Dihadron production in DIS at NLO: the real corrections

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ABSTRACT: By using the formalism of the light-cone wave function along with the colour glass condensate effective theory, we consider next-to-leading order (NLO) corrections to the production of a pair of hadrons in electron-proton, or electron-nucleus, collisions at small Bjorken $x$. To the order of interest, the process involves the fluctuation of a virtual photon into a quark-antiquark pair, followed by the emission of a gluon from either the quark, or the antiquark. For the case of a virtual photon with transverse polarization, we compute the real NLO corrections, where the emitted gluon is present in the final state. We first compute the tree-level cross-section for the production of the quark-antiquark-gluon system and then deduce the real NLO corrections to dihadron production by integrating out the kinematics of the gluon. We verify in detail that, in the limit where the gluon is soft, our calculation reproduces the (real piece of the) B-JIMWLK evolution of the leading-order cross-section for quark-antiquark production. Similarly, in the limit where the gluon is collinear with its emitter, we recover the real terms in the DGLAP evolution of the fragmentation function. The virtual NLO corrections to dihadron production will be presented by one of us in a subsequent publication.

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1 Introduction

One of the main discoveries of the HERA experimental program on deeply inelastic electron-proton scattering is the fact that the gluon density inside a hadron sharply rises as one explores smaller and smaller values of Bjorken $x$ at a fixed virtuality $Q^2$ [1, 2]. While conceptually anticipated on the basis of the perturbative evolution in QCD with increasing energy, as encoded in the celebrated BFKL equation [3–5], this rapid rise challenges our understanding of QCD scattering in the vicinity of the unitarity limit. The natural solution to this problem, as also emerging from perturbative QCD, is the phenomenon of \textit{gluon saturation}, which is a consequence of the non-linear dynamics of the highly occupied gluon...
modes. Predicted already before the advent of the HERA data [6, 7], this phenomenon finds its modern, pQCD-based, formulation in the effective theory for the color glass condensate (CGC) [8–12]. The saturated gluon matter dubbed as the CGC [13] is believed to be a universal form of hadronic matter which controls the QCD scattering amplitudes for sufficiently high energies. The properties of this matter and the high-energy amplitudes can be reliably computed within pQCD as soon as the characteristic transverse-momentum scale, the saturation momentum squared $Q_s^2$ (a measure of the gluon density per unit transverse area), is sufficiently hard: $Q_s^2 \gg \Lambda_{\text{QCD}}^2$. This scale rises, roughly, as a power of $1/x$ and also (for a large nucleus) as a power of the nuclear mass number $A$: $Q_s^2(x,A) \propto A^{1/3}(1/x^{\lambda})$ with $\lambda \simeq 0.2$. Hence pQCD should be a good tool for studies of gluon saturation at sufficiently small values of $x$ and/or large values of $A$.

This motivated the use of perturbative techniques for computing the high-energy evolution of the gluon correlation functions — a non-linear generalisation of the BFKL equation known as the B-JIMWLK evolution1 [13–19] — as well as the hard impact factors which enter the CGC calculation of hadronic cross-sections for “dilute-dense” collisions, like electron-nucleus ($eA$) or proton-nucleus ($pA$). These ingredients have been originally computed to leading order (LO) in the QCD running coupling $\alpha_s$. However, one needs (at least) next-to-leading order (NLO) estimates in order to reach a perturbative accuracy of about 10%, as required for realistic predictions for the phenomenology. And indeed, over the last years one assists at strenuous efforts aiming at promoting the CGC effective theory to NLO accuracy.

For the high-energy evolution, this program resulted not only in the NLO versions of the BK [21] and B-JIMWLK [22–27] equations, but also in collinearly-improved equations [28–35] which resum to all orders the radiative corrections enhanced by large transverse logarithms (thus curing an instability of the strict NLO approximation [36–38] and partially including the effects of the DGLAP evolution [39–41]). Concerning the “impact factors” (the scattering matrix elements void of evolution), the NLO corrections have been computed for a few “dilute-dense” collisions. In the context of $pA$ collisions, this includes the single inclusive hadron (or jet) production in $pA$ collisions [42–54], photon production at central rapidities [55], and the “real” NLO corrections to dihadron production [56, 57] — i.e. the NLO effects associated with a 3-parton final state, out of which only 2 are measured. For electron-hadron ($ep$ or $eA$) deep inelastic scattering (DIS) at small Bjorken $x$, one knows by now the NLO corrections to the inclusive structure functions (including massive quarks) [58–64], to the exclusive (diffractive) production of (light or heavy) vector mesons and dijets [65–70], to the inclusive photon + dijet production in $eA$ collisions at small $x$ [71], to the inclusive dijet (or dihadron) production [72–75] (see also [76–78] for an earlier calculation of polarized trijet production), and to semi-inclusive DIS (single inclusive hadron production) [79]. One has recently computed the real NLO corrections to the diffractive structure function [80], as well as the diffractive production of 3 jets [81–83].

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1This is truly a functional renormalisation-group equation [13–18], equivalent to an infinite hierarchy of coupled equations [19], and applies to gauge-invariant correlators built with products of Wilson lines. In the limit of a large number of colors ($N_c \gg 1$), the first equation in this hierarchy reduces to a closed, non-linear, equation for the dipole amplitude, known as the Balitsky-Kovchegov equation [19, 20].
Among the processes above listed, the inclusive production of a pair of jets or hadrons is particularly interesting, as this is sensitive to gluon saturation via the azimuthal correlations among the two measured particles [84–98]. Multiple scattering should lead a broadening in their azimuthal distribution around the back-to-back peak. Such a broadening has been observed in d+Au collisions at RHIC [99, 100], although at this level it looks difficult to distinguish the effects of saturation from those of the final state radiation (the “Sudakov factor” [101]). The kinematical conditions for studying this effect are expected to be better at the Electron-Ion Collider (EIC) [102]. This explains the interest in having accurate theoretical predictions for this process in $ep$, or $eA$ collisions.

It is our purpose here to present a calculation of the real NLO corrections to di-hadron production at forward rapidities in electron-hadron collisions in the CGC formalism. (The corresponding virtual corrections will be presented by one of us (Y.M.) in a separate paper.) At LO, the respective cross-section involves the production of a quark-antiquark pair, which is initiated by the decay of the virtual photon and put on mass-shell by the scattering off the hadronic target [86]. The real NLO corrections involve the emission of an additional gluon, by either the quark or the antiquark, which is produced too in the final state, but it is not measured. As aforementioned, there are already several independent calculations of these NLO corrections in the literature [72–74, 76–78], with results which appear to agree with each other. So, one may wonder why there is need for yet another calculation. In our opinion, there are in fact several reasons for that.

First, these are complex and tedious calculations, which are technically involved and conceptually subtle. So, it is indeed useful to have several independent calculations, in order to cross check the previous results. Second, the previous calculations generally use different techniques and/or consider different final states: we use the light-cone wave function (LCWF) formalism together with light-cone perturbation theory (LCPT) and look at di-hadron final states. The same formalism has been used in ref. [73], which however considered dijet final states. Ref. [72] uses momentum-space (Feynman-rule) perturbation theory together with dijet final states, while ref. [74] focuses on dihadrons (like ourselves), but uses the spinor helicity method for computing Feynman graphs. Moreover, our results and those in ref. [74] are complementary to each other, in the sense that they refer to different polarisation states for the virtual photon — transverse polarisation in our analysis and, respectively, longitudinal polarisation in [74]. Very recently, after the first version of our preprint was released, the authors of ref. [74] have extended their NLO calculation to the case of a transverse photon [103].

Despite such formal differences, we have checked that our results for the real NLO corrections fully agree with the corresponding results in the literature, whenever direct comparisons were possible. In particular, we found complete agreement with the results of ref. [73] both at the level of the tri-parton (quark, antiquark, gluon) component of the virtual photon LCWF and at the level of the cross-section for three parton production. We

\[\text{Given our results, it would be straightforward to deduce the NLO corrections for the case where the unmeasured parton is the quark, or the antiquark: we will present explicit results for three-parton (qqg) production and the NLO contributions to di-hadron production can be deduced by integrating out the kinematics of the unmeasured parton.}\]
also found agreement with other results in the literature which refer to the tri-parton cross-section, notably those in refs. [72, 103], whenever these results were sufficiently explicit to allow for comparisons.

But even when the results appear to agree with each other, we still believe that our analysis may offer new insights, not only because of the use of different methods, but also because of the different emphasis that we have put on various aspects of the calculation and the associated physical discussions. Notably, we feel that a strong point of our analysis is the particularly transparent discussion of the two special limits, which represent important checks of the NLO results: the soft gluon limit, in which we shall recover the B-JIMWLK evolution of the LO cross-section, and the limit where the gluon becomes collinear with its emitter, where we shall recover the DGLAP evolution of the fragmentation function for the final quark, or antiquark.

As compared to previous studies in the literature, we shall analyse the soft gluon limit diagram by diagram and we shall thus demonstrate that only a particular class of Feynman graphs contribute in this limit — those where the gluon is emitted sufficiently close to the scattering with the nuclear target. There are 16 such graphs and their respective contributions are found to be in an one-to-one correspondence with the (real) terms of the B-JIMWLK equation for the quadrupole [104–106].

Furthermore, we shall use an original strategy (that we introduced in the context of proton-nucleus collisions [57]) to verify the emergence of the DGLAP evolution in the collinear limit. While all the other approaches in the literature rely on the transverse momentum representation for that purpose (see e.g. [74] for a discussion in a similar context), we have reformulated the collinear limit directly in term of transverse coordinates (the natural representation for constructing the LCWF in the presence of multiple scattering). While the two representations should be eventually equivalent, we feel that our method is more efficient in practice, as it allows one to easily recognise the diagrams which contribute to the collinear limit and to extract the DGLAP evolution already before the final Fourier transform to momentum space.

This paper is structured as follows. In section 2 we schematically describe the results of the LCPT for the virtual photon light-cone wavefunction to the order of interest — that is, for the quark-antiquark ($q\bar{q}$) and the quark-antiquark-gluon ($q\bar{q}g$) Fock space components. Notice that by “LCWF” we truly mean the outgoing state that would be probed by the detector and which also includes the effects of the collision with the hadronic target. The general but formal results from section 2 are subsequently used to derive fully explicit expressions for the respective wavefunctions and scattering amplitudes. Specifically, in section 3 we deduce the amplitude and the cross-section for the leading order (LO) process, that is, the exclusive production of a $q\bar{q}$ pair. Then, in section 4 we present the results for the quark-antiquark-gluon Fock component, where the gluon can be emitted from either the quark or anti-quark. In section 5 we deduce the real NLO corrections to the dihadron cross-section by integrating out the kinematics of one (anyone) of the three final partons. In particular, we explain the simplifications which occur in the colour structure (i.e. in the partonic $S$-matrices) due to the fact that one of the partons is not measured. In section 6 we show that in the limit where the unmeasured parton is a soft gluon, our NLO results
reproduce, as expected, the (real part of the) JIMWLK evolution for the LO dijet cross-
section. Then in section 7, we consider the limit where the unmeasured gluon is produced
by a collinear splitting. We show that, in this limit, our NLO results develop collinear
singularities, that we isolate to leading logarithmic accuracy and verify that they can be
interpreted as one-step in the DGLAP evolution of the fragmentation function. Finally in
appendix A we summarise our final results for the tri-parton LCWF and the corresponding
cross-section in fully explicit notations (as opposed to the compact, but somehow symbolic,
notations that we use in the main text), in order to facilitate their comparison with the
corresponding results in the literature.

2 The virtual photon light-cone wavefunction

We consider high-energy electron-proton or electron-nucleus scattering in a Lorentz frame
where the virtual photon $\gamma^*$ is a relativistic right-mover, with 4-momentum $q^\mu = (q^+, -Q^2/2q^+, q)$ in light-cone notations, whereas the nucleus is an ultrarelativistic
left-mover, which for the present purposes can be simply treated as a Lorentz-contracted
shockwave.

We describe this collision using the light-cone wavefunction (LCWF) formalism, which
aims at constructing the wavefunction of the virtual photon as a superposition of Fock
states built with “bare quanta” (the eigenstates of the free piece of the Hamiltonian). This
superposition is evolving with time, due to the radiation of new quanta, and is constructed
via time-dependent perturbation theory. The bare quanta are assumed to be on their mass
shall: a bare quark or gluon carries a 4-momentum $k^\mu = (k^+, k^-, k_\perp)$, where $k^+ > 0$ and
the minus component is the LC energy: $k^- = E_k = k_\perp^2/2k^+$, with $k_\perp = |k|$. A bare
single-particle state is specified by the particle 3-momentum $(k^+, k_\perp)$ and the corresponding
quantum numbers, like spin, polarisation, and colour. E.g., a bare quark state is written
as $|q_\alpha^\lambda\rangle$, where $\lambda = \pm 1/2$ denotes the helicity and $\alpha = 1, 2, \ldots, N_c$ is a colour index
in the fundamental representation. Whenever no confusion is possible, we shall denote
bare single-particle states simply as $|q\rangle$, $|g\rangle$, and $|\gamma\rangle$ for a quark, a gluon, and the original
photon, respectively.

We would like to construct the final (“outgoing”) state at time $x^+ \to \infty$ for the case
where the incoming state at $x^+ \to -\infty$ is a bare, space-like, photon. We work in the
projectile light-cone (LC) gauge $A^+ = 0$ and in the interaction representation. As already
mentioned, the hadronic target is a shockwave localised near $x^+ = 0$, so we can factorise
the scattering from the “initial-state” and the “final-state” evolutions — the quantum
evolutions occurring before ($x^+ < 0$) and respectively after ($x^+ > 0$) the collision. We can
then write (in compact notations)

$$|\gamma\rangle_{out} = U(\infty, 0) \hat{S} U(0, -\infty) |\gamma\rangle$$

(2.1)

where $|\gamma\rangle$ is the bare photon state, $\hat{S}$ is the scattering operator (or “$S$-matrix”) describing
the collision with the shockwave in the eikonal approximation, and the evolution operator
is built with the interaction piece $H_{\text{int}} = H_{\text{int}}^{\text{QCD}} + H_{\text{int}}^{\text{QED}}$ of the full (QCD plus QED)
Hamiltonian:

\[ U(x^+, x_0^+) = T \exp \left\{ -i \int_{x_0^+}^{x^+} dy^+ H_{\text{int}}(y^+) \right\}. \] (2.2)

(The time-dependence in \( H_{\text{int}}(y^+) \) is governed by the free Hamiltonian \( H_0 = H_{\text{QCD}}^0 + H_{\text{QED}}^0 \), as standard in the interaction representation.) The QED Hamiltonian \( H_{\text{QED}}^\text{int} \) generates the splitting of the virtual photon into a quark-antiquark pair in a colour singlet state (a “colour dipole”), whereas the QCD Hamiltonian \( H_{\text{QCD}}^\text{int} \) governs the evolution of this \( q \bar{q} \) pair via gluon emissions.

The quark and the antiquark can be put on-shell by their scattering off the nuclear shockwave and thus emerge as two “jets” in the final state. Our ultimate goal is to compute the cross-section for dijet production to lowest order in the QED coupling, i.e. to \( \mathcal{O}(\alpha_{\text{em}}) \), and to next-to-leading order in the QCD coupling, i.e. to \( \mathcal{O}(\alpha_s) \) (as usual, we have \( \alpha_{\text{em}} = e^2/4\pi \) and \( \alpha_s = g^2/4\pi \)). To that aim, it is sufficient to compute single gluon emissions,\(^3\) which can be either real — the gluon exists in the final state, albeit it is not measured —, or virtual — the gluon is emitted and reabsorbed at the level of the amplitude, meaning that the final state is just a (bare) \( q \bar{q} \) state.

To the accuracy of interest, one can expand the evolution operators in eq. (2.1) to linear order in\(^4\) \( H_{\text{QED}}^\text{int} \equiv H_{\text{QED}} \), for instance

\[
U(0, -\infty) = 1 - i \int_{-\infty}^{0} \text{d}x^+ T \left\{ e^{-i \int_{-\infty}^{0} \text{d}y^+ H_{\text{QCD}}(y^+) H_{\text{QED}}(x^+)} \right\} \\
\quad \rightarrow 1 - i \int_{-\infty}^{0} \text{d}x^+ U_{\text{QCD}}(0, x^+) H_{\text{QED}}(x^+),
\] (2.3)

with \( U_{\text{QCD}} \) given by eq. (2.2) with \( H_{\text{int}}(y^+) \rightarrow H_{\text{QCD}}(y^+) \). The simpler expression in the second line is sufficient when acting on the photon state \( |\gamma\rangle \). Using this last expression for \( U(0, -\infty) \) together with the corresponding one for \( U_I(\infty, 0) \), one finds that eq. (2.1) reduces to

\[
|\gamma\rangle_{\text{out}} = |\gamma\rangle - iU_{\text{QCD}}(\infty, 0) \hat{S} \int_{-\infty}^{0} \text{d}x^+ U_{\text{QCD}}(0, x^+) H_{\text{QED}}(x^+) |\gamma\rangle \\
\quad - i \int_{0}^{\infty} \text{d}x^+ U_{\text{QCD}}(\infty, x^+) H_{\text{QED}}(x^+) |\gamma\rangle.
\] (2.4)

The physical interpretation of the above decomposition is quite transparent. The first corrective term, where \( x^+ \) takes negative values, describes a process which starts with the photon decay \( \gamma \rightarrow q \bar{q} \) at some time \( x^+ < 0 \), then the \( q \bar{q} \) state evolves according to QCD until it hits the nuclear target at time \( x^+ = 0 \); the scattering between the dressed \( q \bar{q} \) state and the shockwave is quasi-instantaneous (at least, within the eikonal approximation that we employ here) and is described by the \( S \)-matrix \( \hat{S} \); finally, the partonic state emerging from the scattering evolves up to the time of measurement \( x^+ \rightarrow \infty \).

The second corrective term in eq. (2.4) describes the situation where the photon decays after crossing the shockwave, at some time \( x^+ > 0 \). In this case there is no scattering,\(^3\)

\(^3\)Two real emissions would contribute to \( \mathcal{O}(g^4) \) for the outgoing state, but only to \( \mathcal{O}(g^6) \) for the dijet cross-section.

\(^4\)From now on we omit the upper script “int” on the Hamiltonians, to simplify notations.
but merely the evolution of the $q\bar{q}$ state from $x^+$ up to infinity. It is convenient for what follows to observe that the effect of this second term is to subtract the no-scattering limit ($\hat{S} \to 1$) of the first term; that is, eq. (2.4) is equivalent to

$$|\gamma\rangle_{\text{out}} = |\gamma\rangle - iU_{\text{QCD}}(\infty, 0) (\hat{S} - 1) \int_{-\infty}^{0} dx^+ U_{\text{QCD}}(0, x^+) H_{\text{QED}}(x^+) |\gamma\rangle.$$

Indeed, in the absence of scattering, the virtual photon must eventually return to its initial state, since a space-like photon cannot decay into a system of on-shell partons. This means that $|\gamma\rangle_{\text{out}} \to |\gamma\rangle$ as $\hat{S} \to 1$, or $U(\infty, -\infty)|\gamma\rangle = |\gamma\rangle$, which to linear order in $H_{\text{QED}}$ implies the result in eq. (2.5).

The perturbative expansion of the outgoing LCWF to the order of interest can now be obtained by expanding the evolution operators in eq. (2.5) to second order in $H_{\text{QCD}}$:

$$U_{\text{QCD}}(x^+, x_0^+) = 1 - i \int_{x_0^+}^{x^+} dy^+ H_{\text{QCD}}(y^+) - \int_{x_0^+}^{x^+} dy_2^+ \int_{x_0^+}^{y_2^+} dy_1^+ H_{\text{QCD}}(y_2^+) H_{\text{QED}}(y_1^+) + \cdots.$$

(2.6)

To evaluate the action of the interaction Hamiltonian, we shall work in the basis of bare multi-partonic Fock states, i.e. energy-momentum eigenstates of the free Hamiltonian with fixed numbers of free, on-shell, partons (quarks, antiquarks, and gluons). This not only matches the time-dependence of $H_{\text{QCD}}(y^+)$, but is also convenient for computing particle production in the final state: indeed, for asymptotically large times, where the interactions are adiabatically switched off, the bare Fock states are also eigenstates of the full Hamiltonian. If $|i\rangle$ and $|j\rangle$ are two generic Fock states, then

$$\langle j | H_{\text{QCD}}(y^+) | i \rangle = e^{i(E_j - E_i)y^+ - \epsilon^|y^+|} \langle j | H_{\text{QCD}} | i \rangle,$$

(2.7)

with $E_i$ and $E_j$ the LC energies of the two states (the sum of the LC energies of the constituent partons). The adiabatic factor $e^{-\epsilon^|y^+|}$ was introduced in order to turn off the interactions at large (positive or negative) times. This factor is useful in the intermediate calculations — e.g. it ensures the convergence of time integrations like that in eq. (2.5) —, but the final, physical, results must have a finite limit when $\epsilon \to 0$. Similarly,

$$H_{\text{QED}}(x^+) |\gamma\rangle = e^{i(E_q - E_q^\gamma)x^+ - \epsilon|x^+|} |q\bar{q}\rangle \langle q\bar{q} | H_{\text{QED}} |\gamma\rangle,$$

(2.8)

where $|q\bar{q}\rangle$ is a generic quark-antiquark bare state and the sum over all such states, i.e. over the momenta and the quantum numbers of the two bare fermions, is implicitly understood. Of course, this sum is constrained by the conservation properties encoded in the matrix element of $H_{\text{QED}}$; in particular, the $q\bar{q}$ state created by the decay of the photon must be a colour-singlet (or “colour dipole”). In what follows, we shall systematically use this convention, that repeated indices must be summed over.

