ON A CONFORMAL GAUSS-BONNET-CHERN INEQUALITY
FOR LCF MANIFOLDS AND RELATED TOPICS

HAO FANG

ABSTRACT. In this paper, we prove the following two results:

First, we study a class of conformally invariant operators \( P \) and their related conformally invariant curvatures \( Q \) on even-dimensional Riemannian manifolds. When the manifold is locally conformally flat (LCF) and compact without boundary, \( Q \)-curvature is naturally related to the integrand in the classical Gauss-Bonnet-Chern formula, i.e., the Pfaffian curvature. For a class of even-dimensional complete LCF manifolds with integrable \( Q \)-curvature, we establish a Gauss-Bonnet-Chern inequality.

Second, a finiteness theorem for certain classes of complete LCF four-fold with integrable Pfaffian curvature is also proven. This is an extension of the classical results of Cohn-Vossen and Huber in dimension two. It also can be viewed as a fully non-linear analogue of results of Chang-Qing-Yang in dimension four.

1. INTRODUCTION

Let \( M \) be a Riemannian manifold of even dimension \( n \), with a Riemannian metric \( g \). Denote \( [g] = \{ e^{2w} g; \ w \in C^\infty(M) \} \) as the conformal metric class determined by \( g \). It is known (Cf. \[Br2\] \[FG2\]) that there exist local curvatures \( Q_{2k} \) (with \( 2k \leq \text{dim} M \)) that satisfy certain conformal transformation laws if the Riemannian metric varies in the conformal metric class \( [g] \). The most interesting one is \( Q_n = Q_{\text{dim} M} \). Denoted also as \( Q \) for future convenience, it satisfies the following transformation law:

\[
Q_w \ dv_{g_w} = (Q_0 + P_n w) \ dv_{g_0},
\]

where \( g_0 \) and \( g_w = e^{2w} g_0 \) are both in \([g]\) and \( P_n \) is an \( n \)-th order linear elliptic operator. In recent years, significant progress has been made in the study of \( P_n \) and \( Q_n \), for \( n = 2 \) and \( n = 4 \). It is known to be closely related to the theory of partial differential operators and spectral invariants. For more details and background, see Section 2.

For a Riemannian manifold \((M, g)\) of dimension \( 2m \), an important characteristic class, Pfaffian invariant, is defined as the \( m \)-th Chern class, \( c_m(M) \), hence it can be represented as a curvature invariant by the standard Chern-Weil theory. If we assume that \( M \) is locally conformally flat (LCF), it is an interesting fact that \( Q_n \) is a multiple of the Pfaffian of the metric modulo a divergence term. This can be proved by applying a result of Branson-Gilkey-Pohjanpelto in invariant theory (Cf. \[BrGP\]). Thus, if \( M \) is compact without boundary, the

\[\text{Date: February 20, 2004.}\]
Gauss-Bonnet-Chern theorem gives:

\[ C_n \int_M Q_g dv = \text{Euler}(M), \]

with \( C_n = \frac{1}{((n-2)!!)^2 |S^{n-1}|} \). (Notice that \((2k)!! = (2k)(2k-2)\cdots2\) for a positive integer \(k\) and \(0!! = 1\); \(|S^{n-1}|\) denotes the volume of the standard \((n-1)\)-sphere of radius 1.) Here Euler\((M)\) denotes the Euler Characteristic of \(M\), which is a topological invariant of the manifold.

One of the goals of this paper is to extend the above-mentioned formula to certain complete LCF manifolds. There are several known results in low dimensional cases. In dimension two, a classical result for complete open surfaces by Cohn-Vossen \([CV]\) and Huber \([H]\) shows a Gauss-Bonnet-Chern-type inequality is valid for complete surfaces with integrable \(Q_2\) (which is exactly the Gaussian curvature in dimension two). In dimension four, Chang, Qing and Yang \([CQY1, CQY2]\) extended this inequality to certain complete LCF manifolds with integrable \(Q_4\). In this paper, the general even dimensional case will be considered and the following will be proven:

**Theorem 1.1.** Assume \((M, g)\) is a complete LCF manifold with finitely many conformally flat ends with

\[ \int_M |Q_g| dv < \infty. \]

If, near the ends, the scalar curvature \(R_g\) satisfies

\[ R_g \geq 0, \]

then

\[ C_n \int_M Q_g dv \leq \text{Euler}(M), \]

where \(C_n = \frac{1}{((n-2)!!)^2 |S^{n-1}|}\).

With the exception of the work of Cheeger-Gromov \([CG]\) on manifolds with bounded geometry, and the work of Greene-Wu \([GW]\) on the complete 4-folds with positive sectional curvature, there is little known about extensions of the original Gauss-Bonnet-Chern formula for complete manifolds in higher dimensions. Theorem 1.1 suggests that \(Q_n\), obtained by adding a divergence term to the Pfaffian curvature, should be the right integrand to consider for complete LCF manifolds.

Theorem 1.1 is proved by first analyzing the model problem where \(M = \mathbb{R}^n\). For the model problem, a geometric averaging argument further reduces the metrics to rotationally symmetric metrics on \(\mathbb{R}^n\), for which a uniqueness result of the conformal factor is proved by solving the ODE induced from 1.1. The general case is then derived from the model problem by a gluing argument.

Theorem 1.1 can be applied to study the conformal compactification of certain LCF manifolds, as in \([H]\) and \([CQY2]\). See \([F]\) for some details. This will be addressed in a separate paper.
During the course of proving Theorem 1.1 we closely study the conformal transformation law (1.1) of the $Q$ curvature, which is a linear elliptic PDE with respect to the background metric. Interestingly, some of the techniques we employ are also effective for various non-linear problems. In particular, we study the Pfaffian curvature of a complete LCF four-fold, which satisfies a fully non-linear conformal transformation law of Monge-Ampere type. (See Section 4 for more details.)

Hence, the second part of this paper is a generalization of the main result of [H, CQY2] in another direction; namely, we consider the compactification of LCF manifolds. Under some local curvature conditions, we prove the following finiteness result in dimension four:

**Theorem 1.2.** Let $\Omega$ be a four-fold with a LCF metric. If there exist constants $C$ and $C'$ such that

\[ C \geq R_g \geq C' > 0, \quad \|\nabla g R_g\|_g \leq C, \quad \text{Ric}_g \geq -C g \]

and

\[ \int_{\Omega} |\text{Pfaff}_g| dv_g < \infty, \]

then $\Omega = S^4 \setminus \{p_1, \ldots, p_k\}$ for some $p_i \in S^4$ ($i = 1, \ldots, k$).

In proving Theorem 1.2, the local divergence structure of the Pfaffian plays an important role, overcoming the difficulties caused by the lack of linear transformation laws of the Pfaffian.

We would like to comment that there have been extensive studies on the geometric significance of the corresponding fully non-linear equation in general metric situation. Especially in dimension four, Chang-Gursky-Yang proved a conformal sphere theorem [CGY2]. See also [CY6] for references on further developments.

In a separate paper, we would like to address the general dimensional cases of this compactness problem.

This paper is organized as follows: in Section 2 some preliminary facts about conformally invariant operators and curvatures are given. In Section 3, we prove Theorem 1.1. In Section 4 we prove Theorem 1.2.

**Acknowledgment.** This material represents part of the author’s doctoral dissertation at Princeton University, 2001. The author would like to thank his thesis advisor, Alice Chang, for support and guidance. He also wishes to thank Paul Yang, Jeff Viaclovsky and many others for their interest in the work and for their helpful discussions.

## 2. Conformally invariant operators and curvatures

In this Section we give a review on the conformally invariant operators and curvatures.
2.1. The general metric case. Let $M$ represent a Riemannian $n$-fold with a fixed Riemannian metric $g_0$. Any metric $g$ in the conformal class $[g_0]$ can be expressed as $g = g_w = e^{2w} g_0$, where $w$ is a smooth function on $M$. It is therefore true that the metrics in $[g_0]$ can be endowed with an affine structure modelled after $C^\infty(M)$, the linear space of smooth functions on $M$. Let $R_g$ and $\text{Ric}_g$ be the scalar curvature and the Ricci curvature of $g$, respectively. For future convenience, we define a symmetric quadratic form, the Schouten tensor:

\[
A_g = \frac{1}{n-2} (\text{Ric}_g - \frac{R_g}{2(n-1)} g).
\]

Denote $\sigma_k = \sigma_k(A_g)$ to be the $i$-th symmetric polynomial of the eigenvalues of $A_g$. That is, if we denote $\lambda_1, \cdots, \lambda_n$ as eigenvalues of $A_g$,

\[
\sigma_k = \sigma_k(A_g) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} (\prod_{i=1}^{k} \lambda_{i_j}).
\]

In particular, we have $\sigma_1 = \frac{R_g}{2(n-1)}$, which is also denoted as $J$ for future convenience.

A conformally invariant operator, $P_{2k}$, is a $2k$-th order partial differential operator acting on $C^\infty(M)$, such that under a conformal change of the metric $g_w = e^{2w} g_0$, it obeys the following transformation law:

\[
P_{2k,w} f = e^{-\alpha w} P_{2k,0} (e^{\beta w} f),
\]

for some real $\alpha$ and $\beta$. Note that we use subscripts to indicate the metrics used.

