On extending the quantum measure

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Abstract
We point out that a quantum system with a strongly positive quantum measure or decoherence functional gives rise to a vector-valued measure whose domain is the algebra of events or physical questions. This gives an immediate handle on the question of the extension of the decoherence functional to the sigma algebra generated by this algebra of events. It is on the latter that the physical transition amplitudes directly give the decoherence functional. Since the full sigma algebra contains physically interesting questions, like the return question, extending the decoherence functional to these more general questions is important. We show that the decoherence functional, and hence the quantum measure, extends if and only if the associated vector measure does. We give two examples of quantum systems whose decoherence functionals do not extend: one is a unitary system with finitely many states, and the other is a quantum sequential growth model for causal sets. These examples fail to extend in the formal mathematical sense and we speculate on whether the conditions for extension are unphysically strong.

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1. Introduction

The need for a measurement-independent interpretation of quantum theory is perhaps most keenly felt when constructing a theory of quantum cosmology. In describing the physics of the very early universe, we are confronted with the dilemma of how to interpret the quantum formalism in the absence of external measurements. The standard interpretation places an emphasis on the state vector and gives primary status to external measurements but it is questions of spacetime form that are of interest in quantum cosmology, e.g. how likely is it for a homogeneous and isotropic universe to arise from an initial big-bang-type singularity?

In the quantum measure theory formulation of quantum theory [1–6] measuring devices play no fundamental role. Quantum theory is viewed as a generalization of classical stochastic theory, where conceptual primacy is given to the sample space of histories or spacetime configurations, \( \Omega \), which is summed over in the path integral. In analogy with classical
stochastic systems a quantum system is described by a quantum measure space, \((\Omega, \mathcal{A}, \mu)\), where \(\mathcal{A}\) is an event algebra or set of propositions about the system and dynamical information is contained in the quantum measure \(\mu : \mathcal{A} \rightarrow \mathbb{R}^+\) which is given by the path integral. \(\mu\) obeys the quantum sum rule \([1]\)

\[
\mu(\alpha \cup \beta \cup \gamma) = \mu(\alpha \cup \beta) + \mu(\alpha \cup \gamma) + \mu(\beta \cup \gamma) - \mu(\alpha) - \mu(\beta) - \mu(\gamma)
\]

for all pairwise disjoint sets \(\alpha, \beta, \gamma \in \mathcal{A}\). \(\mu\) is not in general a probability measure (even with the normalization \(\mu(\Omega) = 1\)) since it does not satisfy the Kolmogorov sum rule in the presence of quantum interference: \(\exists \alpha, \beta \in \mathcal{A}\) with \(\alpha \cap \beta = \emptyset\) and \(\mu(\alpha \cup \beta) \neq \mu(\alpha) + \mu(\beta)\). Thus, quantum measure theory is a genuine generalization of classical stochastic dynamics. This measure theoretic expression of quantum theory moreover admits an interpretation which is independent of measuring devices as described in \([7, 8]\). In particular, as in classical stochastic theory, an ‘observable’ is simply a measurable set in \(\Omega\), i.e. an element of \(\mathcal{A}\). As we will explore in this work, in models of quantum cosmology, this histories-based approach can give rise to genuine covariant observables \([9]\).

In classical stochastic theories, the probability measure on the event algebra is often defined indirectly in terms of transition probabilities from one momentary state to another. The classical random walk is a standard example. The transition probabilities define, directly, the probability of events which are limited in time, for example: ‘is the walker at site \(x\) at time \(t\)?’ Given an initial position at say \(t = 0\), one calculates the probabilities of each of the walks with \(t\) steps that end at \(x\) using the transition probabilities and adds them together. The probability of certain physically interesting events, however, cannot be directly calculated in this way. These questions involve arbitrarily long times and are epitomized by the return question: ‘does the walker ever return to the origin?’ In order to find the probability of such events, one must be able to extend the probability measure on the finite-time events to infinite-time events. For non-negative measures this is guaranteed by the Carathéodory–Kolmogorov extension theorem, which gives a unique extension of the measure on the algebra of finite-time events to the sigma algebra it generates. The sigma algebra is closed under countable unions and intersections and one can show that the ‘return event’ is an element of this algebra \([10, 11]\).

Similarly, in most quantum systems the quantum measure derives from transition amplitudes from one momentary state to another. To make predictions about infinite-time events, like the quantum analogue of the return question, requires an extension of the quantum measure to an algebra that includes such events. There is however no known analogue of the Carathéodory–Kolmogorov extension theorem for quantum measures. In this paper we take first steps in investigating this issue. The technical development that helps this analysis is the histories Hilbert space construction from quantum measures which derives from a decoherence functional \([12]\). We show that the quantum measure is equivalent to a derived vector measure, i.e. a measure valued in this histories Hilbert space. Unlike the quantum measure, vector measures are additive and have been studied extensively in the literature \([13]\). In this paper, we address the question of extension of the quantum measure by studying the derived vector measure for a class of systems in which the histories Hilbert space is finite dimensional.

In section 2, we define the quantum vector ‘pre-measure’ on the algebra of finite-time events after reviewing the histories Hilbert space construction of \([12]\). In section 3, we examine finite-dimensional unitary systems which evolve in discrete time steps and show that for a generic evolution, the quantum vector pre-measure does not extend to a quantum vector measure on the sigma algebra. This suggests that infinite time questions may not be observables in these theories.
One of the main motivations for studying quantum measure theory is to be able to address questions in quantum cosmology and in section 4 we examine a ‘complex percolation’ dynamics of causal set quantum gravity. In causal set theory, the spacetime continuum is replaced by a discrete substructure, the causal set which is a locally finite partially ordered set. The elements of the causal set represent spacetime events and the partial order between the elements represents the causal relationships between these events [14]. A generic causal set has no continuum spacetime approximation, however, and it is only via an appropriate choice of dynamics on the set of all possible causal sets that one expects a continuum spacetime to emerge. Complex percolation dynamics generalizes the classical transitive percolation sequential growth dynamics studied in [9, 15, 16]. Here, a causal set is grown element by element starting from a single element, and amplitudes are assigned to each transition. This growth process generates, stochastically, a causal set in which each element carries the label of which stage of the growth it was born at. A straightforward implementation of label invariance (the discrete analogue of general covariance) is possible if the quantum vector pre-measure extends. Although the labelled dynamics is well defined, we find that the quantum vector pre-measure does not extend except when the amplitudes are real and non-negative. The requirement of existence of an extension for quantum sequential growth models could therefore be viewed as an additional constraint on the quantum dynamics.

In section 5 we discuss the implications of our results. Using an example of a finite unitary system we show that the lack of an extension can be related to other pathologies. In particular, we show that every element of the finite time event algebra, $\mathfrak{A}$, except $\Omega$ itself, is contained in an element of $\mathfrak{A}$ of zero measure. Therefore, if we adopt the predictive rule that any event contained in a set of zero measure does not occur, this means that no finite time event can occur in this model. For the complex percolation-type models, the question of an extension is tied closely to the construction of covariant observables. For real amplitudes this gives rise to the same set of observables as those defined in [9]. We also discuss the possibility that the requirement for an extension to the full sigma algebra may be too strong a constraint. The extension requires an unconditional convergence of the quantum measure, but it seems possible to use the structure of the event algebra $\mathfrak{A}$ to define a weaker conditional convergence. Whether such a program can be carried out rigorously is a matter of future research.

It is the purpose of this work to lay some of the groundwork for future investigations into physically realistic examples for which the histories Hilbert space is infinite dimensional. These include the Schrödinger particle and the quantum random walk [4, 12]. The existence of an extension of the quantum vector pre-measure in this infinite-dimensional case requires a weaker condition than in the finite-dimensional case, but there are indications that even this is not satisfied for the Schrödinger particle [17]. This may suggest that time of passage questions, notoriously difficult to pose in the standard approach to quantum theory, are not observables even within the quantum measure approach.

2. The quantum vector measure

In this section we show that a quantum measure space in which the quantum measure derives from a strongly positive decoherence functional is equivalent to a vector measure which takes its values in a Hilbert space.

In order to formulate quantum dynamics as a measure space, the sample space $\Omega$ is taken to be the set of histories summed over in the path integral. For a single particle in $\mathbb{R}^3$ whose evolution starts at some initial time, this is the space of all trajectories that are infinite to the future, while for a scalar field in $\mathbb{R}^3 \times \mathbb{R}$ it is the set of all spacetime field configurations. An event algebra $\mathfrak{A}$ over the sample space $\Omega$ is a collection of subsets of $\Omega$ that forms an algebra.
or field of sets over \( \Omega \). Thus, (i) \( \alpha \in \mathcal{A} \Rightarrow \alpha^c \in \mathcal{A} \), where \( \alpha^c \) is the complement of \( \alpha \) in \( \Omega \) and (ii) \( \alpha \cap \beta \in \mathcal{A} \) and \( \alpha \cup \beta \in \mathcal{A} \) for any \( \alpha, \beta \in \mathcal{A} \). A sigma algebra \( \mathcal{S} \) satisfies, in addition to (i) and (ii), closure under countable unions. The sigma algebra \( \mathcal{S}_\Omega \) generated by an algebra \( \mathcal{A} \) is defined to be the (unique) smallest sigma algebra containing \( \mathcal{A} \). In what follows, in order to distinguish between a measure on an algebra and that on a sigma algebra, we will refer to the former as a \textit{pre-measure} and the latter as a measure (thus, a pre-measure is a measure if the algebra on which it is defined is a sigma algebra).

