On the elements of the Earth’s ellipsoid of inertia

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Abstract - By using the data for the known geopotential models by means of artificial satellite, the central moments of inertia of the Earth are determined. For this purpose, it was used the value $H = 0.00327369 \pm 9.8 \cdot 10^{-8}$ for dynamical flattening of the Earth \cite{7}. The results obtained indicate that the pole of inertia is located near the Conventional International Origin (CIO). Also, the orientation of the triaxial ellipsoid of inertia for nine geopotential models considered is given. Our results improve the ones obtained by Erzhanov and Kalybaev \cite{3}.

Key Words and Phrases: Geopotential, Earth’s moments of inertia, Earth’s rotation, Dynamical flattening, Harmonic coefficients.

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1 Introduction

The first artificial satellite of the Earth, Sputnik-1, was launched on 4 October 1957. Studying the trajectory of the following satellites (Sputnik-2 and Sputnik-3), D. King-Hele has determined the zonal coefficient $J_2$. The value $J_2 = 1.084 \cdot 10^{-3}$ was quite close to that calculated by terrestrial measurements. The US satellite, Vanguard-1, launched in March 1958, made it possible for the first time the assessment of discrepancy between the ellipsoid and geoid. The value of $J_4$ was obtained in the same year and the first odd zonal term in 1959 by Y. Kozai. In 1961, W. Kaula produced a complete model of degree 4, involving all the coefficients $C_{lm}$ and $S_{lm}$ of the associated Lagrange function $P_{22}$. From this moment, the information about the gravitational field of the Earth has become more numerous and accurate.

The first data from satellites have been used in the development of geopotential models from the early 1970. The SAO-SE model (Smithsonian Astrophysical Observatory - Standard Earth), established in 1966, used in 1972 the first laser-ranging measurements to establish satellite distances. The GEM model (Goddard Earth Model) was established by NASA’s GSFC (Goddard Space Flight Center) in the United States as a reaction to the classified US military models. The first model, GEM-1, was published in 1972, expanding the potential to degree 12. He then followed the whole series of geopotential models to the GEM-10 (developed up to order 20). Subsequently it was developed the model EGM, as a result of the collaboration between GSFC-NASA, NIMA (National
Imagery and Mapping Agency) and OSU (Ohio State University). In 1996 came EGM96S, of degree and order 70, with data provided solely by satellites, and EGM96, of degree and order 360, adjoining geophysical data [1].

In this paper, using the data provided by the SE-2 geopotential models from SE series, GEM-5 to GEM-10 models from GEM series and the EGM96 model, the Earth’s moments of inertia are calculated. Although at most six digits are accurate, to compare the results more easily with the values obtained by Erzhanov and Kalybaev [3], the calculations are performed with nine digits. In order to determine the polar moment $C'$ it was used the equation obtained by Prof. Ieronim Mihaila [6]. In fact, for all the nine models used, the value of $C'$ in Tables 4 and 6 coincides with the value of $C'$ obtained by considering

$$H' = \frac{1}{2C'}[2C' - (A' + B')] = H$$

like in [3], where the dynamical flattening of the Earth

$$H = \frac{1}{2C}[2C - (A + B)]$$

is obtained from the constant of precession. We used here for dynamical flattening the value $H = 0.00327369 \pm 9.8 \cdot 10^{-8}$ [7]. Thus, it demonstrates that the choice made in [3], namely $H' = H$, is valid until the order of $10^{-9}$. The orientation of the ellipsoid of inertia is described in Table 7. The data from satellites on the gravitational potential indicate that the equatorial principal moments of inertia of the Earth are not equal (see Table 6) and it also shows us that the polar axis does not coincide with the axis of rotation. The pole of inertia $P_i$ remains near the Conventional International Origin.

2 Representations of the geopotential

In the theory of the movement of the Earth’s artificial satellites it is chosen as a reference system the geocentric system $O\xi\eta\zeta$, the axis $O\zeta$ of the system being given by the position of the Conventional International Origin. The origin plan for longitude, $O\xi\zeta$, is the plan of the Greenwich meridian.

