Some new results on bar visibility of digraphs

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Abstract

Visibility representation of digraphs was introduced by Axenovich, Beveridge, Hutchinson, and West (SIAM J. Discrete Math. 27(3) (2013) 1429–1449) as a natural generalization of \( t \)-bar visibility representation of undirected graphs. A \( t \)-bar visibility representation of a digraph \( G \) assigns each vertex at most \( t \) horizontal bars in the plane so that there is an arc \( xy \) in the digraph if and only if some bar for \( x \) "sees" some bar for \( y \) above it along an unblocked vertical strip with positive width. The visibility number \( b(G) \) is the least \( t \) such that \( G \) has a \( t \)-bar visibility representation. In this paper, we solve several problems about \( b(G) \) posed by Axenovich et al. and prove that determining whether the bar visibility number of a digraph is 2 is NP-complete.

Keywords: bar visibility number, graph representation, transitive tournament, NP-complete.

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1 Introduction

Visibility representation of graphs has been studied extensively in computational geometry and has important application in VLSI design, computer vision, etc.; for a book devoted to the topic, see Ghosh [4]. Among various types of visibility representations of graphs, we focus here on bar visibility representation in the plane.

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A graph $H$ is a **bar visibility graph** if each vertex can be assigned a horizontal line segment in the plane (called a *bar*) so that vertices are adjacent if and only if the corresponding bars can see each other along an unblocked channel, where a *channel* is a vertical strip of positive width. The assignment of bars is a **bar visibility representation** of $H$. Tamassia and Tollis [10] and Wismath [14] characterized bar visibility graphs (see Hutchinson [6] for another proof).

**Theorem 1.1** ([10] [14]). A graph $H$ has a bar visibility representation if and only if $H$ can be embedded in the plane so that all cut-vertices appear on the boundary of one face.

Chang, Hutchinson, Jacobson, Lehel, and West [3] extended this concept to all graphs by introducing $t$-bar visibility representations of graphs. A **$t$-bar visibility representation** of a graph $H$ assigns each vertex up to $t$ bars in the plane so that two vertices are adjacent if and only if some bar for one vertex can see some bar for the other via an unblocked channel. The least $t$ such that $H$ has a $t$-bar visibility representation is called the **bar visibility number** of $H$, denoted by $b(H)$.

Axenovich, Beveridge, Hutchinson, and West [2] introduced an analogue for directed graphs. A **$t$-bar visibility representation** of a digraph $G$ assigns each vertex at most $t$ bars in the plane so that there is an arc $xy$ in the digraph if and only if some bar for $x$ sees some bar for $y$ above it via an unblocked channel. The **bar visibility number** $b(G)$ of a digraph $G$ is the least $t$ such that $G$ has a $t$-bar visibility representation. Digraphs with bar visibility number 1 are **bar visibility digraphs**.

In a digraph, a vertex is a *source* or a *sink* if it has indegree 0 or outdegree 0, respectively. A **consistent cycle** is an oriented cycle with no source or sink. Tomassia and Tollis [10] and independently Wismath [15] characterized bar visibility digraphs.

**Theorem 1.2** ([10] [15]). Let $G$ be a digraph, and let $G'$ be the digraph formed from $G$ by adding two vertices $s$ and $t$, an arc $sv$ for every source vertex $v$ in $G$, an arc $wt$ for every sink vertex $w$, and the arc $st$. A digraph $G$ is a bar visibility digraph if and only if $G'$ is planar and has no consistent cycle.

Thus planarity is necessary but not sufficient for $b(G) = 1$. Axenovich, Beveridge, Hutchinson, and West [2] showed that $b(G) \leq 4$ when $G$ is a planar digraph, $b(G) \leq 2$ when $G$ is outerplanar, and in general $b(G) \leq (|V(G)| + 10)/3$. For outerplanar digraphs, West and Wise [12] gave a forbidden substructure characterization for those with $b(G) = 1$.

A **tournament** is an orientation of a complete graph. A tournament $T$ is transitive if $xz$ is an arc whenever $xy$ and $yz$ are arcs. In particular, $T$ is transitive if and only if there is a linear ordering of the vertices such that $xy$ is an arc if and only if $x$ precedes $y$ in the ordering. Up to isomorphism, there is only one transitive tournament on $n$ vertices, denoted by $T_n$. In [2], the authors gave the exact value of $b(T_n)$ for $1 \leq n \leq 15$ except for $n \in \{11, 12\}$, and they gave two upper bounds for $b(T_n)$ by using Steiner systems.
Axenovich et al. [2] posed two open problems and two conjectures that we address here.

**Problem 1.5** ([2]). What is the least \( \alpha \) such that always \( b(T_n) \leq \alpha n + c \) for some fixed \( c \)?

**Problem 1.6** ([2]). What is \( \lim_{n \to \infty} b(T_n)/n \) (if the limit exists)?

**Conjecture 1.7** ([2]). \( b(T_{11}) = 3 \).

**Conjecture 1.8** ([2]). If \( G \) is an orientation of an undirected graph \( \hat{G} \), then \( b(G) \leq 2b(\hat{G}) \).

In Section 2, we present a simple construction proving \( b(T_n) \leq \lceil n/4 \rceil \). This does not improve the upper bound when \( n \) is sufficiently large but is valid for all \( n \), improving statement (1) of Theorem 1.4. In Section 3, we prove that \( \lim_{n \to \infty} b(T_n)/n \) exists and is at least \( (3-\sqrt{7})/2 \), about 0.177124. This improves the easy lower bound of 1/6, mentioned in [2], that follows from Euler’s Formula. As a consequence of our lower bound, we prove Conjecture 1.7 in particular, \( b(T_{11}) = b(T_{12}) = 3 \) and \( b(T_{17}) = 4 \). In Section 4, we disprove Conjecture 1.8 for \( b(\hat{G}) = 1 \) but in general observe \( b(G) \leq 4b(\hat{G}) \). Finally, in Section 5 we prove that determining whether \( b(G) \leq 2 \) is NP-complete.

