GROUPS OF TYPE $E_7$ OVER ARBITRARY FIELDS

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Abstract. Freudenthal triple systems come in two flavors, degenerate and nondegenerate. The best criterion for distinguishing between the two which is available in the literature is by descent. We provide an identity which is satisfied only by nondegenerate triple systems. We then use this to define algebraic structures whose automorphism groups produce all adjoint algebraic groups of type $E_7$ over an arbitrary field of characteristic $\neq 2, 3$.

The main advantage of these new structures is that they incorporate a previously unconsidered invariant (a symplectic involution) of these groups in a fundamental way. As an application, we give a construction of adjoint groups with Tits algebras of index 2 which provides a complete description of this involution and apply this to groups of type $E_7$ over a real-closed field.

A useful strategy for studying simple (affine) algebraic groups over arbitrary fields has been to describe such a group as the group of automorphisms of some algebraic object. We restrict our attention to fields of characteristic $\neq 2, 3$. The idea is that these algebraic objects are easier to study, and their properties correspond to properties of the group one is interested in. Weil described all groups of type $A_n$, $B_n$, $C_n$, $D_n$, and $G_2$ in this manner in \cite{Wei60}. Similar descriptions were soon found for groups of type $F_4$ (as automorphism groups of Albert algebras) and $G_2$ (as automorphism groups of Cayley algebras). Recently, groups of type $^2D_4$ and $^6D_4$ have been described in \cite{KMRT98, §43} as groups of automorphisms of triality central simple algebras. The remaining groups are those of types $E_6$, $E_7$, and $E_8$.

As an attempt to provide an algebraic structure associated to groups of type $E_7$, Freudenthal introduced a new kind of algebraic structure in \cite{Fre54, §4}, which was later studied axiomatically in \cite{Mey68}, \cite{Bro69}, and \cite{Fer72}. These objects, called Freudenthal triple systems, come in two flavors: degenerate and nondegenerate. The automorphism groups of the nondegenerate ones provide all simply connected groups of type $E_7$ with trivial Tits algebras over an arbitrary field. (The Tits algebras of a group $G$ are the endomorphism rings of certain irreducible representations of $G$. They were introduced in \cite{Tit66}, or see \cite{KMRT98, §27.A} for another treatment.) In fact, more is true: they are precisely the $G$-torsors for $G$ simply connected split of type $E_7$.

One issue that has not been addressed adequately in the study of triple systems is how to distinguish between the two kinds. A triple system consists of a 56-dimensional vector space endowed with a nondegenerate skew-symmetric bilinear form and a quartic form (see \cite{L} for a complete definition), and we say that the triple system is nondegenerate precisely when this quartic form is irreducible when we extend scalars to a separable closure of the

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base field. There seems to be essentially no criterion available in the literature to distinguish between the two types other than checking the definition. In Section 2, we show that one of the identities which nondegenerate triple systems are known to satisfy is not satisfied by degenerate ones, thus providing a means for differentiating between the two types which doesn’t require enlarging the base field.

In Section 3, we define algebraic structures whose groups of automorphisms produce all groups of type $E_7$ up to isogeny. Thanks to the preceding result distinguishing between degenerate and nondegenerate triple systems, no Galois descent is needed for this definition. We call these structures gifts (short for generalized Freudenthal triple systems). They are triples $(A, \sigma, \pi)$ such that $A$ is a central simple $F$-algebra of degree 56, $\sigma$ is a symplectic involution on $A$, and $\pi : A \to A$ is an $F$-linear map satisfying certain axioms (see 3.2 for a full definition). We also show an equivalence of categories between the category of gifts over an arbitrary field $F$ and the category of adjoint (equivalently, simply connected) groups of type $E_7$ over $F$. A description of the flag (a.k.a. homogeneous projective) varieties of an arbitrary group of type $E_7$ is then easily derived in Section 4.

The main strength of these gifts is that they include the involution $\sigma$ in an intrinsic way. Specifically, as mentioned above every adjoint group $G$ of type $E_7$ is isomorphic to $\text{Aut}(A, \sigma, \pi)$ for some gift $(A, \sigma, \pi)$. The inclusion of the split adjoint group of type $E_7$ in the split adjoint group of type $C_{28}$ gives via Galois cohomology that the pair $(A, \sigma)$ is an invariant of the group $G$. We give a construction (in 3.3) of gifts with algebra component $A$ of index 2 (which is equivalent to constructing groups of type $E_7$ with Tits algebra of index 2 and corresponds to Tits’ construction of analogous Lie algebras in [Tit66a], or see [Jac71, §10]) and we describe the involution $\sigma$ explicitly in this case. In the final section, we use this construction to prove some facts about simple groups of type $E_7$ over a real-closed field.

Notations and conventions. All fields that we consider will have characteristic $\neq 2, 3$.

For $g$ an element in a group $G$, we write $\text{Int}(g)$ for the automorphism of $G$ given by $x \mapsto gxg^{-1}$.

For $X$ a variety over a field $F$ and $K$ any field extension of $F$, we write $X(K)$ for the $K$-points of $X$.

When we say that an affine algebraic group (scheme) $G$ is simple, we mean that it is absolutely almost simple in the usual sense (i.e., $G(F_s)$ has a finite center and no noncentral normal subgroups for $F_s$ a separable closure of the ground field).

We write $\mathbb{G}_{m, F}$ for the algebraic group whose $F$-points are $F^*$ and $\mu_n$ for its subgroup group of $n$th roots of unity.

We will also follow the usual conventions for Galois cohomology and write $H^i(F, G) := H^i(\text{Gal}(F_s/F), G(F_s))$ for $G$ any algebraic group over $F$, and similarly for the cocycles $Z^1(F, G)$. For more information about Galois cohomology, see [Ser79] and [Ser94].

We follow the notation in [Lam73] for quadratic forms.

For $I$ a right ideal in a central simple $F$-algebra $A$, we define the rank of $I$ to be $(\dim_F I)/\deg A$. Thus when $A$ is split, so that we may write $A \cong \text{End}_F(V)$ for some
$F$-vector space $V$, $I = \text{Hom}_F(V,U)$ for some subspace $U$ of $V$ and the rank of $I$ is precisely the dimension of $U$.

1. Background on triple systems

Definition 1.1. (See, for example, [Fer72, p. 314] or [Gar99, 3.1]) A (simple) Freudenthal triple system is a 3-tuple $(V,b,t)$ such that $V$ is a 56-dimensional vector space, $b$ is a nondegenerate skew-symmetric bilinear form on $V$, and $t$ is a trilinear product $t: V \times V \times V \rightarrow V$.

We define a 4-linear form $q(x,y,z,w) := b(x,t(y,z,w))$ for $x, y, z, w \in V$, and we require that

\begin{itemize}
  \item \textbf{FTS1:} $q$ is symmetric,
  \item \textbf{FTS2:} $q$ is not identically zero, and
  \item \textbf{FTS3:} $t(t(x,x,x),x,y) = b(y,x)t(x,x,x) + q(y,x,x,x)$ for all $x,y \in V$.
\end{itemize}

We say that such a triple system is nondegenerate if the quartic form $v \mapsto q(v,v,v,v)$ on $V$ is absolutely irreducible (i.e., irreducible over a separable closure of the base field) and degenerate otherwise.

Note that since $b$ is nondegenerate, FTS1 implies that $t$ is symmetric.

