Braids and crossed modules

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Abstract

For $n \geq 4$, we describe Artin’s braid group on $n$ strings as a crossed module over itself. In particular, we interpret the braid relations as crossed module structure relations.

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1 Introduction

The braid group $B_n$ on $n$ strings was introduced by Emil Artin in 1926, cf. [21] and the references there. It has various interpretations, specifically, as the group of geometric braids in $\mathbb{R}^3$, as the mapping class group of an $n$-punctured disk, etc. The group $B_1$ is the trivial group and $B_2$ is infinite cyclic. Henceforth we will take $n \geq 3$. The group $B_n$ has $n - 1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2,$$

$$(1.1)$$

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}, \quad 1 \leq j \leq n - 2.$$ 

$$(1.2)$$

The main result of the present paper, Theorem 5.1 below, says that, as a crossed module over itself, (i) the Artin braid group $B_n$ has a single generator, which can be taken to
be any of the Artin generators $\sigma_1, \ldots, \sigma_{n-1}$, and that, furthermore, (ii) the kernel of the surjection from the free $B_n$-crossed module $C_n$ in any one of the $\sigma_j$’s onto $B_n$ coincides with the second homology group $H_2(B_n)$ of $B_n$, well known to be cyclic of order 2 when $n \geq 4$ and trivial for $n = 2$ and $n = 3$. Thus, the crossed module property corresponds precisely to the relations (1.2), that is, these relations amount to Peiffer identities (see below). When $C_n$ is taken to be the free $B_n$-crossed module in $\sigma_1$, for $n \geq 4$, the element $\sigma_3 \sigma_1 \sigma_1^{-1}$ of $C_n$ has order 2 and generates the copy of $H_2(B_n) \cong \mathbb{Z}/2$ in $C_n$; the element $\sigma_3 \sigma_1 \sigma_1^{-1}$ of $C_n$ being the generator of the kernel of the projection from $C_n$ to $B_n$, it is then manifest how this central element forces the relations (1.1).

The braid groups are special cases of the more general class of Artin groups [3], [11]. However, for a general Artin group presentation, there does not seem to be an obvious way to concoct a crossed module of the same kind as that for the braid group presentation, and the results of the present paper seem to be special for braid groups.

More recently augmented racks [13] and quandles have been explored in the literature, and there is an intimate relationship between augmented racks, quandles, crossed modules, braids, tangles, etc. [13]. See also Remark 5.2 below. Is there a single theory, yet to be uncovered, so that augmented racks, quandles, links, braids, tangles, crossed modules, Hecke algebras, etc. are different incarnations thereof? In Section 6 below we explain how operads are behind our approach and, perhaps, operads can be used to build such a single theory. We hope our paper contributes to the general understanding of the situation.

It is a pleasure to acknowledge the stimulus of conversation with L. Kaufman at a physics meeting in Varna (2007) which made me rethink old ideas of mine related with braids, identities among relations, and crossed modules. I am indebted to J. Stasheff for a number of comments on a draft of the paper which helped improve the exposition and to F. Cohen for discussion about the generating cycle of $H_2(B_n)$, see Remark 5.3 below; this discussion prompted me to work out the relationship between polyhedral geometry and identities among relations in Section 6 below.

2 Historical remarks

The defining relations (1.1) and (1.2) for $B_n$ are intimately connected with Hecke algebra relations. V. Jones discovered new invariants of knots [18] by analyzing certain finite-dimensional representations of the Artin braid group into the Temperley-Lieb algebra [32], that is, a von Neumann algebra defined by a kind of Hecke algebra relations.

In the late 1970’s, working on the unsettled problem of J. H. C. Whitehead’s [35] whether any subcomplex of an aspherical 2-complex is itself aspherical [17], I had noticed that the relations (1.2) strongly resemble the Peiffer identities (cf. [26], [27]) defining a free crossed module, but it seems that this similarity has never been made explicit in the literature nor has it been used to derive structural insight into braid groups or into related groups such as knot or link groups.

That kind of similarity gets even more striking when one examines Whitehead’s original proof [35] of the following result of his: Adding 2-cells to a pathwise connected space $K$ so that the space $L$ results yields the $\pi_1(K)$-crossed module $\pi_2(L, K)$ that is free on the
2-cells that have been attached to form $L$. A precise form of this result will be spelled out below as Theorem \[3.1\]. The crucial step of that proof proceeds as follows: Let $B^2$ denote the 2-disk, $S^1$ its boundary circle, and denote by $o$ the various base points. To prove that the obvious map from the free $\pi_1(K)$-crossed module on the 2-cells in $L \setminus K$ to $\pi_2(L,K)$ is injective, Whitehead considers a null-homotopy $(B^2 \times I, S^1 \times I, o \times I) \to (L, K, o)$; when this null-homotopy is made transverse, the pre-images of suitable small disks in the interiors of the 2-cells in $L \setminus K$ form a link in the solid cylinder $B^2 \times I$ (more precisely: a partial link based at the 2-disk $B^2$ at the bottom of the cylinder) which truly arises from a kind of tangle or braid; Whitehead then argues in terms of the exterior of that link and in particular interprets the Wirtinger relations corresponding to the crossings in terms of the identities defining a free crossed module, that is, in terms of the Peiffer identities. In the late 1970’s I learnt from R. Peiffer (private communication) that Reidemeister had also been aware of the appearance of this kind of link resulting from a null homotopy. A more recent account of Whitehead’s argument can be found in [5]. See also the survey article [4], where the relationship between identities among relations and links is explained in detail, in particular in terms of the notion of picture which is a geometric representation of an element of a crossed module; a geometric object equivalent to a picture had been exploited already in [26] under the name Randwegaggregat. These links very much look like the tangles that have been used to describe the Temperley-Lieb algebra, cf. [19] (p. 8) which, in turn, leads, via a trace, to a description of the Jones polynomial. The formal relationship between Peiffer identities and this kind of link was also exploited in [30].

In the early 1980’s, some of my ideas related with identities among relations and crossed modules merged into the papers [7], [9], and [17]; in particular, pictures are crucially used in these papers.

3 Crossed modules

We will denote the identity element of a group by $e$. Let $C$ and $G$ be groups, $\partial: C \to G$ a homomorphism, and suppose that the group $C$ is endowed with an action of $G$ which we write as

$$G \times C \longrightarrow C, \ (x, c) \longmapsto x^c, \ x \in G, \ x \in C.$$ 

These data constitute a crossed module provided

- relative to the conjugation action of $G$ on itself $\partial$ is a homomorphism of $G$-groups, and
- in $C$ the identities

$$aba^{-1} = \partial ab$$

are satisfied.

