Gluing Delaunay ends to minimal $n$-noids using the DPW method

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Abstract

we construct constant mean curvature surfaces in euclidean space by gluing $n$ half Delaunay surfaces to a non-degenerate minimal $n$-noid, using the DPW method.

1 Introduction

In [3], Dorfmeister, Pedit and Wu have shown that surfaces with non-zero constant mean curvature (CMC for short) in euclidean space admit a Weierstrass-type representation, which means that they can be represented in terms of holomorphic data. This representation is now called the DPW method. In [18], we used the DPW method to construct CMC $n$-noids: genus zero, CMC surfaces with $n$ Delaunay type ends. These $n$-noids can be described as a unit sphere with $n$ half Delaunay surfaces with small necksizes attached at prescribed points. They had already been constructed by Kapouleas in [11] using PDE methods.

In the case $n = 3$, Alexandrov-embedded CMC trinoids have been classified by Große Brauckman, Kusner and Sullivan in [9]. In particular, equilateral CMC trinoids form a 1-parameter family, parametrized on an open interval. On one end, equilateral trinoids degenerate like the examples described above: they look like a sphere with 3 half Delaunay surfaces with small necksizes attached at the vertices of a spherical equilateral triangle. On the other end, equilateral trinoids limit, after suitable blow-up, to a minimal 3-noid: a genus zero minimal surface with 3 catenoidal ends (see Fig. 1).

It seems natural to ask if one can generalize this observation and construct CMC $n$-noids by gluing half Delaunay surfaces with small necksizes to a minimal $n$-noid. This is indeed the case, and has been done by Mazzeo and Pacard in [14] using...
PDE methods. In this paper, we propose a quite simple and natural DPW potential to construct these examples. We prove:

**Theorem 1** Let $n \geq 3$ and let $M_0$ be a non-degenerate minimal $n$-noid. There exists a smooth family of CMC surfaces $(M_t)_{0 < |t| < \epsilon}$ with the following properties:

1. $M_t$ has genus zero and $n$ Delaunay ends.
2. $\frac{1}{t} M_t$ converges to $M_0$ as $t \to 0$.
3. If $M_0$ is Alexandrov-embedded, all ends of $M_t$ are of unduloid type if $t > 0$ and of nodoid type if $t < 0$. Moreover, $M_t$ is Alexandrov-embedded if $t > 0$.

Non-degeneracy of a minimal $n$-noid will be defined in Sect. 2. The two surfaces $M_t$ and $M_{-t}$ are geometrically different: if $M_t$ has an end of unduloid type then the corresponding end of $M_{-t}$ is of nodoid type. See Proposition 6 for more details. Of course, a minimal $n$-noid is never embedded if $n \geq 3$ so the surfaces $M_t$ are not embedded. Alexandrov-embedded minimal $n$-noids whose ends have coplanar axes have been classified by Cosin and Ros in [2], and Alexandrov-embedded CMC $n$-noids whose ends have coplanar axes have been classified by Große-Brauckmann, Kusner and Sullivan in [10].

As already said, these surfaces have already been constructed in [14]. Our motivation to construct them with the DPW method is to answer the following questions:

1. How can we produce a DPW potential from the Weierstrass data $(g, \omega)$ of the minimal $n$-noid $M_0$?
2. How can we prove, with the DPW method, that $\frac{1}{t} M_t$ converges to $M_0$?

The answer to Question 2 is Theorem 4 in Sect. 4, a general blow-up result in the context of the DPW method. In [19], we use the DPW method to construct higher genus CMC surfaces with small necks. Theorem 4 is used to ensure that the necks have asymptotically catenoidal shape.

**Remark 1** As the referee pointed out, the relation between minimal surfaces and CMC-1 surfaces in the DPW framework has already been investigated by Brander and Dorfmeister [1]. In that paper, the authors propose a DPW potential from the Weierstrass data of a minimal surface $M$. The Monodromy Problem is not addressed, however, so the resulting CMC-1 surfaces do not close, unless $M$ is simply connected.
I would like to thank the referee for his helpful comments and for providing the references [7,8].

2 Non-degenerate minimal $n$-noids

A minimal $n$-noid is a complete, immersed minimal surface in $\mathbb{R}^3$ with genus zero and $n$ catenoidal ends. Let $M_0$ be a minimal $n$-noid and $(\Sigma, g, \omega)$ its Weierstrass data. This means that $M_0$ is parametrized on $\Sigma$ by the Weierstrass Representation formula:

$$
\psi(z) = \text{Re} \int_{z_0}^{z} \left( \frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right)
$$

Without loss of generality, we can assume that $\Sigma = \mathbb{C} \cup \{ \infty \} \setminus \{ p_1, \ldots, p_n \}$, where $p_1, \ldots, p_n$ are complex numbers and $g \neq 0, \infty$ at $p_1, \ldots, p_n$ (by rotating $M_0$ if necessary). Then $\omega$ needs a double pole at $p_1, \ldots, p_n$ so has $2n - 2$ zeros, counting multiplicity. Since $\omega$ needs a zero at each pole of $g$, with twice the multiplicity, it follows that $g$ has $n - 1$ poles so has degree $n - 1$. Hence we may write

$$
g = \frac{A(z)}{B(z)} \quad \text{and} \quad \omega = \frac{B(z)^2}{\prod_{i=1}^{n}(z - p_i)^2} dz
$$

where

$$
A(z) = \sum_{i=1}^{n} a_i z^{n-i} \quad \text{and} \quad B(z) = \sum_{i=1}^{n} b_i z^{n-i}.
$$

We are going to deform this Weierstrass data, so we see $a_i, b_i$ and $p_i$ for $1 \leq i \leq n$ as complex parameters. We denote by $x \in \mathbb{C}^{3n}$ the vector of these parameters, and by $x_0$ the value of the parameters corresponding to the minimal $n$-noid $M_0$.

Let $\gamma_i$ be the homology class of a small circle centered at $p_i$ and define the following periods for $1 \leq i \leq n$ and $0 \leq k \leq 2$, depending on the parameter vector $x \in \mathbb{C}^{3n}$:

$$
P_{i,k}(x) = \int_{\gamma_i} g^k \omega
$$

$$
P_i(x) = (P_{i,0}(x), P_{i,1}(x), P_{i,2}(x)) \in \mathbb{C}^3
$$

$$
Q_i(x) = \int_{\gamma_i} \left( \frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right) \in \mathbb{C}^3.
$$

Then

$$
Q_i(x) = \left( \frac{1}{2}(P_{i,0}(x) - P_{i,2}(x)), \frac{i}{2}(P_{i,0}(x) + P_{i,2}(x)), P_{i,1}(x) \right).
$$
The components of $Q_i(x_0)$ are imaginary because the Period Problem is solved for $M_0$. This gives

$$P_{i,2}(x_0) = P_{i,0}(x_0) \quad \text{and} \quad P_{i,1}(x_0) \in i\mathbb{R} \quad (3)$$

Moreover, $\text{Im}(Q_i(x_0)) = -\phi_i$ where $\phi_i$ is the flux vector of $M_0$ at the end $p_i$. By the Residue Theorem, we have for all $x$ in a neighborhood of $x_0$:

$$\sum_{i=1}^{n} P_i(x) = 0$$

Let $P = (P_1, \ldots, P_{n-1})$ and $Q = (Q_1, \ldots, Q_{n-1})$.

**Definition 1** $M_0$ is non-degenerate if the differential of $P$ (or equivalently, $Q$) at $x_0$ has complex rank $3n - 3$.

**Remark 2** If $n \geq 3$, we may (using Möbius transformations of the sphere) fix the value of three points, say $p_1, p_2, p_3$. Then “non-degenerate” means that the differential of $P$ with respect to the remaining parameters is an isomorphism of $\mathbb{C}^{3n-3}$.

This notion is related to another standard notion of non-degeneracy:

**Definition 2** $M_0$ is non-degenerate if its space of bounded Jacobi fields has (real) dimension 3.

**Theorem 2** If $M_0$ is non-degenerate in the sense of Definition 2, then $M_0$ is non-degenerate in the sense of Definition 1.

**Proof** Assume $M_0$ is non-degenerate in the sense of Definition 2. Then in a neighborhood of $M_0$, the space $\mathcal{M}$ of minimal $n$-noids (up to translation) is a smooth manifold of dimension $3n - 3$ by a standard application of the Implicit Function Theorem. Moreover, if we write $\phi_i \in \mathbb{R}^3$ for the flux vector at the $i$-th end, then the map $\phi = (\phi_1, \ldots, \phi_n)$ provides a local diffeomorphism between $\mathcal{M}$ and the space $V$ of vectors $v = (v_1, \ldots, v_n) \in (\mathbb{R}^3)^n$ such that $\sum_{i=1}^{n} v_i = 0$. (All this is proved in Section 4 of [2] in the case where all ends are coplanar. The argument goes through in the general case.) Hence given a vector $v \in V$, there exists a deformation $M_t$ of $M_0$ such that $M_t \in \mathcal{M}$ and $\frac{d}{dt} \phi(M_t)|_{t=0} = v$. We may write the Weierstrass data of $M_t$ as above and obtain a set of parameters $x(t)$, depending smoothly on $t$, such that $x(0) = x_0$. Then $dQ(x_0) \cdot x'(0) = -i v$. Since $Q$ is holomorphic, its differential is complex-linear so $dQ(x_0)$ has complex rank equal to $\dim V = 3n - 3$. 

If all ends of $M_0$ have coplanar axes, then $M_0$ is non-degenerate in the sense of Definition 2 by Proposition 2 in [2]. In particular, the (most symmetric) $n$-noids of Jorge-Meeks are non-degenerate. This implies that generic $n$-noids in the component of the Jorge-Meeks $n$-noid are non-degenerate.
3 Background

In this section, we recall standard notations and results used in the DPW method. We work in the “untwisted” setting.

3.1 Loop groups

A loop is a smooth map from the unit circle $S^1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$ to a matrix group. The circle variable is denoted $\lambda$ and called the spectral parameter. The unit disk is denoted $\mathbb{D}$. For $\rho > 1$, we denote $D_\rho$ the disk $|\lambda| < \rho$, $D^*_\rho = D_\rho \setminus \{0\}$ and $A_\rho$ the annulus $\frac{1}{\rho} < |\lambda| < \rho$.

- If $G$ is a matrix Lie group (or Lie algebra), $\Lambda G$ denotes the group (or algebra) of smooth maps $\Phi : S^1 \to G$.
- $\Lambda_{+}SL(2, \mathbb{C}) \subset \Lambda SL(2, \mathbb{C})$ is the subgroup of maps $B$ which extend holomorphically to $\mathbb{D}$ with $B(0)$ upper triangular.
- $\Lambda_{+}^{\mathbb{R}}SL(2, \mathbb{C}) \subset \Lambda_{+}SL(2, \mathbb{C})$ is the subgroup of maps $B$ such that $B(0)$ has positive entries on the diagonal.

**Theorem 3** (Iwasawa decomposition) The multiplication $\Lambda SU(2) \times \Lambda_{+}^{\mathbb{R}}SL(2, \mathbb{C}) \to \Lambda SL(2, \mathbb{C})$ is a diffeomorphism. The unique splitting of an element $\Phi \in \Lambda SL(2, \mathbb{C})$ as $\Phi = FB$ with $F \in \Lambda SU(2)$ and $B \in \Lambda_{+}^{\mathbb{R}}SL(2, \mathbb{C})$ is called Iwasawa decomposition. $F$ is called the unitary factor of $\Phi$ and denoted $\text{Uni}(\Phi)$. $B$ is called the positive factor and denoted $\text{Pos}(\Phi)$.

