An improved upper bound for critical value of the contact process on $\mathbb{Z}^d$ with $d \geq 3$

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Abstract: In this paper we give an improved upper bound for critical value $\lambda_c$ of the basic contact process on the lattice $\mathbb{Z}^d$ with $d \geq 3$. As a direct corollary of our result,

$$\lambda_c \leq 0.340$$

when $d = 3$.

Keywords: contact process, critical value, upper bound, linear system.

1 Introduction

In this paper we are concerned with the basic contact process on $\mathbb{Z}^d$ with $d \geq 3$. First we introduce some notations. For each $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$, we use $||x||$ to denote the $l_1$-norm of $x$, i.e.,

$$||x|| = \sum_{i=1}^{d} |x_i|.$$  

For any $x, y \in \mathbb{Z}^d$, we write $x \sim y$ when and only when $||x - y|| = 1$, i.e., $x \sim y$ means that $x$ and $y$ are neighbors on $\mathbb{Z}^d$. For $1 \leq i \leq d$, we use $e_i$ to denote the $i$th elementary unit vector of $\mathbb{Z}^d$, i.e.,

$$e_i = (0, \ldots, 0, 1_{\text{ith}}, 0, \ldots, 0).$$  

We use $O$ to denote the origin of $\mathbb{Z}^d$.

The contact process $\{\eta_t\}_{t \geq 0}$ on $\mathbb{Z}^d$ is a spin system with state space $\{0, 1\}^{\mathbb{Z}^d}$ (see the definition of the spin system in Chapter 3 of [4]). The flip rates function of $\{\eta_t\}_{t \geq 0}$ is given by

$$c(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \sum_{y \sim x} \eta(y) & \text{if } \eta(x) = 0 \end{cases}$$

for any $(\eta, x) \in \{0, 1\}^{\mathbb{Z}^d} \times \mathbb{Z}^d$, where $\lambda > 0$ is a constant called the infection rate. That is to say, the state of the process flips from $\eta$ to $\eta^x$ at rate $c(x, \eta)$, where

$$\eta^x(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ 1 - \eta(x) & \text{if } y = x. \end{cases}$$

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Intuitively, the contact process describes the spread of an epidemic on the graph. Vertices in state 1 are infected while that in state 0 are healthy. An infected vertex waits for an exponential time with rate 1 to become healthy while an healthy one is infected at rate proportional to the number of infected neighbors.

The contact process is introduced by Harris in [2]. For a detailed survey of the study of the contact process, see Chapter 6 of [4] and Part one of [6].

In this paper we are mainly concerned with the critical value of the contact process. Assuming that $\eta_0(x) = 1$ for any $x \in \mathbb{Z}^d$, then the critical value $\lambda_c$ is defined as

$$\lambda_c = \sup \{ \lambda : \lim_{t \to +\infty} P_\lambda(\eta_t(O) = 1) = 0 \}, \tag{1.3}$$

where $P_\lambda$ is the probability measure of the contact process with infection rate $\lambda$. The definition of $\lambda_c$ is reasonable according to the following property of the contact process. For $\lambda_1 \geq \lambda_2$ and $t > s$, conditioned on all the vertices are in state 1 at $t = 0$,

$$P_{\lambda_1}(\eta_s(O) = 1) \geq P_{\lambda_2}(\eta_t(O) = 1). \tag{1.4}$$

A rigorous proof of Equation (1.4) is given in Section 6.1 of [4].

When $d = 1$, it is shown in Section 6.1 of [4] that $\lambda_c(1) \leq 2$. Liggett improves this result in [5] by showing that $\lambda_c(1) \leq 1.94$. For $d \geq 3$, it is shown in [3] that

$$\lambda_c(d) \leq \alpha_1(d) = \frac{1}{\gamma_d} - 1$$

while it is shown in [1] that

$$\lambda_c(d) \leq \alpha_2(d) = \frac{1}{2d(2\gamma_d - 1)},$$

where $\gamma(d) > 1/2$ is the probability that the simple random walk on $\mathbb{Z}^d$ starting at $O$ never returns to $O$. Both these two results lead to the conclusion that