Using the construction of Fefferman-Graham [FG], Graham-Jenne-Mason-Sparling [GJMS] showed that if $\text{dim} M = n$ is odd, $P_{2k}$ exists for $(\alpha, \beta) = (\frac{1}{2}n + k, \frac{1}{2}n - k)$, with $k$ being any positive integer. If $n$ is even, $P_{2k}$ exists for $(\alpha, \beta) = (\frac{1}{2}n - k, \frac{1}{2}n + k)$, with $k \leq \frac{1}{2}n$. In either case, $P_{2k}$ can be assigned the same symbol as that of $\Delta^k$. Furthermore, if $g$ is locally Euclidean,

\[
P_{2k} = \Delta^k.
\]

Except for low order cases, the general expression of $P_{2k}$ is unknown. Explicit formulae of $P_{2k}$ on $S^n$ has appeared in Branson [Br1]; see also Beckner [Be]. For inductive expressions of the conformally invariant operators, see [GJMS, Wu1, Wu2].

Recently, Alexakis has given some general description of the conformal invariant operators, see [A].

If $(\alpha, \beta) \neq (\text{dim} M, 0)$, we define

\[
Q_{2k} \equiv P_{2k,1}.
\]

It is clear that $Q_{2k}$ depends only locally on the Riemannian metric. Hence, by invariance theory, it is fully determined by the curvature tensor and its covariant derivatives.

For example, if $n > 2$, $P_2$ is the well-known conformal Laplacian; $Q_2$ is then a multiple of scalar curvature. The famous Yamabe problem studies the existence of constant scalar curvature metric in any conformal metric class on a compact closed manifold. It was settled by Yamabe, Trudinger, Aubin and Schoen by
using techniques of calculus of variations and studying the PDE induced from (2.5) (for $k = 1$). See [LP] for more details and complete references.

However, the most interesting case occurs when $\dim M = n = 2m$ is even and $(\alpha, \beta) = (n, 0)$. From physical considerations (Cf. [De], for example), it is natural to ask if there exists a local curvature invariant $Q_n$ satisfying the following conformal transformation law:

$$Q_{n,w} = e^{-nw}(Q_{n,0} + P_{n,0}w).$$

In dimension two, it is easy to see that

$$P_2 = \Delta g, \quad Q_2 = \frac{1}{2}R$$

satisfy (2.3) and (2.6). There have been extensive studies for the geometry of the scalar curvature of closed Riemann surfaces. The Nirenberg problem asks which functions on a Riemann surface can be prescribed as the scalar curvature (i.e. $Q_2$) of a metric in a given conformal class. On the other hand, from the view of calculus of variations, $P_2$ and $Q_2$ are closely related to extremals of the zeta functional determinant of the Laplacian (i.e. $P_2$). With the delicate analytic tools developed, many deep geometric results have been obtained by Onofri [O], Trudinger [T], Moser [Mo], Kazdan-Warner [KW], Chang-Yang [CY1, CY2], Chang-Liu [CL], Osgood-Phillips-Sarnak [OPS1, OPS2] and others. See also [C] for a survey.

In dimension four, Paneitz [P] proved that the following $P_4$ and $Q_4$ satisfy (2.7) and (2.6):

$$P_4 = \Delta^2 g + \delta(2R_g g - 2\text{Ric}_g)d,$$

$$Q_4 = \frac{1}{6}(-3\|\text{Ric}\|^2 + R_g^2 - \Delta_g R_g),$$

with $\delta$ being the adjoint operator of $d$ with respect to $g$. In analogy to the two-dimensional case, it is interesting to study the problem of prescribing $Q_4$ curvature for a given four-manifold as well as the properties of $Q_4$, a 4-th order linear elliptic operator. Furthermore, $P_4$ and $Q_4$ naturally appear in the variation of the functional determinants of certain conformally invariant operators. Extensive studies on the analysis and the geometry of the $P_4$ and $Q_4$ have been carried out by Beckner [Be], Branson-Chang-Yang [BrCY], Chang-Yang [CY3], Chang-Gursky-Yang [CGY], Gursky [Q] and many others. See also [CY5] and [CY6] for surveys.

In [Br2], Branson proved the existence of $Q_n$ curvature satisfying (2.6) for arbitrary even dimensions. Recently, Graham and Zworski [GZ] has given a different proof, which was later greatly simplified in [FG2]. The new approach, which is partly based on the fundamental work of Fefferman and Graham [FG] on the construction of ambient metric, has also inspired many related works [FH].

However, due to the complicated nature of the expression of $Q_n$ (as well as that of $P_n$) for $n$ large, except for a discussion on $Q_n$ for metrics in the standard
conformal metric class of $S^n$ \cite{CY4}, few results have been obtained. See \cite{GP} for dimension eight computation by using the tractor calculus technique.

Notice that the pair $(P_n, Q_n)$ is not unique for $n \geq 4$ in general. For example, denote $W_g$ as the Weyl tensor of the metric. Given a pair $(P_n, Q_n)$ satisfying the above-mentioned relationship, it is easy to check that $(P_n + c_1 \|W\|^\frac{1}{2}, Q_n + c_2 \|W\|^\frac{1}{2})$ satisfies the same relations for arbitrary real $c_1$ and $c_2$. See \cite{A} for a structure theorem of general conformally invariant curvatures.

2.2. The LCF metric case. We now restrict to the case where the metric is locally conformally flat (LCF). That means, in local coordinates, the metric can be represented as $g = e^{2w} g_0$, where $g_0$ is the standard Euclidean metric. Since the curvature tensor of the flat metric $g_0$ vanishes, one gets $Q_{n,0} = 0$. Thus, by (2.4) and (2.6), we have

\begin{align*}
P_{n,w} &= e^{-nw} \Delta^m, \\
Q_{n,w} &= e^{-nw} \Delta^m w,
\end{align*}

where $\Delta$ is the Laplace operator with respect to the flat metric. $Q_n$ is thus uniquely determined when the metric is LCF. However, as mentioned in the previous Subsection, the explicit expressions of $P_n$ and $Q_n$ using the Riemann curvature tensor and its covariant derivatives are difficult to obtain for higher dimensional cases.

We fix local coordinates $\{x_1, \ldots, x_n\}$ such that $g_{0,ij} = \delta_{ij}$. Hence, locally $g_{ij} = e^{2w} \delta_{ij}$. Under this coordinate system, the following well-known formulae hold:

\begin{align*}
(2.9) \quad R_g &= -2(n-1)e^{-2w}(\Delta w + \frac{n-2}{2} \|\nabla w\|^2);
\\
(2.10) \quad \text{Ric}_{g,ij} &= (2-n)w_{ij} - \Delta w \delta_{ij} + (n-2)(w_iw_j - \|\nabla w\|^2 \delta_{ij});
\\
(2.11) \quad A_{g,ij} &= -w_{ij} + w_iw_j - \frac{1}{2} \|\nabla w\|^2 \delta_{ij},
\end{align*}

where $w_i = \frac{\partial}{\partial x_i} \quad w = \nabla_i w$ is the derivative with respect to the flat metric.

We also denote Pfaff $= c_m(M, g)$ as the Pfaffian of the metric $g$, with the normalization so that for a closed manifold $M$, the Gauss-Bonnet-Chern theorem reads:

\begin{align*}
(2.12) \quad \int_M \text{Pfaff}_g \ dv_g = \text{Euler}(M).
\end{align*}

Pfaffian invariants for LCF metrics can be expressed as a contraction of the Schouten tensor as follows (Cf. \cite{V}):

\begin{align*}
(2.13) \quad \text{Pfaff}_g &= \frac{1}{6 ((n-2)!!)^2 |S^{n-1}|} \sigma_m(A_g).
\end{align*}
Notice that the right hand side of (2.13), when viewed as an expression of the conformal factor \( w \) by (2.11), is fully non-linear.

The following result is a consequence of a theorem of Branson, Gilkey and Pohjanpelto [BrGP]:

**Proposition 2.1.** If \((M, g)\) is an LCF manifold, then

\[
(2.14) \quad \frac{1}{((n-2)!!)^2 |S^{n-1}|} Q_n = \text{Pfaff} + \delta_g B,
\]

where \( B \) is a 1-form depending locally on the metric \( g \). Hence, by (2.13), the following Gauss-Bonnet-Chern formula holds if \( M \) is closed:

\[
(2.15) \quad C_n \int_M Q_n \, dv_g = \text{Euler}(M),
\]

where \( C_n = \frac{1}{((n-2)!!)^2 |S^{n-1}|} \).

Proposition 2.1 is one of the few results on the global properties of the \( Q_n \) curvature. See [Br2] for more details. (See also [A] for a generalization to the non-LCF case.) It establishes the integral of \( Q_n \) as a topological quantity of a closed LCF manifolds. It is thus desirable to extend this link to a more general class of LCF manifolds, which is one of the motivations of this paper.

### 3. A conformal Gauss-Bonnet-Chern inequality

In this Section, we focus on conformal metrics on even dimensional LCF spaces. Assume \( n = 2m \) is a positive even number and \( g_0 \) is the standard Euclidean metric on \( \mathbb{R}^n \). A locally conformally flat metric \( g \) can be represented locally as \( g = g_w = e^{2w} g_0 \), with \( w \) being smooth.