3.2 The matrix model of $\mathbb{R}^3$

In the DPW method, one identifies $\mathbb{R}^3$ with the Lie algebra $\mathfrak{su}(2)$ by

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \leftrightarrow X = -i \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathfrak{su}(2).$$

We have $\det(X) = ||x||^2$. The group $SU(2)$ acts as linear isometries on $\mathfrak{su}(2)$ by conjugation: $H \cdot X = HXH^{-1}$.

3.3 The DPW method

The input data for the DPW method is a quadruple $(\Sigma, \xi, z_0, \phi_0)$ where:

- $\Sigma$ is a Riemann surface.
- $\xi = \xi(z, \lambda)$ is a $\Lambda sl(2, \mathbb{C})$-valued holomorphic 1-form on $\Sigma$ called the DPW potential. More precisely,

$$\xi = \begin{pmatrix} \alpha & \lambda^{-1} \beta \\ \gamma & -\alpha \end{pmatrix} \quad (4)$$

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where $\alpha(z, \lambda)$, $\beta(z, \lambda)$, $\gamma(z, \lambda)$ are holomorphic 1-forms on $\Sigma$ with respect to the $z$ variable, and are holomorphic with respect to $\lambda$ in the disk $\mathbb{D}_\rho$ for some $\rho > 1$.

- $z_0 \in \Sigma$ is a base point.
- $\phi_0 \in \Lambda SL(2, \mathbb{C})$ is an initial condition.

Given this data, the DPW method is the following procedure.

- Let $\tilde{\Sigma}$ be the universal cover of $\Sigma$ and $\tilde{z}_0 \in \tilde{\Sigma}$ be an arbitrary element in the fiber of $z_0$. Solve the Cauchy Problem on $\tilde{\Sigma}$:

\[
\begin{cases}
  d\Phi(z, \lambda) = \Phi(z, \lambda) \xi(z, \lambda) \\
  \Phi(\tilde{z}_0, \lambda) = \phi_0(\lambda)
\end{cases}
\]

(5)

to obtain a solution $\Phi: \tilde{\Sigma} \rightarrow \Lambda SL(2, \mathbb{C})$.

- Compute the Iwasawa decomposition $(F(z, \cdot), B(z, \cdot))$ of $\Phi(z, \cdot)$.
- Define $f: \tilde{\Sigma} \rightarrow su(2) \sim \mathbb{R}^3$ by the Sym-Bobenko formula:

\[
f(z) = -2i \frac{\partial F}{\partial \lambda}(z, 1) F(z, 1)^{-1} =: \text{Sym}(F(z, \cdot)).
\]

(6)

Then $f$ is a CMC-1 (branched) conformal immersion. $f$ is regular at $z$ (meaning unbranched) if and only if $\beta(z, 0) \neq 0$. Its Gauss map is given by

\[
N(z) = -i F(z, 1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F(z, 1)^{-1} =: \text{Nor}(F(z, \cdot)).
\]

(7)

The DPW method actually constructs a moving frame for $f$. The differential of $f$ is given by

\[
df(z) = 2i B_{11}(z, 0)^2 F(z, 1) \begin{pmatrix} 0 & \beta(z, 0) \\ \beta(z, 0) & 0 \end{pmatrix} F(z, 1)^{-1}.
\]

(8)

Equation (8) can also be obtained by differentiation of the Sym-Bobenko formula (6).

**Remark 3** In [18], I have opposite signs in Eqs. (6) and (7). This is unfortunate because it makes the basis $(f_x, f_y, N)$ negatively oriented. Equation (6) is the right formula, which one obtains by untwisting the standard Sym-Bobenko formula in the twisted case. See [13] or [17].

### 3.4 The monodromy problem

Assume that $\Sigma$ is not simply connected so its universal cover $\tilde{\Sigma}$ is not trivial. Let $\text{Deck}(\tilde{\Sigma}/\Sigma)$ be the group of fiber-preserving diffeomorphisms of $\tilde{\Sigma}$. Let $\Phi$ be the solution of the Cauchy Problem (5). For $\gamma \in \text{Deck}(\tilde{\Sigma}/\Sigma)$, let

\[
\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\gamma(z), \lambda) \Phi(z, \lambda)^{-1}
\]
be the monodromy of $\Phi$ with respect to $\gamma$ (which is independent of $z \in \tilde{\Sigma}$). The standard condition which ensures that the immersion $f$ descends to a well defined immersion on $\Sigma$ is the following system of equations, called the Monodromy Problem.

$$\forall \gamma \in \text{Deck}(\tilde{\Sigma}/\Sigma) \quad \begin{cases} \mathcal{M}_\gamma(\Phi) \in \Lambda SU(2) & (i) \\ \mathcal{M}_\gamma(\Phi)(1) = \pm I_2 & (ii) \\ \frac{\partial \mathcal{M}_\gamma(\Phi)}{\partial \lambda}(1) = 0 & (iii) \end{cases}$$ (9)

One can identify $\text{Deck}(\tilde{\Sigma}/\Sigma)$ with the fundamental group $\pi_1(\Sigma, z_0)$ (see for example Theorem 5.6 in [5]), so we will in general see $\gamma$ as an element of $\pi_1(\Sigma, z_0)$. Under this identification, the monodromy of $\Phi$ with respect to $\gamma \in \pi_1(\Sigma, z_0)$ is given by

$$\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\tilde{\gamma}(1), \lambda)\Phi(\tilde{\gamma}(0), \lambda)^{-1}$$

where $\tilde{\gamma} : [0, 1] \to \tilde{\Sigma}$ is the lift of $\gamma$ such that $\tilde{\gamma}(0) = \tilde{z}_0$.

3.5 Gauging

**Definition 3** A gauge on $\Sigma$ is a map $G : \Sigma \to \Lambda_+ SL(2, \mathbb{C})$ such that $G(z, \lambda)$ depends holomorphically on $z \in \Sigma$ and $\lambda \in \mathbb{D}_\rho$ for some $\rho > 1$.

Let $\Phi$ be a solution of $d\Phi = \Phi \xi$ and $G$ be a gauge. Let $\hat{\Phi} = \Phi \times G$. Then $\hat{\Phi}$ and $\Phi$ define the same immersion $f$. This is called “gauging”. The gauged potential is

$$\hat{\xi} = \hat{\Phi}^{-1}d\hat{\Phi} = G^{-1}\xi G + G^{-1}dG$$

and will be denoted $\xi \cdot G$, the dot denoting the action of the gauge group on the potential.

3.6 Functional spaces

We need to introduce a functional space for functions on the unit circle. We need that space to be a Banach algebra, and functions in that space should extend holomorphically to a neighborhood of the unit circle. The following choice is natural. We decompose a smooth function $f : \mathbb{S}^1 \to \mathbb{C}$ in Fourier series

$$f(\lambda) = \sum_{i \in \mathbb{Z}} f_i \lambda^i$$

Fix some $\rho > 1$ and define

$$\|f\| = \sum_{i \in \mathbb{Z}} |f_i| \rho^{|i|}$$
Let $\mathcal{W}_\rho$ be the space of functions $f$ with finite norm. This is a Banach algebra, owing to the fact that the weight $\rho^{\pm 1}$ is submultiplicative (see Section 4 in [8]). Functions in $\mathcal{W}_\rho$ extend holomorphically to the annulus $A_\rho$.

We define $\mathcal{W}_\rho^{\geq 0}$, $\mathcal{W}_\rho^{> 0}$, $\mathcal{W}_\rho^{\leq 0}$ and $\mathcal{W}_\rho^{< 0}$ as the subspaces of functions $f$ such that $f_i = 0$ for $i < 0$, $i \leq 0$, $i > 0$ and $i \geq 0$, respectively. Functions in $\mathcal{W}_\rho^{\geq 0}$ extend holomorphically to the disk $D_\rho$ and satisfy $|f(\lambda)| \leq \|f\|$ for all $\lambda \in D_\rho$. We write $\mathcal{W}^0 \sim \mathbb{C}$ for the subspace of constant functions, so we have a direct sum $\mathcal{W}_\rho = \mathcal{W}_\rho^{\geq 0} \oplus \mathcal{W}_\rho^{> 0} \oplus \mathcal{W}_\rho^{< 0}$. (The Banach algebra $\mathcal{W}_\rho$ is said to be decomposable, see [7] page 70.) A function $f$ will be decomposed as $f = f^- + f^0 + f^+$ with $(f^-, f^0, f^+) \in \mathcal{W}_\rho^{\geq 0} \times \mathcal{W}_\rho^{> 0} \times \mathcal{W}_\rho^{< 0}$.

We define the star operator by

$$f^*(\lambda) = \overline{f(1/\lambda)} = \sum_{i \in \mathbb{Z}} f_{-i} \lambda^i$$

The involution $f \mapsto f^*$ exchanges $\mathcal{W}_\rho^{\geq 0}$ and $\mathcal{W}_\rho^{\leq 0}$. We have $\lambda^* = \lambda^{-1}$ and $c^* = \overline{c}$ if $c$ is a constant. A function $f$ is real on the unit circle if and only if $f = f^*$. We extend the star involution to loops by $M^*(\lambda) = M(1/\lambda)^T$, so a loop $F$ is unitary if and only if $F^* F = I_2$.

If $L$ is a loop group, we denote $L_\rho \subset L$ the subgroup of loops whose entries are in $\mathcal{W}_\rho$. The loop groups $\Lambda SL(2, \mathbb{C})_\rho$, $\Lambda SU(2)_\rho$ and $\Lambda_{\mathbb{R}^+} SL(2, \mathbb{C})_\rho$ are Banach Lie groups, moreover:

**Proposition 1** Iwasawa decomposition restricts to an analytic diffeomorphism between the Banach Lie groups $\Lambda SL(2, \mathbb{C})_\rho$ and $\Lambda SU(2)_\rho \times \Lambda_{\mathbb{R}^+} SL(2, \mathbb{C})_\rho$.

**Proof** let $\Phi \in \Lambda SL(2, \mathbb{C})_\rho$ and let $(F, B)$ be its Iwasawa decomposition. We want to prove that $F$ and $B$ have entries in $\mathcal{W}_\rho$. Since $F$ is unitary, we have $\Phi^* \Phi = B^* B$. Now $B^*$ has only non-positive powers of $\lambda$, so $(B^*, B)$ is a Birkhoff-type decomposition of $M = \Phi^* \Phi$. Since $M \in \Lambda SL(2, \mathbb{C})_\rho$, it is known that both factors of its Birkhoff decomposition have entries in $\mathcal{W}_\rho$, owing to the fact that $\mathcal{W}_\rho$ is decomposable (see Theorem 1.4 in [7]). So $B \in \Lambda_{\mathbb{R}^+} SL(2, \mathbb{C})_\rho$ and $F \in \Lambda SU(2)_\rho$ follows.

