QCTL model-checking with QBF solvers

Akash Hossain  François Laroussinie
IRIF, Univ. Paris Diderot  IRIF, Univ. Paris Diderot

October 8, 2020

Abstract

Quantified CTL (QCTL) extends the temporal logic CTL with quantifications over atomic propositions. This extension is known to be very expressive: QCTL allows us to express complex properties over Kripke structures (it is as expressive as MSO). Several semantics exist for the quantifications: here, we work with the structure semantics, where the extra propositions label the Kripke structure (and not its execution tree), and the model-checking problem is known to be PSPACE-complete in this framework. We propose a new model-checking algorithm for QCTL based on a reduction to QBF. We consider several reduction strategies and we compare them with a prototype (based on several QBF solvers) on different examples.

1 Introduction

Temporal logics have been introduced in computer science in the late 1970’s by Pnueli [24]; they provide a powerful formalism for specifying correctness properties of evolving systems. Various kinds of temporal logics have been defined, with different expressive power and algorithmic properties. For instance, the Computation Tree Logic (CTL) expresses properties of the computation tree of the system under study (time is branching: a state may have several successors), and the Linear-time Temporal Logic (LTL) expresses properties of one execution at a time (a system is viewed as a set of executions).

Temporal logics allow model checking, i.e. the automatic verification that a finite state system satisfies its expected behavioural specifications [25][7]. It is well known that CTL model-checking is PTIME-complete and LTL model-checking (based on automata techniques) is PSPACE-complete. Verification tools exist for both logics and model-checking is now commonly used in the design of critical reactive systems. The main limitation to this approach is the state-explosion problem: symbolic techniques (for example with BDD), SAT-based approaches, or partial order reductions have been developed and they are impressively successful. The SAT-based model-checking consists in using SAT-solvers in the decision procedures. It was first developed for bounded model-checking (to search
for executions whose length is bounded by some integer, satisfying some temporal property) which can be reduced to some satisfiability problem and then can be solved by a SAT-solver [5]. SAT approaches have also been extended to unbounded verification and combined with other techniques [22]. Many studies have been done in this area, and it is widely considered as an important approach in practice, which complements other symbolic techniques like BDD ones (see [4] for a survey).

In terms of expressiveness, CTL (or LTL) still has some limitations: in particular, it lacks the ability of counting. For instance, it cannot express that an event occurs (at least) at every even position along a path, or that a state has two successors. In order to cope with this, temporal logics have been extended with propositional quantifiers [27]: those quantifiers allow for adding fresh atomic propositions in the model before evaluating the truth value of a temporal-logic formula. That a state has at least two successors can then be expressed (in quantified CTL, hereafter written QCTL) by saying that it is possible to label the model with atomic proposition $p$ in such a way that there is a successor that is labelled with $p$ and one that is not.

Different semantics for QCTL have been studied in the literature depending on the definition of the labelling: either it refers to the finite-state model – it is the structure semantics – or it refers to the execution tree – it is the tree semantics. Both semantics are interesting and have been extensively studied [16, 13, 23, 14, 9, 17]. While the tree semantics allow us to use the tree automata techniques to get decision procedures (model-checking and satisfiability are TOWER-complete [17]), the situation is quite different for the structure semantics: in this framework, model-checking is PSPACE-complete and satisfiability is undecidable [13].

In this paper, we focus on the structure semantics. We first motivate this choice by showing that QCTL may encode many logics, for example we explain how to reduce model-checking for SML (Sabotage Modal Logic) [29] to the QCTL model-checking problem. Then we propose a model-checking algorithm based on a reduction to QBF (propositional logic augmented with quantifiers): given a Kripke structure $K$ and a QCTL formula $\Phi$, we show how to build a QBF formula $\hat{\Phi}^K$ which is valid iff $K \models \Phi$. It is natural to use QBF quantifiers to deal with propositional quantifiers of QCTL. Of course, QBF-solvers are not as efficient as SAT-solvers, but still much progress has been made and QBF-solvers have already been considered for model-checking, as in [11, 8]. Here we propose several reductions depending on the way of dealing with nested temporal modalities, and we compare them with a prototype we implemented (connected to different solvers: Z3 [10], qfm [8], cqesto [15] and qfun [26]). As far as we know, it is the first implementation of a model-checker for QCTL.

Here, our first objective is to use the QBF-solver as a tool to check complex properties over limited size models, and this is therefore different from the classical use of SAT-based techniques which are precisely applied to solve verification problems for very large systems.

The outline of the paper is as follows: we begin with setting up the necessary formalism in order to define QCTL and to discuss its semantics. We then devote
Section 3 to the different reductions to QBF. Finally, Section 4 contains several practical results and examples.

2 Definitions

2.1 Kripke structures

Let $AP$ be a finite set of atomic propositions.

**Definition 1** A Kripke structure is a tuple $K = \langle V, E, \ell \rangle$, where $V$ is a finite set of vertices (or states), $E \subseteq V \times V$ is a set of edges (we assume that for any $x \in V$, there exists $x' \in V$ s.t. $(x, x') \in E$), and $\ell : V \to 2^{AP}$ is a labelling function.

An infinite path (also called an execution) in a Kripke structure is an infinite sequence $\rho = x_0x_1x_2\ldots$ such that for any $i$ we have $x_i \in V$ and $(x_i, x_{i+1}) \in E$. We write $\text{Path}_\omega K$ for the set of infinite paths of $K$ and $\text{Path}_\omega K(x)$ for the set of infinite paths issued from $x \in V$. Given such a path $\rho$, we use $\rho_{\leq i}$ to denote the $i$-th prefix $x_0\ldots x_i$, $\rho_{\geq i}$ for the $i$-th suffix $x_ix_{i+1}\ldots$, and $\rho(i)$ for the vertex $x_i$. The size of $K$ is $|V| + |E|$.

Given a set $P \subseteq AP$, two Kripke structures $K = (V, E, \ell)$ and $K' = (V', E', \ell')$ are said $P$-equivalent (denoted by $K \equiv_P K'$) if $V = V'$, $E = E'$, and for every $x \in V$ we have: $\ell(x) \cap P = \ell'(x) \cap P$.

2.2 QCTL

This section is devoted to the definition of the logic QCTL, which extends the classical branching-time temporal logic CTL with quantifications over atomic propositions.

**Definition 2** The syntax of QCTL is defined by the following grammar:

$$ QCTL \ni \varphi, \psi ::= q | \neg \varphi | \varphi \lor \psi | EX \varphi | EU \psi | A \varphi U \psi | \exists p. \varphi $$

where $q$ and $p$ range over $AP$.

QCTL formulas are evaluated over states of Kripke structures:

**Definition 3** Let $K = (V, E, \ell)$ be a Kripke structure, and $x \in V$. The semantics of QCTL formulas is defined inductively as follows:

- $K, x \models p$ iff $p \in \ell(x)$
- $K, x \models \neg \varphi$ iff $K, x \not\models \varphi$
- $K, x \models \varphi \lor \psi$ iff $K, x \models \varphi$ or $K, x \models \psi$
Let $\mathcal{K}, x \models \text{EX} \varphi$ iff $\exists (x, x') \in E$ s.t. $\mathcal{K}, x' \models \varphi$

$\mathcal{K}, x \models \text{EF} \varphi$ iff $\exists \rho \in \text{Path}_K(x)$, $\forall i \geq 0$ s.t. $\mathcal{K}, \rho(i) \models \psi$ and for any $0 \leq j < i$, we have $\mathcal{K}, \rho(j) \models \varphi$

$\mathcal{K}, x \models \text{A} \varphi \psi$ iff $\forall \rho \in \text{Path}_K(x)$, $\exists i \geq 0$ s.t. $\mathcal{K}, \rho(i) \models \psi$ and for any $0 \leq j < i$, we have $\mathcal{K}, \rho(j) \models \varphi$

$\mathcal{K}, x \models \exists p. \varphi$ iff $\exists K' \equiv_{\text{AP}(\rho)} K$ s.t. $K', x \models \varphi$

In the sequel, we use standard abbreviations such as $\top$, $\bot$, $\land$, $\Rightarrow$ and $\iff$. We also use the additional temporal modalities of $\text{CTL}$: $\text{AX} \varphi = \neg \text{EX} \neg \varphi$, $\text{EE} \varphi = \text{E} \land \text{U} \varphi$, $\text{AF} \varphi = \text{A} \land \text{U} \varphi$, $\text{EG} \varphi = \neg \text{AF} \neg \varphi$, $\text{AG} \varphi = \neg \text{EF} \neg \varphi$, $\text{E} \varphi \text{W} \psi = \neg \text{A} \neg \psi \text{U} (\neg \varphi \land \neg \varphi)$ and $\text{A} \varphi \text{W} \psi = \neg \text{E} \neg \psi \text{U} (\neg \varphi \land \neg \varphi)$.

Moreover, we use the following abbreviations related to quantifiers over atomic propositions: $\forall p. \varphi = \neg \exists p. \neg \varphi$, and for a set $P = \{p_1, \ldots, p_k\} \subseteq \text{AP}$, we write $\exists P.\varphi$ for $\exists p_1, \ldots, \exists p_k.\varphi$ and $\forall P.\varphi$ for $\forall p_1, \ldots, \forall p_k.\varphi$.

The size of a formula $\varphi \in \text{QCTL}$, denoted $|\varphi|$, is defined inductively by: $|\varnothing| = 1$, $|\neg \varphi| = |\exists \varphi| = |\text{EX} \varphi| = 1 + |\varphi|$, $|\varphi \lor \psi| = |\text{E} \varphi \text{U} \psi| = |\text{A} \varphi \text{U} \psi| = 1 + |\varphi| + |\psi|$. The temporal height of $\varphi$, denoted $\text{th}(\varphi)$, is the maximum number of nested temporal modalities in $\varphi$: $\text{th}(q) = 0$, $\text{th}(\varphi \land \psi) = \text{th}(\varphi \lor \psi) = \max(\text{th}(\varphi), \text{th}(\psi))$, $\text{th}(\neg \varphi) = \text{th}(\exists \varphi)$, $\text{th}(\text{EX} \varphi) = 1 + \text{th}(\varphi)$, and $\text{th}(\text{E} \varphi \text{U} \psi) = \text{th}(\text{A} \varphi \text{U} \psi) = 1 + \max(\text{th}(\varphi), \text{th}(\psi))$. And given a subformula $\psi$ in $\varphi$, the temporal depth of $\psi$ in $\varphi$ (denoted $\text{td}_\varphi(\psi)$) is the number of temporal modalities having $\psi$ in their scope in $\varphi$.

Two $\text{QCTL}$ formulas $\varphi$ and $\psi$ are said to be equivalent (written $\varphi \equiv \psi$) iff for any structure $\mathcal{K}$, any state $x$, we have $\mathcal{K}, x \models \varphi$ iff $\mathcal{K}, x \models \psi$. This equivalence is substitutive.

A formula $\varphi$ is said to be in Negation Normal Form (NNF) when the negations are only applied to atomic propositions in $\varphi$: any $\text{QCTL}$ formula is equivalent to some formula in NNF built from operators in $\{\land, \lor, \exists, \forall, \text{EX}, \text{AX}, \text{E} \text{U}, \text{A} \text{U}, \text{E} \text{W}, \text{A} \text{W}\}$ and literals $p$ and $\neg p$ with $p \in \text{AP}$.

### 2.3 Discussion on the semantics.

The semantics we defined is classically called the structure semantics (or Kripke semantics in [13]): a formula $\exists p. \varphi$ holds true in a Kripke structure $\mathcal{K}$ iff there exists a $p$-labelling of the structure $\mathcal{K}$ such that $\varphi$ is satisfied. Another well-known semantics coexists in the literature for propositional quantifiers, the tree semantics: $\exists p. \varphi$ holds true when there exists a labelling by $p$ of the execution tree (the infinite unfolding) of the Kripke structure under which $\varphi$ holds. If, for $\text{CTL}$, interpreting formulas over the structure or the execution tree is equivalent, this is not the case for $\text{QCTL}$. For example, $\forall p. (p \Rightarrow \text{EX} p)$ specifies the existence of a self-loop in the current state when interpreted in the structure semantics, and it is never true in the tree semantics. Finally note that there is also the amorphous semantics [13], where $\exists p. \varphi$ holds true at a state $x$ in some Kripke

---

1 If $\varphi \equiv \psi$, replacing the subformula $\varphi$ by $\psi$ in a formula $\varphi$ does not change the truth value of $\varphi$. 

---

4
structure $\mathcal{K}$ if, and only if, there exists some Kripke structure $\mathcal{K}'$ with a state $x'$ such that $x$ and $x'$ are bisimilar, and for which there exists a $p$-labelling making $\varphi$ hold true at $x'$. With this last semantics, the logic is insensitive to unwinding, and more generally it is bisimulation-invariant (contrary to the two previous semantics). Now we compare the tree and the structure semantics.

