A Shrinking Factor for Unitarily Invariant Norms under a Completely Positive Map

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A relation between values of a unitarily invariant norm of Hermitian operator before and after action of completely positive map is studied. If the norm is jointly defined on both the input and output Hilbert spaces, one defines a shrinking factor under the restriction of given map to Hermitian operators. As it is shown, for any unitarily invariant norm this shrinking factor is not larger than the maximum of two values for the spectral norm and the trace norm.

Keywords: Ky Fan’s maximum principle, symmetric gauge function, Choi-Kraus representation, Ky Fan’s norm

I. INTRODUCTION

In many disciplines, linear maps on a space of operators provide key tools for treatment of the subject. For several reasons the class of completely positive maps (CP-maps) is especially valuable [1]. The recent advances in quantum information theory have led to a renewed interest in this area [10]. In effect, it seems that all possible changes of quantum state is covered by contractive completely positive maps [8], though the monotonicity of relative entropy can be proved in more general framework [10]. Anyway, all the important examples are actually completely positive. Thus, studies of used quantitative measures under action of CP-maps form a very actual issue. Of course, other properties of CP-maps with respect to certain norms are subjects of active research [4, 7]. As a rule, distance measures are metrics induced by norms with handy properties. Unitarily invariant norms are very useful in this regard [6]. At the same time, some variety of measures is typically needed with respect to themes of interest. So, a question on contractivity of given map with respect to applied norm is significant in many different fields of physics (see [12] and references therein). Hence we may be interested in general results on a problem of contractivity without explicit specification of the measure. Below the result of such a kind will be given for the class of unitarily invariant norms. Namely, the norm of image of Hermitian operator is not greater than the norm of operator itself multiplied by some shrinking factor. For given CP-map and any norm from the considered class, this factor does not exceed the maximum of two exact values of shrinking factor for the spectral norm and the trace norm. The discussion is carried out entirely in finite dimensional setting.

II. DEFINITION AND NOTATION

Let \( H \) be \( d \)-dimensional Hilbert space. We denote by \( L(H) \) the space of all linear operators on \( H \), and by \( L_{s.a.}(H) \) the space of self-adjoint (Hermitian) operators on \( H \). For any \( X \in L(H) \) the operator \( X^\dagger X \) is positive semidefinite, and its unique positive square root is denoted by \( |X| \). The eigenvalues of \(|X|\) counted with multiplicities are the singular values of operator \( X \), in signs \( \sigma_i(X) \) [6]. Each unitarily invariant norm is generated by some symmetric gauge function of the singular values, i.e. \(||X|||_g = g(\sigma_1(X), \ldots, \sigma_d(X))\) (see, e.g., theorem 7.4.24 in [6]). The determining properties for a symmetric gauge function are listed in [6]. The two families, the Schatten norms and the Ky Fan norms, are most widely used. For any real \( p \geq 1 \), the Schatten \( p \)-norm is defined as [6]

\[
||X||_p := \left( \sum_{i=1}^{d} \sigma_i(X)^p \right)^{1/p}.
\]

This family recovers the trace norm \(||X||_1\) for \( p = 1 \), the Frobenius norm \(||X||_2\) for \( p = 2 \), and the spectral norm \(||X||_{\infty}\) for \( p \to \infty \) [6]. Let us use these signs, although \(||X||_{\infty} = ||X||_{(1)}\) and \(||X||_{1r} = ||X||_{(d)}\) as the Ky Fan norms though. For integer \( k \geq 1 \), the Ky Fan \( k \)-norm is defined by [6]

\[
||X||_{(k)} := \sum_{i=1}^{d} \sigma_i^k(X) \equiv g(k)(\sigma_1(X), \ldots, \sigma_d(X)),
\]

where the arrows down show that the singular values are put in the decreasing order. In terms of the norms (1), the partitioned trace distances have been introduced [15]. These measures enjoy similar properties to the trace norm distance. In the following, we will assume that \(||X||_{(k)} = ||X||_{1r}\) for \( k \geq d \). We shall now define the main object treated in this paper.
**Definition 2.1.** Let \( \Phi_{s.a.} \) be the restriction of CP-map \( \Phi : L(\mathcal{H}_A) \to L(\mathcal{H}_B) \) to Hermitian operators. Its shrinking factor with respect to given unitarily invariant norm \( \|X\|_g \) is defined as

\[
\eta_g(\Phi_{s.a.}) := \sup \left\{ \|\Phi(X)\|_g : X \in L_{s.a.}(\mathcal{H}_A), \|X\|_g = 1 \right\}.
\]

