Concentration in vanishing adiabatic exponent limit of solutions to the Aw-Rascle traffic model

Shouqiong Sheng, Zhiqiang Shao

College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350108, China

Abstract

In this paper, we study the phenomenon of concentration and the formation of delta shock wave in vanishing adiabatic exponent limit of Riemann solutions to the Aw-Rascle traffic model. It is proved that as the adiabatic exponent vanishes, the limit of solutions tends to a special delta-shock rather than the classical one to the zero pressure gas dynamics. In order to further study this problem, we consider a perturbed Aw-Rascle model and proceed to investigate the limits of solutions. We rigorously proved that, as $\gamma$ tends to one, any Riemann solution containing two shock waves tends to a delta-shock to the zero pressure gas dynamics in the distribution sense. Moreover, some representative numerical simulations are exhibited to confirm the theoretical analysis.

MSC: 35L65; 35L67

Keywords: Aw-Rascle traffic model; Riemann solutions; Delta shock wave; Vanishing adiabatic exponent limit; Zero pressure gas dynamics; Weighted Dirac-measure; Numerical simulations

1. Introduction

The celebrated Aw-Rascle (AR) model of traffic flow reads (cf. [1]):

$$\left\{ \begin{array}{l}
\rho_t + (\rho u)_x = 0, \\
(\rho(u + p(\rho)))_t + (\rho u(u + p(\rho)))_x = 0,
\end{array} \right. \tag{1.1}$$

where $\rho$ and $u$ represent the traffic density and velocity of the cars located at position $x$ at time $t$, respectively; $p$ is the velocity offset and called as the “pressure” inspired from gas dynamics. The model (1.1) is now widely used to study the formation and dynamics of traffic jams. It was proposed by Aw and Rascle [1] to remedy the deficiencies of second order models of car traffic pointed out by Daganzo [6] and had also been independently derived by Zhang [30]. Since its introduction, it had received extensive attention (see [18, 20, 23, 28], etc.).

In this paper, we are concerned with the “pressure” function

$$p(\rho) = \rho^\gamma, \quad 0 < \gamma < 1. \tag{1.2}$$
The Riemann solutions of (1.1) with classical pressure \( p(\rho) = \rho^\gamma \) \( (\gamma > 0) \) were obtained at low densities by Aw and Rascle [1]. Lebacque, Mammar, and Salem1 [13] also solved the Riemann problem of (1.1) with classical pressure \( p(\rho) = \rho^\gamma \) \( (\gamma > 0) \) with an extended fundamental diagram for all possible initial data. Sun [28] studied the interactions of elementary waves to system (1.1).

We are interested in the Riemann problem for (1.1)-(1.2) with initial data

\[
(\rho, u)(0, x) = \begin{cases} 
(\rho_-, u_-), & x < 0, \\
(\rho_+, u_+), & x > 0,
\end{cases}
\]  

(1.3)

where \( \rho_\pm > 0 \) and \( u_\pm \) are given constant states. We assume that \( u_+ < u_- \).

System (1.1)-(1.2) is just like a hyperbolic system for conservation laws of the form

\[
\partial_t U + \partial_x F(U) = 0,
\]  

(1.4)

with

\[
U = \begin{pmatrix} \rho \\ \rho u + \rho^{\gamma+1} \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + up^{\gamma+1} \end{pmatrix} = 0.
\]

When \( \gamma \to 0 \), the limiting system of (1.1)-(1.2) formally becomes the zero pressure gas dynamics,

\[
\begin{cases} 
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + p) = 0,
\end{cases}
\]  

(1.5)

which can be used to describe the process of the motion of free particles sticking under collision and depict the formation of large scale in the universe. The solutions to the zero pressure gas dynamics were widely studied by many scholars (see [2-3, 7-9, 15-16, 26], etc.). In particular, the existence of measure solutions of the Riemann problem was first proved by Bouchut [2] and the existence of the global weak solution was obtained by Brenier and Grenier [3] and E, Rykov and Sinai [7]. Sheng and Zhang [26] discovered that the \( \delta \)-shocks and vacuum states do occur in the Riemann solutions to the zero pressure gas dynamics (1.5) by the vanishing viscosity method. Huang and Wang [9] proved the uniqueness of the weak solution for the case when the initial data is a Radon measure.

A distinctive feature for (1.5) is just that the \( \delta \)-shocks and vacuum states do occur in the Riemann solutions. In paper [23], Shen and Sun studied the limits of Riemann solutions of (1.1) with classical pressure \( p(\rho) = \varepsilon \rho^\gamma \) \( (\gamma > 0) \) as \( \varepsilon \to 0^+ \). They identified a special \( \delta \)-shock in the limit of solutions, whose the propagation speed and the strength are different from those of the zero pressure gas dynamics (1.5). Then, they analyzed a perturbed Aw-Rascle model and proved that the limit of Riemann solutions to the perturbed Aw-Rascle model are those of (1.5) when \( \varepsilon \to 0^+ \). The idea of vanishing pressure limits dates back to early works of Li [14], Chen and Liu [4,5], and the vanishing pressure limit method was also applied to other systems [17-20, 22, 24-25, 29].

Let us turn to the Euler system of power law in Eulerian coordinates,

\[
\begin{cases} 
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + p(\rho))_x = 0,
\end{cases}
\]  

(1.6)

When the pressure tends to zero or a constant, the Euler system (1.6) formally tends to the zero pressure gas dynamics. In earlier seminal papers, Chen and Liu [4] first showed the formation of
\(\delta\)-shocks and vacuum states of the Riemann solutions to the Euler system (1.6) for polytropic gas by taking limit \(\varepsilon \to 0^+\) in the model \(p(\rho) = \varepsilon \rho^\gamma / \gamma\) (\(\gamma > 1\)), which describe the phenomenon of concentration and cavitation rigorously in mathematics. Further, they also obtained the same results for the Euler equations for nonisentropic fluids in [5]. The same problem for the Euler equations (1.6) for isothermal case (\(\gamma = 1\)) was studied by Li [14]. Recently, Muhammad Ibrahim, Fujun Liu and Song Liu [10] showed the same phenomenon of concentration also exists in the model \(p(\rho) = \rho^\gamma\) (\(0 < \gamma < 1\)) as \(\gamma \to 0\), which is the case that the pressure goes to a constant. Namely, they showed rigorously the formation of delta wave with the limiting behavior of Riemann solutions to the Euler equations (1.6).

Motivated by [10], for the Aw-Rascle model (1.1) with classical pressure (1.2), we show the same phenomenon of concentration also exists in the case \(0 < \gamma < 1\) and \(u_+ < u_-\) as \(\gamma \to 0\). We can see that, as \(\gamma \to 0\), the Riemann solution converges to a special delta shock solution, whose the propagation speed and the strength are different from those of the PGD model (1.5), which means the Riemann solution of (1.1)-(1.2) don’t converge to the delta shock solution of (1.5).

In order to solve this problem, we motivated by [23], adding a suitable perturbation in the pressure term in the Aw-Rascle model (1.1)-(1.2). That is we consider the perturbed Aw-Rascle (PAR) model as follows:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
\left( \rho u + \frac{1}{2} \rho^\gamma \right)_t + \left( \rho u^2 + u \rho^\gamma \right)_x &= 0,
\end{align*}
\]

where \(1 < \gamma < 3\). For convenience and conciseness, we replace \(\rho p(\rho)\) with \(p(\rho)\) in (1.1) and take \(p(\rho) = \rho^\gamma\) for \(\gamma \in (1, 3)\). In the system (1.7), \(p(\rho) = \rho^\gamma\) can be regarded as the traffic pressure term and \(1 < \gamma < 3\) is analogous with the adiabatic exponent \(0 < \gamma < 2\) in the Aw-Rascle model (1.1)-(1.2). It is proved that when \(\gamma \to 1\), the limit of the Riemann solutions containing two shock waves of the perturbed Aw-Rascle model is exactly a delta shock solution of the zero pressure gas dynamics (1.5).

Finally, by using the fifth-order weighted essentially non-oscillatory scheme and third-order Runge-Kutta method [12, 27], some representative numerical simulations are exhibited, which are completely consistent with theoretical analysis.

The rest of the paper is organized as follows. For the sake of completeness, in Section 2, we briefly review the delta shock wave and vacuum state in the Riemann solutions of the zero pressure gas dynamics (1.5). In Section 3, we display some results on the Riemann solutions of (1.1)-(1.2) when \(0 < \gamma < 1\). In Section 4, we discuss the limits of Riemann solutions of (1.1)-(1.2) as the adiabatic exponent vanishes. In Section 5, we display some results on the Riemann solutions of (1.7) when \(1 < \gamma < 3\). In Section 6, we show rigorously the formation of delta shock wave with the limiting behavior of Riemann solutions of (1.7) as \(\gamma \to 1\). In section 7, we present the numerical results.

2. Preliminaries

For the sake of completeness, in this section we briefly recall the delta shock wave and vacuum state in the Riemann solutions of the zero pressure gas dynamics (1.5). More details can be found in [26, 24, 16, 11].

The system (1.5) has a double eigenvalue \(\lambda = u\) and only one right eigenvector \(\vec{v} = (1, 0)^T\). The
system is obviously nonstrictly hyperbolic, and $\lambda$ is linearly degenerate by $\nabla \lambda \cdot \mathbf{F} \equiv 0$, in which $\nabla$ denotes the gradient with respect to $(\rho, u)$. Therefore, in classical sense, the associated elementary waves involve only contact discontinuities. It can be seen from previous works [11, 16, 24, 26] that the Riemann problem for (1.5) with initial data (1.3) can be solved by contact discontinuities, vacuum or delta shock wave connecting two constant states $(\rho_{\pm}, u_{\pm})$.

When $u_- < u_+$, there is no characteristic passing through the region $u_- t < x < u_+ t$ and the vacuum appears in this region. The solution can be expressed as

$$
(\rho, u)(t, x) = \begin{cases}
(\rho_-, u_-), & -\infty < x < u_- t, \\
(0, \frac{x}{t}), & u_- t \leq x \leq u_+ t, \\
(\rho_+, u_+), & u_+ t < x < +\infty.
\end{cases}
$$

When $u_- = u_+$, the constant states $(\rho_{\pm}, u_{\pm})$ can be connected by a contact discontinuity. The solution can be expressed as

$$
(\rho, u)(t, x) = \begin{cases}
(\rho_-, u_-), & -\infty < x < u_- t, \\
(\rho_+, u_+), & u_- t < x < +\infty.
\end{cases}
$$

When $u_- > u_+$, the characteristic lines from initial data will overlap, so the Riemann solution cannot be constructed by using the classical waves, we seek a solution containing a weighted Dirac delta function with the support on a line.

To do so, a two-dimensional weighted delta function $w(t)\delta_S$ supported on a smooth curve $S = \{(t(s), x(s)) : a < s < b\}$ is defined by

$$
\langle w(t)\delta_S, \varphi(t, x) \rangle = \int_a^b w(t(s))\varphi(t(s), x(s))ds,
$$

for all test functions $\varphi(t, x) \in C^\infty(0, +\infty) \times (-\infty, +\infty)$.