A simple observation is helpful in studying \( b(T_n) \) for small \( n \).

**Lemma 1.9.** \( b(T_n) \leq b(T_{n+1}) \leq b(T_{n+2}) \leq b(T_n) + 1 \).

*Proof.* Because \( T_n \) is transitive, removing bars from a visibility representation of \( T_n \) cannot add any unwanted visibility. Thus we can obtain an \( m \)-bar visibility representation of \( T_n \) from one for \( T_{n+1} \) by removing the bars for one vertex, and similarly \( b(T_{n+1}) \leq b(T_{n+2}) \).

To complete the proof, we obtain a \((k+1)\)-bar visibility representation of \( T_{n+2} \) from a \( k \)-bar visibility representation of \( T_n \). Draw the representation of the smaller tournament with vertices \( v_1, \ldots, v_n \) in the left half-plane. In the right half-plane, we will add one bar for each of \( v_0, \ldots, v_{n+1} \), representing all arcs involving the two new vertices \( v_0 \) and \( v_{n+1} \) at a cost of adding one new bar for each old vertex.

Index the vertices so that \( v_0 \) is a source and \( v_{n+1} \) is a sink in \( T_{n+2} \), making \( T_{n+2} \) indeed transitive. For \( 1 \leq i \leq n \), assign to \( v_i \) the bar from the point \((i-1,1)\) to the point \((i,1)\). Assign to \( v_0 \) the bar from \((0,0)\) to \((n+1,0)\), and assign \( v_{n+1} \) the bar from \((0,2)\) to \((n+1,2)\). This generates arcs from \( v_0 \) to all of \( v_1, \ldots, v_{n+1} \) and from all of \( v_0, \ldots, v_n \) to \( v_{n+1} \). \( \square \)
2 An upper bound on \(b(T_n)\)

In this section, we prove an upper bound on \(b(T_n)\) for general \(n\) by using decompositions of the complete graph. A well-known result about complete graphs of even order is that they decompose into spanning paths.

**Lemma 2.1** \([II]\). The complete graph \(K_{2m}\) with vertex set \(\{x_1, \ldots, x_{2m}\}\) decomposes into spanning paths \(P_1, \ldots, P_m\) given by

\[
P_i = \langle x_i x_{i+1} x_{i-1} x_{i+2} x_{i-2} \cdots x_{i+(m-1)} x_{i-(m-1)} x_{i+m} \rangle
\]

for \(1 \leq i \leq m\), with subscripts on \(x\) taken modulo \(2m\).

For \(1 \leq i \leq m\), the central edge of \(P_i\) as specified above is \(x_{i+[m/2]} x_{i-[m/2]}\), which we designate as \(e_i\). Note that \(e_1, \ldots, e_m\) is a perfect matching in \(K_{2m}\). The example with \(m = 4\) decomposes \(K_8\) into the spanning paths \(P_1, \ldots, P_4\), where

\[
\begin{align*}
P_1 &= \langle x_1 x_2 x_3 x_4 x_5 x_6 \rangle, \\
P_2 &= \langle x_2 x_3 x_4 x_5 x_6 x_7 \rangle, \\
P_3 &= \langle x_3 x_4 x_5 x_6 x_7 x_8 \rangle, \\
P_4 &= \langle x_4 x_5 x_6 x_7 x_8 x_9 \rangle.
\end{align*}
\]

The matching consisting of the central edges is \(\{x_3 x_7, x_4 x_8, x_5 x_1, x_6 x_2\}\). Note also that every orientation of a path is a bar visibility digraph.

**Theorem 2.2.** The bar visibility number of the transitive tournament \(T_n\) is at most \(\lceil n/4 \rceil\).

\begin{proof}
By Lemma \([I.9]\) it suffices to prove \(b(T_n) \leq m\) when \(n = 4m\). We aim to decompose \(T_n\) into \(m\) bar visibility digraphs, each represented using one bar per vertex; this yields \(b(T_n) \leq m\). Index the vertices of \(T_n\) as \(v_1, \ldots, v_n\) so that the arcs are \(\{v_i v_j : i < j\}\). Partition the vertex set into two sets \(A\) and \(B\), where \(A = \{v_1, \ldots, v_m\} \cup \{v_{3m+1}, \ldots, v_{4m}\}\) and \(B = \{v_{m+1}, \ldots, v_{3m}\}\).

The subtournaments \(T_n[A]\) and \(T_n[B]\) induced by \(A\) and \(B\) are isomorphic to \(T_{2m}\). By Lemma \([2.1]\) they decompose into orientations of \(m\) paths, which we call \(P_1, \ldots, P_m\) in \(T_n[A]\) and \(Q_1, \ldots, Q_m\) in \(T_n[B]\). These paths inherit orientations from \(T_{4m}\). In order to express them in the form \([I]\), in \(P_1, \ldots, P_m\) we view \(v_1, \ldots, v_m\) as \(x_1, \ldots, x_m\) and \(v_{3m+1}, \ldots, v_{4m}\) as \(x_{m+1}, \ldots, x_{2m}\). In \(Q_1, \ldots, Q_m\), we view \(v_{m+1}, \ldots, v_{3m}\) as \(x_1, \ldots, x_2\) in order.

The remaining arcs form an orientation of the complete bipartite graph \(K_{2m,2m}\) with parts \(A\) and \(B\). The arcs are oriented from \(v_1, \ldots, v_m\) in \(A\) to all of \(B\) and from all of \(B\) to \(v_{3m+1}, \ldots, v_{4m}\) in \(A\).

