CLASSIFICATION OF DEGREE TWO CURVES IN $C^{(2)}$ WITH POSITIVE SELF-INTERSECTION

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Abstract

In this paper we give a precise classification of the curves in the symmetric square of a curve $C$ of degree two and positive self-intersection. We prove that there are no such curves with geometric genus less than $2g(C) - 2$ or greater than $2g(C)$. We give a constructible argument for the classification. We study the singularities and self-intersection of any degree two curve in the symmetric square of a curve. Moreover, we give examples of low genus curves with positive self-intersection on the usual product.

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1 Introduction

In this paper we study curves \( \tilde{B} \) in \( C^{(2)} \) with \( g(C) \geq 2 \) of degree two \( (\tilde{B} \cdot C_p = 2) \) and we give a complete classification of those with positive self-intersection. In particular, we prove that there are no curves of low genus with positive self-intersection and we study all possible cases for higher genus.

A fundamental tool for this study is the main result in [13] where it is proven a characterication of the curves in \( C^{(2)} \) with certain degree:

\textbf{Theorem 1.1.} Let \( B \) be an irreducible smooth curve such that there are no non-trivial morphisms \( B \rightarrow C \). A morphism of degree one from the curve \( B \) to the surface \( C^{(2)} \) exists, with image \( \tilde{B} \) of degree \( d \) if, and only if, there exists a smooth irreducible curve \( D \) and a diagram

\[
\begin{array}{ccc}
D & \xrightarrow{(d:1)} & B \\
\downarrow{(2:1)} & & \downarrow{(d:1)} \\
C & \rightarrow & \end{array}
\]

which does not reduce.

When \( d \) is prime that the diagram does not reduce is equivalent to the property that the diagram does not complete, that is, that does not exist a curve \( H \) and maps such that we obtain a commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
\downarrow{(d:1)} & & \downarrow{(d:1)} \\
C & \rightarrow & \end{array}
\]

In [10] Question 7.6] the authors wonder if there exists a curve \( B \) in a surface \( S \) with \( q(S) < p_a(B) < 2q(S) - 1 \) and \( B^2 > 0 \) They call such a curve a low genus curve. We
consider this question for curves in $C^{(2)}$ using the previous results and find a bound on the possible degree depending on the genus of $C$ and $\sigma$ (see [1]). When the genus is big this bound suggests that such a curve should have low degree. This fact motivates the study of low degree curves in the symmetric square. In this paper we study the degree two case and in a future paper we will consider the degree three case.

For a curve of degree two, a diagram as in Theorem [1.1] is given by the action of two involutions in a curve $D$. In order to study such diagrams we prove a proposition that allows us to find diagrams that do not complete or decide if a given diagram completes. That is

**Proposition 1.2.** Let $D$ be a projective smooth irreducible curve with the action of a finite group $G$. Let $\alpha, \beta \in G$ with orders $o(\alpha) = d \geq 2$ and $o(\beta) = e \geq 2$. Consider the diagram

\[
\begin{array}{ccc}
D & \overset{(e:1)}{\longrightarrow} & D/\langle \beta \rangle = B \\
\downarrow^{(d:1)} & & \downarrow \\
D/\langle \alpha \rangle = C
\end{array}
\]

Then,

1. If the order of $\langle \alpha, \beta \rangle$ equals $e \cdot d$ then the diagram completes.

2. If the order of $\langle \alpha, \beta \rangle$ is strictly greater than $e \cdot d$ then the diagram does not complete.

If the order of $\langle \alpha, \beta \rangle$ is strictly less than $e \cdot d$ anything can happen.

Next, we study the geometry of degree two curves. In particular, we study the possible singularities that can appear. We have two different cases depending on the irreducibility of $\pi^*_C(\tilde{B})$. If $\nu(\alpha)$ denotes the number of points in $C$ fixed by $\alpha \in Aut(C)$, we prove the following for the reducible case:

**Proposition 1.3.** A curve $\tilde{B}$ of degree two in $C^{(2)}$ such that $\pi^*_C(\tilde{B})$ is reducible exists if and only if there exists an automorphism of $C$, $\alpha$, of degree at least three. Moreover,

\[
\tilde{B} = \{ x + \alpha(x) \mid x \in C \}
\]

and $\tilde{B}$ has $\frac{1}{2}(\nu(\alpha^2) - \nu(\alpha))$ nodal singularities and normalization the curve $C$.

In the non-reducible case, with $D_n$ denoting the dihedral group of order $2n$, we prove that:

**Theorem 1.4.** A curve $\tilde{B}$ of degree two in $C^{(2)}$, such that $\pi^*_C(\tilde{B}) =: \check{D}$ is irreducible and $B$ denotes its normalization, exists if and only if there exists a curve $D$ and two
Degree two curves in $C^{(2)}$ involutions $i$ and $j$ in $\text{Aut}(D)$ with $\langle i, j \rangle = D_n$, $n \geq 3$, such that $C = D/\langle j \rangle$, $B = D/\langle i \rangle$ and $D$ is the normalization of $\pi_C^*(\tilde{B})$.

Moreover, $\tilde{B}$ has $\frac{1}{2}(\nu((ij)^2) - \nu(ij))$ nodal singularities and $\tilde{D}$ has $\frac{1}{2}\nu((ij)^2)$ nodal singularities.

Next, we concentrate in degree two curves with positive self-intersection. We study all such curves giving the following Classification Theorem:

**Theorem 1.5 (Classification).** All pairs of smooth curves $(C, B)$ with $B \overset{(1:1)}{\longrightarrow} C^{(2)}$ and image $\tilde{B}$ such that $\tilde{B}^2 > 0$ and $\tilde{B} \cdot C_p = 2$ fall in one of the following cases:

0. $C$ is a curve of genus 2 with an action of an automorphism of order 10, $\alpha$, such that $\nu(\alpha) = 1$, $\nu(\alpha^2) = 3$, $\nu(\alpha^5) = 6$ and $\tilde{B}$ is the symmetrization of the graph of $\alpha$. There is a finite number of isomorphism classes of curves $C$ with such an automorphism.

1. There is a curve $D$ with an action of $D_{10}$ such that $C$ and $B$ are the quotients of $D$ by certain involutions $i, j \in D_{10}$.

There are three topological types of actions on $D$, giving three families with the following properties:

| $g(D)$ | $g(C)$ | $g(B)$ | $\tilde{B}^2$ | Moduli dim. of $D$ | Moduli dim. of $C$ | Other properties |
|--------|--------|--------|--------------|-------------------|-------------------|------------------|
| 5      | 2      | 3      | 1            | 1                 | 1                 | $D$ hyperelliptic |
|        |        |        |              |                   |                   | $\tilde{B}$ smooth | (D6.1)           |
| 4      | 2      | 2      | 1            | 1                 | 1                 | $D$ hyperelliptic |
|        |        |        |              |                   |                   | $\tilde{B}$ has 1 node | (D6.2)         |
| 6      | 2      | 3      | 1            | 2                 | 2?                | $D$ bielliptic    |
|        |        |        |              |                   |                   | $\tilde{B}$ smooth | (D6.3)           |

Furthermore, in all three families the curve $B$ is hyperelliptic and $p_a(\tilde{B}) = 2g(C) - 1$.

2. There is a curve $D$ with an action of $D_6$ such that $C$ and $B$ are the quotients of $D$ by certain involutions $i, j \in D_6$. There are ten topological types of actions on $D$.

The first one gives a family such that

| $g(D)$ | $g(C)$ | $g(B)$ | $\tilde{B}^2$ | Moduli dim. of $D$ | Moduli dim. of $C$ | Other properties |
|--------|--------|--------|--------------|-------------------|-------------------|------------------|
| 5      | 2      | 3      | 2            | 2                 | 2                 | $\tilde{B}$ has 1 node | (D10.1)    |

Furthermore, in this family the curves $D$ and $B$ are hyperelliptic and $p_a(\tilde{B}) = 2g(C)$.

The other nine topological types give nine families with the following properties
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| $g(D)$ | $g(C)$ | $g(B)$ | $\tilde{B}^2$ | Moduli dim. of $D$ | Moduli dim. of $C$ | Other properties |
|--------|--------|--------|---------------|------------------|------------------|-----------------|
| 7      | 3      | 4      | 1             | 2                | 2                | $\tilde{B}$ has 1 node $(D10.2)$ |
| 9      | 3      | 5      | 1             | 3                | 3                | $\tilde{B}$ smooth $(D10.3)$ |
| 5      | 2      | 3      | 2             | 1                | 1                | $\tilde{B}$ smooth $(D10.4)$ |
| 6      | 3      | 3      | 1             | 2                | 2                | $\tilde{B}$ has 2 nodes $(D10.5)$ |
| 8      | 3      | 4      | 1             | 3                | 3                | $\tilde{B}$ has 1 node $(D10.6)$ |
| 4      | 2      | 2      | 2             | 1                | 1                | $\tilde{B}$ has 1 node $(D10.7)$ |
| 6      | 2      | 3      | 2             | 2                | 2                | $\tilde{B}$ smooth $(D10.8)$ |
| 5      | 2      | 2      | 2             | 2                | 2                | $\tilde{B}$ has 1 node $(D10.9)$ |
| 7      | 2      | 3      | 2             | 3                | 2                | $\tilde{B}$ smooth $(D10.10)$ |

Furthermore, in all nine families $B$ and $C$ are bielliptic and $p_a(\tilde{B}) = 2g(C) - 1$.

3. There is a curve $D$ with an action of $D_4$ such that $C$ and $B$ are the quotients of $D$ by certain involutions $i, j \in D_4$. There are three families of topological types of actions on $D$ with the following characteristics:

| $g(D)$ | $g(C)$ | $g(B)$ | $\tilde{B}^2$ | Moduli dim. of $D$ | Other properties |
|--------|--------|--------|---------------|------------------|-----------------|
| $-1 + s + \frac{1}{2}k$ | $\frac{s+k}{2}$ | $\frac{2s+k-1}{4}$ | 4 | $\frac{2s+k-4}{4}$ | $\tilde{B}$ has $\frac{k}{4}$ nodes $s + k \geq 8$ $(D4.1)$ |
| $-2 + s + \frac{1}{2}k$ | $\frac{s+k-2}{2}$ | $\frac{2s+k-1}{4}$ | 4 | $\frac{2s+k-4}{4}$ | $\tilde{B}$ has $\frac{k}{4}$ nodes $s + k \geq 10$ $(D4.2)$ |
| $-3 + s + \frac{1}{2}k$ | $\frac{s+k-4}{2}$ | $\frac{2s+k-1}{4}$ | 4 | $\frac{2s+k-4}{4}$ | $\tilde{B}$ has $\frac{k}{4}$ nodes $s + k \geq 12$ $(D4.3)$ |

Furthermore, in all three families $C$ is hyperelliptic, with any possible genus, and $p_a(\tilde{B}) = 2g(C)$.

When considering the preimages of these curves in $C \times C$ we find some examples of low genus curves with positive self-intersection. Hence, answering the question in [10] as it has been done in [3]. We note that all such curves have arithmetic genus $2g(C) - 2$, that is, the greatest possible low genus.

In Section 2 we compute the self-intersection of a curve $\tilde{B}$ in $C^{(2)}$ of degree $d$ defined by a diagram as in Theorem 11 using the data given by the morphisms in the diagram. Moreover, we study the conditions for such a curve to be a low genus curve with positive self-intersection.
In Section 3 we introduce a method to construct diagrams that do not complete using the action of a finite group on a curve (Proposition 1.2). By Theorem 1.1, it is also a method to find curves of certain degrees in square symmetric products of curves. When the order of the group generated by the two automorphisms equals the product of the orders, the quotient curve by this group completes the diagram. When the order is greater, we prove that a curve completing the diagram does not exist using the cardinality of the fibers and the order of the group. In the final part of the section we give some background on group actions that we use in the following sections.

