The Zero-Difference Properties of Functions and Their Applications

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Abstract
A function $f$ from group $(A, +)$ onto group $(B, +)$ is $(n, m, S)$ zero-difference (ZD), if $S$ is the minimal one of those sets $T$ such that for any non-zero element $a \in A$, $|\{x \in A \mid f(x + a) - f(x) = 0\}| \in T$, where $n = |A|$ and $m = |B|$. ZD is a generalization of differentially $\delta$-vanishing, zero-difference balanced and near zero-difference balanced. In this paper, the framework of obtaining the ZD property of coset index function over algebraic rings is proposed. Then the ZD properties of coset index functions over residual class rings $\mathbb{Z}_{p^k}$ are shown, where $p$ is a prime number and $k \geq 2$ is a positive integer. Finally, DSS-ZD functions and FHS-ZD functions are proposed in order to obtain optimal difference systems of sets (DSS) and optimal frequency-hopping sequences (FHS).

Keywords: Near Zero-Difference Balanced, Zero-Difference, Coset Index Function, Difference Systems of Sets, Frequency-Hopping Sequences

1. Introduction
Let $(A, +)$ and $(B, +)$ be two finite Abelian groups. Denote $A^* = A \setminus \{0\}$. A function from $A$ onto $B$ is an $(n, m, \lambda)$ zero-difference balanced (ZDB) function, if there exists a constant number $\lambda$ such that for any element $a \in A^*$,

$$||x \in A \mid f(x + a) - f(x) = 0|| = \lambda,$$

where $n = |A|$ and $m = |B|$. Carlet and Ding first proposed the concept of ZDB function in 2004 [3]. Some optimal combinatorics objects, such as constant composition codes, constant weight codes, difference systems of sets and frequency-hopping sequences, can be obtained by ZDB functions. Hence many researchers have been working on constructing ZDB functions (see [1–3, 7, 8, 11, 20, 26, 28, 31–33] and the references therein). In 2014, Carlet et al. proposed another concept called differentially $\delta$-vanishing [4]. A function from $A$ to $B$ is differentially $\delta$-vanishing, if for any element $a \in A^*$,

$$1 \leq ||x \in A \mid f(x + a) - f(x) = 0|| \leq \delta.$$

Any $(n, m, \lambda)$ ZDB function is differentially $\lambda$-vanishing ($\lambda$-DV). But there had been little research on this concept, until Jiang and Liao proposed a related concept called generalized zero-difference balanced (G-ZDB) function in 2016 [17]. A function from $A$ onto $B$ is an $(n, m, S)$ generalized zero-difference balanced function, if there exists a constant set $S \subseteq \mathbb{N}$ such that for any element $a \in A^*$,

$$||x \in A \mid f(x + a) - f(x) = 0|| \in S,$$

where $n = |A|$ and $m = |B|$. Then some combinatorics objects can be obtained by G-ZDB functions [16, 17, 22], but they are not optimal. The reason is that the size of $S$ is too large, which implies they are not really zero-difference balanced. Therefore, in 2018, Xu et al. gave another concept called near zero-difference balanced (N-ZDB) function...
A function from $A$ onto $B$ is an $(n, m, \lambda, t)$ near zero-difference balanced function, if there exist a constant number $\lambda$ and a $t$-subset $T$ of $A^*$ such that for any element $a \in A^*$,

$$ ||x \in A \mid f(x + a) - f(x) = 0|| = \begin{cases} \lambda, & a \in T \\ \lambda + 1, & a \notin T \end{cases}, $$

where $n = |A|$ and $m = |B|$.

The four concepts ZDB, $\lambda$-DV, G-ZDB and N-ZDB, are close related to each other. Among them, G-ZDB captures most useful information. However the concept of G-ZDB would lead to the misunderstanding that such G-ZDB functions are really "balanced". In this paper, the concept of zero-difference (ZD) is used instead of G-ZDB.

**Definition 1.** A function $f$ from $A$ onto $B$ is $(n, 1, \lambda)$ ZD if and only if $f$ is $(n, (\lambda, \lambda))$ ZD; if $f$ is $(n, m, S)$ ZD, then $f$ is $\lambda$-DV where $\lambda = \max_{i \in S} i$; if the set $S$ in G-ZDB is changed to be the minimal set, then $(n, m, S)$ ZD is equivalent to $(n, m, S)$ G-ZDB; if $f$ is $(n, m, \lambda, t)$ N-ZDB, then $f$ is $(n, m, (\lambda, \lambda + 1))$ ZD.

It was shown that ZD property can be used to construct the same combinatorics objects as ZDB property can be [15–17, 22]. Although, some of the objects are not optimal with respect to some bounds, Xu et al. shown that optimal frequency-hopping sequences and optimal difference systems of sets can be obtained via N-ZDB functions [27].

Our contributions are as follows:

- We propose the framework of obtaining the ZD properties of coset index functions (defined in (3)) over rings;
- We generalize the work in [27] and propose many N-ZDB functions with good applications;
- We define two new concepts, DSS-ZD and FHS-ZD, to capture the characteristic of ZD functions with good applications;
- We construct optimal frequency-hopping sequences and optimal difference systems of sets, respectively;

The rest of this paper is organized as follows. In Section 2, some properties of ZD functions are given and the framework of obtaining the ZD properties of coset index functions over algebraic rings is proposed. In Section 3, the ZD properties of coset index functions over residual class rings are shown. In Section 4, as the applications of ZD property, the conditions of the constructed combinatorics objects being optimal is studied. In Section 5, conclusions are made.

2. On Algebraic Ring

2.1. Some Properties of ZD functions

In this subsection, let $f$ be an $(n, m, S)$ ZD function from $(A, +)$ onto $(B, +)$. To avoid the trivial cases, it always assume that $m \geq 2$ throughout this paper since constant functions are $(n, 1, m)$ ZD. Define $n_0 = ||x \in A \mid f(x) = b||$, for every $b \in B$. Define $n_{\alpha} = ||x \in A \mid f(x + \alpha) = f(x)||$, for every $\alpha \in A^*$. Denote $\lambda = \max_{i \in S} i$ and $\mu = \min_{i \in S} i$.

The following lemma follows directly from the definition of ZD property.
Lemma 1. Notations are as above. We have
\[ \sum_{b \in B} r_b (r_b - 1) = \sum_{\alpha \in A^*} \lambda_{\alpha}. \]

Proof. Consider the following set
\[ \{(x, \alpha) \in A \times A^* \mid f(x + \alpha) - f(x) = 0\}. \]
On one hand, given \(\alpha \in A^*\), the number of \(x\) satisfying the equation is \(\lambda_{\alpha}\). It equals the right hand side when \(\alpha\) runs over \(A^*\). On the other hand, given \(b \in B\), the number of \(x\) satisfying \(f(x) = b\) is \(r_b\) and the number of \(\alpha\) satisfying the equation is \(r_b - 1\) for each \(x\). It leads to the left hand side when \(b\) runs over \(B\).

Let \(\lambda\) be the arithmetic average of the multi-set \(\{\lambda_{\alpha} \mid \alpha \in A^*\}\), i.e.,
\[ \lambda = \frac{1}{n-1} \sum_{\alpha \in A^*} \lambda_{\alpha}. \tag{1} \]

Based on Lemma 1, we have the following lower bound on \(\lambda\).

Lemma 2. Notations are as above. We have
\[ \lambda \geq \frac{(n - \epsilon)(n - \epsilon - m)}{m(n - 1)} + \lambda - \lambda, \tag{2} \]
where \(n = km + \epsilon\) with \(0 \leq \epsilon < m\). In particular,
\[ \lambda = \frac{(n - \epsilon)(n + \epsilon - m)}{m(n - 1)} + \lambda - \lambda, \]
if and only if, for \(b \in B\), \(r_b = k\) for \(m - \epsilon\) times and \(r_b = k + 1\) for the other \(\epsilon\) times.

Proof. According to Lemma 1, we have
\[ \sum_{b \in B} r_b^2 - n = \sum_{\alpha \in A^*} \lambda_{\alpha} = (n - 1)\lambda = (n - 1)(\lambda - \lambda + \lambda). \]

Moreover, we have
\[ \sum_{b \in B} r_b^2 - n \geq \min_{r_b, b \in B} \sum_{b \in B} r_b^2 - n \]
Note that \(\sum_{b \in B} r_b = n\). By integral programming, \(\{r_b \mid b \in B\}\) attains the minimum value, if and only if, \(f\) is as balanced as possible. Note that \(n = km + \epsilon\). We have
\[ \min_{r_b, b \in B} \sum_{b \in B} r_b^2 - n = (m - \epsilon)^2 + \epsilon(k + 1)^2 - n = \frac{(n - \epsilon)(n + \epsilon - m)}{m}. \]

It completes the proof by some simple calculations.

Using Lemma 1, we can also obtain the bounds on the size of preimage sets. The sizes of all preimage sets are important in some applications, such as construct constant composition codes and difference systems of sets.

Lemma 3. Notations are as above. For each \(b \in B\), we have
\[ \frac{n - \sqrt{\Delta}}{m} \leq r_b \leq \frac{n + \sqrt{\Delta}}{m}, \]
where \(\Delta = (n + \lambda n - \lambda)m^2 - (n^2 + n + \mu n - \mu)m + n^2\).
Proof. For any $b \in B$, we have

$$0 \leq \sum_{b_1, b_2 \in B \setminus \{b\}, b_1 \neq b_2} (r_{b_1} - r_{b_2})^2$$

$$= \sum_{b_1, b_2 \in B \setminus \{b\}, b_1 \neq b_2} r_{b_1}^2 + r_{b_2}^2 - 2r_{b_1}r_{b_2}$$

$$= 2(m - 2) \sum_{b \in B \setminus \{b\}, b_1 \neq b_2} r_{b_1}^2 - 2 \sum_{b_1, b_2 \in B \setminus \{b\}, b_1 \neq b_2} r_{b_1}r_{b_2}.$$ 

It then follows that

$$(m - 2) \sum_{b \in B \setminus \{b\}} r_{b_0}^2 \geq \sum_{b_1, b_2 \in B \setminus \{b\}, b_1 \neq b_2} r_{b_1}r_{b_2}.$$ 

By Lemma 1, we have

$$n + (n - 1)\mu \leq \sum_{b \in B} r_{b_0}^2$$

$$= (\sum_{b \in B \setminus \{b\}} r_{b_0}^2) + r_{b_0}^2$$

$$= (\sum_{b \in B \setminus \{b\}} r_{b_0}^2) + (n - \sum_{b \in B \setminus \{b\}} r_{b_0})^2$$

$$\leq m(\sum_{b \in B \setminus \{b\}} r_{b_0}^2) + nr_{b_0}^2 + n^2 - 2n(n - r_0)$$

Finally, solving the above inequality of $r_0$ completes the proof. \(\square\)

In a similar way, it also has the following lemmas.

