A NEW TOPOLOGICAL GENERALIZATION OF
DESCRIPTIVE SET THEORY

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Abstract. We introduce a new topological generalization of the \(\sigma\)-
projective hierarchy, not limited to Polish spaces. Earlier attempts have
replaced \(\omega\) by \(\kappa\), for \(\kappa\) regular uncountable, or replaced countable by
\(\sigma\)-discrete. Instead we close the usual \(\sigma\)-projective sets under continu-
ous images and perfect preimages together with countable unions. The
natural set-theoretic axiom to apply is \(\sigma\)-projective determinacy, which
follows from large cardinals. Our goal is to generalize the known results
for \(K\)-analytic spaces (continuous images of perfect preimages of \(\omega\))
to these more general settings. We have achieved some successes in the
area of Selection Principles — the general theme is that nicely defined
Menger spaces are Hurewicz or even \(\sigma\)-compact. The \(K\)-analytic results
are true in \(ZFC\); the more general results have consistency strength of
only an inaccessible.

1. Introduction

Classical descriptive set theory studies sets of reals, or, more generally,
subsets of a Polish (i.e. separable completely metrizable) space, which are
in some sense definable — think of Borel, analytic, etc. See [Keč95]. Due to
its intrinsic interest and its usefulness in many areas of mathematics, there
have been a number of attempts to generalize it, e.g. Choquet [Cho59],
Frolík [Fro63], Sion [Sio60], Stone [Sto63], etc. Choquet’s attempt retained
the central role of the space \(P\) of irrationals, but weakened continuous maps
to upper semi-continuous compact-valued ones (defined later). Frolík, in-
stead, retained continuity but weakened \(P\) to any Lindelöf Čech-complete
space. Jayne (see [RJ80]) observed that, in fact, these two approaches were
equivalent, yielding the \(K\)-analytic spaces:

Definition 1.1. [RJ80] A space is \(K\)-analytic (we say \(K-\Sigma^1_1\)) if it is the
continuous image of a Lindelōf Čech-complete space.

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\(K\)-analytic, generalized descriptive set theory, Menger, Hurewicz, \(\sigma\)-compact, \(\sigma\)-
projective determinacy, upper semi-continuous compact-valued multifunction.
The $K$-analytic spaces have been applied widely in functional analysis—see [KKL11].

Stone and others went in a different direction, generalizing “countable” to “$\sigma$-discrete”, while retaining metrizability. This effort was less successful. Recently, set theorists have generalized the role of $\mathcal{P}$ in its equivalent $\omega\omega$ form to $\kappa\kappa$, for an arbitrary regular cardinal $\kappa$. Although this leads to interesting set theory, it is unlikely to find applications outside of logic. But set theorists have also generalized classical descriptive set theory to consider “definable” sets of reals more complicated than Borel or analytic ones. The $\sigma$-projective sets are obtained by extending the Borel sets by using continuous real-valued functions, countable unions, and complementation within $\mathbb{R}$. Here is the definition.

**Definition 1.2.** [Kec95] A subset of reals is analytic ($\Sigma^1_1$) if it is a continuous image of $\mathcal{P}$ (equivalently, of any Borel set). It is co-analytic ($\Pi^1_1$) if its complement is analytic.

**Definition 1.3.** [Kec95] In general, a subset of reals is $\Pi^1_\xi$, for $\xi < \omega_1$, if its complement is $\Sigma^1_\xi$ and a subset of reals is $\Sigma^1_{\xi+1}$ if it is the continuous image of a $\Pi^1_\xi$ set. For $\alpha \leq \omega_1$ a limit ordinal, a subset of reals $X$ is $\Sigma^1_\alpha$ if there is a $Y_i \in \Sigma^1_{\xi_i}, i < \omega, \xi_i < \alpha$ such that $X = \bigcup_{i \in \omega} Y_i$.

The class $\Sigma^1_\omega$ is known as the projective sets and the class $\Sigma^1_{\omega_1}$ as the $\sigma$-projective sets. Since properties such as analytic have been used to refer to subspaces of Polish spaces as well as to general topological spaces, we will use the word sets to refer to subspaces of Polish spaces.

We continue the standard sloppiness of interchangeably speaking of $\mathcal{P}$, $\mathbb{R}$, $[0, 1]$, and $[0, 1]^\omega$, since their Borel, projective, etc., structures are isomorphic.

An even more far-reaching extension of the collection of “definable” sets of reals is those in $L(\mathbb{R})$ (see [Kan09]). It is easy to show that the $\sigma$-projective sets are all in $L(\mathbb{R})$. In fact, the $\sigma$-projective sets are precisely those sets of reals which are in $L_{\omega_1}(\mathbb{R})$ [AMS21].

These more extensive collections of definable sets of reals were not easily handled by classical descriptive set theory, but the advent of determinacy axioms consistent with ZFC has made them quite tractable. We will often be assuming such axioms. The novelty of our approach is that we follow Choquet, Frolík, and Rogers-Jayne, but replace $\mathcal{P}$ with $\sigma$-projective sets. Like Frolík, we do not require our continuous maps to be real-valued. There is much to be investigated using a functional analysis lens. We have so far mainly looked at these theories from the viewpoint of Selection Principles.

There has been much work in Selection Principles considering the successively stronger Lindelöf properties: Menger, Hurewicz, $\sigma$-compact. We will give the definitions, but read [Tsa11] for a thorough introduction and survey.
Definition 1.4. A topological space \( X \) is Menger if given any countable sequence of open covers \( \langle U_1, U_2, \ldots, U_n, \ldots \rangle \) of \( X \) there are finite subsets \( V_i \subseteq U_i \) such that \( \bigcup_{i \in \omega} V_i \) is also an open cover of \( X \).

Definition 1.5. A topological space \( X \) is Hurewicz if given any countable sequence of open covers \( \langle U_1, U_2, \ldots, U_n, \ldots \rangle \) of \( X \), none containing a finite subcover, there are finite subsets \( V_i \subseteq U_i \) such that \( \bigcup_{i \in \omega} V_i \) is a \( \gamma \)-cover of \( X \). An open cover \( \mathcal{O} \) is a \( \gamma \)-cover of \( X \) if \( \mathcal{O} \) is infinite and every element of \( X \) appears in all but finitely members of \( \mathcal{O} \).

(Note Arhangel’skii uses “Hurewicz” for the property we now call “Menger”.) Menger analytic sets are \( \sigma \)-compact \([Hur26]\); for \( \sigma \)-projective sets and, indeed, for sets of reals in \( L(\mathbb{R}) \), it is consistent from an inaccessible cardinal \([Fen93]\, [DT98]\) and follows from larger cardinals \([MS94]\, [Agu22]\) that determinacy axioms hold which imply such Menger sets are \( \sigma \)-compact.

There has been a series of papers showing that nicely defined Menger sets of reals need not be \( \sigma \)-compact if \( V = L \) is assumed \([MF88]\), but are if determinacy is assumed instead \([MF88]\), and then considering the situation for more general Menger spaces; \([TT17]\, [TTT21]\, [Tal20]\). Assuming the Axiom of Choice, however, there are Menger sets of reals that are not \( \sigma \)-compact, indeed not even Hurewicz. See the survey \([Tsa11]\) for references.

One would expect it to be easy to construct a Menger space that is not Hurewicz but the second author noticed that, whenever he tried to define a Menger space that was not \( \sigma \)-compact, it wound up being Hurewicz. He therefore conjectured that “definable” Menger spaces are Hurewicz. Indeed, he proved that \( K \)-analytic Menger spaces are Hurewicz \([Tal20]\). However, \( K \)-analytic Menger spaces need not be \( \sigma \)-compact \([Tal20]\).

Just as the \( K \)-analytic spaces generalize analytic sets, we can define spaces generalizing the \( \sigma \)-projective sets. Every analytic set is a \( K \)-analytic space; a \( K \)-analytic metrizable space is analytic \([RJ80]\), see Corollary 5.8. (We don’t have to require separability, because \( K \)-analytic spaces are Lindelöf and Lindelöf metrizable spaces are separable.) Our generalizations of \( \sigma \)-projective will have analogous properties.

Continuous functions (maps) are integral to topology; also very important are the perfect maps, which are continuous, send closed sets to closed sets, and for which the inverses of points are compact. Many topological properties are preserved by perfect maps and/or their inverses. A topological property is called perfect if it is preserved by both perfect maps and their inverses.

Definition 1.6. Given a family \( \Gamma \) of subsets of reals, \( \Gamma \) determinacy (\( \text{Det}(\Gamma) \)) is the statement, “given \( A \in \gamma \), the perfect information game of countable length and payoff set \( A \) has a winning strategy for one of the players”. Then \( \sigma \)-projective determinacy—for short, \( \sigma \)-PD—is \( \text{Det}(\{ X \subseteq \mathbb{R} : X \text{ is } \sigma \text{-projective} \}) \).

We expect—and it is true—that:
Proposition 1.7. $\sigma$-PD implies every Menger $\sigma$-projective set is $\sigma$-compact.

This was proved for “projective” in [TT17]; the same proof works.

Definition 1.8. Given a topological property $P$, a space $X$ is projectively $P$ if given any continuous function $f : X \to \mathbb{R}$, $f(X)$ has property $P$.

We will be interested in spaces which are projectively $\sigma$-compact, projectively $\sigma$-projective, or projectively countable. Some authors replace “projectively” by “functionally”. Some replace $\mathbb{R}$ by $[0,1]$ or any separable metrizable space. These are all equivalent.

In analogy to one of the original definitions of $K$-analyticity, let us consider the class $\mathcal{K}$ of images of $\sigma$-projective sets under Upper Semi-Continuous Compact-Valued multifunctions. Below is the definition of USCCV multifunctions. We call members of $\mathcal{K}$ $K$-projective.

Definition 1.9. Given a set $X$ we will denote by $\mathcal{P}(X)$ its power set, the collection of all subsets of $X$. We say that a function $F : X \to \mathcal{P}(Y)$ (a multifunction) is Upper Semi-Continuous if given a closed set $C \subseteq Y$, its outer inverse

$$F^{-1}(C) = \{x \in X : F(x) \cap C \neq \emptyset\}$$

is closed.

Equivalently, given an open set $U \subseteq Y$, its inner inverse

$$F_{in}^{-1}(U) = \{x \in X : F(x) \subseteq U\}$$

is open.

$F$ is compact-valued if $F(x)$ is compact for all $x \in X$. $F$ is USCCV if it is upper semi-continuous and compact-valued.

This may be a bit difficult to absorb, but more pleasantly we have,

Definition 1.10. If the space $Y$ is the continuous image of a space $Z$ such that there is a perfect surjective function $f : Z \to X$, then we say that $Y$ is a CIPP (Continuous Image of a Perfect Preimage) of $X$.

Rogers and Jayne [RJ80] proved:

Corollary 3.7. Given spaces $X$ and $Y$, $Y$ is a USCCV image of $X$ if and only if it is a CIPP of $X$.

An easy corollary is:

Corollary 1.11 (Folklore). USCCV images of Lindelöf spaces are Lindelöf,

Proof. Continuous functions preserve Lindelöf. Lindelöf is a perfect property. $\square$

Rogers and Jayne [RJ80] proved that

Theorem 4.7. A space is $K$-analytic if and only if it is a USCCV image of $\mathbb{P}$.
Not so easy is

**Theorem 5.7.** All members of $\mathcal{H}$ are projectively $\sigma$-projective.

We will postpone the proof until Section 5.

