Revisiting event horizon finders

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Abstract
Event horizons are the defining physical features of black hole spacetimes, and are of considerable interest in studying black hole dynamics. Here, we reconsider three techniques to find event horizons in numerical spacetimes: integrating geodesics, integrating a surface, and integrating a level-set of surfaces over a volume. We implement the first two techniques and find that straightforward integration of geodesics backward in time is most robust. We find that the exponential rate of approach of a null surface towards the event horizon of a spinning black hole equals the surface gravity of the black hole. In head-on mergers we are able to track quasi-normal ringing of the merged black hole through seven oscillations, covering a dynamic range of about $10^3$. Both at late times (when the final black hole has settled down) and at early times (before the merger), the apparent horizon is found to be an excellent approximation of the event horizon. In the head-on binary black hole merger, only some of the future null generators of the horizon are found to start from past null infinity; the others approach the event horizons of the individual black holes at times far before merger.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
The two-body problem in general relativity has been the focus of extensive work for many years and, because there is no analytic solution, it must be solved numerically. Binary black hole mergers are expected to be one of the most astrophysically common sources of gravitational radiation for detectors such as LIGO [1, 2]. Recent advances in simulating binary black hole mergers include the development of the generalized harmonic evolution system [3] and the moving punctures technique [4, 5]. In the last several years the field has reached a stage where binary black hole simulations are becoming routine. Numerical simulations have been remarkably successful in expanding our understanding of binary black holes, but challenges remain.
One particular challenge is to be able to more accurately locate the holes during the merger. There are two useful concepts to describe the location of black holes in a spacetime, apparent horizons (AH) and event horizons (EH). An EH is the true surface of a black hole: it is defined as the boundary of the region of the spacetime that is causally connected to future null infinity. Because the definition of the EH involves global properties of the spacetime, one must know the full future evolution of the spacetime before the EH can be determined exactly. This difficulty has led researchers to instead identify black holes with apparent horizons, which are defined in terms of the expansion of null congruences. Indeed, AH finders are highly developed and have been the subject of extensive work (see, e.g., the review [6]). Unlike an EH, an AH can be located from data on a single spacelike hypersurface, i.e., on each time step of a numerical evolution, without knowing the future evolution of the spacetime. The AH is often an effective substitute for the EH for several reasons. First, according to the cosmic censorship conjecture, if an AH is present, it must be surrounded by an EH. Second, if an AH is present on a spacelike hypersurface through a stationary spacetime, it coincides with the EH. Finally, in numerical simulations, apparent horizons generally show behaviour attributed to event horizons: For instance, the area of the AH typically does not decrease and it is usually almost constant whenever the spacetime is only mildly dynamic. In fact, apparent horizons have motivated the development of ‘isolated’ and ‘dynamical’ horizons (see [7] for a review). These surfaces satisfy analogues of the laws of black hole thermodynamics, although they are defined quasi-locally, rather than globally.

However, using the AH to locate the holes is not always appropriate. For instance, the AH is slicing dependent, while the EH is not. Indeed, the Schwarzschild spacetime can be sliced in such a way that no AH exists [8]. Furthermore, even on slicings on which an AH is present, there are few precise mathematical statements about how ‘close’ AH and EH are. Finally, AH and EH behave qualitatively differently during a black hole merger: the EH around each black hole expands continuously until the two components of the EH join into one, whereas a common apparent horizon appears discontinuously quite some time after the EHs have merged. The common AH encompasses the two individual AHs, which continue to exist as surfaces of zero outgoing null expansion for some time after the merger.

Early EH finders [9, 10] followed null geodesics forward in time and determined whether or not each geodesic eventually escapes to infinity. Following geodesics forward in time is unstable in that slightly perturbed geodesics will diverge from the EH and either escape to infinity or fall into the singularity. Furthermore, a large number of geodesics with different directions must be sampled at each point and at each time step to determine if one of these succeeds in escaping to infinity [10]. To reduce the number of sampling points, the EH search in [10] was performed on a series of time slices proceeding backward from late to early times; to find the EH on each time slice, they integrated geodesics forward in time, using the already-located EH at the later time as an initial guess.

Since outgoing null geodesics diverge from the event horizon when going forward in time, when going backward in time they will converge onto the event horizon [11, 12]. All recent EH finders use this observation, and follow null geodesics or null surfaces backward in time [6, 11–19].

Several algorithms have been developed to follow null geodesics backward in time. These can be divided into three types, which we shall refer to as the ‘geodesic method’, the ‘surface method’ and the ‘level-set method’. The geodesic method works by simply integrating the geodesic equation, as done by Libson et al [12]. Libson et al express concerns that the

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1 More precisely, we define AH as the outermost marginally outer-trapped surface, where an outer-trapped surface is a topological 2-sphere with zero expansion along outgoing null normals.

2 More precisely, the 2-surface formed by the intersection of the spatial slice and the EH.
geodesic method may be susceptible to tangential ‘drifting’ of the geodesics. However, this is not evident when the method is applied to the science applications in that paper, nor do we find tangential drifting in our simulations. To avoid any issues with drifting, Libson et al introduced the surface method: a complete null surface (rather than individual geodesics) is evolved backward in time. In [11, 12] this surface was parametrized based on axisymmetry (although the parametrization of [11, 12] cannot handle generic axisymmetric situation, cf section 2.2 below), and many interesting results on the structure of caustics and the geometry of the horizon for axisymmetric spacetimes were obtained in [13, 15]. Diener [19] and Caveny et al [16–18] independently introduced the level-set method by recasting the surface method in a way that does not assume symmetry: rather than evolving a single 2D surface, they evolve a volume-filling series of surfaces given as the level sets of a spacetime function \( f(t, x^i) \). To avoid exponentially steepening gradients of \( f \), Caveny et al introduce an artificial diffusive term, whereas Diener reinitializes \( f \) whenever necessary.

This paper re-examines these techniques for event horizon finding in the context of the Caltech/Cornell Spectral Einstein Code (SpEC), which provides an infrastructure for highly accurate simulations of Einstein’s equations for single and binary black holes. Recent work includes highly accurate computations of gravitational waveforms from inspiraling binaries [20–22]. The availability of high accuracy binary evolutions motivates the development of very precise event horizon finding techniques in order to extract all possible physics from these simulations. Therefore, this paper reconsiders the three techniques mentioned above in the context of general binary black hole mergers without any symmetries.

We implement the geodesic method, and generalize the surface method to arbitrary situations without symmetries. Both methods are then applied to single Kerr black holes, and a head-on binary black hole merger. In both cases, the geodesic method is found to be more robust. We encounter two fundamental problems with the level-set method, and therefore halted our efforts to implement it in SpEC.

This paper is organized as follows: in section 2, we explain the three methods in more detail and give details of our numerical implementation. Section 3 presents results for a single Kerr black hole, and in section 4, we apply the techniques to a head-on BBH merger, where we extract ringdown behaviour and the behaviour of the individual event horizons before merger. We close with a conclusion in section 5.

2. Methods

All EH-finding techniques considered here proceed backward in time and must therefore be performed after the numerical evolution of the spacetime has been completed. We assume that we have access to the spacetime metric in a 3+1 decomposition

\[
\text{d} s^2 = -N^2 \text{d} t^2 + \gamma_{ij}(\text{d} x^i + \beta^i \text{d} t)(\text{d} x^j + \beta^j \text{d} t),
\]

where \( N \) is the lapse, \( \beta^i \) is the shift, and \( \gamma_{ij} \) is the 3-metric on the slice. Latin indices \( i, j, \ldots = 1, 2, 3 \) denote spatial dimensions; below we will use Greek indices to denote spacetime dimensions, \( \alpha, \beta \ldots = 0, 1, 2, 3 \). The time \( t \) in (1) represents the coordinate time of the numerical evolution. Typically, the metric data \( \gamma_{ij}, \beta^i \), and \( N \) are available at discrete times and at discrete spatial grid points. Evaluating the values of the metric components elsewhere requires interpolation.

A black-hole merger exhibits several characteristic features of relevance to EH finders, as illustrated in figure 1 (cf [12, 18, 15]). At times sufficiently far prior to merger, the EH and

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3 While figure 1 is meant as an illustration, it presents actual data from the head-on binary black hole merger discussed in section 4. The time given at each frame of figure 1 will aid in the discussion in section 4.
AH are expected to coincide closely (and indeed, we confirm this below for our simulation). The green dashed curves in figure 1 represent future generators of the event horizon, i.e. null geodesics that will merge onto the event horizon through cusps in the individual event horizons. These cusps are clearly visible at time \( t = 13.5M \) where the individual EHs have diverged significantly from their respective AHs. At \( t_{\text{CEH}} = 14.6M \) the two previously disjoint components of the event horizon join. We shall refer to this time \( t_{\text{CEH}} \) as the merger of the black hole binary. After the merger, the event horizon of the merged black hole can be seen relaxing towards its final time-independent shape. The common apparent horizon appears at \( t_{\text{CAH}} = 17.8M \), and approaches the event horizon as the evolution proceeds; at \( t = 80M \), the AH coincides almost exactly with the event horizon.

