The openness condition for a coadjoint orbit projection of the semidirect product Lie group $M((n, p), \mathbb{R}) \rtimes GL(n, \mathbb{R})$

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Abstract. Let $G(n, p)$ be the semidirect product Lie group of the vector space $K := M((n, p), \mathbb{R})$ of $n \times p$ real matrices and the Lie group $L := GL(n, \mathbb{R})$ of $n \times n$ real invertible matrices. Moreover, we denote by $\mathfrak{g}(n, p)$ the Lie algebra of $G(n, p)$ whose the dual vector space is $\mathfrak{g}^*(p, n)$. In this paper, we study the projection of a coadjoint orbit of $G(n, p)$ from $\mathfrak{g}^*(p, n)$ to $K'$. The main purpose is to give necessary and sufficient conditions for the openness of a coadjoint orbit projection. In this research, we applied the study literature method by studying the openness of a coadjoint orbits. As the main result, we proved the openness condition for coadjoint orbits projections in $K'$. For the future research, the openness of coadjoint orbits of $G(n, p)$ still needs to be investigated more.

1. Introduction

The notion of the orbit method was discussed by Kirillov in the context of harmonic analysis and symplectic geometry [8]. In the orbit method, we arise to a coadjoint orbit notion. It means a Lie group $G$ acts on a space $\mathfrak{g}^*$ which is a dual space to a vector space $\mathfrak{g}$ of the Lie group $G$. The coadjoint orbit gives some advantages in explanation of a unitary dual of $G$ as a topological space. In particular, it give an easier alternative way in construction of a representation of Lie groups. To construct a unitary irreducible representation $\pi$ of the Lie group $G$ corresponding to the orbit method, we can choose a point $\delta_0$ contained in a coadjoint orbit $\Omega_{\delta_0} \subseteq \mathfrak{g}^*$ and we extend a 1-dimensional unitary irreducible representation $\chi_H$ of a Lie subgroup $H$ of a maximal dimension subalgebra $\mathfrak{h}$ which is subordinate to $\delta$ to obtain the representation $\pi$ [7]. In other words, we induce the 1-dimensional irreducible unitary representation $\chi_H$ of a Lie subgroup $H$. In this term, we denote it by $\pi := \text{Ind}_{\mathfrak{h}\times H}^{G\times H}$. Our explanations show the significance of the orbit method in theory of Lie group representations.

In addition, coadjoint orbits of Lie groups shall give a type of its representation. It is well known that the dimension of coadjoint orbits is always even and it has a closed differential 2-form. Another property of coadjoint orbits is openness. This property has implication for representation of Lie groups. For example, each representation of Lie groups of dimension 4 whose coadjoint orbits are open is square-integrable. In such a semidirect product Lie group denoted by $G(n, p) := M((n, p), \mathbb{R}) \rtimes GL(n, \mathbb{R})$, before we observe the openness of coadjoint orbits of $G(n, p)$, it shall be interesting to observe the coadjoint orbit projection of $G(n, p)$. This is very important since the openness of coadjoint orbit projection shall give a necessary condition for the openness of a coadjoint orbits of $G(n, p)$. In the other words, the openness of a coadjoint orbit projection is a necessary condition for the openness of coadjoint orbits of $G(n, p)$. Thus, for future research, our result of the
openness of a coadjoint orbit projection can be applied to consider the openness of coadjoint orbits of \( G(n, p) \). Furthermore, based on this openness of coadjoint orbits, we also can consider whether its representation of the Lie group \( G(n, p) \) is square-integrable or not. In general, we can also apply the orbit method not only for matrix Lie groups but also for any type of a Lie group \( G \).

The research of coadjoint orbits of Lie groups have been done by many researchers. We found about geometrical coadjoint orbit structures, for instance in Mykytyuk’s results [13]. In particular, Mykytyuk’s result proved that a coadjoint orbit \( \Omega \subset g^* \) can be considered as fibre bundles which contain some affine subspace \( \mathcal{E} \subset \Omega \subset g^* \). In addition, we assume that \( G \) is a connected Lie group and \( H \) is a connected normal subgroup of \( G \). Let \( g \) and \( h \) be Lie algebras of \( G \) and \( H \), respectively, whose dual vector spaces are \( \mathfrak{g}^* \) and \( \mathfrak{h}^* \). For \( \sigma \in \mathcal{E} \), let \( \eta \) be an element of \( \mathfrak{h}^* \) as a restriction of \( \sigma \) by \( h \). The stabilizer \( H_\eta \) acts transitively on the affine subspace \( \mathcal{E} \). Furthermore, we also found the quantum of coadjoint orbits in case MD4-groups [16] and topological and geometrical aspects of coadjoint orbits in case of semisimple Lie groups ([3], [4]). Eventhough the research in [4] just concerns to matrix Lie groups but it seems interesting to generalize for general Lie groups.

