BIPARTITE DIVISOR GRAPHS FOR INTEGER SUBSETS

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Abstract. Inspired by connections described in a recent paper by Mark L. Lewis, between the common divisor graph $\Gamma(X)$ and the prime vertex graph $\Delta(X)$, for a set $X$ of positive integers, we define the bipartite divisor graph $B(X)$, and show that many of these connections flow naturally from properties of $B(X)$. In particular we establish links between parameters of these three graphs, such as number and diameter of components, and we characterise bipartite graphs that can arise as $B(X)$ for some $X$. Also we obtain necessary and sufficient conditions, in terms of subconfigurations of $B(X)$, for one $\Gamma(X)$ or $\Delta(X)$ to contain a complete subgraph of size 3 or 4.

Bipartite graph, common divisor graph, prime vertex graph.

1. Introduction

We introduce the bipartite divisor graph $B(X)$, for a non-empty subset $X$ of positive integers, that contains information about two previously studied graphs, namely the prime vertex graph and the common divisor graph for $X$. Our work was inspired by a recent paper [3] by Mark L. Lewis which provides a fascinating overview of various graphs associated with groups. Surprisingly strong combinatorial information available for these graphs leads to structural information about the groups and their representations. Lewis distilled and unified many results concerning these ‘group graphs’ (from the 78 references in his bibliography) by first defining two graphs associated with an arbitrary non-empty subset $X$ of positive integers:

1. the prime vertex graph $\Delta(X)$ has, as vertex set $\rho(X)$, the set of primes dividing some element of $X$, and two such primes $p, q$ are joined by an edge if and only if $pq$ divides some $x \in X$;
2. the common divisor graph $\Gamma(X)$ has vertex set $X^* := X \setminus \{1\}$, and $x, y \in X^*$ form an edge if and only if $\gcd(x, y) > 1$.

Although, in the group setting, the subset $X$ is usually the set of irreducible character degrees, or the set of conjugacy class sizes, or conjugacy class indices, of a finite group, Lewis showed that, even for arbitrary integer sets $X$, the prime vertex graph and the common divisor graph share very similar combinatorial properties, for example, they have the same number of connected components, and similar diameters (where by the diameter Lewis means the maximum diameter of a connected component).

The first author wishes to thank Yazd University Research Council for financial support during his study leave, and The School of Mathematics and Statistics, University of Western Australia, in particular Prof. Praeger, for their hospitality during his visit and for the facilities and help provided. The second author was supported by a Federation Fellowship of the Australian Research Council.
The bipartite divisor graph $B(X)$, for an arbitrary non-empty subset $X$ of positive integers, has as vertex set the disjoint union $\rho(X) \cup X^*$, and its edges are the pairs $\{p, x\}$ where $p \in \rho(X)$, $x \in X^*$ and $p$ divides $x$. Thus $B(X)$ is bipartite, and $\{\rho(X) \mid X^*\}$ forms a bi-partition of the vertex set, that is unique if $B(X)$ is connected (and in the group cases, $B(X)$ is often connected, see for example [3] Theorems 4.1, 5.2, 7.1, 8.1, 9.8], key references being [2, 4, 5, 6]). Moreover the ‘distance 2-graph’, derived from $B(X)$ by replacing the edges of $B(X)$ by the set of pairs $\{u, v\}$ that have distance 2 in $B(X)$, contains $\Delta(X)$ and $\Gamma(X)$ as the subgraphs induced on $\rho(X)$ and $X^*$, respectively. Thus it is not surprising that several combinatorial properties of $\Delta(X)$ and $\Gamma(X)$ can be derived from similar properties for $B(X)$. We study such properties as the diameter, girth, number of connected components, and clique number for these three graphs, obtaining precise links relating these parameters for the various graphs, which we summarise below. Our findings lead to interesting new questions in the ‘group case’, some of which are explored in a forthcoming paper [1] of the authors with Bubboloni and Dolfi.

In particular, in [3, Lemma 3.3], Lewis proved that every graph $G$ is isomorphic to $\Delta(X)$ and to $\Gamma(Y)$ for some sets of positive integers $X$ and $Y$. Our main result characterises those bipartite graphs that arise as $B(X)$ for some $X$. The proof in Section 2 gives an explicit construction of a subset $X$, for a given bipartite graph $G$.

**Theorem 1.1.** A bipartite graph $G$ is isomorphic to $B(X)$, for some non-empty set of positive integers $X$, if and only if $G$ is non-empty and has no isolated vertices.

**Comment on Notation:**

(a) We call a graph bipartite if there is a bipartition $\{V_1 \mid V_2\}$ of its vertex set with both $V_1, V_2$ non-empty, such that each edge joins a vertex of $V_1$ to a vertex of $V_2$. An empty graph is a graph with at least one vertex and no edges, and a vertex in a graph is isolated if it lies on no edge.