Recall that the central arcs \(e_1, \ldots, e_m\) of the paths \(P_1, \ldots, P_m\) form a perfect matching on \(A\). Let \(G_i\) be the digraph obtained by joining both endpoints of \(e_i\) to all the vertices of \(Q_i\), inheriting the orientation from \(T_n\). As illustrated in Figure \([I]\) \(G_i\) is a planar digraph: we place the vertices of \(Q_i\) on a horizontal axis between the vertices of \(e_i\), with the rest of \(P_i\) extending from the central arc \(e_i\). Figure \([I]\) shows the decomposition \(\{G_1, \ldots, G_4\}\) for \(T_{16}\).
\end{proof}
To show that $G_i$ is a bar visibility digraph, we apply Theorem 1.2. Note first that each arc $e_i$ has one endpoint in $v_1, \ldots, v_m$ and one endpoint in $v_{3m+1}, \ldots, v_{4m}$. This means that every vertex in $B$ is neither a source nor a sink in $G_i$. In the figure, we add $s$ to the left and $t$ to the right. Since sources and sinks in $G_i$ lie along $P_i$, we can add arcs from $s$ to the sources and from the sinks to $t$, plus the arc $st$, while maintaining planarity. Hence by Theorem 1.2 $G_i$ is a bar visibility graph, as desired.

![Diagram](a) The subgraph $G_1$

![Diagram](b) The subgraph $G_2$

![Diagram](c) The subgraph $G_3$

![Diagram](d) The subgraph $G_4$

Figure 1: A decomposition of $T_{16}$ in which each subgraph is a bar visibility digraph.

Theorem 2.2 yields $b(T_{11}) \leq b(T_{12}) \leq 3$, which also follows from the construction in [2] for $b(T_{15}) \leq 3$. Proving Conjecture 1.7 that $b(T_{11}) = 3$ requires the lower bound, which will follow from our results in the next section. They also yield $b(T_{17}) \geq 4$, which with Lemma 1.9 and $b(T_{15}) \leq 3$ from [2] implies $b(T_{17}) = 4$. It remains open whether $b(T_{16})$ is 3 or 4.
3 \( b(T_n)/n \): Convergence and a Lower Bound

In this section we prove that \( b(T_n)/n \) converges as \( n \to \infty \) and derive a nontrivial lower bound on it that implies \( b(T_{11}) \geq 3 \) and \( b(T_{17}) \geq 4 \).

**Observation 3.1** ([2]). If \( G \) is a digraph with underlying graph \( \hat{G} \), then \( b(G) \geq b(\hat{G}) \).

**Proof.** A \( t \)-bar representation of \( G \) is also a \( t \)-bar representation of its underlying graph. \( \Box \)

Chang et al. [3] proved that the complete graph \( K_n \) has bar visibility number \( \lceil n/6 \rceil \) for \( n \geq 7 \); thus also \( b(T_n) \geq \lceil n/6 \rceil \) for \( n \geq 7 \). With Theorem 1.4, it follows that \( 1/6 \leq b(T_n)/n \leq 3/14 + O(1/n) \). To prove that \( b(T_n)/n \) converges, we need the following lemma, which also yields the upper bound of \( 3/14 \) from \( b(T_{15}) = 3 \).

**Lemma 3.2** ([2]). If \( b(T_l) = t \) for some \( l \), then \( b(T_n) \leq \frac{tn}{l-1} + O(1) \) for sufficiently large \( n \).

This lemma is based on the famous result of Wilson [13] implying that when \( n \) is sufficiently large, there exists \( m \) with \( n \leq m \leq n + c_l \) such that \( K_m \) decomposes into copies of \( K_l \) (called a Steiner system). In \( T_n \), the vertex sets of these copies induce copies of \( T_l \), which has a bar visibility representation using at most \( t \) bars per vertex. In the decomposition, each copy containing a vertex \( v \) uses \( l-1 \) of the edges incident to it in \( K_m \), so each vertex appears in \( (m-1)/(l-1) \) copies in the decomposition. We thus obtain a representation of \( T_m \) using at most \( t(m-1)/(l-1) \) bars per vertex, and then deleting bars for any \( m-n \) vertices does not introduce unwanted visibilities. Thus \( b(T_n) \leq tn/(l-1) + O(1) \).

**Theorem 3.3.** \( b(T_n)/n \) converges.

**Proof.** Let \( a = \lim \inf b(T_n)/n \) and \( b = \lim \sup b(T_n)/n \). If \( b(T_n)/n \) does not converge, then \( a < b \). By the definitions of \( \lim \inf \) and \( \lim \sup \), there is a positive integer \( l \) with \( l > \frac{3(a+b)}{b-a} + 1 \) such that \( b(T_l)/l = c \), where \( a \leq c < (a+b)/2 \). That is, \( b(T_l) = cl \). By Lemma 3.2, \( b(T_n) \leq \frac{tn}{l-1} + O(1) \) for sufficiently large \( n \). For sufficiently large \( n \), we then have

\[
\frac{b(T_n)}{n} \leq \frac{c \cdot l}{l-1} + o(1) < \frac{a+b}{2} + \frac{a+b}{2} \cdot \frac{1}{l-1} < \frac{a+b}{2} + \frac{a+b}{2} \cdot \frac{\frac{3(a+b)}{b-a} + 1}{1-1} = \frac{a+2b}{3} < b,
\]

which contradicts \( \lim \sup b(T_n)/n = b \). \( \Box \)

The lower bound uses an undirected graph associated with a \( t \)-bar visibility representation.
Definition 3.4. The derived graph of a \( t \)-bar visibility representation is a plane graph obtained by introducing an edge for each pair of bars that see each other along an unblocked channel (omitting loops) and then shrinks each bar to a point, keeping its edges.

Given a \( k \)-bar visibility representation of \( K_n \), the derived graph is a planar graph \( H \) with at most \( kn \) vertices and at least \( \binom{n}{2} \) edges. Euler’s Formula then requires \( \binom{n}{2} \leq 3kn - 6 \), which simplifies to \( k > \frac{(n - 1)}{6} \) since \( k \) is an integer. We improve on this lower bound by showing that at least \( k^2 - k \) of the edges in \( H \) duplicate visibilities and hence are wasted.

