Vertex Operators and Solitons of Constrained KP Hierarchies

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Abstract. We construct the vertex operator representation for the Affine Kac-Moody $\hat{s}l(M+K+1)$ algebra, which is relevant for the construction of the soliton solutions of the constrained KP hierarchies. The oscillators involved in the vertex operator construction are provided by the Heisenberg subalgebras of $\hat{s}l(M+K+1)$ realized in the unconventional gradations. The well-known limiting cases are the homogeneous Heisenberg subalgebra of $\hat{s}l(M+1)$ and the principal Heisenberg subalgebra of $\hat{s}l(K+1)$. The explicit example of $M = K = 1$ is discussed in detail and the corresponding soliton solutions and tau-functions are given.

1 Introduction

We consider a generalized Drinfeld-Sokolov hierarchy (Drinfel’d and Sokolov (1985), de Groot, et al (1992), Burroughs et al (1993)), based on a Kac-Moody algebra $\hat{G} = \hat{s}l(M+K+1)$ and defined by the following matrix eigenvalue problem:

$$L \Psi = 0 ; \quad L \equiv (D - A - E) ; \quad D \equiv I \frac{\partial}{\partial x}$$

(1.1)

The constant $E$ is a non-regular and semisimple element of $\hat{G}$ given by

$$E = \sum_{a=1}^{K} E_{\alpha M+a}^{(0)} + E_{-(\alpha M+1+\hdots+\alpha M+K)}^{(1)}$$

(1.2)

The potential $A$ in (1.1) contains the dynamical variables of the model, namely $q_i$, $r_i$, $U_a$ and $\nu$, and is given by (Aratyn et al (1997a), McIntosh (1993)):

$$A = \sum_{i=1}^{M} (q_i P_i + r_i P_{-i}) + \sum_{a=1}^{K} U_{M+a} (\alpha_{M+a} \cdot H^{(0)}) + \nu \hat{c}$$

(1.3)

where $P_{\pm i} = E_{\pm (\alpha_i + \alpha_{i+1} + \hdots + \alpha_M)}^{(0)}$, $i = 1, 2, \hdots, M$, and $\hat{c}$ is a central element of $\hat{G}$. We are using the Cartan-Weyl basis for $\hat{G}$ with generators $H^{(n)}$ and $E^{(n)}$, with $i = 1, 2, \hdots, \text{rank}\, \hat{G}$, $n \in \mathbb{Z}$ being the eigenvalues of the standard derivation $d$ of $\hat{G}$, and $\alpha$ being roots of the finite algebra $s(l(M+K+1))$, associated to the affine Kac-Moody algebra $\hat{G} = \hat{s}l(M+K+1)$. 
The integral gradations of $\hat{G}$ (Kac and Peterson (1985)) play an important role in the construction of integrable hierarchies (de Groot, et al (1992)). They are labeled by sets of rank $\hat{G}$ + 1 co-prime non-negative integers $s = (s_0, s_1, \ldots, s_r)$. The corresponding grading operators, in the case of $\hat{sl}(M+1)$, are given by

$$Q_s \equiv \sum_{a=1}^{M+K} s_a \lambda_a \cdot H^{(0)} + d \sum_{i=0}^{M+K} s_i$$

(1.4)

where $\lambda_a$ are the fundamental weights of $\hat{sl}(M+1)$. The relevant gradation to the hierarchy associated to (1.1) is

$$s_{\text{cKP}} = (1, 0, \ldots, 0, 1, \ldots, 1) \quad \leftrightarrow \quad Q_{s_{\text{cKP}}} \equiv \sum_{a=1}^{K} \lambda_{M+a} \cdot H^{(0)} + (K+1)d$$

(1.5)

Notice that $E$ and $A$ have grades 1 and 0 respectively, w.r.t. (1.5).

The gradation (1.5) interpolates between the principal $s_{\text{principal}} = (1, \ldots, 1)$ and the homogeneous one $s_{\text{homogeneous}} = (1, 0, \ldots, 0)$. As it is well-known these two limits define the KdV (de Groot, et al (1992)) and AKNS hierarchies (Fordy and Kulish (1983), Aratyn et al (1995)), respectively with

$$E_{\text{KdV}} = E^{(1)}_{-(\alpha_1 + \cdots + \alpha_K)} + \sum_{a=1}^{K} E^{(0)}_{\alpha_a}$$

(1.6)

and

$$E_{\text{AKNS}} = \lambda_M H^{(1)}$$

(1.7)