If \( \mathcal{H}_A = \mathcal{H}_B \) then on both the spaces a norm \( \|X\|_g \) is defined by the same symmetric gauge function. When \( \dim(\mathcal{H}_A) \neq \dim(\mathcal{H}_B) \), we append zero singular values so that the vectors \( \sigma(X) \) and \( \sigma(\Phi(X)) \) have the same dimensionality equal to \( \max\{d_A, d_B\} \). In this regard, our consideration is related to those symmetric gauge functions that are not changed by adding zeros. Only under this condition the same unitarily invariant norm is correctly defined on the spaces of different dimensionality. The needed property is provided by all the functions assigned to the Ky Fan gauge function that enjoys this property. Indeed, the symmetric gauge functions, providing the above property, form a convex set.

In Definition 2.1 the supremum is taken over Hermitian inputs \( X \). First, self-adjoint operators are very important in many applications including quantum information topics. Say, the difference between two density matrices is traceless Hermitian, and the restriction to such operators deserves attention. Second, a consideration of Hermitian \( X \) allows to simplify analysis. Third, some relations with positive or self-adjoint operators have later been extended to more general ones. So, our definition is suitable for such a generalization.

In the seminal paper \([5]\), Ky Fan obtained important results with respect to extremal properties of eigenvalues. One of his formulations is now known as Ky Fan’s maximum principle. The present author have applied this power to simplify analysis. Third, some relations with positive or self-adjoint operators have later been extended to more general ones \([2, 18]\). So, our definition is suitable for such a generalization.

In the seminal paper \([5]\), Ky Fan obtained important results with respect to extremal properties of eigenvalues. One of his formulations is now known as Ky Fan’s maximum principle. The present author have applied this power for stating the basic properties of the partial fidelities \([14]\), which were originally introduced by Uhlmann. The present author have applied this power for stating the basic properties of the partial fidelities \([14]\), which were originally introduced by Uhlmann\([17]\), and the partitioned trace distances \([15]\). Changing the proof of theorem 1 in \([2]\), we can merely prove

\[
\sum_{i=1}^{k} \lambda_i^k(X) = \max \{ \text{Tr}(PX) : 0 \leq P \leq I, \text{Tr}(P) = k \},
\]

where the maximum is taken over those positive operators \( P \) with trace \( k \) that satisfy \( P \leq I \). Alternately, the maximization may be over all projectors of rank \( k \), as in the original statement \([2]\). If operator \( X \) is positive semidefinite then the maximum can be taken under the condition \( \text{Tr}(P) \leq k \) or, for projectors, \( \text{rank}(P) \leq k \). Using the Jordan decomposition, we have the following result.

**Lemma 2.2.** For any \( X \in L_{s.a.}(\mathcal{H}) \) and \( k \geq 1 \), there exist two mutually orthogonal projectors \( P_Q \) and \( P_R \) such that \( \text{rank}(P_Q + P_R) \leq k \) and

\[
\|X\|_{(k)} = \text{Tr}[(P_Q - P_R)X].
\]

**Proof.** First, we suppose that \( k \leq d \). We write \( X = Q - R \) with positive semidefinite \( Q \) and \( R \) whose supports are orthogonal. These operators are positive and negative parts of \( X \) respectively. Putting the spectral decomposition

\[
|X| = Q + R = \sum_q q u_q u_q^\dagger + \sum_r r v_r v_r^\dagger,
\]

we see that \( \{q\} \cup \{r\} = \{\sigma_i(X)\} \). For given \( k \), we define two subspaces, namely

\[
\mathcal{K}_Q := \text{span}\{u_q : q \in \{\sigma_1^k, \sigma_2^k, \ldots, \sigma_k^k\}\}, \quad \mathcal{K}_R := \text{span}\{v_r : r \in \{\sigma_1^k, \sigma_2^k, \ldots, \sigma_k^k\}\}.
\]

If \( P_Q \) is projector onto \( \mathcal{K}_Q \) and \( P_R \) is projector onto \( \mathcal{K}_R \), then we at once get \( (P_Q - P_R)X = P|X| \) for projector \( P = P_Q + P_R \) of rank \( k \). By construction, the trace of \( P|X| \) sums just \( k \) largest singular values of \( X \). The case \( k > d \) is reduced to the trace norm for that the needed projectors are already built and \( \text{rank}(P_Q + P_R) = d < k \). \( \square \)