We refer to the identities \[3.1\] or, equivalently,

$$aba^{-1}(\partial ab)^{-1} = e,$$ \[3.2\]

as Peiffer identities. Morphisms of crossed modules are defined in the obvious way. Thus crossed modules form a category.

Let $\partial: C \to G$ be a crossed module. For later reference, we recall the following facts:
The image $\partial(C) \subseteq G$ is a normal subgroup; we will write the quotient group as $Q$.

- The kernel $\ker(\partial)$ of $\partial$ is a central subgroup of $C$ and acquires a $Q$-module structure.

- Via the $G$-action, the abelianized group $C_{Ab}$ acquires a $Q$-module structure.

Let $\varphi: Y \rightarrow C$ be a map. The crossed module $\partial: C \rightarrow G$ is said to be free on $\varphi$ provided the following universal property is satisfied:

Given a crossed module $\delta: D \rightarrow H$, a homomorphism $\alpha: G \rightarrow H$, and a (set) map $b: Y \rightarrow D$ such that $\alpha \circ \partial \circ \varphi = \delta \circ b$, there is a unique group homomorphism $\beta: C \rightarrow D$ with $\beta \circ \varphi = b$ such that the diagram

$$
\begin{array}{ccc}
C & \rightarrow & G \\
\downarrow \beta & & \downarrow \alpha \\
D & \rightarrow & H
\end{array}
$$

is a morphism of crossed modules from $\partial$ to $\delta$.

When $\partial: C \rightarrow G$ is free on $\varphi: Y \rightarrow C$, in view of the universal property of a free crossed module, the abelianized group $C_{Ab}$ is then the free $Q$-module having (a set in bijection with) $Y$ as a $Q$-basis. Consequently, the map $\varphi$ is then injective and, somewhat loosely, one says that $\partial: C \rightarrow G$ is free on $Y$.

Given the crossed module $\partial: C \rightarrow G$, we will refer to $C$ as a $G$-crossed module; when $\partial: C \rightarrow G$ is a free crossed module, we will refer to $C$ as a free $G$-crossed module.

Let $L$ be a space obtained from the pathwise connected space $K$ by the operation of attaching 2-cells. In [35], J.H.C. Whitehead attempted an algebraic description of the second homotopy group $\pi_2(L)$. In [36], he reformulated his earlier results in terms of a precise algebraic description of the second relative homotopy group $\pi_2(L, K)$. In [37], he finally codified the result as follows:

**Theorem 3.1 (Whitehead).** Via the homotopy boundary map from $\pi_2(L, K)$ to $\pi_1(K)$, the group $\pi_2(L, K)$ is the free crossed $\pi_1(K)$-module in the 2-cells that have been attached to $K$ to form $L$.

Recall that the geometric realization of a presentation $\langle X, R \rangle$ of a group is the 2-complex $K(X, R)$ with a single vertex whose 1- and 2-cells are in bijection with $X$ and $R$, respectively, in such a way that, for $r \in R$, the attaching map of the associated 2-cell corresponds to the word $\xi_r$ in the generators $X$ when the relator $r$ is written out as a word in the generators $X$.

Let $\langle X, R \rangle$ be a presentation of a group, let $F$ be the free group on $X$, and let $N$ be the normal closure in $F$ of the relators $R$. An identity among relations [26] (synonymously: identity sequence) is an $m$-tuple $(q_1, \ldots, q_m)$ ($m$ may vary) of pairs of the kind $q_j = (\xi_j, r_j^s)$ where $\xi_j \in F$, $r_j \in R$, $\epsilon_j = \pm 1$, such that evaluation of $\prod \xi_j r_j^s \xi_j^{-1}$ in $F$ yields the identity element $e$ of $F$. Among the observations made in [26], we quote only the following ones: (i) Given two relators $r$ and $s$, the quadruple

$$
((e, r), (e, s), (e, r^{-1}), (\xi_r, s^{-1})), \tag{3.4}
$$
where $\xi_r \in F$ refers to the evaluation of $r$ in $F$, is always an identity among relations, independently of a particular presentation. In the context of group presentations, these identities are referred to in the literature as *Peiffer identities*.

(ii) The group of identities among relations modulo that of identities of the kind (3.4) is isomorphic to the second homotopy group of the geometric realization of the presentation $\langle X, R \rangle$. Here the term “second homotopy group” is to be interpreted in the language of homotopy chains, which here captures the second homology group of the universal covering space of the geometric realization of the presentation under discussion.

(iii) The combinatorial structure of a 3-manifold can be characterized in terms of identities among relations; in particular, the operation of attaching a 3-cell admits an interpretation in terms of killing a certain identity among relations.

In [27], some of these results are rediscussed and the idea of a free crossed module is introduced without a name. A modern account of identities among relations and crossed modules can be found in [4].

**Example 3.2.** Given the group $G$, with respect to the adjoint action, the identity homomorphism $G \to G$ is a crossed module in an obvious manner. In the present paper, we will explore this crossed module for the special case where $G$ is the braid group $B_n$ on $n$ strings.

**Example 3.3.** The injection of a normal subgroup into a group is a crossed module in an obvious way.

**Example 3.4.** Given a group $G$, the obvious map $G \to \text{Aut}(G)$ which sends a member of $G$ to the inner automorphism determined by it is a crossed module. This observation has been explored in [16].

**Example 3.5.** Given the central extension $1 \to A \to E \to G \to 1$ of (discrete) groups, where $A$ is an abelian group, conjugation in $E$ induces a crossed module structure on $E \to G$.

**Example 3.6.** Let $\langle X, R \rangle$ be the Wirtinger presentation of a knot in $\mathbb{R}^3$, let $K$ be the geometric realization of $\langle X, R \rangle$, pick a generator $x \in X$, and let $r$ be the relator $r = x$. The normal closure of $r$ is well known to be the entire knot group. Let $L = K \cup e^2$ be the geometric realization of the resulting presentation $\langle X, R \cup \{r\} \rangle$ of the trivial group. By construction, $\pi_1(K)$ is the group of the knot and, in view of Whitehead’s theorem (Theorem 3.1 above), the homotopy boundary map $\partial: \pi_2(L, K) \to \pi_1(K)$ is a free crossed module, the $\pi_1(K)$-crossed module $\pi_2(L, K)$ being free on the single generator $[r]$. Since $L$ is actually contractible, $\partial: \pi_2(L, K) \to \pi_1(K)$ is an isomorphism. Thus the group of the knot is a free crossed module over itself in a single generator. Suitably interpreted, this kind of observation is also valid for a geometrically indecomposable link.

