Observable subgroups of algebraic monoids

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Abstract

A closed subgroup $H$ of the affine, algebraic group $G$ is called observable if $G/H$ is a quasi-affine algebraic variety. In this paper we define the notion of an observable subgroup of the affine, algebraic monoid $M$. We prove that a subgroup $H$ of $G$ is observable in $M$ if and only if $H$ is closed in $M$ and there are “enough” $H$-semiinvariant functions in $\mathbb{k}[M]$. We show also that a closed, normal subgroup $H$ of $G$ (the unit group of $M$) is observable in $M$ if and only if it is closed in $M$. In such a case there exists a determinant $\chi: M \to \mathbb{k}$ such that $H \subseteq \ker(\chi)$. As an application, we show that in this case the affinized quotient $M/\operatorname{aff} H$ of $M$ by $H$ is an affine algebraic monoid scheme with unit group $G/H$.

1. Introduction

A closed subgroup $H$ of the affine algebraic group $G$ is called an observable subgroup if the homogeneous space $G/H$ is a quasi-affine variety. Such subgroups have been researched extensively, notably by F. Grosshans, see \cite{5} for a survey on this topic, and Theorem 2.12 below for other useful characterizations of observable subgroups. In \cite{10} the authors presented the notion of an observable action of $G$ on the affine variety $X$, together to its basic properties. In this paper we develop further the notion of observable actions. In particular, we investigate the situation where $M$ is an affine algebraic monoid with unit group $G$, and $H$ is a closed subgroup of $G$, such that the action of

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$H$ on $M$ by left multiplication is observable. In this case, we say that $H$ is observable in $M$.

We describe now the organization of this paper.

In Section 2 we provide the basic definitions and properties of observable actions and affinized quotients. In Section 3 we give several characterizations of observable subgroups. In Theorem 3.3 we deduce a number of useful consequences from the assumption that $H$ is observable in $M$. In particular it follows that $H$ is an observable subgroup of $G$, and that it is closed in $M$. In Theorem 3.4 we characterize the observable subgroups of $M$ in terms of semiinvariants. In Theorem 3.5 we show that $H$ is an observable subgroup of $M$ if and only if $H$ is the isotropy group of some vector $v \in V$ in some rational representation $\rho : M \to \text{End}(V)$ of $M$. In the final section (Section 4) we use the results of the previous section to study the affinized quotient of an affine algebraic monoid by a closed normal subgroup. In Theorem 4.4 we show that if $H$ is a closed, normal subgroup of $G$, closed in $M$, then $H$ is observable in $M$. Whether this is true for nonnormal closed subgroups of $G$ is an open question. See Remark 4.7. As an application, we show that the affinized quotient of an affine algebraic monoid $M$ by a normal subgroup $H$, closed in $M$, is an algebraic monoid, with unit group $G/H$.

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2. Preliminaries

Let $\mathbb{k}$ be an algebraically closed field. We work with affine algebraic varieties $X$ over $\mathbb{k}$. An algebraic group is assumed to be a smooth, affine, group scheme of finite type over $\mathbb{k}$. If $X$ is an affine variety over $\mathbb{k}$ we denote by $\mathbb{k}[X]$ the ring of regular functions on $X$. If $I \subset \mathbb{k}[X]$ is an ideal, we denote by $V(I) = \{x \in X : f(x) = 0 \ \forall \ f \in I\}$. If $Y \subset X$ is a subset, we denote by $I(Y) = \{f \in \mathbb{k}[X] : f(y) = 0 \ \forall \ y \in Y\}$. Morphisms $\varphi : X \to Y$ between affine varieties correspond to morphisms of algebras $\mathbb{k}[Y] \to \mathbb{k}[X]$, by $\varphi \mapsto \varphi^* : \mathbb{k}[X] \to \mathbb{k}[Y]$, with $\varphi^*(f) = f \circ \varphi$. If $X$ is irreducible we denote by $\mathbb{k}(X)$ the field of rational functions on $X$. If $A$ is any integral domain we denote by $[A]$ its quotient field. Thus if $X$ is an irreducible affine variety, then $\mathbb{k}(X) = [\mathbb{k}[X]]$.

Let $G$ be an affine algebraic group and let $X$ be an algebraic variety. A (regular) action of $G$ on $X$ is a morphism $\varphi : G \times X \to X$, denoted by $\varphi(g, x) = g \cdot x$, such that $(ab) \cdot x = a \cdot (b \cdot x)$ and $1 \cdot x = x$ for all $a, b \in G$ and $x \in X$. Since all the actions we work with are regular, we drop the adjective regular. The orbit of $x \in X$ is denoted by $O(x) = \{g \cdot x : g \in G\}$.
If \( G \times X \to X \) is a regular left action of \( G \) on \( X \) we consider the induced right action of \( G \) on \( k[X] \), defined as follows. If \( f \in k[X] \) and \( g \in G \), then \( (f \cdot g)(x) = f(gx) \). It is well known that \( G \)-stable closed subset of \( X \) correspond to \( G \)-stable radical ideals of \( k[X] \). We say that \( f \in k[X] \) is \( G \)-invariant if \( f \cdot g = f \) for any \( g \in G \). The set of \( G \)-invariants \( G\langle k[X] \rangle \) forms a \( k \)-subalgebra of \( k[X] \), possibly non-finitely generated. Analogous considerations can be made if we start with a right action \( X \times G \to X \).

A finite dimensional (rational) \( G \)-module is a finite dimension \( k \)-vector space \( V \) together with a left action of \( G \) on \( V \) by linear automorphisms. A right action of \( G \) on \( V \) defines, in a similar way, the notion of a right \( G \)-module.

Recall that an algebraic monoid \( M \) is an algebraic variety together with an associative product \( m : M \times M \to M \) with neutral element \( 1 \), such that \( m \) is a morphism of algebraic varieties. We denote the set of idempotent elements of \( M \) by \( E(M) = \{ e \in M : e^2 = e \} \). We denote the unit group of \( M \) by \( G(M) = \{ g \in M : \exists g^{-1} \in M , \ g^{-1}g = gg^{-1} = 1 \} \). It is known that \( G(M) \) is an algebraic group, open in \( M \) (see [9] and [11]).

2.1. Characters of affine algebraic monoids

In this section we establish the basic facts about extendible characters for the case of linear algebraic monoids.

**Definition 2.1.** Let \( M \) be an algebraic monoid. A character of \( M \) is a morphism of algebraic monoids \( M \to k \). We denote the monoid of characters of \( M \) by

\[
\mathcal{X}(M) = \{ \chi \in k[M] : \chi(ab) = \chi(a)\chi(b) \ \forall \ a, b \in M , \ \chi(1) = 1 \}.
\]

If \( G = G(M) \), then restriction induces an injective morphism of (abstract) monoids \( \mathcal{X}(M) \hookrightarrow \mathcal{X}(G) \).