**III. MAIN RESULTS**

In this section, we will study a change of unitarily invariant norms under action of a CP-map. Since they are positive-valued, upper bounds are usually indispensable. Let \( \Phi : L(\mathcal{H}_A) \to L(\mathcal{H}_B) \) be a completely positive linear map. We shall use the Choi-Kraus representation \([2, 8]\).

\[
\Phi(X) = \sum_n E_n X E_n^\dagger, \quad E_n : \mathcal{H}_A \to \mathcal{H}_B.
\]

From the physical viewpoint, this result is examined in \([10]\). In the context of Stinespring’s dilation theorem, it is discussed in \([1]\). The Choi-Kraus representation is not unique, but a freedom is unitary in character (see theorem 8.2 in \([10]\)). Two sets \( \{E_n\} \) and \( \{G_m\} \) determine the same CP-map if and only if

\[
G_m = \sum_n v_{mn} E_n.
\]
where numbers $v_{mn}$ are entries of some unitary matrix of proper dimensionality. Then for given CP-map the two positive semidefinite operators

$$
M := \sum_n E_n^* E_n, \quad W := \sum_n E_n^T E_n
$$

are not dependent on a choice of the set $\{E_n\}$. The second operator has been used for another definition of the trace norm distance via extremal properties of contractive CP-maps [15].

**Theorem 3.1.** Let $\Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ be a CP-map. For every $X \in \mathcal{L}_{s.a.}(\mathcal{H}_A)$ there holds

$$
|||\Phi(X)|||_k \leq \eta|||X|||_k, \quad k = 1, 2, \ldots, \max\{d_B, d_A\},
$$

where the factor $\eta := \max\{|||M|||_\infty, |||W|||_\infty\}$.

**Proof.** First, we assume that $d_B \leq d_A$. Let $X = Q - R$ be the Jordan decomposition of $X$, then $\Phi(X) = \Phi(Q) - \Phi(R)$. It follows from $\Phi(X)^\dagger = \Phi(X)$, Lemma 2.2 and properties of the trace that

$$
|||\Phi(X)|||_k = \text{Tr}_B \left[ (\Pi_Q - \Pi_R)(\Phi(Q) - \Phi(R)) \right] \leq \text{Tr}_B \left[ (\Pi_Q + \Pi_R)(\Phi(Q) + \Phi(R)) \right] = \eta \text{Tr}_A [(S + T)|X|]
$$

for two mutually orthogonal projectors with $\text{rank}(\Pi_Q + \Pi_R) \leq k$. In [15] we use $Q + R = |X|$ and positive semidefinite operators

$$
S = \eta^{-1} \sum_n E_n^T Q_n E_n, \quad T = \eta^{-1} \sum_n E_n^T R_n E_n.
$$

Denoting $\mu \equiv |||M|||_\infty$ and $\nu \equiv |||W|||_\infty$, we obviously write $\mu^{-1}M \leq I_B$ and $\nu^{-1}W \leq I_A$. Combining the former with properties of the trace, we have

$$
\text{Tr}_A (S + T) = \eta^{-1} \text{Tr}_B \left[ (\Pi_Q + \Pi_R)M \right] \leq \text{Tr}_B \left[ (\Pi_Q + \Pi_R) \mu^{-1}M \right] \leq k.
$$

Using $\Pi_Q + \Pi_R \leq I_B$ and $\nu^{-1}W \leq I_A$, we also obtain

$$
\langle u, (S + T)u \rangle = \eta^{-1} \sum_n \langle u, E_n^T (\Pi_Q + \Pi_R) E_n u \rangle \leq \langle u, \nu^{-1}W u \rangle \leq \langle u, \nu^{-1}W u \rangle \leq \langle u, u \rangle
$$

for each $u \in \mathcal{H}_A$. This implies $S + T \leq I_A$ and the truth of using Ky Fan’s principle for the right-hand side of [13]. So, the relations [13] and [11] provide the claim. When $d_B > d_A$, the calculations [15] remain valid for $k > d_A$, hence the right-hand side of [13] is not greater than $\eta|||X|||_\infty$.