**Example 3.7.** Let $G$ be a group. A $G$-rack or augmented rack [13] or $G$-automorphic set [2] is a $G$-set $Y$ together with a map $\partial: Y \to G$ such that

$$\partial(xy) = x\partial(y)x^{-1}, \quad x \in G, \quad y \in Y. \tag{3.5}$$

The universal object associated with the $G$-rack $Y$ is a crossed module $\partial: C_Y \to G$ [13] where the notation $\partial$ is abused. In particular, when $Y$ is the free $G$-set $G \times S$ for a set $S$,
the crossed module \( \partial : C_Y \to G \) is the free crossed module on \( S \). Notice that the identities \([3.5]\) yield again a version of the Peiffer identities.

4 The main technical result

**Theorem 4.1.** Let \( G \) be a group, let \( S \subseteq G \) be a subset of \( G \) whose normal closure is all of \( G \), let \( \partial : C \to G \) be the free crossed module on \( S \), and suppose that the induced map \( C_{\text{Ab}} \to G_{\text{Ab}} \) is an isomorphism of abelian groups. Then \( \partial \) fits into a central extension of the kind

\[
0 \longrightarrow H_2(G) \longrightarrow C \overset{\partial}{\longrightarrow} G \longrightarrow 1. \tag{4.1}
\]

**Proof.** Let \( \langle X, R \rangle \) be a presentation of \( G \), let \( F \) be the free group on \( X \), and let \( N \) be the normal closure of \( R \) in \( F \). Abusing notation, let \( S \subseteq F \) be a subset which under the projection from \( F \) to \( G \) is mapped bijectively onto \( S \subseteq G \). Then \( \langle X, R \cup S \rangle \) is a presentation of the trivial group.

Let \( K \) be the geometric realization of the presentation \( \langle X, R \rangle \) and \( L \) that of the presentation \( \langle X, R \cup S \rangle \) of the trivial group. By construction, the fundamental group \( \pi_1(K) \) of \( K \) is \( G \). In view of Whitehead’s theorem (Theorem \[3.1\] above), the homotopy boundary map \( \partial : \pi_2(L, K) \to G \) is a free crossed module, the \( G \)-crossed module \( C = \pi_2(L, K) \) being free on the set \( S \).

Since \( L \) is simply connected, the Hurewicz map \( \pi_2(L) \to H_2(L) \) is an isomorphism and the exact homotopy sequence of the pair \((L, K)\) takes the form

\[
\pi_2(K) \longrightarrow H_2(L) \longrightarrow \pi_2(L, K) \overset{\partial}{\longrightarrow} G \longrightarrow 1. \tag{4.2}
\]

It is a classical fact that the cokernel of the Hurewicz map \( \pi_2(K) \to H_2(K) \) is the second homology group \( H_2(G) \) of \( G \) \([15]\). The group \( H_2(L, K) \) is free abelian on a basis in bijection with the set \( S \), and inspection of the exact homology sequence

\[
0 \longrightarrow H_2(K) \longrightarrow H_2(L) \longrightarrow H_2(L, K) \overset{H\partial}{\longrightarrow} H_1(K) \longrightarrow 0 \tag{4.3}
\]

of the pair \((L, K)\) shows that the canonical map \( H_2(K) \to H_2(L) \) is an isomorphism if and only if the homology boundary \( H_2(L, K) \to H_1(K) \cong H_1(G) \) is an isomorphism.

The requirement that the induced map \( C_{\text{Ab}} \to G_{\text{Ab}} \) be an isomorphism of abelian groups says precisely that the homology boundary \( H_2(L, K) \to H_1(K) \cong H_1(G) \) is an isomorphism. Consequently the exact homotopy sequence \([4.2]\) induces the asserted central extension \([4.1]\). \( \square \)

**Example 4.2.** Let \( A \) be the free abelian group on a set \( S \). Then \( H_2(A) \) is the exterior square \( A \wedge A \) and the free crossed \( A \)-module \( C \) on \( S \) fits into a central extension

\[
0 \longrightarrow A \wedge A \longrightarrow C \overset{\partial}{\longrightarrow} A \longrightarrow 1. \tag{4.4}
\]

This extension is the universal central extension of \( A \) that corresponds to the universal \( A \wedge A \)-valued skew-symmetric 2-cocycle on \( A \); the skew-symmetricity determines this 2-cocycle uniquely. In Section \[7\] we will characterize this kind of central extension by a universal property.
5 Artin’s braid group

The braid group $B_n$ on $n$ strings has $n - 1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i - j| \geq 2,$$

and

$$\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}, \quad 1 \leq j \leq n - 2.$$  

Instead of (5.2), we take the relations

$$\sigma_j\sigma_{j+1}\sigma_j^{-1}\sigma_{j+1}^{-1} = e.$$  

Adding the single relation $\sigma_1 = e$ plainly kills the entire group, that is, the normal closure of $\sigma_1$ (or any other $\sigma_j$) in $B_n$ is all of $B_n$.

For $1 \leq j \leq n - 2$, let

$$r_{j,j+1} = x_jx_{j+1}x_j^{-1}x_{j+1}^{-1}x_j^{-1}x_{j+1}^{-1}$$

and let $\hat{R} = \{r_{1,2}, r_{2,3}, \ldots, r_{n-2,n-1}\}$. For $1 \leq j, k \leq n - 1$ with $j < k - 1$, let

$$r_{j,k} = [x_j, x_k] = x_jx_kx_j^{-1}x_k^{-1},$$

let $\tilde{R}$ denote the finite set of these $r_{j,k}$, and let $R = \hat{R} \cup \tilde{R}$. Then

$$\langle X, R \rangle = \langle x_1, x_2, \ldots, x_{n-1}; r_{1,2}, r_{2,3}, \ldots, r_{n-2,n-1}, r_{1,3}, \ldots, r_{n-3,n-1} \rangle$$

is a presentation of $B_n$. We suppose the notation being adjusted in such a way that, for $1 \leq j \leq n - 1$, the generator $\sigma_j$ of $B_n$ is represented by $x_j$. Adding the relator $r = x_1$ or, more generally, a relator of the kind $r = x_j, 1 \leq j \leq n - 1$, we obtain the presentation $\langle X, R \cup \{r\} \rangle$ of the trivial group.

