GSV-INDEX FOR HOLOMORPHIC PFAFF SYSTEMS

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Abstract. In this work we introduce the GSV-index for varieties invariant by a holomorphic Pfaff system on complex manifolds. We work with Pfaff systems not necessarily locally decomposable. We prove a non-negativity property for the index. As an application, we prove that the non-negativity of the GSV-index gives us the obstruction to the solution of the Poincaré problem for Pfaff systems on projective spaces.

1. Introduction

The GSV-index for vector fields tangent to hypersurfaces with isolated singularities was introduced by X. Gómez-Mont, J. Seade and A. Verjovsky [15] generalizing the Poincaré-Hopf index. The concept of GSV-index for vector fields tangent to complete intersections was extended by J. Seade and T. Suwa in [22, 23] and J.-P. Brasselet, J. Seade and T. Suwa in [3].

D. Lehmann, M. G. Soares and T. Suwa [16] introduced the virtual index for vector fields on complex analytic varieties via Chern-Weil theory. They showed that this index coincides with the index GSV-index if the variety is a local complete intersection variety with isolated singularities. X. Gómez-Mont in [14] defined the homological index, via homological algebra, which also coincides with the GSV-index. Recently, T. Suwa in [24] gave a new interpretation of GSV-index as a residue arising from a certain localization of the Chern class of the ambient tangent bundle.

M. Brunella in [5] introduced the GSV-index in terms of the germs of 1-forms inducing the foliation and establish a relation between the GSV-index with the Khanedani-Suwa variational index [21] and Camacho-Sad index [2]. Moreover, he showed that the non-negativity of the GSV-index is the obstruction for the solution for the Poincaré problem in complex compact surfaces. We recall that the Poincaré problem is a question proposed by H. Poincaré in [13] of bounding the degree of algebraic solutions of an algebraic differential equation on the complex plane. Many authors have been working on Poincaré problem and its generalizations for Pfaff systems, see for instance the papers by Cerveau and Lins Neto [11], M. G. Soares [26], M. Brunella and L. G. Mendes [6], E. Esteves and S. L. Kleiman [13], V.
In [25] T. Suwa proposed generalize certain formulas of GSV-index for vector fields for higher dimensional case. In this work, our main goal is define the GSV-index for Pfaff systems on complex manifolds. We also demonstrate some of its properties. As application, we prove that the non-negativity of the GSV-index gives us the obstruction to the solution of the Poincaré Problem for Pfaff systems on projective complex space.

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2. Preliminaries

2.1. Holomorphic Pfaff Systems.

Definition 2.1. Let $X$ be an $n$-dimensional complex manifold. A holomorphic Pfaff system of rank $p$ $(1 \leq p \leq n)$ on $X$ is a non-trivial section $\omega \in H^0(X, \Omega^p_X \otimes \mathcal{N})$, where $\mathcal{N}$ is a holomorphic line bundle on $X$. The singular set of $\omega$ is defined by $\text{Sing}(\omega) = \{z \in X; \omega(z) = 0\}$.

Given a Pfaff system $\omega$ of rank $p$ on $X$, then $\omega$ is determined by the following:

(i) a open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of $X$;
(ii) holomorphics $p$-forms $\omega_\alpha \in \Omega^p_{U_\alpha}$ satisfying

$$\omega_\alpha = (h_{\alpha \beta}) \omega_\beta \quad \text{on} \quad U_\alpha \cap U_\beta \neq \emptyset,$$

where $h_{\alpha \beta} \in \mathcal{O}(U_\alpha \cap U_\beta)^*$ determines the cocycle representing $\mathcal{N}$.

Definition 2.2. We say that an analytic subvariety $V$ of $X$ is invariant by a Pfaff system $\omega$ if $i^*\omega \equiv 0$, where $i : V \hookrightarrow X$ is the inclusion.