Before we turn to a study of the NLO corrections, let us derive the leading-order (LO) result for the $q\bar{q}$ Fock space component, as obtained by replacing both QCD evolution
operators by unity:

\[
|\gamma\rangle^{(0)}_{\text{out}qq} = -i(\hat{S} - 1) \int_{-\infty}^{0} dx^+ e^{i(E_qq-E_\gamma)x^+ + \epsilon x^+} |q\bar{q}\rangle \langle q\bar{q}|H_{\text{QED}}|\gamma\rangle,
\]

\[
= -|q_1\bar{q}_1\rangle \langle q_1\bar{q}_1|\hat{S} - 1|q\bar{q}\rangle \langle q\bar{q}|H_{\text{QED}}|\gamma\rangle.
\]

(2.9)

In the final result, one can neglect the \(\epsilon\) prescription in the denominator, since the space-like photon cannot be degenerate with the on-shell quark-antiquark pair.

### 2.1 Real gluon emissions

A single gluon emission by either the quark or the antiquark produces a \(|q\bar{q}g\rangle\) Fock component in the final state, which contributes to \(\mathcal{O}(g)\) to the outgoing wavefunction and hence to \(\mathcal{O}(g^2)\) to the dijet cross-section. Such an emission is generated by the term linear in \(H_{\text{QCD}}\) in the expansion (2.6) of the QCD evolution operator. With reference to eq. (2.5), it is clear that there are two possible time-orderings: the gluon can be emitted either before, or after, the scattering between the \(qq\) pair and the nuclear shockwave. When the emitter is the antiquark, these two possibilities are represented by graphs (a) and (b) in figure 1. As a matter of facts, there is also a third graph, illustrated in figure 1.c, where the photon decays after crossing the shockwave. But as already explained, this case is implicitly included in eq. (2.5), via the subtraction of the no-scattering limit from the \(S\)-matrix.

Consider the initial-state emission first, cf. figure 1.a. The corresponding contribution to eq. (2.5) reads (cf. eqs. (2.6) and (2.8))

\[
(-i)^2(\hat{S} - 1) \int_{-\infty}^{0} dx^+ \int_{-\infty}^{0} dy^+ e^{i(E_{q\bar{q}}-E_\gamma)y^+ + \epsilon y^+} |q\bar{q}\rangle \langle q\bar{q}|H_{\text{QCD}}(y^+)H_{\text{QED}}(x^+)|\gamma\rangle
\]

\[
= -(\hat{S} - 1) \int_{-\infty}^{0} dy^+ \int_{-\infty}^{0} dx^+ e^{i(E_{q\bar{q}}-E_\gamma)x^+ + \epsilon x^+} |q\bar{q}\rangle \langle q\bar{q}|H_{\text{QCD}}(y^+)H_{\text{QED}}(x^+)|\gamma\rangle
\]

\[
e^{i(E_{q\bar{q}}-E_\gamma)x^+ + \epsilon x^+} \langle q\bar{q}|H_{\text{QED}}|\gamma\rangle
\]

\[
= |q_2\bar{q}_2g_2\rangle \langle q_2\bar{q}_2g_2|\hat{S} - 1|q_1\bar{q}_1g_1\rangle \langle q_1\bar{q}_1g_1|H_{\text{QCD}}|q\bar{q}\rangle \langle q\bar{q}|H_{\text{QED}}|\gamma\rangle \langle q\bar{q}|H_{\text{QED}}|\gamma\rangle.
\]

(2.10)
When inserting intermediate Fock states in the second line, we used the fact that the action of $H_{\text{QCD}}$ on the $|q\bar{q}\rangle$ state consists in a gluon emission, leading to a state of the form $|q_1\bar{q}_1g_1\rangle$, whereas the subsequent action of the $S$-matrix cannot change the partonic content of that state, but only modify the momenta and the discrete quantum numbers (polarisation and colour) of the 3 partons. By energy-momentum conservation, the energy denominators in eq. (2.10) cannot vanish, hence one can safely let $\epsilon \rightarrow 0$ in the final result.

Consider similarly the final-state emission, cf. figure 1.b. By expanding the final evolution operator in eq. (2.5) to linear order, one finds the respective contribution as
\[ (-i)^2 \int_0^\infty dy^+ H_{\text{QCD}}(y^+) (S - 1) \int_{-\infty}^0 dx^+ H_{\text{QED}}(x^+) |\gamma\rangle \]
\[ = - |q_2\bar{q}_2g_2\rangle \frac{\langle q_2\bar{q}_2g_2|H_{\text{QCD}}|q_1\bar{q}_1\rangle}{E_{q_2\bar{q}_2g_2} - E_{q_1\bar{q}_1}} \frac{\langle q_1\bar{q}_1|S - 1|q\bar{q}\rangle}{E_{q\bar{q}} - E_\gamma} \frac{\langle q\bar{q}|H_{\text{QED}}|\gamma\rangle}{E_{q\bar{q}} - E_\gamma}, \] (2.11)
where we have omitted the intermediate steps as well as the $i\epsilon$ prescription (since unnecessary).

To summarise, the $q\bar{q}g$ Fock space component of the outgoing LCWF of the virtual photon reads
\[ |\gamma\rangle_{out}^{\text{virtual}} = |q_2\bar{q}_2g_2\rangle \left\{ \frac{\langle q_2\bar{q}_2g_2|S - 1|q_1\bar{q}_1g_1\rangle}{E_{q_2\bar{q}_2g_2} - E_{q_1\bar{q}_1g_1}} \frac{\langle q_1\bar{q}_1g_1|H_{\text{QCD}}|q\bar{q}\rangle}{E_{q\bar{q}} - E_\gamma} \right. \]
\[ + \frac{\langle q_2\bar{q}_2g_2|H_{\text{QCD}}|q_1\bar{q}_1\rangle}{E_{q_2\bar{q}_2g_2} - E_{q_1\bar{q}_1}} \frac{\langle q_1\bar{q}_1|S - 1|q\bar{q}\rangle}{E_{q\bar{q}} - E_\gamma} \left. \right\} \frac{\langle q\bar{q}|H_{\text{QED}}|\gamma\rangle}{E_{q\bar{q}} - E_\gamma}, \] (2.12)
where the first (second) line describes one initial-state (final-state) gluon emission.

It is interesting to notice the structure of the energy denominators in eq. (2.12). For the initial-state decays, they involve the difference between the LC energies of the intermediate state and of the initial state, respectively; e.g. $E_{q_1\bar{q}_1g_1} - E_\gamma$. For the final-state emissions, on the other hand, the energy of the initial state is replaced by that of the final state, e.g. $E_{q_2\bar{q}_2} - E_{q_2\bar{q}_2g_2}$. These rules are in fact general and we shall see other examples in what follows.

## 2.2 Virtual corrections

We now turn to the virtual corrections, that is, the one-loop contributions to the amplitude which are generated by the second-order terms in the expansion of the evolution operators in eq. (2.5). These can be either self-energy graphs, where the gluon is emitted and reabsorbed by the same fermion, or gluon exchange graphs, where the gluon is emitted by the quark and reabsorbed by the anti-quark, or vice-versa. For both topologies, we encounter three types of time-orderings: initial-state evolution (cf. figure 2), final-state evolution (cf. figure 3), and mixed evolution, where the gluon is emitted before the scattering with the shockwave and is reabsorbed after that scattering (cf. figure 4).

We start with the last case, where the gluon crosses the shockwave and thus can interact with it. We have already shown in eq. (2.10) the $q\bar{q}g$ state generated by first emitting a gluon and then scattering with the shockwave. To obtain the “crossing” virtual
Figure 2. The 4 possible topologies for virtual corrections occurring in the initial state.

Figure 3. The 4 possible topologies for virtual corrections occurring in the final state.

Figure 4. The 4 possible topologies for virtual corrections in which the gluon crosses the shockwave.
corrections, one must reabsorb this gluon after the scattering, which can be done by acting with the QCD Hamiltonian (the linear term in the expansion of $U_{QCD}(\infty, 0)$) on the $q\bar{q}$ state in eq. (2.10). Using

$$-i\int_0^\infty dy^+ \langle q_3\bar{q}|H_{QCD}(y^+)|q_2\bar{q}_2g\rangle = -i\int_0^\infty dy^+ e^{i(E_{q_3}\bar{q} - E_{q_2}\bar{q}_2)}y^+ - i\epsilon \langle q_3\bar{q}|H_{QCD}|q_2\bar{q}_2g\rangle$$

one deduces (the upper label $C$ stays for “crossing”)

$$|\gamma|^C_{out} = -|q_3\bar{q}| \frac{\langle q_3\bar{q}|H_{QCD}|q_2\bar{q}_2g\rangle}{E_{q_2}\bar{q}_2 - E_{q_3}} \langle q_2\bar{q}_2g|\hat{S} - 1|q_1\bar{q}_1g\rangle \frac{\langle q_1\bar{q}_1g|H_{QCD}|q\bar{q}\rangle}{E_{q_2}\bar{q}_2 - E_{q_1}\bar{q}_1 - i\epsilon} \frac{\langle q\bar{q}|H_{QCD}|\gamma\rangle}{E_{q\bar{q}} - E_{\gamma}}$$

where the energy denominators cannot vanish.

Consider now the one-loop graphs associated with initial-state evolution (cf. figure 2). To that aim, one must use the second-order term in the expansion of $U_{QCD}(0, -\infty)$, cf. eq. (2.6), to first emit a gluon by the $q\bar{q}$ pair and then reabsorb it. The relevant matrix element reads (compare to eq. (2.10))

$$(-i)^3\int_{-\infty}^0 dy^2_1 \int_{-\infty}^{y_1^2} dy^1_2 \int_{-\infty}^{y_1^1} dx^+ \langle q_2\bar{q}_2|H_{QCD}(y_2^+)H_{QCD}(y_1^+)H_{QED}(x^+)|\gamma\rangle$$

$$= -\frac{\langle q_2\bar{q}_2|H_{QCD}|q_1\bar{q}_1g_1\rangle}{E_{q_2}\bar{q}_2 - E_{\gamma} - 3i\epsilon} \frac{\langle q_1\bar{q}_1g_1|H_{QCD}|q\bar{q}\rangle}{E_{q_1}\bar{q}_1 - E_{\gamma} - 2i\epsilon} \frac{\langle q\bar{q}|H_{QED}|\gamma\rangle}{E_{q\bar{q}} - E_{\gamma} - i\epsilon}.$$  

(2.15)

After also adding the scattering with the nuclear target, one finds (the upper label $B$ stays for “before”)

$$|\gamma|^B_{out} = -|q_3\bar{q}| \langle q_3\bar{q}|\hat{S} - 1|q_2\bar{q}_2\rangle \frac{\langle q_2\bar{q}_2|H_{QCD}|q_1\bar{q}_1g\rangle}{E_{q_2}\bar{q}_2 - E_{\gamma}} \frac{\langle q_1\bar{q}_1g|H_{QCD}|q\bar{q}\rangle}{E_{q_1}\bar{q}_1 - E_{\gamma}} \frac{\langle q\bar{q}|H_{QED}|\gamma\rangle}{E_{q\bar{q}} - E_{\gamma}}$$

(2.16)

where the $i\epsilon$ prescription was again ignored, since unimportant.

The remaining case, that of the virtual corrections generated via final-state evolution (cf. figure 3), turns out to be more subtle. To understand the difficulty, let us first present the formal respective result, as obtained after expanding the evolution operator $U_{QCD}(\infty, 0)$ to second order; this reads (the upper label $A$ stays for “after”)

$$|\gamma|^A_{out} = -|q_3\bar{q}| \frac{\langle q_3\bar{q}|H_{QCD}|q_3\bar{q}_3\rangle}{E_{q_3}\bar{q}_3 - E_{q_3}} \frac{\langle q_2\bar{q}_2|H_{QCD}|q_1\bar{q}_1\rangle}{E_{q_2}\bar{q}_2 - E_{\gamma}} \frac{\langle q_1\bar{q}_1|\hat{S} - 1|q_2\bar{q}_2\rangle}{E_{q_1}\bar{q}_1 - E_{q_1}} \frac{\langle q\bar{q}|H_{QED}|\gamma\rangle}{E_{q\bar{q}} - E_{\gamma}}.$$  

(2.17)

At a first sight, this contribution has the expected structure: e.g., the energy denominators associated with the QCD transitions are built as differences between the energy of the state prior to the parton branching and that of the final state $|q_3\bar{q}_3\rangle$. Consider however the case of a self-energy correction, where the gluon is emitted and reabsorbed by the same
Figure 5. (a) The quark-antiquark fluctuation of the virtual photon. (b,c) The 2 graphs contributing to the leading-order scattering amplitude in coordinate space.

fermion — quark or antiquark. In that case, the kinematics of the emitter is clearly the same before and after the loop, hence the intermediate state \(|q_1\bar{q}_1\rangle\) is degenerate with the final state, \(E_{q_1\bar{q}_1} = E_{q_3\bar{q}_3}\), and the respective denominator vanishes. This explains why we have carefully kept the \(i\epsilon\) pieces in the QCD energy denominators in eq. (2.17).

This difficulty is in fact well known: it is related to the proper definition of the final state (the “wavefunction renormalisation”). A prescription to circumvent this difficulty has been proposed in [107]. For the physical problem at hand, it will be further discussed in the subsequent paper [108], which will be fully devoted to the virtual corrections. From now on, in this paper we shall restrict ourselves to real gluon emissions alone.

3 Dihadron production in DIS at leading order

As a warm-up, in this section we shall derive the well-known result for the dijet (or dihadron) production in deep inelastic scattering at leading order [86], by following the LCWF formalism outlined in section 2. The corresponding Feynman graphs are shown in figure 5.

3.1 The leading order photon outgoing state

The outgoing state at leading order (LO) is shown in compact but formal notations in eq. (2.9). In this section, we shall explicitly compute this state, via the following strategy: first, we shall construct the quark-antiquark Fock state generated by the decay of the virtual photon, by working in momentum space, where the Feynman rules of LCPT are most conveniently formulated; that is, we shall evaluate the graph in figure 5.a. Then, we shall perform a Fourier transform to the transverse coordinate representation, which is more convenient for computing the action of the \(S\)-matrix in the eikonal approximation; we shall thus deduce the contributions of the graphs in figure 5.b and c.

To LO, the \(q\bar{q}\) component of the virtual photon LCWF reads

\[
|\gamma_{\lambda}(Q,q)\rangle_{\gamma q}^{(0)} = \int_0^\infty \frac{dp^+}{2\pi} \frac{dk^+}{2\pi} \int \frac{d^2p}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} |q_1^\alpha(p)q_1^\beta(k)\rangle \frac{\langle q_3^\gamma(k)\bar{q}_3^\delta(p) | H_{\gamma\rightarrow q\bar{q}} | \gamma_{\lambda}(q)\rangle}{E_{q}(p) + E_{\bar{q}}(k) - E_{\gamma}(q,Q)},
\]  

(3.1)
with the following notations: $q = (q^+, \mathbf{q})$ is the 3-momentum of the incoming photon with virtuality $Q$, whereas $k \equiv (k^+, \mathbf{k})$ and $p \equiv (p^+, \mathbf{p})$ similarly refer to the final quark and anti-quark. Furthermore, $\lambda = T$ or $L$ denotes the photon polarisation state, $\alpha, \beta$ are the fermion colour indices and $\lambda_1, \lambda_2$ are the respective helicity states. The matrix element in eq. (3.1) involves delta-functions for 3-momentum conservation, which imply $p = q - k$ and $p^+ = q^+ - k^+$. The energy denominator is computed as

$$E_q(p) + E_\gamma(k) - E_\gamma(q, Q) = \frac{p^2}{2p^+} + \frac{k^2}{2k^+} - \frac{q^2 - Q^2}{2q^+} = \frac{\tilde{k}^2 + \tilde{Q}^2}{2\vartheta(1 - \vartheta)q^+} ,$$

where $\vartheta$ is the quark longitudinal momentum fraction, $\tilde{k}$ is the transverse momentum of the quark relative to its parent photon, and $\tilde{Q}^2$ is a measure of the virtuality of the $q\bar{q}$ pair (see below):

$$\vartheta \equiv \frac{k^+}{q^+}, \quad \tilde{k} \equiv k - \vartheta q, \quad \tilde{Q}^2 \equiv \vartheta(1 - \vartheta)Q^2. \quad (3.3)$$

We shall consider in more detail the case of a virtual photon with transverse polarisation. The relevant matrix element reads (for a quark flavour with electric charge $e_f$)

$$\langle q_\lambda^a(p)\bar{q}_\lambda^b(k) | H_{\gamma \rightarrow q\bar{q}} | \gamma^j_\lambda(q) \rangle = (2\pi)^3 \delta^{(3)}(q - k - p) e_f^a \delta_{\alpha\beta} \frac{\sigma \cdot k}{k^+} \frac{2q^i}{q^+} \left[ \chi_{\alpha}^{\dagger} \bar{q}_\lambda^j \epsilon^{\alpha\beta} \chi_{\beta} \chi_\lambda \right] \epsilon_\lambda^f \chi_\lambda \epsilon_\lambda^b \chi_\lambda \chi_{\lambda_2} \left[ (2\vartheta - 1)\delta_{ij} + i\varepsilon_{ij} \vartheta \right], \quad (3.4)$$

where $\chi_\lambda$ with $\lambda = \pm 1/2$ are the usual helicity states and in obtaining the second line we have used the following identity ($\sigma^i$ with $i = 1, 2, 3$ are the usual Pauli matrices)

$$\frac{2q^i}{q^+} \frac{\epsilon \cdot k}{k^+} - \sigma^i (q - k) \frac{q^+ - k^+}{q^+} = \frac{2(2\vartheta - 1)\delta_{ij} + i\varepsilon_{ij} \vartheta}{\vartheta(1 - \vartheta)q^+} (k^j - \vartheta q^j), \quad (3.5)$$

together with the following definition:

$$\varphi_{\lambda_1\lambda_2}^{ij}(\vartheta) \equiv \chi_{\lambda_1} \left[ (2\vartheta - 1)\delta_{ij} + i\varepsilon_{ij} \vartheta \right] \chi_{\lambda_2} = \delta_{\lambda_1\lambda_2} \left[ (2\vartheta - 1)\delta_{ij} + 2i\varepsilon_{ij} \lambda_1 \right]. \quad (3.6)$$

The coefficient $2\vartheta - 1$ multiplying $\delta_{ij}$ in eq. (3.6) arises as the difference between the longitudinal momentum fractions, $\vartheta$ and $1 - \vartheta$, of the quark and the antiquark produced by the photon decay.

After inserting (3.4) and (3.2) into eq. (3.1), using 3-momentum conservation and replacing $k \rightarrow \tilde{k}$ as the integration variable, one finds

$$\left[ \gamma^j_\lambda(Q, q^+, \mathbf{q}) \right]^{(0)}_{\tilde{q}} = \int_0^1 d\vartheta \int d^2\tilde{k} e_f \varphi_{\lambda_1\lambda_2}^{ij}(\vartheta) \sqrt{q^+} \tilde{k}^j \left( \bar{q}_\lambda^j ((1 - \vartheta)q^+, (1 - \vartheta)q - \tilde{k}) q_\lambda^a (\vartheta q^+, \tilde{k} + \vartheta \mathbf{q}) \right). \quad (3.7)$$
Its Fourier transform to the transverse coordinate representation is readily obtained as
\[
\left| \gamma_T^5(Q, q^+, w) \right|_{q\bar{q}}^{(0)} = \int \frac{d^2 q}{(2\pi)^2} e^{-i w \cdot q} \left| \gamma_T^5(Q, q^+, q) \right|_{q\bar{q}}^{(0)} = \int_{x,y} \int_0^1 d\vartheta \frac{ie f \varphi_{\lambda_1 \lambda_2}^i(\vartheta) \sqrt{q^+}}{(2\pi)^2 \sqrt{2}} \frac{R^i}{R} Q K_1(\bar{Q}R) 
\times \delta^{(2)}(w - c) \left| \bar{q}^\alpha_{\lambda_2} ((1 - \vartheta)q^+, y) q^\alpha_{\lambda_1} (\vartheta q^+, x) \right). \tag{3.8}
\]

Here \( x \) and \( y \) are the transverse coordinates of the quark and the antiquark, respectively, \( R \equiv x - y \) is their relative separation, \( R = |R| \), \( c = \vartheta x + (1 - \vartheta)y \) is their center of energy; we have used the shorthand notation \( \int_x \equiv \int d^2 x \) for the transverse integrations.

At this level, it is straightforward to include the effect of the scattering with the nuclear target (the shockwave) in the eikonal approximation: the transverse coordinates, the longitudinal momenta, and the helicity states of the quark and the antiquark remain unchanged, but their colour states get rotated by the interaction with the colour field generated by the left-moving partons from the target:
\[
\hat{S} \left| \bar{q}^\alpha_{\lambda_2} (p^+, y) q^\alpha_{\lambda_1} (k^+, x) \right> = \left( V(x) V^\dagger(y) \right)_{\alpha\beta} \left| \bar{q}^\beta_{\lambda_2} (p^+, y) q^\beta_{\lambda_1} (k^+, x) \right> \tag{3.9}
\]
where \( V(x) \) and \( V^\dagger(y) \) are Wilson lines in the fundamental representation of the colour group:
\[
V(x) = T \exp \left\{ ig \int dx^+ t^a A_a^-(x^+, x) \right\}, \quad U(x) = T \exp \left\{ ig \int dx^+ T^a A_a^- (x^+, x) \right\}. \tag{3.10}
\]

We have also introduced here the Wilson line \( U(x) \) in the adjoint representation, for later convenience (this describe the colour precession of a gluon). In these equations, \( A_a^- \) is the colour field representing Coulomb exchanges between the quark or the antiquark from the dipole and colour sources (quark and gluons) from the target. In the CGC effective theory, this field is random and must be averaged out at the level of the cross-section (see below).