Theorem 3.5. The transitive tournament \( T_n \) on \( n \) vertices satisfies
\[
b(T_n) \geq \frac{3n - 5 - \sqrt{7n^2 - 28n + 25}}{2} > \frac{3 - \sqrt{7}}{2}n + \sqrt{7} - \frac{5}{2}.
\]
Therefore
\[
\lim b(T_n)/n \geq \frac{(3 - \sqrt{7})}{2} \approx 0.177124.
\]

Proof. Let \( k = b(T_n) \); by Lemma 1.9 \( k \leq n/2 \). Begin with a \( k \)-bar visibility representation of \( T_n \) giving \( k \) bars to each vertex. Index the vertices as \( v_1, \ldots, v_n \) so that all arcs are oriented from \( v_i \) to \( v_j \) with \( i < j \). With this vertex ordering, we can shift bars vertically so that each bar has vertical coordinate equal to the index of its assigned vertex. We can also combine the bars for \( v_1 \) into a single bar and those for \( v_n \) into a single bar and extend each to have the leftmost left endpoint and rightmost right endpoint among all bars.

The result is again a \( k \)-bar visibility representation of \( T_n \), using altogether \( kn - 2 \) + 2 bars. Its derived graph \( H \) is a planar graph with at most \( 3k(n - 2) \) edges. To derive a lower bound on \( |E(H)| \), we begin by selecting \( \binom{n}{2} \) edges consisting of one each from level \( i \) to level \( j \) for all \( i \) and \( j \) such that \( 1 \leq i < j \leq n \). Next we find extra edges.

For each \( j \) with \( 2 \leq j \leq k \), there are \( k \) bars at level \( j \). Because we have extended the bar at level 1 to be leftmost and rightmost, each bar at level \( j \) is seen by at least one bar from below. Hence the vertices representing \( v_j \) in \( H \) together have at least \( k \) edges entering them in \( H \). However, in \( T_n \) there are only \( j - 1 \) edges entering \( v_j \), and we have already counted that many. Hence there are at least \( k - j + 1 \) extra edges entering the vertices at level \( j \).

Summing over all such \( j \), there are at least \( k(k - 1)/2 \) extra edges entering these vertices.

A symmetric argument applies to edges leaving vertices near the top. For each \( i \) with \( n-k+1 \leq i \leq n-1 \), there are at least \( k - (n - i) \) extra edges leaving the vertices at level \( i \). Summing over all such \( i \), we have found at least \( k(k - 1)/2 \) extra edges from vertices at level \( i \) to higher vertices. Since \( k \leq n/2 \), these two sets of extra edges are disjoint, and \( H \) has at least \( k(k - 1) \) edges beyond those selected to represent actual edges of \( T_n \).

We now have the inequality
\[
\frac{n(n - 1)}{2} + k^2 - k \leq 3k(n - 2),
\]
which simplifies to \(2k^2 - (6n - 10)k + n^2 - n \leq 0\). Solving the quadratic inequality yields
\[
k \geq \frac{3n - 5 - \sqrt{7n^2 - 28n + 25}}{2} > \frac{3 - \sqrt{7}}{2}n + \sqrt{7} - \frac{5}{2}.
\]
This completes the proof. \(\square\)

**Corollary 3.6.** \(b(T_{11}) = b(T_{12}) = 3\) and \(b(T_{17}) = 4\).

**Proof.** Set \(n = 11\) and \(n = 17\) in the formula of Theorem 3.5 to obtain the lower bounds. We have \(b(T_{11}) \geq \frac{3\cdot 11 - 5 - \sqrt{7\cdot 11^2 - 28\cdot 11 + 25}}{2} = 14 - \sqrt{141} \approx 2.1257\) and \(b(T_{17}) \geq \frac{3\cdot 17 - 5 - \sqrt{7\cdot 17^2 - 28\cdot 17 + 25}}{2} = 23 - \sqrt{393} \approx 3.1758\), so \(b(T_{11}) \geq 3\) and \(b(T_{17}) \geq 4\). For the upper bounds, we cite Lemma 1.9 and the known values \(b(T_{13}) = b(T_{15}) = 3\) from Theorem 1.3. \(\square\)

### 4 Bar visibility numbers of a graph and its orientations

In this section, we provide an undirected graph \(\hat{G}\) having a orientation \(G\) with \(b(G) > 2b(\hat{G})\), thereby disproving Conjecture 1.8. Nevertheless, upper bounds in terms of \(b(\hat{G})\) do hold.

The tool we use for the construction is the interval number of a graph. A \(t\)-interval representation of a graph \(H\) assigns to each vertex \(v \in V(H)\) at most \(t\) intervals in \(\mathbb{R}\) so that \(uv \in E(H)\) if and only if some interval assigned to \(u\) intersects some interval assigned to \(v\). The interval number \(i(H)\) of \(H\) is the minimum \(t\) such that \(H\) has a \(t\)-interval representation. A \(t\)-interval representation of \(H\) has depth 2 if no point on the real line lies in intervals assigned to more than two vertices. The proof of the following theorem published originally in [9] was flawed, but a different and shorter proof has now been published by Guégan, Knauer, Rollin, and Ueckerdt [5].

**Theorem 4.1 (5).** Every planar graph has interval number at most 3, and this is sharp.

In [9], Scheinerman and West showed that the planar graph \(\hat{G}_1\) in Figure 2 has interval number 3. This graph arises by adding a pendent edge at each vertex in the larger part of the complete bipartite graph \(K_{2,9}\).