Corresponding of coadjoint orbits, we have a notion of Frobenius Lie algebras whose coadjoint orbits are open. Namely, a Lie algebra \( \mathfrak{g} \) is called to be Frobenius if its Lie group \( G \) admits an open coadjoint orbit. The recent researches of open coadjoint orbits have been done in [10] and [11]. In those previous researches, we computed all open coadjoint orbits of Lie groups of real Frobenius Lie algebra of dimension 4 and we also obtained that their representations associated to open coadjoint orbits are unitary irreducible. Furthermore, in this context, we proved these representations of Lie groups are square-integrable and we computed their Duflo-Moore operators for these representations explicitly.

In general, let \( G \) be a semidirect product of \( C \cong \mathbb{R}^n \) and \( D \) of connected subgroup of \( \text{GL}(C) \) of invertible isomorphism group on \( C \). We denote it by \( G \cong C \rtimes D \) whose Lie algebra is \( \mathfrak{g} := \mathfrak{c} \oplus \mathfrak{d} \) with \( \mathfrak{g}^* := C^* \oplus \mathfrak{d}^* \) its dual vector space to \( \mathfrak{g} \). Let \( \psi : \mathfrak{g}^* \rightarrow C^* \) be a canonical projection. In this case, we showed that the projection of a coadjoint orbit is open in \( C^* \) [10]. Indeed, we see coadjoint orbit structures in others Lie group types ([1], [6], [17]). These results motivated us to study coadjoint orbit projection in a semidirect product Lie group type.

In this research, we apply our previous results to the semidirect product Lie group \( G(n, p) := \mathbb{M}((n, p), \mathbb{R}) \rtimes \text{GL}(n, \mathbb{R}) \) where \( p \) divides \( n \) and \( \mathbb{M}((n, p), \mathbb{R}) \) is \( n \times n \) real matrices space and \( \text{GL}(n, \mathbb{R}) \) is a Lie group of \( n \times n \) real invertible matrices. Let \( \mathfrak{g}(n, p) := \mathbb{M}((n, p), \mathbb{R}) \oplus \mathfrak{gl}(n, \mathbb{R}) \) be the Lie algebra of \( G \cong \mathbb{M}((n, p), \mathbb{R}) \rtimes \text{GL}(n, \mathbb{R}) \) with dual vector space \( \mathfrak{g}^*(p, n) := \mathbb{M}((p, n), \mathbb{R}) \oplus \mathfrak{gl}(n, \mathbb{R}) \). From now, we denote by \( K \) the vector space \( \mathbb{M}((n, p), \mathbb{R}) \) and by \( L \) the Lie group \( GL(n, \mathbb{R}) \) whose the Lie algebra is \( \mathfrak{g} \). Furthermore, the dual space of \( K \) is denoted by \( K^* := \mathbb{M}((p, n), \mathbb{R}) \). Let \( \mathcal{E} \) be decomposed by a subspace \( I \subset \mathcal{E} \) such that \( \mathcal{E} = I \oplus \mathcal{L}_{\alpha_0} \) where \( \mathcal{L}_{\alpha_0} \) is the stabilizer of \( \mathcal{E} \) at a point \( \alpha_0 \in K \). In general setting, we can investigate the openness of coadjoint orbits of Lie group \( G(n, p) \) by computing them directly. However, the computations are complicated. Therefore, one of alternatives is to compute the openness of coadjoint orbit projection in the space \( K \). Therefore, our result can motivate to prove the openness of a coadjoint orbit of \( G(n, p) \) in the space \( \mathfrak{g}^*(p, n) \) and we can generalize the one to one correspondence of the adjoint and coadjoint orbits for this Lie group \( G(n, p) \) applying the results in [2]. As the main result, our purpose is to show the following proposition and we shall prove it in the next section.

