Using Collocation Fuzzy Wavelet like Operator to Approximate the Solution of Fuzzy Fredholm Integral Equation

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Abstract  
In this paper, first we introduce a numerical method for approximating the solution of linear Fuzzy Fredholm Integral Equations of the second kind (FFIE-2). This method is based on the fuzzy wavelet like operator. Then, we discuss and investigate the convergence and error analysis of the proposed method. Finally, to show the accuracy of the proposed method, we present two numerical examples.

Keywords: Fredholm Integral Equation, Fuzzy Linear System, Fuzzy Wavelet like Operator

1. Introduction

Integral equations are very useful for solving many problems in several applied fields like mathematical economics and optimal control theory. The concept of integration of fuzzy functions was first introduced by Dubois and Prade1. Alternative approaches were later suggested by Goetschel and Voxman; Kaleva; Matloka; Nanda and others. Also, Kaleva defined the integral of fuzzy function using the Lebesgue type concept for integration. The authors of defined the Henstoch integral of fuzzy-valued functions. In, the fuzzy Riemann integral and its numerical integration was investigated. The numerical methods for fuzzy differential equations have been studied by Abbasbandy and Allahviranloo8-10, and others.

Fixed point theorems for fuzzy mappings, an important tool for demonstrating existence and uniqueness of solutions to fuzzy differential and integral equations, have recently been used by various authors, see11-14. Since fuzzy integral equations usually cannot be solved explicitly, so it is required to obtain approximate solutions. To this end, many authors applied numerical methods for solving such equations. The topics of numerical methods for solving fuzzy differential and integral equations are presented in15. In16, the authors introduced some quadrature rules for the integral of fuzzy-number-valued mappings. Friedman et al. solved the integral equations by using the embedding method17. Solving nonlinear Hammerstein fuzzy integral equations by using fuzzy wavelet like operator is done in18. In19, the authors used a numerical method based on Quadrature rules and the successive approximations method for solving nonlinear fuzzy Fredholm integral equations. Recently, Bica et al.20 introduced an iterative numerical method to obtain numerical solution of nonlinear fuzzy Hammerstein Volterra delay integral equations. To study the other methods for solving FFIE-2, see21-32.

Wavelet theory is a relatively new and an emerging area in mathematical research. Also, wavelets are the suitable and powerful tool for approximating functions based on wavelet basis functions. So, recently several researchers have attempted to develop “fuzzy wavelets” based models and systems. Haung and Zeng23 developed a fuzzy wavelet algorithm based on fuzzy transforms and wavelets but were used separately. In24, the author defined some fuzzy wavelet-like operators and presented their pointwise and uniform convergence with rates to the fuzzy unit operator as I. Recently, the authors of25 constructed fuzzy Haar wavelet and applied it on solving linear FFIE-2.
Here, by using fuzzy wavelet like operator, we propose a numerical approach for approximating the solution of linear FFIE-2.

\[ f(x) = g(x) \otimes \lambda \int K(x,t) \otimes f(t) dt \]  

(1)

where \( K(x,t) \) is an arbitrary kernel function over the square \([a,b] \times [a,b]\) and \( g(x) \) is a fuzzy real valued function of \( x \). Also, we present the error analysis of the proposed method. We should mention that, in this paper, as we do not suppose any kind orthogonality condition on the scaling function as \( \phi \) and the operators act on fuzzy valued continuous functions over real numbers \( R \).

The rest of the paper is organized as follows: In Section 2, we review some necessary backgrounds and notions of fuzzy sets theory. In Section 3, we present proposed method to approximate the solution of (1) by using fuzzy wavelet like operator. The error estimation of this method is proved in Section 4. Finally in Section 5, two numerical examples are presented to show the validity of the results obtained by proposed method.

2. Preliminaries

In this section, we review some necessary backgrounds and notions of fuzzy sets theory.

2.1 Definition

A fuzzy number is a function \( u: R \rightarrow [0,1] \) with the following properties:

- \( u \) is normal, that is there exists \( x_0 \in R \) such that \( u(x_0) = 1 \),
- \( u \) is fuzzy convex set
- (i.e., \( \lambda x+(1-\lambda)y \geq \min \{u(x),u(y)\} \) \( \forall x,y \in R, \lambda \in [0,1] \),
- \( u \) is upper semicontinuous on \( R \),
- The \( \{x \in R: u(x) > 0\} \) is compact set.

