TRANSVERSAL KILLING AND TWISTOR SPINORS ASSOCIATED TO THE BASIC DIRAC OPERATORS

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Abstract. We study the interplay between basic Dirac operator and transversal Killing and twistor spinors. In order to obtain results for general Riemannian foliations with bundle-like metric we consider transversal Killing spinors that appear as natural extension of the harmonic spinors associated with the basic Dirac operator. In the case of foliations with basic-harmonic mean curvature it turns out that this type of spinors coincide with the standard definition. We obtain the corresponding version of classical results on closed Riemannian manifold with spin structure, extending some previous results.

Keywords: Riemannian foliations; basic Dirac operator; transversal Killing spinors; transversal twistor spinors.

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1. Introduction

On differentiable closed Riemannian manifolds the interplay between the spectrum of associated natural differential operators such as Dirac operator and its square, the Laplace operator- on a side, and the geometry of the underlying manifold- on the other side, has already become a classical research subject.

If we consider a foliated structure on our manifold such that the metric tensor field is bundle-like (i.e. the manifold can be locally described as a Riemannian submersion [1]), then similar operators can be defined in this particular setting, standing as important tools for the study of the transverse spectral geometry of the foliations [2]. Furthermore, on the foliated manifold can be considered the existence of a symplectic structure or a CR-submanifold (see e.g. [3, 4, 5, 6, 7]) which is also known to interact with the canonical differential operators [8].

Concerning the Dirac-type operators, the transversal Dirac operator for Riemannian foliations was introduced in [9]. In the particular case of a Riemannian foliation with basic mean curvature form, this operator is used to define the basic Dirac operator, which is a symmetric, essentially self-adjoint and transversally elliptic [9]. As pointed out by the authors, Dirac-type operators defined in this particular framework are relevant at least in $\mathbb{R}^4$, when the Yang-Mills equations can be dimensionally reduced, yielding magnetic monopole equations.

In general the procedure of reducing a dynamical system to two or more systems of lower dimensions with respect to foliations of the manifold of which the dynamics take place has proved to be an useful tool in order to relate different integrable systems together with their associated symmetries. Sometimes the reverse of the reduction procedure is used to investigate difficult dynamical systems. It is also

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possible to sort of unfold the initial dynamics by imbedding it in a larger one which is easier to integrate and then projecting the solution back to the initial manifold [10].

On the other side, concerning the relevance of foliated manifolds, among a number of possible applications we mention the problem of incorporating physical chiral spinors in four dimensions in the framework of $N = 1, D = 11$ model of supergravity [11]. In this respect, a $G$-foliated model of the 7-dimensional internal manifold resulting from $N = 1, D = 11$ model of supergravity was proposed in [12]. The ground-state compactification of the 11-dimensional manifold $M_{11}$ into $M_7 \times M_4$ is the result of a global submersion $M_{11}$ into $M_4$, giving rise to a foliation of dimension 7 and codimension 4.

Now, regarding the spectral properties of a Dirac operator in the standard framework of a closed, Riemannian manifold we refer to [13, 14, 15]; in this setting the so called Killing spinors are known to be related to the spectrum of Dirac operator in a particular way; beside the fact that they are a tool which allow us to construct Killing vector fields (i.e. the corresponding flow is represented by local isometries) the Killing spinors are exactly the eigenspinors corresponding to the lower eigenvalue of Dirac operator [13, 15], so the geometrical conditions that insure the existence of Killing spinors and the realization of the limiting case for the Dirac spectrum are the same. A generalization of the Killing spinors is represented by twistor spinors [13, 16].

Concerning the transverse Killing spinors, for the first time they appear in [17], when the specific equation is written using derivatives in the transverse directions with respect to a vector field associated to a parallel 1-form. Moreover, the spinors are parallel in the direction of the vector field. In the framework represented by Riemannian foliations with arbitrary dimension, the transverse Killing spinors were introduced under this name in [18]. They verify a differential equation similar to the classical case; in order to investigate their properties using Dirac type equations and Lichnerowicz type formulas it is convenient to consider, as above, spinors parallel in the leafwise directions, i.e. basic spinors [19, 9, 20]. Recently, for the setting of Riemannian foliations with one-dimensional leaves (the so called Riemannian flows), in [21] the authors introduced a more flexible concept. They consider spinors that are not basic anymore, but the first derivatives of the spinor field can behave differently along the transverse and leafwise distributions. Regarding the transverse twistor spinors, they were defined and studied in [19].

The main goal of this paper is to obtain the corresponding interplay between transversal Killing spinors and basic Dirac operator in the framework of Riemannian foliations. In order to made such an achievement in the general setting of Riemannian foliations we define transversal Killing spinors as natural extension of basic spinors transversally parallel with respect to a modified connection associated with the basic Dirac operator (see Section 2). We also introduce in a natural way a class of twistor spinors.

For general Riemannian foliations these definitions differs from [20, 18, 19], but for the particular case of Riemannian foliation with basic-harmonic curvature, which is the most convenient setting, this definition coincide with the previous definition [20, 18, 19], and our results turn out to be generalization of [18, 19]. Moreover, in the standard manner [9, 18], the absolute case (when the manifold is foliated by points) becomes a generalization of the case of closed Riemannian manifolds.
In the second section we introduce the main geometrical object we are dealing with. In the third section we derive the main results and point out the specific features of the above Killing spinors, while in the fourth section of the paper we study twistor spinors in the presence of a foliated structure. In the final part of the paper we point out some possible applications of the results and some physical considerations.

2. Geometric objects related to the transverse geometry of Riemannian foliations

The framework of this paper is represented by a smooth, closed (i.e. compact, without boundary) Riemannian manifold $M$ with a foliated structure $\mathcal{F}$. We also consider a metric tensor $g$ which is bundle-like [1]. The leafwise distribution tangent to leaves will be denoted by $T\mathcal{F}$; a corresponding transverse distribution $Q = T\mathcal{F}^\perp \simeq TM/T\mathcal{F}$ is obtained. Let us assume $\dim M = n$, $\dim T\mathcal{F} = p$ and $\dim Q = q$, with $p + q = n$.

