Stringy Mirror Symmetry

Jyh-Haur Teh

Abstract

We prove that the mirror pairs constructed by Batyrev and Borisov have stringy mirror symmetry.

1 Introduction

The topological mirror symmetry test predicts that if two smooth $n$-dimensional Calabi-Yau manifolds $V$ and $V^*$ form a mirror pair, then their Hodge numbers satisfy the relations

$$h^{p,q}(V) = h^{n-p,q}(V^*), \quad 0 \leq p, q \leq n$$

But many mirror pairs $(V, V^*)$ found by physicists are Calabi-Yau varieties with Gorenstein abelian quotient singularities and these relations do not hold if $V$ or $V^*$ is not smooth. To formulate a correct mirror symmetry, Batyrev and Dais introduced (see [5]) the notion of stringy Hodge numbers $h^{p,q}_{st}$ for Calabi-Yau varieties with at worst log-terminal singularities and Batyrev modified the topological mirror symmetry test (see [1]) to

$$h^{p,q}_{st}(V) = h^{n-p,q}_{st}(V^*), \quad 0 \leq p, q \leq n$$

He and Borisov (see [3]) proposed a general construction of mirror pairs which includes mirror constructions of physicists for rigid Calabi-Yau manifolds, and in [4], they showed that their mirror pairs satisfy the modified topological mirror symmetry test.

In [7], we modified the construction of motivic integration and introduced the notion of stringy invariants. One of the most interesting invariants to us is given by the dimension of subspaces of cohomology groups which are generated by algebraic cycles. In particular, we showed that for two birational Calabi-Yau manifolds, their corresponding subspaces generated by algebraic cycles have the same dimensions. As a consequence, if the Hodge conjecture or Grothendieck standard conjecture is true in one of them, then it is true for another one. We conjectured (see [7, Conjecture 3]) that the Batyrev-Borisov’s mirror pairs satisfy a stringy mirror symmetry test. The main result in this paper is to show that this actually follows directly from Batyrev-Borisov’s result.

2 Main result

We recall briefly notations from [7]. Let $K_0(Var)$ be the Grothendieck group of complex algebraic varieties and $S = \{\mathbb{L}^i\}_{i \geq 0}$ where $\mathbb{L} = \mathbb{C}$ and we denote by $\mathbb{L}^0 = 1 = \text{point}$. Let $\mathcal{L} = \mathbb{Z}\{\mathbb{L}^i | i \in \mathbb{Z}_{\geq 0}\}$ be the free abelian group generated by $\mathbb{L}^i$ for all nonnegative $i$. Then $K_0(Var)$ is canonically a $\mathcal{L}$-module. Let $\mathcal{N} = S^{-1}K_0(Var)$ be the localization of $K_0(Var)$ with respect to $S$. Let $F^k\mathcal{N}$ be
the subgroup of \( \mathcal{N} \) generated by elements of the form \([X]/\mathbb{L}^i\) where \( i - \dim X \geq k \). Then we have a decreasing filtration

\[
\cdots \supset F^k \mathcal{N} \supset F^{k+1} \mathcal{N} \supset \cdots
\]

of abelian subgroups of \( \mathcal{N} \). The Kontsevich group of varieties is defined to be \( \hat{\mathcal{N}} := \lim_{\leftarrow} \mathcal{N}/F^k \mathcal{N} \) and \( \hat{\mathcal{N}} \) is the image of the canonical map \( \mathcal{N} \to \hat{\mathcal{N}} \). Then \( \hat{\mathcal{N}} \) is canonically a \( \mathcal{L} \)-submodule. A motivic invariant is a group homomorphism from \( \mathcal{N}[\{1/L_i \mid i \geq 1\}] \) to \( \mathbb{Z} \) and we say that a family of motivic invariants \( \phi = \{ \phi_{j,n} \mid j, n \in \mathbb{Z} \} \) is of type \( (a, b) \in \mathbb{Z} \times \mathbb{Z} \) if \( \phi_{j,n}(X \times \mathbb{L}^k) = \phi_{j-ak,n-bk}(X) \) for any \( j, n \) and any variety \( X \). We say that \( \phi \) is bounded if \( \phi_{j,n}(X) \) vanishes for \( |j|, |n| \) large enough, depending on \( X \).