For the Riemann problem with $u_+ < u_-$, we can construct a dirac-measured solution with parameter $\sigma$ as follows,

$$
\rho(t, x) = \rho_0(t, x) + w(t)\delta_S, \quad u(t, x) = u_0(t, x),
$$

where $S = \{(t, \sigma t) : 0 \leq t < +\infty\}$,

$$
\rho_0(t, x) = \begin{cases}
\rho_-, & x < \sigma t, \\
\rho_+, & x > \sigma t,
\end{cases}
$$

$$
u_0(t, x) = \begin{cases}
u_-, & x < \sigma t, \\
\sigma, & x = \sigma t, \\
u_+, & x > \sigma t,
\end{cases}
$$

and

$$
w(t) = t(\sigma[\rho] - [\rho u]),
$$
in which \([q] = q_+ - q_-\) denotes the jump of function \(q\) across the discontinuity discontinuity. The dirac-measured solution \((\rho, u)\) constructed above is called a delta shock solution of (1.5) in the sense of distributions if
\[
\langle \rho, \varphi_t \rangle + \langle \rho u, \varphi_x \rangle = 0,
\]
\[
\langle \rho u, \varphi_t \rangle + \langle \rho u^2, \varphi_x \rangle = 0,
\]
hold for any test function \(\varphi(t, x) \in C_0^\infty([0, +\infty) \times (-\infty, +\infty))\), where
\[
\langle \rho, \varphi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0(t, x) \varphi(t, x) dx dt + \langle w(t) \delta_S, \varphi(t, x) \rangle,
\]
\[
\langle \rho u, \varphi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0(t, x) u_0(t, x) \varphi(t, x) dx dt + \langle \sigma w(t) \delta_S, \varphi(t, x) \rangle.
\]
Then the following generalized Rankine-Hugoniot relation
\[
\begin{align*}
\frac{dx}{dt} &= \sigma, \\
\frac{dw(t)}{dt} &= \sigma [\rho] - [\rho u], \\
\frac{d(w(t))_\sigma}{dt} &= \sigma [\rho u] - [\rho u^2],
\end{align*}
\]
holds, where \([\rho] = \rho_+ - \rho_-\), with initial data
\[
(x, w)(0) = (0, 0).
\]
To guarantee uniqueness, the delta shock should satisfy the entropy condition:
\[
u_+ < \sigma < u_-,
\]
which means that all the characteristic lines on both sides of the discontinuity are incoming. So it is a overcompressive condition.

Solving (2.10) with initial data (2.11) under the entropy condition (2.12), we have
\[
\begin{align*}
w(t) &= \sqrt{\rho_-} u_+ (u_- - u_+) t, \\
\sigma &= \frac{\sqrt{\rho_+} u_+ + \sqrt{\rho_-} u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}}.
\end{align*}
\]
Therefore, a delta shock solution defined by (2.4) with (2.5), (2.6) and (2.13) is obtained.

3. Riemann solutions of the AR model (1.1)-(1.2)

In this section, we review the Riemann solutions of (1.1)-(1.2) with initial data (1.3), for which the detailed investigations can be found in Sun [28].

The system (1.1)-(1.2) has two eigenvalues
\[
\lambda_1 = u - \gamma \rho^\gamma, \quad \lambda_2 = u,
\]
with the corresponding right eigenvectors
\[
\mathbf{\tau}_1 = (1, -\gamma \rho^{\gamma-1})^T, \quad \mathbf{\tau}_2 = (1, 0)^T.
\]
satisfying
\[ \nabla \lambda_1 \cdot \mathbf{v}_1 = -\gamma(\gamma + 1)\rho^{\gamma-1} < 0, \]
and
\[ \nabla \lambda_2 \cdot \mathbf{v}_2 \equiv 0. \]

Therefore, system (1.1)-(1.2) is strictly hyperbolic for \( \rho > 0 \), and \( \lambda_1 \) is genuinely nonlinear for \( \rho > 0 \) and the associated wave is either shock wave or rarefaction wave, while \( \lambda_2 \) is always linearly degenerate and the associated wave is the contact discontinuity.

Since (1.1), (1.2) and the Riemann data (1.3) are invariant under stretching of coordinates: \((t, x) \rightarrow (\tau t, \tau x) \) (\( \tau \) is constant), we seek the self-similar solution \((\rho, u)(t, x) = (\rho, u)(\xi)\), \( \xi = \frac{x}{t} \).

Then the Riemann problem (1.1), (1.2) and (1.3) is reduced to the following boundary value problem of the ordinary differential equations:

\[
\begin{aligned}
-\xi \rho \xi + (\rho u) \xi &= 0, \\
-\xi (\rho u + \rho^{\gamma+1}) \xi + (\rho u^2 + u \rho^{\gamma+1}) \xi &= 0,
\end{aligned}
\]
with \((\rho, u)(\pm \infty) = (\rho_\pm, u_\pm)\).

For any smooth solution, system (3.2) can be written as

\[
\begin{pmatrix}
 u - \xi \\
 (u - \xi)(u + (\gamma + 1)\rho^\gamma) - \xi \rho + 2\rho u + \rho^{\gamma+1}
\end{pmatrix} \begin{pmatrix}
 \rho \\
 u \xi
\end{pmatrix} = 0.
\]

Besides the constant solution
\[ (\rho, u)(\xi) = \text{constant} \quad (\rho > 0), \]
it provides a rarefaction wave which is a continuous solution of (3.3) in the form \((\rho, u)(\xi)\). Then, for a given left state \((\rho_-, u_-)\), the rarefaction wave curves in the phase plane, which are the sets of states that can be connected on the right by a 1-rarefaction wave, are as follows:

\[
R(\rho_-, u_-) : \left\{ \begin{array}{l}
\xi = \lambda_1 = u - \gamma\rho^\gamma, \\
 u - u_- = -(\gamma^\gamma - \rho_-^\gamma), \\
 \rho < \rho_-, u > u_-.
\end{array} \right.
\]

Differentiating the second equation of (3.4) with respect to \( \rho \) yields
\[ u_\rho = -\gamma\rho^{\gamma-1} < 0, \]
and
\[ u_{\rho\rho} = -\gamma(\gamma - 1)\rho^{\gamma-2} > 0, \]
which mean that for \( 0 < \gamma < 1 \), the rarefaction wave curve \( R(\rho_-, u_-) \) is monotonic decreasing and convex in the \((\rho, u)\) phase plane \((\rho > 0)\). Moreover, it can be concluded from (3.4) that \( \lim_{\rho \to 0^+} u = u_- + \rho_-^\gamma \)
for the rarefaction wave curve \( R(\rho_-, u_-) \), which implies that \( R(\rho_-, u_-) \) intersects the \( u \)-axis at the point \((0, \bar{u}_*)\), where \( \bar{u}_* \) is determined by \( \bar{u}_* = u_- + \rho_\gamma^- \).

For a bounded discontinuity at \( \xi = \sigma \), the Rankine-Hugoniot relation

\[
\begin{cases}
-\sigma [\rho] + [\rho u] = 0, \\
-\sigma [\rho u + \rho^{\gamma+1}] + [\rho u^2 + u\rho^{\gamma+1}] = 0,
\end{cases}
\]

holds, where \([\rho] = \rho - \rho_-\), etc. Eliminating \( \sigma \) from (3.5), we obtain

\[
[\rho][\rho u^2] - ([\rho u])^2 = -[\rho][\rho u^{\gamma+1}] + [\rho u][\rho^{\gamma+1}].
\]

Simplifying (3.6) yields

\[
(u - u_-)^2 = -(u - u_-)(\rho\gamma - \rho_-^\gamma).
\]

If \( u - u_- \neq 0 \), we have

\[
(u - u_-) = -(\rho\gamma - \rho_-^\gamma) \quad \text{and} \quad \sigma = u - \frac{\rho_-(\rho\gamma - \rho_-^\gamma)}{\rho - \rho_-},
\]

where \( \sigma, (\rho_-, u_-) \) and \( (\rho, u) \) are the shock speed, the left state and the right state, respectively.

Otherwise, for case \( u = u_- \) (i.e., \([u] = 0\)), we have

\[
\sigma = u = u_-.
\]

The classical Lax entropy conditions imply that the propagation speed \( \sigma \) for the 1-shock wave has to be satisfied with

\[
\sigma < \lambda_1(\rho_-, u_-), \quad \lambda_1(\rho, u) < \sigma < \lambda_2(\rho, u).
\]

From the first equation of (3.5), we obtain

\[
\sigma = \frac{\rho u - \rho_- u}{\rho - \rho_-} = u_- + \frac{\rho}{\rho - \rho_-}(u - u_-).
\]

If \( u > u_- \), then from (3.7), we have \( \rho < \rho_- \), and

\[
\sigma - u_- = \frac{\rho}{\rho - \rho_-}(u - u_-) = -\frac{\rho(\rho\gamma - \rho_-^\gamma)}{\rho - \rho_-} = -\rho\gamma^- \rho^{-\gamma-1},
\]

for some \( \tilde{\rho} \in (\rho, \rho_-) \). By direct calculation, we have

\[
\gamma \rho_-^\gamma - \rho\gamma^- \rho^{-\gamma-1} > \gamma(\rho_-^\gamma - \rho_-^\gamma) > 0,
\]

which implies that

\[
\sigma - u_- > -\gamma \rho_-^\gamma.
\]

This contradicts with \( \sigma < \lambda_1(\rho_-, u_-) \). Then, given a left state \((\rho_-, u_-)\), the possible states that can be connected to \((\rho_-, u_-)\) on the right by shock wave in the 1-family are as follows:

\[
S(\rho_-, u_-) : \left\{ \begin{array}{l}
\sigma = u - \frac{\rho_-(\rho\gamma - \rho_-^\gamma)}{\rho - \rho_-}, \\
\sigma = u - \frac{\rho_-(\rho\gamma - \rho_-^\gamma)}{\rho - \rho_-}, \\
\rho > \rho_-, u < u_-
\end{array} \right.
\]

(3.8)
Differentiating \( u \) with respect to \( \rho \) in the second equation of (3.8) gives that for \( \rho > \rho_- \),
\[
 u_\rho = -\gamma \rho^{\gamma-1} < 0 \quad \text{and} \quad u_{\rho\rho} = -\gamma(\gamma - 1)\rho^{\gamma-2} > 0,
\]
which means that the shock wave curve \( S(\rho_-, u_-) \) is monotonic decreasing and convex in the \((\rho, u)\) phase plane \((\rho > \rho_-)\). It can also be derived from (3.8) that \( \lim_{\rho \to +\infty} u = -\infty \) for the shock wave curve \( S(\rho_-, u_-) \), which indicates that the shock wave curve intersects with the \( \rho \)-axis at a point.

Since \( \lambda_2 \) is linearly degenerate, the set of states \((\rho, u)\) can be connected to a given left state \((\rho_-, u_-)\) by a contact discontinuity on the right if and only if
\[
 J : \xi = u = u_-.
\]

In the \((\rho, u)\) phase plane \((\rho, u \geq 0)\), through a given point \((\rho_-, u_-)\), we draw the elementary wave curves. We find that the elementary wave curves divide the quarter phase plane \((\rho > \rho_-)\) into three regions, \( I = \{(\rho, u)|u < u_-\} \), \( II = \{(\rho, u)|u_- < u < u_*\} \), and \( III = \{(\rho, u)|u > u_*\} \), where \( u_* = u_- + \rho_+^\gamma \), see Fig. 1. According to the right state \((\rho_+, u_+)\) in the different regions, one can construct the unique global Riemann solution connecting two constant states \((\rho_\pm, u_\pm)\) as follows: (1) \((\rho_+, u_+) \in I(\rho_-, u_-) : S + J\), (2) \((\rho_+, u_+) \in II(\rho_-, u_-) : R + J\), (3) \((\rho_+, u_+) \in III(\rho_-, u_-) : R + \text{Vac} + J\) (see Fig. 1), where “+” means “followed by”.

