Statistical inference for expectile-based risk measures

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Abstract

Expectiles were introduced by Newey and Powell [43] in the context of linear regression models. Recently, Bellini et al. [6] revealed that expectiles can also be seen as reasonable law-invariant risk measures. In this article, we show that the corresponding statistical functionals are continuous w.r.t. the 1-weak topology and suitably functionally differentiable. By means of these regularity results we can derive several properties such as consistency, asymptotic normality, bootstrap consistency, and qualitative robustness of the corresponding estimators in nonparametric and parametric statistical models.

Keywords: Expectile-based risk measure; 1-weak continuity; Quasi-Hadamard differentiability; Statistical estimation; Weak dependence; Strong consistency; Asymptotic normality; Bootstrap consistency; Qualitative robustness; Functional delta-method

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1. Introduction

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be an atomless probability space and use \( L^p = L^p(\Omega, \mathcal{F}, \mathbb{P}) \) to denote the usual \( L^p \)-space. The \( \alpha \)-expectile of \( X \in L^2 \), with \( \alpha \in (0, 1) \), can uniquely be defined by

\[
\rho_\alpha(X) := \arg\min_{m \in \mathbb{R}} \left\{ \alpha \mathbb{E}[(X - m)^+] + (1 - \alpha) \mathbb{E}[(m - X)^+] \right\} = \arg\min_{m \in \mathbb{R}} \mathbb{E}[V_\alpha(X - m)]
\]  

(see Proposition 1 and Example 4 in [6]), where

\[
V_\alpha(x) := \begin{cases} 
\alpha x^2 , & x \geq 0 \\
(1 - \alpha) x^2 , & x < 0 
\end{cases}, \quad x \in \mathbb{R}.
\]

Expectiles were introduced by Newey and Powell [43] in the context of linear regression models. On the one hand, (1) generalizes the expectation of \( X \) which coincides with \( \rho_{\alpha}(X) \) when specifically \( \alpha = 1/2 \). On the other hand, (1) is similar to the \( \alpha \)-quantile of \( X \) which can be obtained by replacing \( x^2 \) by \( |x| \) in the definition of \( V_\alpha \). This motivates the name \( \alpha \)-expectile.

For every \( X \in L^2 \) the mapping \( m \mapsto \mathbb{E}[V_\alpha(X - m)] \) is convex and differentiable with derivative given by \( m \mapsto -2U_\alpha(X)(m) \), where

\[
U_\alpha(X)(m) := \mathbb{E}[U_\alpha(X - m)], \quad m \in \mathbb{R}
\]  

(2)

with

\[
U_\alpha(x) := \begin{cases} 
\alpha x , & x \geq 0 \\
(1 - \alpha) x , & x < 0 
\end{cases}, \quad x \in \mathbb{R}.
\]

Moreover, for \( X \in L^1 \) the mapping \( m \mapsto U_\alpha(X)(m) \) is well defined and bijective; cf. Lemma A.1 (Appendix A). These observations together imply that for \( X \in L^2 \) the \( \alpha \)-expectile admits the representation

\[
\rho_\alpha(X) = U_\alpha(X)^{-1}(0),
\]  

(3)

where \( U_\alpha(X)^{-1} \) denotes the inverse function of \( U_\alpha(X) \). In particular, (3) can be used to define a map \( \rho_\alpha : L^1 \rightarrow \mathbb{R} \) which is compatible with (1). For every \( X \in L^1 \) the value in (3) will be called the corresponding \( \alpha \)-expectile.

Recently, Bellini et al. [6] revealed that expectiles can be also seen as reasonable risk measures when \( 1/2 \leq \alpha < 1 \). In Proposition 6 in [6], they prove that the map \( \rho_\alpha : L^2 \rightarrow \mathbb{R} \) provides a coherent risk measure if (and only if) \( 1/2 \leq \alpha < 1 \). Recall that a map \( \rho : \mathcal{X} \rightarrow \mathbb{R} \), with \( \mathcal{X} \) a subspace of \( L^0 \), is said to be a coherent risk measure if it is

- monotone: \( \rho(X_1) \leq \rho(X_2) \) for all \( X_1, X_2 \in \mathcal{X} \) with \( X_1 \leq X_2 \),
- cash-invariant: \( \rho(X + m) = \rho(X) + m \) for all \( X \in \mathcal{X} \) and \( m \in \mathbb{R} \),
- subadditive: \( \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2) \) for all \( X_1, X_2 \in \mathcal{X} \),
• positively homogenous: $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \geq 0$.

It is shown in the Appendix A (Proposition A.2) that even the map $\rho_\alpha : L^1 \rightarrow \mathbb{R}$ provides a coherent risk measure if (and only if) $1/2 \leq \alpha < 1$. For $0 < \alpha < 1/2$ the map $\rho_\alpha : L^1 \rightarrow \mathbb{R}$ is at least monotone, cash-invariant, and positively homogeneous. For this reason we will henceforth refer to $\rho_\alpha : L^1 \rightarrow \mathbb{R}$ as expectile-based risk measure at level $\alpha \in (0, 1)$. It is worth mentioning that $\rho_\alpha$ already appeared implicitly in an earlier paper by Weber [49]. As Ziegel [51] pointed out that $\rho_\alpha$ satisfies a particularly desirable property of risk measures in the context of backtesting, $\rho_\alpha$ attracted special attention in the field of monetary risk measurement in the last few years [1, 5, 6, 22, 24, 51]. For pros and cons of expectile-based risk measures and of other standard risk measures see, for instance, the discussions by Acerbi and Szekely [1], Bellini and Di Bernardino [5], and Emmer et al. [24].

This article is concerned with the statistical estimation of expectile-based risk measures. The goal is the estimation of $\rho_\alpha(X)$ for some $X \in L^1$ with unknown distribution function $F$. Let $\mathcal{F}_1$ be the class of all distribution functions on $\mathbb{R}$ satisfying $\int |x| \, dF(x) < \infty$. Note that $\mathcal{F}_1$ coincides with the set of the distribution functions of all elements of $L^1$, because the underlying probability space was assumed to be atomless. Also note that $F \in \mathcal{F}_1$ if and only if $\int_{-\infty}^{0} F(x) \, dx < \infty$ and $\int_{0}^{\infty} (1 - F(x)) \, dx < \infty$ hold. Since $\rho_\alpha$ is law-invariant (i.e. $\rho_\alpha(X_1) = \rho_\alpha(X_2)$ when $\mathbb{P} \circ X_1^{-1} = \mathbb{P} \circ X_2^{-1}$), we may associate with $\rho_\alpha$ a statistical functional $\mathcal{R}_\alpha : \mathcal{F}_1 \rightarrow \mathbb{R}$ via

$$\mathcal{R}_\alpha(F_X) := \rho_\alpha(X), \quad X \in L^1,$$

where $F_X$ denotes the distribution function of $X$. That is,

$$\mathcal{R}_\alpha(F) = \mathcal{U}_\alpha(F)^{-1}(0) \quad \text{for all } F \in \mathcal{F}_1,$$

where

$$\mathcal{U}_\alpha(F)(m) := \int U_\alpha(x - m) \, dF(x), \quad m \in \mathbb{R}.\quad (6)$$

Then, if $\hat{F}_n$ is a reasonable estimator for $F$, the plug-in estimator $\mathcal{R}_\alpha(\hat{F}_n)$ is typically a reasonable estimator for $\rho_\alpha(X) = \mathcal{R}_\alpha(F)$.

In a nonparametric framework, a canonical example for $\hat{F}_n$ is the empirical distribution function

$$\hat{F}_n := \frac{1}{n} \sum_{i=1}^{n} 1_{[X_i, \infty)}$$

of $n$ identically distributed random variables $X_1, \ldots, X_n$ drawn according to $F$. In this case we have

$$\mathcal{R}_\alpha(\hat{F}_n) = \mathcal{U}_\alpha(\hat{F}_n)^{-1}(0) = \text{unique solution in } m \text{ of } \sum_{i=1}^{n} U_\alpha(X_i - m) = 0.\quad (8)$$
That is, the plug-in estimator is nothing but a simple Z-estimator (M-estimator). For Z-estimators (M-estimators) there are several results concerning consistency and the asymptotic distribution in the literature. A classical reference is Huber’s seminal paper [30]; see also standard textbooks as [31, 46, 47, 48]. Recently Holzmann and Klar [29] used results of Arcones [3] and Van der Vaart [47] to derive asymptotic properties of the Z-estimator in (8). They restricted their attention to i.i.d. observations but allowed for observations without finite second moment.

On the other hand, even in the nonparametric setting the estimator \( \hat{F}_n \) may differ from the empirical distribution function so that the plug-in estimator need not be a Z-estimator. See, for instance, Section 3 in [7] for estimators \( \hat{F}_n \) being different from the empirical distribution function. Also, in a parametric setting the estimator \( \hat{F}_n \) will hardly be the empirical distribution function. For these reasons, we will consider a suitable linearization of the functional \( R_\alpha \) in order to be in the position to derive several asymptotic properties of the plug-in estimator \( R_\alpha(\hat{F}_n) \) in as many as possible situations.

Linearizations of Z-functionals have been considered before, for instance, by Clarke [17, 18]. However, these results do not cover the particular Z-functional \( R_\alpha \), because the function \( U_\alpha \) is unbounded. By using the concept of quasi-Hadamard differentiability as well as the corresponding functional delta-method introduced by Beutner and Zähle [7, 8] we will overcome the difficulties with the unboundedness of \( U_\alpha \). Quasi-Hadamard differentiability of the functional \( R_\alpha \) will in particular admit some bootstrap results for the plug-in estimator \( R_\alpha(\hat{F}_n) \) when \( \hat{F}_n \) is the empirical distribution function of \( X_1, \ldots, X_n \).

It is worth mentioning that Heesterman and Gill [28] also considered a linearization approach to Z-estimators. However (boiled down to our setting) they did not consider a linearization of the functional \( R_\alpha \) (to be evaluated at the estimator \( \hat{F}_n \) of \( F \)) but only of the functional that provides the unique zero of a strictly decreasing and continuous function tending to \( \pm \infty \) as its argument tends to \( \pm \infty \) (as the function \( U_\alpha(\hat{F}_n) \)). To some extent this approach is less flexible than our approach. Especially parametric estimators cannot be handled by this approach without further ado.

The rest of this article is organized as follows. In Section 2 we will establish a certain continuity and the above-mentioned differentiability of the functional \( R_\alpha \). In Sections 3–4 we will apply the results of Section 2 to the nonparametric and parametric estimation of \( R_\alpha(F) \). In Section 5 we will prove the main result of Section 2 and in Section 6 we will verify two examples and a lemma presented in Sections 3–4. The Appendix provides some auxiliary results. In particular, in Section B of the Appendix we formulate a slight generalization of the functional delta-method in the form of Beutner and Zähle [8].

## 2. Regularity of the functional \( R_\alpha \)

In this section we investigate the functional \( R_\alpha : F_1 \to \mathbb{R} \) defined in (5) for continuity and differentiability. We equip \( F_1 \) with the 1-weak topology. This topology is defined
to be the coarsest topology for which the mappings $\mu \mapsto \int f \, dF$, $f \in C_1$, are continuous, where $C_1$ is the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $|f(x)| \leq C_f (1 + |x|)$ for all $x \in \mathbb{R}$ and some finite constant $C_f > 0$. A sequence $(F_n) \subseteq F_1$ converges 1-weakly to some $F_0 \in F_1$ if and only if $\int f \, dF_n \to \int f \, dF_0$ for all $f \in C_1$; cf. Lemma 3.4 in [33]. The set $F_1$ can obviously be identified with the set of all Borel probability measures $\mu$ on $\mathbb{R}$ satisfying $\int |x| \, \mu(dx) < \infty$. In this context the 1-weak topology is sometimes referred to as $\psi_1$-weak topology; see, for instance, [34]. But for our purposes it is more convenient to work with the $F_1$-terminology.

Let $L_0$ be the space of all Borel measurable functions $v : \mathbb{R} \to \mathbb{R}$ modulo the equivalence relation of $\ell$-almost sure identity. Note that $F_1 \subseteq L_0$, and let $L_1 \subseteq L_0$ be the subspace of all $v \in L_0$ for which

$$
\|v\|_{1, \ell} := \int |v(x)| \, \ell(dx)
$$

is finite. Here, and henceforth, $\ell$ stands for the Borel Lebesgue measure on $\mathbb{R}$. Note that $F_1 - F_2 \in L_1$ for $F_1, F_2 \in F_1$. It is well-known that $\| \cdot \|_{1, \ell} : L_1 \to \mathbb{R}_+$ provides a complete and separable norm on $L_1$ and that

$$
d_{W,1}(F_1, F_2) := \|F_1 - F_2\|_{1, \ell}
$$

defines the Wasserstein-1 metric $d_{W,1} : F_1 \times F_1 \to \mathbb{R}_+$ on $F_1$. Also note that $d_{W,1}$ metrizes the 1-weak topology on $F_1$; cf. Remark 2.9 in [34].

### 2.1. Continuity

Since the Wasserstein-1 metric $d_{W,1}$ metrizes the 1-weak topology on $F_1$, the following theorem is an immediate consequence of a recent result by Bellini et al. [6, Theorem 10].

**Theorem 2.1** The functional $\mathcal{R}_\alpha : F_1 \to \mathbb{R}$ is continuous for the 1-weak topology.

Theorem 2.1 can also be obtained by combining Theorem 4.1 in [16] with the Representation theorem 3.5 in [34]. Indeed, these two theorems together imply that the risk functional associated with any law-invariant coherent risk measure on $L^1$ is 1-weakly continuous. For $1/2 \leq \alpha < 1$ the functional $\mathcal{R}_\alpha$ itself is derived from a law-invariant coherent risk measure (see Proposition A.2 in Appendix A). So it is 1-weakly continuous. For $0 < \alpha < 1/2$ the map $\hat{\rho}_\alpha : L^1 \to \mathbb{R}$ defined by $\hat{\rho}_\alpha(X) := -\rho_\alpha(-X)$ provides a law-invariant coherent risk measure (cf. Proposition A.2 in Appendix A), so that the associated statistical functional $\mathcal{R}_\alpha : F_1 \mapsto \mathbb{R}$, $\mathcal{R}_\alpha(F) = -\mathcal{R}_\alpha(\hat{F})$, is 1-weakly continuous. Here $\hat{F}$ stands for the distribution function derived from $F$ via $\hat{F}(x) := 1 - F((-x) -)$. Since for any sequence $(F_n)_{n \in \mathbb{N}} \subseteq F_1$, $F_n \to F_0$ 1-weakly if and only if $\hat{F}_n \to \hat{F}_0$ 1-weakly, it follows that also the functional $\mathcal{R}_\alpha$ is 1-weakly continuous.

By the 1-weak continuity of $\mathcal{R}_\alpha$ we are in the position to easily derive strong consistency of the plug-in estimator $\mathcal{R}_\alpha(\hat{F}_n)$ for $\mathcal{R}_\alpha(F)$ in several situations; see Sections 3.1 and 4.4.
2.2. Differentiability

We will use the notion of quasi-Hadamard differentiability introduced in [7, 8]. Quasi-Hadamard differentiability is a slight (but useful) generalization of the conventional tangential Hadamard differentiability. The latter is commonly acknowledged to be a suitable notion of differentiability in the context of the functional delta-method (see e.g. the bottom of p. 166 in [28]), and it was shown in [7, 8] that the former is still strong enough to obtain a functional delta-method. Let \( L_1 \) be equipped with the norm \( \| \cdot \|_{1,\ell} \).

**Definition 2.2** Let \( \mathcal{R} : F_1 \to \mathbb{R} \) be a map and \( L^0_1 \) be a subset of \( L_1 \). Then \( \mathcal{R} \) is said to be quasi-Hadamard differentiable at \( F \in F_1 \) tangentially to \( L^0_1 \) with \( \mathcal{L}_1 \) \( \ell \)-norm if there exists a continuous map \( \dot{\mathcal{R}}_F(v) : L^0_1 \to \mathbb{R} \) such that

\[
\lim_{n \to \infty} \left| \dot{\mathcal{R}}_F(v) - \frac{\mathcal{R}(F + \varepsilon_nv) - \mathcal{R}(F)}{\varepsilon_n} \right| = 0
\]

holds for each triplet \((v, (v_n), (\varepsilon_n))\), with \( v \in L^0_1 \), \( (\varepsilon_n) \subseteq (0, \infty) \) satisfying \( \varepsilon_n \to 0 \), \( (v_n) \subseteq L_1 \) satisfying \( \|v_n - v\|_{1,\ell} \to 0 \) as well as \( (F + \varepsilon_nv) \subseteq F_1 \). In this case the map \( \dot{\mathcal{R}}_F \) is called quasi-Hadamard derivative of \( \mathcal{R} \) at \( F \) tangentially to \( L^0_1 \) with \( \mathcal{L}_1 \) \( \ell \)-norm.

Note that even when \( L^0_1 = L_1 \), quasi-Hadamard differentiability of \( \mathcal{R} \) at \( F \) tangentially to \( L^0_1 \) is not the same as Hadamard differentiability of \( \mathcal{R} \) at \( F \) tangentially to \( L_1 \) (with \( L_0 \) regarded as the basic linear space containing both \( F_1 \) and \( L_1 \)). Indeed, \( \| \cdot \|_{1,\ell} \) does not impose a norm on all of \( L_0 \) (but only on \( L_1 \)), so that Hadamard differentiability w.r.t. the norm \( \| \cdot \|_{1,\ell} \) is not defined.

**Theorem 2.3** Let \( F \in F_1 \) and assume that it is continuous at \( \mathcal{R}_\alpha(F) \). Then the functional \( \mathcal{R} : F_1 \to \mathbb{R} \) is quasi-Hadamard differentiable at \( F \) tangentially to \( L^0_1 \) with linear quasi-Hadamard derivative \( \dot{\mathcal{R}}_{\alpha,F} : L^0_1 \to \mathbb{R} \) given by

\[
\dot{\mathcal{R}}_{\alpha,F}(v) := -\frac{(1 - \alpha) \int_{(-\infty,0)} v(x + \mathcal{R}_\alpha(F)) \ell(dx) + \alpha \int_{(0,\infty)} v(x + \mathcal{R}_\alpha(F)) \ell(dx)}{(1 - 2\alpha)F(\mathcal{R}_\alpha(F)) + \alpha}. \tag{11}
\]

Note that \((1 - 2\alpha)F(\mathcal{R}_\alpha(F)) + \alpha = (1 - \alpha)F(\mathcal{R}_\alpha(F)) + \alpha(1 - F(\mathcal{R}_\alpha(F))) > 0\) holds so that the denominator in (11) is strictly positive. Also note that quasi-Hadamard differentiability is already known form Theorem 2.4 in [33]. However, in [35] the derivative was not specified explicitly. The proof of Theorem 2.3 can be found in Section 5.

