Parameter estimation from noisy and one-bit quantized data has become an important topic in signal processing, as it offers low cost and low complexity in the implementation. On the one hand, Direction-of-Arrival (DoA) estimation using Sparse Linear Arrays (SLAs) has recently gained considerable interest in array processing due to their attractive capability of providing enhanced degrees of freedom. In this paper, the problem of DoA estimation from one-bit measurements received by an SLA is considered and a novel framework for solving this problem is proposed. The proposed approach first provides an estimate of the received signal covariance matrix through minimization of a constrained weighted least-squares criterion. Then, MUSIC is applied to the spatially smoothed version of the estimated covariance matrix to find the DoAs of interest. Several numerical results are provided to demonstrate the superiority of the proposed approach over its counterpart already propounded in the literature.

Index Terms— One-bit quantization, Sparse linear arrays, direction of arrival (DoA) estimation, weighted least-squares.

1. INTRODUCTION

The problem of Direction of Arrival (DoA) estimation is of central importance in the field of array processing with many applications in radar, sonar, and wireless communications [1–3]. Estimating DoAs using Uniform Linear Arrays (ULAs) is well investigated in the literature; a number of algorithms such as Maximum Likelihood (ML) estimation, MUSIC, ESPRIT and subspace fitting have been presented and their performance thoroughly analyzed [4–6]. However, it is widely known that ULAs are not capable of identifying more sources than the number of physical elements in the array [2, 6].

To transcend this limitation, exploitation of Sparse Linear Arrays (SLAs) with particular geometries, such as Minimum Redundancy Arrays (MRAs) [7], co-prime arrays [8] and nested arrays [9] has been proposed. These architectures can dramatically boost the degrees of freedom of the array for uncorrelated source signals such that a significantly larger number of sources than the number of physical elements in the array can be identified. In addition, the enhanced degrees of freedom provided by these SLAs can improve the resolution performance appreciably compared to ULAs [9]. These features have spurred further research on DoA estimation using SLAs in recent years. A detailed study on DoA estimation via SLAs through an analysis of the Cramér-Rao Bound (CRB) has been conducted in [10]. Further, several approaches for DoA estimation via SLAs have been proposed in the literature. In general, existing estimators can be classified under two main groups: 1. Sparsity-Based Methods (SBMs); 2. Augmented Covariance-Based Methods (ACBMs). SBMs first discretize the angular domain into a grid and then estimate DoAs by imposing sparsity constraints on source profiles and exploiting the compressive sensing recovery techniques [11–13]. In the second approach, DoAs are estimated by applying conventional subspace methods such as MUSIC, ESPRIT on an Augmented Sample Covariance Matrix (ASCM) obtained from the original sample covariance matrix by exploiting the difference co-array structure [9, 14, 15].

Quantization of signals of interest is an essential step in digital signal processing. While high-resolution amplitude quantization at the Nyquist rate is typically desired, it may either be impractical or be impossible in many applications due to limitations on power consumption and production cost of Analog-to-Digital Converters (ADCs) [16]. One-bit quantizers allow for an extremely high sampling rate at a low cost and low power consumption while maintaining good performance. Recently, the use of one-bit quantized data has gained considerable interest in different applications such as massive MIMO systems [17–19], radar [20, 21] and array processing [22–26]. Using one-bit quantized data for DoA estimation has been considered in [22–25] for ULAs and it has been shown that one-bit quantization leads to a moderate performance loss compared to the case where unquantized data is used. Further, the problem of DoA estimation via SLAs and one-bit data has been addressed in [26]; it has been demonstrated that the performance degradation due to one-bit quantization can, to some extent, be compensated using SLAs.

In this paper, we propose a new framework for DoA estimation via SLAs using one-bit quantized measurements. Contrary to [26] which uses the Bussgang theorem, here the covariance matrix of unquantized data is recovered from one-bit measurements through the solution of a constrained optimization problem. Then, by providing representative numerical results, it is shown that the estimated covariance matrix from the proposed approach leads to a better performance compared to the Bussgang-aided method given in [26].

