**PT spectroscopy of the Rabi problem**

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We investigate the effects of a time-periodic, non-hermitian, PT-symmetric perturbation on a system with two (or few) levels, and obtain its phase diagram as a function of the perturbation strength and frequency. We demonstrate that when the perturbation frequency is close to one of the system resonances, even a vanishingly small perturbation leads to PT symmetry breaking. We also find a restored PT-symmetric phase at high frequencies, and at moderate perturbation strengths, we find multiple frequency windows where PT-symmetry is broken and restored. Our results imply that the PT-symmetric Rabi problem shows surprisingly rich phenomena absent in its hermitian or static counterparts.

**Introduction.** A two-level system coupled to a sinusoidally varying potential is a prototypical example of a time-dependent, exactly solvable Hamiltonian, with profound implications to atomic, molecular, and optical physics [1]. When the frequency of perturbation \( \omega \) is close to the characteristic frequency \( \Delta \) of the two-level system - near resonance - the system undergoes complete population inversion for an arbitrarily small strength \( \gamma \) of the potential [2]. The implications of this result to spin magnetic resonance, Rabi flopping [3], and its generalization, namely the Jaynes-Cummings model [4, 5], have been extensively studied over the past half century [6, 7]. Surprisingly, the quantum Rabi problem, where the full quantum nature of the perturbing bosonic field is taken into account, has only been recently solved [8].

The two-level model is useful because it is applicable to many-level systems when the perturbation frequency is close to or resonant with a single pair of levels. As the detuning away from resonance \( |\Delta - \omega| \) increases, the perturbation strength necessary for population inversion increases linearly with it; in a many-level system, with increased potential strength, transitions to other levels have to be taken into account and the resultant problem is not exactly solvable. Therefore, understanding the behavior of a system in the entire parameter space \((\gamma, \omega)\) requires analytical and numerical approaches. All of these studies are restricted to hermitian potentials.

In recent years, discrete Hamiltonians with a hermitian tunneling term \( H_0 \) and a non-hermitian perturbation \( V \) that are invariant under combined parity and time-reversal (PT) operations have been extensively investigated [9, 10]. The spectrum \( \epsilon_x \) of a PT-symmetric Hamiltonian is real when the strength \( \gamma \) of the non-hermitian perturbation is smaller than a threshold \( \gamma_{PT} \) set by the hermitian tunneling term. Traditionally, the emergence of complex-conjugate eigenvalues that occurs when the threshold is exceeded, \( \gamma > \gamma_{PT} \), is called PT symmetry breaking [17, 19]. It is now clear that PT-symmetric Hamiltonians represent open systems with balanced gain and loss, and PT symmetry breaking is a transition from a quasiequilibrium state (PT-symmetric state) to a state with broken reciprocity (PT-broken state) [20]. Recent experiments on optical waveguides [21, 23] and resonators [24] with amplification and absorption have shown that PT systems display a wealth of novel phenomena [25, 26] that are absent in closed or purely dissipative systems. We note that all experiments and most of the theoretical work, with few exceptions [27, 29], have only explored systems with static gain and loss potentials.

And what of a system perturbed by a time-periodic gain-loss potential \( V(t) \)? What is the criterion for PT-symmetry breaking? What is the analog of Rabi flopping in such a case? We answer these questions by investigating small, \( N \)-site lattices perturbed by a pair of balanced gain-loss potentials \( \pm i \gamma \cos(\omega t) \) located at parity symmetric sites. Such systems can be realized in coupled waveguides with a complex refractive index [22, 23, 28] that varies along the propagation direction, or in coupled resonators [24].