Let $\Lambda sl(2, \mathbb{C})_\rho$, $\Lambda su(2)_\rho$ and $\Lambda_{\mathbb{R}^+} sl(2, \mathbb{C})_\rho$ be the Banach Lie algebras of respectively $\Lambda SL(2, \mathbb{C})_\rho$, $\Lambda SU(2)_\rho$ and $\Lambda_{\mathbb{R}^+} SL(2, \mathbb{C})_\rho$. The following decomposition is standard (the factors can be written explicitly in term of Fourier coefficients):

$$\Lambda sl(2, \mathbb{C})_\rho = \Lambda su(2)_\rho \oplus \Lambda_{\mathbb{R}^+} sl(2, \mathbb{C})_\rho.$$ 

By the inverse mapping theorem, the multiplication $\Lambda SU(2)_\rho \times \Lambda_{\mathbb{R}^+} SL(2, \mathbb{C})_\rho \rightarrow \Lambda SL(2, \mathbb{C})_\rho$ is an analytic local diffeomorphism in a neighborhood of $(I_2, I_2)$, and in a neighborhood of any element $(F, B)$ using left multiplication by $F$ and right multiplication by $B$. Since we already know the multiplication is bijective, it is a global diffeomorphism. \hfill \Box
4 A blow-up result

In this section, we consider a one-parameter family of DPW potential \( \xi_t \) with solution \( \Phi_t \) and assume that \( \Phi_0(z, \lambda) \) is independent of \( \lambda \). Then its unitary part \( F_0(z, \lambda) \) is independent of \( \lambda \). The Sym Bobenko formula yields that \( f_0 \equiv 0 \), so the family \( f_t \) collapses to the origin as \( t = 0 \). The following theorem says that the blow-up \( \frac{1}{t} f_t \) converges to a minimal surface whose Weierstrass data is explicitly computed.

**Theorem 4** Let \( \Sigma \) be a Riemann surface, \( (\xi_t)_{t \in I} \) a family of DPW potentials on \( \Sigma \) and \( (\Phi_t)_{t \in I} \) a family of solutions of \( d\Phi_t = \Phi_t \xi_t \) on the universal cover \( \tilde{\Sigma} \) of \( \Sigma \), where \( I \subset \mathbb{R} \) is a neighborhood of 0. Fix a base point \( z_0 \in \tilde{\Sigma} \). Assume that

1. \((t, z) \mapsto \xi_t(z, \cdot) \) and \( t \mapsto \Phi_t(z_0, \cdot) \) are \( C^1 \) maps into \( \text{SL}(2, \mathbb{C})_\rho \) and \( \text{ASL}(2, \mathbb{C})_\rho \), respectively.
2. For all \( t \in I \), \( \Phi_t \) solves the Monodromy Problem (9).
3. \( \Phi_0(z, \lambda) \) is independent of \( \lambda \):

\[
\Phi_0(z, \lambda) = \left( \begin{array}{cc} a(z) & b(z) \\ c(z) & d(z) \end{array} \right)
\]

Let \( f_t = \text{Sym(Uni}(\Phi_t)) : \Sigma \to \mathbb{R}^3 \) be the CMC-1 immersion given by the DPW method. Then

\[
\lim_{t \to 0} \frac{1}{t} f_t(z) = \psi(z)
\]

where \( \psi : \Sigma \to \mathbb{R}^3 \) is a (possibly branched) minimal immersion with the following Weierstrass data:

\[
g(z) = \frac{-a(z)}{c(z)} \quad \text{and} \quad \omega = 4c(z)^2 \partial_t \xi_t|_{t=0}^{(-1)}.
\]

The limit is for the uniform \( C^1 \) convergence on compact subsets of \( \Sigma \).

Here \( \xi_t^{(-1)} \) denotes the coefficient of \( \lambda^{-1} \) in the upper right entry of \( \xi_t \). In case \( \omega = 0 \), the minimal immersion degenerates into a point and \( \psi \) is constant.

**Proof** by standard ODE theory, \((t, z) \mapsto \Phi_t(z, \cdot) \) is a \( C^1 \) map into \( \text{ASL}(2, \mathbb{C})_\rho \). Let \((F_t, B_t)\) be the Iwasawa decomposition of \( \Phi_t \). By Proposition 1, \((t, z) \mapsto F_t(z, \cdot) \) and \((t, z) \mapsto B_t(z, \cdot) \) are real analytic maps into \( \text{SU}(2) \rho \) and \( \text{ASL}(2, \mathbb{C})_\rho \), respectively. At \( t = 0 \), \( \Phi_0 \) is constant with respect to \( \lambda \), so its Iwasawa decomposition is the standard \( QR \) decomposition:

\[
F_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \left( \begin{array}{cc} a & -c \\ c & a \end{array} \right) \quad B_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \left( \begin{array}{cc} |a|^2 + |c|^2 & ab + \bar{c}d \\ 0 & 1 \end{array} \right).
\]
The Sym-Bobenko formula (6) yields \( f_0 = 0 \). Let \( \mu_t = B^0_{t;11} \) and \( \beta_t = \xi_t^{(-1)} \). By Eq. (8), we have

\[
d f_t(z) = 2i \mu_t(z)^2 F_t(z, 1) \begin{pmatrix} 0 & \beta_t(z) \\ \beta_t(z) & 0 \end{pmatrix} F_t(z, 1)^{-1}.
\]

Hence \((t, z) \mapsto d f_t(z)\) is a \( C^1 \) map. At \( t = 0 \), \( \xi_0 \) is constant with respect to \( \lambda \), so \( \beta_0 = 0 \). Define \( \tilde{f}_t(z) = \frac{1}{t} f_t(z) \) for \( t \neq 0 \). Then \( d \tilde{f}_t(z) \) extends at \( t = 0 \), as a continuous function of \((t, z)\), by

\[
d \tilde{f}_0 = 2i \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \begin{pmatrix} 0 & \beta' \\ \beta' & 0 \end{pmatrix} \begin{pmatrix} a & -c \\ c & a \end{pmatrix}
\]

where \( \beta' = \frac{d}{dt} \beta_t|_{t=0} \). In euclidean coordinates, this gives

\[
d \tilde{f}_0 = 4 \Re \left[ \frac{1}{2} (c^2 - a^2) \beta', \frac{i}{2} (c^2 + a^2) \beta', -ac \beta' \right].
\]

Writing \( g = \frac{-ac}{c} \) and \( \omega = 4c^2 \beta' \), we obtain

\[
\tilde{f}_0(z) = \tilde{f}_0(z_0) + \Re \int_{z_0}^{z} \left[ \frac{1}{2} (1 - g^2) \omega, \frac{i}{2} (1 + g^2) \omega, g \omega \right]
\]

and we see that \( \tilde{f}_0 \) is a minimal surface with Weierstrass data \((g, \omega)\). The last statement of Theorem 4 comes from the fact that \( d \tilde{f}_t \) converges uniformly to \( d \tilde{f}_0 \) on compact subsets of \( \Sigma \).

\( \square \)

### 4.1 Example

As an example, we consider the family of Delaunay surfaces given by the following DPW potential in \( \mathbb{C}^* \):

\[
\xi_t(z, \lambda) = \begin{pmatrix} 0 & \lambda^{-1} r + s \\ \lambda r + s & 0 \end{pmatrix} \frac{dz}{z}
\]

with initial condition \( \Phi_1(1) = I_2 \). As \( t \to 0 \), we have \((r, s) \to (0, \frac{1}{2})\). We have

\[
\Phi_0(z, \lambda) = \exp \left( \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \log z \right) = \frac{1}{2\sqrt{z}} \begin{pmatrix} z + 1 & z - 1 \\ z - 1 & z + 1 \end{pmatrix}
\]

\[
\frac{\partial \xi_t}{\partial t}|_{t=0} = \begin{pmatrix} 0 & 2\lambda^{-1} \\ 2\lambda & 0 \end{pmatrix} \frac{dz}{z}.
\]
Theorem 4 applies and gives

\[ g(z) = \frac{1 + z}{1 - z} \quad \text{and} \quad \omega(z) = 4 \left( \frac{z - 1}{2\sqrt{z}} \right)^2 \frac{2\,dz}{z} = 2 \left( \frac{z - 1}{z} \right)^2 \, dz. \]

This is the Weierstrass data of a horizontal catenoid of waist-radius 4 and axis \( Ox_1 \), with \( x_1 \to +\infty \) at the end \( z = 0 \).

## 5 The DPW potential

We now start the proof of Theorem 1. Let \((g, \omega)\) be the Weierstrass data of the given minimal \( n \)-noid \( M_0 \), written as in Sect. 2. We introduce \( 3n \) \( \lambda \)-dependent parameters \( a_i, b_i \) and \( p_i \) for \( 1 \leq i \leq n \) in the functional space \( \mathcal{W}^{\geq 0}_{\lambda \rho} \). The vector of these parameters is denoted \( x \in (\mathcal{W}^{\geq 0}_{\lambda \rho})^{3n} \). The parameter \( x \) is in a neighborhood of a (constant) central value \( x_0 \in (\mathcal{W}^{\geq 0}_{\lambda \rho})^{3n} \) which correspond to the Weierstrass data of \( M_0 \), written as in Sect. 2. We define

\[
A_x(z, \lambda) = \sum_{i=1}^{n} a_i(\lambda) z^{n-i},
\]

\[
B_x(z, \lambda) = \sum_{i=1}^{n} b_i(\lambda) z^{n-i},
\]

\[
g_x(z, \lambda) = \frac{A_x(z, \lambda)}{B_x(z, \lambda)} \quad \text{(10)}
\]

\[
\omega_x(z, \lambda) = \frac{B_x(z, \lambda)^2 \, dz}{\prod_{i=1}^{n} (z - p_i(\lambda))^2}. \quad \text{(11)}
\]

For \( t \) in a neighborhood of 0 in \( \mathbb{R} \), we consider the following DPW potential:

\[
\xi_{t,x}(z, \lambda) = \begin{pmatrix} 0 & \frac{1}{4} t (\lambda - 1)^2 \lambda^{-1} \omega_x(z, \lambda) \\ d g_x(z, \lambda) & 0 \end{pmatrix}.
\]

We fix a base point \( z_0 \), away from the poles of \( g \) and \( \omega \), and we take the initial condition

\[
\phi_0(\lambda) = \begin{pmatrix} g_x(z_0, \lambda) & 1 \\ -1 & 0 \end{pmatrix}.
\]

These choices are motivated by the following observations:

1. At \( t = 0 \), we have

\[
\xi_{0,x}(z, \lambda) = \begin{pmatrix} 0 & 0 \\ d g_x(z, \lambda) & 0 \end{pmatrix}.
\]
The solution of the Cauchy Problem (5) is given by

\[ \Phi_{0, x}(z, \lambda) = \begin{pmatrix} g_x(z, \lambda) & 1 \\ -1 & 0 \end{pmatrix} \]  

(12)

which is well-defined, so the Monodromy Problem (9) is solved at \( t = 0 \).

2. The same conclusion holds if \( \lambda = 1 \) instead of \( t = 0 \). In particular, Items (ii) and (iii) of the Monodromy Problem (9) are automatically solved.

3. At \( x = x_0 \), we have \( g_{x_0} = g \) so \( \Phi_{0, x_0}(z, \lambda) \) is independent of \( \lambda \). Moreover,

\[ \frac{\partial \xi(-1)}{\partial t} \mid_{t=0} = \frac{\omega}{4}. \]

Provided the Monodromy Problem is solved for all \( t \) in a neighborhood of 0, Theorem 4 applies and the limit minimal surface has Weierstrass data \((g, \omega)\) so is the minimal \( n \)-noid \( M_0 \), up to translation (see details in Sect. 7.1).

