Determination of the Total Accelerated Electron Rate and Power Using Solar Flare Hard X-Ray Spectra

Eduard P. Kontar1, Natasha L. S. Jeffrey1, and A. Gordon Emslie2
1 School of Physics & Astronomy, University of Glasgow, G12 8QQ, Glasgow, UK
2 Department of Physics & Astronomy, Western Kentucky University, Bowling Green, KY 42101, USA

Received 2018 May 5; revised 2018 December 20; accepted 2018 December 21; published 2019 February 4

Abstract
Solar flare hard X-ray (HXR) spectroscopy serves as a key diagnostic of the accelerated electron spectrum. However, the standard approach using the collisional cold thick-target model poorly constrains the lower-energy part of the accelerated electron spectrum, hence the overall energetics of the accelerated electrons are typically constrained only to within one or two orders of magnitude. Here, we develop and apply a physically self-consistent, warm-target approach that involves the use of both HXR spectroscopy and imaging data. This approach allows an accurate determination of the electron distribution low-energy cutoff, and hence the electron acceleration rate and the contribution of accelerated electrons to the total energy released, by constraining the coronal plasma parameters. Using a solar flare observed in X-rays by RHESSI, we demonstrate that using the standard cold-target methodology, the low-energy cutoff (hence the energy content in electrons) is essentially undetermined. However, the warm-target methodology can determine the low-energy electron cutoff with ~7% uncertainty at the 3σ level, hence it permits an accurate quantitative study of the importance of accelerated electrons in solar flare energetics.

Key words: Sun: activity – Sun: flares – Sun: X-rays, gamma rays

1. Introduction
Over the last 50 years or so, solar flares have been observed over a very broad range of frequencies, from radio waves to gamma-rays. Hard X-ray (HXR) emission remains the key diagnostic to quantitatively determine the properties of accelerated electrons (see, e.g., Benz 2008; Holman et al. 2011 for reviews) and provides the essential properties (e.g., magnitude, shape) of the electron distribution (e.g., Petsionian 2016). This is because (i) HXR production is linearly related to the emitting electron spectrum, thus providing a relatively simple expression linking the electron and HXR spectra (see, e.g., Kontar et al. 2011, for a review) and (ii) unlike radio waves or optical emission, the HXRs produced throughout the entire flare volume (including both the corona and chromosphere) are very weakly affected by propagation effects, so there is a very close relation between the radiation produced and the radiation observed.

Imaging spectroscopy observations of X-ray coronal sources and chromospheric footpoints (see e.g., Emslie et al. 2003; Kontar et al. 2008; Saint-Hilaire et al. 2010), using data from RHESSI (Lin et al. 2002), have remarkably confirmed the validity of the so-called “thick-target” model (e.g., Brown 1971; Syrovatskii & Shmeleva 1972). In this model, electrons are accelerated in the relatively tenuous corona and propagate downward to the chromosphere, emitting electron–ion bremsstrahlung as they proceed, particularly at the dense chromospheric footpoints (see, e.g., Holman et al. 2011, for a review). Using HXR emission as a diagnostic of solar flare electrons requires knowledge not only of the bremsstrahlung cross section, but also of the energy evolution of the bremsstrahlung-producing electrons (see, e.g., Brown et al. 2003). In general, the energy evolution of accelerated electrons is determined not only by binary Coulomb collisions with ambient electrons (and, to a lesser extent, ions), but also through scattering by plasma turbulence (e.g., Bian et al. 2017) and collective effects such as return current ohmic losses (e.g., Knight & Sturrock 1977; Emslie 1980; Zharkova & Gordovskyy 2006; Alaoui & Holman 2017) and Langmuir wave generation in the inhomogeneous plasma (e.g., Kontar et al. 2012).

In a thick-target model, the electrons completely stop in the bremsstrahlung target and consequently the relationship between the magnitudes and shapes of the electron and HXR spectra depends only on the (energy-dependent) rate of energy loss of individual electrons within the target. For energies $E \gg k_B T$ (where $k_B$ is the Boltzmann constant and $T$ the temperature), the target is said to be “cold” (Section 2), and this has been the basis for much of the modeling of the HXR and electron spectral relationship in solar flares (e.g., Brown 1971; Syrovatskii & Shmeleva 1972; Lin & Hudson 1976). However, Emslie (2003) has pointed out that the energy loss rate drops rapidly from its cold-target value to a value near zero as the accelerated electron energy approaches a few $k_B T$, and that recognition of this effect can significantly modify the number of electrons necessary to produce a given HXR burst (see Section 3). Galloway et al. (2005) have further noted the importance of the collisional energy diffusion of the few $k_B T$ electrons. However, all previous treatments have not considered the transport of thermalized electrons. Recently, Kontar et al. (2015) have developed a “warm-target approach” and demonstrated that these near-thermal electrons serve as an important constraint on the overall accelerated electron spectrum; they also pointed out the need to take into account their spatial diffusive motion.

As accelerated electron spectra are typically rather steep power laws with spectral indices of $\delta \approx 4$ or larger (e.g., Dennis 1988; Petsionian et al. 2002), the energy content in accelerated electrons is strongly dependent on the value of the lowest electron energy in the distribution, the low-energy cutoff, $E_c$; underestimating the value of $E_c$ by even a factor of two can result in order of magnitude or greater overestimates of the energy content in accelerated electrons. Applying the
cold-target model (Section 2) to observed solar flare HXR spectra strongly suggests that accelerated electrons account for a considerable fraction of the total magnetic energy released in the flare (Emslie et al. 2004, 2005, 2012). However, because the lower-energy end of the HXR spectrum is dominated by thermal bremsstrahlung, previous works (e.g., Holman et al. 2003; Emslie et al. 2012) could not determine \( E_c \), but could provide only an upper limit, and hence a lower limit to the energy contained in nonthermal electrons. In an attempt to provide a better estimate of the energy content in accelerated electrons, Aschwanden et al. (2016, 2017) included an approximate (Kontar et al. 2015; see Section 3 below) treatment of warm-target effects in their analysis of the relationship between the HXR and accelerated electron spectra. Their analysis led to the inference of very significant energy contents in accelerated electrons, and, for some events, produced the particularly intriguing result that the inferred energy content in accelerated electrons was larger than (and in some cases, over an order of magnitude greater than) the estimate of the magnetic energy available. Therefore, despite these numerous efforts, the details of how the energy is partitioned in solar flares remains essentially unknown.