The fact that $E$ is semisimple means it decomposes $\hat{G}$ into the kernel and image of its adjoint action, i.e. $\hat{G} = \text{Ker (ad } E) + \text{Im (ad } E)$. Then one can gauge transform $L$ into Ker (ad $E$), $L \rightarrow L_0 \equiv U L U^{-1}$, with $U$ being an exponentiation of generators of negative $s_{\text{cKP}}$-grade of $\hat{G}$. One introduces a flow equation for each element $b^{(N)}$, in the center of Ker (ad $E$), with positive $s_{\text{cKP}}$-grade $N$, as follows

$$\frac{dL}{dt_{N}} = \frac{dA}{dt_{N}} \equiv [L, B_{N}]$$

(1.8)

where $B_{N} \equiv (U^{-1} b^{(N)} U)_{s_{\text{cKP}} \geq 0}$. We shall choose $b^{(1)} \equiv E$, $B_{1} \equiv E + A$ and $t_{1} \equiv x$.

We are interested in discussing the soliton solutions for the model defined by equation (1.1), following the method of (Ferreira et al (1997)). According to (Ferreira et al (1997)) the basic ingredient for the appearance of soliton solutions is that there must exist one or several “vacuum solutions”, such that the Lax operators, when evaluated on them, should lie in some abelian subalgebra of $\hat{G}$, and in addition the corresponding components of it should be constant. Accordingly, one requires

$$B_{N}^{(\text{vac.})} \equiv \epsilon_{N} = \sum_{i=0}^{N} \epsilon_{N}^{i} a_{i} ; \quad [a_{i}, a_{j}] = i \beta_{i} \delta_{i+j,0}$$

(1.9)
where, $\beta_i$ is some constant, and the oscillators $a_i$ have $s_{i\text{KP}}$-grade $i$. Then one can write

$$B_N^{(\text{vac})} = \frac{\partial \psi^{(\text{vac})}}{\partial t_N} \psi^{-1(\text{vac})} ; \quad \text{with} \quad \psi^{(\text{vac})} = \exp \left( \sum_N \varepsilon_N t_N \right) \quad (1.10)$$

The soliton solutions are constructed using the dressing transformation method as follows. Choose a constant element $h$ which is an exponentiation of the generators of $\hat{G}$, and perform the generalized Gauss decomposition

$$\psi^{(\text{vac})} h \psi^{-1(\text{vac})} = \left( \psi^{(\text{vac})} h \psi^{-1(\text{vac})} \right)_{s_{i\text{KP}}<0} \times \left( \psi^{(\text{vac})} h \psi^{-1(\text{vac})} \right)_{s_{i\text{KP}}>0} \quad (1.11)$$

Define the group element

$$\psi^h = \begin{pmatrix} \left( \psi^{(\text{vac})} h \psi^{-1(\text{vac})} \right)_{s_{i\text{KP}}<0} \end{pmatrix}^{-1} \psi^{(\text{vac})} h \quad (1.12)$$

Then, one can easily verify that

$$B_N^h = \frac{\partial \psi^h}{\partial t_N} \psi^{h-1} \quad (1.13)$$

has the same grading structure as $B_N$. Since $B_N^h$ is a flat connection it automatically satisfies the Lax (or flow) equation, and therefore by equating $B_N$ to $B_N^h$ one gets a solution for the hierarchy for each choice of $h$.

The soliton solutions correspond to those $h$’s which are products of exponentials of eigenvectors of the oscillators $a_i$’s, i.e.

$$h = e^{F_1} e^{F_2} \cdots e^{F_n}, \quad [\varepsilon_N, F_k] = \omega_N^{(k)} F_k, \quad k = 1, 2, \ldots, n. \quad (1.14)$$

Notice, that the $F_k$’s do not have to be eigenvectors of all $\varepsilon_N$’s. The soliton is a solution of a given flow equation, associated to a time $t_N$, only if the $F_k$’s are eigenvectors of the corresponding $\varepsilon_N$ and also $\varepsilon_1$.

We also define, following (Ferreira et al (1997) ), the generalized Hirota tau-function as

$$\tau_{\mu_0, \mu'_0}(t) = \langle \mu'_0 | \psi^{(\text{vac})} h \psi^{(\text{vac})}^{-1} | \mu_0 \rangle$$

$$= \langle \mu'_0 | \sum_N \varepsilon_N t_N h e^{-\sum_N \varepsilon_N t_N} | \mu_0 \rangle, \quad (1.15)$$

where $| \mu_0 \rangle$ and $| \mu'_0 \rangle$ are suitably chosen states in an integrable highest weight representation of $\hat{G}$, which are annihilated by all generators with positive $s_{i\text{KP}}$-grade.

The truncation of the Hirota’s formal expansion is now understood from the nilpotency of the operators $F_k$’s. Such property is most easily verified using the vertex operator construction of the integrable highest weight representation
of \( \mathcal{G} \). In this paper, we give the explicit construction of the vertex operator representation relevant for the soliton solutions of the hierarchy (1.1).