$K$-analytic spaces are USCCV images of $P$; Lindelöf $\Sigma$-spaces are USCCV images of any separable metrizable space. Lindelöf $\Sigma$-spaces show up in many contexts, which is why Tkachuk [Tk10] calls them “an omnipresent class”. In particular, the $K$-countably determined spaces of [RJ80] are precisely the Lindelöf $\Sigma$ spaces. $\mathcal{H}$ fits neatly between the class of $K$-analytic spaces and the class of Lindelöf $\Sigma$-spaces. Assuming $\sigma$-PD, it is closer to the former class.

Most of the applications of $K$-analyticity use Lindelöfness. Thus $\mathcal{H}$ is a worthy generalization of the class of $K$-analytic sets, especially assuming $\sigma$-PD. However, Menger spaces are trivially Lindelöf, so in the context of getting Menger spaces to be Hurewicz, much more inclusive classes than $\mathcal{H}$ can be considered. Indeed,

**Theorem 6.2.** In ZFC, Menger projectively analytic spaces are Hurewicz. $\sigma$-PD implies Menger projectively $\sigma$-projective spaces are Hurewicz.

We will then have:

**Corollary 6.4.** $\sigma$-PD implies Menger spaces in $\mathcal{H}$ are Hurewicz.

The $\sigma$-projective subsets of any Polish space are closed under continuous real-valued images, countable unions, and complements. Can we get a reasonable class $\mathcal{L}$ containing all $K$-analytic spaces and closed under continuous images, countable unions and some sort of restricted complement? We discuss this question in Section 7.

2. **Definitions**

All topological spaces that we work with will be completely regular.

**Definition 2.1.** Given a compactification $\mu X$ of $X$ the remainder of $X$ in $\mu X$ is $\mu X \setminus X$.

**Definition 2.2.** We will denote by $\beta X$ the Stone-Čech compactification of $X$ and by $X^*$ its Stone-Čech remainder, $\beta X \setminus X$.

**Definition 2.3.** A space is $\check{C}ech$-complete if it is $G_\delta$ in its Stone-Čech compactification. Equivalently, if its Stone-Čech remainder is $\sigma$-compact.

In addition to $\mathcal{H}$, we will consider two proper subclasses of it;

**Definition 2.4.** Given a class of topological spaces $\Gamma$, $\mathcal{C}(\Gamma)$ is the class of all topological spaces that are continuous images of elements in $\Gamma$.

**Definition 2.5.** Given a class of topological spaces $\Gamma$, $\mathcal{P}(\Gamma)$ is the class of all topological spaces that are perfect preimages of elements in $\Gamma$. 
Definition 2.6. Given a class of topological spaces \( \Gamma \), \( \mathcal{K}(\Gamma) \) is the class of all topological spaces that are USCCV images of elements in \( \Gamma \).

We call \( \sigma\)-P the set of all \( \sigma\)-projective sets of reals (or subsets of \([0,1]\), etc.). When referring to the classes \( \mathcal{K}(\sigma\text{-P}) \), \( \mathcal{P}(\sigma\text{-P}) \) or \( \mathcal{C}(\sigma\text{-P}) \) we may also call them \( \mathcal{K} \), \( \mathcal{P} \) or \( \mathcal{C} \) respectively.

Definition 2.7. A topological space \( X \) is perfect if every closed set of \( X \) is a \( G_\delta \). This is standard terminology. There is no relationship between perfect spaces and perfect functions.

3. A powerful tool: USCCV multifunctions

In [Why65] Whyburn refers to multifunctions (functions from a set to the power set of some set) as a useful approach to encompass a variety of topological results. Whyburn presents Upper Semi-Continuous Compact-Valued (USCCV) multifunctions as the right analogue of continuous functions in multifunctions settings. A first important thing to notice about USCCV multifunctions is that every continuous function is (a single-valued) USCCV multifunction. Also that the outer (or inner) inverse of a multifunction is also a multifunction. Together with these, Whyburn showed that:

Proposition 3.1. [Why65]

1) Images of compact sets under USCCV are compact (hence the composition of USCCV multifunctions is USCCV).

2) A multifunction \( F : X \rightarrow Y \) is closed and has compact inverse values for singletons if and only if \( F^{-1} \) is USCCV.

Notice that, in 2), \( F \) does not need to be Upper Semi-Continuous and that \( F^{-1} \) is the outer inverse.

As mentioned in Section 1, \( K \)-analytic spaces can be defined in terms of USCCV multifunctions. We are interested in working with the Stone-Čech remainders of topological spaces in order to understand \( \mathcal{K} \) and its variations better. We therefore work on generalizing some classic results by replacing “continuous” with “USCCV” and “\( \Sigma_\beta^1 \) (\( \Pi_\beta^1 \))” with “\( \mathcal{K}(\Sigma_\beta^1) \) (\( \mathcal{K}(\Pi_\beta^1) \))”.

Proposition 3.2.

1) Given a continuous function \( f : X \rightarrow K \) with \( K \) compact and \( f(X) \) dense in \( K \), there is a unique continuous function \( \hat{f} : \beta X \rightarrow K \) extending \( f \) such that \( \hat{f}|X = f \) and \( \hat{f}(\beta X) = K \) (generalized in Theorem 3.14).

2) Given a perfect map \( f : X \rightarrow K \) with \( K \) compact and \( \hat{f} \) the extension given by the Stone-Čech compactification, we have that \( \hat{f}(\beta X \setminus X) \subseteq K \setminus f(X) \) (generalized in Theorem 3.14).

3) Perfect real-valued images of co-analytic sets are co-analytic (generalized in Corollary 5.4 and Lemma 7.12).
4) Given a surjective continuous function \( f : Y \to G \) with \( Y \) Polish, \( G \) separable metrizable and \( X \subseteq G \) a \( \Sigma^1_\beta (\Pi^1_\beta) \) set, then \( f^{-1}(X) \) is \( \Sigma^1_\beta (\Pi^1_\beta) \) (generalized in Theorem 5.13).

These results are all in the literature. 1) and 2) can be found in [PW88] and [Eng89]. 3) is implicit in [Tal20]. 4) is done for projective sets in [Kec95] and the same proof works.

To achieve the corresponding generalizations, it is useful to think that the co-domain of a multifunction is the space itself and not its power set. With this in mind, we have the following definitions:

**Definition 3.3.** We say that a multifunction \( F : X \to Y \) is closed (open) if for any closed (open) set \( A \subseteq X \) we have that \( F(A) = \bigcup_{a \in A} F(a) \) is closed (open) in \( Y \).

**Definition 3.4.** We say that \( F \) is a perfect USCCV multifunction if \( F \) is a USCCV closed multifunction with compact point-inverses.

**Proposition 3.5.** The outer inverse of a perfect USCCV multifunction is a perfect USCCV multifunction.

**Proof.** Let \( F : X \to Y \) be a surjective perfect USCCV multifunction. Using Proposition 3.1 point 2), we know that \( F^{-1} \) is USCCV. Since \( F \) is USCCV, given a closed set \( C \subseteq Y \),

\[
\bigcup_{b \in C} F^{-1}(b) = \{ x : (\exists b \in C)(b \in F(x)) \} = \{ x : F(x) \cap C \neq \emptyset \} = F^{-1}(C)
\]

is closed.

Finally, given \( x \in X \)

\[
(F^{-1})^{-1}(x) = (F^{-1})^{-1}(\{x\}) = \{ y : \{x\} \cap F^{-1}(y) \neq \emptyset \} = \{ y : x \in F^{-1}(y) \} = \{ y : y \in F(x) \} = F(x).
\]

Since \( F \) is USCCV, we have that \( (F^{-1})^{-1}(x) \) is compact and \( F^{-1} \) is a perfect USCCV multifunction. \( \square \)

Rogers and Jayne proved:

**Proposition 3.6.** [RJ80] Given a USCCV multifunction \( F : X \to A \), the graph of \( F \) (the space \( \text{graph}(F) = \bigcup_{x \in X}\{x\} \times F(x) \)) is a perfect preimage of \( X \) (the perfect function being the projection to \( X \)) and the projection \( \pi_A : \text{graph}(F) \to A \) is a continuous single-valued function.

**Corollary 3.7.** Given spaces \( X \) and \( Y \), \( Y \) is a USCCV image of \( X \) if and only if it is a CIPP of \( X \).

**Proof.** If \( Y \) is a CIPP of \( X \) then, we know that every continuous function is USCCV and, by Proposition 3.1 point 2), the (outer) inverse multifunction of a perfect function is also USCCV. Furthermore, Proposition 3.1 point 1) tells us that the composition of USCCV multifunctions is USCCV. This shows that \( Y \) is a USCCV image of \( X \). If \( Y \) is a USCCV image of \( X \), then,
by Proposition 3.6, $Y$ is the continuous image of $\text{graph}(F)$ which is a perfect preimage of $X$. So $Y$ is a CIPP of $X$. □

Although the definition of CIPP is easier to digest, the use of USCCV multifunctions is a better tool when dealing with compositions. In other words, it is not obvious that the CIPP of a CIPP of a space $X$ is a CIPP of $X$. Nevertheless, thanks to Whyburn’s Proposition 3.1, it is straightforward that the USCCV image of a USCCV image of $X$ is also a USCCV image of $X$.

**Definition 3.8.** Given multifunctions $F_i : X_i \rightarrow Y_i$ with $i \in I$, we say that $F : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$ defined as $F(f) = \prod_{i \in I} F_i(f(i))$ is the product multifunction of $\{F_i : i \in I\}$.

**Proposition 3.9.** The product of USCCV multifunctions is USCCV.

*Proof.* Let $F_i : X_i \rightarrow Y_i$, $i \in I$ be USCCV multifunctions and $F : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$ be such that $F(f) = \prod_{i \in I} F_i(f(i))$. It is immediate to see that $F(f)$ is compact whenever (and only when) for each $i \in I$ we have that $F_i(f(i))$ is compact.

Notice that, given $f \in \prod_{i \in I} X_i$, $f(i) \subseteq \prod_{i \in I} A_i \subseteq \prod_{i \in I} Y_i$ if and only if $F(f(i)) \subseteq A_i$ for all $i \in I$. Equivalently, for every $i \in I$, $f(i) \in (F_i)^{−1}(A_i)$. This means that $F_i^{−1}(\prod_{i \in I} A_i) = \prod_{i \in I} (F_i)^{−1}(A_i)$.

Let $\prod_{i \in I} V_i \subseteq \prod_{i \in I} Y_i$ be a basic open set, i.e. each $V_i \subseteq Y_i$ is open and $V_i = Y_i$ for all but finitely many $i \in I$. Since $F_i$ is USCCV for every $i \in I$, $(F_i)^{−1}(V_i)$ is open. We also have that $(F_i)^{−1}(V_i) = (F_i)^{−1}(Y_i) = X_i$ for all but finitely many $i \in I$. Therefore, $F^{−1}(\prod_{i \in I} V_i) = \prod_{i \in I} (F_i)^{−1}(V_i)$ is (basic) open. This shows that $F$ is upper semi-continuous. □

**Definition 3.10.** A space $X$ is powerfully Lindelöf if $X^\omega$ is Lindelöf.

**Corollary 3.11.** USCCV multifunctions preserve powerful Lindelöfness.

*Proof.* Assume that $X$ is powerfully Lindelöf and let $F : X \rightarrow Y$ be a surjective USCCV multifunction. Using Proposition 3.9 we know that $\prod_{i \in \omega} F : X^\omega \rightarrow Y^\omega$ is USCCV and that, since $X^\omega$ is Lindelöf, we have that $Y^\omega = \prod_{i \in \omega} F(X^\omega)$ is Lindelöf. □

Now we focus on the following classic result.

