A \textit{K}-THEORETIC NOTE ON GEOMETRIC QUANTIZATION

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\textbf{Abstract.} We show that the results of the paper \textit{Symplectic Reduction and Riemann-Roch for Circle Actions} \cite{DG} of Duistermaat, Guillemin, Meinrenken and Wu can be expressed entirely in \textit{K}-theory. We show that their quantization is simply a pushforward in \textit{K}-theory, and use Lerman’s symplectic cutting and the localization theorem in equivariant \textit{K}-theory to prove that quantization commutes with reduction. Only the case where the action is free on the zero level set of the moment map is addressed.

1. \textbf{Introduction}

In their paper, \textit{Symplectic Reduction and Riemann-Roch for Circle Actions} \cite{DG}, H. Duistermaat, V. Guillemin, E. Meinrenken and S. Wu use E. Lerman’s symplectic cutting technique \cite{Lerman} to prove that quantization commutes with reduction, in the case of a circle action (see \cite{DG}). They define the quantization of a compact symplectic manifold $M$ via index theory, as the index of the Spin$^C$ operator (the “Riemann-Roch number”) associated to an almost-complex structure compatible with the symplectic structure. If $M$ is a Hamiltonian $G$-space, this is a virtual $G$-module, i.e. an element of $R(G)$. In section 2 we show that this can be expressed as the pushforward $p_! L$ of the prequantum line bundle $L$ by the map $p : M \to \ast$, provided we use the correct $K$-orientation. This is a straightforward use of the index theorem; however, since the arguments in section 4 depend heavily on signs, we go through the construction slowly to make sure that the orientation is correct.

The rest of the paper is devoted to showing that the proof in \cite{DG} can be directly translated into $K$-theory. In particular, the index theorem is not necessary at this point, and the crucial ingredient is the localization theorem in equivariant $K$-theory, which we review in section 3. This relates the pushforward on the whole manifold $M$ (i.e. $Q(M)$) to the pushforwards from the fixed point sets of the circle action. Since the fixed point sets are trivial $G$-spaces, their equivariant $K$-theory splits up as

$$K_G(F) \cong K(F) \otimes R(G) \cong K(F) \otimes \mathbb{Z}[z, z^{-1}]$$

so we can treat $K$-classes on the fixed point sets as Laurent polynomials with coefficients in $K(F)$. It turns out that we only need to know a few basic facts about the $z$-dependence of these polynomials. In section 4 we state without proof the properties of symplectic cutting which we need; in particular, that the reduced space $M_G$ embeds into each of the “cut” spaces as a component of the fixed point set.

In section 4 we use these tools to prove that quantization commutes with reduction (Theorem 3). We show that $Q(M)^G = Q(M_G)$ by equating both to $Q(M_+)$, where $M_+$ is one of the “cuts” of $M$ (Props. 1 and 2). We prove both of these propositions by embedding the rings $K(F)[z, z^{-1}]$ into two different rings of formal
Laurent series:
\[ K(F)[[z]]_z \text{ and } K(F)[[z^{-1}]]_{z^{-1}}. \]

These embeddings correspond to the limit arguments in section 2 of [3].

In a forthcoming paper [13] we will extend the ideas in section 4 to generalized \( S^1 \) equivariant cohomology theories. That work will generalize the result of Kalkman [9] on localization for manifolds with boundary.

To fix notation and conventions, let \( G \) be a compact Lie group, and let \( M \) be a Hamiltonian \( G \)-space with symplectic form \( \omega \) and moment map \( \phi : M \to \mathfrak{g}^* \) (say \((M, \omega, \phi)\) for short). (In section 3 we require \( G \) to be topologically cyclic; in sections 4.1 and 4 we specialize to the case \( G = S^1 \).) Choosing a Riemannian metric \( g \) gives an almost-complex structure \( J \), unique up to isotopy, by the requirement \( g(v, w) = \omega(Jv, w) \). We assume \( M \) is prequantizable, with prequantum line bundle \( L \) and connection \( \nabla \), and that the action of \( G \) extends to an equivariant action on \((M, L)\). Then the infinitesimal action of \( G \) on sections of \( L \) is given by the formula of Kostant [8]:
\[
D_v(s) = \nabla_v s - i\langle \phi, v \rangle s
\]
where \( v \in \mathfrak{g} \) and \( s \in \Gamma(L) \). Let \( F_r \) be the fixed point set of the \( G \)-action, with connected components \( F_r = \bigcup_r F_r \). Then on \( F_r \) the moment map has fixed value \( \phi_r \), and Kostant’s formula reduces to
\[
D_v(s) = -i\langle \phi_r, v \rangle s
\]
so \( \phi_r \) must be a weight vector, and the action of \( G \) on \( L|F_r \) has weight \(-\phi_r \). We assume in sections 4.1 and 4 that the action is free on the level set \( Z = \phi^{-1}(0) \).

