LINEAR OPERATORS WITH COMPACT SUPPORTS, PROBABILITY MEASURES AND MILITYUTIN MAPS

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Abstract. The notion of a regular operator with compact supports between function spaces is introduced. On that base we obtain a characterization of absolute extensors for 0-dimensional spaces in terms of regular extension operators having compact supports. Milyutin maps are also considered and it is established that some topological properties, like paracompactness, metrizability and \( k \)-metrizability, are preserved under Milyutin maps.

1. Introduction

In this paper we assume that all topological spaces are Tychonoff. The main concept is that one of a linear map between function spaces with compact supports. Let \( u : C(X, E) \rightarrow C(Y, E) \) be a linear map, where \( C(X, E) \) is the set of all continuous functions from \( X \) into a locally convex linear space \( E \). We say that \( u \) has compact supports if for every \( y \in Y \) the linear map \( T(y) : C(X, E) \rightarrow E \), defined by \( T(y)(h) = u(h)(y), h \in C(X, E) \), has a compact support in \( X \). Here, the support of a linear map \( \mu : C(X, E) \rightarrow E \) is the set \( s(\mu) \) of all \( x \in \beta X \) such that for every neighborhood \( U \) of \( x \) in \( \beta X \) there exists \( h \in C(X, E) \) with \( (\beta h)(\beta X - U) = 0 \) and \( \mu(h) \neq 0 \). Recall that \( \beta X \) is the Čech-Stone compactification of \( X \) and \( \beta h : \beta X \rightarrow \beta E \) the extension of \( h \). Obviously, \( s(\mu) \subset \beta X \) is closed, so compact. When \( s(\mu) \subset X \), \( \mu \) is said to have a compact support.

In a similar way we define a linear map with compact supports when consider the bounded function sets \( C^*(X, E) \) and \( C^*(Y, E) \) (if \( E \) is the real line \( \mathbb{R} \), we simply write \( C(X) \) and \( C^*(X) \)). If all \( T(y) \) are regular linear maps, i.e., \( T(y)(h) \) is contained in the closed convex hull \( \text{convh}(X) \) of \( h(X) \) in \( E \), then \( u \) is called a regular operator.

Haydon [19] proved that Dugundji spaces introduced by Pelczynski [26] coincides with the absolute extensors for 0-dimensional compact spaces (br., \( X \in AE(0) \)). Later Chigogidze [10] provided a more general definition of \( AE(0)- \) spaces in the class of all Tychonoff spaces. The notion of linear operators with
compact supports arose from the attempt to find a characterization of $AE(0)$-spaces similar to the Pelczynski definition of Dugundji spaces. Here is this characterization (see Theorems 4.1-4.2). For any space $X$ the following conditions are equivalent: (i) $X$ is an $AE(0)$-space; (ii) for every $C$-embedding of $X$ in a space $Y$ there exists a regular extension operator $u: C(X) \to C(Y)$ with compact supports; (iii) for every $C$-embedding of $X$ in a space $Y$ there exists a regular extension operator $u: C^*(X) \to C^*(Y)$ with compact supports; (iv) for any $C$-embedding of $X$ in a space $Y$ and any complete locally convex space $E$ there exists a regular extension operator $u: C^*(X, E) \to C^*(Y, E)$ with compact supports.

It is easily seen that $u: C(X, E) \to C(Y, E)$ (resp., $u: C^*(X, E) \to C^*(Y, E)$) is a regular extension operator with compact supports iff there exists a continuous map $T: Y \to P_c(X, E)$ (resp., $T: Y \to P^*_c(X, E)$) such that $T(y)$ is the Dirac measure $\delta_y$ at $y$ for all $y \in X$. Here, $P_c(X, E)$ (resp., $P^*_c(X, E)$) is the space of all regular linear maps $\mu: C(X, E) \to E$ (resp., $\mu: C^*(X, E) \to E$) with compact supports equipped with the pointwise convergence topology (we write $P_c(X)$ and $P^*_c(X)$ when $E = \mathbb{R}$). Section 2 is devoted to properties of the functors $P_c$ and $P^*_c$ (actually, $P^*_c$ is the well known functor $P_\beta$ [9] of all probability measures on $\beta X$ whose supports are contained in $X$). It appears that $P_c(X)$ is homeomorphic to the closed convex hull of $e_X(X)$ in $\mathbb{R}^{C(X)}$ provided $X$ is realcompact, where $e_X$ is the standard embedding of $X$ into $\mathbb{R}^{C(X)}$ (Proposition 2.4), and $P_c(X)$ is metrizable iff $X$ is a metric compactum (Proposition 2.5(ii)).

In Section 3 we consider regular averaging operators with compact support and Milyutin maps. Milyutin maps between compact spaces were introduced by Pelczynski [26]. There are different definitions of Milyutin maps in the non-compact case, see [1], [28] and [37]. We say that a surjection $f: X \to Y$ is a Milyutin map if $f$ admits a regular averaging operator $u: C(X) \to C(Y)$ having compact supports. This is equivalent to the existence of a map $T: Y \to P_c(X)$ such that $f^{-1}(y)$ contains the support of $T(y)$ for all $y \in Y$. It is shown, for example, that for every product $Y$ of metric spaces there is a 0-dimensional product $X$ of metric spaces and a perfect Milyutin map $f: X \to Y$ (Corollary 3.10). Moreover, every $p$-paracompact space is an image under a perfect Milyutin map of a 0-dimensional $p$-paracompact space (Corollary 3.11).

In the last Section 5 we prove that some topological properties are preserved under Milyutin maps. These properties include paracompactness, collection-wise normality, (complete) metrizability, stratifiability, $\delta$-metrizability and $k$-metrizability. In particular, we provide a positive answer to a question of Shchepin [31] whether every $AE(0)$-space is $k$-metrizable (see Corollary 5.5).

Some of the result presented here were announced in [33] without proofs.
Everywhere in this section $E, F$ stand for locally convex linear topological spaces and $C(X, E)$ is the set of all continuous maps from a space $X$ into $E$. By $C^*(X, E)$ we denote the bounded elements of $C(X, E)$. Let $\mu: C(X, E) \to F$ (resp., $\mu: C^*(X, E) \to F$) be a linear map. The support of $\mu$ is defined as the set $s(\mu)$ (resp., $s^*(\mu)$) of all $x \in \beta X$ such that for every neighborhood $U$ of $x$ in $\beta X$ there exists $f \in C(X, E)$ (resp., $f \in C^*(X, E)$) with $(\beta f)(\beta X - U) = 0$ and $\mu(f) \neq 0$, see [36]. Obviously, $s(\mu)$ and $s^*(\mu)$ are closed in $\beta X$, so compact. Let us note that in the above definition $(\beta f)(\beta X - U) = 0$ is equivalent to $f(X - U) = 0$. We also use $s^*(\mu)$ to denote the support of the restriction $\mu|C^*(C, E)$ when $\mu$ is defined on $C(X, E)$ (in this case we have $s^*(\mu) \subset s(\mu)$).

**Lemma 2.1.** Let $\mu$ be a linear map from $C(X, E)$ (resp., from $C^*(X, E)$) into $F$, where $E$ and $F$ are norm spaces.

(i) If $V$ a neighborhood of $s(\mu)$ (resp., $s^*(\mu)$), then $\mu(f) = 0$ for every $f \in C(X, E)$ (resp., $f \in C^*(X, E)$) with $(\beta f)(V) = 0$.

(ii) If the restriction $\mu|C^*(X, E)$ is continuous when $C^*(X, E)$ is equipped with the uniform topology, then $\mu(f) = 0$ provided $f \in C(X, E)$ (resp., $f \in C^*(X, E)$) and $(\beta f)(s(\mu)) = 0$ (resp., $(\beta f)(s^*(\mu)) = 0$).

(iii) In each of the following two cases $s(\mu)$ coincides with $s^*(\mu)$: either $s(\mu) \subset X$ or $\mu$ is a non-negative linear functional on $C(X)$.

**Proof.** When $\mu$ is a linear map on $C(X, E)$, items (i) and (ii) were established in [36, Lemma 2.1]: the case when $\mu$ is a linear map on $C^*(X, E)$ can be done by similar arguments. To prove (iii), we first suppose that $s(\mu) \subset X$. Then $s^*(\mu)$ is the support of the restriction $\mu|C^*(X, E)$ and $s^*(\mu) \subset s(\mu)$. So, we need to show that $s(\mu) \subset s^*(\mu)$. For a given point $x \in s(\mu)$ and its neighborhood $U$ in $\beta X$ there exists $g \in C(X, E)$ with $g(X - U) = 0$ and $\mu(g) \neq 0$. Because $g(s(\mu)) \subset E$ is compact, we can find $\epsilon > 0$ such that $s(\mu)$ is contained in the set $W = \{y \in X : ||g(y)|| < \epsilon\}$, where $||.||$ denotes the norm in $E$. Let $B_{\epsilon} = \{z \in E : ||z|| \leq \epsilon\}$ and $r: E \to B_{\epsilon}$ be a retraction (i.e., a continuous map with $r(z) = z$ for every $z \in B_{\epsilon}$). Then $h(y) = g(y)$ for every $y \in W$, where $h = r \circ g$. Hence, choosing an open set $V$ in $\beta X$ such that $V \cap X = W$, we have $(\beta(h - g))(V) = 0$. Since $V$ is a neighborhood of $s(\mu)$, by (i), $\mu(h) = \mu(g) \neq 0$. Therefore, we found a map $h \in C^*(X, E)$ such that $\beta h(\beta X - U) = 0$ and $\mu(h) \neq 0$. This means that $x \in s^*(\mu)$. So, $s(\mu) = s^*(\mu)$.

Now, let $E = F = \mathbb{R}$ and $\mu$ be a non-negative linear functional on $C(X)$. Suppose there exists $x \in s(\mu)$ but $x \notin s^*(\mu)$. Then, for some neighborhood $U$ of $x$ in $\beta X$, we have

\begin{equation}
(1) \quad \mu(h) = 0 \text{ for every } h \in C^*(X) \text{ with } h(X - U) = 0.
\end{equation}
Since \( x \in s(\mu) \), there exists \( f \in C(X) \) such that \( f(X - U) = 0 \) and \( \mu(f) \neq 0 \). Now, we use an idea from [21, proof of Theorem 1]. We represent \( f \) as the sum \( f^+ + f^- \), where \( f^+ = \max\{f, 0\} \) and \( f^- = \min\{f, 0\} \). Since both \( f^+ \) and \( f^- \) are 0 outside \( U \) and \( \mu(f) = \mu(f^+) + \mu(f^-) \neq 0 \) implies that at least one of the numbers \( \mu(f^+) \) and \( \mu(f^-) \) is not 0, we can assume that \( f \geq 0 \). By (1), \( f \) is not bounded. Therefore, there is a sequence \( \{x_n\} \subset X \) such that \( \{t_n = f(x_n) : n \geq 1\} \) is an increasing and unbounded sequence. We set \( t_0 = 0 \) and for every \( n \geq 1 \) define the function \( f_n \in C^*(X) \) as follows: \( f_n(x) = 0 \) if \( f(x) \leq t_{n-1} \), \( f_n(x) = f(x) - t_{n-1} \) if \( t_{n-1} < f(x) \leq t_n \) and \( f_n(x) = t_n - t_{n-1} \) provided \( f(x) > t_n \). Let also \( h_n = t_n \cdot f_n \) and \( h = \sum_{n=1}^{\infty} h_n \). Then \( h \) is continuous and for every \( n \geq 1 \) we have
\[
(2) \quad t_n(f - f_1 - f_2 - ... - f_n) \leq h - h_1 - h_2 - ... - h_n.
\]

Since all \( f_n \) and \( h_n \) are bounded and continuous functions satisfying \( f_n(X - U) = h_n(X - U) = 0 \), it follows from (1) that \( \mu(h_n) = \mu(f_n) = 0 \), \( n \geq 1 \). So, by (2), \( t_n \cdot \mu(f) \leq \mu(h) \) for every \( n \). Hence, \( \mu(f) = 0 \) which is a contradiction. Therefore, \( s(\mu) = s^*(\mu) \).

We say that a linear map \( \mu \) on \( C(X, E) \) (resp., on \( C^*(X, E) \)) has a compact support if \( s(\mu) \subset X \) (resp., \( s^*(\mu) \subset X \)). If \( \mu \) takes values in \( E \), then it is called regular provided \( \mu(f) \) belongs to the closure of the convex hull \( \text{conv } f(X) \) of \( f(X) \) for every \( f \in C(X, E) \) (resp., \( f \in C^*(X, E) \)). Below, \( C_k(X, E) \) (resp., \( C^*_k(X, E) \)) stands for the space \( C(X, E) \) (resp. \( C^*(X, E) \)) with the compact-open topology.