**Lemma 4.** Notations are as above. We have

$$\sum_{b \in B} r_{b_0}^2 = (n - 1)\lambda + n.$$ 

**Lemma 5.** Notations are as above. We have

$$\lambda \geq \frac{(n - \epsilon)(n + \epsilon - m)}{m(n - 1)},$$

where $n = km + \epsilon$ with $0 \leq \epsilon < m$. In particular,

$$\lambda = \frac{(n - \epsilon)(n + \epsilon - m)}{m(n - 1)},$$

if and only if, for $b \in B$, $r_b = k$ for $m - \epsilon$ times and $r_b = k + 1$ for the other $\epsilon$ times.

**Lemma 6.** Notations are as above. For each $b \in B$, we have

$$\frac{n - \sqrt{\Delta}}{m} \leq r_b \leq \frac{n + \sqrt{\Delta}}{m},$$

where $\Delta = (n + \lambda n - \lambda)m^2 - (\lambda^2 + n + \lambda n - \lambda)m + n^2$.

**Remark 2.** As a comparison, the properties of ZD functions in Lemmas 1, 2 and 3, are generalizations of those properties of ZDB functions in [26], and of those properties of N-ZDB functions in [27].
2.2. The ZD Property of Coset Index Function

In this paper, we assume that \((R, +, \times)\) is a ring with identity. Note that \((R, \times)\) is a monoid and partitions can be formed by a subgroup.

**Lemma 7.** [12, pp. 8-10] Let \((R, +, \times)\) be a ring of order \(n\), and let \(G\) be a subgroup of \((R, \times)\). Then

\[ D_G = \{ rG \mid r \in R \} \]

is a partition of \(R\), where \(rG = \{ rg \mid g \in G \}\) is called a coset of \(G\) for any \(r \in R\).

Given a subgroup \(G\) of the multiplicative monoid \((R, \times)\), a partition \(D_G\) of \(R\) is obtained by Lemma 7. Now we define a function \(g_G\) from \(R\) to \(D_G\): \(g_G(x) = rG\) where \(rG\) contains \(x\), and there exists a bijective function \(h_G\) from \(D_G\) to \(\mathbb{Z}_{|D_G|}\). So by function composition, a function \(f_G\) from \(R\) to \(\mathbb{Z}_{|D_G|}\) is constructed, i.e.,

\[ f_G(x) = h_G(g_G(x)). \]  

(3)

The function \(f_G(x)\) from \(R\) onto \(\mathbb{Z}_{|D_G|}\) is called the coset index function induced by \(G\).

**Remark 3.** Lemma 7 guarantees that \(g_G\) and \(f_G\) are well-defined.

Let \(x\) be an unknown in \(R\). Here are some notations needed to describe our framework:

- \(S(G, a)\) denotes the union of the solution sets of equations \(x(g - 1) = a\) for every \(g \in G\) and \(N(G, a)\) denotes the size of \(S(G, a)\);
- \(d(G)\) denotes the set of possible sizes of all cosets \(rG\) when \(r\) runs over \(R\), and \(M(G, a)\) denotes the number of elements \(r \in R\) such that the size of the coset \(rG\) is \(a\);
- \(R^*\) denotes the set of all invertible elements in \((R, \times)\) and \(R^*\) denotes the set of all nonzero elements in \(R\).

Now we propose our framework as follows.

**Theorem 8.** Let \((R, +, \times)\) be a ring of order \(n\), and let \(G\) be a subgroup of \((R, \times)\). \(f_G(x)\) denotes the coset index function induced by \(G\). Then \(f_G(x)\) from \((R, +)\) to \((\mathbb{Z}_m, +)\) is an \((n, m, S)\) ZD function, where \(m = \sum_{a \in d(G)} \frac{M(G, a)}{a}\), \(S = \{N(G, a) \mid a \in R^*\}\).

**Proof.** Consider the three parameters of ZD property of \(f_G\). The first parameter is obvious. The second parameter can be obtained by Lemma 7. The following is to show that the third parameter is correct. Since \(h_G\) is bijective, we have

\[ \{ x \in R \mid f_G(x + a) = f_G(x) \} = \{ x \in R \mid g_G(x + a) = g_G(x) \}. \]

Then it suffices to show that for any \(a \in R^*\),

\[ \{ x \in R \mid g_G(x + a) = g_G(x) \} = \bigcup_{g \in G} \{ x \in R \mid x(g - 1) = a \}. \]

On one hand, for any \(x \in \{ x \in R \mid g_G(x + a) = g_G(x) \}\), there exists an element \(r \in R\) and two elements \(g_1, g_2 \in G\) such that

\[ \begin{cases} x + a = rg_1 \\ x = rg_2. \end{cases} \]

It implies

\[ x + a = rg_1 = rg_2g_2^{-1}g_1 = xg_2^{-1}g_1. \]

Then we have \(x(g_2^{-1}g_1 - 1) = a\). Therefore \(x \in \bigcup_{g \in G} \{ x \in R \mid x(g - 1) = a \}\).

On the other hand, for any \(x \in \bigcup_{g \in G} \{ x \in R \mid x(g - 1) = a \}\), there exists an element \(g \in G\) such that

\[ x(g - 1) = a. \]
It implies

\[ x + a = xy \in xG. \]

Then we have \( g_G(x + a) = g_G(x) = xG. \) Therefore \( x \in \{ x \in R \mid g_G(x + a) = g_G(x) \}. \)

Finally, we have

\[ \{ x \in R \mid g_G(x + a) = g_G(x) \} = \bigcup_{g \in G} \{ x \in R \mid x(g - 1) = a \}. \]

\[ \square \]

Note that \( 1 \in G \) and \( a \neq 0. \) So when \( g = 1, \{ x \in R \mid x(g - 1) = a \} = \emptyset. \) If the group \( G \) happens to satisfy the condition

\[ (G - 1) \setminus \{ 0 \} \subset R^c, \]

then the equation \( x(g - 1) = a \) has exactly one solution for each \( g \in G \setminus \{ 1 \}. \) So the function constructed by Theorem 8 is also a ZDB function. Corollary 9 is almost the same as Theorem 1 in [28]. However \( R \) is not required to be commutative in Corollary 9.

**Corollary 9.** Let \((R, +, \times)\) be a ring of order \( n, \) and let \( G \) be a subgroup of \((R, \times), f_G(x) \) is the coset index function. If \( G \) satisfies the condition

\[ (G - 1) \setminus \{ 0 \} \subset R^c, \]

then \( f_G(x) \) is an \((n, \left\lceil \frac{n}{k} \right\rceil + 1, k - 1)\) ZDB function from \((R, +)\) to \((\mathbb{Z}_n, +),\) where \( m = \left\lceil \frac{n}{k} \right\rceil + 1, k = |G|, \) and \( G - 1 = \{ a - 1 \mid a \in G \}. \)

**Remark 4.** Let \( R \) be residual class ring \( \mathbb{Z}_n \) or the product of finite fields \( \mathbb{F}_q, \) then the ZDB functions in [1, 11, 31] can be retrieved by our method. So Corollary 9 can be viewed as a generalization of those results.

If the coset index function \( f_G \) is modified a little bit, then a ZD function \( f^0_G \) can be obtained by Corollary 10.

**Corollary 10.** Using the notations in Corollary 9, define a function

\[ f^0_G(x) = \begin{cases} f_G(x), & x \neq 0, \\ f_G(1), & x = 0. \end{cases} \]

Then \( f^0_G \) is an \((n, \left\lceil \frac{n}{k} \right\rceil, k - 1, k)\) ZD function.

**Proof.** Obviously \( |\text{Im}(f_G)| = |\text{Im}(f^0_G)| = 1 = \left\lceil \frac{n}{k} \right\rceil \), since there does not exist \( x \in R \) such that \( f^0_G(x) = f_G(0). \) Ro solve the equation \( f^0_G(x + a) = f^0_G(x), \) consider two special cases.

- **Case \( a \in G: \)** We assert that \( x = 0 \) is a solution of the equation \( f^0_G(x + a) = f^0_G(x). \) Since
    \[ f^0_G(0 + a) = f^0_G(a) = f_G(a) = h_G(G), \]
    and
    \[ f^0_G(0) = f_G(1) = h_G(G), \]
    we have \( f^0_G(0 + a) = f^0_G(0), \) where \( h_G \) is the bijection in (3).

