We present a gauge invariant digitization of \((1 + 1)d\) scalar quantum electrodynamics for an arbitrary spin truncation for qudit-based quantum computers. We provide a construction of the Trotter operator in terms of a universal qudit-gate set. The cost savings of using a qutrit based spin-1 encoding versus a qubit encoding are illustrated. We show that a simple initial state could be simulated on current qutrit based hardware using noisy simulations for two different native gate set.

I. INTRODUCTION

Quantum computing offers a natural way to simulate the dynamics of quantum field theories. While classical Monte Carlo simulations of lattice gauge theories have been able to extract static quantities to high precision \([1, 2]\), classical Monte Carlo simulations encounter problems with determinations of dynamic quantities due to a sampling to noise issue known as the sign problem. While work has been done to begin tackling these problems \([3, 4]\) using classical computers, quantum computing still offers another path forward.

Many quantum field theories (QFT) have continuous symmetries such as Quantum Electrodynamics (QED) which has a \(U(1)\) symmetry; Quantum Chromodynamics (QCD), which has a \(SU(3)\) symmetry, and Scalar \(\phi^4\) which has continuous values for the field. While classical computers can truncate these continuous symmetries to machine precision, in order to store the values of the nine matrix elements for a gauge link in QCD to double-precision would require \(O(1000)\) qubits. This is clearly infeasible for noisy intermediate scale quantum (NISQ) hardware and in practice truncations of these symmetries will be necessary. These truncations of field or group elements can take various forms. The scalar fields in \(\phi^4\) can be approximated with even distributions of the field values and imposing field cutoffs \([5, 6]\). Compact QED can be mapped to a \(Z_n\) or \(U(1)\) symmetries \([7–14]\). The non-Abelian groups \(SU(2)\) and \(SU(3)\) can be digitized in various ways \([15–19]\). Other methods such as imbedding the theory into higher dimensions using quantum link models \([20, 21]\) and D-Theory \([22, 23]\) is also possible. Since truncations lead to a different theory being simulated on the computer, understanding how these truncations distort the physics is an interesting question but regardless these distortions must be removed \([24]\). Understanding how to return to the continuous symmetries is its own problem \([14, 15, 25–30]\). Digitizations of \(U(1)\) for Quantum Electrodynamics typically use Hilbert spaces that have an odd integer states per site or link \([7–10, 13, 14, 31]\); similar issues will arise for \(SU(3)\) \([15, 17]\). Because these digitizations do not nicely map onto Hilbert spaces of dimension \(2^n\) there are states that will not be used and will complicate the circuit structure.

While simulations of Quantum Chromodynamics are still many years off, digital quantum simulations of \(1 + 1\) and \(2 + 1\) dimensional field theories are already in progress \([32–43]\). Simulations of the Transverse Ising model (TIM) \([36, 44–56]\) and some simpler gauge theories such as the Schwinger model have been a major focus of qubit based computers \([15, 24, 33, 38, 57]\). Compact scalar Quantum Electrodynamics (sQED) in \((1 + 1)d\) has implementations proposed for optical lattices \([8, 9]\). This model is also called the Abelian Higgs model, however we will refer to it as sQED in this work. sQED is a natural first step for simulations on near term qudit NISQ computers because it is a \((1 + 1)d\) gauge theory with a continuous symmetry that is coupled matter. In addition this theory’s Hamiltonian can be written in an explicitly local gauge invariant way \([7–10]\). The fact that this is a gauge-matter theory
and the algebra for its Hamiltonian can naturally be represented by qudits makes this model amenable to simulations on near term NISQ computers using qudit-based architectures. This is not the only model amenable to qudit-based machines; O(N) spin models and spin-1 Ising models are also possible [58] as well as (2+1)d U(1) [31, 59, 60]. Simulations of dynamics for sQED would be timely given the recent interest in algorithms [61–68], testing [69–72] and development of qutrit based hardware [70, 73–80].

This paper is laid out as follows. Sec. II discusses the Abelian Higgs model and its Hamiltonian formulation. In Sec. III, we discuss the systematic errors introduced by spin truncation. Sec. IV discusses how the Hamiltonian can be digitized on qudit base hardware. We walk through the choice of observable, the methods of state preparation, and the simulation using a noise model of a qutrit based quantum computer in Sec. V. Finally, Sec. VI highlights the results and a road map of future models of interest.

