ON LANDAU–SIEGEL ZEROS AND HEIGHTS OF SINGULAR MODULI

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Abstract. Let $\chi_D$ be the Dirichlet character associated to $\mathbb{Q}(\sqrt{D})$ where $D < 0$ is a fundamental discriminant. Improving Granville–Stark [7], we show that

$$
\frac{L'}{L}(1, \chi_D) = \frac{1}{6} \text{height}(j(\tau_D)) - \frac{1}{2} \log |D| + C + o_{D \to -\infty}(1),
$$

where $\tau_D = \frac{1}{2}(-\delta + \sqrt{D})$ for $D \equiv \delta \pmod{4}$ and $j(\cdot)$ is the $j$-invariant function with $C = -1.057770 \ldots$. Assuming the “uniform” abc-conjecture for number fields, we deduce that $L(\beta, \chi_D) \neq 0$ with $\beta \geq 1 - \sqrt{5} \varphi + o(1)$ where $\varphi = (1 + \sqrt{5})/2$, which we improve for smooth $D$.

1. Introduction

In 2000, Granville and Stark [7] showed that the uniform abc-conjecture for number fields (Conjecture 5.2 (iii)) implies that there are no “Siegel zeros” for odd characters, by deducing that, under uniform abc, the class number of $\mathbb{Q}(\sqrt{D})$ satisfies

$$
h(D) \geq (1 + o(1)) \frac{\pi}{3} \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)}^{(D)} \frac{1}{a},
$$

where the sum runs over the reduced binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ of fundamental discriminant $D < 0$. Comparing this lower bound to the unconditional, more traditional type of lower bound:

$$
h(D) = \left(1 + O\left(\frac{\log \log |D|}{\log |D|}\right)\right) \frac{\pi}{3} \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)}^{(D)} \frac{1}{a},
$$

where $\chi_D$ is the corresponding quadratic character, one derives from uniform abc that $|\frac{L'}{L}(1, \chi_D)| \ll \log |D|$, which is equivalent to $L(s, \chi_D)$ having no “Siegel zeros”. Our main theorems imply a slightly more precise estimate than (1.2), and the existence of a subsequence of $D$’s for which $\frac{L'}{L}(1, \chi_D) = o_{D \to -\infty}(\log |D|)$.

For $D < 0$ we need the generator of the ring of integers of $\mathbb{Q}(\sqrt{D})$,

$$
\tau_D = \begin{cases} 
\frac{1}{2}(-1 + \sqrt{D}) & \text{if } D \equiv 1 \pmod{4}, \\
\frac{1}{2}\sqrt{D} & \text{if } D \equiv 0 \pmod{4};
\end{cases}
$$

and let $j(\cdot)$ be the classical $j$-invariant function. Define the height of $m/n \in \mathbb{Q}$ with $(m, n) = 1$ by

$$
\text{ht}(m/n) = \log \max\{|m|, |n|\}.
$$
We will give the height function for algebraic numbers in subsection 2.3. We improve on (1.2) by using Duke’s equidistribution theorem [6] (Lemma 4.2):

Theorem 1.1. For fundamental discriminants $D < 0$, we have

\[
\frac{L'(1, \chi_D)}{L(1, \chi_D)} = \frac{1}{6} \mathrm{ht}(j(\tau_D)) - \frac{1}{2} \log |D| + C + o_{D \to -\infty}(1),
\]

where $C := \kappa_1 - \kappa_2 + \kappa_3 + \gamma - \log 2 = -1.057770 \ldots$, and $\gamma = 0.577215 \ldots$ is Euler–Mascheroni’s constant. In particular, this implies that

\[
h(D) = \left( \frac{1 + O(1/\log |D|)}{1 + 2 \frac{L'(1, \chi_D)}{L(1, \chi_D)} / \log |D|} \right) \frac{\pi}{3} \sqrt{|D|} \sum_{(a,b,c)} \frac{1}{a}.
\]