The set of all fuzzy numbers denoted by \( R_f \).

2.2 Definition

Suppose that \( u \in R_f \). The \( r \)-level set of \( u \) is denoted by \([u]_r = [u_l(r), u_r(r)]\) and defined \([u]_0 = \{x \in R: u(x) \geq r\}\), where \( 0 < r \leq 1 \). Also, \([u]_0^\circ = \{x \in R: u(x) > 0\}\). It follows that the level sets of \( u \) are closed and bounded intervals in \( R^d \).

It is well-known that, for \( u, v \in R_f \) and \( \lambda \in R \), we can define uniquely the sum \( u \oplus v \) and the product \( \lambda \otimes u \), \( \lambda \in R \) by

\[ [u \oplus v]' = [u]' \oplus [v]', \quad [\lambda \otimes u]' = \lambda [u]', \quad \forall r \in [0,1], \]

Where \([u]' \oplus [v]'\) means the usual addition of two intervals (as subset of \( R \)) and \([\lambda u]'\) means the usual product between a scalar and a subset of \( R \). We use the same symbol \( \Sigma \) both for the sum of real numbers and for the sum \( \oplus \) (when the terms are fuzzy numbers).

2.3 Definition

An arbitrary fuzzy number in represented, in parametric form, by an ordered pair of functions \( u(r), \bar{u}(r) \), \( 0 \leq r \leq 1 \), which satisfy the following requirements:

- \( u(r) \) is a bounded left continuous non decreasing function over \([0,1]\),
- \( \bar{u}(r) \) is a bounded left continuous non increasing function over \([0,1]\),
- \( u(r) \leq \bar{u}(r), \quad 0 \leq r \leq 1 \),
- The addition and scalar multiplication of fuzzy numbers in \( R_f \) are defined as follows:

\[ u \oplus v = (u(r) + v(r), \bar{u}(r) + \bar{v}(r)) \]
\[ \lambda \otimes u = \begin{cases} (\lambda u(r), \lambda \bar{u}(r)) \quad \lambda \geq 0, \\ (\lambda \bar{u}(r), \lambda u(r)) \quad \lambda < 0 \end{cases} \]

2.4 Definition

For arbitrary fuzzy numbers \( u = ( \underline{u}, \bar{u} ) \) and \( v = ( \underline{v}, \bar{v} ) \), the quantity

\[ D(u,v) = \sup_{r \in [0,1]} \left[ \| u'(r) - v'(r) \| + \| \bar{u}'(r) - \bar{v}'(r) \| \right] \]

is called the distance between \( u \) and \( v \). It is proved that \((R_f,D)\) is a complete metric space with the following properties:

- \( D(u \oplus w, v \oplus w) = D(u,v), \forall u, v, w \in R_f, \)
- \( D(k \otimes u, k \otimes v) = |k| D(u,v), \forall u, v \in R_f, \forall k \in R, \)
- \( D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \forall u, v, w, e \in R_f, \)

2.5 Definition

Let \( f, g: [a,b] \rightarrow R_f \) be fuzzy real number valued functions. The uniform distance between is defined by

\[ D \ast (f,g) = \sup \{ D(f(x), g(x)); x \in [a,b] \} \]

2.6 Definition

Let \( f: [a,b] \rightarrow R_f \), \( f \) is a fuzzy-Riemann integrable to \( J \in R_f \) if for any \( \varepsilon > 0 \), there exist \( \delta > 0 \) such that for any division
2.11 Definition
Let \( f: R \to R \) be a uniformly continuous fuzzy real number valued function, if and only if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) whenever \( |x-y| < \delta \); \( x, y \in R \) implies that 
\[
D(f(x), g(x)) \leq \varepsilon.
\]
One denotes it as \( f \in C^U_f (R) \).

2.12 Definition
Let \( f: R \to R \) be a bounded function, then function \( \omega_{f} (f, \delta) = \sup \{ D(f(x), g(x)) | x, y \in R, |x-y| \leq \delta \}, \)
where \( R^+ \) is the set of positive real numbers, is called
the modulus of continuity of on \( R \).

