ZEROS AND ROOTS OF UNITY IN CHARACTER TABLES

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Dedicated to the memory of Patrick X. Gallagher

Abstract. For any finite group $G$, Thompson proved that, for each $\chi \in \text{Irr}(G)$, $\chi(g)$ is a root of unity or zero for more than a third of the elements $g \in G$, and Gallagher proved that, for each larger than average class $g^G$, $\chi(g)$ is a root of unity or zero for more than a third of the irreducible characters $\chi \in \text{Irr}(G)$. We show that in many cases “more than a third” can be replaced by “more than half”.

1. Introduction

For any finite group $G$, let

$$\theta(G) = \min_{\chi \in \text{Irr}(G)} \frac{\{|g \in G : \chi(g) \text{ is a root of unity or zero}\}}{|G|}$$

and let

$$\theta'(G) = \min_{|g^G| \geq |G|/|\text{Cl}(G)|} \frac{|\{\chi \in \text{Irr}(G) : \chi(g) \text{ is a root of unity or zero}\}|}{|\text{Irr}(G)|}.$$

Burnside proved that each $\chi \in \text{Irr}(G)$ with $\chi(1) > 1$ has at least one zero, P. X. Gallagher proved that each $g \in G$ with $|g^G| > |G|/|\text{Cl}(G)|$ is a zero of at least one $\chi \in \text{Irr}(G)$, J. G. Thompson proved that $\theta(G) > 1/3$, and Gallagher proved that $\theta'(G) > 1/3$.

The proofs run by taking the relations $\sum_{g \in G} |\chi(g)|^2 = |G| (\chi \in \text{Irr}(G))$ and $\sum_{\chi \in \text{Irr}(G)} |\chi(g)|^2 = |G|/|G^G|$, applying the elements $\sigma$ of the Galois group $G = \text{Gal}(Q(e^{2\pi i/|G|})/Q)$, averaging over $G$, and using that the average over $G$ of $|\sigma(\alpha)|^2$ is $\geq 1$ for any nonzero algebraic integer $\alpha \in Q(e^{2\pi i/|G|})$, or using the fact, due to C. L. Siegel, that the average over $G$ of $|\sigma(\alpha)|^2$ is $\geq 3/2$ for any algebraic integer $\alpha \in Q(e^{2\pi i/|G|})$ which is neither a root of unity nor zero, cf. [1, 3, 4, 11]. For certain groups, there are also strong asymptotic results about zeros due to Gallagher, M. Larsen, and the author [5, 8, 10].

Are the lower bounds of $1/3$ for $\{\theta(G) : |G| < \infty\}$ and $1/3$ for $\{\theta'(G) : |G| < \infty\}$ the best possible?

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**Question 1.** What is the greatest lower bound of \(\{\theta(G) : |G| < \infty\}\)?

**Question 2.** What is the greatest lower bound of \(\{\theta'(G) : |G| < \infty\}\)?

The author suspects that the answers to these questions are both \(1/2\). In particular, we propose the following:

**Conjecture 1.** \(\theta(G)\) and \(\theta'(G)\) are \(\geq 1/2\) for every finite group \(G\).

We establish the conjecture for all finite nilpotent groups by establishing a much stronger result about zeros for this family of groups, which includes all \(p\)-groups. The number of \(p\)-groups of order \(p^n\) was shown by G. Higman [6] and C. C. Sims [12] to equal \(p^{\frac{2}{27}n^3 + O(n^{8/3})}\) with \(n \to \infty\), and it is a folklore conjecture that almost all finite groups are nilpotent in the sense that the number of nilpotent groups of order at most \(n\)

\[
\frac{\text{the number of nilpotent groups of order at most } n}{\text{the number of groups of order at most } n} = 1 + o(1),
\]

which, in view of our result, would mean that Conjecture 1 holds for almost all finite groups.

Conjecture 1 is readily verified for rational groups, such as Weyl groups, and all groups of order \(< 2^9\), and although \(\theta(G) = 1/2\) for certain dihedral groups, the second inequality is strict in all known cases. The author suspects that both inequalities are strict for all finite simple groups:

**Conjecture 2.** \(\theta(G)\) and \(\theta'(G)\) are \(> 1/2\) for every finite simple group \(G\).

We verify Conjecture 2 for \(A_n, L_2(q), Suz(2^{2n+1}), Ree(3^{2n+1})\), all sporadic groups, and all simple groups of order \(\leq 10^9\). We also show that both \(\theta(Suz(2^{2n+1}))\) and \(\theta'(Suz(2^{2n+1}))\) tend to \(1/2\) as \(n \to \infty\). In particular, the answers to Questions 1 and 2 must lie between 1/3 and 1/2.

### 2. Nilpotent groups

We begin with our results on finite nilpotent groups.