Expressing $\frac{L'(1, \chi_D)}{L(1, \chi_D)}$ in terms of $\mathrm{ht}(j(\tau_D))$ allows us to understand $L$-function values using Diophantine geometry techniques: The classical abc-conjecture (the Masser–Oesterlé conjecture) states that for coprime integers $a, b, c$ satisfying $a + b = c$, we have

\[
\log \max \{|a|, |b|, |c|\} \leq (1 + \varepsilon) \sum_{p|abc} \log p + O\varepsilon(1),
\]

where the implied constant depends only on $\varepsilon > 0$. Several important results in number theory follow from this statement, including “asymptotic Fermat” (see Example 5.5.2, p. 71–72 of Vojta [18]). Granville and Stark used a uniform extension of the abc-conjecture to number fields. The tricky part is deciding what to conjecture about the contribution of ramified primes in this inequality: thus, if $\text{rd}_K := |\Delta_K|^{1/[K:Q]}$ then we study two versions, where one includes the error term

\[
O(\log(\text{rd}_K)) \text{ (O-weak uniformity) or } o(\log(\text{rd}_K)) \text{ (weak uniformity)},
\]

given by Conjectures 5.2 (i) and (ii), respectively. With minor modifications to the argument, we prove the following version of Granville–Stark’s theorem:

Theorem 1.2. Assuming the abc-conjecture for number fields with

- O-weak uniformity we have $|\frac{L'(1, \chi_D)}{L(1, \chi_D)}| \ll \log |D|$. 
- weak uniformity we have $\limsup_{D \to -\infty} \frac{1}{\log |D|} \frac{L'(1, \chi_D)}{L(1, \chi_D)} = 0$.

1.1. Traditional approach to $\frac{L'(1, \chi_D)}{L(1, \chi_D)}$. Let $\varrho(\chi)$ be the set of non-trivial zeros of $L(s, \chi)$, each included in the set with multiplicity. We are interested in the zeros of $L(s, \chi)$ inside the region

\[
R_A = R_A(q) := \left\{ s \in \mathbb{C} \mid \sigma \geq 1 - \frac{A}{\log q}, \ |t| \leq 1 \right\},
\]

for any given $A \in \mathbb{R}_{\geq 1}$ with $q > e^A$. A Siegel zero is a real zero $\beta$ of $L(s, \chi)$ for some primitive quadratic $\chi \pmod{q}$, with $\beta \geq 1 - \frac{A}{\log q}$ for some given $A > 0$.

1See Lemmas 4.3, 4.4, 4.5 for the definition of $\kappa_1, \kappa_2, \kappa_3$, respectively.

2Formulated so as to be consistent between different field extensions.
Theorem 1.3. For every primitive character \( \chi \) (mod \( q \)) we have the bound \( |\varrho(\chi) \cap \mathcal{R}_A| \ll e^{3A} \) (with an absolute implicit constant), and

\[
\sum_{\varrho(\chi) \cap \mathcal{R}_A} \Re\left( \frac{1}{1 - \varrho} \right) < \left( 1 - \frac{1}{\sqrt{5}} \right) \frac{1}{2} \log q + \Re\left( \frac{L'(1, \chi)}{L(1, \chi)} \right) + O(e^{3A}).
\]

An integer \( q \) is \( y \)-smooth if all its prime factors are \( \leq y \). Chang [1] established wide zero-free regions for such \( q \) (see subsection 3.3) so we deduce:

Theorem 1.4. For every \( \delta > 0 \) there is an \( N_\delta > 0 \) such that if \( q > N_\delta \) is \( q_\delta \)-smooth then the only possible element of \( \varrho(\chi) \cap \mathcal{R}_A \) with \( A \ll 1/\delta \), is the potential Siegel zero \( \beta = \beta_\chi := \max\{\sigma \in \mathbb{R} \mid L(\sigma, \chi) = 0\} \). In that case

\[
\Re\left( \frac{L'(1, \chi)}{L(1, \chi)} \right) = \frac{1}{1 - \beta} + O(\delta^{1/2} \log q).
\]

Theorem 1.4 implies that, for every fixed \( \delta > 0 \), if \( D < 0 \) is a \( |D|^{\delta} \)-smooth fundamental discriminant then

\[
L'(1, \chi_D) \geq -M\delta^{1/2} \log |D|
\]

for some \( M \in \mathbb{R}_{>0} \) not depending on \( \delta \), provided \( |D| \) is sufficiently large.