**Remark 2.4** As a direct consequence of Theorem 2.3 we obtain that the functional \( \mathcal{R}_\alpha \) is also quasi-Hadamard differentiable at \( F \) (being continuous at \( \mathcal{R}_\alpha(F) \)) tangentially to any subspace of \( L_1 \) that is equipped with a norm being at least as strict as the norm \( \| \cdot \|_{1,\ell} \).

\[ \Diamond \]
Example 2.5 To illustrate Remark 2.4 let \( \phi : \mathbb{R} \to [1, \infty) \) be a continuous function that is non-increasing on \((-\infty, 0]\) and non-decreasing on \([0, \infty)\). Let \( F_\alpha \) be the set of all distribution functions \( F \) on \( \mathbb{R} \) for which \( \| F - \mathbb{1}_{[0,\infty)} \|_\phi < \infty \), where \( \| v \|_\phi := \sup_{x \in \mathbb{R}} |v(x)| \phi(x) \). Let \( D \) be the space of all bounded càdlàg functions on \( \mathbb{R} \) and \( D_\phi \) be the subspace of all \( v \in D \) satisfying \( \| v \|_\phi < \infty \) and \( \lim_{|x| \to \infty} v(x) = 0 \). If \( C_\phi := \int 1/\phi \, d\ell < \infty \), then \( D_\phi \subseteq L_1 \) and \( F_\phi \subseteq F_1 \). On the space \( D_\phi \), the norm \( \| \cdot \|_\phi \) is stricter than \( \| \cdot \|_{1,\ell} \), because

\[
\| v \|_{1,\ell} = \int |v(x)| \, \ell(dx) \leq C_\phi \| v \|_\phi \quad \text{for every } v \in D_\phi.
\]

Therefore \( R_\alpha \) is also quasi-Hadamard differentiable at \( F \) tangentially to \( D_\phi(D_\phi) \) with linear quasi-Hadamard derivative \( \tilde{R}_{\alpha,F} : D_\phi \to \mathbb{R} \) given by (11) restricted to \( v \in D_\phi \), where \( D_\phi \) is equipped with the norm \( \| \cdot \|_\phi \). \( \Diamond \)

The established quasi-Hadamard differentiability of \( R_\alpha \) brings us in the position to easily derive results on the asymptotics of \( R_\alpha(\hat{F}_n) \); see Sections 3.2, 3.3, and 4.2. In Section 4.2 we combine Theorem 2.3 with a central limit theorem (by Dede [19]; cf. Theorem [4,3] below) for the empirical process in the space \( (L_1, \| \cdot \|_{1,\ell}) \) in order to obtain the asymptotic distribution of \( R_\alpha(\hat{F}_n) \) in a rather general nonparametric setting. In view of Example 2.5 one can alternatively use central limit theorems for the empirical process in the space \( (D_\phi, \| \cdot \|_\phi) \) to obtain the asymptotic distribution of \( R_\alpha(\hat{F}_n) \). See, for instance, Examples 4.4–4.5 in [8] as well as references cited there.

### 3. Nonparametric estimation of \( R_\alpha(F) \)

In this section we consider nonparametric statistical models. We will always assume that the sequence of observations \( (X_i) \) is a strictly stationary sequence of real-valued random variables. In addition we will mostly assume that \( (X_i) \) is ergodic; see Section 6.1 and 6.7 in [14] for the definition of a strictly stationary and ergodic sequence. Recall that every sequence of i.i.d. random variables is strictly stationary and ergodic. Moreover a strictly stationary sequence is ergodic when it is mixing in the ergodic sense, and it is mixing in the ergodic sense when it is \( \alpha \)-mixing; see Section 2.5 in [13]. For illustration, also note that many GARCH processes are \( \alpha \)-mixing; cf. [11, 42].

Throughout this section the estimator for the marginal distribution function \( F \) of \( (X_i) \) is assumed to be the empirical distribution function \( \hat{F}_n \) of \( X_1, \ldots, X_n \) as defined in (7). Note that the mapping \( \Omega \to F_1, \omega \mapsto \hat{F}_n(\omega, \cdot) \), is \( (\mathcal{F}, \mathcal{B}(F_1)) \)-measurable for the Borel \( \sigma \)-algebra \( \mathcal{B}(F_1) \) on \( (F_1, d_{W,1}) \), because the mapping \( \mathbb{R}^n \to F_1, (x_1, \ldots, x_n) \mapsto \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(x_i, \infty)} \) is \( (\| \cdot \|_1, d_{W,1}) \)-continuous. Hence by continuity of \( R_\alpha \) w.r.t. \( d_{W,1} \), it follows that \( R_\alpha(\hat{F}_n) \) is a real-valued random variable on \( (\Omega, \mathcal{F}, \mathbb{P}) \).
3.1. Strong consistency

For $1/2 \leq \alpha < 1$ the following theorem is a direct consequence of Theorem 2.6 in [34]. In the general case, Theorem 3.1 ensures that one can follow the lines in the proof of Theorem 2.6 in [34] to obtain the assertion of Theorem 3.1. We omit the details.

**Theorem 3.1** Let $(X_i)$ be a strictly stationary and ergodic sequence of $L^1$-random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $F$ the distribution function of the $X_i$. Let $\hat{R}_n$ be the empirical distribution function of $X_1, \ldots, X_n$. Then the plug-in estimator $R_{\alpha}(\hat{R}_n)$ is strongly consistent for $R_{\alpha}(F)$ in the sense that

$$R_{\alpha}(\hat{R}_n) \to R_{\alpha}(F) \quad \text{P-a.s.}$$

If $X_1, X_2, \ldots$ are i.i.d. random variables, then strong consistency can also be obtained from classical results on Z-estimators as, for example, Lemma A in Section 7.2.1 of [46]. Moreover, it was shown recently by Holzmann and Klar [29, Theorem 2] that in the i.i.d. case one even has $\sup_{\alpha \in [\alpha_\ell, \alpha_u]} |R_{\alpha}(\hat{R}_n) \to R_{\alpha}(F)| \to 0 \text{ P-a.s.}$ for any $\alpha_\ell, \alpha_u \in (0, 1)$ with $\alpha_\ell < \alpha_u$.

3.2. Asymptotic distribution

Dedecker and Prieur [20] introduced the following dependence coefficients for a strictly stationary sequence of real-valued random variables $(X_i) \equiv (X_i)_{i \in \mathbb{N}}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$:  

$$\tilde{\phi}(n) := \sup_{k \in \mathbb{N}} \sup_{x \in \mathbb{R}} \| \mathbb{P}[X_{n+k} \in (-\infty, x]|\mathcal{F}_i^k]\cdot - \mathbb{P}[X_{n+k} \in (-\infty, x)]\|_{\infty}, \quad (13)$$

$$\tilde{\alpha}(n) := \sup_{k \in \mathbb{N}} \sup_{x \in \mathbb{R}} \| \mathbb{P}[X_{n+k} \in (-\infty, x]|\mathcal{F}_i^k]\cdot - \mathbb{P}[X_{n+k} \in (-\infty, x)]\|_{1}. \quad (14)$$

Here $\mathcal{F}_i^k := \sigma(X_1, \ldots, X_k)$ and $\| \cdot \|_p$ denotes the usual $L^p$-norm on $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \in [1, \infty]$. Note that by Proposition 3.22 in [12] the usual $\phi$- and $\alpha$-mixing coefficients $\phi(n)$ and $\alpha(n)$ can be represented as in (13)–(14) with $\sup_{x \in \mathbb{R}}$ and $(-\infty, x]$ replaced by $\sup_{A \in \mathcal{B}(\mathbb{R})}$ and $A$, respectively. In particular, $\tilde{\phi}(n) \leq \phi(n)$ and $\tilde{\alpha}(n) \leq \alpha(n)$. It is worth mentioning that in [20] the starting point is actually a strictly stationary sequence of random variables indexed by $\mathbb{Z}$ (rather than $\mathbb{N}$) and that therefore the definitions of the above dependence coefficients are slightly different. However, it is discussed in detail in the Appendix that any strictly stationary sequence $(X_i) \equiv (X_i)_{i \in \mathbb{N}}$ can be extended to a strictly stationary sequence $(Y_i)_{i \in \mathbb{Z}}$ satisfying $\tilde{\phi}(n) = \phi(n)$ and $\tilde{\alpha}(n) = \alpha(n)$, where $\phi(n)$ and $\alpha(n)$ are the dependence coefficients of $(Y_i)_{i \in \mathbb{Z}}$ as originally introduced in [20].

It is also discussed in the Appendix that if $(X_i)$ is in addition ergodic, then $(Y_i)_{i \in \mathbb{Z}}$ is ergodic too.

Let us denote by $Q_{|X_1|}$ the càdlàg inverse of the tail function $x \mapsto \mathbb{P}[|X_1| > x]$. Let us write $N_{0,s^2}$ for the centered normal distribution with variance $s^2$. Moreover, let us use $\sim$ to denote convergence in distribution.
Theorem 3.2 Let \((X_i)\) be a strictly stationary and ergodic sequence of real-valued random variables on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Denote by \(F\) the distribution function of the \(X_i\), and assume that \(F\) is continuous at \(R_\alpha(F)\) and that \(\int \sqrt{F(1-F)} \, d\ell < \infty\) (in particular \(F \in \mathcal{F}_1\)). Let \(\hat{F}_n\) be the empirical distribution function of \(X_1, \ldots, X_n\) as defined in [7]. Finally assume that one of the following two conditions holds:

\[
\sum_{n \in \mathbb{N}} n^{-1/2} \phi(n)^{1/2} < \infty, \tag{15}
\]

\[
\sum_{n \in \mathbb{N}} n^{-1/2} \int_{(0,1]} (u-1/2) \ell(u) \, du < \infty. \tag{16}
\]

Then

\[
\sqrt{n}(R_\alpha(\hat{F}_n) - R_\alpha(F)) \overset{d}{\to} Z_F \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}))
\]

for \(Z_F \sim N_{0,s^2}\) with

\[
s^2 = s^2_{\alpha,F} := \int_{\mathbb{R}^2} f_{\alpha,F}(t_0) C_F(t_0, t_1) f_{\alpha,F}(t_1) (\ell \otimes \ell)(d(t_0, t_1)), \tag{17}
\]

where

\[
f_{\alpha,F}(t) := \frac{1}{(1-2\alpha)F'(R_\alpha(F)) + \alpha} \Big( (1-\alpha)1_{(-\infty, R_\alpha(F))}(t) + \alpha 1_{[R_\alpha(F), \infty)}(t) \Big), \tag{18}
\]

\[
C_F(t_0, t_1) := F(t_0 \wedge t_1)(1 - F(t_0 \vee t_1)) + \sum_{i=0}^\infty \sum_{k=2}^\infty \text{Cov}(1_{\{X_i \leq t_i\}}, 1_{\{X_k \leq t_{k-1}\}}). \tag{19}
\]

Proof Theorem 2.3 shows that \(R_\alpha\) is quasi-Hadamard differentiable at \(F\) tangentially to \(L_1(L_1)\) (w.r.t. the norm \(\| \cdot \|_{1,t}\)) with quasi-Hadamard derivative \(\dot{R}_{\alpha,F}\) given by (11). The functional delta-method in the form of Theorem B.3(i) and Theorem C.3 then imply that \(\sqrt{n}(R_\alpha(\hat{F}_n) - R_\alpha(F))\) converges in distribution to \(\dot{R}_{\alpha,F}(B_F)\), where \(B_F\) is an \(L_1\)-valued centered Gaussian random variable with covariance operator \(\Phi_{B_F}\) given by (50). Now, \(\dot{R}_{\alpha,F}(B_F) = -\int f_{\alpha,F}(x) B_F(x) \ell(\, dx)\) for the \(L_\infty\)-function \(f_{\alpha,F}\) given by (18). Since \(B_F\) is a centered Gaussian random element of \(L_1\), and since \(f_{\alpha,F}\) represents a continuous linear functional on \(L_1\), the random variable \(\dot{R}_{\alpha,F}(B_F)\) is normally distributed with zero mean and variance \(\text{Var}[\dot{R}_{\alpha,F}(B_F)] = \mathbb{E}[\dot{R}_{\alpha,F}(B_F)^2] = \Phi_{B_F}(f_{\alpha,F}, f_{\alpha,F})\), and the latter expression is equal to the right-hand side in (17). \(\square\)

Note that when \(X_1, X_2, \ldots\) are i.i.d. random variables, then (15) and (16) are clearly satisfied and the expression for the variance \(s^2\) in (17) simplifies insofar as the sum \(\sum_{i=0}^1 \sum_{k=2}^\infty (\cdots)\) in (19) vanishes, so that \(s^2 = \mathbb{E}[U_\alpha(X_1 - R_\alpha(F))^2]/d_F(\alpha)^2\) with \(d_F(\alpha) := (1-2\alpha)F'(R_\alpha(F)) + \alpha\). The latter may be seen by applying Hoeffding’s variance formula (cf., e.g., Lemma 5.24 in [40]) to calculate \(\text{Var}[(X_1 - R_\alpha(F))^+]\) and \(\text{Var}[(X_1 - R_\alpha(F))^+]\) (take into account that by (3) we have \(\mathbb{E}[U_\alpha(X_1 - R_\alpha(F))^2] = \mathbb{E}[U_\alpha(X_1 - R_\alpha(F))^2] = \text{Var}[U_\alpha(X_1 - R_\alpha(F))^+]\), and obviously \(\text{Cov}((X_1 - R_\alpha(F))^+, (X_1 - R_\alpha(F))^-) = 0\)). But even in this case the
variance $s^2$ depends on the unknown distribution function $F$ in a fairly complex way. So, for the derivation of asymptotic confidence intervals the bootstrap results of Section 3.3 are expected to lead to a more efficient method than the method that is based on the nonparametric estimation of $s^2 = s^2_{α,F}$.

**Remark 3.3** In the i.i.d. case Theorem 3.2 can also be obtained from classical results on Z-estimators as, for example, Theorem A in Section 7.2.2 of [46]. Recently Holzmann and Klar [29, Theorem 7] showed that, still in the i.i.d. case, continuity of $F$ at $R_α(F)$ is even necessary in order to obtain a normal limit. It is also worth mentioning that the integrability condition on $F$ in Theorem 3.2 is slightly stronger than needed, at least in the i.i.d. case. Holzmann and Klar [29, Corollary 4] only assumed that $F$ possesses a finite second absolute moment which is slightly weaker than assuming our integrability condition.

**Remark 3.4** The following assertions illustrate the assumptions of Theorem 3.2.

(i) The integrability condition $\int \sqrt{F(1 - F)} dℓ < \infty$ holds if $\int φ^2 dF < \infty$ for some continuous function $φ : \mathbb{R} \to [1, \infty)$ satisfying $\int 1/φ dℓ < \infty$ and being strictly decreasing and strictly increasing on $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively.

(ii) Condition (15) holds if $\bar{φ}(n) = O(n^{-b})$ for some $b > 1$.

(iii) Condition (16) implies condition (15) with $\bar{φ}(n)$ replaced by $\bar{α}(n)$.

(iv) Condition (16) is equivalent to

$$\sum_{n \in \mathbb{N}} n^{-1/2} \int_{(0, \infty)} \bar{α}(n)^{1/2} \wedge \mathbb{P}[|X_1| > x]^{1/2} ℓ(\,dx\,) < \infty.$$ 

(v) Condition (16) holds if $\int \sqrt{F(1 - F)} dℓ < \infty$ and $\bar{α}(n) = O(n^{-b})$ for some $b > 1$.

See Section 6.1 for the proofs of these assertions.

### 3.3. Bootstrap consistency

In this section we present two results on bootstrap consistency in the setting of Theorem 3.2. In the following Theorem 3.5 we will assume that the random variables $X_1, X_2, \ldots$ are i.i.d. In Theorem 3.6 ahead we will assume that the sequence $(X_i)$ is $β$-mixing. We will use $g_{\text{BL}}$ to denote the bounded Lipschitz metric on the set of all Borel probability measures on $\mathbb{R}$; see the Appendix B for the definition of the bounded Lipschitz metric. By $\mathbb{P}_ξ'$ we will mean the law of a random variable $ξ$ under $\mathbb{P}'$, and as before $N_{0,s^2}$ refers to the centered normal distribution with variance $s^2$. 
Theorem 3.5 Let \((X_i)\) be a sequence of i.i.d. real-valued random variables on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Denote by \(F\) the distribution function of the \(X_i\), and assume that \(F\) is continuous at \(\mathcal{R}_\alpha(F)\) and that \(\int \phi^2 dF < \infty\) for some continuous function \(\phi: \mathbb{R} \to [1, \infty)\) satisfying \(\int 1/\phi \, dl < \infty\) (in particular \(F \in F_1\)). Let \(\hat{F}_n\) be the empirical distribution function of \(X_1, \ldots, X_n\) as defined in \([7]\). Let \((W_{ni})\) be a triangular array of nonnegative real-valued random variables on another probability space \((\Omega', \mathcal{F}', \mathbb{P}')\). Set 
\[
(\Omega, \mathcal{F}, \mathbb{P}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')
\]
and define the map \(\hat{F}_n^*: \Omega \to F_1\) by
\[
\hat{F}_n^*(\omega, \omega') := \frac{1}{n} \sum_{i=1}^n W_{ni}(\omega') \mathbb{I}_{[X_i(\omega), \infty]}.
\]
Finally assume that one of the following two settings is met:

(a) (Efron’s bootstrap) The random vector \((W_{n1}, \ldots, W_{nn})\) is multinomially distributed according to the parameters \(n\) and \(p_1 = \cdots = p_n = \frac{1}{n}\) for every \(n \in \mathbb{N}\).

(b) (Bayesian bootstrap) \(W_{ni} = Y_i / \overline{Y}_n\) for every \(n \in \mathbb{N}\) and \(i = 1, \ldots, n\), where \(\overline{Y}_n := \frac{1}{n} \sum_{j=1}^n Y_j\) and \((Y_j)\) is any sequence of nonnegative i.i.d. random variables on \((\Omega', \mathcal{F}', \mathbb{P}')\) with distribution \(\mu\) which satisfies \(\int_0^\infty \mu([x, \infty))^{1/2} \, dx < \infty\) and whose standard deviation coincides with its mean and is strictly positive.

Then
\[
\lim_{n \to \infty} \mathbb{P}[\{\omega \in \Omega : g_{BL}(\mathbb{P}'_{\sqrt{n}(\mathcal{R}_\alpha(\hat{F}_n^*(\omega)))-\mathcal{R}_\alpha(\hat{F}_n(\omega))}, N_{0, s^2}) \geq \delta\}] = 0 \quad \text{for all } \delta > 0, \tag{21}
\]
where \(s^2 = s^2_{\alpha,F}\) is given by \([17]\) (with \(C_F(t_0, t_1) = F(t_0 \wedge t_1)(1 - F(t_0 \vee t_1))\)).

Proof First of all note that \(\mathcal{R}_\alpha(\hat{F}_n^*)\) may be verified to be \((\mathcal{F}, \mathcal{B}(\mathbb{R}))\)-measurable in a similar way like \(\mathcal{R}_\alpha(\hat{F}_n)\). Theorem 2.3 shows that \(\mathcal{R}_\alpha\) is quasi-Hadamard differentiable at \(F\) tangentially to \(L_1(L_1)\) (w.r.t. the norm \(\| \cdot \|_{l,1}\)) with linear quasi-Hadamard derivative \(\hat{R}_\alpha,F\) given by \([11]\). The functional delta-method in the form of Theorem B.3(ii) along with Theorems 3.2 and C.4 then implies that \((21)\) with \(N_{0,s^2}\) replaced by the law of \(\hat{R}_\alpha,F(B_F)\) holds, where \(B_F\) is an \(L_1\)-valued centered Gaussian random variable with covariance operator \(\Phi_{BF}\) given by \([60]\) (with \(C_F(t_0, t_1) = F(t_0 \wedge t_1)(1 - F(t_0 \vee t_1))\)).

As in the proof of Theorem 3.2 we obtain \(\hat{R}_\alpha,F(B_F) \sim N_{0,s^2}\). For the application of Theorem 3.2 we note that \(\int \sqrt{F(1 - F)} \, dt < \infty\) is ensured by the assumption \(\int \phi^2 dF < \infty\); cf. Remark 3.4(i).