Our primary results are as follows: i) The system can be in "PT-broken phase" even if the spectrum \( \epsilon_x(t) \) of the Hamiltonian \( H(t) \) is real at all times. ii) Near every resonance, the PT symmetric threshold is reduced from its static value \( \gamma_{PT} \) to the detuning, \( \gamma_{PT}(\omega) \propto |\omega - \Delta| \); in particular, it vanishes at the resonance. iii) For any gain-loss strength including \( \gamma \gg \gamma_{PT} \), the PT-symmetric phase is restored at high frequencies \( \omega > \omega_c \propto \gamma \). iv) At intermediate strengths, \( \gamma \sim \gamma_{PT} \), the PT symmetry is broken in multiple windows in the frequency domain. Thus, a harmonic, PT-symmetric perturbation provides a new spectroscopic tool for investigating the level structure of a system.

**PT phase diagram.** The Hamiltonian for an N-site lattice with constant tunneling is

\[
H_0 = -\hbar J \sum_{x=1}^{N-1} (|x\rangle\langle x+1| + |x+1\rangle\langle x|),
\]

where \(|x\rangle\) is a normalized state localized on site \( x \), \( J \) is the tunneling rate, and \( \hbar = h/(2\pi) \) is the scaled Planck’s constant. The action of the parity operator on the lattice
is given by $x \rightarrow \bar{x} = N + 1 - x$ and the antilinear time-reversal operator acts as $i \rightarrow -i$. The spectrum of $H_0$ is given by $\epsilon_n = -2\hbar J \cos(k_n) = -\epsilon_n$ and the normalized eigenfunctions are $\psi_n(x) = |n\rangle = \sin(k_n x) / \sqrt{1 + 1/N}$ where $k_n = n\pi/(N + 1)$ with $1 \leq n \leq N$. The energy differences $\hbar\Delta_{nm} = \epsilon_n - \epsilon_m > 0$ define the possible resonances for this $N$-level system. Motivated by the Rabi problem, here we will only consider $N = \{2, 3, 4\}$. This system is perturbed by a balanced gain-loss potential

$$
V(t) = i\hbar \gamma \cos(\omega t) \langle x_0 | x_0 \rangle - |x_0 \rangle \langle x_0 | \neq V(t). \tag{2}
$$

Eq. (2) implies that at time $t = 0$, $x_0$ is the gain or amplification site and $\bar{x}_0$ is the loss or absorption site. The non-hermitian potential satisfies $PTV(t)PT = V(t)$. The total Hamiltonian $H(t) = H_0 + V(t)$ is periodic in time, i.e. $H(t + 2\pi/\omega) = H(t)$, and its properties are best analyzed via its Floquet counterpart, $H = -i\hbar \partial_t + H$. In the frequency domain, the non-Hermitian, $PT$-symmetric Floquet Hamiltonian is given by

$$
H_{x,x'}^{p,q} = -i\hbar \omega \delta_{p,q} \delta_{x,x'} - \delta_{p,q} \hbar J (\delta_{x,x'+1} + \delta_{x,x'-1})
+ i\hbar \gamma \delta_{x,x'} (\delta_{p,q+1} + \delta_{p,q-1}) (\delta_{x_0,x} - \delta_{x_0,x'}) \tag{3}
$$

where $p, q \in \mathbb{Z}$ denote the Floquet band indices. In practice, a truncated Floquet Hamiltonian with $|p| \leq N_f$ is used, so that $H$ is an $(2N_f + 1) \times (2N_f + 1)$ matrix.

We define $PT$-symmetry breaking as the emergence of complex-conjugate eigenvalues for the Floquet Hamiltonian $H$. As we will show below, this can occur even if the instantaneous eigenvalues $\epsilon_\lambda(t)$ of the time-dependent Hamiltonian $H(t)$ are purely real over the entire period $T_\omega = 2\pi/\omega$.