**Theorem 1.** For each finite nilpotent group \(G\), and each \(\chi \in \text{Irr}(G)\) with \(\chi(1) > 1\), \(\chi(g) = 0\) for more than half of the elements \(g \in G\).

**Theorem 2.** Let \(G\) be a finite nilpotent group, and let \(g \in G\). If \(|g^G| > \frac{|G|}{|\text{Cl}(G)|}\), then \(\chi(g) = 0\) for more than half of the nonlinear \(\chi \in \text{Irr}(G)\). If \(|g^G| = \frac{|G|}{|\text{Cl}(G)|}\), then \(\chi(g) = 0\) for at least half of the nonlinear \(\chi \in \text{Irr}(G)\).

**Corollary 3.** \(\theta(G)\) and \(\theta'(G)\) are \(> 1/2\) whenever \(G\) is nilpotent.

The key ingredient in the proof of Theorems 1 and 2 is Theorem 8, which will replace the result of Siegel used by Thompson and Gallagher. Its proof relies on some auxiliary results of independent interest and is based on arithmetic in cyclotomic fields.

For each positive integer \(k\), we denote by \(\zeta_k\) a primitive \(k\)-th root of unity. For any algebraic integer \(\alpha\) contained in some cyclotomic field, we denote by
\[ f(\alpha) \text{ the least integer } l \text{ such that } \alpha \text{ is a sum of } l \text{ roots of unity, by } f(\alpha) \text{ the least positive integer } k \text{ such that } \alpha \in \mathbb{Q}(\zeta_k), \text{ and by } m(\alpha) \text{ the normalized trace} \]

\[ \frac{1}{[\mathbb{Q}(|\alpha|^2) : \mathbb{Q}]} \text{Tr}_{\mathbb{Q}(|\alpha|^2)/\mathbb{Q}}(|\alpha|^2), \]

so for any cyclotomic field \( \mathbb{Q}(\zeta) \) containing \( \alpha \),

\[ m(\alpha) = \frac{1}{[\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})]} \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} |\sigma(\alpha)|^2. \]

Lemma 4. Let \( a_1, a_2, \ldots, a_l \) and \( b_1, b_2, \ldots, b_m \) be rational integers, and let \( \alpha_1, \alpha_2, \ldots, \alpha_l \) and \( \beta_1, \beta_2, \ldots, \beta_m \) be \( p^n \)-th roots of unity with \( p \) prime and \( n \) nonnegative. If

\[ \sum_{j=1}^{l} a_j \alpha_j = \sum_{k=1}^{m} b_k \beta_k, \]

then

\[ \sum_{j=1}^{l} a_j \equiv \sum_{k=1}^{m} b_k \pmod{p}. \]

Proof of Lemma 4. If \( n = 0 \), then there is nothing to prove, so assume \( n \geq 1 \). Let \( \zeta \) be a primitive \( p^n \)-th root of unity. For each \( \alpha_j \) and \( \beta_k \), let \( r_j \) and \( s_k \) be nonnegative integers such that \( \alpha_j = \zeta^{r_j} \) and \( \beta_k = \zeta^{s_k} \). Put

\[ P(x) = \sum_{j=1}^{l} a_j x^{r_j} - \sum_{k=1}^{m} b_k x^{s_k}. \]

Then \( P(\zeta) = 0 \), so \( P(x) \) is divisible in \( \mathbb{Z}[x] \) by the cyclotomic polynomial

\[ \Phi_{p^n}(x) = \Phi_p(x^{p^{n-1}}). \]

Hence \( P(1) \equiv 0 \pmod{p} \). \( \square \)

Proposition 5. Let \( G \) be a finite group, let \( \chi \in \text{Irr}(G) \), and let \( g \) be an element of \( G \) with order a power of a prime \( p \). If \( p = 2 \) or \( \chi(1) \neq \pm 2 \pmod{p} \), then either \( \chi(g) = 0 \), \( \chi(g) \) is a root of unity, or \( m(\chi(g)) \geq 2 \).

Proof of Proposition 5. Suppose that \( p = 2 \) or \( \chi(1) \neq \pm 2 \pmod{p} \). Let \( p^n \) be the order of \( g \), and let \( \zeta \) be a primitive \( p^n \)-th root of unity, so \( \chi(g) \in \mathbb{Q}(\zeta) \). Let \( \alpha = \zeta^m \chi(g) \) with \( m \) such that

\[ f(\alpha) = \min_k f(\zeta^k \chi(g)). \]

We will show that either \( \alpha = 0 \), \( \alpha \) is a root of unity, or \( m(\alpha) \geq 2 \).