A decoherence functional is a complex function \( D: \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C} \) which represents the quantum interference between two events. \( D \) is [4]

1. Hermitian: for all \( \alpha, \beta \in \mathcal{A} \), we have \( D(\alpha, \beta) = D^*(\beta, \alpha) \).
2. Finitely bi-additive: for any \( \alpha \in \mathcal{A} \) and \( m \) mutually disjoint \( \beta_i \in \mathcal{A} \), we have
   \[
   D(\alpha, \bigcup_{i=1}^m \beta_i) = \sum_{i=1}^m D(\alpha, \beta_i).
   \]
3. Normalized: \( D(\Omega, \Omega) = 1 \).
4. Strongly positive: for any finite collection of \( \{\alpha_n\} \) in \( \mathcal{A} \), the matrix \( M_{ij} \equiv D(\alpha_i, \alpha_j) \) is positive semi-definite, i.e. it has non-negative eigenvalues.

The quantum pre-measure \( \mu: \mathcal{A} \rightarrow \mathbb{R}^+ \) derives from the decoherence functional via \( \mu(\alpha) \equiv D(\alpha, \alpha) \) and we see that the bi-additivity of \( D \) means that \( \mu \) satisfies the quantum sum rule (1) but \( \mu(\alpha \cup \beta) \neq \mu(\alpha) + \mu(\beta) \) if \( \text{Re}(D(\alpha, \beta)) \neq 0 \) for disjoint \( \alpha \) and \( \beta \). In what follows we will refer interchangeably to both \( D \) and \( \mu \) as the \textit{quantum pre-measure} [5]. The construction in [12] of a Hilbert space from the event algebra \( \mathcal{A} \) and the decoherence functional \( D \) implies that the quantum measure is equivalent to a Hilbert space-valued measure which is additive, unlike the quantum measure [5]. This gives us a useful ‘vector measure’ avatar of the quantum measure. We now briefly review vector measures and pre-measures.

A \textit{vector pre-measure} [13] \( \eta \) is a function from an algebra \( \mathcal{A} \) over \( \Omega \) to a Banach space \( \mathcal{B} \) which is \textit{finitely additive}, i.e. for every disjoint pair \( \alpha, \beta \in \mathcal{A} \)

\[
\eta(\alpha \cup \beta) = \eta(\alpha) + \eta(\beta).
\]
If \( \mathcal{S} \) is a sigma algebra, a \textit{vector measure} \( \eta_\mathcal{S}: \mathcal{S} \rightarrow \mathcal{B} \) is moreover required to be \textit{countably additive}

\[
\eta_\mathcal{S}\left( \bigcup_{n=1}^{\infty} \alpha_n \right) = \sum_{n=1}^{\infty} \eta_\mathcal{S}(\alpha_n)
\]
in the norm topology of \( \mathcal{B} \) for \textit{all} sequences \( \alpha_n \) of pairwise disjoint members of \( \mathcal{S} \). The sum \( \sum_{n=1}^{\infty} \eta_\mathcal{S}(\alpha_n) \) must therefore converge unconditionally in the norm.

The Banach space of interest to us is the histories Hilbert space \( \mathcal{H} \) of [12] and we briefly review this construction below. Let \( V \) be the space of complex-valued functions on \( \mathcal{A} \) which are non-zero only on a finite number of elements of \( \mathcal{A} \). \( V \) is the free vector space over \( \mathcal{A} \) and the decoherence functional provides an inner product

\[
\langle u, v \rangle_V \equiv \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{A}} u^*(\alpha) v(\beta) D(\alpha, \beta).
\]
\( V \) is itself not a Hilbert space since it contains non-zero zero-norm vectors and may not be complete. The histories Hilbert space \( \mathcal{H} \) is constructed by taking the set of Cauchy sequences \( \{u_i\} \) in \( V \) and quotienting by the equivalence relation

\[
[u_i] \sim [v_i] \quad \text{if} \quad \lim_{i \rightarrow \infty} \|u_i - v_i\|_V = 0,
\]
3 While \( \mu \) has no standard measure theoretic analogue, since it is not additive, the decoherence functional belongs to the class of ‘bi-additive complex-valued pre-measures’, also called ‘bi-measures’ or ‘poly-measures’ [18].
4 We thank Rafael Sorkin for this observation.
Moreover, the countable additivity of $D(\Omega_1, \Omega_1)$ is said to be an extension of the decoherence functional $D$ on $\mathcal{A} \times \mathcal{A}$ to a decoherence functional on $\mathcal{A} \times \mathcal{A}$. $D$ is Hermitian, since

$$
\langle \varepsilon_\alpha(\alpha), \varepsilon_\alpha(\beta) \rangle = \langle \varepsilon_\alpha(\beta), \varepsilon_\alpha(\alpha) \rangle^*.
$$

Moreover, the countable additivity of $\varepsilon$ and the countable bi-additivity of the inner product implies countable bi-additivity of $D$. In addition, $D$ is strongly positive. Finally, since $\Omega \in \mathcal{A}$, $D(\Omega, \Omega) = D(\Omega, \Omega) = 1$. Thus, the extension $\mu$ can be used to construct a countably bi-additive, positive and Hermitian decoherence functional on $\mathcal{A}$. If $\varepsilon$ is the unique extension
of $\mu_\nu$, then so is $D$ the unique extension of $D$. (On the other hand, a unique extension $D$ of $D$ yields a $\mu_\nu$ determined only up to an overall phase which itself can be determined if $\mu_\nu$ is also known.)

For a non-negative scalar pre-measure $\mu : \mathcal{A} \to \mathbb{R}^+$, the Carathéodory extension theorem [10, 11] guarantees the existence of a unique extension. For complex vector measures, however, a unique extension is not guaranteed and one requires additional conditions on the vector pre-measure\(^5\). In the case of finite-dimensional vector measures, one of these conditions is that the vector pre-measure should be of bounded variation. The total variation $|\mu_\nu|$ of a vector pre-measure $\mu_\nu$ is given by

$$|\mu_\nu|(\alpha) = \sup_{\pi(\alpha)} \sum_{\rho} \|\mu_\nu(\alpha_\rho)\|,$$

where the supremum is over all finite partitions $\pi(\alpha) = \{\alpha_\rho\}$ of $\alpha$ (note that $|\mu_\nu|(\alpha)$ is not just $|\mu_\nu(\alpha)|$). $|\mu_\nu|$ is itself a non-negative finitely additive pre-measure on $\mathcal{A}$ and is countably additive iff $\mu_\nu$ is (proposition 9, chapter 1.1, [13]). $\mu_\nu$ is said to be of bounded variation if $|\mu_\nu(\alpha)| < \infty$ for all $\alpha \in \mathcal{A}$.

We note that in any basis the components $\mu_\nu^{(i)}$, $i = 1, \ldots, n$, of a vector pre-measure $\mu_\nu : \mathcal{G} \to \mathbb{C}^n$ are themselves complex-valued pre-measures on $\mathcal{G}$.

**Claim 1.** Let $\mu_\nu : \mathcal{A} \to \mathbb{C}^n$ be a vector pre-measure and $\mu_\nu^{(i)} : \mathcal{A} \to \mathbb{C}$, $i = 1, \ldots, n$, be the components of $\mu_\nu$ in an orthonormal basis. Then $\mu_\nu$ is of bounded variation iff $\mu_\nu^{(i)}$ is of bounded variation.

**Proof.** Let $\|\cdot\|$ and $|\cdot|$ be the $\mathbb{C}^n$ and $\mathbb{C}$ norms, respectively. For any $\alpha \in \mathcal{A}$ and any $i \in \{1, \ldots, n\}$ since $\|\mu_\nu^{(i)}(\alpha)\| \geq |\mu_\nu^{(i)}(\alpha)|$

$$|\mu_\nu^{(i)}(\alpha)| \leq \sum_{i=1}^{n} |\mu_\nu^{(i)}(\alpha)|.$$  

(15)

Therefore, if $\mu_\nu$ is of bounded variation then so is $\mu_\nu^{(i)}$. From the triangle inequality

$$\|\mu_\nu(\alpha)\| \leq \sum_{i=1}^{n} |\mu_\nu^{(i)}(\alpha)|.$$  

(16)

Thus, for any finite partition $\{\alpha_\rho\}$ of $\alpha$, $\rho \in \{1, \ldots, m < \infty\}$

$$\sum_{\rho=1}^{m} \|\mu_\nu(\alpha_\rho)\| \leq \sum_{\rho=1}^{m} \sum_{i=1}^{n} |\mu_\nu^{(i)}(\alpha_\rho)| \leq \sum_{i=1}^{n} |\mu_\nu^{(i)}(\alpha)|.$$  

(17)

Clearly, if $|\mu_\nu^{(i)}(\alpha)| \leq b_i < \infty$, i.e. it is bounded for each $i$, then

$$\sum_{\rho=1}^{m} \|\mu_\nu(\alpha_\rho)\| \leq \sum_{i=1}^{n} b_i$$

for every finite partition of $\alpha$. Hence $\mu_\nu$ is also bounded. \hfill \Box

A complex measure $\mu_\nu : \mathcal{G} \to \mathbb{C}$ is required to be countable additive in an unconditionally convergent sense [19]: for any $\alpha \in \mathcal{G}$, $\mu_\nu(\alpha) = \sum_{i=1}^{\infty} \mu_\nu(\alpha_i)$ for every partition $\{\alpha_i\}$ of $\alpha$. This unconditional convergence implies that its total variation is bounded [19]. Hence for a countably additive complex pre-measure on $\mathcal{A}$ to extend to a complex measure on $\mathcal{G}_\mathcal{A}$, it must also be of bounded variation. Now, the components $\mu_\nu^{(i)}$, $i = 1, \ldots, n$, of a countably additive

5 The Carathéodory–Hahn–Kluvanek extension theorem [13] gives necessary and sufficient conditions for a vector pre-measure on $\mathcal{A}$ to extend to a vector measure on $\mathcal{G}_\mathcal{A}$.