In this paper, we present in Section 3 a self-consistent warm-target algorithm to correctly and accurately calculate a value of \( E_c \) for a given HXR spectrum and for given thermal properties of the corona. The procedure, which involves the use of both X-ray spectroscopy and imaging data, allows an accurate determination of the electron spectrum at both high (nonthermal) and low (thermal) energies, and hence an accurate evaluation of the contribution of electrons to the overall energetics of a flare (Section 4). We discuss the results and their implications in Section 5.

2. Accelerated Electrons in a Cold Thick-target Model

The spatially integrated HXR spectrum (photons \( \text{cm}^{-2} \text{s}^{-1} \text{keV}^{-1} \)) observed at Earth is given by

\[
I(\epsilon) = \frac{1}{4\pi R^2} \int_{\epsilon}^{\infty} Q(\epsilon, E) \langle nVF(E) \rangle \ dE, \tag{1}
\]

where \( \epsilon \) (keV) is the photon energy, \( E \) (keV) is the electron kinetic energy, \( R \) (cm) is the distance from the Sun to Earth, and \( Q(\epsilon, E) \) (cm\(^2\) keV\(^{-1}\)) is the bremsstrahlung cross section, differential in electron energy \( E \). The quantity \( \langle nVF(E) \rangle \) (electrons cm\(^{-2}\) s\(^{-1}\) keV\(^{-1}\)) is the density-weighted mean electron flux spectrum, integrated over the flaring region (Brown et al. 2003), and it should be noted that it is determined solely from observations of \( I(\epsilon) \) and knowledge of the bremsstrahlung cross section \( Q(\epsilon, E) \). However, to proceed further to the determination of the injected (or accelerated) electron spectrum \( N(E_0) \) (electrons s\(^{-1}\) keV\(^{-1}\)) from the mean electron flux spectrum \( \langle nVF(E) \rangle \) requires a model for the electron dynamics in the flare. This is often based on the simple collisional cold thick-target model (Brown 1971), which assumes that the electrons lose all their kinetic energy in Coulomb collisions with much less energetic electrons in the ambient plasma. For such a model, the electron energy loss rate is \( dE/dt = -Knv/E \), where \( n \) (cm\(^{-3}\)) is the ambient density, \( v \) (cm s\(^{-1}\)) the electron velocity, and \( K = 2\pi e^4 \ln \Lambda \) (cm\(^2\) keV\(^2\)), with \( e \) (esu) the electronic charge and \( \ln \Lambda \) the Coulomb logarithm. It follows (see Brown et al. 2003) that the relationship between the mean electron flux spectrum and the injected rate spectrum \( \dot{N}(E_0) \) (electrons s\(^{-1}\) keV\(^{-1}\)) is

\[
\langle nVF(E) \rangle = \frac{E}{K} \int_{E}^{\infty} \dot{N}(E_0) \ dE_0, \quad \dot{N}(E_0) = -K \frac{d}{dE} \left( \frac{\langle nVF(E) \rangle}{E} \right) \big|_{E=E_0}. \tag{2}
\]

Application of expressions (1) and (2) to the power-law form \( I(\epsilon) \propto \epsilon^{-\gamma} \), which is often (e.g., Holman et al. 2011) an excellent approximation to observed HXR spectra, shows (e.g., Brown 1971) that the accelerated electron spectrum is also a power law:

\[
\dot{N}(E) = \frac{\delta - 1}{\epsilon_c} \left( \frac{E}{\epsilon_c} \right)^{-\delta}, \tag{3}
\]

with \( \delta = \gamma + 1 \). Here, \( \epsilon_c \) (keV) is formally an arbitrary reference energy, and in practice it is taken to be a low-energy cutoff in the injected electron spectrum. \( \dot{N}_0 \) (s\(^{-1}\)) is the total rate of electron acceleration \( \dot{N}_0 = \int_{E_0}^{\infty} N(E_0) \ dE_0 \). The associated total power \( P \) (keV s\(^{-1}\)) in the nonthermal electrons is

\[
P = \int_{E_0}^{\infty} E \dot{N}(E) \ dE_0 = \frac{1}{\delta - 2} E_c^{2-\delta} \dot{N}_0 \epsilon_c, \tag{4}
\]

so that for a prescribed accelerated spectrum, \( \dot{N}(E_0) \), \( P \) is strongly dependent on the value of the low-energy cutoff \( \epsilon_c \).

A typical solar flare X-ray spectrum also contains very substantial thermal bremsstrahlung emission at energies \( \epsilon \approx (5-30) \) keV. This emission, produced in plasma heated to temperatures \( k_B T \approx 2 \) keV (e.g., Doschek et al. 1980; Phillips 2004; Sylwester et al. 2014), effectively masks the bremsstrahlung produced by the accelerated electrons at lower energies, making the determination of \( E_c \), and so \( P \), a significant challenge. It also means that while the cold thick-target model may be quite valid in the relatively cool chromosphere, it incorrectly describes the evolution of electrons in the hotter corona, especially electrons with energies up to a few times the thermal energy \( k_B T \) (Emslie 2003; Galloway et al. 2005; Jeffery et al. 2014; Kontar et al. 2015). As we have noted, because of the steeply decreasing form of the accelerated electron spectrum, it is these very electrons that carry the bulk of the energy. Therefore, to correctly model the electron spectrum in this critical range, we require the warm-target equivalent of Equation (2), which should be consistent with the form of the X-ray spectrum at both low (thermal) and high (nonthermal) energies. We consider just such a model in Section 3 below.

To demonstrate the extent to which the collisional cold thick-target model cannot accurately constrain the parameters of the accelerated electron distribution, we performed a typical spectral fit to the RHESSI X-ray spectrum of flare SOL2013-05-13T02:12, occurring on 2013 May 13, using the Object Spectral Executive (OSPEX; Schwartz et al. 2002). We created the count spectrum using the combined RHESSI front detectors (excluding the inadequately operating Detectors 2 and 7; Smith et al. 2002; Holman et al. 2011), during the impulsive phase period from 02:09–02:10 UT. The count spectrum was fitted between the energies of 10–100 keV with the following
components: an isothermal fit (using the function $f_{\text{vth}}$, which accounts for the low energy ($<30$ keV) thermal bremsstrahlung); a cold thick-target model (using the function $f_{\text{thick}}$, which accounts for the nonthermal bremsstrahlung produced by the accelerated electrons); an auxiliary functions representing a pile-up correction; and one Gaussian line at 10 keV that accounts for an instrumental feature. As the flare is located at the limb where albedo effects are minimal, we do not add the albedo function. The spectral range chosen avoids the majority of other line and instrumental features present at lower energies, and also the dominant background at higher energies. We fit the spectrum multiple times using different fixed values of $E_c$, ranging between 5 and 40 keV.

