FROM DYADIC $\Lambda_\alpha$ TO $\Lambda_\alpha$

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Abstract. In this paper we show how to compute the $\Lambda_\alpha$ norm, $\alpha \geq 0$, using the dyadic grid. This result is a consequence of the description of the Hardy spaces $H^p(R^N)$ in terms of dyadic and special atoms.

Recently, several novel methods for computing the BMO norm of a function $f$ in two dimensions were discussed in [9]. Given its importance, it is also of interest to explore the possibility of computing the norm of a BMO function, or more generally a function in the Lipschitz class $\Lambda_\alpha$, using the dyadic grid in $R^N$. It turns out that the BMO question is closely related to that of approximating functions in the Hardy space $H^1(R^N)$ by the Haar system. The approximation in $H^1(R^N)$ by affine systems was proved in [2], but this result does not apply to the Haar system. Now, if $H^A(R)$ denotes the closure of the Haar system in $H^1(R)$, it is not hard to see that the distance $d(f, H^A)$ of $f \in H^1(R)$ to $H^A$ is $\sim \int_0^\infty f(x) \, dx$, see [1]. Thus, neither dyadic atoms suffice to describe the Hardy spaces, nor the evaluation of the norm in BMO can be reduced to a straightforward computation using the dyadic intervals. In this paper we address both of these issues. First, we give a characterization of the Hardy spaces $H^p(R^N)$ in terms of dyadic and special atoms, and then, by a duality argument, we show how to compute the norm in $\Lambda_\alpha(R^N)$, $\alpha \geq 0$, using the dyadic grid.

We begin by introducing some notations. Let $\mathcal{J}$ denote a family of cubes $Q$ in $R^N$, and $\mathcal{P}_d$ the collection of polynomials in $R^N$ of degree less than or equal to $d$. Given $\alpha \geq 0$, $Q \in \mathcal{J}$, and a locally integrable function $g$, let $p_Q(g)$ denote the unique polynomial in $\mathcal{P}_[\alpha]$ such that $|g - p_Q(g)| \chi_Q$ has vanishing moments up to order $[\alpha]$.

For a locally square-integrable function $g$, we consider the maximal function $M_{\alpha, \mathcal{J}}^{1,2}g(x)$ given by

$$M_{\alpha, \mathcal{J}}^{1,2}g(x) = \sup_{x \in Q, Q \in \mathcal{J}} \frac{1}{|Q|^{\alpha/N}} \left( \frac{1}{|Q|} \int_Q |g(y) - p_Q(g)(y)|^2 \, dy \right)^{1/2}.$$
The Lipschitz space $\Lambda_{\alpha,J}$ consists of those functions $g$ such that $M^{\alpha,2}_{n,J}g$ is in $L^\infty$; $\|g\|_{\Lambda_{\alpha,J}} = \|M^{\alpha,2}_{n,J}g\|_\infty$; when the family in question contains all cubes in $R^N$, we simply omit the subscript $J$. Of course, $\Lambda_0 = BMO$.

Two other families, of dyadic nature, are of interest to us. Intervals in $R$ of the form $I_{n,k} = [(k-1)2^n, k2^n]$, where $k$ and $n$ are arbitrary integers, positive, negative or 0, are said to be dyadic. In $R^N$, cubes which are the product of dyadic intervals of the same length, i.e., of the form $Q_{n,k} = I_{n,k_1} \times \cdots \times I_{n,k_N}$, are called dyadic, and the collection of all such cubes is denoted $D$.

There is also the family $D_0$. Let $I_{n,k}' = [(k-1)2^n, (k+1)2^n]$, where $k$ and $n$ are arbitrary integers. Clearly $I_{n,k}'$ is dyadic if $k$ is odd, but not if $k$ is even.

Now, the collection $\{I_{n,k} : n,k \text{ integers}\}$ contains all dyadic intervals as well as the shifts $[(k-1)2^n + 2\cdot 2^n, k2^n + 2\cdot 2^n)$ of the dyadic intervals by their half length. In $R^N$, put $D_0 = \{Q_{n,k}' : Q_{n,k}' = I_{n,k_1}' \times \cdots \times I_{n,k_N}'\}$; $Q_{n,k}'$ is called a special cube. Note that $D_0$ contains $D$ properly.