We now turn to the case where the observations \(X_1, X_2, \ldots\) may be dependent. We focus on the so-called circular bootstrap \([41, 45]\), which is only a slight modification of the moving blocks bootstrap \([15, 37, 39, 41]\). To ensure that in the following \(\hat{F}_n^*\) is the distribution function of a probability measure, we assume that \(n\) ranges only over \(N_m := \{m^k : k = 1, 2, \ldots\}\) for some arbitrarily fixed integer \(m \geq 2\). Let \((\ell_n)\) be a sequence in \(\mathbb{N}\) such that \(\ell_n < n\) is a divisor of \(n\) and \(\ell_n \nearrow \infty\) as \(n \to \infty\), and set
imply

At an informal level this means that given a sample as above and the map \( \hat{\kappa} \) where

\[
\text{define the map}
\]

\[
\text{One can argue as in the proof of Theorem 3.5 (with Theorem C.5 in place of Proof.)}
\]

Consider the nonparametric statistical infinite product model

3.4. Qualitative robustness

Consider the nonparametric statistical infinite product model

\[
(\Omega, \mathcal{F}, \{\mathbb{P}^\theta : \theta \in \Theta\}) := (\mathbb{R}^N, \mathcal{B}(\mathbb{R})^\otimes N, \{P^\otimes F : F \in \mathcal{F}_1\}),
\]

where \( P_F \) is the Borel probability measure on \( \mathbb{R} \) associated with the distribution function \( F \). Let \( X_i \) be the \( i \)-th coordinate projection on \( \Omega = \mathbb{R}^N \), and note that \( X_1, X_2, \ldots \) are i.i.d. with distribution function \( F \) under \( \mathbb{P}^F \) for every \( F \in \mathcal{F}_1 \). Let \( \hat{F}_n \) be the empirical
distribution function of $X_1, \ldots, X_n$ as defined in (7) and set $\hat{\mathcal{R}}_n := \mathcal{R}_\alpha(\hat{F}_n)$. We will say that the sequence of estimators $(\hat{\mathcal{R}}_n)$ is qualitatively robust on a given set $G \subseteq \mathcal{F}_1$ if for every $F \in G$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $n \in \mathbb{N}$

$$\mathcal{G} \subseteq \mathcal{G}, \quad \varrho_p(P_F, P_G) \leq \delta \quad \implies \quad \varrho_p(\mathbb{P}^F \circ \hat{\mathcal{R}}^{-1}_n, \mathbb{P}^G \circ \hat{\mathcal{R}}^{-1}_n) \leq \varepsilon,$$

where $\varrho_p$ refers to the Prohorov metric on the set of all Borel probability measures on $\mathbb{R}$. Theorem 3.7 ahead shows that the sequence of estimators $(\hat{\mathcal{R}}_n)$ is qualitatively robust on every so-called w-set in $\mathcal{F}_1$. Following [36], we say that a subset $G \subseteq \mathcal{F}_1$ is a w-set in $\mathcal{F}_1$ if the relative 1-weak topology and the relative weak topology coincide on $G$. Several characterizations and examples of w-sets in $\mathcal{F}_1$ have been worked out in [36]. The examples include the class of distribution functions of all normal distributions, the class of distribution functions of all Gamma distributions with location parameter 0, the class of distributions functions of all Pareto distributions on $[c, \infty)$ with shape parameter $\alpha \in [\alpha, \infty)$ for any fixed $\alpha > 1$ and $\overline{c} > 0$, and others.

The following theorem is an immediate consequence of Theorem 3.8 and Lemma 3.4 in [50] and Theorems 2.1 and 3.1.

**Theorem 3.7** The sequence of estimators $(\hat{\mathcal{R}}_n)$ is qualitatively robust on every w-set in $\mathcal{F}_1$.

### 3.5. Comparison to other empirical risk measures

As already mentioned in the introduction, the expectile-based risk measure $\rho_\alpha$ has recently attracted some attention as a tool of quantitative risk management. Already established alternatives are the Value at Risk $V_{\alpha}(X) := F_X^{\alpha}(-\infty)$ and the Average Value at Risk $AV_{\alpha}(X) := \frac{1}{\alpha} \int_{(\alpha, 1)} F_X^\alpha(s) \ell(ds)$ at level $\alpha \in (0, 1)$, where $F_X^\alpha$ denotes the left-continuous quantile function of the distribution function $F_X$ of $X$. Whereas $V_{\alpha}$ may be evaluated at any random variable on the underlying probability space, $AV_{\alpha}$ is restricted to $L^1$-random variables just as $\rho_\alpha$. Since they both are law-invariant, they may be associated with statistical functionals in the same way as expectile-based risk measures. The corresponding nonparametric empirical estimators can be obtained by evaluating these functionals at the empirical distribution function as defined in (7).

In Table 1 below these estimators are compared to the empirical expectile-based risk measure. To keep the discussion tight, we shall restrict considerations to i.i.d. samples.

Let $\mathcal{F}_0$ be the set of all distribution functions on $\mathbb{R}$, $\mathcal{F}_2$ be the subset of all $F \in \mathcal{F}_0$ satisfying $\int x^2 dF(x) < \infty$, and $\mathcal{F}_0^\alpha$ be the subset of all $F \in \mathcal{F}_0$ having a unique $\alpha$-quantile. Moreover let $\mathcal{F}_2^{\alpha,e}$ be the sets of all $F \in \mathcal{F}_2$ being continuous at $\mathcal{R}_\alpha(F)$, and $\mathcal{F}_2^{\alpha,a}$ be the analogue for the Average Value at Risk at level $\alpha$. The first two columns of the first two lines of Table 1 can be derived by means of the classical theory of $L$-statistics as presented in [36, 47]. It is moreover known from [25] that strong consistency of the empirical $\alpha$-quantile ($V_{\alpha}$) does not hold for $F \in \mathcal{F}_0 \setminus \mathcal{F}_0^\alpha$. The third column
of the first two lines is known from the results of [29]; see also our elaborations above. The results of [29] moreover show that asymptotic normality of the empirical \( \alpha \)-expectile cannot be obtained for \( F \in F_2 \setminus F_2^{\alpha,e} \). The first and the second column of the third line are known from the discussion at the end of Section 2 in [36] and Section 4.3 in [36], respectively. The third column of the third line is justified by Theorem 3.7 above. It follows by Hampel’s theorem that robustness of the empirical \( \alpha \)-quantile (\( \text{V@R}_\alpha \)) cannot be obtained on sets larger than \( F_0^a \). On the other hand, it is not clear to us whether or not robustness of the empirical Average Value at Risk and the empirical expectile can be obtained on sets that are not w-sets in \( F_1 \).

|                      | \( \text{V@R}_\alpha \) | \( \text{AV@R}_\alpha \) | \( \rho_\alpha \) |
|----------------------|-------------------------|-------------------------|------------------|
| strong consistency   | for \( F \in F_0^a \)   | for \( F \in F_1 \)      | for \( F \in F_1 \) |
| asymptotic normality | for \( F \in F_0^a \)   | for \( F \in F_2^{\alpha,a} \) | for \( F \in F_2^{\alpha,e} \) |
| qualitative robustness | on \( F_0^a \)          | on w-sets in \( F_1 \)   | on w-sets in \( F_1 \) |

Table 1: Comparison of empirical estimators of \( \text{V@R}_\alpha \), \( \text{AV@R}_\alpha \), and \( \rho_\alpha \).

4. Parametric estimation of \( \mathcal{R}_\alpha(F) \)

In this section we consider a parametric statistical model \((\Omega, \mathcal{F}, \{P^\theta : \theta \in \Theta\})\), where the parameter set \( \Theta \) is any topological space. In Section 4.2 we will also impose some additional structure on \( \Theta \). For every \( n \in \mathbb{N} \) we let \( \hat{\theta}_n : \Omega \to \Theta \) be any map, which should be seen as an estimator for \( \theta \). For every \( \theta \in \Theta \) we fix a distribution function \( F_\theta \in F_1 \), which can be seen as a characteristic derived from the parameter \( \theta \). In particular, \( \hat{F}_n := F_{\hat{\theta}_n} \) can be seen as an estimator for \( F_\theta \).

4.1. Strong consistency

Here we need no further assumptions on the topological space \( \Theta \). The following Theorem 4.1 is an immediate consequence of Theorem 2.1.

**Theorem 4.1** Let \( \theta_0 \in \Theta \) and assume that the mapping \( \theta \mapsto F_\theta \) is 1-weakly sequentially continuous at \( \theta_0 \). Moreover assume that \( \hat{\theta}_n \to \theta_0 \) \( P_{\theta_0} \)-a.s. Then, under \( P_{\theta_0} \), the estimator \( \mathcal{R}_\alpha(F_{\hat{\theta}_n}) \) is strongly consistent for \( \mathcal{R}_\alpha(F_{\theta_0}) \) in the sense that

\[
\mathcal{R}_\alpha(F_{\hat{\theta}_n}) \to \mathcal{R}_\alpha(F_{\theta_0}) \quad P_{\theta_0} \text{-a.s.}
\]

Note that in Theorem 4.1 the concept of strong consistency is used as a purely analytical property of the sequence \( (\mathcal{R}_\alpha(F_{\hat{\theta}_n})) \) without further measurability condition on \( \mathcal{R}_\alpha(F_{\hat{\theta}_n}) \). Example 4.3 will illustrate the conditions of Theorem 4.1.
Remark 4.2 Recall from Lemma 3.4 in [33] that a sequence $(F_n) \subseteq F_1$ converges 1-weakly to some $F_0 \in F_1$ if and only if $\int f \, dF_n \to \int f \, dF_0$ for all $f \in C_1$. Thus the mapping $\theta \mapsto F_\theta$ is 1-weakly sequentially continuous at $\theta_0$ if and only if for every sequence $(\theta_n) \subseteq \Theta$ with $\theta_n \to \theta_0$ we have $\int f \, dF_{\theta_n} \to \int f \, dF_{\theta_0}$ for all $f \in C_1$. □

Example 4.3 Let $\Theta := \mathbb{R} \times (0, \infty)$ and $F_{(m,s^2)}$ be the distribution function of the log-normal distribution $\text{LN}_{(m,s^2)}$ with parameters $(m, s^2) \in \Theta$. Recall that $\text{LN}_{(m,s^2)}$ possesses the Lebesgue density
\[
f_{(m,s^2)}(x) := \begin{cases} (2\pi s^2)^{-1/2} e^{-(\log(x) - m)^2/(2s^2)} & , x > 0 \\ 0 & , x \leq 0 \end{cases}.
\]
It is shown in Section 6.2 ahead that the mapping $(m, s^2) \mapsto F_{(m,s^2)}$ is 1-weakly sequentially continuous at every $(m_0, s_0^2) \in \Theta$. Further, in the corresponding infinite statistical product model $(\mathbb{R}^N, \mathcal{B}(\mathbb{R})^\otimes N, \{\text{LN}_{(m,s^2)} : (m, s^2) \in \Theta\})$ a maximum likelihood estimator $(\hat{m}_n, \hat{s}_n^2)$ for $(m, s^2)$ is given by
\[
\begin{align*}
\hat{m}_n(x_1, x_2, \ldots) & := \left\{ \frac{1}{n} \sum_{i=1}^{n} \log(x_i) , \min_{i=1,\ldots,n} x_i > 0 , \min_{i=1,\ldots,n} x_i \leq 0 \right\}, \\
\hat{s}_n^2(x_1, x_2, \ldots) & := \left\{ \frac{1}{s^2} \sum_{i=1}^{n} (\log(x_i) - \hat{m}_n(x_1, x_2, \ldots))^2 , \min_{i=1,\ldots,n} x_i > 0 , \min_{i=1,\ldots,n} x_i \leq 0 \right\}
\end{align*}
\]
for any fixed $\underline{m} \in \mathbb{R}$ and $\underline{s}^2 > 0$. By using the classical strong law of large numbers, $(\hat{m}_n, \hat{s}_n^2)$ is easily shown to be strongly consistent. □

Example 4.4 Let $\Theta := (1, \infty)$ and $F_a$ be the distribution function of the Pareto distribution $\text{Par}_{a,\overline{x}}$ with unknown tail-index $a > 0$ and known location parameter $\overline{x} > 0$. Recall that $\text{Par}_{a,\overline{x}}$ possesses the Lebesgue density
\[
f_a(x) := \begin{cases} a \overline{x}^{a+1} x^{-(a+1)} & , x > \overline{x} \\ 0 & , x \leq \overline{x} \end{cases}.
\]
It may be verified very easily that the mapping $a \mapsto F_a$ is 1-weakly sequentially continuous at every $a_0 \in \Theta$. Further, in the corresponding infinite statistical product model $(\mathbb{R}^N, \mathcal{B}(\mathbb{R})^\otimes N, \{\text{Par}_{a,\overline{x}} : a \in \Theta\})$ a maximum likelihood estimator $\hat{a}_n$ for the tail-index $a$ is given by
\[
\hat{a}_n(x_1, x_2, \ldots) := \left\{ \frac{1}{\overline{x}} \sum_{i=1}^{n} (\log(x_i) - \log(\overline{x}))^{-1} , \min_{i=1,\ldots,n} x_i > \overline{x} , \min_{i=1,\ldots,n} x_i \leq \overline{x} \right\}
\]
for any fixed $\overline{x} > 1$. Since the expectation of the logarithm of a $\text{Par}_{a,\overline{x}}$-distributed random variable equals $\log(\overline{x}) + 1/a$, it follows easily by the classical strong law of large numbers that this estimator is strongly consistent.
A popular alternative estimator for the tail-index \( a \) is the so-called Hill estimator

\[
\hat{a}_{n,k_n}^H(x_1, x_2, \ldots) := \left\{ \frac{1}{k_n} \sum_{i=1}^{k_n} \log(x_{n:n-i+1}) - \log(x_{n:n-k_n}) \right\}^{-1}, \quad \min_{i=1,\ldots,k_n} x_i > \tau \\
\min_{i=1,\ldots,k_n} x_i \leq \tau
\]

for any fixed \( \tau > 1 \), where \( n \geq 2, k_n \in \{1, \ldots, n-1\} \), and \( x_{n:1} \leq \ldots \leq x_{n:n} \) denotes an increasing ordering of \( x_1, \ldots, x_n \). It is known from [21] that \( (\hat{a}_{n,k_n}^H) \) is strongly consistent for \( a \) whenever \( k_n/\log(\log(n)) \to 0 \) and \( k_n/n \to 0 \) as \( n \to \infty \).

\[\Diamond\]

### 4.2. Asymptotic distribution

In this section we assume that \( \Theta \) and \( \Upsilon_0 \) are subsets of a vector space \( \Upsilon \) equipped with a separable norm \( \| \cdot \|_{\Upsilon} \). We denote by \( \mathcal{B}(\Upsilon) \) the Borel \( \sigma \)-algebra on \( (\Upsilon, \| \cdot \|_{\Upsilon}) \) and by \( \mathcal{B}_1 \) the Borel \( \sigma \)-algebra on \( (\mathcal{L}_1, \| \cdot \|_{1,\ell}) \). Moreover we define a map \( \hat{\mathcal{F}} : \Theta \to \mathcal{F}_1(\subseteq \mathcal{L}_0) \) by

\[
\hat{\mathcal{F}}(\theta) := F_\theta.
\]

In the following theorem we assume that \( \hat{\mathcal{F}} \) is Hadamard differentiable at some \( \theta_0 \in \Theta \) tangentially to \( \Upsilon_0 \) with trace \( \mathcal{L}_1 \) (in the sense of Definition [B.1] and Remark [B.2](iii)). This means that there exists a continuous map \( \hat{\mathcal{F}}_{\theta_0} : \Upsilon_0 \to \mathcal{L}_1 \) (the Hadamard derivative) such that

\[
\lim_{n \to \infty} \left\| \hat{\mathcal{F}}_{\theta_0}(\tau) - \frac{F_{\theta_0 + \varepsilon_n \tau_n} - F_{\theta_0}}{\varepsilon_n} \right\|_{1,\ell} = 0
\]

holds for each triplet \( (\tau, (\tau_n), (\varepsilon_n)) \) with \( \tau \in \Upsilon_0 \), \( (\tau_n) \subseteq \Upsilon \) satisfying \( (\theta_0 + \varepsilon_n \tau_n) \subseteq \Theta \) as well as \( \|\tau_n - \tau\|_{\Upsilon} \to 0 \), and \( (\varepsilon_n) \subseteq (0, \infty) \) satisfying \( \varepsilon_n \to 0 \). Recall that \( F_1 - F_2 \in \mathcal{L}_1 \) holds for every \( F_1, F_2 \in \mathcal{F}_1 \).

**Theorem 4.5** Let \( \theta_0 \in \Theta \) and \( (a_n) \) be a sequence of positive real numbers tending to \( \infty \). Let \( \hat{\theta}_n : \Omega \to \Theta \) be any map such that \( a_n(\hat{\theta}_n - \theta_0) \) is \( (\mathcal{F}, \mathcal{B}(\Upsilon)) \)-measurable and \( a_n(\hat{\theta}_n - \theta_0) \sim Y_{\theta_0} \) under \( \mathbb{P}_{\theta_0} \) for some \( (\Upsilon, \mathcal{B}(\Upsilon)) \)-valued random variable \( Y_{\theta_0} \) taking values only in \( \Upsilon_0 \). Assume that \( a_n(F_{\hat{\theta}_n} - F_{\theta_0}) \) is \( (\mathcal{F}, \mathcal{B}_1) \)-measurable. Further assume that the map \( \hat{\mathcal{F}} : \Theta \to \mathcal{F}_1(\subseteq \mathcal{L}_0) \) is Hadamard differentiable at \( \theta_0 \) tangentially to \( \Upsilon_0 \) with trace \( \mathcal{L}_1 \), and consider the Hadamard derivative \( \hat{\mathcal{F}}_{\theta_0} : \Upsilon_0 \to \mathcal{L}_1 \). If \( F_{\theta_0} \) is continuous at \( \mathcal{R}_a(F_{\theta_0}) \), then

\[
a_n(\mathcal{R}_a(F_{\hat{\theta}_n}) - \mathcal{R}_a(F_{\theta_0})) \sim \mathcal{R}_a(F_{\theta_0})(\hat{\mathcal{F}}_{\theta_0}(Y_{\theta_0})) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R}))
\]

under \( \mathbb{P}_{\theta_0} \), where \( \mathcal{R}_a_\alpha : F \) is defined by \([14]\).

**Proof** In view of \( a_n(\hat{\theta}_n - \theta_0) \sim Y_{\theta_0} \) under \( \mathbb{P}_{\theta_0} \) and the Hadamard differentiability of \( \hat{\mathcal{F}} \) tangentially to \( \Upsilon_0 \) with trace \( \mathcal{L}_1 \), the functional delta-method in the form of Theorem [B.3] yields \( a_n(F_{\hat{\theta}_n} - F_{\theta_0}) \sim \hat{\mathcal{F}}_{\theta_0}(Y_{\theta_0}) \) in \( (\mathcal{L}_1, \mathcal{B}_1, \| \cdot \|_{1,\ell}) \) under \( \mathbb{P}_{\theta_0} \). Then Theorem [2.3]
and another application of the functional delta-method in the form of Theorem B.3 give (26).

If $\Theta$ is an open subset of a Euclidean space, then we may find a convenient criterion to guarantee the condition of differentiability required for $\mathcal{F}$ in Theorem 4.5. The following lemma provides a criterion in terms of conditions on the map $f : \Theta \times \mathbb{R} \to [0,1]$ defined by

$$ f(\theta, x) := F_\theta(x). $$

In this lemma $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ stand for the Euclidean norm and Euclidean scalar product on $\mathbb{R}^d$, and $\text{grad}_\theta f(\theta, x)$ denotes the gradient of the function $f(\cdot, x)$ at $\theta$ for any fixed $x$.

**Lemma 4.6** Let $\Theta \subseteq \mathbb{R}^d$ be open and $\theta_0 \in \Theta$. Let $V$ denote some open neighbourhood of $\theta_0$ in $\Theta$ such that for every $x$ the map $f(\cdot, x)$ is continuously differentiable on $V$. Furthermore, let $h : \mathbb{R} \to \mathbb{R}$ be an $\ell$-integrable function such that

$$ \sup_{\theta \in V} \| \text{grad}_\theta f(\theta, x) \| \leq h(x) \quad \ell\text{-a.e. } x. $$

Then $\mathcal{F} : \Theta \to \mathcal{F}_1(\subseteq L_0)$ is Hadamard differentiable at $\theta_0$ (tangentially to the whole space $\mathbb{R}^d$) with trace $L_1$, and the Hadamard derivative $\hat{\mathcal{F}}_{\theta_0} : \mathbb{R}^d \to L_1$ is given by

$$ \hat{\mathcal{F}}_{\theta_0}(\tau)(\cdot) := \langle \text{grad}_\theta f(\theta_0, \cdot), \tau \rangle, \quad \tau \in \Theta. $$

The proof of Lemma 4.6 may be found in Section 6.3.