Hence the symplectic reduction \( M_G := Z/G \) is well-defined as a smooth symplectic manifold, with symplectic form \( \omega_G \), and in fact \( M_G \) is prequantizable, with line bundle \( L_G := L|Z/G \).

2. Quantization as a \( K \)-theoretic pushforward

We recall the definition of \( Q(M) \) in [3]. The almost complex structure \( J \) gives a splitting of the complexified cotangent bundle
\[
T^* M \otimes \mathbb{C} = T^* M^{1,0} \oplus T^* M^{0,1}
\]
and hence a bigrading of the exterior algebra
\[
\Lambda^k(T^* M \otimes \mathbb{C}) \cong \sum_{p+q=k} T^* M^{p,q}
\]
and of the deRham complex
\[
\Omega^k(M) = \sum_{p+q=k} \Omega^{p,q}(M)
\]
Define the operator \( \bar{\partial} : \Omega^{0,q}(M) \to \Omega^{0,q+1}(M) \) by
\[
\bar{\partial} = \pi^{0,q+1} \circ d_{0,q}.
\]
This gives a sequence of maps
\[
\cdots \to \Omega^{0,q}(M) \xrightarrow{\bar{\partial}} \Omega^{0,q+1}(M) \xrightarrow{\bar{\partial}} \cdots
\]
(This is not a complex unless \( M \) is a complex manifold.)
Given a Hermitian connection $\nabla$ on the prequantum line bundle $L$, we can form an operator

\[ \bar{\partial}_L : \Omega^{0,q}(M,L) \to \Omega^{0,q+1}(M,L) \]

\[ \bar{\partial}_L := \bar{\partial} \otimes 1 + 1 \otimes (\pi^0 \circ \nabla). \]

This operator has principal symbol

\[ \sigma(\bar{\partial}_L)(x,\alpha)(\beta) = i\alpha^{0,1} \wedge \beta \]

where $x \in M$, $\alpha \in T^*_x M$, and $\beta \in T^*_x M^{0,q} \otimes L$.

We form an elliptic operator $\partial_L$ from $\bar{\partial}_L$ by adding it to its adjoint:

\[ \partial_L : \Omega^{0,\text{even}}(M,L) \to \Omega^{0,\text{odd}}(M,L) \]

\[ \partial_L := \bar{\partial}_L + \bar{\partial}_L^*. \]

The quantization is defined as

\[ Q(M) := \text{a-Ind}(\partial_L) = \ker\partial_L - \text{coker}\partial_L. \]

where I have labeled this “a-Ind” to emphasize the analytical nature of this definition, as opposed to the topological one I will give below.

Given an action of $G$ on $(M,L)$ we can choose $\nabla$ to be preserved by $G$, and hence $Q(M)$ is a virtual representation of $G$.

The Atiyah-Singer index theorem equates the analytical index with the topological index:

\[ \text{a-Ind}(\partial_L) = \text{t-Ind}(\partial_L). \]

The topological index depends only on the symbol of $\partial_L$, as an element of $K_G(T^*M)$, which equals the symbol of $\partial_L$ (493).

To calculate $\text{t-Ind}(\partial_L)$ we push forward the $K$-class $\sigma(\partial_L)$ to a point, in the following manner. First, we use the Riemannian metric $g$ on $M$ to identify $TM$ and $T^*M$.

