3.1 for the case when \( L = H \), the Dziok-Srivastava Operator. Note that the subordination part of Theorem 3.1 holds even if we assume
\[
\text{Re} \left\{ 1 + \frac{z\psi'(z)}{\psi'(z)} \right\} > \max\{0, \text{Re}[\alpha_1(v - \mu) - \mu]\}
\]
instead of “\( \psi(z) \) is convex and \( \text{Re} [\alpha_1(\mu - v) + \mu] \geq 0 \)” and leads to the following corollary to the first part of Theorem 3.1 by taking \( \psi(z) = (1 + Az)/(1 + Bz) \).

Corollary 4.1. Let \( -1 < B < A \leq 1 \) and \( \text{Re}(u - vB) \geq |v - \bar{u}B| \) where \( u = \alpha_1(\mu - v) + \mu + 1 \) and \( v = [\alpha_1(\mu - v) + \mu + 1]B \). If \( f \in \mathcal{A}_p \) satisfies the subordination
\[
\Omega_{H,\mu,v}^{\alpha_1}(f(z)) \left( \mu \Omega_{H,1,1}^{\alpha_1+1}(f(z)) - \frac{\alpha_1 v}{\alpha_1 + 1} \Omega_{H,1,1}^{\alpha_1}(f(z)) \right) = \frac{1}{\alpha_1 + 1} \left( [\alpha_1(\mu - v) + \mu] \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Bz)^2} \right) \ (\alpha_1 \neq -1),
\]
then
\[
\Omega_{H,\mu,v}^{\alpha_1}(f(z)) = \frac{1 + Az}{1 + Bz}
\]
and \( (1 + Az)/(1 + Bz) \) is the best dominant.
Proof. Let
\[
\psi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 < B < A \leq 1),
\] (4.1)
then clearly \(\psi(z)\) is univalent and \(\psi(0) = 1\). Upon logarithmic differentiation of \(\psi\) given by (4.1), we obtain that
\[
z\psi'(z) = \frac{(A - B)z}{(1 + Bz)^2}.
\] (4.2)

Another differentiation of (4.2), yields
\[
1 + z\psi''(z) = \frac{1 - Bz}{1 + Bz}.
\] (4.3)

If \(z = re^{i\theta}, 0 \leq r < 1\), then we have
\[
\Re \left(1 + z\psi''(z)\right) = \frac{1 - B^2r^2}{1 + B^2r^2 + 2Br\cos\theta} \geq 0.
\]

Hence \(\psi(z)\) is convex in \(U\). Also it follows that
\[
[\alpha_1(\mu - \nu) + \mu] + 1 + z\psi''(z) = \frac{[\alpha_1(\mu - \nu) + \mu + 1] + [\alpha_1(\mu - \nu) + \mu - 1]Bz}{1 + Bz}
\]
\[
= \frac{u + vz}{1 + Bz},
\]
where \(u = \alpha_1(\mu - \nu) + \mu + 1\) and \(v = [\alpha_1(\mu - \nu) + \mu - 1]B\). The function \(w(z) = \frac{u + vz}{1 + Bz}\) maps \(U\) into the disk
\[
\left|w - \frac{\bar{u} - \bar{v}B}{1 - B^2}\right| \leq \frac{|v - \bar{u}B|}{1 - B^2}.
\]

Which implies that
\[
\Re \left([\alpha_1(\mu - \nu) + \mu] + 1 + z\psi''(z)\right) \geq \frac{\Re(\bar{u} - \bar{v}B) - |v - \bar{u}B|}{1 - B^2} \geq 0
\]
provided
\[
\Re(u - vB) \geq |v - \bar{u}B|
\]
or
\[
\Re(u - vB) < |v - \bar{u}B|.
\]
Thus the result follows at once by an application of the first part of Theorem \[5.1\]

**Corollary 4.2.** Let \(0 \leq \alpha < 1\) and \(\Re(\alpha_1(\mu - \nu) + \mu) \geq 0\). If
\[
\Omega_{\alpha_1,\mu,\nu}^0(f(z)) = \mu \Omega_{\alpha_1,1,1}^0(f(z)) - \frac{\alpha_1 v}{\alpha_1 + 1} \Omega_{\alpha_1,1,1}^0(f(z))
\]
\[
\leq \frac{1}{(1 - \mu) + 1} \left( \alpha_1(\mu - \nu) + \mu \frac{1 + (1 - 2\alpha)v}{1 - z} + \frac{2(1 - \alpha)}{(1 - z)^2} \right) \quad (\alpha_1 \neq -1),
\]
then
\[
\Re \Omega_{\alpha_1,\mu,\nu}^0(f(z)) > \alpha.
\]

**Proof.** Let
\[
\psi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1),
\] (4.4)
then obviously \(\psi(z)\) is univalent and \(\psi(0) = 1\). By a simple calculation, we have
\[
1 + z\psi''(z) = \frac{1 + z}{1 - z},
\] (4.5)
which clearly indicates that \(\psi(z)\) is convex. If we assume \(\beta = \alpha_1(\mu - \nu) + \mu\) then by hypothesis we have \(\Re \beta \geq 0\). So if we take
\[
w(z) = \beta + \frac{1 + z}{1 - z} = \frac{(1 + \beta) + (1 - \beta)z}{1 - z},
\]
then \(w(z)\) maps the unit disc \(U\) on to \(\Re w > \Re \beta \geq 0\). The result now follows by an application of the subordination part of Theorem \[5.1\]
Note that if \( p = 1, l = m + 1 \text{ and } a_i+1 = \beta_i \ (i = 1, 2, \ldots, m) \), then \( H_1[1]f(z) = f(z), H_1[2]f(z) = zf''(z) \) and \( H_1[3]f(z) = \frac{1}{2}z f''(z) + zf'(z) \). Putting \( \alpha = 1 \) in Corollaries 4.1 and 4.2 we obtain the following corollaries respectively.

**Corollary 4.3.** Let \(-1 < B < A \leq 1\). Let \( \mu \) and \( v \) satisfy \((u-vB) \geq |v-uB|\) where \( u = 2\mu - v + 1 \) and \( v = (2\mu - v - 1)B \). If \( f \in \mathcal{S} \) and satisfies the subordination

\[
(f'(z))^\mu \left( \frac{z'}{f(z)} \right)^v \left( \mu \left( 2 + \frac{zf''(z)}{f'(z)} \right) - v \frac{zf'(z)}{f(z)} \right) < (2\mu - v) \frac{1+Az}{1+Bz} \frac{(A-B)z}{(1+Bz)^2},
\]

then

\[
(f'(z))^\mu \left( \frac{z}{f(z)} \right)^v < \frac{1+Az}{1+Bz}
\]

and \((1+Az)/(1+Bz)\) is the best dominant.