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Footnotes:

5. The Carathéodory–Hahn–Kluvanek extension theorem [13] gives necessary and sufficient conditions for a vector pre-measure on $\mathcal{A}$ to extend to a vector measure on $\mathcal{G}_\mathcal{A}$. 

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References:

[10, 11] guarantees the existence of a unique extension. For complex vector measures, however, a unique extension is not guaranteed and one requires additional conditions on the vector pre-measure\(^5\). In the case of finite-dimensional vector measures, one of these conditions is that the vector pre-measure should be of bounded variation. The total variation $|\mu_\nu|$ of a vector pre-measure $\mu_\nu$ is given by

$$|\mu_\nu|(\alpha) = \sup_{\pi(\alpha)} \sum_{\rho} \|\mu_\nu(\alpha_\rho)\|.$$  

(14)

where the supremum is over all finite partitions $\pi(\alpha) = \{\alpha_\rho\}$ of $\alpha$ (note that $|\mu_\nu|(\alpha)$ is not just $|\mu_\nu(\alpha)|$). $|\mu_\nu|$ is itself a non-negative finitely additive pre-measure on $\mathcal{A}$ and is countably additive iff $\mu_\nu$ is (proposition 9, chapter 1.1, [13]). $\mu_\nu$ is said to be of bounded variation if $|\mu_\nu(\alpha)| < \infty$ for all $\alpha \in \mathcal{A}$.

We note that in any basis the components $\mu_\nu^{(i)}$, $i = 1, \ldots, n$, of a vector pre-measure $\mu_\nu : \mathcal{G} \to \mathbb{C}^n$ are themselves complex-valued pre-measures on $\mathcal{G}$.

**Claim 1.** Let $\mu_\nu : \mathcal{A} \to \mathbb{C}^n$ be a vector pre-measure and $\mu_\nu^{(i)} : \mathcal{A} \to \mathbb{C}$, $i = 1, \ldots, n$, be the components of $\mu_\nu$ in an orthonormal basis. Then $\mu_\nu$ is of bounded variation iff $\mu_\nu^{(i)}$ is of bounded variation.

**Proof.** Let $\|\cdot\|$ and $|\cdot|$ be the $\mathbb{C}^n$ and $\mathbb{C}$ norms, respectively. For any $\alpha \in \mathcal{A}$ and any $i \in \{1, \ldots, n\}$ since $\|\mu_\nu^{(i)}(\alpha)\| \geq |\mu_\nu^{(i)}(\alpha)|$

$$|\mu_\nu^{(i)}(\alpha)| \leq \sum_{i=1}^{n} |\mu_\nu^{(i)}(\alpha)|.$$  

(15)

Therefore, if $\mu_\nu$ is of bounded variation then so is $\mu_\nu^{(i)}$. From the triangle inequality

$$\|\mu_\nu(\alpha)\| \leq \sum_{i=1}^{n} |\mu_\nu^{(i)}(\alpha)|.$$  

(16)

Thus, for any finite partition $\{\alpha_\rho\}$ of $\alpha$, $\rho \in \{1, \ldots, m < \infty\}$

$$\sum_{\rho=1}^{m} \|\mu_\nu(\alpha_\rho)\| \leq \sum_{\rho=1}^{m} \sum_{i=1}^{n} |\mu_\nu^{(i)}(\alpha_\rho)| \leq \sum_{i=1}^{n} |\mu_\nu^{(i)}(\alpha)|.$$  

(17)

Clearly, if $|\mu_\nu^{(i)}(\alpha)| \leq b_i < \infty$, i.e. it is bounded for each $i$, then

$$\sum_{\rho=1}^{m} \|\mu_\nu(\alpha_\rho)\| \leq \sum_{i=1}^{n} b_i$$

for every finite partition of $\alpha$. Hence $\mu_\nu$ is also bounded. \hfill \Box

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Footnotes:

5. The Carathéodory–Hahn–Kluvanek extension theorem [13] gives necessary and sufficient conditions for a vector pre-measure on $\mathcal{A}$ to extend to a vector measure on $\mathcal{G}_\mathcal{A}$.
vector measure $\overline{\mu}_V : \mathcal{G} \to \mathbb{C}^n$ are also countably additive, and hence are of bounded variation. Thus, by the above claim, the countable additivity of $\overline{\mu}_V$ implies that it is of bounded variation.

Bounded variation of a complex measure implies that its restriction $\mu|_\mathcal{A}$ to any subalgebra $\mathcal{A} \subset \mathcal{G}$ is also of bounded variation. Thus, a necessary condition for a complex pre-measure on $\mathcal{A}$ to extend to a measure on $\mathcal{G}|_\mathcal{A}$ is that it is of bounded variation. Along with the above results this means that

**Claim 2.** Bounded variation is a necessary condition for a finite-dimensional vector pre-measure $\mu_V : \mathcal{A} \to \mathbb{C}^N$ to extend to a vector measure $\overline{\mu}_V : \mathcal{G}|_\mathcal{A} \to \mathbb{C}^N$.

For the purposes of this work, therefore, the existence of an extension of the vector pre-measure to a vector measure reduces to whether it is of bounded variation or not.

### 3. Finite unitary systems

In this section we consider the class of finite $N$-dimensional systems which evolve unitarily in discrete unit time steps, $t = 1, 2, 3, \ldots$. In the standard Hilbert space formulation, the Hilbert space at time $t = m$ is $\mathcal{H}_m = \mathbb{C}^N$. The evolution of a state $\psi \in \mathcal{H}_1$ at an initial time $t = 1$ to a state $\psi_m \in \mathcal{H}_m$ at time $t = m$ is governed by the $N \times N$ single-step unitary matrices $U(k + 1, k)$:

$$\psi_m = U(m, m - 1)U(m - 1, m - 2)\ldots U(2, 1)\psi. \quad (18)$$

Let $\{e_1, e_2, \ldots, e_N\}$ be an orthonormal basis for $\mathcal{H}_1$.

To describe this system as a quantum measure space we first identify a history as an infinite string $\gamma = (s_1, s_2, \ldots, s_m)$, where each entry $s_i \in \{1, \ldots, N\}$. The configuration space has $N$ sites, each associated with one of the basis vectors $e_i$. $\Omega$ is the infinite collection of all such strings and the event algebra $\mathcal{A}$ is generated as follows. We associate with every length $m$ finite string $\gamma = (s_1, s_2, \ldots, s_m)$ a cylinder set

$$\text{cyl}(\gamma) \equiv \{\gamma^{(\infty)} \in \Omega | \gamma^{(\infty)}(i) = s_i, i = 1, 2, \ldots, m\}, \quad (19)$$

which is the set of histories for which the first $m$ entries are specified by the string $\gamma$ but are unspecified thereafter. For a fixed $m$ there are $N^m$ such cylinder sets corresponding to the $N^m$ possible length $m$ strings. If $\gamma = (s_1, s_2, \ldots, s_m)$ and $\gamma' = (s'_1, s'_2, \ldots, s'_m)$, with $m' > m$, then

$$\text{cyl}(\gamma') \subset \text{cyl}(\gamma) \quad \text{if} \quad s_i = s'_i \forall i \in \{1, 2, \ldots, m\} \quad (20)$$

$$\text{cyl}(\gamma') \cap \text{cyl}(\gamma) = \emptyset \quad \text{otherwise.} \quad (21)$$

If $\mathcal{A}$ represents the algebra generated from finite unions and intersections of these cylinder sets, then (20) and (21) imply that any $\alpha \in \mathcal{A}$ can be expressed as a finite disjoint union of cylinder sets. In particular, for every $\alpha \in \mathcal{A}$ there exists an $m$ and a $k \leq N^m$ such that $\alpha = \bigcup_{(\gamma)}^k \text{cyl}(\gamma)$, where the $\gamma$ are strings of length $m$ and the cyl($\gamma$) are mutually disjoint.

The decoherence functional for such a unitary system is given by

$$D(\text{cyl}(\gamma), \text{cyl}(\gamma')) = A^*(\gamma)A(\gamma')\delta_{\gamma\gamma'} \quad (22)$$

where $\gamma, \gamma'$ are length $m$ finite strings (or ‘truncated histories’) and $A(\gamma), A(\gamma')$ are complex amplitudes. Assuming an initial state $\psi \in \mathcal{H}_1$,

$$A(\text{cyl}(\gamma)) = U_{s_1s_2}(m, m - 1)U_{s_2s_3}(3, 2)U_{s_3s_1}(2, 1)\psi(s_1). \quad (23)$$
Here $U_{s_{i-1}s_i}(i, i-1)$ is the amplitude to go from $s_{i-1}$ at $t = i-1$ to $s_i$ at $t = i$ and $\psi(s_1) = s_i, \psi$).

The decoherence functional on the full event algebra $\mathcal{A}$ is then obtained using its bi-additivity property.

