THE SCALING LIMIT OF FAIR PEANO CURVES

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ABSTRACT. We study random Peano curves on planar square grids that arise from fair random spanning trees. These are trees that are sampled in such a way as to have the same (if possible) edge probabilities. In particular, we are interested in identifying the scaling limit as the mesh-size of the grid tends to zero. It is known [4] that if the trees are sampled uniformly, then the scaling limit exists and equals SLE$_8$. We show that if we simply follow the same steps as in [4], then fair Peano curves have a deterministic scaling limit.

1. INTRODUCTION

It was shown in [4] that the scaling limit of random Peano curves arising from uniform spanning trees of planar grids exists and is a known Schramm-Loewner evolution stochastic process called SLE$_8$. In this paper, we study random Peano curves which are generated by random spanning trees that are not necessarily uniform. In particular, we are interested in studying the limiting behavior of laws on spanning trees that arise in the context of spanning tree modulus. The latter random trees are called fair trees, because rather than having the same probability of being sampled, they are sampled in such a way as to yield the same (if possible) edge probabilities. In particular, we will be drawing on the two papers [1,2] that initiated the study of fair trees. Fair Peano curves are random Peano curves arising from fair trees. The following is our main result. For a more specific formulation, see Theorem 3.1.

Theorem 1.1. If we simply follow the same scaling-limit construction as in [4], then the resulting fair Peano curves have a deterministic scaling limit.

This is in part due to the fact that in this case the number of fair trees is much smaller than the number of uniform trees.

In Section 2, we summarize the background regarding fair trees that is relevant for our purposes. In Section 3, we formulate and prove Theorem 3.1 (and hence Theorem 1.1). Finally in Section 4 we identify the fair trees in the case of both the modified and standard grids, and hence we give evidence supporting an alternate approach.

2. BACKGROUND ON FAIR TREES AND MODIFIED GRIDS

2.1. Random spanning trees. Consider a connected graph $G = (V, E)$, with vertex-set $V$ and edge-set $E$ that are both finite. We allow multi-edges, but not self-loops. Let $\Gamma_G$ be the set of all spanning trees of $G$. The set of all probability mass functions (pmf’s) $\mu$ on $\Gamma_G$ is denoted with $\mathcal{P}(\Gamma_G)$. A pmf $\mu \in \mathcal{P}(\Gamma_G)$ determines random trees $\gamma$ that satisfy the law:

$$\Pr_\mu (\gamma = \gamma) = \mu(\gamma) \quad \forall \gamma \in \Gamma_G.$$

Key words and phrases. fair trees, uniform spanning trees, random Peano curves, Schramm-Loewner evolution.

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The $\sigma$-weighted uniform spanning trees, write $\sigma$-WUST for short, are random spanning trees obtained by considering some fixed edge-weights $\sigma \in \mathbb{R}_{>0}^E$. These are random trees $\gamma \in \Gamma_G$ whose probability $\mu_\sigma(\gamma)$ is proportional to the product:

$$\prod_{e \in \gamma} \sigma(e).$$

If the edge weights $\sigma$ are constant, then we recover uniform spanning trees and write UST. It is well known, see [5, Chapter 4], that $\sigma$-WUST can be sampled using weighted random walks as in the Aldous-Broder algorithm, or loop-erased weighted random walks as in Wilson’s algorithm. Moreover, a celebrated result of Kirchhoff states that the edge-probabilities for $\sigma$-WUST are related to the effective resistance $\text{eff}\mathcal{R}_\sigma$ for the associated electrical network with edge-conductances given by $\sigma$. More precisely, if $e \in E$, then

$$(2.1) \quad \mathbb{P}_{\mu_\sigma} \left( e \in \gamma \right) = \sigma(e) \text{eff}\mathcal{R}_\sigma(e).$$

2.2. Fair edge usage. As we have seen, the edge-probabilities of UST coincide with the effective resistance of each edge and can therefore vary from edge to edge. Instead of requiring that each spanning tree be equally likely as in the case of UST, we ask for laws that sample spanning trees so as to equalize the edge-probabilities as much as possible. Ideally, we would like all the edge-probabilities to be the same, but this is not always possible. When this happens we say the graph is homogenous. More precisely, the graph $G$ is said to be homogeneous, if it admits a pmf $\mu^* \in \mathcal{P}(\Gamma_G)$ with constant edge probabilities, in which case by [1, Corollary 4.5], the edge-probabilities are uniquely determined as

$$\mathbb{P}_{\mu^*} \left( e \in \gamma \right) \equiv \frac{|V| - 1}{|E|}, \quad \forall e \in E. \quad (2.2)$$

In general, the Fairest Edge Usage (FEU) problem, introduced in [1], tries to minimize the variance of the edge-probabilities:

$$\text{minimize}_{\mu \in \mathcal{P}(\Gamma_G)} \quad \text{Var}(\eta)$$

subject to $\eta(e) = \mathbb{P}_\mu \left( e \in \gamma \right), \quad \forall e \in E. \quad (2.3)$

A random tree $\gamma$ with law $\mu$ (and also the law $\mu$ itself) is fair, if $\mu$ is optimal for the FEU problem (2.3). In [1] it was shown that such fair pmf’s for FEU always exist, but they are in general not unique. On the other hand, the edge probabilities $\eta^*(e) \equiv \mathbb{P}_{\mu^*} \left( e \in \gamma \right)$ corresponding to a fair pmf $\mu^*$, are uniquely determined. In fact, the optimal $\eta^*$ in (2.3) can be computed using the spanning tree modulus algorithms and is related to the family of feasible partitions via Fulkerson duality. More details about these connections can be found in [1].