Some properties of the modulus of continuity are presented below:

2.13 Theorem
The following properties holds:

- \( D(f(x), f(y)) \leq \omega_{[a,b]} (f, |x-y|) \) for any \( x, y \in [a,b] \),
- \( \omega_{[a,b]} (f, \delta) \) is increasing function of \( \delta \),
- \( \omega_{[a,b]} (f, 0) = 0 \),
- \( \omega_{[a,b]} (f, \delta + \delta') \leq \omega_{[a,b]} (f, \delta) + \omega_{[a,b]} (f, \delta') \) for any \( \delta, \delta' \geq 0 \),
and \( f: R \to R \).
- \( \omega_{[a,b]} (f, n \delta) \leq n \omega_{[a,b]} (f, \delta) \) for any \( \delta \geq 0, n \in N \), and \( f: [a,b] \to R \).
- \( \omega_{[a,b]} (f, \lambda \delta) \leq (\lambda + 1) \omega_{[a,b]} (f, \delta) \) for any \( \delta \geq 0, \lambda \geq 0 \),
Where \([.]\) is the ceiling of the number, any \( f: [a,b] \to R \).

2.14 Theorem
Let \( f \in C^U_f (R) \) and the scaling function \( \varphi(x) \) a real-valued bounded function with \( \text{supp} \ varphi(x) \subseteq [-\beta, \beta], 0 < \beta < +\infty, \varphi(x) \geq 0 \), such that 
\[
\sum_{j=-\infty}^{\infty} \varphi(x - j) = 1
\]
On \( R \). For \( k \in Z, x \in R \), put 
\[
(B_k f)(x) = \sum_{j=-\infty}^{\infty} f \left( \frac{j}{2^k} \right) \otimes \varphi(2^k x - j)
\]
which is a fuzzy-wavelet-like operator. Then
\[
D((B_k f)(x), f(x)) \leq \omega_{R^+} \left( f, \frac{\beta}{2^k} \right),
\]
\[
D^* ((B_k f)(x), f(x)) \leq \omega_{R^+} \left( f, \frac{\beta}{2^k} \right)
\]
for all \( x \in R \) and \( k \in Z^+ \). If \( f \in C^U_f (R) \), then as \( k \to +\infty \).
one gets $\omega_n (f, \frac{\beta}{2^k}) \to 0$ and $\lim_{k \to +\infty} B_j f = f$, point wise and uniformly with rates.

3. Presented Method for Solving FFIE-2

Here, we use fuzzy wavelet like operator defined by (2) due to approximate solution of Equation (1). To do this, we approximate the solution of (1) by (2). So, by substituting (2) in (1). We have:

$$\sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k x-j) \equiv g(x)+\lambda \int K(x,t) \otimes \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k t-j) dt$$

By using $2^k t-j= u$, we have

$$\sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k x-j) \equiv g(x)+\lambda \frac{\lambda}{2^k} \sum_{j=2^{2k-1}}^{2^k} f\left(\frac{j}{2^k}\right) \otimes \int K(x, \frac{x+u}{2^k}) \varphi(u) du$$

Now, by using the following scaling function $\varphi(k)$,

$$\varphi(x) = \begin{cases} 1, & \frac{1}{2^k} \leq x < \frac{1}{2^{k+1}} \\ 0, & \text{otherwise} \end{cases}$$

and substituting in (5), we conclude that

$$\sum_{j=2^{2k-1}}^{2^k} f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k x-j) \equiv g(x)+\lambda \frac{\lambda}{2^k} \sum_{j=2^{2k-1}}^{2^k} f\left(\frac{j}{2^k}\right) \otimes A_{jk}(x)$$

For fixed k, suppose that $A_{jk}(x) = \frac{\lambda}{2^k} \int K(x, \frac{x+u}{2^k}) \varphi(u) du, \ j=2^k a, 2^k a+1, \ldots, 2^k b-1,.$

Clearly, we can write Equation (5) in the following form

$$\sum_{j=2^{2k-1}}^{2^k} f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k x-j) \equiv g(x) \otimes \sum_{j=2^{2k-1}}^{2^k} f\left(\frac{j}{2^k}\right) \otimes A_{jk}(x)$$

Now, we use collocation points, $x_i = \frac{i}{2^k}, \ i = 2^k a, 2^k a+1, \ldots, 2^k b-1, in (6)$. Clearly, we have

$$\sum_{j=2^{2k-1}}^{2^k} f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k x-j) = g(x) \otimes \sum_{j=2^{2k-1}}^{2^k} f\left(\frac{j}{2^k}\right) \otimes A_{jk}(x), \ i=2^k a, 2^k a+1, \ldots, 2^k b-1.$$