II. MODEL

Following closely [7–10], 1+1-d compact Scalar QED with the magnitude of the scalar field frozen to unity has the Euclidean lattice action, with similar notation as [7–10] is used for consistency,

\[ S = S_{\text{gauge}} + S_{\text{matter}} \]

\[ S_{\text{gauge}} = -\frac{1}{g^2a_\pi a_\tau} \sum_x \sum_{\nu<\mu} \text{ReTr}(U_{x,\mu\nu}) \]

\[ S_{\text{matter}} = -\kappa_s \sum_x \left( \phi_x^\dagger U_{x,s} \phi_{x+s} + \text{h.c.} \right) - \kappa_r \sum_x \left( \phi_x^\dagger U_{x,\tau} \phi_{x+\tau} + \text{h.c.} \right) \]

where \( \kappa_s = R^2a_\pi/a_\pi \), \( \kappa_r = R^2a_\tau/a_\tau \), and \( R \) is the radial scalar field magnitude and is generally allowed to vary but will be fixed to one in this work. Compact representations of the gauge and matter fields are used:

\[ U_{x,\mu\nu} = U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu,\nu}^\dagger U_{x,\nu}^\dagger \]

\[ U_{x,\mu} = e^{-ia_\mu A_{x,\mu}}, \quad \phi_x = e^{iQ_x} \]

After taking the continuous time limit and using the same notation as in [7–10] we find the following Hamiltonian

\[ \hat{H} = \frac{U}{2} \sum_{i=1}^{N_s} (\hat{L}_i^z)^2 + \frac{Y}{2} \sum_{i=1}^{N_s-1} (\hat{L}_i^z - \hat{L}_{i+1}^z)^2 \]

\[ + \frac{Y}{2} \left( (\hat{L}_1^z)^2 + (\hat{L}_{N_s}^z)^2 \right) - X \sum_{i=1}^{N_s} \hat{U}_i^z \]

where,

\[ \hat{L}_i^z|m\rangle = m|m\rangle \]

\[ \hat{U}^x = \frac{1}{2}(\hat{U}^+ + \hat{U}^-) \]

and \( \hat{U}^{\pm}|m\rangle = |m \pm 1\rangle \).

The coefficients in Eq. (3) are related to the lattice spacing and gauge coupling \( U = g^2 a_\pi \), \( Y = 1/2R^2 a_\pi \), \( X = 2R^2 a_\pi \). In theory the operators \( \hat{L}_i^z \) and \( \hat{U}^x \) are infinite dimensional with the values of \( m \) in Eq. (4) ranging from \( -\infty \) to \( +\infty \). In practice a cut off will be necessary for implementation on quantum hardware so that the Hilbert space is finite. In this case the spins \( m = -n_{\text{max}}, ..., 0, ..., n_{\text{max}} \). The following subsections will highlight the digitization procedure for implementing the Hamiltonian in Eq. (3) on qudit based hardware as well as indicating the difficulties of implementation on qubit based hardware.

III. SYSTEMATIC ERRORS

An important aspect of truncations is examining the size of the truncation versus the lattice spacing. It should be unsurprising that truncation effects will become more significant the closer to the continuum we go. In order to measure the effectiveness of these truncations we will use the following quantity

\[ \chi = \frac{1}{N_s} \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} \langle \Omega | (\hat{L}_i^z - \hat{L}_j^z)^2 | \Omega \rangle \]

where \( |\Omega\rangle \) is the ground state. This quantity measures how correlated the fields are at different
sites. This measure is likely more accurate than comparing eigenvalues of the Hamiltonian because it probes off-diagonal elements and excited states with respect to the eigenbasis of the Hamiltonian.

We expect that at small coupling \( g^2 a^2 \) since the high spin states are easily excited a larger truncation will be necessary. Conversely for strong coupling we should expect that a more coarse truncation will be acceptable. Fig. 1 shows \( \chi \) normalized by the untruncated value as a function of the coupling, \( g^2 a^2 \), for a \( n_s = 4 \) site lattice. The supposition posited earlier carries out here. For couplings on the order of \( g^2 a^2 \leq 10^{-2} \), \( n_{\text{max}} > 4 \) to effectively capture the physics desired. While for couplings of \( g^2 a^2 \approx 1 \), a truncation of \( n_{\text{max}} = 2 \) appears to be sufficient.