Finally, given $I_{n,k}'$, let $I_{n,k}^{L} = [(k-1)2^n, 2^2n]$, and $I_{n,k}^{R} = [2^2n, (k+1)2^n]$. The $2^N$ subcubes of $Q_{n,k}' = I_{n,k_1}' \times \cdots \times I_{n,k_N}'$ of the form $I_{n,k_1}' S_1 \cdots I_{n,k_N}' S_N$, $S_j = L$ or $R$, $1 \leq j \leq N$, are called the dyadic subcubes of $Q_{n,k}'$.

Let $Q_0$ denote the special cube $[-1,1]^N$. Given $\alpha \geq 0$, we construct a family $S_{\alpha}$ of piecewise polynomial splines in $L^2(Q_0)$ that will be useful in characterizing $\Lambda_{\alpha}$. Let $A$ be the subspace of $L^2(Q_0)$ consisting of all functions with vanishing moments up to order $[\alpha]$ which coincide with a polynomial on $P_{[\alpha]}$ on each of the $2^N$ dyadic subcubes of $Q_0$. $A$ is a finite dimensional subspace of $L^2(Q_0)$, and, therefore, by the Graham-Schmidt orthogonalization process, say, $A$ has an orthonormal basis in $L^2(Q_0)$ consisting of functions $p^1, \ldots, p^M$ with vanishing moments up to order $[\alpha]$, which coincide with a polynomial in $P_{[\alpha]}$ on each dyadic subinterval of $Q_0$. Together with each $p^L$ we also consider all dyadic dilations and integer translations given by

$$p^{L}_{n,k,\alpha}(x) = 2^{\alpha(N+\alpha)} p^L(2^\alpha x_1 + k_1, \ldots, 2^\alpha x_N + k_N), \quad 1 \leq L \leq M,$$

and let

$$S_{\alpha} = \{p^{L}_{n,k,\alpha} : n,k \text{ integers}, 1 \leq L \leq M\}.$$

Our first result shows how the dyadic grid can be used to compute the norm in $\Lambda_{\alpha}$.

**Theorem A.** Let $g$ be a locally square-integrable function and $\alpha \geq 0$. Then, $g \in \Lambda_\alpha$ if, and only if, $g \in \Lambda_{\alpha,D}$ and $\Lambda_{\alpha}(g) = \sup_{p \in S_{\alpha}} |\langle g, p \rangle| < \infty$. Moreover,

$$\|g\|_{\Lambda_{\alpha}} \sim \|g\|_{\Lambda_{\alpha,D}} + A_{\alpha}(g).$$

Furthermore, it is also true, and the proof is given in Proposition 2.1 below, that $\|g\|_{\Lambda_{\alpha}} \sim \|g\|_{\Lambda_{\alpha,D_0}}$. However, in this simpler formulation, the tree structure of the cubes in $D$ has been lost.
The proof of Theorem A relies on a close investigation of the predual of $\Lambda_\alpha$, namely, the Hardy space $H^p(R^N)$ with $0 < p = (\alpha + N)/N \leq 1$. In the process we characterize $H^p$ in terms of simpler subspaces: $H^p_D$, or dyadic $H^p$, and $H^p_{S_\alpha}$, the space generated by the special atoms in $S_\alpha$. Specifically, we have

**Theorem B.** Let $0 < p \leq 1$, and $\alpha = N(1/p - 1)$. We then have

$$H^p = H^p_D + H^p_{S_\alpha},$$

where the sum is understood in the sense of quasinormed Banach spaces.

The paper is organized as follows. In Section 1 we show that individual $H^p$ atoms can be written as a superposition of dyadic and special atoms; this fact may be thought of as an extension of the one-dimensional result of Fridli concerning $L^\infty_1$-atoms, see [5] and [1]. Then, we prove Theorem B. In Section 2 we discuss how to pass from $\Lambda_{\alpha,D}$, and $\Lambda_{\alpha,D_0}$, to the Lipschitz space $\Lambda_\alpha$.