**Remark 4** The potential \( \xi_{t,x} \) is inspired from the potential used in [18] to construct CMC \( n \)-noids by perturbation of a sphere. In fact, in the case \( dg_x = dz \), the two potentials are dual to each other. (See Section 3.2.8 of [19] for the definition of duality in the DPW method.)

### 5.1 Regularity

Our potential \( \xi_{t,x} \) has poles at the zeros of \( B_x \) and the points \( p_1, \ldots, p_n \). (At \( \infty \), we have \( \omega_x \sim b_1^2 z^{-2} dz \) which is holomorphic.) We want the zeros of \( B_x \) to be apparent singularities, so we require the potential to be gauge-equivalent to a regular potential in a neighborhood of these points. Consider the gauge

\[ G_x(z, \lambda) = \begin{pmatrix} \frac{g_x(z, \lambda)}{g_x(z, \lambda)} & -1 \\ 0 & g_x(z, \lambda) \end{pmatrix}. \]

The gauged potential is

\[ \hat{\xi}_{t,x} := \xi_{t,x} \cdot G_x = \begin{pmatrix} 0 & \frac{1}{4} t(\lambda - 1)^2 \lambda^{-1} s_x^2 \omega_x \\ g_x^{-2} dg_x & 0 \end{pmatrix}. \]

We have

\[ g_x^{-2} dg_x = \frac{A_x B_x - A_x B_x'}{A_x^2} \]

and

\[ s_x^2 \omega_x = \frac{A_x^2 d \bar{z}}{\prod_{i=1}^n (z - p_i)^2}. \]

Let \( \zeta \) be a zero of \( B_{x_0} \) (recall that \( B_{x_0} \) does not depend on \( \lambda \)). Then \( A_{x_0}(\zeta) \neq 0 \). By continuity, there exists a neighborhood \( U \) of \( \zeta \) such that for \( z \in U, \lambda \in \mathbb{D}_\rho \) and \( x \) close enough to \( x_0 \), \( A_x(z, \lambda) \neq 0 \). So \( \hat{\xi}_{t,x} \) is holomorphic in \( U \times \mathbb{D}_\rho^* \) and moreover, \( \hat{\xi}_{t,x;12} \neq 0 \). This ensures that the immersion extends analytically to \( U \) and is unbranched in \( U \).
6 The monodromy problem

6.1 Formulation of the problem

For $i \in [1, n]$, we denote $p_{i,0}$ the central value of the parameter $p_i$ (so $p_{1,0}, \ldots, p_{n,0}$ are the ends of the minimal $n$-noid $M_0$). We consider the following $\lambda$-independent domain on the Riemann sphere:

$$\Omega = \{ z \in \mathbb{C} : \forall i \in [1, n], |z - p_{i,0}| > \varepsilon \} \cup \{ \infty \}$$

(13)

where $\varepsilon > 0$ is a fixed, small enough number such that the disks $D(p_{i,0}, 8\varepsilon)$ for $1 \leq i \leq n$ are disjoint. As in [18], we first construct a family of immersions $f_t$ on $\Omega$. Then we extend $f_t$ to an $n$-punctured sphere in Proposition 4.

Let $\tilde{\Omega}$ be the universal cover of $\Omega$ and $\Phi_{t,x}(z, \lambda)$ be the solution of the following Cauchy Problem on $\tilde{\Omega}$:

$$\begin{cases} 
    d\Phi_{t,x}(z, \lambda) = \Phi_{t,x}(z, \lambda)\xi_{t,x}(z, \lambda) \\
    \Phi_{t,x}(\tilde{z}_0, \lambda) = \phi_0
\end{cases}$$

(14)

We denote $\gamma_1, \ldots, \gamma_{n-1}$ a set of generators of the fundamental group $\pi_1(\Omega, z_0)$, with $\gamma_i$ encircling the point $p_{i,0}$. We may assume that each $\gamma_i$ is represented by a fixed curve avoiding the poles of $\xi_{t,x}$. Let

$$M_i(t, x) = \mathcal{M}_{\gamma_i}(\Phi_{t,x})$$

be the monodromy of $\Phi_{t,x}$ along $\gamma_i$. By Eq. (12), we have $M_i(0, x) = I_2$. Recall that the matrix exponential is a local diffeomorphism from a neighborhood of 0 in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (respectively $\mathfrak{su}(2)$) to a neighborhood of $I_2$ in $SL(2, \mathbb{C})$ (respectively $SU(2)$). The inverse diffeomorphism is denoted log. For $t \neq 0$ small enough and $\lambda \in \mathbb{D}_\rho \setminus \{1\}$, we define as in [18]

$$\tilde{M}_i(t, x)(\lambda) = \frac{4\lambda}{t(\lambda - 1)^2} \log M_i(t, x)(\lambda).$$

Proposition 2 1. $\tilde{M}_i(t, x)(\lambda)$ extends smoothly at $t = 0$ and $\lambda = 1$, and each entry $\tilde{M}_{1:2}^{1:2}$ is a smooth map from a neighborhood of $(0, x_0)$ in $\mathbb{R} \times (\mathcal{W}_{\rho}^{\geq 0})^3$ to $\mathcal{W}_{\rho}$.

2. At $t = 0$, we have

$$\tilde{M}_i(0, x)(\lambda) = \begin{pmatrix} \mathcal{P}_{i,1}(x) & \mathcal{P}_{i,2}(x) \\ \mathcal{P}_{i,0}(x) & -\mathcal{P}_{i,1}(x) \end{pmatrix}$$

where $\mathcal{P}_{i,k}(x) = \int_{\gamma_i} g_x^k \omega_x$.

(15)

3. The Monodromy Problem (9) is equivalent to

$$\tilde{M}_i(t, x) \in \Delta_{\mathfrak{su}(2)} \text{ for } 1 \leq i \leq n - 1.$$
Proof we follow the proof of Proposition 1 in [18]. We first consider the case where the parameter \( x = (a_i, b_i, p_i)_{1 \leq i \leq n} \) is constant with respect to \( \lambda \), so \( x \in \mathbb{C}^n \). For \( (\mu, x) \) in a neighborhood of \((0, x_0)\) in \( \mathbb{C} \times \mathbb{C}^n \), we define

\[
\widehat{\xi}_{\mu, x}(z) = \begin{pmatrix} 0 & \mu \omega_x(z) \\ d g_x(z) & 0 \end{pmatrix}
\]

where \( \omega_x \) and \( g_x \) are defined by Eqs. (10) and (11), except that \( a_i, b_i, p_i \) are constant complex numbers. Let \( \widehat{\Phi}_{1\mu, x} \) be the solution of the Cauchy Problem \( d \widehat{\Phi}_{1\mu, x} = \widehat{\Phi}_{1\mu, x} \widehat{\xi}_{\mu, x} \) in \( \Omega \) with initial condition \( \widehat{\Phi}_{1\mu, x}(z_0) = \phi_0 \). Let \( N_i(\mu, x) = \mathcal{M}_{\gamma_i}(\Phi_{1\mu, x}) \). By standard ODE theory, each entry of \( N_i \) is a holomorphic function of \( (\mu, x) \). At \( \mu = 0 \), \( \widehat{\Phi}_{0, x} \) is given by Eq. (12), so in particular \( N_i(0, x) = I_2 \). Hence

\[
\widetilde{N}_i(\mu, x) := \frac{1}{\mu} \log N_i(\mu, x)
\]

extends holomorphically at \( \mu = 0 \) with value \( \widetilde{N}_i(0, x) = \frac{\partial N_i}{\partial \mu}(0, x) \). By Proposition 8 in Appendix A of [18] (the same formula appeared before on page 39 of [12]):

\[
\frac{\partial N_i}{\partial \mu}(0, x) = \int_{\gamma_i} \Phi_{0, x} \frac{\partial \widehat{\xi}_{\mu, x}}{\partial \mu} \bigg|_{\mu=0} \Phi_{0, x}^{-1}.
\]

Hence

\[
\widetilde{N}_i(0, x) = \int_{\gamma_i} \begin{pmatrix} g_x & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & g_x \end{pmatrix} = \int_{\gamma_i} \begin{pmatrix} g_x \omega_x & g_x^2 \omega_x \\ -\omega_x & -g_x \omega_x \end{pmatrix}.
\] (17)

For \((t, x)\) in a neighborhood of \((0, x_0)\) in \( \mathbb{R} \times (\mathcal{V}_{\rho}^{\geq 0})^3 \), we have

\[
\xi_{t, x}(z, \lambda) = \widehat{\xi}_{\mu(t, \lambda), x(\lambda)}(z) \quad \text{with} \quad \mu(t, \lambda) = \frac{t(\lambda - 1)^2}{4\lambda}.
\]

Hence

\[
M_i(t, x)(\lambda) = N_i(\mu(t, \lambda), x(\lambda)) \quad \text{and} \quad \widetilde{M}_i(t, x)(\lambda) = \widetilde{N}_i(\mu(t, \lambda), x(\lambda)).
\]

By substitution (see Proposition 9 in Appendix B of [18]), each entry of \( \widetilde{M}_i \) is is a smooth map from a neighborhood of \((0, x_0)\) in \( \mathbb{R} \times (\mathcal{V}_{\rho}^{\geq 0})^3 \) to \( \mathcal{W}_\rho \). Moreover, \( \widetilde{M}_i(0, x) \) is given by Eq. (17). The fact that \( \widetilde{M}_i \) extends holomorphically at \( \lambda = 1 \) implies that Points (ii) and (iii) of Problem (9) are automatically satisfied. Since \( \lambda^{-1}(\lambda - 1)^2 \in \mathbb{R} \) for \( \lambda \in \mathbb{S}^1 \), Equation (i) of Problem (9) is equivalent to Eq. (16). \( \square \)
6.2 Solution of the monodromy problem

Without loss of generality, we may (using a Möbius transformation of the sphere) fix the value of \( p_1, p_2 \) and \( p_3 \). We still denote \( x \in \left( W_{\rho}^{\geq 0} \right)^{3n-3} \) the vector of the remaining parameters.

**Proposition 3** Assume that the given minimal \( n \)-noid is non-degenerate. For \( t \) in a neighborhood of 0, there exists a smooth function \( \bm{x}(t) \in \left( W_{\rho}^{\geq 0} \right)^{3n-3} \) such that \( \tilde{M}_i(t, \bm{x}(t), \cdot) \in A\mathfrak{su}(2) \) for \( 1 \leq i \leq n-1 \). Moreover, \( \bm{x}(0) = \bm{x}_0 \).

**Proof** recalling the definition of \( P_{i,k} \) in Sect. 2 and \( \mathcal{P}_{i,k} \) in Eq. (15), we have

\[
\mathcal{P}_{i,k}(\bm{x})(\lambda) = P_{i,k}(\bm{x}(\lambda)).
\]

Hence \( \mathcal{P}_{i,k} \) is a smooth map from a neighborhood of \( \bm{x}_0 \) in \( \left( W_{\rho}^{\geq 0} \right)^{3n-3} \) to \( W_{\rho}^{\geq 0} \). Moreover, since \( \bm{x}_0 \) is constant, we have for \( X \in \left( W_{\rho}^{\geq 0} \right)^{3n-3} \):

\[
(d\mathcal{P}_{i,k}(\bm{x}_0)X)(\lambda) = dP_{i,k}(\bm{x}_0)X(\lambda).
\]

Let \( \mathcal{P}_i = (\mathcal{P}_{i,0}, \mathcal{P}_{i,1}, \mathcal{P}_{i,2}) \) and \( \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_{n-1}) \). By the non-degeneracy hypothesis and Remark 2, \( d\mathcal{P}(\bm{x}_0) \) is an automorphism of \( \mathbb{C}^{3n-3} \), so \( d\mathcal{P}(\bm{x}_0) \) is an automorphism of \( \left( W_{\rho}^{\geq 0} \right)^{3n-3} \) and restricts to an automorphism of \( \left( W_{\rho}^{\geq 0} \right)^{3n-3} \).