**Lemma 4.2.** If \(H'\) is a spanning subgraph of a triangle-free graph \(H\), then \(i(H') \leq i(H)\).

**Proof.** Let \(uv\) be an edge in \(H\). From an \(i(H)\)-interval representation of \(H\), we obtain an \(i(H)\)-interval representation of \(H - uv\). Consider intervals assigned to \(u\) and \(v\) that intersect. If one is contained in the other, delete the smaller interval from the representation. If they overlap, shorten each by deleting their intersection. Since \(H\) is triangle-free, the points in the intersection do not lie in intervals assigned to any other vertices, so the operation does not delete any other edges from the graph represented. The operation also does not add any edges, and the represented graph remains triangle-free. Thus iteratively deleting the edges of \(E(H) - E(H')\) in this way yields an \(i(H)\)-interval representation of \(H'\). \(\square\)
Figure 2: The graph \( \hat{G}_1 \) with interval number three.

An \( t \)-interval representation of a bipartite graph \( \hat{G} \) with intervals on a horizontal line can be processed from left to right, shifting intervals up or down as needed in becoming bars, to produce a \( t \)-bar visibility representation of any orientation \( G \) of \( \hat{G} \). When \( G \) orients all edges of \( \hat{G} \) from one part to the other, no bar can be placed between bars for two vertices of the other part, and hence the process can be reversed. This yields the following statement.

**Lemma 4.3** ([2]). If \( G \) is an orientation of a bipartite graph \( \hat{G} \), then \( b(G) \leq i(\hat{G}) \), with equality when all edges are oriented from one part to the other.

**Theorem 4.4.** There are digraphs with bar visibility number 3 whose underlying undirected graph is a bar visibility graph.

*Proof.* Let \( \hat{G} \) be the graph shown in Figure 3. This graph is obtained from a cycle with 18 vertices by adding two vertices whose neighborhoods are the even-indexed vertices of the cycle. Since \( \hat{G} \) is a 2-connected planar graph, \( b(\hat{G}) = 1 \), by Theorem 1.1.

Since \( \hat{G} \) is triangle-free and the graph \( \hat{G}_1 \) of Figure 2 is a spanning subgraph of \( \hat{G} \), Lemma 4.2 implies \( i(\hat{G}) \geq i(\hat{G}_1) = 3 \). On the other hand, Theorem 4.1 yields \( i(\hat{G}) \leq 3 \). Hence \( i(\hat{G}) = 3 \). Let \( G \) be an orientation of \( \hat{G} \) that orients all edges from one part to the other. By Lemma 4.3, \( b(G) = i(\hat{G}) = 3 \).

The same argument applies to such orientations of any 2-connected planar bipartite graph; here \( G \) and \( \hat{G} \) provide just one example. \( \square \)

**Theorem 4.4** disproves Conjecture 1.8. Nevertheless, results from [2] yield upper bounds on the bar visibility number of digraphs from the visibility number of the underlying graph.

**Lemma 4.5** ([2]). If \( G \) is a triangle-free planar digraph, then \( b(G) \leq 3 \), and this is sharp.

**Theorem 4.6** ([2]). Let \( G \) be an orientation of a planar graph \( \hat{G} \).

(i) \( b(G) \leq 4 \).

(ii) If \( \hat{G} \) is triangle-free or contains no subdivision of \( K_{2,3} \), then \( b(G) \leq 3 \).

(iii) If \( \hat{G} \) has girth at least 6, then \( b(G) \leq 2 \).
Theorem 4.7. If $G$ is an orientation of a triangle-free graph $\hat{G}$, then $b(G) \leq 3b(\hat{G})$, and this is sharp when $b(\hat{G}) = 1$.

Proof. Let $t = b(\hat{G})$. Because $\hat{G}$ is triangle-free, the derived graph $H$ of a $t$-bar visibility representation of $\hat{G}$ is also triangle-free. Orient the edges of $H$ according to the orientation in $G$, obtaining $\tilde{H}$. By Lemma 4.5, $\tilde{H}$ has a 3-bar visibility representation, which produces a $3t$-bar visibility representation of $G$. Hence $b(G) \leq 3t$.

Sharpness is achieved by be the graph $\hat{G}$ in Figure 3 with $G$ orienting all edges from one part to the other. The proof of Theorem 4.4 shows $b(G) = 3b(\hat{G})$. $\square$

Theorem 4.8. If $G$ is an orientation of a graph $\hat{G}$, then $b(G) \leq 4b(\hat{G})$.

Proof. As in the proof of Theorem 4.7 the claim follows from Theorem 4.6 $\square$

Theorems 4.7 and 4.8 rely on the properties in Lemma 4.5 of being planar and triangle-free. We do not obtain similar conclusions for graphs lacking subdivisions of $K_{2,3}$ or having girth at least 6, because the planar graph $H$ obtain from the $t$-bar visibility representation of such a digraph may have a subdivision of $K_{2,3}$ or a 4-cycle, respectively.

5 NP-Completeness

In this section, we show that recognition of digraphs with bar visibility number 2 is NP-complete, by reduction from the Hamiltonian cycle problem in 3-regular triangle-free graphs.

The digraphs with bar visibility number 1 are the bar visibility digraphs, characterized in Theorem 1.2 by whether a certain auxiliary digraph is planar and has no consistent cycle. There are linear time algorithms for testing planarity (see Section 2.7 in [8]). Also, a digraph has no consistent cycle if and only if it admits a topological sort, and there exist polynomial-time algorithms for finding a topological sort (see [7] for example). Hence there is a polynomial-time algorithm for recognition of digraphs with bar visibility number 1.
In order to study the recognition problem for bar visibility number 2, we define additional aspects of interval representations of graphs, which were introduced in the previous section. A $t$-interval representation of a graph $H$ is a displayed representation if for each vertex, some assigned interval contains an open interval not intersecting any other interval. If the union of a set of intervals in a $t$-interval representation is a single interval, then we say that these intervals appear contiguously. A graph $H$ is $t$-interval tight if $i(H) = t$ and every $t$-interval representation of $H$ assigns $t$ disjoint intervals to each vertex.