$$\lim_{d \to +\infty} 2d\lambda_c(d) = 1.$$  

When $d = 3$, according to the well-known result that $\gamma_3 \approx 0.659$,

$$\alpha_1(3) = 0.517 < \alpha_2(3) = 0.523.$$  

However, $\alpha_2(d) < \alpha_3(d)$ for sufficiently large $d$ according to the fact that

$$\frac{1}{\gamma_d} - 1 = \frac{1}{2d} + \frac{3}{4d^2} + o\left(\frac{1}{d^2}\right)$$

while

$$\frac{1}{2d(2\gamma_d - 1)} = \frac{1}{2d} + \frac{1}{2d^2} + o\left(\frac{1}{d^2}\right).$$

In this paper, we will give another upper bound $\beta(d)$ for the critical value $\lambda_c(d)$ when $d \geq 3$. $\beta(d)$ satisfies that $\beta(d) < \min\{\alpha_1(d), \alpha_2(d)\}$ for each $d \geq 3$. For the precise result, see the next section.
2 Main result

In this section we will give our main result. First we introduce some notations and definitions. From now on we assume that at \( t = 0 \) all the vertices on \( \mathbb{Z}^d \) are in state 1 for the contact process, then let \( \lambda_c \) be the critical value of the contact process defined as in Equation (1.3).

We write \( \lambda_c \) as \( \lambda_c(d) \) when we need to point out the dimension \( d \) of the lattice. We denote by \( \{S_n\}_{n \geq 0} \) the simple random walk on \( \mathbb{Z}^d \), i.e.,

\[
P(S_{n+1} = y | S_n = x) = \frac{1}{2d}
\]

for each \( y \) that \( y \sim x \) and \( n \geq 0 \). We define

\[
\gamma = P(S_n \neq O \text{ for all } n \geq 1 | S_0 = O)
\]

as the probability that the simple random walk never return to \( O \) conditioned on \( S_0 = O \). We write \( \gamma \) as \( \gamma_d \) when we need to point out the dimension \( d \) of the lattice.

The following theorem gives an upper bound of \( \lambda_c(d) \) for \( d \geq 3 \), which is our main result.

**Theorem 2.1.** For each \( d \geq 3 \),

\[
\lambda_c(d) \leq \frac{2 - \gamma_d}{2d\gamma_d}
\]

It is shown in [1] that \( \lambda_c(d) \leq \alpha_2(d) = \frac{1}{2d(2\gamma_d - 1)} \) for each \( d \geq 3 \). Since \( \gamma_d < 1 \),

\[
(2 - \gamma_d)(2\gamma_d - 1) - \gamma_d = -2(\gamma_d - 1)^2 < 0
\]

and hence \( \frac{2 - \gamma_d}{2d\gamma_d} < \alpha_2(d) \) for each \( d \geq 3 \). It is shown in [3] that \( \lambda_c(d) \leq \frac{1}{\gamma_d} - 1 \) for each \( d \geq 3 \). By direct calculation,

\[
1 - \gamma \geq P(S_2 = O | S_0 = O) + P(S_4 = O, S_2 \neq O | S_0 = O)
\]

\[
= \frac{4d^2 + 4d - 3}{8d^2} > \frac{1}{2d - 1}
\]

when \( d \geq 3 \) and hence \( \frac{2 - \gamma_d}{2d\gamma_d} < \alpha_1(d) \) for each \( d \geq 3 \).

For \( d = 3 \), according to the well known result that \( \gamma_3 \approx 0.659 \), we have the following direct corollary.

**Corollary 2.2.**

\[
\lambda_c(3) \leq \frac{2 - \gamma_3}{6\gamma_3} \leq 0.340.
\]

This corollary improves the upper bound of \( \lambda_c(3) \) given by \( \alpha_1(3) \), which is 0.517. According to the example given in Section 3.5 of [4],

\[
\lambda_c(d) \geq \frac{1}{2d - 1}
\]

for each \( d \geq 1 \) and hence \( \lambda_c(d) \in [0.2, 0.340] \).