We write $\Gamma^f_G$ for the set of all fair trees. Trees in $\Gamma_G \setminus \Gamma^f_G$ are called forbidden. In particular, if a $\sigma$-WUST pmf $\mu_\sigma$ is fair, then $G$ does not have any forbidden trees, because $\mu_\sigma$ is supported on all of $\Gamma_G$ by definition.

**Definition 2.1.** A graph $H$ has the all fair property if every spanning tree of $H$ is fair: $\Gamma_H = \Gamma^f_H$. 
2.3. Planar modified grids. The scaling limit for UST was established in [4]. The idea there is to consider a modified grid, see Figure 1, so that each spanning tree gives rise to a normalized (discrete) Peano curve. The authors in [4] show that as the mesh size of the grid goes to zero, these random Peano curves converge to a known stochastic process called $SLE_8$.

A *standard $m$-by-$n$ planar grid* is a squared-grid with vertex set $(\mathbb{Z} \times \mathbb{Z}) \cap ([0,m] \times [0,n])$. A modified grid $G$ consists of a standard $m$-by-$n$ grid $G_0$, together with an extra node $v_0$ connected via an edge to each node on the bottom and the right hand-side of $G_0$. In particular, the bottom-right corner of $G_0$ is connected by two edges to $v_0$. Alternatively, we can think of $G$ as the graph obtained from an $(m+1)$-by-$(n+1)$ grid by identifying every node on the bottom and right hand-sides and removing all the resulting self-loops.

![Figure 1. Modified Grids](image)

We associate to every vertex $v \in V(G_0) = V(G) \setminus \{v_0\}$ in the grid an edge set $E_v \subset E(G)$, consisting of the edge connecting $v$ to its neighbor on the right and the one below $v$. Note that if $v$ belongs to the bottom or the right-side edge of the grid $G_0$, then the neighbor to the right, or the one below $v$, may have to be the special node $v_0$. Then, the family of sets $\{E_v\}_{v \in G_0}$ forms a partition of $E$:

$$E(G) = \bigcup_{v \in V(G) \setminus \{v_0\}} E_v \quad \text{and} \quad E_v \cap E_{v'} = \emptyset, \quad \text{for} \ v \neq v'.$$

Consider the collection of all subgraphs of $G$ that use exactly one edge from each $E_v$:

$$\bar{\Gamma} := \{ T \subset E : |T \cap E_v| = 1, \ \text{for all} \ v \in V(G_0) \}.$$

For instance, the tree on the left in Figure 2 is in $\bar{\Gamma}$, while the tree on the right is not, because the latter tree connects the central node of the grid to both its neighbor on the right and its neighbor below.

![Figure 2. Fair tree on the left and Forbidden tree on the right](image)

Suppose we are given a set $T \in \bar{\Gamma}$. Then, starting from any node $v \in V(G_0)$ one can follow an edge of $T$ and either move to the right or down and eventually reach $v_0$. Thus, every $T \in \bar{\Gamma}$ is connected and spans $V(G)$. Moreover, any such $T$ satisfies

$$|E(T)| = \sum_{v \in V(G_0) \cap T} |E_v\cap T| = \sum_{v \in V(G_0)} 1 = |V(T)| - 1.$$
Therefore, every such $T$ is a spanning tree of $G$, namely $\bar{\Gamma} \subset \Gamma$.

Let $\bar{\mu}$ be the uniform pmf conditioned on $\bar{\Gamma}$:

$$\bar{\mu}(T) := \begin{cases} \frac{1}{|\bar{\Gamma}|} & \text{if } T \in \bar{\Gamma} \\ 0 & \text{otherwise.} \end{cases}$$

A random tree in $\bar{\Gamma}$ with law $\bar{\mu}$ can be sampled by tossing independent fair coins, one for each node $v \in V(G_0)$, and picking either the edge to the right of $v$ or below $v$. In particular, notice that every tree $T \in \bar{\Gamma}$ has a partner tree, i.e., the tree that can be obtained by substituting every edge in $T \cap E_v$ with the other edge in $E_v \setminus T$.

**Lemma 2.2.** The family $\bar{\Gamma}$ is the family of all fair trees for the modified grid, i.e., $\Gamma^f_G = \bar{\Gamma}$. Also, $\bar{\mu}$ is optimal for the FEU problem on $G$.

Moreover, $\bar{\Gamma}$ consists exactly of all the trees $\gamma \in \Gamma_G$ that have the partner tree property, i.e., the trees $\gamma \in \Gamma_G$ such that the complement $E \setminus \gamma$ is also a tree.

For the moment, we will assume Lemma 2.2 and defer its proof to Section 4.5.

### 3. The scaling limit of fair Peano curves

#### 3.1. Scaling limit of fair Peano curves in a square.

We are interested in fair trees on planar grids whose mesh is tending to zero. Let $Q = (0, 1] \times [0, 1)$ be the unit square (including the bottom and right sides). Consider the lattice $L_n = \left[ \frac{1}{n} \mathbb{Z} \times \frac{1}{n} \mathbb{Z} \right] \cap Q$. Once we identify the bottom and right edges of $L_n$, we obtain a modified grid, as in Figure 1. We let $\Gamma_n$ be the set of fair trees on this grid. We can also think of $\Gamma_n$ as the collection of spanning trees $T$ of $L_n$ that are required to include the bottom and right side of $Q$ and that satisfy $|T \cap E_v| = 1$, as in (2.4), for every node $v$ in the interior of $Q$.

