Inelastic Interaction and Blowup New Solutions of Nonlinear and Dispersive Long Gravity Waves

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In this paper, the fractional Broer–Kaup (BK) system is investigated by studying its novel computational wave solutions. These solutions are constructed by applying two recent analytical schemes (modified Khater method and sech–tanh function expansion method). The BK system simulates the bidirectional propagation of long waves in shallow water. Moreover, it is used to study the interaction between nonlinear and dispersive long gravity waves. A new fractional operator is used to convert the fractional form of the BK system to a nonlinear ordinary differential system with an integer order. Many novel traveling wave solutions are constructed that do not exist earlier. These solutions are considered the icon key in the inelastic interaction of slow ions and atoms, where they were able to explain the physical nature of the nuclear and electronic stopping processes. For more illustration, some attractive sketches are also depicted for the interpretation physically of the achieved solutions.

1. Introduction

Studying the energetic atomic projectiles (atoms or ions) is one of the most exciting recent fields which has impacts on surface physics, plasma, and some related applications [1–3]. Moreover, it has an essential role which multi-charged ions play as part of the solar wind in the space environment [4]. These applications include plasma–wall interaction in controlled thermonuclear fusion devices [5], surface analysis [6], single–particle detection [7], and electrical discharges [8]. The projectile energy deposition is connected with the intensity and nature of ion (atom) surface. Thus, both the nature and intensity of the ion depend on the potential and kinetic energies carried by a projectile toward the surface [9]. The energetic ion that can penetrate a solid surface is able to transfer its kinetic energy to target electrons, leading to atomic excitation, collective electron excitations, or ionization [10].

Partial differential equations (PDEs) have been playing an essential role in the energetic atomic projectiles where many nonlinear evolution equations have been derived to describe the dynamical behaviour of several phenomena in atomic and nuclear physics. Thus, partial differential equations (PDEs) have been playing an important role in the emerging technologies where many nonlinear evolution equations have been derived to describe the dynamical behaviour of some distinct phenomena, for example, nonlinear optics, fluid dynamics, Bose–Einstein condensates, and quantum mechanics. However, the inadequacy of the PDEs with an integer order has been clarified because of the nonlocal property where this kind of equation does not explain that kind of properties. Therefore, several nature phenomena have been formulated with nonlinear PDEs with fractional order [11–35]. Thus, many fractional operators have been derived such as conformable fractional derivative, fractional Riemann–Liouville derivatives, Caputo, Caputo–Fabrizio definition, and so on [36–40].
These definitions have been employed to convert the fractional nonlinear partial differential equations to a nonlinear integer-order ordinary differential equation. Then, the computational and numerical schemes can be applied to get various types of solutions for these models and the examples of these schemes.

The \( \mathcal{B} \mathcal{R} \) fractional operator is considered one of the most general recent fractional operators that is derived from avoiding the deficiencies and defects of some other fractional operator. This operator is defined as follows.

**Definition 1.** It is given by [41–44]

\[
\mathcal{B} \mathcal{R}^\alpha_a f(t) = \frac{B(\alpha)}{1 - \alpha} \frac{d}{dt} \int_a^t f(x) \left( \frac{-\alpha (t - a)^\alpha}{1 - \alpha} \right) dx,
\]

(1)

where \( g_a \) stands for the Mittag–Leffler function, given by [45, 46]

\[
g_a \left( \frac{-\alpha (t - a)^\alpha}{1 - \alpha} \right) = \sum_{n=0}^{\infty} \left( \frac{-\alpha(1 - \alpha)^n}{1 - \alpha} \right) \frac{(t - a)^n}{n!} \Gamma(an + 1)
\]

(2)

and \( B(\alpha) \) being a normalization function. Thus,

\[
\mathcal{B} \mathcal{R}^\alpha_a f(x) = \frac{B(\alpha)}{1 - \alpha} \sum_{n=0}^{\infty} \frac{(-\alpha a^{-1})^n}{1 - \alpha} \mathcal{F}^a f(x)
\]

(3)

This research paper studies the fractional BK system that is given by [47–49].

\[
\begin{aligned}
\mathcal{D}^\alpha_t \delta &= \frac{1}{2} (\delta^2 + 2 \mathcal{W} - \delta_x)_x, \\
\mathcal{D}^\alpha_t \mathcal{W} &= \left( \mathcal{D} \mathcal{W} + \frac{1}{2} \mathcal{W}_x \right)_x,
\end{aligned}
\]