**Complexity**  
First note that these two semantics do not have the same algorithmic properties: if QCTL model-checking and satisfiability are TOWER-complete for the tree semantics (the algorithms are based on tree automata techniques), QCTL model-checking is PSPACE-complete for the structure semantics but satisfiability is undecidable (see [17] for a survey).

**Expressive power**  
In both semantics, QCTL is as expressive\(^2\) as the Monadic Second-Order Logic over the finite structures or the infinite trees (depending on the semantics) and as QCTL$^*$ (the extension of the standard CTL$^*$ with state formulas $\exists p.\varphi$). Note also that any QCTL formula is equivalent to a formula in prenex normal form (we will use this result in next sections). All these results are presented in [17].

### 2.4 Motivations and examples for QCTL in the structure semantics

First we present several examples of formulas to illustrate the expressive power of QCTL. Then we will consider a more complex problem showing how QCTL can be useful to encode complex specifications written in other logics.

#### 2.4.1 Examples of QCTL formulas

QCTL allows us to express complex properties over Kripke structures: for example, we can build a characteristic formula (up to isomorphism) of a structure or reduce model-checking problems for multi-player games to QCTL model-checking [18]. Below, we give several examples of counting properties, to illustrate the expressive power of propositional quantifiers.

The first formula below expresses that there exists a unique reachable state satisfying $\varphi$, and the second one states that there exists a unique immediate successor satisfying $\varphi$:

\[
\begin{align*}
\sf{E}_p & = \sf{E} \phi \land \forall p. (\sf{E}p \land \phi) \Rightarrow \sf{A} \phi (\Rightarrow p) \tag{1} \\
\sf{E}_n & = \sf{E} \phi \land \forall p. (\sf{E}p \land \phi) \Rightarrow \sf{A} \phi (\Rightarrow p) \tag{2}
\end{align*}
\]

where we assume that $p$ does not appear in $\varphi$. Consider the formula (1): if there were two reachable states satisfying $\varphi$, then labelling only one of them with $p$ would falsify the $\sf{A} \phi$ subformula. For (2), the argument is similar.  

\(^2\)This would require adequate definitions, since a temporal logic formula may only deal with the reachable part of the model, while MSO has a more global point of view.
The existence of at least \( k \) successors satisfying a given property can be expressed with:

\[
E \geq k X \varphi = \exists p_1 \ldots \exists p_k. \left( \bigwedge_{1 \leq i \leq k} \mathbf{E} (p_i \wedge \bigwedge_{i' \neq i} \neg p_{i'}) \wedge \mathbf{A} X \left( \bigvee_{1 \leq i \leq k} p_i \Rightarrow \varphi \right) \right) \quad (3)
\]

And we can define \( E \geq k X \varphi \) as \( E \geq k X \varphi \wedge \neg E \geq k+1 X \varphi \). Note that these examples show why \( \mathbf{QCTL} \) formulas are not bisimulation-invariant.

When using \( \mathbf{QCTL} \) to specify properties, one often needs to quantify (existentially or universally) over one reachable state we want to mark with a given atomic proposition. To this aim, we add the following abbreviations:

\[
\exists^1 p. \varphi = \exists p. (E = 1 F p) \wedge \varphi \quad \forall^1 p. \varphi = \forall p. ((E = 1 F p) \Rightarrow \varphi)
\]

### 2.4.2 From Sabotage Modal Logic to \( \mathbf{QCTL} \)

\( \mathbf{QCTL} \) can be used to encode (quite easily) many problems for other temporal or modal logics. For example, in [19], \( \mathbf{QCTL} \) is used to decide problems for multi-agent systems (the tree and structure semantics are both used according to the type of strategies allowed in the system). Here we consider another example to motivate the use of \( \mathbf{QCTL} \) with the structure semantics: the model-checking problem for the Sabotage Modal Logic (SML).

SML is a modal logic containing modalities which may delete transitions in the model [29, 1, 20, 21, 2]. SML formulas are built from Boolean connectives, atomic propositions and the following modalities: \( \lozenge \) (and its dual \( \square \)) and \( -\lozenge \) (and its dual \( -\square \)). \( \lozenge \) is equivalent to the \( \mathbf{CTL} \) modality \( \mathbf{E}X \) and \( -\lozenge \) is an edge removal operator. Given a Kripke structure \( \mathcal{K} = (V, E, \ell) \) and a state \( x \in V \), the semantics of these modalities is as follows:

\[
\langle V, E, \ell \rangle, x \models \lozenge \varphi \quad \text{iff} \quad \exists (x, x') \in E \text{ s.t. } \langle V, E, \ell \rangle, x' \models \varphi
\]

\[
\langle V, E, \ell \rangle, x \models \lozenge^c \varphi \quad \text{iff} \quad \exists(y, y') \in E \text{ s.t. } \langle V, E \setminus \{(y, y')\}, \ell \rangle, x \models \varphi
\]

The expressive power of SML is interesting, one can express properties over the frame of the underlying Kripke structure. For example, the formula \( \lozenge \top \wedge \square \lozenge \top \wedge \square \bot \) holds true in \( \mathcal{K}, x \) iff the structure \( \mathcal{K} \) is restricted to a single selfloop from \( x \). Other examples to characterize structures (e.g. cycles of length \( n \)) are given in [2]. We know that SML satisfiability is undecidable and SML model-checking is \( \mathsf{PSPACE} \)-complete [1, 20, 21].

A local variant is also used in literature where the removed transition has to be issued from the current state. This is done with the modality \( \lozenge^c \) defined by:

\[
\langle V, E, \ell \rangle, x \models \lozenge^c \varphi \quad \text{iff} \quad \exists(x, x') \in E \text{ s.t. } \langle V, E \setminus \{(x, x')\}, \ell \rangle, x \models \varphi
\]

We can easily reduce a model-checking instance for SML (including the local modality): \( \mathcal{K}, x \models \Phi \) to a model-checking instance for \( \mathbf{QCTL} \) \( \mathcal{K}', x' \models \Phi' \). The
\[ \overline{p}^n = p \quad \overline{\varphi \land \psi}^n = \overline{\varphi}^n \land \overline{\psi}^n \quad \overline{\neg \varphi}^n = \neg \overline{\varphi}^n \quad \overline{\top}^n = \top \]

\[ \Diamond \varphi^n = \text{EX}(\text{inter} \land \bigwedge_{0 \leq i < n} \neg \text{del}_i \land \text{EX} \overline{\varphi}^n) \]

\[ \Diamond \overline{\varphi}^n = \exists \text{del}_n. \left( \text{EX} \text{EX}(\text{inter} \land \bigwedge_{0 \leq i < n} \neg \text{del}_i \land \text{del}_n) \land \overline{\varphi}^{n+1} \right) \]

\[ \Diamond_{\text{loc}} \varphi^n = \exists \text{del}_n. \left( \text{EX}(\text{inter} \land \bigwedge_{0 \leq i < n} \neg \text{del}_i \land \text{del}_n) \land \overline{\varphi}^{n+1} \right) \]

Table 1: Transformation rules from SML to QCTL

The main idea of the reduction consists in identifying \( \Diamond \)-removed edges by labelling intermediary states along them with some fresh atomic propositions \( \text{del}_i \). If \( k \) is the \( \{ \Diamond, \Diamond_{\text{loc}} \} \)-height of \( \Phi \), we will use the propositions \( \text{del}_0, \ldots, \text{del}_{k-1} \) in the formula \( \Phi' \). Moreover, we can see that any modality \( \Diamond \) allows us to remove any edge in \( K \) (with no relationship with the current state where the formula is interpreted). To encode this, we need to complete the Kripke structure with additional edges connecting any pair of states (including selfloops). Formally \( K' \) is then defined by:

- \( V' = V \cup \{ v_{xy} \mid (x, y) \in E \} \),
- \( E' = \{(x, v_{xy}), (v_{xy}, y) \mid (x, y) \in E \} \cup \{(x, y) \mid x, y \in V \} \)
- \( \ell'(x) = \ell(x) \) for \( x \in V \) and \( \ell(v_{xy}) = \{ \text{inter} \} \) for every \( v_{xy} \) in \( V' \).

where \( \text{inter} \) is a fresh atomic proposition used to mark intermediary states along initial \( K \)'s edges.

Formally we define \( \overline{\varphi}^n \) where \( n \) is the \( \{ \Diamond, \Diamond_{\text{loc}} \} \)-depth of subformula \( \varphi \) in the main formula \( \Phi \) (\( i.e. \varphi \) occurs in the scope of \( n \) nested modalities \( \{ \Diamond, \Diamond_{\text{loc}} \} \) in \( \Phi \)). The definition is given in Table 1 and the correctness of the reduction is stated as follows:

**Proposition 1** Let \( K \) be a Kripke structure \( (V, E, \ell) \), \( x \in V \), and \( \Phi \in \text{SML} \). Given a \( \Phi \)-subformula \( \psi \) such that \( \psi \) occurs at \( \{ \Diamond, \Diamond_{\text{loc}} \} \)-depth \( n \) in \( \Phi \), and given \( E_d \subseteq E \) with \( E_d = \{(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})\} \), we have:

\[ \langle V, E \setminus E_d, \ell \rangle, x \models \psi \iff \langle V', E', \ell'' \rangle, x \models \overline{\psi}^n \]

where \( K' = (V', E', \ell') \) is defined as above, and \( \ell''(v) \) for \( v \in V' \) is defined as follows: if \( v = v_{xy} \) for some \( i \in \{0, \ldots, n-1\} \), then \( \ell''(v) = \{ \text{inter}, \text{del}_i \} \), else \( \ell''(v) = \ell'(v) \).

\( ^3 \)i.e. the maximal number of nested \( \{ \Diamond, \Diamond_{\text{loc}} \} \)-modalities in \( \Phi \).
**Proof:** The proof is done by structural induction on $\psi$. We only consider the modalities $\Diamond$ and $\Diamond\Diamond$:

- $\psi = \Diamond\psi_1$: If $\langle V, E \setminus E_d, \ell \rangle, x \models \psi_1$, there exists $(x, y) \in E \setminus E_d$ such that $\langle V, E \setminus E_d, \ell \rangle, y \models \psi_1$. By i.h., we get $\langle V', E', \ell'' \rangle, y \models \tilde{\psi}_n$, from which we deduce $\langle V', E', \ell'' \rangle, x \models \tilde{\psi}_n$ because the intermediary state $v_{xy}$ is labelled by $\ell''(v_{xy}) = \{\text{inter}\}$ (i.e. no $\text{del}$ is true at $v_{xy}$). The other direction proceeds in the same way.

- $\psi = \Diamond\Diamond\psi_1$: If $\langle V, E \setminus E_d, \ell \rangle, x \models \psi_1$, there exists $(y, y') \in E \setminus E_d$ such that $\langle V, E \setminus (E_d \cup \{(y, y')\}), \ell \rangle, y \models \psi_1$. By i.h., we get $\langle V', E', \ell'' \rangle, x \models \tilde{\psi}_n$ with a labelling $\ell''$ as described in the proposition, in particular we have $\ell''(v_{yy'}) = \{\text{inter}, \text{del} \}$. We can deduce that we have $\langle V', E', \ell'' \rangle, x \models \tilde{\psi}_n$ where $\ell''$ coincides with $\ell''$ except for $v_{yy'}$ that is labelled only by $\text{inter}$. The other direction is similar.

\[ \square \square \]

In particular, we have $\mathcal{K}, x \models \Phi$ iff $\mathcal{K}', x \models \tilde{\Phi}^0$ which provides the reduction.

**Example 1** To illustrate the construction of $\mathcal{K}'$, consider the structure $\mathcal{K}$ in Figure 1 and its corresponding $\mathcal{K}'$. And $\mathcal{K}, x_1 \models \Diamond\Box\Diamond\top$ is reduced to: $\mathcal{K}', x_1 \models \text{EX}\left(\text{inter} \land \text{EX}(\text{inter} \land \text{EX}\left(\exists \text{del}_0.\text{EX}\left(\text{inter} \land \text{del}_0\right)\land \neg \text{EX}(\text{inter} \land \neg \text{del}_0 \land \text{EX}\top))\right)\right))$. Note that the formula holds true because $x_3$ has 2 successors.

![Figure 1: Example of reduction for SML model-checking.](image)

### 3 Model-checking QCTL

Model-checking QCTL is a PSPACE-complete problem, and it is NP-complete for the restricted set of formulas of the form $\exists P. \varphi$, with $P \subseteq \text{AP}$ and $\varphi \in \text{CTL}$ [10]. In this section, we give a reduction from the QCTL model-checking problem to the QBF validity problem.