**Example 4.7** Consider the subset $\Theta := \mathbb{R} \times (0, \infty)$ of $\Upsilon := \mathbb{R}^2$, and let $F_{(m, s^2)}$ be the distribution function of the log-normal distribution $\text{LN}(m, s^2)$ with parameters $(m, s^2) \in \Theta$ as in Example 4.3. Moreover consider the map $\mathcal{F} : \Theta \to \mathcal{F}_1(\subseteq L_0)$ defined by $\mathcal{F}(m, s^2) := F_{(m, s^2)}$. It is shown in Section 6.3 ahead (using Lemma 4.6) that for every fixed $(m_0, s_0^2) \in \Theta$ the map $\mathcal{F}$ is Hadamard differentiable at $(m_0, s_0^2) \in \Theta$ (tangentially to the whole space $\Upsilon = \mathbb{R}^2$) with trace $L_1$ and Hadamard derivative $\hat{\mathcal{F}}_{(m_0, s_0^2)} : \mathbb{R}^2 \to L_1$ given by

$$ \hat{\mathcal{F}}_{(m_0, s_0^2)}(\tau_1, \tau_2)(x) := \begin{cases} -\left( \frac{\tau_1 x + \log(x) - m_0}{s_0} \right) \phi_{(0,1)} \left( \frac{\log(x) - m_0}{s_0} \right), & x > 0 \\ 0, & x \leq 0 \end{cases}, \quad (27) $$

where $\phi_{(0,1)}$ is the standard Lebesgue density of the standard normal distribution.

Further, it may be verified easily that the family $\{\text{LN}_{(m, s^2)} : (m, s^2) \in \Theta\}$ satisfies the assumptions of Theorem 6.5.1 in [38]. Therefore the maximum likelihood estimator $(\hat{m}_n, s_0^2_n)$ given by (23) in the corresponding infinite statistical product model satisfies

$$ \sqrt{n} \left[ \begin{array}{c} \hat{m}_n \\ s_0^2_n \end{array} - \begin{array}{c} m_0 \\ s_0^2 \end{array} \right] \leadsto Y_{(m_0, s_0^2)} \quad \text{in } (\Upsilon, \mathcal{B}(\Upsilon)) $$
under \( P^{(m_0, s_0^2)} \) for every \((m_0, s_0^2) \in \Theta\), where \( Y_{(m_0, s_0^2)} \) is bivariate centered normal with covariance matrix \( \mathcal{I}(m_0, s_0^2)^{-1} \). Here \( \mathcal{I}(m_0, s_0^2) \) denotes the Fisher information matrix at \((m_0, s_0^2)\), and elementary calculations show that

\[
\mathcal{I}(m_0, s_0^2)^{-1} = \begin{bmatrix}
    s_0^2 & 0 \\
    0 & 2(s_0^2)^2 \\
\end{bmatrix}.
\]

Now Theorem 6.2.6 in [38] shows that \( \sqrt{n} (\mathcal{R}_a(F_{(\hat{m}_n, \hat{s}_n^2)}) - \mathcal{R}_a(F_{(m_0, s_0^2)})) \) converges in distribution to \( Z := \mathcal{R}_a: F_{(m_0, s_0^2)}(\mathfrak{F}(m_0, s_0^2)) \). In Section 6.4 it is shown that the limit \( Z \) is centered normal with variance

\[
e^{2m_0+s_0^2} \left( 1 + 2 \left\{ s_0 + \frac{\Phi(0,1)(\varphi(m_0, s_0^2))}{1 - \alpha - (1 - 2\alpha)\Phi(0,1)(\varphi(m_0, s_0^2))} \right\}^2 \right)
\times \frac{1 - \alpha - (1 - 2\alpha)\Phi(0,1)(\varphi(m_0, s_0^2))}{(1 - 2\alpha)\mathcal{I}(m_0, s_0^2)\Phi(0,1)(\varphi(m_0, s_0^2)) + \alpha}^2,
\]

where \( \varphi(m_0, s_0^2) := (m_0 + s_0^2 - \log(\mathcal{R}_a(F_{(m_0, s_0^2)})))/2 \) (note that we may show \( \mathcal{R}_a(F_{(m_0, s_0^2)}) > 0 \); cf. Section 6.4 below) and \( \Phi(0,1) \) denotes the distribution function of the standard normal distribution. Note that \( 1 - \alpha - (1 - 2\alpha)\Phi(0,1)(z) = (1 - \alpha)\Phi(0,1)(-z) + \alpha\Phi(0,1)(z) > 0 \) holds for every \( z \in \mathbb{R} \).

**Example 4.8** Consider the subset \( \Theta := (1, \infty) \) of \( \mathbb{Y} := \mathbb{R} \), and let \( F_a \) be the distribution function of the Pareto distribution with known location parameter \( \tau > 1 \) and unknown tail-index \( a \in \Theta \) as defined in Example 4.4. Moreover consider the map \( \mathfrak{F} : \Theta \to \mathbb{F}_1(\subseteq \mathbb{L}_0) \) defined by \( \mathfrak{F}(a) := F_a \). An easy exercise shows that we may apply Lemma 4.6 to conclude that for every \( a_0 \in \Theta \) the map \( \mathfrak{F} \) is Hadamard differentiable at \( a_0 \in \Theta \) (tangentially to the whole space \( \mathbb{Y} = \mathbb{R} \)) with trace \( \mathbb{L}_1 \) and Hadamard derivative \( \mathfrak{F}_a_0 : \mathbb{R} \to \mathbb{L}_1 \) given by

\[
\mathfrak{F}_{a_0}(y)(x) := \begin{cases}
    y \log(\tau/x) (\tau/x)^{a_0} & , \quad x > \tau \\
    0 & , \quad \tau \leq x
\end{cases}.
\]

We may also verify easily that the family \( \{ \text{Par}_{a, \tau} : a \in \Theta \} \) satisfies the assumptions of Theorem 6.2.6 in [38]. Therefore the maximum likelihood estimator \( \hat{a}_n \) given by (24) in the corresponding infinite statistical product model satisfies

\[
\sqrt{n}(\hat{a}_n - a_0) \sim Y_{a_0} \quad \text{in } (\mathbb{Y}, \mathcal{B}(\mathbb{Y}))
\]

under \( P^{a_0} \) for every \( a_0 \in \Theta \), where \( Y_{a_0} \) is centered normal with \( \text{Var}[Y_{a_0}] = a_0^2 \). For the Hill estimators \( \hat{a}_{n,k_n}^H \) as defined by (25) Theorem 2 in [27] shows that, if \( k_n \to \infty \) and \( k_n = \mathcal{O}(n^\gamma) \) for some \( \gamma < 1 \),

\[
\sqrt{k_n}(\hat{a}_{n,k_n}^H - a_0) \sim Y_{a_0} \quad \text{in } (\mathbb{Y}, \mathcal{B}(\mathbb{Y}))
\]
under \( P^{a_0} \) for every \( a_0 \in \Theta \), where \( Y_{a_0} \) is the same as in (30). That is, up to the rate of convergence, the asymptotic of the maximum likelihood estimator \( \hat{a}_n \) is the same as the asymptotic of the Hill estimator \( \hat{a}^H_{n,k_n} \).

Now we may apply Theorem 4.5 to obtain that both \( \sqrt{n}(\mathcal{R}_\alpha(F_{\hat{a}_n}) - \mathcal{R}_\alpha(F_{a_0})) \) and \( \sqrt{n}(\mathcal{R}_\alpha(F_{\hat{a}^H_{n,k_n}}) - \mathcal{R}_\alpha(F_{a_0})) \) converge in distribution to \( Z := \mathcal{R}_{\alpha,F_{a_0}}(\mathcal{F}_{a_0}(Y_{a_0})) \). In Section 5.3 it is shown that the limit \( Z \) is centered normal with variance

\[
\mathcal{E}^2 = (1 - a_0)\{(1 - 2\alpha)F_{a_0}(\mathcal{R}_\alpha(F_{a_0})) + 2\alpha\} \varphi(a_0, \tau)^2,
\]

where \( \varphi(a_0, \tau) := (\mathcal{R}_\alpha(F_{a_0})/\tau)^{1-a_0}(1 - (1 - a_0)\log(\mathcal{R}_\alpha(F_{a_0})/\tau))(1 - 2\alpha) + \alpha - 1 \).

\[\Box\]

5. Proof of Theorem 2.3

For every \( F \in \mathbf{F}_1 \) the map \( U_\alpha(F)(\cdot) \) given by (6) is real-valued, continuous, strictly decreasing, satisfies \( \lim_{m \to \pm \infty} U(\mathcal{F}(m)) = \mp \infty \), and may be represented as

\[
U_\alpha(F)(m) = -(1 - \alpha) \int (-\infty, m) F(x) \ell(dx) + \alpha \int (m, \infty) (1 - F(x)) \ell(dx) \quad \text{(33)}
\]

\[
= -(1 - \alpha) \int (-\infty, 0) F(x + m) \ell(dx) + \alpha \int (0, \infty) (1 - F(x + m)) \ell(dx). \quad \text{(34)}
\]

This follows by Lemma A.1 and ensures that the functional \( \mathcal{R}_\alpha \) defined by (5), i.e. the mapping \( \mathbf{F}_1 \to \mathbb{R}, F \mapsto \mathcal{R}_\alpha(F) := U_\alpha(F)^{-1}(0) \), is well defined.

5.1. Auxiliary lemmas

**Lemma 5.1** Let \( F \in \mathbf{F}_1 \). Moreover let \( (v, (v_n), (\varepsilon_n)) \) be any triplet with \( v \in \mathbf{L}_1 \), \( (\varepsilon_n) \subseteq (0, \infty) \) satisfying \( \varepsilon_n \to 0 \), and \( (v_n) \subseteq \mathbf{L}_1 \) satisfying \( \|v_n - v\|_{1,\ell} \to 0 \) as well as \( F + \varepsilon_n v_n \in \mathbf{F}_1 \) for every \( n \in \mathbb{N} \). Then the following two assertions hold:

(i) We have

\[
\lim_{n \to \infty} \sup_{m \in \mathbb{R}} \left| \frac{U_\alpha(F + \varepsilon_n v_n)(m) - U_\alpha(F)(m)}{\varepsilon_n} - U_\alpha(v)(m) \right| = 0,
\]

where

\[
U_\alpha(v)(m) := -(1 - \alpha) \int (-\infty, 0) v(x + m) \ell(dx) + \alpha \int (0, \infty) v(x + m) \ell(dx). \quad \text{(35)}
\]

(ii) For any \( \varepsilon > 0 \) there is some \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) such that for every \( n \geq n_0 \) the value \( \mathcal{R}_\alpha(F + \varepsilon_n v_n) \) lies in the open interval \( (\mathcal{R}_\alpha(F) - \varepsilon, \mathcal{R}_\alpha(F) + \varepsilon) \).
It remains to show analogous result has been shown under the additional assumption that tangential Hadamard differentiability in the classical sense as defined in \[26\]; see also Definition \[B.1\] and Remark \[B.2\] below.

\begin{proof}
\begin{enumerate}[\textup{(i)\textup{:}}]
\item In view of \((34)\) and \((35)\), we have
\[
\left| \frac{\mathcal{U}_n(F + \varepsilon_n v_n)(m) - \mathcal{U}_n(F)(m)}{\varepsilon_n} - \mathcal{U}_n(v)(m) \right| = (1 - \alpha) \int_{(-\infty, 0)} |v_n(x + m) - v(x + m)| \ell(dx) + \alpha \int_{(0, \infty)} |v_n(x + m) - v(x + m)| \ell(dx) \leq \|v_n - v\|_{1, \ell}
\]
for every \(m \in \mathbb{R}\). This gives the claim.
\end{enumerate}
\end{proof}

For \(-\infty < a < b < \infty\) let \(\mathcal{B}[a, b]\) denote the space of all bounded Borel measurable functions \(f : [a, b] \to \mathbb{R}\). The space \(\mathcal{B}[a, b]\) will be equipped with the sup-norm \(\| \cdot \|_{\infty}\). Furthermore, let \(\mathcal{B}_b[a, b]\) be the set of all non-increasing \(f \in \mathcal{B}[a, b]\) satisfying the inequalities \(f(a) \geq 0 \geq f(b)\). Then the mapping
\[
\mathcal{I}_{a, b} : \mathcal{B}_b[a, b] \to \mathbb{R}, \quad f \mapsto f^{-}(0)
\]
is well-defined, where \(f^{-}(0) := \sup\{x \in [a, b] : f(x) > 0\}\) with \(\sup \emptyset := a\). Further, for \(x_0 \in (a, b)\) let \(\mathcal{B}_{c, x_0}[a, b]\) denote the linear subspace of \(\mathcal{B}[a, b]\) consisting of all elements of \(\mathcal{B}[a, b]\) which are continuous at \(x_0\). In the following lemma we employ the notion of tangential Hadamard differentiability in the classical sense as defined in \[26\]; see also Definition \[B.1\] and Remark \[B.2\] below.

\begin{lemma*}[\[5.2\]]
Let \(-\infty < a < b < \infty\), and let \(f \in \mathcal{B}_b[a, b]\) be differentiable at some \(x_0 \in (a, b) \cap f^{-1}(\{0\})\) with strictly negative derivative \(f'(x_0)\). Then \(\mathcal{I}_{a, b}\) is Hadamard differentiable at \(f\) tangentially to \(\mathcal{B}_{c, x_0}[a, b]\) with Hadamard derivative \(\mathcal{I}_{a,b,f} : \mathcal{B}_{c, x_0}[a, b] \to \mathbb{R}\) given by
\[
\mathcal{I}_{a,b,f}(w) := -\frac{w(x_0)}{f'(x_0)}, \quad w \in \mathcal{B}_{c, x_0}[a, b].
\]
\end{lemma*}

\begin{proof}
The following proof is inspired by the proof of Lemma 3.9.20 in \[48\], where an analogous result has been shown under the additional assumption that \(f\) is càdlàg. For the convenience of the reader we give a detailed argumentation.

First of all note that \(\mathcal{I}_{a,b,f}(\cdot)\) is obviously continuous w.r.t. the sup-norm \(\| \cdot \|_{\infty}\) (and linear). Now, let \((w_n) \subseteq \mathcal{B}_{c, x_0}[a, b]\) and \((\varepsilon_n) \subseteq (0, \infty)\) be any sequences such that \(f + \varepsilon_n w_n \in \mathcal{B}_b[a, b]\) for all \(n \in \mathbb{N}\), \(\varepsilon_n \to 0\), and \(\|w_n - w\|_{\infty} \to 0\) for some \(w \in \mathcal{B}_{c, x_0}[a, b]\). It remains to show
\[
\lim_{n \to \infty} \left| \mathcal{I}_{a,b,f}(w) - \frac{\mathcal{I}_{a,b}(f + \varepsilon_n w_n) - \mathcal{I}_{a,b}(f)}{\varepsilon_n} \right| = 0.
\]
\end{proof}
For \((36)\) it suffices to show

\[
-w(x_0) \leq \liminf_{n \to \infty} f'(x_0) \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n}, \tag{37}
\]

\[
-w(x_0) \geq \limsup_{n \to \infty} f'(x_0) \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n}. \tag{38}
\]

Since \(f\) is non-increasing with \(f'(x_0) < 0\), we have \(f^{-1}(\{0\}) = \{x_0\}\). That is, \(f(x_0) = 0, f(x) > 0\) for \(x \in [a, x_0)\), and \(f(x) < 0\) for \(x \in (x_0, b]\). Furthermore, in view of \(\|f - (f + \varepsilon_n w_n)\|_\infty = \varepsilon_n \|w_n\|_\infty \to 0\), we may assume without loss of generality that \(I_{a,b}(f + \varepsilon_n w_n) \in (a, b)\) for all \(n \in \mathbb{N}\). Thus we may select a sequence \((\gamma_n)\) in \((0, 1)\) such that \(\gamma_n \leq \varepsilon_n^2\),

\[
a < I_{a,b}(f + \varepsilon_n w_n) - \gamma_n, \quad I_{a,b}(f + \varepsilon_n w_n) + \gamma_n < b, \quad \text{and} \quad I_{a,b}(f + \varepsilon_n w_n) \pm \gamma_n \neq x_0
\]

for all \(n \in \mathbb{N}\). By the definition of \(I_{a,b}\) we then have for every \(n \in \mathbb{N}\)

\[
(f + \varepsilon_n w_n)(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) \geq 0 \geq (f + \varepsilon_n w_n)(I_{a,b}(f + \varepsilon_n w_n) + \gamma_n). \tag{39}
\]

In Step 2 ahead we will show that also the following three assertions hold:

\[
\lim_{n \to \infty} I_{a,b}(f + \varepsilon_n w_n) = x_0 = I_{a,b}(f), \tag{40}
\]

\[
\sup_{n \in \mathbb{N}} \left| \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n} \right| < \infty, \tag{41}
\]

\[
\lim_{n \to \infty} w_n(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) = w(x_0) = \lim_{n \to \infty} w_n(I_{a,b}(f + \varepsilon_n w_n) + \gamma_n). \tag{42}
\]

Before, we will show in Step 1 that \((39)–(42)\) imply \((37)–(38)\).

**Step 1.** Let

\[
a_n := \frac{f(I_{a,b}(f + \varepsilon_n w_n) + \gamma_n) - f(x_0)}{I_{a,b}(f + \varepsilon_n w_n) + \gamma_n - x_0} \quad \text{and} \quad b_n := \frac{f(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) - f(x_0)}{I_{a,b}(f + \varepsilon_n w_n) - \gamma_n - x_0}.
\]

By \(x_0 = I_{a,b}(f)\) and \(f(x_0) = 0\) we have

\[
-b_n \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n} + b_n \frac{\gamma_n}{\varepsilon_n} = -b_n \frac{I_{a,b}(f + \varepsilon_n w_n) - \gamma_n - x_0}{\varepsilon_n} = -\frac{f(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) - f(x_0)}{\varepsilon_n} = -\frac{f(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n)}{\varepsilon_n}.
\]

Moreover, by \((39)\) we have

\[
-w_n(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) \leq \frac{f(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n)}{\varepsilon_n}.
\]
Hence,

\[-(b_n - f'(x_0)) \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n} + b_n \frac{\gamma_n}{\varepsilon_n} - w_n(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) \leq f'(x_0) \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n}.\]  

(43)

Similarly we obtain

\[-(a_n - f'(x_0)) \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n} \leq -(a_n - f'(x_0)) \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n} - a_n \frac{\gamma_n}{\varepsilon_n} - w_n(I_{a,b}(f + \varepsilon_n w_n) + \gamma_n).\]  

(44)

By differentiability of \( f \) at \( x_0 \) and (41) we obtain that both \( (a_n - f'(x_0)) \) and \( (b_n - f'(x_0)) \) converge to zero as \( n \to \infty \). Along with (41) we can conclude that

\[\lim_{n \to \infty} (a_n - f'(x_0)) \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n} = 0,\]
\[\lim_{n \to \infty} (b_n - f'(x_0)) \frac{I_{a,b}(f + \varepsilon_n w_n) - I_{a,b}(f)}{\varepsilon_n} = 0,\]  

(45)

and along with the choice of \( (\gamma_n) \) we can conclude that

\[\lim_{n \to \infty} a_n \frac{\gamma_n}{\varepsilon_n} = f'(x_0) \lim_{n \to \infty} \frac{\gamma_n}{\varepsilon_n} = 0,\]
\[\lim_{n \to \infty} b_n \frac{\gamma_n}{\varepsilon_n} = f'(x_0) \lim_{n \to \infty} \frac{\gamma_n}{\varepsilon_n} = 0.\]  

(46)

Now, (43)–(44) along with (42), (45), and (46) imply (37)–(38).