The main results, displayed in Figure 1, show how the full $\chi^2$ (see Press et al. 1986 for details) and nonthermal electron power $P$ vary with $E_c$. The flat $\chi^2$ for the cold target in Figure 1 demonstrates that the low-energy cutoff $E_c$ cannot be determined at 2$\sigma$ level. For the cold-target model, all values of $E_c$ below $\sim 33$ keV are consistent (reduced $\chi^2 \sim 1 - 2$), with the observed X-ray spectrum at the 2$\sigma$ level, and for this range of $E_c$, $P$ ranges over two orders of magnitude, from $P \approx 3 \times 10^{27}$ erg s$^{-1}$ to $P \approx 3 \times 10^{29}$ erg s$^{-1}$ (blue circles in Figure 1). Hence, the nonthermal electron power $P$ is very poorly constrained by application of the cold thick-target model. The lack of any substantial change in $\chi^2$ for the cold-target fit over a broad range of $E_c$ values (Figure 1) is consistent with the detailed error analysis performed by Ireland et al. (2013), who found that the probability density function for $E_c$ has a long asymmetric tail. As we shall show below, however, the application of a self-consistent warm-target model results in a full $\chi^2$ that has a well-defined range of acceptable $E_c$ values and hence of the nonthermal electron power $P$ (red points in Figure 1). Using the warm-target approach defined below, the $\chi^2$ analysis provides the low-energy cutoff $E_c = (12.1 \pm 0.3)$ keV and $E_c = (12.1 \pm 0.9)$ keV at the 1$\sigma$ (68% confidence) and 3$\sigma$ (99%) levels, respectively.

3. Warm-target Considerations and the Resolution of the Low-energy Cutoff Problem

As they propagate through the 10–30 MK solar corona, electrons with energies of a few $k_B T$ suffer near-elastic collisions with ambient electrons of comparable energy and thus have an energy loss rate that is drastically lower than that in a cold-target approximation (see Emslie 2003 and the simulations in Jeffrey et al. 2014). In a collisional cold thick-target model, the total number of accelerated electrons is not conserved; the electrons essentially “disappear” when they reach $E = 0$. On the other hand, in a warm-target model, the accelerated electrons retain a finite energy, hence their number is conserved. The continued injection of accelerated electrons into a target therefore systematically builds up the number of thermal electrons in that target, a process which ultimately saturates due to the diffusion of thermal electrons into the chromosphere (which, with a temperature $k_B T \lesssim 1$ eV, can accurately be considered a cold target). Therefore, any consistent model of electron transport in collisional plasma must account for both the nonthermal and thermal components of the X-ray spectrum.

Kontar et al. (2015) have shown that if electrons propagate in a region of warm plasma with length $L$ (cm), density $n$ (cm$^{-3}$) and temperature $T$ (K), the mean electron flux spectrum is related to the electron injection rate by (their Equation (3))

$$\langle n V F \rangle(E) = \frac{1}{2K} E e^{-E/k_B T} \int_{E_{\text{min}}}^{E} \frac{e^{E'/k_B T}}{E'} G\left(\frac{E'}{k_B T}\right) \times \int_{E'}^{\infty} N(E_0) dE_0, \quad (5)$$

where $G(x) = [\text{erf}(x) - x \text{erf}'(x)]/2x^2$, and $\text{erf}(x)$ is the error function. The limit $E_{\text{min}}$ is given by

$$E_{\text{min}} \approx 3 k_B T\left(\frac{\lambda}{L}\right)^4, \quad (6)$$

where $\lambda = (k_B T)^2/(2k_B n)$ is the collisional mean-free path, and it is determined by considering the warm plasma properties in the

---

4. (f_pileup_mod) The pile-up component accounts for those photons arriving at the detector at nearly the same time, which are detected as a single count with an energy equal to the sum of the individual photon energies.

5. Note that the subsequent Equation (24) in Kontar et al. (2015) contains an erroneous upper limit of $\infty$ on the first integral.
corona and by the gradual escape of electrons into the cold chromosphere (see Kontar et al. 2015 for details). In general, the shorter the electron mean-free path, the smaller the value of $E_{\text{min}}$ (Equation (6)), so the larger the number of electrons that will thermalize, and hence accumulate, in the corona (Equation (5)).

Equation (5) replaces the cold thick-target result (2). The coronal parameters $T$, $n$, and $L$ that determine the value of $E_{\text{min}}$ hence the form of $\langle nVF \rangle(E)$, can best be obtained from a combination of X-ray spectroscopy and imaging observations: the distance between the coronal source and footpoint straightforwardly gives $L$, while the thermal fit to the HXR spectrum below $\sim 25$ keV gives both the emission measure $EM = n^2V$, and the temperature $T$.

For loop lengths $L \approx 10^7$ cm and coronal densities of order $10^{11}$ cm$^{-3}$ or larger, electrons at quite substantial energies up to $E \approx 20$ keV are thermalized in the corona, and rather than becoming “lost” from the system (as they do in the cold thick-target formulation), they now make a significant, and observable, thermal contribution to the HXR spectrum. This reduces the need for accelerated electrons in this energy range to create the HXR photons observed in this energy range and, as a result, the mean source electron spectrum $\langle nVF \rangle(E)$ needs to extend down only to fairly moderate energies, $E$. This effectively introduces a cutoff energy $E_c$ into the form of $\langle nVF \rangle(E)$, concomitantly reducing the required power in accelerated nonthermal electrons (see Equation (2)). To quantify this effect, we write Equation (5) in a simplified form, obtained by replacing the function $G(\sqrt{E/k_B}T) \simeq \sqrt{E}/\pi k_B T$ (an approximation valid in the range $E \lesssim k_BT$), giving

$$\langle nVF \rangle(E) \approx \frac{1}{2K} E e^{-E/k_BT} \sqrt{\pi k_BT} \int_{E_{\text{min}}}^{E} E^{-3/2} dE' \int_{E_{\text{min}}}^{\infty} \tilde{N}(E_0) dE_0$$

$$\simeq \frac{1}{K} E e^{-E/k_BT} \sqrt{\pi k_BT} E_{\text{min}}^{-1/2} \tilde{N}_0, \quad (7)$$

where we have used the fact that $\tilde{N}(E_0)$ peaks at an energy $E_c$ substantially outside the $E \lesssim k_T$ range of the approximation and have defined

$$\tilde{N}_0 \simeq \int_{E_{\text{min}}}^{\infty} \tilde{N}(E_0) dE_0. \quad (8)$$