**Proposition 3.12.** ([Eng89, 3.7.16]) The extension of a perfect map $g : Y \rightarrow X$ over the Stone-Čech compactification of both spaces, i.e. $\hat{g} : \beta Y \rightarrow \beta X$, is such that $\hat{g}(Y) = g(Y) = X$ and $\hat{g}(\beta Y \setminus Y) \subseteq \beta X \setminus X$. Furthermore, if $g$ is surjective, so is $\hat{g}$, which implies that $\hat{g}(\beta Y \setminus Y) = \beta X \setminus X$. Also, by [Eng89, 3.7.6], for any $B \subseteq X$, $g(g^{−1}(B))$ is perfect.

A corollary of the above proposition is that given a map $g : X \rightarrow Y$, there exists a continuous extension $\hat{g} : \beta X \rightarrow \mu Y$ where $\mu Y$ is any compactification of $Y$. We call this the universal property of the Stone-Čech compactification. Our next objective is to generalize the universal property of the Stone-Čech compactification to USCCV multifunctions:
Proposition 3.13. Given a USCCV multifunction $F : X \to A$, and using the definitions and spaces of Proposition 3.6, we have that:

1) If $F$ has compact point-inverses, then $\pi_A$ has compact point-inverses.
2) If $F$ is a closed multifunction, then $\pi_A$ is a closed function.

In particular, $F$ is a perfect multifunction if and only if $\pi_A$ is a perfect function.

Proof.

1) Given $a \in F(X)$, we know that $F^{-1}(a)$ is compact, so $\pi_A^{-1}(a) = \{(x,a) \in \text{graph}(F) : y \in F(x)\} = F^{-1}(a) \times \{a\}$ is also compact.

2) Since we know that $\pi_A$ is continuous, we just need to show that it is closed. Let $C \subseteq \text{graph}(F) : y \in F(x)$ be a closed set. Let $a \notin \pi_A(C)$ and $y \in F^{-1}(a)$. Let $O_y(a) \times U_y(a) \subseteq X \times A \setminus C$ be an open set such that $(y,a) \in O_y(a) \times U_y(a)$. In other words, $O_y(a)$ and $U_y(a)$ are such that for every $x \in O_y$, $\{x\} \times (F(x) \cap U_y) \cap C = \emptyset$.

Let $O(a) = \bigcup_{y \in F^{-1}(a)} O_y(a)$ and $U(a) = \bigcup_{y \in F^{-1}(a)} U_y(a)$. The open sets $O(a)$ and $U(a)$ still satisfy that for every $x \in O$, $\{x\} \times (F(x) \cap U) \cap C = \emptyset$. Furthermore, notice that $F^{-1}(a) \subseteq O(a)$, so for every $x \notin O(a)$, $a \notin F(x)$. Given that $F$ is closed, we have that $F(\overline{X \setminus O})$ is closed, $a \notin F(\overline{X \setminus O})$ and $V(a) = A \setminus F(\overline{X \setminus O})$ is open. This means that $a \in V(a)$. Furthermore, $a \in V(a) \cap U(a)$ and $V(a) \cap U(a) \cap \pi_A(C) = \emptyset$.

To show the final equivalence, points 1) and 2) show that if a USCCV multifunction is perfect, then $\pi_A$ is also perfect. Now, if $\pi_A$ is perfect, using Propositions 3.5 and 3.6, $\pi_X^{-1}$ is a perfect USCCV multifunction and $F = \pi_A \circ \pi_X^{-1}$ is also perfect. \qed

Theorem 3.14. Given a USCCV multifunction $F : X \to K$ with $K$ compact and $F(X)$ dense in $K$, there is a USCCV multifunction $\hat{F} : \mu X \to K$ extending $F$ such that $\hat{F}|X = F$ and $\hat{F}(\mu X) = K$, where $\mu X$ is any compactification of $X$. Furthermore, if $F$ is a perfect USCCV multifunction, then $\hat{F}(\mu X \setminus X) = K \setminus F(X)$.

Proof. Let $F : X \to K$ with $F$, $X$ and $K$ as described in the statement of the Theorem. Using Proposition 3.6, we know that $Z = \text{graph}(F)$ is a perfect preimage of $X$ and $F(X) \subseteq K$ is a continuous image of a projection from $Z$. Denote by $f : Z \to X$ and $g : Z \to K$ the surjective perfect function and the projection mentioned above.

Using the universal property of $\beta Z$ and Propositions 3.5 and 3.12 we have that the continuous extension $\hat{f} : \beta Z \to \mu X$ is a perfect surjective function and that $\hat{f}^{-1} : \mu X \to \beta Z$ is a perfect surjective USCCV multifunction. Furthermore, since $f$ is perfect, $\hat{f}^{-1}(X) = Z$ and $\hat{f}^{-1}(\mu X \setminus X) = \beta Z \setminus Z$. For the continuous extension of $g$, given that $K$ is compact and $g(Z) = F(X)$ is dense in $K$, $\hat{g} : \beta Z \to K$ is a surjective function. Define $\hat{F} : \mu X \to K$ as $\hat{F} = \hat{g} \circ \hat{f}^{-1}$.
Notice that $\hat{F}$ is a composition of two surjective USCCV multifunctions, so it is also surjective USCCV. On the other hand, since we know the exact definition of $Z$, we know that for each $x \in X$,

$$\hat{F}(x) = \hat{g} \circ \hat{f}^{-1}(x) = g(\{x\} \times F(x)) = F(x).$$

This shows that $\hat{F}$ is a USCCV extension of $F$.

Finally, assume that $F$ was perfect to begin with. Using the same definitions as above and Proposition 3.13 we know that $g$ is perfect. Therefore,

$$\hat{F}(\mu X \setminus X) = \hat{g} \circ \hat{f}^{-1}(\mu X \setminus X) = \hat{g}(\beta Z \setminus Z) = K \setminus g(Z) = K \setminus F(X).$$

□

4. Important old results and some new proofs

As seen in the last section, perfect functions behave in a nice way with respect to the Stone-Čech compactification and its remainder. Because of that, we can find multiple results in the literature about this special kind of function. In addition to Proposition 3.12 here we compile some other results that will be useful:

**Proposition 4.1.** [Eng89] The countable product, countable intersection and the finite union of Čech-complete spaces are Čech-complete.

**Proposition 4.2.** [RJ80] $K$-analytic spaces are closed under countable unions, countable intersections, countable products, and continuous maps.

**Proposition 4.3.** [Eng89, 3.7.6] Given a perfect map $g : X \to Y$ and $B \subseteq Y$, the function $g|g^{-1}(B) : g^{-1}(B) \to B$ is perfect.

**Definition 4.4.** An space $X$ is nowhere locally compact (NLC) if every compact subset of $X$ has empty interior.

**Proposition 4.5.** [van Douwen92] If a space $X$ is nowhere locally compact, then it and its Stone-Čech remainder $X^*$ have the same Stone-Čech compactification and $X^*$ is nowhere locally compact.

**Proposition 4.6.** [Arh13] Let $P$ be a perfect property and $X$ a topological space. If the remainder of $X$ in some compactification has property $P$, then the remainder in any compactification has property $P$.

For the analysis of the hierarchies and classes of our interest, the following results will be useful. We will give proofs of Proposition 4.7 and its Corollary 4.9 that are simpler than the ones in [RJ80].

**Proposition 4.7.** [RJ80] Every USCCV image of $\mathbb{P}$ is $K$-analytic.

**Proof.** Let $F : \omega^\omega \to X$ be a surjective USCCV multifunction. Using Proposition 3.6 $\text{graph}(F)$ is a perfect preimage of $\omega^\omega$, a Lindelöf Čech-complete space. Then, $\text{graph}(F)$ is Lindelöf Čech-complete. By the same proposition, $X$ is a continuous image of it and therefore $X$ is $K$-analytic. □
Proposition 4.8. [RJ80] The perfect preimage of a $K$-analytic space is $K$-analytic.

Corollary 4.9. Being $K$-analytic is a perfect property.

Proof. Since $K$-analyticity is preserved by continuous maps, it is certainly preserved by perfect images. Proposition 4.8 gives us the other part. □

Proposition 4.10. [RJ80] Every $K$-analytic metrizable space is analytic.

Proof. Let $F : \omega^\omega \to X$ be a surjective USCCV multifunction with $X$ a separable metrizable space. Using Proposition 3.6, $\text{graph}(F) \subseteq \omega^\omega \times X$ is a perfect preimage of a Lindelöf Čech-complete space inside a separable metrizable space. Then it is a Polish space, since Čech-completeness is a perfect property [Eng89]. Using Proposition 3.6 again, $X$ is a continuous image of the Polish space $\text{graph}(F)$ and therefore $X$ is analytic. □

Corollary 4.11. [RJ80] $K$-analytic spaces are projectively $\sigma$-projective, indeed, projectively analytic.

As is standard, we can use whatever Polish space is convenient. For our purposes, since we will sometimes want to take a metrizable compactification of a separable metrizable space, it will be convenient to assume our analytic, co-analytic, projective, $\sigma$-projective sets are subsets of $[0,1]$, so their metrizable compactification is just their closure there.

Definition 4.12. A Tychonoff space $X$ is an $s$-space if there exists a countable family $\mathcal{S}$ of open subsets of $\mu X$, a compactification of $X$, such that $X = \bigcup_{i \in I} \bigcap \mathcal{S}_i$ such that $\mathcal{S}_i \subseteq \mathcal{S}$ for all $i \in I$.

Definition 4.13. A space is a Lindelöf $p$-space if it is the perfect preimage of a separable metrizable space.

Definition 4.14. A space is a Lindelöf $\Sigma$-space if it is the continuous image of a Lindelöf $p$-space.

Definition 4.15. A space $X$ has countable type if for any compact subspace $K \subseteq X$ there exists $K'$, a compact subspace of $X$, such that $K \subseteq K'$ and there is a countable base for the open sets including $K'$.

Proposition 4.16. [Arh13] Every Lindelöf $p$-space is a Lindelöf $s$-space.

Proposition 4.17. [Arh13] A space $X$ is an $s$-space if and only if any (some) remainder of $X$ is a Lindelöf $\Sigma$-space.

Proposition 4.18. [Arh13] Every perfect $s$-space is a $p$-space.

Proposition 4.19. [HI58] A space is Lindelöf if and only if its Stone-Čech remainder is of countable type.
5. Properties of $\mathcal{K}$

The main theorem of the next section will show that, under $\sigma$-PD, every Menger space in $\mathcal{K}$ is Hurewicz. In this section, we present several properties of $\mathcal{K}$. We start with unions, products and intersections.

**Theorem 5.1.** Given $\xi \in \omega_1$:

1) $\mathcal{K}(\Pi_{\xi}^1) \cup \mathcal{K}(\Sigma_{\xi}^1) \subseteq \mathcal{K}(\Sigma_{\xi+1}^1) \cap \mathcal{K}(\Pi_{\xi+1}^1)$.

2) The classes $\mathcal{K}(\Pi_{\xi+1}^1)$ are closed under countable unions, countable intersections.

3) $\mathcal{K}(\Sigma_{\xi+1}^1)$ is closed under countable unions, countable intersections and countable products.

4) For $\gamma \in \omega_1$ a limit ordinal, $\mathcal{K}(\Sigma_{\gamma}^1)$ is closed under countable unions, finite intersections and finite products.

5) For $\gamma \in \omega_1$ a limit ordinal, $\mathcal{K}(\Pi_{\gamma}^1)$ is closed under finite unions, countable intersections and countable products.

6) Both $\mathcal{K}(\Pi_{\xi}^1)$ and $\mathcal{K}(\Sigma_{\xi}^1)$ are closed under USCCV multifunctions.