One can linearize FTS3 a little bit to get an equivalent axiom that will be of use later. Specifically, replacing $x$ with $x + \lambda z$, expanding using linearity, and taking the coefficient of $\lambda^2$, one gets the equivalent formula

\begin{equation}
\text{FTS3'} \quad t(t(x,x,z),z,y) = zq(x,x,z,y) + xq(x,z,z,y) + b(y,z)t(x,x,z) + b(y,x)t(x,z,z).
\end{equation}

Example 1.2. (Cf. [Bro69, p. 94], [Mey68, p. 172]) Let $W$ be a 27-dimensional $F$-vector space endowed with a non-degenerate skew-symmetric bilinear form $s$ and set

\begin{equation}
V := \begin{pmatrix} F & W \\ W & F \end{pmatrix}.
\end{equation}

For

\begin{equation}
x := \begin{pmatrix} \alpha \\ j' \\ \beta \end{pmatrix} \quad \text{and} \quad y := \begin{pmatrix} \gamma \\ k' \\ \delta \end{pmatrix}
\end{equation}

set

\begin{equation}
b(x,y) := \alpha \delta - \beta \gamma + s(j,k') + s(j',k).
\end{equation}

We define the determinant map $\det: V \rightarrow F$ by

\begin{equation}
\det(x) := \alpha \beta - s(j,j')
\end{equation}

and set

\begin{equation}
t(x,x,x) := 6 \det(x) \begin{pmatrix} -\alpha & j' \\ -j & \beta \end{pmatrix}.
\end{equation}
Then \((V, b, t)\) is a Freudenthal triple system. Since
\[ q(x, x, x, x) := 12 \det(x)^2, \]

it is certainly degenerate, and we denote it by \(M_s\). By [Bro69, §4] or [Mev68, §4], all degenerate triple systems are forms of one of these, meaning that any degenerate triple system becomes isomorphic to \(M_s\) over a separable closure of the ground field.

**Example 1.5.** For \(J\) an Albert \(F\)-algebra, there is a nondegenerate triple system denoted by \(M(J)\) whose underlying \(F\)-vector space is \(V = (\mathfrak{f}, \mathfrak{p})\). Explicit formulas for the \(b, t,\) and \(q\) for this triple system can be found in [Bro69, §3], [Mev68, §6], [Fer72, §1], and [Gar99, 3.2]. For \(J^d\) the split Albert \(F\)-algebra, we set \(M^d := M(J^d)\). It is called the split triple system because \(\text{Inv}(M^d)\) is the split simply connected algebraic group of type \(E_7\) over \(F\) [Gar99, 3.5]. By [Bro69, §4] or [Mev68, §4] every nondegenerate triple system is a form of \(M^d\).

A *similarity* of triple systems is a map \(f : (V, b, t) \rightarrow (V', b', t')\) defined by an \(F\)-vector space isomorphism \(f : V \rightarrow V'\) such that \(b'(f(x), f(y)) = \lambda b(x, y)\) and \(t'(f(x), f(y), f(z)) = \lambda f(t(x, y, z))\) for all \(x, y, z \in V\) and some \(\lambda \in F^\times\) called the *multiplier* of \(f\). Similarities with multiplier one are called *isometries*. They are the isomorphisms in the obvious category of Freudenthal triple systems. For a triple system \(M\), we write \(\text{Inv}(M)\) for the algebraic group whose \(F\)-points are the isometries of \(M\).

**Remark 1.6.** Although it is not clear precisely what the structure of the automorphism group of a degenerate triple system is, a few simple observations can be made which make it appear to be not very interesting from the standpoint of simple algebraic groups.

Since by definition any element of \(\text{Inv}(M_s)\) must preserve the quartic form \(q\), it must also be a similarity of the quadratic form \(\det\) with multiplier \(\pm 1\). This defines a map \(\text{Inv}(M_s) \rightarrow \mu_2\) which is surjective since \(\overline{\sigma} \in \text{Inv}(M_s)(F)\) maps to \(-1\), where
\[ \overline{\sigma} \left( \begin{array}{ccc} \alpha & \beta & j \\ \alpha' & \beta' & j' \end{array} \right) := \left( \begin{array}{ccc} -\beta & j' & \alpha \\ j & \beta & \alpha' \end{array} \right). \]

So \(\text{Inv}(M_s)\) is not connected.

Also, we can make some bounds on the dimension. Specifically, we define a map \(f : \mathbb{G}_{m,F} \times W \times GL(W) \rightarrow \text{Inv}(M_s)\) by
\[ f(c, u, \phi) \left( \begin{array}{ccc} \alpha & j & \beta \\ \alpha' & j' & \beta' \end{array} \right) := \left( \begin{array}{ccc} 0 & \phi(j) \\ \phi(j') & \phi(j) \end{array} \right), \]

where \(\phi^\dagger = \sigma(\phi)^{-1}\) for \(\sigma\) the involution on \(\text{End}_F(W)\) which is adjoint with respect to \(s\). (So \(s(\phi(w), \phi^\dagger(w')) = s(w, w')\) for all \(w, w' \in W\).) Then \(f\) is an injection of varieties, but
\[ f(c, u, \phi)f(d, v, \psi) = f(cd, du + \phi^\dagger(v), \phi\psi), \]

so it is not a group homomorphism. It does, however, restrict to be a morphism of algebraic groups on \(\mathbb{G}_{m,F} \times \{0\} \times GL(W)\), so \(\text{Inv}(M_s)\) contains a split torus of rank 28. This map \(f\)
is also not surjective since for any \( u \neq 0 \), the map

\[
\varpi f(c, u, \phi) \varpi^{-1}
\begin{pmatrix}
\alpha & j' \\
j' & \beta
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{2} (\alpha - s(\phi(j'), u)) & \beta u + \phi^t(j) \\
\phi(j') & \beta c
\end{pmatrix}
\]

is not in the image of \( f \). For an upper bound, we observe that the identity component of \( \text{Inv} (\mathcal{M}_s) \) is contained in the isometry group of \( \text{det} \) (whose identity component is of type \( D_{28} \), hence is 1540-dimensional). Thus

\[757 < \dim \text{Inv} (\mathcal{M}_s)^+ \leq 1540.\]

2. AN IDENTITY

For a Freudenthal triple system \((V, b, t)\) over \( F \), we define an \( F \)-vector space map \( p : V \otimes_F V \to \text{End}_F(V) \) given by

\[
p(u \otimes v) w := t(u, v, w) - b(w, u)v - b(w, v)u.
\]

In the case where the triple system is nondegenerate, Freudenthal \([Fre54, 4.2]\) also defined a map \( V \otimes_F V \to \text{End}_F(V) \) which he denoted by \( \times \). The obvious computation shows that his map is related to our map \( p \) by

\[
8v \times v' = p(v \otimes v').
\]

**Theorem 2.3.** Let \( \mathfrak{M} := (V, b, t) \) be a Freudenthal triple system with map \( p \) as given above. Then \( \mathfrak{M} \) is nondegenerate if and only if it satisfies the identity

\[
\text{tr}(p(x \otimes x) p(y \otimes y)) = 24 \left( q(x, x, y, y) - 2b(y, x)^2 \right)
\]

for all \( x, y \in V \), where \( \text{tr} \) is the usual trace form on \( \text{End}_F(V) \).

**Proof:** If \( \mathfrak{M} \) is nondegenerate, then the conclusion is \([Fre63, 31.3.1]\) or it can be easily derived from \([Mey68, 7.1]\). So we may assume that \( \mathfrak{M} \) is degenerate and show that it doesn’t satisfy (2.4). Extending scalars, we may further assume that our ground field is separably closed and so that \( \mathfrak{M} = \mathfrak{M}_s \), the degenerate triple system from Example 1.2.

To simplify some of our formulas, we define the *weighted determinant*, \( \text{wdet} : \mathfrak{M}_s \to F \), to be given by

\[\text{wdet}(x) := 3\alpha \beta - s(j, j')\]

for \( x \) and \( y \) as in (1.4).

We first compute the value of the left side of (2.4). For \( x \) and \( y \) as in (1.4), we can directly calculate the action of \( p(x \otimes x) p(y \otimes y) \) on each of the four entries of our matrix as in (1.3). Since we are interested in the trace of this operator, we only record the projection onto the entry that we are looking at.

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto 4 \left[ \text{wdet}(x) \text{wdet}(y) - 4\alpha \delta s(j, k') \right]
\]

(2.5)

\[
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 4 \left[ \text{wdet}(x) \text{wdet}(y) + 4\beta \gamma s(j', k) \right]
\]

(2.6)
(2.7) \[
\begin{pmatrix}
0 & m \\
0 & 0
\end{pmatrix} \mapsto 4 \left[ \text{det}(x) \text{det}(y)m + 4(\alpha\delta - s(k, j')) s(k', m)j \right. \\
+ 2 \text{det}(x)s(k', m)k + 2 \text{det}(y)s(j', m)j \]
\]

(2.8) \[
\begin{pmatrix}
0 & m' \\
m' & 0
\end{pmatrix} \mapsto 4 \left[ \text{det}(x) \text{det}(y)m' - 4(\beta\gamma - s(j, k')) s(k, m')j' \right. \\
- 2 \text{det}(x)s(k, m')k' - 2 \text{det}(y)s(j, m')j'
\]

Since \(s\) is nondegenerate, it induces an identification of \(V\) with its dual \(V^*\) by sending \(x \in V\) to the map \(v \mapsto s(x,v)\). We may also identify \(V \otimes_F V^*\) with \(\text{End}_F(V)\), and combining these two identifications provides an isomorphism \(\varphi_s: V \otimes_F V \xrightarrow{\sim} \text{End}_F(V)\) given by

\[
\varphi_s(x \otimes y)w = xs(y,w),
\]

cf. [KMRT98, 5.1]. One has \(\text{tr}(\varphi_s(x \otimes y)) = s(y,x)\).

