Gravitational vacuum polarization as an alternative to dark matter

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Abstract

It is assumed that the quantum vacuum may be studied as consisting of two contributions, with positive and negative energy respectively, which interact but slightly and may be displaced from each other. Then it is proposed that dark matter may be just an increase of the quantum vacuum energy, with respect to the normal dark energy level, induced by the gravitational field of galaxies or clusters. A simple model is worked out able to reproduce astronomical observations.

An outstanding open problem in present day cosmology is the nature of “dark energy” and “dark matter”, which together contribute more than 95% of the total mass-energy of the universe. The dark energy corresponds to a mass density

\[ \rho_{DE} \sim 10^{-26} \text{kg/m}^3, \]  

and the dark matter to about one third this quantity on the average[1]. A common hypothesis is to identify the dark energy with the cosmological constant introduced by Einstein in 1917. An alternative hypothesis, equivalent in practice, is to postulate a vacuum stress-energy tensor, \( T^{\mu\nu} \), of the form

\[ T^{\mu\nu} = \rho g^{\mu\nu}, \]  

\( g^{\mu\nu} \) being the metric tensor, which amounts to assuming an equation of state for the vacuum of the form

\[ p = -\rho < 0, \ \rho = T^0_0, \ p = -T^1_1 = -T^2_2 = -T^3_3. \]
If the dark energy is due to the quantum vacuum, it is difficult to understand why its density is not either strictly zero or a density at the Planck scale, that is about $10^{123}$ times the observed value, eq.(1). We might assume that the vacuum contains both positive and negative energy contributions which do not cancel completely each other, but then an extremely fine tuning is required in order to get the observed value. In the present paper I propose that the dark energy derives from the fact that the positive energy of long wavelength components of the vacuum fields is not completely cancelled by the other field components whilst its positive pressure is more than compensated by the negative pressure of other field components. This hypothesis reduces substantially the fine tuning needed and, in addition, leads naturally to the hypothesis that the vacuum may be “gravitationally polarized”. That is, vacuum components contributing positive (negative) energy will tend to go to regions with more (less) negative gravitational potential. The purpose of the present paper is to show that this assumption may allow interpreting “dark matter” as a gravitationally polarized “dark energy”.

In more detail the argument is as follows. We should assume that all short-wavelength components of the quantum vacuum are strongly coupled to each other, as shown by the coupling of the corresponding vacuum fluctuations. For instance, virtual photons with Compton wavelength ($\sim 10^{-12} m$) may produce virtual electron-positron pairs and vice versa. This suggests that short-wavelengths of the different vacuum fields could not be separated from each other and their joint contribution to the vacuum energy, either positive or negative, should be not too big in absolute value. In contrast, vacuum field components with long wavelength may be almost uncoupled to those with short wavelength, but either completely cancel the short-wavelength contribution or give a very small, but positive, total vacuum energy. This hypothesis offers the possibility that the total vacuum energy and pressure, in the absence of gravitational field, fulfils the dark energy eqs.(1) to (3) without the need of an extremely fine tuning.

Now it is natural to believe that the long wavelengths part of the vacuum fields may propagate with some independence from the short wavelength components. In order to get a simple model, I shall assume that there is a cut-off, $\lambda_c$, such that field components with longer wavelength are completely decoupled from the remaining vacuum fields. The model so derived is clearly too crude and a more plausible model should include some interaction between the positive energy and the negative energy contributions, thus preventing a too big vacuum polarization. This possibility will be considered
elsewhere. Here I shall pursue with the development of the simple model by fixing the cut-off, $\lambda_c$. It is obvious that it should lie somewhere between the Compton and macroscopic wavelengths, but it is difficult to make a precise estimate. I shall choose $\lambda_c \sim 1 \mu m$, at about the wavelength of visible light, which will lead to good agreement with known dark matter properties (see below). This value corresponds to a particle mass

$$m \sim \frac{hc}{\lambda_c} \sim 1 eV/c^2,$$

not far from neutrino masses. Thus wavelengths greater than $1 \mu m$ are negligible for vacuum fields other than the electromagnetic one.