The restricted evolution from the initial state $\psi \in \mathcal{H}_1$ with respect to a truncated history $\gamma$ is defined to be

$$\tilde{\psi}_\gamma \equiv \tilde{C}_\gamma \psi \in \mathcal{H}_m,$$

(24)

where the class operator

$$\tilde{C}_\gamma \equiv P_{s_m} U(m, m-1) P_{s_{m-1}} U(m-1, m-2) \ldots U(2, 1) P_{s_1}$$

(25)

and $P_s$ is the projector that projects onto the basis states. Evolving $\tilde{\psi}_\gamma$ back to the initial time gives us the state

$$\psi_\gamma \equiv (U(m, m-1) U(m-1, m-2) \ldots U(2, 1))^\dagger \tilde{C}_\gamma \psi \in \mathcal{H}_1,$$

(26)

and it can be shown that the decoherence functional

$$D(\text{cyl}(\gamma), \text{cyl}(\gamma')) \equiv \langle \psi_\gamma, \psi_{\gamma'} \rangle.$$  

(27)

For any $\alpha \in \mathcal{A}$ and a partition into cylinder sets $\alpha = \bigcup_{l=1}^k \text{cyl}(\gamma^l),$ one can also define the state

$$\psi_\alpha = \sum_{l=1}^k \psi_{\gamma^l}$$

(28)

and show that

$$\langle \psi_\alpha, \psi_\beta \rangle = D(\alpha, \beta).$$

(29)

Note that the nested property of cylinder sets (equation (21)) ensures that the state (equation (28)) is independent of the choice of partition. If $\Pi_1 = \{\alpha_1^i\}$ and $\Pi_2 = \{\alpha_2^i\}$ are two distinct partitions of $\alpha$ into cylinder sets, with $i \in I$ and $j \in J$ the respective index sets, then for every $j_1 \in J$ either (i) there is a set $\{j_1, j_2, \ldots, j_s\}$ of (non-repeating) indices in $J$ and an $i \in I$ such that $\alpha_1^i = \alpha_2^j \cup \alpha_2^{j_2} \ldots \cup \alpha_2^{j_s}$ or (ii) there is a set $\{i_1, i_2, \ldots, i_t\}$ of (non-repeating) indices in $J$ such that $\alpha_2^i = \alpha_1^{i_1} \cup \alpha_1^{i_2} \ldots \cup \alpha_1^{i_t}$. Combining it with the linearity of expression (28) suffices to show that it is independent of the choice of partition.

In [12] it was shown that for a finite $N$-dimensional system which evolves unitarily with discrete time steps there is, generically, an explicit, physically meaningful isomorphism $f : \mathcal{H} \to \mathcal{H}_1$ between the histories Hilbert space $\mathcal{H}$ and the standard Hilbert space $\mathcal{H}_1 = \mathbb{C}^N$. Namely,

$$f([\{u_i\}]) \equiv \lim_{i \to \infty} f_0(u_i),$$

(30)

with $f_0 : V \to \mathcal{H}_0$ given by

$$f_0(u) = \sum_{\alpha \in \mathcal{A}} u(\alpha) \psi_\alpha.$$  

(31)

Here $\langle f_0(u), f_0(v) \rangle = \langle u, v \rangle$ and hence $\langle f([\{u_i\}]), f([\{v_i\}]) \rangle = \langle [\{u_i\}], [\{v_i\}] \rangle$. The quantum vector pre-measure $\mu_\psi : \mathcal{A} \to \mathcal{H}$ is therefore mapped to a vector measure $\tilde{\mu}_\psi \equiv f \circ \mu_\psi$ on $\mathcal{H}_1$:

$$\tilde{\mu}_\psi(\alpha) = \sum_{\beta \in \mathcal{A}} \chi_\alpha(\beta) \psi_\beta$$

(32)

Claim 3. The quantum vector measure for a generic (to be defined) finite unitary system with discrete time steps is not of bounded variation.
Proof. In what follows we will first assume that $U$ is time independent so that $U(k+1,k) = U$ for all $k$, where $U$ is the single time-step evolution operator.

For a string of length $m$, $\gamma = (j_1, j_2, \ldots, j_m)$,

$$
\mu_\gamma(\text{cyl}(\gamma)) = (U^\dagger)^m U_{j_m,j_{m-1}} U_{j_{m-1},j_{m-2}} \ldots U_{j_2,j_1} \beta_j e_{j_1}
$$

(33)

where $U_{jk} = \langle e_j, U e_k \rangle$ and $\beta_j = \langle e_j, \psi \rangle$, so that

$$
\|\mu_\gamma(\text{cyl}(\gamma))\| = |\beta_j| |U_{j_m,j_{m-1}} U_{j_{m-1},j_{m-2}} \ldots U_{j_2,j_1}|.
$$

(34)

A string $\gamma'$ of length $m+k$ will be said to be an extension of $\gamma$ if the first $m$ entries of $\gamma'$ are the same, so that $\text{cyl}(\gamma') \subset \text{cyl}(\gamma)$. The set of all $n+k$ extensions $\{\text{cyl}(\gamma'|_{i,i+1})\}$ of $\gamma$ thus provides a partition of $\text{cyl}(\gamma)$ so that

$$
|\mu_\gamma(\text{cyl}(\gamma))| \geq \sum_{i_1,i_2,\ldots,i_k=1}^N \|\mu_\gamma(\text{cyl}(\gamma'|_{i,i+1}))\|
$$

$$
= \left( \sum_{i_1,i_2,\ldots,i_k=1}^N |U_{i_1,i_2-1}||U_{i_2,i_3-1}||\ldots||U_{i_k,1}| \right) \|\mu_\gamma(\text{cyl}(\gamma'))\|.
$$

(35)

Since $U$ is unitary, $\sum_{i=1}^N |U_{ij}|^2 = 1$ and $\sum_{j=1}^N |U_{ij}|^2 = 1$. Thus,

$$
\sum_{i=1}^N |U_{ij}| = 1 + \xi_j
$$

(36)

$$
\sum_{j=1}^N |U_{ij}| = 1 + \eta_i
$$

(37)

where $\xi_j \geq 0$ and $\eta_i \geq 0$. If $\xi_j = 0$ there exists an $i$ such that $|U_{ij}| = 1$ and $U_{i'j} = 0$ for all $i' \neq i$. Similarly, for every $j' \neq j$, $U_{ij'} = 0$. Hence the $i$th row and the $j$th column each have a single non-zero entry $U_{ij}$ which is pure phase.

We now define the genericity assumption for the time-independent case to be that $U$ does not have such entries, i.e. $\xi_j > 0$ for all $j$, and hence $\eta_i > 0$ for all $i$. This excludes dynamics which consist of simple permutations of a number of the site. This assumption means that there is a smallest strictly positive $\xi \leq \xi_j$ for all $j$. This allows us to iteratively bound the term in brackets in (35)

$$
\sum_{i_1,i_2,\ldots,i_k=1}^N \left( \sum_{i_k} |U_{i_k,i_{k-1}}| \right) \times |U_{i_{k-1},i_{k-2}}| \cdots |U_{i_1,1}|
$$

$$
= \sum_{i_1,i_2,\ldots,i_k=1}^N \left( \sum_{i_k=1}^N (1 + \xi_{i_k}) |U_{i_{k-1},i_{k-2}}| \right) \cdots |U_{i_1,1}|
$$

$$
\geq (1 + \xi) \sum_{i_1,i_2,\ldots,i_k=1}^N \left( \sum_{i_k=1}^N |U_{i_{k-1},i_{k-2}}| \right) \cdots |U_{i_1,1}|
$$

$$
\geq (1 + \xi)^k.
$$

(38)

Hence for every $k$,

$$
|\mu_\gamma(\text{cyl}(\gamma))| \geq (1 + \xi)^k \times \|\mu_\gamma(\text{cyl}(\gamma'))\|.
$$

(39)

Since $\xi > 0$, $|\mu_\gamma|$ is not bounded.
For time-dependent $U$ the argument is similar, except that the $\zeta$ are now time dependent.

In particular, instead of extracting a time-independent factor $(1 + \zeta)$ in equation (38) at every step, it becomes time-dependent so that

$$|\tilde{\mu}_r|(\text{cyl}(\gamma)) \geq \prod_{r=1}^{m} (1 + \zeta_r) |\tilde{\mu}_r(\text{cyl}(\gamma))|,$$

(40)

where $\sum_{r=1}^{N} |U_{ij}(r, r-1)| = 1 + \zeta_j(r)$ and $\zeta_j$ is the lowest value of $\zeta_j(r)$ as one varies over $j$. Again assuming that $\zeta_r > 0$, the product $\prod_{r=1}^{m} (1 + \zeta_r)$ converges as $m \to \infty$ only if $\sum_{r=1}^{m} \zeta_r$ does. The genericity condition is therefore that $\sum_{r=1}^{m} \zeta_r$ diverges as $m \to \infty$ in which case (40) diverges and the pre-measure is not of bounded variation.

Thus, the quantum vector pre-measure $\tilde{\mu}_c$ on $\mathcal{A}$ does not extend to a quantum vector measure on $\mathcal{E}_\mathcal{A}$ for generic finite-dimensional unitary systems with discrete time steps.

4. Complex percolation

In causal set theory, the histories space of continuum spacetime geometries is replaced by the collection of causal sets. A causal set $C$, as defined in [14], is a locally finite partially ordered set, namely a countable collection of elements, with an order relation $\prec$ which for all $x, y, z \in C$ is (i) transitive ($x \prec y, y \prec z \Rightarrow x \prec z$), (ii) irreflexive ($x \not\prec x$) and (iii) locally finite, i.e. if $\text{Past}(x) \equiv \{w \in C | w \prec x\}$ and $\text{Fut}(x) \equiv \{w \in C | w \succ x\}$, then the cardinality of the set $\text{Past}(x) \cap \text{Fut}(y)$ is finite. We say that two elements are linked if they are related in the order but there is no element between them in the order.

A causal set is a model for discrete spacetime in which the elements of $C$ represent spacetime events and the partial order represents the causal relationships between events. A generic causal set has no continuum spacetime approximation, however, and it is only via an appropriate choice of dynamics on the set of all possible causal sets that one expects a continuum spacetime to emerge.

The transitive percolation dynamics for causal sets is a classical stochastic dynamics and was studied in detail in [15, 16] and is determined by a single coupling constant $p \in [0, 1]$. Here, a causal set $C$ is ‘grown’ element by element starting with a single element. At stage $n \geq 1$ the $n$th element $e_n$ is born and for each $k = 1, 2, \ldots, n - 1$ independently, $e_n$ is put to the immediate future of $e_k$ with probability $p$ or with probability $1 - p$ left unrelated to $e_k$.