In the following, we assume a Kripke structure $\mathcal{K} = \langle V, E, \ell \rangle$ with $V = \{x_0, \ldots, x_n\}$, an initial state $x_0 \in V$ and a QCTL formula $\Phi$ to be fixed. We
also assume w.l.o.g. that every quantifier ∃ and ∀ in Φ introduces a fresh atomic proposition, and distinct from the propositions used in K. We use $AP^Φ_Q$ to denote the set of quantified atomic propositions in Φ.

These assumptions allow us to use an alternative notation for the semantics of Φ-subformulas: the truth value of $ϕ$ will be defined for a state $x$ in $K$ within an environment $ε: AP^Φ_Q \rightarrow 2^V$, that is a partial mapping associating a subset of vertices to a proposition in $AP^Φ_Q$. We use $K, x \models ε \ s.t. K, x \models_ε \varphi$ to denote that $ϕ$ holds at $x$ in $K$ within $ε$. Therefore the $K$’s labelling $ℓ$ is not modified when a subformula is evaluated, only $ε$ is extended with labellings for new quantified propositions.

Formally the main changes of the semantics are as follows:

$$K, x \models_ε p \iff \left( (p \in AP^Φ_Q \text{ and } x \in ε(p)) \text{ or } (p \notin AP^Φ_Q \text{ and } p \in ℓ(x)) \right)$$

$$K, x \models_ε \exists p. ϕ \iff \exists V' \subseteq V \ s.t. K, x \models_ε [p \mapsto V'] ϕ$$

where $ε[p \mapsto V']$ denotes the mapping which coincides with $ε$ for every proposition in $AP^Φ_Q \setminus \{p\}$ and associates $V'$ to $p$.

We use this new notation in order to better distinguish initial $K$’s propositions and quantified propositions to make proofs simpler. Of course, there is no semantic difference: $K, x \models \Phi$ iff $K, x \models_ε \emptyset$.

In next sections, we consider general quantified propositional formulas (QBF) of the form:

$$\text{QBF} \ni α, β ::= q \mid α \lor β \mid α \land β \mid \neg α \mid \exists q. α$$

We will also use the following classical abbreviations: $α \land β = \neg(\neg α \lor β)$, $α \Rightarrow β = \neg α \lor β$, $α \Leftrightarrow β = (α \Rightarrow β) \land (β \Rightarrow α)$, and $\forall q. α = \neg \exists q. \neg α$. The formal semantics of a formula $α$ is defined over a Boolean valuation for free variables in $α$ (i.e. propositions which are not bound by a quantifier), and it is defined as usual. A formula is said to be closed when it does not contain free variables. In the following, we use the standard notion of validity for closed QBF formulas.

Our aim is then to build a (closed) QBF formula $\hat{Φ}_x^0$ such that $\hat{Φ}_x^0$ is valid iff $Φ$ holds true at $x_0$ in $K$.

### 3.1 Overview

We present several reductions from the QCTL model-checking problem to the QBF validity problem. These reductions are defined as two steps processes: first a pre-processing is carried out on the original QCTL formula and then a syntactic translation into QBF is applied. All these reductions differ only from the pre-processing step, indeed they share the final translation denoted $\hat{Φ}^{x,P}$ for a QCTL formula $Φ$, a vertex $x$ and a subset $P \subseteq AP^Φ_Q$, and defined in Table 2 (its correctness will be established in Theorem 2).

Of course, the construction of the QBF formula $\hat{Φ}^{x,P}$ uses the structure $K$ (the transition relation $E$ and the labelling $ℓ$). The vertex $x$ is the state where the QCTL formula $ϕ$ has to be interpreted. Every quantification $\exists p$ in $ϕ$ is

---

4 We assume w.l.o.g. that every quantifier ∃ or ∀ introduces a new proposition.
replaced by a sequence of quantifications $\exists p_0 \ldots \exists p_n$ in the QBF formula in order to encode the $p$-labelling of $K$ (i.e. a truth value for every state of the structure): the variable $p_i$ is assumed to be true if $x_i$ is labelled by $p$. The set $P$ in $\hat{\varphi}^{x,P}$ is the set of quantified propositions: when evaluating a proposition $p$ at $x$, we need to know whether $p$ belongs to the set of initial atomic propositions of $K$ (and its truth value depends on $\ell(x)$), or $p$ is a quantified proposition introduced by some quantifier $\exists$ or $\forall$ in $\varphi$ (and its truth value is the variable $p^i$). The encoding of the temporal modalities are explained below.

### 3.1.1 Unfolding characterization of the until operators

In Table 2 the temporal modalities are encoded in QBF by unfolding of the transition relation $E$. For example, $EX\varphi$ holds true at $x$ iff there exists some $(x, x') \in E$ such that $x' \models \varphi$, this is precisely the meaning of the corresponding rule in the table with the disjunction over the $x$'s. And the truth value of $EF\varphi$ at $x$ is encoded as a disjunction of the truth values of $\varphi$ at any state $x'$ reachable from $x$ with an arbitrary number of transitions in $E$ ($E^*$ denotes the reflexive and transitive closure of $E$). The rules for $AX$ and $AG$ are similar. Finally the rules for $EU$ and $AU$ can be seen as a depth-first way to look for a path (or a set of paths for $AU$) satisfying the $\mathsf{Until}$ modality. Other temporal modalities ($e.g.$ $EG$, $AF$, $EW$, or $AW$) can be translated into QBF by using their definitions in terms of $EU$ and $AU$ or by using adhoc rules as for $AG$.

Applied in a top down manner, the equivalences in Table 2 define a translation from QCTL to QBF. Before stating the correctness of this translation, we need to associate a Boolean valuation $v_x$ for variables in $\text{AP}^\varphi_Q \times V$ to an environment $\varepsilon$ for $\text{AP}^\varphi_Q$. We define $v_x$ as follows: for any $p \in \text{AP}^\varphi_Q$ and $x \in V$, $v_x(p^x) = \top$ iff $x \in \varepsilon(p)$. Now we have the following theorem which establishes the correctness of the translation $\hat{\varphi}^{x,P}$ (its proof is in \[A\]):

**Theorem 2** Given a QCTL formula $\Phi$, a Kripke structure $K = \langle V, E, \ell \rangle$, a state $x \in V$, an environment $\varepsilon : \text{AP}^\varphi_Q \mapsto 2^V$ and a $\Phi$-subformula $\varphi$, if $\hat{\varphi}^{x,\text{dom}(\varepsilon)}$ is defined inductively w.r.t. the rules of Table 2 we have: $K, x \models_v \varphi$ iff $v_x \models \hat{\varphi}^{x,\text{dom}(\varepsilon)}$.

It is easy to deduce that we have: $K, x_0 \models_0 \Phi$ iff $\hat{\Phi}^{x_0,0}$ is valid. In the sequel we use $\hat{\Phi}^{x_0}$ to denote $\hat{\Phi}^{x_0,0}$. The first reduction (called $\text{UU}$) consists in applying the translation directly on the QCTL formula without any preprocessing:

**Reduction UU:** Given a Kripke structure $K$, a state $x$ in $K$ and a QCTL formula $\Phi$, the reduction $\text{UU}$ (for $K, x \models_0 \Phi$) is defined as the QBF formula $\hat{\Phi}^x$. Its correction is a direct consequence of Theorem 2. The size of the QBF formula is in $O((|\Phi| \cdot |V|)!^{\text{th}(\Phi)})$.

---

5The presence of dedicated rules for $EF$, $AG$ and $AX$ is due to the fact that other reductions eliminate all temporal modalities except $EX$, $AX$, $EF$ and $AG$. Of course the formulas provided by these rules are equivalent modulo Boolean simplifications — to the formulas we could obtain by using the standard definitions $EF = E \cup \mathsf{AW}$, $AG = \neg EF\neg$ and $AX = \neg EX\neg$.  

10
\[ \neg \phi^x,P = \neg \hat{\phi}^x,P \]  
\[ \phi \lor \psi^x,P = \hat{\phi}^x,P \lor \hat{\psi}^x,P \]  
\[ \phi \land \psi^x,P = \hat{\phi}^x,P \land \hat{\psi}^x,P \]  
\[ \exists p.\hat{\phi}^x,P = \exists p^{x_0} \ldots p^{x_n} \land \neg \hat{\phi}^x,P \cup \{p\} \land \hat{\phi}^{x,P} = \begin{cases}  
\hat{\phi}^{x,P} & \text{if } p \in P 
\top & \text{if } p \notin P \text{ and } p \in \ell(x) 
\bot & \text{otherwise} 
\end{cases} \]  
\[ \text{EX} \phi^x,P = \bigvee_{(x,x') \in E} \hat{\phi}^{x',P} \]  
\[ \text{AX} \phi^x,P = \bigwedge_{(x,x') \in E} \hat{\phi}^{x',P} \]  
\[ \text{EF} \phi^x,P = \bigvee_{(x,x') \in E^*} \hat{\phi}^{x',P} \]  
\[ \text{AG} \phi^x,P = \bigwedge_{(x,x') \in E^*} \hat{\phi}^{x',P} \]  
\[ \text{E} \phi \text{U} \psi^x,P = \text{E} \phi^x,P \text{U} \psi^x,P \]  
\[ \text{A} \phi \text{U} \psi^x,P = \text{A} \phi^x,P \text{U} \psi^x,P \]  
\[ \text{E} \phi \text{U} \psi^x,P,X = \text{E} \phi^x,P \text{U} \psi^x,P \cup \{x\} \]  
\[ \text{A} \phi \text{U} \psi^x,P,X = \text{A} \phi^x,P \text{U} \psi^x,P \cup \{x\} \]  

\begin{table}[h]  
\centering  
\begin{tabular}{|c|c|}  
\hline  
\text{Table 2: Translation from QCTL to QBF} & \hline  
\end{tabular}  
\end{table}  

The main drawback of this naive reduction is the size of the QBF formula (any Until modality may induce a formula whose size is in \(O(|V|)!\)). Nevertheless, one can notice that the reduction does not introduce new quantified propositions to encode the temporal modalities, contrary to other methods we will see later. Note also that this method can be adapted to bounded model-checking by fixing a bound on the number of unfoldings.

### 3.2 Fixpoint characterization of the until operators

Here we present the fixpoint method (called FP) for dealing with the modalities \( \text{AU} \) and \( \text{EU} \). Let \( \phi \) and \( \psi \) be two QCTL-formulas. The idea of the method is to build a QCTL formula that is equivalent to \( \text{E} \phi \text{U} \psi \) (or \( \text{A} \phi \text{U} \psi \)) by using only the modalities \( \text{EX}, \text{AX} \) and \( \text{AG} \). We first have the following lemma:

**Lemma 3** For any QCTL formula \( E \phi \psi \), we have:

\[ E \phi \psi \equiv \forall z. \left( \text{AG} (z \leftrightarrow (\psi \lor (\phi \land \text{EX} z))) \Rightarrow z \right) \]

**Proof:** Let \( x \) be a state in a Kripke structure \( K \). Let \( \theta \) be the formula \((\text{AG} (z \leftrightarrow (\psi \lor (\phi \land \text{EX} z))) \Rightarrow z). \)
Assume \( K, x \Vdash E\varphi \psi \). We can use the standard characterization of \( EU \) as fixpoint: \( x \) belongs to the least fixpoint of the equation \( Z = \psi \lor (\varphi \land EX Z) \) where \( \psi \) (resp. \( \varphi \)) is here interpreted as the set of states satisfying \( \psi \) (resp. \( \varphi \)). Therefore any \( z \)-labelling of reachable states from \( x \) corresponding to a fixpoint will have the state \( x \) labelled. This is precisely what is specified by the QCTL formula.

Now if \( K, x \Vdash \theta \) for every \( z \)-labelling corresponding to a fixpoint of the previous equation, this is the case for the \( z \)-labelling of the states reachable from \( x \) and satisfying \( E\varphi \psi \), and we deduce \( K, x \Vdash E\varphi \psi \). □ □

And we have the same result for \( AU \) (whose proof is similar):

**Lemma 4** For any QCTL formula \( A\varphi \psi \), we have:

\[
A\varphi \psi \equiv \forall z. \left( AG(z \iff (\psi \lor (\varphi \land AX z)) \Rightarrow z) \right)
\]

As a direct consequence, we get the following result:

**Proposition 5** For any QCTL formula \( \Phi \), we can build an equivalent QCTL formula \( fpc(\Phi) \) such that: (1) \( fpc(\Phi) \) is built up from atomic propositions, Boolean operators, propositional quantifiers and modalities \( EX \) and \( AG \), and (2) the size of \( fpc(\Phi) \) is in \( O(|\Phi|^{th(\Phi)}) \).

The exponential size of \( fpc(\Phi) \) comes from the fact that \( \iff \) is not considered as a primitive of QBF and then induces a duplication of subformulas when applying the transformation rules based on equivalences of Lemmas 3 and 4 (otherwise its size would be linear in \(|\Phi|\)). Moreover, the temporal height of \( fpc(\Phi) \) is at most \( th(\Phi) + 1 \).