With that notation, the terms in the brackets of (2.7) and (2.8) with coefficient 4 give the maps

\[
(\alpha\delta - s(k, j'))\varphi_s(j \otimes k') \quad \text{and} \quad - (\beta\gamma - s(j, k'))\varphi_s(j' \otimes k)
\]
on \(W\), which have traces

\[
(\alpha\delta - s(k, j'))s(k', j) \quad \text{and} \quad (\beta\gamma - s(j, k'))s(j', k)
\]
respectively. Similarly, the terms with coefficient 2 give the maps

\[
\text{det}(x)\varphi_s(k \otimes k') + \text{det}(y)\varphi_s(j \otimes j') \quad \text{and} \quad -\text{det}(x)\varphi_s(k' \otimes k) - \text{det}(y)\varphi_s(j' \otimes j)
\]
whose sum has trace

\[
2 \left[ \text{det}(x)s(k', k) + \text{det}(y)s(j', j) \right].
\]

Adding all of this up, we see that

\[
\frac{1}{8}\text{tr}(p(x \otimes x)p(y \otimes y)) = \text{wdet}(x)\text{wdet}(y) + 2\beta\gamma s(j', k) - 2\alpha\delta s(j, k') \\
+ 27 \text{det}(x)\text{det}(y) + 2 \left[ (\beta\gamma - s(j, k'))s(j', k) + (\alpha\delta - s(k, j'))s(k', j) \right] \\
+ 2 \left[ \text{det}(x)s(k', k) + \text{det}(y)s(j', j) \right] = 3q(x,y,y) - b(y,x)^2 + 20 \text{det}(x)\text{det}(y) - 5 \text{det}(x,y)^2,
\]

where we have linearized the determinant to define

\[
\text{det}(x,y) := \text{det}(x + y) - \text{det}(x) - \text{det}(y).
\]

Taking the difference of the last expression in (2.10) and one-eighth of the right side of (2.4), we get the quartic polynomial

\[
5b(x,y)^2 + 20 \text{det}(x)\text{det}(y) - 5 \text{det}(x,y)^2.
\]

We plug

\[
x := \begin{pmatrix} 0 & j \\ j' & 0 \end{pmatrix} \quad \text{and} \quad y := \begin{pmatrix} 0 & k \\ k' & 0 \end{pmatrix}
\]

into (2.11), where we have chosen \(j, j', k,\) and \(k'\) such that

\[
s(j, k') = s(j', k) = 0 \quad \text{and} \quad s(j, j') = s(k, k') = 1.
\]
Then \(b(x, y) = 0\) and \(\det(x) = \det(y) = -1\). Since \(\det(x + y) = -2\), we have \(\det(x, y) = 0\). Thus the value of (2.11) is 20 and (2.4) does not hold for degenerate triple systems.

It really was important that we allow \(x \neq y\) in (2.4), since for \(x = y\) (2.11) is identically zero. So all triple systems satisfy the identity

\[
(2.12) \quad \tr(p(x \otimes x)^2) = 24q(x, x, x, x).
\]

3. Gifts

In this section we will define an object we call a gift, such that every adjoint group of type \(E_7\) is the automorphism group of some gift. We must first have some preliminary definitions.

Suppose that \((A, \sigma)\) is a central simple algebra with a symplectic involution \(\sigma\). The sandwich map

\[
\text{Sand}: A \otimes_F A \longrightarrow \End_F(A)
\]

defined by

\[
\text{Sand}(a \otimes b)(x) = axb \quad \text{for} \quad a, b, x \in A
\]
is an isomorphism of \(F\)-vector spaces by [KMRT98, 3.4]. Following [KMRT98, §8.B], we define a map \(\sigma_2\) on \(A \otimes_F A\) which is defined implicitly by the equation

\[
\text{Sand}(\sigma_2(u))(x) = \text{Sand}(u)(\sigma(x)) \quad \text{for} \quad u \in A \otimes A, \ x \in A.
\]

Suppose now that \(A\) is split. Then \(A \cong \End_F(V)\) for some \(F\)-vector space \(V\), and \(\sigma\) is the adjoint involution for some nondegenerate skew-symmetric bilinear form \(b\) on \(V\) (i.e., \(b(fx, y) = b(x, \sigma(f)y)\) for all \(f \in \End_F(V)\)). As in (2.9), we have an identification \(\varphi_b : V \otimes F V \cong \End_F(V)\), and by a straightforward computation (or see [KMRT98, 8.6]), \(\sigma_2\) is given by

\[
(3.1) \quad \sigma_2(\varphi_b(x_1 \otimes x_2) \otimes \varphi_b(x_3 \otimes x_4)) = -\varphi_b(x_1 \otimes x_3) \otimes \varphi_b(x_2 \otimes x_4)
\]

for \(x_1, x_2, x_3, x_4 \in V\).

Finally, for \(f : A \longrightarrow A\) an \(F\)-linear map, we define \(\hat{f} : A \otimes_F A \longrightarrow A\) by

\[
\hat{f}(a \otimes b) = f(a)b.
\]

**Definition 3.2.** A gift \(\mathcal{G}\) over a field \(F\) is a triple \((A, \sigma, \pi)\) such that \(A\) is a central simple \(F\)-algebra of degree 56, \(\sigma\) is a symplectic involution on \(A\), and \(\pi : A \longrightarrow A\) is an \(F\)-vector space map such that

- **G1:** \(\sigma\pi(a) = \pi\sigma(a) = -\pi(a)\),
- **G2:** \(a\pi(a) \neq 2a^2\) for some \(a \in \text{Skew}(A, \sigma)\),
- **G3:** \(\pi(\pi(a)a) = 0\) for all \(a \in \text{Skew}(A, \sigma)\),
- **G4:** \(\hat{\pi} - \hat{\sigma} - \hat{1d} = -\hat{(\pi - \sigma - 1d)}\sigma_2\), and
- **G5:** \(\text{Trd}_A(\pi(a)\pi(a')) = -24 \text{Trd}_A(\pi(a)a')\) for all \(a, a' \in (A, \sigma)\).
By Skew\((A, \sigma)\) we mean the vector space of \(\sigma\)-skew-symmetric elements of \(A\), i.e., those \(a \in A\) such that \(\sigma(a) = -a\).

A definition as strange as 3.2 demands an example.

**Example 3.3.** Suppose that \(\mathcal{M} = (V, b, t)\) is a nondegenerate Freudenthal triple system over \(F\). Set \(\text{End}(\mathcal{M}) := (\text{End}_F(V), \sigma, \pi)\) where \(\sigma\) is the involution on \(\text{End}_F(V)\) adjoint to \(b\). Using the identification \(\varphi_b : V \otimes V \to \text{End}_F(V)\), we define \(\pi : A \to A\) by \(\pi := p\varphi_b^{-1}\), where \(p\) is as in (2.1).

We show that \(\text{End}(\mathcal{M})\) is a gift. A quick computation shows that

\[
-b(\pi(\varphi_b(x \otimes y))z, w) = b(z, \pi(\varphi_b(x \otimes y))w) = -b(\pi(\varphi_b(y \otimes x))z, w),
\]

which demonstrates G1, since \(\sigma \varphi_b(x \otimes y) = -\varphi_b(y \otimes x)\).

Suppose that G2 fails. Then for \(v \in V\), we set \(a := \varphi_b(v \otimes v)\) and observe that \(a^2 = 0\), so

\[
0 = \varphi_b(v \otimes v) \pi(\varphi_b(v \otimes v))v = q(v, v, v, v)v.
\]

Since this holds for all \(v \in V\), \(q\) is identically zero, contradicting FTS2. Thus G2 holds.