**Fig. 1.** \((\rho, u)\)-plane.

### 4. Limit of Riemann solutions of the AR model (1.1)-(1.2)

In this section, we study the limiting behavior of the Riemann solutions of (1.1)-(1.2) with the assumption \( u_+ < u_- \) as \( \gamma \) tends to zero, that is, the formation of delta shock as \( \gamma \to 0 \) in the case \( u_+ < u_- \).

#### 4.1. Formation of delta shock wave

For any fixed \( \gamma \in (0, 1) \), when \( u_+ < u_- \), namely \((\rho_+, u_+) \in I(\rho_-, u_-)\), the Riemann solution of (1.1)-(1.2) is a shock wave \( S \) followed by a contact discontinuity \( J \) with the intermediate state \((\rho_*, u_*)\)
besides two constant states \((\rho_{-}, u_{-})\) and \((\rho_{+}, u_{+})\). They satisfy
\[
S: \begin{cases}
\sigma_{1} = u_{*} - \frac{\rho_{*}(\rho_{-} - \rho_{+})}{\rho_{*} - \rho_{-}}, \\
u_{*} - u_{-} = - (\rho_{*}^{2} - \rho_{-}^{2}), \quad \rho_{*} > \rho_{-},
\end{cases}
\tag{4.1}
\]
and
\[
J: \quad \sigma_{2} = u_{*} = u_{+}, \quad \rho_{*} > \rho_{+},
\tag{4.2}
\]
where \(\sigma_{1}\) and \(\sigma_{2}\) are the propagation speeds of \(S\) and \(J\), respectively. Then we have the following lemmas.

**Lemma 4.1.** \(\lim_{\gamma \to 0} \rho_{*} = \infty\), and \(\lim_{\gamma \to 0} \rho_{*}^{\gamma} =: a = 1 + u_{-} - u_{+}\).

**Proof.** It follows from (4.1) and (4.2) that
\[
u_{-} - u_{+} = \lim_{\gamma \to 0} \rho_{*}^{\gamma} - \rho_{-}^{\gamma}, \quad \rho_{*} > \rho_{\pm}.
\tag{4.3}
\]
Let \(\lim \inf_{\gamma \to 0} \rho_{*} = \alpha\) and \(\lim \sup_{\gamma \to 0} \rho_{*} = \beta\).

If \(\alpha < \beta\), then by the continuity of \(\rho_{*}(\gamma)\), there exists a sequence \(\{\gamma_{k}\}_{k=1}^{\infty} \subseteq (0, 1)\) such that
\[
\lim_{k \to +\infty} \gamma_{k} = 0, \quad \text{and} \quad \lim_{k \to +\infty} \rho_{*}(\gamma_{k}) = c,
\]
for some \(c \in (\alpha, \beta)\). Then substituting the sequence into the right hand side of (4.3), and taking the limit \(k \to +\infty\), we have
\[
u_{-} - u_{+} = \lim_{k \to +\infty} (\rho_{*}(\gamma_{k})^{\gamma_{k}} - \rho_{-}^{\gamma_{k}}) = 0.
\tag{4.4}
\]
This contradicts with the assumption \(u_{-} > u_{+}\). Then we must have \(\alpha = \beta\), which means \(\lim_{\gamma \to 1} \rho_{*}(\gamma) = \alpha\).

If \(\alpha \in (0, +\infty)\), then we can also get a contradiction when taking limit in (4.3). Hence \(\alpha = 0\) or \(\alpha = +\infty\). By the condition \(\rho_{*} > \max\{\rho_{-}, \rho_{+}\}\), it is easy to see that \(\lim_{\gamma \to 0} \rho_{*}(\gamma) = \alpha = +\infty\).

Next taking the limit \(\gamma \to 0\) in (4.3), we have
\[
u_{-} - u_{+} = \lim_{\gamma \to 0} (\rho_{*}^{\gamma} - \rho_{-}^{\gamma}) =: a - 1,
\]
from which we can get \(a = 1 + u_{-} - u_{+}\). The proof is completed. \(\square\)

**Lemma 4.2.**

\[
\lim_{\gamma \to 0} \sigma_{1} = \lim_{\gamma \to 0} \sigma_{2} = \lim_{\gamma \to 0} u_{*} = \sigma,
\]
where \(\sigma = u_{+}\).

**Proof.** From (4.1), (4.2) and Lemma 4.1, we immediately get
\[
\lim_{\gamma \to 0} \sigma_{1} = \lim_{\gamma \to 0} \sigma_{2} = \lim_{\gamma \to 0} u_{*} = u_{-} - \lim_{\gamma \to 0} (\rho_{*}^{\gamma} - \rho_{-}^{\gamma}) = u_{-} - (a - 1) = u_{-} - (u_{-} - u_{+}) = u_{+}.
\]
The proof is completed. \(\square\)

Lemmas 4.1-4.2 show that when \(\gamma\) tends to zero, \(S\) and \(J\) coincide, the intermediate density \(\rho_{*}\) becomes singular.
Lemma 4.3. 
\[
\lim_{\gamma \to 0} \int_{\gamma_{2}}^{\sigma_{2}} \rho_{+} d\xi = \rho_{-}(u_{-} - u_{+}) \neq 0. \tag{4.5}
\]

Proof. From the first equations of the Rankine-Hugoniot relation (3.5) for \( S \) and \( J \), we have
\[
\sigma_{1}(\rho_{-} - \rho_{+}) = \rho_{-} u_{-} - \rho_{+} u_{+}, \tag{4.6}
\]
and
\[
\sigma_{2}(\rho_{+} - \rho_{-}) = \rho_{+} u_{-} - \rho_{-} u_{+}. \tag{4.7}
\]
By (4.6) + (4.7), we get
\[
\lim_{\gamma \to 0} \rho_{-}(\sigma_{2} - \sigma_{1}) = \lim_{\gamma \to 0} (\rho_{-} u_{-} - \sigma_{1} \rho_{-} + \sigma_{2} \rho_{+} - \rho_{+} u_{+}) = \rho_{-}(u_{-} - u_{+}),
\]
which implies that
\[
\lim_{\gamma \to 0} \int_{\sigma_{1}}^{\sigma_{2}} \rho_{+} d\xi = \rho_{-}(u_{-} - u_{+}). \tag{4.8}
\]
The proof is completed. \( \square \)

Lemma 4.3 shows that when \( \gamma \to 0 \), the limit of \( \rho_{+} \) has the same singularity as a weighted Dirac delta function at \( \xi = u_{+} \).

Remark 4.1. It can be concluded from Lemmas 4.1-4.3 that, when \( \gamma \to 0 \), \( S \) and \( J \) coincide to form a new type of nonlinear hyperbolic wave, which is called as the delta shock wave in [45]. Compared with the Riemann solutions of (1.5), it is clear to see that the propagation speed and strength of the delta shock wave here are \( \sigma = u_{+} \) and \( w(t) = \rho_{-}(u_{-} - u_{+}) t \), which are different from those of the classical one to the zero pressure gas dynamics (1.5).

Now, we give the following theorem which give a very nice depiction of the limit of Riemann solutions of (1.1) and (1.2) as \( \gamma \to 0 \) in the case \( u_{+} < u_{-} \).

Theorem 4.4. Let \( u_{+} < u_{-} \). For any fixed \( \gamma \in (0, 1) \), assume that \( (\rho_{\gamma}(t, x), m_{\gamma}(t, x)) = (\rho_{\gamma}(t, x), \rho_{\gamma}(t, x) u_{\gamma}(t, x)) \) is a Riemann solution containing a shock wave and a contact discontinuity of (1.1) and (1.2) with the Riemann initial data (1.3). Then, as \( \gamma \to 0 \), \( (\rho_{\gamma}(t, x), m_{\gamma}(t, x)) \) will converge to
\[
(\rho(t, x), m(t, x)) = (\rho_{0}(t, x) + w_{1}(t) \delta_{\gamma}, \rho_{0}(t, x) u_{0}(t, x) + w_{2}(t) \delta_{\gamma}),
\]
in the sense of distributions, and the singular parts of the limit functions \( \rho(t, x) \) and \( m(t, x) \) are a \( \delta \)-measure with weights
\[
w_{1}(t) = t(\sigma[p] - [\rho u]) = \rho_{-}(u_{-} - u_{+}) t, \quad \text{and} \quad w_{2}(t) = t(\sigma[p]u - [\rho u^{2}])
\]
respectively, where \( \sigma = u_{+} \).

Proof. (1) Set \( \xi = \frac{\sigma}{\gamma} \). Then for any fixed \( \gamma \in (0, 1) \), the Riemann solution containing a shock wave and a contact discontinuity of (1.1) and (1.2) can be written as
\[
(\rho_{\gamma}, u_{\gamma})(\xi) = \begin{cases} 
(\rho_{-}, u_{-}), & \xi < \sigma_{1}, \\
(\rho_{+}, u_{+}), & \sigma_{1} < \xi < \sigma_{2}, \\
(\rho_{+}, u_{+}), & \xi > \sigma_{2}.
\end{cases}
\]
From (3.2), we have the following weak formulations:

$$\int_{-\infty}^{+\infty} \rho_\gamma(\xi)(u_\gamma(\xi) - \xi)\varphi'(\xi)d\xi - \int_{-\infty}^{+\infty} \rho_\gamma(\xi)\varphi(\xi)d\xi = 0, \quad (4.9)$$

$$\int_{-\infty}^{+\infty} \rho_\gamma(\xi)u_\gamma(\xi)(u_\gamma(\xi) - \xi)\varphi'(\xi)d\xi + \int_{-\infty}^{+\infty} (\rho_\gamma(\xi))^{\gamma+1}(u_\gamma(\xi) - \xi)\varphi(\xi)d\xi$$

$$- \int_{-\infty}^{+\infty} (\rho_\gamma(\xi)u_\gamma(\xi) + (\rho_\gamma(\xi))^{\gamma+1}) \varphi(\xi)d\xi = 0, \quad (4.10)$$

for any $\varphi(\xi) \in C_{0}^{+\infty}(R)$.