(b) Usually the set $X$ of positive integers is clear from the context. We therefore suppress $X$ in our notation and write $B, \Delta, \Gamma$ for the graphs $B(X), \Delta(X), \Gamma(X)$ respectively. Similarly, for example, we denote by $n(B), n(\Delta), n(\Gamma)$ the number of connected components of $B, \Delta, \Gamma$ respectively, and, for vertices $x, y$ in the same connected component, we denote by $d_B(x, y), d_\Delta(x, y), d_\Gamma(x, y)$ the distance (length of shortest path) between $x$ and $y$ for the graph $B, \Delta, \Gamma$ respectively. Following Lewis [3], we define the diameter as the maximum distance between vertices in the same connected component, and we denote the diameters of these three graphs by $\text{diam}(B), \text{diam}(\Delta), \text{diam}(\Gamma)$ respectively. Also, if there is a cycle in the graph $B, \Delta$ or $\Gamma$, we denote the girth (the length of the shortest cycle) by $g(B), g(\Delta), g(\Gamma)$, respectively.

**Summary of other results:** (Definitions of the additional graph theoretic concepts are given in the relevant subsection.)

1. $B, \Delta, \Gamma$ have equal numbers of connected components, and the maximum of $\text{diam}(\Delta)$ and $\text{diam}(\Gamma)$ is $\lfloor \frac{\text{diam}(B)}{2} \rfloor$.

2. Any subset of $\{B, \Delta, \Gamma\}$ may be acyclic, and the others not. However, if $B$ contains a cycle of length greater than 4, then all three graphs contain cycles.

3. Both $\Delta$ and $\Gamma$ are acyclic if and only if each connected component of $B$ is a path or isomorphic to $C_4$. (Theorem 4.2)
(4) For \( m = 3, 4 \), at least one of \( \Delta, \Gamma \) contains a clique of size \( m \) (a subgraph \( K_m \)), if and only if \( B \) contains a subgraph in a specified list. (Theorems 4.1 and 4.6)

2. Representing bipartite graphs as \( B(X) \)

In this section we prove Theorem 1.1, giving an explicit construction of a subset \( X \), for a given bipartite graph \( G \). We illustrate the construction with a simple example in Figure 1.

![Figure 1. Examples of \( G \) (left) and \( B(X) \) (right) for Lemma 1.1](image)

**Proof of Theorem 1.1** Suppose that \( G \) is a bipartite graph with vertex bipartition \( \{V_1|V_2\} \). Let \( V_1 = \{v_1, v_2, \ldots, v_m\} \) and \( V_2 = \{u_1, u_2, \ldots, u_n\} \) where \( m \geq 1, n \geq 1 \). Suppose first that \( G \) has no isolated vertices. Let \( p_1, p_2, \ldots, p_m \) be pairwise distinct primes, and let \( M = \{p_1, p_2, \ldots, p_m\} \). Define a bijection \( f : V_1 \rightarrow M \) by \( f(v_i) = p_i \) for each \( i \). For \( 1 \leq j \leq n \) define \( I_j = \{\ell|\{v_i, u_j\} \in E(G)\} \) and set \( x_j = \prod_{\ell \in I_j} p_\ell^i \). Note that \( I_j \neq \emptyset \), since there are no isolated vertices in \( G \). Let \( X = \{x_j|1 \leq j \leq n\} \). The fact that \( \rho(X) = M \) follows because there are no isolated vertices in \( G \). Now \( \{p_i, x_j\} \in E(B) \), the edge set of \( B = B(X) \), if and only if \( p_i \) divides \( x_j = \prod_{\ell \in I_j} p_\ell^i \), that is, if and only if \( i \in I_j \), and this holds if and only if \( \{v_i, u_j\} \in E(G) \). Thus extending \( f \) to a map \( V(G) \rightarrow V(B) \) by \( f(u_i) = x_i \), for each \( i \), defines an isomorphism from \( G \) to \( B(X) \).

Conversely, suppose that \( G \cong B(X) \), for some \( X \). Then, since by the definition of a bipartite graph, \( G \) has at least one vertex, \( X \neq \{1\} \). The fact that \( B(X) \) has no isolated vertices now follows from its definition. \( \square \)

Theorem 1.1 provides an important tool for the proofs in the rest of the paper, by applying the following corollary.

**Corollary 2.1.** For a non-empty set \( X \) of positive integers such that \( X \neq \{1\} \), there exists a second non-empty set \( Y \) of positive integers, and a graph isomorphism \( \phi : B(X) \rightarrow B(Y) \) that induces isomorphisms \( \Delta(X) \cong \Gamma(Y) \) and \( \Gamma(X) \cong \Delta(Y) \).

