Community Structures Are Definable in Networks, and Universal in Real World

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Community detecting is one of the main approaches to understanding networks \textsuperscript{(1)}. However it has been a longstanding challenge to give a definition for community structures of networks. Here we found that community structures are definable in networks, and are universal in real world. We proposed the notions of entropy- and conductance-community structure ratios. It was shown that the definitions of the modularity proposed in \textsuperscript{(2)}, and our entropy- and conductance-community structures are equivalent in defining community structures of networks, that randomness in the ER model \textsuperscript{(3)} and preferential attachment in the PA \textsuperscript{(4)} model are not mechanisms of community structures of networks, and that the existence of community structures is a universal phenomenon in real networks. Our results demonstrate that community structure is a universal phenomenon in the real world that is definable, solving the chal-

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The challenge of definition of community structures in networks. This progress provides a foundation for a structural theory of networks.

We proposed a definition of community structures in networks, solving the fundamental challenge in modern network theory. Our definitions of the entropy- and conductance-community structures are information theoretical and mathematical definitions respectively. Our result of the equivalence of our entropy-, and conductance-community structure ratios, together with the modularity given by physicists shows that the existence of community structures in networks is a phenomenon definable by each of the physical, information theoretical and mathematical approaches, providing a common foundation for the interdisciplinary issue of networks. Our definitions of community structures of networks provide a method to decide both the existence and the quality of community structures of networks. Our discovery of the universality of community structures of real networks predicts that community structures maybe universal in the real world data, and that community structures maybe the key to a structural theory of networks and real world data in general. Our discovery that neither randomness nor preferential attachment is the mechanism of community structures of networks predicts that there must be new mechanisms for real world data. Therefore the definitions and discoveries here not only provide a foundation for a new theory of networks, but also a methodology for rigorous analysis of real world data.

Results

Network has become a universal topology in science, industry, nature and society. Most real networks satisfy a power law degree distribution (4, 5), and a small world phenomenon (6–8).

Community detecting or clustering is a powerful tool for understanding the structures of networks, and has been extensively studied (9–14). Many definitions of communities have been introduced, see (1) for a recent survey. However, the problem is still very hard, not yet satisfactorily solved. The current approaches to community finding take for granted that networks have
community structures. The fundamental questions are thus: Are communities objects naturally formed in a network or simply outputs of a graphic algorithm? Can we really take for granted that networks have community structures? Are community structures definable in networks? What are the natural mechanisms of the community structure of a network, if any?

Here we report our discovery that community structures are robust in networks, in the sense that, the three definitions of community structures based on modularity, entropy and conductance respectively give the same answer to the question whether or not a network has a community structure, that community structures are universal in real networks, and that neither randomness nor preferential attachment is the mechanism of community structures of networks.

**Modularity, Entropy and Conductance Definitions of Community Structure**

The first definition is the modularity community structure (M-community structure, for short). Newman and Girvan (2) defined the notion of modularity to quantitatively measure the quality of community structure of a network. It is built based on the assumptions that random graphs are not expected to have community structure and that a network has a community structure, if it is far from random graphs.

Let $G = (V, E)$ be a network. Given a partition $\mathcal{P}$ of $G$, the modularity of the partition $\mathcal{P}$ of network $G$ with $n$ nodes and $m$ edges is defined by

$$q^\mathcal{P}(G) = \frac{1}{2m} \sum_{i,j} (A_{ij} - P_{ij}) \delta(C_i, C_j),$$

where the summation runs over all pairs of vertices, $A$ is the adjacency matrix, $P_{ij}$ is the expected number of edges between vertices $i$ and $j$ in a null graph, i.e., a random version of $G$. $\delta(C_i, C_j) = 1$ if $C_i = C_j$, and 0 otherwise, $C_k$ is an element of the partition $\mathcal{P}$.

The modularity of $G$ is defined by

$$\sigma(G) = \max_{\mathcal{P}} \{ q^\mathcal{P}(G) \}.$$
Intuitively speaking, the larger $\sigma(G)$ is, the better community structure $G$ has. Therefore we define the modulearity community structure ratio (M-community structure ratio) of $G$ to be the modularity of $G$.

The second definition is based on random walks. The idea is that since random walks from a node in a quality community are not easy to go out of the community, a network can be decomposed into modules by compressing the description of an information flow. Rosvall and Bergstrom (15) proposed a way to use the Huffman code to encode prefix-freely each module and each node (adding an exit code) of a network. This allows us to reuse the codeword of a module-node for a random walk within the module, which compresses the bits of descriptions of random walks by the modules, compared to that of a uniform prefix-free code for all nodes.

Our definition follows the same idea. We consider the shortest average length of codes for a single step of random walks in the case of the standard stationary distribution that the probability of staying at some node $i$ is proportional to the degree of $i$.

