Growth of non-infinitesimal perturbations in turbulence

E. Aurell
Department of Mathematics, Stockholm University S-106 91 Stockholm, Sweden

G. Boffetta
Dipartimento di Fisica Generale, Università di Torino, Via Pietro Giuria 1, I-10125 Torino, Italy
Istituto Nazionale Fisica della Materia, Unità di Torino

A. Crisanti
Dipartimento di Fisica, Università di Roma “La Sapienza”, P.le Aldo Moro 2, I-00185 Roma, Italy
Istituto Nazionale Fisica della Materia, Unità di Roma

G. Paladin
Dipartimento di Fisica Generale, Università di Torino, Via Pietro Giuria 1, I-10125 Torino, Italy
Istituto Nazionale Fisica della Materia, Unità di Torino

A. Vulpiani
Dipartimento di Fisica, Università di Roma “La Sapienza”, P.le Aldo Moro 2, I-00185 Roma, Italy
Istituto Nazionale Fisica della Materia, Unità di Roma

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The standard characterization of the chaotic behavior of a dynamical system is given by the maximum Lyapunov exponent $\lambda_{\text{max}}$, which measures the typical exponential rate of growth of an infinitesimal disturbance \[1\]. It is thus expected that the predictability time is proportional to $\lambda_{\text{max}}^{-1}$, the shortest characteristic time of the system. The underlying point is that the growth of a perturbation is well described by the linear equations for the tangent vector even if this cannot be literally true for non-infinitesimal perturbations. There exist indeed many situations where the Lyapunov analysis has no relevance for the predictability problem and it is necessary to introduce indicators which are able to capture the essential features of a chaotic system. For instance, when two or more characteristic time scales are present a direct identification of the Lyapunov and predictability times leads to paradoxes as recently pointed out in Ref. \[2\].

In this letter, we introduce a measure of the chaoticity degree related to the average doubling time that extends the concept of Lyapunov exponent in the case of non-infinitesimal perturbations. Our indicator is a scale-dependent Lyapunov exponent which becomes particularly useful when there exists a hierarchy of characteristic times such as the eddy turn-over times in three dimensional fully developed turbulence \[3\].

In turbulent flows it is natural to argue that the maximum Lyapunov exponent is roughly proportional to the turnover time $\tau$ of eddies of the size of the Kolmogorov length $\eta$ (the viscous cut-off) that is the shortest characteristic time \[4\]. By dimensional analysis, the turnover time of an eddy of size $\ell$ is $\tau(\ell) \sim \ell^{1-h}$, where $h$ is the scaling exponent of the velocity difference in the eddy

$$v_\ell \equiv |v(\mathbf{x}') - v(\mathbf{x})| \sim \ell^h, \quad \ell = |\mathbf{x}' - \mathbf{x}|. \quad (1)$$

The viscous cut-off vanishes as a power of the Reynolds number $Re$, i.e. $\eta \sim Re^{-(1+h)}$. These relations imply that the maximum Lyapunov exponent should scale as

$$\lambda_{\text{max}} \sim 1 / \tau(\eta) \sim Re^\alpha \quad \text{with} \quad \alpha = \frac{1-h}{1+h}. \quad (2)$$

In the Kolmogorov K41 theory \[5\], $h = 1/3$ for all space points $\mathbf{x}$ so that $\alpha = 1/2$, as first pointed out by Ruelle \[6\]. However, the intermittency of energy dissipation leads to the existence of a spectrum of possible scaling exponents $h$ affecting the value of $\alpha$. In the multifractal approach \[7\], the probability that the velocity difference on scale $\ell$ scales as $v_\ell \sim \ell^h$ is assumed to be $P_\ell(h) \sim \ell^{3-D(h)}$. This ansatz can be tested by measuring the scaling of the structure functions...
\[ \langle v_1 \rangle \sim \ell^p. \] (3)

In the K41 theory, \( \zeta_p = p/3 \) while in the multifractal scenario, \( \zeta_p \) is a non-linear function of \( p \) given by the Legendre transform of the function \( D(h) \), \( \zeta_p = \min_h \{ h p - D(h) + 3 \} \). Moreover, as a consequence of multifractality there is a spectrum of viscous cut-offs, since each \( h \) selects a different damping scale \( \eta(h) \sim R^{-1/(1+h)} \), and hence a spectrum of turnover times \( \tau_h(\eta) \). To find the Lyapunov exponent, we have to integrate over the \( h \)-distribution

\[ \lambda_{\text{max}} \sim \int \tau_h(\eta)^{-1} P_\eta(h) \, dh \sim \int \eta^{h-D(h)+2} \, dh \sim R^\alpha. \] (4)