For pedagogical reasons, in section 2 we review the construction for the pure homogeneous and pure principal gradations corresponding to untwisted and twisted vertex operators related to AKNS and m-KDV hierarchies respectively.

In section 3 we present, following the same line of thought, the vertex functions construction for the intermediate gradation hierarchy. In section 4 we present the vertex operator algebra in terms of OPE (Operator Product Expansion) and show that it, in fact, reproduces the original \( SL(M + K + 1) \) Kac-Moody algebra. In section 5 we specialize for \( K = M = 1 \) and discuss the resulting tau functions (Aratyn et al (1997b)).

## 2 Vertex Operator Construction for the Homogeneous and Principal gradations

Let us first consider the general Kac-Moody algebra in the Cartan-Weyl basis

\[
[H^m_i, H^n_j] = m \delta_{m+n,0} \delta_{i,j} \quad i, j = 1, \ldots, \text{rank } \mathcal{G} \tag{2.16}
\]

\[
[H^m_i, E^m_\alpha] = (\alpha)^i E^{m+n}_\alpha \tag{2.17}
\]

\[
[E^m_\alpha, E^n_\beta] = \left\{ \begin{array}{ll}
\epsilon(\alpha, \beta) E^{m+n}_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\
H_{\alpha+n} & \text{if } \alpha + \beta = 0 \\
0 & \text{otherwise}
\end{array} \right. \tag{2.18}
\]

where \( (H^m_i)^\dagger = H^{-m}_i, (E^m_\alpha)^\dagger = E^{-m}_\alpha \). From eqns ((2.16), (2.17) and (2.18)) we will define an affine Heisenberg subalgebra satisfying

\[
[A^m_i, (A^n_j)^\dagger] = m \delta_{m+n,0} \delta_{i,j} \quad i, j = 1, \ldots, \text{rank } \mathcal{G} \tag{2.19}
\]

### 2.1 Homogeneous Gradation

We associate \( A^m_i = H^m_i \) and \( (A^m_i)^\dagger = (H^m_i)^\dagger \). Define now the rank \( \mathcal{G} \) dimensional Fubini-Veneziano field (see (Goddard et al (1986)))

\[
Q^i(z) = i \sum_{n>0} \frac{A^m_i z^{-n}}{n} \tag{2.20}
\]

\[
(Q^i)^\dagger(z) = -i \sum_{n>0} \frac{(A^n_i)^\dagger z^n}{n} \tag{2.20}
\]

\[
Q^i_0(z) = q^i - ip^i ln z
\]

where \([q^j, p^j] = i \delta_{i,j} \) and \( p^i = H^0_i \).

The construction of step operators satisfying (2.17) and (2.18) is provided by the vertex operator

\[
V^\alpha(z) = e^{i\alpha Q^i(z)} e^{i\alpha Q_0} e^{i\alpha Q(z)} = z^{\frac{\alpha^2}{2}} e^{i\alpha Q^i(z)} e^{i\alpha e^{p^i ln z}} e^{i\alpha Q(z)} \tag{2.21}
\]
where the vector $\alpha$ denotes a root of the algebra $\mathcal{G}$. It follows that

$$H^i(z_1)V^\alpha(z_2) = H^i(z_1)V^\alpha(z_2) : + \frac{(\alpha)^i V^\alpha(z_1)}{z_1 - z_2}$$  \hfill (2.22)

Eqn. ((2.18)) is equivalent to the product

$$V^\alpha(z_1)V^\beta(z_2) = V^\alpha(z_1)V^\beta(z_2) : + \frac{\alpha^i V^\alpha(z_1)}{z_1 - z_2}$$  \hfill (2.23)

where $\ldots$ denotes normal ordering in the sense that $A_n$ are moved to the right of $A^\dagger_{-n}$ and $p$ to the right of $q$ for $\alpha, \beta$ roots of $\mathcal{G}$.

### 2.2 Principal Gradation

In this case the affine Heisenberg subalgebra is defined by

$$A^a_{a+n(K+1)} = \sum_{i=1}^{K+1-a} E_{\alpha_i+\alpha_{i+1}+\cdots+\alpha_{i+a-1}^+} + \sum_{i=1}^{a} E_{-\alpha_i-\alpha_{i+1}+\cdots+\alpha_{i+a-K}}$$  \hfill (2.24)

with $a = 1, 2, \ldots, K$. It is straightforward to verify that

$$[A^a_{a+m(K+1)}, A^b_{b+n(K+1)}] = (a + m(K + 1)) \delta_{m,n}$$  \hfill (2.25)

For instance, take $g = SL(2)$ with

$$A^1_{2n+1} = E_{(n)}^a + E_{-a}^{(n+1)}.$$  \hfill (2.26)

Equation (2.25) is then easily verified.