**Corollary 4.4.** Let \( 0 \leq \alpha < 1 \text{ and } 2\mu \geq v \). If \( f \in \mathcal{S} \) and satisfies

\[
\Re \left( (f'(z))^\mu \left( \frac{z}{f(z)} \right)^v \left( \mu \left( 2 + \frac{zf''(z)}{f'(z)} \right) - v \frac{zf'(z)}{f(z)} \right) \right) > \frac{2(2\mu - v)\alpha - (1-\alpha)}{2},
\]

then

\[
\Re \left( (f'(z))^\mu \left( \frac{z}{f(z)} \right)^v \right) > \alpha.
\]

**Proof.** From Corollary 4.2 we see that

\[
(f'(z))^\mu \left( \frac{z}{f(z)} \right)^v \left( \mu \left( 2 + \frac{zf''(z)}{f'(z)} \right) - v \frac{zf'(z)}{f(z)} \right) < (2\mu - v) \frac{1+(1-2\alpha)z}{1-z} + \frac{2(1-\alpha)z}{(1-z)^2} =: h(z).
\]

We now investigate the image of \( h(U) \). Assuming \( a = 1-2\alpha \) and \( b = 2\mu - v \), we have

\[
h(z) = \frac{b + (1 + a + b + ab)z - abz^2}{(1-z)^2},
\]

where \( h(0) = b \) and \( h(-1) = [2b(1-a) - (1+a)]/4 \). The boundary curve of the image of \( h(U) \) is given by \( h(e^{i\theta}) = u(\theta) + iv(\theta), -\pi < \theta < \pi \), where

\[
u(\theta) = \frac{(1+a)bsin\theta}{2(1-cos\theta)}.
\]

By eliminating \( \theta \), we obtain the equation of the boundary curve as

\[
v^2 = -b^2(1+a) \left( u - \frac{2b(1-a) - (a+1)}{4} \right).
\]

Obviously \((4.6)\) represents a parabola opening towards the left, with the vertex at the point \( \left( \frac{2b(1-a) - (a+1)}{4}, 0 \right) \) and negative real axis as its axis. Hence \( h(U) \) is the exterior of the parabola \((4.6)\) which includes the right half plane

\[
u > \frac{2b(1-a) - (a+1)}{4}.
\]

Hence the result follows at once.

By setting \( \mu = v = 1 \) in Corollary 4.3 we obtain the following example.

**Example 4.1.** Let \(-1 < B < A \leq 1\). If \( f \in \mathcal{S} \) and satisfies the subordination

\[
zf'(z) \left( 2 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right) < \frac{1+Az}{1+Bz} \frac{(A-B)z}{(1+Bz)^2},
\]

then \( f \in S'[A,B] \).

Putting \( \mu = v = 1 \) in Corollary 4.4 we have the following example.

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Example 4.2. Let $0 \leq \alpha < 1$ if $f \in \mathcal{A}$ and satisfies

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \left( 2 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right) \right) > \frac{3\alpha - 1}{2},
\]

then $f \in S'(\alpha)$.

If we take $\mu = 1$ and $\nu = 0$ in the Corollary 4.4, we have the following result.

Example 4.3. Let $0 \leq \alpha < 1$. If $f \in \mathcal{A}$ and satisfies

\[
\text{Re}(2f'(z) + zf''(z)) > \frac{5\alpha - 1}{2},
\]

then $\text{Re} f'(z) > \alpha$.

Remark 4.1. Example 4.3 provides sufficient condition for univalence of $f(z)$ by Noshiro-Warschawski Theorem [11, p.47].

Setting $\mu = 0$ and $\nu = -1$ in Corollary 4.4 we obtain the following result.

Example 4.4. Let $0 \leq \alpha < 1$. If $f \in \mathcal{A}$ and $\text{Re} f'(z) > \frac{3\alpha - 1}{2}$, then $f(z) \in R(\alpha)$.

Remark 4.2. When $\alpha = 1/3$, the above result reduces to [25, Theorem 2].

If we take $\psi(z) = ((1+z)/(1-z))^\eta$ with $0 < \eta \leq 1$ in Theorem 5.1 for the case $L = H$, the Dziok Srivastava operator, then clearly $\psi(z)$ is convex in $U$ and consequently corresponding to the subordination part of the Theorem 5.1 we have the following result.

Corollary 4.5. Let $0 < \eta \leq 1$, $\alpha_i \neq -1$ and $\text{Re}(\alpha_i(\mu - \nu) + \mu) \geq 0$. If $f \in \mathcal{A}_p$ and satisfies

\[
\Omega_{H,\mu,\nu}^{\alpha_i}(f(z)) \left( \mu \Omega_{H,1,1}^{\alpha_i+1}(f(z)) - \frac{\alpha_i\nu}{\alpha_i + 1} \Omega_{H,1,1}^{\alpha_i}(f(z)) \right),
\]

then

\[
\Omega_{H,\mu,\nu}^{\alpha_i}(f(z)) < \left( \frac{1+z}{1-z} \right)^\eta,
\]

and $((1+z)/(1-z))^\eta$ is the best dominant.

By taking $p = 1, l = m + 1, \alpha_1 = 1$ and $\alpha_{i+1} = \beta_i (i = 1, 2, \ldots, m)$, in the above Corollary 4.5 we have the following result.

Corollary 4.6. Let $0 < \eta \leq 1$ and $2\mu \geq \nu$. If $f \in \mathcal{A}$ and satisfies

\[
\left| \arg \left( (f'(z))^{\mu} \left( \frac{z}{f(z)} \right)^{\nu} \left( 2 + \frac{zf''(z)}{f'(z)} - \nu \frac{zf'(z)}{f(z)} \right) \right) \right| < \frac{\delta \pi}{2},
\]

then

\[
\left| \arg \left( (f'(z))^{\mu} \left( \frac{z}{f(z)} \right)^{\nu} \right) \right| < \frac{\eta \pi}{2},
\]

where

\[
\delta = \eta + 1 - \frac{2}{\pi} \arctan \frac{2\mu - \nu}{\eta}.
\]

Proof. In the view of the Corollary 4.5 we have

\[
(f'(z))^{\mu} \left( \frac{z}{f(z)} \right)^{\nu} \left( 2 + \frac{zf''(z)}{f'(z)} - \nu \frac{zf'(z)}{f(z)} \right) < \left( 2\mu - \nu \right) \left( \frac{1+z}{1-z} \right)^\eta = h(z)
\]

implies

\[
(f'(z))^{\mu} \left( \frac{z}{f(z)} \right)^{\nu} < \left( \frac{1+z}{1-z} \right)^\eta
\]

or

\[
\left| \arg \left( (f'(z))^{\mu} \left( \frac{z}{f(z)} \right)^{\nu} \right) \right| < \frac{\eta \pi}{2} \quad (z \in U).
\]
Now we need to find the minimum value of \( \arg h(U) \). Let \( z = e^{i\theta} \). Since \( h(U) \) is symmetrical about the real axis, we shall restrict ourself to \( 0 < \theta \leq \pi \). Setting \( t = \cot \theta / 2 \), we have \( t \geq 0 \) and for \( z = \frac{\tan \theta}{\pi + 1} \), we arrive at

\[
h(e^{i\theta}) = (it)^{\eta - 1} \left[ (2\mu - v)it - \frac{\eta(1 + t^{2})}{2} \right],
\]

where

\[
G(t) = \left[ (2\mu - v)it - \frac{\eta(1 + t^{2})}{2} \right].
\]