We will prove Theorem 2.1 in the next section. A Markov process \( \{\xi_t\}_{t \geq 0} \) with state space \( [0, +\infty)^2 \) will be introduced as a main auxiliary tool for the proof. The definition of \( \{\xi_t\}_{t \geq 0} \) is similar with that of the binary contact path process introduced in [1], except for some modifications in several details.
3 Proof of Theorem 2.1

In this section we give the proof of Theorem 2.1. Throughout this section we assume that the dimension $d$ is fixed and at least 3, which ensures that $\gamma > \frac{1}{2}$. Our aim is to prove the following lemma, Theorem 2.1 follows from which directly.

**Lemma 3.1.** If $a, b > 0$ satisfies

$$2(a + b - 1) - (a^2 + b^2 - 1) - 2ab(1 - \gamma) > 0$$

then

$$\lambda_c \leq \frac{1}{2d(2(a + b - 1) - (a^2 + b^2 - 1) - 2ab(1 - \gamma))}.$$

If we choose $a = b = 1$, then Lemma 3.1 gives the upper bound of $\lambda_c$ the same as that given in [1]. However, the best choices of $a, b$ are $a = b = \frac{1}{2\gamma - 1}$, which gives the following proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let $L(a, b) = 2(a + b - 1) - (a^2 + b^2 - 1) - 2ab(1 - \gamma)$, then

$$\sup \{L(a, b) : a > 0, b > 0\} = L\left(\frac{1}{2\gamma - 1}, \frac{1}{2\gamma - 1}\right) = \frac{\gamma}{2 - \gamma}.$$

As a result, let $a = b = \frac{1}{2\gamma - 1}$, then

$$\lambda_c \leq \frac{1}{2dL(a, b)} = \frac{2 - \gamma}{2d\gamma}$$

according to Lemma 3.1. \(\square\)

The remainder of this paper is devoted to the proof of Lemma 3.1. From now on we assume that $a, b$ are positive constants which satisfies

$$2(a + b - 1) - (a^2 + b^2 - 1) - 2ab(1 - \gamma) > 0.$$

Let $\{\xi_t\}_{t \geq 0}$ be a continuous time Markov process with state space $[0, +\infty)^{2d}$ and generator function given by

$$\Omega f(\xi) = \sum_{x \in \mathbb{Z}^d} \left[ f(\xi^{x,0}) - f(\xi) \right] + \sum_{x \in \mathbb{Z}^d} \sum_{y : y \sim x} \lambda \left[ f(\xi^{x,y}_{a,b}) - f(\xi) \right]$$

$$\quad + \sum_{x \in \mathbb{Z}^d} f'_z(\xi) \left(1 - 2d\lambda[(b - 1) + a]\right) \xi(x)$$

for any $\xi \in [0, +\infty)^{2d}$ and sufficiently smooth function $f$ on $[0, +\infty)^{2d}$, where

$$\xi^{x,0}(y) = \begin{cases} \xi(y) & \text{if } y \neq x, \\ 0 & \text{if } y = x, \end{cases}$$

$$\xi^{x,y}_{a,b}(z) = \begin{cases} \xi(z) & \text{if } z \neq x, \\ b\xi(x) + a\xi(y) & \text{if } z = x \end{cases}$$

and $f'_z$ is the partial derivative of $f(\xi)$ with respect to the coordinate $\xi(x)$.
If \( a = b = 1 \), then \( \{\xi_t\}_{t \geq 0} \) is the binary contact path process introduced in \( \text{[1]} \) after a time-scaling. \( \{\xi_t\}_{t \geq 0} \) belongs to a large crowd of continuous-time Markov processes called linear systems. For the definition and basic properties of the linear system, see Chapter 9 of \( \text{[1]} \).

According to the definition of \( \Omega \), \( \{\xi_t\}_{t \geq 0} \) evolves as follows. For each \( x \in \mathbb{Z}^d \) and each neighbor \( y \) of \( x \), \( \xi_t(x) \) flips to 0 at rate 1 while flips to \( b\xi_t(x) + a\xi_t(y) \) at rate \( \lambda \). Between the jumping moments of \( \{\xi_t(x)\}_{t \geq 0}, \xi_t(x) \) evolves according to the ODE

\[
\frac{d}{dt}\xi_t(x) = \left(1 - 2d\lambda[(b - 1) + a]\right)\xi_t(x). \tag{3.2}
\]

That is to say, if \( \xi(x) \) does not jump during \([t, t + s]\), then

\[
\xi_{t+r}(x) = \xi_t(x) \exp \left\{ \int \left(1 - 2d\lambda[(b - 1) + a]\right) \right\}
\]

for \( 0 < r < s \).