1.2. Main corollaries. We present two results that are consequences of the theorems above. First, by combining Theorems 1.2, 1.3 and 1.4 together, we deduce the following:

Corollary 1.5. Assume the weak uniform abc-conjecture. As \( D \to -\infty \), the function \( L(s, \chi_D) \) has no zeros in the real interval

\[
\left[ 1 - \frac{\sqrt{5}\varphi + o(1)}{\log |D|}, 1 \right]
\]

where \( \varphi := \frac{1 + \sqrt{5}}{2} \), nor in the region

\[
\left\{ s \in \mathbb{C} \mid \sigma \geq 1 - \frac{m\delta^{-1/2} + o_\delta(1)}{\log |D|}, |t| \leq 1 \right\}
\]

(for some absolute constant \( m > 0 \)) when \( D \) is \( |D|^{\delta} \)-smooth for given \( \delta > 0 \).

See subsection 3.4 for a proof. Next we give explicit upper and lower bounds on several key analytic quantities:

Corollary 1.6. For negative fundamental discriminants \( D \) we have:

(i) \( \frac{3}{\sqrt{5}} \log |D| + O(1) \leq \text{ht}(j(\tau_D)) \leq (1 + o(1)) \frac{3}{2} \log |D| \).

(ii) \( \frac{1}{2\sqrt{5}} \log |D| + O(1) \leq \sum_{\varrho(\chi_D)} \frac{1}{\varrho} \leq (1 + o(1)) \frac{3}{2} \log |D| \).

(iii) \( (1 + o(1)) \frac{\pi}{3} \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)} \frac{1}{a} \leq h(D) \leq \left( \sqrt{5} + O\left( \frac{1}{\log |D|} \right) \right) \frac{\pi}{3} \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)} \frac{1}{a} \)
where the starred inequalities “∗” are conditional on the weak uniform abc-conjecture. Moreover, if $D$ is $|D|^{o(1)}$-smooth then these starred inequalities become (asymptotic) starred equalities. If we assume the Generalized Riemann Hypothesis then we obtain these asymptotic equalities for all $D$, with error term a factor of $O(\frac{\log \log |D|}{\log |D|})$.

This corollary follows from Theorems 1.1 and 1.2: part (ii) is obtained from (3.5), and part (iii) from (1.4). (We omit the details.) In each case the upper and lower bounds in Corollary 1.6 differ by an artificial multiplicative factor that is $\sim \sqrt{5}$ arising from our proof (see Lemma 3.4 (ii)), which may be improvable. Data supports part (i) of Corollary 1.6:

![Figure 1. Graph of $\frac{\text{ht}(j(\tau_D))}{3\log |D|}$ for $-10^6 \leq D \leq 0$.](image)

The perceptible slight bias towards $>1$ in Figure 1 can be analyzed by rewriting (1.3) as

$$\frac{\text{ht}(j(\tau_D))}{3\log |D|} = 1 + \frac{2(\frac{L'}{L}(1, \chi_D) - C + o(1))}{\log |D|}.$$ 

Therefore this bias towards $>1$ in Figure 1 must stem from $\frac{L'}{L}(1, \chi_D) - C$ being usually positive. We verify this observation in Figure 2 by noting that $\frac{L'}{L}(1, \chi_D)$ is usually larger than $C = -1.057770 \ldots$.

Lastly, an appendix on the Hadamard factorization of $L$-functions following Lucia [8] is included, justifying a formula we state in section 3.

**Notation.** Let $f, g : [x_0, +\infty) \to \mathbb{R}_+$ be positive real-valued functions defined in $[x_0, +\infty)$ for some $x_0 \in \mathbb{R}$. We write $f \sim g$, $f = o(g)$, or $f = O(g)$, if $|f(x)/g(x)|$ goes to 1, 0, or stays bounded as $x \to +\infty$, respectively. If $f = O(g)$, we also write $f \ll g$. If $f \ll g$ and $g \ll f$, then we write $f \asymp g$.

\[3\text{For example, } \text{ht}(j(\tau_D)) = 3\log |D| + O(\log \log |D|) \text{ assuming the Generalized Riemann Hypothesis.}\]
2. Definitions and notation

We review some notation and results that will be used in this paper:

2.1. Heegner points. We denote the binary, primitive quadratic form

\[ Q(x, y) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y] \]

by \( Q = (a, b, c) \) which has discriminant \( d = b^2 - 4ac \). If \( d < 0 \) then we will assume that \( a > 0 \) so that \( Q \) is positive-definite. Two forms \( Q_1, Q_2 \) are equivalent if

\[ Q_1(x, y) = Q_2(\alpha x + \beta y, \gamma x + \delta y) \]

for some \( (\alpha \beta \gamma \delta) \in \text{SL}_2(\mathbb{Z}) \).

