Note On Jakimovski-Leviatan Operators Preserving $e^{-x}$

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Abstract

In the present article, a modification of Jakimovski-Leviatan operators is presented which reproduce constant and $e^{-x}$ functions. We prove uniform convergence order of a quantitative estimate for the modified operators. We also give a quantitative Voronovskaya type theorem.

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1 Introduction

J. P. King [7] introduced a modification of the well known Bernstein polynomials which preserve constant and the $x^2$ test function. This modification provide better approximation over the usual Bernstein polynomials. In addition, a modification of Szász operators is presented that reproduces the functions $1$ and $e^{2ax}$, $a > 0$ fixed and also are proved uniform convergence, order of approximation via a certain weighted modulus of continuity, and a quantitative Voronovskaya-type theorem by T. Acar [13]. Many different applications of similar type of operators have studied in [7]-[9].

In [1], Jakimovski and Leviatan constructed a new type of operators $P_k$ by using Appell polynomials given as below: $g(u) = \sum_{n=0}^{\infty} a_n u^n$, $g(1) \neq 1$ be an analytic function in the disk $|u < r|$ $(r > 1)$ and $p_k(x) = \sum_{i=0}^{k} a_i x^{i-k!}$, $k \in$
\[ g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \] (1)

Let \( E[0, \infty) \) indicate the class of functions of exponential type on \([0, \infty)\) which satisfy the property \(|f(x)| \leq \beta e^{\alpha x}\) for some finite constants \( \alpha, \beta > 0 \).

In [1], the authors considered the operator \( P_n : E \to C[0, \infty) \)
\[ P_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{p_k(nx)}{k!} f\left(\frac{k}{n}\right). \] (2)

**Remark 1.** If \( g(1) = 1 \) in (1), we obtain \( p_k(x) = \frac{x^k}{k!} \) and we get classical Szász-Mirakjan operator which is given by
\[ S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \]

B. Wood in [6] proved that the operators \( P_n \) are positive if and only if \( \frac{a_n}{g(1)} \geq 0 \) for \( n \in \mathbb{N} \). In [5], Ciupa studied the rate of convergence of these operators. The convergence of these operators in a weighted space of functions on a positive semi-axis and estimate the approximation by using a new type of weighted modulus of continuity introduced by A.D. Gadjiev and A. Aral in [8] were studied in [3].

### 2 Main Results

In this section, we consider the following modified form of generalization of Jakimovski-Leviatan operators
\[ E_n(f; x) = e^{-na_n(x)} \sum_{k=0}^{\infty} \frac{a_k(na_n(x))^k}{k!} f\left(\frac{k}{n}\right) \] (3)

\( x \geq 0, n \in \mathbb{N} \) such that the conditions
\[ E_n(e^{-t}; x) = e^{-x} \] (4)

are satisfied for all \( x \) and all \( n \). Therefore by simple computation, we get
\[ e^{-x} = e^{na_n(x)}[e^{-1/n} - 1]. \]

This implies
\[ a_n(x) = \frac{-x}{n(e^{-1/n} - 1)} \] (5)

in other words we can write as
\[ a_n(x) = \frac{xe^{1/n}}{n(e^{1/n} - 1)}. \]

Thus the operator (3) can be rewritten the following form:
\[ E_n(f; x) = e^{\frac{xe^{1/n}}{n(e^{1/n} - 1)}} \sum_{k=0}^{\infty} a_k \frac{(-1)^k}{k!} \left(\frac{xe^{1/n}}{e^{1/n} - 1}\right)^k f\left(\frac{k}{n}\right) \] (6)

**Lemma 1.** The moment for the operator (3) may be given as
\[ E_n(e^{At}; x) = e^{na_n(x)}[e^{A/n} - 1]. \] (7)
Proof. From (5), we have
\[
E_n(e^{\lambda t};x) = \frac{e^{-na_n(x)}}{g(1)} \sum_{k=0}^{\infty} \frac{a_k (na_n(x))^k}{k!} e^{A_n t}
\]
\[
= \frac{1}{g(1)} \sum_{k=0}^{\infty} \frac{e^{-na_n(x)} (na_n(x)e^{A_n})^k}{k!}
\]
\[
e^{na_n(x)[e^{A_n t} - 1]}
\]

\[\square\]

Lemma 2. If the operator \(E_n\) is defined by (3), then with \(e_r(t) = t^r, r = 0, 1, 2\ldots\) the moments as follows;
\[
E_n(e_0; x) = 1
\]
\[
E_n(e_1; x) = a_n(x)
\]
\[
E_n(e_2; x) = a_n^2(x) + a_n(x)/n
\]
(9)

Lemma 3. By Lemma (2) the central moments for \(E_n\) operator
\[
E_n(\phi^m(t); x) = E_n((t-x)^m; x), m = 0, 1, 2
\]
are given by
\[
E_n(\phi^0(t); x) = 1
\]
\[
E_n(\phi^1(t); x) = a_n(x) - x
\]
\[
E_n(\phi^2(t); x) = (a_n(x) - x)^2 + a_n(x)/n
\]
Furthermore,
\[
\lim_{n \to \infty} n (-x n e^{-1/n - 1} - x) = \frac{x}{2}
\]
\[\text{(12)}\]
\[
\lim_{n \to \infty} n (a_n(x) - x)^2 + \frac{a_n(x)}{n} = x
\]
\[\text{(13)}\]