Let \( P = f(\alpha) \). Using \( \mathbb{Q}(\zeta_k) \cap \mathbb{Q}(\zeta_l) = \mathbb{Q}(\zeta_{k,l}) \), then \( P \) divides \( p^n \). If \( P = 1 \), then \( \alpha \) is rational and the conclusion follows. If \( P \) is divisible by \( p^2 \), then for \( \gamma \) a primitive \( P \)-th root of unity, \( \alpha \) is uniquely of the shape

\[ \alpha = \sum_{k=0}^{p-1} \alpha_k \gamma^k, \quad \alpha_k \in \mathbb{Q}(\zeta_{p^2}), \]

(2)
the $\alpha_k$ are algebraic integers, and a straightforward calculation [2, p. 115] shows that $m(\alpha)$ is at least the number of nonzero $\alpha_k$. By (1), at least two of the $\alpha_k$ are nonzero. Hence $m(\alpha) \geq 2$ if $p^2 \mid P$.

It remains to consider the case $P = p$. Since $Q(\zeta_2) = Q(\zeta_1)$, we must have $p > 2$. If $l(\alpha) = 0$, then $\alpha = 0$; if $l(\alpha) = 1$, then $\alpha$ is a root of unity; and if $l(\alpha) > 2$, then $m(\alpha) \geq 2$ by a result of Cassels [2, Lemma 3]. So assume $l(\alpha) = 2$. Then by [9, Thm. 1(i)], $\alpha$ can be written in the shape

$$\alpha = \epsilon_1 \xi_1^k + \epsilon_2 \xi_2^k, \quad \epsilon_k^2 = 1,$$

where $\xi_1$ and $\xi_2$ are $p$-th roots of unity. If $\xi_1 = \xi_2$, then either $\alpha = 0$ or $m(\alpha) = 4$. So assume $\xi_1 \neq \xi_2$.

By Lemma 4,

$$\epsilon_1 + \epsilon_2 \equiv \chi(1) \pmod{p}. \quad (3)$$

By (3) and the fact that $\chi(1) \not\equiv \pm 2 \pmod{p}$,

$$\epsilon_1 + \epsilon_2 = 0.$$

Hence, for some root of unity $\rho$ and primitive $p$-th root of unity $\xi$,

$$\alpha = (\xi - 1)\rho.$$

Hence

$$m(\alpha) = m(\xi - 1) = 2 - \frac{1}{p-1} \sum_{k=1}^{p-1} (\xi^k + \xi^{-k}) = 2 + \frac{2}{p-1} > 2. \quad \square$$

**Lemma 6.** Let $G$ be a finite group, let $\chi \in \text{Irr}(G)$, and let $g$ be an element of $G$ with order a power of a prime $p$. If $\chi(1) \not\equiv \pm 1 \pmod{p}$, then $\chi(g)$ is not a root of unity.

**Proof of Lemma 6.** Let $p^n$ be the order of $g$, so $\chi(g) \in Q(\zeta_{p^n})$, and suppose that $\chi(g)$ is a root of unity. Since the roots of unity in a given cyclotomic field $Q(\zeta_k)$ are the $l$-th roots of unity for $l$ the least common multiple of 2 and $k$, we then have

$$\chi(g) = \epsilon \xi$$

for some $\epsilon \in \{1, -1\}$ and $p^n$-th root of unity $\xi$. So by Lemma 4, either $\chi(1) \equiv 1 \pmod{p}$ or $\chi(1) \equiv -1 \pmod{p}$. \square

**Lemma 7.** Let $G$ be a finite group of prime-power order, let $g \in G$, and let $\chi \in \text{Irr}(G)$. If $\chi(1) > 1$, then either $\chi(g) = 0$ or $m(\chi(g)) \geq 2$.

**Proof of Lemma 7.** If $|G| = p^n$ with $p$ prime, then each $g \in G$ has order a power of $p$, and each $\chi \in \text{Irr}(G)$ has degree a power of $p$. So if $\chi(1) > 1$, then by Proposition 5 and Lemma 6, for each $g \in G$, $\chi(g) = 0$ or $m(\chi(g)) \geq 2$. \square

For any character $\chi$ of a finite group, let

$$\omega(\chi) = |\{\text{primes dividing } \chi(1)\}|.$$
Theorem 8. Let $G$ be a finite nilpotent group, let $\chi \in \text{Irr}(G)$, and let $g \in G$. Then

$$\chi(g) = 0 \quad \text{or} \quad m(\chi(g)) \geq 2^\omega(\chi).$$

(4)

Proof of Theorem 8. If $|G| = 1$, then $\chi(g) = \chi(1) = 1$, so assume $|G| > 1$. Since $G$ is nilpotent, it is the direct product of its nontrivial Sylow subgroups $P_1, P_2, \ldots, P_n$. Let $g_1, g_2, \ldots, g_n$ be the unique sequence with $g_k \in P_k$ and $g = g_1 g_2 \cdots g_n$.