**Proposition 1.** Let \( \mathcal{L}(\alpha_0, \beta_0) \) be element of \( \mathfrak{g}^*(p, n) := K^* \oplus \mathcal{E} \) ( \( \alpha_0 \in K^*, \beta_0 \in \mathcal{E} \)) and let \( \Omega_{\mathcal{L}(\alpha_0, \beta_0)} \) be a coadjoint orbit of the Lie group \( G(n, p) \). Let

\[
\psi : \mathfrak{g}^* \rightarrow K^*
\]

be a canonical projection of \( \mathfrak{g}^* \) onto \( K^* \). The coadjoint orbit \( \psi(\Omega_{\mathcal{L}(\alpha_0, \beta_0)}) \) is open in \( K^* \) if and only if the map defined by

\[
\Sigma : K \ni U \mapsto U\alpha_0 \in \mathcal{E} \subset \mathcal{L}
\]
is bijective.

We mention here that the notion of the Lie group \( G(n, p) \) was introduced by [15]. Rais proved that the Lie algebra \( \mathfrak{g}(n, p) \) is the Frobenius Lie algebra if \( p \) divides \( n \). Indeed, the Lie group \( G(n, p) \) has an open coadjoint orbit. However, in this paper we discuss the openness of coadjoint orbit projection as stated in Proposition 1 above.

Before going to prove our main result, let us review briefly some basic notions that shall be applied in this paper. Particularly, we are going to review some basic notions of coadjoint orbits, open coadjoint orbits, and Frobenius Lie algebras. We resume some basic notions, for instance in [5], [7], [10], and [12].

Let \( G \) be a Lie group whose Lie algebra is \( \mathfrak{g} \). The action of a Lie group \( G \) on itself is realized by a conjugation. Namely, for each \( x \in G \) we have the following automorphism:

\[
C_x : G \ni g \mapsto xgx^{-1} \in G. \tag{3}
\]

The derivation of \( C_g \) at the fixed point comes to the map written by

\[
\text{Ad}(x) : \mathfrak{g} \to \mathfrak{g}. \tag{4}
\]

Let \( \text{gl}(\mathfrak{g}) \) be the Lie algebra of all isomorphism on \( \mathfrak{g} \). The Lie group \( G \) acts on the Lie algebra \( \mathfrak{g} \) by the adjoin representation

\[
G \ni x \mapsto \text{Ad}(x) \in \text{gl}(\mathfrak{g}). \tag{5}
\]

Keeping in mind this action, we have the notion of the adjoint representation for matrix Lie groups \( G \subset L \subset \text{GL}(n, \mathbb{C}) \) whose Lie algebra \( \mathfrak{g} \) is a subspace of \( n \times n \) matrix space \( \text{Mat}(n, \mathbb{R}) \) as the matrix conjugation which is given by

\[
\text{Ad}(x)Y := xYx^{-1} \quad (x \in G, Y \in \mathfrak{g}). \tag{6}
\]

Furthermore, the Lie group \( G \) acts on the dual vector space \( \mathfrak{g}^* \). In this notion, we get the coadjoint representation dual to adjoint representation, namely for \( x \in G \) we have \( \text{Ad}^*(x) := \text{Ad}(x^{-1})^* \). This coadjoint representation is given by the following formula

\[
\langle \text{Ad}^*(x)\delta, X \rangle = \langle \delta, \text{Ad}(x^{-1})X \rangle \quad (x \in G, X \in \mathfrak{g}, \delta \in \mathfrak{g}^*). \tag{7}
\]

We state again for case the matrix Lie group that the coadjoint representation of \( G \) on the dual vector space \( \mathfrak{g}^* \) can be written of the form

\[
\text{Ad}^*(x) : \mathfrak{g}^* \ni \delta \mapsto \text{Proj}(x\delta x^{-1}) \in \mathfrak{g}^* \quad (x \in G \subset L \subset \text{GL}(n, \mathbb{C})) \tag{8}
\]

where \( \text{Proj} \) is the projection of \( \text{Mat}(n, \mathbb{R}) \) onto \( \mathfrak{v} \equiv \text{Mat}(n, \mathbb{R})/\mathfrak{g}^\perp \). Here \( \mathfrak{g}^\perp \) is the orthogonal complement with respect to trace of matrix multiplication defined as a bilinear form on \( \mathfrak{g} \).