Let $\omega$ be a Pfaff system on $X$ as described above and $V$ an analytic subvariety of $X$ of pure codimension $k$. Suppose that for each $\alpha \in \Lambda$ we have

$$V \cap U_\alpha = \{z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0\},$$

where $f_{\alpha,1}, \ldots, f_{\alpha,k} \in \mathcal{O}(U_\alpha)$. If $V$ is invariant by $\omega$, then for each $i \in \{1, \ldots, k\}$ there exists holomorphic $(p + 1)$-forms $\theta^\alpha_{i1}, \ldots, \theta^\alpha_{ik} \in \Omega^{p+1}_{U_\alpha}$, such that

$$\omega_\alpha \wedge df_{\alpha,i} = f_{\alpha,1}\theta^\alpha_{i1} + \cdots + f_{\alpha,k}\theta^\alpha_{ik}. \quad (1)$$
2.2. Pfaff Systems on $\mathbb{P}^n$. Let $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^k(r))$ be a holomorphic Pfaff system of rank $k$. Now, take a generic non-invariant linearly embedded subspace $i : H \simeq \mathbb{P}^k \hookrightarrow \mathbb{P}^n$. We have an induced non-trivial section $i^* \omega \in H^0(H, \Omega_H^k(r)) \simeq H^0(\mathbb{P}^k, O_{\mathbb{P}^k}(-k - 1 + r))$. The degree of $\omega$ is defined by 

$$\text{deg}(\omega) := \text{deg}(Z(i^* \omega)) = -k - 1 + r.$$ 

In particular, $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^k(\text{deg}(\omega) + k + 1))$. A Pfaff system of degree $d$ can be induced by a polynomial $k$-form on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d + 1$, see for instance [10].

2.3. GSV-index on Surfaces. In this section we present the Brunella’s definition of GSV-index for one-dimensional holomorphic foliations on surfaces, see [5].

Let $X$ be a complex compact surface and $\mathcal{F}$ an one-dimensional holomorphic foliation on $X$. Let $C$ be a reduced curve on $X$. Consider $\omega \in H^0(X, \Omega_X^1 \otimes N_{\mathcal{F}})$ a rank one Pfaff system inducing $\mathcal{F}$. If $C$ is invariant by $\mathcal{F}$ we say that $\mathcal{F}$ is logarithmic along $C$.

Given a point $x \in C$, let $f$ be a local equation of $C$ in a neighborhood $U_\alpha$ of $x$ and let $\omega_\alpha$ be the holomorphic 1-form inducing the foliation $\mathcal{F}$ on $U_\alpha$. Since $\mathcal{F}$ is logarithmic along $C$, it follows from [19, 17, 24] there are holomorphic functions $g$ and $\xi$ defined in a neighborhood of $x$, both does not vanish identically on each irreducible components of $C$, such that

$$g \frac{\omega_\alpha}{f} = \xi \frac{df}{f} + \eta,$$

with $\eta$ being a suitable holomorphic 1-form. M. Brunella in [5] showed that the GSV-index can be defined as follows:

**Definition 2.3** (Brunella [5]). Let $X, C, \mathcal{F}$ and $x$ be as described above. We define

$$\text{GSV}(\mathcal{F}, C, x) = \text{ord}_x \left( \frac{\xi}{g} \big|_C \right),$$

where $\text{ord}_x(\frac{\xi}{g} |_C)$ denoted the vanishes order of $\frac{\xi}{g} |_C$ in $x$.

**Theorem 2.4** (Brunella [5]). Let $X, C, \mathcal{F}$ and $x$ be as described above. Then

$$\sum_{x \in \text{Sing}(\mathcal{F}) \cap C} \text{GSV}(\mathcal{F}, C, x) = N_{\mathcal{F}} \cdot C - C \cdot C.$$

In this work we will generalize the above Theorem.

2.4. Decomposition of meromorphic forms. A. G. Aleksandrov in [1] introduced the concept of multiple residues of logarithmic differentials forms along an analytic hypersurface. This concept is a generalization of the Saito’s decomposition [19]. In this section, we will make a brief presentation about this decomposition of meromorphic forms which will be used in the definition of GSV-index for Pfaff systems.
Let $U$ be a germ of $n$-dimensional complex manifold. Let $D$ be an analytic reduced hypersurface on $U$. Consider the decomposition on irreducible components
\[ D = D_1 \cup \cdots \cup D_k, \]
and suppose that the analytic subvariety $V = D_1 \cap \cdots \cap D_k$ has pure codimension $k$. We assume that
\[ V = \{ z \in U : f_1(z) = \cdots = f_k(z) = 0 \}, \]
with $f_1, \ldots, f_k \in \mathcal{O}(U)$ and for each $i \in \{1, \ldots, k\}$,
\[ D_i = \{ z \in U : f_i(z) = 0 \}. \]

Since $V$ is a reduced variety, then the $k$-form $df_1 \wedge \cdots \wedge df_k$ is not identically zero on each irreducible component of $V$.