1. Characterization of the Hardy spaces $H^p$

We adopt the atomic definition of the Hardy spaces $H^p$, $0 < p \leq 1$, see [6] and [10]. Recall that a compactly supported function $a$ with $[N(1/p - 1)]$ vanishing moments is an $L^2$ $p$-atom with defining cube $Q$ if supp$(a) \subseteq Q$, and

$$|Q|^{1/p} \left( \frac{1}{|Q|} \int_Q |a(x)|^2 dx \right)^{1/2} \leq 1.$$ 

The Hardy space $H^p(R^N) = H^p$ consists of those distributions $f$ that can be written as $f = \sum \lambda_j a_j$, where the $a_j$’s are $H^p$ atoms, $\sum |\lambda_j|^p < \infty$, and the convergence is in the sense of distributions as well as in $H^p$. Furthermore,

$$\|f\|_{H^p} \sim \inf \left( \sum |\lambda_j|^p \right)^{1/p},$$

where the infimum is taken over all possible atomic decompositions of $f$. This last expression has traditionally been called the atomic $H^p$ norm of $f$.

Collections of atoms with special properties can be used to gain a better understanding of the Hardy spaces. Formally, let $A$ be a non-empty subset of $L^2$ $p$-atoms in the unit ball of $H^p$. The atomic space $H^p_A$ spanned by $A$ consists of those $\varphi$ in $H^p$ of the form

$$\varphi = \sum \lambda_j a_j, \quad a_j \in A, \quad \sum |\lambda_j|^p < \infty.$$ 

It is readily seen that, endowed with the atomic norm

$$\|\varphi\|_{H^p_A} = \inf \left\{ \left( \sum |\lambda_j|^p \right)^{1/p} : \varphi = \sum \lambda_j a_j, a_j \in A \right\},$$

$H^p_A$ becomes a complete quasinormed space. Clearly, $H^p_A \subseteq H^p$, and, for $f \in H^p_A$, $\|f\|_{H^p} = \|f\|_{H^p_A}$. 

Two families are of particular interest to us. When \( \mathcal{A} \) is the collection of all \( L^2 \) \( p \)-atoms whose defining cube is dyadic, the resulting space is \( H^p_\mathcal{A} \), or dyadic \( H^p \). Now, although \( \| f \|_{H^p} \leq \| f \|_{H^p_\mathcal{A}} \), the two quasinorms are not equivalent on \( H^p_\mathcal{A} \). Indeed, for \( p = 1 \) and \( N = 1 \), the functions
\[
f_n(x) = 2^n \left[ \chi_{[1-2^{-n},1]}(x) - \chi_{[1,1+2^{-n}]}(x) \right],
\]
satisfy \( \| f_n \|_{H^1} = 1 \), but \( \| f_n \|_{H^1_\mathcal{A}} \sim |n| \) tends to infinity with \( n \).

Next, when \( \mathcal{S}_\alpha \) is the family of piecewise polynomial splines constructed above with \( \alpha = N(1/p - 1) \), in analogy with the one-dimensional results in \( [4] \) and \( [1] \), \( H^p_\mathcal{S}_\alpha \) is referred to as the space generated by special atoms.

We are now ready to describe \( H^p \) atoms as a superposition of dyadic and special atoms.

**Lemma 1.1.** Let \( a \) be an \( L^2 \) \( p \)-atom with defining cube \( Q \), \( 0 < p \leq 1 \), and \( \alpha = N(1/p - 1) \). Then \( a \) can be written as a linear combination of \( 2^N \) dyadic atoms \( a_i \), each supported in one of the dyadic subcubes of the smallest special cube \( Q_{n,k} \) containing \( Q \), and a special atom \( b \) in \( \mathcal{S}_\alpha \). More precisely,
\[
a(x) = \sum_{i=1}^{2^N} d_i a_i(x) + \sum_{L=-n,-k}^{L} c_L p^L(x), \quad |d_i|, |c_L| \leq c.
\]