Thus, finally, the \( q\bar{q} \) outgoing component of the LCWF of a transverse virtual photon reads\(^5\)
\[
\left| \gamma_T^5(Q, q^+, w) \right|_{q\bar{q}}^{(0)} = \int_{x,y} \int_0^1 d\vartheta \frac{ie f \varphi_{\lambda_1 \lambda_2}^i(\vartheta) \sqrt{q^+}}{(2\pi)^2 \sqrt{2}} \frac{R^i}{R} Q K_1(\bar{Q}R) 
\times \left( V(x) V^\dagger(y) - 1 \right)_{\alpha\beta} \delta^{(2)}(w - c) \left| \bar{q}^\alpha_{\lambda_2} ((1 - \vartheta)q^+, y) q^\alpha_{\lambda_1} (\vartheta q^+, x) \right). \tag{3.11}
\]

This is a fully explicit version of the outgoing state (2.9), as written in a mixed Fourier representation (longitudinal momentum and transverse coordinates). We recall that the subtraction of unity from the product of Wilson lines inside the square bracket accounts for the process where the photon decays into a \( q\bar{q} \) pair after crossing the shockwave, in which case there is no scattering (see figure 5.c).

\(^5\)As compared to eq. (2.9), we renounce to the upper label out, to simplify the notation.
Similarly, for a virtual photon with longitudinal polarisation we find the following result:

\[
\left| \gamma_L(Q, q^+, w) \right|^{(0)}_{q\bar{q}} = \int_{x,y} \int_0^1 d\beta \frac{i c e f \sqrt{q^+}}{2\pi^2 \sqrt{2}} Q K_0(\bar{Q} R) \\
\times \left( V(x) V^\dagger(y) - 1 \right)_{\alpha\beta} \delta^{(2)}(w - c) \left[ q_\lambda^2 \right] (1 - \theta q^+, y) q_\lambda^2 (\theta q^+, x). \]

(3.12)

3.2 The dihadron cross-section at leading order

Given the outgoing LCWF as computed in the previous section, we are now in a position to compute the cross-section for dijet (or dihadron) production at leading order. In this approximation, the final “jets” are simply the quark and the antiquark produced by the decay of the virtual photon and which are put on-shell by their scattering off the hadronic target. The corresponding cross-section is obtained by simply counting the number of \( q\bar{q} \) pairs with a given kinematics in the outgoing state. For a transverse photon, we can write

\[
\frac{d\sigma_{\gamma^T \to q\bar{q} + X}}{dk_1^+ d^2k_1 dk_2^+ d^2k_2} = \frac{1}{2} \left( 2\pi \right) \delta(q^+ - k_1^+ - k_2^+) \left( \gamma^T_{q\bar{q}} (Q, q, \bar{q}) \right) \left( N_q(k_1) N_{\bar{q}}(k_2) \right) \left| \gamma^T_{q\bar{q}} (Q, q, \bar{q}) \right|^{(0)}_{q\bar{q}},
\]

(3.13)

where \( N_q(k_1) \) and \( N_{\bar{q}}(k_2) \) are particle number density operators for bare quarks and antiquarks, and the overall factor 1/2 comes from the average over the 2 transverse polarisations. The outgoing photon state in momentum space can be obtained by inverting the Fourier transform in eq. (3.8). This gives

\[
\frac{d\sigma_{\gamma^T \to q\bar{q} + X}}{dk_1^+ d^2k_1 dk_2^+ d^2k_2} = \frac{1}{2} \left( 2\pi \right) \delta(q^+ - k_1^+ - k_2^+) \left( \gamma^T_{q\bar{q}} (Q, q, \bar{q}) \right) \left( N_q(k_1) N_{\bar{q}}(k_2) \right) \left| \gamma^T_{q\bar{q}} (Q, q, \bar{q}) \right|^{(0)}_{q\bar{q}},
\]

(3.14)

where \( w \) and \( \bar{w} \) are the transverse coordinates of the virtual photon in the direct amplitude (DA) and the complex conjugate amplitude (CCA), respectively, and we have also set \( q = 0 \) (this entails no loss of generality). The integrations over \( w \) and \( \bar{w} \) simply remove the delta-functions like \( \delta^{(2)}(w - c) \) in eq. (3.11). To proceed, it is preferable to stick to the transverse coordinate representation, that is, to use eq. (3.11) for the outgoing state together with the Fourier transform of the number density operators. E.g. for the quarks, we write

\[
\hat{N}_q(k^+, \bar{x}) = \frac{1}{(2\pi)^3} b^{\alpha\lambda}_q (k^+, \bar{x}) b_\lambda^{\dagger}(k^+, \bar{x}) = \frac{1}{(2\pi)^3} \int_{\bar{x},x} e^{ik\cdot(x-\bar{x})} b^{\alpha\lambda}_q (k^+, \bar{x}) b_\lambda^{\dagger}(k^+, \bar{x}),
\]

(3.15)

where the (bare) quark creation and annihilation operators satisfy the anti-commutation relation

\[
\left\{ b_\lambda^\alpha (k^+, \bar{x}), b_\lambda^\beta (p^+, y) \right\} = 2\pi \delta_{\lambda_1\lambda_2} \delta^{\alpha\beta} \delta^{(2)}(x - y) \delta(k^+ - p^+).
\]

(3.16)
By using these relations together with the corresponding one for the antiquarks, one finds
\[
\frac{d\sigma_{\gamma A'\to q\bar q+X}}{dk_1^+ dk_2^+} = \frac{2\alpha_{em} N_c}{(2\pi)^3 q^+} \left(\delta^2 + (1 - \vartheta)^2\right) \left(\sum c_j^2\right) \delta(q^+ - k_1^+ - k_2^+) \]
\[
\times \int_{x,y,x,y} e^{-ik_1 \cdot (x - \bar{x}) - ik_2 \cdot (y - \bar{y})} \frac{R \cdot \bar{R}}{R \bar{R}} \tilde{Q}^2 K_1 \left(\tilde{Q} \bar{R}\right) K_1 \left(\tilde{Q} \bar{R}\right) \mathcal{W}(x, y, \bar{y}, \bar{x}),
\]
with $R \equiv x - \bar{y}$ and $\vartheta \equiv \frac{k^+}{q^+}$. The sum over helicities has been performed as (cf. eq. (3.6))
\[
\varphi^{ij}_{\lambda_1 \lambda_2}(\vartheta) \varphi^{j*}_{\lambda_1 \lambda_2}(\vartheta) = 2\delta^{ij} \left[1 + (1 - 2\vartheta)^2\right] = 4\delta^{ij} \left[\vartheta^2 + (1 - \vartheta)^2\right].
\]
(3.18)

In writing eq. (3.17), we have also performed the average over the random colour fields $A_a^-$ in the target. This has generated the function $\mathcal{W}(x, y, \bar{y}, \bar{x})$, which encodes the effects of the scattering:
\[
\mathcal{W}(x, y, \bar{y}, \bar{x}) \equiv Q(x, y, \bar{y}, \bar{x}) - S(x, y) - S(\bar{y}, \bar{x}) + 1.
\]
(3.19)

The 3 non-trivial terms in the r.h.s. are (average) $S$-matrices describing the forward scattering of colourless systems made with up to four partons: a $q\bar{q}$ dipole in the DA,
\[
S(x, y) \equiv \frac{1}{N_c} \left\langle \text{tr}(V(x) V^\dagger(y))\right\rangle,
\]
(3.20)
a similar $q\bar{q}$ dipole, $S(\bar{y}, \bar{x})$, in the CCA, and the $qq\bar{q}q$ quadrupole,
\[
Q(x, y, \bar{y}, \bar{x}) \equiv \frac{1}{N_c} \left\langle \text{tr}(V(x) V^\dagger(y) V(\bar{y}) V^\dagger(\bar{x}))\right\rangle,
\]
(3.21)
where the colour flows connect the $q\bar{q}$ pair in the DA to that in the CCA.

For a virtual photon with longitudinal polarisation one similarly finds (cf. eq. (3.12))
\[
\frac{d\sigma_{\gamma A'\to q\bar q+X}}{dk_1^+ dk_2^+} = \frac{8\alpha_{em} N_c}{(2\pi)^3 q^+} \vartheta(1 - \vartheta) \left(\sum c_j^2\right) \delta(q^+ - k_1^+ - k_2^+)
\]
\[
\times \int_{x,y,x,y} e^{-ik_1 \cdot (x - \bar{x}) - ik_2 \cdot (y - \bar{y})} \tilde{Q}^2 K_0 \left(\tilde{Q} \bar{R}\right) K_0 \left(\tilde{Q} \bar{R}\right) \mathcal{W}(x, y, \bar{y}, \bar{x}).
\]
(3.22)

Eqs. (3.17) and (3.22) are in agreement with previous calculations in the literature [86].

To obtain the corresponding cross-sections for di-hadron production, $\gamma A \to h_1 h_2 + X$, it suffices to convolute the above results for $q\bar{q}$ production with the fragmentation functions describing the probability to find a hadron within the wavefunction of a quark, or antiquark:
\[
\frac{d\sigma_{\gamma A'\to h_1 h_2+X}}{dk_1^+ dk_2^+} = \int \frac{d\zeta_1}{\zeta_1^3} \int \frac{d\zeta_2}{\zeta_2^3} \frac{d\sigma_{\gamma A'\to q\bar q+X}}{dp_1^+ dp_2^+} \bigg|_{p_i = k_i/\zeta_i} \mathcal{D}_{h_1/q}(\zeta_1, \mu^2) \mathcal{D}_{h_2/q}(\zeta_2, \mu^2).
\]
(3.23)

The lower script $p_i = k_i/\zeta_i$ on the $q\bar{q}$ cross-section stands for $p_i^+ = k_i^+ / \zeta_i$ and $p_i = k_i / \zeta_i$, with $i = 1, 2$. 

– 16 –
4 The tri-parton component of the transverse photon outgoing state

In this section, we compute the tri-parton (quark, antiquark, and gluon) Fock-space component \(|\gamma_T\rangle_{qg}\) of the outgoing state produced by the evolution and the scattering of an incoming state representing a virtual photon with transverse polarisation. At leading order, this component involves two parton branchings: the decay of the virtual photon into a quark-antiquark pair \((\gamma_T \rightarrow q\bar{q})\), followed by the emission of a gluon from either the quark \((q \rightarrow qg)\), or the antiquark \((\bar{q} \rightarrow \bar{q}g)\). In practice, it is enough to consider one of these 2 cases — say, gluon emission by the antiquark. The contribution of the other case can then be simply obtained via symmetry operations. Also, as explained in section 2, there is no need to explicitly compute the graphs where the photon decays after crossing the shock-wave (see figure 1.c) — their effects can be accounted for by subtracting the no-scattering limit \(\hat{S} \rightarrow 1\) from the other contributions. So, in practice, it is enough to compute the two graphs in figure 1.a and b, together with the corresponding “instantaneous” graphs, where the intermediate antiquark line is replaced with the instantaneous piece of the fermion propagator (see figure 7).

4.1 Gluon emission by the antiquark

We shall describe in detail the case where the gluon is emitted by the antiquark. We start with the regular graphs in figures 1.a and b. The respective contributions to the outgoing state can be inferred from eq. (2.12):

\[
|\gamma\rangle_{qg} = |q\bar{q}g\rangle \left\{ \langle q_2\bar{q}_2g_2|\hat{S}|q_1\bar{q}_1g_1\rangle \frac{\langle q_1\bar{q}_1g_1|H_{q\rightarrow qg}|q\rangle}{E_{q_1\bar{q}_1g_1} - E_\gamma} + \frac{\langle q_2\bar{q}_2g_2|H_{q\rightarrow qg}|q_1\bar{q}_1\rangle}{E_{q_1\bar{q}_1} - E_{q_2\bar{q}_2g_2}} \langle q_1\bar{q}_1|\hat{S}|q\rangle \frac{\langle q\gamma|H_{\gamma\rightarrow q\bar{q}}\rangle}{E_{q\bar{q}} - E_\gamma} \right\}.
\]  

(4.1)

where the first (second) term within the accolades describes gluon emission prior (after) the shockwave. The QCD matrix elements ensure that \(q_1 = q\) in the first term and, respectively, \(q_2 = q_1\) in the second term. Notice the subscript \(qg\) on the LCWF in the l.h.s.: the antiquark component is shown in boldface to emphasise that this is the fermion which emits the gluon.

As for the leading-order calculation in the previous section, we shall first compute the LCWF in the absence of scattering \((\hat{S} \rightarrow 1)\) and in transverse momentum space. Then we shall construct its Fourier transform to the transverse coordinate representation. Then it will be easy to insert the effects of the collision, in the form of Wilson lines. Notice that, even after replacing \(\hat{S} \rightarrow 1\), the two terms in eq. (4.1) are not identical: they differ in the structure of their energy denominators, which in turn reflects the different time-orderings of the gluon emission w.r.t. \(x^+ = 0\) (the time of scattering).

Our conventions for the momentum-space kinematics are summarised in figure 6.a. Let us first work out the relevant energy denominators. They can all be constructed form the LC energy differences at the emission vertices. For the photon decay \(\gamma \rightarrow q\bar{q}\), we have (we
recall that $\vec{Q}^2 = \vartheta(1 - \vartheta)Q^2$)

$$E_{q\bar{q}} - E_{\gamma} = \frac{1}{2q^+} \left[ \frac{k^2}{\vartheta} + \frac{(q - k)^2}{1 - \vartheta} - q^2 + Q^2 \right] = \frac{(k - \vartheta q)^2 + \vec{Q}^2}{2\vartheta(1 - \vartheta)q^+},$$  \hspace{1cm} (4.2)

whereas for the gluon emission $q \rightarrow qg$,

$$E_{qg} - E_{q\bar{q}} = E_{q\bar{q}} - E_{\gamma} = \frac{1}{2q^+} \left[ \frac{p^2}{\xi} + \frac{(q - k - p)^2}{1 - \vartheta - \xi} - (q - k)^2 \right] = \frac{[\vartheta(1 - \vartheta)p - \xi(q - k)]^2}{2\xi(1 - \vartheta)(1 - \vartheta - \xi)q^+},$$  \hspace{1cm} (4.3)

As expected, this energy difference would vanish for a collinear decay, i.e. in the case where the gluon splitting fraction $z \equiv \xi/(1 - \vartheta)$ controls not only its longitudinal momentum $(p^+ = z(q^+ - k^+))$, but also its transverse momentum $(p = z(q - k))$. The remaining energy denominator involves the sum of the two energy differences written above:

$$E_{qg} - E_{\gamma} = \frac{\vartheta(1 - \vartheta)^2p^2 + \vartheta(1 - \vartheta)(1 - \vartheta - \xi)(\vec{k}^2 + \vec{Q}^2)}{2\vartheta(1 - \vartheta)(1 - \vartheta - \xi)q^+},$$  \hspace{1cm} (4.4)

where we introduced the “shifted” momenta

$$\vec{k} \equiv k - \vartheta q, \quad \vec{p} \equiv p - \frac{\xi}{1 - \vartheta}(q - k),$$  \hspace{1cm} (4.5)

which physically express the deviations from collinearity at the emission vertices and which are convenient to use as integration variables when summing over the final states (see below).

The matrix element for the virtual photon decay has already been shown in eq. (3.4). That for the gluon emission from the antiquark reads [57]

$$\left\langle q_{\lambda_3}^\dagger(u) \bar{\sigma}_{\lambda_4}^\dagger(t) g_{\sigma}^{\dagger}(l) H_{q\rightarrow qg} q_{\lambda_5}^\dagger(s) \bar{\sigma}_{\lambda_2}^\dagger(k) \right\rangle = -(2\pi)^6 \delta^{(3)}(s - u) \delta^{(3)}(k - t - l) \delta_{\gamma \sigma}^\alpha \delta_{\lambda_3 \lambda_1} \frac{gt_{2\delta}}{2\sqrt{2}t^+} \chi_{\lambda_2}^\dagger \left[ \frac{2l^+}{t^+} - \sigma^\dagger \cdot t^+ - \sigma^\dagger \cdot k^+ \sigma^\dagger \right] \chi_{\lambda_4}^\dagger.$$  \hspace{1cm} (4.6)
The structure of this matrix element is consistent with the fact that the antiquark should propagate backwards in time: the colour (δ) and spin (λ₄) indices of the daughter antiquark formally appear as the initial quantum numbers in the matrix element in the r.h.s. of eq. (4.6); similarly, the respective indices β and λ₂ of the emitter appear as final quantum numbers.

For the kinematics shown in figure 6, the above spinorial matrix element reads

\[
\chi^\dagger_\lambda \left[ \frac{2p^i}{p^+} - \sigma \cdot (q - k - p) \frac{\sigma \cdot (q - k)}{q^+ - k^+ - p^+} \right] \chi_\lambda_2 = \frac{\tau^{ij}_{\lambda\lambda_2}(\xi, 1 - \vartheta - \xi) \tilde{p}^j}{\xi (1 - \vartheta - \xi) q^+}. \tag{4.7}
\]

with

\[
\tau^{ij}_{\lambda\lambda_2}(\xi, 1 - \vartheta - \xi) \equiv \chi^\dagger_\lambda \left[ (2(1 - \vartheta) - \xi) \delta^{ij} + i \xi \varepsilon^{ij} \delta \right] \chi_\lambda_2 = \delta_{\lambda\lambda_2} \left[ (2(1 - \vartheta) - \xi) \delta^{ij} + 2i \xi \varepsilon^{ij} \delta \right]. \tag{4.8}
\]

The two arguments of this function are the longitudinal momentum fractions of the daughter partons: the gluon (ξ) and the final antiquark (1 − θ − ξ). The coefficient 2(1 − θ) − ξ multiplying δ^{ij} is recognised as the sum of the momentum fractions, 1 − θ and 1 − θ − ξ, of the antiquark prior and respectively after the gluon emission.

Consider now the first term in eq. (4.1) with \( \tilde{S} = 1 \). This is of course the same as the \( q\bar{q}g \) Fock state at the time of scattering, as shown in figure 6.a. Using the energy denominators (4.2) and (4.4) together with the matrix elements in eqs. (3.4) and (4.6)–(4.7), one finds

\[
\left| \tau^{ij}_{\lambda\lambda_2}(\xi, q^+, q) \right|_{q\bar{q}g}^{(a)} = - \int d^2 \tilde{k} d^2 \tilde{p} \int_0^1 \frac{1 - \theta}{d\theta} \int_0^{\xi} d\xi \left[ \frac{e\varepsilon_{fg} q^+ \vartheta (1 - \theta) \varphi^{ij}_{\lambda\lambda}(\vartheta) \tilde{k}^j \tau^{mn}_{\lambda\lambda_2}(\xi, 1 - \theta - \xi) \tilde{p}^m \tau^a_{\alpha\beta} \xi (1 - \vartheta - \xi) (k^2 + Q^2) \right] \times \left| \tau^{ij}_{\lambda\lambda_2} \left( (1 - \theta - \xi) q^+, q - k - p \right) g^a_m \left( \xi q^+, p \right) q^i_{\lambda_1} \left( \vartheta q^+, k \right) \right|, \tag{4.9}
\]

where the transverse momenta of the final partons can be expressed in terms of the integration variables \( \tilde{k} \) and \( \tilde{p} \) as follows (cf. eq. (4.5))

\[
k = \tilde{k} + \vartheta q, \quad p = \tilde{p} + \xi \left( q - \frac{\tilde{k}}{1 - \vartheta} \right), \quad q - k - p = (1 - \theta - \xi) \left( q - \frac{\tilde{k}}{1 - \vartheta} \right) - \tilde{p}. \tag{4.10}
\]

To prepare the Fourier transform to the transverse coordinate representation, let us first introduce the corresponding representation for the 3-parton final state:

\[
\left| \tau^{ij}_{\lambda\lambda_2} \left( (1 - \theta - \xi) q^+, q - k - p \right) g^a_m \left( \xi q^+, p \right) q^i_{\lambda_1} \left( \vartheta q^+, k \right) \right| = \int_{x,y,z} e^{i \varphi (q - k - p) + i z \cdot p + i m \cdot k} \left| \tau^{ij}_{\lambda\lambda_2} \left( (1 - \theta - \xi) q^+, y \right) g^a_m \left( \xi q^+, x \right) q^i_{\lambda_1} \left( \vartheta q^+, x \right) \right|, \tag{4.11}
\]

Also, we shall use the notations \( \mathbf{w} \) and \( \mathbf{y}' \) for the transverse coordinates of the incoming virtual photon and of the intermediate antiquark, prior to the gluon emission, respectively (see figure 1.a). These are not independent coordinates: energy-momentum conservation
implies that \( \mathbf{w} \) must coincide with the center-of-energy of the \( q \bar{q} \) pair produced by the decay of the virtual photon and also with that of the final 3-parton system; similarly, \( \mathbf{y}' \) must be the same as the center-of-energy of the \( gg \) pair produced by the decay of the intermediate anti-quark. These conditions imply

\[
\mathbf{w} = \vartheta \mathbf{x} + (1 - \vartheta) \mathbf{y}' = \vartheta \mathbf{x} + (1 - \vartheta - \xi) \mathbf{y} + \xi \mathbf{z}, \quad \mathbf{y}' = \frac{(1 - \vartheta - \xi) \mathbf{y} + \xi \mathbf{z}}{1 - \vartheta}.
\]  

(4.12)

Using eq. (4.10) and the above relations for \( \mathbf{w} \) and \( \mathbf{y}' \), one can check that the exponent in eq. (4.11) can be rewritten as

\[
e^{i[\vartheta \mathbf{x} + (1 - \vartheta - \xi) \mathbf{y} + \xi \mathbf{z}] \cdot \mathbf{q} - i(\vartheta - 1)(\vartheta - \xi) \mathbf{p} + i[(1 - \vartheta) \mathbf{x} - (1 - \vartheta - \xi) \mathbf{y} - \xi \mathbf{z}] \cdot \mathbf{k}} = e^{iw \cdot q - iy \cdot \mathbf{p} + iR \cdot \mathbf{k}},
\]  

(4.13)

where we have introduced the notations \( \mathbf{R} \) and \( \mathbf{Y} \) for the transverse separations between the daughter partons after each of the emission vertices:

\[
\mathbf{R} \equiv \mathbf{x} - \mathbf{y}', \quad \mathbf{Y} \equiv \mathbf{y} - \mathbf{z}.
\]  

(4.14)

