Superradiance and multiple scattering of photons in atomic gases

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We study the influence of cooperative effects such as superradiance and subradiance on the scattering properties of dilute atomic gases. We show that cooperative effects lead to an effective potential between two atoms that decays as $1/r$. In the case of superradiance, this potential is attractive for close enough atoms and can be interpreted as a coherent mesoscopic effect. We consider a model of multiple scattering of a photon among superradiant pairs and calculate the elastic mean free path and the group velocity. We study first the case of a scalar wave which allows us to obtain and to understand basic features of cooperative effects and multiple scattering. We then turn to the general problem of a vector wave. In both cases, we obtain qualitatively similar results and derive, for the case of a scalar wave, analytic expressions for the elastic mean free path and for the group velocity for an arbitrary detuning (near resonance).

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I. INTRODUCTION

Coherent multiple scattering of photons in cold atomic gases is an important problem since it presents a path toward the onset of the Anderson localization transition, a long-standing and still open issue. The large resonant scattering cross section of photons reduces the elastic mean free path to values comparable to the photon wavelength, for which the weak-disorder approximation breaks down, thus signaling values comparable to the photon wavelength, for which the cross section of photons reduces the elastic mean free path and the group velocity. We study first the case of a scalar wave which allows us to obtain and to understand basic features of cooperative effects and multiple scattering. We then turn to the general problem of a vector wave. In both cases, we obtain qualitatively similar results and derive, for the case of a scalar wave, analytic expressions for the elastic mean free path and for the group velocity for an arbitrary detuning (near resonance).

wave being scattered by a pair of two-level atoms. Simple expressions for the photon elastic mean free path and group velocity have been derived at resonance and found to be significantly different from those of independent atoms. To be more specific, near resonance, as well as at resonance, the superradiant effect leads to a finite and positive group velocity, unlike the one obtained for light interaction with independent atoms.

In this paper we provide, for the case of a scalar wave, closed expressions for the superradiant contribution to the elastic mean free path and the group velocity for an arbitrary (near resonance) detuning, and calculate the dependence of the transport time on it. In addition, we estimate the maximal interatomic separation in a superradiant pair, which accounts for possible mechanisms that may break the pair. We also compare the effective approach presented in [8] to a more realistic one that takes into account the vectorial nature of the wave.

The paper is organized as follows: In Sec. II we describe the model, which consists of pairs of two-level atoms placed in an external radiation field where the Doppler shift and recoil effects are negligible. In order to investigate the influence of the cooperative effects of such pairs on the multiple scattering of photons we briefly review, in Sec. III, Dicke states and some of their properties. Then we calculate the average interaction potential of a pair of atoms in a Dicke state by averaging over the random orientations of pairs of atoms with respect to the wave vector of a photon incident on the atomic cloud. Next, we study the scattering of a photon by such pairs and, in Sec. IV, compare the results to the case where a classical wave is being scattered by a pair of atoms. This comparison allows us to find an unexpected connection between superradiance and mesoscopic effects. In Secs. V and VI, we consider the multiple scattering of photons by pairs of atoms and calculate the elastic mean free path and the group velocity of photons in the random medium. Finally, our analysis is compared to other approaches in Sec. VII and its results are summarized in Sec. VIII.

II. MODEL

Atoms are taken to be degenerate, two-level systems denoted by $|g\rangle=|j_z=0, m_j=0\rangle$ for the ground state and $|e\rangle=|j_z$,
pared to
tions are satisfied.
level atom approximation is close to reality and not merely a
perfine interactions, etc. But, due to selection rules which

tions, e.g., Coulomb interactions, spin-orbit interactions, hy-

two-level atom neglects the rather complicated energy struc-

width of the excited level is

is the electric dipole mo-

potential and to a modified lifetime as compared to indepen-

ground state leads to a configuration where the two atoms,

is the temperature, so that it is possible to neglect the Dop-

k

l

may be written as

where the atomic raising and lowering operators are

We assume that the typical speed of the atoms, \( v \approx \sqrt{k_BT/\mu} \), is small compared to \( v_{\text{max}} = \Gamma/k \) but large compared to \( v_{\text{min}} = \hbar k/\mu \), where \( \mu \) is the mass of the atom and \( T_0 \)

is the temperature, so that it is possible to neglect the Dop-

pler shift and recoil effects. Indeed, for a temperature of \( T_0 \)

\( =10^{-3} \) K, the typical speed of the atom is \( v = 0.3 \) m/s. Since, for a wave number of \( k = 10^7 \) m\(^{-1}\) and a natural width of \( \Gamma \)

\( =10^7 \) s\(^{-1}\), \( v_{\text{max}} \approx 1 \) m/s and \( v_{\text{min}} \approx 0.01 \) m/s, both assump-

tions are satisfied.