2.1. Geodesic method

The most straightforward way to follow light rays is to simply integrate the geodesic equation [9–12],

\[
\frac{d^2 q^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dq^\alpha}{d\lambda} \frac{dq^\beta}{d\lambda} = 0,
\]

where \( q^\mu(\lambda) \) is the position of the photon on the geodesic, parametrized by an affine parameter \( \lambda \), and \( \Gamma^\mu_{\alpha\beta} \) are the spacetime Christoffel symbols.

Since we have access to our spacetime as a function of the evolution time coordinate \( t \), it is convenient to rewrite (2), replacing \( \lambda \) by \( t \) along the geodesic. Writing \( \dot{q}^\mu = dq^\mu/dt \), and \( a = d\lambda/dt \), we find

\[
\frac{dq^\mu}{d\lambda} = \frac{1}{a} \dot{q}^\mu,
\]

\[
\frac{d^2 q^\mu}{d\lambda^2} = \frac{1}{a^2} \ddot{q}^\mu - \frac{a}{a^2} \dot{q}^\mu.
\]
Substituting into the geodesic equation we obtain
\[ \ddot{q}^\mu = \frac{\dot{a}}{a} \dot{q}^\mu - \Gamma_{\alpha\beta}^{\mu} \dot{q}^\alpha \dot{q}^\beta. \] (5)

The quantity \( a \) is determined by the requirement that \( q^0 = t \), i.e. that at parameter value \( t \) along the geodesic, the geodesic is on the corresponding \( t = \text{const} \) hypersurface of the evolution. This implies \( \dot{q}^0 = 1, \dot{q}^i \) and \( \ddot{q}^i = [0, \dot{q}^i] \). Setting \( \mu = 0 \) in (5) gives \( \frac{\dot{a}}{a} = \Gamma_{\alpha\beta}^{0} \dot{q}^\alpha \dot{q}^\beta \). The spatial components of (5) are the desired evolution equation for the spatial coordinates as a function of coordinate time,
\[ \ddot{q}^i = \Gamma_{\alpha\beta}^{0} \dot{q}^\alpha \dot{q}^\beta \dot{q}^i - \Gamma_{\alpha\beta}^{i} \dot{q}^\alpha \dot{q}^\beta. \] (6)

We convert this set of ordinary differential equations to first-order form by defining \( p^i = \dot{q}^i \), which gives
\[ \dot{q}^i = p^i, \] (7a)
\[ \dot{p}^i = \Gamma_{\alpha\beta}^{0} p^\alpha p^\beta p^i - \Gamma_{\alpha\beta}^{i} p^\alpha p^\beta, \] (7b)
facilitates the use of standard ODE integrators such as Runge–Kutta methods [23, 24].

While integrating geodesics is not new [10, 12], re-expressing the geodesic equation in terms of coordinate time seems to be new. It appears that the primary reason this technique has been phased out in favour of the two techniques described below is the concern that, in a full 3D implementation, slight tangential velocities may be imparted to the outgoing null geodesics through numerical inaccuracies, and that this tangential drift of geodesics could result in unphysical caustics. These concerns are discussed in detail in [12], where the idea of representing the whole surface, rather than individual geodesics, was introduced. This was justified on the basis that for a surface, tangential drift is irrelevant. However, while it is possible that tangential drift can be significant for very coarse, low-resolution simulations, we see no evidence that tangential drift affects our numerical tests of the geodesic method.

We finally like to point out that if one evolves \( p_i = g_{\mu\nu} p^\mu \) instead of \( p^i \) (cf [10]), then the evolution equations depend only on spatial derivatives of the spacetime metric. Evolving \( p_i \) therefore results in computational savings, because the time derivatives of the metric need not be stored or interpolated. This will be investigated in a future work.

2.2. Surface method

The idea of the surface method dates back to Libson et al [12], who used it in axisymmetry. The goal is to evolve a two-dimensional surface \( S_t \) backward in time such that it traces out a null hypersurface \( \mathcal{N} \). The time coordinate \( t \) is inherited from the black hole simulation for which event horizons are to be determined, i.e. \( S_t \) is the intersection of \( \mathcal{N} \) with the spatial hypersurfaces \( \Sigma_t \) of the evolution, as indicated in figure 2. Before the black hole merger, \( t < t_{\text{CEH}} \), the surface \( S_t \) consists not only of the two disjoint parts of the event horizon, but also includes the future generators, which are indicated by the green dashed curves in figure 1. The union of these three components is a smooth self-intersecting surface with the topology of a sphere (as suggested by Kip Thorne [11, 12]).

Let us first consider how to represent the surface to be evolved. Apparent horizon finders often parametrize a surface by giving the radius, relative to a fixed point, as a function of angular coordinates, i.e. \( r = f(\theta, \phi) \). Such a star-shaped surface is insufficient here, because the surface will be self-intersecting for \( t < t_{\text{CEH}} \) and will cease to be star-shaped even before then (see figure 1.1 of [6]). The axisymmetric EH finder presented in [12] parametrized the
surface by $\rho = s(z,t)$, where $z$ is a coordinate along the axis of symmetry, and $\rho$ is the cylindrical radius. This allows some mild form of self-intersection, such as, for instance, the $t = 13.5M$ snapshot in figure 1. However, at earlier times, the locus of future null generators of the horizon ‘bulges outward’ and becomes multivalued when considered as a function of $z$, cf $t = 9M$ in figure 1. In this case, the parametrization of [12] fails even for an axisymmetric configuration. In this paper, we use a parametric representation of $S_t$, i.e. $r^i = r^i(t,u,v)$. The full three-dimensional null hypersurface $\mathcal{N}$ being constructed is represented as a 3-parameter surface in spacetime:

$$r^\mu(t,u,v) = [t, r^i(t,u,v)] .$$

(8)

We wish to find an equation that will allow us to evolve $S_t$ in such a way as to trace out the null 3-surface $\mathcal{N}$. Further, we would like this equation to have the property that for fixed $(u_0, v_0)$, the curve $r^\mu(t,u_0,v_0)$ traces out a null geodesic. This allows us to directly compare the surface obtained by the surface method to the surface obtained for equivalent initial conditions by the geodesic method.

For the curve $r^\mu(t,u,v)$ to be null, its tangent $\partial r^\mu(t,u,v)/\partial t$ must be outgoing and null, i.e.

$$\frac{\partial r^\mu}{\partial t} = \ell^\mu, \quad (9)$$

where $\ell^\mu$ is a null normal to $S_t$. $\ell^\mu$ can be written as

$$\ell^\mu = c(n^\mu + s^\mu) , \quad (10)$$

where $n^\mu$ is the timelike unit normal to $\Sigma_t$, $s^\mu$ is the spatial outward-pointing unit normal to $S_t$ (cf figure 2) and $c$ is an overall scaling. Consistency of (8) and (9) requires that $\ell^\mu$ is normalized such that $\ell^t = 1$. To find the value of $c$ from the condition $\ell^t = 1$, first note that from the 3+1 decomposition,

$$n^\mu = \frac{1}{N} [1, -\beta^i] , \quad (11)$$

where $N$ and $\beta^i$ are the lapse and shift fields. Also, since $s^\mu$ lies within the spatial slice $\Sigma_t$, we may write $s^\mu = [0, s^i]$, so that (10) becomes

$$\ell^\mu = c \left[ \frac{1}{N} s^i - \frac{1}{N} \beta^i \right] . \quad (12)$$

Thus $\ell^t = 1$ implies $c = N$, and we can write our final evolution equation for the spatial components of $r^i$,

$$\frac{\partial r^i}{\partial t} = Ns^i - \beta^i . \quad (13)$$
In order to find the unit normal \( \mathbf{s}' \) to the spatial surface \( S_t \), we follow the standard procedure for a surface parametrized as \( r^i(u, v) \), i.e.,

\[
\mathbf{s}' = \gamma^{ij} \epsilon_{ijk} \frac{\partial r^j}{\partial u} \frac{\partial r^k}{\partial v},
\]

\[
\rho = \sqrt{\gamma_{ij} \mathbf{s}' \mathbf{s}'},
\]

\[
s' = \rho^{-1} \mathbf{s}',
\]

(14a, 14b, 14c)

where \( \epsilon_{ijk} \) is the antisymmetric tensor and where we have chosen the sign of the root such that \( s' \) points outward for a right-handed choice of coordinates.

This evolution equation for the surface method (13) is very different from the evolution equations for the geodesic equations (7a) and (7b). The surface method does not require derivatives of the metric, but derivatives \( \partial u r^i, \partial v r^i \) along the surface; the geodesic method, in contrast, requires derivatives of the metric, but treats each geodesic completely independently. Nevertheless, due to our choice of evolution equation (9), each point on the parametrization of the surface traces its own geodesic; see appendix A for a proof.

2.3. Level-set method

The level-set method [12, 14–19] utilizes a function \( f = f(t, x^i) \) defined on the full spacetime (or at least, a region of spacetime covering the vicinity of the expected location of the EH). The function \( f \) is determined such that \( f = \text{const} \) contours (i.e., level sets) represent null surfaces, i.e., \( g^{\alpha\beta} \partial_\alpha f \partial_\beta f = 0 \). In the 3+1 decomposition, this becomes [17–19]

\[
\partial_t f = \beta^i \partial_i f \mp N \sqrt{\gamma_{ij} \partial_i f \partial_j f},
\]

(15)

where the \( \mp \) accommodates both ingoing and outgoing null surfaces, with the minus sign being appropriate for outgoing null surfaces if the gradient \( \partial_i f \) is outward pointing.