The transitive closure is then taken and the stage $n$ is complete. The procedure of transitive closure automatically takes care of elements which are related to the new element but are not to its immediate past. The resulting causal set $C$ grown in this way is ‘labelled’ by the growth, namely each element is labelled by the stage at which it is born (see figure 1). Thus, the growth process is stochastic and produces a labelled causal set of infinite cardinality in the asymptotic limit $n \to \infty$. Such a causal set is always past finite, i.e. the cardinality of the past set $\text{Past}(x) = \{y | y \prec x\}$ is finite for all $x \in C$. Even though the causal sets produced are labelled, the resultant dynamics satisfies a discrete form of general covariance in that the probabilities of growing, by stage $n$, two labelled causal sets are the same if there is an order-preserving isomorphism between them. In addition, the dynamics satisfies the ‘Bell causality’ condition described in [15]. The labelling of a causal set $C$ produced via a growth is always order preserving namely for any $e, e' \in C$, $e \prec e'$ implies that $l(e) < l(e')$, where $l(e)$ is the label of the element $e$.

Figure 1 show the first few stages of this growth process. Let $C^n_n$ denote the $n$-chain or the totally ordered set of $n$ elements and $C^n_n$ the $n$-antichain or the set of $n$ mutually unrelated elements. Starting with the first element $e_1$ the second element $e_2$ is added with probability
Figure 1. Transition probabilities up to 3-element-labelled causal sets.

$p \in [0, 1]$ to the future of $e_1$, to get the (uniquely) labelled 2-chain $C_2^c$ or is left unrelated to $e_1$ with probability $q$ to get the (uniquely) labelled 2-antichain $C_2^a$. Since for $n = 2$ these are the only possible two element labelled causal sets, $p + q = 1$ (see figure 1). Subsequently, the three 3-element causal sets are grown from $C_2^c$ and $C_2^a$ as shown in figure 1. In the first transition from $C_2^c$ in the figure (from the left), $e_3$ is added to the immediate future of $e_2$ with probability $p$, in the second, $e_3$ is added to the immediate future of $e_1$ but is unrelated to $e_2$ with probability $p \times q$, and in the third, $e_3$ is unrelated to both $e_1$ and $e_2$ with probability $q^2$. In the figure we see that the middle three 3-element-labelled causal sets are different order-preserving relabellings of the same unlabelled causal set.

The probability $P(C^n)$ for an $n$-element-labelled causal set $C^n$ is equal to the product of the transition probabilities. As can be verified from the examples in figure 1, this probability is independent of the labelling: in general, the probability for a labelled causal set is given by $p^L q^R$ where $L$ is the number of links and $R$ the number of relations [15]. The transition probability for going from an $n$-element causal set $C^n$—a parent—to one of its children $C^{n+1}$ is given by a product of $p$'s and $q$'s: if the new element is to the immediate future of (i.e. linked to) $u$ elements in $C^n$ and unrelated to $v$ elements, then the transition amplitude is $p^u q^v$. For a given parent $C^n$ therefore, one can assign an index set $I(C^n)$ of all pairs $(u, v)$, some of which may repeat. For example, when the parent is $C_2^c$, the index set is $I(C_2^c) = \{(1, 0), (1, 1), (0, 2)\}$ while if the parent is $C_2^a$, the index set is $I(C_2^a) = \{(1, 0), (1, 0), (2, 0), (0, 2)\}$ (see figure 1). Note that the set $I(C^n)$ can contain repeated entries.

Importantly, the transition probabilities from a given parent $C^n$ to its children all add up to one [15, 16]. While this can be verified for the simple causal sets in figure 1 it is not obvious that this is true in general. The rules for transitive percolation however can be simply derived from the following more obviously probabilistic process. Starting with a causal set $C^n$ we can add a new element $e_n$ with probability $p$ for it to be related to an existing element in $C^n$ and $q$ for it not to be. The resulting graph does not necessarily satisfy transitive closure. However, because of the simplicity of the rule it is immediately obvious that summing over all such possibilities gives 1. Taking the transitive closure of the graphs produced this way will then give rise to the causal set children of $C^n$. Thus typically, several graphs will contribute to a single child and hence the probability for each child is obtained by adding up the probabilities for these graphs. Consider a transition $C^n \rightarrow C^{n+1}$ and let the new element be related to $r$ elements in $C^n$ and unrelated to $n - r$ elements. If $m \leq r$ is the number of maximal elements in the set of $r$ elements in $C^n$ which are related to the new element, then one has to sum over
all the possible ways to get $C^{n+1}$ after taking the transitive closure. Each contribution is $p^x q^y$ where $x$ ranges from $m$ to $r$ and $y$ from $n - r$ to $n - m$, and each such contribution occurs \( \binom{r-m}{m} \) times. Thus, the probability for this transition is

\[
\left( \sum_{i=0}^{r-m} p^{m+i} q^{r-m-i} \right) q^{n-r} = p^m (p+q)^{r-m} q^{n-r} = p^m q^n.
\]

This is precisely the rule for transitive percolation.

It is useful to point to two special types of transitions which will make their appearance in our analysis below. The first is the ‘timid transition’ in which the new element is added to the future of all the elements in $C^n$. This is precisely the rule for transitive percolation. The second is the ‘gregarious transition’ in which the new element is unrelated to all the existing elements in $C^n$, so that the transition probability is $q^n$.

Let $\Omega$ be the set of all infinite, past finite, labelled causal sets, and let $c^n$ refer to the sub-causal set of $c \in \Omega$ constructed from its first $n$ elements. The cylinder set associated with a finite element causal set $C^n$

\[
\text{cyl}(C^n) \equiv \{ c \in \Omega | c^n \sim C^n, k = 1, \ldots, n \}
\]

is a subset of $\Omega$, where $\sim$ refers to a label-preserving causal set isomorphism. The event algebra $\mathfrak{A}$ is the set of all finite unions of these cylinder sets and it can be shown that every element of $\mathfrak{A}$ is equal to a finite union of mutually disjoint cylinder sets. In particular, $\text{cyl}(C^n)$ is equal to the union of the (disjoint) cylinder sets of all the children of $C^n$: $\text{cyl}(C^n) = \bigcup_j \text{cyl}(C_{i+j}^{n+1})$. However, unlike the finite unitary systems, the number of $(n+1)$-element children depends on the parent $C^n$.

Complex percolation is a natural quantum generalization of transitive percolation in which real probabilities are replaced by complex amplitudes. Thus, the real parameter $p$ of transitive percolation is made complex and gives the transition amplitude for the newly born element to be put to the future of each existing element, while $q$ is the transition amplitude for it to be unrelated (we will see later that $p + q = 1$). The decoherence functional for complex percolation has a simple product form

\[
D(\text{cyl}(C^n), \text{cyl}(C^{n'})) = A^*(C^n)A(C^n')
\]

for cylinder sets, where $A(C^n)$ is the amplitude for the transition from the empty set to the $n$-element causal set $C^n$. Finite bi-additivity of $D$ and finite additivity of $A$ are equivalent and $A$ is a complex measure on $\mathfrak{A}$. The normalization condition $D(\Omega, \Omega) = 1$ implies that $|A(\Omega)| = 1$. We have for $\alpha, \beta \in \mathfrak{A}$

\[
D(\alpha, \beta) = A^*(\alpha)A(\beta).
\]

It is easy to demonstrate that for any such ‘product’ decoherence functional the histories Hilbert space $\mathcal{H} \cong \mathbb{C}$. Choose a vector $v \in V$, with $\|v\| \neq 0$. Such non-zero norm vectors clearly exist, an example being the vector pre-measure of $\text{cyl}(C^n)$. We show that for every $u \in V$ there exists a $\lambda \in \mathbb{C}$ such that $[u] \sim \lambda[v]$, where $\sim$ is the equivalence relation (equation (5)) and $[u], [v]$ are the constant Cauchy sequences for $u$ and $v$, respectively. Then

\[
\|u - \lambda v\|^2 = |S_1|^2 - (\lambda S_1^* S_2 + \lambda^* S_1 S_2^* + |\lambda|^2)|S_2|^2 = |S_1 - \lambda S_2|^2.
\]

where $S_1 = \sum_{\alpha \in \mathfrak{A}} A(\alpha)u(\alpha)$ and $S_2 = \sum_{\alpha \in \mathfrak{A}} A(\alpha)v(\alpha)$. This factorization is possible because of the product form (equation (44)). If we then choose $\lambda = S_1/S_2$ ($S_2 \neq 0$) we have $\|u - \lambda v\| = 0$.

\[ ^6 \text{Since } c^n \text{ is a sub-causal set, it also preserves all the relations between the first } n \text{ elements in } c.\]
In particular, since for any \( \alpha \in \mathfrak{A} \) and a finite partition \( \pi(\alpha) = \{\alpha_{\rho}\} \) of \( \alpha \)
\[
|\mu_{\pi}(\alpha)| \geq \sum_{\rho} \|\mu_{\pi}(\alpha_{\rho})\| = \sum_{\rho} |A(\alpha_{\rho})|.
\]
(46)
Hence \( |\mu_{\pi}| = |A| \), where \( |A| \) the total variation of \( A \), and \( \mu_{\pi} \) is equal to \( A \) up to a phase.

For the special case \( p \) real and \( p \in [0, 1] \), the transition amplitudes are the same as the transition probabilities of classical transitive percolation. However, this real quantum percolation model is distinct from transitive percolation since the quantum measure \( D(\alpha, \alpha) = |A(\alpha)|^2 \) and is therefore non-additive. However, the amplitude measure \( A \) is additive and non-negative, and the Carathéodory–Kolmogorov extension theorem implies that the quantum vector pre-measure extends to the full-sigma algebra. As we will presently see, this special case is the only one which does admit such an extension.