Organization: Section 2 describes the system model. The problem formulation is given in Section 3. Section 4 provides the proposed algorithm for DoA estimation from one-bit measurements. The simulation results and related discussions are included in Section 5. Finally, Section 6 concludes the paper.

Notation: Vectors and matrices are referred to by lower- and upper-case bold-face, respectively. The superscripts *, T, H denote the conjugate, transpose and Hermitian (conjugate transpose) operations, respectively. ∥A∥F stands for the Frobenius norm of A.
\[ [a_i] \] indicates the \( i \)th entry of \( a \). \( \hat{A} \) and \( \hat{a} \) denote the estimate of \( A \) and \( a \), respectively. \((a_1, a_2, \ldots, a_n)\) is an \( n \)-tuple with elements of \( a_1, a_2, \cdots, a_n \). \(|\mathcal{A}|\) represents the cardinality of the set \( \mathcal{A} \). \( \text{diag}(a) \) is a diagonal matrix whose diagonal entries are equal to the elements of \( a \). The \( M \times M \) identity matrix is denoted by \( \mathbf{I}_M \). \( x \) denotes the sign function with \( (x) = 1 \) for \( x \geq 0 \) and \( (x) = -1 \) otherwise. The real and image part of \( a \) are denoted by \( \text{Re}(a) \) and \( \text{Im}(a) \), respectively. \( \mathcal{E}(\cdot) \) stands for the statistical expectation. \( \odot \) and \( \otimes \) represent Kronecker and Khorati-Rao products, respectively. \( \text{vec}((\cdot)) \) denotes the trace, rank and vectorization operations, respectively. \( \hat{A}^\dagger \) and \( \hat{a}^\dagger \) indicates the pseudoinverse and of the full column rank matrix \( A \).

2. SYSTEM MODEL

We consider an SLA with \( M \) elements located at positions \((m_1 \lambda, m_2 \lambda, \cdots, m_M \lambda)\) with \( m_i \in \mathbb{M} \). Here \( \mathbb{M} \) is a set of integers with cardinality \(|\mathbb{M}| = M\), and \( \lambda \) denotes the wavelength of the incoming signals. It is assumed \( K \) narrowband signals with distinct DoAs \( \theta = [\theta_1, \theta_2, \cdots, \theta_K]^T \) impinge on the SLA from far field. While the estimation of the number of sources is an important problem, we assume perfect knowledge of the number of sources here. The unquantized array measurements at time instance \( t \) can be modeled as

\[ y(t) = \mathbf{A}(\theta)s(t) + \mathbf{n}(t) \in \mathbb{C}^{M \times 1}, \quad t = 1, \cdots, N, \quad (1) \]

where \( s(t) \in \mathbb{C}^{K \times 1} \) denotes the vector of source signals, \( \mathbf{n}(t) \in \mathbb{C}^{M \times 1} \) is additive noise, and \( \mathbf{A}(\theta) = [a(\theta_1), a(\theta_2), \cdots, a(\theta_K)] \in \mathbb{C}^{M \times K} \) represents the SLA steering matrix with

\[ a(\theta_i) = [e^{j\pi \sin \theta_{i1}}, e^{j\pi \sin \theta_{i2}}, \ldots, e^{j\pi \sin \theta_{iM}}]^T, \quad (2) \]

being the SLA manifold vector for the \( i \)th signal. Further, the following assumptions are made on source and noise signals:

**A1** \( \mathbf{n}(t) \) follows a zero-mean circular Gaussian distribution with the covariance matrix \( \mathbb{E}(\mathbf{n}(t)\mathbf{n}^H(t)) = \sigma^2 \mathbf{I}_M \).

**A2** The source signal vector is modeled as a zero-mean circular Gaussian random vector with covariance matrix \( \mathbb{E}(s(t)s(t)^H) = \text{diag}(\mathbf{p}) \) where \( \mathbf{p} = [p_1, p_2, \cdots, p_K]^T \in \mathbb{R}^{K \times 1} \) (i.e., \( p_i > 0 \) \&\& \( p_i > 0 \)).