- **Case \( a \in -G: \)** We assert that \( x = -a \) is a solution of the equation \( f^0_G(x + a) = f^0_G(x). \) Since
    \[ f^0_G(-a + a) = f^0_G(0) = f_G(1) = h_G(G), \]
    and
    \[ f^0_G(-a) = f_G(-a) = h_G(G), \]
    we have \( f^0_G(-a + a) = f^0_G(-a). \)
Now for any element \( a \in R^* \), we have

\[
\{ x \in R \mid f_G^0(x + a) = f_G^0(x) \} = \begin{cases} 
\{ x \in R^* \mid f_G(x + a) = f_G(x) \} \cup \{0\}, & a \in G, \\
\{ x \in R^* \mid f_G(x + a) = f_G(x) \} \cup \{-a\}, & a \in -G, \\
\{ x \in R^* \mid f_G(x + a) = f_G(x) \}, & otherwise.
\end{cases}
\]

Note that neither \( x = 0 \) nor \( x = -a \) can be a solution of the equation \( f_G(x + a) = f_G(x) \). Therefore,

\[
||x \in R \mid f_G^0(x + a) = f_G^0(x)|| = \begin{cases} 
k, & a \in G, \\
k, & a \in -G, \\
k - 1, & otherwise.
\end{cases}
\]

It completes the proof. \( \square \)

**Remark 5.** Note that the results in [27, Section 4] are also the main results in [1] and Corollary 9 is a generalization of them. Hence \( f_G^0 \) is a generalization of the exact N-ZDB function in [27, Theorem 4.1]. The technic used in Corollary 10 is called change point technic.

Using the change point technic, we can generalize Corollary 10 a bit.

**Theorem 11.** Let \( f \) be an \( (n, m, \lambda) \) ZDB function from group \((A, +)\) onto group \((B, +)\). Denote \( f(a_0) = b_0 \). For any \( a \neq a_0 \in A \), define a function

\[
g_a(x) = \begin{cases} 
f(x), & x \neq a_0, \\
f(a), & x = a_0.
\end{cases}
\]

If there exist an integer \( e \) such that for any \( b \in B \),

\[
||x \in A \mid f(x) = b|| = \begin{cases} 
k, & b \neq b_0, \\
1, & b = b_0.
\end{cases}
\]

then \( g_a \) is an \( (n, \frac{m}{\lambda}, \lfloor k - 1, k \rfloor) \) ZDB function. Moreover, \( \lambda = k - 1 \).

**Proof.** The proof is similar with that of Corollary 10. Denote \( A_0 = A \setminus \{a_0\} \). For any \( \alpha \in A^* \), we have

\[
\{ x \in A \mid g_a(x + \alpha) = g_a(x) \} = \begin{cases} 
\{ x \in A_0 \mid f(x + \alpha) = f(x) \} \cup \{a_0\}, & \alpha \in f^{-1}(a) - a_0, \\
\{ x \in A_0 \mid f(x + \alpha) = f(x) \} \cup \{a_0 - \alpha\}, & \alpha \in a_0 - f^{-1}(a), \\
\{ x \in A_0 \mid f(x + \alpha) = f(x) \}, & otherwise,
\end{cases}
\]

and

\[
||x \in A \mid g_a(x + \alpha) = g_a(x)|| = \begin{cases} 
\lambda + 1, & \alpha \in f^{-1}(a) - a_0, \\
\lambda + 1, & \alpha \in a_0 - f^{-1}(a), \\
\lambda, & otherwise.
\end{cases}
\]

Therefore, \( g_a \) is an \( (n, \frac{m}{\lambda}, \lfloor \lambda, \lambda + 1 \rfloor) \) ZDB function. Finally we assert that \( \lambda = k - 1 \).

Since \( g_a \) satisfies the condition in Lemma 5, we have

\[
\lambda = \frac{(n - e)(n + e - m)}{m(n - 1)} = k - 1 + \frac{2k}{n - 1}.
\]
where \( m = \frac{2n}{k-1} \) and \( \epsilon = 1 \). Denote \( \delta = |(f^{-1}(a) - a_0) \cap (a_0 - f^{-1}(a))| \). Then \( 0 \leq \delta \leq k \). According to the definition of \( \lambda \), we have

\[
\lambda = \frac{(2k-\delta)(\lambda + 1) + (n - 2k - \delta)\lambda}{n-1} = \lambda + \frac{2k - \delta}{n-1}.
\]

So

\[
k - 1 + \frac{2k}{n-1} = \lambda + \frac{2k - \delta}{n-1},
\]

i.e.,

\[
\lambda = k - 1 + \frac{\delta}{n-1}.
\]

Note that \( \lambda \) and \( k - 1 \) are integers. It must have \( \frac{\lambda}{n-1} = 0 \). Consequently \( \lambda = k - 1 \).

Therefore \( g_a \) is an \((n, \frac{2n}{k-1}, (k-1, k))\) ZD function.

\[\square\]

**Remark 6.** The ZDB functions in Corollary 9 satisfy the conditions in Theorem 11, So 11 is a generalization of Corollary 10.

### 3. On Residual Class Ring

In this section, we consider applying Theorem 8 to the case that \( R = \mathbb{Z}_n \) and \( G = \langle e \rangle \). Since the case that \( n \) is prime, i.e., \( \mathbb{Z}_n \) is a finite field, is already considered in [11], the cases that \( n \) is prime power will be discussed in this section.

We remark that by Theorem 8 the problem of constructing ZD functions comes down to the problem of solving linear equations. So we give two lemmas about solving linear equations in \( \mathbb{Z}_n \) before our constructions.

**Remark 7.** Note that \( \mathbb{Z}_n \) is communicative, so \( ax = b \) and \( xa = b \) are the same. But we just used to write the linear equation as \( ax = b \).

**Lemma 12.** [14] The congruence \( ax \equiv b \pmod{n} \) has solutions if and only if \( d = \gcd(a, n) \mid b \). If \( d \mid b \), then there are exactly \( d \) solutions. If \( x_0 \) is a solution, then the other solutions are given by \( x_0 + n', x_0 + 2n', \ldots, x_0 + (d - 1)n' \), where \( n' = \frac{n}{d} \).

**Lemma 13.** Let \( p \) be a prime number, and let \( k \) be a positive integer. Suppose \( i, a \in \mathbb{Z}_p \setminus \{0\} \). If \( \gcd(i, p) = 1 \) and \( p \mid a \), then for any integer \( t \), the solution of the congruence \((tp^{k-1} - i)x \equiv a \pmod{p^k}\) is independent of \( t \).

**Proof.** Since \( \gcd(i, p) = 1 \), it follow from Lemma 12 that \(-ix \equiv a \pmod{p^k}\) has exactly one solution denoted by \( x_0 \). Note that \( p \mid a \), so it must have \( p \mid x_0 \). Thus we have

\[
(tp^{k-1} - i)x_0 \equiv a \pmod{p^k}
\]

\[
\equiv tp^{k-1}x_0 - ix_0 \equiv a \pmod{p^k}
\]

\[
\equiv -ix_0 \pmod{p^k}
\]

\[
\equiv a \pmod{p^k}
\]

Therefore, \( x_0 \) is also a solution of congruence \((tp^{k-1} - i)x \equiv a \pmod{p^k}\). Since \( \gcd(tp^{k-1} - i, p^k) = 1 \), the congruence \((tp^{k-1} - i)x \equiv a \pmod{p^k}\) has only one solution \( x_0 \). Obviously, \( x_0 \) is independent of \( t \).

\[\square\]

Next we will show some classes of ZD functions by choosing different prime powers \( p^k \), where \( p \) is a prime number and \( k \geq 2 \) is an integer.

#### 3.1. Case \( n = 4 \)

Applying Theorem 8 on \( \mathbb{Z}_4 \), the situations are clear since there are only two subgroups of \( \mathbb{Z}_4^* \). All the ZD functions are listed in Table 1.
Finally, we assert that for any $\alpha$ and $\beta$, \[\alpha \equiv \beta \pmod{2^k - 1}\] if and only if $2^k | \alpha - \beta$. This congruence is known as the equivalence relation associated with the $2^k$-th power residue symbol.

3.2. Case $n = 2^k$

Let $n = 2^k$, $G = \langle 2^{k-1} - 1 \rangle$, where $k > 2$. Then we have the following theorem.

**Theorem 14.** Let $n = 2^k$, where $k > 2$, and let $G = \langle 2^{k-1} - 1 \rangle$ be a subgroup of $Z_n^\times$. Then the coset index function $f_G$ is a $(2^k, 2^{k-1} + 1, [0, 2])$ ZD function.

**Proof.** The proof is consisted by four steps:

1. We claim that $G = \{1, 2^{k-1} - 1\}$ and $|G| = 2$, since

   \[
   \begin{align*}
   (2^{k-1} - 1)^2 & \equiv 2^{2(k-1)} - 2 \cdot 2^{k-1} + 1 \pmod{2^k} \\
   & \equiv 1 \pmod{2^k},
   \end{align*}
   \]

   and

   \[
   2^{k-1} - 1 \not\equiv 1 \pmod{2^k}.
   \]

2. We assert that for any $\alpha \in Z_n^\times$, it has

   \[
   |\alpha G| = \begin{cases} 
   1, & 2^{k-1} | \alpha, \\
   2, & 2^{k-1} \nmid \alpha.
   \end{cases}
   \]

   If $\alpha \equiv \alpha(2^{k-1} - 1) \pmod{2^k}$, then it implies $\alpha \cdot 2 \cdot (2^{k-2} - 1) \equiv 0 \pmod{2^k}$. We have $\gcd(2^{k-2} - 1, 2^k) = 1$, since $k > 2$. So it must have $2^{k-1} | \alpha$.

3. The size of image of $f_G$ can be obtained as follows.

   \[
   |\text{Im}(f_G)| = |D_G| = \frac{2^k - 2}{2} + 2 = 2^{k-1} + 1.
   \]

4. Based on Theorem 8, it suffices to show that for any $\alpha \in Z_n^\times \setminus \{0\}$, it has

   \[
   \bigcup_{g \in G} \{x \in Z_n : (g - 1)x \equiv \alpha \pmod{2^k}\} = \{0, 2\}.
   \]

   Obviously $\bigcup_{g=1}^{|G|} \{x \in Z_n : (g - 1)x \equiv \alpha \pmod{2^k}\} = \{0, 2\}$, hence we only have to show that $\{x \in Z_n : (2^{k-1} - 2)x \equiv \alpha \pmod{2^k}\} = \{0, 2\}$.