Libson et al [12] had previously made use of (15), but parametrized the \( f = 0 \) contour based on axisymmetry. The motivation of evolving (15) directly in the volume is to remove any assumptions of symmetries.

Unfortunately, when trying to implement the level-set method in SpEC, we encountered two fundamental problems. The first difficulty is related to the characteristic speed of the level-set method. Simply put, all \( f = \text{const} \) contours approach the event horizon, therefore new contours need to be filled in at the boundaries of the region in which \( f \) is evolved (i.e., the outer boundary and possibly one or more inner boundaries if black hole excision is employed). To see this, note that the characteristic speed of (15) relative to a spatial direction \( \hat{n}_i \) is

\[
v = N \hat{n}_i \frac{\partial^i f}{\sqrt{\gamma^{ij} \partial_j f \partial_j f}} = -\hat{n}_i \beta^i,
\]

(16)

where the sign of the first term depends on the gradient \( \partial_i f \) being outward pointing. For most coordinate systems of interest, lapse \( N \) and shift \( \beta^i \) behave such that \( v > 0 \) at the outer boundary and at any excision boundaries (if present). When integrating (15) backward in time, well posedness requires boundary conditions at these boundaries. Our preferred numerical techniques are spectral methods because of their promise to achieve exponential convergence for smooth problems. Spectral methods are very sensitive to the existence of an underlying well-posed continuum problem and therefore require boundary conditions. Unfortunately there is no particular physical reasoning to suggest a choice of boundary condition. While essentially any choice of boundary condition that results in \( f \) being continuous rendered our
spectral level-set implementation stable, and convergent to at least first order, we have been unable to find a boundary condition that ensures that \( f \) remains smooth and thus leads to the desired exponential convergence, not even in the single black hole case. A full finite-difference evolution of \( f \) would be less sensitive to the lack of proper boundary conditions (see [19]), but would be much slower for finding an EH in spectral-code metric data (due to interpolations from the spectral to the finite-difference grid) and much less accurate.

The second fundamental difficulty lies in singular behaviour of the function \( f \) in certain cases. Let us consider an equal-mass head-on merger as depicted in figure 1. Assume \( f \) to be smooth, and let us focus on the value of \( f \) at the point of symmetry, marked with \( C \) in figure 1. We assume that \( \partial_\nu f \) is outward pointing near the event horizon. At late times, after the merger, \( f \) will be negative at \( C \), because \( C \) is inside the event horizon. Throughout the whole simulation, \( \partial_\nu f = 0 \) at \( C \) by symmetry, and therefore, (15) implies that \( \partial_t f = 0 \) there, so that \( f \) at \( C \) remains fixed at a finite negative value. At merger, however, the \( f = 0 \) contour passes through \( C \). Therefore, \( f \) must be singular\(^4\). Any method for solving the level-set equations that assumes a smooth and regular solution (including finite-difference methods that do not explicitly treat the singularity) will therefore produce results that differ from the exact solution at the singular point. In [19], one-sided finite-difference stencils are carefully chosen so as not to differentiate across the singularity.

Because of these two issues we have stopped development of a spectral implementation of the level-set method. These problems arise because of properties of the function \( f \), which is merely a tool to represent the actual surface of interest, \( f = 0 \). This surface itself is well behaved and smooth, suggesting it will be possible to evolve this surface directly. Geodesic and surface methods do precisely this, and so we focus on these two methods in the remainder of this paper.

2.4. Numerical implementation

Compared to the implementation of the geodesic method, implementing the surface method is somewhat more complex due to the presence of derivatives along the surface in (14a). Apart from this, geodesic method and surface method share rather uniform implementation details. We shall first discuss those aspects that only apply to the surface method, and then follow with aspects applicable to both methods.

We represent the surface \( \mathbf{r}'(t, u, v) \) with spectral methods (e.g. [25]). These methods approximate a desired function \( U(x, t) \) as a truncated expansion in basis functions \( \phi_k \), for instance Chebyshev polynomials or spherical harmonics:

\[
U(x, t) = \sum_{k=0}^{N-1} \tilde{U}_k(t) \phi_k(x),
\]

where \( N \) is the order of the expansion. The fundamental advantage of spectral methods lies in their fast convergence: for smooth problems and a suitable choice of basis functions, the error of the approximation (17) decreases exponentially with the number of basis functions per dimension [25]. Derivatives of the function \( U \) are computed via the (analytically known) derivatives of the basis functions. Each set of basis functions has an associated set of collocation points \( \mathbf{x}_i \); a matrix multiplication translates between function values at the collocation points, \( U(\mathbf{x}_i) \), and spectral coefficients \( \tilde{U}_k \).

\(^4\) Even with reinitializations of \( f \), as performed in [19], the same argument applies to that time interval between reinitializations during which the topology of the EH changes.
For the surface method, we represent each Cartesian component of $r^i(t, u, v)$ (cf (8)) as an expansion in scalar spherical harmonics,

$$r^i(t, u, v) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \tilde{A}^{i}_{l m}(t)Y^{i}_{l m}(u, v).$$  \hspace{1cm} (18)$$

This expansion assumes that at fixed $t$, the surface has topology $S_2$. Note that (18) allows the surface to intersect itself, as necessary in a binary merger for $t < t_{CEH}$ (cf figure 1). Self-intersection is possible because the coordinates $u$ and $v$ are not assumed to be standard spherical angular coordinates, i.e. relations such as $\cos(u) = z/\sqrt{x^2 + y^2 + z^2}$ will in general not hold.

For spherical harmonics $Y_{lm}(u, v)$, the collocation points form a rectangular grid in $(u, v)$, with the $u$ values chosen so that $\cos(u)$ are the roots of the Legendre polynomial of order $L+1$, and with the $v$ values being uniformly distributed in the interval $[0, 2\pi]$. There are in total

$$N = 2(L + 1)^2$$  \hspace{1cm} (19)$$
collocation points. The evolution equations (13) require derivatives $\partial_r r^i$ and $\partial_v r^i$, which are computed by transformation to spectral coefficients, application of recurrence relations and inverse transform (using the SpherePack library [26]). These derivatives are then substituted into (13) and (14) to compute $\partial_t r^i$, which is evolved at the collocation points.

We represent each Cartesian component $r^i$ as an expansion in scalar spherical harmonics (see (18)) in order to reuse the infrastructure already developed for our spectral evolution code, which represents tensors of arbitrary rank in this manner to simplify our spectral expansions and to simplify communication of tensor quantities across subdomains of different shapes (see, e.g., [27, 28]). An alternative approach would be to represent $r^i$ in terms of vector spherical harmonics, i.e.,

$$r^i(t, u, v) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \tilde{A}^{i}_{lm}(t)Y^{i}_{lm}(u, v).$$  \hspace{1cm} (20)$$

The downside of choosing a scalar spherical harmonic representation is that the equation we impose on the highest order vector spherical harmonics is incorrect, and this leads to an instability. This difficulty with expanding vector quantities in a scalar spherical harmonic basis is cured [27] by performing the following ‘filtering’ operation at each time step: first transform $r^i$ to a vector spherical harmonic basis, then remove the $\ell = L$ and $\ell = L - 1$ coefficients, and then transform back. The removal of both the highest and second highest tensor harmonic modes is necessary, since transforming an $n$th rank tensor from a tensor spherical harmonics to scalar spherical harmonics requires scalar harmonics of up to $L_{\text{scalar}} = L_{\text{tensor}} + n$. We filter two modes because we wish to correctly represent the spatial derivatives of $r^i$ (see (14a)), which are effectively rank 2.

The geodesic method simply evolves the ODEs (7a)–(7b). While each geodesic is evolved independently, we find it nevertheless convenient to represent them as a two-dimensional grid, $q^i(t, u, v)$ where parameters $u$ and $v$ label each geodesic. We use the same parameters $u$ and $v$ for geodesic and surface method, and for this paper, we choose to locate the geodesics at the same $(u, v)$ values as the collocation points of the surface method. We note that this choice is based on convenience to simplify comparison between the two methods; geodesics can be placed at any location, and indeed, we plan as a future upgrade of the geodesic method an adaptive placement of geodesics to help resolve interesting features such as caustics.

Let us now discuss aspects common to the implementation of the geodesic and surface methods: at some late time $t = t_{end}$ long after merger, we initialize the EH surface by choosing
it to be the AH at that time. Our AH finder parametrizes the radius of the AH as a function of standard azimuthal and longitudinal angles on $S^2_r$, $r_{\text{AH}}(t_{\text{end}}, \theta, \phi)$, i.e.,

$$r_{\text{AH}}^i(t_{\text{end}}, \theta, \phi) = r_{\text{AH}}(t_{\text{end}}, \theta, \phi) \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}. \quad (21)$$

When initializing the event horizon surface, we choose $(u, v)$ to coincide with the standard spherical angular coordinates $(\theta, \phi)$, i.e., we set

$$r^i(t_{\text{end}}, u, v) = r_{\text{AH}}^i(t_{\text{end}}, u, v), \quad \text{surface method,} \quad (22)$$

$$q^i(t_{\text{end}}, u, v) = r_{\text{AH}}^i(t_{\text{end}}, u, v), \quad \text{geodesic method.} \quad (23)$$

For the geodesic method we further set $p^i(t_{\text{end}}, u, v) = s_{\text{AH}}^i$, where $s_{\text{AH}}^i$ is the unit normal to the apparent horizon, which is computed similarly to (14a)–(14c). Time stepping is conducted using a fourth-order Runge–Kutta algorithm.