The pullback of $\sigma(\partial_L)$ to $TM$ is given by the complex

\[ \cdots \to \Lambda^q \pi^*_TM \otimes \pi^*_TL \overset{\sigma}{\to} \Lambda^{q+1} \pi^*_TM \otimes \pi^*_TL \to \cdots \]

\[ (v,u) \to v \wedge u \]

where $v \in T_x M$ and $u \in \Lambda^q T^*_x M \otimes L_x$. (Note that I am considering $TM$ as a complex vector bundle, with complex structure $J$, so the above complex $\sigma$ is an element of $K_G(TM).$)

Now $\sigma$ is exactly the Thom isomorphism applied to the vector bundle $L$ (493). We can also express the Thom isomorphism as a push-forward by the zero section, call it $a : M \to TM$:

\[ \sigma = \text{Thom}_{TM}(L) = a_*L \in K_G(TM). \]

The next step in calculating the index is to embed $M$ equivariantly in a trivial $G$-space $\mathbb{C}^n$. We can in fact choose $n$ large enough so that the normal bundle $N_M$ will have a (unique) complex structure $[\mathfrak{2}]$, defined by the exact sequence of complex vector bundles

\[ 0 \to TM \overset{T_j}{\to} j^* T \mathbb{C}^n \to N_M \to 0 \]
We then have the following diagram:

\[ F \xrightarrow{i} M \xrightarrow{j} \mathbb{C}^n \xrightarrow{k} * \]

\[ TM \xrightarrow{T_j} T\mathbb{C}^n \]

Here \( i \) is the inclusion of the fixed point set \( F \) (which will come into the picture soon), \( j \) is the chosen embedding of \( M \) in \( \mathbb{C}^n \), with corresponding tangent map \( T_j \); \( a \) and \( b \) are the zero sections of \( TM \) and \( T\mathbb{C}^n \) respectively; and \( k \) and \( l \) are the inclusions of the origin into \( \mathbb{C}^n \) and \( T\mathbb{C}^n \) respectively.

The topological index of \( \bar{\partial}_L \) is defined to be the pushforward

\[ t\text{-Ind}(\bar{\partial}_L) = (l!)^{-1} (T_j)! \sigma \]

\[ = (l!)^{-1} (T_j)! a_i L \]

It would seem that functoriality of the pushforward immediately gives

\[ Q(M) = t\text{-Ind}(\bar{\partial}_L) = (k_i)^{-1} j_i L =: p_i L \]

where the last equality is the definition of the pushforward of \( L \) by the map \( p : M \to * \). However, we need to be careful about \( K \)-orientations. Each of these pushforwards requires a \( K \)-orientation, for example a complex structure, on the corresponding normal bundle for a precise definition. Let us adopt the temporary notation \( \kappa(f) \) to denote the complex structure on the normal bundle to an embedding \( f : X \to Y \) to be used in the pushforward \( f_i : K_G(X) \to K_G(Y) \). Then in the diagram (19) we know

\[ \kappa(a) = TM \]
\[ \kappa(b) = T\mathbb{C}^n = \mathbb{C}^n \quad \text{ (trivial rank n bundle)} \]
\[ \kappa(T_j) = \pi_{TM}^* N_M \oplus \pi_{TM}^* \overline{N}_M \]
\[ \kappa(k) = \mathbb{C}^n \]
\[ \kappa(l) = \mathbb{C}^{2n} = \kappa(k) \oplus k^* \kappa(b) \]

(See [4] for the identification of \( \kappa(T_j) \).) The last equation shows that the right-hand triangle in the pushforward diagram

\[ K(F) \xrightarrow{i} K_G(M) \xrightarrow{j} K_G(\mathbb{C}^n) \xrightarrow{(k_i)^{-1}} K_G(*) \]

\[ K_G(TM) \xrightarrow{(T_j)^i} K_G(T\mathbb{C}^n) \]

commutes (this is just Bott periodicity); what we need is the correct \( \kappa(j) \) to make the square commute. But if the square is to commute we must have

\[ j^* \kappa(b) \oplus \kappa(j) = \kappa(T_j \circ a) \]
\[ = \kappa(a) \oplus a^* \kappa(T_j) \]
\[ = TM \oplus a^*(\pi^* N_M \oplus \pi^* \overline{N}_M) \]
\[ = TM \oplus N_M \oplus \overline{N}_M \]
\[ = \mathbb{C}^n \oplus \overline{N}_M \]
\[ = j^* \kappa(b) \oplus \overline{N}_M \]
so we must choose \( \kappa(j) = \overline{N}_M \), and not \( N_M \). This gives us our