In Section 4 we study curves of degree two in $C^{(2)}$. We consider first those with preimage in $C \times C$ reducible (Theorem 1.3) and next those with irreducible preimage (Theorem 1.4). Since a degree two morphism of curves is always Galois, we deduce that the second case is always given by the action of a dihedral group on a curve, thus translating the study of the diagrams to the study of group actions on curves. We prove the different pieces of information in the two theorems in some lemmas and propositions during the section. Putting all together we find the two summary results previously stated.

In Section 5 we prove Theorem 1.5. First, we study the possible curves with reducible preimage and positive self-intersection. Next, we consider those with irreducible preimage. We study the numerical conditions determined by our hypothesis on the action of the dihedral group in a curve $D$. We prove first that there are no such curves with low genus, and later, we define, when possible, a generating vector of the corresponding dihedral group acting on a curve $D$. In this way $C$ and $B$ are quotients of $D$ that lie on a diagram which does not complete. For each generating vector we study the moduli dimension of the curves $C$ that appear in this way.

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**Notation:** We denote by $p_a(C) = h^1(C, \mathcal{O}_C)$ the arithmetic genus and when $C$ is smooth by $g(C) = h^0(C, \omega_C)$ the geometric genus (or topological genus). We will call node an ordinary singularity of order two.

For $\alpha \in \text{Aut}(C)$, we denote by $\nu(\alpha)$ the number of points fixed by $\alpha$. We put $\Gamma_\alpha$ for the curve in $C \times C$ given by the graph of $\alpha$, that is, $\Gamma_\alpha = \{(x, \alpha(x)), \ x \in C\}$.

A compact Riemann surface $C$ will be called $\gamma$-hyperelliptic if there is a compact
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Riemann surface $\tilde{C}$ of genus $\gamma$ and a holomorphic mapping of degree 2 $p : C \to \tilde{C}$.

2 Self-intersection of $\tilde{B}$ in $C^{(2)}$

Let $\tilde{B} \subset C^{(2)}$ be a curve of degree $d$ whose immersion is given by a diagram

\[
\begin{array}{c}
D \xrightarrow{(d:1)} B \\
\downarrow g \\
C
\end{array}
\]

that does not reduce. In this section we compute the self-intersection, $\tilde{B}^2$, using the information given by the diagram.

**Lemma 2.1.** Let $\tilde{B} \subset C^{(2)}$ be a curve given by a diagram which does not reduce

\[
\begin{array}{c}
D \xrightarrow{(d:1)} B \\
\downarrow g \\
C
\end{array}
\]

Then,

\[
\tilde{B}^2 = g(D) - 1 - d(2g(C) - 2) + 2(p_a(\tilde{B}) - g(B)) + \frac{\sigma}{2}.
\]

Where $\sigma$ is defined in (7) as the intersection product in $D \times D$ of the graphs of $f$ and $g$.

First, we remind that by [4, VIII D.10] the ramification degree of $f : D \to B$ equals $B \cdot \Delta_D$, considering $B \subset D^{(2)}$ as the set of pairs of points with the same image by $f$. So, by the Riemann-Hurwitz formula we obtain that

\[
B \cdot \frac{\Delta_D}{2} = g(D) - 2g(B) + 1.
\]

Second, from the adjunction formula for $B \subset D^{(2)}$ together with the numerical equivalence $K_{D^{(2)}} \equiv_{num} (2g(D) - 2)D_P - \frac{\Delta_D}{2}$, we deduce that

\[
2g(B) - 2 = B^2 + 2g(D) - 2 - B \cdot \frac{\Delta_D}{2}.
\]

By the equalities (2) and (3) we obtain that the self-intersection of $B$ inside $D^{(2)}$ is

\[
B^2 = 1 - g(D).
\]
Now, from the adjunction formula for \( \tilde{B} \subset C^{(2)} \) together with the numerical class of \( K_{C^{(2)}} \) we deduce that

\[
\tilde{B}^2 = 2p_a(\tilde{B}) - 2 - d(2g(C) - 2) + \tilde{B} \cdot \frac{\Delta_C}{2}.
\]

Furthermore, by the projection formula we obtain that

\[
\tilde{B} \cdot \Delta_C = g^*(B) \cdot \Delta_C = g^*(B \cdot g^* \Delta_C) = B \cdot g^* \Delta_C.
\]

Therefore, it remains to compute \( g^* \Delta_C \cdot B \) to obtain \( \tilde{B}^2 \).

**Lemma 2.2.** Let \( R \) be the divisor in \( D^{(2)} \) defined as \( R = \{ x+y \mid g(x) = g(y) \} \). Then,

\[
g^* \Delta_C = \Delta_D + 2R.
\]

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
D \times D & \xrightarrow{\pi_D} & D^{(2)} \\
\downarrow{g \times g} & & \downarrow{g^{(2)}} \\
C \times C & \xrightarrow{\pi_C} & C^{(2)}
\end{array}
\]

As a set, the preimage of \( \Delta_C \) by \( g^{(2)} \) is clearly formed by the divisors \( \Delta_D \) and \( R \), that is, \( g^{(2)*} \Delta_C = n \Delta_D + mR \) with \( m, n \in \mathbb{Z}_{>0} \). We want to determine \( m \) and \( n \).

We know that \( \pi^*_C(\Delta_C) \) is two times the diagonal \( \Delta_{C \times C} \). Moreover, the divisor \( (g \times g)^*(\Delta_{C \times C}) \) is clearly the diagonal \( \Delta_{D \times D} \) plus a divisor

\[
R_0 = \{ (x, y) \mid g(x) = g(y) \} \setminus \Delta_{D \times D}.
\]

Thus, the preimage of \( g^{(2)*}(\Delta_C) \) in \( D \times D \) is exactly \( 2(\Delta_{D \times D} + R_0) \). Since \( \pi_D \) ramifies with degree two only on the diagonal \( \Delta_{D \times D} \), we deduce that \( n = 1 \) and \( m = 2 \) as claimed.

Notice, moreover, that we have \( R_0 \to \Delta_{C \times C} \) with degree \( d^2 - d \). Since \( R_0 \) is not on the ramified locus of \( g \times g \), there are exactly \( d^2 - d \) points in a general fiber of this map and hence, we deduce that \( R_0 \) is a reduced divisor, and thus also \( R \) is a reduced divisor.

Then, by (2) and (5) we obtain that

\[
\tilde{B} \cdot \Delta_C = B \cdot \Delta_D + 2B \cdot R = 2g(D) - 4g(B) + 2 + \sigma
\]

with

\[
\sigma := 2B \cdot R = \pi_{D*}(D) \cdot R = \pi_{D*}(D \cdot \pi_D^*(R)) = D \cdot \pi_D^*(R)
\]
where the divisor $D \subset D \times D$ is the set of ordered pairs of points with the same image by the morphism $f$. Hence, in a naive sense, we can say that $\sigma$ counts how the fibers of $f$ and $g$ meet inside $D \times D$.

Finally, from formulas (4) and (6) we deduce that

$$(8) \quad \tilde{B}^2 = g(D) - 1 - d(2g(C) - 2) + 2(p_a(\tilde{B}) - g(B)) + \frac{\sigma}{2}. $$

Furthermore, by the adjunction formula for $\tilde{D} \subset C \times C$, we obtain that

$$(9) \quad p_a(\tilde{D}) = 1 + \frac{1}{2}(\tilde{D}^2 + \tilde{D} \cdot K_{C \times C})$$

$$= 1 + \frac{1}{2}(2\tilde{B}^2 + \tilde{D} \cdot ((2g - 2)(C \times P + P \times C)))$$

$$= 1 + \frac{1}{2}(2\tilde{B}^2 + (2g - 2)2d) = 1 + \tilde{B}^2 + d(2g - 2),$$

where the second equality is consequence of

$$(10) \quad \tilde{D}^2 = (\pi^*_C \tilde{B})^2 = \pi^*_C(\tilde{B}^2) = \deg \pi_C \tilde{B}^2 = 2\tilde{B}^2.$$  

Moreover, from (8) and (9) we deduce that

$$(11) \quad p_a(\tilde{D}) - g(D) = 2(p_a(\tilde{B}) - g(B)) + \frac{\sigma}{2}. $$

Notice that, if $\tilde{D}$ is smooth, then by (11) also $\tilde{B}$ will be smooth, but the converse is not true. Summarizing, we have proven Lemma 2.1.

### 2.1 Curves in $C^{(2)}$ with low genus

Now we consider curves $\tilde{B} \subset C^{(2)}$ such that

$$(12) \quad q(C^{(2)}) = g(C) = g < p_a(\tilde{B}) < 2g(C) - 1$$

with positive self-intersection.

We remind that by Castelnuovo-Severi inequality (2), for a diagram of curves as (1) the following inequality is satisfied:

$$(13) \quad g(D) \leq 2g(B) + dg(C) + d - 1.$$ 

Thus, we have a necessary condition for given curves $C, B$ and $D$ to give a diagram as in Theorem 1.1.

Moreover, when $d$ is a prime number, by [1, Theorem 3.2] we have a criteria to decide if a curve completing the diagram can exist. That is, a curve $F$ completing the diagram must satisfy

$$g(D) + 2dg(F) \leq 2g(B) + dg(C) + d - 1.$$
When this inequality is not satisfied, the diagram does not complete and hence by Theorem 1.1 $B$ lays in $C^{(2)}$ (the image of $B$, $\tilde{B}$ could eventually be singular).

By Lemma 2.1 and inequality (13) we get that

$$g(D) = \tilde{B}^2 + 1 + 2d(g(C) - 1) - (p_a(\tilde{D}) - g(D)) \leq 2g(B) + d(g(C) + d - 1$$

$$\Leftrightarrow \tilde{B}^2 \leq 2g(B) + d(3 - g) - 2 + (p_a(\tilde{D}) - g(D)).$$

Since we are assuming that condition (12) is satisfied, necessarily $g(B) \leq p_a(\tilde{B}) \leq 2g - 2$, so we obtain from this inequality that for $g \geq 4$ we have that

$$d \leq \frac{4g - 6 - \tilde{B}^2 + (p_a(\tilde{D}) - g(D))}{g - 3}.$$

Hence, for a fixed $g$, we have a relation between the self-intersection of $\tilde{B}$, its degree and the singularities of $\tilde{D}$.

Remark 2.3. If we assume $\tilde{D}$ smooth, that is, $p_a(\tilde{D}) - g(D) = 0$, since we are considering the case $\tilde{B}^2 \geq 1$, we deduce that $d \leq \frac{4g - 7}{g - 3}$. Thus, for $g \geq 9$ a curve of low genus and positive self-intersection should have degree at most 4.

This inequality motivates the study of curves in $C^{(2)}$ with low degree to look for such a curve.

3 Group actions and non completed diagrams

We consider now group actions on curves and give a method to find diagrams that do not complete or decide if a given diagram completes, depending on the order of the group generated by the two automorphisms defining the diagram. This method is described in Proposition 1.2 that we prove in this section. First, we make an observation:

Remark 3.1. If the order of $\langle \alpha, \beta \rangle$ is strictly less than $e \cdot d$ then anything can happen. For instance, if $\beta = \alpha^k$, we can close the diagram as

$$D \xrightarrow{(e:1)} D/\langle \alpha^k \rangle = B$$

$$\xrightarrow{(d:1)} D/\langle \alpha \rangle = C$$

In that case, the diagram completes if and only if $C$ covers a curve $H$ with degree $e$, because considering the composition of the projection of $B$ onto $C$ with the morphism to $H$ we obtain the completed diagram.
Proof of Proposition 1.2

1. Assume that $|\langle \alpha, \beta \rangle| = e \cdot d$. Let $F$ be the quotient of $D$ by the action of $\langle \alpha, \beta \rangle$. We get a diagram

\[
\begin{array}{ccc}
D^{(e:1)} & \longrightarrow & B \\
\downarrow^{(d:1)} & & \downarrow^{(ed:1)} \\
C & \searrow & D/\langle \alpha, \beta \rangle = F.
\end{array}
\]

Then, we can define morphisms from $B$ and $C$ to $F$ in such a way that the diagram completes, because both $B$ and $C$ are quotients of $D$ by subgroups of $\langle \alpha, \beta \rangle$.