We now define the rank $\mathcal{G}$ dimensional Fubini-Veneziano field

$$Q^a = i \sum_{n=0}^{\infty} A^a_{a+n(K+1)} \frac{z^{-a-n(K+1)}}{a+n(K+1)}$$

$$Q^a = -i \sum_{n=0}^{\infty} A^a_{a+n(K+1)} \frac{z^{a+n(K+1)}}{a+n(K+1)}$$  \hfill (2.27)

Notice that in this case the zero modes are absent.

We now seek within the Kac-Moody algebra $\hat{sl}(K+1)$ the eigenvalues and eigenfunctions of the affine Heisenberg subalgebra defined in (2.25). We therefore find that

$$[A^a_{a+n(K+1)}, F_{b,l}(z)] = \omega^{-al} (\omega^{ab} - 1) z^{a+(K+1)n} F_{b,l}(z)$$  \hfill (2.28)

with $a, b = 1, \ldots, K$, $l = 1, \ldots, K, K+1$, and where $\omega = \exp\frac{2\pi i}{K+1}$. $F_{b,l}$ is a linear combination of Kac-Moody generators of $\hat{sl}(K+1)$ and will be given explicitly in the next section for a more general case. The eigenvalues in equation (1.14) define the roots of $\hat{sl}(K+1)$. Our task is now to find a set of simple roots lying in a $K$ dimensional complex space such that any root defined in (1.14) may be
written as a integer linear combination with all coefficients being positive (or negative). The following set of simple roots satisfy our requirements:

\[ \alpha_{(K)} = \left( \omega^{i-1} - 1, \omega^{2(i-1)} - 1, \ldots, \omega^{K(i-1)} - 1 \right) \]  

for \( i = 1, \ldots, K \). Define now the vertex operator satisfying the eigenvalue equation (1.14) with eigenvalue \( \alpha \) to be

\[ V^\alpha(z) = e^{i\alpha^* Q(z)} e^{i\alpha Q(z)} \]

The OPE of vertices (2.30) is a straightforward calculations yielding

\[ V^\alpha(z_1) V^\beta(z_2) =: V^\alpha(z_1) V^\beta(z_2) : \prod_{p=1}^{K+1} \left( 1 - \frac{z_2 \omega^{-p}}{z_1} \right) \sum_{c=1}^{K+1} \frac{\alpha_c^* \beta_c}{K+1} \]

Since the simple roots in (2.29) satisfy the same addition table as the simple roots of \( \hat{\mathfrak{sl}}(K+1) \) the product of vertices (3.58) closes into the OPE algebra of \( \hat{\mathfrak{sl}}(K+1) \).

3 Vertex Construction for the Intermediate Gradation

In this section we shall discuss the vertex functions for the model defined by eqn. (1.1) with the constant element \( E \) lying in the Kac-Moody algebra \( \mathfrak{sl}(M+K+1) \) given by (1.2) and gradation given by (1.5). Consider the Heisenberg algebra associated to the center of \( \text{Ker}(\text{ad} E) \) and consisting of:

1) Homogeneous part of \( \hat{\mathfrak{sl}}(M) \)

\[ B_i^{(n)} = \alpha_i \cdot H^{(n)} , \quad i = 1, 2, \ldots, M - 1 \]

2) Principal part of \( \hat{\mathfrak{sl}}(K+1) \)

\[ A_{\alpha+n(K+1)} = \sum_{i=1}^{K+1-a} E_{\alpha_i+M+\alpha_{i+M}+\cdots+\alpha_{i+M+a-1}}^{(n)} + \sum_{i=1}^{a} E_{-(\alpha_i+M+\alpha_{i+M}+\cdots+\alpha_{i+M+K-a})}^{(n+1)} , \quad a = 1, 2, \ldots, K \]

3) “A border part”

\[ A_0^{(n)(K+1)} = \sqrt{\frac{M+K+1}{M}} \lambda M \cdot H^{(n)} - \frac{K}{2} \sqrt{\frac{M}{M+K+1}} \delta_{n,0} \]

The above elements of the Heisenberg subalgebra enter the oscillator algebra relations (we put \( c = 1 \)):

\[ \left[ A_{\alpha+n(K+1)} , A_{\beta+m(K+1)}^{\dagger} \right] = (a + (K+1)n) \delta_{n,m} \delta_{a,b} , \quad a, b = 1, \ldots, K \]
with \( A_{a+m(K+1)}^{a+1} = A_{a-m(K+1)}^{K-a+1} \): 
\[
\left[ A_{n(K+1)}^0 , A_{m(K+1)}^0 \right] = (K + 1) n \delta_{n,m} ; \quad \left[ A_n^0 , A_m^a \right] = 0 \tag{3.36}
\]
and 
\[
\left[ B_{i}^{(m)} , B_{j}^{(n)} \right] = K_{i,j} m \delta_{m+n,0} ; \quad i,j = 1, \ldots, M - 1 \tag{3.37}
\]
where \( K_{i,j} \) is the Cartan matrix of \( sl(M) \).