Eq. (4.13) explains why it was more convenient to use the “shifted” transverse momenta \( \mathbf{\tilde{k}} \) and \( \mathbf{\tilde{p}} \) as integration variables, instead of the original momenta \( \mathbf{k} \) and \( \mathbf{p} \): the shifted variables are conjugate to the transverse separations between the daughter partons and thus facilitate the calculation of the Fourier transform to the transverse coordinate representation (defined as in eq. (3.8)). Specifically, by using the following Fourier transform (see e.g. appendix A in [61])

\[
\int \frac{d^2 \mathbf{k}}{2\pi} \frac{d^2 \mathbf{p}}{2\pi} \left( k^2 + Q^2 \right)^{-1} \left[ \alpha(k^2 + Q^2) + p^2 \right] e^{i\mathbf{k} \cdot \mathbf{x} + i\mathbf{p} \cdot \mathbf{z}} = -\frac{x^i z^j}{z^2 \sqrt{x^2 + az^2}} Q K_1 \left( Q \sqrt{x^2 + az^2} \right),
\]  

(4.15)

one finds

\[
\left[ \gamma_T(Q, q^+, \mathbf{w}) \right]^{(a)}_{qg} = -\frac{e e' f g}{2(2\pi)^4} \int_{x,y,z} d\vartheta \int_0^{1-\vartheta} d\xi \frac{\vartheta}{\sqrt{\xi}} \frac{R^i Y^m}{Y^2} (\mathbf{w} - \mathbf{c})
\]

\[
\times \varphi_{\lambda_1 \lambda_2}^{ij} (\vartheta) \tau_{\lambda_2 \lambda_3}^{mn}(\xi, 1 - \vartheta - \xi) t_{\alpha\beta}^a \frac{Q K_1(QD)}{D}
\]

\[
\times \left[ \gamma_{l_{\lambda_3}}((1 - \vartheta - \xi) q^+, \mathbf{y}) g_{\alpha}^a(\xi q^+, \mathbf{z}) q_{\lambda_1}^a(\vartheta q^+, \mathbf{x}) \right],
\]  

(4.16)

where \( \mathbf{c} \) is the center-of-energy of the \( q\bar{q} \) system,

\[
\mathbf{c} \equiv \vartheta \mathbf{x} + (1 - \vartheta - \xi) \mathbf{y} + \xi \mathbf{z},
\]  

(4.17)

while \( D \equiv D(x, y, z, \vartheta, \xi) \) denotes the following transverse distance (below, \( \vartheta' \equiv 1 - \vartheta - \xi \))

\[
D^2(x, y, z, \vartheta, \xi) \equiv (\vartheta \vartheta' (x - y)^2 + (\vartheta + \vartheta') (x - z)^2 + \vartheta' \xi (y - z)^2,
\]  

(4.18)

which can be interpreted as the overall transverse size of the \( q\bar{q} \) partonic fluctuation at the time of scattering \((x^+ = 0)\). This size is limited by the virtuality of the space-like photon: \( QD \lesssim 1 \). (Indeed, the modified Bessel function exponentially vanishes for large
values $QD \gg 1$ of its argument.) It is interesting to notice that $D^2$ can be equivalently rewritten as
\[
D^2 = \vartheta(1 - \vartheta) R^2 + \frac{\xi(1 - \vartheta - \xi)}{1 - \vartheta} Y^2.
\] (4.19)

This rewriting shows that the virtuality $Q^2$ limits both the size $R = |x - y|$ of the $q\bar{q}$ pair produced by the decay of the virtual photon, namely $R \lesssim 1/Q$, and the size $Y = |y - z|$ of the $g\bar{q}$ pair produced by the decay of the antiquark ($\bar{q} \to \bar{g}g$). In particular, when the gluon is soft, $\xi \ll 1$, the virtuality constraint on the gluon emission becomes quite loose: $Y^2 \lesssim 1/(\xi Q^2)$; hence, despite the space-like virtuality, a soft gluon can propagate at relatively large distances from its emitter.

Of course, this whole discussion of the virtuality limits on the transverse size of the partonic fluctuations applies only prior to the scattering ($x^+ \leq 0$). The collision can put the partons on their mass-shell and then they can move from each other arbitrarily far away.

At this point, it is straightforward to add the effects of the scattering in the eikonal approximation and thus deduce our final result for the first term in the r.h.s. of eq. (4.1): it suffices to insert Wilson lines for the 3 partons which are crossing the shockwave. In practice, this amounts to replacing
\[
t^{a}_{\alpha \beta} \to U^{ab}(z) [V(x)t^{b}V^{\dagger}(y)]_{\alpha \beta}
\] (4.20)
in the integrand of eq. (4.16). At this step, it is also convenient to subtract the no-scattering limit ($\hat{S} \to 1$) from the final result\(^6\) and thus directly obtain the first term in the r.h.s. of eq. (2.12). Our final result for this term therefore reads
\[
\left| \gamma_{\gamma}^{(a)}(Q, q^{+}, w) \right|_{q\bar{q}g}^{(a)} = \frac{ee_{f} g q^{+}}{2(2\pi)^{3}} \int_{x,y,z} d^{4}x^{\perp} d^{4}y^{\perp} \frac{R^{j} Y^{n}}{Y^{2}} \delta^{(2)}(w - c)
\]
\[
\times \frac{\varphi_{\lambda_{1}}^{(1)}}{\varphi_{\lambda_{2}}^{(1)}}(\vartheta) \tau^{(1)}_{\lambda_{1}}(\xi, 1 - \vartheta - \xi) \frac{Q K_{1}(QD)}{D} 
\times \left[ U^{ab}(z)V(x)t^{b}V^{\dagger}(y) - t^{a}_{\alpha \beta} \right]_{\alpha \beta}
\times \left\langle \varphi^{(1)}_{\lambda_{2}}[(1 - \vartheta - \xi)q^{+}, y] g^{a}_{\lambda_{1}}(\xi q^{+}, z) q^{a}_{\lambda_{1}}(\vartheta q^{+}, x) \right\rangle.
\] (4.21)

Consider now the second term in eq. (4.1) which, we recall, corresponds to a gluon emission in the final state (cf. figure 1.b). In the absence of scattering ($\hat{S} = 1$), this term differs from the first term there only in one of the energy denominators: the energy difference $E_{q\bar{q}g} - E_{\gamma}$ shown in eq. (4.4) gets replaced by $E_{q\bar{q}} - E_{q\bar{q}g}$, which is shown (up to a sign) in eq. (4.3). So, the corresponding contribution (for $\hat{S} = 1$, once again) can be obtained from eq. (4.9) by replacing
\[
\vartheta(1 - \vartheta)^{2} p^{2} + \xi(1 - \vartheta - \xi)(\hat{k}^{2} + \hat{Q}^{2}) \longrightarrow \vartheta(1 - \vartheta)^{2} \hat{p}^{2},
\] (4.22)

\(^6\)As explained in section 2, this subtraction is a convenient way to include the contribution of the graph in figure 1.c, where the photon decays after crossing the shockwave. In writing eq. (4.21), we keep the upper label (a) although, strictly speaking, this result involves contributions from both graphs (a) and (c) in figure 1.
in the denominator and changing the overall sign. When computing the Fourier transform to transverse coordinates, the integrals over \( \tilde{p} \) and \( \tilde{k} \) factorise from each other. The Wilson lines can then be easily added: in this case, they describe the scattering of the intermediate \( q\bar{q} \) pair, with transverse coordinates \( x \) and \( y' \) (see figure 1.b). After also subtracting the no-scattering limit, one eventually finds\(^7\)

\[
\begin{align*}
\left| \gamma^i_T(Q, q^+, w) \right|_{q\bar{q}}^{(b)} &= \frac{e e_f g q^+}{2(2\pi)^4} \int_{x,y,z} d\theta \int_0^1 d\xi \frac{\partial}{\sqrt{\xi}} \frac{R^i_y Y^n}{Y^2} \delta^{(2)}(w - c) \\
&\quad \times \varphi^{ij}_{\lambda_1,\lambda_2}(\theta, \xi) \tau^{mn}_{\lambda_3,\lambda_4}(\xi, 1 - \theta - \xi) \frac{Q^2 K_1(QR)}{QR} \left[ V(x) V^\dagger(y') t^a - t^a \right]_{\alpha\beta} \\
&\quad \times \left| q_{\lambda_2}^\alpha ((1 - \theta - \xi) q^+ + y) g_m^a(\xi q^+, z) q_{\lambda_1}^a (\theta q^+, x) \right|,
\end{align*}
\]

(4.23)

The above contributions in eqs. (4.21) and (4.23) can be combined with each other by introducing a convenient notation. Specifically, by using (4.12), (4.18), and the following matrix identity relating the adjoint and the fundamental representations,

\[
V^\dagger(y') t^a V(y') = U^{ab}(y') b^b \Longrightarrow U^{ab}(y') V(x) t^b V^\dagger(y') = V(x) V^\dagger(y') t^a,
\]

(4.24)

one sees that the middle line of (4.23) can be obtained from the corresponding line of (4.21) by replacing both \( y \) and \( z \) by \( y' \); indeed, after this replacement, one has \( QD \rightarrow QR \). This replacement is merely formal (it should not be applied to the ensemble of eq. (4.21), but only to its middle line), but it is still useful in that it allows us to introduce a compact notation for the sum of eqs. (4.21) and (4.23):

\[
\begin{align*}
\left| \gamma^i_T(Q, q^+, w) \right|_{q\bar{q}}^{\text{reg}} &= -\frac{e e_f g q^+}{2(2\pi)^4} \int_{x,y,z} d\theta \int_0^1 d\xi \frac{\partial}{\sqrt{\xi}} \frac{R^i_y Y^n}{Y^2} \delta^{(2)}(w - c) \\
&\quad \times \left\{ \Phi^{ijmn}_{\lambda_1,\lambda_2}(\theta, \xi) \frac{Q^2 K_1(QD)}{D} \left[ U^{ab}(z) V(x) V^\dagger(y') - t^a \right]_{\alpha\beta} - (y, z \rightarrow y') \right\} \\
&\quad \times \left| q_{\lambda_2}^\alpha ((1 - \theta - \xi) q^+ + y) g_m^a(\xi q^+, z) q_{\lambda_1}^a (\theta q^+, x) \right|,
\end{align*}
\]

(4.25)

with an “effective vertex” which encompasses the spinor and helicity structure of the whole graph:

\[
\Phi^{ijmn}_{\lambda_1,\lambda_2}(\theta, \xi) \equiv \varphi^{ij}_{\lambda_1,\lambda_2}(\theta, \xi) \tau^{mn}_{\lambda_3,\lambda_4}(\xi, 1 - \theta - \xi).
\]

(4.26)

In our notation in eq. (4.25), we reintroduced the upper script “reg” to recall that this is the contribution of the “regular” graphs which involve the standard, non-local, piece of the intermediate antiquark propagator. On top on that, we also have the instantaneous contribution. This involves only 2 topologies, since the photon decay and the gluon emission now occur at the same time (see figure 7). The respective matrix element reads (see e.g. \([61]\])

\[
\left< q_{\lambda_1}^\alpha(k) q_{\lambda_2}^\beta(s) g_a^\gamma(p) | H_{\gamma \rightarrow q\bar{q}} | \gamma^i_T(q) \right> = -(2\pi)^3 \frac{\delta^{(3)}(q - k - p - s)}{4\sqrt{q^+ p^+}} \frac{g e f g q^+}{2\pi} \frac{R^i_y Y^n}{Y^2} \delta^{ij} \frac{Q^2 K_1(QR)}{QR} \left[ V(x) V^\dagger(y') t^a - t^a \right]_{\alpha\beta}
\]

(4.27)

\(^7\)Since the antiquark formally propagates backwards in time, the colour matrix \( t^a \) associated with the gluon emission vertex is located to the right of the Wilson lines describing the scattering of the \( q\bar{q} \) pair.
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Figure 7. The two instantaneous graphs contributing to the quark-antiquark-gluon outgoing state in the case where the gluon is emitted by the antiquark.

Figure 8. The three possible topologies in the case where the gluon is emitted by the quark.

A straightforward calculation, similar to that leading to eq. (4.21), yields

\[
\left|\gamma_T^q(Q, q^+, w)\right|_{\text{inst}} = \frac{ee_f g}{2(2\pi)^2} \int d\theta \int_0^1 d\xi \sqrt{\xi} \frac{\vartheta(1 - \vartheta - \xi)}{1 - \vartheta} \delta^{(2)}(w - c) \\
\times \left(\delta^{im} + 2i\epsilon^{im}\lambda_1\right) \delta^{\lambda_2\lambda_1} \frac{QK_1(QD)}{D} \left[\tau_{ab}(z)V(x)\delta V^+(y) - \tau^a\right]_{\alpha\beta} \\
\times \left[\varphi_{\lambda_2}^\beta((1 - \vartheta - \xi)q^+, z) g_{\alpha\lambda_1}^\alpha(\vartheta q^+, x) \right].
\] (4.28)

This has the same Wilson line structure as eq. (4.21), as expected given the topology of the contributing diagrams in figure 7. For what follows, it is convenient to observe that one can combine eqs. (4.25) and (4.28) in a unique expression by using a generalised version of the effective vertex (4.26), which reads

\[
\Phi_{\lambda_1\lambda_2}^{ijmn}(x, y, z, \vartheta, \xi) \equiv \varphi_{\lambda_1\lambda_2}(\vartheta) \tau^{mn}(\xi, 1 - \vartheta - \xi) - \delta_{\lambda_1\lambda_2} \delta^{nj} \frac{\xi(1 - \vartheta - \xi)}{1 - \vartheta} \left(\delta^{im} + 2i\epsilon^{im}\lambda_1\right) \frac{Y^2}{R \cdot Y}.
\] (4.29)

It is understood that the second piece in eq. (4.29) vanishes after substituting \(y \to y'\) and \(z \to z'\), so the instantaneous contribution vanishes in this limit, as it should.

4.2 Gluon emission by the quark

Our conventions for the case where the gluon is emitted by the quark are summarised in figure 6.b in transverse momentum space and in figure 8 in transverse coordinate space. (This last figure shows all the possible topologies for the collision with the shock wave and
for the case of the “regular” contributions.) There is in fact no need to explicitly compute these graphs: the respective result can be directly inferred from that in the previous section via suitable changes of variables which amount to exchanging the quark and the antiquark, but such that the final state remains unchanged. Specifically, the contribution of the 3 graphs in figure 8 to the $q\bar{q}g$ Fock space component of the LCWF of the transverse photon can be obtained from eq. (4.25) by (i) exchanging the quark with the antiquark in the final state, (ii) exchanging $\vartheta \leftrightarrow 1 - \vartheta - \xi$, $x \leftrightarrow y$, $\lambda_1 \leftrightarrow \lambda_2$, and $\alpha \leftrightarrow \beta$, (iii) taking the complex conjugate of the integrand, and (iv) changing the overall sign (to account for the fact that the colour charge of the quark is minus that of the antiquark).\footnote{Notice that the leading-order LCWF in eq. (3.11) is invariant under this combined set of symmetry operations, as it should: exchanging the quark and the antiquark has no physical relevance in that case.}

These transformations have the following consequences on the other transverse coordinates and distances in the problem:

$y' \rightarrow \frac{\vartheta x + \xi z}{\vartheta + \xi} \equiv x'$, \quad $R \equiv x - y' \rightarrow y - x' \equiv -\tilde{R}$, \quad $Y \equiv y - z \rightarrow x - z \equiv X$,

\hspace{1cm} (4.30)

(Notice that $x'$ is the transverse coordinate of the intermediate quark, prior to the gluon emission; see figure 8.a). Furthermore, the spinorial structures change as follows (recall eq. (3.6))

$\varphi^{ij}_{\lambda\lambda_1}(\vartheta) \rightarrow \varphi^{ij}_{\lambda\lambda_2}(1 - \vartheta - \xi) = \chi^\dagger_{\lambda_1} \left[(1 - \vartheta - \xi)\delta^{ij} - i\varepsilon^{ij}\sigma^3\right] \chi_{\lambda_2}$

$= -\varphi^{ij}_{\lambda\lambda_2}(\vartheta + \xi) = -\delta_{\lambda\lambda_2} \left[(2(\vartheta + \xi) - 1)\delta^{ij} + 2i\varepsilon^{ij}\lambda\right], \quad (4.31)$

for the photon decay vertex and, respectively (recall eq. (4.8))

$\tau^{mn}_{\lambda\lambda_2}(\xi, 1 - \vartheta - \xi) \rightarrow \tau^{mn*}_{\lambda\lambda_1}(\xi, \vartheta) = \chi^\dagger_{\lambda_1} \left[(2\vartheta + \xi)\delta^{mn} - i\xi\varepsilon^{mn}\sigma^3\right] \chi_{\lambda}$

$= \tau^{mn}_{\lambda\lambda_1}(\xi, \vartheta) = \delta_{\lambda\lambda_1} \left[(2\vartheta + \xi)\delta^{mn} + 2i\xi\varepsilon^{mn}\lambda\right], \quad (4.32)$

for the vertex describing the gluon emission.

Remarkably, these transformations do not change the colour structure (including the Wilson lines) for the diagrams where the parton branchings occur either fully before or fully after the collision. That is, the diagram in figure 8.a has exactly the same colour structure as that in figure 1.a and the same holds for the diagrams in figure 8.c and figure 1.c, respectively. This is so because the vertex for the photon decay has no incidence on the flow of colour, so the matrix $t^a$ describing the gluon emission can be “commuted” through the photon vertex without altering the colour structure. On the other hand, the diagrams where the scattering occurs in the intermediate $q\bar{q}$ state, that is those in figure 8.b and figure 1.b respectively, do have different colour structures, since the matrix $t^a$ does not commute with the Wilson lines describing the collision. Yet, as in eq. (4.25), the $S$-matrix structure of the graph in figure 8.b can be obtained from that in figure 8.a via an appropriate replacement of variables, which follows as well from the symmetry operations above.

After making all these transformations and also changing the overall sign, one finds (note the subscript $q\bar{q}g$ on the outgoing state: the quark label is shown in boldface to
emphasise that this is the source of the gluon emission):

\[
\gamma_T^q(Q, q^+, w)\rangle_{q\bar{q}g} = \frac{eeggq^+}{2(2\pi)^4} \int_{x,y,z} d\theta \int_{0}^{1} d\xi \int_{0}^{1-\theta} d\xi \left[ \frac{1-\vartheta - \xi}{\sqrt{\xi}} \frac{\bar{R}X^n}{X^2} \delta(2)(w - c) \right] 
\times \left\{ \Phi_{\lambda_1 \lambda_2}^{ijm}(x, y, z, \vartheta, \xi) \right\} 
\times \left[ \bar{q}_{\lambda_2}^\beta ((1 - \vartheta - \xi)q^+, y) g_m^\alpha (\xi q^+, z) q_{\lambda_1}^\alpha (\vartheta q^+, x) \right],
\]

(4.33)

where the effective transverse size \(D\) is the same as for an emission by the antiquark, cf. eq. (4.18). (Indeed, this distance is invariant under the symmetry operations exchanging the quark and the antiquark.) When replacing \(x, z \to x'\), the argument of the Bessel function changes as \(QD \to Q\bar{R}\). When \(R = |\bar{R}| = |x' - y|\) and

\[
\tilde{Q}^2 = (\vartheta + \xi)(1 - \vartheta - \xi)Q^2.
\]

(4.34)

The effective vertex \(\tilde{\Phi}\) is obtained via the appropriate transformations from eq. (4.29) and reads

\[
\tilde{\Phi}_{\lambda_1 \lambda_2}^{ijm}(x, y, z, \vartheta, \xi) \equiv \tau_{\lambda_1 \lambda_2}^{ij}(\vartheta + \xi) - \delta_{\lambda_1 \lambda_2} \delta_{i}^{\alpha} \delta_{\lambda_1}^{\beta} \frac{\xi \vartheta}{\vartheta + \xi} (\delta_{i}^{\alpha} - 2i\epsilon^{\alpha \beta} \lambda_1) \frac{X^2}{\bar{R} \cdot X}.
\]

(4.35)

More explicit expressions for the outgoing state (4.33) will be presented in appendix A, where we shall also check that our current results are indeed consistent with a recent calculation of the tri-parton LCWF in ref. [73].

Our final result for the \(q\bar{q}g\) Fock-space component of the LCWF of the transverse photon is the sum of the two contributions, corresponding to emissions by the antiquark and by the quark, respectively:

\[
\gamma_T^q(Q, q^+, w)\rangle_{q\bar{q}g} = \gamma_T^q(Q, q^+, w)\rangle_{q\bar{q}g} + \gamma_T^q(Q, q^+, w)\rangle_{q\bar{q}g}.
\]

(4.36)

5 Next-to-leading order corrections: the real terms

Given our previous results for the tri-parton Fock-space component of the outgoing wavefunction of the transverse virtual photon, it is rather straightforward to deduce the "real" next-to-leading order (NLO) corrections to the cross-section for producing a pair of quark-antiquark jets. This involves two steps: first, one computes the (tree-level) cross-section for the production of three partons (quark, antiquark, gluon); then, one integrates out the kinematics of the gluon which is not measured in the final state.

5.1 The leading-order trijet cross-section

The leading-order cross-section for \(q\bar{q}g\) production in DIS is computed similarly to eq. (3.14), that is, as the expectation value of the product of three number-density operators (themselves built with Fock space operators for bare partons) on the photon LCWF in eq. (4.36):

\[
\frac{d\sigma^{\gamma^A \to q\bar{q}g}}{dk_1^+ dk_2^+ dk_3^+ dk_4^+ dk_5^+ dk_6^+} (2\pi)^4 \delta(q^+ - k_1^+ - k_2^+ - k_3^+)
\]

\[
= \frac{1}{2} \int_{w, \bar{w}} e^{-i q \cdot (w - \bar{w})} \langle q\bar{q}g \rangle \gamma_T^q(Q, q^+, w) \rangle_{q\bar{q}g} \langle N_q(k_1) N_{\bar{q}}(k_2) N_g(k_3) | \gamma_T^q(Q, q^+, w) \rangle_{q\bar{q}g}.
\]

(5.1)
It is natural to distinguish between two types of contributions: direct contributions, where the gluon is emitted and reabsorbed by the same fermion (quark or antiquark), and interference terms, where the gluon is absorbed by the quark in the directed amplitude (DA) and reabsorbed by the antiquark in the complex conjugate amplitude (CCA), or vice-versa.