**Theorem 1.** Let \( p : M \to * \) be the unique map and let \( N_M \) be the stable normal bundle to \( M \), defined by (18) above. Then the quantization of \((M,L)\) is exactly \( p!L \), using the \( K \)-orientation \( N_M \).

The fact that we must use \( \overline{N}_M \) will be significant when we look at localization in section 3. There we will be concerned with the fixed point set \( F \) of the \( G \)-action.

We have a diagram

\[
\begin{array}{ccc}
K_G(F) & \xrightarrow{i_i} & K_G(M) \\
\downarrow{q} & & \downarrow{p} \\
K_G(*) & & \\
\end{array}
\]

which we want to commute. We now know how to precisely define \( p! \) and \( q! \) to agree with the quantization: we use the complex structures \( \kappa(p) = \overline{N}_M \) and \( \kappa(q) = \overline{N}_F \) on the respective stable normal bundles. Letting \( N \) be the usual complex normal bundle of \( F \) in \( M \), defined by

\[
0 \to TF \xrightarrow{T_i} i^*TM \xrightarrow{i} N \xrightarrow{\nu} 0
\]

we must have

\[
\overline{N} \oplus i^*\overline{N}_M = \overline{N}_F
\]

\[
= \kappa(q)
\]

\[
= \kappa(i) \oplus i^*\kappa(p)
\]

\[
= \kappa(i) \oplus i^*\overline{N}_M
\]

so \( \kappa(i) = \overline{N} \). This will be important in getting the signs correct in the next section.

### 3. Localization in Equivariant \( K \)-theory

The key tool we use is the localization theorem of Atiyah and Segal in equivariant \( K \)-theory, which we briefly review. We follow the treatment in [3] except that they are doing index theory and hence work on the tangent bundle, while we work on \( M \) itself.

We wish to calculate \( Q(M) = p!L \in K_G(*) = R(G) \). Since every element of \( R(G) \) is determined by its character, we can specify \( Q(M) \) by evaluating its character at every element \( g \in G \), or even on a dense subset of elements \( g \). For simplicity, assume \( G \) is topologically cyclic. (Of course eventually \( G \) will simply be \( S^1 \).) Fix a (topological) generator \( g \in G \), i.e. let \( (g) \) be dense in \( G \). Then \( M^g = M^G = F \). The localization theorem gives a formula for computing the character of \( p!L \), evaluated at \( g \), in terms of data on \( F \).

We start with the diagram (24). All of these rings are actually \( R(G) \)-algebras, so we can localize at \( g \) (this inverts all characters not vanishing at \( g \)).

\[
\begin{array}{ccc}
K_G(F)_{(g)} & \xrightarrow{\langle i_{(g)} \rangle} & K_G(M)_{(g)} \\
\downarrow{\langle g \rangle} & & \downarrow{(p_{(g)})} \\
R(G)_{(g)} & & \\
\end{array}
\]
**Theorem 2** (3). The map \((i)_g\) is an isomorphism of \(R(G)_g\)-modules.

This allows us to calculate the pushforward by \(p\) in terms of the pushforward by \(q\), at least in the localized ring \(R(G)_g\). This is good enough, since we are interested in evaluating \(p_*L\) at \(g\), and the evaluation map \(ev_g : R(G) \to \mathbb{C}\) factors through the localization \(R(G)_g\). In fact we have the following commutative diagram:

\[
\begin{array}{cccc}
K_G(M) & \xrightarrow{(i)_g} & K_G(M)_g & \xrightarrow{(i)_g^{-1}} & K(F) \otimes R(G)_g \\
\downarrow{p_*} & & \downarrow{(p)_g} & & \downarrow{1 \otimes ev_g} \\
R(G) & \xrightarrow{ev_g} & R(G)_g & \xrightarrow{ev_g} & K(F) \otimes \mathbb{C}
\end{array}
\]

Here we have used the isomorphism \(K_G(F) \cong K(F) \otimes R(G)\) for the trivial \(G\)-space \(F\).