2. Assume that $|\langle \alpha, \beta \rangle| > e \cdot d$. By contradiction we assume that the diagram completes. That is, there exists a curve $H$ giving a commutative diagram

\[
\begin{array}{ccc}
D^{(e:1)} & \longrightarrow & B \\
\downarrow^{(d:1)} & & \downarrow^{(ed:1)} \\
C & \searrow & H.
\end{array}
\]

Hence, the automorphisms $\alpha$ and $\beta$ act on the fibers of $D^{(ed:1)} \to H$, and so the group $\langle \alpha, \beta \rangle$ acts on these fibers.

Therefore, the orbit of a general point of $D$ by the action of $\langle \alpha, \beta \rangle$ must be contained in a fiber of $D^{(ed:1)} \to H$, but the cardinality of the first is strictly greater than the degree of the second, so this inclusion is not possible, and consequently such a curve $H$ cannot exist. \qed

3.1 Background on group actions

To study the geometry of the curves defined by such diagrams we need to remind some basic facts about group actions on curves.

Let $C$ be a curve and let $G \subset \text{Aut}(C)$ be a finite subgroup. For $P \in C$, set

\[G_P = \{ g \in G \mid g(P) = P \}\]

the \textbf{stabilizer} of $P$.

\textbf{Proposition 3.2.} (I.11.7.7) Assume $g(C) \geq 2$. Then $G_P$ is a cyclic subgroup of $\text{Aut}(C)$.

Given $\alpha \in \text{Aut}(C)$, its graph $\Gamma_\alpha$ lies in $C \times C$ and is isomorphic to $C$. With a local computation one can see that
PROPOSITION 3.3. The diagonal in $C \times C$ cuts the graph of an automorphism transversally.

COROLLARY 3.4. Let $\alpha$ and $\beta$ be two automorphisms of a curve $C$. If $\alpha^{-1} \beta \neq 1$, then the graphs of $\alpha$ and $\beta$ in $C \times C$ intersect transversally and moreover, $\Gamma_\alpha \cdot \Gamma_\beta$ equals the number of fixed points $\nu(\alpha^{-1} \beta)$ of the automorphism $\alpha^{-1} \beta$.

PROOF. We have in $C \times C$ the action of $1 \times \alpha^{-1}$. We transform the two considered graphs by this action:

$$
\Gamma_\alpha = \{(x, \alpha(x))\} \xrightarrow{1 \times \alpha^{-1}} \{(x, x)\} = \Delta_{C \times C}
$$

$$
\Gamma_\beta = \{(x, \beta(x))\} \xrightarrow{1 \times \alpha^{-1}} \{(x, \alpha^{-1} \beta(x))\} = \Gamma_{\alpha^{-1} \beta}.
$$

Since the diagonal intersects transversally the graph of any automorphism, we deduce that the two graphs intersect also transversally in $\nu(\alpha^{-1} \beta)$ points. 

Now, we study the action of a group $G$ on the curve $C$ and the resulting orbit space.

LEMMA 3.5. Let $G$ be a finite group of order $n$ acting on a curve $C$. Given a point $P \in C$, let $\alpha$ be a generator of $G_P$. Then we have that

$$
n = |G_P| \cdot |\{\text{conjugates of } G_P\}| \cdot |\{\text{points fixed by } \alpha \text{ in } O_G(P)\}|.
$$

PROOF. By Lagrange Theorem ([12 Theorem 2.27]) and the orbit-stabilizer Theorem ([12 Theorem 5.7]) we obtain that

$$
n = |G| = |G_P| \cdot [G: G_P] = |G_P| \cdot |O_G(P)|.
$$

Now, since the point $P$ has stabilizer $G_P$, given a conjugate of $G_P$, we see that $\alpha G_P \alpha^{-1} = G_{\alpha(P)}$, that is, it is the stabilizer of $\alpha(P)$. Moreover, given any element $\beta \in G$, $\beta(P)$ has stabilizer $G_\beta$ or one of its conjugates. Therefore, in the orbit of $P$ there are the same number of points with stabilizer each conjugate of $G_P$ and all conjugates of $G_P$ are stabilizers of points in the orbit. Hence,

$$
|O_G(P)| = |\{\text{conjugates of } G_P\}| \cdot |\{\text{points fixed by } \alpha \text{ in } O_G(P)\}|.
$$

Theorem 3.6 (Riemann’s Existence Theorem). The group $G$ acts on a curve of genus $g$, with branching type $(g'; m_1, \ldots, m_r)$ if and only if the Riemann-Hurwitz formula is satisfied and $G$ has a $(g'; m_1, \ldots, m_r)$ generating vector.

Where a $(g'; m_1, \ldots, m_r)$ generating vector (or $G$-Hurwitz vector) is a $2g' + r$-tuple

$$
(a_1, b_1, \ldots, a_{g'}, b_{g'}; c_1, \ldots, c_r)
$$
of elements of $G$ generating the group and such that $o(c_i) = m_i$ and $\prod_{j=1}^{g'}[a_i, b_i] \prod_{i=1}^{r} c_i = 1$. We call this last condition the product one condition.

We remark that Riemann’s Existence theorem is not a constructive result. It states the existence of such a curve, nevertheless it gives no further information about it. With the following theorem we will be able to compute the number of fixed points of each element $\gamma \in G$.

**Theorem 3.7.** Let $C$ be a compact Riemann surface and $G$ a group of its automorphisms. Let $(a_1, b_1, \ldots, a_{g'}, b_{g'}; c_1, \ldots, c_r)$ be a $(g'; m_1, \ldots, m_r)$-generating vector of $G$ describing the action of $G$ on $C$. For $1 \neq \gamma \in G$ let $\varepsilon_i(\gamma)$ be 1 or 0 according as $\gamma$ is or is not conjugate to a power of $c_i$.

Then the number $\nu(\gamma)$ of points of $C$ fixed by $\gamma$ is given by the formula

$$\nu(\gamma) = |N_G(\langle \gamma \rangle)| \sum_{i=1}^{r} \frac{\varepsilon_i(\gamma)}{m_i}.$$ 

4 Degree two curves

Now, we study curves of degree two in the symmetric square of a curve. First of all we observe that by the Hodge index theorem a curve $\tilde{B}$ of degree two in $C^{(2)}$ satisfies the inequality $\tilde{B}^2 \leq 4$ and moreover, when $\tilde{B}^2 = 4$ the curve is algebraically equivalent to twice a coordinate curve.

We present first a lemma that will be useful in the discussion that follows. The proof uses basic group theory and the particular group structure of the dihedral groups and is left to the reader.

**Lemma 4.1.** Let $i$ and $j$ be two involutions generating a dihedral group $D_n$, $n \geq 3$. Then, there is no cyclic subgroup containing $(ij)^2$ and one of the involutions $i$ or $j$.

First of all we study those degree two curves $\tilde{B} \subset C^{(2)}$ such that $\pi_C^*(\tilde{B})$ is reducible (Proposition 1.3).

We observe that if $\pi_C^*(\tilde{B})$ reduces, then it consists of two copies of $C$, and the projections onto each factor are isomorphisms. This gives an automorphism of $C$, $\alpha$. Notice that the order of $\alpha$ must be at least 3 because otherwise, $\pi_C^*(\tilde{B})$ would have only one component.

Hence, we have in $C \times C \supset \pi_C^*(\tilde{B}) = C_1 + C_2$ with $C_1 = \Gamma_\alpha$ and $C_2 = \Gamma_{\alpha^{-1}}$ which by $\pi_C$ go to $\tilde{B} = \{x + \alpha(x), \ x \in C\}$. The curve $\tilde{B}$ has normalization $C$ and moreover,

- $\tilde{B} \cdot \Delta_C = 2\nu(\alpha)$. Indeed, consider

$$\tilde{B} \cap \Delta_C = \{x + \alpha(x) \mid \alpha(x) = x\}.$$
The preimage of these points by $\pi_C$ correspond to points were $C_1$ and $C_2$ meet (transversally by Corollary 3.4) over the diagonal. They intersect the diagonal transversally (by Proposition 3.3), and taking local coordinates we see that $\tilde{B}$ and $\Delta_C$ are tangent at $x + x$ for $x$ a point fixed by $\alpha$.

- $|\text{Sing } \tilde{B}| = \frac{1}{2}(\nu(\alpha^2) - \nu(\alpha))$. Indeed, a general curve $C_P$ intersects $\tilde{B}$ in two different points $P + \alpha(P)$ and $P + \alpha^{-1}(P)$. Since $C_P \cdot \tilde{B} = 2$ when these two points are different they are smooth points on $\tilde{B}$. To determine the singularities of $\tilde{B}$ we need to study when these two points coincide. We have two possibilities:

  Either $\alpha(P) = P$ and hence $\tilde{B}$ intersects the diagonal in a smooth (tangent) point as we have just seen.
  
  Or $\alpha(P) = \alpha^{-1}(P) \neq P$, that is, $P$ is fixed by $\alpha^2$ and not by $\alpha$. We observe that if $P$ is fixed by $\alpha^2$, the point $\alpha(P)$ is also fixed by $\alpha^2$, and both give the same singularity $P + \alpha(P) = \alpha(P) + \alpha^2(P)$.

- All singularities of $\tilde{B}$ are nodes. Indeed, consider the normalization morphism

  $$
  C \rightarrow \tilde{B} \subset C^{(2)}
  
  x \rightarrow x + \alpha(x).
  $$

  A singular point $x + \alpha(x)$ with $\alpha^2(x) = x \neq x$ has two preimages by the normalization morphism: $x$ and $\alpha(x)$, hence $\tilde{B}$ has two branches at $x + \alpha(x)$. Moreover, since $C_P \cdot \tilde{B} = 2$, the singularities have order two. Moreover, since the preimage by $\pi_C$ is formed by the graphs of $\alpha$ and $\alpha^{-1}$, which are transversal by Corollary 3.4 and $\pi_C$ is a local isomorphism around these points we conclude that these singularities are nodes.

Therefore, $p_a(\tilde{B}) = g(C) + \frac{1}{2}(\nu(\alpha^2) - \nu(\alpha))$.

Now, we study those curves $\tilde{B} \subset C^{(2)}$, with normalization $B$, which preimage by $\pi_C : C \times C \rightarrow C^{(2)}$ is an irreducible curve $\tilde{D} := \pi_C^*(\tilde{B})$, which has normalization $D$ (Theorem 1.4). Regarding Theorem 1.1 there exists a diagram which do not complete of the form

$$
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & \tilde{B} \\
\downarrow{(2:1)} & f & \downarrow{g} \\
C & & B
\end{array}
$$

defined by such a $\tilde{B} \subset C^{(2)}$. We observe that since both morphisms are of degree two, in $D$ there are two involutions $i$ and $j$ (the changes of sheet) that by Proposition 1.2 generate a group of order at least five, that is, a dihedral group (see 11). Hence, $C = D/(j)$ and $B = D/(i)$. 


We make a detailed study of the action of $D_n$, and specially of the points fixed by certain automorphisms in order to study the singularities of $\tilde{B}$ and $\tilde{D}$.

Since $g$ is the quotient by the action of the involution $j$, clearly $\pi^*_D(R) = \Gamma_j$ (see 2.2). We remind that we consider $D \subset D \times D$ as the set of points $\{(x, f(x))\}$, that is, the graph of the morphism $f$, or which is the same, the graph of the involution $i$.