**Proof.** Suppose first that the defining cube of \( a \) is \( Q_0 \), and let \( Q_1, \ldots, Q_{2^N} \) denote the dyadic subcubes of \( Q_0 \). Furthermore, let \( \{e_1^i, \ldots, e_M^i\} \) denote an orthonormal basis of the subspace \( A_i \) of \( L^2(Q_i) \) consisting of polynomials in \( P_\alpha \), \( 1 \leq i \leq 2^N \). Put
\[
\alpha_i(x) = a(x) \chi_{Q_i(x)} - \sum_{j=1}^{M} \langle a \chi_{Q_i}, e_j^i \rangle e_j^i(x), \quad 1 \leq i \leq 2^N,
\]
and observe that \( \langle \alpha_i, e_j^i \rangle = 0 \) for \( 1 \leq j \leq M \). Therefore, \( \alpha_i \) has \([\alpha]\) vanishing moments, is supported in \( Q_i \), and
\[
\| \alpha_i \|_2 \leq \| a \chi_{Q_i} \|_2 + \sum_{j=1}^{M} \| a \chi_{Q_i} \|_2 \leq (M + 1) \| a \chi_{Q_i} \|_2.
\]
So,
\[
a_i(x) = \frac{2^{N(1/2 - 1/p)}}{M + 1} \alpha_i(x), \quad 1 \leq i \leq N,
\]
is an \( L^2 \) \( p \)-dyadic atom. Finally, put
\[
b(x) = a(x) - \frac{M + 1}{2^{N(1/2 - 1/p)}} \sum_{i=1}^{2^N} a_i(x).
\]
Clearly $b$ has $[\alpha]$ vanishing moments, is supported in $Q_0$, coincides with a polynomial in $P_{[\alpha]}$ on each dyadic subcube of $Q_0$, and

$$
\|b\|^2 \leq \sum_{i=1}^{2^N} \sum_{j=1}^M |(a \chi_{Q_i}, e_j^i)|^2 \leq M \|a\|^2.
$$

So, $b \in A$, and, consequently, $b(x) = \sum_{L=1}^M c_L p^L(x)$, where

$$
|c_L| = \|\langle b, p^L \rangle\| \leq c, \quad 1 \leq L \leq M.
$$

In the general case, let $Q$ be the defining cube of $a$, side-length $Q = \ell$, and let $n$ and $\ell = (k_1, \ldots, k_N)$ be chosen so that $2^{n-1} \leq \ell < 2^n$, and

$$
Q \subset [(k_1 - 1)2^n, (k_1 + 1)2^n] \times \cdots \times [(k_N - 1)2^n, (k_N + 1)2^n].
$$

Then, $(1/2)^{N} \leq |Q|/2^{nN} < 1$.

Now, given $x \in Q_0$, let $a'$ be the translation and dilation of $a$ given by

$$
a'(x) = 2^{-nN/p} a(2^n x - k_1, \ldots, 2^n x - k_N).
$$

Clearly, $[\alpha]$ moments of $a'$ vanish, and

$$
\|a'\|_2 = 2^{nN/p} 2^{-nN/2} \|a\|_2 \leq c |Q|^{1/p} |Q|^{-1/2} \|a\|_2 \leq c.
$$

Thus, $a'$ is a multiple of an atom with defining cube $Q_0$. By the first part of the proof,

$$
a'(x) = \sum_{i=1}^{2^N} d_i a_i(x) + \sum_{L=1}^M c_L p^L(x), \quad x \in Q_0.
$$

The support of each $a_i$ is contained in one of the dyadic subcubes of $Q_0$, and, consequently, there is a $k$ such that

$$
a_i(x) = 2^{-nN/p} a_i(2^{-n} x - k_1, \ldots, 2^{-n} x - k_N)
$$

$a_i$ is an $L^2 p$-atom supported in one of the dyadic subcubes of $Q$. Similarly for the $p_L$'s. Thus,

$$
a(x) = \sum_i d_i a_i(x) + \sum_{L=1}^M c_L p^L_{-n, -k, N(1/p - 1)}(x),
$$

and we have finished. ■

Theorem B follows readily from Lemma 1.1. Clearly, $H^p_D + H^p_{S_n} \hookrightarrow H^p$. Conversely, let $f = \sum_j \lambda_j a_j$ be in $H^p$. By Lemma 1.1 each $a_j$ can be written as a sum of dyadic and special atoms, and, by distributing the sum, we can write $f = f_d + f_s$, with $f_d$ in $H^p_D$, $f_s$ in $H^p_{S_n}$, and

$$
\|f_d\|_{H^p_D}, \|f_s\|_{H^p_{S_n}} \leq c \left( \sum |\lambda_j|^p \right)^{1/p}.
$$