III. DICKE STATES

A. Interaction potential and lifetime

The absorption of a photon by a pair of atoms in their


ground state leads to a configuration where the two atoms,

one excited and the second in its ground state, have multiple

exchange of a photon, giving rise to an effective interaction

potential and to a modified lifetime as compared to indepen-

dent atoms. These two quantities are obtained from the ma-

trix elements of the evolution operator \( U(t) \) between states

such as \( |g_1e_2;0\rangle \). There are six unperturbed and degenerate

states with no photon, given by \( \{|j_1j_2;0\rangle, |e_1g_2;0\rangle\} \) in a

standard basis where \( i,j=-1,0,1 \). The symmetries of the Hamiltonian,

namely, its invariance by rotation around the axis between

the two atoms, and by reflection with respect to a plane con-

taining this axis, allows us to use combinations of

these states that are given by

\[
\phi_i^f = \frac{1}{\sqrt{2}} \left[ |g_1^{\prime}g_2;0\rangle + \epsilon |g_1e_2;0\rangle \right]
\]

with \( \epsilon = \pm 1 \), so that

\[
\langle \phi_i^f | U(t) | \phi_i^f \rangle = \delta_{ij} \delta_{\epsilon \epsilon'} S_i^f(t)
\]

and

\[
S_i^f(t) = \langle e_1g_2;0\rangle U(t)|e_1g_2;0\rangle + \epsilon \langle g_1e_2;0\rangle U(t)|e_1g_2;0\rangle.
\]

The states \( |\phi_i^f\rangle \) may be rewritten in terms of the well-

known Dicke states \( |LM\rangle \), where \( L \) is the cooperation

number and \( M \) is half of the total atomic inversion [3]. For

two atoms, the singlet Dicke state is

\[
|00\rangle = \frac{1}{\sqrt{2}} \left[ |e_1g_2\rangle - |g_1e_2\rangle \right]
\]

and the triplet Dicke states are

\[
|11\rangle = |e_1e_2\rangle,
\]

\[
|10\rangle = \frac{1}{\sqrt{2}} \left[ |e_1g_2\rangle + |g_1e_2\rangle \right],
\]

\[
|1-1\rangle = |g_1g_2\rangle.
\]

The states \( |11\rangle \) and \( |1-1\rangle \) correspond, respectively, to both

atoms in their excited states and both atoms in their ground

states. The singlet state \( |00\rangle \) and the triplet state \( |10\rangle \) both

correspond to one atom in the excited state and the other in

the ground state, but \( |00\rangle \) is antisymmetric where \( |10\rangle \) is sym-

metric under an exchange of the atoms. Therefore, we may

rewrite (6) as \( |\phi_i^f\rangle = |10;0\rangle \) and \( |\phi_i^f\rangle = |00;0\rangle \).

For times such that \( t \gg r/c \), where \( r \) is the distance be-

tween the two atoms, up to second order in the coupling to

the radiation, (8) reads

\[
S_i^f(t) = 1 - \frac{i\hbar}{\Delta E_i^f} - \frac{\hbar\Gamma_i^f}{2}.
\]

The two real quantities \( \Delta E_i^f \) and \( \Gamma_i^f \) are, respectively, the

interaction potential and the probability per unit time of

emission of a photon by the two atoms in the state \( |\phi_i^f\rangle \).

The calculation of these two quantities requires second-order per-

turbation theory with respect to the interaction (2). For this

purpose we define an initial state where one atom is excited

and the other is in its ground state without any photon, and a

final state where the two atoms are exchanged. We also de-

fine intermediate states of two types: both atoms in their

ground states with one virtual photon present and both atoms
in their excited states with one virtual photon present. Summing the corresponding diagrams [11] gives
\[
\Delta E_f = \frac{3\hbar}{4} \left[ -p_i \frac{\cos k_0 r}{k_0 r} + q_i \left( \frac{\cos k_0 r}{(k_0 r)^2} + \frac{\sin k_0 r}{(k_0 r)^2} \right) \right]
\]
(12)
and
\[
\Gamma^f = 1 - \frac{3}{2} \left[ -p_i \frac{\sin k_0 r}{k_0 r} + q_i \left( \frac{\sin k_0 r}{(k_0 r)^2} - \frac{\cos k_0 r}{(k_0 r)^2} \right) \right]
\]
(13)
where we have defined \(k_0 = \omega_0/c\),
\[
p_i = 1 - \hat{\mathbf{r}}^2, \quad q_i = 1 - 3 \hat{\mathbf{r}}^2,
\]
and \(\hat{\mathbf{r}} = (1, \theta, \varphi)\) is a unit vector along the direction joining the two atoms. For a \(\Delta m = m_x - m_y = 0\) transition,
\[
p_0 = \sin^2 \theta, \quad q_0 = 1 - 3 \cos^2 \theta,
\]
while for a \(\Delta m = \pm 1\) transition,
\[
p_{\pm} = \frac{1}{2}(1 + \cos^2 \theta), \quad q_{\pm} = \frac{1}{2}(3 \cos^2 \theta - 1).
\]
(16)
At short distance \(k_0 r \ll 1\), we obtain that \(\Gamma^f \approx 2\Gamma\) for the superradiant state \(|\phi^+\rangle = |10;0\rangle\) and \(\Gamma^f = 0\) for the subradiant state \(|\phi^-\rangle = |00;0\rangle\).

### B. Average interaction potential

For a photon of wave vector \(\mathbf{k}\) incident on an atomic cloud, the potential between the two atoms that we shall denote by \(V_e\) is obtained from (12) by averaging over the random orientations of the pairs of atoms with respect to \(\mathbf{k}\). Since, according to (15) and (16), \(\langle q_i \rangle = 0\) and \(\langle p_i \rangle = 2/3\), we obtain for the average potential \(V_e\)
\[
e V_e(r) = \langle \Delta E_f \rangle = -\frac{\hbar}{2} \Gamma \cos k_0 r
\]
(17)
and the average inverse lifetimes of Dicke states are
\[
\langle \Gamma^f \rangle = \Gamma \left( 1 + \frac{\sin k_0 r}{k_0 r} \right)
\]
(18)
which retain the same features as (13) for \(k_0 r \ll 1\).