In polar coordinates, the expression of the geopotential is [3]

$$U(r, \varphi, \lambda) = G \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} \frac{2}{\delta_m} \frac{(n - m)!}{(n + m)!} \frac{1}{r} P_n^{(m)}(\cos \varphi) \times$$

$$\times \int_V (r')^n \rho(r', \varphi', \lambda') P_n^{(m)}(\cos \varphi') \cos m(\lambda - \lambda') dv$$

or

$$U(r, \varphi, \lambda) = G \sum_{n=0}^{\infty} \frac{Y_n(\varphi, \lambda)}{r^{n+1}},$$
where

\[ Y_n(\phi, \lambda) = \sum_{m=0}^{n} P_n^{(m)}(\cos\phi) [A_{nm}\cos m\lambda + B_{nm}\sin m\lambda]. \tag{1} \]

Here, \( \lambda \) is longitude and \( \phi \) is the geocentric latitude. The symbol \( V \) indicates that the integration should be extended to the whole volume of the Earth. The coefficients \( A_{nm} \) and \( B_{nm} \) from (1) are

\[ A_{nm} = \frac{2}{\delta_m (n+m)!} \int_V (r')^n P_n^{(m)}(\cos\phi') \cos(m\lambda') \rho(r', \phi', \lambda') dv, \tag{2} \]

\[ B_{nm} = \frac{2}{\delta_m (n+m)!} \int_V (r')^n P_n^{(m)}(\cos\phi') \sin(m\lambda') \rho(r', \phi', \lambda') dv, \]

where

\[ \delta_m = \begin{cases} 2, & m \geq 1 \\ 1, & m = 0, \end{cases} \]

while \( P_n \) and \( P_n^{(m)} \) are respectively the conventional zonal harmonics of \( n^{th} \) degree and the associated function of Legendre of \( n^{th} \) degree and \( m^{th} \) order.

We mention that the recommended geopotential form by U.A.I. [8] is

\[ U(r, \phi, \lambda) = \frac{GM}{r} \left[ 1 - \sum_{n=1}^{\infty} \frac{a_e}{r^n} n J_n P_n(\cos\phi) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_e}{r^n} P_n^{(m)}(\cos\phi') (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) \right], \]

where the harmonics coefficients of the geopotential are

\[ J_n = \frac{1}{Ma_e^n} \frac{2(n-m)!}{(n+m)!} \int_V (r')^n P_n^{(m)}(\cos\phi') \cos(m\lambda') \rho(r', \phi', \lambda') dv, \]

\[ C_{nm} = \frac{1}{Ma_e^n} \frac{2(n-m)!}{(n+m)!} \int_V (r')^n P_n^{(m)}(\cos\phi') \cos(m\lambda') \rho(r', \phi', \lambda') dv, \tag{3} \]

\[ S_{nm} = \frac{1}{Ma_e^n} \frac{2(n-m)!}{(n+m)!} \int_V (r')^n P_n^{(m)}(\cos\phi') \sin(m\lambda') \rho(r', \phi', \lambda') dv, \]

while

\[ P_n^{(m)}(\cos\phi) = \frac{1}{\sqrt{h_{nm}}} \frac{\delta_n(n-m)!}{(n+m)!} \frac{d^n P_n(\cos\phi)}{d(\cos\phi)^m} \sin^m \lambda. \]

There are three kinds of Legendrians in use: the conventional Legendrian when \( h_{nm} = \frac{\delta_n(n-m)!}{(n+m)!} \), the normalized Legendrian when \( h_{nm} = 1 \) and the fully
normalized Legendrian when \( h_{nm} = (2n + 1)^{-1} \) \[8\]. We use the conventional Legendrian.

The connection between the harmonics coefficients of the geopotential from \[9\] and the coefficients \[2\] is given by the relations

\[
\begin{align*}
J_n &= -\frac{1}{M a_e^n} A_{n0}, \\
C_{nm} &= \frac{1}{M a_e^n} A_{nm}, \\
S_{nm} &= \frac{1}{M a_e^n} B_{nm},
\end{align*}
\]

where \( M \) is mass of the Earth.