Then Equation (7) can be rewritten in the Maxwellian form

$$\langle nVF \rangle(E) \approx \Delta EM \sqrt{\frac{8}{\pi m_e}} \frac{E}{(kB T)^{3/2}} e^{-E/k_BT}, \quad (9)$$

where the emission measure

$$\Delta EM \approx \frac{\pi}{K} \frac{m_e}{8} (kB T)^2 \tilde{N}_0 E_{\text{min}}^{-1/2}. \quad (10)$$

$\Delta EM$ quantifies the additional contribution to the overall inferred soft X-ray emission measure that results from the thermalization of accelerated electrons; it is a function of $E_{\text{min}}$, a quantity that characterizes the fraction of the accelerated electron distribution that thermalizes in the hot coronal part of the loop. The value of $E_{\text{min}}$ depends on the thermal coronal mean-free path $\lambda$ like $T^3 \lambda^2/L^2$ (Equation (6)), so that $E_{\text{min}}^{1/2} \sim T^{1/2} \lambda^2/L^2 \sim T^{9/2}/n^2 L^2$ and thus $\Delta EM \propto N_0 n^2 L^2 T^{-5/2}$.

Only electrons from the injected distribution with energy $E \lesssim \sqrt{2KnL}$ are essentially considered as thermalized and hence contribute to $\Delta EM$; electrons with energy in excess of this value can be treated as electrons interacting with a cold target in the conventional manner. Evidently, when the low-energy cutoff $E_c < \sqrt{2KnL}$, a substantial contribution $\Delta EM$ is expected.

In the model of Kontar et al. (2015), the growth of emission measure in the thermal plasma due to the thermalization of freshly injected low-energy energetic electrons in the coronal part of a flare loop is balanced with the diffusion of thermal electrons out of the hot coronal part of the loop into the chromosphere. This balance is achieved over a time corresponding to that for electron diffusion along the loop:

$$\tau_{\text{diff}} \approx \frac{L^2}{D}, \quad (11)$$

where $D = k_BT/c_n$ is the thermal diffusion coefficient, and $\tau_{\text{c}} \approx (m_e/k_BT)^{3/2}/\pi e^n \ln \Lambda$ is the thermal electron collision time. Using $n = 10^{11}$ cm$^{-3}$, $L = 10^9$ cm, and $T = 20$ MK, one finds $\tau_{\text{c}} \approx 10^{-2}$ s, $D \approx 3 \times 10^{18}$ cm$^2$ s$^{-1}$, and $\tau_{\text{diff}} \approx 30$ s. This diffusion timescale is comparable to the thermalization time for collisional (Spitzer 1962) thermal conduction, and it is sufficiently rapid that a steady-state balance can be achieved within the time associated with HXR intensity fluctuations. We also note (see above) that $\Delta EM \propto N_0 n^2 T^{-5/2} \propto N_0 n/D$; low diffusion coefficients lead to effective trapping of electrons and hence to a large emission measure from thermalized electrons.

For $E \gg k_BT$, Equation (5) reduces to the cold thick-target form (2). Thus, we can approximate Equation (5) as the sum of a “thermal part” and a “nonthermal part”:

$$\langle nVF \rangle(E) \approx \Delta EM \sqrt{\frac{8}{\pi m_e}} \frac{E}{(kB T)^{3/2}} e^{-E/k_BT}$$

$$+ \frac{E}{K} \int_{E_{\text{min}}}^{\infty} \tilde{N}(E_0) dE_0, \quad (12)$$

and it should be noted that when the low-energy cutoff in the injected distribution $E_c$ is larger than the energy $\sqrt{2KnL}$ that can be effectively stopped within the coronal part of the loop, the contribution $\Delta EM$ of the thermal component becomes negligible and the cold-target approximation is recovered.

Equation (12), with a form of $\tilde{N}(E_0)$ that assumes a power-law form at high energies, has been included in the Solar SoftWare (SSW) and OSPEX routines as the function $f_{\text{thick warm}}$. The convolution of the mean source electron spectrum function $\langle nVF \rangle(E)$ obtained from Equation (12) with the bremsstrahlung cross section $Q(\epsilon, E)$ (e.g., Haug 1997) determines the HXR flux, and by minimizing $\chi^2$, the best-fit parameters can be found. The proper use of the expression (12) will be demonstrated in Section 4. It is instructive to note that Equation (12) is a good approximation to Equation (5) when $E_c < \sqrt{2KnL}$; this is readily seen from Figures 3 through 5 in Kontar et al. (2015).

The above choice of a power-law form is, of course, not unique. For example, Bian et al. (2014) have demonstrated that the stochastic acceleration of electrons in the presence of both
Coulomb collisions and velocity diffusion with a coefficient 
that is inversely proportional to velocity leads to an accelerated 
electron distribution that has the form of a kappa distribution. 
Although the total spatially integrated X-ray flux is not always 
consistent with a single kappa distribution, with an additional 
thermal component (presumably resulting from direct coronal 
heating associated with the magnetic reconnection process) 
required for an acceptable fit, kappa distributions have been 
found in some cases to be consistent with RHESSI observations 
of the overall X-ray spectrum (Kašparová & Karlický 2009; 
Battaglia et al. 2015). In the Appendix, we provide the 
formulae necessary to incorporate a kappa distribution as the 
nonthermal component of the accelerated electron spectrum.

4. Application to Data

Here, we apply the warm-target method outlined in the 
previous section to flare SOL2013-05-13T02:12 to determine 
an acceptable form of $N(E)$, and hence to infer the injected 
electron power, $P$. Flare SOL2013-05-13T02:12 was one of the 
flares analyzed by Aschwanden et al. (2016), who determined 
the nonthermal electron energy by using a simplified applica-
tion of the warm-target model.

As noted above, the parameters for the nonthermal and 
thermal parts of the electron distribution are interrelated within 
the warm-target model. Thus, any algorithm aimed at 
producing a reliable estimate of the nonthermal electron power 
$P$ must also self-consistently determine the corresponding 
parameters (emission measure, temperature) of the thermal part 
of the spectrum. This is a key element of the method described 
below. It should also be noted that the method requires the use 
of both HXR spectroscopy and imaging. There are different 
ways to constrain the coronal thermal parameters. Below, we 
determine the thermal properties of the corona using the short 
time interval of 02:08:52–02:09:00 UT immediately preceding 
the time interval 02:09:00–02:10:00 UT used for the warm-
target analysis. We stress that the following procedure is valid 
regardless of how the thermal properties of the corona, which 
will ultimately constrain the nonthermal electron energetics 
(see subsection 4.3), are determined.