(2) For the first integral on the left-hand side of (4.9), using the method of integration by parts, we can derive

$$\int_{-\infty}^{+\infty} \rho_\gamma(\xi)(u_\gamma(\xi) - \xi)\varphi'(\xi)d\xi = \left( \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{+\infty} \right) \rho_\gamma(\xi)(u_\gamma(\xi) - \xi)\varphi'(\xi)d\xi$$

$$= \rho_- u_- \varphi(\sigma_1) - \rho_+ u_+ \varphi(\sigma_2) - \rho_- \sigma_1 \varphi(\sigma_1) + \rho_+ \sigma_2 \varphi(\sigma_2) + \int_{-\infty}^{\sigma_1} \rho_- \varphi(\xi)d\xi$$

$$+ \int_{\sigma_1}^{+\infty} \rho_+ \varphi(\xi)d\xi + \int_{\sigma_2}^{+\infty} \rho_+ (u_+ - \xi)\varphi'(\xi)d\xi$$

Meanwhile, we have

$$\int_{\sigma_1}^{+\infty} \rho_+(u_+ - \xi)\varphi'(\xi)d\xi = \rho_+ u_+(\varphi(\sigma_2) - \varphi(\sigma_1)) - \rho_+ \sigma_1 \varphi(\sigma_1) + \int_{\sigma_1}^{+\infty} \rho_+ \varphi(\xi)d\xi$$

$$= \rho_+ (\sigma_2 - \sigma_1) \left( u_+ \varphi(\sigma_2) - \varphi(\sigma_1) + \int_{\sigma_1}^{\sigma_2} \varphi(\xi)d\xi \right) - \frac{\sigma_2 \varphi(\sigma_2) - \sigma_1 \varphi(\sigma_1)}{\sigma_2 - \sigma_1}.$$  

Then, by Lemma 4.2-4.3, we can obtain

$$\lim_{\gamma \to 0} \int_{\sigma_1}^{\sigma_2} \rho_+(u_+ - \xi)\varphi'(\xi)d\xi = 0.$$

Hence taking the limit $\gamma \to 0$ in (4.9) leads to

$$\lim_{\gamma \to 0} \int_{-\infty}^{+\infty} (\rho_\gamma(\xi) - \rho_0(\xi))\varphi(\xi)d\xi = (\sigma[\rho] - [\rho u])\varphi(\sigma), \quad (4.11)$$

where $(\rho_0(\xi), u_0(\xi)) = (\rho_\pm, u_\pm)$, $\pm (\xi - \sigma) > 0$.

(3) Similarly, we can obtain for (4.10) that

$$\int_{-\infty}^{+\infty} \rho_\gamma(\xi)u_\gamma(\xi)(u_\gamma(\xi) - \xi)\varphi'(\xi)d\xi$$

$$= (\sigma[\rho u] - [\rho u^2]) \varphi(\sigma) + \int_{-\infty}^{+\infty} \rho_0(\xi)u_0(\xi)\varphi(\xi)d\xi$$

and

$$\int_{-\infty}^{+\infty} (\rho_\gamma(\xi))^{\gamma+1}(u_\gamma(\xi) - \xi)\varphi'(\xi)d\xi = \left( \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{+\infty} + \int_{\sigma_2}^{+\infty} \right) (\rho_\gamma(\xi))^{\gamma+1}(u_\gamma(\xi) - \xi)\varphi'(\xi)d\xi$$

$$= \rho_-^{\gamma+1} u_- \varphi(\sigma_1) - \rho_+^{\gamma+1} u_+ \varphi(\sigma_2) - \rho_-^{\gamma+1} \sigma_1 \varphi(\sigma_1) + \rho_+^{\gamma+1} \sigma_2 \varphi(\sigma_2) + \int_{-\infty}^{\sigma_1} \rho_-^{\gamma+1} \varphi(\xi)d\xi$$

11
\[ + \int_{\sigma_2}^{+\infty} \rho_{\sigma_1}^{-1} \varphi(\xi) d\xi + \rho_{\sigma_1}(\sigma_2 - \sigma_1) \left( u_{\sigma_1} \frac{\varphi(\sigma_1) - \varphi(\sigma_2)}{\sigma_2 - \sigma_1} - \frac{\sigma_2 \varphi(\sigma_2) - \sigma_1 \varphi(\sigma_1)}{\sigma_2 - \sigma_1} + \int_{\sigma_2}^{\sigma_1} \varphi(\xi) d\xi \right), \]

which converges to
\[ (\sigma[\rho - [\rho u]] \varphi) + \int_{-\infty}^{+\infty} \rho_0(\xi) \varphi(\xi) d\xi \]

by Lemma 4.1-4.3.

Thus, following (4.11), we can get
\[ \lim_{\gamma \to 0} \int_{-\infty}^{+\infty} (\rho_\gamma(\xi)u_\gamma(\xi) - \rho_0(\xi)u_0(\xi)) \varphi(\xi) d\xi = (\sigma[\rho - [\rho u]]) \varphi(\sigma). \tag{4.12} \]

(4) Finally, we study the limits of \( \rho_\gamma(t, x) \) and \( \rho_\gamma(t, x)u_\gamma(t, x) \) depending on \( t \) as \( \gamma \to 0 \). Regarding \( t \) as a parameter, we can get from (4.11) that
\[
\lim_{\gamma \to 0} \int_{-\infty}^{+\infty} (\rho_\gamma(\xi) - \rho_0(\xi)) \varphi(t, \xi) d\xi = \lim_{\gamma \to 0} \int_{-\infty}^{+\infty} (\rho_\gamma(x/t) - \rho_0(x/t)) \varphi(t, x) d(x/t) \\
= \frac{1}{t} \lim_{\gamma \to 0} \int_{-\infty}^{+\infty} (\rho_\gamma(t, x) - \rho_0(t, x)) \varphi(t, x) dx = (\sigma[\rho - [\rho u]]) \varphi(t, \sigma t). \tag{4.13} \]

Then multiplying (4.13) by \( t \) and taking integration, we have
\[ \lim_{\gamma \to 0} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} (\rho_\gamma(t, x) - \rho_0(t, x)) \varphi(t, x) dx dt = \int_{0}^{+\infty} t(\sigma[\rho - [\rho u]]) \varphi(t, \sigma t) dt \]

in which by definition (2.3), we have
\[ \int_{0}^{+\infty} t(\sigma[\rho - [\rho u]]) \varphi(t, \sigma t) dt = \langle w_1(\cdot) \delta_s, \varphi(\cdot, \cdot) \rangle. \tag{4.14} \]

where
\[ w_1(t) = t(\sigma[\rho - [\rho u]]) = \rho_-(u_1 - u_0) t. \]

In the same way, we can derive from (4.12) that
\[ \lim_{\gamma \to 0} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} (\rho_\gamma(t, x)u_\gamma(t, x) - \rho_0u_0(t, x)) \varphi(t, x) dx dt = \langle w_2(\cdot) \delta_s, \varphi(\cdot, \cdot) \rangle. \tag{4.15} \]

where
\[ w_2(t) = t(\sigma[\rho u] - [\rho u^2]). \]

The proof is completed. \( \Box \)

5. Riemann solutions of the PAR model (1.7)

In this section, we construct the Riemann solutions of the perturbed Aw-Rascle model (1.7) with initial data (1.3).

The system (1.7) has two eigenvalues
\[ \lambda_1 = u - \sqrt{(\gamma - 1)\rho^{\gamma - 1} u}, \quad \lambda_2 = u + \sqrt{(\gamma - 1)\rho^{\gamma - 1} u}, \tag{5.1} \]

with the corresponding right eigenvectors
\[ \varphi_1 = (\rho, -\sqrt{(\gamma - 1)\rho^{\gamma - 1} u})^T, \quad \varphi_2 = (\rho, \sqrt{(\gamma - 1)\rho^{\gamma - 1} u})^T, \]

\[ \text{initial data (1.3).} \]
satisfying $\nabla \lambda_i \cdot \vec{v}_i \neq 0$ ($i = 1, 2$) for $\rho > 0$ and $(\gamma + 1)\sqrt{u} \neq \sqrt{(\gamma - 1)\rho^{\gamma-1}}$. Thus, this system is strictly hyperbolic and both characteristic fields are genuinely nonlinear for $\rho, u > 0$ and $1 < \gamma < 1 + \gamma_2$ where $\gamma_2 > 0$ is sufficiently small, which means the associated waves are either shock waves or rarefaction waves.

Seeking the self-similar solution

$$(\rho, u)(t, x) = (\rho, u)(\xi), \quad \xi = \frac{x}{t},$$

the Riemann problem (1.7) and (1.3) is reduced to the following boundary value problem of the ordinary differential equations:

$$\begin{cases}
-\xi \rho_{\xi} + (\rho u)_{\xi} = 0, \\
-\xi \left(\rho u + \frac{\rho^2}{\gamma}\right)_{\xi} + (\rho u^2 + \rho^\gamma u)_{\xi} = 0,
\end{cases} \tag{5.2}$$

with $(\rho, u)(\pm \infty) = (\rho_{\pm}, u_{\pm})$.

For any smooth solution, system (5.2) can be written as

$$\begin{pmatrix}
\rho \\
\rho_{\xi} \\
\rho_{\xi} - (\rho u + \frac{\rho^2}{\gamma})_{\xi} + (\rho u^2 + \rho^\gamma u)_{\xi}
\end{pmatrix} = 0. \tag{5.3}$$

Besides the constant solution

$$(\rho, u)(\xi) = \text{constant} \quad (\rho > 0),$$

it provides the 1-rarefaction wave

$$R_1(\rho_{--}, u_{--}) : \begin{cases}
\xi = \lambda_1 = u - \sqrt{(\gamma - 1)\rho^{\gamma-1}u}, \\
\sqrt{u} - \sqrt{u_{--}} = -\sqrt{\frac{1}{\gamma - 1}}\rho^{\gamma-1} + \sqrt{\frac{1}{\gamma - 1}\rho^{\gamma-1}}, \\
\rho < \rho_{--}, u > u_{--},
\end{cases} \tag{5.4}$$

or the 2-rarefaction wave

$$R_2(\rho_{--}, u_{--}) : \begin{cases}
\xi = \lambda_2 = u + \sqrt{(\gamma - 1)\rho^{\gamma-1}u}, \\
\sqrt{u} - \sqrt{u_{--}} = \sqrt{\frac{1}{\gamma - 1}}\rho^{\gamma-1} - \sqrt{\frac{1}{\gamma - 1}\rho^{\gamma-1}}, \\
\rho > \rho_{--}, u > u_{--}.
\end{cases} \tag{5.5}$$

Differentiating the second equation of (5.4) with respect to $\rho$ yields

$$u_\rho = -\sqrt{(\gamma - 1)\rho^{\gamma-3}u} < 0,$$

and

$$u_{\rho\rho} = \frac{1}{2}\sqrt{\gamma - 1} \left(\sqrt{\gamma - 1}\rho^{\gamma-3} - (\gamma - 3)\sqrt{\rho^{\gamma-5}u}\right) > 0,$$

where $\gamma \in (1, 3)$, which mean that for $1 < \gamma < 3$, the rarefaction wave curve $R_1(\rho_{--}, u_{--})$ is monotonic decreasing and convex in the $(\rho, u)$ phase plane $(\rho, u > 0)$.

Moreover, by differentiating $\rho$ and $u$ with respect to $\xi$ in the first equation of (5.4) and combining

$$u_\rho = \frac{u_\xi}{\rho_{\xi}} = -\sqrt{(\gamma - 1)\rho^{\gamma-3}u},$$

we have

$$1 = \left(\frac{\gamma + 1}{2} - \frac{\sqrt{(\gamma - 1)\rho^{\gamma-1}}}{2\sqrt{u}}\right)u_{\xi}. \tag{5.6}$$
Hence, as \( \gamma \in (1, 1 + \gamma_0) \) for \( \gamma_0 \) sufficiently small, we have \( u_\xi > 0 \), i.e., the set \((\rho, u)\) which can be joined to \((\rho_-, u_-)\) by 1-rarefaction wave is made up of the half-branch of \( R_1(\rho_-, u_-) \) with \( u \geq u_- \).

With the same way to compute \( R_2(\rho_-, u_-) \), we can gain \( u_\rho > 0 \), \( u_{\rho\rho} < 0 \), and \( u_\xi > 0 \), which means that it is monotonic crossing and concave for \( 1 < \gamma < 3 \) in the \((\rho, u)\) phase plane \((\rho, u > 0)\) and the set \((\rho, u)\) which can be joined to \((\rho_-, u_-)\) by 2-rarefaction wave is made up of the half-branch of \( R_1(\rho_-, u_-) \) with \( u \geq u_- \).