Let $K = K(X, R)$ be the geometric realization of the presentation $\langle X, R \rangle$ and let $L = K \cup e^2$ be the geometric realization of the presentation $\langle X, R \cup \{r\} \rangle$ of the trivial group. By construction, $\pi_1(K)$ is the group $B_n$. In view of Whitehead’s theorem (Theorem 3.1 above), the homotopy boundary map $\partial: \pi_2(L, K) \to B_n$ is a free crossed module, the $B_n$-crossed module $\pi_2(L, K)$ being free on the single generator $[r]$. We will write $\pi_2(L, K)$ as $C_n$.

**Theorem 5.1.** The kernel of $\partial: C_n \to B_n$ is a central subgroup and is canonically isomorphic to $H_2(B_n)$. Furthermore, for $n = 3$, the group $H_2(B_n)$ is zero and for $n \geq 4$, the group $H_2(B_n)$ is cyclic of order 2, generated as a subgroup of $C_n$ by

$$\sigma_3^3[r] \cdot [r]^{-1} \in C_n.$$  

Consequently, as a crossed module over itself, $B_n$ is generated by $\sigma_1$, subject to the relation

$$\sigma_3^3\sigma_1 = \sigma_1.$$
Thus the crossed module property corresponds precisely to the relations (5.2), that is, these relations amount to Peiffer identities, and the relation (5.6) correspond exactly to the relations (5.1).

N.B. As a crossed module over itself, $B_n$ has a single generator, which can be taken to be any of the generators $\sigma_1, \ldots, \sigma_{n-1}$. In the statement of the theorem, we have written out the relation (5.6) with respect to the generator $\sigma_1$. But we could have written out an equivalent relation in terms of any of the generators $\sigma_1, \ldots, \sigma_{n-1}$.

**Remark 5.2.** In the language of $B_n$-racks, cf. Example 3.7 above, the universal object associated with the $B_n$-rack $B_n \times \{\sigma_1\}$ (with the obvious structure) is the free $B_n$-crossed module on the single generator $\sigma_1$. Thus the universal object associated with the $B_n$-rack $B_n \times \{\sigma_1\}$ recovers $B_n$ itself up to the central extension by $H_2(B_n)$. We shall show in Section 7 below that this central extension is actually universal in a suitable sense.

**Proof.** The first statement is a special case of Theorem 4.1. Indeed, $B_n$ is the normal closure of any of the generators $\sigma_j$, and

$$(C_n)_{\text{Ab}} \cong \pi_2(L, K)_{\text{Ab}} \cong H_2(L, K) \to H_1(K) \cong H_1(B_n) \cong \mathbb{Z}$$

is plainly an isomorphism.

The group $B_2$ is infinite cyclic and hence has zero second homology group. The group $B_3$ is well known to be isomorphic to the trefoil knot group and hence has zero second homology group as well. Henceforth we suppose that $n \geq 4$.

We will now derive the relation (5.6). As usual we denote the 1-skeleton of the cell complex $K$ by $K^1$. The exact homotopy sequence of the pair $(L, K^1)$ takes the form

$$0 \longrightarrow \pi_2(L) \longrightarrow \pi_2(L, K^1) \longrightarrow \pi_1(K^1) \longrightarrow 1. \quad (5.7)$$

Since the $\pi_1(K^1)$-action on $\pi_2(L)$ factors through $\pi_1(L)$ which is the trivial group, the $\pi_1(K^1)$-action on $\pi_2(L)$ is trivial, and the group extension (5.7) is necessarily central. The second homotopy group $\pi_2(L)$ lies in $\pi_2(L, K^1)$ as the group of (classes of) identities among relations in the sense of [20] and [27].

Let $F \cong \pi_1(K^1)$ denote the free group on $X$. Keeping in mind that $\pi_2(L, K^1)$ is the free $F$-crossed module on $R \cup \{r\}$, let

$$\mathcal{I}_{1,3} = ((x_3, r), (e, r^{-1}, (e, r_{1,3}^{-1}))). \quad (5.8)$$

Inspection shows that $\partial_L(\mathcal{I}_{1,3}) = e \in F \cong \pi_1(K^1)$ whence $[\mathcal{I}_{1,3}] \in \pi_2(L) \subseteq \pi_2(L, K^1)$. Indeed, $\mathcal{I}_{1,3}$ is plainly an identity among relations.

Consider the canonical projection from $\pi_2(L, K^1)$ to $C_n \cong \pi_2(L, K)$. Since under this projection each generator in $R$ goes to $e \in \pi_2(L, K)$, the image $[\mathcal{I}_{1,3}] \in \pi_2(L, K)$ of the member $[\mathcal{I}_{1,3}]$ of $\pi_2(L) \subseteq \pi_2(L, K^1)$ can plainly be written as

$$[\mathcal{I}_{1,3}] = \sigma_3[r] \cdot [r]^{-1} \in \pi_2(L, K) \cong C_n. \quad (5.9)$$

Here the bracket notation $[\cdot]$ is slightly abused. Since the 2-complex $L$ is simply connected, the Hurewicz map $\pi_2(L) \to H_2(L)$ is an isomorphism. Furthermore the reasoning
induces the composite of the isomorphism \( \pi \)
the cardinality of the set \( H = 4 \), the group \( H \)
relation (5.6) in Theorem 5.1. These observations complete the proof of Theorem 5.1.

Left-hand map \( H \langle K \rangle \form of a surjection
Consequently \( I \)
isomorphism of abelian groups, the target group \( H \)
Inspection of the commutative diagram
we note that the rank of the free abelian group \( \pi \)
we shall not need the explicit form of these identities among relations. For intelligibility
we can construct an identity among relations \( I \)

\[
\begin{align*}
0 & \longrightarrow H_2(K) \longrightarrow \pi_2(L, K^1) \xrightarrow{\partial L} F \longrightarrow 1 \\
0 & \longrightarrow H_2(B_n) \longrightarrow C_n \xrightarrow{\partial} B_n \longrightarrow 1
\end{align*}
\]

(5.10)

reveals that \([I_{1,3}] \in H_2(B_n)\).
Likewise, for \( 1 \leq j, k \leq n - 1 \) with \( k - j \geq 2 \), corresponding to the relator

\[ r_{j,k} = [x_j, x_k] = x_jx_kx_j^{-1}x_k^{-1}, \]

we can construct an identity among relations \( I \)
\( I \)
is free abelian in the set of 2-cells in \( K \setminus \widehat{K} \) or, equivalently, in the set \( \widehat{R} \).