We define the following smooth maps with value in \( W_{\rho} \) (the star operator is defined in Sect. 3.6)

\[
\mathcal{F}_i(t, \bm{x}) = \tilde{M}_{i,11}(t, \bm{x}) + \tilde{M}_{i,11}(t, \bm{x})^*
\]

\[
\mathcal{G}_i(t, \bm{x}) = \tilde{M}_{i,12}(t, \bm{x}) + \tilde{M}_{i,21}(t, \bm{x})^*
\]

Problem (16) is equivalent to \( \mathcal{F}_i = \mathcal{G}_i = 0 \). Actually, by definition, \( \mathcal{F}_i = \mathcal{F}_i^* \), so Problem (16) is equivalent to

\[
\mathcal{F}_i(t, \bm{x})^+ = 0, \quad \text{Re}(\mathcal{F}_i(t, \bm{x})^0) = 0 \quad \text{and} \quad \mathcal{G}_i(t, \bm{x}) = 0 \quad \text{for } 1 \leq i \leq n-1.
\]

At \( t = 0 \), we have by Eq. (15):

\[
\mathcal{F}_i(0, \bm{x}) = \mathcal{P}_{i,1}(\bm{x}) + \mathcal{P}_{i,1}(\bm{x})^*
\]

\[
\mathcal{G}_i(0, \bm{x}) = \mathcal{P}_{i,2}(\bm{x}) - \mathcal{P}_{i,0}(\bm{x})^*
\]

Equation (3) tells us precisely that that at the central value, we have \( \mathcal{F}_i(0, \bm{x}_0) = 0 \) and \( \mathcal{G}_i(0, \bm{x}_0) = 0 \). We have for \( X \in \left( W_{\rho}^{\geq 0} \right)^{3n-3} \):

\[
d\mathcal{F}_i(0, \bm{x}_0)X = d\mathcal{P}_{i,1}(\bm{x}_0)X + (d\mathcal{P}_{i,1}(\bm{x}_0)X)^*
\]

\[
d\mathcal{G}_i(0, \bm{x}_0)X = d\mathcal{P}_{i,2}(\bm{x}_0)X - (d\mathcal{P}_{i,0}(\bm{x}_0)X)^*
\]
Projecting on $\mathcal{W}_\rho^>^0$ and $\mathcal{W}_\rho^<^0$ we obtain:

$(dF_i(0, x_0)X)^+ = dP_{i, 1}(x_0)X^+$

$(dG_i(0, x_0)X)^+ = dP_{i, 2}(x_0)X^+$

$(dG_i(0, x_0)X)^- = -(dP_{i, 0}(x_0)X^+)^*$

$(dG_i(0, x_0)X)^-^* = -dP_{i, 0}(x_0)X^+.$

Hence the operator

$$[dF_i(0, x_0)^+, dG_i(0, x_0)^+, dG_i(0, x_0)^-^*]_{1 \leq i \leq n-1}$$

only depends on $X^+$ and is an automorphism of $(\mathcal{W}_\rho^>^0)^{3n-3}$ because $dP(x_0)$ is. Projecting on $\mathcal{W}^0$ we obtain:

$$(dF_i(0, x_0)X)^0 = 2 \Re (dP_{i, 1}(x_0)X^0)$$

$$(dG_i(0, x_0)X)^0 = dP_{i, 2}(x_0)X^0 - dP_{i, 0}(x_0)X^0.$$ 

Hence the $\mathbb{R}$-linear operator

$$[\Re(dF_i(0, x_0)^0), dG_i(0, x_0)^0]_{1 \leq i \leq n-1}$$

only depends on $X^0$ and is surjective from $\mathbb{C}^{3n-3}$ to $(\mathbb{R} \times \mathbb{C})^{3n-3}$. This implies that the differential of the map $(F_i^+, G_i^+, G_i^-^*)$, $\Re(F_i^0), G_i^0_{1 \leq i \leq n-1}$ is surjective from $(\mathcal{W}_\rho^>^0)^{3n-3}$ to $(\mathcal{W}_\rho^>^0)^3 \times \mathbb{R} \times \mathbb{C}^{n-1}$. Proposition 3 follows from the Implicit Function Theorem. \hfill \Box

**Remark 5** The kernel of the differential has real dimension $3n - 3$ so we have $3n - 3$ free real parameters. These parameters correspond to deformations of the flux vectors of the minimal $n$-noid.

### 7 Geometry of the immersion

From now on, we assume that $x(t)$ is given by Proposition 3. We write $a_{i, t}, b_{i, t}$ and $p_{i, t}$ for the value of the corresponding parameters. (These parameters are in the space $\mathcal{W}_\rho^>^0$ so are functions of $\lambda$.) For ease of notation, we write $g_t, \omega_t, \xi_t$ and $\Phi_t$ for $\xi X(t)$, $\omega X(t)$, $\xi X(t)$ and $\Phi X(t)$, respectively. Let $F_t = \text{Uni}(\Phi_t)$. Since the Monodromy Problem is solved, the Sym-Bobenko formula (6) defines a CMC-1 immersion $f_t : \Omega \rightarrow \mathbb{R}^3$, where $\Omega$ is the (fixed) domain defined by Eq. (13).

**Proposition 4** The immersion $f_t$ extends analytically to

$$\Sigma_t := \mathbb{C} \cup \{\infty\} \setminus \{p_{1, t}(0), \cdots, p_{n, t}(0)\}$$

where $p_{i, t}(0)$ is the value of $p_{i, t}$ at $\lambda = 0$. \hfill \$$\Sigma$ Springer
We omit the proof which is exactly the same as the proof of Point 1 of Proposition 4 in [18]. It relies on Theorem 3 in [18] which allows for $\lambda$-dependent changes of variables in the DPW method.

### 7.1 Convergence to the minimal $n$-noid

**Proposition 5** \( \lim_{t \to 0} \frac{1}{t} f_t = \psi \) where $\psi$ is (up to translation) the conformal parametrization of the minimal $n$-noid given by Eq. (1). The limit is the uniform $C^1$ convergence on compact subsets of $\Sigma_0 = \mathbb{C} \cup \{ \infty \} \setminus \{ p_{1,0}, \cdots, p_{n,0} \}$.

**Proof** at $t = 0$, we have $g_0 = g$ and $\omega_0 = \omega$. By Eq. (12) and definition of the potential, we have

\[
\Phi_0(z, \lambda) = \begin{pmatrix} g(z) & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \xi_{t;12}^{(-1)}}{\partial t}\bigg|_{t=0} = \frac{\omega}{4}.
\]

By Theorem 4, \( \frac{1}{t} f_t \) converges to a minimal surface with Weierstrass data $(g, \omega)$ on compact subsets of $\Sigma_0$ minus the poles of $g$. In a neighborhood of the poles of $g$, we use the gauge introduced in Sect. 5.1. With the notations of this section and writing $\Phi_t = \Phi_t G_{\chi(t)}$, we have

\[
\Phi_0(z, \lambda) = \begin{pmatrix} 1 & 0 \\ -g(z) & -1 \end{pmatrix} \quad \text{and} \quad \frac{\partial \xi_{t;12}^{(-1)}}{\partial t}\bigg|_{t=0} = \frac{g^2 \omega}{4}.
\]

By Theorem 4 again, \( \frac{1}{t} f_t \) converges to a minimal surface with Weierstrass data $(g, \omega)$ in a neighborhood of the poles of $g$. The two limit minimal surfaces are of course the same, since they coincide in a neighborhood of $z_0$. \( \square \)

### 7.2 Delaunay ends

In this section, we prove that the immersion $f_t$ has Delaunay ends. Delaunay ends in the DPW method have been studied in [4,13]. Following [17], we gauge our potential to a perturbation of the standard Delaunay potential and we use the results in [13].

We denote $N_0$ the Gauss map of the minimal $n$-noid $M_0$. For $1 \leq i \leq n$, we denote $C_i$ the catenoid to which $M_0$ is asymptotic at $p_{i,0}$ and $\tau_i > 0$ the necksize of $C_i$.

**Definition 4** We say that $N_0$ points to the inside in a neighborhood of $p_{i,0}$ if it points to the component of $\mathbb{R}^3 \setminus C_i$ containing the axis of $C_i$.

**Proposition 6** For $1 \leq i \leq n$ and $t \neq 0$:

1. The immersion $f_t$ has a Delaunay end at $p_{i,t}$. If we denote $w_{i,t}$ its weight then

\[
\lim_{t \to 0} t^{-1} w_{i,t} = \pm 2\pi \tau_i
\]
where the sign is + if \( N_0 \) points to the inside in a neighborhood of \( p_{1,0} \) and − otherwise.

2. Its axis converges as \( t \to 0 \) to the half-line through the origin directed by the vector \( N_0(p_{1,0}) \).

3. If \( N_0 \) points to the inside in a neighborhood of \( p_{1,0} \), there exists a uniform \( \varepsilon > 0 \) such that for \( t > 0 \) small enough, \( f_t(D^*(p_{1,0}, \varepsilon)) \) is embedded.

**Proof** in a neighborhood of the puncture \( p_{1,t} \), we may use \( w = g_t(z) - g_t(p_{1,t}) \) as a local coordinate. Note that \( p_{1,t} \in \mathcal{W}_\rho^{>0} \) so, as a function of \( \lambda \), extends holomorphically to \( \mathbb{D}_\rho \). Thus the coordinate \( w \) depends holomorphically on \( \lambda \in \mathbb{D}_\rho \). This is not a problem by Theorem 3 in [18]. Consider the gauge

\[
G(w) = \begin{pmatrix} k \sqrt{w} & -1 \frac{1}{2k \sqrt{w}} \\ 0 & \frac{1}{2k \sqrt{w}} \end{pmatrix}.
\]

Here we can take \( k = 1 \), but later on we will take another value of \( k \) so we do the computation for general values of \( k \neq 0 \). The gauged potential is

\[
\hat{\xi}_t := \xi_t \cdot G = \begin{pmatrix} 0 & \frac{d}{dw} - \frac{w(t(\lambda-1)^2}{4k^2\lambda} \alpha_{i,t}(\lambda) \\ k^2 \frac{dw}{w} & 0 \end{pmatrix}.
\]

Since \( \omega_t \) has a double pole at \( p_{1,t} \), \( \hat{\xi}_t \) has a simple pole at \( w = 0 \) with residue

\[
A_{i,t}(\lambda) = \begin{pmatrix} 0 & \frac{1}{4k^2} + \frac{t(\lambda-1)^2}{4k^2\lambda} \alpha_{i,t}(\lambda) \\ k^2 & 0 \end{pmatrix}
\]

where

\[
\alpha_{i,t} = \text{Res}_{p_{1,t}}(w\omega_t) = \text{Res}_{p_{1,t}}(g_t(z) - g_t(p_{1,t}))\omega_t.
\]

**Claim 1** For \( t \) small enough, \( \alpha_{i,t} \) is a real constant (i.e. independent of \( \lambda \), possibly depending on \( t \)).