The linear system \( \{\xi_t\}_{t \geq 0} \) and the contact process \( \{\eta_t\}_{t \geq 0} \) have the following relationship.

**Lemma 3.2.** For any \( x \in \mathbb{Z}^d \) and \( t \geq 0 \), let

\[
\hat{\eta}_t(x) = \begin{cases} 
1 & \text{if } \xi_t(x) > 0, \\
0 & \text{if } \xi_t(x) = 0,
\end{cases}
\]

then \( \{\hat{\eta}_t\}_{t \geq 0} \) is a version of the contact process introduced in Equation \( \text{[1,2]} \).

**Proof of Lemma 3.2.** ODE (3.2) can not make \( \{\xi_t(x)\}_{t \geq 0} \) flip from 0 to a positive value or flip from a positive value to 0, hence \( \hat{\eta}_t(x) \) stays its value between jumping moments of \( \xi(x) \). If \( \hat{\eta}_t(x) = 1 \), i.e, \( \xi_t(x) > 0 \), then \( \hat{\eta}_t(x) \) flips to 0 when and only when \( \xi_t(x) \) flips to 0 at some jumping moment. As a result, \( \hat{\eta}_t(x) \) flips from 1 to 0 at rate 1. If \( \hat{\eta}_t(x) = 0 \), i.e, \( \xi_t(x) = 0 \), then \( \hat{\eta}_t(x) \) flips to 1 when and only when \( \xi_t(x) \) flips to

\[
b\xi_t(x) + a\xi_t(y) = a\xi_t(y)
\]

for a neighbor \( y \) with \( \xi_t(y) > 0 \) at some jumping moment. As a result, \( \hat{\eta}_t(x) \) flips from 0 to 1 at rate

\[
\lambda \sum_{y:y \sim x} 1_{(\xi_t(y) > 0)} = \lambda \sum_{y:y \sim x} \hat{\eta}_t(y),
\]

where \( 1_A \) is the indicator function of the event \( A \). In conclusion, \( \{\hat{\eta}_t\}_{t \geq 0} \) evolves in the same way as a contact process evolves according to the flip rates function given in Equation \( \text{[1,2]} \).

By Lemma 3.2 from now on we assume that \( \{\eta_t\}_{t \geq 0} \) and \( \{\xi_t\}_{t \geq 0} \) are coupled under the same probability space such that \( \eta_0(x) = \xi_0(x) = 1 \) for each \( x \in \mathbb{Z}^d \) and \( \eta_t(x) = 1 \) when and only when \( \xi_t(x) > 0 \).

The following two lemmas about expectations of \( \xi_t(x) \) and \( \xi_t(x)\xi_t(y) \) are important for the proof of Lemma 3.1.

**Lemma 3.3.** If \( \xi_0(x) = 1 \) for any \( x \in \mathbb{Z}^d \), then

\[
E\xi_t(x) = 1
\]

for any \( x \in \mathbb{Z}^d \) and \( t \geq 0 \).
Lemma 3.4. For any $x \in \mathbb{Z}^d$ and $t \geq 0$, let $F_t(x) = E[\xi_t(O)\xi_t(x)]$, then conditioned on $\xi_0(x) = 1$ for all $x \in \mathbb{Z}^d$,

$$
\frac{d}{dt}F_t = \left( \frac{d}{dt}F_t(x) \right)_{x \in \mathbb{Z}^d} = G_\lambda F_t,
$$

(3.3)

where $G_\lambda$ is a $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix that

$$
G_\lambda(x, y) = \begin{cases} 
-4a\lambda d & \text{if } x \neq 0 \text{ and } x = y, \\
2a\lambda & \text{if } x \neq 0 \text{ and } x \sim y, \\
1 - 4d\lambda(b - 1) - 4d\lambda a + 2d\lambda(b^2 - 1) + 2d\lambda a^2 & \text{if } x = y = 0, \\
4abd\lambda & \text{if } x = 0 \text{ and } y = e_1, \\
0 & \text{otherwise}
\end{cases}
$$

and $e_1$ is defined as in Equation (1.1).