The association

\[ I_{j,k} \longmapsto r_{j,k} \quad (1 \leq j, k \leq n - 1, \ j < k - 1) \]

induces the composite of the isomorphism \( \pi_2(L) \rightarrow H_2(K) \) with the isomorphism from
\( H_2(K) \) to \( H_2(K, \widehat{K}) \).

For \( n \geq 4 \), the group \( H_2(B_n) \) is well known to be cyclic of order 2, cf. [8], and the
left-hand map \( H_2(K) \rightarrow H_2(B_n) \) in the diagram (5.10) is surjective by construction. For
\( n = 4 \), the group \( H_2(K) \cong \mathbb{Z} \) is free abelian in the relator \( r_{1,3} \), and that map takes the
form of a surjection

\[ \mathbb{Z} \rightarrow H_2(B_4), \quad r_{1,3} \mapsto [I_{1,3}]. \] (5.11)

Consequently \([I_{1,3}] \in H_2(B_4)\) is non-zero and, the obvious map \( H_2(B_4) \rightarrow H_2(B_n) \) being
an isomorphism, we conclude that \([I_{1,3}] \in H_2(B_n)\) is non-zero for any \( n \geq 4 \).

Finally, the explicit description (5.9) of the element \([I_{1,3}] \in C_n\) justifies at once the relation (5.6) in Theorem 5.1. These observations complete the proof of Theorem 5.1 \( \square \)

Remark 5.3. The subgroup of \( B_n \ (n \geq 4) \) generated by \( x_1 \) and \( x_3 \) is well known to
be free abelian of rank 2; it is convenient to think of this group as the direct product
\( B_2 \times B_2 \), viewed as a subgroup of \( B_n \). The geometric realization \( K(x_1, x_3; r_{1,3}) \) of the
presentation \( \langle x_1, x_3; r_{1,3} \rangle \) of \( B_2 \times B_2 \) is an ordinary torus which is plainly aspherical. Furthermore, \( \langle x_1, x_3; x_1, r_{1,3} \rangle \) is then a presentation of the free cyclic group \( C \) generated by
\( x_3 \); the geometric realization \( K(x_1, x_3; x_1, r_{1,3}) \) thereof is a spine for a solid torus having
fundamental group \( C \), the identity among relations \( I_{1,3} \) is defined over the presentation
\( \langle x_1, x_3; x_1, r_{1,3} \rangle \) of \( C \) and, indeed, records the attaching map for a 3-cell to form the solid torus from the spine. Playing the same game as in the previous proof, but with the geometric realizations \( K(x_1, x_3; r_{1,3}) \) and \( K(x_1, x_3; x_1, r_{1,3}) \) rather than with \( K \) and \( L \), respectively, we find that \( \pi_2(K(x_1, x_3; x_1, r_{1,3})) \) is the free \( C \)-module freely generated by \( \langle I_{1,3} \rangle \); the induced map from \( H_2(K(x_1, x_3; r_{1,3})) \) to \( H_2(K(x_1, x_3; x_1, r_{1,3})) \) is an isomorphism, both groups are infinite cyclic, the group \( H_2(K(x_1, x_3; r_{1,3})) \) amounts to the second homology group \( H_2(B_2 \times B_2) \cong \mathbb{Z} \) of \( B_2 \times B_2 \), and under the Hurewicz map from \( \pi_2(K(x_1, x_3; x_1, r_{1,3})) \) to \( H_2(K(x_1, x_3; x_1, r_{1,3})) \) the free \( C \)-module generator \( [I_{1,3}] \) goes to a generator of \( H_2(K(x_1, x_3; x_1, r_{1,3})) \). Thus the surjection (5.11) comes down to the map

\[
H_2(B_2 \times B_2) \longrightarrow H_2(B_4)
\]

induced by the injection of \( B_2 \times B_2 \) into \( B_4 \). Hence the map (5.12) yields the generator of \( H_2(B_4) \) and, more generally, of \( H_2(B_n) \) for \( n \geq 4 \). This observation goes back at least to [8] where it is pointed out that this kind of construction, suitably extended, yields the entire integral homology of \( B_n \).

The universal covering space of \( K(x_1, x_3; x_1, r_{1,3}) \) is an infinite cylinder together with infinitely many 2-disks attached to the cylinder, more precisely, when the cylinder is realized in ordinary 3-space, the disks are attached to the interior of the cylinder. Thus the identity among relations \( I_{1,3} \) over the presentation \( \langle x_1, x_3; x_1, r_{1,3} \rangle \) of the free cyclic group \( C \) is realized geometrically as a tesselated sphere arising from the boundary surface of a solid cylinder having top and bottom a disk labelled \( r = x_1 \) and wall (lateral surface) labelled \( r_{1,3} \); the cylinder wall arises from glueing a rectangle labelled \( r_{1,3} \) along the edge labelled \( x_3 \), and the sphere arises from glueing the top and bottom disks along the circles labelled \( x_1 \). Indeed, the universal covering space of the spine \( K(x_1, x_3; x_1, r_{1,3}) \) arises from glueing infinitely many copies of that tesselated sphere along the top and bottom disks. That tesselated sphere can be seen as a cycle representing the generator of \( H_2(B_n) \).

### 6 Parallelohedral geometry and identities among relations

Using the language of identities among relations, we will now construct, under the circumstances of Theorem 5.1, a complete system of \( \pi_1(K) \)-module generators of \( \pi_2(K) \) and thereafter describe the induced map \( \pi_2(K) \to \pi_2(L) \cong H_2(K) \) so that the cokernel \( H_2(B_n) \cong \mathbb{Z}/2 \) thereof becomes explicitly visible. We will in particular exploit the geometry that underlies the hyperplane arrangements associated with various root systems. This will provide additional geometric insight into the braid group \( B_n \) and will in particular explain why the group \( H_2(B_n) \) is cyclic of order 2 and how the relation (5.6) arises. It will also make our reasoning self-contained. For ease of the reader, we note that the method of pictures developed in [4], [9] and [17] yields a straightforward procedure for reading off the identity sequences (6.2), (6.5), (6.8) and (6.10) below.