Let us now characterize the interaction potential \(V_e\). Whereas for a single pair of atoms the potential (12) is anisotropic and decays at short distance as \(1/r^4\), a behavior that originates from the transverse part of the photon propagator, we obtain that, on average over angular configurations, the potential (17) between two atoms in a Dicke state \(|L0\rangle\) in vacuum becomes isotropic and decays as \(1/r\). This behavior coincides with the one obtained by considering the interaction of two-level atoms with a scalar wave. This could have been anticipated since in that case the transverse contribution \(q_i\) to the photon propagator averages to 0. A related behavior for the orientation average interaction potential has been also obtained for the case of an intense radiation field [9], and it has recently been investigated in order to study effects of a long-range and attractive potential between atoms in a Bose-Einstein condensate for far-detuned light [10]. This latter potential, which is fourth order in the coupling to the radiation, corresponds to the interaction energy between two atoms in their ground states in the presence of at least one photon. The average potential \(V_e\) we have obtained is different from that case: it is second order in the coupling to the radiation and it corresponds to the interaction energy of Dicke states \(|L0\rangle\) in vacuum.

### C. Scattering properties

In order to study the scattering properties of Dicke states we introduce the collision operator \(T(z) = V + VG(z)V\), where \(V\) is given by (2) and \(G(z) = (z - H)^{-1}\) is the resolvent where the Hamiltonian \(H\) is the sum of (1) and (2). The matrix element that describes the transition amplitude from the initial state \(|i\rangle = |1 - 1; m_x \hat{\mathbf{e}}\rangle\), where the two atoms are in their ground states in the presence of a photon of frequency \(\omega = c|\mathbf{k}|\) and polarization \(\hat{\mathbf{e}}\), to the final state \(|f\rangle = |1 - 1; m_x \hat{\mathbf{e}}'\rangle\) is
\[
T = \langle f | T(z) = h(\omega - \omega_0) | i \rangle
\]
(19)
where \(|\mathbf{k}| = |\mathbf{k}'|\). By using the closure relation we may write \(T\) as the sum of a superradiant and a subradiant contribution, \(T = T^+ + T^-\) [12], with
\[
T^\pm = \langle f | V | \phi^\pm \rangle \langle \phi^\pm | G(z = h(\omega - \omega_0)) | \phi^\pm \rangle \langle \phi^\pm | V | i \rangle,
\]
(20)
where \(|\phi^\pm\rangle\) are the Dicke states \(|L0\rangle\) in vacuum. The two matrix elements in (20) represent the absorption and the emission of a real photon by the pair of atoms. They are easily obtained from (2)–(5) and lead to the following expressions for the scattering amplitudes:
\[
T^+ = A e^{i(k - k') \cdot \mathbf{r}} \cos \left( \frac{k \cdot \mathbf{r}}{2} \right) \cos \left( \frac{k' \cdot \mathbf{r}}{2} \right) G^+\]
(21)
and
\[
T^- = A e^{i(k - k') \cdot \mathbf{r}} \sin \left( \frac{k \cdot \mathbf{r}}{2} \right) \sin \left( \frac{k' \cdot \mathbf{r}}{2} \right) G^-.
\]
(22)
We have defined \(\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2\), \(\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2\), and
\[
A = \frac{\hbar \omega}{e_0 \Omega} d^2 \langle \hat{d} \cdot \hat{e} | \hat{d}^\mp \rangle.
\]
(23)
where the reduced matrix element and the corresponding unit vector are
\[
d = \langle j_0 | d | j_0 \rangle, \quad \hat{d} = \frac{1}{d} \langle j_m | d | j_m \rangle.
\]
(24)
The propagators \(G^\pm\) are the expectation values of the resolvent in the Dicke states \(|\phi^\pm\rangle\), namely, \(G^\pm = \langle \phi^\pm | G(h \delta) | \phi^\pm \rangle\), where close to resonance \(\delta = \omega - \omega_0 \ll \omega_0\). The propagators result from the sum of an infinite series of virtual photon exchanges between the two atoms in the pair and are given in terms of (12) and (13) by
\[
G^\pm = \left( \frac{\hbar}{2} + i\hbar \Gamma^\pm \right)^{-1}.
\]
(25)
The average propagator is then obtained by averaging $G^\pm$ over the random orientations of the pairs of atoms with respect to the wave vector $\mathbf{k}$ of the incident photon. However, we shall consider in a first stage the effective propagator obtained for the case of a scalar wave. This amounts to writing for the effective propagator the expression

$$G^e = \left[ \frac{\hbar}{\delta + i\frac{\Gamma}{2} \pm \frac{i \hbar r}{2 k_0 r}} \right]^{-1},$$

where we have used (17) and (18) for the average potential and for the average inverse lifetimes. This expression constitutes a priori a rough approximation of the exact average. We shall calculate later, in Sec. VI, the exact expression of the average propagator and show that it is rather complicated, whereas the approximate expression using a scalar wave gives similar qualitative results. Therefore, it allows for a better understanding of relevant physical quantities such as the elastic mean free path and group velocity. From now on, we thus use the scalar wave approximation in order to provide, in a rather simple way, the main features of multiple scattering by superradiant pairs.

With the help of (26), the scattering amplitudes are

$$T^e_+ = A e^{i(k \cdot r)/2} \cos \left( \frac{k \cdot r}{2} \right) G^+_e$$

and

$$T^e_- = A e^{i(k \cdot r)/2} \sin \left( \frac{k \cdot r}{2} \right) G^-_e.$$  \hspace{1cm} (27)

At short distances $k_0 r \ll 1$, the subradiant amplitude $T^-_e$ becomes negligible as compared to the superradiant term $T^+_e$. Therefore, the potential (17) is attractive and decays as $1/r$. More precisely, at short distances the effective propagator $G^+_e$ diverges for $\delta/\Gamma = 1/(2k_0 r)$ and $G^-_e$ is purely imaginary for $\delta/\Gamma = -i/(2k_0 r)$. Thus, for $\delta/\Gamma < 1/(2k_0 r)$ the imaginary part of the subradiative term (28) is negligible as compared to the imaginary part of the superradiative term (27) and for $\delta/\Gamma < 1/(2k_0 r)$ both the real part and the imaginary parts of (28) are negligible as compared to (27).