Let \( G(t) = U(t) + iV(t) \), where \( U(t) = -\frac{\eta(1 + t^{2})}{2} \) and \( V(t) = (2\mu - v)t \), there arises two cases namely \( 2\mu > v \) and \( 2\mu = v \). If \( 2\mu > v \), then a calculation shows that \( \min_{t \geq 0} \arg G(t) \) occurs at \( t = 1 \) and

\[
\min_{t \geq 0} \arg G(t) = \pi - \arctan \frac{2\mu - v}{\eta}.
\]

Thus

\[
\min_{|z| < 1} \arg h(z) = \frac{(\eta + 1)\pi}{2} - \arctan \frac{2\mu - v}{\eta}.
\]

If \( 2\mu = v \), then \( \arg G(t) = \pi \) and \( \min_{|z| < 1} \arg h(z) = (\eta + 1)\pi/2 \). Thus for \( 2\mu \geq v \), we have

\[
\min_{|z| < 1} \arg h(z) = \min \left\{ \frac{(\eta + 1)\pi}{2}, \frac{(\eta + 1)\pi}{2} - \arctan \frac{2\mu - v}{\eta} \right\}
\]

\[
= \frac{(\eta + 1)\pi}{2} - \arctan \frac{2\mu - v}{\eta}.
\]

This completes the proof of the corollary.

By taking \( \mu = v = 1 \) in the above Corollary 4.6, we obtain the following example.

**Example 4.5.** Let \( 0 < \eta \leq 1 \). If \( f \in \mathcal{A} \) and satisfies

\[
\left| \arg \left\{ \frac{zf''(z)}{f''(z)} \left( 2 - \frac{zf''(z)}{f''(z)} + \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \frac{(\eta + 1)\pi}{2} - \arctan \frac{1}{\eta},
\]

then \( f \in S S^* \). Let

\[
\|z'\| < \frac{\eta\pi}{2}.
\]

By setting \( \mu = 1 \) and \( v \) in Corollary 4.6, we have the following example.

**Example 4.6.** Let \( 0 < \eta \leq 1 \). If \( f \in \mathcal{A} \) and satisfies

\[
\left| \arg \left\{ f'(z) \left( 2 + \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \frac{(\eta + 1)\pi}{2} - \arctan \frac{2}{\eta},
\]

then \( \|z'\| < \frac{\eta\pi}{2} \). By taking \( \mu = 0 \) and \( v = -1 \) in Corollary 4.6, we get the following example.

**Example 4.7.** Let \( 0 < \eta \leq 1 \). If \( f \in \mathcal{A} \) and satisfies

\[
\left| \arg f'(z) \right| < \frac{(\eta + 1)\pi}{2} - \arctan \frac{1}{\eta},
\]

then \( \|z'\| < \frac{\eta\pi}{2} \).

We now enlist a few applications of Theorem 3.1 for the operator \( L = H \), the Dziok Srivastava operator, by taking \( \psi(z) = \sqrt{1 + z} \) as dominant. Obviously \( \psi(z) \) is a convex function in the open unit disk \( U \) with \( \psi(0) = 1 \). The subordination part of Theorem 3.1 leads to the following result.

**Corollary 4.7.** Let \( \alpha \not= -1 \) and \( \text{Re} \left[ \alpha \left( \mu - v \right) + \mu \right] \geq 0 \). If \( f \in \mathcal{A}_\rho \) and satisfies the subordination

\[
\Omega_{n,\mu,v}^{\alpha_1} \left( \mu \Omega_{n,1,1}^{\alpha_1 + 1} (f(z)) - \frac{\alpha_1 v}{\alpha_1 + 1} \Omega_{n,1,1}^{\alpha_1} (f(z)) \right)
\]

\[
< \frac{1}{\alpha_1 + 1} \left[ \left( \alpha_1 \left( \mu - v \right) + \mu \right) \sqrt{1 + z} + \frac{z}{2\sqrt{1 + z}} \right],
\]

then
then
\[ \Omega_{h,\mu,v}(f(z)) < \sqrt{1+z} \]
and \( \sqrt{1+z} \) is the best dominant.

By taking \( p = 1, l = m + 1, \alpha_1 = 1 \) and \( \alpha_{l+1} = \beta_i \) (i = 1, 2, ..., m) in Corollary 4.7 we obtain the following result.

**Corollary 4.8.** Let \( 2\mu \geq v \). If \( f \in \mathcal{A} \) and satisfies the subordination
\[ (f'(z))^\mu \left( \frac{z}{f(z)} \right)^v \left( \mu \left( 2 + \frac{zf''(z)}{f'(z)} \right) - v\frac{zf'(z)}{f(z)} \right) < (2\mu - v)\sqrt{1+z} + \frac{z}{2\sqrt{1+z}}, \]
then
\[ (f'(z))^\mu \left( \frac{z}{f(z)} \right)^v < \sqrt{1+z} \]
and \( \sqrt{1+z} \) is the best dominant.

We obtain the following example from Corollary 4.8.

**Example 4.8.** If \( f \in \mathcal{A} \) and satisfies
\[ \left| \frac{zf'(z)}{f(z)} \left( 2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right| < \sqrt{1.22} \approx 1.10, \]
then \( f \in \mathcal{S} \mathcal{L} \).

**Proof.** Putting \( \mu = v = 1 \) in Corollary 4.8 we have
\[ \frac{zf'(z)}{f(z)} \left( 2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < \sqrt{1+z} + \frac{z}{2\sqrt{1+z}} =: h(z), \]
implies
\[ \frac{zf'(z)}{f(z)} < \sqrt{1+z}. \]
The dominant \( h(z) \) can be written as
\[ h(z) = \frac{3z + 2}{2\sqrt{1+z}}. \]
Writing \( h(e^{i\theta}) = u(e^{i\theta}) + iv(e^{i\theta}), -\pi < \theta < \pi \), we have
\[ u(\theta) = \frac{3\cos(3\theta/4) + 2\cos(\theta/4)}{2\sqrt{2}\cos(\theta/2)} \]
and
\[ v(\theta) = \frac{3\sin(3\theta/4) - 2\sin(\theta/4)}{2\sqrt{2}\cos(\theta/2)}. \]
A simple calculation gives
\[ u^2(\theta) + v^2(\theta) = \frac{13 + 12\cos(\theta)}{8\cos(\theta/2)} =: k(\theta). \]
A computation shows that \( k(\theta) \) has minimum at \( \theta = \arccos(\sqrt{1/24}) \) and \( k(\theta) \geq \frac{3}{\sqrt{2}} \approx 1.22 \). Since \( h(0) = 1 \) and \( h(-1) = -\infty \), by a computation we come to know that the image of \( h(\mathbb{U}) \) is the interior of the domain bounded by parabola opening towards left which contains the interior of the circle \( u^2 + v^2 = 1.22 \). Hence the result follows at once.