4.1. Description of the Warm-target Fitting Procedure and 
Illustrative Application

We now enunciate the general steps in the method and, by 
way of illustration, apply it to the flare SOL2013-05-13T02:12. 
We have chosen to apply Equation (12) with an assumed 
power-law form for $N(E)$; the use of other forms for $N(E)$ 
(e.g., a kappa distribution) involves straightforward adjust-
ments at the pertinent steps.

1. First, we evaluate the plasma parameters in the flaring 
loop. We have chosen to evaluate the thermal parameters 
by using the short time interval of 02:08:52–02:09:00 UT 
just before the studied interval of 02:09:00–02:10:00 UT. 
The HXR spectrum during the 02:08:52–02:09:00 UT interval 
was fitted with a combined thermal function plus 
a cold thick-target nonthermal function (e.g., using the 
SSW/OSPEX functions $f_{\text{vth}} + f_{\text{thick}}$). Although 
both nonthermal and thermal components are included in 
the modeling, this step is primarily used to estimate the 
thermal plasma parameters of the coronal loop-top 

source, namely the temperature $T$ (keV) and the emission 
measure $EM_0$ (cm$^{-3}$); the nonthermal parameters will be 
determined more accurately later. The minimum value of 
the reduced $\chi^2$ gives the best estimates of $T$ and $EM_0$. 
Even though the fitting is performed with an assumed 
power-law form for the cold thick-target nonthermal 
component, we have found that using another form (such 
as a kappa distribution) does not substantially affect the 
inferred values of the thermal parameters $EM_0$ and $T$.

(For SOL2013-05-13T02:12, the relative abundance 
parameter was fixed at the default value of unity, 
and we determined best-fit values of $T = (2.52 \pm 
0.03)$ keV and $EM_0 = (6.51 \pm 0.30) \times 10^{10}$ cm$^{-3}$ 
for 02:08:52–02:09:00 UT. Note that throughout this work, 
1σ OSPEX uncertainties are used, unless stated 
otherwise.)

2. With the values of $EM_0$ and $T$ now determined, we fit the 
HXR spectrum during our study time interval of 
02:09:00–02:10:00 UT. The HXR spectrum during this 
time interval is fitted with $f_{\text{vth}}$ and the warm-target 
fitting function $f_{\text{thick}}$. Note that the total 
emission measure $EM = EM_0 + \Delta EM$ obtained by this 
fit includes any contribution $\Delta EM$ from the thermaliza-
tion of injected electrons. Any additional parameters/
contributions pertinent to the observed flux (e.g., elemental abundances, albedo, pulse pile-up, lines, drm_mod, etc.) should also be included in this fit (Figure 2). The function $f_{\text{thick \_ warm}}$ contains 10 parameters: the first six parameters are exactly the same as for $f_{\text{thick2}}$ (i.e., the electron acceleration rate $N_0$ (s$^{-1}$), the spectral index (low) $\delta_{\text{low}}$, the break energy $E_B$ (keV), the spectral index (high) $\delta_{\text{high}}$, the low-energy cutoff $E_c$ (keV), and the high energy cutoff $E_H$ (keV)). The other four parameters relate to the properties of the thermal properties of the corona: the loop temperature $T_{\text{loop}}$ (keV), loop density $n_{\text{loop}}$ (cm$^{-3}$), and half-length of the loop $L$ (cm), and the relative elemental abundances.

a. As discussed in Step 1, an isothermal estimate of the loop-top temperature, $T$, is determined using $f_{\text{vth}}$ during 02:08:50–02:09:00 UT, and we fix $T_{\text{loop}} = 2.52$ keV.

b. To determine the density $n_{\text{loop}}$, we use the estimate of EM$_0$ obtained in Step 1. The number density can then be estimated using $n_{\text{loop}} = \sqrt{\text{EM}_0/V}$, where $V$ is the volume of the emitting plasma, found from X-ray imaging (see Figure 3).

(For SOL2013-05-13T02:12, the volume was estimated by applying the imaging algorithm Visibility Forward Fitting (VIS_FWDFIT; Schmahl et al. 2007). Using VIS_FWDFIT, the volume of the coronal source varies with energy (see Jeffrey et al. 2015), so we determined the mean volume over the energy range of 10–25 keV (Figure 3), yielding a value of $V = (0.86 \pm 0.20) \times 10^{27}$ cm$^3 \approx (0.9 \pm 0.2) \times 10^{27}$ cm$^3$. Using the inferred values of EM$_0$ and $V$ (and their associated uncertainties) gives $n_{\text{loop}} = (8.7 \pm 2.1) \times 10^{10}$ cm$^{-3}$.)

c. The half-length $L$ of the loop (the length from the coronal source to the chromospheric footpoint), is estimated directly from the image (see Figure 3).

(For SOL2013-05-13T02:12, we estimate from the X-ray image (blue curve in Figure 3 left) that $L = (24 \pm 5)'' \approx (17 \pm 3.5)$ Mm.)

3. Once the plasma parameters $n_{\text{loop}}$, $T_{\text{loop}}$, and $L$ have been determined and fixed, then the remaining nonthermal parameters of the $f_{\text{thick \_ warm}}$ fit, namely $(N_0, \delta_{\text{low}}, E_B, \delta_{\text{high}}, E_c, E_H)$, can be determined during 02:09:00–02:10:00 UT. As usual, we suggest that $E_H$ be fixed at the high default value, and that $E_B$ and $\delta_{\text{high}}$ are left free only if required (otherwise keep them fixed with $E_B$ above the highest energy in the fit).

(For SOL2013-05-13T02:12, we find $E_c = (12.1 \pm 0.3)$ keV, $N_0 = (12.8 \pm 0.6) \times 10^{35}$ electrons s$^{-1}$ and $\delta_{\text{low}} = (4.25 \pm 0.02).$)

4. The nonthermal electron power can now be determined as $P = (\delta - 1)/(\delta - 2) N_0 E_c$, as for the cold thick-target case (Equation (4)), as we assume a power-law form for the accelerated electrons.

(For SOL2013-05-13T02:12, we find $P = (35.8 \pm 1.9) \times 10^{27}$ erg s$^{-1}$.)

All of the major fit parameters for $f_{\text{vth}}$ and $f_{\text{thick \_ warm}}$ for SOL2013-05-13T02:12 are shown in Table 1. Using the derived plasma parameters, we calculate a value of $\tau_{\text{diff}} \sim 30$ s, showing that the 1 min observation time is well justified here.