**Proposition 2.2.** Let \( E \) be a norm space. A regular linear map \( \mu \) on \( C(X, E) \) (resp., \( C^*(X, E) \)) has a compact support in \( X \) if and only if \( \mu \) is continuous on \( C_k(X, E) \) (resp., \( C^*_k(X, E) \)).

**Proof.** We consider only the case when \( \mu \) is a map on \( C(X, E) \), the other one is similar. Suppose \( s(\mu) = H \subset X \). Since \( \mu \) is regular, \( \mu(f) \in \text{conv } f(X) \) for every \( f \in C(X, E) \). This yields \( ||\mu(f)|| \leq ||f|| \), \( f \in C^*(X, E) \). Hence, the restriction \( \mu|C^*(X, E) \) is continuous with respect to the uniform topology. So, by Lemma 2.1(iii), for every \( f \in C(X, E) \) the value \( \mu(f) \) depends only on the restriction \( f|H \). Therefore, the linear map \( \nu : C(H, E) \to E, \nu(g) = \mu(\tilde{g}) \), where \( \tilde{g} \in C(X, E) \) is any continuous extension of \( g \), is well defined. Note that such an extension \( \tilde{g} \) always exists because \( H \subset X \) is compact. Moreover, the restriction map \( \pi_H : C_k(X, E) \to C_k(H, E) \) is surjective and continuous. Since \( \mu = \nu \circ \pi_H \), \( \mu \) would be continuous provided \( \nu : C_k(H, E) \to E \) is so. Next claim implies that for every \( g \in C(H, E) \) we have \( \nu(g) \in \text{conv } g(H) \) and \( ||\nu(g)|| \leq ||g|| \), which guarantee the continuity of \( \nu \).
Claim 1. $\mu(f) \in \text{conv } f(H)$ for every $f \in C(X, E)$

Indeed, if $\mu(f) \not\in \text{conv } f(H)$ for some $f \in C(X, E)$, then we can find a closed convex neighborhood $W$ of $\text{conv } f(H)$ in $E$ and a function $h \in C(X, E)$ such that $\mu(f) \not\in W$, $h(X) \subset W$ and $h(x) = f(x)$ for all $x \in H$. As it was shown above, the last equality implies $\mu(f) = \mu(h)$. Hence, $\mu(f) = \mu(h) \in \text{conv } h(X) \subset W$, which is a contradiction.

Now, suppose $\mu: C_k(X, E) \to E$ is continuous. Then there exists a compact set $K \subset X$ and $\epsilon > 0$ such that $||\mu(f)|| < 1$ for every $f \in C(X, E)$ with $\sup\{||f(x)|| : x \in K\} < \epsilon$. We claim that $s(\mu) \subset K$. Indeed, otherwise there would be $x \in s(\mu) - K$, a neighborhood $U$ of $x$ in $\beta X$ with $U \cap K = \emptyset$, and a function $g \in C(X, E)$ such that $g(X - U) = 0$ and $\mu(g) \neq 0$. Choose an integer $k$ with $||\mu(kg)|| \geq 1$. On the other hand, $kg(x) = 0$ for every $x \in K$. Hence, $||\mu(kg)|| < 1$, a contradiction. \hfill \Box

Now, for every space $X$ and a locally convex space $E$ let $P_c(X, E)$ (resp., $P^*_c(X, E)$) denote the set of all regular linear maps $\mu: C(X, E) \to E$ (resp., $\mu: C^*(X, E) \to E$) with compact supports equipped with the weak (i.e. pointwise) topology with respect to $C(X, E)$ (resp., $C^*(X, E)$). If $E$ is the real line, we write $P_c(X)$ (resp., $P^*_c(X)$) instead of $P_c(X, \mathbb{R})$ (resp., $P^*_c(X, \mathbb{R})$). It is easily seen that a linear map $\mu: C(X) \to \mathbb{R}$ (resp., $\mu: C^*(X) \to \mathbb{R}$) is regular if and only if $\mu$ is non-negative and $\mu(1) = 1$. If $h: X \to Y$ is a continuous map, then there exists a map $P_c(h): P_c(X) \to P_c(Y)$ defined by $P_c(h)(\mu)(f) = \mu(f \circ h)$, where $\mu \in P_c(X)$ and $f \in C(Y)$. Considering functions $f \in C^*(Y)$ in the above formula, we can define a map $P_c^*(h): P^*_c(X) \to P^*_c(Y)$. It is easily seen that $s(P_c(h)(\mu)) \subset h(s(\mu))$ (resp., $s^*(P_c^*(h)(\mu)) \subset h(s^*(\mu))$) for every $\mu \in P_c(X)$ (resp., $\mu \in P^*_c(X)$). Moreover, $P_c(h_2 \circ h_1) = P_c(h_2) \circ P_c(h_1)$ and $P^*_c(h_2 \circ h_1) = P^*_c(h_2) \circ P^*_c(h_1)$ for any two maps $h_1: X \to Y$ and $h_2: Y \to Z$. Therefore, both $P_c$ and $P^*_c$ are covariant functors in the category of all Tychonoff spaces and continuous maps. Let us also note that if $X$ is compact then $P_c(X)$ and $P^*_c(X)$ coincide with the space $P(X)$ of all probability measures on $X$.

For every $x \in X$ we consider the Dirac’s measure $\delta_x \in P_c(X, E)$ defined by $\delta_x(f) = f(x)$, $f \in C(X, E)$. In a similar way we define $\delta_x^* \in P^*_c(X, E)$. We also consider the maps $i_X: X \to P_c(X, E)$, $i_X(x) = \delta_x$, and $i^*_X: X \to P^*_c(X, E)$, $i^*_X(x) = \delta_x^*$. Next proposition is an easy exercise.

Proposition 2.3. Let $h: X \to Y$ be a map.

(i) The map $i_X: X \to P_c(X)$ is a closed $C$-embedding, and $i^*_X: X \to P^*_c(X)$ is a closed $C^*$-embedding;

(ii) The map $P_c(h)$ is a (closed) $C$-embedding provided $h$ is a (closed) $C$-embedding;
(iii) The map $P^*_c(h)$ is a (closed) $C^*$-embedding provided $h$ is a (closed) $C^*$-embedding.

There exists a natural embedding $e_X: X \rightarrow \mathbb{R}^{C(X)}$, $e_X(x) = (f(x))_{f \in C(X)}$. Denote by $M^+(X)$ the set of all regular linear functionals on $C(X)$ with the pointwise topology and consider the map $m_X: M^+(C) \rightarrow \mathbb{R}^{C(X)}$, $m_X(\mu) = (\mu(f))_{f \in C(X)}$. It easily seen that $m_X$ is also an embedding extending and $m_X(M^+(X))$ is a closed convex subset of $\mathbb{R}^{C(X)}$. Moreover, $P_c(X) \subset M^+(X)$. It is well known that for compact $X$ the space $P(X)$ is homeomorphic with the convex closed hull of $e_X(X)$ in $\mathbb{R}^{C(X)}$. A similar fact is true for $P_c(X)$.

**Proposition 2.4.** If $X$ is realcompact, then $P_c(X)$ is homeomorphic to the closed convex hull of $e_X(X)$ in $\mathbb{R}^{C(X)}$.

**Proof.** Obviously, $m_X(P_c(X))$ is a convex subset of $\mathbb{R}^{C(X)}$ containing the set $\text{conv} \ e_X(X)$. It suffices to show that $m_X(P_c(X))$ coincides with the set $B = \text{conv} \ e_X(X)$. Suppose $\mu \in P_c(X)$. By Lemma 2.1(ii) and Proposition 2.2, for every $f \in C(X)$ the value $\mu(f)$ is determined by the restriction $f|s(\mu)$. So, there exists an element $\nu \in P(s(\mu))$ such that $\mu(f) = \nu(f|s(\mu))$, $f \in C(X)$ (see the proof of Proposition 2.2). Since the set $P_f(s(\mu))$ of all measures from $P(s(\mu))$ having finite supports is dense in $P(s(\mu))$ [17], there is a net $\{\nu_\alpha\}_{\alpha \in A} \subset P_f(s(\mu))$ converging to $\nu$ in $P(s(\mu))$. Each $\nu_\alpha$ can be identified with the measure $\mu_\alpha \in P_c(X)$ defined by $\mu_\alpha(f) = \nu_\alpha(f|s(\mu))$, $f \in C(X)$. Moreover, the net $\{\mu_\alpha\}_{\alpha \in A}$ converges to $\nu$ in $P_c(X)$. Then $\{m_X(\mu_\alpha)\}_{\alpha \in A} \subset \text{conv} \ e_X(X)$ and converges to $m_X(\mu)$ in $\mathbb{R}^{C(X)}$. So, $m_X(\mu) \in B$. In this way we obtained $m_X(P_c(X)) \subset B$.

On the other hand, since $m_X(M^+(X))$ is a closed and convex subset of $\mathbb{R}^{C(X)}$ containing $e_X(X)$, $B \subset m_X(M^+(X))$. So, the elements of $B$ are of the form $m_X(\mu)$ with $\mu$ being a regular linear functional on $C(X)$. Since $X$ is realcompact, according to [21, Theorem 18], any such a functional has a compact support in $X$. Therefore, $B \subset m_X(P_c(X))$. □

There exists a natural continuous map $j_X: P_c(X) \rightarrow P^*_c(X)$ assigning to each $\mu \in P_c(X)$ the measure $\nu = \mu|C^*(X)$. By Lemma 2.1 and Proposition 2.2, $s(\mu) = s^*(\nu)$ and $\mu(f)$ and $\nu(g)$ depend, respectively, on the restrictions $f|s(\mu)$ and $g|s^*(\nu)$ for all $f \in C(X)$ and $g \in C^*(X)$. This implies that $j_X$ is one-to-one. Using again Lemma 2.1 and Proposition 2.2, one can show that $j_X$ is surjective. According to next proposition, $j_X$ is not always a homeomorphism.

A subset $A$ of a space $X$ is said to be bounded if $f(A) \subset \mathbb{R}$ is bounded for every $f \in C(X)$. This notion should be distinguished from the notion of a bounded set in a linear topological space.

**Proposition 2.5.** For a given space $X$ we have:

(i) The map $j_X$ is a homeomorphism if and only if $X$ is pseudocompact;
(ii) $P_c(X)$ is metrizable if and only if $X$ is compact and metrizable.
Proof. (i) Obviously, if $X$ is pseudocompact, then $C(X) = C^*(X)$ and $j_X$ is the identity on $P_c(X)$. Suppose $X$ is not pseudocompact and choose $g \in C(X)$ and a discrete countable set $\{x(n) : n \geq 1\}$ in $X$ such that $\{g(x(n)) : n \geq 1\}$ is unbounded and discrete in $\mathbb{R}$. For every $n \geq 2$ define the measures $\mu_n \in P_c(X)$ and $\nu_n \in P^*_c(X)$ as follows: $\mu_1 = \delta_{x(1)}$, $\mu_n = (1 - 1/n)\delta_{x(1)} + \sum_{k=2}^{n+1}(1/n)^2\delta_{x(k)}$ and $\nu_1 = \delta^*_{x(1)}$, $\nu_n = (1 - 1/n)\delta^*_{x(1)} + \sum_{k=2}^{n+1}(1/n)^2\delta^*_{x(k)}$. Obviously, $j_X(\mu_n) = \nu_n$ for all $n \geq 1$ and $s(\mu_n) = s^*(\nu_n) = \{x(1), x(2), \ldots, x(n+1)\}$, $n \geq 2$. So, $g(\bigcup_{n=1}^{\infty} s(\mu_n))$ is unbounded in $\mathbb{R}$. This, according to [35, Proposition 3.1] (see also [3]), means that the sequence $\{\mu_n\}_{n \geq 1}$ is not compact. On the other hand, it is easily seen that $\{\nu_n\}_{n \geq 2}$ converges in $P^*_c(X)$ to $\nu_1$. Consequently, $j_X$ is not a homeomorphism.

(ii) First we prove that $P_c(\mathbb{N})$ is not metrizable, where $\mathbb{N}$ is the set of the integers $n \geq 1$ with the discrete topology. For every $n \geq 1$ let $K(n) = P_c(\{1, 2, \ldots, n\})$. Obviously, every $K(n)$ is homeomorphic to a simplex of dimension $n - 1$ and $K(n) \subset K(m)$ for $n \leq m$. Moreover, $P_c(\mathbb{N}) = \bigcup_{n \geq 1} K(n)$.

Claim 2. $P_c(\mathbb{N})$ is nowhere locally compact.

