Two-Center Black Holes, Qubits and Elliptic Curves

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(Dated: January 18, 2013)

Abstract

We relate the U-duality invariants characterizing two-center extremal black hole solutions in the $stu$, $st^2$ and $t^3$ models of $N = 2$, $d = 4$ supergravity to the basic invariants used to characterize entanglement classes of four-qubit systems. For the elementary example of a D0D4-D2D6 composite in the $t^3$ model we illustrate how these entanglement invariants are related to some of the physical properties of the two-center solution. Next we show that it is possible to associate elliptic curves to charge configurations of two-center composites. The hyperdeterminant of the hypercube, a four-qubit polynomial invariant of order 24 with 2894276 terms, is featuring the $j$ invariant of the elliptic curve. We present some evidence that this quantity and its straightforward generalization should play an important role in the physics of two-center solutions.

PACS numbers: 03.67.-a, 03.65.Ud, 03.65.Ta, 02.40.-k
I. INTRODUCTION

The aim of the present paper is to show that it is possible to relate the entanglement measures usually used in studies concerning four-qubit systems to the U-duality invariants found recently by Ferrara et. al. characterizing extremal two-center black hole solutions in the $stu$, $st^2$ and $t^3$ models. Interestingly as a byproduct of establishing this correspondence one can also come across an interesting connection between charge configurations of such black holes and a special class of elliptic curves.

Multi center black hole solutions provide an interesting research direction within the rapidly evolving field of black hole solutions in supergravity, string and M-theory. For such solutions the attractor mechanism has been generalized giving rise to split attractor flows. For two-center solutions the latter term refers to the situation, when in moduli space after crossing walls of marginal stability the attractor flows are separately evolving to the attractor points of the constituent single center solutions. Recently these developments have triggered activity in a variety of new research fields such as attractor flow trees, entropy enigmas, microstate counting, and bound state recombination.

For single center solutions it has become obvious that the notion of duality charge orbits and their invariants are useful concepts for classifying black hole solutions together with their supersymmetry properties. Except for a special case for multicenter solutions the corresponding structure of orbits and invariants is still unknown. In order the remedy this situation in a recent paper Ferrara et al. conducted a systematic study on the structure of invariants characterizing the charge configurations and invariants of two center solutions. In the case of the $stu$, $st^2$ and $t^3$ models looking at the structure of such invariants one immediately notices structural similarities to the well known sets of four-qubit invariants discussed in the seemingly unrelated field of quantum information.

Since the advent of the black hole qubit correspondence such coincidences should not come as a surprise. It is well-known by now that few qubit entangled systems are capable of providing interesting new insight into the structure of black hole solutions and their attractor flows. The occurrence of these qubits is related to the presence of tensor products of the spin $\frac{1}{2}$ irreducible representations of $SL(2)$ groups. Such products of $SL(2)$s show up as subgroups of $U$-duality groups governing the entangled web of dualities of supergravity models giving rise to black hole solutions. Initially mathematical coincidences were recorded
merely for two and three qubit systems and the corresponding axion-dilaton and \textit{stu} black holes\cite{18}, however evidence for \textit{n}-qubit systems with \textit{n} \textgt{ 3} to make their presence in this context started to accumulate.

In this line of development the relevance of four qubit systems to black hole solutions in supergravity was first pointed out in our paper\cite{20} where the isomorphism

\begin{equation}
\text{so}(4,4) \simeq \text{sl}(2)^4 \oplus (2,2,2,2) \tag{1}
\end{equation}

has been used to describe $7 \times 16$ of the 133 $E_{7(7)}$ generators of $N = 8, d = 4$ supergravity. These generators describing seven copies of four-qubit states are not belonging to the \text{sl}(2)^7 subalgebra of $E_{7(7)}$. A suitable incidence geometry accounting for the relationship between the 7 groups of 16 generators is that of the dual Fano plane giving rise to a geometry, dual to the one describing the ”tripartite entanglement of seven qubits” interpretation\cite{20,21} of the quartic $E_{7(7)}$ black hole entropy formula. The isomorphism of Eq.(1) was also discussed in the review paper of Borsten et. al.\cite{22} providing further interesting examples of simple qubit systems.

In our next paper\cite{23} in a four-qubit entanglement based formalism the structure of extremal stationary spherically symmetric black-hole solutions in the STU model of $N = 2, d = 4$ supergravity was described. The basic idea facilitating this interpretation was the fact that stationary solutions in $d = 4$ supergravity can be described by dimensional reduction along the time direction\cite{24}. In this $d = 3$ picture the global symmetry group $\text{SL}(2, \mathbb{R})^3$ of the STU model is extended by the Ehlers $\text{SL}(2, \mathbb{R})$ accounting for the fourth qubit. One can then introduce a four-qubit state depending on the charges, the moduli, and the warp factor. Here it was also noticed that in the terminology of four-qubit entanglement extremal black-hole solutions should correspond to nilpotent, and nonextremal ones to semisimple states. The upshot of these considerations was the emerging possibility of relating the entanglement properties of such and similar states to different classes of black-hole solutions in the STU model. The challenge of elaborating on this idea was recently taken up in the papers of Borsten et.al.\cite{25} In these papers the authors applied the black-hole qubit correspondence to the problem of classifying four-qubit entanglement. The key technical ingredient was the Kostant-Sekiguchi theorem which establishes the link between nilpotent orbits of extremal black holes and four qubit entanglement types. The emerging picture is: we have 31 entanglement families which reduce to nine up to permutations of the qubits. These nice
papers confirmed once again that the input coming from string theory can be useful in establishing results in a different field, since the literature until now on four qubit entanglement classification was confusing and seemingly contradictory.

In this paper we would like to show that the charge orbit classification of two-center black hole solutions in the \textit{stu} model is another arena where four-qubit systems naturally make their appearance. As a first possible step in this direction here we establish a correspondence between the U-duality invariants of Ferrara et.al.\cite{1} and the four-qubit invariants showing up in classification schemes of entanglement types in quantum information. Establishing this correspondence simplifies some of the invariants proposed so far, clarifies their geometric and algebraic roles, and provides hints for further generalizations outside the framework of $N = 2, d = 4$ supergravity. As an extra bonus the four-qubit picture also hints at a basic physical role these invariants are playing in the theory of two-center solutions. For one of the invariants not fully appreciated yet, our considerations establish a special role. It is the $SL(2)^4$ and permutation invariant hyperdeterminant of type $2 \times 2 \times 2 \times 2$. This is a polynomial of order 24 in the 16 amplitudes of the four-qubit state. Mapping the 16 amplitudes to the 16 charges characterizing two-center solutions in the \textit{stu} model, for a special case we show that the structure of this hyperdeterminant seem to govern issues of consistency in the realm of two-center solutions. These ideas also suggest a natural way for associating an elliptic curve of a special kind to a particular charge configuration. The coefficients of our elliptic curve are the algebraically independent four-qubit invariants, and its discriminant is just the hyperdeterminant. We also present some evidence for the conjecture that the structure of the $j$ invariant of the elliptic curve should play an important role in the physical properties of the two-center solution. The idea that elliptic functions and the $j$ invariant might play some role in four-qubit systems and the black hole qubit correspondence was first suggested by P. Gibbs\cite{26} some related discussion appeared in the paper of Bellucci et.al.\cite{27}.

The organization of this paper is as follows. In Section II. we summarize the background material on four-qubit invariants, reduced density matrices, and the structure of the hyperdeterminant of the hypercube. We introduce a quartic polynomial featuring the algebraically independent four-qubit $SL(2, \mathbb{C})$ invariants. In Section III. we are discussing extremal two center black hole charge states in a four-qubit based picture. Here we work out a dictionary between the invariants found by Ferrara et. al.\cite{1} in the so called Calabi-Visentini basis.
and the algebraically independent four-qubit ones in the "special coordinates" basis. We give some of the invariants used in Ref. a simpler appearance, and connect other invariants of physical meaning to properties of four-qubit reduced density matrices. Here we also show that the set of algebraically independent polynomial invariants in both the four-qubit and the Ferrara et.al. description are based on two seemingly different quartic polynomials however, with the same resolvent cubic.

Section IV. is devoted to a case study featuring BPS $D0D4 - D2D6$ composites in the $t^3$ model. In the paper of Bates and Denef this elementary example has already turned out to be a good playing ground for investigating the basic properties of two center solutions, hence we opted for illustrating the physical role of our four qubit invariants in the very same setting. These considerations relate the (necessary) consistency condition, guaranteeing the BPS composite to exist, to the positivity of the hyperdeterminant and to the extra constraint that the two nonzero invariants of the $t^3$ model are having the same sign. It turns out that precisely these conditions are the ones guaranteeing the fundamental quartic polynomial to have real roots. We then associate an elliptic curve of Weierstrass canonical form to the resolvent cubic of this quartic and show how physical properties are nicely encapsulated in the structure of its $j$ invariant.

In Section V. we examine the status of our rather ad hoc assignment: two-center charge configuration-elliptic curve, more thoroughly. By switching to the most general Tate form of an elliptic curve we show that our association of elliptic curves to two-center black hole charge configurations in the $stu$ model is a natural one. This means that the nonzero coefficients $a_j$ with $j = 1, 2, 3, 4$ appearing in the Tate form are algebraically independent four-qubit invariant homogeneous polynomials of order $2j$. These coefficients have important physical meaning: $a_1$ is just the canonical symplectic pairing between the charge vectors of the two centers. Vanishing of $a_4$ gives rise to the $st^2$, and a further vanishing of $a_2$ results in the $t^3$ truncation. One can also see that in the $st^2$ and $t^3$ models sending $a_3$ to zero corresponds to the limit when our elliptic curve degenerates. One of the nontrivial coefficients $a_6$ in the Tate form is always zero for the $stu$ model. This is related to the vanishing of a nontrivial polynomial constraint of homogeneous degree 12 valid in the $stu$ model, already observed by Ferrara et.al. Based on the latest results of Andrianopoli et. al we conjecture that we should be able to generalize our correspondence between charge orbits and elliptic curves also for the case of maximal $N = 8, d = 4$ supergravity. In this case the $stu$ model should
arise as an $a_6 \neq 0$ truncation implemented by the vanishing of a polynomial of order 12.

The aim of our last speculative Section VI. is to draw the readers attention to some interesting structural similarities showing up in a variety of physical contexts where our four-qubit invariants parametrizing elliptic curves might play a crucial role. Here we give a new look and interpretation to a triality invariant curve originally introduced by Seiberg and Witten\textsuperscript{14}. Now this curve is parameterized by four-qubit invariants also displaying permutation invariance. In this new setting we also invoke the F-theory interpretation of this curve as was given by Sen\textsuperscript{15}. Finally our conclusions and some comments are left for Section VI.

II. FOUR QUBIT SYSTEMS

In order to facilitate a four-qubit description of the two center charge configurations in the \textit{stu} model our aim in this subsection is to review the background material on four-qubit states and their entanglement measures. A four qubit state can be written in the form

$$|\Lambda\rangle = \sum_{i_0 i_1 i_2 i_3} A_{i_0 i_1 i_2 i_3} |i_0 i_1 i_2 i_3\rangle, \quad |i_0 i_1 i_2 i_3\rangle \equiv |i_0\rangle \otimes |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \in V_0 \otimes V_1 \otimes V_2 \otimes V_3$$

where $V_{0,1,2,3} \equiv \mathbb{C}^2$. Let the subgroup of stochastic local operations and classical communication\textsuperscript{28} representing admissible fourpartite manipulations on the qubits be just $SL(2, \mathbb{C})^\otimes 4$ acting on $|\Lambda\rangle$ as

$$|\Lambda\rangle \mapsto (S_0 \otimes S_1 \otimes S_2 \otimes S_3)|\Lambda\rangle, \quad S_\alpha \in SL(2, \mathbb{C}), \quad \alpha = 0, 1, 2, 3.$$ \hspace{1cm} (3)

Our aim is to give a unified description of four-qubit states taken together with their SLOCC transformations and their associated invariants. As we will see states and transformations taken together can be described in a unified manner using the group $SO(8, \mathbb{C})$.

Let us discuss the structure of four-qubit $SL(2, \mathbb{C})^\times 4$ invariants\textsuperscript{16,17,31,32}. The number of algebraically independent four-qubit invariants is four. We have one quadratic, two quartic, and one sextic invariant. In our recent paper\textsuperscript{17} we investigated the structure of these invariants in the special frame where two of our qubits played a distinguished role. As we will see this scenario is just the one needed in the two-center STU black hole context since in this setting one of the special qubits (the one labelled by the number 0) will be associated to the
horizontal $SL_h(2, \mathbb{R})$ of Ferrara et al. and the other (the one labelled by the number 1) is arising as the first factor from the structure $SL(2, \mathbb{R}) \times SO(2, 2)$ known from the STU model. Indeed such structure is the one arising as a special case of the infinite Jordan symmetric sequence of $N = 2$ $d = 4$ supergravity theories.

To an arbitrary state $|\Lambda\rangle$ we can also associate a $4 \times 4$ matrix

$$L \equiv \begin{pmatrix} \Lambda_{0000} & \Lambda_{0001} & \Lambda_{0010} & \Lambda_{0011} \\ \Lambda_{0100} & \Lambda_{0101} & \Lambda_{0110} & \Lambda_{0111} \\ \Lambda_{1000} & \Lambda_{1001} & \Lambda_{1010} & \Lambda_{1011} \\ \Lambda_{1100} & \Lambda_{1101} & \Lambda_{1110} & \Lambda_{1111} \end{pmatrix} \equiv \begin{pmatrix} A^1 & A^2 & A^3 & A^4 \\ B^1 & B^2 & B^3 & B^4 \\ C^1 & C^2 & C^3 & C^4 \\ D^1 & D^2 & D^3 & D^4 \end{pmatrix}.$$  \hspace{1cm} (4)

or four four-vectors. The splitting of the amplitudes of $|\Lambda\rangle$ into such four-vectors reflects our special choice for the distinguished qubits compatible with our conventions. We will also need the matrices

$$M = \begin{pmatrix} A^1 & A^2 & B^1 & B^2 \\ C^1 & C^2 & D^1 & D^2 \\ A^3 & A^4 & B^3 & B^4 \\ C^3 & C^4 & D^3 & D^4 \end{pmatrix}, \quad N = \begin{pmatrix} A^1 & A^3 & B^1 & B^3 \\ A^2 & A^4 & B^2 & B^4 \\ C^1 & C^3 & D^1 & D^3 \\ C^2 & C^4 & D^2 & D^4 \end{pmatrix}.$$ \hspace{1cm} (5)

Notice that in four-qubit notation the matrices $M$ and $N$ are arising from $L$ by the permutations $(012)(3)$ and $(0)(123)$ meaning that the index structure of these matrices is

$$L \leftrightarrow \Lambda_{i_0i_1i_2i_3}, \quad M \leftrightarrow \Lambda_{i_1i_2i_0i_3}, \quad N \leftrightarrow \Lambda_{i_0i_2i_3i_1}.$$ \hspace{1cm} (6)

Notice that these matrices are entering in the reduced density matrices as

$$\varrho_{01} \equiv \text{Tr}_3|\Lambda\rangle\langle\Lambda| = LL^\dagger, \quad \varrho_{12} \equiv \text{Tr}_0|\Lambda\rangle\langle\Lambda| = MM^\dagger, \quad \varrho_{02} \equiv \text{Tr}_1|\Lambda\rangle\langle\Lambda| = NN^\dagger.$$ \hspace{1cm} (7)

$$\overline{\varrho}_{23} \equiv \text{Tr}_2|\Lambda\rangle\langle\Lambda| = L^\dagger L, \quad \overline{\varrho}_{03} \equiv \text{Tr}_1|\Lambda\rangle\langle\Lambda| = M^\dagger M, \quad \overline{\varrho}_{13} \equiv \text{Tr}_0|\Lambda\rangle\langle\Lambda| = N^\dagger N.$$ \hspace{1cm} (8)

where overline denotes complex conjugation.