5.1.1 Direct contributions

Two particular diagrams describing direct emissions — one by the quark, the other one by the antiquark — are illustrated in figure 9. For each type of emitter, there are nine distinct topologies, corresponding to the product of the three graphs shown in figure 1 (for gluon emission by the antiquark) and their complex conjugates. (For simplicity, we explicitly discuss only the case of the regular graphs; the instantaneous contributions are eventually added by modifying the effective vertices as shown in eqs. (4.29) and (4.35).) Recall however that in our calculation of the LCWF we have grouped the three graphs in figure 1 (or in figure 8) into two terms: the last graph has been used as a subtraction term for the two previous ones, thus making clear that this particular LCWF component vanishes in the absence of the scattering. Hence, the direct contribution to the 3-jet cross-section will in fact contain only 4 terms.

As in the previous section, we first present our result for the emission by the antiquark. A direct calculation using eq. (5.1) together with the LCWF in eq. (4.25) yields (as before, we show the antiquark label in boldface, to recall that this is the gluon emitter)

\[
\frac{d\sigma^{\gamma A\rightarrow q\bar{q}g}}{dk_1^+ d^2k_1 dk_2^+ d^2k_2 dk_3^+ d^2k_3} = \frac{\alpha_s \alpha_e C_F N_c g^2 Q^2}{2(2\pi)^{10}(q^+)^2\xi} \left( \sum \epsilon_j^2 \right) \delta(q^+ - k_1^+ - k_2^+ - k_3^+) \\
\times \int_{x,y,z,x,y,z} e^{-ik_1(x-x)} - ik_2(y-y) - ik_3(z-z) \frac{R^1 Y^{imn} R^1\bar{Y}^{imn}}{Y^2\bar{Y}^2} \\
\times \left[ k_1^{imjn}(x,y,z,x,y,z,\vartheta,\xi) W(x,y,z,x,y,z) \right. \\
\left. - (y,z \rightarrow y') - (x,y \rightarrow x) + (y,z \rightarrow y' & x,y \rightarrow x') \right].
\]  

(5.2)

The notations here are similar to those in eq. (4.25) (see also figure 9). Transverse coordinates with a bar refer to the CCA. The longitudinal momentum fractions \( \vartheta \) and \( \xi \) are the same in the DA and in the CCA, since they are fully fixed by the kinematics of the final state, as follows:

\[
\vartheta = \frac{k_1^+}{q^+}, \quad \xi = \frac{k_3^+}{q^+}.
\]  

(5.3)

The \( \delta \)-function enforcing longitudinal momentum conservation implies \( k_2^+ = (1 - \vartheta - \xi)q^+ \), which in particular requires \( \vartheta + \xi \leq 1 \).

The tensorial kernel \( k_1^{imjn} \) is defined as

\[
k_1^{imjn}(x,y,z,x,y,z,\vartheta,\xi,\Omega) \equiv \frac{\tilde{d}^{irn}(x,y,z,\vartheta,\xi) \tilde{D}^{jrn}(x,y,z,\vartheta,\xi)}{K_1(QD(x,y,z,\vartheta,\xi)) K_1(QD(x,y,z,\vartheta,\xi))} \frac{D(x,y,z,\vartheta,\xi) D(x,y,z,\vartheta,\xi)}{D(x,y,z,\vartheta,\xi) D(x,y,z,\vartheta,\xi)}. 
\]  

(5.4)
Figure 9. Two direct contributions to the leading-order cross-section for $q\bar{q}g$ production. The gluon is emitted and reabsorbed by the same fermion: the antiquark in figure (a) and the quark in figure (b).

The effective vertex $\Phi_{lir}\lambda_1\lambda_2$ has been defined in eq. (4.29) as the sum of two pieces, corresponding to regular and instantaneous gluon emission vertices. Hence the product of effective vertices occurring in eq. (5.4) involves four pieces, that will be explicitly presented in appendix A. Here we only show the first piece, built with the regular vertices alone (cf. eq. (4.26)). Indeed, this piece will be important for discussing the soft and collinear limits of our results in the next sections. It reads

$$
\Phi_{\lambda_1\lambda_2}^{lir}(\vartheta, \xi) \Phi_{\lambda_1\lambda_2}^{*ir}(\vartheta, \xi) \bigg|_{\text{reg} \times \text{reg}} = \chi_\lambda^\dagger \left[ (2 - 2\vartheta - \xi)^2 + \xi^2 \right] \delta^{mn} + 2i\xi(2 - 2\vartheta - \xi)\varepsilon^{mn}\sigma^3 \right)
\left[ 1 + (1 - 2\vartheta)^2 \right] \delta^{ij} + 2i(2\vartheta - 1)\varepsilon^{ij}\sigma^3 \right] \chi_\lambda \\
= 8 \left[ (1 - \vartheta - \xi)^2 + (1 - \vartheta)^2 \right] \left[ \vartheta^2 + (1 - \vartheta)^2 \right] \delta^{ij}\delta^{mn} - 8\xi(2\vartheta - 1)(2 - 2\vartheta - \xi)\varepsilon^{ij}\varepsilon^{mn},
$$

(5.5)

where we have also used $\chi_\lambda^\dagger \chi_\lambda = 2$ and $\chi_\lambda^\dagger \sigma^3 \chi_\lambda = 2\lambda_1\delta_{\lambda_1\lambda_2}$. The first piece in the last equation, proportional to $\delta^{ij}\delta^{mn}$, generates the scalar products of the transverse separations in the DA and, respectively, the CCA, whereas the second piece, proportional to $\varepsilon^{ij}\varepsilon^{mn}$, generates their vectorial products, followed by a final scalar
The product between the 2 vectors previously generated:

$$R^i Y^m R^j Y^n \delta^{ij} \delta^{mn} = (R \cdot R)(Y \cdot Y), \quad R^i Y^m R^j Y^n \varepsilon^{ij} \varepsilon^{mn} = (R \times R)(Y \times Y).$$  \hfill (5.6)

The structure of the squared vertex in eq. (5.5) is consistent with that reported in refs. [78] (see eq. (2.27) there) and [73] (see eqs. (9.31-32)). It furthermore agrees with eq. (B.5) in [72], which refers to the case where the gluon is emitted the final state (i.e. after the collision with the shockwave) in both the DA and the CCA. \hfill (10) For the other cases, the sums over spins and helicities were not explicitly computed in [72], so it is difficult to directly compare the results.) More detailed comparisons with previous results in the literature, as well as a more explicit (but less compact) rewriting of the cross-section (5.2), are presented in appendix A.

The effects of the collision are encoded in the function $W$, defined as the following linear combination of partonic $S$-matrices:

$$W(x, y, z, \mathbf{x}, \mathbf{y}, \mathbf{z}) \equiv S_{qgqg}(x, y, z, \mathbf{x}, \mathbf{y}, \mathbf{z}) - S_{qgq}(x, y, z) - S_{qgq}^\dagger(\mathbf{x}, \mathbf{y}, \mathbf{z}) + 1. \hfill (5.7)$$

The first term in the r.h.s., with the most complex structure, corresponds to the topology depicted in figure 9a: all three partons scatter in both the DA and the CCA. So, this is built with a total of six Wilson lines — two in the adjoint representation and four in the fundamental one. Using Fierz identities like (4.24), it is possible to express all the adjoint $S$-matrices:

$$S_{qgqg}(x, y, z, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{1}{C_F N_c} \left\langle \left[ U^a(\mathbf{x}) U(z) \right]^{ab} \text{tr} \left[ V(y) t^a V^\dagger(\mathbf{x}) V(x) t^b V^\dagger(y) \right] \right\rangle \simeq Q(x, z, \mathbf{x}, \mathbf{y}) Q(z, y, \mathbf{y}, \mathbf{z}),$$ \hfill (5.8)

where the approximate equality in the second line holds at large $N_c$: in this limit, the 6-parton $S$-matrix of interest reduces to the product of two fundamental quadrupoles.

Furthermore, the second term $S_{qgq}(x, y, z)$ in eq. (5.7) corresponds to a diagram where the DA describes initial-state evolution (both parton branchings occur prior to the scattering), whereas the CCA describes final-state evolution (the two parton branchings occur after the scattering). Hence, Wilson lines must be attached only to the three partons from the DA, yielding

$$S_{qgq}(x, y, z) = \frac{1}{C_F N_c} \left\langle U^{ba}(z) \text{tr} \left[ V(x) t^a V^\dagger(y) t^b \right] \right\rangle \simeq S(x, z) S(z, y).$$ \hfill (5.9)

A similar interpretation holds for the third term, with however the DA and the CCA being interchanged with each other. Finally the unit term in eq. (5.7) represents the case where the evolution is restricted to the final state for both the DA and the CCA.

---

9 Notice that $\varepsilon^{ij} = \varepsilon^{ij}^\dagger$ for any $i, j = 1, 2$ and that a vector product between 2 transverse vectors, like $R \times \mathbf{R}$, is a 3-dimensional vector which is oriented along the third axis, i.e. the collision axis.

10 As we shall demonstrate in section 7, the piece proportional to $\varepsilon^{ij} \varepsilon^{mn}$ does not contribute to the cross-section in this particular case (gluon emission in the final state), in agreement with the results in [72, 73].
The above considerations also show that at large \( N_c \) the overall \( S \)-matrix structure can be fully expressed in terms of dipoles and quadrupoles in the fundamental representation of the colour group:

\[
W(x, y, z, \bar{\xi}, \bar{\eta}, \bar{z}) \simeq Q(x, z, \bar{\xi}, \bar{\eta})Q(z, y, \bar{\eta}, \bar{z}) - S(x, z)S(z, y) - S(\bar{\xi}, \bar{\eta})S(\bar{\eta}, \bar{z}) + 1. 
\]

The remaining three terms within the square brackets in eq. \( (5.2) \), for which we use the concise notation introduced in eq. \( (4.25) \), refer to situations where the scattering in the 3-parton final state gets replaced by scattering in the \( q\bar{q} \) intermediate state (in either the DA, or the CCA, or both). These terms will be explicitly presented in appendix A. Here, we only comment on their colour structure. Consider the second term, symbolically denoted as \( (y, z \to y') \). The respective \( S \)-matrix structure is obtained by simultaneously replacing \( y \to y' \) and \( z \to y' \) in eq. \( (5.10) \), which yields

\[
W(x, y', y', \bar{\xi}, \bar{\eta}, \bar{z}) \simeq Q(x, y', \bar{\xi}, \bar{\eta})S(\bar{\eta}, \bar{z}) - S(x, y') - S(\bar{\xi}, \bar{\eta})S(\bar{\eta}, \bar{z}) + 1. 
\] \hspace{1cm} (5.11)

For the third term, denoted as \( (\bar{\eta}, \bar{z} \to \bar{\eta}') \), one similarly finds

\[
W(x, y, z, \bar{\xi}, \bar{\eta}', \bar{z}') \simeq Q(x, z, \bar{\xi}, \bar{\eta}')S(z, y) - S(x, z)S(z, y) - S(\bar{\xi}, \bar{\eta}') + 1. 
\] \hspace{1cm} (5.12)

Finally, for the fourth term \( (y, z \to y') \) and \( (\bar{\eta}, \bar{z} \to \bar{\eta}') \), one finds

\[
W(x, y', y', \bar{\xi}, \bar{\eta}', \bar{z}') \simeq Q(x, y', \bar{\xi}, \bar{\eta}') - S(x, y') - S(\bar{\xi}, \bar{\eta}') + 1. 
\] \hspace{1cm} (5.13)

Incidentally, we recognise here the same colour structure as at leading order, cf. eq. \( (3.19) \). This should not be a surprise: this last term corresponds to Feynman graphs where the gluon is emitted in the final state, i.e. after the scattering with the target.

At this level, it is easy to write down the other direct contribution, where the gluon is emitted and reabsorbed by the quark (see figure 9b). This is obtained from eq. \( (5.2) \) by replacing \( \vartheta \to 1 - \vartheta - \xi \), \( \bar{R} \to \bar{R}, \ Y \to X \), \( y' \to x' \), and similarly for the transverse coordinates with a bar. The corresponding kernel \( \tilde{K}^{mn}_{ij} \) is defined as in eq. \( (5.4) \), but with the effective vertex \( \tilde{\Phi}^{ijmn}_{\lambda_1\lambda_2} \) from eq. \( (4.35) \). The identity \( (5.5) \) remains true, but with \( \vartheta \to 1 - \vartheta - \xi \), of course. The “subtracted” terms representing the topologies where the shockwave is inserted in the intermediate \( q\bar{q} \) state, are now obtained via the replacements \( (x, z \to x') \) or/and \( (\bar{\xi}, \bar{\eta} \to \bar{\xi}') \).

The \( S \)-matrix structure in eq. \( (5.7) \) remains unchanged, for the reasons explained in section 4.2. This means that the colour structure for the particular graph depicted in figure 9b is exactly the same as for that in figure 9a. But this is not true anymore for the other graphs, where the shockwave is inserted in the intermediate \( q\bar{q} \) state. For instance, for the piece denoted as \( (x, z \to x') \), one finds

\[
W(x', y, x', \bar{\xi}, \bar{\eta}, \bar{z}) \simeq S(z, \bar{\xi})Q(x', y, \bar{\eta}, \bar{z}) - S(x', y) - S(z, \bar{\xi})S(\bar{\eta}, \bar{z}) + 1, \hspace{1cm} (5.14)
\]

which is indeed different from the corresponding result for an emission from the antiquark, that is, the term \( (y, z \to y') \), as shown in eq. \( (5.11) \).
5.1.2 Interference terms

Figure 10 shows a particular interference term, in which the gluon is emitted by the quark in the DA and reabsorbed by the antiquark in the CCA. As for the direct terms, there are 9 possible topologies, but only 4 terms due to our peculiar way of regrouping terms. The other type of diagrams, where the gluon is emitted by the antiquark and absorbed by the quark, are related to those of the first type via complex conjugation. Hence the overall contribution of the interference graphs reads

\[
\frac{d\sigma^\gamma A \to q\bar{q}}{dk_1^+d^2k_1 dk_2^+d^2k_2 dk_3^+d^2k_3} = -\frac{\alpha_s\alpha_em C_F N_c}{2(2\pi)^{10}(q^+)^2\xi} \left( \sum c_f^2 \right) \delta(q^+-k_1^+-k_2^+-k_3^+)
\times 2\Re e \int_{x,y,z,\bar{x},\bar{y},\bar{z}} e^{-ik_1(x-\bar{x})-ik_2(y-\bar{y})-ik_3(z-\bar{z})} \frac{\tilde{R}^i X^m \bar{R}^j Y^n}{X^2 Y^2} \times [K_2^{imjn}(x, y, z, \bar{x}, \bar{y}, \bar{z}, \bar{\vartheta}, \bar{\xi}, Q) W(x, y, z, \bar{x}, \bar{y}, \bar{z}) - (x, z \to x') - (y, \bar{z} \to \bar{y'}) + (x, z \to x' \& y, \bar{z} \to \bar{y'})].
\]  

(5.15)

where the new kernel is defined as

\[
K_2^{imjn}(x, y, z, \bar{x}, \bar{y}, \bar{z}, \bar{\vartheta}, \bar{\xi}, Q) = \tilde{\Phi}^{lrm}_{\lambda\lambda}(x, y, z, \bar{\vartheta}, \bar{\xi}) \Phi^{ljrn}_{\lambda\lambda}(\bar{x}, \bar{y}, \bar{z}, \bar{\vartheta}, \bar{\xi})
\times \frac{K_1(QD(x, y, z, \bar{\vartheta}, \bar{\xi})) K_1(QD(\bar{x}, \bar{y}, \bar{z}, \bar{\vartheta}, \bar{\xi}))}{D(x, y, z, \bar{\vartheta}, \bar{\xi}) D(\bar{x}, \bar{y}, \bar{z}, \bar{\vartheta}, \bar{\xi})}.
\]

(5.16)

The product of effective vertices with the instantaneous pieces excluded is evaluated similarly to eq. (5.5) and reads

\[
\tilde{\Phi}^{lrm}_{\lambda\lambda}(\vartheta, \xi) \Phi^{ljrn}_{\lambda\lambda}(\vartheta, \xi) \bigg|_{\text{reg.} \times \text{reg.}} = 8[1 - \xi - 2\vartheta(1 - \xi - \vartheta)] \left[ \xi(1 - \xi) + 2\vartheta(1 - \xi - \vartheta) \right] \delta^{ij} \delta^{mn} - 8\xi(1 - 2\vartheta - \xi)^2 \varepsilon^{ij} \varepsilon^{mn}.
\]

(5.17)

Like in eq. (5.5), the “squared” vertex is the sum of two contributions, one generating scalar products of transverse vectors, the other one giving rise to vectorial products. The structure in eq. (5.17) is indeed consistent with the corresponding results in the literature [72, 73, 78] (see eq. (2.32) in [78], eq. (B.7) in [72], and eq. (9.11) in [73]). The $S$-matrix (colour) structure in the first term in eq. (5.15) is the same as for the direct contributions, that is, it is given by the function $W$ shown in eq. (5.7).

Once again, we reserve more details to appendix A, where we present the complete structure of the product $\tilde{\Phi}\Phi^*$ of effective vertices, together with fully explicit expressions for the four terms contributing to the cross-section (5.15) and a detailed comparison with previous results in the literature.

5.2 The real NLO corrections to quark-antiquark production in DIS

The real NLO corrections to (forward) dihadron (or dijet) production in DIS at small $x$ are obtained from the leading-order trijet results in the previous subsection, by integrating
Figure 10. A particular interference graph contributing to the leading-order cross-section for $q \bar{q} g$ production. The gluon is emitted in the initial state (prior to the scattering) in both the DA and the CCA.

out the kinematics of the parton that is not measured in the final state. In what follows we shall explicitly consider only the case where the unmeasured parton is the gluon (so the "dijet" is a $q \bar{q}$ pair, as at leading order). This is indeed the most interesting case, in that it overlaps with both the JIMWLK and the DGLAP evolutions and it corresponds to the virtual corrections to be discussed later on. The results for the two other cases, where one measures either a quark-gluon pair, or an antiquark-gluon one, can be simply obtained via similar manipulations.

One therefore has

$$\frac{d\sigma}{dk^+_1 d^2 k_1 dk^+_2 d^2 k_2} = \int dk^+_3 d^2 k_3 \frac{d\sigma}{dk^+_1 d^2 k_1 dk^+_2 d^2 k_2 dk^+_3 d^2 k_3},$$

(5.18)

where the subscript "rNLO" stays for real next-to-leading order corrections. The trijet cross-section in the r.h.s. is the sum of direct and interference contributions, as shown in eqs. (5.2) and (5.2). By inspection of these expressions, it is clear that the integral over $k_3^+$ can be trivially performed by using the $\delta$-function for longitudinal momentum conservation (which leaves a step-function $\Theta(q^+ - k_1^+ - k_2^+)$ constraining the final momenta), whereas the integral over $k_3$ yields a factor $(2\pi)^2 \delta(2)(z - \bar{z})$, which allows one to identify the coordinates of the unmeasured gluon in the DA and the CCA, respectively. The result of eq. (5.18) can be succinctly written as

$$\frac{d\sigma}{dk^+_1 d^2 k_1 dk^+_2 d^2 k_2} = (2\pi)^2 \frac{d\sigma}{dk^+_1 d^2 k_1 dk^+_2 d^2 k_2 dk^+_3 d^2 k_3} \bigg|_{k_3^+ = q^+ - k_1^+ - k_2^+, \ z = \bar{z}},$$

(5.19)

where it is understood that the trijet cross-section in the r.h.s. is given by expressions like (5.2), but without the $\delta$-function expressing the conservation of longitudinal momentum. It is understood that the longitudinal momentum fractions of $\vartheta$ and $\xi$ must now be expressed in terms of the respective momenta of the two measured particles, that is (cf. eq. (5.3))

$$\vartheta = \frac{k_1^+}{q^+}, \quad \xi = 1 - \frac{k_1^+ + k_2^+}{q^+},$$

(5.20)

where it is understood that $\vartheta + \xi \leq 1$.

The fact that the gluon is not measured brings some simplifications in the structure of the $S$-matrix in eq. (5.7): when the gluon scatters in both the DA and the CCA, as is the
case for the first term in the r.h.s. of eq. (5.7) (recall figure 9a), the associated Wilson lines compensate each other by unitarity, $U(z)U(z) = 1$, and then the product of quadrupoles appearing in the second line of eq. (5.8) reduces to a product of dipoles (we consider large $N_c$, for simplicity):}

$$S_{qqgqg}(x, y, z, \overline{x}, \overline{y}, \overline{z} = z) \simeq S(x, \overline{x})S(y, \overline{y}). \tag{5.21}$$

But for the other terms in eq. (5.7), the identification $z = \overline{z}$ brings no special simplification, because the gluon is not anymore interacting on both sides of the cut. Altogether, eq. (5.10) gets replaced by

$$W(x, y, z, \overline{x}, \overline{y}, \overline{z} = z) \simeq S(x, \overline{x})S(y, \overline{y}) - S(x, z)S(z, y) - S(z, \overline{x})S(y, \overline{z}) + 1. \tag{5.22}$$

6 The soft gluon limit: recovering the B-JIMWLK evolution

In this section, we will show that in the limit where the gluon emission is soft ($\xi \to 0$), our previous results for the real NLO corrections reproduce the expected part of the B-JIMWLK evolution of the leading-order cross-section (3.17). By the “expected part” we mean those terms in the B-JIMWLK equation for the quadrupole $Q(x, y, \overline{y}, \overline{x})$ in which the soft gluon is emitted by the leg at $x$ or at $y$ (i.e. in the DA) and is reabsorbed by the leg at $\overline{x}$ or at $\overline{y}$ (in the CCA). All the other pieces of the evolution equation for the quadrupole, as well as the complete evolution equations for the dipole $S$-matrices $S(x, y)$ and $S(y, x)$ which enter the LO colour structure in eq. (3.19) should be generated by the virtual NLO corrections to the dijet cross-section.

