ON CALABI–YAU THREEFOLDS ASSOCIATED TO A WEB OF QUADRICS

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ABSTRACT. We study the geometry of the birational map between an intersection of a web of quadrics in \( \mathbb{P}_7 \) that contains a plane and the double octic branched along the discriminant of the web.

INTRODUCTION

It is a classical fact that there is a correspondence between the base locus \( S \) of a net of quadrics in \( \mathbb{P}_5 \) and the double sextic branched along the discriminant of the net. The latter is the moduli space of certain rank-2 sheaves on the former (see [26]). Moreover, if the base locus contains a line \( L \), then the two surfaces are birational. More general conditions for the existence of a birational map were given by Nikulin and Madonna (see [22] and its sequels).

A precise description of the birational map between the surface \( S \) and the double sextic can be found in [7]. In this case, \( S \) is the blow-up of the double sextic along rank-4 quadrics in the net. The latter results from the fact that the map defined by the linear system \( |2H - 3L - \sum L_i| \), where \( H \) is the hyperplane section in \( \mathbb{P}_5 \) and \( L_i \) are the lines on \( S \) that meet \( L \) (see [7, Thm 3.3]), is hyperelliptic. Moreover, one can show that the birational map factors through another K3 surface (a space quartic that contains a twisted cubic) and its geometry (e.g. the contracted curves) is governed by the behaviour of the lines \( L_i \). The birational map between the two surfaces can be also constructed via an incidence variety ([18]). The latter construction was adopted in [24] to the case of a generic web \( W = \text{span}(Q_0, Q_1, Q_2, Q_3) \) in \( \mathcal{O}_{\mathbb{P}_7}(2) \), such that its base locus \( X_{16} \) contains a fixed plane \( \Pi \). More precisely, using Bertini-type and computer algebra arguments, Michałek proved that if we put \( S_8 \) (resp. \( X_8 \)) to denote the discriminant surface of the web \( W \) (resp. the double cover of the web \( W \) branched along the discriminant surface \( S_8 \)) and \( W \) is generic enough, then the Calabi-Yau varieties \( X_{16} \) and \( X_8 \) are birational. However, the approach of [24] gives neither explicit sufficient condition for birationality of \( X_{16} \) and \( X_8 \) nor a method to study the geometry of the map.

In this paper, for the matrices \( q_0, \ldots, q_3 \) that give the quadrics \( Q_0, \ldots, Q_3 \in \mathcal{O}_{\mathbb{P}_7}(2) \) such that \( Q_0 \cap \ldots \cap Q_3 \) contains a plane \( \Pi \) we define two auxiliary matrices \( a, A \) and use them to obtain a surface \( B \subset \mathbb{P}_4 \) and a three-dimensional quintic \( X_5 \subset \mathbb{P}_4 \) that contains the surface \( B \). Then, under the assumptions

[A1]: \( X_{16} \) has exactly 10 singularities on \( \Pi \) and is smooth away from the plane \( \Pi \),
[A2]: no 4 singular points of \( X_{16} \) lie on a line,
[A3]: the set \( \{ x \in B : \text{rank}(A(x)) \leq 2 \} \) consists of 46 points ,
[A4]: the discriminant surface \( S_8 \) has only isolated singularities,

we show that there is a birational map \( X_{16} \dashrightarrow X_8 \) that factors as the composition

\[
X_{16} \xrightarrow{\sigma^{-1}} \tilde{X}_{16} \xrightarrow{\pi} X_5 \xrightarrow{\psi^{-1}} \tilde{X}_5 \xrightarrow{\phi} X_8 ,
\]

where \( \sigma, \psi \) are certain blow-ups, \( \pi \) is resolution of the projection from \( \Pi \) and \( \tilde{\phi} \) is obtained via Stein factorization from restriction of the so-called Bordiga conic bundle to the blow-up of the quintic

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In particular, under the above assumptions \( B \) is the so-called (smooth) Bordiga sextic. Bordiga sextic and Bordiga conic bundle have been studied already by the Italian school (see [30], [2] and the bibliography in the latter), so the above factorization enables us to give a precise description of the geometry of the birational map in question. In particular, we are able to show that the map has no two-dimensional fibers, describe the contracted curves (Thm 3.6), classify the singularities of the discriminant of the web (and prove that all of them admit a small resolution) and give an upper bound of their number (see Cor. 4.7).

Our considerations yield that the assumptions \([A1], \ldots, [A4]\) are fulfilled by a generic web of quadrics such that its base locus contains a fixed plane. Careful analysis of our arguments shows that one can assume less in order to obtain a birational map \( X_{16} \to X_8 \), but once one omits the above assumptions the geometry of the birational map changes. For instance, if \([A2]\) is not satisfied, the surface in \( \mathbb{P}_4 \) one obtains as a result of the projection is no longer the Bordiga surface, without \([A1]\) (resp. \([A3]\)) the threefold \( X_{16} \) (resp. \( X_5 \)) has higher singularities etc. Still, the main strategy we use can be applied to study those degenerations - we do not follow this path in order to maintain the paper compact.

Our motivation is twofold. First, it seems a natural question to ask under what assumptions a three-dimensional Calabi-Yau analogue of the well-known result on K3 surfaces holds. Second, we obtain a very precise description of a map between certain Calabi-Yau manifolds that (with help of a computer algebra system applied to a given example) could be of interest on its own, for instance as a source of examples of small resolutions.

The paper is organized as follows. In Sect. 1 we study the singularities of the threefold \( X_{16} \) and Hodge numbers of its blow-up \( \tilde{X}_{16} \). Sect. 2 is devoted to properties of projection from the plane \( \Pi \). In the next section we describe the behaviour of the restriction of Bordiga conic bundle to the blow-up of the quintic \( X_5 \) we defined in Sect. 2. Finally, the last part (Sect. 4) contains a classification of singularities of the discriminant of the web and proof of main results of the paper.

**Convention:** In this note we work over the base field \( \mathbb{C} \). By an abuse of notation we use the same symbol to denote a homogeneous polynomial and its zero–set in projective space.

## 1. Singularities of the intersection of four quadrics and a small resolution

Let \( Q_0, Q_1, Q_2, Q_3 \subset \mathbb{P}_7 \) be linearly independent quadrics that contain a (fixed) plane \( \Pi \) and let

\[
X_{16} := Q_0 \cap Q_1 \cap Q_2 \cap Q_3
\]

be their (scheme-theoretic) intersection.

Without loss of generality we can assume that \( \Pi := \{(x_0 : \ldots : x_7) : x_0 = \ldots = x_4 = 0\} \), which implies that each \( Q_i \) is given by the matrix

\[
q_i = \begin{bmatrix}
q_i & b_i^T \\
0 & 0 & 0 \\
b_i & 0 & 0 \\
2 & 0 & 0 
\end{bmatrix}
\]
where \( \underline{q}_i \) is a \( 5 \times 5 \) matrix, \( b_i := \begin{bmatrix} l_i \\ m_i \\ n_i \end{bmatrix} \) and \( l_i, m_i, n_i \in \mathbb{C}^5 \) are row-vectors. Moreover, in order to simplify our notation we put \( b(y) := \sum_i y_i b_i \) and
\[
c(x_5, x_6, x_7) := x_5 \begin{bmatrix} l_0^T & l_1^T & l_2^T & l_3^T \end{bmatrix} + x_6 \begin{bmatrix} m_0^T & m_1^T & m_2^T & m_3^T \end{bmatrix} + x_7 \begin{bmatrix} n_0^T & n_1^T & n_2^T & n_3^T \end{bmatrix}.
\]

We have (compare \([24\text{, Prop. 1.8}]\))

**Lemma 1.1.**

\[
sing(X_{16}) \cap \Pi = \{(0 : \cdots : x_5 : x_6 : x_7) : \text{rank}(c(x_5, x_6, x_7)) \leq 3\}
\]

In particular, if the set \( sing(X_{16}) \cap \Pi \) is finite, then it consists of at most 10 points.

**Proof.** Observe that the intersection \( X_{16} \) is singular at a point \( x \) iff the differentials \( dQ_i(x) = (q_i(x))^T \) of quadratic forms \( Q_i \) at \( x \) are linearly dependent, that is if there exists \( (y_0 : \cdots : y_3) \in \mathbb{P}_3 \) such that
\[
\sum_{i=0}^{3} y_i q_i(x) = 0.
\]

For \( x = (0 : \cdots : 0 : x_5 : x_6 : x_7) \in \Pi \) the above condition reduces to \( \sum y_i (x_5 l_i^T + x_6 m_i^T + x_7 n_i^T) = 0 \). We can rewrite the latter as
\[
b(y)^T (x_5, x_6, x_7)^T = 0.
\]

For a fixed \( y \in \mathbb{P}_3 \) there exists a point in \( \Pi \) satisfying the above relation iff \( \text{rank}(b(y)) \leq 2 \).

Moreover, for every \((x_5, x_6, x_7)\) and \( y \) we have
\[
c(x_5, x_6, x_7)y = b(y)^T (x_5, x_6, x_7)^T.
\]

Therefore, \((0, \ldots, 0, x_5, x_6, x_7)\) is a singularity of \( X_{16} \) iff there exist \( y \in \mathbb{P}_3 \) such that \( c(x_5, x_6, x_7)y = 0 \) or equivalently
\[
\text{rank}(c(x_5, x_6, x_7)) \leq 3.
\]

Finally, suppose that the set \( sing(X_{16}) \cap \Pi \) is finite. Then, the number of its elements does not exceed the degree of the determinantal variety of \( 4 \times 5 \) matrices of rank \( \leq 3 \). The latter is 10 by \([14\text{, Ex. 14.4.14}]\) (see also \([19\text{, 27}]\)). \( \square \)

From now on we make the following **assumption:**

[\( A1 \):] \( X_{16} \) has exactly 10 singularities on \( \Pi \) and is smooth away from the plane \( \Pi \),

As an immediate consequence of [\( A1 \)] we obtain

\[\text{Remark 1.2.}\] For each \( y \in \mathbb{P}_3 \) we have \( \text{rank}(b(y)) \geq 2 \). Indeed, we assumed that \( X_{16} \) has only isolated singularities on \( \Pi \). Therefore, for a fixed \( y \in \mathbb{P}_3 \), there exists at most one point in \( \Pi \) satisfying the relation \( (1) \), so \( \text{rank}(b(y)) \) cannot be lower than 2.

**Lemma 1.1** and \([6]\) support the following conjecture.

\[\text{Conjecture 1.3.}\] a) A nodal complete intersection of four quadrics in \( \mathbb{P}_7 \) with at most nine nodes is \( \mathbb{Q} \)-factorial.

b) A nodal complete intersection of four quadrics in \( \mathbb{P}_7 \) with exactly ten nodes that is not \( \mathbb{Q} \)-factorial contains a plane \( \Pi \).
Lemma 1.4. Suppose that \([A1]\) holds.

a) The ideal of the set \(\text{sing}(X_{16}) \cap \Pi\) is generated by all \(4 \times 4\) minors of the matrix \(c(x_5, x_6, x_7)\). In particular, the ideal in question contains no cubics.

b) For each \(x \in \text{sing}(X_{16})\) there exists precisely one quadric in \(W\) such that \(x\) is its singularity.

c) There exist three quadrics in the web \(W\) that meet transversally.

d) The set \(\{ y \in \mathbb{P}_3 : \text{rank}(b(y)) = 2\}\) consists of precisely 10 points.

Proof. a) Recall that the determinantal variety \(P(\mathcal{V}_{10}) \subset \mathbb{P}_{19}\) given by the condition

\[
\text{rank}
\begin{bmatrix}
  z_0 & \ldots & z_4 \\
  & \vdots & \\
  z_{15} & \ldots & z_{19}
\end{bmatrix}
\leq 3
\]

has dimension 17 and degree 10. Moreover, the ideal generated by \(4\) minors of the above matrix is perfect by \([12]\) (see also \([5, \text{Cor. 2.8}]\)). Therefore, the ring \(\mathbb{C}[z_0, \ldots, z_{19}] / (I(\mathcal{V}_{10}) + I(\mathcal{P}))\) is 1-dimensional Cohen-Macaulay and the ideal \(I(\mathcal{V}_{10}) + I(\mathcal{P})\) coincides with its radical.

b) The plane \(P(\mathcal{P}) \subset \mathbb{P}_{19}\) meets the variety \(P(\mathcal{V}_{10})\) in exactly ten points, so none of the latter belongs to \(\text{sing}(P(\mathcal{V}_{10}))\). But, as one can check by direct computation (see also \([30]\)), all points of \(\mathcal{V}_{10}\) that satisfy the condition

\[
\text{rank}
\begin{bmatrix}
  z_0 & \ldots & z_4 \\
  & \vdots & \\
  z_{15} & \ldots & z_{19}
\end{bmatrix}
\leq 2
\]

are its singularities. The latter implies that

\[
\forall x \in \text{sing}(X_{16}) \quad \text{rank}(c(x_5, x_6, x_7)) = 3.
\]

Consequently, there exists precisely one \(y \in \mathbb{P}_3\) that lies in the kernel of the matrix \(c(x_5, x_6, x_7)\). By \([2]\), the latter is equivalent to the condition \((0 : \ldots : x_5 : x_6 : x_7) \in \text{sing}(Q(y))\). In this way we have shown the claim b).

c) follows from b) by standard arguments.

d) Suppose that a point \(y \in \mathbb{P}_3\) satisfies the relation (11) for two various points in \(\Pi\). Then, the line spanned by both points in question lies in the kernel of the matrix \(b(y)\) and \(\text{rank}(b(y)) < 2\), which is impossible by Remark [12]. In this way we have shown that

\[
\# \{ y \in \mathbb{P}_3 : \text{rank}(b(y)) = 2 \} \geq \# \text{sing}(X_{16}).
\]

The other inequality has been shown in the proof of part b).