In the limit \( \text{Re} \to \infty \), the integral can be estimated by the saddle point and gives \( \alpha = \max_h \{ D(h) - 2 - h \}/(1 + h) \). By using the function \( D(h) \) obtained with the random beta model fit, one has \( \alpha = 0.459... \)

In the predictability problem, we are interested in defining the growth of an error on the velocity field. As usual we consider the Euclidean norm

\[ \delta v(t) = \left( \int d^3 x |v(t) - \delta v(t)|^2 \right)^{1/2}. \] (5)

to introduce the notion of distance between two velocity fields \( v \) and \( v' \).

Then, the predictability time \( T_p \) is the time necessary for an initial error \( \delta v(0) \equiv \delta_0 \) to become larger than a given but arbitrary threshold value \( \Delta \):

\[ T_p = \max_{t \geq 0} \{ \delta v(t') \leq \Delta \text{ for } t' < t \}. \] (6)

In a first approximation, neglecting the non-linear terms of the evolution equation for the error growth and assuming that both \( \delta_0 \) and \( \Delta \) are infinitesimal, one obtains

\[ T_p \sim \lambda_{\text{max}}^{-1} \ln(\Delta/|\delta_0|) \approx \lambda_{\text{max}}^{-1}. \] (7)

In turbulence, such a relation would imply that the predictability time decreases with the Reynolds number as \( \text{Re}^{-\alpha} \). This is contradictory with the quite intuitive expectation that the predictability time should be roughly proportional to the turn-over time of the energy containing eddies on the large scales, and so practically independent of the Reynolds number.

The paradox stems from assuming the validity of the Lyapunov analysis for perturbations \( \delta v \) that are much larger than the typical velocity difference \( v_\eta \sim \eta/\tau(\eta) \) on the Kolmogorov length scale \( \eta \). In this case, the error growth is non-exponential as it can be understood by simple heuristic arguments. The problem can be faced by generalizing the concept of maximum Lyapunov exponent to the case of non-infinitesimal perturbations.

The generalization is particularly useful in systems with many characteristic time-scales.

For this purpose, it is convenient to consider the time \( T_r(\delta v) \) necessary for a perturbation to grow from \( \delta v \) to \( r \delta v \), for a generic \( r > 1 \). For \( r = 2 \) this is the doubling time of a perturbation, usually studied in atmospheric predictability experiments. After performing an average over different realizations of the flow or, equivalently, a time average along a trajectory \( v(t) \) in the phase space, we introduce the scale-dependent Lyapunov exponent

\[ \lambda(\delta v) = \left\langle \frac{1}{T_r(\delta v)} \right\rangle \ln r. \] (8)

Such a definition is consistent with the request of recovering the maximum Lyapunov exponent in the limit of infinitesimal error, since

\[ \lim_{\delta v \to 0} \lambda(\delta v) = \lambda_{\text{max}}. \] (9)

It is easy to estimate the scaling of \( \lambda(\delta v) \) when the perturbation is in the inertial range \( v_\eta \leq \delta v \leq v_L \), \( L \) being the size of the energy containing eddies. In this case, following the phenomenological ideas of Lorenz, the doubling predictability time of an error of magnitude \( \delta v \) can be identified with the turn-over time \( \tau(\ell) \) of an eddy with typical velocity difference \( v_\ell \sim \delta v \). Since \( \tau(\ell) \sim \ell^{1-h} \sim \ell^{1-1/h} \), one has

\[ \lambda(\delta v) \sim \delta v^{-\beta}, \quad \beta = 1/h - 1. \] (10)

Neglecting intermittency, i.e. using the Kolmogorov value \( h = 1/3 \), gives \( \beta = 2 \). In the dissipative range \( \delta v < v_\eta \), the error can be considered infinitesimal, implying \( \lambda(\delta v) = \lambda_{\text{max}} \).