Now we introduce on the vector space $\mathbb{C}^4 \simeq \mathbb{C}^2 \times \mathbb{C}^2$ corresponding to the third and fourth qubit a symmetric bilinear form $g : \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C}$ with matrix representation

$$g = \varepsilon \otimes \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ \hspace{1cm} (9)
This means that we have an $SL(2, \mathbb{C})^\times 2$ invariant quantity with the explicit form

$$g(A, B) \equiv A \cdot B = g_{\alpha \beta} A^\alpha B^\beta = A_\alpha B^\alpha = A^4 - A^2 B^3 - A^3 B^2 + A^4 B^1. \tag{10}$$

We can also introduce a dual four-qubit state

$$|\lambda\rangle = \sum_{i_0 i_1 i_2 i_3 = 0, 1} \lambda_{i_0 i_1 i_2 i_3} |i_0 i_1 i_2 i_3\rangle \tag{11}$$

with the associated matrix

$$l \equiv \begin{pmatrix} 
\lambda_{0000} & \lambda_{0001} & \lambda_{0010} & \lambda_{0011} \\
\lambda_{0100} & \lambda_{0101} & \lambda_{0110} & \lambda_{0111} \\
\lambda_{1000} & \lambda_{1001} & \lambda_{1010} & \lambda_{1011} \\
\lambda_{1100} & \lambda_{1101} & \lambda_{1110} & \lambda_{1111} 
\end{pmatrix} \equiv \begin{pmatrix} 
a^1 & a^2 & a^3 & a^4 \\
b^1 & b^2 & b^3 & b^4 \\
c^1 & c^2 & c^3 & c^4 \\
d^1 & d^2 & d^3 & d^4 
\end{pmatrix}, \tag{12}$$

where

$$a^\alpha = \epsilon^{\alpha \beta \gamma \delta} B_\beta C_\gamma D_\delta, \quad b^\beta = \epsilon^{\alpha \beta \gamma \delta} A_\alpha C_\gamma D_\delta \quad c^\gamma = \epsilon^{\alpha \beta \gamma \delta} A_\alpha B_\beta D_\delta \quad d^\delta = \epsilon^{\alpha \beta \gamma \delta} A_\alpha B_\beta C_\gamma. \tag{13}$$

Here $\epsilon^{1234} = +1$, and indices are lowered by the matrix of $g$. Notice that the amplitudes of the dual four-qubit state are cubic in the original ones.

Using these definitions we define the quadratic and sextic invariants as

$$I_1 \equiv \frac{1}{2}(A \cdot D - B \cdot C), \quad I_3 \equiv \frac{1}{2}(a \cdot d - b \cdot c). \tag{14}$$

The explicit form of the sextic invariant in terms of the dot product of Eq.(10) is

$$2I_3 = \text{Det} \begin{pmatrix} 
A \cdot A & A \cdot B & A \cdot D \\
A \cdot C & B \cdot C & C \cdot D \\
A \cdot D & B \cdot D & D \cdot D 
\end{pmatrix} - \text{Det} \begin{pmatrix} 
A \cdot B & B \cdot B & B \cdot C \\
A \cdot C & B \cdot C & C \cdot C \\
A \cdot D & B \cdot D & C \cdot D 
\end{pmatrix}. \tag{15}$$

We also recall that the explicit form of $I_1$ is hiding its permutation invariance. Moreover, though the expression of $I_3$ of Eq.(14) is similar to the one of $I_1$ the invariant $I_3$ is not invariant under the permutation of the qubits.

Now we turn to the structure of quartic invariants. We have two independent of such invariants and the simplest of them is the obvious expression

$$I_4 \equiv \text{Det} \mathcal{L} \tag{16}$$
i.e. the determinant of the 4 × 4 matrix of Eq. (1). In order to present the definition of the second one we define separable bivectors of the form

\[ \Pi_{\mu\nu\alpha\beta} \equiv \Lambda_{\mu\alpha} \Lambda_{\nu\beta} - \Lambda_{\mu\beta} \Lambda_{\nu\alpha}, \quad \alpha, \beta, \mu, \nu = 1, 2, 3, 4. \] (17)

Here our labelling convention \( \Lambda_{\mu\alpha} \) indicates that \( \mu = 1, 2, 3, 4 \) identifies the four-vector in question (i.e. \( A, B, C \) or \( D \) of Eq.(1)), and the label \( \alpha = 1, 2, 3, 4 \) refers to the component of the particular vector. Now our last invariant is the quartic combination

\[ I_2 = \frac{1}{6} \Pi_{\mu\nu\alpha\beta} \Pi^{\mu\nu\alpha\beta}. \] (18)

Obviously the symmetric nondegenerate bilinear form \( g \) of Eq.(10) acting on four vectors like \( A, B, C, D \in \mathbb{C}^4 \) is inducing a corresponding symmetric nondegenerate bilinear form on the space of bivectors \( \Lambda^2 \mathbb{C}^4 \). By an abuse of notation we use again the symbol \( \cdot \) for this new bilinear form with the definition

\[ (A \wedge B) \cdot (C \wedge D) \equiv 2((A \cdot C)(B \cdot C) - (A \cdot D)(B \cdot C)). \] (19)

Now \( I_2 \) can also be written in the equivalent form

\[ I_2 = \frac{1}{6} \left[ (A \wedge B) \cdot (C \wedge D) + (A \wedge C) \cdot (B \wedge D) - \frac{1}{2}(A \wedge D)^2 - \frac{1}{2}(B \wedge C)^2 \right]. \] (20)

An important comment here is in order. Let us have a look at \( I_4 \) and also at the determinants of the matrices of Eq.(5)

\[ L \equiv I_4 = \text{Det}\mathcal{L}, \quad M \equiv \text{Det}\mathcal{M}, \quad N \equiv \text{Det}\mathcal{N}. \] (21)

Then one can prove\(^{16,30}\)

\[ L + M + N = 0. \] (22)

One also has the constraint

\[ M - N = 3I_2 - 2I_1^2, \] (23)

that we will need later.

It is known\(^{16}\) that the minimal set of algebraically independent \( SL(2)^{\times 4} \) invariants is consisting of a quadratic, two quartic and one sextic invariant. Our choice for this set will be\(^{17}\) \( I_1, I_2, I_4 \) and \( I_3 \). Let us now present the reason for this choice. Let us consider the matrix

\[ \Omega \equiv \mathcal{L}g\mathcal{L}^T g. \] (24)
Then its characteristic polynomial is
\[ \Sigma_4(\Lambda_{i_0i_1i_2i_3}, t) \equiv \text{Det}(1t - \Omega) = t^4 - 4I_1t^3 + 6I_2t^2 - 4I_3t + I_4, \] (25)

Clearly by Newton’s identities we have
\[ I_1 = \frac{1}{4} \text{Tr}\Omega, \quad I_2 = \frac{1}{12} [(\text{Tr}\Omega)^2 - \text{Tr}\Omega^2], \]
\[ I_3 = \frac{1}{24} [(\text{Tr}\Omega)^3 - 3\text{Tr}\Omega\text{Tr}\Omega^2 + 2\text{Tr}\Omega^3], \quad (I_4)^2 = \text{Det}\Omega. \] (27)

This form of writing our invariants is related to the fact that there is a \(1 - 1\) correspondence between the \(SL(2, \mathbb{C})^\otimes 4\) orbits of four-qubit states and the \(SO(4, \mathbb{C}) \times SO(4, \mathbb{C})\) ones of \(4 \times 4\) matrices.

The polynomial of Eq. (25) in the four-qubit context appeared in our recent paper\(^{17}\) its role as a characteristic polynomial has been emphasized in Ref.\(^{31}\). The discriminant of this fourth order polynomial is the hyperdeterminant\(^{37}\) \(D_4\) of the \(2 \times 2 \times 2 \times 2\) hypercube \(\Lambda_{i_0i_1i_2i_3}\). It is a polynomial of degree 24 in the 16 amplitudes and has 2894276 terms\(^{33}\). \(D_4\) can be expressed\(^{17}\) in terms of our fundamental invariants as
\[ 256D_4 = S^3 - 27T^2 \] (28)
where
\[ S = (I_4^2 - I_2^2) + 4(I_2^2 - I_1I_3), \quad T = (I_4^2 - I_2^2)(I_4^2 - I_2) + (I_3 - I_1I_2)^2. \] (29)

For an alternative form of \(D_4\) see the papers of Refs.\(^{16,32}\).

In closing this section we briefly discuss some results on the classification of entanglement classes for four qubits\(^{31,34}\). By entanglement classes we mean orbits under \(SL(2, \mathbb{C})^\times 4 \cdot \text{Sym}_4\) where \(\text{Sym}_4\) is the symmetric group on four symbols. The basic result states that four qubits can be entangled in nine different ways\(^{31,34}\). It is to be contrasted with the two entanglement classes\(^{28}\) obtained for three qubits. For a refined classification of four qubit entanglement motivated by the black hole qubit correspondence see the papers of Borsten et.al.\(^{25}\)

Let us consider the matrix
\[ \mathcal{R}_A \equiv \begin{pmatrix} 0 & \Lambda g \\ -\Lambda^T g & 0 \end{pmatrix} \] (30)
which now can be regarded as an element of the Lie algebra of \(SO(8, \mathbb{C})\). If the matrix \(\mathcal{R}_A\) is diagonalizable under the action
\[ \mathcal{R}_A \mapsto S\mathcal{R}_AS^{-1}, \quad S = \begin{pmatrix} S_0 \otimes S_1 & 0 \\ 0 & S_2 \otimes S_3 \end{pmatrix}, \quad S_\alpha \in SL(2, \mathbb{C}) \] (31)
we say that the corresponding four-qubit state $|\Lambda\rangle$ is semisimple. If $R_{\Lambda}$ is nilpotent then we call the corresponding state $|\Lambda\rangle$ nilpotent too. It is known that a nilpotent orbit is conical i.e. if $|\Lambda\rangle$ is an element of the orbit then $\lambda|\Lambda\rangle$ is also an element for all nonzero complex numbers $\lambda$. Hence a nilpotent orbit is also a $GL(2, \mathbb{C})^4$ orbit. It is clear that for nilpotent states all of our algebraically independent invariants are zero.

A semisimple state of four qubits can always be transformed to the form

$$|G_{abcd}\rangle = \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle),$$

(32)

where $a, b, c, d$ are complex numbers. This class corresponds to the so called GHZ class found in the three-qubit case. For this state the reduced density matrices obtained by tracing out all but one of the qubits are proportional to the identity. This is the state with maximal four-partite entanglement. Another interesting property of this state is that it does not contain true three-partite entanglement. A straightforward calculation shows that the values of our invariants $(I_1, I_2, I_3, I_4)$ occurring for the state $|G_{abcd}\rangle$ representing the generic class are

$$I_1 = \frac{1}{4}[a^2 + b^2 + c^2 + d^2], \quad I_2 = \frac{1}{6}[(ab)^2 + (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2 + (cd)^2],$$

$$I_3 = \frac{1}{4}[(abc)^2 + (abd)^2 + (acd)^2 + (bcd)^2], \quad I_4 = abcd$$

(33)

hence the values of the invariants $(4I_1, 6I_2, 4I_3, I_4^2)$ are given in terms of the elementary symmetric polynomials in the variables $(t_1, t_2, t_3, t_4) = (a^2, b^2, c^2, d^2)$. For the semisimple states $|G_{abcd}\rangle$ the value of $D_4$ can be expressed as

$$D_4 = \frac{1}{256}\prod_{i<j}(t_i - t_j)^2, \quad (t_1, t_2, t_3, t_4) \equiv (a^2, b^2, c^2, d^2).$$

(34)

Notice that for the states $|G_{abcd}\rangle$ with $D_4$ nonvanishing $(t_i \neq t_j)$ the corresponding matrix of Eq. (30) belongs to a Cartan subalgebra of $SO(8, \mathbb{C})$. The stabilizer of such states corresponds to the Weyl group of $SO(8, \mathbb{C})$. This stabilizer is the Klein group generated by the four elements $I \otimes I \otimes I \otimes I$ and $\sigma_a \otimes \sigma_a \otimes \sigma_a \otimes \sigma_a$ for $a = 1, 2, 3$.

III. TWO CENTER EXTREMAL BLACK HOLES AS FOUR QUBITS SYSTEMS.

In the paper of Ferrara et.al. in order to describe the structure of the U-duality invariant polynomials associated to the two-center extremal black hole solutions the Calabi-Visentini
(CV) basis has been used. Here by $U$-duality we mean the continuous limit valid for large values of the charges of the usual-nonperturbative string theory symmetries. First we describe the connection of the CV basis to the one making the four-qubit structures explicit. Next we turn to an entanglement based understanding of the structure of the two-center $U$-duality invariants.

In the CV basis the two-center black hole solutions are characterized by a pair of real charge vectors $Q_1, Q_2 \in \mathbb{R}^8$

\[
Q_1 \equiv (P^0, P^1, P^2, P^3, Q_0, Q_1, Q_2, Q_3)^T, \quad Q_2 \equiv (p^0, p^1, p^2, p^3, q_0, q_1, q_2, q_3)^T. \tag{36}
\]

As we see these charge vectors are containing two four-vectors each namely $P^I, Q_I \equiv \eta_{IJ}Q^J$, and $p^I, q_I \equiv \eta_{IJ}q^J I, J = 0, 1, 2, 3$ where the raising and lowering of the indices $I$ and $J$ are effected by the metric $\eta_{IJ}$ and $\eta^{IJ}$ of $SO(2, 2)$ answering the symmetric bilinear form $h$ acting on the charge four-vectors $P^I$ and $Q^I$ in the CV basis as

\[
h(P, Q) \equiv P \circ Q = \eta_{IJ}P^IQ^J = -P^0Q^0 - P^1Q^1 + P^2Q^2 + P^3Q^3. \tag{37}
\]

Let us now relate the 16 component charge vector in the CV basis characterizing a particular two center extremal black hole solution in the STU model to a real unnormalized four-qubit pure state $|\Lambda\rangle$ by relating the four-vectors $P^I, Q^I, p^I, q^I \in \mathbb{R}^4$ of Eq.\((36)\) and $A^a, B^a, C^a, D^a \in \mathbb{R}^4$ of Eq.\((41)\) as follows.

\[
\begin{pmatrix}
P^0 \\
1 \\
2 \\
P^3
\end{pmatrix}_{CV} = \frac{1}{\sqrt{2}} \begin{pmatrix}
A^1 - A^4 \\
A^2 + A^3 \\
-A^1 - A^4 \\
-A^2 + A^3
\end{pmatrix}, \quad \begin{pmatrix}
Q^0 \\
Q^1 \\
Q^2 \\
Q^3
\end{pmatrix}_{CV} = \frac{1}{\sqrt{2}} \begin{pmatrix}
B^1 - B^4 \\
B^2 + B^3 \\
-B^1 - B^4 \\
-B^2 + B^3
\end{pmatrix} \tag{38}
\]

\[
\begin{pmatrix}
p^0 \\
p^1 \\
p^2 \\
p^3
\end{pmatrix}_{CV} = \frac{1}{\sqrt{2}} \begin{pmatrix}
C^1 - C^4 \\
C^2 + C^3 \\
-C^1 - C^4 \\
-C^2 + C^3
\end{pmatrix}, \quad \begin{pmatrix}
q^0 \\
q^1 \\
q^2 \\
q^3
\end{pmatrix}_{CV} = \frac{1}{\sqrt{2}} \begin{pmatrix}
D^1 - D^4 \\
D^2 + D^3 \\
-D^1 - D^4 \\
-D^2 + D^3
\end{pmatrix} \tag{39}
\]