One key feature that is evident is that there seems to be a stark difference between the spin-1 and spin-2 truncation at all couplings; this was seen as well in [81]. The discrepancy is not unexpected, for \( Z_n \) theories there is a marked discrepancy between \( n \leq 4 \) and \( n \geq 5 \) accurate representations of \( U(1) \) [82].

The key point that should be understood is that in the strongly coupled regime a spin-2 (qupet) will likely be sufficient but in the weakly coupled regime a spin-4 to spin-6 truncation will be necessary to capture the desired physics [10]. For nearer term devices a qupet may not yet be feasible but qutrits are actively being studies. This will provide a good foundation for benchmarking and developing tools for higher truncations even if it does not accurately represent the physics of the theory.

IV. ENCODING

A. qudit representation

As previously mentioned, implementation on physical hardware requires a maximal spin cut off denoted \( n_{\text{max}} \). For a given maximal integral spin, the operators defined in Eq. (4) behave as follows,

\[
(\hat{L}^z)_{i,j} = (n - i)\delta_{i,j}
\]  
(6)

where \( 0 \leq i,j < 2n + 1 \) and \( U^x \) can either truncate at \( m = \pm n_{\text{max}} \) or have highest and lowest states wrap around like a \( Z_{n_{\text{max}}} \) theory.

The Hamiltonian can be separated into two non-commuting parts. The terms containing \( L^z \) and the terms containing \( U^x \). This spin-\( n_{\text{max}} \) system is most naturally mapped onto a \((2n_{\text{max}} + 1)\)-qudit architecture. A universal basis for this machine will require a collection of \( 2n_{\text{max}} + 1 \) \( SU(2) \) rotations that couple the energies levels together and allow an arbitrary \( SU(2n_{\text{max}} + 1) \) rotation. We can define these operators that compose the \( SU(2) \) sub-algebras as generalizations of the Pauli matrices which will be defined as \( \hat{X}^{a,b} \), \( \hat{Y}^{a,b} \), and \( \hat{Z}^{a,b} \). The behavior on these operators is given by

\[
\hat{\sigma}^x_{a,b}|c\rangle = \begin{cases} 
|b\rangle & c = a \\
|a\rangle & c = b \\
0 & c \neq a, b
\end{cases},
\]  
(7)

\[
\hat{\sigma}^y_{a,b}|c\rangle = \begin{cases} 
-i|b\rangle & c = a \\
-i|a\rangle & c = b \\
0 & c \neq a, b
\end{cases},
\]  
(8)

FIG. 1. \( \chi \) normalized by an infinite bound extrapolation as a function of the coupling and spin truncation for 4 sites.
\[
\hat{\sigma}_{a,b}^z |c\rangle = \begin{cases} 
|a\rangle & c = a \\
-|b\rangle & c = b \\
0 & c \neq a, b 
\end{cases}, \quad (9)
\]

where \(1 \leq a, b, c \leq 2n_{\text{max}} + 1\). We can also define rotations,

\[
\hat{R}^\alpha_{(a,b)}(\theta) = e^{i\theta \hat{\sigma}^z_{(a,b)}}, \quad (10)
\]

where \(\alpha = x, y, z\), and \(a\) and \(b\) indicate the states for the Pauli sub-algebra to mix between. For an arbitrary spin-\(n_{\text{max}}\) system we can write the operators of Equation (3) as follows:

\[
\hat{L}_z = \sum_{j=1}^{n_{\text{max}}} (n_{\text{max}} - j + 1)(\hat{\sigma}^z_{j,n_{\text{max}}} + \hat{\sigma}^z_{n_{\text{max}},n_{\text{max}} - j + 1}) \quad (11)
\]

and

\[
\hat{U}_x = \frac{1}{2} \left( c_{\text{bound}} \hat{\sigma}^x_{(1,2n_{\text{max}}+1)} + \sum_{j=1}^{2n_{\text{max}}-1} \hat{\sigma}^x_{(j,j+1)} \right) \quad (12)
\]

where \(c_{\text{bound}}\) is 1 if a \(Z_n\) model is desired or 0 if a \(U(1)\)-truncation is desired.