First of all we observe that the points in $\pi^*_D(R) \cap D$ are pairs of different points in $D \subset D \times D$ with the same image in $\tilde{D}$, so their images by $g \times g$ are singularities of $\tilde{D}$. Now, we are going to see that their images in $\tilde{B}$ are smooth points.

**Lemma 4.2.** The image in $\tilde{B}$ by $\pi_C|D$ of a point $(g \times g)(x, i(x))$ with $ij(x) = x$ is a smooth point where $\tilde{B}$ is tangent to the diagonal.

**Proof.** To begin with, we are going to study the singularities on $\tilde{D}$ of the form $(g \times g)(x, i(x))$.

Consider the morphism $g \times g : D \times D \to C \times C$. It is Galois with group $(1 \times j, j \times 1)$. The preimage of $\tilde{D}$ by $g \times g$ consists in four divisors:

\[
\begin{align*}
D_0 &= (1 \times 1)(D) = \{(x, i(x))\}, \\
D_1 &= (1 \times j)(D) = \{(x, ji(x))\}, \\
D_2 &= (j \times 1)(D) = \{(j(x), i(x))\} = \{(x, ij(x))\} \quad \text{and} \\
D_3 &= (j \times j)(D) = \{(j(x), ji(x))\} = \{(x, jij(x))\}.
\end{align*}
\]

The points $(x, i(x))$ and $(i(x), x)$ with $ij(x) = x$ are two intersections of the curves $D_0$ and $D_3$ with the same image by $g \times g$. Since $g \times g$ is not ramified in these points and the divisors $D_0$ and $D_3$ are transversal by Corollary 3.4, we deduce that $\tilde{D}$ is transversal on the image, and therefore it has nodes.

Now, we are interested in the image by $\pi_C$ of one of these points $(g(x), g(i(x))) = (g(x), g(x))$. Since it is over the diagonal, which is the ramification divisor of $\pi_C$, there are no other points with the same image. Moreover, the points $(x, i(x))$ and $(i(x), x)$ have the same image by $\pi_D$, and hence $g(x) + g(x)$ has only one preimage in $B$, which is the normalization of $\tilde{B}$, so $\tilde{B}$ has only one branch in $g(x) + g(x)$.

Consequently, we have a nodal singularity $(g(x), g(x))$ in $\tilde{D} \subset C \times C$ which image in $\tilde{B} \subset C^{(2)}$ has a single branch. We want to see that this branch is smooth.

Let $(z_1, z_2)$ be a system of local coordinates in $C \times C$ with both $z_i$ a local coordinate in $C$ around $g(x)$. Using them, $\pi_C$ is written locally as $(z_1, z_2) \to (z_1 + z_2, z_1z_2) = (z, t)$ with $(z, t)$ local coordinates in $C^{(2)}$ centered in $g(x) + g(x)$, making a local computation and using that in $(g(x), g(x))$ there is a node we obtain that $g(x) + g(x)$ is a smooth point of $\tilde{B}$. 
Since the intersection multiplicity in \((g(x), g(x))\) of \(\tilde{D}\) and \(\Delta_{C \times C}\) is two, also the intersection multiplicity in \(g(x) + g(x)\) of \(\tilde{B}\) and \(\Delta_{C}\) is two, and therefore it is a tangent point of these two curves.

Now, we are going to study the rest of singularities of \(\tilde{B}\) and \(\tilde{D}\).

**Proposition 4.3.**

\[
|\text{Sing } \tilde{B}| = \frac{1}{4} \left( \nu((ij)^2) - \nu(ij) \right).
\]

**Proof.** First, we want to know when two points of \(B(x + y\) and \(z + t)\) have the same image in \(\tilde{B}\) by \(g(2)\), where \(i(x) = y, i(z) = t\) and we assume that \(j(x) = z\) and \(j(y) = t\). Then, we obtain that

\[
ijij(x) = iji(z) = ij(t) = i(y) = x.
\]

And in a similar way \(y, z\) and \(t\) are fixed by \((ij)^2\). Conversely, we are going to take one point fixed by \((ij)^2\) and see when it gives a singularity in \(\tilde{B}\).

Let \(x\) be a point fixed by \((ij)^2\). Since there is no cyclic subgroup of \(D_n\) containing \(i\) and \((ij)^2\), we deduce that \(x\) is not fixed by \(i\). Let \(y = i(x) \neq x\). We want to study the fiber of \(g(2)\). We begin considering the image of \(y\) by the automorphism \(j\) and see that the only possibility to have a singularity is that \(j(y) = z \notin \{x, y\}\). Then, \(i(z) = t \notin \{x, y\}\) and we deduce that \(j(t) = x\). So, we have two different points in \(B\) with the same image in \(\tilde{B}\), that is, a singularity in \(\tilde{B}\). Notice that \(x, y, z, t\) are four points fixed by \((ij)^2\), and not by \(ij\). They are giving one singularity in \(\tilde{B}\) and two different singularities in \(\tilde{D}\).

Therefore, \(|\text{Sing } \tilde{B}| \geq \frac{1}{4} \left( \nu((ij)^2) - \nu(ij) \right)\). We are going to see that there are no more.

For a point in \(B\) outside the ramification locus of \(g(2)\), its image will be a singularity only if there is another point with the same image, because \(g(2)\) is a local isomorphism. We have already studied this case in the above discussion. Hence, it remains to consider the ramification points of \(g(2)\).

In \(B\) there are two types of points where \(g(2)\) ramifies: those in \(R\) (see Lemma 2.2) and those in a coordinate curve \(D_Q\) with \(Q\) a ramification point of \(g\). We have seen in Lemma 4.2 that the image of a point in \(B \cap R\) is always smooth, so it remains only to study those in \(D_Q\) for \(Q \in \text{Ram}(g)\).

To do this, we study the intersection of \(\tilde{B}\) with a coordinate curve \(C_p, P \in \text{Branch}(g)\) and see that \(\tilde{B}\) is smooth in all these points.

Since \(2 = \tilde{B} \cdot C_p\), the multiplicity of a singular point is at most two, and if there are two different points in \(\tilde{B} \cap C_p\) they are smooth points of \(\tilde{B}\).

We consider the set of points

\[
C_p \cap \tilde{B} = \{P + Q \mid \exists x, y \in D \text{ with } g(x) = P, g(y) = Q \text{ and } f(x) = f(y)\}.
\]
Since \( P \in \text{Branch}(g) \) then \( \exists x \in D \) such that \( j(x) = x \) and \( g(x) = P \). We take the point \( y = i(x) \neq x \) (\( i \) and \( j \) have no common fixed point), and deduce that \( g(y) \neq P \).

Hence, we obtain that \( \tilde{B} \cap C_P = P + g(y) \) with multiplicity 2. To prove that this is a smooth point of \( \tilde{B} \) we consider the other coordinate curve passing through this point, that is, now we consider \( \tilde{B} \cap C_{g(y)} \).

If \( j(y) = y \), since \( i(x) = y \), then \( y \) would be fixed by \( (ij)^2 \), but \( j \) and \( (ij)^2 \) have no common fixed points. Hence, \( j(y) = z \neq y \) and in particular, \( g(y) = g(z) \). Let \( t = i(z) \neq x, y, z \). We obtain that \( \tilde{B} \cap C_{g(y)} = \{g(y) + P, g(y) + g(t)\} \), two smooth different points of \( \tilde{B} \). Hence, we have proved the proposition.

**Corollary 4.4.**

\[ |\text{Sing} \tilde{D}| = \frac{1}{2} \nu((ij)^2). \]

**Proof.** By the arguments in the proof of Proposition 4.3 we obtain the inequality

\[ |\text{Sing} \tilde{D}| \geq \frac{1}{2} \nu((ij)^2). \]

The curve \( \tilde{D} \) could have other singularities if there were other tangencies between \( \tilde{B} \) and \( \Delta_C \). We remind that \( g^{(2)*}(\Delta_C) = \Delta_D + 2R \) (Lemma 2.2). We have already seen that those intersections of \( \tilde{B} \) and \( \Delta_C \) which preimage in \( D^{(2)} \) is a point in \( R \) correspond to singularities in \( \tilde{D} \). It remains to consider those with preimage in \( \Delta_D \).

The preimage of a point \( x + x \) with \( i(x) = x \) by \( \pi_D \) is an intersection between \( D(= \Gamma_i) \) and \( \Delta_{D \times D} \) which intersect transversally by Proposition 3.3. Hence, the intersection multiplicity is one, and in the image by \( \pi_D \), the curves \( B \) and \( \Delta_D \) intersect also with multiplicity one, and therefore, they are transversal. And since \( g^{(2)} \) is not ramified also \( \tilde{B} \) and \( \Delta_C \) are transversal.

Now, we study these singularities to determine their contribution to the arithmetic genus of \( \tilde{D} \) and \( \tilde{B} \).

**Proposition 4.5.** All singularities in \( \tilde{D} \) and \( \tilde{B} \) are nodes.

**Proof.** We will begin studying the singularities of \( \tilde{D} \) and later their images in \( \tilde{B} \).

Consider the morphism \( g \times g : D \times D \to C \times C \). It is Galois with group \( \langle 1 \times j, j \times 1 \rangle \). The preimage of \( \tilde{D} \) by this morphism consists in the four divisors described in (14).

We have just seen in Corollary 4.3 that the singularities of \( \tilde{D} \) come from points fixed by \( (ij)^2 \). Let \( x_0 \) be such that \( (ij)^2(x_0) = x_0 \). In \( D \times D \) it gives two points in \( D_0 \cap D_3 \).

These two points have the same image by \( g \times g \), and are not ramified. Since \( D_0 \) and \( D_3 \) are transversal and the morphism \( g \times g \) is a local isomorphism around these points, \( \tilde{D} \) is transversal on the image. Therefore, all singularities in \( \tilde{D} \) are nodes.
Since we have seen that the image of a singularity over the diagonal is a smooth point of $\tilde{B}$ and $\pi_C$ is a local isomorphism out of the diagonal we deduce that all singularities in $\tilde{B}$ are nodes.

Therefore, by Propositions 4.3 and 4.5 we obtain that

**Corollary 4.6.**

\[
\begin{align*}
p_a(\tilde{B}) - g(B) &= \frac{1}{4}(\nu((ij)^2) - \nu(ij)) \\
p_a(\tilde{D}) - g(D) &= \frac{1}{2}\nu((ij)^2).
\end{align*}
\]

5 **Proof of Theorem 1.5**

In this section we study degree two curves with $\tilde{B}^2 > 0$ in order to prove Theorem 1.5.

5.1 **Reducible case**

First of all we study those degree two curves $\tilde{B} \subset C^{(2)}$ such that $\pi_C^*(\tilde{B})$ is reducible. We begin with an example of a special case:

**Example 5.1.** We consider the $(0; 10, 5, 2)$-generating vector of $\mathbb{Z}/10$ given by $(\alpha, \alpha^4, \alpha^5)$ where $\alpha$ denotes a generator of the group. It gives a morphism $C \rightarrow \mathbb{P}^1$, where $C$ is a curve with an automorphism $\alpha \in Aut(C)$ of order 10 with $\nu(\alpha) = 1$, $\nu(\alpha^2) = 3$ and $\nu(\alpha^5) = 6$ (see Lemma 3.5). By the Riemann-Hurwitz formula we obtain that

\[2g(C) - 2 = 10(-2) + 1 \cdot 9 + 2 \cdot 4 + 5 \cdot 1 \Rightarrow g(C) = 2.\]

Consider now the graph of $\alpha$, $\Gamma_\alpha$, in $C \times C$. The image of $\Gamma_\alpha$ by $\pi_C$ is a curve $\tilde{B}$ of degree two in $C^{(2)}$ such that $\pi_C^*(\tilde{B}) = \Gamma_\alpha + \Gamma_{\alpha^{-1}}$.

In the following proposition we prove that it is the only curve of degree two in $C^{(2)}$ with reducible preimage by $\pi_C$ and positive self-intersection.