Taking the infimum over the decompositions of $f$ we get $\|f\|_{H^p_D + H^p_{S_n}} \leq c \|f\|_{H^p}$, and $H^p \hookrightarrow H^p_D + H^p_{S_n}$. This completes the proof.
The meaning of this decomposition is the following. Cubes in $D$ are contained in one of the $2^N$ non-overlapping quadrants of $R^N$. To allow for the information carried by a dyadic cube to be transmitted to an adjacent dyadic cube, they must be connected. The $p_{n,k,\alpha}$'s channel information across adjacent dyadic cubes which would otherwise remain disconnected. The reader will have no difficulty in proving the quantitative version of this observation: Let $T$ be a linear mapping defined on $H^p$, $0 < p \leq 1$, that assumes values in a quasinormed Banach space $X$. Then, $T$ is continuous if, and only if, the restrictions of $T$ to $H^p_D$ and $H^p_{S_\alpha}$ are continuous.

2. Characterizations of $\Lambda_\alpha$

Theorem A describes how to pass from $\Lambda_{\alpha,D}$ to $\Lambda_\alpha$, and we prove it next. Since $(H^p)^* = \Lambda_\alpha$ and $(H^p_D)^* = \Lambda_{\alpha,D}$, from Theorem B it follows readily that $\Lambda_\alpha = \Lambda_{\alpha,D} \cap (H^p_{S_\alpha})^*$, so it only remains to show that $(H^p_{S_\alpha})^*$ is characterized by the condition $A_\alpha(g) < \infty$.

First note that if $g$ is a locally square-integrable function with $A_\alpha(g) < \infty$ and $f = \sum_{j,L} c_{j,L} p_{n_j, k_j, \alpha}$, since $0 < p \leq 1$,

$$|\langle g, f \rangle| \leq \sum_{j,L} |c_{j,L}| |\langle g, p_{n_j, k_j, \alpha} \rangle| \leq A_\alpha(g) \left[ \sum_{j,L} |c_{j,L}|^p \right]^{1/p},$$

and, consequently, taking the infimum over all atomic decompositions of $f$ in $H^p_{S_\alpha}$, we get $g \in (H^p_{S_\alpha})^*$ and $\|g\|_{(H^p_{S_\alpha})^*} \leq A_\alpha(g)$.

To prove the converse we proceed as in [3]. Let $Q_n = [-2^n, 2^n]^N$. We begin by observing that functions $f$ in $L^2(Q_n)$ that have vanishing moments up to order $[\alpha]$ and coincide with polynomials of degree $[\alpha]$ on the dyadic subcubes of $Q_n$ belong to $H^p_{S_\alpha}$ and

$$\|f\|_{H^p_{S_\alpha}} \leq |Q_n|^{1/p-1/2}\|f\|_2.$$

Given $\ell \in (H^p_{S_\alpha})^*$, for a fixed $n$ let us consider the restriction of $\ell$ to the space of $L^2$ functions $f$ with $[\alpha]$ vanishing moments that are supported in $Q_n$. Since

$$|\ell(f)| \leq \|\ell\|_{H^p_{S_\alpha}} \|f\|_{H^p_{S_\alpha}} \leq \|\ell\|_{Q_n} |Q_n|^{1/p-1/2}\|f\|_2,$$

this restriction is continuous with respect to the norm in $L^2$ and, consequently, it can be extended to a continuous linear functional in $L^2$ and represented as

$$\ell(f) = \int_{Q_n} f(x) g_n(x) \, dx,$$
where \( g_n \in L^2(Q_n) \) and satisfies \( \|g_n\|_2 \leq \|\ell\| |Q_n|^{1/p-1/2} \). Clearly, \( g_n \) is uniquely determined in \( Q_n \) up to a polynomial \( p_n \) in \( \mathcal{P}_{[\alpha]} \). Therefore,

\[
g_n(x) - p_n(x) = g_m(x) - p_m(x), \quad \text{a.e. } x \in Q_{\min(n,m)}.
\]