Time evolution of a state \(|\psi\rangle\) is carried out via the traditional operator \(e^{-iH\hat{t}}\). In order to implement this on a quantum computer we need to Trotterize [83] the Hamiltonian and split it into non-commuting terms:

\[
\hat{U}_{\text{tr}}(\hat{t}) = \left( e^{-i\hat{t}(U+2Y)/2} \sum (\hat{L}_z)^2 \right) \quad (13)
\]

\[
e^{-i\hat{t}Y} \sum \hat{L}_z^{i} \hat{L}_{i+1} \hat{U}^x \left( \prod_{j=1}^{2n_{\text{max}}-1} \hat{R}^z_{j,j+1}(\theta_{j}) \right) C_{\text{sum}}. \quad (16)
\]

A diagram of this circuit for any spin truncation is shown in Fig. 2. This Trotterization is straightforward; three types of terms will be present:

1. one qudit rotation \(e^{-i\hat{t}(U+2Y)/2(\hat{L}_z)^2}\)

2. one qudit rotation \(e^{i\hat{t}X\hat{U}^x}\)

3. two qudit rotations \(e^{i\hat{t}Y\hat{L}_z \hat{L}_{i+1}^z}\)

The \((\hat{L}_z)^2\) rotations are relatively straightforward to implement:

\[
e^{i\theta(\hat{L}_z)^2} = \prod_{j=1}^{2n_{\text{max}}} e^{i a_{j,j+1} \theta \hat{\sigma}^z_{j,j+1}} \quad (14)
\]

where the \(a_{j,j+1}\) terms are found by solving the linear equation,

\[
(\hat{L}_z)^2 = a_0 \mathbb{1} + \sum_{j=1}^{2n_{\text{max}}-1} a_{j,j+1} \hat{\sigma}^z_{j,j+1} \quad (15)
\]

for the coefficients \(a_{j,j+1}\) and \(a_0\). The values for these operators are found to scale quadratically with respect to the spin truncation \(n_{\text{max}}\). With more details provided in Appendix A.

The \(\hat{L}_z \hat{L}_{i+1}\) term is also relatively straightforward as well and involves solving a similar set of equations. This operator can be written for arbitrary \(n\) as,

FIG. 2. Quantum circuit for \(U_{\text{tr}}(\hat{t})\) defined in Eq. (13).
The \( \hat{U} \) on single qutrit rotations indicates they are applied on the target qutrit of the \( C_{\text{sum}} \) gates. This coupled \( \hat{L}^z \hat{L}^z \) rotation then can be written in terms of at most \( 2n_{\text{max}} + 1 \) two-qudit gates and \( 4n_{\text{max}}^2 \) one qudit diagonal rotations which are expected to be relatively noiseless [74]. The \( C_{\text{sum}} \) operator is a generalization of the CNOT gate and shifts the state \( |a\rangle \) to \( |a+1\rangle \); it is given by
\[
C_{\text{sum}} = \sum_{k=1}^{2n_{\text{max}}+1} |k\rangle \langle k| \otimes \hat{X}^k, \tag{17}
\]
where
\[
\hat{X} = \sum_{k=1}^{2n_{\text{max}}+1} |k\rangle \langle \text{mod}_{2n_{\text{max}}+1}(k+1)|. \tag{18}
\]

The \( \hat{U}_i^x \) rotations can be implemented in one of two ways. The naive way is to Trotterize the components of \( \hat{U}^x \),
\[
e^{i\theta \hat{U}^x} \approx e^{i\theta \text{bound}\theta_x^{(1,2n_{\text{max}}+1)}} \prod_{j=1}^{2n_{\text{max}}} e^{i\theta \delta_x^{(j,j+1)}}, \tag{19}
\]
which results in \( 2n_{\text{max}} + 1 \) single qudit rotations. A slightly more complicated but exact way involves finding the set of rotations \( \{ R_{(a,b)}^x(\theta_j) \} \) which implement \( e^{i\theta \hat{U}^x} \). This is relatively simple and tractable for most qudit based architectures because it involves diagonalizing a \( 2n_{\text{max}} + 1 \) dimensional matrix where \( n_{\text{max}} \) will be less than 12 for any approximation of \( U(1) \). This will require at most \( (2n_{\text{max}} + 1)^2 - 1 \) rotations. In the case where \( \text{bound} = 1 \) this involves constructing a generalization of the Hadamard gate to a qudit [74, 77],
\[
\hat{H} = \sum_{k=0}^{2n} \sum_{j=0}^{2n} |j\rangle \langle k| e^{i(k+j)\pi/(2n+1)}. \tag{20}
\]