Lemma 1.5. Assume that \(Z_P = \{ f(y_1, \ldots, y_4) = 0 \} \subset \mathbb{C}^4\) is a three-dimensional isolated hypersurface singularity that contains the germ of the plane \(\{ y_1 = y_2 = 0 \}\). If the ideal

\[
\langle \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_4}, f, y_1, y_2 \rangle \subset \mathcal{O}_{\mathbb{C}^4, P}
\]

is maximal, then \(Z_P\) is a node.

Proof. We are to show that hessian of \(f\) in \(P\) does not vanish. Let \(f_1, f_2 \in \mathcal{O}_{\mathbb{C}^4, P}\) satisfy the condition \(f = y_1 \cdot f_1 + y_2 \cdot f_2\). By direct computation we have

\[
\langle f_1, f_2, y_1, y_2 \rangle = \langle y_1, y_2, y_3, y_4 \rangle.
\]
Consider the linear parts $f_i^{(1)} = \sum_{j=1}^4 f_{i,j}^{(1)} y_j$ for $i = 1, 2$. Then hessian of $f$ in $P$ is given by

$$
\det \begin{bmatrix}
\frac{f_{1,1}^{(1)}}{f_{1,2}^{(1)} + f_{1,3}^{(1)}} & \frac{f_{1,1}^{(1)} f_{1,2}^{(1)}}{f_{1,2}^{(1)}} & \frac{f_{1,1}^{(1)} f_{1,3}^{(1)}}{f_{1,2}^{(1)}} & \frac{f_{1,1}^{(1)} f_{1,4}^{(1)}}{f_{1,2}^{(1)}} \\
\frac{f_{2,1}^{(1)}}{f_{2,2}^{(1)} + f_{2,3}^{(1)}} & \frac{f_{2,1}^{(1)} f_{2,2}^{(1)}}{f_{2,2}^{(1)}} & \frac{f_{2,1}^{(1)} f_{2,3}^{(1)}}{f_{2,2}^{(1)}} & \frac{f_{2,1}^{(1)} f_{2,4}^{(1)}}{f_{2,2}^{(1)}} \\
\frac{f_{3,1}^{(1)}}{f_{3,2}^{(1)} + f_{3,3}^{(1)}} & \frac{f_{3,1}^{(1)} f_{3,2}^{(1)}}{f_{3,2}^{(1)}} & \frac{f_{3,1}^{(1)} f_{3,3}^{(1)}}{f_{3,2}^{(1)}} & 0 \\
\frac{f_{4,1}^{(1)}}{f_{4,2}^{(1)} + f_{4,3}^{(1)}} & \frac{f_{4,1}^{(1)} f_{4,2}^{(1)}}{f_{4,2}^{(1)}} & \frac{f_{4,1}^{(1)} f_{4,3}^{(1)}}{f_{4,2}^{(1)}} & 0
\end{bmatrix}
= - \det \begin{bmatrix}
\frac{f_{1,1}^{(1)} f_{1,2}^{(1)}}{f_{1,2}^{(1)}} & \frac{f_{1,1}^{(1)} f_{1,3}^{(1)}}{f_{1,2}^{(1)}} & \frac{f_{1,1}^{(1)} f_{1,4}^{(1)}}{f_{1,2}^{(1)}} \\
\frac{f_{2,1}^{(1)} f_{2,2}^{(1)}}{f_{2,2}^{(1)}} & \frac{f_{2,1}^{(1)} f_{2,3}^{(1)}}{f_{2,2}^{(1)}} & \frac{f_{2,1}^{(1)} f_{2,4}^{(1)}}{f_{2,2}^{(1)}} \\
\frac{f_{3,1}^{(1)} f_{3,2}^{(1)}}{f_{3,2}^{(1)}} & \frac{f_{3,1}^{(1)} f_{3,3}^{(1)}}{f_{3,2}^{(1)}} & 0 \\
\frac{f_{4,1}^{(1)} f_{4,2}^{(1)}}{f_{4,2}^{(1)}} & \frac{f_{4,1}^{(1)} f_{4,3}^{(1)}}{f_{4,2}^{(1)}} & 0
\end{bmatrix}^2.
$$

To show that the right-hand side of the latter equality does not vanish put $y_1 = y_2 = 0$ in (1).

**Lemma 1.6.** If [A1] holds, then all singularities of $X_{16}$ are nodes (i.e. $A_1$ points).

*Proof.* Without loss of generality we can assume that all singularities of $X_{16}$ lie in the affine chart $x_7 \neq 0$ and the variety $Y := Q_0 \cap Q_1 \cap Q_2$ is smooth (see Lemma 1.4). By abuse of notation we use the same symbol to denote a quadric and the dehomogenization of its equation (i.e. $x_7 = 1$).

Observe that putting $x_0 = x_1 = \cdots = x_4 = 0$ in the ideal $(\wedge^4 \operatorname{Jac}(Q_0, \ldots, Q_3), Q_0, \ldots, Q_3)$ we get the ideal in $\mathbb{C}[x_5, x_6]$ generated by $4 \times 4$ minors of the matrix $c(x_5, x_6, 1)$. In particular, (see Lemma 1.1) we can compute the dimension of the $\mathbb{C}$-vector space

$$
\dim(\mathbb{C}[x_0, \ldots, x_6]/(\wedge^4 \operatorname{Jac}(Q_0, \ldots, Q_3), Q_0, \ldots, Q_3, x_0, \ldots, x_4)) = 10.
$$

Moreover, the assumption [A1] yields an isomorphism

$$
\bigoplus_{P \in \operatorname{sing}(X_{16})} \mathcal{O}_{C^7, P}/(\wedge^4 \operatorname{Jac}(Q_0, \ldots, Q_3), Q_0, \ldots, Q_3, x_0, \ldots, x_4) \simeq \mathbb{C}[x_0, \ldots, x_6]/(\wedge^4 \operatorname{Jac}(Q_0, \ldots, Q_3), Q_0, \ldots, Q_3, x_0, \ldots, x_4)
$$

Therefore, for each $P \in \operatorname{sing}(X_{16})$, we have

$$
(5) \quad \dim(\mathcal{O}_{C^7, P}/(\wedge^4 \operatorname{Jac}(Q_0, \ldots, Q_3), Q_0, \ldots, Q_3, x_0, \ldots, x_4) \mathcal{O}_{C^7, P}) = 1.
$$

Fix a point $P \in \operatorname{sing}(X_{16})$ and assume that the germ of $Y$ near $P$ can be (analytically) parametrized as the graph of a map $(x_4(x_0, \ldots, x_3), \ldots, x_6(x_0, \ldots, x_3))$. Let $\tilde{Q}_3$ be the composition of the above parametrization with (the dehomogenized equation of) the quadric $Q_3$. By direct computation, (5) implies that the ideal

$$
(\tilde{Q}_3, \frac{\partial \tilde{Q}_3}{\partial x_0}, \ldots, \frac{\partial \tilde{Q}_3}{\partial x_3}) + I(\Pi) \subset \mathcal{O}_{Y, P}
$$

is maximal. By Lemma 1.5 the point $P$ is an $A_1$ singularity of $X_{16}$. \qed

We introduce the following notation:

$$
(6) \quad \sigma : \tilde{X}_{16} \to X_{16}
$$

is the blow-up of $X_{16}$ along the plane $\Pi$ and $S$ stands for the strict transform of the plane $\Pi$ under the blow-up $\sigma$. The variety $\tilde{X}_{16}$ is smooth and the blow-up in question replaces the 10 nodes with 10 disjoint smooth rational curves

$$
(7) \quad E_1, \ldots, E_{10} \subset S.
$$

**Convention:** In the sequel, we shall identify smooth points of $X_{16}$ with their images in $\tilde{X}_{16}$, i.e. write $P$ instead of $\sigma(P)$ whenever it leads to no ambiguity.

In the next section we will use the following lemma.
Lemma 1.7. The variety $\tilde{X}_{16}$ is a projective Calabi–Yau manifold with the following Hodge diamond

\[
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & 2 & 0 & \\
1 & 56 & 56 & 1 \\
0 & 2 & 0 & \\
& & & 1
\end{array}
\]

Proof. By Lemma 1.4b we can assume that $Y = Q_0 \cap Q_1 \cap Q_2$ is smooth. Let $\sigma : \tilde{Y} \to Y$ be the blow-up of $Y$ along $\Pi$ with exceptional divisor $E$. We have

$$\sigma_* \mathcal{O}_Y(kE) = \mathcal{O}_Y,$$
$$R^1 \sigma_* \mathcal{O}_Y(E) = 0,$$
$$R^1 \sigma_* \mathcal{O}_Y(2E) = \mathcal{O}_\Pi(-1).$$

Since $\mathcal{O}_Y(\tilde{X}_{16}) = \sigma^* \mathcal{O}_Y(X) \otimes \mathcal{O}_\Pi(-E)$ using the projection formula we get

$$\sigma_* \mathcal{O}_Y(-k\tilde{X}_{16}) = \mathcal{O}_Y(-kX),$$
$$R^1 \sigma_* \mathcal{O}_Y(-\tilde{X}_{16}) = 0,$$
$$R^1 \sigma_* \mathcal{O}_Y(-2\tilde{X}_{16}) = \mathcal{O}_\Pi(-5).$$

The Leray spectral sequence and the Kodaira vanishing imply

$$H^i(\mathcal{O}_Y(-\tilde{X}_{16})) = 0 \text{ for } i \leq 3, \quad H^4(\mathcal{O}_Y(-\tilde{X}_{16})) \cong \mathbb{C}.$$

Since

$$H^i(\mathcal{O}_Y(-2Y)) = 0, \text{ for } i \leq 3,$$
$$H^4(\mathcal{O}_Y(-2X)) \cong H^0(\mathcal{O}_Y(2)) \cong \mathbb{C}^{33},$$
$$H^4(\mathcal{O}_Y(-2\tilde{X}_{16})) \cong H^0(\mathcal{O}_Y(\tilde{X}_{16})) \cong H^0(\mathcal{O}_Y(X) \otimes I(\Pi)) \cong \mathbb{C}^{27},$$
$$H^i(R^1 \sigma_* (\mathcal{O}_Y(-2\tilde{X}_{16}))) = 0, \text{ for } i = 0, 1$$
$$H^2(R^1 \sigma_* (\mathcal{O}_Y(-2\tilde{X}_{16}))) \cong H^2(\mathcal{O}_\Pi(-5)) \cong \mathbb{C}^6$$

the Leray spectral sequence implies

$$H^i(\mathcal{O}_Y(-2\tilde{X}_{16})) = 0, \text{ for } i \leq 3$$

and consequently

$$H^i(\mathcal{N}^{\nu}_{\tilde{X}_{16}|\tilde{Y}}) = 0 \text{ for } i \leq 2.$$\n
From the exact sequence

$$0 \to \sigma^* \Omega^1_Y \to \Omega^1_{\tilde{Y}} \to \Omega^1_{E/\Pi} \to 0$$

we get

$$\sigma_* \Omega^1_Y = \Omega^1_{\tilde{Y}}, \quad R^1 \sigma_* \Omega^1_Y = \mathcal{O}_\Pi$$

and so

$$H^1 \Omega^1_{\tilde{Y}} \cong \mathbb{C}^2.$$\n
Similarly, the exact sequence

$$0 \to \sigma^* (\Omega^1_Y(-X)) \otimes \mathcal{O}_\tilde{Y}(E) \to \Omega^1_{\tilde{Y}}(-\tilde{X}_{16}) \to \Omega^1_{E/\Pi}(-1) \otimes \sigma^* \mathcal{O}_Y(-X) \to 0$$