We denote by $\Omega^q_U(D_i)$ the $\mathcal{O}_U$-module of meromorphic differential $q$-forms with simple poles on the $\hat{D}_i = D_1 \cup \cdots \cup D_{i-1} \cup D_{i+1} \cup \cdots \cup D_k$, for each $i = 1, 2, \ldots, k$.

**Theorem 2.5 (Aleksandrov [1])**. Let $U$, $D$ and $V$ as described above. If $\omega \in \Omega^q_U(D)$ is a meromorphic $q$-form such that
\[ df_j \wedge \omega \in \sum_{i=1}^k \Omega^{q+1}_{U}(\hat{D}_i), \quad j = 1, \ldots, k, \]
then, there exists a holomorphic function $g$, which is not identically zero on every irreducible component of $V$, a holomorphic $(q-k)$-form $\xi \in \Omega^{q-k}_U(D)$ and a meromorphic $q$-form $\eta \in \sum_{i=1}^k \Omega^q_U(\hat{D}_i)$ such that the following decomposition hold
\[ g\omega = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_k}{f_k} \wedge \xi + \eta. \]

Since $g$ is not identically zero on every irreducible component of $V$, the restriction
\[ \sum_{i=1}^k \Omega^{q+1}_{U}(\hat{D}_i), \quad j = 1, \ldots, k, \]
is well defined and it is called of multiple residue of meromorphic $q$-form $\omega$.

**Proposition 2.6 (Aleksandrov [1])**. On the conditions of Theorem 2.5, the multiple residues of differential form $\omega$ do not depend on decomposition (3).

**Remark 2.7**. We notice that (see [1]) in decomposition (3), the function $g$ belong to the ideal (of $\mathcal{O}(U)$) generated by all $k \times k$ minors of Jacobian matrix $\text{Jac}(f_1, \ldots, f_k)$ of map
\[ z \in U \mapsto (f_1(z), \ldots, f_k(z)) \in \mathbb{C}^k. \]
In other words, if for each multi-index $I = (i_1, \ldots, i_k)$, $1 \leq i_1, \ldots, i_k \leq n$, we denote the corresponding minor of $\text{Jac}(f_1, \ldots, f_k)$ by
\[ \Delta_I = \det \left[ \frac{\partial f_i}{\partial z_r} \right], \quad 1 \leq i, r \leq k, \]
then \( g \) can be written as

\[
g = \sum_{|I|=k} \lambda_I \Delta_I, \quad \lambda_I \in \mathcal{O}(U). \tag{4}
\]

Moreover, if the meromorphic \( q \)-form \( \omega \) is represented in \( U \) by

\[
\omega = \frac{1}{f_1 \cdots f_k} \sum_{|J|=q} a_J(z) dZ_J, \quad a_J \in \mathcal{O}(U),
\]

then for each minor \( \Delta_I \), hold

\[
\Delta_I \sum_{|J|=q} a_J dZ_J = 
\]

\[
df_1 \wedge \cdots \wedge df_k \wedge \left( \sum_{|J|=q-k} a_{(I,J^I)} dZ_{J^I} \right) + (f_1 \cdots f_k) \eta. \tag{5}
\]

Thus, for the special case where \( q = k \), we have that \( \xi \in \mathcal{O}(U) \) is given by

\[
\xi = \sum_{|I|=k} \lambda_I a_I. \tag{6}
\]

**Proposition 2.8.** Let \( \omega \in H^0(X, \Omega_X^p \otimes \mathcal{N}) \) a Pfaff system, of rank \( p \), on a complex manifold \( X \) and \( V \) a reduced local complete intersection subvariety of codimension \( k \). If \( V \) is invariant by \( \omega \), then for all local representation of \( \omega_\alpha = \omega|_{U_\alpha} \) of \( \omega \), and all local expression of \( V \) in \( U_\alpha \)

\[
V \cap U_\alpha = \{ z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0 \},
\]

there exist a holomorphic function \( g_\alpha \in \mathcal{O}(U_\alpha) \), a holomorphic \((p-k)\)-form \( \xi_\alpha \in \Omega_{U_\alpha}^{p-k} \) and a holomorphic \( p \)-form \( \eta_\alpha \in \Omega_{U_\alpha}^p \), such that

\[
g_\alpha \omega_\alpha = df_{\alpha,1} \wedge \cdots \wedge df_{\alpha,k} \wedge \xi_\alpha + \eta_\alpha. \tag{7}
\]

Moreover, \( g_\alpha \) is not identically zero on every irreducible component of \( V \) and \( \eta_\alpha \) is given by

\[
\eta_\alpha = f_{\alpha,1} \eta_{\alpha,1} + \cdots + f_{\alpha,k} \eta_{\alpha,k},
\]

where each \( \eta_{\alpha,i} \in \Omega_{U_\alpha}^p \) is a holomorphic \( p \)-form.