**B. spin 1: Qubit vs. Qutrit**

Up until this point the work has been generalized to qudits. While high spin truncations are needed to simulate actual quantum electrodynamics [7, 10], qudits become more difficult to control the more states that are included [62, 68]. From this point on we will specialize to the spin-1 truncation throughout this work and set \( \text{bound} = 0 \). The fundamental operators from the Hamiltonian defined in Eq. (3) \( \hat{L}^z \) and \( \hat{U}^x \). For a spin-1 (3 state) truncation the \( \hat{L}^z \) operator is defined as
\[
\hat{L}^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{21}
\]
and can be embedded into a two qubit Hilbert space with the following encoding:
\[
\hat{L}^z = (\hat{Z}_2 + \hat{Z}_1 \otimes \hat{Z}_2)/2, \tag{22}
\]
where the \( \hat{Z}_i \) correspond to the Pauli-z matrix on qubit i. By extension, the operator \( (\hat{L}^z)^2 \) is given by
\[
(\hat{L}^z)^2 = (1 + \hat{Z}_1)/2. \tag{23}
\]

Similarly the \( \hat{U}^x \) operator is given by
\[
\hat{U}^x = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{24}
\]
and can be embedded into the Hilbert space of two qubits with the following linear combination of tensor products
\[
\hat{U}^x = \hat{X}_1 \otimes (1 + \hat{X}_2 + \hat{Z}_2)/2 + \hat{Y}_1 \otimes \hat{Y}_2/2. \tag{25}
\]
The Trotterization can be broken up into the two one-qudit rotations,
\[
e^{-i\delta t(U/2+Y)\hat{L}^z} \text{ and } e^{i\delta tXU}^z, \tag{26}
\]
and the two-qudit rotation,
\[
e^{i\delta tL^z_{L_1}L^z_{L_2}}. \tag{27}
\]

In the case of physical qutrits, since both of the single qutrit rotations are an element
of SU(3) they can easily be broken up into 8 rotations [84] defined in Eq. (10) as follows:

$$\hat{V} = e^{i\alpha_1 \hat{\sigma}_0^1} e^{i\alpha_2 \hat{\sigma}_0^2} e^{i\alpha_3 \hat{\sigma}_0^3} e^{i\alpha_4 \hat{\sigma}_0^4} e^{i\alpha_5 \hat{\sigma}_0^5} e^{i\alpha_6 \hat{\sigma}_0^6} e^{i\alpha_7 \hat{\sigma}_0^7} e^{i\alpha_8 \hat{\sigma}_0^8},$$

(28)

where $\hat{V}$ is an arbitrary SU(3) rotation.

Using this Euler decomposition we find that these operators can be written with the following rotations,

$$e^{-i\delta t (U/2 + Y)} = R_{0,1}^z (-\delta t (U/6 + Y/3)) R_{1,2}^z (\delta t (U/6 + Y/3)),$$

(29)

and

$$e^{i\delta t \tilde{X} \tilde{U}^x} = \hat{R}_{0,1}^y (-\frac{\pi}{4}) \hat{R}_{0,2}^y (\frac{\pi}{4}) \hat{R}_{0,1}^y \left( \frac{\delta t X \sqrt{2}}{2} \right) \hat{R}_{0,2}^y (-\frac{\pi}{4}) \hat{R}_{0,1}^y (\frac{\pi}{4}).$$

(30)

Given the native gate set of [74, 77], these two rotations together can be implemented in at most 15 one-qutrit rotations and likely fewer depending on the angles in the of the $\sigma_y^{ab}$ rotations; while the implementation of [71] can implement this with at most 5 rotations for the $\hat{U}^x$ term and 2 for the $(\hat{L}^z)^2$ term, the extra gates from [74] come from the non-continuous parameterization of the $X$ and $Y$ rotations. Given that $\sigma_z^{ab}$ rotations are done virtually they are effectively noiseless and at most only 8 noisy gates are present.