Let $\left( \left( \frac{1}{2n} + \frac{1}{n} \mathbb{Z} \right) \times \left( \frac{1}{2n} + \frac{1}{n} \mathbb{Z} \right) \right) \cap Q$ be the dual grid to $L_n$. For each element $\gamma \in \Gamma_n$, there is a unique spanning tree $\hat{\gamma}$ of the dual grid so that $\gamma \cap \hat{\gamma} = \emptyset$, as illustrated in Figure 3. Thus given $\gamma \in \Gamma_n$, we can define a curve $\zeta = \zeta(\gamma)$ in $Q$ from 0 to $1 + i$ that winds between $\gamma$ and $\hat{\gamma}$. Except for near 0 and $1 + i$, we take $\zeta$ to be on the lattice $\left( \frac{1}{4n} + \frac{1}{2n} \mathbb{Z} \right) \times \left( \frac{1}{4n} + \frac{1}{2n} \mathbb{Z} \right)$. We call $\zeta$ the Peano curve associated with $\gamma$, and we define $A_n = \{ \zeta(\gamma) : \gamma \in \Gamma_n \}$ to be the collection of all such Peano curves.

Putting the uniform pmf on $\Gamma_n$ corresponds to the uniform pmf on $A_n$, and we consider elements of $A_n$ as random Peano curves. We are interested in the limit of these random curves as $n \to \infty$. Since $\zeta_n \in A_n$ passes within distance $\frac{1}{4n}$ of every vertex in the lattice $L_n$, ...
Figure 4. The Hilbert spacefilling curve is a continuous function $h$ from $[0, 1]$ onto the closed unit square. It is defined as the limit of approximating curves, two of which are shown (left and middle). The right image shows $h[0, 3/4]$ in black and $h[3/4, 3/4 + \epsilon]$ for $\epsilon = 3/64$ in grey.

the limit of the path traced out by the full curve $\zeta_n$ will converge to $\overline{Q}$. We are interested in whether $\zeta_n$ converges as a curve to a limit spacefilling curve or not, and whether the limit is a random or deterministic object.

Theorem 3.1 below will answer both of these questions, but let us pause for a moment to discuss the notion of spacefilling curves. A spacefilling curve in $\mathbb{Q}$ is a continuous function $\phi : [0, T] \to \mathbb{Q}$ so that $\phi[0, T] = \mathbb{Q}$. Since the image of the path fills up the entire square, a spacefilling curve is best visualized by an approximation. For an example, see Figure 4 which shows two curves that approximate the Hilbert spacefilling curve. We can view a spacefilling curve as a way to grow $\mathbb{Q}$ in a continuous fashion. The continuity implies that we can always make the diameter of $\phi[t_0, t_0 + \epsilon]$ small by taking $\epsilon$ small enough.

In Theorem 3.1, we will see that the limit of the random Peano curves $\zeta_n$ exists and is deterministic. Further, this limit gives us a way to grow $\overline{Q}$ in a continuous fashion. The continuity implies that we can always make the diameter of $\phi[t_0, t_0 + \epsilon]$ small by taking $\epsilon$ small enough.

Theorem 3.1. When $n \to \infty$, the fair Peano curve $\zeta_n[0, t_b]$ converges to $Q_b$ (in the Hausdorff metric) for all $b \in [0, 2]$ almost surely.

In order to prove this, we first prove a result which roughly says that the branches of the random trees $\gamma \in \Gamma_n$ become more and more “diagonal” as $n$ increases. Let $\gamma \in \Gamma_n$ and let $m \in \mathbb{N}$. Let $z_1, \ldots, z_m \in \mathbb{Q} \cap \{x + iy : x = 1/n \text{ or } y = 1 - 1/n\}$ be spaced out as evenly as possible along the left and top vertices of the lattice $\mathcal{L}_n$. We assume these points are ordered clockwise. See Figure 5. We define the random set $\beta_n(m) \subset \gamma$ to be the the union of paths that start at $z_j$ for $j \in \{1, 2, \cdots m\}$ and take steps either to the right or down following $\gamma$ until reaching the bottom or right side of $Q$. Let $L_n(m)$ be the union of the line segments $l_j = \{x + iy \in \overline{Q} : x + y = \text{Re}(z_j) + \text{Im}(z_j)\}$ for $j \in \{1, 2, \cdots m\}$.

If $m$ is not an integer, we define $\beta_n(m)$ and $L_n(m)$ using the integer part of $m$.

Lemma 3.2. Let $\beta_n$ and $L_n$ be defined as above. Then a.s. there exists $N = N(\omega)$ so that for $n > N$,

$$d_H(\beta_n(n^{1/4}), L_n(n^{1/4})) \leq \frac{1}{2} n^{-1/4},$$

where $d_H$ refers to Hausdorff distance.
Let us first consider the case when \( m = 1 \). Then \( \beta_n(1) \) is a path that starts at \( w = 1/n + (1 - 1/n)i \) and moves down or to the left with probability 1/2 at each step, until reaching the boundary of \( Q \). Applying the transformation \( \phi(z) = e^{\frac{\pi}{4}i(z - w)} \) to \( \beta_n(1) \) yields the graph of a 1-dimensional simple random walk which has step size \( \sqrt{\frac{2}{2n}} \) and at most 2n steps. Even when \( m \neq 1 \), we will still be able to relate \( \beta_n(m) \) to a simple random walk, and then the lemma will follow from a well-known result about the running maximum of a simple random walk.

**Proof of Lemma 3.2** We begin by relating \( \beta_n(m) \) to a simple random walk. To that end, let \( S_k \) be a 1-dimensional simple random walk with step size 1. We can create \( \beta_n(m) \) from \( S_k \) by chopping the walk into \( m \) pieces of length \( \sqrt{\frac{2}{2n}} \), and then scaling, rotating, shifting, and truncating these pieces. In particular, assume that we have created curves \( \alpha_1, \ldots, \alpha_{j-1} \) for \( j \leq m \). Then to create \( \alpha_j \) we start at \( z_j \) and follow the path of \( \sqrt{\frac{2}{2n}} e^{-\frac{\pi}{4}i(S_k - 2n(j-1) + z_j)} \) for \( k \geq 2n(j-1) \) until reaching the boundary of \( Q \) or until hitting \( \alpha_{j-1} \) (and in this latter case, we then follow \( \alpha_{j-1} \) to the boundary of \( Q \)). Then \( \beta_n(m) \) and \( \cup_{j=1}^m \alpha_j \) are equal in distribution. In what follows we will take \( m = n^{1/4} \).