   Note that $\gcd(2^{k-1} - 2, 2^k) = 2$. By Lemma 12, the congruence $(2^{k-1} - 2)x \equiv \alpha \pmod{2^k}$ has solutions if and only if $2 | \alpha$, and if $2 | \alpha$, there are exactly 2 solutions. So

   \[
   N(2^{k-1} - 1, \alpha) = \begin{cases} 
   2, & 2 | \alpha, \\
   0, & 2 \nmid \alpha.
   \end{cases}
   \]

Finally, $f_G$ is a $(2^k, 2^{k-1} + 1, [0, 2])$ ZD function.
3.3. Case \( n = p^2 \)

Let \( n = p^2 \), \( G = \langle p - 1 \rangle \), where \( p \) is an odd prime. We have the following theorem.

**Theorem 15.** Let \( n = p^2 \), where \( p \) is an odd prime, and let \( G = \langle p - 1 \rangle \) be a subgroup of \( \mathbb{Z}_n^* \). Then the coset index function \( f_G \) is a \( (p^2, p, \{p, p^2 - p + 1\}) \) ZD function.

**Proof.** The proof is consisted by four steps:

1. We claim that \( G = \langle (p - 1)^{t-1}(tp - 1) \mod p^2 \mid t = 0, 1, \ldots, (2p - 1) \rangle \), and \( |G| = 2p \). We have
   \[
   (p - 1)^{2p} \equiv 1 \pmod{p^2}, \\
   (p - 1)^2 \equiv -2p + 1 \not\equiv 1 \pmod{p^2}, \\
   \text{and } (p - 1)^p \equiv -1 \not\equiv 1 \pmod{p^2}.
   \]
   Hence the multiplicative order of \( p - 1 \) is \( 2p \), i.e., \( |G| = 2p \). Furthermore,
   \[
   G = \langle (p - 1)^t \mod p^2 \mid t = 0, 1, \ldots, (2p - 1) \rangle = \langle (p - 1)^{t-1}(tp - 1) \mod p^2 \mid t = 0, 1, \ldots, (2p - 1) \rangle.
   \]

2. We assert that for any \( \alpha \in \mathbb{Z}_{p^2} \), it has
   \[
   |\alpha G| = \begin{cases} 1, & \alpha = 0, \\ 2, & p \mid \alpha \text{ and } \alpha \neq 0, \\ 2p, & p \nmid \alpha. \end{cases}
   \]
   If \( \alpha = 0 \), then obviously \( |\alpha G| = ||0|| = 1 \).
   If \( p \nmid \alpha \), then \( \alpha \) has a multiplicative inverse. As a result, \( |\alpha G| = |G| = 2p \).
   If \( p \mid \alpha \text{ and } \alpha \neq 0 \), then it must have
   \[
   \alpha G = \langle \alpha, \alpha(p - 1) \mod p^2 \rangle,
   \]
   since
   \[
   \alpha(p - 1)^2 \equiv \alpha p^2 - 2p\alpha + \alpha \equiv \alpha \pmod{p^2}.
   \]
   Moreover, if \( \alpha \equiv \alpha(p - 1) \pmod{p^2} \), i.e., \( \alpha(p - 2) \equiv 0 \pmod{p^2} \), then \( \alpha = 0 \) which is a contradiction.

3. The size of image of \( f_G \) can be obtained as follows.
   \[
   |\text{Im}(f_G)| = |D_G| = \frac{p(p - 1)}{2p} + \frac{p - 1}{2} + 1 = p.
   \]

4. We make a partition of the group \( G \) as \( \{G_1, G_2\} \), where
   \[
   G_1 = \langle (p - 1)^{t-1}(tp - 1) \mod p^2 \mid t = 0, 2, 4, \ldots, (2p - 2) \rangle = \langle (p - 1)^t \mid t = 0, 2, 4, \ldots, (2p - 2) \rangle, \\
   G_2 = \langle (p - 1)^{t-1}(tp - 1) \mod p^2 \mid t = 1, 3, 5, \ldots, (2p - 1) \rangle = \langle (tp - 1) \mid t = 1, 3, 5, \ldots, (2p - 1) \rangle.
   \]
Due to Theorem 8, it suffices to show that for any $\alpha \in \mathbb{Z}_{p^2} \setminus \{0\}$,
\[
| \bigcup_{g \in G} \{ x \in \mathbb{Z}_{p^2} \mid (g - 1)x \equiv \alpha \pmod{p^2} \} | \in \{ p, p^2 - p + 1 \}.
\]

For $G_1$,
\[
S(G_1, \alpha) = \bigcup_{g \in G_1} \{ x \in \mathbb{Z}_{p^2} \mid (g - 1)x \equiv \alpha \pmod{p^2} \}
\]
\[
= \bigcup_{i=0}^{p-1} \{ x \in \mathbb{Z}_{p^2} \mid ((-2ip + 1) - 1)x \equiv \alpha \pmod{p^2} \}
\]
\[
= \bigcup_{i=0}^{p-1} \{ x \in \mathbb{Z}_{p^2} \mid -2ipx \equiv \alpha \pmod{p^2} \}.
\]

Note that $\alpha \neq 0$. Hence $\{ x \in \mathbb{Z}_{p^2} \mid 0x \equiv \alpha \pmod{p^2} \} = \emptyset$. It follows from Lemma 12 that
\[
S(G_1, \alpha) = \bigcup_{i=0}^{p-1} \{ x \in \mathbb{Z}_{p^2} \mid -2ipx \equiv \alpha \pmod{p^2} \}.
\]

where $[\frac{a}{n}]_n$ denotes a solution of the congruence $ax \equiv b \pmod{n}$. So we have
\[
N(G_1, \alpha) = |S(G_1, \alpha)|
\]
\[
= \left\{ \begin{array}{ll}
p^2 - p, & p \mid \alpha, \\
0, & p \nmid \alpha. 
\end{array} \right.
\]

For $G_2$,
\[
S(G_2, \alpha) = \bigcup_{g \in G_2} \{ x \in \mathbb{Z}_{p^2} \mid (g - 1)x \equiv \alpha \pmod{p^2} \}
\]
\[
= \bigcup_{i=0}^{p-1} \{ x \in \mathbb{Z}_{p^2} \mid ((2i + 1)p - 1) - 1)x \equiv \alpha \pmod{p^2} \}
\]
\[
= \bigcup_{i=0}^{p-1} \{ x \in \mathbb{Z}_{p^2} \mid ((2i + 1)p - 2)x \equiv \alpha \pmod{p^2} \}
\]
\[
= \bigcup_{i=0}^{p-1} \{ \frac{\alpha}{(2i + 1)p - 2} \}.
\]

Due to Lemma 13, we have
\[
N(G_2, \alpha) = |S(G_2, \alpha)|
\]
\[
= \left\{ \begin{array}{ll}
1, & p \mid \alpha, \\
p, & p \nmid \alpha.
\end{array} \right.
\]
Moreover, for any \( x \in \bigcup_{i=0}^{p-1} \left[ \frac{q}{i+1}p \right] \), it must have \( p \mid x \), by Lemma 13. So \( x \not\in \mathbb{Z}_{p^2}^\times \).

Finally, for any \( \alpha \in \mathbb{Z}_{p^2} \setminus \{0\} \), it has

\[
S(G_1, \alpha) \bigcap S(G_2, \alpha) = \begin{cases} \mathbb{Z}_{p^2}^\times \cap \bigcup_{i=1}^{p-1} \left[ \frac{q}{i+1}p \right] & p \mid \alpha, \\ \emptyset & p \nmid \alpha, \end{cases}
\]

Consequently,

\[
N(G, \alpha) = N(G_1, \alpha) + N(G_2, \alpha) = \begin{cases} p^2 - p + 1, & p \mid \alpha, \\ p, & p \nmid \alpha. \end{cases}
\]

Finally, \( f_G \) is a \( (p^2, p, \{p, p^2 - p + 1\}) \) ZD function.

3.4. Case \( n = p^k \)

Let \( n = p^k \), \( G = \langle p^{k-1} - 1 \rangle \) where \( p \) is an odd prime number and \( k > 2 \). We have the Theorem 15 as follow.

**Theorem 16.** Let \( n = p^k \), where \( p \) is an odd prime number. Let \( G = \langle p^{k-1} - 1 \rangle \) be a subgroup of \( \mathbb{Z}_n^\times \). Then the coset index function \( f_G \) is a \( (p^k, \frac{2^{k-1} - 2^{k-1}}{2}, \{1, p, p^k - p^{k-1} + 1\}) \) ZD function.

**Proof.** The proof is similar with that of Theorem 15. We have

1. \( G = \{(1)^t, p^{k-1} - 1 \pmod{p^k} | t = 0, 1, \ldots, (2p - 1)\} \) and \( |G| = 2p; \)

2. for any \( \alpha \in \mathbb{Z}_{p^k} \), it has

\[
|\alpha G| = \begin{cases} 1, & \alpha = 0, \\ 2, & p \mid \alpha \text{ and } \alpha \neq 0, \\ 2p, & p \nmid \alpha; \end{cases}
\]

3. \( |\text{Im}(f_G)| = p^{k-1} - \frac{p^{k-1}}{2}; \)

4. for any \( \alpha \in \mathbb{Z}_{p^k} \setminus \{0\} \), it has

\[
N(G, \alpha) = \begin{cases} p^k - p^{k-1} + 1, & p^{k-1} \mid \alpha, \\ p, & p \nmid \alpha, \\ 1, & p^{k-1} \nmid \alpha \text{ and } p \mid \alpha. \end{cases}
\]

Finally, \( f_G \) is a \( (p^k, \frac{2^{k-1} - 2^{k-1}}{2}, \{1, p, p^k - p^{k-1} + 1\}) \) ZD function.
3.5. Others

In this subsection, we propose more ZD functions by choosing different moduli \( n \) and groups \( G = \langle e \rangle \), according to Theorem 8.