Step 2. It remains to show (40)–(42). First we show (40). The inequalities in (39) imply

\[f(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) \geq -\varepsilon_n \|w_n\|_\infty \quad \text{and} \quad f(I_{a,b}(f + \varepsilon_n w_n) + \gamma_n) \leq \varepsilon_n \|w_n\|_\infty.\]

In particular,

\[\liminf_{n \to \infty} f(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) \geq 0 \geq \limsup_{n \to \infty} f(I_{a,b}(f + \varepsilon_n w_n) + \gamma_n).\]  

(47)

Now let \( \varepsilon > 0 \) be such that \( a < x_0 - \varepsilon < x_0 + \varepsilon < b \). Then \( f(x_0 + \varepsilon/2) < 0 < f(x_0 - \varepsilon/2) \), and in view of (47) we may find some \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \) we have \( \gamma_n < \varepsilon/2 \) and

\[f(I_{a,b}(f + \varepsilon_n w_n) - \gamma_n) > -|f(x_0 + \varepsilon/2)| = f(x_0 + \varepsilon/2),\]
\[f(I_{a,b}(f + \varepsilon_n w_n) + \gamma_n) < f(x_0 - \varepsilon/2).\]

Since \( f \) is non-increasing, this means that \( |I_{a,b}(f + \varepsilon_n w_n) - x_0| < \varepsilon \) for all \( n \geq n_0 \). That is, (40) indeed holds.
Next, \((12)\) is an immediate consequence of \((10)\), \(\|w_n - w\|_\infty \to 0, \gamma_n \to 0\), and the continuity of \(w\) at \(x_0\).

Finally we will show by way of contradiction that \((11)\) holds. So let us first assume that 
\[
(\mathcal{I}(f_0 + \varepsilon_{i(n)} w_{i(n)}) - x_0)/\varepsilon_{i(n)} \to -\infty
\]
holds for some subsequence \((i(n)) \subseteq (n)\). By \(f_0(x_0) = 0\) we have
\[
(f + \varepsilon_{i(n)} w_{i(n)})(\mathcal{I}_{a,b}(f + \varepsilon_{i(n)} w_{i(n)}) + \gamma_{i(n)})
\]
\[
\varepsilon_{i(n)}
\]
\[
= \frac{(f + \varepsilon_{i(n)} w_{i(n)})(\mathcal{I}_{a,b}(f + \varepsilon_{i(n)} w_{i(n)}) + \gamma_{i(n)}) - f(x_0)}{\varepsilon_{i(n)}}
\]
\[
= a_{i(n)} \frac{\mathcal{I}_{a,b}(f + \varepsilon_{i(n)} w_{i(n)}) - x_0}{\varepsilon_{i(n)}} + a_{i(n)} \frac{\gamma_{i(n)}}{\varepsilon_{i(n)}} + w_{i(n)}(\mathcal{I}_{a,b}(f + \varepsilon_{i(n)} w_{i(n)}) + \gamma_{i(n)})
\]
for every \(n \in \mathbb{N}\). Since \(f\) is differentiable at \(x_0\) with strictly negative derivative, we obtain from \((10)\)
\[
\lim_{n \to \infty} a_{i(n)} \frac{\mathcal{I}_{a,b}(f + \varepsilon_{i(n)} w_{i(n)}) - x_0}{\varepsilon_{i(n)}} = \infty
\]
and
\[
\lim_{n \to \infty} a_{i(n)} \frac{\gamma_{i(n)}}{\varepsilon_{i(n)}} = f'(x_0) \lim_{n \to \infty} \frac{\gamma_{i(n)}}{\varepsilon_{i(n)}} = 0.
\]
Therefore in view of \((12)\), we may conclude
\[
\lim_{n \to \infty} \frac{(f + \varepsilon_{i(n)} w_{i(n)})(\mathcal{I}_{a,b}(f + \varepsilon_{i(n)} w_{i(n)}) + \gamma_{i(n)})}{\varepsilon_{i(n)}} = \infty
\]
which contradicts \((39)\). In a similar way we obtain a contradiction when supposing that 
\((\mathcal{I}_{a,b}(f + \varepsilon_{j(n)} w_{j(n)}) - x_0)/\varepsilon_{j(n)} \to \infty\) for some subsequence \((j(n)) \subseteq (n)\), using \((b_{j(n)})\). □

5.2. Main part of the proof

Let \(F \in \mathbf{F}_1\) and assume that it is continuous at \(\mathcal{R}_\alpha(F)\). First of all note that the functional \(\mathcal{R}_{\alpha,F} : \mathbf{L}_1 \to \mathbb{R}\) defined by \((11)\) is easily seen to be continuous.

Now, let \((v_n) \subseteq \mathbf{L}_1\) and \((\varepsilon_n) \subseteq (0, \infty)\) be any sequences such that \(F + \varepsilon_n v_n \in \mathbf{F}_1\) for all \(n \in \mathbb{N}\), \(\varepsilon_n \to 0\), and \(\|v_n - v\|_{1,\ell} \to 0\) for some \(v \in \mathbf{L}_1\). In view of part (ii) of Lemma 35 we may assume without loss of generality that \((\mathcal{R}_\alpha(F + \varepsilon_n v_n))\) is a sequence in \([a, b]\) with \(a := \mathcal{R}_\alpha(F) - \varepsilon\) and \(b := \mathcal{R}_\alpha(F) + \varepsilon\) for some \(\varepsilon > 0\). Setting \(f := \mathcal{U}_\alpha(F)\rangle[a,b]\) and \(f_n := \mathcal{U}_\alpha(F + \varepsilon_n v_n)\rangle[a,b]\) for \(n \in \mathbb{N}\), this means that \((f_n)_{n \in \mathbb{N}_0}\) is a sequence in \(\mathbf{B}_{\perp,0}[a, b]\), and
\[
\frac{\mathcal{R}_\alpha(F + \varepsilon_n v_n) - \mathcal{R}_\alpha(F)}{\varepsilon_n} = \frac{\mathcal{I}_{a,b}(f_n) - \mathcal{I}_{a,b}(f)}{\varepsilon_n} = \frac{\mathcal{I}_{a,b}(f + \varepsilon_n \frac{f_n - f}{\varepsilon_n}) - \mathcal{I}_{a,b}(f)}{\varepsilon_n}
\]
for all \( n \in \mathbb{N} \). Taking the identity \((1 - 2\alpha)F(\mathcal{R}_\alpha(F)) + \alpha = (1 - \alpha)F(\mathcal{R}_\alpha(F)) + \alpha(1 - F(\mathcal{R}_\alpha(F)))\) and the definition of \(\dot{\mathcal{R}}_{\alpha,F}\) by (11) into account, it thus remains to show

\[
\lim_{n \to \infty} \frac{\mathcal{I}_{a,b}(f + \varepsilon_n \frac{I_{\alpha-F}}{\varepsilon_n} - \mathcal{I}_{a,b}(f)} = \frac{\dot{\mathcal{U}}_{\alpha}(v)(\mathcal{R}_\alpha(F))}{(1 - \alpha)F(\mathcal{R}_\alpha(F)) + \alpha(1 - F(\mathcal{R}_\alpha(F)))}, \tag{49}
\]

where \(\dot{\mathcal{U}}_{\alpha}\) is given by (35).

We intend to apply Lemma 5.2 in order to verify (49). By part (i) of Lemma 5.1 we have

\[
\lim_{n \to \infty} \sup_{m \in [a,b]} |w_n(m) - w(m)| = 0
\]

for

\[
w_n(\cdot) := \frac{f_n(\cdot) - f(\cdot)}{\varepsilon_n} \quad \text{and} \quad w(\cdot) := \dot{\mathcal{U}}_{\alpha}(v)(\cdot).
\]

According to (33) we have

\[
f(m) = -(1 - \alpha) \int_{(-\infty,m)} F(x) \ell(dx) + \alpha \int_{(m,\infty)} (1 - F(x)) \ell(dx) \quad \text{for all } m \in [a,b].
\]

Since \(F\) is continuous at \(\mathcal{R}_\alpha(F)\) by assumption, we may apply the second fundamental theorem of calculus for regulated functions to conclude that the function \(f\) is differentiable at \(x_0 := \mathcal{R}_\alpha(F) = \mathcal{I}_{a,b}(f) \in (a, b)\) with derivative

\[
f'(x_0) = -(1 - \alpha)F(\mathcal{R}_\alpha(F)) - \alpha(1 - F(\mathcal{R}_\alpha(F))) < 0.
\]

Below we will show that \(w_n\) and \(w\) are continuous on \([a,b]\). So Lemma 5.2 implies (49).

It remains to show the continuity of \(w_n\) and \(w\). In view of the definition of \(w(\cdot) = \dot{\mathcal{U}}_{\alpha}(v)\) due to (35), we obtain by change of variable formula

\[
|w(m_1) - w(m_2)| = |\dot{\mathcal{U}}_{\alpha}(v)(m_1) - \dot{\mathcal{U}}_{\alpha}(v)(m_2)|
\leq (1 - \alpha) \int_{(m_1,m_2)} v(x) \ell(dx) + \alpha \int_{(m_1,m_2)} v(x) \ell(dx)
\leq \left| \int_{(m_1,m_2)} v(x) \ell(dx) \right|.
\]

Since \(v\) as an element of \(L_1\) is Lebesgue integrable, it follows that \(w\) is continuous on \([a,b]\). Further, in view of (33) we have

\[
w_n(m) = (1 - \alpha) \int_{(-\infty,m)} v_n(x) \ell(dx) + \alpha \int_{(m,\infty)} v_n(x) \ell(dx) \quad \text{for all } m \in [a,b],
\]

and thus we may show continuity of \(w_n\) in the same way as we have done for \(w\). This completes the proof of Theorem 2.3.
6. Remaining proofs

6.1. Proof of Remark 3.4

For (i) note that finiteness of the integral \( \int \phi^2 \, dF \) implies that there exists a constant \( C > 0 \) such that \( F(y) \leq C \phi(y)^{-2} \) for \( y < 0 \) and \( 1 - F(y) \leq C \phi(y)^{-2} \) for \( y > 0 \). In view of \( \int \phi \, d\ell < \infty \), it follows that the integral \( \int F(1 - F) \, d\ell \) is finite. Assertion (ii) is trivial. Condition (16) implies that \( \tilde{\alpha}(n) \leq 1/2 \) for sufficiently large \( n \), and \( Q_{1}\,X_{1}(u)^{-1/2} \geq Q\,X_{1}(1/2)/\sqrt{2} \) holds for \( u \in (0,1/2) \) anyway. Thus assertion (iii) follows easily. Concerning assertion (iv) note that by application of Fubini’s theorem we may observe

\[
\frac{1}{2} \int_{(0,\infty)} \tilde{\alpha}(n)^{1/2} \, \mathbb{P}(|X_1| > x)^{1/2} \, \ell(dx) = \int_{(0,\infty)} \int_{[0,\mathbb{P}(|X_1| > x))] \mathbbm{1}_{(0,\tilde{\alpha}(n))}(u) \, u^{-1/2} \, \ell(du) \, \ell(dx)
\]

Finally, assertion (v) is a consequence of (iv) and the equivalence of the integrability conditions \( \int \sqrt{F(1-F)} \, d\ell < \infty \) and \( \int_{(0,\infty)} \mathbb{P}(|X_1| > x) \, \ell(dx) < \infty \).

\[ \square \]

6.2. Proof of Example 4.3

For every \((m_0,s_0^2), (m_n, s_n^2) \in \Theta\) and \( f \in C_1 \) we have

\[
| \int f \, dF_{(m_n,s_n^2)} - \int f \, dF_{(m_0,s_0^2)} | \\
\leq \int_{(0,\infty)} \frac{|f(x)|}{x} \left| \frac{1}{\sqrt{2\pi s_n^2}} e^{-\frac{\{\log(x) - m_n\}^2}{2s_n^2}} - \frac{1}{\sqrt{2\pi s_0^2}} e^{-\frac{\{\log(x) - m_0\}^2}{2s_0^2}} \right| \ell(dx)
\]

\[
\leq C_f \int_{(0,\infty)} \frac{1}{\sqrt{2\pi s_n^2}} e^{-\frac{\{\log(x) - m_n\}^2}{2s_n^2}} - \frac{1}{\sqrt{2\pi s_0^2}} e^{-\frac{\{\log(x) - m_0\}^2}{2s_0^2}} \ell(dx)
\]

for some finite constant \( C_f > 0 \) depending on \( f \). If \((m_n, s_n^2) \to (m_0, s_0^2)\), then

\[
\lim_{n \to \infty} \frac{1}{\sqrt{2\pi s_n^2}} e^{-\frac{\{\log(x) - m_n\}^2}{2s_n^2}} = \frac{1}{\sqrt{2\pi s_0^2}} e^{-\frac{\{\log(x) - m_0\}^2}{2s_0^2}} \quad \text{for all} \ x \in \mathbb{R}
\]

and

\[
\lim_{n \to \infty} \int_{(0,\infty)} \frac{1}{\sqrt{2\pi s_n^2}} e^{-\frac{\{\log(x) - m_n\}^2}{2s_n^2}} \ell(dx) = \lim_{n \to \infty} \int_{\mathbb{R}} x \, dF_{(m_n,s_n^2)}(x) = \lim_{n \to \infty} e^{m_n+s_n^2/2} = e^{m_0+s_0^2/2} = \int_{\mathbb{R}} x \, dF_{(m_0,s_0^2)}(x) = \int_{(0,\infty)} \frac{1}{\sqrt{2\pi s_0^2}} e^{-\frac{\{\log(x) - m_0\}^2}{2s_0^2}} \ell(dx).
\]
Since \( \frac{1}{\sqrt{2 \pi s_n^2}} e^{-\frac{(\log(x) - m_n)^2}{2s_n^2}} \geq 0 \) for all \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \), Lemma 21.6 in [4] yields
\[
\lim_{n \to \infty} \int_{(0,\infty)} \left| \frac{1}{\sqrt{2 \pi s_n^2}} e^{-\frac{(\log(x) - m_n)^2}{2s_n^2}} - \frac{1}{\sqrt{2 \pi s_0^2}} e^{-\frac{(\log(x) - m_0)^2}{2s_0^2}} \right| \ell(dx) = 0.
\]
Along with (50) and Remark 4.2 this shows that the mapping \((m, s^2) \mapsto F(m, s^2)\) is 1-weakly sequentially continuous at every \((m_0, s_0^2) \in \Theta\).

### 6.3. Proof of Lemma 4.6

Consider any triplet \((\tau, (\tau_n), (\varepsilon_n))\) with \(\tau \in \mathbb{R}^d\), \((\tau_n) \subseteq \mathbb{R}^d\) satisfying \((\theta_0 + \varepsilon_n \tau_n) \subseteq \Theta\) as well as \(\|\tau_n - \tau\| \to 0\), and \((\varepsilon_n) \subseteq (0,\infty)\) satisfying \(\varepsilon_n \to 0\). Since \(V\) is an open subset of \(\mathbb{R}^d\) containing \(\theta_0\), we may assume without loss of generality that \((\theta_0 + \varepsilon_n \tau_n) \in V\) for every \(n \in \mathbb{N}\). Since \(f(\cdot, x)\) is continuously differentiable at \(\theta_0\) for every \(x \in \mathbb{R}\), we may conclude
\[
\lim_{n \to \infty} \frac{F_{\theta_0 + \varepsilon_n \tau_n}(x) - F_{\theta_0}(x)}{\varepsilon_n} = \lim_{n \to \infty} \frac{f(\theta_0 + \varepsilon_n \tau_n, x) - f(\theta_0, x)}{\varepsilon_n} = \langle \text{grad}_\theta f(\theta_0, x), \tau \rangle
\]
for every \(x \in \mathbb{R}\). Moreover, by the mean value theorem in several variables,
\[
\left| \frac{F_{\theta_0 + \varepsilon_n \tau_n}(x) - F_{\theta_0}(x)}{\varepsilon_n} \right| \leq \sup_{\theta \in V} \|\text{grad}_\theta f(\theta, x)\| \|\tau_n\| \leq h(x) \sup_{n \in \mathbb{N}} \|\tau_n\|
\]
for all \(n \in \mathbb{N}\) and \(x \in \mathbb{R}\). By assumption, the majorant \(h\) is \(\ell\)-integrable. Thus an application of the dominated convergence theorem yields
\[
\lim_{n \to \infty} \int_{\mathbb{R}} \left| \frac{F_{\theta_0 + \varepsilon_n \tau_n}(x) - F_{\theta_0}(x)}{\varepsilon_n} - \langle \text{grad}_\theta f(\theta_0, x), \tau \rangle \right| \ell(dx) = 0.
\]
Thus \(\mathcal{F}\) satisfies the claimed differentiability property. \(\square\)

### 6.4. Proof of Example 4.7

For the first assertion we intend to apply Lemma 4.6. To this end we consider the map \(f : \Theta \times \mathbb{R} \to [0,1]\) defined by
\[
f((m, s^2), x) := F(m, s^2)(x),
\]
where \(\Theta := \mathbb{R} \times (0,\infty)\). For every \((m, s^2) \in \Theta\), the distribution function \(F(m, s^2)\) satisfies \(F(m, s^2)(x) = 0\) if \(x \leq 0\), and \(F(m, s^2)(x) = \Phi_{(0,1)}((\log(x) - m)/s)\) for \(x > 0\). So, obviously, for any \(x \in \mathbb{R}\) the map \(f(\cdot, x)\) is continuously differentiable on \(\Theta\) with gradient
\[
\text{grad}_{(m, s^2)} f((m, s^2), x) = \begin{cases} \frac{1}{s} \frac{\log(x) - m}{2 s^3} \Phi_{(0,1)}(\frac{\log(x) - m}{s}) & , \ x > 0 \\ (0,0) & , \ x \leq 0 \end{cases}.
\]
(51)
Let \((m_0, s_0^2) \in \mathbb{R} \times (0, \infty)\), and define the map \(h : \mathbb{R} \to \mathbb{R}\) by

\[
h(x) := \begin{cases} 
0, & x \leq 0 \\
\frac{1}{\sqrt{\pi(s_0^2-\delta)}} \left( 1 + \frac{2}{\delta} \right), & 0 < x \leq e^{m_0-2\delta} \\
\frac{1}{\sqrt{\pi}} \left( \frac{1}{\sqrt{s_0^2-\delta}} + \frac{C+|m_0|+\delta}{\sqrt{s_0^2-\delta}} \right), & e^{m_0-2\delta} \leq x \leq e^{m_0+\delta} \\
\frac{1}{\sqrt{\pi}} e^{-\frac{(\log(x)-m_0)^2}{2(s_0^2+\delta)^2}} \left( \frac{1}{\sqrt{s_0^2-\delta}} + \frac{\log(x)-m_0+\delta}{\sqrt{s_0^2-\delta}} \right), & x \geq e^{m_0+\delta}
\end{cases}
\]

We will now show that

\[
\|\text{grad}_{(m,s^2)} f((m, s^2), x)\| \leq h(x) \text{ for all } ((m, s^2), x) \in ((m_0-\delta, m_0+\delta) \times (s_0^2-\delta, s_0^2+\delta)) \times \mathbb{R}
\]

for some sufficiently small \(\delta > 0\). Let us choose \(\delta > 0\) such that \(s_0^2 - 2\delta > 0\). In particular \((m, s^2) \in \Theta\) if \(\| (m, s^2) - (m_0, s_0^2) \| < 2\delta\). For \(x \in (0, e^{m_0-2\delta})\) and \((m, s^2) \in (m_0 - \delta, m_0 + \delta) \times (s_0^2 - \delta, s_0^2 + \delta)\), we obtain

\[
\|\text{grad}_{(m,s^2)} f((m, s^2), x)\| \leq \sqrt{\frac{2}{2\pi}} e^{-\frac{(\log(x)-m)^2}{2s^2}} \left( \frac{1}{s} + \frac{|\log(x)-m|}{s^3} \right) \leq \sqrt{\frac{1}{\pi}} \left( \frac{2s^2}{s^3} |\log(x)-m| \right) \leq \sqrt{\frac{1}{\pi}} \left( 1 + \frac{2}{s^3} |\log(x)-m| \right) \leq \sqrt{\frac{1}{\pi}} \left( 1 + \frac{2}{\delta} \right). \tag{53}\]

If \(x > e^{m_0+\delta}\) and \((m, s^2) \in (m_0 - \delta, m_0 + \delta) \times (s_0^2 - \delta, s_0^2 + \delta)\), then the inequalities \(|\log(x)-(m_0+\delta)| \leq |\log(x)-m| \leq |\log(x)-(m_0-\delta)|\) hold, and thus

\[
\|\text{grad}_{(m,s^2)} f((m, s^2), x)\| \leq \sqrt{\frac{2}{2\pi}} e^{-\frac{(\log(x)-m)^2}{2s^2}} \left( \frac{1}{s} + \frac{|\log(x)-m|}{s^3} \right) \leq \sqrt{\frac{1}{\pi}} e^{-\frac{(\log(x)-m_0)^2}{2s^2}} \left( \frac{1}{s} + \frac{|\log(x)-m_0+\delta|}{s^3} \right) \leq \sqrt{\frac{1}{\pi}} e^{-\frac{(\log(x)-m_0-\delta)^2}{2(s_0^2+\delta)^2}} \left( \frac{1}{\sqrt{s_0^2-\delta}} + \frac{|\log(x)-m_0+\delta|}{\sqrt{s_0^2-\delta}} \right). \tag{54}\]

Now, let \(C := \sup_{x \in [e^{m_0-2\delta}, e^{m_0+\delta}]} |\log(x)|\). Then for \(x \in [e^{m_0-2\delta}, e^{m_0+\delta}]\) and \((m, s^2) \in (m_0 - \delta, m_0 + \delta) \times (s_0^2 - \delta, s_0^2 + \delta)\), we may observe

\[
\|\text{grad}_{(m,s^2)} f((m, s^2), x)\| \leq \sqrt{\frac{2}{2\pi}} e^{-\frac{(\log(x)-m)^2}{2s^2}} \left( \frac{1}{s} + \frac{|\log(x)-m|}{s^3} \right) \leq \sqrt{\frac{1}{\pi}} \left( \frac{1}{s} + \frac{|\log(x)| + |m|}{s^3} \right) \leq \sqrt{\frac{1}{\pi}} \left( \frac{1}{\sqrt{s_0^2-\delta}} + \frac{C + |m_0| + \delta}{\sqrt{s_0^2-\delta}} \right). \tag{55}\]
By \((53)\)–\((55)\) the function \(h\) indeed satisfies \((52)\).