The next step is to explicitly identify the map \((i)_g^{-1}\). This turns out to be simple. Recall that the \(K\)-orientation for the map \(i\) was \(\kappa(i) = N\). We have (4, 493)

\[
i^*i_1 u = N \Lambda N \cdot u := \left( \sum (-1)^k \Lambda^k N \right) \cdot u
\]

so when we localize at \(g\), the inverse is simply

\[
(i)_g^{-1} L = \frac{i^*L}{\Lambda N}
\]

Using this explicit inverse we can write down the localization formula giving the result of evaluation at \(G\):

\[
(p_*L)(g) = q \left( \frac{i^*L(g)}{\Lambda N(g)} \right)
\]

where the quantity in parentheses is in \(K(F) \otimes \mathbb{C}\), and the evaluations \(i^*L(g)\) and \(\Lambda N(g)\) are defined by the composite map

\[
K_G(F)_g \cong K(F) \otimes R(G)_g \xrightarrow{1 \otimes ev_g} K(F) \otimes \mathbb{C}
\]

In other words, to use this formula, we need to represent \(i^*L\) and \(\Lambda N(g)\) as sums of \(G\)-fixed bundles tensored with characters of \(G\), and then evaluate at \(g\) by the prescriptions

\[
\begin{align*}
u \otimes \chi & \mapsto u \cdot \chi(g) \\
u \otimes \chi/\psi & \mapsto u \cdot \chi(g)/\psi(g)
\end{align*}
\]

4. Quantization Commutes With Reduction

4.1. Symplectic Cutting. From here on we deal only with the case \(G = S^1\). In [11] E. Lerman defines an operation on a Hamiltonian \(S^1\)-space called *symplectic cutting*. (See also [3].) Cutting produces from a Hamiltonian \(S^1\)-space \((M, \omega, \phi)\) a pair of Hamiltonian \(S^1\)-spaces \((M_+, \omega_+, \phi_+)\) and \((M_-, \omega_-, \phi_-)\) with the following properties:
• The reduced space $M_{S^1}$ embeds in both $M_+$ and $M_-$ as a component of the
fixed point set.
• $M_+ \setminus M_{S^1}$ is equivariantly, symplectically isomorphic to $\phi^{-1}(\mathbb{R}_+)$. 
• $M_- \setminus M_{S^1}$ is equivariantly, symplectically isomorphic to $\phi^{-1}(\mathbb{R}_-)$. 
• $\phi_+(M_{S^1}) = \phi_-(M_{S^1}) = 0$.

Further, if $M$ is prequantizable, with prequantum line bundle $L$ and prequantizing 
connection $\nabla$, both $M_+$ and $M_-$ are prequantizable, with line bundles $L_+$ and $L_-$,
and the restriction of these bundles to the reduced space is just the reduced line 
bundle:

\begin{align}
L_+|_{M_{S^1}} &\cong L_{S^1}, \\
L_-|_{M_{S^1}} &\cong L_{S^1}.
\end{align}

(37)

4.2. The Main Results. Since symplectic cutting embeds the reduced space $M_{S^1}$
into pieces of the original space $M$ as a fixed point set, we can apply the $K_{S^1}$-
localization theorem. We can prove our main result, Theorem 3 by com paring 
$M$, $M_+$, and $M_{S^1}$.

In sections 4.3 and 4.4 we use Laurent series expansions to prove the following 
two propositions.