Figure 1 is the $PT$-symmetric phase diagram of a two-level system in the $(\gamma/J, \omega/J)$ plane. Figure 1(a) shows the largest imaginary part of the spectrum of $H$. The static threshold $\gamma_{PT}/J = 1$ is suppressed down to zero when the perturbation frequency matches a resonance, $\omega/J = 2$. (b) A close-up of the area marked by the white rectangle in panel (a) shows that for moderate $\gamma/J \sim 1$, multiple $PT$-symmetric (PTS) and $PT$-broken (PTB) regions occur when $\omega$ is varied. (c) Maximum of the net intensity $I(t) = \langle \psi(t)|G(t)|\psi(t)\rangle$ at two cutoff times $t = 10\pi/\omega$ (open red circles) and $t = 20\pi/\omega$ (solid blue squares), on horizontal logarithmic axis, as a function of $\omega$ on the vertical axis, at $\gamma/J = 0.9$ (white dot-dashed line in panel (b)). PTB regions are distinguished by a clear dependence of $I_{max}$ on the cutoff.

A complementary method to distinguish the $PT$-broken (PTB) region from the $PT$-symmetric (PTS) region is to track the net intensity $I(t) = \langle \psi(t)|G(t)|\psi(t)\rangle = \langle \psi(t) | G(t) G(t) | \psi(0) \rangle$ of an initially normalized state $|\psi(0)\rangle$. We obtain the non-unitary time-evolution operator

$$
G(t) = T e^{-\frac{i}{\hbar} \int_0^t dt' H(t')} \approx \prod_{k=1}^{t/\delta t} e^{-\frac{i}{\hbar} \delta t H(k\delta t)}, \tag{4}
$$

Figure 1 shows the $PT$-symmetric phase diagram of a two-level system in the $(\gamma/J, \omega/J)$ plane. Figure 1(a) shows the largest imaginary part of the spectrum of the Hamiltonian $H$ with 101 Floquet bands. The region with zero imaginary part (dark blue) is the $PT$-symmetric phase and the region with nonzero imaginary part (all other colors) is the $PT$-broken phase; the static, $\omega = 0$, threshold is given by $\gamma_{PT}/J = 1$. Figure 1(a) has three universal features that appear in systems with larger $N$ as well. The first is a vanishingly small $PT$-symmetric threshold $\gamma_{PT}/(\omega)$ that occurs when $\omega$ is close to the single resonance frequency for the system, $\Delta_{21} = 2J$. The second is the emergence of $PT$-symmetric phase that occurs at large frequencies. $\omega > \omega_c \propto \gamma$ for any gain-loss strength including $\gamma/J \geq 1$. The third feature is the presence of multiple windows along the frequency axis such that the $PT$-symmetry is broken within each window. Figure 1(b) is a higher resolution close-up of the parameter space marked by the white rectangle in Fig. 1(a). It shows that for a fixed $\gamma$, as the frequency of the $PT$ potential is changed, multiple $PT$-symmetric and $PT$-broken regions emerge. These regions are present both below and above the static threshold.
where the discretization time-step $\delta t/T_\omega \ll 1$ is chosen to ensure that $G(t)$ is independent of it. Figure 2(c) shows the maximum intensity $I_{\text{max}}$ reached before time $t$ at cutoff times $t = 5T_\omega$ (open red circles) and $t = 10T_\omega$ (solid blue squares), as a function of the $\mathcal{PT}$-perturbation frequency $\omega$ on the vertical axis. These results are for $\gamma/J = 0.9$, initial state localized on the first site, and a time-step $\delta t/T_\omega = 10^{-5}$. In the $\mathcal{PT}$-symmetric region, $I(t)$ undergoes bounded oscillations and therefore $I_{\text{max}}$ is the same for the two time-cutoffs. In a sharp contrast, in the $\mathcal{PT}$-broken region, $I(t)$ increases exponentially with time and $I_{\text{max}}$ doubles on the logarithmic scale when the cutoff is doubled. Thus, both approaches show the existence of multiple frequency windows where $\mathcal{PT}$ symmetry is broken; note that the phase-boundaries in Fig. 2(c) do not exactly match those in Fig. 2(b) due to the finite time-cutoff.