For each $P_k$, let $\chi_k \in \text{Irr}(P_k)$ be the unique irreducible constituent of the restriction of $\chi$ to $P_k$. Then

$$\chi(g) = \chi_1(g_1)\chi_2(g_2) \cdots \chi_n(g_n), \quad \chi(1) = \chi_1(1)\chi_2(1) \cdots \chi_n(1),$$

(5)

$$\chi_k(1) \text{ divides } |P_k|,$$

(6)

$$|P_j|, |P_k| = 1 \quad \text{for} \quad j \neq k,$$

(7)

and

$$\chi_k(g_k) \in \mathbb{Q}(\zeta_{|P_k|}).$$

(8)

For any algebraic integers $\alpha \in \mathbb{Q}(\zeta_l)$ and $\beta \in \mathbb{Q}(\zeta_m)$ with $(l, m) = 1$, we have $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_l)\mathbb{Q}(\zeta_m)$ and $\mathbb{Q}(\zeta_l) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}$, and hence

$$m(\alpha \beta) = m(\alpha)m(\beta).$$

(9)

By (5), (7), (8), and (9),

$$m(\chi(g)) = m(\chi_1(g_1))m(\chi_2(g_2)) \cdots m(\chi_n(g_n)).$$

(10)

By (10) and Lemma 7,

$$\chi(g) = 0 \quad \text{or} \quad m(\chi(g)) \geq 2^w,$$

where $w$ denotes the number of characters $\chi_k$ with $\chi_k(1) > 1$. From (5), (6), and (7), $w$ is equal to the number of prime divisors of $\chi(1)$. \hfill \Box

Proposition 9. For each finite nilpotent group $G$, and each $\chi \in \text{Irr}(G)$,

$$\frac{|\{g \in G : \chi(g) = 0\}|}{|G|} \geq 1 - \frac{1}{2^\omega(\chi)} \left( \frac{|G| - \chi(1)^2 + 2^\omega(\chi)}{|G|} \right).$$

(11)

Proof of Proposition 9. Let $G$ be a finite nilpotent group, and let $\chi \in \text{Irr}(G)$. By Theorem 8, for each $g \in G$,

$$\chi(g) = 0 \quad \text{or} \quad m(\chi(g)) \geq 2^\omega(\chi).$$

(12)

Now take the relation

$$|G| = \sum_{g \in G} |\chi(g)|^2,$$

apply the elements $\sigma$ of the Galois group $G = \text{Gal}(\mathbb{Q}(\zeta_{|G|})/\mathbb{Q})$, and average over $G$. This gives

$$|G| = \sum_{g \in G} m(\chi(g)).$$

(13)
From (12) and (13),
\[ |G| \geq \chi(1)^2 + 2^{\omega(\chi)}|\{g \in G : \chi(g) \neq 0\}| - 2^{\omega(\chi)}. \] (14)
By (14), we have (11). □

Proof of Theorem 1. By Proposition 9. □

Proof of Theorem 2. Taking the relation
\[ \frac{|G|}{|g^G|} = \sum_{\chi \in \text{Irr}(G)} |\chi(g)|^2, \]
applying the elements $\sigma$ of the Galois group $G = \text{Gal}(\mathbb{Q}(\zeta_{|G|})/\mathbb{Q})$, and averaging over $G$, we have
\[ \frac{|G|}{|g^G|} = \sum_{\chi \in \text{Irr}(G)} m(\chi(g)). \] (15)
So for $L = \{\chi \in \text{Irr}(G) : \chi(1) = 1\}$ and $N = \text{Irr}(G) - L$,
\[ \frac{|G|}{|g^G|} = |L| + \sum_{\chi \in N} m(\chi(g)). \] (16)
By Theorem 8, for each $\chi \in N$,
\[ \chi(g) = 0 \text{ or } m(\chi(g)) \geq 2. \] (17)
From (16) and (17),
\[ \frac{|G|}{|g^G|} \geq |L| + 2|\{\chi \in N : \chi(g) \neq 0\}|. \] (18)
By (18), if $|\text{Cl}(G)| = |G|/|g^G|$, then $|\{\chi \in N : \chi(g) = 0\}| \geq |N|/2$, and if $|\text{Cl}(G)| > |G|/|g^G|$, then $|\{\chi \in N : \chi(g) = 0\}| > |N|/2$. □

Proof of Corollary 3. By Theorem 1 and Theorem 2. □

3. Simple groups

We now establish Conjecture 2 for several families of simple groups.