Often, the geopotential is defined by the next expression:

\[
U(r, \varphi, \lambda) = \frac{GM}{r} [1 + \sum_{n=1}^{\infty} \left(\frac{a_e}{r}\right)^n I_n P_n(\cos \varphi) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{a_e}{r}\right)^n I_{nm} P_{nm}(\cos \varphi) \cos m(\lambda - \lambda_{nm})],
\]

where the relations between the coefficients \( I_n, I_{nm} \) and the constants \( \lambda_{nm} \) with the coefficients \[3\] are (see \[3\])

\[
\begin{align*}
I_n &= -J_n, \\
I_{nm} &= \sqrt{C_{nm}^2 + S_{nm}^2}, \\
\lambda_{nm} &= \frac{1}{m} \arctan \frac{S_{nm}}{C_{nm}}.
\end{align*}
\]

If instead the \( P_{n}^{(m)} \) Legendre polynomials we consider the functions \( \mathcal{P}_{nm} \), with

\[
\mathcal{P}_{nm}(x) = \sqrt{2(n-m)!(2n+1)} \frac{n!(n+m)!}{(n+m)!} P_{n}^{(m)}(x),
\]

then the series \[3\] becomes (see \[8\])

\[
U(r, \varphi, \lambda) = \frac{GM}{r} [1 - \sum_{n=1}^{\infty} \left(\frac{a_e}{r}\right)^n J_n P_n(\cos \varphi) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{a_e}{r}\right)^n \mathcal{P}_{nm}(\cos \varphi)(A_{nm} \cos m\lambda + B_{nm} \sin m\lambda)],
\]

where

\[
C_{nm} = A_{nm} \sqrt{\frac{2(n-m)!(2n+1)}{(n+m)!}}.
\]
\[ S_{nm} = B_{nm} \sqrt{\frac{2(n-m)!(2n+1)}{(n+m)!}}. \]

If we note the following terms
\[ q_{n0} = \sqrt{2n+1}, \]
\[ q_{nm} = \sqrt{\frac{2(n-m)!(2n+1)}{(n+m)!}}, \]
then the polynomials \( P_n \) and \( P_{nm} \) may be written as follows
\[ P_n(x) = -q_{n0}P_n(x), \]
\[ P_{nm}(x) = q_{nm}P_{m}^{(m)}(x) \]
and the series (3) becomes (see [8])
\[ U(r, \varphi, \lambda) = \frac{GM}{r}[1 - \]
\[ - \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( \frac{ae}{r} \right)^n P_{nm}(\cos \varphi)(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)], \]
where
\[ q_{n0}C_{n0} = I_n, \]
\[ q_{nm}C_{nm} = C_{nm}, \]
\[ q_{nm}S_{nm} = S_{nm}. \]

The geopotential models are characterized by some constant values (see Table 1), called the universal constants of geopotential. They include the equatorial radius of the Earth \( (a_e) \), the geocentric gravitational constant \( (GM) \) and the geometrical flattening of the Earth \( (f_e) \).

In the following, it is necessary to know the dynamical flattening of the Earth \( (H) \). We use the value of \( H \) calculated from the constant of precession [7]
\[ H = 0.00327369 \pm 9.8 \cdot 10^{-8}. \] (5)

| MODEL      | \( a_e [m] \) | \( f_e \) | \( GM \cdot 10^{-14} [m^3 \cdot s^{-2}] \) |
|------------|---------------|-----------|-----------------------------------------------|
| SE-2       | 6378155.0     | 1/298.255 | 3.986013                                       |
| GEM-5, GEM-6 | 6378155      | 1/298.255 | 3.986013                                       |
| GEM-7, GEM-8 | 6378137.8    | 1/298.7925 | 3.9860013                                      |
| GEM-9, GEM-10 | 6378139.1  | 1/299.7925 | 3.9860064 ± 0.02                              |
| EGM96      | 6378136.3     | 1/298.257 | 3.986004415                                    |