Now we can formally define the reduction \( FP \):

**Reduction \( FP \):** Given a Kripke structure \( K \), a state \( x \) in \( K \) and a QCTL formula \( \Phi \), the reduction \( FP \) is defined as the QBF formula \( \widehat{fpc}(\Phi)^x \). Its correction is a direct consequence of Proposition 5 and Theorem 2. The size of the QBF formula is \( O((|\Phi| \cdot |K|)^{th(\Phi)+1}) \).

The exponential size comes from the nesting of temporal modalities and each one may provide a QBF formula of size \((|V| + |E|)\). Note also that the number of propositional variables in the QBF formula is bounded by \(|\Phi| \cdot |V|\).

### 3.3 Reduction via flat formulas (\( FFP \))

To avoid the size explosion of \( \widehat{fpc}^x \), one can use an alternative approach for prenex QCTL formulas. Remember that any QCTL formula can be translated into an equivalent QCTL formula in prenex normal form whose size is linear in the size of the original formula [17].

---

6Labelling other states does not matter.

7The temporal height can be increased by 1 if \( \Phi \) has an until operator whose subformulas are Boolean combinations of atomic propositions.
In the sequel, we use $S_{\Phi}$ to denote the set of temporal subformulas occurring in $\Phi$ at a temporal depth greater than or equal to 1.

A CTL formula is said to be basic when it is of the form $\text{EX} \alpha$, $\text{E} \alpha \text{U} \beta$ or $\text{A} \alpha \text{U} \beta$ where $\alpha$ and $\beta$ are Boolean combinations of atomic propositions (a basic formula is then formula starting with a temporal modality and whose temporal height is 1). It is easy to observe that any CTL formula can be translated into a QCTL formula with a temporal depth less or equal to 2:

**Proposition 6** For any CTL formula $\Phi$, we can build an equivalent QCTL formula $\Psi$ of the form: $\exists \kappa_1 \ldots \exists \kappa_m. \left( \Phi_0 \land \bigwedge_{1 \leq i \leq m} \text{AG}(\kappa_i \leftrightarrow \theta_i) \right)$ where $\Phi_0$ is a Boolean combination of basic CTL formulas and every $\theta_i$ is a basic CTL formula (for any $1 \leq i \leq m$). Moreover, $|\Psi|$ is in $O(|\Phi|)$.

**Proof:** Consider a CTL formula $\Phi$ built with temporal modalities in $\{\text{EX}, \text{EU}, \text{AU}\}$. The proof is done by induction over the size of $S_{\Phi}$. If $|S_{\Phi}| = 0$, the original formula is a Boolean combination of basic CTL formulas and it satisfies the property. Now, assume $|S_{\Phi}| > 0$. $\Phi$ must have at least one basic (strict) subformula $\theta_1$. And $\Phi$ is equivalent to the formula $\exists \kappa_1. (\Phi[\theta_1 \leftarrow \kappa_1] \land \text{AG}(\kappa_1 \leftrightarrow \theta_1))$, where $\kappa_1$ is a fresh atomic proposition, and $\phi[\alpha \leftarrow \beta]$ is $\phi$ where every occurrence of $\alpha$ is replaced by $\beta$. Indeed, any state reachable from the current state $x$ will be labelled by $\kappa_1$ iff $\theta_1$ holds true at that state (NB: the states that are not reachable from $x$ do not matter for the truth value of $\Phi$), and this enforces the equivalence. We have $|S_{\Phi[\theta_1 \leftarrow \kappa_1]}| < |S_{\Phi}|$, thus we can apply induction hypothesis to get:

$$\Phi[\theta_1 \leftarrow \kappa_1] = \exists \{\kappa_2 \ldots \kappa_m\}. \left( \Phi_0 \land \bigwedge_{2 \leq i \leq m} \text{AG}(\kappa_i \leftrightarrow \theta_i) \right) = \Psi'$$

where $\kappa_2 \ldots \kappa_m$ are fresh atomic propositions, $\Phi_0$ is a Boolean combination of basic CTL formulas and every $\theta_i$ is a basic CTL formula. Then we have:

$$\Phi = \exists \kappa_1. \left[ \left( \exists \{\kappa_2 \ldots \kappa_m\}. \left( \Phi_0 \land \bigwedge_{2 \leq i \leq m} \text{AG}(\kappa_i \leftrightarrow \theta_i) \right) \right) \land \text{AG}(\kappa_1 \leftrightarrow \theta_1) \right]$$

$$= \exists \{\kappa_1 \ldots \kappa_m\}. \left( \Phi_0 \land \bigwedge_{1 \leq i \leq m} \text{AG}(\kappa_i \leftrightarrow \theta_i) \right) = \Psi$$

Note that the last equivalence comes from the fact that no $\kappa_i$ with $i > 1$ occurs in $\theta_1$. By i.h. the size of $\Psi'$ is linear in $|\Phi[\theta_1 \leftarrow \kappa_1]|$, and the size of $\Phi[\theta_1 \leftarrow \kappa_1]$ is smaller than that of $\Phi$, therefore the size of $\Psi$ is linear in $|\Phi|$. \hfill $\Box$ $\Box$

And then we have:

**Proposition 7** For any QCTL formula $\Phi$, we can build an equivalent QCTL formula $\text{flat}_1(\Phi)$ of the form: $Q. \left( \Phi_0 \land \bigwedge_{1 \leq i \leq m} \text{AG}(\kappa_i \leftrightarrow \theta_i) \right)$ where $Q$ is a sequence of quantifications, $\Phi_0$ is a Boolean combination of basic CTL formulas, every $\kappa_i$ is an atomic proposition, and every $\theta_i$ (for $i = 1, \ldots, m$) is a basic CTL formula. $\text{And } |\text{flat}_1(\Phi)|$ is in $O(|\Phi|)$. 

13
Proof: From [17], we know how to build a prenex formula $\text{Prenex}(\Phi) = Q.\Gamma$ whose size is linear in $|\Phi|$. $\Gamma$ belongs to CTL, and then Proposition 6 allows us to build a QCTL formula $Q.\exists \kappa_1 \ldots \exists \kappa_m.\left(\Phi_0 \land \bigwedge_{1 \leq i \leq m} \text{AG}(\kappa_i \Leftrightarrow \theta_i)\right)$ equivalent to $\Phi$. Note that we have $\text{th}(\text{flat}_1(\Phi)) \leq 2$. □

We can define a new reduction:

Reduction FFP: Given a Kripke structure $\mathcal{K}$, a state $x$ in $\mathcal{K}$ and a QCTL formula $\Phi$, the reduction FFP is defined as the QBF formula $\text{fpc}(\text{flat}_1(\Phi))$. Its correction is a direct consequence of Proposition 7, Proposition 5 and Theorem 2. The size of the QBF formula is $O((|\Phi| \cdot |K|)^3)$. Therefore this reduction provides a PSPACE algorithm for QCTL model-checking. But there are two disadvantages to this approach. First, putting the formula into prenex normal form may increase the number of quantified atomic propositions and the number of alternations (which is in fine linear in the number of quantifiers in the original formula) [17]. For example, when extracting a quantifier $\forall$ from some EX modality, we need to introduce two propositions, this can be seen for the formula $\text{EX}((\forall p. (\text{AX}p \lor \text{AX}\neg p))$ which is translated as:

$$\exists z. \forall p. \forall z'. \left( (\text{EX}(z \land z') \Rightarrow \text{AX}(z \Rightarrow z')) \land \text{EX}(z \land (\text{AX}p \lor \text{AX}\neg p)) \right)$$

where the proposition $z$ is used to label a state, and $z'$ is used to enforce that only at most one successor is labelled by $z$. Of course, these two remarks may have a strong impact on the complexity of the decision procedure. Finally, note also that the resulting QBF formula is not in prenex normal form because $\text{fpc}$ may insert new quantifications.

Example 2 To illustrate the reduction, we consider the following (CTL) formula $\Phi = \text{EF}(\text{Eu}b \land \text{EX}(\text{Ec}))$. The formula $\text{fpc}(\text{flat}_1(\Phi))$ will provide the following QCTL formula:

$$\exists k_1, k_2, k_3. \left( [\forall k_4. \text{AG}(k_4 \Leftrightarrow (k_1 \land k_3) \lor \text{EX} k_4) \Rightarrow k_4] \land \text{AG}\left(k_1 \Leftrightarrow [\forall k_5. \text{AG}(k_5 \Leftrightarrow (b \lor (a \land \text{EX} k_5)) \Rightarrow k_5)] \land \text{AG}((k_2 \Leftrightarrow \text{EX} c) \land \text{AG}(k_3 \Leftrightarrow \text{EX} k_2) \right) \right)$$

where $k_1$ is used to label states satisfying $\text{Eu}b$, $k_2$ is for $\text{EX}c$, $k_3$ for $\text{EXEX}c$ and $k_4$ is used for the fixpoint encoding of the main $\text{EF}$ modality.

3.4 Method PNF

We now propose a reduction to get a QBF formula in prenex normal form. It is also based on a flattening of the formula (but slightly different from the one used for FFP), and uses a different technique to eliminate temporal modalities EU and AU.
First, we consider a QCTL formula \( \Phi \) under negation normal form (NNF): This transformation causes \( \Phi \) to be built from temporal modalities in \( S_{tmod} = \{ \text{EX}, \text{AX}, \text{EU}, \text{AU}, \text{EW}, \text{AW} \} \). Due to the NNF, we can consider a new flattening process where equivalences are replaced by implications: we just need to ensure that if a \( \kappa_i \) is true in some state \( x \) (and possibly makes some subformula true), then the formula \( \theta_i \) associated with \( \kappa_i \) in the flattening actually holds true at \( x \). We have the following proposition, whose proof is in [B].

**Proposition 8** For any QCTL formula \( \Phi \), we can build an equivalent QCTL formula \( \text{flat}_2(\Phi) \) in NNF and of the form:

\[
Q \exists \kappa_1 \ldots \exists \kappa_m. \left( \Phi_0 \land \bigwedge_{1 \leq i \leq m} \text{AG}(\kappa_i \Rightarrow \theta_i) \right)
\]

where \( Q \) is a sequence of quantifications, \( \Phi_0 \) is a CTL formula containing only the temporal modalities \( \text{EX}, \text{AX}, \text{EF} \) or \( \text{AG} \), and whose temporal height is less than or equal to 1, and every \( \theta_i \) is a basic CTL formula (with \( 1 \leq i \leq m \)). Moreover, \( |\text{flat}_2(\Phi)| \) is in \( O(|\Phi|) \) and \( \text{th}(\text{flat}_2(\Phi)) \leq 2 \).

From the previous proposition, we derive a new reduction based on a transformation (denoted \( \text{Replace}_{U,W}(\cdot) \)) of \( \text{flat}_2(\Phi) \): first we will add a new quantified proposition \( \chi \) to handle least fixpoints in \( \text{flat}_2(\Phi) \) and then we will rewrite every \( \text{flat}_2(\Phi) \)-subformulas of the form \( \text{AG}(\kappa_i \Rightarrow \theta_i) \) depending on the type of \( \theta_i \) in order to get a formula built from temporal modalities in \( \{ \text{EX}, \text{AX}, \text{EF}, \text{AG} \} \).

Assume \( \Psi = Q \exists \kappa_1 \ldots \exists \kappa_m. \left( \Phi_0 \land \bigwedge_{1 \leq i \leq m} \text{AG}(\kappa_i \Rightarrow \theta_i) \right) \) with the same form as in Proposition 8 we define \( \text{Replace}_{U,W}(\Psi) \) as:

\[
Q \exists \{\kappa_1 \ldots \kappa_m, \forall \chi \}. \left( \Phi_0 \land \bigwedge_{1 \leq i \leq m} \text{AG}(\kappa_i \Rightarrow \theta_i) \right)
\]

where the transformation \( \widehat{\theta}_i \) is defined by:

\[
\begin{align*}
\text{AG}(\kappa_i \Rightarrow \text{EX} \psi) &= \text{AG}(\kappa_i \Rightarrow \text{EX} \psi) \\
\text{AG}(\kappa_i \Rightarrow \text{AX} \psi) &= \text{AG}(\kappa_i \Rightarrow \text{AX} \psi) \\
\text{AG}(\kappa_i \Rightarrow \text{EU} \psi) &= \text{AG}(\kappa_i \Rightarrow (\psi \lor (\varphi \land \text{EX} \kappa_i))) \\
\text{AG}(\kappa_i \Rightarrow \text{AU} \psi) &= \text{AG}(\kappa_i \Rightarrow (\psi \lor (\varphi \land \text{AX} \kappa_i))) \\
\text{AG}(\kappa_i \Rightarrow \text{EU} \varphi) &= \text{EF}((\psi \lor (\varphi \land \text{EX} \chi)) \land \neg \chi) \lor \text{AG}(\kappa_i \Rightarrow \chi) \\
\text{AG}(\kappa_i \Rightarrow \text{AU} \varphi) &= \text{EF}((\psi \lor (\varphi \land \text{AX} \chi)) \land \neg \chi) \lor \text{AG}(\kappa_i \Rightarrow \chi)
\end{align*}
\]

The correctness of this transformation is stated as follows:

**Proposition 9** For any QCTL formula \( \Phi \), we have: \( \Phi \equiv \text{Replace}_{U,W}(\text{flat}_2(\Phi)) \).
Proof: We have to prove that the substitutions are correct. The proof is easy for weak Until modalities: for example, consider a state labelled by \( \kappa_i \) associated with some \( E \varphi W \psi \), then the substitution ensures that \( x \) satisfies either (1) \( \psi \) or (2) \( \varphi \) with a successor satisfying \( \kappa_i \) which ensures. Clearly such states labelled by \( \kappa_i \) belong to the corresponding greatest fixpoint.