implies

$$\sigma_* \Omega^1_Y(-\tilde{X}_{16}) \cong \Omega^1_Y(-X) \text{ and } R^1 \sigma_* \Omega^1_Y(-\tilde{X}_{16}) \cong \mathcal{N}^{\nu}_{\Pi|Y} \otimes \mathcal{O}_Y(-X).$$
Twisting the exact sequence
\[ 0 \to N_{\Pi|Y} \to N_{\Pi|P^7} \to N_{Y|P^7}|\Pi \to 0 \]
with \( \mathcal{O}_Y(-X) \cong \mathcal{O}_Y(-2) \) we get
\[ H^0 N_{\Pi|Y} \otimes \mathcal{O}_Y(-X) = H^0 N_{\Pi|Y} \otimes \mathcal{O}_Y(-X) = 0 \quad \text{and} \quad H^1 N_{\Pi|Y} \otimes \mathcal{O}_Y(-X). \]
Since \( H^3(\Omega^1_Y(-X)) \cong H^1(\mathcal{T}_Y) = 36 \), while \( H^3(\Omega^1_X(-X_{16})) \cong H^1(\mathcal{T}_Y) = 33 \) the Leray spectral sequence yields
\[ H^3\Omega^1_X(-X_{16}) = 0, \text{ for } i = 0, 1, 2. \]
From the exact sequence
\[ 0 \to \Omega^1_Y(-X_{16}) \to \Omega^1_Y \to \Omega^1_Y \otimes \mathcal{O}_{X_{16}} \to 0 \]
we conclude
\[ H^1(\Omega^1_Y \otimes \mathcal{O}_{X_{16}}) \cong H^1\Omega^1_Y \cong \mathbb{C}^2. \]
Finally, the exact sequence
\[ 0 \to N_{X_{16}|Y} \to \Omega^1_Y \otimes \mathcal{O}_{X_{16}} \to \Omega^1_{X_{16}} \to 0 \]
yields
\[ H^1\Omega^1_{X_{16}} \cong H^1(\Omega^1_Y \otimes \mathcal{O}_{X_{16}}) \cong \mathbb{C}^2. \]
The standard computation with help of [14, Example 3.2.12] yields that the Euler number \( e(\tilde{X}_{16}) = -108 \) (see also [24, Prop. 1.14]), so we can compute \( h^{1,2}(\tilde{X}_{16}) \).

As another consequence of [A1] we obtain the following simple observation.

**Remark 1.8.** The web \( W \) contains no rank-4 quadrics.

**Proof.** Suppose that \( Q_0 \in W \) is a rank-4 quadric. Then it is a cone through the 3-space \( \text{sing}(Q_0) \) over a smooth quadric in \( \mathbb{P}^3 \). The latter contains no planes, so the 3-space \( \text{sing}(Q_0) \) and the plane \( \Pi \) meet. On the other hand, since each point in \( \text{sing}(Q_0) \cap Q_1 \cap Q_2 \cap Q_3 \) is a singularity of \( X_{16} \), the assumption [A1] implies that \( \text{sing}(Q_0) \) meets \( \Pi \) in exactly one point \( P \in \text{sing}(X_{16}) \). Moreover, we have \( \text{sing}(Q_0) \cap Q_1 \cap Q_2 \cap Q_3 = \{ P \} \).

Lemma [13, b] yields that the quadrics \( Q_1, Q_2, Q_3 \) are smooth in \( P \). By Bézout the intersection multiplicity of \( \text{sing}(Q_0), Q_1, Q_2, Q_3 \) in the point \( P \) is 8. The latter exceeds the product of multiplicities of the varieties in question in the point \( P \). From [11, Thm 6.3] we obtain the inequality:

\[ \dim(\text{sing}(Q_0) \cap T_PQ_1 \cap T_PQ_2 \cap T_PQ_3) \geq 1. \]

To complete the proof, suppose that \( \text{sing}(Q_0) \) is the zero set of the coordinates \( x_0, x_1, x_6, x_7 \). Recall that \( \Pi \) is given by vanishing of \( x_0, \ldots, x_4 \), so we have \( P = (0 : \ldots : 1 : 0 : 0) \) and only 12 entries in the matrix \( q_0 \) do not vanish.

The point \( P \) is a node on \( X_{16} \), so \( \dim(T_PQ_1 \cap T_PQ_2 \cap T_PQ_3) = 4 \). Consider the affine chart \( x_5 = 1 \). The inequality (8c) implies that there exists a nonzero \( v := (0, 0, v_2, v_3, v_4, 0, 0) \) in the 4-dimensional intersection of the tangent spaces. Furthermore, all quadrics in question contain \( \Pi \), so the 4-space contains the vectors \((0, \ldots, 0, 1, 0)\) and \((0, \ldots, 0, 1)\). Consequently, a parametrization of \( T_PQ_1 \cap T_PQ_2 \cap T_PQ_3 \) is given by the map
\[ (\lambda_1, \ldots, \lambda_4) \mapsto \lambda_1v + \lambda_2w + \lambda_3(0, \ldots, 1, 0) + \lambda_3(0, \ldots, 1), \]
where \( w := (w_0, \ldots, w_4, 0, 0) \).

Finally, direct computation shows that intersection of the tangent cones \( C_PQ_0, T_PQ_1, T_PQ_2, T_PQ_3 \) consists of two planes. The latter is impossible because we assumed the point \( P \) to be a node of \( X_{16} \). Contradiction. \( \square \)
2. Projection from the plane

Here we maintain the notation of the previous section. Moreover, we assume that [A1] holds and [A2]: no 4 singular points of $X_{16}$ lie on a line.

In view of Lemma 1.3a it seems natural to ask whether the assumption [A1] implies [A2]. The example below shows that this is not the case.

Example 2.1. Consider the following $8 \times 8$ symmetric matrices

$$q_0 := \begin{bmatrix} 0 & -4 & 4 & 0 & 1 & -2 & 0 & 1 \\ -4 & 4 & 4 & 3 & -3 & 2 & 2 & -2 \\ 4 & -4 & 4 & 1 & -1 & 0 & -1 & 0 \\ 0 & 3 & 1 & -2 & -1 & -2 & -1 & 2 \\ 1 & -3 & -1 & -1 & 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$q_1 := \begin{bmatrix} -2 & 2 & -1 & -3 & 0 & 0 & 0 & -2 \\ 2 & 0 & -4 & 1 & 1 & 4 & -3 & 2 \\ -1 & -4 & 2 & 3 & 1 & 1 & 0 & -1 \\ -3 & 1 & 3 & -2 & -3 & 1 & -3 & 1 \\ 0 & 1 & 1 & -3 & 2 & -2 & 0 & 0 \\ 0 & 4 & 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$q_2 := \begin{bmatrix} -4 & 1 & 1 & 1 & -2 & -1 & -1 & -1 \\ 1 & -1 & -1 & -3 & -3 & 0 & 1 \\ 1 & -1 & 2 & -4 & 0 & 2 & 2 & 1 \\ 1 & -1 & -4 & 2 & -1 & -1 & 1 & 1 \\ -2 & -3 & 0 & -1 & -4 & -2 & 0 & 0 \\ -1 & -3 & 2 & -1 & -2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$q_3 := \begin{bmatrix} -4 & -1 & -4 & 3 & -1 & 4 & 1 & 0 \\ -1 & 4 & -4 & 3 & 0 & 3 & -1 & 0 \\ -4 & -4 & 0 & 1 & 0 & 1 & 1 & 1 \\ 3 & -3 & 1 & 2 & 2 & 1 & 0 & -2 \\ -1 & 0 & 0 & 2 & 4 & 3 & 0 & 0 \\ 4 & 3 & 1 & 1 & 3 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By direct computation with help of [15], the intersection in $\mathbb{P}_7$ of the quadrics defined by the above matrices has 10 isolated singularities on the plane $\Pi$ and is smooth elsewhere. In the same way one checks that 4 singular points of the intersection in question lie on the line $(0 : \ldots : 0 : x_6 : x_7)$ and are given by the equation

$$19x_6^4 + 102x_6^3x_7 + 189x_6^2x_7^2 + 137x_6x_7^3 + 27x_7^4 = 0.$$

In this section we study the projection $X_{16} \setminus \Pi \ni (x_0 : \ldots : x_7) \mapsto (x_0 : \ldots : x_4) \in \mathbb{P}_4$ from the plane $\Pi$. Observe that the map in question lifts to a regular map

$$\pi : \tilde{X}_{16} \rightarrow \mathbb{P}_4$$

given by the linear system $|H - S|$, where $H$ is the pullback of a hyperplane section under the blow-up $\sigma : \tilde{X}_{16} \rightarrow X_{16}$, and $S$ stands for the strict transform of $\Pi$.

Lemma 2.2. We have the following intersection numbers:

$$H^3 = 16,$$
$$H^2 \cdot S = 1,$$
$$H \cdot S^2 = -3,$$
$$S^3 = -1,$$
$$\langle H - S \rangle^3 = 5.$$

Proof. The first two statements are obvious. The intersection number $H \cdot S^2$ equals the intersection number in $S$ of the restrictions $H|_S$, $S|_S$. Since $S$ is a blow-up of the plane $\Pi$ in 10 points, the restriction $H|_S$ is the pullback $l$ of a line in $\Pi$. Moreover, $S|_S$ is the normal bundle of $S$ in the
Calabi–Yau manifold $\tilde{X}_{16}$. Hence it is the canonical divisor $K_S = -3l + \sum_{i=1}^{10} E_i$, where $E_1, \ldots, E_{10}$ are the 10 exceptional curves (see (7)). Finally, we have

$$H \cdot S^2 = (l \cdot (-3l + \sum_{1}^{10} E_i))_S = -3.$$ Similarly, $S^3 = ((-3l + \sum_{1}^{10} E_i)^2)_S = 9 - 10 = -1$. The last statement follows from Newton’s formula.

To simplify our notation we put $x := (x_0 : \ldots : x_4) \in \mathbb{P}^4$ and define the following matrices :

$$a(x) := \begin{bmatrix} l_0x & l_1x & l_2x & l_3x \\ m_0x & m_1x & m_2x & m_3x \\ n_0x & n_1x & n_2x & n_3x \end{bmatrix}, \quad \mathfrak{A}(x) := \begin{bmatrix} x^T \, q_0x \\ x^T \, q_1x \\ x^T \, q_2x \\ x^T \, q_3x \end{bmatrix}.$$

Observe that the following equality holds (cf. [2, p. 30])

$$a(x) y = b(y) x.$$ Let $Q_i$ be the quadratic form associated to the matrix $a(x)$ and let $C_i$ denote the cubic given by the degree-3 minor of the matrix $a(x)$ obtained by deleting its $i$-th column, e.g.

$$C_0 := \det \begin{bmatrix} l_1x & l_2x & l_3x \\ m_1x & m_2x & m_3x \\ n_1x & n_2x & n_3x \end{bmatrix}.$$

**Lemma 2.3.**

a) The image of $\tilde{X}_{16}$ under $\pi$ is the quintic $X_5$ given by the equation

$$\det(\mathfrak{A}(x)) = C_0 \cdot Q_0 - C_1 \cdot Q_1 + C_2 \cdot Q_2 - C_3 \cdot Q_3 = 0.$$  

b) The image of $S$ under $\pi$ is the (smooth) Bordiga sextic $B \subset \mathbb{P}_4$ given by vanishing of the cubics $C_0, \ldots, C_4$ (i.e. all $3 \times 3$ minors of the matrix $a(x)$). Moreover, the map $\pi|_{S} : S \to B$ is an isomorphism.

**Proof.** Obviously, the restriction of the quadric $\sum_{0}^{3} \alpha_i Q_i$ to the 3-space

$$\text{span}\{x, \Pi\} = \{ (\mu_0 x_0 : \ldots : \mu_0 x_3 : \mu_0 x_4 : \mu_1 : \mu_2 : \mu_3) \mid (\mu_0 : \mu_1 : \mu_2 : \mu_3) \in \mathbb{P}_3 \}$$

is given by the polynomial

$$\sum_{0}^{3} \alpha_i (l_i x) \mu_0^2 + 2 \left( \sum_{0}^{3} \alpha_i (l_i x) \right) \mu_0 \mu_1 + 2 \left( \sum_{0}^{3} \alpha_i (m_i x) \right) \mu_0 \mu_2 + 2 \left( \sum_{0}^{3} \alpha_i (n_i x) \right) \mu_0 \mu_3.$$  

a) Observe that $x \in \mathbb{P}_4 \setminus \pi(S)$ lies in the image of $X_{16}$ under the projection from $\Pi$ iff the planes residual to $\Pi$ in the intersections of the quadrics $Q_i$ with the 3-space $\text{span}\{x, \Pi\}$ intersect. By (13), the latter is equivalent to the vanishing $\det(\mathfrak{A}(x)) = 0$. Laplace formula completes the proof.

b) From (13) we obtain that the condition

$$\sum_{0}^{3} \alpha_i (l_i x) = \sum_{0}^{3} \alpha_i (m_i x) = \sum_{0}^{3} \alpha_i (n_i x) = 0$$

is satisfied iff the restriction $(\sum_{0}^{3} \alpha_i Q_i)|_{\text{span}\{x, \Pi\}}$ is the double plane $2\Pi$. The latter holds precisely when $x$ lies in the image of $\Pi$ under the projection in question.