Since elements \(a\) like in the preceding paragraph span \(\text{Skew}(A, \sigma)\), in order to prove G3 we may show that

\[
(3.4) \quad \pi(\pi(a)a' + \pi(a')a)y = 0,
\]

where

\[
(3.5) \quad a = \varphi_b(x \otimes x) \text{ and } a' = \varphi_b(z \otimes z).
\]

A direct expansion of the left-hand side of (3.4) shows that it is equivalent to FTS3′.

Using just the bilinearity and skew-symmetry of \(b\) and the trilinearity of \(t\), G4 is equivalent to

\[
t(x, y, x') = t(x, x', y) \text{ for all } x, x', y \in V.
\]

Thus G4 holds by FTS1.

Finally, consider G5. If \(a\) is symmetric, then by G1 \(\pi(a) = 0\) and the identity holds. If \(a'\) is symmetric then the left-hand side of G5 is again zero by G1. Since \(\sigma\) and \(\text{Trd}_A\) commute, we have

\[
\text{Trd}_A(\pi(a)a') = \sigma(\text{Trd}_A(\pi(a)a')) = -\text{Trd}_A(a'\pi(a)) = -\text{Trd}_A(\pi(a)a'),
\]

so the right-hand side of G5 is also zero. Consequently, by the bilinearity of G5, we may assume that \(a\) and \(a'\) are skew-symmetric, and we may further assume that \(a\) and \(a'\) are as given in (3.5). Then G5 reduces to (2.4).

It turns out that this construction produces all gifts with split central simple algebra component.

**Lemma 3.6.** Suppose that \(\mathfrak{G} = (A, \sigma, \pi)\) is a gift over \(F\). Then \(\mathfrak{G} \cong \text{End}(\mathcal{M})\) for some nondegenerate Freudenthal triple system over \(F\) if and only if \(A\) is split.
Proof: One direction is done by Example 3.3, so suppose that \( (A, \sigma, \pi) \) is a gift with \( A \) split. Then we may write \( A \cong \text{End}_F(V) \) for some \( 56 \)-dimensional \( F \)-vector space \( V \) such that \( V \) is endowed with a nondegenerate skew-symmetric bilinear form \( b \) and \( \sigma \) is the involution on \( A \) which is adjoint to \( b \). We define \( t: V \times V \times V \rightarrow V \) by

\[
t(x, y, w) := \pi(\phi_b(x \otimes y))w + b(w, x)y + b(w, y)x.
\]

Observe that \( t \) is trilinear. We define a \( 4 \)-linear form \( q \) on \( V \) as in FTS2.

The proof that FTS3′ implies G3 in Example 3.3 reverses to show that G3 implies FTS3′. Similarly, G4 implies that \( t(x, y, z) = t(x, z, y) \) for all \( x, y, x' \in V \), so \( q(w, x, y, z) = q(w, x, y, z) \). G1 implies that \( q(w, x, y, z) = q(z, x, y, w) = q(w, y, x, z) \). Since the transpositions (3 4), (1 4), and (2 3) generate \( S_4 \) (= the symmetric group on four letters) \( q \) is symmetric.

Next, suppose that FTS2 fails, so that \( q \) is identically zero. Then since \( b \) is nondegenerate, \( t \) is also zero. Then for \( v, v', z \in V \) and \( a := \varphi_b(\varphi \otimes v) \) and \( a' := \varphi_b(\varphi \otimes v') \),

\[
(a\pi(a') + a'\pi(a))z = 2(b(v', v')b(v', z)v + b(v', v)b(v, z)) = 2(aa' + a'a)z.
\]

Since elements of the same form as \( a \) and \( a' \) span \( \text{Skew}(A, \sigma) \), this implies that G2 fails, which is a contradiction. Thus FTS2 holds and \( (V, b, t) \) is a Freudenthal triple system.

Finally, writing out G5 in terms of \( V \) gives (2.4), which shows that \( (V, b, t) \) is nondegenerate.

Remark 3.7. The astute reader will have noticed that our definition of \( \text{End}(\mathcal{M}) \) almost works if \( \mathcal{M} \) is degenerate, in that the only problem is that the resulting \( (A, \sigma, \pi) \) doesn’t satisfy G5. That example and the proof of 3.6 make it clear that if we remove the axiom G5 from the definition of a gift, then we would get an analog to Lemma 3.6 where the Freudenthal triple system is possibly degenerate.

Remark 3.8. Observe that in the isomorphism \( \mathfrak{G} \cong \text{End}(\mathcal{M}) \) from the preceding lemma, \( \mathcal{M} \) is only determined up to similarity. To wit, for a nondegenerate triple system \( \mathcal{M} = (V, b, t) \) and \( \lambda \in F^* \), we define a similar structure \( \mathcal{M}_\lambda = (V, \lambda b, \lambda t) \). Then \( \mathcal{M}_\lambda \) is also a nondegenerate triple system and \( \text{End}(\mathcal{M}) = \text{End}(\mathcal{M}_\lambda) \). The only potential difficulty with this last equality would be if the \( \pi \) produced by \( \mathcal{M}_\lambda \), which we shall denote by \( \pi_\lambda \), is different from the \( \pi \) produced by \( \mathcal{M} \). However, we see that

\[
\pi_\lambda(\varphi_{\lambda b}(x \otimes y))w = \lambda t(x, y, w) - \lambda b(w, x)y - \lambda b(w, y)x
= \lambda\pi(\varphi_b(x \otimes y))w
= \pi(\varphi_{\lambda b}(x \otimes y))w.
\]

Isometries and derivations.

Definition 3.9. An isometry in a gift \( \mathfrak{G} := (A, \sigma, \pi) \) is an element \( f \in A \) such that \( \sigma(f)f = 1 \) and \( \pi(faf^{-1}) = f\pi(a)f^{-1} \) for all \( a \in A \) (this ensures that \( \text{Int}(f) \) is an automorphism of \( \mathfrak{G} \)). The motivation for the name is that then the isometries in \( \text{End}(\mathcal{M}) \) are just the isometries of \( \mathcal{M} \). We set \( \text{Iso}(\mathfrak{G}) \) to be the algebraic group whose \( F \)-points are the group of isometries in \( \mathfrak{G} \).
A derivation in $\mathfrak{G}$ is an element $f \in \text{Skew}(A, \sigma)$ which satisfy

GD: $\pi(fa) - \pi(af) = f\pi(a) - \pi(a)f$ for all $a \in A$.

We define $\text{Der}(\mathfrak{G})$ to be the vector space of derivations in $\mathfrak{G}$. The name derivation has the same sort of motivation: We will see in the proof of Proposition 3.11 that the derivations in $\text{End}(V, b, t)$ are precisely the maps in $\text{End}_F(V)$ which are traditionally called derivations of the triple system $(V, b, t)$.

3.10. The description of the isometries in $\text{End}(V, b, t)$ combined with Lemma 3.6 shows that $\text{Iso}(\mathfrak{G})$ is simple simply connected of type $E_7$. Since any automorphism of $\mathfrak{G}$ is also an isomorphism of $A$, it must be of the form $\text{Int}(f)$ for some $f \in A^*$ such that $\sigma(f)f \in F^*$. Thus the map $\text{Iso}(\mathfrak{G}) \rightarrow \text{Aut}(\mathfrak{G})$ is a surjection over a separable closure of the ground field. Since the kernel of this map is the center of $\text{Iso}(\mathfrak{G})$, $\text{Aut}(\mathfrak{G})$ is simple adjoint of type $E_7$.

We can actually say more. The group $\text{Iso}(\mathfrak{G})$ is a subgroup of the symplectic group $\text{Sp}(A, \sigma)$, whose $F$-points are the elements $f \in A$ such that $\sigma(f)f = 1$. It is this embedding combined with the vector representation of $\text{Sp}(A, \sigma)$ (= the natural embedding of $\text{Sp}(A, \sigma)$ in $A^*$) which gives rise to the Tits algebra of $\text{Iso}(\mathfrak{G})$, and so the Tits algebra of $\text{Iso}(\mathfrak{G})$ is the same as the Tits algebra of $\text{Sp}(A, \sigma)$, which is just $A$.

