0. Introduction

One of the main outcomes of the theory of quantum groups has been the discovery, by Kashiwara and Lusztig, of certain distinguished bases in quantized enveloping algebras with very favorable properties: positivity of structure constants, compatibility with all highest weight integrable representations, etc. These canonical bases, which are in some ways analogous to the Kazhdan–Lusztig bases of Hecke algebras, have proven to be a powerful tool in the study of the representation theory of (quantum) Kac–Moody algebras, especially in relation to solvable lattice and vertex models in statistical mechanics (see e.g. [JMMO], [KMS]). They also sometimes carry representation-theoretic information themselves, encoding character formulas and decomposition numbers ([A], [LLT], [VV]). At the same time, and in another direction, they have found some applications in algebraic geometry ([E]) and have inspired some recent deep work in combinatorics (e.g. [FZ] and subsequent work).

The construction of the canonical basis $B$ of a quantized enveloping algebra $U_v^+(g)$ in $L_u^+$ is based on Ringel’s earlier discovery that $U_v^+(g)$ may be realized in the Hall algebra of the category of representations of a quiver $Q$ whose underlying graph is the Dynkin diagram of $g$ ([E]). The set of isomorphism classes of representations of $Q$ of a given class $d$ in $K_0(\text{Rep } Q)$ is the set of orbits of a reductive group...
G₃ on a vector space E₃; Lusztig realizes U₊(g) geometrically as a convolution algebra of (G₃-equivariant, semi-simple) constructible sheaves on E₃, and obtains the canonical basis as the set of all simple perverse sheaves appearing in this algebra.

The present work is an attempt to carry out the above constructions in the situation when g is no longer a Kac-Moody algebra but a loop algebra of a Kac-Moody algebra. A n a generalized version of Ringel's construction was already considered in [51]: the Hall algebra of the category of coherent sheaves on a weighted projective line (in the sense of [41]) provides a realization of a positive part of the (quantum) a quantization U₊(Lg) of a Kac-Moody algebra g associated to a star Dynkin diagram. In the case of a projective line with no weight, which was first considered by Kapranov [34], this gives the positive part of U₊(B₃) in Drinfeld's presentation.

In this paper, we propose a construction of a canonical basis B (for the positive part U₊(Ln)) of U₊(Lg), when g is associated to a star Dynkin diagram. Rather than categories of coherent sheaves on a weighted projective line we use the (equivalent) categories of coherent sheaves on a smooth projective curve X, equivariant with respect to a fixed group G Aut(X) satisfying X=ℂ'P¹. We consider a convolution algebra U₈ of (equivariant, semi-simple) constructible sheaves on various Quot schemes, and take as a basis of U₈ the set of all simple perverse sheaves appearing in this way. Note that we need the (smooth part) of the whole Quot scheme and not only its (semi)stable part. One difference between the coherent sheaves and the quiver situation is that in the former all regular indecomposables (torsion sheaves) lie at the top of the category (i.e., they admit no nonzero homomorphism to a nonregular indecomposable). For instance, when X = ℂ'P¹, this makes the picture essentially of finite type, although the corresponding quantum group is a quantization of a Lie algebra (in particular, it is easy to explicitly describe all simple perverse sheaves occurring in the canonical basis). On the other hand, unlike the quiver case, the lattice of submodules has an infinite depth in general, and it is necessary to consider an inductive limit of Quot schemes (or the category of perverse sheaves on such an inductive limit). As a consequence, our canonical basis B naturally lives in a completion B₈ of U₈.

We conjecture that the algebra B₈ C(v) is isomorphic to a completion B₊(Ln) of U₊(Ln) and prove this conjecture in the finite type case (when X = ℂ'P¹, i.e., when U₊(Lg) is a quantum affine algebra), and in the tame case (when X is an elliptic curve and U₊(Lg) is a quantum toroidal algebra).

For instance, in the simplest case X = ℂ'P¹; G = Id, we get a canonical basis of (a completion of) the positive part of U₊(B₃) in Drinfeld's presentation. More interestingly, B₈ C(v) and B₊(Ln) are isomorphic when X is of genus one and G = ℂ₂; Z=32; Z'=42 or Z=62. This implies the existence of a canonical basis in quantum toroidal algebras of type D₄; E₆; E₇; E₈, and provides a first example of a canonical basis in a non Kac-Moody setting. We give a parametrization of this canonical basis as well as a description of all the corresponding simple perverse sheaves.

In [53], we will also construct a canonical basis for HVERTISE for HVERTISE when X = E is an elliptic curve and G = Id. The algebra HVERTISE is an elliptic analog of a Heisenberg algebra and its canonical basis should have an interesting combinatorial description.
Of course, when g is of finite type, the loop algebra Lg is again a Kac-Moody algebra, and there is already a canonical basis B for (the Kac-Moody positive part of) the quantum group U_q(Lg) by Kashiwara and Lusztig. Since B and \( \mathcal{B} \) are bases of subalgebras corresponding to two different Borel subalgebras, it is not possible to compare them directly. However, we conjecture that the basis \( \mathcal{B} \) is also compatible with the highest weight integrable representations, and we show in some examples (for \( g = s_0 \) and the fundamental representation \( 0 \)) that B and \( \mathcal{B} \) project to the same bases in such a module. The action of \( \mathcal{B} \) on \( 0 \) is particularly well-suited for the \"sem in finite\" description of \( 0 \) given by Feigin and Stoyanovskii. \[ FS \] : the natural action of \( \mathcal{B} \) (constructed from the Harder-Narasimhan stratification) induces the action of \( 0 \) by \"principal subspaces\" considered in \[ FS \]. The coincidence of B and \( \mathcal{B} \) in \( 0 \) suggests that B and \( \mathcal{B} \) are incarnations of the same \"canonical basis\", which, despite its numerous known realisations (algebraic, topological, K-theoretic, …) should be essentially unique. The canonical basis B of a Kac-Moody algebra is well known to be compatible with (left and right) ideals generated by elements \( E_i \). We formulate (at least for \( B_2 \)) a conjecture stating that the same is true for \( \mathcal{B} \) for the ideal generated by the Fourier modes of some suitably defined \"quantum currents\" : \( E(z)^{1} \): (see Section 13.3). In a similar spirit, our whole construction is invariant under twisting by line bundles, and hence the Picard group \( P = (\mathcal{O}(X); o) \) acts on \( \mathcal{B}_\gamma (L(n)) \) by automorphisms preserving the canonical basis.

The methods of proof of both the present work and \[ BS \] are adapted from \[ FS \] and rely on the Harder-Narasimhan stratification and on the theory of mutations. These methods do not seem to extend to the general case where \( X \) is of genus at least two. In \[ Lu0 \], wild quivers are dealt with by exploiting the freedom of choice of an orientation, via a Fourier-Deligne transform. It would be interesting to nd a proper substitute in the category of coherent sheaves on a curve, as this would likely be an important step towards the various conjectures in this paper.

The plan of the paper is as follows. In Section 1 we recall the definitions of loop Kac-Moody algebras, of their quantum cousins and of the cyclic Hall algebras. In Section 2 we consider the category \( C_{CH}(X) \) of \( G \)-equivariant coherent sheaves on a curve \( X \) and construct the relevant Quot schemes. Section 3 introduces inductive limits of Quot schemes and the category of perverse sheaves on them. In Sections 4,5 we define the convolution algebra \( B_\gamma \) following the method in \[ Lu0 \]. Our main theorem and conjecture are in Section 5A. Section 6 is dedicated to the proof of the first part of this theorem. Sections 7 and 8 provide a description of the categories \( C_{CH}(X) \) when \( X \) is of genus at most one and collect some technical results pertaining to the Harder-Narasimhan stratification of Quot schemes. Our main theorem is proved in Sections 9 and 10. Section 12 relates the algebras \( B_\gamma \) and \( \mathcal{B}_\gamma (L(n)) \), and gives some \"first\" properties of the canonical basis \( \mathcal{B} \). Section 13 provides some formulas for \( \mathcal{B} \) when \( g = B_2 \) as well as some formulas for its action on the fundamental representation \( 0 \), and ends with some conjectures concerning the compatibility of \( \mathcal{B} \) with some ideals.

1. Loop algebras and quantum groups.

1.1. Loop algebras. Let be a star shaped Dynkin diagram with branches \( J_i : : : = J_0 \) of respective lengths \( p_i \). Let \( n \) denote the central node and for \( i = 1; : : : ; N \) and \( j = 1; : : : ; p_i \), let \( (i; j) \) be the jth node of \( J_i \), so that \( i \) and \( (i; 1) \) are adjacent for all \( i \). To is associated a generalized Cartan matrix
\[ A = (a_{i,j})_{i,j} \] and a Kac-Moody algebra \( g \). We will consider an \( n \)-algebra \( Lg \) of \( g \) generated by some elements \( h_{j+1} \) for \( j \in \mathbb{Z} \) and \( c \) subject to the following set of relations:

\[
\begin{align*}
[a_{i,j} h_{j+1}] &= k_{ij} a_{i} c; \\
[a_{i,j} f_{j+1}] &= f_{j+1} a; \\
[a_{i,j} e_{j+1}] &= a_{i+1,j} a; \\
[e_{j+1} f_{j+1}] &= [e_{j+1} f_{j+1}] = [f_{j+1} e_{j+1}] = [f_{j+1} e_{j+1}] = 0 \text{ if } j = 0.
\end{align*}
\]

The Lie algebra \( g \) is embedded in \( Lg \) as the subalgebra generated by elements \( h_{j+1} \) for \( j \in \mathbb{Z} \). When \( g \) is a simple Lie algebra (resp. an affine Kac-Moody algebra), \( Lg \) is the corresponding affine Lie algebra (resp. the corresponding toroidal algebra) with in finite dimensional center, see [S2].

We write (resp. \( Q \)) for the root system (resp. the root lattice) of \( g \), so that the root system of \( Lg \) is \( \mathfrak{b} = ( + \mathbb{Z} ) \mathbb{Z} \) and the weight lattice is \( \mathfrak{F} = Q \mathbb{Z} \). Here \( G \) is the extra imaginary root. The lattices \( Q \) and \( \mathfrak{F} \) are both equipped with standard symmetric bilinear forms \( ( , ) \) with values in \( \mathbb{Z} \). We write \( 2 \) for the simple root corresponding to a vertex

Let \( g \) be the subalgebras of \( g \) associated to the branches \( J_1, \ldots, J_N \). For \( i = 1, \ldots, N \) the map \( e_{i,j} f_{i,j} \mapsto h_{i,j} \) gives rise to an isomorphism between the subalgebra \( Lg_i \) of \( Lg \) generated by \( e_{i,j} f_{i,j} \) for \( j \in \mathbb{Z} \) and \( h_{i,j} \). We denote by \( h_i, e_i, f_i \) the standard nilpotent subalgebra, generated by \( f_i, h_i, e_i \) for \( i = 2, 3, \ldots \), where \( f_i \) is a lowest root vector of \( g \). Finally, we call the subalgebra of \( Lg \) generated by \( h_1 \) for \( i = 1, \ldots, N \), \( f_1, h_1 \) a Lee root and \( f_{1,1} \). The set of weights occurring in \( Lg \) defines a positive cone \( \mathfrak{F}^+ \). A root \( \alpha \) belongs to \( \mathfrak{F}^+ \) if and only if \( \alpha_i > 0 \) or \( \alpha_i = 0 \) for all \( i \). It is known that \( Lg \) is an extension

\[
\begin{array}{c}
0 \longrightarrow K \\
\longrightarrow Lg \\
\longrightarrow g[t; t^{-1}] \\
\longrightarrow 0.
\end{array}
\]

Let \( p_i \) be the positive maximal parabolic subalgebra associated to the central node \( ? \), and let \( l_i \) be its Levi subalgebra, resp. its nilpotent radical. The projection of \( \mathfrak{L} \) to \( g[t; t^{-1}] \mathbb{C}^+ \) is equal to \( l_i \). The \( l_i \) is the (positive) nilpotent subalgebra of \( l_i \).

1.2. All algebras. Fix \( n \geq 2 \). The cyclic quiver of type \( A_n^{(1)} \) is the oriented graph with set of vertices \( Z = \mathbb{Z} \) and set of arrows \( (i, j) \). \( Z = \mathbb{Z} \) graded vector space \( V = V \mathbb{Z} \) is a \( Z \)-graded \( k \)-vector space and \( x = (x_i) \) belongs to the space

\[
N_i^{(n)} = f(x_i) \mathbb{Z} \quad \text{Hom}(V_i, V_i), \quad x_i \mathbb{Z} \quad \text{for any } i \text{ and } N \cdot 0.
\]

The set of all nilpotent representations of \( A_n^{(1)} \) forms an abelian category of global dimension one. We briefly recall the definition of the Hall algebra \( \mathbb{H} \). Let us take \( k \) to be a finite field with \( q \) elements. For any \( Z = \mathbb{Z} \) graded vector space \( V \), let \( \mathbb{C}_G \{ N_i^{(n)} \} \) be the set of \( G \)-invariant functions \( N_i^{(n)} \) of the group \( G = \mathbb{Q}_1 GL(V_1) \) naturally acts on \( N_i^{(n)} \) by conjugation. For each \( d \geq 2 \) \( N = \mathbb{Z} \), let
Proposition 1.1 \( S_1 \) Lemmata 4.3, it can naturally be viewed as a quantum deformation of \( U \). Let us, for given \( f \in C_\circ \left( \mathbb{N}_d^{(n)} \right) \) and \( g \in C_\circ \left( \mathbb{N}_d^{(n)} \right) \), set

\[
(1.1) \quad f \quad g = \frac{b \delta \nu'' \delta \nu''}{a \delta \nu'' \delta \nu''} \left( \mathbb{N}_d^{(n)} \right);
\]

where \( b \in C_\circ \left( \mathbb{N}_d^{(n)} \right) \) is the unique function satisfying \( p_2 (h) = p_1 (f) \) and \( \frac{b \delta \nu'' \delta \nu''}{a \delta \nu'' \delta \nu''} = \left( \begin{array}{c} b \delta \nu'' \delta \nu'' \\ a \delta \nu'' \delta \nu'' \end{array} \right) \) is the natural projection. For any \( f \in C_\circ \left( \mathbb{N}_d^{(n)} \right) \) and \( g \in C_\circ \left( \mathbb{N}_d^{(n)} \right) \), we put

\[
(1.2) \quad h_1 = \frac{1}{\nu''} \sum_{i=1}^{n} X \left( \begin{array}{c} 1 \nu'' \\ 0 \end{array} \right) ; \quad n \left( \begin{array}{c} 1 \nu'' \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \nu'' \\ 1 \nu'' \end{array} \right);
\]

By \( S_1 \), Lemma 4.3, \( H \) is generated by \( E_1 \) and \( h_1 \) for \( i \in \mathbb{N}_d \), where \( h_1 \) and \( i \in \mathbb{N}_d \) satisfy

\[
0 \quad V_d ^{\circ} \xrightarrow{a \delta \nu''} V_d ^{\circ} \xrightarrow{b \delta \nu''} V_d ^{\circ} \xrightarrow{d \delta \nu''} 0
\]

and let \( F \) be the subset of representations \( \mathbb{N}_d^{(n)} \) of \( V_d ^{\circ} \). Consider the diagram

\[
\mathbb{N}_d^{(n)} \xrightarrow{F} \mathbb{N}_d^{(n)} \xrightarrow{G} \mathbb{N}_d^{(n)}
\]

where \( F \) denotes the embedding and \( \left( \begin{array}{c} 0 \nu'' \delta \nu'' \\ a \delta \nu'' \delta \nu'' \end{array} \right) \). For any \( f \in C_\circ \left( \mathbb{N}_d^{(n)} \right) \) we put

\[
(1.3) \quad f = \frac{b \delta \nu'' \delta \nu''}{a \delta \nu'' \delta \nu''} (f) ; \quad d \delta \nu'' (f) = \nu'' \delta \nu'' (f) ; \quad (f) 2 C_\circ \left( \mathbb{N}_d^{(n)} \right) \subset C_\circ \left( \mathbb{N}_d^{(n)} \right);
\]

Proposition 1.1 \( S_1 \). We have \( H = U^+ \left( \mathbb{N}_d^{(n)} \right) \subset C_\circ \left( \mathbb{N}_d^{(n)} \right) \) and

\[
(1.4) \quad \mathbb{N}_d^{(n)} \xrightarrow{F} \mathbb{N}_d^{(n)} \xrightarrow{G} \mathbb{N}_d^{(n)}
\]

where \( z_1 \in H \) is central element of degree (1; 1) satisfying \( z_1 = z_i = z_i \), 1 + 1 = 2.
Remark. The convention $v^2 = q$ differs from the one used in [S2], by the change of variables $v \to v^2$ (this is also valid for Section 1.3).

1.3. Quantum groups. The enveloping algebra $U(L\mathfrak{g})$ admits a well-known deformation which is due to Drinfeld (see [D]). We will be concerned here with a certain deformation of its positive part $U(L\mathfrak{n})$. For each branch $J_i$ of we consider one copy of the algebra $H_{p_i}$ and we denote by $E_{(i,j)}$, resp. $H_{i}$, the generators of $H_{p_i}$, considered in Section 1.2, corresponding to the $j$th vertex of $A^{(1)}_{p_i-1}$ or to a positive integer $l$. We let $U_\nu(L\mathfrak{n})$ be the $C$-algebra generated by $H_{p_i}$ for $i=1;\ldots;N$, $H_{?;l}$ for $12N$ and $E_{?;k}$ for $2Z$, subject to the following set of relations:

i) For all $i, H_{?;l} = v^{k^2}h_{i;l}$, and $[ E_{(i,j)}; E_{(i';j')} ] = 0$ if $i' \neq i$.

ii) We have

$$H_{?;l}; E_{?;m} = (v^2 + v^{-1})E_{?;l+1};$$

$$v^2 E_{?;l+1} E_{?;m} = E_{?;m} E_{?;l+1} = E_{?;m+1} E_{?;l};$$

iii) $a_{?;l} = 0 \quad \forall E_{?;l}; E_{?;m} = 0; \quad E_{?;l} E_{(i,j)} = v E_{(i,j)} E_{?;l} = 0; \quad p_2 > 2 \quad \forall [ E_{(i,j)}; E_{(i';j')} ] = 0;$$

iv) For $1 = (?;l)$ with $i=1;\ldots;N$, and $J_1; J_2; 2 N$ and $J_1; J_2 = 2 N$, $t 2 Z$ we have

$$E_{?;m+1}; E_{?;l} = v E_{?;m}; E_{?;l} = E_{?;m+1}; E_{?;l} = v E_{?;m}; E_{?;l} = 0;$$

$$E_{?;m+1}; E_{?;l} = v E_{?;m}; E_{?;l} = E_{?;m+1}; E_{?;l} = v E_{?;m}; E_{?;l} = 0;$$

v) For $1 = (?;l)$ with $i=1;\ldots;N$ and $J_1; J_2 = 2 N$, $t 2 Z$ we have

$$E_{?;m+1}; E_{?;l} = v E_{?;m}; E_{?;l} = E_{?;m+1}; E_{?;l} = v E_{?;m}; E_{?;l} = 0;$$

$$E_{?;m+1}; E_{?;l} = v E_{?;m}; E_{?;l} = E_{?;m+1}; E_{?;l} = v E_{?;m}; E_{?;l} = 0;$$

It is convenient to introduce another set of elements of $U_\nu(L\mathfrak{n})$ : denote $\nu$ for $1$ by the formal relation $1+1 = \exp(1+1 H_{?;l}(s))$. The sets $f; g$ and $fH_{?;l}(g)$ span the same subalgebra of $U_\nu(L\mathfrak{n})$.

Remark. The relations are slightly renormalized from those in [S2]. In particular, the element $H_{?;l}$ corresponds to $H_{?;l}=1$ in [S2]. The elements $1$ are sometimes denoted $F_1$ in the literature.

2. Coherent sheaves and Quot schemes

2.1. Equivariant coherent sheaves. Let $(X;G)$ be a pair consisting of a smooth projective curve $X$ over a field $k$ and a finite group $G \subseteq \text{Aut}(X)$ such that $X = G \cdot F_1$. The category of $G$-equivariant coherent sheaves on $X$ will be denoted by $C_{coh}(X)$. It is an abelian category of global dimension one. The quotient map $X \to F_1$ is ramified at points, say $1;\ldots;N$ $2 F_1$ with respective indices of ramification $p_1;\ldots;p_N$. We assume that all the points $1$ lie in $X(k)$. Let $L = (X;G)$ be the star-shaped Dynkin diagram with branches of length $p_1;\ldots;p_N$ (if $(X;G) = (F_1;1G)$ then $G$ is of type $A_1$). We also put $= F_1;\ldots;N \subseteq G \cdot F_1$.

Any $F \subseteq C_{coh}(X)$ has a canonical torsion subsheaf $(F)$ and locally free quotient $(F')$ and there is a (noncanonical) isomorphism $F' / (F) \to (F')$.

Let us first describe the torsion sheaves in $C_{coh}(X)$. Let $O_X$ be the structure sheaf of $X$ with the trivial $G$-structure. For each closed point $x \in X$, there exists,
up to isomorphism, a unique simple torsion sheaf \( x \) with support in \( \mathcal{X} \), and there holds

\[
\dim \text{Ext}(x; x) = \dim \text{Ext}(x; \mathcal{O}_X) = \dim \text{Hom}(\mathcal{O}_X; x) = \deg(x)
\]

The full subcategory \( \mathcal{T}_x \) of torsion sheaves supported at \( x \) is equivalent to the category of nilpotent representations (over the residue \( \mathcal{E}_x \)) at \( x \) of the cyclic quiver \( A_0^{(1)} \) with one vertex and one loop.

On the other hand, for each ramification point \( j \in \mathbb{Z} \) of index \( p_i \), there exists \( p_i \) simple torsion sheaves \( S_i^{(1)} \), \( j \in \mathbb{Z} = p_i \mathbb{Z} \) with support in \( \mathcal{X} \). We have

\[
\dim \text{Ext}(S_i^{(1)}; S_i^{(1)}) = \dim \text{Hom}(\mathcal{O}_X; S_i^{(1)}) = p_i:
\]

Hence, the category \( \mathcal{T}_x \) is equivalent to the category of nilpotent representations (over \( \mathcal{E}_x \)) of the cyclic quiver \( A_0^{(1)} \). Under this equivalence, the sheaf \( S_i^{(1)} \) goes to the simple module \( (V_j; x) \) where \( V_j = \mathcal{E}_x \) and \( V_h = f \mathcal{E}_x \) if \( j \) is not in \( \mathbb{Z} \). We denote by \( \mathcal{E}_x = f(V_j; x) \) for \( j = 1; \ldots ; N \) and \( Z = p_i \mathcal{E}_x \) the set of simple exceptional torsion sheaves.

The Picard group of \( (\mathcal{O}_X; G) \) is also easy to describe. Let \( L \) (resp. \( L_i \)) be the nontrivial extension of \( \mathcal{O}_x \) by a generic simple torsion sheaf \( x \) for the \( \mathcal{O}_x \) (resp. by the simple torsion sheaf \( S_i^{(1)} \)). Then \( L \) and \( L_i \) for \( i = 1; \ldots ; N \) generate \( \text{Pic}(\mathcal{O}_X; G) \) with the relations \( L_i \wedge L = \mathcal{L} \). Thus

\[
\text{Pic}(\mathcal{O}_X; G) = \mathcal{L} := \mathbb{Z} L \wedge L_i = \mathbb{Z} = p_i \mathbb{Z}:
\]

In particular, \( \text{Pic}(\mathcal{O}_X; G) \) is of rank one.

The category \( \text{Ch}_c(\mathcal{O}_X) \) inherits Serre duality from the category \( \text{Ch}(\mathcal{O}_X) \). Let \( \mathcal{K} \) be the canonical sheaf of \( \mathcal{X} \) with its natural \( G \)-structure. Then for any \( F; G \), \( \mathcal{K} \) there exists a \((\text{functorial})\) isomorphism \( \text{Ext}(F; G) \wedge \text{Hom}(\mathcal{G}; F \wedge \mathcal{K}) \). It is known that (see [5])

\[
(2) \quad \text{for any } X \in \mathcal{O}_X \text{ and } N \in \mathcal{N}, \quad (\text{natural} \ O : \mathcal{O}_X) = \mathbb{Z} \mathcal{L}_i \wedge L_i \quad \text{from which it follows that } H^0(\mathcal{K}) = 0.
\]

Write \( K(\mathcal{O}_X) \) for the Grothendieck group of \( \text{Ch}_c(\mathcal{O}_X) \). This group is equipped with the Euler form \( h_M \wedge h_N = \dim \text{Hom}(M; N) \) and its symmetrized version \( h_M \wedge h_N = \dim \text{Hom}(M; N) + \dim \text{Hom}(N; M) \). The set of classes \( 2 K(\mathcal{O}_X) \) for which there exists a simple sheaf \( F \) with \( [F] = 2 \mathbb{Z} \) is the cone of \( K(\mathcal{O}_X) \) denoted \( K^+(\mathcal{O}_X) \). Then (see [2], Prop 5.1)

\[
\text{Lemma 2.1. There is a canonical isomorphism } h : K(\mathcal{O}_X) \wedge \mathcal{O}_X \text{ compatible with the symmetric form } s(\mathcal{O}_X). \text{ It maps } K^+(\mathcal{O}_X) \text{ to the positive weight lattice } \mathcal{O}_X^+.
\]

Under this isomorphism, we have \( h(\mathcal{O}_X) = \gamma, h([\mathcal{X}]) = \gamma([\mathcal{X}]) \) for any closed point \( \mathcal{X} \) of degree one, \( h((S_i^{(1)})) = (\Sigma j) \) for \( j \not\in \mathbb{Z} \) and \( h((S_i^{(0)})) = j(\Sigma j) \). In particular, the Picard group \( \text{Pic}(\mathcal{O}_X; G) \) is in correspondence with the set of roots of \( G \) in the form \( \gamma \wedge \Sigma j(\Sigma j) \). Define a finite set

\[
S = \{ \mathcal{X} \gamma \wedge \Sigma j(\Sigma j) \} \quad \text{finite set}
\]
For $s \rightarrow S$ we denote by $L_s$ the line bundle corresponding to $s$. Observe that by Lemma 2.2 we have

\[(2.2) \quad \text{Ext}(L_s; L_s) = 0 \quad \text{for any } s; s' \rightarrow S.\]

For any $F \rightarrow C \rightarrow C$ we put $F(n) = F \cdot L^n$. 

Lastly, we denote the degree map $\deg : K(X) \rightarrow \mathbb{Z}[D_X] = \mathbb{T} + \mathbb{Z}[D_X]$ and write $\deg(\cdot)$ for $\deg(\cdot) 2 u_1 N [u_1^{(3)}] + \mathbb{Z}[D_X]$. We set $! = \deg([u_1^{(3)}])$. We will also need a degree map taking values in $Q$: if $\text{deg}(\cdot) 2 K(X)$ and $\text{deg}(\cdot) = u_2 \text{deg}(u_2)$ we put $j = p^m(u_2) 2 Q$, where $p = i(x_2) (p_2)$. We extend both of these notations to a coherent sheaf $F$ by setting $\text{deg}(F) = \text{deg}(F^I)$ and similarly for $j$. 

Remark. If $G$ is any smooth projective curve $X$ (over $C$) and a group $G$ of automorphisms of $X$ such that $X \rightarrow X \rightarrow G$ is an isomorphism $D$. This can be proved as follows. From Galois theory, it is enough to find a finite group $G$ generated by elements $y_i$ of order $p_i$ such that $y_i \cdot N \neq 1$. Consider matrices in $SL(2; F)$

\[
\begin{align*}
\gamma_1 &= \begin{pmatrix} p_i & x_i \\
0 & p_i^{-1} \end{pmatrix} \quad \text{for } i = 1, \ldots, N \quad 2; \\
\gamma_N &= \begin{pmatrix} p_n & 0 \\
x_n & p_n^{-1} \end{pmatrix} \quad \gamma_N^{-1} = \gamma_1 \quad N \neq 1
\end{align*}
\]

where $i$ is a primitive $i^{th}$ root of unity in some big enough finite field $F$. Each $y_i$, $i = 1, \ldots, N$, is of order $p_i$, and the product $y_i \cdot N \neq 1$ has determinant one and trace equal to a certain nonconstant polynomial in the $x_i$'s. In particular, replacing $F$ by a finite extension if necessary, we can choose $x_i, i = 1, \ldots, N$ such that $\text{Tr}(y_i^{-1}) = p_n + p_n^{-1}$. But then $\gamma_N$ is diagonalizable and of order $p_n$ as desired. 

iii) The category $Coh_X$ admits an equivalent description as the category of $D$-parabolic coherent sheaves on $P^1$ where $D = \sum D_i$ is the ramification divisor of the quotient map $X \rightarrow P^1$ (see [3]). It is also equivalent to the category of coherent sheaves on a weighted projective line in the sense of Geigel and Lenzing (see [5]) where this category was first singled out and studied in detail; in fact the notion of a weighted projective line (for $D$-parabolic coherent sheaves) is slightly more general but it also covers the case of diagrams of type $\lambda_0$, with arbitrary node chosen as the central node $2$. All the constructions and results in this paper could (and maybe should!) have been written in this more general setting. 

2.2. Equivariant Quot schemes. We will say that $F \rightarrow C \rightarrow C$ is generated by a collection of sheaves $f; g_{ij}; i$ if the natural map

\[
\text{Mor}(T_i; F) \quad T_i \rightarrow F
\]

is surjective. 

Lemma 2.2. For any $E \rightarrow C \rightarrow C$ there exists $2 Z$ such that for any $m \geq 1$ we have

i) $E$ is generated by $f L_s (m) g_{ij}; i$; $E = 0$ for any $s \rightarrow S$. 

Moreover, this also implies that

ii) $\text{Ext}(L_s (m); E) = 0$ for any $s \rightarrow S$. 

