The Fifth Coefficients of Strongly Convex Functions

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Abstract. Let the function $f$ be analytic in $D = \{z : |z| < 1\}$ and given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For $0 < \alpha \leq 1$, let $C(\alpha)$ be the class of strongly convex functions defined by $\left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi \alpha}{2}$, $z \in D$. We give sharp bounds for the fifth coefficient $|a_5|$.

1. Introduction
Let $S$ be the class of analytic functions $f$ defined on the unit disc $D = \{z : |z| < 1\}$. $f \in S$ can be written in Taylor series expansion as follows.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \quad (1.1)$$

The function (1.1) is said to be normalised univalent function if $z_0 = 0, a_0 = 0, a_1 = 1$, and $a_n \in \mathbb{C}$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.2)$$

Definition 1.1. Suppose that $f$ is analytic in $D$ and given by (1.2). For $0 < \alpha \leq 1$, $f$ is said to be strongly convex of order $\alpha$ ($C(\alpha)$) if and only if

$$\left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi \alpha}{2}, \quad z \in D. \quad (1.3)$$

In [1], sharp bounds of the coefficients $|a_2|, |a_3|$, and $|a_4|$ of strongly convex function were given. Not only those coefficients, but also sharp bounds for the initial coefficients of the inverse function, invariance results for the second Hankel determinant $H_2 = |a_2 a_4 - a_3^2|$, the first three coefficient of log $\left(f(z)/z\right)$, and Fekete-Szegö theorems.

We note that strongly convex function is a subset of convex function. In [2], we note that convex function is the smallest subset of $S$, which means the convex function is subset of starlike function. Meanwhile, the fifth coefficients of starlike functions were given in [3] as follows.

$$|a_5| \leq \left\{ \begin{array}{ll}
\frac{\alpha}{2}, & 228\alpha^4 - 194\alpha^3 + 2\alpha^2 + 39\alpha - 9 \leq 0, \\
\frac{\alpha^2}{9} (38\alpha^2 + 7), & 38\alpha^3 - 30\alpha^2 + 16\alpha \geq \frac{\alpha}{2}.
\end{array} \right. \quad (1.4)$$
Thus, the fifth coefficients of strongly convex functions must be less then or equal to (1.4). It is the purpose of this paper to give some sharp bounds for the modulus of the fifth coefficients of strongly convex function.

2. Preliminaries

We shall use the following lemmas concerning the class \( P \) of the functions with positive real part in \( \mathbb{D} \).

**Lemma 2.1.** [4] Suppose that \( p \in P \), the class of functions satisfying \( \text{Re} \ p(z) > 0 \) for \( z \in \mathbb{D} \) with coefficients given by

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,
\]

then \( |c_n| \leq 2 \) for \( n \geq 1 \).

**Lemma 2.2.** [5] Let \( p \in P \) with coefficients \( c_n \) given by (2.1), then

\[
|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.
\]

**Lemma 2.3.** [3] Let \( p \in P \) with coefficients \( c_n \) given by (2.1), then

\[
|c_4 - c_1 c_3 - \mu c_2^2 + \mu^2 c_1 c_2| \leq \begin{cases} 2, & 0 \leq \mu \leq 1, \\ \frac{2}{2\mu - 1}, & \mu \geq 1. \end{cases}
\]

**Lemma 2.4.** [6] If the functions

\[
\phi_1(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad \text{and} \quad \phi_2(z) = 1 + \sum_{n=1}^{\infty} p_n z^n
\]

belong to \( P \), then the same is true of the function

\[
\phi_3(z) = 1 + \frac{1}{2} \sum_{n=1}^{\infty} c_n p_n z^n.
\]

**Lemma 2.5.** [6] Let

\[
h(z) = 1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \cdots
\]

and

\[
1 + G_1(z) = 1 + b_1' z^n + b_2' z^{2n} + b_3' z^{3n} + \cdots
\]

be functions of \( P \). If \( A \) is defined by

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \gamma_{n-1}}{n} G_1^n(z) = \sum_{n=1}^{\infty} A_n z^n,
\]

where

\[
\gamma_n = \frac{1}{2^n} \left( 1 + \frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} \alpha_k \right), \gamma_0 = 1,
\]

then

\[
|A_n| \leq 2.
\]
3. The Fifth Coefficients

In this section we prove the following.