**Proof.** Let \( G = B(X) \) with vertex bipartition \( \{\rho(X)|X^*\} \). By definition, \( G \) is non-empty and has no isolated vertices. We apply the proof of Theorem 1.1 to the reverse bipartition \( \{X^*|\rho(X)\} \). This produces a non-empty subset \( Y \) of positive integers and a graph isomorphism \( \phi : G \rightarrow B(Y) \), that induces a graph isomorphism from \( \Delta(Y) \) to the distance 2 graph induced on the first part \( X^* \) of the bipartition (which by definition of \( G \) is \( \Gamma(X) \)), and a graph isomorphism from \( \Gamma(Y) \) to the
distance 2 graph induced on the second part $\rho(X)$ of the bipartition (which by
definition of $G$ is $\Delta(X)$).

Thus if we wish to prove that a certain relationship holds between $B(X)$ and
$\Delta(X)$, for all $X$, and also between $B(X)$ and $\Gamma(X)$, for all $X$, it is often sufficient
to prove only one of these assertions.

3. RELATING THE PARAMETERS OF $B, \Delta, \Gamma$

In this section we study certain parameters for the three graphs, namely distance,
diameter, girth, and number of components. Throughout the section let $X$ denote a
non-empty subset of positive integers with $X \neq \{1\}$, so that $X^* \neq \emptyset$. As mentioned
above we simplify our notation and write $B := B(X), \Delta := \Delta(X), \Gamma := \Gamma(X)$. We
denote the vertex sets of these graphs by $V(B), V(\Delta), V(\Gamma)$, and the edge sets by
$E(B), E(\Delta), E(\Gamma)$, respectively.

3.1. Distance, diameter, and numbers of components. A key technical result
in Lewis’s paper, namely \cite{3} Lemma 3.1 and Corollary 3.2, can be interpreted as a
1-1 correspondence between the (connected) components of $\Delta$ and $\Gamma$ leading to the
consequence that the diameters of $\Delta$ and $\Gamma$ differ by at most 1. We extend these
results to give analogous information about the graph $B$ from which the facts about
$\Delta$ and $\Gamma$ may be deduced. We note that, although in many of the ‘group cases’
the graphs $\Delta$ and $\Gamma$ have at most 2 components and diameter at most 3 (see for example \cite{3} Corollary 4.2, Theorems 7.1, 8.1, 8.3)), for general $X$ these parameters
may be arbitrarily large.

For $u \in V(B)$, let $[u]_B$ denote the connected component of $B$ containing $u$, and
similarly define $[u]_\Delta, [u]_\Gamma$ if $u \in V(\Delta)$ or $u \in V(\Gamma)$ respectively.

Lemma 3.1. Let $p, q \in \rho(X)$ and $x, y \in X^*$ such that $[p]_B = [q]_B$ and $[x]_B = [y]_B$.
Then,

(a) $d_B(p, q) = 2d_\Delta(p, q), d_B(x, y) = 2d_\Gamma(x, y)$;
(b) if $p$ divides $x$ and $q$ divides $y$, then $[p]_B = [x]_B = [p]_\Delta \cup [x]_\Gamma$ and $d_B(p, q) -
\quad d_B(x, y) \in \{-2, 0, 2\}$;
(c) $n(B) = n(\Delta) = n(\Gamma)$;
(d) either
\quad (i) $diam(B) = 2\max\{diam(\Delta), diam(\Gamma)\}$, and $|diam(\Delta) - diam(\Gamma)| \leq 1$,
or
\quad (ii) $diam(\Delta) = diam(\Gamma) = \frac{1}{2}(diam(B) - 1)$.

Table \ref{tab:parameters} gives simple examples to show that all possibilities for the diameters of
$B, \Delta, \Gamma$ given by Lemma 3.1(d) arise.

Proof. (a) Suppose that $d_\Delta(p, q) = k$. Then there exists a shortest path $P_\Delta =
(p_0, p_1, \ldots, p_k)$ in $\Delta$ with $p = p_0$ and $q = p_k$. Now $\{p_i, p_{i+1}\}$ is an edge of $\Delta$ if and
only if $d_\Delta(p_i, p_{i+1}) = 2$, and hence there exists a path $P_B = (p_0, x_1, p_1, \ldots, x_{k-1}, p_k)$
in $B$ of length $2k$. Thus $d_B(p, q) \leq 2k$, and as $p, q$ are in the same part of the
bipartition of $B$, we have $d_B(p, q) = 2\ell \leq 2k$. If $P_B' = (p_0', x_1', p_1', \ldots, x_\ell', p_\ell')$ is a
shortest path in $B$ with $p = p_0$ and $q = p_\ell'$, then $P_\Delta' = (p_0', p_1', \ldots, p_\ell')$ is a path
of length $\ell$ in $\Delta$, so $k = d_\Delta(p, q) \leq \ell$, and hence $d_B(p, q) = 2k = 2d_\Delta(p, q)$. An analogous proof shows that $d_B(x, y) = 2d_T(x, y)$.