We can interpret these results by saying that, at short distances ($k_0 r \ll 1$), the time evolution of the initial state

$$|\psi(0)\rangle = |e_1 g_2; 0\rangle = \frac{1}{\sqrt{2}} [\{\phi^+\} + \{\phi^-\}],$$

corresponds, for times shorter than $1/\Gamma$, to a periodic exchange of a virtual photon between the two atoms at the Rabi frequency

$$\Omega_R = \frac{\langle \Delta E^- \rangle - \langle \Delta E^+ \rangle}{\hbar},$$

which is much larger than $\Gamma$ since, with the help of (17),

$$\Omega_R = \frac{\Gamma}{k_0 r}.$$  \hspace{1cm} (30)

For larger times, the two atoms return to their ground states and a real photon $(k e')$ is emitted. At large distances ($k_0 r \gg 1$), the Rabi frequency becomes smaller than $\Gamma$, so that the excitation energy makes only a few oscillations between the two atoms, thus leading to a negligible interaction potential.

We finally notice that the angular distribution of the light scattered by two atoms in a superradiant state is nearly identical to that of a single atom. This follows from the fact that at short distance $k_0 r \ll 1$, we can neglect higher-order multipolar corrections so that the corresponding additional phase shift $k_0 r \cos \theta$ between waves emitted by the two atoms becomes negligible ($\theta$ is the angle between the direction of the emitted photon and the axis between the two atoms).

### IV. COOPERATIVE EFFECTS AND COHERENT BACKSCATTERING

It is interesting to derive the previous results in another way that emphasizes the analogy with coherent backscattering [1,2]. To that purpose, we write the scattering amplitude $T$ defined previously in (19) as a superposition of two “classical,” scalar amplitudes $T_1$ and $T_2$ [13], each of them being a sum of single-scattering and double-scattering contributions, that is,

$$T_1 = \frac{t}{1 - t^2 G_0} (e^{i(k \cdot r_1) + tG_0 e^{i(k \cdot r_2 - k \cdot r_1)})}$$  \hspace{1cm} (32)

and

$$T_2 = \frac{t}{1 - t^2 G_0} (e^{i(k \cdot r_2) + tG_0 e^{i(k \cdot r_2 - k \cdot r_1)})}. (33)$$

Here

$$t = \frac{4\pi}{k_0} \frac{\Gamma/2}{\delta + i \Gamma/2}$$  \hspace{1cm} (34)

is the amplitude of a scalar wave scattered by a single atom at the origin, and the prefactor $t/(1 - t^2 G_0)$, where

$$G_0 = -\frac{e^{i k_0 r}}{4\pi r}$$  \hspace{1cm} (35)

accounts for the summation of the series of virtual photon exchange between the two scatterers. We can single out in the total amplitude $T = T_1 + T_2$ the single-scattering contribution $T_1$ and write the intensity associated with the higher-order scattering term shown in Fig. 1 as

$$|T - T_1|^2 = 2 \left| \frac{t^2 G_0}{1 - t^2 G_0} \right|^2 \left[ 1 + \cos(k + k') \cdot (r_1 - r_2) \right].$$  \hspace{1cm} (36)

The structure of relation (36) is very reminiscent of that of the so-called coherent backscattering intensity, which occurs in the multiple elastic scattering of light. But although they are analogous, (36) differs from coherent backscattering. In the latter case, averaging over the spatial positions $r_1$ and $r_2$ makes the interference term $\cos(k + k') \cdot (r_1 - r_2)$ vanish in general, with two exceptions:

1. $k + k' = 0$. In the direction exactly opposite to the direction of incidence, the intensity is twice the classical value.
This phenomenon is known as coherent backscattering.

(2) \( \mathbf{r}_1 = \mathbf{r}_2 \). These are closed multiple-scattering trajectories which are at the origin of the phenomenon of weak localization.

In (36) the interference term, i.e., the second term in the square brackets, reaches its maximum value 1 for \( \mathbf{r}_1 = \mathbf{r}_2 \) so that we obtain from (32), (33), and (27) that \( T_1 = T_2 \approx (1/2)T^*_0 \), up to a proportionality factor \([13]\). Thus, the total amplitude is given by the superradiant term with no subradiant contribution.

V. MULTIPLE SCATTERING AND COOPERATIVE EFFECTS

A. Effective self-energy

We consider now multiple scattering of a photon by superradiant pairs built out of atoms separated by a distance \( r \) and coupled by the attractive interaction potential \( V_e \). This situation corresponds to a dilute gas that is assumed to satisfy

\[
 r \ll \lambda_0 \ll n_1^{1/3},
\]

where \( n_1 \) is the density of pairs and \( \lambda_0 = 2\pi/k_0 \) is the atomic transition wavelength. The limiting case (37) corresponds to a situation where the two atoms that form a superradiant pair, through exchange of a virtual photon, constitute an effective scatterer, and cooperative interactions between otherwise well-separated pairs are negligible. Let us stress that we study here a simplified model where only pairs of atoms have been taken into account. A more realistic model should include higher-order terms that account for cooperative effects between more than two atoms, but we do not consider such higher-order terms, i.e., including superradiant clusters of three or more atoms. The purpose of the current model is to examine the contribution of superradiant pairs to the transport properties of the gas. We use the Edwards model \([1,14]\) to describe the medium as a discrete collection of \( N_j \) superradiant pairs in a volume \( \Omega \). Each pair, located at \( \mathbf{R}_k \), is characterized by its scattering potential \( u(\mathbf{R} - \mathbf{R}_k) \). Therefore, the disorder potential is given by

\[
 u = \sum_{k} \sum_{j} u(\mathbf{R} - \mathbf{R}_k).
\]