Then using the definitions above we clearly have for example

\[
P \circ Q = A \cdot B = g(A, B) = g_{\alpha\beta}A^\alpha B^\beta \tag{40}
\]

with the bilinear form $g$ defined as in Eq.\((11)\).
Let us also give the connection between the Calabi-Visentini basis and the one usually used in special geometry, i.e. the special coordinates (SC) symplectic frame. This frame yields the usual set of electric and magnetic charges i.e. \( (P^i, Q_i)_{SC} \) and \( (p^I, q_I)_{SC} \). In the following we will use these charges so it is important to clarify their relationship to the components of our four qubit state \( |\Lambda\rangle \) (see Eq. (4)).

\[
\begin{pmatrix}
\Lambda_{0000} \\
\Lambda_{0001} \\
\Lambda_{0010} \\
\Lambda_{0011}
\end{pmatrix} = \begin{pmatrix} A^1 \\ A^2 \\ A^3 \\ A^4 \end{pmatrix} \equiv \begin{pmatrix} P^0 \\ P^2 \\ P^3 \\ Q_1 \end{pmatrix}_{SC}, \quad \begin{pmatrix} \Lambda_{0100} \\
\Lambda_{0101} \\
\Lambda_{0110} \\
\Lambda_{0111}
\end{pmatrix} = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \\ B^4 \end{pmatrix} = \begin{pmatrix} P^1 \\ Q_3 \\ Q_2 \\ -Q_0 \end{pmatrix}_{SC} \tag{41}
\]

\[
\begin{pmatrix} \Lambda_{1000} \\
\Lambda_{1001} \\
\Lambda_{1010} \\
\Lambda_{1011}
\end{pmatrix} = \begin{pmatrix} C^1 \\ C^2 \\ C^3 \\ C^4 \end{pmatrix} \equiv \begin{pmatrix} p^0 \\ p^2 \\ p^3 \\ q_1 \end{pmatrix}_{SC}, \quad \begin{pmatrix} \Lambda_{1100} \\
\Lambda_{1101} \\
\Lambda_{1110} \\
\Lambda_{1111}
\end{pmatrix} = \begin{pmatrix} D^1 \\ D^2 \\ D^3 \\ D^4 \end{pmatrix} = \begin{pmatrix} p^1 \\ q_3 \\ q_2 \\ -q_0 \end{pmatrix}_{SC} \tag{42}
\]

We note here however, that our conventions are slightly different from the ones used in Ref.1. The charges \( p^1, p^2, p^3, q_0 \) and \( P^1, P^2, P^3, Q_0 \) in the SC basis used by us are the negatives of the corresponding ones in the SC basis as used in Ref.1. see Eqs. (38-39), Eqs. (41-42) and Eq. (B.3) of that paper.

Notice also that in our four-qubit state \( |\Lambda\rangle \) with amplitudes \( \Lambda_{i_0 i_1 i_2 i_3} \) sit two three-qubit states with amplitudes \( \Lambda_{0i_1 i_2 i_3} \) and \( \Lambda_{1i_1 i_2 i_3} \). The first set of amplitudes labelled by \( i_0 = 0 \) corresponds to the charge configuration of the first black hole and the second labelled by \( i_0 = 1 \) describes the second black hole. As we see the first label plays a distinguished role with an extra \( SL(2, \mathbb{R})_0 \) (dubbed by Ferrara et.al. the horizontal one) acting on. This group represents the generalized exchange symmetry between the centers.

Let us now define three bivectors

\[
X \equiv A \wedge B, \quad Y \equiv C \wedge D, \quad Z \equiv \frac{1}{2}(A \wedge D - B \wedge C). \tag{43}
\]

In component notation we have for example

\[
X_{\alpha \beta} = A_\alpha B_\beta - A_\beta B_\alpha. \tag{44}
\]

Switching to the Calabi-Visentini basis these objects are the \( T \)-tensors used in the paper of Ferrara et.al. As we can see the bivectors \( X \) and \( Y \) are separable i.e. they are precisely the
ones satisfying the Plücker relations
\[X_{12}X_{34} - X_{13}X_{23} + X_{14}X_{23} = 0, \quad Y_{12}Y_{34} - Y_{13}Y_{23} + Y_{14}Y_{23} = 0\]
(45)
on the other hand the bivector \(Z\) is entangled i.e. in the nomenclature of fermionic entanglement\(^{40}\) it has Slater rank two.

Now we introduce the shorthand notation for the product of two separable bivectors as defined in Eq.(19)
\[X \cdot Y \equiv (A \wedge B) \cdot (C \wedge D)\]
(46)Notice that in the notation of Eq.(41)-(42)
\[X^2 \equiv X \cdot X = 2(A^2B^2 - (A \cdot B)^2), \quad Y^2 \equiv Y \cdot Y = 2(C^2D^2 - (C \cdot D)^2)\]
(47)are just two times the quartic invariants of the charges characterizing the two black holes
\[\frac{1}{2}X^2 = I_4(Q_1) = -D_3(\Lambda_{0i_1i_2i_3}), \quad \frac{1}{2}Y^2 = I_4(Q_2) = -D_3(\Lambda_{1i_1i_2i_3}).\]
(48)Here \(D_3(\Lambda_{i_1i_2i_3})\) is Cayley’s hyperdeterminant\(^{36}\).

With these definitions we can define the quantities
\[I_{+2} = \frac{1}{2}X^2, \quad I_{+1} = \frac{1}{2}X \cdot Z, \quad I_0 = \frac{1}{6}(2Z^2 - X \cdot Y), \quad I_{-1} = \frac{1}{2}Y \cdot Z, \quad I_{-2} = \frac{1}{2}Y^2\]
(49)and the ones
\[I' = \frac{1}{2}X \cdot Y, \quad I'' = \frac{3}{2}Z^2.\]
(50)When reinterpreted in the CV basis these are precisely the \(SL(2, \mathbb{R})_1 \times SL(2, \mathbb{R})_2 \times SL(2, \mathbb{R})_3\) invariants of Ferrara et.al.\(^1\) The important property of the invariants of Eq.(49) is that they are covariants with respect to \(SL(2, \mathbb{R})_0\) acting on the distinguished (horizontal) qubit. Indeed, they are sitting in the \(5\) (spin 2) irreducible representation of this group.

Now we elucidate another aspect of this important property of the set of invariants of Eq.(49). In order to do this we first recall that for two-qubits the canonical measure of pure state entanglement is the concurrence\(^{35}\)
\[C = 2|D_2(\Lambda_{i_1i_2})| = 2|\Lambda_{00}\Lambda_{11} - \Lambda_{01}\Lambda_{10}|\]
(51)which is related to the determinant of an ordinary \(2 \times 2\) matrix. For three-qubits the basic quantity characterizing genuine three-qubit entanglement\(^{28}\) is the three-tangle
\[\tau = 4|D_3(\Lambda_{i_0i_1i_2})|\]
(52)
where now $D_3$ is Cayley’s hyperdeterminant\textsuperscript{35,36}. According to the method of Schlaffli\textsuperscript{37} $D_3$ is related to the discriminant $\Delta_2$ of the quadratic polynomial
\begin{equation}
\Pi_2(\Lambda_{i_0i_1i_2}, t) \equiv D_2(\Lambda_{i_0i_1i_2} t + \Lambda_{i_1i_2}) = (\Lambda_0 \cdot \Lambda_0)t^2 + 2(\Lambda_0 \cdot \Lambda_1)t + (\Lambda_1 \cdot \Lambda_1)
\end{equation}
where $\Lambda_0$ and $\Lambda_1$ are four-vectors with components $(\Lambda_{000}, \Lambda_{001}, \Lambda_{010}, \Lambda_{011})$ and $(\Lambda_{100}, \Lambda_{101}, \Lambda_{110}, \Lambda_{111})$ and the $\cdot$ product is the usual one of Eq.(10). Moreover, due to permutation invariance of $D_3$ we obtain the same expression whenever the first or the second qubit plays a distinguished role. Obviously the quantities $J_{+1} = (\Lambda_0 \cdot \Lambda_0)$, $J_0 = (\Lambda_0 \cdot \Lambda_1)$, and $J_{-1} = (\Lambda_1 \cdot \Lambda_1)$ are $SL(2)$ singlets, however they are covariants under the "horizontal" $SL(2)_0$. Indeed, the triple $J_{+1}, J_0, J_{-1}$ transforms according to the irreducible representation 3 of spin 1 of this "horizontal” group which can be regarded as some sort of generalized exchange symmetry working between the two two-qubit systems. It is also clear that searching singlets with respect to this "horizontal” symmetry group can reveal some new properties of our pair of two qubit systems, namely that they are secretly comprising a system having a higher degree of symmetry.

Now proceeding by analogy we define the polynomial
\begin{equation}
\Pi_4(\Lambda_{i_0i_1i_2i_3}, t) \equiv D_3(\Lambda_{i_0i_1i_2i_3} t + \Lambda_{i_1i_2i_3}) = I_{+2}t^4 + 4I_{+1}t^3 + 6I_0t^2 + 4I_{-1}t + I_{-2}.
\end{equation}
A straightforward calculation shows that the coefficients of this polynomial are precisely the covariants of Eq.(49). Now according to theorem 14.4.1 and corollary 14.2.10 of Ref.\textsuperscript{37} the discriminant of this quartic polynomial $\Delta_4$ divided by 256 is just the hyperdeterminant $D_4(\Lambda_{i_0i_1i_2i_3})$ of the hypercube of format $2 \times 2 \times 2 \times 2$. Besides being a singlet with respect to the horizontal $SL(2)_0$, $D_4$ is also an invariant under $S_4$ the permutation group of the four qubit system.

At this point one can notice\textsuperscript{17} that the polynomials $\Sigma_4$ and $\Pi_4$ of Eqs.(25) and (54) are having the same discriminants $\Delta_4$ hence both can be used to obtain an expression for $D_4$. Notice that $\Sigma_4$ is a polynomial with its coefficients $I_1, I_2, I_3, I_4$ also being $SL(2)_0$ singlets however, $\Pi_4$ is a polynomial with coefficients $I_{\pm2}, I_{\pm1}, I_0$ being merely $SL(2)_0$ covariants. Using this observation we can construct new $SL(2)_0$ singlets from the quantities of Eq.(49) by relating them to the known algebraically independent four-qubit ones namely $I_1, I_2, I_3$ and $I_4$.

In order to relate the $SL(2)_0$ singlets found by Ferrara et.al. to our four algebraically independent four-qubit invariants one just has to compare the relevant expressions. In fact
many of their invariants and the constraints satisfied by them take in this four-qubit setting a much simpler and instructive form. In particular their complete set of invariants with corresponding degrees $2, 4, 6, 8$ denoted by $\mathcal{W}, \chi, I_6, \text{Tr}(\mathcal{J}^2)$ is related to ours as

$$\mathcal{W} = 2I_1, \quad \chi = 3I_2 - 2I_1^2, \quad I_6 = -I_3. \quad (55)$$

By virtue of Eqs. (23) and (50) we also have the relations

$$I' - I'' = \frac{3}{2}I_2, \quad \chi = M - N. \quad (56)$$

We still have to account for the invariant $\text{Tr}(\mathcal{J}^2)$ of order 8 built from $\mathcal{J}$ the symmetric traceless matrix comprising the covariants $I_{\pm 2}, I_{\pm 1}, I_0$. The 5 independent components of this matrix are transforming according to the 5 of $SL(2)_0$. In order to reveal the meaning of this invariant and also an extra one $\text{Tr}(\mathcal{J}^3)$ of order 12 let us reconsider the awkward looking polynomial constraint of Eq. (5.6) of Ref. 1.

$$\mathcal{P}_{12} \equiv I_6^2 + \mathcal{W}\chi I_6 + \text{Tr}(\mathcal{J}^3) + \frac{\text{Tr}(\mathcal{J}^2)\mathcal{W}^2}{12} - \frac{\text{Tr}(\mathcal{J}^2)\chi}{3} - \frac{\mathcal{W}^6}{432} + \frac{\mathcal{W}^4\chi}{36} + \frac{5\mathcal{W}^2\chi^2}{36} + \frac{4\chi^3}{27} = 0. \quad (57)$$

Using the dictionary of Eq. (55) we can cast this constraint in the nice form

$$\text{Tr}(\mathcal{J}^3) = [4(I_2^2 - I_1 I_3) - \text{Tr}(\mathcal{J}^2)](I_1^2 - I_2) - (I_3 - I_1 I_2)^2. \quad (58)$$

There is one more invariant of order 8 which is directly related to $\text{Tr}(\mathcal{J}^2)$

$$\mathcal{P}_8 \equiv -12\text{Tr}(\mathcal{J}^2) + 24I_6\mathcal{W} + (\mathcal{W}^2 + 2\chi)^2. \quad (59)$$

Recall that the constraint $\mathcal{P}_8 = 0$ implements the reduction of the $stu$ model to the $st^2$ model in a manifestly $SL(2)_0$ invariant manner. Using again Eq. (55) the new form of $\mathcal{P}_8$ is

$$\mathcal{P}_8 = 12(3I_2^2 - 4I_1 I_3 - \text{Tr}(\mathcal{J}^2)). \quad (60)$$

Putting this into Eq. (58) and recalling Eq. (29) one obtains the simple expressions

$$\text{Tr}(\mathcal{J}^2) = S, \quad \text{Tr}(\mathcal{J}^3) = -T, \quad (61)$$

provided

$$\mathcal{P}_8 = -12I_1^2 = -12L^2. \quad (62)$$
The first result of these considerations is that \(-\frac{1}{12}P_8\) is really the square of the basic fourth order invariant \(L = I_4\). According to Eqs. (4) and (21) \(L\) is just the determinant of the matrix \(\mathcal{L}\) we have started our four-qubit considerations with. Moreover, according to Eqs. (7-8) we also see that for unnormalized four-qubit states we have
\[-\frac{1}{12}P_8 = \text{Det} \varrho_{01} = \text{Det} \varrho_{12},\]

(63)
hence this invariant is related to the determinant of the reduced density matrices of our four-qubit state |Λ⟩ corresponding to the bipartite split of the form: (01)(23). We can also conclude that in the four-qubit picture the reduction of the \(stu\) model to the \(st^2\) is effected by sending one of the eigenvalues of the reduced density matrices corresponding to the (01)(23) split to zero. Notice however, that the remaining density matrices corresponding to the remaining two splits (02)(13) and (03)(12) are generally not sharing this property. This means that for the \(st^2\) model we have
\[L = 0, \quad M \neq 0, \quad N \neq 0.\]

(64)

Recall now that a suitable further reduction to the \(t^3\) model is obtained by employing the following two \(SL(2)\) invariant constraints:
\[\chi = 0, \quad P_8 = 0.\]

(65)

By virtue of Eq. (56) and the constraint \(L + M + N = 0\) these constraints can be described in the compact form
\[L = M = N = 0.\]

(66)

This means that for the \(t^3\) reduction of the \(stu\) model all of the reduced density matrices of Eqs. (7-8) of the four qubit state |Λ⟩ have a zero eigenvalue.