Now consider the case of the transformation of \( AG(\kappa_i \Rightarrow E \varphi U \psi) \). The formula \( EF((\psi \lor (\varphi \land EX \chi)) \land \neg \chi) \lor AG(\kappa_i \Rightarrow \chi) \) specifies that either the \( \chi \)-labeling does not mark all states satisfying \( E \varphi U \psi \) (because there exists a reachable state which is not labelled by \( \chi \) when it should because it satisfies \( \psi \lor (\varphi \land EX \chi) \)) or every reachable state labelled by \( \kappa_i \) is also labelled by \( \chi \).

Now if we consider that this property should be true for every \( \chi \)-labelling (cf. the \( \forall \) quantifier in the definition of \( \text{Replace}_{U,W}(\text{flat}_2(\Phi)) \)), then we can deduce that every state labelled by \( \kappa_i \) has to belong to the least fixpoint (cf. Prop. 1, page 97 in [28]) and this ensures that \( E \varphi U \psi \) holds true for every state labelled by \( \kappa_i \).

Example 3 If we consider the formula \( \Phi = EF(EaUb \land AcW(EXd)) \). We have:

\[
\text{flat}_2(\Phi) = \exists k_1 k_2 k_3. (EF(k_1 \land k_3) \land AG(k_1 \Rightarrow EaUb) \land AG(k_2 \Rightarrow EXd) \land AG(k_3 \Rightarrow AcWk_2)).
\]

And then we have:

\[
\text{Replace}_{U,W}(\text{flat}_2(\Phi)) = \exists k_1 k_2 k_3. \forall \chi. \left( EF(k_1 \land k_3) \land \left[ EF((b \lor (a \land EX(k_4))) \land \neg k_4)) \lor AG(k_1 \Rightarrow k_4) \right] \land AG(k_2 \Rightarrow EXd) \land AG(k_3 \Rightarrow (k_2 \lor (c \land AX(k_3)))) \right)
\]

And we get a new reduction providing a smaller formula. Indeed, translating formulas of the form \( AG(p \Rightarrow EXq) \) in QBF provide a formula whose size is in \( O(|V| + |E|) \), that is in \( O(|K|) \).

Reduction PNF: Given a Kripke structure \( K \), a state \( x \) in \( K \) and a QCTL formula \( \Phi \), the reduction PNF is defined as the prenex QBF formula \( \text{Replace}_{U,W}(\text{flat}_2(\Phi))^x \). Its correction is a direct consequence of Proposition 9 and Theorem 2. The size of the QBF formula is \( O(|K| \cdot |\Phi|) \).

Again this provides another algorithm in \( \text{PSPACE} \).

3.5 Method based on BitVec to encode distance (FBV)

In the previous reduction, the modalities \( EU \) and \( AU \) may introduce an alternation of quantifiers: an atomic proposition \( \kappa \) is introduced by an existential quantifier, and then a universal quantifier introduces a variable \( \chi \) to encode the fixpoint characterisation of \( U \). We propose another reduction in order to avoid this alternation: for this, we will use bit vectors (instead of single Boolean values) to encode the distance (in terms of number of transitions) from the current state to a state satisfying the right-hand side of the Until modality. To build
such a formula, we have to fix the size of the bit vectors and then the maximum value we can represent, therefore the reduction is parameterised by a value $N$ and the bit vectors will be able to encode values from 0 to $N$ ($N$ will be set by $|V|$ when the reduction is applied over a Kripke structure). Therefore contrary to previous reductions, the correctness of the preprocessing will depend on the size (the number of states) of the structure we will consider.

As for PNF, we consider a QCTL formula $\Phi$ under negation normal form (NNF) and we use the flat$_2$ transformation, but we define a new transformation, called $\text{Replace}^2_{U,W}(\Psi, N)$, to replace temporal modalities.

For modalities $\textbf{E}W$ and $\textbf{A}W$, we use the same idea as in $\text{Replace}_{U,W}$ for method PNF. For the Until-based modalities, corresponding to least fixpoints, we use bit vectors $\kappa_i$ instead of a single Boolean value $\kappa_i$ to encode the truth value of $\textbf{E}\varphi U \psi$ or $\textbf{A}\varphi U \psi$. For a formula $\theta_i = \textbf{E}_\varphi U \psi$, the value $\kappa_i$ encodes in binary the distance from $x$ to a state satisfying $\psi$ along some path satisfying $\varphi U \psi$, and for $\theta_i = \textbf{A}_\varphi U \psi$, the value $\kappa_i$ encodes an overapproximation of the distance before reaching a state satisfying $\psi$ (and verifying $\varphi U \psi$) along any path issued from $x$.

In the new reduction we need to compare the values encoded by bit vectors with integer values encoded in binary. These comparisons will be done by propositional formulas. Consider a bit vector $\kappa_i$ instead of a single Boolean value $\kappa_i$ to encode the truth value of $\textbf{E}\varphi U \psi$ or $\textbf{A}\varphi U \psi$. For a formula $\theta_i = \textbf{E}_\varphi U \psi$, the value $\kappa_i$ encodes in binary the distance from $x$ to a state satisfying $\psi$ along some path satisfying $\varphi U \psi$, and for $\theta_i = \textbf{A}_\varphi U \psi$, the value $\kappa_i$ encodes an overapproximation of the distance before reaching a state satisfying $\psi$ (and verifying $\varphi U \psi$) along any path issued from $x$.

In the new reduction we need to compare the values encoded by bit vectors with integer values encoded in binary. These comparisons will be done by propositional formulas. Consider a bit vector $\kappa_i = \kappa_i^{k_{BV}} \ldots \kappa_i^0$ and an integer value $d$ encoded in binary with $k_{BV}$ bits $d^{k_{BV}} \ldots d^0$ where $\kappa_i^0$ and $d^0$ are the least significant digits of the values. In the following we identify 1 with $\top$ and 0 with $\bot$. We define two types of formulas $[\kappa_i = d]$ (i.e. the value encoded by $\kappa_i$ equals to $d$) and $[\kappa_i < d]$ (i.e. the value encoded by $\kappa_i$ is less than $d$) as follows:

$$[\kappa_i = d] = \bigwedge_{0 \leq j < k_{BV}} \kappa_i^j \Leftrightarrow d^j$$
$$[\kappa_i < d] = \bigvee_{0 \leq m < k_{BV}} (\lnot \kappa_i^m \land d^m) \land \bigwedge_{m < j < k_{BV}} (\kappa_i^j \Leftrightarrow d^j)$$

Assume $\Psi = \exists \kappa_1 \ldots \exists \kappa_m (\Phi_0 \land \bigwedge_{1 \leq i \leq m} \textbf{AG}(\kappa_i \Rightarrow \theta_i))$ with the same form as in Proposition 8, we define $\text{Replace}^2_{U,W}(\Psi, N)$ as:

$$\exists \{\hat{\kappa}_1 \ldots \hat{\kappa}_m\} \left(\tilde{\Phi}_0 \land \bigwedge_{1 \leq i \leq m} \textbf{AG}(\hat{\kappa}_i \Rightarrow \hat{\theta}_i)\right)$$

where:

- $k_{BV} = \lceil \log(N + 1) \rceil$,
- $\hat{\kappa}_i$ is a Boolean proposition (resp. a vector of $k_{BV}$ Boolean propositions) if $\theta_i$ is of the form $\textbf{E}_\psi$, $\textbf{A}_\psi$, $\textbf{E}_\varphi W \psi$ or $\textbf{A}_\varphi W \psi$ (resp. if $\theta_i$ is of the form $\textbf{E}_\varphi U \psi$ or $\textbf{A}_\varphi U \psi$),

---

8The value of $k_{BV}$ will depend on $N$. 17
Proof: Consider the formula

\[ \text{Replace size of the formula by a factor of } N \]

\[ \text{with } |x \in \text{Proposition } 10 \]

For any state \( x \) in a state \( \text{A} \) which ensures that

\[ \text{for all states along the prefix to the distance to } y, \]

For the other direction, the definition of a

\[ \text{ensured and in both case, the subformula will not be assumed to be true.} \]

Finally we can see that the transformation \( \text{Replace}_2^2 (\Phi), N) \) increases the size of the formula by a factor of \( N \cdot [\log(N + 1)] \). And then we have:

- the transformation \( \tilde{\theta}_i \) is defined as follows:

\[
\begin{align*}
\text{AG}(\kappa_i \Rightarrow \text{EX}\psi) &= \text{AG}(\kappa_i \Rightarrow \text{EX} \tilde{\psi}) \\
\text{AG}(\kappa_i \Rightarrow \text{AX}\psi) &= \text{AG}(\kappa_i \Rightarrow \text{AX} \tilde{\psi}) \\
\text{AG}(\kappa_i \Rightarrow \text{EU}\text{W}\psi) &= \text{AG}(\kappa_i \Rightarrow (\tilde{\psi} \lor (\tilde{\varphi} \land \text{EX} \kappa_i))) \\
\text{AG}(\kappa_i \Rightarrow \text{EU}\text{W}\psi) &= \text{AG}(\kappa_i \Rightarrow (\tilde{\psi} \lor (\tilde{\varphi} \land \text{AX} \kappa_i))) \\
\text{AG}(\kappa_i \Rightarrow \text{EU}\text{U}\psi) &= \text{AG} \left[ ((\mu_i = 0) \Rightarrow \tilde{\psi}) \land \bigwedge_{1 \leq d < N} (\mu_i = d) \Rightarrow (\tilde{\varphi} \land \text{EX} [\mu_i = d - 1]) \right] \\
\text{AG}(\kappa_i \Rightarrow \text{AU}\psi) &= \text{AG} \left[ ((\mu_i = 0) \Rightarrow \tilde{\psi}) \land \bigwedge_{1 \leq d < N} (\mu_i = d) \Rightarrow (\tilde{\varphi} \land \text{AX} [\mu_i < d]) \right]
\end{align*}
\]

- \( \tilde{\alpha} \) denotes the formula \( \alpha \) where every occurrence of \( \kappa_i \) is replaced by the formula \( [\mu_i < N] \) for every \( i \) such that \( \kappa_i \) is a bit vector (i.e. is associated with \( \text{EU} \) or \( \text{AU} \) in the flattening step),

And we have the following proposition which specifies that \( \text{Replace}_2^2 (\Phi), N) \) is equivalent to \( \Phi \) for every structure whose size is at most \( N \):

**Proposition 10** For any QCTL formula \( \Phi \), any Kripke structure \( K = (V, E, \ell) \) with \( |V| \leq N \) and any \( x \in V \), we have: \( K, x \models \Phi \) if and only if \( K, x \models \text{Replace}_2^2 (\Phi), N) \).

**Proof:** Consider the formula \( \text{EU}\text{U}\psi \) and some bit vector \( \mu_i \) associated with it. If \( K, x \models \text{EU}\text{U}\psi \), then there exists a path \( \rho \) from \( x \) satisfying \( \varphi\text{U}\psi \). The length of the prefix of \( \rho \) from \( x \) to the state \( y \) where \( \psi \) is true is at most \( |V| - 1 \) (we can assume the path to be simple). Therefore we can fix the values of \( \mu_i \) for all states along the prefix to the distance to \( y \) and we have the result. And for the states that do not satisfy \( \varphi\text{U}\psi \), we can set \( \mu_i \) to the value \( N \).

For the other direction, the definition of a \( \mu_i \) ensures that if the value \( v \) encoded by \( \mu_i \) in a given state \( x \) is less than \( N \), then \( v \) corresponds actually to the distance from \( x \) to a state satisfying \( \psi \) along a path satisfying \( \varphi\text{U}\psi \) (it decreases down to 0): this distance is maybe not the length of a shortest path but it is sufficient to conclude that \( \text{EU}\text{U}\psi \) holds true at \( x \). And indeed when the truth value of \( \text{EU}\text{U}\psi \) in a state \( x \) is needed, we interpret \( [\mu_i < N] \) at \( x \).