We first show that \( q \) must equal \( 1 - p \). The normalization condition on \( D \) means that \( |A(\Omega)| = 1 \), so that \( A(\Omega) = \exp(i\Phi) \) and we will choose this phase to be 1. If \( C^1 \) denotes the single element causal set then \( \text{cyl}(C^1) = \Omega \). We also have \( \text{cyl}(C^1_a) = \text{cyl}(C^1_2) \cup \text{cyl}(C^1_{\emptyset}) \) and \( \text{cyl}(C^1_a) \cap \text{cyl}(C^1_{\emptyset}) = \emptyset \), where, as before \( C^1_a \) and \( C^1_{\emptyset} \) are the 2-chain and 2-antichain, respectively. This means
\[
A(\text{cyl}(C^1)) = A(\text{cyl}(C^1_a)) + A(\text{cyl}(C^1_{\emptyset})) = A(\text{cyl}(C^1)) \times (p + q) \implies p + q = 1.
\]
As shown above, the histories Hilbert space for a product decoherence functional is one dimensional. This means that the quantum vector measure is, up to an overall phase, just the amplitude \( A: \mathfrak{A} \to \mathbb{C} \).

We now show that

**Lemma 1.** The quantum vector measure of complex percolation is not of bounded variation when the parameter \( p \) is not real.

We begin by considering the set of all labelled \( n \)-element causal sets \( \{C^n_i\} \), where \( i = 1, 2, \ldots, I(n) \) where \( I(n) \) is the number of \( n \) element-labelled causal sets. For example, for \( n = 2, I(2) = 2 \) so that \( i = 1, 2 \) and for \( n = 3, I(3) = 7 \), so that \( i = 1, \ldots, 7 \). Since \( A(\Omega) = \sum_{i=1}^{I(n)} A(\text{cyl}(C^n_i)) \), and \( |A(\Omega)| = 1 \), the triangle inequality implies that
\[
\sum_{i=1}^{I(n)} |A(C^n_i)| \geq 1,
\]
(47)
where for brevity of notation we have replaced \( \text{cyl}(C^n) \) with \( C^n \). The equality is satisfied only if all the \( A(C^n_i) \) are collinear. We see that

**Claim 4.** For any \( n \geq 2 \), the equality in (47) is satisfied only when the parameter \( p \) is real.

**Proof.** Let \( p = |p| \exp(i\phi), \ q = |q| \exp(i\Phi) \). First, for \( n = 2 \) the equality means that \( |p| + |q| = 1 \), which combined with \( p + q = 1 \) means that \( p \) is real and non-negative. For \( n > 2 \), consider the following two \( n \) element causal sets, (a) the \( n \) chain \( C^n_c \) and (b) \( C^n_{\emptyset} \), an \( n - 2 \) chain topped with a ‘\( V \)’, i.e. with \( e_n, e_{n-1} \) to the immediate future of the maximal element \( e_{n-2} \) of the \( n - 2 \) chain \( C^n_{e_{n-2}} \), and unrelated to each other (see figure 2). The amplitudes for these causal sets are
\[
A(C^n_c) = p^{n-1}, \text{ and } A(C^n_{\emptyset}) = p^{n-1}q.
\]
Requiring collinearity of these amplitudes is therefore equivalent to requiring that \( q \) and hence \( p \) is real and non-negative. \( \square \)

Thus, for \( p \) non-real
\[
\sum_{i=1}^{I(n)} |A(C^n_i)| > 1
\]
(48)
Thus, for all \( n \geq 2 \). For \( n = 2 \) it is useful to express the inequality as
\[
|p| + |q| = 1 + \zeta, \quad \zeta > 0
\] (49)

**Claim 5.** If \( |p| > 1 \) or \( |q| > 1 \), the quantum vector measure is not of bounded variation.

**Proof.** Consider any partition of \( \Omega \) which contains \( \text{cyl}(C^n) \). Since \( A(C^n) = p^{n-1} \), \( |A(\Omega)| > 1 \) is a strict inequality for any \( n \) if \( q \neq 0 \). Similarly, consider any partition of \( \Omega \) which contains \( \text{cyl}(C^n) \). Since \( A(C^n) = q^{n(n-1)} \), \( |A(\Omega)| > |q|^{n(n-1)} \) for any \( n \), if \( p \neq 0 \). Thus, \( A \) is not of bounded variation if either \( |p| > 1 \) or \( |q| > 1 \).

Thus, we may restrict our attention to \( |p| < 1, |q| < 1 \). Consider an \( n \)-element causal set \( C^n \). Let \( \{C^h_{j_1}\} \) be the set of its \( n + 1 \) element (or immediate) descendants and \( a(j_1) \) the associated transition amplitude. The index \( j_1 = 1, 2, \ldots, J_1(C^n) \) where \( J_1(C^n) \) are the number of immediate descendants of \( C^n \). In turn, let \( C^{n+1}_{j_1j_2} \) denote the \( n + 2 \) element descendant of \( C^n \), and \( a(j_1j_2) \) the associated transition amplitude and so on. The index \( j_2 \) depends on \( j_1 \) since \( j_2 = 1, 2, \ldots, J_2(C^h_{j_1}) \) where \( J_2(C^h_{j_1}) \) are the number of immediate descendants of \( C^{n+1}_{j_1} \) and so on. \( j_2 \) thus carries a hidden index \( j_1 \), but we will not include it explicitly in the expressions below. The set \( \Pi \equiv \{C^{n+s}_{j_1j_2\ldots j_s}\} \) of \( n + s \) element descendants of \( C^n \) gives rise to a disjoint partition of \( \text{cyl}(C^n) \) where the maximum value \( A(C^{n+s}_{j_1j_2\ldots j_s}) \) of each \( j_s \) is determined by its parent \( C^{n+s-1}_{j_1j_2\ldots j_{s-1}} \). Using a shorthand notation for the maximum values of \( j_1, j_2, \ldots, j_s \), the total variation takes the form
\[
|A(C^n) \geq \sum_{j_1=1}^{J_1} \left( \sum_{h=1}^{J_2(j_1)} \left( \sum_{j_2=1}^{J_2(j_1j_2j_3)} \left( \sum_{h=1}^{J_3(j_2j_3j_4)} A(C^{n+s}_{j_1j_2\ldots j_s}) \right) \right) \right)
\] (50)

where
\[
A(C^{n+s}_{j_1j_2\ldots j_s}) = A(C^n) \times a(j_1) a(j_2) \ldots a(j_{s-1}), \quad (51)
\]

Thus
\[
|A(C^n) \geq |A(C^n)| \times \left( \sum_{j_1} |a(j_1)| \left( \sum_{j_2} |a(j_2)| \left( \sum_{j_3} |a(j_3)| \left( \sum_{j_4} |a(j_4)| \right) \right) \right),
\] (52)
where we have suppressed the dependences of the \( j_i \)’s. We now show that
\[
\sum_j |a_{j_i}(j_i)| \geq 1 + \zeta 
\]
for every \( i \).

The final sum within the nested brackets of (52)
\[
\sum_j |a_{j_i}(j_i)| \geq 1
\]

since
\[
\sum_j a_{j_i}(j_i) = 1.
\]

Now, as in transitive percolation, each term in (55) is of the form \( p^w q^v \), with \((u, v) \in \mathcal{I}\), where we have suppressed the dependence of the index set \( \mathcal{I} \) on the parent \( C_{j_1; j_2; \ldots; j_t} \). If \( m \) is the number of maximal elements in \( C_{j_1; j_2; \ldots; j_t} \), and \( \mathcal{I}' \) is the index set which excludes \((u, v) = (m, 0)\), then
\[
\sum_j a_{j_i}(j_i) = \sum_{(u, v) \in \mathcal{I}} p^w q^v = p^m + \sum_{(u, v) \in \mathcal{I}'} p^u q^v = 1 + \sum_{w \in W} c_w q^w
\]
for some appropriate index set \( W \) and coefficients \( c_w \). Since the above sum is always equal to 1 and is true for all \( q \), this means that \( c_w = 0 \). Thus,
\[
\sum_j |a_{j_i}(j_i)| = |p|^m + \sum_{(u, v) \in \mathcal{I}'} |p|^u |q|^v
\]
\[
= (1 + \zeta - |q|)^m + \sum_{(u, v) \in \mathcal{I}'} (1 + \zeta - |q|)^u |q|^v
\]
\[
= (1 - |q|)^m + \sum_{(u, v) \in \mathcal{I}'} (1 - |q|)^u |q|^v
\]
\[
+ \binom{m}{1}(1 - |q|)^{m-1}\zeta + \binom{m}{2}(1 - |q|)^{m-2}\zeta^2 + \ldots + \zeta^m
\]
\[
+ \sum_{(u, v) \in \mathcal{I}'} \left( \binom{u}{1}(1 - |q|)^{u-1}\zeta + \binom{u}{2}(1 - |q|)^{u-2}\zeta^2 + \ldots + \zeta^u \right)|q|^v,
\]
where \( \mathcal{I}' \) is the index set which excludes both \((u, v) = (m, 0)\) and \((u, v) = (0, n + s - 1)\). The first two terms are of the same form as expression (56) and hence equal to 1 since it is independent of the choice of \(|q|\). Since \( 0 \leq |q| \leq 1 \), each of the terms in the above expression is positive. For \(|q| \neq 1\) it therefore suffices to focus on the terms linear in \( \zeta \). We see that this can be simplified to
\[
\left( (1 - |q|)^m + \sum_{(u, v) \in \mathcal{I}'} (1 - |q|)^u |q|^v \right)(1 - |q|)^{-1}
\]
\[
+ \binom{m - 1}{0}(1 - |q|)^{m-1} + \sum_{(u, v) \in \mathcal{I}'} ((u - 1)(1 - |q|)^u |q|^v)
\]
\[
> (1 - |q|^{m+s-1})(1 - |q|)^{-1}.
\]
For any $n + s - 1 \geq 1$ the above expression is $> 1$ for $|q| < 1$. Thus, each nested sum
\[ \sum_{j_k} |a_{j_k-1}(j_k)| \geq (1 + \zeta) \]
for any $j_{k-1}$ and hence from (52) we see that
\[ |A(C^n) | \geq |A(C^n) | \times (1 + \zeta)^s. \]
Since $s$ can be made arbitrarily large, this means that $|A|\langle C^n \rangle$ is not bounded. When $|q| = 1$, expression (57) simplifies to
\[ \sum_{j_k} |a_{j_k-1}(j_k)| = 1 + \zeta^m + \sum_{(u,v)\in I''} \zeta^u. \]
If the number of maximal elements $m = 1$, then equation (59) is satisfied. If $m > 1$, then there is a transition in which the new element is added to the immediate future of only one single element and unrelated to some $r \geq 1$ others. This means that $(u, v) = (1, r) \in I''$ and again equation (59) is satisfied. Hence, so is equation (60).