**Proof** the proof is similar to the proof of Point 2 of Proposition 4 in [18]. We use the standard theory of Fuchsian systems. Fix \( t \neq 0 \) and \( \lambda \in \mathbb{S}^1 \setminus \{1\} \). Assume that \( \alpha_{i,t}(\lambda) \neq 0 \). Let \( \Phi_t = \Phi_t G \). The eigenvalues of \( A_{i,t} \) are \( \pm \Lambda_{i,t} \) with

\[
\Lambda_{i,t}(\lambda)^2 = \frac{1}{4} + \frac{t(\lambda-1)^2}{4k^2\lambda} \alpha_{i,t}(\lambda).
\]

Provided \( t \neq 0 \) is small enough, \( \Lambda_{i,t} \notin \mathbb{Z}/2 \) so the system is non resonant and \( \Phi_t \) has the following standard \( z^A P \) form in the universal cover of \( D(0, \varepsilon)^* \):

\[
\Phi_t(w, \lambda) = V(\lambda) \exp(A_{i,t}(\lambda) \log w) P(w, \lambda)
\]
where \( P(w, \lambda) \) descends to a well defined holomorphic function of \( w \in D(0, \varepsilon) \) with \( P(0, \lambda) = I_2 \). Consequently, its monodromy is

\[
\mathcal{M}_{\gamma_i}(\Phi_t) = V(\lambda) \exp(2\pi i \Lambda_{i,t}) V(\lambda)^{-1}
\]

with eigenvalues \( \exp(\pm 2\pi i \Lambda_{i,t}(\lambda)) \). Since the Monodromy Problem is solved, the eigenvalues are unitary complex numbers, so \( \Lambda_{i,t}(\lambda) \in \mathbb{R} \) which implies that \( \alpha_{i,t}(\lambda) \in \mathbb{R} \). This of course remains true if \( \alpha_{i,t}(\lambda) = 0 \). Hence \( \alpha_{i,t} \) is real on \( S^1 \setminus \{1\} \). Since all the parameters involved in the definition of \( \omega_t \) are in \( \mathcal{W}^\geq_0 \), \( \alpha_{i,t} \) is holomorphic in the unit disk. Hence it is constant.

\[
\square
\]

Returning to the proof of Proposition 6, let \((r, s) \in \mathbb{R}^2\) be the solution of

\[
\begin{cases}
rs = \frac{1}{4} t \alpha_{i,t} \\
r + s = \frac{1}{2} \\
r < s
\end{cases}
\]

(20)

Since \( r < s \), \( \sqrt{r\lambda + s} \) is well defined and does not vanish for \( \lambda \in \mathbb{D} \). We take \( k = \sqrt{r\lambda + s} \) in the definition of the gauge \( G \). Using Eq. (20), we have:

\[
(r\lambda^{-1} + s)(r\lambda + s) = \frac{1}{4} + rs(\lambda - 1)2\lambda^{-1} = \frac{1}{4} + \frac{1}{4} t(\lambda - 1)^2\lambda^{-1}\alpha_{i,t}.
\]

So the residue of \( \hat{\xi}_t \) becomes

\[
A_{i,t} = \begin{pmatrix}
0 & \frac{1}{r\lambda + s} \left( \frac{1}{4} + \frac{t(\lambda - 1)^2}{4\lambda}\alpha_{i,t} \right) \\
r\lambda + s & 0
\end{pmatrix}
= \begin{pmatrix}
0 & \frac{1}{r\lambda + s} \left( \frac{1}{4} + \frac{t(\lambda - 1)^2}{4\lambda}\alpha_{i,t} \right) \\
r\lambda + s & 0
\end{pmatrix}
\]

which is the residue of the standard Delaunay potential. By [13], the immersion \( f_t \) has a Delaunay end at \( p_{i,t} \) of weight \( w_{i,t} = 8\pi rs = 2\pi i \alpha_{i,t} \). It remains to relate \( \alpha_{i,0} \) to the logarithmic growth \( \tau_i \). For ease of notation, let us write \( p_i = p_{i,0} \). Assume that \( N_0 \) points to the inside in a neighborhood of \( p_i \). The flux of \( M_0 \) along \( \gamma_i \) is equal to

\[
\phi_i = 2\pi \tau_i N_0(p_i) = 2\pi \frac{\tau_i}{|g(p_i)|^2 + 1} \left( 2 \text{Re}(g(p_i)), 2 \text{Im}(g(p_i)), |g(p_i)|^2 - 1 \right)
\]

On the other hand, we have seen in Sect. 2 that the flux is equal to

\[
\phi_i = -2\pi \text{Res}_{p_i} \left( \frac{1}{2} (1 - g^2) \omega, \frac{i}{2} (1 + g^2) \omega, g \omega \right)
\]

Comparing these two expressions for \( \phi_i \), we obtain

\[
\text{Res}_{p_i} (g \omega) = -\tau_i \frac{|g(p_i)|^2 - 1}{|g(p_i)|^2 + 1} \quad \text{and} \quad \text{Res}_{p_i} \omega = -2\tau_i \frac{g(p_i)}{|g(p_i)|^2 + 1}
\]
Using Eq. (19), this gives

\[ \alpha_{i,0} = \text{Res}_{p_i}(g \omega) - g(p_i) \text{Res}_{p_i} \omega = \tau_i \]

If \( N_0 \) points to the outside in a neighborhood of \( p_i \), then \( \phi_i = -2\pi \tau_i N_0(p_i) \), so the same computation gives \( \alpha_{i,0} = -\tau_i \). This proves Point 1 of Proposition 6.

To prove Point 2, we use Theorem 5 in Appendix A. We need to compute \( \hat{\Phi}_0 \) at \( w = 1 \). At \( t = 0 \), we have \( k = \frac{1}{\sqrt{2}} \) so

\[ G(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}. \]

At \( t = 0 \), we have \( w = g(z) - g(p_i) \), so \( w = 1 \iff g(z) = g(p_i) + 1 \). Using Eq. (12),

\[ \hat{\Phi}_0(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} g(p_i) + 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} g(p_i) + 1 & -g(p_i) + 1 \\ -1 & 1 \end{pmatrix}. \]

Fix \( 0 < \alpha < 1 \). By Theorem 5 (using \( t\alpha_{i,t} \) as the time parameter), there exists \( \epsilon > 0 \), \( T > 0 \) and \( c \) such that for \( 0 < |t| < T \):

\[ \| f_i(z) - f^{D}_{i,t}(z) \| \leq c|t| |z - p_{i,t}|^\alpha \text{ in } D^*(p_{i,t}, \epsilon) \]

where \( f^{D}_{i,t} : \mathbb{C}\{p_{i,t}\} \to \mathbb{R}^3 \) is a Delaunay immersion. We compute the limit axis of \( f^{D}_{i,t} \) using Point 3 of Theorem 5:

\[ \hat{\Phi}_0(1) H = \begin{pmatrix} g(p_i) & 1 \\ -1 & 0 \end{pmatrix} = \Phi_0(p_i). \]

\[ Q = F_0(p_i) \]

\[ Qe_3 Q^{-1} = \text{Nor}(F_0(p_i)) = N_0(p_i). \]

This proves Point 2 of Proposition 6. If \( N_0 \) points to the inside in a neighborhood of \( p_{i,0} \), then for \( t > 0, t\alpha_{i,t} > 0 \) so Point 3 follows from Point 2 of Theorem 5. \( \square \)

### 7.3 Alexandrov-embeddedness

We recall from [2,9] the definition of Alexandrov-embeddedness in the non-compact case:

**Definition 5** A surface \( M \) of finite topology is Alexandrov-embedded if \( M \) is properly immersed, if each end of \( M \) is embedded, and if there exists a compact 3-manifold \( \overline{W} \) with boundary \( \partial \overline{W} = \overline{S} \), \( n \) points \( q_1, \cdots, q_n \in \overline{S} \) and a proper immersion \( F : W = \overline{W} \setminus \{q_1, \cdots, q_n\} \to \mathbb{R}^3 \) whose restriction to \( S = \overline{S} \setminus \{q_1, \cdots, q_n\} \) parametrizes \( M \).

**Lemma 1** Let \( M \) be an Alexandrov-embedded minimal surface with \( n \) catenoidal ends. With the notations of Definition 5, we equip \( W \) with the flat metric induced by \( F \), so...
\( F \) is a local isometry, and we denote \( N \) the inside normal to \( S \). Then there exists a flat 3-manifold \( W' \) containing \( W \), a local isometry \( F' : W' \to \mathbb{R}^3 \) extending \( F \) and \( r > 0 \) such that the tubular neighborhood \( \text{Tub}_r S \) is embedded in \( W' \). In other words, the map \((x,s) \mapsto \exp_x(sN(x))\) from \( S \times (-r, r) \) to \( W' \) is well defined and is a diffeomorphism onto its image.

**Proof** since \( M \) has catenoidal ends, there exists \( r > 0 \) such that the inside tubular neighborhood map

\[
g : S \times (0, r) \to W, \quad g(x, s) = \exp_x(sN(x))
\]

is a diffeomorphism onto its image. Since \( F \) is a local isometry, we have

\[
F(g(x, s)) = F(x) + s \, dF(x)N(x) \quad \text{for} \quad (x, s) \in S \times (0, r).
\]  (21)

We define \( W' \) as the disjoint union \((S \times (-r, r)) \sqcup W\) where we identify \((x, s) \in S \times (0, r)\) with its image \(g(x, s) \in W\). We define \( F' : W' \to \mathbb{R}^3 \) by \( F' = F \) in \( W \) and

\[
F'(x, s) = F(x) + s \, dF(x)N(x) \quad \text{for} \quad (x, s) \in S \times (-r, r).
\]

The map \( F' \) is well defined by Eq. (21). We equip \( S \times (-r, r) \) with the flat metric induced by the local diffeomorphism \( F' \), which extends the metric already defined on \( S \times (0, r) \) by identification with \( W \). Since

\[
dF'(x, 0)(X, T) = dF(x)X + T dF(x)N(x)
\]

the metric restricted to \( S \times \{0\} \) is the product metric, so the normal to \( S \times \{0\} \) in \( S \times (-r, r) \) is \( N(x, 0) = (0, 1) \). Since \( F' \) is a local isometry, we have for \((x, s) \in S \times (-r, r)\)

\[
F'(\exp_{(x,0)} sN(x, 0)) = F'(x, 0) + s \, dF'(x, 0)(0, 1)
\]

\[
= F(x) + s \, dF(x)N(x) = F'(x, s)
\]

Hence \( \exp_{(x,0)} sN(x, 0) = (x, s) \) so \( \text{Tub}_r(S \times \{0\}) \) is embedded in \( S \times (-r, r) \). \( \square \)

We now return to the proof of Theorem 1. We orient the minimal \( n \)-noid \( M_0 \) so that its Gauss map points to the inside in a neighborhood of \( p_1 \). For \( 0 < |t| < \epsilon \), we denote \( M_t \) the image of the immersion \( f_t \) that we have constructed.

**Proposition 7** If \( M_0 \) is Alexandrov embedded, then for \( t > 0 \) small enough, \( M_t \) is Alexandrov embedded.

**Proof** our strategy is to cut \( M_t \) by suitable planes into pieces which are either close to \( M_0 \) or Delaunay surfaces (see Fig. 2). Then we prove that each piece, together with flat disks in the cutting planes, is the boundary of a domain, using the Jordan Brouwer Theorem.
Fig. 2 Decomposition of a 4-noid into pieces. Only one Delaunay end is represented, and $F(W_{0,t})$ is represented as an embedded domain for clarity, but in general it will be immersed.