Table 1. Universal constants of geopotential
3 Determination of the moments of inertia of the Earth

Using the expressions (2) of the harmonics coefficients of the geopotential, one finds the relations between the first coefficients and the moments of inertia of the Earth $A', B', C', D', E', F'$ in the system $O\xi\eta\zeta$, namely

\[
A_{20} = \frac{A' + B'}{2} - C',
\]
\[
A_{21} = E',
\]
\[
B_{21} = D',
\]
\[
A_{22} = \frac{B' - A'}{4},
\]
\[
B_{22} = \frac{F'}{2}.
\]

(6)

For the moments of inertia of the Earth, the following notations were used:

\[
A' = \int_V \rho(\xi, \eta, \zeta)(\eta^2 + \zeta^2)dv,
\]
\[
B' = \int_V \rho(\xi, \eta, \zeta)(\xi^2 + \zeta^2)dv,
\]
\[
C' = \int_V \rho(\xi, \eta, \zeta)(\xi^2 + \eta^2)dv,
\]
\[
D' = \int_V \rho(\xi, \eta, \zeta)\eta\eta dv, \quad E' = \int_V \rho(\xi, \eta, \zeta)\zeta\zeta dv, \quad F' = \int_V \rho(\xi, \eta, \zeta)\xi\eta dv,
\]

where $\rho$ is the density. On the other hand, from (4) and (6) the following relations between the coefficients $I_n, C_{nm}, S_{nm}$ and the moments of inertia of the Earth are obtained:

\[
2Ma^2C_{21} = E',
\]
\[
2Ma^2S_{21} = D',
\]
\[
Ma^2I_2 = C' - \frac{A' + B'}{2},
\]
\[
4Ma^2C_{22} = B' - A',
\]
\[
2Ma^2S_{22} = F'.
\]

(8)

The five relations (8) are insufficient to determine the six moments of inertia (7) of the Earth. The system would be complete if another independent equation is added.

Erzhanov and Kalybaev [3] had the idea to use the following expression for the sixth equation of the system

\[
H' = \frac{1}{2C'}[2C' - (A' + B')],
\]

(9)
taking the $H' = H$ without a motivation for this approximation. To avoid it, Prof. I. Mihaila deducted an equation for calculating the polar moment $C'$, using for this purpose the expression of $H$ [7].

With this equation of the polar moment:

$$ax^3 + bx^2 + cx + d = 0,$$

(9)

where the coefficients $a$, $b$, $c$ and $d$ are respectively

$$a = 8H^3,$$
$$b = 8H^3a',$$
$$c = -2H(3 - 4H)a^2 + 2H(3 - 2H)^2\left(\frac{a'^2 - b'^2}{4} - D'^2 - E'^2 - F'^2\right),$$
$$d = -2(1 - H)a'^3 + (3 - 2H)^2a'\left(\frac{a'^2 - b'^2}{4} - D'^2 - E'^2 - F'^2\right) +$$
$$+(3 - 2H)^3(2D'E'F' + \frac{a' + b'}{2}E'^2 + \frac{a' - b'}{2}D'^2),$$
$$a' = -2J_2Ma'^2,$$
$$b' = B' - A' = 4Ma'^2C_{22},$$

the system of six independent algebraic equations (8) and (9) for the six searched moments of inertia $A', B', C', D', E', F'$ is obtained.

If the values of harmonic coefficients of order $n = 2$ for the geopotential and the dynamical flattening of the Earth are known, then the normalized moments of inertia $\overline{A}', \overline{B}', \overline{C}', \overline{D}', \overline{E}', \overline{F}'$ can be determined easily. We note here $\overline{A} = A'/Ma'^2$, etc. The $SE - 2$ model is completed by the $SE - 2'$ model, where the harmonics coefficients $\overline{C}_{21} = -0.001196 \cdot 10^{-6}$ and $\overline{S}_{21} = -0.003466 \cdot 10^{-6}$ were calculated by Erzhanov and Kalybaev [3].