(b) Suppose now that $p$ divides $x$ and $q$ divides $y$. If the path $P_B'$ above can be chosen with $x'_1 = x$ and $x'_1 = y$ then $d_B(p, q) - d_B(x, y) \geq 2$, and a similar argument to the above shows that equality holds; on the other hand if one of these equalities, but not the other can be achieved then we find $d_B(p, q) = d_B(x, y)$, while if neither can hold on a shortest path $P_B'$ then $d_B(x, y) - d_B(p, q) = 2$. This proves that $d_B(p, q) - d_B(x, y) = 2$ and that the component $[p]_B$ contains all of $p, q, x, y$, and the fact that $[p]_B = [p]_\Delta \cup [x]_T$ follows from the proof of part (a).

(c) The assertion about numbers of connected components follows from the assertion in part (b) about components.

(d) Let $m = \max\{\text{diam}(\Delta), \text{diam}(\Gamma)\}$. Then it follows from part (a) that $\text{diam}(B) \geq 2m$. Let $M = \text{diam}(B)$, so that $M \geq 2m$, and choose $a, b \in V(B)$ such that $d_B(a, b) = M$. If $a$ and $b$ are both in $\rho(X)$ (or both in $X^\ast$), then $M = d_B(a, b) = 2d_\Delta(a, b) \leq 2\text{diam}(\Delta) \leq 2m$ (respectively, $M \leq 2\text{diam}(\Gamma) \leq 2m$) and in either case we conclude that $M = 2m$. On the other hand (without loss of generality) suppose that $a \in \rho(X)$ and $b \in X^\ast$, so $M$ is odd and hence $M \geq 2m + 1$.

Let $p \in \rho(X)$ be the vertex adjacent to $b$ on a path $P_B$ of length $M$ from $a$ to $b$. Then the sub-path of $P_B$, from $p$ to $a$, must be a shortest path between these two vertices, by definition of $M$, and hence $M - 1 = d_B(a, p) = 2d_\Delta(a, p)$, by part (a), and this is at most $2\text{diam}(\Delta)$. A similar argument using the vertex adjacent to $a$ on $P_B$ yields $M \leq 2\text{diam}(\Gamma) + 1$. It follows that $\text{diam}(\Delta) = \text{diam}(\Gamma) = \frac{M - 1}{2}$.

To prove the last assertion of (i) we may assume that $\text{diam}(B) = 2m$. Let $j = \text{diam}(\Delta)$ and let $p_0, p_j \in \rho(X)$ be such that $d_\Delta(p_0, p_j) = j$. Then by part (a), there exists a path $P_B'$ of length $2j$ in $B$ from $p_0$ to $p_j$. Let $x_0, x_1$ be vertices on $P_B'$ adjacent to $p_0$ and $p_j$, respectively. Then the sub-path of $P_B'$ from $x_0$ to $x_1$ of length $2j - 2$ is a shortest path in $B$ between these two vertices. Thus $d_B(x_0, x_1) = 2j - 2$.

By part (a), $d_T(x_0, x_1) = j - 1$ and therefore $\text{diam}(\Gamma) \geq \text{diam}(\Delta) - 1$. A similar argument shows that $\text{diam}(\Delta) \geq \text{diam}(\Gamma) - 1$. Hence $|\text{diam}(\Delta) - \text{diam}(\Gamma)| \leq 1$.\hfill $\square$

### 3.2. Cycles and girth

A graph $G$ is said to be acyclic if it contains no cycles, that is, it contains no closed paths of length at least 3. On the other hand, recall that, if $G$ contains a cycle then the minimum length of its cycles is called its girth and denoted $g(G)$. For any subset $K \subseteq \{B, \Delta, \Gamma\}$ there exists $X$ such that the graphs in $K$ are acyclic and each graph not in $K$ contains a cycle. Examples of subsets $X$ are provided in Table 2 for the seven non-empty subsets $K$ of $\{B, \Delta, \Gamma\}$.

| $X$        | $\text{diam}(\Delta)$ | $\text{diam}(\Gamma)$ | $\text{diam}(\overline{B})$ |
|-----------|----------------------|-----------------------|-----------------------------|
| $\{6, 10, 15\}$ | 1                    | 1                     | 2                           |
| $\{2, 10\}$    | 1                    | 1                     | 3                           |
| $\{2, 3, 6\}$   | 1                    | 2                     | 4                           |
| $\{6, 15\}$    | 2                    | 1                     | 4                           |