**Lemma 5.2.** The only curves $\tilde{B} \subset C^{(2)}$ of degree two, such that $\pi_C^*(\tilde{B})$ is reducible and $\tilde{B}^2 > 0$, are the symmetrization of the graph of $\alpha$ on $C$, where $C$ is a curve of genus 2 and $\alpha$ is an automorphism of order 10 such that $\nu(\alpha) = 1$, $\nu(\alpha^2) = 3$ and $\nu(\alpha^5) = 6$.

**Proof.** Let $\tilde{B}$ be such a curve. We are going to describe it in order to prove the proposition.

From Proposition we have that $p_a(\tilde{B}) = g(C) + \frac{1}{2}(\nu(\alpha^2) - \nu(\alpha))$. Since necessarily $p_a(\tilde{B}) > g(C)$ we obtain that $\tilde{B}$ is singular, so $o(\alpha)$ must be even (different from 2).

Moreover, by [7, V.1.5] we know that

\[\nu(\alpha^2) \leq 2 + \frac{2g}{o(\alpha^2) - 1}.\]
We call \( g = g(C) \), \( s = \nu(\alpha) \) and \( r = \nu(\alpha^2) - \nu(\alpha) \). From the adjunction formula we deduce that
\[
\tilde{B}^2 = 2p_a(\tilde{B}) - 2 - 2(2g - 2) + \tilde{B} \cdot \Delta_C = -2g + 2 + r + s > 0 \iff 2g - 2 < r + s = \nu(\alpha^2).
\]

Next, we study separately the different possibilities for \( o(\alpha^2) \).

Assume first that \( o(\alpha^2) \geq 4 \) (so \( o(\alpha) \geq 8 \)), then, from (15) and (16) we deduce that
\[
2 + \frac{2g}{3} \geq 2 + \frac{2g}{o(\alpha^2) - 1} \geq r + s > 2g - 2.
\]

This implies that \( 3 > g \), so it remains to consider \( g = 2 \) with \( r + s = 3 \). From (17) we deduce that \( 5 \geq o(\alpha^2) \geq 4 \). We consider the morphism \( C \to C/\langle \alpha \rangle \) and notice that the points fixed by \( \alpha^2 \) not fixed by \( \alpha \) come by pairs, and hence \( r \) is even. Therefore, we have that \( r = 2 \) and \( s = 1 \) (since we have seen that \( r \neq 0 \)). There could be other ramification points coming from points fixed by \( o(\alpha^2) \) that would come by groups of \( o(\alpha^2) \) (see Lemma 3.5). Considering the Riemann-Hurwitz formula for this morphism with all our data we obtain that \( o(\alpha^2) = 4 \) is not possible and that for \( o(\alpha^2) = 5 \) we obtain that \( C/\langle \alpha \rangle = \mathbb{P}^1 \) and \( \nu(\alpha^5) = 5 + 1 = 6 \), that is, the case described in Example 5.1.

Now, it remains to take care of the cases \( o(\alpha) = 6 \) and \( o(\alpha) = 4 \). Considering the Riemann-Hurwitz formula for the morphism \( \to C/\langle \alpha \rangle \) we see that such an action does not exist.

And therefore, the only case for \( \tilde{B} \) satisfying the hypothesis of the proposition is the one described in Example 5.1.

We notice that the Hurwitz space of such morphisms \( C \to \mathbb{P}^1 = C/\langle \alpha \rangle \) has dimension 0 because there are only three branch points, that by the action of the automorphisms of \( \mathbb{P}^1 \) we can consider \( 0, 1, \infty \). Therefore, in moduli, we have a finite number of such curves \( C \) (see [15]). Hence, we have proven point 0. of the Theorem.

### 5.2 Irreducible case

Now, we are going to study the possible \( \tilde{B} \subset C^{(2)} \) with \( \pi_C^*(\tilde{B}) \) irreducible.

As we have seen in Section 4, the curve \( B \) lies in a diagram
\[
\begin{align*}
D \xrightarrow{(2:1)} & \quad B \\
\downarrow & \quad \downarrow \text{(2:1)} \\
C
\end{align*}
\]

that does not complete and there exist two involutions \( i \) and \( j \) (the changes of sheet) such that \( B = D/\langle i \rangle \) and \( C = D/\langle j \rangle \) with \( \langle i, j \rangle = D_n \).
We use the following notation for the number of fixed points of the involved automorphisms of the curve $D$:

$$s = \nu(j), \quad t = \nu(i), \quad r = \nu(ij) \quad \text{and} \quad r + k = \nu((ij)^2).$$

Let $b = g(B)$, $g = g(C)$ and $h = g(D)$.

The strategy is to find the restrictions on the numbers $s, t, k, b, g$ and $h$ given by our hypothesis.

First, by the Riemann-Hurwitz formula for the morphism $D \to C$ we obtain

$$g = \frac{2h + 2 - s}{4}. \quad (19)$$

Second, by the Riemann-Hurwitz formula for the morphism $D \to B$ and Corollary 4.6 we deduce that

$$p_a(\tilde{B}) = b + \frac{1}{4}(r + k - r) = \frac{1}{4}(2h + 2 - t + k). \quad (20)$$

Third, by (19) and (20) the condition $g < p_a(\tilde{B})$ translates into

$$\frac{2h + 2 - s}{4} < \frac{2h + 2 - t}{4} + \frac{1}{4}k \Leftrightarrow t < s + k. \quad (21)$$

And from Lemma 2.1 and Corollary 4.6 together with (19) we deduce that

$$\tilde{B}^2 = -h + 1 + s + \frac{1}{2}(r + k) > 0 \Leftrightarrow h \leq s + \frac{1}{2}(r + k). \quad (22)$$

**Note:** We can assume that $\langle i, j \rangle = D_{2l}$. Indeed, if $\langle i, j \rangle = D_{2l+1}$ then the involutions $i$ and $j$ would be conjugate, and so $t = s$. Since 2 and $2l+1$ are coprime, the automorphisms $ij$ and $(ij)^2$ would have the same fixed points and thus $k = 0$, contradicting (21).

Let $\gamma = g(D/D_{2l})$. By the Riemann-Hurwitz formula for group quotients applied to $\tau : D \to D/D_{2l}$, we have that

$$h - 1 = 2l(2\gamma - 2) + 2l \sum_{P \in B_1} \left(1 - \frac{1}{m_P}\right) \quad (23)$$

where $m_P = e_Q - 1$ with $\tau(Q) = P$ (we remind that since $\tau$ is Galois, it is totally ramified, and we call $m_P$ the order of the branch point $P$).

We count the number of branch points of $\tau$ using Lemma 3.5 repetitively: since $i$ and $j$ are non conjugated of order two we have $\frac{l}{2}$ branch points of order 2 corresponding to the conjugacy class of the stabilizer $\langle i \rangle$ and $\frac{l}{2}$ branch points of order 2 corresponding to the conjugacy class of $\langle j \rangle$, which will be different from the previous. Moreover, we will have $\frac{l}{2}$ branch points of order $2l$ corresponding to $\langle ij \rangle$ and $\frac{1}{4}$ branch points of order $l$. 

corresponding to \((ij)^2\). We could also have other branch points coming from powers of \(ij\) that do not generate the whole \((ij)\). Note that \(s, t\) and \(r\) are even and \(k\) is multiple of 4.

All together, this gives that
\[
2l \sum_{P \in Br} \left(1 - \frac{1}{m_P}\right) \geq 2l\left(\frac{l+1}{2} \cdot \frac{1}{2} + \frac{r}{2}(1 - \frac{1}{2}) + \frac{k}{4}(1 - \frac{1}{2})\right)
= \frac{l}{2}(s + t) + r\frac{2l-1}{2} + k\frac{l-1}{2}.
\]

Then, by (23) we have that
\[
(24) \quad h - 1 \geq 2l(2\gamma - 2) + (s + t)\frac{l}{2} + r\frac{2l-1}{2} + k\frac{l-1}{2}.
\]

Claim: \(\gamma = 0, t + r \leq 6\)

Proof of the Claim: We prove that otherwise we would have
\[
(25) \quad 2l(2\gamma - 2) + (s + t)\frac{l}{2} + r\frac{2l-1}{2} + k\frac{l-1}{2} \geq s + \frac{1}{2}(r + k)
\]

which would contradict (22).

We observe that (25) is equivalent to
\[
(26) \quad 2l(2\gamma - 2) + t\frac{l}{2} + s\left(\frac{l}{2} - 1\right) + r\left(2l - 2\right) + k\frac{2(l - 2)}{l} \geq 0.
\]

Since \(l \geq 2\) it is always satisfied for \(\gamma > 0\), and when \(\gamma = 0\) the inequality (26) becomes
\[
l(t + r - 8) + (l - 2)(s + r + k) \geq 0.
\]

The second addend is always positive, so for \(t + r \geq 8\) it is satisfied. ♦

Note that as a consequence we need to impose always that
\[
(27) \quad l(t + r - 8) + (l - 2)(s + r + k) < 0.
\]

5.2.1 Curves with low genus

At this point we add the hypothesis of \(\tilde{B}\) being a low genus curve. The goal of this section is to prove

**Theorem 5.3.** There are no degree two curves \(\tilde{B}\) lying in \(C^{(2)}\) with \(g(C) < p_a(\tilde{B}) < 2g(C) - 1\) and \(\tilde{B}^2 > 0\).

**Proof.** Let \(\tilde{B}\) be a curve in the symmetric square of a curve \(C\) of genus \(g\) with \(\tilde{B} \cdot C_p = 2\) such that \(g < p_a(\tilde{B}) < 2g - 1\). Let \(B\) be the normalization of \(\tilde{B}\). Notice that since \(\Delta_C^2 < 0\) we can assume \(\tilde{B} \neq \Delta_C\).
By Proposition 5.2, we know that if $\pi_C^*(\tilde{B})$ is reducible then the self-intersection of $\tilde{B}$ is not positive, except for the case described in Example 5.1 that does not satisfy the inequality for the genus. Hence, it remains to study the case $\tilde{D} = \pi_C^*(\tilde{B})$ irreducible.

First of all we impose $g \geq 3$ because otherwise there is no possible $p_a(\tilde{B})$ with $g < p_a(\tilde{B}) < 2g - 1$. So, by (19) we have that

\begin{equation}
\label{eq:28}
g = \frac{2h + 2 - s}{4} \geq 3 \Leftrightarrow s \leq 2h - 10.
\end{equation}

By (20) and (28), the inequality $p_a(\tilde{B}) \leq 2g - 2$ is equivalent to

\begin{equation}
\label{eq:29}
\frac{2h + 2 - t}{4} + \frac{1}{4}k \leq 2\frac{2h + 2 - s}{4} - 2 \Leftrightarrow 6 + 2s + k \leq 2h + t.
\end{equation}

By (22) and (29) we obtain that

\begin{equation}
\frac{6 + 2s + k - t}{2} \leq h \leq s + \frac{1}{2}(r + k)
\end{equation}

and hence $t + r \geq 6$ and therefore, by the claim, $t + r = 6$.

Finally, we want to see that (22) is not possible.

**Claim:** Necessarily $l = 2$.

**Proof of the Claim**

Now, from (19) and (28) we deduce that $h = s + \frac{1}{2}(6 - t + k)$ and $t + 4 \leq s + k$.

Finally, we observe that the inequality (24) with these new restrictions becomes

\begin{equation}
(2 - l)(s - t + k) \geq -10 + 4l.
\end{equation}

Which together with $s + k - t \geq 4 > 0$ implies that $l = 2$. ◊

Considering again (24) for $l = 2$, $\gamma = 0$, $h = s + \frac{1}{2}(6 - t + k)$ and $t + r = 6$ we observe that we do not have equality. Then, in order to satisfy the Riemann-Hurwitz formula, there should be more branch points ($n'$ of such) satisfying

\begin{equation}
1 = 4 \sum \left(1 - \frac{1}{m_i}\right) \geq 4n' \frac{1}{2}
\end{equation}

which is not possible, and hence such a morphism $D \to D/D_{2l}$ does not exist.