For more generality, we shall start with the case where the soft gluon is measured as well, that is, with the LO cross-section for producing a $q\bar{q}g$ triplet, but such that $\xi \to 0$. In this slightly more general case, we will recognise the action of the “production” version of the JIMWLK Hamiltonian [109–111]. This is a generalised version of the JIMWLK Hamiltonian which, when acting on the cross-section for particle production in dilute-dense ($pA$ or $eA$) collisions, leads to the emission of an additional, soft, gluon, which is measured in the final state. The (relevant part of the) standard B-JIMWLK equation for the quadrupole [104–106] will be eventually obtained after identifying $z = \overline{z}$.

The soft gluon limit is obtained from our previous result by keeping only the leading non-trivial terms in the limit $\xi \to 0$. One then encounters several types of simplifications.

First one can neglect the recoil of the emitter (quark or antiquark) at the respective emission vertex, meaning that its transverse coordinate is not modified by the soft emission.
In practice this means that we can neglect the difference between “primed” and “unprimed” coordinates: \( y' \simeq y \) for an emission by the antiquark (recall figure 1) and similarly \( x' \simeq x \) for an emission by the quark (cf. figure 8). This also implies that the transverse size of the \( q\bar{q} \) is not affected by a soft gluon emission; indeed, when \( \xi \to 0 \), eqs. (4.14), (4.18) and (4.30) imply \( R \simeq \tilde{R} \simeq |x - y| \), whereas \( D^2 \simeq \vartheta(1 - \vartheta)(x - y)^2 \). From now on, in this section we shall use the notation \( R \equiv |x - y| \) (and similarly \( \tilde{R} \equiv |x - \tilde{y}| \) in the CCA). One also has \( \tilde{Q}^2 \simeq \bar{Q}^2 = \vartheta(1 - \vartheta)Q^2 \).

Furthermore, one can take the limit \( \xi \to 0 \) within the emission kernels (5.4) and (5.16), which entails important simplifications — both in the effective vertices and in the energy denominators. In particular, the contributions of the instantaneous graphs to the effective vertices vanish in this limit, cf. eqs. (4.29) and (4.35), and the same is true for the pieces proportional to \( \varepsilon^{ij}\varepsilon^{mn} \) in the squared vertices (5.5) and (5.17) (indeed, all these contributions are suppressed by additional factors of \( \xi \)). If one combines the respective limits of the emission kernels with the explicit factors involving \( \xi \) and \( \vartheta \) in equations like (5.2) and (5.15), one finds that the dependence upon the longitudinal fractions \( \xi \) and \( \vartheta \) becomes the same for all the 3 contributions to the trijet cross-section — the direct contribution of the antiquark (5.2), the similar one due to the quark, and the interference terms (5.15) —, and it is such that the photon decay vertex factorises out from the gluon emission:

\[
\frac{\vartheta^2}{\xi} K_{1}^{imjn} \simeq \frac{(1 - \vartheta)^2}{\xi} K_{1}^{imjn} \simeq \frac{\vartheta(1 - \vartheta)}{\xi} K_{2}^{imjn} \\
\simeq 16\delta^{ij}\delta^{mn} \vartheta(1 - \vartheta) \frac{k_{1}(Q\bar{R})k_{1}(\tilde{Q}\tilde{R})}{\xi RR},
\]

(6.1)

So, the only difference between these three channels which subsists in the soft gluon limit refers the gluon emission vertex, which involves the appropriate transverse separation, \( X = x - z \) or \( Y = y - z \), between the gluon and its emitter.

The “degeneracy” in eq. (6.1) has yet another important consequence: it implies that the four terms contributing to any of the three channels under consideration (i.e. the four terms within the square brackets in eq. (5.2), or those within the square brackets in eq. (5.15)) are now multiplied by exactly the same emission kernel. Consider e.g. eq. (5.2), which we recall represents direct emissions by the antiquark: after the simplifications in eq. (6.1), the operation \( (y, z \to y') \) now reduces to \( z \to y \) (since \( y \simeq y' \) anyway) and it has no effect on the kernel, but only on the S-matrix structure exhibited in eq. (5.7) (and similarly for the other operation \( (\tilde{y}, \tilde{z} \to \tilde{y}') \)). Accordingly, the colour structures associated with the four terms in eq. (5.2) are simply added to each other and many of the individual S-matrices (12 over a total of 16) cancel in their sum. The 4 surviving terms correspond to the 4 graphs shown in figure 11 for the case of eq. (5.2). The 12 missing terms, which in fact correspond to only 5 distinct topologies (due to our rearrangement of the three amplitude graphs in figure 1), are those where the photon has crossed the shockwave prior to its decay, in either the DA, or the CCA, or both. That is, these are the contributions to the cross-section which involve the graph in figure 1.c on at least one side of the cut (two such diagrams are shown in figure 12).
Figure 11. The four diagrams which give the dominant contribution to the piece of the cross-section in (5.2) (direct emission by the antiquark) in the limit where the emitted gluon is soft ($\xi \ll 1$).

Figure 12. Two diagrams which contribute to the $q\bar{q}g$ cross-section (5.2) in general, but do not survive in the soft limit $\xi \ll 1$. Altogether there are five such graphs.

Physically, these simplifications can be understood as follows: the soft gluon with longitudinal momentum $p^+ = \xi q^+$ and transverse momentum $p$ has a short formation time $\tau_p = 2\xi q^+/p^2$, hence in order to affect the scattering (and the ensuing particle production), it must occur relatively close to the shockwave, with a distance $\Delta x^+ \sim \tau_p$ away from it. The diagrams shown in figure 12 are suppressed in this limit since the photon decay must occur even closer to the shockwave (at least, on one side of the cut), hence the longitudinal phase-space for such a diagram is considerably reduced compared to that available to those in figure 11, where the $x^+$-position of the photon vertex is unconstrained.

This physical interpretation becomes transparent by inspection of the energy denominators corresponding to the three graphs in figure 1. For the first 2 graphs, in figure 1.a and 1.b respectively, there is only one energy denominator involving the gluon. In the soft limit $\xi \ll 1$, the respective energy difference is dominated by the gluon light-cone energy...
\[ p^- = p^2/2ξq^+ \text{ (recall eqs. (4.3) and (4.4)):} \]

\[ E_{q\bar{q}g} - E_{\bar{q}g} \simeq E_{q\bar{q}g} - E_\gamma \simeq \frac{p^2}{2q^+ξ}, \]

which shows that the longitudinal phase-space for the soft gluon emission is of order \( τ_p \sim ξ \), as anticipated. However, the denominator corresponding to the graph in figure 1c involves the product \( (E_{q\bar{q}} - E_{q\bar{q}g})(E_\gamma - E_{q\bar{q}g}) \) of two large energy differences, hence that terms scales like \( τ_p^2 \sim ξ^2 \) and is strongly suppressed as \( ξ \to 0 \). By combining the amplitudes in figure 1.a and 1.b with similar graphs in the CCA, one generates the four cross-section graphs depicted in figure 11.

After exploiting all the aforementioned simplifications and adding together the direct and interference contributions, the soft gluon limit of the trijet cross-section is obtained as

\[ \frac{dσ^{γA→q\bar{q}g}}{dk_1^+ dk_2^+ dk_3^+ dk_4^+} \bigg|_{ξ→0} = \frac{2α_{em}N_c}{(2π)^6q^+} \left( q^2 + (1 - q)^2 \right) \left( \sum e^2_j \right) \delta(q^+ - k_1^+ - k_2^+) \]

\[ \times \int_{x,y,z} e^{-ik_1(ξ-ξ)} e^{-ik_2(y-ξ)} R \frac{R}{RR} Q^2 K_1(QR)K_1(Q\bar{R}) \]

\[ \times \frac{4}{(2π)^4} ξq^+ \int_ξ \Delta_{real} Q(x,y,z,ξ) Q(z,y,ξ) \]

where we have also ignored the soft momentum \( k_3^+ \) in the \( δ \)-function for longitudinal momentum conservation. In the first two lines of this formula, we have factorised the would-be leading-order cross-section (3.17), but without the respective S-matrix structure (3.19). The third line encodes the information about the soft gluon emission: the emission kernels in transverse coordinate space and the colour structures corresponding to all possible topologies. Specifically, one finds (for any \( N_c \))

\[ \Delta_{real} Q(x,y,z,ξ) \equiv \left[ \frac{X \cdot X}{X^2X^2} + \frac{Y \cdot Y}{Y^2Y^2} - \frac{X \cdot Y}{X^2Y^2} - \frac{Y \cdot X}{Y^2X^2} \right] Q(x,z,ξ) Q(z,y,ξ) \]

\[ + \left[ \frac{X \cdot X}{X^2X^2} + \frac{Y \cdot Y}{Y^2Y^2} \right] Q(x,y,ξ) \]

\[ - \left[ \frac{X \cdot Y}{X^2Y^2} + \frac{Y \cdot X}{Y^2X^2} \right] S(x,y)S(ξ,ξ) \]

\[ - \left[ \frac{X \cdot X}{X^2X^2} - \frac{X \cdot Y}{X^2Y^2} \right] Q(x,y,ξ) S(ξ,ξ) \]

\[ - \left[ \frac{Y \cdot Y}{Y^2Y^2} - \frac{X \cdot Y}{Y^2X^2} \right] Q(x,y,ξ) S(ξ,ξ) \]

\[ - \left[ \frac{X \cdot X}{X^2X^2} - \frac{Y \cdot X}{Y^2X^2} \right] S(x,z) Q(z,y,ξ) \]

\[ - \left[ \frac{Y \cdot Y}{Y^2Y^2} - \frac{X \cdot Y}{Y^2X^2} \right] S(x,z) Q(z,y,ξ) \]

The diagrammatic interpretation of the various terms in this lengthy expression can be easily traced back by inspection of their respective emission kernel and S-matrix structure. For instance the 4 terms proportional to \( \left( Y \cdot Υ \right)/\left( Y^2Y^2 \right) \) correspond to the 4
graphs in figure 11. The associated colour structures are recognised as the first terms in eqs. (5.10), (5.11), (5.12), and (5.13), respectively (with \( y = y' \), of course).

The expression in the third line of eq. (6.3) represents indeed the action of the “production” JIMWLK Hamiltonian on the original quadrupole \( Q(x, y, \mathbf{p}, \mathbf{q}) \) from eq. (3.19). This can be verified e.g. by comparing eq. (6.4) with eq. (A.5) in ref. [11]. In making such a comparison, it is useful to recall that at large \( N_c \) one can approximate \( 2\alpha_s C_F \simeq \alpha_s N_c \equiv \pi \bar{\alpha}_s \).

Furthermore, the standard version of the B-JIMWLK equation for the quadrupole [104–106] is easily recovered after integrating out both the transverse momentum \( k_3 \) (which leads to the identification \( z = z' \)) and the longitudinal momentum \( k_3^+ = \xi q^+ \) of the soft gluon. The result can be written as an evolution equation with \( \ln(1/\xi) \), namely,

\[
\xi \frac{\partial Q(x, y, \mathbf{p}, \mathbf{q})}{\partial \xi} \Bigg|_{\text{real}} = \frac{\bar{\alpha}_s}{2\pi} \int_z \Delta_{\text{real}} Q(x, y, \mathbf{p}, \mathbf{q}; z = z'),
\]

(6.5)

whose r.h.s. is easily checked to precisely coincide with all the relevant terms in e.g. eq. (2.17) from ref. [105] (the terms which are missing in this comparison will be generated by the virtual NLO corrections to dijet production, as we shall demonstrate in [108]).

At this point, a careful reader may observe that the above integral over \( z \) develops logarithmic divergences, coming from the limit \( |z| \to \infty \), that is, from gluon emissions which occur arbitrarily far away from the \( q\bar{q} \) pair. (In momentum space, this corresponds to \( k_{3\perp} \equiv |k_3| \to 0 \).) To explicitly see such a divergence, consider any of the emission kernels in eq. (6.4), e.g.

\[
\frac{\mathbf{Y} \cdot \mathbf{Y}}{\mathbf{Y}^2 \mathbf{Y}^2} = \frac{(y - z) \cdot (\mathbf{y} - \mathbf{z})}{(y - z)^2 (\mathbf{y} - \mathbf{z})^2} = \frac{1}{2} \left[ \frac{1}{(y - z)^2} + \frac{1}{(\mathbf{y} - \mathbf{z})^2} - \frac{(y - \mathbf{y})^2}{(y - z)^2 (\mathbf{y} - \mathbf{z})^2} \right].
\]

(6.6)

When \( z \equiv |z| \to \infty \), the first two terms (the “tadpoles”) decay like \( 1/z^2 \) and give rise to logarithmic divergences within an integral like that in eq. (6.5), while the last term (the “dipole kernel”) decays faster, like \( 1/z^4 \), and gives a convergent contribution. Clearly, when taken individually, all the 16 terms in the r.h.s. of eq. (6.4) develop such divergences, generally referred to as “soft and infrared” (or “soft and collinear”) in the literature. Indeed, they appear when both the longitudinal momentum of the emitted gluon and its transverse momentum are arbitrarily small: \( \xi \to 0 \) and \( k_{3\perp} \to 0 \). Such divergences are unphysical (e.g. they cannot be absorbed in some kind of quantum evolution) and must cancel in the final result for the cross-section. And they actually do so, as we shall explain in what follows. Interestingly, for the problem at hand, there are two mechanisms which contribute to these cancellations: besides the expected “real-virtual” cancellations, there is also the effect of gauge symmetry, namely, the fact that the \( q\bar{q} \) pair emitting the gluon is a colour singlet (a dipole).

For instance, consider the contribution in the first line of eq. (6.4), in which the scattering operator reduces to \( S(x, \mathbf{x})S(y, \mathbf{y}) \) after using \( z = z' \). This contribution is the sum of the four graphs in which the gluon is emitted prior to the scattering with the shockwave in both the DA and the CCA: two direct-emission graphs (like the first diagram in figure 11) and two interference ones. Using decompositions similar to (6.6) for all the emission kernels within the square brackets, one easily sees that the tadpoles cancel between the direct...
emissions and the interference terms. The ensuing kernel can therefore be written as a linear combination of dipole kernels alone (with $\mathcal{M}_{xyz} \equiv \frac{(x-y)^2}{(x-z)^2(y-z)^2}$):

$$\frac{X \cdot \bar{X}}{X^2\bar{X}^2} + \frac{Y \cdot \bar{Y}}{Y^2\bar{Y}^2} - \frac{X \cdot \bar{Y}}{X^2\bar{Y}^2} - \frac{Y \cdot \bar{X}}{Y^2\bar{X}^2} = -\frac{1}{2} [\mathcal{M}_{x\bar{x}z} + \mathcal{M}_{y\bar{y}z} - \mathcal{M}_{x\bar{y}z} - \mathcal{M}_{y\bar{x}z}].$$

(6.7)

This cancellation is easy to understand: the $q\bar{q}$ pair is a colour dipole, hence the gluon field created by the pair at very large $z$ (much larger than the transverse extent of the dipole) decays much faster than the individual Coulomb fields of the two sources.

Similar cancellations occur within all the contributions in the r.h.s. of eq. (6.4), except for two: those where the gluon is emitted after the scattering with the shockwave in both the DA and the CCA. In that case, the direct emissions and the interference terms are weighted by different scattering operators — $Q(x, y, y, x)$ and, respectively, $S(x, y)S(y, x)$ —, so the infrared divergences do not cancel among the real NLO terms alone. To see the cancellations, one must include the corresponding virtual corrections [72–74, 103]. A fully NLO calculation of the virtual corrections goes beyond the scope of this paper, yet it is not truly needed for the present purposes: we need these corrections only in the soft gluon limit $\xi \ll 1$, where they are much simpler and also well known, as their are a part of the B-JIMWLK evolution. Using the experience with the latter (see especially [104–106] for the evolution of the quadrupole), we shall qualitatively explain here the relevant real-virtual cancellations.

Consider first the direct emissions, where the gluon is emitted and reabsorbed by the same quark in both the DA and the CCA, and in the final state (i.e. after the scattering). In figure 13 we show the relevant diagrams for the case of an emission by the antiquark. The first graph is the “real” contribution and is responsible for the term proportional to $[(Y \cdot \bar{Y})/(Y^2\bar{Y}^2)]Q(x, y, y, x)$ in eq. (6.4). The two other graphs exhibit the anti-quark self-energies, in the DA and the CCA respectively. When $\xi \ll 1$, each such a self-energy graph is simply a tadpole and has a symmetry factor $-1/2$ relative to the real graph. Hence the self-energy graphs simply cancel the tadpoles in the decomposition (6.6) of the kernel for real emissions. Similar cancellations occur when the emitter is the quark. Accordingly, the emission kernel multiplying the quadrupole $Q(x, y, \bar{y}, \bar{x})$ gets replaced by

$$\frac{X \cdot \bar{X}}{X^2\bar{X}^2} + \frac{Y \cdot \bar{Y}}{Y^2\bar{Y}^2} \rightarrow -\frac{1}{2} [\mathcal{M}_{x\bar{x}z} + \mathcal{M}_{y\bar{y}z}],$$

(6.8)

which gives a finite result when integrated over $z$. 

Figure 13. Three diagrams — one real and two virtual — for which the double, soft and infrared, divergences cancel in their sum. All diagrams involve a gluon emission in the final state. The first graph is a direct emission by the antiquark and the two others are self-energy corrections.
Figure 14. Four interference diagrams — two real and two virtual — for which the double, soft and infrared, divergences cancel in their sum. All diagrams involve a gluon emission in the final state.

The respective interference graphs are shown in figure 14: two “real” graphs and two “virtual” ones. (Each of the virtual graphs shown in figure 14 is truly the sum of two diagrams in light-cone perturbation theory, corresponding to the two possible time-orderings of the gluon endpoints. Each of these diagrams has a symmetry factor $-1/2$.) When $\xi \ll 1$, the virtual graphs involve the same kind of emission kernel as the real ones, except for a change of sign. Hence, after including the virtual graphs, the emission kernel multiplying $S(x, y)S(y, x)$ in the r.h.s. of eq. (6.4) gets modified as follows:

$$\left[-\frac{X \cdot Y + Y \cdot X}{X^2Y^2} + \frac{Y \cdot X}{Y^2X^2}\right] \rightarrow -\left[\frac{X \cdot Y}{X^2Y^2} + \frac{Y \cdot X}{Y^2X^2} - \frac{X \cdot Y}{X^2Y^2} - \frac{X \cdot Y}{X^2Y^2}\right]$$

$$= \frac{1}{2} [M_{xyz} + M_{yzx} - M_{xyz} - M_{xzy}]. \quad (6.9)$$

The final result is expressed in terms of dipole kernels alone, thus yielding a finite integral over $z$.

Of course there are additional virtual corrections besides those shown in figures 13 and 14, but they do not matter for the cancellation of soft and infrared divergences. The remaining graphs — quark self-energies and gluon exchanges between the quark and the antiquark, where the gluon is either emitted and reabsorbed in the initial state (prior to scattering), or it is crossing the shockwave — mutually cancel their individual divergences, due to the colour neutrality of the $q \bar{q}$ pair. The full evolution equation for the quadrupole, including all real and virtual corrections, can be found in [104–106].
The collinear limit: recovering the DGLAP evolution

An another interesting limit of our previous results for the “real” NLO corrections to the dihadron production is the limit where the gluon is emitted in the final state and it is nearly collinear with its emitter. In this limit, the cross-section is expected to develop a logarithmic divergence from the integral over the transverse momentum of the gluon relative to its emitter. Physically, this divergence signals the fact that the gluon is emitted longtime after the interaction and should be viewed as a part of the wavefunction of the measured parton. For this interpretation to be correct, the collinear divergence should factorise from the “hard process” — here, the photon decay into a $q\bar{q}$ pair and the scattering between this pair and the nuclear target — and be recognised as the first step in the DGLAP evolution of the fragmentation function of the parent parton. In this section, we shall verify that this is indeed the case for the gluon emission by the antiquark.

Since our results for the cross-sections are expressed in the transverse coordinate representation, it is convenient to also formulate the collinear limit in this representation. For definiteness, let us consider the case where the gluon is emitted by the antiquark. In coordinate space, the collinear limit for this emission is the limit where the transverse separation $Y = y - z$ between the daughter partons (the antiquark and the gluon) is much larger than the separation $R = x - y'$ between the $q\bar{q}$ pair produced by the photon decay: $|Y| \gg |R|$. To see this, it is convenient to change the transverse momentum variables for the final $\bar{q}g$ pair from $k_2$ and $k_3$, to $K$ and $P$, defined as

$$K = k_2 + k_3, \quad P = \frac{(1 - \vartheta - \xi)k_3 - \xi k_2}{1 - \vartheta} \quad \Rightarrow \quad e^{-i k_2 \cdot y - i k_3 \cdot z} = e^{-i K \cdot y' + i P \cdot Y}. \quad (7.1)$$

Clearly, $K$ is the total transverse momentum of the $\bar{q}g$ pair and is conjugated to the transverse position $y'$ of its center-of-energy (cf. eq. (4.12)), whereas $P$ is the relative transverse momentum of that pair and is conjugated to the transverse separation $Y$. The collinear limit corresponds to $P \to 0$. (Indeed, when $P = 0$, the gluon takes a transverse momentum fraction equal to its longitudinal splitting fraction $\xi/(1 - \vartheta)$, meaning it is collinear with the antiquark; recall the discussion after eq. (4.3).) The structure of the phases in eq. (7.1) confirms that the collinear limit $P \to 0$ corresponds to large values of $Y = |Y|$, as anticipated. In turn, this implies that the gluon must be emitted after the scattering with the nuclear shockwave, in order to escape the virtuality barrier.