Let $G = (V, E)$ be a graph with $n$ nodes and $m$ edges, and $\mathcal{P}$ be a partition of $V$. We use $L^U(G)$ to denote the minimum average number of bits to represent a step of random walk (in the stationary distribution) with a uniform code in $G$, and $L^P(G)$ to denote the minimum average number of bits to represent a step of random walk in $G$ with a code of modules given by $\mathcal{P}$ in $G$. By information theoretical principle, we have

$$L^U(G) = -\sum_{i=1}^{n} \frac{d_i}{2m} \cdot \log_2 \frac{d_i}{2m},$$

(1)

where $d_i$ is the degree of node $i$.

$$L^P(G) = -\sum_{j=1}^{L} \sum_{i=1}^{n_j} \frac{d_i^{(j)}}{2m} \cdot \log_2 \frac{d_i^{(j)}}{V_j} - \frac{m_g}{m} \left( \sum_{j=1}^{L} \frac{V_j}{2m} \cdot \log_2 \frac{V_j}{2m} \right),$$

(2)

where $L$ is the number of modules in partition $\mathcal{P}$, $n_j$ is the number of nodes in module $j$, $d_i^{(j)}$ is the degree of node $i$ in module $j$, $V_j$ is the volume of module $j$, and $m_g$ is the number of edges
crossing two different modules.

We define the entropy community structure ratio of $G$ by $\mathcal{P}$ by

$$
\tau^\mathcal{P}(G) = 1 - \frac{L^\mathcal{P}(G)}{L^U(G)}
$$

We define the entropy community structure ratio of $G$ (E-community structure ratio of $G$) by

$$
\tau(G) = \max_{\mathcal{P}} \{ \tau^\mathcal{P}(G) \}.
$$

Both the modularity and the entropy community structure ratio of a graph $G$ depend on randomness, the first is in the null version of the graph, and the second is in random walks in the graph. The two definitions are not convenient to measure the quality of overlapping communities, instead of a partition of the graph.

Here we introduce a mathematical definition based on conductance. Given a graph $G = (V, E)$, and a subset $S$ of $V$, the conductance of $S$ is given by

$$
\Phi(S) = \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}},
$$

where $E(S, \bar{S})$ is the set of edges with one endpoint in $S$ and the other in the complement of $S$, i.e. $\bar{S}$, $\text{vol}(X)$ is the summation of degrees $d_x$ for all $x \in X$.

We say that a set $X \subset V$ is a possible community if: (i) the induced subgraph of $X$, $G_X$ is connected, (ii) the size $|X|$ of $X$ is not less than $\log n$ (i.e., not too small), and (iii) the size of $X$ is less than $\sqrt{n}$ (i.e., not too large), where $n$ is the size of $V$.

(i) is a basic condition. (ii) and (iii) avoid trivial communities that are either not well-evolved, or is essentially a significant part of the whole network.
Suppose that $\mathcal{X} = \{X_1, X_2, \ldots, X_l\}$ is a set of possible communities of $G$. Let $X = \bigcup_j X_j$. For a community $X_j$, we use $1 - \Phi(X_j)$ to define the quality of the community.

For every $x \in X$, suppose that $X'_1, X'_2, \cdots, X'_p$ are all $X_j$’s that contain $x$, then define

$$a^X(x) = \frac{1}{p} \sum_{j=1}^{p} (1 - \Phi(X'_j)),$$

where $a^X(x)$ represents the average quality of all the communities containing $x$.

We define the conductance community structure ratio of $G$ by $\mathcal{X}$ (or C-community structure ratio, for short) by

$$\theta^X(G) = \frac{1}{n} \sum_{x \in X} a^X(x),$$

where $n$ is the number of nodes in $G$.

We define the conductance community structure ratio of $G$ by

$$\theta(G) = \max_{\mathcal{X}} \{\theta^X(G)\}.$$

Let $\mathcal{A}$ be an algorithm, and $G$ be a network. Suppose that $\mathcal{X}$ is the set of all possible communities found in $G$ by $\mathcal{A}$. Then define the conductance community structure ratio of $G$ by $\mathcal{A}$ by

$$\theta^A(G) = \theta^X(G).$$

This gives rise to a way to measure the quality of a community detecting algorithm. Intuitively, for two algorithms $\mathcal{A}$ and $\mathcal{B}$, if $\theta^A(G) > \theta^B(G)$, then $\mathcal{A}$ is better than $\mathcal{B}$ in finding the community structure of $G$. Clearly $\theta(G)$ characterizes the community structure of $G$.

Now we have three definitions of community structure of networks, the M-, E-, and C-community structure ratios. Intuitively speaking, the M-, E- and C-community structure ratios
capture the quality of community structure of $G$ from the viewpoints of physics, information theory and mathematics respectively.