Consequently, the self-intersection of $\tilde{B}$ is not positive. □

### 5.2.2 Curves of higher genus

Now we concentrate in curves with higher genus. We remind that we assume $\gamma = 0$ and $t + r \leq 6$. Moreover, for $l \geq 4$ the inequality (27) implies

\begin{equation}
\frac{s + r + k}{8 - (t + r)} < \frac{l}{l - 2} \leq 2.
\end{equation}
This inequality will help us to determine the possible values of our parameters. We are going to study separately the cases \( l \geq 4, l = 3 \) and \( l = 2 \).

We want to give a generating vector of a dihedral group \( D_{2l} \) in such a way that it defines an action on a curve \( D \) with exactly the number of fixed points determined by \( (t, r, s, k) \) and possibly some other coming from other powers of the automorphism \( ij \). Once we have such an action, we can construct a non completing diagram \([13]\) satisfying all our conditions with \( C = D/\langle j \rangle \) and \( B = D/\langle i \rangle \). Since we have imposed that \( D/D_{2l} = \mathbb{P}^1 \), the type of the vector will be \( (0; m_1, \ldots, m_n) \).

Once we have such a generating vector, and hence the curves \( D \) giving diagrams characterizing curves \( \tilde{B} \subset C^{(2)} \) with our conditions, we are going to study the properties of the curves involved in each case. In particular, we compute the genus of \( D, C \) and \( B \), the arithmetic genus of \( \tilde{B} \) and its self-intersection. Moreover, we compute the dimension of each family, with special attention to the dimension of the image in the moduli space of curves of genus \( g(C) \) for each family.

To begin with, we prove the following lemma:

**Lemma 5.4.** Let \( D \) be a curve with an action of \( D_{2l} \). Let \( i \) and \( j \) be two involutions generating this dihedral group. Denote by \( C = D/\langle j \rangle \). Since \( (ij)^l \) is in the center of \( D_{2l} \), its action descends to \( C \). Then, if \( \beta \) is the induced automorphism in \( C \), \( \nu(\beta) = \frac{1}{2}(\nu(i) + \nu((ij)^l)) \) if \( l \) is odd, and \( \nu(\beta) = \frac{1}{2}(\nu(j) + \nu((ij)^l)) \) if \( l \) is even.

**Proof.** Consider \( C \) embedded in \( D^{(2)} \) as \( \{P + j(P), P \in D\} \). In this way, the action of \( \beta \) on \( C \) is just

\[
\beta(P + j(P)) = (ij)^l(P) + (ij)^l(j(P)) = (ij)^l(P) + j((ij)^l(P)).
\]

A point will be fixed by \( \beta \) when either \( P = (ij)^l(P) \) or \( P = (ij)^lj(P) \). From the former we obtain \( \frac{1}{2} \nu((ij)^l) \) points of \( C \) fixed by \( \beta \) and from the later \( \frac{1}{2} \nu((ij)^lj) \). Notice that when \( l \) is odd \( (ij)^lj \) is conjugated to \( i \) in \( D_{2l} \), so \( \nu((ij)^lj) = \nu(i) \) and when \( l \) is even \( (ij)^lj \) is conjugated to \( j \) in \( D_{2l} \) and hence \( \nu((ij)^lj) = \nu(j) \). \( \Box \)

We will denote by \( \beta_C \in \text{Aut}(C) \) the action on \( C \) induced by \( (ij)^l \) and by \( \beta_B \) the action induced on \( B \).

By the discussion in Section 4 we are able to compute the genus of \( D, C \) and \( B \), the arithmetic genus of \( p_0(\tilde{B}) \) and the self-intersection of \( \tilde{B} \) once we know the number of fixed points corresponding to the different conjugacy classes in \( D_{2l} \). We do not include all these computations since they are straightforward from the discussion in Section 4. In a similar way, the conclusions about \( C \) and \( B \) being hyperelliptic or bielliptic come from the action of \( \beta_C \) or \( \beta_B \) and are omitted since they are based in a repeated use of Riemann-Hurwitz Theorem and Lemma 5.4.
As it is seen in [6], the curves $D$ with the prescribed action of $D_n$ are parametrized by an algebraic variety of dimension $3(g' + 1) - d$ where $g' = g(D/D_n)$ and $d$ is the number of branch points in the projection morphism to the quotient curve $D/D_n$. We call this variety $\mathcal{D}$. The image of $\mathcal{D}$ in the moduli space $\mathcal{M}_h$, given by forgetting the action, is an irreducible variety of the same dimension. We want to study the morphism $\eta$ from $\mathcal{D}$ to $\mathcal{M}_g$ that sends $(D, \rho)$ to $[C]$, and we wonder in which cases it has positive dimensional fibers.

We are going to study each numerical case $(t, r, s, k)$ separately to finish the proof of our theorem. We give some details in the first case and omit them for the rest of cases.

Before, we make some general remarks.

First, we consider the morphism $q : D/\langle j, (ij)^l \rangle \to D/D_{2l} \cong \mathbb{P}^1$. We observe that it is not Galois since $\langle j, (ij)^l \rangle$ is not normal in $D_{2l}$ for $l \neq 2$, and it is Galois for $l = 2$.

Second, we note that to give a curve $D$ with an action of $D_{2l}$, is equivalent to give $\mathbb{P}^1$ with a certain number, $n$, of marked points, and the branching data for the map. To avoid automorphisms, we can fix three of these points to be 0, 1 and $\infty$. As we change the rest of points, we change the pair $(D, \rho)$ in the family $\mathcal{D}$ of dimension $n - 3$.

Third, we are going to see that in all cases the curve $C$ is $\gamma$-hyperelliptic for $\gamma = 0, 1$ with $\beta_C = i_\gamma$, hence, to give the curve $C$ is equivalent to give $\mathbb{P}^1$ or the curve $E$, with the branch points of $p$ (the $\gamma$-hyperelliptic morphism) marked $(m$ points).

We have the following diagram of curves for each described action $\rho$ of $D_{2l} = \langle i, j \rangle$ on a curve $D$:

We observe that for $l \neq 2$ the curves $D/\langle j, (ij)^l \rangle$ and $D/\langle i, (ij)^l \rangle$ are isomorphic because $\langle j, (ij)^l \rangle$ and $\langle i, (ij)^l \rangle$ are conjugate.

Hence, $\eta$ is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, \ldots, x_{n-3}\}$ together with the monodromy description, to the curve determined by $\{F; x_1, \ldots, x_m\}$ were $F$ is the genus $\gamma$ curve given by the quotient of $C$ by its $\gamma$-hyperelliptic involution. Therefore, study the fibers of $\eta$ is equivalent to study the fibers of the morphism $\mathcal{M}_{0,n} \times \{\rho\} \to \mathcal{M}_{\gamma,m}$ defined by the previous correspondence.
Finally, given a curve $[C]$ in the image of $\eta$, we consider the data determined by its $\gamma$-hyperelliptic involution, which is unique for $\gamma = 0$ and there is at most a finite number of possibilities for $\gamma = 1$. If we know the morphism $q$, we can recover the data which determines $(D, \rho)$ taking the images of the branch points of $p$ together with the rest of branch points of $q$. Therefore, we can translate the question on the dimension of the fiber of $\eta$ to a question on the number of possible morphisms $q$ for a given curve $[C]$ in the image of $\eta$. We want to determine if given $\mathbb{P}^1$ (respectively $E$) with $m$ marked points and some information on the branching type of $q$, then there are a finite number of possible $q$’s, and hence a finite number of curves $[C]$, or otherwise, there is a positive dimension family of $q$’s and hence $\eta$ has positive dimensional fibers.

Assume first that $l \geq 4$. By the conditions $t + r \leq 6$, (22) and (30) together with the parities of $s$ and $k$, the possibilities for $t, r, s$ and $k$ are just a few 4-tuples. By the Riemann Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$ together with the conditions given by Riemann-Existence theorem we reduce this case to three possibilities (see [14] for a detailed discussion):

$$(D10.1) \ t = 0, \ r = 2, \ s = 4, \ k = 0.$$ 

By the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we find that there is another branch point, with $m = 2$ and that $l = 5$. Thus, the additional branch point needs to come from points fixed by $(ij)^5 \in D_{10}$. Consequently we impose one branch point coming from $ij$ (image of points fixed by it), 2 branch points coming from $j$ (and its conjugates $(ij)^2 \in D_{10}$) and one coming from $(ij)^5$.

We take the $(0; 10, 10, 2, 2)$-generating vector of $D_{10}$ given by $(ij, (ij)^5, (ij)^4 j, j)$.

We observe that $\mathcal{D}$ is the 1-dimensional family of all curves of genus 5 with maximal dihedral symmetry (see [5]).

Claim: The map $\eta$ is finite.

Proof of the claim: First, notice that the morphism $q : \mathbb{P}^1 \to \mathbb{P}^1$ has 3 branch points, and therefore, there are a finite number of such morphisms modulo automorphisms of $\mathbb{P}^1$.

Second, we observe that $D/D_{10} \cong \mathbb{P}^1$ has four marked points. To avoid automorphisms we fix $x$ to be the branch point associated to (image of) points fixed by $(ij)^5$, 0 to be associated to the points fixed by $ij$ and $\{1, \infty\}$ to be associated to $j$.

Third, $C/\langle \beta_C \rangle \cong \mathbb{P}^1$ has six marked points, the points where the hyperelliptic morphism, $p$, is branched. We observe that since $p$ is the projection given by the action of $\beta_C$, five of the branch points of $p$ are a fiber of $q$, in particular, the images of points fixed by $(ij)^5$ in $D$, and the sixth has ramification index 5 in $q$, in particular, the image of the points fixed by $ij$.

Therefore, $\eta$ is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x\}$ to the one determined by $\{\mathbb{P}^1; q^{-1}(0), q^{-1}(x)\}$, where $q$ is the morphism $D/\langle j, (ij)^5 \rangle \cong \mathbb{P}^1 \to$
Finally, given a curve \([C]\) in the image of \(D\), we recover \(\eta^{-1}([C])\) taking the image of the branch points of its hyperelliptic involution by a suitable \(q\) (they will be 0 and \(x\)) together with the other two branch points of \(q\) (they will be 1 and \(\infty\)). By a suitable \(q\) we mean that one of the branch points of \(p\) is a ramification point of \(q\), and the other five are a fiber of \(q\).

Such a morphism \(q\) exists because \(C\) is in the image of \(D\), and therefore it is the quotient of a \(D\). Moreover, there are only a finite number of possibilities for \(q\), and hence, we can recover at most a finite number of \((D, \rho) \in D\). ♦

\((D10.2)\) \(t = 2, r = 0, s = 2, k = 4\).

By the Riemann-Hurwitz formula for \(D \to D/D_{2l} = \mathbb{P}^1\), we find that there is one more branch point with \(m = 2\) and that \(l = 5\). We take the \((5, 2, 2, 2)\)-generating vector of \(D_{10}\) given by \(((ij)^2, (ij)^5, (ij)^2i, j)\) to construct \(D\).

We observe that \(D\) is the 1-dimensional family of all curves of genus 4 with maximal dihedral symmetry (see \([5]\)).

Moreover, since \(q\) has three branch points we deduce, as in point \((D10.1)\), that the map \(\eta\) is finite.

\((D10.3)\) \(t = 2, r = 0, s = 6, k = 0\).

By the Riemann-Hurwitz formula for \(D \to D/D_{2l} = \mathbb{P}^1\), we find that there is another branch point with \(m = 2\) and that \(l = 5\). We take the \((2, 2, 2, 2)\)-generating vector of \(D_{10}\) given by \(((ij)^5, (ij)^4i, j, j, j)\) to construct \(D\).