Suppose \( M \) is a manifold with a LCF metric \( g \). Let \( P = P_n \) be the conformally invariant operator defined in (2.3). Let \( Q = Q_n \) be the corresponding conformal curvature invariant defined by (2.6). By the discussion in Section 2, we have

\[
(3.1) \quad P_w = e^{-nw} \Delta^m,
\]

\[
(3.2) \quad Q_g \equiv Q_{2m,g} = e^{-nw} \Delta^m w
\]

Here without further notice the operators are all with respect to the flat metric.

We study a LCF manifold \((M, g)\) satisfying the following assumptions:

- **(A1)** \( g \) is complete;
- **(A2)** \( R_g \geq 0 \) near the end;
- **(A3)**

\[
\int_M |Q_g| \, dv_g < \infty.
\]

Our goal of this Section is to prove Theorem 1.1.

We first consider a model case, where \( M = \mathbb{R}^n \), and the metric is rotationally symmetric. We prove the following:
Theorem 3.1. Assume $g = e^{2w}g_0$ is a metric on $\mathbb{R}^n$ such that $w(x) = w(\|x\|)$. If $g$ satisfies assumptions (A1), (A2) and (A3), then

$$C_n \int_{\mathbb{R}^n} Q_g \, dv_g \leq 1,$$

with $C_n = \frac{1}{(n-2)!!(S^{n-1})}$. To prove Theorem 3.1 we construct the Green’s function as in [CQY1] and Theorem 3.1 is proved by solving an ODE and establishing a uniqueness result for the conformal factor $w$.

A geometric averaging procedure is then applied to prove the following:

Theorem 3.2. Assume $g = e^{2w}g_0$ is a metric on $\mathbb{R}^n$. Suppose $g$ satisfies assumptions (A1), (A2) and (A3). Then

$$C_n \int_{\mathbb{R}^n} Q_g \, dv_g \leq 1.$$  

Applying a gluing argument, we prove Theorem 1.1 which is re-stated below for convenience:

Theorem 3.3. Assume $(M, g)$ is a LCF manifold satisfying assumptions (A1), (A2) and (A3). If $M$ has only finitely many complete ends, then

$$C_n \int_M Q_g \, dv_g \leq \text{Euler}(M).$$

This is a generalization of the Gauss-Bonnet-Chern inequality proved by Huber [H] in the two-dimensional case and Chang, Qing and Yang [CQY1] in the four-dimensional case.

This Section is organized as follows. In 3.1, we prove Theorem 3.1 in 3.2, we prove Theorem 3.2 in 3.3, we prove Theorem 3.3.

3.1. $\mathbb{R}^n$—the rotationally symmetric case. Let $g = e^{2w}g_0$ be a conformal metric on $\mathbb{R}^n$ which satisfies assumptions (A1), (A2) and (A3). In this Subsection, we make the assumption that $w$ is rotationally symmetric; in other words, if $r = \|x\|$, then

$$w(x) = w(r).$$

By (3.5), we define

$$f(x) \equiv Qe^{nw(x)} = \Delta^m w.$$  

Then $f \in L^1(\mathbb{R}^n)$ due to (A3).

To treat the PDE (3.6), we notice that the standard theory of elliptic PDE does not apply directly since the Calderon-Zygmund theory does not cover the $L^1$ case. However, a Green’s function defined in [CQY1] can still give us a basic solution. More specifically, because $f(x)$ is integrable, the following

$$v(x) \equiv C_n \int_{\mathbb{R}^n} \ln\left(\frac{\|y\|}{\|x - y\|}\right) f(y) \, dy$$

is a solution.
is well defined and smooth. It is easy to confirm that

\[(3.8) \quad \Delta^m v(x) = f(x).\]

Therefore,

\[
\Delta^m (w - v) = 0.
\]

We will now study the uniqueness for solutions for \((3.6)\) under the rotational symmetry condition. \((3.6)\) then can be viewed as an ODE. We prove the following simple lemma:

**Lemma 3.4.** If \(u\) is a smooth rotationally symmetric function on \(\mathbb{R}^n \setminus \{0\}\), and satisfies the differential equation

\[(3.9) \quad \Delta^m u = 0,\]

then

\[u(x) = c_0 + c_1 \ln r + c_2 r^2 + c_4 r^4 + \cdots + c_{n-2} r^{n-2} + c'_2 r^{-2} + c'_4 r^{-4} + \cdots + c'_{n-2} r^{-2-n}.\]

**Proof.** Since \(u\) is rotationally symmetric, \((3.9)\) reduces to a linear ODE of \(n\)-th order. \(1, \ln r, r^2, \ldots, r^{n-2}, r^{-2}, \ldots, r^{-2-n}\) are seen to be \(n\) linearly independent solutions of this ODE. Hence, the general solution is the linear combination of these expressions. \(\blacksquare\)

**Proposition 3.5.** Given \(f, v\) as above,

\[
\lim_{r \to 0} r \dot{v}(r) = 0,
\]

\[
\lim_{r \to \infty} r \dot{v}(r) = -C_n \int_{\mathbb{R}^n} f(y) dy,
\]

where dot denotes the derivative with respect to \(r\).

**Proof.** Let \(s \equiv \|y\|\). Clearly,

\[
\frac{d}{dr} \|x - y\|^2 = \frac{1}{r} (r^2 - s^2 + \|x - y\|^2).
\]

Thus,

\[
r \dot{v}(r) = -C_n \int_{\mathbb{R}^n} \frac{r^2 - s^2 + \|x - y\|^2}{2 \|x - y\|^2} f(y) dy.
\]

The first part of the proposition is then straightforward. The second part is equivalent to the fact that

\[(3.10) \quad I(x) \equiv \int_{\mathbb{R}^n} \frac{r^2 - s^2}{\|x - y\|^2} f(y) dy \to \int_{\mathbb{R}^n} f(y) dy,
\]

when \(r \to \infty\).

Since \(f\) is rotationally symmetric, \(I\) depends only on \(r = \|x\|\). Hence,

\[(3.11) \quad I(r) = \int_{\mathbb{R}^n} \left( \int_{\|x\|=r} \frac{r^2 - s^2}{\|x - y\|^2} dS_x \right) f(y) dy.
\]
Here $dS_x$ is the volume form on the standard $S^{n-1}$ and
\[ \int_{S^{n-1}} F \, dS_x = \frac{\int_{S^{n-1}} F \, dS_x}{\int_{S^{n-1}} 1 \, dS_x} \]
for a function $F$ defined on $S^{n-1}$. Define
\[ II(r, s) \equiv \int_{\|x\| = r} \frac{1}{\|x - y\|^2} \, dS_x. \]

We now prove the following technical lemma:

**Lemma 3.6.** There exists a positive $C$ such that:
\[ |r^2II(r, s) - 1| \leq C \frac{s^2}{r^2} \quad \text{for } s \leq r; \]
\[ II(r, s) < \frac{C}{s^2} \quad \text{for } s > r. \]

**Proof.** Taking Laplacian with respect to $y$ to $II$, we get
\[ \Delta^{m-2}II = C \int_{\|x\| = r} \frac{1}{\|x - y\|^{n-2}} \, dS_x. \]
Since \( \frac{1}{\|x - y\|^{n-2}} \) is a multiple of the Green’s function for the Laplacian on $\mathbb{R}^n$, we see that
\[ \Delta^{m-2}II = \frac{C}{s^{n-2}}, \]
for $s > r$;
\[ \Delta^{m-2}II = \frac{C}{r^{n-2}} \]
for $s \leq r$, with $C$ depending only on $n$. Hence, when $s < r$, it is easy to see from the bounded-ness of $II$ and the proof of Lemma 3.4 that
\[ II(r, s) = c_0(r) + c_2(r)s^2 + \cdots + c_{n-4}s^{n-4} \frac{s^{n-4}}{r^{n-2}}. \]

Notice the homogeneity in (3.12), one has $r^2II(r, s)$ depending only on $\frac{s}{r}$. Combining the fact that $II(r, 0) = \frac{1}{r^2}$, we have
\[ r^2II(r, s) = 1 + p\left(\frac{s^2}{r^2}\right), \]
where $p$ is a polynomial of degree $m-1$ with no constant terms. We have proved the lemma when $s < r$. When $s > r$, from (3.14) and the Hölder’s Inequality,
\[ \int_{\|x\| = r} \frac{1}{\|x - y\|^2} dS_x \leq \left( \int_{\|x\| = r} \frac{1}{\|x - y\|^{n-2}} dS_x \right)^{\frac{1}{m-1}} = \frac{C}{s^2}, \]
Lemma 3.6 is then proved.
We now continue the proof of Proposition 3.5. Taking into account (3.11), (3.12) and Lemma 3.6,
(3.17)
\[ |I(r) - \int_{\mathbb{R}^n} f(y) dy| \leq \int_{\|y\| \leq r} C \frac{s^2}{r^2} f(y) dy + C \int_{\|y\| \geq r} |f(y)| dy. \]
Thus, for any \( \epsilon > 0 \), there is a positive \( R \) large enough such that
\[ \int_{s > \epsilon R} |f(y)| dy \leq \epsilon. \]
Next, we see that when \( r = \|x\| > R \),
\[ |\int_{s \leq \epsilon r} C \frac{s^2}{r^2} f(y) dy| \leq C \epsilon^2 \int_{\mathbb{R}^n} |f(y)| dy; \]
\[ |\int_{s > \epsilon r} C \frac{s^2}{r^2} f(y) dy| \leq C \int_{s > \epsilon R} |f(y)| dy < C \epsilon. \]
Combining these and (3.17), we have
\[ |I(x) - \int_{\mathbb{R}^n} f(y) dy| \leq C (1 + \int_{\mathbb{R}^n} |f(y)| dy) \epsilon, \]
the integrability of \( f \) then leads to (3.10). The proof of Proposition 3.5 is completed.