4.2. Comparison with Simple Analytic Estimates

A crude estimate of the cutoff energy, $E_c$, can be obtained by considering the energy at which the systematic energy loss rate vanishes in the Fokker–Planck equation governing the evolution of the nonthermal electron spectrum. Kontar et al. (2015)
used this method to obtain the approximate result $E_c^* \simeq \delta \times k_B T$. Although this provides a useful, and easily applied, estimate of $E_c$, and hence of the electron power $P$ (Equation (4)), it should be stressed that this simplified expression corresponds simply to the value at which the nonthermal component of $(nVF)(E) = 0$; below this value of $E_c^*$, the value of $(nVF)(E)$ is not zero, but is in fact negative. Using $T_{loop} \simeq 2.5$ keV and $\delta = 4.2$, the simple estimate $E_c^* = \delta \times T_{loop}$ [keV] gives $E_c^* \simeq 10$ keV; this was the approach used by Aschwanden et al. (2016). Here, $E_c^*$ is close to $E_c$, determined by the warm-target, forward-fit method. However, this will not be true of all flares, hence we stress that the nonthermal electron parameters should only be determined using the warm-target, forward-fit method described in Section 4.1.

The $E_c^*$ approximation was used by Aschwanden et al. (2016, 2017) to compute the energy content in accelerated electrons. In some cases, using the low-energy cutoff estimate $E_c^* \simeq \delta \times k_B T$ (Kontar et al. 2015) might lead to an increase in the energy content of nonthermal electrons and could explain the high values of energy in nonthermal electrons estimated. This is in part because of the use of the rather simple way in which $E_c^*$ is estimated; however, it is also because of the low values of the temperature $T$ ($\lesssim 0.8$ keV) used by Aschwanden et al. (2016) in applying this simple estimate. In deriving the approximate formula $E_c^* = \delta \times k_B T$, Kontar et al. (2015) stress that the value of $T$ used must be the value of $T$ corresponding to the Maxwellian thermal plasma in the loop.

### Table 1

| Parameter       | Fit Status                                      |
|-----------------|-------------------------------------------------|
| $f_{\text{vth}}$| $E_{\text{c0}} = (6.51 \pm 0.30) \times 10^{48} \text{ cm}^{-3}$ Fixed, found from $f_{\text{vth}}$ at 02:08:52–02:09:00 UT. |
|                 | $T = (2.52 \pm 0.03)$ keV Fixed, found from $f_{\text{vth}}$ at 02:08:52–02:09:00 UT. | |
| $f_{\text{thick, warm}}$ | $N_0 = (12.8 \pm 0.6) \times 10^{35} \text{ s}^{-1}$ Free, found from warm-target fitting at 02:09:00–02:10:00 UT. |
|                 | $\delta_{\text{thick}} = 4.25 \pm 0.02$ Free, found from warm-target fitting at 02:09:00–02:10:00 UT. |
|                 | $E_c = (12.1 \pm 0.3)$ keV Free, found from warm-target fitting at 02:09:00–02:10:00 UT. |
|                 | $n_{\text{loop}} = (8.7 \pm 2.1) \times 10^{10} \text{ cm}^{-3}$ Fixed using $E_{\text{c0}}$ & $V$ (from imaging at 02:08:00–02:10:00 UT). |
|                 | $T_{\text{loop}} = (2.52 \pm 0.03)$ keV Fixed and equal to $T$ (from $f_{\text{vth}}$ at 02:08:52–02:09:00 UT). |
|                 | $L = (17.0 \pm 3.5)$ Mm Fixed from imaging at 02:08:00–02:10:00 UT. |
|                 | $P = (35.8 \pm 1.9) \times 10^{27} \text{ erg s}^{-1}$ Determined from $N_0$, $\delta_{\text{thick}}$, $E_c$. |

Note. Only the $f_{\text{thick, warm}}$ parameters related to the accelerated electron distribution (i.e., $N_0$, $\delta$, $E_c$) are left free during warm-target fitting. The plasma parameters (i.e., $E_{\text{c0}}$, $T_{\text{loop}}$, $n_{\text{loop}}$, $L$) are determined using a combination of X-ray spectroscopy and imaging, and then are fixed during fitting.

---

1. As $L$ becomes smaller (decreasing from 40 to 5 Mm), $E_c$ decreases from $\sim 20$ to $\sim 5$ keV and the power $P$ correspondingly increases. This is because the smaller column density in the corona means that fewer accelerated electrons thermalize there. The estimated uncertainty in $L$ of $\pm 3.5$ Mm leads to an uncertainty of $\pm 2$ keV in $E_c$ that is not accounted for in OSPEX.

2. Decreasing the plasma number density $n_{\text{loop}}$ similarly lowers the coronal column density, hence the number of accelerated electrons thermalized there; a similar variation in $E_c$ hence results. Again, the estimated uncertainty in $n_{\text{loop}}$ of $\pm 2.1 \times 10^{10} \text{ cm}^{-3}$ leads to an uncertainty of $\pm 2$ keV in $E_c$ that is not accounted for in OSPEX.

3. Keeping the temperature, $T$, from $f_{\text{vth}}$ fixed, but varying the $f_{\text{thick, warm}}$ parameter $T_{\text{loop}}$ ($\approx T$) between $T_{\text{loop}} = 4.0$ keV and $T_{\text{loop}} = 1.5$ keV, decreases $E_c$ from $\sim 16$ keV down to $\sim 7$ keV. This is because, once again, a lower value of $T_{\text{loop}}$ results in a smaller number of accelerated electrons blending into the thermal background, so that the nonthermal tail now extends to lower values of $E_c$.

We note that an isothermal fit was used to determine the plasma parameters in the flaring corona, even though it has been shown by Jeffrey et al. (2015), using a combination of X-ray spectroscopy and imaging, that for SOL2013-05-13T02:12 the coronal source cannot be strictly isothermal. Therefore, the plasma temperature should be viewed as an average temperature throughout the volume in which the electrons propagate. The number density $n_{\text{loop}}$ should also be viewed as an average value.