Let \( M_k = \max\{|S_0|, |S_1|, \ldots, |S_k|\} \) be the running maximum of \( |S_k| \), which is known to satisfy

\[
\limsup_{k \to \infty} \frac{M_k}{\sqrt{2k \log \log k}} = 1 \text{ a.s..}
\]

This follows from the law of the iterated logarithm; see for instance Exercise 2 in Section 7.5 of [3]. Instead of \( M_k \), we are interested in \( \sqrt{\frac{2}{2n}} M_{2n^{5/4}} \), the running maximum of the random walk scaled by \( \sqrt{\frac{2}{2n}} \) and run until time \( 2nm = 2n^{5/4} \). Note that

\[
\frac{\sqrt{2}}{2n} M_{2n^{5/4}} = Cn^{-3/8} \sqrt{\log \log(2n^{5/4})} \cdot \frac{1}{\sqrt{4n^{5/4} \log \log(2n^{5/4})}}.
\]

Since the lim sup of the factor on the right is 1 a.s., then for almost every \( \omega \), we can take \( n \) large enough so that

\[
\frac{\sqrt{2}}{2n} M_{2n^{5/4}}(\omega) < \frac{1}{2}n^{-1/4}.
\]
Assume that we have an \( \omega \) such that (3.1) holds. Then when \( n \) is large enough, each path \( \alpha_j \) will be at most distance \( \frac{1}{2} n^{-1/4} \) from the corresponding line segment \( l_j \). In other words, the path \( \alpha_j \) must remain in a tube of diameter \( n^{-1/4} \) centered around \( l_j \). Since we have \( n^{1/4} \) of these tubes, (roughly) evenly spaced within a distance of \( \sqrt{2} \) along the diameter of \( Q \), we are guaranteed that none of the paths \( \alpha_j \) will intersect each other. In this case,

\[
d_H \left( \beta_n(n^{1/4}), L_n(n^{1/4}) \right) \leq \frac{\sqrt{2}}{2n} M_n^{5/4} < \frac{1}{2} n^{-1/4}
\]

and the lemma follows.

We are now ready for the proof of Theorem 3.1.

Proof of Theorem 3.1. Note that \( \zeta_n[0,t_b] \subset Q_b \). To prove the theorem, we will show that a.s. there exists \( N \) so that for all \( b \in [0,2] \), every point of \( Q_b \) is within distance \( 3n^{-1/4} \) of \( \zeta_n[0,t_b] \) when \( n > N \).

Consider the points \( z_1, \cdots, z_m \) associated with \( \beta_n(m) \) for \( m = \) the integer part of \( n^{1/4} \). Let \( j = j(b) \) be the largest index so that \( z_j \) is at least distance \( \frac{1}{2} n^{-1/4} \) below the line \( \{ x + iy \in \overline{Q} : x + y = b \} \). Note that since the \( n^{1/4} \) points \( z_1, \cdots, z_m \) are evenly spaced, \( z_j \) is at most distance \( 2n^{-1/4} \) from the line \( \{ x + iy \in \overline{Q} : x + y = b \} \). Let \( \alpha_j \) be the path that starts at \( z_j \) and follows \( \gamma \) down and to the right until reaching the boundary of \( Q \).

Note that when \( \zeta_n \) passes within \( \frac{1}{4n} \) of \( z_j \), it will not return to the region below \( \alpha_j \), and so it must have already passed within distance \( \frac{1}{4n} \) of every vertex of \( \mathcal{L}_n \) that is below \( \alpha_j \).
Applying Lemma 3.2, almost surely there exists $N$ so that for $n > N$, $\alpha_j$ is within distance $\frac{1}{2}n^{-1/4}$ of $l_j$. In this situation, when $\zeta_n$ passes within $\frac{1}{4n}$ of $z_j$, then $\zeta_n$ has not yet reached the line $\{x + iy \in \overline{Q} : x + y = b\}$ but $\zeta_n$ has passed within distance $\frac{1}{4n}$ of every vertex below a line that has a distance at most $\frac{5}{2}n^{-1/4}$ from $\{x + iy \in \overline{Q} : x + y = b\}$. □

See Figure 6 for a simulation of a fair tree with $n = 200$. Each edge in the tree has a unique closest vertex in the bottom/right side of $Q$. Edges that share the same closest vertex are colored the same color. This coloring highlights the diagonal nature of fair trees.

4. Fair trees for modified grids and for standard grids

In this section we give more details about the FEU problem and then prove Lemma 2.2. Also, we show that standard grids might be better suited for this kind of scaling limit.

4.1. The 1-densest subgraph. Define the measure of 1-density for a graph $G$ as:

$$\theta(G) := \frac{|E|}{|V| - 1}.$$  

Hence, by (2.2), when $G$ is homogeneous, the edge probabilities are all equal to $\theta(G)^{-1}$. Let $\mathcal{H}(G)$ be the family of all subgraphs $H \subset G$ such that $H$ is:

(i) non-trivial, meaning that it has at least one edge;
(ii) connected;
(iii) vertex-induced.