A first consequence is the splitting of the tangent and the cotangent vector bundles associated with $M$

$$TM = Q \oplus T\mathcal{F},$$

$$TM^* = Q^* \oplus T\mathcal{F}^*.$$ 

The canonical projection operator associated with the distributions $Q$ will be denoted by $\pi_Q$.

For local investigation of our manifold we will use local vector fields $\{e_i, f_a\}$ defined on a neighborhood of a point $x \in M$ inducing an orthonormal basis at any point where they are defined, $\{e_i\}_{1 \leq i \leq q}$ spanning the distribution $Q$ and $\{f_a\}_{1 \leq a \leq p}$ spanning the distribution $T\mathcal{F}$.

A convenient tool for the study of the basic geometry of the Riemannian foliated manifold $(M, \mathcal{F}, g)$, is the so called Bott connection (see e.g. [2], which is a linear, metric and torsion-free connection). If we denote by $\nabla^g$ the canonical Levi-Civita connection, then on the transverse distribution $Q$ we can define the connection $\nabla$ by the following relations

$$\nabla_U X := \pi_Q([U, X]),$$

$$\nabla_Y X := \pi_Q(\nabla^\nabla_Y X),$$

for any smooth sections $U \in \Gamma(T\mathcal{F})$, $X, Y \in \Gamma(Q)$. In a standard manner we can associate to $\nabla$ the transversal Ricci operator $\text{Ric}^\nabla$, the transversal scalar curvature $\text{Scal}^\nabla$ and the transversal gradient of a basic function $f \text{ grad}^\nabla$.

It is also convenient to employ basic (projectable) local vector field $\{e_i\}_{1 \leq i \leq q}$, parallel on the leafwise directions with respect to the above Bott connection, so we use this type of transverse orthonormal basis throughout the paper. We denote by $\Gamma_b(Q)$ the set of basic vector fields.

The classical de Rham complex of differential forms $\Omega(M)$ is restricted to the complex of basic differential forms, defined as [2]

$$\Omega_b(M) := \{ \omega \in \Omega(M) \mid \iota_U \omega = \mathcal{L}_U \omega = 0 \},$$

where $U$ is again an arbitrary leafwise vector field, $\mathcal{L}$ being the Lie derivative along $U$, while $\iota$ stands for interior product. The corresponding basic exterior derivative $d_b$ comes as a restriction of the classical de Rham derivative, $d_b := d|_{\Omega_b(M)}$. Let us
notice that basic de Rham complex is defined independent of the metric structure \( g \). The adjoint operator, namely the basic co-derivative \( \delta_b \), can also be considered (see e.g. [22]).

One differential form of particular importance, which may not be necessarily a basic differential form, is the mean curvature form. In order to define it, first of all we set \( k^\sharp := \pi_{Q} \left( \sum_a \nabla^g f_a \right) \) to be the mean curvature vector field associated with the distribution \( TF \). Then, \( k \) will be the mean curvature form which is subject to the condition

\[
k(U) = \langle k^\sharp, U \rangle,
\]

\( \sharp \) being the musical isomorphism and \( \langle \cdot, \cdot \rangle \) the scalar product in \( TM \).

The de Rham complex can be decomposed as a direct sum [22, Theorem 2.1]

\[
\Omega(M) = \Omega_b(M) \bigoplus \Omega_b(M)^\perp,
\]

with respect to the \( C^\infty \)-Frechet topology. Consequently, on any Riemannian foliation the mean curvature form can be written as

\[
k = k_b + k_o,
\]

where \( k_b \in \Omega_b(M) \) is the basic component of the mean curvature, \( k_o \) being the orthogonal complement. In the following we denote \( \tau := k_b^\sharp \).

**Remark 1.** The above co-derivative operator \( \delta_b \) can be calculated using the vector field \( \tau \) and the Bott connection (see [2, 22])

\[
\delta_b = -\sum_i \iota_{e_i} \nabla e_i + \iota_{\tau}.
\]

With the above notations, at any point \( x \) on \( M \) we consider the Clifford algebra \( Cl(Q_x) \) which, with respect to the orthonormal basis \( \{e_i\} \) is generated by 1 and the vectors \( \{e_i\} \) over the complex field, being subject to the relations \( e_i \cdot e_j = -2\delta_{i,j} \), \( 1 \leq i, j \leq q \), where dot stands for Clifford multiplication. The resulting bundle \( Cl(Q) \) of Clifford algebras will be called the Clifford bundle over \( M \), associated with \( Q \).

The additional assumptions for the foliation \( F \) is the transverse orientability and the existence of a transverse spin structure. This means that there exists a principal \( \text{Spin}(q) \)-bundle \( \tilde{P} \) which is a double sheeted covering of the transverse principal \( SO(q) \)-bundle of oriented orthonormal frames \( P \), such that the restriction to each fiber induces the covering projection \( \text{Spin}(q) \rightarrow SO(q) \); such a foliation is called spin foliation (see e.g. [23]). If \( \Delta_q \) is the spin irreducible representation associated with \( Q \), then the foliated spinor bundle \( S := \tilde{P} \times_{\text{Spin}(q)} \Delta_q \) can be constructed [24]. Furthermore, in a classical way a smooth bundle action can be considered

\[
\Gamma(Cl(Q)) \otimes \Gamma(S) \rightarrow \Gamma(S).
\]

We denote this action also with Clifford multiplication; it verifies the condition

\[
(u \cdot v) \cdot s = u \cdot (v \cdot s),
\]

for \( u, v \in \Gamma(Cl(Q)), s \in \Gamma(S) \).

It is easily seen that \( S \) becomes a bundle of Clifford modules [24].

**Remark 2.** The above transverse Clifford action is obtained from the standard case of the tangent bundle as a restriction of the Clifford action from \( TM \) to the distribution \( Q \) (for the particular case of vector bundles see also [25]).
The lifting of the Riemannian connection on $P$ can be used to introduce canonically a connection on $S$, which will be denoted also by $\nabla$. The compatibility between the Clifford action and the connection $\nabla$ is expressed in the relation

$$\nabla_U (u \cdot s) = (\nabla_U u) \cdot s + u \cdot \nabla_U s,$$

for any $U \in \Gamma(TM)$, $u \in \Gamma(Cl(Q))$, $s \in \Gamma(S)$, extending canonically the connection $\nabla$ to $\Gamma(Cl(Q))$.