(4)

where \([0 < \alpha < 1], \mathcal{D} = \mathcal{D}(x, t), \mathcal{W} = \mathcal{W}(x, t)\) is unknown functions in time \((t)\) and place \((x)\). System (3) is used to simulate the bidirectional propagation of long waves in shallow water. Moreover, it studies the interaction between dispersive and nonlinear long gravity waves. System (3) is an integrable system and is related to the classical Boussinesq equation that is given by [50]

\[
\mathcal{B}_t - \lambda^2 \mathcal{B}_{xx} - \beta \mathcal{B}_{xxxx} - \alpha (\mathcal{B}^3)_{xx} = 0,
\]

(5)

where \([\lambda, \alpha, \beta] \) are arbitrary constants. Equation (5) is a model of nonlinear dispersive waves. Employing the next wave transformation (the \( \mathcal{B} \mathcal{R} \) fractional operator)

\[
Y = ((1 - \alpha)(c t^{-n})/B(\alpha)\sum_{n=0}^{\infty}(-\alpha(1 - \alpha)^n) / n! T(1 - an)) + x,
\]

\( \delta = \delta(Y), \mathcal{W} = \mathcal{W}(Y) \), where \((c)\) is an arbitrary constant to the system (3); then, differentiate the results once and substitute the second equation of the system into the first equation leading to convert the system into the next equation

\[
4 \epsilon^2 \delta^4 - 6 \epsilon \delta^2 + 2 \delta^3 - \delta'' = 0.
\]

(6)

Applying the homogeneous balance principle to Equation (6), yields \((m = 1)\).

The rest of research paper is organized as follows: Section 2 applies the modified Khater method and sech–tanh functions expansion method to the suggested model to get novel solitary wave solutions of it. Section 3 explains the physical interpretation of the shown sketches in our paper. Section 4 shows the novelty of our paper by comparing our results with those obtained in previous research papers. Section 5 explains the conclusion of all the steps of our paper in detail.

## 2. Application

Here, in this section, the modified Khater method and sech–tanh functions expansion method are applied to the \( \mathcal{B} \mathcal{R} \) equation to simulate the bi–directional propagation of long waves in shallow water.

2.1. The Modified Khater Method. Applying the modified Khater method to Equation (6) leads to formulate the general solution of this model in the following formula

\[
\delta(Y) = \sum_{i=1}^{m} a_i \mathcal{H}^{i\mathcal{H}(Y)} + \sum_{i=1}^{m} b_i \mathcal{H}^{-i\mathcal{H}(Y)} + a_0
\]

(7)

\[
= a_1 \mathcal{H}^{i\mathcal{H}(Y)} + a_0 + b_1 \mathcal{H}^{-i\mathcal{H}(Y)},
\]

where \([a_0, a_1, b_1]\) are arbitrary constants to be determined later. Additionally, \( \mathcal{F}(Y) \) is the solution function of the following ordinary differential equation

\[
\mathcal{F}^{i\mathcal{H}} = \frac{1}{\ln(\mathcal{H})} \left[ \delta \mathcal{H}^{i\mathcal{H}(Y)} + \xi \mathcal{H}^{-i\mathcal{H}(Y)} + \chi \right],
\]

(8)

where \([\delta, \xi, \chi]\) are arbitrary constants. Substituting Equation (7) (along (7)) into Equation (6) and collecting all terms with the same power of \([\mathcal{H}^{i\mathcal{H}(Y)}, i = -5, -4, \ldots, 4, 5]\), give a system of the algebraic equation. Using the Mathematica 12 program for solving this system, yields

**Family 2.**

\[
\begin{aligned}
a_0 &\rightarrow \frac{1}{2} \sqrt{X^2 - 4\delta Q} + \chi, a_1 &\rightarrow \delta, b_1 \\
&\rightarrow 0, c &\rightarrow \frac{1}{2} \sqrt{X^2 - 4\delta Q}.
\end{aligned}
\]

(9)

Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas
For $[\chi^2 - 4\delta Q > 0 \& \delta \neq 0]$, 
\[
\delta_1(x, t) = -\frac{1}{2} \sqrt{\chi^2 - 4\delta Q} \left( \frac{1}{2} \chi \sqrt{\chi^2 - 4\delta Q} \right) \left( \frac{1}{2} \chi \sqrt{\chi^2 - 4\delta Q} \right) - 1, \\
\delta_2(x, t) = -\frac{1}{2} \sqrt{\chi^2 - 4\delta Q} \left( \frac{1}{2} \chi \sqrt{\chi^2 - 4\delta Q} \right) - 1.
\]
(10)

For $[\delta Q < 0 \& \delta \neq 0 \& \chi = 0]$,
\[
\delta_3(x, t) = -\sqrt{-\delta Q} \left( \frac{1}{B(\alpha) \sum_{n=0}^{\infty} (-\alpha/(1-\alpha))^n \gamma'(1-an)} \right), \\
\delta_4(x, t) = -\sqrt{-\delta Q} \left( \frac{1}{B(\alpha) \sum_{n=0}^{\infty} (-\alpha/(1-\alpha))^n \gamma'(1-an)} \right).
\]
(12)

