Anti De Sitter Geometry And Strongly Coupled Gauge Theories

Jaume Gomis

Department of Physics and Astronomy
Rutgers University
Piscataway, NJ 08855–0849

jaume@physics.rutgers.edu

We propose supergravity duals to non-trivial fixed points of the renormalization group in three dimensions with ADE global symmetries. All the fixed point symmetries are identified with space-time symmetries.
1. Introduction

Maldacena [1] has recently conjectured a very interesting duality relation between conformal field theories and space-time backgrounds. The precise equivalence between these rather different theories has not yet been spelled out. An important test of the duality comes from comparing the symmetries of the field theories with those of space-time [13,14]. In [15,13] a conjectured equivalence between correlation functions of the conformal field theory and supergravity actions in AdS backgrounds is given. Calculations supporting this correspondence can be found in [16]. In this note we identify the symmetries between three dimensional strongly coupled gauge theories and M-theory backgrounds. These backgrounds have an $AdS_4 \times K$ geometry, where $K$ is a compact seven dimensional manifold and correspond to the near horizon geometry of $N$ M2-branes at an ADE singularity. Related theories to those we analyze have recently been considered in [17,18,19,20].

The results of [4,13], are essential to get complete agreement between symmetries. In [4,13] it was found that gauge symmetries in the AdS space-time background correspond to global symmetries of the theory on the boundary of AdS. This will enable us to relate the ADE global symmetry group of the strongly coupled gauge theory with ADE gauge symmetry in space-time.

In section two we briefly discuss the three dimensional gauge theories whose non-trivial fixed points have global ADE symmetries. We realize these theories as brane configurations and identify the space time theory corresponding to the fixed point to be that of $N$ M2-branes at an ADE singularity. We carefully analyze the near horizon geometry of such membrane configurations and find out that it contains an $AdS_4$ part and that the manifold $K$ can be written as a $S^3$ fibration over an ADE quotiented four disk $D_4$. In section three we map all the symmetries. Section 4 contains conclusions.

2. Interacting fixed points and space-time geometry

Three dimensional gauge theories are asymptotically free and can be at non-trivial fixed points of the renormalization group in the infrared [21]. It is believed that non-trivial fixed points of the renormalization group are not only scale invariant but also conformal invariant. We will be interested in three dimensional $N = 4$ gauge theories (theories with

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1 The near horizon geometry correspondence with branes appeared in [2,3,4,5, 6,7,8,9,10,11,12].

2 In section 2 the geometry will be made precise.
8 real supercharges) which have non-trivial fixed points in the infrared with ADE global symmetries. These conformal field theories exhibit universality since they have dual short distance descriptions. These theories are related by three dimensional mirror symmetry \cite{22}.

These gauge theories can be realized on branes in string theory. One of the mirrors can be constructed\footnote{When $\Gamma$ acts in the regular representation.} as the theory on $N$ D2-branes of Type IIA sitting at a $C^2/\Gamma$ singularity, where $\Gamma$ is an ADE discrete subgroup of $SU(2)$ \cite{23,24}. The content of the gauge theory can be conveniently summarized by a quiver diagram. The gauge group can be read from the nodes of the extended Dynkin diagram of the corresponding ADE Lie algebra. The hypermultiplets are given by the links of the extended Dynkin diagram and are in the bifundamental representation of the adjacent nodes\footnote{See \cite{23,24} for more details.}. These theories have a quantum moduli space of vacua with two branches. In \cite{21} these theories were solved using string theory considerations and in \cite{22,25} they were analyzed by field theory methods. The Coulomb branch is the moduli space of instantons of the corresponding ADE gauge group. The Higgs branch, obtained as a hyper-Kahler quotient, is an ADE singularity. The two branches intersect at the origin of the ADE singularity, and the theory there is at a non-trivial fixed point of the renormalization group with ADE global symmetry \cite{21}. It is important to point out that the nontrivial fixed point is in the strongly coupled regime $g_{YM}^2 \to \infty$ \cite{21}.

The space-time theory corresponding to this fixed point is modified. To tune to the fixed point one needs to send the Yang-Mills coupling constant to infinity. By open string considerations $g_{YM}^2 \propto g_s$, where $g_s$ is the coupling constant of the Type IIA theory on which the D2-branes are embedded. Thus, to tune to the fixed point we must push the space-time picture to M-theory. Therefore, the space-time description of the fixed point is that of $N$ M2-branes sitting at an ADE singularity.

Following the AdS-CFT correspondence we wish to identify all the symmetries of the three dimensional interacting fixed points with those of M-theory on the near horizon geometry of $N$ M2-branes on an ADE singularity. In order to identify the near horizon geometry corresponding to this brane configuration it is instructive to write down the near horizon geometry of M2-branes in flat space. The metric is given by

$$ds^2 = \frac{r^4}{R^4} d\vec{x}^2 + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_7^2,$$  \hspace{1cm} (2.1)
where $R^6 = 2^5 \pi^2 N \ell^3_p$ and $r^2 = x_3^2 + \ldots x_{10}^2$. The geometry is that of $AdS_4 \times S^7$. The original conjecture by Maldacena is that the three dimensional conformal field theory corresponding to the theory of $N$ M2-branes is dual to M-theory on $AdS_4 \times S^7$. The conformal group $SO(3,2)$ is identified with the Anti de Sitter group in space-time and the $SO(8)$ R-symmetry group of the field theory with the isometry group of $S^7$.