For every non-zero discriminant \( d \equiv 0 \) or 1 (mod 4), we have the principal form

\[ Q_{1,d}(x, y) := \begin{cases} x^2 - \frac{d}{4}y^2 & \text{if } d \equiv 0 \pmod{4}, \\ x^2 + xy + \frac{1-d}{4}y^2 & \text{if } d \equiv 1 \pmod{4}. \end{cases} \]

For \( d < 0 \), we say that \( (a, b, c) \) is reduced if \(-a < b \leq a < c \) or \( 0 \leq b \leq a = c \); every quadratic form of discriminant \( d \) is equivalent to a single reduced form. If \( Q \) is reduced, then \( a \leq \sqrt{|d|}/3 \).

Each binary quadratic form \((a, b, c)\) with \( d < 0 \) is associated with a CM point

\[ \tau_{(a,b,c)} = \frac{-b + \sqrt{d}}{2a} \in \mathcal{H} := \{ z \in \mathbb{C} \mid \Im(z) > 0 \}. \]

\((a, b, c)\) is reduced if and only if \( \tau_{(a,b,c)} \) lies in the fundamental domain, \( \mathcal{F} \), depicted on the right. Heegner points are the CM points associated to the reduced forms of fundamental discriminants \( D < 0 \); i.e., \( D \)'s which are the discriminant of some
quadratic number field. Write \( \Lambda_D \) for the set of Heegner points of \( D \), and \( C\ell(D) \) for the ideal class group of \( \Q(\sqrt{D}) \). By the classical correspondence

\[
C\ell(D) \ni \mathcal{A} \leftrightarrow \text{reduced } (a, b, c) \ (b^2 - 4ac = D) \leftrightarrow \tau = \frac{-b + \sqrt{D}}{2a} \in \Lambda_D \ (\subseteq \mathcal{F}),
\]

we may index Heegner points by \( C\ell(D) \), namely \( \Lambda_D = \{ \tau_\mathcal{A} \mid \mathcal{A} \in C\ell(D) \} \). The Heegner point associated to the trivial ideal class in \( C\ell(D) \), which corresponds to the principal form \( \mathcal{A} \), is denoted \( \tau_D \).

2.2. Singular moduli. For \( \tau \in \mathfrak{h} \), write \( q = q_\tau := e^{2\pi i \tau} \). The \( q \)-expansion of the \( j \)-invariant function has the form

\[
j(\tau) := \frac{1 + 240 \sum_{n \geq 1} (\sum_{d|n} d^3) q^n}{q \prod_{n \geq 1} (1 - q^n)^{24}} = \frac{1}{q} + \sum_{n \geq 0} c(n) q^n,
\]

with \( c(0) = 744 \), \( c(1) = 196884 \), and \( \mathbb{Z}_{\geq 1} \ni c(n) \sim \frac{1}{\sqrt{2}} e^{4\pi \sqrt{n}} n^{-3/4} \). This is the unique modular function with respect to \( \text{SL}_2(\mathbb{Z}) \) of weight 0, holomorphic in \( \mathfrak{h} \), satisfying \( j(e^{2\pi i/3}) = 0 \), \( j(i) = 1728 \), and having a simple pole at \( i\infty \). The values taken by \( j(\tau) \) at CM points \( \tau \in \mathfrak{h} \) are called singular moduli.

For \( D < 0 \) a fundamental discriminant, write \( H_D \) for the Hilbert class field of \( \Q(\sqrt{D}) \), which is the maximal unramified abelian extension of \( \Q(\sqrt{D}) \). Then:

- \( H_D = \Q(\sqrt{D}, j(\tau_D)) \) (and \( [H_D : \Q(\sqrt{D})] = [\Q(j(\tau_D)) : \Q] = h(D) \));
- \( \{ j(\tau) \mid \tau \in \Lambda_D \} \) is the complete set of \( \text{Gal}(\overline{\Q}/\Q(\sqrt{D})) \)-conjugates of \( j(\tau_D) \);
- \( j(\tau_D) \) is an algebraic integer.