The transverse metric induces a hermitian structure on $S$; if we denote it by $(\cdot | \cdot)$, we have that $(X \cdot s_1 | s_2) = -(s_1 | X \cdot s_2)$, for any $X \in \Gamma(Q)$, $s_1, s_2 \in \Gamma(S)$. Then $\nabla$ becomes a metric connection, similar to the classical case of spin manifolds (see e.g. [13, Chapter 1]) and we have

$$X (s_1 | s_2) = (\nabla_X s_1 | s_2) + (s_1 | \nabla_X s_2).$$

In order to define the basic Dirac operator, we need first to introduce the transversal Dirac operator,

$$D_{tr} := \sum_i e_i \cdot \nabla e_i,$$

and the basic spinors or holonomy invariant sections, in accordance with [9]

(2) $\Gamma_b (S) := \{ s \in \Gamma(S) | \nabla_U s = 0, \text{ for any } U \in \Gamma(TF) \}.$

The transversal Dirac operator may not be formally self-adjoint; consequently, in the definition of the basic Dirac operator an auxiliary term related to the basic component of the mean curvature form is added

(3) $D_b := \sum_i e_i \cdot \nabla e_i - \frac{1}{2} \tau,$

and the domain of this operator is restricted to the above set of basic spinors [23, 9].

**Remark 3.** As in the standard setting, the above basic Dirac operator does not depend on the local framework $\{e_i\}_{1 \leq i \leq q}$. It is a transversally elliptic and essentially self-adjoint differential operator with respect to the inner product canonically associated with the hermitian structure. We emphasize the fact that the spectrum $\sigma(D_b)$ is discrete [9].

Another way to construct the basic Dirac operator will be described in the following. We start out by modifying the Bott connection.

**Definition 1.** The modified connection on the space of basic sections $\Gamma_b (S)$ is given by [26]

(4) $\tilde{\nabla}_X s := \nabla_X s - \frac{1}{2} (X, \tau) s,$

for any $X \in \Gamma(TM)$ and $s \in \Gamma(S)$. 
A key property of $\nabla$ is that it can be used to construct the basic Dirac operator $D$:

$$
\sum e_i \cdot \nabla e_i = \sum e_i \cdot \left( \nabla e_i - \frac{1}{2} (e_i, \tau) \right) = \sum a e_i \cdot \nabla e_i - \frac{1}{2} \tau = D_b.
$$

**Remark 4.** It is easy to see that the modified connection on $S$ is also compatible with the Bott connection, i.e.

$$\nabla_X (Y \cdot s) = \nabla_X Y \cdot s + Y \cdot \nabla_X s,$$

for any $X, Y \in \Gamma(Q)$. On the other side we must observe that the modified connection does not share other classical properties. Unlike the canonically connection $\nabla$, the modified connection is not a metric connection, and in general this aspect has impact on the formal computation.

Now, as we have already defined the basic Dirac operator, we introduce a category of spinors intimately related to this differential operator. In order to consolidate the motivation let us study the following example.

We consider the torus $T^2 := \mathbb{R}^2/\mathbb{Z}$ with the metric $g = e^{2f(y)} dx^2 + dy^2$, for some periodic function $f$ (see e.g. [23]). As a consequence, $\{\partial_y, \frac{\partial}{\partial f(y)}\}$ will be an orthonormal basis at any point, $Q = \text{span}\{\partial_y\}$, $TF = \text{span}\{\partial_x/e^{f(y)}\}$. The spin structure considered on the transverse circles will be the trivial one (see e.g. [16]); the transverse Clifford multiplication by $\partial_y$ will be represented by multiplication with the purely imaginary unit $i$. We use the Koszul formula to calculate the mean curvature vector field and the mean curvature form.

$$
\left\langle \nabla_g \frac{\partial_x}{e^{f(y)}}, \partial_y \right\rangle = \frac{1}{2} \left( \left\langle \left[ \partial_y, \frac{\partial_x}{e^{f(y)}} \right], \frac{\partial_x}{e^{f(y)}} \right\rangle + \left\langle \left[ \partial_y, \frac{\partial_x}{e^{f(y)}} \right], \frac{\partial_x}{e^{f(y)}} \right\rangle \right) = -f'(y).
$$

Consequently, we get $k = -f'(y) dy$; the mean curvature form is obviously basic, as it does not depend on $x$, so $k \equiv k_y$ and we obtain

$$\tau = -f'(y) \partial_y.$$

On the other side it is interesting to note that

$$\delta_b k = \left( -i \partial_y \nabla \partial_y + i f'(y) \partial_b \right) (-f'(y) dy) = f''(y) + (f'(y))^2 ,$$

so $\delta_b k$ does not necessarily vanish, i.e. $k$ is not a basic-harmonic differential 1-form.

Finally, we calculate the basic Dirac operator

$$D_b = i \partial_y - \frac{1}{2} \left( -f'(y) \right) i = i \left( \partial_y + \frac{1}{2} f'(y) \right).$$

We investigate the harmonic basic spinors. It is easy to see that the basic solutions of the equation $D_b s_1 = 0$ have the form $s_1 = ce^{-\frac{1}{2}f(y)}$, $c \in \mathbb{C}$. 

Remark 5. From the above calculations we see that $\bar{\nabla}_{\partial_s}s_1 = 0$ and $\nabla_{\partial_s}s_1 \neq 0$. So, even for the above simple example of Riemannian foliation, we see that the harmonic spinors of the basic Dirac operator are parallel spinors with respect to the above modified connection $\bar{\nabla}$ but not with respect to the classical connection $\nabla$; also, the above spinors are not transverse Killing spinors with the definitions from [18, 19, 20, 21].

In the classical setting, a category of spinors naturally related to parallel spinors and eigenspinors associated to the lower eigenvalue is represented by Killing spinors. In our particular framework we introduce a similar type of spinors with respect to our connection $\bar{\nabla}$.