| MODEL  | $C_{20} \cdot 10^6$ | $C_{21} \cdot 10^6$ | $S_{21} \cdot 10^6$ | $C_{22} \cdot 10^6$ | $S_{22} \cdot 10^6$ |
|--------|---------------------|---------------------|---------------------|---------------------|---------------------|
| $SE - 2$ | -484.16596        | 0.000000           | 0.000000           | 2.41290            | -1.36410            |
| $SE - 2'$ | -484.16596       | -0.001196         | -0.003466         | 2.41290             | -1.36410             |
| GEM - 5  | -484.16620        | -0.00120          | -0.00870          | 2.42820            | -1.36040            |
| GEM - 6  | -484.16610        | -0.00090          | -0.00120          | 2.42510            | -1.38830            |
| GEM - 7  | -484.16460        | -0.00310          | -0.00090          | 2.43030            | -1.39460            |
| GEM - 8  | -484.16460        | -0.00010          | -0.00030          | 2.43450            | -1.39530            |
| GEM - 9  | -484.16555        | -0.00021          | -0.00406          | 2.43400            | -1.39786            |
| GEM - 10 | -484.16544        | -0.00104          | -0.00243          | 2.43404            | -1.39907            |
| EGM96   | -484.16537        | -0.000187         | 0.001195          | 2.43914            | -1.40017            |

Table 2. Normalized harmonics coefficients of the geopotential (see[3], [4])
In Tables 3, using the models of geopotential from Table 2, we evaluate these normalized moments.

Further, if the mass $M$ and the equatorial radius of the Earth are known, then one can determine the central moments of inertia $A'$, $B'$, $C'$, $D'$, $E'$, $F'$. In Table 4.a. and Table 4.b., using the data from Table 1 and the value of the gravitational constant given by the IAU (1976) System of Astronomical Constants, namely $G = 6.672 \cdot 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$, these moments are evaluated.

Once the values of the moments (7) known, the principal moments of inertia $A$, $B$, $C$ can be determined by solving the secular equation (see [2], [9])

$$\Delta(q) = \begin{vmatrix} A' - q & -F' & -E' \\ -F' & B' - q & -D' \\ -E' & -D' & C' - q \end{vmatrix} = 0 \quad (10)$$

The roots $q_1$, $q_2$, $q_3$ of equation (10) represent the principal moments of inertia $A$, $B$, respectively $C$. 

| MODEL | $\overline{A}' \cdot 10^6$ | $\overline{B}' \cdot 10^6$ | $\overline{C}' \cdot 10^6$ |
|-------|----------------|----------------|----------------|
| SE-2  | 0.329619974    | 0.329626204    | 0.330705717    |
| SE-2' | 0.329619974    | 0.329626204    | 0.330705717    |
| GEM-5 | 0.329620259    | 0.329626529    | 0.330706023    |
| GEM-6 | 0.329620507    | 0.329625671    | 0.330705717    |
| GEM-7 | 0.329619038    | 0.329625314    | 0.330704801    |
| GEM-8 | 0.329619033    | 0.329625319    | 0.330704801    |
| GEM-9 | 0.329619643    | 0.329625927    | 0.330705412    |
| GEM-10| 0.329619643    | 0.329625927    | 0.330705412    |
| EGM96 | 0.329619636    | 0.329625934    | 0.330705412    |

Tables 3.a. The normalized moments of inertia of the Earth
The values of the Earth’s normalized principal moments of inertia $\bar{A}$, $\bar{B}$, $\bar{C}$, where $\bar{A} = \frac{A}{A'}$, etc., obtained by the solving of the equation (10) with the geopotential models from Tables 3 are presented in Table 5. In Table 6, using the data from Table 5 and Table 1, are evaluated the principal moments of inertia $A$, $B$, $C$. 