**Lemma 5.1** ([1]). The graph $K_{t^2+t-1,t+1}$ is $t$-interval tight. If $K_{t^2+t-1,t+1}$ is an induced subgraph of a graph $H$, then in every $t$-interval representation of $H$ the intervals for vertices of $K_{t^2+t-1,t+1}$ appear contiguously. If $u$ and $v$ are any specified vertices from opposite parts of $K_{m,n}$, then $K_{m,n}$ has a displayed $i(K_{m,n})$-interval representation in which $u$ and $v$ are assigned the leftmost and rightmost intervals in the representation, respectively.

We introduce concepts for $t$-bar visibility representations of digraphs analogous to those for $t$-interval representations of graphs. A $t$-bar visibility representation of a digraph $G$ has depth 2 if every channel intersects at most two bars in the representation. The representation is displayed if for each $v \in V(G)$ there is an unbounded channel that intersects some bar for $v$ and no other bar. If the derived graph of the representation is connected, then we say that the bars in the representation appear contiguously. A digraph $G$ is $t$-bar tight if $b(G) = t$ and in every $t$-bar visibility representation each vertex is assigned $t$ bars.

**Lemma 5.2** ([2]). A digraph $G$ has a depth-2 $t$-bar visibility representation if and only if its underlying graph $\hat{G}$ has a depth-2 $t$-interval representation.

Let $\tilde{K}_{m,n}$ denote an orientation of $K_{m,n}$ that orients all its edges from one part to the other. Both such orientations have the same bar visibility number.

**Lemma 5.3.** The digraph $\tilde{K}_{t^2+t-1,t+1}$ is $t$-bar tight. If $\tilde{K}_{t^2+t-1,t+1}$ is an induced subgraph of a digraph $G$, then in any $t$-bar visibility representation of $G$ the bars for vertices of $\tilde{K}_{t^2+t-1,t+1}$ appear contiguously. Furthermore, if $u$ and $v$ are any specified vertices from opposite parts of $\tilde{K}_{m,n}$, then $\tilde{K}_{m,n}$ has a displayed $b(K_{m,n})$-bar visibility representation in which $u$ and $v$ are assigned the leftmost and rightmost bars in the representation, respectively.

**Proof.** Since each vertex in $\tilde{K}_{m,n}$ is a source or a sink, all $b(\tilde{K}_{m,n})$-bar visibility representations have depth 2. Since $K_{m,n}$ is triangle-free, every $i(K_{m,n})$-interval representation has depth 2. By Lemma 5.2, $b(\tilde{K}_{m,n}) = i(K_{m,n})$. By Lemma 5.1, $\tilde{K}_{t^2+t-1,t+1}$ is $t$-bar tight. The claim about specified vertices follows by symmetry.

Our reduction involves transforming a 3-regular triangle-free graph $H$ into a digraph $G$ such that $H$ has a Hamiltonian cycle if and only if $b(G) = 2$. 

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Definition 5.4. Given a 3-regular triangle-free graph $H$, define a test digraph $f(H)$ as follows. Begin with an arbitrary orientation of $H$. Add three copies of $\tilde{K}_{5,3}$, denoted $H_1$, $H_2$, and $H_3$. Choose sinks $s_1 \in V(H_1)$ and $s_2 \in V(H_2)$, choose sources $t_2 \in V(H_2)$ and $t_3 \in V(H_3)$, and add the arcs $s_1 t_2$ and $s_2 t_3$. For each vertex $v \in V(H)$, add a copy $M_v$ of $\tilde{K}_{5,3}$ and an arc $vv'$ for one vertex $v' \in V(M_v)$. Also add the arcs $s_1 v$ and $v t_2$. Finally, for one special vertex $z \in V(H)$, add an arc from $z$ to each vertex in $H_2$ and $H_3$ (we already have $z t_2$; no need for an extra copy). Also add $s_1 x$ and $x t_2$ for each $x \in V(M_z)$.

Figure 4: The graph $f(H)$.

Figure 4 illustrates the test digraph $f(H)$ obtained from $H$. If $H$ has $n$ vertices, then $H$ has $3n/2$ edges, and $f(H)$ has $9n + 24$ vertices and $39n/2 + 78$ edges. The test digraph can be produced in time polynomial in the size of $H$.

For two disjoint subgraphs $D_1$ and $D_2$ of a (di)graph $D$, we denote by $D_1 + D_2$ the subgraph of $D$ induced by $V(D_1) \cup V(D_2)$. It contains $D_1$, $D_2$, and the arcs with endpoints in $V(D_1)$ and $V(D_2)$.

Lemma 5.5. If $H$ is a 3-regular triangle-free Hamiltonian graph, then $b(f(H)) = 2$. 

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Proof. Let $G = f(H)$. Since $G$ contains $\tilde{K}_{5,3}$ as a subgraph, $G$ is non-planar. Hence Theorem 1.2 yields $b(G) \geq 2$. Our approach is to develop special 2-bar visibility representations for subgraphs of $G$ and combine them when $H$ has a spanning cycle to obtain a 2-bar visibility representation of $G$.

**Step 1.** Construct displayed 2-bar visibility representations of $H_1$, $H_2$, and $H_3$ so that in $H_1$ vertex $s_1$ is assigned a rightmost bar, in $H_2$ vertices $s_2$ and $t_2$ are assigned a rightmost bar and a leftmost bar, respectively, and in $H_3$ vertex $t_3$ is assigned a leftmost bar. These representations are guaranteed by Lemma 5.3. Combine the representations of $H_1$, $H_2$, and $H_3$ as shown in Figure 5. This incorporates the arcs $s_1t_2$ and $s_2t_3$ without using any extra bar and also does not introduce any unwanted arc.