**The Modularity, Entropy and Conductance Definitions of Community Structure Are Equivalent**

Are there any relationships among the three definitions of quality of community structures of networks, i.e., the M-, E-, and C-community structure ratios? Do the three definitions give the same answer to the question whether or not a network has a community structure? We conjecture that the answer is yes. For this, we propose the following hypothesis.

**Community structure hypothesis:** Given a network $G$, the following properties are equivalent,

1) $G$ has an M-community structure,

2) $G$ has an E-community structure, and

3) $G$ has a C-community structure.

We verify the community structure hypothesis by computing the M-, E-, and C-community structure ratios for networks of classical models. The first model is the ER model (3). In this model, we construct graph as follows: Given $n$ nodes $1, 2, \ldots, n$, and a number $p$, for any pair $i, j$ of nodes $i$ and $j$, we create an edge $(i, j)$ with probability $p$. The second is the PA model (4). In this model, we construct a network by steps as follows: At step 0, choose an initial graph $G_0$. At step $t > 0$, we create a new node, $v$ say, and create $d$ edges from $v$ to nodes in $G_{t-1}$, chosen with probability proportional to the degrees in $G_{t-1}$, where $G_{t-1}$ is the graph constructed at the end of step $t - 1$, and $d$ is a natural number.

We depict the curves of the M-, E-, and C-community structure ratios of networks of the ER model and the PA model in Figures 1 and 2 respectively.

From Figures 1 and 2 we observe that:
(1) The curves of the M-, E-, and C-community structure ratios of networks generated from the ER model are similar.

(2) The curves of the M-, E-, and C-community structure ratios of networks generated from the PA model are similar.

(1) and (2) show that the community structure hypothesis holds for all networks generated from the classic ER and PA models. We notice that every network essentially uses the mechanisms of both the ER and the PA models. Our results here imply that the community structure hypothesis may hold for most real networks.

**Empirical Criterions of Community Structures**

By observing the experiments in Figures 1 and 2, we have that for a network $G$ of either the ER model or the PA model, the following three properties (1), (2) and (3) either hold simultaneously or fail to hold simultaneously:

(1) the E-community structure ratio of $G$, $\tau(G)$, is greater than 0,

(2) the M-community structure ratio of $G$, $\sigma(G)$, is greater than 0.3, and

(3) the C-community structure ratio of $G$, $\theta(G)$, is greater than 0.3.

This result suggests an empirical criterion for deciding whether or not a network has a community structure. Let $G$ be a network, then

1. We say that $G$ has a community structure if the E-, M-, and C-community structure ratios of $G$ are greater than 0, 0.3 and 0.3 respectively.

2. The values $\sigma(G)$, $\tau(G)$ and $\theta(G)$ measure the quality of community structure of $G$, the larger they are, the better community structure $G$ has.
Randomness and Preferential Attachment Are Not Mechanisms of Community Structure

By the empirical criterion and by observing the experiments in Figures 1 and 2, we have that

1. For a network $G$ generated from the ER model, if $p < \frac{1}{2000}$ (in which case, the expected average number of edges is $< 5$), then $G$ has a community structure, and if $p > \frac{1}{2000}$, then $G$ fails to have a community structure.

2. For a network $G$ generated from the PA model, if $d < 5$, then $G$ has a community structure, and if $d > 5$, then $G$ fails to have a community structure.

This shows that the existence of community structure of networks of the ER and PA models depends on the density of the networks, that only networks with average number of edges $< 5$ may have a community structure, and that nontrivial networks of the ER and PA models fail to have a community structure. This is an interesting and useful discovery. It explains some mysterious phenomena: usually people believe that networks generated from the ER and PA models fail to have a community structure (although a proof is apparently needed), but sometimes people found graphs of the ER and PA models having extremely high modularity ($16$); in evolutionary games, some people implemented experiments on networks of the PA model with particular average number of edges $d = 4$ without any explanation ($17, 18$). Now we know that a network of the ER or PA model has a community structure only if the average number of edges is less than a small constant, 5 say, and that community structure of a network plays an essential role in networks.

Community Structures Are Universal in Real Networks

By using the empirical criterion of community structure of networks, we are able to decide whether or not a given network has a community structure.
We implemented the experiments of the entropy-, modularity- and conductance-community structure ratios, i.e., $\tau(G)$, $\sigma(G)$ and $\theta(G)$, for 22 real networks, which are given in Table 1. By observing the table, we have the following results: For every network $G$,

1. Then:
   - $\tau(G) > 0$,
   - $\sigma(G) > 0.3$, and
   - $\theta(G) > 0.3$.

2. $\tau(G) \leq \sigma(G)$ and $\tau(G) \leq \theta(G)$.