**Theorem 17.** Suppose \( s \geq 1 \) is an integer. Let \( n = p^k \), where \( p \) is a prime number and \( k \geq 2s \). Let \( G = \langle p^{k-s} + 1 \rangle \) be a subgroup of \( \mathbb{Z}_n^* \). Then the coset index function \( f_G \) is a \( (p^k, (sp + p - s)p^{k-x-1}, i, \{0, \sum_{j=0}^{s} \varphi(p^{k-x})\}) \) ZD function, where \( \varphi \) is the Euler function.

**Proof.** The proof is similar with that of Theorem 15. We have

1. \( G = \{1 + tp^{k-x} \pmod{p^k} | t = 0, 1, \ldots, (p^s - 1) \} \) and \( |G| = p^s \);
2. for any \( \alpha \in \mathbb{Z}_n \), it has
   \[
   |\alpha G| = \begin{cases} 
   1, & \alpha = 0, \\
   p^s, & \alpha \neq 0, \\
   \sum_{j=0}^{s} \varphi(p^{k-x}), & \alpha \neq 0.
   \end{cases}
   \]
   where \( p^s \) if \( p^s \) is the largest power of \( p \) dividing \( \alpha \).
3. \( |\text{Im}(f_G)| = \sum_{j=0}^{s} \varphi(p^{k-x}) + \sum_{j=0}^{s} \varphi(p^{k-x}) + 1=(sp + p - s)p^{k-x-1}; \)
4. for any \( \alpha \in \mathbb{Z}_{p^s} \setminus \{0\} \), it has
   \[
   N(G, \alpha) = \begin{cases} 
   0, & \alpha = 1, \\
   \frac{s(p^{k-x})}{p^s}, & \alpha \neq 1.
   \end{cases}
   \]
Finally, \( f_G \) is a \( (p^k, (sp + p - s)p^{k-x-1}, \cup_{j=0}^{s} \{0, \sum_{j=0}^{s} \varphi(p^{k-x})\}) \) ZD function. \( \square \)

In Theorem 17, let \( s = 1 \), we obtain

**Proposition 18.** Let \( n = p^k \), where \( p \) is a prime number and \( k \geq 2 \). Let \( G = \langle p^{k-x} + 1 \rangle \) be a subgroup of \( \mathbb{Z}_n^* \). Then the coset index function \( f_G \) is a \( (p^k, 2p^{k-x} - p^{k-x}, \{0, p^s - p^{k-x}\}) \) ZD function.

**Theorem 19.** Let \( n = p_1 p_2 \), where \( p_1, p_2 \) are two distinct prime numbers. Suppose \( s_i, t_i \) are two integers such that \( s_i t_i = p_i - 1 \) (\( i = 1, 2 \)). Denote \( d = \gcd(s_1, s_2) \). Let \( e \) be an integer determined by the following system of equations:

\[
\begin{align*}
\frac{e}{p_1} &\equiv g_1^1 \pmod{p_1} \\
\frac{e}{p_2} &\equiv g_2^2 \pmod{p_2}
\end{align*}
\]

where \( g_i \) is a generator of \( \mathbb{Z}_{p_i}^* \) (\( i = 1, 2 \)). Let \( G = \langle e \rangle \) be a subgroup of \( \mathbb{Z}_n^* \). Then the coset index function \( f_G \) is a \( (p_1 p_2, 1 + t_1 + t_2 + dt_1, \{a_0, a_1, a_2 \}) \) ZD function, where \( a_0 = \frac{1}{d}(s_1 s_2 - s_1 - s_2) + 1, a_1 = \frac{(p_1 - 1)s_1 s_2}{d} - p_1 + s_2, a_2 = \frac{(p_2 - 1)s_1 s_2}{d} - p_2 + s_1 \).

**Proof.** The proof is similar with that of Theorem 15. Let \( r = \text{lcm}(s_1, s_2) = \frac{p_1 p_2}{\gcd(s_1, s_2)} = \frac{p_1 p_2}{d} \). We have

1. \( G = \{e^r \pmod{p_1 p_2} | r = 0, 1, \ldots, (r - 1) \} \) and \( |G| = r \);
2. for any \( \alpha \in \mathbb{Z}_n \), it has
   \[
   |\alpha G| = \begin{cases} 
   1, & \alpha = 0, \\
   p_1 l, & \alpha \neq 0, \\
   p_2 l, & \alpha \neq 0, \\
   r, \quad \gcd(n, \alpha n) = 1.
   \end{cases}
   \]
3. \( |\text{Im}(f_G)| = 1 + \frac{p_1 - 1}{s_2} + \frac{p_2 - 1}{t_2} + \frac{(p_1 - 1)(p_2 - 1)}{r} = 1 + t_1 + t_2 + dt_1 t_2 \);
Finally, \( f_{ZG} \) is a \((p_1p_2, 1+t_1+t_2+dt_1t_2, \{a_0, a_1, a_2\})\) ZD function, where \( a_0 = \frac{1}{d}(s_1s_2 - s_1 - s_2) + 1 \), \( a_1 = \frac{(p_1-1)s}{d} - p_1 + s_2 \), and \( a_2 = \frac{(p_2-1)s}{d} - p_2 + s_1 \).

**Theorem 20.** Let \( n = mp \), where \( m \) is a positive integer and \( p \) is a prime number such that \( \gcd(m, p) = 1 \). Suppose \( s, t \) are two integers such that \( st = p - 1 \). Let \( e \) be an integer determined by the following system of equations:

\[
\begin{align*}
\frac{e}{p} &\equiv 1 \pmod{m} \\
\frac{e}{p} &\equiv g^i \pmod{p},
\end{align*}
\]

where \( g \) is a generator of \( \mathbb{Z}_p^* \). Let \( G = \langle e \rangle \) be a subgroup of \( \mathbb{Z}_n^* \). Then the coset index function \( f_{ZG} \) is an \((mp, m(1 + t), [0, m(s - 1)])\) ZD function.

**Proof.** The proof is similar with that of Theorem 14. When calculating the size of coset \( \alpha G \) and solving the equations in Theorem 8, we use the trick of Chinese Remainder Theorem and will have the following results.

1. \( G = \{ e^i \pmod{mp} \mid i = 0, 1, \ldots, (s-1) \} \) and \( |G| = s \);
2. For any \( \alpha \in \mathbb{Z}_n \), it has

\[
|\alpha G| = \begin{cases} 
1, & p \mid \alpha, \\
1, & p \nmid \alpha.
\end{cases}
\]
3. \( |\text{Im}(f_{ZG})| = \frac{(p-1)m}{d} + m = m(1 + t) \);
4. For any \( \alpha \in \mathbb{Z}_n^* \) and \( e^i \in G \) \((1 \leq i \leq s - 1)\), we have

\[
N_{\alpha}^{e^i}(G, \alpha) = |\{ x \in \mathbb{Z}_m \mid (e^i - 1)x \equiv \alpha \pmod{m} \}|
\]

\[
= |\{ x \in \mathbb{Z}_m \mid 0x = \alpha \pmod{m} \}|
\]

\[
= \begin{cases} 
m, & m \mid \alpha, \\
0, & m \nmid \alpha.
\end{cases}
\]

and

\[
N_{\alpha}^{p}(G, \alpha) = |\{ x \in \mathbb{Z}_p \mid (e^i - 1)x \equiv \alpha \pmod{p} \}|
\]

\[
= |\{ x \in \mathbb{Z}_p \mid (g^d - 1)x = \alpha \pmod{p} \}|
\]

\[
= |\{ x \in \mathbb{Z}_p \mid x = (g^{dj} - 1)^{-1}x \pmod{p} \}|
\]

\[
= 1.
\]

Note that for \( \alpha \in \mathbb{Z}_n^* \), \( p \nmid \alpha \) if \( m \mid \alpha \). It implies that \((g^d - 1)^{-1}\alpha \neq (g^{dj} - 1)^{-1}\alpha \pmod{p} \) for \( 1 \leq i \neq j \leq s - 1 \). Hence, we have

\[
N(G, \alpha) = \begin{cases} 
m(s-1), & p \nmid \alpha \text{ and } m \mid \alpha, \\
0, & \text{otherwise}.
\end{cases}
\]

Finally, \( f_{ZG} \) is an \((mp, m(1 + t), [0, m(s - 1)])\) ZD function.
4. Applications

Once ZD functions are constructed, many code objects can be obtained. In this section, we present some applications and propose two new concepts, namely, DSS-ZD and FHS-ZD.

4.1. Constant Composition Codes

An \((n, M, d, [w_0, w_1, \ldots, w_{q-1}])\) constant composition code (CCC) is a code over an Abelian group \([b_0, b_1, \ldots, b_{q-1}]\) with length \(n\), size \(M\) and minimum Hamming distance \(d\) such that in every codeword the element \(b_i\) appears exactly \(w_i\) times for every \(i\). Let \(A_q(n, d, [w_0, w_1, \ldots, w_{q-1}])\) be the maximum size of an \((n, M, d, [w_0, w_1, \ldots, w_{q-1}])\) CCC. A CCC is optimal if the bound in Lemma 21 is met.

Lemma 21. [23] If \(nd - n^2 + \sum_{i=0}^{q-1} w_i^2 > 0\), then

\[
A_q(n, d, [w_0, w_1, \ldots, w_{q-1}]) \leq \frac{nd}{nd - n^2 + \sum_{i=0}^{q-1} w_i^2}.
\]

In 2016, Jiang and Liao showed that ZD can be used to construct CCCs.

Proposition 22 (Theorem 4.1 in [17]). Denote

\[A = \{a_0, a_1, \ldots, a_{n-1}\}, B = \{b_0, b_1, \ldots, b_{m-1}\}.\]

Let \(f\) be a function from \(A\) to \(B\). If \(f\) is an \((n, m, S)\) ZD function, then

\[C_f = \{(f(a_0 + a_i), \ldots, f(a_{n-1} + a_i)) \mid 0 \leq i \leq n-1\}\]

is an \((n, n-\lambda, [w_0, w_1, \ldots, w_{m-1}])\) CCC over \(B\), where \(w_i = |\{x \in A \mid f(x) = b_i\}|\) and \(\lambda = \max_{x \in S} x\).

