Lower Bounds for Eigenvalues of Elliptic Operators: By Nonconforming Finite Element Methods

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Abstract The paper is to introduce a new systematic method that can produce lower bounds for eigenvalues. The main idea is to use nonconforming finite element methods. The conclusion is that if local approximation properties of nonconforming finite element spaces are better than total errors (sums of global approximation errors and consistency errors) of nonconforming finite element methods, corresponding methods will produce lower bounds for eigenvalues. More precisely, under three conditions on continuity and approximation properties of nonconforming finite element spaces we analyze abstract error estimates of approximate eigenvalues and eigenfunctions. Subsequently, we propose one more condition and prove that it is sufficient to guarantee nonconforming finite element methods to produce lower bounds for eigenvalues of symmetric elliptic operators. We show that this condition hold for most low-order nonconforming finite elements in literature. In addition, this condition provides a guidance to modify known nonconforming elements in literature and to propose new nonconforming elements. In fact, we enrich locally the Crouzeix-Raviart element such that the new element satisfies the condition; we also propose a new nonconforming element for second order elliptic operators and prove that it will yield lower bounds for eigenvalues. Finally, we prove the saturation condition for most nonconforming elements.

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1 Introduction

Finding eigenvalues of partial differential operators is important in the mathematical science. Since exact eigenvalues are almost impossible, many papers and books investigate their bounds from above and below. It is well known that the variational principle (including conforming finite element methods) provides upper bounds. But the problem of obtaining lower bounds is generally considerably more difficult. Moreover, a simple combination of lower and upper bounds will produce intervals to which exact eigenvalue belongs. This in turn gives reliable a posteriori error estimates of approximate eigenvalues, which is essential for the design of the coefficient of safety in practical engineering. Therefore, it is a fundamental problem to achieve lower bounds for eigenvalues of elliptic operators. In fact, the study of lower bounds for eigenvalues can date back to remarkable works of Forsythe [17,18] and Weiberger [39,40], where lower bounds of eigenvalues are derived by finite difference methods for second order elliptic eigenvalue problems. Since that finite difference methods in some sense coincide with standard linear finite element methods with mass lumping, one could expect that finite element methods with mass lumping give lower bounds for eigenvalues of operators, we refer interested readers to Armentano and Duran [1], Hu et al. [20] for this aspect.

Nonconforming finite element methods are alternative possible ways to produce lower bounds for eigenvalues of operators. In deed, the lower bound property of eigenvalues by nonconforming elements are observed in numerics, see, Zienkiewicz et al. [49], for the nonconforming Morley element, Rannacher [33], for the nonconforming Morley and Adini elements, Liu and Yan [31], for the nonconforming Wilson [36,42], $EQ_{1}^{\text{rot}}$ [29], and $Q_{1}^{\text{rot}}$ [34] elements. See, Boffi [7], for further remarks on possible properties of discrete eigenvalues produced by nonconforming methods.

However, there are a few results to study the lower bound property of eigenvalues by nonconforming elements. One early result in this direction is analyzed in a remarkable paper by Armentano and Duran [2] for the Laplacian operator. The analysis is based on an identity for errors of eigenvalues. It is proved that the nonconforming linear element of Crouzeix and Raviart [13] leads to lower bounds for eigenvalues provided that eigenfunctions $u \in H^{1+r}(\Omega) \cap H^0_0(\Omega)$ with $0 < r < 1$. The idea is generalized to the enriched nonconforming rotated $Q_1$ element of Lin et al. [29] in Li [25], and to the Wilson element in Zhang et al. [48]. See Yang et al. [46] for a survey of earlier works. The extension to the Morley element can be found in Yang et al. [47]. However, all of those papers are based on the saturation condition of approximations by piecewise polynomials for which a rigorous proof is missed in literature. We refer interested readers to Lin et al. [27,28], Lin and Lin [30], Yang [43], Yang et al. [46], Zhang et al. [48] for expansion methods based on superconvergence or extrapolation, which analyzes the lower bound property of eigenvalues by nonconforming elements on uniform rectangular meshes.

The aim of our paper is to introduce a new systematic method that can produce lower bounds for eigenvalues. The main idea is to use nonconforming finite element methods. However, some numerics from the literature demonstrate that some nonconforming elements produce upper bounds of eigenvalues though some other nonconforming elements yield lower bounds, see Liu and Yan [31], Rannacher [33]. We find that the general condition lies...
in that local approximation properties of nonconforming finite element spaces $V_h$ should be better than total errors (sums of global approximation errors and consistency errors) of nonconforming finite element methods. Then corresponding nonconforming methods will produce lower bounds for eigenvalues of elliptic operators. More precisely, first, we shall analyze errors of discrete eigenvalues and eigenfunctions. Second, we shall propose a condition on nonconforming element methods and then under the saturation condition prove that it is sufficient for lower bounds for eigenvalues. With this result, to obtain lower bound for eigenvalue is to design nonconforming element spaces with enough local degrees of freedom when compared to total errors. This in fact results in a systematic method to approximate eigenvalues from below. As one application of the theory, we check that this condition holds for most used nonconforming elements, e.g., the Wilson element [36,42], the nonconforming linear element by Crouzeix and Raviart [13], the nonconforming rotated $Q_1$ element by Rannacher and Turek [34], Shi and Wang [36], and the enriched nonconforming rotated $Q_1$ element by Lin et al. [29] for second order elliptic operators, the Morley element [32,36] and the Adini element [24,36] for fourth order elliptic operators, and the Morley-Wang-Xu element [38] for $2m$th order elliptic operators. As another important application, we follow this guidance to enrich locally the Crouzeix-Raviart element such that the new element satisfies the sufficient condition. In the final application, we propose a new nonconforming element method for second order elliptic operators and show that it actually produces lower bounds for eigenvalues. As an indispensable and important part of the paper, we prove the saturation condition for most of these nonconforming elements.

The paper is organized as follows. In the following section, we shall present symmetric elliptic eigenvalue problems and their nonconforming element methods in an abstract setting. In Sect. 3, based on three conditions on discrete spaces, we analyze error estimates for both discrete eigenvalues and eigenfunctions. In Sect. 4, under one more condition, we prove an abstract result that eigenvalues produced by nonconforming methods are smaller than exact ones. In Sects. 5–6, we check these conditions for various nonconforming methods in literature and we also propose two new nonconforming methods that admit lower bounds for eigenvalues in Sect. 7. In Sect. 8, we analyze the saturation condition for piecewise polynomial approximations. We end this paper by Sect. 9 where we give some comments, which is followed by the appendix where we give comments on the saturation condition when eigenfunctions are singular.

2 Eigenvalue Problems and Nonconforming Finite Element Methods

Let $V \subset H^m(\Omega)$ denote some standard Sobolev space on some bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$ with a piecewise flat boundary $\partial \Omega$. $2m$th order elliptic eigenvalue problems read:

\begin{equation}
(\lambda, u) \in \mathbb{R} \times V \text{ such that }
\end{equation}

\begin{equation}
\begin{array}{l}
a(u, v) = \lambda(\rho u, v)_{L^2(\Omega)} \text{ for any } v \in V \text{ and } \|\rho^{1/2}u\|_{L^2(\Omega)} = 1, \\
with a positive function $\rho \in L^\infty(\Omega)$. The bilinear form $a(u, v)$ is symmetric, bounded, and coercive in the following sense:
\end{array}
\end{equation}

\begin{equation}
\begin{array}{l}
a(u, v) = a(v, u), |a(u, v)| \lesssim \|u\|_V \|v\|_V, \text{ and } \|v\|_V^2 \lesssim a(v, v) \text{ for any } u, v \in V,
\end{array}
\end{equation}

with the norm $\|\cdot\|_V$ over the space $V$. Throughout the paper, an inequality $A \lesssim B$ replaces $A \leq C B$ with some multiplicative mesh-size independent constant $C > 0$ that depends only on the domain $\Omega$, and the shape (e.g., through the aspect ratio) of elements. Finally, $A \approx B$ abbreviates $A \lesssim B \lesssim A$. Springer
Under the conditions (2.2), we have that the eigenvalue problem (2.1) has a sequence of eigenvalues

\[ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \not\to +\infty, \]

and corresponding eigenfunctions

\[ u_1, u_2, u_3, \ldots, \]

which can be chosen to satisfy

\[ (\rho u_i, u_j)_{L^2(\Omega)} = \delta_{ij}, \quad i, j = 1, 2, \ldots. \]

We define

\[ E_\ell = \text{span}\{u_1, u_2, \ldots, u_\ell\}. \]

Then, eigenvalues and eigenfunctions satisfy the following well-known minimum-maximum principle:

\[ \lambda_k = \min_{\dim V_k = k, V_k \subset V} \max_{v \in V_k} \frac{a(v, v)}{\| (\rho v, v)_{L^2(\Omega)} \|} = \max_{u \in E_k} \frac{a(\rho u, u)}{\| (\rho u, u)_{L^2(\Omega)} \|}. \tag{2.3} \]

For any eigenvalue \( \lambda \) of (2.1), we define

\[ M(\lambda) := \{ u : u \text{ is an eigenfunction of (2.1) to } \lambda \}. \tag{2.4} \]

We shall be interested in approximating the eigenvalue problem (2.1) by finite element methods. To this end, we suppose we are given a discrete space \( V_h \) defined over a regular triangulation \( T_h \) of \( \Omega \) into (closed) simplexes or \( n \)-rectangles [9].

We need the piecewise counterparts of differential operators with respect to \( T_h \). For any differential operator \( \mathcal{L} \), we define its piecewise counterpart \( \mathcal{L}_h \) in the following way: we suppose that \( v_K \) is defined over \( K \in T_h \) and that the differential action \( \mathcal{L} v_K \) is well-defined on \( K \) which is denoted by \( \mathcal{L}_K v_K \) for any \( K \in T_h \); then we define \( v_h \) by \( v_h|_K = v_K \) where \( v_h|_K \) denotes its restriction of \( v_h \) over \( K \); finally we define \( \mathcal{L}_h v_h \) by \( (\mathcal{L}_h v_h)|_K = \mathcal{L}_K v_K \).

We consider the discrete eigenvalue problem: Find \((\lambda_h, u_h) \in \mathbb{R} \times V_h\) such that

\[ a_h(u_h, v_h) = \lambda_h (\rho u_h, v_h)_{L^2(\Omega)} \;
\]

for any \( v_h \in V_h \) and \( \| \rho^{1/2} u_h \|_{L^2(\Omega)} = 1 \). \tag{2.5}

Here and throughout of this paper, \( a_h(\cdot, \cdot) \) is the piecewise counterpart of the bilinear form \( a(\cdot, \cdot) \) where differential operators are replaced by their discrete counterparts. Conditions on the approximation and continuity properties of discrete spaces \( V_h \) are assumed as follows, respectively.

(H1) \( \| \cdot \| := a_h(\cdot, \cdot)^{1/2} \) defines a norm over discrete spaces \( V_h \).

(H2) Discrete spaces \( V_h \) have the following approximation properties:

\[ \inf_{v_h \in V_h} \| v - v_h \|_h \lesssim h^{r_a} |v|_{H^{m+r_a}(\Omega)}, \]

for any \( v \in V \cap H^{m+r_a}(\Omega) \) with \( 0 < r_a \leq 1 \).