It is easy to see that $\text{Der}(\mathfrak{G})$ is actually a Lie subalgebra of $\text{Skew}(A, \sigma)$, where the bracket is the usual commutator. In fact, by formal differentiation as in [Bor91, 3.21] or [Jac59, §4], $\text{Der}(\mathfrak{G})$ is the Lie algebra of $\text{Iso}(\mathfrak{G})$.

**Proposition 3.11.** For $\mathfrak{G} := (A, \sigma, \pi)$ a gift, $\text{im} \pi = \text{Der}(\mathfrak{G})$.

**Proof:** Since $\text{im} \pi$ and $\text{Der}(\mathfrak{G})$ are both vector subspaces of $A$, it is equivalent to prove this over a separable closure. Thus we may assume that $A$ is split, so that $\mathfrak{G} = \text{End}(M)$ for some nondegenerate triple system $M := (V, b, t)$ over $F$ by Lemma 3.6.

Consider the vector subspace $D$ of $\text{Skew}(A, \sigma)$ consisting of elements $d$ such that

$$dt(u, v, w) = t(du, v, w) + t(u, dv, w) + t(u, v, dw)$$

for all $u, v, w \in V$. (These are known as the derivations of $M$.) The obvious computation shows that $\text{im} \pi \subseteq D$, which one can find in [Mey68, p. 166, Lem. 1.3]. Conversely, the reverse containment holds by [Mey68, p. 185, S. 8.3]. (He has an “extra” hypothesis that the characteristic of $F$ is not 5 because he is also considering triple systems of dimensions 14 and 32, but that is irrelevant for our purposes.)

For $d \in \text{Skew}(A, \sigma)$, consider the element

$$(3.12) \quad \pi(da) - \pi(ad) - d\pi(a) + \pi(a)d$$

in $A$, which is zero if and only if $d$ satisfies GD. Since $\pi$ is linear, 3.12 is zero for all $a$ if and only if it is zero when $a = \varphi_b(u \otimes v)$. Applying the endomorphism 3.12 to $w \in V$ and expanding out, we obtain

$$t(du, v, w) + t(dv, u, w) + t(u, v, dw) - dt(u, v, w).$$

So $d$ satisfies GD if and only if it lies in $D = \text{im} \pi$. \hfill $\blacksquare$
A category equivalence. We will now show that there is an equivalence of categories between the category of adjoint groups of type $E_7$ over $F$ and the category of gifts over $F$, where both categories have isomorphisms for morphisms (i.e., they are groupoids). We use the notation and vocabulary of \cite{KMRT98} §26 with impunity.

**Theorem 3.13.** The automorphism group of a gift defined over a field $F$ is an adjoint group of type $E_7$ over $F$. This provides an equivalence between the groupoid of gifts over $F$ and the groupoid of adjoint groups of type $E_7$ over $F$.

**Proof:** Let $\mathcal{C}_\mathfrak{G}(F)$ denote the groupoid of gifts over $F$ and let $\mathcal{C}_{E_7}(F)$ denote the groupoid of adjoint groups of type $E_7$ over $F$. Let

$$S(F): \mathcal{C}_\mathfrak{G}(F) \longrightarrow \mathcal{C}_{E_7}(F)$$

be the functor induced by the map on objects given by $\mathfrak{G} \mapsto \text{Aut}(\mathfrak{G})$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}_\mathfrak{G}(F) & \xrightarrow{S(F)} & \mathcal{C}_{E_7}(F) \\
i & & \downarrow j \\
\mathcal{C}_\mathfrak{G}(F_\text{s}) & \xrightarrow{S(F_\text{s})} & \mathcal{C}_{E_7}(F_\text{s})
\end{array}
$$

where $F_\text{s}$ is a fixed separable closure of $F$ and $i, j$ are the obvious scalar extension maps. These maps are both “$\Gamma$-embeddings” for $\Gamma$ the Galois group of $F_\text{s}$ over $F$, in that there is a $\Gamma$-action on the morphisms in the category over $F_\text{s}$ with fixed points the morphisms coming from $F$. Since the diagram is commutative and is compatible with the $\Gamma$-action on the morphisms in the categories over $F_\text{s}$, $S(F_\text{s})$ is said to be a “$\Gamma$-extension” of $S(F)$.

Since any nondegenerate triple system is a form of $\mathfrak{M}^d$ \cite{IT66}, every nondegenerate gift is a form of $\text{End}(\mathfrak{M}^d)$ by Lemma \cite{IT66}. Thus $\mathcal{C}_\mathfrak{G}(F_\text{s})$ is connected. Since $\mathcal{C}_{E_7}(F_\text{s})$ is also connected and any object in either category has automorphism group the split adjoint group of type $E_7$, $S(F_\text{s})$ is an equivalence of groupoids. (For $G$ adjoint of type $E_7$, $\text{Aut}(G) \cong G$ by \cite{IT66} 1.5.6.)

By \cite{KMRT98}, 26.2 we only need to show that $i$ satisfies the descent condition, i.e., that 1-cocycles in the automorphism group of some fixed element of $\mathcal{C}_\mathfrak{G}(F_\text{s})$ define objects in $\mathcal{C}_\mathfrak{G}(F)$. Let $(A, \sigma, \pi)$ be a gift over $F$ and set

$$W := \text{Hom}_F(A \otimes_F A, A) \oplus \text{Hom}_F(A, A) \oplus \text{Hom}_F(A, \text{Skew}(A, \sigma)).$$

Here $(m, \sigma, \pi) \in W$ gives the structure of $(A, \sigma, \pi)$. The rest of the argument is as in 26.9, 26.12, 26.14, 26.15, 26.18, or 26.19 of \cite{KMRT98}. We let $\rho$ denote the natural map $GL(A)(F_\text{s}) \longrightarrow GL(W)(F_\text{s})$ and observe that elements of the orbit of $w$ under $\text{im} \rho(F_\text{s})$ define all possible gifts over $F_\text{s}$ and the objects of $\mathcal{C}_\mathfrak{G}(F)$ can be identified with the set of all $w' \in W$ such that $w'$ is in the orbit of $w$ over $F_\text{s}$. Then $i$ satisfies the descent condition by \cite{KMRT98}, 26.4. \qed
4. Applications to flag varieties

Definition 4.1. An inner ideal of a gift $\mathcal{G} = (A, \sigma, \pi)$ is a right ideal $I$ of $A$ such that $\pi(I\sigma(I)) \subseteq I$. A singular ideal of $\mathcal{G}$ is a right ideal of $A$ such that $\pi(I\sigma(I)) = 0$.

If $A$ is split so that $\mathcal{G} \cong \text{End}(\mathcal{M})$ for some triple system $\mathcal{M} = (V, b, t)$, there is a bijection between subspaces $U$ of $V$ and right ideals $\text{Hom}_F(V, U)$ of $\text{End}_F(V)$. In this bijection, inner ideals of $\mathcal{M}$ (i.e., those subspaces $U$ such that $t(U, U, V) \subseteq U$) correspond to inner ideals of $\mathcal{G}$. The same statement holds for singular ideals where a singular ideal of $\mathcal{M}$ is defined to be a subspace $U$ such that $t(u, u', v) = b(v, u)u' + b(v, u')u$ for all $u, u' \in U$ and $v \in V$.

Since all inner ideals in a nondegenerate Freudenthal triple system are totally isotropic with respect to the skew-symmetric bilinear form [Fer72, 2.4], any singular or inner ideal in a gift $\mathcal{G}$ is isotropic, i.e., $\sigma(I)I = 0$.

Now all of the flag varieties (a.k.a. homogeneous projective varieties) for an arbitrary group $\text{Aut}(\mathcal{G})$ of type $E_7$ can easily be described in terms of the singular and rank 12 inner ideals of the gift $\mathcal{G}$, by translating the corresponding description for the flag varieties for such groups with trivial Tits algebras from [Gar99, §7]. Specifically, one simply takes the statement of [Gar99, 7.5] and replaces every instance of “$n$-dimensional” with “rank $n$” as well as replacing $\text{Inv}(\mathcal{B})$ with $\text{Aut}(\mathcal{G})$.