The mass density associated to the electromagnetic zero-point field (energy $1/2h\nu$ per normal mode) cut-off at $\lambda_c$ is positive and has the value

$$\rho_0 = \frac{1}{c^2} \int_0^{c/\lambda_c} \frac{1}{2} h\nu \frac{8\pi \nu^2}{c^3} d\nu = \frac{\pi h}{c\lambda_c^4} \sim 4 \times 10^{-16} kg/m^3. \quad (4)$$

The essential hypothesis of the model here proposed is that the vacuum contribution due to long wavelengths, with density eq.(4), and the remaining vacuum contributions, with negative density, behave as two non-interacting fluids. In particular in a static space-time, the only case to be studied here, each part should be in hydrostatic equilibrium in the gravitational field.

For simplicity I shall consider a static space-time with spherical symmetry, where I should study the hydrostatic equilibrium of three non-interacting fluids, ruled by

$$\frac{dp_j}{dr} = -\Phi'(\rho_j + p_j), \quad j = 1, 2, 3, \quad (5)$$

where $j = 1$ corresponds to the long wavelengths vacuum contribution, $j = 2$ to the remaining vacuum fields and $j = 3$ to the baryonic matter. (Units $c = G = 1$ will be used throughout this paper.) The gravitational potential, $\Phi$, is given by

$$\Phi' \equiv \frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r^2 - 2mr}, \quad m(r) = \int_0^r 4\pi \rho(r) r^2 dr, \quad (6)$$

where $\rho = \rho_1 + \rho_2 + \rho_3$ is the total mass density and $p = p_1 + p_2 + p_3$ is the total pressure at $r$ (all measured in the local frame). It may be realized that the sum in $j$ of the three eqs.(5), combined with eq.(6) gives the standard Tolman-Oppenheimer-Volkoff equation of general relativity, namely

$$\frac{dp}{dr} = \frac{d\Phi}{dr} (\rho + p) \equiv -\frac{m + 4\pi r^3 p}{r^2 - 2mr} (\rho + p). \quad (7)$$
It is easy to relate the density and pressure of each component of the vacuum with the potential $\Phi$ if we know the corresponding equation of state. For the long-wavelength electromagnetic zero-point field I must assume the radiation equation of state, that is

$$p_1 = \frac{1}{3} \rho_1,$$  \hspace{1cm} (8)

which, inserted in the first eq.(5), leads to

$$\frac{1}{3} \frac{d\rho_1}{dr} = -\Phi \frac{4}{3} \rho_1 \Rightarrow \rho_1 = \rho_{10} \exp(-4\Phi),$$  \hspace{1cm} (9)

$\rho_{10}$ being the contribution to mass density at infinity (where $\Phi \to 0$). As said above the contribution of the remaining vacuum fields to both the energy density and the pressure should be negative, in order to cancel almost completely the positive contribution of the long-wavelength (electromagnetic) field. I shall assume that the second contribution, seen as a fluid, is more rigid that the first one so that the pressure as a function of the energy density is more steep than eq.(8). As a simple equation of state which allows analytical solutions I propose

$$p_2 = \frac{\gamma (\rho_2)^2}{\rho_{20}}, \quad \rho_{20} < 0,$$  \hspace{1cm} (10)

where $\gamma$ is a dimensionless parameter and $\rho_{20}$ is the negative contribution to the mass density at infinity. Putting this into the first eq.(5) I obtain

$$\gamma \frac{1}{\rho_{20}} \frac{d(\rho_2)^2}{dr} = -\Phi' \left[ \rho_2 + \frac{\rho_2}{\rho_{20}} \right] \Rightarrow \rho_2 = \frac{\rho_{20}}{\gamma} \left[ (1 + \gamma) \exp \left( -\frac{1}{2} \Phi \right) - 1 \right].$$  \hspace{1cm} (11)