3 A Quantitative Result

In this section, we represent the rate of uniform convergence for the \(E_n\) operators. In [12], the uniform convergence estimate for any sequence of positive linear operators were established by Boyanov and Veselinov. In [11], Holhoš presented the following theorem:

Theorem 1. ([11]) If a sequence of linear positive operators \(L_n : C^* [0, \infty) \to C^* [0, \infty)\) satisfy the equalities
\[
\|L_ne_0 - 1\|_{[0, \infty)} = \alpha_n
\]
\[
\|L_n(e^{-t}) - e^{-x}\|_{[0, \infty)} = \beta_n
\]
\[
\|L_n(e^{-2t}) - e^{-2x}\|_{[0, \infty)} = \gamma_n
\]
(14)
then, for \(f \in C^* [0, \infty)\), we have
\[
\|L_nf - f\|_{[0, \infty)} \leq \alpha_n \|f\|_{[0, \infty)} + (2 + \alpha_n) \omega^*(f, \sqrt{\alpha_n + 2\beta_n + \gamma_n}),
\]
(15)
where the modulus of continuity is defined as:
\[
\omega^*(f, \delta) = \sup_{|t-x| \leq \delta, x \geq 0} |f(t) - f(x)|.
\]
(16)
Now, we will prove the quantitative estimate for $E_n$ operators defined by (3) which preserve $e^{-x}$ function:

**Theorem 2.** For $f \in C^+[0, \infty)$, we get

$$\|E_n f - f\|_{[0, \infty)} \leq 2 \omega^*(f; \sqrt{2\beta_n + \gamma_n}),$$

where

$$\beta_n = \|E_n(e^{-t}) - e^{-x}\|_{[0, \infty)}$$

$$\gamma_n = \|E_n(e^{-2t}) - e^{-2x}\|_{[0, \infty)}.$$

Here, $\beta_n$ and $\gamma_n$ tend to zero as $n$ goes to infinity, therefore $E_n$ converges uniformly to $f$.

**Proof.** By using the equalities (3) and (5), we get

$$E_n(e^{-\lambda t}; x) = e^{-na_n(x)} \sum_{k=0}^{\infty} a_k \frac{(na_n(x))^k}{k!} e^{-\lambda \frac{k}{n}} = e^{-na_n(x)} e^{na_n(x)e^{-\lambda/n}} = e^{na_n(x)}(e^{-\lambda/n} - 1) = e^{\lambda/n}(e^{1/n} - 1).$$

Firstly, let take $\lambda = 1$. Using the inequality

$$\frac{u - v}{\ln u - \ln v} < \frac{u + v}{2}$$

for $0 < v < u$, we get

$$e^{-u_n} - e^{-x} < \frac{1 - u_n}{2}(xe^{-u_n} + xe^{-x}).$$

On the other hand, since

$$\max_{x>0} x e^{-bx} = \frac{1}{eb}$$

for every $b > 0$, we can write as

$$e^{-u_n} - e^{-x} < \frac{1 - u_n}{2} \left( \frac{1}{eu_n} + \frac{1}{e} \right) = \frac{1 - u_n^2}{2eu_n}$$

where $u_n = -\frac{1}{e^{1/n}} \left( e^{1/n} - 1 \right)$. Thus

$$\|E_n(e^{-t}; x) - e^{-x}\|_{[0, \infty)} = \beta_n < \frac{1 - u_n^2}{2eu_n} \to 0,$$

as $n \to \infty$.

For $\lambda = 2$, we have

$$e^{-v_n} - e^{-2x} < \frac{2 - v_n}{2}(xe^{-v_n} + xe^{-2x}) < \frac{2 - v_n}{2} \left( \frac{1}{ev_n} + \frac{1}{2e} \right) = \frac{4 - v_n^2}{4ev_n}.$$
where \( v_n = -\frac{1}{e^{2n}} \left( \frac{e^{2/n} - 1}{e^{2/n} - n} \right) \). Therefore

\[
\| E_n(e^{-2x}; x) - e^{-2x} \|_{(0, \infty)} = \gamma_n < \frac{4 - v_n^2}{4e^{2n}} \to 0,
\]
as \( n \to \infty \). Hence the proof is complete.

\[\Box\]

4 A Quantitative Voronovskaya type theorem

Now, we will proof a Quantitative Voronovskaya type theorem for \( E_n \) operators.