We begin with recalling some polyhedral geometry. A parallelohedron is a convex polyhedron whose translates by a lattice have disjoint interior and cover \( n \)-dimensional real affine space. This notion is implicit in [25] and was introduced in [33] and [34]. In dimension 2, the only parallelohedra are the parallelogram and the hexagon. In dimension
3, there are exactly five parallelohedra: the cube, hexagonal prism, truncated octahedron, rhombic dodecahedron, and the elongated rhombic dodecahedron. As a cellular complex, the permutahedron $P_n$ of order $n$ is the geometric realization of the poset of ordered partitions of the set $\{1, 2, \ldots, n\}$. Equivalently, when the symmetric group $S_n$ on $n$ letters acts on $\mathbb{R}^n$ via permutation of coordinates, the permutahedron is the convex hull of the $S_n$-orbit of a generic point. Thus $S_n$ acts on the permutahedron of order $n$, the action being free on the set of vertices. In particular, the 1-skeleton of the permutahedron of order $n$ is the Cayley complex of $S_n$ relative to the standard generators; here the convention is that any generator that is an involution is represented by a single unoriented edge rather than by a pair of oppositely oriented edges. Permutahedra were explored already in [29] (not under this name which seems to have been introduced much later). A permutahedron is a special kind of parallelohedron. Permutahedra are explored e.g. on p. 65/66 of [10]. The permutahedra form an operad [1, 22] (p. 98). This fact is behind the cellular models for iterated loop spaces developed in [23]. The permutahedron $P_3$ is a hexagon and the permutahedron $P_4$ is the truncated octahedron, that is, the Archimedean solid having as faces eight hexagons and six squares, the corresponding tesselation of real affine 3-space being the bitruncated cubic honeycomb. In general, $P_n$ is an omnitruncated $(n - 1)$-cell.

Given a general hyperplane arrangement, we will refer to the convex hull $A$ of a Weyl group orbit of a point in the interior of a chamber as a Coxeter polytope; when the Weyl group is finite, a classifying space for the corresponding Artin group arises from $A$ by the operation of suitably identifying faces of the polytope $A$. For the braid group, this construction was written down in [6]; the resulting cell complex is actually dual to the cell complex constructed in [14] and shown there to be aspherical. This latter cell complex, in turn, arises from a kind of lexicographic cell decomposition of the configuration space of $n$ particles moving in the plane yielding a poset that provides a classifying space for the pure braid group; the operation of taking orbits relative to the symmetric group on $n$ letters then furnishes a classifying space for the ordinary braid group. For a general Artin group, the classifying space arising from a Coxeter polytope can be found in [28] and is also lurking behind the approach in [31]. For the braid group, a suitable comparison of various operads of the same kind as the permutahedra operad mentioned above also leads to this classifying space [11]. The true reason for the quotient $K$ of the Coxeter polytope with faces suitably identified to have the homotopy groups $\pi_j$ zero for $j \geq 2$ is that, when $n \geq 2$ is fixed, the universal covering space of $K$ is appropriately tiled by translates of the permutahedron $P_n$; indeed this universal covering space is a product of $n - 1$ trees (a copy of the real line for $n = 2$).

For example, for $j = 1, 2, 3$, the convex hull $A_j$ of a point in the interior of a chamber of the root system $A_j$ is an interval for $j = 1$, a hexagon for $j = 2$, and a truncated octahedron for $j = 3$. Thus $A_n$ is exactly the permutahedron $P_{n+1}$. Henceforth we will use the notation $A_n$. The braid group $B_n$ is the Artin group associated with the Coxeter graph $A_{n-1}$. In this case, the faces are precisely polytopes associated with the full subgraphs of $A_{n-1}$ with $n - 2$ nodes, and the operation of suitably identifying faces of the Coxeter polytope yields a cell complex whose 2-skeleton is the geometric realization $K(X, R)$ of the Artin presentation (5.4) of $B_n$. 

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We have already noted that the association

\[ I_{j,k} \mapsto r_{j,k} \quad (1 \leq j, k \leq n - 1, \; j < k - 1) \]

induces the composite

\[ \pi_2(L) \rightarrow H_2(K, \hat{K}) \quad (6.1) \]

of the isomorphism \( \pi_2(L) \rightarrow H_2(K) \) with the isomorphism from \( H_2(K) \) to \( H_2(K, \hat{K}) \).

**Proposition 6.1.** As a \( B_n \)-module, \( \pi_2(K) \) has a complete system of generators

\[ [J_{j,k,\ell}] \in \pi_2(K) \subseteq \pi_2(K, K^1), \; 1 \leq j < k < \ell \leq n - 1 \]

given by suitable identities \( J_{j,k,\ell} \) among the relations of the Artin presentation (5.4) of \( B_n \). Furthermore, the composite of the induced map from \( \pi_2(K) \) to \( \pi_2(L) \) with (6.1) is given by the associations

\[ [J_{j,j+1,j+2}] \mapsto 2r_{j,j+2}, \; [J_{j,j+1,k}] \mapsto r_{j+1,k} - r_{j,k}, \; [J_{j,k-1,k}] \mapsto r_{j,k} - r_{j,k-1}, \]

in particular, that map is zero on generators of the kind \([J_{j,k,\ell}]\) with \( k - j \geq 2 \) and \( \ell - k \geq 2 \). Consequently \( H_2(B_n) \) is cyclic of order 2 and each free generator of \( H_2(K) \) corresponding to a generator of \( H_2(K, \hat{K}) \) of the kind \( r_{j,k} \) for \( k - j \geq 2 \) goes to the single non-zero element of \( H_2(B_n) \).

We begin with the preparations for a formal proof of this proposition: Let \( n \geq 4 \) and consider the Artin presentation (5.4) of \( B_n \). The possible 3-dimensional parallelohedra that can arise from the full subgraphs of \( A_{n-1} \) with three nodes are the truncated octahedron (Coxeter graph \( A_3 \)), the hexagonal prism (Coxeter graph \( A_1 \times A_2 \)), and the cube (Coxeter graph \( A_1 \times A_1 \times A_1 \)). We will now realize each of these parallelohedra by an identity among relations. We will proceed for general \( n \geq 4 \):