**Definition 2.1.** Suppose that \( \phi = \{ \phi_{j,n} \mid j, n \in \mathbb{Z} \} \) is a family of bounded motivic invariants of type \( (a, b) \). We define

\[
\phi(X; u, v) := \sum_{j, n} \phi_{j,n}(X) u^j v^n
\]

and

\[
\phi(L^{-1}; u, v) = (u^a v^b)^{-1} \phi(L^0; u, v)
\]

then we have a group homomorphism

\[
\phi : \hat{\mathcal{N}}[\{1/L_i \mid i \geq 1\}] \to \mathbb{Z}[[u, v, (u^a v^b)^{-1}]]
\]

If \( X \) is a normal irreducible algebraic variety with at worst log-terminal singularities, and \( \rho : Y \to X \) is a resolution of singularities such that the relative canonical divisor \( D = \sum_i a_i D_i \) has simple normal crossings. Then the stringy \( \phi \)-function of type \( (a, b) \) associated to \( \phi \) is defined to be

\[
\phi^{st}(X; u, v) := \sum_{J \subseteq I} \phi(D^0_J; u, v) \prod_{j \in J} \frac{u^a v^b - 1}{(u^a v^b)^{a_j + 1} - 1}
\]

where \( I = \{1, ..., r\} \) and

\[
D_J = \begin{cases} 
\cap_{j \in J} D_j, & \text{if } J \neq \emptyset; \\
X, & \text{if } J = \emptyset
\end{cases}
\]

and \( D^0_J := D_J - \bigcup_{i \notin J} D_i \).

**Remark 2.2.** In [7], Definition 13, \( \phi(L^{-1}; u, v) \) was defined to be \((u^a v^b)^{-1}\). This is not totally correct since we want to make it a group homomorphism, we need to modify it to \((u^a v^b)^{-1} \phi(L^0; u, v)\). Also, we need to modify Conjecture 3 in [7] to the following statement which is the main result of this paper.

**Theorem 2.3.** If \((V, W)\) is a Batyrev-Borisov’s mirror pair and \( \phi^{st} \) is a stringy \( \phi \)-function of type \( (a, b) \), then

\[
\phi^{st}(V; u, v) = (-u^a)^n \phi^{st}(W; u^{-1}, v)
\]

where \( n \) is the dimension of \( V \) and \( W \).
To prove this result, we first proceed as in the proof of [1, Theorem 4.3] and then use Batyrev-Borisov’s result of the mirror symmetry of stringy E-functions of their mirror pairs.

Let $X$ be a normal $d$-dimensional $\mathbb{Q}$-Gorenstein toric variety associated with a rational polyhedral fan $\Sigma \subset N_\mathbb{R} = N \otimes \mathbb{R}$ where $N$ is a free abelian group of rank $d$. For a cone $\sigma \in \Sigma$, let $\sigma^0$ be the relative interior of $\sigma$. The property that $X$ is $\mathbb{Q}$-Gorenstein is equivalent to the existence of a continuous function $\varphi_K : N_\mathbb{R} \to \mathbb{R}_{>0}$ satisfying

1. $\varphi_K(e) = 1$, if $e$ is a primitive integral generator of a 1-dimensional cone $\sigma \in \Sigma$.
2. $\varphi_K$ is linear on each cone $\sigma \in \Sigma$.

The following desingularization result can be found in [6, Proposition 5-2-2].

**Proposition 2.4.** Let $\rho : X' \to X$ be a toric desingularization of $X$, which is defined by a subdivision $\Sigma'$ of the fan $\Sigma$. Then the irreducible components $D_1, ..., D_r$ of the exceptional divisor $D$ of the birational morphism $\rho$ have only normal crossings and they one-to-one correspond to primitive integral generators $e'_1, ..., e'_r$ of those 1-dimensional cones $\sigma' \in \Sigma'$ which do not belong to $\Sigma$. Moreover, in the formula

$$K_{X'} = \rho^*K_X + \sum_{i=1}^r a_i D_i$$

one has $a_i = \varphi_K(e'_i) - 1$ for all $i \in \{1, ..., r\}$.

**Lemma 2.5.** If $\phi^{st}$ is a stringy function of type $(a, b)$, then

$$\phi^{st}(X; u, v) = E^{st}(X; u^a, v^b)\phi(L^0; u, v)$$

where $E^{st}(X; u, v)$ is the stringy E-function of $X$.

**Proof.** Since the proof follows exactly as the proof [1, Theorem 4.3], we briefly sketch the main idea of the proof and refer the reader to the original paper of Batyrev. Let $\Sigma'(J)$ be the star of the cone $\sigma_J \subset \Sigma'$, i.e., $\Sigma'(J)$ consists of the cones $\sigma' \in \Sigma'$ such that $\sigma' \supset \sigma_J$. Denote by $\Sigma_0(J)$ the subfan of $\Sigma'(J)$ consisting of those cones $\sigma' \in \Sigma'(J)$ which do not contain any $e'_i$ where $i \notin J$. Then the fan $\Sigma'(J)$ defines the toric subvariety $D_J \subset X'$ and the fan $\Sigma_0(J)$ defines the open subset $D_0^J \subset D_J$.

The canonical stratification by torus orbits

$$X' = \bigcup_{\sigma' \in \Sigma'} X'_{\sigma'}$$

induces the following stratifications

$$D_0^J = \bigcup_{\sigma' \in \Sigma_0(J)} X'_{\sigma'}, \quad \emptyset \neq J \subset I$$

Then apply the stringy $\phi$-function, we get

$$\phi^{st}(D_0^J; u, v) = \sum_{\sigma' \in \Sigma_0(J)} (u^a v^b - 1)^{d-\dim \sigma'} \phi(L^0; u, v)$$

for $\emptyset \neq J \subset I$. 