Indeed, otherwise there would be $\mu \in P_c(\mathbb{N})$ and its open neighborhood $O(\mu)$ in $P_c(\mathbb{N})$ with $\overline{O(\mu)}$ being compact. Then, by [35, Proposition 3.1], $S = \cup\{s(\nu) : \nu \in O(\mu)\}$ is a bounded subset of $\mathbb{N}$. Hence, $S \subset \{1, 2, \ldots, p\}$ for some $p \geq 1$. The last inclusion means that $O(\mu) \subset K(p)$, so $\dim O(\mu) \leq p - 1$. Therefore, $O(\mu)$ being open in $P_c(\mathbb{N})$ is also open in each $K(n)$, $n \geq p$. Since every open subset of $K(n)$ is of dimension $n - 1$, we obtain that $\dim O(\mu) > p - 1$, a contradiction.

Now, suppose $P_c(\mathbb{N})$ is metrizable and fix $\mu \in P_c(\mathbb{N})$. Since $P_c(\mathbb{N})$ is nowhere locally compact and $K(n)$, $n \geq 1$, are compact, $U(\mu) - K(n) \neq \emptyset$ for all $n \geq 1$ and all neighborhoods $U(\mu) \subset P_c(\mathbb{N})$ of $\mu$. Using the last condition and the fact that $\mu$ has a countable local base (as a point in a metrizable space), we can construct a sequence $\{\mu_n\}_{n \geq 1}$ converging to $\mu$ in $P_c(\mathbb{N})$ such that $\mu_n \notin K(n)$ for all $n$. Consequently, $s(\mu_n) \notin \{1, 2, \ldots, n\}$, $n \geq 1$. To obtain a contradiction, we apply again [35, Proposition 3.1] to conclude that $s(\mu) \cup \bigcup_{n \geq 1} s(\mu_n)$ is a bounded subset of $\mathbb{N}$ because $\{\mu, \mu_n : n \geq 1\}$ is a compact subset of $P_c(\mathbb{N})$. Therefore, $P_c(\mathbb{N})$ is not metrizable.

Let us complete the proof of (ii). If $X$ is compact metrizable, then $P_c(X)$ is metrizable (see, for example [17]). Suppose $P_c(X)$ is metrizable. Then, by Proposition 2.3(i), $X$ is also metrizable. If $X$ is not compact, it should contain a $C$-embedded copy of $\mathbb{N}$ and, according to Proposition 2.3(ii), $P_c(X)$ should contain a copy of $P_c(\mathbb{N})$. So, $P_c(\mathbb{N})$ would be also metrizable, which is not possible. Therefore, $X$ is compact and metrizable provided $P_c(X)$ is metrizable. □
Proposition 2.6. If one of the spaces $P_c(X)$ and $P_c^*(X)$ is Čech-complete, then $X$ is pseudocompact.

Proof. We prove first that none of the spaces $P_c[0,\infty)$ and $P_{c}^{*}(\mathbb{Z})$ is Čech-complete. Indeed, suppose $P_c[0,\infty)$ is Čech-complete. Since $P_c[0,\infty)$ is Lindelöf (as the union of the compact sets $K(n) = P_c[0,\infty)(\{1, 2, \ldots, n\})$), it is a $\sigma$-paraconsistent in the sense of Arhangel’skii [2]. So, there exists a perfect map $g$ from $P_c[0,\infty)$ onto a separable metric space $Z$. Then the diagonal product $q = g\triangle j_n \colon Z \times P_{c}^{*}(\mathbb{Z})$ is perfect (because $g$ is perfect) and one-to-one (because $j_n$ is one-to-one). Thus, $q$ is a homeomorphism. Since $P_{c}^{*}(\mathbb{Z})$ is second countable [9], $Z \times P_{c}^{*}(\mathbb{Z})$ is metrizable. Consequently, $P_c[0,\infty)$ is metrizable, a contradiction (see Proposition 2.5(ii)).

Suppose now that $P_{c}^{*}(\mathbb{Z})$ is Čech-complete, so it is a Polish space. Since $P_{c}^{*}(\mathbb{Z})$ is the union of the compact sets $K^*(n) = P_{c}^{*}(\{1, 2, \ldots, n\})$, $n \geq 1$, there exists $m > 1$ such that $K^*(m)$ has a non-empty interior. Then $K(m) = P_c[0,\infty)(\{1, 2, \ldots, m\})$ has a non-empty interior in $P_c[0,\infty)$ because $K(m) = j_{n}^{-1}(K^*(m))$. According to Claim 2, this is again a contradiction.

If $X$ is not pseudocompact, there exists a function $g \in C(X)$ and a discrete set $A = \{x_n : n \geq 1\}$ in $X$ such that $g(x_n) \neq g(x_m)$ for $n \neq m$ and $g(A)$ is a discrete unbounded subset of $\mathbb{R}$. Since $g(A)$ is $C$-embedded in $\mathbb{R}$, it follows that $A$ is also $C$-embedded in $X$. So, $A$ is a $C$-embedded copy of $\mathbb{N}$ in $X$. Then, by Proposition 2.3, $P_c(X)$ contains a closed copy of $P_c[0,\infty)$ and $P_{c}^{*}(X)$ contains a closed copy of $P_{c}^{*}(\mathbb{Z})$. Since non of $P_c[0,\infty)$ and $P_{c}^{*}(\mathbb{Z})$ is Čech-complete, non of $P_c(X)$ and $P_{c}^{*}(X)$ can be Čech-complete. This completes the proof. $\square$

We say that an inverse system $S = \{X_\alpha, p_\beta^\alpha, A\}$ is factorizing [11] if for every $h \in C(X)$, where $X$ is the limit space of $S$, there exists $\alpha \in A$ and $h_\alpha \in C(X_\alpha)$ with $h = h_\alpha \circ p_\alpha$. Here, $p_\alpha \colon X \rightarrow X_\alpha$ is the $\alpha$-th limit projection. According to [9], $P_{c}^{*}$ is a continuous functor, i.e. for every factorizing inverse system $S$ the space $P_{c}^{*} (\text{lim} S)$ is the limit of the inverse system $P_{c}^{*}(S) = \{P_{c}^{*}(X_\alpha), P_{c}^{*}(p_\beta^\alpha), A\}$. The same is true for the functor $P_{c}$.

Proposition 2.7. $P_c$ is a continuous functor.

Proof. Let $S = \{X_\alpha, p_\beta^\alpha, A\}$ be a factorizing inverse system with a limit space $X$ and let $\{\mu_\alpha : \alpha \in A\}$ be a thread of the system $P_c(S)$. For every $\alpha \in A$ we consider the measure $\nu_\alpha = j_{X_\alpha}(\mu_\alpha)$. Here, $j_{X_\alpha} : P_c(X_\alpha) \rightarrow P_{c}^{*}(X_\alpha)$ is the one-to-one surjection defined above. It is easily seen that $\{\nu_\alpha : \alpha \in A\}$ is a thread of the system $P_{c}^{*}(S)$, so it determines a unique measure $\nu \in P_{c}^{*}(X)$ (recall that $P_{c}^{*}$ is a continuous functor). There exists a unique measure $\mu \in P_c(X)$ with $j_X(\mu) = \nu$. One can show that $P_c(p_\alpha)(\mu) = \mu_\alpha$ for all $\alpha$. Hence, the set $P_c(X)$ coincides with the limit set of the system $P_c(S)$. It remains to show that for every $\mu^0 \in P_c(X)$ and its neighborhood $U$ in $P_c(X)$ there exists $\alpha \in A$ and a neighborhood $V$ of $\mu^0 = P_c(p_\alpha)(\mu^0)$ in $P_c(X_\alpha)$ such that $P_c(p_\alpha)^{-1}(V) \subset U$. We can suppose that $U = \{\mu \in P_c(X) : |\mu(h_i) - \mu^0(h_i)| < \epsilon, i = 1, 2, \ldots, k\}$.
for some $\epsilon > 0$ and $h_i \in C(X)$, $i = 1, 2, \ldots, k$. Since $S$ is factorizing, we can find $\alpha \in A$ and functions $g_i \in C(X_\alpha)$ such that $h_i = g_i \circ p_\alpha$ for all $i = 1, \ldots, k$. Then $V = \{\mu_\alpha \in P_c(X_\alpha) : |\mu_\alpha(g_i) - \mu_\alpha^0(g_i)| < \epsilon, i = 1, 2, \ldots, k\}$ is the required neighborhood of $\mu_\alpha^0$.

\section{Milyutin maps and linear operators with compact supports}

For every linear operator $u : C(X, E) \to C(Y, E)$, where $E$ is a locally convex linear space, and $y \in Y$ there exists a linear map $T(y) : C(X, E) \to E$ defined by $T(y)(g) = u(g)(y)$, $g \in C(X, E)$. We say that $u$ has compact supports (resp., $u$ is regular) if each $T(y)$ has a compact support in $X$ (resp., each $T(y)$ is regular). In a similar way we define a linear operator with compact supports if $u : C(X, E) \to C^*(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$ or $u : C^*(X, E) \to C(Y, E)$). Let us note that a linear map $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) is regular and has compact supports iff the formula

\[(3) \quad T(y)(g) = u(g)(y) \text{ with } g \in C(X, E) \text{ (resp., } g \in C^*(X, E))\]

produces a continuous map $T : Y \to P_c(X_\alpha, E)$ (resp., $T : Y \to P_c^*(X, E)$). If $f : X \to Y$ is a surjective map, then a liner operator $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) is called an averaging operator for $f$ if $u(\varphi \circ f) = \varphi$ for every $\varphi \in C(Y, E)$ (resp., $\varphi \in C^*(Y, E)$). It is easily seen that $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) is a regular averaging operator for $f$ with compact supports if and only if the map $T : Y \to P_c(X, E)$ (resp., $T : Y \to P_c^*(X, E)$) defined by (3), has the following property: the support of every $T(y)$, $y \in Y$, is contained in $f^{-1}(y)$. Such a map $T$ will be called a map associated with $f$. It is also clear that if $T : Y \to P_c(X, E)$ (resp., $T : Y \to P_c^*(X, E)$) is a map associated with $f$, then the equality (3) defines a regular averaging operator $u : C(X, E) \to C(Y, E)$ (resp., $u : C^*(X, E) \to C^*(Y, E)$) for $f$ with compact supports.

A surjective map $f : X \to Y$ is said to be Milyutin if $f$ admits a regular averaging operator $u : C(X) \to C(Y)$ with compact supports, or equivalently, there exists a map $T : Y \to P_c(X)$ associated with $f$. A surjective map $f : X \to Y$ is called weakly Milyutin (resp., strongly Milyutin) if there exists a map $T : Y \to P^*_c(X)$ (resp., $T : P_c(Y) \to P_c(X)$) such that $s^*(g(y)) \subset f^{-1}(y)$ for all $y \in Y$ (resp., $s(g(\mu)) \subset f^{-1}(s(\mu))$ for all $\mu \in P_c(Y)$). Obviously, every strongly Milyutin map is Milyutin. Moreover, if $T : Y \to P_c(X)$ is a map associated with $f$, then the map $j_X \circ T : Y \to P^*_c(X)$ is witnessing that Milyutin maps are weakly Milyutin. One can also show that if $f : X \to Y$ is weakly Milyutin, then its Čech-Stone extension $\beta f : \beta X \to \beta Y$ is a Milyutin map.

We are going to establish some properties of (weakly) Milyutin maps.
Proposition 3.1. Let \( f : X \to Y \) be a weakly Milyutin map and \( E \) a complete locally convex space. Then \( f \) admits a regular averaging operator \( u : C^*(X, E) \to C^*(Y, E) \) with compact supports.

Proof. Let \( T : Y \to P^*_c(X) \) be a map associated with \( f \). For every \( g \in C^*(X, E) \) let \( B(g) = \text{conv} \ g(X) \) and consider the map \( P^*_c(g) : P^*_c(X) \to P^*_c(B(g)) \). Since \( B(g) \) is a closed and bounded in \( E \) and \( E \) is complete, by [5, Theorem 3.4 and Proposition 3.10], there exists a continuous map \( b : P^*_c(B(g)) \to B(g) \) assigning to each measure its barycenter. The composition \( e(g) = b \circ P^*_c(g) : P^*_c(X) \to E \) is a continuous extension of \( g \) (we consider \( X \) as a subset of \( P^*_c(X) \)). Now, we define \( u : C^*(X, E) \to C^*(Y, E) \) by \( u(g) = e(g) \circ T \). This a linear operator because \( e(g)(\mu) = \int_X g d\mu \) for every \( \mu \in P^*_c(X) \). Since \( e(g) \) is a map from \( P^*_c(X) \) into \( B(f) \), the linear map \( \Lambda(y) : C^*(X, E) \to E, \Lambda(y)(g) = u(g)(y) \), is regular for all \( y \in Y \).