The second result of our considerations finally clarifies the role played by the invariants \(\text{Tr}(\mathcal{J}^2)\) and \(\text{Tr}(\mathcal{J}^3)\). In particular looking at Eqs. (28), (29) and Eq. (61) we see that that the four-qubit hyperdeterminant \(256D_4\) which is just the discriminant of our polynomial \(\Sigma_4\) of Eq. (25) can be expressed with the help of these invariants of order 8 and 12 as \(S^3 - 27T^2\). Moreover, since the discriminant of a quartic is the same as the discriminant of its resolvent cubic one can also show that the two quartic equations \(\Sigma_4 = 0\) and \(\Pi_4 = 0\) of Eqs. (25) and (54) are having the same resolvent cubics. Indeed, a straightforward calculation shows that
the corresponding resolvent qubics in both cases are of the form

\[ u^3 - Su - 2T = 0. \tag{67} \]

In the first case we get back to the known expressions for \( S \) and \( T \) of Eq.\( \text{(29)} \), and in the second one we get\(^1\)

\[
S = 3I_0^2 - 4I_+I_- + I_+I_-,
T = I_0^3 + I_+I_- + I_+I_- - I_+I_0I_- - 2I_+I_0I_-.
\tag{68}
\]

IV. THE D2D6 − D0D4 SPLIT

A. Invariants

In order to uncover the role of our four-qubit invariants playing in the theory of two center black hole solutions let us consider a special class of two center black hole solutions in the \( t^3 \) model featuring a \( D0D4 − D2D6 \) split in the type \( IIA \) duality frame. In this case the vectors of Eq. \( \text{(41)}-\text{(42)} \) in the special coordinate (SC) basis are

\[
\begin{pmatrix}
A^1 \\
A^2 \\
A^3 \\
A^4
\end{pmatrix} = \begin{pmatrix}
0 \\
P \\
P \\
0
\end{pmatrix}, \quad \begin{pmatrix}
B^1 \\
B^2 \\
B^3 \\
B^4
\end{pmatrix} = \begin{pmatrix}
P \\
0 \\
0 \\
-U
\end{pmatrix},
\tag{69}
\]

corresponding to the first black hole with charge configuration \( Q_1 \) of a BPS \( D0D4 \) system and

\[
\begin{pmatrix}
C^1 \\
C^2 \\
C^3 \\
C^4
\end{pmatrix} = \begin{pmatrix}
v \\
0 \\
0 \\
q
\end{pmatrix}, \quad \begin{pmatrix}
D^1 \\
D^2 \\
D^3 \\
D^4
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
q \\
0
\end{pmatrix},
\tag{70}
\]

corresponding to the second black hole with charge configuration \( Q_2 \) of a BPS \( D2D6 \) system.

For BPS configurations in the first case we should have

\[
-D_3(Q_1) = (A \cdot A)(B \cdot B) - (A \cdot B)^2 = 4UP^3 > 0 \tag{71}
\]

and in the second the corresponding constraint is

\[
-D_3(Q_2) = (C \cdot C)(D \cdot D) - (C \cdot D)^2 = -4vq^3 > 0. \tag{72}
\]
Notice that our charge split for the special values of $U = 4$, $P = q = 1$ and $v = -4$ incorporates the illustrative example of Bates and Denef\(^4\) (in that paper $v$ is related to ours via a sign flip). The four-vectors $A^\alpha$, $B^\beta$, $C^\gamma$ and $D^\delta$ are comprising the 16 amplitudes of a real four-qubit state as displayed in Eqs. (2) and (4). Now using Eqs. (14)-(15), (16) and (20) the algebraically independent four-qubit invariants $I_1, I_2, I_3$ and $I_4$ can be calculated. The explicit forms of these invariants are

$$I_1 = \frac{1}{2}(Uv - 3Pq), \quad I_2 = \frac{2}{3}I_1^2, \quad I_3 = -Pq(Pq + Uv)^2, \quad I_4 = 0. \quad (73)$$

Notice that by virtue of Eqs. (22)-(23) our $D_0D_4 - D_2D_6$ example illustrates the constraints we have already discussed in connection with the $t^3$ model, namely the ones $L = M = N = 0$. Since $L = I_4 = 0$ and $3I_2 = 2I_1^2$ ($\chi = 0$) the quartic equation $\Sigma_4 = 0$ arising from the polynomial of Eq. (25) now reduces to a cubic one of the following form

$$t^3 - 4I_1t^2 + 4I_1^2t - 4I_3 = 0 \quad (74)$$

The discriminant of this cubic equation is

$$\Delta = 4I_3 \left(I_3 - \left(\frac{2I_1}{3}\right)^3\right). \quad (75)$$

According to Eqs. (28)-(29) the hyperdeterminant $D_4$ is also related to this discriminant and is of the form

$$256D_4 = 27I_3^3 \left(\left(\frac{2I_1}{3}\right)^3 - I_3\right) = -(UvPq)(Pq)^2(Uv + 9Pq)^2(Uv + Pq)^6. \quad (76)$$

**B. Consistency condition**

Let us now consider the necessary condition\(^3\) for our two center charge configuration supporting a corresponding two center stationary extremal BPS black hole solution. As it is well-known this condition is of the form

$$|x_1 - x_2| = \langle Q_1, Q_2 \rangle \frac{|Z_1 + Z_2|_{r=\infty}}{2\text{Im}(Z_2Z_1)_{r=\infty}} \quad (77)$$

where $x_{1,2}$ are the locations of the centers, $r = |x|$, $Z_{1,2}$ are the central charges corresponding to the charges $Q_{1,2}$, and the symplectic product of the charge vectors $\langle Q_1, Q_2 \rangle$ is related to our quadratic four-qubit invariant as

$$I_1 = \frac{1}{2}\langle Q_1, Q_2 \rangle. \quad (78)$$
The explicit forms of the central charges for our centers are

\[
Z_1 = e^{K/2}(U - 3P\tau^2), \quad Z_2 = e^{K/2}(3q\tau + v\tau^3).
\]

(79)

Here \(\tau\) is as usual the complex scalar field of the \(t^3\) model

\[
\tau = x - iy, \quad y > 0
\]

(80)

with negative imaginary part\(^{42,45}\), and \(K = -\log(8y^3)\) is the Kähler potential.

Now the constraint dictated by Eq.\((77)\) is

\[
I_1 \text{Im}(\overline{Z}_2 Z_1)_\infty > 0.
\]

(81)

Explicitly we have

\[
8y^2 \text{Im}(\overline{Z}_2 Z_1)_\infty = -3Pv(x^2_\infty + y^2_\infty)^2 + (9Pq + 3Uv)x^2_\infty + (9Pq - Uv)y^2_\infty + 3Uq.
\]

(82)

We are interested in the structure of BPS \(D0D4 - D2D6\) composites hence according to Eqs.\((71)-(72)\) we should have

\[
UP > 0, \quad vq < 0.
\]

(83)

For all possible sign combinations satisfying these constraints we have

\[
(-Pv)I_1 < 0.
\]

(84)

Let us now write our consistency condition as

\[
8y^2 \left( -\frac{I_1}{Pv} \right) (-Pv\text{Im}(\overline{Z}_2 Z_1)_\infty) \equiv \left( -\frac{I_1}{Pv} \right) \mathcal{P} > 0.
\]

(85)

Now a calculation shows that by virtue of Eq.\((84)\) consistency demands that

\[
\mathcal{P} = 3(Pv(x^2_\infty + y^2_\infty) + I_1 - Uv)^2 + UP(2vy^2_\infty)^2 - \frac{3}{4}(Uv + Pq)(Uv + 9Pq) < 0.
\]

(86)

Let us now look at the expression for our hyperdeterminant of format \(2 \times 2 \times 2 \times 2\) as given by Eq.\((76)\). Clearly positivity of \(D_4\) implies that \(-UvPq > 0\) and neither \(Uv = -Pq\) nor \(Uv = -9Pq\). Notice that all of these conditions are compatible with our physically interesting situation. For \(-UvPq > 0\) is compatible with our choice of signs supporting a pair of BPS configurations, and in order to talk about a \(D0D4 - D2D6\) split none of the four charges can be zero. Moreover, since the first term of Eq.\((86)\) is positive, the second
one for BPS configurations is positive as well \((UP > 0)\), in order to satisfy this condition \((Uv + Pq)(Uv + 9Pq)\) has to be positive. Hence we see that all of the physically relevant conditions are encoded into the structure of the hyperdeterminant \(D_4\). In particular \(D_4 > 0\) and \(-PvI_1 < 0\) of Eq. (84) gives a necessary condition for the consistency condition to hold. Unfortunately the second of our conditions is featuring \(-Pv\) which is not coming from any of our four-qubit invariants.

In order to eliminate this shortcoming let us now take another look at the form of our polynomial of Eq. (25). By employing the substitution

\[
t = I_1 + y
\]

this polynomial can be transformed to the reduced form

\[
y^4 + ay^2 + by + c
\]

where

\[
a = 6(I_2 - I_1^2), \quad b = 12I_1I_2 - 4I_3 - 8I_1^3, \quad c = -3(I_1^2 - I_2)^2 + S
\]

and \(S\) is our well-known quantity defined by Eq. (29) or alternatively by Eq. (68). Now the conditions

\[
\Delta_4 > 0, \quad a < 0, \quad c < \frac{a^2}{4}
\]

imply that our original polynomial Eq. (25) featuring the fundamental 4-qubit invariants is having only real roots. Note that here \(\Delta_4\) is the discriminant of Eq. (25), and we also recall that \(\Delta_4 = 256D_4\). A calculation for the degenerate cases \((I_4 = 0)\) shows that these conditions are

\[
D_4 > 0, \quad I_1^2 > M, \quad I_1I_3 > M(I_1^2 - M)
\]

yielding the \(st^2\) model and the further specialization \(M = 0\) results in

\[
D_4 > 0 \quad I_1I_3 > 0
\]

corresponding to the \(t^3\) model. Now by virtue of Eq. (73) it is easy to see that a further specification to the case of our BPS \(D0D4 - D2D6\) split renders our condition \(-PvI_1 < 0\) equivalent to the one \(I_1I_3 > 0\). Hence we obtained the nice result that for \(D0D4 - D2D6\) splits the conditions encapsulated in the positivity of three-qubit (i.e. U-duality) invariants of Eqs. (71)-(72) and the positivity of the four-qubit ones of Eq. (92) provide a necessary condition for the consistency condition for such two-center composites to hold.
Let us also verify explicitly that the aforementioned criteria indeed provide real roots of our polynomial \(^{(25)}\) featuring the algebraically independent four-qubit invariants. For the \(t^3\) model one of the roots is zero due to the vanishing of the invariant \(I_4\). For the remaining three roots we have to look at the solutions of Eq.\(^{(74)}\). After the substitution

\[
s = t - 2 \left( \frac{2I_1}{3} \right)
\]

and the definitions

\[
\mu = -3 \left( \frac{2I_1}{3} \right)^2, \quad \nu = 2 \left( \frac{2I_1}{3} \right)^3 - 4I_3
\]

this cubic equation and its discriminant takes the form

\[
s^3 + \mu s + \nu = 0, \quad \Delta = \left( \frac{\mu}{2} \right)^2 + \left( \frac{\nu}{3} \right)^3
\]

where for the explicit form of \(\Delta\) see Eq.\(^{(75)}\). According to Eq.\(^{(76)}\) \(\Delta < 0\), this yields for Cardano’s formula the case of ”casus irreducibilis”\(^{47}\) with explicit solutions \(s_{j+1}, j = 0, 1, 2\).

Transforming back these solutions by using Eq.\(^{(93)}\) to the variables \(t_{j+1}\) we obtain the final solutions

\[
t_{j+1} = 4 \left( \frac{2I_1}{3} \right) \sin^2 \left( \frac{\Phi}{6} + \frac{\pi j}{3} \right), \quad j = 0, 1, 2
\]

where

\[
\sin^2 \left( \frac{\Phi}{2} \right) = \frac{I_3}{\left( \frac{2I_1}{3} \right)^3}
\]

a quantity clearly positive by virtue of Eqs.\(^{(73)}\) i.e. \(I_1I_3 > 0\) for all sign combinations compatible with Eq.\(^{(83)}\). Since according to Eqs.\(^{(75)}\) and \(^{(76)}\) the sign of \(D_4\) is just the opposite of the sign of \(\Delta\) we see that in the special case of the BPS \(D0D4 - D2D6\) split in the \(t^3\) model the real roots we have obtained are in accord with our conditions of Eq.\(^{(92)}\) used in a more general context.

C. Real roots and canonical forms

Our explicit formulas for the real roots of the fundamental polynomial of Eq.\(^{(25)}\) enables an explicit construction of the canonical forms of the four qubit states associated to the charge configurations describing two-center solutions. Let us give just a few examples for the BPS \(D0D4 - D2D6\) split. First we write the roots of the fundamental polynomial of
Eq. (25) for the BPS $D0D4 - D2D6$ splits of the $t^3$ model in the following form

$$t_{j+1} = 4(e - f) \sin^2 \left( \frac{\Phi}{6} + \frac{\pi}{3}j \right), \quad j = 0, 1, 2 \quad t_4 = 0 \quad (98)$$

$$\cos \Phi = \frac{e(e + 3f)^2 + f(f + 3e)^2}{e(e + 3f)^2 - f(f + 3e)^2}, \quad e = \frac{Uv}{3}, \quad f = Pq. \quad (99)$$

Note that for BPS solutions we have $ef < 0$. Notice also that for our special case $2I_1 = \langle Q_1, Q_2 \rangle = 3(e - f)$ cannot be zero for BPS splits so this charge configuration is mutually nonlocal.

As our first example let us consider the nontrivial cases $f \neq 0$, and $e \neq 0$ when $D_4 = 0$. The first case is characterized by the constraint $e + 3f = 0$. In this case $I_3 \neq 0$ and from Eq. (99) we get $\Phi = \pi$ hence

$$t_1 = t_3 = \frac{2I_1}{3}, \quad t_2 = 4 \left( \frac{2I_1}{3} \right), \quad t_4 = 0, \quad I_1 = \frac{4Uv}{3}. \quad (100)$$

Using now Eq. (35) up to permutations one can associate a canonical form to this configuration as shown in Eq. (32). This situation arises for example when $U = -v = 3$ and $P = q = 1$.

Notice that this is a highly degenerate case since now $D_3(Q) = 0$, and $W = 3(1 - \tau)(1 + \tau^2)$, hence $Z = 0$ for $\tau = \pm 1$ which is on the boundary of the lower half plane. One can also see that for $e + 3f = 0$ the superpotential $W$ can be written in a factorized form.

Our second example is associated with the case when $f + 3e = 0$ producing the other nontrivial zero for $D_4$. In this case $I_3 = 0$, and $\Phi = 0$ hence we have

$$t_1 = 0, \quad t_2 = t_3 = 3 \left( \frac{2I_1}{3} \right), \quad t_4 = 0, \quad I_1 = 2Uv. \quad (101)$$

From Eq. (32) one can see that the canonical state is proportional to the one $|0101\rangle + |1010\rangle$.

Our last example is a one with $\Phi = \frac{\pi}{2}$. In this case $e(e + 3f)^2 + f(f + 3e)^2 = 0$ which is of the form

$$\lambda^3 + 5\lambda^2 + \frac{5}{3}\lambda + \frac{1}{27} = 0, \quad \lambda = \frac{Pq}{Uv}. \quad (102)$$

It is easy to check that $\lambda = -1/3$ i.e. $e + f = 0$ is a solution, hence $(\lambda + 1/3)(\lambda^2 + 14\lambda/3 + 1/9)$ is a factorized form of our polynomial, yielding the solutions

$$\lambda_1 = -\frac{1}{3}, \quad \lambda_{2,3} = \frac{1}{3}(-7 \pm 4\sqrt{3}). \quad (103)$$

Let us consider the rational solution $\lambda = -\frac{1}{3}$. In this case we get

$$t_1 = \frac{4I_1}{3}, \quad t_{2,3} = \frac{I_1}{3}(1 + \sqrt{3})^2, \quad t_4 = 0, \quad I_1 = Uv. \quad (104)$$

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Now the real roots are all different hence $D_4 \neq 0$. However, now $Uv = -3Pq$ hence $(Uv + PQ)(Uv + 9PQ) < 0$. A consequence of this is that the consistency condition of Eq. (86) cannot be satisfied so no charge configuration of this kind supports a two-center solution. The canonical form can again be read off from Eqs. (32)-(35). Notice also that for all three cases $t_1 + t_2 + t_3 + t_4 = 4I_1$ as it has to be, moreover the corresponding values for $\lambda$ can be written in the form $\lambda_k = -1/3^k$ for $k = 0, 1, 2$.