5. A construction

The rest of this section is taken up with a construction which produces groups of type $E_7$ with Tits algebras of index 2, and in particular all gifts (hence all groups of type $E_7$) over a real-closed field. (We say that a field $F$ is real if $-1$ is not a sum of squares in $F$ and that it is real-closed if it is real and no algebraic extension is real, please see [Lam73, Ch. 9, §§1, 2] for more information.) Since one of these groups is anisotropic, this is the first construction of an anisotropic group of type $E_7$ directly in terms of this 56-dimensional form. (Groups of this type have been implicitly constructed as the automorphism groups of Lie algebras given by the Tits construction.)

A diversion to symplectic involutions. Let $Q := (\alpha, \beta)_F$ be the quaternion algebra over $F$ generated by elements $i$ and $j$ such that $i^2 = \alpha$, $j^2 = \beta$, and $ij = -ji$. Our construction will involve taking a 56-dimensional skew-symmetric bilinear form from a triple system over $F$ and twisting it to get a symplectic involution $\sigma$ on $M_{28}(Q)$. However, if our triple system is of the form $\mathcal{M}(J)$ for some $J$, then the skew-symmetric form is of a very special kind, namely it is obtainable from a quadratic form $q$ (specifically, $(1) \perp T$ where $T$ is the trace on the Jordan algebra), and we use this fact to get an explicit description of $\sigma$.

More generally, suppose that $(V, q)$ is a nondegenerate quadratic space over $F$ with associated symmetric bilinear form $b_q$ such that $b_q(x, x) = q(x)$. Let $K = F(\sqrt{\alpha})$ with nontrivial $F$-automorphism $\iota$ and fix some square root of $\alpha$ in $K$. Then for $W_K = (V \oplus V) \otimes K$, we set $s$ to be the skew-symmetric bilinear form given by

$$s(w_1, w_2) := \sqrt{\alpha} [b_q(v_1, v'_2) - b_q(v_2, v'_1)],$$
where \( w_i = (v_i, v'_i) \in W_K \) for \( i = 1, 2 \). Since \( q \) is nondegenerate, \( s \) is as well.

We define a twisted ction of \( \iota \) on \( W_K \) by setting \( 'w = (\psi_c \otimes \iota)w \) where the \( \iota \) on the right denotes the usual action and \( \psi_c \) is given by \( \psi_c(v, v') = (cv', (\beta/c)v) \) for some \( c \in F^* \). Then the fixed subspace is a vector space over \( F \) which we denote by \( W \). Note that \( W \otimes_F K \cong W_K \). Since \( s('w_1, 'w_2) = \beta s(w_1, w_2) \), the action of \( \iota \) on \( W_K \) induces an \( \iota \)-semilinear automorphism \( \text{Int}(\psi_c) \otimes \iota \) of \( (\text{End}_K(W_K), \sigma_s) \).

**Lemma 5.1.** Suppose that for \( 1 \leq i \leq n \), \((V_i, q_i)\) is a nondegenerate quadratic space over \( F \) isomorphic to \( \langle a_i \rangle \), \( K = F(\sqrt{\alpha}) \), and \( (W_K, i_s) \) is the associated symplectic space defined over \( K \) as described above, and for \( c_i \in F^* \), \((A, \sigma)\) is fixed subalgebra defined by descent from \((\text{End}_K(W_K), \sigma_s)\) by \( \text{Int}(\psi_{c_1} \oplus \cdots \oplus \psi_{c_n}) \otimes \iota \). Then \( A \) is Brauer-equivalent to \( Q = (\alpha, \beta)_F \) and the involution \( \sigma \) is adjoint to the hermitian form

\[
\langle c_1a_1, \ldots, c_na_n \rangle
\]

over \( Q \).

There is a unique symplectic involution on \( Q \) which we will denote by \( \gamma \). It is defined by setting \( \gamma(i) = -i \) and \( \gamma(j) = -j \). In the statement of the lemma, hermitian means with respect to the involution \( \gamma \).

**Proof:** Consider the two isomorphisms

\[
(Q \otimes_F K) \otimes_F M_n(F) \xrightarrow{f} M_2(K) \otimes_F M_n(F) \xrightarrow{g} M_{2n}(F) \otimes_F K.
\]

We fix a square root of \( \alpha \) in \( K \) and let \( E_{rs} \) be the matrix whose only nonzero entry is a 1 in the \((r, s)\)-position. We define \( f \) by

\[
f(1 \otimes E_{rs}) := \begin{pmatrix} 1 & c_r \\ c_r & \alpha \end{pmatrix} \otimes E_{rs},
\]

\[
f(i \otimes 1) := \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -c_r \sqrt{\alpha} \end{pmatrix} \otimes 1,
\]

and

\[
f(j \otimes E_{rs}) := \begin{pmatrix} 0 & c_s \\ \beta/c_r & 0 \end{pmatrix} \otimes E_{rs}.
\]

We define \( g \) by dividing \( M_{2n}(K) \) into \( n \times n \) blocks and setting \( g(x \otimes E_{rs}) \) to be the matrix with \( x \) in the \((r, s)\) block.

Set \( m \) in \( M_{2n}(F) \otimes K \) to be the matrix with diagonal blocks \( C_1, \ldots, C_n \) for \( C_i := \begin{pmatrix} 0 & c_i \\ \beta/c_i & 0 \end{pmatrix} \). Then \( A \) is the \( F \)-subalgebra of \( M_{2n}(F) \otimes_F K \) fixed by \( \text{Int}(m) \otimes \iota \), for \( \iota \) the nontrivial \( F \)-automorphism of \( K \). Since \( g^{-1}(\text{Int}(m) \otimes \iota)g \) fixes \( f(Q \otimes F \otimes M_n(F)) \), \( gf \) provides an \( F \)-isomorphism \( Q \otimes_F F \otimes_F M_n(F) \cong A \).
The involution \( \tau \) on \( M_{2n}(F) \otimes K \) which is adjoint relative to \( s_1 \oplus \cdots \oplus s_n \) is
\[
\tau := \text{Int} \left( \begin{pmatrix} A_1^{-1} & & \\ & \ddots & \\ & & A_n^{-1} \end{pmatrix} \right) \circ \text{Int} \left( \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} \right) \circ t,
\]
where \( A_i = \sqrt{\alpha}a_i \text{Id}_2, J = (0_i 1_i), \) and \( t \) is the transpose \( [KMRT98, \text{p. 24}] \). For \( \gamma \) the unique symplectic involution on \( Q \), we have an involution on \( M_{2n}(F) \otimes K \) given by
\[
(gf)(\gamma \otimes \text{Id}_K \otimes t)(gf)^{-1} = \left( \text{Int} \left( \begin{pmatrix} c_1^{-1}J & & \\ & \ddots & \\ & & c_n^{-1}J \end{pmatrix} \right) \oplus \text{Id}_K \right).
\]
So the involution \( \sigma \) induced by \( \sigma \) on \( Q \otimes K \otimes M_n(F) \) satisfies
\[
(gf)\sigma(\gamma \otimes \text{Id}_K \otimes t)(gf)^{-1} = \tau(gf)(\gamma \otimes \text{Id}_K \otimes t)(gf)^{-1} = \text{Int} \left( \begin{pmatrix} c_1A_1^{-1} & & \\ & \ddots & \\ & & c_nA_n^{-1} \end{pmatrix} \right).
\]
Since \( (gf)^{-1}(c_iA_i^{-1}) = c_i(\sqrt{\alpha}a_i)^{-1} \), the lemma is proven. \( \square \)

**The construction.** In characteristic \( \neq 5 \) (and as always \( \neq 2,3 \)), we can define a *Brown algebra* to be a 56-dimensional central simple structurable algebra with involution \((B, -)\) such that the space of skew-symmetric elements is one-dimensional. In characteristic 5, a different definition is needed at the moment due to insufficiently strong classification results in that characteristic. See \([Gar99, \text{§2}]\) for a full definition and \([5.2]\) below for examples.

The relevant point is that given a Brown \( F \)-algebra \( \mathcal{B} \), one can produce a nondegenerate Freudenthal triple system \( \mathfrak{M} := (V, b, t) \) in a relatively natural way, see \([AF84, \text{§2}]\) or \([Gar99, \text{§4}]\). This triple system is determined only up to similarity (i.e., for every \( \lambda \in F^\times, (V, \lambda b, \lambda t) \) is also a triple system associated to \( \mathcal{B} \), and these are all of them). Also, this construction produces all nondegenerate Freudenthal triple systems over \( F \) by \([Gar99, \text{4.14}]\).