As it is known, the role of particular symmetric gauge functions $g(k)(\cdot)$ is that norm inequalities can sometimes be extended to all unitarily invariant norms. Let $u, v \in \mathbb{C}^d$ be given vectors with $d = \max\{d_A, d_B\}$. In accordance with theorem 7.4.45 in [15], the inequality $g(u) \leq g(v)$ holds for all symmetric gauge functions $g(\cdot)$ on $\mathbb{C}^d$ if and only if $g(k)(u) \leq g(k)(v)$ for $k = 1, 2, \ldots, d$. By Theorem 3.1, for any symmetric gauge function we then obtain

$$
g(\sigma_i(\Phi(X))) \leq \eta g(\sigma_i(X)),$$

or merely $|||\Phi(X)|||_g \leq \eta|||X|||_g$, whenever $X \in \mathcal{L}_{s.a.}(\mathcal{H}_A)$. In terms of shrinking factors, the norm inequality can be reformulated as follows.

**Theorem 3.2.** For each unitarily invariant norm $|||\cdot|||_g$, defined on both the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, a corresponding shrinking factor satisfies

$$
\eta_g(\Phi_{s.a.}) \leq \max\{|||M|||_\infty, |||W|||_\infty\}.
$$

In the next section we will show that $|||M|||_\infty$ is the exact value of shrinking factor for the spectral norm and $|||W|||_\infty$ is the one for the trace norm. So, a degree of non-contraction of $\Phi_{s.a.}$ is quite revealed by these two values.

**IV. THE SPECTRAL NORM AND TRACE NORM**

Let $X$ be Hermitian operator such that $|||X|||_1 = 1$. Using the Jordan decomposition $X = Q - R$, we get

$$
|||\Phi(X)|||_\infty = \text{Tr}_B \left[ \Pi(\Phi(Q) - \Phi(R)) \right] \leq \text{Tr}_B \left[ \Pi(\Phi(Q) + \Phi(R)) \right]
$$

(6)
for corresponding projector $\Pi$ of rank one. Due to $|X| \leq I_A$ and Ky Fan’s maximum principle \cite{2}, the right-hand side of \eqref{10} can be treated as

\[ \text{Tr}_A \left( \sum_n E_n^\dagger \Pi E_n |X| \right) \leq \text{Tr}_A \left( \sum_n E_n^\dagger \Pi E_n \right) = \text{Tr}_B (M \Pi) \leq ||M||_\infty . \]

So, we have $||\Phi(X)||_\infty \leq ||M||_\infty$ for any $X \in \mathcal{L}_{s.a.}(\mathcal{H}_A)$ with $||X||_\infty = 1$. Noting $\Phi(I_A) = M$, the inequality between norms is saturated. Hence we obtain the exact value of shrinking factor

\[ \eta_\infty(\Phi_{s.a.}) = ||M||_\infty . \]  

Note that this is a particular case of the Russo-Dye theorem (see, e.g., corollary 2.9 in \cite{11}). The above calculation is given here due to its simplicity and illustration of the method.

In line with \eqref{9}, for the trace norm there holds

\[ ||\Phi(X)||_{tr} \leq \text{Tr}_B \left[ (\Pi_Q + \Pi_R) (\Phi(Q) + \Phi(R)) \right] = \text{Tr}_A (W|X|) . \]  

since $\Pi_Q + \Pi_R = I_B$ by rank($\Pi_Q + \Pi_R$) = $d_B$. If $||X||_{tr} = 1$ then the right-hand side of \eqref{8} does not exceed $||W||_\infty$. This value can actually be reached. Let $Y$ be projector onto 1-dimensional eigenspace corresponding to the largest eigenvalue of operator $W$. Then $\Phi(Y)$ is positive semidefinite and

\[ ||\Phi(Y)||_{tr} = \text{Tr}_B (\Phi(Y)) = \text{Tr}_A (WY) = ||W||_\infty . \]

In other words, the exact value of shrinking factor is given by

\[ \eta_{tr}(\Phi_{s.a.}) = ||W||_\infty . \]  

Thus, for the spectral and trace norms the exact value of shrinking factor is simply calculated. For other norms a task is more difficult but the bound of Theorem 3.2 is useful for many aims. So, this bound can be rewritten as

\[ \eta_{tr}(\Phi_{s.a.}) \leq \max \{ \eta_\infty(\Phi_{s.a.}), \eta_{tr}(\Phi_{s.a.}) \} . \]  

To sum up, we have a valuable conclusion. If the restriction $\Phi_{s.a.}$ is contractive with respect to both the spectral and trace norm then it is contractive with respect to all unitarily invariant norms. Moreover, a degree of non-contractivity can be measured by using these two norms.

Finally, we apply our results to the operation of partial trace. This operation is especially important in the context of quantum information processing. Hence we are interested in relations between norms before and after partial trace.