Proposition 1. Let $M, N$ be prequantizable Hamiltonian $S^1$-spaces with moment 
maps $\phi, \psi$. Assume that 0 is not the maximum value of $\phi$ or of $\psi$. If $\phi^{-1}(\mathbb{R}_+)$ is 
equivariantly symplectomorphic to $\psi^{-1}(\mathbb{R}_+)$, then

$$Q(M)^{S^1} = Q(N)^{S^1}.$$ 

Proposition 2. Let $M$ be a prequantizable Hamiltonian $S^1$-space with moment 
map $\phi$ and line bundle $L$. Assume that 0 is the minimum value of $\phi$. Let $F_0 = 
\phi^{-1}(0)$ and consider the maps $i_0 : F_0 \to M$, $q_0 : F_0 \to \ast$. Then

$$Q(M)^{S^1} = (q_0)_* i_0^* L \in K(\ast) \cong \mathbb{Z}.$$ 

Assuming these propositions we can prove that quantization commutes with
reduction. Consider our Hamiltonian $S^1$-space $M$. Since 0 is a regular value of $\phi$ it
is certainly not the maximum of $\phi$ or of $\phi_+$. Applying Prop. 4 to $M$ and $M_+$ gives

$$Q(M)^{S^1} = Q(M_+)^{S^1}. \tag{38}$$

Now 0 is the minimum value of $\phi_+$, so we can apply Prop. 2 to $M$ and $\phi^{-1}(0) = 
M_{S^1}$. Here $i_0^* L = L_{S^1}$ and $(q_0)_* L_{S^1} = Q(M_{S^1})$, so

$$Q(M_+)^{S^1} = Q(M_{S^1}). \tag{39}$$

Putting these together gives our main theorem.

Theorem 3. Let $(M, \omega, \phi)$ be a prequantizable Hamiltonian $S^1$-space with prequan-
tum line bundle $L$ and assume that the action is free on the zero level set $\phi^{-1}(0)$.
Then the quantization $Q(M) = p_!(L)$ commutes with reduction:

$$Q(M)^{S^1} = Q(M_{S^1}).$$
4.3. Expansion in Laurent Series. It remains to prove Props. 3 and 4. We will use the localization theorem, and two different expansions in Laurent Series, which correspond to the limit arguments (“z → 0” and “z → ∞”) in 4.3.

In the case of a Hamiltonian circle action, we can express the localization formula (34) in the following way. First, we recall that the fixed point set (40) breaks up into connected components $F_r$, on each of which the action of $S^1$ on $L$ has weight $-\phi_r$. The localization formula becomes

$$ (p_t L)(g) = q_1 \left( \sum_r i_r^* L(g) \right) $$

It turns out that we need to know very little about the quantity $p_t L$ to prove the propositions. This allows us to do everything within $K_{G(F)} \otimes \mathbb{C} \cong K(F) \otimes \mathbb{C}[z, z^{-1}]$, without actually evaluating the pushforward. (We tensor with $\mathbb{C}$ so that later operations involving tensors will be exact; since the final result $(p_t L)(g)$ is in $\mathbb{C}$ this is sufficient.)

We want to consider the contribution of each component of the fixed point set in turn. So fix $r$, and let $l_r = i_r^* L$ with the trivial action of $G$. Then as an element of $K_{G(F)} \otimes \mathbb{C} \cong K(F_r) \otimes R(G) \otimes \mathbb{C} \cong K(F_r) \otimes \mathbb{C}[z, z^{-1}]$, we have

$$ i_r^* L = l_r \cdot z^{-\phi_r}. $$

Let $S_r$ be the set of weights of the action of $G$ on $N_r$. Then we can write $\overline{N}_r \in K(F_r) \otimes \mathbb{C}[z, z^{-1}]$ as

$$ \overline{N}_r = \bigoplus_{k \in S_r} \mathcal{N}_{r,k} z^{-k} $$

where the $\mathcal{N}_{r,k}$ are vector bundles with trivial $G$-action. (Note they are not necessarily line bundles. In fact we will not need to use a splitting principle.) Let $n_{r,k} = \text{rank} \mathcal{N}_{r,k}.$