Proof. Assume $E^0$ and $E^\circ$ both satisfy i) and ii) with corresponding integers $I^0; I^\circ$. Let $0 \to E^0 \to E \to E^\circ \to 0$ be a short exact sequence and $x \mapsto \in H^1(M; I^\circ)[g]$. Applying $\text{Hom}(L_s(\mathfrak{m}); \cdot)$ and using ii) we obtain an exact sequence

$$0 \to \text{Hom}(L_s(\mathfrak{m}); E^0) \to \text{Hom}(L_s(\mathfrak{m}); E) \to \text{Hom}(L_s(\mathfrak{m}); E^\circ) \to 0$$

and $\text{Ext}(L_s(\mathfrak{m}); E) = 0$. The image of the natural map

$$M : \text{Hom}(L_s(\mathfrak{m}); E) \to L_s(\mathfrak{m}); E$$

contains $E^0$ and maps surjectively on $E^\circ$ so that it is surjective. Hence if i); ii) hold for two sheaves then it also holds for any extension between them.

Any torsion sheaf is obtained as an extension of the simple sheaves $O_x$ for $x \in \mathbb{Z}$ and $S_1^{(i)}$ for $(i,j) \neq (0,0)$. Moreover, for any $x \in \mathbb{Z}$ there is an exact sequence $0 \to O_x \to O_x \to 0$ and likewise for any $(i,j)$ there is an exact sequence $0 \to L_s \to L_s \to S_1^{(i)} \to 0$ for suitable $s > 0$. Thus the statement of the lemma holds for all torsion sheaves. Note that ii) holds for an arbitrary torsion sheaf and arbitrary $m \in \mathbb{Z}$. Similarly, by [11], Proposition 2.5, any vector bundle is an extension of line bundles and, in turn, for any line bundle $L$ there exists a line bundle $L_0$ and an extension of $O_x$ by a torsion sheaf. In particular, any vector bundle is generated by $L_s(\mathfrak{m})g$ for some $m$. Conversely, if a vector bundle $E$ is generated by $L_s(\mathfrak{m})g$ then it can be obtained as an extension of line bundles $V_i$ for $i = 1, \ldots, r$ with $\text{deg}(V_i) = \text{deg}(L_s(\mathfrak{m}))$ for any $s > 0$. But then using (2.2) we deduce that $\text{Ext}(L_s(\mathfrak{m}); V_i) = 0$ for all $s > 0$ and the lemma follows.

Lem 2.3. Let $F \subset \text{ Coh}_0(X)$ be generated by $L_s(n_0)_g$, for $x \in 2\mathbb{Z}$. There exists an integer $n = q(\ ; n_0) \in 2\mathbb{Z}$ depending only on $n_0$; and $|F|$ such that any subsheaf $G$ of $F$ with $|G| = s$ satisfies

i) $G$ is generated by $L_s(m)$,

ii) $\text{Ext}(L_s(\mathfrak{m}); G) = 0$ for any $s > 0$.

Proof. If $F$ is generated by $L_s(n_0)_g$, then so is $F^\circ$. In particular, $F^\circ$ possesses a filtration

$$0 = V_1 \subset \cdots \subset V_r = \mathcal{E}^\circ$$

with successive quotients $V_i = V_{i-1} / V_i$ being line bundles of degree at least $\text{deg}(V_i) = n_0$. Thus we have $\text{deg}(F^\circ) = \text{deg}(F) + \text{deg}(\mathcal{E})$ and $\text{rank}(F^\circ) = \text{rank}(F)^n$. A similar reasoning shows that

$$(2.3) \quad \text{Hom}(L_s(n_0); \mathcal{E}) \to 0 \to \text{deg}(\mathcal{E}) \to (\text{rank}(F^\circ) = \text{rank}(F)^n) / 3.$$

Now $\mathcal{E} = \mathcal{F}^\circ$, and therefore $\text{deg}(\mathcal{E}) = \text{deg}(\mathcal{F})$. Consider a filtration

$$0 = G_1 \subset \cdots \subset G_r$$

of $G$ with the property that $G_i = G_{k-1}$ is a line subbundle of $G = G_{k-1}$ of maximal possible degree.

Claim 2.1. We have $\text{deg}(G_i = G_{k-1}) = \text{deg}(\mathcal{F}) - [\text{rank}(\mathcal{F}) - 1]n_0 + (i-1)(3 + i)$.

Proof of claim. We have $\text{deg}(G_i) = \text{deg}(\mathcal{F}) - [\text{rank}(\mathcal{F}) - 1]n_0$ by (2.3). Now let $0 \to H^0 \to H \to H^\circ \to 0$ be a short exact sequence where $H^0, H^\circ$ are line bundles. Assume that

$$(2.4) \quad \text{deg}(H^0) > \text{deg}(H^\circ) + 3.$$
Applying $\text{Hom}(\mathcal{O}(1); \cdot)$ we obtain a sequence

$$0 \to \text{Hom}(\mathcal{O}(1); H) \to \text{Hom}(\mathcal{O}(1); H^0)\text{ Ext}(\mathcal{O}(1); H^0):$$

By Serre duality and $\mathcal{H}$ we have

$$\dim \text{Ext}(\mathcal{O}(1); H^0) = \dim \text{Hom}(\mathcal{O}(1); H^0)$$

Thus $\text{Hom}(\mathcal{O}(1); H) \neq 0$. In particular, if $\mathcal{H}$ holds then $H^0$ is not a maximal degree line subbundle of $H$. The claim easily follows by induction.

From the claim and from $\mathcal{P}$ we deduce that

$$\deg(G_i \cap G_{i+1}) = \deg \text{ deg}(\cdot) \text{ rank}(\cdot) \text{ rank}(\cdot) \text{ rank}(\cdot) \text{ rank}(\cdot) n_0$$

Let $Q(\cdot; n_0)$ denote the right hand side of the above inequality. Let $q(\cdot; n_0)$ be the greatest integer such that $q(\cdot; n_0) < Q(\cdot; n_0)$. Then each $G_i = G_{i+1}$ is generated by $F_{L_s}(q(\cdot; n_0))g_{s+1}$ and $\text{Ext}(L_s(q(\cdot; n_0)); G_{i+1}) = 0$ for all $s \leq S$. The Lemma follows.

We assume from now on the ground field $k$ to be algebraically closed. Let $E$ be a coherent, flat sheaf; $F$ is a $G$-equivariant sheaf.

**Proposition 2.1.** The functor $\mathcal{H}(\cdot; n_0)$ is represented by a projective scheme $E \mathcal{H}(\cdot; n_0)$.

**Proof.** This result is known in the (equivalent) situation of parabolic bundles on a curve (see $\mathcal{H}$ and Remark 1.1). We will only sketch a construction of $\mathcal{H}(\cdot; n_0)$ in characteristic zero. By Lemma 2.2, we have $\text{Ext}(L_s(n); G_{i+1}) = 0$ for all $s \leq S$. By Lemma 2.3 we have $\text{Ext}(L_s(n); G_{i+1}) = 0$ for all $s \leq S$, so that $\text{dim} \text{Hom}(L_s(n); G_{i+1}) = \text{dim} \text{Hom}(L_s(n); G_{i+1})$.

This defines a map to a Grassmannian

$$Y: \mathcal{H}(\cdot; n_0) \to \text{Gr}(L_s(n); E) \to \text{Gr}(L_s(n); E) \text{ flat}$$

Conversely, define a map

$$Q_s: \text{Gr}(L_s(n); E) \to \text{Gr}(L_s(n); E)$$

by assigning to $(V_s)_s$ the image of the composition

$$M \to V_s \text{ L}_s: \text{Hom}(L_s(n); E) \text{ L}_s:$$

By Lemma 2.4, every subsheaf of $E$ of class $E$ is generated by $F_{L_s}(n_0)g_{s+1}$ and therefore $= \text{Id}$. Thus is injective.

The image of $Q_s$ can be explicitly described. Let $H$ be the subset of the Grassmannian $\text{Gr}(L_s(n); E)$ consisting of elements $(V_s)_s$ such that the image of the composition

$$M \to \text{Hom}(L_s(n); E) \text{ L}_s: \text{Hom}(L_s(n); E) \text{ L}_s:$$

is of dimension $\text{dim} \text{Hom}(L_s(n); E)$ for all $s \geq S$ and $n \leq n$. It is clear that $\mathcal{H}(E)$ is $H$. Conversely, if $(V_s)_s$ then for all $n \leq n$ and $s \geq S$ we have $\text{dim} \text{Hom}(L_s(n); (V_s)_s) = \text{dim} \text{Hom}(L_s(n); E)$...

On the other hand, by
Lemma 2.4. The stabilizer of any point induces isomorphisms.

Proof.

Lemma 2.2.

Define an element from which we conclude Ext(Ker) follows. Now let \( g_s \) be an element in the stabilizer of \( \phi \). By definition, \( (g_s)Ker = Ker \), hence \( (g_s) \) induces an automorphism of \( F \). This gives a map \( \text{Stab} \to \text{Aut}(F) \). The inverse map is constructed as follows. As above, an automorphism \( u \in \text{Aut}(F) \) induces automorphism \( u \circ \text{Aut}(\text{Hom}(L(n);F)) \). Set \( (u) = (\text{Id} u) \). Then \( = \text{Id} \) and \( = \text{Id} \), and the conclusion follows.

Lemma 2.5. The variety \( Q_n \) is smooth.

Proof. By an equivariant version of [LP], Corollary 8.10, a point \( E_n \) is smooth if \( \text{Ext}(\text{Ker} F) = 0 \). Observe that if \( 2Q_n \) then for all \( s \), we have \( h^1(L(n);F) = \dim \text{Hom}(L(n);F) \), hence \( \text{Ext}(L(n);F) = 0 \). Applying Hom ( \( IF \) ) to the exact sequence \( 0 \to \text{Ker} F \to E_n \to F \to 0 \) yields an exact sequence

\[
\text{Ext}(F,F) \to \text{Ext}(E_n,F) \to \text{Ext}(\text{Ker} F,F)
\]

from which we conclude \( \text{Ext}(\text{Ker} F,F) = 0 \) as desired.
3. Transfer functors

In this section we set up the construction of the $\ind lim$ of the varieties $Q_n$, as $n$ tends to 1. More precisely, we denote the category of (equivariant) perverse sheaves and complexes on such an inductive limit.

3.1. Notations. We use notations of Chapter 8 regarding perverse sheaves. In particular, for an algebraic variety $X$ we denote by $D(X)$ the derived category $D_{\text{perverse}}(X)$ of $\text{Q}_1$-constructible sheaves and complexes of geometric origin on $X$. The Verdier dual of a complex $P$ is denoted by $D(P)$. If $G$ is a connected algebraic group acting on $X$ then we denote by $Q_G(X)$ the category of $G$-equivariant complexes. Recall that if $H \subset G$ is a subgroup and if $Y$ is an $H$-variety then there are canonical (inverse) equivalences

$$\text{ind}_G^H : \text{Q}_H(Y) \cong \text{Q}_G(Y) \text{mod } G$$

and

$$\text{res}_G^H : \text{Q}_G(Y) \rightarrow \text{Q}_H(Y) \text{mod } H$$

which commute with Verdier duality. These are characterized (up to isomorphism) as follows: let

$$Y \xrightarrow{\gamma} G \xrightarrow{\varphi} G_H Y$$

be the natural diagram; if $P \in \text{Q}_H(Y) \text{mod } G$ then $\varphi \text{ind}_G^H (P)$ and $\gamma \text{res}_G^H (P)$. Conversely, if $P \in \text{Q}_G(Y)$ then $\text{res}_G^H (P) = i \varphi$ where $i : Y \rightarrow G_H Y$ is the canonical embedding. Finally, if $Y \rightarrow X$ then $\overline{Y}$ stands for the Zariski closure of $Y$ in $X$.

3.2. Transfer functors. Fix $2 K^+(X)$. We will denote, for each pair of integers $n < m$, an exact functor $\Gamma_{nm} : Q_m(Q_n)$ ! $Q_n(Q_m)$. We first set certain notations and data. For all $n$ we put

$$E_{n,m+1}^* = \text{Hom} (L(n); L(n+1))$$

and we set $E_{n,m+1}^* = \text{Hom} (L(n)); L(n+1))$. There is a canonical (evaluation) surjective map $\Gamma_{nm} : E_{n,m+1}^* \rightarrow E_{n,m+1}$. We also set $u_n : E_{n,m+1}^* \rightarrow E_{n,m+1}$ and set $u_n = u_n^s : E_{n,m+1}^* \rightarrow E_{n,m+1}$. Consider the following open subvariety of $Q_n$,

$$Q_{n+1} = \{ f : E_{n,m+1} \rightarrow \text{F} \}$$

and let us denote by $j : Q_{n+1} \rightarrow Q_n$ the embedding. We will denote the open subvariety $H \text{ind}_{E_{n,m+1}}$ by

$$R_{n,m+1} = \{ f : E_{n,m+1} \rightarrow \text{F} \}$$

for all $S$ and $F$ is generated by $L(n+1)$.

Composing with $\Gamma_{nm} : E_{n,m+1}^* \rightarrow E_{n,m+1}$, for all $S$ and $F$ is generated by $L(n+1)$. Similary, composing with $u_n$ gives an embedding $v_{n,m+1} : Q_{n+1} \rightarrow H \text{ind}_{E_{n,m+1}}$.

Set $G_{n,m+1} = Q_{GL}(E_{n,m+1}^*)$. Let $P_{n,m+1}$ be the stabilizer of $K$ for $u_n^s$. Finally, put $P = \sigma P_{n,m+1}$ and $G_{n,m+1} = \text{Aut}(E_{n,m+1})$. The group $\text{Aut}(E_{n,m+1})$ (and thus $G_{n,m+1}$) naturally acts on $H \text{ind}_{E_{n,m+1}}$. The group $P$ acts on $Q_n$ via the quotient induced by $u_n^s$.

$$P_{n,m+1} \rightarrow \text{GL}(L(n); \\ G_n$$
By construction, there is a natural $G_{n,m+1}$-equivariant isomorphism

\[(3.1) \quad G_{n,m+1} \cong Q_{n+1} \rightarrow R_{n,m+1}.
\]

Note that there is a canonical embedding $GL(d_{\omega}(n+1);1) \rightarrow GL(E_{\mu,n+1})$, giving rise to an embedding

\[ G_{n+1} = GL(d_{\omega}(n+1);1) \rightarrow GL(E_{\mu,n+1}) = G_{n,m+1};
\]

Lemma 3.1. The map $n,m+1$ induces a canonical $G_{n,m+1}$-equivariant surjective morphism

\[ n,m+1 : G_{n,m+1} \rightarrow Q_{n+1} \rightarrow R_{n,m+1};
\]

Proof. Let us first show that the image of $n,m+1$ belongs to $R_{n,m+1}$.

Claim 3.1. If $F$ is generated by $FL_{s}(m)g_{S_{2}}$ with $m > n$ then the natural map

\[ a : \ Hom(L_{s}(n);L_{s}(m)) \rightarrow Hom(L_{s}(m);F) \rightarrow Hom(L_{s}(n);F)
\]

is surjective for every $s \geq 2$.

Proof of claim. We argue by induction on the rank. Since $T(1) \rightarrow T$ for any torsion sheaf, it is enough to prove the claim for vector bundles. The statement is clear for line bundles. Let $F_{1} = F$ be a subsheaf isomorphic to $L_{s}(m)$. Applying $Hom(L_{s}(n);\_)$ to the exact sequence $0 \rightarrow F_{1} \rightarrow F \rightarrow F_{1} \rightarrow 0$ yields a sequence

\[ 0 \rightarrow Hom(L_{s}(n);F_{1}) \rightarrow Hom(L_{s}(n);F) \rightarrow Hom(L_{s}(n);F_{1}) \rightarrow Ext(L_{s}(n);F_{1}).
\]

Using Serre duality, we have

\[ \dim Ext(L_{s}(n);L_{s}(m)) = \dim Hom(L_{s}(n);L_{s}(m)) = H^{0}(L_{s}(n)) = 0:
\]

Now, it is clear that $Hom(L_{s}(n);F_{1})$. Hence it is enough to show that $Hom(L_{s}(n);F_{1})$. We have a commutative diagram of canonical maps

\[ L_{s} : Hom(L_{s}(n);L_{s}(m)) \rightarrow Hom(L_{s}(m);F) \rightarrow Hom(L_{s}(n);F)
\]

The map $c$ is surjective by the computation above and, by induction hypothesis, so is $a^{0}$. Thus $c^{0}$ is onto as desired.

Next, $x : E_{n+1}(F) \rightarrow 2Q_{n+1}$ and consider the following commutative diagram,

\[ L_{s} : Hom(L_{s}(n);L_{s}(n+1)) \rightarrow Hom(L_{s}(n+1);F) \rightarrow Hom(L_{s}(n);F)
\]

By the claim $a$ is surjective and by hypothesis is an isomorphism. Hence $x$ is also surjective, and $n,m+1$ maps $Q_{n+1}$ into $R_{n,m+1}$. Observe that $n,m+1$ is $G_{n+1}$-equivariant so that $n,m+1$ is well-defined and $G_{n,m+1}$-equivariant. Finally, $G_{n,m+1}$-orbits are parametrized on both sides by isomorphism classes of coherent sheaves of class , generated by $FL_{s}(n+1)g_{S_{2}}$. Hence $n,m+1$ is onto.
Lemma 3.2. The map $n, m + 1$ is smooth with connected fibers of dimension $\dim (\mathcal{F} = \mathcal{G}_n) = X \dim (\mathcal{G}_n) = c_0 (n; n + 1; ) c_0 (n; n + 1; ) c_0 (n; ))$.

Definition. Introduce the functor $e_{n, m + 1} : \mathcal{G}_n (Q_n) ! \mathcal{G}_{n + 1} (Q_{n + 1})$

\[ F \mapsto e_{n, m + 1} (n, m + 1) \quad \text{ind}_{\mathcal{G}_n}^{\mathcal{G}_{n + 1}} (n, m + 1) \quad \text{ind}_{\mathcal{G}_n}^{\mathcal{G}_{n + 1}} (n, m + 1) \]

and put $n, m + 1 = e_{n, m + 1} (\text{Hom} \mathcal{G}_n \mathcal{G}_{n + 1})$. We extend this definition to arbitrary $n < m$ by setting

\[ e_{n, m} = e \, m \, 1 \, m \quad e_{n, m + 1} \quad n, m = m \, 1 \, m \quad n, m + 1 \]

Note that $e_{n, m + 1} (Q_n) = (n, m + 1) \quad \text{ind}_{\mathcal{G}_n}^{\mathcal{G}_{n + 1}} (n, m + 1)$. From the properties of $\text{ind}_{\mathcal{G}_n}^{\mathcal{G}_{n + 1}}$ one deduces that $G = e_{n, m + 1} (Q_n)$ is characterized, up to isomorphism, by the property that there exists an element $H \in \mathcal{G}_n (Q_{n + 1})$ such that $v_{n, m + 1} H = j F$ and $n, m + 1, H = G$.

Using Lemma 3.2 and the fact that $j$ is an open embedding, we see that $n, m + 1$ is exact, preserves permutivity and commutates with Verdier duality.

The functors $e_{n, m}$ can be given a direct definition as follows: let us put

\[ M = \mathcal{G}_{n + 1} \]

\[ E_{n, m} = \text{Hom} (L_{n + 1} (L_s (j)) L_{n + 1} (j + 1)) \quad E_{n, m} \]

\[ \begin{align*}
E_{n, m} = & \quad M \\
E_{n, m} = & \quad E_{n, m} \quad L_s (j) \\
E_{n, m} = & \quad E_{n, m} \quad L_s (j)
\end{align*} \]

where $s = f (s) + 1$ and $j_2 S_g$. We have a canonical (evaluation) map $c_{n, m} : E_{n, m} E_{n, m}$ and a composition $u_{n, m} = u_{n, m} = u_{n, m} = u_{n, m} = u_{n, m} E_{n, m}$. Putting

\[ R_{n, m} = f \quad E_{n, m} \quad F \quad j_2 S_g \quad \text{Hom} (L_s (j); F) \quad \text{is surjective} \]

for all $s 2 S$ and $F$ is generated by $f L_s (j) g_2 S$. We thus obtain embeddings $n, m : Q_n \mathcal{G}_{n + 1} \quad R_{n, m}$ and $v_{n, m} : Q_n ! \mathcal{G}_{n + 1} \quad R_{n, m}$. Finally, the group $G_{n, m} = \mathcal{G}_{n + 1} ! \mathcal{G}_{n + 1}$ naturally acts on $R_{n, m}$. If $F 2 Q_{n + 1} (Q_n)$ then $G = e_{n, m} (Q_n)$ is characterized by the condition that there exists $H 2 Q_{n + 1} (Q_{n + 1})$ such that $v_{n, m} H = j F$ and $n, m H = G$.

By construction, we have

Lemma 3.3. Let $2 K^+ (X)$ and $n < m < l$. There is a canonical natural transformation $m, l \quad n, m ! n, l$. 
3.3. The category \( \mathcal{Q} \). We will now define a triangulated category \( \mathcal{Q} \) as a projective limit of the system \( \langle \mathcal{Q}_{n}, Q_{n} \rangle \), \( n \in \mathbb{N} \), \( n \geq 2 \). Let \( \mathcal{Q} \) be the additive category with

- \( \mathcal{Q}(Q_n, Q_m) \) is the set of collections \( (F_n; x_{m})_{n \leq m \leq 2n} \) where \( F_n : Q_n \rightarrow Q_m \) and \( x_{m} : n \rightarrow (F_n) ! F_{n} \) satisfy the conditions

\[
\begin{align*}
F_{n} & = F_{m} \quad \text{for every } n \leq m \leq 2n;
F_{n} & = F_{m} \quad \text{for every } n \leq m \leq 2n.
\end{align*}
\]

- For any two objects \( F = (F_n; x_{m}) ; F \rightarrow (F_{n} ; x_{m}) \), we have

\[
\text{Hom}_{\mathcal{Q}}(F,F) = f(\varnothing; n) \quad \text{and} \quad \text{Hom}_{\mathcal{Q}}(F,F) = x_{m} \quad \text{if } n < m < 2n.
\]

The collection of translation functors \( F \rightarrow F[1] \) of \( \mathcal{Q}_{n}, Q_{n} \) give rise to an automorphism \( F \rightarrow F[1] \) of \( \mathcal{Q} \). Let us call a triangle \( F \rightarrow G \rightarrow H \rightarrow F[1] \) distinguished if all the corresponding triangles \( F_n \rightarrow G_n \rightarrow H_n \rightarrow F_n[1] \) are.

Let \( m \) a 3.4. The category \( \mathcal{Q} \), equipped with the automorphism \( T \) and the above collection of distinguished triangles, is a triangulated category.

Remarks. i) Assume that \( \mathcal{Q} \) is the class of a torsion sheaf. Then it is easy to see that \( \mathcal{Q}_{n}, Q_{n} \) for any two \( n \leq m \leq 2n \). Hence in this case \( \mathcal{Q} \sim \mathcal{Q}_{n}, Q_{n} \).

ii) Now assume that \( \mathcal{Q} \) is the class of a sheaf of rank at least one. Then from Section 2, one deduces that \( Q_{n} \) is empty for \( n \) big enough.

We will still call objects of \( \mathcal{Q} \) semisimple complexes. Note that since Verdier duality commutes with the translation functors \( n \rightarrow m \), it gives rise to an involutive functor on \( \mathcal{Q} \), which we still call Verdier duality and denote again by \( D \). The category \( \mathcal{Q} \) is closed under certain finite sums: let us call a countable collection \( \mathcal{F} \) of objects of \( \mathcal{Q} \) admissible if for any \( n \geq 2 \) the set of \( \mathbb{2} \) for which \( \mathcal{F}_{n} \neq 0 \) is finite. It is clear that if \( \mathcal{F} \) is a sum of \( \mathcal{F}_{n} \neq 0 \) is a well-defined object of \( \mathcal{Q} \). Let us call an object \( \mathcal{F} \) simple if \( \mathcal{F}_{n} \neq 0 \) is a simple perverse sheaf for all \( n \). Note that if \( \mathcal{F} \) and \( \mathcal{F}_{n} \neq 0 \) for every \( n \geq 2 \) then \( \mathcal{F} \) and \( \mathcal{F}_{n} \) are isomorphic. Indeed, for any \( n \leq m \), \( n, m \) induces an equivalence \( \mathcal{Q}_{n}, Q_{n} \sim \mathcal{Q}_{n}, Q_{n} \) thus \( \mathcal{F}_{n} \) is \( \mathcal{F}_{n} \) for all \( n \). But \( \mathcal{F}_{n} \) and \( \mathcal{F}_{n} \neq 0 \) are simple perverse sheaves, they are determined up to isomorphism by their restriction to the open set \( Q_{n} \).

Corollary 3.1. For any object \( H \in \mathcal{Q} \) there exists an admissible collection of simple objects \( \mathcal{F}_{n} \) and integers \( d_{i} \) such that \( H \sim \bigoplus_{n \geq 2} \mathcal{F}_{n} \).

4. Induction and restriction functors

4.1. Induction functor. As in [1], we consider, for all \( n \geq 2 \) and \( K \neq 0 \), a diagram:

\[
\begin{array}{c}
\mathcal{Q}_{n} \xrightarrow{F_{n}} \mathcal{E}_{0} \xrightarrow{F_{n}} \mathcal{E}_{0} \xrightarrow{F_{n}} \mathcal{Q}_{n}^{+}
\end{array}
\]

where we used the following notations:

- \( \mathcal{Q}_{n} \) is the variety of tuples \( (V_{n}, a_{n}, b_{n})_{n \geq 2} \) with

- \( (V_{n}, a_{n}, b_{n})_{n \geq 2} \) is a subspace of dimension \( d_{n} \).

The minimal \( M \) such that

\[
\begin{array}{c}
\bigoplus_{n \geq 2} V_{n} \otimes \mathbb{L}_{n}(n)
\end{array}
\]
\(- a_s : V_s \to E_n^{j^s}, b_s : E_n^{j^s} = V_s \to E_n^{j^s}, \)
\(- E^{0} \) is the variety of tuples \((i; (V_s)_s)_{s \in S}\) as above,
\(- p_1 : (i; (V_s)_s, a_s, b_s)_{s \in S} = \{(b; \gamma_0, \gamma) : a \equiv \gamma \} \) where \(\gamma = L_a V_s L_s (n)\),
\(- p_2, p_3 \) are the projections.

Lemma 4.1. The map \(p_1 \) is smooth.

Proof. Let \(X \) be the variety of tuples \((V_s; a_s, b_s)_{s} \) as above \((a \ G_n^+ \) -homogeneous space\). The map \(p_1 \) factors as

\[ E^{0} \overset{p_0}{\to} X \overset{p_1}{\to} \mathbb{Q}_n^{n+1} \overset{Q_n^{1,0}}{\to} \mathbb{Q}_n \]

where \(p_0 : (i; (V_s)_s, a_s, b_s)_{s} = (\gamma; \gamma) \) and \(p_1^{0} \) is the projection. We claim that \(p_0^{0} \) is a vector bundle of rank \(hE_n^{1} \) \( \to \) \( i \). Indeed \( \) the fiber of \(p_0^{0} \) over a point \((i; E_n^{1} F_2); (j; E_n^{1} F_1)\) is canonically isomorphic to \(\text{Hom}(\ker(i); F_1)\). Let us set \(K_i = \ker(i) \) for \(i = 1, 2\). From the sequence \(0 \to K_2 \to E_n \to F_2 \to 0 \) we have

\[ \text{Ext}(F_2; E_n) = \text{Ext}(E_n; E_n) = 0 \]

But since \( \text{Ext}(L_s(n); L_s(n)) \to 0 \) for all \(s, s^0 \),

\(4.2\) \[ \text{Ext}(E_n; E_n) = \text{Ext}(K_2; E_n) = 0 \]

Similarly, applying \(\text{Hom}(K_2; \) \( \to 0 \) \( K \to E_n \to F_1 \to 0 \) yields a sequence \(\text{Ext}(K_2; K_1) \to 0 \to \text{Ext}(K_2; E_n) \to 0 \). Using \(4.2\), we now obtain \(\text{Ext}(K_2; F_1) = 0 \) and hence \(\dim \text{Hom}(K_2; F_1) = \dim hE_n^{1} F_1) \) \( i \). \( X \)

From the above proof it follows that \(E^{0} \) is smooth. Note that \(p_2 \) is a principal \(G_n^+ \to \mathbb{Q}_n^{1,0} \) bundle, and hence \(E^{0} \) is also smooth. Unfortunately, \(p_3 \) is not proper in general, as one can readily see when \(X = \mathbb{P}^1 \) and \(G = \text{Id} \). However, the following weaker property will be sufficient for us.

Lemma 4.2. Fix \(m \geq 2 \) such that \(n \equiv 0 \) \((\sim + m) \). Then \(p_3 : p_3^{-1}(Q_n^{1, m}) \to Q_n^{1, m} \) is proper.

Proof. Let \((i; E_n^{1} F) \to Q_n^{1, m} \). Since \(n \equiv 0 \) \((\sim + m) \), Lemma 4.2 implies that any subsheaf \(F \to Q_n^{1, m} \). Therefore, we have a canonical commutative square

\[
\begin{array}{ccc}
L_n \text{Hom}(L_s(n); G) & \xrightarrow{G} & G \\
\downarrow & & \downarrow \\
L_n \text{Hom}(L_s(n); F) & \xrightarrow{F} & F
\end{array}
\]

and composing with the isomorphism \(\text{Hom}(L_s(n); F) \to E_n^{1} F \) \( \to \) \( \sim G \). \( \to \) \( E_n^{1} F \) yields a collection of subsheaves \(FV_s \to E_n^{1} F \) such that \((i; (V_s)) \to \) \( \sim G \). Conversely, any such collection of subsheaves \(FV_s \to E_n^{1} F \) arises in this way. In other term \(s, p_3^{-1}(Q_n^{1, m}) \)

represents the functor \(H_n^{0} \) \( \to \) \( \sim + , \) from the category of smooth schemes over \(k \) to the category of sets which assigns to the set of all \(G \to E_n^{1} F \)

where \(G \to E_n^{1} F \) are \(- \Delta, G \to \) the set of \(G \to E_n^{1} F \) \( \to \)

and is generated by \(fL_s(n); g) \) for any closed point \(2 \).
- The projection map $E_n^+ \to E_n^+ \to G_+$ induces isomorphisms
  
  $$j : E_n^+ \to G_+ \to \text{Hom}(E_n^+; E_n^+) = G_+$$

  for any closed point $j$.