So, it remains to show that the support of each \( \Lambda(y) \) is compact and it is contained in \( f^{-1}(y) \). Because \( T \) is associated with \( f \), \( K(y) = s^*(T(y)) \) is a compact subset of \( f^{-1}(y), y \in Y \). We are going to show that if \( h|K(y) = g|K(y) \) with \( h, g \in C^*(X, E) \), then \( \Lambda(y)(h) = \Lambda(y)(g) \). That would imply the support of \( \Lambda(y) \) is contained in \( K(y) \subset f^{-1}(y) \), and hence it should be compact. To this end, observe that \( T(y) \) can be considered as an element of \( P(K(y)) \) - the probability measures on \( K(y) \). So, \( T(y) \) is the limit of a net \( \{\mu_\alpha \} \subset P(K(y)) \) consisting of measures with finite supports. Each \( \mu_\alpha \) is of the form \( \sum_{i=1}^{k(\alpha)} \lambda_i^\alpha \delta_{x_i^\alpha} \),

where \( x_i^\alpha \in K(y) \) and \( \lambda_i^\alpha \) are positive reals with \( \sum_{i=1}^{k(\alpha)} \lambda_i^\alpha = 1 \). Then \( \{e(g)(\mu_\alpha)\} \) converges to \( e(g)(T(y)) \) and \( \{e(h)(\mu_\alpha)\} \) converges to \( e(h)(T(y)) \). On the other hand, \( e(h)(\mu_\alpha) = \int_X h d\mu_\alpha = \sum_{i=1}^{k(\alpha)} \lambda_i^\alpha h(x_i^\alpha) \) and \( e(g)(\mu_\alpha) = \sum_{i=1}^{k(\alpha)} \lambda_i^\alpha g(x_i^\alpha) \). Since \( h|K(y) = g|K(y) \), \( h(x_i^\alpha) = g(x_i^\alpha) \) for all \( \alpha \) and \( i \). Hence, \( e(h)(T(y)) = e(g)(T(y)) \) which means that \( \Lambda(y)(h) = \Lambda(y)(g) \). Therefore, \( u \) is a regular averaging operator for \( f \) and has compact supports. \( \square \)

Corollary 3.2. Let \( X \) be a complete bounded convex subset of a locally convex space and \( f : X \to Y \) be a weakly Milyutin map such that \( f^{-1}(y) \) is convex for every \( y \in Y \). Then there exists a map \( g : Y \to X \) such that \( g(y) \in f^{-1}(y) \) for all \( y \in Y \).

Proof. Let \( T : Y \to P^*_c(X) \) be a map associated with \( f \). By [5, Proposition 3.10], the barycenter \( b(\mu) \) of each measure \( \mu \in P^*_c(X) \) belongs to \( X \) and the map \( b : P^*_c(X) \to X \) is continuous. Since the support of each \( T(y) \), \( y \in Y \), is compact subset of \( f^{-1}(y) \) and \( \text{conv} \ s^*(T(y)) \subset f^{-1}(y) \) (recall that \( f^{-1}(y) \) is convex), \( b(T(y)) \in f^{-1}(y) \). So, the map \( g = b \circ T \) is as required. \( \square \)

Recall that a set-valued map \( \Phi : X \to Y \) is lower semi-continuous (br., lsc) if for every open \( U \subset Y \) the set \( \Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\} \) is open in \( X \).
Lemma 3.3. For every space $X$ and a linear space $E$ the set-valued map $\Phi_X: P_c(X,E) \to X$, (resp., $\Phi_X^*: P_c^*(X,E) \to X$) defined by $\Phi_X(\mu) = s(\mu)$, (resp., $\Phi_X^*(\mu) = s^*(\mu)$) is lsc.

Proof. A similar statement was established in [4, Lemma 1.2.7], so we omit the arguments. □

Proposition 3.4. Let $f: X \to Y$ be a weakly Milyutin map. Then we have:

(i) $\beta f: \beta X \to \beta Y$ is a Milyutin map;

(ii) $f$ is a Milyutin map provided $f$ is perfect.

Proof. Let $T: Y \to P_c^*(X)$ be a map associated with $f$. To prove (i), observe that $P_c(i): P_c^*(X) \to P_c(\beta X)$ is an embedding, where $i: X \to \beta X$ is the standard embedding (see Proposition 2.3(iii)). Because $P_c(\beta X) = P(\beta X)$ is compact, we can extend $T$ to a map $\tilde{T}: \beta Y \to P(\beta X)$. It suffices to show that $\tilde{T}$ is a map associated with $\beta f$. To this end, consider the lsc map $\Phi = \beta f \circ \Phi_{\beta X} \circ \tilde{T}: \beta Y \to \beta Y$. Since $\Phi$ is lsc and $\Phi(y) = y$ for all $y \in Y$, $\Phi(y) = y$ for any $y \in \beta Y$. This means that the support of any $\tilde{T}(y), y \in \beta Y$, is contained in $(\beta f)^{-1}(y)$. So, $\beta f$ is a Milyutin map.

The proof of (ii) follows from (i) and the following result of Choban [12, Proposition 1.1]: if $\beta f$ admits a regular averaging operator and $f$ is perfect, then $f$ admits a regular averaging operator $u: C(X) \to C(Y)$ such that

$$ \inf \{ h(x) : x \in f^{-1}(y) \} \leq u(h)(y) \leq \sup \{ h(x) : x \in f^{-1}(y) \} $$

for every $h \in C(X)$ and $y \in Y$. This implies that the support of each linear map $T(y): C(X) \to \mathbb{R}$, $y \in Y$, defined by (3), is contained in $f^{-1}(y)$. Hence, $s(T(y))$ is compact because so is $f^{-1}(y)$ (recall that $f$ is perfect). Therefore, $f$ is a Milyutin map. □

Proposition 3.5. Let $f: X \to Y$ be a Milyutin map. Then, in each of the following cases $f$ is strongly Milyutin: (i) $f^{-1}(K)$ is compact for every compact set $K \subset Y$; (ii) every closed and bounded subset of $X$ is compact.

Proof. Let $u: C(X) \to C(Y)$, $u(h)(y) = g(y)(h)$, be a corresponding regular averaging operator with compact supports, where $g: Y \to P_c(X)$ is a map associated with $f$. We are going to extend $g$ to a map $\tilde{g}: P_c(Y) \to P_c(X)$ such that $s(\tilde{g}(\mu)) \subset f^{-1}(s(\mu))$ for all $\mu \in P_c(Y)$. Let $\mu \in P_c(Y)$ and $K = s(\mu) \subset Y$. Then $g(K)$ is a compact subset of $P_c(X)$. Hence, by [35, Proposition 3.1], $H = \bigcup \{ s(g(y)) : y \in K \}$ is a bounded and closed subset of $X$. Since $s(g(y)) \subset f^{-1}(y)$ for all $y \in Y$, $H \subset f^{-1}(K)$. So, in each of the cases (i) and (ii), $H$ is compact. Define $\tilde{g}(\mu): C(X) \to \mathbb{R}$ to be the linear functional $\tilde{g}(\mu)(h) = \mu(u(h)), h \in C(X)$. One can check that $\tilde{g}(\mu)(h) = 0$ provided $h(H) = 0$. This means that the support of $\tilde{g}(\mu)$ is a compact subset of $H$, so $\tilde{g}(\mu) \in P_c(X)$. Moreover, $\tilde{g}$, considered as a map from $P_c(Y)$ to $P_c(X)$
is continuous and satisfies the inclusions \( s(\tilde{g}(\mu)) \subset f^{-1}(s(\mu)), \mu \in P_c(Y) \). Therefore, \( f \) is strongly Milyutin.

A map \( f: X \to Y \) is said to be 0-invertible [20] if for any space \( Z \) with \( \dim Z = 0 \) and any map \( p: Z \to Y \) there exists a map \( q: Z \to X \) such that \( f \circ q = p \). Here, \( \dim Z = 0 \) means that \( \dim \beta Z = 0 \). We say that \( f: X \to Y \) has a metrizable kernel if there exists a metrizable space \( M \) and an embedding \( X \subset Y \times M \) such that \( \pi_Y|X = f \), where \( \pi_Y: Y \times M \to Y \) is the projection.

Next theorem is a generalization of [13, Theorem 3.4] and [20, Corollary 1].

**Theorem 3.6.** Let \( f: X \to Y \) be a surjection with a metrizable kernel and \( Y \) a paracompact space. Then the following conditions are equivalent:

1. \( f \) is (weakly) Milyutin;
2. The set-valued map \( f^{-1}: Y \to X \) admits a lsc compact-valued selection;
3. \( f \) is 0-invertible.

**Proof.** \((i) \Rightarrow (ii)\) Let \( f \) be weakly Milyutin and \( T: Y \to P_c^*(X) \) is a map associated with \( f \). By Lemma 3.3, the map \( \Phi_X^*: P_c^*(X) \to X \) is lsc, so is the map \( \Phi_X^* \circ T \). Moreover, \( \Phi_X^*(T(y)) = s^*(T(y)) \subset f^{-1}(y) \) for all \( y \in Y \). Hence, \( \Phi_X^* \circ T \) is a compact-valued selection of \( f^{-1} \).

\((ii) \Rightarrow (iii)\) Suppose \( M \) is a metrizable space such that \( X \subset Y \times M \) and \( \pi_Y|X = f \). Suppose also that \( f^{-1} \) admits a compact-valued lsc selection \( \Phi: Y \to X \). To show that \( f \) is 0-invertible, take a map \( p: Z \to Y \) with \( \dim Z = 0 \), and let \( Z_1 = (\beta p)^{-1}(Y) \). Then \( Z_1 \) is paracompact (as a perfect preimage of \( Y \)) and \( \dim Z_1 = 0 \) because \( \beta Z_1 = \beta Z = 0 \) is 0-dimensional. The set-valued map \( \pi_M \circ \Phi \circ p_1: Z_1 \to M \) is lsc and compact-valued, where \( \pi_M: Y \times M \to M \) is the projection and \( p_1 = (\beta p)|Z_1 \). According to [23], \( \pi_M \circ \Phi \circ p_1 \) admits a (single-valued) continuous selection \( q_1: Z_1 \to M \). Finally, the map \( q: Z \to X \), \( q(z) = (p(z), q_1(z)) \) is the required lifting of \( p \), i.e. \( f \circ q = p \).

\((iii) \Rightarrow (i)\) By [28], there exists a perfect weakly Milyutin map \( p: Z \to Y \) with \( Z \) being a 0-dimensional paracompact. Then, by Proposition 3.4(ii), \( p \) is a Milyutin map. Since \( f \) is 0-invertible, there exists a map \( g: Z \to X \) with \( f \circ g = p \). If \( T: Y \to P_c(Z) \) is a map associated with \( p \), then \( \tilde{T} = P_c(g) \circ T: Y \to P_c(X) \) is a map associated with \( f \) because \( s(\tilde{T}(y)) \subset g(p^{-1}(y)) \subset f^{-1}(y) \) for all \( y \in Y \). Hence, \( f \) is a Milyutin map.

**Corollary 3.7.** Let \( f: X \to Y \) be a surjective map such that either \( X \) and \( Y \) are metrizable or \( f \) is perfect. Then the following are equivalent: \( i) \) \( f \) is weakly Milyutin; \( ii) \) \( f \) is Milyutin; \( iii) \) \( f \) is strongly Milyutin.

**Proof.** If \( X \) and \( Y \) are metrizable, this follows from Proposition 3.5 and Theorem 3.6. In case \( f \) is perfect, we apply Propositions 3.4 and 3.5.

A space \( Z \) is called a \( k_2 \)-space if every function on \( Z \) is continuous provided it is continuous on every compact subset of \( Z \).
Theorem 3.8. The product $f$ of any family $\{f_\alpha : X_\alpha \to Y_\alpha, \alpha \in A\}$ of weakly Milyutin maps is also weakly Milyutin. If, in addition, $Y = \prod\{Y_\alpha : \alpha \in A\}$ is a $k_\mathbb{R}$-space and for every $\alpha \in A$ the closed and bounded subsets of $X_\alpha$ are compact, then $f$ is Milyutin provided each $f_\alpha$ is Milyutin.