Performing the limit \( \rho \to 0 \) in the second equation in (5.4) yields

\[
\lim_{\rho \to 0} \sqrt{u} = \sqrt{u_-} - \lim_{\rho \to 0} \sqrt{\frac{\rho^{\gamma-1}}{\gamma - 1} + \frac{\rho^{\gamma-1}}{\gamma - 1}} = \sqrt{u_-} + \sqrt{\frac{\rho^{\gamma-1}}{\gamma - 1}}.
\]

Then we have

\[
\lim_{\rho \to 0} u = \left( \sqrt{u_-} + \sqrt{\frac{\rho^{\gamma-1}}{\gamma - 1}} \right)^2 =: u_0^\gamma.
\]

Thus we conclude that there exists \( u_0^\gamma \) such that the 1-rarefaction wave curve \( R_1(\rho_-, u_-) \) intersects the \( u \)-axis at the point \((0, u_0^\gamma)\).

Performing the limit \( \rho \to +\infty \) of the second equation in (5.5) yields

\[
\lim_{\rho \to +\infty} \sqrt{u} = \sqrt{u_-} + \lim_{\rho \to +\infty} \left( \sqrt{\frac{\rho^{\gamma-1}}{\gamma - 1}} - \sqrt{\frac{\rho^{\gamma-1}}{\gamma - 1}} \right) = +\infty,
\]

which implies that \( \lim_{\rho \to +\infty} u = +\infty \).

For a bounded discontinuity at \( \xi = \bar{\rho} \), the Rankine-Hugoniot relation

\[
\begin{cases}
-\bar{\rho}[\rho] + [\rho u] = 0, \\
-\bar{\rho}[\rho u + \frac{1}{2}\rho^2] + [\rho u^2 + u\rho^\gamma] = 0,
\end{cases}
\]

holds, where \([\rho] = \rho - \rho_-\), etc. Eliminating \( \sigma \) from (5.9), we obtain

\[
[\rho][\rho u^2] - ([\rho u])^2 = -[\rho][u\rho^\gamma] + [\rho u][\frac{1}{\gamma}\rho^\gamma].
\]

Simplifying (5.10) yields

\[
(u_- - u_-)^2 = \left( \frac{1}{\rho_-} - \frac{1}{\rho^\gamma} \right) (u\rho^\gamma - u_- \rho^\gamma_-) - \frac{1}{\gamma \rho \rho_-} (\rho u - \rho_- u_-) (\rho^\gamma - \rho^\gamma_-),
\]

i.e.,

\[
(u_- - u_-)^2 = \frac{\rho - \rho_-}{\rho \rho_-} u_- (\rho^\gamma - \rho^\gamma_-) + \frac{\rho - \rho_-}{\rho \rho_-} \rho^\gamma (u_- - u_-) - \frac{1}{\gamma \rho \rho_-} (\rho u - \rho_- u_-) (\rho^\gamma - \rho^\gamma_-) - \frac{1}{\gamma \rho \rho_-} \rho (u_- - u_-) (\rho^\gamma - \rho^\gamma_-).
\]

Therefore,

\[
\left( \frac{u_- - u_-}{\rho - \rho_-} \right)^2 = \left( \frac{\rho - \rho_-}{\gamma \rho \rho_-} \rho^\gamma (\rho - \rho_-) - \frac{1}{\gamma \rho \rho_-} (\rho^\gamma - \rho^\gamma_-) \right) + \frac{u_- - u_-}{\rho - \rho_-} \left( \frac{\rho^{\gamma-1}}{\rho_-} - \frac{1}{\gamma \rho \rho_-} \rho^\gamma (\rho - \rho_-) \right).
\]

Set \( \alpha = \frac{u_- - u_-}{\rho - \rho_-} \). Then (5.12) can be simplified as

\[
\alpha^2 = \left( \frac{\rho^{\gamma-1}}{\rho_-} - \frac{1}{\gamma \rho \rho_-} \rho^\gamma (\rho - \rho_-) \right) \alpha - 1 - \frac{u_- - u_-}{\rho - \rho_-} \left( \frac{\rho^{\gamma-1}}{\rho_-} - \frac{1}{\gamma \rho \rho_-} \rho^\gamma (\rho - \rho_-) \right) = 0.
\]
This is a quadratic form in \(\alpha\) and we can solve this to obtain
\[
\frac{u - u_-}{\rho - \rho_-} = \frac{1}{2\rho_{\rho_-}} \left( \rho^\gamma - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right) \pm \sqrt{\frac{1}{4\rho^2\rho_-} \left( \rho^\gamma - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right)^2 + \left( 1 - \frac{1}{\gamma} \right) \rho_{\rho_-}^{-1} (\rho^\gamma - \rho_-^\gamma)},
\]
where \((\rho_-, u_-)\) and \((\rho, u)\) are the shock speed, the left state and the right state, respectively.

1-shock wave \(S_1(\rho_-, u_-)\):

The classical Lax entropy conditions imply that the propagation speed \(\overline{\sigma}\) for the 1-shock wave has to be satisfied with
\[
\overline{\lambda}_1(\rho, u) < \overline{\sigma} < \overline{\lambda}_1(\rho_-, u_-).
\]
From the first equation of (5.9), we have
\[
\overline{\sigma} = \frac{\rho u - \rho_- u_-}{\rho - \rho_-} = u_- + \frac{\rho}{\rho_-} (u - u_-).
\]
Then, it follows from the right inequality of (5.14) that
\[
\frac{\rho}{\rho - \rho_-} (u - u_-) < -\sqrt{(\gamma - 1) \rho_-^{\gamma-1} u_-} < 0,
\]
which implies that \(u - u_-\) and \(\rho - \rho_-\) have different signs. Similarly, for the left inequality of (5.14), we can gain
\[
\frac{\rho}{\rho - \rho_-} (u - u_-) > -\sqrt{(\gamma - 1) \rho_-^{\gamma-1} u_-}.
\]
Combining (5.15) and (5.16), it is easy to get
\[-\sqrt{(\gamma - 1) \rho_-^{\gamma+1} u} < \frac{\rho_{\rho_-}}{\rho - \rho_-} (u - u_-) < -\sqrt{(\gamma - 1) \rho_-^{\gamma+1} u_-},
\]
which indicates that \(\rho > \rho_- u > u\), and the minus sign is taken in (5.13) for 1-shock wave. Hence given a left state \((\rho_-, u_-)\), the 1-shock wave curve \(S_1(\rho_-, u_-)\) in the phase plane which is the set of states that can be connected on the right by a 1-shock is as follows
\[
u - u_- = \frac{1}{2\rho_{\rho_-}} \left( \rho^\gamma (\rho - \rho_-) - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right) - \sqrt{\frac{1}{4\rho^2\rho_-} \left( \rho^\gamma (\rho - \rho_-) - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right)^2 + \left( 1 - \frac{1}{\gamma} \right) \rho_{\rho_-}^{-1} (\rho^\gamma - \rho_-^\gamma), \quad \rho > \rho_-.
\]

2-shock wave \(S_2(\rho_-, u_-)\):

The propagation speed \(\overline{\sigma}\) for the 2-shock wave should satisfy
\[
\overline{\lambda}_2(\rho, u) < \overline{\sigma} < \overline{\lambda}_2(\rho_-, u_-).
\]
With the similar calculations to the 1-shock wave, we have the the 2-shock curve \(S_2(\rho_-, u_-)\):
\[
u - u_- = \frac{1}{2\rho_{\rho_-}} \left( \rho^\gamma (\rho - \rho_-) - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right) - \sqrt{\frac{1}{4\rho^2\rho_-} \left( \rho^\gamma (\rho - \rho_-) - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right)^2 + \left( 1 - \frac{1}{\gamma} \right) \rho_{\rho_-}^{-1} (\rho^\gamma - \rho_-^\gamma), \quad \rho < \rho_-.
\]
Differentiating \(u\) with respect to \(\rho\) in the second equation of (5.11) gives that for \(\rho > \rho_-\),
\[
\rho_{\rho_-} I_1 u_{\rho} = I_2,
\]
where
\[
I_1 = 2(u - u_-) - \frac{\gamma - 1}{\gamma} \left( \frac{1}{\rho_-} - \frac{\rho}{15 \rho} \right) \rho^\gamma + \frac{1}{\gamma} \left( \rho^{\gamma-1} - \rho_-^{\gamma-1} \right) < 0,
\]
This is a quadratic form in \(\alpha\) and we can solve this to obtain
\[
\frac{u - u_-}{\rho - \rho_-} = \frac{1}{2\rho_{\rho_-}} \left( \rho^\gamma - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right) \pm \sqrt{\frac{1}{4\rho^2\rho_-} \left( \rho^\gamma - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right)^2 + \left( 1 - \frac{1}{\gamma} \right) \rho_{\rho_-}^{-1} (\rho^\gamma - \rho_-^\gamma)},
\]
\[ I_2 = \frac{\gamma - 1}{\gamma \rho} \left( (\rho^\gamma - \rho_-^\gamma) \rho_- u_- + \gamma (\rho - \rho_-) \rho_-^\gamma u_- \right), \]

which gives \( u_\rho < 0 \) for \( \gamma \in (1, 1 + \gamma_0) \) where \( \gamma_0 \) sufficiently small, which indicates that the 1-shock wave curve \( S_1(\rho_-, u_-) \) is monotonic decreasing in the region \( \rho > \rho_- \) in the \((\rho, u)\) phase plane. Moreover, letting \( u = 0 \) in (5.11), it is easy to get

\[ u_- = \frac{1}{\gamma} \left( u_- (\rho_\gamma^{-1} - \rho_-^{-1}) - (\gamma - 1) \left( \frac{1}{\rho_-} - \frac{1}{\rho} \right) \rho_-^\gamma u_- \right). \] (5.21)

Setting

\[ f(\rho) = u_- - \frac{1}{\gamma} \left( u_- (\rho_\gamma^{-1} - \rho_-^{-1}) - (\gamma - 1) \left( \frac{1}{\rho_-} - \frac{1}{\rho} \right) \rho_-^\gamma u_- \right). \]

Then \( f(\rho) f(+\infty) < 0 \), and \( f(\rho) \) is continuous with respect to \( \rho \). Therefore, there exists \( \rho_0 \in (\rho_-, +\infty) \) such that \( f(\rho_0) = 0 \), which implies that the 1-shock wave curve \( S_1(\rho_-, u_-) \) intersects with the \(\rho\)-axis at a point.

Similarly, we can get \( u_\rho > 0 \) for the 2-shock wave for for \( \gamma \in (1, 1 + \gamma_0) \) where \( \gamma_0 \) sufficiently small, which indicates that the 2-shock wave curve \( S_2(\rho_-, u_-) \) is monotonic increasing in the region \( \rho < \rho_- \) in the \((\rho, u)\) phase plane. From (5.19), it is not difficult to check that that \( \lim_{\rho \to 0^+} u = -\infty \) for the 2-shock wave curve \( S'_2(\rho_-, u_-) \), which implies that curve \( S'_2(\rho_-, u_-) \) has the \(u\)-axis as its asymptotic line.