We show that the CCC constructed in Proposition 22 is optimal only if the ZD function is a ZDB function.

Proposition 23. In Proposition 22, if \(C_f\) is optimal, then \(S = \{\lambda\}\), i.e., \(f\) is a ZDB function.

Proof. If \(C_f\) is optimal, then

\[n = \frac{nd}{nd - n^2 + \sum_{i=0}^{m-1} w_i^2},\]

where \(d = n - \lambda\) and \(q = m\). We have

\[\sum_{i=0}^{m-1} w_i^2 = n\lambda + n - \lambda.\]

It follows from Lemma 1 that

\[n\lambda - \lambda = \sum_{\alpha \in A'} \lambda_{\alpha},\]

where \(\lambda_{\alpha} = |\{x \in A \mid f(x + \alpha) = f(x)\}|\) for every \(\alpha \in A'\). It implies

\[n - 1 = \sum_{\alpha \in A'} \frac{\lambda_{\alpha}}{\lambda} = 1,\]

We have \(0 \leq \frac{\lambda_{\alpha}}{\lambda} \leq 1\), since \(\lambda_{\alpha} \in S\) and \(\lambda = \max_{x \in S} x\). Therefore, \(\frac{\lambda_{\alpha}}{\lambda} = 1,\) for every \(\alpha \in A'\). It implies that \(S = \{\lambda\}\), i.e., \(f\) is a ZDB function.
4.2. Difference Systems of Sets

Difference systems of sets (DSS) are related with comma-free codes, authentication codes and secret sharing schemes [13, 24]. Let \( \{D_0, D_1, \ldots, D_{q-1}\} \) be disjoint subsets of an abelian group \((G, +)\). Denote \(|G| = n\) and \(|D_i| = w_i\) for every \(i\). Then \( \{D_0, D_1, \ldots, D_{q-1}\} \) is said to be an \((n, [w_0, w_1, \ldots, w_{q-1}], \lambda)\) DSS if the multi-set
\[
\{ x - y \mid x \in D_i, y \in D_j, 0 \leq i \neq j \leq q - 1 \}
\]
contains every non-zero element \(g \in G\) at least \(\lambda\) times. Moreover, a DSS is perfect if every non-zero element \(g\) appears exactly \(\lambda\) times in the multi-set just mentioned above. It is required that
\[
\tau_q(n, \lambda) = \sum_{i=0}^{q-1} |D_i|
\]
as small as possible. A DSS is called optimal if the bound in Lemma 24 is met.

**Lemma 24.** [25] For an \((n, [w_0, w_1, \ldots, w_{q-1}], \lambda)\) DSS, we have
\[
\tau_q(n, \lambda) \geq \sqrt{\text{SQUARE}(\lambda(n-1) + \lceil \frac{m-1}{q-1} \rceil)} ,
\]
where \(\text{SQUARE}(x)\) denotes the smallest square number that is no less than \(x\) and \(\lceil x \rceil\) denotes the smallest integer that no less that \(x\).

**Remark 8.** DSSs on non-cyclic groups are related to authentication codes and secret sharing schemes[13, 24].

In 2016, Jiang and Liao gave a method to construct DSSs by ZD functions.

**Proposition 25.** [17, Theorem 4.2] Denote \(B = \{b_0, b_1, \ldots, b_{m-1}\}\). If \(f\) is an \((n, m, S)\) ZD function from \(A\) to \(B\), then
\[
D = \{D_i \mid 0 \leq i \leq q - 1 \}
\]
is an \((n, [w_0, w_1, \ldots, w_{m-1}], n - \lambda)\) DSS, where \(D_i = \{x \in A \mid f(x) = b_i\}\), \(w_i = |D_i|\), \(\lambda = \max_{x \in S} x\). Specially, \(D\) is perfect if and only if \(f\) is a ZDB function.

To obtain optimal DSSs, we need following lemmas to prove Theorem 28.

**Lemma 26.** Let \(a\) be a positive integer, and let \(b\) be a real number. \([x]\) denotes the ceiling function. Then \(a < [b]\), if and only if, \(a < b\).

**Proof.** For some \(0 \leq \varepsilon < 1\), we have
\[
a < [b] \iff a \leq [b] - 1 \iff a \leq b - \varepsilon - 1 \iff a \leq b - (1 - \varepsilon) \iff a < b.
\]
\[
\square
\]

Equivalently, the following lemma holds.

**Lemma 27.** Let \(a\) be a positive integer, and let \(b\) be a real number. \([x]\) denotes the ceiling function. Then \(a \geq [b]\), if and only if, \(a \geq b\).

**Theorem 28.** In Proposition 25, the DSS \(D\) is optimal, if and only if \(n \geq m\lambda - m + 2\).
Proof. Note that \( m \geq 2 \). If \( n \geq m\lambda - m + 2 \), then \( n - 1 > m(\lambda - 1) \). That is \( \frac{n - \lambda}{m} > \lambda - 1 \). Hence, we have

\[
\rho(n - 1) + \left\lceil \frac{\rho(n - 1)}{q - 1} \right\rceil \geq (n - \lambda)(n - 1) + \frac{(n - \lambda)(n - 1)}{m - 1} = (n - 1)(n - \lambda + \lambda - 1) = (n - 1)^2,
\]

where \( q = m \) and \( \rho = n - \lambda \). Since \( \tau_q(n, \rho) = n \), it follows from Lemma 24 that

\[
\sqrt{\text{SQUARE}(\rho(n - 1) + \left\lceil \frac{\rho(n - 1)}{q - 1} \right\rceil)} = n.
\]

Therefore, \( D \) meets the bound in Lemma 24. Conversely, if \( D \) is optimal, then we have

\[
\sqrt{\text{SQUARE}((n - \lambda)(n - 1) + \left\lceil \frac{(n - \lambda)(n - 1)}{m - 1} \right\rceil)} = n.
\]

That is

\[
(n - \lambda)(n - 1) + \left\lceil \frac{(n - \lambda)(n - 1)}{m - 1} \right\rceil > (n - 1)^2.
\]

According to Lemma 26, we can get

\[
(n - \lambda)(n - 1) + \frac{(n - \lambda)(n - 1)}{m - 1} > (n - 1)^2,
\]

which leads to \( n - 1 > m(\lambda - 1) \). Finally we have \( n \geq m\lambda - m + 2 \).

Remark 9. Since N-ZDB function is a special case of ZD function, Theorem 28 is a generalization of Theorem 5.12 in [27].

Remark 10. When the ZD function \( f \) is also a ZDB function, the condition of the constructed DSSs being optimal in this paper is weaker than that in [8] (see also [33, Lemma 6]). If \( m \geq 2 \), then \( n \geq m\lambda - m + 2 \) implies \( n \geq m\lambda \) which is the condition in [8].

As a result, it improves Theorem 7 in [28] a bit.

Lemma 29. [28, Theorem 7] Let \( f \) be an \( (n, \frac{m}{m+2} + 1, m - 1) \) ZDB function of Theorem 3 in [28]. Then the DSS constructed from \( f \) by the method in Proposition 25 is optimal if \( n \geq (m - 1)^2 \).

Corollary 30. Let \( f \) be an \( (n, \frac{m}{m+2} + 1, m - 1) \) ZDB function of Corollary 9. Then the DSS constructed from \( f \) by the method in Proposition 25 is optimal if \( n \geq \frac{m(m - 1)}{2} + 1 \).

Example 11. Using the notations in Corollary 9, put \( R = \mathbb{Z}_{11} \) and \( G = (4) \). Then the group \( G \) of order 5 satisfies Condition (4). Hence there is an \( (11, 3, 4) \) ZDB function \( f \) which lead to an \( (11, [1, 5, 5], 7) \) DSS. Obviously this DSS is optimal. It easy to check that \( f \) satisfies the condition in Corollary 30, but not the condition in Lemma 29.

The following propositions imply that, to get optimal DSSs, \( f \) may be neither a ZDB function nor a N-ZDB function.

Proposition 31. There exists an optimal \( (2^k, [1, 1, 2, 2, \ldots, 2], 2^k - 2) \) DSS where \( k \geq 3 \).
Proof. It follows from Theorem 14 that \((2^k, 2^{k-1} + 1, [0, 2])\) ZD functions exist. For \(k \geq 3\), we have
\[
2^k = 2(2^{k-1} + 1) - 2 \geq 2(2^{k-1} + 1) - (2^{k-1} + 1) + 2.
\]
It completes the proof by Theorem 28.

Proposition 32. There exists an optimal \((2p, [1, 1, 2, 2, \ldots, 2], 2p - 2)\) DSS where \(p\) is an odd prime number. \(\square\)

Proof. It follows from Theorem 20 that \((mp, m(1 + i), [0, m(s - 1)])\) ZD functions exist, where \(p\) is a prime number, \(\gcd(m, p) = 1\) and \(s, t\) are two positive integers such that \(p - 1 = st\). Let \(m = 2\), and let \(p \geq 3\) be an odd prime number. Then we can put \(s = 2\), and we obtain \((2p, p + 1, [0, 2])\) ZD functions. We can check that
\[
2p = 2(p + 1) - 2 \geq 2(p + 1) - (p + 1) + 2.
\]
It completes the proof by Theorem 28. \(\square\)

It makes us to propose a new concept to describe those ZD functions which can obtain optimal DSSs.

Definition 12. An \((n, m, S)\) ZD function \(f\) is an \((n, m, \lambda)\) DSS-ZD function if \(n \geq m\lambda - m + 2\), where \(\lambda = \max_{x \in S} x\).

Besides Proposition 31 and Proposition 32, Theorem 5.13, Theorem 5.14 and Theorem 5.15 in [27] also gave some DSS-ZD functions. So it is interesting to construct new DSS-ZD functions.

Due to Theorem 11, we have the following N-ZDB functions which are also DSS-ZD functions.

Theorem 33. The \((n, \frac{am}{k}, k - 1, k)\) ZD functions \(f\) in Theorem 11 are DSS-ZD functions.