The intermittency of energy dissipation reflects the dynamical intermittency of the chaoticity degree, so that our arguments based on dimensional analysis cannot be fully correct. In the framework of the multifractal approach, our indicator scales as

\[ \lambda(\delta v) \sim \int dh \, \delta v^{[3-D(h)]/h} \, \delta v^{1-1/h} \] (11)

where we have used arguments similar to those leading to and the scaling factor \( \ell \sim \delta v^{1/h} \). From the inequality \( D(h) \leq 3h + 2 \), which is the analogous for turbulence of the standard inequality \( f(\alpha) \leq \alpha \) in multifractals, we have

\[ \frac{2 + h - D(h)}{h} \geq -2 \quad \text{for all } h. \] (12)

Equality holds for \( h = h_3 \), the exponent that realizes the minimum in the Legendre transform for the exponent of the third-order structure function \( \zeta_3 = \min_h \{ 3h + 3 - D(h) \} = 1 \). Therefore a saddle point estimation of \( (11) \) gives
\[-\beta = \min_h \left[ \frac{2 + h - D(h)}{h} \right] = -2. \quad (13)\]

An important consequence of multifractality follows from the existence of a spectrum of dissipative cut-offs \(\eta(h)\) which reduces the effective inertial range where the scaling of \(\lambda(\delta v)\) holds. To be more specific, the multifractal approach leads to \(\eta(h) \sim \delta u \sim \eta(\delta v) \sim 1 + h(\delta v)\), and the integral \(\langle|\delta u|\rangle\) has to be performed for \(h(\delta v) \leq h \leq h_{\text{max}}\), where \(h(\delta v)\) is given by

\[
\bar{h}(\delta v) \sim \frac{\bar{\lambda}(\delta v)}{1 + h(\delta v)}.
\quad (14)
\]

As a consequence, the scaling \(\lambda(\delta v) \sim \delta v^{-\beta}\) holds only for \(\delta v > \delta u \sim \eta(\delta v) \sim 1 + h(\delta v)\), i.e., the inertial range is reduced by intermittency. In the range \(\delta u < \delta v < \delta u \sim \eta(\delta v) \sim 1 + h(\delta v)\), we expect a non-trivial shape of \(\lambda(\delta v)\) depending on \(D(h)\).

In order to test our results we have numerically studied the GOY shell model \(\eta(h)\) for the energy cascade in fully developed turbulence. The model is an approximation of the Navier-Stokes equations obtained by dividing the Fourier space into shells of wavenumbers \(k_n < |k| < k_{n+1}\). A complex scalar \(u_n\) is associated with the \(n^{\text{th}}\) shell individuated by \(k_n = k_0 2^n\). It represents the velocity difference over a length scale \(\ell \sim k_n^{-1}\). Since the energy cascade in turbulence is believed to be local in the \(k\)-space with an exponentially decreasing interaction among shells, it is reasonable to consider only the interactions of a shell with its nearest and next-nearest neighbors. The Navier-Stokes equations are then approximated by a set of ordinary differential equations:

\[
\frac{d}{dt} u_n = g_n - \nu \gamma_n^2 u_n + f \delta_n.4
\]

\[
g_n = i a_n k_n u_{n+1}^* u_{n+2}^* + i b_n k_n u_{n-1} u_{n+1} + i c_n k_n u_{n-2} u_{n-1} + \gamma \nu_k^2 - 2a_n^2 - 2a_{n-1}^2
\]

(16)

with 

\[
b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = 0.
\]

The coefficients of the nonlinear term obey \(a_n + b_{n+1} + c_{n+2} = 0\) to ensure energy conservation for \(f = \nu\). With the standard choice for three dimensional turbulence \(a_n = 1, b_n = -1/2, c_n = -1/2\), there is a second conserved quantity \(\sum_n (-1)^n k_n |u_n|^2\), which in the shell model plays the role of helicity \(h(\delta v)\).

The shell model exhibits non-linear exponents \(\zeta\) for the structure functions \(h(\delta v)\), as found in experimental data \(\eta(h)\). We have determined the scale dependent Lyapunov exponent by a numerical integration of the GOY model starting form two different initial conditions. The distance between the two velocity fields is Euclidean, i.e., \(\delta u = (\sum_n |u_n - u_n'|^2)^{1/2}\), and we have computed a time average over the trajectory \(\{u_n(t)\}\) of the quantity \(T_r(\delta u)^{-1}\) with \(r = 21/2\).