Note that when we say $F_1 = GF_2$ for functions $F_1, F_2$ on $\mathbb{Z}^d$ and $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix $G$, we mean

$$
F_1(x) = \sum_{y \in \mathbb{Z}^d} G(x, y)F_2(y)
$$

for each $x \in \mathbb{Z}^d$, as the product of finite-dimensional matrices.

The proofs of Lemmas 3.3 and 3.4 rely heavily on Theorems 9.1.27 and 9.3.1 of [4]. These two theorems can be seen as the extension of the Hille-Yosida Theorem for the linear system, which ensures that we can execute the calculation

$$
\frac{d}{dt}S(t)f = S(t)\Omega f
$$

(3.4)

for a linear system with generator $\Omega$ and semi-group $\{S_t\}_{t \geq 0}$ when $f$ has the form $f(\xi) = \xi(x)$ or $f(\xi) = \xi(x)\xi(y)$.

Proof of Lemma 3.3. By the generator $\Omega$ of $\{\xi_t\}_{t \geq 0}$ and Theorem 9.1.27 of [4] (i.e., Equation (3.3) for $f(\xi) = \xi(x)$),

$$
\frac{d}{dt}E\xi_t(x) = -E\xi_t(x) + \lambda \sum_{y \sim x} \left[(b - 1)E\xi_t(x) + aE\xi_t(y)\right] + \left(1 - 2d\lambda[(b - 1) + a]\right)E\xi_t(x)
$$

for each $x \in \mathbb{Z}^d$. Since $\xi_0(x) = 1$ for all $x \in \mathbb{Z}^d$, $E\xi_t(x)$ does not depend on the choice of $x$ according to the spatial homogeneity of $\{\xi_t\}_{t \geq 0}$. Therefore,

$$
\frac{d}{dt}E\xi_t(x) = -E\xi_t(x) + \lambda \sum_{y \sim x} \left[(b - 1)E\xi_t(x) + aE\xi_t(y)\right] + \left(1 - 2d\lambda[(b - 1) + a]\right)E\xi_t(x)
$$

$$
= -E\xi_t(x) + 2d\lambda(a + b - 1)E\xi_t(x) + \left(1 - 2d\lambda(a + b - 1)\right)E\xi_t(x) = 0.
$$

As a result, $E\xi_t(x) \equiv E\xi_0(x) = 1$. 

\qed

Proof of Lemma 3.4. According to the generator $\Omega$ of $\{\xi_t\}_{t \geq 0}$ and Theorem 9.3.1 of [4] (i.e., Equation (3.3) for $f(\xi) = \xi(x)\xi(y)$),

$$
\frac{d}{dt}F_t(x) = -2F_t(x) + \lambda \sum_{y \sim x} \left[(b - 1)F_t(0) + aE[\xi_t(y)\xi_t(x)]\right]
$$

$$
+ \lambda \sum_{y \sim x} \left[(b - 1)F_t(0) + aF_t(y)\right] + 2\left(1 - 2d\lambda(a + b - 1)\right)F_t(x)
$$

(3.5)
when \( x \neq O \) while

\[
\frac{d}{dt} F_t(O) = -F_t(O) + \lambda \sum_{y \sim O} 2abF_t(y) + 2d\lambda(b^2 - 1)F_t(O) + \lambda \sum_{y \sim O} a^2 E[\xi_t^2(y)] \\
+ 2(1 - 2d\lambda(a + b - 1))F_t(O).
\]

(3.6)