(H3) Consistency errors of nonconforming finite element methods have the following convergence:

\[ \sup_{0 \neq v_h \in V_h} \frac{a_h(v, v_h) - (A v, v_h)_{L^2(\Omega)}}{\| v_h \|_h} \lesssim h^{r_c} |v|_{H^{m+r_c}(\Omega)}, \]

for any \( v \in V \cap H^{m+r_c}(\Omega) \) with \( 0 < r_c \leq r_a \leq 1 \). Let \( V' \) be the continuous functional space of \( V \). Differential operators \( A \) are defined as: Given \( u \in V \), \( Au \in V' \) reads

\[ \langle Au, v \rangle = a(u, v) \text{ for any } v \in V. \tag{2.6} \]
To assure the lower bound property of discrete eigenvalues, we need one more condition.

(H4) Let \( u \) and \( u_h \) be eigenfunctions of problems (2.1) and (2.5), respectively. Assume that there exists an interpolation \( \Pi hu \in V_h \) with the following properties:

\[
\begin{align*}
    a_h(u - \Pi hu, u_h) &= 0, \\
    \|\rho^{1/2}u\|_{L^2(\Omega)}^2 - \|\rho^{1/2}\Pi hu\|_{L^2(\Omega)}^2 &\lesssim h^{2r_c + \Delta r_a}, \\
    \|\rho^{1/2}(\Pi hu - u)\|_{L^2(\Omega)} &\lesssim h^{r_a + \Delta r_a},
\end{align*}
\]

provided that \( u \in V \cap H^{m+r_a}(\Omega) \) for two constants \( \Delta r_c > 0 \) and \( \Delta r_a \geq 0 \) with \( r_a - r_c + \Delta r_a > 0 \).

Let \( N = \dim V_h \). Under the condition (H1), the discrete problem (2.5) admits a sequence of discrete eigenvalues

\[ 0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{N,h}, \]

and corresponding eigenfunctions

\[ u_{1,h}, u_{2,h}, \ldots, u_{N,h}. \]

In the case where \( V_h \) is a conforming approximation in the sense \( V_h \subset V \), it immediately follows from the minimum-maximum principle (2.3) that

\[ \lambda_k \leq \lambda_{k,h}, k = 1, 2, \ldots, N, \]

which indicates that \( \lambda_{k,h} \) is an approximation above \( \lambda_k \).

We define the discrete counterpart of \( E_\ell \) by

\[ E_{\ell,h} = \text{span}\{u_{1,h}, u_{2,h}, \ldots, u_{\ell,h}\}. \]

Then, we have the following discrete minimum-maximum principle:

\[
\lambda_{k,h} = \min_{\dim V_{k,h} = k, V_{k,h} \subset V_h} \max_{v \in V_{k,h}} \frac{a_h(v, v)}{\rho v, v}_{L^2(\Omega)} = \max_{u \in E_{\ell,h}} \frac{a_h(u, u)}{(\rho u, u)_{L^2(\Omega)}}. \tag{2.8}
\]

3 Error Estimates of Eigenvalues and Eigenfunctions

In this section, we shall analyze errors of discrete eigenvalues and eigenfunctions by nonconforming methods. We refer to Babuška and Osborn [5], Rannacher [33] for some alternative analysis in the functional analysis setting. We would like to stress the analysis is a nontrivial extension to nonconforming methods of the analysis for conforming methods in Strang and Fix [37]. For simplicity of presentation, we only consider the case where \( \lambda_\ell \) is an eigenvalue of multiplicity 1 and also note that the extension to the multiplicity \( \geq 2 \) case follows by using notations and concepts, for instance, from [10, Page 406].

Given any \( f \in L^2(\Omega) \), let \( u_f \) be the solution to the dual problem: Find \( u_f \in V \) such that

\[
a(u_f, v) = (\rho f, v)_{L^2(\Omega)} \text{ for any } v \in V. \tag{3.1}
\]

Generally speaking, the regularity of \( u_f \) depends on, among others, regularities of \( f \) and \( \rho \), elliptic operators under consideration, the shape of the domain \( \Omega \) and the boundary condition imposed. To fix the main idea and therefore avoid too technical notation, throughout this
paper, without loss of generality, assume that \(u_f \in V \cap H^{m+r_c}(\Omega)\) with \(0 < r_c \leq 1\) in the sense that

\[
\|u_f\|_{H^{m+r_c}(\Omega)} \lesssim \|\rho^{1/2}f\|_{L^2(\Omega)}.
\]

In order to analyze \(L^2\) error estimates of eigenfunctions, define quasi-Ritz-projections \(P_h u_\ell \in V_h\) by

\[
a_h(P_h' u_\ell, v_h) = \lambda_\ell (\rho u_\ell, v_h)_{L^2(\Omega)} \text{ for any } v_h \in V_h.
\]

The analysis also needs Galerkin projection operators \(P_h : V \to V_h\) by

\[
a_h(P_h v, w_h) = a_h(v, w_h) \text{ for any } w_h \in V_h, v \in V.
\]

**Remark 3.1** We note that \(P_h'\) is identical to \(P_h\) for conforming methods, which indicates the difference between conforming elements analyzed in Strang and Fix [37] and nonconforming elements under consideration.

Under the conditions (H1), (H2), and (H3), a standard argument for nonconforming finite element methods, see, for instance, Brenner and Scott [9], proves

\[
\|\rho^{1/2}(v - P_h v)\|_{L^2(\Omega)} + h^{r_c} \|v - P_h v\|_{V_h} \lesssim h^{2r_c} |v|_{H^{m+r_c}(\Omega)},
\]

provided that \(v \in V \cap H^{m+r_c}(\Omega)\) with \(0 < r_c \leq 1\).

Throughout this paper, \(u_\ell, u_j,\) and \(u_i\) are eigenfunctions of the problem (2.1), while \(u_{\ell,h}, u_{j,h},\) and \(u_{i,h}\) are discrete eigenfunctions of the discrete eigenvalue problem. Note that \(P_h' u_\ell\) is the finite element approximation of \(u_\ell\). Under the conditions (H1)–(H3), a standard argument for nonconforming finite element methods, see, for instance, Brenner and Scott [9], proves

**Lemma 3.2** Suppose that the conditions (H1)–(H3) hold. Then,

\[
\|\rho^{1/2}(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)} + h^{r_c} \|u_\ell - P_h' u_\ell\|_{V_h} \lesssim h^{2r_c} |u_\ell|_{H^{m+r_c}(\Omega)},
\]

provided that \(u_\ell \in V \cap H^{m+r_c}(\Omega)\) with \(0 < r_c \leq 1\).

From \(P_h' u_\ell \in V_h\) we have

\[
P_h' u_\ell = \sum_{j=1}^N (\rho P_h' u_\ell, u_{j,h}) u_{j,h}.
\]

For the projection operator \(P_h'\), we have the following important property

\[
(\lambda_{j,h} - \lambda_\ell)(\rho P_h' u_\ell, u_{j,h})_{L^2(\Omega)} = \lambda_\ell (\rho (u_\ell - P_h' u_\ell), u_{j,h})_{L^2(\Omega)}.
\]

In fact, we have

\[
\lambda_{j,h}(\rho P_h' u_\ell, u_{j,h})_{L^2(\Omega)} = a_h(u_{j,h}, P_h' u_\ell) = \lambda_\ell (\rho u_\ell, u_{j,h})_{L^2(\Omega)}.
\]

Suppose that \(\lambda_\ell \neq \lambda_j\) if \(\ell \neq j\). Then there exists a separation constant \(d_\ell\) with

\[
\frac{\lambda_\ell}{|\lambda_{j,h} - \lambda_\ell|} \leq d_\ell \text{ for any } j \neq \ell,
\]

provided that the meshsize \(h\) is small enough.

**Lemma 3.3** Let \(u_\ell\) and \(u_{\ell,h}\) be eigenfunctions of (2.1) and (2.5), respectively, and \(P_h' u_\ell\) be the solution of problem (3.3). There holds that

\[
\|\rho^{1/2}(u_\ell - u_{\ell,h})\|_{L^2(\Omega)} \leq 2(1 + d_\ell) \|\rho^{1/2}(u_\ell - P_h' u_\ell)\|_{L^2(\Omega)}.
\]
Thus we have provided that $u$

Assume that forming finite element methods, see, for instance, Brenner and Scott [9], proves

This leads to

This completes the proof. $\blacksquare$

Note that

Since both $u_\ell$ and $u_{\ell,h}$ are unit vectors, we can choose them such that $\beta_\ell \geq 0$. Hence we have $|\beta_\ell - 1| \leq \|\rho^{1/2}(u_\ell - \beta_\ell u_{\ell,h})\|_{L^2(\Omega)}$. Thus, we obtain

This completes the proof. $\blacksquare$

**Theorem 3.4** Let $u_\ell$ and $u_{\ell,h}$ be eigenfunctions of (2.1) and (2.5), respectively. Suppose that the conditions (H1)–(H3) hold. Then,

$$\|\rho^{1/2}(u_\ell - u_{\ell,h})\|_{L^2(\Omega)} \lesssim h^{2r_c} |u_\ell|_{H^{m+r_c}(\Omega)},$$

provided that $u_\ell \in V \cap H^{m+r_c}(\Omega)$ with $0 < r_c \leq 1$.

**Proof** It is a direct consequence of lemmas 3.2 and 3.3. $\blacksquare$

Next we analyze errors of eigenvalues. To this end, define $\tilde{u}_{\ell,h} \in V$ by

$$a(\tilde{u}_{\ell,h}, v) = \lambda_{\ell,h}(\rho u_{\ell,h}, v)_{L^2(\Omega)} \text{ for any } v \in V.$$  

(3.12)

It follows from (2.1) and (2.5) that

$$\langle \rho(\tilde{u}_{\ell,h} - u_{\ell,h}), u_\ell \rangle_{L^2(\Omega)} = \lambda_{\ell,h}^{-1} \lambda_{\ell,h}(\rho u_{\ell,h}, u_\ell)_{L^2(\Omega)} - (\rho u_{\ell,h}, u_\ell)_{L^2(\Omega)} = \frac{(\lambda_{\ell,h} - \lambda_\ell)(\rho u_{\ell,h}, u_\ell)_{L^2(\Omega)}}{\lambda_\ell}.$$  

(3.13)

Thus we have

$$\lambda_{\ell,h} - \lambda_\ell = \frac{\lambda_\ell (\rho(\tilde{u}_{\ell,h} - u_{\ell,h}), u_\ell)_{L^2(\Omega)}}{(\rho u_{\ell,h}, u_\ell)_{L^2(\Omega)}}.$$  

(3.14)

Assume that $\tilde{u}_{\ell,h} \in V \cap H^{m+r_c}(\Omega)$ with $0 < r_c \leq 1$ in the sense that

$$\|\tilde{u}_{\ell,h}\|_{H^{m+r_c}(\Omega)} \lesssim \lambda_{\ell,h}.$$  

(3.15)

Note that $u_{\ell,h}$ is the finite element approximation of $\tilde{u}_{\ell,h}$. A standard argument for nonconforming finite element methods, see, for instance, Brenner and Scott [9], proves
Lemma 3.5 Suppose that the conditions (H1)-(H3) hold. Then,
\[
\|\rho^{1/2}(u_{\ell,h} - \tilde{u}_{\ell,h})\|_{L^2(\Omega)} + h^{r_c} \|u_{\ell,h} - \tilde{u}_{\ell,h}\|_h \lesssim \lambda_{\ell,h} h^{2r_c},
\]
provided that \(\tilde{u}_{\ell,h} \in H^{m+r_c}(\Omega)\).