5. Discussion

We have seen that for a class of finite-dimensional systems the quantum vector pre-measure on $\mathfrak{A}$ does not admit an extension to the sigma algebra $\mathcal{S}_A$ generated by $\mathfrak{A}$. We now discuss the implications of these results.

To start with, the lack of an extension does not by itself imply that no physical observables exist. For the finite unitary systems all the events in $\mathfrak{A}$ are measurable and hence are physical observables. The lack of an extension simply means that while finite time questions are observables, not all infinite time questions are observables, not all infinite time questions are.

For causal sets on the other hand, one is interested in covariant or label invariant events—only these correspond to 'observables'. The growth process generates labelled causal sets and a covariant event, $A$, would have the property that if a labelled causet $\gamma$ is an element of $A$ then so is every relabelling of $\gamma$. It can be shown that none of the events in the algebra, $\mathfrak{A}$, of finite unions of cylinder sets are covariant. Covariant events [9] are found only in the sigma algebra $\mathcal{S}_A$ generated by $\mathfrak{A}$. Let us define the sub-sigma algebra of these covariant events, $\mathcal{F}'\mathcal{S}_A$ over $\mathfrak{A}$. There is a natural identification of these covariant events with subsets of the set of unlabelled past-finite causal sets $\Omega_1'$, and $\mathcal{F}'\mathcal{S}$ can be considered as a sigma algebra over $\Omega_1'$. For the classical stochastic growth models of [15] since the extension of the probability pre-measure to $\mathcal{S}_A$ is guaranteed starting from a given probability pre-measure on $\mathfrak{A}$, this procedure gives rise to a unique covariant probability measure space $(\Omega_1', \mathcal{F}'\mathcal{S}, \mu')$, where $\mu'$ is the extended measure restricted to the subalgebra $\mathcal{F}'\mathcal{S}$. If a quantum vector pre-measure were to extend to a vector measure on $\mathcal{S}_A$, the same procedure provides a collection of covariant quantum observables.

When $p$ is real, i.e. for real quantum percolation, the pre-measure does extend to $\mathcal{S}_A$, and hence covariant quantum observables can be constructed along the lines of [9]. In particular, from the product form of the decoherence functional (equation (44)) it is trivial to see that all the covariant observables of classical transitive percolation are also observables for real quantum percolation. In [9] it was shown that the covariantly defined and physically accessible stem sets generate a sub-sigma algebra $\mathcal{S}'_A \subset \mathcal{S}'$. A stem $c$ in a causal set $C$ is a sub-causal set of $C$ which contains its own past. The stem set, $\text{stem}(c) \subset \Omega_1'$, is then the set of (unlabelled)
causal sets which contains a stem isomorphic to $c$. Since the sets of measure zero in transitive
percolation are also sets of measure zero in real quantum percolation, the results of [9] imply
that for $p > 0$, $\mathcal{S}'$ is $\mathcal{S}$ up to sets of measure zero. Thus, at least in this simple example of
quantum dynamics, one recovers a complete set of covariant observables, since the model is
essentially equivalent to classical transitive percolation.

As we have seen, for complex percolation with $p$ not real, $\mu$ does not extend to $\mathcal{S}_\mathbb{A}$, and
hence the construction of covariant observables, if at all possible, requires a different approach.
As discussed in section 2 countable additivity of the measure requires an unconditional
convergence of the right-hand side of equation (3). It is this that implies bounded variation
for finite-dimensional vector measures. One option then is to require that the measure on $\mathcal{S}_\mathbb{A}$ satisfy only a conditional convergence, with the conditionality, somehow, determined
by the structure of the cylinder sets. The cylinder sets have a nested or ‘filtered’ structure
(equations (20) and (21)) which arises due to the physics of the model of dynamical evolution
of the system in terms of transitions between states. This structure could lead to a ‘canonical
representation’ of at least some of the elements of $\mathcal{S}_\mathbb{A}$ as countable, ordered series of set
operations on cylinder sets and this order could be used to define the measure of the limiting
events. One simple example is to (partially) order the cylinder sets by the cardinality of the
truncated history and then for a countable disjoint union of cylinder sets the order of summation
for the measure could be that cardinality. In order to be confident that physically meaningful
measures are obtained in this way, the summation order must somehow be canonical.

If successful, such an approach would be in keeping with the attitude one often adopts in
physics. Namely, the failure of a quantum measure to have a ‘mathematical extension’ does
not mean that it cannot have a ‘physical extension’ of the sort described above. This happens
in physics all the time: even though one often comes across non-convergent expressions, we
can make sense of them by applying a physically meaningful cutoff and using the limit as the
cutoff is taken away to define the quantity. We can do the same thing here: define limits only
in a physically meaningful way. Even if quantities are only conditionally convergent, these
‘conditions’ are physically determined.

A alternative approach is to start with classical transitive percolation with parameter $p$,
calculate the probabilities for covariant events and then complexify them by taking the $p$
complex if possible. In other words, if $P(\alpha, p)$ is the probability for the covariant event
$\alpha \in \mathcal{S}_\mathbb{A}$ in transitive percolation, and $P$ is an analytic function of $p$ then one defines the
$A(\alpha) := P(\alpha, p)$ for the $p$ complex to be the amplitude for $\alpha$ in complex percolation. As a
concrete example, consider the event $\alpha_0 \in \mathcal{S}_\mathbb{A}$ corresponding to the existence of an element
which is to the past of all other elements\footnote{A causal set with an element to the past of all other elements is referred to as ‘originary’}. The probability of such an event for transitive
percolation is given by the Euler function

$$\phi(q) = \prod_{i=1}^{\infty} (1 - q^i).$$

Since $\phi(q)$ is analytic for all $q \in \mathbb{C}$, $|q| < 1$, we can define $D(\alpha_0, \alpha_0) \equiv |\phi(q)|^2$. It might be
possible, in this way, to define the quantum measure on a large collection of covariant events.
It is even possible that the probabilities of all the covariant events in transitive percolation
are analytic in $p$ in which case this procedure would result in a genuine finitely bi-additive
quantum measure on $\mathcal{S}_\mathbb{A}$.

On the other hand, the failure of bounded variation could have problematic implications
for physical predictions. In classical probability theory, an event of zero measure almost
surely does not occur—we will say it is precluded—as is any event that is contained in a
set of measure zero. Without going into the details of any specific interpretational scheme for quantum measure theory (see [7, 8] for such schemes) there is a case for taking that predictive rule over, wholesale, to the quantum case: any event contained in a set of zero quantum measure is precluded, almost surely it will not occur. (Note, however, that sets of zero quantum measure can contain events of non-zero quantum measure [1, 5] and so this rule says that some events of non-zero quantum measure are nevertheless precluded.) We will adopt the rule and explore its implications.

We now show that bounded variation is sufficient, for the finite-dimensional unitary systems of section 3, to ensure that not all elements of \( \mathfrak{A} \) are contained in sets of measure zero: bounded variation ensures that not all finite time events are precluded.

**Claim 6.** Consider a finite-dimensional unitary system as described in section 3 and let \( \mathfrak{A} \) be the event algebra of finite time events generated by the cylinder sets. If \( \mu_v \) is of bounded variation, then not every \( \alpha \in \mathfrak{A} \) can be contained in a set of measure zero.

**Proof.** Assume the contrary, i.e. that every \( \alpha \in \mathfrak{A}, \alpha \subset \Omega \) is either of measure zero or contained in a set of measure zero. Let \( \Gamma(n) \) denote the set of length \( n \) strings. For any \( \gamma \in \Gamma(n) \) define \( \Gamma = \Gamma(n) \setminus \gamma \) and a disjoint union of cylinder sets \( \alpha \gamma = \bigcup_{\gamma \in \Gamma} \text{cyl}(\gamma) \in \mathfrak{A} \). Then there exists a \( \beta \gamma \supseteq \alpha \gamma \) which is of quantum measure zero, i.e. \( \| \mu_v(\beta \gamma) \|^2 = D(\beta \gamma, \beta \gamma) = 0 \). This means that \( \mu_v(\beta \gamma) \) is a zero vector in \( H \), so that \( \mu_v(\Omega) = \mu_v(\beta \gamma) \) where \( \beta \gamma \subseteq \text{cyl}(\gamma) \), by the additivity of \( \mu_v \). Thus, \( \| \mu_v(\beta \gamma) \| = 1 \). Now consider the following partition of \( \Omega \). For each \( \gamma \in \Gamma(n) \), express \( \text{cyl}(\gamma) \) as the disjoint union \( \beta \gamma \cup (\text{cyl}(\gamma) \setminus \beta \gamma) \), so that \( \Omega \) can be expressed as the disjoint union

\[
\Omega = \bigcup_{\gamma \in \Gamma(n)} \beta \gamma \cup (\text{cyl}(\gamma) \setminus \beta \gamma). \tag{63}
\]

Then,

\[
|\mu_v|(\Omega) \geq \sum_{\gamma \in \Gamma(n)} \| \mu_v(\beta \gamma) \| = N^n \tag{64}
\]

for each \( n \) and is hence unbounded. \( \square \)

While this does not prove that bounded variation is a necessary condition, the following is an example which is not of bounded variation, and for which every set in \( \mathfrak{A} \) is contained in a set of measure zero and hence is precluded. In other words, the only event in \( \mathfrak{A} \) that does occur is \( \Omega \) itself! This system belongs to the class of generic unitary systems studied in section 3 and hence is not of bounded variation.

