Limits on stochastic magnetic fields: A defense of our paper [1]

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In their recent paper “Faraday rotation of the cosmic microwave background polarization by a stochastic magnetic field”, Kosowsky et al. [2] have commented about our paper [1], in which we derived very strong limits on the amplitude of a primordial magnetic field from gravitational wave production. They argue that our limits are erroneous. In this short comment we defend our result.

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In Ref. [1] we have shown that, if a magnetic field is present on super-horizon scales in the early universe, its power is very efficiently converted into gravitational waves during its evolution from super- to sub-horizon scales. We used this fact to derive stringent limits on the amplitude of a magnetic field created before the nucleosynthesis epoch.

In their recent paper [2], Kosowsky et al. state that our limits are not valid. In the discussion section, they argue “... the expansion rate of the universe is the same whether energy density is converted from magnetic fields into gravity waves or not, since the energy density of both scale the same way with the expansion of the universe. So the actual constraint is on the total radiation energy density in the magnetic field, which is constrained to be about 1% of the total energy density in the usual manner... The corresponding limit on the total comoving mean magnetic field strength is around $10^{-8}$ Gauss, not the $10^{-27}$ Gauss claimed in [1].”

We now explain why this conclusion is wrong. We employ the same notation convention as [1], and we always consider the comoving amplitude of the magnetic field. For magnetic fields with spectral index $n > -3$, the magnetic field energy at wave number $k$ is given by

$$\frac{d\Omega_B(k)}{d\log(k)} = \frac{B_\lambda^2}{8\pi \rho_c} \frac{(k\lambda)^{n+3}}{2^{n+5}/\Gamma(\frac{n+5}{2})},$$

where $B_\lambda$ is the magnetic field amplitude at some fixed reference scale $\lambda$. This energy spectrum is always blue, and therefore dominated by its value at the upper cutoff $k_e$. This cutoff scale is time dependent, $k_e(\eta)$. We set the magnetic field to zero on scales which are already sub-horizon at the time $\eta_*$ of formation of the magnetic field, since we cannot be sure that its spectrum is a power law on these very small scales. Therefore, the upper cutoff at the time of formation of the magnetic field is given by $k_e = \eta_*^{-1}$ (where $\eta$ denotes conformal time). This assumption is a conservative one for the derivation of our result. At later times, the magnetic field is damped on scales smaller than a time dependent damping scale, which gives us the cutoff $k_D(\eta)$ [3, 4]. We therefore obtain the cutoff function

$$k_e(\eta) = \min(k_*, k_D(\eta)).$$

Of course at formation $k_* = 1/\eta_*$, while at later time $k_D(\eta)$ is decreasing, and eventually becomes smaller that $k_*$. The magnetic field energy density at a given time $\eta$ is therefore given by

$$\Omega_B(\eta) = \Omega_B(k_e(\eta)) = \int_0^{k_e(\eta)} \frac{dk}{k} \frac{d\Omega_B(k)}{d\log(k)} = \frac{B_\lambda^2}{8\pi \rho_c} \frac{(k_e(\eta)\lambda)^{n+3}}{2^{n+5}/\Gamma(\frac{n+5}{2})}.$$ (3)

In our paper [1], we have shown that at the time a given scale crosses the horizon, and for the maximally allowed magnetic fields ($\Omega_B \sim \Omega_{rad}$, see Eqs. (24) and (26) in Ref. [1]), a considerable fraction$^1$ of the magnetic field energy density is converted into gravitational wave energy density:

$$\left. \frac{d\Omega_G(k, \eta)}{d\log(k)} \right|_{k=1/\eta} \sim \frac{d\Omega_B(k)}{d\log(k)}.$$

As time goes on, the magnetic field is damped on sub-horizon scales $k > k_D(\eta) \gg 1/\eta$, while the gravitational waves are not damped, since after formation they no longer interact with the matter and the radiation in the universe. This is the main point which Kosowsky et al. have missed. The magnetic field density parameter at nucleosynthesis is given by

$$\Omega_B(\eta_{nuc}) = \frac{B_\lambda^2}{8\pi \rho_c} \frac{(k_D(\eta_{nuc})\lambda)^{n+3}}{2^{n+5}/\Gamma(\frac{n+5}{2})},$$ (4)

$^1$ For some values of the spectral index we obtain more energy in gravity waves than in the magnetic field. This comes from the fact that we linearize the problem and therefore do not take into account backreaction. We expect the correct fraction of the energy in gravity waves to lie between 50% and 100% of the magnetic field energy density.
where we have integrated up to the cutoff at nucleosynthesis, $k_D(\eta_{\text{nuc}})$; while the gravitational wave density parameter is

$$\Omega_G \simeq \Omega_B(\eta_*) = \frac{B^2}{8\pi\rho_c} \frac{(k_\lambda)^{n+3}}{2^{n+5}/\Gamma(\frac{n+5}{2})},$$  \hspace{1cm} (5)$$

where we integrate up to the cutoff corresponding to the time of creation of the magnetic field: gravitational wave production for a magnetic mode $k$ takes place before horizon crossing, before the magnetic field is damped by interaction with the cosmic plasma. A considerable part of the magnetic energy is therefore “stored” in gravitational waves.