If \( M \) is an irreducible affine algebraic monoid, then there exists \( n \geq 0 \) and a morphism of algebraic monoids \( \rho : M \hookrightarrow M_n(k) \), such that \( \rho \) is closed immersion (see for example [9, Theorem 3.8]). This motivates the following definition.

**Definition 2.2.** Let \( M \) be an irreducible affine algebraic monoid. A character \( \chi \in \mathcal{X}(M) \) is called a determinant if \( \chi^{-1}(0) = M \setminus G(M) \).

By the considerations above, determinants always exists.
Remark 2.3. (1) Observe that if \( \det M \to k \) is a determinant, then \( G(M) = M_{\det} \). In particular, \( k[G(M)] = k[M]_{\det} \).

(2) Clearly, \( \mathcal{X}(M) \subset \mathcal{X}(G(M)) \) is a unital submonoid that generates \( \mathcal{X}(G(M)) \) as a group. Indeed, let \( \det : M \to k \) be a determinant and \( \chi \in \mathcal{X}(G(M)) \). Then there exists \( f \in k[M] \) and \( n \geq 0 \) such that \( \chi = f \frac{1}{\det^n} \). It follows that \( f|G \) is a character, and hence, by continuity, \( f \in \mathcal{X}(M) \).

The notion of an extendible character is very useful in the study of observable subgroups of algebraic groups. We extend this notion to the setting of algebraic monoids.

Definition 2.4. Let \( M \) be an affine algebraic monoid and let \( H \subset G(M) \) be a closed subgroup. A non-trivial character \( \chi \in \mathcal{X}(H) \) is extendible if there exists a non-zero semiinvariant element of weight \( \chi \); that is, there exists \( f \in k[M] \) such that \( f \cdot x = \chi(x)f \) for all \( x \in H \). Such an element is called an extension of \( \chi \). We denote the monoid of extendible characters of \( H \) by \( \mathcal{E}_M(H) \).

Clearly, restriction to \( G = G(M) \) induces an injective homomorphisms of (abstract) monoids \( \mathcal{E}_M(H) \hookrightarrow \mathcal{E}_G(H) \).

Remark 2.5. Observe that if \( \chi \) is an extendible character and \( f \) is an extension of \( \chi \), we can suppose that \( f(1) = 1 \). Then \( f(x) = (f \cdot x) = (1)\chi(x)f(1) = \chi(x) \) for all \( x \in N \), and thus \( f \) is an extension of \( \chi \) to a regular function of \( M \).

Definition 2.6. Let \( M \) be an affine algebraic monoid and \( V \) a finite rational \( M \)-module. For every \( \alpha \in V^* \) and \( v \in V \) we define \( \alpha|v : M \to k \) as \( (\alpha|v)(x) = \alpha(x \cdot v) \) for all \( x \in M \). A function of this form is called a \( V \)-representative function or simply a representative function.

Definition 2.7. Let \( G \) be an algebraic group and let \( \chi \in \mathcal{X}(G) \) be a character. Let \( V \) be a right \( G \)-module, with action \( \varphi : V \times G \to V \), \( \varphi(v, g) = v \cdot g \). Then the twisted representation \( V_\chi \) is defined as follows. As a vector space \( V = V_\chi \), and the action \( V_\chi \times G \to V_\chi \) is given by \( v \star g = \chi(g)(v \cdot g) \), for every \( g \in G, v \in V_\chi \).

The following theorem is a straightforward generalization of the corresponding (well known) result for algebraic groups (see [4, Theorem 7.2.3]). We include a proof for the sake of completeness.
Theorem 2.8. Let $M$ be an affine algebraic monoid with unit group $G$, $H \subset G$ a closed subgroup and $V$ a finite dimensional rational right $H$-module. There exists a finite dimensional rational right $M$-module $W$, an extendible character $\chi : H \to G_m$ and an injective morphism $\iota : V \to (W|_H)_{\chi^{-1}}$.

Proof. Given $V$ as above, we proceed as in the proof of [4, Theorem 7.2.3]. It is well known that there exists an injective morphism of $H$-modules $\theta : V \to \bigoplus I k[M]$, where $I$ is a finite set of indexes (see for example [4, Theorem 4.3.13]). Consider the $H$-morphism $\alpha = \bigoplus \pi : \bigoplus I k[M] \to \bigoplus I k[H]$, where $\pi : k[M] \to k[H]$ is the canonical projection.

Call $V' = \alpha^{-1} (\theta(V)) \subset \bigoplus I k[M]$ and $\beta$ the restriction of $\alpha$ to the $H$-submodule $V'$:

\[
\begin{array}{c}
V' \xrightarrow{\beta} \bigoplus I k[M] \\
\downarrow \alpha \\
V \xrightarrow{\theta} \bigoplus I k[H]
\end{array}
\]

Let $F$ be a finite $k$-linear basis of $V$ and let $F_0 \subset V'$ be a finite set such that $\beta(F_0) = F$. Call $R$ the finite dimensional $G$-submodule of $\bigoplus I k[M]$ generated by $F_0$; then $R$ is an $M$-module. Let $S$ be finite dimensional $H$-submodule of $R|_H$ (contained in $V'$) generated by $F_0$. In this way, we produce an exact sequence of rational $H$–modules $0 \to U \to S \to V \to 0$. Call $n = \dim_k U$ and consider the commutative diagram below

\[
\begin{array}{cccccc}
0 & \to & U \otimes \bigwedge^n U & \to & S \otimes \bigwedge^n U & \to & V \otimes \bigwedge^n U & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \phi \\
0 & \to & U \wedge \bigwedge^n U = 0 & \to & S \wedge \bigwedge^n U & & & &
\end{array}
\]

where all the solid arrows are the canonical ones, the first row is exact and in the second row the term $U \wedge \bigwedge^n U$ equals zero by dimensional reasons — notice that all the exterior products are taken inside the exterior algebra $\bigwedge R$. One can now prove that $\phi$ is bijective (see the proof of [4, Theorem 7.2.3]).