Proof. Let $T_\alpha : Y_\alpha \to P_c^*(X_\alpha)$ be a map associated with $f_\alpha$ for each $\alpha$. Then, by Proposition 3.4, $\beta f_\alpha$ is a Milyutin map and $\beta T_\alpha : \beta Y_\alpha \to P(\beta X_\alpha)$ is associated with $\beta f_\alpha$. So, $u_\alpha : C(\beta X_\alpha) \to C(\beta Y_\alpha)$, $u_\alpha(h)(y) = \beta T_\alpha(y)(h)$, $y \in \beta Y_\alpha$ and $h \in C(\beta X_\alpha)$, is a regular averaging operator for $\beta f_\alpha$. Let $X = \prod\{X_\alpha : \alpha \in A\}$, $\tilde{X} = \prod\{\beta X_\alpha : \alpha \in A\}$, $\tilde{Y} = \prod\{\beta Y_\alpha : \alpha \in A\}$ and $\tilde{f} = \prod\{\beta f_\alpha : \alpha \in A\}$. According to [26], there exists a regular averaging operator $u : C(\tilde{X}) \to C(\tilde{Y})$ for $\tilde{f}$ such that $u(h \circ p_\alpha) = u_\alpha(h) \circ q_\alpha$, $\alpha \in A$, $h \in C(\beta X_\alpha)$, where $p_\alpha : \tilde{X} \to \beta X_\alpha$ and $q_\alpha : \tilde{Y} \to \beta Y_\alpha$ are the projections. This implies that, if $\tilde{T} : \tilde{Y} \to P(\tilde{X})$ is the map associated to $\tilde{f}$ and generated by $u$, we have $s(\tilde{T}(y)) \subset \prod\{s(T_\alpha(q_\alpha(y))) : \alpha \in A\}$, $y \in Y$. Hence, $s(\tilde{T}(y)) \subset f^{-1}(y)$ for every $y \in Y$. So, $\tilde{T}$ maps $Y$ into the subspace $H$ of $P(\tilde{X})$ consisting of all measures $\mu \in P(\tilde{X})$ with $s(\mu) \subset X$. Now, let $\pi : \beta X \to \tilde{X}$ be the natural map and $P(\pi) : P(\beta X) \to P(\tilde{X})$. Then, $\theta = P(\pi)|P_c^*(X) : P_c^*(X) \to H$ is a homeomorphism (for more general result see [9, Proposition 1]). Therefore, $T = \theta^{-1} \circ (\tilde{T}|Y) : Y \to P_c^*(X)$ is a map associated with $f$. Thus, $f$ is weakly Milyutin.

Suppose now that $Y$ is a $k_\mathbb{R}$-space, $f_\alpha$ are Milyutin maps and the closed and bounded subsets of each $X_\alpha$ are compact. We already proved that there exists a regular averaging operator $u : C^*(X) \to C^*(Y)$ for $f$ and a corresponding to $u$ map $T : Y \to P_c^*(X)$ associated with $f$ such that $s^*(T(y)) \subset \prod\{s(T_\alpha(q_\alpha(y))) : \alpha \in A\} \subset f^{-1}(y)$ for every $y \in Y$. Here, each $T_\alpha : Y_\alpha \to P_c(X_\alpha)$ is a map associated with $f_\alpha$ (recall that $f_\alpha$ are Milyutin maps). For any $h \in C(X)$ and $n \geq 1$ define $h_n \in C^*(X)$ by $h_n(x) = h(x)$ if $|h(x)| \leq n$, $h_n(x) = n$ if $h(x) \geq n$ and $h_n(x) = -n$ if $h(x) \leq -n$. Since for every $y \in Y$ the support $s^*(T(y)) \subset X$ is compact, $h|s^*(T(y)) = h_n|s^*(T(y))$ with $n \geq n_0$ for some $n_0$. Hence, the formula $v(h)(y) = \lim u(h_n)(y)$, $y \in Y$, defines a function on $Y$. Let us show that $v(h)$ is continuous. Since $Y$ is a $k_\mathbb{R}$-space, it suffices to prove that $v(h)$ is continuous on every compact set $K \subset Y$. Then each of the sets $T_\alpha(K_\alpha) \subset P_c(X_\alpha)$ is compact, where $K_\alpha = q_\alpha(K)$. By [35, Proposition 3.1], $Z_\alpha = \cup\{s(\mu) : \mu \in T_\alpha(K_\alpha)\}$ is bounded in $X_\alpha$ and, hence compact (recall that all closed and bounded subsets of $X_\alpha$ are compact). Let $Z$ be the closure in $X$ of the set $\cup\{s^*(\mu) : \mu \in T(K)\}$. Since $Z \subset \prod\{Z_\alpha : \alpha \in A\}$, $Z$ is also compact. So, there exists $m$ such that $h|Z = h_m|Z$ for all $n \geq m$. This implies that $v(h)|K = u(h_m)|K$. Hence, $v(h)$ is continuous on $K$. Since for every $y \in Y$ the support of $T(y)$ is compact and each $u(h)(y)$, $h \in C^*(X)$, depends on $h|s^*(T(y))$, $v : C(X) \to C(Y)$ is linear and the support of $T'(y) \in P_c(X)$ is contained in $s^*(T(y)) \subset f^{-1}(y)$, where $T' : Y \to P_c(X)$ is defined by $T'(y)(h) = \cdots$
v(h)(y), h ∈ C(X), y ∈ Y. Moreover, it follows from the definition of v that it is regular and v(ϕ ◦ f) = ϕ for every ϕ ∈ C(Y). Therefore, v is a regular averaging operator for f with compact supports.

**Corollary 3.9.** A product of perfect Milyutin maps is also Milyutin.

**Proof.** Since any product of perfect maps is perfect, the proof follows from Corollary 3.7 and Theorem 3.8.

**Corollary 3.10.** Let \( Y = \prod \{Y_\alpha : \alpha \in A \} \) be a product of metrizable spaces. Then there exists a 0-dimensional product \( X \) of metrizable spaces space and a 0-invertible perfect Milyutin map \( f : X \to Y \).

**Proof.** By [12, Theorem 1.2.1], for every \( \alpha \in A \) there exists a 0-dimensional metrizable space \( X_\alpha \) and a perfect Milyutin map \( f_\alpha : X_\alpha \to Y_{\alpha} \). Then, by Corollary 3.9, \( f = \prod \{f_\alpha : \alpha \in A \} \) is a perfect Milyutin map from \( X = \prod \{X_\alpha : \alpha \in A \} \) onto \( Y \). It is easily seen that \( f \) is 0-invertible because each \( f_\alpha \) is 0-invertible (see Theorem 3.6). Moreover, since \( \dim X_\alpha = 0 \) for each \( \alpha \), \( \dim X = 0 \).

Recall that \( X \) is a \( p \)-paracompact space [2] if it admits a perfect map onto a metrizable space.

**Corollary 3.11.** For every \( p \)-paracompact space \( Y \) there exists a 0-dimensional \( p \)-paracompact space \( Y \) and a perfect 0-invertible Milyutin map \( f : X \to Y \).

**Proof.** Since \( Y \) is \( p \)-paracompact, it can be considered as a closed subset of \( M \times \mathbb{I}^r \), where \( M \) is metrizable and \( r \geq \aleph_0 \). There exist perfect Milyutin maps \( g : M_0 \to M \) and \( h : M_0 \to \mathbb{I}^r \) with \( g \) being the Cantor set [26] and \( M_0 \) a 0-dimensional metrizable space. [12, Theorem 1.2.1]. Then the product map \( \Phi = g \times h : M_0 \times \mathbb{C}_r \) is a perfect 0-invertible Milyutin map (see Corollary 3.10), and let \( T : M \times \mathbb{I}^r \to P_c(M_0 \times \mathbb{C}_r) \) be a map associated with \( \Phi \). Define \( X = \Phi^{-1}(Y) \) and \( f = \Phi(x) \). Since \( X \) is closed in \( M_0 \times \mathbb{C}_r \), it is a 0-dimensional \( p \)-paracompact. Since \( \Phi \) is 0-invertible (as a product of 0-invertible maps, see Theorem 3.6), so is \( f \). To show that \( f \) is Milyutin, observe that \( X \) is \( C \)-embedded in \( M_0 \times \mathbb{C}_r \). So, \( P_c(X) \) is embedded in \( P_c(M_0 \times \mathbb{C}_r) \) such that \( T(y) \in P_c(X) \) for all \( y \in Y \). This means that \( T|Y \) is a map associated with \( f \). Hence, \( f \) is Milyutin.

Now, we provide a specific class of Milyutin maps. Suppose \( B \subset Z \) and \( g : B \to D \). We say that \( g \) is a \( Z \)-normal map provided for every \( h \in C(D) \) the function \( h \circ g \) can be continuously extended to a function on \( Z \). A map \( f : X \to Y \) is called 0-soft [10] if for any 0-dimensional space \( Z \), any two subspaces \( Z_0 \subset Z_1 \subset Z \), and any \( Z \)-normal maps \( g_0 : Z_0 \to X \) and \( g_1 : Z_1 \to Y \) with \( f \circ g_0 = g_1|Z_0 \), there exists a \( Z \)-normal map \( g : Z_1 \to X \) such that \( f \circ g = g_1 \).

**Proposition 3.12.** Every 0-soft map is Milyutin.
Proof. Let $f : X \to Y$ be 0-soft. Consider $Y$ as a $C$-embedded subset of $\mathbb{R}^{C(Y)}$ and let $\varphi : Z \to \mathbb{R}^{C(Y)}$ be a perfect Milyutin map with $\dim Z = 0$ (see Corollary 3.10). Since $Y$ is $C$-embedded in $\mathbb{R}^{C(Y)}$, $g_1 = \varphi|Z_1 : Z_1 \to Y$ is a $Z$-normal map, where $Z_1 = \varphi^{-1}(Y)$. Because $f$ is 0-soft, there exists a $Z$-normal map $g : Z_1 \to X$ with $f \circ g = g_1$. Now, for every $h \in C(X)$ choose an extension $e(h) \in C(Z)$ of $h \circ g$ (such $e(h)$ exist since $g$ is $Z$-normal). Define $v : C(X) \to C(Y)$ by $v(h) = u(e(h))|Y$, where $u : C(Z) \to C(\mathbb{R}^{C(Y)})$ is a regular averaging operator for $\varphi$ having compact supports. The map $v$ is linear because for every $y \in Y$ $u(e(h))(y)$ depends on the restriction $e(h)|\varphi^{-1}(y)$. By the same reason $v$ has compact supports. Moreover, $v$ is a regular averaging operator for $f$. Hence, $f$ is Milyutin.

\[ \Box \]

4. $AE(0)$-spaces and Regular Extension Operators with Compact Supports

Let $X$ be a subspace of $Y$. A linear operator $u : C(X, E) \to C(Y, E)$ is said to be an extension operator provided each $u(f)$, $f \in C(X, E)$ is an extension of $f$. One can show that such an extension operator $u$ is regular and has compact supports if and only if there exists a map $T : Y \to P_c(X, E)$ such that $T(x) = \delta_x$ for every $x \in X$. Sometimes a map $T : Y \to P_c(X, E)$ satisfying the last condition will be called a $P_c$-valued retraction. The connection between $u$ and $T$ is given by the formula $T(y)(f) = u(f)(y)$, $f \in C(X, E)$, $y \in Y$.

Pełczyński [26] introduced the class of Dugundji spaces: a compactum $X$ is a Dugundji space if for every embedding of $X$ in another compact space $Y$ there exists an extension regular operator $u : C(X) \to C(Y)$ (note that $u$ has compact supports because $X$ is compact). Later Haydon [19] proved that a compact space $X$ is a Dugundji space if and only if it is an absolute extensor for 0-dimensional compact spaces (br., $X \in AE(0)$). The notion of $X \in AE(0)$ was extended by Chigogidze [10] in the class of all Tychonoff spaces as follows: a space $X$ is an $AE(0)$ if for every 0-dimensional space $Z$ and its subspace $Z_0 \subset Z$, every $Z$-normal map $g : Z_0 \to X$ can be extended to the whole of $Z$.

We show that an analogue of Haydon’s result remains true and for the extended class of $AE(0)$-spaces.

**Theorem 4.1.** For any space $X$ the following conditions are equivalent:

(i) $X$ is an $AE(0)$-space;

(ii) For every $C$-embedding of $X$ in a space $Y$ there exists a regular extension operator $u : C(X) \to C(Y)$ with compact supports;

(iii) For every $C$-embedding of $X$ in a space $Y$ there exists a regular extension operator $u : C^*(X) \to C^*(Y)$ with compact supports.

**Proof.** (i) $\Rightarrow$ (ii) Suppose $X$ is $C$-embedded in $Y$ and take a set $A$ such that $Y$ is $C$-embedded in $\mathbb{R}^A$. It suffices to show there exists a regular extension
operator $u: C(X) \rightarrow C(\mathbb{R}^A)$ with compact supports, or equivalently, we can find a map $T: \mathbb{R}^A \rightarrow P_c(X)$ with $T(x) = \delta_x$ for all $x \in X$. By Corollary 3.10, there exists a 0-dimensional space $Z$ and a Milyutin map $f: Z \rightarrow \mathbb{R}^A$. This means that the map $g: \mathbb{R}^A \rightarrow P_c(Z)$ associated with $f$ is an embedding. Since $X$ is $C$-embedded in $\mathbb{R}^A$, the restriction $f|f^{-1}(X)$ is a $Z$-normal map. So, there exists a map $g: Z \rightarrow X$ extending $f|f^{-1}(X)$ (recall that $X \in AE(0)$). Then $T = P_c(g) \circ g: \mathbb{R}^A \rightarrow P_c(X)$ has the required property that $T(x) = \delta_x$ for all $x \in X$.