Inserting the above estimate into (3.14) proves:

Theorem 3.6 Let \(\lambda_{\ell}\) and \(\lambda_{\ell,h}\) be eigenvalues of (2.1) and (2.5), respectively. Suppose that (H1)-(H3) hold. Then,
\[
|\lambda_{\ell,h} - \lambda_{\ell}| / \lambda_{\ell} \lesssim \lambda_{\ell,h} h^{2r_c} \left| \frac{1}{1 + (\rho(u_{\ell,h} - u_{\ell}), u_{\ell})_{L^2(\Omega)}} \right|.
\]

Proof First we have
\[
\frac{(\rho u_{\ell}, u_{\ell})_{L^2(\Omega)}}{(\rho u_{\ell,h}, u_{\ell})_{L^2(\Omega)}} = \frac{1}{1 + (\rho(u_{\ell,h} - u_{\ell}), u_{\ell})_{L^2(\Omega)}}.
\]
It follows from (3.14) that
\[
\lambda_{\ell,h} - \lambda_{\ell} = \frac{\lambda_{\ell}(\rho(\tilde{u}_{\ell,h} - u_{\ell,h}), u_{\ell})_{L^2(\Omega)}}{(\rho u_{\ell}, u_{\ell})_{L^2(\Omega)}} = \frac{\lambda_{\ell}(\rho(\tilde{u}_{\ell,h} - u_{\ell,h}), u_{\ell})_{L^2(\Omega)}}{(\rho u_{\ell,h}, u_{\ell})_{L^2(\Omega)}}.
\]
This leads to
\[
|\lambda_{\ell,h} - \lambda_{\ell}| \leq \lambda_{\ell} \|\rho^{1/2}(\tilde{u}_{\ell,h} - u_{\ell,h})\|_{L^2(\Omega)} \left| \frac{1}{1 + (\rho(u_{\ell,h} - u_{\ell}), u_{\ell})_{L^2(\Omega)}} \right|.
\]
This and Lemma 3.5 yield
\[
|\lambda_{\ell,h} - \lambda_{\ell}| \lesssim \lambda_{\ell,h} \lambda_{\ell} h^{2r_c} \left| \frac{1}{1 + (\rho(u_{\ell,h} - u_{\ell}), u_{\ell})_{L^2(\Omega)}} \right|
\]
which completes the proof. \(\square\)

Finally we can have error estimates in the energy norm of eigenfunctions.

Theorem 3.7 Let \(u_{\ell}\) and \(u_{\ell,h}\) be eigenfunctions of (2.1) and (2.5), respectively. Suppose that the conditions (H1)-(H3) hold. Then,
\[
\|u_{\ell,h} - u_{\ell}\|_h \lesssim h^{r_c} \|u_{\ell}\|_{H^{m+r_c}(\Omega)},
\]
provided that \(u_{\ell} \in V \cap H^{m+r_c}(\Omega)\) with \(0 < r_c \leq 1\).

Proof In order to bound errors of eigenfunctions in the energy norm, we need the following decomposition:
\[
a_h(u_{\ell,h} - u_{\ell,h}, u_{\ell} - u_{\ell,h}) = a(u_{\ell}, u_{\ell}) + a_h(u_{\ell,h}, u_{\ell,h}) - 2a_h(u_{\ell}, u_{\ell,h})
\]
\[
= \lambda_{\ell}\|\rho^{1/2}(u_{\ell} - u_{\ell,h})\|_{L^2(\Omega)}^2 + \lambda_{\ell,h} - \lambda_{\ell} + 2\lambda_{\ell}(\rho u_{\ell}, u_{\ell,h} - u_{\ell}) - 2a_h(u_{\ell}, u_{\ell,h} - u_{\ell}).
\]
(3.19)
Then, the desired result follows from Theorem 3.6, (3.3), and the condition (H3). \(\square\)
4 Lower Bounds for Eigenvalues: An Abstract Theory

This section proves that the conditions (H1)–(H4) are sufficient conditions to guarantee nonconforming finite element methods to yield lower bounds for eigenvalues of elliptic operators.

**Theorem 4.1** Let \((\lambda, u)\) and \((\lambda_h, u_h)\) be solutions of problems (2.1) and (2.5), respectively. Assume that \(u \in V \cap H^{m+r}(\Omega)\) and that \(\beta h^{2r_c} \leq \|u - u_h\|^2_h\) with \(0 < r_c \leq r_a \leq 1\), for a positive constant \(\beta\) which is independent of \(h\). If the conditions (H1)–(H4) hold, then

\[ \lambda_h \leq \lambda, \quad (4.1) \]

provided that \(h\) is small enough.

**Proof** Let \(\Pi_h\) be the operator in the condition (H4). A similar argument of Armentano and Duran [2] proves

\[
\lambda - \lambda_h = \|u - u_h\|^2_h - \lambda_h \|\rho^{1/2}(\Pi_h u - u_h)\|^2_{L^2(\Omega)}
\]

\[+ \lambda_h \left(\|\rho^{1/2} \Pi_h u\|^2_{L^2(\Omega)} - \|\rho^{1/2} u\|^2_{L^2(\Omega)}\right).\]

(4.2)

(We refer interested readers to Zhang et al. [48] for an identity with full terms). From the abstract error estimate (3.11) it follows that

\[\|\rho^{1/2}(u - u_h)\|_{L^2(\Omega)} \lesssim h^{2r_c}.\]

This, the third term in the condition (H4) plus the triangle inequality yield

\[\lambda_h \|\rho^{1/2}(\Pi_h u - u_h)\|^2_{L^2(\Omega)} \leq C_2 (h^{4r_c} + h^{2(r_a + \Delta r_a)})\]

for a positive constant \(C_2\) which is independent of \(h\). In addition, the second term in the condition (H4) reads

\[\lambda_h \left(\|\rho^{1/2} u\|^2_{L^2(\Omega)} - \|\rho^{1/2} \Pi_h u\|^2_{L^2(\Omega)}\right) \leq C_3 h^{2r_c + \Delta r_c}\]

for a positive constant \(C_3\) which is independent of \(h\). Since \(0 < r_c \leq r_a\), the above two equations and the saturation condition leads to

\[\lambda - \lambda_h \geq h^{2r_c} (\beta - C_2 (h^{2r_c} + h^{2(r_a + \Delta r_a - r_c))} - C_3 h^{\Delta r_c}).\]

Since the positive constants \(\beta\) and \(r_c, \Delta r_c, r_a - r_c + \Delta r_a\) are independent of \(h\), this proves that the right-hand side of the above equation will be nonnegative when the mesh size \(h\) is small enough.

The condition that \(\beta h^{2r_c} \leq \|u - u_h\|^2_h\) is usually referred to as the saturation condition in the literature. The condition is closely related to the inverse theorem in the context of the approximation theory by trigonometric polynomials or splines. For the approximation by conforming piecewise polynomials, the inverse theorem was analyzed in Babuška et al. [3], Widlund [41]. For nonconforming finite element methods, the saturation condition was first analyzed in Shi [35] for the Wilson element by an example, which was developed by Chen and Li [12] by an expansion of the error. See Křížek et al. [23] for lower bounds of discretization errors by conforming linear/bilinear finite elements. Babuska and Strouboulis [4] analyzed Lagrange finite element methods for elliptic problems in one dimension. In Sect. 8 and the appendix, we shall analyze the saturation condition for most of nonconforming finite element methods under consideration. To our knowledge, it is the first time to analyze systematically this condition for nonconforming methods.
Since Galerkin projection operators from (3.4) or their high order perturbations of nonconforming spaces $V_h$ can be taken as interpolation operators $\Pi_h$, the terms $\|\rho^{1/2}u\|_{L^2(\Omega)}^2 - \|\rho^{1/2}\Pi_h u\|_{L^2(\Omega)}^2$ are dependent on only local approximation properties but not consistency properties of $V_h$ for most practical cases. However total errors $\|u - u_h\|_h^2$ depend on both global approximation and consistency properties of $V_h$. The local approximation errors $\|\rho^{1/2}u\|_{L^2(\Omega)}^2 - \|\rho^{1/2}\Pi_h u\|_{L^2(\Omega)}^2$ will be either of high order or negative when we have enough many local degrees of freedom (compared to total errors), namely,

$$\|u - u_h\|_h^2 \geq \|\rho^{1/2}u\|_{L^2(\Omega)}^2 - \|\rho^{1/2}\Pi_h u\|_{L^2(\Omega)}^2.$$ 

If this happens we say local approximation properties are better than total errors. Hence, Theorem 4.1 states that corresponding methods of eigenvalue problems will produce lower bounds for eigenvalues for such a case. Thus, to get a lower bound for an eigenvalue is to design nonconforming finite element spaces with enough local degrees of freedom when compared to total errors. This in fact provides a systematic tool for the construction of lower bounds for eigenvalues of operators in mathematical science.

Since total errors depend on both global approximation and consistency properties (see Theorems 3.4 and 3.7), there are two methods to modify or construct nonconforming finite element spaces which can produce lower bounds of eigenvalues:

- Keep the global approximation properties but weaken the continuity of nonconforming finite element spaces which will have better local approximation properties but not so good consistency properties, see Sect. 7 for more details.
- Keep continuity of nonconforming finite element spaces but locally enrich shape function spaces such that the zero order moments on each element of $\Pi_h u$ and $u$ are identical. As a result, the nonconforming finite element spaces will have better local approximation properties but not so good global approximation properties (possibly consistency properties either), see Sect. 7 for more details.

5 Nonconforming Elements of Second Order Elliptic Operators

This section presents some nonconforming schemes of second order elliptic eigenvalue problems that the conditions (H1)–(H4) proposed in Sect. 2 are satisfied. Let the boundary $\partial \Omega$ be divided into two parts: $\Gamma_D$ and $\Gamma_N$ with $|\Gamma_D| > 0$, and $\Gamma_D \cup \Gamma_N = \partial \Omega$. For ease of presentation, assume that (2.1) is the Poisson eigenvalue problem imposed general boundary conditions.

Let $T_h$ be regular $n$-rectangular triangulations of domains $\Omega \subset \mathbb{R}^n$ with $2 \leq n$ in the sense that $\bigcup_{K \in T_h} K = \hat{\Omega}$, two distinct elements $K$ and $K'$ in $T_h$ are either disjoint, or share an $\ell$-dimensional hyper-plane, $\ell = 0, \ldots, n - 1$. Let $\mathcal{H}_h$ denote the set of all $n - 1$ dimensional hyper-planes in $T_h$ with the set of interior $n - 1$ dimensional hyper-planes $\mathcal{H}_h(\Omega)$ and the set of boundary $n - 1$ dimensional hyper-planes $\mathcal{H}_h(\partial \Omega)$, $\mathcal{N}_h$ is the set of nodes of $T_h$ with the set of internal nodes $\mathcal{N}_h(\Omega)$ and the set of boundary nodes $\mathcal{N}_h(\partial \Omega)$.