Since $M_0$ is Alexandrov embedded, $N_0$ points to the inside in a neighborhood of each end, so $M_t$ has embedded ends by Proposition 6. Let $\varepsilon > 0$ be the number given by our application of Theorem 5 in Sect. 7.2 and $f_i^D : \mathbb{C}\setminus \{p_{i,t}\} \to \mathbb{R}^3$ be the Delaunay immersion which approximates $f_i$ in $D^*(p_{i,t}, \varepsilon)$. Recall that $f_i(D^*(p_{i,t}, \varepsilon))$ is embedded. Let $\tilde{f}_i = \frac{1}{t} f_i$. By Proposition 5, $\tilde{f}_i$ converges to $\psi$ on compact subsets of $\Sigma_0$, where $\psi : \Sigma_0 \to \mathbb{R}^3$ is a parametrization of $M_0$. Since $M_0$ has catenoidal ends, we may assume (taking $\varepsilon$ smaller if necessary) that $\psi(D^*(p_{i,0}, \varepsilon))$ is embedded and $N_0 \neq N_0(p_{i,0})$ in $D^*(p_{i,0}, \varepsilon)$.

Let $h_i : \mathbb{R}^3 \to \mathbb{R}$ be the height function in the direction $N_0(p_{i,0})$, defined by

$$h_i(x) = \langle x, N_0(p_{i,0}) \rangle.$$  

We shall cut $M_t$ by the plane $h_i = \delta$ where $\delta > 0$ is a fixed, large enough number such that for $1 \leq i \leq n$,

$$\delta > \max_{C(p_{i,0}, \varepsilon)} h_i \circ \psi.$$  

Since $\lim_{z \to p_{i,0}} h_i \circ \psi(z) = +\infty$, we may fix a positive, small enough $\varepsilon' < \varepsilon$ such that

$$\min_{C(p_{i,0}, \varepsilon')} h_i \circ \psi > \delta.$$  

Let $A_{i,t}$ be the annulus defined by $\varepsilon' \leq |z - p_{i,t}| \leq \varepsilon$. Since $N_0 \neq N_0(p_{i,0})$ in $A_{i,t}$,

$$\min_{A_{i,t}} \|N_0(z) - N_0(p_{i,0})\| > 0.$$  

For $t > 0$ small enough:

$$\max_{C(p_{i,t}, \varepsilon)} h_i \circ \tilde{f}_i < \delta \quad (22)$$

$$\min_{C(p_{i,t}, \varepsilon')} h_i \circ \tilde{f}_i > \delta \quad (23)$$

$$\min_{A_{i,t}} \|N_t(z) - N_0(p_{i,0})\| > 0. \quad (24)$$
Hence the function $h_t \circ \tilde{f}_t$ has no critical point in the annulus $A_{i,t}$. So $h_t \circ \tilde{f}_t = \delta$ defines a regular closed curve $\gamma_{i,t}$ in $A_{i,t}$. At $t = 0$, $h_t \circ \psi = \delta$ is a single curve around $p_{i,0}$, so $\gamma_{i,t}$ has only one component and is not contractible in $A_{i,t}$. Let $D_{i,t} \subset \mathbb{C}$ be the topological disk bounded by $\gamma_{i,t}$ and $D^*_{i,t} = D_{i,t} \setminus \{p_{i,t}\}$. Let $\Delta_{i,t}$ be the closed topological disk bounded by $\tilde{f}_t(\gamma_{i,t})$ in the plane defined by $h_t(x) = \delta$.

**Claim 2** For $t > 0$ small enough, $\tilde{f}_t(D^*_{i,t}) \cap \Delta_{i,t} = \emptyset$.

**Proof** of course, $h_t \circ \tilde{f}_t > \delta$ in $D^*_{i,t} \cap A_{i,t}$. What we need to prove is that $\tilde{f}_t(D^*(p_{i,t}, \varepsilon'))$ does not intersect $\Delta_{i,t}$. We do this by comparison with the Delaunay surface. Let $\Pi_i = N_0(p_{i,0})$ and $\pi_i = \mathbb{R}^3 \to \Pi_i$ be the orthogonal projection. Since $\psi$ has a catenoidal end at $p_{i,0}$, $\psi(A_{i,t})$ is a graph over an annulus in the plane $\Pi_i$, with inside boundary circle $\pi_i \circ \psi(C(p_{i,t}, \varepsilon))$ and outside boundary circle $\pi_i \circ \psi(C(p_{i,t}, \varepsilon'))$. Moreover, $N_0$ is close to $N_0(p_{i,t})$. Since $\tilde{f}_t$ is $C^1$ close to $\psi$ in $A_{i,t}$, for $t > 0$ small enough, $\tilde{f}_t(A_{i,t})$ is a graph over an annulus in the plane $\Pi_i$, with inside boundary circle $\pi_i \circ \tilde{f}_t(C(p_{i,t}, \varepsilon))$ and outside boundary circle $\pi_i \circ \tilde{f}_t(C(p_{i,t}, \varepsilon'))$.

Now we go back to the original scale. Since $f_t$ is $C^1$ close to $f^D_{i,t}$ in $D^*(p_{i,t}, \varepsilon)$, we conclude that $f^D_{i,t}(A_{i,t})$ is a graph over an annulus in the plane $\Pi_i$, with inside boundary circle $\pi_i \circ f^D_{i,t}(C(p_{i,t}, \varepsilon))$ and outside boundary circle $\pi_i \circ f^D_{i,t}(C(p_{i,t}, \varepsilon'))$. Then from the geometry of Delaunay surfaces, there exists a curve $\gamma_{i,t,0}$ in $D^*(p_{i,t}, \varepsilon')$ such that $f^D_{i,t}(\gamma_{i,t,0})$ is a closed curve in the plane $h_t = \frac{1}{2}$. Let $D_{i,t,0}$ be the disk bounded by $\gamma_{i,t,0}$ and $A_{i,t,0}$ be the closed annulus bounded by $\gamma_{i,t}$ and $\gamma_{i,t,0}$. Then $h_t \circ f^D_{i,t} > \frac{1}{2}$ in $D_{i,t,0}$ and $f^D_{i,t}(A_{i,t,0})$ is a graph over an annulus in the plane $\Pi_i$. Since $f_t$ is $C^1$ close to $f^D_{i,t}$ in $D^*(p_{i,t}, \varepsilon)$, we conclude that $h_t \circ f_t > \frac{1}{4}$ in $D^*_{i,t,0}$ and $f_t(A_{i,t,0})$ is a graph over an annulus in the plane $\Pi_i$.

Back to the scale $\frac{1}{t}$, $\tilde{f}_t(D_{i,t,0} \cap A_{i,t,0})$ is a graph over an annulus in the plane $\Pi_i$ whose inside boundary circle is $\pi_i \circ \tilde{f}_t(\gamma_{i,t}) = \pi_i(\partial \Delta_{i,t})$, so $\tilde{f}_t(D_{i,t,0} \cap A_{i,t,0}) \cap \Delta_{i,t} = \emptyset$. Moreover, $h_t \circ \tilde{f}_t > \frac{1}{4t} \gg \delta$ in $D^*_{i,t,0}$ so $\tilde{f}_t(D^*_{i,t,0}) \cap \Delta_{i,t} = \emptyset$. 

**Claim 3** For $t > 0$ small enough, $\tilde{f}_t(D^*_{i,t}) \cup \Delta_{i,t}$ is the boundary of a cylindrically bounded domain $W_{i,t} \subset \mathbb{R}^3$.

**Proof** since $f_t$ is close to $f^D_{i,t}$ in $D^*_{i,t}$, we can find an increasing diverging sequence $(R_{t,k})_{k \in \mathbb{N}}$ such that $f_t(D^*_{i,t})$ intersects the plane $h_t = R_{t,k}$ transversely along a closed curve $f_t(\gamma_{i,t,k})$. (Explicitly, we can take $R_{t,k} = \frac{1}{2} + k h_t(T_t)$ where $T_t \in \mathbb{R}^3$ is the period of the Delaunay surface $f^D_{i,t}$.) Let $A_{i,t,k}$ be the annulus bounded by $\gamma_{i,t}$ and $\gamma_{i,t,k}$. Let $\Delta_{i,t,k}$ be the closed disk bounded by $\tilde{f}_t(\gamma_{i,t,k})$ in the plane $h_t = t^{-1}R_{t,k}$. Then $\tilde{f}_t(A_{i,t,k}) \cup \Delta_{i,t} \cup \Delta_{i,t,k}$ is topologically a sphere: the image of $S^2$ by an injective continuous map. By the Jordan Brouwer Theorem, it is the boundary of a bounded domain $W_{i,t,k}$. Clearly, $W_{i,t,k} \subset W_{i,t,k+1}$. We take $W_{i,t} = \bigcup_{k \in \mathbb{N}} W_{i,t,k}$. 

Let $\Omega_t = C \cup \{\infty\} \setminus (D_{1,t} \cup \cdots \cup D_{n,t})$. Let $W'$ be the flat 3-manifold given by Lemma 1 and denote $F : W' \to \mathbb{R}^3$ its developing map (instead of $F'$). (Here $W'$ is an open manifold, meaning not a manifold-with-boundary.)
Claim 4 For \( t > 0 \) small enough, there exists a compact domain \( W_{0,t} \) in \( W' \) such that

\[
F(\partial W_{0,t}) = \tilde{f}_t(\Omega_i) \cup \Delta_{1,t} \cup \cdots \cup \Delta_{n,t}.
\]

Proof by definition, \( \psi \) lifts to a diffeomorphism \( \hat{\psi} : \Sigma_0 \rightarrow S \subset W' \) such that \( F \circ \hat{\psi} = \psi \). Since \( M_0 \) has catenoidal ends, there exists domains \( V_1, \ldots, V_n \) in \( W' \) such that for \( 1 \leq i \leq n \):

- \( F : V_i \rightarrow F(V_i) \subset \mathbb{R}^3 \) is a diffeomorphism,
- \( V_i \) is foliated by flat disks on which \( h_i \circ F \) is constant (in particular, \( h_i \circ F \) is constant on \( \partial V_i \)),
- \( \hat{\psi}(D^*(p_{i,0}, \varepsilon)) \subset V_i \) (which might require taking a smaller \( \varepsilon > 0 \)),
- \( h_i < \delta \) on \( V_i \cap \hat{\psi}(\Sigma_0 \setminus \bigcup_{i=1}^n D(p_{i,0}, \varepsilon)) \) (which might require taking a larger \( \delta \)).