It is well known that, for a generic $4 \times 3$ matrix whose entries are linear forms in five variables, the surface given by the vanishing of $3 \times 3$ minors is $\mathbb{P}_2$ blown-up in 10 points (see e.g. [2]). Still, it is not always the case (see e.g. [30]). To see that our surface is indeed the (smooth)
Bordiga sextic, observe that the linear system $|H - S|$ restricts on $S$ to the complete linear system $|4l - \sum_{i=1}^{10} E_i|$. We apply [1] Lemma 2.9.1 to show that the system in question embeds $S$ into $\mathbb{P}_4$ as the (smooth) Bordiga sextic. By Lemma 1.4.a no cubic contains all singularities of $X_{16}$. Suppose that 8 singularities of $X_{16}$ lie on a conic. Then its product with the line through the remaining two singular points is a cubic containing $\text{sing}(X_{16})$. Consequently the existence of such a conic is ruled out by Lemma 1.4.a. Finally no 4 singularities lie on a line by the assumption [A2]. \[\square\]

Remark 2.4. a) Observe that, since the (scheme–theoretic) intersection $B$ of the zeroes of the degree-3 minors of the matrix $a(\bar{x})$ is smooth, we have

$$\text{rank}(a(\bar{x})) = 2 \quad \text{for every } \bar{x} \in B.$$ b) The rational curves $E_1, \ldots, E_{10} \subset X_{16}$ are mapped by $\pi$ to lines in $\mathbb{P}_4$ contained in the Bordiga sextic. Indeed, we have $(H - S) \cdot E_j = ((4l - \sum E_i) \cdot E_j)_S = 1$ for $j = 1, \ldots, 10$. Geometrically, points on such a line $\subset B$ correspond to the 3-spaces in the 4-space $T_P X_{16}$, where $P$ is a node of $X_{16}$, that contain the plane $\Pi$.

We introduce the following notation:

$$U := X_{16} \setminus (S \cup \bigcup_{V \text{ linear}, V \subset X_{16}, V \cap \Pi \neq \emptyset} \sigma^{-1}(V)).$$

Lemma 2.5. Suppose that [A1], [A2] hold.

a) The map $\pi|_U$ is an isomorphism onto the image and we have the equality $\pi(U) = (X_5 \setminus B)$.

b) The inclusion $\text{sing}(X_5) \subseteq B$ holds. In particular, the quintic $X_5$ is normal.

Proof. a) Fix $P \in U$. Then $\sigma(P) \notin \Pi$. Since $X_{16}$ is an intersection of quadrics we have the equality

$$\text{span}(\sigma(P), \Pi) \cap X_{16} = \Pi \cup \{\sigma(P)\},$$

which implies that $\pi|_U$ is injective and the linear map $d_P \pi$ is an isomorphism.

We claim that

$$\pi(X_{16} \setminus U) = B.$$ 

Let $V \subset X_{16}$, $V \not\subset \Pi$ be a linear subspace such that $V \cap \Pi \neq \emptyset$. Let $\sigma(P_1) \in (V \setminus \Pi)$ and let $\sigma(P_2) \in (V \cap \Pi)$. By definition of $\pi$ all points from $\text{span}(\sigma(P_1), \sigma(P_2)) \setminus \{\sigma(P_2)\}$ lie in one fiber of $\pi$. On the other hand, the proper transform of the line $\text{span}(\sigma(P_1), \sigma(P_2))$ under $\sigma$ meets $S$. Since $\pi$ maps that proper transform of the line in question to one point and $\pi(P_2) \in B$ we have $\pi(P_1) \in B$, and we obtain the claim.

It remains to show the inclusion

$$\pi(U) \subset (X_5 \setminus B).$$

Suppose that $\pi(P_3) = \pi(P_4)$, where $P_3 \in X_{16} \setminus U$ and $P_4 \in U$. If $\sigma(P_3) \in \text{reg}(X_{16})$, then the line $\text{span}(\sigma(P_3), \sigma(P_4))$ is tangent to $X_{16}$ in $\sigma(P_3)$ and meets it in $\sigma(P_4)$. In particular, it is contained in each quadric of the system $W$, so $\text{span}(\sigma(P_3), \sigma(P_4)) \subset X_{16}$ and $P_4 \notin U$. Contradiction. Similar argument yields contradiction when $\sigma(P_3) \in \text{sing}(X_{16})$.

b) By [A1] and part a) we know that $\text{sing}(X_5) \subset B$. Suppose that $\text{sing}(X_5) = B$. Since $B$ is smooth, Lemma 2.3.a implies that $\text{det}(\mathfrak{A}(\bar{x})) \in I(B)^2$. The latter is impossible because the ideal $I(B)$ is generated by the cubics $C_0, C_1, C_2, C_3$.

Finally $X_5$ is a 3-dimensional hypersurface with at most 1-dimensional singularities, so it is normal. \[\square\]

After those preparations we can study higher-dimensional fibers of $\pi$. 

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**Lemma 2.6.** a) The map $\pi$ has no two-dimensional fibers and its only one-dimensional fibers are proper transforms of lines on $X_{16}$ that meet $\Pi$ but are not contained in $\Pi$.  

b) The following equality holds

$$\text{sing}(X_5) := \{ \overline{a} \in B : \text{rank}(\mathfrak{A}(\overline{a})) \leq 2 \}.$$ 

c) The map $\pi$ has only finitely many one-dimensional fibers.

**Proof.** a) As we have already shown in the proof of Lemma 2.5 the proper transform of each line on $X_{16}$ that meets $\Pi$ but is not contained in $\Pi$ lies in a fiber of $\pi$.

The regular map $\pi$ is birational and its image is normal, so we can apply Zariski’s Main Theorem [17, Thm 5.2] to see that the map $\pi$ has connected fibers. Moreover, by Lemma 2.3.b

$$\text{(15)}$$ 

each fiber of $\pi$ meets the surface $S$ in at most one point.

Let $F$ be a fiber of $\pi$ such that $\dim(F) \geq 1$. Let $P_1, P_2 \in (F \setminus S)$. Then the 3-spaces $\text{span}(\sigma(P_1), \Pi)$, $\text{span}(\sigma(P_2), \Pi)$ coincide, so the line $\text{span}(\sigma(P_1), \sigma(P_2))$ meets the plane $\Pi$. Obviously, the intersection point does not coincide with $P_1$, $P_2$. Since $X_{16}$ is intersection of quadrics, we have $\text{span}(\sigma(P_1), \sigma(P_2)) \subset X_{16}$, which implies that

$$\text{span}(\sigma(P_1), \sigma(P_2)) \subset \sigma(F).$$

Suppose that the fiber $F$ contains a point $P_3 \notin S$ such that $\sigma(P_3) \notin \text{span}(\sigma(P_1), \sigma(P_2))$. Then, arguing as in (2), we show that $\text{span}(\sigma(P_1), \sigma(P_3))$ is a line contained in $\sigma(F)$ and meeting the plane $\Pi$. But, (15) implies that the proper transforms (under the blow-up $\sigma$) of two lines meeting $\Pi$ in different points do not lie in the same fiber of $\pi$. Consequently, by (15), the image $\sigma(F)$ is a plane in $X_{16}$ that intersects $\Pi$ in precisely one point. Observe that the planes $\sigma(F)$, $\Pi$ meet in a singularity of $X_{16}$. Let $H$ be the pullback of a hyperplane section under the blow-up $\sigma$ and let $\overline{\sigma(F)}$ denote the proper transform of $\sigma(F)$. If we put $\overline{l}$ (resp. $\overline{m}$) to denote the proper transform of a line in $\sigma(F)$ (resp. in $\Pi$) that runs through no singularities of $X_{16}$, then we obtain the following table of intersection numbers.

| $\overline{\sigma(F)}$ | $S$ | $H$ |
|------------------------|-----|-----|
| $\overline{l}$         | $-3$| $0$ |
| $\overline{m}$         | $0$ | $-3$| $1$ |
| $H^2$                  | $1$ | $1$ | $16$ |

The resulting matrix has non-zero determinant, so Picard number of $\tilde{X}_{16}$ is at least 3, which is impossible by Lemma 1.7. This contradiction shows that the fiber $F$ coincides with the proper transform of the line $\text{span}(\sigma(P_1), \sigma(P_2))$.

b) As in the proof of Lemma 2.3, we see that the line through the points $(\overline{a}, x_5, x_6, x_7)$ and $(0, x'_5, x'_6, x'_7)$ is contained in $X_{16}$ if and for any $\lambda \in \mathbb{C}$ and $i = 0, \ldots, 3$ we have

$$\overline{x}^T a \overline{x} + 2(l_{\overline{x}}. m_{\overline{x}}. n_{\overline{x}})(x_5, x_6, x_7)^T + 2\overline{x}^T(l_{\overline{x}}. m_{\overline{x}}. n_{\overline{x}})(x'_5, x'_6, x'_7)^T = 0.$$ 

Fix $\overline{a} \in B$. From Remark 2.4 we know that $\text{rank}(a(\overline{a})) = 2$. Consequently, there exist points $(x_5, x_6, x_7)$ and $(x'_5, x'_6, x'_7)$ such that the line spanned by $(\overline{a}, x_5, x_6, x_7)$ and $(0, x'_5, x'_6, x'_7)$ is contained in $X_{16}$ if and only if $\text{rank}(A(\overline{a})) = 2$.

c) Assume to the contrary that the map $\pi$ contracts infinitely many lines. Then there is a ruled surface $G \subset \tilde{X}_{16}$ such that the fibers of $G$ are contracted by $\pi$. Let $l$ (resp. $E_i$) be the class of a (general) fiber of $G$, (resp. of an exceptional curve of the blow-up $\sigma$). We have the following
intersection numbers

\[ (16) \]

\[
\begin{array}{c|c|c|c}
S & G & H \\
\hline
l & 1 & -2 & 1 \\
E_1 & -1 & \nu & 0 \\
\end{array}
\]

The above table yields immediately that \( H \) and \( S \) are linearly independent in \( \text{Pic}(\tilde{X}_{16}) \otimes \mathbb{Q} \). Since \( h^{1,1}(\tilde{X}_{16}) = 2 \), we can find \( d_H, d_S \in \mathbb{Q} \) such that \( G \sim_{\text{num}} d_H H + d_S S \). From (16) we obtain

\[ G \sim_{\text{num}} (\nu - 2) H - \nu S. \]

Therefore Lemma 2.2 yields the equality

\[ (H - S)^2 \cdot G = 5\nu - 22. \]

As the divisor \( G \) is contracted by \( \pi \) we conclude that \( \nu = \frac{22}{5} \), which is impossible by (16). \( \square \)

In particular, Lemma 2.6 implies that the map \( \pi : \tilde{X}_{16} \to X_5 \) is a resolution of singularities of the quintic \( X_5 \). As \( \pi \) contracts only finitely many curves (i.e. the singular locus of \( X_5 \) is zero-dimensional), it is in fact a small resolution that introduces exactly one copy of \( \mathbb{P}_1 \) over each singularity.

The lemma below gives a simple criterion when the quintic \( X_5 \) is nodal.

**Lemma 2.7.** All singularities of the quintic \( X_5 \) are nodes iff the set \( \text{sing}(X_5) \) consists of 46 points.