We now give some interesting applications of Theorem 3.2 for the case \( L = H \). Note that if we replace the statement "\( \psi(z) \) is convex in the open unit disc \( \mathbb{U} \) and \( \Re((\mu - v)\alpha_1 + \mu) \geq 0 \)" by
\[ \Re \left( 1 + \frac{z\psi''(z)}{\psi'(z)} \right) > \max \{ 0, \Re((v - \mu)\alpha_1 - \mu) \} \]
in the hypothesis of Theorem 3.2 still the subordination part of the result holds so we obtain the following corollary as a straight forward consequence to the first part of Theorem 3.2 by taking \( \psi(z) = (1 + Az)/(1 + Bz) \).
Corollary 4.9. Let \(-1 < B < A \leq 1\) and \(\mathrm{Re}(u - vB) \geq |v - \bar{u}B|\) where \(u = (\mu - v)\alpha_i + \mu + 1\) and \(v = [(\mu - v)\alpha_i + \mu - 1]B\). If \(f \in \mathcal{A}_p\), \(F\) as defined in (3.8) and

\[
\Omega^{\alpha_i}_{H, \mu, v}(F(z))(\mu(\alpha_i + 1)\Omega^{\alpha_i}_{H, 1, 0}(f(z), F(z)) - \nu \alpha_i \Omega^{\alpha_i}_{H, \mu, -1}(f(z), F(z)))< ((\mu - v)\alpha_i + \mu) \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Bz)^2},
\]

then

\[
\Omega^{\alpha_i}_{H, \mu, v}(F(z)) \prec \frac{1 + Az}{1 + Bz}
\]

and \((1 + Az)/(1 + Bz)\) is the best dominant.

Corollary 4.10. Let \(0 \leq \alpha < 1\) and \(\mathrm{Re}[(\mu - v)\alpha_i + \mu] \geq 0\). If \(f \in \mathcal{A}_p\), \(F\) as defined in (3.8) and

\[
\Omega^{\alpha_i}_{H, \mu, v}(F(z))(\mu(\alpha_i + 1)\Omega^{\alpha_i}_{H, 1, 0}(f(z), F(z)) - \nu \alpha_i \Omega^{\alpha_i}_{H, \mu, -1}(f(z), F(z)))< ((\mu - v)\alpha_i + \mu) \frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{2(1 - \alpha)z}{(1 - z)^2},
\]

then

\[
\Omega^{\alpha_i}_{H, \mu, v}(F(z)) \prec \frac{1 + (1 - 2\alpha)z}{1 - z}
\]

and \((1 + (1 - 2\alpha)z)/(1 - z)\) is the best dominant.

Putting \(p = 1, l = m + 1, \alpha_i = 1\) and \(\alpha_{i+1} = \beta_i (i = 1, 2, \ldots m)\) in Corollaries 4.9 and 4.10 we obtain the following results respectively.

Corollary 4.11. Let \(-1 < B < A \leq 1\) and \(\mathrm{Re}(u - vB) \geq |v - \bar{u}B|\) where \(u = 2\mu - v + 1, v = [2\mu - v - 1]B\). If \(f \in \mathcal{A}_p\), \(F\) as defined in (3.8) and

\[
(F'(z))^\mu \left(\frac{z}{F(z)}\right)^v \left(2\mu \frac{f'(z)}{F'(z)} - \nu \frac{f(z)}{F(z)}\right) < (2\mu - v) \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Bz)^2},
\]

then

\[
(F'(z))^\mu \left(\frac{z}{F(z)}\right)^v \prec \frac{1 + Az}{1 + Bz}
\]

and \((1 + Az)/(1 + Bz)\) is the best dominant.

Corollary 4.12. Let \(0 \leq \alpha < 1\) and \(2\mu \geq v\). If \(f \in \mathcal{A}_p\), \(F\) as defined in (3.8) and

\[
\mathrm{Re}\left\{ (F'(z))^\mu \left(\frac{z}{F(z)}\right)^v \left(2\mu \frac{f'(z)}{F'(z)} - \nu \frac{f(z)}{F(z)}\right) \right\} < \frac{2(2\mu - v)\alpha - (1 - \alpha)}{2},
\]

then

\[
\mathrm{Re}\left[ (F'(z))^\mu \left(\frac{z}{F(z)}\right)^v \right] > \alpha.
\]

Proof. From Corollary 4.10 we see that

\[
(F'(z))^\mu \left(\frac{z}{F(z)}\right)^v \left(2\mu \frac{f'(z)}{F'(z)} - \nu \frac{f(z)}{F(z)}\right) \prec (2\mu - v) \frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{2(1 - \alpha)z}{(1 - z)^2} \equiv h(z)
\]

implies

\[
\mathrm{Re}\left[ (F'(z))^\mu \left(\frac{z}{F(z)}\right)^v \right] > \alpha.
\]

Let \(z = e^{i\theta}, -\pi \leq \theta \leq \pi\). Then

\[
\mathrm{Re}(h(e^{i\theta})) = \mathrm{Re}\left\{ (2\mu - v) \frac{1 + (1 - 2\alpha)e^{i\theta}}{1 - e^{i\theta}} + \frac{2(1 - \alpha)e^{i\theta}}{(1 - e^{i\theta})^2} \right\} = (2\mu - v)\alpha - \frac{(1 - \alpha)}{2} \left( \frac{1}{\sin^2(\theta/2)} \right) =: k(\theta).
\]

A calculation shows that \(k(\theta)\) attains its maximum at \(\theta = \pi\) and

\[
\max_{|\theta| \leq \pi} k(\theta) = \frac{2(2\mu - v)\alpha - (1 - \alpha)}{2}.
\]

Hence the result follows at once.
Taking $\mu = v = 1$ in the Corollary 4.11 we have the following example.

**Example 4.9.** Let $-1 < B < A \leq 1$. If $f \in \mathcal{A}$, $F$ as defined in (3.8) and

$$\frac{zF'(z)}{F(z)} \left( 2 \frac{f'(z)}{F'(z)} - \frac{f(z)}{F(z)} \right) < \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Bz)^2},$$

then $F \in S^*[A, B]$.

Putting $\mu = v = 1$ in the Corollary 4.12 we obtain the following example.