Instead of \( B_{i}^{(m)} \), it is more convenient to work with 
\[
K_{i}^{(n)} = \sum_{p=1}^{i} p B_{i}^{(n)} \tag{3.38}
\]
such that 
\[
\text{Tr} \left( K_{i}^{(n)} K_{j}^{(m)} \right) = \delta_{i,j} \delta_{m+n,0} \text{ and } \tag{3.39}
\]
\[
\left[ K_{i}^{(m)} , K_{j}^{(n)} \right] = m \delta_{m+n,0} \delta_{i,j} ; \quad i,j = 1, \ldots, M - 1
\]

To summarize, we had parametrized the Heisenberg subalgebra in terms of elements:
\[
A_{a+n(K+1)}^a , \quad A_{n(K+1)}^0 , \quad K_{i}^{(n)} ; \quad a = 1,2,\ldots,K ; \quad i = 1, 2, \ldots, M - 1 \tag{3.40}
\]

with \( n = 0 \) constitute a Cartan subalgebra of \( sl(M + K + 1) \).

We work with the eigenstates of this subalgebra having the following form 
\[
F_{a,l} = \frac{\hat{c}}{(w^a - 1)} + \sum_{n \in \mathbb{Z}} z^{-n(K+1)} \sum_{i=1}^{K} w_{M+i}^{a(i-p)} \sum_{p=1}^{i} z^{-p} E_{\alpha+iM+\alpha+i+M+1+\cdots+\alpha+i+M+b-1}^{(n)} + \sum_{b=1}^{K} \sum_{n \in \mathbb{Z}} z^{-(b+n(K+1))} \sum_{i=1}^{K+1-b} w_{M+i}^{a(i-1)} E_{\alpha+iM+\alpha+i+M+1+\cdots+\alpha+i+M+b-1}^{(n)} \tag{3.41}
\]
where \( a = 1, \ldots, K , \quad l = 1, \ldots, K, K+1 \) and \( \omega \) is a non-trivial \( K+1 \)-th root of unity \((\omega^{K+1} = 1)\).

\[
\bar{F}_{r,l} = \sum_{n \in \mathbb{Z}} z^{-n(K+1)} \sum_{p=0}^{K} z^{-p} E_{\alpha+iM+\alpha+i+M+1+\cdots+\alpha+i+M+b}^{(n)} \tag{3.41}
\]
where \( r = 1, \ldots, M \), \( l = 1, \ldots, K, K+1 \), and

\[
F_{a,j} = \sum_{n \in \mathbb{Z}} z^{-n} E_{\alpha_j}^{(n)} ; \quad j = 1, \ldots, M - 1 \tag{3.42}
\]
One finds the following form of algebraic relations for the eigenstates of the Heisenberg subalgebra:

\[
\begin{align*}
\left[ A^{a}a_{a} + n(K + 1), F_{b,l}(z) \right] &= \omega^{-al} (\omega^{ab} - 1) z^{a+(K+1)n} F_{b,l}(z) \\
\left[ A^{0}_{n(K+1)}, F_{b,l}(z) \right] &= 0 \\
\left[ K^{(n)}_{i}, F_{b,l}(z) \right] &= 0
\end{align*}
\]

(3.43)

where \(a, b = 1, \ldots, K\) and \(l = 1, \ldots, K, K + 1\),

\[
\begin{align*}
\left[ A^{a}_{n(n(K+1))}, \bar{F}_{r,l}(z) \right] &= \omega^{-al} z^{a+n(K+1)} \bar{F}_{r,l}(z) \\
\left[ A^{0}_{n(K+1)}, \bar{F}_{r,l}(z) \right] &= \sqrt{\frac{K + 1}{\lambda_{M}^{2}}} z^{n(K+1)} \bar{F}_{r,l}(z) \\
\left[ K^{(n)}_{i}, \bar{F}_{r,l}(z) \right] &= \frac{1}{N_{i}} \left( \sum_{p=1}^{i} \delta_{r,p} - \delta_{r,i+1} \right) z^{n(K+1)} \bar{F}_{r,l}(z)
\end{align*}
\]

(3.44)

where \(N_{i} = \sqrt{i(i+1)}, r = 1, \ldots, M\) and \(l = 1, \ldots, K, K + 1\).