Indeed, as discussed in relation with eq. (4.19), the amplitude for a gluon emission occurring prior to the collision — as described by the LCWF (4.21) (or, equivalently, by the first term in eq. (4.25)) — is proportional to $K_1(QD)$, which implies $Y^2 \lesssim 1/(\xi Q^2)$. On the other hand, there is no such a restriction for a gluon emitted after the collision, cf. figure 1.b — the respective LCWF eq. (4.21) (or the second term in eq. (4.25)) is instead proportional to $K_1(QR)$, which is independent of $Y$. This argument also shows that the instantaneous piece of the effective vertex (4.29) does not contribute to the collinear limit. Clearly, a similar argument holds for gluon emissions by the quark.

From the experience with the DGLAP evolution, we expect the interference terms (cf. section 5.1.2) not to contribute in the collinear limit. The respective argument is
quite obvious in momentum space, since the gluon cannot be simultaneously collinear to both the quark and the antiquark (except for special configurations of zero measure). It is interesting to see how this argument works in coordinate space. Consider a final-state gluon which is emitted by the antiquark in the direct amplitude (DA) and by the quark in the complex-conjugate amplitude (CCA). Once again, it is convenient to replace $k_2$ and $k_3$ by the variables $K$ and $P$ defined in eq. (7.1). The variable $P$ is conjugated to the transverse separation between the gluon and the final antiquark, that is, to $Y = y - z$ in the DA and to $Y = \bar{y} - \bar{z}$ in the CCA. The collinear limit for an emission by the antiquark corresponds to $P \to 0$, which implies that $Y$ and $\bar{Y}$ are simultaneously large. In the DA, where the transverse position $y'$ of the intermediate antiquark is different from the final one $y$, the limit $Y \to \infty$ brings no constraint on the transverse size $R = |x - y'|$ of the $q\bar{q}$ pair in the intermediate state\(^{11}\) (prior to the gluon emission). This size will therefore take its natural value $R \sim 1/\bar{Q}$, as imposed by the photon decay (via the modified Bessel function $K_1(\bar{Q}R)$ in eq. (4.23)). But in the CCA, where the gluon is emitted by the quark, one has $y' = \bar{y}$, while the transverse coordinate $x'$ of the intermediate quark coincides with the center of energy of the final quark-gluon pair (cf. eq. (4.30)). So, the distance $\bar{R} \equiv |x' - \bar{y}|$ is typically commensurable with $\bar{Y}$ and becomes arbitrarily large when $\bar{Y} \to \infty$. Accordingly, this interference term is strongly damped by the Bessel function $K_1(\bar{Q}\bar{R})$ (recall the discussion after eq. (4.33)).

Note that the recoil of the emitter was essential for the validity of the above argument: the fact that $x'$ is different from $x$, and that their difference $x' - x$ has a component proportional to the transverse coordinate $z$ of the gluon (cf. eq. (4.30)), ensures that $\bar{R}$ becomes large simultaneously with $\bar{Y}$. Hence, this argument ceases to apply in the soft gluon limit $\xi \ll 1$, where the recoil is negligible and one can set $x' = x$ and $y' = \bar{y}$ in both the DA and the CCA. Hence, it looks like the interference terms become important in the soft ($\xi \to 0$) and collinear ($P \to 0$) limit. Note however that $\xi \to 0$ also implies $P \to k_3$, cf. (7.1), so the soft & collinear limit is tantamount to the soft & infrared limit discussed towards the end of section 6. From that discussion, we know that the respective contributions cancel between real and virtual corrections, cf. figure 14, so this double limit needs not be considered anymore.

To summarise, the dominant contributions to the tri-parton cross-section in the collinear limit come from direct gluon emissions (by either the quark, or the antiquark) in the final state (i.e. after the collision with the SW). E.g., when the emitter is the antiquark, these contributions correspond to the 4 graphs shown in figure 15. Their sum is contained in the last term, denoted as $(y, z \to y') \& (\bar{y}, \bar{z} \to \bar{y})$, in eq. (5.2). The associated colour structure, shown in eq. (5.13), is the same as at leading-order, cf. eq. (3.19), which is a necessary condition for factorisation.

To complete the proof of factorisation in the collinear limit, one must demonstrate that the tensorial kernel $K_1^{imjn}$ multiplying the fourth term in eq. (5.2) factorises in this limit.

\(^{11}\)The coordinate $y'$ of the intermediate antiquark is the same as the center of energy of the final quark-gluon pair, cf. eq. (4.12). So $y'$ can be close to $z$ despite the fact that the final antiquark and the gluon are very far away from each other: in general, they are also far away from the quark and from the intermediate antiquark.
To start with, let us check that the last piece, proportional to $\varepsilon^{ij}\varepsilon^{mn}$, in the product (5.5) of the effective vertices does not contribute to this limit. To that aim, it is convenient to change the integration variables in eq. (5.2) from $x, y, z$ to $x, y', Y$, so that the Fourier phases take the form in eq. (7.1) (and similarly in the CCA). This is convenient since, in these new variables and for final-state gluon emissions, the Fourier transforms over $Y$ and $Y$ factorise from the other integrations\footnote{This is perhaps most obvious by inspection of the respective amplitude in eq. (4.23): the variable $Y$ enters this amplitude only via the gluon emission kernel $\propto Y^n/Y^2$.} and are easily computed as

$$
\int_{Y,Y'} e^{iP\cdot(Y-Y')} Y^m Y^n / Y^2 = (2\pi)^2 \frac{P^m P^n}{P^2}. \tag{7.2}
$$

When this structure is multiplied with the squared vertex (5.5), the last term in the latter, which is antisymmetric in $m$ and $n$, will clearly yield a vanishing contribution. Using the remaining part of eq. (5.5), which is symmetric in $m$ and $n$, together with the fact that $QD \rightarrow \bar{Q}R$ for final state emissions, one finds

$$
\frac{\partial^2}{\xi K_{1 imjn}} \bigg|_{4th\, term} = 32\delta^{ij}\delta^{mn} \partial P_{\gamma \rightarrow q}(\theta) P_{q \rightarrow g} \left( \frac{\xi}{1-\theta} \right) \frac{K_1(\bar{Q}R)K_1(\bar{Q}R)}{R^2}, \tag{7.3}
$$

where

$$
P_{\gamma \rightarrow q}(z) = \frac{z^2 + (1-z)^2}{2}, \quad P_{q \rightarrow g}(z) = \frac{1 + (1-z)^2}{2z}, \tag{7.4}
$$

are the DGLAP splitting functions for the processes $\gamma \rightarrow q\bar{q}$ and $q \rightarrow qg$, respectively.

The factorisation of the collinear gluon emission from the hard process is now manifest: first, the virtual photon fluctuates into a $q\bar{q}$ pair with transverse coordinates $x$ and $y'$, and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure15.png}
\caption{The four graphs surviving in the collinear limit for the gluon emission by the antiquark. These graphs are void of final-state interactions.}
\end{figure}
respectively. Then this pair scatters off the nuclear SW, with the S-matrix shown in eq. (5.13). Finally, the antiquark emits a gluon with splitting fraction \( z = \xi / (1 - \vartheta) \).

Putting things together, one finds (after also integrating over the phase-space of the unmeasured gluon, which removes the \( \delta \)-function for longitudinal momentum conservation and identifies \( z = \bar{z} \))

\[
\frac{d\sigma_{NLO}^{A \to q\bar{q}+X}}{dk_1^+ d^2k_1 dk_2^+ d^2k_2} \Bigl|_{(a), \text{coll}} \approx \frac{4\alpha_{em} N_c}{(2\pi)^6 (q^+)^2 (1 - \vartheta)} P_{\gamma \to q}(\vartheta) \left( \sum e_j^2 \right) \\
\times \int_{\bar{z}, y, x, y'} e^{-ik_1(x-\bar{x})-ik_2(y-\bar{y})} \frac{R \cdot \bar{R}}{R \bar{R}} \bar{Q}^2 K_1(\bar{Q} R) K_1(\bar{Q} \bar{R}) \\
\times \left[ Q(x, y', \bar{y}', \bar{x}) - S(x, y') - S(\bar{y}', \bar{x}) + 1 \right] \\
\times \frac{4\alpha_s C_F}{(2\pi)^2} P_{q \to g}(z_2) \int_z (y' - z) \cdot (\bar{y} - z) \\
\times \left[ (y' - z)^2 (\bar{y} - z)^2 \right],
\]

where the longitudinal fractions \( \vartheta \) and \( \xi \) are given by eq. (5.20).

At this level, it is convenient to replace the integration variables \( y \) and \( \bar{y} \) by \( y' \) and \( \bar{y}' \), respectively (since the latter are the actual arguments of both the photon decay probability (e.g. \( R = x - y' \)) and the S-matrix structure), and also to use \( \vartheta \) and \( z_2 \) as independent longitudinal momentum fractions, instead of \( \vartheta \) and \( \xi \). Here, \( z_2 \) is the splitting fraction of the daughter antiquark at that \( \bar{q} \to \bar{q}g \) vertex:

\[
z_2 \equiv 1 - \frac{\xi}{1 - \vartheta} = \frac{k_2^+}{q^+ - k_1^+} \implies y' = z_2 y + (1 - z_2) z. \tag{7.6}
\]

One thus finds

\[
\frac{d\sigma_{NLO}^{A \to q\bar{q}+X}}{dk_1^+ d^2k_1 dk_2^+ d^2k_2} \Bigl|_{(a), \text{coll}} \approx \frac{4\alpha_{em} N_c}{(2\pi)^6 (z_2 q^+)^2 (1 - \vartheta)} P_{\gamma \to q}(\vartheta) \left( \sum e_j^2 \right) \\
\times \int_{\bar{z}, y, x, y'} e^{-ik_1(x-\bar{x})-ik_2(y-\bar{y})/z_2} \frac{R \cdot \bar{R}}{R \bar{R}} \bar{Q}^2 K_1(\bar{Q} R) K_1(\bar{Q} \bar{R}) \\
\times \left[ Q(x, y', \bar{y}', \bar{x}) - S(x, y') - S(\bar{y}', \bar{x}) + 1 \right] \\
\times \frac{4\alpha_s C_F}{(2\pi)^2} P_{q \to g}(z_2) \int_z (y' - z) \cdot (\bar{y} - z) \\
\times \left[ (y' - z)^2 (\bar{y} - z)^2 \right],
\]

where we have also used \( P_{q \to g}(1 - z_2) = P_{q \to g}(z_2) \). Via these manipulations, the collinear divergence has been isolated in the last term and also factorised from the leading-order cross-section for the production of a pair of jets (cf. (3.17)): a quark with 3-momentum \( k_1 = (k_1^+, k_1) \) and an antiquark with 3-momentum \( p = (p^+, p) \), where \( p^+ = k_2^+ / z_2 \) and \( p = k_2 / z_2 \).

The collinear divergence refers to the large-\( |z| \) limit of the last integral in eq. (7.7). To exhibit this singularity, it is convenient to make an excursion through momentum space and introduce a low-momentum cutoff \( \Lambda \) on the transverse momentum of the unmeasured gluon, meaning an upper cutoff \( \sim 1/\Lambda \) on the transverse separations \( |y' - z| \) and \( |\bar{y}' - z| \).
One thus finds
\[ \int \frac{(\mathbf{y}' - \mathbf{z}) \cdot (\mathbf{y}' - \mathbf{z})}{(\mathbf{y}' - \mathbf{z})^2} = \int \frac{d^2 \mathbf{p}}{p^2} e^{-i \mathbf{q} \cdot (\mathbf{y}' - \mathbf{y})} \Theta(\mathbf{p} - \Lambda) = \pi \ln \frac{1}{r^2 \Lambda^2} + \pi \ln \frac{1}{r^2 \mu^2} + \pi \ln \frac{\mu^2}{\Lambda^2}, \]
\[(7.8)\]
where \( r \equiv |\mathbf{y}' - \mathbf{y}| \) and it is understood that \( \Lambda \ll 1/r \) (since one is eventually interested in the limit \( \Lambda \to 0 \)). In the last step, we have split the result between a would-be divergent piece which is independent of \( r \) and which will be shortly absorbed into the DGLAP evolution and an \( r \)-dependent piece which is a part of the NLO correction to the hard factor. The factorisation scale \( \mu \) is \textit{a priori} arbitrary (within the range \( \Lambda < \mu < 1/r \)), but ideally it should be chosen of order \( 1/r \sim |k_2^2|/z^2 \), to minimise the NLO corrections. In practice though, it is more convenient to use \( \mu^2 = \max(k_1^2, k_2^2) \), so that the same factorisation scale also applies to a collinear emission by the quark.

The would-be singular contribution to the last line in eq. (7.7) describes the evolution of the final antiquark state via the emission of a quasi-collinear gluon. This piece must be subtracted from the NLO correction and absorbed into the DGLAP evolution of the (anti)quark fragmentation function into hadrons:
\[ D_{h/q}(\zeta, \mu^2) = \int_1^\zeta \frac{dz}{z^3} \left[ \delta(1 - z) + \frac{\alpha_s C_F}{\pi} \frac{1}{z} P_{q\to q}(z) \ln \frac{\mu^2}{\Lambda^2} \right] D_{h/q}^{(0)} \left( \frac{\zeta}{z} \right), \]
\[(7.9)\]
where \( D_{h/q}^{(0)}(\zeta) \) is the (non-perturbative) fragmentation function of a bare quark. Hence, after adding the collinearly-divergent piece of eq. (7.7) to the LO dihadron cross-section, one finds an approximate version of eq. (3.23) where the (first step in the) DGLAP evolution is included only on the final antiquark state:
\[ \frac{d\sigma_{\gamma A \to h_1h_2+X}}{dk^1_1 d^2k_1 dk_2^2 d^2k_2^{(0)}} \approx \int \frac{d\zeta_1}{\zeta_1^3} \int \frac{d\zeta_2}{\zeta_2^3} \frac{d\sigma_{\gamma A \to q\bar{q}+X}}{dp_1^2 d^2p_1 dp_2^2 d^2p_2^{(0)}} \bigg|_{p_i^{(0)} = k_i/\zeta_i} \frac{D_{h_1/q}^{(0)}(\zeta_1) D_{h_2/q}(\zeta_2, \mu^2)}{D_{h/q}(\zeta_1, \mu^2)}. \]
\[(7.10)\]
In particular, the \( \delta \)-function for longitudinal momentum conservation inherent in the partonic cross-section, cf. eq. (3.17), reproduces the normalisation factor in eq. (7.7), via the following identity:\[13\]
\[ \int \frac{dz}{z^3} \delta \left( q^+ - k_1^+ - \frac{k_2^+}{z} \right) = \frac{1}{z^2 q^+(1 - \vartheta)}, \]
\[(7.11)\]
with \( \vartheta = k_1^+/q^+ \) and \( z_2 \) as defined in eq. (7.6).

Clearly, the case of a collinear emission by the quark can be similarly treated and leads to the analog of eq. (7.10) in which one step in the DGLAP evolution is included on the final quark state.

\[13\]To make contact with the result in eq. (7.7), which is written at partonic level, one can use \( D_{h/q}^{(0)}(\zeta) \to \delta(1 - \zeta) \).
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A Explicit results for the tri-jet amplitude and cross-section

In presenting our results for the tri-parton ($q\bar{q}g$) light-cone wavefunction (LCWF) in section 4 and later on for the tri-jet cross-section in section 5, we have used compact but rather symbolic notations, which allowed us to encode in a same formula contributions coming from different topologies — i.e. from different time-orderings between the gluon emission and the collision with the shockwave (SW), and also from the regular and the instantaneous vertices. Examples are eq. (4.33) for the $q\bar{q}g$ outgoing state, eq. (5.2) for the tri-jet cross-section corresponding to direct emissions, and eq. (5.15) for the respective contribution of the interference terms. In what follows, we shall rewrite these results in more explicit forms, by renouncing to some of the symbolic notations. This is especially useful in view of comparisons with the corresponding results in the literature [72–74, 76–78]. To keep the final expressions relatively simple, we shall employ the large $N_c$ approximation for the colour structures expressing $S$-matrices; accordingly, the latter are exclusively built with colour dipoles and quadrupoles. To have more streamlined notations, we rewrite the longitudinal momentum fractions as $\vartheta_i = k_i^+/q^+$; the relation with the previous notations is as follows: $\vartheta_1 = \vartheta$, $\vartheta_2 = \xi$, and $\vartheta_3 = 1 - \vartheta - \xi$.

A.1 The tri-jet amplitude

Our results for the tri-parton outgoing state in section 4 are rather explicit in the case where the gluon is emitted by the antiquark, cf. section 4.1, but they are rather compactly written in the case where the gluon is emitted by the quark, cf. eq. (4.33). For more clarity, we shall here rewrite eq. (4.33) as the sum of three explicit contributions, by using the new notations for the longitudinal fractions:

(a) gluon emitted in the initial state (i.e. prior to the scattering with the SW). The corresponding result appears as the first term in eq. (4.33), but without the instantaneous
piece of the effective vertex (this piece will be separately shown). This is rewritten here as
\[
\left| \gamma_T^b(Q, q^+, q = 0) \right|_{\text{qg}}^{(a)} = \frac{e e g q^+}{2(2\pi)^2} \int_{x, y, z} d\vartheta_1 d\vartheta_2 d\vartheta_3 \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \frac{\tilde{R}^j X^n}{X^2} \\
\times \sqrt{\vartheta_1 \vartheta_2 \vartheta_3} \tau^{nm}_{\lambda_1 \lambda_2}(\vartheta_3, \vartheta_1) \varphi^{ij}_{\lambda_1 \lambda_2}(\vartheta_1 + \vartheta_3) \frac{Q K_1(Q D)}{D} \left[ U^{ab}(z)V(x)\epsilon^b V^+(y) - \epsilon^a \right]_{\alpha \beta} \\
\times \left[ g_\alpha^\beta(\vartheta_2 q^+, y) g_\beta^\alpha(\vartheta_3 q^+, z) q_\alpha^\nu(\vartheta_1 q^+, x) \right].
\]

Note that the delta-function \(\delta^{(2)}(w - c)\) originally present in eq. (4.33) has been removed by specialising to the case where the incoming photon has zero transverse momentum, \(q = 0\). We recall that \(\tilde{R}\) and \(X\) are the transverse separations between the intermediate quark and the antiquark and, respectively, between the final quark and the gluon (cf. eq. (4.30))
\[
\tilde{R} = x' - y, \quad X = x - z, \quad x' = \frac{\vartheta_1 x + \vartheta_3 z}{\vartheta_1 + \vartheta_3}.
\]

Furthermore, \(D \equiv D(x, y, z, \vartheta_i)\) is the effective transverse size of the intermediate \(qg\) partonic system (compare to eq. (4.18)),
\[
D^2(x, y, z, \vartheta_i) \equiv \vartheta_1 \vartheta_2 (x - y)^2 + \vartheta_1 \vartheta_3 (x - z)^2 + \vartheta_2 \vartheta_3 (y - z)^2.
\]

The functions \(\varphi^{ij}_{\lambda_1 \lambda_2}(\vartheta)\) and \(\tau^{nm}_{\lambda_1 \lambda_2}(\vartheta_3, \vartheta_1)\) encode the spinorial structure at the vertices for the photon decay and, respectively, for the gluon emission by the aquark (cf. eqs. (3.6) and (4.8)):
\[
\varphi^{ij}_{\lambda_1 \lambda_2}(\vartheta) = \delta_{\lambda_1 \lambda_2} \left[ (2\vartheta - 1)\delta^{ij} + 2i\varepsilon^{ij}\lambda_1 \right],
\]
\[
\tau^{nm}_{\lambda_1 \lambda_2}(\vartheta_3, \vartheta_1) = \delta_{\lambda_1 \lambda_2} \left[ (2\vartheta_1 + \vartheta_3)\delta^{ij} + 2i\vartheta_3 \varepsilon^{ij}\lambda_1 \right].
\]

(b) gluon emitted in the final state (after the scattering). This case corresponds to the term symbolically denoted as \((x, z \rightarrow x')\) in eq. (4.33). We recall that this symbol means that one should replace \(x \rightarrow x'\) and \(z \rightarrow x'\), with \(x'\) from eq. (A.2), in all the terms within the curly brackets in eq. (4.33) — that is, in the effective vertex, in the argument of the first Bessel function, and in the colour structure expressing the \(S\)-matrix. After this replacement,
\[
D^2 \rightarrow \vartheta_2 (\vartheta_1 + \vartheta_3)(y - x')^2 = \vartheta_2 (1 - \vartheta_2)\tilde{R}^2,
\]
so the respective contribution to the outgoing state is found as
\[
\left| \gamma_T^b(Q, q^+, q = 0) \right|_{\text{qg}}^{(b)} = -\frac{e e g q^+}{2(2\pi)^2} \int_{x, y, z} d\vartheta_1 d\vartheta_2 d\vartheta_3 \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \frac{\tilde{R}^j X^n}{X^2} \\
\times \sqrt{\vartheta_1 \vartheta_2 \vartheta_3} \tau^{nm}_{\lambda_1 \lambda_2}(\vartheta_3, \vartheta_1) \varphi^{ij}_{\lambda_1 \lambda_2}(\vartheta_1 + \vartheta_3) \frac{Q K_1(\tilde{Q} \tilde{R})}{\tilde{R}} \left[ \varepsilon^a V(x')V^+(y) - \varepsilon^a \right]_{\alpha \beta} \\
\times \left[ g_\alpha^\beta(\vartheta_2 q^+, y) g_\beta^\alpha(\vartheta_3 q^+, z) q_\alpha^\nu(\vartheta_1 q^+, x) \right].
\]
(c) the contribution of the instantaneous gluon emission by the quark, as encoded in the second piece of the effective vertex in eq. (4.35):

\[
\left| \gamma_{\ell}^{g}(Q, q^+, q = 0) \right|_{qg}^{(c)} = \frac{\alpha_s e_q g}{2(2\pi)^4} \int_{x,y,z} d\vartheta_1 d\vartheta_2 d\vartheta_3 \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \\
\times \frac{\vartheta_1}{\sqrt{\vartheta_3}} \frac{\vartheta_1 \vartheta_3}{\vartheta_1 + \vartheta_3} \left( e^{im - 2ie^{im}\lambda_1} \frac{QK_1(QD)}{D} [U^{ab}(z)V(x)h^{b}(y) - t^a]_{\alpha\beta} \\
\times \delta_{\lambda_1\lambda_2} \bar{\chi}_{\lambda_3}(\vartheta_2 q^+, y) q^n_\alpha(\vartheta_3 q^+, z) q^n_\beta(\vartheta_1 q^+, x) \right). \tag{A.7}
\]

At this level, we can compare our above results to the corresponding ones in the literature — namely to those in ref. [73] which also uses the LCWF formalism, so its results and ours can be compared already of the level of the amplitudes. It is easy to check by inspection that our results in eq. (A.1) and eq. (A.6) are indeed consistent with eqs. (3.24)–(3.25) and, respectively, eqs. (3.26)–(3.27) of ref. [73].