As $S$ and $U$ are $H$-submodules of $R|_H$, we can view $\phi$ as an injective $H$–morphism $\varphi_1 : V \otimes \bigwedge^n U \to \bigwedge^{n+1} R|_H$. Call $\chi$ the rational character associated to the one dimensional $H$-module $\bigwedge^n U$, i.e. the character defined by the formula $x \cdot u = \chi(x) u$ for all $x \in H$. The map $\varphi_2 : V \to \bigwedge^{n+1} R$,
\[ \varphi_2(m) = \varphi_1(m \otimes u), \] satisfies that for all \( x \in H \),

\[ \varphi_2(m \cdot x) = \varphi_1((m \cdot x) \otimes u) = \varphi_1(\chi^{-1}(x)(m \cdot x \otimes u \cdot x)) = \chi^{-1}(x)\varphi_1((m \otimes u) \cdot x) = \chi^{-1}(x)\varphi_1(m \otimes u) \cdot x = \chi^{-1}(x)\varphi_2(m) \cdot x. \]

Hence, if we let \( W = \bigwedge^{n+1} R \) and \( \iota = \varphi_2 \), the proof of the theorem will be complete once we prove that the character \( \chi \) is extendible. Consider the \( M \)-module \( \bigwedge^n R \) and let \( \alpha \in (\bigwedge^n V)^* \) be such that \( \alpha(u) = 1 \). Then the representative function \( \alpha|u \in \mathbb{k}[M] \) is an extension of \( \chi \). Indeed, if \( x \in H \) and \( m \in M \), then

\[ ((\alpha|u) \cdot x)(m) = (\alpha|u)(xm) = \alpha(u \cdot (xm)) = \chi(x)\alpha(u \cdot m) = \chi(x)(\alpha|u)(m) \]

and \( (\alpha|u)(1) = \alpha(u) = 1 \). So that \( \alpha|u \) is an extension of \( \chi \). \( \square \)

Let \( M \) be an affine algebraic monoid with unit group \( G \). Many results about extendible characters for subgroups of \( G \) apply to the situation of closed subgroups of \( M \).

**Lemma 2.9.** Let \( M \) be an affine algebraic monoid with unit group \( G \), and \( H \subset G \) a subgroup, closed in \( M \).

1. Let \( \rho \in E_M(H) \) be an extendible character and let \( f \in k[M] \) be a semi-invariant of weight \( \rho \). Consider the left action \( M \times k[M] \to k[M] \) given by \( (x \cdot f)(m) = f(mx), x, m \in M \). Then for any \( x \in M, x \cdot f \) is also a semi-invariant of weight \( \rho \).

2. If \( \pi : k[M] \to k[H] \) is the canonical projection and \( \rho \in E_M(H) \) is an extendible character, there exists an extension \( f \in k[M] \) of \( \rho \) such that \( \pi(f) = \rho \).

3. A character \( \rho \in \chi(H) \) is extendible if and only if there exists a rational right \( M \)-module \( V \) and an injective morphism of right \( H \)-modules \( \iota : \mathbb{k} \rho \to V \). In other words \( \rho \) is extendible if and only if there exists a rational right \( M \)-module \( V \) and a non-zero element \( v \in V \) such that \( v \cdot x = \rho(x)v \) for all \( v \in H \). Moreover, the right \( M \)-module \( V \) can be taken to be a finite dimensional right \( M \)-submodule of \( k[M] \).

4. For any \( \gamma \in \chi(H) \) there exists \( \rho \in E_M(H) \) such that \( \gamma \rho \in E_M(H) \).

**Proof.** Many of these results are straightforward generalizations of the corresponding results for algebraic groups. See [4, Lemma 7.2.8]. For the sake of completeness we include the proofs of (3) and (4).
To prove (3) suppose that $\rho$ is extendible, let $f \in k[M]$ be an extension and call $W$ the rational right $G$-submodule of $k[M]$ generated by $f$. Then the map $\iota : k\rho \to W$, $\iota(\rho) = f$, does the job. Conversely, if one has an injective morphism $\iota : k\rho \to W|_H$ and call $n = \iota(\rho)$, then $n \cdot x = \iota(\rho) \cdot x = \iota(\rho \cdot x) = \rho(x)\iota(\rho) = \rho(x)n$. Take now $\alpha \in N^*$ such that $\alpha(n) = 1$ and consider $f = \alpha|n$. Then, $f \cdot x(m) = (\alpha|n)(xm)\alpha(n \cdot (xm)) = \rho(x)(\alpha|n)(m) = \rho(x)f$ for all $x \in H$, and $f(1) = (\alpha|n)(1) = \alpha(n) = 1$.

To prove (4) consider $V = k\gamma$. By Theorem 2.8 there exists an extendible character $\rho$, a right $M$-module $W$ and an injective $H$-morphism $\iota : k\gamma \to (W|_H)_{\rho^{-1}}$. Call $n = \iota(\gamma)$, then for any $x \in H$ we have that $n \cdot x = \iota(\gamma) \cdot x = \rho(x)\iota(\gamma \cdot x) = \rho(x)\gamma(x)\iota(\gamma) = \rho(x)\gamma(x)n$. From (3) we conclude that $\rho \gamma \in E_M(H)$.

2.2. Observable actions of algebraic groups

Observable subgroups were introduced by Bialynicki-Birula, Hochschild and Mostow in [1]. Since then they have been researched extensively, notably by F. Grosshans. See [5] for a survey on this topic. This basic notion has recently been generalized and reformulated in the context of geometric invariant theory by the authors. See [10].

**Definition 2.10.** Let $G$ be an affine algebraic group and let $H \subset G$ be a closed subgroup. The subgroup $H$ is observable in $G$ if and only if for any nonzero $H$-stable ideal $I \subset k[G]$ we have that $I^G \neq (0)$, where we consider the right action of $H$ on $k[G]$ given by $(f \cdot h)(a) = f(aha)$, for all $f \in k[G]$, $h \in H$ and $a \in G$.

Analogously, one can define the notion of a right observable subgroup, by considering the action $H \times k[G] \to k[G]$, $(h \cdot f)(a) = f(ah)$ for $f \in k[G]$, $a \in G$, $h \in H$.

**Example 2.11.** (1) If $U \subset G$ is a closed unipotent subgroup, then $U$ is observable, since any $U$-module has non-zero invariant elements.

(2) If $H \subset G$ is a normal closed subgroup, then $H$ is observable. This follows from condition (2) of Theorem 2.12 below.

(3) Let $H \subset G$ be a closed subgroup, such that $\chi(H) = 1$. Then $H$ is observable in $G$. This follows for example from condition (6) of Theorem 2.12 below.

We now present a collection of equivalent definitions of observability. The proofs can be found in [4, Thms. 10.2.9 and 10.5.5]. We give a different
proof for the fact that $H$ is observable in $G$ if and only if $E_G(H)$ is a group (equivalence $(1) \iff (7)$ of Theorem 2.12). This will provide some insight into the more general situation of algebraic monoids given in Theorem 3.4.

Theorem 2.12. Let $G$ be an affine algebraic group and $H \subset G$ a closed subgroup. Then the following conditions are equivalent:

1. The subgroup $H$ is observable in $G$.
2. The homogeneous space $G/H$ is a quasi-affine variety.
3. For an arbitrary proper and closed subset $C \subset G/H$, there exists an non-zero invariant regular function $f \in k[G]^H$ such that $f(C) = 0$.

Moreover, if $G$ is connected the above conditions are equivalent to any of the following.