For \( \text{AU}\text{U}\psi \), we can see from the definition that if the value \( v \) encoded by \( \mu_i \) in \( x \) is less than \( N \), then all successors of \( x \) have an encoded value less than \( v \), which ensures that \( \text{AU}\text{U}\psi \) holds true at \( x \).

Note that when the value encoded by \( \mu_i \) is greater than \( N \), then nothing is ensured and in both case, the subformula will not be assumed to be true. \( \square \)

Finally we can see that the transformation \( \text{Replace}_2^2 (\-N) \) increases the size of the formula by a factor of \( N \cdot [\log(N + 1)] \). And then we have:
**Reduction FBV:** Given a Kripke structure $\mathcal{K}$, a state $x$ in $\mathcal{K}$ and a QCTL formula $\Phi$, the reduction FBV is defined as the prenex QBF formula $Replace_{U,W}(\text{flat}_2(\Phi), |V|)^x$. Its correction is a direct consequence of Proposition 10 and Theorem 2. The size of the QBF formula is in $O(|V| \cdot |\mathcal{K}| \cdot \lceil \log(|V| + 1) \rceil \cdot |\Phi|)$.

This provides another PSPACE algorithm but the size of the resulting QBF formula is larger than that obtained by the previous method. Finally one can notice that this approach would allow us to easily adapt the algorithm to bounded model-checking: instead of considering values from 1 to $|V|$ for $d$ in the definition of Until modalities, one could restrict the range to a smaller interval, to get a smaller QBF formula to check.

### 3.6 Dealing with $\exists^1$ and $\forall^1$

The quantifiers $\exists^1$ and $\forall^1$ are very useful in many specifications. It can be interesting to develop ad-hoc algorithms in order to improve the generated QBF formulas and to be able to choose the encoding of these operators. Formally we defined $\exists^1 p.\varphi$ as $\exists p.((E_{=1} F p) \land \varphi)$. In the following, we use the abbreviation $\text{uniq}(p)$ to denote $E_{=1} F p$ and we will see several methods to deal with it during the translation into QBF.

There are three possible encodings of $\text{uniq}(p)$:

1. by using its QCTL definition ($E_{=1} F p$): and translate this formula with the rules described above for the different methods.

2. by an explicit disjunction in the QBF formula: given a state $x$, $\text{uniq}(p)$ is equivalent to: $\left( \bigvee_{(x,y) \in E^*} (p^y \land \bigwedge_{z \neq y} \neg p^z) \right)$.

3. by Bit vectors: the quantifier ($\exists$ or $\forall$) associated with $p$ introduces a bit vector $\overline{p}$ of size $\lceil \log(|V| + 1) \rceil$ to store the number of the state selected by the quantifier: in the QBF formula, verifying that a state $x$ is labelled by $p$ consists in verifying that the number of $x$ equals to the value $\overline{p}$ (which can be encoded as a propositional formula). And $\text{uniq}(p)$ consists in verifying that there exists one reachable state whose number is $\overline{p}$. Note that this method reduces the number of quantified propositions (instead of having $|V|$ propositions as in the two first methods, we have only $\lceil \log(|V| + 1) \rceil$ propositions with this encoding).

Note that the encoding may provide a formula that is not anymore in PNF. To avoid this, we can gather all subformulas $\text{uniq}(-)$ in the main part of the QCTL formula in order to keep the prenex form. Indeed we can first observe the following equivalences when $\varphi$ does not depend on $p$:

$$
\varphi \land \exists p.\psi \equiv \exists p. (\varphi \land \psi) \quad \varphi \land \forall p.\psi \equiv \forall p. (\varphi \land \psi) \\
\varphi \Rightarrow \exists p.\psi \equiv \exists p. (\varphi \Rightarrow \psi) \quad \varphi \Rightarrow \forall p.\psi \equiv \forall p. (\varphi \Rightarrow \psi)
$$

19
Now consider a Prenex QCTL formula $Q \Phi$ where $Q$ contains $\exists^1$ or $\forall^1$ quantifications for $k$ propositions $p_1, \ldots, p_k$. Let $P_{\exists^1} \subseteq \{1, \ldots, k\}$ be the set of indexes such that $i \in P_{\exists^1}$ iff $p_i$ is introduced by some quantifier $\exists^1$ in $\Phi$. Let $P_{\forall^1}$ be $\{1, \ldots, k\} \setminus P_{\exists^1}$. The previous equivalences allow us to get a formula of the form:

$$\Phi' = \tilde{Q}(\text{uniq}(p_1) \ op_1 (\text{uniq}(p_2) \ op_2 (\ldots \text{uniq}(p_k) \ op_k \Phi)))$$

where (1) $\tilde{Q}$ is $Q$ where $\exists^1$ (resp. $\forall^1$) is replaced by $\exists$ (resp. $\forall$), and (2) $op_i$ is $\land$ (resp. $\Rightarrow$) if $i \in P_{\exists^1}$ (resp. $i \in P_{\forall^1}$). And it remains to see that the formula $\Phi'$ is equivalent to:

$$\Phi'' = Q \left( (\bigwedge_{i \in P_{\forall^1}} \text{uniq}(p_i)) \Rightarrow \left( (\bigwedge_{i \in P_{\exists^1}} \text{uniq}(p_i)) \land \Phi \right) \right)$$

The correctness of this equivalence is based on Proposition 11 in C.

Finally, the three previous encodings can be used to get a QBF formula. Note that the last two (explicit disjunction and bit vectors) will then provide a formula in prenex normal form. Moreover the first and third encodings use the structure $K$.

Note also that we could also consider a variant with $\exists \leq^1$ where the associated proposition cannot label more than one state. The same encodings described above can be done.

## 4 Experimental results

In this section, we consider four examples to evaluate and compare the different reductions. These problems can be solved efficiently without a QCTL model-checker, but they provide valuable insights on the performance of the reduction strategies, and the properties to be checked cannot be expressed with classical temporal logics. In every case we consider graphs that can easily be scaled up by tweaking a few parameters.

For these experiments, we use our prototype qctlmc\(^9\) to translate model-checking instances for QCTL into QBF instances. The reductions produce either a QBF formula in QCIR-G14 format\(^10\) or in the format used by the tool Z3. We considered the following QBF solvers\(^11\):

- Z3: it is a powerful SMT solver\(^10\) which also handles QBF instances.
  
  There are many features in Z3 but we only use the restricted part for QBF. Its admits any kind of QBF formula in an adhoc format (called Z3 format here). We used Z3 4.8.7 for the tests.

\(^9\)Our tool is available online [https://www.irif.fr/~francoisl/qctlmc.html](https://www.irif.fr/~francoisl/qctlmc.html), it implements the different reductions.

\(^10\)See [http://www.qbflib.org](http://www.qbflib.org)

\(^11\)Many QBF solvers handle only prenex CNF formulas, but here we need non CNF formula and we didn’t want to add extra formula rewritings, this explains our choice of solvers.
Figure 2: Structures for the reset property and the resources distribution.

- **qfm**: it is a QBF solver [8] with counterexample guided refinement (based on sat-solver cadical [3] or minisat [12]), it allows us to deal with general QBF formulas in QCIR format.

- **qfun**: it is a QBF solver based on Recursive Abstraction Refinement and machine learning [26]. It requires prenex formulas in QCIR format.

- **cqesto**: it is a QBF solver based on clause selection [15], it requires a prenex formula in QCIR format.

The solvers *cqesto* and *qfun* require prenex formulas and then can only be used with the reductions *PNF* and *FBV*. There are none of these restrictions for *Z3* and *qfm*. All results are presented in Subsection 4.3.

### 4.1 Reset property

We consider the *reset property*: the existence of a set of (at most) \( m \) states such that from any reachable state it is possible to reach at least one of the selected states. For this we can use \( \Lambda_m = \exists^1 p_1 \ldots \exists^1 p_m. (\Box (E \bigvee_{1 \leq i \leq m} p_i) \Box) \)

which selects \( m \) states by using the propositions \( p_i \)'s (the use of quantifier \( \exists^1 \) ensures that at most \( m \) states are selected).

Now given two parameters \( n, k \in \mathbb{N} \), we define the Kripke structure \( V_{n,k} = (V, E, \ell) \) that contains a root \( r \) and \( n \) different cycles of length \( k \) (and no atomic proposition). It is depicted at Figure 2. We then clearly have \( V_{n,k} \models \Lambda_m \) iff \( m \geq n \). The main characteristics of this example are: a simple temporal formula, the use of the \( \exists^1 \) operator and a Kripke structure with a low branching degree.
4.2 \( k \)-connectivity

Here, we consider an undirected graph, and we want to check whether there exist (at least) \( k \) internally disjoint paths\(^{14}\) from a vertex \( x \) to some vertex \( y \). A classical result in graph theory due to Menger ensures that, given two vertices \( x \) and \( y \) in a graph \( G \), the minimum number of vertices whose deletion makes that there is no more paths between \( x \) and \( y \) is equal to the maximum number of internally disjoint paths between these two vertices \(^6\).

We can encode these two ideas with the following QCTL formulas (interpreted in \( x \)):

\[
\Phi_k = \exists p_1 \ldots \exists p_{k-1}. \left( \bigwedge_{1 \leq i < k} \text{EX} (E(p_i \land \bigwedge_{j \neq i} \neg p_j) U y) \land \text{EX} \left( \bigwedge_{1 \leq i < k} \neg p_i \right) U y \right)
\]

(4)

\[
\Psi_k = \forall^1 p_1 \ldots \forall^1 p_{k-1}. \text{EX} \left( \bigwedge_{1 \leq i < k} \neg p_i \right) U y
\]

(5)

\( \Phi_k \) uses the labelling by the \( p_i \)’s to mark the internal vertices of \( k \) paths between the current position and the vertex \( y \). The modality \( \text{EX} \) is used to consider only the intermediate states (and not the starting state). The formula \( \Psi_k \) proceeds differently: the idea is to mark exactly \( k-1 \) states with \( p_1, \ldots, p_{k-1} \) and to verify that there still exists at least one path leading to \( y \) without going through the states labelled by some \( p_i \). By Menger’s Theorem, we know that these formulas are equivalent over undirected graphs.

We interpret these formulas over Kripke structures \( S_{n,m} \) with \( n \geq m \) (see Figure 2) which correspond to two kinds of grids \( n \times n \) connected by \( m \) edges (these edges are of the form \((q_{i,n}, r_{1,i})\) or \((q_{n,i}, r_{i,1})\)). The initial state is \( q_{1,1} \) and when evaluating \( \Phi_k \) or \( \Psi_k \) we assume the state \( r_{n,n} \) to be labelled by \( y \). In this context, we clearly have that \( \Phi_k \) and \( \Psi_k \) hold for true at \( q_{1,1} \) iff \( k \leq m \).

The first lesson of this example is that \( \Phi_k \) is much more difficult to verify than \( \Psi_k \): The number of temporal modalities is the main explanation. The way in which the formula is written is of great importance for the successful handling of even small examples.

4.3 Nim game

Nim game is a turn-based two-player game. A configuration is a set of heaps of objects and a boolean value indicating whose turn it is. At each turn, a player has to choose one non-empty heap and remove at least one object from it. The aim of each player is to remove the last object. Given a configuration \( c \) and a Player-\( J \) with \( J \in \{1, 2\} \), we can build a finite Kripke structure \( S_J \), where \( x_c \) is a state corresponding to the configuration \( c \) and use a QCTL formula \( \Phi_J^{\text{win}} \) such that \( S, x_c \models \Phi_J^{\text{win}} \) iff Player-\( J \) has a winning strategy from \( c \). Note that there is

\(^{14}\) Two paths \( \text{src} \leftrightarrow r_1 \leftrightarrow \ldots \leftrightarrow r_k \leftrightarrow \text{dest} \) and \( \text{src} \leftrightarrow r'_1 \leftrightarrow \ldots \leftrightarrow r'_{k'} \leftrightarrow \text{dest} \) are internally disjoint iff \( r_i \neq r'_j \) for any \( 1 \leq i \leq k \) and \( 1 \leq j \leq k' \). And note that with this definition, if there is an edge \((x, y)\), there exist \( k \) internally disjoint paths from \( x \) to \( y \) for any \( k \).
a simple and well-known criterion over the numbers of objects in each heap to decide who has a winning strategy, but we consider this problem just because it is interesting to illustrate what kind of problem we can solve with QCTL.

Each configuration corresponds to a state in $S_J$. Every move of Player-$J$ from a configuration $c$ to a configuration $c'$ provides a transition $(x_c, x_{c'})$ in $S_J$. However, a move of Player-$J$ from $c$ to $c'$ is encoded as two transitions $x_c \rightarrow x_{c,c'} \rightarrow x_{c'}$ where $x_{c,c'}$ is an intermediary state we use to encode a strategy for Player-$J$ (marking $x_{c,c'}$ by an atomic proposition will correspond to Player-$J$ choosing $c'$ from $c$). We assume that every state $x_c$ is labelled by $t_1$ if it’s Player-$1$’s turn to play at $c$, and by $t_2$ otherwise. Every intermediary state $x_{c,c'}$ is labelled by int. We also label empty configurations by $w_1$ or $w_2$, depending on which player made the last move.