We will next show that \(h\) is also \(\ell\)-integrable. For any \(\gamma > e^{m_0 + \delta}\), an application of the change of variable formula yields

\[
\int_{(e^{m_0 + \delta}, \gamma)} \frac{1}{\sqrt{\pi}} e^{-\frac{(\log(x) - m_0 - \delta)^2}{2(s_0^2 + \delta)}} \left( \frac{1}{\sqrt{s_0^2 - \delta}} + \frac{|\log(x) - m_0 + \delta|}{\sqrt{s_0^2 - \delta^3}} \right) \ell(dx)
= \int_{m_0 + \delta}^{\log(\gamma)} \frac{1}{\sqrt{\pi}} e^{-\frac{(y - m_0 - \delta)^2}{2(s_0^2 + \delta)}} \left( \frac{1}{\sqrt{s_0^2 - \delta}} + \frac{|y - m_0 + \delta|}{\sqrt{s_0^2 - \delta^3}} \right) e^y dy
= \int_{0}^{\log(\gamma) - m_0 - \delta} \frac{1}{\sqrt{\pi}} e^{-\frac{(z - m_0 - \delta)^2}{2(s_0^2 + \delta)}} \left( \frac{1}{\sqrt{s_0^2 - \delta}} + \frac{|z + 2\delta|}{\sqrt{s_0^2 - \delta^3}} \right) e^{z + m_0 + \delta} dz
= e^{(2m_0 + s_0^2 + \delta)/2} \int_{0}^{\log(\gamma) - m_0 - \delta} \frac{1}{\sqrt{\pi}} e^{-\frac{(z - m_0 - \delta)^2}{2(s_0^2 + \delta)}} \left( \frac{1}{\sqrt{s_0^2 - \delta}} + \frac{|z + 2\delta|}{\sqrt{s_0^2 - \delta^3}} \right) dz.
\]

Denoting by \(Z\) any normally distributed random variable with mean \(s_0^2 - \delta\) and variance \(s_0^2 + \delta\) and by \(f_Z\) its standard Lebesgue density, we end up with

\[
\int_{(e^{m_0 + \delta}, \infty)} |h(x)| \ell(dx)
= \lim_{\gamma \to \infty} \int_{e^{m_0 + \delta}}^{\gamma} \frac{1}{\sqrt{\pi}} e^{-\frac{(\log(x) - m_0 - \delta)^2}{2(s_0^2 + \delta)}} \left( \frac{1}{\sqrt{s_0^2 - \delta}} + \frac{|\log(x) - m_0 + \delta|}{\sqrt{s_0^2 - \delta^3}} \right) dx
\leq \sqrt{2(s_0^2 + \delta)} e^{(2m_0 + s_0^2 + \delta)/2} \int_{0}^{\infty} f_Z(z) \left( \frac{1}{\sqrt{2(s_0^2 - \delta)}} + \frac{|z + 2\delta|}{\sqrt{s_0^2 - \delta^3}} \right) dz
\leq \sqrt{2(s_0^2 + \delta)} e^{(2m_0 + s_0^2 + \delta)/2} \mathbb{E} \left[ \frac{1}{\sqrt{2(s_0^2 - \delta)}} + \frac{|Z + 2\delta|}{\sqrt{s_0^2 - \delta^3}} \right]
< \infty.
\]

By definition of \(h\) this implies that \(h\) is indeed \(\ell\)-integrable.

Now, Lemma \((4.6)\) along with \((51)\) and \((52)\) shows that the map \(\hat{f}\) is Hadamard differentiable at \((m_0, s_0^2)\) with trace \(L_1\) and that the Hadamard derivative \(\hat{f}_{(m_0, s_0^2)} : \Theta \to L_1\) is given by \((27)\). This proves the first assertion in Example \((4.7)\).

For the last assertion in Example \((4.7)\) we first of all note that \(F_{(m_0, s_0^2)}\) is a continuous function. In particular, it is continuous at \(R_\alpha(F_{(m_0, s_0^2)})\). It follows by \((11)\) that

\[
\hat{R}_{\alpha,F_{(m_0, s_0^2)}}(\hat{f}_{(m_0, s_0^2)})(\tau_1, \tau_2)
= (1 - \alpha) \int_{(0, R_\alpha(F_{(m_0, s_0^2)}))}^{} \phi_{(0, 1)} \left( \frac{\log(x) - m_0}{s_0} \right) \left( \frac{\tau_1}{s_0} + \frac{\log(x) - m_0}{s_0^2} \right) \ell(dx)
= (1 - 2\alpha) F_{(m_0, s_0^2)}(R_\alpha(F_{(m_0, s_0^2)})) + \alpha \int_{(R_\alpha(F_{(m_0, s_0^2)}), \infty)}^{} \phi_{(0, 1)} \left( \frac{\log(x) - m_0}{s_0} \right) \left( \frac{\tau_1}{s_0} + \frac{\log(x) - m_0}{s_0^2} \right) \ell(dx)
= (1 - 2\alpha) F_{(m_0, s_0^2)}(R_\alpha(F_{(m_0, s_0^2)})) + \alpha \int_{(R_\alpha(F_{(m_0, s_0^2)}), \infty)}^{} \phi_{(0, 1)} \left( \frac{\log(x) - m_0}{s_0} \right) \left( \frac{\tau_1}{s_0} + \frac{\log(x) - m_0}{s_0^2} \right) \ell(dx)
\]

(56)
for all \((\tau_1, \tau_2) \in \Theta\). Let \(a := R_\alpha(F_{(m_0,s_0^2)})\). For \(b \leq 0\) and any random variable \(W\) with distribution function \(F_{(m_0,s_0^2)}\) we have \(a \geq (1-\alpha)\mathbb{E}[(b-W)^+]\) due to (3). We may apply several times the change of variable formula to obtain

\[
\int_{(a,\infty)} \phi_{(0,1)} \left( \frac{\log(x) - m_0}{s_0} \right) \left( \frac{\tau_1}{s_0} + \frac{(\log(x) - m_0)\tau_2}{s_0^3} \right) \ell(dx)
= \int_{-\infty}^{\infty} \phi_{(0,1)} \left( \frac{t - m_0}{s_0} \right) \left( \frac{\tau_1}{s_0} + \frac{(t - m_0)\tau_2}{s_0^3} \right) e^t dt
= \int_{(\log(a)-m_0)/s_0}^{\infty} \phi_{(0,1)}(u) \left( \frac{\tau_1}{s_0} + \frac{u\tau_2}{s_0^3} \right) e^{s_0u+m_0} du
= e^{m_0+s_0^2/2} \int_{(\log(a)-m_0)/s_0}^{\infty} \frac{e^{-(u-s_0)^2/2}}{\sqrt{2\pi}} \left( \frac{\tau_1}{s_0} + \frac{u\tau_2}{s_0^3} \right) du
= e^{m_0+s_0^2/2} \int_{(\log(a)-m_0-s_0^2)/s_0}^{\infty} \frac{e^{-w^2/2}}{\sqrt{2\pi}} \left( \frac{\tau_1}{s_0} + \frac{(w+s_0)\tau_2}{s_0^3} \right) dw
= \frac{e^{m_0+s_0^2/2}}{s_0} \Phi_{(0,1)}(\psi(m_0,s_0^2,a)) \tau_1
+ \frac{e^{m_0+s_0^2/2}}{s_0^2} \left( s_0 \Phi_{(0,1)}(\psi(m_0,s_0^2,a)) + \phi_{(0,1)}(\psi(m_0,s_0^2,a)) \right) \tau_2,
\]

where \(\psi(m_0,s_0^2,a) := (m_0 + s_0^2 - \log(a))/s_0\). In the same way we may calculate

\[
\int_{(0,a)} \phi_{(0,1)} \left( \frac{\log(x) - m_0}{s_0} \right) \left( \frac{\tau_1}{s_0} + \frac{(\log(x) - m_0)\tau_2}{s_0^3} \right) \ell(dx)
= \int_{-\infty}^{\log(a)} \phi_{(0,1)} \left( \frac{t - m_0}{s_0} \right) \left( \frac{\tau_1}{s_0} + \frac{(t - m_0)\tau_2}{s_0^3} \right) e^t dt
= \frac{e^{m_0+s_0^2/2}}{s_0} \Phi_{(0,1)}(-\psi(m_0,s_0^2,a)) \tau_1
+ \frac{e^{m_0+s_0^2/2}}{s_0^2} \left( s_0 \Phi_{(0,1)}(-\psi(m_0,s_0^2,a)) + \phi_{(0,1)}(\psi(m_0,s_0^2,a)) \right) \tau_2.
\]

Hence in view of (50) we end up with

\[
\mathcal{R}_{\alpha,F_{(m_0,s_0^2)}}(\mathcal{S}_{(m_0,s_0^2)}(\tau_1, \tau_2))
= \frac{e^{m_0+s_0^2/2}}{s_0} \left( 1 - \alpha - (1 - 2\alpha)\Phi_{(0,1)}(\psi(m_0,s_0^2,a)) \right) \left( \mathcal{R}_{\alpha,F_{(m_0,s_0^2)}}(\psi(m_0,s_0^2,a)) \right) \tau_1
+ \frac{e^{m_0+s_0^2/2}}{s_0^2} \phi_{(0,1)}(\psi(m_0,s_0^2,a)) \tau_2.
\]

Therefore, \(\mathcal{R}_{\alpha,F_{(m_0,s_0^2)}}(\mathcal{S}_{(m_0,s_0^2)}(Y_{(m_0,s_0^2)}))\) is centered normal with variance as in (28). □
6.5. Proof of Example 4.8

For $b \leq \tau$ and any random variable $W$ with distribution function $F_{a_0}$ we have
\[ \alpha \mathbb{E}[(W - b)^+] - (1 - \alpha) \mathbb{E}[(b - W)^+] = \alpha (\mathbb{E}[W] - b) = \frac{\tau}{a - 1} > 0. \]

Thus $\mathcal{R}_a(F_{a_0}) > \tau$ for every $a_0 \in \Theta$ due to (3). Now, invoking (11), we may draw on (20) to observe that for any $a_0 \in \Theta$
\[
\mathcal{R}_{\alpha;F_{a_0}}(\hat{\alpha}_{a_0}(Y_{a_0})) = -Y_{a_0} \frac{(1 - \alpha) \int_{\mathbb{R}} \log((\tau/x) (\tau/x)^{a_0}) dx + \alpha \int_{\mathbb{R}} \log((\tau/x) (\tau/x)^{a_0}) dx}{(1 - 2\alpha) F_{a_0}(\mathcal{R}_{\alpha}(F_{a_0})) + \alpha}.
\]

Routine calculations yield
\[
\int_{\beta}^{\gamma} \log((\tau/x) (\tau/x)^{a_0}) dx = \frac{\tau}{(1 - a_0)^2} (\gamma/\tau)^{1-a_0} (1 - (1 - a_0) \log(\gamma/\tau)) - \frac{\tau}{(1 - a_0)^2} (\beta/\tau)^{1-a_0} (1 - (1 - a_0) \log(\beta/\tau))
\]
for $a_0 \in \Theta$ and $\tau \leq \beta < \gamma < \infty$. Hence for every $a_0 \in \Theta$ the random variable $
\mathcal{R}_{\alpha;F_{a_0}}(\hat{\alpha}_{a_0}(Y_{a_0}))$ is centered normal with variance given by (32).

A. Expectiles as risk measures on $L^1$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and use $L^1 = L^1(\Omega, \mathcal{F}, \mathbb{P})$ to denote the usual $L^1$-space. Pick $\alpha \in (0, 1)$ and let $\mathbb{U}_a$ be as in (2).

Lemma A.1 For every $X \in L^1$, the mapping $m \mapsto \mathbb{U}_\alpha(X)(m)$ is real-valued, continuous, strictly decreasing, and satisfies $\lim_{m \to \pm \infty} \mathbb{U}_\alpha(X)(m) = \mp \infty$. In addition it may be represented by
\[
\mathbb{U}_\alpha(X)(m) = -(1 - \alpha) \int_{(\infty, m)} F_X(x) \ell(dx) + \alpha \int_{(m, \infty)} (1 - F_X(x)) \ell(dx)
\]
for all $m \in \mathbb{R}$, where $F_X$ is the distribution function of $X$.

Proof First of all note that the expectation $\mathbb{E}[\mathbb{U}_\alpha(X - m)]$ exists for every $m \in \mathbb{R}$, because $X \in L^1$. Further, we have
\[
\mathbb{U}_\alpha(X)(m)
= \alpha \mathbb{E}[(X - m)^+] - (1 - \alpha) \mathbb{E}[-(X - m)^+]
= \alpha \int_{(m, \infty)} (1 - F_X(x)) \ell(dx) - (1 - \alpha) \int_{(-\infty, m)} (1 - F_X(x)) \ell(dx)
= \alpha \int_{(m, \infty)} (1 - F_X(x)) \ell(dx) - (1 - \alpha) \int_{(-\infty, m)} F_X(x) \ell(dx)
\]
for all $m \in \mathbb{R}$. 

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That is, (57) holds. Applying Change of Variable to both integrals yields (58).

For every \( m \in \mathbb{R} \) and \( \varepsilon > 0 \), we obtain by (57) that
\[
U_\alpha(X)(m + \varepsilon) - U_\alpha(X)(m) = -\int_{(m,m+\varepsilon)} \left( (1-\alpha)F_X(x) + \alpha(1-F_X(x)) \right) \ell(dx) < 0.
\]
Thus, \( m \mapsto U_\alpha(X)(m) \) is strictly decreasing. By the dominated convergence theorem we also have
\[
\lim_{m \to m} \mathbb{E}[(X - \tilde{m})^+] = \mathbb{E}[(X - m)^+] \quad \text{and} \quad \lim_{m \to m} \mathbb{E}[(\tilde{m} - X)^+] = \mathbb{E}[(m - X)^+]
\]
for every \( m \in \mathbb{R} \), implying continuity of \( m \mapsto U_\alpha(X)(m) \). By the Monotone Convergence theorem, we have
\[
\lim_{m \to -\infty} \mathbb{E}[(X - m)^+] = \infty \quad \text{and} \quad \lim_{m \to \infty} \mathbb{E}[(m - X)^+] = \infty,
\]
and by the dominated convergence theorem we obtain
\[
\lim_{m \to -\infty} \mathbb{E}[(X - m)^+] = 0 \quad \text{and} \quad \lim_{m \to \infty} \mathbb{E}[(m - X)^+] = 0.
\]
This gives
\[
\lim_{m \to -\infty} U_\alpha(X)(m) = \infty \quad \text{and} \quad \lim_{m \to \infty} U_\alpha(X)(m) = -\infty.
\]
The proof is complete. \( \square \)

Lemma [A.1] ensures that (3) defines a map \( \rho_\alpha : L^1 \to \mathbb{R} \). The following proposition shows that this map is a coherent risk measure when \( 1/2 \leq \alpha < 1 \).

**Proposition A.2** The map \( \rho_\alpha : L^1 \to \mathbb{R} \) is monotone, cash-invariant, positively homogeneous, and continuous w.r.t. the \( L^1 \)-norm \( \| \cdot \|_1 \). It is subadditive (and thus coherent) if and only if \( 1/2 \leq \alpha < 1 \). If \( 0 < \alpha < 1/2 \), then the map \( \bar{\rho}_\alpha : L^1 \to \mathbb{R} \) defined by \( \bar{\rho}_\alpha(X) := -\rho_\alpha(-X) \) provides a \( \| \cdot \|_1 \)-continuous coherent risk measure.

**Proof** In view of Lemma [A.1] it may be verified easily that the map \( \rho_\alpha \) is monotone, cash-invariant, and positively homogeneous. Concerning subadditivity for \( 1/2 \leq \alpha < 1 \), we want to show that \( \rho_\alpha \) is a convex mapping. For this purposes let \( X_1, X_2 \in L^1 \) and \( \lambda \in [0,1] \). By convexity of \( U_\alpha \) we may observe
\[
0 \geq \lambda U_\alpha(X_1)(\rho_\alpha(X_1)) + (1-\lambda)U_\alpha(X_2)(\rho_\alpha(X_2))
= \lambda \mathbb{E}[U_\alpha(X_1) - \rho_\alpha(X_1)] + (1-\lambda)\mathbb{E}[U_\alpha(X_2) - \rho_\alpha(X_2)]
\geq \mathbb{E}[U_\alpha(\lambda X_1 - \rho_\alpha(X_1)) + (1-\lambda)(X_2 - \rho_\alpha(X_2))]
= U_\alpha(\lambda X_1 + (1-\lambda)X_2)(\lambda \rho_\alpha(X_1) + (1-\lambda)\rho_\alpha(X_2)).
\]
Since by Lemma [A.1] the mapping \( m \mapsto U_\alpha(X)(m) \) is strictly decreasing for any \( X \in L^1 \), we may conclude
\[
\lambda \rho_\alpha(X_1) + (1-\lambda)\rho_\alpha(X_2) \geq \rho_\alpha(\lambda X_1 + (1-\lambda)X_2).
\]
This shows convexity of $\rho_\alpha$ which along with positive homogeneity implies subadditivity. Hence $\rho_\alpha$ is a coherent risk measure on $L^1$, so that it is also continuous w.r.t. $\|\cdot\|_1$ due to Theorem 4.1 in [16]. Moreover, by Proposition 6 in [6], the restriction of $\rho_\alpha$ to $L^2$ is a coherent risk measure if and only if $1/2 \leq \alpha < 1$. This proves the first part of Proposition [A.2] except the $\|\cdot\|_1$-continuity for $0 < \alpha < 1/2$.