**Lemma 3.7.** Let \( v \) as defined as above, we have, for some positive constant \( C \),
(3.18)
\[ r |\dot{v}(r)| \leq C; \]
(3.19)
\[ r^2 |\Delta v| \leq C. \]

**Proof.** The first part follows simply from Proposition 3.5. For the second part, notice that
(3.20)
\[ \Delta v = C \int_{\mathbb{R}^n} \frac{1}{\|x - y\|^2} f(y) dy. \]
Again, since \( f(y) \) is rotationally symmetric, we can replace \( \frac{1}{\|x - y\|^2} \) in the integrand by \( II(r, s) \), which is defined in (3.12). Apply Lemma 3.6 to prove that
\[ r^2 II(r, s) \leq C. \]
Thus,
\[ |r^2 \Delta v| \leq C \int_{\mathbb{R}^n} |f(y)| dy. \]

It is now possible to prove the following uniqueness result:
**Theorem 3.8.** Let the conditions be as in Theorem 3.1. We have

\[ w(x) = v(x) + c, \]

where \( c \) is a constant.

**Proof.** By Lemma 3.4 we have

\[ w(x) = v(x) + c_0 + c_1 \ln r + c_2 r^2 + c_3 r^4 \cdots + c_{n-2} r^{n-2} + c_{n-2}' r^{-2} + c_4 r^{-4} \cdots + c_{n-2}' r^{-2n}. \]

Since \( w(x) \) is smooth at the origin, \( (r \dot{w}(r))|_{r=0} = 0 \), by Proposition 3.5 we have that \( c_2 = c_1 = 0 \).

From (2.9), the non-negativity of the scalar curvature is equivalent to

\[ \Delta w + (m-1) \|\nabla w\|^2 \leq 0. \]

We prove that \( c_2 = \cdots = c_{n-2} = 0 \) by a contradiction argument. Let \( k \geq 1 \) be the largest index such that \( c_{2k} \neq 0 \). Combined with Lemma 3.7 it is clear that near infinity, \( \Delta w + (m-1) \|\nabla w\|^2 \) has the leading term as \( (m-1) c_{2k}^2 r^{2k-2} > 0 \), which contradicts with (3.21). Thus, all the \( c_i \)'s are vanishing. The proof is thus completed.

Finally, we are ready to give the following

**Proof of Theorem 3.1** We need to apply the completeness condition of the metric. Notice that if

\[ \lim_{r \to \infty} r \dot{w}(r) = a \]

exists, near \( \infty \) we have that \( e^{w(x)} \propto r^a \). For the metric \( e^{2w(x)} \) to be complete, it has to be true that

\[ a \geq -1. \]

Applying Proposition 3.5 we have proved the inequality.

From the argument above, we also have the following:

**Corollary 3.9.** Let the conditions given as in Theorem 3.1. If further we assume \( re^{w(x)} \) is bounded, we have the equality in (3.21).

3.2. \( \mathbb{R}^n \)-the general case. We now describe the geometric averaging procedure to reduce Theorem 3.2 to Theorem 3.1.

Consider the spherical coordinate for \( \mathbb{R}^n \). Namely, denote \( x \in \mathbb{R}^n \) as

\[ x = (r, \theta), \quad r \geq 0, \quad \theta \in S^{n-1}, \]

where \( S^{n-1} \) is the standard sphere (with radius 1) in \( R^n \).

Assume that \( g = e^{2w(x)} g_0 \) is a conformal metric on \( \mathbb{R}^n \). Denote \( \bar{g} = e^{2\bar{w}} g_0 \), with

\[ \bar{w}(x) = \bar{w}(r) \equiv \int_{S^{n-1}} \bar{w}(r, \theta) d\theta, \]

here we use \( \int_{S^{n-1}} \) \( d\theta \) to represent the average of a function over \( S^{n-1} \). To study the relation between \( g \) and \( \bar{g} \), denote \( \nabla_{\theta}, \Delta_{\theta} \) as the covariant derivative and the...
Laplacian on $S^{n-1}$, respectively. The following relations are obvious:

\[
\nabla = \nabla_{\mathbb{R}^n} = (\partial_r, \frac{1}{r} \nabla_\theta),
\]

\[
\Delta = \Delta_{\mathbb{R}^n} = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_\theta.
\]

We realize that

\[
\|\nabla_\theta w\|^2 = r^2 (\|\nabla w\|^2 - |\partial_r w|^2).
\]

**Proposition 3.10.** If $g$ satisfies assumptions (A1), (A2) and (A3), then $\bar{g}$ satisfies assumptions (A2) and (A3).

**Proof.** Assumption (A3) for $\bar{g}$ metric is easy to verify since we actually have

\[
\int |Q_{n,\bar{g}}| dv_{\bar{g}} = \int |Q_{n,g}| dv_g.
\]

This is because $Q_{n,\bar{g}} dv_{\bar{g}} = \Delta^m \bar{w} \, dx$, $Q_{n,g} dv_g = \Delta^m w \, dx$, and

\[
\int_{|x|=r} \Delta^m \bar{w} \, dx = \int_{|x|=r} \Delta^m w \, dx.
\]

To verify (A2) for $\bar{g}$, by (2.9), $R_g \geq 0$ is equivalent to $\Delta w + (m-1) \|\nabla w\|^2 \leq 0$. Since

\[
\Delta \bar{w} = \int \Delta w \, d\theta,
\]

and

\[
\|\nabla \bar{w}\|^2 = (\int \partial_r w \, d\theta)^2 \leq \int \|\nabla w(r, \theta)\|^2 \, d\theta.
\]

Hence it is apparent that $\Delta \bar{w} + (m-1) \|\nabla \bar{w}\|^2 \leq 0$, which implies that $R_{\bar{g}}$ is non-negative.

**Proposition 3.11.** If $g$ satisfies assumptions (A1), (A2) and (A3), then $\bar{g}$ is complete.

**Proof.** As before, we define

\[
f(x) \equiv Qe^{nw(x)} = \Delta^m w,
\]

(3.26)

\[
v(x) \equiv C_n \int_{\mathbb{R}^n} \ln(\frac{\|y\|}{\|x-y\|}) f(y) \, dy
\]

(3.27)

\[u(x) = w(x) - v(x).
\]

Then $f \in L^1(\mathbb{R}^n)$ and $v, u \in C^\infty(\mathbb{R}^n)$.

We first show two intermediate results. The first one is a generalization of Theorem 3.8

**Claim 3.12.** $u(x)$ is a constant function.
Proof. Similar to the rotationally symmetric case, we have

$$\Delta^m u = 0.$$  

To show the uniqueness result, we proceed to consider the rotational symmetrization procedure with respect to a fixed point $P \in \mathbb{R}^n$:

$$\bar{w}_P(x) = \int_{||y-P||=||x-P||} w(y) d\theta$$

$$\bar{v}_P(x) = \int_{||y-P||=||x-P||} v(y) d\theta$$

$$\bar{u}_P(x) = \int_{||y-P||=||x-P||} u(y) d\theta.$$  

Notice that

$$v(x) = C_n \int_{\mathbb{R}^n} \ln(\frac{||y-P||}{||x-y||}) f(y) dy + C_n \int_{\mathbb{R}^n} \ln(\frac{||y||}{||P-y||}) f(y) dy$$

$$= C_n \int_{\mathbb{R}^n} \ln(\frac{||y-P||}{||x-y||}) f(y) dy + C,$$

apply Proposition 3.10, $g_{\bar{w},P}$ satisfies conditions (A2) and (A3). Apply Lemma 3.4 and Theorem 3.8, we have

$$\bar{u}_P(x) = c.$$  

In particular, it implies that

$$\Delta u(P) = \Delta \bar{u}_P(P) = 0.$$  

Hence, we have shown that $u$ is harmonic over $\mathbb{R}^n$. We can finish the prove of Claim 3.12 by following an argument of [CQY1]: because $u$ is harmonic, so is $u_i(x) = \frac{\partial u}{\partial x^i}(x)$. It leads to

$$|u_i(P)|^2 = \left| \int_{||x-P||=r} u_i d\theta \right|^2$$

$$= \left| \int_{||x-P||=r} u_i d\theta \right|^2$$

$$\leq \int_{||x-P||=r} ||\nabla u||^2 d\theta$$

$$\leq \int_{||x-P||=r} (||\nabla w||^2 + ||\nabla v||^2) d\theta \leq \frac{C}{r^2} \to 0,$$

as $r \to \infty$. In the last step we have applied Lemma 3.7 for (3.28) and the fact that
\[
\int_{||x-P||=r} ||\nabla w||^2 d\theta \\
= \frac{1}{m-1} \int_{||x-P||=r} (\Delta w - e^w J_w) d\theta \\
\leq \frac{1}{m-1} \Delta \bar{w}_P(r) = 0.
\]

We thus have proved that \( u_i = 0 \) all any \( x_i \); hence, \( u \) is a constant.

The second intermediate result is the following analogue of Lemma 3.2 of [CQY1].

**Claim 3.13.** If
\[
w(x) = C_n \int_{\mathbb{R}^n} \ln(\frac{||y||}{||x-y||}) \Delta^m w(y) \, dy + C,
\]
then
\[e^{-\bar{w}} \int_{||x||=r} e^w \to 1,\]
as \( r \to \infty \).