Definition 2. A spinor $s \in \Gamma_b(S)$ which satisfies the equation

$$\bar{\nabla}_X s + \frac{\lambda}{q} X \cdot s = 0,$$

for any $X \in \Gamma_b(Q)$ is called transversal Killing spinor associated with the connection $\bar{\nabla}$ or $\tau$-Killing spinor.

Remark 6. As in the standard setting, it is easy to see that a basic spinor parallel with respect to $\bar{\nabla}$ is a $\tau$-Killing spinor. Also, each $\tau$-Killing spinor is an eigenspinor of the basic Dirac operator.

The concept of Killing spinors can be extended to twistor spinors in a classical manner [13, 16].

Definition 3. We denote by $\tau$-twistor spinors (twistor spinors with respect to the modified connection) a basic spinor satisfying the following equation

$$\bar{\nabla}_X s + \frac{1}{q} X \cdot D_b s = 0.$$

Remark 7. We will see in the next sections that these definitions extend previous basic spinors existing in the particular case of Riemannian foliations with basic-harmonic mean curvature [20 18 19]. As the linear connection employed to define $\tau$-Killing spinors in not metric, it will be interesting to point out that they do not have constant length, unlike the previous definition.

Now, as we have defined all necessary transverse geometric objects, in the final part of this section we shortly present some results that we employ in order to study spectral properties of basic Dirac operators. It turns out that, the most convenient setting is represented by the case of Riemannian foliations with basic-harmonic mean curvature, that is $d_b k = 0, \delta_b k = 0$. Then, in order to obtain results in the general case, we need a sequence of metric changes that leave the transverse metric on the normal bundle intact.

A first relevant result in this direction is due to Domínguez.

Theorem 1. [27] The bundle-like metric can be transformed (leaving the transverse metric on the normal bundle intact) such that the orthogonal part $k_o$ of the mean curvature vanishes while the basic part of the mean curvature $k_b$ holds; consequently the new bundle-like metric has basic mean curvature.

The above metric transformation is based on a change of the transversal sub-bundle $Q$, and a conformal change of the leafwise metric (see the proof of [27]}
Theorem 9.18]). These are in fact the fundamental metric changes needed in order to study the basic component of the mean curvature [22].

Secondly, we have the following result obtained by Mason.

**Theorem 2.** [28] Furthermore, the above bundle-like metric can be transformed (leaving the transverse metric on the normal bundle intact) into a metric with basic-harmonic mean curvature.

This metric change is in fact a conformal change of the leafwise metric which use the theory of stochastic flows.

Finally, an important spectral rigidity result is due to Habib and Richardson.

**Theorem 3.** [23] The spectrum \( \sigma(D_b) \) is invariant with respect to any metric change that leaves the transverse metric on the normal bundle intact.

As a consequence, we can study first of all the spectrum of basic Dirac operator in the most convenient framework of Riemannian foliations with basic-harmonic mean curvature, then pull back the results in the initial general case using the spectral rigidity result [23]. For the case \( q \geq 2 \) the authors obtain that the lower bound estimate for an eigenvalue \( \lambda \) of \( D_b \)

\[
\lambda^2 \geq \frac{1}{4} \frac{q}{q-1} \text{Scal}_b^\nabla,
\]

where \( \text{Scal}_b^\nabla := \min_{x \in M} \text{Scal}_x^\nabla \), as an extension of [18].

While the above method is very useful when studying the eigenvalues of \( D_b \), in general the corresponding eigenspinors are not invariant with respect to all the above changes of the metric.

3. **Basic Dirac operators and transversal \( \tau \)-Killing spinors**

In the framework of closed Riemannian manifold with spin structure, the interplay between Dirac operator and Killing spinors provide some remarkable and very interesting results [13, 16]. In this section we study this interplay between basic Dirac operator and transversal \( \tau \)-Killing spinors on Riemannian foliations, obtaining corresponding results in our specific setting, generalizing also known results from basic-harmonic Riemannian foliations [18, 19]. We start out by considering a Riemannian foliation with a bundle-like metric, with an arbitrary, non-necessary basic-harmonic mean curvature form, defined on a closed manifold.

**Definition 4.** For any real valued basic function \( f \) we define on \( S \) the linear connections \( \nabla^f_X \) and \( \bar{\nabla}^f_X \),

\[
\nabla^f_X := \nabla_X + fX, \\
\bar{\nabla}^f_X := \bar{\nabla}_X + fX.
\]

for any \( X \in \Gamma_b(Q) \).

**Proposition 1.** In the above setting, considering the corresponding curvature operators, we have the equality \( \bar{\nabla}^f_{[X,Y]} = \nabla^f_{[X,Y]} \) for \( X, Y \in \Gamma_b(Q) \).

**Proof.** Using the standard definition of the curvature operator, we start with the relation

\[
\bar{\nabla}^f_{[X,Y]} = \nabla^f_Y \nabla^f_X - \nabla^f_X \nabla^f_Y - \nabla^f_{[X,Y]}.
\]
Furthermore, we get
\[
\bar{\nabla}_X^f \bar{\nabla}_Y^f = \left( \bar{\nabla}_X^f - \frac{1}{2} \langle X, \tau \rangle \right) \left( \bar{\nabla}_Y^f - \frac{1}{2} \langle Y, \tau \rangle \right)
\]
\[
= \bar{\nabla}_X^f \bar{\nabla}_Y^f - \frac{1}{2} \langle \bar{\nabla}_X Y, \tau \rangle - \frac{1}{2} \langle Y, \bar{\nabla}_X \tau \rangle - \frac{1}{2} \langle Y, \langle X, \tau \rangle \rangle \bar{\nabla}_X - \frac{1}{2} \langle X, \tau \rangle \bar{\nabla}_Y + \frac{1}{4} \langle X, \tau \rangle \langle Y, \tau \rangle.
\]

We also have the corresponding relation for the second term of (6); for the third term we have
\[
\bar{\nabla}_{[X,Y]}^f = \bar{\nabla}_{[X,Y]}^f - \frac{1}{2} \langle [X,Y], \tau \rangle.
\]