**Example 4.10.** Let $0 \leq \alpha < 1$. If $f \in \mathcal{A}$, $F$ as defined in (3.3) and satisfies

$$\text{Re} \left\{ \frac{zF'(z)}{F(z)} \left( 2 \frac{f'(z)}{F'(z)} - \frac{f(z)}{F(z)} \right) \right\} < \frac{(3\alpha - 1)}{2},$$

then $F \in S^*(\alpha)$.

Putting $\mu = v = -1$ and assuming $f \in S^*$ in Corollary 4.12 we get the following example.

**Example 4.11.** Let $0 \leq \alpha < 1$. If $f \in \mathcal{S}$, $F$ as defined in (3.3) and

$$\text{Re} \left\{ \frac{F(z)}{z^2F'(z)} \left( \frac{f(z)}{F(z)} - 2 \frac{f'(z)}{F'(z)} \right) \right\} < - \frac{(\alpha + 1)}{2},$$

then $F \in S^*_p(\alpha)$.

Putting $\mu = 1$ and $v = 0$ in Corollary 4.12 we obtain the following example.

**Example 4.12.** Let $0 \leq \alpha < 1$. If $f \in \mathcal{S}$, $F$ as defined in (3.3) and

$$\text{Re} f'(z) < \frac{5\alpha - 1}{4},$$

then $\text{Re} F'(z) > \alpha$.

Putting $\mu = 0$ and $v = -1$ in Corollary 4.12 we have the following example.

**Example 4.13.** Let $0 \leq \alpha < 1$. If $f \in \mathcal{S}$, $F$ as defined in (3.3) and

$$\text{Re} \left\{ \frac{f(z)}{z} - \frac{3\alpha - 1}{2} \right\},$$

then $F \in R(\alpha)$.

By taking $\psi(z) = ((1 + z)/(1 - z))^\eta$ in the subordination part of Theorem 5.2 for the case $L = H$, the Dzoik Srivastava operator, we have the following result.

**Corollary 4.13.** Let $0 < \eta \leq 1$ and $\text{Re}[(\mu - v)\alpha_1 + \mu] \geq 0$. If $f \in \mathcal{S}_p$, $F$ as defined in (3.8) and satisfies the subordination

$$\Omega^{\eta}_{H, \mu, v}(F(z)) \left( (\alpha_1 + 1)\mu \Omega^{\eta}_{H, 1, 0}(f(z), F(z)) - \nu \alpha_1 \Omega^{\eta}_{H, 0, -1}(f(z), F(z)) \right) \prec \left( (\mu - v)\alpha_1 + \mu + \frac{2\eta z}{1 - z^2} \right) \left( \frac{1 + z}{1 - z} \right) \eta,$$

then

$$\Omega^{\eta}_{H, \mu, v}(F(z)) \prec \left( \frac{1 + z}{1 - z} \right) \eta,$$

and $((1 + z)/(1 - z))^\eta$ is the best dominant.

By putting $p = 1, l = m + 1, \alpha_1 = 1$ and $\alpha_{i+1} = \beta_i$ ($i = 1, 2, \ldots, m$) in the above Corollary 4.13 we obtain the following result.

**Corollary 4.14.** Let $0 < \eta \leq 1$ and $2\mu \geq v$. If $f \in \mathcal{S}_p$, $F$ as defined in (3.8) and

$$\left| \arg \left\{ \left( f'(z) \right)^\mu \left( \frac{z}{F(z)} \right)^\nu \left( 2\mu \frac{f'(z)}{F'(z)} - \nu \frac{f(z)}{F(z)} \right) \right\} \right| < \frac{(\eta + 1)\pi}{2} - \arctan \frac{(2\mu - v)}{\eta},$$

then

$$\left| \arg \left\{ \left( f'(z) \right)^\mu \left( \frac{z}{F(z)} \right)^\nu \right\} \right| < \frac{\eta \pi}{2}. $$
Corollary 4.15. Let 0 < \eta \leq 1. If \( f \in \mathcal{A} \), \( F \) as defined in (3.8) and
\[
\left| \arg \left\{ \frac{zF'(z)}{F(z)} \left( 2 \frac{f'(z)}{F'(z)} - \frac{f(z)}{F(z)} \right) \right\} \right| < \frac{(\eta + 1)\pi}{2} - \arctan \left( \frac{1}{\eta} \right),
\]
then \( F(z) \in S^{\eta}(\eta) \).

Taking the dominant \( \psi(z) = \sqrt{1 + z} \), which is a convex function in the open unit disc \( U \), in the subordination part of Theorem 3.2 we have the following corollary for the operator \( L = H \), the Dzoiko Srivastava operator.

Corollary 4.15. Let 0 < \eta \leq 1 and \( F \) as defined in (3.8) and
\[
\Omega_{1,1,0}^{\eta}(f(z)) \left( (\alpha_1 + 1)\mu \Omega_{1,1,0}^{\eta}(f(z)) - \alpha_1 \nu \Omega_{1,1,0}^{\eta}(f(z)) \right) \prec (\alpha_1 \mu - \nu) \sqrt{1 + z} + \frac{z}{2\sqrt{1 + z}},
\]
then
\[
\Omega_{1,1,0}^{\eta}(F(z)) \prec \sqrt{1 + z}
\]
and \( \sqrt{1 + z} \) is the best dominant.

Putting \( p = 1, l = m + 1, \alpha_1 = 1 \) and \( \alpha_{i+1} = \beta_i \) (\( i = 1, 2, \ldots, m \)) in Corollary 4.15 we obtain the following result.

Corollary 4.16. Let 0 < \eta \leq 1 and \( 2\mu \geq \nu \). If \( f \in \mathcal{A} \), \( F \) as defined in (3.8) and
\[
(F'(z))^{\mu} \left( \frac{z}{F(z)} \right)^{\nu} \left( 2\mu \frac{f'(z)}{F'(z)} - \nu \frac{f(z)}{F(z)} \right) \prec (2\mu - \nu) \sqrt{1 + z} + \frac{z}{2\sqrt{1 + z}},
\]
then
\[
(F'(z))^{\mu} \left( \frac{z}{F(z)} \right)^{\nu} \prec \sqrt{1 + z}
\]
and \( \sqrt{1 + z} \) is the best dominant.

Putting \( \mu = \nu = 1 \) in the above Corollary 4.16 we have the following example.

Example 4.15. Let 0 < \eta \leq 1. If \( f \in \mathcal{A} \), \( F \) as defined in (3.8) and
\[
\left| \frac{zF'(z)}{F(z)} \left( 2 \frac{f'(z)}{F'(z)} - \frac{f(z)}{F(z)} \right) \right| < \sqrt{1.22} \approx 1.10,
\]
then \( F \in \mathcal{L} \).

Proof. The above result can be proved using the technique adopted in the proof of the Example 4.8 and hence it is omitted here.