The sum of the two contributions will give the total energy density and pressure of the vacuum, namely

$$\rho = \rho_{10} \exp(-4\Phi) + \frac{\rho_{20}}{\gamma} \left[ (1 + \gamma) \exp \left( -\frac{1}{2} \Phi \right) - 1 \right],$$  \hspace{1cm} (12)

$$p = \frac{1}{3} \rho_{10} \exp(-4\Phi) + \frac{\rho_{20}}{\gamma} \left[ (1 + \gamma) \exp(-\frac{1}{2} \Phi) - 1 \right]^2.$$  \hspace{1cm} (13)

Now it is trivial to find values of $\rho_{10}$, $\rho_{20}$ and $\gamma$ such that, when $\Phi = 0$, we get the dark energy values, that is $\rho = -p = \rho_{DE}$ with $\rho_{DE}$ given by eq.(11).
I get
\[ \rho_{10} = (K + 1) \rho_{DE}, \quad \rho_{20} = -K \rho_{DE} \text{ with } K \equiv \frac{4}{3\gamma - 1} >> 1. \quad (14) \]

Assuming \( \rho_{10} \sim -\rho_{20} \sim \rho_0 \) as given in eq.(11) I obtain \( K \sim 4 \times 10^{10} \), a very high value which means that the density and pressure of the two assumed vacuum components cancel quite accurately.

Eqs.(12) and (13) may be simplified taking into account that, in the cases of interest for us, the potential is very weak, that is \( |\Phi| \ll 1 \). To second order in the potential the equations become
\[
\rho = \rho_0 \left[ \exp(-4\Phi) + 3 - 4 \exp\left(-\frac{1}{2}\Phi\right) \right] \approx -2\rho_0\Phi + \frac{15}{2}\rho_0\Phi^2 + \rho_{DE}, \quad (15)
\]
\[
p = \frac{1}{3}\rho_0 \exp(-4\Phi) - 3\rho_0 \left[ \frac{4}{3} \exp\left(-\frac{1}{2}\Phi\right) - 1 \right]^2 - \rho_{DE} \approx \rho_0\Phi^2 - \rho_{DE}. \quad (16)
\]

Typically \( \Phi \sim -10^{-6} \) in galaxies so that, identifying \( \rho_0 \) with the value eq.(4) I obtain \(-2\rho_0\Phi \sim 10^{-21} kg/m^3 >> \rho_{DE} \). In contrast the dark matter pressure predicted by the model is of the same order as the dark energy pressure, \(-\rho_{DE} \), but has opposite sign. The comparison with observations is as follows.

Recent measurements in the Milky Way [2] give a mass \( M(<60kpc) = (4.0 \pm 0.7) \times 10^{11}M_\odot \) so that the potential is \( \Phi(60kpc) \sim -M/R = -3.3 \times 10^{-7} \), consistent with the above estimate for \( \Phi \) within the galaxy. On the other hand it is known [3] that the local density, near the solar system, is \(0.2 - 0.5 Gev/cm^3 \sim (4 - 9) \times 10^{-22} kg/m^3 \), in agreement with our model estimate.

Eq.(15) allows writing the “vacuum polarization law” in the simple form
\[ \nabla \rho \approx -2\rho_0 \nabla \Phi = 2\rho_0 g, \quad (17) \]
g being the gravitational field. But I stress that this relation should have limited validity. Indeed, in strong fields the vacuum polarization would suffer some saturation so that \( |\nabla \rho| \) is smaller than predicted by eq.(17). On the other hand for very small potential, \( \Phi \), eqs.(15) and (16) might be modified because, being small differences of big quantities, they would be rather sensitive to any change in the model.

We see that the mass density, eq.(15), and pressure, eq.(16), due to the polarized vacuum energy mimic a fluid of cold matter with equation of state
\[ p_{DM} = \frac{1}{4\rho_0} \rho_{DM}^2 << \rho_{DM}, \quad (18) \]
where $\rho_{DM} = \rho - \rho_{DE}$ is the density above the dark energy level, and similarly for the pressure. Thus in the study of dark matter we might use eq.\((18)\) as the equation of state of a hypothetical fluid. For instance if eq.\((18)\) is put in the hydrostatic equilibrium equation (see eq.\((5)\)) I get eqs.\((15)\) and \((16)\) to leading order in $\Phi$, that is

$$\rho_{DM} \approx -2\rho_0 \Phi, \quad p_{DM} \approx \rho_0 \Phi^2.$$  \hfill (19)

Of course eqs.\((15)\) to \((18)\) are specific for our simple model, in particular they rest upon eq.\((10)\). However the qualitative behaviour of the polarized vacuum energy mimicking cold matter is likely valid for any reasonable model, e. g. with equation of state different from eq.\((10)\).