The map $p_3$ is induced by the natural forgetful map $(G; G_+) \to G_+$. Hence the
  berm at a point $F : E_n^+ \to F$ represents the usual quotient functor $\text{Hom}(E_n^+; F)$, and
  from Proposition 2.1 we deduce that $p_3$ is proper.

X

If $n q(j; + ; m)$ we set

$$\text{Ind}_{n,m} = E_n^+ \to \text{Hom}(E_n^+; E_n^+) = G_+$$

and $\text{Ind}_{n,m} = \text{Ind}_{n,m} \circ \text{Hom}(G_n^+; F)$, with this notation,

$$\text{Ind}_{n,m} \text{ commutes with Verdier duality. Note that from Lemma } 1.12 \text{ and the (equivariant version of the) Decomposition theorem } BBD, \text{ it follows that } \text{Ind}_{n,m} \text{ actually takes values in } Q_{n,m}^+ (Q_{m,+}) \text{ (recall that all perverse sheaves considered are of geometric origin).}$$

Lemma 4.3. Let $n \leq m$. There is a canonical isomorphism

$$\text{Ind}_{n,m} : Q_{n,m} (Q_n) = Q_{n,m} (Q_m)$$

Proof. It is enough to prove both formulas with $G$ and $\text{Ind}_{n,m}$. Statement 1) follows directly from Lemma 1.12 and the definitions. Now let $n ; m ; l$ be as in 1). From Lemma 3.3, it is enough to show the commutativity of the following diagrams of functors

\[
\begin{array}{ccc}
Q_{n,m} (Q_n) & \xrightarrow{P_2} & Q_{n,m} (Q_m) \\
\downarrow & & \downarrow \\
Q_{n,m} (Q_n) & \xrightarrow{P_1} & Q_{n,m} (Q_m) \\
\end{array}
\]

Since $n ; m \leq n ; m$, we have, as in the proof of Lemma 4.2, canonical identifications

$$E_n^+ H \text{Hom}_{E_n^+} ; + ; l$$

This allows one to define transfer functors $n ; m \to Q_{n,m} (Q_n) = Q_{n,m} (Q_n)$, and

$$n ; m : Q_{n,m} (Q_n) \to Q_{n,m} (Q_m)$$

in the same manner as in Section 3, and therefore complete the diagram with two middle vertical arrows. The commutativity of each ensuing square follows from standard base change arguments. We leave the details to the reader.

Definition. Let $F \to Q ; H \to Q$. By Lemma 4.3 1), for each $x ; m \geq 2$ Z, the complexes $\text{Ind}_{n,m} (H_n, E_n)$ are all canonically isomorphic to $n \leq m$. Furthermore, for each $l > m$ the complexes $\text{Ind}_{n,m} (H_n, E_n)$ and $\text{Ind}_{n,m} (H_n, E_{n+l})$ are canonically isomorphic by Lemma 4.3 1). The collection of
completes \( \text{Ind}_{n, m}(W_n F_n)\); \( n \), \( q(\cdot; +; m)g \) thus gives rise to an object of \( Q^+ \) which is unique up to isomorphism. We denote this functor by

\[
\text{Ind}^i : Q \to Q^+ \to Q^+
\]

and call it the induction functor.

Lemma 4.4 (associativity of induction). For each triple \( ; ; 2 K^+ \to X \) there are (canonical) natural transformations \( \text{Ind}^i \circ \text{Ind}^j \to \text{Ind}^j \circ \text{Ind}^i \).

Proof. It is enough to show that, for any \( n \) and \( m \) we have a canonical natural transformation

\[
(4.4) \quad \text{Ind}_{m, n}^i \circ \text{Ind}_{n, m}^j \circ \text{Ind}_{m, n}^i \to \text{Ind}_{n, m}^j \circ \text{Ind}_{m, n}^i \circ \text{Ind}_{m, n}^i(\text{Id}_{n, m})
\]

Consider the functor

\[
\text{Ind}_{n, m}^i : Q_{n, m}(Q_n) \to Q_{n, m}(Q_n) \to Q_{n, m}(Q_n)
\]

induced by the diagram \( 4.4 \). Using Lemma 4.3 we have

\[
\text{Ind}_{m, n}^i \circ \text{Ind}_{n, m}^j \circ \text{Ind}_{m, n}^i = (\text{Id}_{n, m}) \circ \text{Ind}_{m, n}^i \circ \text{Ind}_{n, m}^j \circ \text{Ind}_{m, n}^i(\text{Id}_{n, m})
\]

and a similar expression for the right-hand side of \( 4.4 \). Hence it is enough to prove that \( \text{Ind}_{n, m}^i \circ \text{Ind}_{n, m}^j \circ \text{Ind}_{m, n}^i = (\text{Id}_{n, m}) \circ \text{Ind}_{n, m}^i \circ \text{Ind}_{m, n}^j \circ \text{Ind}_{m, n}^i(\text{Id}_{n, m}) \). Define the following triple analogue of diagram \( 4.4 \)

\[
Q_n \xrightarrow{\phi} Q_n \xrightarrow{\phi} Q_n \xrightarrow{\phi} Q_n
\]

where

\( F^0 \) is the variety of tuples \( (s; W_s; V_s; a_s; b_s)_{s \in S} \) with

\[
\begin{align*}
&- (s; E^+_s; +) \in Q^+_n, \\
&- W_s \subseteq V_s \subseteq E^+_s, \quad E^+_s \text{ is a chain of subspaces of respective dimensions} \\
&- c_s(n; +) = M_s, \quad c_s(n; +) = +
\end{align*}
\]

\( F^0 \) is the variety of tuples \( (s; W_s; V_s; a_s; b_s; c_s)_{s \in S} \) as above,

\( F_\circ \) is the variety of tuples \( (s; M_s; V_s; a_s; b_s; c_s)_{s \in S} \) as above,

\[
\begin{align*}
&- q_s : W_s \to s, c_s : E^+_s = V_s, E^+_s = V_s, E^+_s = V_s, \\
&- q_s = M_s, W_s, V_s, a_s, b_s, c_s
\end{align*}
\]

\( q_s, q_\circ, q_\circ \) are the projections.

Standard arguments now show that

\[
\text{Ind}_{n, m}^i \circ \text{Ind}_{n, m}^j \circ \text{Ind}_{m, n}^i = (\text{Id}_{n, m}) \circ \text{Ind}_{n, m}^i \circ \text{Ind}_{m, n}^j \circ \text{Ind}_{m, n}^i(\text{Id}_{n, m})
\]

as wanted.

Using Lemma 4.4, we may now define an iterated induction map

\[
\text{Ind}^{i,j,z} : Q \to Q^+ \to Q^+ \to Q^+
\]
4.2. Restriction functors. Let \( n \geq 2 \), \( Z \), \( 2 \mathbb{K}^+(X) \). For \( s \geq 2 \) we x a subspace \( V_s \subseteq E_n^+ \) of dimension \( d_s(n) \) and we x isomorphism \( s_{as} : V_s ! E_n^+ \), \( \lambda_s : E_n^+ \to V_s ! E_n^+\). We now consider the diagram

\[
\begin{array}{ccc}
V_n^+ & \xrightarrow{i} & F_n \\
\downarrow & & \downarrow \\
V_n & \xrightarrow{j} & Q_n \\
\end{array}
\]

where

\[
-F \text{ is the subvariety of } Q_n^+ \text{ consisting of } (\cdot ; E_n^+; F) \text{ such that } \]

\[
M \left[ \begin{array}{c}
V_s \\
\lambda_s(n) \\
\end{array} \right] = \]

\[-(\cdot) = (\cdot) \in \mathbb{P} \lambda_s^{-1} : \text{ where we put } V_s = V_s \lambda_s(n)
\]

Observe that by the proof of Lemma 4.1, \( \mathbb{P} \lambda_s^{-1} \) is a vector bundle of rank \( h \mathfrak{E}_n \) \( \cdot 1 \).

We define a functor

\[
\underline{\mathfrak{R}} \underline{e}_n^j = i : Q_n \to (Q_n^+)! D(Q_n, Q_n)
\]

and set \( \mathfrak{R} \underline{e}_n^j = \underline{\mathfrak{R}} \underline{e}_n^j [h ; 1] \).

Lemma 4.5. Assume that \( \mathfrak{R} \underline{e}_n^j (P) \) 2 \( Q_n \to (Q_n^+) \) \( Q_n \) for some \( P \geq 2 \) \( Q_n^+ \). Then, for any \( n \geq m \geq 2 \) \( Z \) there is a canonical isomorphism of functors \( \mathfrak{R} \underline{e}_m^j (Q_n) \to \mathfrak{R} \underline{e}_m^j (Q_n) \).

Proof. Given the definitions of \( \mathfrak{R} \underline{e}_n^j \) and \( \mathfrak{R} \underline{e}_m^j \) it is enough to construct a natural transformation

\[
(e_{n,m}^j e_{n,m}^j) \underline{\mathfrak{R}} \underline{e}_n^j \to (e_{n,m}^j e_{n,m}^j) \underline{\mathfrak{R}} \underline{e}_m^j
\]

Also we may assume that \( m = n + 1 \). Consider the inclusion diagram

\[
\begin{array}{ccc}
V_n^+ & \xrightarrow{i} & F_n \\
\downarrow & & \downarrow \\
V_n & \xrightarrow{j} & Q_n \\
\end{array}
\]

Claim 4.1. Both squares in (4.6) are cartesian. In particular, we have

\[
j_! \underline{\mathfrak{R}} \underline{e}_n^j = \underline{\mathfrak{R}} \underline{e}_n^j [h ; 1] : \]

Proof of claim. This is clear for the first square. Now let \( F \to G \) be coherent sheaves generated by \( fL_s(m)g \) and let 0 ! \( F \to H \to G \) 0 be an extension. By Lemma 2.1, \( \text{Ext}(L_s(m); F) = \text{Ext}(L_s(m); G) = 0 \) for all \( s \geq 2 \). Then the proof of Lemma 2.1 shows that \( H \) is also generated by \( fL_s(m)g \), and hence that the second square is also cartesian.

Let \( \mathfrak{V}_n^m \subseteq E_n^+ \) be as in the definition of \( \mathfrak{R} \underline{e}_n^j \). Let \( P_n^j G_n^+ \) be the stabilizer of

\[
\mathfrak{V}_n^m = M \frac{\mathfrak{V}_n^m}{\mathfrak{V}_n^m} L_s(m)
\]

and let \( P_n^j G_n^+ \) be the stabilizer of

\[
\mathfrak{V}_n^m = M \frac{\mathfrak{V}_n^m}{\mathfrak{V}_n^m} L_s(m)
\]
There is a natural surjection $P_{n,m}^i \rightarrow G_{n,m} \rightarrow G_{n,m}$. For simplicity we set $G_{n,m} = G_{n,m} \rightarrow G_{n,m}$. Now consider the following diagram

\[ \begin{array}{cccc}
Q_{n,m} & \rightarrow & F_{n,m} & \rightarrow \\
p_1 & & p_2 & \\
G_{n,m} & \rightarrow & G_{n,m} & \rightarrow \\
\downarrow & & \downarrow & \\
P_{n,m}^i & \rightarrow & F_{n,m}^i & \rightarrow \\
\end{array} \]

where all squares are commutative. The rightmost square being cartesian, we have $p_3 \circ q_0 = 1 \circ p_2 \circ = 1 \circ p_2 : P_{n,m}^i \rightarrow$. Further one, since $p_2^0$ is an adaption,

\[ \begin{align*}
1! &= 1 \circ p_2^0 \circ p_3^0 \\
&= 1 \circ p_2^0 \circ [2 \dim (P_{n,m}^i \rightarrow G_{n,m}^i)] \\
&= 0 \circ p_2^0 \circ [2 \dim (P_{n,m}^i \rightarrow G_{n,m}^i)]
\end{align*} \]

so that

(4.8) \[ p_3 \circ q_0 = 0 \circ q_0 \circ p_3 [2 \dim (P_{n,m}^i \rightarrow G_{n,m}^i)] : \]

Next, there are compatible surjective maps

(4.9) \[ \begin{array}{cccc}
0 & \rightarrow & V_{n,m} & \rightarrow \\
u & & u_{n,m} & \\
E_{n,m}^* & \rightarrow & W_{n,m} & \rightarrow \\
\downarrow & & \downarrow & \\
0 & \rightarrow & W_{n,m} & \rightarrow \\
\end{array} \]

where we put $W_{n,m} = E_{n,m}^* = V_{n,m}$ and $W_{n,m} = E_{n,m}^* = V_{n,m}$. Let $T^* G_{n,m}$ (resp. $T G_{n,m}$, resp. $T G_{n,m}$) be the stabilizer of $\ker u_{n,m}$ (resp. of $\ker u$, resp. of $\ker u$) and put $T \cup = T \cup \setminus P_{n,m}^i$. By construction, we have a natural surjection $T \cup T \cup T$. We also have a section $T \cap T ! T$. There is a projection diagram with maps induced by $T$.

\[ \begin{array}{cccc}
G_{n,m} & \rightarrow & Q_{n,m} & \rightarrow \\
\rightarrow & & \rightarrow & \\
P_{n,m}^i & \rightarrow & F_{n,m}^i & \rightarrow \\
\downarrow & & \downarrow & \\
G_{n,m}^i \cap & \rightarrow & G_{n,m} & \rightarrow \\
\rightarrow & & \rightarrow & \\
F_{n,m}^i & \rightarrow & F_{n,m}^i & \rightarrow \\
\end{array} \]

The rightmost square being cartesian, we have $q_0 \circ q_0 = 0 \circ q_0 \circ q_0 = 0 \circ q_0 \circ q_0$. Further one, $q_0$ is a vector bundle so

\[ 2 \circ q_0 = 2 \circ q_0 \circ q_0 \]

\[ = 0 \circ q_0 \circ q_0 \circ q_0 \]

\[ = 2 \circ q_0 \circ q_0 \circ q_0 [2 \dim (T \cap T) = T \cap T] \]

\[ = 0 \circ q_0 \circ q_0 \circ q_0 [2 \dim (T \cap T) = T \cap T] \]
and finally

\[ q_0 \frac{\partial}{\partial s} + \text{2dim} \left( T^1; \mathcal{M} \right) = q_0 \frac{\partial}{\partial s} + \text{2dim} \left( \mathcal{M} \right) \]

Now, if \( P^0 = \text{ind}_{T^1}^{G_{n,m}}(\mathcal{P}) \) is characterized (up to isomorphism) by the relation \( q_0 \frac{\partial}{\partial s} + \text{2dim} \left( \mathcal{M} \right) \) by the relation \( q_0 \frac{\partial}{\partial s} + \text{2dim} \left( \mathcal{M} \right) \) (and similarly for \( \text{ind}_{T}^{G_{n,m}} \), etc.). Thus, putting (4.2) and (4.10) together gives

\[ \text{ind}_{T}^{G_{n,m}}(\mathcal{P}) = \text{ind}_{T^1}^{G_{n,m}}(\mathcal{P}) \times 2 \left( \text{dim} \left( \mathcal{M} \right) \right) \]

As in Section 3 one sees that \( R_{n,m}^j \) is isomorphic to the subvariety of \( R_{n,m}^j \) defined by

\[ R_{n,m}^j = f( :E_n^j \rightarrow M 2 R_{n,m}^j \}

and that the natural map

\[ M \rightarrow \text{Hom} \left( L_n(n); L_{n,m}(m) \right) \]

induces embeddings \( F_m \rightarrow R_{n,m}^j \) as well as a surjective morphism

\[ 2 : P_{n,m}^j \rightarrow F_m \rightarrow R_{n,m}^j \]

Next observe the diagram

\[ \begin{array}{ccc}
G_{n,m}^n & \stackrel{Q_{n,m}^j}{\rightarrow} & F_{n,m}^j \\
\downarrow & & \downarrow \\
G_{n,m}^n & \stackrel{Q_{n,m}^j}{\rightarrow} & F_{n,m}^j \\
\end{array} \]

For simplicity, let us label \( A_{11}, A_{12}, A_{21}, A_{22} \) the vertices of the rightmost square in (4.12). A direct computation shows that the natural map \( \frac{\partial}{\partial s} : A_{21} \rightarrow A_{11}, A_{12}, A_{22} \) is a vector bundle of rank \( r = \text{dim} \left( \mathcal{M} \right) \) \( \text{dim} \left( \mathcal{M} \right) \) \( \text{dim} \left( \mathcal{M} \right) \) \( \text{dim} \left( \mathcal{M} \right) \). Hence a computation similar to (4.13) gives

\[ \text{Res}_{n,m}^j \left( \mathcal{M} \right) \]

Finally, again as in (4.11), we have an isomorphism of functors

\[ \text{Res}_{n,m}^j \left( \mathcal{M} \right) \]

where

\[ Q_m^j \\
\text{Res}_{n,m}^j \left( \mathcal{M} \right) \]

is the diagram for \( \text{Res}_{n,m}^j \left( \mathcal{M} \right) \).
Combining (4.10), (4.11), (4.13) and (4.14) yields (using simplified notations)

\[
\begin{align*}
(e_{n,m}^i e_{n,m})^i & = \text{res} \quad \text{ind} \quad i \quad j \\
& = \text{res} \quad \text{ind} \quad j [2] \quad d_s (n; ) (d_s (n;m ; ) d_s (n; )) ] \\
& = \text{res} \quad \text{ind} \quad j [2] \quad d_s (n;m ; ) d_s (n;m ; ) d_s (n; ) d_s (n; )) ] \\
& = \text{res} \quad \text{ind} \quad j [2] \quad d_s (n;m ; ) d_s (n;m ; ) d_s (n; ) d_s (n; )) ] \\
& = \text{res} \quad \text{ind} \quad j [2] \quad d_s (n;m ; ) d_s (n;m ; ) d_s (n; ) d_s (n; )) ] \\
& = e_{n,m}^i e_{n,m}^j \quad \text{as desired.}
\end{align*}
\]

\[X\]

5. The algebra \( U_\lambda \)

5.1. Some simple quot schemes. We start by describing the varieties \( Q_n \) in some important examples. We will need the following pieces of notation. For \( i = 1; \ldots; N \) there exists a unique additive function \( \text{deg}_i : \text{Pic}(X, G) \to \mathbb{Z} \) satisfying

\[
\text{deg}_i ( (u_{ij}) ) = 1; \quad \text{deg}_i ( ) = \text{deg}_i ( (u_{ij}) ) = 0 \quad \text{if } k \notin \{i\}
\]

(recall that \( (u_{ij}) \) is the class of the simple torsion sheaf \( S_1 \) while \( (u_{ij}) \) is that of a generic simple torsion sheaf). For \( s \in 2 \) we have \( h[L_s]^i \) \( (u_{ij}) \) if \( i = \text{deg}_i (s) \) and \( h[L_s]^i \) \( (u_{ij}) \) otherwise.

For \( i \leq 1; \ldots; N \) and \( g \) \( G \), denote as in Section 1 by \( N \) \( (e_1) \) the space of all nilpotent representations of the cyclic quiver \( A_{1} \) of dimension \( (1) \) \( N \) \( (e_1) \), on which the group \( G = \bigotimes_{j} ^{N} \text{GL}(j) \) naturally acts. For \( 1 \leq i \leq \lambda \) \( N \) and \( 1 \leq j \leq (1) \) \( G \), as above we have embeddings

\[
\begin{align*}
\gamma^i : \text{GL}(1) & \to \text{Aut}(L_s \delta_0 (i_1)) = \text{Aut}(L_s (1)) \\
\gamma^i (x) : (x; x; \ldots; x) & \to \text{Aut}(L_s (i_1)) = \text{Aut}(L_s (\delta_0 (i_1))) \\
\gamma^i (x) & \to x^{-\text{deg}_i (n)}
\end{align*}
\]

Finally, in general let \( l_1; \ldots; l_2 \) \( N \) and let \( l_1; \ldots; l_2 \) \( s \) be as above (for corresponding values of the index \( l_1; \ldots; l_2 \)). Setting \( \gamma^i : \text{GL}(l_1) \to \mathbb{C}_{G} \), we denote by

\[
\begin{align*}
\gamma^i & : \text{GL}(l_1) \to \text{GL}(l_1) \\
\gamma^i & : \text{GL}(l_1) \to \mathbb{C}_{G}
\end{align*}
\]

the composition of the embeddings

\[
\begin{align*}
\gamma^i & : \gamma^i \gamma^i : \text{GL}(l_1) \gamma^i \gamma^i \gamma^i \text{Aut}(L_s (l_1)) \gamma^i \text{Aut}(L_s (\delta_0 (l_1))) \\
j = j & \quad k = k \quad j = j \quad k = k
\end{align*}
\]
with the inclusion of the Levi factor

\[ \begin{array}{c c c c}
Y & Y^\circ & \text{Aut}(L_\mathfrak{g}_),& \text{Aut}(L_\mathfrak{g}^{d_\mathfrak{g}}(\mathfrak{g}_0))  \\
\mathfrak{g} & \mathfrak{g}^\circ & G_0 & ! G_0 \\
\end{array} \]

Such an embedding is clearly well-defined up to inner automorphism of \( G_0 \).

We now proceed with the description of the Quot schemes \( \mathcal{Q}_n \) in some simple cases. Recall that if \( G \) is the class of a torsion sheaf then \( \mathcal{Q}_n \) for any \( n > 0 \). For simplicity, we will simply write \( \mathcal{Q} : G \); for \( \mathcal{Q} \), etc. Also, by the support of a sheaf \( F \), we mean the support of sheaf \( \mathcal{Q} \), where \( \mathcal{Q} : X ! P^1 \) is the quotient map.

Assume that \( \mathcal{P} = \mathcal{P} \) for some \( i \), and that \( j = 0 \) for at least one \( j \). Then any sheaf \( F \) with \( \mathcal{P} = \mathcal{P} \) is supported at \( i \). In this case we have

\[ \mathcal{Q} : G = N^{(\mathcal{P})}; \]

In particular, \( \mathcal{Q} = \mathcal{Q} \) if \( \mathcal{Q} = (\mathcal{P}) \) with \( j \neq 0 \) and \( \mathcal{Q} = PGL(\mathfrak{M}) \) where \( M = 1 \times \mathfrak{M} \) if \( k \neq 1 \) if \( \mathcal{Q} = (\mathcal{P}) \).

Next, assume that \( \mathcal{Q} = \mathcal{Q} \). We have \( E = L^\mathfrak{g}_e \mathfrak{g}_I \) so that \( G = C \). The assignment \( (E \mapsto \mathcal{Q} : F) \) induces an automorphism \( : \mathcal{Q} : P^1 \) and we have

\[ \begin{align*}
\mathcal{Q} & : G = N^{(\mathcal{P})}; \\
\mathcal{Q} & : G = N^{(\mathcal{P})}; \\
\end{align*} \]

Observe that \( N^{(\mathcal{P})} \) has \( p_i \) open orbits, and hence \( \mathcal{Q} = N^{(\mathcal{P})} \) has \( p_i \) irreducible components. In the simplest case \( \mathcal{K} : G = (P^1; \text{Id}) \) we have \( \mathcal{Q} : P^1 \).

More generally, assume \( \mathfrak{g} = \mathfrak{g} \) for some \( 1 < 2 \). We have \( E^\mathfrak{g}_I = L^\mathfrak{g}_e \mathfrak{g}_I \) and \( G^\mathfrak{g}_I = GL(\mathfrak{g}_I) \). Let us consider the bers of the support map \( 1 : Q^1 ! S^1 \) (here the support is counted with the multiplicity given by the class in \( \mathcal{K} (\mathfrak{g}_I) \)). If \( t_2 P^1 \) then

\[ \begin{align*}
\mathcal{Q} & : G = N^{(\mathcal{P})}; \\
\mathcal{Q} & : G = N^{(\mathcal{P})}; \\
\end{align*} \]

where \( N_1 = N^{(\mathcal{P})} \) is the nilpotent cone, while

\[ \begin{align*}
\mathcal{Q} & : G = N^{(\mathcal{P})}; \\
\mathcal{Q} & : G = N^{(\mathcal{P})}; \\
\end{align*} \]

The ber at a general point looks like a product of bers of the previous type for smaller values of \( i \). Namely, if \( t = (t_1; t_2; \ldots; t_k) \mid (n_1; n_2; \ldots; n_k) \) then \( S^1 \) with \( l_j + n_k = 1 \) then

\[ \begin{align*}
\mathcal{Q} & : G = N^{(\mathcal{P})}; \\
\mathcal{Q} & : G = N^{(\mathcal{P})}; \\
\end{align*} \]

where \( H = (n_1; n_2; \ldots; n_k) \) and \( N^{(\mathcal{P})} \) the graph of \( \mathcal{g} \), the support map restricts to \( 1 : \mathcal{Q} :! S^1 \mathfrak{g}_I \).

As an example of a variety \( Q^1 \), let us again assume that \( \mathcal{K} : G = (P^1; \text{Id}) \). We then have \( Q^1 = e \mathfrak{g} e(\mathfrak{g}_I) \). Via the embedding \( \mathcal{Q} : \mathcal{Q} : G \mathfrak{g} e(\mathfrak{g}_I) \), the graph of \( \mathcal{g} \), the support map restricts to \( 1 : \mathcal{Q} :! S^1 \mathfrak{g}_I \).

We will need the following subvarieties of \( \mathcal{Q} \), denoted for an arbitrary torsion class. Recall that if \( \mathcal{Q} = (\mathcal{P}) \) then \( j = \frac{\mathcal{Q}}{\mathcal{P}_i} \). If \( \mathcal{Q} \) is a torsion sheaf
then \( F = F^0 \) and \( \text{supp}(F^0) \cap P^1 \) and \( \text{supp}(F) \). We also put

\[
U_d = f( :E_0 \ F) \cup \text{supp}(F) \ \text{and} \ U_d = f( :E_0 \ F) \cup \text{supp}(F)
\]

It is clear that \( U_d \) determines a decreasing filtration of \( Q^1 \), and that

\[
U_0 = U_0 \cup U_d = 1 \ (S^1 \ P^1) \cup \text{supp}(F)
\]

is nonempty if and only if \( d = 1 \) for some \( u \), in which case \( U_d \) is an open set in \( Q^1 \). We construct a stratification \( \mathbb{P}^1 \) as follows. Let \( \_ = ( \_1, \_2, \ldots, \_n ) \) be an \( r \)-tuple of partitions such that \( \_j \) \( \in \mathbb{P}^1 \) and \( \text{supp}(F) \). We put

\[
U_1 = f( :E_1 \ F) \cup \_j \quad M \bigcup \_k \quad M \bigcup \_k \quad M \bigcup \_k
\]

for some distinct \( x_1, \ldots, x_r \in \mathbb{P}^1 \).

We denote by \( Q_x^{(n)} \) the indecomposable torsion sheaf supported at \( x \) of length \( n \). Observe that \( U_1^{(1,1,1,1)} \) is a smooth open subvariety of \( Q^1 \).

Finally, let \( \_ = ( \_1, \ldots, \_n ) \) be the class of a line bundle \( L \) of degree \( 0 \leq d \leq 1 \) and \( \text{supp}(F) \). We put \( \_ = ( \_1, \ldots, \_n ) \) and \( \_ = ( \_1, \ldots, \_n ) \). For each \( \_1, \ldots, \_n \), let us choose a subspace \( V_{\_1, \ldots, \_n} \) of dimension \( \mathbb{P}^1 \) together with morphisms \( \_1, \ldots, \_n \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]

where \( p : F = F \) is the canonical surjection. Then, denoting by \( \_ = ( \_1, \ldots, \_n ) \) the stabilizer of \( \_ \), we have \( Q_n(\_1, \ldots, \_n) \) and \( \_ = ( \_1, \ldots, \_n ) \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]

where \( p : F = F \) is the canonical surjection. Then, denoting by \( \_ = ( \_1, \ldots, \_n ) \) the stabilizer of \( \_ \), we have \( Q_n(\_1, \ldots, \_n) \) and \( \_ = ( \_1, \ldots, \_n ) \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]

where \( p : F = F \) is the canonical surjection. Then, denoting by \( \_ = ( \_1, \ldots, \_n ) \) the stabilizer of \( \_ \), we have \( Q_n(\_1, \ldots, \_n) \) and \( \_ = ( \_1, \ldots, \_n ) \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]

where \( p : F = F \) is the canonical surjection. Then, denoting by \( \_ = ( \_1, \ldots, \_n ) \) the stabilizer of \( \_ \), we have \( Q_n(\_1, \ldots, \_n) \) and \( \_ = ( \_1, \ldots, \_n ) \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]

where \( p : F = F \) is the canonical surjection. Then, denoting by \( \_ = ( \_1, \ldots, \_n ) \) the stabilizer of \( \_ \), we have \( Q_n(\_1, \ldots, \_n) \) and \( \_ = ( \_1, \ldots, \_n ) \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]

where \( p : F = F \) is the canonical surjection. Then, denoting by \( \_ = ( \_1, \ldots, \_n ) \) the stabilizer of \( \_ \), we have \( Q_n(\_1, \ldots, \_n) \) and \( \_ = ( \_1, \ldots, \_n ) \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]

where \( p : F = F \) is the canonical surjection. Then, denoting by \( \_ = ( \_1, \ldots, \_n ) \) the stabilizer of \( \_ \), we have \( Q_n(\_1, \ldots, \_n) \) and \( \_ = ( \_1, \ldots, \_n ) \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]

where \( p : F = F \) is the canonical surjection. Then, denoting by \( \_ = ( \_1, \ldots, \_n ) \) the stabilizer of \( \_ \), we have \( Q_n(\_1, \ldots, \_n) \) and \( \_ = ( \_1, \ldots, \_n ) \). Finally, we put \( \_ = ( \_1, \ldots, \_n ) \) and

\[
S_n(\_1, \ldots, \_n) = f( :E_0 \ F) \cup \text{supp}(F) \cup \_1, \ldots, \_n
\]
For any $a$ we define a simple object (5.1) in the category of perverse sheaves. For $2 K^*(\mathbb{X})$, the constant sheaf $Q_{\mathbb{X}}$ belongs to $Q_{\mathbb{X}}$ and for each $n$ we have $Q_{\mathbb{X}} = Q_{\mathbb{X}}$. This gives rise to a well-de ned simple object $Q_{\mathbb{X}}$.