**Theorem 3.1.** Let \( f \in C(\alpha) \) with coefficients given by (1.2). Then

\[
|a_5| \leq \begin{cases} \frac{1}{10}\alpha, & 0 < \alpha \leq \alpha_0, \\ \frac{1}{45}\alpha^2 \left(38\alpha^2 + 7\right), & \frac{9}{22} \leq \alpha \leq 1, \end{cases}
\]

(3.1)

where \( \alpha_0 = 0.350162 \ldots \) is the unique positive root of the equation

\[
\varphi(\alpha) = 228\alpha^4 - 194\alpha^3 + 2\alpha^2 + 39\alpha - 9.
\]

We note that the result for \(|a_5|\) is not complete. We also note that the first condition in (3.1) holds for \(0 < \alpha \leq \alpha_0\) and the second condition holds for \(\frac{9}{22} \leq \alpha \leq 1\).

**Proof.** Since \( f \in C(\alpha) \), it follows from (1.3) that there exist \( p \in \mathcal{P} \) such that

\[
1 + \frac{zf''(z)}{f'(z)} = (p(z))^\alpha = \left(1 + \sum_{n=1}^{\infty} c_n z^n\right)^\alpha.
\]

(3.2)

Equating coefficients in (3.2) gives

\[
\begin{align*}
a_2 &= \frac{\alpha c_1}{2}, \\
a_3 &= \frac{1}{12}\alpha \left(2c_2 + (3\alpha - 1)c_1^3\right), \\
a_4 &= \frac{1}{144}\alpha \left(12c_3 + 6(5\alpha - 2)c_1c_2 + (17\alpha^2 - 15\alpha + 4)c_1^3\right), \\
a_5 &= \frac{1}{720}\alpha \left(36c_4 + 18(2\alpha - 1)c_2^2 + 12(7\alpha - 3)c_1c_3 + 12(10\alpha^2 - 10\alpha + 3)c_1^2c_2 \\
&\quad + (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)c_1^4\right).
\end{align*}
\]

(3.3)

Suppose first that \( \alpha_1 \leq \alpha \leq 1 \), where \( \alpha_1 = 0.630333 \ldots \) is the unique positive root of

\[38\alpha^3 - 60\alpha^2 + 37\alpha - 9 = 0.\]

Since all the coefficients are positive, then from (3.3) we apply Lemma 2.1 to obtain

\[|a_5| \leq \frac{1}{45}\alpha^2 \left(38\alpha^2 + 7\right),\]

which is the first inequality for \(|a_5|\) when \( \alpha_1 \leq \alpha \leq 1 \).

Next suppose that \( \frac{1}{2} \leq \alpha \leq \alpha_1 \). Then from (3.3) we have

\[
\begin{align*}
a_5 &= \frac{1}{720}\alpha \left(36c_4 + 18(2\alpha - 1)c_2^2 + 12(7\alpha - 3)c_1c_3 + 2(38\alpha^3 - 23\alpha + 9)c_1^2c_2 \\
&\quad + 2(-38\alpha^3 + 60\alpha^2 - 37\alpha + 9)c_1^4\eta_1\right),
\end{align*}
\]

where

\[
\eta_1 = c_2 - \frac{c_1^2}{2}.
\]
Since all the coefficients are positive, we apply Lemma 2.1 and Lemma 2.2 to obtain

\[
|a_5| \leq \frac{1}{720} \alpha \left( 36(2) + 18(2\alpha - 1)(2)^2 + 12(7\alpha - 3)|c_1|(2) + 2(38\alpha^3 - 23\alpha + 9)|c_1|^2(2) \\
+ 2(-38\alpha^3 + 60\alpha^2 - 37\alpha + 9)|c_1|^2 \left( 2 - \frac{|c_1|^2}{2} \right) \right)
= \frac{1}{720} \left( 144\alpha + 24(7\alpha - 3)|c_1| + 24(10\alpha^2 - 10\alpha + 3)|c_1|^2 \\
+ (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)|c_1|^4 \right).
\]