The only difficult step is inverting $\overline{N}_r$. To formally invert polynomials, it is useful to embed the polynomial ring in the larger ring of formal power series. In our case, we need to embed our ring $K(F) \otimes \mathbb{C}[z, z^{-1}]$ of formal Laurent polynomials with coefficients in $K(F)$ (localized at $g$), into two different rings of formal Laurent series, depending on which proposition we want to prove:

$$ K(F) \otimes \mathbb{C}[z, z^{-1}] \to K(F) \otimes \mathbb{C}[z] $$

$$ K(F) \otimes \mathbb{C}[z, z^{-1}] \to K(F) \otimes \mathbb{C}[z^{-1}]. $$

Here $\mathbb{C}[z]_z$ is the ring of Laurent series “at $z = 0$,” i.e. allowing an infinite number of nonzero terms with positive powers of $z$, but only a finite number of nonzero terms with negative powers of $z$. Similarly $\mathbb{C}[z^{-1}]_{z^{-1}}$ is the ring of Laurent series “at $z = \infty$.” It is not hard to see that these maps really are injective, since they are derived from the basic inclusion $\mathbb{C}[z] \subset \mathbb{C}[z]$ by localization and tensoring, and both operations are exact over a field. Hence we lose no information in this process.

Let $S_{r+} := S_r \cap \mathbb{Z}_+$ and $S_{r-} := S_r \cap \mathbb{Z}_-$. Note $S_r = S_{r+} \cup S_{r-}$ since the zero weight doesn’t appear in the normal bundle. Also note that $S_{r+} = \emptyset$ iff $\phi_r$ is the
maximum of $\phi$, and $S_{r-} = \emptyset$ iff $\phi_r$ is the minimum of $\phi$. (For a general manifold, these would only be statements about local minima and maxima, but since this is a Hamiltonian $S^1$-space there are no local maxima or minima except the global max and min. See [7], [10].) Then

$$\Lambda N_r = \Lambda \bigoplus_{k \in S_r} N_{r,k} z^{-k}$$

$$= \prod_{k \in S_r} \Lambda(N_{r,k} z^{-k})$$

$$= \prod_{k \in S_r} \sum_{j=0}^{n_{r,k}} (-1)^j (\Lambda^j N_{r,k}) z^{-jk}$$

$$= P_r(z) Q_r(z^{-1})$$

where $P_r$ and $Q_r$ are polynomials with constant term 1 and invertible leading coefficient:

$$\text{Leading coeff. of } P_r = \prod_{k \in S_{r-}} (-1)^{n_{r,k}} \det N_{r,k}$$

$$\text{Leading coeff. of } Q_r = \prod_{k \in S_{r+}} (-1)^{n_{r,k}} \det N_{r,k} \cdot$$

We have

$$P_r = 1 \iff S_{r-} = \emptyset \iff \phi_r = \phi_{\min}$$

$$Q_r = 1 \iff S_{r+} = \emptyset \iff \phi_r = \phi_{\max}.$$

Hence we can invert $\Lambda N$ in the formal Laurent rings according to the results in the appendix.

1. (“Limit as $z \to 0$”) In the ring $K(F) \otimes \mathbb{C}[[z]] z^{-1}$ we have

$$\Lambda N_r^{-1} = P_r^{-1} Q_r^{-1}$$

Satisfy

$$\left\{
\begin{array}{l}
1 + O(z) \quad \text{if } \phi_r = \phi_{\max} \\
O(z) \quad \text{if } \phi_r \neq \phi_{\max}.
\end{array}
\right.$$

Here $O(z^k)$ indicates a term that has no nonzero coefficients below the $k$th power.

2. (“Limit as $z \to \infty$”) In $K(F) \otimes \mathbb{C}[[z^{-1}]] z^{-1}$ we have

$$\Lambda N_r^{-1} = P^{-1} Q^{-1}$$

Satisfy

$$\left\{
\begin{array}{l}
1 + o(z^{-1}) \quad \text{if } \phi_r = \phi_{\min} \\
o(z^{-1}) \quad \text{if } \phi_r \neq \phi_{\min}.
\end{array}
\right.$$

Here $o(z^k)$ indicates a term that has no nonzero coefficients above the $k$th power.

4.4. **Proof of the Propositions.**

*Proof of Prop. 1.* The multiplicity of the trivial representation in $Q(M)$ is just the constant term in the polynomial

$$Q(M)(z) = q! \left( \sum_r l_r z^{-\phi_r} P_r^{-1} Q_r^{-1} \right).$$
We will show that the constant term depends only on the components $F_r$ with $\phi_r > 0$.