Let $P$ be the set of all simple objects of $Q$ appearing (possibly with a shift) in an induction product $\text{Ind}_i \circ \text{Res}_i : (\ldots, i_1, i_2, \ldots)\mapsto 2 P$. We define $P$ as the full subcategory of $Q$ consisting of all admissible sums of objects of the form $P$ with $P$ and $P$.

Lemma 5.1. For any $a$ and $b$ with $a + 1 = b$ and $i$, let $E_i = f( :E_1 \quad F \quad V_{E_1} \quad V_{E_1} \quad E_1)$. Then $E_i$ is smooth.

Proof. The statement concerning the induction functor is clear from the definitions and from Section 4.1. We prove the second statement. Let us rst show that for any $P = F(\mathbb{X})$ in $U^+$ and any $n$ we have $\text{Res}_i(P) = Q_{\mathbb{X}}(Q_n)$, where $i = i_1, \ldots, i_k$ and $i_k = b$. Set $E_i = f( :E_1 \quad F \quad V_{E_1} \quad V_{E_1} \quad E_1)$ and let $E_i$ be the projection. By Lemma 4.1, the variety $E_i$ is smooth. Using the definitions (see Lemma 4.2), we have $P_n'$ in $\text{Res}_i(P)$ for $i = b$. Now, let us x a subsheaf $V_{E_1}$ as well as identi cations $V_{E_1}$ and $E_1$. Let us consider the two varieties $F_i = f( :E_1 \quad F \quad V_{E_1} \quad V_{E_1} \quad E_1)$. These t together in a diagram (5.1)

The (singular) variety $F_1$ is smooth and is constructed as follows. Fix $a = (i_1; \ldots; i_k)$ and $b = (i_1; \ldots; i_k)$ such that $i_1 + \ldots + i_k = 1$. The subvariety $F_1(\mathbb{X}; a) = f( :E_1 \quad F \quad V_{E_1} \quad V_{E_1} \quad E_1)$ is smooth.
is smooth by Lemma 4.1, and we clearly have $F_1^0 = S_{\neq \phi}F_1^0(\phi;\alpha)$. Furthermore, the restriction of $p_1^0$ to $F_1^0(\phi;\alpha)$ factors as a composition

$$F_1^0(\phi;\alpha) \xrightarrow{b_{\phi}} E_1^0(\phi) \xrightarrow{E_1^0(\alpha) b_{\phi}} Q_1 \xrightarrow{1}$$

where $E_1^0(\phi)$ (resp. $E_1^0(\alpha)$) is defined as $E_1^0$ by replacing $i$ by and $j$ by $i$ (resp. by $j$). Let $b_{\phi} : (V_i, i) \mapsto (V_i, i)$ such that $b_{\phi} : i \to i$. Then $b_{\phi} : p_1^0(Q_{E_1^0(\phi)}^0) = p_1^0(Q_{E_1^0(\alpha)}^0)$ and $b_{\phi} : p_1^0(Q_{E_1^0(\phi)}^0) \to p_1^0(Q_{E_1^0(\alpha)}^0)$. Note that the restriction of $p_1^0$ to $F_1^0(\phi;\alpha)$ is still a vector bundle and that, for $1 \leq n$, the restriction of $p_1^0$ to $E_1^0(\alpha)$ is supported on $Q_{E_1^0(\phi)}^0$. We may assume that $E_1^0(\phi) \to E_1^0(\alpha)$ is proper. Using Lemma 8.1.6, we deduce that

$$\text{Res}_{E_1^0(\phi)} \to \text{Res}_{E_1^0(\alpha)}$$

where $\text{Res}_{E_1^0(\phi)}$ is the rank of $(\phi;\alpha)$. Now, applying Lemma 4.1, we obtain

$$\text{Res}_{E_1^0(\phi)} \to \text{Res}_{E_1^0(\alpha)}$$

Finally, notice that $\text{Res}_{E_1^0(\phi)}(Q_{E_1^0(\alpha)}^0) = \text{Ind}_{E_1^0(\phi)}(Q_{E_1^0(\alpha)}^0)$. We conclude the proof of the Lemma by showing the following result.

**Lemma 5.2.** Let be of rank at most one. Then $1$ belongs to $U$.

**Proof.** If $Q$ is the class of a vector bundle then $2f$ and there is nothing to prove. So we may assume that $Q$ is the class of a torsion sheaf. Let us first consider the case where $6$ is supported at some point $i$, i.e., there is a point $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, N\}$ with $j \neq 0$ for at least one $j$, such that $Q_{i,j}$ is not empty. In this case $Q_{i,j}$ is described in terms of the category of representations of the cyclic quiver, and the product corresponds to the usual Hall algebra product. It is then easy to see that $1$ appears in a product of the form $\text{Ind}_{(i,j)}^{(i,j)}(1) \otimes (1, i,j)$. Now let us assume that $1 = 1$ for some $1 > 0$. Let us consider the product $A = \text{Ind}_{(i,j)}^{(i,j)}(1)$. The only subspaces of class of a sheaf $F$ is $U_{i+1}^0$ corresponding to a point of $U_{i+1}^0$ (so that $x_i, \ldots, x_i$ are distinct) and the $O_{x_i}$. It follows that the stalk of $A$ over such points is of rank $1$, and that the monodromy representation of $U_{i+1}^0$ on that space factors through the composition $\text{Res}_{U_{i+1}^0} \to \text{Res}_{U_{i+1}^0}$. In particular, $U_{i+1}^0$ is equal to the regular representation of $S_1$. Now let us consider the product $A = \text{Ind}_{(i,j)}^{(i,j)}(1)$. The only subspaces of class of a sheaf $F$ is $U_{i+1}^0$ corresponding to a point of $U_{i+1}^0$. In the general case, we may write $A = F_{i+1}^0, i + 1$ where $i + 1$ is supported at $i$. We consider the product $B = \text{Ind}_{(i,j)}^{(i,j)}(1) \otimes \text{Res}_{E_1^0(\phi)}(Q_{E_1^0(\alpha)}^0)$. The set $U$ of points of $Q$ corresponding to sheaves isomorphic to $F_{i+1}^0, i + 1$ has a dense open set. Observe that $F_{i+1}^0, i + 1$ is the only subsheaf of class $i$ of such a sheaf. As before, this implies that $\text{Res}_{U_{i+1}^0} \to \text{Res}_{U_{i+1}^0}$. As for the induction product (see Lemma 4.1), there exists a canonical natural transformation $\text{Res}_i^{\phi} \to \text{Res}_i^{\phi}$. For any triple $(; ; )$ 2
5 The algebra $U$. Set $A = \mathbb{Z}[v; v^{-1}]$. Following Lusztig, we consider the free $A$-module $U_A$ generated by elements $b_p$ where $P$ runs through $P = P_f P_g$, modulo the relation $b_{P} b_{P'} = v_{P} b_{P+P'}$. The $A$-module has a natural $A^+$-gradation $U_A = \bigoplus_{n} U_A[n]$. Let us say that an element $b_p$ is of $h$-degree $n$ if $P = (p_n)_{n \in \mathbb{Z}}$ with $P_n = 0$ for $m > n$ and $P_n \in \mathbb{O}$. We denote by $\hat{U}_A$ the completion of $U_A$ with respect to the $h$-adic topology. We also extend by linearity the notation $b_p$ to an arbitrary (admissible) complex $P U$; $2 K^{+}(X)$. Finally, if $2 K^{+}(X)$ and $\mathbb{Z}$ then we set $b_1 = b_1$.

We endow the space $\hat{U}_A$ with an algebra and a coalgebra structure as follows. First, by Lemma 5.1 the functors $\text{Ind}^i$ and $\text{Res}^i$ give rise to well-defined $A$-linear maps

$$m; : U_A[1] U_A[1] \hat{U}_A[1] \hat{U}_A[1];$$

$$; : U_A[1] U_A[1] \hat{U}_A[1] \hat{U}_A[1];$$

Furthem ore, these maps are continuous by Lemma 5.2 and Lemma 5.3, and we may extend them to continuous maps

$$m; : \hat{U}_A[1] \hat{U}_A[1] \hat{U}_A[1] \hat{U}_A[1];$$

$$; : \hat{U}_A[1] \hat{U}_A[1] \hat{U}_A[1] \hat{U}_A[1];$$

The associativity of $m = \text{Ind}^i$, $m$; follows from Lemma 5.4 and the coassociativity of $= \text{Res}^i$; $; ;$ is proved in a similar way. Note that $\hat{U}_A$ is an algebra morphism only after a suitable twist, as in [Lu0].

By definition, $\hat{U}_A$ comes equipped with a (topological) basis $\bigoplus_{P \in \mathbb{P}} \hat{U}_A$. The following key result of the paper will be proved in Section 6 and Sections 9-10. Let $K^{+}(X)^{\text{tor}} = \bigoplus_{(i; j) \in \mathbb{N}} (i; j)$ be the set of classes of torsion sheaves.

**Theorem 5.1.** i) The subalgebra $U_A^{\text{tor}} = \bigoplus_{(i; j) \in \mathbb{N}} U_A[1]$ is generated by elements $b_{1}(i; j)$ and $b_1$ for $(i; j) \neq 1$.

ii) Assume that $X$ is of genus at most one. Then $\hat{U}_A$ is topologically generated by $U_A^{\text{tor}}$ and the elements $b_{1 \text{c}}(i; j)$ where $F = O_X(n)$ for $n \geq 2$ and $12 N$; i.e. the subalgebra generated by $U_A^{\text{tor}}$ and the elements $b_{1 \text{c}}(i; j)$ is dense in $\hat{U}_A$.

We conjecture that the statement ii) holds with no restrictions on the genus of the curve $X$. 

K $^+(X)^{3}$. This allows us to define an iterated restriction functor

$$\text{Res}^i: : U_1^i \wedge U_1^i.$$
6. Proof of Theorem 5.3.1

6.1. Let \( P_d \) (resp. \( P_{>d} \)) be the set of perverse sheaves of \( P \) with support contained in \( U_d \) (resp. in \( U_{>d} \)), and set \( P_d = P_{d\mid P_{>d}} \). For simplicity, let us temporarily denote by \( W \) the subalgebra of \( \mathbb{U}_A^{\text{or}} \) generated by \( l_{1,(i)} \) and \( l_{1} \) for \( (i;j) \neq (1;1) \) and \( 1 \leq n \leq 2 \). We will prove by induction on \( d \) that for any torsion class, any element \( b_F \) with \( P \not\subset P_d \) belongs to \( W \). The following two extreme cases will serve as a basis for our induction.

Lemma 6.1. Any \( P \not\subset P_{0} \) is isomorphic to a sheaf of the form \( \mathbb{IC}(U_{(1);(1);(1)});j \) for some irreducible representation of \( S_{1} \).

Proof. Since \( \mathbb{U}_{0} = U_{1} \) for \( \mathbb{E} \) we may assume that \( 1 \) for some \( E \). Next, it is clear that any perverse sheaf in \( P \) appearing in \( \text{Ind}(\mathbb{IF}(1,1)),(1) \) with \( k \neq 1 \) \( (l;j) \neq (1;2) \) for at least one \( k \) actually lies in \( P_{1} \). Therefore \( P \) appears in a product \( A = \text{Ind}(\mathbb{IF}(1,1)) \) by definition, where we have \( A = p_{j}(\mathbb{IF}(1,1)) \) where \( \mathbb{IF}(1,1) \) is the natural projection to \( \mathbb{Q}^{1} \). Since by assumption \( U_{1} \not\subset \text{supp}(P) \) is dense in \( \text{supp}(P) \) we may restrict ourselves to the open set \( p_{j}(U_{1}) \). We claim that the map \( p_{j} \) is seen isomorphism \( U_{1} \) \( (\mathbb{IF}(1,1)) \) being the unique relevant strata. Indeed, let \( e = (1);(1);(1) \) be an \( r \)-tuple of partitions such that \( \mathbb{IF}(1,1);(1) \) for any point \( \mathbb{IF}(1,1) \) in \( U_{1} \) the sheaf \( \mathbb{IF}(1,1) \) of torsion sheaves supported at distinct points, and by Lemma 6.2, the dimension of the corresponding \( G^{1} \)-orbit is equal to \( j_{1}^{2} \) \( \dim \text{Aut} F = j_{1}^{2} + k \dim \text{Aut} F_{k} \). Now, we may view a torsion sheaf supported at a generic point as an element of a space \( N(1) \) of nilpotent representations of the quiver with one loop (or nilpotent cone). In particular, we have

\[
\dim \text{Aut} F_{k} = \prod_{j}^{\mathbb{IF}(1,1)} j_{1}^{2} \dim N_{(\alpha)} \prod_{j}^{\mathbb{IF}(1,1)} j_{1}^{2} \dim N_{(\alpha)} = \prod_{j}^{\mathbb{IF}(1,1)} j_{1}^{2} \dim N_{(\alpha)} + 2 \dim B_{(\alpha)}
\]

where \( N_{(\alpha)} \) is the nilpotent orbit of \( \text{gl}_{\alpha} \), associated to \( (\alpha), \) and \( B_{(\alpha)} \) is the Springer fiber over that orbit. Since the isomorphism classes of sheaves \( F \) of the above type forms a smooth \( r \)-dimensional family, we obtain

\[
\dim U_{1} = j_{1}^{2} + k \dim \text{Aut} F_{k} = j_{1}^{2} + k \dim B_{(\alpha)}
\]

\[
(6.1) \quad \text{codim} U_{1} = (l_{1};(1) + 2 \dim B_{(\alpha)}
\]

Finally, there is a canonical finite morphism \( \mathbb{IF}(1,1) \) of any point \( u \) in \( U_{1} \) to \( \text{b} \), \( \text{dim} \mathbb{IF}(1,1) = \dim B_{(\alpha)} \). The claim now follows by comparing with \( (6.1) \).

From the decomposition theorem of [BBD] (see also [BM]) we thus obtain (up to a global shift)

\[
(6.2) \quad j_{1} \mathbb{IC}(U_{(1);(1);(1)});1 \mathbb{IC}(U_{(1);(1);(1)});1 V_{1} = \bigoplus_{\text{Irrep } S_{1}} \mathbb{IC}(U_{(1);(1);(1)});1 V_{1}
\]

where \( j_{1} : U_{1} \not\subset Q^{1} \) is the open embedding, and \( \text{Irrep } S_{1} \) denotes the set of irreducible representations of \( S_{1} \). The Lemma follows.
The above proof implies the following generalization of (6.2).

Corollary 6.1. Fix integers \( n_1; \ldots; n_r \) such that \( n_i \neq 1 \) and let \( \lambda \) be a representation of \( S_n \). We have

\[
\text{Ind}_{n_1}^{n_r} (IC(U_{((1; \ldots; 1)}; 1)) = IC(U_{((1; \ldots; 1)}; 1));\]

where \( r \) is known to have many induction hypotheses, we deduce that

\[
\text{Corollary 6.1.}
\]

Lemma 6.2. If \( P \neq 2 \) then \( b_{P} \neq 2 \).

Proof. By assumption we have

\[
\text{supp}(P) = f : E \rightarrow \text{supp}(F) \quad \text{g}.
\]

Note that \( U_{j} \) is a disjoint union of varieties \( Y_{i;j;u} \) where \( i = 1 \), where

\[
Y_{i;j;u} = f : E \rightarrow \text{supp}(F) \quad \text{f} + 1 \text{g}; \quad \text{f} \Rightarrow \text{g}.
\]

Since \( P \) is simple, we have \( \text{supp}(P) \) \( Y_{i;j;u} \) for some \( i; j; u \).

1) Let us first assume that there exists \( i \) such that \( i \neq 0 \) (and \( j = 0 \) for \( j \neq i \)). Then

\[
\text{Lemma 6.2.}
\]

Following Lusztig (see [Lu2]), consider the \( (v) \)-linear map to the Hall algebra

\[
i : U_{i} \rightarrow H_{p_{i}}.
\]

\[
b_{P} \in U_{i} \quad \forall \text{dim}(G) \quad \text{dim}(G) \quad \times \quad \text{dim}(G) \quad (1) \text{dim}_H^i \quad \text{dim}_H^j \quad (IC(G)) \quad v \quad i
\]

which is a \( (\mathfrak{g}) \)-module isomorphism (in the above, we view \( N_{p_{i}} \) as a subvariety of \( Y_{i;j} \) via (6.4)). Further, \( H_{p_{i}} \) is equipped with a canonical basis \( B_{H_{p_{i}}} \) and \( \text{dim}_H^i \), which consists of the functions

\[
\times \quad \text{dim}_H^i \quad (IC(G)) \quad v \quad i
\]

where \( \mathfrak{o} \) runs through the set of all \( G \)-orbits in \( N_{p_{i}} \), for all \( \mathfrak{g} \). In particular, up to a power of \( v \), we have \( i(B_{P}) \in B_{H_{p_{i}}} \) for any \( P \neq 2 \) (b).

We now prove by induction on \( j \) that \( b_{P} \neq 2 \).

If \( \mathfrak{b} \) then \( i(B_{P}) \in U_{i} \setminus \{1\} \) and in the second case \( i(B_{P}) \in U_{i} \setminus \{1\} \) by [Lu1], Section 12.3. Now assume that the result is proved for all \( \mathfrak{g} \) with \( j \neq j \) and that \( \lambda \neq . By \text{Lemma } 5.1 \), we have \( \text{Res}_{P}(F) \in U_{i} \) any \( \mathfrak{g} \) such that \( + \neq . Using \text{the induction hypothesis, we deduce that}

\[
b_{\text{Res}}(g) = ; \quad B_{P} \neq 2 \quad \text{W}
\]

We conclude using the following result, which is proved in the appendix.

Lemma 6.3. If \( b_{P} \neq 2 \), then \( b_{P} \) is of degree \( \mathfrak{g} \) and if \( i(B_{P}) \in U_{i} \setminus \{1\} \) \( U_{i} \setminus \{1\} \) for all \( 0 < \mathfrak{g} \) such that \( + \neq \) then \( b_{P} \neq 2 \).
ii) We now treat the general case. Let $P \rightarrow P_{jj}$ with $\text{supp}(P) \rightarrow Y_{1,\cdots,1}$, with arbitrary $i$. It is clear that $Q = \text{Res}_1^{1,\cdots,1}(P)$ is supported on the disjoint union $(\cup i)^{\oplus_i}_Y \rightarrow Y_{1,\cdots,1}(X_i)$. Let $Q_0$ be the restriction of $Q$ to $Y_{1,\cdots,1}$, $Y \times \text{supp} \rightarrow Y_{1,\cdots,1}$. (We have $\text{supp} \rightarrow Y_{1,\cdots,1}$.) Hence by the above result we have $p^0_2 \rightarrow W$. But it is easy to see that, up to a shift, $P = \text{Ind}_1^{1,\cdots,1}(Q_0)$, so that $p^0_2 \rightarrow W$ as desired.

Lemma A.4. Let $P \rightarrow P_{jj}^{+1}$ such that $\text{Ind}_1^{1,\cdots,1}(P_{jj}) = P \rightarrow P_{jj}$ with $P_{jj}$ being a sum of (shifts) of sheaves in $P_{jj}^{+1}$. There exists a complex $T \rightarrow P_{jj}$ such that $\text{Ind}_1^{1,\cdots,1}(P_{jj}) = T \rightarrow T_{jj}$.

**Proof.** It is enough to prove the following statement: there exists a complex $T \rightarrow P_{jj}$ and integers $i \rightarrow 2$ such that $\text{Ind}_1^{1,\cdots,1}(P_{jj}) = T \rightarrow T_{jj}$ with $P_{jj}$ being a sum of (shifts) of sheaves in $P_{jj}^{+1}$. Indeed, since $P$ is simple there exists $i$ such that $\text{Ind}_1^{1,\cdots,1}(P_{jj}) = P \rightarrow P_{jj}$ satisfying the same conditions as $P_1$. The fact that $\text{Ind}$ commutes with Verdier duality then forces $i = 0$.

We consider the diagram

$$
\begin{array}{c}
Q_{jj}^{+1} \leftarrow F_{jj} \rightarrow Q^{1} \\
\downarrow \quad \downarrow \\
U_{jj}^{1} \rightarrow Q
\end{array}
$$

where the top row is the restriction of the diagram for $\text{Res}_1^{1,\cdots,1}$ to the closed subvariety $Q_{jj}^{+1}$, and $U_{jj}^{1}$ is the embedding. By construction, the stalk of $\text{Res}_1^{1,\cdots,1}(P)$ at a point $(2 : E \rightarrow F_{jj} ; 1 : E \rightarrow F_1)$ is zero if $\text{supp}(P) \rightarrow j_{jj}$ and $\text{supp}(P) \rightarrow j_{jj}$. Hence $\text{supp}(U_{jj}^{1}) = Q_{jj}^{+1}$. If $Q = j_{jj}(U_{jj}^{1})$ (a direct summand of $\text{Res}_1^{1,\cdots,1}(P)$). Note that $\text{supp}(Q) \rightarrow Q_{jj}^{+1}$. Also, by Lemma A.5, we have $Q \rightarrow U_{jj}^{1}$ so there exists $T \rightarrow P_{jj}$ and integers $i \rightarrow 2$ such that $Q = j_{jj}(T)$. We claim that, up to a shift, $Q$ satisfies the requirements of the Lemma. To see this, first consider the diagram

$$
\begin{array}{c}
Q_{jj}^{+1} \rightarrow E_{jj}^{0} \rightarrow E_{jj}^{0} \rightarrow U_{jj}^{1} \\
\downarrow \quad \downarrow \quad \downarrow \\
Q_{jj}^{+1} \rightarrow E_{jj}^{0} \rightarrow E_{jj}^{0} \\
\downarrow \quad \downarrow \quad \downarrow \\
Q_{jj}^{+1} \rightarrow Q_{jj}^{+1}
\end{array}
$$

where the bottom row is the restriction of the induction diagram to the closed subvariety $Q_{jj}^{+1} = Q_{jj}^{+1}$, where $E_{jj}^{0} = (\text{Res}_1^{1,\cdots,1}(Q_{jj}^{+1}))$. Note that all the squares in the above diagram are cartesian. On the other hand, if $(i_1 : E \rightarrow F) \rightarrow Q_{jj}^{+1}$ then there exists a unique subsheaf $G \rightarrow F$ such that $G = G$ and $|\text{E}| = 0$. Hence $p^0_2$ is an isomorphism. Standard base change arguments yield

$$(6.5) \quad \text{Ind}_1^{1}(Q) = j_{jj}p^0_1(p^0_2(j_{jj}(Q)) = p^0_2p^0_2p^0_2(j_{jj}(Q)).$$
Next, we look at the diagram

\[
\begin{array}{c}
Q^+_{jj} \xrightarrow{p^0} F_{jj} \xrightarrow{0} U^1_0 \xrightarrow{U} Q_{jj} \\
\downarrow h \quad \downarrow \quad \downarrow h \\
Q^+_{jj} \xrightarrow{i} F_{jj} \xrightarrow{1} Q^1 \xrightarrow{Q} Q
\end{array}
\]

where \( F_{jj} = (U^1_0 \cdot Q_{jj}) \). Similar arguments as above give

\[
6.6 \quad j^0_1 : i (P) = j_1 (Q) = 0^0_1 \cdot j_0 (P): \]

Finally, consider the diagram

\[
\begin{array}{c}
Q^+_{jj} \xrightarrow{p^0} F_{jj} \xrightarrow{0} U^1_0 \xrightarrow{U} Q_{jj} \\
\downarrow a \quad \downarrow \quad \downarrow a \\
E^1_0 \xrightarrow{p^0} P \xrightarrow{P} \end{array}
\]

where \( a : F_{jj} \mapsto E^1_0 \) is the canonical embedding and \( p^0_0 = p^0_0 p^0_0 \). Recall that we have fixed a subspace \( V \supseteq E^1_0 \) in the definition of \( F_{jj} \). Let \( P \subseteq G^+ \) be the stabilizer of \( V \) and let \( U \) be its unipotent radical. There are canonical identifications \( Q^+_{jj} \mapsto E^1_0 \cdot G^+ \cdot F_{jj} \) and \( E^1_0 \cdot G^+ \cdot F_{jj} \). Furthermore, for any \( (j : E^1_0 \rightarrow F_{jj} : 1 : E \mapsto F_{jj} \cdot U \rightarrow Q_{jj} \) we have \( Ext(F_1; F_2) = 0 \) since \( F_1 \) and \( F_2 \) have disjoint supports. This implies that \( F_{jj} \mapsto P \subseteq G^+ \).

Now, \( 0^0_1 \cdot j_1 (P) \) is \( P \)-equivalent hence there exists \( P^0 \subseteq Q_0 \mapsto Q^+_{jj} \mapsto Q_0 \) such that \( 0^0_1 \cdot j_1 (P) = 0^0_1 \cdot j_1 (P) \). Then \( 0^0_1 \cdot j_1 (P) = 0^0_1 \cdot j_1 (P) = 0^0_1 \cdot j_1 (P) \) where \( d = \dim (P = G^+ \cdot F_{jj}) \) and

\[
a p^0_1 \cdot j_0_1 (P) = a p^0_1 \cdot j_1 (P) \mapsto [2d] \\
= 0^0_1 \cdot j_1 (P) \mapsto [2d] \\
= 0^0_1 \cdot j_0 (P) \mapsto [2d] \\
= 0^0_1 \cdot j_0 (P) \mapsto [2d]:
\]

But \( a : Q_0 \mapsto Q^+_{jj} \) is an equivalence, hence \( p^0_1 \cdot j_0_1 (P) = p^0_1 \cdot j_0 (P) \mapsto [2d] \) and

\[
6.7 \quad p^0_1 \cdot j_0_1 (P) = j_0 (P) \mapsto [2d]:
\]

Combining 6.7 with 6.6 and 6.3 finally yields \( j_1 \cdot \text{Ind}^*_i (Q, [2d]) = j_0 (P), \) i.e.

\[
\text{Ind}^*_i (R_1 T_1 [k_1 + 2d], P) \mapsto P_1
\]

with \( \supp (P_1) \mapsto Q^+_{jj} \). We are done.

6.2. We may now prove that \( \text{U}^*_H = W \). We argue by induction. Fix \( 2 \cdot K^+ (X) \) and let us assume that \( \text{U}^*_H (P) = W \cdot V \) for all \( i < 2 \). We will prove by descending induction on \( \text{b}_P (2 W) \) for all \( P \subseteq 2 P_{jj} \). If \( i = \) this follows from Lemma 6.2. So assume that \( \text{b}_P (2 W) \) for all \( P \subseteq 2 P_{jj} \) for some \( P \neq W \) such that \( i = 1 \),
and let $P \neq P_{j_1}$. Applying Lemma $6.4$ gives

$$b_\mathcal{P} 2 b_R b_T + \left. M \right|_{P_2 P_{j_1}}$$

for some $T \neq P_{j_1} R \neq P_0$. If $n \leq 0$ then by the induction hypothesis the right-hand side belongs to $W$ and hence so does $b_\mathcal{P}$. Now assume that $n = 0$, so that by Lemma $6.4$ we have $P = \mathcal{P} \cup \{ \{1, \ldots, n_1\} \}$ for some $2$-Irrep $S_1$. It is well-known that the Grothendieck group $K_0(\mathcal{P}_1)$ is spanned by the class of the trivial representation and the set of classes $\text{Ind}_{S_1}^{G_1}(\chi_{S_1})$ for $2$-Irrep $S_1$. This is a set of genus $\text{atmosphere} X$.

We assume that $\mathcal{P}$ is stable and $\mathcal{P}'$ is a subcategory of $\mathcal{P}$. If the same holds with $\mathcal{P}'$ replaced by $\mathcal{P}$, then we say that $\mathcal{P}$ is stable. Clearly, any line bundle is stable and any torsion sheaf is semistable. The following facts can be found in \cite{Friedlander}, Section 5.

**Proposition 7.1** (\cite{Friedlander}). The following set of assertions hold:

1. If $F$ is semistable of slope $i$ and if $F_1 > F_2$ then $\text{Hom}(F_1; F_2) = 0$.
2. If $F_1$ and $F_2$ are semistable and $F_1 > F_2$ then $\text{Ext}(F_1; F_2) = 0$.
3. Any indecomposable sheaf is semistable and if $X = P$ then any indecomposable vector bundle is stable.
4. The full subcategory $C$ of $\mathcal{C}_0(\mathcal{P}_1)$ consisting of the zero sheaf together with all semistable sheaves of slope $i$ is abelian, artinian and closed under extensions. Its objects are all of finite length and the simple objects are formed by the stable sheaves of slope $i$. 

**Remark**. In this section we describe the categories $\text{Coh}_G(\mathcal{P}_1)$ and collect certain results pertaining to the Harder-Narasimhan filtration for objects of $\text{Coh}_G(\mathcal{P}_1)$ which will be used in the proof of Theorem 5.1. We assume until the end of the paper that $\mathcal{P}$ is of genus $a$ at most one.
The assertion 9 is true for an arbitrary curve $X$; it follows from Serre duality together with the fact that $j_X^*j = 0$ when $X$ is of genus at most one.

Any coherent sheaf $F$ possesses a unique \textit{irrational} $0$-\textit{stratification} $F_0, F_1, \ldots, F_r$ such that $F_i = F_0$ is semistable of slope $i$ for $i = 1, \ldots, r$, and $i > 2 > \lambda$. Furthermore, by Proposition 7.4.7 this stratification splits (noncanonically), i.e., we have $F = F_0 \oplus F_1 \oplus \cdots \oplus F_r$. Write $H(N(\mathcal{F})) = \{F_0 \cup F_1 \cup \cdots \cup F_r \}$ and call this sequence the $HN$ type of $F$. For any $2Q \{ f \}$, we write $F = F_{i_1} \oplus \cdots \oplus F_{i_r}$.

The following result is an easy consequence of the existence of the $HN$ stratification and of Proposition 7.4.1.

Lemma 7.1. If $Q_0 \neq 0$ then $(\ )$.

For any $(1, \ldots, 1)$ $2(Q^+ X)^* = (1, \ldots, 1)$, so that $P_i = 1$ and for any $n \in \mathbb{Z}$, let us consider the subfunctor of $\text{Hom}_{\mathcal{M}_n}$ denoted by $\mathcal{F} \mapsto \mathcal{F}(\ )$ for all closed points $2g^0$.

This subfunctor is represented by a subsheaf $E^0_n = \text{HN}^1_n(1, \ldots, 1)$, and we have $Q_n = \text{HN}^1_n(1, \ldots, 1)$.