We define a gift \( \text{End}(\mathcal{B}) \) by setting \( \text{End}(\mathcal{B}) := \text{End}(\mathfrak{M}) \). Although \( \mathfrak{M} \) is only determined up to similarity by \( \mathcal{B}, \text{End}(\mathcal{B}) \) is still well-defined, as observed in Remark 3.8.

Our construction will need to use a specific kind of Brown algebra explicitly.

**Example 5.2.** (\([Gar99, 2.3, 2.4], \text{cf. } [All90, 1.9] \)) The principal examples of Brown \( F \)-algebras are denoted by \( \mathcal{B}(J, \Delta) \) for \( J \) an Albert \( F \)-algebra and \( \Delta \) a quadratic étale \( F \)-algebra. We set \( \mathcal{B}(J, F \times F) \) to be the \( F \)-vector space \((J^\times F_+)^2\) with multiplication given by
\[
\begin{pmatrix} \alpha_1 & j_1 \\ j_1' & \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & j_2 \\ j_2' & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + T(j_1, j_2') & \alpha_2 j_2 + \beta_2 j_1 + j_1' \times j_2' \\ \alpha_1 j_2' + \beta_1 j_2 + j_1 \times j_2 & \beta_1\beta_2 + T(j_2, j_1') \end{pmatrix},
\]
where \( T \) is the bilinear trace form on \( J \) and \( \times \) is the Freudenthal cross product (see \([KMRT98, \text{§38}]\) for more information about these maps). The involution \( - \) on \( \mathcal{B}(J, F \times F) \) is given by
\[
\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \beta & j \\ j' & \alpha \end{pmatrix}.
\]
The map \( \varpi \) defined by
\[
\varpi \left( \begin{array}{cc}
\alpha & j \\
 j' & \beta \\
\end{array} \right) = \left( \begin{array}{cc}
\beta & j' \\
 j & \alpha \\
\end{array} \right)
\]
is an automorphism of \( \mathcal{B}(J, F \times F) \) as an algebra with involution. For \( \Delta \) a quadratic field extension of \( F \) and \( \iota \) the nontrivial \( F \)-automorphism of \( \Delta \), we define \( \mathcal{B}(J, \Delta) \) to be the \( F \)-subspace of \( \mathcal{B}(J, F \times F) \otimes_F \Delta \) fixed by the \( \iota \)-semilinear automorphism \( \varpi \otimes \iota \). Note that this construction is compatible with scalar extension in that for any field extension \( K \) of \( F \), \( \mathcal{B}(J, \Delta) \otimes_F K \cong \mathcal{B}(J \otimes_F K, \Delta \otimes_F K) \).

**Construction 5.3.** Suppose that \( K \) is a quadratic field extension of \( F \), \( Q \) is a quaternion algebra over \( F \) which is split by \( K \), and \( J \) is an Albert \( F \)-algebra which is also split by \( K \). Then there is a gift \( \mathfrak{g} := (M_{28}(Q), \sigma, \pi) \) such that

1. \( \text{Iso}(\mathfrak{g}) \) is split by \( K \).
2. \( \text{Der}(\mathfrak{g}) \) is isomorphic to the Lie algebra resulting from the Tits construction with \( Q \) and \( J \) as inputs.
3. If \( Q \) is split, then \( \mathfrak{g} \cong \text{End}(\mathcal{B}(J, K)) \).
4. For \( \gamma \) the unique symplectic involution on \( Q \), the involution \( \sigma \) is adjoint to the \( \gamma \)-hermitian form \( (1) \perp T \) for \( T \) the trace on \( J \).

A thorough description of the Tits construction can be found in [Jac71, §10].

**Proof:** Consider the Brown algebras \( \mathcal{B} := \mathcal{B}(J, K) \) and \( \mathcal{B}^q := \mathcal{B}(J^d, K) \). Their spaces of skew-symmetric elements are both spanned by some element \( s_0 \) such that \( F[s_0] \cong K \), and they are isomorphic as algebras with involution over \( K \), so \( \mathcal{B} \) corresponds to a 1-cocycle \( \psi \) in \( Z^1(K/F, \text{Aut}^+(\mathcal{B}^q)) \), which must be given by
\[
\psi \left( \begin{array}{cc}
\alpha & j \\
 j' & \beta \\
\end{array} \right) = \left( \begin{array}{cc}
\alpha & \psi(j) \\
 \psi(j') & \beta \\
\end{array} \right)
\]
for some \( \psi \in GL(J^d)(K) \) which preserves the norm on \( J^d \). Since \( \psi \) is a 1-cocycle, \( \psi \iota \psi = \text{Id}_K \), so we note that \( \psi \psi^\dagger \iota = \text{Id}_{J^d} \) for the unique \( \psi \in GL(J^d)(K) \) such that \( T(\psi(j), \psi^\dagger(j')) = T(j, j') \) for all \( j, j' \in J^d \).

Since \( Q \) is split by \( K \), it is isomorphic to a quaternion algebra \( (a, b)_F \) for some \( a, b \in F^* \) such that \( K = F(\sqrt{-a}) \). Consider \( t \in \text{End}_F(\mathcal{B}^q) \) given by
\[
t \left( \begin{array}{cc}
\alpha & j \\
 j' & \beta \\
\end{array} \right) = \left( \begin{array}{cc}
\alpha/b & b\psi(j) \\
 \psi(j') & b^2\beta \\
\end{array} \right).
\]

By [Gar99, 5.10], the split triple system \( \mathcal{M}^q \) is the unique triple system associated to \( \mathcal{B}^q \). The map \( t \) is a similarity of \( \mathcal{M}^q \) with multiplier \( b \), so \( \text{Int}(t) \) is an automorphism of the gift associated to \( \text{End}(\mathcal{B}^q) \), which is the split gift \( \mathfrak{g}^q \). Also,
\[
t t t t(x) = bx
\]
for all \( x \in \mathcal{B}^q \). Thus setting \( \eta_t = \text{Int}(t) \) defines a 1-cocycle \( \eta \in Z^1(K/F, \text{Aut}(\mathfrak{g}^q)) \), which defines a nondegenerate gift \( (A, \sigma, \pi) \) over \( F \) which is split over \( K \).
The map $H^1(K/F, \text{Aut} (\mathfrak{G}^d)) \to H^2(K/F, \mu_2)$ sends the class of $\mathfrak{G}$ to the class of $A$ (which is the Tits algebra associated to $\text{Iso}(\mathfrak{G})$), where we are identifying $H^2(K/F, \mu_2)$ and the subgroup of the Brauer group of $F$ consisting of algebras which are split by $K$. The image of $\eta$ under this map is the 2-cocycle $f$ given by $f_{i,i} = b$. This 2-cocycle determines the quaternion algebra $Q$ by [Spr59, pp. 250, 251], so $\eta$ determines a gift $G$ of the form $(M_{28}(Q), \sigma, \pi)$. The Lie algebras resulting from the Tits construction where $Q$ and $J$ are split by $K$ are described in terms of Galois descent in [Jac71, p. 87], which immediately gives (2). If $Q$ is split, we may set $b = 1$, so part (3) is clear. Thus we are left with proving part (4).

For the Freudenthal triple system $(V, s, t)$ associated to $\mathcal{B}(J^d, K)$ over $K$ we take $V = (F \otimes F)^d \otimes F K$ and

$$s \left( \left( \begin{array}{c} \alpha_1 \\ j_1 \\ \beta_1 \end{array} \right), \left( \begin{array}{c} \alpha_2 \\ j_2 \\ \beta_2 \end{array} \right) \right) = \sqrt{a} \left( \left( \alpha_1 \beta_2 - \alpha_2 \beta_1 + (T(j_1, j_2') - T(j_1', j_2)) \right) \right).$$

Note that $s$ is defined over $F$, where the action of $\iota$ on $V$ is by $\tilde{\sigma} \otimes \iota$. This formula for the skew-symmetric bilinear form comes from [AF84, pp. 192, 195], where we have taken $s_0 := \left( \begin{array}{c} \sqrt{a} \\ -\sqrt{a} \end{array} \right) \in \mathcal{B}(J^d, K) \otimes_F K$.