**Claim 7.** For a two-state system whose dynamics is determined by the unitary operator

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{65}
\]

and whose initial state is \( e_1 \), every element in \( \mathfrak{A} \), except for \( \Omega \), is contained in a set of measure zero.

**Proof.** For a two-dimensional system, \( s_1, s_2 \in \{1, 2\} \). If \( e_1, e_2 \) are the orthonormal basis vectors of \( \mathcal{H}_1 \), the trivial evolution of this basis (i.e. via the identity map) to time \( t = m \) gives the

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9 This example is due to Rafael Sorkin.
orthonormal basis vectors $e_1(m), e_2(m)$ of $\mathcal{H}_m$. For $U$ given by equation (65) the restricted evolution $\tilde{\psi}_\gamma$ for any $m$-length truncated history $\gamma = (s_1, s_2, \ldots, s_m)$ is

$$\tilde{\psi}_\gamma = C_\gamma \psi = \left( \frac{1}{\sqrt{2}} \right)^{m-1} (i)^f e_{s_1} \delta_{s_1, 1},$$

(66)

where $f \in \{0, 1, \ldots, m-1\}$ denotes the number of ‘flips’, i.e. transitions from 1 to 2 or 2 to 1 in $\gamma$. For example, if $\gamma = (1, 2, 1)$, then $f = 2$. For this choice of initial state, $\tilde{\psi}_\gamma$ is identically zero on all histories with $s_1 = 2$, and moreover for $f$ even, $e_{s_m} = e_1(m)$ while for $f$ odd, $e_{s_m} = e_2(m)$. Thus, truncated histories with the same $m, f \mod 2$ values have the same $\tilde{\psi}_\gamma$.

If $\Gamma(m)$ denotes the set of all $m$-length strings, then the unrestricted evolution $\tilde{\psi}$ from $\psi$ is

$$\tilde{\psi} = \sum_{\gamma \in \Gamma(m)} \tilde{\psi}_\gamma = \sum_{\gamma \in \Gamma(m)} C_\gamma \psi = a_1(m) e_1(m) + a_2(m) e_2(m).$$

(67)

$\tilde{\psi}$ satisfies the normalization $\langle \tilde{\psi}, \tilde{\psi} \rangle = 1$. Contributions to $a_1(m)$ come only from truncated histories with even $f$ and those to $a_2(m)$ only from truncated histories with odd $f$. The number of strings with precisely $f$ flips is $\binom{m-1}{f}$, and hence summing over even and odd $f$ respectively we find

$$a_1(m) = \left( \frac{1}{\sqrt{2}} \right)^{m-1} \sum_{j=0}^{j_{\text{max}}} (-1)^j \binom{m-1}{2j},$$

$$a_2(m) = \left( \frac{1}{\sqrt{2}} \right)^{m-1} \sum_{k=0}^{k_{\text{max}}} (-1)^k \binom{m-1}{2k+1},$$

(68)

where for $m$ even $j_{\text{max}} = k_{\text{max}} = \frac{1}{2}(m-2)$ and for $m$ odd $j_{\text{max}} = \frac{1}{2}(m-1)$ and $k_{\text{max}} = \frac{1}{2}(m-3)$.

For $m = 4q + 1$ the binomial expansion for $(1 + i)^{4q}$ gives

$$a_1(4q + 1) = \left( \frac{1}{\sqrt{2}} \right)^{4q} (-1)^q 2^{2q} = (-1)^q,$$

$$a_2(4q + 1) = 0.$$  

(69)

(70)

Now, for every $\gamma \in \Gamma(4q + 1)$ whose first entry is $s_1 = 1$ and which has precisely $2q$ flips, the last entry $s_{4q+1} = 1$. Hence the strings with $2q$ flips contribute only to $a_1(4q + 1)$, each with an amplitude $\left( \frac{1}{\sqrt{2}} \right)^{4q} (-1)^q$. Moreover, the set $\Gamma_1 \subset \Gamma(4q + 1)$ of truncated histories with $f = 2q$ has cardinality $\binom{4q}{2q}$. Since $\binom{4q}{2q} > 2^{2q}$, it is therefore possible to pick a subset $\Gamma_2$ of $\Gamma_1$ so that

$$\tilde{\psi} = \sum_{\gamma \in \Gamma_2} C_\gamma \psi.$$  

(71)

The complement of the set $\bigcup_{\gamma \in \Gamma_1} \text{cyl}(\gamma)$ is therefore of measure zero.

We now show that every cylinder set $\text{cyl}(\tilde{\gamma})$ with $\tilde{\gamma} = (s_1, s_2, \ldots, s_{\tilde{m}})$ contains an event $\alpha$ such that its complement is of zero measure. Since every event except $\Omega$ is in the complement of some cylinder set, it also belongs to the complement of $\alpha$ which suffices to prove our result.

Let $\tilde{f}$ be the number of flips in $\tilde{\gamma}$. Any event which is a subset of $\text{cyl}(\tilde{\gamma})$ shares the first $\tilde{m}$ entries with $\tilde{\gamma}$ and hence has at least $\tilde{f}$ flips. Choose a $q$ such that $2q > \tilde{f}$ and $4q + 1 > \tilde{m}$ and let $\Gamma_\gamma \subset \text{cyl}(\tilde{\gamma})$ be the set of truncated histories of length $4q + 1$ with precisely $2q$ flips.
The cardinality of this set is \((\frac{4q+1}{2} - \tilde{m})\) and each truncated history in it contributes a factor of \((\frac{4q}{2} - \tilde{m} + 1)\). Since \(\tilde{f} \in \{0, 1, \ldots, \tilde{m} - 1\}\), if \(q \geq 3\) and \(q \geq \tilde{m}\)

\[
\left(\frac{4q}{2} - \tilde{m} + 1\right) > \left(\frac{4q - \tilde{m} + 1}{2} - \tilde{f}\right) > \frac{3q}{q} > 2^q.
\]

Thus, there exists a \(\Gamma_1 \subset \Gamma_2 \subset \Gamma_1\) for which equation (71) is satisfied so that the complement of the event \(\bigcup_{\tilde{f} \in \tilde{m}}\) is of measure zero. Thus, every event in the complement of cyl(\(\tilde{\gamma}\)) is contained in a set of measure zero.

Since this is true for all cylinder sets, it means that every event in \(\mathcal{A}\) is contained in a set of measure zero. \(\Box\)

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References

[1] Sorkin R D 1994 Mod. Phys. Lett. A 9 3119 (arXiv:gr-qc/9401003)
[2] Sorkin R D 1994 Quantum measure theory and its interpretation 4th Drexel Symposium on Quantum Nonintegrability (8–11 September 1994, Philadelphia) (arXiv:gr-qc/9507057)
[3] Salgado R B 2002 Mod. Phys. Lett. A 17 711 (arXiv:gr-qc/9903015)
[4] Martin X, O’Connor D and Sorkin R D 2005 Phys. Rev. D 71 024029 (arXiv:gr-qc/0403085)
[5] Surya S and Walden P 2010 Found. Phys. 40 585–606 (arXiv:0809.1951)
[6] Gudder S 2009 J. Math. Phys. 50 123509 (arXiv:0909.2203)
[7] Sorkin R D 2007 J. Phys. Conf. Ser. 67 012018 (arXiv:quant-ph/0703276)
[8] Dowker F and Ghazi-Tabatabai Y 2008 J. Phys. A: Math. Theor. 41 105301 (arXiv:0711.0894 [quant-ph])
[9] Brightwell G, Dowker H F, Garcia R S, Henson J and Sorkin R D 2003 Phys. Rev. D 67 084031 (arXiv:gr-qc/0210061)
[10] Kac M 1959 Probability and Related Topics in Physical Sciences (Lectures in Applied Mathematics) (London: Interscience)
[11] Halpern P R 1978 Measure Theory (Graduate Texts in Mathematics vol 18) (Berlin: Springer)
[12] Dowker H F, Johnston S and Sorkin R 2010 J. Phys. A: Math. Theor. 43 275302 (arXiv:1002.0589)
[13] Dieudonné J and Uhl J J 1977 Vector Measures (Mathematical Surveys no 15) (Providence, RI: American Mathematical Society)
[14] Bombelli L., Lee J H, Meyer D and Sorkin R 1987 Phys. Rev. Lett. 59 521
[15] Rideout D P and Sorkin R D 2000 Phys. Rev. D 61 024002 (arXiv:gr-qc/9904062)
[16] Rideout D P 2001 Dynamics of causal sets PhD Thesis Syracuse University (arXiv:gr-qc/0212064)
[17] Geroch R Path Integrals Unpublished (available at http://physics.syr.edu/~sorkin/lecture.notes/)
[18] Kluson V 1991 Proc. Am. Math. Soc. 111 233–9
[19] Rudin W 1987 Real and Complex Analysis (New York: McGraw-Hill)
[20] Penrose R 1972 Techniques of differential topology in relativity (Society for Industrial and Applied Mathematics, Philadelphia, PA, 1973) Hawking S and Ellis G 1973 Large scale structure of spacetime (Cambridge: Cambridge University Press)