Differential Invariants of Infinite-Dimensional Algebras That Are Equivalence Algebras of Classes of PDE

Irina YEHORCHENKO

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs’ka Str., 01601 Kyiv-4, Ukraine
E-mail: iyegorch@imath.kiev.ua

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Abstract

We describe differential invariants of infinite-dimensional algebras being equivalence algebras of some classes of PDE and study structure of these algebras.

1 Introduction

We consider infinite-dimensional algebras in the framework of the applications of the Lie theory and symmetry analysis of differential equations. We will discuss finding and full description of differential invariants for such algebras (for basic concepts on differential invariants of Lie algebras see [1]). The methods and concepts presented here for infinite-dimensional algebras are based on similar concepts for finite-dimensional Lie algebras ([2], [3], [4]). Some aspects of differential invariants for pseudogroups (another term for infinite-dimensional counterparts of local Lie transformation groups) were considered in [5].

Here we are studying a special class of infinite-dimensional - equivalence algebras of classes of differential equations (algebras of equivalence transformations, see e.g. [4]).

As a simple but illustrative example we will be considering a class of equations

\[ u_{tt} - u_{xx} = f(u, u_t^2 - u_x^2), \]  

where \( u = u(t, x) \),

\[
  u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2},
\]

\( f \) is an arbitrary function of its arguments (parameter function of the class of equations).

Definition 1. An equivalence transformation for a class of equations (1) is a change of variables of the form

\[ t' = \tau(t, x, u), \quad x' = \kappa(t, x, u), \quad u' = \omega(t, x, u), \quad f' = \varphi(t, x, u, f) \]  

that transforms each equation of the form (1) into an equation of the same form

\[ u'_{tt'} - u'_{xx'} = f'(u', u'^2_{tt'}, u'^2_{xx'}). \]  

The equations (1) and (3) are said to be equivalent.

Operators of the equivalence algebra for the class of equation (1) where found in [6] (the notation \( \sigma = u_t^2 - u_x^2 \) is used):
\( Y_0 = x\partial_t + t\partial_x - u_x\partial_{u_t} - u_t\partial_{u_x}, \)
\( Y_1 = \partial_t, \quad Y_2 = \partial_x, \)
\( Y_3 = x\partial_x + t\partial_t - 2f\partial_f - \sigma\partial_\sigma - 2f\sigma\partial_{f\sigma}, \)
\( Y_\phi = \phi\partial_u + 2\phi'\sigma\partial_\sigma + 2(\phi'f + \phi''\sigma)\partial_f \)
\( +(\phi''f + \phi'''\sigma - 2\sigma f\phi')\partial_{f\sigma} + (\phi'' - f\phi')\partial_{f\sigma}, \)
where \( \phi = \phi(u) \) is an arbitrary function, \( \phi', \phi'' \) and \( \phi''' \) are its first, second and third derivatives.

In this paper we will use the algebra (4) as an example for consideration of two aspects related to infinite-dimensional algebras whose basis operators include arbitrary functions - description of their differential invariants and investigation of structure of such algebras. For both aspects we suggest using discretisation of basis operators of such algebras.

Practical finding of differential invariants of infinite-dimensional algebras was considered in many papers, e.g. in a number of papers by N.H. Ibragimov and his coauthors (see e.g. [8, 9]). Finding of the differential invariants of the algebra (4) was considered in [6], where such invariants were used for characterisation of linearisable equations from the class (1).

However, the ad-hoc approaches used in earlier papers did not allow full rigorous description of functional bases of such differential invariants. The algorithm we proposed in [10] gives such description that is essential for applications.

Here we use this method for an equivalence algebra of a class of the nonlinear wave equations. The obtained description of functional bases of the differential invariants allows using them for characterisation of equations - that is, for each pair of equations from the class (1) we can determine whether these equations are equivalent under transformations of the type (2). In the case of ODE it had been already shown that knowledge of such invariants gives both necessary and sufficient conditions of equivalence [11].