Theorem 10. Let $n > 0$.
I. For $G = A_n$, we have
\[ \theta(G), \theta'(G) > \begin{cases} 1/2 & \text{if } n < 9, \\ 3/4 & \text{if } n \geq 9. \end{cases} \] (19)
II. For $G = \text{Suz}(q)$ with $q = 2^{2n+1}$, we have
\[ \theta(G) = \frac{1}{2} + \frac{(q+1)(q^2+2)}{2q^2(q^2+1)} \] (20)
and
\[ \theta'(G) = \frac{1}{2} + \frac{5}{2(q+3)}. \] (21)
so \( \theta(G), \theta'(G) > 1/2 \) and
\[
\theta(G), \theta'(G) \to 1/2 \quad \text{as} \quad q \to \infty.
\] (22)

III. For \( G = L_2(q) \) with \( q = p^n \) a prime power, we have \( \theta(G), \theta'(G) > 1/2 \).

IV. For \( G = \text{Ree}(3^{2n+1}) \), we have \( \theta(G), \theta'(G) > 1/2 \).

V. For each sporadic group \( G \), we have \( \theta(G), \theta'(G) > 1/2 \).

VI. For each finite simple group \( G \) of order \( \leq 10^9 \), we have \( \theta(G), \theta'(G) > 1/2 \).

**Corollary 11.** \( \inf \{ \theta(G) : |G| < \infty \}, \inf \{ \theta'(G) : |G| < \infty \} \in [1/3, 1/2] \).

**Proof of Corollary 11.** Thompson and Gallagher give the lower bound of 1/3. The upper bound of 1/2 follows from part II of Theorem 10. \( \square \)

**Verification of I.** (19) holds up to \( n = 14 \), so assume \( n \geq 15 \). In the character table of \( A_n \), the values are rational integers, except some values \( \chi(g) \) with
\[
|\chi(g)|^2 = \frac{1 + \lambda_1 \lambda_2 \ldots}{4}
\]
for some partition \( \lambda \) of \( n \) into distinct odd parts \( \lambda_1 > \lambda_2 \). Since \( n \geq 15 \), it follows that each pair \( (\chi, g) \in \text{Irr}(G) \times G \) satisfies
\[
\chi(g) = 0, \quad |\chi(g)| = 1, \quad \text{or} \quad |\chi(g)|^2 \geq 4.
\] (23)

Using (23) and the fact that simple groups do not have irreducible characters of degree 2, we get that each nonprincipal \( \chi \in \text{Irr}(G) \) satisfies
\[
|G| > |\{ g \in G : |\chi(g)| = 1 \}| + 4|\{ g \in G : |\chi(g)| \neq 0, 1 \}|,
\] (24)
and from (24) it follows that \( \theta(G) > 3/4 \). Similarly, for any class \( g^G \) with \( |g^G| \geq |G|/|\text{Cl}(G)| \), we have
\[
|\text{Cl}(G)| \geq |\{ \chi \in \text{Irr}(G) : |\chi(g)| = 1 \}| + 4|\{ \chi \in \text{Irr}(G) : |\chi(g)| \neq 0, 1 \}|,
\]
and hence \( \theta'(G) > 3/4 \). \( \square \)

**Verification of II.** Let \( n \geq 2, r = 2^n, q = 2^{2n-1} \), and \( G = \text{Suz}(q) \), so
\[
|G| = q^2(q - 1)(q^2 + 1) = q^2(q - r + 1)(q + r + 1).
\]
Maintaining the notation of Suzuki [13], there are elements \( \sigma, \rho, \xi_0, \xi_1, \xi_2 \) such that each element of \( G \) can be conjugated into exactly one of the sets
\[
1^G, \sigma^G, \rho^G, (\rho^{-1})^G, A_0 - \{ 1 \}, A_1 - \{ 1 \}, A_2 - \{ 1 \},
\]
where \( A_i = \langle \xi_i \rangle (i = 1, 2, 3) \), and the irreducible characters of \( G \) are given by the following table [13, Theorem 13]:

| \( 1 \) | \( \sigma \) | \( \rho, \rho^{-1} \) | \( \xi_0^t \neq 1 \) | \( \xi_1^t \neq 1 \) | \( \xi_2^t \neq 1 \) |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| \( X \) | \( q^2 \) | 0 | 0 | 1 | -1 | -1 |
| \( X_i \) | \( q^2 + 1 \) | 1 | 1 | \( \xi_0^t (\xi_0^t) \) | 0 | 0 |
| \( Y_j \) | \( (q - r + 1)(q - 1) \) | \( r - 1 \) | -1 | 0 | -\( \xi_1^t (\xi_1^t) \) | 0 |
| \( Z_k \) | \( (q + r + 1)(q - 1) \) | \( -r - 1 \) | -1 | 0 | 0 | -\( \xi_2^t (\xi_2^t) \) |
| \( W_l \) | \( \frac{r(q - 1)}{2} \) | \( -\frac{r}{2} \) | \( \pm \frac{\sqrt{4r - 1}}{2} \) | 0 | 1 | -1 |
Here, \(1 \leq i \leq \frac{q}{2} - 1, 1 \leq j \leq \frac{q+r}{4}, 1 \leq k \leq \frac{q-r}{4}, 1 \leq l \leq 2,\)

\[
e^i e^j e^k e^l = \zeta^i + \zeta^{-i}, \quad \zeta = e^{2\pi\sqrt{-1}/(q-1)},
\]