**Proof.** Let \( \mu(\cdot) \) stand for the Milnor number. Lemma 2.5 yields that the regular map \( \pi : \tilde{X}_{16} \to X_5 \) is birational. By Lemma 2.6 it contracts only the lines in \( X_{16} \) that intersect the plane \( \Pi \). The contracted lines are pairwise disjoint, so we obtain

\[ -108 - \#(\text{sing}(X_5)) = e(X_5) = -200 + \sum_{P \in \text{sing}(X_5)} \mu(P, X_5), \]

where the second equality results from [10, Cor. 5.4.4]. To complete the proof recall that the Milnor number of a singularity is 1 iff the singularity in question is an \( A_1 \) point. \( \square \)

3. **Restriction of the Bordiga conic bundle**

In this section we maintain the assumptions and notation of the previous one, i.e. we assume that \([A1], [A2]\) hold. In particular, the scheme-theoretic intersection of the zeroes of the degree-3 minors of the matrix \( a(x) \) is smooth (see [10]) and the locus \( \{ y \in \mathbb{P}_4 : \text{rank}(b(y)) = 2 \} \) consists of 10 points. Moreover, we make the following assumption:

**[A3]:** the set \( \{ x \in B : \text{rank}(A(x)) \leq 2 \} \) consists of 46 points.

One can show (see [2, Ex. 3 on p. 35]) that the rational map

\[ (17) \quad \mathbb{P}_4 \setminus B \ni x \mapsto (C_0(x) : -C_1(x) : C_2(x) : -C_3(x)) \in \mathbb{P}_3 \]

lifts to a regular map (so-called Bordiga conic bundle - see [2, Ex. 3 on p. 35])

\[ \Phi : \text{Bl}_B \mathbb{P}_4 \to \mathbb{P}_3. \]

that is generically a conic-bundle ([ibid., Prop. 2.1]). The map \( \Phi \) is the projection onto the second factor from the closure of the graph of the rational map defined by (17) (see also (11)) i.e. from the set

\[ (x, y) \in \mathbb{P}_4 \times \mathbb{P}_3 : b(y)x = 0. \]
By Lemma 1.4 it has exactly ten 2-dimensional fibers over the points $y \in \mathbb{P}_3$ such that $\text{rank}(b(y)) = 2$. Such a fiber is the plane

\begin{equation}
\Phi^{-1}(y) = \{(x, y) : b(y)x = 0\}.
\end{equation}

Observe that restrictions of the cubics polynomials $\mathcal{C}_i$ to the plane $\{b(y)x = 0\}$ are proportional, so the plane cuts $\mathcal{B}$ along a cubic curve (see also [2, Ex. 3 on p. 35]).

The remaining fibers $\Phi^{-1}(y)$ are 3-secant lines to $\mathcal{B}$. They are given by (19) with $\text{rank}(b(y)) = 3$.

In Sect. 1 we studied the map $\tilde{X}_{16} \rightarrow X_5$. By Lemma 2.7 the quintic $X_5$ admits another small resolution of singularities

\begin{equation}
\psi : \tilde{X}_5 \rightarrow X_5
\end{equation}

obtained by blowing-up the Bordiga surface $\mathcal{B}$. The strict transform $S_1$ of $\mathcal{B}$ is a plane blown-up in 56 points (some of the 46 points that are centers of the second blow-up may lie on the exceptional curves of the first blow-up). We put $F_1, \ldots, F_{46}$ to denote the exceptional curves of the small resolution in question. Then, the two resolutions differ by flops of the 46 smooth rational curves $L_1, \ldots, L_{46} \subset \tilde{X}_{16}$ and $F_1, \ldots, F_{46} \subset \tilde{X}_5$.

The restriction of the conic bundle $\Phi$ induces the regular map

$$\phi : \tilde{X}_5 \rightarrow \mathbb{P}_3.$$

This regular map is given by the linear system $|3H_1 - S_1|$ on $\tilde{X}_5$, where $H_1$ is pullback of the hyperplane section $\mathcal{O}_{\mathbb{P}_4}(1)$. We have the following intersection numbers

**Lemma 3.1.**

$$H_1^3 = 5,$$

$$H_1^2 : S_1 = 6,$$

$$H_1 : S_1^2 = -2,$$

$$S_1^3 = -47,$$

$$(3H_1 - S_1)^3 = 2.$$

**Proof.** The first two statements follow from the fact that $\text{deg}(X_5) = 5$ and $\text{deg}(\mathcal{B}) = 6$. The others can be obtained from the equalities

\begin{equation}
H_1|_{S_1} = 4l - \sum_{1}^{10} \psi^*(\pi(E_i)) , \quad S_1|_{S_1} = -3l + \sum_{1}^{10} \psi^*(\pi(E_i)) + \sum_{1}^{46} F_j.
\end{equation}

where $l$ is the pull-back of $\mathcal{O}_{\mathbb{P}_4}(1)$ under both blow-ups. Recall (Remark 2.4.b) that the curves $\pi(E_1), \ldots, \pi(E_{10})$ are lines on $\mathcal{B}$. \hfill \square

Since $\phi$ is surjective, as an immediate consequence of Lemma 3.1 we obtain

**Corollary 3.2.** The mapping $\phi$ is generically 2:1.

In order to obtain a precise description of fibers of $\phi$ we will need the following lemma (compare [24]):

**Lemma 3.3.** A point $z \in \tilde{X}_5$ is mapped by $\phi$ to $y \in \mathbb{P}_3$ iff the 3-space span$(\langle \psi(z) : 0 : 0 : 0 \rangle, \Pi)$ is contained in the quadric $Q(y) := \sum y_i Q_i$.

**Proof.** Observe that for any $x = (x : x_5 : x_6 : x_7) \in \text{span}(\langle x : 0 : 0 : 0 \rangle, \Pi)$ we have

\begin{equation}
x^Tq(y)x = x^Tq(y)x + 2(x_5, x_6, x_7)b(y)x
\end{equation}

($\Leftarrow$): Put $x = \psi(z)$ in (22) to obtain

$$\psi(z)^Tq(y)\psi(z) = -2(x_5, x_6, x_7)b(y)\psi(z) \quad \text{for all } x_5, x_6, x_7 \in \mathbb{C}.$$
The latter implies \( b(y)\psi(z) = 0 \) and (see (19)) the equality \( \phi(z) = y \).

\((\Rightarrow)\): Suppose that \( z \in \tilde{X}_5 \setminus S_1 \). From \( \phi(z) = y \) we get \( b(y)\psi(z) = 0 \). By (22) we have

\[
x^T q(y)x = \psi(z)^T q(y)\psi(z) \quad \text{for all } x = (\psi(z) : x_5 : x_6 : x_7) \in \text{span}(\psi(z), \Pi).
\]

But (see (17)), we can assume that \( y = (C_0(\psi(z)) : \ldots : -C_3(\psi(z))) \). Therefore, Lemma (23) a yields the equalities \( \psi(z)^T q(y)\psi(z) = \det(\mathcal{A}(\psi(z))) = 0 \). In this way we have shown the inclusion

\[
\{(x, y) \in \tilde{X}_5 : b(y)x = 0\} \subset \{(x, y) \in \mathbb{P}_3 \times \mathbb{P}_3 : \text{span}(x : 0 : 0 : 0, \Pi) \subset Q(y)\},
\]

which completes the proof. □

Recall, that we have the map \( (\psi \circ (\pi|_S)^{-1} \circ \sigma) : S_1 \to \mathcal{B} \simeq S \to \Pi \). In the lemma below we put \( \hat{l} \) (resp. \( \hat{E}_1, \ldots, \hat{E}_{10} \)) to denote the pullback of \( \mathcal{O}_\Pi(1) \) (resp. of the exceptional divisors (7)) to \( S_1 \).

**Lemma 3.4.** An irreducible curve \( D \subset S_1 \) is contracted by \( \phi \) iff (up to a relabelling of the divisors \( \hat{E}_1, \ldots, \hat{E}_{10} \)) it belongs to one of the following linear systems

\[
a) \ |\hat{E}_1 - F_1 - F_2 - F_3 - F_4|,
b) \ |\hat{l} - \hat{E}_1 - \hat{E}_2 - \hat{E}_3 - F_1 - F_2 - F_3|,
c) \ |2\hat{l} - \hat{E}_1 - \ldots - \hat{E}_7 - F_1 - F_2|,
d) \ |3\hat{l} - 2\hat{E}_1 - \hat{E}_2 - \ldots - \hat{E}_9 - F_1 - \ldots - F_5|.
\]

In the cases (a)–(c) the curve in question is the proper transform of a line in \( \mathcal{B} \), whereas the case (d) corresponds to a conic in the intersection of \( \mathcal{B} \) with the plane \( \{b(y)x = 0\} \), where \( \text{rank}(b(y)) = 2 \).

In particular, if the intersection \( \mathcal{B} \cap \{b(y)x = 0\} \) is an irreducible cubic, then its proper transform is not contracted by \( \phi \).

**Proof.** Recall that \( \phi = \Phi|_{\tilde{X}_5} \) and the fibers of \( \Phi \) are lines and planes given by (19).

Before we prove the claim, we study two-dimensional fibers of \( \Phi \). Let \( \text{sing}(X_{16}) = \{P_1, \ldots, P_{10}\} \).

By (13) for each singularity \( P_i \) there exists a unique point \( y^{(i)} \in \mathbb{P}_3 \) such that \( c(P_i)y^{(i)} = 0 \). Then, by (2), we have \( \text{rank}(b(y^{(i)})) = 2 \).

Lemma (14) a yields that for each \( i \in \{1, \ldots, 10\} \) there is a unique degree-three curve \( C_i \subset \Pi \) such that \( P_j \in C_i \), for \( j \neq i \). Let \( \tilde{C}_i := \sigma^* C_i - \sum_{j \neq i} E_j \in |3\hat{l} - \sum_{j \neq i} E_j| \) be the corresponding curve on \( S \). By direct computation the following equality holds

\[
\pi(\tilde{C}_i) = \mathcal{B} \cap \{x \in \mathbb{P}_4 : b(y^{(i)}x = 0\}
\]

In general, cubics \( C_i \) are smooth, and the curves \( \pi(\tilde{C}_i) \subset \mathcal{B} \) are also smooth planar cubics. We have the following possible degenerations:

(i) The curve \( C_i \) is irreducible, but \( \text{sing}(C_i) = \{P_{j_0}\} \) for a \( j_0 \neq i \). Then the exceptional curve \( E_{j_0} \) is a component of the curve \( \tilde{C}_i := \sigma^* C_i - \sum_{j \neq i} E_j \) and the curve \( \tilde{C}_i - E_{j_0} \) is irreducible. By Remark (27) a the image \( \pi(E_{j_0}) \) is a line on \( \mathcal{B} \), whereas \( \pi(\tilde{C}_i) \) is a smooth conic. In this way we obtain a decomposition of \( \mathcal{B} \cap \{x \in \mathbb{P}_4 : b(y^{(i)}x = 0\} \). Observe that for a given integer \( i \neq j_0 \) there exists at most one cubic in \( |\mathcal{O}_\Pi(3) - \sum_{j \neq i} E_j - E_{j_0}| \).

(ii) The cubic \( \tilde{C}_i \) is union of a line and a smooth conic. Then, by [A2] and Lemma (14) a the line contains two (resp. three) singularities of \( X_{16} \) and the conic contains 7 (resp. 6) of them.

(iii) The curve \( \tilde{C}_i \) can be union of three lines. The assumption [A2] yields that each line contains three singularities of \( X_{16} \).

In this way (up to a permutation of the points in \( P_1, \ldots, P_{10} \)), we obtain the following possibilities
for the decomposition of the cubic $\phi$ for $i = 10$:

\[
(3l - 2E_1 - E_2 - \cdots - E_9) + E_1, \\
(l - E_1 - E_2) + (2l - E_3 - \cdots - E_9),
\]

(24)

\[
(l - E_1 - E_2 - E_3) + (2l - E_4 - \cdots - E_9), \\
(l - E_1 - E_2 - E_3) + (2l - E_3 - \cdots - E_9) + E_3, \\
(l - E_1 - E_2 - E_3) + (l - E_4 - E_5 - E_6) + (l - E_7 - E_8 - E_9).
\]

After those preparations we can prove the lemma. Assume that an irreducible curve $D \subset S_1$ is contained in $\phi^{-1}(y)$ for a point $y \in \mathbb{P}_3$. The map $\phi|_{S_1} : S_1 \to \mathbb{P}_3$ is given by the linear system

(25)

\[
|15 \hat{l} - 4 \sum_{1}^{10} \hat{E}_i - \sum_{1}^{46} F_j|,
\]

so $D \neq F_j$ for each $j \leq 46$.

Suppose that $\text{rank}(b(y)) = 2$. We can assume that $D \subset \phi^{-1}(y^{(10)})$. Then $\psi(D) \subset B$ is a component of (23). If $\psi(D)$ is image under $\pi$ of a curve from the system $|3l - 2E_1 - E_3 - \cdots - E_9|$, then we have

\[
\deg(\psi(D)) = (3l - 2E_1 - E_3 - \cdots - E_9) \cdot (4l - \sum_{1}^{10} E_i) = 12 - 2 - 8 = 2.
\]

Let $\text{sing}(X_5) \cap \psi(D) = \{\psi(F_1), \ldots, \psi(F_p)\}$. Since $D$ coincides with the proper transform of $\psi(D)$ under the blow-up $\psi$, we have

\[
D \subset |3l - 2 \hat{E}_1 - \hat{E}_2 - \cdots - \hat{E}_9| - F_1 - \cdots - F_p|.
\]

and, by (25), the degree of $\phi(D)$ is $(5 - p)$. Consequently, the curve $D$ is contracted by $\phi$ iff $p = 5$.