Thus, we obtain the following, $n \times n, \ n=2^k(b-a)$, fuzzy linear system of equations:

$$C \otimes X = Y \otimes B \otimes X$$

where

$$X = (f(a), f(a+1), \ldots, f(b-1))^t,$$

$$Y = (g(a), g(a+1), \ldots, g(b-1))^t,$$

$$B = (B_i)_{i=a}^{b}, \ B_{i} = A_{i,k}(x_i),$$

$$C = (C_{i,j})_{i=a}^{b} \ C_{i,j} = \varphi(2^k x_j).$$

Here, we suppose that $b-a \in N$, where is the set of natural numbers. Clearly, the above system is a dual fuzzy linear system. By solving this system and using Equation (2), we can present the approximate solution of (1). For more details about solving Equation (7), see $^{39}$.

4. Error Estimation

4.1 Theorem

Consider linear FFIE-2 as follows:

$$f(x) = g(x) \otimes \lambda \int K(x,t) \otimes f(t) dt,$$

where $K(x,t)$ is an arbitrary continuous kernel function over the square $a \leq x, t \leq b$ and $g(x) \neq 0$ is a continuous real valued fuzzy function over $a \leq x \leq b$. Under the hypothesis of Theorem 2.14, we have:

$$D(f(x), B_j f(x)) \leq M |\lambda| (b-a) \omega_{[a,b]} \left(f, \frac{\beta}{2^k}\right)$$

where $M = \max |K(x,t)|, \ a \leq x, t \leq b$, and $B_j f(x) = \sum f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k x-j)$

is the approximate solution of Equation (8).

Proof: We would like to estimate

$$|D(f(x), B_j f(x))| = D(g(x) \otimes \lambda \int K(x,t) \otimes f(t) dt, g(x) \otimes \lambda \int K(x,t) \otimes (B_j f(t)) dt)$$

$$\leq |\lambda| \int M D\left(f, \beta \right) (f, t) dt \leq M |\lambda| (b-a) \omega_{[a,b]} \left(f, \frac{\beta}{2^k}\right)$$

It is obvious that

$$(B_j f)(x) = \sum_{j=2^{2k-1}}^{2^k} f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k x-j),$$

$$(B_j f)(x) = \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \otimes \varphi(2^k x-j),$$

Now, we consider two cases as follows:

(a) $K(x,t) \geq 0$,

(b) $K(x,t) < 0$.

In case (a), we have:

$$\tilde{e} = \tilde{f}(x) - (B_j f)(x) = [-\tilde{g}(x) + \lambda \int K(x,t) \tilde{f}(t) dt] - [-\tilde{g}(x) + \lambda \int K(x,t) \tilde{f}(t) dt]$$

$$= \lambda \int K(x,t) \tilde{f}(t) dt - \lambda \int K(x,t) \tilde{f}(t) dt = \lambda \int K(x,t) \tilde{f}(t) dt - \tilde{f}(t) dt.$$
\[ e = \int (x) - \hat{B}_f(x) = \int \hat{f}(x) + \lambda \int \hat{f}(x) \hat{f}(t) dt - \int \hat{f}(x) + \lambda \int \hat{f}(x) \hat{f}(t) dt \]

\[ = \lambda \int \hat{f}(x) \hat{f}(t) dt - \lambda \int \hat{f}(x) \hat{f}(t) dt = \lambda \int \hat{f}(x) \hat{f}(t) dt - \lambda \int \hat{f}(x) \hat{f}(t) dt. \]

Hence
\[ e = e + e \]
\[ = \lambda \int \hat{f}(x) \hat{f}(t) dt - \lambda \int \hat{f}(x) \hat{f}(t) dt + \lambda \int \hat{f}(x) \hat{f}(t) dt - \lambda \int \hat{f}(x) \hat{f}(t) dt. \]

\[ = \lambda \int \hat{f}(x) \hat{f}(t) dt + \lambda \int \hat{f}(x) \hat{f}(t) dt. \]

And
\[ |e| = |e + e| \leq |\lambda| \int \hat{f}(x) \hat{f}(t) dt + |\lambda| \int \hat{f}(x) \hat{f}(t) dt. \]

Since \( K(x,t) \) is an arbitrary continuous kernel function over the square \( a \leq x, t \leq b \), there exist \( M > 0 \) such that \( M = \max |K(x,t)| \). So, we can write
\[ e \leq |\lambda| \int \hat{f}(x) \hat{f}(t) dt + |\lambda| \int \hat{f}(x) \hat{f}(t) dt. \]