For each $K \in T_h$, introduce the following affine invertible transformation

$$F_K : \hat{K} \rightarrow K, x_i = h_{x_i,K} \xi_i + x_{i0}$$

with the center $(x_{01}^0, x_{02}^0, \ldots, x_{0n}^0)$ and the lengths $2h_{x_i,K}$ of $K$ in the directions of the $x_i$-axis, and the reference element $\hat{K} = [-1, 1]^n$. In addition, set $h = \max_{1 \leq i \leq n} h_{x_i}$. Springer
Over the above mesh $T_h$, we shall consider two classes of nonconforming element methods for the eigenvalue problem (2.1), namely, the Wilson element in any dimension, the enriched nonconforming rotated $Q_1$ element in any dimension.

Let $V_h$ be discrete spaces of aforementioned nonconforming element methods. The finite element approximation of Problem (2.1) is defined as in (2.5).

For all the elements, one can use continuity and boundary conditions for discrete spaces $V_h$ given below to verify the conditions (H1)–(H3), see Lin et al. [29], Rannacher and Turek [34], Shi and Wang [36], Wilson et al. [42] for further details. Let $V_h$ be discrete spaces of aforementioned nonconforming element methods. The finite element defined by $V_h$, given below to verify the conditions (H1)–(H3), see Lin et al. [29], Rannacher and Turek [34], Shi and Wang [36], Wilson et al. [42] for further details. Let $V_h$ be discrete spaces of aforementioned nonconforming element methods. The finite element defined by

\[ V_h := \{ v \in L^2(\Omega) : v|_K \circ F_K \in Q_{Wil}(\hat{K}) \text{ for each } K \in T_h, v \text{ is continuous at internal nodes, and vanishes at boundary nodes on } \Gamma_D \}. \]

The degrees of freedom read

\[ v(a_j), 1 \leq j \leq 2^n \text{ and } \frac{1}{|K|} \int_K \frac{\partial^2 v}{\partial x_i^2} \, dx, \quad 1 \leq i \leq n, \]

where $a_j$ denote vertices of element $K$.

In order to show the condition (H4), let $P_h$ be the Galerkin projection operator defined in (3.4). The approximation property of the operator $P_h$ reads

\[ h\|u - P_h u\|_h + \|u - P_h u\|_{L^2(\Omega)} \lesssim h^2 \|u\|_{H^2(\Omega)}, \]

provided that $u \in V \cap H^2(\Omega)$. This plus (5.1) lead to

\[ \lambda_h(\rho(P_h u - u), P_h u + u)_{L^2(\Omega)} = \lambda(\rho(P_h u - u), P_h u + u)_{L^2(\Omega)} + O(h^4) \]

\[ = 2\lambda(\rho(P_h u - u), u)_{L^2(\Omega)} + O(h^4). \]

To analyze the term $\lambda(\rho(P_h u - u), u)_{L^2(\Omega)}$, let $I_h$ be the canonical interpolation operator for the Wilson element, which admits the following error estimates:

\[ h\|u - I_h u\|_h + \|u - I_h u\|_{L^2(\Omega)} \lesssim h^{1+r_o} \|u\|_{H^{1+r_o}(\Omega)}, \]

provided that $u \in V \cap H^{1+r_o}(\Omega)$ with $1 \leq r_o \leq 2$. Since $\|u - P_h u\|_h \lesssim h^{r_o}$ provided that $u \in V \cap H^{1+r_o}(\Omega)$ with $1 \leq r_o \leq 2$,

\[ \lambda(\rho u, P_h u - I_h u)_{L^2(\Omega)} - a_h(u, P_h u - I_h u) \lesssim h \|u\|_{H^2(\Omega)} \|P_h u - I_h u\|_h \lesssim h^{1+r_o} \|u\|_{H^{1+r_o}(\Omega)}^2. \]
This and (5.5) state

\[
\lambda(\rho(P_h u - u), u)_{L^2(\Omega)} = \lambda(\rho u, P_h u - u)_{L^2(\Omega)} - a_h(u, P_h u - u) + a_h(u - P_h u, P_h u - u) \\
+ \lambda(\rho u, I_h u - I_h u)_{L^2(\Omega)} - a_h(u, I_h u - u) \\
= a_h(u - I_h u, u) + O(h^{1+c_0}).
\] (5.6)

To analyze the term \(a_h(u - I_h u, u)\), let \(I_K\) denote the restriction of \(I_h\) on element \(K\). Then we have the following result.

**Lemma 5.1** For any \(u \in P_3(K)\) and \(v \in P_1(K)\), it holds that

\[
(\nabla(u - I_K u), \nabla v)_{L^2(K)} = -\sum_{i=1}^{n} \sum_{j \neq i} \frac{h_{x_i, K}^3}{3} \int_{K} \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial v}{\partial x_j} \, dx.
\] (5.7)

**Proof** The definition of the interpolation operator \(I_K\) leads to

\[
u - I_K u = \sum_{i=1}^{n} h_{x_i, K}^3 \frac{\partial^3 u}{\partial x_i^2} (\xi_j^3 - \xi_i) + \sum_{i=1}^{n} \sum_{j \neq i} \frac{h_{x_i, K} h_{x_j, K}}{2} \frac{\partial^3 u}{\partial x_i^2 \partial x_j} (\xi_j^3 - \xi_j).
\]

A direct calculation proves

\[
(\nabla(u - I_K u), \nabla v)_{L^2(K)} = -\sum_{i=1}^{n} \sum_{j \neq i} \frac{h_{x_i, K}^3}{3} \int_{K} \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial v}{\partial x_j} \, dx,
\]

which completes the proof. \(\square\)

Given any element \(K\), define \(J_{\ell, K} v \in P_1(K)\) by

\[
\int_{K} \nabla^i J_{\ell, K} v \, dx = \int_{K} \nabla^i v \, dx, \quad i = 0, \ldots, \ell,
\] (5.8)

for any \(v \in H^\ell(K)\). Note that the operator \(J_{\ell, K}\) is well-defined. Let \(\Pi^0_K\) denote the constant projection operator over \(K\), namely,

\[
\Pi^0_K v := \frac{1}{|K|} \int_{K} v \, dx \text{ for any } v \in L^2(K).
\]

The property of operator \(J_{\ell, K}\) reads

\[
\|\nabla^i (v - J_{\ell, K} v)\|_{L^2(K)} \leq h_{K}^{\ell - i} \|\nabla^\ell (v - J_{\ell, K} v)\|_{L^2(K)} \text{ and } \nabla^\ell J_{\ell, K} v = \Pi^0_K \nabla^\ell v \text{ for any } v \in H^\ell(K).
\] (5.9)

**Lemma 5.2** For uniform meshes, it holds that

\[
(\nabla_h(u - I_h u), \nabla u)_{L^2(\Omega)} = \sum_{i=1}^{n} \sum_{j \neq i} \frac{h_{x_i, K}^3}{3} \int_{K} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \, dx + o(h^2),
\] (5.10)

provided that \(u \in H^3(\Omega)\) and the meshsize is small enough.
Proof A combination of (5.5) and (5.9) leads to
\[
(\nabla_h (u - I_h u), \nabla u)_{L^2(\Omega)} = \sum_{K \in T_h} (\nabla (u - I_K u), \nabla u)_{L^2(K)}
\]
\[
= \sum_{K \in T_h} (\nabla (u - I_K u), \nabla J_{1,K} u)_{L^2(K)} + O(h^3).
\]

The operator $J_{3,K}$ yields the following decomposition
\[
\sum_{K \in T_h} (\nabla (u - I_K u), \nabla J_{1,K} u)_{L^2(K)} = \sum_{K \in T_h} (\nabla (I - I_K)(I - J_{3,K}) u, \nabla J_{1,K} u)_{L^2(K)} + O(h^3),
\]
\[(5.11)\]

It follows from (5.5) and (5.9) that the second term on the right-hand side of the above equation can be estimated as
\[
\sum_{K \in T_h} (\nabla (I - I_K)(I - J_{3,K}) u, \nabla J_{1,K} u)_{L^2(K)} \lesssim \sum_{K \in T_h} h_K^2 \| (I - \Pi_K^0) \nabla^3 u \|_{L^2(K)} \| \nabla u \|_{L^2(K)} = o(h^2),
\]

since piecewise constant functions are dense in the space $L^2(\Omega)$ when the meshsize is small enough. The first term on the right-hand side of (5.11) can be analyzed by (5.7), which reads
\[
\sum_{K \in T_h} (\nabla (I - I_K)(I - J_{3,K}) u, \nabla J_{1,K} u)_{L^2(K)} = - \sum_{K \in T_h} \sum_{i=1}^n \sum_{j \neq i} h_{3,i,K}^2 \int_K \frac{\partial^3 J_{3,K} u}{\partial x_i^2 \partial x_j} \frac{\partial J_{1,K} u}{\partial x_j} dx + o(h^2),
\]
when the meshsize is small enough. Since the mesh is uniform and $\frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_j}$ vanish on the boundary which is perpendicular to $x_i$ axises, elementwise integrations by parts complete the proof. \(\square\)

A summary of (5.4), (5.6) and (5.10) proves that
\[
\lambda_h (\rho (P_h u - u), P_h u + u)_{L^2(\Omega)} \geq 0
\]
when the meshsize is small enough and $u \in H^3(\Omega)$. In Sect. 8, we prove that $h \lesssim \| u - u_h \|_h$ when $u \in H^{1+r_\alpha}(\Omega)$. Therefore, the condition (H4) holds with $\Pi_h = P_h$, $r_\alpha = 2$, $\Delta r_\alpha = 0$, $r_c = 1$, $\Delta r_c = +\infty$ for the Wilson element provided that $r_\alpha = 2$ and the mesh is uniform and is fine enough.