To realize the truncated octahedron we note that, for each \( 1 \leq j \leq n - 3 \), the subgroup of \( B_n \) generated by \( x_j, x_{j+1}, x_{j+2} \) is a copy of \( B_4 \). Accordingly, consider the identity

\[ J_{j,j+1,j+2} = (q_1, q_2, q_3, q_4, q_5, \ldots, q_{13}, q_{14}) \quad (6.2) \]

among relations arising from reading off an identity sequence from a truncated octahedron, each hexagon being labelled with the relator \( r_{j,j+1} \) or \( r_{j+1,j+2} \) and each square being
labelled with the relator \( r_{j,j+2} \):

\[
\begin{align*}
q_1 &= (x_j^{-1} x_{j+1}^{-1} x_{j+2}^{-1} x_j^{-1}, r_{j,j+2}) \\
q_2 &= (x_j^{-1} x_{j+1}^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+2}, r_{j,j+1}) \\
q_3 &= (x_j^{-1} x_{j+1}^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1}, r_{j+1,j+2}) \\
q_4 &= (x_j^{-1} x_{j+1}^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1} x_{j+2}, r_j^{-1}) \\
q_5 &= (x_j^{-1} x_{j+1}^{-1} x_{j+2}^{-1} x_j^{-1}, r_{j,j+1}) \\
q_6 &= (x_j^{-1} x_{j+1}^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1}, r_{j+1,j+2}) \\
q_7 &= (x_j^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1}^{-1} x_j^{-1}, r_{j,j+2}) \\
q_8 &= (x_j^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1}^{-1} x_j^{-1} x_{j+2}, r_{j,j+2}) \\
q_9 &= (x_j^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1}^{-1} x_j^{-1} x_{j+2} x_j, r_{j+1,j+2}) \\
q_{10} &= (x_j^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1}^{-1} x_j^{-1}, r_{j,j+2}) \\
q_{11} &= (x_j^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1}^{-1} x_j^{-1} x_j x_{j+1}, r_{j+1,j+2}) \\
q_{12} &= (x_j^{-1} x_{j+2}^{-1} x_j^{-1} x_{j+1}^{-1} x_j^{-1} x_j x_{j+2}, r_{j,j+2}) \\
q_{13} &= (x_j^{-1} x_{j+2}^{-1} x_j^{-1} x_j^{-1}, r_{j,j+1}) \\
q_{14} &= (x_j^{-1} x_{j+2}^{-1} x_j^{-1}, r_{j,j+2})
\end{align*}
\]

For \( B_4 \), the construction stops at this stage. In particular, attaching a 3-cell to the geometric realization \( K(x_1, x_2, x_3; r_{1,2}, r_{2,3}, r_{1,3}) \) via the attaching map given by \( L_{1,2,3} \) yields a classifying space of \( B_4 \). This amounts precisely the operation of identifying faces of the truncated octahedron mentioned earlier. Thus the kernel of \( \pi_2(K, K^1) \to Z[B_4] \langle x_1, x_2, x_3 \rangle \) is the free \( B_4 \)-module freely generated by \( [A] \) and the resulting \( B_4 \)-chain complex is a free resolution of the integers over \( B_4 \); cf. e. g. [31]. Consequently

\[
\begin{array}{cccc}
0 & \longrightarrow & \pi_2(K) & \longrightarrow & \pi_2(K, K^1) & \longrightarrow & \pi_1(K^1) & \longrightarrow & B_4 & \longrightarrow & 1
\end{array}
\]

(6.4)
is then a free crossed resolution of \( B_4 \); cf. [10] for the notion of free crossed resolution.

When \( n \geq 5 \), additional identities among relations arise, according to the possible full subgraphs \( A_2 \times A_1 \) of \( A_{n-1} \): For each pair \( (j, k) \) with \( 1 \leq j < k \leq n-1 \) and \( j < k-2 \), the subgroup of \( B_n \) generated by \( x_j, x_{j+1}, x_k \) is isomorphic to the direct product \( B_3 \times B_2 \) of a copy of \( B_3 \) with a copy of \( B_2 \). Accordingly, consider the identity

\[
J_{j,j+1,k} = (q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8)
\]

(6.5)
among relations arising from reading off an identity sequence from a prism having base and top a hexagon labelled \( r_{j,j+1} \) and having, furthermore, six lateral squares, each square
being labelled with the relators \( r_{j,k} \) or \( r_{j+1,k} \):

\[
\begin{align*}
q_1 &= (x_{j+1}^{-1}x_j^{-1}x_{j+1}^{-1}r_{j,j+1}^{-1}) \\
q_2 &= (x_{j+1}^{-1}x_j^{-1}x_{j+1}^{-1}x_jx_k^{-1}r_{j,j+1}^{-1}) \\
q_3 &= (x_{j+1}^{-1}x_j^{-1}x_{j+1}^{-1}x_j^{-1}r_{j,k}^{-1}) \\
q_4 &= (x_{j+1}^{-1}x_j^{-1}x_{j+1}^{-1}x_k^{-1}r_{j,k}^{-1}) \\
q_5 &= (x_{j+1}^{-1}x_j^{-1}x_k^{-1}x_{j+1}^{-1}x_jx_{j+1}r_{j,j+1}) \\
q_6 &= (x_{j+1}^{-1}x_j^{-1}x_j^{-1}r_{j,j+1}^{-1}) \\
q_7 &= (x_{j+1}^{-1}x_j^{-1}x_k^{-1}r_{j,k}) \\
q_8 &= (x_{j+1}^{-1}x_k^{-1}r_{j+1,k}).
\end{align*}
\]

(6.6)

With the notation \( F_{3,2} \) for the free group on \( x_j, x_{j+1}, x_k \) and \( C_{3,2} \) for the free crossed \( F_{3,2} \)-module on the relators \( r_{j,j+1}, r_{j,k}, r_{j+1,k} \), the sequence

\[
0 \longrightarrow \mathbb{Z}[B_3 \times B_2] \langle \mathcal{J}_{j,j+1} \rangle \longrightarrow C_{3,2} \xrightarrow{\partial} F_{3,2} \longrightarrow B_3 \times B_2 \longrightarrow 1 \tag{6.7}
\]

is a free crossed resolution of \( B_3 \times B_2 \).