[Table 1: Illustration of all cases of Lemma 3.1(d)]

Illustration of all cases of Lemma 3.1(d)
and if $X = X_2 \cup X_3 \cup X_4$, with the $X_i$ as in Table 2, then all three graphs contain cycles. In this last example, $B$ has girth 4, while the other two graphs have girth 3. However, once the graph $B$ contains a cycle of length greater than 4, we prove that all three of the graphs contain cycles. Even in this case it is possible for one or both of $\Delta$ or $\Gamma$ to have girth 3 regardless of the size of $g(B)$, simply by adding to $X$ an analogue of the subset $X_2$ or $X_3$ of Table 2 (invoking suitable primes). However if the girths of $\Delta$ or $\Gamma$ are greater than 3, we show that there is a tight link between these girths and the minimum length of cycles in $B$ with more than 4 vertices.

In Table 2 we denote a complete graph and a cycle on $m$ vertices by $K_m$ and $C_m$, respectively, and if $B = B(X)$ is a complete bipartite graph with $|\rho(X)| = m$ and $|X^*| = n$, then we denote $B$ by $K_{m,n}$.

**Lemma 3.2.** Suppose that $B$ contains a cycle of length greater than 4. Then each of $\Delta$ and $\Gamma$ also contains a cycle. Moreover, for $\Phi \in \{\Delta, \Gamma\}$, either $g(\Phi) = 3$ or $g(\Phi) = \frac{g'(B)}{2}$, where $g'(B)$ is the minimum length of cycles of $B$ with more than four vertices.

**Proof.** Since $B$ is bipartite, $g'(B) = 2k$ for some $k \geq 3$. Let $P_B = (p_1, x_1, \ldots, p_k, x_k)$ be a closed path of length $2k$ in $B$ with the $p_i \in \rho(X)$ and the $x_i \in X^*$. By the definition of $B$, $p_i$ divides $x_i$ and $x_{i-1}$, for $i = 1, \ldots, k$, reading the subscripts modulo $k$. Hence there exist closed paths of length $k$ in both $\Delta$ and $\Gamma$. This implies that both $\Delta$ and $\Gamma$ contain cycles and $g(\Delta) \leq k, g(\Gamma) \leq k$.

If $g(\Delta) = \ell < k$, then there exists a closed path $P_\Delta = (p'_1, p'_2, \ldots, p'_\ell)$ in $\Delta$. By the definition of $\Delta$, for each $i$, there exists $x'_i \in X^*$ that is divisible by both $p'_i$ and $p'_{i+1}$, reading subscripts modulo $\ell$. If the $x'_i$ are pairwise distinct, then $P_B = (p'_1, x'_1, \ldots, p'_\ell, x'_\ell)$ is a closed path in $B$ of length $2\ell$, and $6 \leq 2\ell < 2k = g'(B)$, which is a contradiction. Hence the $x'_i$ are not all distinct. Let $i, j$ be such that $1 \leq i < j \leq \ell$ and $x'_i = x'_j$. Then in $\Delta$ the induced subgraph on the subset $\{p'_1, p'_{i+1}, p'_j, p'_{j+1}\}$ is a complete graph (of order 3 or 4) and hence $\ell = g(\Delta) = 3$. Thus either $g(\Delta) = 3$ or $g(\Delta) = k$. A similar proof shows that either $g(\Gamma) = 3$ or $g(\Gamma) = k$. 

We consider further, in Section 4, the case where both $\Gamma$ and $\Delta$ are acyclic, characterising the graphs $B$ in this case.

| $i$ | $X_i$ | $B$ | $\Delta$ | $\Gamma$ |
|-----|-------|-----|-----------|-----------|
| 1   | $\{2\}$ | $K_2$ | $K_1$ | $K_1$ |
| 2   | $\{2, 4, 8\}$ | $K_{1,3}$ | $K_1$ | $K_3$ |
| 3   | $\{105\}$ | $K_{3,1}$ | $K_3$ | $K_1$ |
| 4   | $\{11 \cdot 13, 11^2 \cdot 13\}$ | $C_4$ | $K_2$ | $K_2$ |
| 5   | $X_2 \cup X_3$ | $K_{1,3} + K_{3,1}$ | $K_1 + K_3$ | $K_1 + K_4$ |
| 6   | $X_2 \cup X_3$ | $K_{1,3} + C_4$ | $K_1 + K_2$ | $K_2 + K_3$ |
| 7   | $X_3 \cup X_4$ | $K_{3,1} + C_4$ | $K_2 + K_3$ | $K_1 + K_2$ |
4. Subgraphs of $B, \Delta, \Gamma$

In this section we prove several results that link existence of certain subgraphs in $B$ with the existence of related subgraphs in $\Delta$ and $\Gamma$. Let $G = (V,E)$ be a graph with vertex set $V$ and edge set $E$. By a subgraph of $G$, we mean a graph $G_0 = (V_0,E_0)$ where $V_0 \subseteq V$ and $E_0 \subseteq E \cap V_0^{(2)}$. If $E_0 = E \cap V_0^{(2)}$, then $G_0$ is called an induced subgraph.