2.3. Heights and conductors. For a number field \( K/\Q \) and a place \( v \) on \( K \), write \( K_v \) for the completion of \( K \) with respect to \( v \). For non-archimedean \( v \), write \( \mathfrak{p}_v \) for the prime ideal in \( \mathcal{O}_K \) associated to \( v \), \( p_v \) for the positive rational prime below \( \mathfrak{p}_v \), and \( f_v \) for the degree of the residual extension \( K_v/\Q_{p_v} \). If \( v \) is archimedean, let \( \iota_v : K \hookrightarrow \C \) denote an embedding which induces \( v \). For \( x \in K^\times \), define the normalized absolute value \( \| \cdot \|_v \) as:

\[
\| x \|_v := \begin{cases} 
|\iota_v(x)| & \text{if } K_v \simeq \R, \\
|\iota_v(x)|^2 & \text{if } K_v \simeq \C, \\
\pi^{f_v - \text{ord}_{\mathfrak{p}_v}(x)} & \text{if } v \text{ is non-archimedean},
\end{cases}
\]

where \( \text{ord}_{\mathfrak{p}_v}(x) (= v(x)) \) denotes the power of \( \mathfrak{p}_v \) appearing in the prime factorization of the principal fractional ideal \( (x) \subseteq K \), and \( | \cdot | \) is the usual absolute value in \( \C \).

Let \( \mathcal{M}_K \) denote the set of inequivalent places of \( K \) satisfying the product formula \( \prod_{v \in \mathcal{M}_K} \| x \|_v = 1 \) for all \( x \in K^\times \), and write \( \mathcal{M}_K^{\text{non}} \subseteq \mathcal{M}_K \) for the subset of the non-archimedean places. For a point \( P = [x_0 : \ldots : x_n] \in \mathbb{P}_K^n \) in the projective \( n \)-space over \( K \), we define its (naive, absolute, logarithmic) height by

\[
\text{ht}(P) := \frac{1}{[K : \Q]} \sum_{v \in \mathcal{M}_K} \log \max_{i \leq n} \{ \| x_i \|_v \},
\]
and its \((\text{logarithmic})\) conductor by
\[
N_K(P) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in \mathcal{M}_K^{\text{non}} \atop \exists i,j \leq n \text{ s.t. } v(x_i) \neq v(x_j)} f_v \log(p_v).
\]

Neither \(ht\) nor \(N\) depend on the choice of representatives for \(P \in \mathbb{P}_K^n\), since, for any \(c \in K^\times\), we have \(ht(P) = ht(cP)\) from the product formula, and \(N_K(P) = N_K(cP)\) since the sum runs over the same places. Moreover, the height does not depend on the choice of the base field (provided its \(\mathbb{P}_n\) contains \(P\)); however, the conductor does.

Finally, for \(x \in K^\times\), write \(ht(x) := ht([x : 1])\). If \(\alpha \in \overline{\mathbb{Q}}^\times\) is integral, then
\[
(2.4) \quad ht(\alpha) = \frac{1}{|\mathcal{A}|} \sum_{\alpha^* \in \mathcal{A}} \log^+ |\alpha^*|,
\]
where \(\log^+ x := \log \max\{1, x\}\) for \(x \in \mathbb{R}_{>0}\), \(\mathcal{A}\) is the complete set of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-conjugates of \(\alpha\), and the absolute value is being taken on some fixed embedding \(\iota: \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}\) (to which the value of \(ht(\alpha)\) is independent).