Proof. Let us choose some decomposition $E_n = E_n$ and denote by $Q_n = \text{HN}^1_n(1, \ldots, 1)$. The corresponding closed embedding. Then, since $Q_n^{(1)}$ is open in $Q_n^{(1)}, \text{HN}^1_n(1, \ldots, 1)$ is open in the image of the natural map $G \to Q_n^{(1)}$, $G_n^{(1)} = G_n$. In particular, $\text{HN}^1_n(1, \ldots, 1)$ is also constructible.

By Lemma 7.3, $Q_n^{(1)}$ is empty if $(1) < n$, i.e., if $j < n$. For any $x \in X$, there exists only one rational number $i$, where all $i$ satisfy the requirement $(1) < n$. This proves the second part of the Lemma.

The natural statement is now obvious.

Example. If $X$ is of class of a line bundle then the decomposition $Q_n = \mathcal{F} Q_n [L]$ in Section 5.1 is the $HN$ stratification: we have $Q_n [L] = \text{HN}^1_n(1, \ldots, 1)$.

A priori, the subsheaves $\text{HN}^1_n(1, \ldots, 1)$ only provide a naive stratification of $Q_n$ in the broad sense, i.e., the Zariski closure of a stratum in general may not be a union of strata. Hence, we cannot define a partial order on the set of possible $HN$ types directly using inclusion of strata closures. Instead we use the following combinatorial order: we say that $(1, \ldots, 1) \preceq (i, \ldots, i)$ if there exists $1$ such that $1 = 1$ is $1$ and $(1) < (i)$ or $(i) = (1)$. Note that $\text{HN}^1$ $\mathcal{F} \to \mathcal{F} \mathcal{G}$ if and only if there exists $t \in \mathbb{R}$ such that $\mathcal{F} \mathcal{G}^{(t)} = (\mathcal{F} \mathcal{G})^{(t)}$ for any $t > t$. Finally observe that $Q_n \mathcal{F} \mathcal{G}^{(t)} = \mathcal{F} \mathcal{G}^{(t)}$. For any $t$. If $X$ is of class of a line bundle then the decomposition $Q_n = \mathcal{F} Q_n [L]$ in Section 5.1 is the $HN$ stratification: we have $Q_n [L] = \text{HN}^1_n(1, \ldots, 1)$.

If $X$ is of class of a line bundle then the decomposition $Q_n = \mathcal{F} Q_n [L]$ in Section 5.1 is the $HN$ stratification: we have $Q_n [L] = \text{HN}^1_n(1, \ldots, 1)$.
Lemma 7.3. The following hold:

i) Assume that $HN(F) = (1; \ldots ; r)$ and suppose that $F \leq G$. Then $HN(G) = (1; \ldots ; r)$.

ii) Let $F$ be a coherent sheaf and let $0 = F_1 \leq \cdots \leq F_r = F$ be a filtration such that $F_i = F_{i+1}$ if $i < j$. Then $HN(F) = (F_1; \ldots ; F_r)$ with equality if and only if $F_i = F_{i+1}$ is semistable for all $i$.

Proof. Statement i) is obvious. We prove statement ii). Let us set $G_i = F_i/F_{i+1}$. If $G_i$ is semistable for all $i$ then it is clear that $HN(F) = (1; \ldots ; r)$. Suppose on the contrary that $G_i$ is semistable for $i < j$ but that $G_j$ isn't. In particular, we have $G_{j+1} > 0$. There is a unique sheaf $G_j$ with $F_j = G_j'F$ and $G_j = G_j'/G_{j+1}$. It is easy to see that $HN(G_j') = HN(G_j)$. Hence by $\emptyset$ we also have $HN(F) = (F_1; \ldots ; F_r)$ as wanted.

Now let $F = (F_n)_n$ be a simple object in $Q$. There exists $n \geq 0$ such that $F_n \leq G$ is a semistable perverse sheaf on $Q$. By Lemma 7.2 there exists a unique $HN$ type $(1; \ldots ; r)$ with $G_n = G$ such that $supp(F_n)$.

$HN^{-1}(1; \ldots ; r)$ while $supp(F_n) \setminus HN^{-1}(1; \ldots ; r)$ is dense in $supp(F_n)$. The sequence $(1; \ldots ; r)$ is independent of $n$. We call this the generic $HN$ type of $F$ and denote it by $HN(F)$.

Finally, we record the following result for future use.

Lemma 7.4. Let $F$ be a coherent sheaf in $CoC (X)$, and let $2 Q \{ f, g \}$. If $G \leq F$ is a sub sheaf satisfying $[G] = [F]$ then $G = F$.

Proof. We argue by induction on the number of indecomposable summands of $G$ (for all $F$ and simultaneously, if $G$ is indecomposable then by Proposition 7.8 ii) $G$ is semistable of slope $(F)$). There is a (non-canonical) splitting $F = F' \oplus F''$. By the $HN$ filtration we have $Hom(G; F'') = 0$. Hence $G \leq F$. But $[G] = [F]$ and so $G = F$.

Now assume that the statement is true for all $F' \oplus F''$ where $G'$ has at most $k$ indecomposable summands, and let us assume that $G = G_1 \oplus \cdots \oplus G_k$. If $G_i$ indecomposable and $H = (G_i)$, then we have $G_i = (G_i)^{\oplus k}$, so arguing as above we get $G_i = F$. Now set $G'' = G_1$ and $F' = F_0 = F = G_1$. We have $F'' = F = G_1 = [F] = [G_1] = [G_1] = [G_1]$ and $G''$ has $k$ indecomposable summands. Hence by the induction hypothesis, $G'' \leq F''$, and so $G \leq F''$, and finally $G = F$.

8. Some Lemmas.

This section contains several technical results needed later in the course of the proof of Theorem 5.1 ii).

8.1. We begin with the following

Lemma 8.1. Let $V \leq U \leq ^{\wedge} U$, where $(U)$.

Then we have $supp(Ind^{\wedge}(V)) = HN^{-1}(\emptyset)$.

Proof. Consider the (iterated) induction diagram (see Section 4.1. for notations)

$$
\begin{array}{ccc}
Q_1 & \stackrel{P_1}{\longrightarrow} & Q_2 \\
\downarrow & & \downarrow \\
E_1 & \stackrel{P_2}{\longrightarrow} & E_2 \\
\downarrow & & \downarrow \\
E_0 & \stackrel{P_3}{\longrightarrow} & E_0 \\
\downarrow & & \downarrow \\
Q_1 & \stackrel{P_1}{\longrightarrow} & Q_2
\end{array}
$$
It is enough to show that \( p_3(\mathcal{E}^0) \rightarrow S \rightarrow HN^1(\cdot) \). To see this, let \((G_2; \ldots; G_1) \rightarrow Q_1 \rightarrow 1 \) and \((\nu; : E_1 \rightarrow F) \rightarrow 2 \). Putting \( F_1 = (\nu;_1) \) we have \( F_1 \rightarrow F \) and \((\nu;_1)^{-1}(F_1) \rightarrow (\nu;)^{-1}(F_1) \). Thus, from Lemma 7.3, ii) we obtain \( HN(F) \).

8.2. Let us now consider the following situation. Let \( \nu;_1 : 2 \rightarrow K^+ \mathcal{E}(\nu;_1) \) such that \( = \rightarrow + \) and let \( S_1 \rightarrow Q_1 \rightarrow S_2 \rightarrow Q_2 \rightarrow \ldots \) be locally closed subsets invariant under the corresponding group, and satisfying the following properties:

i) Denoting by \( Q_n \rightarrow F \rightarrow Q_n \rightarrow Q_n \) the restriction diagram as in Section 4.2, we have \( S_n \setminus F = S_n \setminus S_n \),

ii) For any closed point \((\cdot; : E_1 \rightarrow F) \rightarrow 2 \) \( S_n \) there exists a unique subsheaf \( G \) of \( F \) of class \( (\cdot;_n) \), and we have \( \text{Ext}(F = G; G) = 0 \).

The set \( S_n \) gives rise, via the induction functors \( m \) and \( n \), to locally closed subsets \( S_m \) for any \( m \geq 2 \). The same is true for \( S_n \) and \( S_m \), and properties i) and ii) above are satisfied for all \( m \). Let us denote by \( j_m : S_n \rightarrow Q_m \) the embedding.

Lemma 8.2. Under the above conditions, we have, for any \( P = (\mathcal{E}_m)_m \rightarrow 2 \) \( Q \), such that \( \text{supp}(\mathcal{E}_m) \rightarrow S_m \) for all \( m \),

\[ j(\text{Ind} \quad \nu;_i \mid \text{Res}^i(\mathcal{E}_m) \quad j \mid (\cdot)) \rightarrow j \mid (\cdot) \]

Proof. Consider the following commutative diagram, where we have set \( F = F_\mathcal{E}(\nu;_1) \), \( \nu;_1 \) the vertical arrow are canonical embeddings and where the notations are otherwise as in Section 4.2.:

\[
\begin{array}{ccc}
S_1 & \rightarrow & F \\
\downarrow g_1 & & \downarrow g_2 \\
Q_1 & \rightarrow & Q_1
\end{array}
\]

By the assumption i), the rightmost square is Cartesian. We deduce by base change that

\[ \nu; \rightarrow 0 \rightarrow \nu;_1. \]

Next look at the diagram

\[
\begin{array}{ccc}
S_1 & \rightarrow & E^0 \\
\downarrow & & \downarrow & & \downarrow \\
\nu;_1 & \rightarrow & \nu; & \rightarrow & \nu;_1 \\
\downarrow & & \downarrow & & \downarrow \\
Q_1 & \rightarrow & E^0 & \rightarrow & Q_1
\end{array}
\]

where \( E^0 = \nu;_1^{-1}(S_1) \) and \( E^0 = \nu;^{-1}(E^0) \). By assumption i) the rightmost square is Cartesian, while by assumption ii) \( \nu;_1 \) is an isomorphism. Hence, by base change we get

\[ \nu;_1 \rightarrow 0 \rightarrow \nu;_1. \]

Now let us set \( R = \text{Ind} \quad \nu;_i \mid \text{Res}^i(\mathcal{E}_m) \). Combining (8.1) with (8.2) and applying this to \( R \) yields

\[ \nu;_1(\mathcal{E}_m)_m = \nu;_1 \text{Ind} \quad \nu;_1 \mid \text{Res}^i(\mathcal{E}_m)_m \quad [d_m] \]

for any \( m \), where \( d_m = \dim(\mathcal{E}_m)_m \).
Recall that a choice of a subsheaf $E_1/E_1$ is implicit in the definition of $F$. Let us denote by $P_0$ the stabilizer. Let us also $x$ a splitting $E_1/E_1$. This choice defines a Levi subgroup $L$ of $G_1$ isomorphic to $G_0 = G_1/G_1$, and induces a section $s: Q_1 \to F$.

We claim that $F' = F^L = (S_1, S_1)$. Indeed, by assumption ii), if

\[(2: E_i, F_2; (1: E_i, F_1)) 2 S_1 S_1\]

then $G' = F_2$. Hence, using Lemma 2.4, we have

\[F^L = \big| S_1 \big| S_1 \setminus F^L = \big| S_1 \big| S_1\]  

Similarly, it is easy to check that $S_1 \to G_1/F$ and that $E_0$ is a principal $G_0$-bundle over $S_1$. To sum up, we have a commutative diagram

\[
\begin{array}{c}
S_1 \\
E_0 \downarrow \quad \downarrow \quad \downarrow \\
S_1 \\
\end{array}
\]

and equivalences

\[0 : Q_{G^0} (S_1)! \to Q_0 (F^L) \dim (G_1 = F)\];

\[p_0^0 : Q_{G^0} (S_1)! \to Q_{G^0} (E_0) \dim (L)\];

\[0 : Q_{G^0} (S_1)! \to Q_0 (F^L) \dim (P = L)\];

\[p_0^0 : Q_{G^0} (S_1)! \to Q_{G^0} (E_0) \dim (G_1)\].

Let $T$ be an object in $Q_{G^0} (S_1)$. By the above set of equivalences, there exists $T_0 : Q_{G^0} (S_1)! \to Q_0 (T_0)$ such that $T_0 = T$. Hence, $T_0 = T$. Moreover, $p_0^0 = p_0^0$ so that $p_0^0 = T$. Altogether we obtain $p_0^0$. Applying this to $Q_1$ yields the desired result. The Lemma is proved.

Corollary 8.1. Let $i_1, \ldots, i_n$ be $K^+(X)$ such that $= P_i$. Assume given locally closed subsets $S_i, S_i$ satisfying the following conditions:

i) Denoting by $Q_n \leftarrow F \rightarrow Q_n$ the iterated restriction diagram, we have $S_n \setminus F = \left\{(S_n, r \in E^F) \quad \left| \left(\! \left. \left( G_1 \right) \right| r \in E^F \right) \right\}$ of $F$ such that $G_1 = G_1$ and we have $\text{Ext}(G_1 = G_1; G_1 = G_1) = 0$ for any $j > 1$.

Then, for any $P_m = (F_m)_{m} 2 Q$ such that $\text{supp}(P_m) S_m$ and $\text{supp}(P_m) \setminus S_m \in \mathcal{E}$; for any $m$, we have

\[j \left( \text{Ind}^{G_0}_{G_1} : \text{Res}^{G_0}_{G_1} (P) \right) \in \left( \text{Res}^{G_0}_{G_1} (P) \right)\]

where $j : S_1 \to Q_1$ is the embedding.

**Proof.** Use induction and Lemma 8.2.
8.3. Recall that \(O_1^n\) denotes the \(G_n\)-orbit in \(Q_n\) corresponding to a sheaf \(H\) of class \(\mathcal{O}_n\). If \(H\) is not generated by \(\mathcal{F}_n\), for such a sheaf \(H\), the collection of perverse sheaves \(\mathcal{IC}(\mathcal{O}_n,1)\) defines a simple object of \(Q\), which we denote by \(\mathcal{IC}(\mathcal{O}_n)\).

Lemma 8.3. Let \(F : G\) be coherent sheaves of class \(\mathcal{O}\) respectively. Assume that there exists a collection of smooth locally closed \(G_1^+\)-subvarieties \(S_1\) satisfying \(\mu_1(S_1) = S_1\), and such that

1) in the induction diagram

\[
\begin{array}{c}
\begin{array}{c}
Q_1 \quad Q_1 \\
\end{array} \quad \begin{array}{c}
p_1 \quad p_1 \\
\end{array} \quad \begin{array}{c}
E \quad E \quad E \\
\end{array} \quad \begin{array}{c}
p_3 \quad p_3 \quad p_3 \\
\end{array} \quad Q_1^+ \\
\end{array}
\]

\(S_1\) is a dense open subset of \(p_1p_3^{-1}(O_1^G \times_O^F 1)\),

2) for any sheaf \(K\) such that \(\mathcal{O}_1^K\) \(S_1\), there exists a unique subsheaf \(\mathcal{O}_1^0\) \(\mathcal{K}\) such that \(\mathcal{O}_1^0\) \(\mathcal{G}\) and \(K = \mathcal{O}_1^0\) \(\mathcal{F}\).

Then there exists a collection of \(G_1^+\)-stable dense open subsets \(V_1 \subset S_1\) satisfying \(\mu_{1,1}(V_1) = V_1\) and such that, for \(m \geq 0\), \(\text{Ind}^1_i\) \((\mathcal{IC}(\mathcal{O}_1^0) \cap \mathcal{IC}(\mathcal{O}_1^F))\) = \(\mathcal{IC}(\mathcal{V}_m) \cap \mathcal{P}\), where \(\mathcal{P}\) has support in \(V_m \cap V_{m-1}\). In particular, if there exists a sheaf \(K\) such that \(\mathcal{O}_1 = \mathcal{O}_1^K\) then \(\text{Ind}^1_i\) \((\mathcal{IC}(\mathcal{O}_1^0) \cap \mathcal{IC}(\mathcal{O}_1^F))\) = \(\mathcal{IC}(\mathcal{O}_1^K) \cap \mathcal{P}\), where \(\mathcal{P}\) has support in \(\mathcal{O}_1^K \cap \mathcal{O}_1\).

Proof. From the proof of Lemma 4.4, the fibers of \(p_1\) in \(\mathcal{E}\) are irreducible. Hence \(p_1p_3^{-1}(O_1^G \times_O^F 1)\) is irreducible. Since \(p_1\) is continuous, \(p_3^{-1}(O_1^G \times_O^F 1)\) is open in \(p_3^{-1}(O_1^G \times_O^F 1)\) and since \(p_3p_2\) is continuous,

\[
\text{Ind}^1_i\) \((\mathcal{IC}(\mathcal{O}_1^0) \cap \mathcal{IC}(\mathcal{O}_1^F))\) \(\mathcal{S}_1\) for \(m \geq 0\).

The Lemma will be proved once we show that, for suitable open subsets \(V_1 \subset S_1\), we have \(j(\text{Ind}^1_i\) \((\mathcal{IC}(\mathcal{O}_1^0) \cap \mathcal{IC}(\mathcal{O}_1^F))\) = \(V_1\), where \(j : V_1 \subset S_1\), stands for the inclusion. By the second assumption, the map \(p_1 : p_1^{-1}(S_1) \cap p_3^{-1}(O_1^G \times_O^F 1) \subset \mathcal{S}_1\) is an isomorphism, hence \(\dim (p_1^{-1}(O_1^G \times_O^F 1)) = \dim S_1\). Next, observe that since \(p_1\) is smooth and \(p_2\) is a principal bundle, we have

\[
\dim p_2^{-1}(O_1^G \times_O^F 1) < \dim p_2^{-1}(O_1^G \times_O^F 1) = \dim S_1;
\]

and therefore \(p_2p_3^{-1}(O_1^G \times_O^F 1)\) is a proper closed subset of \(S_1\). We choose \(V_1\) to be its complement. Now, we compute, for \(m \geq 0\),

\[
j(\text{Ind}^1_i\) \((\mathcal{IC}(\mathcal{O}_1^0) \cap \mathcal{IC}(\mathcal{O}_1^F))\)\(m\) = \(m, j p_1 p_2(p_1 h) (\mathcal{IC}(\mathcal{O}_1^0) \cap \mathcal{IC}(\mathcal{O}_1^F))\)
\[
= m, j p_1 p_2(p_1 h (I_{O_i^G} L_{O_i^F}) = m, h (I_{O_i^G} L_{O_i^F}) = m, h (I_{V_1}) = 1_{V_1},
\]

as desired. The last statement of the Lemma is obvious as \(O_1^K\) is a \(G_1^+\)-orbit.
9. Proof of Theorem 5.1 ii) (finite type)

In this section we prove Theorem 5.1 ii) when \( X \) is of genus zero. We first collect some more facts about the HN stratification under this hypothesis.

9.1. HN stratification for \( X \neq \mathbb{P}^1 \). Assume that \( X \neq \mathbb{P}^1 \). In that case, \( G = SO(3) \) is associated to a Dynkin diagram of type ADE via the McKay correspondence and \( L_g \) is the affine Lie algebra of the same type. Recall that by Lemma 2.1, for any pair \( (X,G) \) there is a canonical isomorphism \( h : K(X) \to \Phi \) restricting to an isomorphism \( K^*(X) \to \Phi^* \). The following result is proved in [19] (see also [22], Section 7):

Lemma 9.1. There is an indecomposable sheaf \( F \) of class \( 2 K^*(X) \) if and only if \( h(F) \) is a root of \( L_g \). Moreover, such a sheaf is unique up to isomorphism if and only if \( h(F) \) is a real root.

From the explicit description of \( h \) we easily deduce that \( G \) has only finitely many simple (i.e., stable) objects if \( c < 1 \). In addition, from \( f_X < 0 \) and Serre duality we conclude that for any two semi-stable sheaves \( F_1 \) and \( F_2 \) of the same slope \( 2 \), \( \dim \text{Ext}^1(F_1,F_2) = \dim \text{Hom}(F_2,F_1(l_X)) = 0 \). Hence, for any \( c < 1 \), \( C \) is a semistable category with finitely many simple objects.

Lemma 9.2. Assume \( G \) is not a torsion class. Then the connected components of \( Q_n(1) \) are the orbits \( Q_n(F) \), where \( F \) runs through the set of isomorphism classes of semi-stable sheaves of class \( C \) generated by \( \text{fl}_n(n)g_{B_2}B \).

Proof. Let \( G_1, \ldots, G_r \) be a complete collection of distinct simple objects in \( C \). Since \( C \) is a semisimple, we have \( F \twoheadrightarrow \bigotimes l G_i \cdot \text{Hom}(G_i;F) \) for any \( F \in C \), and

\[
(9.1) \quad \dim \text{Hom}(G_i;F) \quad [g] = : \]

Consider the tautological sheaf \( F \) on the universal family \( X = X \cdot Q_n \) and its restriction to \( X^0 = X \cdot Q_n(1) \). By definition, we have \( F_{X^0} \to \) \( F \) if \( t = (0) : E_n \to G \). By the semicontinuity theorem (I, III Thm. 12.8) the function \( \dim \text{Hom}(G_i;F_X) \) is upper semicontinuous in \( t \). Using Lemma 2.1 we see that it is also lower semicontinuous in \( t \) for \( t \) small, and thus locally constant on \( Q_n(1) \). We deduce that \( F_{X^0} \to F_{X^0} \) if \( t \) and \( t_0 \) belong to the same connected component of \( Q_n(1) \). Finally, using Lemma 2.2 we conclude that each connected component of \( Q_n(1) \) is formed by a single \( G \cdot F \)-orbit. The Lemma follows.

The next remark will be of importance for us.

Lemma 9.3. Let \( H \) be an indecomposable vector bundle, let \( d \leq 2 \) and set \( F = H^d \). There exists sheaves \( G \) and \( K \) with \( \text{rank}(G) < \text{rank}(F) \) or \( \text{rank}(G) = \text{rank}(F) \) and \( f < j(j) < f(j) \) satisfying the following property: there exists a unique subsheaf \( G^0 \) of \( F \) such that \( G^0 \to G \) and \( G = G^0 \to K \). Moreover, if \( H \) is generated by \( \text{fl}_n(n)g \) and \( H \subseteq \text{O}_n(n) \) then \( G \) and \( K \) can also be chosen to be generated by \( \text{fl}_n(n)g \).

Proof. The statement is invariant under twisting by a line bundle. If \( H = L_1 \) then we have \( \text{Hom}(\text{O}_n(L_1)) = 1 \) and we may take \( G = \text{O}_n^d \) and \( K = \text{O}_n(0) \cdot \text{fl}_n(n)^d \). Thus the Lemma is true for all line bundles. Now let us assume that \( H \) is of rank at least two. By inspection of all the cases, using Lemma 2.3 and the definition of \( \Phi \), one
checks that, up to twisting by a line bundle, the class of $H$ can be written as
\[ X = \text{rank}(H) \mathcal{O}_X + n^{(l)} \cdot 1 \]
with \( n^{(l)} = 0 \) and \( n^{(b)} = 0 \) for all $i$. Note that $[H] \not\in \text{rank}(H) \mathcal{O}_X$ since $H$ is indecomposable, hence $\text{rank}(H) > 0$. Therefore, using Proposition 5.1, we have $\text{dim} \text{Hom}(\mathcal{O}_X; H) = h^0 \mathcal{O}_X H = \text{rank}(H)$. In particular, the natural map $\text{Hom}(\mathcal{O}_X; H) \to H$ is neither zero nor surjective. The same is true of the map $d : \text{Hom}(\mathcal{O}_X; F) \to H$ if $F$. We claim that $G = \text{Im} d$ and $K = \text{Coker} d$ satisfy the requirements of the Lemma. Indeed, let
\[ 0 \to G \to F \to K \to 0 \]
be an exact sequence. The map $a$ induces a linear map $c : \text{Hom}(\mathcal{O}_X; F) \to \text{Hom}(\mathcal{O}_X; F)$ such that the diagram
\[
\begin{array}{ccc}
G & \xrightarrow{a} & F \\
\text{Hom}(\mathcal{O}_X; F) & & \text{Hom}(\mathcal{O}_X; F) \\
\text{Hom}(\mathcal{O}_X; F) & \xrightarrow{c} & \text{Hom}(\mathcal{O}_X; F)
\end{array}
\]
is commutative, and thus $\text{Im} a = \text{Im} d$. But from $\text{Coker} a = \text{Coker} d$ we deduce that $\text{Im} a = \text{Im} d$ as wanted. The last assertion of the Lemma is clear from the above construction.

9.2. We start the proof of Theorem 5.1. Assume that $X \not\subset \mathbb{P}^1$. Let us denote again by $W$ the subalgebra of $B_A$ generated by $U_{A}^{\alpha \beta}$ and the elements $1_{\mathcal{D}_X \otimes 1_n}$ for $n = 2, 12, 12$ and $N$. We have to show the following statement:

a) For any $n = 2, 12$ and for any $P = (P_m)_m$, there exists $S_1, S_2, 0, U$ such that $S_1 = P_m$, $S_2$ and $b_s = 2, 2, W$.

It will be convenient for us to prove at the same time as a) the following:

b) For any vector bundle $F$, we have $IC(\mathcal{O}_X; F) = 2, P$.

We consider $x = 2, 12$ and argue by induction on the rank of $P$. If $P_m = 0$ there is nothing to prove and if $P = 2, P$ satisfies $P_m = 0$ then (1) $m \leq j \leq n$ rank $(\mathcal{O}_X)$ and, hence, for a fixed rank, we may either argue by induction on $j$. The case of rank $(\mathcal{O}_X) = 0$ is the subject of Section 6. So let us $x$ of rank at least one and let us assume that $a$) is proved for all $x$ with rank $(\mathcal{O}_X) < \text{rank}(\mathcal{O}_X)$ or rank $(\mathcal{O}_X) = \text{rank}(\mathcal{O}_X)$ and $j < j$ and that $b)$ holds for all $F$ generated by $f_{x, (m, n)} \otimes \mathcal{O}_X$ satisfying the same conditions. Let $P = 2, P$. We will show that $a)$ holds for $P$. We argue once again by induction, this time on the generic $\text{HN}$ type of $P$ with respect to the order \\ "denoted in Section 7.1. So we may assume in addition that $a)$ is proved for all $F$ with $\text{HN}(P)$ for $\text{HN}(P)$. Let us write $\text{HN}(P) = (1; \ldots; r) = 0$ and $\ldots = (1; \ldots; 1)$.

Let us now suppose that $r > 1$. Consider the complex
\[ R = (\mathcal{E}_m)_m = \text{Ind}^{-1} \otimes \mathcal{E}(\mathcal{O}_X) \to U : \\
\]
By construction, $(1) > \ldots > (r)$. Thus, by Lemma 3.1, we have
\[ \text{supp} \mathcal{E}_m \subset \text{HN}_m^{-1}(\mathcal{O}_m) \]
for any $m = 2, 12$. Let $j_m : H N_1(\mathcal{O}_m) \to \mathcal{O}_m$ be the embedding. Note that by repeated applications of Lemma 3.2, it follows that for any $(\mathcal{E}_m \to F) \otimes H N_1(\mathcal{O}_m)$, there...
exists a unique extension $G_1 \oplus F$ of $F$ such that $[G_1 \oplus G_1] = 1$. Moreover, we have $\text{Ext}(G_1 \oplus G_1; G_1 \oplus G_1) = 0$ for any $j > 1$ since $(j) > (1)$. Hence we may use Corollary $6.\text{a}$ with $U_0 = Q^{(1)}_1$, and we deduce that for any $m \geq 2$ there holds

$$\text{(9.3)}$$

$$I_m (R_m) \cdot I_m (P_m);$$

By Lemma $5.\text{a}$, we have $\text{Res} (P) = 2 U \cdot \langle U \rangle$. Let us first assume that

$$\text{(9.4)}$$

$$S_n = \text{Res} (P)_n \cdot S_n;$$

and $b_{\delta} \cdot b_{\delta} W \cdot \langle W \rangle$. By continuity of the induction product (see Lemma $4.\text{a}$), we can pick $n$ small enough so that $\text{Ind} (S)_n \cdot \text{Ind} (P)_n \cdot \text{Ind} (S)_n$. Next, by $\text{(9.2)}$ and $\text{(9.3)}$, we have

$$\text{(9.5)}$$

$$\text{Ind} (P)_n \cdot \text{Res} (P)_n \cdot P \cdot R \cdot S_0;$$

for some $R \cdot 2 U$ with $\text{supp} (P)$ $\subseteq H_n \cdot \langle 1 \rangle$. By the second induction hypothesis there exists $W; W \cdot \langle 0 \rangle$ such that $W_n = R_n \cdot W_n$ and $b_{\delta} \cdot b_{\delta} W \cdot \langle W \rangle$. Gathering terms, we finally obtain

$$\text{Ind} (S)_n \cdot W_n = P_n \cdot \text{Ind} (S)_n \cdot W_n;$$

where $b_{\text{Ind} (S)_n} b_{\text{Ind} (P)_n} + b_{\text{Ind} (P)_n} W_n$. This closes the induction step when $r > 1$ and $\delta$ $\delta (1;1)$. In that last case, $\text{rank} (1) = \text{rank} (1)$ and we give the following different argument. Let us write $\text{Res} (P) = \langle T_1 \rangle \cdot U_1$. By Theorem $5.\text{l}$ $U_0; b_{U_0} W$ for any $i$. By the induction hypothesis, there exists $S_0; S_0$ such that $(S_0)_n = (T_1)_n = (S_0)_n$. Thus, there exists a complex $V_1; V_1$ satisfying $(V_1)_n = (V_1)_n = 0$ and

$$\text{Ind} (S)_n \cdot U_1 \cdot \text{Res} (P)_n \cdot \text{Ind} (S)_n \cdot U_1;$$

From $(V_1)_n = 0$ we deduce easily $\text{Ind} (V_1; U_1)_n = 0$, and similarly for $V_1$. Thus

$$\text{Ind} (S)_n \cdot U_1 \cdot \text{Res} (P)_n \cdot \text{Ind} (S)_n \cdot U_1;$$

The rest of the argument runs as in the case $\delta (1;1)$ above. This closes the induction step when $r > 1$.