**Proof.** Let us consider for each \( i \in \{1, \ldots, k\} \)

\[
\mathcal{D}_i = \{ z \in U_\alpha : f_{\alpha,i}(z) = 0 \}.
\]

and

\[
\hat{\mathcal{D}}_i = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{i-1} \cup \mathcal{D}_{i+1} \cup \cdots \cup \mathcal{D}_k.
\]

Since \( V \) is invariant by \( \omega \) it follows from the expression (7) that for each \( i \in \{1, \ldots, k\} \), there exist differentials \((p+1)\)-forms \( \theta_{i1}^\alpha, \ldots, \theta_{ik}^\alpha \in \Omega_{U_\alpha}^{p+1} \), such that

\[
\omega_\alpha \wedge df_{\alpha,i} = f_{\alpha,1} \theta_{i1}^\alpha + \cdots + f_{\alpha,k} \theta_{ik}^\alpha.
\]
With this, we deduce that

\[ df_{\alpha,j} \wedge \frac{\omega_{\alpha}}{f_{\alpha}} \in \sum_{i=1}^{k} \Omega_{U_{\alpha}}^{p} (\hat{D}_i), \quad j = 1, \ldots, k. \]

Hence, the meromorphic \( p \)-form \( \frac{\omega_{\alpha}}{f_{\alpha}} \) satisfies the hypotheses of the Theorem 2.5 and the decomposition (7) follows from decomposition (3). □

The decomposition (7) will be called by a Aleksandrov-Saito’s decomposition of \( \omega \) in \( U_{\alpha} \).

### 3. GSV-index for Pfaff system

In this section we will define the GSV-index for Pfaff systems.

Consider a Pfaff system \( \omega \in H^0(X, \Omega_X^p \otimes \mathcal{N}) \) rank \( p \) and \( V \) a reduced local complete intersection subvariety of codimension \( k \) invariant by \( \omega \). Let us denote \( \text{Sing}(\omega, V) := \text{Sing}(\omega) \cap V \). We will also assume that the rank of \( \omega \) coincides with the codimension of \( V \), i.e., \( p = k \). Fixed an irreducible component \( S_i \) of \( \text{Sing}(\omega, V) \), let us take \( \omega_{\alpha} = \omega|_{U_{\alpha}} \), a local representation of \( \omega \), such that \( U_{\alpha} \cap S_i \neq \emptyset \). Assume that \( \omega_{\alpha} \) is defined by

\[ \omega_{\alpha} = \sum_{|I|=k} a_I(z) dZ_I, \quad a_I \in \mathcal{O}(U_{\alpha}). \]

Consider also a local expression of \( V \) in \( U_{\alpha} \), given by

\[ V \cap U_{\alpha} = \{ z \in U_{\alpha} : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0 \}. \]

and let us take an Aleksandrov-Saito’s decomposition of \( \omega \) in \( U_{\alpha} \),

\[ g_{\alpha} \omega_{\alpha} = (df_{\alpha,1} \wedge \cdots \wedge df_{\alpha,k}) \xi_{\alpha} + \eta_{\alpha}, \quad (8) \]

with \( \eta_{\alpha} = f_{\alpha,1} \eta_{\alpha,1} + \cdots + f_{\alpha,k} \eta_{\alpha,k} \), where \( \eta_{\alpha,i} \in \Omega_{U_{\alpha}}^{k} \) and, furthermore, \( \xi_{\alpha} \) being a holomorphic function.