It is interesting to realize that all these special charge configurations giving rise to special four-qubit canonical forms are outside the domain of legitimacy two-center solutions. Later when we connect the special values of $e$ and $f$ to properties of the $j$ function we have something more to say about this phenomenon.

In order to get charge configurations supporting BPS two-center solutions we have to choose the asymptotic moduli from the eligible region boundend by the usual wall of marginal stability

$$4\xi x^2_\infty = (x^2_\infty + y^2_\infty + \xi)(x^2_\infty + y^2_\infty + \eta), \quad \xi = \frac{U}{3P}, \quad \eta = -\frac{3q}{v}. \quad (105)$$

D. Splitting of invariants

Notice that the quartic invariant for the $t^3$ model has the explicit form

$$-D_3(Q_1 + Q_2) = -(Uv)^2 + 3(Pq)^2 - 6UvPq + 4UP^3 - 4vq^3. \quad (106)$$

Now $I_{+2} = 4UP^3$ and $I_{-1} = -4vq^3$ hence according to Eq. (54) we can write

$$-D_3(Q_1 + Q_2) = \Pi_4(A_{i_0i_1i_2i_3}, 1) = [4I_{+1} + 6I_0 + 4I_{-1}] + I_{+2} + I_{-2}. \quad (107)$$

From this expression we see that the relationship between the quantities $-D_3(Q_1 + Q_2)$, $-D_3(Q_{1,2})$ is governed by the combination $4I_{+1} + 6I_0 + 4I_{-1}$ of $SL(2)_0$ covariants. Such relationships are needed for studying situations when the two-center solution is BPS, but the corresponding single center one is not.

For $D0D4 - D2D6$ splits the quantity above can also be written in two equivalent forms featuring quantities related to factors of $SL(2)_0$ invariants

$$-D_3(Q_1 + Q_2) = -8UvPq - 2I_1(Uv + Pq) - D_3(Q_1) - D_3(Q_2), \quad (108)$$

or

$$-D_3(Q_1 + Q_2) = -24(Pq)^2 - 2I_1(Uv + 9Pq) - D_3(Q_1) - D_3(Q_2). \quad (109)$$
Hence if both of the constituents are BPS i.e. \(-D_3(Q_1) > 0\) and \(-D_3(Q_2) > 0\), the conditions for the single centered system to be BPS too i.e. \(-D_3(Q_1 + Q_2) > 0\) are again governed by a four qubit invariant \(I_1\) or by the factors of another four qubit one namely \(D_4\). It is amusing to realize that all of the factors of \(D_4\) are appearing in these two equations.

As an example one can see that if \(I_1 < 0\) and \(D > 0\) and moreover \(Uv + Pq > 0\) (which is a factor of \(D_4\)), then two BPS configurations again yield a BPS one. In the example of Bates and Denef\(^4\) \(U = -v = 4\) and \(P = q = 1\), \(-D_3(Q_1) > 0\) and \(-D_3(Q_2) > 0\), hence \(D_4 > 0\) and \(I_1 < 0\) but \(Uv + Pq < 0\) hence \(-D_3(Q_1 + Q_2) = -125 < 0\). In this case a single center BPS solution does not exist, although the corresponding two-center one does.

E. The \(j\) invariant

It should be obvious by now that the basic mathematical object giving rise to our observations is the fundamental polynomial of Eq.(25) featuring all of our algebraically independent four-qubit invariants. For no matter what kind of two center charge configuration we have we can construct this quartic polynomial. As a next step we can calculate its resolvent cubic of Eq.(67). After the substitution \(u = 2x\) this polynomial takes the form \(2(4x^3 - Sx - T)\). As a next step to this resolvent cubic we associate an elliptic curve of Weierstrass canonical form as

\[
y^2 = 4x^3 - Sx - T. \tag{110}
\]

The \(j\) invariant of this curve is defined as

\[
\[1728S^3 \over S^3 - 27T^2]. \tag{111}
\]

In particular for the special case of the \(t^3\) model we get

\[
\[2I_1\] \over I_3 ((2I_1)^3 - 24I_3). \tag{112}
\]

After recalling the definitions of Eq.(99) a further specification to our case of the BPS \(D0D4 - D2D6\) split one obtains

\[
1728S^3 = (2I_1)^3((2I_1)^3 - 24I_3)^3 = 27(e - f)^3[27e(3f + e)^2 - 3f(3e + f)^2]^3, \tag{113}
\]

\[
256D_4 = S^3 - 27T^2 = -27[e(3f + e)^2][f(3e + f)^2]^3. \tag{114}
\]
Now
\[ 27e(3f + e)^2 - 3f(3e + f)^2 = (3e - \xi_1 f)(3e - \xi_2 f)(3e - \xi_3 f) \tag{115} \]
where
\[ \xi_{j+1} = -5 + 4\sqrt{2}\omega_j, \quad \omega_j = e^{2\pi j/3}, \quad j = 0, 1, 2. \tag{116} \]

Interestingly the number \( \xi_1 \) can also be written in the form
\[ \xi_1 = -5 + 4\sqrt{2} = 3 \left( \frac{1 - 3\sqrt{2}}{1 + \sqrt{2}} \right)^2 \tag{117} \]
moreover, one can check that \( \xi_1 + \xi_2 + \xi_3 = -15, \xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3 = 75 \) and \( \xi_1\xi_2\xi_3 = 3. \)

Finally one obtains for the \( j \) invariant the expression
\[ j = \frac{(f - e)^3}{[e(3f + e)^2][f(3e + f)^2]^3}(3e - \xi_1 f)(3e - \xi_2 f)(3e - \xi_3 f)^3. \tag{118} \]

Clearly the denominator of \( j \) is zero precisely when \( D_4 \) is vanishing. According to the results of the previous subsections for BPS \((ef < 0)\) composites when this happens the consistency condition cannot be satisfied. The numerator on the other hand can vanish either when \( 2I_1 = \langle Q_1, Q_2 \rangle = 0 \) i.e. the charge configuration is local, or when the remaining real factor vanishes i.e. when \( Uv = \xi_1 Pq. \) Since according to Eq.(117) \( \xi_1 \) is positive, for BPS composites the latter two conditions cannot be satisfied. Hence the \( j \) function is not having any pathological behavior for the physically legitimate cases of BPS \( D0D4 - D2D6 \) composites.

Notice however, that the \( j \) function can be made to vanish for non-BPS \( D0D4 - D2D6 \) composites. In this case we have \( ef > 0, \) and one can try to combine the elementary non-BPS solutions obtained for the \( D0D4 \) and \( D2D6 \) systems. For such non-BPS composites with \textit{local} charge configurations a generalization of the consistency condition has recently been given.

\textbf{V. ELLIPTIC CURVES}

\textbf{A. Elliptic curves and four-qubit states}

Let us try to justify that our association of an elliptic curve to the resolvent cubic Eq.(67) of our fundamental polynomial of Eq.(25) is a natural one. For this it is useful to regard our
three-qubit charge states \(|Q_1\rangle\) and \(|Q_2\rangle\) with amplitudes given by Eqs. (36), (41)-(42) as the ones embedded into more general unnormalized four-qubit ones with complex amplitudes. (We will have something more to say about this embedding later.) Hence we can use the algebraically closed field of complex numbers to put our considerations in a more general setting.

Let us now notice that according to Luque and Thibon\(^{16}\) we can write our key quantity \(256D_4 = S^3 - 27T^2\) in yet another form by rewriting the polynomial invariants \(S\) and \(T\) as

\[
12S = U^2 - 2V, \quad 216T = U^3 - 3UV + 216D^2
\]

where

\[
U = H^2 + 4(M - L), \quad V = 12(HD - 2LM).
\]

The algebraically independent invariants\(^{16}\) \(H, L, M, D\) showing up in these expressions are related to our set \(I_1, I_2, I_3, I_4\) as\(^{17}\)

\[
I_1 = \frac{1}{2}H, \quad I_2 = \frac{1}{6}(H^2 + 2L + 4M), \quad I_3 = D + \frac{1}{2}HL, \quad I_4 = L.
\]

It is known that a nonsingular projective curve of genus 1 is isomorphic to a plane cubic curve of the Tate form\(^{51}\)

\[
x_0x_2^2 + a_1x_0x_1x_2 + a_3x_0^2x_2 - x_1^3 - a_2x_0x_1^2 - a_4x_0^2x_1 - a_6x_0^3 = 0.
\]

Here the coefficients \(a_1, a_2, a_3, a_4, a_6\) can be taken from an algebraically closed field \(\mathbb{F}\) of arbitrary characteristic and \((x_0, x_1, x_2)\) are homogeneous coordinates of the projective plane \(\mathbb{P}^2\). (Of course for qubits we are merely interested in the complex case i.e. our concern will be curves in \(\mathbb{C}P^2\).) Using inhomogeneous coordinates \((x, y) \equiv (x_1/x_0, x_2/x_0)\) this can be rewritten as

\[
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.
\]

Now the \(j\) invariant and discriminant \(\Delta\) of a plane cubic curve of this form is defined\(^{51}\) as

\[
j = \frac{c_4^3}{\Delta} = 1728\frac{c_4^3}{c_4^3 - c_6^2}
\]

where

\[
c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^3 + 36b_2b_4 - 216b_6
\]

\[
b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6.
\]
A necessary and sufficient condition for our curve to be nonsingular is \( \Delta \neq 0 \).

Now looking at Eqs. (119)-(120) one can easily check that four-qubit states provide a particularly nice parametrization for a special class of plane cubic curves with \( a_6 \equiv 0 \). Indeed setting

\[
b_2 = U = H^2 + 4(M - L), \quad b_4 = \frac{1}{12}V = HD - 2LM, \quad b_6 = D^2 \quad (127)
\]
yields

\[
c_4 = 12S, \quad c_6 = -216T. \quad (128)
\]

Moreover, one can also see that

\[
a_1 = H = 2I_1, \quad a_2 = M - L, \quad a_3 = D = I_3 - I_1I_4, \quad a_4 = -LM, \quad a_6 = 0 \quad (129)
\]

hence the family of such cubic curves for the stu model \((H \neq 0, D \neq 0, M \neq 0, L \neq 0)\) is

\[
y^2 + Hxy + Dy = x^3 + (M - L)x^2 - LMx \quad (130)
\]

for the st\(^2\) model \((H \neq 0, D \neq 0, M \neq 0, L = 0)\)

\[
y^2 + Hxy + Dy = x^3 + Mx^2 \quad (131)
\]

and finally for the t\(^3\) model \((H \neq 0, D \neq 0, L = 0, M = 0)\) it is

\[
y^2 + Hxy + Dy = x^3. \quad (132)
\]

For all of these curves the hyperdeterminant is given by the usual formula \(256D_4 = S^3 - 27T^2\) with the \(j\) invariant having the form of Eq. (111). As an example let us take a look at the hyperdeterminant \(D_4\) for the st\(^2\) model which is given by the expression

\[
256D_4 = I_3^2 \left( I_3[(2I_1)^2 - 27I_3] + 36M[I_2^2 - 2I_1I_3] \right), \quad 2M = 3I_2 - 2I_1^2. \quad (133)
\]

One can see that in accordance with our association of cubic curves to the stu model and its truncations for \(M = 0\) we get back to Eq. (76) i.e. the expression for the t\(^3\) model.

Recall also that if the characteristic of the field is not 2 or 3 via further projective transformations (i.e. completing the square on the left hand side and then the cube on the right hand side of Eq. (130)) one obtains the Weierstrass canonical form of Eq. (110) which is simply related to the resolvent cubic of our polynomial \(\Sigma_4\) of Eq. (25) we have...
started our considerations with. Notice however, that although due to the fact that now \( F = \mathbb{C} \) the Weierstrass form in our case can always be reached, this form is not showing the basic differences between the structures of the cubic curves associated to the \( stu \), \( st^2 \) and \( t^3 \) models, hence we prefer the Tate form.

In the following as usual by an elliptic curve we will mean a nonsingular plane cubic curve as given by Eq.\((122)\) and the "point at infinity" \((x_0, x_1, x_2) = (0, 0, 1)\). Since for these curves we have only \((0, 0, 1)\) as a point at infinity by an abuse of notation one can call Eq.\((123)\) as the defining equation for the corresponding elliptic curve. An elliptic curve is defined over a subfield \( F' \) of \( F \) if all the coefficients \( a_1, a_2, a_3, a_4, a_6 \) are taken from \( F' \).

In the case of the discrete version of the continuous \( U \)-duality group, as nonperturbative string symmetries, the coefficients of this curve are the algebraically independent four-qubit invariants \( H, L, M, D \) (related to the ones \( I_1, I_2, I_3, I_4 \) via Eq.\((121)\)) taken from the ring of integers of the subfield \( \mathbb{Q} \). In this case the arithmetic aspects of the theory of elliptic curves within the context of two-center black hole solutions becomes important.

The upshot of these considerations is that to a particular charge configuration of two-center black holes of the \( stu \) model we can indeed associate quite naturally an elliptic curve of special kind (i.e. \( a_6 = 0 \)). This elliptic curve is of the form of Eq.\((130)\).

Notice that the invariants \( H, L, M \) play an important physical role. Since \( H = \langle Q_1, Q_2 \rangle \) the vanishing of this invariant gives rise to local charge configurations, in this case the \( xy \) term of Eq.\((130)\) is missing. The vanishing conditions on \( L \) and \( M \) give rise to truncations to the \( st^2 \) and \( t^3 \) models. Notice also that since for \( L = 0 \) we have \( D = I_3 \) hence in the \( st^2 \) and \( t^3 \) models sending \( D \) to zero corresponds to taking the limit when our elliptic curve degenerates (see Eqs.\((76)\), \((121)\) and \((133)\)).

In establishing the correspondence between two center charge configurations and elliptic curves we tacitly assumed that our unnormalized charge states with real amplitudes (transforming according to the corresponding representation of the continuous \( U \)-duality group valid in the supergravity approximation) are regarded as ones embedded in unnormalized four-qubit states with complex amplitudes. However, until this point we did not specify what kind of complex states we are having in mind.

In a recent series of papers it was shown that for single center extremal stationary spherically symmetric BPS and non-BPS black hole solutions apart from charge states i.e. unnormalized three-qubit states with real amplitudes, we can also introduce unnormalized
complex three-qubit states which are depending on the charges and also on the moduli fields. (For a more general class of such states even the warp factor can be included.) It has been shown that the complex amplitudes $\psi_{000}, \psi_{001}, \ldots, \psi_{111}$ of such states $|\psi\rangle$ are related to the central charge and its covariant derivatives with respect to the moduli. For example the amplitudes featuring the central charge have the form

$$\psi_{000} = \sqrt{2}Z, \quad \psi_{111} = -\sqrt{2}Z.$$ 

Moreover, recently it has also been demonstrated that such complex three-qubit states are just special types of four-qubit ones. It is natural to expect that for two center solutions complex four-qubit states of similar type should show up. For example in the consistency condition of Eq.(77) the term $\text{Im}(Z_2Z_1)$ can be reinterpreted as a truncation of a pairing between the two complex amplitudes $\Psi_{0000} \equiv \sqrt{2}Z_1$ and $\Psi_{1111} = -\sqrt{2}Z_2$ of such four-qubit states. In this respect recall the formalism developed by Denef in the type IIB duality frame where apart from the usual antisymmetric, topological moduli independent intersection product (our four-qubit invariant $I_1$ in the $stu$ model) the importance of the symmetric, positive definite, moduli dependent Hodge product is emphasized (in the $stu$ model giving rise to the interpretation of the black hole potential as the norm of a three-qubit state). Such structures are the natural ones appearing in the multicenter black hole context. In the $stu$ case it is easy to establish a correspondence between this formalism and the one where complex three and four-qubit states appear. We conjecture that such moduli dependent complex states and their associated invariants featuring elliptic curves could be the relevant mathematical objects hiding behind the considerations of this paper based merely on charge states. Note that moduli dependent curves of that kind would also display explicit dependence on extra complex parameters giving rise to elliptic fibrations. We will have something more to say about this possibility in Section V.