Now we shall observe an action of \( \mathfrak{g} \) on its dual vector space \( \mathfrak{g}^* \). We take a derivation on the coadjoint representation to obtain the action of \( \mathfrak{g} \) on its dual vector space \( \mathfrak{g}^* \). We denote it by \( \text{ad}^*(X) \), \( (X \in \mathfrak{g}) \). Namely, we obtain
\[ \langle \text{ad}^* (X) \delta, Y \rangle = \langle \delta, \text{ad}(-X)Y \rangle = -\langle \delta, [X,Y] \rangle \quad (X,Y \in \mathfrak{g}, \delta \in \mathfrak{g}^*). \] (9)

For the matrix Lie groups, the eqs.(8) has the form

\[ \text{ad}^* (X) \delta = \text{Proj}([X, \delta]). \] (10)

**Definition 2** [7]. Let \( G \) be a Lie group whose Lie algebra is \( \mathfrak{g} \) and its dual vector space of \( \mathfrak{g} \) is a space \( \mathfrak{g}^* \). The coadjoint orbit at a point \( \delta \in \mathfrak{g}^* \) is given in the following formula

\[ \Omega_\delta := \{ \text{Ad}^* (x) \delta \quad ; \quad x \in G \} \subset \mathfrak{g}^*. \] (11)

**Definition 3** [7]. Let \( G \) be a Lie group whose Lie algebra is \( \mathfrak{g} \) and its dual vector space of \( \mathfrak{g} \) is a space \( \mathfrak{g}^* \). The stabilizer of \( G \) at a point \( \delta \in \mathfrak{g}^* \) is given in the following formula

\[ G_\delta := \{ x \in G \quad ; \quad \text{Ad}^* (x) \delta = \delta \} \subset G \] (12)

whose the Lie algebra of \( G_\delta \) takes the form as follows:

\[ \mathfrak{g}_\delta := \{ X \in \mathfrak{g} \quad ; \quad \text{ad}^* (X) \delta = 0 \} \subset \mathfrak{g}. \] (13)

**Definition 4** [14]. A Lie algebra \( \mathfrak{g} \) is said to be Frobenius if its Lie group \( G \) has an open coadjoint orbit. In other words, there exists a linear functional \( \delta \in \mathfrak{g}^* \) such that a stabilizer \( G_\delta = \{ 0 \} \).

2. Experimental Method

In this research, we apply the axiomatic method by reviewing some relevant literatures [5], [7], [10], and [12] corresponding to coadjoint orbits. We study the notion of coadjoint orbits and its openness of the Lie group \( G(n, p) \) which is mapped by a projection map in eqs. (1).

![Figure 1](https://example.com/fishbone-diagram.png)

**Figure 1.** Fishbone Diagram to obtain openness condition of coadjoint orbits.
3. Result and Discussion

Before going to prove Proposition 1, let us recall some realizations of the Lie group \( G(n, p) \), the Lie algebra \( g(n, p) \), and its dual vector space \( g^*(p, n) \). We denote by \( g(A, x) \) element in \( G(n, p) \), where \( A \in K \) and \( x \in L \) and we realize in the following matrix

\[
g(A, x) = \begin{bmatrix} x & A \\ 0 & 1 \end{bmatrix}.
\]

whose invers is given by the following matrix

\[
g(A, x)^{-1} = \begin{bmatrix} x^{-1} & -x^{-1}A \\ 0 & 1 \end{bmatrix}.
\]

For elements in \( g \), we denote by \( K(U, X) \) where \( U \in K \) and \( X \in L \) and we realize in the following matrix

\[
K(U, X) = \begin{bmatrix} X & U \\ 0 & 0 \end{bmatrix}.
\]

Furthermore, elements in \( g^* \) are denoted by \( L^*(\alpha, \beta) \) where \( \alpha \in K^* \) and \( \beta \in L^* \) and we also realize it in the matrix of the form

\[
L^*(\alpha, \beta) = \begin{bmatrix} \beta & * \\ \alpha & * \end{bmatrix}.
\]