(ii) $\Rightarrow$ (iii) Let $X$ be $C$-embedded in $Y$ and $u: C(X) \rightarrow C(Y)$ a regular extension operator with compact supports. Then $u(f) \in C^*(Y)$ for all $f \in C^*(X)$ because $u$ is regular. Hence, $u|C^*(X): C^*(X) \rightarrow C^*(Y)$ is a regular extension operator with compact supports.

(iii) $\Rightarrow$ (i) Suppose $X$ is $C$-embedded in $\mathbb{R}^A$ for some $A$ and $u: C^*(X) \rightarrow C^*(\mathbb{R}^A)$ is a regular extension operator with compact supports. So, there exists a map $T: \mathbb{R}^A \rightarrow P_c(X)$ with $T(x) = \delta_x$, $x \in X$. Assume that $A$ is the set of all ordinals $\{\lambda: \lambda < \omega(\tau)\}$, where $\omega(\tau)$ is the first ordinal of cardinality $\tau$.

For any sets $B \subset D \subset A$ we use the following notations: $\pi_B: \mathbb{R}^A \rightarrow \mathbb{R}^B$ and $\pi^D_B: \mathbb{R}^D \rightarrow \mathbb{R}^B$ are the natural projections, $X(B) = \pi_B(X)$, $p_B = \pi_B|X$ and $p^B = \pi^D_B|X(D)$. A set $B \subset A$ is called $T$-admissible if for any $x \in X$ and $y \in \mathbb{R}^A$ the equality $\pi_B(x) = \pi_B(y)$ implies $p_c^*(p_B)(\delta_x) = p_c^*(p_B)(T(y))$. Let us note that if $B$ is $T$-admissible, then there exists a map

$$(4) \quad T_B: \mathbb{R}^B \rightarrow p^*_c(X(B)) \text{ such that } T_B(z) = \delta_x \text{ for all } z \in X(B).$$

Indeed, take an embedding $i: \mathbb{R}^B \rightarrow \mathbb{R}^A$ such that $\pi_B \circ i$ is the identity on $\mathbb{R}^B$, and define $T_B = p^*_c(p_B) \circ T \circ i$.

Claim 3. For every countable set $B \subset A$ there exists a countable $T$-admissible set $D \subset A$ containing $B$.

We construct by induction an increasing sequence $\{D(n)\}_{n \geq 1}$ of countable subsets of $A$ such that $D \subset D(1)$ and for all $n \geq 1$, $x \in X$ and $y \in \mathbb{R}^A$ we have

$$(5) \quad p^*_c(p_{D(n)}(\delta_x)) = p^*_c(p_{D(n)})(T(y)) \text{ provided } \pi_{D(n+1)}(x) = \pi_{D(n+1)}(y).$$

Suppose we have already constructed $D(1), \ldots, D(n)$. Since $D(n)$ is countable, the topological weight of $X(D(n))$ is $\aleph_0$. So is the weight of $p^*_c(X(D(n)))$ [9]. Then the map $p^*_c(p_{D(n)}(\delta_x)) \circ T: \mathbb{R}^A \rightarrow p^*_c(X(D(n)))$ depends on countable many coordinates (see, for example [27]). This means that there exists a countable set $D(n + 1)$ satisfying (5). We can assume that $D(n + 1)$ contains $D(n)$, which completes the induction. Obviously, the set $D = \bigcup_{n \geq 1} D(n)$ is countable. Let us show it is $T$-admissible. Suppose $\pi_D(x) = \pi_D(y)$ for some $x \in X$ and $y \in \mathbb{R}^A$. Hence, for every $n \geq 1$ we have $\pi_{D(n+1)}(x) = \pi_{D(n+1)}(y)$ and, by (5), $p^*_c(p_{D(n)})(\delta_x) = p^*_c(p_{D(n)})(T(y))$. This means that the support of each measure $p^*_c(p_{D(n)})(T(y))$ is the point $p_{D(n)}(x)$. The last relation implies that the support
of $P_c^*(p_D)(T(y))$ is the point $p_D(x)$. Therefore, $P_c^*(p_D)(T(y)) = P_c^*(p_D)(\delta_x)$ and $D$ is $T$-admissible.

**Claim 4.** Any union of $T$-admissible sets is $T$-admissible.

Suppose $B$ is the union of $T$-admissible sets $B(s)$, $s \in S$, and $\pi_B(x) = \pi_B(y)$ with $x \in X$ and $y \in \mathbb{R}^A$. Then $\pi_B(s)(x) = \pi_B(s)(y)$ for every $s \in S$. Hence, $P_c^*(p_B)(T(y)) = P_c^*(p_B(s))(\delta_x)$, $s \in S$. So, the support of each $P_c^*(p_B)(T(y))$ is the point $p_B(x)$ because $p_B(x) = \bigcap \{ (g_{B(s)})^{-1}(p_B(x)) : s \in S \}$. This means that $B$ is $T$-admissible.

**Claim 5.** Let $B \subset A$ be $T$-admissible. Then we have:

(a) $X(B)$ is a closed subset of $\mathbb{R}^B$;

(b) $P_B(V)$ is functionally open in $X(B)$ for any functionally open subset $V$ of $X$.

Since $B$ is $T$-admissible, according to (4) there exists a map $T_B : \mathbb{R}^B \to P_c^*(X(B))$ such that $T_B(z) = \delta_z$ for all $z \in X(B)$. To prove condition (a), suppose $\{z_\alpha : \alpha \in \Lambda \}$ is a net in $X(B)$ converging to some $z \in \mathbb{R}^B$. Then $\{T_B(z_\alpha)\}$ converges to $T_B(z)$. But $T_B(z_\alpha) = \delta_{z_\alpha} \in i_{X(B)}(X(B))$ for every $\alpha$ and, since $i_{X(B)}(X(B))$ is a closed subset of $P_c^*(X(B))$ (see Proposition 2.3(i)), $T_B(z) \in i_{X(B)}(X(B))$. Hence, $T_B(z) = \delta_y$ for some $y \in X(B)$. Using that $i_{X(B)}$ embeds $X(B)$ in $P_c^*(X(B))$, we obtain that $\{z_\alpha\}$ converges to $y$, so $y = z \in X(B)$.

To prove (b), let $V$ be a functionally open subset of $X$ and $g : X \to [0, 1]$ a continuous function with $V = g^{-1}((0, 1])$. Then $u(g) \in C^*(\mathbb{R}^A)$ with $0 \leq u(g)(y) \leq 1$ for all $y \in \mathbb{R}^A$ and let $W = u(g)^{-1}((0, 1])$. Since $\pi_B(W)$ is functionally open in $\mathbb{R}^B$ (see, for example [34]), $\pi_B(W) \cap X(B)$ is functionally open in $X(B)$. So, it suffices to show that $p_B(V) = \pi_B(W) \cap X(B)$. Because $u(g)$ extends $g$, we have $V \subset W$. So, $p_B(V) \subset \pi_B(W) \cap X(B)$. To prove the other inclusion, let $z \in \pi_B(W) \cap X(B)$. Choose $x \in X$ and $y \in W$ with $\pi_B(x) = \pi_B(y)$. Then $P_c^*(p_B)(T(y)) = P_c^*(p_B)(\delta_x) = \delta_z$ (recall that $B$ is $T$-admissible). Hence, $s^*(T(y)) \subset p_B^{-1}(z)$. Since $y \in W$, $T(y)(g) = u(g)(y) \in (0, 1]$. This implies that $s^*(T(y)) \cap V = \emptyset$ (otherwise $T(y)(g) = 0$ because $g(X - V) = 0$, see Proposition 2.1(iii)). Therefore, $z \in p_B(V)$, i.e. $\pi_B(W) \cap X(B) \subset p_B(V)$. The proof of Claim 5 is completed.

Let us continue the proof of (iii) $\Rightarrow$ (i). Since $A$ is the set of all ordinals $\lambda < \omega(\tau)$, according to Claim 3, for every $\lambda$ there exists a countable $T$-admissible set $B(\lambda) \subset A$ containing $\lambda$. Let $A(\lambda) = \bigcup \{ B(\eta) : \eta < \lambda \}$ if $\lambda$ is a limit ordinal, and $A(\lambda) = \bigcup \{ B(\eta) : \eta \leq \lambda \}$ otherwise. By Claim 4, every $A(\lambda)$ is $T$-admissible. We are going to use the following simplified notations:

$$X_\lambda = X(A(\lambda)), \ p_\lambda = p_{A(\lambda)} : X \to X_\lambda \text{ and } p_\lambda^\eta : X_\eta \to X_\lambda \text{ provided } \lambda < \eta.$$
Since $A$ is the union of all $A(\lambda)$ and each $X_\lambda$ is closed in $\mathbb{R}^{A(\lambda)}$ (see Claim 5(a)), we obtain a continuous inverse system $S = \{X_\lambda, p^0_\lambda, \lambda < \eta < \omega(\tau)\}$ whose limit space is $X$. Recall that $S$ is continuous if for every limit ordinal $\gamma$ the space $X_\gamma$ is the limit of the inverse system $\{X_\lambda, p^0_\lambda, \lambda < \gamma\}$. Because of the continuity of $S$, $X \subseteq AE(0)$ provided $X_1 \subseteq AE(0)$ and each short projection $p^{\lambda+1}_\lambda$ is 0-soft. The space $X_1$ being a closed subset of $\mathbb{R}^{A(1)}$ is a Polish space, so an $AE(0)$ [10]. Hence, it remains to show that all $p^{\lambda+1}_\lambda$ are 0-soft.

We fix $\lambda < \omega(\tau)$ and let $E(\lambda) = A(\lambda) \cap (B(\lambda) \cup B(\lambda + 1))$. Since $E(\lambda)$ is countable, there exists a sequence $\{\beta_n\} \subseteq A(\lambda)$ such that $\beta_n \leq \lambda$ for each $n$ and $E(\lambda) \subseteq C(\lambda) \subseteq A(\lambda)$, where $C(\lambda) = \cup \{B(\beta_n) : n \geq 1\}$. By Claim 4, the sets $C(\lambda)$ and $D(\lambda) = B(\lambda) \cup B(\lambda + 1) \cup C(\lambda)$ are countable and $T$-admissible. Consider the following diagram:

\[
\begin{array}{ccc}
X_{\lambda+1} & \xrightarrow{p^{\lambda+1}_D} & X_\lambda \\
\downarrow{p^{A(\lambda+1)}_D} & & \downarrow{p^{A(\lambda)}_C} \\
X(D(\lambda)) & \xrightarrow{p^{D(\lambda)}_C} & X(C(\lambda))
\end{array}
\]

We are going to prove first that the diagram is a cartesian square. This means that the map $g: X_{\lambda+1} \to Z$, $g(x) = (p^{A(\lambda+1)}_D(x), p^{\lambda+1}_A(x))$, is a homeomorphism. Here $Z = \{(x_1, x_2) \in X(D(\lambda)) \times X_\lambda : p^{D(\lambda)}_C(x_1) = p^{A(\lambda)}_C(x_2)\}$ is the fibered product of $X(D(\lambda))$ and $X_\lambda$ with respect to the maps $p^{D(\lambda)}_C$ and $p^{A(\lambda)}_C$. Let $z = (x(1), x(2)) \in Z$. Since $(D(\lambda) - C(\lambda)) \cap (A(\lambda) - C(\lambda)) = \emptyset$ and $A(\lambda + 1) = (D(\lambda) - C(\lambda)) \cup (A(\lambda) - C(\lambda)) \cup C(\lambda)$, there exists exactly one point $x \in \mathbb{R}^{A(\lambda+1)}$ such that $\pi^{A(\lambda+1)}_{D(\lambda)}(x) = x(1)$ and $\pi^{A(\lambda+1)}_{A(\lambda)}(x) = x(2)$. Choose $y \in \mathbb{R}^A$ with $\pi_{A(\lambda+1)}(y) = x$. Since $D(\lambda)$ and $A(\lambda)$ are $T$-admissible, $P_\gamma^* (p^{D(\lambda)}(T(y))) = \delta_{x(1)}$ and $P_\gamma^* (p^{A(\lambda)}(T(y))) = \delta_{x(2)}$. Consequently, $p^{A(\lambda+1)}_{D(\lambda)}(H) = x(1)$ and $p^{A(\lambda+1)}_{A(\lambda)}(H) = x(2)$, where $H$ is the support of the measure $P_\gamma^* (p^{A(\lambda+1)}(T(y)))$. Hence, $H = \{x\}$ is the unique point of $X_{\lambda+1}$ with $g(x) = z$. Thus, $g$ is a surjective and one-to-one map between $X_{\lambda+1}$ and $Z$. To prove $g$ is a homeomorphism, it remains to show that $g^{-1}$ is continuous. The above arguments yield that $x = g^{-1}(z)$ depends continuously from $z \in Z$. Indeed, since $D(\lambda) \cap A(\lambda) = C(\lambda)$, we have

\[
x(1) = (a, b) \in \mathbb{R}^{D(\lambda) - C(\lambda)} \times \mathbb{R}^{C(\lambda)} \text{ and } x(2) = (b, c) \in \mathbb{R}^{C(\lambda)} \times \mathbb{R}^{A(\lambda) - C(\lambda)},
\]

where $z = (x(1), x(2)) \in Z$. Hence, $g^{-1}(z) = (a, b, c)$ is a continuous function of $z$.