We remind the reader that when $\gamma/J \leq 1$, the $2 \times 2$ Hamiltonian $H(t) = \text{i}\hbar \gamma \cos(\omega t)(\sigma_x - \hbar J \sigma_z)$ has a purely real spectrum $\epsilon_n(t) = \pm \hbar [J^2 - \gamma^2 \cos^2(\omega t)]^{1/2}$ at all times, and yet, the norm of the time-evolution operator $G(t)$ is either bounded or exponential-in-time at different frequencies [27].

Figure 2 shows the $\mathcal{PT}$-symmetric phase diagram of a three-level system, shown is the base-10 log of the maximum imaginary part of the spectrum of $\mathcal{H}$. Its three features - a linearly vanishing $\mathcal{PT}$-threshold at the resonance $\omega/J = \sqrt{2} \approx 1.41$, a restored $\mathcal{PT}$-symmetric phase at high frequencies, and multiple frequency windows with $\mathcal{PT}$-symmetric and $\mathcal{PT}$-broken phases at moderate $\gamma/J$ - are universal.

Figure 2 shows the $\mathcal{PT}$ phase diagram of a three-level system obtained by using 81 Floquet bands. We plot the base-10 logarithm of the largest imaginary part of eigenvalues of $\mathcal{H}$ to easily distinguish the $\mathcal{PT}$-symmetric phase (blue) and $\mathcal{PT}$-broken phase (red). For a three-level system, the unperturbed spectrum is given by $\epsilon_n = \{0, \pm \sqrt{2}J\}$, and has two resonance frequencies $\Delta_{21} = \sqrt{2}J = \Delta_{32}$ and $\Delta_{31} = 2\sqrt{2}J$. Fig. 2 shows a linearly vanishing $\gamma_{\mathcal{PT}}(\omega)$ at the first resonance $\omega/J = \sqrt{2}$, a $\mathcal{PT}$-restored phase at high frequencies, and a number of frequency windows where $\mathcal{PT}$ symmetry is broken at moderate values of $\gamma/J \leq 1$. There is no $\mathcal{PT}$-broken region near the second resonance, $\omega/J = 2\sqrt{2}$ (not shown) because $V(t)$ does not connect states with same parity. Thus, the phase diagram for a three-level system shares the fundamental characteristics of that for a two-level system.

Lastly, we consider a four-level system, $N = 4$. In this case, the resonances between states with opposite parity occur at $\Delta_{21}/J = 1 = \Delta_{34}/J, \Delta_{23}/J = (\sqrt{5} - 1) \approx 1.236$, and $\Delta_{34}/J = (\sqrt{5} + 1) \approx 3.236$. There are two inequivalent locations for the gain-loss potential, $x_0 = 1$ and $x_0 = 2$, and both have the same static threshold $\gamma_{\mathcal{PT}}/J = 1$. In the first case, only center-two of the four eigenvalues become complex at the threshold, whereas in the second case all four simultaneously become complex [33]. Figure 3 shows the $\mathcal{PT}$ phase diagram obtained with 41 Floquet bands. Both panels, (a) and (b), demonstrate the three salient features discussed earlier for both two and three-level systems.

Understanding the phase boundaries. We now derive the $\mathcal{PT}$-phase boundaries at occur at high frequencies or vanishingly small $\gamma$ near a resonance. The natural time-scale for an unperturbed system is proportional to $1/J$ and its static $\mathcal{PT}$ breaking threshold is also set by $J$. At high frequencies $\omega/J \gg 1$, the rapidly varying potential $\gamma \cos(\omega t)$ is replaced by its average over the characteristic time-scale, $\gamma_{\text{av}} \propto (\gamma/\omega)J$. For any gain-loss strength $\gamma$, no matter how large, increasing the frequency reduces the effective strength $\gamma_{\text{av}}$ and thus restores the
PT-symmetric phase. The slope of the linear phase-boundary in the region $\omega/J \gg 1$ will depend upon the number of levels $N$ and the location $x_0$ of the gain-loss potential, but the linear behavior of the phase boundary is universal.