Note that we will often identify a subset of edges in $E(G)$ with the corresponding edge-induced subgraph of $G$. Consider the following 1-densest subgraph problem: Find $H \in \mathcal{H}(G)$ such that

$$\theta(H) = \max_{H' \in \mathcal{H}(G)} \theta(H').$$  

In [1], it was shown that the spanning tree modulus algorithm can provide a solution to this combinatorial problem. Namely, the modulus algorithm computes the optimal density $\eta^*$ in (2.3). Next, if $H$ is a connected component of the set of edges in $E$ where the minimum value of $\eta^*$ is attained, then $H \in \mathcal{H}(G)$, $H$ is homogeneous by construction and $H$ solves the 1-densest subgraph problem. Such $H$ is called a homogeneous core of $G$.

4.2. Reducibility and the deflation process. A proper subgraph $H \in \mathcal{H}(G)$ has the restriction property with respect to $G$, if for every fair tree $\gamma \in \Gamma^f_G$, the restriction $\gamma \cap H$ is a spanning tree in $\Gamma_H$.

Definition 4.1. The graph $G$ is said to be reducible, if it admits a proper subgraph $H \in \mathcal{H}(G)$ with the restriction property with respect to $G$. A graph that is not reducible is called irreducible.

When the homogeneous core $H$ defined in Section 4.1 is a proper subgraph of $G$ (and hence $G$ is inhomogeneous), then $H$ also has the restriction property, see [1, Theorem 5.2]. Moreover, by [1, Theorem 5.7], in this case, not only will the restriction of a fair tree $\gamma \in \Gamma^f_G$ to $H$ be a spanning tree of $H$, but it will also be a fair tree of $H$. Secondly, the forest $\gamma \setminus H$ is mapped to a fair tree of the quotient graph $G/H$ by the quotient map. And finally, given a fair tree for $H$ and a fair tree for $G/H$, they can be glued together to yield a fair tree of $G$. This is the essence of the deflation process, namely that arbitrary fair trees of $G$ can be
built from fair trees of homogeneous graphs. We summarize and rephrase the main results of [1] as follow.

**Theorem 4.2 ( [1] ).** Let \( G = (V, E) \) be a graph with multi-edges, but no loops. Then, either

(i) \( G \) is homogeneous, and \( \theta(H) \leq \theta(G) \) for all \( H \in \mathcal{H}(G) \); or

(ii) \( G \) is inhomogeneous, and hence admits a homogeneous core \( H_0 \in \mathcal{H}(G) \), \( H_0 \subsetneq G \), that solves the 1-densest subgraph problem (4.1) with \( \theta(H_0) > \theta(G) \), and that has the restriction property with respect to \( G \).

Moreover, for any fair tree \( \gamma \) of \( G \), the projection of \( \gamma \setminus H_0 \) onto \( G/H_0 \) is a fair spanning tree of \( G/H_0 \). And, fair trees of the core \( H_0 \) can be coupled with fair trees of the quotient graph \( G/H_0 \) to produce fair trees of \( G \).

4.3. **Minimal 1-densest subgraphs.** We say \( H \in \mathcal{H}(G) \) solves the minimal 1-densest subgraph problem, if \( H \) is a 1-densest subgraph and no proper subgraph of \( H \) is also a 1-densest subgraph. In other words, we look for a solution of the 1-densest subgraph problem that has the following property:

**Definition 4.3.** A graph \( H \) has the strict 1-density property, if

\[ \theta(H') < \theta(H) \]

for all proper subgraphs \( H' \in \mathcal{H}(H) \).

When \( H \in \mathcal{H}(G) \) solves the minimal 1-densest problem, we say that \( H \) is a minimal core for \( G \). In the next result, we show that the deflation process described in Theorem 4.2 can also be phrased in terms of minimal cores instead.

**Theorem 4.4.** Let \( G = (V, E) \) be a graph with multi-edges, but no loops. Then, either

(i) \( G \) has the strict 1-density property (and hence is homogeneous), or

(ii) \( G \) admits a minimal core, namely, a subgraph \( H_0 \in \mathcal{H}(G) \), \( H_0 \subsetneq G \), that is a minimal 1-densest subgraph. Also, in this case,

(a) \( H_0 \) has the restriction property with respect to \( G \);

(b) for any fair tree \( \gamma \) of \( G \), the restriction \( \gamma \cap H_0 \) onto \( H_0 \) is a fair spanning tree of \( H_0 \);

(c) for any fair tree \( \gamma \) of \( G \), the projection of \( \gamma \setminus H_0 \) onto \( G/H_0 \) is a fair spanning tree of \( G/H_0 \);

(d) moreover, fair trees of the core \( H_0 \) can be coupled with fair trees of the quotient graph \( G/H_0 \) to produce fair trees of \( G \).

Note that if \( G \) satisfies Theorem 4.4 (ii), then \( G \) may be homogeneous or not. Indeed, if \( G \) is homogeneous and not strictly 1-dense, then it must admit a proper subgraph with the same 1-density.

**Proof of Theorem 4.4.** First note that if \( G \) is inhomogeneous, then by [1, Theorem 5.2], \( G \) admits a homogeneous core \( H_0 \in \mathcal{H}(G) \), such that \( H_0 \subsetneq G \), and by [1, Theorem 5.8], \( \theta(H_0) > \theta(G) \), so \( G \) is not strictly 1-dense. Thus, if \( G \) is strictly 1-dense, then \( G \) must be homogeneous. This is the (i) alternative.

Now, assume \( G \) is not strictly 1-dense. Let \( H_0 \in \mathcal{H}(G) \) solve the minimal 1-densest subgraph problem for \( G \). Then, \( H_0 \) is a proper subgraph of \( G \) and \( \theta(H_0) \geq \theta(G) \). By minimality, \( H_0 \) has the strict 1-density property. Therefore, by Theorem 4.4 (i), \( H_0 \) is homogeneous.
Let $H$ be a homogeneous core of $G$, as described in Theorem 4.2, so that $H$ solves the 1-densest subgraph problem and satisfies $\theta(H)^{-1} = \min_{e \in G} \eta^*(e)$. Since both $H$ and $H_0$ solve the 1-densest subgraph problem, we have $\theta(H) = \theta(H_0)$.