To prove the second part, let $0 < \alpha < 1/2$. In this case the mapping $m \mapsto U_\alpha(m)$ is concave, which implies

\[
0 \leq \lambda U_\alpha(X_1)(\rho_\alpha(X_1)) + (1 - \lambda)U_\alpha(X_2)(\rho_\alpha(X_2)) \\
= \lambda \mathbb{E}[U_\alpha(X_1 - \rho_\alpha(X_1))] + (1 - \lambda)\mathbb{E}[U_\alpha(X_2 - \rho_\alpha(X_2))] \\
\leq \mathbb{E}[\lambda(X_1 - \rho_\alpha(X_1)) + (1 - \lambda)(X_2 - \rho_\alpha(X_2))] \\
= U_\alpha(\lambda X_1 + (1 - \lambda)X_2)(\lambda \rho_\alpha(X_1) + (1 - \lambda)\rho_\alpha(X_2))
\]

for any $X_1, X_2 \in L^1$ and $\lambda \in [0, 1]$. Again by Lemma [A.1] the mapping $m \mapsto U_\alpha(X)(m)$ is strictly decreasing for any $X \in L^1$. We thus obtain

\[
\lambda \rho_\alpha(X_1) + (1 - \lambda)\rho_\alpha(X_2) \leq \rho_\alpha(\lambda X_1 + (1 - \lambda)X_2) \quad \text{for all } X_1, X_2 \in L^1, \lambda \in [0, 1].
\]

Therefore the map $\tilde{\rho}_\alpha : L^1 \to \mathbb{R}$ defined by $\tilde{\rho}_\alpha(X) := -\rho_\alpha(-X)$ is convex. It is also monotone, cash-invariant, positively homogeneous, and thus a coherent risk measure; note again that subadditivity follows from convexity under the condition of positive homogeneity. As a coherent risk measure on $L^1$, the map $\tilde{\rho}_\alpha$ is $\|\cdot\|_1$-continuous, drawing again on [16] Theorem 4.1. Then, obviously, $\rho_\alpha$ is $\|\cdot\|_1$-continuous too which completes the proof.

\[\square\]

**B. Quasi-Hadamard differentiability and functional delta-method**

Let $V$ and $\tilde{V}$ be vector spaces, and let $E \subseteq V$ and $\tilde{E} \subseteq \tilde{V}$ be subspaces equipped with norms $\|\cdot\|_E$ and $\|\cdot\|_{\tilde{E}}$, respectively.

**Definition B.1** Let $H : V_H \to \tilde{V}$ be a map defined on a subset $V_H \subseteq V$, and $E_0$ be a subset of $E$. Then $H$ is said to be quasi-Hadamard differentiable at $x \in V_H$ tangentially to $E_0(\tilde{E})$ with trace $\tilde{E}$ if $H(x) - H(y) \in \tilde{E}$ for all $y \in V_H$ and there exists a continuous map $\tilde{H}_x : E_0 \to \tilde{E}$ such that

\[
\lim_{n \to \infty} \left\| \tilde{H}_x(x_0) - \frac{H(x + \varepsilon_n x_n) - H(x)}{\varepsilon_n} \right\|_{\tilde{E}} = 0 \tag{59}
\]

holds for each triplet $(x_0, (x_n), (\varepsilon_n))$, with $x_0 \in E_0$, $(\varepsilon_n) \subseteq (0, \infty)$ satisfying $\varepsilon_n \to 0$, $(x_n) \subseteq E$ satisfying $\|x_n - x_0\|_E \to 0$ as well as $(x + \varepsilon_n x_n) \subseteq V_H$. In this case the map $\tilde{H}_x$ is called quasi-Hadamard derivative of $H$ at $x$ tangentially to $E_0(\tilde{E})$ with trace $\tilde{E}$.
Remark B.2 (i) When $\hat{V} = \hat{E}$, then $H(x) - H(y) \in \hat{E}$ automatically holds for all $x, y \in V_H$ and the notion of quasi-Hadamard differentiability of $H$ at $x \in V_H$ tangentially to $E_0(E)$ with trace $\hat{E}$ coincides with the notion of quasi-Hadamard differentiability of $H$ at $x$ tangentially to $E_0(E)$ as introduced in [7, 8].

(ii) When $\hat{V} = E$, $E_0 = E$, and $\| \cdot \|_E$ provides a norm on all of $V$, then the notion of quasi-Hadamard differentiability of $H$ at $x$ tangentially to $E_0(E)$ coincides with the classical notion of Hadamard differentiability at $x$ tangentially to $E$ as defined in [20]. However, in general the Hadamard derivative of $H$ at $x$ tangentially to $E$ is not necessarily the same as the quasi-Hadamard derivative of $H$ tangentially to $E(E)$, because in the latter case the norm $\| \cdot \|_E$ may be defined only on $E$ (and not on all of $V$).

(iii) When $E_0 = E$ and $\| \cdot \|_E$ provides a norm on all of $V$ then we skip the prefix “quasi” and speak of Hadamard differentiability of $H$ at $x$ tangentially to $E$ with trace $\hat{E}$, and when even $E_0 = E = V$ then we in addition skip the suffix “tangentially to $E$”.

The discussion in part (ii) of the preceding remark shows in particular that quasi-Hadamard differentiability is a weaker notion of “differentiability” than the classical (tangential) Hadamard differentiability. However, Theorem B.3 shows that this notion is still strong enough to obtain a functional delta-method (even for the bootstrap).

Denote by $B^o$ the $\sigma$-algebra on $E$ that is generated by the open balls. Convergence in distribution in $E$ will be considered for the open-ball $\sigma$-algebra. More precisely, let $(\xi_n)$ be a sequence of $(E, B^o)$-valued random variables on a probability space $(\Omega, F, P)$, and $\xi$ be an $(E, B^o)$-valued random variable on some probability space $(\hat{\Omega}, \hat{F}, \hat{P})$. Then $(\xi_n)$ is said to converge in distribution to $\xi$, in symbols $\xi_n \overset{\circ}{\sim} \xi$, if $\int f d\hat{P}_\xi_n \to \int f d\hat{P}_\xi$ for all bounded, continuous and $(B^o, B(\mathbb{R}))$-measurable functions $f : E \to \mathbb{R}$. Note that, whenever $\xi$ concentrates on a separable measurable set, we have $\xi_n \overset{\circ}{\sim} \xi$ if and only if $\varrho^o_{BL}(P_{\xi_n}, P_{\xi}) \to 0$, where $\varrho^o_{BL}$ is the bounded Lipschitz distance defined by

$$\varrho^o_{BL}(\mu, \nu) := \sup_{f \in BL^o_1} \left| \int f \, d\mu - \int f \, d\nu \right|$$

with $BL^o_1$ the set of all $(B^o, B(\mathbb{R}))$-measurable $f : E \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq \|x - y\|_E$ for all $x, y \in E$ and $\sup_{x \in E} |f(x)| \leq 1$. If $(E, \| \cdot \|_E)$ is separable, then $B^o$ coincides with the Borel $\sigma$-algebra $B$ on $E$. Then the notion of convergence $\xi_n \overset{\circ}{\sim} \xi$ boils down to the conventional notion of convergence in distribution, because every continuous function $f : E \to \mathbb{R}$ is $(B, B(\mathbb{R}))$-measurable. We then also write $\xi_n \sim \xi$ and $\varrho_{BL}$ instead of $\xi_n \overset{\circ}{\sim} \xi$ and $\varrho^o_{BL}$, respectively.

Now, let $(\Omega, F, P)$ be a probability space, and $(\hat{T}_n)$ be a sequence of maps $\hat{T}_n : \Omega \to V$. Regard $\omega \in \Omega$ as a sample drawn from $P$, and $\hat{T}_n(\omega)$ as a statistic derived from $\omega$. Let $\theta \in V$, and $(a_n)$ be a sequence of positive real numbers tending to $\infty$. Let $(\Omega', F', P')$ be another probability space and set $(\hat{\Omega}, \hat{F}, \hat{P}) := (\Omega \times \Omega', F \otimes F', P \otimes P')$. The probability measure $P'$ represents a random experiment that is run independently of the random
Then the following assertions hold:

Let $(\hat{T}_n^*)$ be a sequence of maps $\hat{T}_n^* : \Omega \to V$. Finally denote by $\tilde{B}$ and $\tilde{g}_{BL}$ the Borel $\sigma$-algebra on $E$ and the bounded Lipschitz distance on $E$, respectively. The following theorem is a slight generalization of Theorem 3.1 in [3]; one can use the same proof with the obvious (minor) modifications.

**Theorem B.3** Let $H : V_H \to V$ be a map defined on a subset $V_H \subseteq V$. Let $E_0 \subseteq E$ be a separable subspace and assume that $E_n \in B^\circ$. Assume that $(E, \| \cdot \|_E)$ is separable, let $(a_n)$ be a sequence of positive real numbers tending to $\infty$, and consider the following conditions:

(a) $a_n(\hat{T}_n - \theta)$ takes values only in $E$, is $(\mathcal{F}, B^\circ)$-measurable, and satisfies

$$a_n(\hat{T}_n - \theta) \xrightarrow{\mathbb{P}} \xi \quad \text{in } (E, B^\circ, \| \cdot \|_E)$$

for some $(E, B^\circ)$-valued random variable $\xi$ on some probability space $(\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ with $\xi(\Omega) \subseteq E_0$.

(b) $a_n(H(\hat{T}_n) - H(\theta))$ takes values only in $\tilde{E}$ and is $(\mathcal{F}, \tilde{B})$-measurable.

(c) The map $H$ is quasi-Hadamard differentiable at $\theta$ tangentially to $E_0(\xi)$ with trace $\tilde{E}$ and quasi-Hadamard derivative $\hat{H}_\theta$.

(d) The quasi-Hadamard derivative $\hat{H}_\theta$ can be extended from $E_0$ to $E$ such that the extension $\hat{H}_\theta : E \to \tilde{E}$ is linear and $(B^\circ, \tilde{B})$-measurable. Moreover, the extension $\hat{H}_\theta : E \to \tilde{E}$ is continuous at every point of $E_0$.

(e) $a_n(H(\hat{T}_n^*) - H(\hat{T}_n))$ takes values only in $\tilde{E}$ and is $(\mathcal{F}, \tilde{B})$-measurable.

(f) $a_n(\hat{T}_n^* - \theta)$ and $a_n(\hat{T}_n^* - \hat{T}_n)$ take values only in $E$ and are $(\mathcal{F}, B^\circ)$-measurable, and

$$a_n(\hat{T}_n^*(\omega, \cdot) - \hat{T}_n(\omega)) \xrightarrow{\mathbb{P}} \xi \quad \text{in } (E, B^\circ, \| \cdot \|_E), \quad \mathbb{P}\text{-a.e. } \omega.$$

(f') $a_n(\hat{T}_n^* - \theta)$ and $a_n(\hat{T}_n^* - \hat{T}_n)$ take values only in $E$ and are $(\mathcal{F}, B^\circ)$-measurable, and

$$\lim_{n \to \infty} \mathbb{P}^{\mathbb{P}}[\{\omega \in \Omega : \tilde{g}_{BL}(\mathbb{P}'_{a_n(\hat{T}_n^*(\omega, \cdot) - \hat{T}_n(\omega))}, \tilde{P}_\theta(\xi) \geq \delta)\} = 0 \quad \text{for all } \delta > 0.$$

Then the following assertions hold:

(i) If conditions (a)–(c) hold, then $\hat{H}_\theta(\xi)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable and

$$a_n(H(\hat{T}_n) - H(\theta)) \xrightarrow{\mathbb{P}} \hat{H}_\theta(\xi) \quad \text{in } (\tilde{E}, \tilde{B}, \| \cdot \|_{\tilde{E}}).$$

(ii) If conditions (a)–(f) hold, then $\hat{H}_\theta(\xi)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable and

$$\lim_{n \to \infty} \mathbb{P}[\{\omega \in \Omega : \tilde{g}_{BL}(\mathbb{P}'_{a_n(H(\hat{T}_n^*(\omega, \cdot) - H(\hat{T}_n(\omega))))}, \tilde{P}_\theta(\xi) \geq \delta)\} = 0 \quad \text{for all } \delta > 0.$$
(iii) Assertion (ii) still holds when assumption (f) is replaced by (f').

For (f) and (f’) in the preceding theorem note that the mapping \( \omega' \mapsto a_n(\hat{T}_n^*(\omega, \omega') - \hat{T}_n(\omega)) \) is \((\mathcal{F}', \mathcal{B}')\)-measurable for every fixed \( \omega \in \Omega \), because \( a_n(\hat{T}_n^* - \hat{T}_n) \) is \((\mathcal{F}, \mathcal{B})\)-measurable with \( \mathcal{F} = \mathcal{F} \otimes \mathcal{F}' \). That is, \( a_n(\hat{T}_n^*(\omega, \cdot) - \hat{T}_n(\omega)) \) can be seen as an \((\mathcal{E}, \mathcal{B}')\)-valued random variable on \((\Omega', \mathcal{F}', \mathcal{B}')\) for every fixed \( \omega \in \Omega \). Analogously, we can regard \( a_n(H(\hat{T}_n^*(\omega, \cdot)) - H(\hat{T}_n(\omega))) \) as a real-valued random variable on \((\Omega', \mathcal{F}', \mathcal{B}')\) for every fixed \( \omega \in \Omega \). This matters for the formulation of part (ii) in the preceding theorem.

C. Convergence in distribution of the empirical process regarded as an \( L_1 \)-valued random variable

By definition \( L_1 \) is the set of all Borel measurable functions \( v : \mathbb{R} \to \mathbb{R} \) with \( \|v\|_{1, \ell} < \infty \) modulo the equivalence relation of almost sure identity, where \( \| \cdot \|_{1, \ell} \) is defined in (9). It is known that \( (L_1, \| \cdot \|_{1, \ell}) \) is a separable Banach space; cf. Theorem 4.1.3 and Corollary 4.2.2 in [10]. Denote by \( \mathcal{B}_1 \) the Borel algebra on \( L_1 \) w.r.t. the norm \( \| \cdot \|_{1, \ell} \). Let \( \xi \) be a real-valued stochastic process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with index set \( \mathbb{R} \). That is, \( \xi : \Omega \times \mathbb{R} \to \mathbb{R} \) is any map such that the coordinate \( \omega \mapsto \xi(t, \omega) \) is \((\mathcal{F}, \mathcal{B}(\mathbb{R}))\)-measurable for every \( t \in \mathbb{R} \). The process \( \xi \) is said to be measurable if \( \xi : \Omega \times \mathbb{R} \to \mathbb{R} \) is \((\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))\)-measurable.

Lemma C.1 If the stochastic process \( \xi \) is measurable and \( \xi(\omega, \cdot) \in L_1 \) for all \( \omega \in \Omega \), then \( \omega \mapsto \xi(\omega, \cdot) \) is an \((\mathcal{F}, \mathcal{B}_1)\)-measurable mapping from \( \Omega \) to \( L_1 \). In particular, \( \xi \) can be seen as an \((L_1, \mathcal{B}_1)\)-valued random variable.

Proof The family of all open balls generate \( \mathcal{B}_1 \), because \((L_1, \mathcal{B}_1)\) is separable. Thus it suffices to show that \( \xi^{-1}(B_r(v)) \in \mathcal{F} \) for every \( r > 0 \) and \( v \in L_1 \), where \( B_r(v) \) denotes the \( \| \cdot \|_{1, \ell}\)-open ball with radius \( r > 0 \) around \( v \in L_1 \). Let \( r > 0 \) and \( v \in L_1 \). By assumption the map \( \xi : L_1 \times \mathbb{R} \to \mathbb{R} \) is \((\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))\)-measurable. Since the mapping \( t \mapsto v(t) \) is Borel measurable for every \( v \in L_1 \), it follows that also the map \( \xi_v : L_1 \times \mathbb{R} \to \mathbb{R} \) defined by \( \xi_v(\omega, t) := v(t) \) is \((\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))\)-measurable. By Fubini’s theorem we obtain in particular that the map \( \mathcal{I}_v : \Omega \to \mathbb{R} \) defined by \( \mathcal{I}_v(\omega) := \int |\xi_v(\omega, t) - \xi(\omega, t)| \ell(dt) \) is \((\mathcal{F}, \mathcal{B}(\mathbb{R}))\)-measurable. Along with

\[
\xi^{-1}(B_r(v)) = \left\{ \omega \in \Omega : \int |\xi_v(\omega, t) - \xi(\omega, t)| \ell(dt) < r \right\} = \mathcal{I}_v^{-1}([0, r)),
\]

this implies \( \xi^{-1}(B_r(v)) \in \mathcal{F} \).

Remark C.2 It is well known that every real-valued stochastic process \( \xi \) with right-continuous paths is measurable. In particular, the process \( \sqrt{n}(F_{\hat{\theta}_n} - F) \) is measurable when \( F \) is a distribution function and \( F_{\hat{\theta}_n} \) is a process with right-continuous paths. It
follows that $\sqrt{n}(F_{\hat{a}_n} - F)$ can be seen as an $L_1$-valued random variable when $F \in F_1$ and $F_{\hat{a}_n}$ takes values only in $F_1$.

The following Theorem C.3 recalls the statements of Propositions 3.2 and 3.5 in [19]. Here $L_\infty$ refers to the space of all bounded Borel measurable functions from $\mathbb{R}$ to $\mathbb{R}$ modulo the equivalence relation of $\ell$-almost sure identity.

**Theorem C.3** With the notation and under the assumptions of Theorem 3.2 (except the continuity of $F$ at $\mathcal{R}_\alpha(F)$),

$$\sqrt{n}(\hat{F}_n - F) \leadsto B_F \quad \text{in } (L_1, \mathcal{B}_1, \| \cdot \|_{1, \ell})$$

for an $L_1$-valued centered Gaussian random variable $B_F$ with covariance operator

$$\Phi_{B_F}(f,g) = \int_{\mathbb{R}^2} f(s)C_F(s,t)g(t)\,d(s,t) \quad \text{for all } f, g \in L_\infty, \quad (60)$$

where $C_F(s,t)$ is defined by (14).

Recall from [2] that an $(L_1, \mathcal{B}_1)$-valued random variable $B$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be an $L_1$-valued Gaussian random variable if $\Lambda(B)$ is a real-valued Gaussian random variable for each $\| \cdot \|_{1, \ell}$-continuous linear functional $\Lambda : L_1 \to \mathbb{R}$, i.e., if $\int f(t)B(t)\,\ell(dt)$ is a real-valued Gaussian random variable for every $f \in L_\infty$. The covariance operator of such an $L_1$-valued Gaussian random variable $B$ is the mapping $\Phi_B : L_\infty \times L_\infty \to \mathbb{R}$ defined by

$$\Phi_B(f,g) := \mathbb{E}\left[ \left( \int f(s)(B(s) - \mathbb{E}[B(s)])\,\ell(ds) \right) \left( \int g(t)(B(t) - \mathbb{E}[B(t)])\,\ell(dt) \right) \right].$$

**Theorem C.4** With the notation and under the assumptions of Theorem 3.3 (except the continuity of $F$ at $\mathcal{R}_\alpha(F)$),

$$\sqrt{n}(\hat{F}_n^\ast(\omega, \cdot) - \hat{F}_n(\omega)) \leadsto B_F \quad \text{in } (L_1, \mathcal{B}_1, \| \cdot \|_{1, \ell}), \quad \mathbb{P} \text{-a.e. } \omega, \quad (61)$$

where $B_F$ is as in Theorem C.3 (with $C_F(t_0, t_1) = F(t_0 \land t_1)(1 - F(t_0 \lor t_1))$).