**A3** Source and noise vectors are mutually independent.

**A4** There is no temporal correlation between the snapshots, i.e.,

\[ \mathbb{E}(\mathbf{n}(t_1)\mathbf{n}^H(t_2)) = \mathbb{E}(s(t_1)s(t_2)) = 0 \]

when \( t_1 \neq t_2 \) and \( \mathbf{0} \) is an all zero matrix of appropriate dimensions.

Based on the above assumptions, the covariance matrix of \( y(t) \) is expressed as:

\[ \mathbb{R} = \mathbb{E}(y(t)y^H(t)) = \mathbf{A}(\theta)\text{diag}(\mathbf{p})\mathbf{A}^H(\theta) + \sigma^2 \mathbf{I}_M \in \mathbb{C}^{M \times M}. \quad (3) \]

Following [9, 10, 14], the difference co-array model of the SLA is obtained by vectorizing the covariance matrix, which results in

\[ \text{vec}(\mathbb{R}) = (\mathbf{A}^H(\theta) \odot \mathbf{A}(\theta))\mathbf{p} + \sigma^2 \text{vec}(\mathbf{I}_M), \]

\[ = J\mathbf{A}_d(\theta)\mathbf{p} + \sigma^2 \text{vec}(\mathbf{I}_M) \in \mathbb{C}^{M^2 \times 1}, \quad (4) \]

where \( \mathbf{A}_d(\theta) \in \mathbb{C}^{(2D-1)\times K} \) is the steering matrix of the difference co-array whose elements are located at \((\pm D - \frac{1}{2}, \cdots, \pm D - \frac{1}{2})\) with \( \ell \in \mathbb{D} = [m_p - m_q, m_p, m_q] \in \mathbb{M} \) and \( D = |\mathbb{D}|. \) Further, the selection matrix \( J \) is defined as follows

**Definition 1.** The binary matrix \( J \in \{0, 1\}^{M^2 \times (2D-1)} \) is defined as [10]

\[ J = \left[ \text{vec}(\mathbf{L}_{D-1}^T) \cdots \text{vec}(\mathbf{L}_0) \cdots \text{vec}(\mathbf{L}_{D-1}) \right], \quad (5) \]

where \( \mathbf{L}_n \) is \( p \times q \) with \( J_{pq} = \begin{cases} 1, & \text{if } m_p - m_q = \ell_n, \quad 1 \leq p, q \leq M \text{ and } 0 \leq n \leq D - 1, \\ 0, & \text{otherwise,} \end{cases} \)

In practice, the array measurements are quantized by ADCs. In the most extreme form of quantization, the measurements are directly converted into binary data by a comparator measuring the sign of the real and imaginary parts of the received signal. One-bit quantization allows for an extremely high sampling rate at a low cost and low power consumption. Assuming one-bit ADCs are used at the SLA, the quantized one-bit measurements are denoted as [17, 27]

\[ z(t) = Q(y(t)), \quad (6) \]

where the \( i \)th element of \( Q(y(t)) \) is given by \( Q(y(t))_i = \frac{1}{\sqrt{2} \text{Re}(y(t)_i)} + \frac{1}{\sqrt{2} \text{Im}(y(t)_i)} \). In this paper, we provide an algorithm to estimate DoAs from \( z(t) \).

3. PROBLEM FORMULATION

In this section, we formulate an optimization problem from which the covariance matrix of \( y(t) \), i.e., \( \mathbb{R} \), can be recovered from one-bit observations. Once \( \mathbb{R} \) is obtained, Co-Array-Based MUSIC (CAB-MUSIC), described in [9, 14], can be used to estimate DoAs. It can be readily checked that \( \mathbb{R} \) is a structured matrix with only \( 2D - 1 \) free parameters, i.e.,

\[ \mathbb{R} = \mathbf{u}_0 \mathbf{L}_0 + \sum_{i=1}^{D-1} \mathbf{u}_i \mathbf{L}_i + \sum_{i=1}^{D-1} \mathbf{u}_i^* \mathbf{L}_i^T, \quad (7) \]