**Proof.** Points 1) to 5) follow from the classic results of the $\sigma$-projective hierarchy and Proposition 3.9.

Point 6) follows from their respective definitions and the fact that the composition of USCCV multifunctions is USCCV. □

**Corollary 5.2.** Every space in $\mathcal{K}$ is powerfully Lindelöf.

**Corollary 5.3.** For every $\beta \in \omega_1$, $\mathcal{K}(\Sigma_{\beta+1}^1) = \mathcal{K}(\Pi_{\beta}^1)$.

**Proof.** Both results come from the facts that $\Pi_{\beta}^1 \subseteq \Sigma_{\beta+1}^1$ and that, by definition, every member in $\Sigma_{\beta+1}^1$ is the continuous image of a member in $\Pi_{\beta}^1$. □

In light of the above result, we will only refer to the levels of $\mathcal{K}$ as $\mathcal{K}(\Sigma_{\beta}^1)$, with $\beta \in \omega_1$. The choice of $\Sigma$ over $\Pi$ is not arbitrary, since the $\mathcal{K}(\Pi_{\beta}^1)$ classes do not encompass $\mathcal{K}(\Sigma_{\gamma}^1)$ with $\gamma$ a limit ordinal.

**Corollary 5.4.** For every $\beta \in \omega_1$, being a $\mathcal{K}(\Sigma_{\beta}^1)$ ($\mathcal{K}(\Pi_{\beta}^1)$) space is a perfect property.

**Proof.** Recall that if $f : A \to B$ is a perfect function, then both $f$ and $f^{-1}$ are USCCV multifunctions. By point (6) of Theorem 5.1 every $\mathcal{K}(\Sigma_{\beta}^1)$ ($\mathcal{K}(\Pi_{\beta}^1)$) class is closed under USCCV images. □

It follows immediately that

**Corollary 5.5.** Being in $\mathcal{K}$ is a perfect property.

**Lemma 5.6.** For every $\beta \in \omega_1$, $\mathcal{K}(\Sigma_{\beta}^1)$ is closed hereditary.

**Proof.** Assume that $B \in \mathcal{K}(\Sigma_{\beta}^1)$ and that $F : A \to B$ is a surjective USCCV multifunction with $A \in \Sigma_{\beta}^1$. Also, let $C \subseteq B$ a closed set. On the one hand,
we know that $F^{-1}(C)$ is closed in $A$ and, as shown by classical descriptive set theory, that implies that $F^{-1}(C) \in \Sigma_1^\beta$. On the other hand, we can see $F$ as a function with co-domain $\beta B$. With this point of view, if we let $D$ be the (compact) closure of $C$ in $\beta B$, we have that $D \cap B = C$ and that, for every $a \in F^{-1}(C)$, $F(a) \cap D = F(a) \cap C$ is compact. Then the function $G : F^{-1}(C) \to C$ defined as $G(a) = F(a) \cap C$ is a USCCV surjective multifunction. This shows that $C \in \mathcal{K}(\Sigma_1^\beta)$. \hfill \Box

Now we can show that $\mathcal{K}$ is projectively $\sigma$-projective.

**Theorem 5.7.** Let $Z \in \mathcal{K}$ and let $f : Z \to Y \subseteq \mathbb{R}$ be a surjective continuous function. Then $Y$ is a $\sigma$-projective set.

**Proof.** Assume that $Z \in \mathcal{K}(\Pi^1_\gamma)$. Let $X$ be a $\Pi^1_\gamma$ set and $G : X \to Z$ a USCCV surjective multifunction. Let $F = f \circ G$. Notice that $F : X \to Y$ is a USCCV surjective multifunction.

Without loss of generality, assume $X, Y \subseteq [0,1]$. This implies that $\overline{X}$ and $\overline{Y}$ are metric compactification of $X$ and $Y$. By Theorem 5.1 we can extend $F$ to a surjective multifunction $\hat{F} : \overline{X} \to \overline{Y}$.

Using Proposition 5.6 notice that graph($\hat{F}$) $\subseteq [0,1]^2$ is a separable metrizable space. Furthermore, graph($\hat{F}$) maps perfectly onto $\overline{X}$ and maps continuously onto $\overline{Y}$. By Theorem 5.1 point 6), $\pi_{\overline{X}}^{-1}(X)$ is $\Pi^1_\gamma$. This means that $\pi_{\overline{Y}}(\pi_{\overline{X}}^{-1}(X)) = Y$ is $\Sigma^1_{1+\gamma}$.

The result for $\Sigma^1_{\gamma+1}$ follows immediately from the $\Pi^1_\gamma$ case and the proof for $\Sigma^1_\gamma$ when $\gamma$ is a limit follows from the $\Sigma^1_{\delta+1}$ results. \hfill \Box

**Corollary 5.8.** If $Y \in \mathcal{K}(\Sigma^1_\gamma)$ is a metrizable space then $Y \in \Sigma^1_\gamma$.

**Proof.** $Y$ is separable because Lindelöf metrizable spaces are separable. The proof is analogous to Theorem 5.7 but assuming that $Y \subseteq [0,1]^\omega$. (Separable metrizable spaces need not embed in $[0,1]$ but do embed in $[0,1]^\omega$.)

If one does not like pretending that $[0,1]$ and $[0,1]^\omega$ are the same, one can use the fact (see e.g. [Kec95]) that every separable metrizable space is a perfect image of a subspace of the Cantor space and hence of $\mathbb{R}$. (Incidentally, in [Tal20] the second author followed [RJ80] a little too closely, unnecessarily privileging $\mathbb{R}^\omega$ over $\mathbb{R}$ in the table of equivalents on page 11.)

**Corollary 5.9.** For every $\alpha < \beta \in \omega_1$, there is a space $B$ such that $B$ is $\mathcal{K}(\Sigma^1_\beta)$ but is not $\mathcal{K}(\Sigma^1_\alpha)$.

**Proof.** It is well-known that the $\sigma$-projective sets form a hierarchy of length $\omega_1$ [Mos99]. Let $B \in \Sigma^1_\beta \setminus \Sigma^1_\alpha$. We know that $B$ is $\mathcal{K}(\Sigma^1_\beta)$. Since $B \notin \Sigma^1_\alpha$, by 5.8 $B \notin \mathcal{K}(\Sigma^1_\alpha)$.

Although we presented a simpler proof for Proposition 4.10, the technique presented in Rogers and Jayne’s Theorem 5.5.1 of [RJ80] gives other characterizations of metrizability for $K$-analytic spaces. These can be generalized
to the whole $\mathcal{K}$ hierarchy. Analyzing their proof carefully, it can be generalized to prove the following assertion:

**Theorem 5.10.** Given a class of topological spaces $\Gamma$ and a pair of topological properties $P$ and $Q$ such that

1. Every member of $\Gamma$ with property $Q$ is a USCCV image of a space with property $P$.
2. For $X \in \Gamma$ with property $Q$, $X \times X$ is Lindelöf and $X$ has a $G_\delta$ diagonal.
3. $\Gamma$, property $Q$ and property $P$ are closed-hereditary.
4. Every countable product of spaces with property $P$ has property $P$.
5. Finite unions of spaces with property $P$ have property $P$.

Then every member of $\Gamma$ with property $Q$ is a continuous image of a space with property $P$.

**Proof.** Let $P$, $Q$ and $\Gamma$ be as described in the Theorem. Furthermore, let $X \in \Gamma$ have property $Q$. We shall use $\Delta$ to denote the diagonal of $X$, $\Delta = \{(x, x) : x \in X\}$. Then since $X \times X$ is Lindelöf and $X$ has a $G_\delta$ diagonal, $(X \times X) \setminus \Delta$ is an $F_\sigma$ and, therefore, also Lindelöf.

As a consequence of $(X \times X) \setminus \Delta$ being Lindelöf, there exist open $G^0_i, G^1_i \subseteq X$, for $i \in \omega$, such that $(X \times X) \setminus \Delta = \bigcup_{i \in \omega} G^0_i \times G^1_i$. For each $(j, i) \in \{(0, 1) \times \omega\}$, let $A^j_i = X \setminus G^j_i$. This sequence of pairs of sets has some interesting properties. First, since all of these sets are closed, by hypothesis, each $A^j_i \in \Gamma$ and has property $Q$. Also, since $G^0_i \times G^1_i \cap \Delta = \emptyset$, $G^0_i \cap G^1_i = \emptyset$ and $A^0_i \cup A^1_i = X$. Finally, given $x, y \in X$ such that $x \neq y$, there is an $i(x, y)$ such that $(x, y) \in G^0_{i(x,y)} \times G^1_{i(x,y)}$. Notice that $x \in G^0_{i(x,y)}$, $x \notin G^1_{i(x,y)}$, $y \in G^1_{i(x,y)}$ and $y \notin G^0_{i(x,y)}$.

By the first point in the hypothesis, for each $(j, i) \in \{(0, 1) \times \omega\}$, there is a surjective USCCV multifunction $L^j_i : B^j_i \to A^j_i$ such that $B^j_i$ has property $P$. By the fifth and fourth points in our hypothesis, we have that

$$\prod_{i \in \omega} \left((B^0_i \times \{0\}) \cup (B^1_i \times \{1\})\right)$$

has property $P$. Define, for each $i \in \omega$, $\alpha(i) = (x^\alpha_i, b^\alpha_i)$ and $N_i = (B^0_i \times \{0\}) \cup (B^1_i \times \{1\})$.

Since each $L^j_i$ is USCCV and surjective over $A^j_i$, Proposition 3.9 implies that the multifunction $M : \prod_{i \in \omega} N_i \to X^\omega$ with $M(\alpha) = \prod_{i \in \omega} L^\alpha_i (B^\alpha_i)$ is surjective USCCV. Notice that the subspace

$$\tilde{X} = \{\xi \in X^\omega : (\forall i, n \in \omega)(\xi(i) = \xi(n))\}$$

inside $X^\omega$ is homeomorphic to $X$ with its induced topology. Furthermore, $\tilde{X}$ is closed in $X^\omega$. To see this, define the open set $D^i_{i,j} = X$ when $i \neq j$
and \( D^x_{i,j} = X \setminus \{x\} \), for each \( i, j \in \omega \) and \( x \in X \). We can see that \( X^\omega \setminus \vec{X} = \bigcup_{x \in X} \bigcup_{i \in \omega} \prod_{j \in \omega} D^x_{i,j} \). This implies that \( C = M^{-1}(\vec{X}) \) is closed in \( \prod_{i \in \omega} N_i \), so it has property \( P \).

To finish the proof, we will show that \( M \upharpoonright C : C \to \vec{X} \) is single-valued. This will show that \( X \) is homeomorphic to a continuous image of a space with property \( P \). An element of \( \vec{X} \) will be denoted by \( \vec{x} \).

Assume that there is an \( \alpha \in C \) and \( x, y \in X \) such that \( \vec{x}, \vec{y} \in M(\alpha) \) and \( x \neq y \). In particular, we know that \( x, y \in L^\alpha_{i(x,y)}(B^\alpha_{i(x,y)}) \subseteq A^\alpha_{i(x,y)} \). But, as mentioned when we defined \( i_{(x,y)} \), it is not possible for both \( x \) and \( y \) to be in \( A^\alpha_{i(x,y)} \). Therefore, for each \( \alpha \in C \), \( |M(\alpha) \cap X^\omega| = 1 \). This shows that \( M \upharpoonright C \) is a continuous surjective single-valued function. \( \square \)

As we can see, when following the [RJ80] proof one ends up with a single-valued function to the desired space, \( X \), from the product of countably many spaces that were, originally, the domain of countably many USCCV multifunctions to subspaces of \( X \).

