ON THE CLASSIFICATION OF TORIC SINGULARITIES

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Abstract. For a toric log variety with standard coefficients, we show that the minimal log discrepancy at a closed invariant point bounds the Cartier index of a neighbourhood.

1. Introduction

The class of log canonical singularities appears naturally in the birational classification of algebraic varieties. The main invariants of such singularities are the index and the minimal log discrepancy, and we expect that these two invariants separate the singularities into series. In this note we show that the two invariants are equivalent up to finitely many values, in the special case of toric singularities.

To describe the invariants, suppose \( P \in X \) is an isolated log canonical singularity. Here \( X \) is a normal variety. We denote by \( K \) a canonical Weil divisor of \( X \) and suppose \( nK \sim 0 \) for a positive integer \( n \). Let \( n \) be minimal with this property, called the index of \( P \in X \). By Hironaka’s resolution of singularities, there is a birational modification \( \mu: X' \to X \) such that \( X' \) is nonsingular, \( \mu^{-1}(P) = \Sigma \) is a divisor with simple normal crossings, and \( \mu: X' \setminus \Sigma \to X \setminus P \) is an isomorphism. Let \( m \) be the smallest multiplicity of the general member of \( |nK_{X'} + n\Sigma| \) along the prime components of \( \Sigma \). Then \( \frac{m}{n} \) is independent of the choice of \( \mu \) and is called the minimal log discrepancy of \( X \) at \( P \), denoted \( a(P; X) \). For the relevance of these invariants to the birational classification of algebraic varieties, see [2].

Suppose the index is fixed. Then \( a(P; X) \) is a rational number with fixed denominator. As it is expected that \( a(P; X) \leq \dim X \), it would follow that \( a(P; X) \) could take only finitely many values. Conversely, suppose \( a(P; X) \) is fixed. Then, according to Shokurov, we expect that the index is bounded. First, this is the analog for singularities of the boundedness in terms of volume of canonically polarized varieties. Second, there is some evidence for this conjecture. A surface germ \( P \in X \) with \( a(P; X) = 0 \) has index 1, 2, 3, 4 or 6 (Shokurov [7]). A similar statement holds in dimension 3 (Ishii [4]). If \( P \in X \) is a terminal 3-fold singularity, then the index is the denominator of \( a(P; X) \) (Kawamata [7]).

It is increasingly becoming clear that in order to classify algebraic varieties we must allow not only certain singularities, but even certain boundary divisors, to measure ramification. These boundaries are crucial in the study of singularities and they unify the theories of open and closed manifolds. In this note we only allow boundaries with so called standard coefficients.

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2. The bound

We refer to Oda [6] for standard notions on toric varieties. For more details on toric log varieties, see [1]. Let \( X \) be an affine toric variety of dimension \( d \), let \( P \in X \) be the unique closed point fixed by the torus. Let \( \{ H_a \} \) be the invariant prime divisors of \( X \). Let \( B = \sum_a b_a H_a \) be a \( \mathbb{Q} \)-divisor with the following properties:

- \( nK + nB \sim 0 \) for some positive integer \( n \). Suppose \( n \) is minimal with this property.
- \( \{ b_a \} \subset \{ \frac{l-1}{l}; l \in \mathbb{Z}_{\geq 1} \} \cup \{1\} \).

By Lemma 2.3, \( \vol \{ \} \) is positive constant depending on \( \Lambda \). Let \( \Lambda = \{ \} \). By Lemma 2.2, this is equivalent to \( \{ b_a \} \subset \{ \frac{l-1}{l}; l \in \mathbb{Z}_{\geq 1} \} \cup \{1\} \).

It follows that \( \langle X, B \rangle \) has log canonical singularities and \( a(P; X, B) \geq 0 \) is a rational number.

**Theorem 2.1.** Let \( q \) be the denominator of \( a(P; X, B) \). Then \( n \leq c_d q^d \), where \( c_d \) is a positive constant depending on \( d \) only.

In particular, if \( q \) is fixed then the coefficients of \( B \) belong to a finite set.