Now we discuss some applications of Theorem 3.1 when \( L = I \), the Integral transform. The subordination part of Theorem 3.1 yields the following corollary by taking \( \psi(z) = (1 + Az)/(1 + Bz) \) and
\[
\text{Re} \left( 1 + \frac{z\psi'(z)}{\psi(z)} \right) > \max \{0, \text{Re}[(\nu - \mu)(\lambda + p)]\}
\]
instead of taking \( \psi \) is convex and \( \text{Re}[(\nu - \mu)(\lambda + p)] \geq 0 \).

Corollary 4.17. Let \(-1 < B < A \leq 1 \) and \( \lambda \neq -p \) be a complex number. Let \( \text{Re}(\mu - \nu B) \geq |\nu - \mu B| \) where \( u = (\mu - \nu)(\lambda + p) + 1, v = [(\mu - \nu)(\lambda + p) - 1]B \). If \( f \in \mathcal{A} \), \( F \) as defined in (3.8) and
\[
\Omega_{1,1,1}^{\mu}(f(z)) \left( \mu \Omega_{1,1,1}^{\mu}(f(z)) - \nu \Omega_{1,1,1}^{\nu}(f(z)) \right) \prec (\mu - \nu) \frac{1 + Az}{1 + Bz} + \frac{1}{\lambda + p} \frac{(A - B)z}{(1 + Bz)^2},
\]
then
\[
\Omega_{1,1,1}^{\mu}(F(z)) < \frac{1 + Az}{1 + Bz}
\]
and \( (1 + Az)/(1 + Bz) \) is the best dominant.
Corollary 4.18. Let $0 \leq \alpha < 1$, $\lambda \neq -p$ be a complex number and $\text{Re}[(\mu - v)(\lambda + p)] \geq 0$. If $f \in \mathcal{A}$ and
\[
\Omega_{f, \mu, v}(f(z)) \left( \mu \Omega_{f, 1, 1}^†(f(z)) - v \Omega_{f, 1, 1}^†(f(z)) \right) \prec (\mu - v) \frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{1}{\lambda + p} \frac{2(1 - \alpha)z}{(1 - z)^2},
\]
then
\[
\Omega_{f, \mu, v}(f(z)) \prec \frac{1 + (1 - 2\alpha)z}{1 - z}
\]
and $(1 + (1 - 2\alpha)z)/(1 - z)$ is the best dominant.

Note that for $p = 1$, $\lambda = 0$ and $r = 0$, we have $I_1(0,0)f(z) = f(z), I_1(1,0)f(z) = zf''(z), I_1(2,0)f(z) = z(zf''(z) + f'(z))$. Putting these values in Corollary 4.17, we have the following result.

Corollary 4.19. Let $-1 < B < A \leq 1$. Let $(\mu - vB) \geq |v - uB|$, where $u = \mu - v + 1$ and $v = (\mu - v - 1)B$. If $f \in \mathcal{A}$ and
\[
(f'(z))^\mu \left( \frac{z}{f(z)} \right)^\nu \left( \mu \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \nu \frac{zf''(z)}{f'(z)} \right) \prec \frac{1 + Az}{1 + Bz} + \frac{(A-B)z}{(1 + Bz)^2},
\]
then
\[
(f'(z))^\mu \left( \frac{z}{f(z)} \right)^\nu \prec \frac{1 + Az}{1 + Bz}
\]
and $(1 + Az)/(1 + Bz)$ is the best dominant.

Corollary 4.20. Let $0 \leq \alpha < 1$ and $\mu \geq v$. If $f \in \mathcal{A}$ and satisfies
\[
\text{Re} \left[ (f'(z))^\mu \left( \frac{z}{f(z)} \right)^\nu \left( \mu \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \nu \frac{zf''(z)}{f'(z)} \right) \right] > \frac{2(\mu - v)\alpha - (1 - \alpha)}{2},
\]
then
\[
\text{Re} \left[ (f'(z))^\mu \left( \frac{z}{f(z)} \right)^\nu \right] > \alpha.
\]

Proof. The proof is similar to that of the Corollary 4.4 hence omitted here.

Putting $\mu = v = 1$ in Corollary 4.19 we have the following result.

Example 4.16. Let $-1 < B < A \leq 1$. If $f \in \mathcal{A}$ and satisfies
\[
\frac{zf'(z)}{f(z)} \left( \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \frac{zf''(z)}{f'(z)} \right) \prec \frac{(A-B)z}{(1 + Bz)^2},
\]
then $f \in S^*[A,B]$.

Setting $\mu = v = 1$ in Corollary 4.20 we have the following result:

Example 4.17. Let $0 \leq \alpha < 1$. If $f \in \mathcal{A}$ satisfies the differential subordination
\[
\text{Re} \left[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf'(z)}{f'(z)} + \frac{zf''(z)}{f'(z)} \right) \right] > \frac{\alpha - 1}{2},
\]
then $f \in S^*(\alpha)$.

Remark 4.3. In fact for $\alpha = 0$ the above Example 4.17 reduces to the result [28, Corollary 2] due to Owa and Obradović.

Putting $\mu = 1$ and $v = 0$ in Corollary 4.20 we have the following result.

Example 4.18. Let $0 \leq \alpha < 1$. If $f \in \mathcal{A}$ and satisfies
\[
\text{Re} \left[ f'(z) + zf''(z) \right] > \frac{3\alpha - 1}{2},
\]
then $\text{Re} f'(z) > \alpha$.

Remark 4.4. 1. The above Example 4.18 extends the result [8, Theorem 5] due to Chichra.
2. Corollary 4.20 reduces to [25, Theorem 2] when $\mu = 0, v = -1$ and $\alpha = 1/3$.

If we take $\psi(z) = ((1 + z)/(1 - z))^\eta$ with $0 < \eta \leq 1$, for the case $L = I$, then clearly $\psi(z)$ is convex in the open unit disc $\mathbb{U}$ and we have the following corollary from the subordination part of Theorem 3.11.
Corollary 4.21. Let \( 0 < \eta \leq 1, \lambda \neq -p \) be a complex number and \( \Re[(\mu - \nu)(\lambda + p)] \geq 0 \). If \( f \in \mathcal{A}_p \), and satisfies the subordination

\[
\Omega^\mu_{\lambda, \nu}(f(z)) \left( \mu \Omega^\mu_{1,1,1}(f(z)) - \nu \Omega^\nu_{1,1,1}(f(z)) \right) \prec \left( \mu - \nu \right) \left( \frac{2 \eta z}{(\lambda + p)(1 - z)} \right) \left( 1 + z \right)^{\eta},
\]

then

\[
\Omega^\mu_{\lambda, \nu}(f(z)) \prec \left( \frac{1 + z}{1 - z} \right)^{\eta}
\]

and \( \left( \frac{1 + z}{1 - z} \right)^{\eta} \) is the best dominant.

Putting \( p = 1, \lambda = 0 \) and \( r = 0 \) in Corollary 4.21, we obtain the following corollary.