In the following I will study the dark matter problem in a galaxy assumed spherical. The study of more realistic galaxies and clusters would be similar although more involved. For this study I may neglect the dark energy contribution, which is very small whenever the dark matter is relevant. In order to get the spatial distribution of dark matter I shall solve eq.\((6)\). For this purpose the Newtonian approximation is good enough, that is

$$\Phi' \approx \frac{m}{r^2} \Rightarrow \frac{d}{dr} (r^2 \Phi') = \frac{dm}{dr} = 4\pi r^2 (\rho + \rho_B) = -8\pi r^2 \rho_0 \Phi + 4\pi r^2 \rho_B.$$  \hfill (20)

where $\rho_B (r)$ is the baryonic mass density of the galaxy and I have approximated $\rho$ by the first term of the right hand side of eq.\((15)\). In the external region, where baryonic matter is negligible, the solution of eq.\((20)\) is

$$\Phi = \frac{C}{r} \sin (\mu r + \delta) \Rightarrow \rho = \frac{2C\rho_0}{r} \sin (\mu r + \delta),$$  \hfill (21)

where

$$\mu = \sqrt{8\pi \rho_0 G/c^2}, \quad \mu^{-1} = R_{DM} \sim 3 \times 10^{20} m \sim 10 kpc.$$  \hfill (22)

$R_{DM}$ is the typical radius of the dark matter distribution predicted by the model and I have used for $\rho_0$ the value \((4)\). The solution in the internal region cannot be found without knowing the distribution of baryonic matter. As a simple model I shall consider a distribution consisting of a constant density, $\rho_B$, within a sphere of radius $R$, zero outside. Then the regular solution of eq.\((20)\) for $r < R$ is

$$\Phi = \frac{\rho_B}{2\rho_0} - \frac{A}{r} \sin (\mu r) \Rightarrow \rho = \frac{2A\rho_0}{r} \sin (\mu r) - \rho_B.$$  \hfill (23)
The condition that both $\Phi$ and $\Phi'$ are continuous at $r = R$ leads to

\[
A \sin (\mu R) - C \sin (\mu R + \delta) = \frac{\rho_B R}{2\rho_0} = \frac{4\pi \rho_B R}{\mu^2},
\]

\[
A \cos (\mu R) - C \cos (\mu R + \delta) = \frac{\rho_B}{2\rho_0 \mu} = \frac{4\pi \rho_B}{\mu^3},
\]

(24)

where eq.(22) has been taken into account. These equations allow getting two of the parameters $A$, $C$ and $\delta$ in terms of one of them. Thus I will write the dark matter distribution in terms of $\delta$ as follows. Multiplying the first eq.(24) times $\cos (\mu R)$ and the second one times $\sin (\mu R)$ and subtracting I get

\[
C \sin \delta = \frac{4\pi R}{\mu^2} \rho_B \left[ \frac{\sin (\mu R)}{\mu R} - \cos (\mu R) \right] \approx \frac{4}{3} \pi R^3 \rho_B = M_B, \quad (25)
\]

$M_B$ being the total baryonic mass of the galaxy. In the second eq.(25) I have taken into account that the typical radius of dark matter in galaxies is much larger than the radius of baryonic matter, i.e. $\mu R = R_{\text{baryonic}}/R_{\text{DM}} \ll 1$. Putting eq.(25) into the first eq.(24) I obtain

\[
A = M_B \left( \frac{1}{\tan \delta} + \frac{3}{2\mu R} + \frac{3}{\mu^3 R^3} \right), \quad (26)
\]

where I have again neglected terms of order $\mu R$. Using the expressions obtained for $A$ and $C$ in eqs.(21) and (23), respectively, I get the potential, to order $\mu R$, and the dark matter density in terms of the parameter $\delta$, namely

\[
\Phi \approx - \frac{M_B}{r \sin \delta} \sin (\mu r + \delta), \quad \rho \approx \frac{2M_B \rho_0}{r \sin \delta} \sin (\mu r + \delta), \quad r > R.
\]

(28)

Provided that $\pi/2 > \delta > \mu R$, as I shall assume (see next paragraph), the latter expression shows that, beyond the typical galaxy radius $R$, the dark matter density decreases as $1/r$ at the beginning and more steeply later on, going to zero at $r = (\pi - \delta)/\mu$. This behaviour roughly agrees with observations.