Likewise, for each pair \((j, k)\) with \(1 \leq j < k \leq n - 1\) and \(j < k - 2\), the subgroup of \( B_n \) generated by \( x_j, x_{k-1}, x_k \) is isomorphic to the direct product \( B_2 \times B_3 \) of a copy of \( B_2 \) with a copy of \( B_3 \). Accordingly, consider the identity

\[
\mathcal{J}_{j,k-1} = (q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8) \tag{6.8}
\]

among relations arising from reading off an identity sequence from a prism having base and top a hexagon labelled \( r_{k-1,k} \) and having, furthermore, six lateral squares, each square being labelled with the relators \( r_{j,k-1} \) or \( r_{j,k} \):

\[
\begin{align*}
q_1 &= (x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}r_{k-1,k}^{-1}) \\
q_2 &= (x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}r_{j,k}) \\
q_3 &= (x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}x_j^{-1}r_{j,k-1}) \\
q_4 &= (x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}x_j^{-1}r_{j,k}) \\
q_5 &= (x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}r_{j,k-1}) \\
q_6 &= (x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}x_j^{-1}r_{j,k-1}) \\
q_7 &= (x_k^{-1}x_k^{-1}x_k^{-1}x_k^{-1}r_{j,k-1}) \\
q_8 &= (x_k^{-1}x_k^{-1}r_{j,k}).
\end{align*}
\]

(6.9)

When \( n \geq 6 \), more identities among relations arise, according to the possible full subgraphs \( A_1 \times A_1 \times A_1 \) of \( A_{n-1} \): For each triple \((j, k, \ell)\) with \(1 \leq j < k < \ell \leq n - 1\), \( k - j \geq 2\), and \( \ell - k \geq 2\), the subgroup of \( B_n \) generated by \( x_j, x_k, x_\ell \) is free abelian of rank three. The familiar commutator identity

\[
[x, y] \cdot y[x, z] \cdot [y, z] \cdot z[y, x] \cdot [z, x] \cdot x[z, y] = e
\]

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valid in any group entails the following identity
\[ \mathcal{J}_{j,k,\ell} = (q_1, q_2, q_3, q_4, q_5, q_6) = ((e, r_{i,j}), (x_j, r_{i,k})(e, r_{j,k}), (x_k, r_{i,j}^{-1})(e, r_{j,k}^{-1}),(x_i, r_{j,k}^{-1})) \] (6.10)

among relations. This identity among relations also arises from reading off an identity sequence from a cube whose pairs of opposite lateral squares are labelled with the relators \( r_{j,k}, r_{j,\ell}, \) or \( r_{k,\ell} \). With the notation \( F_{2,2,2} \) for the free group on \( x_j, x_k, x_\ell \) and \( C_{2,2,2} \) for the free crossed \( F_{2,2,2} \)-module on the relators \( r_{j,k}, r_{j,\ell}, r_{k,\ell} \), the resulting sequence
\[
0 \rightarrow \mathbb{Z}[B_2^{\times 3}][\mathcal{J}_{j,k,\ell}] \rightarrow C_{2,2,2} \xrightarrow{\partial} F_{2,2,2} \rightarrow B_2^{\times 3} \rightarrow 1 \] (6.11)
is then a free crossed resolution of \( B_2 \times B_2 \times B_2 \).

We can now prove Proposition 6.1. When \( n \geq 4 \), as a \( B_n \)-module, \( \pi_2(K) \) is generated by the elements
\[ [\mathcal{J}_{j,k,\ell}] \in \pi_2(K) \subseteq \pi_2(K, K^1), \ 1 \leq j < k < \ell \leq n - 1. \]

Indeed, the induced map \( \pi_2(K) \rightarrow \pi_2(K, K^1)_{\text{Ab}} \) is still injective and \( \pi_2(K, K^1)_{\text{Ab}} \) amounts to the free \( B_n \)-module freely generated by the 2-cells of \( K \) or, equivalently, by the set \( R \) of relators. Let \( \mathbb{Z}[B_n] \) denote the integral group ring of \( B_n \) and let \( \mathbb{Z}[B_n]\langle X \rangle \) be the free \( B_n \)-module freely generated by \( X \). The map \( \pi_2(K, K^1)_{\text{Ab}} \rightarrow \mathbb{Z}[B_n]\langle X \rangle \) induced by \( \partial: \pi_2(K, K^1) \rightarrow \pi_1(K^1) \) has kernel the isomorphic image of \( \pi_2(K) \) and, together with the obvious map \( \mathbb{Z}[B_n]\langle X \rangle \rightarrow \mathbb{Z}[B_n] \), yields the beginning of the familiar free resolution of the integers in the category of \( \mathbb{Z}[B_n] \)-modules, well explored at various places in the literature [11, 12, 28, 31]. These remarks establish Proposition 6.1. The 14-gon and the octagons discussed above are related with the geometry of posets of maximal chains of the symmetric group on \( n \) letters, cf. e. g. [20].

### 7 Universal central extensions

The Schur multiplicator of a group (later identified as the second homology group) was originally discovered by Schur in the search for central extensions to realize projective representations by linear ones. In this spirit, we will now push further the situation of Example 3.5. The starting point is the (familiar) observation that, given the group \( G \) with \( H_1(G) \) free abelian or zero, for any abelian group \( A \), the universal coefficient map
\[
H^2(G, A) \longrightarrow \text{Hom}(H_2(G), A) \] (7.1)
is an isomorphism whence the central extensions of \( G \) by the abelian group \( A \) are then classified by \( \text{Hom}(H_2(G), A) \).

We will refer to a central extension
\[
0 \rightarrow H_2(G) \rightarrow U \rightarrow G \rightarrow 1 \] (7.2)
as being \textit{universal} provided it has the following universal property: Given the central extension
\[
e: 0 \rightarrow A \rightarrow E \xrightarrow{\pi} G \rightarrow 1,
\]
the identity of $G$ lifts to a commutative diagram

$$
\begin{array}{c}
0 & \longrightarrow & \text{H}_2(G) & \longrightarrow & U & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow \vartheta & & \downarrow \Theta & & \downarrow \text{Id} & & & & \\
0 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & G & \longrightarrow & 1
\end{array}
$$

(7.3)

of central extensions in such a way that the assignment to $e$ of $\vartheta$ induces the universal coefficient map (7.1).

When $\text{H}_1(G)$ is free abelian or zero (that is, $G$ is perfect), it is straightforward to construct a universal central extension of $G$ and standard abstract nonsense reveals that a universal extension is then unique up to isomorphism. Here the notion of universal central extension is to be interpreted with a grain of salt, though, since this notion is lingua franca only for the case where $G$ is perfect [24] and can then be characterized in other ways.

**Theorem 7.1.** Under the circumstances of Theorem 4.1 the central extension (4.1) is the universal central extension of $G$.

**Proof.** This is an immediate consequence of the freeness of the $G$-crossed module $C$. We leave the details to the reader. 

Thus, in particular, the free $B_n$-crossed module $C_n$ in Theorem 5.1 above yields the universal central extension of $B_n$. Likewise the central extension (4.4) is the universal central extension of the free abelian group $A$ on the set $S$. Furthermore, the map (5.11) is the universal map induced from the injection of $A$ in $B_4$ (or more generally $B_n$) via the universal property of the universal central extension.

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