The formulas of an adjoint representation, coadjoint representation, derivation of coadjoint representation can be found in [15]. For our importance, we rewrite in the following formulas using eqs. (6), (8), and (10)

\[
\text{Ad}(g(A, x))K(U, X) = g(A, x)K(U, X)g(A, x)^{-1} = K(xU - xXx^{-1}A, xXx^{-1}) \in g
\]

where \( A, U \in K, x \in L \), and \( X \in L \). For \( K(U, X) \in g \) and \( L^*(\alpha, \beta) \in g^* \), we have a value of linear functional \( L^*(\alpha, \beta) \) on \( K(U, X) \) by the following formula

\[
\langle L^*(\alpha, \beta), K(U, X) \rangle := \text{trace}(\alpha U) + \text{trace}(\beta X).
\]

Then we obtain the coadjoint representation formula as follows.

\[
\text{Ad}^*(g(A, x))L^*(\alpha, \beta) = \text{Proj}(g(A, x)L^*(\alpha, \beta)g(A, x)^{-1}) = L^*(ax^{-1}, x\beta x^{-1} + A\alpha x^{-1}) \in g^*
\]
where $A \in K, \alpha \in K^*, x \in L$, and $\beta \in \Lambda$.

We obtain the latter formula as follows

$$\text{ad}^*(K(U,X))L^*(\alpha, \beta) = \text{Proj}([L^*(\alpha, \beta), K(U,X)]) = L^*(-\alpha X, [X, \beta] + U\alpha) \in \mathfrak{g}^*$$  \hspace{1cm} (20)

**Remark 5**[9]. Let $\delta \in \mathfrak{g}^*$. A coadjoint orbit $\Omega_\delta$ is diffeomorphic to a quotient space $G/G_\delta$.

**Lemma 6**[10]. Let $G$ be a Lie group of semidirect product $C \rtimes D$ with $V$ is isomorphic to $\mathbb{R}^n$ and $D$ is a connected subgroup of $GL(C)$. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ its dual. For $\delta_0 := \delta(p_0, \alpha_0) \in \mathfrak{g}^*$ ($p_0 \in C^*$ and $\alpha_0 \in \Omega^*$), the coadjoint orbit $\Omega_{\delta_0}$ is open if and only if the following map

$$\psi : \mathfrak{g} \ni K(U,X) \mapsto \text{ad}^*(K(U,X))\delta_0 \in \mathfrak{g}^*$$  \hspace{1cm} (21)

is bijective.

**Proof of Proposition 1.** Let $I$ be a subspace of $\mathfrak{L}$ and for $\alpha_0 \in K^*$ we have the stabilizer $\mathfrak{L}_{\alpha_0}$ such that

$$\mathfrak{L} := I \oplus \mathfrak{L}_{\alpha_0}.$$  \hspace{1cm} (22)

We first assume that the coadjoint orbit $\psi(\Omega_{L^*(\alpha_0, \beta_0)})$ is open in $K^*$ where $\psi$ is the projection map defined by eqs. (1). We shall show that the map $\Sigma$ defined by eqs. (2) is bijective. By direct computations using eqs. (1) and (19), then we obtain the following formula

$$\psi(\Omega_{L^*(\alpha_0, \beta_0)}) = \psi(\text{Ad}^*(g(A,x))L^*(\alpha_0, \beta_0)) = \psi(\{L^*(\alpha_0 x^{-1}, x\beta_0 x^{-1} + A\alpha_0 x^{-1}) ; A \in K, \quad x \in L\}) = \{L^*(\alpha_0 x^{-1}, 0) ; \quad x \in L\} \cong K/K_{\alpha_0}. \hspace{1cm} (23)$$

Since $\psi(\Omega_{L^*(\alpha_0, \beta_0)})$ is open in $K^*$ and $\psi(\Omega_{L^*(\alpha_0, \beta_0)}) \cong K/K_{\alpha_0}$, then we have the dimension equality as follows.

$$\dim K^* = \dim \mathfrak{L} - \dim \mathfrak{L}_{\alpha_0} \hspace{1cm} (24)$$

and by eqs. (22) we also have

$$\dim \mathfrak{L} = \dim I + \dim \mathfrak{L}_{\alpha_0}.$$  \hspace{1cm} (25)
Thus, \( \dim K^* = \dim \mathfrak{l} \).