2 Differential Invariants: Background

Investigation of differential invariants usually stems from the problem of description of equations that are invariant under certain algebras (see e.g. [12]). Equations invariant under certain algebras can be presented as functions of absolute differential invariants or as relative differential invariants.

The theory related to searching of differential invariants of finite-dimensional Lie algebras is based on the classical results by S. Lie [1]. The relevant definitions and theorems can be found e.g. in [3, 2].

We consider description of differential invariants for Lie algebras \( Q \) that consist of infinitesimal operators of the form \( Q_i = \xi^k_i(x, u)\partial_{x^k} + \eta^k_i(x, u)\partial_{u^k}. \) Here \( x = (x^1, x^2, \ldots, x^n), u = (u^1, \ldots, u^m). \) We mean summation over the repeated indices.

**Definition 2.** The function

\[ F = F(x, u, u_1, \ldots, u_l), \]
where \( u \) is the set of all \( k \)-th order partial derivatives of the function \( u \) is called a differential invariant for the Lie algebra \( L \) with basis elements \( Q_i \) of the form \( (0.1) \) \( (L = \langle Q_i \rangle) \) if it is an
invariant of the \((l - r)\)-th prolongation of this algebra:

\[
\tilde{Q}_s F(x, u, u_1, \ldots, u_l) = \lambda_s(x, u, u_1, \ldots, u_l) F,
\]

where the \(\lambda_s\) are some functions; when \(\lambda_i = 0\), \(F\) is called an absolute invariant; when \(\lambda_i \neq 0\), it is a relative invariant.

For the prolongation formulae for infinitesimal operators see [3, 2].

We will identify the order of a differential invariant of an equivalence algebra by the order of its highest derivative of any of the arbitrary functions in the respective class of equations.

The order of the prolongation needed to find \(l\)th-order differential invariants may be smaller than \(l\), when coefficients of the basis operators of the equivalence algebra include derivatives of arbitrary functions in the class of equations.

When speaking of differential invariants we will always mean absolute differential invariants.

**Definition 3.** A maximal set of functionally independent invariants of order \(l_1 \leq l\) of the Lie algebra \(L\) is called a functional basis of the \(l\)th-order differential invariants for the algebra \(L\).

All absolute invariants of a particular order can be presented as functions of invariants from a functional basis. The number of invariants (up to a certain particular order \(r\)) in a functional basis is determined as difference between the number of all derivatives up to the \(l\)-th order and both dependent and independent variables, and of the general rank of \(l\)-th Lie prolongation of the basis operators of the algebra under consideration.

The case of infinite-dimensional algebras appears more complicated than a finite-dimensional one, as the bases of such algebras contain infinite (countable) number of operators (e.g. Virasoro and Kac-Moody algebras), and/or infinitesimal operators having arbitrary functions as coefficients (e.g. the equivalence algebra [1]).

However, it appears that despite the name “infinite-dimensional” and actually having infinite-dimensional bases, such algebras, for a fixed rank of their Lie prolongations and for the purpose of finding their differential invariants can be treated as finite-dimensional. The ranks of \(l\)-th Lie prolongations of basis operators appear to be finite for each fixed \(l\). Though, unlike finite-dimensional algebras, these ranks do not stabilise, or do not reach any fixed value with increase of the prolongation rank.

Finiteness of such rank is discussed in [13], where finiteness of a functional basis of differential invariants was proved.

### 3 Discretisation of the basis operators containing arbitrary functions

Here we present a systematic procedure (see [10]) that considerably simplifies previously utilised calculations of differential invariants for the infinite-dimensional equivalence algebras. Instead of arbitrary functions in the coefficients of the basis operators we use expansions of these functions into Taylor series.

Here we deal with arbitrary functions, and in principle it may not be always possible to expand them into Taylor series. These functions have to be infinitely differentiable (due to commutation condition in the definition of the Lie algebra), but that does not mean that they are analytical.
However, for the purpose of calculation of differential invariants of infinite-dimensional algebras we can reasonably limit ourselves with consideration of only analytical functions in coefficients of basis operators, and with finite number of such arbitrary functions in coefficients of basis operators.