The writers of \cite{9} resolved a question for those unitarily invariant norms that are multiplicative over tensor products. The explicit Choi-Kraus representation of partial trace is given in \cite{10}. However, the operators $M$ and $W$ can be found directly. Let us take $\mathcal{H}_A = \mathcal{H}_B \otimes \mathcal{H}_C$ with partial tracing over $\mathcal{H}_C$, that is

\[ \Psi(X) := \text{Tr}_C(X) \]  

for any $X \in \mathcal{L}(\mathcal{H}_B \otimes \mathcal{H}_C)$. First, this operation preserves trace, because

\[ \text{Tr}_B (\Psi(X)) = \text{Tr}_B \{\text{Tr}_C(X)\} = \text{Tr}_A(X) . \]

Combining this with $\text{Tr}_B (\Psi(X)) = \text{Tr}_A (W|X|)$ finally gives $W = I_A$. Second, the right-hand side of definition for $M$ is rewritten as

\[ \sum_n E_n I_A E_n^\dagger = \Phi(I_B \otimes I_C) = I_B \text{Tr}_C(I_C) . \]

So we obtain $M = d_C I_B$, where $d_C = \text{dim}(\mathcal{H}_C)$. Because $||M||_\infty = d_C$ and $||W||_\infty = 1$, the statement of Theorem 3.2 gives

\[ ||\Psi(X)||_g \leq d_C ||X||_g \]  

for $X \in \mathcal{L}_{s.a.}(\mathcal{H}_A)$ and any unitarily invariant norm. For the spectral norm this relation coincides with the one given in \cite{3}. For the Frobenius norm the method of \cite{9} provides a more precise bound. On the other hand, the validity of \cite{12} is not restricted to those norms that are multiplicative over tensor product.
[1] Bhatia, R.: *Positive Definite Matrices*. Princeton: Princeton University Press, 2007
[2] Bhatia, R.: Some inequalities for norm ideals. Commun. Math. Phys. **111**, 33–39 (1987)
[3] Choi, M.-D.: Completely positive linear maps on complex matrices. Linear Algebra Appl. **10**, 285-290 (1975)
[4] Devetak, I., Junge, M., King, C. and Ruskai, M.B.: Multiplicativity of completely bounded $p$-norms implies a new additivity result. Commun. Math. Phys. **266**, 37–63 (2006)
[5] Fan, K.: On a theorem of Weyl concerning eigenvalues of linear transformations. I. Proc. Nat. Acad. Sci. USA **35**, 652–655 (1949)
[6] Horn, R.A. and Johnson, C.R.: *Matrix Analysis*. Cambridge: Cambridge University Press, 1985
[7] Jenčová, A.: A relation between completely bounded norms and conjugate channels. Commun. Math. Phys. **266**, 65-70 (2006)
[8] Kraus, K.: General state changes in quantum theory. Ann. Phys. **64**, 311-335 (1971)
[9] Lidar, D.A., Zanardi, P. and Khodjasteh, K.: Distance bounds on quantum dynamics. Phys. Rev. A **78**, 012308 (2008)
[10] Nielsen, M.A. and Chuang, I.L.: *Quantum Computation and Quantum Information*. Cambridge: Cambridge University Press, 2000
[11] Paulsen, V.: *Completely Bounded Maps and Operator Algebras*. Cambridge: Cambridge University Press, 2002
[12] Peres-Garsia, D., Wolf, M.M., Petz, D. and Ruskai, M.B.: Contractivity of positive and trace preserving maps under $L_p$ norms. J. Math. Phys. **47**, 083506 (2006)
[13] Rastegin, A.E.: Trace distance from the viewpoint of quantum operation techniques. J. Phys. A: Math. Theor. **40**, 9533–9549 (2007)
[14] Rastegin, A.E.: Some properties of partial fidelities. Quantum Inf. Comput. **9**, 1069–1080 (2009)
[15] Rastegin, A.E.: Partitioned trace distances. Quantum Inf. Process. DOI 10.1007/s11128-009-0128-7
[16] Uhlmann, A.: Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory. Commun. Math. Phys. **54**, 21–32 (1977)
[17] Uhlmann, A.: On ”partial” fidelities. Rep. Math. Phys. **45**, 407–418 (2000)
[18] Watrous, J.: Notes on super-operator norms induced by Schatten norms. Quantum Inf. Comput. **5**, 58-68 (2005)