In the \((\rho, u)\) phase plane \((\rho, u \geq 0)\), through a given point \((\rho_-, u_-)\), we draw the elementary wave curves. We find that the elementary wave curves divide the quarter phase plane \((\rho, u \geq 0)\) into five regions, see Fig. 2. According to the right state \((\rho_+, u_+)\) in the different regions, one can construct the unique global solution to the Riemann problem (1.7) and (1.3) as follows:

1. \((\rho_+, u_+) \in I(\rho_-, u_-) : R_1 + R_2;\)
2. \((\rho_+, u_+) \in II(\rho_-, u_-) : S_1 + R_2;\)
3. \((\rho_+, u_+) \in III(\rho_-, u_-) : R_1 + S_2;\)
4. \((\rho_+, u_+) \in IV(\rho_-, u_-) : S_1 + S_2;\)
5. \((\rho_+, u_+) \in V(\rho_-, u_-) : R_1 + \text{Vac} + R_2,\)

where “+” means “followed by”.

![Fig. 2. Curves of elementary waves.](image)
6. Limits of Riemann solutions of (1.7)

In this section, we study the limiting behavior of the Riemann solutions of (1.7) as $\gamma$ goes to one, that is, the formation of delta shock and the vacuum states as $\gamma \to 1$, respectively in the case $u_- > u_+$ and in the case $u_- < u_+$.

6.1. Formation of delta shock wave

In this subsection, we study the formation of $\delta$-shock in the Riemann problem (1.7) and (1.3) when $u_- > u_+$ as $\gamma \to 1$.

Lemma 6.1. If $u_+ < u_-$, then there is a sufficiently small $\gamma_0 > 0$ such that $(\rho_+, u_+) \in IV(\rho_-, u_-)$ as $1 < \gamma < 1 + \gamma_0$.

Proof. If $\rho_+ = \rho_-$, then $(\rho_+, u_+) \in IV(\rho_-, u_-)$ for any $\gamma \in (1, 3)$. Thus, we only need to consider the case $\rho_+ \neq \rho_-$. It can be derived from (5.17) and (5.19) that all possible states $(\rho, u)$ that can be connected to the left state $(\rho_-, u_-)$ on the right by a 1-shock wave $S_1$ or a 2-shock wave $S_2$ should satisfy

\begin{align*}
S_1: u &= u_- + \frac{1}{2\rho_-} \left( \rho^\gamma (\rho - \rho_-) - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right) \\
&\quad - (\rho - \rho_-) \sqrt{\frac{1}{4\rho_+^2 \rho_-^2} \left( \rho^\gamma - \frac{\rho}{\gamma} \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho - \rho_-} \right) \right)^2 + (1 - \frac{1}{\gamma}) \frac{u_-}{\rho_-} \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho - \rho_-} \right), \quad \rho > \rho_-, \quad (6.1)
\end{align*}

\begin{align*}
S_2: u &= u_- + \frac{1}{2\rho_-} \left( \rho^\gamma (\rho - \rho_-) - \frac{\rho}{\gamma} (\rho^\gamma - \rho_-^\gamma) \right) \\
&\quad + (\rho - \rho_-) \sqrt{\frac{1}{4\rho_+^2 \rho_-^2} \left( \rho^\gamma - \frac{\rho}{\gamma} \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho - \rho_-} \right) \right)^2 + (1 - \frac{1}{\gamma}) \frac{u_-}{\rho_-} \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho - \rho_-} \right), \quad \rho < \rho_-. \quad (6.2)
\end{align*}

If $\rho_+ \neq \rho_-$ and $(\rho_+, u_+) \in IV(\rho_-, u_-)$, then from Fig. 1, (6.1) and (6.2), we have

\begin{align*}
\left( -\rho - \rho_- \right) \sqrt{\frac{1}{4\rho_+^2 \rho_-^2} \left( \rho^\gamma - \frac{\rho}{\gamma} \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho + \rho_-} \right) \right)^2 + (1 - \frac{1}{\gamma}) \frac{u_-}{\rho_+ \rho_-} \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho + \rho_-} \right), \quad \rho_+ > \rho_-, \quad (6.3)
\end{align*}

\begin{align*}
\left( \rho_+ - \rho_- \right) \sqrt{\frac{1}{4\rho_+^2 \rho_-^2} \left( \rho^\gamma - \frac{\rho}{\gamma} \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho + \rho_-} \right) \right)^2 + (1 - \frac{1}{\gamma}) \frac{u_-}{\rho_+ \rho_-} \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho + \rho_-} \right), \quad \rho_+ < \rho_-, \quad (6.4)
\end{align*}

which implies that

\begin{align*}
\left( -\frac{1}{2} \rho - \frac{1}{\rho_+} \left( \frac{\rho^\gamma}{\rho_+ - \rho_-} - \frac{\rho_+ (\rho^\gamma - \rho_-^\gamma)}{1 + (\rho_+ - \rho_-)^2} \right) \left( \frac{\rho^\gamma - \rho_-^\gamma}{\rho + \rho_-} \right), \quad \rho_+ > \rho_-, \quad (6.5)
\end{align*}
Since 
\[ \lim_{\gamma \to 1} \left( \sqrt{\frac{1}{4\rho^2 \rho^2} \left( \frac{\rho^2 - \rho \gamma (\rho^2 - \rho^2)}{\rho - p^2} \right)^2} + \left(1 - \frac{1}{\gamma} \right) \frac{u_{-}}{\rho_{+} \rho_{-}} \left( \frac{\rho^2 - \rho \gamma}{\rho_{+} - p^2} \right) \right) \]
\[ \ddot{\rho}_{+} \ddot{\rho}_{-} \left( \frac{\rho^2 - \rho \gamma (\rho^2 - \rho^2)}{\rho_{+} - p^2} \right) \]
\[ \frac{1}{2} \left( \frac{1}{\rho_{+} - p_{-}} \left( \frac{\rho^2 - \rho \gamma (\rho^2 - \rho^2)}{\rho_{+} - p^2} \right) \right) = 0, \]  
(6.6)

it follows that there exists \( \gamma_0 > 0 \) small enough such that, when \( 1 < \gamma < 1 + \gamma_0 \), we have

\[ \sqrt{\frac{1}{4\rho^2 \rho^2} \left( \frac{\rho^2 - \rho \gamma (\rho^2 - \rho^2)}{\rho_{+} - p^2} \right)^2} + \left(1 - \frac{1}{\gamma} \right) \frac{u_{-}}{\rho_{+} \rho_{-}} \left( \frac{\rho^2 - \rho \gamma}{\rho_{+} - p^2} \right) \leq \frac{u_{-} - u_{+}}{\rho_{+} - \rho_{-}}, \]

Then, it is obvious that \((\rho_{+}, u_{+}) \in IV(\rho_{-}, u_{-})\) when \( 1 < \gamma < 1 + \gamma_0 \). The proof is completed. \( \square \)

According to the relation (5.11), for a given state \((\rho_{-}, u_{-})\), the shock curves \(S_1(\rho_{-}, u_{-})\) and \(S_2(\rho_{-}, u_{-})\) can also be expressed as below:

\[ u_{-} - u_{+} = \frac{1}{\gamma} \left( u_{-} - u_{+} \right) \left( \rho^2 \rho_{+} - \rho_{-} \rho_{+} \right) + \left( u_{-} - u_{+} \right) \left( \rho^2 - \rho_{+} \right) \left( \rho^2 - \rho_{+} \right), u_{-} > u_{+}, \]  
(6.7)

with \( \rho > \rho_{-} \) for a 1-shock curve \(S_1(\rho_{-}, u_{-})\), and \( \rho < \rho_{-} \) for a 2-shock curve \(S_2(\rho_{-}, u_{-})\).

When \( 1 < \gamma < 1 + \gamma_0 \), namely \((\rho_{+}, u_{+}) \in IV(\rho_{-}, u_{-})\), suppose that \((\rho_{+}, u_{+})\) is the intermediate state connected with \((\rho_{-}, u_{-})\) by a 1-shock wave \(S_1\) with the speed \(\sigma_1\), and \((\rho_{+}, u_{+})\) by a 2-shock wave \(S_2\) with the speed \(\sigma_2\), then it follows from (6.7) that

\[ u_{+} - u_{-} = \frac{1}{\gamma} \left( u_{+} - u_{-} \right) \left( \rho^2 \rho_{-} + \rho_{-} \rho_{+} \right) + \left( u_{+} - u_{-} \right) \left( \rho^2 + \rho_{-} \right) \left( \rho^2 + \rho_{-} \right), \]  
(6.8)

\[ u_{+} - u_{-} = \frac{1}{\gamma} \left( u_{+} - u_{-} \right) \left( \rho^2 \rho_{-} + \rho_{-} \rho_{+} \right) + \left( u_{+} - u_{-} \right) \left( \rho^2 + \rho_{-} \right) \left( \rho^2 + \rho_{-} \right), \]  
(6.9)

with the shock speed

\[ \sigma_1 = \frac{\rho_{+} u_{+} - \rho_{-} u_{-}}{\rho_{+} - \rho_{-}}, \quad \sigma_2 = \frac{\rho_{+} u_{+} - \rho_{-} u_{-}}{\rho_{+} - \rho_{-}}, \]  
(6.10)

respectively. In this case, the Riemann solution is

\[ (\rho, u)(t, x) = \begin{cases}
(\rho_{-}, u_{-}), & x < \sigma_1 t, \\
(\rho_{+}, u_{+}), & \sigma_1 t < x < \sigma_2 t, \\
(\rho_{+}, u_{+}), & x > \sigma_2 t.
\end{cases} \]  
(6.11)

Based on (6.8) and (6.9), we can get that

\[ u_{-} - u_{+} = \frac{1}{\gamma} \left( u_{-} - u_{+} \right) \left( \rho^2 \rho_{+} - \rho_{-} \rho_{+} \right) + \left( u_{-} - u_{+} \right) \left( \rho^2 + \rho_{-} \right) \left( \rho^2 + \rho_{-} \right), \]  
(6.12)
Then we have the following lemmas.

**Lemma 6.2.** \( \lim_{\gamma \to +\infty} \rho_* = +\infty \), and \( \lim_{\gamma \to 1} (\gamma - 1)\rho_* u_* =: a = \left( \frac{\sqrt{p_- - p_+}}{\sqrt{p_-} + \sqrt{p_+}} (u_- - u_+) \right)^2 \).

**Proof.** Let \( \lim \inf \rho_* = \alpha \), and \( \lim \sup \rho_* = \beta \).

If \( \alpha < \beta \), then by the continuity of \( \rho_*(\gamma) \), there exists a sequence \( \{\gamma_n\}_{n=1}^\infty \subseteq (1, 3) \) such that

\[
\lim_{n \to +\infty} \gamma_n = 1, \quad \text{and} \quad \lim_{n \to +\infty} \rho_*(\gamma_n) = c,
\]

for some \( c \in (\alpha, \beta) \). Then substituting the sequence into the right hand side of (6.12), taking the limit \( n \to +\infty \), and noting \( u_+ < u_* < u_- \) in mind, we have

\[
\lim_{n \to +\infty} \frac{1}{\gamma_n} (\gamma_n - 1) \left( \frac{1}{\rho_+} - \frac{1}{\rho_*} \right) \left( (\rho_*(\gamma_n))^{\gamma_n} (u_* - \rho_+ u_+ + (u_- - u_*) (\rho_*(\gamma_n))^{\gamma_n - 1} - \rho_+^{\gamma_n - 1}) \right) = 0.
\]

(6.13)

Thus, we can obtain from (6.12) that

\[
u_- - u_+ = 0,
\]

which contradicts with the assumption \( u_- > u_+ \). Then we must have \( \alpha = \beta \), which means \( \lim_{\gamma \to 1} \rho_*(\gamma) = \alpha \).