We expect the map \(\eta\) to be finite. Indeed, there should be only a finite number of possibilities for \(q\). Since \(q\) is a degree five morphism from \(\mathbb{P}^1\) to \(\mathbb{P}^1\), it is given, in homogeneous coordinates by two degree five polynomials. Given five of the branch points of \(p\), we assume that their image is \(0 \in \mathbb{P}^1\) and we have one of the polynomials determined. Assuming that the sixth point has image \(\infty \in \mathbb{P}^1\), we obtain one factor of the other polynomial. When we impose the branching type \((1, 2, 2)\) in the four branch points, we obtain a system of twelve equations with five unknowns.

The resolution of this system of equations has a very high computational cost because of the high degree of the equations involved. We were not able to finish it. Probably a more refined algorithm would be needed. Nevertheless, the high number of equations compared to the number of unknowns takes us to conjecture that this system of equations has a finite number of solutions. If so, we could recover at most a finite number of \((D, \rho) \in D\).

Finally notice that in all three cases \(p_a(\tilde{B}) = 2g(C) - 1\).

System specifications: Processor: Intel Xeon W3520 @2.67GHz. 4 GB RAM. Using Windows 64 bits and Wolfram Mathematica 9
Assume now that $l = 3$.

We are going to consider now the case $\langle i, j \rangle = D_6$. Since $s, r, k$ and $t$ are even integers, to condition (27) is equivalent to

\[(31)\quad s + 4r + k + 3t \leq 22.\]

In $D_6$ there are six conjugacy classes: $[Id], [i], [j], [ij], [(ij)^2]$ and $[(ij)^3]$. We denote by $p = \nu((ij)^3)$. Thus, the Riemann-Hurwitz formula for $D \to D/D_6$ reads

\[h = \frac{1}{2}(-22 + 3s + 3t + 5r + 2k + p)\]

If we consider the second condition in (22) with these values, then we deduce that

\[(32)\quad s + 3t + 4r + k + p \leq 22.\]

Notice that if it is satisfied, then also (31) is satisfied.

Now, we observe that we can embed $D_6$ in $S_6$ in such a way that $i$ is odd and $j$ is even (thus $ij$ is odd, $(ij)^2$ is even and $(ij)^3$ is odd). Since we will need the product one condition for the generating vector, we need to impose $t + r + \frac{p}{3}$ to be even, or which is the same, $t + r + \frac{p}{3}$ multiple of four. Furthermore, we can also embed $D_6$ in $S_6$ in such a way that $i$ is even and $j$ is odd and hence we need to impose $s + r + \frac{p}{3}$ multiple of four. By this, inequality (32), our previous conditions and the conditions from the Riemann-Existence theorem we find the following possibilities:

(D6.1) $t = 0, r = 0, p = 12, s = 4, k = 4$.

We define the generating vector $((ij)^3, (ij)^3, (ij)^2, (ij)^4, j, j)$ to construct $D$.

Since $q$ has three branch points, following the proof of the analogous claim in case (D10.1) we deduce that the map $\eta$ is finite.

(D6.2) $t = 0, r = 2, p = 6, s = 4, k = 4$.

We define the generating vector $(ij, (ij)^3, (ij)^2, j, j)$ to construct $D$.

Claim: The map $\eta$ is finite.

Proof of the claim: Given a curve $[C]$ in the image of $D$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable $q$ together with the other three branch points of $q$. By a suitable $q$ we mean that one of the branch points of $p$ has ramification index 3 in $q$, and the other 3 are a fiber of $q$.

Such a morphism $q$ exists by construction and moreover, there are only a finite number of possibilities for $q$. Indeed, since the elliptic curve is given, one of the branch points determine the immersion of $E$ in $\mathbb{P}^2$ in such a way that it is an inflexion point, and at least for one of the four options, the other three points will be over a line. Then, the projection point will be the intersection of the tangent to the inflexion and the line containing the other three. ♦

(D6.3) $t = 0, r = 2, p = 6, s = 8, k = 0$. 

We define the generating vector \((ij, (ij)^3, (ij) j, j, j)\) to construct \(D\).

**Claim:** The map \(\eta\) is finite.

**Proof of the claim:** Given a curve \(C\) in the image of \(D\), we recover \(\eta^{-1}([C])\) taking the image of the branch points of its bielliptic involution by a suitable \(q\) together with the other four branch points of \(q\). By a suitable \(q\) we mean that one of the branch points of \(p\) has ramification order 3 in \(q\), and the other three are a fiber of \(q\).

As in case (D6.2) such a morphism \(q\) exists and there are only a finite number of possibilities for it. ♦

(D6.4) \(t = 0, r = 4, p = 0, s = 4, k = 0\).

We define the generating vector \((ij, ji, j, j)\) to construct \(D\).

**Claim:** The map \(\eta\) is finite.

**Proof of the claim:** Given a curve \([C]\) in the image of \(D\), we recover \(\eta^{-1}([C])\) taking the image of the branch points of its bielliptic involution by a suitable \(q\), together with the other two branch points of \(q\). By a suitable \(q\) we mean that the two branch points of \(p\) have ramification index 3 in \(q\).

Such a morphism \(q\) exists because \(C\) is in the image of \(D\), and therefore it is the quotient of a \(D\). Moreover, there are only a finite number of possibilities for \(q\). Indeed, since we have the elliptic curve given as the quotient of \(C\) by its bielliptic involution, one of the branch points of \(p\) determine the immersion of \(E\) in \(\mathbb{P}^2\) in such a way that it is an inflexion point, and then necessarily the other point will be another inflexion. The projection point will be then the intersection of the respective tangent lines. Thus, we can recover at most a finite number of \((D, \rho) \in D\). ♦

(D6.5) \(t = 2, r = 0, p = 6, s = 2, k = 8\).

We define the generating vector \((j, i, (ij)^3, (ij)^2, (ij)^2)\) to construct \(D\).

**Claim:** The map \(\eta\) is finite.

**Proof of the claim:** Given a curve \([C]\) in the image of \(D\), we recover \(\eta^{-1}([C])\) taking the image of the branch points of its bielliptic involution by a suitable \(q\) together with the other three branch points of \(q\). By a suitable \(q\) we mean that one of the branch points of \(p\) is not ramified but lies over a branch point and the other 3 are a fiber of \(q\).

Such a morphism \(q\) exists by construction and moreover, there are only a finite number of possibilities for \(q\). Indeed, since the elliptic curve is given, taking three of the branch points we determine the immersion of \(E\) in \(\mathbb{P}^2\), and taking a line passing through the fourth and tangent to \(E\) but not on this point (a finite number of such), we obtain a finite number of candidates for the projection point. Only those with two points with ramification index 3 are possible \(q\)'s. ♦

(D6.6) \(t = 2, r = 0, p = 6, s = 6, k = 4\).

We define the generating vector \(((ij)^3, (ij)^2, i, j, j, j)\) to construct the curve \(D\).
**Claim:** The map $\eta$ is finite.

**Proof of the claim:** Given a curve $[C]$ in the image of $D$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic morphism by a suitable $q$ together with the other four branch points of $q$. By a suitable $q$ we mean that one of the branch points of $p$ is non ramified lying over a branch point and the other three are a fiber of $q$.

As in (D6.5) such a morphism $q$ exists and there is a finite number of such. ♦

(D6.7) $t = 2, r = 2, p = 0, s = 2, k = 4$.

We define the generating vector $(i, j, ij, (ij)^4)$ to construct $D$.

**Claim:** The map $\eta$ is finite.

**Proof of the claim:** Given a curve $[C]$ in the image of $D$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable $q$ and the other two branch points of $q$. By a suitable $q$ we mean that one of the branch points of $p$ has ramification order 3, and the other is non ramified lying over a branch point of $q$.

Such a morphism $q$ exists by construction and moreover, there are only a finite number of possibilities for $q$. Indeed, since the elliptic curve is given, one of the branch points determines the immersion on $\mathbb{P}^2$ in such a way that it is an inflexion, and taking a line passing through the other and tangent to $E$, but not on this point (a finite number of such), we obtain a finite number of candidates for the projection point. Only those with two points with ramification index 3 would be possible $q$'s. ♦

(D6.8) $t = 2, r = 2, p = 0, s = 6, k = 0$.

We define the generating vector $(j, j, i, ij)$ to construct the curve.

**Claim:** The map $\eta$ is finite.

**Proof of the claim:** Given a curve $[C]$ in the image of $D$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable $q$ together with the other three branch points of $q$. By a suitable $q$ we mean that one of the branch points of $p$ has ramification order 3, and the other is non ramified lying over a branch point of $q$.

As in (D6.7) such a morphism $q$ exists and there are only a finite number of possibilities for it. ♦

(D6.9) $t = 4, r = 0, p = 0, s = 4, k = 4$.

We define the generating vector $(i, i, (ij)^4, (ij)^2j, j)$ to construct $D$.

**Claim:** The map $\eta$ is finite.

**Proof of the claim:** Given a curve $[C]$ in the image of $D$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable $q$ together with the other three branch points of $q$. By a suitable $q$ we mean that the branch points of $p$ have images different branch points of $q$ being non ramified. Such a morphism $q$ exists by construction.

Moreover, there are only a finite number of possibilities for $q$. Indeed, let $x$ and $y$ be
the branch points of \( p \). A suitable \( q \) can be described by the immersion of \( E \) in \( \mathbb{P}^2 \) given by the linear series of the fibers followed by the projection from a point not belonging to the image of \( E \). Assume that we have such an immersion. The point with ramification index three is an inflection of the curve in \( \mathbb{P}^2 \) and the projection point lies over the tangent in this point. Moreover, the lines linking \( x \) and \( y \) with the projection point are tangent to the curve in certain points \( x' \) and \( y' \) respectively.

Assume by contradiction that there is a positive dimensional family of pairs \((x', y')\) as above. Given a particular immersion determined by one such \( q \), we remind that the projection point is determined by the intersection of the tangent to an inflexion point and the lines through \( x \) and \( y \) tangent to the curve. If we move \( x \) and \( y \) by a point \( z \in E \subset \mathbb{P}^2 \), we change the immersion of the curve, but we keep the same planar equation. If there is a one dimensional family of suitable morphisms \( q \), then the point where the new tangents through \( x+z \) and \( y+z \) intersect should be over the tangent to the inflexion point, giving another morphism \( q \) in the family. Doing the effective computations we find that for a general \( z \) it does not happen, and hence, there are only a finite number of suitable \( q \)'s.

\[ (D6.10) \quad t = 4, \quad r = 0, \quad p = 0, \quad s = 8, \quad k = 0. \]

We define the generating vector \((i, i, j, j, j, j)\) to construct \( D \).

**Claim:** The map \( \eta \) has 1-dimensional fibers.

**Proof of the claim:** Given a curve \([C]\) in the image of \( D \), we recover \( \eta^{-1}([C]) \) taking the image of the branch points of its bielliptic morphism by a suitable \( q \) together with the other four branch points of \( q \). By a suitable \( q \) we mean that the branch points of \( p \) are non-ramified lying over a branch point, and \( q \) has generic ramification.

Such a morphism \( q \) exists by construction and moreover, we claim that there is a one dimensional family of possibilities for \( q \).

Indeed, the elliptic curve \( E \) is given, with two marked points \( x \) and \( y \), and we are looking for a \( q : E \to \mathbb{P}^1 \) with generic ramification and the two marked points over a branch point but non-ramified.

Each immersion of \( E \) in \( \mathbb{P}^2 \) is given by a line bundle \( a \in Pic^3(E) \). If we consider the projection \( \pi : E^{(3)} \to Pic^3(E) \), the fibers of this morphism are \( \mathbb{P}^2 \)'s given by the linear series. A morphism to \( \mathbb{P}^1 \) of order 3 can be seen a line in this \( \mathbb{P}^2 \) with no base point, that is, not contained in a divisor \( E_x \).