The proof of Claim 3.13 is identical to the proof of Lemma 3.2 of [CQY1], which treats dimension four case. We omit it here.

Now we can continue the proof of Proposition 3.11. We only need to show that \( \int_0^\infty e^{\bar{w}} \, dr \) is divergent. Since for a fixed \( \theta \), \( \int_0^\infty e^{w(r,\theta)} \, dr \) is divergent because of completeness of metric \( g \), this can be proved by applying Claims 3.12 and 3.13.

Thus we have completed the proof of Proposition 3.11.

**Corollary 3.14.** If \( g = e^{2w} g_0 \) is a conformal metric on \( \mathbb{R}^n \) such that \( R_g \geq 0 \), and \( P_n = 0 \), then \( w \) is a constant.

**Proof.** One constructs the metric \( \tilde{g} = e^{2\bar{w}} \) as in (3.23). From the proof of Proposition 3.10, it is true that \( \tilde{g} \) has non-negative scalar curvature and vanishing \( Q \) curvature. The conclusion then follows from Theorem 3.8.

We now give the proof of Theorem 3.2. But this is a straightforward application of Propositions 3.10, 3.11 and Theorem 3.1.

**3.3. LCF manifolds with finitely many ends.** In this Subsection, we give the proof of Theorem 3.3 which is an extension of Theorem 1.2 of [CQY2] in higher dimensional case. We will take advantage of the topological invariance of \( \int Qdv \) and give a doubling argument. Our approach is more geometrical, comparing to the approach of Chang-Qing-Yang, which is more analytical.

First we prove the following simplified result:
Proposition 3.15. Assume $\Omega$ is a domain in $S^n$ with a conformal metric $g$ satisfying assumptions (A1), (A2) and (A3). If $\Lambda = S^n \setminus \Omega$ is a finite set of $k$ points, then

\begin{equation}
C_n \int_{\Omega} Q \, dv_g \leq (2 - k).
\end{equation}

Proof. Let $\Lambda = \{p_1, \cdots, p_k\}$. A stereographic projection from $S^n$ to $\mathbb{R}^n$ can be chosen so that $p_1$ is sent to infinity. Without confusion, we identify the images of $\Lambda$ under the projection with itself. There is a function $w$ smooth away from $\Lambda$ such that the metric can be represented as $g = e^{2w} g_0$, where $g_0$ is the Euclidean metric on $\mathbb{R}^n$. We fix a partition of unity,

\begin{equation}
1 = l_1(x) + \cdots + l_k(x),
\end{equation}

such that $l_i(x)$ is a smooth function supported near $p_i$ and $l_i = 1$ near $p_i$. Let $w_i(x) = w(x) l_i(x)$. We consider the metric $g_i = e^{2w_i} g_0$.

Note that $g_1$ satisfies assumptions (A1), (A2) and (A3). Theorem 3.2 then gives the follows:

\begin{equation}
C_n \int_{\mathbb{R}^n} Q g_1 \, dv_{g_1} = C_n \int_{\mathbb{R}^n} \Delta^m w_1 \, dx \leq 1.
\end{equation}

For a fixed $i \geq 2$, without loss of generality, we assume that $p_i$ is just the origin. $w_i$ has compact support and the metric $g_k$ also satisfies assumptions (A1), (A2) and (A3). We construct

\begin{align*}
\bar{w}_i &\equiv \int_{S_{n-1}} w_i(r, \theta) \, d\theta \\
&\text{and} \\
v_i(x) &\equiv C_n \int_{\mathbb{R}^n} \ln(\|y\|) \Delta^m \bar{w}_i(y) \, dy.
\end{align*}

Notice that Proposition 3.5 and Proposition 3.10 can still be applied to the metric $g_i$. Tracing the argument in the proof of Theorem 3.8, we see that

\begin{equation}
\bar{w}_i(x) = v_i(x) + c_{1,i} \ln r + c_{0,i}.
\end{equation}

Note that $\bar{w} = 0$ for $\|x\|$ large, by Proposition 3.5

\begin{equation}
c_{1,i} = C_n \int_{\mathbb{R}^n} Q(g_i)\, dv_{g_i} = C_n \int_{\mathbb{R}^n} \Delta^m \bar{w} \, dx = C_n \int_{\mathbb{R}^n} \Delta^m w_i \, dx.
\end{equation}

Follow the proof of Theorem 3.1, instead of getting (3.22), the completeness of $\bar{g}_i$ near the origin shows

\begin{equation}
c_{1,i} \leq -1.
\end{equation}

Combine (3.25), (3.31), (3.32), (3.33) with the fact that $w = \sum w_k$, we have

\begin{equation}
C_n \int_{\Omega} Q_g \, dv_g = C_n \int_{\mathbb{R}^n} \Delta^m w \, dx = \sum_i C_n \int_{\mathbb{R}^n} \Delta^m w_i \, dx \leq 1 + (k - 1)(-1) = 2 - k.
\end{equation}

Thus, Proposition 3.15 is proven. □
Define \( d(\cdot) \) to be the distance function to \( \Lambda \) on \( S^{n-1} \). We have the following easy extension of Corollary 3.9.

**Corollary 3.16.** Let the conditions be those of Proposition 3.15. If the conformal factor \( w \) satisfies that \( d(p)e^w(p) \leq C \) for any \( p \in \Omega \) and some positive constant \( C \), then the equality holds in (3.30).

We are ready to give the proof of Theorem 3.3. Suppose a complete LCF manifold \( M \) has \( k \) disjoint ends \( E_1, \ldots, E_k \). We can thus choose a local coordinate chart for each \( E_i \) such that the metric is represented as \( e^{2w_i(x)}g_0 \), \( g_0 \) being the \( n \)-dimensional Euclidean metric and \( \|x\| > 1 \). Our approach is following: First we modify the conformal metric so that each end links the manifold \( M \) in a strict tubular fashion; then, we cut off each ends to get manifolds with boundary; finally, in order to estimate \( \int Q dv \), we double the compact piece and apply Gauss-Bonnet-Chern formula and extend each ends naturally to apply Proposition 3.15.

First, we do a compact perturbation of the metric near the ends. Let \( \eta \) be a cut off function such that \( \eta(x) = 1 \) for \( 2 < \|x\| < 3 \) and \( \eta(x) = 0 \) for \( \|x\| < 1 \) and \( \|x\| > 4 \). Define a new conformal metric

\[
g' = \begin{cases} 
    e^{2\eta(x)(-w_i(x)-\ln\|x\|)}g & \text{on } E_i; \\
    g & \text{on } M \cup \bigcup E_i.
\end{cases}
\]

By the choice of \( \eta \), \( g' \) is well-defined and smooth. Hence,

\[
(3.34) \quad \int_M Q_{g'} dv_{g'} - \int_M Q_g dv_g = \sum_i \int_{\frac{5}{2} \leq \|x\| \leq \frac{9}{2}} \Delta^m[\eta(x)(-w_i(x)-\ln\|x\|)] dv_{g_0} = 0.
\]

Second, for each \( i \), define \( E'_i \subset E_i = \{x; \|x\| > \frac{5}{2}\} \). If \( M_1 \equiv M \cup \bigcup E'_i \), then \( M_1 \) is a compact manifold with boundary. \( \partial M_1 \) has \( k \) components and near each of them the metric \( g' \) is a locally product metric due to the construction of \( g' \). Hence, we can glue two pieces of \( (M_1, g') \) together to get a closed manifold \( M_2 \). Still referring to the gluing metric on \( M_2 \) as \( g' \), we apply the Gauss-Bonnet-Chern formula for closed manifolds to get

\[
(3.35) \quad C_n \int_{M_1} Q(g') dv_{g'} = \frac{1}{2} C_n \int_{M_2} Q(g') dv_{g'} = \frac{1}{2} \text{Euler}(M_2) = \text{Euler}(M_1).
\]

Next, the metric \( g' \) on \( E'_i \) is equal to \( \frac{1}{\|x\|^2} g_0 \) near the boundary of \( E'_i \), can be extended in the coordinate chart to region \( E''_i = \{\|x\| > 0\} \), still denoted as \( g' \), so that \( g' = \frac{1}{\|x\|^2} g_0 \) for \( x \in E''_i - E'_i \). Notice for \( x \in E''_i - E', Q(g') = \|x\|^n \Delta^m(-\ln\|x\|) = 0 \). \((E''_i, g')\) satisfies assumptions (A1), (A2) and (A3). Proposition 3.15 is applied to \((E''_i, g')\) to get

\[
(3.36) \quad \int_{E'_i} Q(g')dv_{g'} = \int_{E''_i} Q(g')dv_{g'} \leq 0.
\]

Finally, combine (3.35), (3.35) and (3.36), we prove Theorem 3.3.
In the previous Sections, we proved the Gauss-Bonnet-Chern-type inequality for certain LCF manifolds with integrable $Q$ curvature. As we have seen, the conformal variation of $Q$ is just $P$ operator, which is linear elliptic. This fact is crucial in our study in Section 2. However, it is interesting that some of the techniques developed in the previous Sections are also applicable to study the Pfaffian curvature of certain complete LCF manifolds of dimension four. As in [CQY2], we pose the following stronger assumption for the curvature:

\[
(A4) \quad C_1 \geq R_g \geq C > 0, \quad \|\nabla_g R_g\|_g \leq C, \quad \text{Ric}_g \geq -C g
\]

for some positive constants $C$ and $C_1$.