In the above method, we determined $E_{\text{c0}}$ from a time interval before the interval studied, but this approach is not the only way to determine the thermal properties of the corona. As an illustrative example, another approach, motivated by the fact that the plasma parameters change as the flare evolves (i.e., chromospheric evaporation, conduction, and nonthermal electron heating), is to estimate the plasma parameters using the average values inferred from time interval both before (i.e., 02:08:52–02:09:00 UT) and after (i.e., 02:10:00–02:10:08 UT) the studied interval. This gives average thermal properties of $E_{\text{c0}} = (7.37 \pm 0.45) \times 10^{48} \text{ cm}^{-3}$, $T = (2.52 \pm 0.04)$ keV, and $n_{\text{loop}} = (9.3 \pm 2.2) \times 10^{10} \text{ cm}^{-3}$. Using these values in the main interval of 02:09:00–02:10:00 UT gives electron parameters of $\delta = 4.14 \pm 0.03$, a significantly larger...
c = \pm (\text{keV})

The inferred electron power is now

\[ P = \frac{4.8 \pm 0.8}{10^{27}} \text{ erg s}^{-1}. \]

Thus, this is an excellent example that demonstrates how important it is to determine the thermal properties of the corona as accurately as possible with future X-ray instrumentation, so that the nonthermal electron parameters are determined as accurately as possible using the warm-target fitting method provided.

### 5. Summary and Discussion

The procedure presented in this paper allows us to infer from HXR spectra the characteristics of accelerated electrons, including the electron acceleration rate and the nonthermal electron power. This is achieved by self-consistently considering the evolution of near-thermal electrons, the electrons that carry the bulk of the energy in solar flares. Previous flare energy estimates based on a cold thick-target model provide only an upper limit on the cutoff energy \( E_c \), hence a lower limit on the power \( P \) and the energy contained in nonthermal electrons. Furthermore, the low-energy cutoff \( E_c \) is very poorly constrained by the cold thick-target model, so that the nonthermal electron power \( P \) (Figure 1, bottom panel) can range over two orders of magnitude.

By contrast, our warm-target fitting procedure (available as \texttt{f\_thick\_warm.pro} in OSPEX) provides not only an acceptable fit in terms of \( \chi^2 \) (reduced \( \chi^2 \approx 1 \)), but provides estimates of the electron power \( P \) that lie within a range of \( \pm 6\% \) at the 1\( \sigma \) level, as shown in Figure 1. The electron acceleration rate (s\(^{-1}\)) and power (erg s\(^{-1}\)) are now well constrained because the warm-target approach preserves the number of electrons, which all contribute to various parts of the observed X-ray spectrum. An important feature of the warm-target model is that the density and temperature of the corona as accurately as possible with future X-ray instrumentation, so that the nonthermal electron parameters are determined as accurately as possible using the warm-target fitting method provided.
to whether the temperatures and densities derived from RHESSI data are representative of the conditions at the field lines about which the electrons spiral, and allows the user to determine and constrain the electron properties in different plasma conditions and flare scenarios.

The self-consistent consideration of both thermal and nonthermal electrons provides more realistic estimates for the energy contained in accelerated electrons in solar flares. The detailed procedure explained herein (and the availability of the f_thick_warm.pro function in OSPEX) enables such calculations to be made straightforwardly. While the methodology based on the cold-target model cannot meaningfully constrain the low-energy cutoff and hence determine the energy in accelerated electrons with acceptable precision, the use of the warm-target procedure presented here does constrain the low-energy cutoff to within $\sim (7-14)\%$ uncertainty at the 3$\sigma$ confidence level; this allows the quantitative study of the contribution of accelerated electrons to overall solar flare energetics with uncertainties less than or comparable with those associated with other energetic components.

The authors are thankful to the referee for insightful comments, and to Alexander Warmuth for useful discussions. E.P.K. and N.L.S.J. were supported by a STFC consolidated grant ST/P000533/1. A.G.E. was supported by grant NNX17AI16G from NASA’s Heliophysics Supporting Research program.

Appendix

Kappa-function Injected Spectrum

Bian et al. (2014) have demonstrated that the stochastic acceleration of electrons in the presence of Coulomb collisions leads naturally to an accelerated electron distribution (electrons cm$^{-3}$ (cm s$^{-1}$)$^{-3}$) that takes the form of a kappa distribution (their Equation (17)):

$$f_k(v) = \frac{n_k}{\pi^{3/2} v_{le}^3 \kappa^{3/2}} \frac{\Gamma(k)}{\Gamma(k - \frac{3}{2})} \left(1 + \frac{v^2}{\kappa v_{le}^2}\right)^{-k}, \quad (13)$$

where $v_{le} = \sqrt{2k_BT_e/m_e}$ is the thermal speed, and $\kappa = \Gamma_{col}/2D_0$ is the ratio of the collisional and diffusional coefficients in the Fokker–Planck equation describing electron transport. Equation (13) is valid at all velocities; there is no “low-energy cutoff” in the accelerated electron spectrum. For a collision-dominated environment, $\kappa$ is large and the distribution (Equation 13) approximates a Maxwellian form $\exp(-v^2/v_{le}^2)$, while for a turbulence-dominated environment, $\kappa$ is small and the distribution (Equation 13) approximates a power-law form $\sim v^{-2\kappa}$. Figure 5 demonstrates the similarities and the differences between two spectra from Equation (18) and from Equation (12). At high energies $E \gg k_BT$, the distributions look like a power law $N_k(E) \propto E^{-\kappa+1}$, but they behave differently in the deka-keV range (Figure 5).

The accelerated electron spectrum $N_k(E)$ corresponding to the form (13) is

$$\langle n_{\text{VF}}(E) \rangle \approx \Delta E m_e \frac{8}{\pi \kappa} \left(\frac{E}{k_BT_e}\right)^{3/2} e^{-E/k_BT_e}$$

$$= \frac{E}{K} \int_0^\infty N_k(E_0) dE_0$$

$$= \Delta E m_e \frac{8}{\pi \kappa} \left(\frac{E}{k_BT_e}\right)^{3/2} e^{-E/k_BT_e}$$

$$+ \frac{N_0}{K} \left[1 + \left(1 - \frac{1}{\kappa}\right) \frac{E}{k_BT_e}\right]. \quad (17)$$

where, using Equations (6) and (10) and the fact that the kappa distribution is flat at low energies,

$$\Delta E m_e \approx \frac{\pi}{K} \sqrt{\frac{m_e}{8}} (k_BT_e)^2 \frac{N_0}{E_{\text{min}}^{3/2}} \approx \frac{\pi}{K} \sqrt{\frac{m_e}{8}} (k_BT_e)^2 \frac{N_0}{E_{\text{min}}^{3/2}}$$

$$\approx \frac{\pi}{K} \sqrt{\frac{m_e}{24 \pi L}} (k_BT_e)^{3/2} N_0, \quad (18)$$

where $N_0$ (s$^{-1}$) is the total injected rate. Equations 8 (17) and (18) (see Equations (12) and (10)) give the mean electron

\[ E = \text{Electron energy, [keV]} \]