(4) $H = \{x \in G : x \cdot f = f, \forall f \in k[G]^H\}$.
(5) $[k[G]]^H = [k[G]^H]$.
(6) $E_G(H) = \mathcal{X}(H)$.
(7) $E_G(H)$ is a group.

Proof. To prove that $(1) \implies (7)$, let $H$ be observable in $G$ and $\chi \in E_G(H)$ an extendible character. Let $f \in k[G]$ be a semiinvariant of weight $\chi$. Then the ideal $I = f k[G]$ is $H$-stable, and hence there exists $g \in k[G]$ such that $(fg) \cdot x = fg$ for all $x \in H$. It follows that $g \cdot x = \chi^{-1}(x)g$ for all $x \in H$. Thus $\chi^{-1} \in E_G(H)$.

To prove that $(7) \implies (1)$, let $I \subset k[G]$ be a non-zero $H$-stable ideal, and consider $H_{\text{uni}} = R_u(H)[H,H]$. Then $H_{\text{uni}}$ is normal in $H$, with $\mathcal{X}(H_{\text{uni}}) = \{1\}$, and such that $H_{\text{uni}} \setminus H$ is a torus; in particular $\mathcal{X}(H) = \mathcal{X}(H_{\text{uni}} \setminus H)$, since $\mathcal{X}(H_{\text{uni}}) = \{1\}$. It follows that $H_{\text{uni}}$ is observable in $G$, and thus $I^{H_{\text{uni}}} \neq (0)$ is a $H$-module. Then $I^{H_{\text{uni}}}$ is a right $(H_{\text{uni}} \setminus H)$-module, and hence

$$(0) \neq I^{H_{\text{uni}}} = \bigoplus_{\chi \in \mathcal{X}(H_{\text{uni}} \setminus H)} I^{H_{\text{uni}}} = \bigoplus_{\chi \in \mathcal{X}(H)} I^{H_{\text{uni}}}_{\chi}$$

Hence, there exists $\chi \in \mathcal{X}(H)$ and $f \in I \setminus \{0\}$ such that $f \cdot x = \chi(x)f$ for all $x \in H$. Thus $\chi \in E_G(H)$. Since $E_G(H)$ is a group, then $\chi^{-1} \in E_G(H)$ and hence there exists $g \in k[G]$ such that $g \cdot x = \chi^{-1}(x)g$ for all $x \in H$. It follows that $0 \neq fg \in I^H$. □

Definition 2.13. Let $G$ be an affine algebraic group acting on an affine variety $X$. We say that the action is observable if for any nonzero $G$-stable ideal $I \subset k[X]$ we have that $I^G \neq (0)$. 8
Example 2.14. (1) The action of an unipotent group on an affine variety is always observable, since for every right $U$-module $M$, we have that $M^U \neq 0$.

(2) Let $G$ be an algebraic group and $H$ a closed subgroup. Then the action of $H$ on $G$ by left translations is observable if and only if $H$ is observable in $G$, in the sense of Definition 2.10, if and only if the action by right translations is observable.

Observable actions have been studied in detail in [10], where several characterizations of this important property were presented. We collect here the ones that we need in what follows. We begin by recalling some notation.

Definition 2.15. Let $G$ be an affine group acting on an affine variety $X$. We let

$$\Omega(X) = \{ x \in X : \text{dim} \mathcal{O}(x) \text{ is maximal and } \overline{\mathcal{O}(x)} = \mathcal{O}(x) \}.$$  

That is, $\Omega(X)$ is the set of orbits of maximal dimension that are closed. The reader should be aware that $\Omega(X)$ can be empty.

Theorem 2.16. Let $G$ be a connected affine algebraic group acting on an irreducible affine variety $X$. Then the following are equivalent

1. The action is observable.
2. (i) $[k[X]^G] = [k[X]^G]$ and (ii) $\Omega(X)$ contains a non-empty open subset.
3. There exists a nonzero invariant $f \in k[X]^G$ such that the action of $G$ on $X_f$ is observable.

Proof. See [10, Proposition 3.2 and Theorem 3.10].

If $G$ is a reductive group acting on an affine variety $X$, then a result of Popov (see [4, Theorem 4]) guarantees that if $\Omega(X)$ is nonempty then it is an open set. In [10] this result was used to prove that in this case, condition (2ii) of Theorem 2.16 is sufficient to guarantee observability.

Proposition 2.17. Let $G$ be a reductive group acting on an affine variety $X$. Then the action is observable if and only if $\Omega(X) \neq \emptyset$.

Proof. See [10, Theorem 4.7].
2.3. The Affinized quotient

Let $G$ be an affine algebraic group acting on the affine variety $X$. It is well known that the categorical quotient does not necessarily exist, even when $\mathbb{k}[X]^G$ is finitely generated. However, if $\mathbb{k}[X]^G$ is finitely generated, then $\text{Spec}(\mathbb{k}[X]^G)$ satisfies a universal property in the category on the affine algebraic varieties.

**Definition 2.18.** Let $G$ be an affine algebraic group acting on an affine variety $X$, in such a way that $\mathbb{k}[X]^G$ is finitely generated. The **affinized quotient** of the action is the morphism $\pi : X \to X/\text{aff}G = \text{Spec}(\mathbb{k}[X]^G)$.

It is clear that $\pi$ satisfies the following universal property.

Let $Z$ be an affine variety and $f : X \to Z$ a morphism constant on the $G$-orbits. Then there exists a unique $\tilde{f} : X/\text{aff}G \to Z$ such that the following diagram is commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow & & \downarrow \\
X/\text{aff}G & \xrightarrow{\tilde{f}} & Z
\end{array}
\]

Indeed, it is clear that the induced morphism $f^* : \mathbb{k}[Z] \to \mathbb{k}[X]$ factors trough $\mathbb{k}[X]^G$.

**Remark 2.19.** It is clear that the morphism $\pi : X \to X/\text{aff}G$ is dominant. However, $\pi$ is not necessarily surjective. For example, consider a semisimple group $X = H$ and its maximal unipotent subgroup $G$ acting on $X$ by left translation.

3. Observable subgroups of algebraic monoids

We now adapt the notion of observability to the situation of subgroups of algebraic monoids.

**Definition 3.1.** Let $M$ be an algebraic monoid with unit group $G$, and let $H \subset G$ be a closed subgroup. We say that $H$ is (left) observable in $M$ if the action by left multiplication $H \times M \to M$, $h \cdot m = hm$, is observable in the sense of Definition 2.13.

Similarly, we say that $H$ is right observable in $M$ if the action by right multiplication $M \times H \to M$, $m \cdot h = mh$, is observable.
Example 3.2. (1) Since the action of a unipotent group on an affine variety is observable (all orbits are closed), it follows that any unipotent subgroup \( U \) of \( G(M) \) is observable in \( M \).

(2) If \( M = G(M) \) is an algebraic group, then Definitions 2.10 and 3.1 coincide.