(25)

and \(e_1^j\) and \(e_2^k\) are certain characters on \(A_1\) and \(A_2\). The \(A_i\)'s satisfy

\[
|A_0| = q - 1, \quad |A_1| = q + r + 1, \quad |A_2| = q - r + 1,
\]

(26)

and denoting by \(G_i\) \((i = 0, 1, 2)\) the set of elements \(g \in G\) that can be conjugated into \(A_i - \{1\}\), we have

\[
|G_i| = \frac{|A_i| - 1}{|G|/|A_i|},
\]

(27)

where \(l_0 = 2\) and \(l_1 = l_2 = 4\).

Let \(\gamma_s = \zeta^s + \zeta^{-s}\) with \(\zeta = e^{2\pi\sqrt{-1}/(q-1)}\) and \(s \in \mathbb{Z}\). Then

\[
|\gamma_s| = 1 \Leftrightarrow 6s \pm (q - 1) \equiv 0 \pmod{3(q - 1)}
\]

and

\[
|\gamma_s| = 0 \Leftrightarrow 4s \pm (q - 1) \equiv 0 \pmod{2(q - 1)}.
\]

Since \(q - 1 \equiv 1 \pmod{3}\), and \(q - 1 \equiv 1 \pmod{2}\), it follows that

\[
|\gamma_s| \not\in \{0, 1\} \text{ for all } s \in \mathbb{Z}.
\]

(28)

So for any \(X_i\),

\[
|\{g \in G : X_i(g) \text{ is a root of unity or zero}\}| = |G| - |G_0| - 1,
\]

(29)

and for any \(g \in G_0\),

\[
|\{\chi \in \text{Irr}(G) : \chi(g) \text{ is a root of unity or zero}\}| = \frac{q}{2} + 4.
\]

(30)

By (29) and (26)–(27),

\[
\theta(G) \leq \frac{1}{2} + \frac{(q + 1)(q^2 + 2)}{2q^2(q^2 + 1)}.
\]

(31)

Equality must hold in (31) because

\[
|\{g \in G : W_i(g) \in \{0, 1, -1\}\}| = |G_0| + |G_1| + |G_2| > |G| - |G_0| - 1
\]

and for any \(\chi \in \text{Irr}(G) - \{X_i\} - \{W_i\},\)

\[
|\{g \in G : \chi(g) \in \{0, 1, -1\}\}| \geq 2|\rho^G| + |G_0| + |G_2| > |G| - |G_0| - 1.
\]

By (30) and the fact that, for any \(g \in G_0\), \(|C_G(g)| = q - 1 < q + 3 = |C(G)|\), we have

\[
\theta'(G) \leq \frac{1}{2} + \frac{5}{2(q + 3)}.
\]

(32)

Equality must hold in (32) because \(1^G, \sigma^G, \text{ and } \rho^G\) have size \(|G|/|C(G)|\), and for any \(g \in G_1 \cup G_2,\)

\[
|\{\chi \in \text{Irr}(G) : \chi(g) \in \{0, 1, -1\}\}| \geq \frac{3q - r + 12}{4} \geq \frac{q}{2} + 4. \quad \Box
\]
Verification of III. Let \( q = p^n \) with \( p \) prime, \( G = L_2(q) \), let \( R \) and \( S \) be as in [7, pp. 402–403], and let \( G_0 \) (resp. \( G_1 \)) be the set of nonidentity elements \( g \in G \) that can be conjugated into \( \langle R \rangle \) (resp. \( \langle S \rangle \)).

Assuming first \( p \neq 2 \), then

\[
|G| = \frac{q(q^2 - 1)}{2}, \quad |\text{Cl}(G)| = \frac{q + 5}{2}, \tag{33}
\]

\[
|G_0| = \frac{q(q + 1)(q - 3)}{4}, \quad |G_1| = |G_0| + q = \frac{q(q - 1)^2}{4}, \tag{34}
\]

and \( G - G_0 \cup G_1 \) consists of 3 classes: \( 1^G, a^G, b^G \), with \( |C_G(a)| = |C_G(b)| = q \).