9.3. From now on we assume that $r = 1$, i.e. that $P$ is generically semistable: $\text{supp} (P_0) \setminus Q^{(1)}_0$ is open in $\text{supp} (P_0)$. Since $P_0$ is simple and $G_0$-equivariant it follows from Lemma $5.\text{a}$ that $\text{supp} (P_0) \setminus Q^{(1)}_0$ is a single $G_0$-orbit corresponding to some isoclass $F = C (1)$. Since in addition $P_0$ is a $G_0$-equivariant perverse sheaf, we deduce that $P_0 = \text{IC} (Q^{(1)}_0; 1)$, and nally that $P = \text{IC} (Q^{(1)}_0) \cdot (s)$ for some vector bundle $F$. Let us decompose $F$ into isotypical components $F = G_1$ where $G_1, H_1, H_1$ and $H_1$ are distinct indecomposable sheaves. Note that $\text{Ext} (H_1; H_1) = 0$ for any $i$. Let us rst assume that $s > 1$. In that case, we have $\text{rank} (G_1) < \text{rank} (F)$ for all $i$ and all $G_1$ are generated by $F_{\delta}$. By induction hypothesis (b), $\text{IC} (Q^{(1)}_1) \cdot 2 P \cdot 1$. By induction hypothesis a), there exists, for any $n \geq 2$, a complex $S_0; S_0$ such that
b_S b_S', 2 W ' W and S_n: IC O^{G'}_n). By continuity, we can choose n_0 so that

\[ \text{Ind}^\mathcal{S}_h, \text{Ind}^\mathcal{S}_h, \text{Ind}^\mathcal{S}_h \]

Since any extension of the G_i is necessarily trivial and isomorphic to F, and since Hom(G_i,G_j) = 0 for i \neq j, we may apply Lemma repeatedly to deduce

\[ \text{Ind}^\mathcal{S}_h(\mathcal{O}^{G'}_n) = \mathcal{O}^F \]

where supp(R) = \mathcal{O}^F n_0 \mathcal{F} H N \rho (\rho). By the induction hypothesis, there exists complexes W, W_0 such that b_w = b_w', 2 W and W_n = W_n. We deduce that

\[ \text{Ind}^\mathcal{S}_h(\mathcal{O}^{G'}_n) = \mathcal{O}^F \]

This settles the case s > 1.

It remains to consider the situation s = 1, which corresponds to the case F = H where H is indecomposable. If H = O_X (n) then b_w = 2 W by definition. Otherwise, let G/K be the sheaves supplied by Lemma: there is a unique subsheaf G_0' F such that G_0' = G/K. By the induction hypothesis b), IC O(G/2) P; IC O(K) 2 P. Consider as usual the induction diagram

\[ \begin{array}{ccc}
Q_1^1 & \rightarrow & Q_1^2 \\
\text{E} & \rightarrow & \text{E} \\
\text{P}_1 & \rightarrow & \text{P}_2
\end{array} \]

We have

\[ p_3 p_2 p_1 (O^K O^G) = p_3 p_2 p_1 (O^K O^G) = p_3 p_2 p_1 (O^K O^G) \]

thus p_3 p_2 p_1 (O^K O^G) is irreducible. Hence p_3 p_2 p_1 (O^K O^G) \setminus Q_1^1 is also irreducible, and contains O^F. By Lemma, we deduce that O^F is a dense open subset of p_3 p_2 p_1 (O^K O^G). We may now apply Lemma to obtain

\[ \text{Ind}^\mathcal{S}_h(\mathcal{O}^{G'}_n) = \mathcal{O}^F \]

where supp(R) = \mathcal{O}^F n_0 \mathcal{F} H N \rho (\rho). We conclude in exactly the same manner as in the case s > 1 above. Finally, note that this also closes the induction for statement b). Theorem 5.1 ii) is proved.

10. Proof of Theorem 5.1 ii) (tame type)

This section contains the proof of Theorem 5.1 ii) when X is an elliptic curve, which we assume is the case throughout this Section. We again start with a few preliminary results.

10.1 H N Iteration and mutations. It is well-known (see e.g. [5], Chap. IV) that the ramification index is then one of the following: (2;2;2); (3;3;3); (2;4;4) or (2;3;3). Accordingly, Lg is an elliptic Lie algebra of type D_2 \oplus A_1; E_6 \oplus A_1; E_7 \oplus A_1 or E_8 \oplus A_1 (see [5]) or [9]; Section 1). Recall that \rho denotes the LCM of the ramification indices of \mathcal{O} \mathcal{Xj} (so that \rho = 2;3;4 or 6 respectively). Observe that \mathcal{Xj} = 0 and \mathcal{Xj} = 0. Furthermore, if F \mathcal{Xj} 2 C then F \mathcal{Xj} 2 C.

Lemma 10.1. There is an indecomposable sheaf \mathcal{F} of class \mathcal{F} K^+ (X) if and only if h(\mathcal{F}) is a root of Lg. Moreover, such a sheaf is unique up to isomorphism if and only if \mathcal{F} is real.
Proof. This follows from [LM] (see also [S2], Section 8).

We will call an indecomposable sheaf \( F \) real if it is imaginary according to the type of the root \( \rho \).

The categories \( \text{coh}_C(\mathcal{X}) \) and \( C \) can be nicely described via the concept of mutations (see [LM]) which we briefly recall. A family \( A = (A_1, \ldots, A_p) \) of stable sheaves in \( \text{coh}_C(\mathcal{X}) \) will be called cyclic if \( \text{End}(A_i) = k, A_i = A_i \mod \rho \) for all \( i \) and \( H^0(A_i; A_j) = 0 \) if \( i \neq j \). For any \( F \) in \( \text{coh}_C(\mathcal{X}) \) we define the left mutation \( L_A F \) of \( F \) with respect to \( A \) via the canonical exact sequence

\[
\text{Hom}(A_i; F) \to A_i ! F \to L_A F \to 0
\]

and the right mutation \( R_A F \) via a universal extension sequence

\[
0 \to \text{Ext}(F; A_i) \to A_i ! R_A F \to F ! 0
\]

Given a cyclic family \( A \), denote full subcategories \( A \) and \( A' \) of \( \text{coh}_C(\mathcal{X}) \) by the following conditions:

\[
A = \text{ff} 2 \text{coh}_C(\mathcal{X}) j \text{Hom}(F; A_j) = 0 \text{ for all } j
\]

and

\[
A' = \text{ff} 2 A j \text{Hom}(A_i; F) A_i ! F \text{ is a monomorphism} g
\]

Theorem 10.1 ([LM], Th. 4.4). The functor of left mutations \( L_A \) induces an equivalence of categories \( L : A' \to A \), with inverse given by the right mutation \( R_A \).

From this, Lenzing and Melzer deduce that for any two \( 1, 2 \) \( Q \) \{ \text{if g}, there is a canonical equivalence of categories \( A, C \). As a consequence, all stable in imaginary indecomposables of slope \( \rho \) have the same class in \( K^*(\mathcal{X}) \), which we denote \( \Lambda \).

We will need a slight (equivalent) variant of their construction. Let \( F_m, m \geq 1 \) be the \( m \)-th Farey sequence: we have \( F_0 = \frac{1}{1} \) and \( F_m = \frac{a_i}{b_i} \ldots \frac{a_1}{b_1} \) \{ \text{if g}, then

\[
F_{m+1} = \frac{a_i}{b_i + 1} \frac{a_i + a_1}{b_i + a_1 + b_1} \ldots \frac{1}{b_1 + 1} = \frac{1}{g}
\]

Every positive rational number belongs to \( F_m \) for \( m \) big enough. Moreover, any pair of rationals \( \frac{a}{b}, \frac{c}{d} \) with \( abc \neq 2 \) \{ satisfying \( ad = 1 \) appears as consecutive entries in some \( F_m \). To unburden the notation, we write \( (1; 2) = 1 \) if \( 1 = \frac{a}{b} < \frac{c}{d} = 2 \) and \( ad = 1 \), and \( 1?2 = \frac{a+bc}{b+d} \).

Proposition 10.1 ([S2], Prop. 8.8). Let \( 1, 2 \) be positive rationals such that \( (1; 2) = 1 \) and let \( A_1 = (A_1, \ldots, A_p) \) be a cyclic family of \( C \). The right mutation \( R_A \) \{树脂, the left mutation \( L_A \) \} restricts to an exact equivalence \( C, C \). Moreover, for any simple \( (i, \text{stable}) \) sheaf \( F \in C \) and any \( i = 1, \ldots, p \) we have \( \dim \text{Ext}(F; A_i) \).

In particular, if \( (1; 2) = 1 \) then \( R_A (A_1) = A_1 \).

Using the above proposition, we will construct inductively a cyclic family \( A \) in \( C \) for every \( 2 \) \( Q \). We set \( A_0 = \text{ff} X ; X \ldots ; X \rho \) \{ and \( A_1 = \text{ff} X ; X \ldots ; X \rho \) \{ for some such that \( p_1 = \rho \), assume that \( A \) is already defined for \( A_1 \) \{ belonging to \( F_m \), and let \( 1, 2 \) be consecutive entries in \( F_m \). We put \( A_{1?2} = R_A (A_1) \). This allows us to define \( A \) for any positive \( i \). Finally,
observe that $F \to F(1)$ is an equivalence $C \to C_{+1}$ for any; for arbitrary we set $A = A_{+N} \in (N)$ if $N + > 0$. It is easy to check that this does not depend on the choice of $N$. The equivalence may now be obtained as follows. If $(; 2) = 1$ then we put $; 2i = L_{A_{+1}}$. For any positive $2 Q$ there exists a unique chain of such pairs, $f_1$; $g_2$; $f_3$; $g_4$; $; 2$ such that $; 1 + 2 = \frac{1}{2}$ and $21 = 32 + 2$. We do not need $1$ as the com position

$$; 1 = 1; 2 \quad \quad ; 2 
$$

If is negative then we put

$$; 1 = 1; + \in (Q \in N)) for N = 0 (again, this is independent of the choice of $N$). Finally, for arbitrary $1; 2 Q$ we set $; 1 + 2 = 1; 2 \quad ; 1 + 2$. We have $; 1 + 2$ for any three rational numbers $1; 2; 3$.

The open subfunctor $H \mathbb{B}_{E_{n}}^{0}$, $H \mathbb{B}_{E_{n}}^{3}$, given by

$$E f (f_2 \in E_{n})\quad 2 H \mathbb{B}_{E_{n}}^{0} j \quad 2; 8 \in S
$$

is represented by an open subscheme $Q_{n; 0}^{(2)} Q_{n}^{(2)}$. In addition, there is a well-defined geometric quotient $Q_{n; 0}^{(2)} Q_{n}^{(2)} = G_{n}$. The equivalences $; 1 + 2$ have some nice implications for the schemes $Q_{n; 0}^{(2)}$. Namely, if $; 1; 2$ are positive integers such that $(; 1; 2) > 1$ then com position with the projection $F \to F_{1}$ induces a natural transformation from $H \mathbb{B}_{E_{n}}^{0}$ to the functor $X, ; 2$, defined by

$$E f (f_2 \in E_{n})\quad 2 H \mathbb{B}_{E_{n}}^{1} j \quad 2; 8 \in S
$$

Fixing an embedding $e : E_{n} \to E_{n}$ provides an identification between the schemes representing $X, ; 2$, $G_{n}^{-}$, where $F$ is the stabilizer of $\text{Im}(e)$.

The natural transformation above now induces a morphism $Q_{n; 0}^{(2)} G_{n}^{-} Q_{n; 0}^{(2)}$, which restricts to a morphism $Q_{n; 0}^{(1)} G_{n}^{-} Q_{n; 0}^{(1)}$ and descends to a morphism

$$; 1: Q_{n; 0}^{(1)} G_{n}^{-} Q_{n; 0}^{(1)} G_{n}^{-}. The existence of the equivalence $; 1$ implies that $; 1$ is an isomorphism. Considering these isomorphisms and arguing as in the previous discussion, we deduce

Corollary 10.1. For any two $1; 2 Q$ and any $n > 0$ there is a canonical isomorphism $; 1: Q_{n; 0}^{(1)} G_{n}^{-} Q_{n; 0}^{(1)} G_{n}^{-}$. In particular, $; 1: Q_{n; 0}^{(1)} G_{n}^{-} Q_{n; 0}^{(1)} G_{n}^{-}$.

Finally, we present two more corollaries of the existence of the equivalence $; 1; 2$.

Corollary 10.2. Asume that $; 1; 2$ are positive and $(; 1; 2) = 1$. Let $F$ be a simple sheaf in $C_{2}$. There exists a unique subsheaf $G \to R_{A_{+1}} F$ such that $G = \text{Ext}^{1}(F; A_{1})$. Moreover, we have $R_{A_{+1}} F = G$. For some $F$.

Proof. By Proposition 10.1 and Serre duality, we have $\dim \text{Hom}(A_{1; 2}; R_{A_{+1}} F) = \dim \text{Ext}(R_{A_{+1}} F; A_{1}) 2 f \in 1 g$ (in fact $\dim \text{Hom}(A_{1; 2}; R_{A_{+1}} F) = \dim \text{Ext}(A_{1; 2}; F)$). The result follows.

Corollary 10.3. Let again $1 = \frac{1}{2}$; $2 = \frac{1}{2}$ be as above and let $F$ be a simple sheaf in $C_{2}$ of class $2$. Fix an integer $12 f; \text{pg}$ and denote $K$ by the universal
sequence

\[(10.1) \quad 0 \to A_1 \xrightarrow{\xi} K \xrightarrow{g} F \to 0;\]

Then \(K\) is the (unique) simple real sheaf of class \(\xi + [A_1]\).

**Proof.** Since \(\text{Ext}(F; A_j) = 1\) and \(K\) is given by a universal sequence, \(K \in A_1 \quad F\). From (10.1) it follows that for every indecomposable summand \(G\) of \(K\) we have \(1 < (G) < 2\). Observe that \(K\) is of rank \(b + pd\) and degree \(a + pc\), and that \(\gcd(a + pc; b + pd) = 1\) since \(ad - bc = 1\). Thus either \(K\) is a stable real sheaf of slope \(\frac{a + pc}{b + pd}\), or \(K\) splits as a nontrivial sum \(K = K^0 \oplus K^0\) where \(K^0 < \frac{a + pc}{b + pd}\) and \(K^0 > \frac{a + pc}{b + pd}\). But since \((a + pc)(b + pd)c = 1\), any sheaf of slope satisfying \(\frac{a + pc}{b + pd} < \frac{1}{c}\) is of rank at least \(b + pd\). This rules out the second possibility and proves the Corollary.

10.2. More Lemmas. \(\text{Recall that for any two } \xi, \eta \in \mathbb{Q}\) [1, 2] \(Q\{\xi\} + \eta\) we have a canonical equivalence \(_{\xi, \eta}: C_{\xi, \eta} \cong C_{\xi, \eta}\). Also, if \(\xi, \eta \in \mathbb{Q}\{\xi\} + \eta\) then \(2 \in K^+(\xi + \eta)\) denotes the class of a generic real sheaf of slope \(\xi\). For \(F\) a semistable sheaf of slope \(\xi\) let us set \(\supp(F) = \supp((1, F)) = \supp(F)\), and \(\pi = \pi((1, F))\). If \(\pi\) is the class of a sheaf \(F\), then we write \(j : j = j\downarrow\pi\) which is independent of the choice of \(\pi\). Finally, we introduce a decreasing filtration of \(Q\{\xi\}\) by locally closed subsets by setting

\[Q_{n,j} = \{E_n \cap F \mid 2 C_{1, j + F} \leq n\},\]

and we put

\[P_{n,j} = \{E_n \mid 2 C_{1, j + F} \leq n\}\]

We denote \(Q_{n,j}\) and \(P_{n,j}\) similarly and set \(Q_{n,j} = Q_{n,j} \ominus Q_{n,j} \ominus Q_{n,j} \ominus Q_{n,j}\). Note that \(P_{n,j} \neq P_{n,j}\) if and only if \(P\) is supported on a component of the semistable locus \(Q_{n,j}\). Lastly, we let \(U_{n,j} = \{U_{n,j}, j, n\}\) stand for the open subset of \(Q_{n,j}\) consisting of points \(E_n \leq 1\) where \(F\) splits as a direct sum of \(1\) distinct stable sheaves (in particular, if \(E_n = 1\) then \(U_{n,j} = U_{n,j}, j, n\) is defined in Section 5.1). As in Section 5.1, there is a canonical map \(1: U_{n,j} \to U_{n,j} \to U_{n,j}\).

**Lemma 10.2.** Let \(F\) be a stable real sheaf of slope \(\xi\). Then \(IC(O^\xi \cdot) = P\) for any \(d \leq 1\).

**Proof.** It is enough to prove this for \(d > 0\), and since \(\text{Ext}(F; \mathcal{O}^\xi) = 0\) for any real stable \(F\), we may further reduce ourselves to the case \(d = 1\). We use once again an argument by induction. Observe that \(IC(O^\xi) = P\) for any \(F = \mathcal{O}_C\) \((\mathcal{O}^\xi) = \mathcal{O}_X, \mathcal{O}_Y, \ldots, \mathcal{O}_X^1\) since such \(F\) is a line bundle. Now assume that \(IC(O^\xi) = P\) whenever \(G\) is a real stable sheaf of slope \(\xi\) with \(2 E_n\), and let \(A_1, A_2\) be two consecutive entries in \(F_n\). Let \(A_1 = (\mathcal{O}_{A_1}, \ldots, \mathcal{O}_{A_1})\) be the canonical cyclic family in \(C\). Note that \(A_1\) is real stable, hence \(IC(O^\xi) = P\) for all \(i\). Let \(F = \mathcal{O}_{A_1} \oplus \mathcal{O}_{A_2}\) be a stable sheaf. By Proposition 10.1, there is an universal exact sequence

\[0 \to L \to A_1 \xrightarrow{a} F \xrightarrow{b} L_{A_1} \to F \to 0\]

where \(J = f_1; \ldots; pg\) is a proper subset. Applying Lemma 8.2 we deduce that \(IC(O_{\mathcal{O}^\xi}^{A_1}) = P\) (see [82], Claim 8.4). Applying Lemma 8.2 again, we obtain \(\text{Ind}((F^A_\lambda)) = IC(O_{\mathcal{O}^\xi}^{A_1})) = IC(O^\xi)\). This closes the induction and proves the Lemma.
The aim of the rest of this section is to prove the following proposition.

Proposition 10.2. For any $2 \mathcal{Q} \{ f \}$, $g$ and $l$, we have

$$P_0^1 = \text{Imm}(U(l)^{(1)}; \tau(l)) \in \text{Irr}(S_0(g)).$$

Proof. Twisting by $O_X(l)$ if necessary shows that it is enough to prove the statement for strictly positive. We first consider the case when $l = 1$. We argue by induction on the order of the Farey sequence in which $l$ appears. The statement is clear for $l = 1$. Assume that it is proved for all appearing in $F$ and let $l; \tau$ be consecutive entries in $F$. We will prove the statement of the Lemma for $l = 1$. First observe that $P_0 \in \tau$; as the restriction of $\text{Ind}(l_2 \tau l_1)$ to $Q^{(1)}_{n,0}$ is nonzero for $n = 1$. So let $P = (P)\in l$ be a perverse sheaf in $F\in \tau$, and let us modify $n = 0$. By assumption, $P_n$ is the middle extension of some simple perverse sheaf on $Q^{(1)}_{n,0}$, which in turn is associated to an irreducible local system $L$ on some $G_n$-invariant open subset $U$ of $Q^{(1)}_{n,0}$. We have $U_n = 1 \cup P(1)^{n,0}$ for some finite set $\tau$, and as $L$ is $G_n$-invariant, it is the pullback under $\tau$ of some irreducible local system $L_0$ on $P^1n$.

Let us now consider the restriction diagram

$$Q_n \xrightarrow{F} Q_n^2 \xrightarrow{Q_n^1}.$$

Let $f A_1; \cdots; A_m$ be the canonical cyclic family in $C$, and let us take any point $x_0 \in Q_n^1$ in the orbit corresponding to $\tau X A_1$. We denote by $f : x_0g ! Q_n^1$ the embedding. Observe that as $\text{Res}(z_2 : P) \in 2 \cup \tau$ and $\text{Res}(z_1 : P_n)$, we have $(f) : (P)$. Finally, we put

$$F^0 = (f x_0g ! Q_n^1)^{(1)}; \quad U^0_n = F^0 \setminus Q_n^1.$$

If $F$ is a stable sheaf of slope, then it follows from the definition that $(f x_0g ! Q_n^1)^{(1)}$. We thus deduce from Corollary 10.3 that the diagram

$$\begin{array}{ccc}
U^0_n & \xrightarrow{f x_0g} & Q_n^1 \\
\downarrow & & \downarrow \\
P^1n & \xrightarrow{1} & P^1n
\end{array}$$

is commutative. The smooth subvariety $G_n \in F$ contains the open set $Q^{(1)}_{n,0}$ and by (10.2), the map $1 : Q^{(1)}_{n,0} \to P^1n^+$ extends to a $G_n$-invariant map $G_n \xrightarrow{f} P^1n^+$. Hence, $P_{n,0} = (L_0)$, and $P_{n,0} = 1 : (L_0)$. Denoting $g : F^0 \to Q_n^1$ and $l : 1 \cup P(1)^n$, the embeddings, we now compute by base change

$$1 : (E_n) = : g ((P) = 1 \cup (L_0) = 1 \cup (L_0)[2d])$$

where $d$ is the dimension of the fiber of $\tau$. Hence, $1 \cup (L_0) \in 2 \cup (P_n)$, $f \in F$, and by the induction hypothesis, we deduce that $1 \cup (L_0) = Q_{1[l]}$ for some $\tau$, and finally $L_0 = Q_{1}$ as desired. This proves that $P = 1$ and closes the induction.

Let us now consider a perverse sheaf $P F \in P^1n^+$, where $l$ is arbitrary.

Claim 10.1. The complex $P$ appears (up to shift) in $\text{Ind}^{\text{null}} \in \text{Res}^{\text{null}}(P)$. 

Proof of claim. Let us rst observe that there is a geometric quotient \( \frac{1}{2} ; \): 
\( U_{n,l}(1) ; U_{n,l}(1) \) ! \( S^1(\mathbb{P}^1 n) \) \( n \) \( \text{where } \) \( = ffz 1 ; ; ; ; z_2 ; g \) \( j_2 = z_2 \) for some \( i \) \( s \) \( g \) is the generalized diagonal. By our assumption on \( P \), there exists a simple complex \( P_0 \) on \( S^1(\mathbb{P}^1 n) \) \( n \) \( \text{such that } P = ( \frac{1}{2} , 1 ) ( P_0 ) \). Consider a commutative diagram where the top row is the restriction diagram on \( U_{n,l}(1) ; U_{n,l}(1) \):

\[
\begin{array}{c}
U_{n,l}(1) ; U_{n,l}(1) \\
\downarrow \downarrow \\
S^1(\mathbb{P}^1 n) \times n \xrightarrow{P} \mathbb{P}^1 n \times n
\end{array}
\]

where \( Q_{n,l}^{(n)} \) \( n \) \( \rightarrow ) \) \( ( \frac{1}{2} , 1 ) ( \mathbb{P}^1 n ) \times n \) and where \( p \) is the natural projection map. By base change, we have

\[
\text{Res} \mathbb{P}^1 ( \mathbb{P} ) \left( \mathbb{P}^1 ( \mathbb{P} ) \right)^{\times n} = : ( \frac{1}{2} , 1 ) ( \mathbb{P}_0 ) = ( \frac{1}{2} , 1 ) ( \mathbb{P}_0 ) [ \text{2d} ]
\]

where \( d \) is the rank of the vector bundle. Similarly, the induction diagram on \( U_{n,l}(1) ; U_{n,l}(1) \)
reads

\[
\begin{array}{c}
U_{n,l}(1) ; U_{n,l}(1) \\
\downarrow \downarrow \\
S^1(\mathbb{P}^1 n) \times n \xrightarrow{P} \mathbb{P}^1 n \times n
\end{array}
\]

where \( E_{0} = p_{1} \left( Q_{n,l}^{(n)} \right) \) \( n \) \( \left( \frac{1}{2} , 1 \right) \) \( \mathbb{P}^1 ( \mathbb{P} ) \) \( = p_{2} \left( E_{0} \right) = p_{3} \left( Q_{n,l}^{(n)} \right) \). We have

\[
\text{Ind} \mathbb{P}^1 \left( \text{Res} \mathbb{P}^1 ( \mathbb{P} ) \right) p_{1}^{-1} \left( Q_{n,l}^{(n)} \right) n = p_{3} p_{2} p_{1} \left( \frac{1}{2} , 1 \right) ( \mathbb{P}_0 ) [ \text{2d} ]:
\]

There is a natural morphism \( h : E_{0} \left( \mathbb{P}^1 n \right)^{\times n} \) tting into a cartesian diagram

\[
\begin{array}{c}
U_{n,l}(1) ; U_{n,l}(1) \\
\downarrow \downarrow \\
S^1(\mathbb{P}^1 n) \times n \xrightarrow{P} \mathbb{P}^1 n \times n
\end{array}
\]

We deduce \( p_{3} p_{2} p_{1} \left( \frac{1}{2} , 1 \right) ( \mathbb{P}_0 ) [ \text{2d} ] = p_{3} p_{2} p_{1} ( \mathbb{P}_0 ) [ \text{2d} ] = 1 ; p_{3} p_{2} p_{1} ( \mathbb{P}_0 ) [ \text{2d} ].
\]

But since \( p \) is an \( n \) \( \text{ft} \), there is a nonzero morphism \( P_0 ! \) \( p \), \( p \mathbb{P}^1 ( \mathbb{P} ) \).

Hence \( p_{3} p_{2} p_{1} \left( \frac{1}{2} , 1 \right) ( \mathbb{P}_0 ) \) appears in \( \text{Ind} \mathbb{P}^1 \left( p_{1}^{-1} \left( Q_{n,l}^{(n)} \right) n \right) = p_{3} p_{2} p_{1} \left( \frac{1}{2} , 1 \right) ( \mathbb{P}_0 ) [ \text{2d} ].
\]

the claim follows.

To nish the proof of the Proposition, it remains to notice that, by the rst part of the Proposition, \( \text{Res} \mathbb{P}^1 \left( \mathbb{P} ight) p_{1}^{-1} \left( Q_{n,l}^{(n)} \right) n \) is a direct sum of shifts of complexes \( \left( \mathbb{P}^1 n \right) \) \( 1 \) \( \mathbb{P}^1 n \) \( 1 \) and that, by the same argument as in Lemma 6.1, we have

\[
\text{Ind} \mathbb{P}^1 \left( \mathbb{P}^1 n \right) n = M \quad 
\]

the claim follows.

To nish the proof of the Proposition, it remains to notice that, by the rst part of the Proposition, \( \text{Res} \mathbb{P}^1 \left( \mathbb{P} ight) p_{1}^{-1} \left( Q_{n,l}^{(n)} \right) n \) is a direct sum of shifts of complexes \( \left( \mathbb{P}^1 n \right) \) \( 1 \) \( \mathbb{P}^1 n \) \( 1 \) and that, by the same argument as in Lemma 6.1, we have

\[
\text{Ind} \mathbb{P}^1 \left( \mathbb{P}^1 n \right) n = M \quad 
\]

the claim follows.
Lemma [5]b) when $X$ is of genus one. Let us temporarily denote again by $W$ the subalgebra of $G$ generated by $U^n$ and the elements $1_{B^{+}(n)}$ for $n \geq 2$ and $L$. We will show that:

a) For any $n \geq 2$, $2 \times K^1(X)$ and for any $P = (P_n, n) \geq 2$ there exists $S^0 \otimes 2 U$ such that $S^0 \otimes P_n \otimes S_n$ and $b_S \otimes b^0 \otimes 2 W$.

The proof starts in exactly the same way as in the finite type case: we have $n \geq 2$ and goes by induction on the rank, the degree, and then the $HN$ type. Statement a) is true when $F$ is of rank zero by virtue of Theorem 5.19. So let $F < 2$ and let us assume that a) holds for all $0 < 2$ with $rank(F) < rank(P)$ or rank $(F) = rank(G)$ and $j > j_i$ or $HN(P) = HN(F)$. Arguing just as in the finite type case (Section 9.2), we reduce ourselves to the case of a generically semistable $P$ of slope, say $(HN(P) = (0) and (1) = (0)$. By Proposition 7.4, it easily follows that any extension of two sheaves of $HN$ type $(0)$, with one of them being of $HN$ type $(1)$, is again of $HN$ type $(0)$. As a consequence, $I = M \otimes A^0_P$.

If $L$ is an ideal of $J = \theta_A(J)$, by our induction hypothesis, it is enough to work in the quotient space $U_A = \theta_A(J)$. Let us now consider the case $J = (1) = \theta_A(J)$, i.e., to show that:

a) For any $n \geq 2$ and for any $P$ such that $HN(P) = (0) and (1) = (0)$ there exists a complex $S^0 \otimes 2 U$ such that $S^0 \otimes P_n \otimes S_n$ and $b_S \otimes b^0 \otimes 2 W + I$.

Let us assume that $S^0 \otimes (1) \otimes 2 \otimes 1$ for some $(i, j)$, $2 \otimes 1$ and $(i, j)$ is the class of a $(m, 2)$-schem of a realizable sheaf. Then $P = IC(O^F \otimes \phi)$ where $F = \phi \otimes \phi$. If $F$ is a line bundle then $b_S \otimes 2 W$ by definition, and we may assume that $F$ is of rank at least two. By Proposition 10.1, there exists a complex $S^0 \otimes 2 U$ such that $(i, j) = 1$ and $(i, j) = (0)$ and we have $F' \otimes R_A \otimes G$ where $G = (1, 2) \otimes (S \otimes (i, j))$, i.e., $F$ is obtained as the universal extension

$$0 \otimes L \otimes J \otimes F \otimes G \otimes 0$$

for $J$ a proper subset of $\phi_1, \ldots, \phi_p$ and $A_1 = \phi_1 \otimes \cdots \otimes \phi_p$. By Lemma [10.2], $IC(O^G \otimes \phi_1)$ and $IC(O^\phi_1)$, $i = 1; \ldots; p$ all belong to $F$. As $J$ is proper, reasoning similar to [7.2], Claim 8.4 shows that $IC(O^L \otimes A_1 \otimes \phi_1)$ $2 \otimes 1$. Observe that $A_i, i = 1; \ldots; p$ and $G$ are all of strictly smaller rank than $F$. Hence, by the induction hypothesis, there exists, for every $n \geq 2$, a complex $S^0 \otimes 2 U$ such that $S^0 \otimes 2 U$.