4.1. Triangles in $\Delta$ and $\Gamma$. First we look at the existence of triangles (that is, 3-cycles, closed paths of length 3) in the graphs $\Delta$ and $\Gamma$.

**Theorem 4.1.** At least one of $\Delta, \Gamma$ contains a triangle if and only if $B$ contains $C_6$ or $K_{1,3}$ as an induced subgraph.

**Proof.** Suppose first that $g(\Gamma) = 3$ and let $P = (x_1,x_2,x_3)$ be a cycle in $\Gamma$. If there exists a prime $p$ which divides $x_i$, for all $i = 1, 2, 3$, then the set $\{p,x_1,x_2,x_3\}$ induces a subgraph $K_{1,3}$ of $B$. So we may assume that no such prime exists. Then, since $P$ is a cycle in $\Gamma$, there are distinct primes $p_1,p_2,p_3$ such that, for each $i$, $p_i$ divides $x_{i-1}$ and $x_i$, writing subscripts modulo 3, and the set $\{p_1,x_1,p_2,x_2,p_3,x_3\}$ induces a subgraph $C_6$ of $B$. Thus $g(\Gamma) = 3$ implies that $B$ contains an induced subgraph isomorphic to either $C_6$ or $K_{1,3}$. By Corollary 2.1 it follows that $g(\Delta) = 3$ implies that $B$ contains an induced subgraph isomorphic to either $C_6$ or $K_{1,3}$.

Conversely, if $\{p_1,x_1,p_2,x_2,p_3,x_3\}$ induces a subgraph $C_6$ in $B$, where the $p_i \in \rho(X)$ and the $x_i \in X^*$, then $(p_1,p_2,p_3)$ and $(x_1,x_2,x_3)$ are cycles in $\Delta$ and $\Gamma$ respectively, so $g(\Delta) = g(\Gamma) = 3$. Similarly if $B$ contains an induced subgraph $K_{1,3}$, then at least one of $\Delta, \Gamma$ contains a triangle. This completes the proof. $\Box$

4.2. Acyclic graphs. Next we characterise the cases where both $\Delta$ and $\Gamma$ are acyclic.

**Theorem 4.2.** Both the graphs $\Gamma$ and $\Delta$ are acyclic if and only if each connected component of $B$ is a path or a cycle of length 4.

**Proof.** Suppose first that $\Delta, \Gamma$ are both acyclic. If some vertex of $B$ lies on at least three edges, then one of $\Delta, \Gamma$ contains a 3-cycle, which is a contradiction. Thus each vertex of $B$ lies on at most two edges in $B$. Since $B$ is bipartite, this means that each connected component of $B$ is a path, or a cycle $C_{2k}$ of even length $2k \geq 4$.

Moreover, in the case of a component $C_{2k}$, it follows from Lemma 3.2 that $k = 2$.

Conversely, suppose that each component of $B$ is a path or isomorphic to $C_4$. For a component $C_4$ of $B$, the corresponding component of $\Delta, \Gamma$ is isomorphic to $K_2$. Consider a component $B'$ of $B$ which is a path. Suppose that $P_\Delta = (p_1,p_2,\ldots,p_\ell)$ is a cycle in the corresponding component of $\Delta$ of length $\ell \geq 3$. By the definition of $\Delta$, for each $i$, there exists $x_i \in X^*$ that is divisible by both $p_i$ and $p_{i+1}$, reading subscripts modulo $\ell$. If the $x_i$ are pairwise distinct, then $P_B = (p_1,x_1,\ldots,x_\ell)$ is a cycle in $B'$, which is a contradiction. Hence there exist $i,j$ such that $1 \leq i < j \leq \ell$ and $x_i = x_j$. This however implies that $x_i$ is joined to at least three vertices in $B'$, contradicting the fact that $B'$ is a path. Hence the component of $\Delta$ corresponding to $B'$ is acyclic. A similar proof shows that the component of $\Gamma$ corresponding to $B'$ is also acyclic. $\Box$
We have the following immediate corollary for the case where both $\Delta$ and $\Gamma$ are trees, where by a tree we mean a connected acyclic graph.