(3) If follows from Proposition 2.17 that if \( H \subset G(M) \) is reductive, and closed in \( M \), then \( H \) is observable in \( M \). Indeed, if \( H = \overline{H} \), then \( \Omega(M) \neq \emptyset \).

Theorem 3.3. Let \( M \) be an affine irreducible algebraic monoid with unit group \( G \), and let \( H \subset M \) be an observable subgroup. Then

(1) The subgroup \( H \) is observable in \( G \).

(2) \( H \) closed in \( M \).

(3) \( [k[M]]^H = [k[G]]^H = [k[G]^H] = [k[M]^H] \).

(4) The subgroup \( H \) satisfies

\[
H = \{ x \in G : f \cdot x = f, \forall f \in [k[G]^H] \} = \\
\{ x \in G : f \cdot x = f, \forall f \in [k[M]^H] \} = \\
\{ x \in M : f \cdot x = f, \forall f \in [k[M]^H] \}^0.
\]

Recall that if \( N \) is an algebraic monoid, then \( N^0 \) is the unique irreducible component containing 1, see for example \([12, \text{Thm. 4}]\).

Proof. (1) Let \( C \subset G \) be a \( H \)-stable closed subset. Then \( C \subset M \) is \( H \)-stable, and it follows that there exists a \( f \in \mathcal{V}(C)^H \setminus \{0\} \subset k[M] \). Since \( G \) is open in \( M \), it follows that \( f \in \mathcal{V}(C)^H \setminus \{0\} \subset [k[G]] \).

(2) Since \( H \) is observable in \( M \), it follows from Theorem 2.16 that \( \Omega(M) \) contains a non-empty open subset. Since \( G = \bigcup Hg \), \( G \) is contained in \( M_{\text{max}} \), and hence there exists \( g \in G \) such that \( g \in \Omega(M) \cap G \), i.e. such that \( Hg \) is closed in \( M \). Since multiplication by an element of \( g \) is an isomorphism, it follows that \( H \) is closed in \( M \).

(3) First observe that \( \overline{[k[M]^H]} \subset \overline{[k[G]^H]} = \overline{[k[G]]^H} \). Let \( g \in \overline{[k[M]^H]} \), and consider the ideal \( I = \{ f \in [k[M]^H] : fg \in [k[M]] \} \). Then \( I \) is a non-zero \( H \)-stable ideal, and hence there exists \( h \in I^H \setminus \{0\} \). It follows that \( r = hg \in k[M]^H \) and hence \( g = \frac{r}{h} \in k[M]^H \). The remaining equality follows from Theorem 2.12.

(4) The first equality follows from Theorem 2.12. It is clear that \( A = \{ x \in G : f \cdot x = f, \forall f \in [k[G]^H] \} \subset B = \{ x \in G : f \cdot x = f, \forall f \in [k[M]^H] \} \). Let \( x \in B \) and \( f \in k[G]^H \). By (3), it follows that \( f \in \overline{[k[G]^H]} = \overline{[k[M]^H]} \). Hence, there exist \( g, h \in k[M]^H \) such that \( f = \frac{g}{h} \). It follows that \( f \cdot x = \frac{g \cdot x}{h} = \frac{g}{h} \).
In order to prove the last equality, we first observe that \( N = \mathcal{V}(\{ x \mapsto (f \cdot x)(m) - f(m), \ m \in M, \ f \in k[M]^H \}) \) is a closed submonoid of \( M \). Since \( G \) is open in \( M \), it follows that 

\[
H = \overline{H} = \overline{H} \cap G = N^0.
\]

\[\square\]

**Theorem 3.4.** Let \( M \) be an irreducible affine algebraic monoid with unit group \( G \) and let \( H \subset G \) be a subgroup, closed in \( M \). Then the following conditions are equivalent.

1. The subgroup \( H \) is observable in \( M \).
2. \( E_M(H) \) is a group. That is, for every \( \rho \in E_M(H) \), \( \rho^{-1} \in E_M(H) \).
3. \( E_M(H) = \mathcal{X}(H) \), i.e. every rational character is extendible.
4. For every finite dimensional rational right \( H \)-module \( V \) there exists a finite dimensional rational right \( M \)-module \( W \) and an injective morphism of \( H \)-modules \( \xi : V \to W |_H \).
5. \( H \) is observable in \( G \) and for any (some) determinant \( \det : M \to k \), then \( \frac{1}{\det |_H} \in E_M(H) \).

**Proof.** In order to prove that (1) implies (2), assume that \( H \) is observable in \( M \), and let \( \chi \in E_M(H) \) be an extendible character. Let \( g \) be an extension of \( \chi \). Then the ideal \( gk[M] \) is a nonzero \( H \)-stable ideal and hence there exists \( f \in k[M] \) such that \( (gf) \cdot x = gf \) for all \( x \in H \). It follows that \( f \cdot x = \chi^{-1}(x)f \) for all \( x \in H \). That is, \( \chi^{-1} \in E_M(H) \).

Since \( E_M(H) \) generates \( \mathcal{X}(H) \) as a group, it is clear that conditions (2) and (3) are equivalent.

To prove that (2) implies (1), assume that \( E_M(H) \) is a group and let \( I \subset k[M] \) be a non-zero \( H \)-stable ideal. Let \( H_{\text{uni}} = R_\alpha(H)[H,H] \), as in the proof of Theorem 2.12. Then \( H_{\text{uni}} \) is observable in \( H \). Consider a determinant \( \det : M \to k \). Since \( \mathcal{X}(H_{\text{uni}}) = 1 \), it follows that \( H_{\text{uni}} \subset \det^{-1}(1) \). Thus, we have found an \( H_{\text{uni}} \)-invariant regular function \( \det \in k[M]^{H_{\text{uni}}} \), such that \( k[G] = k[M]_{\det} \). It follows from Theorem 2.16 that \( H_{\text{uni}} \) is observable in \( M \), and thus the right \( H \)-module \( I^{H_{\text{uni}}} \) is not trivial. Then, as in the proof of Theorem 2.12, it follows that \( I^{H_{\text{uni}}} \) is a right \( (H_{\text{uni}} \setminus H) \)-module, and hence

\[
(0) \neq I^{H_{\text{uni}}} = \bigoplus_{\chi \in \mathcal{X}(H_{\text{uni}} \setminus H)} I^{H_{\text{uni}}} = \bigoplus_{\chi \in \mathcal{X}(H)} I^{H_{\text{uni}}}.
\]

Hence, there exists \( \chi \in \mathcal{X}(H) \) and \( f \in I \setminus \{0\} \) such that \( f \cdot x = \chi(x)f \) for all \( x \in H \). Thus \( \chi \in E_M(H) \). Since \( E_M(H) \) is a group, it follows that

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\[ \chi^{-1} \in E_M(H). \] Thus, there exists \( g \in k[G] \) such that \( g \cdot x = \chi^{-1}(x)g \) for all \( x \in H \). It follows that \( 0 \neq fg \in I^H \).