In the following table we collect data on each curve considered in (24). In particular, the integer in the last column is the number of singularities of $X_5$ that lie on $\psi(D)$ provided $D$ is contracted by the map $\phi$:

| $\pi^{-1}(\psi(D))$ | $\deg(\psi(D))$ | $\#(\text{sing}(X_5) \cap \psi(D))$ |
|----------------------|------------------|-------------------------------------|
| $3l - 2E_1 - E_2 - \cdots - E_9$ | 2 | 5 |
| $2l - E_1 - \cdots - E_6$ | 2 | 6 |
| $2l - E_1 - \cdots - E_7$ | 1 | 2 |
| $l - E_1 - E_2$ | 2 | 7 |
| $l - E_1 - E_2 - E_3$ | 1 | 3 |
| $E_1$ | 1 | 4 |

Finally, observe that for a point $y^{(i)} \in \mathbb{P}_3$, where $i = 1, \ldots, 10$, the intersection

(26)

\[
X_5 \cap \{z \in \mathbb{P}_4 : b(y^{(i)})|_z = 0\}
\]

is a degree–5 planar curve, so it is union of the cubic considered above and a conic (possibly reducible) that does not lie on $B$. The points $\psi(F_j)$ are singular points of $X_5$, so they are also singular points of the quintic curve (26), which yields some extra constrains on the possible arrangements. Since a line contained in $\pi^{-1}(\psi(D))$ intersects the residual quartic in four points, the line of the type $(l - E_1 - E_2)$ is never contracted. Similar argument rules out the conic $(2l - E_1 - \cdots - E_6)$. In this way we arrive at the cases (a)–(d) of the lemma.

Assume that $\text{rank}(b(y)) = 3$. Then $D$ is the strict transform of a line $l_y \subset B$. In particular, there exist $d, m_i, n_j \in \mathbb{Z}$ such that $D \subset |dl - \sum_{i=1}^{10} m_i \hat{E}_i - \sum_{j=1}^{46} n_j F_j|$. Since the curve $D$ is smooth
and rational, we have $n_j = 0$ or $1$. Moreover, by the genus formula
\[
(d\tilde{l} - \sum_{i=1}^{10} m_i \tilde{E}_i - \sum_{j=1}^{46} n_j F_j) \cdot ((d-3)\tilde{l} - \sum_{i=1}^{10} (m_i - 1) \tilde{E}_i - \sum_{j=1}^{46} (n_j - 1) F_j) = d^2 - 3d - \sum_{i=1}^{10} (m_i^2 - m_i) = -2.
\]
Furthermore, the equality $4d - \sum_{i=1}^{10} m_i = 1$ holds because $l_y$ is a line on $B$ (see also Lemma 2.3b). Finally, since $D$ is contracted by the map given by the linear system $|3H_1 - S_1|$ we have
\[
(15\tilde{l} - 4\sum_{i=1}^{10} \tilde{E}_i - \sum_{j=1}^{46} F_j) \cdot (d\tilde{l} - \sum_{i=1}^{10} m_i \tilde{E}_i - \sum_{j=1}^{46} n_j F_j) = 15d - 4\sum_{i=1}^{10} m_i - \sum_{j=1}^{46} n_j = 0.
\]
From the above we obtain the following equations
\[
\sum m_i^2 = d^2 + d + 1, \\
\sum m_i = 4d - 1, \\
4 - d = \sum n_j,
\]
where $n_j = 0, 1$. The solution $d = 3, m_1 = 2, m_i = 1$ for $i > 1$ is excluded by Lemma 1.4a. The others correspond to the cases (a)–(c) of the lemma. □

Now we are in position to prove

**Lemma 3.5.** Let $y \in \mathbb{P}_3$ be a point such that $\text{rank}(b(y)) = 3$. Then the fiber $\phi^{-1}(y)$ is $1$-dimensional iff $\text{rank}(q(y)) = 6$.

**Proof.** By abuse of notation we put $\psi$ to denote the blow-up $\text{Bl}_B \mathbb{P}_4 \to \mathbb{P}_4$.

Assume that the line $\Phi^{-1}(y)$ is contracted by $\phi$. Then the set $\psi(\Phi^{-1}(y)) = \{ x \in \mathbb{P}_4 : b(y)x = 0 \}$ is a line on $X_5$. Observe that the linear space $\text{span}(\{(x : 0 : 0 : 0) : x \in \psi(\Phi^{-1}(y))\})$ is 4-dimensional. By Lemma 3.3 the quadric $Q(y)$ contains the 4-space $\text{span}(\{(x : 0 : 0 : 0) : x \in \psi(\Phi^{-1}(y))\})$, which yields $\text{rank}(q(y)) \leq 6$. Finally $\text{rank}(q(y)) = 6$, because $\text{rank}(b(y)) = 3$.

On the other hand, if $\text{rank}(q(y)) = 6$, then $\text{sing}(Q(y))$ is a line. Since $\text{rank}(b(y)) = 3$, the line $\text{sing}(Q(y))$ does not meet the plane $\Pi$. Put $L$ to denote the image of the line $\text{sing}(Q(y))$ under the projection from the plane $\Pi$. Then $\text{span}(\{(x : 0 : 0 : 0) : x \in L\}) \subset Q(y)$ for every $x \in L$. From Lemma 3.3 we obtain that the proper transform of the line $L$ under the blow-up $\psi$ is contracted by $\phi$. □

In the theorem below we identify curves in $\mathbb{P}_4$ with their proper transforms under the blow-up $\psi$: whenever we say a line (resp. a conic) we mean its proper transform.

**Theorem 3.6.** There are four types of fibers $\phi^{-1}(y)$ of the map $\phi : \tilde{X}_5 \to \mathbb{P}_3$:

a) union of the conic residual to the cubic $B \cap \Phi^{-1}(y)$ in the planar quintic $X_5 \cap \Phi^{-1}(y)$ with the components of the cubic that satisfy the conditions of Lemma 3.4 iff $\text{rank}(q(y)) \in \{5, 6, 7\}$ and $\text{rank}(b(y)) = 2$ (i.e. a singularity of $Q(y)$ lies on $\Pi$),

b) a line in $\mathbb{P}_4$ iff $\text{rank}(q(y)) = 6$ and $\text{rank}(b(y)) = 3$ (equivalently $\text{sing}(Q(y)) \cap \Pi = \emptyset$),

c) one point iff $\text{rank}(q(y)) = 7$ and $\text{rank}(b(y)) = 3$,

d) two points iff $\text{rank}(q(y)) = 8$.

**Proof.** Suppose that $\text{rank}(b(y)) = 3$. Then the linear space $\text{span}(\{(x : 0 : 0 : 0) : x \in \psi(\Phi^{-1}(y))\})$ is 4-dimensional and $\text{sing}(Q(y)) \cap \Pi = \emptyset$. In view of Lemma 3.5 we can assume that $\text{rank}(q(y)) \geq 7$ and the line $\psi(\Phi^{-1}(y)) = \{ x : b(y)x = 0 \}$ is not contained in $X_5$. Moreover, by (22), for every point $x = (x, x_5, x_6, x_7) \in \text{span}(\{(x : 0 : 0 : 0) : x \in \psi(\Phi^{-1}(y))\})$, we have
\[
x^T q(y)x = x^T q(y)x.
\]
Observe, that the quadratic form given by \( q(y) \) does not vanish identically on the line \( \{ x : b(y)x = 0 \} \) because the latter is not contained in \( X_5 \). Consequently, intersection of \( Q(y) \) with the linear 4-space \( \text{span}(\{ x : 0 : 0 : 0 \} : x \in \psi(\Phi^{-1}(y))) \), \( \Pi \) consists of either one or two 3-spaces.

Lemma 3.3 implies that the fibre \( \phi^{-1}(y) \) consists of a unique point iff the restriction \( Q(y)|_{\text{span}(\{ x : 0 : 0 : 0 \} : x \in \psi(\Phi^{-1}(y)))}\), \( \Pi \) is a full square.

Suppose that the fibre in question is one point. From (27) there exists a point \( v \in \mathbb{P}_5 \), such that
\[
b(y)v = 0 \quad \text{and} \quad q(y)v = 0
\]
which means that \( (v : 0 : 0 : 0) \in \text{sing}(Q(y)) \) and \( \text{rank}(q(y)) < 8 \).

Assume that \( \text{rank}(q(y)) < 8 \). Then \( Q(y) \) is a cone with the unique vertex \( (v : v_5 : v_6 : v_7) \) away from the plane \( \Pi \). The latter yields \( v \neq 0 \). Moreover, since the tangent space to \( Q(y) \) in each point contains the vertex we have \( b(y)v = 0 \) and
\[
(\mathbf{x} : v_5 : v_6 : v_7) \in \text{span}(\{ x : 0 : 0 : 0 \} : x \in \psi(\Phi^{-1}(y))) \text{,} \Pi
\]
Now \( \mathbf{x} : v_5 : v_6 : v_7 \) is a singularity of the restriction (28), so the polynomial \( \mathbf{x}^T q(y) \mathbf{x} \) has a unique double root on the line \( \{ x : b(y)x = 0 \} \) and (28) is a full square.

Assume that \( y \in \mathbb{P}_3 \) is a point such that \( \text{rank}(b(y)) = 2 \), and maintain the notation of the proof of Lemma 3.3 Then \( y = q^{(i)} \) for an \( i \in \{1, \ldots, 10\} \). By definition of the map \( \phi \), the proper transform under the blow-up \( \psi \) of the (possibly reducible) conic residual to (28) in the quintic (20) is always contracted by \( \phi \). Moreover, a component of (28) is contracted iff it satisfies the conditions of Lemma 3.3.

Observe that rank of the quadric \( Q(y^{(i)}) \) does not exceed 7 because we have \( \text{rank}(b(y^{(i)})) = 2 \). □

Remark 3.7. By Lemma 1.4.d there are exactly ten fibers of \( \phi \) of the type a). The number of fibers of type b) will be discussed in the next section (see Cor. 1.7).

4. Discriminant of the web \( W \)

In this section we maintain the notation and the assumptions of the previous ones. In particular, we assume that \( [A1] \), \( [A2] \), \( [A3] \) hold. Let \( S_8 \) stand for the discriminant surface of the web \( W \). From now on we assume that

[A4]: the discriminant surface \( S_8 \) has only isolated singularities .

To simplify notation we put
\[
\mathbb{I}_l := [a_{i,j}]_{i,j=0, \ldots, 7}, \text{ where } a_{i,i} = 1 \text{ for } i = 1, \ldots, l \text{ and } a_{i,j} = 0 \text{ otherwise.}
\]

At first we give conditions when a singularity of \( S_8 \) is a node:

Lemma 4.1. Let \( Q_0 \) be a rank-7 quadric in the web \( W \).

a) The quadric \( Q_0 \) is a smooth point of \( S_8 \) iff \( \text{sing}(Q_0) \notin X_{16} \).

b) The quadric \( Q_0 \) is a node of \( S_8 \) iff \( \text{sing}(Q_0) \in X_{16} \).

Proof. Let \( q_k := [g_{i,j}]_{i,j=0, \ldots, 7} \) and let \( Q^{(k)} := (q_{0,7}^{(k)}, \ldots, q_{0,7}^{(k)}) \). After an appropriate change of coordinates we can assume that \( q_0 = \mathbb{I}_7 \). In particular, \( \text{sing}(Q_0) = \{ (0 : \ldots : 0 : 1) \} \).

Let \( \mathbf{G} := [g_{i,j}]_{i,j=1,2,3} \), where \( g_{i,j} := \langle Q^{(i)}, Q^{(j)} \rangle \) and \( \langle \cdot, \cdot \rangle \) stands for the bilinear form defined by the identity matrix. By direct computation we have
\[
\det(q_0 + \sum_{k=1}^{3} \mu_k \cdot q_k) = \left( \sum_{k=1}^{3} \mu_k \cdot q_{7,7}^{(k)} \right) - \left( (\mu_1, \mu_2, \mu_3) \cdot \mathbf{G} \cdot (\mu_1, \mu_2, \mu_3)^T \right) + \text{(terms of degree } \geq 3). \]
a) Obviously, \((1 : 0 : 0 : 0)\) is a smooth point of \(S_8\) iff the vector \((q_{7,7}^{(1)}, q_{7,7}^{(2)}, q_{7,7}^{(3)})\) does not vanish. The latter holds iff \((1 : 0 : 0 : 0) \not\in X_{16}\), which concludes the proof.

b) \(\Rightarrow\): the implication in question results immediately from the part a).