**Proof** Theorem 5.2 in [3] shows that the imposed assumptions imply that (61) with $\leadsto$ and $(L_1, \mathcal{B}_1, \| \cdot \|_{1, \ell})$ replaced by $\leadsto^\circ$ and $(D_\phi, \mathcal{D}_\phi, \| \cdot \|_\phi)$ holds. Here $D_\phi$ is the space of all càdlàg functions $v : \mathbb{R} \to \mathbb{R}$ with $\|v\|_\phi := \sup_{x \in \mathbb{R}} |v(x)|\phi(x) < \infty$ and $\mathcal{D}_\phi$ is the open-ball $\sigma$-algebra on $(D_\phi, \| \cdot \|_\phi)$. Since $D_\phi \subseteq L_1$ and $\| \cdot \|_{1, \ell} \leq C_\phi \| \cdot \|_\phi$ with $C_\phi := \int 1/\phi \,d\ell < \infty$, the natural embedding $D_\phi \to L_1$, $v \mapsto v$, is $( \| \cdot \|_\phi, \| \cdot \|_{1, \ell})$-continuous. Thus the continuous mapping theorem in the form of [3] Theorem 6.4 ensures that (61) itself holds too. \hfill \Box
Theorem C.5 With the notation and under the assumptions of Theorem 3.6 (except the continuity of \( F \) at \( R_\alpha(F) \)),

\[ \sqrt{n}(\hat{F}_n^\alpha(\omega, \cdot) - \hat{F}^\alpha(\omega)) \underset{\text{in } (L_1, B_1, \| \cdot \|_{1,e})}{\sim} B_F \quad \text{P-a.e. } \omega, \tag{62} \]

where \( B_F \) is as in Theorem C.3.

Proof One can argue as in the proof of Theorem C.4 (with Theorem 5.4 of [8] in place of Theorem 5.2 in [8]). \qed

D. A note on the dependence coefficients \( \tilde{\phi} \) and \( \tilde{\alpha} \)

Let \((X_i)_{i \in \mathbb{N}}\) be a strictly stationary and ergodic sequence of real-valued random variables on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). At the beginning of Section 3.2 we claimed that Dedecker and Prieur [20] introduced the following dependence coefficients:

\[ \tilde{\phi}(n) := \sup_{k \in \mathbb{N}} \sup_{x \in \mathbb{R}} \| \mathbb{P}[X_{n+k} \in (-\infty, x)|\mathcal{F}_n^k(\cdot)] - \mathbb{P}[X_{n+k} \in (-\infty, x)] \|_\infty, \]

\[ \tilde{\alpha}(n) := \sup_{k \in \mathbb{N}} \sup_{x \in \mathbb{R}} \| \mathbb{P}[X_{n+k} \in (-\infty, x)|\mathcal{F}_n^k(\cdot)] - \mathbb{P}[X_{n+k} \in (-\infty, x)] \|_1, \]

where \( \mathcal{F}_n^k := \sigma(X_1, \ldots, X_k) \) and \( \| \cdot \|_p \) denotes the usual \( L^p \)-norm on \( L^p = L^p(\Omega, \mathcal{F}, \mathbb{P}) \), \( p \in [1, \infty] \). Note, however, that in [20] the starting point is a strictly stationary and ergodic sequence \((Y_i)_{i \in \mathbb{Z}}\), and the above dependence coefficients are actually defined by

\[ \tilde{\phi}(n) := \sup_{x \in \mathbb{R}} \| \mathbb{P}[Y_n \in (-\infty, x)|\mathcal{F}_0^0(\cdot)] - \mathbb{P}[Y_n \in (-\infty, x)] \|_\infty, \]

\[ \tilde{\alpha}(n) := \sup_{x \in \mathbb{R}} \| \mathbb{P}[Y_n \in (-\infty, x)|\mathcal{F}_0^0(\cdot)] - \mathbb{P}[Y_n \in (-\infty, x)] \|_1, \]

where \( \mathcal{F}_0^0 := \sigma(\{Y_i : i \leq 0\}) \). In the following we will discuss that the strictly stationary and ergodic sequence \((X_i)_{i \in \mathbb{N}}\) can be extended to a strictly stationary sequence \((Y_i)_{i \in \mathbb{Z}}\) being again ergodic and satisfying \( \tilde{\phi}(n) = \tilde{\phi}(n) \) and \( \tilde{\alpha}(n) = \tilde{\alpha}(n) \). More precisely, we may define a strictly stationary and ergodic sequence \((Y_i)_{i \in \mathbb{N}_0}\) by \( Y_i := X_{i+1} \), and Lemma 9.2 of [32] shows that this sequence can be extended to a strictly stationary sequence \((Y_i)_{i \in \mathbb{Z}}\). Lemmas D.3 and D.2 hence show that \((Y_i)_{i \in \mathbb{Z}}\) is again ergodic and that \( \tilde{\phi}(n) = \tilde{\phi}(n) \) and \( \tilde{\alpha}(n) = \tilde{\alpha}(n) \), \( n \in \mathbb{N} \).

For any random variable \( X \) on \((\Omega, \mathcal{F}, \mathbb{P})\) and any sub-\( \sigma \)-algebra \( \mathcal{A} \subseteq \mathcal{F} \), the following dependence coefficients have been introduced in [20]:

\[ \phi(\mathcal{A}, X) := \sup_{x \in \mathbb{R}} \| \mathbb{P}[X \in (-\infty, x)|\mathcal{A}] - \mathbb{P}[X \in (-\infty, x)] \|_\infty, \tag{63} \]

\[ \alpha(\mathcal{A}, X) := \sup_{x \in \mathbb{R}} \| \mathbb{P}[X \in (-\infty, x)|\mathcal{A}] - \mathbb{P}[X \in (-\infty, x)] \|_1. \tag{64} \]

Lemma D.1 Let \((\mathcal{A}_i)_{i \in \mathbb{N}_0}^{\mathbb{N}}\) be an increasing sequence of sub-\( \sigma \)-algebras of \( \mathcal{F} \) satisfying \( \mathcal{A}_\infty = \sigma(\bigcup_{i \in \mathbb{N}} \mathcal{A}_i) \). Then the following assertions hold:
(i) $\phi(A_i, X) \leq \phi(A_{i+1}, X)$ holds for $i \in \mathbb{N}$, and $\lim_{i \to \infty} \phi(A_i, X) = \phi(A_\infty, X)$.

(ii) $\alpha(A_i, X) \leq \alpha(A_{i+1}, X)$ holds for $i \in \mathbb{N}$, and $\lim_{i \to \infty} \alpha(A_i, X) = \alpha(A_\infty, X)$.

**Proof** Let us start by representations of the dependency coefficients established in [20]. For this purpose let $BV_1$ denote the space of all left continuous functions $f : \mathbb{R} \to \mathbb{R}$ with total variation bounded above by 1. According to Lemmas 4 and 1 in [20] we have

$$\phi(A_i, X) = \sup \{\|\text{Cov}(Y, h(X))\| : Y \text{ is } A_i\text{-measurable, } \|Y\|_1 \leq 1, h \in BV_1\} \quad (65)$$

$$\alpha(A_i, X) = \sup \{\|E[h(X)|A_i] - E[h(X)]\|_1 : h \in BV_1\} \quad (66)$$

for every $i \in \mathbb{N} \cup \{\infty\}$. We may observe immediately from (65) that

$$\phi(A_i, X) \leq \phi(A_{i+1}, X) \leq \phi(A_\infty, X) \quad \text{for all } i \in \mathbb{N}. \quad (67)$$

Furthermore, for any $h \in BV_1$ and every $i \in \mathbb{N}$, we may observe

$$\|E[h(X)|A_i] - E[h(X)]\|_1 = E[\|E[h(X)|A_i+1] - E[h(X)]\||A_i]\|
\leq E[\|E[h(X)|A_i+1] - E[h(X)]\| |A_i] = \|E[h(X)|A_{i+1}] - E[h(X)]\|_1. \quad (68)$$

In view of (68) this implies

$$\alpha(A_i, X) \leq \alpha(A_{i+1}, X) \leq \alpha(A_\infty, X) \quad \text{for all } i \in \mathbb{N}. \quad (69)$$

For every fixed $h \in BV_1$ we obtain by (68) and Theorem 10.5.1 in [23] (a version of Doob's martingale convergence theorem) that

$$\lim_{i \to \infty} E[h(X)|A_i] = E[h(X)|A_\infty] \quad \mathbb{P}\text{-a.s.}$$

Since $h$ as an element of $BV_1$ is bounded, it follows by the dominated convergence theorem that

$$\lim_{i \to \infty} \|E[h(X)|A_i] - E[h(X)|A_\infty]\|_1 = \lim_{i \to \infty} E[\|E[h(X)|A_i] - E[h(X)|A_\infty]\|] = 0. \quad (70)$$

For arbitrary $\varepsilon > 0$ we may find by (68) some $h \in BV_1$ such that the inequality $\alpha(A_\infty, X) - \varepsilon < \|E[h(X)|A_\infty] - E[h(X)]\|_1$ holds. Then by (68) along with (66)

$$\alpha(A_\infty, X) \geq \lim_{i \to \infty} \sup \alpha(A_i, X)
\geq \lim_{i \to \infty} \inf \alpha(A_i, X)
\geq \lim_{i \to \infty} \|E[h(X)|A_i] - E[h(X)]\|_1
= \|E[h(X)|A_\infty] - E[h(X)]\|_1
\geq \alpha(A_\infty, X) - \varepsilon.$$

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Hence \( \lim_{i \to \infty} \alpha(A_i, X) = \alpha(A_\infty, X) \). This completes the proof of statement (b).

Now, in addition of \( h \in BV_1 \) let us fix any \( \mathbb{P}_{|A_\infty} \)-integrable random variable \( Y \) with \( \|Y\|_1 = \mathbb{E}[|Y|] \leq 1 \). Firstly, \( Y_i := \mathbb{E}[Y|A_i] \) and \( Z_i := \mathbb{E}[|Y||A_i] \) define martingales adapted to the filtered probability space \((\Omega, (A_i)_{i \in \mathbb{N} \cup \{\infty\}}, A_\infty, \mathbb{P}_{|A_\infty})\) and they satisfy \( \|Y_i\|_1 = \mathbb{E}[|Y_i|] \leq 1 \). and \( \mathbb{E}[][|Z_i|] \leq 1 \) for every \( i \in \mathbb{N} \cup \{\infty\} \). Hence we may draw on Theorem 10.5.1 of [23] to observe \( \lim_{i \to \infty} Y_i = Y \) \( \mathbb{P} \)-a.s. and \( \lim_{i \to \infty} Z_i = Z_\infty \) \( \mathbb{P} \)-a.s. Furthermore, \( \lim_{i \to \infty} \mathbb{E}[Z_i] = \mathbb{E}[|Y|] \) and any \( Z_i \) is nonnegative. Therefore, \( (Z_i) \) is uniformly \( \mathbb{P} \)-integrable which implies that \( \{Y_i\} \) is uniformly \( \mathbb{P} \)-integrable because \( |Y_i| \leq Z_i \) \( \mathbb{P} \)-a.s. holds for every \( i \in \mathbb{N} \cup \{\infty\} \). Thus \( \lim_{i \to \infty} \mathbb{E}[|Y_i - Y|] = \lim_{i \to \infty} \mathbb{E}[|Y_i|] = \mathbb{E}[|Y|] \). and \( \lim_{i \to \infty} \mathbb{E}[Y_i] = \mathbb{E}[Y] \) since \( h(X) \) is a bounded random variable. In particular \( \text{Cov}(Y_i, h(X)) \to \text{Cov}(Y, h(X)) \). If we now take \( (65) \), we may verify \( \phi(A_i, X) \to \phi(A_\infty, X) \) in a similar way as we established \( \alpha(A_i, X) \to \alpha(A_\infty, X) \). This shows the full statement (a) and completes the proof. \( \square \)

Now let \( (X_i)_{i \in \mathbb{N}} \) be any strictly stationary sequence on \((\Omega, \mathcal{F}, \mathbb{P})\), and let \((Y_i)_{i \in \mathbb{Z}}\) be the strictly stationary extension of \((X_i)_{i \in \mathbb{N}}\) as introduced above. In the following lemma we describe the mixing coefficients \( \tilde{\phi}(n) \) and \( \tilde{\alpha}(n) \) in terms of the dependence coefficients defined in \((63)-(64)\) and the sequence \((Y_i)_{i \in \mathbb{Z}}\). Let \( \mathbb{Z}_- := \{0, -1, -2, \ldots\} \).

**Lemma D.2** For \( n \in \mathbb{N} \) we have

\[
\tilde{\phi}(n) = \phi(\sigma(\{Y_i : i \in \mathbb{Z}_-\}), Y_n) \quad \text{and} \quad \tilde{\alpha}(n) = \alpha(\sigma(\{Y_i : i \in \mathbb{Z}_-\}), Y_n).
\]

**Proof** Set \( A_k := \sigma(Y_{-k+1}, \ldots, Y_0) \) for \( k \in \mathbb{N} \), and \( A_\infty := \sigma(\{Y_i : i \in \mathbb{Z}_-\}) \). By strict stationarity of \((Y_i)_{i \in \mathbb{Z}}\), the random vector \((Y_{-k+1}, \ldots, Y_0, Y_n)\) has the same distribution as \((Y_0, \ldots, Y_{k-1}, Y_{n-k}, Y_{n+k-1}) = (X_1, \ldots, X_k, X_n, X_{n+k})\) for every \( k, n \in \mathbb{N} \). In particular, for every \( k, n \in \mathbb{N} \), the random variables \( Y_n \) and \( X_n, X_{n+k} \) are identically distributed and \( \mathbb{P}_{Y_n|A_k} = \mathbb{P}_{X_{n+k}|F_i} \). Hence we may observe

\[
\tilde{\phi}(n) = \sup_{k \in \mathbb{N}} \phi(A_k, Y_n) \quad \text{and} \quad \tilde{\alpha}(n) = \sup_{k \in \mathbb{N}} \alpha(A_k, Y_n) \quad \text{for every} \; k, n \in \mathbb{N}.
\]

Finally, \( A_k \subseteq A_{k+1} \) holds for \( k \in \mathbb{N} \), and \( A_\infty \) is generated by \( \bigcup_{k=1}^{\infty} A_k \). Then the statement of Lemma D.2 follows immediately from Lemma D.1 \( \square \)

**Lemma D.3** The sequence \((Y_i)_{i \in \mathbb{Z}}\) is ergodic if the sequence \((X_i)_{i \in \mathbb{N}}\) is ergodic.

**Proof** For \( I = \mathbb{Z} \) or \( I = \mathbb{N} \), denote by \( \mathcal{B}(\mathbb{R}^I) \) the standard Borel \( \sigma \)-algebra on \( \mathbb{R}^I \) (generated by the standard product topology on \( \mathbb{R}^I \)) and let

\[
S_I : \mathbb{R}^I \to \mathbb{R}^I, \quad (x_i)_{i \in I} \mapsto (x_{i+1})_{i \in I}
\]

be the (one-step) shift operator. Furthermore let \( \mathcal{M}_I(S_I) \) be the set of all probability measures \( \mu \) on \( \mathcal{B}(\mathbb{R}^I) \) satisfying \( \mu = \mu \circ S_I^{-1} \), that is, the set of all probability measures
\(\mu\) on \(\mathcal{B}(\mathbb{R}^I)\) under which the shift operator \(S_I\) is measure-preserving. Recall that \(\mu \in \mathcal{M}(S_I)\) is said to be \((S_I,\mathcal{I})\)-ergodic if the corresponding invariant \(\sigma\)-algebra \(\mathcal{I}\) (i.e. the set of all \(A \in \mathcal{B}(\mathbb{R}^I)\) with \(A = S_I^{-1}(A)\)) is trivial (i.e. \(\mu[A] \in \{0,1\}\) for all \(A \in \mathcal{I}\)). It is known that

\[
\{\mu \in \mathcal{M}_I(S_I) : \mu \text{ is } S_I\text{-ergodic} \} = \{\mu \in \mathcal{M}_I(S_I) : \mu \text{ is extreme point of } \mathcal{M}_I(S_I)\};
\]

see, for instance, Theorem 9.12 of [32]. The sequence \((Y_i)_{i \in \mathbb{Z}}\) is strictly stationary and so its distribution \(P_{(Y_i)_{i \in \mathbb{Z}}}\) belongs to \(\mathcal{M}_I(S_Z)\). Thus it suffices to show that \(P_{(Y_i)_{i \in \mathbb{Z}}}\) is an extreme point of \(\mathcal{M}_I(S_Z)\). This will be done by way of contradiction.

Suppose that \(P_{(Y_i)_{i \in \mathbb{Z}}}\) is not an extreme point of \(\mathcal{M}_I(S_Z)\). Then there exist different \(\mu, \nu \in \mathcal{M}_I(S_Z)\) and some \(\lambda \in (0,1)\) such that \(P_{(Y_i)_{i \in \mathbb{Z}}} = \lambda \mu + (1 - \lambda) \nu\). For the Borel-measurable mapping

\[
\Pi_N : \mathbb{R}^\mathbb{Z} \longrightarrow \mathbb{R}^\mathbb{N}, \quad (x_i)_{i \in \mathbb{Z}} \longmapsto (x_i)_{i \in \mathbb{N}}
\]

we have \(S_N \circ \Pi_N = \Pi_N \circ S_Z\). Thus \((\bar{\mu} \circ \Pi_N^{-1}) \circ S_N^{-1} = (\bar{\mu} \circ S_Z^{-1}) \circ \Pi_N^{-1} = \bar{\mu} \circ \Pi_N^{-1}\) for every \(\bar{\mu} \in \mathcal{M}_I(S_Z)\), and so \(\bar{\mu} \circ \Pi_N^{-1} \in \mathcal{M}_I(S_N)\) for every \(\bar{\mu} \in \mathcal{M}_I(S_Z)\). Further, we clearly have \(P_{(Y_i)_{i \in \mathbb{Z}}} \circ \Pi_N^{-1} = P_{(X_i)_{i \in \mathbb{N}}} \circ S_N^{-1}\) for the distribution \(P_{(X_i)_{i \in \mathbb{N}}}\) of \((X_i)\), and by the stationarity of \((X_i)\) we also have \(P_{(X_i)_{i \in \mathbb{N}}} \in \mathcal{M}_I(S_N)\), i.e. \(P_{(X_i)_{i \in \mathbb{N}}} \circ S_N^{-1} = P_{(X_i)_{i \in \mathbb{N}}}\). Thus \(P_{(Y_i)_{i \in \mathbb{Z}}} \circ \Pi_N^{-1} = P_{(X_i)_{i \in \mathbb{N}}}\). In particular, \(P_{(Y_i)_{i \in \mathbb{Z}}} \circ \Pi_N^{-1} = \lambda \mu \circ \Pi_N^{-1} + (1 - \lambda) \nu \circ \Pi_N^{-1}\). Since \(P_{(Y_i)_{i \in \mathbb{Z}}} \circ \Pi_N^{-1} = P_{(X_i)_{i \in \mathbb{N}}^{-1}}\) is an extreme point of \(\mathcal{M}_I(S_N)\) by assumption and \([69]\), it follows that \(\mu \circ \Pi_N^{-1} = \nu \circ \Pi_N^{-1}\). Since \(\mu\) and \(\nu\) belong to \(\mathcal{M}_I(S_Z)\), we may conclude that \(\mu\) and \(\nu\) have identical finite-dimensional marginal distributions. In particular \(\mu = \nu\), which contradicts the assumption \(\mu \neq \nu\). \(\square\)

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