Given two points \( x, y \in E \), for each \( \mathbb{P}^2 = \pi^{-1}(a) \) we have four points of type \( x+2x' \) and four of type \( y+2y' \), hence, there are 16 lines that contain one of each type. In this same fiber of \( \pi \) there are 9 points of type \( 3Q \). Therefore, for a general, at least one of the 16 lines through \( x+2x' \) and \( y+2y' \) will not contain a point of type \( 3Q \), and therefore, we deduce that given \( x, y \in E \) there is a 1-dimensional family \( (dim Pic(E) = 1) \) of morphisms of degree 3 from \( E \) to \( \mathbb{P}^1 \) with generic branching type and \( x, y \) non ramified but with image
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a branch point.

Hence, we can recover a one dimensional family of $(D, \rho) \in D$. And for each $D$ we find a different $B$, therefore, we have a one dimensional family of curves $\tilde{B} \subset C^{(2)}$ for each $[C] \in \eta(D)$.

Assume finally that $l = 2$.

We consider now the case $\langle i, j \rangle = D_4$.

First, we observe that we can embed $D_4$ in $S_4$ in such a way that $i$ is odd and $j$ is even (thus $ij$ is odd and $(ij)^2$ is even). Since we will need the product one condition when constructing the generating vector, we need to impose $\frac{t}{2} + \frac{r}{2}$ to be even, or what is the same, $t + r$ to be multiple of four. Moreover, we can also embed $D_4$ in $S_4$ in such a way that $i$ is even and $j$ is odd hence we impose $s + r$ to be multiple of four.

Since in $D_4$ we have five conjugacy classes $[Id], [i], [j], [ij]$ and $[(ij)^2]$, all branch points in $D \to D/D_4$ will be considered in either $t, s, r$ or $k$, thus the Riemann-Hurwitz formula reads $2h - 2 = -16 + 2s + 2t + 3r + k$.

The condition $g \geq 2$ is equivalent to $s \leq 2h - 6$, which with the expression of $h$ becomes

$$s \leq -14 + 2s + 2t + 3r + k - 6 \Leftrightarrow 20 \leq s + 2t + 3r + k.$$  

We remind that we have seen that necessarily $t + r \leq 6$. By our previous conditions and the conditions from the Riemann-Existence theorem we find the following possibilities:

(D4.1) $t = 0, r = 4$. We need that $s > 0$ to obtain a generating vector. Since $s + r$ should be multiple of four, we obtain that $s$ is multiple of four. From (33) we deduce that $8 \leq s + k$.

Depending on the parity of $\frac{k}{4}$ we define the generating vector as:

- If $\frac{k}{4}$ is even: $(ij, ji, j, \ldots, j, (ij)^2, \ldots, (ij)^2)$.  
  \hspace{0.5cm} s/2 \hspace{1cm} k/4

- If $\frac{k}{4}$ is odd: $(ij, ji, j, \ldots, j, j(ij)^2, (ij)^2, \ldots, (ij)^2)$.  
  \hspace{0.5cm} s/2-1 \hspace{1cm} k/4

We note that $\nu(\beta_B) = \frac{k+4}{2}$ and hence, we obtain that $g(B/\langle \beta_B \rangle) = \frac{s}{4} \geq 1$.

(D4.2) $t = 2, r = 2$. We need that $s > 0$ to be able to have the product one condition. Since $s + r$ should be multiple of four, we obtain that $s = 4\alpha + 2$ with $\alpha \in \mathbb{Z}_{\geq 0}$. From (33) we deduce that $10 \leq s + k$.

Depending on the parity of $\frac{k}{4}$ we define the generating vector as:

- If $\frac{k}{4}$ is even: $(i, ij, j, \ldots, j, (ij)^2, \ldots, (ij)^2)$.  
  \hspace{0.5cm} s/2 \hspace{1cm} k/4

We note that $\nu(\beta_B) = \frac{k+4}{2}$ and hence, we obtain that $g(B/\langle \beta_B \rangle) = \frac{s}{4} \geq 1$. 

(D4.3) $t = 1, r = 3$. We need that $s > 0$ to be able to have the product one condition. Since $s + r$ should be multiple of four, we obtain that $s = 4\alpha + 2$ with $\alpha \in \mathbb{Z}_{\geq 0}$. From (33) we deduce that $10 \leq s + k$.

Depending on the parity of $\frac{k}{4}$ we define the generating vector as:

- If $\frac{k}{4}$ is even: $(i, ij, j, \ldots, j, (ij)^2, \ldots, (ij)^2)$.  
  \hspace{0.5cm} s/2 \hspace{1cm} k/4

We note that $\nu(\beta_B) = \frac{k+4}{2}$ and hence, we obtain that $g(B/\langle \beta_B \rangle) = \frac{s}{4} \geq 1$. 

(D4.4) $t = 0, r = 6$. We need that $s > 0$ to be able to have the product one condition. Since $s + r$ should be multiple of four, we obtain that $s = 4\alpha + 2$ with $\alpha \in \mathbb{Z}_{\geq 0}$. From (33) we deduce that $10 \leq s + k$.

Depending on the parity of $\frac{k}{4}$ we define the generating vector as:

- If $\frac{k}{4}$ is even: $(i, ij, j, \ldots, j, (ij)^2, \ldots, (ij)^2)$.  
  \hspace{0.5cm} s/2 \hspace{1cm} k/4

We note that $\nu(\beta_B) = \frac{k+4}{2}$ and hence, we obtain that $g(B/\langle \beta_B \rangle) = \frac{s}{4} \geq 1$. 

(D4.5) $t = 1, r = 5$. We need that $s > 0$ to be able to have the product one condition. Since $s + r$ should be multiple of four, we obtain that $s = 4\alpha + 2$ with $\alpha \in \mathbb{Z}_{\geq 0}$. From (33) we deduce that $10 \leq s + k$.

Depending on the parity of $\frac{k}{4}$ we define the generating vector as:

- If $\frac{k}{4}$ is even: $(i, ij, j, \ldots, j, (ij)^2, \ldots, (ij)^2)$.  
  \hspace{0.5cm} s/2 \hspace{1cm} k/4

We note that $\nu(\beta_B) = \frac{k+4}{2}$ and hence, we obtain that $g(B/\langle \beta_B \rangle) = \frac{s}{4} \geq 1$.
• If $\frac{k}{4}$ is odd: $(i, j, \ldots, j, \frac{ij}{s/2}, (ij)^2, \ldots, (ij)^2).$

We note that $\nu(\beta_B) = \frac{k+2+2}{2}$ and hence, we obtain that $g(B/\langle \beta_B \rangle) = \frac{s-2}{4} \geq 0.$

(D4.3) $t = 4, r = 0.$ We need that $s > 0$ to generate. Since $s + r$ should be multiple of four, we obtain that $s$ is multiple of four. From (33) we deduce that $12 \leq s + k.$

Depending on the parity of $\frac{k}{4}$ we define the generating vector as:

• If $\frac{k}{4}$ is even: $(i, i, j, \ldots, j, \frac{ij}{s/2}, (ij)^2, \ldots, (ij)^2).$

• If $\frac{k}{4}$ is odd: $(i, i, j, \ldots, j, \frac{ij}{s/2-1}, (ij)^2, (ij)^2, \ldots, (ij)^2).$

We note that $\nu(\beta_B) = \frac{k+4}{2}$ and hence, we obtain that $g(B/\langle \beta_B \rangle) = \frac{s-4}{4} \geq 0.$

We remark that for these three families we have that $\tilde{B}^2 = 4,$ the maximum possibility by the Hodge index theorem. Therefore, $\tilde{B}$ is algebraically equivalent to two times a coordinate curve. Moreover, since a coordinate curve has positive self-intersection, the restriction of $Pic^0(C^{(2)})$ to $Pic^0(C_P)$ is injective. Then, if we consider the restriction of $\tilde{B}$ to a coordinate curve, we have two points, and hence it is linearly equivalent to the restriction of the sum of two coordinate curves. Then, if we consider the divisor $\tilde{B} - (C_{p_1} + C_{p_2})$ it is on $Pic^0(C^{(2)})$ and restricted to $Pic^0(C_P)$ it is zero, therefore, since this morphism is injective, we deduce that $\tilde{B}$ is linearly equivalent to the sum of two coordinate curves, and hence $h^0(C^{(2)}, \mathcal{O}_{C^{(2)}}(\tilde{B})) \geq 2.$

We have, therefore, 3 more families of morphisms giving pairs $(B, C)$ with $B \to C^{(2)}$ of degree one.

These are all the possible cases of degree two curves in $C^{(2)}$ and positive self-intersection. Therefore we have finished the proof of Theorem 1.5.

5.3 Further comments

Remark 5.5. Notice that in all cases $\tilde{B}^2 \leq p_a(\tilde{B}) - g(C) + 2,$ thus satisfying the inequality in Corollary 4.7 of [10] even if in the cases with $g(C) = 2$ the surface is not of general type, and therefore the hypothesis are not fulfilled.

We can say something more about the curves $D$ in relation to the curves $C$ and $B$ using the following lemma:

Corollary 5.6. We have the following isogeny for the Jacobian variety of one of the curves $D$ that appear on the theorem:

$$J_D \approx J_C \times J_B \times J_{D/(ij)}.$$
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**Proof.** We can decompose $G = D_n = \langle i, j \rangle$, with $i$ and $j$ two involutions, as

$$D_n = \langle ij \rangle \cup \langle i \rangle \cup \langle (ij)^2 i \rangle \cup \cdots \cup \langle (ij)^{n-1} i \rangle.$$ 

We remind that in our case $D/D_n \cong \mathbb{P}^1$, so by [8] with $n = t + 1$ we obtain that

$$J_n^D \approx J_n^{D/\langle ij \rangle} \times J_n^{D/\langle i \rangle} \times J_n^{D/\langle (ij)^2 i \rangle} \times \cdots \times J_n^{D/\langle (ij)^{n-1} i \rangle}.$$ 

Since $i$ is conjugated of all $(ij)^{2k} i$ and $j$ is conjugated of all $(ij)^{2k+1} i$ we deduce that

$$J_{D/\langle i \rangle} \approx J_{D/\langle (ij)^{2k} i \rangle} \approx J_B \quad J_{D/\langle ij \rangle} \approx J_{D/\langle (ij)^{2k+1} i \rangle} \approx J_C.$$ 

Therefore,

$$J_n^D \approx J_n^{D/\langle ij \rangle} \times J_B \times J_C$$

and applying Poincaré duality we obtain the stated isogeny.

**5.3.1 Low genus curves in $C \times C$**

We remark finally that some of the curves $\tilde{D} \subset C \times C$ are, in fact, low genus curves with positive self-intersection. Indeed, since by definition $\tilde{D}$ is the preimage of $\tilde{B}$ by $\pi_C$, we have that

$$(34) \quad \tilde{D}^2 = (\pi_C^{-1}(\tilde{B}))^2 = \deg \pi_C \tilde{B}^2 = 2\tilde{B}^2$$

and moreover

$$(35) \quad q(C \times C) = 2g$$

$$(36) \quad p_a(\tilde{D}) = g(D) + \frac{1}{2}(r + k).$$

Therefore, as soon as $\tilde{B}$ has positive self-intersection, so does $\tilde{D}$. And moreover, if $2g \leq h + \frac{1}{2}(r + k) \leq 4g - 2$ then $\tilde{D}$ is low genus curve with positive self-intersection.

We look at the results of Theorem [1.5] and note that this happens in all cases defined by the action of $D_{10}$, the four cases defined by the action of $D_6$ with $g(C) = 3$ and all the cases defined by the action of $D_4$.

Therefore, we have found some examples of low degree curves in an irregular surface. We note that all our examples, as well as those found in [3] are such that $p_a(\tilde{D}) = 2q(S) - 2$. That is, the curves are exactly at the boundary of the range of low genus curves.
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