where \( \mathbf{u}_0 = \sum_{k=1}^{K} p_k + \sigma^2 \) and \( \mathbf{u}_i = \sum_{k=1}^{K} p_k e^{j\pi \sin \theta_{ik}}. \) Hence, to estimate \( \mathbb{R} \), we need only to estimate the complex vector \( \mathbf{u} = [\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_{D-1}]^T. \) In case the unquantized data are available, minimization of the following weighted least squares criterion yields
a large-snapshot maximum likelihood (ML) estimate of the structured covariance matrix in (7) [28, 29]:

\[
(\text{vec}(YY^H) - \text{vec}(R))^H(\text{vec}(YY^H) - \text{vec}(R)) = \|R^{-\frac{1}{2}}(YY^H - R)R^{-\frac{1}{2}}\|^2_F, \tag{8}
\]

where \( Y = \frac{1}{\sqrt{N}}[y_1 \ldots y(N)] \). Unfortunately, the above criterion is non-convex in \( R \). Hence, inspired by (8), the following alternative convex criterion was proposed in [30]

\[
\|R^{-\frac{1}{2}}(YY^H - R)(YY^H)^{-\frac{1}{2}}\|^2_F = \text{tr}(R^{-\frac{1}{2}}(YY^H)^{-1}R^{-\frac{1}{2}}) + \text{tr}(R^{-\frac{1}{2}}(YY^H)R^{-\frac{1}{2}}), \tag{9}
\]

which is convex in \( R \). The above objective function is shown to converge to the ML criterion for a growing number of snapshots. Using the objective function (9), an optimization problem for the covariance matrix recovery from the one-bit measurements can be cast as follows:

\[
\begin{aligned}
\text{minimize} & \quad \text{tr}(R^{-\frac{1}{2}}(YY^H)^{-1}R^{-\frac{1}{2}}) + \text{tr}(R^{-\frac{1}{2}}(YY^H)R^{-\frac{1}{2}}) \\
\text{subject to} & \quad R > 0, \\
& \quad \text{vec}(\text{Re}\{Z\}) \odot \text{vec}(\text{Re}\{Y\}) \geq 0, \\
& \quad \text{vec}(\text{Im}\{Z\}) \odot \text{vec}(\text{Im}\{Y\}) \geq 0,
\end{aligned} \tag{10}
\]

where \( Z = [z(1) \ldots z(N)] \). The above optimization problem is non-convex due to its dependence on \( Y \), but it can be recast as a Semi-Definite Programming (SDP) by introducing some slack variables.

### 4. ONE-BIT DOA ESTIMATION

We consider the slack variables \( X, W, T \) and \( \Phi \) such that \( X = R^{-\frac{1}{2}}(YY^H)^{-1}R^{-\frac{1}{2}}, W \succeq X^{-1}, T = YY^H \) and \( \Phi = R^{-\frac{1}{2}} \). Then it can be shown that optimization problem (10) is equivalent to the following one

\[
\begin{aligned}
\text{minimize} & \quad \text{tr}(X) + \text{tr}(W) \\
\text{subject to} & \quad R > 0, \\
& \quad \text{vec}(\text{Re}\{Z\}) \odot \text{vec}(\text{Re}\{Y\}) \geq 0, \\
& \quad \text{vec}(\text{Im}\{Z\}) \odot \text{vec}(\text{Im}\{Y\}) \geq 0, \\
& \quad X = \Phi T^{-1} \Phi, \\
& \quad R = \Phi \Phi, \\
& \quad T = YY^H.
\end{aligned} \tag{11}
\]

Optimization problem (11) is an SDP with three equality constraints, which are non-convex. It is however possible to replace the equality constraints in (11) with three rank constraints on semi-definite matrices using the following Lemma.