Let \( r > 0 \) be the radius of the embedded tubular neighborhood of \( S \) in \( W' \) constructed in Lemma 1. For \( t > 0 \) small enough, \( ||\tilde{f}_t - \psi|| < r \) in \( \Omega_i \), so \( \tilde{f}_t \) lifts to \( \hat{f}_t : \Omega_i \rightarrow W' \) such that \( F \circ \hat{f}_t = \tilde{f}_t \). (Explicitly, \( \hat{f}_t(z) = \exp_{\hat{\psi}(z)}(\tilde{f}_t(z) - \psi(z)) \).) From the properties of \( V_i \) and the convergence of \( \hat{f}_t \) to \( \psi \) on compact subsets of \( \Sigma_0 \), we have for \( t > 0 \) small enough

\[
\hat{f}_t(\Omega_i \cap D(p_{i,0}, \varepsilon)) \subset V_i \quad (25)
\]
\[
h_i < \delta \quad \text{on} \quad V_i \cap \hat{f}_t(\Sigma_0 \setminus \bigcup_{i=1}^n D(p_{i,0}, \varepsilon)). \quad (26)
\]

By Eq. (25), \( \tilde{f}_t(\gamma_{i,t}) \subset V_i \) so \( \Delta_{i,t} \) lifts to a closed disk \( \tilde{\Delta}_{i,t} \subset V_i \) such that \( \partial \tilde{\Delta}_{i,t} = \tilde{f}_t(\gamma_{i,t}) \) and \( F(\tilde{\Delta}_{i,t}) = \Delta_{i,t} \). Since \( F \) is a diffeomorphism on \( V_i \), \( \tilde{f}_t(\Omega_i \cap D(p_{i,t}, \varepsilon)) \) is disjoint from \( \tilde{\Delta}_{i,t} \). By (26), \( \tilde{f}_t(\Omega_i) \setminus \bigcup_{i=1}^n D(p_{i,t}, \varepsilon) \) is disjoint from \( \tilde{\Delta}_{i,t} \). Hence \( \tilde{f}_t(\Omega_i) \cap \tilde{\Delta}_{i,t} = \emptyset \). Then \( \tilde{f}_t(\Omega_i) \cup \tilde{\Delta}_{1,t} \cup \cdots \cup \tilde{\Delta}_{n,t} \) is a topological sphere in \( W' \). Since \( M_0 \) has genus zero, \( W' \) is homeomorphic to \( \mathbb{R}^3 \). By the Jordan Brouwer Theorem, \( \tilde{f}_t(\Omega_i) \cup \tilde{\Delta}_{1,t} \cup \cdots \cup \tilde{\Delta}_{n,t} \) is the boundary of a compact domain \( W_{0,t} \subset W' \). \( \square \)

Returning to the proof of Proposition 7, let \( W_t \) be the abstract 3-manifold with boundary obtained as the disjoint union \( \overline{W}_{0,t} \cup \overline{W}_{1,t} \cup \cdots \cup \overline{W}_{n,t} \), identifying \( \overline{W}_{0,t} \) and \( \overline{W}_{i,t} \) along their boundaries \( \tilde{\Delta}_{i,t} \) and \( \Delta_{i,t} \) via the map \( F \) for \( 1 \leq i \leq n \). Let \( F_t : W_t \rightarrow \mathbb{R}^3 \) be the map defined by \( F_t = F \) in \( \overline{W}_{0,t} \) and \( F_t = \text{id} \) in \( \overline{W}_{i,t} \) for \( 1 \leq i \leq n \). Then \( F_t \) is a proper local diffeomorphism whose boundary restriction parametrizes \( M_t \). Moreover, since each \( \overline{W}_{i,t} \) is homeomorphic to a closed ball minus a boundary point, we may compactify \( W_t \) by adding \( n \) points. This proves that \( M_t \) is Alexandrov-embedded. \( \square \)

Appendix: On Delaunay ends in the DPW method

As already said, Delaunay ends in the DPW method have been studied in [13], where it is proved that if \( \xi \) is a holomorphic perturbation of the standard Delaunay potential, the resulting immersion has a Delaunay end at \( z = 0 \) and is close to a Delaunay surface in a disk \( D(0, \varepsilon) \). In case the potential \( \xi_0 \) depends on a parameter \( t \) and the Delaunay residue has weight \( \sim t \), their result has been refined by Raujouan in [15], yielding a
uniform \( \varepsilon \) as \( t \to 0 \). This is delicate because the corresponding Fuchsian system is resonant at \( t = 0 \). The result of [15] is the key to proving embeddedness of the CMC \( n \)-noids constructed in [18].

We consider the standard Delaunay residue for \( t \leq 1/16 \):

\[
A_t(\lambda) = \begin{pmatrix} 0 & \lambda^{-1}r + s \\ \lambda r + s & 0 \end{pmatrix}
\]

where

\[
\begin{aligned}
 r + s &= \frac{1}{2} \\
 rs &= t \\
 r &< s
\end{aligned}
\]

In particular, in the limit case \( t = 0 \), we have

\[
A_0 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}
\]

Definition 6 [15] A perturbed Delaunay potential is a family of DPW potentials \( \xi_t \) of the form

\[
\xi_t(z, \lambda) = A_t(\lambda) \frac{dz}{z} + R_t(z, \lambda) dz
\]

where \( R_t \) is of class \( C^2 \) with respect to \( (t, z, \lambda) \in (-T, T) \times D(0, \varepsilon) \times \mathbb{A}_\rho \) for some positive \( \varepsilon \) and \( T \), and satisfies \( R_0 = 0 \). In particular, \( \xi_0 = A_0 \frac{dz}{z} \).

Let \( (e_1, e_2, e_3) \) represent the canonical basis of \( \mathbb{R}^3 \) in the \( su(2) \) model.

Theorem 5 Let \( \xi_t \) be a perturbed Delaunay potential. Let \( \Phi_t(z, \lambda) \) be a family of solutions of \( d\Phi_t = \Phi_t \xi_t \) in the universal cover of the punctured disk \( D^*(0, \varepsilon) \). Assume that \( \Phi_t(z, \lambda) \) depends continuously on \( (t, z, \lambda) \) and that the Monodromy Problem for \( \Phi_t \) is solved. Let \( f_t = \text{Sym} (\text{Uni}(\Phi_t)) \) be the immersion given by the DPW method. Finally, assume that \( \Phi_0(1, \cdot) \) is constant (i.e. independent of \( \lambda \)).

Given \( 0 < \alpha < 1 \), there exists uniform positive numbers \( \varepsilon' \leq \varepsilon \), \( T' \leq T \), \( c \) and a family of Delaunay immersions \( f_t^D : \mathbb{C}^* \to \mathbb{R}^3 \) such that:

1. For \( 0 < |t| < T' \) and \( 0 < |z| < \varepsilon' \):

\[
\|f_t(z) - f_t^D(z)\| \leq c |t| |z|^{\alpha}.
\]

2. For \( 0 < t < T' \), \( f_t : D^*(0, \varepsilon') \to \mathbb{R}^3 \) is an embedding.

3. The end of \( f_t^D \) at \( z = 0 \) has weight \( 8\pi t \) and its axis converges when \( t \to 0 \) to the half-line spanned by the vector \( Q e_3 Q^{-1} \) where

\[
Q = \text{Uni} (\Phi_0(1) H) \quad \text{and} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]

Thomas Raujouan has proved this result in [15], Theorem 3, in the case \( \Phi_0(1, \lambda) = I_2 \). He proves that the limit axis is spanned by \( e_1 \). (In fact, he finds that the limit axis is
−e_1, but this is because he has the opposite sign in the Sym-Bobenko formula. See Remark 3.) Then in Sect. 2 of [15], he explains, in the case r > s, how to extend his result to the case where \( \Phi_0(1, \lambda) \) is constant. We adapt his method to the case r < s.

**Lemma 2** Under the assumptions of Theorem 5, there exists a gauge G(z) and a change of variable h(z) with h(0) = 0 such that \( \tilde{\xi}_t = (h^* \xi_t) \cdot G \) is a perturbed Delaunay potential (with residue \( A_t \)) and \( \tilde{\Phi}_t = (h^* \Phi_t) \times G \) satisfies at \( t = 0 \)

\[
\tilde{\Phi}_0(1, \lambda) = Q H^{-1} \in \Lambda SU(2). \tag{27}
\]

**Proof** we follow the method explained in Section 2 of [15]. We take the change of variable in the form

\[
h(z) = \frac{z}{pz + q}
\]

where \( p, q \) are complex numbers (independent of \( t \)) to be determined, with \( q \neq 0 \). We consider the following gauge:

\[
G(z) = \frac{1}{\sqrt{q(pz + q)}} \begin{pmatrix}
 pz + q & pz \\
 0 & q
\end{pmatrix}.
\]

It is chosen so that

\[
G(0) = I_2 \quad \text{and} \quad dG = GA_0 \frac{dz}{z} - A_0 G \frac{dh}{h}. \tag{28}
\]

(In fact, the gauge G is found as the only solution of Problem (28) which is upper triangular.) We have

\[
\tilde{\xi}_t = (h^* \xi_t) \cdot G = G^{-1} \left( A_t(\lambda) \frac{dh}{h} + R_t(h, \lambda)dh \right) G + G^{-1} dG.
\]

Since \( G(0) = I_2 \), \( \tilde{\xi}_t \) has a simple pole at \( z = 0 \) with residue \( A_t \). Using Eq. (28), we obtain at \( t = 0 \):

\[
\tilde{\xi}_0 = G^{-1} A_0 \frac{dh}{h} G + G^{-1} dG = A_0 \frac{dz}{z}.
\]

Hence \( \tilde{\xi}_t \) is a perturbed Delaunay potential. It remains to compute \( \tilde{\Phi}_0(1) \). The matrix \( H \) diagonalises \( A_0 \):

\[
A_0 = H \begin{pmatrix}
 -1 & 0 \\
 0 & \frac{1}{2}
\end{pmatrix} H^{-1}
\]
Gluing Delaunay ends to minimal \( n \)-noids using the DPW method

Hence

\[
\Phi_0(z) = \Phi_0(1) z^{A_0} = \Phi_0(1) H \begin{pmatrix} \frac{1}{\sqrt{z}} & 0 \\ 0 & \sqrt{z} \end{pmatrix} H^{-1}
\]

\[
\widetilde{\Phi}_0(1) = \Phi_0(h(1)) G(1)
\]

\[
= \Phi_0(1) H \begin{pmatrix} \sqrt{p+q} & 0 \\ 0 & \frac{1}{\sqrt{p+q}} \end{pmatrix} H^{-1} \frac{1}{\sqrt{q}} \begin{pmatrix} p + q & p \\ 0 & q \end{pmatrix} H^{-1}
\]

We decompose \( \Phi_0(1) H = QR \) with \( Q \in SU(2) \) and \( R = \begin{pmatrix} \rho & \mu \\ 0 & \frac{1}{\rho} \end{pmatrix} \). Then

\[
\widetilde{\Phi}_0(1) = Q \begin{pmatrix} \rho & \mu \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \sqrt{q} & \frac{p}{\sqrt{q}} \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix} H^{-1}
\]

We take \( q = \frac{1}{\rho^2} \) and \( p = -\frac{\mu}{\rho} \) to cancel the two matrices in the middle and obtain Eq. (27).

We can now prove Theorem 5. Let

\[
\widehat{\Phi}_t(z, \lambda) = H Q^{-1} \Phi_t(z, \lambda) = H Q^{-1} \Phi_t(h(z), \lambda) G(z, \lambda)
\]

Since \( \widehat{\Phi}_0(1, \lambda) = I_2 \), we can apply Theorem 3 in [15] which says that the resulting immersion \( \widehat{f}_t \) satisfies Points 1 and 2 of Theorem 5 and its limit axis is spanned by \( e_1 \). We have

\[
f_t \circ h = Q H^{-1} \widehat{f}_t H Q^{-1}
\]

so \( f_t \circ h \) and \( \widehat{f}_t \) differ by a rotation and the limit axis of \( f_t \) is spanned by the vector \( Q H^{-1} e_1 H Q^{-1} \). Now

\[
H^{-1} e_1 H = -\frac{i}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = -i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = e_3.
\]

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