Clearly, the size of $S$ will depend on the number of objects of each heap in the initial configuration. The formula $\Phi_{\text{win}}$ depends only on $J$:

$$\Phi_{\text{win}}^J = \exists m. \left( \mathbf{AG}(t_J \Rightarrow \mathbf{EX}m) \land \mathbf{AF}(w_J \lor (\text{int} \land \neg m)) \right)$$

This formula holds true in a state corresponding to some configuration $c$ iff there exists a labelling by $m$ such that every reachable configuration where it’s Player-$J$’s turn, has a successor labelled by $m$ (thus a possible choice to do) and every execution from the current state leads to either a winning state for Player-$J$ or a non-selected intermediary state, therefore all outcomes induced by the underlying strategy have to verify $Fw_J$. Note that in this example, the Kripke structure is acyclic (except the self-loops on the ending states). There is no $\exists^1$ or $\forall^1$ operator, so the encoding of uniq does not matter. Moreover the formula is already flat, so the methods $\text{FP}$ and $\text{FFP}$ are similar.
### 4.4 Resources distribution

The last example is as follows: given a Kripke structure $S$ and two integers $k$ and $d$, we aim at choosing at most $k$ states (called targets in the following) such that every reachable state (from the initial one) can reach a target in less than $d$ transitions. This problem can be encoded with the following QCTL formula where $d$ modalities $\text{EX}$ are nested:

$$
\Phi_{\text{res}}^{k,d} = \exists^1 c_1 \ldots \exists^k c_k. \ AG \left( \bigvee_{1 \leq i \leq k} c_i \lor \text{EX} \left( \bigvee_{1 \leq i \leq k} c_i \lor \left( \ldots \lor \text{EX} \left( \bigvee_{1 \leq i \leq k} c_i \right) \right) \right) \right)
$$

For experimental results, we consider the grid $K_{n,m}$ described at Figure 2 where nodes have high branching degrees. Here the interesting point is the nesting of temporal modalities $\text{EX}$: we will see that only the methods based on flat formulas are successful. Moreover due to the structure of the formula, the methods FBV and PNF are similar (no Until is used except the outermost $\text{AG}$).

### 4.5 Overview of experimental results

The main experimental results are given in Tables 3 and 4. We distinguish the time to build the QBF formula and the solver’s part (a timeout is set to 600 seconds). For example, for $V_{10,30} \models \Lambda_{12}$, the result is $38 + 2$, it means that

| Problem | Size | $U$/$Z_3$ | $FP$/$Z_3$ | $FP$/$qfm$ | $FFP$/$Z_3$ |
|---------|------|----------|-------------|-------------|-------------|
| Reset property: | | | | | |
| 1 $V_{10,30} \models \Lambda_{12}$ | 301 | 76+0.7 | 38+2 | 40+7 | 42+2 |
| 2 $V_{15,100} \models \Lambda_{16}$ | 1501 | X | X | X | X |
| 3 $V_{6,10} \not\models \Lambda_{5}$ | 61 | 0.4+25 | 0.5+42 | 0.5+122 | X |
| $k$-Connectivity: | | | | | |
| 4 $S_{10,5} \models \Psi_{4}$ | 200 | X | 2+0.6 | 3+2 | X |
| 5 $S_{15,5} \models \Psi_{4}$ | 450 | X | 15+4 | 16+5 | X |
| 6 $S_{15,7} \models \Psi_{6}$ | 450 | X | 22+112 | 25+56 | X |
| 7 $S_{30,6} \models \Psi_{4}$ | 1800 | X | 188+126 | 196+100 | X |
| 8 $S_{10,4} \not\models \Psi_{5}$ | 200 | X | 3+0.7 | 3+1 | 84+X |
| Nim game: | | | | | |
| 9 $[3,4,5] \models \Phi_{\text{win}}^{1}$ | 96 | 40+0.2 | 0.1+0.1 | 0.1+X | (FP) |
| 10 $[2,3,4,4] \models \Phi_{\text{win}}^{1}$ | 124 | 157+0.3 | 0.1+0.1 | 0.1+X | (FP) |
| 11 $[3,4,5,6] \models \Phi_{\text{win}}^{1}$ | 330 | X | 0.2+0.2 | 0.2+X | (FP) |
| 12 $[2,4,8,14] \not\models \Phi_{\text{win}}^{1}$ | 1556 | X | 3+3 | 3+X | (FP) |
| Resources distribution: | | | | | |
| 13 $L_{10,10} \models \Phi_{\text{res}}^{8,6}$ | 100 | X | X | X | 14+0.3 |
| 14 $L_{12,12} \models \Phi_{\text{res}}^{8,6}$ | 144 | X | X | X | 30+0.7 |
| 15 $L_{12,12} \models \Phi_{\text{res}}^{8,6}$ | 144 | X | X | X | 28+0.6 |
| 16 $L_{20,20} \models \Phi_{\text{res}}^{8,6}$ | 400 | X | X | X | 153+3 |

Table 3: Overview of experimental results (1).
Table 4: Overview of experimental results (2).

| z pb | PNF/Z3 | PNF/qfm | PNF/cqesto | PNF/qfun | FBV/cqesto |
|------|--------|---------|------------|----------|------------|
| Reset property: | | | | | |
| 1 | 1+4 | 1+1 | 1+0.2 | 1+0.3 | 110+12 |
| 2 | 7+X | 7+9 | 8+2 | 8+2 | X |
| 3 | 0.4+25 | 0.7+94 | 0.1+156 | 0.2+30 | 3+X |
| k-Connectivity: | | | | | |
| 4 | 0.2+5 | 0.2+46 | 0.2+0.1 | 0.2+5 | 48+X |
| 5 | 0.4+16 | 0.4+X | 0.4+6 | 0.4+487 | 507+X |
| 6 | 0.7+X | 0.7+X | 0.7+X | 0.7+X | X |
| 7 | 2+587 | 2+9 | 4+4.3 | 4+X | X |
| 8 | 0.2+2 | 0.2+10 | 0.2+19 | 0.2+8 | 131+X |
| Nim game: | | | | | |
| 9 | 0.1+123 | 0.1+1 | 0.1+1 | 0.1+31 | 212+28 |
| 10 | 0.1+49 | 0.1+18 | 0.1+1 | 0.1+X | 317+50 |
| 11 | 0.3+X | 0.3+X | 0.3+X | 0.3+X | X |
| 12 | 4+X | 4+X | 4+X | 4+X | X |
| Resources distribution: | | | | | |
| 13 | 16+0.4 | 17+0.3 | 18+0.2 | 17+0.2 | (PNF) |
| 14 | 34+0.6 | 35+0.6 | 35+0.3 | 36+0.4 | (PNF) |
| 15 | 39+0.7 | 35+0.5 | 39+0.3 | 35+0.3 | (PNF) |
| 16 | 207+3 | 190+3 | 180+2 | 179+3 | (PNF) |

the construction of the QBF formula took 38s and the QBF solver needed 2s to decide whether the formula was valid or not. A result of the form 7 + X means that the formula was built in 7s but the solver didn’t give the result in less than 600s. And a result of the form X means that the QBF formula was not yet built after 600s. Moreover (A) means that this reduction is similar to the reduction A (i.e. the two preprocessing steps provide the same formula). Best results for a case are in bold.

The first remark is that we can successfully apply our techniques and get answers for QCTL verification problems. In the tables, we just selected the most significant results. The encoding of \( \exists^1 \) and \( \forall^1 \) is always done with bit vectors: indeed the two other encodings presented in section 3.6 are never the best solution on the examples we consider.

The results show that the two most interesting methods seem to be the FP reduction (associated with Z3) and the PNF reduction (associated with cqesto). As soon as the formula contains nested temporal modalities the PNF reduction is better. For the Nim game, the reduction FP is much more efficient than any other reduction and give the solution for structures with more than 1500 states. For the resource distribution (with a high branching degree of the structure and a high temporal nesting in the formula), the flattening is mandatory (PNF or FFP).

The results also depend on the choice of the solver: a QBF formula can be
solved very easily by a solver but requires a very long execution time for others.

5 Conclusion

We have presented several reductions from \textsc{QCTL} model-checking to \textsc{QBF}. This provides a first tool for \textsc{QCTL} model-checking with the structure semantics. These first results are rather interesting and encouraging. We have seen the importance of writing "good" \textsc{QCTL} formulas for which the solver will be able to provide a result (this problem already exists for classical temporal logics, but it is more significant here due to the complexity induced by the quantifications). The examples also show that there is no "one best strategy" and no "one best solver": the best choices depend on the structure of the considered formula, the structure of the model. Two reductions seem to be the most interesting: \textsc{FP} for simple \textsc{QCTL} formulas (with few nesting of temporal modalities) and \textsc{PNF} for more complex formulas. Considering several solvers is an important point: the heuristics used by the solvers may affect significantly the execution times. In the future, we plan to continue to work on reduction strategies for \textsc{QCTL}. Considering special methods for timed modalities of the form $\text{EF} \prec d \varphi$ (i.e. "a state satisfying $\varphi$ is reachable in less than $d$ transitions") would be very useful in practice and the reduction \textsc{FBV} could be useful for it. Considering other solvers (based on prenex CNF formulas) would be also an interesting work. Finally we also plan to consider other logics for which such reductions to \textsc{QBF} are possible to get decision procedures.

Acknowledgement We would like to strongly thank Mikolás Janota for his help and advice about the \textsc{QBF} solvers, as well as Yann Regis-Gianas who helped us with the experiments part, and the anonymous reviewers for their helpful suggestions to improve the paper.

References

[1] Carlos Areces, Raul Fervari, and Guillaume Hoffmann. Relation-changing modal operators. \textit{Logic Journal of the IGPL}, 23(4):601–627, 2015.

[2] Guillaume Aucher, Johan van Benthem, and Davide Grossi. Modal logics of sabotage revisited. \textit{J. Log. Comput.}, 28(2):269–303, 2018.

[3] Armin Biere. CaDiCaL at the SAT Race 2019. In Marijn Heule, Matti Järvisalo, and Martin Suda, editors, \textit{Proc. of SAT Race 2019 – Solver and Benchmark Descriptions}, volume B-2019-1 of \textit{Department of Computer Science Series of Publications B}, pages 8–9. University of Helsinki, 2019.

[4] Armin Biere, Alessandro Cimatti, Edmund M. Clarke, Ofer Strichman, and Yunshan Zhu. Bounded model checking. \textit{Advances in Computers}, 58:117–148, 2003.
[5] Armin Biere, Alessandro Cimatti, Edmund M. Clarke, and Yunshan Zhu. Symbolic model checking without bdds. In Tools and Algorithms for Construction and Analysis of Systems, 5th International Conference, TACAS ’99, Amsterdam, The Netherlands, March 22-28, 1999, Proceedings, volume 1579 of Lecture Notes in Computer Science, pages 193–207. Springer, 1999.

[6] J. Adrian Bondy and Uppaluri S. R. Murty. Graph Theory. Graduate Texts in Mathematics. Springer, 2008.

[7] Edmund M. Clarke and E. Allen Emerson. Design and synthesis of synchronization skeletons using branching-time temporal logic. In Dexter C. Kozen, editor, Proceedings of the 3rd Workshop on Logics of Programs (LOP ’81), volume 131 of Lecture Notes in Computer Science, pages 52–71. Springer-Verlag, 1982.

[8] Simon Cooksey, Sarah Harris, Mark Batty, Radu Grigore, and Mikolas Janota. Pridemm: Second order model checking for memory consistency models. In Pre-proceedings of TAPAS 2019, 10th Workshop on Tools for Automatic Program Analysis, pages 7–26, 2019.

[9] Arnaud Da Costa, François Laroussinie, and Nicolas Markey. Quantified CTL: Expressiveness and model checking. In Maciej Koutny and Irek Ulidowski, editors, Proceedings of the 23rd International Conference on Concurrency Theory (CONCUR’12), volume 7454 of Lecture Notes in Computer Science, pages 177–192. Springer-Verlag, September 2012.

[10] Leonardo Mendonça de Moura and Nikolaj Bjørner. Z3: an efficient SMT solver. In Tools and Algorithms for the Construction and Analysis of Systems, 14th International Conference, TACAS 2008, Budapest, Hungary, 2008. Proceedings, volume 4963 of Lecture Notes in Computer Science, pages 337–340. Springer, 2008.

[11] Nachum Dershowitz, Ziyad Hanna, and Jacob Katz. Bounded model checking with QBF. In Theory and Applications of Satisfiability Testing, 8th International Conference, SAT 2005, St. Andrews, UK, June 19-23, 2005, Proceedings, volume 3569 of Lecture Notes in Computer Science, pages 408–414. Springer, 2005.