Here is an easy consequence of Theorem 5.10:

**Corollary 5.11.** Every Lindelöf \( \Sigma \)-space with a \( G_\delta \) diagonal is a continuous image of a separable metrizable space.

**Proof.** Let \( \Gamma \) be the class of all topological spaces. Let property \( Q \) be “being a Lindelöf \( \Sigma \)-space with a \( G_\delta \) diagonal”. Let property \( P \) be “being a separable metrizable space”. \( \square \)

Jayne and Rogers use \( \Gamma \) as the \( K \)-analytic spaces, \( Q \) as “separable metrizable space” and \( P \) as “analytic set” (they use \( \omega^\omega \), but analytic is enough). With this strategy, they showed that for \( K \)-analytic spaces “analytic space” is equivalent to “spaces such that \( (X \times X) \setminus \Delta \) is Lindelöf”. This partially generalizes if we replace “analytic” by “\( \Sigma^1_\beta \)”.

**Theorem 5.12.** Given \( X \in \mathcal{K}(\Sigma^1_\beta) \), if \( X \) has a \( G_\delta \) diagonal then \( X \) is a continuous image of a \( \Sigma^1_\beta \) set.

**Proof.** Since \( X \) is powerfully Lindelöf and \( X \) has a \( G_\delta \) diagonal, then \( (X \times X) \setminus \Delta \) is Lindelöf. Using Theorem 5.10 with \( \Gamma = \mathcal{K}(\Sigma^1_\beta) \), \( P \) the property “being a \( \Sigma^1_\beta \) set” and \( Q \) the (trivial) property “\( (X \times X) \setminus \Delta \) is Lindelöf”; we have that every space \( X \) in \( \mathcal{K}(\Sigma^1_\beta) \) such that \( (X \times X) \setminus \Delta \) is Lindelöf is a continuous image of a \( \Sigma^1_\beta \) set. This finishes the proof. \( \square \)

Theorem 5.10 gives us a new proof of a result that we proved earlier.

**Corollary 5.8.** If \( X \in \mathcal{K}(\Sigma^1_\beta) \) is a metrizable space, then \( X \in \Sigma^1_\beta \).
Proof. As before, we note $X$ is separable. Using Theorem 5.10 with $\Gamma = \mathcal{K}(\Sigma^1_\delta)$, $P$ the property “being a $\Sigma^1_\delta$ set” and $Q$ the property “$X$ is a separable metrizable space”; we have that every space $X$ in $\mathcal{K}(\Sigma^1_\delta)$ that is metrizable is a continuous image of a $\Sigma^1_\delta$ set. This implies that $X$ is a $\Sigma^1_\delta$ set. \hfill\Box

Note that point 2) in Theorem 5.10 creates a path for constructing single-valued functions from USCCV multifunctions even when not working in the separable metrizable space context (but using a property $Q$ of interest).

Finally, we would like to know whether the levels of the hierarchy are preserved by preimages.

**Theorem 5.13.** Let $Y$ be a $\mathcal{K}(\Sigma^1_\xi)$ space, $X \subseteq G$ a $\mathcal{K}(\Sigma^1_\xi)$ space and $F : Y \to G$ a surjective USCCV multifunction such that $\hat{F} \circ \hat{F}^{-1}(X) = X$ for $\hat{F} : \mu Y \to \beta G$ a USCCV extension of $F$. Then $F^{-1}(X)$ is $\mathcal{K}(\Sigma^1_\xi)$.

**Proof.** Let $Y$, $G$, $X$ and $F$ be as described in the theorem. Consider the function given by Theorem 3.14, $\hat{F} : \mu Y \to \beta G$. Since $\hat{F}$ is a perfect USCCV multifunction, by Proposition 3.5 $\hat{F}^{-1}$ is also perfect. Since $\hat{F} \circ \hat{F}^{-1}(X) = X$, $\hat{F}^{-1}|X : X \to \hat{F}^{-1}(X)$ is a surjective perfect function. By Theorem 5.1 point 6), $\hat{F}^{-1}(X)$ is $\mathcal{K}(\Sigma^1_\xi)$. Since $Y$ is $\mathcal{K}(\Sigma^1_\xi)$, using a different point of Theorem 5.1 we have,

$$F^{-1}(X) = \hat{F}^{-1}(X) \cap Y$$

is $\mathcal{K}(\Sigma^1_\xi)$. \hfill\Box

**Theorem 5.14.** Let $X$ be a $\mathcal{K}(\Sigma^1_\xi)$ space. Let $f : Y \to X$ be a surjective continuous function. Then $Y$ is $\mathcal{K}(\Sigma^1_\xi)$.

**Proof.** Let $\hat{f} : \beta Y \to \beta X$ the continuous extension of $f$ given by the Stone-Čech compactification. We know that $f \circ \hat{f}^{-1}(X) = X$. Using Theorem 5.13 we conclude that $Y$ is $\mathcal{K}(\Sigma^1_\xi)$.

\hfill\Box

6. APPLICATIONS TO SELECTION PRINCIPLES

Now we shall consider Selection Principles for $\mathcal{K}$ and related classes.

We don’t need the following alternative definition of “Hurewicz”, but it may be of interest to those unfamiliar with Hurewicz spaces.

**Proposition 6.1.** [Tal11] A topological space $X$ is Hurewicz if and only if, given a Čech-complete $G$ such that $X \subseteq G$, there exists a $\sigma$-compact space $F$ such that $X \subseteq F \subseteq G$.

**Theorem 6.2.** $\sigma$-PD implies every Menger projectively $\sigma$-projective space is Hurewicz. In ZFC, Menger projectively analytic spaces are Hurewicz.

**Proof.** This follows from

**Proposition 6.3.** [Tal13] Lindelöf projectively $\sigma$-compact spaces are Hurewicz.
Corollary 6.4. \( \sigma\text{-PD} \) implies Menger spaces in \( \mathcal{K} \) are Hurewicz.

As mentioned in the introduction, this conclusion cannot easily be improved since there is a \( K \)-analytic space that is Hurewicz but not \( \sigma \)-compact. Also, \( \mathcal{K} \) is not the same as the class of all projectively \( \sigma \)-projective spaces.

Example 8.7. A Hurewicz \( K \)-analytic space that is not \( \sigma \)-compact: Okunev’s space.

Example 8.9. There are projectively \( \sigma \)-projective spaces that are not in \( \mathcal{K} \): Peng’s L-space.

Nevertheless, demanding a little more structure on the space lets us get a stronger conclusion. The following results come from [KKL11] or are direct generalizations of results of Rogers and Jayne in [JR79]. For the following proofs it will be more convenient to use \( \omega^\omega \) rather than \( \mathbb{R} \) or \([0,1]\).

Proposition 6.5. [KKL11] For every perfect Lindelöf space \( X \) that is not \( \sigma \)-compact, there exists a non-empty closed subset \( N \subseteq X \) such that for every open set \( U \) of \( X \), \( U \cap N \) is not \( \sigma \)-compact.

Lemma 6.6. For every perfect Lindelöf space \( X \) that is not \( \sigma \)-compact, there exist \( L_n, n \in \omega \), pairwise disjoint closed sets that are not \( \sigma \)-compact and such that \( \bigcup_{n \in \omega} L_n \) is closed.

Proof. Let \( X \) be a perfect topological Lindelöf space that is not \( \sigma \)-compact and \( N \) be the closed subspace obtained from Proposition 6.5. We will work with the induced topology on \( N \) from this point onward.

Since \( N \) is not compact but is Lindelöf, it is not countably compact, so there is a sequence of countably many decreasing closed sets, \( F_n \subseteq N \), such that \( \bigcap_{n \in \omega} F_n = \emptyset \) and \( F_0 = N \). Since \( X \) and \( N \) are both perfect, for each \( n \in \omega \) there exist countably many decreasing open sets \( O_{n}^{m} \subseteq N \), such that \( \bigcap_{m \in \omega} O_{n}^{m} = F_n \) and such that \( O_{n}^{m+1} \subseteq O_{n}^{m+1} \subseteq O_{n}^{m} \).

Let \( E^{(i)} = \bigcap_{n, m \leq i} O_{n}^{m} \) and \( U^{(i)} = \bigcap_{n, m \leq i} O_{n}^{m} \). For each \( i \in \omega \), \( E^{(i)} \) is closed and \( U^{(i)} \) is open. Furthermore, \( U^{(i+1)} \subseteq E^{(i+1)} \subseteq U^{(i)} \) and

\[
\bigcap_{i \in \omega} U^{(i)} = \bigcap_{i \in \omega} E^{(i)} = \bigcup_{n \in \omega} F_n = \emptyset.
\]

By definition, \( O_{0}^{0} = O_{0}^{m} = E^{(0)} = U^{(0)} = N \). Therefore, there must be an \( i_1 \) such that \( N \setminus U^{(i_1)} \) is not \( \sigma \)-compact. Otherwise,

\[
N = N \setminus \emptyset = N \setminus \bigcap_{i \in \omega} U^{(i)} = \bigcup_{i \in \omega} N \setminus U^{(i)}
\]

would be \( \sigma \)-compact.

Let \( i_0 = -1 \) and for all \( n \in \omega \), \( n \geq 1 \), define \( i_n \) such that \( E^{(i_n+1)} \setminus U^{(i_n+1)} \) is not \( \sigma \)-compact. This number always exists. To see this, we can use the same
argument we used for $N$ with an induction. To simplify notation, we will write $E^{(n)}$ instead of $E^{(i_n)}$ (we substitute for $E^{(i)}$ the subsequence $E^{(i_n)}$).

Now we can define our desired closed sets. For each $n \in \omega$, let $L_n = E^{(i_n+1)} \setminus U^{(i_n+1)}$. We know that each $L_n$ is closed and not $\sigma$-compact. Furthermore, since $E^{(i+1)} \subseteq U^{(i)}$, we know that they are pairwise disjoint. To finish the proof, let $x \in \bigcap_{n \in \omega} L_n$. Since $\bigcap_{i \in \omega} E^{(i)} = \emptyset$ and the sets $E^{(i)}$ are decreasing, there is an $m_x \in \omega$ such that for all $n \geq m_x$, $x \notin E^{(n)} \subseteq E^{(m_x)}$.

Let $G_x$ be an open neighborhood of $x$ that is disjoint from $E^{(m_x)}$. Notice that for all $n \geq m_x$, $G_x \cap L_n = \emptyset$. Then,

$$G_x \cap \bigcup_{n \in \omega} L_n = G_x \cap \bigcup_{n < m_x} L_n.$$  

This means that $x \in \bigcap_{n < m_x} L_n = \bigcap_{n < m_x} L_n$. So $x \in \bigcup_{n \in \omega} L_n$ and $\bigcup_{n \in \omega} L_n$ is closed.

We need some new notation before continuing.

**Definition 6.7.** Given $N \subseteq Y$, $F : A \rightarrow Y$ a surjective multifunction, and $D \subseteq A$, let $D_N = \{a \in A : F(a) \cap N \neq \emptyset\}$. Let $F_N : D_N \rightarrow N$ be defined by $F_N(a) = F(a) \cap N$.

Recall that for $s \in \omega^{<\omega}$,

$$[s] = \{a \in \omega^\omega : s \subseteq a\}$$

is (basic) open and for $T \subseteq \omega^{<\omega}$,

$$[T] = \{a \in \omega^\omega : (\forall n \in \omega)(a \upharpoonright n \in T)\}$$

denotes the set of all branches of $T$ and is closed.