We show that

\[
\frac{1}{720} \alpha \left( 144\alpha + 24(7\alpha - 3)|c_1| + 24(10\alpha^2 - 10\alpha + 3)|c_1|^2 + (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)|c_1|^4 \right)
\]
attains its maximum value at |c_1| = 2 on [0, 2]. To do it, write |c_1| = x and consider the function

\[
g(x) = \frac{1}{720} \alpha \left( 144\alpha + 24(7\alpha - 3)x + 24(10\alpha^2 - 10\alpha + 3)x^2 + (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)x^4 \right),
\]
where x ∈ [0, 2]. It is easy to show that

\[
g'(x) = \frac{1}{720} \alpha \left( 24(7\alpha - 3) + 48(10\alpha^2 - 10\alpha + 3)x + 4(38\alpha^3 - 60\alpha^2 + 37\alpha - 9)x^3 \right) > 0,
\]
where x ∈ [0, 2]. Thus we have

\[
|a_5| \leq \frac{1}{720} \left( 144\alpha + 24(7\alpha - 3)|c_1| + 24(10\alpha^2 - 10\alpha + 3)|c_1|^2 + (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)|c_1|^4 \right)
\]
\[
\leq \frac{1}{720} \left( 144\alpha + 24(7\alpha - 3)(2) + 24(10\alpha^2 - 10\alpha + 3)(2)^2 + (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)(2)^4 \right)
\]
\[
= \frac{1}{45} \alpha^2 \left( 38\alpha^2 + 7 \right).
\]
Which is the second inequality for |a_5| when \( \frac{1}{2} \leq \alpha \leq 1 \).

Next we suppose that \( \frac{9\alpha}{16} \leq \alpha \leq \frac{1}{2} \). Then from (3.3) we have

\[
a_5 = \frac{1}{720} \alpha \left( 9(3 + 2\alpha)c_4 + 9(1 - 2\alpha)\eta_2 + 3(22\alpha - 9)c_1c_3 + 2(38\alpha^3 - 5\alpha)c_1^2c_2 \\
+ 2(-38\alpha^3 + 60\alpha^2 - 37\alpha + 9)c_1^2 \eta_1 \right),
\]
where

\[
\eta_1 = c_2 - \frac{c_1^2}{2},
\]
\[
\eta_2 = c_4 - c_1c_3 - 2c_2^2 + 2c_1^2c_2.
\]

Since all the coefficients are positive, we apply Lemma 2.1, Lemma 2.2, and Lemma 2.3 to obtain

\[
|a_5| \leq \frac{1}{720} \beta \left( 3 + 2\alpha)(2) + 9(1 - 2\alpha)(2(2(2) - 1)) + 3(22\alpha - 9)|c_1|(2) \\
+ 2(38\alpha^3 - 5\alpha)|c_1|^2(2) + 2(-38\alpha^3 + 60\alpha^2 - 37\alpha + 9)|c_1|^2 \left( 2 - \frac{|c_1|^2}{2} \right) \right)
\]
\[
= \frac{1}{720} \beta \left( 108 - 72\alpha + 6(22\alpha - 9)|c_1| + 12(20\alpha^2 - 14\alpha + 3)|c_1|^2 \\
+ (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)|c_1|^4 \right).
\]
(3.4)
Because there’s no critical point inside $[0,2]$, we only need to check the maximum value of (3.4) at $|c_1|=0$ and $|c_1|=2$. After a simple calculation, we find that (3.4) attains its maximum value at $|c_1|=2$. Thus we have

$$ |a_5| \leq \frac{1}{720} \beta \left( 108 - 72\alpha + 6(22\alpha - 9)|c_1| + 12(20\alpha^2 - 14\alpha + 3)|c_1|^2 
+ (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)|c_1|^4 \right) $$

$$ \leq \frac{1}{720} \beta \left( 108 - 72\alpha + 6(22\alpha - 9)(2) + 12(20\alpha^2 - 14\alpha + 3)(2)^2 
+ (38\alpha^3 - 60\alpha^2 + 37\alpha - 9)(2)^4 \right) $$

$$ = \frac{1}{45} \alpha^2 (38\alpha^2 + 7). $$

Which is the third inequality for $|a_5|$ when $\frac{9}{22} \leq \alpha \leq \frac{1}{2}$.