B. Degeneracies

The discriminant $\Delta$ of our elliptic curve is related to the hyperdeterminant $D_4$ as $256D_4 = \Delta$. For nonvanishing $D_4$ the curves are nonsingular. Since our curves are of genus 1 they are topologically tori. Vanishing of $D_4$ results in different degeneracies of these tori (for example one of their homologically nontrivial cycles can contract to a point). One can illustrate this
in the $stu$ and $st^2$ models when both of the two BPS constituents are having vanishing Bekenstein-Hawking entropy i.e $I_{\pm 2} = 0$ for both centers.

Indeed, let us use the parametrization as introduced by Sen\cite{Sen} for such two-center composites $Q = Q_1 + Q_2$

$$
\begin{pmatrix}
Q \\
P
\end{pmatrix} = \begin{pmatrix}
adQ - abP \\
cdQ - bcP
\end{pmatrix} + \begin{pmatrix}
abP - bcQ \\
adP - cdQ
\end{pmatrix}, \quad \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in SL(2, \mathbb{Z}).
$$

(135)

Here $P$ and $Q$ are four-vectors related to the ones of Eq.\(\text{(38)-(42)}\) as follows

$$
A = a(dQ - bP), \quad B = c(dQ - bP), \quad C = b(aP - cQ), \quad D = d(aP - cQ).
$$

(136)

Since the vectors $A$ and $B$ and $C$ and $D$ are now proportional the bivectors of $X$ and $Y$ of Eq.\(\text{(43)}\) are zero hence $I_{\pm 2} = I_{\pm 1} = 0$ though $I_0 \neq 0$. Now Eq.\(\text{(68)}\) shows that

$$
S = 3I_2^2, \quad T = I_3^2,
$$

(137)

hence $S^3 - 27T^2 = 0$ i.e. $D_4 = 0$. The nonseparable bivector is $Z = \frac{1}{2}Q \wedge P$ hence

$$
I_0 = \frac{1}{3}Z^2 = \frac{1}{12}(Q \wedge P) \cdot (Q \wedge P) = -\frac{1}{6}D_3(Q).
$$

(138)

Alternatively one can directly check that $I_3 = I_4 = 0$ and $6I_2 = -6I_0 = D_3(Q)$ in accordance with Eq.\(\text{(56)}\), hence $S = 3I_2^2$ and $T = I_3^2$ yielding again $D_4 = 0$.

In order to obtain the Tate form of our elliptic curve we have to calculate the quantities $H, L, M, D$. One can show that

$$
H = \hat{P} \cdot \hat{Q}, \quad L = 0, \quad M = -\frac{1}{4} \hat{P}^2 \hat{Q}^2, \quad D = 0
$$

(139)

where we introduced the notation

$$
\begin{pmatrix}
\hat{Q} \\
\hat{P}
\end{pmatrix} = \begin{pmatrix}
d & -b \\
-c & a
\end{pmatrix} \begin{pmatrix}
Q \\
P
\end{pmatrix}.
$$

(140)

Now the elliptic curve of Eq.\(\text{(130)}\) is of the form

$$
y^2 + \hat{P} \cdot \hat{Q} xy = x^3 - \frac{1}{4} \hat{P}^2 \hat{Q}^2 x^2.
$$

(141)

Completing the square on the left hand side (and by an abuse of notation using $y$ for the new variable again) yields

$$
y^2 = x^2 \left(x + \frac{1}{4}D_3(Q)\right).
$$

(142)
Notice that the quantities $\hat{P}$ and $\hat{Q}$ hence the Tate form of the cubic curve is featuring the parameters $a, b, c, d$ characterizing the two-center split, however the latter form is not. This is due to the fact that $D_3(Q) = (P \cdot Q)^2 - P^2 Q^2 = (\hat{P} \cdot \hat{Q})^2 - \hat{P}^2 \hat{Q}^2$. This leads us to the important observation that at least for this singular example two-center charge configurations labelled by elements of $SL(2, \mathbb{Z})$ correspond to the same singular curve with the canonical form. Hence apart from the four copies of $SL(2)$s related to four-qubit systems there seems to be a fifth $SL(2)$ parametrizing a hidden torus. We will return to this important point later.

Eq. (142) represents a cubic curve with a node at the point $(0, 0)$. The tangent lines at $(0, 0)$ are obtained from the equation $y^2 - (D_3/4)x^2 = 0$. Such lines are of the form

$$\left(y - \frac{1}{2}\sqrt{D_3}\right)\left(y + \frac{1}{2}\sqrt{D_3}\right) = 0. \quad (143)$$

It is important to recall that we assumed that all of our curves are over the complex numbers, i.e. $\mathbb{F} = \mathbb{C}$. Hence for a cubic in the canonical form $y^2 = x^3 + ax + b$ we are still free to use transformations of the form $(x, y) \mapsto (\mu^2 x, \mu^3 y)$ and $(a, b) \mapsto (\mu^4 a, \mu^6 b)$, where $\mu \in \mathbb{C}^\times$. Using the transformation $(x, y) \mapsto (-x, -iy)$, i.e. $\mu = i$ transforms Eq. (142) to the form

$$y^2 = x^2 \left(x - \frac{1}{4}D_3(Q)\right). \quad (144)$$

As we see such transformations are similar to the ones changing a BPS charge configuration ($D_3 < 0$) to a non-BPS one ($D_3 > 0$). Interestingly as shown by Eq. (143) (and a similar one obtained from Eq. (144)) the pair of lines at the node are parametrized by $\sqrt{-D_3}$ for the BPS-case or $\sqrt{D_3}$ for the non-BPS one, i.e. quantities proportional to the Bekenstein-Hawking entropy.

Notice also that Eq. (144) can also be parametrized as

$$y^2 = 3t(x' - t)^2 + (x' - t)^3 = (x')^3 - 3t^2(x') - 2t^3, \quad t = -\frac{D_3}{12} \quad (145)$$

Using Eqs. (137)–(138) we can write this as

$$y^2 = x'^3 - \frac{1}{4}Sx' - \frac{1}{4}T. \quad (146)$$

Now a further transformation $(u, v) = (x/2, x/8)$ i.e. $\mu^2 = 1/2$ transforms this to the form $v^2 = u^3 - Su - 2T$ of Eq. (67), with the left hand side being just the resolvent cubic of
$\Sigma_4 = 0$. The procedure discussed here illustrates how the general Tate form of our cubic curve of Eq. (141) can be transformed to a resolvent cubic form.

It is important to realize that if the domain of definition of our curve is the subfield $\mathbb{Q}$ from the general Tate form of Eq. (130) merely the form

$$y^2 = x^3 - \frac{1}{4}Sx + \frac{1}{4}T$$

(147)

can be reached. Further reduction is possible only if $\mu$ is a square in $\mathbb{Q}$. In the case of our singular curve of Eq. (142) if $D_3$ is not a square we cannot represent the tangent lines in the form of Eq. (143) hence the tangent lines in this case are not rational.

Due to permutation symmetry we have no parametrization of (135) type for the $t^3$ model available. However, as a degenerate example in this case we can consider the example of Bates and Denef instead by sending one of the charges e.g. $P$ to zero. In this case one of the black holes is a small one, and the two center composite is characterized by the charges $U, v, q$. Now one checks that $I_3 = 0$, hence according to Eq. (76) we get again $D_4 = 0$. The explicit form of the corresponding degenerate elliptic curve is

$$y^2 + Hxy = x^3, \quad H = \langle Q_1, Q_2 \rangle = Uv,$$

(148)

with $j = \infty$ just like in the previous example.

C. Invariance properties

Since the coefficients $a_j$ of the elliptic curve of Eq. (130) are invariants the same curve is associated to equivalent two-center charge configurations in the $stu$ model. In other words our curve is clearly invariant under the action of the group $SL(2, \mathbb{R})_0 \times G_4$, where $SL(2, \mathbb{R})$ is the horizontal symmetry group of generalized exchange transformations of the centers and $G_4$ is the $d = 4$ continuous $U$-duality group of the $stu$ model which is $SL(2, \mathbb{R}) \times 3$.

However, as far as physics is concerned in a special case of the $t^3$ model we have also found some connection between the structure of $SL(2, \mathbb{R})_0 \times G_4$ (four-qubit) invariants and the structure of the consistency condition of Eq. (77). This connection at first sight is not surprising since the consistency condition contains a four-qubit invariant $2I_1 = \langle Q_1, Q_2 \rangle$, however more importantly it is also featuring the quantity $\text{Im}(\overline{Z}_2 Z_1)$ which is not invariant with respect to the full group of $SL(2, \mathbb{R})_0 \times G_4$ transformations.
In order to show this let us observe however, that this quantity is invariant under transformations belonging to the horizontal subgroup $SL(2,\mathbb{R})_0$. For let us recall our definition of Eq.(36) and then write the central charges as

$$Z_1 = e^{K/2}(Q_0 + Q_1\tau_1 + Q_2\tau_2 + Q_3\tau_3 - P_1\tau_2\tau_3 - P_2^2\tau_1\tau_3 - P_3^3\tau_1\tau_2 + P_0^0\tau_1\tau_2\tau_3) \quad (149)$$

$$Z_2 = e^{K/2}(q_0 + q_1\tau_1 + q_2\tau_2 + q_3\tau_3 - p_1^1\tau_2\tau_3 - p_2^2\tau_1\tau_3 - p_3^3\tau_1\tau_2 + p_0^0\tau_1\tau_2\tau_3), \quad (150)$$

then for $a, b, c, d \in \mathbb{R}$ we have $Z_1' = a Z_1 + b Z_2$, and $Z_2' = c Z_1 + d Z_2$, hence $\text{Im}(Z_2'Z_1') = \text{Im}(Z_2Z_1)$ due to $ad - bc = 1$. However, this quantity is invariant merely under a special subgroup $54$ of $G_4$. Indeed, we have

$$Z(K\tau_1, K\tau_2, K\tau_3; KQ_i) = Z(\tau_1, \tau_2, \tau_3; Q_i) \quad (151)$$

where $K$ is the subgroup of $G_4$ transformations of the form

$$\begin{pmatrix} 1 & 0 \\ k_1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ k_2 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ k_3 & 1 \end{pmatrix} |Q_i\rangle, \quad \tau_a \mapsto \tau_a + k_a \quad k_a \in \mathbb{R}, \quad a = 1, 2, 3. \quad (152)$$

Notice that the group $K$ with $k_a \in \mathbb{Z}$ is precisely the stabilizer of the cusps in the three copies of the fundamental domain of the modular group, i.e. the stabilizer of $\tau_a = -i\infty$ for $a = 1, 2, 3$.

Since the walls of marginal stability are determined by the central charges and for the four qubit invariant $2I_1$ we have $\langle KQ_1, KQ_2 \rangle = \langle Q_1, Q_2 \rangle$ clearly wall-crossing does not obstruct the $K$ subgroup. Therefore we see that unlike our elliptic curves the consistency condition is left invariant (in the above sense) merely with respect to the subgroup $SL(2,\mathbb{R})_0 \times K$. Hence the correspondence between the physics of two center solutions and our special class of elliptic curves should be refined (see in this respect the comments at the end of Section V.A).

Now we turn to another important issue we have not discussed yet. Based on our experience with the degenerate case studied in the previous subsection we expect that for the nondegenerate case our mapping of two center charge configurations characterized by four-qubit invariants to elliptic curves should be many to one. Moreover, since in the Tate form we should have $a_6 = 0$ our mapping in the $stu$ case cannot be onto either. We also know from the theory of elliptic functions that if we work in the algebraic closure of our field $\mathbb{F}$ i.e. $\mathbb{F}_{\text{alg}}$ two elliptic curves are isomorphic if and only if their $j$ invariants are the
same. Hence in the case of \( \mathbb{R} \) working with the complex numbers charge configurations with different four-qubit invariants could be mapped to isomorphic elliptic curves with the same \( j \) invariant. In the singular case according to Eqs. (139)–(140) the values of \( H \) and \( M \) are clearly different for different splits, however \( j = \infty \) in all cases.

Now over \( \mathbb{C} \) every elliptic curve \( E \) is isomorphic to an elliptic curve \( E(\Lambda) \) where \( \Lambda \) is a lattice in the complex plane and the mapping between \( \mathbb{C}/\Lambda \) (a torus) and \( E(\Lambda) \) is an isomorphism provided by the usual map defined by the Weierstrass \( \mathcal{P} \) function. Explicitly \( E(\Lambda) \) is of the form

\[
x_0x_2^2 = 4x_1^3 - g_2(\hat{\tau})x_1x_0^2 - g_3(\hat{\tau})x_0^3
\]

(153)

where \( g_2(\hat{\tau}) = 60G_4(\hat{\tau}) \) and \( g_3(\hat{\tau}) = 140G_6(\hat{\tau}) \) with \( G_k(\hat{\tau}) \) are the Eisenstein series

\[
G_{2k}(\hat{\tau}) = \sum_{(m,n) \in \mathbb{Z}^2, (m,n) \neq (0,0)} \frac{1}{(m+n\hat{\tau})^{2k}}
\]

(154)

and if \( x \equiv x_1/x_0 \) and \( y = x_2/x_0 \) then \( x = \mathcal{P}(z) \) and \( y = \mathcal{P}'(z) \). Here the lattice vectors of \( \Lambda \) in \( \mathbb{C} \) are \( \omega_1 = 1 \) and \( \omega_2 = \hat{\tau} \) with \( \hat{\tau} \in \mathbb{H} \) being the modular parameter of the torus and the Weierstrass \( \mathcal{P} \) function is defined as

\[
\mathcal{P}(z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2, (m,n) \neq (0,0)} \left[ \frac{1}{(z + m + n\hat{\tau})^2} - \frac{1}{(m + n\hat{\tau})^2} \right].
\]

(155)

Here we have used the notation \( \hat{\tau} \) for the modular parameter in order not to confuse it with the moduli \( \tau_1, \tau_2, \tau_3 \) of the \( stu \) and with the moduli \( \tau \) of the \( t^3 \) model. Now the \( j \) invariant gives rise to the \( j \) function

\[
j(\hat{\tau}) = 1728 \frac{g_2(\hat{\tau})^3}{g_2(\hat{\tau})^3 - 27g_3(\hat{\tau})^2}.
\]

(156)

Comparing this with the expression of the \( j \) invariant as given in terms of the four-qubit invariants \( S \) and \( T \) of Eq. (111) to a two center charge configuration we can associate a torus with modular parameter \( \hat{\tau} \). Now \( j : \mathbb{H} \to \mathbb{C} \) is an automorphic function i.e.

\[
j \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = j(\hat{\tau}), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}).
\]

(157)

It can be regarded\(^{55}\) as a map providing a holomorphic universal covering of the orbifold \( \mathcal{H}/PSL(2, \mathbb{Z}) \) (i.e. the fundamental domain for \( PSL(2, \mathbb{Z}) \)). Its Fourier series at \( \hat{\tau} = i\infty \) is provided by the variable \( u \equiv \exp 2\pi i\hat{\tau} \) as

\[
j(\hat{\tau}) - 744 = \frac{1}{u} + \sum_{n=1}^{\infty} c_n u^n, \quad c_n \in \mathbb{Z}^+ \cup \{0\}.
\]

(158)
From this we see that the two-center charge configurations with discriminant $D_4 = 0$ of the previous subsection with $j = \infty$ should correspond to the modular parameter $\hat{\tau} = i\infty$ i.e. the cusp of the Riemann surface $\mathcal{H}/\text{PSL}(2,\mathbb{Z})$. The different possible two-center splits labelled by elements of $SL(2,\mathbb{Z})$ not changing the value of the $j$ invariant seem to be related to the modular property of the $j$-function as shown by Eq.(157). It is also intriguing to recall that the tangent lines at the node of the degenerate elliptic curve are parametrized by the Bekenstein-Hawking entropy (see Eq.(143)).