Since \( \xi_0(x) = 1 \) for any \( x \in \mathbb{Z}^d \), according to the spatial homogeneity of \( \{\xi_t\}_{t \geq 0} \),

\[
E[\xi_t(x)\xi_t(y)] = F_t(y - x) = F_t(x - y)
\]

for any \( x, y \in \mathbb{Z}^d \) and

\[
F_t(e_i) = F_t(-e_i) = F_t(e_1)
\]

for \( 1 \leq i \leq d \). Therefore, by Equations (3.5) and (3.6),

\[
\frac{d}{dt} F_t(x) = \begin{cases} 
-4ad\lambda F_t(x) + 2a\lambda \sum_{y \sim x} F_t(y) & \text{if } x \neq O, \\
[1 - 4d\lambda(a + b - 1) + 2d\lambda(b^2 - 1) + 2da^2 \lambda]F_t(O) + 4abcd\lambda F_t(e_1) & \text{if } x = O.
\end{cases}
\]

(3.7)

Lemma 3.4 follows from Equation (3.7) directly.

The following lemma shows that if \( \lambda \) ensures the existence of an positive eigenvector of \( G_\lambda \) with respect to the eigenvalue 0, then \( \lambda \) is an upper bound of \( \lambda_c \), which is crucial for us to prove Lemma 3.1.

**Lemma 3.5.** If there exists \( K : \mathbb{Z}^d \to [0, +\infty) \) that

\[
\inf_{x \in \mathbb{Z}^d} K(x) > 0
\]

\( G_\lambda K = 0 \) (here \( 0 \) means the zero function on \( \mathbb{Z}^d \)),

where \( G_\lambda \) is defined as in Lemma 3.4, then

\[
\lambda \geq \lambda_c.
\]

We give the proof of Lemma 3.5 at the end of this section. Now we show how to utilize Lemma 3.5 to prove Lemma 3.1.

**Proof of Lemma 3.7.** Let \( \{S_n\}_{n \geq 0} \) be the simple random walk on \( \mathbb{Z}^d \) as we have introduced in Section 2, then we define

\[
H(x) = P(S_n = O \text{ for some } n \geq 0 | S_0 = x)
\]

for any \( x \in \mathbb{Z}^d \). Then \( H(O) = 1 \) and

\[
H(x) = \frac{1}{2^d} \sum_{y : y \sim x} H(y)
\]

(3.8)

for any \( x \neq O \). According to the spatial homogeneity of the simple random walk,

\[
\gamma = P(S_n \neq O \text{ for all } n \geq 1 | S_0 = O) \\
= P(S_n \neq O \text{ for all } n \geq 0 | S_0 = e_1) = 1 - H(e_1).
\]

(3.9)

For \( a, b > 0 \) that

\[
2(a + b - 1) - (a^2 + b^2 - 1) - 2ab(1 - \gamma) > 0
\]

For \( a, b > 0 \) that

\[
2(a + b - 1) - (a^2 + b^2 - 1) - 2ab(1 - \gamma) > 0
\]
and \( \lambda > \frac{1}{2d[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)]} \), we define

\[
K(x) = H(x) + \frac{2d\lambda[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)] - 1}{1 + 2d\lambda(a+b-1)^2}
\]

for each \( x \in \mathbb{Z}^d \). Then,

\[
\inf_{x \in \mathbb{Z}^d} K(x) \geq \frac{2d\lambda[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)] - 1}{1 + 2d\lambda(a+b-1)^2} > 0
\]

and \( G_\lambda K = 0 \) according to Equations (3.8), (3.9) and the definition of \( G_\lambda \). As a result, by Lemma 3.5, \( \lambda \geq \lambda_c \) for any \( \lambda > \frac{1}{2d[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)]} \) and hence

\[
\lambda_c \leq \frac{1}{2d[2(a+b-1)-(a^2+b^2-1)-2ab(1-\gamma)]}.
\]

At last we give the proof of Lemma 3.5.