Expansion of coefficients into series allows replacement of operators with arbitrary functions with infinite series of infinitesimal operators without such arbitrary functions. That allows straightforward calculation of the prolongations’ ranks (in some cases the rank is equal to the number of variables and derivatives, and then it is easy to see without any further calculations that there are no absolute invariants of the respective order).

**Statement.** For a fixed order $l$ there exists a functional basis of any Lie algebra, including infinite-dimensional algebras with finite number of such arbitrary functions in coefficients of basis operators or with countable infinite sequences of basis operators with no arbitrary functions.

For our example of an equivalence algebra we get the following representation of the basis operators:

$$Y_0 = x \partial_t + t \partial_x, Y_1 = \partial_t, Y_2 = \partial_x,$$

$$Y_3 = x \partial_x + t \partial_t - 2f \partial_f - \sigma \partial_\sigma - 2f_u \partial_{f_u},$$

$$Y_0^k = u^k \partial_u + 2ku^{k-1}\sigma \partial_\sigma + 2(ku^{k-1}f + k(k - 1)u^{k-2} \sigma) \partial_f +$$

$$(k(k - 1)u^{k-2}f + k(k - 1)(k - 2)u^{k-3}\sigma - 2\sigma f_u k(k - 1)u^{k-2}) \partial_{f_u} +$$

$$(k(k - 1)u^{k-2} - f_\sigma ku^{k-1}) \partial_{f_\sigma},$$

$$Y_k^\phi = u^k \partial_u + 2ku^{k-1}\sigma \partial_\sigma + 2(ku^{k-1}f + k(k - 1)u^{k-2}\sigma) \partial_f +$$

$$2((k(k - 1)u^{k-2}f + k(k - 1)(k - 2)u^{k-3}\sigma - 2\sigma f_u k(k - 1)u^{k-2}) \partial_{f_u} +$$

$$(k(k - 1)u^{k-2} - f_\sigma ku^{k-1}) \partial_{f_\sigma}).$$

### 4 Structure of the Discretised Equivalence Algebra

The commutation relations for basis operators of the algebra (6) are as follows:

$$[Y_0^n, Y_\phi^m] = (m - n)Y_\phi^{m+n-1},$$

$$[Y_0, Y_\phi^k] = [Y_1, Y_\phi^k] = [Y_2, Y_\phi^k] = [Y_3, Y_\phi^k] = [Y_0, Y_3] = [Y_1, Y_2] = 0,$$

$$[Y_0, Y_1] = -Y_2, [Y_0, Y_2] = -Y_1, [Y_1, Y_3] = Y_1, [Y_2, Y_3] = Y_2.$$  

It is easy to see that the largest $k$ for which the set $\{Y_0, Y_1, Y_2, Y_3, Y_\phi^k\}$ forms a finite-dimensional algebra is equal to 2.

### 5 Differential Invariants for an Equivalence Algebra

Coefficients of the basis operators of the algebra (6) contain first derivatives of the arbitrary function $f$, so the basis operators themselves would give first-order differential invariants (if exist), and their first prolongations will give second-order differential invariants.

To find differential invariants for the algebra with the discretised basis operators (6) we find the so-called minimal generating set of operators, that is the minimal finite set of operators from the set (6) having the same general rank as the whole set (6). The number of invariants in the functional basis of the order $l$ is equal to the difference of the number of variables entering these
differential invariants (dependent and independent variables, and derivatives of the relevant order), and of the general rank of the needed prolongation of the basis operators.

The basis operators include first derivatives of the function $f$ and may in principle give first-order differential invariants. However, we have 7 variables ($t, x, u, \sigma, f, f_u, f_\sigma$), and the general rank of the operators (6) is equal to 7. We have no absolute differential invariants of the first order, but one relative differential invariant

$$R = \sigma f_\sigma - f,$$

representing a special manifold for the algebra (6) where its rank is equal to 6.

A functional basis of the second-order absolute differential invariants may be taken as

$$R_1 = \frac{\sigma f_{\sigma \sigma}}{\sigma f_\sigma - f}, R_2 = \frac{-2\sigma^2 f f_{\sigma \sigma} + \sigma (f_u - \sigma f_{\sigma u}) + f (\sigma f_\sigma - f)}{(\sigma f_\sigma - f)^2}$$

(with $\sigma f_\sigma - f \neq 0$).