From these investigations it is natural to conjecture that similar invariance properties of other two-center splits with $D_4 \neq 0$ should hold. It would be nice to uncover the physical role of this hidden torus and the extra $SL(2)$ symmetry associated with it. Some speculations on how the extra modular parameter $\hat{\tau}$ should be implemented into a four-qubit picture will be given in Section VI.

**D. A conjecture**

Let us consider the elliptic curve of Eq.(130) associated to two-center charge configurations of the $stu$ model. As we have noticed this curve is of the Tate form satisfying the special constraint $a_6 = 0$. The coefficients $a_j$ are four-qubit polynomials with definite degree of homogeneity $2j$. Indeed, $a_1 = H$ is a quadratic, $a_3 = D$ is a sextic, $a_2 = M - L$ is a quartic, and $a_4 = -LM$ is an octic polynomial in the amplitudes of the four-qubit charge state. (The variables $x$ and $y$ should be assigned the degrees four and six respectively.) Now according to this observation the constraint $a_6 = 0$ should be arising from the vanishing condition of a polynomial of degree 12. It is easy to identify this polynomial constraint. It is just the constraint $\mathcal{P}_{12} = 0$ of Eq.(57), or alternatively of Eq.(58), hence in a redundant notation for the $stu$ model we can write

$$y^2 + Hxy + Dy = x^3 + (M - L)x^2 - LMx + \mathcal{P}_{12}.$$ (159)

Since the $stu$ model can be regarded as a consistent truncation of $N = 8$, $d = 4$ maximal supergravity with continuous $\mathbb{R}$ $U$-duality group $G_4 \equiv E_7(7)$ one might conjecture that it should be possible to substantially generalize our considerations concerning two-center charge configurations by studying elliptic curves of the form

$$y^2 + P_2xy + P_6y = x^3 + P_4x^2 + P_8x + P_{12}.$$ (160)
where $P_{2j}$ are polynomial invariants of the group $SL(2)_0 \times G_4$ where $SL(2)_0$ refers to the horizontal symmetry group acting as a generalized exchange symmetry group on the two centers. Now we have two sets of 56 component charge vectors where each of these vectors is transforming according to the symplectic irreducible representation of $G_4$ which is now the fundamental of $E_7(7)$. According to an analysis of two centered magical charge orbits working in the complexification of $G_4$ one discovers that the two centered charge orbits correspond to different real forms of the quotient of the complex groups $E_7/\text{SO}(8)$. In particular we have two $\frac{1}{8}$-BPS two-centered charge orbits with one of them being $E_7(7)/\text{SO}(4, 4)$.

Recall now Eq. (11) displaying in its structure the group theoretical reason for our occurrence of four-qubit states. According to this equation one should be able to obtain our four-qubit invariants as special cases of $SL(2)_0 \times G_4$ ones. Hence we conjecture that an elliptic curve of the (160) form should display this reduction procedure via implementing the constraint $P_{12} = 0$ via a suitable reduction of some unknown $SL(2)_0 \times G_4$ polynomial invariant $P_{12}$.

In fact many pieces of such invariants are already at our disposal. For example the $N = 8$ analogues $J_0, J_{\pm 1}, J_{\pm 2}$ of the four-qubit covariants $I_0, I_{\pm 1}, I_{\pm 2}$ of Eq. (49) arising from the polarization of Cayley’s hyperdeterminant can now be obtained from the polarization of Cartan’s quartic invariant. Using these explicit expressions one can construct the quantities answering the similar ones of Eq. (68)

$$S = 3J_0^2 - 4J_1J_{-1} + J_{+2}J_{-2}, \quad \mathcal{T} = J_0^3 + J_1^2J_{-2} + J_{-1}^2J_{+2} - J_0J_{-2}^2 - 2J_1J_0J_{-1}. \quad (161)$$

Indeed precisely these polynomials of order 8 and 12 has been suggested as obvious candidates for members of a complete basis for $SL(2)_0 \times G_4$ invariants. Now one can define an elliptic curve

$$y^2 = x^3 - \frac{1}{4}Sx + \frac{1}{4}\mathcal{T} \quad (162)$$

as the one corresponding to Eq. (147). As was commented there this form can always be obtained from the Tate form if the characteristic of the field is neither 2 nor 3. Hence we conclude that our Tate form Eq. (160) featuring the unknown invariants $P_{2j}$ should reduce to Eq. (162) after completing the square on the left and the cube on the right hand side. Explicitly this process amounts to the transition from using the coefficients $a_j = P_{2j}$ to using the ones $c_2 = -\frac{1}{4}S$ and $c_4 = \frac{1}{4}\mathcal{T}$ of Eqs. (125)-(126).

As a solid piece of evidence it is also obvious that $P_2 = \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle$ with $\langle , \rangle$ being the usual symplectic product of charge vectors which is a singlet with respect to $SL(2)_0 \times G_4$. Hence
for mutually local charge configurations our cubic curve of Eq. (160) falls short of the term proportional to $xy$ just like in the $stu$ truncation. Moreover, as shown by Eqs. (12)-(14) in the four-qubit case our quadratic and sextic invariants $I_1$ and $I_3$ are duals of each other. Hence we expect that the invariant $P_6$ should be related to a quadratic combination of the Freudenthal duals of the corresponding charge vectors which are cubic in terms of the original charges. Moreover, these dual quantities should be antisymmetric with respect to the exchange of the centers. Luckily a quantity $J_6$, satisfying these criteria is also available (see Eq. (3.23) of Andrianopoli et al.).

Unfortunately since according to Eq. (121) $D = I_3 - I_1 I_4$ in the $stu$ case this quantity is probably not directly related to our invariant $P_6$. Hence one is still left with the problem of finding the invariants $P_4, P_8$ and $P_{12}$.

In order to gain some insight into the physical meaning of the invariant $P_4$ let us look again at the $stu$ (four-qubit) case. In this case we know that the corresponding invariant is $M - L$. As a first step we would like to somehow relate this quantity to some invariant known within the context of $N = 8, d = 4$ supergravity.

For the $stu$ truncation let us suppose that we have chosen a particular two-center charge configuration parametrized by the four real numbers $a, b, c, d$ of the canonical form of Eq. (32). Now the explicit form of the invariants $H, M, L, D$ is

$$H = \frac{1}{2}(a^2 + b^2 + c^2 + d^2), \quad M = \left[ \left( \frac{c-d}{2} \right)^2 - \left( \frac{a-b}{2} \right)^2 \right] \left[ \left( \frac{a+b}{2} \right)^2 - \left( \frac{c+d}{2} \right)^2 \right]$$  \hspace{1cm} (163)

$$D = \frac{1}{4}(ad - bc)(cd - ab)(ac - bd), \quad L = abcd.$$  \hspace{1cm} (164)

Consider now the quartic $E_7(7)$ invariant expressed in terms of the matrix of central charge $Z$ of $N = 8$ supergravity in $d = 4$. This matrix is an $8 \times 8$ complex antisymmetric one which can be brought to the canonical form after using a suitable $U(8)$ transformation $Z \mapsto U^T Z U$. This canonical form is

$$U^T Z U = \begin{pmatrix} z_0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_3 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  \hspace{1cm} (165)
Now Cartan's quartic invariant is

\[ J_4(Z) \equiv \text{Tr}(Z \overline{Z} \overline{Z} \overline{Z}) - \frac{1}{4} (\text{Tr}(Z \overline{Z}))^2 + 4(\text{Pf}(Z) + \text{Pf}(\overline{Z})) \]  

(166)

where the Pfaffian is

\[ \text{Pf}(Z) = \frac{1}{2^4 4!} \varepsilon^{ABCDG}Z_{AB}Z_{CD}Z_{EF}Z_{GH}, \quad A, B, \cdots = 1, 2, \ldots 8 \]  

(167)

Subscripts \(A, B \ldots\) label an of 8 \(SU(8)\). Using the canonical form one gets the well-known expression

\[ J_4 = (|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2)^2 - 4(\sum_{n<m} |z_n z_m|^2) + 8\text{Re}(z_0 z_1 z_2 z_3) \]  

(168)

It is also known that by an \(SU(8)\) transformation it is possible to remove three phases, so for instance \(z_1, z_2, z_3\) can be chosen to be real (or else having the same phase). Hence the canonical form can be characterized by four real numbers \(r_n = |z_n|\) and a phase \(\phi\). The form of \(J_4\) reflecting these considerations is

\[ J_4 = [(r_0 + r_1)^2 - (r_2 + r_3)^2][(r_0 - r_1)^2 - (r_1 - r_2)^2] + 8r_0 r_1 r_2 r_3 (\cos \phi - 1) \]

\[ = -16M + 8L(\cos \phi - 1) = 8(N - M) + 8L \cos \phi \]

where we have used the identity \(L + M + N = 0\) of Eq.(22) and we made the identifications \(r_0 = a, r_1 = b, r_2 = c \) and \(r_3 = d\). Hence though we did not manage to get directly to the invariant \(M - L\) however for \(\phi = \frac{\pi}{2}\) we get \(N - M\) instead which is related to \(M - L\) via a permutation of the qubits labelled by \(i_1 i_2 i_3\) carrying the \(G_4\) labels. Notice also that the permutation group \(S_3\) responsible for this triality symmetry at the \textit{stu} model level is related to triality of the complex group \(SO(8, \mathbb{C})\) whose real form \(SO(4, 4)\) is featuring the tripartite entanglement of seven qubits interpretation of Cartan's quartic invariant\(^{20,21}\) of \(N = 8 d = 4\) supergravity and the corresponding \textit{stu} truncation. Notice also that according to Eq.(56) \(M - N\) is just the invariant \(\chi\) of Ferrara et.al\(^1\). These considerations show that the unknown invariant \(P_4\) should truncate to \(\chi = N - M\) after some suitable permutations.

It is also interesting to notice that for \(z_n\) \textit{real} i.e. \(\phi = 0\) we have

\[ J_4 = -16M = 16\tilde{r}_0 \tilde{r}_1 \tilde{r}_2 \tilde{r}_3. \]  

(169)
Here
\[ \tilde{r}_n = \sum_{m=0}^{3} (H \otimes H)_{nm} r_m, \]
\[ \begin{pmatrix} \tilde{r}_0 \\ \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} \] (170)

where \( H \) is the Hadamard matrix used for implementing discrete Fourier transformation in quantum information theory. Notice that with the definition \( Q_n = 2\tilde{r}_n \) we have \( J_4 = Q_0 Q_1 Q_2 Q_3 \) the standard expression giving rise to the Bekenstein-Hawking entropy \( S = \pi \sqrt{4Q_0 Q_1 Q_2 Q_3} \) of the \( D2 - D2 - D2 - D6 \) brane configuration in the type IIA duality frame.\(^{58}\)

There still remained the task to understand the meaning of \( P_8 \) and \( P_{12} \). Since in the \( N = 2, d = 4 \) context the vanishing of the corresponding invariants identifies the \( stu, st^2 \) and \( t^3 \) truncations in terms of the structure of the \( [160] \) elliptic curve it would be important to know their explicit forms. Finally the discriminant \( S^3 - 27T^2 \) as a polynomial invariant of order 24 should play the role of some sort of generalization of the hyperdeterminant of type \( 2 \times 2 \times 2 \times 2 \). It would be interesting to clarify what kind of role this discriminant and the associated \( j \) function plays in the physics of two-centered black hole solutions.

VI. A TRIALITY SYMMETRIC CURVE

The aim of this speculative section is to draw the readers attention to some interesting structural similarities showing up in a variety of physical contexts where our four-qubit invariants parametrizing elliptic curves might play a crucial role.

In the previous investigations our basic philosophy for the association of elliptic curves to two-center charge configurations was based on the resolvent cubic of the fundamental polynomial of Eq. (25). Is there any other physically appealing way for this association which retains the fundamental role of this polynomial and at the same time also featuring an extra modular parameter \( \hat{\tau} \) accounting for a hidden torus? In order to show that the answer to this question is yes let us rewrite Eq. (25) in the form

\[ \Sigma_4(A_{i_1i_2i_3i_4}, \hat{\tau}, -x) = x^4 + (\alpha \beta)(4I_1)x^3 + (\alpha \beta)^2(6I_2)x^2 + (\alpha \beta)^3(4I_3)x + (\alpha \beta)^4I_4. \] (171)
The quantities displaying explicit dependence on \( \hat{\tau} \) are defined as

\[
\alpha = e_2 - e_1, \quad \beta = e_3 - e_1
\]

(172)

where

\[
4(x - e_1)(x - e_2)(x - e_3) = 4x^3 - g_2x - g_3
\]

(173)

with \( g_2 \) and \( g_3 \) given by Eqs. (153)-(154). Explicitly we have

\[
e_1 - e_2 = \theta_3^4(0, \hat{\tau}), \quad e_3 - e_2 = \theta_1^4(0, \hat{\tau}), \quad e_1 - e_3 = \theta_2^4(0, \hat{\tau})
\]

(174)

where

\[
\theta_1(0, \hat{\tau}) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+1/2)^2}
\]

(175)

\[
\theta_2(0, \hat{\tau}) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2}
\]

(176)

\[
\theta_3(0, \hat{\tau}) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2}
\]

(177)

with \( q = e^{2\pi i \hat{\tau}} \). One also has

\[
e_1 = -\frac{2}{3} + 16q + 16q^2 + \ldots
\]

(178)

\[
e_2 = -\frac{1}{3} - 8q^{1/2} - 8q - 32q^{3/2} - 8q^2 + \ldots
\]

(179)

\[
e_3 = -\frac{1}{3} + 8q^{1/2} - 8q + 32q^{3/2} - 8q^2 + \ldots
\]

(180)

Notice that \( e_1 + e_2 + e_3 = 0 \) similarly to the property of four qubit quartic invariants \( L + M + N = 0 \). In the limit \( \hat{\tau} \to i\infty \) we have \( \alpha \beta \to 1 \) hence from Eq. (171) we get back to our usual polynomial of Eq. (25). Clearly introducing the factor \( \alpha \beta \) amounts to rescaling the amplitudes of our four-qubit state of Eq. (2) from \( \Lambda \) to the \( \hat{\tau} \) dependent ones \( \sqrt{\alpha \beta \Lambda} \).

Let us now introduce an extra complex parameter \( u \) to be specified later and consider the following family of elliptic curves

\[
y^2 = x(x - \alpha u)(x - \beta u) - (\alpha - \beta)^2 I_1x^2 + \frac{1}{2}(\alpha - \beta)\alpha \beta [(\alpha + \beta)I_4 - 3(\alpha - \beta)I_2]x
\]

\[
- (\alpha - \beta)\alpha^2 \beta^2 I_4 u - (\alpha - \beta)^2 \alpha^2 \beta^2 I_3.
\]

(181)
Now the right hand side is a quadratic polynomial in $u$. Its discriminant turns out to be just $(\alpha - \beta)^2\Sigma_4(\Lambda, \tau, -x)$. This gives the desired clue for yet another way of associating an elliptic curve to two-center charge configurations.