To prove that (2) implies (4), we follow the idea of the proof of [4, Theorem 10.2.9]. If \( V \) is a finite dimensional rational right \( H \)-module, using Theorem 2.8 we deduce the existence of an extendible character \( \rho \), a finite dimensional rational right \( M \)-module \( W \) and an injective map \( \iota : V \to W \) such that \( \iota(m \cdot x) = \rho^{-1}(x) \iota(m) \cdot x \) for all \( x \in H \). By hypothesis the character \( \rho^{-1} \) is extendible, so if we take \( f \in k[M] \) that extends \( \rho^{-1} \) and call \( W_0 \) the right \( M \)-submodule of \( k[M] \) generated by \( f \), we can define an injective map \( \xi : V \to W \otimes W_0 \), given as \( \xi(m) = \iota(m) \otimes f \). If we endow \( W \otimes W_0 \) with the diagonal right \( M \)-module structure, the following computation shows that \( \xi \) is \( H \)-equivariant. Let \( x \in H \). Then

\[
\xi(m \cdot x) = \iota(m \cdot x) \otimes f = \rho^{-1}(x) (\iota(m) \cdot x) \otimes f = (\iota(m) \cdot x) \otimes \rho^{-1}(x)f = (\iota(m) \cdot x) \otimes (f \cdot x) = \xi(m) \cdot x.
\]

We now prove that (4) implies (3). Let \( \gamma \in \mathcal{X}(H) \) be a rational character of \( H \) and consider the rational \( H \)-module \( V = k\gamma \). If \( \xi \) and \( W \) are as in condition (4) we conclude, using Lemma 2.9 that \( \gamma \) is extendible.

Assume now that (1) holds. Then by Theorem 3.3 \( H \) is observable in \( G \), and it follows from (2) that \( \frac{1}{\operatorname{det}} \in E_M(H) \). Hence, (5) holds.

Finally, if (5) holds, then by Theorem 2.12, \( \mathcal{X}(H) = E_G(H) \). Let \( \operatorname{det} \in \mathcal{X}(M) \) be a determinant such that \( \frac{1}{\operatorname{det}} \vert_H \in E_M(H) \), and let \( f \in k[M] \) be an extension of \( \frac{1}{\operatorname{det}} \vert_H \). If \( \chi \in \mathcal{X}(H) \), let \( g \in k[G] \) be an extension of \( \chi \). Then there exists \( l \in k[M] \) and \( n \geq 0 \) such that \( g = \frac{l}{\operatorname{det}^n} \). Therefore, for every \( x \in H \) and \( a \in G \), we have that

\[
\chi(x) \frac{l(a)}{\operatorname{det}^n(a)} = \chi(x)g(a) = (g \cdot x)(a) = \frac{(l \cdot x)(a)}{\operatorname{det}^n(x) \operatorname{det}^n(a)}.
\]

It follows that \( l \cdot a = \chi(x) \operatorname{det}^n(x)l \); that is \( l \in k[M] \) is an extension of \( \chi \operatorname{det}^n \in \mathcal{X}(H) \). Then \( lI \in k[M] \) is an extension of \( \chi \).

\[\square\]

**Theorem 3.5.** Let \( M \) be an irreducible affine algebraic monoid with unit group \( G \) and let \( H \subset G \) be a subgroup, closed in \( M \). Then the following conditions are equivalent.

(1) The subgroup \( H \) is observable in \( M \).

(2) There exists a finite dimensional rational right \( M \)-module \( V \) and \( v \in V \) such that \( G_v = H \).

In particular, if condition (2) holds, then \( G_v = M_v \) is closed in \( M \).
and consider the stabilizer submodule generated by \( H \) on \( M \) and hence the action by right multiplication of \( H \) on \( M \) and hence induces an inclusion \( \iota \) such that \( \{ \frac{w_0}{u_0}, \ldots, \frac{w_n}{u_n} \} \) generates \( k[M] \) over \( k \). Let \( W \subset k[M] \) be the finite dimensional rational right \( M \)-submodule generated by \( u_0, \ldots, u_n \). Let \( V = \bigoplus_{i=0}^n W_i \). We conclude that \( f \cdot a = f \) for all \( f \in k[M] \) and hence, \( a \in H \), as follows by example from [4, Corollary 7.3.4].

To prove that (2) implies (1), let \( v, V \) be as in (2) and consider the following commutative diagram

\[
\begin{array}{ccc}
M & \rightarrow & z \cdot M \\
\downarrow & & \downarrow \\
G & \rightarrow & v \cdot G
\end{array}
\]

Since \( G/H \) is quasi-affine, it follows that \( H \) is observable in \( G \), and hence \( [k[G]^H] = [k[G]]^H \). Thus,

\[
[k[v \cdot M]] = k(v \cdot G) \cong [k[G]^H] = [k[G]]^H.
\]

On the other hand, the orbit morphism \( M \rightarrow v \cdot M \) is dominant, and hence induces an inclusion \( \iota : k[v \cdot M] \hookrightarrow k[M] \). Since \( G_v = H \), it follows that \( f(v \cdot (hm)) = f(v \cdot m) \) and hence \( \iota(k[v \cdot M]) \subset k[M]^H \). Thus,

\[
[k[M]^H] = [k[G]^H] = [k[v \cdot M]] \subset [k[M]^H] \subset [k[M]]^H,
\]

and hence the action by right multiplication of \( H \) on \( M \) is observable.

Finally, recall that observable subgroups are closed in \( M \). \( \square \)

**Remark 3.6** (Open question). It is clear that all the results of this section remain valid when considering right observability simply by adapting the statements and proofs in an obvious way.

If \( M = G(M) \) is an algebraic group, it is well known that \( H \subset G \) is (left) observable in \( G \) if and only if \( H \) is right observable. Indeed, in this case the antipode \( S : k[G] \rightarrow k[G], S(f)(x) = f(x^{-1}) \), induces an isomorphism
In the more general case of an algebraic monoid \( M \), this line of reasoning cannot be applied, since \( S(\mathbb{k}[M]) \) is not included in \( \mathbb{k}[M] \). This raises the following question.

**Q1** Let \( M \) be an algebraic monoid with unit group \( G \), and \( H \subset G \) a closed subgroup, (left) observable in \( M \). Is \( H \) right observable in \( M \)?

### 4. Quotients of monoids by normal subgroups

Let \( M \) be an algebraic monoid of unit group \( G \) and let \( H \subset G \) a closed subgroup, observable in \( M \). Our goal is to prove that if \( H \) is normal in \( G \), then the affinized quotient \( M/_{\text{aff}} H \) is an affine monoid with unit group \( G/H \). See Theorem 4.4. Before doing so, we present a general result about the affinized quotient of a monoid by an observable subgroup.