FIG. 2. Perturbative expansion of the self-energy in a power series in the parameter \( n_1 u_0^2 \). Solid lines account for the free photon Green’s function \( g_0 \). Pairs of dotted lines, connected by \( \times \), stand for the two-point correlation function \( B \). The first term \( \Sigma_1 \), proportional to \( n_1 u_0^2 \), accounts for independent scattering events, while the second term \( \Sigma_2 \), proportional to \( n_1^2 u_0^4 \), describes interference effects between pairs of scatterers.

\[
 U(\mathbf{R}) = \sum_{i=1}^{N_i} u(\mathbf{R} - \mathbf{R}_i).
\]
The latter quantity is obtained by averaging \( A \) in (27) over Zeeman sublevels \( m \) that appear in its definition given by (23) and (24).

The additional average, denoted by \( \cdots \), is taken over distances \( r \) up to a maximal value \( r_m \) which accounts for all possible mechanisms that may break the pairs.

The value of \( r_m \) can be estimated by comparing the kinetic energy \( K \) of a superradiant pair to its average potential energy \( V_e \). We have \( K = \hbar^2/\mu r^2 \) and from (17) we obtain that \( V_e = -\hbar \Gamma/2k_0r \). Minimizing the average energy

\[
E(r) = \frac{\hbar^2}{\mu r^2} - \frac{\hbar \Gamma}{2k_0r},
\]

with respect to \( r \) yields

\[
k_0r_m = 4\frac{\hbar k_0^2}{\mu \Gamma} \tag{46}
\]

or

\[
k_0r_m = 4\frac{v_{\text{min}}}{v_{\text{max}}}, \tag{47}
\]

where the speeds \( v_{\text{min}} \) and \( v_{\text{max}} \), have been defined in Sec. II. For typical values \( \Gamma = 10^7 \text{ s}^{-1} \) and \( k_0 = 10^7 \text{ m}^{-1} \) we obtain that \( k_0r_m \approx 0.05 \). Thus, we can use the results obtained in Sec. III C and consider the superradiant term only.

For \( j_0 = 0 \) and \( j_1 = 1 \), we rewrite (26) as

\[
\Sigma = \frac{6\pi n_1}{k_0} \int_0^{r_m} \frac{dr}{\delta \Gamma + 1/(2k_0r)} + i. \tag{48}
\]

We stress again that, in our approach, a pair of atoms in a superradiant state is considered as a single scatterer, and the effective medium parameters are derived from \( \Sigma \) as will be shown in the next sections. In contrast to our treatment, others [16,19] consider multiple scattering of a real photon by independent atoms and use the second term \( \Sigma_2 \), which describes interference effects between the scatterers, to calculate corrections to the elastic mean free path and to the refractive index of the medium. A further comparison between these two points of view is given in Sec. VII.

**B. Elastic mean free path**

The elastic mean free path \( l_e \) is obtained from the imaginary part of the self-energy, namely,

\[
l_e = -\text{Im} \Sigma_1. \tag{49}
\]

Let us stress that (49) is equivalent, in the case of a dilute gas, to the known formula

\[
l_e = \frac{1}{n_\sigma_e}, \tag{50}
\]

where the total cross section \( \sigma_e \) is obtained for \( k_0r \ll 1 \) from (27) by means of the optical theorem.

![FIG. 3. Ratio between the elastic mean free paths \( l_0 \) and \( l_e \) as a function of the reduced detuning \( \delta \Gamma \) for \( k_0r_m = 0.05, 0.07, \) and 0.1. Away from resonance, for blue detuning, the elastic mean free path \( l_e \) becomes smaller than \( l_0 \) in a ratio roughly given by \( 1/(k_0r_m)^2 \). At resonance, the ratio between the elastic mean free paths is given by (57).](image)

\[
\sigma_e = -\frac{2\Omega}{\hbar c} \text{Im} \langle \mathcal{T}_e (k = k', \dot{e} = \dot{e}') \rangle_m \tag{51}
\]

and \( \langle \cdots \rangle_m \) represents an averaging over Zeeman sublevels. The equivalence in this case is proven easily if one uses (44) and the usual expression for the inverse lifetime

\[
\Gamma = \frac{d^2k_0^3}{3\pi\epsilon_1^2\hbar^2}, \tag{52}
\]

where the reduced matrix element is defined in (24). Therefore, from (48) and (49) we obtain that

\[
\frac{1}{l_e(\delta)} = \frac{6\pi n_1}{k_0^2} f_1 \left( k_0r_m, \frac{\delta}{\Gamma} \right) \tag{53}
\]

where we have defined the function

\[
f_1(u,v) = \frac{1}{2u} \int_0^{2u} \frac{dx}{1 + (u + 1/x)^2}. \tag{54}
\]

The integral is easily carried out analytically and the explicit expression is given in Appendix A. It is interesting to compare \( l_e \) to the elastic mean free path \( l_0 \) that corresponds to near-resonant elastic scattering of a photon by a single atom. The latter quantity is obtained by replacing \( \Gamma \) by \( \Gamma/2 \) in (53) (since the inverse lifetime of a single atom is half the one related to a superradiant pair) and \( 1/x \) by 0 in (54) (since the interatomic distance is taken to be infinite for a single atom) and it is given by

\[
l_0(\delta) = \frac{k_0^2}{6\pi n_1} \left[ 1 + \left( \frac{2\delta}{\Gamma} \right)^2 \right]. \tag{55}
\]

In Fig. 3 the ratio between these two quantities is plotted as
a function of the reduced detuning $\delta/\Gamma$ from resonance for several values of $k_0r_m$.