Furthermore, we also have

\[
\begin{align*}
\left[ A^{a}_{n+n(K+1)}, F_{\alpha_{j}}(z) \right] &= 0 \\
\left[ A^{0}_{n(K+1)}, F_{\alpha_{j}}(z) \right] &= 0 \\
\left[ K^{(n)}_{i}, F_{\alpha_{j}}(z) \right] &= \frac{1}{N_{i}} \left( \sum_{p=1}^{i} 2p\delta_{p,j} - p\delta_{p,j-1} - p\delta_{p,j+1} \right) z^{n} F_{\alpha_{j}}(z)
\end{align*}
\]

(3.45)

where \(i, j = 1, \ldots, M - 1\).

Since, \(F_{j,l}, \bar{F}_{r,l}\) and \(F_{\alpha_{j}}\) are step operators associated with the Cartan subalgebra defined by the Heisenberg subalgebra they correspond to the roots of \(sl(M + K + 1)\). The simple roots defining these steps operators are:

\[
\begin{align*}
\alpha_{i} &= (\alpha_{(M)}, 0, 0_{(K)}), \quad i = 1, \ldots, M - 1 \\
&= \left( 0_{(M-2)}, \frac{1 - M}{N_{M-1}}, \sqrt{\frac{K + 1}{\lambda_{M}^{2}}}, 1_{(K)} \right) \quad \text{for } i = M \\
&= (0_{(M-1)}, 0, \alpha_{(K)}), \quad \text{for } i = M + j, \quad j = 1, \ldots, K
\end{align*}
\]

(3.46)

where \(0_{(N)}\) denotes the \(N\)-dimensional null vector, \(1_{(N)}\) is the \(N\)-dimensional vector with \(N\) unit components. The root vectors \(\alpha_{(M)}\), are

\[
\begin{align*}
\alpha_{(M)}_{1} &= \left( \frac{2}{N_{1}}, 0_{(M-2)} \right) \\
\alpha_{(M)}_{2} &= \left( -\frac{1}{N_{1}}, \frac{3}{N_{2}}, 0_{(M-3)} \right)
\end{align*}
\]
the general vertex operator in the normal ordered form:

\[ \alpha_{(M)}(z) = \left( 0_{(i-2)}, \frac{(i-1)}{N_{i-1}}, \frac{(i+1)}{N_i}, 0_{(M-1-i)} \right), i = 3, \ldots, M - 1 \] (3.47)

and \( \alpha_{(K)} \) are given in eqns. (2.29).

The step operators associated to the simple roots of (3.47), (2.29) are nothing but the eigenstates of the Heisenberg algebra (3.40) and the association is

\[ E_{\alpha_i} \leftrightarrow F_{\alpha_i} \quad (= E_{\alpha_i} \text{ of } sl(M)) \quad i = 1, 2, \ldots, M - 1 \] (3.48)

\[ E_{\alpha_M} \leftrightarrow F_{M,K+1} \] (3.49)

\[ E_{\alpha_{M+a}} \leftrightarrow F_{1,K-a+2} \quad a = 1, 2, \ldots, K \] (3.50)

The remaining eigenstates are associated to the general roots of the form \( \alpha = \alpha_1 + \ldots + \alpha_j \). For instance \( F_{2,0} \leftrightarrow E_{\alpha_{M+1} + \alpha_{M+2}} \) with \( \alpha_{M+1} + \alpha_{M+2} = (0, \ldots, 0, \omega^2 - 1, \ldots, \omega^{2K} - 1) \).

Define now the Fubini-Veneziano operators

\[ Q_0^i(z) = q^i - ip^i \ln z \quad Q_i(z) = i \sum_{n=1}^{\infty} \frac{\kappa_i(n) z^{-n}}{n} \quad i = 1, \ldots, M - 1 \] (3.51)

\[ Q_0^M(z) = q^M - ip^M \ln z \quad Q_M(z) = i \sum_{n=1}^{\infty} \frac{A_0^0(z)}{n(K+1)} z^{-n(K+1)} \] (3.52)

\[ Q^{M+a}(z) = i \sum_{n=0}^{\infty} \frac{A_a^n z^{a+n(K+1)}}{a + n(K+1)} \quad a = 1, 2, \ldots, K \] (3.53)

where \( p_M \) is equal to \( A_0^0 \) from expression (3.34) and the zero modes satisfy \([p^i, q^j] = -i\delta^{ij} \).