**Lemma 1.** Let \( C_1 \in \mathbb{R}^{m \times m}, C_2 \in \mathbb{R}^{n \times m} \) and \( C_{12} \in \mathbb{R}^{m \times n} \). If \( C_1 \succeq 0 \), then the equality \( C_2 = C_{12}^HC_{12}^{-1}C_{12} \) is equivalent to the following rank and semi-definite inequalities

\[
\text{rank}(C_1) \leq m \quad \text{and} \quad \text{rank}(C_{12}) \geq 0. \tag{12}
\]

**Proof.** It is readily confirmed that \( C_2 = C_{12}^HC_{12}^{-1}C_{12} \) if and only if \( \text{rank}(C_2 - C_{12}^HC_{12}^{-1}C_{12}) = 0 \). Since \( C_1 \) is positive definite, \( \text{rank}(C_2 - C_{12}^HC_{12}^{-1}C_{12}) = 0 \) can be equivalently expressed as \( \text{rank}(C_1) + \text{rank}(C_2 - C_{12}^HC_{12}^{-1}C_{12}) \leq m \). Further, it follows from the Guttman rank additivity formula [31] that \( \text{rank}(C_1) + \text{rank}(C_2 - C_{12}^HC_{12}^{-1}C_{12}) = \text{rank}(C_1)C_{12}^{-1}C_{12} \). This gives the rank condition in (12). Moreover, it follows from \( C_2 - C_{12}^HC_{12}^{-1}C_{12} = 0 \) and \( C_1 \succeq 0 \) that \( C_{12} \) has to be positive semi-definite. 

By making use of Lemma 1, it is possible to recast optimization problem (11) as follows:

\[
\begin{aligned}
\text{minimize} & \quad \text{tr}(X) + \text{tr}(W) \\
\text{subject to} & \quad R > 0, \\
& \quad \text{vec}(\text{Re}\{Z\}) \odot \text{vec}(\text{Re}\{Y\}) \geq 0, \\
& \quad \text{vec}(\text{Im}\{Z\}) \odot \text{vec}(\text{Im}\{Y\}) \geq 0, \\
& \quad \text{rank}(\text{Re}\{T\}) \leq M, \\
& \quad \text{rank}(\text{Im}\{Y\}) \leq N.
\end{aligned} \tag{13}
\]

The above optimization problem is an SDP with three rank constraints on semi-definite matrices, which can be solved iteratively using Algorithm 1 where the sequential problem at iteration \( k \) is formulated as

\[
\begin{aligned}
\text{minimize} & \quad \text{tr}(X_k) + \text{tr}(W_k) + w_k e_k \\
\text{subject to} & \quad R_k > 0, \\
& \quad \text{vec}(\text{Re}\{Z\}) \odot \text{vec}(\text{Re}\{Y\}) \geq 0, \\
& \quad \text{vec}(\text{Im}\{Z\}) \odot \text{vec}(\text{Im}\{Y\}) \geq 0, \\
& \quad \text{rank}(\text{Re}\{T\}) \leq M, \\
& \quad \text{rank}(\text{Im}\{Y\}) \leq N.
\end{aligned} \tag{14}
\]

where \( V_{k-1}, F_{k-1} \) and \( G_{k-1} \) are the eigenvectors corresponding to the \( M \) smallest eigenvalues of \( \begin{bmatrix} T_{k-1} & \Phi_{k-1} \\ \Phi_{k-1} & R_{k-1} \end{bmatrix} \)
and \( \begin{bmatrix} I_N & Y_{k-1} H_k \\ Y_{k-1} & T_{k-1} \end{bmatrix} \) obtained at the previous iteration. Indeed, the optimization problem which should be solved at each step of Algorithm 1 is an SPD that can be solved efficiently. Further, to obtain \( V_0, F_0 \) and \( G_0 \), it is possible to use the relaxed solution of (13) by dropping the rank constraints. It is proved in \[32\] that Algorithm 1 converges to at least a local minimum of (13). Once \( R \) is obtained from Algorithm 1, Co-Array-Based MUSIC (CAB-MUSIC), described in \[9, 14\], can be employed to estimate DoAs.