At resonance, we obtain from (53) that

$$I_s(0) = \frac{k_0^2}{8\pi n_i(k_0r_m)^2}$$

and hence

$$\frac{I_0(0)}{I_s(0)} = \frac{4}{3}(k_0r_m)^2 \ll 1.$$  (57)

Away from resonance, for blue detuning, the elastic mean free path $l_s$ becomes smaller than $l_0$ in a ratio roughly given by $1/(k_0r_m)^2$. This is a direct consequence of the existence of the attractive potential $V_r$.

C. Group velocity

Another important physical quantity that characterizes multiple scattering of a photon is its group velocity $v_g$ given in terms of the refractive index $n$ by the usual relation

$$\frac{c}{v_g} = \eta + \frac{d\eta}{d\omega}.$$  (58)

The refractive index for a dilute medium is

$$\eta = (1 + n_i \Re \alpha)^{1/2},$$  (59)

where the dynamic atomic polarizability $\alpha$ is proportional to the self-energy

$$\alpha = -\frac{1}{n_i} \left( \frac{c}{\omega} \right)^2 \Sigma_1.$$  (60)

Thus, we obtain that

$$\eta = \left[ 1 - \left( \frac{c}{\omega} \right)^2 \Re \Sigma_1 \right]^{1/2}.$$  (61)

Substituting (61) into (58) yields

$$\frac{c}{v_g} = \frac{\eta}{\eta} \left( 1 - \left( \frac{c^2}{2\omega d\omega} \Re \Sigma_1 \right) \right).$$  (62)

From the self-energy (48), we notice that $\eta = 1$ for all values of the detuning $\delta/\Gamma$ and in a large range of densities $n_i$, so that

$$\frac{c}{v_g} = 1 - \frac{n_i}{n_c} f_2(k_0r_m^2 \frac{\delta}{\Gamma}),$$  (63)

where we have defined the characteristic density

$$n_c = \frac{k_0^3 \Gamma}{6\pi \omega_0}.$$  (64)

and the function

$$f_2(u,v) = \frac{1}{2u} \int_0^{2u} dx \left[ \frac{1 - (v + x/2)^2}{(v + x/2)^2} \right].$$  (65)

The integration is easily performed and the explicit expression is given in Appendix A. By replacing $\Gamma$ by $\Gamma/2$ in (63)

FIG. 4. Group velocities $v_g$ (solid line) and $v_0$ (dotted line) as a function of the reduced detuning $\delta/\Gamma$ for $n_i/n_c = 10^5$ and $k_0r_m = 0.1$. The group velocity $v_0$ diverges at two symmetric values of order unity of the reduced detuning and it takes negative values in between. The group velocity $v_g$, near resonance, remains finite and positive.

and $1/x$ by 0 in (65), we obtain the group velocity $v_0$ of light interacting with independent two-level atoms,

$$\frac{c}{v_0(\delta)} = 1 - \frac{n_i}{n_c} \frac{1 - (2\delta\Gamma)^2}{[1 + (2\delta\Gamma)^2]^2}. $$  (66)

For the typical values $\Gamma = 10^7$ s$^{-1}$, $k_0 = 10^7$ m$^{-1}$, and $n_i = 10^{10}$ cm$^{-3}$, we obtain that $n_i/n_c \approx 10^7$.

Figure 4 displays the group velocities $v_g$ and $v_0$ plotted as a function of the reduced detuning $\delta/\Gamma$ for $n_i/n_c = 10^5$ and $k_0r_m = 0.1$.

$v_g$ appears to diverge at quite a large and negative value of the detuning $\delta/\Gamma = -1/(2k_0r_m)$. But near resonance it is well behaved, meaning that it remains finite and positive. At resonance, according to (63), the group velocity is

$$\frac{c}{v_g(0)} = 1 + 4\pi \frac{n_i}{k_0^2 \Gamma} (k_0r_m)^2.$$  (67)

This expression of $v_g$ differs substantially from the one obtained for $v_0$. For densities $n_i > n_c$, the group velocity $v_g$ diverges at two symmetric values of order unity of the detuning and it takes negative values in between (i.e., also at resonance), as can be seen in Fig. 4. This problem was recognized a long time ago [15] and an energy velocity has been defined which describes energy transport through a diffusive medium [16,17]. However, the diffusion coefficient, which will be discussed in the next section, is derived from the group velocity and not from the energy velocity [1]. Moreover, a closed expression for the energy velocity $v_E$ has been obtained only for the case of resonant Mie scattering [18]. The expression is similar to (67) and is given by
It is then interesting to notice that the inclusion of cooperative effects even at the lowest order, i.e., taking into account superradiant pairs, allows one to obtain a group velocity that is well behaved at resonance, unlike the case of resonant scattering by independent atoms.