Next we derive the cone-shaped phase boundary that occurs at small $\gamma$ in the neighborhood of a resonance $\omega \sim \Delta_{nm}$. For a state $|\psi(t)\rangle = \sum_{n} c_n(t) \exp(-i \epsilon_n t/\hbar) |n\rangle$, the interaction-picture equation of motion for the level-occupation coefficients $c_n(t)$ is given by

$$i\partial_t c_n(t) = \sum_{m=1}^{N} V_{nm}(t) e^{i \Delta_{nm} t} c_m(t), \quad (5)$$

$$V_{nm}(t) = i\gamma \cos(\omega t)|1 - (-1)^{n+m}|m\rangle \langle n| \langle x_0| \langle x_0|m\rangle. \quad (6)$$

For a two-level system, when $\omega \approx \Delta_{21} = 2J$, averaging over high-frequency terms simplifies Eq.(5) to

$$\partial_t^2 c_{1,2}(t) + \left[(\omega/2 - J)^2 - (\gamma/2)^2\right] c_{1,2}(t) = 0. \quad (7)$$

Eq.(7) implies that when $|\omega - 2J| > \gamma$, the coefficients $c_1(t)$ and $c_2(t)$ oscillate in time and remain bounded, and the system is in the $PT$-symmetric phase. When $|\omega - 2J| < \gamma$, $c_{1,2}(t)$ increase with time exponentially, and the system is in the $PT$-broken phase. Thus, $PT$-symmetric phase boundary is given by $\gamma_{PT}(\omega) = |\omega - 2J|$, and the threshold gain-loss strength vanishes as $\omega \rightarrow \Delta_{21} = 2J$. A visual inspection of Fig. 1(a) shows that, indeed, the slope of the phase-boundary lines fanning away from $\omega = 2J$ is one. Eq.(7) also implies that along the cone-shaped phase boundary, the net intensity $I(t)$ grows quadratically with time [20, 21, 24].

When $N = 3$ a similar analysis with three equidistant levels implies that near resonance $c_2(t)$ satisfies a third-order differential equation, $\partial_t^3 c_2(t) + \left[(\omega - \sqrt{2}J)^2 - (3\gamma/4)^2\right] \partial_t c_2(t) = 0$. Therefore, the phase-boundary separating the $PT$-broken region from the $PT$-symmetric region is given by $\gamma_{PT}(\omega) = (4/3)|\omega - \sqrt{2}J|$. This, too, can be verified by a visual inspection of Fig. 2. Our analysis also predicts that along this phase boundary, the net intensity of an initially normalized state increases quartically with time, i.e. $I(t) \propto t^4$, because $\partial_t^4 c_2(t) = 0$. In an $N$-level system, our analysis is applicable to a pair of levels $m, n$ when the $PT$-perturbation frequency is close to the resonance that connects those levels, $\omega \sim \Delta_{nm}$.

Thus, a vanishingly small $PT$ perturbation induces $PT$-symmetry breaking when the frequency of the perturbation is close to a resonance. Near resonance, the spatial oscillations of a state match the gain-loss temporal oscillations; as a result, it spends most of the time on the gain-medium site, leading to an exponential growth in the net intensity.

**Conclusion.** In this paper, we have proposed the $PT$-symmetric Rabi model. We have shown that a harmonic, gain-and-loss perturbation leads to a rich $PT$ phase diagram with three salient features. Among them is the existence of multiple frequency windows in which $PT$-symmetry is broken and restored. Time-dependent $PT$ potentials have been extensively investigated in continuum one-dimensional optical structures [25, 26]. Our results show that such potentials are a surprising spectroscopic probe, where the phase of the system - $PT$-broken or $PT$-symmetric - denotes the proximity of the perturbation frequency to a resonance of the system.

Although we have focused only on few-level systems here, our results are applicable to larger lattices, particularly in the vicinity of a resonance. Deep in the $PT$-broken phase, at long times, nonlinear effects also become relevant, although they do not affect our findings.

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