For $[\chi = 0 \& \delta Q = -\delta]$,
\[
\delta_5(x, t) = \sqrt{\chi^2 - \delta Q} \coth \left( \frac{1}{2} \frac{\chi}{\sqrt{\chi^2 - \delta Q}} - \frac{(\alpha - 1) \sqrt{\chi^2 - \delta Q}}{B(\alpha) \sum_{n=0}^{\infty} (-\alpha/(1-\alpha))^n \gamma'(1-an)} \right).
\]
(13)

For $[\chi = 0 \& \delta Q = 0]$,
\[
\delta_6(x, t) = \frac{1}{2} \left( \frac{1}{\delta} \exp \left( \frac{4}{\delta \exp \left( \chi - (\alpha - 1) \sqrt{\chi^2 - \delta Q}/2B(\alpha) \sum_{n=0}^{\infty} (-\alpha/(1-\alpha))^n \gamma'(1-an)) \right) - 2 \right) + \sqrt{\chi^2} \right).
\]
(15)

Family 3.

Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas

For $[\chi^2 - 4\delta Q > 0 \& \delta \neq 0]$, 
\[
\delta_7(x, t) = \frac{1}{2} \left( \frac{4\delta Q}{\sqrt{\chi^2 - 4\delta Q} \tanh \left( \frac{1}{2} \chi \sqrt{\chi^2 - 4\delta Q} - (\alpha - 1) \sqrt{\chi^2 - 4\delta Q}/4B(\alpha) \sum_{n=0}^{\infty} (-\alpha/(1-\alpha))^n \gamma'(1-an) \right) + \chi \right) + \sqrt{\chi^2 - 4\delta Q} + \chi.
\]
(17)

Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas

For $[\chi^2 - 4\delta Q > 0 \& \delta \neq 0]$, 
\[
\delta_8(x, t) = \frac{1}{2} \left( \frac{4\delta Q}{\sqrt{\chi^2 - 4\delta Q} \coth \left( \frac{1}{2} \chi \sqrt{\chi^2 - 4\delta Q} - (\alpha - 1) \sqrt{\chi^2 - 4\delta Q}/4B(\alpha) \sum_{n=0}^{\infty} (-\alpha/(1-\alpha))^n \gamma'(1-an) \right) + \chi \right) + \sqrt{\chi^2 - 4\delta Q} + \chi.
\]
(18)
For \([\delta_q < 0 \& q \neq 0 \& \delta \neq 0 \& \chi = 0]\),

\[
\mathcal{S}_{10}(x, t) = \sqrt{-\delta_q} \left( \coth \left( \frac{(a - 1)\delta_q r^\alpha}{B(a) \sum_{n=0}^{\infty} (-\alpha/(1 - \alpha)) n \Gamma(1 - an)} + x \sqrt{-\delta_q} \right) + 1 \right),
\]

\[
\mathcal{S}_{11}(x, t) = \sqrt{-\delta_q} \left( \tanh \left( \frac{(a - 1)\delta_q r^\alpha}{B(a) \sum_{n=0}^{\infty} (-\alpha/(1 - \alpha)) n \Gamma(1 - an)} + x \sqrt{-\delta_q} \right) + 1 \right). 
\]

(19)

For \([\chi = 0 \& q = -\delta]\),

\[
\mathcal{S}_{12}(x, t) = q \tanh \left( \rho \left( x - \frac{(a - 1)\sqrt{q^2 r^\alpha}}{B(a) \sum_{n=0}^{\infty} (-\alpha/(1 - \alpha)) n \Gamma(1 - an)} \right) \right) + \sqrt{q^2}.
\]

(20)

For \([\chi = q/2 = \kappa \& \delta = 0]\),

\[
\mathcal{S}_{13}(x, t) = \frac{1}{2} \left( \frac{4\kappa}{\exp \left( \kappa \left( x - (a - 1)\sqrt{\kappa^2 r^\alpha}/2B(a) \sum_{n=0}^{\infty} (-\alpha/(1 - \alpha)) n \Gamma(1 - an) \right) \right) - 2 + \sqrt{\kappa^2 + \kappa} \right).
\]

(21)

For \([\delta = 0 \& \chi \neq 0 \& q \neq 0]\),

\[
\mathcal{S}_{14}(x, t) = \frac{1}{2} \left( \frac{2\chi q}{q - \chi \exp \left( \chi \left( x - (a - 1)\sqrt{\chi^2 r^\alpha}/2B(a) \sum_{n=0}^{\infty} (-\alpha/(1 - \alpha)) n \Gamma(1 - an) \right) \right) \right) + \sqrt{\chi^2 + \chi} \right).
\]