D. Diffusion coefficient and transport time

Diffusive transport of photons through a gas is characterized by the photon diffusion coefficient

\[ D(\delta) = \frac{1}{3} \frac{v_g(\delta)}{v_g(\delta)} I_r(\delta) \]

which combines the elastic mean free path and the group velocity, both derived from the complex-valued self-energy (48). The diffusion coefficient \( D \) is of great importance since it enters into expressions of various measured physical quantities, such as the transmission and the reflection coefficients of a disordered medium [1]. In addition to these average quantities, an incident pulse that probes a nearly static distribution of bright and dark spots. This snapshot, known as a speckle pattern, can be characterized by the angular-correlation function and the time-correlation function of the light intensity (diffusing wave spectroscopy). In the first case, the correlation function of the transmission coefficient between two distinct directions of the transmitted wave is measured. In the second case, the intensity of the transmitted wave is measured at different times, so that the motion of the scatterers must be taken into account. As pointed out before, in both cases the diffusion coefficient plays an important role, as it enters in the relevant expressions. Moreover, the critical behavior of transport close to the Anderson localization transition at strong disorder is also obtained from the scaling form of \( D \).

Its expression, deduced from (53) and (63), depends on the range \( r_m \) and on the detuning \( \delta / \Gamma \). Since the group velocity and the elastic mean free path are significantly modified for superradiant states, we thus expect the diffusion coefficient to be different from its value obtained for independent atoms.

We define the transport time by

\[ \tau_r(\delta) = \frac{l_o(\delta)}{v_g(\delta)}. \]

At resonance and for \( n_r \gg n_r \), it can be rewritten with the help of (56) and (67) as

\[ \tau_r(0) = \frac{1}{2 \Gamma}. \]

in accordance with our assumption of superradiant states. Near resonance, the transport time depends weakly on the detuning. But, away from it, \( \tau_r \) depends on the detuning and thus on frequency, as can seen from Fig. 5 where the inverse of the transport time \( \tau_r^{-1} / \Gamma \) is plotted as a function of the reduced detuning \( \delta / \Gamma \) for \( n_r = 10^{10} \text{ cm}^{-3} \), \( \Gamma = 10^7 \text{ s}^{-1} \), and \( k_0 = 10^7 \text{ m}^{-1} \) for several values of \( k_0 r_m \).

VI. AVERAGE SELF-ENERGY

So far, we have used the effective approach introduced in Sec. III C, where we have considered the case of a scalar wave being scattered by a pair of two-level atoms. In this simple approach, the propagator of a scalar wave (26) has been calculated and the self-energy (43) has been obtained by averaging (26) over the distance between the two atoms in a pair. This effective approach leads to simple expressions for the elastic mean free path (53) and the group velocity (63) of the wave. In this section we calculate these quantities for a given \( \Delta m \) transition and \( k_0 l \ll 1 \), while taking into account the vectorial nature of the wave. With this purpose, we average the propagator (25) over the random orientations of the pairs of atoms (with respect to the wave vector of the incident photon) as well as over the distance between the two atoms in a pair. Therefore, the average self-energy is now given by

\[ \Sigma^r = \frac{6 \pi n_r}{k_0} \frac{1}{4 \pi r_m^2} \int h \Gamma G^+(r) dr, \]

where the averaging is over the interatomic axis \( r \) (over both magnitude and orientations). The evaluation of (72) for a \( \Delta m = 0 \) transition is rather cumbersome and it is presented in Appendix B. By following the procedure described in the previous section, we obtain the corresponding elastic mean free path \( l'_r \) and the group velocity \( v'_g \). In Fig. 6 the ratio between \( l_0 \) given by (55) and \( l'_r \) is plotted as a function of the reduced detuning \( \delta / \Gamma \) for several values of \( k_0 r_m \).

As in the effective approach, at resonance \( l'_r \) is found to be larger than \( l_0 \), but away from resonance it becomes smaller.

In Fig. 7 the group velocity \( v'_g \) is plotted as a function of the reduced detuning \( \delta / \Gamma \) for \( n_r/l_r = 10^5 \) and \( k_0 r_m = 0.1 \).

Around resonance, the group velocity \( v'_g \) is finite and positive, as in the scalar case, but much larger as compared to...
(63) and it is close to $c$. Thus, we may conclude that in both approaches the superradiant effect leads to a finite and positive group velocity, unlike the one obtained for light interaction with independent atoms. However, the group velocity of a scalar wave is much smaller compared to the one of a photon.

VII. DISCUSSION

In this section we compare our analysis to other approaches [16,19] where resonant multiple scattering of light has been considered. There, using a multiple-scattering expansion for the calculation of the self-energy up to second order in $n_i \mu_0^2$, a correction to the elastic mean free path and to the refractive index has been obtained. In the latter approach, no distinction has been made between the external photon that performs multiple scattering on all atoms and virtual photons exchanged between two atoms in a superradiant state, leading to the average interaction potential $V_c$. This distinction needs to be made for dilute enough atomic gases since in that case the average distance $n_i^{-1/3}$ between atoms is large. Moreover, in this case, the dipole-dipole interaction induced by the external photon depends on the detuning, a situation that corresponds to the case of intense radiation presented in [9] but not to the current experiments made on cold atomic clouds [20].

VIII. CONCLUSIONS

We have considered multiple scattering of a photon by pairs of atoms that are in a superradiant state. On average over disorder configurations, an attractive interaction potential builds up between close enough atoms, which decays as $1/r$. The contribution of superradiant pairs, resulting from this potential, to scattering properties is significantly different from that of independent atoms. This shows up in the behaviors of the group velocity, the elastic mean free path and the diffusion coefficient which are different from their values obtained for independent atoms. We have considered the case of a scalar wave and have shown that it allows to define an effective long-range and attractive potential for pairs of atoms in a superradiant state. Then, we have studied the case of a vector wave and have shown that the results obtained in the scalar case remain qualitatively valid. We have considered a simplified model where only pairs of atoms have been taken into account. A more realistic model should include higher-order terms that account for cooperative effects between more than two atoms [21]. The purpose of the current model is to show that already for a dilute gas in the weak-disorder limit, cooperative effects modify significantly the transport properties of light.