**Definition 3.1.** We define

\[ \text{GSV}(\omega, V, S_i) := \text{ord}_{S_i} \left( \frac{\xi_{\alpha}}{g_{\alpha}|_V} \right). \]

**Theorem 3.2.** \( \text{GSV}(\omega, V, S_i) \) depends only on \( \omega, V \) and \( S_i \). Moreover, the following formula holds

\[ \sum_i \text{GSV}(\omega, V, S_i)[S_i] = c_1([\mathcal{N} \otimes \text{det}(\mathcal{N}_{V/X})^{-1}])|_V \sim [V] \]
Proof. Firstly, it follows from the definition and Proposition 2.6 that GSV(ω, V, S_i) do not depend on chosen decomposition of ω. Now, we will prove that GSV(ω, V, S_i) does not depend on the local representation ω, and does not depend on local expression for V:

\[ V \cap U_\alpha = \{ z \in U_\alpha : f_{\alpha,1}(z) = \cdots = f_{\alpha,k}(z) = 0 \}. \]

In fact, if we consider another local representation ω, such that \( U_\beta \cap S_i \neq \emptyset \) and other local expression for V

\[ V \cap U_\beta = \{ z \in U_\beta : f_{\beta,1}(z) = \cdots = f_{\beta,k}(z) = 0 \}, \]

we obtain the Aleksandrov-Saito’s decomposition of ω in \( U_\beta \)

\[ g_\beta \omega = (df_{\beta,1} \wedge \cdots \wedge df_{\beta,k}) \xi_\beta + \eta_\beta \]

and the decomposition of ω in \( U_\alpha \)

\[ g_\alpha \omega = (df_{\alpha,1} \wedge \cdots \wedge df_{\alpha,k}) \xi_\alpha + \eta_\alpha \]

where \( \eta_\beta |_V = \eta_\alpha |_V = 0 \). Now, in the intersection \( U_\alpha \cap U_\beta \neq \emptyset \) we have that

\[ (df_{\alpha,1} \wedge \cdots \wedge df_{\alpha,k}) = m_{\alpha \beta} (df_{\beta,1} \wedge \cdots \wedge df_{\beta,k}) + \theta_{\alpha \beta} \]

where \( m_{\alpha \beta} |_V \in O_V(U_\alpha \cap U_\beta \cap V)^* \) is the cocycle of the determinant of normal bundle \( \det(N_{V/X}) \) on \( V \) and \( \theta_{\alpha \beta} |_V = 0 \). Also, in \( U_\alpha \cap U_\beta \) we have that

\[ \omega_\alpha = h_{\alpha \beta} \omega_\beta, \]

where \( h_{\alpha \beta} \in O(U_\alpha \cap U_\beta)^* \) is the cocycle of the line bundle \( N \). On the one hand, by using (12) and (13) in (11), we obtain

\[ (g_\alpha h_{\alpha \beta}) \omega_\beta = (m_{\alpha \beta} \xi_\alpha)(df_{\beta,1} \wedge \cdots \wedge df_{\beta,k}) + \eta_\alpha + \theta_{\alpha \beta} \]

On the other hand, by using (10) in (14), we obtain

\[ (h_{\alpha \beta} \xi_\beta g_{\alpha \beta} - m_{\alpha \beta} \xi_\alpha g_{\beta})(df_{\beta,1} \wedge \cdots \wedge df_{\beta,k}) = \eta_\beta + g_{\beta} \eta_\alpha + g_{\beta} \theta_{\alpha \beta}. \]

Since \( \eta_\beta + g_{\beta} \eta_\alpha + g_{\beta} \theta_{\alpha \beta} |_V \equiv 0 \), we conclude that

\[ (h_{\alpha \beta} \xi_\beta g_{\alpha \beta} - m_{\alpha \beta} \xi_\alpha g_{\beta})(df_{\beta,1} \wedge \cdots \wedge df_{\beta,k}) \equiv 0 \mod (f_{\beta,1}, \ldots, f_{\beta,k}) \]

It follows from (20) that there exist \( r \in \mathbb{Z} \), with \( r \geq 1 \), such that

\[ \mathcal{D}^r(h_{\alpha \beta} \xi_\beta g_{\alpha \beta} - m_{\alpha \beta} \xi_\alpha g_{\beta}) \in df_{\beta,1} \wedge \Omega_{U_\beta}^{k-1} + \cdots + df_{\beta,k} \wedge \Omega_{U_\beta}^{k-1}, \]

where \( \mathcal{D} \) is the ideal of \( O_{U_\beta} \) generated by all minors of maximal order of the Jacobian matrix \( \text{Jac}(f_{\beta,1}, \ldots, f_{\beta,k}) \). Now, as observed in [1] Proposition 2] the image \( \text{Im}(\mathcal{D}) \) of \( \mathcal{D} \) in the ring \( O_{U_\beta} \) is not equal to \( \text{Ann}(O_{U_\beta}) \). In fact, since \( V \) is reduced, then \( df_{\beta,1} \wedge \cdots \wedge df_{\beta,k} \) does not vanish identically on each irreducible component of \( V \). Therefore, it follows from [7] Theorem 2.4. (1)] that the \( O_{U_\beta} \)-depth of the
ideal $\mathcal{D}$ is at least 1. Then, there is $D \in \mathcal{D}$ which is not a zero-divisor in $\mathcal{O}_{V \cap U_\beta}$ and