[12] Niklas Eén and Niklas Sörensson. An extensible sat-solver. In Theory and Applications of Satisfiability Testing, 6th International Conference, SAT 2003. Santa Margherita Ligure, Italy, May 5-8, 2003 Selected Revised Papers, volume 2919 of Lecture Notes in Computer Science, pages 502–518. Springer, 2003.

[13] Tim French. Decidability of quantified propositional branching time logics. In Markus Stumptner, Dan Corbett, and Mike Brooks, editors, Proceedings of the 14th Australian Joint Conference on Artificial Intelligence.
(AJCAI’01), volume 2256 of Lecture Notes in Computer Science, pages 165–176. Springer-Verlag, December 2001.

[14] Tim French. Quantified propositional temporal logic with repeating states. In Proceedings of the 10th International Symposium on Temporal Representation and Reasoning and of the 4th International Conference on Temporal Logic (TIME-ICTL ’03), pages 155–165. IEEE Comp. Soc. Press, July 2003.

[15] Mikolás Janota. Circuit-based search space pruning in QBF. In Theory and Applications of Satisfiability Testing - SAT 2018 - 21st International Conference, SAT 2018, Oxford, UK, July 9-12, 2018, Proceedings, volume 10929 of Lecture Notes in Computer Science, pages 187–198. Springer, 2018.

[16] Orna Kupferman. Augmenting branching temporal logics with existential quantification over atomic propositions. In Pierre Wolper, editor, Proceedings of the 7th International Conference on Computer Aided Verification (CAV’95), volume 939 of Lecture Notes in Computer Science, pages 325–338. Springer-Verlag, July 1995.

[17] François Laroussinie and Nicolas Markey. Quantified CTL: expressiveness and complexity. Logical Methods in Computer Science, 10(4), 2014.

[18] François Laroussinie and Nicolas Markey. Augmenting ATL with strategy contexts. Inf. Comput., 245:98–123, 2015.

[19] François Laroussinie, Nicolas Markey, and Arnaud Sangnier. ATLsc with partial observation. In Proceedings Sixth International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2015, Genoa, Italy, 21-22nd September 2015, volume 193 of EPTCS, pages 43–57, 2015.

[20] Christof Löding and Philipp Rohde. Model checking and satisfiability for sabotage modal logic. In FST TCS 2003: Foundations of Software Technology and Theoretical Computer Science, 23rd Conference, Mumbai, India, December 15-17, 2003, Proceedings, volume 2914 of Lecture Notes in Computer Science, pages 302–313. Springer, 2003.

[21] Christof Löding and Philipp Rohde. Solving the sabotage game is pspace-hard. In Mathematical Foundations of Computer Science 2003, 28th International Symposium, MFCS 2003, Bratislava, Slovakia, August 25-29, 2003, Proceedings, volume 2747 of Lecture Notes in Computer Science, pages 531–540. Springer, 2003.

[22] Kenneth L. McMillan. Applying SAT methods in unbounded symbolic model checking. In Computer Aided Verification, 14th International Conference, CAV 2002, Copenhagen, Denmark, July 27-31, 2002, Proceedings, volume 2404 of Lecture Notes in Computer Science, pages 250–264. Springer, 2002.
A Proof of Theorem 2

Proof: We assume $\Phi$ to be fixed and we prove the property by structural induction over the subformula $\varphi$. Boolean operators are omitted.

- $\varphi = p$: if $\mathcal{K}, x \models p$, then either $p$ is a quantified proposition and $x$ belongs to $\varepsilon(p)$ and thus $v_{\varepsilon} \models p^x$ by definition of $v_{\varepsilon}$, or $p$ belongs to $\ell(x)$. In both cases we have $v_{\varepsilon} \models \hat{p}^x, \text{dom}(c)$. The converse is similar.

- $\varphi = \exists p.\psi$: We have $\mathcal{K}, x \models \exists p.\psi$ iff there exists $V' \subseteq V$ s.t. $\mathcal{K}, x \models_{\varepsilon[p \mapsto V']} \psi$, iff (by i.h.) there exists $V' \subseteq V$ s.t. $v_{\varepsilon[p \mapsto V']} \models \hat{\psi}^x, \text{dom}(c) \cup \{p\}$ which is equivalent to $v_{\varepsilon} \models \exists p^x : p^x \ldots p^x, \hat{\psi}^x, \text{dom}(c) \cup \{p\}$ (by definition of $v_{\varepsilon}$).

- $\varphi = \text{EX}\psi$: $\mathcal{K}, x \models \text{EX} \psi$ iff there exists $(x, x') \in E$ s.t. $\mathcal{K}, x' \models \psi$, iff (by i.h.) there exists $(x, x') \in E$ s.t. $v_{\varepsilon} \models \hat{\psi}^{x'}, \text{dom}(c)$ which is equivalent to $v_{\varepsilon} \models \text{EX} \psi$.
• $\varphi = \mathbf{AX}\psi$: Similar to $\mathbf{EX}$ with a conjunction to ensure that all successors satisfy $\psi$.

• $\varphi = \mathbf{EF}\psi$ or $\varphi = \mathbf{AG}\psi$: Similar to $\mathbf{EX}$ or $\mathbf{AX}$ except that we consider any reachable state $x'$ instead of immediate successors (thus we use the reflexive and transitive closure $E^*$ of $E$).

• $\varphi = \mathbf{E}\psi_1 \mathbf{U}\psi_2$: The definition of $\hat{\varphi}^{x,\operatorname{dom}(\varepsilon)}$ corresponds to a finite unfolding of the expansion law that characterizes the $\mathbf{EU}$ modality. Assume $K, x \models_{\varepsilon} \varphi$. There exists a path $\rho \in \operatorname{Path}_K^\varepsilon(x)$ and a position $i \geq 0$ s.t. $\rho(i) \models_{\varepsilon} \psi_2$ and $\rho(k) \models_{\varepsilon} \psi_1$ for any $0 \leq k < i$. The finite prefix $x = \rho(0) \cdots \rho(k)$ can be assumed to be simple, and then $k < |V|$. By using i.h., we get $v_x \models \psi_2$ and $v_x \models \psi_1$ for any $0 \leq k < i$. From this point, the reader can easily verify by induction (starting at $i$, down to 0) that $v_x \models \mathbf{E}\psi_1 \mathbf{U}^\varepsilon \psi_2^\rho(k,\operatorname{dom}(\varepsilon),\{\rho(j)\mid j \leq k\})$ for all $k \leq i$. This makes $\hat{\varphi}^{x,\operatorname{dom}(\varepsilon)}$ to be satisfied by $v_x$.

Conversely, assume $v_x \models \hat{\varphi}^{x,\operatorname{dom}(\varepsilon)}$, i.e. $v_x \models \mathbf{E}^{x,\operatorname{dom}(\varepsilon)} \mathbf{U}^\varepsilon \psi_2^\rho(k,\operatorname{dom}(\varepsilon),\{\rho(j)\mid j \leq k\})$. Given the definition, there exists a sequence of states $x_0, \ldots, x_i$ s.t. (1) $x_0 = x$, (2) $v_x \models \psi_2$ and (3) for any $0 \leq j < i$ we have $v_x \models \psi_1$ s.t. $x_j \in E$ and $x_j+1 \notin \{x_0, \ldots, x_j\}$. And by i.h., we can deduce that $x_0 \ldots x_i$ is a path in $K$ satisfying $\psi_1 \mathbf{U} \psi_2$.

• $\varphi = \mathbf{A}\psi_1 \mathbf{U}\psi_2$: this case is similar to the previous one, except that we have to consider loops. Assume $K, x \models_{\varepsilon} \varphi$. Then any path issued from $x$ satisfies $\psi_1 \mathbf{U} \psi_2$: either there is a simple prefix witnessing $\psi_1 \mathbf{U} \psi_2$ (and ending with a state satisfying $\psi_2$), or there is a loop from some point. In the latter case, one of the states in the loop has to verify $\psi_2$. In both cases, the definition of $\hat{\varphi}^{x,\operatorname{dom}(\varepsilon)}$ gives the result.

\[ \square \]

\[ \square \]

### B Proof of Proposition 8

**Proof:** Consider w.l.o.g. a QCTL formula $\Phi$ in prenex normal form and NNF. We can define $\Phi_0$ and the basic formulas $\theta_i$s as in Proposition 8 except that $\Phi_0$ contains only $\mathbf{EX}$, $\mathbf{AX}$, $\mathbf{EF}$ or $\mathbf{AG}$ modalities, and every other modality is associated with some quantified proposition $\kappa$ and a subformula $\mathbf{AG}(\ldots)$ in the main conjunction of $\Psi$. Every $\theta_i$ starts with a modality in $S_{\text{mod}}$. Let $\varphi_i$ be the original $\Phi$-subformula associated with $\theta_i$. Note that $\Psi$ is in NNF, and $\kappa_i$ occurs only once in $\Psi$ in the scope of a negation, and it happens in the subformula $\mathbf{AG}(\kappa_i, \Rightarrow \theta_i)$. We now have to show that $\Phi$ is equivalent to $\Psi$. Consider the formula $\hat{\Psi}$ where every $\Rightarrow$ is replaced by $\Leftrightarrow$: by following the same arguments of Proposition 8 we clearly have $\Phi \equiv \hat{\Psi}$, and $\Psi \Rightarrow \hat{\Psi}$. It remains to prove the opposite direction.
To prove $\Psi \Rightarrow \tilde{\Psi}$, it is sufficient to show that this is true for the empty $Q$ (as equivalence is substitutive). Assume $\mathcal{K}, x \models_\varepsilon \Psi$. Then there exists an environment $\varepsilon'$ from $\{\kappa_1, \ldots, \kappa_m\}$ to $2^V$ such that $\mathcal{K}, x \models_{\varepsilon'\varepsilon} \Phi_0 \land \bigwedge_i \text{AG}(\kappa_i \Rightarrow \theta_i)$.

Now we have:

$$\forall i, \mathcal{K}, x \models_{\varepsilon'\varepsilon} \theta_i \Rightarrow \mathcal{K}, x \models_\varepsilon \varphi_i \quad \text{and} \quad \mathcal{K}, x \models_{\varepsilon'\varepsilon} \Phi_0 \Rightarrow \mathcal{K}, x \models_{\varepsilon'\varepsilon} \tilde{\Phi}_0$$

Indeed, assume that it is not true and $\mathcal{K}, x \models_{\varepsilon'\varepsilon} \theta_i$ and $\mathcal{K}, x \not\models_\varepsilon \varphi_i$. Consider such a formula $\varphi_i$ with the smallest temporal height. The only atomic propositions $\kappa_j$ occurring in $\theta_i$ are then associated with some $\theta_j$ and $\varphi_j$ which verify the property and thus any state satisfying such a $\theta_j$, also satisfies $\varphi_j$. Therefore any state labelled by such a $\kappa_j$ is correctly labelled (and satisfies $\varphi_j$). And the states that are not labelled by $\kappa_j$ cannot make $\theta_i$ to be wrongly evaluated to true (because $\kappa_j$ is not in the scope of a negation). Therefore $\varphi_i$ holds true at $x$. The same holds for $\Phi_0$ and $\tilde{\Phi}_0$. As a direct consequence, we have $\Psi \Rightarrow \tilde{\Psi}$. □ □

C Proposition for Section 3.6

NB: we assume that every quantifier introduces a fresh atomic proposition.

**Proposition 11** Let $Q$, $Q'$ and $Q''$ be three blocks of quantifiers. We have the following equivalence:

$$Q \exists p_1 Q' \forall p_2 Q'' \left( \text{uniq}(p_1) \land (\text{uniq}(p_2) \Rightarrow \Phi) \right) \equiv Q \exists p_1 Q' \forall p_2 Q'' \left( \text{uniq}(p_2) \Rightarrow (\text{uniq}(p_1) \land \Phi) \right) \quad (6)$$

**Proof:**

- $(1) \Rightarrow (2)$: there exists a labelling for $p_1$ s.t. $\text{uniq}(p_1)$ is true and for any $p_2$-labelling, if $\text{uniq}(p_2)$ holds true, then we have $\Phi$. Now choose the same $p_1$-labelling to evaluate the right-hand side formula, we know that $\text{uniq}(p_1)$ is true, and moreover for every $p_2$-labelling satisfying $\text{uniq}(p_2)$, $\varphi$ is satisfied. This provides the result.

- $(2) \Rightarrow (1)$: Choose the $p_1$-labelling. We know that there exists a $p_2$-labelling such that $\text{uniq}(p_2)$ is true (e.g. by labelling only the current state with $p_2$), this implies that we have $\text{uniq}(p_1)$ and $\Phi$. And therefore, for every $p_2$-labelling satisfying $\text{uniq}(p_2)$, we have $\Phi$. □ □