(22)

2.2. The Sech–Tanh Function Expansion Method. Applying the sech–tanh function expansion method to Equation (6) leads to formulate the general solution of this model in the following formula:

\[
\mathcal{S}(Y) = \sum_{i=1}^{m} \sec h^{-1}(Y) (a_i \sech(Y) + b_i \tanh(Y)) + a_0
\]

\[
= a_1 \sech(Y) + a_0 + b_1 \tanh(Y),
\]

(23)

where \([a_0, a_1, b_1]\) are arbitrary constants to be determined later. Substituting Equation (23) into Equation (6) and collecting all terms with the same power of \([\sech(Y), \tanh(Y)]\), give a system of the algebraic equation. Using the Mathematica 12 program for solving this system, yields

**Family 4.**

\[
[a_0 \longrightarrow -1, a_1 \longrightarrow 0, b_1 \longrightarrow 1, c \longrightarrow -1]
\]

(24)

Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas:

\[
\mathcal{S}_{15}(x, t) = \tanh \left( \frac{(a - 1)f^\alpha}{B(a) \sum_{n=0}^{\infty} (-\alpha/(1 - \alpha)) n \Gamma(1 - an)} + x \right) - 1.
\]

(25)

**Family 5.**

\[
[a_0 \longrightarrow 1, a_1 \longrightarrow 0, b_1 \longrightarrow 1, c \longrightarrow 1]
\]

(26)
Thus, the explicit wave solutions of Equation (4) are formulated in the following formulas:

\[
S_{16}(x,t) = \tanh \left( x - \frac{1}{B(\alpha)} \sum_{n=0}^{\infty} \frac{(\alpha - 1)^{n}}{n!} T(1 - \alpha n) \right) + 1.
\]

(27)

3. Figure Interpretation

This section gives the physical interpretation of the shown figures in our paper. All our obtained solutions are considered traveling wave solutions. In the following lines, we give the physical interpretation of the shown figures:

(1) Figure 1 shows the bell wave solution (8) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the \( x \)-axis and the contour plot (b) to explain the overhead view of the solution when \( \alpha = (1/2), \delta = 1, \chi = 3, q = 2 \).

(2) Figure 2 shows the singular wave solution (13) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the \( x \)-axis and the
Figure 3: Numerical simulation of Equation (17) in two-dimensional and contour sketches.

Figure 4: Numerical simulation of Equation (18) in two-dimensional and contour sketches.

(3) Figure 3 shows the W-wave solution (14) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the contour plot (b) to explain the overhead view of the solution when $\alpha = (1/2), \delta = 1, \chi = 3.$

(4) Figure 4 shows the rouge-wave solution (15) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the contour plot (b) to explain the overhead view of the solution when $\alpha = (1/2), \delta = 1, \chi = 5, \varrho = 6.$

(5) Figure 5 shows the Breath-wave solution (21) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the contour plot (b) to explain the overhead view of the solution when $\alpha = (1/2).$

(6) Figure 6 shows the kink-wave solution (22) in the two-dimensional plot (a) to explain the wave propagation pattern of the wave along the x-axis and the contour plot (b) to explain the overhead view of the solution when $\alpha = (1/2).$
contour plot (b) to explain the overhead view of the solution when \( \alpha = (1/2) \).

**4. Results and Discussion**

This section shows the novelty of this research paper by explaining the comparison between our obtained solutions and those obtained in previous paper.

**4.1. Computational Schemes.** This paper investigated the analytical solutions of the fractional BK system by using two recent computational schemes (modified Khater method and sech–tanh function expansion method). These methods are considered recent analytical schemes in this field, and they were not applied to this model yet.

**4.2. Obtained Computational Wave Solutions**

(i) Equation (10) is equal to Equation (25) when \( \chi^2 - 4\eta \delta = 4 \)

(ii) All our obtained solutions are different from those obtained in [47–49] where the authors of [47–49] applied different methods to solve the BK system with an integer order. On the other hand, we investigated the fractional form of this model

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**Figure 5:** Numerical simulation of Equation (25) in two–dimensional and contour sketches.

**Figure 6:** Numerical simulation of Equation (27) in two–dimensional and contour sketches.
5. Conclusion

In this paper, we investigated new soliton wave solutions of the nonlinear fractional BK system by using two recent analytical schemes. A new fractional operator is used to convert the fractional form of this model to a nonlinear partial differential equation with an integer order. The modified Khater method and sech–tanh function expansion method were applied to this model. Some new soliton wave solutions were obtained, and some of them were explained by plotting them in two–dimensional and contour plots of these solutions. The novelty of our paper was shown by making the comparison between our obtained solutions and that was purchased in previously published articles.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Ethical Approval

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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