![Figure 5: Representation of $H_1 + H_2 + H_3$.](image)

**Step 2.** Since $H$ is Hamiltonian, it decomposes into a spanning cycle $C$ and a perfect matching. Construct a displayed representation of $C$ using one bar for each vertex other than $z$ and two bars for $z$ (the leftmost and rightmost bars). Lemma 5.3 provides a displayed 2-bar visibility representation of $M_z$ with $z'$ assigned the rightmost bar. Place this representation on the left of the representation of $C$ to incorporate the arc $zz'$ as shown in Figure 6.

![Figure 6: Representation of $C + M_z$.](image)

**Step 3.** Place the displayed representation of $C + M_z$ between the bars for $s_1$ and $t_2$ to incorporate all arcs from $s_1$ to vertices in $C$ and $M_z$, arcs from vertices in $C$ and $M_z$ to $t_2$. By extending the rightmost $z$-bar, we can represent all arcs from $z$ to vertices in $H_2$ and $H_3$. See Figure 7.

**Step 4.** For each arc $uw$ in the perfect matching $E(H) - E(C)$ with $z \notin \{u, w\}$, construct a displayed 2-bar visibility representation of $M_u + M_w + uw$ using two bars for each vertex in $M_u$ and $M_w$, one bar for $u$, and one bar for $w$, as shown in Figure 8. If $z \in \{u, w\}$, say...
Figure 7: Representation of $H_1 + H_2 + H_3 + C + M_z$.

$z = w$ by symmetry, then construct a displayed 2-bar visibility representation of $M_u + u$ and place it on the right side of the representation of $H_1 + H_2 + H_3 + C + M_z$ (Figure 7) to incorporate the arc $uz$ without introducing an extra bar for $z$.

Steps 1–4 complete a 2-bar visibility representation of $G$. □

Figure 8: Representation of $M_u + M_w + uw$.

**Definition 5.6.** A bar $A$ covers a bar $B$ (above or below) if the projection of $B$ on the horizontal axis is contained in the projection of $A$ on the axis.

**Lemma 5.7.** Let $H$ be a 3-regular triangle-free graph. If $b(f(H)) = 2$, then $H$ is Hamiltonian.

**Proof.** Let $G = f(H)$, and let $\Psi$ be a 2-bar visibility representation of $G$. For $F \subseteq G$, let $\Psi(F)$ denote the set of bars representing $V(F)$ in $\Psi$. By Lemma 5.3, each $H_i$ is 2-bar tight, and in any 2-bar visibility representation of $G$ the bars for vertices of $H_i$ appear contiguously. By left-right symmetry, we may assume that bars for $H_1$ are to the left of those for $H_2$ and $H_3$. Since there is only one arc $s_it_{i+1}$ joining $H_i$ and $H_{i+1}$, avoiding unwanted visibilities requires this arc to be represented by a visibility between the rightmost bar in $\Psi(H_i)$ and the leftmost bar in $\Psi(H_{i+1})$. Now $H_1 + H_2 + H_3$ must be represented in $\Psi$ as shown in Figure 7.
Note that $\Psi(H_1 + H_2 + H_3)$ is contiguous, and the bars establishing $s_1t_2$ and $s_2t_3$ are inner bars (not leftmost or rightmost bars) in $\Psi(H_1 + H_2 + H_3)$.

For each $v \in V(H) - \{z\}$, in order to avoid unwanted visibilities, we need two bars for $v$: one bar $\overline{v}$ between bars for $s_1$ and $t_2$ in $\Psi(H_1 + H_2 + H_3)$ to establish arcs $s_1v$ and $vt_2$, and one bar $\overline{v}$ outside the horizontal extent of $\Psi(H_1 + H_2 + H_3)$ to establish the arc $vv'$ (because $v'$ is not incident to any vertex in $H_i$). Two claims about the bars that can be seen by $\overline{v}$ and $\overline{v}$ will enable us to extract a spanning cycle in $H$ using the bars of the form $\overline{v}$.

Claim 1: If $v \in V(H) - \{z\}$, then $\overline{v}$ sees a bar for at most one vertex in $H$.

Suppose that $\overline{v}$ sees a bar $p$ for vertex $p \in V(H) - \{z\}$. Also $\overline{v}$ and $p$ must see bars for $v'$ and $p'$. These must be end bars in $\Psi(M_v)$ and $\Psi(M_p)$, since $v$ has no neighbor in $M_v$ other than $v'$, and similarly for $p$. One end of $\overline{v}$ and one end of a bar for $v'$ establish $vv'$, and similarly, one end of $p$ and one end of a bar for $p'$ establish $pp'$. The other ends of $\overline{v}$ and $p$ must be used to represent $vp$ (or $pv$). Thus $\overline{v}$ and $p$ are inner bars in $\Psi(M_v + M_p + vp)$ (or $\Psi(M(v) + M(p) + pv)$), which appears contiguously (see Figure 8).

Suppose that $\overline{v}$ also sees a bar for $q \in V(H)$. If $q \neq z$, then by the preceding paragraph $\overline{q}$ introduces an unwanted visibility with something in $M_v + vv'$ or $M_p + pp'$.

Hence we may assume $q = z$. Since $\overline{v}$ is an inner bar in $\Psi(M_v + M_p + vp)$, and among those vertices $z$ can only be adjacent to $v$ and $p$ (but not both, since $H$ has no triangle), the bar for $z$ seen by $\overline{v}$ must be covered by $\overline{v}$ and see only $\overline{v}$. Now the other bar $\hat{z}$ for $z$ must see bars for $z'$ and for two other neighbors of $z$ in $H$. Since $H$ is triangle-free, the inside bar $\overline{w}$ for some vertex $w \in V(H)$ must now be covered by $\hat{z}$. This obstructs the visibility between $\overline{w}$ and the bar for one of $s_1$ or $t_2$, preventing one of $s_1w$ and $wt_2$ from being established. The contradiction completes the proof of Claim 1.

Claim 2: If $v \in V(H) - \{z\}$, then $\overline{v}$ sees bars for at most two vertices in $H$.