If \( \alpha \in (0, +\infty) \), then we can also get a contradiction when taking limit in (6.12). Hence \( \alpha = 0 \) or \( \alpha = +\infty \). By the condition \( \rho_* > \max\{\rho_-, \rho_+\} \), it is easy to see that \( \lim_{\gamma \to 1} \rho_*(\gamma) = \alpha = +\infty \).

Next taking the limit \( \gamma \to 1 \) in (6.12), we have

\[
u_- - u_+ = \sqrt{\lim_{\gamma \to 1} (\gamma - 1) \rho_*^2 u_* \left( \frac{1}{\rho_-} - \frac{1}{\rho_*} \right) + \lim_{\gamma \to 1} (\gamma - 1) \rho_*^2 u_* \left( \frac{1}{\rho_+} - \frac{1}{\rho_*} \right) =: \left( \frac{1}{\rho_-} + \frac{1}{\rho_+} \right) \sqrt{a},
\]

from which we can get \( a = \left( \frac{\sqrt{p_- - p_+}}{\sqrt{p_-} + \sqrt{p_+}} (u_- - u_+) \right)^2 \). The proof is completed. \( \square \)

**Lemma 6.3.**

\[
\lim_{\gamma \to 1} \overline{\sigma}_1 = \lim_{\gamma \to 1} \overline{\sigma}_2 = \lim_{\gamma \to 1} u_* = \sigma, \quad (6.14)
\]

and

\[
\lim_{\gamma \to 1} \int_{\overline{\sigma}_1}^{\overline{\sigma}_2} \rho_* d\xi = \sigma [p] - [pu], \quad (6.15)
\]

where \( \sigma = \frac{\sqrt{u_- - u_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}} \).

**Proof.** From (6.8)-(6.10) and Lemma 6.2, we immediately get

\[
\lim_{\gamma \to 1} u_* = u_- - \lim_{\gamma \to 1} \frac{1}{\gamma} \left( \gamma - 1 \right) \left( \frac{1}{\rho_-} - \frac{1}{\rho_*} \right) \left( \rho_*^2 u_* - \rho_*^2 u_- + (u_- - u_*) (\rho_*^{\gamma_n - 1} - \rho_+^{\gamma_n - 1}) \right)
\]

\[
= u_- - \sqrt{\frac{\alpha}{\rho_-}} = u_- - \frac{\sqrt{p_- - p_+} (u_- - u_+)}{\sqrt{\rho_-} (\sqrt{\rho_-} + \sqrt{\rho_+})} = \sigma,
\]

\[
\lim_{\gamma \to 1} \overline{\sigma}_1 = \lim_{\gamma \to 1} \frac{\rho_* u_* - \rho_- u_+}{\rho_* - \rho_-} = u_+ + \lim_{\gamma \to 1} \frac{\rho_* (u_- - u_*)}{\rho_- - \rho_*} = u_- - \frac{\sqrt{p_- - p_+} (u_- - u_+)}{\sqrt{\rho_-} (\sqrt{\rho_-} + \sqrt{\rho_+})} = \sigma.
\]
and
\[
\lim_{\gamma \to 1} \sigma_2 = \lim_{\gamma \to 1} \frac{\rho_+ u_+ - \rho_* u_*}{\rho_+ - \rho_*} = u_+ + \lim_{\gamma \to 1} \frac{\rho_*}{\rho_+ - \rho_*} (u_+ - u^\star) = u_+ + \sqrt{\frac{a}{\rho_+}} u_+ + \frac{\sqrt{\rho_+ - \rho_*}}{\sqrt{\rho_+ (\rho_+ + \sqrt{\rho_+})}} = \sigma.
\]

From the first equations of the Rankine-Hugoniot relation (5.9) for \( S_1 \) and \( S_2 \), we have
\[
\bar{\sigma}_1 (\rho_+ - \rho_1) = \rho_1 - \rho_+ u^\star,
\]
and
\[
\bar{\sigma}_2 (\rho_+ - \rho_1) = \rho_1 u_+ - \rho_+ u^\star.
\]
By (6.14), (6.16) and (6.17), we get
\[
\lim_{\gamma \to 1} \rho_1 (\bar{\sigma}_2 - \bar{\sigma}_1) = \lim_{\gamma \to 1} (\rho_1 - \bar{\sigma}_1 + \bar{\sigma}_2) \rho_1 - (\rho_1 - \rho_1 u_+ + \rho_1 u_+) = \sigma[\rho] - \lfloor \rho u \rfloor,
\]
which implies that
\[
\lim_{\gamma \to 1} \int \bar{\sigma}_2 \rho_1 d\xi = \sigma[\rho] - \lfloor \rho u \rfloor.
\]

The proof is completed. \( \Box \)

**Remark 6.1.** Lemmas 6.2-6.3 show that when \( \gamma \) tends to one, the two shock curves \( S_1 \) and \( S_2 \) coincide to form a new delta shock wave, and the delta shock wave speed \( \sigma \) is the limit of both the particle velocity \( u_\ast \) and two shocks’ speed \( \bar{\sigma}_1, \bar{\sigma}_2 \). What is more, the intermediate density \( \rho_\ast \) tend to singular as \( \gamma \to 1 \).

What is more, we will further derive that, when \( \gamma \to 1 \), the limit of Riemann solutions of (1.7) with the Riemann initial data (1.3) under the assumption \( u_+ < u_- \) is a delta shock wave solution of the zero pressure gas dynamics (1.5) with the same Riemann initial data \( (\rho_\ast, u_\ast) \) in the sense of distributions.

**Theorem 6.4.** Let \( u_+ < u_- \). For any fixed \( \gamma \in (1, 3) \), assume that \( (\rho_\gamma(t,x), m_\gamma(t,x)) = (\rho_\gamma(t,x), \rho_\gamma(t,x) u_\gamma(t,x)) \) is a Riemann solution containing two shocks \( S_1 \) and \( S_2 \) of (1.7) with the Riemann initial data (1.3) constructed in Section 5. Then, as \( \gamma \to 1 \), \( (\rho_\gamma(t,x), m_\gamma(t,x)) \) will converge to
\[
(\rho(t,x), m(t,x)) = (\rho_0(t,x) + w_1(t) \delta_0, \rho_0(t,x) u_0(t,x) + w_2(t) \delta_0),
\]
in the sense of distributions, and the singular parts of the limit functions \( \rho(t,x) \) and \( m(t,x) \) are a \( \delta \)-measure with weights
\[
w_1(t) = t(\sigma[\rho] - \lfloor \rho u \rfloor), \ \text{and} \ \ w_2(t) = t(\sigma[\rho u] - \lfloor \rho u^2 \rfloor),
\]
respectively, which form a delta shock solution of (1.5) with the same Riemann data (1.3). Here
\[
\sigma = \frac{\sqrt{\rho_0} - \sqrt{\rho_+}}{\sqrt{\rho_+} + \sqrt{\rho_\gamma}}.
\]
**Proof.** (1) Set \( \xi = \frac{x}{\gamma} \). Then for any fixed \( \gamma \in (1, 3) \), the Riemann solution containing two shocks \( S_1 \) and \( S_2 \) of (1.7) with the Riemann initial data (1.3) can be written as
\[
(\rho_\gamma, u_\gamma)(\xi) = \begin{cases} 
(\rho_-, u_-), & \xi < \bar{\sigma}_1, \\
(\rho_-, u_\ast), & \bar{\sigma}_1 < \xi < \bar{\sigma}_2, \\
(\rho_\gamma, u_\gamma), & \xi > \bar{\sigma}_2.
\end{cases}
\]
From (5.2), we have the following weak formulations:

\[ \int_{-\infty}^{+\infty} \rho_{\gamma}(\xi)(u_{\gamma}(\xi) - \xi)\varphi'(\xi) d\xi - \int_{-\infty}^{+\infty} \rho_{\gamma}(\xi)\varphi(\xi) d\xi = 0, \]

for any \( \varphi(\xi) \in C_{0}^{+\infty}(R). \)

(2) For the first integral on the left-hand side of (6.19), using the method of integration by parts, we can derive

\[ \int_{-\infty}^{+\infty} \rho_{\gamma}(\xi)(u_{\gamma}(\xi) - \xi)\varphi'(\xi) d\xi = \left( \int_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \right) \rho_{\gamma}(\xi)(u_{\gamma}(\xi) - \xi)\varphi'(\xi) d\xi \]

\[ = \rho_{\gamma}u_{\gamma}(\sigma_{1}) - \rho_{\gamma}u_{\gamma}(\sigma_{2}) - \rho_{\gamma}u_{\gamma}(\sigma_{1})\sigma_{1} - \rho_{\gamma}u_{\gamma}(\sigma_{2}) + \int_{-\infty}^{+\infty} \rho_{\gamma}\varphi(\xi) d\xi \]

\[ + \int_{-\infty}^{+\infty} \rho_{\gamma}\varphi(\xi) d\xi + \int_{-\infty}^{+\infty} \rho_{\gamma}(u_{\gamma} - \xi)\varphi'(\xi) d\xi \]

Meanwhile, we have

\[ \int_{-\infty}^{+\infty} \rho_{\gamma}(u_{\gamma} - \xi)\varphi'(\xi) d\xi = \rho_{\gamma}(u_{\gamma}(\sigma_{2} - \sigma_{1}) - \rho_{\gamma}(\sigma_{2} - \sigma_{1})) + \int_{-\infty}^{+\infty} \rho_{\gamma}\varphi(\xi) d\xi \]

\[ = \rho_{\gamma}(\sigma_{2} - \sigma_{1}) \left( u_{\gamma}(\sigma_{2} - \sigma_{1}) - \int_{-\infty}^{+\infty} \frac{\rho_{\gamma}\varphi(\xi) d\xi}{\sigma_{2} - \sigma_{1}} \right). \]

Then, by Lemma 6.2-6.3, we can obtain

\[ \lim_{\gamma \to 1} \int_{-\infty}^{+\infty} \rho_{\gamma}(u_{\gamma} - \xi)\varphi'(\xi) d\xi = 0. \]

Hence taking the limit \( \gamma \to 1 \) in (6.19) leads to

\[ \lim_{\gamma \to 1} \int_{-\infty}^{+\infty} (\rho_{\gamma}(\xi) - \rho_{0}(\xi))\varphi(\xi) d\xi = (\sigma[\rho] - [\rho u])\varphi(\sigma), \]

where \( (\rho_{0}(\xi), u_{0}(\xi)) = (\rho_{\pm}, u_{\pm}), \quad (\xi - \sigma) > 0. \)

(3) Similarly, we can obtain for (6.20) that

\[ \lim_{\gamma \to 1} \int_{-\infty}^{+\infty} \rho_{\gamma}(\xi)(u_{\gamma}(\xi) - \xi)\varphi'(\xi) d\xi \]

\[ = \left( \sigma[\rho u] - [\rho u^{2}] \right)\varphi(\sigma) + \int_{-\infty}^{+\infty} \rho_{0}(\xi)u_{0}(\xi)\varphi(\xi) d\xi, \]

and

\[ \int_{-\infty}^{+\infty} (\rho_{\gamma}(\xi))^{\gamma}(u_{\gamma}(\xi) - \frac{1}{\gamma} \xi)\varphi'(\xi) d\xi = \left( \int_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \right) (\rho_{\gamma}(\xi))^{\gamma} \left( u_{\gamma}(\xi) - \frac{1}{\gamma} \xi \right) \varphi'(\xi) d\xi \]

\[ = \rho_{\gamma}u_{\gamma}(\sigma_{1}) - \rho_{\gamma}u_{\gamma}(\sigma_{2}) - \frac{1}{\gamma} \rho_{\gamma}u_{\gamma}^{2}(\sigma_{1}) + \frac{1}{\gamma} \rho_{\gamma}u_{\gamma}^{2}(\sigma_{2}) + \int_{-\infty}^{+\infty} \rho_{0}(\xi)\varphi(\xi) d\xi. \]
which converges to

\[(\sigma[\rho] - [p\upsilon])\varphi(\sigma) + \int_{-\infty}^{+\infty} \rho_0(\xi)\varphi(\xi) d\xi\]

by Lemma 6.2-6.3.