Notice that in the limit $\hat{\tau} \to i\infty$ our curve boils down to $y^2 = x(x-u)^2$ a curve similar to the one of Eq. (142) with $j$ invariant $\infty$. Moreover, via the transformation $v = u + (\alpha - \beta)\frac{1}{2}H$ one can also obtain a new form

$$
y^2 = x(x - \alpha v)(x - \beta v) + \beta(\alpha - \beta)H x^2 - \alpha\beta(\alpha - \beta)(\alpha M + \beta N)x$$

$$- \alpha\beta(\alpha - \beta)Hvx - \alpha^2\beta^2(\alpha - \beta)Lv - \alpha^2\beta^2(\alpha - \beta)^2D.
\tag{182}
$$

Though not in the Tate form this curve is displaying similar quantities than our previous curve of Eq.(130). In particular for truncations from $stu$ charge configurations to $st^2$ and $t^3$ ones the structure of the curve is getting simpler step by step. Though this curve now shows an explicit dependence on a modular parameter $\tau$ and also featuring a new complex variable $v$, now its discriminant cannot obviously be related to a hyperdeterminant of type $2 \times 2 \times 2 \times 2 \times 2$. However, curves like Eq.(182) have other appealing properties which we would like to discuss.

Actually the curve of Eq.(181) is related to a one introduced by Seiberg and Witten in their study of $N = 2$ supersymmetric $SU(2)$ gauge theory with four quark flavours. The parameter $u$ in that case was the gauge invariant modulus representing the square of the Higgs expectation value and $\tau$ was the complex coupling constant $\theta/\pi + 8\pi i/g^2$ of Montonen-Olive duality. In that context their curve was parametrized not by algebraically independent four-qubit invariants but by the squares of the four quark masses $m_1, m_2, m_3$ and $m_4$. However, after comparing the relevant expressions (see in particular Eq. 17.58 of Ref.14) it is easy to show that the quark masses squared in that case correspond to the squares of the parameters $a, b, c$ and $d$ of our four-qubit canonical form of Eqs.(32)-(34).

Apart from the symmetry $SL(2,\mathbb{R})_0 \times G_4$ where $G_4 = SL(2,\mathbb{R})^\times 3$ is the continuous $U$-duality group in the supergravity approximation, the $stu$ model also has an important triality symmetry. In our four-qubit description this $S_3$ permutation symmetry should manifest itself in representing somehow elliptic curves in a way displaying four qubit invariants that are also invariant under permutation. In analogy with the Seiberg-Witten curve there is also the possibility to present a triality invariant form of that kind. For this purpose we have to chose from four algebraically independent four-qubit invariants which are also invariant.
under the full permutation group $S_4$. Such invariants were first constructed by Schl"afli\textsuperscript{156} in 1852. Let us denote these invariants as $H, \Gamma, \Sigma$ and $\Pi$. They are of order $2, 6, 8$ and $12$ respectively. The new quantities $\Gamma, \Sigma$ and $\Pi$ are defined as

$$
\Gamma = D + E + F, \quad \Sigma = L^2 + M^2 + N^2, \quad \Pi = (L-M)(M-N)(N-L).
$$

Here $E$ and $F$ are defined as

$$
HL = E - D, \quad HM = D - F, \quad HN = F - E.
$$

Notice in particular that in terms of the quantities $H, L, M, D$ used to describe the Tate form of the elliptic curve of Eq.(130) the permutation invariant version of the sextic invariant reads as

$$
\Gamma = 3D + H(L-M).
$$

Let us now consider the triality symmetric curve\textsuperscript{14}

$$
y^2 = w_1w_2w_3 + \frac{\lambda}{3}[(M-N)(e_2-e_3)w_1 + (N-L)(e_3-e_1)w_2 + (L-M)(e_1-e_2)w_3] - \frac{\lambda^2}{3} \Gamma
$$

where

$$
w_i = x - e_i\tilde{u} - e_i^2H, \quad \lambda = (e_1 - e_2)(e_2 - e_3)(e_3 - e_1) \quad i = 1, 2, 3.
$$

Notice that after using Eq.(185) and the canonical form for our four qubit states of Eq.(32) and (35) with $(t_1, t_2, t_3, t_4) = (a^2, b^2, c^2, d^2)$ for the polynomial $\Gamma$ of sixth order we obtain the expression

$$
\frac{1}{3} \Gamma = \frac{3}{16} \sum_{i>j>k} t_it_jt_k - \frac{1}{96} \sum_{i\neq j} t_it_j^2 + \frac{1}{96} \sum_i t_i^3.
$$

This is precisely the sixth order invariant of Eq. (16.36) of Seiberg and Witten\textsuperscript{14} provided we make the identification $t_i \equiv m_i^2$ i.e. the canonical four-qubit parameters are identified with the quark masses of the four flavours. Similar calculations verify that the remaining invariants of that paper namely $R, T_1, T_2$ and $T_3$ can be identified with the invariants $H, \frac{1}{3}(M-N), \frac{1}{3}(N-L)$ and $\frac{1}{3}(L-M)$ in the canonical parametrization. The important property of the curve of Eq.(186) is that after making the transformations

$$
u = \tilde{u} + \frac{1}{2}e_1H, \quad x \rightarrow x - \frac{1}{2}e_1u + \frac{1}{2}e_1^2H
$$

and performing the (weak coupling\textsuperscript{14}) limit $\tau \rightarrow i\infty$ it boils down to the form

$$
y^2 = x^2(x - u). \quad (190)
$$
Moreover, in order to make contact with our original curve of Eq. (181) one just has to perform the change of variables in Eq. (186)

\[ x \mapsto x - e_1 \tilde{u} - e_1^2 H, \quad u = \tilde{u} + \frac{1}{2} e_1 H. \]  

(191)

It is also interesting to realize that our identification of the quark masses squared with the canonical four-qubit parameters automatically incorporates the special cases when some of the masses are zero with the singular cases in the four-qubit classification scheme of Verstraete. Moreover, switching to the two-center \( stu \) context there the canonical parameters would rather be identified with the the parameters \( z_1, z_2, z_3 \) and \( z_4 \) of the canonical form of the central charge matrix of \( N = 8 \ d = 4 \) supergravity (see also the discussion of the previous subsection). Recall also in this respect the observation of Seiberg and Witten that triality symmetry of \( SO(8) \) is connected to the permutation symmetry \( S_3 \) as the mod 2 reduction of \( SL(2, \mathbb{Z}) \). Comparing Eqs. (17.34)-(17.35) of Ref. 14 with our Eq. (170) related to the structure of the matrix \( \mathcal{M} \) and a similar expression related to the matrix \( \mathcal{N} \) that are in turn related to the structure of four-qubit reduced density matrices (see Eqs. (7)-(8)) we see that triality symmetry is intrinsically related not only to the structure of four-qubit entanglement but to the structure of the manifold \( X \) described by the curve of Eq. (186). In this context it is especially instructive to recall the arguments of Seiberg and Witten on the structure of the cohomology of \( X \). In particular it is tempting to reinterpret the triality invariant expressions for the cohomology classes of the periods as genuine unnormalized four-qubit states in canonical form (see Eqs. 17.36 and 17.37 of that paper). The redundancy in arriving at such an expression can be attributed to the action of the Weyl group of \( SO(8) \) on the canonical parameters (i.e. the action of the Klein group on the canonical form of \( |\Lambda\) in accord with a comment of Luque and Thibon in the four-qubit context. (See also the last paragraph of Section II. in this respect.)

Finally it is amusing to recall yet another context where elliptic curves parametrized by four-qubit invariants reveal some intriguing structural similarities with interesting physics. First of all it is well-known that in F-theory an extra \( SL(2, \mathbb{Z}) \) and a hidden torus also makes its presence in a spectacular way. In this case one considers an elliptic fibration \( M \) with some basis manifold \( B \) and fiber a two dimensional torus with modular parameter \( \hat{\tau} \). Now \( F \)-theory is defined on \( M \) as type \( IIB \) string theory on \( B \) with the axion dilaton modulus of type \( IIB \) string theory identified with the modular parameter of the two-torus.
In the original setting\textsuperscript{59} an elliptically fibered $K3$ surface was considered of the form $y^2 = x^3 + f(u)x + g(u)$ where $f$ and $g$ are polynomials of degree eight and degree twelve in $u \in \mathbb{C}P^1$. This curve describes a torus for each point of the Riemann sphere $\mathbb{C}P^1$ labelled by the complex coordinate $u$. It is interesting to realize that this elliptically fibered $K3$ surface is of the form of Eq. (147) where $S$ and $T$ are polynomials of order eight and twelve. However, according to Eq. (32) when writing $S$ and $T$ in the canonical form we are having four complex coordinates instead of the one $u$. The points where the torus degenerates corresponds to the vanishing of the discriminant of the cubic which is a polynomial of order 24. In the four qubit parametrization this just corresponds to the vanishing of our hyperdeterminant of order 24 related to the quantity $S^3 - 27T^2$. In the original $F$-theory context the compactification described by this elliptical fibration corresponds to a configuration of 24 seven branes of type IIB theory located at the zeros of the discriminant. On the other hand in the four-qubit canonical parametrization the explicit expression for the discriminant of Eq. (35) is similar to the usual ones describing coincident seven brane configurations\textsuperscript{61}, provided we are regarding one of the canonical parameters to be special.

Notice also that the study of $F$-theory compactifications\textsuperscript{60} is effected by looking at elliptic fibrations described by elliptic curves of the Tate form similar to the one of Eq. (160) where now the coefficients are polynomials in the coordinates of the base manifold $B$. An important limit studied in these investigations is the Sen limit\textsuperscript{61} or orientifold limit. This is achieved by demanding that the axio-dilaton to be constant almost everywhere in type IIB space-time. Explicitly this limit is realized by setting

\begin{equation}
    a_3 \mapsto \varepsilon a_3, \quad a_4 \mapsto \varepsilon a_4, \quad a_6 \mapsto \varepsilon^2 a_6,
\end{equation}

in the Tate-form. In the two center $stu$-context $a_i$ are polynomial invariants of homogeneous degree $2i$ and $a_6 = 0$. For the $stu$ model we have $a_4 = -LM$, $a_3 = D$. For the $st^2$ and $t^3$ models $L = 0$ hence $I_3 = D$ and a corresponding limit is achieved by sending the invariant $D$ to zero by e.g. scaling down one of the charges to zero. This is precisely what happened in Section V.B. and also in the situation described by Eq. (148).

Finally our triality symmetric Seiberg-Witten curve of Eq. (186) also makes its presence in the $F$-theory context\textsuperscript{45}. Here the basic idea is to deform away from the special point in moduli space where the orientifold picture applies. This important deformation is effected by a curve of the form $y^2 = x^3 + \tilde{f}(u)x + \tilde{g}(u)$ where $\tilde{f}$ and $\tilde{g}$ are polynomials in $u$ of
degree two and three. Sen has shown that this curve can be cast in the form of Eq. (186) provided we make a suitable mapping of the parameters involved in the two different physical contexts. Notice that in the original papers the relevant curves were not parametrized by the four-qubit invariants $H, L, M, N$ and $\Gamma$ as in Eq. (186). Instead in these papers four parameters were employed, which are easily identified with the four canonical parameters of four-qubit states of Eq. (32). In particular in the F-theory context the relevant deformation is a one corresponding to splitting of the six coincident zeros of the discriminant away from each other, i.e in the orientifold picture moving the four coincident seven branes away from the orientifold plane. The collective coordinates of the background describing such a physical situation are that of an $N = 1$ supersymmetric $SO(8)$ gauge theory in eight dimensions with moduli space characterized by a complex scalar field $\Phi$. At a generic point in moduli space the vacuum expectation value of this field has the form similar to the canonical form of the central charge of Eq. (165) we used in Section V. in the different context of $N = 8$ supergravity. Now one can check that the four canonical complex parameters $c_1, c_2, c_3$ and $c_4$ of $\langle \Phi \rangle$ are behaving exactly like the canonical parameters of a four-qubit state. Again the polynomial expressions used by Seiberg and Witten in their curve of the Eq. (186) form are the ones for the four-qubit invariants $H, L, M, N$ and $\Gamma$ now with the simple identification $c_i = m_i$. Our four-qubit analysis as presented in this paper shows similar symmetry properties (permutation symmetry of the stu truncation connected to triality of $SO(8)$ within $N = 8$ supergravity). This might indicate that the physics of two-center black holes could be another arena where this curve plays a basic role. Of course these structural similarities could be superficial however, in any case the possible physical ramifications should be explored further.

VII. CONCLUSIONS

In this paper we have shown how the $U$-duality invariants introduced by Ferrara et al. characterizing two-center extremal black hole charge configurations in the $stu$, $st^2$ and $t^3$ models of $N = 2$, $d = 4$ supergravity can be understood as entanglement invariants of four-qubit systems. In this entanglement based picture the geometric and algebraic meaning of these invariants is displayed in a nice and unified manner. For one of the entanglement invariants that have not yet made its debut to the supergravity literature we have found a
distinguished role. It is the hyperdeterminant of type $2 \times 2 \times 2 \times 2$ which is the generalization of Cayley’s hyperdeterminant featuring the macroscopic black hole entropy formula in the $stu$ model. For the special example of the BPS $D0D4 – D2D6$ split in the $t^3$ model we have demonstrated that this polynomial invariant of order 24 seems to govern important issues of consistency for the two center solutions.

We have also introduced a quartic polynomial featuring the algebraically independent four-qubit invariants. For our simple example we have shown that the property that this polynomial has real roots provides a necessary condition for the consistency condition to hold. The resolvent cubic of this fundamental polynomial can be cast into a cubic of Weierstrass canonical form which in turn can be used to define an elliptic curve associated to the two-center charge configuration. After switching to the more convenient Tate form of this curve we have shown that this association is natural, meaning that the Tate form is displaying combinations of the algebraically independent invariants of physical meaning. Indeed, the Tate form falls short of terms step by step as we perform truncations to the $st^2$ and $t^3$ models, restrict attention to mutually local charge configurations, or perform the degenerate limit resulting in splits into small black holes. The discriminant of this elliptic curve is just the hyperdeterminant, a quantity also featuring the $j$ invariant of the curve. For our example the structure of the $j$ invariant nicely encapsulates the basic properties of the $D0D4 – D2D6$ split.

The mapping from two center black hole charge configurations to elliptic curves is many to one. Moreover, for the $stu$ model only a special class of curves with $a_6 = 0$ in Eq.(130) can be reached. We observed that the vanishing of $a_6$ can be attributed to the vanishing of a polynomial of degree 12 characterizing the $stu$ model. Based on this we conjectured that our $N=2$ picture can be substantially generalized also incorporating two center charge configurations of $N=8$ supergravity. Here Cartan’s quartic invariant and its polarizations should make their presence together with the usual symplectic invariant and a new invariant of order 6 introduced recently\cite{13}. The discriminant of this curve of more general type then would play the role of a generalization of our hyperdeterminant, with probably similar physical meaning than its $stu$ descendent.

Finally we presented other physical contexts where our four-qubit invariants parametrizing elliptic curves also play an important role. Here we have given a new look to the triality invariant curve originally introduced by Seiberg and Witten\cite{14}. In its new form this curve
is featuring four-qubit invariants also displaying permutation invariance. In this new setting we have also invoked the F-theory interpretation of this curve as was given by Sen\textsuperscript{15}. Taken together with the two-center black hole context studied in this paper the main unifying theme in these scenarios seems to be the presence of a hidden torus and its extra set of $PSL(2,\mathbb{Z})$ modular transformations. Note that in our investigations elliptic curves appeared merely as natural mathematical objects nicely encapsulating the information on the structure of two-center $U$-duality invariants. However, apart from this we have also presented some evidence that the hyperdeterminant and the $j$ invariant associated to these curves might contain useful information on issues of marginal stability, domain walls and split attractor flows. The physical background underlying the use of elliptic curves in the F-theory and $N=2$ supersymmetric $SU(2)$ gauge theory context is well-known. Is there any deeper physical reason also accounting for the occurrence of elliptic curves related to the structure of multi-center black hole solutions?

VIII. ACKNOWLEDGEMENT

This work was supported by the New Hungary Development Plan (Project ID: TÁMOP-4.2.1/B-09/1/KMR-2010-0002).

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