Note that $\overline{v}$ cannot see both bars for a vertex $w \in H - \{z\}$, since only one bar for $w$ is in $\Psi(H_1 + H_2 + H_3)$. If $\overline{v}$ sees both bars for $z$, then $\overline{v}$ and the two bars for $z$ must occur as shown in Figure 8. The left bar for $z$ must see one end of $\overline{v}$, because we use this bar to see $z' \in M_z$, and no other vertex in $M_z$ is adjacent to $z$. The right bar for $z$ must see the other end of $\overline{v}$, because we use this $z$-bar to incorporate the arcs from $z$ to $V(H_2) \cup V(H_3)$. Since $H$ is 3-regular, and by Claim 1 $\overline{v}$ establishes at most one arc involving a neighbor of $v$ besides $z$, at least one neighbor $x$ of $v$ in $H$ still needs $\overline{v}$ to establish an arc involving $x$ and $v$. Because $H$ is triangle-free, $xz \notin E(H)$, so $\overline{v}$ will be blocked by $\overline{v}$ from establishing the arc $s_1x$ or the arc $xt_2$. Hence $\overline{v}$ cannot see both bars for one other vertex of $H$.

If $\overline{v}$ sees bars for three distinct vertices in $H$, then one of them (say $\overline{y}$) must be covered by $\overline{v}$ (because $H$ is triangle-free), which prevents $\overline{v}$ from establishing one of $\{s_1y, yt_2\}$.

Conclusion: Since $H$ is 3-regular, the consequence of Claims 1 and 2 is that for $v \in V(H) - \{z\}$, the inside bar $\overline{v}$ sees bars for exactly two neighbors of $v$ in $H$, and the outside
bar $v$ sees a bar for exactly one neighbor of $v$ in $H$. We can therefore follow the inside bars from left to right as a path, with both ends of this path of bars being bars for $z$. This produces a Hamiltonian cycle in $H$.

\textbf{Lemma 5.8 ([11]).} Determining whether a 3-regular triangle-free graph contains a Hamilton cycle is NP-complete.

\textbf{Theorem 5.9.} Determining whether a digraph has the bar visibility number 2 is NP-complete.

\textit{Proof.} From Lemmas 5.5 and 5.7, $b(f(H)) = 2$ if and only if $H$ is Hamiltonian. The claim then follows from Lemma 5.8.

Furthermore, testing $b(G) \leq t$ for digraphs is NP-complete for any fixed $t$ with $t \geq 2$. By constructing a digraph $\tilde{G}$ from $G$ such that $\tilde{G}$ has a $t$-bar visibility representation if and only if $G$ has a $(t - 1)$-bar visibility representation, one can reduce $(t - 1)$-bar visibility representation to $t$-bar visibility representation. The claim then follows from Theorem 5.9.

Let $G$ be an arbitrary digraph, and let $\tilde{G}$ be a digraph whose underlying graph is obtained from $G$ by adding three copies of $K_{t^2+t-1,t+1}$ for each vertex $v \in V(G)$, with one edge joining $v$ to the central copy and one edge joining each of the two other copies to the central copy, their endpoints in the central copy being adjacent (see Figure 5 and Theorem 2 of [11]). In $\tilde{G}$, edges in each copy of $K_{t^2+t-1,t+1}$ are oriented from one part to the other, and other edges are oriented arbitrarily.

\textbf{References}

[1] B. Alspach, The wonderful Walecki construction, \textit{Bull. Inst. Combin. Appl.} \textbf{52} (2008), 7–20.

[2] M. Axenovich, A. Beveridge, J.P. Hutchinson, and D.B. West, Visibility number of directed graphs, \textit{SIAM J. Discrete Math.} \textbf{27}(3) (2013), 1429–1449.

[3] Y.W. Chang, J.P. Hutchinson, M.S. Jacobson, J. Lehel, and D.B. West, The bar visibility number of a graph, \textit{SIAM J. Discrete Math.} \textbf{18}(3) (2004), 462–471.

[4] S.K. Ghosh, \textit{Visibility algorithms in the plane}, (Cambridge University Press, 2007).
[5] G. Guégan, K. Knauer, J. Rollin, and T. Ueckerdt, The interval number of a planar graph is at most three, *J. Combin. Theory (B)* **146** (2021), 61–67.

[6] J.P. Hutchinson, A note on rectilinear and polar-visibility graphs, *Discrete Appl. Math.* **148** (2005), 263–272.

[7] A.B. Kahn, Topological sorting of large networks, *Communications of the ACM* **5** (11) (1962), 558–562.

[8] B. Mohar and C. Thomassen, *Graphs on Surfaces*, (The John Hopkins University Press, Baltimore, 2001).

[9] E.R. Scheinerman and D.B. West, The interval number of a planar graph: Three intervals suffice, *J. Combin. Theory (B)* **35**(3) (1983), 224–239.

[10] R. Tamassia and I.G. Tollis, A unified approach to visibility representations of planar graphs, *Discrete Comput. Geom.* **1** (1986), 321–341.

[11] D.B. West and D.B. Shmoys, Recognizing graphs with fixed interval number is NP-complete, *Discrete Math.* **8** (1984), 295–305.

[12] D.B. West and J.I. Wise, Bar visibility numbers for hypercubes and outerplanar digraphs, *Graphs and Combin.* **33** (2017), 221–231.

[13] R.M. Wilson, An existence theory for pairwise balanced designs, iii: proof of the existence conjectures, *J. Combin. Theory (A)* **18** (1975), 71–79.

[14] S.K. Wismath, Characterizing bar line-of-sight graphs, *Proc. 1st ACM Symp. Comput. Geom., Baltimore, MD, 1985*, (ACM, 1985), 147–152.

[15] S.K. Wismath, Bar-representable visibility graphs and a related network flow problem, Ph.D. thesis, University of British Columbia, Vancouver, BC, Canada, 1989.