On the other hand, we have:
\[ \hat{f}(t) - \hat{f}(t) = \hat{f}(t) - \sum_{j=-\infty}^{\infty} \frac{1}{2} \varphi(2^j t - j) - \sum_{j=-\infty}^{\infty} \frac{1}{2} \varphi(2^j t - j). \]

Therefore
\[ |\hat{f}(t) - \hat{f}(t)| = \left| \sum_{j=-\infty}^{\infty} \frac{1}{2} \varphi(2^j t - j) \right| \]
\[ \leq \sum_{j=-\infty}^{\infty} \left| \hat{f}(t) - \frac{1}{2} \varphi(2^j t - j) \right| \leq \sum_{j=-\infty}^{\infty} \left| \hat{f}(t) - \frac{1}{2} \varphi(2^j t - j) \right| \]
\[ \leq \sum_{j=-\infty}^{\infty} \left| \hat{f}(t) - \frac{1}{2} \varphi(2^j t - j) \right| \leq |t - \frac{1}{2} \varphi(2^j t - j)|. \]

Notice that, under hypotheses of Theorem 4.1, we conclude that
\[ -\beta \leq 2^j t - j \leq \beta \Rightarrow -\frac{\beta}{2} \leq t - \frac{j}{2} \leq \frac{\beta}{2} \Rightarrow |t - \frac{j}{2}| \leq \frac{\beta}{2}. \]

Hence
\[ (9) \leq \sum_{j=-\infty}^{\infty} \varphi(2^j t - j) \omega(\frac{\beta}{2^j}) = \omega(\frac{\beta}{2^j}). \]

Therefore, as \( k \to +\infty \) we get
\[ |\hat{f}(t) - \hat{f}(t)| \to 0. \]

Similarly, we conclude
\[ |\hat{f}(t) - \hat{f}(t)| \to 0, as k \to +\infty. \]

So, we obtain that
\[ |e| = |\hat{e} + e| \to 0, as k \to +\infty. \]

In case (b)
\[ \hat{f}(x,t) < 0. \]

So
\[ \hat{e} = \int (\hat{g}(x) + \lambda \int \hat{f}(x) \hat{f}(t) dt - \int \hat{g}(x) + \lambda \int \hat{f}(x) \hat{f}(t) dt) \]
\[ = \lambda \int \hat{f}(x) \hat{f}(t) dt - \lambda \int \hat{f}(x) \hat{f}(t) dt = \lambda \int \hat{f}(x) \hat{f}(t) dt - \lambda \int \hat{f}(x) \hat{f}(t) dt. \]

The following result is proved similarly
\[ e = \lambda \int \hat{f}(x) \hat{f}(t) dt. \]

Hence
\[ |e| = |\hat{e} + e| \leq |\lambda| \int \hat{f}(x) \hat{f}(t) dt \]
\[ \leq |\lambda| \int \hat{f}(x) \hat{f}(t) dt \]
\[ \leq |\lambda| \int \hat{f}(x) \hat{f}(t) dt \]
\[ \leq \int \hat{f}(x) \hat{f}(t) dt. \]

So, we obtain that
\[ |e| = |\hat{e} + e| \to 0, as k \to +\infty. \]

5. Numerical Examples

To illustrate the efficiency of the presented method in Section 3, we present two Examples. Also, we compare the numerical solution obtained by using the proposed method with the exact solutions.

Through this section, we suppose that
\[ \varphi(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} < x \leq 1 \end{cases} \]

5.1 Example

Consider the following linear FFIE-2
\[ g(x,r) = rx + \frac{3}{26} - \frac{3r}{26} - \frac{1}{13} x^2 - \frac{1}{13} x^2 r, \]
\[ \bar{g}(x,r) = 2x - rx + \frac{3r}{26} + \frac{1}{13} x^2 r - \frac{3}{26} - \frac{3}{13} x^2 \]
\[ K(x,t) = \frac{x^2 + t^2 - 2}{13}, 0 \leq x, t \leq 2. \]

Also, let \( a = 0, b = 1 \). The exact solution in this case is given by
\[ \hat{f}(x,r) = rx, \]
\[ \bar{f}(x,r) = (2 - r)x. \]