One can also find an explicit formula for the quartic form $q$ there, which then implicitly specifies $t$, but we won’t be using that.

Consider the embedding $\text{Aut}(\text{End}(\mathcal{B}(J^d, K))) \hookrightarrow \text{Aut}(V, s)$. Since all nondegenerate skew-symmetric forms over $V$ of the same dimension are isomorphic, there is some $\varphi \in \text{Aut}(V, s)(K)$ such that

$$\varphi \left( \begin{array}{c} \alpha \\ j \\ \beta \end{array} \right) = \left( \begin{array}{c} \alpha \\ \varphi(j) \\ \beta \end{array} \right)$$

and $\varphi^{-1} \eta \iota \varphi = \text{Id}_V$. Then

$$\varphi^{-1} \eta \iota \varphi \left( \begin{array}{c} \alpha \\ j' \\ \beta \end{array} \right) = \left( \begin{array}{c} \alpha/b \\ \beta/b \end{array} \right).$$

Since $s$ is the skew-symmetric bilinear form constructed from (1) with $c = b^{-1}$ and $T$ with $c = b$, by Lemma 5.1 we get that $\sigma$ is adjoint to $b^{-1} \perp bT \cong b(1) \perp T$, which is similar to (1) $\perp T$. \hfill \Box

**Example 5.4.** Let $Q$ be a nonsplit quaternion algebra over a field $F$ with splitting field $K$ a quadratic extension of $F$, and set $\mathfrak{G}$ to be the gift over $F$ constructed in 5.3 from $K$, $Q$, and the split Albert $F$-algebra $J^d$. Then $\mathfrak{G}$ contains a rank 6 maximal singular ideal corresponding to the 6-dimensional maximal singular ideal in $\mathcal{B}(J^d, \Delta)$ given in [Gar99, 7.6], so by the description of the flag varieties from Section 4 the non-end vertex of the length 2 arm of the Dynkin diagram of $\text{Iso}(\mathfrak{G})$ is circled. Since the Tits algebra of $\text{Iso}(\mathfrak{G})$ is Brauer-equivalent to $Q$ and hence nonsplit, the end vertex of the long arm is not circled, so $\text{Iso}(\mathfrak{G})$ must be of type $E_7^9$, in the notation of [Tit66b, p. 59].
6. Groups of type $E_7$ over real-closed fields

It is well-known that the Tits construction produces all Lie algebras of type $E_7$ over a real-closed field $R$ (i.e., $R$ is real and $R(\sqrt{-1})$ is algebraically closed), see [Jac71, pp. 120, 121]. There are four of them, and they are obtained by letting the quaternion algebra in the construction be split ($M_2(R)$) or nonsplit (where $Q = H := (-1, -1)_R$, the unique nonsplit quaternion algebra) and the Albert algebra $J$ be split ($J^d$) or “minimally split”, so that $J = \mathfrak{S}_3(O, 1)$, where $O$ is the Cayley division algebra over $R$. (For a definition and a general discussion of Cayley algebras, please see [Sch66, Ch. III, §4] or [KMRT98, §33.C].) These quaternion and Albert algebras also fill the hypotheses of our construction, and so our construction produces all the groups of type $E_7$ over $R$.

For each of these four groups, we write $G = \text{Iso}(M_{28}(Q), \sigma, \pi)$ and would like to say something about $\sigma$. Since $\sigma$ is symplectic, it is hyperbolic whenever $Q$ is split, so there is only an issue when $Q = H$. We have a very explicit description of $\sigma$ coming from the construction, and here we use that to calculate a particular invariant of $\sigma$, the Witt index $w(M_{28}(Q), \sigma)$. We know that $\sigma$ is adjoint to the hermitian form $h = \langle 1 \rangle \perp T$ over some 28-dimensional $H$-vector space $V$ where $T$ is the trace form of the Albert algebra $J$ in the construction, and the Witt index is defined to be the number $2n$ such that $h$ is isomorphic to an anisotropic form plus $n$ hyperbolic planes. Equivalently, it is the maximum rank of an isotropic right ideal $I$ of $(M_{28}(H), \sigma)$. To compute the Witt index, we apply a trick to reduce to quadratic forms: We consider $V$ as a $(4 \cdot 28)$-dimensional vector space over $R$ endowed with a quadratic form $q$ defined by $q(v) := h(v, v)$. We say that $q$ is the quadratic trace form of $h$. This map (for general $V$) is an injection from the Witt group of hermitian forms over $H$ to the Witt group of quadratic forms over $R$ [Sch85, 10.1.7], and in particular a single hyperbolic plane has trace form four hyperbolic planes and $\langle 1 \rangle$ has trace form the reduced norm of $H$, which is $4 \langle 1 \rangle$. So the Witt index of $\sigma$ is just half of the usual Witt index of the quadratic trace form $q$. Since $q \cong 4(\langle 1 \rangle \perp T)$ and our base field is real-closed, the Witt index of $\sigma$ is twice that of the Witt index of the the quadratic form $\langle 1 \rangle \perp T$.

Looking up the description of the trace form on $J$ from [KMRT98, 37.9], we find that the trace form on $J^d$ is $3 \langle 1 \rangle \perp 12H$ for $H$ denoting a hyperbolic plane, and the trace form on $\mathfrak{S}_3(O, 1)$ is $27 \langle 1 \rangle$. Thus we have the following table:

| $Q$     | $J$  | $w(M_{28}(Q), \sigma)$ | type          | comments               |
|---------|------|------------------------|---------------|------------------------|
| $M_2(R)$ | $J^d$ | 28                     | $E_7^{9}$     | split                  |
| $M_2(R)$ | $\mathfrak{S}_3(O, 1)$ | 28 | $E_{7,3}^{28}$ |                |
| $H$     | $J^d$ | 24                     | $E_{7,4}^{9}$ |                |
| $H$     | $\mathfrak{S}_3(O, 1)$ | 0  | $E_{7,0}^{133}$ | anisotropic/compact   |

The entries in the “type” column refer to the types from Tits’ classification of simple groups [Tit66b, p. 59]. The association of types with choices of $Q$ and $J$ follow from the classification in [Jac71] and Example 5.4 or the fact that one can read off from the Tits indices in [Tit66b] that the groups of type $E_{7,3}^{28}$ have trivial Tits algebra.
6.2. One immediate consequence of this has to do with the Galois cohomology of these groups. Specifically, let $G = \text{Iso}(M_{28}(H), \sigma, \pi)$ be a simple simply connected group of type $E_7$ over $R$ and let $\overline{G} = \text{Aut}(M_{28}(H), \sigma, \pi)$ be its associated adjoint group. Clearly $G$ is a subgroup of $\text{Sp}(M_{28}(H), \sigma)$ and $\overline{G}$ is a subgroup of $\text{Aut}(M_{28}(H), \sigma)$, and the short exact sequence $1 \rightarrow Z(G) \rightarrow G \rightarrow \overline{G} \rightarrow 1$ and the corresponding one for the symplectic groups induce a commutative diagram

\[
\begin{array}{ccc}
\overline{G}(R) & \xrightarrow{\partial} & H^1(R, \mu_2) \\
\downarrow & & \downarrow \cong \\
\text{Aut}(M_{28}(H), \sigma) & \xrightarrow{} & H^1(R, \mu_2) \\
& & R^*/R^{*2}
\end{array}
\]

That is, the image of $\partial$ is contained in the image of the bottom arrow, which is known [KMRT98, p. 425]. It is the group of similarity factors of the hermitian form $h$ (the hermitian form for which $\sigma$ is adjoint), i.e., the elements $\mu \in F^*$ such that $\mu h \sim h$. The construction of the quadratic trace form $q$ from $h$ makes it clear that any similarity factor of $h$ must also be a similarity factor of $q$, so since $q$ is not hyperbolic and our ground field is real-closed, the image of $\partial$ must be trivial.

6.3. In the remaining two cases (where $Q$ is split), $\partial$ is certainly surjective. In fact, it is surjective over an arbitrary field $F$ whenever $G$ is isomorphic to $\text{Inv}(\mathfrak{M}(J))$ for some Albert $F$-algebra $J$. This is because $\mathfrak{M}(J)$ has a similarity with multiplier $\lambda$ for all $\lambda \in F^*$ by the construction in [Fer72, p. 327].

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