We can express

\[ \sum_r l_r z^{-\phi_r} P_r^{-1} Q_r^{-1} \]

in the Laurent series ring $K(F)[[z]]z$ using (46). The terms in the sum (50) with $\phi_r < 0$ are of the form $O(z)$ by (46), so they do not contribute to the constant term. The terms with $\phi_r = 0$ are also of the form $O(z)$ since we are assuming $\phi_{\text{max}} \neq 0$.

Since the constant terms in $Q(M)(z)$ and $Q(N)(z)$ only depend on the fixed point sets in $\phi^{-1}(\mathbb{R}_+)$ and $\psi^{-1}(\mathbb{R}_+)$ respectively, and these portions of $M$ and $N$ are symplectomorphic, we have

\[ Q(M)^{S^1} = Q(N)^{S^1}. \]

**Proof of Prop. 2.** $(M, \omega, \phi)$ is a prequantizable Hamiltonian $S^1$-space with line bundle $L$. Since 0 is the minimum value of $\phi$, $F_0 = \phi^{-1}(0) \subset M^{S^1}$. The localization theorem gives

\[ Q(M)(z) = q_0 \left( \sum_{\phi_r > 0} l_r z^{-\phi_r} P_r^{-1} Q_r^{-1} \right) + (q_0)_! (i_0^* L P_0^{-1} Q_0^{-1}) \]

where $q_0 : F \to *$ and $i_0 : F \to M$.

In $K(F)[[z^{-1}]]z^{-1}$, the terms in the summation

\[ \sum_{\phi_r > 0} l_r z^{-\phi_r} P_r^{-1} Q_r^{-1} \]

have only negative powers, by (48), so they do not contribute to the constant term. The contribution of $F$ is

\[ i_0^* L \cdot (1 + o(z^{-1})) \]

again by (48), so the constant term is just the pushforward from $F_0$,

\[ Q(M)^{S^1} = (q_0)_! i_0^* L. \]

**5. Appendix: Inverting Polynomials in Laurent Series Rings**

Here we write down some elementary facts about inverting polynomials in rings of formal Laurent Series, needed in section 4.3 above.

Let $R$ be a ring. We want to formally invert Laurent polynomials, i.e. elements of $R[z, z^{-1}]$. For our purposes we only need to know the most basic facts about the dependence of these inverses on $z$, and for that purpose, it is useful to embed $R[z, z^{-1}]$ into the two rings of formal Laurent series, $R[[z]]z$ and $R[[z^{-1}]]z^{-1}$.

We look at the case of $R[[z]]z$, formal Laurent series in at $z = 0$. 
1. Let \( a(z) = a_0 + a_1 z + \ldots + a_n z^n \) be a polynomial over \( R \). Suppose \( a_0 \) is invertible. Then we can invert \( a \) in \( R[[z]] \), hence \( a \) is invertible in \( R[[z]]_z \):
\[
a^{-1} = a_0^{-1} (1 + (a_1/a_0) z + \ldots + (a_n/a_0) z^n)^{-1}
\]
\[
= a_0^{-1} \sum_{l=0}^{\infty} ((a_1/a_0) z + \ldots + (a_n/a_0) z^n)^l
\]
\[
= a_0^{-1} + O(z).
\]

2. Let \( b(z) = b_0 + b_1 z^{-1} + \ldots + b_m z^{-m} \) be a Laurent polynomial over \( R \). Suppose \( b_m \) is invertible. Then we can invert \( b \) in \( R[[z]]_z \):
\[
b = b_m z^{-m} (1 + (b_{m-1}/b_m) z + \ldots + (b_0/b_m) z^m)
\]
\[
b^{-1} = b_m^{-1} z^m \sum_{l=0}^{\infty} ((b_{m-1}/b_m) z + \ldots + (b_0/b_m) z^m)^l
\]
\[
= b_m^{-1} z^m + O(z^{m+1}).
\]

The case of \( R[[z^{-1}]]_{z^{-1}} \) is exactly parallel; simply exchange \( z \) with \( z^{-1} \) and \( O \) with \( o \).

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