Corollary 4.22. Let \( 0 < \eta \leq 1 \) and \( \mu \geq v \). If \( f \in \mathcal{A}_p \) and satisfies

\[
\left| \arg \left\{ (f'(z))^\mu \left( \frac{\frac{z}{f(z)}}{\left( 1 + \frac{zf''(z)}{f'(z)} \right)} \right) \right\} \right| < \frac{\delta \pi}{2},
\]

where

\[
\delta = \eta + 1 - \frac{2}{\pi} \arctan \frac{\mu - \nu}{\eta},
\]

then

\[
\left| \arg \left\{ (f'(z))^\mu \left( \frac{\frac{z}{f(z)}}{\left( 1 + \frac{zf''(z)}{f'(z)} \right)} \right) \right\} \right| < \frac{\eta \pi}{2}.
\]

Proof. The proof of the above Corollary 4.22 is much akin to the proof of Corollary 4.20 hence it is left here.

The following example is obtained by taking \( \mu = v = 1 \) in the above Corollary 4.22.

Example 4.19. If \( f \in \mathcal{A}_p \) and satisfies

\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \frac{(\eta + 1)\pi}{2},
\]

then \( f \in SS^\nu(\eta) \).

Taking \( \psi(z) = \sqrt{1 + z} \), convex function in the open unit disc \( \mathbb{U} \), as dominant in the subordination part of the Theorem 3.1, we obtain the following corollary.

Corollary 4.23. Let \( \lambda \neq -p \) be a complex number and \( \Re[(\mu - \nu)(\lambda + p)] \geq 0 \). If \( f \in \mathcal{A}_p \), and satisfies the subordination

\[
\Omega^\mu_{\lambda, \nu}(f(z)) \left( \mu \Omega^\mu_{1,1,1}(f(z)) - \nu \Omega^\nu_{1,1,1}(f(z)) \right) \prec (\mu - \nu)\sqrt{1 + z} + \frac{z}{2(\lambda + p)\sqrt{1 + z}},
\]

then

\[
\Omega^\mu_{\lambda, \nu}(f(z)) \prec \sqrt{1 + z}
\]

and \( \sqrt{1 + z} \) is the best dominant.

Putting \( p = 1, \lambda = 0 \) and \( r = 0 \) in Corollary 4.23, we have the following corollary.

Corollary 4.24. Let \( \mu \geq v \). If \( f \in \mathcal{A}_p \) and satisfies the subordination

\[
(f'(z))^\mu \left( \frac{\frac{z}{f(z)}}{\left( 1 + \frac{zf''(z)}{f'(z)} \right)} \right) \prec (\mu - \nu)\sqrt{1 + z} + \frac{z}{2\sqrt{1 + z}},
\]

then

\[
(f'(z))^\mu \left( \frac{\frac{z}{f(z)}}{\left( 1 + \frac{zf''(z)}{f'(z)} \right)} \right) \prec \sqrt{1 + z}
\]

and \( \sqrt{1 + z} \) is the best dominant.

Example 4.20. If \( f \in \mathcal{A}_p \) and satisfies

\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \left( 1 - \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \frac{1}{2\sqrt{2}} \approx 0.35,
\]

then \( f \in \mathcal{A}_p \).
Proof. Putting \( \mu = \nu = 1 \) in Corollary 4.24 and using the technique used in the proof of Example 4.8 the proof follows at once.

Now we will derive some applications of Theorem 3.2 when \( L = I \), the Integral transform. The following corollary is obtained from the subordination part of Theorem 3.2 by taking \( \psi(z) = (1 + Az)/(1 + Bz) \) and

\[
\text{Re} \left( 1 + z \frac{\psi''(z)}{\psi'(z)} \right) > \max\{0, \text{Re}[(\nu - \mu)(\lambda + p)]\}
\]

instead of taking \( \psi \) is convex and \( \text{Re}[(\mu - \nu)(\lambda + p)] \geq 0 \).

**Corollary 4.25.** Let \(-1 < B < A \leq 1 \) and \( \text{Re}(u - vB) \geq |v - uB| \) where \( u = (\mu - \nu)(\lambda + p) + 1, v = (\mu - \nu)(\lambda + p) - 1 \) \( B \) and \( \lambda \neq -p \) be a complex number. If \( f \in \mathcal{A}_p \) and satisfies the subordination

\[
\Omega_{\mu, \nu}^I(f(z)) \left( \mu \Omega_{\mu, \nu}^I(f(z), F(z)) - \nu \Omega_{\mu, \nu}^I(f(z), F(z)) \right) < (\mu - \nu) \frac{1 + Az}{1 + Bz} + \frac{1}{\lambda + p} \frac{(A - B)z}{(1 + Bz)^2},
\]

then

\[
\Omega_{\mu, \nu}^I(F(z)) \prec \frac{1 + Az}{1 + Bz}
\]

where \( F \) is defined as in (3.8) and \((1 + Az)/(1 + Bz)\) is the best dominant.

Putting \( p = 1, \lambda = 0 \) and \( r = 0 \), in the above Corollary 4.25 it reduces to Corollary 4.11

**Corollary 4.26.** Let \( 0 \leq \alpha < 1, \lambda \neq -p \) be a complex number and \( \text{Re}[(\mu - \nu)(\lambda + p)] \geq 0 \). If \( f \in \mathcal{A}_p \) and satisfies the subordination

\[
\Omega_{\mu, \nu}^I(f(z)) \left( \mu \Omega_{\mu, \nu}^I(f(z), F(z)) - \nu \Omega_{\mu, \nu}^I(f(z), F(z)) \right) < (\mu - \nu) \frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{2(1 - \alpha)}{\lambda + p} \frac{z}{(1 - z)^2},
\]

where \( F \) is defined as in (3.8), then

\[
\text{Re} \Omega_{\mu, \nu}^I(f(z)) > \alpha.
\]

On setting \( p = 1, \lambda = 0 \) and \( r = 0 \) the above Corollary 4.26 reduces to Corollary 4.12. Taking \( \psi(z) = (1 + z)/(1 - z)^\eta, 0 < \eta \leq 1 \), as dominant in the subordination part of Theorem 3.2 for the Integral operator \( I = L \), we have the following corollary. Further on setting \( p = 1, \lambda = 0 \) and \( r = 0 \), it reduces to Corollary 4.14

**Corollary 4.27.** Let \( 0 < \eta \leq 1, \lambda \neq -p \) be a complex number and \( \text{Re}[(\mu - \nu)(\lambda + p)] \geq 0 \). If \( f \in \mathcal{A}_p \), \( F \) as defined in (3.8) and satisfies the subordination

\[
\Omega_{\mu, \nu}^I(f(z)) \left( \mu \Omega_{\mu, \nu}^I(f(z), F(z)) - \nu \Omega_{\mu, \nu}^I(f(z), F(z)) \right) < \left( \mu - \nu \right) + \frac{2\eta z}{(\lambda + p)(1 - z^2)} \left( \frac{1 + z}{1 - z} \right)^\eta,
\]

then

\[
\Omega_{\mu, \nu}^I(f(z)) \prec \left( \frac{1 + z}{1 - z} \right)^\eta
\]

and \((1 + z)/(1 - z)^\eta\) is the best dominant.