Consequently, if \( g(x) = g_n(x) - p_n(x) \), \( x \in Q_n \),

\( g(x) \) is well defined a.e. and, if \( f \in L^2 \) has \( [\alpha] \) vanishing moments and is supported in \( Q_n \), we have

\[
\ell(f) = \int_{\mathbb{R}^N} f(x) g_n(x) \, dx
= \int_{\mathbb{R}^N} f(x) [g_n(x) - p_n(x)] \, dx
= \int_{\mathbb{R}^N} f(x) g(x) \, dx.
\]

Moreover, since each \( 2^nN/p \cdot pL(2^n \cdot +k) \) is an \( L^2 \) \( p \)-atom, \( 1 \leq L \leq M \), it readily follows that

\[
A_\alpha(g) = \sup_{1 \leq L \leq M} \sup_{n,k \in \mathbb{Z}} |\langle g, 2^{-n/p} pL(2^n \cdot +k) \rangle |
\leq \|\ell\| \sup_L \|p^L\|_{H^p} \leq \|\ell\|,
\]

and, consequently, \( A_\alpha(g) \leq \|\ell\|, \) and \( (H^p_S^\alpha)^* \) is the desired space. ■

The reader will have no difficulty in showing that this result implies the following: Let \( T \) be a bounded linear operator from a quasinormed space \( X \) into \( \Lambda_\alpha, \mathcal{D} \). Then, \( T \) is bounded from \( X \) into \( \Lambda_\alpha \) if, and only if, \( A_\alpha(Tx) \leq c \|x\|_X \) for every \( x \in X \).

The process of averaging the translates of dyadic BMO functions leads to BMO, and is an important tool in obtaining results in BMO once they are known to be true in its dyadic counterpart, BMO\(_d\), see [7]. It is also known that BMO can be obtained as the intersection of BMO\(_d\) and one of its shifted counterparts, see [8]. These results motivate our next proposition, which essentially says that \( g \in \Lambda_\alpha \) if, and only if, \( g \in \Lambda_{\alpha,D} \) and \( g \) is in the Lipschitz class obtained from the shifted dyadic grid. Note that the shifts involved in this class are in all directions parallel to the coordinate axis and depend on the side-length of the cube.

**Proposition 2.1.** \( \Lambda_\alpha = \Lambda_{\alpha,D_0} \), and \( \|g\|_{\Lambda_\alpha} \sim \|g\|_{\Lambda_{\alpha,D_0}} \).

**Proof.** It is obvious that \( \|g\|_{\Lambda_{\alpha,D_0}} \leq \|g\|_{\Lambda_\alpha} \). To show the other inequality we invoke Theorem A. Since \( D \subset D_0 \), it suffices to estimate \( A_\alpha(g) \), or, equivalently, \( |\langle g, p \rangle| \) for \( p \in S_\alpha \), \( \alpha = N(1/p - 1) \). So, pick \( p = p^L_{n,k,\alpha} \in S_\alpha \). The defining cube \( Q \) of \( p^L_{n,k,\alpha} \) is in \( D_0 \), and, since \( p^L_{n,k,\alpha} \) has \( [\alpha] \) vanishing moments,
\[ \langle p_{n,k,\alpha}^L, p_Q(g) \rangle = 0. \] Therefore,

\[
| \langle g - p_Q(g), p_{n,k,\alpha}^L \rangle | \\
\leq \| p_{n,k,\alpha}^L \|_2 \| g - p_Q(g) \|_{L^2(Q)} \\
\leq |Q|^\alpha/N |Q|^{1/2} \| p_{n,k,\alpha}^L \|_2 \| g \|_{\Lambda_\alpha, \nu_\alpha}.
\]

Now, a simple change of variables gives \( |Q|^\alpha/N |Q|^{1/2} \| p_{n,k,\alpha}^L \|_2 \leq 1 \), and, consequently, also \( A_{\alpha}(g) \leq \| g \|_{\Lambda_\alpha, \nu_\alpha}. \)

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