### 3. On \(\Re(L'(1, \chi))\) and zeros near \(s = 1\)

Consider primitive characters \(\chi \pmod{q}\) for \(q \geq 2\), and write \(\varrho(\chi)\) for the set of non-trivial zeros of \(L(s, \chi)\) counted with multiplicity. In this section we are going to prove the following two propositions:

**Proposition 3.1.** Let \(S\) be any finite subset of non-trivial zeros of \(L(s, \chi)\) \((\text{including the empty set})\). Then:
\[
\sum_{\varrho \in S} \Re\left(\frac{1}{1 - \varrho^*}\right) < \left(1 - \frac{1}{\sqrt{5}}\right) \frac{1}{2} \log q + \Re\left(\frac{L'(1, \chi)}{L(1, \chi)}\right) + \left(1 + \frac{1}{\sqrt{5}}\right) \frac{2|S| + 3}{2} - 1.
\]

**Proposition 3.2.** Consider the region
\[
(3.1) \quad \mathcal{B}_f(q) := \left\{ s \in \mathbb{C} \mid \sigma > 1 - \frac{1}{f(q)}, \quad |t| < \frac{1}{\sqrt{f(q)}} \right\},
\]
where \(f: \mathbb{Z}_{\geq 2} \rightarrow \mathbb{R}\) satisfies \(2 \leq f(q) \leq 4 \log q\). Then:
\[
\left| \Re\left(\frac{L'(1, \chi) - \sum_{\varrho(\chi) \cap \mathcal{B}_f(\varrho)} \frac{1}{1 - \varrho}}{L(1, \chi)}\right) \right| < \left(14.5 + 2|\varrho(\chi) \cap \mathcal{B}_f(\varrho)|\right) \sqrt{f(q) \log q}.
\]

Theorem 1.3 follows directly from Proposition 3.1 together with the classical log-free zero-density estimate (see Chapter 18 of Iwaniec–Kowalski [11]), which in particular states that
\[
\varrho(\chi) \cap \left\{ s \in \mathbb{C} \mid \sigma \geq 1 - \frac{A}{\log q}, \quad |t| \leq 1 \right\} \ll e^{cA},
\]
where we can take \(c = 3\). Theorem 1.4 on the other hand will be proved using Proposition 3.2 in subsection 3.3.
3.1. Lemmas. Our starting point is the following formula:

\[
\frac{L'}{L}(s, \chi) = \left( \sum_{\chi} \frac{1}{s - \varrho(\chi)} \right) - \frac{1}{2} \log \left( \frac{q}{\pi} \right) - \frac{1}{2} \Gamma' \left( \frac{s + a}{2} \right),
\]

where \( a_{\chi} := \frac{1}{2} (1 - \chi(-1)) \) (see Appendix A for details). From the functional equation of \( L(s, \chi) \), we have that, if \( \varrho \in \{ s \in \mathbb{C} \mid 0 < \sigma < 1 \} \) is a zero of \( L(s, \chi) \), then \( \overline{\varrho}, 1 - \varrho \) are zeros of \( L(s, \overline{\chi}) \), and \( 1 - \overline{\varrho} \) is a zero of \( L(s, \chi) \). Thus, by noting that

\[
\sum_{\chi} \varrho(\chi) \left( s - \varrho \right) - \frac{1}{2} \log \left( \frac{q}{\pi} \right) - \frac{1}{2} \Gamma' \left( s + a \chi \right),
\]

we get

\[
\sum_{\chi} \varrho(\chi) \Pi_{\sigma-1}(\varrho) = \frac{1}{2} \log \left( \frac{q}{\pi} \right) + \Re \left( \frac{L'}{L}(\sigma, \chi) \right) + \frac{1}{2} \Gamma' \left( \frac{\sigma + a}{2} \right),
\]

where \( \Pi \) is the pairing-up function:

\[
\Pi_{\varepsilon}(s) := \frac{1}{s + \varepsilon} + \frac{1}{\overline{s} + \varepsilon} + \frac{1}{1 - s + \varepsilon} + \frac{1}{1 - \overline{s} + \varepsilon},
\]

defined for \( s, \varepsilon \in \mathbb{C} \) such that \( \varepsilon \neq -s, -\overline{s}, -1 + s, -1 + \overline{s} \). In this notation, by setting \( \sigma = 1 \) in (3.3) and using the special values \( \Gamma'(1) = -\gamma \) and \( \Gamma'(\frac{1}{2}) = -\gamma - 2 \log 2 \), it follows that

\[
\sum_{\chi} \varrho(\chi) \Pi_{0}(\varrho) = \frac{1}{2} \log q + \Re \left( \frac{L'}{L}(1 + \chi) \right) - \frac{1}{2} \left( \gamma + \log 2\pi + \chi(-1) \log 2 \right).
\]

The next lemma estimates (3.5) for small perturbations of \( \varepsilon \) in \( \Pi_{\varepsilon}(\varrho) \).