$IC(O_n \otimes A_1 \otimes \phi_1)$, $\otimes T^0_0 \otimes S^0 \otimes T^0_0 \otimes b^0 \otimes b^0 \otimes 2 W$. By continuity of the product (Lemma [10.3]), we may choose $n^0$ small enough so that

$$(10.3) \quad Ind(IC(O^G \otimes \phi_1)) \otimes IC(O^L \otimes A_1 \otimes \phi_1) \otimes Ind(S^0 \otimes T \otimes T^0 \otimes T^0) = Ind(S^0 \otimes T \otimes S \otimes T)$$

Since $F$ (and thus $F^1$) is a universal extension, Lemma [10.3] yields

$$Ind(IC(O^G \otimes \phi_1)) \otimes IC(O^L \otimes A_1 \otimes \phi_1) \otimes IC(O^F \otimes \phi_1) \otimes R$$

where $supp(R) = O^F \otimes n^0 \otimes F$. Note that $O^F \otimes n^0 \otimes F \otimes S \otimes H^1 \otimes (HN(1))$ since $F$ is realizable. Thus, by the induction hypothesis, there exists a complex $W^0 \otimes P^0$ such that $W^0 \otimes S_n \otimes W_n$ and $b_S \otimes b^0 \otimes 2 W$. Substituting this in (10.3) yields

$$IC(O^F \otimes \phi_1) \otimes Ind(S^0 \otimes T \otimes T^0 \otimes T^0) \otimes W_n \otimes Ind(S^0 \otimes T \otimes S \otimes T)$$

for $J$ a proper subset of $\phi_1, \ldots, \phi_p$ and $A_1 = \phi_1 \otimes \cdots \otimes \phi_p$. By Lemma [10.2], $IC(O^G \otimes \phi_1)$ and $IC(O^\phi_1)$, $i = 1; \ldots; p$ all belong to $F$. As $J$ is proper, reasoning similar to [7.2], Claim 8.4 shows that $IC(O^L \otimes A_1 \otimes \phi_1)$ $2 \otimes 1$. Observe that $A_i, i = 1; \ldots; p$ and $G$ are all of strictly smaller rank than $F$. Hence, by the induction hypothesis, there exists, for every $n \geq 2$, a complex $S^0 \otimes 2 U$ such that $S^0 \otimes 2 U$.

$IC(O_n \otimes A_1 \otimes \phi_1)$, $\otimes T^0_0 \otimes S^0 \otimes T^0_0 \otimes b^0 \otimes b^0 \otimes 2 W$. By continuity of the product (Lemma [10.3]), we may choose $n^0$ small enough so that

$$(10.3) \quad Ind(IC(O^G \otimes \phi_1)) \otimes IC(O^L \otimes A_1 \otimes \phi_1) \otimes Ind(S^0 \otimes T \otimes T^0 \otimes T^0) = Ind(S^0 \otimes T \otimes S \otimes T)$$

Since $F$ (and thus $F^1$) is a universal extension, Lemma [10.3] yields

$$Ind(IC(O^G \otimes \phi_1)) \otimes IC(O^L \otimes A_1 \otimes \phi_1) \otimes IC(O^F \otimes \phi_1) \otimes R$$

where $supp(R) = O^F \otimes n^0 \otimes F$. Note that $O^F \otimes n^0 \otimes F \otimes S \otimes H^1 \otimes (HN(1))$ since $F$ is realizable. Thus, by the induction hypothesis, there exists a complex $W^0 \otimes P^0$ such that $W^0 \otimes S_n \otimes W_n$ and $b_S \otimes b^0 \otimes 2 W$. Substituting this in (10.3) yields

$$IC(O^F \otimes \phi_1) \otimes Ind(S^0 \otimes T \otimes T^0 \otimes T^0) \otimes W_n \otimes Ind(S^0 \otimes T \otimes S \otimes T)$$
where \( b_W = b _W + ( b_S - b _S ) ( b _T - b _T ) \). This concludes the case of the form of the lemma.

By Lemma 10.2, IC \( ( O ^ F ^ - 1 ) \) 2 P for any real stable sheaf \( F \). Hence, by the above, IC \( ( O ^ F ^ - 1 ) \) satisfies statement \( a' \) in particular for any real stable sheaf \( F \) of slope \( s \). Repeating verbatim the proof of Lemma 6.1, and using the continuity of the induction product as in the previous paragraph, we deduce that statement \( a' \) holds for every simple complex \( P \) belonging to \( \mathcal{P} \) (that is, generically supported on the exceptional set \( Q ^ { ( i )} _{0} \) for some \( i \) satisfying \( ( ) = 0 \).

Now suppose that \( s = 1 \), i.e. \( s \) is the \( 2 \)-multiple of a class of an imaginary stable sheaf, and that \( P = IC ( U ^ { ( i )} _{0} ) = 1 \). Recall that by Proposition 10.1, there exists rational \( \frac{1}{2} \) satisfying \( ( \frac{1}{2} ) = 1 \) and \( ( \frac{1}{2} ) = 0 \), and the right multiplication \( R _{A} \) with respect to \( A _{1} = A _{1} ; \cdots ; A _{p} \) defines an equivalence \( C _{2} \sim C _{2} \). Let \( K \) denote the (unique) real simple sheaf of class \( ^{+} \) \( [ A _{1} ] \). From Corollaries 10.2 and 10.3 it follows that for any simple sheaf \( P \) of class \( ^{+} \) \( [ A _{1} ] \), there exists a unique subsheaf \( G \) isomorphic to \( P ^{+} _{1} A _{1} \) sitting in an exact sequence

\[ 0 \rightarrow G \rightarrow P \rightarrow K \rightarrow 0. \]

We deduce, using Lemma 6.2, that

\[ \text{Ind} ( IC ( O _{K} ^{1} ) ) \rightarrow IC ( O _{P ^{+} _{1} A _{1} ^{1} } ) ) ) = 1 \]

for some dense open subset \( V _{1} \). But by Proposition 10.2, this implies that

\[ (10.4) \quad \text{Ind} ( IC ( O _{K} ^{1} ) ) \rightarrow IC ( O _{P ^{+} _{1} A _{1} ^{1} } ) ) = 1 \]

where \( supp ( R _{n} ) \setminus Q ^{ ( i )} _{0} \) \( Q ^{ ( i )} _{0} \), i.e. where all simple summands of \( R \) are either of strictly smaller \( HN \) type, or belong to \( P _{1} ^{+} \). An argument entirely similar to that in Section 6.2, based on Lemma 6.2 and Proposition 10.2, shows that for any simple complex \( T = P _{1} ^{+} \), \( b _{T} \) belongs to the subalgebra \( 
\begin{align*}
\text{of} \quad \text{generated by} \quad b _{T} 
\end{align*}
\]

\[ \begin{align*}
2 \rightarrow r \rightarrow ( \frac{1}{2} ) \rightarrow ( \frac{1}{2} ) \rightarrow ( 1 ) \quad g \quad ( f ; ; ; ; ( 1 ) \quad g. \quad As \quad for \quad all \quad these \quad values \quad of \quad r, \quad \text{rank} \quad ( ) < \text{rank} \quad ( 1 ) \quad g, \quad \text{statement} \quad \text{a} \quad \text{holds} \quad \text{for} \quad 1, \quad \text{for} \quad n ^{0} \quad 2 \quad Z. \quad \text{We} \quad \text{deduce,} \quad \text{using} \quad \text{the} \quad \text{continuity} \quad \text{of} \quad \text{the} \quad \text{induction} \quad \text{product} \quad \text{as} \quad \text{in} \quad \text{the} \quad \text{previous} \quad \text{paragraph,} \quad \text{that} \quad \text{a} \quad \text{holds} \quad \text{for} \quad \text{all} \quad P = P _{1} ^{+} \text{, and hence for} \quad R. \quad \text{Similarly,} \quad K \quad \text{and} \quad P = P _{1} ^{+} \quad \text{are of rank strictly less than} \quad 1, \quad \text{so} \quad \text{a} \quad \text{holds} \quad \text{for} \quad \text{them,} \quad \text{for} \quad \text{any} \quad n ^{0} \quad 2 \quad Z \quad \text{and hence for their induction product. But then from} \quad (10.4) \quad \text{it follows} \quad \text{that} \quad \text{a} \quad \text{holds} \quad \text{for} \quad 1 \quad \text{as well as wanted.}
\end{align*}
\]

We finally conclude the proof of the induction step, and thus of the theorem. Notice that, by Proposition 10.2, for any \( P = P _{1} ^{+} \), \( b _{P} \) is in the subalgebra \( U _{A} \) generated by \( b _{1} \) for \( 1 \). On the other hand, a cut-and-paste of the argument in Lemma 6.2 shows that for any \( P = P _{1} ^{+} \), \( b _{P} \) belongs to the subalgebra \( U _{A} \) generated by \( b _{1} ; ( \frac{1}{2} ) \) and \( b _{1} \) for \( 1 \) and \( ( \frac{1}{2} ) \). As \( a' \) holds for these complexes, it also holds for \( P \).

11. Corollaries

11.1. Let us put

\[ P ^{vb} = fP \bigcup P _{1} \quad \text{or} \quad ( 1 ; ; ; ; ( 1 ) \quad 6; 1; g. \]

\]
(in other words, \( P^{vb} \) is the set of all simple perverse sheaves whose support generally consists of vector bundles). We set \( \Theta_A^{vb} = T_{P^{vb}} A \) (here \( T \) denotes a possibly in finite, but admissible sum).

Corollary 11.1. The following hold:

1) For any simple perverse sheaf \( P \) which is neither in \( P^{vb} \) nor in \( P^{tor} \) there exists unique simple perverse sheaves \( R_1 \in P^{vb} \) and \( T_1 \in P^{tor} \) such that

\[
\text{Ind}(R_1 \otimes T_1) = P^{vb} \]

with \( \text{supp}(P^{vb}) = H \) where \( H \) is the set of all integers \( i \) such that

\[
\text{Res}^{i}(\text{Ind}(R_1 \otimes T_1)) = P^{vb}
\]

with \( P^{vb} \) satisfying the same conditions as above. As \( P \) is simple, \( P \) appears in a product \( \text{Ind}(R_1 \otimes T_1) \) for some fixed \( i \). Furthermore, from the fact that \( \text{Ind} \) commutes with Verdier duality, we deduce that \( \text{Ind}(R_1 \otimes T_1) \) is a sheaf on \( P^{vb} \). To prove the unicity, we apply the restriction functor to \( P^{vb} \) and we obtain \( \text{Res}(P^{vb}) = P^{vb} \). The existence of the multiplication map \( \Theta_A^{vb} \cdot P^{tor} = \Theta_A^{vb} \cdot P^{tor} \) now follows from \( i \) and an induction argument on the \( H \)

The elements of \( P^{tor} \) are described explicitly in Section 6. Thus Corollary 11.1 provides, when \( X = P^{vb} \), a complete description of the elements in \( P^{vb} \).

Proof. By Section 9.2, there exists some \( R_1 \in P^{vb} \) and integers \( i \) such that

\[
\text{Ind}(R_1 \otimes T_1) = P^{vb}
\]

with \( P^{vb} \) satisfying the same conditions as above. As \( P \) is simple, \( P \) appears in a product \( \text{Ind}(R_1 \otimes T_1) \) for some fixed \( i \). Furthermore, from the fact that \( \text{Ind} \) commutes with Verdier duality, we deduce that \( \text{Ind}(R_1 \otimes T_1) \) is a sheaf on \( P^{vb} \). To prove the unicity, we apply the restriction functor to \( P^{vb} \) and we obtain \( \text{Res}(P^{vb}) = P^{vb} \). The existence of the multiplication map \( \Theta_A^{vb} \cdot P^{tor} = \Theta_A^{vb} \cdot P^{tor} \) now follows from \( i \) and an induction argument on the \( H \)

Finally, we show \( ii \). Let \( P \in P^{vb} \) be of class \( i \) and put \( H \) such that \( H \in \mathbb{Z} \). There exists \( m \) large enough such that \( P_m \) be of class \( i \). There exists only one \( m \) many isomorphism classes of vector bundles of class \( m \) generated by \( f \) for \( g \). Let there be only one \( m \) many \( G_m \)-orbits in \( Q_m \). In particular, there is a unique dense open orbit \( O_m \) in \( \text{supp}(P_m) \). Since \( P_m \) is simple and \( G_m \)-equivariant, we must have \( P_m = \text{IC}(O_m) \) and hence \( P = \text{IC}(O_m) \). Conversely, by Section 9.2, \( \text{Corollary 11.2} \), there is a natural isomorphism \( \text{P'}^{vb} \Rightarrow P^{tor} \).
11.2. A new type. Let us assume that $X$ is an elliptic curve. For any rational number let $P \in \mathbb{P}$ be the subset of simple perverse sheaves which are generically semistable of slope $\lambda$. Recall the definition of the quotient $U_A$ of $\Theta_A$ from Section 10.3. We have

Corollary 11.3. The assignment $b \rightarrow \mu(b)$ for $2f \in J(\mu)$ $j(\mu) \in \Theta_1$ $1g \in \{ f \}$ $j \in \Theta_1$ $g$ extends to an isomorphism of algebras $\mu : U_A^{tor} \rightarrow U_A^{tor}$. Furthermore, the morphism $\mu$ maps the basis $fb \in JP \rightarrow \Theta_1 P^{tor}$ $g$ onto the basis $fb \in JP \rightarrow \Theta_1 P^{tor}$ $g$, and thus induces a canonical bijection $\mu : P^{tor} \rightarrow P$.

Corollary 11.4. Let $P \in \mathbb{P}$ be of $\text{HN}$ type $(\lambda; \cdots; \lambda)$ and set $i = (\lambda)$. There exists unique simple perverse sheaves $P_i \in \mathbb{P}$ such that

$$\text{Ind}(P_1 \otimes \cdots \otimes P_t) = P \otimes P_0$$

where $\text{supp}(P_0) = \text{supp}(f) \in \text{HN}$ $(\lambda)$. 

Proof. The proof is similar to that of Corollary 11.1. 

Combining Corollaries 11.3 and 11.4 gives us a description and a parametrization of the simple perverse sheaves in $P$:

Corollary 11.5. There is a natural isomorphism $P \rightarrow \bigotimes_{i=1}^{Q_0} \mathbb{P}$, where $Q_0$ means a product of finitely many terms.

12. Canonical basis of $U_v(Ln)$

In this section, we use Theorem 5.2 to construct a (topological) canonical basis of $U_v(Ln)$ when $X$ is of genus at most one. We will first describe the precise link between $U_v(Ln)$ and $\Theta_A$. We will be working over the field $C(v)$ rather than $\mathbb{A}$, and we will denote simply by $\Theta_A^{tor}$, etc., the $C(v)$-algebras $\Theta_A^{tor}$, $\text{CoH}_c(X)$, etc. In [S2], we showed that the Hall algebra of the category $\text{CoH}_c(X)$ over a finite field provides a realization of $U_v(Ln)$, so we first relate $\Theta_A$ and $\text{Hall algebra}$.

12.1. Hall algebra. Let us assume that $X$ is defined over the finite field $F_q$ and put $k = F_q$. For every $l \geq 1$, let $H_{\text{CoH}_c}(X \otimes F_q)$ be the Hall algebra of the category of $G$-equivariant coherent sheaves on $X$ over $F_q$, as studied in [S2]. Recall that this is an associative $C$-algebra with basis $f[G]$ $j \in J$ $g$ indexed by the set $I$ of isoclasses of $G$-coherent sheaves on $X$ over $F_q$. There is, for every $2k^+(X)$, a natural map

$$P_{un} : C_{\text{un}}(Q_0) \rightarrow H_{\text{CoH}_c}(X \otimes F_q)$$

where $x_g \in F_q$ is any point in $O_{\text{un}}(Q_0)$. The spaces $C_{\text{un}}(Q_n)$ form an inductive system for $n \geq 2$, and the applications $P_{un}$ give rise to a map

$$P_t = \lim_{n \to \infty} P_{un} : \lim_{n \to \infty} C_{\text{un}}(Q_n) \rightarrow \hat{H}_{\text{CoH}_c}(X \otimes F_q)$$

where $\hat{H}_{\text{CoH}_c}(X \otimes F_q)$ is the completion of $H_{\text{CoH}_c}(X \otimes F_q)$ consisting of possibly infinite linear combinations of elements $[G]$ which are admissible in the sense of Section 3.
Next consider the map
\[ n : U_A \rightarrow C_{G_n}(Q_n) \]
\[ b_p \times X \xrightarrow{\dim G_n} \mathbb{P}^{\dim G_n} \]
\[ \mathbb{P}^{\dim G_n} \times (1)^{\dim G_n} \rightarrow (\dim G_n)^{\dim G_n} \mathbb{P}^{\dim G_n} \]

In the limit, we obtain a map
\[ \lim_{n} n : \mathbb{P}^{\dim G_n} \rightarrow \lim_{n} C_{G_n}(Q_n) \]

Proposition 12.1. For every \( K(\mathfrak{X}) \), the map is injective.

Proof. Let \( x \in 2 K_0 \) be homogenoeous of class \( \mathfrak{X} \). The proof goes in three steps.

i) Let us assume that \( \mathfrak{X} \) is a torsion class and set \( i = j \). We may, by Theorem 5.1.3, write
\[ X = \mathbb{P}_k \oplus \cdots \oplus \mathbb{P}_k \]

where \( \mathbb{P}_k \) is a noncommutative polynomial in \( \mathfrak{X} \) and \( T_k \) belongs to the algebra generated by \( b_{\alpha_i} \) for \( (i,j) \in \mathfrak{X} \). Set \( U_A = \mathbb{P}_2 \oplus \mathbb{P}_2 \). We may, by Lemma 6.1, deduce that \( U_A = \oplus_{(i,j)} U_{A_{ij}} b_{(i,j)} \). By Lemma 6.1, we further have \( \mathbb{P}_k \) for any \( i \) and \( k \). From this one sees that it is possible to reduce \( X \) to a normal surface.

Rearranging the sum, we may further assume that \( \mathbb{P}_k \) is a maximal for \( k = 1, \ldots, m \), and we denote by \( \mathfrak{X} \) the common value. Now, let \( x \) be a point on \( 2 \mathbb{P}^1 \). Let \( F \) be a torsion sheaf of length \( \mathfrak{X} \) supported at \( z \) and \( G \) be a torsion sheaf supported on the exceptional locus. As argued in Section 6.2, and using the maximality of \( \mathfrak{X} \), one checks that, up to a power of \( x \),

\[ (12.1) \quad 0 = \sum \mathfrak{X} \]

Define two closed subsets of \( \mathbb{P}^1 \), where \( a \in \mathbb{C} \) and \( R_i \) are all (distinct) generically semistable simple perverse sheaves. We may also assume that \( x \) is a torsion class. If \( X \) is \( F^1 \mathfrak{X} \) then by Corollary 11.1, there exists an isomorphism \( \mathfrak{X} : U_A^{\mathfrak{X}} \). We have, for \( n \Rightarrow 0 \)

\[ 0 = (x)_{A_1} = \sum_{i} c_i l_{A_i} \]

from which we deduce that \( c_i = 0 \) for all \( i \), and thus \( x = 0 \). Now suppose that \( x \) is an elliptic curve. Let \( \mathfrak{X} \) denote the projection of \( x \) to \( U_A^{\mathfrak{X}} \) and recall that there is a canonical isomorphism \( (x)_{A_1} \). We have, for \( n \Rightarrow 0 \)

\[ 0 = (x)_{A_1} = \sum_{i} c_i l_{A_i} \]

and hence \( (x)_{A_1} = 0 \), and \( \mathfrak{X} = 0 \). As \( R_i \) is generically semistable for all \( i \), this also implies that \( x = 0 \), as desired.
iii) Finally, by Corollaries 11.1 and 11.4, any \( x \in \mathcal{B}_A \) may be written as an admissible sum

\[
x = \sum_{i} \alpha_i b_{r_{(i,1)}'} - r_{(i,1)}'
\]

where \( R_{(i,1)}' \) is simple and generically semistable of class \( (i_j) \), and for any \( x \in \mathcal{B}_A \), \( \alpha_i \) are \( i \)-maximal. Restricting to the component of that \( x \) \( \mathcal{B}_A \) type, we get, for \( n \)

\[
0 = (x)_{\mathcal{B}_A} ( (i_1) ; \ldots ; (i_k) )
\]

and case ii) above, we deduce that \( \alpha_i = 0 \) for \( i = 1 ; \ldots ; k \), and hence \( x = 0 \). The proposition is proved.

Conjecture 12.1. The composition

\[
P_1 = \prod_{i=1}^{N} \mathcal{B}_A \rightarrow \mathcal{B}_A \rightarrow \mathcal{B}_C = H_{\mathcal{A}} \rightarrow \mathcal{B}_C = H_{\mathcal{A}}
\]

is an algebra isomorphism.

As in [12.2], this statement is equivalent to a precise computation of the Frobenius eigenvalues of the cohomology of the complexes appearing in induction products in \( \mathcal{B}_A \).

12.2. The morphism \( \varphi : U_{(\mathcal{B}_A)} \rightarrow \mathcal{B}_A \). Motivated by Conjecture 12.1, we are at last in position to give the perverse sheaf analogue of the main result in [22].

We introduce a few more notations regarding cyclic quivers. Fix \( 1 \leq i \leq N \). Recall the subalgebra \( U_{\mathcal{B}_A}^0 \) of \( U_{\mathcal{B}_A} \) corresponding to complexes of sheaves supported at the exceptional point \( i \). (see Lemma 12.2 for the precise definition), and the algebra isomorphism

\[
\mathbb{A} : U_{\mathcal{B}_A}^0 \rightarrow H_{\mathcal{B}_A}
\]

\[
U_{\mathcal{B}_A}^0 \subset U_{\mathcal{B}_A} \rightarrow H_{\mathcal{B}_A}
\]

De ne elements \( (i) \in H_{\mathcal{B}_A} \) by the relation \( 1 + \prod_{(i)} g^{(i)} = \exp( \prod_{i} h_{(i)} s^{(i)} ) \).

Keeping the notations of 12.2, we have \( (i) = (i) \) \( i \neq 1 \) \( \mathcal{C} \) (see [22], Section 6.3.).

We postpone the proof of this technical Lemma to the appendix

Lemma 12.1. There exists unique elements \( u_{\mathcal{B}_A}^{(i)} \) \( U_{\mathcal{B}_A}^0 \rightarrow H_{\mathcal{B}_A} \) for \( i = 1 \) satisfying the following set of relations:

\[
u^{(i)} = u^{(i)} + u^{(i)} u_{\mathcal{B}_A}^{(i)} + u_{\mathcal{B}_A}^{(i)} u_{\mathcal{B}_A}^{(i)} u^{(i)}.
\]

Finally, we define inductively some elements \( u_{\mathcal{B}_A}^{(i)} \) \( U_{\mathcal{B}_A}^0 \rightarrow H_{\mathcal{B}_A} \) by the relations

\[
u^{(i)} = u^{(i)} + u^{(i)} u_{\mathcal{B}_A}^{(i)} + u_{\mathcal{B}_A}^{(i)} u_{\mathcal{B}_A}^{(i)} u^{(i)}.
\]
Theorem 12.1. For \( (i_1, \ldots, i_n) \neq (2, \ldots, 2) \) we set \( d((\cdot)) = hD_X b_{i_1} \ldots b_{i_n} \).

The assignment

\[
\begin{align*}
(12.6) & \quad E_{(i_j)}: b_{(i_j)}; \quad \text{for } (i_j) \neq 0; \\
(12.7) & \quad 1 \rightarrow X; \quad \text{for } l \leq 1; \\
(12.8) & \quad E_{\gamma} \rightarrow \begin{cases}
0 & \text{for } m \leq 2, \\
1 & \text{for } m > 2.
\end{cases}
\end{align*}
\]

extends to an algebra homomorphism \( U \rightarrow U \).

Proof. The computations needed to prove this are well-defined, i.e., that \( E_{(i_j)} \), \( (\cdot) \) and \( E_{\gamma} \) satisfy relations (9) and (10) in Section 3.2. An equivalence between the category of torsion sheaves supported on some exceptional point \( x \) and the category of nilpotent representations of the quiver \( A_{p+1} \). Let \( C_1 = \text{Coh}_C(X) \) be the full subcategory consisting of all such torsion sheaves which correspond to representations \( V(x) \) of \( A_{p+1} \) where \( x ; \ldots; x_p \) are all isomorphism classes. Also, for \( 12 \leq n \) and \( n > 2 \), we put

\[
z_i = f : X \rightarrow Y.
\]

We claim that

\[
(12.9) \quad n(\cdot) \rightarrow (\cdot) = \begin{cases}
1 \text{ for } n(\cdot) = 1, \\
0 \text{ for } n(\cdot) = 0.
\end{cases}
\]

The first equality is obvious. The second equality is proved by induction on \( n \), using the defining property \( (12.5) \) of \( U \). Finally, we compute

\[
n(\cdot) \rightarrow (\cdot) = \begin{cases}
1 \text{ for } n(\cdot) = 1, \\
0 \text{ for } n(\cdot) = 0.
\end{cases}
\]

where \( A = f : X \rightarrow Y \). In particular, \( A \) denotes the hyperquot scheme parametrizing \( \gamma \)-equivariant functors from \( \gamma \) to \( A \) with subquotients of given classes \( (\cdot) \). If \( A \) is an algebraic torsion sheaf on \( Y \), then \( A \) is a line bundle and \( F \) is a torsion sheaf. Now observe that \( Quot_{\gamma}^{\omega \mu} = \{ \text{Quot}_{\gamma}^{\omega \mu} \} \). Hence the set of such equivalence classes can be checked directly, in a way entirely similar to \( (12.3) \), for all \( n(\cdot) \rightarrow (\cdot) \).

Each of the relations (9) and (10) in Section 3.2. Proof may be written as an equality \( b_{(i_j)} = b_{(i_j)} \) for certain complexes \( R_1; R_2 \) in \( U \), and \( 2 \text{ for } (\cdot) \). By Proposition \( (12.1) \), such an equality follows from the equality of functions \( \gamma(\cdot, \cdot) = \gamma(\cdot, \cdot) \).

The set of such equivalences can be checked directly, in a way entirely similar to \( (12.3) \), for all \( n \). We leave the details to the reader.

Remark. Note that from \( (12.2) \) and \( (12.3) \), Theorem 5.1 follows that the composition of the assignments \( (12.2) \), \( (12.3) \) and \( (12.4) \) with \( p_1 \) extends to an algebra homomorphism \( U \rightarrow U \). Thus, as \( \gamma(\cdot, \cdot) = \gamma(\cdot, \cdot) \). Hence the set of such equivalences can be checked directly, in a way entirely similar to \( (12.3) \), for all \( n \). We leave the details to the reader.

Theorem 12.2. Assume that \( X \) is of genus at most one. Then the map \( \gamma \) is injective and \( \gamma \) is dense in \( \Theta \).

Proof. We will now prove the density of \( \gamma(\cdot, \cdot) \). We begin by observing the following result.

Lemma 12.2. The algebra \( \gamma \) is topologically generated by \( U_{\gamma} \) and the elements \( b_{(i_j)} \) for all \( t \geq 2 \).


Proof. By Theorem 5.1, we only need to show that for any $F = O_X(t)$, the complex $IC(O^F)$ belongs to the closure of the subalgebra generated by $U^{tor}$ and $fb_{P_X(t)}$ at $2Z$. We denote this subalgebra, and we argue by induction. The claim is evident for $l = 1$. For $l \geq 12N + 2Z$ and let us assume that $IC(O^F) = 2A$ for all $F = O_X(t)$, with $t \geq 2Z$ and $t < 1$. As $\text{Ext}(O_X(t))O_X(t) = 0$, we have for $n \geq 0$:

$$\text{Ind}(P_{P_X(t)})_{P_{P_X(t)}} = Q_{\text{dim } O^F_n}$$

where $H(F \Omega)$ is the graded total comodule of the variety of complete flags in $k^n$. Since $O^F_n$ is open in $Q_n^F$, $R_{P^1} = 0$ for all $R \geq 2P$ in $\text{IC}(O^F)$. Hence, from (12.10) we get:

$$b^{P_{P_X(t)}} = b^{\text{IC}(O^F)} + b^{P_{P_X(t)}}$$

where $\text{supp}(R_{P^1}^0) = Q_n^P \cap O^F_n$. From this and from Theorem 5.1 (ii) it follows that $b_{P_{P_X(t)}}$ belongs to the subalgebra generated by $U^{tor}$ and $fb_{P_X(t)}$ if $G \cdot O_X(t)$, with $t \geq 2Z$ and $t < 1$. Thus, by our induction hypothesis, $b = 2A$, and from (12.11) we get $b^{\text{IC}(O^F)} = 2A$ as wanted. The Lemma is proved.

By construction, $b_{P_X(t)}$ belongs to $\dim$ for any $(i,j) \neq 0$ and $L$. From this and the definition of $\dim$ it is also clear that $b_{P_X(t)} \geq \dim$. Hence, by the above Lemma, it only remains to see that $b_{P_X(t)} \geq \dim$ for all $t \geq 2Z$. For this, we define for any line bundle $L^0$ and element $E_{L^0}$ in $U^{tor}(2U\ell(L^n))$ as follows: $E_{L^0}(t) = E_{L^0}(t)$ and if $[L^0] = [L^0] + (i,j)$ with $\dim (E_{L^0}(i,j)) = 1$ then we put $E_{L^0}(v) = v^1E_{L^0}(i,j)E_{L^0}(E_{L^0}(i,j))$. With this definition, we have, using (12.11), Section 6A,

$$b^{P_{P_X(t)}}(E_{L^0}) = v^1b_{P_X(t)}^{P_{P_X(t)}} = \dim$$

for $n \geq 0$.

We claim that:

$$b_{P_{P_X(t)}} = (E_{L^0}) + \psi^{P_{P_X(t)}}b_{P_{P_X(t)}}^{P_{P_X(t)}}(E_{L^0})b_{P_{P_X(t)}}^{P_{P_X(t)}}$$

Indeed, it is enough to check the corresponding equality obtained after applying $b_{P_{P_X(t)}}$, and this follows from (12.12). Finally, note that the elements $b_{P_{P_X(t)}}$ are in $U^{tor}(\dim)$. This proves that $\dim = \emptyset$.