**Definition 6.8.** Given $s \in \omega^{<\omega}$, $T \subseteq \omega^{<\omega}$, and $A \subseteq \omega^\omega$ we define $[s]_A = [s] \cap A$ and $[T]_A = [T] \cap A$.

**Definition 6.9.** Given $T \subseteq \omega^{<\omega}$ and $s \in \omega^{<\omega}$ we define $(T)_s$ to be the set of all successor nodes of $s$ that are in $T$.

The sets $[s]_A$ and $[T]_A$ define the induced topology on $A$ (see chapter I.2.B of [Kec95]).

**Definition 6.10.** We say that a family $\{N_s : s \in \omega^{<\omega}\}$ is a disjoint Souslin scheme if

1) If $s \subseteq t$, $N_t \subseteq N_s$.
2) If $s \not\subseteq t$ and $t \not\subseteq s$, $N_s \cap N_t = \emptyset$.

Furthermore, we say that a disjoint Souslin scheme is closed if

3) For all $s \in \omega^{<\omega}$, $N_s \neq \emptyset$ and $N_s$ is closed.
4) For each $n \in \omega$, $\bigcup_{s \in \omega^\omega} N_s$ is closed.

**Definition 6.11.** Given $A \subseteq \omega^\omega$, the generating tree of $A$ is

$$T_A = \{s \in \omega^{<\omega} : (\exists b \in A)(\exists n \in \omega)(b \upharpoonright n = s)\}.$$
Given $U \subseteq \omega^{<\omega}$, the tree generated by $U$ is
\[ T_U = \{ s \in \omega^{<\omega} : (\exists t \in U)(\exists n < |t|)(t|n = s) \}. \]

Notice that if $A \subseteq B \subseteq \omega^\omega$ then $[T_A]_B = A$ if and only if $A$ is closed in $B$.

**Lemma 6.12.** Assume that $X$ is a perfect Lindelöf $\Sigma$-space. If $X$ is not $\sigma$-compact, then there is a disjoint closed Souslin scheme such that for every $s \in \omega^{<\omega}$, $N_s$ is a non-empty closed non-$\sigma$-compact subset of $N$.

**Proof.** Let $X$ be a Lindelöf $\Sigma$-space, let $B \subseteq \omega^\omega$ and let $F : B \to X$ a surjective USCCV multifunction. Let $N$ be as in Proposition 6.5. For each $s \in \omega^{<\omega}$ we will recursively define $N_s \subseteq N$. First, let $N_0 = N$. Now, assume that we have defined $N_s$. Let $L_i, i \in \omega$ be the countably many closed sets given by Lemma 6.6 applied to $N_s$. For each $i \in \omega$, let $N_s^{-i} = L_i$.

To finish the proof, we sketch the induction argument used to show that for each $n \in \omega$, $\bigcup_{s \in \omega^n} N_s$ is closed. First, $N_0 = N$. Assuming the result for $n$, if $|s| = n$, then, by the properties of $L_i$, $\bigcup_{i \in \omega} N_s^{-i} \subseteq N_s$ is closed. Then, using the same properties, $\bigcup_{s \in \omega^n} \bigcup_{i \in \omega} N_s^{-i} = \bigcup_{i \in \omega^{n+1}} N_i$ is closed. \(\square\)

It is important to remark that, although we were able to create inside $X$ the sets $N_s$ for each $s \in \omega^{<\omega}$, that does not mean that for every $r \in \omega^\omega$ we have that $\bigcap_{s \in r} N_s \neq \emptyset$. This last statement would create a copy of $\omega^\omega$ inside $X$, which is not true in general.

**Lemma 6.13.** Assume that $X$ is a perfect Lindelöf $\Sigma$-space and let $B \subseteq \omega^\omega$ be such that $X$ is the USCCV image of $B$ by $F$. If $X$ is not $\sigma$-compact, then there is a non-$\sigma$-compact closed subset $N \subseteq X$ and a closed set $A \subseteq B$ such that $F_N|A$ is surjective and the USCCV image of every non-empty open subset of $A$ is non-$\sigma$-compact.

**Proof.** Let $X$ be a Lindelöf $\Sigma$-space, let $B \subseteq \omega^\omega$ and let $F : B \to X$ be a surjective USCCV multifunction. Let $N \subseteq X$ be the closed non-$\sigma$-compact subspace coming from Proposition 6.5. Since USCCV multifunctions preserve compactness, $A = F^{-1}(N) \subseteq B$ is a closed subset of $B$ that is not $\sigma$-compact. Furthermore, there are only countably many basic open sets $O \subseteq A$ such that $F(O) \cap N$ is $\sigma$-compact. If $O$ is the collection of all basic open subsets of $A$ such that their images are $\sigma$-compact in $N$, then $A \setminus \bigcup O$ and $N \setminus F(\bigcup O)$ are non-empty and non-$\sigma$-compact. With this, we can assume that for any open subset of $A$, the image of that open subset is not $\sigma$-compact. Finally, let $F_N : A \to N$ be such that $F_N(b) = F(b) \cap N$ for all $b \in A$. \(\square\)

**Lemma 6.14.** Assume that $X$ is a perfect Lindelöf $\Sigma$-space and let $B \subseteq \omega^\omega$ be such that $X$ is the USCCV image of $B$. If $X$ is not $\sigma$-compact, then there is a non-$\sigma$-compact closed subset $A$ of $B$, a $g : \omega^{<\omega} \to T_A$, and a disjoint closed Souslin scheme such that:

1) For every $s \in \omega^{<\omega}$, $(T_{g(\omega^{<\omega})})_{g(s)}$ is infinite and $F_N([g(s)]_A) \cap N_s$ is a closed non-empty non-$\sigma$-compact subset of $N$. 
2) $[T_{g(\omega^\omega)}]_A$ is a closed non-empty non-$\sigma$-compact subset of $A$.
3) If $b \in [T_{g(\omega^\omega)}]_A$ and $g(s) \subseteq b$ then $F_N(b) \cap N_s \neq \emptyset$.

Proof. Let $X$ be a Lindelöf $\Sigma$-space, let $B \subseteq \omega^\omega$ and let $F : B \to X$ be a surjective USCCV multifunction. Furthermore, let $N$ and $A$ be as in Lemma 6.12.

We know that all compact sets of $\omega^\omega$ are the branches of finite branching trees [Kec93]. Since no open subset of $A$ is $\sigma$-compact, we know that for any $s \in A$ there is a $t \in A$, with $s \subseteq t$, such that $(T_A)_t = \{i \in \omega : t^\frown i \in A\}$ is infinite.

For each $s \in \omega^{<\omega}$ we will recursively define $N_s \subseteq N$ in a similar way to what we did in the proof of Lemma 6.12. As before, let $N_0 = N$. Now, assume that we have defined $N_s$ and $g(s) \in A$ for $s \in \omega^{<\omega}$. Let $L_i$ be the countably many closed sets given by Lemma 6.6. We will find $\{t_i : i \in \omega\} \subseteq (T_A)_{g(s)}$ such that if $i \neq j$, $t_i \neq t_j$, and such that $F_N([t_i]_A) \cap L_i$ is non-empty and non-$\sigma$-compact.

For each $i \in \omega$, $L_i = L_i \cap F_N([g(s)]) = L_i \cap \bigcup_{t \in (T_A)_{g(s)}} F_N([t]_A) = \bigcup_{t \in (T_A)_{g(s)}} F_N([t]_A) \cap L_i$. If $F_N([t]_A) \cap L_i$ were $\sigma$-compact for all $t \in (T_A)_{g(s)}$, then $L_i$ would be $\sigma$-compact. On the other hand, if there are only finitely many $t \in (T_A)_{g(s)}$ such that $F_N([t]_A) \cap L_i$ is non-empty and non-$\sigma$-compact, then $F_N([g(s)]) \setminus \bigcup_{i \in \omega} L_i = \bigcup_{t \in (T_A)_{g(s)}} F_N([t]_A) \setminus \bigcup_{i \in \omega} L_i$ is non-empty and non-$\sigma$-compact. So we can use Lemma 6.6 again. In this manner we can find the desired $\{t_i : i \in \omega\} \subseteq (T_A)_{g(s)}$. Furthermore, notice that we will still have the property of the union of closed sets being closed, since the proof only requires that the $L_i$ form a countable collection of closed sets with empty intersection.

For each $i \in \omega$, let $g(s^\frown i) \in T_A$ be such that $t_i \subseteq g(s^\frown i)$ and $(T_A)_{g(s^\frown i)}$ is infinite. Finally, define $N_{s^\frown i} = F_N([t_i]_A) \cap L_i$ to be the closed set given by Proposition 6.5. Remember that $N_{s^\frown i}$ is obtained from $F_N([t_i]_A) \cap L_i$ by subtracting a $\sigma$-compact open set, and $F_N([s]_A)$ is non-$\sigma$-compact for all $s \in A$. With this, elements from the proof of Lemma 6.12 and the above construction, we can verify that the sets $N_s$ form a disjoint closed Souslin scheme that satisfies points 1) and 3). To finish the proof, we will show point 2) by contradiction.

Assume that $[T_{g(\omega^\omega)}]_A$ is $\sigma$-compact. Then there is an $f \in \omega^\omega$ such that for all $b \in [T_{g(\omega^\omega)}]_A$ there is an $n_b \in \omega^\omega$ such that $b(m) < f(m)$ for all $m \geq n_b$. Take $a \in [T_{g(\omega^\omega)}]_A$ such that $a(n_b) < f(m)$; by construction, there is an $s_a \in \omega^{<\omega}$ such that $a \subseteq s_a$ and $(T_{g(\omega^\omega)})_{s_a}$ is infinite. This means that we can find a $b_f \in [T_{g(\omega^\omega)}]_A$ such that $b_f(s_a) + 1 \in (T_{g(\omega^\omega)})_{s_a}$ and $b_f(s_{a^\frown i}) > f(s_{a^\frown i})$. This contradicts the assumption of $[T_{g(\omega^\omega)}]_A$ being $\sigma$-compact.
Theorem 6.15. Assume that $X$ is a perfect Lindelöf $\Sigma$-space and let $B \subseteq \omega^\omega$ be such that $X$ is the USCCV image of $B$. If $X$ is not $\sigma$-compact, then there is a non-$\sigma$-compact closed subset of $X$ that is the perfect preimage of a closed subset of $B$.

Proof. Let $X$ be a Lindelöf $\Sigma$-space, let $B \subseteq \omega^\omega$ and let $F : B \to X$ be a surjective USCCV multifunction. Let $N, g, A$ and $N_s$ for each $s \in \omega^\omega$ be the sets and functions coming from Lemma 6.14.

For each $b \in [T_g(\omega^\omega)]_A$, let $H(b) = \bigcap_{g(s) \subseteq b} N_s \cap F_N(b)$. Since $N_s$ is closed, \{ $N_s : g(s) \subseteq b$ \} is nested, and $F_N(b)$ is compact, we know that $H(b)$ is a non-empty compact set. Also, notice that

$$N' = \{ H(b) : b \in [T_g(\omega^\omega)]_A \} = \bigcap_{n \in \omega} \bigcup_{s \in \omega^n} N_s$$

is closed. Then, to finish the proof, we need to show that

$$H : [T_g(\omega^\omega)]_A \to N'$$

is a USCCV closed multifunction and that $H^{-1}$ is single-valued. With this, we will have that $H^{-1}$ is a perfect function.