Lemma 3.3. For \( 0 < \varepsilon < .85 \), we have:

(i) \( \sum_{\chi} \Pi_{\varepsilon}(\varrho) < \frac{1}{2} \log q + \frac{1}{\varepsilon} \);

(ii) \( \left| \sum_{\chi} \frac{\Pi_{\varepsilon}(\varrho)}{4} - \frac{1}{2} \log q + \frac{1}{2} \left( \gamma + \log 2\pi + \chi(-1) \log 2 \right) \right| < 1 + \frac{1}{\varepsilon} \).

Proof. For \( 0 < \sigma < 1.425 \), the function \( \frac{\Gamma'}{\Gamma}(\sigma) \) is strictly increasing and negative. Thus, since \( \Pi_{\varepsilon}(\varrho) > 0 \), we have from (3.3) that

\[
0 < \sum_{\chi} \frac{\Pi_{\varepsilon}(\varrho)}{4} < \Re \left( \frac{L'}{L}(1 + \varepsilon, \chi) \right) + \frac{1}{2} \log q.
\]

Moreover, since \( -\frac{\zeta'}{\zeta}(\sigma) < (\sigma - 1)^{-1} \) for \( 1 < \sigma \leq 2 \), it follows that

\[
\Re \left( \frac{L'}{L}(1 + \varepsilon, \chi) \right) \leq \left| \frac{\zeta'}{\zeta}(1 + \varepsilon) \right| < \frac{1}{\varepsilon},
\]

which proves part (i). For the second part, we deduce from (3.3) that

\[
\left| \sum_{\chi} \frac{\Pi_{\varepsilon}(\varrho)}{4} - \frac{1}{2} \log q + \frac{1}{2} \left( \gamma + \log 2\pi + \chi(-1) \log 2 \right) \right| < 1 + \frac{1}{\varepsilon}.
\]
\[
< \left| \frac{\zeta'}{\zeta}(1 + \varepsilon) \right| + \frac{1}{2} \left( \gamma + \log 4 \right) < \frac{1}{\varepsilon} + 1,
\]

concluding the proof. \qed

Now, let \( M \geq 2 \) be a fixed real number, and take the following partition of the critical strip \( \{ s \in \mathbb{C} \mid 0 < \sigma < 1 \} \):

\[
R_1 := \{ s \in \mathbb{C} \mid 0 < \sigma < 1, \ |t| \geq 1 \}
\]

\[
R_2 := \left\{ s \in \mathbb{C} \mid \frac{1}{M} \leq \sigma \leq 1 - \frac{1}{M}, \ |t| < 1 \right\}
\]

\[
R_3 := \left\{ s \in \mathbb{C} \mid 0 \leq \sigma(1 - \sigma) \leq \frac{1}{M} (1 - \frac{1}{M}), \ M^{-1/2} \leq |t| < 1 \right\}
\]

\[
\tilde{B} := \{ s \in \mathbb{C} \mid 0 < \sigma < 1 \} \setminus (R_1 \cup R_2 \cup R_3)
\]

Note that, in the notation of (3.1), \( \tilde{B} = B_{1/M} \cup (1 - B_{1/M}) \). The goal of the next lemma is to bound \( \Pi_0(s) \) in terms of \( \Pi_\varepsilon(s) \) for \( s \) sufficiently far away from 0 and 1, which in our case means “s outside of \( \tilde{B} \”).

**Lemma 3.4 (Pairing-up lemma).** The following hold:

(i) For \( 0 < \Re(s) < 1 \) and \( \varphi := \frac{1 + \sqrt{5}}{2} \), we have \( \Pi_0(s) > \frac{\Pi_{\varphi-1}(s)}{2 \varphi - 1} \);

(ii) For \( s \in R_1 \cup R_2 \cup R_3 \) and \( 0 < \varepsilon < 1 \), we have

\[
|\Pi_0(s) - \Pi_\varepsilon(s)| < 5M \varepsilon \Pi_\varepsilon(s).
\]

**Remark 3.5.** This lemma was initially inspired by an argument attributed to U. Vorhauer used to estimate \(-\sum_{\chi} \Re(1/\chi)\). Although we were unable to find the original source, the argument is outlined in Exercise 8, Section 10.2 of Montgomery–Vaughan [15].