Since $D(\lambda)$ and $C(\lambda)$ are countable and $T$-admissible sets, both $X(D(\lambda))$ and $X(C(\lambda))$ are Polish spaces and $p^{D(\lambda)}_C$ is functionally open (see Claim 5(b)).
Hence, $P_{C(A)}^{D(\lambda)}$ is 0-soft [10]. This yields that $P_{\lambda}^{\lambda+1}$ is also 0-soft because the above diagram is a cartesian square. □

Next proposition provides a characterization of $AE(0)$-spaces in terms of extension of vector-valued functions. This result was inspired by [7].

**Theorem 4.2.** A space $X \in AE(0)$ if and only if for any complete locally convex space $E$ and any $C$-embedding of $X$ in a space $Y$ there exists a regular extension operator $C^*(X,E) \to C^*(Y,E)$ with compact supports.

**Proof.** Suppose $X \in AE(0)$ and $X$ is $C$-embedded in a space $Y$. Then by Theorem 4.1(iii), there exists a regular extension operator $v: C^*(X) \to C^*(Y)$ with compact supports. This is equivalent to the existence of a $P_e^*$-valued retraction $T: Y \to P_e^*(X)$. We can extend each $f \in C^*(X,E)$ to a continuous bounded map $e(f): P_e^*(X) \to E$. Indeed, let $B(f) = \text{conv } f(X)$ and consider the map $P_e^*(f): P_e^*(X) \to P_e^*(B(f))$. Obviously, $B(f)$ is a bounded convex closed subset $E$, so it is complete. Then, by [5, Theorem 3.4 and Proposition 3.10], there exists a continuous map $b: P_e^*(B(f)) \to B(f)$ assigning to each measure $\nu \in P_e^*(B(f))$ its barycenter $b(\nu)$. The composition $e(f) = b \circ P_e^*(f): P_e^*(X) \to B(f)$ is a bounded continuous extension of $f$. We also have

(6) \[ e(f)(\mu) = \int_X f d\mu \] for every $\mu \in P_e^*(X)$.

Finally, we define $u: C^*(X,E) \to C^*(Y,E)$ by $u(f) = e(f) \circ T$, $f \in C^*(X,E)$. The linearity of $u$ follows from (6). Moreover, for every $y \in Y$ the linear map $\Lambda(y): C^*(X,E) \to E$, $\Lambda(y)(f) = u(f)(y)$, is regular because $\Lambda(y)(f) \in \text{conv } f(X)$. Using the arguments from the proof of Proposition 3.1 (the final part), we can show that each $\Lambda(y)$, $y \in Y$, has a compact support which is contained in $K(y) = s^*(T(y)) \subset X$. Therefore, $u$ is a regular extension operator with compact supports.

The other implication follows from Theorem 4.1. Indeed, since $\mathbb{R}$ is complete, there exists a regular extension operator $u: C^*(X) \to C^*(Y)$ provided $X$ is $C$-embedded in $Y$. Hence, by Theorem 4.1(iii), $X \in AE(0)$. □

Recall that a space $X$ is an absolute retract [10] if for every $C$-embedding of $X$ in a space $Y$ there exists a retraction from $Y$ onto $X$.

**Corollary 4.3.** Let $X$ be a convex bounded and complete subset of a locally convex topological space. Then $X$ is an absolute retract provided $X \in AE(0)$.

**Proof.** Suppose $X$ is $C$-embedded in a space $Y$. According to [5, Theorem 3.4 and Proposition 3.10], the barycenter of each $\mu \in P_e(X)$ belongs to $X$ and the map $b: P_e(X) \to X$ is continuous. Since $X \in AE(0)$, by Theorem 4.1, there exists a $P_e$-valued retraction $T: Y \to P_e(X)$. Then $r = b \circ T: Y \to X$ is a retraction. □
Lemma 4.4. Let $X \subset Y$ and $u: C(X) \to C(Y)$ be a regular extension operator with compact supports. Suppose every closed bounded subset of $X$ is compact. Then there exists a map $T_c: P_c(Y) \to P_c(X)$ (resp., $T_c^*: P_c^*(Y) \to P_c^*(X)$) such that $P_c(i) \circ T_c$ (resp., $P_c^*(i) \circ T_c^*$) is a retraction, where $i: X \to Y$ is the embedding of $X$ into $Y$.

Proof. For every $\mu \in P_c(Y)$ define $T_c(\mu): C(X) \to \mathbb{R}$ by $T_c(\mu)(f) = \mu(u(f))$, $f \in C(X)$. Obviously, each $T_c(\mu)$ is linear. Let us show that $T_c(\mu) \in P_c(X)$ for all $\mu \in P_c(Y)$. Since $u$ has compact supports, the map $T: Y \to P_c(X)$ generated by $u$ is continuous. Hence, $T(s(\mu))$ is a compact subset of $P_c(X)$ (recall that $s(\mu) \subset Y$ is compact). Then by [2] (see also [35, Proposition 3.1]), $H(\mu) = \cup \{ s(T(y)) : y \in s(\mu) \}$ is closed and bounded in $X$, and hence compact. Let us show that the support of $T_c(\mu)$ is compact. That will be done if we prove that $s(T_c(\mu)) \subset H(\mu)$. To this end, let $f(H(\mu)) = 0$ for some $f \in C(X)$. Consequently, $T(y)(f) = 0$ for all $y \in s(\mu)$. So, $u(f)(s(\mu)) = 0$. The last equality means that $T_c(\mu)(f) = 0$. Hence, each $T_c(\mu)$ has a compact support and $T_c$ is a map from $P_c(Y)$ to $P_c(X)$. It is easily seen that $P_c(i)(T_c(\mu)) = \mu$ for all $\mu \in P_c(i)(P_c(X))$. Therefore, $P_c(i) \circ T_c$ is a retraction from $P_c(i)$ onto $P_c(i)(P_c(X))$.

Now, we consider the linear operators $T_c^*(\nu): C^*(X) \to \mathbb{R}$, $T_c^*(\nu)(h) = \nu(u(h))$ with $\nu \in P_c^*(Y)$ and $h \in C^*(X)$. Observed that $u(h) \in C^*(Y)$ for $h \in C^*(X)$ because $u$ is a regular operator, so the above definition is correct. To show that $T_c^*$ is a map from $P_c^*(Y)$ to $P_c^*(X)$, for every $\nu \in P_c^*(Y)$ take the unique $\mu \in P_c(Y)$ with $j_Y(\mu) = \nu$. Then $s(\mu) = s(\nu)$ according to Proposition 2.1. Hence, $T_c^*(\nu)(h) = 0$ provided $h \in C^*(X)$ with $h|s(T_c(\mu)) = 0$. So, the support of $T_c^*(\nu)$ is contained in $s(T_c(\mu))$. This means that $T_c^*$ maps $P_c^*(Y)$ into $P_c^*(X)$. Moreover, one can show that $P_c^*(i) \circ T_c^*$ is a retraction.

Ditor and Haydon [14] proved that if $X$ is a compact space, then $P(X)$ is an absolute retract if and only if $X$ is a Dugundji space of weight $\leq \aleph_1$. A similar result concerning the space of all $\sigma$-additive probability measures was established by Banakh-Chigogidze-Fedorchuk [6]. Next theorem shows that the same is true when $P_c(X)$ or $P_c^*(X)$ is an AR.

Theorem 4.5. For a space $X$ the following are equivalent:

(i) $P_c(X)$ (resp., $P_c^*(X)$) is an absolute retract;
(ii) $P_c(X)$ (resp., $P_c^*(X)$) is an AE(0);
(iii) $X$ is a Dugundji space of weight $\leq \aleph_1$.

Proof. (i) $\Rightarrow$ (ii) This implication is trivial because every AR is an AE(0).

(ii) $\Rightarrow$ (iii) It suffices to show that $X$ is compact. Indeed, then both $P_c(X)$ and $P_c^*(X)$ are AE(0) and coincide with $P(X)$. So, by Corollary 4.3, $P(X)$ is an AR. Applying the mentioned above result of Ditor-Haydon, we obtain that $X$ is a Dugundji space of weight $\leq \aleph_1$. 

Suppose $X$ is not compact. Since $P_c(X)$ (resp., $P_c^*(X)$) is an $AE(0)$-space, it is realcompact. Hence, so is $X$ as a closed subset of $P_c(X)$ (resp., $P_c^*(X)$). Consequently, $X$ is not pseudocompact (otherwise it would be compact), and there exists a closed $C$-embedded subset $Y$ of $X$ homeomorphic to $\mathbb{N}$ (see the proof of Proposition 2.6). Since $Y$ is an $AE(0)$, according to Theorem 4.1, there exists a regular extension operator $u: C(Y) \to C(X)$ with compact supports.

Then, by Lemma 4.4, $P_c(Y)$ (resp., $P_c^*(Y)$) is homeomorphic to a retract of $P_c(X)$ (resp., $P_c^*(X)$). Hence, one of the spaces $P_c(Y)$ and $P_c^*(Y)$ is an $AE(0)$ (as a retract of an $AE(0)$-space). Suppose $P_c^*(Y) \in AE(0)$. Since $P_c^*(Y)$ is second countable, this implies $P_c^*(Y)$ is Čech-complete. Hence, by Proposition 2.6, $Y$ is pseudocompact, a contradiction. If $P_c(Y) \in AE(0)$, then $P_c(Y)$ is metrizable according to a result of Chigogidze [10] stating that every $AE(0)$-space whose points are $G_\delta$-sets is metrizable (the points of $P_c(Y)$ are $G_\delta$ because $j_Y: P_c(Y) \to P_c^*(Y)$ is an one-to-one surjection and $P_c^*(Y)$ is metrizable). But by Proposition 2.5(ii), $P_c(Y)$ is metrizable only if $Y$ is compact and metrizable. So, we have again a contradiction.

(iii) $\Rightarrow$ (i) This implication follows from the stated above result of Ditor and Haydon [14].

5. Properties preserved by Milyutin maps

In this section we show that some topological properties are preserved under Milyutin maps. Let $\mathcal{F}$ be a family of closed subsets of $X$. We say that $X$ is collectionwise normal with respect to $\mathcal{F}$ if for every discrete family $\{F_\alpha : \alpha \in A\} \subset \mathcal{F}$ there exists a discrete family $\{V_\alpha : \alpha \in A\}$ of open in $X$ sets with $F_\alpha \subset V_\alpha$ for each $\alpha \in A$. When $X$ is collectionwise normal with respect to the family of all closed subsets, it is called collectionwise normal.

**Theorem 5.1.** Every weakly Milyutin map preserves paracompactness and collectionwise normality.

**Proof.** Let $f: X \to Y$ be a weakly Milyutin map and $u: C^*(X) \to C^*(Y)$ a regular averaging operator for $f$ with compact supports.

Suppose $X$ is collectionwise normal, and let $\{F_\alpha : \alpha \in A\}$ be a discrete family of closed sets in $Y$. Then $\{f^{-1}(F_\alpha) : \alpha \in A\}$ is a discrete collection of closed sets in $X$. So, there exists a discrete family $\{V_\alpha : \alpha \in A\}$ of open sets in $X$ with $f^{-1}(F_\alpha) \subset V_\alpha$, $\alpha \in A$. Let $V_0 = X - \bigcup\{f^{-1}(F_\alpha) : \alpha \in A\}$ and $\gamma = \{V_\alpha : \alpha \in A\} \cup \{V_0\}$. Since $\gamma$ is a locally finite open cover of $X$ and $X$ is normal (as collectionwise normal), there exists a partition of unity $\xi = \{h_\alpha : \alpha \in A\} \cup \{h_0\}$ on $X$ subordinated to $\gamma$ such that $h_\alpha(f^{-1}(F_\alpha)) = 1$ for every $\alpha$. Observe that $h_{\alpha(1)}(x) + h_{\alpha(2)}(x) \leq 1$ for any $\alpha(1) \neq \alpha(2)$ and any $x \in X$. So, $u(h_{\alpha(1)})(y) + u(h_{\alpha(2)})(y) \leq 1$ for all $y \in Y$. This yields that $\{u(h_\alpha)^{-1}(\{1/2, 1\}) : \alpha \in A\}$ is a disjoint open family in $Y$. Moreover,
Let \( X \) be paracompact and \( \omega \) an open cover of \( Y \). So, there exists a locally finite open cover \( \gamma \) of \( X \) which an index-refinement of \( f^{-1}(\omega) \). Let \( \xi \) be a partition of unity on \( X \) subordinated to \( \gamma \). It is easily seen that \( u(\xi) \) is a partition of unity on \( Y \) subordinated to \( \omega \). Hence, by [24], \( Y \) is paracompact. \( \square \)

**Corollary 5.2.** Let \( f : X \to Y \) be a weakly Milyutin map and \( X \) a (completely) metrizable space. Then \( Y \) is also (completely) metrizable.