In this Section, we prove Theorem 1.2, which is rephrased here for readers’ convenience:

**Theorem 4.1.** Let $\Omega$ be a manifold with a LCF metric satisfying assumptions (A1), (A2), and (A4). If further

\[
\int_{\Omega} |\text{Pfaff}_g| dv_g < \infty,
\]

then $\Lambda = S^4 \setminus \Omega$ is a finite set.

Comparing to the situation treated in [CQY2], we replace $Q$ curvature by the Pfaffian of the manifold. Hence, it is a non-linear extension of the main result of [CQY2].

By the curvature condition and the extension map construction of Schoen-Yau [SY], we can view $\Omega$ as a domain in $S^4$. As in before, we identify $\Omega$ with its image in $\mathbb{R}^n$ under a stereographic projection and write $g = e^{2w} g_0$ with $g_0$ being the Euclidean metric. Again, we use the upper-bar to denote geometric quantities with respect to $g$ metric.

First we study the fully non-linear transformation law of the Pfaffian. For $g$, by (2.7) and (2.9), and (2.10) and (2.14), we have the following:

\[
(4.1) \quad \text{Pfaff}_g = C_4 (Q_g + \bar{\Delta} J).
\]

\[
(4.2) \quad \frac{\text{Pfaff}}{C_4} = e^{-4w} ((\Delta w)^2 - \|\nabla^2 w\|^2 + 2 \nabla^2 w (\nabla w, \nabla w) + \|\nabla w\|^2 \Delta w),
\]

where all the operators are with respect to the Euclidean metric and $\nabla^2$ denotes the Hessian.

It is an interesting observation that

\[
(4.3) \quad \text{Pfaff} = C_4 \sigma_2,
\]

where $\sigma_2$ is defined in (2.2). This is actually a special case of a more general fact that for any LCF metric on a $2m$-dimensional manifold, the Pfaffian is a constant multiple of $\sigma_m$ (Cf. [V], for example).
In this section, we first give a $C^0$ estimate of the conformal factor $w$; then we give an estimate of the size of $S^4 \setminus \Omega$.

Notice that in this Section, we fix $n = 4$ and $m = 2$, though many arguments work for general dimensions. See [F] for more general statements.

4.1. $C^0$ estimate. In this Subsection, we give the key estimates of the conformal factor.

First we quote a lemma of Yau (Cf. [SY]), which is a special case of the gradient estimate for positive harmonic functions on a complete manifold.

**Lemma 4.2.** For a manifold $M$ with a LCF complete metric $g = e^{2w} g_0$ satisfying the following: the scalar curvature $R_g$ and the Ricci curvature $\text{Ric}_g$ satisfy the following point-wise estimates near the complete end:

$$C \geq R_g \geq 0, \quad \|\nabla_g R_g\|_g \leq C, \quad \text{Ric}_g \geq -C g$$

for some positive constant $C$, then there exists a constant $C$ such that

$$\|\nabla_g w\|_g \leq C.$$

We then prove a non-existence result which is an analogue of Theorem 3.8.

**Lemma 4.3.** There is no metric $g = e^{2w} g_0$ on $\mathbb{R}^4$ satisfying (A1), (A2), (A4) and (4.4) $\text{Pfaff} = 0$.

Notice now that the averaging method we applied to prove Theorem 3.8 does not work since unlike the Q curvature, the Pfaffian does not satisfy a linear transformation law. An integral estimate is applied instead.

**Proof.** We assume there exists such a metric. First, we prove that the Ricci curvature is bounded. Because $0 = \sigma_2 = \frac{1}{2}(-\|A\|^2 + J^2)$, by (2.1) and (A2),

$$\|\text{Ric}_g\|^2 = \|(n-2)A_g + J\|^2 \leq C(R^2_g + \|A_g\|^2) \leq C'$$

for some positive constants $C$ and $C'$.

It is clear that Lemma 4.2 is applicable. Thus, for some positive $C$,

$$\|\nabla w\| \leq C e^w.$$

For any region $D \subset \Omega$ and a positive $\alpha < \frac{1}{2}$, we claim that there is some positive constant $C$ such that

$$\alpha \int_D e^{(4+\alpha)w} dv \leq C \int_{\partial D} e^{(3+\alpha)w} dv',$$

with $dv$, $dv'$ denoting the Euclidean volume forms on $D$ and $\partial D$, respectively.

We now prove the claim. Notice that by (4.2) and (4.3), $e^{4w} \sigma_2$ can be re-written as a divergence form:

$$e^{4w} \sigma_2 = \delta((\Delta w + \|\nabla w\|^2 - \nabla^2 w)dw).$$
It follows that
\begin{equation}
\int_D \sigma_2 e^{(4+\alpha)w} dv = \int_D (-\Delta w - \|\nabla w\|^2 + \nabla^2 w)(dw, \alpha e^{\alpha w} dw) dv \\
+ \int_{\partial D} e^{\alpha w} (\Delta w + \|\nabla w\|^2 - \nabla^2 w)(\partial_n w) dv'.
\end{equation}

Since \( \sigma_2 = \text{Pfaff} = 0 \),
\begin{equation}
0 = \int_D e^{\alpha w} (-\Delta w - \|\nabla w\|^2 - \nabla^2 w) (\partial_n w) dv + \int_{\partial D} (\Delta w + \|\nabla w\|^2 - \nabla^2 w)(\partial_n w) dv'.
\end{equation}

By (A2),
\begin{equation}
|\Delta w + \|\nabla w\|^2| = |J| e^{2w} \leq C e^{2w},
\end{equation}
for some constant \( C \). Combine (4.11) and (4.6) we get
\begin{equation}
|\Delta w| \leq C e^{2w}
\end{equation}

Combine (4.5), (4.6) and (4.12), it is not hard to see that
\begin{equation}
\|\nabla^2 w\| \leq C e^{2w}
\end{equation}

Therefore, from (4.6), (4.12) and (4.13), we have
\begin{equation}
\int_{\partial D} e^{\alpha w}[(\Delta w + \|\nabla w\|^2 - \nabla^2 w)(\partial_n w) + \frac{\|\nabla w\|^2}{2}(\alpha \partial_n w)] dv' \leq C \int_{\partial D} e^{(3+\alpha)w} dv'.
\end{equation}

On the other hand,
\begin{equation}
\int_D e^{\alpha w} \|\nabla w\|^2[\Delta w - \|\nabla w\|^2 - \frac{\alpha}{2} \|\nabla w\|^2 - \Delta w] dv \geq C \int_D e^{\alpha w} \|\nabla w\|^2 J e^{2w} dv
\end{equation}

\begin{equation}
\geq C \int_D e^{\alpha w} \|\nabla (e^w)\|^2 dv.
\end{equation}

Since
\[ J = -e^{-3w} \Delta e^w \]
in dimension four, through integration by part,
\[
\int_D e^{\alpha w} \|\nabla (e^w)\|^2 dv = \int_D (e^{\alpha w} J e^{4w} - \alpha e^{\alpha w} \|\nabla (e^w)\|^2) dv + \int_{\partial D} e^{(2+\alpha)w} \partial_n w \ dv'.
\]

With \( \alpha \leq \frac{1}{2} \), it is true that
\[
(4.16) \quad \int_D e^{\alpha w} \|\nabla (e^w)\|^2 dv \leq C \int_D e^{(4+\alpha)w} dv - \alpha C \int_{\partial D} e^{(3+\alpha)w} dv'.
\]

Combine (4.10), (4.16), (4.15) and (4.14), we reach the proof of the claim (4.7).

Choose the domain \( D \) as \( B(0, r) \), the ball centered at the origin with radius \( r \), and define
\[
(4.17) \quad F(r) \equiv \int_{B(0, r)} e^{(4+\alpha)w} dv.
\]

It is clear that
\[
(4.18) \quad F'(r) = \int_{\partial B(0, r)} e^{(4+\alpha)w} dv'.
\]

By Hölder Inequality,
\[
(4.19) \quad \int_{\partial D} e^{(3+\alpha)w} dv' \leq \left[ \int_{\partial D} e^{(4+\alpha)w} dv' \right]^{\frac{3+\alpha}{4+\alpha}} \left[ \int_{\partial D} dv' \right]^{\frac{1}{4+\alpha}}.
\]

Substitute (4.17), (4.18) and (4.19) into (4.7), we have that
\[
(4.20) \quad \alpha F(r) \leq C[F'(r)]^{\frac{3+\alpha}{4+\alpha}} \cdot r^{\frac{3}{4+\alpha}},
\]

which implies that
\[
(4.21) \quad (-[F(r)]^{\frac{1}{3+\alpha}})' \geq C\alpha^{\frac{7+2\alpha}{3+\alpha}} (r^{\frac{3}{3+\alpha}})'.
\]

Then for any \( b > 1 \), integrate (4.21) over \([1, b] \) we get
\[
[F(b)]^{\frac{1}{3+\alpha}} \leq C\alpha^{\frac{7+2\alpha}{3+\alpha}} (1 - b^{\frac{3}{3+\alpha}}) + [F(1)]^{\frac{1}{3+\alpha}}.
\]

Let \( b \) tends to \( \infty \) we would get the absurd conclusion that \( F(b) \leq 0 \). We thus have finished the proof of Lemma 4.3.