First, if $a, b \in [T_g(\omega^\omega)]_A$ and $a \neq b$, we have that $H(a) \cap H(b) = \emptyset$. This shows that $H^{-1}$ is single-valued. Also, if $J \subseteq [T_g(\omega^\omega)]_A$ is closed, then $J = [V][T_g(\omega^\omega)]_A$ for some subtree $V$ of $\omega^\omega$. By Lemma 6.12 we know that $\bigcup_{s \in \omega^n \cap V} N_s$ is closed for all $n \in \omega$. Therefore,

$$H(J) = H([V][T_g(\omega^\omega)]_A) = \bigcap_{n \in \omega} \bigcup_{s \in \omega^n \cap V} N_s$$

is closed. This shows that $H$ is a closed multifunction.

We know that $H(b)$ is compact for each $b \in [T_g(\omega^\omega)]_A$. Let $U \subseteq [T_g(\omega^\omega)]_A$ be open and let $H(b) \subseteq U$ for some $b \in [T_g(\omega^\omega)]_A$. Notice that, since $\bigcap_{g(s) \subseteq b} N_s \cap F_N(b) = H(b)$, there must be a $s_{b,U} \in \omega^\omega$ such that $g(s_{b,U}) \subseteq b$ and $N_{s_{b,U}} \subseteq U$. Then,

$$b \in [g(s_{b,U})][T_g(\omega^\omega)]_A \subseteq H^{-1}_U(N_{s_{b,U}}) \subseteq H^{-1}_U(U).$$

This shows that $H^{-1}_U(U)$ is open and so $H$ is USCCV. \qed

[TT17] proved that $K$-analytic perfect Menger spaces are $\sigma$-compact; we will extend that result here.

Theorem 6.16. (Hurewicz Dichotomy) $\sigma$-PD implies that every perfect space in $\mathcal{M}$ that is not $\sigma$-compact has a closed subset that is a perfect preimage of $\omega^\omega$.

Proof. Let $X$ be a perfect space in $\mathcal{M}$ that is not $\sigma$-compact, $B \subseteq \omega^\omega$ a $\sigma$-projective set, and $F : B \to X$ a USCCV surjective multifunction. By Theorem 6.15, we can find a non-$\sigma$-compact closed set $N' \subseteq X$ and $A \subseteq B$ such that there is a surjective perfect function $f : N' \to A$. 

□
Since $\sigma$-projectivity is a closed hereditary property, we know that $A$ is $\sigma$-projective. Then, using $\sigma$-PD, since $A$ is not $\sigma$-compact there is a closed subset of $A$ homeomorphic to $\omega^\omega$ [Kec77]. Call this closed subset $W$. Notice that $f^{-1}(W)$ is a closed subset of $N'$ and $X$ and that $f|f^{-1}(W)$ is perfect. Therefore, $f^{-1}(W) \subseteq X$ is a closed subset that is a perfect preimage of $\omega^\omega$.

**Corollary 6.17.** $\sigma$-PD implies that every perfect Menger space in $\mathcal{X}$ is $\sigma$-compact.

**Proof.** By Theorem 6.15, if $X \in \mathcal{X}$ is perfect and not $\sigma$-compact, then $X$ has a closed set, say $C$, that is the perfect preimage of a non-$\sigma$-compact $\sigma$-projective space, say $D$. If $X$ is Menger, then $C$ and $D$ are also Menger. But $D$ contradicts $\sigma$-PD (since that implies that every Menger $\sigma$-projective set is $\sigma$-compact).

In [KL11] the authors prove a version of Hurewicz Dichotomy for hereditarily Lindelöf $K$-analytic subspaces of hereditarily paracompact spaces. They call their Theorem 3.13 the Calbrix-Hurewicz Theorem. We expect we can prove a $\sigma$-PD version of that theorem, but have not tried.

**Definition 6.18.** A space $X$ is cosmic if it is the continuous image of a separable metrizable space.

Arhangel’skiı-Calbrix [AC99] proved

**Theorem 6.19.** Every Menger cosmic analytic space is $\sigma$-compact.

We can now see that the key fact is that cosmic spaces are hereditarily Lindelöf and hence perfect. We saw earlier (Theorem 5.12) that Lindelöf $\Sigma$-spaces with $G_\delta$ diagonals are cosmic. Thus

**Corollary 6.20.** $K$-analytic Menger spaces with $G_\delta$ diagonals are $\sigma$-compact [Tal20] and $\sigma$-PD implies Menger members of $\mathcal{X}$ with $G_\delta$ diagonals are $\sigma$-compact.

**Proof.** By Theorem 6.16 and Corollary 6.17.

7. Variations on $\mathcal{X}$

In this section, we discuss subclasses of $\mathcal{X}$ and possible enlargements of it. For example, let $\mathcal{C}$ be the class of continuous images of $\sigma$-projective sets. Not surprisingly, $\mathcal{C}$ is a proper subclass of $\mathcal{X}$—see Example 8.1. We could also consider $\mathcal{P}$; the class of perfect preimages of $\sigma$-projective sets. Also not surprisingly, $\mathcal{P}$ is a proper subclass of $\mathcal{X}$.

**Theorem 7.1.** $\sigma$-PD implies Menger members of $\mathcal{C}$ are $\sigma$-compact.

**Proof.** Members of $\mathcal{C}$ are cosmic, hence perfect, so 6.17 applies.

**Theorem 7.2.** Menger perfect preimages of analytic sets are $\sigma$-compact. $\sigma$-PD implies Menger members of $\mathcal{P}$ are $\sigma$-compact.
Proof. Let $X$ be a Menger space, $B \subseteq [0, 1]$ a $\sigma$-projective set and $f : X \to B$ a perfect surjective function. Since being Menger is preserved by continuous functions, we have that $f(X) = B$ is Menger. By $\sigma$-PD, $B$ is $\sigma$-compact. Therefore, its perfect preimage, $f^{-1}(B) = X$, is also $\sigma$-compact. The analytic case is analogous. □

Okunev’s space (see Example 8.7) is $K$-analytic—and hence is in $\mathcal{K}$—but is not in $\mathcal{P}$ since it is Menger but not $\sigma$-compact.

In generalizing classical descriptive set theory beyond the real line or separable metrizable spaces, a crucial obstacle is how to generalize the natural idea that the complement in $\mathbb{R}$ or $[0, 1]$ of a nice set is another nice set, worthy of being included in our hierarchy of nice sets. When we are no longer working within a fixed space such as $\mathbb{R}$, taking a complement suddenly catapults us onto the shaky ground of classes. One way of dealing with this classic problem is to make the relatively mild assumption that there are arbitrarily large inaccessible cardinals $\kappa$ and take for $X \in V_\kappa$ its complement in $V_\kappa$. A more plausible approach to dealing with this problem is to take the complement of a space $X$ to be the Stone-Čech remainder $X^*$. It is convenient to have $\beta X^* = \beta X$; this will happen for nowhere locally compact (NLC) spaces, i.e. spaces such that no point has a compact neighbourhood [van Douwen92]. This is not such an onerous restriction: for any $X$, $X \times \mathbb{Q}$ and $X \times \mathbb{P}$ are NLC. By counting ultrafilters, one can prove that $|\beta X| \leq 2^{2^{|X|}}$. For $\kappa$ inaccessible, we then have that $V_\kappa$ is closed under Stone-Čech remainders, i.e. for any space $X$ in $V_\kappa$, $X^*$ is in $V_\kappa$, since $V_\kappa$ is a model of ZFC and thus satisfies Power Set and Separation axioms. Similarly, $V_\kappa$ is closed under continuous images, since they do not increase cardinality and $V_\kappa$ satisfies Replacement.

The assumption that there is an inaccessible cardinal is much weaker than the large cardinal assumption needed to imply $\sigma$-PD, and is exactly the consistency strength of what is needed to get $\sigma$-projective Menger sets to be $\sigma$-compact [TTT21]. If one wants an axiomatic approach, rather than working in a particular $V_\kappa$, one can assume there are arbitrarily large inaccessible cardinals. Then the Stone-Čech remainder and all continuous images of a space will be sets rather than classes. Again, the assumption that there are arbitrarily large inaccessible cardinals is a much weaker assumption than is required to get $\sigma$-PD. Making this inaccessible assumption, we can then attempt to form an $\omega_1$-length hierarchy by closing under Stone-Čech remainders, continuous images, and countable unions, possibly restricting ourselves to NLC spaces. Does this make sense? Are spaces in this hierarchy projectively $\sigma$-projective, assuming $\sigma$-PD, and hence Menger ones Hurewicz? We made this claim in a seminar in January, 2022 at the Fields Institute, but that was premature and we withdraw it. There are difficulties.

Notice that our otherwise well-behaved class $\mathcal{K}$ is NOT closed under Stone-Čech remainder, even for nowhere locally compact spaces:
Definition 7.3. A space is co-$K$-analytic ($K^{-\Pi_1^1}$) if its Stone-Čech remainder is a $K$-analytic space.

A different reasonable definition of a space $X$ being co-$K$-analytic is that $X = Y^*$, for some $K$-analytic $Y$. By Proposition 4.5, these two definitions coincide for NLC spaces.

Example 8.5. For every cardinal $\kappa > \aleph_0$, if $D$ is the discrete space of size $\kappa$, then $D \times \omega^\omega$ is a Čech-complete NLC space that is not Lindelöf but is co-$K$-analytic.

Example 8.7. A $K$-analytic space with non-Lindelöf Stone-Čech remainder: Okunev’s space.

If we try to close the $K$-analytic spaces under Stone-Čech remainders and continuous images, we quickly get every space!

Theorem 7.4. Every topological space is the continuous image of an NLC space with $\sigma$-compact Stone-Čech remainder.

Proof. Let $X$ be a topological space of size $\kappa$ and let $D$ be the discrete space of size $\kappa$. The space $D \times \omega^\omega$ is a Čech-complete NLC space. This means that its Stone-Čech remainder is NLC $\sigma$-compact. We know that $X$ is a continuous image of $D$ and that $D$ is a continuous image of $D \times \omega^\omega$. □

Is there a more modest approach? Perhaps we should confine ourselves to Lindelöf spaces, which is a very plausible restriction, since Lindelöfness is used in most $K$-analytic applications. We thus have the following definitions:

Definition 7.5. A topological space is $L-\Sigma_1^1$ if it is $K$-analytic. A topological space is $L-\Pi_1^1$ if it is a Lindelöf space and its Stone-Čech remainder is an $L-\Sigma_1^1$. In general, a topological space is $L-\Sigma_1^{\xi+1}$, $1 \leq \xi < \omega_1$ if it is the continuous image of an $L-\Pi_1^{\xi+1}$. A topological space is $L-\Pi_1^\xi$, $1 \leq \xi < \omega_1$ if it is Lindelöf and its Stone-Čech remainder is an $L-\Sigma_1^\xi$.

For $\alpha < \omega_1$ a limit ordinal, a topological space $X$ is $L-\Sigma_1^\alpha$ if there exists $Y_i \in L-\Sigma_1^{\xi_i}$, $i < \omega$, $\xi_i < \alpha$ such that $X = \bigcup_{i \in \omega} Y_i$.

We will refer to $L-\Sigma_1^{\omega_1}$ as the class $\mathcal{L}$.

Theorem 7.4 shows we need to add something like this Lindelöf restriction. Note that, as far as the conjecture that “definable” Menger spaces are $\sigma$-compact (or Hurewicz) is concerned, this is not so crucial, since Menger spaces are Lindelöf, but if we want to generalize other properties of $K$-analytic spaces, this is a problem. The remainders of $K$-analytic spaces —even NLC ones— need not be Lindelöf.

We conjecture that restricting ourselves to members of $\mathcal{L}$ gives us a nicer class to work with, but we have still been unable to prove that such spaces are projectively $\sigma$-projective. Posing a concrete problem:

Question 7.6. Does $\sigma$-PD (actually, just Det($\Pi_1^1$)) imply that Lindelöf co-$K$-analytic spaces are projectively $\sigma$-projective?
Should we resign ourselves to adding extra conditions that are not necessarily satisfied by $K$-analytic spaces? One obvious try is to require countable type. Important examples to contend with are Okunev's space and its remainder.