Now, let us emphasize that \( k_b \) is a closed 1-form [22, Corollary 3.5], so
\[
\langle Y, \bar{\nabla}_X \tau \rangle = \langle X, \bar{\nabla}_Y \tau \rangle.
\]

Also, the Bott connection is torsion-free, so we get
\[
\langle \bar{\nabla}_X Y - \bar{\nabla}_X Y - [X,Y], \tau \rangle = 0.
\]

Summing up, we observe that all terms containing \( \tau \) vanish and we obtain the above relation. \( \square \)

**Remark 8.** For \( f \equiv 0 \) we derive that \( \bar{R}_{X,Y} = R_{X,Y} \).

Now let us study the spin curvature operator applied on \( \Gamma_b(S) \). A Riemannian foliation can be locally identified with a Riemannian submersion [1], and we can consider a local transversal, i.e. a local transverse base manifold \( T \), so all the geometric objects we use are basic (projectable to the local transversal), and we can locally identify the transverse spin curvature operator with the spin curvature operator on the transverse manifold \( T \). Then, similar to the classical case [13, Equation 1.13], we get the following corresponding local identity for the framework of Riemannian foliations [18, Equation (4.4)], which will be useful in our further considerations.

\[
(7) \quad \sum_i e_i \cdot \text{Ric} \bar{\nabla} (e_i) \cdot s = -\text{Scal} \bar{\nabla} s.
\]

We also have
\[
(8) \quad \sum_i e_i \cdot R_{X,e_i} s = -\frac{1}{2} \text{Ric} \bar{\nabla} (X) \cdot s.
\]

In the following let us denote the spectrum of the basic Dirac operator defined on a Riemannian foliation with \( \sigma(D_b) = \{ \lambda_k \}_{k \geq 1} \), such that \( |\lambda_1| \leq |\lambda_2| \leq \ldots \).

**Proposition 2.** On a closed Riemannian foliation of codimension \( q \geq 2 \) endowed with a transverse spin structure we assume the existence of a basic spinor \( s_1 \in \Gamma_b(S) \) which verifies the equation
\[
(9) \quad \bar{\nabla}_X s_1 + \frac{f}{q} X \cdot s_1 = 0,
\]
for any \( X \in \Gamma_b(Q) \), \( f \) being a basic real-valued function. Then \( f \) is constant \( f = \pm \lambda_1 \), (as a consequence the spinor is transverse \( \tau \)-Killing), the foliation is
transversally Einstein, the transversal scalar curvature $\text{Scal}^\nabla$ is a positive constant function $\text{Scal}^\nabla \equiv \text{Scal}^\nabla_0 > 0$, and

$$\lambda_1^2 = \frac{1}{4}\frac{q}{(q-1)}\text{Scal}^\nabla_0.$$ 

**Proof.** From (9) we have

$$0 = \sum_i e_i \cdot \tilde{R}^\nabla_{X,e_i} s_1 = \sum_i e_i \cdot R^\nabla_{X,e_i} s_1. $$

On the other side

$$R^\nabla_{X,e_i} s_1 = \nabla^\nabla_X e_i - \nabla^\nabla e_i - \nabla^\nabla Q([X,e_i]) s_1.$$  

As $s \in \Gamma_b(S)$ is a basic spinor parallel in the leafwise direction, we can adjust accordingly the last term in the expression of curvature operator. Consequently

$$R^\nabla_{X,e_i} s_1 = R_{X,e_i} s_1 + f^2 q^2 X \cdot e_i \cdot e_i \cdot s_1 - f^2 q^2 \sum_i e_i \cdot X \cdot e_i \cdot s_1 + f^2 q X \cdot e_i \cdot s_1 - f^2 q X \cdot s_1 + f^2 q X \cdot e_i \cdot s_1.$$  

We notice that the last term vanishes, being the torsion of the Bott connection [2, Proposition 3.8]. Then

$$\sum_i e_i \cdot \tilde{R}^\nabla_{X,e_i} s_1 = \sum_i e_i \cdot R_{X,e_i} s_1 - qX \left( f^2 q \right) s_1 - \sum_i e_i \left( f^2 q \right) e_i \cdot X \cdot s_1$$

$$+ f^2 q^2 \sum_i e_i \cdot X \cdot e_i \cdot s_1 + q f^2 q X \cdot s_1.$$  

Let us notice that

$$\sum_i e_i \left( f^2 q \right) e_i = \text{grad}^\nabla \left( f^2 q \right),$$

and

$$\sum_i e_i \cdot X \cdot e_i = \sum_i -2 \langle e_i, X \rangle e_i - \sum_i X \cdot e_i \cdot e_i$$

$$= -2X + qX$$

$$= (q-2)X.$$
Finally, using also [7] we end up with the relation [18] (for the classical framework of a closed Riemannian manifold see [15, Equation 3.5])

$$-\frac{1}{2} \sum_i R_{X,e_i} e_i \cdot s_1 - qX(\frac{f}{q})s_1 - \text{grad}^\nabla \left( \frac{f}{q} \right) X \cdot s_1 + \frac{f^2}{q^2} (q-1) X \cdot s_1 = 0,$$

Now, the conclusion comes from [18]. □

So, as a remark, the above fundamental properties of Killing spinor still hold for our particular definition.

We present in the following the main result of our paper.

**Theorem 4.** Let us consider on a closed manifold a Riemannian foliation with a transverse spin structure; assume also that the mean curvature form is non-necessarily basic. Then the lower bound of the eigenvalues of the basic Dirac operator is attained

$$\lambda^2 = \frac{1}{4} \frac{q}{q-1} \text{Scal}^\nabla,$$

if and only if there exist a transverse \(\tau\)-Killing spinor associated to one of the real numbers \(\pm \frac{1}{\sqrt{\frac{1}{4} \frac{q}{q-1} \text{Scal}^\nabla}}\) or \(-\frac{1}{\sqrt{\frac{1}{4} \frac{q}{q-1} \text{Scal}^\nabla}}\).