Both methods require interpolation of certain quantities such as the spatial metric $\gamma_{ij}$ onto the grid points of the surface $r^i(t, u, v)$. For the spectral evolutions of the Caltech–Cornell group [20, 21, 28], the evolution data are represented as spectral expansions in space (for each fixed time $t$) and spatial interpolation is performed by evaluating the appropriate spectral expansions (17) at the desired spatial coordinates $r^i$. Evolution data are available at discrete evolution times $t_n$ and temporal interpolation is performed with sixth-order Lagrange interpolation (i.e., utilizing three time slices on either side of the required time).

Finally, we define an area element $\sqrt{h}$ on the surface as the root of the determinant of the induced metric,

$$h = \frac{1}{\sin^2 u} \det \begin{pmatrix} \gamma_{ij} \partial_u r^i \partial_u r^j & \gamma_{ij} \partial_u r^i \partial_v r^j \\ \gamma_{ij} \partial_v r^i \partial_u r^j & \gamma_{ij} \partial_v r^i \partial_v r^j \end{pmatrix}. \quad (24)$$

The area of the evolved surface is then given by

$$A(t) = \int \sqrt{h(t, u, v)} \sin u \, du \, dv. \quad (25)$$

Explicitly pulling out the factor $\sin u$ in (24) and (25) ensures that $\sqrt{h}$ is a constant for a coordinate sphere in Euclidean space; this will simplify figure 11. Since all the geodesics (or surface grid points) are on a Legendre–Gauss grid, we compute the derivatives in (24) spectrally, and we evaluate (25) by Legendre–Gauss quadrature. For binary hole mergers before merger, we sometimes evaluate $h$ based on finite-difference derivatives $\partial_u r^i$ and $\partial_v r^i$. This is discussed in detail in section 4.4.

### 3. Application to Kerr spacetime

Initial tests of the event horizon finder are conducted using the Kerr spacetime in Kerr–Schild coordinates (See section 33.6 of [29]):

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H l_\mu l_\nu. \quad (26)$$

Here $H$ is a scalar function of the coordinates, $\eta_{\mu\nu}$ is the Minkowski metric, and $l^\mu$ is a null vector. In Cartesian coordinates $(t, x, y, z)$, the functions $H$ and $l^\mu$ for a black hole of mass $M$ and dimensionless spin parameter $a/M$ in the $z$ direction are

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5 The spectral spatial interpolation is computationally more expensive than temporal Lagrangian interpolation. Whenever the domain decomposition for the Einstein evolution is identical for all time steps involved in a temporal interpolation, the time interpolation is performed before the spatial interpolation. In that case, only one spectral spatial interpolation is necessary (on the time-interpolated data), rather than six, speeding up the computation.
\[ H = \frac{M r_{BL}^3}{r_{BL}^4 + a^2 z^2}, \quad (27a) \]
\[ l_\mu = \left( 1, \frac{x r_{BL} + a y}{r_{BL}^2 + a^2}, \frac{y r_{BL} - a x}{r_{BL}^2 + a^2}, \frac{z}{r_{BL}} \right), \quad (27b) \]

where \( r_{BL}(x, y, z) \) is the Boyer–Lindquist radial coordinate, defined by
\[ r_{BL}^2 = \frac{1}{2}(x^2 + y^2 + z^2 - a^2) + \left( \frac{1}{4}(x^2 + y^2 + z^2 - a^2)^2 + a^2 z^2 \right)^{1/2}. \quad (28) \]

If we define the Kerr–Schild spherical coordinates in the straightforward way (\( r = \sqrt{x^2 + y^2 + z^2}, \cos(\theta) = z/r, \) etc), we find that the event horizon of the Kerr black hole in these coordinates is given by
\[ r_{Kerr} = \sqrt{r^4 + r^2 a^2 \cos^2 \theta}, \quad (29) \]

where \( r_+ \equiv M + \sqrt{M^2 - a^2} \). The surface area of the event horizon is given by
\[ A_{Kerr} = 8\pi M(M + \sqrt{M^2 - a^2}). \quad (30) \]

For our tests on Kerr spacetime we choose the same initial surface for both the surface and geodesic methods: a coordinate sphere of radius \( r = 2.5M \), which does not coincide with the horizon. The evolution begins at \( t_{end} = 0 \) and proceeds backward in time towards negative \( t \). Because we choose to place geodesics coincident with the collocation points of the surface method (see section 2.4), we can use the highest angular index \( L \) as a measure of resolution. The total number of geodesics or grid points is given by (19). The choice of spin in the \( z \) direction is for convenience. We have repeated the numerical tests below for spins of several different orientations, and we find no substantial difference in either stability or accuracy.

In order to test our methods of finding an EH, we use two measures of error. The first measures the error in the coordinate location of the event horizon. We define
\[ \Delta r(u, v) = r(u, v) - r_{Kerr}(\theta(u, v), \phi(u, v)). \quad (31) \]

where \( r(u, v), \theta(u, v), \) and \( \phi(u, v) \) are the Kerr–Schild radial and angular coordinates of the surface, which are found from either the surface-method variables \( r^i(u, v) = [x(u, v), y(u, v), z(u, v)] \) or the geodesic-method variables \( q^i(u, v) = [x(u, v), y(u, v), z(u, v)] \) in the usual way, e.g., \( x(u, v) = r(u, v) \sin \theta(u, v) \cos \phi(u, v) \). Specifically, we will use the root-mean-square of \( \Delta r \) over all grid points or geodesics, which we shall denote by \( ||\Delta r|| \), as a global measure of the error.

Our second error measure is the deviation of the area of our surface from the Kerr value,
\[ \Delta A = A(t) - A_{Kerr}. \quad (32) \]

where \( A(t) \) is determined by equation (25).

Figure 3 shows errors in the AH surface as computed using the geodesic method for a Kerr black hole. The error measure \( ||\Delta r|| \), (31), does not change with \( L \) because the evolution of each geodesic is independent of the total number of geodesics. The error measure \( ||\Delta A|| \), (32), does depend on \( L \), but only because the computation of the surface area depends on all geodesics. It is clear from figure 3 that the geodesic method can stably model Kerr black holes of any spin.

At \( t_{end} = 0 \), we start the EH finder with an initial surface that does not coincide with the EH of Kerr. Therefore, figure 3 shows initial transients as the surface being followed by the EH finder approaches the EH of Kerr. Figure 4 shows an enlargement of this phase. We find
Figure 3. Geodesic method applied to a Kerr black hole. The top panels show the area difference between the computed and exact solution, normalized by the area of the exact solution. The bottom panels show the difference between the computed and exact location of the EH, as measured by (31). These data are shown for two series of runs: in the left panels we keep the dimensionless spin of the black hole fixed at $a/M = 0.6$ and vary the resolution $L$ of the EH finder. In the right panels we vary the spin parameter $a/M$ at fixed resolution. In all cases, the EH finder starts at $t = 0$ and the geodesics are evolved backward in time.

Figure 4. Approach of the tracked null surface onto the event horizon of Kerr black holes with various spins. The symbols show the numerical data (the same data as in the lower-right panel of figure 3), and the solid lines are representative least-squares fits. Table 1 compares the numerically computed e-folding time to the surface gravity of the black hole.
that the tracked surface approaches the Kerr EH exponentially when integrating backward in time,
\[ ||\Delta r|| \propto e^{t/\tau}. \]  \hspace{1cm} (33)

The time scale \( \tau \) depends on the spin of the Kerr background. It has been shown in a number of coordinate systems [12, 18, 19] that the e-folding time for a non-spinning black hole is \( \tau = 4M \). This is not true in all coordinate systems: for example, in Schwarzschild coordinates \( \tau = 2M \). In appendix B, we generalize this result to show that null geodesics, perturbed from the Kerr EH, diverge from the EH exponentially with an e-folding time equal to \( 1/g_H \), where
\[ g_H = \frac{\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})}, \] \hspace{1cm} (34)
is the surface gravity of the horizon in Kerr–Schild coordinates. In table 1, we compare the numerically computed e-folding time \( \tau \) (obtained by least-squares fits) to \( g_H \), and find excellent agreement.

We now turn our attention to the surface method. For a Schwarzschild black hole, the surface method with the standard tensor spherical harmonic filtering is stable, as shown by the ‘\( F = 0 \)’ line in the left panel of figure 5. However, the method is unstable for spinning black holes and fails within about \( 10M \) for spin \( a/M = 0.6 \) (see the ‘\( F = 0 \)’ line in the right panel of figure 5).