**Proposition 4.1.** Let \( M \) be an algebraic monoid with unit group \( G \) and let \( H \subset G \) be a closed subgroup, observable in \( M \). Assume that \( \mathbb{k}[M]^H \) is finitely generated. Then \( M/_{\text{aff}} H \) is an affine embedding of \( G/H \). That is, \( M/_{\text{aff}} H \) is an affine \( G \)-variety, with an open orbit isomorphic to \( G/H \).

**Proof.** Consider the following commutative diagram

\[
\begin{array}{ccc}
G & \longrightarrow & M \\
\downarrow & & \downarrow \pi \\
G/H & \xrightarrow{\varphi} & M/_{\text{aff}} H
\end{array}
\]

where the existence of \( \varphi \) follows from the universal property of the quotient. Since \( H \) is observable in \( M \), it follows from Theorem 3.3 that \( \mathbb{k}(M/_{\text{aff}} H) = [\mathbb{k}(M)^H] = [\mathbb{k}[M]]^H = [\mathbb{k}[G]]^H = [\mathbb{k}[G]^H] \). Hence, \( \varphi \) is a birational dominant \( G \)-morphism. Since \( G/H \) is a homogeneous \( G \)-space, it follows that \( \varphi \) is an open immersion. \( \square \)

**Proposition 4.2.** Let \( M \) be a reductive monoid with unit group \( G \), and let \( H \subset G \) be a normal subgroup, closed in \( M \). Then the categorical quotient \( M/\!/H \) is an affine algebraic monoid with unit group \( G/H \). Furthermore, \( \pi : M \to M/\!/H \) is a morphism of algebraic monoids.

**Proof.** First of all, we prove that \( H \) is observable in \( M \). Indeed, \( H \) is a normal subgroup of \( G \) and thus it is a reductive group. Since \( H \subset M \) is
closed, it follows that \(gH\) is closed in \(M\) for all \(g \in G\), and we deduce from Proposition 2.17 that \(H\) is observable in \(M\).

Let \((G \times G) \times \mathbb{k}[M] \to \mathbb{k}[M]\), \(((a, b) \cdot f)\) \((m) = f(a^{-1} mb)\) \(a, b \in G\), \(m \in M\), be the canonical action. If \(f \in \mathbb{k}[M]^H\) and \(c \in H\), then

\[
(c \cdot ((a, b) \cdot f))(m) = ((c, 1) \cdot ((a, b) \cdot f))(m) = f(a^{-1}c^{-1}mb) = \\
f(la^{-1}mb) = f(a^{-1}mb) = ((a, b) \cdot f)(m)
\]

where \(l \in H\) is such that \(a^{-1}c^{-1} = la^{-1}\). If follows that \(\mathbb{k}[M]^H\) is a \((G \times G)\)-submodule of \(\mathbb{k}[M]\). Let now \((c, d) \in H \times H\). Then \((1, d) \cdot f = f\). Indeed, if \(g \in G\), then \(gd = lg\) for some \(l \in H\), and hence \(f(gd) = f(lg) = f(g)\). In other words, \((1, d) \cdot f|_G = f|_G\), and thus \((1, d) \cdot f = f\). Hence, \(M//H\) is an affine \((G/H \times G/H)\)-variety and the coproduct \(m^* : \mathbb{k}[M] \to \mathbb{k}[M] \otimes \mathbb{k}[M]\) is such that

\[
m^*(\mathbb{k}[M]^H) \subset (\mathbb{k}[M]^H \otimes \mathbb{k}[G]) \cap (\mathbb{k}[G] \otimes \mathbb{k}[M]^H) = \mathbb{k}[M]^H \otimes \mathbb{k}[M]^H.
\]

Moreover, since \(\mathbb{k}[M]^H \subset \mathbb{k}[G]^H = \mathbb{k}[G/H]\), it follows that \(M//H\) is an algebraic monoid and that we have a commutative diagram

\[
\begin{array}{ccc}
G \rightarrow & M \\
\downarrow & \downarrow \pi \\
G/H & \varphi \rightarrow M//H
\end{array}
\]

where the existence of \(\varphi\) follows from the universal property of the quotient. Then \(\varphi\) is a dominant morphism of algebraic monoids, and hence \(G(M//H) = \varphi(G/H)\). Since \(H\) is observable in \(M\) it follows that

\[
\mathbb{k}(G/H) = [\mathbb{k}[G]^H] = [\mathbb{k}[M]^H] = [\mathbb{k}[M]^H] = \mathbb{k}(M//H).
\]

Thus, \(\varphi : G/H \to M//H\) is an injective birational \(G\)-morphism, and it follows that \(\varphi : G/H \to G(M//H)\) is an open immersion. Hence, \(\varphi\) is an isomorphism. \(\square\)

The following result (with the exception of the last assertion) follows from [8, Theorem 2.5] and [9, Theorem 6.1].

**Theorem 4.3.** Let \(M\) be an affine algebraic monoid with unit group \(G\). Then \(\mathbb{k}[M]^{R_u(G)}\) is a finitely generated algebra. Moreover, \(M/_{\text{aff}} R_u(G) = \text{Spec}(\mathbb{k}[M]^{R_u(G)})\) is an affine algebraic monoid with unit group \(G/R_u(G)\). The morphism \(\pi : M \to M/_{\text{aff}} R_u(G)\) is a surjective morphism of algebraic
monoids, and satisfies the following universal property. For any morphism $f : M \to N$, of algebraic monoids, such that $f(R_u(G)) = \{1_N\}$ there exists a unique morphism $\tilde{f} : M/\text{aff} \to N$ such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{\pi} & & \downarrow{\tilde{f}} \\
M/\text{aff} & \xrightarrow{\pi} & N
\end{array}
\]

commutes. Moreover, the subgroup $H \subset G$ is closed in $M$ if and only if $\pi(H) = (HR_u(G))/R_u(G) \subset G/R_u(G)$ is closed in $M/\text{aff} R_u(G)$.

**Proof.** We provide a proof of the last assertion, since it is not proved in [8] or [9].

It follows from [7, Theorem 3.18] that $H = \overline{H}$ if and only if $E(\overline{H}) = \{1\}$. But by [7, Corollary 6.10], $E(\overline{H}) = \bigcup_{a \in H} aE(S)a^{-1}$, where $S \subset H$ is a maximal tours of $H$. Hence, $H = \overline{H} \subset M$ if and only if $S = \overline{S} \subset M$ for any (some) maximal torus $S \subset H$.