**Example 8.7.** A co-$K$-analytic NLC space, $(O \times \mathbb{Q})^*$, that is not Lindelöf. $O$ is Okunev's space.

Okunev's space is an example of a $K$-analytic space that is not of countable type. If we require countable type, we will have remainders of Lindelöf spaces Lindelöf. But continuous images of even NLC $K$-analytic spaces of countable type need not be of countable type. For example, both $O$ and $O \times \mathbb{Q}$ are images of spaces of countable type (see Example 8.7 below).

There is one case in which we can add a not-too-onerous condition to a co-$K$-analytic space and get Menger such spaces to be $\sigma$-compact.

**Theorem 7.7.** $\text{Det}(\Pi^1_1)$ implies perfect co-$K$-analytic Menger spaces are $\sigma$-compact.

**Proof.** Let $X$ be a perfect co-$K$-analytic Menger space. By definition, the Stone–Čech remainder of $X$ is $K$-analytic, hence, a Lindelöf $\Sigma$-space. But Proposition 4.17 $X$ is an $s$-space, but Proposition 4.18 tells us that perfect $s$-spaces are $p$-spaces. In [Tal20] the second author proved:

**Proposition 7.8.** $\text{Det}(\Pi^1_1)$ implies Menger co-$K$-analytic $p$-spaces are $\sigma$-compact.

However, we have been unable to extend Theorem 7.7.

**Question 7.9.** Are perfect Menger continuous images of co-$K$-analytic spaces $\sigma$-compact?

Not every Menger $p$-space is $\sigma$-compact—consider a Menger non-$\sigma$-compact subset of $\mathbb{R}$. But what about Menger co-$K$-analytic $p$-spaces? We answered that in Proposition 7.8 above. In fact,

**Definition 7.10.** We will say $X$ is co-$\mathcal{H}$ if $X^* \in \mathcal{H}$. We similarly define $X$ being co-$\mathcal{P}$.

**Theorem 7.11.** $\sigma$-PD implies Menger co-$\mathcal{H}$ $p$-spaces are $\sigma$-compact.

**Proof.** We first prove a generalization of [Tal20, Lemma 1.9]:

**Lemma 7.12.** Perfect images of elements of co-$\mathcal{H}$ are elements of co-$\mathcal{H}$.

**Proof.** Let $f$ be a perfect map mapping an $X \in \mathcal{H}$ onto $Y$. Extend $f$ to $\hat{f}$ mapping $\beta X$ onto $\beta Y$. Then by Proposition 8.12 $\hat{f}(X^*) = Y^*$ and $\hat{f} | \hat{f}^{-1}(Y^*) = \hat{f} | X^*$ is perfect. But $X^* \in \mathcal{H}$ and so $Y^* \in \mathcal{H}$. \qed
Now to prove the theorem, let $X$ be Menger co-$\mathcal{H}$. $X$ is a $p$-space so there is a perfect function $f$ mapping $X$ onto a separable metrizable space and hence into $[0,1]^\omega$. By the Lemma, $(f(X))^* \in \mathcal{H}$. Extend $f$ to $\hat{f}$ mapping $\beta X$ onto the closure of $f(X)$ in $[0,1]^\omega$. By the Lemma, $f(X)$ is co-$\mathcal{H}$. Then $(f(X))^* \in \mathcal{H}$. Then $(\hat{f}(X))^*$ is co-$\mathcal{H}$. Then $(\hat{f}(X))^* \in \mathcal{H}$. Extend $\hat{f}$ to $\bar{f}$ mapping $\beta X$ onto the closure of $\hat{f}(X)$ in $[0,1]^\omega$. Being in $\mathcal{H}$ is a perfect property, so by Proposition 4.6, the remainder of $f(X)$ in $[0,1]^\omega$, i.e., $[0,1]^\omega \setminus f(X)$ is in $\mathcal{H}$. But then $[0,1]^\omega \setminus f(X)$ is $\sigma$-projective, so $f(X)$ is $\sigma$-projective. $f(X)$ is also Menger, so by $\sigma$-PD it is $\sigma$-compact, but $f$ is perfect, so $X$ is $\sigma$-compact. □

**Corollary 7.13.** Menger co-$\mathcal{P}$ spaces are $\sigma$-compact.

*Proof.* They are co-$\mathcal{H}$ $p$-spaces. □

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**8. Examples and counterexamples**

**Example 8.1. A compact space in $\mathcal{H}$ not in $\mathcal{C}$.**

Given that all the $\sigma$-projective sets are inside separable metrizable spaces, they are, at most, of size $c$. Since given $X \in \mathcal{C}$ there exists $B \subseteq \mathbb{R}$ and $f : B \rightarrow X$ a continuous surjective function, we have that $|X| \leq |B| \leq |\mathbb{R}| = c$.

On the other hand, every compact space is a USCCV image of any $\sigma$-projective set. In particular, any ordinal $\omega + 1$ of cardinality strictly bigger than $c$ (with the order topology) is a Hausdorff compact space. These spaces are not in $\mathcal{C}$.

For an example of a space in $\mathcal{H}$ which is not in $\mathcal{P}$, see Okunev’s space in 8.7. It is $K$-analytic but not of countable type, so its remainder is not Lindelöf $p$, so it is not in $\mathcal{P}$.

**Definition 8.2.** We say that an space is $k$-discrete if it is the disjoint union of open compact subspaces.

Clearly,

**Lemma 8.3.** A space is $k$-discrete if and only if it is the perfect preimage of a discrete space.

**Example 8.4. A $K$-analytic space with $k$-discrete Stone-Čech remainder.**

Let $D$ be an uncountable discrete space, $C$ its one-point compactification. Name the new point $\ast$. Notice that the space $[0,1] \times C$ is compact and let $X = ((0,1) \times D) \cup ([0,1] \times \{\ast\})$.

Notice that $X$ is not compact since the intersection of the sequence of decreasing closed sets $(0,1/n) \times \{d\}$ is empty. Nevertheless, $X$ is $K$-analytic (actually, $\sigma$-compact) since

$$X = ((0,1) \times D) \cup ([0,1] \times \{\ast\}) = ((0,1) \times C) \cup ([0,1] \times \{\ast\}) =$$

$$= \left( \bigcup_{n \in \omega} [1/n, 1 - 1/n] \times C \right) \cup ([0,1] \times \{\ast\}).$$
On the other hand, \([0,1] \times C\) is compact and \(X\) is dense in it. Furthermore, \([0,1] \times (C \setminus X) = \{0,1\} \times D\), a discrete space homeomorphic to \(D\). Thus \(X\) is a \(K\)-analytic space such that, in one of its compactifications, the remainder is homeomorphic to \(D\).

By the universal property of the Stone-Čech compactification, there is a function \(f : \beta X \to [0,1] \times C\) such that \(f|X^*\) is perfect and \(f(X^*) = ([0,1] \times C) \setminus X\) which is homeomorphic to \(D\). Therefore \(X^*\) is a perfect preimage of the uncountable discrete space \(D\).

**Example 8.5. A Čech-complete NLC space that is not Lindelöf.**

Let \(D\) be the discrete space of size \(\kappa\) for some \(\kappa > \omega\). Since \(D\) is locally compact, we can define its one-point compactification \(\alpha D\). Using the universal property of the Stone-Čech compactification, we can extend the identity function to a surjective function \(i : \beta D \to \alpha D\) with \(i(\beta D \setminus D) = \alpha D \setminus D\). Since \(\alpha D \setminus D\) is a single point, \(\beta D \setminus D = i^{-1}(\alpha D \setminus D)\) is compact (a closed set in a compact space). This shows that \(D\) is Čech-complete since its Stone-Čech remainder is a compact space.

By Proposition 4.1, \(D\times \omega\) is a Čech-complete space. It is an NLC space that is not Lindelöf since \(D\) is uncountable. Therefore, it is not \(K\)-analytic, but its Stone-Čech remainder is NLC \(K\)-analytic since the remainder is \(\sigma\)-compact.

**Definition 8.6.** A topological space \(X\) is Rothberger if given any countable sequence of open covers of \(X\) \(\langle U_1, U_2, ..., U_n, ... \rangle\) there are \(V_i \in U_i\) such that \(\{V_i : i \in \omega\}\) is also an open cover of \(X\).

**Example 8.7. A \(K\)-analytic space with a non-Lindelöf remainder.**

Let \(A(\mathbb{P})\) be the Alexandroff duplicate of the irrational numbers \(\mathbb{P}\). In other words, \(A(\mathbb{P})\) is the set \(\mathbb{P} \times \{0,1\}\) with the topology generated by \(\{(r,1)\}\) for all \(r \in \mathbb{P}\) and \((U \times \{0\}) \cup (\mathbb{P} \setminus A) \times \{1\}\) where \(U\) is open in the usual topology of \(\mathbb{P}\) and \(A \subseteq \mathbb{P}\) is finite. We will first show that this space is of countable type. Given a compact set \(K \subseteq A(\mathbb{P})\), we know that \(K \cap (\mathbb{P} \times \{1\})\) is compact, open and finite (so it has countable character). On the other hand, \(K \cap (\mathbb{P} \times \{0\})\) is a compact subset of \(\mathbb{P}\) which also has countable character. Therefore, \(K\) has countable character in \(A(\mathbb{P})\).

**Okunev’s space** \(X\) is the quotient space \(A(\mathbb{P})/(\mathbb{P} \times \{0\})\). This is the topological space obtained by collapsing the non-discrete copy of the irrational numbers to a single point. Okunev’s space is a \(K\)-analytic (see e.g. [Tal20]) – hence Lindelöf —space which is not of countable type, since it is \(K\)-analytic, Rothberger, and not \(\sigma\)-compact (see [Tal20]), but

**Proposition 8.8.** [Tal20] \(K\)-analytic Rothberger spaces of countable type are \(\sigma\)-compact.

Since \(X \times \mathbb{Q}\) is also \(K\)-analytic, Rothberger, and not \(\sigma\)-compact, it is not of countable type, but also is nowhere locally compact.

Using Proposition 4.19, we know that \((X \times \mathbb{Q})^*\) has a \(K\)-analytic remainder but is not Lindelöf.
Example 8.9. A Hereditarily Lindelöf projectively countable space with non-Lindelöf square

Moore [Moo05] has an example of a hereditarily Lindelöf space $X$ such that $X^2$ is not Lindelöf (proved in [Pen15]), but $X$ is projectively countable, hence—in particular—projectively $\sigma$-projective. Notice that this space is not in $\mathcal{K}$ because $X^2$ is not Lindelöf.

9. Future projections and questions

Although we have concentrated on the $\sigma$-projective sets of reals, most of our results extend to those sets of reals that are in $L(\mathbb{R})$. Somewhat larger large cardinals than needed for $\sigma$-PD yield determinacy for $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. All the results about Menger implying $\sigma$-compact or Hurewicz go through unchanged. One point that is not completely obvious, but is true, is that $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is closed under countable intersections, countable unions and continuous real-valued images. The point is that each of these can be coded by a real. That is clear for the first two; for the third, it is because continuous real-valued functions on the reals are determined by their values on rational arguments, so again this can be coded by a real.

If one is satisfied with consistency rather than direct implication, it should be noted that all our “Menger implies $\sigma$-compact” results, both for the $\sigma$-projective sets and for $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, are equiconsistent with an inaccessible cardinal. For discussion, see [TTT21].

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