Proof. Note that \(\frac{am}{k} \geq 1\). It is straight to check that
\[
n = k \cdot \frac{n - 1}{k} + 1 \geq k \cdot \frac{n - 1}{k} = \frac{n - 1}{k} + 2.
\]
It completes the proof by Theorem 28. \(\square\)

Corollary 34. Let \(n\) be an integer with the following factorization
\[
n = \prod_{i=1}^{r} p_i^{e_i},
\]
where \(p_i\) are prime numbers, \(e_i\) are positive integers \((i = 1, 2, \ldots, r)\). Let \(e \geq 2\) be an integer such that \(e | p_i^{e_i} - 1\) for every \(i\). Then there exist optimal \((n, [e, e, \ldots, e, e + 1], n - e)\) DSSs over \((\prod_{i=1}^{r} p_i^{e_i}, +)\).

Proof. Note that there exist such \((n, \frac{am}{k} + 1, e - 1)\) ZDB functions according to Theorem 1 in [11], which are special cases in Corollary 9. It completes the proof by Corollary 10 and Theorem 33. \(\square\)

Remark 13. Corollary 34 is a generalizations of Theorem 5.13 in [27] and provide more optimal DSSs with new parameters. In the following, Example 14 can not be obtained by Theorem 5.13 in [27].

Example 14. In Theorem 33, put \(n = 3^2 \times 5 = 45\) and \(e = 4\). Then we obtain an optimal \((45, [4, 4, \ldots, 4, 5], 41)\) DSS.

The ZDB functions in [2, Theorem 1] and [29, Corollary 2] satisfy the conditions in Theorem 11. So two class of optimal DSSs can be obtained over cyclic groups and non-cyclic groups.
Corollary 35. Let $n$ be an positive integer with the following factorization

$$n = \prod_{i=1}^{r} p_i^{e_i},$$

where $p_i$ are prime numbers, $e_i$ are positive integers $(i = 1, 2, \ldots, r)$. Let $e \geq 2$ be an integer such that $e \mid p_i - 1$ for every $i$. Then there exist optimal $(en, [e - 1, e - 1, \ldots, e - 1, e], en - e + 1)$ DSSs over ($\mathbb{Z}_{en}, +$).

Corollary 36. Let $n$ be an integer with the following factorization

$$n = \prod_{i=1}^{r} p_i^{e_i},$$

where $p_i$ are prime numbers, $e_i$ are positive integers $(i = 1, 2, \ldots, r)$. Let $e \geq 2$ be an integer such that $e \mid p_i^2 - 1$ for every $i$. Then there exist optimal $(en, [e - 1, e - 1, \ldots, e - 1, e], en - e + 1)$ DSSs over ($\mathbb{Z}_{e} \times \prod_{i=1}^{r} \mathbb{Z}_{p_i^2}, +$).

We summarize some known optimal DSSs in Table 2 where $p$ and $q$ denote a prime number and a prime power, respectively.

4.3. Frequency-hopping Sequences

For any two sequences $X, Y$ of length $n$ over an alphabet $B$. The Hamming correlation $H_{X,Y}$ is defined as

$$H_{X,Y}(t) = \sum_{i=0}^{n-1} h[x_i, y_{(i+t)} \mod n], 0 \leq t < n$$

where $h[a, b] = 1$ if $a = b$, and 0 otherwise. Frequency-hopping sequences (FHS) are the sequences such that the hamming autocorrelation is as small as possible. Lempel and Greenberger gave a lower bound in 1974 [19]. Let $(n, m, \lambda)$ denote an FHS $X$ of length $n$ over an alphabet of size $m$ with $\lambda = H(X)$. An FHS is optimal if the bound in Lemma 37 is met.

Lemma 37. [19] For any FHS $X$ of length $n$ over an alphabet of size $m$, define

$$H(X) = \max_{t \in \mathbb{Z}/n\mathbb{Z}} H_{X,X}(t),$$

then

$$H(X) \geq \left\lceil \frac{(n - e)(n + e - m)}{m(n - 1)} \right\rceil,$$

(7)

where $e$ is the least nonnegative residue of $n$ modulo $m$ and $[x]$ denotes the smallest integer that no less that $x$.

In 2016, Liu and Liao gave a method to construct FHSs by ZD functions.

Proposition 38 (Theorem 13 in [22]). Let $f$ be an $(n, m, \lambda)$ ZD function from cyclic group $(A, +)$ to group $(B, +)$. Then $T = \{f(i\alpha)\}_{i=0}^{n-1}$ is an $(n, m, \lambda)$ FHS, where $\alpha$ is a generator of $A$ and $\lambda = \max_{x \in S} x$.

Remark 15. The criteria of optimality in this subsection are different from those in [22]. In this subsection, we call an FHS is optimal with respect to the bound in Lemma 37. In [22], they call an FHS is optimal with respect to a set of FHSs (see Lemma 12 in [22]).

Now the conditions of the constructed FHSs being optimal is given as follows.

Theorem 39. In proposition 38, the FHS $T$ is optimal, if and only if, the following conditions hold:

1. $[C + \lambda - \lambda] = [C].$
Corollary 30

[7, Proposition 7]

Corollary 36

[33, Corollary 1]

Table 2: Some known optimal DSSs with parameters \((n, [w_0, w_1, \ldots, w_{m-1}], \rho)\)

| \(n\) | \(w_0, w_1, \ldots, w_{m-1}\) | \(\rho\) | Constraints | Reference |
|-------|-----------------|-------|-------------|-----------|
| \(q^2 + 1\) | \(w_0, w_1, \ldots, w_{q-1}\) | \(q^2 - q\) | \(m \in \mathbb{Z}^+\) and \(q = 2^m\) | [7, Proposition 10] |
| \(q\) | \(w_0, w_1, \ldots, w_{d-1}\) | \(q - \frac{q}{d}\) | \(d \mid q\) | [7, Proposition 7] |
| \(p^2\) | \(2p - 1, p - 1, \ldots, p - 1\) | \(p^2 - p\) | | [8, Corollary 8] |
| \(q^m\) | \(w_0, w_1, \ldots, w_{q-1}\) | \(q^m - d + 1\) | \(d \mid q - 1\) | [9, Proposition 14] |
| \(q^2 + q + 1\) | \(w_0, w_1, \ldots, w_{q-1}\) | \(q^2 + 2\) | | [9, Proposition 22] |
| \(q^m - 1\) | \(w_0, w_1, \ldots, w_{q-1}\) | \(q^m - q^{m-s}\) | \(1 \leq s \leq m\) | [33, Theorem 6] |
| \(t^{\frac{q^m-1}{N}}\) | \(w_0, w_1, \ldots, w_{q-1}\) | \(t^{\frac{q^m-1}{N}}\) | \(N \mid q - 1, \gcd(N, m) = 1\) | [33, Theorem 6] |
| \(n\) | \(1, e, e, \ldots, e\) | \(n - e + 1\) | \(n = \prod_{i=1}^{k} p_i^e_i > 3\), \(p_i\) is odd prime and \(e \mid (p_i - 1)\) for all \(i\) | [1, Theorem 1] |
| \(ev\) | \(1, e - 1, e - 1, \ldots, e - 1\) | \(ev - e + 2\) | \(v = \prod_{i=1}^{k} p_i^e_i\), \(p_i\) is odd prime and \(e(e - 1) \mid (p_i - 1)\) for all \(i\) | [2, Theorem 1] |
| \(n\) | \(1, e, e, \ldots, e\) | \(n - e + 1\) | \(n = \prod_{i=1}^{k} p_i^e_i\), \(n \geq (m - 1)^2\) and \(e \mid (p_i^e_i - 1)\) for all \(i\) | [28, Theorem 7] |
| \(n\) | \(e + 1, e, e, \ldots, e\) | \(n - e\) | \(n = \prod_{i=1}^{k} p_i^e_i\), \(p_i\) is odd prime and \(e \mid (p_i - 1)\) for all \(i\) | [27, Theorem 5.13] |
| \(ev\) | \(1, e - 1, e - 1, \ldots, e - 1\) | \(ev - e + 2\) | \(v = \prod_{i=1}^{k} p_i^e_i\), \(p_i\) is odd prime, \(e \mid (p_i - 1)\) and \(e \mid (p_i^e_i - 1)\) for all \(i\) | [27, Theorem 5.14] |
| \(ev\) | \(e - 1, e - 1, \ldots, e - 1\) | \(ev - e + 2\) | \(v = \prod_{i=1}^{k} p_i^e_i\), \(p_i\) is odd prime, \(e \mid (p_i - 1)\) and \(e \mid (p_i^e_i - 1)\) for all \(i\) | [27, Theorem 5.15] |
| \(n\) | \(1, e, e, \ldots, e\) | \(n + 1 - e\) | \(n = \prod_{i=1}^{k} p_i^e_i\), \(n \geq \frac{m(m-1)}{2} + 1\) and \(e \mid (p_i^e_i - 1)\) for all \(i\) | Corollary 30 |
| \(n\) | \(e + 1, e, e, \ldots, e\) | \(n - e\) | \(n = \prod_{i=1}^{k} p_i^e_i\), \(p_i\) is odd prime and \(e \mid (p_i^e_i - 1)\) for all \(i\) | Corollary 34 |
| \(ev\) | \(e, e - 1, e - 1, \ldots, e - 1\) | \(ev - e + 1\) | \(v = \prod_{i=1}^{k} p_i^e_i\), \(p_i\) is odd prime and \(e \mid (p_i - 1)\) for all \(i\) | Corollary 35 * |
| \(ev\) | \(e, e - 1, e - 1, \ldots, e - 1\) | \(ev - e + 1\) | \(v = \prod_{i=1}^{k} p_i^e_i\), \(p_i\) is odd prime and \(e \mid (p_i - 1)\) for all \(i\) | Corollary 36 * |

*: the difference between these two cases is that one is defined on cyclic group and the other is not.