By [8, Theorem 2.5], it follows that, if $T \subset G$ is a maximal torus of $G$, then $\pi|_T : T \to \overline{\pi(T)} \subset M/\text{aff} R_u(G)$ is an isomorphism. Let now $S \subset H$ be a maximal torus of $H$ and $T \subset G$ a maximal torus of $G$ such that $S \subset T$. Then $\pi|_S : S \to \overline{\pi(S)} \subset \overline{\pi(T)}$ is an isomorphism. In particular, $S = \overline{S} \subset M$ if and only if $\pi(S) = \overline{\pi(S)} \subset M/\text{aff} R_u(G)$. Since $\pi(S) \subset \overline{\pi(H)}$ is a maximal tours, it follows that $H = \overline{H} \subset M$ if and only if $\pi(H) = \overline{\pi(H)} \subset M/\text{aff} R_u(G)$.

We conclude the proof by showing that the affinized quotient $\pi : M \to M/\text{aff} R_u(G)$ is a surjective morphism of algebraic monoids (this fact is implicit in [8,9] but was not stated or proved). By construction, $\pi$ is a morphism of algebraic monoids. Since $M/\text{aff} R_u(G)$ is a reductive monoid, it follows that

$$M/\text{aff} R_u(G) = G/R_u(G)E(M/\text{aff} R_u(G))G/R_u(G),$$

see for example [8, Theorem 4.2]. The surjectivity of $\pi$ follows now from the fact that $\pi(E(M)) = E(M/\text{aff} R_u(G))$. □

**Theorem 4.4.** Let $M$ be an algebraic monoid with unit group $G$, and let $H \subset G$ be a normal subgroup, closed in $M$. Then the action $H \times M \to M$ is observable. Moreover, if $k[M]^H$ is finitely generated, then the affinized quotient $M/\text{aff} H$ is an affine algebraic monoid, with unit group $G/H$.

**Proof.** Let $\pi : M \to M/\text{aff} R_u(G)$ be the affinized quotient as in Theorem 4.3. By [ibid] $M/\text{aff} R_u(G)$ is a reductive algebraic monoid with unit group
We use $\pi$ to help us find a determinant function on $M$ (the function $\mu$ below) with suitable properties. Consider the normal subgroup $\pi(H) = \pi(H R_u(G)) \subset G/R_u(G)$. Then $\pi(H)$ is closed in $M/\text{aff } R_u(G)$ and hence by Proposition 4.2 it follows that $N = (M/\text{aff } R_u(G))/\pi(H)$ is an affine algebraic monoid. Let $\rho : M/\text{aff } R_u(G) \to N$ be the affinized quotient and $\chi : N \to k$ a character such that $\chi^{-1}(0) = N \setminus G(N) = (G/R_u(G))/\pi(H) \cong G/H R_u(G)$. Then $\mu : \chi \circ \rho \circ \pi : M \to k$ is a character such that $\mu(H) = 1$ and $\mu^{-1}(0) = M \setminus G$. In particular $\mu \in k[M]^H$. Since $H$ is normal in $G H$, the action of $H$ on $G$ is observable in $G$. In other words, the action of $H$ on $G = M_\mu$ is observable. It follows from Theorem 2.16 that the action of $H$ on $M$ is observable.

To prove the last assertion we use the same arguments as in the proof of Proposition 4.2. Since $m^*(k[M]^H) \subset k[M]^H \otimes k[M]^H$, it follows that $M/\text{aff } H$ is an algebraic monoid and that we have a commutative diagram

\[
\begin{array}{ccc}
G & \rightarrow & M \\
\downarrow \pi & & \downarrow \\
G/H & \varphi \rightarrow & M/\text{aff } H
\end{array}
\]

where the existence of $\varphi$ follows from the universal property of the quotient. Then $\varphi$ is a dominant morphism of algebraic monoids, and hence $G(M/\text{aff } H) = \varphi(G/H)$. Since the action of $H$ on $M$ is observable, it follows from Theorem 3.3 that $[k[M]^H] = [k[M]^H]^H$, and hence $\varphi$ is a birational morphism. Since $\rho$ is $G$-equivariant, to follows that $\varphi$ is an open immersion, and thus $G/H \cong G(M/\text{aff } H)$. \hfill $\Box$

If $H$ is normal in $G(M)$ then the affinized quotient $M/\text{aff } H$ satisfies a universal property in the category of algebraic monoids.

**Proposition 4.5.** Let $N$ be an algebraic monoid and $f : M \to N$ a morphism of algebraic monoids such that $f(H) = \{1_N\}$. Then there exists a morphism of algebraic monoids $\tilde{f} : M/\text{aff } H \to N$ such that the following diagram commutes.

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow \pi & & \downarrow \\
M/\text{aff } H & \xrightarrow{\tilde{f}} & N
\end{array}
\]

**Proof.** Let
be the Chevalley decomposition of \( N \), where \( \alpha \) is the Albanese morphism (see \[2\], Theorem 3.2.1) and \[3\], Theorem 1.1). Since \( M \) is affine, it follows that form the property of the Albanese morphism that \( f(M) \subset L \) (see for example \[3\], Theorem 5.1). Since \( f \) is a morphism of algebraic monoids, such that \( f^* : k[L] \rightarrow k[M] \) factors through \( k[M]^H \). In other words, there exists \( \tilde{f} : M/_{aff} H \rightarrow L \) such that \( f = \tilde{f} \circ \pi \). \( \square \)

**Example 4.6.** Let

\[
H = \left\{ \begin{pmatrix} a^2 & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in k^* \right\} \subset \text{GL}_2(k) \subset M_{2 \times 2}(k) = M.
\]

Then \( H \) is observable in \( \text{GL}_2(k) = G(M) \) and closed in \( M \), but there exists no character \( \chi : M \rightarrow k \) such that \( H \subset \chi^{-1}(1) \) and \( \chi^{-1}(0) = M \setminus G(M) \). Since \( H \) is closed in \( G \) and since \( \ell_g : M \rightarrow M, \ell_g(m) = g \cdot m \), is an isomorphism, it follows that \( gH \subset M \) is closed in \( M \) for all \( g \in \text{GL}_2(k) \). Since \( H \) is reductive, it follows from Proposition 2.17 that the action of \( H \) on \( M \) is observable.

**Remark 4.7 (Open question).** Example 4.6 shows that the condition of \( H \subset G \) being a normal subgroup is crucial in Theorem 4.4. However, in that example the action of \( H \) on \( M \) is observable. This raises the following question.

**Q2** Let \( M \) be an algebraic monoid with unit group \( G \), and let \( H \subset G \) an observable group, closed in \( M \). Is the action \( H \times M \rightarrow M \) observable?

**Remark 4.8.** If Q2 has a positive answer, then Q1 (see Remark 3.6) has a positive answer. Indeed, assume that Q2 has a positive answer and let \( H \subset M \) be a left observable subgroup. Then by Theorem 3.3 it follows that \( H \) is observable in \( G \), and hence \( H \) is right observable in \( G \). If then follows from Q2 that \( H \) is right observable in \( M \).

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