We want to find the near horizon geometry of $N$ M2-branes at a $C^2/\Gamma$ singularity. For concreteness we will concentrate on the case $\Gamma = A_{k-1} = Z_k$, but the generalization to the other ADE discrete groups is straightforward. Let $z_1, z_2$ be the complex coordinates of the $C^2/Z_k$ singularity. The orbifold action is given by

\[
\begin{align*}
    z_1 &\to e^{\frac{2\pi i}{k}} z_1 \\
    z_2 &\to e^{-\frac{2\pi i}{k}} z_2.
\end{align*}
\]

This action leaves the $AdS_4$ geometry invariant but it acts non-trivially on the $S^7$ geometry. Let us analyze carefully the orbifold action on $S^7$. Consider $S^7 \subset C^4$ defined by

\[
|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = a^2.
\]

The $Z_k$ orbifold group acts as in (2.2) in the $z_1, z_2$ coordinates and leaves $z_3, z_4$ invariant. Let $\pi : C^4 \to C^2(z_1, z_2)$ denote the projection onto the $(z_1, z_2)$ plane. Then the restriction $\pi|_{S^7}$ exhibits $S^7$ as an $S^3$-fibration over the disk $D_4$ of radius $a$ in $C^2(z_1, z_2)$. The radius of the $S^3$ fiber varies over the base such that it shrinks to zero size at the boundary $\partial D_4$. The fibration is not globally trivial. The orbifold action identifies the $S^3$ fibers over a $Z_k$ orbit through any arbitrary point $(z_1, z_2) \neq (0,0)$. At the same time, it leaves the $S^3$ fiber over the origin invariant. Therefore the quotient is an $S^3$ fibration over the singular space $D_4/Z_k$. Note that the singular locus of the total space consists of an $A_{k-1}$ singularity fibered over the central $S^3$ fiber.

Now that the near horizon geometry has been found we can identify the symmetries of three dimensional fixed points with those of M-theory on $AdS_4 \times S^3 \times f D_4/\Gamma$.

\[5\] The membranes lie in the 012 plane, so that the coordinates $x_i, i = 3, \ldots 10$ are transverse to the membranes.

\[6\] The following geometrical analysis was done in collaboration with D.-E. Diaconescu.
3. Space-Time Symmetries and Fixed Point Symmetries

The identification of the conformal field theory symmetries with the space-time symmetries can be done as follows. Since the field theory is at a non-trivial fixed point of the renormalization group the scale symmetry is enlarged to the conformal group in three dimensions, $SO(3,2)$. This can be identified with the Anti de Sitter symmetry group in the space-time theory. For this it is crucial for the orbifold action not to affect the $AdS_4$ geometry. The gauge theories we started with have $\mathcal{N} = 4$ supersymmetry in three dimensions, and the $SO(4)$ R-symmetry group can be identified with the isometry group of the $S^3$ fiber.

Moreover, the CFT has an ADE global symmetry arising at the origin of the moduli space of ADE instantons. As shown in \cite{13}, global symmetries in the CFT correspond to gauge symmetries in the $AdS$ geometry. We will show that the ADE global symmetry can be identified with ADE gauge symmetry of M-theory on $AdS_4 \times S^3 \times_f D_4/\Gamma$. In M-theory we can wrap membranes on the shrunken cycles of the ADE singularity. This gives rise to ADE gauge bosons in the remaining directions. In order to be left with ADE gauge symmetry in $AdS_4$ we need to reduce the gauge fields on $S^3$, and since $\dim(H^0(S^3)) = 1$, one is left with ADE gauge bosons in $AdS_4$. This ADE gauge symmetry is the one we want to identify with the ADE global symmetry of the fixed point. Therefore, by analyzing the space-time geometry corresponding the fixed point we have identified all the symmetries in the two theories.

4. Conclusions

A preliminary step towards understanding the AdS-CFT correspondence is to identify the symmetries of both theories. In this note we do that for three dimensional conformal field theories with ADE global symmetries by identifying the space-time description of these fixed points. Some of the symmetries of the field theory arise as geometrical symmetries of the near horizon geometry. The ADE global symmetries correspond to ADE gauge symmetries in space-time.

It would be interesting to perform similar test for other non-trivial fixed points of the renormalization group. Eventually, one might hope to be able to describe all such fixed points by theories in space-time. Along the lines of \cite{13,12,26} it would be interesting to compute some correlation functions and anomalous dimensions of these conformal field theories from M-theory considerations.
After this work was completed the paper [24] appeared with similar subject.

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