3
Let \( \Sigma'(\emptyset) \) be the subfan of \( \Sigma' \) consisting for those cones \( \sigma' \in \Sigma' \) which do not contain any element of \( \{e'_1, \ldots, e'_r\} \). Then \( \Sigma'(\emptyset) \) defines the canonical stratification of

\[
X' \setminus D = \sum_{\sigma' \in \Sigma'(\emptyset)} X'_{\sigma'}
\]

Then we have

\[
\phi(X' \setminus D; u, v) = \sum_{\sigma' \in \Sigma'(\emptyset)} (u^a v^b - 1)^{d - \dim \sigma'}
\]

We may write

\[
\sum_{n \in \sigma'_j \cap N} (u^a v^b - \varphi_K(n)) = \prod_{j \in J} \frac{(u^a v^b - \varphi_K(c'_j))}{1 - (u^a v^b - \varphi_K(c'_j))} = \prod_{j \in J} \frac{1}{1 - (u^a v^b)^{a_j + 1} - 1}
\]

Therefore

\[
\prod_{j \in J} \frac{u^a v^b - 1}{(u^a v^b)^{a_j + 1} - 1} = (u^a v^b - 1)^{|J|} \sum_{n \in \sigma'_j \cap N} (u^a v^b - \varphi_K(n))
\]

So by the resolution of singularities \( \rho : X' \to X \), we have

\[
\phi^{st}(X; u, v) = \phi(X \setminus D; u, v) + \sum_{\emptyset \neq J \subseteq I} \phi(D^0_J; u, v)(u^a v^b - 1)^{|J|} \left( \sum_{n \in \sigma'_j \cap N} (u^a v^b - \varphi_K(n)) \right)
\]

\[
= \sum_{\sigma' \in \Sigma'(\emptyset)} (u^a v^b - 1)^{d - \dim \sigma'} + \sum_{\emptyset \neq J \subseteq I} \sum_{\sigma' \in \Sigma'_0(J)} \left( \sum_{n \in \sigma'_j \cap N} (u^a v^b - 1)^{d + |J| - \dim \sigma'} \right) \left( \sum_{n \in \sigma'_j \cap N} (u^a v^b - \varphi_K(n)) \right)
\]

By the second part of the computation \([1, \text{Theorem 4.3}]\), we have

\[
\bigcup_{\sigma \in \Sigma} \sigma^0 \cap N = \bigcup_{\sigma' \in \Sigma'} (\sigma')^0 \cap N
\]

This gives us the equality

\[
\phi^{st}(X; u, v) = (u^a v^b - 1)^d \sum_{\sigma \in \Sigma} \sum_{n \in \sigma^0 \cap N} (u^a v^b - \varphi_K(n)) \phi(\mathbb{L}^0; u, v) = E_{st}(X; u, v) \phi(\mathbb{L}^0; u, v)
\]

\[\square\]

Proof. Now Theorem \([2, \text{Theorem 2.3}]\) is an easy consequence of the above computation and Batyrev-Borisov’s result. By Batyrev-Borisov’s result, \((V, W)\) satisfies the mirror symmetry \(E_{st}(V; u, v) = (-u)^n E_{st}(W; u^{-1}, v)\). So \(\phi^{st}(V; u, v) = E_{st}(V; u^a, v^b) \phi(\mathbb{L}^0; u, v) = (-u^a)^n E_{st}(W; (u^a)^{-1}, v^b) \phi(\mathbb{L}^0; u, v) = (-u^a)^n \phi^{st}(W; u^{-1}, v)\).

\[\square\]
References

[1] V. V. Batyrev, *Stringy Hodge numbers of varities with Gorenstein canonical singularities*, Proc. Taniguchi Symposium 1997, In 'Integrable Systems and Algebraic Geometry, Kobe/Kyoto 1997', World Sci. Publ. (1999), 1-32.

[2] V. V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, New Trends in Algebraic geometry, Euroconference on Algebraic Geometry(Warwick 1996), London Math. Soc. Lecture Note Ser. 264, K. Hulek et al Ed., CUP, 1999, 1-11.

[3] V. V. Batyrev and L. A. Borisov, *Dual cones ad mirror symmetry for generalized Calabi-Yau manifolds*, in Mirror Symmetry II, (eds. S.-T. Yau), (1995), pp 65-80.

[4] V. V. Batyrev and L. A. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math., Vol. 126 (1996), no. 1, 183-203.

[5] V. V. Batyrev and D. Dais, *Strongy McKay Correspondence, String-Theoretic Hodge Numbers and Mirror Symmetry*, Topology, 35 (1996), 901-929.

[6] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the Minimal Model Program*, Adv. Studies in Pure Math., 10 (1987), 283-360.

[7] J.H. Teh, *Motivic integration and projective bundle theorem in morphic cohomology*, Math. Proc. Camb. Phil. Soc., 147 (2009), 295-321.