We will prove the injectivity separately for the finite and affine cases.

1) Assume that $X = P^1$. In order to obtain the injectivity of , we choose some integer $l \geq 2Z$ and consider the subalgebra $U_{L^n}$ of $U_{L^n}$ generated by $E_{L^0}$ and $P_{P_X(t)}$. For $l > 1$ and $P_{P_X(t)}$, $j = 1, \ldots, N$. Since $U_{L^n}$ is the limit of $U_{L^n}$ as $n$ tends to $1$, it is enough to show that the restriction of $U_{L^n}$ to $L^n$ is injective. Let $U_{L^n}$ be the $(v)$-span of elements $b_{vP}$ where $P = (P_m)$ is such that $P_m \notin 0$. By Corollary 11 we have $U_{L^n} = U^{vb}_{L^n}$, $U^{tor}$ where $U^{vb}_{L^n} = \bigwedge_{P^{vb}}C(v)b_{P}$ with $P^{vb} = \text{IC}(O^F)F$ is a vector bundle generated by $P_{L^n}(n)g$.

Thus by (12.2) Section 7.1, we have $\dim U_{L^n}(L^n)(f) = \dim U_{L^n}(f)$ for any $f$, and it is enough to prove that $\dim (U_{L^n}(L^n)(f)) = \dim U_{L^n}(f)$ for any $f$. Now, by construction the algebra $U_{L^n}(L^n)$ contains all elements $E_{L^n}$ with $L$ being a line bundle generated by $P_{L^n}(n)g$, and we have $E_{L^n} = b_{P_{L^n}}$ for some unique (virtual) complex $P_{L^n}$ satisfying $P_{L^n}$, $P^{vb}_{L^n}$ isomorphic. By (12.2) Prop. 7.2, any indecomposable vector bundle $F$ generated by $P_{L^n}(n)g$ can be obtained as a successive extension of line bundles also generated by $P_{L^n}(n)g$. It follows that for any $F$ as above, $IC(O^F)$ appears in some induction product $\text{Ind}(P_{L^n})$.
where all $L^i$ are generated by $fL_x(n)g$. Thus $E_{L} = b_{P_{L}}$ where $P_{L}$ is a (virtual) complex satisfying

$$\tag{12.14} \left(P_{L}\right)_{n} \neq 0 \quad \text{for} \quad m; \quad \text{form} \quad 0;$$

More generally, we write any vector bundle generated by $fL_x(n)g$ as $F = G_{L}^{n}$, where $G_{L}$ is an indecomposable vector bundle and $(G_{L})_{n} 
eq 0$. Then the complex $P_{L}$ defined by $b_{P_{L}} = b_{G_{L}}^{n}$ satisfies $(12.14)$ as well. Finally, observe that $U^{tor}(U_{v}^{n}(L_{n}))$. Combining with $U = U^{gb}$, we get the desired inequality of dimensions.

(12.14) $U^{tor}(U_{v}^{n}(L_{n}))$. Combining with $U = U^{gb}$, we get the desired inequality of dimensions.

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By a graded dimension argument as in [32], Section 8 we have $\dim U_{v}^{n}(L_{n})[1] = \dim U^{n}[1]$ for any $X$, and hence it is enough to show that $\dim (U_{v}^{n}(L_{n}))[1] = \dim U^{n}[1]$. We have $(U_{v}^{n}(L_{n}))^{tor}$, and by construction there exists, for each $L A_{n}$, a (virtual) complex $P_{L}$ such that $(E_{L}) = b_{P_{L}}$ and $(P_{L})_{n} = 0$. Using the theory of mutations and arguing as in [32], Section 8, we deduce from this that, for any $P_{2} P$, with $> n$ and $2 K + X$, there exists a virtual complex $\Phi$ such that

$$\tag{12.15} \left(b_{P_{2}} 2 \left(U_{v}^{n}(L_{n})\right)_{n}\right)_{n} = \left(P_{2} \right)_{n}^{n}.$$

From this it follows in turn using Lemma [32] and Corollary [117] that there exists a virtual complex $\Phi$ satisfying $(12.15)$ for any $P_{2} P^{> n}$. Finally, as $b_{P_{2}} 2 \left(U_{v}^{n}(L_{n})\right)$ for $L A_{n}$, the same also holds for any generically semistable $P_{2} P$ appearing in $U^{n}$. The inequality of dimensions follows. This finishes the proof of Theorem X.

Define a completion $\tilde{\mathbf{V}}_v(Ln)\otimes \tilde{\mathbf{V}}_v(Ln)[1]$ and

$$\tilde{\mathbf{V}}_v(Ln)[1] = f \sigma_{ab} ja_{i} 2 U_{v}^{n}(L_{n})[i]; b_{2} 2 U_{v}^{n}(L_{n})[i]; i + 1; \quad \deg(i) = 1; \quad f; \quad i = 0.$$

This completion is still an algebra. Moreover, Theorem [122] may be restated as follows.

Corollary 12.1. Assume $X$ is of genus at most one. Then the map extends to an isomorphism from $\tilde{\mathbf{V}}_v(Ln)$ to $\mathbf{V}_v(Ln)$.

Again, we conjecture that this result remains valid for arbitrary data $(X; G)$.

Remarks. (i) The algebra $\mathbf{V}_v$ is naturally equipped with an integral form $\mathbf{V}_v$. It is likely that the corresponding integral form in $\tilde{\mathbf{V}}_v(Ln)$ is the $A$-algebra generated by $H_{P}$, and the divided powers $E_{\gamma}^{n} = E_{\gamma}^{n}$. (i) The proof of the existence of the map is valid for an arbitrary pair $(X; G)$. 
12.3. Canonical basis of $\mathfrak{H}_v(L_n)$. By Corollary 11.3, we may use, to transport the basis $b_F$ of $\mathfrak{H}$ to a (topological) basis $\mathfrak{B}$ of $\mathfrak{H}_v(L_n)$, whose elements we still denote by $b_F$. In particular, for $2 K \langle X \rangle$ for we denote by $b_2 \mathfrak{B}$ the canonical basis element corresponding to $1$; if $IC(D^F)2 P$ for some vector bundle $F$ then we simply denote by $b_F$ the corresponding canonical basis element of $\mathfrak{H}_v(L_n)$ (note that by Corollary 11.1 iii, this is true for any vector bundle if $X \neq P$). The following Lemma a is clear.

Lemma 12.3. i) For any $l$, let $B_1 = B_{l;1}$ denote the usual canonical basis of $U^+_l(B_{l,1})$ $H_{pl}$, then $B_1 \mathfrak{B}$ and $B_1$ corresponds to the set of simple perverse sheaves in $P$ with support in $f:E_{nl} \rightarrow F$: if $f$ supps $(V)$, then $f$ is for $F$.

ii) For any line bundle $L \rightarrow P \in \mathcal{O}(X;G)$, every element of $\mathfrak{B}$ belongs to the canonical basis $\mathfrak{B}$.

Since Verdier duality commutes with all the functors Ind$_{n,m}$, it descends to a semilinear involution $\mathfrak{H}_v$, and thus gives rise to a semilinear involution $x$ of $\mathfrak{H}_v(L_n)$. From the definition of one gets

\[
E_{ij;k} = E_{ij;k} - \frac{1}{n+1} \left( \begin{array}{c} n+1 \\ n_1 \end{array} \right) \frac{1}{U_{n_1}} \frac{1}{U_{n_2}};
\]

where $d_{ij} = \text{Ind}_{j;1}^{n+1}$. Observe that the natural action of $P \in \mathcal{O}(X;G)$ on the cotangent bundle induces an action on $\mathfrak{H}_v$, and hence an action on $\mathfrak{H}_v(L_n)$. For instance, twisting by $O_X(1)$ gives rise to the automorphism mapping $E_{ij;k}$ to $E_{ij;k+1}$ and all other generators $H_{ij;k}$, $E_{ij;k}$ to themselves.

Proposition 12.2. The canonical basis $\mathfrak{B}$ satisfies the following properties:

i) $b b^\circ 2 b = b + b \mathfrak{B}$ for any $b \in \mathfrak{B}$.

ii) $\overline{b} = b$ for any $b \in \mathfrak{B}$.

iii) The action of $P \in \mathcal{O}(X;G)$ on $\mathfrak{H}_v(L_n)$ preserves $\mathfrak{B}$. In particular, $b$ is $b \mathfrak{B}$ if and only if $(b^2)$ is $\mathfrak{B}$.

iv) For any $(i;j) \not\in \mathfrak{B}$ and $n > 0$ there exists subsets $\mathfrak{B}_{(i;j);n} \subset \mathfrak{B}$ such that $\mathfrak{H}_v(L_n) = \bigcup_{n \geq 0} \mathfrak{B}_{(i;j);n} C(v)$.b.

Proof. Statements i) and iii) are obvious. If $P \in \mathcal{O}(x)$ then $P = IC(D^F)$ and thus $D(P) = P$. The same is true if $P$ is $P$ (see Sections 5 and 8 for the notations). On the other hand, if $P \in \mathcal{O}(x)$ then by Lemma 12.3. we have $P = IC(U_{(i);j;1}(1))$ for some irreducible representation of the symmetric group $S_1$. This is also self
dual since any representation of $S_1$ is self-dual. Now let $P^+_j$ be an arbitrary torsion simple perverse sheaf. By Lemma 5.4 there exists simple perverse sheaves $R^+_j$ and $T^+_j$ such that $\text{Ind}^+_j(R^+_j, T^+_j) = P^+_j$ with $P^+_j$ being a sum of shifts of complexes in $P^+_j$. Since Verdier duality commutes with $\text{Ind}$, we obtain $D(P) = D(P^+_1) = P^+_1$ from which it follows that $D(P) = P$. The case of an arbitrary $P$ now follows in the same manner using Corollary 11.4. This proves (ii). Finally, (iv) is obtained by combining [10], Theorem 14.32 with Corollary 11.4 and the description of $Q^n_{\lambda_j}$ given in Section 5.1.

13. Examples and conjectures.

13.1. Canonical basis for $B_k$. Let us assume that $X = P^1$ and $G = \text{Id}$, so that $U_{\lambda}(L(n))$ is the 'Wahorn' positive part of the quantum algebra $U_{\lambda}(B_k)$. In that situation, the elements of the canonical basis corresponding to torsion sheaves are indexed by partitions; let us identify the subalgebra of $U_{\lambda}(L(n))$ generated by $f^j_\lambda$ by $M$ accadon's ring of symmetric functions via the assignment $j \mapsto P^j_\lambda$. By Lemma 5.4 we deduce that

$$b_j^i = (C \cup j^j_{(1)} \cup \cdots) = (s)$$

where $s$ are respectively the irreducible $S_j$-module and the Schur function associated to $j$. In particular $b^1 = b^{(1)}$ belongs to $B^1$.

In rank one we have

$$X \quad b_0 = E_t + \sqrt{V_1 E_t}$$

(for simplicity we omit the symbol $\lambda$). As an example of canonical basis elements in rank two, we give

$$b_0 = E_t + \sqrt{2 E_t E_{t^2}} + \sqrt{V_{t^2} E_{t^2}} + (1) = E_t + \sqrt{V_1 E_t} + \sqrt{V_{t^2}}$$

$$b_0 = E_t + \sqrt{2 E_t E_{t^2}} + \sqrt{V_{t^2} E_{t^2}} + (1) = E_t + \sqrt{V_1 E_t} + \sqrt{V_{t^2}}$$

The above examples of canonical basis elements correspond to perverse sheaves $1$ for some $2k(X)$. Similar formulas give the canonical basis elements corresponding to $0(\lambda)_m \circ 0(\lambda+1)_m$ for any $n \geq 2$.

Finally, the bar involution $\chi$ is also easy to describe explicitly: it is given by $1 = 1$ and

$$E_t = E_t + \lambda_1^n + \lambda_1^{n+1} + \cdots$$

$$n > 0, l_1 > 0$$
Remark. From (12.8) and the relations

\begin{align}
(13.4) & \quad \mathcal{D}(n) \mathcal{D}(1) = \mathcal{V} \mathcal{D}(n) & n \\
(13.5) & \quad \mathcal{D}(n) \mathcal{D}(1) = \mathcal{V}^{\mathcal{D}(n)} & n \\
(13.6) & \quad n \mathcal{M} (n+m) = n \\
\end{align}

we may compute the coproduct on $\mathfrak{g}_A = \mathfrak{g}_V (L)$. By definition,

\[ \mathcal{E} = \mathcal{V} \mathcal{D}(n) \]

where \( n = \mathcal{P} \mathcal{D}(n) + \mathcal{P} \mathcal{D}(1) \). By (13.5), \( n \mathcal{M} (n+m) = n m \) and hence

\[ \mathcal{D}(n) \mathcal{D}(m) \mathcal{E} = \mathcal{V}^{\mathcal{M}(n)} \mathcal{D}(n) \mathcal{D}(m) \mathcal{E} = \mathcal{V}^{\mathcal{M}(n+m)} \mathcal{E} \]

One finds that \( \mathcal{D}(n) \mathcal{D}(m) \mathcal{E} = \mathcal{D}(n+m) \mathcal{E} \), so that \( \mathcal{D}(n) \mathcal{D}(1) = \mathcal{D}(n+1) \). A similar computation gives

\[ \mathcal{D}(n) \mathcal{D}(m) \mathcal{E} = \mathcal{V}^{\mathcal{M}(n+m)} \mathcal{E} = \mathcal{V}^{\mathcal{M}(n)} \mathcal{V}^{\mathcal{M}(m)} \mathcal{E} \]

Hence, setting \( \mathcal{L} = \mathcal{P} \mathcal{D}(n) \mathcal{D}(m) \mathcal{E} = \mathcal{V}^{\mathcal{M}(n+m)} \mathcal{E} \), we obtain

\[ \mathcal{E} = \mathcal{E} \mathcal{V}^{\mathcal{M}(n+m)} \mathcal{E} = \mathcal{V}^{\mathcal{M}(n)} \mathcal{V}^{\mathcal{M}(m)} \mathcal{E} \]

Finally, one checks that the elements \( \mathcal{L} \) may be characterized through the relation

\[ \mathcal{L} \mathcal{D}(1) = \mathcal{D}(1) \mathcal{D}(1) \mathcal{E} = \mathcal{D}(1) \mathcal{E} = \mathcal{E} \mathcal{D}(1) \mathcal{E} \]

This result is a quantum analogue of [FS], Theorem 2.2.1. Let \( \mathcal{I}_0 \) be the ideal of $\mathfrak{g}_V (L)$ generated by the above elements. For \( t < 0 \) we have

\[ \mathcal{D}(n)(t) \mathcal{D}(m)(t) = \mathcal{V}^{\mathcal{D}(n+m)}(t) \]

It may be explicitly checked that these expressions vanish, using (for instance) the description of the action of $U_V (\mathfrak{g}_V)$ on $L_0$ in [V]. This implies that \( \mathcal{I}_0 \) contains \( \mathcal{I}_0 \). On the other hand, $\mathcal{W}_0$ has finite rank weight spaces, so using [FS] Theorem 2.2.1 we have for any weight

\[ \dim (\mathfrak{g}_V (L)) = [.] = \dim (\mathcal{W}_0) = \dim (\mathfrak{g}_V (L)) = [.] \]

13.2. Action on the principal subspace. Let us again consider the case of $\mathfrak{g}_V$. Let $L_0$ be the irreducible highest weight $U_V (\mathcal{I}_0)$ module generated by the highest weight vector $\mathcal{E}$, where $\mathcal{H}_0 = 0$; $\mathcal{O}_0 = 0$. Following [FS], we consider the subalgebra $U_V (\mathcal{I}_0) = U_V (\mathfrak{g}_V)$ generated by $E$ for $t \neq 0$ and put $\mathcal{W}_0 = U_V (\mathfrak{g})$. We observe that by definition $E_1 \mathcal{V}_0 = H_0 \mathcal{V}_0 = 0$ for any $t$, and that therefore $\mathfrak{g}_V (L_n)$ acts on $L_0$. Moreover, we also have $\mathcal{W}_0 = \mathfrak{g}_V (L_n) \mathcal{V}_0$. Let $\mathcal{I}_0$ be the annihilator in $\mathfrak{g}_V (L_n)$ of $\mathcal{V}_0$. 

Proposition 13.1. The ideal $\mathcal{I}_0$ of $\mathfrak{g}_V (L_n)$ is generated by the elements $b_0 (t)$ for $t > 0$ and the elements $b_0 (t) \circ (t)$ for $t < 0$.

Proof (sketch). This result is a quantum analogue of [FS], Theorem 2.2.1. Let \( \mathcal{I}_0 \) be the ideal of $\mathfrak{g}_V (L_n)$ generated by the above elements. For $t > 0$ we have

\[ \mathcal{D}(n)(t) \mathcal{D}(m)(t) = \mathcal{V}^{\mathcal{D}(n+m)}(t) \]

It may be explicitly checked that these expressions vanish, using (for instance) the description of the action of $U_V (\mathcal{I}_0)$ on $L_0$ in [V]. This implies that $\mathcal{I}_0$ contains $\mathcal{I}_0$. On the other hand, $\mathcal{W}_0$ has finite rank weight spaces, so using [FS] Theorem 2.2.1 we have for any weight

\[ \dim (\mathfrak{g}_V (L)) = [.] = \dim (\mathcal{W}_0) = \dim (\mathfrak{g}_V (L)) = [.] \]
We deduce that $I^0 = I_0$ as desired.

Set $C_0 = fF_j f' O (l_i) \cap \omega (l_i) < l_i < 0; l_{i+1} f 2 g$

Conjecture 13.1. We have $L_0 = \text{canon} (v) b_p$, where

$$P_{W_0} = fF j f 2 C_0$$

Assuming Conjecture 13.1, we may equip $W_0$ with a canonical basis $B_0 = fF_j f' 2 C_0$. Recall that $L_0$ is already equipped with a canonical basis $B^+$, which is obtained by application on $v_0$ of the canonical basis $B$ of $U_V (B_2)$ (as in [Le]). More precisely, let $L_0$ be the level one Fock space and let $fF_j f_2 g$ be its Leclerc-Thibon canonical basis (see, e.g., [VW]), which is indexed by the set of all partitions. It is known that $L_0 = \text{canon} (v) b^+$ where

$$0 = fF_j f 6 i, i \geq 0$$

One calculates directly that $b_0 (1) v_0 = b^+_2 (2n+1)$. More computational evidence suggests that in fact

Conjecture 13.2. We have $B_0 \subseteq B^+$. In particular, the canonical basis $B^+$ of $L_0$ is compatible with the principal subspace $W_0$.

Note that the last statement in the conjecture is not obvious from the definition of $B^+$ since the subalgebra of $U_V (B_2)$ generated by $fF (2n) j f n < 0$ is not compatible with the canonical basis $B$ of $U_V (B_2)$.

The whole space $L_0$ may be recovered from the principal subspace $W_0$ by a limiting process as follows. The affine Weyl group $W_0 = Z_2 \circ Z$ contains a subalgebra of translations $T_0, n \geq 2$. The weight spaces of $L_0$ of weight $T_0 (0)$ are one-dimensional and there exists a unique collection of vectors $v_n = \text{canon} (v) b^+ 2 C_0$ such that $b_0 (\langle 2n+1 \rangle) = v_1 = 2n+1 \qquad v_1 = v_0$ (see [FS]). Explicitly, we have

$$v_0 = b^+_2 (2n+1) \quad \text{if } n > 0 \quad \text{and} \quad v_n = b^+_2 (2n+1) \quad \text{if } n < 0$$

We set $W_n = \text{canon} (v) (L n) \quad y = U_V (B_2) \quad y$. The subspaces $W_n$ for $n \geq 2$ form an exhaustive filtration of $W_0$. Finally, if $n > 0$, we set $C_n = fF (2n) j f 2 C_0$ and

$$P_{W_n} = fF \text{canon} (G) (2n+1) \quad O (2n+3) \quad O (2n+3) \quad O (1) \quad j f 2 C_0$$

Lemma 13.1. Fix $n > 0$. Then

i) For any $G 2 C_n$, we have

$$b_0 (\langle 2n+1 \rangle) b_0 (\langle 2n+3 \rangle) \quad O (1)$$

$$2 b_0 (\langle 2n+1 \rangle) O (\langle 2n+3 \rangle) \quad O (\langle 1 \rangle)$$

ii) We have

$$\text{canon} (v) (L n) b_0 (\langle 2n+1 \rangle) b_0 (\langle 2n+3 \rangle) \quad O (1)$$

$$= \text{canon} (v) b_0 (\langle 2n+1 \rangle) b_0 (\langle 2n+3 \rangle) \quad O (1) \quad P_{W_n} \quad C (v) b_0 (\langle 2n+1 \rangle) b_0 (\langle 2n+3 \rangle) \quad O (1) \quad P_{2 P_{W_n}} \quad C (v) b_0 (\langle 2n+1 \rangle) b_0 (\langle 2n+3 \rangle) \quad O (1)$$

Proof. This is shown using arguments similar to those of Section 9.1. We omit the details.
param etrized by the collection $C_1$ of sem i-in nite sequences $(l_1 < l_2 < \ldots)$ where, for $N$ odd and $l_{2k + 1} = l_k + 2$.

The natural generalization of Conjecture 13.2 is

**Conjecture 13.3.** The basis $B_1$ coincides with the Leclerc-Thibon basis $B'$. 

Remark. According to Corollary 13.2, the basis $B'$ is param etrized by pairs $(\mathcal{F}'; \mathcal{G})$, where $\mathcal{F}$ is a vector bundle and $\mathcal{G}$ is a partition. We may also de ne a PBW-type basis defined as follows. For $\mathcal{F} = O_X$, let $d_i$, $g_i$ be such that $l_k < n \leq l_{k+1}$. Then the collection of elements $B_{PB}^n = \mathcal{F}_n \mathcal{G}$ where $\mathcal{F}$ runs through the set of vector bundles and $\mathcal{G}$ runs through the set of partitions is a quantum analogue of the monomial basis constructed in [FS].

13.3. **Conjecture.** We still assume that $(\mathcal{F}; \mathcal{G}) = (\mathcal{F}'; \mathcal{G})$. Observe that by 13.4, 13.2, 13.3, the ideal $I_0$ in Section 13.2 is generated by $E; t \in \mathcal{F}$, $0$, $f \mathcal{G}$, and the Fourier coe cients of $\mathcal{G}$ are ordered* quantum current

$$E(z) = \frac{E_{t_k} x t_k}{t_k}$$

where $E = E_{t_k}$ if $t_k = t_0$, and $E = E_{t_k} x t_k$ otherwise. More generally, we de ne for any $12 N$

$$E(z) = \frac{E_{t_k} x t_k}{t_k}$$

where $E_{t_k} x t_k = \nu^t E_{t_k} x t_k$, and $x(t) = \mu^t = \mu + \nu^t$. Let $E(z) = E^{(t)}$.

**Conjecture 13.4.** For any $12 N$ and $r \in \mathcal{G}$, the canonical basis $B'$ is compatible with the left ideal generated by $E; t \in \mathcal{F}$, $r$, $f \mathcal{G}$, and the Fourier coe cients of $E(z)$.

This conjecture is equivalent to saying that $B'$ descends to a basis of the principal subspace $W_0$ of $L'$.

14. Appendix

A.1. **Proof of Lemma 3.** For simplicity we write $p$ for $p_i$. Recall that the canonical basis $B_{PB}$ is naturally indexed by the set $G$-orbits in $N$ for all $p$. Such an orbit is determined by the collection of integers $d_i(O) = \dim \ker x_k + 1$ for $k \in \mathcal{P}$ and $k \in \mathcal{L}$, any point. Put $m(O) = d_i(O) + d_{i-1}(O) + d_{i-2}(O)$ and $m(O) = m(O)$.

Where $e_{\mu l}$ are formal variables (the element $\mu$ detemines $d_{\mu l}$). Let
us call aperiodic an element \( m \) (or the corresponding orbit \( O_m \)) for which the following statement is true: for every \( t \geq 1 \) there exists a \( Z = \mathbb{Z} \)-basis such that \( m_{ir} = 0 \). By \((14.3)\), we have

\[
U_A^+ (B_m) = \bigcap_{v \in 1^{20} \cdot 0} C [v] v^{-1} 1_{O_m} ;
\]

where \( O_{ap} \) is the set of aperiodic orbits. Similarly, we call \( m \) and \( O_m \) totally periodic if \( m_{ij} = m_{ji} \) for all \( i \neq j \). Now let \( b_m \) be such that \( b_m \neq 0 \). Using the proposition \((14.1)\), we deduce directly that \( b_m \in 2 U_A^+ (B_m) \) if \( \leq 6 (x; y; z; r) \) for some \( r \neq 2 \), and that there exists \( c \in C \) such that

\[
b_m \in c 2 U_A^+ (B_m)
\]

if \( = (x; y; z; r) \). Let \( s : H \to \) be the automorphism of order \( p \) induced by the cyclic permutation of the quiver. It is clear that \( s (U_A^+ (B_m)) = U_A^+ (B_m) \) and we have \( s (z_r) = z_r \). Hence, if \( b_m \) satisfies \((14.2)\), then either \( c = 0 \) and \( m \) is aperiodic, or \( c \neq 0 \) and \( m \) is xed by \( s \), hence totally periodic. The set of integers \( f_{i;j} \) for any \( x e \) then forms a partition independent of \( i \). Let \( b \) be the last nonzero part of \( x e \) and \( d = (0; 0; 0; 0) \). Using the lemma \((3.4)\), we have

\[
d_{m; \leq} (v^{\dim \infty} 1_{O_m} )_{p_{n; \leq}} \bigcap_{v \in 2 \mathbb{N} [v]}
\]

for any \( n \neq 0 \)

\[
d_{m; \leq} (v^{\dim \infty} 1_{O_m} )_{p_{n; \leq}} \bigcap_{v \in 2 \mathbb{N} [v]}
\]

We deduce

\[
(14.3)
\]

By hypothesis \( d_{m; \leq} (O_m)_{p_{n; \leq}} \bigcap_{v \in 2 \mathbb{N} [v]}
\]

for any \( n \neq 0 \) and \( m \). But since the orbit under \( s \) of any aperiodic \( n \) is of size at least \( 2 \), together with the fact that \( b_m \in 2 \mathbb{N} [v] \) \( b_m \) implies that all nonzero coefficients of \( d_{m; \leq} (O_m)_{p_{n; \leq}} \bigcap_{v \in 2 \mathbb{N} [v]}
\]

are greater than two. This contradicts \((14.3)\), and hence proves that \( c = 0 \) in \((14.2)\). The claim follows.

A 2. Proof of Lemma \((12.1)\). We again drop the index \( \leq \) throughout, and write \( 1 \) for \( 1_{n; \leq} \) and \( \leq \) for \( (1; \leq 2; 1; \leq 3; \leq m; \leq m) \). Finally, the computations below are conducted up to a global power of \( v \). We prove the lemma by induction. The result is true for \( r = 0 \). Assume that \( u_1; \leq \leq u_r \) satisfying \((12.4)\) are already known and belong to \( U_A^+ (B_m) \), and let us put \( u_r \in 1_{r; \leq 1} u_{r-1} \leq 1 u_r \). Substituting, we obtain

\[
u_r = \bigcap_{n \in (1; \leq 2; \leq m; \leq h) 2} V
\]

where \( J = f (n_0; \leq \leq n_r) \) \( n_r = r g \). A simple calculation using the relations \( \in \cap (i; \leq j) = 1_{n; \leq m} \leq 1 \leq m \) yields \( \in \cap (i; \leq j) = u_1 \leq u_r \leq 1 \leq U_A^+ (B_m) \) \( U_A^+ (B_m) \) for all \( r \neq 0 \). Proposition \((14.1)\) now implies that there exists \( c \in C \) and \( u_0^0 \leq 2 U_A^+ (B_m) \) such that \( u_r = c x_r + u_0 \). It remains to prove that \( c = 0 \). Put \( H_0^0 = 1 E H_0 \). Since \( x \leq H_0 \) but \( u_r \leq H_0 \) for all \( r \), it is enough to show that

\[
(14.4)
\]
We argue once more by induction, this time on $p$. If $p = 2$ then $l_r r = 1_A$ where $A = f(x_1; x_2)$ and $f$ is $N_{(x_1;x_2)}$. Let us set $A_1 = f(x_1; x_2) \text{adj} \ker x_2$. If $p = 2$ then $1_r r = 1_A$. It is easy to see that, up to a power of $v$, $l_{1_A} = 1_{(y)}$ where $A_0 = f(x_1; x_2)$ and $f$ is surjective $N_{(x_1;x_2)}$. Since $l_{1_{(y)}}$ is proportional to $E_1 \Gamma_1$, $E_1$ divides $l_{1_A}$, and hence also $l_r r$. Next, assume that $l_{1_{(y)}}$ is established for some $p_0 > 2$. Consider the algebra homomorphism $a : H_{p_0} \rightarrow H_{p_0+1}$ induced by the collection of embeddings $a^0 : N_{(y)} \rightarrow N_{(y+1)}$ defined by $W_1 = W_2 = V_1$, $W_1 = V_{p_0}$ for $i = p_0 + 1$ and $a^0(x_1; : : : ; x_{p_0}) = (x_1; \text{Id}; x_2; : : : ; x_{p_0})$. Note that $a(1_{p_0}) = 1_{p_0+1}$ since $a^0(E_1) = v_1 E_2 E_1 E_2$ and $a^0(E_i) = E_{i+1}$ for $i \neq 1$. Thus $l_B r = a(l_r r) 2 H_{p_0+1}$, where $B = f(y_1; : : ; y_{p_0+1}) \text{adj} \ker y_2$. This is proved in the same way as in the case $p = 2$. Thus $l_r r 2 H_{p_0+1}$ and the induction is closed.

Acknowledgements

I would like to thank Susumu Ariki, Gaétan Chenevier, Bangming Deng, Sergey Loktev, Eric Vasserot and Jie Xiao for many very helpful discussions. In particular, the work [FS] was pointed out to me by Eric Vasserot. Parts of this work were done at Yale University, Tsinghua University and RIMS, and I warmly thank these institutions for their hospitality.
\[ U_A, 47 \]
\[ U^\text{iso}, 27 \]
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