Now we are ready to show the \( C^1 \) estimate as follows:

**Lemma 4.4.** For a LCF metric \( g \) on \( \Omega \subset S^n \) satisfying (A1), (A2), and (A4), we have
\[
(4.22) \quad C < e^{w(x)} d(x) \leq C',
\]

for some positive \( C \) and \( C' \).

**Proof.** The left-hand side of the inequality is a direct consequence of Lemma 4.2.

To prove the right-hand side of the inequality, we run a blow-up argument, following Schoen [S] and [CQY2]. For simplicity, denote
\[
(4.23) \quad u(x) = e^{(m-1)w(x)}.
\]
If the claim is not true, we would have a sequence of \( \{x_i\} \in \Omega \), such that
\[
A_i = u(x_i) d^{m/2-1}(x_i) \to \infty.
\]

For simplicity, we define the following quantities:
\[
\sigma_i \equiv \frac{1}{2} d(x_i), \quad f_i(y) \equiv (\sigma_i - d(y, x_i))^{m/2-1} u(y).
\]

It follows from (4.24) that
\[
f_i(x_i) = \sigma_i^{m/2-1} u(x_i) = \frac{1}{2} A_i \to \infty
\]
when \( i \to \infty \) and \( f_i(y) = 0 \) for \( y \in \partial B(x_i, \sigma_i) \). Thus, there exists some point \( y_i \) such that
\[
f(y_i) = \max \{ f_i(y) : y \in B(x_i, \sigma_i) \}.
\]

Set
\[
\lambda_i \equiv u(y_i),
\]
and
\[
v_i(x) \equiv \lambda_i^{-1} u \left( \lambda_i^{-\frac{1}{m-1}} x + y_i \right)
\]
for \( x \in B(0, R_i) \). Hence,
\[
v_i(0) = 1.
\]

Let \( r_i = \frac{1}{2} (\sigma_i - d(x_i, y_i)) \) and \( R_i = r_i u(y_i) \); then \( x \in B(0, R_i) \) if and only \( y = \lambda_i^{\frac{1}{m-1}} x + y_i \in B(x_i, r_i) \). Notice that \( R_i \to \infty \) as \( i \to \infty \).

Notice that
\[
0 < v_i(x) = \frac{u(y)}{u(y_i)} \leq \left( \frac{\sigma_i - d(x_i, y_i)}{\sigma_i - d(x_i, y)} \right)^{m/2-1} \leq \left( \frac{\sigma_i}{\sigma_i - r_i} \right)^{m/2-1} \leq 2^{m/2-1}.
\]

By (2.3), \( g_{v_i} \) satisfies the following
\[
J_i(x) = -\Delta v_i(x) v_i(x)^{-\frac{m+2}{m-2}},
\]
for \( x \in B(0, R_i) \). Note that the boundedness of \( v_i \) will also give the boundedness of \( |\nabla_x J_i(x)| \).

It thus follows that, taking a subsequence if necessary,
\[
J_i \to J_\infty \in C^0_{\text{loc}}(\mathbb{R}^n)
\]
for some \( J_\infty \geq C > 0 \). Hence a subsequence of \( v_i \) converges uniformly on compact sets in \( C^{1,\alpha}(\mathbb{R}^n) \). Let the limit function be \( v_\infty \). By the standard elliptic theory, we show that \( v_\infty \in C^{2,\alpha}(\mathbb{R}^n) \), and
\[
-\Delta v_\infty(x) = J_\infty(x) v_\infty(x)^{\frac{m+1}{m-1}}.
\]

Applying the elliptic theory again, \( v_\infty \) is actually smooth. Applying the maximum principle and the fact that \( v_\infty(0) = 1 \), we derive \( v_\infty(x) > 0 \). If \( w_i(x) \equiv \ln v_i(x) \) and \( w_\infty(x) \equiv \ln v_\infty(x) \), by passing to a subsequence, we conclude
\[
w_i \to w_\infty \text{ in } C^{2,\alpha}(\mathbb{R}^n).
\]
This implies that for \( g_\infty = e^{2u_\infty}g_0 \),
\[
Pfaff g_\infty = 0.
\]

It is easy to see that \( g_\infty \) satisfies the condition of Lemma 4.3, hence we have a contradiction. We have thus completed the proof of Lemma 4.4.

**Remark 4.5.** By the way we present the proof of Lemma 4.4, the blowup argument we used works also for general dimensions.

4.2. **Proof of Theorem 4.1.** Following [CQY2], we define subsets of \( M \):
\[
U_\lambda = \{ x : e^{w(x)} \geq \lambda \},
S_\lambda = \{ x : e^{w(x)} = \lambda \}.
\]

Then, if \( \mathbf{n} \) is the outward normal vector of \( S_\lambda \) as the boundary of \( U_\lambda \), \( \partial_n w \geq 0 \).

We work with the level sets of \( w \) from now on.

We begin with a technical result:

**Lemma 4.6.** Given any LCF metric \( g \) on \( \Omega \), and any \( f \in C^\infty(\Omega) \), if \( f = f(\lambda) \), then
\[
\int_{U_\lambda} (\Delta_g f) dv_g = \lambda \frac{d}{d\lambda} \left[ \int_{U_\lambda} (\Delta_g w) f \ dv_g - \int_{S_\lambda} (\partial_n w) f \ dv_g' \right],
\]
where \( \partial_n \) is the unit outward normal derivative with respect to \( g \), and \( v_g' \) is the induced volume form on \( S_\lambda \).

**Proof.** This is proved through direct computation. Because \( f \) is constant on \( S_\lambda \),
\[
\Delta_g f = \partial_n^2 f + H \partial_n f
\]
on \( S_\lambda \), where \( H \) is the mean curvature of \( S_\lambda \subset U_\lambda \) with respect to \( g \). Also notice that on \( S_\lambda \),
\[
\frac{d}{d\lambda} dv_g' = H dv_g'.
\]
Thus, if \( S_\lambda \) is smooth, which is true for almost all \( \lambda \),
\[
\frac{d}{d\lambda} \int_{S_\lambda} (\partial_n \lambda) f \ dv_g' = \int_{S_\lambda} \frac{\partial_n^2 \lambda f + (\partial_n \lambda)(\partial_n f) + (\partial_n \lambda)f H}{\partial_n \lambda} \ dv_g'
\]
(4.39)

by (4.37) and (4.38). Using Stokes’ Theorem and the co-area formula, we get
\[
\int_{U_\lambda} \Delta_g f \ dv_g = -\int_{S_\lambda} (\partial_n f) \ dv_g' = -\frac{d}{d\lambda} \int_{S_\lambda} (\partial_n \lambda) f \ dv_g' + \int_{S_\lambda} \frac{\Delta \lambda}{\partial_n \lambda} f \ dv_g'
\]
(4.40)

by (4.37) and (4.38). Using Stokes’ Theorem and the co-area formula, we get
The co-area formula also leads to

\begin{equation}
\int_{S_\lambda} (\frac{\partial n}{\lambda}) f \, dv'_g = \lambda \frac{d}{d\lambda} \int_{U_\lambda} (\frac{(\partial n_\lambda)^2}{\lambda^2}) f \, dv_g.
\end{equation}

Finally, we have

\begin{equation}
\int_{U_\lambda} \Delta_g f \, dv_g = \lambda \frac{d}{d\lambda} \left[ \int_{U_\lambda} (\frac{\Delta_g \lambda}{\lambda}) f \, dv_g - \int_{S_\lambda} (\partial n_\lambda) \Delta_g f \, dv'_g \right]
\end{equation}

by the fact that \( \lambda = e^w \) on \( S_\lambda \).

We are now in the position to prove Theorem 4.1. This is an interesting generalization of the main results of [CQY2], where we explore heavily the (local) divergence structure of the Pfaffian curvature.

**Proof of Theorem 4.1**  As in the proof of Lemma 4.3, we first apply Lemma 4.2 to show

\begin{equation}
\| \nabla w \| \leq C e^w.
\end{equation}

For notational simplicity, we use upper-bar to to denote geometric objects constructed with \( g_w \) metric.