Inspecting Jordan’s table [7, p. 402], each \( \chi \in \text{Irr}(G) \) satisfies either

(i) \( \chi(g) \in \{0, 1, -1\} \) on \( G_0 \cup G_1 \), or
(ii) \( \chi(g) \in \{1, -1\} \) on \( a^G \cup b^G \) and \( \chi(g) = 0 \) on \( G_0 \) or \( G_1 \).

If \( q > 3 \), then

\[
|G_0| + |G_1| > |G|/2 \quad \text{and} \quad |a^G| + |b^G| + |G_0| > |G|/2, \tag{35}
\]

and if \( q = 3 \), then \( G \cong A_4 \). So \( \theta(G) > 1/2 \). Similarly,

\[
|\{\chi \in \text{Irr}(G) : \chi(a), \chi(b) \in \{0, 1, -1\}\}| = \frac{q + 1}{2},
\]

and for \( g \in G_0 \) (resp. \( g \in G_1 \)) and \( \chi \in \text{Irr}(G) \), we have \( \chi(g) \in \{0, 1, -1\} \) away from the \( \leq (q - 3)/4 \) irreducible characters of degree \( q + 1 \) (resp. the \( \leq (q - 1)/4 \) characters of degree \( q - 1 \)), from which it follows that \( \theta(G') > 1/2 \).

For \( p = 2 \), we have \( |G| = q(q^2 - 1), |\text{Cl}(G)| = q + 1, \)

\[
|G_0| = \frac{q(q + 1)(q - 2)}{2}, \quad |G_1| = \frac{q^2(q - 1)}{2}, \tag{36}
\]

and \( G - G_0 \cup G_1 \) consists of 2 classes: \( 1^G \) and \( a^G \) with \( |C_G(a)| = q \). The irreducible characters of \( G \) are given by Jordan [7, p. 403]. There is the principal character, 1 character of degree \( q \), \( q/2 \) characters of degree \( q - 1 \), and \( q/2 - 1 \) characters of degree \( q + 1 \). All the characters satisfy \( \chi(g) \in \{0, 1, -1\} \) on \( a^G \), the character of degree \( q \) is \( \pm 1 \) on \( G_0 \) and \( G_1 \), the characters of degree \( q - 1 \) vanish on \( G_0 \), and the characters of degree \( q + 1 \) vanish on \( G_1 \). From this, it follows that \( \theta(G) \) and \( \theta'(G) \) are \( > 1/2 \). \( \square \)

Verification of IV. Let \( n \) be a positive integer, \( m = 3^n \), \( q = 3^{2n+1} \), and \( G = \text{Ree}(q) \), so

\[
|G| = q^3(q - 1)(q + 1)(q^2 - q + 1), \quad |\text{Cl}(G)| = q + 8.
\]

The irreducible characters of \( G \) are given by Ward [14] in a 16-by-16 table, with the last 6 rows being occupied by 6 families of exceptional characters, the sizes of which are, from top to bottom,

\[
\frac{q - 3}{4}, \quad \frac{q - 3}{4}, \quad \frac{q - 3}{24}, \quad \frac{q - 3}{8}, \quad \frac{q - 3m}{6}, \quad \frac{q + 3m}{6}.
\]
From Ward’s table, we find that for any class \( g^G \notin \{1^G, X^G, J^G\} \), \( \chi(g) \in \{0, 1, -1\} \) for more than half of the irreducible characters \( \chi \) of \( G \). Since the classes \( 1^G, X^G, J^G \) all have size \(< |G|/|\text{Cl}(G)|\), we conclude that \( \theta'(G) > 1/2 \).

The first step in verifying \( \theta(G) > 1/2 \) is to compute the following table:

### Table 1

| \( \Omega \subset G \) | \( |\cup_{g \in G} g \Omega g^{-1}| \) |
|----------------------|-----------------------------|
| \{1\}                | 1                           |
| \( \langle R \rangle - \{1\} \) | \( q^3(q-3)(q^3+1) \)       |
| \( \langle S \rangle - \{1\} \) | \( q^3(q-1)(q-3)(q^2-q+1) \) |
| \( M^- - \{1\} \)    | \( q^3(q-1)(q-3)(q^2-q-3m) \) |
| \( M^+ - \{1\} \)    | \( q^3(q-1)(q-3)(q^2+q+3m) \) |
| \{X\}                | \( |G|/q^3 \)                |
| \{Y\}                | \( |G|/3q \)                |
| \{T\}                | \( |G|/2q^2 \)               |
| \{T\}                | \( |G|/2q^2 \)               |
| \{YT\}               | \( |G|/3q \)                |
| \{Y\}                | \( |G|/3q \)                |
| \{JT\}               | \( |G|/2q \)                |
| \{JT\}               | \( |G|/2q \)                |
| \( J\langle R \rangle - \{J\} \) | \( q^3(q-3)(q^3+1) \)       |
| \( J\langle S \rangle - \{J\} \) | \( q^3(q-1)(q-3)(q^2-q+1) \) |
| \{J\}                | \( |G|/q(q^2 - 1) \)        |