**Proof.** If we assume the existence of a transversal Killing spinor \(s_1\) associated with \(\nabla\), then

$$D_b s_1 = \sum_i e_i \cdot \nabla e_i s_1$$

$$= \mp \left( \sum_i e_i \cdot \frac{1}{q} \sqrt{\frac{1}{4} \frac{q}{q-1} \text{Scal}^\nabla} e_i \right) s_1$$

$$= \pm \sqrt{\frac{1}{4} \frac{q}{q-1} \text{Scal}^\nabla} s_1,$$

so the lower bound estimate is attained, in accordance with [18] [23].

For the converse statement, as we use a Lichnerowicz type formula, let us observe that the basic mean curvature vector field \(\tau\) that appears in the definition of the basic Dirac operator is actually obtained via the non-trivial Hodge type decomposition [11] (see also [22, Theorem 2.1]). Then, for arbitrary Riemannian foliations is difficult to express \(\tau\) and use it in the standard calculation in order to get a Lichnerowicz formula (for an example a little bit more complicated than we presented in Section 2 see [23, Example 2]; compare also with the case of transverse Dirac operator, for instance [3, Theorem 4]). In turn, we change the metric into a new bundle-like metric with basic mean curvature, we obtain the result in this particular setting, then we pull-back the result in the general case, basically using the method from [26]. Consequently, let us consider a deformation of the metric as in [24]. If the lower bound estimate of the spectrum is realized for the initial metric, using the spectral rigidity result [23], the lower bound estimate will be attained also for the case of basic mean curvature form. In this particular setting \(\tau \equiv k^2\) and we have the Lichnerowicz type formula [25]

\begin{equation}
\|D_b s_1\|^2 = \|\nabla s_1\|^2 + \frac{1}{4} \int_M \text{Scal}^\nabla |s_1|^2,
\end{equation}
for \( s_1 \in \Gamma_b(S) \), where \(|s_1|^2 = (s \mid s_1)\), \(||\cdot||\) being the \(L^2\) norm associated with the hermitian structure.

In what follows, let us consider the connection

\[
\nabla^\lambda_X := \nabla_X + \frac{\lambda_1}{q} X \cdot
\]

when \( \lambda_1 \) is the eigenvalue for which the lower bound is attained, \( s_1 \) being the corresponding eigenspinor. In the calculations below we follow closely in our particular setting the main steps from the original approach of Friedrich [14] in order to get the lower bound estimate of the Dirac spectrum. Comparing with the corresponding approach of Jung on Riemannian foliations [18, Theorem 4.2], as we use a different Lichnerowicz formula, let us notice the vanishing of the mean curvature term in the final estimate. Using then standard arguments, in the limiting case we obtain necessary conditions for the eigenspinors.

Using (10), after calculations we get [26]

\[
\int_M \left| D_b s_1 - \frac{\lambda_1}{q} s_1 \right|^2 = \int_M \sum_i \left| \nabla^\lambda_{e_i} s_1 \right|^2 + \frac{1}{4} \int_M \text{Scal}^\nabla |s_1|^2 + \int_M (1 - q) \frac{\lambda_1^2}{q^2} |s_1|^2,
\]

and, consequently [26]

\[
\int_M \left( \frac{q - 1}{q} \lambda_1^2 - \frac{1}{4} \text{Scal}^\nabla \right) |s_1|^2 = \int_M \sum_i \left| \nabla^\lambda_{e_i} s_1 \right|^2.
\]

From here, it turns out that

\[
\int_M \sum_i \left| \nabla^\lambda_{e_i} s_1 \right|^2 = 0 \quad \text{and} \quad \lambda_1^2 = \frac{q}{4q - 1} \text{Scal}^\nabla,
\]

so

\[
\nabla_X s_1 + \frac{\lambda_1}{q} X \cdot s_1 = 0,
\]

for any \( X \in \Gamma(Q) \), i.e. the spinor \( s \) is a transversal \( \tau - \)Killing spinor.

As we noticed above, the metric change described in [27] leaves the transverse metric and the basic part \( k_b \) of the mean curvature intact, so the action of the modified connection \( \nabla \) on \( \Gamma_b(S) \) does not change. On the other side the Clifford multiplications, related to the transverse metric, always agree. As a consequence, the spinor \( s \) will be a transversal Killing spinor associated with \( \nabla \) for the initial metric tensor, and we extend our result to arbitrary Riemannian foliations using [27].

**Proposition 3.** If a Riemannian foliation defined on a closed manifold with \( q \geq 2 \) admits a transversal Killing spinor with respect to the connection \( \nabla \), then the foliation is taut.

**Proof.** Assuming that the Riemannian foliations admit a transversal Killing spinor with respect to the connection \( \nabla \), we obtain that there is an eigenspinor \( s_1 \) for which the lower bound estimate of the spectrum of the basic Dirac operator is attained; from the Proposition [2] we also obtain that in the case \( q \geq 2 \) the foliation should be transversally Einstein, and the transversal scalar curvature must be non-negative and constant. Using now once again the rigidity of the spectrum \( \sigma(D_b) \), we obtain that using a sequence of metric changes that hold the transverse part of the metric
we end up with a Riemannian foliation with a basic-harmonic mean curvature for
which the above lower bound is also realized and the square of the first eigenvalue
is again $\frac{4}{q-1} \mathrm{Scal}_0$. As $q \geq 2$, from [18, Theorem 5.2] we get that with respect to
the deformed metric the foliation should be minimal, i.e. $k \equiv 0$. As a consequence,
by the above sequences of metric changes we obtain a Riemannian foliation with
minimal leaves; as this is the standard definition for a taut foliation (see e. g. [22]),
the conclusion follows. □

Remark 9. For foliations with basic-harmonic mean curvature the existence of a
classical transversal Killing spinor is restricted by the condition $k \equiv 0$ [18]; as a
result $\nabla \equiv \nabla$, and a spinors is transversally Killing with respect to $\nabla$ and $\nabla$ in
the same time. Consequently, the above results are natural generalization of [18
Theorem 5.3] for basic spinors and [19, Corollary 4.5].

4. Transversal $\tau-$twistor spinors

Within this section, in the same framework of Riemannian foliations with non-
necessarily basic-harmonic mean curvature, using our previous method we study
the main properties of the transversal $\tau-$twistor spinors introduced in Section 2.