Thus, from (6.21), we can get

\[
\lim_{\gamma \to 1} \int_{-\infty}^{+\infty} (\rho_\gamma(\xi) - \rho_0(\xi))\varphi(\xi, t) d\xi = \lim_{\gamma \to 1} \int_{-\infty}^{+\infty} (\rho_\gamma(x/t) - \rho_0(x/t))\varphi(x/t, t) dx/t\]

\[
= \frac{1}{t} \lim_{\gamma \to 1} \int_{-\infty}^{+\infty} (\rho_\gamma(t, x) - \rho_0(t, x))\varphi(t, x) dx = (\sigma[\rho] - [p\upsilon])\varphi(t, \sigma t).\]

Then multiplying (6.23) by t and taking integration, we have

\[
\lim_{\gamma \to 1} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} (\rho_\gamma(t, x) - \rho_0(t, x))\varphi(t, x) dx dt = \int_{0}^{+\infty} t(\sigma[\rho] - [p\upsilon])\varphi(t, \sigma t) dt
\]

in which by definition (2.3), we have

\[
\int_{0}^{+\infty} t(\sigma[\rho] - [p\upsilon])\varphi(t, \sigma t) dt = \langle w_1(\cdot) \delta_S, \varphi(\cdot, \cdot) \rangle.
\]

where

\[
w_1(t) = t(\sigma[\rho] - [p\upsilon]).
\]

In the same way, we can derive from (6.22) that

\[
\lim_{\gamma \to 1} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} (\rho_\gamma(t, x)u_\gamma(t, x) - \rho_0u_0(t, x))\varphi(t, x) dx dt = \langle w_2(\cdot) \delta_S, \varphi(\cdot, \cdot) \rangle.
\]

where

\[
w_2(t) = t(\sigma[\rho u] - [p\upsilon^2]).
\]

The proof is completed. □

7. Numerical results

In this section, we use the fifth-order weighted essentially non-oscillatory scheme and third-order Runge-Kutta method [12, 27] with the mesh 400 points to present some groups of representative numerical results for the Aw-Rascle traffic model (1.1)-(1.2) and the perturbed Aw-Rascle model (1.7) as \( \gamma \) decreases. A number of iterative numerical trials are executed to guarantee what we demonstrate are not numerical objects. The numerical simulations are consistent with the theoretical analysis.

7.1. Formation of delta-shocks in (1.1)-(1.2)
The numerical simulations are corresponding to the theoretical analysis in Section 4. When 
\((\rho_+, u_+) \in I(\rho_-, u_-)\), we take the initial data as follows:

\[
(\rho, u)(0, x) = \begin{cases} 
(3.5, 6), & x < 0, \\
(2, 4), & x > 0,
\end{cases}
\] (7.1)

and compute the solution of the Riemann problem of (1.1)-(1.2) up to \(t = 0.4\), the numerical simulations 
for different choices of \(\gamma\), starting with \(\gamma=0.6\), then \(\gamma=0.3\), and finally \(\gamma=0.01\), are presented in Figs. 3-5 
which show the process of concentration and formation of the delta shock wave in vanishing adiabatic exponent limit of solutions containing a shock wave and a contact discontinuity.

![Fig. 3. Density (left) and velocity (right) for \(\gamma = 0.6\).](image)

![Fig. 4. Density (left) and velocity (right) for \(\gamma = 0.3\).](image)
From these numerical results, we can clearly observe that, when \( \gamma \) decreases, the locations of the shock wave and contact discontinuity become closer and closer, and the density of the intermediate state increases dramatically, while the velocity becomes a piecewise constant function. In the end, as \( \gamma \to 0 \), along with the intermediate state, the shock wave and the contact discontinuity coincide to form a delta-shock, while the velocity keeps a step function. The numerical simulations are in complete agreement with the theoretical analysis in Section 4.

### 7.2. Formation of delta-shocks in (1.7)

The numerical simulations are corresponding to the theoretical analysis in Section 6. When \((\rho_+, u_+) \in S_1 S_2 (\rho_-, u_-)\), we take the initial data as follows:

\[
(\rho, u)(0,x) = \begin{cases} 
(3,4), & x < 0, \\
(2.5,2), & x > 0, 
\end{cases}
\]

(7.2)

and compute the solution of the Riemann problem of (1.7) up to \( t = 0.4 \), the numerical simulations for different choices of \( \gamma \), starting with \( \gamma = 1.4 \), then \( \gamma = 1.04 \), and finally \( \gamma = 1.001 \), are presented in Figs. 6-8 which show the process of concentration and formation of the delta shock wave in the pressureless limit of solutions containing two shocks.

Fig. 6. Density (left) and velocity (right) for \( \gamma = 1.4 \).
From these numerical results, we can clearly observe that, as $\gamma$ decreases, the locations of the two shocks become closer and closer, and the density of the intermediate state increases dramatically, while the velocity becomes a piecewise constant function. In the end, as $\gamma \to 1$, along with the intermediate state, the two shocks coincide to form the delta shock wave of the zero pressure gas dynamics (1.5), while the velocity keeps a step function. The numerical simulations are in complete agreement with the theoretical analysis in Section 6.
References

[1] A. Aw, M. Rascle, Resurrection of “second order” models of traffic flow, SIAM J. Appl. Math. 60 (2000) 916-938.

[2] F. Bouchut, On zero pressure gas dynamics, in: Advances in Kinetic Theory and Computing, in: Ser. Adv. Math. Appl. Sci., vol. 22, World Scientific Publishing, River Edge, NJ, 1994, pp. 171-190.

[3] Y. Brenier, E. Grenier, Sticky particles and scalar conservation laws, SIAM J. Numer. Anal. 35 (1998) 2317-2328.

[4] G.Q. Chen, H. Liu, Formation of $\delta$-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids, SIAM J. Math. Anal. 34 (2003) 925-938.

[5] G.-Q. Chen, H. Liu, Concentration and cavitation in the vanishing pressure limit of solutions to the Euler equations for nonisentropic fluids, Phys. D 189 (2004) 141-165.

[6] C. Daganzo, Requiem for second order fluid approximations of traffic flow, Transportation Res. Part B 29 (1995) 277-286.

[7] W. E, Yu.G. Rykov, Ya.G. Sinai, Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics, Comm. Math. Phys. 177 (1996) 349-380.

[8] S. Ha, F. Huang, and Y. Wang, A global unique solvability of entropic weak solution to the one-dimensional pressureless Euler system with a flocking dissipation, J. Differ. Equations 257 (2014) 1333-1371.

[9] F. Huang, Z. Wang, Well posedness for pressureless flow, Comm. Math. Phys. 222 (2001) 117-146.

[10] M. Ibrahim, F. Liu, S. Liu, Concentration of mass in the pressureless limit of Euler equations for power law, Nonlinear Anal. Real World Appl. 47 (2019) 224-235.

[11] K.T. Joseph, A Riemann problem whose viscosity solutions contain $\delta$-measures, Asymptot Anal. 7(1993) 105-120.

[12] A. Kurganov, E. Tadmor, New high-resolution central schemes for nonlinear conservation laws and convection diffusion equations, J. Comput. Phys. 160 (2000) 241-282.

[13] J. Lebacque, S. Mammar, and H. Salem, The Aw-Rascle and Zhangs model: Vacuum problems, existence and regularity of the solutions of the Riemann problem, Transp. Res. Part B 41 (2007) 710-721.

[14] J. Li, Note on the compressible Euler equations with zero temperature, Appl. Math. Lett. 14 (2001) 519-523.

[15] J. Li, H. Yang, Delta-shocks as limits of vanishing viscosity for multidimensional zero-pressure gas dynamics, Quart. Appl. Math. 59 (2) (2001) 315-342.

[16] J. Li, T. Zhang, S. Yang, The two-dimensional Riemann problem in gas dynamics, Vol. 98 of Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman, Harlow, 1998.
[17] H. Li, Z. Shao, Delta shocks and vacuum states in vanishing pressure limits of solutions to the relativistic Euler equations for generalized Chaplygin gas, Commun. Pure Appl. Anal. 15 (2016) 2373-2400.

[18] J. Liu, W. Xiao, Flux approximation to the Aw-Rascle model of traffic flow, Journal of Mathematical Physics 59, 101508 (2018); doi: 10.1063/1.5063469.

[19] D. Mitrovic, M. Nedeljkov, Delta-shock waves as a limit of shock waves, J. Hyperbolic Differ. Equ. 4 (2007) 629-653.

[20] L. Pan, X. Han, The Aw-Rascle traffic model with Chaplygin pressure, J. Math. Anal. Appl. 401 (2013) 379-387.

[21] S.F. Shandarin, Ya.B. Zeldovich, The large-scale structure of the universe: turbulence, intermittency, structures in a self-gravitating medium, Rev. Modern Phys. 61 (1989) 185-220.

[22] C. Shen, The limits of Riemann solutions to the isentropic magnetogasdynamics, Appl. Math. Lett. 24 (2011) 1124-1129.

[23] C. Shen, M. Sun, Formation of delta shocks and vacuum states in the vanishing pressure limit of Riemann solutions to the perturbed Aw-Rascle model, J. Differential Equations 249 (2010) 3024-3051.

[24] C. Shen, M. Sun, Z. Wang, Limit relations for three simple hyperbolic systems of conservation laws, Math. Meth. Appl. Sci. 33 (2010) 1317-1330.

[25] W. Sheng, G. Wang, G. Yin, Delta wave and vacuum state for generalized Chaplygin gas dynamics system as pressure vanishes, Nonlinear Anal. Real World Appl. 22 (2015) 115-128.

[26] W. Sheng, T. Zhang, The Riemann problem for the transportation equations in gas dynamics, in: Mem. Amer. Math. Soc., 137, AMS, Providence, 1999.

[27] C. W. Shu, Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws, in Advanced Numerical Approximation of Nonlinear Hyperbolic Equations, Lecture Notes in Mathematics Vol. 1697 (Springer Berlin Heidelberg, 1998), pp. 325-432.

[28] M. Sun, Interactions of elementary waves for the Aw-Rascle model, SIAM J. Appl. Math. 69 (2009) 1542-1558.

[29] G. Yin, W. Sheng, Delta shocks and vacuum states in vanishing pressure limits of solutions to the relativistic Euler equations for polytropic gases, J. Math. Anal. Appl. 355 (2009) 594-605.

[30] H. Zhang, A non-equilibrium traffic model devoid of gas-like behavior, Transportation Res. Part B 36 (2002) 275-290.