$$D' (h_{\alpha \beta} \xi_\beta g_\alpha - m_{\alpha \beta} \xi_\alpha g_\beta) \in df_{\beta,1} \wedge \Omega_{U_\beta}^{k-1} + \cdots + df_{\beta,k} \wedge \Omega_{U_\beta}^{k-1}.$$ 

Thus, the class $D' (h_{\alpha \beta} \xi_\beta g_\alpha - m_{\alpha \beta} \xi_\alpha g_\beta) = 0 \in \Omega^k_{V \cap U_\beta}$. Then

$$(h_{\alpha \beta} \xi_\beta g_\alpha - m_{\alpha \beta} \xi_\alpha g_\beta)|_V = 0$$

since $D \in \mathcal{D}$ is not a zero-divisor. We conclude that

$$(15) \quad \frac{\xi_\alpha}{g_\alpha}|_V = h_{\alpha \beta} (m_{\alpha \beta})^{-1} \frac{\xi_\beta}{g_\beta}|_V.$$

Since $h_{\alpha \beta}$ and $(m_{\alpha \beta})^{-1}$ are non-vanishing holomorphic functions we have

$$\text{ord}_{S_i} \left( \frac{\xi_\alpha}{g_\alpha}|_V \right) = \text{ord}_{S_i} \left( \frac{\xi_\beta}{g_\beta}|_V \right).$$

Now, we will prove the formula (9). Indeed, consider the isomorphism determined by the duality of Poincaré

$$P_V : H^2(V, \mathbb{C}) \rightarrow H_{2n-2k-2}(V, \mathbb{C}).$$

In particular, we have

$$(16) \quad P_V (c_1([N \otimes \det(N_{V/X})^{-1}])|_V) = c_1([N \otimes \det(N_{V/X})^{-1}])|_V \sim [V].$$

Note that, by (15) the family of meromorphic functions $s = \{(\xi_\alpha/g_\alpha)|_V\}_{\alpha \in \Lambda}$ defines a meromorphic section of the line bundle $[N \otimes \det(N_{V/X})^{-1}]|_V$. Therefore, we have the divisor associated to the section $s$ given by

$$(s)_0 = \sum_i \text{GSV}(\omega, V, S_i)[S_i].$$

That is

$$\mathcal{O}((s)_0) \cong [N \otimes \det(N_{V/X})^{-1}]|_V.$$ 

Thus,

$$c_1([N \otimes \det(N_{V/X})^{-1}])|_V = c_1(\mathcal{O}((s)_0))$$

$$= c_1 \left( \mathcal{O} \left( \sum_i \text{GSV}(\omega, V, S_i)[S_i] \right) \right)$$

$$= \sum_i \text{GSV}(\omega, V, S_i)P_V^{-1}([S_i]),$$
where each \( P^{-1}_V([S_i]) \) is the dual of the cycle \([S_i] \in H_{2n-2k-2}(V, \mathbb{C}) \) determined by \( S_i \). Consequently,

\[
P_V(c_1([N \otimes \det(N_{V/X})^{-1}])|_V) = \sum_i \text{GSV}(\omega, V, S_i)[S_i].
\]

Now, using (17) in equality (16), we obtain

\[
\sum_i \text{GSV}(\omega, V, S_i)[S_i] = c_1([N \otimes \det(N_{V/X})^{-1}])|_V \sim [V].
\]

□

Next result gives us an alternative way to calculate the GSV-index. This formula can be compared with the Suwa’s formula in [25, Proposition 5.1].

**Theorem 3.3.** Let \( \omega, V \) and \( S_i \) be as described above. For each multi-index \( I \), with \(|I|=k\), the following formula holds

\[
\text{GSV}(\omega, V, S_i) = \text{ord}_{S_i}(a_I|_V) - \text{ord}_{S_i}(\Delta_I|_V).
\]

where \( \Delta_I \) is the \( k \times k \) minors of Jacobian matrix \( \text{Jac}(f_{\alpha,1}, \ldots, f_{\alpha,k}) \), corresponding to the multi-index \( I \).