By using the proposed method in Section 3, we can obtain the approximate solution for this example. To compare the numerical results with the exact solution in Tables 1 and 2.
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### Table 1.
Numerical results for Example 5.1 in $x=1$ for $k=1, j=2$

| r-cut | left r-cut error | right r-cut error |
|-------|------------------|-------------------|
| 0.1   | 0.000433092      | 0.199775          |
| 0.2   | 0.0115076        | 0.1887            |
| 0.3   | 0.0225821        | 0.177625          |
| 0.4   | 0.0336567        | 0.166551          |
| 0.5   | 0.0447312        | 0.155476          |
| 0.6   | 0.0558057        | 0.144402          |
| 0.7   | 0.0668802        | 0.133327          |
| 0.8   | 0.0779548        | 0.122253          |
| 0.9   | 0.0890293        | 0.111178          |
| 1.0   | 0.100104         | 0.100104          |

### Table 2.
Numerical results for Example 5.1 in $x=1$ for $k=6, j=64$

| r-cut | left r-cut error | right r-cut error |
|-------|------------------|-------------------|
| 0.1   | 0.0143795        | 0.0132534         |
| 0.2   | 0.0132534        | 0.0121272         |
| 0.3   | 0.0121272        | 0.0110011         |
| 0.4   | 0.0110011        | 0.00987495        |
| 0.5   | 0.00987495       | 0.0087488         |
| 0.6   | 0.0087488        | 0.00762265        |
| 0.7   | 0.00762265       | 0.00649651        |
| 0.8   | 0.00649651       | 0.00537036        |
| 0.9   | 0.00537036       | 0.00524421        |
| 1.0   | 0.00524421       | 0.00524421        |

### Table 3.
Numerical results for Example 5.2 in $x=0.0078125$ for $k=7, j=1$

| r-cut | left r-cut error | right r-cut error |
|-------|------------------|-------------------|
| 0.1   | 0.0126965        | 0.0108777         |
| 0.2   | 0.0113869        | 0.00956806        |
| 0.3   | 0.0100772        | 0.00825838        |
| 0.4   | 0.00876749       | 0.0069487         |
| 0.5   | 0.00745781       | 0.00563901        |
| 0.6   | 0.00614861       | 0.00432933        |
| 0.7   | 0.00483845       | 0.00301965        |
| 0.8   | 0.00352876       | 0.00170997        |
| 0.9   | 0.00221908       | 0.000400283       |
| 1.0   | 0.000909399      | 0.000909399       |

### Table 4.
Numerical results for Example 5.2 in $x=0.0078125$ for $k=7, j=1$

| r-cut | left r-cut error | right r-cut error |
|-------|------------------|-------------------|
| 0.1   | 0.0126965        | 0.0108777         |
| 0.2   | 0.0113869        | 0.00956806        |
| 0.3   | 0.0100772        | 0.00825838        |
| 0.4   | 0.00876749       | 0.0069487         |
| 0.5   | 0.00745781       | 0.00563901        |
| 0.6   | 0.00614861       | 0.00432933        |
| 0.7   | 0.00483845       | 0.00301965        |
| 0.8   | 0.00352876       | 0.00170997        |
| 0.9   | 0.00221908       | 0.000400283       |
| 1.0   | 0.000909399      | 0.000909399       |

5.2 Example
Consider the following linear FFIE-

\[ g(x,t) = \left(\frac{r+3}{4}\right) e^{x^3} - \frac{1}{9}(2e^x + 1)x \]

\[ \bar{g}(x,t) = \left(\frac{5-r}{4}\right) e^{x^3} - \frac{1}{9}(2e^x + 1)x \]

\[ K(x,t) = x.t \quad 0 \leq x, t \leq 1 \]

Also, let \( a = 0, b = 1 \). The exact solution in this case is given by

\[ f(x, r) = \left(\frac{r+3}{4}\right) e^x, \]

\[ \bar{f}(x, r) = \left(\frac{5-r}{4}\right) e^x. \]

By using the proposed method in Section 3, we can present the approximate solution for this example. To compare the numerical results with the exact solution in Tables 3 and 4.

### Figure 1.
Comparing exact and approximate solutions for Example 5.1 in $x=1$ for $k=1, j=2$. 
6. Conclusion

To approximate the solution of FFIE-2, a new approach based on fuzzy wavelet like operator via a real-valued scaling function and collocation method is proposed. Convergence analysis of the proposed method is investigated by using the modulus of continuity in one theorem. To illustrate the efficiency of the presented method, two examples are given. Comparing the numerical solutions with the exact solution show that the proposed method can be a suitable method for solving FFIE-2 numerically.

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