Figure 2 shows the scaling of \((1/T_r(\delta v))\) as a function of \(\delta v\) in the GOY model with \(N = 27\) shells and viscosity \(\nu = 10^{-9}\). For comparison we have also computed the eddy turn-over times

\[
\tau_n^{-1} = k_n \langle|u_n|^2\rangle^{1/2}.
\]

We thus provide a clear evidence that there is a large range of small scales where \(\lambda(\delta v) = \lambda_{\text{max}}\) and \(\tau_n \sim \nu\), as a consequence of the reduction of the inertial range for the scale dependent Lyapunov exponent. Note that in the GOY model \(\lambda_{\text{max}} \sim 10^{-2} \tau(\eta)^{-1}\), though the dependence of the two quantities on the Reynolds number is the same.

In Figure 2 we have plotted the rescaled quantity \(\lambda(\delta v)/\delta u/\delta u\) versus \(\delta u/\delta u\) using the Kolmogorov values \(h = 1/3, \alpha = (1 - h)/(1 + h) = 1/2\) and \(\gamma = h/(1 + h) = 1/4\). The collapse of the data obtained at different Reynolds numbers is a pretty confirmation of our scaling arguments. The reduction of the inertial range for the scale dependent Lyapunov exponent reveals itself in the small range where \(\lambda(\delta v) \sim \delta u^{-2}\) holds.

We conclude by noting that our scale-dependent Lyapunov exponent \(\lambda(\delta v)\) has some similarity with the concept of the \(\varepsilon\)-entropy recently discussed by Gaspard and Wang \(\eta(h)\) for the treatment of experimental data. We stress that in chaotic systems the maximum Lyapunov exponent is often more relevant and more easily computed than the Kolmogorov-Sinai entropy. Therefore we believe that our \(\lambda(\delta v)\) will often be more relevant and more easily computable than the \(\varepsilon\)-entropy. Moreover, since we use the evolution law and not experimental data, in our case there are no particular limitations as to the number of degrees of freedom involved.

In conclusion, when the perturbations are non-infinitesimal it is necessary to extend the definition of Lyapunov exponent to make it physically consistent. The generalization proposed in this letter is particular useful when many characteristic time scales are present. Our result allows one to get a quantitative control of the growth of perturbations which are non-infinitesimal, looking at the average of the inverse doubling time. By this definition one has the two advantages of maintaining the link with the forecast limitation of a system and of recovering the maximum Lyapunov exponent in the limit of infinitesimal perturbations. The scale-dependent Lyapunov exponent thus is an important tool of investigation of highly dimensional dynamical systems and, far from being limited to the predictability problem of turbulent flows in geophysics \(\eta(h)\), it can assume a great relevance in the characteristic of very different chaotic phenomena.

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FIG. 1. $\langle 1/T_r(\delta v) \rangle$ (diamond) as a function of $\delta v$ for the GOY model with $N = 27$, $k_0 = 0.05$, $f = (1+i) \times 0.005$ and $\nu = 10^{-9}$. The crosses are the inverse of the eddy turn-over times $\tau^{-1}(\delta v) = k_n (|u_n|^2)^{1/2}$ versus $\delta v = (|u_n|^2)^{1/2}$. The straight line has slope $-2$.

FIG. 2. $\ln \left[ \left( \frac{1}{T_r(\delta v)} \right)/\text{Re}^{1/2} \right]$ versus $\ln [\delta v/\text{Re}^{-1/4}]$ at different Reynolds numbers $\text{Re} = \nu^{-1}$. The results are obtained in the GOY model for $k_0 = 0.05$, $f = (1+i) \times 0.005$ and: (diamond) $N = 24$ and $\nu = 10^{-8}$; (plus) $N = 27$ and $\nu = 10^{-9}$; (square) $N = 32$ and $\nu = 10^{-10}$; (cross) $N = 35$ and $\nu = 10^{-11}$. The straight line has slope $-2$. 
\begin{align*}
\ln \left( \frac{1}{T'} \right) / Re^{1/2} \\
\ln \left( \frac{\delta v}{Re^{-1/4}} \right)
\end{align*}

Fig. 2