Algorithm 1: Iterative rank minimization for solving (13)

Input: The problem information \( w_0, t, \epsilon_1 \) and \( \epsilon_2 \).
Output: A local minimum of (13).

Begin

1. initialization: Set \( k = 0 \). Solve the relaxed problem in (13) by dropping the rank constraints to obtain \( V_0, F_0 \) and \( G_0 \).
2. while: \( \epsilon_k \geq \epsilon_1 \) and \(|\text{tr}(X_k) + \text{tr}(W_k) - \text{tr}(X_{k-1}) + \text{tr}(W_{k-1})| \geq \epsilon_2 \).
3. Solve the sequential problem (14).
4. Update \( X_{k-1}, F_{k-1} \) and \( G_{k-1} \) and set \( k = k + 1 \).
5. Update \( w_k = w_{k-1} \times t \).
6. end while

End

5. SIMULATION RESULTS

In this section, we provide some numerical results to compare the performance of the proposed approach to that of the Bussgang-aided method given in \[26\]. In all experiments, each simulated point has been computed by 1000 Monte Carlo repetitions. In addition, it is assumed that the \( K \) independent sources are located at \(-60^\circ + 120^\circ (k - 1)/(K - 1)|k = 0, 1, \ldots, K - 1 \). All sources have an equal power, i.e., \( p_k = p \) for all \( k \), and the SNR is defined as \( 10 \log \frac{p}{\sigma^2} \). Throughout this section, we use a nested array with \( M = 12 \) physical elements and the following geometry:

\[ \mathbf{M}_{\text{nested}} = \{1, 2, 3, 4, 5, 6, 7, 14, 21, 28, 35, 42\} \]  \( (15) \)

Fig. 2 depicts the Root-Mean-Squares-Error (RMSE) for \( \theta_2 \) in degree versus SNR. The number of snapshots is considered to be \( N = 500 \). Given \( M = 12 \), two different scenarios are considered: (a) \( K = 3 < M \), and (b) \( K = 14 > M \). It is observed that the proposed approach presents a better performance compared to the Bussgang-aided method in both cases and shows an RMSE very close to that of DoA estimates obtained from the unquantized data. As an illustrative example, the proposed algorithm is able to improve the quantization loss, defined as \( 10 \log (\text{MSE}_{\text{quantized}}/\text{MSE}_{\text{unquantized}}) \), about 2 to 2.5 dB compared to the Bussgang-aided method at SNR = 5 dB. Fig. 3 plots the RMSE for \( \theta_2 \) in degree versus the number of snapshots for SNR = 3 dB and \( K = 5 \). Fig. 3 indicates that the proposed algorithm performs very closely to the unquantized case when an adequate large number of snapshots is available. However, the performance of DoA estimation via one-bit measurements starts deviating from the performance acquired using the unquantized data when the number of snapshots reduces.

6. CONCLUSION

A novel framework for estimating DoAs from one-bit measurements obtained by an SLA was proposed in this paper. The proposed algorithm provides an alternative approach to estimating the covariance matrix of unquantized data beside Bussgang-aided approach already propounded in the literate. The provided simulation results showed that the proposed algorithm leads to better performance compared to the Bussgang-aided approach.

In the future, it is of considerable interest to investigate the one-bit DoA estimation problem given in this paper when the thresholds at ADCs vary in time as well as across the array elements. Using the varying thresholds is expected to significantly enhance the performance of the proposed algorithm. In addition, it will be interesting to investigate the performance bounds of DoA estimation via SLAs and one-bit measurements. The computational complexity of the proposed techniques also need to be further evaluated.

Fig. 2. RMSE in degree for \( \theta_2 \) versus SNR for a nested array with \( M = 12 \) elements and configuration given in (15), \( N = 500 \), and: (a) \( K = 3 < M \); (b) \( K = 14 > M \).

Fig. 3. RMSE in degree for \( \theta_2 \) versus the number of snapshots for a nested array with \( M = 12 \) elements and configuration given in (15), SNR = 0 dB, and \( K = 5 \).
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