The sharpness of the bound for $|a_5|$ in interval $\frac{9}{22} \leq \alpha \leq 1$ follows from (3.3) when

$$ p(z) = 1 + 2z + 2z^3 + 2z^3 + 2z^4 + \cdots. $$

This establishes the second inequality in (3.1).

For the first inequality, let

$$ a_5 = \frac{1}{20} \alpha \xi, $$

(3.5)

where

$$ \xi = c_4 + \frac{2\alpha - 1}{2} c_2^2 + \frac{7\alpha - 3}{3} c_1 c_3 + \frac{10\alpha^2 - 10\alpha + 3}{3} c_1^2 c_2 + \frac{38\alpha^3 - 60\alpha^2 + 37\alpha - 9}{36} c_1^4. $$

(3.6)

We proceed to show that $|\xi| \leq 2$ for $0 < \alpha \leq \alpha_0$. Replacing the function (2.3) with (2.2) in Lemma 2.5, inequality (2.4) for $n=4$ yields $|A_4| \leq 2$, where

$$ A_4 = \frac{1}{2} \gamma_0 p_4 c_4 - \frac{1}{4} \gamma_1 (p_2^2 c_2^2 + 2p_1 p_3 c_1 c_3) + \frac{3}{8} \gamma_2 p_1^2 c_2 c_1^2 - \frac{1}{16} \gamma_3 p_1^4 c_1^4. $$

(3.7)

Choosing $p_2 = p_4 = 2$ and $p_1 = p_3$, that is for $\alpha \in (0,\alpha_0)$, let

$$ g(z) = \frac{1 + 2\delta z + z^2}{1 - z^2}, $$

and comparing (3.7) with (3.6), we need to take

$$ \gamma_1 = \frac{1 - 2\alpha}{2} = \frac{1}{2} \left( 1 + \frac{1}{2} \beta_1 \right), $$

$$ \gamma_2 = \frac{(1 - 2\alpha)(10\alpha^2 - 10\alpha + 3)}{3(3 - 7\alpha)} = \frac{1}{4} \left( 1 + \beta_1 + \frac{1}{2} \beta_2 \right), $$

$$ \gamma_3 = \frac{(1 - 2\alpha)^2 (9 - 37\alpha + 60\alpha^2 - 80\alpha^3)}{4(3 - 7\alpha)^2} = \frac{1}{8} \left( 1 + \frac{3}{2} \beta_1 + \frac{3}{2} \beta_2 + \frac{1}{2} \beta_3 \right). $$

From these equations we obtain

$$ \beta_1 = -4\alpha, \beta_2 = \frac{2(3 - 7\alpha + 36\alpha^2 - 80\alpha^3)}{3(3 - 7\alpha)^2}, \beta_3 - \beta_1 + \beta_2 = \frac{4(1 - 2\alpha)}{3(3 - 7\alpha)^2} \varphi(\alpha) + 2. $$
Since $\beta_2 \leq 2$, we must have $20\alpha + \alpha - 3 \leq 0$. In addition, for $\alpha$ within this range, the inequality $\left|\frac{\beta_1}{\beta_2}\right| \leq 1$ holds.

We need to demonstrate a function $h \in \mathcal{P}$. We claim that one such function $h$ can be written in the form

$$h(z) = \frac{\beta_2}{2} \frac{1 + 2 \left(\frac{\beta_1}{\beta_2}\right) z + z^2}{1 - z^2} + \left(1 - \frac{\beta_2}{2}\right) \frac{1 + bz^3}{1 - bz^3},$$

where $2b + \beta_1 - b\beta_2 = \beta_3$. Since $5 - 27\alpha + 16\alpha^2 + 112\alpha^3 - 152\alpha^4 > 0$ in $[0, \alpha_0]$, the condition $|b| \leq 1$ gives $\beta_3 - \beta_1 + \beta_2 \leq 2$, which in turn yields $\varphi(\alpha) \leq 0$.

We now apply Lemma 2.5 on (3.5) to obtain

$$|\alpha_5| \leq \frac{1}{20} \alpha(2) = \frac{1}{10} \alpha.$$

This establishes the first inequality in (3.1).

4. Remark
From (1.4) and (3.3), we note that the fifth coefficients of strongly convex function is less than the fifth coefficients of strongly starlike function.

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