**Theorem 3.** Let \( f', f'' \in C^\ast[0, \infty) \). Then, we have

\[
\left| n [ E_n(f; x) - f(x) ] - \frac{X}{2} f'(x) - \frac{X}{2} f''(x) \right| \leq | r_n(x) | \left| f'(x) \right| + | q_n(x) | \left| f''(x) \right| + 2 \left( 2g_n(x) + x + s_n(x) \right) \omega^* \left( f''; 1/\sqrt{n} \right)
\]

where

\[
\begin{align*}
  r_n(x) &= nE_n(\phi^1_x(t); x) - \frac{X}{2} \\
  q_n(x) &= \frac{1}{2} \left( nE_n(\phi^2_x(t); x) - x \right) \\
  s_n(x) &= \sqrt{n^2 E_n((e^{-t} - e^{-t})^4; x)} \sqrt{n^2 E_n((x - t)^4; x)}
\end{align*}
\]

**Proof.** By the Taylor’s expansion of \( f \), we have

\[
f(t) = f(x) + f'(x)(t - x) + \frac{f''(x)}{2}(t - x)^2 + h(t, x)(t - x)^2,
\]

where

\[
h(t, x) = \frac{f''(\eta) - f''(x)}{2}
\]

is a continuous function and \( \eta \) is a number between \( x \) and \( t \). Applying the \( E_n \) operator to both sides of equality (18), we get

\[
\begin{align*}
  &E_n(f; x) - f(x) - f'(x)E_n(\phi^1_x(t); x) - \frac{f''(x)}{2} E_n(\phi^2_x(t); x) \\
  &\leq E_n(h(t, x) \phi^2_x(t); x).
\end{align*}
\]

\[
\begin{align*}
  &\left| n [ E_n(f; x) - f(x) ] - \frac{X}{2} f'(x) - \frac{X}{2} f''(x) \right| \leq \left| nE_n(\phi^1_x(t); x) - \frac{X}{2} f'(x) \right| \\
  &\quad + \frac{1}{2} \left| nE_n(\phi^2_x(t); x) - x \right| \left| f''(x) \right| + \left| nE_n(h(t, x) \phi^2_x(t); x) \right|.
\end{align*}
\]

Take \( r_n(x) = nE_n(\phi^1_x(t); x) - \frac{X}{2} \) and \( q_n(x) = \frac{1}{2} \left( nE_n(\phi^2_x(t); x) - x \right) \). Thus, we get

\[
\left| n [ E_n(f; x) - f(x) ] - \frac{X}{2} f'(x) - \frac{X}{2} f''(x) \right| \leq \left| r_n(x) \right| \left| f'(x) \right| + \left| q_n(x) \right| \left| f''(x) \right| + \left| nE_n(h(t, x) \phi^2_x(t); x) \right| \quad (19)
\]

In order to complete the proof of the theorem, we must find the last term of the inequality (19). Here, we also know that the equalities given in (12) and (13) \( r_n(x) \to 0, q_n(x) \to 0 \) as \( n \to \infty \) at any point \( x \in [0, \infty) \).

Using the property

\[
| f(t) - f(x) | \leq \left( 1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2} \right) \omega^* (f; \delta), \delta > 0
\]

\[\$\text{sciende}\]
we have

\[ |h(t,x)| \leq \left( 1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2} \right) \omega^*(f'', \delta). \]

If \( |e^{-x} - e^{-t}| \leq \delta \), then \( |h(t,x)| \leq 2\omega^*(f'', \delta) \). In case \( |e^{-x} - e^{-t}| > \delta \), then we get \( |h(t,x)| \leq 2\frac{(e^{-t} - e^{-x})^2}{\delta^2} \omega^*(f'', \delta) \). Hence

\[ |h(t,x)| \leq 2 \left( 1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2} \omega^*(f'', \delta) \right). \]

By using this inequality and applying Cauchy-Schwarz inequality, we get

\[
n.E_n\left( |h(t,x)| \phi^2(t); x \right) \leq 2n\omega^*(f'', \delta)E_n\left( \phi(t); x \right) \\
+ \frac{2n}{\delta^2} \omega^*(f'', \delta) \sqrt{E_n \left( (e^{-x} - e^{-t})^4; x \right)} \sqrt{E_n \left( (t-x)^4; x \right)}.
\]

Choosing \( \delta = 1/\sqrt{n} \) and letting

\[ s_n(x) = \sqrt{n^2E_n \left( (e^{-x} - e^{-t})^4; x \right)} \sqrt{n^2E_n \left( (t-x)^4; x \right)}, \]

we obtain our result which was claim in the theorem.

\[ \square \]

**Corollary 1.** Let \( f, f'' \in C^4[0, \infty) \). Then we get

\[ \lim_{n \to \infty} n\left[ E_n(f; x) - f(x) \right] = \frac{1}{2} [f'(x) + f''(x)] \]

for any \( x \in [0, \infty) \).

5 Conclusion

In this paper, it is studied the theoretical aspects of Jakimovski-Leviatan operators which reproduce constant and \( e^{-x} \) functions. A theorem for determining uniform convergence order of a quantitative estimate for the modified operators are presented. We also prove a quantitative Voronovskya type theorem. For the following studies, the convergence of the operators by illustrative graphics in Maple to certain functions are investigated.

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