From [4] and [5], one can see that the ratio between the two energy densities is $\Omega_B(\eta_{\text{nuc}})/\Omega_G \simeq (k_D(\eta_{\text{nuc}})/k_\lambda)^{n+3}$. Only for $n \approx -3$ or $\eta_* \simeq \eta_{\text{nuc}}$ this factor is of order unity; in this case, one can apply the nucleosynthesis bound indifferently to $\Omega_G$ or $\Omega_B$, and one gets the same constraint. In most cases instead, the ratio is rather huge. Let us consider the example of magnetic field generation at the electroweak phase transition. In [4] we calculate $k_D(\eta_{\text{nuc}}) \simeq 6 \times 10^{-7}\text{sec}^{-1} \sim 60 \text{ pc}^{-1}$, and $\eta_{\text{ew}} \simeq 4 \times 10^4 \text{ sec}$. Taking into account that electroweak magnetic field generation is causal (not inflationary), and therefore $n=2$ (see [3]), we obtain

$$\frac{\Omega_B(\eta_{\text{nuc}})}{\Omega_G} \bigg|_{\eta_* = \eta_{\text{ew}}} \simeq (k_D(\eta_{\text{nuc}})/\eta_{\text{ew}})^5 \simeq 8 \times 10^{-9},$$  \hspace{1cm} (6)$$

and by no means one! If the magnetic field is generated during inflation, one is no longer forced to have $n = 2$, but can have arbitrary values of $n > -3$. If we take $n \simeq 0$, for an inflation scale of $10^{15}$ GeV, we have $k_* \simeq 1/\eta_{\text{inf}} \simeq 10^{13}/\eta_{\text{ew}}$, and we obtain

$$\frac{\Omega_B(\eta_{\text{nuc}})}{\Omega_G} \bigg|_{\eta_* = \eta_{\text{ew}}} \simeq (k_D(\eta_{\text{nuc}})/\eta_{\text{inf}})^3 \simeq 10^{-43}.$$  \hspace{1cm} (7)$$

We can conclude that, applying the nucleosynthesis bound on $\Omega_G$, we find much stronger constraints on $B_\lambda$, due to the fact that a sizable fraction of the magnetic field energy is converted into gravitational waves before the damping process.

Apart from not taking into account this damping, there is a second point that has been missed in Kosowsky et al. We specifically pronounce a limit for the amplitude of the stochastic magnetic field smoothed over a scale $\lambda \sim 0.1$ Mpc. On the contrary, they talk about ‘the total comoving mean magnetic field’, which is largely dominated by its value on small scales, hence $B(k_D(\eta_{\text{nuc}}))$. The value of the field at this scale is limited to a few $10^{-8}$ Gauss by the constraint $\Omega_B < 0.1 \Omega_{\text{rad}}$ at nucleosynthesis. But this field value has no relevance, for two reasons. First of all, because the damping scale will grow, which means that $B(k_D(\eta_{\text{nuc}}))$ will be damped away before it can ever give rise to magnetic fields in galaxies. Typically, the highest mode which survives damping is $k_D(\eta_{\text{nuc}}) \sim 10^8$ Mpc$^{-1}$, much smaller than $k_D(\eta_{\text{nuc}})$.

Secondly, the scale relevant for magnetic fields in galaxies and clusters is $\lambda \sim 0.1\ldots1$ Mpc, and we have thus formulated limits for this scale. If $B(k_D(\eta_{\text{nuc}}))k_D(\eta_{\text{nuc}})^{3/2} \equiv B_{\text{inf}} \lesssim 10^{-8}$ Gauss, the limit on the scale $\lambda \gg 1/k_D(\eta_{\text{nuc}})$ is much smaller, namely $B_\lambda = B(k = 1/\lambda)\lambda^{-3/2} = B_{\text{inf}}(k_D(\eta_{\text{nuc}})\lambda)^{-(n+3)/2} \lesssim 10^{-8}$ Gauss $\times 10^{-6(n+3)/2}$. For a spectral index $n = 2$, for example, $B_\lambda$ is smaller than the maximal field by a factor of $\sim 10^{15}$, namely $B_\lambda \lesssim 10^{-23}$ Gauss.

In conclusion, even if the magnetic field and gravitational wave energy densities scale in the same way with the expansion if the universe, applying the nucleosynthesis bound on the induced gravitational wave energy density gives a much stronger constraint on the amplitude of the magnetic field. Both the magnetic field and gravitational wave energy spectra are blue, and therefore dominated by their value at the upper cutoff. However, the upper cutoff for the gravitational wave spectrum is much higher than the one for the magnetic field spectrum at the epoch of nucleosynthesis: $\eta_{\text{inf}}^{-1} \gg k_D(\eta_{\text{nuc}})$. The reason for this being, that the conversion of magnetic field energy into gravitational wave energy takes place when a given mode enters the horizon, before the magnetic field is dissipated by interaction with the cosmic fluid.

Furthermore, the interesting limit is not the one on the ‘mean magnetic field’ which is dominated by the value at the smallest scale, but the limit on the field amplitude at some scale $\lambda$ which is relevant for galactic magnetic fields, and certainly has to be larger than the damping scale at the redshift of galaxy formation.

Of course, these limits apply for magnetic fields generated before the epoch of nucleosynthesis.

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