Therefore, we perform additional filtering for spinning black holes. After each time step, we compute
\[ R(u, v) = \sqrt{\delta_{ij} r^i(u, v) r^j(u, v)}, \] \hspace{1cm} (35)
expand \( R(u, v) \) in scalar spherical harmonics,
\[ R(u, v) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \tilde{R}_{\ell m} Y_{\ell m}(u, v), \] \hspace{1cm} (36)
and truncate the highest \( F \) modes of this expansion:
\[ \tilde{R}_{\ell m} \to 0, \quad \text{for} \quad \ell > L - F. \] \hspace{1cm} (37)
From these filtered coefficients, we reconstruct the filtered radius function \( R_F(u, v) \) and replace
\[ r^i \to \frac{R_F}{R} r^i. \] \hspace{1cm} (38)

The right panel shows that with appropriate choice of \( F \), the horizon of a Kerr black hole with spin \( a/M = 0.6 \) can be followed for thousands of \( M \). Unfortunately, we do not understand

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**Table 1.** Exponential approach of the null surface to the correct event horizon location. \( M_{gH} \) represents the (dimensionless) surface-gravity of a Kerr black hole with spin \( a/M \). \( M/\tau \) is the numerical rate of approach as determined by fits to the data shown in figure 4.

| \( a/M \) | \( M_{gH} \) | \( M/\tau \) | \( M_{gH} - M/\tau \) |
|---|---|---|---|
| 0.0 | 1/4 = 0.25 | 0.249998 | 2 \times 10^{-6} |
| 0.2 | 0.247449 | 0.247440 | 9 \times 10^{-6} |
| 0.4 | 0.239110 | 0.239093 | 1.7 \times 10^{-5} |
| 0.6 | 0.222222 | 0.222212 | 1.0 \times 10^{-5} |
| 0.8 | 3/16 = 0.1875 | 0.187500 | <10^{-6} |
| 0.9 | 0.151784 | 0.151784 | <10^{-6} |
| 0.99 | 0.061814 | 0.061814 | <10^{-6} |
the effect of $F$ on stability, and therefore a parameter search through possible values for $F$ is required.

With this additional filtering in place, we now examine the convergence and accuracy of the surface method. Figure 6 shows the convergence behaviour of the surface method. From the top plots, we can see that for a black hole of moderate spin ($a/M = 0.6$), the surface method is accurate and convergent, although long-term stability issues remain. Also, the surface area computed by the surface method appears to be more accurate than the location of the surface, cf upper versus lower panels of figure 6. This arises, because for a small change $\delta \tilde{A}_{lm}$ in an expansion coefficient $\tilde{A}_{lm}$ in (18) with $\ell \neq 0$, the change in $||\Delta r||$ is linear in $\delta \tilde{A}_{lm}$, whereas the change in area is quadratic. The high accuracy of $A_{EH}$ is a welcome feature, since the EH area is one of the most important results of an EH finder. Unfortunately, the surface method is not capable of tracking the horizon for spins $a/M \gtrsim 0.8$ for a useful length of time.

While the geodesic method appears superior in these Kerr tests, there are two main benefits to implementing the surface method. First, it is computationally more efficient. The bulk of processing time is spent on interpolating the metric data from the simulation, and the surface method requires the metric only (ten components) whereas the geodesic method requires the metric, as well as its spatial and time derivatives (50 components). Second, the surface method can be used to check the errors in the geodesic method in circumstances where the surface method performs well, i.e., lower spins.

For these tests, the initial set of geodesics (or surface) is chosen to be a sphere of radius 2.5$M$. In this case it requires a time $\gtrsim 100M$ for either method to converge onto the actual event horizon. This shows that for cases in which the actual EH is unknown, it is important to have a near-stationary situation at the end of the simulation, so that the initial guess (generally taken to be the AH) has time to converge onto the true EH. The length of this interval will depend on the desired accuracy, the quality of the initial guess and the spin of the black hole. For example, during a time $\Delta t = 10/g_H$ (i.e., 40$M$ for $a/M = 0$, but 160$M$ for $a/M = 0.99$) the tracked surface will have approached the EH to a fraction $e^{-10} \simeq 5 \times 10^{-5}$ of the distance between the initial guess and the EH.
4. Head-on binary black hole merger

4.1. Details of BBH evolution

When looking for a straightforward dynamical spacetime where tracking the event horizon is of interest, one of the standard scenarios is the head on merger of two equal-mass non-spinning black holes [11–13, 15, 18]. First, the SpEC code is utilized to evolve the solution of Einstein’s equations for the head-on merger. Initially the holes are at rest, \( r \approx 4.5M \) apart, where \( M = M_A + M_B \) is the total mass at \( t = 0 \) (because the black holes are non-spinning, we take the irreducible mass as the black hole mass, \( M_{irr,A/B} = \sqrt{\frac{A_{AH,A/B}}{16\pi}} \)). Initial data are constructed by solving the conformal thin sandwich equations [30, 31] with the same setup as in [28], but setting the orbital frequency \( \Omega_0 = 0 \). These data are then evolved with the SpEC code using the dual coordinate frame technique described in [28] and with a domain decomposition with two excision spheres. A common apparent horizon appears at \( t = t_{CAH} = 17.83M \). Shortly thereafter, at \( t = t_{grid} = 18.96M \), the original domain decomposition with two excision boundaries is replaced by a set of concentric spherical shells with one larger excision boundary. The new excision boundary lies somewhat inside the common apparent horizon, but outside the original excision boundaries. The region very close
Figure 7. Evolution of a head-on BBH merger: normalized constraint violations. The left panel shows the complete evolution. The right panel enlarges the time around the merger, with formation of a common apparent horizon and time of regridding indicated by ‘CAH’ and ‘regrid’, respectively. The discontinuity at $t_{\text{CAH}}$ arises because the constraints are computed only outside the common AH for $t > t_{\text{CAH}}$. At $t_{\text{regrid}}$, the constraints jump because of the different numerical truncation error of the ringdown domain decomposition.

to the original excision boundaries, and between them, is dropped, and is no longer evolved. Data are interpolated from the highest resolution merger run onto three resolutions of this new domain decomposition. The simulation is continued up to $t = 95M$ and the final mass of the merged black hole is $M_{\text{final}} = 0.9493M$.

The simulation is performed at three progressively higher resolutions, named ‘N0’ through ‘N2’. The SpEC code does not strictly enforce the Hamiltonian or momentum constraints, nor the artificial constraints that arise from the first-order reduction of the generalized harmonic formulation of Einstein’s equations [32]. As such, it is important to monitor the values of these constraints during the simulation, as shown in figure 7. We normalize the constraints by an appropriate norm of the derivatives of the evolved variables (see (71) of [32] for the precise definition) and integrate constraint violations and normalization only outside the two individual apparent horizons or the common apparent horizon for this run.

4.2. EH finder behaviour

Since the EH finder follows the EH backward in time, we begin our discussion with the ringdown phase of the head-on merger. Initial data for both the geodesic and surface methods are taken from the apparent horizon at $t = 81.24M$, about 60M after appearance of a common AH.

We run both the geodesic and surface methods for angular resolutions $L = 7, 15, 23, \ldots, 47$ and compute the area $A(t)$ of the tracked surface for these runs. We do not employ filtering as per (37) for the surface method.

Figure 8 plots the relative differences between $A(t)$ computed with different angular resolutions. This plot exhibits several noteworthy features, which we discuss in the next few paragraphs.

During the ringdown phase, $t \gtrsim 20M$, both the surface and geodesic methods perform admirably: even at low resolution $L = 7$, the area is computed to better than $10^{-16}$ and this error drops rapidly below $10^{-12}$ as $L$ is increased. The rapid convergence with $L$ in the ringdown regime is not too surprising, because the angular resolution of the merger simulation
is $L_{\text{evolution}} = 25$. Therefore, angular modes $\ell > 25$ of the EH finder carry only information about the way in which the surface parameters $(u, v)$ deviate from the $(\theta, \phi)$ coordinates of the simulation. As can be seen from the excellent convergence for $t \gtrsim 20M$ in figure 8, such deviations are not very important. We also note that the long-term instability exhibited by the surface method during the Kerr test is not apparent.

Close to merger and before merger, $t \lesssim 20M$, the tracked surface becomes very distorted and therefore requires much higher angular resolution. This is apparent in the comparatively larger errors in $A(t)$ for $t_{\text{CEH}} < t \lesssim 20M$. In this time interval, the errors in the surface method grow more rapidly than those of the geodesic method. We attribute this to a degradation of the convergence rate of the spectral expansion (18). The surface method relies on the spectral expansion in an essential way to compute the derivatives that enter into (14a). In contrast, evolution of geodesics is independent of the spectral expansion and the spectral series is used only to compute the surface area via (25).

At the point of merger, when the surface being tracked by the EH finders intersects itself for the first time, the error in the area computation suddenly increases drastically in either method. The reasons for this are quite different for the two methods: the geodesic method evolves individual geodesics perfectly fine through $t_{\text{CEH}}$. The large errors in figure 8 arise because of the use of spectral integration to compute the surface area: at a caustic, the surface-area element $\sqrt{h}$, (24), tends to zero, resulting in a non-smooth integrand in the area integral (25), destroying exponential convergence of the spectral area integration. Below, we will explain how we employ finite-difference integration instead. We shall address area calculation for $t < t_{\text{CEH}}$ in section 4.4, where we also discuss how to compute the area of the EH excluding the future generators of the EH.