3. For most networks $G$, $\sigma(G) \approx \tau(G) + \alpha$ for some number $\alpha$ in the interval $[0.2, 0.3]$, and $\sigma(G) \approx \theta(G)$.

The experiments in Table 1 show that the community structure hypothesis holds for real networks, that community structures are universal in most real networks, and that the existence of community structures in real networks is independent of which definition of the M-, E- and C-community structures is used.

By observing all the curves in Figures 1 and 2 and all experiments in Table 1 again, we have the following conclusions: (1) The three definitions of modularity-, entropy- and conductance-community structures are equivalent in defining community structures of networks. This implies that the physical, information theoretical, and mathematical definitions of community structures of networks are equivalent, and that the existence of community structures of networks is a phenomenon independent of which one of the physical, information theoretical and mathematical definitions of community structures is used, and independent of algorithms for finding them. (2) There exists an empirical criterion for deciding the existence and quality of community
structure of a network. This also solves an important open question to test the quality of community finding algorithms. (3) Neither randomness nor preferential attachment is a mechanism of community structures of networks. (4) Community structures are universal in real networks. Together with (1) above, this implies that the existence of community structures is a universal phenomenon of real networks, for which we have to explain the reason why. Together with (3) above, this implies that there must be new mechanisms for the existence of community structures of real networks other than the well-known mechanisms of randomness and preferential attachment for classic models of networks.

**Discussions**

Our results above show that the physical, information theoretical and mathematical definitions of community structures of networks are equivalent in characterizing the existence and quality of community structures of networks, that nontrivial networks of classic ER and PA models fail to have a community structure, and that most real networks do have a community structure. The significance of our results are four folds: 1) the existence of community structures is a natural phenomenon definable in networks, by one of the physical, information theoretical and mathematical definitions, 2) community structures are universal in real world data, 3) mechanisms of classic models are not mechanisms of community structures of networks, and 4) the existence and quality of community structures of network data can be tested by our definitions and criterions. This progress poses fundamental questions: What are the mechanisms of community structures of real networks? What roles do the community structures play in networks? What are the new algorithms and applications based on structures of networks and big data, in general? Answering these questions would build a new theory of networks, the structural theory of networks, which is of course a grand challenge in network science.

**Methods**

The data of real networks can be found from the websites: [http://snap.standford.edu](http://snap.standford.edu)
Figure 1: This figure gives the E-, M- and C-community structure ratios (denoted by e-, m- and c-ratios respectively) of networks, for $n = 10,000$, and for $p$ up to 0.005 of the ER model.

or http://www-personal.umich.edu/~mejn/netdata

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Table 1: The entropy, modularity and conductance community structure ratios of real networks, written by $\tau(G)$, $\sigma(G)$ and $\theta(G)$ respectively.

| network     | $\tau(G)$ | $\sigma(G)$ | $\theta(G)$ |
|-------------|-----------|-------------|-------------|
| cit_heph    | 0.22      | 0.56        | 0.37        |
| cit_hepht   | 0.2       | 0.53        | 0.36        |
| col_astroph | 0.24      | 0.51        | 0.49        |
| col_condmat | 0.37      | 0.64        | 0.76        |
| col_grqc    | 0.44      | 0.79        | 0.89        |
| col_hephh   | 0.26      | 0.58        | 0.7         |
| col_hepth   | 0.39      | 0.69        | 0.83        |
| email_enron | 0.21      | 0.5         | 0.63        |
| email_euall | 0.39      | 0.73        | 0.76        |
| p2p4        | 0.11      | 0.38        | 0.36        |
| p2p5        | 0.11      | 0.4         | 0.36        |
| p2p6        | 0.12      | 0.39        | 0.38        |
| p2p8        | 0.15      | 0.46        | 0.46        |
| p2p9        | 0.15      | 0.46        | 0.42        |
| p2p24       | 0.21      | 0.47        | 0.48        |
| p2p25       | 0.23      | 0.49        | 0.5         |
| p2p30       | 0.24      | 0.5         | 0.53        |
| p2p31       | 0.25      | 0.5         | 0.52        |
| roadnet_ca  | 0.67      | 0.99        | 0.98        |
| roadnet_pa  | 0.66      | 0.99        | 0.98        |
| roadnet_tx  | 0.67      | 0.99        | 0.98        |
Figure 2: This figure gives the E-, M- and C-community structure ratios (denoted by e-, m- and c-ratios respectively) of networks, for $n = 10,000$, and for $d \leq 50$ of the PA model.

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Community Structure Ratios

c-ratios
e-ratios
m-ratios
Community Structure Ratios

c-ratios
e-ratios
m-ratios