To also compare the instantaneous contributions, we need to combine together our respective results for emissions by the antiquark (cf. eq. (4.28)) and by the quark (cf. eq. (A.7)). Indeed, it is this combination which is shown in eqs. (3.28)–(3.29) of ref. [73]. This sum has the same structure as eq. (A.7), except for the replacement of its results and ours can be compared already of the level of the amplitudes. It is easy to check by inspection that our results in eq. (A.1) and eq. (A.6) are indeed consistent with eqs. (3.24)–(3.25) and, respectively, eqs. (3.26)–(3.27) of ref. [73]. Since our respective results agree already at the level of the amplitudes, they surely lead to equivalent results for the tri-jet cross-section.

A.2 The tri-jet cross-section: direct emissions

For the direct emissions, we consider again the case where the emitter is the antiquark. The corresponding cross-section, as compactly shown in eq. (5.2), can be naturally decomposed into four pieces, corresponding to different time orderings between the vertices for gluon emission and the scattering with the shockwave (SW):

(i) gluon emitted in the initial state (i.e. prior to the scattering with the SW) in both the DA and the CCA:

\[
\frac{d\sigma^{\gamma A\rightarrow qg}_{\ell}(x)}{d\vartheta_1^2 d^2k_1 d\vartheta_2^2 d^2k_2 d\vartheta_3^2 d^2k_3} = \frac{\alpha_s e_q g}{2(2\pi)^4} \left( \sum_{\ell_f} \epsilon_f^2 \right) \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \\
\times \int_{x,y,z} \mathcal{M}_{\ell_f \ell_r \lambda} \mathcal{M}_{\ell_r \ell_f \lambda} \mathcal{M}_{\ell_r \ell_f \lambda} \mathcal{M}_{\ell_f \ell_r \lambda} \frac{Q^2 K_1(QD)K_1(QD)}{D D} \\
\times \left[ Q(x, z, \bar{z}, \bar{x}), Q(z, y, y, \bar{z}) - S(x, z) S(z, y) - S(z, \bar{y}) S(\bar{y}, \bar{z}) + 1 \right]. \tag{A.9}
\]
Here, $R$ and $Y$ are the transverse separations between the quark and the intermediate antiquark and, respectively, between the final antiquark and gluon (cf. eqs. (4.12) and (4.14)):

$$ R = x - y', \quad Y = y - z, \quad y' = \frac{\vartheta_3 y + \vartheta_3 z}{\vartheta_2 + \vartheta_3}. $$

(A.10)

The function $D \equiv D(x, y, z, \vartheta_i)$ appears in eq. (A.3), while $\overline{D} \equiv D(x, y, z, \vartheta_i)$ is similarly defined in terms of the transverse coordinates in the CCA.

The product of the instantaneous pieces yields

$$ \Phi_{ijmn}^{\text{inst}} \equiv \Phi_{ijmn}^{\text{inst}}(\vartheta_1, \vartheta_2, \vartheta_3) \chi_{ijmn} \left( \delta^{\mu \nu} + 2ie_{ijm}^\lambda \right) \frac{\gamma^2 \overline{R} \cdot \overline{Y}}{R \cdot Y}. $$

(A.11)

In the CCA, $\Phi_{ijmn}^{\text{inst}}$ is obtained by replacing $R \to \overline{R}$ and $Y \to \overline{Y}$ in the above. Note that the coordinate dependence only matters for the instantaneous piece.

Clearly, the product of effective vertices which enters the cross-section (A.9) involves four pieces. One of these pieces, that built with regular terms alone, was already evaluated in eq. (5.5), and implies

$$ \Phi_{ijmn}^{\text{reg}} \equiv \Phi_{ijmn}^{\text{reg}}(\vartheta_1, \vartheta_2, \vartheta_3) \chi_{ijmn} \left( \delta^{\mu \nu} + 2ie_{ijm}^\lambda \right) \frac{\gamma^2 \overline{R} \cdot \overline{Y}}{R \cdot Y}. $$

(A.12)

The product of the instantaneous pieces yields

$$ \Phi_{ijmn}^{\text{inst}} \equiv \Phi_{ijmn}^{\text{inst}}(\vartheta_1, \vartheta_2, \vartheta_3) \chi_{ijmn} \left( \delta^{\mu \nu} + 2ie_{ijm}^\lambda \right) \frac{\gamma^2 \overline{R} \cdot \overline{Y}}{R \cdot Y}. $$

(A.13)

while that between a regular piece and an instantaneous one gives

$$ \Phi_{ijmn}^{\text{reg}} \Phi_{ijmn}^{\text{inst}} \overline{R} \cdot \overline{Y} \cdot Y^2 = \frac{8\vartheta_2^2 \vartheta_3^2}{(\vartheta_2 + \vartheta_3)^2} \gamma^2 \overline{R} \cdot \overline{Y} \cdot Y^2. $$

(A.14)

The reverse product “inst. x reg.” yields a similar contribution in which $(R \cdot Y) Y^2 \to (\overline{R} \cdot \overline{Y}) Y^2$.

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(ii) gluon emitted in the final state (after the scattering) in the DA and absorbed in the initial state (prior to the scattering) in the CCA. This case corresponds to the term symbolically denoted as \((y, z \rightarrow y')\) in eq. (5.2). We recall that this symbol means that one should replace \(y \rightarrow y'\) and \(z \rightarrow y'\), with \(y'\) from eq. (A.10), in all the terms within the curly brackets in eq. (A.9) — that is, in the effective vertices, in the argument of the first Bessel function, and in the colour structure expressing the \(S\)-matrix. After this replacement,

\[
D^2 \rightarrow \vartheta_1(\vartheta_2 + \vartheta_3)(x - y')^2 = \vartheta_1(1 - \vartheta_1)R^2, \tag{A.15}
\]

while the instantaneous piece of the effective vertex vanishes (for the DA only), so we are left with

\[
\frac{d\sigma^{\gamma_A \rightarrow \gamma y}}{d\vartheta_1^2 d\vartheta_2^2 d\vartheta_3^2 d^2 k_3} = \frac{\alpha_s\alpha_{em} C_F N_c}{2(2\pi)^{10}} \left( \sum e_f^2 \right) \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \times \frac{1}{\vartheta_1^2(1 - \vartheta_1)} \times \frac{\vartheta_1}{1 - \vartheta_1)2} \frac{\Phi_{\gamma A \lambda}}{\Phi_{\gamma A \lambda}} \frac{Q_K(\bar{Q} R) Q_K(Q \bar{D})}{R D} \times \left[ Q(x, y', z, \bar{x}) S(y, z) - S(x, y') - S(y', z) S(y, z) + 1 \right]. \tag{A.16}
\]

where we recall that \(Q^2 = \vartheta_1(1 - \vartheta_1)Q^2\). The product of the effective vertices now involves just two pieces: the “reg. x reg.” piece in eq. (A.12) and the “reg. x inst.” piece in eq. (A.14).

(iii) gluon emitted in the initial state (prior to the scattering) in the DA and absorbed in the final state (after the scattering) in the CCA. This case, which corresponds to the piece denoted as \((\bar{y}, \bar{z} \rightarrow \bar{y}')\) in eq. (5.2), is obtained via obvious substitutions in eq. (A.16). In particular, the corresponding colour structure is shown in eq. (5.12).

(iv) gluon emitted in the final state (after the scattering) in both the DA and the CCA. This case, which corresponds to the piece denoted as \((y, z \rightarrow y') \& (\bar{y}, \bar{z} \rightarrow \bar{y}')\) in eq. (5.2), leads to

\[
\frac{d\sigma^{\gamma_A \rightarrow \gamma y}}{d\vartheta_1^2 d\vartheta_2^2 d\vartheta_3^2 d^2 k_3} = \frac{\alpha_s\alpha_{em} C_F N_c}{2(2\pi)^{10}} \left( \sum e_f^2 \right) \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \times \frac{1}{\vartheta_1^2(1 - \vartheta_1)} \times \frac{\vartheta_1}{1 - \vartheta_1)2} \frac{\Phi_{\gamma A \lambda}}{\Phi_{\gamma A \lambda}} \frac{Q_K(\bar{Q} R) K_1(\bar{Q} R)}{R D} \times \left[ Q(x, y', \bar{y}, \bar{x}) - S(x, y') - S(y', \bar{x}) + 1 \right]. \tag{A.17}
\]
We are now in a position to compare with the corresponding results in the literature. Once again, it is simpler to compare with ref. [73], which used the same (LCWF) formalism like us and presented fully explicit results for the tri-jet cross-section, albeit specialised to the photoproduction limit $Q^2 \to 0$ (but the comparison remains non-trivial even in that limit). Notice that the final results in [73] are directly shown for the quark-antiquark dijet cross-section — the kinematics of the gluon is integrated over —, so for the purposes of this comparison one should let $z = \pi$ in our above formulae. The authors of ref. [73] compared their results with those from the earlier paper [72] and concluded to their equivalence in all the situations where ref. [72] provided explicit results (see appendix A.2 in [73] for details). We shall also present a partial comparison with the very recent work in ref. [103].

With this in mind, it is quite straightforward to check that the sum of the “reg. × reg.” pieces in the above cases (i), (ii) and (iii) indeed coincides (up to normalisation conventions) with the result show in eq. (9.29) from ref. [73]. Furthermore, the pieces involving just one instantaneous vertex (in either the DA, or the CCA), as contained too in the above cases (i), (ii) and (iii), coincide with eqs. (9.37)–(9.38) in [73]. Finally, the contribution involving the product of the instantaneous pieces, which enters only case (i), is identical to eqs. (9.33)–(9.34) in [73]. The only comparison that we cannot perform is that concerning our case (iv) above — gluon emissions in the final state —, since the respective result in [73] is strongly modified by the requirement to measure two actual jets (whose definition involves a jet radius parameter $R$).

Concerning ref. [103], we have checked in detail the agreement between our above result (A.17) (gluon emission by the antiquark in the final state) and eq. (18) in that paper. As for the other terms, which involve gluon emissions in the initial state, it is straightforward to check the agreement between our results and those in ref. [103] so far as the colour structure and the emission kernels are concerned. On the other hand, the dependence upon the longitudinal fractions $\vartheta_i$ is very differently presented in [103] (compare e.g. eq. (21) there with our above (A.9)) and so far we have not been able to verify their mutual consistence.

A.3 The tri-jet cross-section: interference terms

The interference terms represent graphs where the gluon is emitted by the quark in the DA and by the antiquark in the CCA, or vice-versa. In that case too, there are four types of topologies, corresponding to different time orderings between the emission of the gluon and the scattering off the shockwave. The general result is schematically shown in eq. (5.15) and will be decomposed here into four terms that are shown in full detail.

---

14 The authors of ref. [73] systematically sum up the instantaneous-vertex contributions due to emissions by the quark and the antiquark. Accordingly, some of the pieces encoded in eqs. (9.33)–(9.34) and (9.37)–(9.38) from [73] also include “interference” terms according to our current terminology. Those will be discussed in the next subsection.

15 Notice that the integration variables referring to the antiquark in eq. (18) from [103] are the coordinates in the intermediate state — denoted as $y'$ and $y''$ in our (A.17) — and not the final respective coordinates $y$ and $\bar{y}$. This change of variables introduces a relative factor $(1 - \vartheta_1)/(2\vartheta_2^2)$ between the two results.
(i) Gluon emitted in the initial state in both the DA and the CCA:

\[
\frac{d\sigma^{\gamma_A\to q\bar{q}}_\text{inter}(\lambda)}{d^{7}k_1dq_1d^{7}k_2dq_2d^{7}k_3} = -\frac{\alpha_s\alpha_{em} C_F N_c}{(2\pi)^4} \left( \sum_j c_j^2 \right) \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \times 2 \Re e \int_{x,y,z,\bar{x},\bar{y},\bar{z}} e^{-ik_1(x-x)} - ik_2(y-y) - ik_3(z-z) \frac{R^i X^m R^j Y^n}{X^2 Y^2} \times \left\{ \vartheta_1 \vartheta_2 \Phi^{\text{ijmn}}_{\lambda_1\lambda_2} \Phi^{\text{ijrn}}_{\lambda_1\lambda_2} \frac{Q^2 K_1(QD)K_1(Q\bar{D})}{D \bar{D}} \right\} \right\}. \quad (A.18)
\]

This equation also involves the effective vertex \( \tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2} \) which refers to the case where the gluon is emitted by the quark, cf. eq. (4.35):

\[
\tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2} = \tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2 \text{ reg.}} + \tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2 \text{ inst.}}
\]

\[
\tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2 \text{ reg.}}(\vartheta_1) = \tau^{\text{mn}}_{\lambda_1\lambda_2}(\vartheta_3, \vartheta_1) \tilde{\varphi}^i_{\lambda_2}(\vartheta_1 + \vartheta_3),
\]

\[
\tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2 \text{ inst.}}(\vartheta_1; R, Y) = -\delta_{\lambda_1\lambda_2} \delta^{nj} \frac{\vartheta_1 \vartheta_3}{\vartheta_1 + \vartheta_3} \left( \lambda^{lm} - 2i \epsilon^{lm} \lambda_1 \right) \frac{X^2}{R \cdot X}. \quad (A.19)
\]

The product of effective vertices involves four pieces, as in the case of direct emissions. The "reg. x reg." piece has already been evaluated in eq. (5.17), which in the new notations reads

\[
\tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2} \Phi^{\text{ijrn}}_{\lambda_1\lambda_2 \text{ reg.}} \frac{R^i X^m R^j Y^n}{X^2 Y^2} = 8 \left[ \vartheta_1 + \vartheta_2 - 2 \vartheta_1 \vartheta_2 \right] \left[ \vartheta_3(1 - \vartheta_3) + 2 \vartheta_1 \vartheta_2 \right] \left( \bar{R} \cdot R \right) (X \cdot Y)
\]

\[
- 8 \vartheta_3(\vartheta_1 - \vartheta_2)^2 \left( \bar{R} \times R \right) (X \times Y), \quad (A.20)
\]

The product of the instantaneous pieces is readily obtained as

\[
\tilde{\Phi}^{\text{ijrn}}_{\lambda_1\lambda_2} \Phi^{\text{ijrn}}_{\lambda_1\lambda_2 \text{ inst.}} \frac{R^i X^m R^j Y^n}{X^2 Y^2} = \frac{8 \vartheta_1 \vartheta_2 \vartheta_3^2}{(\vartheta_1 + \vartheta_3)(\vartheta_2 + \vartheta_3)} X^2 Y^2, \quad (A.21)
\]

whereas the two mixed combinations require some more work:

\[
\tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2} \Phi^{\text{ijrn}}_{\lambda_1\lambda_2 \text{ reg.}} \frac{R^i X^m R^j Y^n}{X^2 Y^2}
\]

\[
= - \left( \delta^{ir} - 2i \epsilon^{ir} \lambda_1 \right) \tau^{\text{nr}}_{\lambda_1\lambda_2}(\vartheta_3, \vartheta_1) \tilde{\varphi}_2(\vartheta_1 + \vartheta_3) \delta^{nj} \frac{\vartheta_2 \vartheta_3}{\vartheta_2 + \vartheta_3} \frac{R^i X^m Y^2}{X^2 Y^2}
\]

\[
= -8 \frac{\vartheta_2 \vartheta_3(\vartheta_1 + \vartheta_3)^2}{\vartheta_2 + \vartheta_3} (\bar{R} \cdot X) Y^2, \quad (A.22)
\]

and similarly

\[
\tilde{\Phi}^{\text{ijmn}}_{\lambda_1\lambda_2} \Phi^{\text{ijrn}}_{\lambda_1\lambda_2 \text{ inst.}} \frac{R^i X^m R^j Y^n}{X^2 Y^2} = -8 \frac{\vartheta_1 \vartheta_3(\vartheta_2 + \vartheta_3)^2}{\vartheta_1 + \vartheta_3} (\bar{R} \cdot Y) X^2. \quad (A.23)
\]
(ii) Gluon emitted in the final state in the DA and absorbed in the initial state in the CCA. This is obtained by replacing $x \to x'$ and $z \to x'$ in all the terms within the curly brackets in eq. (A.18). After this replacement, $D$ simplifies as shown in eq. (A.5), while the instantaneous piece of the effective vertex $\tilde{\Phi}_{\lambda_1, \lambda}^{\text{irm}}$ vanishes. Hence we are left with
\[
\frac{d\sigma^{\gamma^A-h\bar{q}q}_{\text{inter}(14)}}{d\theta_1^+ d^2 k_1 d\theta_2^+ d^2 k_2 d\theta_3^+ d^2 k_3} = \frac{\alpha_s \alpha_{em} C_F N_c}{2(2\pi)^{10}} \left( \sum c_j^2 \right) \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \\
\times 2 \Re e \int_{x,y,z,x',y'} e^{-ik_1(x-x')-ik_2(y-y')-ik_3(z-z')} \frac{\tilde{R}^i X^m \bar{R}^j \bar{Y}^n}{X^2 \bar{Y}^2} \\
\times \left\{ \frac{\vartheta_1}{(1 - \vartheta_2)\vartheta_3} \tilde{\Phi}_{\lambda_1, \lambda}^{\text{irm}} \Phi_{\lambda_1, \lambda}^{\text{irm} \ast} \frac{\tilde{Q}K_1(\tilde{Q} R) QK_1(\tilde{Q} \bar{R})}{R} \bar{D} \\
\times \left[ S(z, x) Q(x', y, y', z) - S(x', y) - S(z, x) S(y, z) + 1 \right] \right\}, \tag{A.24}
\]
where the product $\tilde{\Phi} \bar{F}$ includes the “reg.×reg.” piece (A.20) and the “reg.×inst.” piece (A.22).

(iii) Gluon emitted in the initial state in the DA and absorbed in the final state in the CCA. This case, which corresponds to the piece denoted as $(\tilde{y}, \tilde{z} \to \tilde{y}')$ in eq. (5.15), explicitly reads (we recall that $\tilde{R} \equiv |x - y'|$ and $\tilde{Q}^2 = \vartheta_1(1 - \vartheta_1)Q^2$)
\[
\frac{d\sigma^{\gamma^A-h\bar{q}q}_{\text{inter}(14)}}{d\theta_1^+ d^2 k_1 d\theta_2^+ d^2 k_2 d\theta_3^+ d^2 k_3} = \frac{\alpha_s \alpha_{em} C_F N_c}{2(2\pi)^{10}} \left( \sum c_j^2 \right) \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \\
\times 2 \Re e \int_{x,y,z,x',y'} e^{-ik_1(x-x')-ik_2(y-y')-ik_3(z-z')} \frac{\tilde{R}^i X^m \bar{R}^j \bar{Y}^n}{X^2 \bar{Y}^2} \\
\times \left\{ \frac{\vartheta_2}{(1 - \vartheta_1)\vartheta_3} \tilde{\Phi}_{\lambda_1, \lambda}^{\text{irm}} \Phi_{\lambda_1, \lambda}^{\text{irm} \ast} \frac{QK_1(QD) QK_1(Q \bar{R})}{D} \bar{R} \\
\times \left[ Q(x, z, y', x) S(z, y) - S(x, z) S(z, y) - S(y', x) + 1 \right] \right\}, \tag{A.25}
\]
where the product $\tilde{\Phi} \bar{F}$ includes the “reg.×reg.” piece (A.20) and the “inst.×reg.” piece (A.23).

(iv) Gluon emitted in the final state in both the DA and the CCA. This case, which corresponds to the piece denoted as $(x, z \to x') k(y, z \to y')$ in eq. (5.15), leads to
\[
\frac{d\sigma^{\gamma^A-h\bar{q}q}_{\text{inter}(14)}}{d\theta_1^+ d^2 k_1 d\theta_2^+ d^2 k_2 d\theta_3^+ d^2 k_3} = -\frac{\alpha_s \alpha_{em} C_F N_c}{2(2\pi)^{10}} \left( \sum c_j^2 \right) \delta(1 - \vartheta_1 - \vartheta_2 - \vartheta_3) \\
\times 2 \Re e \int_{x,y,z,x',y'} e^{-ik_1(x-x')-ik_2(y-y')-ik_3(z-z')} \frac{\tilde{R}^i X^m \bar{R}^j \bar{Y}^n}{X^2 \bar{Y}^2} \\
\times \left\{ \frac{1}{(1 - \vartheta_1)(1 - \vartheta_2)\vartheta_3} \tilde{\Phi}_{\lambda_1, \lambda}^{\text{irm}} \Phi_{\lambda_1, \lambda}^{\text{irm} \ast} \frac{\tilde{Q}K_1(\tilde{Q} R) QK_1(\tilde{Q} \bar{R})}{R} \bar{D} \\
\times \left[ S(\tilde{y}', \tilde{x}) S(x', y) - S(x', y) - S(\tilde{y}', \tilde{x}) + 1 \right] \right\}. \tag{A.26}
\]
Once again, we shall compare our above results for the interference terms with the corresponding ones in ref. [73]. The “reg. × reg.” pieces in the above cases coincides (up to normalisation conventions) with the result show in eq. (9.30)–(9.32) in [73]. The interference terms involving (one or two) instantaneous vertices are exquivalently contained in eqs. (9.37)–(9.38) and (9.33)–(9.34) from [73].

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