The surface method exhibits additional, more fundamental, problems at $t_{\text{CEH}}$, when the surface being tracked intersects itself in a caustic with $\sqrt{h} \to 0$. At such a point, the tangents to the surface, $\partial_r r^i$ and $\partial_t r^i$ are either no longer linearly independent, or one of them is zero (cf (24)). Therefore the surface normal $s^i$ in (14a) is ill defined.
Figure 9. Error estimates for the surface and geodesic methods, with surface resolution $L = 47$. The left panels show the root-mean-square pointwise deviation between the different runs, whereas the right panels show the differences in the surface area. The lines labelled ‘G:N#–N#’ (‘S:N#–N#’) in the upper panels show the difference between the geodesic method (surface method) when applied to merger simulations of different resolution N. The lines labelled ‘N#:S–G’ in the lower panels show the differences between the surface and geodesic methods for a given N (where N0, N1, and N2 are resolutions of the merger simulation). Note that the time scale of all plots change at $t = 25M$.

While the surface method presently cannot evolve through merger, it nevertheless yields valuable consistency checks with the geodesic method during the ringdown phase. Figure 9 presents such a comparison between the two methods and examines the effect of varying the resolution of the underlying binary black hole simulation. The top panels show differences between the results of the geodesic method applied to evolutions with different resolutions (labelled ‘G:N#–N#’). As the underlying resolution is increased, the differences become smaller. Likewise, the lines labelled ‘S:N#–N#’ show the analogous differences when running the surface method. When the surface method works, $t \gtrsim 15M$, it is more accurate than the geodesic method. For times close to the formation of the common event horizon, $t \lesssim 15M$, errors in the surface method grow very rapidly and render our current implementation essentially useless. The bottom panels of figure 9 show differences between surface and geodesic method at the same resolution of the evolved data. This difference decreases with increasing $N$, as it should. During ringdown, $t \gtrsim 15M$, the difference is essentially equal to the error in the geodesic method; for $t \lesssim 15M$ it is dominated by errors in the surface method.
Figure 10. Surface area differences between the EH, AH, and the final area, normalized by the final area. Also plotted are error estimates for $A_{EH}(t)$ and $A_{AH}(t)$.

The right panels in figure 9 examine the surface area $A(t)$. No clear convergence is apparent for $t \gtrsim 20M$, perhaps because the surface area of the event horizon can be calculated with great accuracy even at low values of $N$. Given the lack of clear convergence, we shall take as our error estimate for the post-merger area the square sum of the following three error measures: (a) the change in $A(t)$ between the geodesic method applied to the head-on simulation at the two highest resolutions (i.e., ‘G:N1–N2’), (b) the change in $A(t)$ between the geodesic and surface methods (i.e., ‘N2:S–G’) and finally, (c) the change in $A(t)$ in the geodesic method at $L = 47, N2$ when doubling the timestep (from $0.056 M$ to $0.112 M$; the effect of this is small and not shown in figure 9). This combined error estimate is plotted in figure 10.

4.3. Quasinormal modes during ringdown

After the merger, the distorted merged black hole rings down into a stationary black hole. During this phase, the area of the event horizon, $A_{EH}$, will approach its final value $A_{\text{Final}}$, and one expects that the apparent horizon approaches the event horizon. This is explored in figure 10. This plot also contains the error estimates obtained from figures 8 and 9. Figure 10 shows that the areas of the common AH and EH differ by about 10% when the common AH first appears, though this difference drops to 0.1% within about 3$M$. After this rapid initial drop, the ringdown is clearly apparent. The areas of both EH and AH approach their final areas exponentially, and this approach is resolved through about five orders of magnitude. A least-squares fit of $\log[A_0 - A_{EH}(t)]$ to the function $C - \lambda_{\text{obs}}t$ for $30M \leq t \leq 70M$, yields $\lambda_{\text{obs}} = 0.181 M^{-1}_{\text{final}}$. There are furthermore periodic features visible in the EH and AH areas, with seven periods clearly distinguishable. The period of oscillation is found to be $\tau_{\text{osc}} = 8.00M$, therefore $\omega_{\text{obs}} = 0.745 M^{-1}_{\text{final}}$. 

19
Decay rate $\lambda_{\text{obs}}$ and frequency $\omega_{\text{obs}}$ can be related to quasi-normal modes of a Schwarzschild black hole as follows: the quasi-normal mode parameters of a perturbed black hole are typically defined with reference to oscillations in the metric fields, which can be written as

$$\delta g_{\mu \nu} \propto e^{-\lambda t} \sin(\omega t),$$

where $\lambda$ is the decay coefficient and $\omega$ is the angular frequency of the metric oscillation. Therefore

$$\delta g_{\mu \nu} \propto -\lambda e^{-\lambda t} \sin(\omega t) + \omega e^{-\lambda t} \cos(\omega t).$$

The energy flux through the horizon, and therefore the change of its mass is $\dot{M} \propto |\delta g_{\mu \nu}|^2$, so we have

$$\frac{\dot{A}}{A} \propto \dot{M} \propto e^{-2\lambda t} \left[ \lambda^2 + \omega^2 + (\omega^2 - \lambda^2) \cos(2\omega t) - \lambda \omega \sin(2\omega t) \right].$$

Thus the observed values $(\lambda_{\text{obs}}, \omega_{\text{obs}})$ should be twice the values $(\lambda, \omega)$ of a quasi-normal mode. Indeed, the lowest quasi-normal mode of a perturbed Schwarzschild black hole is the $\ell = 2, n = 0$ mode, with $\lambda_{20} = 0.08896 M_{\text{final}}^{-1}$ and $\omega_{20} = 0.37367 M_{\text{final}}^{-1}$. Consistent with (41), we find that $\lambda_{\text{obs}} - 2\lambda_{20} = 0.003 M_{\text{final}}^{-1}$, and $\omega_{\text{obs}} - 2\omega_{20} = 0.002 M_{\text{final}}^{-1}$.

4.4. Treatment of the merger

Before examining the merger phase in detail, we must develop tools to analyse the topology change the event horizon undergoes during merger. As seen in figure 1, prior to merger, the surface found by the event horizon finder is the union of the two individual event horizons and the set of future generators of the joint event horizon. The event horizon itself consists of two topological spheres. At merger, $t = t_{\text{CEH}}$, the topology of the event horizon changes to a sphere. For $t < t_{\text{CEH}}$, generators of the event horizon continuously enter the event horizon at the cusps on the event horizons of the two approaching holes. The geodesic method traces geodesics perfectly fine through merger back to the start of the head-on binary black-hole evolution, and the trajectories of the geodesics are convergent as the resolution of the underlying evolution is increased, see the top left panel of figure 9. In this section, we address two questions relevant to analysing the output of the geodesic method: first, when going toward earlier times, some geodesics leave the event horizon; how does one decide whether a given geodesic is still on the event horizon, or whether it is merely a future generator of the event horizon? Second, how can one compute the area of the event horizon (i.e., not counting the area of the locus of future generators)?

Let us first consider the area element $\sqrt{h}$ of the EH surface, with $h$ given by (24), which requires derivatives $\partial_u$, $\partial_v$ along the surface, thus connecting the neighbouring geodesics. Because we place the geodesics at a $(u, v)$ grid consistent with spherical harmonic basis functions, we can use spectral differentiation to compute these derivatives (and have done so, up to this point in the paper). Convergence of this spectral expansion, however, becomes increasingly slow for $t \lesssim t_{\text{CEH}}$, and therefore, we compute henceforth the derivatives $\partial_u r^i$ and $\partial_v r^i$ with second-order finite difference stencils.

Figure 11 plots the area element $\sqrt{h}$ as a function of time for a few representative geodesics. This figure was obtained from our highest resolution run using 20 000 geodesics. To reduce the CPU cost, these geodesics were initialized at $t = 19.8 M$ from the $L = 47$ run of the surface method. For some geodesics in figure 11, $\sqrt{h}$ approaches zero at a certain time.
This feature can be used to determine whether a given geodesic is still on the horizon: we first note that the change of area element along a given null geodesic (i.e., for fixed $u$, $v$) is proportional to the expansion of this particular geodesic:

$$\partial_t \log \sqrt{h} = \frac{\partial_t (\sqrt{h})}{\sqrt{h}} \propto \theta. \quad (42)$$

The constant of proportionality depends on the parametrization of the null geodesic. Note that by Raychaudhuri’s equation, the expansion of a generator of the event horizon must be non-negative, $\theta \geq 0$. Figure 11 shows the area element as a function of time for a few representative geodesics.