Then with Table 1 and Ward’s table in hand, a straightforward check establishes that, for each \( \chi \in \text{Irr}(G) \),

\[
|\{g \in G : \chi(g) \in \{0, 1, -1\}\}| > |G|/2.
\]

Hence \( \theta(G) > 1/2 \). \( \square \)

**Verification of V and VI.** Here, in Tables 2 and 3, we report the values of \( \theta \) and \( \theta' \) for sporadic groups and simple groups of order \( \leq 10^9 \). All values are rounded to the number of digits shown.
Table 2. The sporadic groups.

| G     | $\theta(G)$ | $\theta'(G)$ |
|-------|-------------|--------------|
| $M_{11}$ | 0.7290      | 0.8000       |
| $M_{12}$ | 0.7955      | 0.8667       |
| $M_{22}$ | 0.7117      | 0.8333       |
| $M_{23}$ | 0.6827      | 0.7647       |
| $M_{24}$ | 0.6913      | 0.7692       |
| $J_1$   | 0.5583      | 0.6000       |
| $J_2$   | 0.6373      | 0.6190       |
| $J_3$   | 0.5840      | 0.7143       |
| $J_4$   | 0.6925      | 0.7903       |
| $Co_1$  | 0.8739      | 0.8515       |
| $Co_2$  | 0.8347      | 0.8333       |
| $Co_3$  | 0.7528      | 0.8333       |
| $F_i^{22}$ | 0.8029      | 0.8769       |
| $F_i^{23}$ | 0.8328      | 0.8469       |
| $F_i^{24}$ | 0.8808      | 0.8056       |
| $HS$    | 0.7853      | 0.8750       |
| $McL$   | 0.6722      | 0.8333       |
| $He$    | 0.7088      | 0.7576       |
| $Ru$    | 0.8517      | 0.8333       |
| $Suz$   | 0.8141      | 0.8372       |
| $ON$    | 0.6830      | 0.8667       |
| $HN$    | 0.6362      | 0.7593       |
| $Ly$    | 0.7879      | 0.8491       |
| $Th$    | 0.7978      | 0.8750       |
| $B$     | 0.8812      | 0.8587       |
| $M$     | 0.8855      | 0.8711       |

Table 3. The simple groups of order $\leq 10^9$ that are not cyclic, $A_n$, $L_2(q)$, $Suz(2^{2n+1})$, $Ree(3^{2n+1})$, or sporadic.

| G     | $\theta(G)$ | $\theta'(G)$ |
|-------|-------------|--------------|
| $L_3(3)$  | 0.6736      | 0.8333       |
| $U_3(3)$  | 0.7049      | 0.8571       |
| $L_3(4)$  | 0.6000      | 0.8000       |
| $S_3(3)$  | 0.8713      | 0.9000       |
| $U_3(4)$  | 0.6892      | 0.7273       |
| $L_3(5)$  | 0.7103      | 0.8571       |
| $L_3(5)$  | 0.6754      | 0.8667       |
| $S_4(4)$  | 0.6433      | 0.7037       |
| $S_6(2)$  | 0.8867      | 0.8333       |
| $L_3(7)$  | 0.6235      | 0.7273       |
| $U_3(3)$  | 0.7121      | 0.9000       |
| $S_4(3)$  | 0.8321      | 0.9130       |
| $S_4(5)$  | 0.6501      | 0.6471       |
| $U_3(8)$  | 0.5701      | 0.6786       |
| $U_3(7)$  | 0.6741      | 0.7586       |
| $L_4(3)$  | 0.6911      | 0.8621       |
| $L_3(5)$  | 0.8736      | 0.8333       |
| $U_3(2)$  | 0.8041      | 0.9149       |
| $L_3(8)$  | 0.6650      | 0.7083       |
| $2F_4(2)'$ | 0.7006      | 0.8182       |
| $L_3(9)$  | 0.5488      | 0.6000       |
| $U_3(9)$  | 0.6237      | 0.6739       |
| $U_3(11)$ | 0.5494      | 0.6250       |
| $S_4(7)$  | 0.7341      | 0.7308       |
| $O_n^+(2)$ | 0.8555      | 0.9245       |
| $O_n^-(2)$ | 0.7578      | 0.8462       |
| $3D_4(2)$ | 0.6920      | 0.6571       |
| $L_3(11)$ | 0.6660      | 0.6970       |
| $G_2(4)$  | 0.6449      | 0.7500       |
| $L_3(13)$ | 0.5354      | 0.5938       |
| $U_3(13)$ | 0.6662      | 0.6957       |
| $L_4(4)$  | 0.6020      | 0.5714       |

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