\(\Leftarrow\): Assume that \((q_{7,7}^{(1)}, q_{7,7}^{(2)}, q_{7,7}^{(3)}) = 0\). Then, \(Q_0 = (1 : 0 : 0 : 0) \in \text{sing}(S_8)\) is a node iff the matrix \(\mathfrak{S}\) has maximal rank, i.e. \(Q^{(1)}, Q^{(2)}, Q^{(3)}\) are linearly independent. Moreover, we have \((0 : \ldots : 0 : 1) \in \text{sing}(X_{16})\).

Suppose that \(\text{rank}(\mathfrak{S}) < 3\). Then, the last row in a matrix obtained as a non-trivial linear combination of the matrices \(q_1, q_2, q_3\) vanishes, which means that the point \((0 : \ldots : 0 : 1)\) is a singularity of a quadric that belongs to \(\text{span}(\{Q_1, Q_2, Q_3\})\). In particular, the quadric in question does not coincide with \(Q_0\). The latter is impossible by Lemma 1.4.b. Contradiction.

In the rank-6 case we have the following characterization.

**Lemma 4.2.** Let \(Q_0\) be a rank-6 quadric in the web \(W\).

a) The quadric \(Q_0\) is a node of \(S_8\) iff \(\text{sing}(Q_0) \not\subseteq Q\) for all \(Q \neq Q_0\), \(Q \in W\).

b) \(Q_0\) is an \(A_m\) singularity, where \(m \geq 2\), iff \(\text{sing}(Q_0) \cap \Pi = \emptyset\) and there exists a quadric \(Q \in W\), \(Q \neq Q_0\) such that \(\text{sing}(Q_0) \subseteq Q\).

c) The quadric \(Q_0\) is a double point of the surface \(S_8\).

**Proof.** As in the proof of Lemma 4.1 we change the coordinates in such a way that \(q_0 = I_6\). Then, the line \(\text{sing}(Q_0)\) is the set of zeroes of the coordinates \(x_0, \ldots, x_5\). Let \((\cdot, \cdot)_-\) be the bilinear form on \(\mathbb{C}^6\) given by the formula:

\[(29) \quad \langle (q^{(1)}_{6,6}, q^{(1)}_{6,7}, q^{(1)}_{7,7}), (q^{(2)}_{6,6}, q^{(2)}_{6,7}, q^{(2)}_{7,7}) \rangle_- := 1/2 \cdot (q^{(1)}_{6,6} \cdot q^{(2)}_{7,7} + q^{(1)}_{6,7} \cdot q^{(2)}_{6,7} - 2q^{(1)}_{6,7}q^{(2)}_{6,6})\]

and let \(\mathfrak{F} := [h_{i,j}]_{i,j=1,2,3}\), where \(h_{i,j} := \langle (q^{(i)}_{6,6}, q^{(i)}_{6,7}, q^{(i)}_{7,7}), (q^{(j)}_{6,6}, q^{(j)}_{6,7}, q^{(j)}_{7,7}) \rangle_-\). By direct computation we have

\[(30) \quad \det(q_0 + \sum_{k=1}^{3} \mu_k \cdot q_k) = (\mu_1, \mu_2, \mu_3) \cdot \mathfrak{F} \cdot (\mu_1, \mu_2, \mu_3)^T + \text{(terms of degree} \geq 3)\).

a) Observe that, by (30), the quadric \(Q_0\) is a node of \(S_8\) iff \(\text{rank}(\mathfrak{F}) = 3\).

\(\Rightarrow\): Suppose that there exists a quadric \(Q \neq Q_0\), \(Q \in W\) such that \(\text{sing}(Q_0) \subseteq Q\). If \(Q\) is given by the matrix \([q_{i,j}]_{i,j=0,\ldots,7}\), then \(q_{6,6}, q_{6,7}, q_{7,7}\) vanish, which yields that \(\text{rank}(\mathfrak{F}) < 3\).

\(\Leftarrow\): If \(\text{rank}(\mathfrak{F}) < 3\), then we can find a matrix \(q = [q_{i,j}]_{i,j=0,\ldots,7}\) such that \(q \in \text{span}(\{q_1, q_2, q_3\})\) and the entries \(q_{6,6}, q_{6,7}, q_{7,7}\) vanish. The latter means that the quadric \(Q\) given by \(q\) contains the line \(\text{sing}(Q_0)\). We have \(Q \neq Q_0\) because \(Q_0 \not\in \text{span}(\{Q_1, Q_2, Q_3\})\).

b) By part a) we can assume that \(\text{sing}(Q_0) \subseteq Q_1\), which implies that the entries \(q_{6,6}^{(1)}, q_{6,7}^{(1)}, q_{7,7}^{(1)}\) of the matrix \(q_1\) vanish. Moreover, by (30), the quadric \(Q_0\) is an \(A_m\) singularity, where \(m \geq 2\), iff \(\text{rank}(\mathfrak{F}) = 2\) (see e.g. [9] Prop. 8.14).

\(\Rightarrow\): Suppose that \(P \in \text{sing}(Q_0) \cap \Pi\). Then \(P \in \text{sing}(X_{16})\) and there exists a quadric in the pencil \(\text{span}(\{Q_2, Q_3\})\) that meets the line \(\text{sing}(Q_0)\) only in the point \(P\). In particular we can assume that \(Q_2 \cap \text{sing}(Q_0) = \{P\}\) and \(P := (0 : \ldots : 0 : 1)\). The latter yields

\[
q_{6,6}^{(2)} = 1 \quad \text{and} \quad q_{6,7}^{(2)} = q_{7,7}^{(2)} = 0.
\]

Furthermore, since \(P \in Q_3\) we have \(q_{7,7}^{(3)} = 0\). Then

\[
(\mu_1, \mu_2, \mu_3) \cdot \mathfrak{F} \cdot (\mu_1, \mu_2, \mu_3)^T = -(q_{6,7}^{(3)})^2 \cdot \mu_3^2,
\]

which implies that \(Q_0\) is not an \(A_m\) singularity of the octic surface \(S_8\).

\(\Leftarrow\): By Lemma 4.2.a we have \(\text{rank}(\mathfrak{F}) \leq 2\), so it suffices to show that \(\text{rank}(\mathfrak{F}) \notin \{0, 1\}\).
Assume that \( \text{rank}(\mathcal{I}) = 1 \). This means that
\[
\text{rank} \begin{bmatrix} h_{2,2} & h_{2,3} \\ h_{3,2} & h_{3,3} \end{bmatrix} = 1.
\]
Suppose that the vectors \((q_{6,6}^{(2)}, q_{6,7}^{(2)}, q_{7,7}^{(2)}), (q_{6,6}^{(3)}, q_{6,7}^{(3)}, q_{7,7}^{(3)})\) are linearly independent. By replacing \(q_2\) with an appropriate linear combination of \(q_2, q_3\) we can assume that the first column of the matrix (31) vanishes. Then, from (28) and \(h_{2,2} = 0\) we obtain the equality \(\text{rank}([q_{i,j}]_{i,j=6,7}) = 1\).
Performing an appropriate change of coordinates on the line \(\text{sing}(Q_0)\) we arrive at
\[
q_{6,6}^{(2)} = 1 \quad \text{and} \quad q_{6,7}^{(2)} = q_{7,7}^{(2)} = 0.
\]
Then, the equality \(h_{3,2} = 0\) yields \(q_{7,7}^{(3)} = 0\). The latter implies that
\[
(0 : \ldots : 0 : 1) \in \text{sing}(Q_0) \cap \text{sing}(X_{16}).
\]
Finally, the assumption [A1] gives \(P \in \text{sing}(Q_0) \cap \Pi\).
Suppose that (31) holds and the vectors \((q_{6,6}^{(2)}, q_{6,7}^{(2)}, q_{7,7}^{(2)}), (q_{6,6}^{(3)}, q_{6,7}^{(3)}, q_{7,7}^{(3)})\) are linearly dependent. Then, we can assume that the entries \(q_{6,6}^{(2)}, q_{6,7}^{(2)}, q_{7,7}^{(2)}\) vanish, which implies \(\text{sing}(Q_0) \subset Q_2\). Finally, since the line \(\text{sing}(Q_0)\) is contained in the quadrics \(Q_1, Q_2, \text{each point in the intersection } Q_3 \cap \text{sing}(Q_0)\) is a singularity of \(X_{16}\). By [A1] we have \(\text{sing}(Q_0) \cap \Pi \neq \emptyset\).
In the same way the equality \(\text{rank}(\mathcal{I}) = 0\) implies \(\text{sing}(Q_0) \cap \Pi \neq \emptyset\). We omit the details.

By parts a) and b) we can assume that \(\text{sing}(Q_0) \subset Q_1\) and \(\text{sing}(Q_0) \cap \Pi \neq \emptyset\). Suppose that \(\mathcal{I} = 0\). From \(h_{2,2} = 0\) we obtain (32). Then \(h_{3,2} = 0\) yields \(q_{7,7}^{(3)} = 0\), and by \(h_{3,3} = 0\) the entry \(q_{6,6}^{(3)}\) vanishes. By replacing \(q_3\) with \((q_3 - q_2)\) we obtain the inclusion \(\text{sing}(Q_0) \subset Q_3\).
To complete the proof we assume, as in Section II (see the proof of Remark 1.8), that the plane \(\Pi\) (resp. the line \(\text{sing}(Q_0)\)) is given by vanishing of the coordinates \(x_0, \ldots, x_4\) (resp. \(x_0, \ldots, x_3\) and \(x_6, x_7\)). Observe that the point \(P = (0 : \ldots : 0 : 1 : 0 : 0) \in \text{sing}(Q_0) \cap \Pi\) is a singularity of \(X_{16}\). Therefore, Lemma 1.4 b yields that the quadrics \(Q_1, Q_2, Q_3\) are smooth in \(P\). By direct computation, there exist \(v_1, \ldots, v_4 \in \mathbb{C}\) such that the intersection of the tangent spaces \(T_P Q_1, T_P Q_2, T_P Q_3\) is parametrized by the map
\[
(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (\lambda_1 v_1, \lambda_1 v_2, \lambda_1 v_3, \lambda_1 v_4, \lambda_2, \lambda_3, \lambda_4).
\]
Substituting the above parametrization to (dehomogenized) \(Q_0\) we see that the tangent cone \(C_{P} X_{16}\) is contained in union of two 3-planes, so the point \(P \in X_{16}\) is not a node. Contradiction (see Lemma 1.6).

**Remark 4.3.** Direct computation with help of [15], gives examples of webs of quadrics such that the assumptions [A1], [A2], [A3] [A4] are fulfilled and the quadric \(Q_0\) satisfies the conditions of Lemma 4.2 b. One can check that for generic choice of the quadrics one obtains an A3 singularity of the discriminant octic \(S_8\).

To complete the description of singularities of \(S_8\) we prove the following lemma.

**Lemma 4.4.** A quadric \(Q_0 \in W\) is a point of multiplicity at least 3 on \(S_8\) iff \(\text{rank}(q_0) = 5\).

**Proof.** (\(\Rightarrow\)): Lemmata 4.1, 4.2 imply that \(\text{rank}(Q) \leq 5\). Remark 1.8 completes the proof.
(\(\Leftarrow\)): Assume that \(q_0 = I_5\) and compute the determinant \(\det(q_0 + \sum_{k=1}^{3} \mu_k \cdot q_k)\).