Now we define a map given by

\[
\tau : \mathfrak{l} \ni Y \mapsto -\alpha_0 Y \in K^*.
\] (26)

This map \( \tau \) defined in eqs. (26) is bijective. To see this, let \( Y \in \mathfrak{l} \) with \( \tau(Y) = -\alpha_0 Y = 0 \). Then \( Y \in \mathfrak{g}_{\alpha_0} \). Therefore, \( Y \in \mathfrak{l} \cap \mathfrak{g}_{\alpha_0} \). But, since \( \mathfrak{l} \cap \mathfrak{g}_{\alpha_0} = \{0\} \) then \( Y = 0 \). Thus, the map \( \tau \) is injective. Furthermore, since \( \dim K^* = \dim \mathfrak{l} \), then the map \( \tau \) is bijective.

Now we shall show that \( \Sigma \) is injective. Let \( U \) be arbitrary element of \( K \) with \( \Sigma(U) = 0 \) that is \( U\alpha_0 = 0 \). For every \( X \in \mathfrak{l} \) and by eqs. (18), we obtain the following formula

\[
\langle U\alpha_0, X \rangle = \langle -\alpha_0 X, U \rangle = 0.
\] (27)

This implies \( U = 0 \). Therefore, \( \Sigma \) is injective. Furthermore, since \( \dim K = \dim \mathfrak{l}^* \), then \( \Sigma \) is bijective.

Now we assume that \( \Sigma \) is bijective. We shall show that \( \psi(\Omega_{L^*(\alpha_0\beta_0)}) \) is open in \( K^* \). In other words, we shall show that the map defined in eqs.(26) is bijective. Since \( \Sigma \) is bijective then \( \dim K = \dim \mathfrak{l}^* \). Let \( Y \in \mathfrak{l} \) with \( \tau(Y) = -\alpha_0 Y = 0 \). For every \( U \in K \) we have

\[
\langle -\alpha_0 Y, U \rangle = \langle U\alpha_0, Y \rangle = 0.
\] (28)

This implies that \( \tau \) is injective. Furthermore, since \( \dim \mathfrak{l} = \dim K^* \) then \( \tau \) is bijective. Thus, by Lemma 6 the coadjoint orbit \( \psi(\Omega_{L^*(\alpha_0\beta_0)}) \) is open in \( K^* \) as required.

As discussion, we mentioned before that our result motivate to investigate further of the openness of a coadjoint orbit \( \Omega_{L^*(\alpha_0\beta_0)} \) in \( g^*(p,n) \). For the future research, we consider the following conjecture stated as follows:

**Conjecture 7.** A coadjoint orbit \( \Omega_{L^*(\alpha_0\beta_0)} \) of the Lie group \( G(n,p) \) is open in the space \( g^*(p,n) \) if and only if it satisfies Proposition 1 and a stabilizer \( \mathfrak{g}_{\alpha_0} \) is trivial.

Furthermore, motivated by the result [2], we also consider the following second conjecture

**Conjecture 8.** In the Lie group \( G(n,p) \), there is a bijective map of adjoint orbit of \( G(n,p) \) and its coadjoint orbit.
4. Conclusion
We considered necessary and sufficient conditions for the coadjoint orbit projection \( \psi(\Omega_{L^*(\alpha_0, \beta_0)}) \) to be open in \( K^* \). In other words, we proved that for any element \( L^*(\alpha_0, \beta_0) \in g^* := K^* \oplus L \) where \( \alpha_0 \in K^* \), \( \beta_0 \in L \), the coadjoint orbit \( \psi(\Omega_{L^*(\alpha_0, \beta_0)}) \) is open in \( K^* \) if and only if the map \( \Sigma : K^* \ni U \mapsto U\alpha_0 \in L \) is bijective where \( \psi : g^* \to K^* \) is a canonical projection of \( g^* \) onto \( K^* \). Our result is important for the future research since the coadjoint orbit projection can motivate to study the openness of a coadjoint orbit \( \Omega_{L^*(\alpha_0, \beta_0)} \) in \( g^* \) as stated in Conjecture 7. Furthermore, we can also investigate a bijection between an adjoint orbit and a coadjoint orbit as stated in Conjecture 8.

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