The invariants (8) were found in [6]. Our approach allows proving that they form a functional basis of the second-order absolute differential invariants and using them for characterisation of the equations from the class (1).

We will be looking for invariants in the form $F = F(t, x, u, \sigma, f, f_u, f_\sigma, f_{uu}, f_{u\sigma}, f_{\sigma \sigma})$ - depending on 10 variables. Here invariants depending on the second derivatives of $u$ may be separated, and we do not need them for our purpose of characterisation of the equations.

We will not write down the first prolongation of the operators (6) as it is quite cumbersome, but by direct computation it is possible to determine that it general rank is equal to 8, and as we have 10 variables, a functional basis will contain 10-8=2 invariants. It is also easy to check that the invariants (8) are functionally independent, so they indeed form a functional basis.

The steps for calculation of functional bases of absolute differential invariants for infinite-dimensional algebras with arbitrary functions in basis operators are as follows:

1. Expand the arbitrary functions in the coefficients of the operators into Taylor series.
2. Transform the set of operators with arbitrary functions into discrete infinite set without arbitrary functions.
3. Find needed prolongations of the basis operators of the equivalence algebra.
4. Calculate rank of the prolongation of the algebra.
5. Find a minimal “generating set” of operators with the rank of their prolongation equal to that of the prolongation of the algebra.
6. Find a functional basis using the “generating set”.

6 Characterisation of the differential equations

In our study of characterisation of partial differential equations we will follow the ideas presented in [11] where a similar problem was considered for ODE.

For particular forms of equations of the form (1) values of the invariants of the equivalence algebra will be some functions of $u$ and $\sigma$ - $R_1(u, \sigma, f) = \rho_1(u, \sigma)$, $R_2(u, \sigma, f) = \rho_2(u, \sigma)$. We
can prove that values of all higher order differential invariants can be presented as functions of
\( \rho_1(u, \sigma), \rho_2(u, \sigma) \) (that means that they form a fundamental basis for such algebra), and thus
values of these invariants will determine equivalence classes for the class of equations (1).

**Statement.** Two equations

\[
 u_{tt} - u_{xx} = f_1(u, \sigma), \tag{9}
\]

and

\[
 u_{tt} - u_{xx} = f_2(u, \sigma), \tag{10}
\]

are equivalent if and only if the values of the differential invariants (8) for these equations are
the same:

\[
 R_1(u, \sigma, f_1) = R_1(u, \sigma, f_2), \quad R_2(u, \sigma, f_1) = R_2(u, \sigma, f_2). \tag{11}
\]

Treating the expressions

\[
 R_1 = \frac{\sigma f_{\sigma \sigma}}{\sigma f_{\sigma} - f} = \rho_1(u, \sigma),
\]

\[
 R_2 = \frac{-2\sigma^2 f f_{\sigma \sigma} + \sigma (f_{u} - \sigma f_{\sigma u}) + f (\sigma f_{\sigma} - f)}{(\sigma f_{\sigma} - f)^2} = \rho_2(u, \sigma)
\]
as the system of partial differential equations on the function \( f \), it is possible to describe all
equations in a particular equivalence class.

### 7 Conclusions

We presented an approach for investigation of infinite-dimensional algebras of the first-order
differential operators on a particular example of an equivalence algebra of a class of PDE (1).
The presented approach for finding differential invariants of infinite-dimensional algebras allows
characterisation of equivalent differential equations and description of equations invariant under
certain infinite-dimensional algebras.

For the class of equations (1) we have found the fundamental basis of the second-order
differential invariants of its equivalence algebra and showed that knowing these invariants is
sufficient for characterisation of equations from this class, that is to tell whether two equations
from the class (1) are equivalent.

This approach is very promising for characterisation of equivalence classes represented by
many interesting equations, and description all equations equivalent (reducible by local trans-
formations) to specific equations.

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