**Proof.** Let \( T : Y \to P^*_c(X) \) be a map associated with \( f \). Then \( \phi = \Phi^*_X \circ T : Y \to X \) is a lsc compact-valued map (see Lemma 3.3 for the map \( \Phi^*_X \)) such that \( \phi(y) \subset f^{-1}(y) \) for every \( y \in Y \). Since \( Y \) is paracompact (by Theorem 4.1), we can apply Michael’s selection theorem [25] to find an upper semi-continuous (br., usc) compact-valued selection \( \psi : Y \to X \) for \( \phi \) (recall that \( \psi \) is usc provided the set \( \{y \in Y : \psi(y) \cap F \neq \emptyset\} \) is closed in \( Y \) for every closed \( F \subset X \)). Then \( f|X_1 : X_1 \to Y \) is a perfect surjection, where \( X_1 = \bigcup\{\psi(y) : y \in Y\} \). Hence, \( Y \) is metrizable as a perfect image of a metrizable space.

If \( X \) is completely metrizable, then so is \( Y \). Indeed, by [1, Theorem 1.2], there exists a closed subset \( X_0 \subset X \) such that \( f|X_0 : X_0 \to X \) is an open surjection. Then \( Y \) is complete (as a metric space being an open image of a complete metric space). \( \square \)

**Proposition 5.3.** Let \( f : X \to Y \) be a weakly Milyutin map with \( X \) being a product of metrizable spaces. Then we have:

(i) The closure of any family of \( G_\delta \)-sets in \( X \) is a zero-set in \( X \);

(ii) \( X \) is collectionwise normal with respect to the family of all closed \( G_\delta \)-sets in \( X \).

**Proof.** Let \( X = \prod\{X_\gamma : \gamma \in \Gamma\} \), where each \( X_\gamma \) is metrizable. Suppose \( u : C^*(X) \to C^*(Y) \) is a regular averaging operator for \( f \) with compact supports.

(i) Let \( G \) be a union of \( G_\delta \)-sets in \( Y \). Then so is \( f^{-1}(G) \) in \( X \) and, by [22, Corollary], there exists \( h \in C^*(X) \) with \( h^{-1}(0) = f^{-1}(G) \). Since \( h(T(y)) = 0 \) for each \( y \in G \), \( u(h)(G) = 0 \). On the other hand, \( \inf\{h(x) : x \in T(y)\} > 0 \) for every \( y \notin \overline{G} \). Hence, \( u(h)(y) > 0 \) for any \( y \notin \overline{G} \). Consequently, \( u(h)^{-1}(0) = \overline{G} \).

(ii) Let \( \{F_\alpha : \alpha \in A\} \) be a discrete family of closed \( G_\delta \)-sets in \( Y \). Then so is the family \( \{H_\alpha = f^{-1}(F_\alpha) : \alpha \in A\} \) in \( X \). Moreover, by (i), each \( F_\alpha \) is a zero-set in \( Y \), hence \( H_\alpha \) is a zero-set in \( X \).

We can assume that \( \Gamma \) is uncountable (otherwise \( X \) is metrizable and the proof follows from Theorem 5.1). Consider the \( \Sigma \)-product \( \Sigma(a) \) of all \( X_\gamma \) with a base-point \( a \in X \). Since \( \Sigma(a) \) is \( G_\delta \)-dense in \( X \) (i.e., every \( G_\delta \)-subset of \( X \) meets \( \Sigma(a) \)), \( \Sigma(a) \) is \( C \)-embedded in \( X \) [32] and
Proof. We consider only the case $k = 1$. Let $W_0 = \Sigma(a) - \{W_a : a \in A\}$. Choose a partition of unity $\{h_a : a \in A\} \cup \{h_0\}$ in $\Sigma(a)$ subordinated to the locally finite cover $W_0$ of $\Sigma(a)$ such that $h_0(W_a \cap \Sigma(a)) = 1$ for each $a$. Since $\Sigma(a)$ is C-embedded in $X$, each $h_a$ can be extended to a function $g_a$ on $X$. Because of (7), $g_\alpha(H_a) = 1$, $\alpha \in A$. The density of $\Sigma(a)$ in $X$ implies that $g_\alpha(x) = g_\alpha(x')$ for any $x \in X$. As in the proof of Theorem 5.1, this implies that $F_a \subset U_a = u(g_\alpha)^{-1}((1/2, 1])$ and the family $\{V_\alpha : \alpha \in A\}$ is disjoint. Then, as in the proof of [16, Theorem 5.1.17], there exists a discrete family $\{V_\alpha : \alpha \in A\}$ of open subsets of $Y$ with $F_a \subset V_\alpha$, $\alpha \in A$.

A space $X$ is called $k$-metrizable [29] if there exists a $k$-metric on $X$, i.e., a non-negative real-valued function $d$ on $X \times \mathcal{R}(X)$, where $\mathcal{R}(X)$ denotes the family of all regularly closed subset of $X$ (i.e., closed sets $F \subset X$ with $F = \text{int}_X(F)$) satisfying the following conditions:

(K1) $d(x, F) = 0$ iff $x \in F$ for every $x \in X$ and $F \in \mathcal{R}(X)$;

(K2) $F_1 \subset F_2$ implies $d(x, F_2) \leq d(x, F_1)$ for every $x \in X$;

(K3) $d(x, F)$ is continuous with respect to $x$ for every $F \in \mathcal{R}(X)$;

(K4) $d(x, \bigcup\{F_\alpha : \alpha \in A\}) = \inf\{d(x, F_\alpha) : \alpha \in A\}$ for every $x \in X$ and every increasing linearly ordered by inclusion family $\{F_\alpha\}_{\alpha \in A} \subset \mathcal{R}(X)$.

If $\mathcal{K}(X)$ is a family of closed subsets of $X$, then a function $d : X \times \mathcal{K}(X) \rightarrow \mathcal{R}$ satisfying conditions (K1) – (K3) with $\mathcal{R}(X)$ replaced by $\mathcal{K}(X)$ is called a monotone continuous annihilator of the family $\mathcal{K}(X)$ [15]. When $\mathcal{K}(X)$ consists of all zero sets in $X$, then any monotone continuous annihilator is said to be a $\delta$-metric on $X$ [15]. The well known notion of stratifiability [8] can be express as follows: $X$ is stratifiable iff there exists a monotone continuous annihilator on $X$ for the family of all closed subsets of $X$.

A space $X$ is perfectly $k$-normal [30] provided every $F \in \mathcal{R}(X)$ is a zero-set in $X$.

Theorem 5.4. Every weakly Milyutin map $f : X \rightarrow Y$ preserves the following properties: stratifiability, $\delta$-metrizability, and perfectly $k$-normality. If, in addition, $\text{cl}_X(f^{-1}U) = f^{-1}(\text{cl}_Y(U))$ for every open $U \subset Y$, then $f$ preserves $k$-metrizability.

Proof. We consider only the case $f$ satisfies the additional condition which is denoted by (s) (the proof of the other cases is similar). Let $u : C^*(X) \rightarrow C^*(Y)$ be a regular averaging operator for $f$ having compact supports, and $d(x, F)$ be a $k$-metric on $X$. We may assume that $d(x, F) \leq 1$ for any $x \in X$ and $F \in \mathcal{R}(X)$, see [29]. Let $F_G = \text{cl}_X(f^{-1}(\text{int}_Y(G)))$ for each $G \in \mathcal{R}(Y)$,
and define \( h_G(x) = d(x, F_G) \). Consider the function \( \rho : Y \times \mathcal{RC}(Y) \to \mathbb{R} \), \( \rho(y, G) = u(h_G)(y) \). We are going to check that \( \rho \) is a k-metric on \( Y \).

Suppose \( G(1), G(2) \in \mathcal{RC}(Y) \) and \( G(1) \subset G(2) \). Then \( F_{G(1)} \subset F_{G(2)} \), so \( h_{G(2)} \leq h_{G(1)} \). Consequently, \( \rho(y, G(2)) \leq \rho(y, G(1)) \) for any \( y \in Y \). On the other hand, obviously, \( \rho(y, G) \) is continuous with respect to \( y \) for every \( G \in \mathcal{RC}(Y) \). Hence, \( \rho \) satisfies conditions (K2) and (K3).

Suppose \( G \in \mathcal{RC}(Y) \). Then \( s^*(T(y)) \subset f^{-1}(y) \subset F_G \) for every \( y \in \text{int}_Y(G) \), where \( T : Y \to P^e(X) \) is the associated map to \( f \) generated by \( u \). Consequently, \( h_G(s^*(T(y))) = 0 \) which implies \( u(h_G)(y) = 0 \), \( y \in \text{int}_Y(G) \). On the other hand, if \( y \notin G \), then \( s^*(T(y)) \cap F_G = \emptyset \) and \( h_G(x) > 0 \) for all \( x \in s^*(T(y)) \). Since \( u(h_G)(y) \geq \inf \{ h_G(x) : x \in s^*(T(y)) \} \) (recall that \( u \) is an averaging operator for \( f \) ), \( u(h_G)(y) > 0 \). Hence, \( u(h_G)(y) = \rho(y, G) = 0 \) iff \( y \in G \), so \( \rho \) satisfies condition (K1).

To check condition (K4), suppose \( \{ G(\alpha) : \alpha \in A \} \subset \mathcal{RC}(Y) \) is an increasing linearly ordered by inclusion family and \( G = \text{cl}_Y(\cup \{ G(\alpha) : \alpha \in A \}) \). Using that \( f \) satisfies condition (s), we have \( F_G = \text{cl}_X(\cup \{ F_{G(\alpha)} : \alpha \in A \}) \). Since \( \{ F_{G(\alpha)} : \alpha \in A \} \) is also increasing and linearly ordered by inclusion, according to condition (K4), \( h_G(x) = \inf \{ h_{G(\alpha)}(x) : \alpha \in A \} \) for every \( x \in X \). Let \( y \in Y \) and \( \epsilon > 0 \). Then for every \( x \in X \) there exists \( \alpha_x \in A \) such that \( h_{G(\alpha_x)}(x) < h_G(x) + \epsilon \). Choose a neighborhood \( V(x) \) of \( x \) in \( X \) such that \( h_{G(\alpha_x)}(z) < h_G(z) + \epsilon \) for all \( z \in V(x) \). Since \( s^*(T(y)) \) is compact, it can be covered by finitely many \( V(x(i)) \), \( i = 1, \ldots, n \), with \( x(i) \in s^*(T(y)) \). Let \( \beta = \max \{ \alpha_x(i) : i \leq n \} \). Then \( h_{G(\beta)}(x) < h_G(x) + \epsilon \) for all \( x \in s^*(T(y)) \). The last equality yields \( \rho(y, G(\beta)) \leq \rho(y, G) + \epsilon \) because \( u(h_G)(y) \) and \( u(h_{G(\beta)})(y) \) depend only on the restrictions \( h_{G(\beta)}|s^*(T(y)) \) and \( h_G|s^*(T(y)) \), respectively. Thus, \( \inf \{ \rho(y, G(\alpha)) : \alpha \in A \} \leq \rho(y, G) \). The inequality \( \rho(y, G) \leq \inf \{ \rho(y, G(\alpha)) : \alpha \in A \} \) is obvious because \( G \) contains each \( G(\alpha) \), so \( \rho \) satisfies condition (K4). Therefore, \( Y \) is k-metrizable. \( \square \)

Next corollary provides a positive answer to a question of Shchepin [31].

**Corollary 5.5.** Every \( AE(0) \)-space is k-metrizable.

**Proof.** Let \( X \) be an \( AE(0) \)-space of weight \( \tau \). By [10, Theorem 4], there exists a surjective 0-soft map \( f : \mathbb{N}^\tau \to X \). Since \( \mathbb{N}^\tau \in AE(0) \) (as a product of \( AE(0) \)-space) and every 0-soft map between \( AE(0) \)-spaces is functionally open [10, Theorem 1.15], \( f \) satisfies condition (s) from the previous theorem. On the other hand, \( \mathbb{N}^\tau \) is k-metrizable as a product of metrizable spaces [29, Theorem 15]. Hence, the proof follows from Proposition 3.12 and Theorem 5.4. \( \square \)

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