At late time $t = t_{\text{end}}$ where the final black hole has settled down, we start with geodesics on the apparent horizon, which will be very close to the event horizon. Therefore, we assume that at $t_{\text{end}}$ all tracked geodesics are generators of the event horizon. Consistently with this assumption, figure 11 shows that $\partial_t \log \sqrt{h}$ starts out very close to zero, and increases as we approach the dynamical time region around the merger. If a generator remains on the event horizon, $\partial_t \log \sqrt{h}$ will eventually decrease again and approach zero at very early times before the merger. Generators leaving the event horizon must do so at points where the generators cross, according to a theorem by Penrose [34, 29]. For the head-on merger, at such a point nearby geodesics cross and pass through each other. Just after the geodesic enters the horizon, the horizon generators diverge from each other and their expansion is positive (and so is $\partial_t \log \sqrt{h}$). Just before the caustic points, nearby future generators of the event horizon converge towards the caustic point with negative expansion. In fact, at the caustic, $\partial_t \sqrt{h}$ changes sign discontinuously, as can be seen in figure 11.
Figure 12. Convergence of the surface area of the event horizon during merger. The lower plot shows results for placement of the geodesic pole parallel to the axis of symmetry (i.e., consistent with axisymmetry), the upper plot has a geodesic axis perpendicular to the axis of symmetry of the merger. In both cases, geodesics are tracked using the geodesic method; derivatives for $\sqrt{h}$ (cf (24)) are computed with finite differences; geodesics are removed from the event horizon based on (43). Lines are the difference between each resolution and the next highest.

Therefore, the largest time at which the expansion of a geodesic passes through zero will be the time it joins the event horizon,

$$\partial_t \log \sqrt{h} \begin{cases} \leq 0, & t = t_{\text{join}}, \\ > 0, & t > t_{\text{join}}. \end{cases}$$  \hspace{1cm} (43)

In practice, we keep track of (43) with a mask function $f_M(u, v)$, which is initially identical to unity. As we evolve backward in time, we evaluate $\partial_t \log \sqrt{h}$ at each time step, and if it drops below some tolerance $-\text{tol}$ for a point $(u_0, v_0)$ we set $f_M(u_0, v_0) = 0$ for that geodesic. The tolerance $\text{tol}$ is necessary to avoid misidentifications due to numerical truncation error at very early or late times, where $\partial_t \log \sqrt{h} \to 0$ for event horizon generators. Because $\partial_t \log \sqrt{h}$ changes so rapidly at a caustic, the precise value for tol is not very important; we use $\text{tol} = 10^{-3}$.

For generic situations, generators can also leave the EH at points where finitely separated generators cross (a ‘cross-over point’ in the language of Husa & Winicour [35]). At such points, $\sqrt{h}$ remains positive, and criterion (43) reduces to a necessary but not sufficient condition that a generator has left the horizon, i.e. $t_{\text{join}}$ from (43) will be a lower bound for the actual time when a particular geodesic leaves the horizon. Cross-over points could be diagnosed by monitoring the minimal distance between every pair of followed geodesics, and we shall discuss this point in more detail in a future publication.

The area of the event horizon (consisting of the two disjoint components for $t < t_{\text{CEH}}$) is found by multiplying $\sqrt{h}$ by the mask function $f_M$, and integrating

$$A_{\text{EH}} = \int f_M(u, v) \sqrt{h(u, v)} \sin u \, du \, dv.$$  \hspace{1cm} (44)
Figure 13. Area of the event horizon and of the apparent horizons before merger and during merger. The vertical dotted lines indicate the formation of a common event horizon and appearance of a common apparent horizon; the inset shows an enlargement for early time.

For \( t < t_{\text{CEH}} \), there are two major sources of error in this integral: first, each geodesic can either be on or off the horizon. When \( f_M \) changes discontinuously from 1 to 0 for a geodesic, the area of the event horizon will change discontinuously. Note that this will occur at different times for different resolutions. The severity of this effect will depend on how many geodesics enter the horizon simultaneously, as illustrated in figure 12. This figure shows the convergence of the event horizon area with increasing number of geodesics, and for two distinct orientations of the geodesics. In either case, the geodesics are initialized at \( t = 19.86M \) from the \( L = 47 \) surface method determining the event horizon during ringdown, and in either case the geodesics are placed on a rectangular \((u, v)\) grid as detailed in section 2.4. In the lower panel of figure 12, the geodesics are oriented respecting the axisymmetry (i.e., the \( u = 0 \) polar axis is aligned with the axis of symmetry), whereas in the upper panel the \( u = 0 \) axis is perpendicular to the axis of symmetry. The lower panel of figure 12, with geodesics respecting the symmetry, shows much larger variations in the area as the resolution is increased. This arises because due to the symmetry, a full ring of geodesics leaves simultaneously, thus amplifying the discontinuity of \( A_{\text{EH}}(t) \). For perpendicular orientation of the geodesics, individual geodesics leave the horizon, resulting in smaller jumps; this is the configuration we will use in the next section to examine the physics of the black hole merger.

The second source of error in the evaluation of (44) arises because the integrand is not smooth once geodesics have left the horizon. For fixed \( t < t_{\text{CEH}} \), \( \sqrt{h} \) approaches zero linearly toward the caustic; off the horizon, \( f_M \sqrt{h} \equiv 0 \) by virtue of the mask function, so overall, the integrand is only continuous, and we cannot expect exponential convergence of the integral, despite using a Gauss-quadrature formula to evaluate (44).\

4.5. Analysis of merger phase

When evolving geodesics backward, we find that the first geodesic leaves the horizon at \( t_{\text{CEH}} = 14.58M \), the time of merger. However, it should be noted that the point at which an

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6 For \( t > t_{\text{CEH}} \), \( A_{\text{EH}} \) in figure 12 is limited by the finite-difference derivatives used to compute \( \sqrt{h} \). Better accuracy can be obtained using spectral derivatives, as can be seen from the right panels of figure 9. For the analysis of the merger below, this difference is invisible.
observer sees the EH change topology is not invariant because the curve traced by the cusps of the two black holes is spacelike [36]. Figure 13 shows the surface area of the EH and the common and individual AHs during the merger phase. The common apparent horizon forms at $t_{CAH} = 17.8M$, and we track the individual apparent horizons up to $t = 18.8M$. The area of the individual apparent horizons is remarkably constant. Up to the formation of the common event horizon, its fractional increase is less than $10^{-5}$; up to the common apparent horizon, its fractional increase is $5 \times 10^{-5}$, and even when we stop tracking the inner horizons, their area has increased by only $1.6 \times 10^{-4}$. In contrast, $A_{EH}$ varies significantly more and at significantly earlier times, as can be seen from the inset.

To examine the relation between individual apparent horizons and event horizons, we plot in figure 14 the difference $\Delta A \equiv A_{EH} - (A_{AH,A} + A_{AH,B})$. For times $6 \lesssim t/M \lesssim 14$, $\Delta A$ grows exponentially with an e-folding time of $1.95M$. This e-folding time is within a few percent of the surface gravity of a black hole with the initial mass of the black holes in the head-on simulation. This confirms that as geodesics are integrated backwards in time, the individual components of the event horizon approach the individual apparent horizons with the expected rate. If our code were free from all numerical errors, the curve in figure 14 would continue to decrease exponentially as one proceeds backwards in time. Instead, this curve saturates at $\Delta A/A \approx 0.1\%$ at $t = 0$, and in addition, a feature in $\Delta A$ appears at $t \approx 5M$ because the EH area falls below $A_{AH,A} + A_{AH,B}$ and therefore $\Delta A$ changes sign. These effects are due to numerical errors, particularly finite-difference errors in the computation of $\sqrt{h}$ and the use of a finite number of geodesics. If one wishes to achieve much better than $0.1\%$ accuracy of the event horizon surface area at very early times when the two holes are widely separated, the EH must be split into two individual surfaces to be evolved separately.

5. Conclusion

This paper examines three different methods for locating event horizons in dynamical black hole spacetimes, the geodesic method, the surface method and the level-set method. All three methods rely on the principle that outgoing geodesics exponentially approach the event horizon when followed backward in time. We implement both the geodesic and surface methods, the latter one implemented without the assumption of axisymmetry as done in earlier work [12].
Overall, we find that the geodesic method is more robust, with the capability to accurately follow highly spinning black holes (tested up to $a/M = 0.99$), as well as the merger of two black holes. For the head-on merger, we find that the surface-area element $\sqrt{h}$ of the geodesic congruence is an excellent diagnostic of whether and when a geodesic joins the event horizon, cf (43). In more generic situations, this criterion might have to be amended by a second test for crossing of geodesics that are initially (at $t = t_{\text{end}}$) separated by a finite amount. Errors due to tangential drift of the geodesics—as explained in [12]—are not apparent in our simulations. The observed good properties of the geodesic method might be related to the improvements in accuracy of the spacetime metric since the early tests [12], as well as the ability to interpolate the metric spectrally to the geodesic locations. Because each geodesic is evolved independently, the geodesic method parallelizes trivially. Tracking of the cusp of the disjoint components of the event horizon before merger, as well as computation of $A_{\text{EH}}$ is currently not highly accurate, as comparatively few geodesics cover the region close to the cusps. Our current scheme switches from the surface method to a large number of geodesics some time after merger where the surface method is still very accurate, say at $t_0$. We plan on adaptively placing additional geodesics at $t_0$ based on where cusps occur.