First of all we see that this concept is in fact an extension of [19]; there the
twistor spinors are defined in the particular case of Riemannian foliations with basic
mean curvature as basic spinors satisfying the classical equation written using the
connection $\nabla$ on the spinor bundle $\mathcal{S}$ and the basic Dirac operator

$$\nabla_X s + \frac{1}{q} X \cdot D_b s = 0$$

for any $X \in \Gamma_b(Q)$. Indeed, by [19 Theorem 3.2], the twistor spinors exist only on
minimal foliations. As for minimal foliations $\tau = 0$ and $\nabla \equiv \nabla$, we see that our
Definition 3 agrees with [19] for this particular class of Riemannian foliations. As
the parallel spinors with respect to the connection $\nabla$ considered in the Section 2 are
obviously $\tau-$twistor spinor defined on a taut, non-minimal foliation, the Definition
3 is in fact an extension of [19]; however, the fact that the $\tau-$twistor spinors exist
only on taut foliation (as well as $\tau-$Killing spinors) does not seem to be a direct
consequence.

In the calculations below let us assume $q > 2$. We use the relations (8), (7), the
fact that Bott connection has vanishing torsion and is compatible with the modified
connection $\nabla$, as well as the fact that the basic spinors are parallel on the leafwise
directions.
\[
\frac{1}{2} \text{Ric}^\nabla(X) \cdot s = - \sum_i e_i \cdot R_{X,e_i} s = - \sum_i e_i \cdot R_{X,e_i} s = - \sum_i e_i \cdot (\nabla_X \nabla_{e_i} - \nabla_{e_i} \nabla_X - \nabla_{\pi_Q([X,e_i])}) s.
\]

\[
= - \sum_i e_i \left( \nabla_X \left( \frac{-1}{q} e_i \cdot D_b s \right) - \nabla_{e_i} \left( \frac{-1}{q} X \cdot D_b s \right) \right) - \frac{1}{q} \pi_Q([X,e_i]) \cdot D_b s.
\]

Now, as the transverse part of the torsion tensor associated to the Bott connection vanishes, using standard computation (see e. g. [16, Appendix A]), we get

\[
\bar{\nabla}_X D_b s = - \frac{q}{2(q-2)} \text{Ric}^\nabla(X) \cdot s + \frac{1}{q-2} X \cdot D_b^2 s.
\]

From here

(11) \[ D_b^2 s = \frac{q}{4(q-1)} \text{Scal}^\nabla s, \]

and we obtain the corresponding upgrading

(12) \[ \bar{\nabla}_X D_b s = \frac{q}{(q-2)} \left( \frac{-\text{Ric}^\nabla(X)}{2} \cdot s + \frac{1}{4(q-1)} \text{Scal}^\nabla X \cdot s \right). \]

Remark 10. The relations (11) and (12) represent the corresponding version of some classical results concerning twistor spinors in the classical setting of Riemannian manifolds [16, Appendix A]. Restricting the framework to Riemannian foliations with basic mean curvature, it is also possible to obtain the corresponding results written using only the spin connection \( \nabla \) (associated to the so called W-twistor spinors) see [19, Proposition 3.4.].

In the last part of the paper we prove an interesting property of the zeros of twistor spinors.

First of all let us consider the bundle \( E := S \bigoplus S \), endowed with the connection

\[
\nabla^E_X \left( \begin{array}{c} s_1 \\ s_2 \end{array} \right) := \left( \begin{array}{cc} \nabla_{\pi_Q(X)} & \frac{1}{q} \pi_Q(X) \\ \frac{q}{(q-2)} \left( \text{Ric}^\nabla(X) + \frac{1}{q-1} \text{Scal}^\nabla X \right) \cdot \nabla_{\pi_Q(X)} & \end{array} \right) \left( \begin{array}{c} s_1 \\ s_2 \end{array} \right),
\]

for any \( s_1, s_2 \in \Gamma(S) \), as a generalization of the basic-harmonic case [19] (for the classical case see e.g. [13]). It is easy to see that if \( s \) is a twistor spinor, then the smooth section \( \left( \begin{array}{c} s \\ D_b s \end{array} \right) \) of \( E \) is in fact parallel with respect to \( \nabla^E \), as a
consequence of \([5], [12]\) and the definition of \(\nabla^E\). Considering arguments similar to \([13]\) for the transverse directions, as the spinors \(s\) and \(D_b s\) are basic spinors, parallel along the leaves, defined by \([2]\), we see that if the manifold \(M\) is connected and at a point \(x \in M\) we have \(s_x = (D_b s)_x = 0\), then \(s \equiv 0\) all over the compact manifold \(M\).

For any basic function \(f\) we define the basic Hessian associated to the connection \(\nabla\)

\[
\text{Hess}_x^\nabla (f)(X, Y) := X (Y (f)) - \nabla_X Y (f)
\]

for any \(X, Y \in \Gamma_b (Q)\).

**Remark 11.** As above, if \(T\) is a local transverse manifold, as basic geometric objects in our framework can be locally projected on \(T\), it is easy to see that \(\text{Hess}_x^\nabla\) is just the standard Hessian on the transverse manifold.

We are now able to prove the corresponding version of a classical property of the zeros of a twistor spinor \([13, 16]\).

**Proposition 4.** On a connected Riemannian foliation of arbitrary codimension \(q\) endowed with a nontrivial transversal \(\tau\)-twistor spinor \(s\), the leaves where \(s\) vanishes are isolated on the quotient set \(M/F\).

**Proof.** Let us assume \(s\) is a nontrivial basic twistor spinor such that \(s_x = 0\) at \(x \in M\). In the following we calculate \(\text{Hess}_x^\nabla (|s|^2)\).

We investigate the two components of the basic Hessian defined by \([13]\).

\[
X \left( Y \left( |s|^2 \right) \right) = X \left( 2 \Re (\nabla Y s \mid s) \right) = 2 \Re (\nabla_X \nabla Y s \mid s) + 2 \Re (\nabla Y s \mid \nabla_X s),
\]

\[
\nabla_X Y \left( |s|^2 \right) = 2 \Re (\nabla \nabla_X Y s \mid s).
\]

As \(s_x = 0\), the only term we need to study is \(\Re (\nabla Y s \mid \nabla_X s)\).