It remains $\delta$ as a free parameter which cannot be obtained from our model. Thus the normalization (amplitude) of the dark matter distribution
is arbitrary and, furthermore, the model predicts bound states of dark matter alone, without any amount of real (e.g., baryonic) matter. This possibility seems unplausible. Consequently we should assume that such bound states are not stable, but stability cannot be studied within our time-independent model. Improvements of the model will be studied elsewhere, but in the present paper I will simply fix the amplitude of the dark matter distribution by comparison with observations. The total dark matter mass may be obtained integrating the density eqs. (27) and (28) in the region where $\rho > 0$, that is

$$M_{DM} = \int 4\pi r^2 dr \approx \frac{MB}{\sin \delta} \int_{\mu R}^{\pi - \delta} x \sin (x + \delta) dx \approx MB \left( \frac{\pi - \delta}{\sin \delta} + 1 \right),$$

(29)

where the dark matter mass within the sphere of radius $R$ has been neglected, it being of order $(\mu R)^2$ (1/tan$\delta$ is of order $\mu R$, see below). As the dark matter mass is typically more than 10 times the baryonic mass in galaxies, the choice $\sin \delta \lesssim 0.3$ seems appropriate.

For large $r$, the density eq. (28) oscillates between positive and negative values, which seems unphysical. We cannot use the expedient of substituting $\rho = 0$ for the predicted density (28) when $r$ is larger than the first zero of the density function. Indeed this would contradict the relation eq. (15) between dark matter density and potential. Actually when $r$ is large, and therefore $|\Phi|$ small, we should take into account several corrections, e.g., deviations from sphericity, the influence of neighbour galaxies, or the presence of some baryonic matter in the form of gas. I shall study just one correction, namely the dark energy term $\rho_{DE}$ which should be added to $\rho$. In order to see whether this term makes the density positive everywhere we should take into account that the minimum of the density eq. (28) will happen near $\mu r \sim 3\pi/2$ with the value

$$\rho_{\min} \sim -\frac{2\rho_0\mu MB}{(3\pi/2 - \delta) \sin \delta} \sim -0.1\rho_0\mu R^2 \frac{MB}{R} \frac{1}{\sin \delta}.$$

Thus with $\rho_0$ given by eq. (4), $\mu R \sim 0.1, \sin \delta \sim 0.3, M_B/R \sim 10^{-6}$, I get $\rho_{\min} \sim -10^{-23} kg/m^3$, far from being cancelled by the dark energy density, eq. (11). Thus the predicted total density, $\rho + \rho_{DE}$, is still negative in some regions, which shows that the model requires some modifications. These would be most important for very small densities where the cancellation between the $\rho_1$ and $\rho_2$ is delicate (see eq. (12)).

A prediction of the model here proposed is the universal value of the
dark matter radius, $\mu^{-1}$, estimated in about 10$kpc$. Thus the extension of the region occupied by dark matter is ruled by the parameter $\mu$, with the result that it will be concentrated in the central region in clusters (whose radius is larger than $\mu^{-1}$) whilst it would extend well beyond the region of baryonic matter in galaxies, which roughly agrees with observations. Of course this property is also true if dark matter consists of a gas of particles with an equation of state like eq.(18).

In summary I have shown that both dark matter and dark energy might be interpreted as vacuum mass-energy provided that the vacuum may be (slightly) polarized by the presence of gravity. The particular model here proposed roughly agrees with observations. It is to be seen whether a more sophisticated model may reproduce all the observed properties of dark matter.

References

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