To prove (a) in Theorem 4.4 (ii), assume that $H_0$ does not have the restriction property with respect to $G$. Then there is a fair tree $\gamma_0 \in \Gamma_{G}'$, such that $\gamma_0 \cap H_0$ is not a spanning tree of $H_0$. Since $\gamma_0 \cap H_0$ is acyclic, it must be a forest and we must have $|\gamma_0 \cap H_0| < |V(H_0)| - 1$. Also, since $\gamma_0$ is a fair tree of $G$, it must be in the support of some fair pmf $\mu^*$. For every other $\gamma'$ in the support of $\mu^*$, we also have that $\gamma' \cap H_0$ is acyclic, so we at least get the bound $|\gamma' \cap H_0| \leq |V(H_0)| - 1$. Therefore, we can write

$$\theta(H_0)^{-1} = \theta(H)^{-1} = \min_{e \in G} \eta^*(e) \leq \min_{e \in H_0} \eta^*(e)$$

$$\leq \frac{1}{|E(H_0)|} \sum_{e \in H_0} \eta^*(e) = \frac{1}{|E(H_0)|} \sum_{e \in H_0} \mathbb{P}_{\mu^*}(e \in \gamma)$$

$$= \frac{1}{|E(H_0)|} \sum_{e \in H_0} \sum_{\gamma \in \Gamma_G} \mu^*(\gamma) \mathcal{N}(\gamma, e) = \frac{1}{|E(H_0)|} \sum_{\gamma \in \Gamma_G} \mu^*(\gamma) \sum_{e \in H_0} \mathcal{N}(\gamma, e)$$

$$= \frac{1}{|E(H_0)|} \sum_{\gamma \in \Gamma_G} \mu^*(\gamma) |\gamma \cap H_0| < \frac{|V(H_0)| - 1}{|E(H_0)|} \sum_{\gamma \in \Gamma_G} \mu^*(\gamma) = \theta(H_0)^{-1}.$$

The last inequality is strict, because it is strict when $\gamma = \gamma_0$. This leads to a contradiction. Therefore $H_0$ has the restriction property with respect to $G$. This proves part (a) in Theorem 4.4 (ii). Moreover, the same calculation shows that equality must hold everywhere. In particular, the minimum of $\eta^*$ on $H_0$ must coincide with the the average of $\eta^*$ on $H_0$. Thus,

$$\eta^*(e) \equiv \theta(H_0)^{-1} \quad \forall e \in H_0.$$

Finally parts (b), (c), and (d) follow from part (a), because Theorem 5.5, Lemma 5.6 and Theorem 5.7, in [1] only depend on the fact that $H_0$ has the restriction property with respect to $G$. \hfill \square

4.4. Irreducibility, the all fair property, and uniformizability. Finally we make the following definition.

**Definition 4.5.** A graph $H$ is uniformizable, if there is a set of edge-weights $\sigma \in \mathbb{R}_{>0}^{E(H)}$ so that the corresponding $\sigma$-WUST pmf $\mu_\sigma$ is fair.

The results in [2] show that the fair trees of an arbitrary graph $G$ can be built from the fair trees of uniformizable graphs. In particular, one can use the Aldous-Broder or the Wilson algorithms to sample them. We summarize the results from [2] in the next theorem.

**Theorem 4.6 ([2]).** Let $G = (V, E)$ be a graph, possibly with multi-edges, but no loops. Assume $G$ is biconnected. Then, the following are equivalent:

(a) $G$ is uniformizable (see Definition 4.5);

(b) $G$ is irreducible (see Definition 4.1);

(c) $G$ has the strict 1-density property (see Definition 4.3);

(d) $G$ has the all fair property (see Definition 2.1).

Moreover, in this case the weights $\sigma$ are unique and the corresponding $\sigma$-WUST $\mu_\sigma$ is the fair pmf of maximal entropy among all the fair pmf’s.
In the next two sections we use this theory to describe the fair trees for modified grids and for regular grids.

4.5. **Fair trees for modified planar grids.** We begin by computing the fair trees for the modified grids, by proving Lemma 2.2.

**Proof of Lemma 2.2.** To see that \( \bar{\Gamma} \subset \Gamma \) it is enough to show that \( \bar{\mu} \) is optimal for the FEU problem on \( G \). Notice that, when \( T \in \bar{\Gamma} \), each edge \( e \in E(T) \) lies in \( T \cap E_v \) for some vertex \( v \in V(G_0) \). Also, by construction, exactly half of the trees of \( \bar{\Gamma} \) use any given \( e \). Therefore, the edge-usage probabilities for \( \bar{\mu} \) are:

\[
\mathbb{P}_{\bar{\mu}}(e \in T) = \frac{1}{|\bar{\Gamma}|} \sum_{T \in \bar{\Gamma}} \mathcal{N}(T, e) = \frac{1}{2},
\]

where \( \mathcal{N}(T, e) = 1_{\{e \in T\}} \) gives the usage of \( e \) by \( T \). Coincidentally, this also proves that \( G \) is homogeneous.

Before proving the other inclusion, we examine the reducibility of \( G \). Note that \( G \) has \( mn + 1 \) vertices and \( 2mn \) edges. Therefore, it has 1-density

\[
\theta(G) = 2.
\]

Let \( H \) be the subgraph of \( G \) induced by the special vertex \( v_0 \) and the bottom-right corner of the standard grid \( G_0 \). Thus, \( H \) consists of two nodes connected by a double edge. We refer to such graphs as *digons*. In particular, \( \theta(H) = \theta(G) = 2 \). By Theorem 4.4, \( G \) is not strictly 1-dense and hence \( G \) is reducible. Moreover, since \( G \) is homogeneous, Theorem 4.2 shows that every subgraph in \( \mathcal{H}(G) \) has 1-density at most 2. This shows that \( H \) is a solution of the Minimal 1-Densest Subgraph (MDS) problem for \( G \) and by Theorem 4.4 (ii), we can deflate \( G \) with respect to \( H \).