**Proof.** By the expression (5) we have

\[
(\Delta_I \sum_{|J|=k} a_J dZ_J)|_V = [(df_{\alpha,1} \wedge \ldots \wedge df_{\alpha,k}) a_I]|_V.
\]

Since

\[
df_{\alpha,1} \wedge \ldots \wedge df_{\alpha,k} = \sum_{|J|=k} \Delta_J dZ_J,
\]

for each \( J \), with \(|J|=k\), we get

\[
(\Delta_I a_J)|_V = (\Delta_J a_I)|_V.
\]

Thus,

\[
\text{GSV}(\omega, V, S_i) = \text{ord}_{S_i} \left( \frac{\xi_\alpha}{g_\alpha} |_V \right)
\]

\[
= \text{ord}_{S_i} \left( \frac{\xi_\alpha}{g_\alpha} |_V \right) + \text{ord}_{S_i}(\Delta_I|_V) - \text{ord}_{S_i}(\Delta_I|_V)
\]

\[
= \text{ord}_{S_i} \left( \frac{\xi_\alpha \Delta_I}{g_\alpha} |_V \right) - \text{ord}_{S_i}(\Delta_I|_V).
\]

By (6) we have

\[
\xi_\alpha = \sum_{|J|=k} \lambda_J a_J.
\]
Hence, by using (18) we obtain

$$\xi_\alpha \Delta_I|_V = \sum_{|J|=k} \lambda_J (a_J \Delta_I)|_V = \sum_{|J|=k} \lambda_J (\Delta_J a_J)|_V$$

$$= \sum_{|J|=k} \lambda_J \Delta_J a_J|_V = g_\alpha a_I|_V,$$

where in the last step we have used (4). Thus,

$$\text{ord}_{S_i} \left( \frac{\xi_\alpha \Delta_I|_V}{g_\alpha} \right) = \text{ord}_{S_i} (a_I|_V).$$

Then

$$\text{GSV}(\omega, V, S_i) = \text{ord}_{S_i} (a_I|_V) - \text{ord}_{S_i} (\Delta_I|_V).$$

\[\square\]

**Corollary 3.4.** Let $\omega$, $V$ and $S_i$ be as described above. If $S_i \cap \text{Sing}(V) = \emptyset$, then $\text{GSV}(\omega, V, S_i) \geq 0$.

**Proof.** By Theorem 3.3 for all multi-index $I$, with $|I| = k$, we have

\begin{equation}
\text{GSV}(\omega, V, S_i) = \text{ord}_{S_i} (a_I|_V) - \text{ord}_{S_i} (\Delta_I|_V).
\end{equation}

Let $h_i \in \mathcal{O}_V(U_\alpha \cap V)$ be a function which defines locally $S_i$ in $U_\alpha \cap V$, i.e.,

$$(U_\alpha \cap V) \cap S_i = \{ z \in U_\alpha \cap V : h_i(z) = 0 \}.$$

If $\text{GSV}(\omega, V, S_i) < 0$ then, by (19), for each multi-index $I$, with $|I| = k$,

$$\text{ord}_{S_i} (\Delta_I|_V) = \delta_I > 0$$

Thus,

$$\Delta_I|_V = h_\alpha^{\delta_I} \mu_I|_V,$$

for some function $\mu_I \in \mathcal{O}_V(U_\alpha \cap V)$. This implies that for each multi-index $I$, with $|I| = k$, the following inclusion occurs

$$\{ z \in U_\alpha \cap V : h_i(z) = 0 \} \subset \{ z \in U_\alpha \cap V : \Delta_I(z) = 0 \}.$$

Thus, we conclude that

$$(U_\alpha \cap V) \cap S_i = \{ z \in U_\alpha \cap V : h_i(z) = 0 \} \subset \bigcap_{|I| = k} \{ z \in U_\alpha \cap V : \Delta_I(z) = 0 \} = U_\alpha \cap \text{Sing}(V).$$

Therefore, we obtain $S_i \cap \text{Sing}(V) \neq \emptyset$. $\square$

**Corollary 3.5.** Let $\omega$ and $V$ be as described above. If $V$ is smooth, then for all $S_i$

$$\text{GSV}(\omega, V, S_i) > 0.$$
Proof. If \( V \) is smooth, then there exist some \( k \times k \) minor \( \Delta_I \) (of Jacobian matrix \( \text{Jac}(f_{\alpha,1}, \ldots, f_{\alpha,k}) \)) which is not zero along \( V \). Therefore,

\[
\text{ord}_S((\Delta_I|_V) = 0.
\]