Figure 5. Electron injection spectrum $N(E)$ for a kappa distribution with $\kappa = 4$, $k_BT = 1.8$ keV (Equation (16); solid line) and a power law with $\delta = 3$ and a 10 keV low-energy cutoff (Equation (3); red dashed line).

where $A$ is the cross-sectional area of the loop. The total rate of electron injection is

$$N_0 = \int_0^\infty N_k(E) dE = 2 A \frac{2k_BT_e}{m_e} \frac{1}{\kappa^{1/2}}$$

$$\times \left(\frac{k - \frac{3}{2}}{\kappa}\right) \left(1 + E/k_BT_e\right)^{\kappa-1/2}, \quad (15)$$

Figure 5. Electron injection spectrum $N(E)$ for a kappa distribution with $\kappa = 4$, $k_BT = 1.8$ keV (Equation (16); solid line) and a power law with $\delta = 3$ and a 10 keV low-energy cutoff (Equation (3); red dashed line).
flux corresponding to the injection of a kappa distribution, and it should be reemphasized that both the thermal and nonthermal parts of Equation (17) are determined by the form of $N(E_0)$.

**ORCID iDs**

Eduard P. Kontar @ https://orcid.org/0000-0002-8078-0902

Natasha L. S. Jeffrey @ https://orcid.org/0000-0001-6583-1989

A. Gordon Emslie @ https://orcid.org/0000-0001-8720-0723

**References**

Alaoui, M., & Holman, G. D. 2017, ApJ, 851, 78

Aschwanden, M. J., Caspi, A., Cohen, C. M. S., et al. 2017, ApJ, 836, 17

Aschwanden, M. J., Holman, G., O’Flannagain, A., et al. 2016, ApJ, 832, 27

Bai, T., & Ramaty, R. 1978, ApJ, 219, 705

Battaglia, M., Motorina, G., & Kontar, E. P. 2015, ApJ, 815, 73

Benz, A. O. 2008, LRSP, 5, 1

Bian, N. H., Emslie, A. G., & Kontar, E. P. 2017, ApJ, 835, 262

Bian, N. H., Emslie, A. G., Stackhouse, D. J., & Kontar, E. P. 2014, ApJ, 796, 142

Brown, J. C. 1971, SoPh, 18, 489

Brown, J. C., Emslie, A. G., & Kontar, E. P. 2003, ApJL, 595, L115

Christe, S., Shih, A. Y., Krucker, S., et al. 2017, AGU Fall Meeting, #SH44A-07

Dennis, B. R. 1988, SoPh, 118, 49

Doschek, G. A., Feldman, U., Kreplin, R. W., & Cohen, L. 1980, ApJ, 239, 725

Emslie, A. G. 1980, ApJ, 235, 1055

Emslie, A. G. 2003, ApJL, 595, L119

Emslie, A. G., Dennis, B. R., Holman, G. D., & Hudson, H. S. 2005, IGRA, 110, 11103

Emslie, A. G., Denniss, B. R., Shih, A. Y., et al. 2012, ApJ, 759, 71

Emslie, A. G., Kontar, E. P., Krucker, S., & Lin, R. P. 2003, ApJL, 595, L107

Emslie, A. G., Kucharek, H., Dennis, B. R., et al. 2004, IGRA, 109, 10104

Galloway, R. K., MacKinnon, A. L., Kontar, E. P., & Helander, P. 2005, A&A, 438, 1107

Hang, E. 1997, A&A, 326, 417

Holman, G. D., Aschwanden, M. J., Aurass, H., et al. 2011, SSRv, 159, 107

Holman, G. D., Sui, L., Schwartz, R. A., & Emslie, A. G. 2003, ApJL, 595, L97

Ireland, J., Tolbert, A. K., Schwartz, R. A., Holman, G. D., & Dennis, B. R. 2013, ApJ, 769, 89

Jeffrey, N. L. S., Kontar, E. P., Bian, N. H., & Emslie, A. G. 2014, ApJ, 787, 86

Jeffrey, N. L. S., Kontar, E. P., & Dennis, B. R. 2015, A&A, 584, A89

Kašparová, J., & Karlický, M. 2009, A&A, 497, L13

Knight, J. W., & Sturrock, P. A. 1977, ApJ, 218, 306

Kontar, E. P., Brown, J. C., Emslie, A. G., et al. 2011, SSRv, 159, 301

Kontar, E. P., Hannah, I. G., & MacKinnon, A. L. 2008, A&A, 489, L7

Kontar, E. P., Jeffrey, N. L. S., Emslie, A. G., & Bian, N. H. 2015, ApJ, 809, 35

Kontar, E. P., MacKinnon, A. L., Schwartz, R. A., & Brown, J. C. 2006, A&A, 446, 1157

Kontar, E. P., Ratcliffe, H., & Bian, N. H. 2012, A&A, 539, A43

Langer, S. H., & Petrosian, V. 1977, ApJ, 215, 666

Lin, R. P., Dennis, B. R., Hurford, G. J., et al. 2002, SoPh, 210, 3

Lin, R. P., & Hudson, H. S. 1976, SoPh, 50, 153

Petrosian, V. 2016, ApJ, 830, 28

Petrosian, V., Donaghy, T. Q., & McTiernan, J. M. 2002, ApJ, 569, 459

Phillips, K. J. H. 2004, ApJ, 605, 921

Press, W. H., Flannery, B. P., & Teukolsky, S. A. 1986, Numerical Recipes. The Art of Scientific Computing (Cambridge: Cambridge Univ. Press)

Saint-Hilaire, P., Krucker, S., & Lin, R. P. 2010, ApJ, 721, 1933

Schmahl, E. J., Pernak, R. L., Hurford, G. J., Lee, J., & Bong, S. 2007, SoPh, 240, 241

Schwartz, R. A., Csillaghy, A., Tolbert, A. K., et al. 2002, SoPh, 210, 165

Smith, D. M., Lin, R. P., Turin, P., et al. 2002, SoPh, 210, 33

Spitzer, L. 1962, Physics of Fully Ionized Gases (New York: Interscience)

Sylwester, B., Sylwester, J., Phillips, K. J. H., Kępka, A., & Mrozek, T. 2014, ApJ, 787, 122

Syrovatskii, S. I., & Shmeleva, O. P. 1972, SvA, 16, 273

Zharkova, V. V., & Gordovskyy, M. 2006, ApJ, 651, 553