We now apply \([5]\) and \([4]\) and obtain that

\[
(\nabla Y s \mid \nabla_X s) = (\nabla Y s \mid \nabla_X s) + \frac{1}{2} \langle X, \tau \rangle (s \mid \nabla_X s) + \frac{1}{2} \langle X, \tau \rangle (s \mid \nabla_X s) + \frac{1}{4} \langle X, \tau \rangle \langle X, \tau \rangle |s|^2
\]

\[
= \frac{1}{q^2} \langle X \cdot D_b s, Y \cdot D_b s \rangle + \frac{1}{2} \langle Y, \tau \rangle (s \mid \nabla_X s) + \frac{1}{4} \langle Y, \tau \rangle \langle X, \tau \rangle |s|^2.
\]

From \([13\ p. 15]\) we get furthermore that

\[
\langle X \cdot D_b s, Y 
abla_X s \rangle = \langle X, Y \rangle |D_b s|^2,
\]

so at the point \(x\), as \(s_x = 0\), we obtain

\[
\text{Hess}_x^\nabla (|s|^2)(X_x, Y_x) = 2 \Re ((\nabla Y s)_x \mid (\nabla_X s)_x) = \frac{2}{q^2} \langle X_x, Y_x \rangle |D_b s|^2_x.
\]
As \((D_b s)_x \neq 0\) (otherwise, in accordance with the above considerations the twistor spinor \(s\) would vanish everywhere), we get that the basic Hessian of the local function obtained on the local transverse manifold \(T\) by projecting the basic function \(|s|^2\) is positive defined at \(x\); as the basic functions are constant along the leaves of the foliation, the conclusion follows.

\[\square\]

5. Some physical considerations

The Dirac operators in the presence of Riemannian foliations have attracted much attention in physics. We do not intend to give any kind of extensive introduction in spin geometry, but some motivating examples that may lead to possible applications are of interest.

In the last time Sasakian manifolds, as an odd-dimensional cousin of Kähler manifolds, have become of high interest in connection with many modern studies in physics. One of their principal applications in physics has been in higher-dimensional supergravity, string theory and \(M\)-theory where they can provide backgrounds for reduction to lower-dimensional spacetimes. AdS/CFT conjecture \([29]\) relates quantum gravity, in certain backgrounds, to ordinary quantum field theory without gravity. In particular the AdS/CFT correspondence relates Sasaki-Einstein geometry, in dimensions five and seven, to superconformal field theory in dimensions four and three, respectively.

The foliation generated by the Reeb vector field \(\xi\) has a transverse Kähler structure. If the orbits of \(\xi\) are closed, the Sasakian structure is called quasi-regular. The Reeb field generates a locally free \(S^1\)-action such that the leaf space is an orbifold and the transverse Kähler structure projects to it. There are examples of Sasakian structures which are not quasi-regular \([30]\). In the opposite case, if the orbits of \(\xi\) do not all close, the Sasakian structure is said to be irregular.

The properties of Sasaki-Einstein spaces can be obtained from an alternative definition of Sasaki-Einstein manifold connected with the existence of a Killing spinor \([31]\). The geometric features of generic supergravity solutions with unbroken supersymmetry and fluxes, so the relation between Killing spinor and geometry that admits such a spinor needs to be further elucidated \([32]\). On the other side, as pointed out in \([16]\), the Killing spinors are highly relevant for the investigation of supersymmetric models for string theory in dimension 10. From this point of view we hope that our results concerning transversal Killing spinors would be helpful for the investigation of this geometrical objects in the particular framework represented by Riemannian foliations.

Another interesting example is represented by the Euclidean Taub-Newman-Unti-Tamburino (Taub-NUT) space which appears in various problems. Hawking \([33]\) has suggested that the Euclidean Taub-NUT metric might give rise to the gravitational analogue of the Yang-Mills instantons. Also this metric is the space part of the line element of the Kaluza-Klein monopole.

Iwai and Katayama \([34]\) generalized the Taub-NUT metrics in the following way. Let us consider a metric \(\tilde{g}\) on an open interval \(U\) in \((0, +\infty)\) and a family of Berger metrics \(\tilde{g}(r)\) on \(S^3\) indexed by \(U\). Then the twisted product metrics \(g = \tilde{g} + \tilde{g}(r)\) on the annulus \(U \times S^3 \subset \mathbb{R}^4 \setminus \{0\}\) is called a generalized Taub-NUT metric.

The Taub-NUT metrics have been drawing wide interest. In particular, from the viewpoint of dynamical systems, the symmetry of the dynamical system associated
with that metric is similar to that for the Coulomb/Kepler problem. The four-dimensional problem is reduced to an $S^1$ action when the associated momentum mapping takes nonzero fixed values. If the original Hamiltonian system admits a symmetry group that is commutative with the group used for the reduction, the reduced Hamiltonian system admits the same symmetry group \[35\]. In the case of the Taub-NUT metrics the reduced Hamiltonian system is the three-dimensional Kepler problem along with a centrifugal potential and Dirac’s monopole field.

The importance of anomalous Ward identities in particle physics is well known. The anomalous divergence of the axial vector current in a background gravitational field is directly related to the index theorem. Namely, the axial anomaly is interpreted as the index of the chiral Dirac operator. The difference between the number of null states of positive and of negative chirality on a ball or annular domain, may become nonzero for suitable choices of the parameters of the metric and of the domain when one imposes the Atiyah-Patodi-Singer spectral condition at the boundary. In the case of the standard Taub-NUT space, which is hyperKähler and therefore scalar-flat, it can be proved that there are no harmonic $L^2$ spinors using the Lichnerowicz identity and the infiniteness of the volume \[36\]. Moreover there do not exist $L^2$ harmonic spinors on $\mathbb{R}^4$ for the generalized Taub-NUT metrics. In particular, the $L^2$ index of the Dirac operator vanishes \[37\].

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