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APPENDIX A

In this appendix, we establish expressions (53) and (63) for the elastic mean free path and the group velocity. At resonance, simple expressions for the elastic mean free path (56) and the group velocity (67) are obtained by a perturbative expansion with respect to the small parameter $k_0 r_m$.

1. Elastic mean free path

The elastic mean free path is given by (53) in terms of the function $f_1$ defined in (54). The integral in (54) is easily carried out analytically, and it leads to
The group velocity is given by
\[ A_4 = \ln \left( \frac{1 - \frac{1}{4} r_m^2}{a + b x_m + \frac{1}{4} r_m^2} \right), \quad (A3) \]
and
\[ B_4 = \frac{\pi}{2} - \tan^{-1}\left( b + \frac{1}{2} r_m \right), \quad (A4) \]
and
\[ C_4 = \frac{1}{x_m} = k_{0r_m} \ll 1. \quad (A5) \]

At resonance $\delta=0$ we have $a=1$, $b=0$ and by expanding (A4) with respect to $k_{0r_m}$ we obtain
\[ B_4 = \frac{2}{x_m} \left( 1 - \frac{4}{3x_m^2} \right). \quad (A6) \]
Thus,
\[ l_s(0) = \frac{k_0^2}{8\pi n_1 (k_{0r_m})^2} \quad (A7) \]
as given in (56).

2. Group velocity

The group velocity is given by (63) in terms of the function $f_2$ defined in (65). The integral in (65) is easily carried out analytically and it yields
\[ \frac{c}{v_s(\delta)} = 1 - \frac{n_i}{n_e} \frac{F_1}{a^2 C_1}, \quad (A8) \]
where
\[ F_1 = b \left( 1 - \frac{1}{4} \right) A_1 + \frac{a - 2}{4} A_1' + \left( \frac{3}{2} - \frac{2}{a} \right) B_1 \]
\[ - bB_1' + \left( 1 - \frac{a}{2} \right) C_1, \quad (A9) \]
\[ A_1' = - \frac{b + \frac{1}{2} r_m}{a + b x_m + \frac{1}{4} r_m^2}, \quad (A10) \]
and
\[ B_1' = - \frac{1}{2} \frac{1}{1 + \left( b + \frac{1}{2} r_m \right)^2}. \quad (A11) \]

At resonance $\delta=0$ we have $a=1$, $b=0$ and by expanding (A10) and (A4) with respect to $k_{0r_m}$ we obtain
\[ A_1' = \frac{2}{x_m} \left( \frac{4}{x_m^2} - 1 \right) \quad (A12) \]
and
\[ B_1 = \frac{2}{x_m} \left( 1 - \frac{4}{3x_m^2} \right). \quad (A13) \]
Thus,
\[ \frac{c}{v_s(0)} = 1 + \frac{2 n_i}{3 n_e} (k_{0r_m})^2, \quad (A14) \]
as given in (67).

APPENDIX B

The aim of this appendix is to calculate the average self-energy (72) for a $\Delta m=0$ transition in the case where $k_{0r} \ll 1$. First, we average the superradiative propagator (25) over the orientation of the interatomic axis and obtain analytical expressions for its real and imaginary parts. Then, by averaging over the interatomic distance up to $r_m$ we obtain the average self-energy (72).

For a $\Delta m=0$ transition and $k_{0r} \ll 1$, the superradiative propagator (25) may be written with the help of (12) and (13) as
\[ \hbar \Gamma^+ = \left[ \frac{\delta}{F} + \frac{3}{4} \left( \frac{3 \cos^2 \theta - 1}{(k_{0r})^3} + \frac{1}{2} \frac{(1 + \cos^2 \theta)}{k_{0r}} \right) + i \right], \quad (B1) \]
where the interatomic axis is $\mathbf{r}=(r, \theta, \varphi)$. Averaging over the orientations
\[ \hbar \Gamma(\Gamma^+) = \frac{1}{4\pi} \int \hbar \Gamma^+ d \cos \theta d \varphi \quad (B2) \]
yields for the imaginary part
\[ \hbar \Gamma \text{Im}(\Gamma^+) = - \frac{P + Q}{\beta^2} \quad (B3) \]
and for the real part
\[ \hbar \Gamma \text{Re}(\Gamma^+) = W_P + W_Q, \quad (B4) \]
where we have defined
\[ P = \frac{1}{8A_2 \beta \cos(\gamma/2)} \ln \left( 1 + 2\beta \cos(\gamma/2) + \beta^2 \right), \quad (B5) \]
Finally, we average (B3) and (B4) over the interatomic distance up to \( r_m \),

\[
h \Gamma \text{Im}(G^*) = -\frac{1}{r_m} \int_0^{r_m} dr \frac{P + Q}{\beta^2} \tag{B12}
\]

and

\[
h \Gamma \text{Re}(\overline{G^*}) = \frac{1}{r_m} \int_0^{r_m} dr (W_P + W_Q). \tag{B13}
\]

The integrals can be evaluated numerically and give the average self-energy (72) since

\[
\frac{1}{4 \pi r_m} \int h \Gamma G^*(r) dr = h \Gamma \langle \overline{G^*} \rangle. \tag{B14}
\]

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