**Proof of Lemma 3.5** For any \( x, y \in \mathbb{Z}^d \), we define

\[
G_\lambda^0(x, y) = \sum_{u \in \mathbb{Z}^d} G_\lambda(x, u)G_\lambda(u, y).
\]

It is easy to check that the sum in the right-hand side converges since only finite terms are not zero. By induction, if \( G_\lambda^n \) is well-defined for \( 1 \leq k \leq n \), then we define

\[
G_\lambda^{n+1}(x, y) = \sum_{u \in \mathbb{Z}^d} G_\lambda^n(x, u)G_\lambda(u, y).
\]

It is easy to check that \( G_\lambda^n \) is well-defined for each \( n \geq 1 \) according to the definition of \( G_\lambda \) and

\[
\sup_{x,y \in \mathbb{Z}^d} \sum_{n=0}^{+\infty} \frac{t^n|G_\lambda^n(x, y)|}{n!} < +\infty
\]

for any \( t \geq 0 \), where \( G_\lambda^0(x, y) = 1_{\{x=y\}} \). Then, it is reasonable to define the \( \mathbb{Z}^d \times \mathbb{Z}^d \) matrix \( e^{tG_\lambda} \) as

\[
e^{tG_\lambda}(x, y) = \sum_{n=0}^{+\infty} \frac{t^nG_\lambda^n(x, y)}{n!}
\]

for \( x, y \in \mathbb{Z}^d \) and \( t \geq 0 \). Since \( K \) satisfies \( G_\lambda K = 0 \),

\[
G_\lambda^n K = G_\lambda^{n-1}G_\lambda K = 0
\]

for each \( n \geq 1 \) and hence

\[
(e^{tG_\lambda} K)(x) = \sum_{y \in \mathbb{Z}^d} e^{tG_\lambda}(x, y)K(y) = \sum_{y \in \mathbb{Z}^d} G_\lambda^0(x, y)K(y) = K(x)
\]

(3.10)

for each \( x \in \mathbb{Z}^d \) and \( t \geq 0 \), i.e., \( K \) is the eigenvector of \( e^{tG_\lambda} \) with respect to the eigenvalue 1.
For any $\xi \in (-\infty, +\infty)^{Z^d}$, we define
\[ \|\xi\|_\infty = \sup_{x \in Z^d} |\xi(x)|. \]
Furthermore, we define
\[ W = \{\xi \in (-\infty, +\infty)^{Z^d} : \|\xi\|_\infty < +\infty\}, \]
then $W$ is a Banach space with norm $\| \cdot \|_\infty$. By the definition of $G_\lambda$, it is easy to check that there exists $M > 0$ that
\[ \|G_\lambda(\xi_1 - \xi_2)\|_\infty \leq M \|\xi_1 - \xi_2\|_\infty \]
for any $\xi_1, \xi_2 \in W$, i.e., ODE (3.3) satisfies Lipschitz condition. As a result, according to the theory of the linear ODE on the Banach space, ODE (3.3) has the unique solution that
\[ F_t = e^{tG_\lambda}F_0 \]
for any $t \geq 0$. Since $F_0(x) = 1$ for any $x \in Z^d$,
\[ F_t(O) = \sum_{y \in Z^d} e^{tG_\lambda}(O, y)F_0(y) = \sum_{y \in Z^d} e^{tG_\lambda}(O, y). \]
Since $G_\lambda(x, y) \geq 0$ when $x \neq y$, $e^{tG_\lambda}(x, y) \geq 0$ for any $x, y \in Z^d$. Therefore, by Equation (3.10),
\[ E(\xi_t^2(O)) = F_t(O) \leq \sum_{y \in Z^d} e^{tG_\lambda}(O, y) \frac{K(y)}{\inf_{x \in Z^d} K(x)} = \frac{K(O)}{\inf_{x \in Z^d} K(x)} \]
for any $t \geq 0$. According to Lemmas 3.2, 3.3, Equation (3.11) and Cauchy-Schwartz inequality,
\begin{align*}
\lim_{t \to +\infty} P_\lambda(\eta_t(O) = 1) &= \lim_{t \to +\infty} P_\lambda(\xi_t(O) > 0) \\
&\geq \lim_{t \to +\infty} \sup_{t \to +\infty} \frac{(E\xi_t(O))^2}{E(\xi_t^2(O))} = \lim_{t \to +\infty} \frac{1}{E(\xi_t^2(O))} \\
&\geq \frac{\inf_{x \in Z^d} K(x)}{K(O)} > 0.
\end{align*}
As a result,
\[ \lambda \geq \lambda_c \]
for any $\lambda$ that there exists $K$ which satisfies $\inf_{x \in Z^d} K(x) > 0$ and $G_\lambda K = 0$.

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