If we do that, the resulting quotient graph \( G/H \) still maintain the structure whereby there is a special node \( v_0 \) and the edge-set \( E \) can be partitioned as \( \bigcup_{v \in V \setminus \{v_0\}} E_v \) with \( |E_v| = 2 \), as before. In particular, since the edge count decreases by 2 and the node count decreases by 1, \( G/H \) still has 1-density equal to 2. Moreover, this creates additional digons and \( G/H \) is again homogeneous, so these digons solve the MDS problem, which means that we can repeat the deflation process iteratively. For instance, we can apply the deflation process going from right to left, row by row, until we are left with a single digon graph. Finally, observe that at each step the two edges that get contracted are exactly the pairs of edges in the sets \( E_v \) for \( v \in V(G) \setminus \{v_0\} \) defined above.

Now assume that \( T \in \Gamma \setminus \bar{\Gamma} \). Then, by definition of \( \bar{\Gamma} \), and the Pigeonhole principle, there exists vertices \( v^*, v^*_+ \in V(T) \), such that \( |T \cap E_{v^*}| = 2 \) and \( |T \cap E_{v^*_+}| = 0 \). Indeed, \( T \) must choose \( mn \) edges from \( E \), which is partitioned into \( mn \) pairs of edges of the form \( E_v \). However, this means that in the deflation process described above, we are guaranteed to get to a point where the two edges in, say, \( E_{v^*_+} \), connect \( v^*_+ \) to a single other node \( u^*_+ \), so as to form a subgraph \( H_+ \) of 1-density two in a larger homogeneous graph \( G_+ \) of the same 1-density. Let \( T_+ \) be the tree obtained from \( T \) by removing all the pairs \( E_v \) that have been involved in the deflation thus far. Then, \( T_+ \) does not restrict to a spanning tree of \( H_+ \), and thus \( T \) is forbidden. \( \square \)

4.6. **Fair trees for regular planar grids.** One reason the scaling limit of the Peano curves in the case of modified grids turned out to be deterministic is because the family of
fair spanning trees is much smaller than the family of all spanning trees. To explore a setting that may have a richer collection of spanning trees, we return to regular planar grids.

Recall the strict 1-denseness property as defined in Definition 4.3.

Lemma 4.7. The $m$-by-$n$ planar grid $G_{m,n}$ has the strict 1-density property.

In particular, by Lemma 4.7, Theorem 4.6, and Definition 4.5, there is a unique set of edge weights $\sigma > 0$ on $E(G_{m,n})$ so that the corresponding $\sigma$-WUST $\mu_\sigma$ is fair. This suggests a different approach to the scaling limit of fair Peano curves. Namely, sample fair $\sigma$-WUST trees from the grid $G_{m,n}$ and then condition on such trees containing the bottom and right hand-side of the grid. This should give rise to a much richer family of spanning trees that may have a non-deterministic scaling limit. We will pursue these considerations in a separate paper.

Proof of Lemma 4.7. First note that:

$$|V(G_{m,n})| = mn \quad \text{and} \quad |E(G_{m,n})| = (m-1)n + m(n-1) = 2mn - m - n$$

So

$$\theta(G_{m,n}) = \frac{2mn - m - n}{mn - 1} = 2 - \frac{m + n + 2}{mn - 1}$$

In particular,

$$\frac{\partial \theta(G_{m,n})}{\partial m} = -\frac{(mn - 1) - n(m + n + 2)}{(mn - 1)^2} = \frac{(n + 1)^2}{(mn - 1)^2} > 0,$$

and by symmetry the same holds for the derivative in $n$, meaning that the 1-density of rectangular grids is strictly increasing with the size of the grid, is always less than 2, and in fact tends to 2 as $m,n \to \infty$.

Now let $H$ be a proper connected vertex-induced subgraph of $G = G_{m,n}$. We first consider the smallest rectangular grid containing $H$. Namely, let

$$i_{\min} := \min \{ i : \exists j, (i,j) \in V(H) \}$$

and similarly define $i_{\max}, j_{\min},$ and $j_{\max}$. Then, define the translated grid

$$\tilde{G} := (i_{\min}, j_{\min}) + G_{i_{\max} - i_{\min}, j_{\max} - j_{\min}}.$$ 

If $H = \tilde{G}$, then $\theta(H) = \theta(\tilde{G}) < \theta(G)$ and we are done. If $H$ is a proper subgraph of $\tilde{G}$, then since $\theta(\tilde{G}) \leq \theta(G)$, it will be enough to show that $\theta(H) < \theta(\tilde{G})$. In particular, without loss of generality we can assume that $\tilde{G} = G_{m,n}$.