Remark 5.3 For two and three dimensions, the Wilson element were analyzed in Li [26], Yang and Bi [44]. The analysis herein improves the result therein by weakening the regularity of eigenfunctions.
5.2 The Enriched Nonconforming Rotated $Q_1$ Element in Any Dimension

Denote by $Q_{EQ}(K)$ the enriched nonconforming rotated $Q_1$ element space defined by Lin et al. [29]

$$Q_{EQ}(K) := \mathcal{P}_1(K) + \text{span}\{x_1, x_2, \ldots, x_n\}. \quad (5.13)$$

The enriched nonconforming rotated $Q_1$ element space $V_h$ is then defined by

$$V_h := \left\{ v \in L^2(\Omega) : v|_K \in Q_{EQ}(K) \text{ for each } K \in \mathcal{T}_h, \int f v df = 0, \right. \left. \text{ for all internal } n-1 \text{ dimensional hyper-planes } f, \right. \left. \text{ and } \int_{f} v df = 0 \text{ for all } f \text{ on } \Gamma_D \right\}. \quad (5.14)$$

Here and throughout this paper, $[v]$ denotes the jump of $v$ across $f$. For the enriched nonconforming rotated $Q_1$ element, we define the interpolation operator $\Pi_1^h : H^1_D(\Omega) \to V_h$ by

$$\int f \Pi_1^h v df = \int f v df \text{ for any } v \in H^1_D(\Omega), f \in \mathcal{H}_h,$$

$$\int K \Pi_1^h v dx = \int K v dx \text{ for any } K \in \mathcal{T}_h. \quad (5.14)$$

For this interpolation operator, we have

**Lemma 5.4** There holds that

$$\|u - \Pi_1^h u\|_{L^2(K)} \lesssim h^{1+r_a}|u|_{H^{1+r_a}(K)} \text{ for any } u \in H^{1+r_a}(K) \text{ with } 0 < r_a \leq 1 \text{ and } K \in \mathcal{T}_h. \quad (5.15)$$

**Proof** Since $u - \Pi_1^h u$ has vanishing mean on $K$, it follows from the Poincare inequality that

$$\|u - \Pi_1^h u\|_{L^2(K)} \lesssim h_K \|\nabla(u - \Pi_1^h u)\|_{L^2(K)}.$$ 

Then the desired result follows from the usual interpolation theory and the interpolation space theory for the singular case $u \in H^{1+r_a}(K)$. $\square$

**Lemma 5.5** For the enriched nonconforming rotated $Q_1$ element, it holds the condition $(H4)$.

**Proof** We define the space $Q_K = \begin{pmatrix} a_{11} + a_{12} x_1 \\ a_{21} + a_{22} x_2 \\ \vdots \\ a_{n1} + a_{n2} x_n \end{pmatrix}$ with free parameters $a_{11}, a_{12}, \ldots, a_{n1}, a_{n2}$.

From the definition of the operator $\Pi_h$, we have

$$(\nabla(u - \Pi_h u), \psi)_{L^2(K)} = 0, \text{ for any } \psi \in Q_K. \quad (5.16)$$

Let $\nabla_h$ be the piecewise gradient operator which is defined element by element. Since $\nabla_h \Pi_h u|_K \in Q_K$, this leads to

$$(\nabla_h \Pi_h u)|_K = P_K(\nabla u)|_K, \quad (5.17)$$

with the $L^2$ projection operator $P_K$ from $L^2(K)$ onto $Q_K$. This proves $a_h(u - \Pi_h u, u_h) = 0$.

It remains to show estimates in $(H4)$. To this end, let $\Pi^0$ be the piecewise constant projection.
operator (defined by \( \Pi^K_0 = \Pi^K_0 \) for element \( K \)). Without loss of generality, we assume that \( \rho \) is piecewise constant. It follows from the definition of the interpolation operator \( \Pi_h \) that
\[
\| \rho^{1/2} \Pi_h u \|_{L^2(\Omega)}^2 - \| \rho^{1/2} u \|_{L^2(\Omega)}^2 = (\rho (\Pi_h u - u), \Pi_h u + u)_{L^2(\Omega)}
\]
\[
= (\rho (\Pi_h u - u), \Pi_h u + u - \Pi(0)(\Pi_h u + u))_{L^2(\Omega)}
\]
\[
\lesssim h \| \rho^{1/2} (\Pi_h u - u) \|_{L^2(\Omega)} \| \nabla_h (\Pi_h u + u) \|_{L^2(\Omega)},
\]
(5.18)
which completes the proof of (H4) with \( r_c = r_a, \Delta r_c = 2 - r_a, \) and \( \Delta r_a = 1 \), provided that \( u \in H^1_0(\Omega) \cap H^{1+r_a}(\Omega) \) with \( 0 < r_a \leq 1 \).

In Sect. 8 and the appendix, we show that \( h \lesssim \| u - u_h \|_h \) for the case \( r_a = r_c = 1 \), and that there exist meshes such that \( h^{r_a} \lesssim \| u - u_h \|_h \) holds for the case \( 0 < r_a = r_c < 1 \), respectively. Therefore, we have that the result in Theorem 4.1 holds for this class of elements.

### 6 Morley-Wang-Xu Elements for 2mth Order Operators

This section studies 2mth order elliptic eigenvalue problems defined over the bounded domain \( \Omega \subset \mathbb{R}^n \) with \( 1 < n \) and \( m \leq n \). Let \( \kappa = (\kappa_1, \ldots, \kappa_n) \) be the multi-index with \( |\kappa| = \sum_{i=1}^n \kappa_i \), we define the space
\[
V := \{ v \in L^2(\Omega), \frac{\partial^\kappa v}{\partial x^\kappa} \in L^2(\Omega), |\kappa| \leq m, v|_{\partial \Omega} = \frac{\partial^\ell v}{\partial v^\ell} |_{\partial \Omega} = 0, \ell = 1, \ldots, m - 1 \},
\]
(6.1)
with \( v \) the unit normal vector to \( \partial \Omega \). The partial derivatives \( \frac{\partial^\kappa v}{\partial x^\kappa} \) are defined as
\[
\frac{\partial^\kappa v}{\partial x^\kappa} := \frac{\partial |\kappa| v}{\partial x^{\kappa_1} \cdots \partial x^{\kappa_n}}.
\]
(6.2)

Let \( D^\ell v \) denote the \( m \)th order tensor of all \( \ell \)th order derivatives of \( v \), for instance, \( \ell = 1 \) the gradient, and \( \ell = 2 \) the Hessian matrix. Let \( C \) be a positive definite operator with the same symmetry as \( D^m v \), the bilinear form \( a(u, v) \) reads
\[
a(u, v) := (\sigma, D^m v)_{L^2(\Omega)} \text{ and } \sigma := CD^m u,
\]
(6.3)
which gives rise to the energy norm
\[
\| u \|^2_V := a(u, u) \text{ for any } u \in V,
\]
(6.4)
which is equivalent to the usual \( |u|_{H^m(\Omega)} \) norm for any \( u \in V \).

2mth order elliptic eigenvalue problems read: Find \( (\lambda, u) \in \mathbb{R} \times V \) with
\[
a(u, v) = \lambda(\rho u, v)_{L^2(\Omega)} \text{ for any } v \in V \text{ and } \| \rho^{1/2} u \|_{L^2(\Omega)} = 1,
\]
(6.5)
with some positive function \( \rho \in L^\infty(\Omega) \).

Consider Morley-Wang-Xu elements in Wang and Xu [38] and apply them to eigenvalue problems under consideration. Let \( T_h \) be some shape regular decomposition into \( n \)-simplex of the domain \( \Omega \). Denote by \( \mathcal{H}_{n-i, h} \) for \( i = 1, \ldots, n \), all \( n - i \) dimensional sub-simplices of \( T_h \) with \( v_{n-i, f} \) any one of unit normal vectors to \( f \in \mathcal{H}_{n-i, h} \). Let \( \lbrack \cdot \rbrack \) denote the jump of piecewise functions over \( f \). For any \( n - i \) dimensional boundary sub-simplex \( f \), the jump \( \lbrack \cdot \rbrack \) denotes the trace restricted to \( f \). As usual, \( h_K \) is the diameter of \( K \in T_h \), and \( h_f \) the diameter.
of \( f \in \mathcal{H}_{n-i,h} \). Given \( K \in T_h \), let \( \partial K \) denote the boundary of \( K \). Morley-Wang-Xu element spaces are defined in Wang and Xu [38], which read

\[
V_h := \{ v \in L^2(\Omega), \forall f | v \in \mathcal{P}_m(K), \int_f \frac{\partial^{m-i} v}{\partial n_{n-i,f}} df = 0, \forall f \in \mathcal{H}_{n-i,h}, i = 1, \ldots, m \}. \tag{6.6}
\]

Define the discrete stress \( \sigma_h = CD_h^m u_h \), the broken versions \( a_h(\cdot, \cdot) \) and \( \| \cdot \|_{c_h} \) follow, respectively,

\[
a_h(u_h, v_h) := (\sigma_h, D_h^m v_h)_{L^2(\Omega)}, \text{ for any } u_h, v_h \in V + V_h,
\]

\[
\|u_h\|_h^2 := a_h(u_h, u_h) \text{ for any } u_h \in V + V_h,
\]

where \( D_h^m \) is defined elementwise with respect to the partition \( T_h \).

The discrete eigenvalue problem reads: Find \((\lambda_h, u_h) \in \mathbb{R} \times V_h\), such that

\[
a_h(u_h, v_h) = \lambda_h h^{m} u_h, v_h \|_{L^2(\Omega)} \text{ for any } v_h \in V_h \text{ and } \|\rho^{1/2}u_h\|_{L^2(\Omega)} = 1. \tag{6.7}
\]

The canonical interpolation operator for the spaces \( V_h \) is defined by: Given any \( v \in V \), the interpolation \( \Pi_h v \in V_h \) is defined as

\[
\int_f \frac{\partial^{m-i} \Pi_h v}{\partial n_{n-i,f}} df = \int_f \frac{\partial^{m-i} v}{\partial n_{n-i,f}} df, \text{ for any } f \in \mathcal{H}_{n-i,h}, i = 1, \ldots, m. \tag{6.8}
\]

For this interpolation, we have the following approximation

\[
\|\rho^{1/2}(u - \Pi_h u)\|_{L^2(\Omega)} \lesssim h^{m+r_{a}} |u|_{H^{m+r_{a}}(\Omega)} \text{ for any } u \in V \cap H^{m+r_{a}}(\Omega) \text{ with } 0 < r_{a} \leq 1. \tag{6.9}
\]

It is straightforward to see that conditions (H1)–(H3) hold for this class of elements, see Shi and Wang [36], Wang and Xu [38]. Then, it follows from Theorems 3.4, 3.6, and 3.7 that

\[
\|u - u_h\|_h \lesssim h^{r_{a}} \text{ and } \|u - u_h\|_{L^2(\Omega)} \lesssim h^{2r_{a}}, \tag{6.10}
\]

provided that eigenfunctions \( u \in V \cap H^{m+r_{a}}(\Omega) \) with \( 0 < r_{a} \leq 1 \).

**Theorem 6.1** Let \((\lambda, u)\) and \((\lambda_h, u_h)\) be solutions of problems (6.5) and (6.7), respectively. Then,

\[
\lambda_h \leq \lambda, \tag{6.11}
\]

provided that \( h \) is small enough.

**Proof** The definition of \( \Pi_h \) in (6.8) yields \( a_h(u - \Pi_h u, v_h) = 0 \) for any \( v_h \in V_h \). The condition (H4) follows immediately from (6.9). In addition, in Sect. 8 and the “Appendix”, we show that \( h \lesssim |u - u_h|_h \) for the case \( r_{a} = 1 \) and that there exist meshes such that \( h^{r_{a}} \lesssim \|u - u_h\|_h \) for the case \( 0 < r_{a} < 1 \), respectively. Then, the desired result follows from Theorem 4.1 for \( m \geq 2 \). \( \square \)

### 7 New Nonconforming Elements

In this section, we shall follow the condition (H4) and the saturation condition in Theorem 4.1 to propose two new nonconforming finite elements for second order operators. This is of two fold, one is to modify a nonconforming element in literature such that the modified one will meet the condition (H4), the other is to construct a new nonconforming element.
7.1 The Enriched Crouzeix-Raviart Element

To fix the idea, we only consider the case where \( n = 2 \) and note that the results can be generalized to any dimension. Let \( T_h \) be some shape regular decomposition into triangles of the polygonal domain \( \Omega \subset \mathbb{R}^2 \). Here we restrict ourselves to the case where the bilinear form \( a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)} \) with the mixed boundary condition \( |\Gamma_N| \neq 0 \).