The surface method is less robust and exhibits a long-term instability when applied to Kerr black holes with spins $a/M \lesssim 0.6$, and rapid blow-up for larger spins. Nevertheless during the ringdown phase $t > t_{\text{CEH}}$ of the head-on merger, the surface method locates the event horizon with comparable accuracy to the geodesic method and provides an important independent test of the geodesic method. However, when the surface being tracked self-intersects in a caustic point, our current method for defining the normal breaks down because $\partial r^i / \partial v = 0$ in (14a)–(14c), and thus our current implementation of the surface method fails.

The level-set method, finally, is not implemented in this paper. It requires boundary conditions for the level-set function $f$; furthermore $f$ can become singular during a black hole merger. Both reasons made it unduly difficult to implement this method in our spectral code. In conclusion, we find that the geodesic method, the oldest of the three methods considered, to be the most accurate and useful in our tests.

Turning our attention to applications of the event horizon finders, figure 4 presents a new quantitative test of event horizon finders: when finding the EH of a Kerr black hole starting away from the true horizon, does the tracked null surface approach the true event horizon with the correct rate, namely the surface gravity $g_H$? Table 1 confirms this for the geodesic method. For the head-on merger, both geodesic and surface methods perform admirably during the ringdown phase, where we are able to clearly observe the quasi-normal ringing of the single merged black hole. For both the event and apparent horizons, the frequency and damping time of the ringing matches the $(\ell = 2, n = 0)$ mode of the Schwarzschild quasi-normal ringing spectrum to within 2% for the decay rate and 0.3% for the frequency.

Furthermore, we find that the apparent horizons provide an excellent approximation to the event horizon for the head-on merger very early before the merger, and very late after the ringdown. Thus, while in principle the apparent horizon is slice dependent and there is no guarantee that it should coincide with the event horizon, in practice no such behaviour is found.

Finally, perhaps surprisingly, for the head-on binary black hole merger only some of the future null generators of the horizon start at past null infinity. A significant fraction of the generators rather start close to the individual event horizons of the black holes before merger. This can be seen in the spacetime diagram in figure 15, most clearly for the geodesic pointed to with an arrow. These geodesics begin to diverge from the individual event horizon as the second black hole approaches. The increased gravity of both black holes causes such geodesics then to ‘turn around’ and join the event horizon at the seam of the pair of pants.
Figure 15. Spacetime diagram of the head-on merger. The pale lines denote geodesics that will join the event horizon. Some of these geodesics come from past null infinity, but others come from a region close to the individual event horizons (cf the arrow and the circled geodesics on the far black hole).

In the future we plan to study event horizons in the more diverse black hole scenarios currently being simulated: mergers of inspiraling black holes; spinning and/or non-equal mass binary black holes, as well as black hole neutron star mergers.

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Appendix A. Points in the surface method follow geodesics

Consider a two-dimensional family of null geodesics, \( q^\mu(t, u, v) \), where \( u, v \) label different geodesics. Assume the parameter \( t \) along the geodesic coincides with the coordinate time of the underlying black hole simulation, i.e. \( q^t(t, u, v) = t \). This family of geodesics traces out a three-dimensional null surface \( \mathcal{N} \), parametrized by coordinates \( t, u, v \): \( q^\mu(t, u, v) \), where \( t \) is the parameter along each null curve, and \( u, v \) are the parameters relating each null curve to nearby null curves. In this parametrization, we can write the outgoing null normal \( \ell^\mu = \partial q^\mu/\partial t \big|_{u,v} \), i.e. a coordinate derivative \( \ell = \partial_t \) within the \((t, u, v)\) coordinates of \( \mathcal{N} \). Displacement vectors that relate each null curve to its neighbours are given by \( \tilde{m} = \partial/\partial u \), \( \tilde{n} = \partial/\partial v \). Since coordinate derivatives commute, we have

\[
\ell^\mu \nabla_\mu m^\nu = m^\mu \nabla_\mu \ell^\nu. \tag{A.1}
\]
Let us consider the rate of change of the inner product $\ell^\mu m_\mu$ as we change the time $t$ along a geodesic (i.e., for fixed $u$ and $v$):

$$\partial_t(\ell^\mu m_\mu) = \ell^\nu \nabla_\nu (\ell^\mu m_\mu) = m_\mu \ell^\nu \nabla_\nu (\ell^\mu) + \ell^\mu \ell^\nu \nabla_\nu (m_\mu). \quad (A.2)$$

From (A.1), the second term of (A.2) vanishes,

$$\ell^\mu \ell^\nu \nabla_\nu (m_\mu) = \ell^\mu m_\nu \nabla_\nu (\ell^\mu) = \frac{1}{2} m_\nu \nabla_\nu (\ell^\mu \ell^\mu) = 0. \quad (A.3)$$

Substituting the formula for parallel transport of $\ell^\mu$ along the geodesics, $\ell^\nu \nabla_\nu \ell^\mu = \kappa \ell^\mu$ (with $\kappa = 0$ if $t$ is affine), (A.2) finally becomes

$$\partial_t(\ell^\mu m_\mu) = m_\mu \ell^\nu \nabla_\nu (\ell^\mu) = \kappa m_\mu \ell^\mu. \quad (A.4)$$

A similar calculation results in $\partial_t(\ell^\mu n_\nu) = \kappa \ell^\mu n_\mu$.

So far, this appendix only discusses the geodesic method. We now use the results just obtained to show that surface and geodesic methods will construct the same null surface $N$. Both methods start with the same two-dimensional surface at some late time $t_0$, and the tangent $\dot{q}^\mu(t_0, u, v)$ to the geodesics at $t_0$ is chosen to be normal to the 2-surface. Therefore, at $t_0$, $\ell^\mu = \dot{q}^\mu$, and the surfaces resulting from evolving both the geodesic and surface methods will coincide at times infinitesimally near $t_0$. Because $\ell^\mu m_\mu = \ell^\mu n_\mu = 0$ initially, (A.4) implies that $\ell^\mu m_\mu = \ell^\mu n_\mu = 0$ at all other times. Thus, the tangent to the geodesics always remains orthogonal to the surface described by the positions of all the geodesics at a given time $t$. Since $\dot{q}^\mu$ is normal to that surface, null, outgoing, and has $\dot{q}^0 = 1$, it is identical at all times to $\ell^\mu$ as constructed by the surface method. Therefore, we see that the surfaces obtained by the geodesic and surface methods agree, and both techniques trace out the same $N$ given the same initial conditions.

Appendix B. Proof of surface gravity conjecture

We consider a null geodesic $q^\mu(t)$ that asymptotes to a horizon generator $q_H^\mu(t)$ for $t \to -\infty$, i.e.,

$$q^\mu(t) = q_H^\mu(t) + \delta q^\mu(t) \quad (B.1)$$

with $\delta q^\mu(t) \to 0$ as $t \to -\infty$. In the discussion of figure 4 we have asserted that

$$\delta q^\mu(t) \propto e^{g_H t}. \quad (B.2)$$

where $g_H$ is the surface gravity of the black hole, and where the coordinates $x^\mu$ are Kerr–Schild coordinates, cf (26)–(28). To confirm this assertion, one can substitute (B.1) into the geodesic equation and expand to linear order in $\delta q^\mu$ (where we assume that $\delta q^\mu$, $\delta \dot{q}^\mu$, and $\delta \ddot{q}^\mu$ are of the same order). One then needs to show that the resulting linear equation indeed has the solution (B.2).

The linearization of the geodesic equation is most easily performed in adopted coordinates. We have performed the analysis in ‘rotating spheroidal Kerr–Schild coordinates’ $x^\mu = (t, r_{BL}, \theta, \phi)$, related to the standard Kerr–Schild coordinates of (26)–(28) by the coordinate transformation

$$x = \sqrt{r_{BL}^2 + a^2 \sin \theta \cos(\phi + \Omega_H t)}, \quad (B.3)$$

$$y = \sqrt{r_{BL}^2 + a^2 \sin \theta \sin(\phi + \Omega_H t)}, \quad (B.4)$$

$$z = r_{BL} \cos \theta. \quad (B.5)$$
The time $t$ is not transformed. Horizon generators have the form $q^{\mu'} = [t, r, \theta_0, \phi_0]$, with $r = M + \sqrt{M^2 - a^2}$ and $\theta_0, \phi_0$ constants, i.e., $q^{\mu'} \propto [1, 0, 0, 0]$. In these coordinates, we have considered the geodesic equation in affine parametrization, (2) and have indeed confirmed

$$\delta q^{\mu'} \propto e^{\delta \alpha t} \tag{B.6}$$

to leading order in $\delta q^{\mu'}$. Exponential divergence from a horizon generator—as in (B.6)—is a property present in a quite general class of coordinate systems. For instance, consider the coordinate transformation

$$t' = t + f(x^i), \quad x^i' = x^i(x^i), \tag{B.7}$$

where the Jacobian $\partial x^i / \partial x^i$ and its inverse are finite in a neighbourhood of the horizon. In this case, $\delta q^{\mu'}$ and $\delta q^{\mu}$ are related merely by a multiplication by the Jacobian, so the exponential behaviour $e^{\delta \alpha t}$ is the same in both coordinate systems. The coordinate transformation (B.3)–(B.5) falls into this class, and therefore (B.6) implies (B.2).

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