Table 4.a. The moments of inertia of the Earth

| MODEL | $A \cdot 10^{-37} [kg \cdot m^2]$ | $B \cdot 10^{-37} [kg \cdot m^2]$ | $C \cdot 10^{-37} [kg \cdot m^2]$ |
|-------|----------------------------------|----------------------------------|----------------------------------|
| SE-2  | 8.010992630                     | 8.011144042                     | 8.03780227                      |
| SE-2' | 8.010992630                     | 8.011144042                     | 8.03780227                      |
| GEM-5 | 8.010999557                     | 8.011151941                     | 8.03780227                      |
| GEM-6 | 8.011005584                     | 8.01131088                      | 8.03780227                      |
| GEM-7 | 8.010903161                     | 8.01155690                      | 8.03791025                      |
| GEM-8 | 8.010903040                     | 8.011055812                     | 8.03791025                      |
| GEM-9 | 8.010931380                     | 8.011084104                     | 8.037919434                     |
| GEM-10| 8.010931380                     | 8.011084104                     | 8.037919434                     |
| EGM96 | 8.010920187                     | 8.01073251                      | 8.03703875                      |

Table 4.b. The moments of inertia of the Earth

| MODEL | $D' \cdot 10^{-30} [kg \cdot m^2]$ | $E' \cdot 10^{-29} [kg \cdot m^2]$ | $F' \cdot 10^{-32} [kg \cdot m^2]$ |
|-------|----------------------------------|----------------------------------|----------------------------------|
| SE-2  | 0.329619513                      | 0.32962665                       | 0.330705717                     |
| SE-2' | 0.329619513                      | 0.32962665                       | 0.330705717                     |
| GEM-5 | 0.329619801                      | 0.32962629                       | 0.330705717                     |
| GEM-6 | 0.329619939                      | 0.32962629                       | 0.330705717                     |
| GEM-7 | 0.329618555                      | 0.329626597                      | 0.330704801                     |
| GEM-8 | 0.329618549                      | 0.329626580                      | 0.330704801                     |
| GEM-9 | 0.329619167                      | 0.329626403                      | 0.330705412                     |
| GEM-10| 0.329619160                      | 0.329626410                      | 0.330705412                     |
| EGM96 | 0.329619148                      | 0.329626422                      | 0.330705412                     |
As seen from Tables 3 and 5 or from Tables 4 and 6, it is noticed that $C'$ coincides with $C$. For all the nine geopotential models used, the value of $C'$ found here coincides with the value of $C'$ obtained by considering

$$H' = \frac{1}{2C'}[2C' - (A' + B')] = H$$

as in the work [3]. It is thus demonstrated that the choice made by Erzhanov and Kalybaev, namely $H' = H$, is valid until $10^{-9}$.

### 4 Orientation of the ellipsoid of inertia

Let $Oxyz$ be the system of the Earth’s principal axes of inertia, whose coordinate axes are chosen so that

\[
A = \int_V \rho(x, y, z)(y^2 + z^2)dv, \\
B = \int_V \rho(x, y, z)(z^2 + x^2)dv, \\
C = \int_V \rho(x, y, z)(x^2 + y^2)dv,
\]

\[
\int_V \rho(x, y, z)xydv = \int_V \rho(x, y, z)xzdv = \int_V \rho(x, y, z)yzdv = 0.
\]

The orientation of the system with respect to $O\xi\eta\zeta$ may be given by the Euler angles. We use the notations from [3]

\[
\beta = (O\xi, ON), \alpha = (ON, Ox), \gamma = (O\zeta, Oz),
\]

where $ON$ is the intersection between the plans $O\xi\eta$ and $Oxy$, called the line of nodes.
Let \((p_\xi, p_\eta, p_\zeta), (q_\xi, q_\eta, q_\zeta), (r_\xi, r_\eta, r_\zeta)\) be the direction cosines of the axes \(Ox, Oy, Oz\) in respect with \(O\xi\eta\zeta\). They are the projections of the unit vectors \(p, q, r\) of the principal axes in the system \(O\xi\eta\zeta\). On the other hand, the \(O\xi\eta\zeta\) system overlaps \(Oxyz\) by three rotations \(R_\beta, R_\gamma, R_\alpha\). The direction cosine have the following expressions:

\[
\begin{align*}
p_\xi &= \cos(x, \xi) = \cos\beta \cos\alpha - \sin\beta \sin\alpha \cos\gamma, \\
p_\eta &= \cos(x, \eta) = \sin\beta \cos\alpha + \cos\beta \sin\alpha \cos\gamma, \\
p_\zeta &= \cos(x, \zeta) = \sin\alpha \sin\gamma, \\
q_\xi &= \cos(y, \xi) = -\cos\beta \sin\alpha - \sin\beta \cos\alpha \cos\gamma, \\
q_\eta &= \cos(y, \eta) = -\sin\beta \sin\alpha + \cos\beta \cos\alpha \cos\gamma, \\
q_\zeta &= \cos(y, \zeta) = \cos\alpha \sin\gamma, \\
r_\xi &= \cos(z, \xi) = \sin\beta \sin\gamma, \\
r_\eta &= \cos(z, \eta) = -\cos\beta \sin\gamma, \\
r_\zeta &= \cos(z, \zeta) = \cos\gamma.
\end{align*}
\]

(12)

and satisfy the orthogonality conditions

\[
\begin{align*}
p \cdot p &= q \cdot q = r \cdot r = 1, \\
p \cdot q &= p \cdot r = q \cdot r = 0.
\end{align*}
\]

(13)

The direction cosines are given by the relations

\[
\frac{\gamma_{i1}}{\delta_{i1}} = \frac{\gamma_{i2}}{\delta_{i2}} = \frac{\gamma_{i3}}{\delta_{i3}} = \frac{1}{\sqrt{\delta_{i1}^2 + \delta_{i2}^2 + \delta_{i3}^2}},
\]

(14)

where \(\delta_{i1}, \delta_{i2}, \delta_{i3}\), with \(i=1,2,3\), are the cofactors of the elements in row \(i\) of the determinant \(\Delta\) which appears in the secular equation (10), \(q\) being successively replaced by the principal moments of inertia \(A, B\) and respectively \(C\) (see [2], [5]). In the relation (13), we have \(\gamma_{11}, \gamma_{12}, \gamma_{13}\) = \((p_\xi, p_\eta, p_\zeta), \gamma_{21}, \gamma_{22}, \gamma_{23}\) = \((q_\xi, q_\eta, q_\zeta), \gamma_{31}, \gamma_{32}, \gamma_{33}\) = \((r_\xi, r_\eta, r_\zeta)\).

In determining the orientation of the principal axes of inertia, besides the orthogonality conditions (13), it is also required to fulfill the condition that the system \(Oxyz\) to have the same orientation as the system \(O\xi\eta\zeta\), namely (see [2])

\[
\begin{vmatrix}
p_\xi & p_\eta & p_\zeta \\
q_\xi & q_\eta & q_\zeta \\
r_\xi & r_\eta & r_\zeta
\end{vmatrix} = 1.
\]

(15)

Once determined the direction cosines, it can be obtained the three Euler’s angles by the relations (12). Thus, for determining the angle \(\gamma\) it is used the following relation

\[
\cos\gamma = r_\zeta.
\]
and to obtain the angle $\alpha$, the following relations are used

$$p_\zeta = \sin \alpha \sin \gamma,$$
$$q_\zeta = \cos \alpha \sin \gamma.$$

For the angle $\beta$ we have the following formulae

$$\sin \beta \sin \gamma = p_\eta q_\zeta - q_\eta p_\zeta,$$
$$\cos \beta \sin \gamma = p_\xi q_\zeta - q_\xi p_\zeta.$$

If $\gamma = 0$, then the axes $Oz$ and $O\zeta$ coincide and the plan $Oxy$ coincides with $O\xi\eta$. The problem of the orientation for the system $Oxyz$ is reduced in this case to the problem of the orientation for the plan system $Oxy$ in relation to $O\xi\eta$. We have from (13):

$$p_\xi = \cos \beta \cos \alpha - \sin \beta \sin \alpha = \cos (\beta + \alpha) = \cos \lambda,$$
$$p_\eta = \sin \beta \cos \alpha + \cos \beta \sin \alpha = \sin (\beta + \alpha) = \sin \lambda,$$
$$q_\xi = -\cos \beta \sin \alpha - \sin \beta \cos \alpha = -\sin (\beta + \alpha) = -\sin \lambda,$$
$$q_\eta = -\sin \beta \sin \alpha + \cos \beta \cos \alpha = \cos (\beta + \alpha) = \cos \lambda,$$

(16)

where it was noted by $\lambda$ the angle between axes $Ox$ and $O\xi$. The expressions (16) give us the ellipse of inertia orientation in the plan $O\xi\eta$.