Note that the original Crouzeix-Raviart element can only guarantee theoretically lower bounds of eigenvalues for the singular case in the sense that \( u \in H^{1+r_a}(\Omega) \) with \( 0 < r_a < 1 \). To produce lower bounds of eigenvalues for both the singular case \( u \in H^{1+r_a}(\Omega) \) and the smooth case \( u \in H^2(\Omega) \), we propose to enrich the shape function space by \( x_1^2 + x_2^2 \) on each element. This leads to the following shape function space

\[
Q_{ECR}(K) = P_1(K) + \text{span}\{x_1^2 + x_2^2\} \quad \text{for any } K \in T_h.
\]

The enriched Crouzeix-Raviart element space \( V_h \) is then defined by

\[
V_h := \left\{ v \in L^2(\Omega) : v|_K \in Q_{ECR}(K) \text{ for each } K \in T_h, \int_f [v] df = 0, \text{ for all internal edges } f, \text{ and } \int_f v df = 0 \text{ for all edges } f \text{ on } \Gamma_D \right\}.
\]

For the enriched Crouzeix-Raviart element, we define the interpolation operator \( \Pi_h : H^1_D(\Omega) \to V_h \) by

\[
\int_f \Pi_h v df = \int_f v df \text{ for any } v \in H^1_D(\Omega) \text{ for any edge } f,
\]

\[
\int_K \Pi_h v dx = \int_K v dx \text{ for any } K \in T_h.
\]

For this interpolation operator, a similar argument of Lemma 5.4 leads to:

**Lemma 7.1** There holds that

\[
\|u - \Pi_h u\|_{L^2(K)} \lesssim h^{1+r_a} |u|_{H^{1+r_a}(K)} \text{ for any } u \in H^{1+r_a}(K) \text{ with } 0 < r_a \leq 1 \text{ and } K \in T_h.
\]

**Lemma 7.2** For the enriched Crouzeix-Raviart element, it holds the condition (H4).

**Proof** We follow the idea in Lemma 5.5 to define the space \( Q_K = \left( a_{11} + a_{12} x_1 \right) a_{21} + a_{12} x_2 \) with free parameters \( a_{11}, a_{21}, a_{12} \). From the definition of the operator \( \Pi_h \), we have

\[
(\nabla (u - \Pi_h u), \psi)_{L^2(K)} = 0 \text{ for any } \psi \in Q_K.
\]

Indeed, we integrate by parts to get

\[
(\nabla (u - \Pi_h u), \psi)_{L^2(K)} = -(u - \Pi_h u, \text{div } \psi)_{L^2(K)} + \sum_{f \in \partial K} \int_f (u - \Pi_h u) \psi \cdot v_f ds.
\]

Since both \( \text{div } \psi \) and \( \psi \cdot v_f \) (on each edge \( f \)) are constant, then (7.4) follows from (7.2). Since \( \nabla_h \Pi_h u|_K \in Q_K \), the identity (7.4) leads to

\[
(\nabla_h \Pi_h u)|_K = P_K (\nabla u)|_K,
\]

(7.5)
with the $L^2$ projection operator $P_K$ from $L^2(K)$ onto $Q_K$. Then a similar argument of Lemma 5.5 completes the proof.

In Sect. 8 and the “Appendix”, we have proven that $h \lesssim \|u - u_h\|_h$ for the case $r_a = r_c = 1$, and that there exist meshes such that $h^{r_a} \lesssim \|u - u_h\|_h$ for the case $0 < r_a = r_c < 1$. Hence, the result in Theorem 4.1 holds for this class of elements.

7.2 A New First Order Nonconforming Element

With the condition from Theorem 4.1, a systematic method obtaining the lower bounds for eigenvalues is to design nonconforming finite element spaces with good local approximation property but not so good total errors. To make the idea clearer, we propose a new nonconforming element that admits lower bounds for eigenvalues. Let $\mathcal{T}_h$ be some shape regular decomposition into triangles of the polygonal domain $\Omega \subset \mathbb{R}^2$. We define

$$V_h := \left\{ v \in L^2(\Omega) : v|_K \in P_2(K) \text{ for each } K \in \mathcal{T}_h, \int_f [v] df = 0, \text{ for all internal edges } f, \text{ and } \int_f v df = 0 \text{ for all edges } f \text{ on } \Gamma_D \right\}.$$ 

Since the conforming quadratic element space on the triangle mesh is a subspace of $V_h$, the usual dual argument proves

$$\|u - P_h u\|_{L^2(\Omega)} \lesssim h^{1 + r_a} |u|_{H^{1 + r_a}(\Omega)},$$

provided that $u \in V \cap H^{1 + r_a}(\Omega)$ with $1 < r_a \leq 2$. In Sect. 8, it is shown that $h \lesssim \|\nabla_h (u - u_h)\|_{L^2(\Omega)}$, which in fact implies the condition (H4) for this case. For the singular case $u \in V \cap H^{1 + r_a}(\Omega)$ with $0 < r_a < 1$, a similar argument of the enriched Crouzeix-Raviart element is able to show the condition (H4).

8 The Saturation Condition

In this section, we shall prove, for some cases, the saturation condition which is used in Theorem 4.1. The error basically consists of two parts: approximation errors and the consistency errors. In this section, we analyze the case where approximation errors are dominant and the case where consistency errors are dominant; in the “Appendix”, we give some comments for the case where eigenfunctions are singular.

8.1 The Saturation Condition Where Approximation Errors are Dominant

Let $u \in V \cap H^m(\Omega)$ be eigenfunctions of some $2m$th order elliptic operator. Let $V_h$ be some $k$th order conforming or nonconforming approximation spaces to $H^m(\Omega)$ over the mesh $\mathcal{T}_h$ in the following sense:

$$\sup_{0 \neq v \in H^{m+k}(\Omega) \cap V} \inf_{v_h \in V_h} \frac{\|D^m_h (v - v_h)\|_{L^2(\Omega)}}{|v|_{H^{m+k}}} \lesssim h^k \text{ for some positive integer } k. \quad (8.1)$$

Then the following condition is sufficient for the saturation condition:

(H5) At least one fixed component of $D^{m+k}_h v_h$ vanishes for all $v_h \in V_h$ while the $L^2$ norm of the same component of $D^{m+k}_h u$ is nonzero.
Let $J_m$ denote the $m$th order tensor of all $m$th order derivatives of $v$, for instance, $m = 1$ the gradient, and $m = 2$ the Hessian matrix, and that $D^m_v$ are the piecewise counterparts of $D^m$ defined element by element.

Theorem 8.1 Under the condition (H5), there holds the following saturation condition:

\[ \sum_{k \in \mathcal{H}} \left\| \frac{\partial^k v}{\partial x^k} \right\|_{L^2(\Omega)}^2 = \sum_{k \in \mathcal{H}} \sum_{K \in \mathcal{T}_h} \left\| \frac{\partial^k (v - J_m, v)}{\partial x^k} \right\|_{L^2(K)}^2 \leq 2 \sum_{k \in \mathcal{H}} \sum_{K \in \mathcal{T}_h} \left( \left\| \frac{\partial^k (u - J_m, u)}{\partial x^k} \right\|_{L^2(K)}^2 + \left\| \frac{\partial^k (J_m, u - J_m, v)}{\partial x^k} \right\|_{L^2(K)}^2 \right) \leq \| D^m (u - J_m, v) \|_{L^2(\Omega)}^2 + h^{-2k} \| D^m (J_m, u - J_m, v) \|_{L^2(\Omega)}^2. \]

The estimate of (8.3) and the triangle inequality lead to

\[ \sum_{k \in \mathcal{H}} \left\| \frac{\partial^k v}{\partial x^k} \right\|_{L^2(\Omega)}^2 \lesssim \| D^m (u - J_m, v) \|_{L^2(\Omega)}^2 + h^{-2k} \| D^m (J_m, u - J_m, v) \|_{L^2(\Omega)}^2. \]
Finally it follows from (8.6) that

$$h^{2k} \sum_{\kappa \in \Omega} \| \frac{\partial^{k} u}{\partial x^{k}} \|_{L^2(\Omega)}^2 \lesssim \| D^m_h(u - u_h) \|_{L^2(\Omega)}^2$$  (8.11)

when the meshsize is small enough, which completes the proof.

Remark 8.2 Under the condition (H5), a similar argument can prove the following general saturation conditions:

$$h^{k+m-\ell} \lesssim \| D^\ell_h(u - u_h) \|_{L^2(\Omega)}^2, \quad \ell = 0, 1, \ldots, m.$$

Next, we prove the condition (H5) for various elements in literature.

(1) The Morley-Wang-Xu element. Since $D^m_h v_h \equiv 0$ for all $v_h \in V_h$ for this family of elements and $v \equiv 0$ if $D^m v \equiv 0$ for any $v \in V \cap H^{m+1}(\Omega)$, the condition (H5) holds.

(2) The enriched Crouzeix-Raviart element. Let $\partial_{12,h}^2 v$ denote the piecewise counterpart of the differential operator $\frac{\partial^2}{\partial x \partial y}$. We have $\partial_{12,h}^2 v \equiv 0$ for any $v_h \in V_h$. We only consider the case where $\Omega = [0, 1]^2$ and $u \in H^1_0(\Omega)$. If $\| \frac{\partial^2 v}{\partial x \partial y} \|_{L^2(\Omega)}$ vanishes for $v \in V \cap H^2(\Omega)$. Then, $v$ should be of the form $v(x, y) = f(x) + g(y)$, where $f(x)$ is some function of the variable $x$ and $g(y)$ is some function of the variable $y$. Now the homogenous boundary condition indicates that $f(x) \equiv C_1$ and $g(y) \equiv C_2$ for some constants $C_1$ and $C_2$, which in turn concludes that $v \equiv 0$. This proves the condition (H5).

(3) The same argument applies to the nonconforming rotated $Q_1$ element, the enriched nonconforming rotated $Q_1$ element, and the conforming $Q_1$ element in any dimension.

8.2 The Saturation Condition Where Consistency Errors are Dominant

In this subsection, we prove the saturation condition for where consistency errors are dominant. As usual it is very complicated to give an abstract estimate for consistency errors in a unifying way. Therefore, for ease of presentation, we shall only consider the new first order nonconforming element proposed in this paper. However, the idea can be extended to other nonconforming finite element methods.