**Corollary 4.3.** Both graphs $\Gamma$ and $\Delta$ are trees if and only if either $B$ is a path or $B \cong C_4$.

### 4.3. Incidence graphs of complete graphs.

As preparation for our final theorem, we study the existence of incidence graphs of complete graphs as subgraphs of $B$.

**Definition 4.4.** Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. Then the incidence graph $\text{Inc}(G)$ of $G$, is the bipartite graph with vertex set $V \cup E$ such that $\{v, e\}$ forms an edge if and only if $v \in V$, $e \in E$ and $v$ is incident with $e$ in $G$.

**Lemma 4.5.** The graph $B$ contains a subgraph isomorphic to $\text{Inc}(K_\ell)$ if and only if one of the following conditions (i) or (ii) holds.

(i) $\Gamma$ contains a complete subgraph $K_\ell$ with vertices $\{x_1, x_2, \ldots, x_\ell\}$, and there are $\binom{\ell}{2}$ pairwise distinct primes $p_{ij}$, for $1 \leq i < j \leq \ell$, such that $p_{ij}$ divides $\gcd(x_i, x_j)$.

(ii) $\Delta$ contains a complete subgraph $K_\ell$ with vertices $\{p_1, p_2, \ldots, p_\ell\}$ and there are $\binom{\ell}{2}$ pairwise distinct numbers $x_{ij} \in X^*$, for $1 \leq i < j \leq \ell$, such that $p_ip_j$ divides $x_{ij}$.

**Proof.** Suppose that condition (i) holds and let $\mathcal{G}$ be the subgraph of $B$ with $V(\mathcal{G}) = \{x_1, x_2, \ldots, x_\ell\} \cup \{p_{ij}, 1 \leq i < j \leq \ell\}$ and edges $\{p_{ij}, x_i\}$ and $\{p_{ij}, x_j\}$ for each $i, j$. (Note that $\mathcal{G}$ may not be an induced subgraph.) Let $K_\ell$ be the complete graph on $V = \{1, 2, \ldots, \ell\}$ with edge set $E = \{e_{ij} = \{i, j\}|1 \leq i < j \leq \ell\}$. Define $\Psi : \text{Inc}(K_\ell) \rightarrow \mathcal{G}$ by $\Psi : i \rightarrow x_i, e_{ij} \rightarrow p_{ij}$. It is straightforward to check that $\Psi$ is a graph isomorphism. A similar isomorphism can be constructed if condition (ii) holds.

Conversely suppose that $B$ contains a subgraph $\mathcal{M}$ isomorphic to $\text{Inc}(K_\ell)$. Then $\mathcal{M}$ is connected and bipartite, and hence one of its bipartite halves, say $V$, has size $\ell$, and each pair of its elements are at distance two in $B$. Moreover $V$ must be contained in $X^*$ or $\rho(X)$, and hence induce a complete subgraph $K_\ell$ of $\Gamma$ or $\Delta$, respectively. It is straightforward to check the remaining assertions of the condition (i) or (ii) respectively. $\square$

### 4.4. Complete subgraphs $K_4$ of $\Delta$ and $\Gamma$.

In this final subsection we show in Theorem 4.6 that existence of a complete subgraph $K_4$ of $\Delta$ or $\Gamma$ is equivalent to existence of at least one of a small number of possible subgraphs of $B$.

To demonstrate that each of the cases of Theorem 4.6 does indeed occur, we give in Tables 3 and 4 examples of small subsets $X$ for the various cases. In Table 3, $p, p_1, \ldots, p_6$ denote primes such that $p_i \neq p_j$ for $i \neq j$. Also $L(K_4)$ denotes the line graph of $K_4$, with vertex set $E(K_4)$ and two vertices adjacent if and only if the corresponding edges of $K_4$ have a vertex in common. We denote by $K_{a,b}$ a complete bipartite subgraph of $B(X)$ with $a$ vertices in $\rho(X)$ and $b$ vertices in $X^*$. Finally, recall the definition of $\text{Inc}(K_\ell)$ from Definition 4.4 and let $\mathcal{K}, \mathcal{G}$ denote the graphs in Figure 2.
Table 3. Small examples for each case of Theorem 4.6 with $\Gamma(X) = K_4$.

| $X$          | $B$         | $\Gamma$ | $\Delta$ |
|--------------|-------------|-----------|----------|
| $\{p, p^2, p^3, p^4\}$ | $K_{4,4}$ | $K_4$     | $K_1$    |
| $\{p_1p_2, p_1p_3, p_1p_4, p_2p_3\}$ | $K$       | $K_4$     | $K_3$    |
| $\{p_1p_2, p_1p_3, p_1p_4, p_2p_3p_4\}$ | $\mathcal{G}$ | $K_4$     | $K_4$    |
| $\{p_1p_2p_3, p_1p_4p_5, p_2p_4p_6, p_3p_5p_6\}$ | $\text{Inc}(K_4)$ | $K_4$     | $L(K_4)$ |