The two qutrit $L^z \otimes L^z$ rotation is implementable with 3 controlled sum gates,

$$C_{\text{sum}} = \sum_{k=0}^{2} |k\rangle \langle k| \otimes (X^{01} X^{12})^k,$$

(31)

and 4 single qutrit $\sigma_z$ rotations. This implementation is,

$$e^{i\delta L^x \otimes L^z} = C_{\text{sum}} R_{0,1}^{z:t} (2\theta/3) R_{1,2}^{z:t} (\theta/3) C_{\text{sum}} R_{0,1}^{z:t} (2\theta/3) R_{0,1}^{z:t} (\theta/3) C_{\text{sum}}.$$

(32)

The controlled sum is a generalization of the qubit-CNOT gate to qudit based architecture [85] and is realizable on current qutrit based hardware [74, 77]. The $t$ on the single qutrit rotations indicates that they are applied on the target qutrit of the $C_{\text{sum}}$ gate.

In the context of the implementing on physical qubits, the $U^x$ operator will require 3 CNOTs to implement [86]. The $L^z \otimes L^z$ term will require approximately 16 CNOTs to couple the all the 2-, 3-, and 4-qubit rotations (shown in Fig. 3).

At this point, it is clear the qutrit formulation clearly is better than the qubit formation because of the reduced number of entangling gates but a qubit formulation is possible. The circuit depth in the qutrit formulation is 6 two-qutrit gates deep per Trotter step, while the qubit formulation is 19 CNOTs deep per Trotter step.
| gate type | qubit encoding | qutrit encoding |
|-----------|----------------|----------------|
| $\hat{U}^z$ | 1 qubit 2 qubit | 1 qutrit 2 qutrit |
| $(\hat{L}^z)^2$ | 1 0 2 0 |
| $\hat{L}^z \otimes \hat{L}^z$ | 4 8 4 3 |

TABLE I. Gate Costs assuming native Qiskit gates compared to decompositions shown in Eqs. (29), (30), and (32)

V. SIMULATION

A. State Preparation and Time Evolution

We will work in a regime $g^2a_s^2 = 5$ and $n_s = 4$. In this regime the ground state accurately represented by the iterative tensor product of the lowest eigenstate of the matrix

$$A = \frac{1}{2} \begin{pmatrix} g^2a_s^2 + 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & g^2a_s^2 + 1 \end{pmatrix},$$

where we have set $Y = 1/2$ and $X = 2$. The lowest energy eigenstate of this operator can be written

$$|\Psi_0\rangle = \frac{1}{\mathcal{N}} \left(|0\rangle_q + b|1\rangle_q + |2\rangle_q\right),$$

where

$$\mathcal{N} = \sqrt{2 + b^2}$$

and

$$b = g^2a_s^2 + 1 - \sqrt{(g^2a_s^2 - 1)^2 + 32/4}.$$

The subscript $q$ indicates these are represented in the qutrit state values rather than the $\hat{L}^z$ spin values. The overlap of the state

$$|\Gamma\rangle = \prod_{i=1}^{4} (|\Psi_0\rangle)^\otimes$$

as a function of the couplings and lattice sizes is shown in Fig. 4. We will demonstrate measuring the correlator,

$$C = \langle \Gamma | e^{itH} \hat{U}_- e^{-itH} \hat{U}_+ |\Gamma\rangle. \quad (37)$$

In order to measure this correlator, we need to be able to prepare the states $|\Gamma\rangle$ and $\hat{U}_+ |\Gamma\rangle$. This is relatively straightforward to accomplish. The state $|\Psi_0\rangle$ in Eq. (34) can be prepared from the state $|0\rangle_q$ with two one qutrit rotations,

$$\hat{V}_g = \hat{R}_{1,2}^\dagger(-\rho_2) \hat{R}_{0,1}^\dagger(\rho_1) \quad (38)$$

where $\rho_1 = \arccos(-1/\sqrt{\mathcal{N}})$ and $\rho_2 = \arcsin(-1/\sqrt{\mathcal{N}})$. Preparing the superposition of $|\Psi_0\rangle$ and $\frac{1}{\mathcal{N}} \hat{U}_+ |\Psi_0\rangle$ is slightly harder. Using an ancilla, 2 $C_{sum}$ gates, and 7 one qutrit gates, we can prepare this superposition of states