As an embedded planar graph $G_{m,n}$ has $(m-1)(n-1)$ bounded faces. Whenever such a face has all four vertices in $V(H)$ we fill it in so that $H$ becomes a connected compact set $\hat{H}$ in the plane. In particular, the complement of $\hat{H}$ is an open set consisting of finitely many bounded components $\Omega_j, j = 1, \ldots, k$, and one unbounded component $\Omega_0$. The boundary of each one of these components can be parametrized by a curve $\Gamma_j, j = 0, \ldots, k$. This can be seen by thickening $\hat{H}$ a little bit, for instance by defining

$$\hat{H}_\epsilon := \bigcup_{z \in \hat{H}} \{ w \in \mathbb{C} : |w - z| \leq \epsilon \}.$$ 

For each $\epsilon$, the boundary of $\hat{H}_\epsilon$ consists of $C^1$-smooth Jordan curves $\Gamma_j, j = 0, \ldots, k$. If we parametrize each $\Gamma_j, j = 0, \ldots, k$ with its arc-length parametrization, then, as $\epsilon$ tends to 0, they converge uniformly to the curves $\Gamma_j$. 
We begin by looking at the boundary of the unbounded component \( \Omega_0 \) and think of it as being parametrized counter-clockwise. In particular, \( \Gamma_0 \) can be taken to be piecewise linear so that the derivative \( \Gamma_0' \) is well defined away from the nodes of \( \mathbb{Z}^2 \). Since \( \Gamma_{0,\epsilon} \) bounds the simply connected domain \( \Omega_{0,\epsilon} \cup \{\infty\} \), by the argument principle, the change in argument for the derivative \( \Gamma_{0,\epsilon}' \), as the curve completes one full loop, is \( 2\pi \). As \( \epsilon \) tends to 0, this property is inherited by \( \Gamma_0 \), as long as the argument of the derivative \( \Gamma_0' \) is properly defined. At any moment when \( \Gamma_0 \) is not at a node of \( \mathbb{Z}^2 \), the right hand of a walker traveling along with \( \Gamma_0 \) will always be touching \( \Omega_0 \). In particular, the only changes in the argument of \( \Gamma_0' \) that are allowed when \( \Gamma_0 \) passes through a node of \( \mathbb{Z}^2 \) are

\[
0, \frac{\pi}{2}, \pi, \text{ and, } -\frac{\pi}{2}.
\]

In words, either the walker goes straight, turns 90° left, does a 180° turn in the positive direction, or turns 90° right. This can be verified by looking at the unbounded component \( \Omega_{0,\epsilon} \) of the thickened \( \tilde{H}_\epsilon \). Indeed, if \( \Gamma_0 \) were to do a 180° turn in the negative direction, then while walking along \( \Gamma_{0,\epsilon} \), the right hand would be touching the thickened neighborhood of an edge, and this neighborhood would disappear as \( \epsilon \) tends to 0, so that edge would not be part of the boundary of \( \Omega_0 \), which leads to a contradiction.

Now assume \( v \in \mathbb{Z}^2 \) is a node where \( \Gamma_0 \) makes a right turn (90° in the negative direction), and let \( u \in \mathbb{Z}^2 \) be the node visited by \( \Gamma_0 \) just before \( v \), and \( w \in \mathbb{Z}^2 \) the one visited just after \( v \). Then, \( u, v, w \) bound a face \( f \) of \( G_{m,n} \) and the fourth corner \( t \) does not belong to \( H \). To see why, assume by contradiction that all four corners are in \( V(H) \), then the face would be contained in \( \tilde{H} \). But then the right hand would not be touching the unbounded component \( \Omega_0 \) in this case. Also, note that since \( H \) is vertex-induced the two sides of \( f \) incident at \( t \) also do not belong to \( E(H) \).

Now, if we add \( t \) to \( V(H) \), to get a new vertex-induced graph \( H' \). Since one of the coordinates of \( t \) is equal to one of the coordinates of either \( u \) or \( w \), we see that \( H' \) is still a subgraph of \( G_{m,n} \). Moreover, we are guaranteed that

\[
E(H') \geq E(H) + 2.
\]

In particular,

\[
\theta(H') = \frac{E(H')}{V(H') - 1} \geq \frac{E(H) + 2}{V(H)}.
\]

Since \( H \) is a subgraph of the grid \( G_{m,n} \), we always have \( \deg_H(x) \leq 4 \). Moreover, at least one vertex of \( H \) has \( \deg_H(x) < 4 \). Therefore, by the Handshake Lemma,

\[
2E(H) = \sum_{x \in V(H)} \deg_H(x) < 4V(H).
\]

This implies that \( \theta(H') > \theta(H) \) because

\[
\frac{E(H) + 2}{V(H)} > \frac{E(H)}{V(H) - 1} \iff (E(H) + 2)(V(H) - 1) > E(H)V(H) \iff 2V(H) > E(H).
\]

If \( H' = G_{m,n} \), we are done. Otherwise, we can repeat the same argument with \( H \) replaced by \( H' \). This process has to end, so without loss of generality, we can assume that \( \Gamma_0 \) never makes any right turns. In particular, the argument of \( \Gamma_0' \) can only change by 0, \( \pi/2 \), or \( \pi \). So since there has to be an even number of argument changes that are positive, and the sum
must be $2\pi$, we either get two changes by $\pi$ or four changes by $\pi/2$. In either case, we see that $\Gamma_0$ describes the boundary of a rectangular grid, and by assumption this grid has to be $G_{m,n}$.

Finally, consider a bounded component $\Omega_j$ with $j \geq 1$. Assume that the boundary of $\Omega_j$ is described in the clockwise direction by a curve $\Gamma_j$, constructed as $\Gamma_0$ above, using the thickened set $\hat{H}_\epsilon$. Once again the allowed argument changes at a grid node are $0, \pi/2, \pi$, and $-\pi/2$. Also, when walking along $\Gamma_j$ the right-hand touches $\Omega_j$ and the total argument change must equal $-2\pi$. In particular, there must always be at least four right turns with argument change $-\pi/2$. By repeating the argument above we see that each right turn identifies a vertex in $\Omega_j$ that can be added to $H$ in a way to increase the 1-density, and therefore after finitely many steps we get that the component $\Omega_j$ has been filled in. Hence, after finitely many steps, $H = G_{m,n}$. This proves the lemma. \hfill \Box

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