The orientation of the ellipsoid of inertia corresponding to geopotential models is given in Table 7.a and Table 7.b. From the model $SE - 2$ with $\gamma = 0$ was determined the orientation of $Oxy$ in relation to $O\xi\eta$. This orientation is used as a standard for the other models.

In Table 7.b, the longitudes and the geocentric latitudes of the ellipsoid of inertia axes are also given. For the determination of the coordinates $\lambda_A$ and $\varphi_A$ of $Ox$, we have the following relations

$$p_\xi = \cos \lambda_A \cos \varphi_A,$$
$$p_\eta = \sin \lambda_A \cos \varphi_A,$$
$$p_\zeta = \sin \varphi_A.$$

Similarly, for the determination of the coordinates $\lambda_B$ and $\varphi_B$ of $Oy$, we have the following relations

$$q_\xi = \cos \lambda_B \cos \varphi_B,$$
$$q_\eta = \sin \lambda_B \cos \varphi_B,$$
$$q_\zeta = \sin \varphi_B.$$

For $Oz$, the longitude $\lambda_C$ is determined from the following formula

$$\lambda_C = 360^\circ - (90^\circ + \beta),$$

where $\beta$ is given by (11).
As it is observed in Table 7.b, for the geopotential models considered, the longitude of $Ox$ axis of the triaxial ellipsoid of inertia is about $-15^\circ$ and the longitude of $Oy$ axis has a value close to $75^\circ$, except for the model GEM-6 for which $\lambda_A \simeq -17^\circ$ and $\lambda_B \simeq 73^\circ$. From the values obtained, except for the model GEM-5, it is determined the mean ellipsoid of inertia with the principal moments $A = 8.01095639 \cdot 10^{37} \text{kg} \cdot \text{m}^2$, $B = 8.011108377 \cdot 10^{37} \text{kg} \cdot \text{m}^2$, $C = 8.03733747 \cdot 10^{37} \text{kg} \cdot \text{m}^2$ and with the orientation $\lambda_A = -15^\circ.2$, $\lambda_B = 74^\circ.8$. The value of $H$ obtained from the mean values of the principal moments of inertia, namely $H = 0.00327369$, is the same as in [7].

In Figure 1, the position of the inertial pole on the surface of the Earth is represented with respect to the conventional international pole $P_0$ for the nine models of geopotential considered. Since the angle $\gamma$ is a small angle, then, in a Cartesian reference $XP_0Y$ in the tangent plan at $P_0$, with the axis $P_0X$ tangent to the Greenwich meridian, the pole of inertia $P_i$ will have the polar coordinates ($\lambda_C$, $\gamma$).

It is remarked that the pole of the ellipsoid SE-2 when $\gamma = 0$ coincides with the conventional international pole $P_0$, the other positions of the pole remaining in the neighborhood, except for the pole of inertia corresponding to GEM-5. The coordinates of the mean pole $P$ are $\lambda_C = 209^\circ.7$ and $\gamma = 0^\circ.5$. Therefore
the mean polar axis differs from the rotation axis by 0°.5. The mean pole $\overline{P}$ deviates approximately by 15 meters from the pole of rotation.

Figure 1. The position of the inertial pole on the surface of the Earth

The results obtained show that the approximation made in the paper [3] is satisfied and the improved values for the principal moments of inertia $A$, $B$, $C$ are obtained. On the other hand, it is better emphasized the fact that the polar axis of inertia is located in the neighborhood of the Earth’s rotation axis. For the geopotential models considered, the longitudes of the axes $Ox$ and $Oy$ of the triaxial ellipsoids of inertia have concordant values.

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