The example below shows that the bound of Remark 1.8 is sharp, and the discriminant octic \(S_8\) can have triple points.
Example 4.5. We define the following matrices:

\[
q_0 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & -4 & 0 & -2 & 1 \\
0 & 0 & 0 & 4 & 3 & 0 & 2 -4 \\
0 & 0 & -4 & 3 & 8 & 0 & -5 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 2 & -5 & 0 & 0 0 \\
0 & 0 & 1 & -4 & 0 & 0 & 0
\end{bmatrix}
\]

\[
q_1 := \begin{bmatrix}
-4 & -4 & 2 & -1 & 0 & -1 -3 \\
-4 & 2 & 0 & 0 & 4 & -2 & 0 -1 \\
2 & 0 & 0 & -1 & 2 & -2 & 4 2 \\
-1 & 0 & -1 & 2 & 3 & -1 & 3 -2 \\
0 & 4 & 2 & 3 & -4 & -2 & 0 1 \\
-1 & -2 & -2 & -1 & -2 & 0 & 0 0 \\
-1 & 0 & 4 & 3 & 0 & 0 & 0 0 \\
-3 & -1 & 2 & -2 & 1 & 0 & 0 0
\end{bmatrix}
\]

\[
q_2 := \begin{bmatrix}
4 & -3 & -3 & -2 & 1 & -3 -3 & -1 \\
-3 & -2 & -3 & -4 & 1 & 4 3 1 \\
-3 & -3 & 4 & 1 & 0 & 1 1 1 \\
-2 & 4 & 1 & 2 & -2 & 0 1 4 \\
1 & 1 & 0 & -2 & 4 & -1 0 -1 \\
-3 & 4 & 1 & 1 & 0 & 0 0 0 \\
-3 & 3 & 1 & 1 & 0 & 0 0 0 \\
-1 & 1 & 1 & 4 & -1 & 0 0 0
\end{bmatrix}
\]

\[
q_3 := \begin{bmatrix}
4 & 1 & 2 & 2 & -2 & 1 -2 & 0 \\
-1 & 2 & 2 & -3 & -1 & 4 -2 & 4 \\
2 & 2 & -2 & -1 & 1 & 3 2 -1 \\
2 & -3 & -1 & -2 & 0 & 1 3 -2 \\
-2 & -1 & 1 & 0 & -4 & 4 1 -1 \\
-1 & -4 & 3 & 1 & 4 & 0 0 0 \\
-2 & -2 & 2 & 3 & 1 & 0 0 0 \\
0 & 4 & -1 & -2 & -1 & 0 0 0
\end{bmatrix}
\]

By direct computation with help of \([15]\), the intersection in \(\mathbb{P}_7\) of the quadrics defined by the above matrices satisfies the assumptions \([A1], \ldots, [A4]\). As one can easily see, we have \(\text{rank}(q_0) = 5\).

We put \(\pi_2 : X_8 \rightarrow W\) to denote the double cover of the web \(W\) branched along the discriminant surface \(S_8\). We have the following theorem (compare \([24, \text{Thm 3.1}]\)).

**Theorem 4.6.** Assume that \([A1], \ldots, [A4]\) hold.

a) There exists a (small) resolution \(\hat{\phi} : \hat{X}_5 \rightarrow X_8\) of singularities of the double octic \(X_8\) such that the following diagram commutes:

\[
\begin{array}{ccc}
\hat{X}_5 & \xrightarrow{\phi} & \mathbb{P}_3 \\
\downarrow{\hat{\phi}} & & \downarrow{\pi_2} \\
X_8 & \xrightarrow{\pi_1} & \mathbb{P}_3
\end{array}
\]

b) Let \(\pi\) be the map induced by the projection from the plane \(\Pi\) (see \([9]\)) and let \(\sigma\) (resp. \(\psi\)) be the blow up defined by \([8]\) (resp. \([21]\)). Then the composition

\[
X_{16} \xrightarrow{\sigma^{-1}} \hat{X}_{16} \xrightarrow{\pi} X_5 \xrightarrow{\psi^{-1}} \hat{X}_5 \xrightarrow{\hat{\phi}} X_8
\]

is a birational map between the base locus of the web \(W\) and its double cover branched along the discriminant surface \(S_8\). In particular, the base locus \(X_{16}\) and the discriminant double octic \(X_8\) are birational to the quintic 3-fold \(X_5\) (see \([12]\)) that contains Bordiga sextic.

**Proof.** a) Consider Stein factorization of the map \(\phi : \hat{X}_5 \rightarrow \mathbb{P}_3\):

\[
\phi = \hat{\phi} \circ \phi'
\]

where \(\phi'\) is finite and \(\hat{\phi}\) has connected fibers. By Cor. \([82]\) the map \(\phi'\) is a (ramified) double cover of \(\mathbb{P}_3\). Thm \([8, \text{Thm 3.6}]\) and the assumption \([A4]\) imply the equality \(\phi' = \pi_2\). Then the map \(\hat{\phi} : \hat{X}_5 \rightarrow X_8\) is birational (see e.g. \([9\text{, p. 11}]\)). Thm \([8, \text{Thm 3.6}]\) implies that the set of 1-dimensional fibers of the latter map coincides with \(\hat{\phi}^{-1}(\text{sing}(X_8))\). This completes the proof.

b) We have just shown that the map \(\hat{\phi}\) is birational. The claim follows from Lemma \([2, \text{Lemma a}]\).
In the case of the double sextic defined by a net of quadrics that contain a (fixed) line the discriminant curve has only nodes as singularities (see [21, Thm 3.3]).

In the corollary below we discuss the singularities of the discriminant surface $S_8$.

**Corollary 4.7.** Assume that $[A1], \ldots, [A4]$ hold.

a) The equality $\sum_{P \in \text{sing}(X_8)} (\mu(P, X_8) + 1) = 188$ holds, where $\mu(P, X_8)$ stands for the Milnor number of $X_8$ in the point $P$.

b) A quadric $Q_0 \in W$ is a singularity of $S_8$ of the type given in the first column of the table below iff it satisfies the conditions listed in the other column.

| Type of singularity | Conditions |
|---------------------|------------|
| smooth point        | rank($q_0$) | 7 |
| $A_1$               | sing($Q_0$) $\cap X_{16} = \emptyset$ |
| $A_m, m \geq 3, m$  | 6 |
| odd                | $\{Q \in W : Q \neq Q_0, \text{sing}(Q_0) \subset Q\} = \emptyset$ |
| double point of corank 2 | 6 |
| $k$-fold point, $k \geq 3$ | 5 |
|                      | $\{Q \in W : Q \neq Q_0, \text{sing}(Q_0) \subset Q\} \neq \emptyset$ |

**Proof.** a) To compute the sum of Milnor numbers of singularities of $X_8$ we compare topological Euler numbers of $X_5$ and $X_8$. By the assumption $[A3]$ and Lemma 2.7 we have $e(X_5) = -108$. On the other hand, by Chern class argument the Euler number of a smooth octic in $\mathbb{P}_3$ is 304, so [10, Cor. 5.4.4] implies $e(X_8) = -296 + \sum_{P \in \text{sing}(X_8)} \mu(P, S_8)$. Observe that in our set-up the equality $\mu(P, S_8) = \mu(P, X_8)$ holds. From Thm 4.6.a we get

$$-108 + \#(\text{sing}(X_8)) = -296 + \sum_{P \in \text{sing}(X_8)} \mu(P, X_8).$$

that yields the claim.

b) By Thm 4.6.b and [28, Cor. 1.16] the octic $S_8$ has no $A_m$ points with $m$ even. The claim follows now directly from Lemmata 4.1, 4.2 and Lemma 4.4. \hfill $\square$

**Remark 4.8.** Under the assumptions $[A1], \ldots, [A4]$ the following inequality holds

$$\# \{P \in \text{sing}(S_8) : P \text{ is not an } A_m \text{ point, where } m \geq 1\} \leq 10.$$  

**Proof.** By Lemmata 4.1, 4.2 each double point $Q_0 \in \text{sing}(S_8)$ that is not an $A_m$ singularity is a singular quadric and its singular locus meets the plane $\Pi$. The same holds for rank-5 quadrics in the web $W$ (see Thm 3.6). Therefore, the inequality results from Remark 3.7. \hfill $\square$

**Final remarks:** a) According to [21, Thm 4.1] the normal bundle a smooth rational curve that is contracted on a 3-fold is one of the following: $(\mathcal{O}_{\mathbb{P}_3}(-1) \oplus \mathcal{O}_{\mathbb{P}_3}(-1))$, $(\mathcal{O}_{\mathbb{P}_3}(-2) \oplus \mathcal{O}_{\mathbb{P}_3})$, $(\mathcal{O}_{\mathbb{P}_3}(-3) \oplus \mathcal{O}_{\mathbb{P}_3}(1))$. Remark 1.3 and Ex. 4.1 show that all such bundles can come up in our set-up. For the conditions imposed on the equation of a (smooth) 3-fold quintic in $\mathbb{P}_4$ by the normal bundle of a contracted curve the reader should consult [20, App. A, B].

b) Assume that all singularities of $S_8$ are A-D-E points. By [3, Thm 1.1] the Hodge diamond of any small Kähler resolution of the double octic $X_8$ coincides with the one given in Lemma [17]. In view of [29, Cor. 5.1] and [ibid., Prop. 6.1], the latter implies that the assumptions $[A1], \ldots, [A4]$ determine position of singularities of $S_8$ with respect to sections of $\mathcal{O}_{\mathbb{P}_3}(8)$ (compare [24, Prop. 2.13]).
c) In Thm 3.6 we describe components of $\Phi^{-1}(y)$ when $\text{rank}(b(y)) = 2$. Since all singularities of $X_8$ admit a small resolution, [25 Thm 5.5] can be applied to obtain a more precise description of such fibers. We omit details because of lack of space.

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References

[1] N. Addington, *The Derived Category of the Intersection of Four Quadrics*, preprint arXiv:0904.1764

[2] A. Alzati, F. Russo, *Some extremal contractions between smooth varieties arising from projective geometry*, Proc. Lond. Math. Soc. (3) 89 (2004), 25–53.

[3] V. Batyrev, *Birational Calabi–Yau n-folds have equal Betti numbers*, in New trends in algebraic geometry (Warwick, 1996), 1–11, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999.

[4] R. Braun, *On a geometric property of the normal bundle of surfaces in $\mathbb{P}_4$*, Math. Z. 206 (1991), 535–550.

[5] W. Bruns, U. Vetter, *Determinantal rings*. Springer Lecture Notes 1327, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[6] I. Cheltsov, *Factorial threefold hypersurfaces*. J. Algebraic Geom. 19 (2010), 781791.

[7] S. Cynk, S. Rams, *On a map between two K3 surfaces associated to a net of quadrics*. Arch. Math. (Basel) 88 (2007), 353-359.

[8] O. Debarre, *Higher-Dimensional Algebraic Geometry*. Springer 2001.

[9] A. Dimca, *Topics in real and complex singularities*. Vieweg Advanced Lectures in Mathematics, Vieweg, Braunschweig/Wiesbaden 1987.

[10] A. Dimca, *Singularities and topology of hypersurfaces*. Springer-Verlag, New York, 1992.

[11] R. Draper, *Intersection theory in analytic geometry*. Math. Ann. 180 (1969), 175-204.

[12] J. A. Eagon, D. G. Northcott, *Ideals defined by matrices and a certain complex associated to them*, Proc. Roy. Soc. London Ser. A 269 (1962), 188-204.

[13] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics 150, Springer-Verlag, New York, 1999.

[14] W. Fulton, *Intersection theory*. Springer-Verlag, New York, 1984.

[15] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, *SINGULAR 3-1-2 — A computer algebra system for polynomial computations*. http://www.singular.uni-kl.de (2010).

[16] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*. John Wiley and Sons Inc., 1978.

[17] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Mathematics 52, Springer-Verlag, New York, 1977.

[18] R. Janik, *Birational maps between K3 surfaces*. masterthesis (in Polish) Cracow (1998).

[19] T. Jozefiak, A. Lasoux, P. Pragacz, *Classes of determinantal varieties associated with symmetric and skew-symmetric matrices*. Math. USSR Izv. 18 (1982), 575-586.

[20] S. Katz, *On the finiteness of rational curves on quintic threefolds*. Compositio Mathematica, 60 (1986), 151–162.

[21] H. B. Laufer, *On $\mathbb{CP}_1$ as an exceptional set*, in Recent developments in several complex variables, Proc. Conf. Princeton Univ. 1979, Ann. Math. Stud. 100 (1981), 261–275.

[22] C. Madonna, V.V. Nikulin, *On a classical correspondence between K3 surfaces I*. Proc. Steklov Math. Inst. Vol. 241 (2003), 120-153.

[23] C. Madonna, V.V. Nikulin, *On a classical correspondence between K3 surfaces II*. In *Strings and Geometry*, Clay Math. Proc. Vol. 3 (2003), 285-300.

[24] M. Michalek, *Birational maps between Calabi-Yau manifolds associated to webs of quadrics*. preprint (2009), arXiv: math.AG/0904.4404v4.

[25] D. R. Morrison, *The birational geometry of surfaces with rational double points*. Math. Ann. 271 (1985), no. 3, 415438.

[26] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*. Invent. Math. 77 (1984), 101–116.

[27] P. Pragacz, *Cycles of isotropic subspaces and formulas for symmetric degeneracy loci*. in Topics in Algebra, Banach Center Publications, vol. 26 no. 2 (1990), 189-199.

[28] M. Reid, *Minimal models of canonical 3-folds*, Algebraic varieties and analytic varieties, Proc. Symp., Tokyo 1981, Adv. Stud. Pure Math. 1 (1983), 131–180.

[29] S. Rams, *Defect and Hodge numbers of hypersurfaces*. Adv. Geom. 8 (2008), 257-288.

[30] T. G. Room, *The geometry of determinantal loci*. Cambridge University Press, Cambridge 1938.