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\Psi_0\rangle |0\rangle_a + \frac{1}{\mathcal{N}} \left(|1\rangle + a|2\rangle\right) |1\rangle_a\right). \quad (39)$$

The circuit, $\hat{V}_{\text{prep}}$, which constructs this state is shown in Fig. 5.
After this it is relatively straightforward to measure the correlator. The Trotterized time evolution operator from Eq. (13) is applied to the working qubits followed by $C_\text{sum}$ controlled on the ancilla to the first qutrit. Then by measuring the $\hat{\sigma}^x_{0,1}$ on the ancilla and the states on the working qubits we extract the real part of the correlator. If $\hat{\sigma}^y_{0,1}$ is measured on the ancilla, then the imaginary part of the correlator is extracted instead. Diagrammatically this circuit is shown in Fig. 6. The correlator is then given by the following quantum operations

\[
C = \left( \langle \hat{C}^\dagger (\hat{\sigma}^x_{0,1})_a \hat{C} \rangle + \langle \hat{C}^\dagger (\hat{\sigma}^y_{0,1})_a (\hat{Z}_2) \hat{C} \rangle \right) + i \left( \langle \hat{C}^\dagger (\hat{\sigma}^y_{0,1})_a \hat{C} \rangle + \langle \hat{C}^\dagger (\hat{\sigma}^y_{0,1})_a (\hat{Z}_2) \hat{C} \rangle \right) / 2,
\]

where $|$ represents the state $|0000\rangle \otimes (|0\rangle_a + |1\rangle_a)$. The operator $\hat{Z}_2$ can be applied classically after measuring the qutrit state by apply a $-1$ to any measurement of the first working qutrit that is in the state $|2\rangle$.

**B. Noisy Emulations**

While many noise channels are available such as decay channels that correspond to spontaneous decays from higher excited states [87] and Pauli channels [88] which correspond to “bit” flips, phase flips. Here we use the following Pauli channel noise model for qutrit,

\[
\mathcal{E}(\rho; q) = (1 - \sum_{i<j} \alpha (\rho_{i,j}^\alpha)^g) \rho + \sum_{i<j} (\sigma_{i,j}^\alpha)^g \rho (\sigma_{i,j}^\alpha)^g,
\]

where $\alpha = x, y, z$, $(i, j)$ indicates the mixing between qutrit states $|i\rangle$ and $|j\rangle$, $\rho_{i,j}^\alpha$ is the prob-
ability of such error occurring, and \( q \) indicates which qutrit to apply the noise operation on. This noise model is inspired by the way the fidelities of [77] are reported. This should not be surprising as it looks like an extension of the qubit version of a Pauli noise model used by many [45, 50, 51, 89]. The two qutrit noise model is easily extendable from this using a tensor product of all the Pauli terms for the two qutrits.

Simulations using the noise model described in Eq. (41) are discussed here. The probabilities for the Pauli errors used are listed in Table II which were found for a recent randomized benchmark for a Transmon based qutrit system [77]. The noise model was applied assuming that \( R_x, R_y \) and two-qutrit rotations are noisy and that \( R_z \) rotations are noiseless [74].

The results of noisy emulations of the observable \( \mathcal{C} \) are shown in Fig. 7 for Trotter step size \( \delta t = 0.39 \) (additional simulations at \( \delta t = 0.235 \) and 0.31 are shown in Appendix B). These steps sizes were chosen because they balanced the Trotter fidelity with the emulated noise in the gates to allow time dynamics to be observed. While the computer tested in [77] had the controlled sum (\( C_{sum} \)) as the native two-qutrit gate, we also consider a case where a \( e^{-i\theta L_z \otimes L_z} \) rotation can be used as a native gate assuming the same Pauli errors.