Taking \( \psi = \sqrt{1 + z} \) in the subordination part of Theorem 3.2 we have the following corollary corresponding to the integral operator \( I = L \), which finally reduces to Corollary 4.16 when \( p = 1, \lambda = 0 \) and \( r = 0 \).

**Corollary 4.28.** Let \( \lambda \neq -p \) be a complex number and \( \text{Re}[(\mu - \nu)(\lambda + p)] \geq 0 \). If \( f \in \mathcal{A}_p \), \( F \) as defined in (3.8) and

\[
\Omega_{\mu, \nu}^I(f(z)) \left( \mu \Omega_{\mu, \nu}^I(f(z), F(z)) - \nu \Omega_{\mu, \nu}^I(f(z), F(z)) \right) < (\mu - \nu) \sqrt{1 + z} + \frac{z}{2(\lambda + p) \sqrt{1 + z}},
\]

where \( F \) is defined as in (3.8), then

\[
\Omega_{\mu, \nu}^I(f(z)) \prec \sqrt{1 + z}
\]

and \( \sqrt{1 + z} \) is the best dominant.
References

[1] Rosihan M. Ali, Nak Eun Cho, S. Sivaprasad Kumar, and V. Ravichandran, First Order Differential Subordinations for Functions Associated with the Lemniscate of Bernoulli, *Taiwanese Journal of Mathematics*, accepted.

[2] R. Aghalary, S.B. Joshi, R.N. Mohapatra and V. Ravichandran, Subordinations for analytic functions defined by the Dziok-Srivastava linear operator, *Appl. Math. Comput.* 187 (2007), no. 1, 13–19.

[3] H. A. Al-Kharsani and N. M. Al-Areefi, On classes of multivalent functions involving linear operator and multiplier transformations, *Hacet. J. Math. Stat.* 37 (2008), no. 2, 115–127.

[4] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* 135 (1969), 429–446.

[5] T. Bulboacă, A class of superordination-preserving integral operators, *Indag. Math. (N.S.)* 13 (2002), no. 3, 301–311.

[6] T. Bulboaca, Classes of first-order differential superordinations, *Demonstratio Math.* 35 (2002), no. 2, 287–292.

[7] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.* 15 (1984), no. 4, 737–745.

[8] P. N. Chichra, New subclasses of close-to-convex functions, *Proc. Amer. Math. Soc.* 62 (1976), no. 1, 37–43 (1977).

[9] N. E. Cho and H. M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, *Mathematical and Computer Modelling*, 37 (2003) 39–49.

[10] N. E. Cho and T. H. Kim, Multiplier transformations and strongly close-to-convex functions, *Bull. Korean Math. Soc.*, 40 (2003), 399–410.

[11] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

[12] J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transform. Spec. Funct.*, 14 (2003) 7-18.

[13] Yu. E. Hohlov, Operators and operations in the class of univalent functions, *Izv. Vysš. Učebn. Zaved. Mat.*, 10 (1978), 83–89.

[14] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.* 23 (1970/1971), 159–177.

[15] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.* 176 (1993), no. 1, 138–147.

[16] O. S. Kwon and N. E. Cho, Inclusion properties for certain subclasses of analytic functions associated with the Dziok-Srivastava operator, *J. Inequal. Appl.*, Art. ID 51079, 10 pp.

[17] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, 16 (1965), 755–758.

[18] J.-L. Liu, Notes on Jung-Kim-Srivastava integral operator. Comment on: “The Hardy space of analytic functions associated with certain one-parameter families of integral operators” [J. Math. Anal. Appl. 176 (1993), no. 1, 138–147] by I. B. Jung, Y. C. Kim and H. M. Srivastava, *J. Math. Anal. Appl.* 294 (2004), no. 1, 96–103.

[19] J.-L. Liu and S. Owa, On a class of univalent functions involving certain linear operator, *Indian J. Pure Appl. Math.* 33 (2002), no. 11, 1713–1722.

[20] A. E. Livingston, On the radius of univalence of certain analytic functions, *Proc. Amer. Math. Soc.*, 17 (1966), 352–357.

[21] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications. Pure and Applied Mathematics No. 225, Marcel Dekker, New York, (2000).

[22] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, *Complex Var. Theory Appl.* 48 (2003), no. 10, 815–826.

[23] K. I. Noor, On some analytic functions defined by a multiplier transformation, *Int. J. Math. Math. Sci.* 2007, Art. ID 92439, 9 pp.

[24] K. I. Noor and M. A. Noor, On integral operators, *J. Math. Anal. Appl.* 238 (1999), no. 2, 341–352.

[25] M. Obradović, Estimates of the real part of $f(z)/z$ for some classes of univalent functions, *Mat. Vesnik* 36 (1984), no. 4, 266–270.

[26] S. Owa, On the distortion theorems I, *Kyungpook Math. J.*, 18 (1978), 53–58.

[27] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, 39 (1987), 1057–1077.

[28] S. Owa and M. Obradović, An application of differential subordinations and some criteria for univalency, *Bull. Austral. Math. Soc.*, 41 (1990), no. 3, 487–494.

[29] V. Ravichandran and S. S. Kumar, On sufficient conditions for starlikeness, *Southeast Asian Bull. Math.*, 29 (2005), no. 4, 773–783.

[30] St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, 49 (1975), 109–115.

[31] G. St. Ruscheweyh, Subclasses of univalent functions, in *Complex Analysis: Fifth Romanian-Finnish Seminar*, Part I (Bucharest, 1981), Lecture Notes in Mathematics, Vol. 1013, Springer-Verlag, Berlin and New York, 1983, 362–372.

[32] S. Sivaprasad Kumar, V. Ravichandran and H. C. Taneja, Differential sandwich theorems for linear operators, *International Journal of Mathematical Modeling, Simulation and Applications*, 2 (2009), no. 4, 490–507.

[33] S. Sivaprasad Kumar, H. C. Taneja and V. Ravichandran, Classes of multivalent functions defined by Dziok-Srivastava linear operator and multiplier transformation, *Kyungpook Math. J.*, 46 (2006), no. 1, 97–109.

[34] J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* No. 19 (1996), 101–105.

[35] H. M. Srivastava, Some families of fractional derivative and other linear operators associated with analytic, univalent, and multivalent functions, in *Analysis and its applications* (Chennai, 2000), 209–243, Allied Publ., New Delhi

[36] B. A. Uralcgaddu and C. Somancatha, Certain classes of univalent functions, in *Current topics in analytic function theory*, 371–374, World Sci. Publ., River Edge, NJ.