The corresponding conjugated Fubini-Veneziano operators \( Q^i(z) \) are obtained from (3.51)-(3.53) by taking into consideration rules \( \kappa_i(n)^\dagger = \kappa_i(-n)^\dagger \), \( (A_{(n)(K+1)})^\dagger = A_{(-n)(K+1)}^a \).

The total number of \( Q^i \)'s equals \( M + K \) which is the rank of \( sl(M + K + 1) \).

Putting together the simple root structure from eqns. (3.47), (2.29) with the Fubini-Veneziano operators enables us to write down a compact expression for the general vertex operator in the normal ordered form:

\[ V^\alpha(z) = z^{(\alpha(M))^2} \times \]
\[ \times \exp \left( i\alpha^* \cdot Q^\dagger(z) \right) \exp \left( i\alpha \cdot Q \ln z \right) \exp \left( i\alpha \cdot Q(z) \right) \] (3.54)

where \( (\alpha(M))^2 = \sum_{j=1}^{M} (\alpha_j(M))^2 \) i.e. the sum of squares of components in the \( M - 1 \) subspace, and

\[ \omega = (Q_1, Q_2, \ldots, Q_M, Q_{M+1}, \ldots, Q_{M+K}) \] (3.55)

\[ q = (q_1, q_2, \ldots, q_M, 0_{(K)}) \] (3.56)

\[ p = (p_1, p_2, \ldots, p_M, 0_{(K)}) \] (3.57)
The product of two vertex operators is found to be:

\[ V^\alpha(z_1) V^\beta(z_2) =: V^\alpha(z_1) V^\beta(z_2) : = \sum_{j=1}^{M} (\alpha_j^2 + 2\alpha_j^\beta) \frac{1}{2} \sum_{j=1}^{M} \beta_j^\beta j \]

\[ \times \prod_{p=0}^{K} \left( 1 - \frac{z_2}{z_1} \right)^{(\alpha M + \beta M + \alpha^* + \beta^*/(K+1))} \]

where the * stands for complex conjugation. Similarly for the more complicated products of vertices.

It is also straightforward to see that the square of a vertex vanishes. In order to consider the product of several vertex operators we first define the generalized scalar product for \( M + K \) component vectors

\[ (a \circ b)_p = \delta_p,0 \sum_{i=1}^{M-1} a_i b_i^* + \frac{a_i + b_i^*}{K+1} \sum_{i=1}^{K} a_i + \frac{b_i^*}{K+1} \omega^p \]

(3.59)

With this new notation, eqn ((3.58)) becomes

\[ V^\alpha(z_1) V^\beta(z_2) =: V^\alpha(z_1) V^\beta(z_2) : = \sum_{i=1}^{M} a_i b_i^* \prod_{p=0}^{K} \left( 1 - \frac{z_2}{z_1} \omega^p \right)^{(\alpha \circ \beta)_p} \]

(3.60)

The general formula for the product of several vertices is therefore given as

\[ V^{\gamma_1}(z_1) \cdots V^{\gamma_N}(z_N) = : V^{\gamma_1}(z_1) \cdots V^{\gamma_N}(z_N) : \prod_{m=1}^{N-1} z_m^{\gamma_m+\gamma_{m+2}+\cdots+\gamma_N} \times \prod_{r=2}^{N} \prod_{i<r} \prod_{p=0}^{K} \left( 1 - \frac{z_r \omega^p}{z_{i-r}} \right)^{\gamma_r \circ \gamma_{i-r}} \]

(3.61)

4 The OPE Algebra of Vertex Operators

We have already proposed a set of \( M + K \) simple roots satisfying the same addition properties of those of \( sl(M+K+1) \). We now use the product of vertices given in (3.58) to obtain the OPE to show that, in fact, it reproduces the algebra \( sl(M+K+1) \). Let us associate the Kac-Moody currents to the vertex operators via

\[ e^{(M)}_\alpha = V^\alpha(z) \]

(4.62)

for roots in the \( sl(M) \) sector, i.e. \( \alpha = (\alpha_M,0,0,(K)) \) where \( \alpha_M \) denote a root in the pure \( sl(M) \) subalgebra,

\[ e^{(K)}_\alpha = \frac{V^\alpha(z)}{1 - \omega^\alpha} \]

(4.63)
for roots in the $sl(K + 1)$ sector, i.e. $\alpha = (0(M), 0, \alpha_K(M, K))$ where $\alpha_K$ is a root in the $sl(K + 1)$ sector and $\#$ denote the number of simple roots in $\alpha_K$,

$$e^{\alpha(MK)} = \frac{V^\alpha(z)}{(K + 1)^{\#}}$$

for roots of the form $\alpha = (\alpha_M, \alpha_M, \alpha_K(M))$, where $\alpha_M$ denote the $M$-th component of the root $\alpha$.