A clear feature is that the native \( L_z^2 \) rotation allows for a 8 to 9 Trotter steps before the gate noise completely suppresses the signal while the native \( C_{sum} \) allows for 4 to 5 Trotter steps before the signal is lost. This suggests that near term qutrit based machines such as those tested by [74, 77] may be able to simulate short term dynamics of this model and allow for early benchmarking of a more complicated field theory than the Transverse Ising Model.

| Term                  | (0, 1) | (0, 2) | (1, 2) |
|-----------------------|--------|--------|--------|
| one qutrit term       | 0.00038 | 0.00143 | 0.00068 |
| two qutrit term       | 0.003  | 0.003  | 0.003  |

TABLE II. Pauli errors for one and two qutrit Pauli terms. The two qutrit term applies to each element of the Pauli noise channel, i.e., each \( \sigma_i^{\alpha}, \sigma_k^{\beta} \) has that probability of occurring.
VI. CONCLUSIONS

This work has shown how to implement the Abelian Higgs model on a qutrit based digital quantum computer. We show how to measure a two point correlation function and demonstrate that a few Trotter steps are feasible using current estimates for the Pauli channel noise on transmon qutrits [77].

Simulations of this model on qutrit based quantum computers would be a significant step toward real time simulations of Quantum Electrodynamics and other field theories with continuous or larger symmetry groups. These simulations will help pave the way toward understanding the dynamics from an ab initio perspective inelastic scattering processes.

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Appendix A: Rotation Decompositions

The coefficients \(a_{j,j+1}\), \(b_{j,j+1}\), and \(c_{j,j+1}\) scale according to the following equations,

\[
\begin{align*}
    a_{j,j+1} &= \frac{2}{3} j + \frac{2}{3} \\
    b_{j,j+1} &= -j^2 - \frac{4}{3} j - \frac{1}{3} \\
    c_{j,j+1} &= \frac{1}{3} j^3 + \frac{1}{2} j^2 + \frac{1}{6} j.
\end{align*}
\]

Using Eq. (A1), we can find an closed form expression for the angles \(\alpha_{j,j+1}\),

\[
\alpha_{j,j+1}(n) = \frac{2j + 2}{3} n^2 - \frac{3j^2 - 4j - 1}{3} n + \frac{2j^3 + 3j^2 + j}{6}.
\]

\[
\begin{tabular}{|c|c|c|c|}
\hline
angle & \(a_{j,j+1}\) & \(b_{j,j+1}\) & \(c_{j,j+1}\) \\
\hline
\(\alpha_{0,1}\) & \(2/3\) & \(-1/3\) & 0 \\
\(\alpha_{1,2}\) & \(4/3\) & \(-8/3\) & 1 \\
\(\alpha_{2,3}\) & 2 & -7 & 5 \\
\(\alpha_{3,4}\) & \(8/3\) & \(-40/3\) & 14 \\
\(\alpha_{4,5}\) & \(10/3\) & \(-65/3\) & 30 \\
\(\alpha_{5,6}\) & 4 & -32 & 55 \\
\(\alpha_{6,7}\) & \(14/3\) & 133/3 & 91 \\
\(\alpha_{7,8}\) & \(16/3\) & \(-176/3\) & 140 \\
\(\alpha_{8,9}\) & 6 & -75 & 204 \\
\hline
\end{tabular}
\]

TABLE III. Angles \(\alpha_{i,i+1}\) for the \(e^{-i\theta(L_z)^2}\) rotations to provide rotations up to a spin truncation \(n = 9\). There is an anti-symmetry of the angles after passing the \(\alpha_{(n-1)/2,(n-1)/2+1}\) where the angles are then follow the reverse pattern and are negative, e.g. for \(n = 1\) \(\alpha_{0,1} = -\alpha_{1,2}\), and for \(n = 2\) \(\alpha_{0,1} = -\alpha_{3,4}\) and \(\alpha_{1,2} = -\alpha_{2,3}\).

Appendix B: Additional Correlators

Here we show the noisy simulations of the time evolution of \(\tilde{C}\) for 2 additional Trotter steps, \(\delta t = 0.235\) and \(\delta t = 0.31\). These Trotter step sizes were chosen because they are highly faithful as demonstrated in Fig. 8. In addition we can see a signal for these smaller Trotter step sizes using both native gate sets.
FIG. 8. Time evolution of the correlator $C$ for 3 different Trotter step sizes, $\delta t = 0.235$ (left), $\delta t = 0.31$ (center), and $\delta t = 0.39$ (right). Using two different the native gate sets.

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