Table 4. Small examples for each case of Theorem 4.6 with $\Delta(X) = K_4$.

| $X$          | $B(X)$       | $\Gamma(X)$ | $\Delta(X)$ |
|--------------|--------------|--------------|--------------|
| $\{p_1p_2p_3p_4\}$ | $K_{4,4}$ | $K_1$        | $K_4$        |
| $\{p_1p_2p_3, p_1p_4, p_2p_3p_4\}$ | $K$       | $K_3$        | $K_4$        |
| $\{p_1p_2p_3, p_2p_4, p_3p_4, p_1p_4\}$ | $\mathcal{G}$ | $K_4$        | $K_4$        |
| $\{p_1p_2, p_1p_3, p_1p_4, p_2p_3, p_2p_4, p_3p_4\}$ | $\text{Inc}(K_4)$ | $L(K_4)$     | $K_4$        |

**Figure 2.** The graph $\mathcal{K}$ (to the left) and $\mathcal{G}$ (to the right) of Theorem 4.6.

**Theorem 4.6.** (i) If $\Delta$ has a subgraph $K_4$, then $B$ contains a subgraph isomorphic to one of $K_{4,4}$, $\text{Inc}(K_4)$, $\mathcal{K}$ or $\mathcal{G}$.

(ii) If $\Gamma$ has a subgraph $K_4$, then $B$ contains a subgraph isomorphic to one of $K_{4,4}$, $\text{Inc}(K_4)$, $\mathcal{K}$ or $\mathcal{G}$.

(iii) If $B$ contains a subgraph isomorphic to one of $K_{4,4}$, $K_{4,4}$, $\text{Inc}(K_4)$, $\mathcal{K}$ or $\mathcal{G}$, then at least one of $\Delta$ or $\Gamma$ has a subgraph $K_4$.

**Proof.** (i) Suppose that $\pi = \{p_1, p_2, p_3, p_4\} \subseteq \rho(X)$ induces a subgraph $K_4$ of $\Delta$. If there exists $x \in X$ divisible by $\prod_{i=1}^{4} p_i$, then the subgraph of $B$ induced on $\pi \cup \{x\}$ is $K_{4,4}$. Thus we may assume that no such $x$ exists. By the definition of $\Delta$, for each $i, j$ satisfying $1 \leq i < j \leq 4$, there exists $x_{ij} \in X$ such that $p_ip_j$ divides $x_{ij}$.

Suppose next that some element $x \in X$ is divisible by three of the $p_i$, without loss of generality, that $x$ is divisible by $p_1p_2p_3$. If $x_{14}, x_{24}, x_{34}$ are all distinct, then the subgraph of $B$ induced on $\pi \cup \{x, x_{14}, x_{24}, x_{34}\}$ contains the graph $\mathcal{G}$ of Figure 2. If this is not the case then, without loss of generality, $x_{14} = x_{24}$, and this number is therefore divisible by $p_1p_2p_4$. By our assumption, $p_3$ does not divide $x_{14}$, and
hence \( x_{34} \neq x_{14} \), and the subgraph of \( B \) induced on \( \pi \cup \{ x, x_{14}, x_{34} \} \) contains the graph \( K \) of Figure 2.

Thus we may assume that no element of \( X \) is divisible by more than two primes in \( \pi \), and hence that the \( x_{ij} \) are pairwise distinct. It now follows from Lemma 4.5 that \( B \) contains a subgraph isomorphic to \( \text{Inc}(K_4) \).

(ii) Suppose that \( X_0 \{ x_1, x_2, x_3, x_4 \} \subseteq X^* \) induces a subgraph \( K_4 \) of \( \Gamma \). By Corollary 2.1 there is a set \( Y \) and a graph isomorphism \( \phi : B(X) \rightarrow B(Y) \) that induces an isomorphism \( \Gamma(X) \cong \Delta(Y) \). Thus \( \Delta(Y) \) has an induced subgraph \( K_4 \), and hence part (ii) follows from part (i).

(iii) Finally suppose that \( B \) contains a subgraph \( H \) isomorphic to one of \( K_{1,1} \), \( K_{4,1} \), \( \text{Inc}(K_4) \), \( K \) or \( G \). Then \( H \) is connected and bipartite, and the distance 2 graph induced on one of its bipartite halves is isomorphic to \( K_4 \). Thus \( \Delta \) or \( \Gamma \) has a subgraph isomorphic to \( K_4 \). \( \square \) \( \square \)

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