**Proof.** Let \( X = T \mathbb{N} \) \( \text{emb}(\sigma) \) for a strongly rational polyhedral cone \( \sigma \subset \mathbb{N}_\mathbb{R} \). Let \( \{ e_a \} \) be the primitive points of \( \mathbb{N} \) on the rays of \( \sigma \). By assumption, there exists \( \psi \in \mathbb{N}_\mathbb{Q}^* \) such that \( \langle \psi, e_a \rangle = 1 - b_a \) for every \( \alpha \), and \( n \geq 1 \) is smallest with \( n\psi \in \mathbb{N}_\mathbb{Z}^* \). Since \( B \) has one standard coefficient, the sublattice \( \langle \psi, N \rangle \subset \mathbb{R} \) contains 1. Therefore \( \langle \psi, N \rangle = \frac{1}{n}\mathbb{Z} \). The minimal log discrepancy at \( P \) is computed as follows:

\[
a(P; X, B) = \min_{e \in \mathbb{N} \cap \text{int}(\sigma)} \langle \psi, e \rangle.
\]

Let \( \Lambda = N \cap \psi^\perp \). Choose \( e \in N, \langle \psi, e \rangle = \frac{1}{n} \). Define \( \square = \Lambda \mathbb{Z} \cap (\sigma - e) \). \( e_a = (n - nb_a)(v_a + e) \), where \( \{ v_a \} \) are the vertices of \( \square \). Since \( \Lambda \oplus \mathbb{Z}e = N \), we have

- \( na = \min\{i \geq 1; \Lambda \cap \text{int}(i\square) \neq \emptyset\} \).
- \( n - nb_a = \min\{i \geq 1; iv_a \in \Lambda \} \) for every vertex \( v_a \) of \( \square \).

Since \( B \) is standard, \( n\square \) has vertices in \( \Lambda \). Denote \( na = j \), \( S = j\square \). Then

- \( j = \min\{i \geq 1; \Lambda \cap \text{int}(\frac{j}{q}S) \neq \emptyset\} \).
- \( S \) has vertices in \( \frac{1}{q}\Lambda \).

**Step 1:** Let \( z \in \Lambda \cap \text{int}(S) \). We may shrink \( S \) until \( \{z\} = \frac{1}{q}\Lambda \cap \text{int}(S) \). Since \( S \) has vertices in \( \frac{1}{q}\Lambda \), it follows by Hensley [3] that there exists a positive constant \( \gamma = \gamma_{d-1} \), depending only on \( d - 1 \), such that \( \gamma(S - S) \subset S \).

**Step 2:** By definition, the cone over \( \{j\} \times S \) with vertex 0 contains no point of \( \mathbb{Z} \times \Lambda \) in its interior. Let \( C \subset \mathbb{R} \times \Lambda \) be the cone over \( \{j\} \times (z + \gamma(S - S)) \) with vertex 0. Let \( C' \) be its reflexion about the lattice point \( (j, z) \). Then \( P = C \cup C' \) is a convex body symmetric about \( \{(j, z)\} = \text{int}(P) \cap \mathbb{Z} \times \Lambda \). By Minkowski’s first theorem (see [5] for example), \( \vol_{\mathbb{Z} \times \Lambda}(P) \leq 2^d \).

By Lemma 2.2, this is equivalent to

\[
2 \cdot \frac{j}{d} \vol_{\Lambda}(z + \gamma(S - S)) \leq 2^d.
\]

By Lemma 2.3, \( \vol_{\Lambda}(z + \gamma(S - S)) = \gamma^{d-1} \vol_{\Lambda}(S - S) \geq \gamma^{d-1} \frac{2^{d-1}}{(d-1)!q^{d-1}} \). Therefore \( j \leq \frac{d}{\gamma^{d-1}} q^{d-1} \). Therefore the claim holds for \( c_d = \frac{d}{\gamma^{d-1}} \).

**Lemma 2.2.** Let \( \square \subset \Lambda \mathbb{R}, h > 0 \). Let \( C \subset \mathbb{R} \times \Lambda \) be the cone over \( \{h\} \times \square \) with vertex 0. Then \( \vol_{\mathbb{Z} \times \Lambda}(C) = \frac{h}{\dim \Lambda + 1} \vol_{\Lambda}(\square) \).
Lemma 2.3. Let $\Box \subset \Lambda^d_R$ be a lattice convex body. Then $\text{vol}_\Lambda(\Box - \Box) \geq \frac{2^d}{d!}.$

Proof. We may assume $\Box$ is a simplex with one vertex at the origin, with vertices $0, v_1, \ldots, v_d.$ $
\Box - \Box$ contains the convex hull $H$ of $\pm v_1, \ldots, \pm v_d.$ For $f \in \{0, 1\}^d,$ denote by $C_f$ the convex hull of $0, (-1)^{f(1)}v_1, \ldots, (-1)^{f(d)}v_d.$ The $C_f$’s cover $H$ and have no interior points in common. Each $C_f$ is a lattice convex body, hence $\text{vol}_\Lambda(C_f) \geq \frac{1}{d!}.$ Their cardinality is $2^d,$ hence $\text{vol}_\Lambda(\Box - \Box) \geq \text{vol}_\Lambda(H) = \sum_f \text{vol}_\Lambda(C_f) \geq \frac{2^d}{d!}.$

Remark 2.4. We may take $c_1 = 1, c_2 = 2.$

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