In order to give lower bounds of consistency errors, given any edge (boundary and interior) $e$, we construct functions $v_e \in V_h$ such that:

(1) $v_e$ vanishes on $\Omega \setminus \omega_e$;
(2) $v_e$ vanishes on two Gauss-Legendre points of the other four edges than $e$ of $\omega_e$;
(3) $v_e$ vanishes at two interior points of $\omega_e$, see points $(\frac{1}{4}, \frac{1}{4})$ and $(-\frac{1}{4}, \frac{1}{4})$ in Fig. 1 for examples of the reference edge patch;
(4) $\int_e [v_e] ds = O(h^2) \neq 0$. 
See Fig. 1 for the reference edge patch and degrees of freedom for \( v_e \). Note that such a function can be found. In fact, for the reference edge patch in Fig. 1, a direct calculation shows that there exists a function \( v_e \in V_h \) such that

\[
\int_e \left[ v_e \right] s ds = 0.
\]

Let \( \Pi_1^1 \) be the \( L^2 \) projection from \( L^2(e) \) to \( P_1(e) \). Since \( \int_e [v_h] ds = 0 \) for any edge \( e \) of \( T_h \) and \( v_h \in V_h \), it follows that

\[
\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial}{\partial \tau} (\Pi_1^1 \frac{\partial u}{\partial \nu}) [v_h] s ds = \sum_e \int_e \frac{\partial}{\partial \tau} (\Pi_1^1 \frac{\partial u}{\partial \nu}) [v_h] s ds + \sum_e \int_e (I - \Pi_1^1 \frac{\partial u}{\partial \nu}) [v_h] ds. \tag{8.13}
\]

Define

\[
v_h = \sum_e v_e \frac{\partial}{\partial \tau} (\Pi_1^1 \frac{\partial u}{\partial \nu}). \tag{8.14}
\]

Since \( \frac{\partial}{\partial \tau} (\Pi_1^1 \frac{\partial u}{\partial \nu}) \) are constants, definitions of \( v_e \) yield

\[
\sum_e \int_e \frac{\partial}{\partial \tau} (\Pi_1^1 \frac{\partial u}{\partial \nu}) [v_h] s ds \geq C h \sum_e \frac{\partial}{\partial \tau} (\Pi_1^1 \frac{\partial u}{\partial \nu}) \| [v_h] \|_{L^2(e)},
\]

and

\[
\| \nabla_h v_h \|_{L^2(\Omega)} \leq C h^{-1/2} \left( \sum_e \frac{\partial}{\partial \tau} (\Pi_1^1 \frac{\partial u}{\partial \nu}) \| [v_h] \|_{L^2(e)}^2 \right)^{1/2}.
\]

A substitution of these two inequalities into (8.13) leads to

\[
\sup_{0 \neq v_h \in V_h} \frac{\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial \nu} v_h ds}{\| \nabla_h v_h \|_{L^2(\Omega)}} \geq C_1 h \| \nabla^2 u \|_{L^2(\Omega)} - C_2 h^{r_a} |u|_{H^{1+r_a}(\Omega)}, \tag{8.15}
\]

provided that \( u \in H^{1+r_a}(\Omega) \) with \( 1 < r_a \leq 2 \) for some positive constants \( C_1 \) and \( C_2 \). Since \( \| \nabla^2 u \|_{L^2(\Omega)} \) can not vanish, this proves the saturation condition.

Remark 8.3 Thanks to two nonconforming bubble functions in each element, a similar argument is able to show a corresponding result for the Wilson element [36,42].

9 Conclusion and Comments

In this paper, we propose a systematic method that can produce lower bounds for eigenvalues of elliptic operators. With this method, to obtain lower bounds is to design nonconforming finite element spaces with enough local degrees of freedom when compared to total errors. We check that several nonconforming methods in literature possess this promising property. We also propose some new nonconforming methods with this feature. In addition, we study systematically the saturation condition for both conforming and nonconforming finite element methods.

Certainly, there are many other nonconforming finite elements which are not analyzed herein. Let mention several more elements and give some short comments on applications of the theory herein to them. The first one is the nonconforming rotated \( Q_1 \) element from Rannacher and Turek [34]. For this element, discrete eigenvalues are smaller than exact ones when eigenfunctions are singular, see more details from Yang et al. [46]. The same comments

\[\square\]
applies for the Crouzeix–Raviart element of Crouzeix and Raviart [13], see more details from Armentano and Duran [2], Yang et al. [45]. The last one is the Adini element [24,36] for fourth order problems. For this element, by an expansion result of [22, Lemma] and a similar identity like that of Lemma 4.1 therein, a similar argument for the Wilson element is able to show that discrete eigenvalues are smaller than exact ones provided that eigenfunctions \( u \in H^4(\Omega) \), which will improve the results from Yang [43] by weakening the regularity of eigenfunctions.

10 Appendix A: The Comment for the Saturation Condition of the Singular Case

We need the concept of the interpolation space. Let \( X, Y \) be a pair of normed linear spaces. We shall assume that \( Y \) is continuously embedded in \( X \) with \( Y \subset X \) and \( \| \cdot \|_X \lesssim \| \cdot \|_Y \). For any \( t \geq 0 \), we define the \( K \)-functional

\[
K(f, t) = K(f, t, X, Y) = \inf_{g \in Y} \| f - g \|_X + t|g|_Y, \quad (10.1)
\]

where \( \| \cdot \|_X \) is the norm on \( X \) and \( | \cdot |_Y \) is a semi-norm on \( Y \). The function \( K(f, .) \) is defined on \( \mathbb{R}^+ \) and is monotone and concave (being the pointwise infimum of linear functions). If \( 0 < \theta < 1 \) and \( 1 < q \leq \infty \), the interpolation space \( (X, Y)_{\theta, q} \) is defined as the set of all functions \( f \in X \) such that \( u \in H_{m+\epsilon}^{s+\epsilon} (\Omega) \) with \( 0 < s + \epsilon \leq 1 \).
Proof We assume that the saturation condition \( h^s \lesssim \| D_h^m(u - u_h) \|_{L^2(\Omega)} \) does not hold for any mesh \( T_h \) with the meshsize \( h \). In other word, we have
\[
\| D_h^m(u - u_h) \|_{L^2(\Omega)} \lesssim h^{s+\epsilon},
\]
for some \( \epsilon > 0 \). In the following, we assume that \( s + \epsilon \leq 1 \). By the condition (H8), we have
\[
\inf_{v \in V_{m+1, h}} \| D^m(u - v) \|_{L^2(\Omega)} \lesssim \| D_h^m(u - \Pi^c u_h) \|_{L^2(\Omega)} \lesssim h^{s+\epsilon}.
\]

Take \( X = H^m(\Omega) \) and \( Y = H^{m+s+\epsilon}(\Omega) \). The inequality (10.7) is the usual Jackson inequality and the inequality (10.3) is the Berstein inequality in the context of the approximation theory [15, 16]. We can follow the idea of [16, Theorem 5.1, Chapter 7] to estimate terms like \( \psi_k = \varphi_k - \varphi_{k-1}, k = 1, 2, \ldots, \psi_0 = \varphi_0 = 0 \). Then we have
\[
\| D^m \psi_k \|_{L^2(\Omega)} \leq \| D^m(u - \varphi_k) \|_{L^2(\Omega)} + \| D^m(u - \varphi_{k-1}) \|_{L^2(\Omega)} \lesssim 2^{-k(s+\epsilon)}.
\]
Since \( \varphi_\ell = \sum_{k=0}^{\ell} \psi_k \) and \( |\psi_0|_{H^{m+s+\epsilon}}(\Omega) = 0 \), it follows from (10.3), (10.7) and (10.8) that
\[
K(u, 2^{-(s+\epsilon)\ell}) \leq \| u - \varphi_\ell \|_{H^m(\Omega)} + 2^{-(s+\epsilon)\ell} \sum_{k=1}^{\ell} 2^{k(s+\epsilon)} |\psi_k|_{H^m(\Omega)}
\]
\[
\lesssim 2^{-(s+\epsilon)\ell} + 2^{-(s+\epsilon)\ell} \sum_{k=1}^{\ell} 2^{k(s+\epsilon)} |\psi_k|_{H^m(\Omega)}
\]
\[
\lesssim \ell 2^{-(s+\epsilon)\ell}.
\]
\[
|u|_{(H^m(\Omega), H^{m+s+\epsilon}(\Omega)), 2} = \left( \sum_{k=0}^{\infty} [2^{k(s+\epsilon)} K(u, 2^{-k(s+\epsilon)})]^2 \right)^{1/2}
\]
\[
\lesssim \left( \sum_{k=0}^{\infty} [2^{k(s+\epsilon)(\theta-1)}]^2 \right)^{1/2}.
\]
Let \( \theta = 1 - \epsilon_0 \) with \( \epsilon_0 > 0 \) such that \( \epsilon - (s + \epsilon)\epsilon_0 > 0 \). This leads to
\[
|u|_{(H^m(\Omega), H^{m+s+\epsilon}(\Omega)), 2} \lesssim \left( \sum_{k=0}^{\infty} [k 2^{-k(s+\epsilon)}] \right)^{1/2} < \infty.
\]
This proves that \( u \in H^{m+(1-\epsilon_0)(s+\epsilon)}(\Omega) \) which is a proper subspace of \( H^{m+s}(\Omega) \) since \( \epsilon - (s + \epsilon)\epsilon_0 > 0 \), which contradicts with the fact that we only have the regularity \( u \in H^{m+s}(\Omega) \).

10.2 Proofs for the Conditions (H6)–(H8)

It follows from Davydov [14] that there exist piecewise polynomial spaces \( V_{m+1, h}^c \) with nodal basis over \( T_h \) such that \( V_{m+1, h}^c \) are nested and conforming in the sense that \( V_{m+1, h}^c \subset V_{m+1, h/2}^c \subset H^m(\Omega) \) for any \( 1 \leq n \) and \( m \leq n \).

This result actually proves the conditions (H6) and (H7). The proof of (H8) needs the interpolation of \( V_h \) into the conforming finite element space. To this end, we introduce the projection average interpolation operator of Brenner [8], Shi and Wang [36].

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Let $V^c_{m+1,h}$ be a conforming finite element space defined by $(K, P^c_K, D^c_K)$, where $D^c_K$ is the vector functional and the components of $D^c_K$ are defined as follows: for any $v \in C^k(K)$

$$d_{i,K}(v) := \begin{cases} \frac{1}{|F_i,K|} \int_{F_i,K} D_{i,K} v df & 1 \leq i \leq k_1, \\ \frac{1}{|k_i|} \int_{k_i} D_{i,K} v dx & k_1 < i \leq k_2, \\ \end{cases}$$

where $a_{i,K}$ are points in $K$, $F_{i,K}$ are non-zero-dimensional faces of $K$, $\kappa := \max_{1 \leq i \leq L} k(i)$ where $k(i)$ orders of derivatives used in degrees of freedom $D_{i,K} := \sum_{|\alpha| = k(i)} \eta_{\alpha,K} \partial^\alpha$, $1 \leq i \leq L$, $\eta_{\alpha,K}$ are constants which depend on $i, \alpha$, and $K$.