(2) For \(b \in B\), \(w_b = k\) for \(m - \epsilon\) times and \(w_b = k + 1\) for the other \(\epsilon\) times,

where \(\overline{\lambda}\) is defined in (1), \(C = \frac{(n - x)(n + x - m)}{m(n - 1)}\), \(w_b = \|x \in A \mid f(x) = b\|\) and \(n = km + \epsilon\) with \(0 \leq \epsilon < m\)

Proof. If the two conditions hold, then we have

\[\lambda = C + \lambda - \overline{\lambda} = [C + \lambda - \overline{\lambda}] = [C].\]

The first equation follows from Lemma 2. Therefore the FHS \(T\) is optimal.

Conversely, if the FHS \(T\) is optimal, then we have

\[H(T) = \lambda = [C].\]

Note that \([C + \lambda - \overline{\lambda}] \geq [C]\). It follows from Lemma 2 that

\[\lambda \geq [C + \lambda - \overline{\lambda}].\]

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So we have 
\[ \lambda = [C + \lambda - \overline{\lambda}] = [C]. \]

Due to Lemma 27, the equality in (2) holds and \( f \) is as balanced as possible.

**Remark 16.** Since N-ZDB function is a special case of ZD function, Theorem 39 is a generalization of Lemma 5.4 in [27].

Theorem 5.5, Theorem 5.6 and Theorem 5.7 in [27] imply that, a ZD function \( f \) is not necessary be a ZDB function to obtain optimal FHS. In order to characterize such ZD functions, we propose some concepts.

**Definition 17.** A function \( f \) from \( A \) onto \( B \) is almost balanced (AB) if for \( b \in B, w_b = k \) for \( m - \varepsilon \) times and \( w_b = k + 1 \) for the other \( \varepsilon \) times, where \( w_b = |\{x \in A \mid f(x) = b\}|, n = |A|, m = |B| \) and \( n = km + \varepsilon \) with \( 0 \leq \varepsilon < m. \)

In Theorem 39, the number \( C \) is constant once the function \( f \) is given, and is the lower bound of \( \overline{\lambda} \). Therefore, we call the number \( C \) the zero-difference degree (ZDD) of function \( f \).

**Definition 18.** An \((n, m, S)\) ZD function \( f \) defined on cyclic group is an \((n, m, \lambda)\) FHS-ZD function if \( f \) is AB and \([C + \lambda - \overline{\lambda}] = [C]\), where \( C \) is the ZDD of \( f \), \( \lambda = \max_{i \in S} \lambda \) and \( \overline{\lambda} \) is defined in (1).

However it is not easy to check the condition \([C + \lambda - \overline{\lambda}] = [C]\) since it involves three variables \( C, \lambda \) and \( \overline{\lambda} \). So we show that it only depends on two variables \( \lambda \) and \( \overline{\lambda} \) if the function is AB.

**Proposition 40.** Using the same notations in Definition 18, let \( f \) be an \((n, m, S)\) ZD and AB function. Then \( f \) is an \((n, m, \lambda)\) FHS-ZD function if and only if \( \lambda - \overline{\lambda} < 1 \).

**Proof.** According to Lemma 5, the fact that \( f \) is AB implies \( \overline{\lambda} = C. \) Then \([C + \lambda - \overline{\lambda}] = [C]\) if and only if \( \lambda = [C] \).

If \( \lambda - \overline{\lambda} < 1 \), then \( \lambda < \overline{\lambda} + 1 = C + 1 \). On one hand, from Lemma 26 we have \( \lambda < [C] + 1 \). That is \( \lambda \leq [C] \). On the other hand, note that \( \lambda \geq \overline{\lambda} = C. \) From Lemma 27, we have \( \lambda \geq [C]. \) So \( \lambda = [C]. \) Consequently \( f \) is an FHS-ZD function.

Conversely, if \( f \) is an FHS-ZD function, then \( \lambda = [C] = [\overline{\lambda}] \). Obviously, we have \( \lambda \leq [\overline{\lambda}], \) i.e., \( \lambda < [\overline{\lambda}] + 1 \). Using Lemma 26, we obtain \( \lambda - \overline{\lambda} < 1 \).

Now we prove that the ZD functions in Theorem 11 are FHS-ZD functions.

**Theorem 41.** The \((n, \frac{n-1}{k-1}, k)\) ZD functions \( f \) in Theorem 11 are \((n, \frac{n-1}{k-1}, k)\) FHS-ZD functions.

**Proof.** Obviously \( f \) is AB. From the proof of Theorem 11, we have \( \overline{\lambda} = k + 1 + \frac{2k}{n-1} \) and \( \lambda = k \). Then \( \lambda - \overline{\lambda} = 1 - \frac{2k}{n-1} < 1 \).

It completes the proof by Proposition 40.

**Remark 19.** Since N-ZDB function is a special case of ZD function, Theorem 41 is a generalization of Theorem 5.5, Theorem 5.6 and Theorem 5.7 in [27].

Using the above theorem, it is easy to obtain optimal FHSs from ZD functions.

**Corollary 42.** Let \( n \) be an positive integer with the following factorization

\[ n = \prod_{i=1}^{r} p_i^{e_i}, \]

where \( p_i \) are prime numbers, \( e_i \) are positive integers \( (i = 1, 2, \ldots, r). \) Let \( e \geq 2 \) be an integer such that \( e \mid p_i - 1 \) for every \( i. \) Then there exist optimal \((en, \frac{n-1}{e-1}, e-2)\) FHSs over \( \mathbb{Z}_{en^+}. \)

**Proof.** Note that according to Theorem 1 in [2], there exist such \((en, \frac{n-1}{e-1} + 1, e-2)\) ZDB functions over \( \mathbb{Z}_{en}. \) It completes the proof by Theorem 11 and Theorem 41.

To end this subsection, we summarize some known optimal FHSs in Table 3.
Table 3: Some known optimal FHSs with parameters \((n, m, i)\)

| \(n\) | \(m\) | \(\lambda\) | Constraints | Reference |
|-------|-------|-----------|-------------|-----------|
| \(p\) | \(e\) | \(f\) | \(p = ef + 1\) is prime, \(e\) is even and \(f\) is odd | [5, Corollary 2] |
| \(p\) | \(e + 1\) | \(f - 1\) | \(p = ef + 1\) is prime and \(2 \leq f \leq e + 2\) | [5, Corollary 3] |
| \(p\) | \(L\) | \(2g\) | \(p = 2Lg + 1\) is an odd prime number and \(p \equiv 3 \pmod{4}\) | [6, Theorem 7] |
| \(p\) | \(L + 1\) | \(2g - 1\) | \(p \equiv 3 \pmod{4}, g\) is odd and \(3 \leq g \leq \frac{L + 3}{2}\) | [6, Theorem 9] |
| \(p^2\) | \(p\) | \(p\) | \(p\) is prime | [18, Theorem 2] |
| \(p^t - 1\) | \(p^k\) | \(p^{t-k} - 1\) | \(p\) is prime and \(1 \leq k \leq t\) | [19, Theorem 2] |
| \(p^e\) | \(e\) | \(f\) | \(p = ef + 1\) is an odd prime number and \(f\) is odd | [21, Theorem 3.1] |
| \(p^e\) | \(e\) | \(f - 1\) | \(p = ef + 1\) is an odd prime number | [21, Theorem 3.2] |
| \(q - 1\) | \(e\) | \(f\) | \(q = ef + 1\) is a prime power and \(f\) is even | [10, Theorem 4] |
| \(q - 1\) | \(e + 1\) | \(f - 1\) | \(q = ef + 1\) is a prime power | [10, Theorem 5] |
| \(\frac{q - 1}{e}\) | \(q\) | \(\frac{q - 1}{e}\) | \(q\) is a prime power, \(\lfloor (q - 1) \pmod{e}\rfloor = 1\) | [10, Theorem 5] |
| \(n\) | \(\frac{n - 1}{e} + 1\) | \(e - 1\) | \(e \mid p_i - 1\), for \(1 \leq i \leq k\), \(k > 1\) or \(k = 1\) with \(\frac{n - 1}{e}\) is odd and prime | [30, Corollary 2] |
| \(n\) | \(\frac{n - 1}{e}\) | \(e\) | \(e\) is odd and \(e \mid (p_i - 1)\), for \(1 \leq i \leq k\). | [30, Theorem 3] |
| \(ev\) | \(\frac{ev - 1}{e - 1}\) | \(e - 2\) | \(v = \prod_{i=1}^{k} p_i\), \(p_i\) are odd and prime. | [27, Theorem 5.5] |
| \(ev\) | \(\frac{(e-1)n - 1}{e - 2}\) | \(e - 2\) | \(e \geq 3, e \mid (p_i - 1)\) and \(e - 2 \mid (p_i - 1)\), for \(1 \leq i \leq k\) | [27, Theorem 5.6] |
| \(ev\) | \(\frac{(e-1)n - 1}{e - 2}\) | \(e - 2\) | \(e \geq 3, e \mid (p_i - 1)\) and \(e - 2 \mid (p_i - 1)\), for \(1 \leq i \leq k\) | [27, Theorem 5.7] |
| \(ev\) | \(\frac{ev - 1}{e - 1}\) | \(e - 2\) | \(v = \prod_{i=1}^{k} p_i\), \(p_i\) are odd and prime. | Corollary 42 |

5. Conclusion

In this paper, we have proposed two concepts called zero-difference and coset index function. Then we discussed the zero-difference properties of coset index functions. By change point technic, we get ZD functions from ZDB functions. Finally, we show that these ZD functions are DSS-ZD functions and FHS-ZD functions, i.e., optimal DSSs and optimal FHSs can be obtained by them.

Yi et al. gave the conditions of the DSSs and FHSs constructed by ZDB functions being optimal [28, Theorem 7 and Theorem 8]. It is interesting that by our change point technic, the ZD functions obtained by those ZDB function can construct optimal DSSs and FHSs without any conditions.

In the future work, we are expected to construct more DSS-ZD functions and FHS-ZD functions.

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