Notice by (4.1) and (4.3),

\begin{equation}
\sigma_2 = e^{-4w} \Delta^2 w + \Delta J
\end{equation}

Apply Lemma 4.6 with the metric \( g_w \), we have

\begin{equation}
\int_{U_\lambda} \Delta^2 w dv = \lambda \frac{d}{d\lambda} \left[ \int_{U_\lambda} (\Delta w)^2 dv - \int_{S_\lambda} (\partial n_w) \Delta w dv' \right]
\end{equation}

It’s a simple observation that Lemma 4.6 can also be applied with respect to the metric \( g = g_w \) to get

\begin{equation}
\int_{U_\lambda} \bar{\Delta} J dv_g = \lambda \frac{d}{d\lambda} \left[ \int_{U_\lambda} \bar{\Delta} w J dv_g - \int_{S_\lambda} (\bar{\n} w) J dv'_g \right],
\end{equation}

with \( \bar{\n} = e^{-w} n \) being the unit normal vector of \( S_\lambda = \partial U_\lambda \) with respect to \( g \). Notice that \( J = -e^{-2w}(\Delta w + \| \nabla w \|^2) \) and \( \bar{\n} w = e^{-w} \nabla \bar{\n} w \), we get, from (4.3), (4.41) and (4.45),
\[
\int_{U_\lambda} \sigma_2 dv_g = \lambda \frac{d}{d\lambda} \int_{U_\lambda} [(\Delta w)^2 - (\Delta w + 2\|\nabla w\|^2)(\Delta w + \|\nabla w\|^2)] dv \\
- \lambda \frac{d}{d\lambda} \int_{S_\lambda} \partial_n w[\Delta w - (\Delta w + \|\nabla w\|^2)] dv'.
\]

(4.46)

\[
= \lambda \frac{d}{d\lambda} \left[ \int_{U_\lambda} (3Je^{2w} + \|\nabla w\|^2)\|\nabla w\|^2 dv + \int_{S_\lambda} (\partial_n w)\|\nabla w\|^2 dv' \right].
\]

Define

\[
F(\lambda) \equiv \int_{U_\lambda} (3Je^{2w} + \|\nabla w\|^2)\|\nabla w\|^2 dv + \int_{S_\lambda} (\partial_n w)\|\nabla w\|^2 dv'.
\]

Notice \(J\) and \(\partial_n w\) both being non-negative,

\[
F(\lambda) \geq C \int \|\nabla w\|^2 e^{2w} dv = C \int \|\nabla e^w\|^2 dv = C \int (\lambda - 1)dv + \int_{S_\lambda} (\partial_n \lambda \cdot \lambda dv' \]

(4.47)

\[
= C \int J\lambda^4 dv + \int_{S_\lambda} \partial_n \lambda \cdot \lambda dv' \geq C \int e^{4w} dv.
\]

Following [CQY2], we now apply Lemma 4.4 and Lemma 2.6 of [CQY2] to get

(4.48) \( F(\lambda) \leq -C(\lambda^{\frac{1}{4}\text{dim}\Lambda} - 1), \) if \( \text{dim} \Lambda > 0; \)

(4.49) \( F(\lambda) \leq -N \ln \lambda, \) if \( \text{dim} \Lambda = 0, H^0(\Lambda) = \infty, \)

for any large integer \( N. \) Hence, there is a sequence of \( \lambda_i, \) such that \( \lambda_i \to \infty \) and \( \lambda_i \frac{d}{d\lambda} F(\lambda_i) \to -\infty \) as \( i \) tends to infinity. But by (4.46) this contradicts with assumption (A4). Hence we have proved that \( S^4 \setminus \Omega \) is of Hausdorff dimension 0 and it is actually finite. The proof is finished.

\[\blacksquare\]

References

[A] S. Alexakis, preprint, 2004.

[Be] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, Annals of Math., 138 (1993), pp. 213-242.

[Br1] T. Branson, Group representations arising from Lorentz conformal geometry, J. Funct. Anal., 74 (1987), pp. 199–291.

[Br2] T. Branson, Sharp inequalities, the functional determinant, and the complementary series, Trans. Amer. Math. Soc. 347 (1995), pp. 361–374.

[Br3] T. Branson, The functional determinant, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Lecture Notes Series, 4 (1993).

[BrCY] T. Branson, A. S.-Y. Chang and P. C. Yang, Estimates and extremals for zeta function determinants on four-manifolds, Comm. Math. Phys., 149 (1992), pp. 241–262.
T. Branson and P. Gilkey, The Asymptotics of the Laplacian on a Manifold with Boundary, Comm. Partial Differential Equations, 15 (1990), pp. 245–272.

T. Branson, P. Gilkey and J. Pohjanpelto, Invariants of locally conformally flat manifolds, Trans. Amer. Math. Soc., 347 (1995), pp. 939–953.

J. Cheeger and M. Gromov, On the characteristic numbers of complete manifolds of bounded curvature and finite volume, Differential Geometry and Complex Analysis, Springer (1985), pp. 115–154.

K. C. Chang and J. Q. Liu, A Morse-theoretic approach to the prescribing Gaussian curvature problem, Variational methods in nonlinear analysis (Erice, 1992), Gordon and Breach (1995), pp. 55–62.

S.-Y. A. Chang, The Moser-Trudinger inequality and applications to some problems in conformal geometry, Nonlinear partial differential equations in differential geometry (Park City, UT, 1992), IAS/Park City Math. Ser., 2, Amer. Math. Soc.(1996), pp. 65–125.

S.-Y. A. Chang, M. Gursky and P. C. Yang, Regularity of a fourth order nonlinear PDE with critical exponent, Amer. J. Math. 121 (1999), no.2, pp. 215–257.

S.-Y. A. Chang, M. Gursky and P. C. Yang, An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature, Ann. of Math. (2), 144 (2002), no. 3, pp. 709–787.

S.-Y. A. Chang, J. Qing and P. C. Yang, On the Chern-Gauss-Bonnet integral for conformal metrics on $\mathbb{R}^4$, Duke Math. J., 103 (2000), pp. 523-544.

S.-Y. A. Chang, J. Qing and P. C. Yang, Compactification of a class of conformally flat 4-manifold, Invent. Math., 142 (2000), pp. 65–93.

S.-Y. A. Chang and P. C. Yang, Prescribing Gaussian curvature in $S^2$, Acta Math., 159 (1987), pp. 215–259.

S.-Y. A. Chang and P. C. Yang, Conformal deformations of metrics in $S^2$, J. Diff. Geom., 27 (1988), pp. 259–296.

Chang, Sun-Yung A. and Yang, Paul C., Extremal metrics of zeta function determinants on 4-manifolds, Ann. of Math. (2), 142 (1995), pp. 171-212.

S.-Y. A. Chang and P. C. Yang, On uniqueness of solutions of nth order differential equations in conformal geometry, Math. Res. Lett., 4 (1997), pp. 91–102.

S.-Y. A. Chang and P. C. Yang, On a fourth order curvature invariant, Spectral problems in geometry and arithmetic (Iowa City, 1997), Contemp. Math., 237, Amer. Math. Soc., Providence, RI (1999), pp.9–28.

S.-Y. A. Chang and P. C. Yang, Non-linear partial differential equations in conformal geometry, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), Higher Ed. Press, Beijing, (2002), pp. 189–207.

S. Cohn-Vossen, Kürneste Wege und Totalkrümmung auf Flächen, Compositio Math., 2 (1935), pp. 69–133.

S. Deser, Geometric classification of conformal anomalies in arbitrary dimensions, Phys. Lett. B, 309 (1993), pp. 279–284.

H. Fang, Ph.D. Thesis, Princeton University, 2001.

C. Fefferman and C. R. Graham, Conformal Invariants, Astérisque, Numero Hors Serie (1984), pp. 95–116.

C. Fefferman and C. R. Graham, Q-curvature and Poincaré metrics, Math. Res. Lett., 9 (2002), pp. 139–151.

C. Fefferman and K. Hirachi, Ambient metric construction of $Q$-curvature in conformal and CR geometries, Math. Res. Lett., 10 (2003), pp. 819–831.

M. Gursky, The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE, Comm. Math. Phys., 207 (1999), pp. 131–143.
[GJMS] C. R. Graham, R. Jenne, L. J. Mason and G. A. J. Sparling, *Conformally invariant powers of the Laplacian. I. Existence* J. London Math. Soc., 46 (1992), pp. 557–565.

[GP] R. Gover and L. Peterson, *Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus*, Comm. Math. Phys., 235 (2003), pp. 339–378.

[GW] R. Greene and H. Wu, *C∞ convex functions and manifolds of positive curvature*, Acta Math., 137 (1976), pp. 209-245.

[GZ] C.R. Graham and M. Zworski, *Scattering matrix in conformal geometry*, Invent. Math., 152 (2003), pp. 89–118.

[H] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv., 32 (1957), pp. 13–72.

[KW] J. Kazdan and F. Warner, *Prescribing curvature*, Proc. Symp. Pure Math., vol. 27 (1975), part II, pp. 309-320.

[LP] J. M. Lee and T. H. Parker, *The Yamabe problem*, Amer. Math. Soc. Bulletin (New Series), 17 (1987), pp. 37–91.

[Mo] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana U. Math. J., 20 No.11 (1971), pp. 1077–1092.

[O] E. Onofri, *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Physics., 86 (1982), pp. 321-326.

[OPS1] B. Osgood, R. Phillips and P. Sarnak, *Extremals of determinants of Laplacians*, J. Funct. Anal., 80, (1988), pp 148–211.

[OPS2] B. Osgood, R. Phillips and P. Sarnak, *Compact isospectral sets of surfaces*, J. Funct. Anal., 80, (1988), pp 212–234.

[P] S. Paneitz, *A quadratic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.

[S] R. Schoen, *The existence of weak solutions with prescribed singular behaviour for a conformally scalar equation*, Comm. Pure and Appl. Math., XLI (1998), pp. 317-392.

[SY] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, International Press (1994).

[T] N. Trudinger, *On embedding into Orlicz spaces and some applications*, J. Math. Mach., 17 (1967), pp. 473–483.

[V] J. Viaclovsky *Conformal geometry, contact geometry, and the calculus of variations*, Duke Math. J., 101 (2000), pp. 283–316.

[Wü1] V. Wünsch, *On Conformally invariant differential operators*, Math. Nachr., 129 (1986), pp. 269–281.

[Wü2] V. Wünsch, *Some new conformal covariants*, Z. Anal. Anwendungen, 19 (2000), pp. 339–357.

Courant Institute of Mathematical Sciences, New York University

E-mail address: haofang@cims.nyu.edu