Therefore, it follows from Theorem 3.3 that

\[
\text{GSV}(\omega, V, S) = \text{ord}_S((a_I|_V) - \text{ord}_S((\Delta_I|_V) = \text{ord}_S((a_I|_V) > 0.
\]

\[ \square \]

4. An Application: Poincaré Problem for Pfaff systems

In this section, we will consider the GSV-index for Pfaff systems on a projective space \( \mathbb{P}^n \). We will see that the non-negativity of the GSV-index gives us the obstruction to the solution of the Poincaré Problem for Pfaff systems.

**Theorem 4.1.** Let \( \omega \) and \( S \) be as described above. If \( V \) is a complete intersection variety of codimension \( k \) and multidegree \( (d_1, \ldots, d_k) \), which is invariant by \( \omega \) on \( \mathbb{P}^n \). Then

\[
\sum_i \text{GSV}(\omega, V, S_i) \deg(S_i) = \left[ d + k + 1 - (d_1 + \cdots + d_k) \right] \cdot (d_1 \cdots d_k).
\]

In particular, if \( \text{GSV}(\omega, V, S_i) \geq 0 \), for all \( i \), we have

\[
d_1 + \cdots + d_k \leq d + k + 1.
\]

**Proof.** By applying the Theorem 3.2 and take degrees we obtain

\[
\sum_i \text{GSV}(\omega, V, S_i) \deg(S_i) = \deg(c_1([N \otimes \text{det}(N_{V/\mathbb{P}^n})^{-1}]|_V \sim [V])).
\]

On the one hand, the normal bundle of \( V \) is given by

\[
N_{V/\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_k)|_V.
\]

Thus

\[
\text{det}(N_{V/\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(d_1 + \cdots + d_k)|_V.
\]

On the other hand, we have that

\[
\mathcal{N} = \mathcal{O}_{\mathbb{P}^n}(d + k + 1).
\]

Now, since

\[
[V] = (d_1 \cdots d_k) \ c_1(\mathcal{O}(1))^k
\]

we get

\[
\deg([\mathcal{N} \otimes \text{det}(N_{V/\mathbb{P}^n})^{-1}]|_V \sim [V]) = [(d + k + 1) - (d_1 + \cdots + d_k)] \cdot (d_1 \cdots d_k).
\]
We obtain,
\[
\sum_i GSV(\omega, V, S_i) \deg(S_i) = [d + k + 1 - (d_1 + \cdots + d_k)] \cdot (d_1 \cdots d_k).
\]
Now, if \( GSV(\omega, V, S_i) \geq 0 \), for all \( i \), we have
\[
0 \leq \sum_i GSV(\omega, V, S_i) \deg(S_i) = [d + k + 1 - (d_1 + \cdots + d_k)] \cdot (d_1 \cdots d_k).
\]
This implies that
\[
d_1 + \cdots + d_k \leq d + k + 1.
\]

In the next result we obtain a similar bound due to Esteves–Cruz \[12\] and Corrêa–Jardim \[9\].

**Corollary 4.2.** Let \( \omega, V \) and \( S \) be as described above with \( V \) invariant by \( \omega \). Suppose that for all \( i \) we have \( S_i \cap \text{Sing}(V) = \emptyset \), then
\[
d_1 + \cdots + d_k \leq d + k + 1.
\]
Moreover, if \( V \) is regular we have
\[
d_1 + \cdots + d_k \leq d + k.
\]

**Proof.** The result follows from Corolary 3.5 and Corolary 3.4.

Now, we will give a optimal example.

**Example 4.3.** Consider the Pfaff system \( \omega \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d + k + 1)) \) given by
\[
\omega = \sum_{0 \leq j \leq k} (-1)^j d_j f_j df_0 \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k,
\]
where \( f_j \) is a homogeneous polynomial of degree \( d_j \). We can see that
\[
d_0 + d_1 + \cdots + d_k = d + k + 1.
\]
Suppose that \( \deg(f_0) = d_0 = 1 \) and that \( V = \{ f_1 = \cdots = f_k = 0 \} \) is smooth. We have that \( V \) is invariant by \( \omega \) and
\[
d_1 + \cdots + d_k = d + k.
\]

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