Let $\omega(a)$ denote the union of elements that share point $a$ and $\omega(F)$ the union of elements having in common the face $F$. Let $N(a)$ and $N(F)$ denote the number of elements in $\omega(a)$ and $\omega(F)$, respectively. For any $v \in V_h$, define the projection average interpolation operator $\Pi^c : V_h \rightarrow V^c_{m+1,h}$ by

1. For $1 \leq i \leq k_1$, if $a_{i,K} \in \partial \Omega$ and $d_{i,K}(\phi) = 0$ for any $\phi \in C^k(\overline{\Omega}) \cap V$, then $d_{i,K}(\Pi^c v | K) := 0$; otherwise $d_{i,K}(\Pi^c v | K) = \frac{1}{N(a_{i,K})} \sum_{K' \in \omega(a_{i,K})} D_{i,K}(v | K')(a_{i,K})$;

2. For $k_1 < i \leq k_2$, if $F_{i,K} \subset \partial \Omega$ and $d_{i,K}(\phi) = 0$ for any $\phi \in C^k(\overline{\Omega}) \cap V$, then $d_{i,K}(\Pi^c v | K) := 0$; otherwise $d_{i,K}(\Pi^c v | K) = \frac{1}{N(F_{i,K})} \sum_{K' \in \omega(F_{i,K})} \frac{1}{|F_{i,K}|} \int_{F_{i,K}} D_{i,K}(v | K')(a_{i,K}) df$;

3. For $k_2 < i \leq L$ $d_{i,K}(\Pi^c v | K) := \frac{1}{|K|} \int_{K} D_{i,K}(v | K) dx$.

**Lemma 10.2** For all nonconforming element spaces under consideration, there exists $r \in \mathbb{N}$, $r \geq m$ such that $V^c_h | K \subset P^r(K) \subset P^c_K$. Then, for $m < k \leq \min(r + 1, 2m)$, $0 \leq l \leq m$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, it holds that

$$|| D^m_h (v_h \ominus \Pi^c v_h) ||^2_{L^2(\Omega)} \lesssim \sum_{K \in T_h} \left( \frac{k-1}{2} \sum_{j=m}^{k-j} h^2(j-m) + \sum_{F \subset \partial K \cap \partial \Omega} \sum_{|\alpha| = j} || \partial^\alpha v_h ||^2_{0,F} \\
+ h_K \sum_{F \subset \partial K \cap \partial \Omega} \sum_{|\alpha| = m, \alpha_1 < m} \| \partial_1^{\alpha_1} v_h \|_{1,F} \cdot \tau_{F,2} \cdot \partial_1 \tau_{F,n} \cdot \tau_{F,n} \cdot \tau_{0,F} \right),$$

where $\tau_{F,2}, \ldots, \tau_{F,n}$ are $n - 1$ orthonormal tangent vectors of $F$.

**Proof** Since $V^c_h | K \subset P^r(K) \subset P^c_K$, a slight modification of the argument in [36, Lemma 5.6.4] can prove the desired result; see also Brenner [8] for the proof of the nonconforming linear element with $m = 1$. □
The remaining proof is based on bubble function techniques, see Carstensen and Hu [11] for a posteriori error analysis of second order problems, see Gudi [19], Hu and Shi [21] for a posteriori error analysis of fourth order problems. Let $v_h = u_h$ in the above lemma. Such an analysis leads to

$$
\| D_h^m (\Pi^c u_h - u_h) \|_{L^2(\Omega)} \lesssim \| D_h^m (u - u_h) \|_{L^2(\Omega)} \lesssim h^{s+\epsilon}.
$$

(10.12)

References

1. Armentano, M.G., Duran, R.G.: Mass-lumping or not mass-lumping for eigenvalue problem. Numer. Methods PDEs 19, 653–664 (2003)
2. Armentano, M.G., Duran, R.G.: Asymptotic lower bounds for eigenvalues by nonconforming finite element methods. ETNA 17, 93–101 (2004)
3. Babuška, I., Kellogg, R.B., Pitkäranta, J.: Direct and inverse error estimates for finite elements with mesh refinements. Numer. Math. 33, 447–471 (1979)
4. Babuška, I., Strouboulis, T.: The finite element method and its reliability. Oxford Science Publications, Oxford (2000)
5. Babuška, I., Osborn, J.E.: Eigenvalue problems. In: Ciarlet, P.G., Lions, J.L. (eds.) Handbook of Numerical Analysis, VII: Finite Element Methods (Part I). Elsevier, Amsterdam (1991)
6. Bergh, J., Löfström, J.: Interpolation Spaces: An Introduction. Springer, Berlin (1976)
7. Boffi, D.: Finite element approximation of eigenvalue problems. Acta Numerica 19, 1–120 (2010)
8. Brenner, S.: Poincaré-Friedrichs inequality for piecewise $H^1$ functions. SIAM J. Numer. Anal. 41, 306–324 (2003)
9. Brenner, S.C., Scott, R.L.: The Mathematical Theory of Finite Element Methods. Springer, Berlin (1996)
10. Carstensen, C., Gedicke, J.: An oscillation-free adaptive FEM for symmetric eigenvalue problem. Numer. Math. 118, 401–427 (2011)
11. Carstensen, C., Hu, J.: A unifying theory of a posteriori error control for nonconforming finite element methods. Numer. Math. 107, 473–502 (2007)
12. Chen, H.S., Li, B.: Superconvergence analysis and error expansion for the Wilson element. Numer. Math. 69, 125–140 (1994)
13. Crouzeix, M., Raviart, P.-A.: Conforming and nonconforming finite element methods for solving the stationary Stokes equations. RAIRO Anal. Numér. 7, 33–76 (1973)
14. Davydov, O.: Stable local bases for multivariate spline space. J. Approx. Theory 111, 267–297 (2001)
15. DeVore, R.A.: Nonlinear approximation. Acta Numerica 7, 51–150 (1998)
16. DeVore, R.A., Lorentz, G.G.: Constructive Approximation. Springer, Berlin (1993)
17. Forsythe, G.E.: Asymptotic lower bounds for the frequencies of certain polygonal membranes. Pacific J. Math. 4, 467–480 (1954)
18. Forsythe, G.E.: Asymptotic lower bounds for the fundamental frequency of convex membranes. Pacific J. Math. 4, 691–702 (1954)
19. Gudi, T.: A new error analysis for discontinuous finite element methods for linear elliptic problems. Math. Comput. 79, 2169–2189 (2010)
20. Hu, J., Huang, Y.Q., Shen, H.M.: The lower approximation of eigenvalue by lumped mass finite element methods. J. Comput. Math. 22, 545–556 (2004)
21. Hu, J., Shi, Z.C.: A new a posteriori error estimate for the Morley element. Numer. Math. 112, 25–40 (2009)
22. Hu, J., Shi, Z.C.: The lower bound of the error estimate in the $L^2$ norm for the Adini element of the biharmonic equation. arXiv:1211.4677 [math.NA] (2012)
23. Křížek, M., Roos, H., Chen, W.: Two-sided bounds of the discretizations error for finite elements. ESAIM: M2AN 45, 915–924 (2011)
24. Lascaux, P., Lesaint, P.: Some nonconforming finite elements for the plate bending problem. RAIRO Anal. Numer. 1, 9–53 (1975)
25. Li, Y.A.: Lower approximation of eigenvalue by the nonconforming finite element method. Math. Numer. Sin. 30, 195–200 (2008)
26. Li, Y.A.: The analysis of the lower approximation of eigenvalue by the Wilson element. Adv. Appl. Math. Mech. 3, 598–610 (2011)
27. Lin, Q., Huang, H.T., Li, Z.C.: New expansions of numerical eigenvalues for $-\Delta u = \lambda \rho u$ by nonconforming elements. Math. Comput. 77, 2061–2084 (2008)
28. Lin, Q., Huang, H.T., Li, Z.C.: New expansions of numerical eigenvalues by Wilson’s element. J. Comput. Appl. Math. 225, 213–226 (2009)
29. Lin, Q., Tobiska, L., Zhou, A.: On the superconvergence of nonconforming low order finite elements applied to the Poisson equation. IMA J. Numer. Anal. 25, 160–181 (2005)
30. Lin, Q., Lin, J.: Finite Element Methods: Accuracy and Improvements. Science Press, Beijing (2006)
31. Liu, H.P., Yan, N.N.: Four finite element solutions and comparison of problem for the poisson equation eigenvalue. Chinese J. Numer. Meth. Comput. Appl. 2, 81–91 (2005)
32. Morley, L.S.D.: The triangular equilibrium element in the solutions of plate bending problem. Aero. Quart. 19, 149–169 (1968)
33. Rannacher, R.: Nonconforming finite element methods for eigenvalue problems in linear plate theory. Numer. Math. 33, 23–42 (1979)
34. Rannacher, R., Turek, S.: Simple nonconforming quadrilateral Stokes element. Numer. Methods PDEs 8, 97–111 (1992)
35. Shi, Z.C.: A remark on the optimal order of convergence of Wilson nonconforming element. Math. Numer. Sin. 8, 159–163 (1986). (in Chinese)
36. Shi, Z.C., Wang, M.: The Finite Element Method. Science Press, Beijing (2010). (in Chinese)
37. Strang, G., Fix, G.: An Analysis of the Finite Element Method. Prentice-Hall, Englewood Cliffs (1973)
38. Wang, M., Xu, J.C.: Minimal finite-element spaces for 2m-th order partial differential equations in $\mathbb{R}^n$. Math. Comp. 82, 25–43 (2013)
39. Weiberger, H.F.: Upper and lower bounds by finite difference methods. Comm. Pure Appl. Math. 9, 613–623 (1956)
40. Weiberger, H.F.: Lower bounds for higher eigenvalues by finite difference methods. Pacific J. Math. 8, 339–368 (1958)
41. Widlund, O.: On best error bounds for approximation by piecewise polynomial functions. Numer. Math. 27, 327–338 (1977)
42. Wilson, E.L., Taylor, R.L., Doherty, W.P., Ghaboussi, J.: Incompatible displacement methods. In: Fenves, S.J. (ed.) Numerical and Computer Methods in Structural Mechanics, pp. 43–57. Academic Press, New York (1973)
43. Yang, Y.D.: A posteriori error estimates in Adini finite element for eigenvalue problems. J. Comput. Math. 18, 413–418 (2000)
44. Yang, Y.D., Bi, H.: Lower spectral bounds by Wilson’s brick discretization eigen-value. Appl. Numer. Math. 60, 782–787 (2010)
45. Yang, Y.D., Lin, F.B., Zhang, Z.M.: N-simplex Crouzeix-Raviart element for second order elliptic/eigenvalue problems. Int. J. Numer. Anal. Model. 6, 615–626 (2009)
46. Yang, Y.D., Zhang, Z.M., Lin, F.B.: Eigenvalue approximation from below using nonforming finite elements. Science in China: Mathematics 53, 137–150 (2010)
47. Yang, Y.D., Lin, Q., Bi, H., Li, Q.: Lower eigenvalues approximation by Morley elements. Adv. Comput. Math. 36, 443–450 (2012)
48. Zhang, Z., Yang, Y., Chen, Z.: Eigenvalue approximation from below by Wilson’s elements. Chinese J. Num. Math. Appl. 29, 81–84 (2007)
49. Zienkiewicz, O.C., Cheung, Y.K.: The Finite Element Method in Structural and Continuum Mechanics. McGraw-Hill, New York (1967)