It follows from the complex structure of the simple roots given in the previous section, that the OPE algebra displaying the most singular part of the product of vertices is given by

$$V^\alpha(z_1)V^\beta(z_2) = \begin{cases} 
  z_1V^{\alpha + \beta}(z_1) & \text{if } \alpha + \beta \text{ is a root} \\
  z_1^{\alpha}H(z_1) - z_1 \frac{d}{dz_1}(\frac{z_1}{z_1 - z_2}) & \text{if } \alpha + \beta = 0 \\
  0 & \text{otherwise}
\end{cases}$$

(4.65)

5 Special Case $M=K=1$; Tau Functions

We consider now the special case of $M = K = 1$ with $\hat{G} = \hat{sl}(3)$ and the Lax matrix operator from (1.1) with $A$ from (1.3) and $E$ from (1.2) being given by (see (Aratyn et al 1997b))

$$L = D - \left( \begin{array}{ccc} 0 & q & 0 \\ r & U_2 & 1 \\ 0 & \lambda & -U_2 \end{array} \right) - \nu \hat{c}$$

(5.66)

where $\lambda$ is the usual loop parameter. We now describe two different two-soliton solutions obtained from the above vertex construction.

The first one has $r = 0$ and $q \neq 0$ and equal to

$$q = -\sqrt{2}z_2e^{(t_3 z_2^3 + t z_2 + x z_2)} \left( 1 + \frac{1}{2}e^{-2x z_1 - 2t z_1} \left( \frac{z_1 + z_2}{z_1 - z_2} \right) \right) / \tau_0^{(0)}$$

(5.67)

with the tau-functions $\tau_0^{(0)}$ and $\tau_2^{(0)}$

$$\tau_0^{(0)} = 1 - \frac{1}{2} e^{-2x z_1 - 2t z_1} \quad ; \quad \tau_2^{(0)} = 1 + \frac{1}{2} e^{-2x z_1 - 2t z_1}$$

(5.68)

from which we obtain $U_2$ and $\nu$ by using:

$$U_2 = -\partial_x \ln \left( \frac{\tau_0^{(0)}}{\tau_2^{(0)}} \right) \quad ; \quad \nu = -\partial_x \ln \left( \tau_0^{(0)} \right)$$

(5.69)

Another two-soliton solution for which this time both $q \neq 0$ and $r \neq 0$ is:

$$\tau_0^{(0)} = 1 + (-1)^{(\sigma/2)} \frac{z_1 + \sigma/2 z_2}{z_1 - \sigma/2} e^{(xz_1 + tz_1^2 + xz_2^1 + tz_2^2 - tz_3^2 + t z_3^3)} / \tau_0^{(0)}$$

(5.70)
with $\sigma = 0, 2,$ and

$$r = \frac{\sqrt{2} z_2 e^{-t z_2^2 + x z_2 + t_3 z_2^3}}{\tau_2^{(0)}} \quad ; \quad q = \frac{\sqrt{2} z_1 e^{(t z_1^2 + x z_1 + t_3 z_1^3)}}{\tau_0^{(0)}}$$

(5.71)

and again $U_2$ and $\nu$ can be obtained from (5.69).

In the above examples we only kept the times $t_n$ with $n \leq 3$ for which we verified validity of the relevant evolution equations.

The novel feature of the above soliton solutions is that they mix exponentials

$$\exp \left( \sum_{n=1}^{\infty} t_n z_j^n \right)$$

which represent a typical time dependence for the KP solutions with pure KdV time dependence of the type $\exp \left( \sum_{n=0}^{\infty} t_{2n+1} z_j^{2n+1} \right)$ involving only odd times (Aratyn et al (1997b)).

6 APPENDIX

We now provide some useful relations extensively used in obtaining the formulas given in the text:

$$\sum_{n=0}^{\infty} \frac{x^{a+(K+1)n}}{a+(K+1)n} = \frac{-1}{K+1} \sum_{p=1}^{K+1} \omega^{ap} \ln(1-x\omega^{-p}) \quad ; \quad a = 1, \ldots, K$$

(6.72)

for

$$\omega^{K+1} = 1.$$

(6.73)

It also follows that

$$1 + \omega + \cdots + \omega^K = 0$$

(6.74)

and

$$(1 - x^{K+1}) = \prod_{p=1}^{K+1} (1 - \omega^p x)$$

(6.75)

from where we obtain, after using L'Hopital's rule

$$K + 1 = \prod_{p=1}^{K} (1 - \omega^p)$$

(6.76)

and

$$\sum_{p=1}^{K} \frac{1}{1-\omega^p} = \frac{K}{2}$$

(6.77)

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