**Proof of Lemma 3.4.** Let \( s = \sigma + it \) and \( \varepsilon > 0 \). Writing \( \bar{\sigma} := \sigma(1 - \sigma) \) and \( \bar{\sigma}_\varepsilon := (\sigma + \varepsilon)(1 - \sigma + \varepsilon) = \bar{\sigma} + \varepsilon(1 + \varepsilon) \), we have

\[
\frac{\Pi_\varepsilon(s)}{2} = \frac{\sigma + \varepsilon}{(\sigma + \varepsilon)^2 + t^2} + \frac{1 - \sigma + \varepsilon}{(1 - \sigma + \varepsilon)^2 + t^2}
\]

\[
= \left( \frac{\bar{\sigma}_\varepsilon + (1 + \bar{\sigma}_\varepsilon)t^2 + t^4}{\bar{\sigma}^2 + ((1 + 2\varepsilon)^2 - 2\bar{\sigma}_\varepsilon)t^2 + t^4} \right) \frac{1 + 2\varepsilon}{1 + t^2}
\]

\[
= \left( 1 + \frac{\bar{\sigma}(1 - \bar{\sigma}) + 3\bar{\sigma}t^2 + \varepsilon(1 + \varepsilon)(1 - 2\bar{\sigma} - \varepsilon(1 + \varepsilon) - t^2)}{\bar{\sigma}^2 + (1 - 2\bar{\sigma})t^2 + t^4 + \varepsilon(1 + \varepsilon)(2\bar{\sigma} + \varepsilon(1 + \varepsilon) + t^2)} \right) \frac{1 + 2\varepsilon}{1 + t^2}.
\]
Since \( \varepsilon(1 + \varepsilon)(2\tilde{\sigma} + \varepsilon(1 + \varepsilon) + 2t^2) > 0 \), we get

\[
\Pi_0(s) - \frac{\Pi_\varepsilon(s)}{1 + 2\varepsilon} \geq \frac{-1 + 2\tilde{\sigma} + \varepsilon(1 + \varepsilon) + t^2}{2\left(\tilde{\sigma}^2 + (1 - 2\tilde{\sigma})t^2 + t^4 + \varepsilon(1 + \varepsilon)(2\tilde{\sigma} + \varepsilon(1 + \varepsilon) + 2t^2)\right)} \varepsilon(1 + \varepsilon) + t^2.
\]

(3.7)

Since \( \varepsilon(1 + \varepsilon) = 1 \) for \( \varepsilon = \varphi - 1 \), part (i) follows immediately from (3.7).

For part (ii), we divide the proof into three steps. Our starting point is the fact that, if \( 0 < \sigma < 1 \), then, for \( \eta > -1 \),

\[
\frac{\sigma}{\sigma^2 + t^2} \leq (1 + \eta)\frac{\sigma + \varepsilon}{(\sigma + \varepsilon)^2 + t^2} \iff \eta \geq \left(\frac{\sigma - t^2/(\sigma + \varepsilon)}{\sigma^2 + t^2}\right) \varepsilon.
\]

• Step 1: \( \frac{\Pi_\varepsilon(s)}{1 + 2\varepsilon} < \Pi_0(s) < \left(1 + \frac{\varepsilon^2}{1 + \varepsilon}\right) \Pi_\varepsilon(s) \) for \( s \in \mathcal{R}_1 \).

For the lower bound, it suffices to note that, for every \( s \in \mathcal{R}_1 \), we have

\[-1 + 2\tilde{\sigma} + \varepsilon(1 + \varepsilon) + t^2 > 0,\]

and thus, from (3.7), we obtain \( \Pi_0(s) - \Pi_\varepsilon(s)/(1 + 2\varepsilon) > 0 \). For the upper bound, we use (3.8). For \( s \in \mathcal{R}_1 \), we have

\[
\left(\frac{\sigma - t^2/(\sigma + \varepsilon)}{\sigma^2 + t^2}\right) \varepsilon < \left(\frac{1}{t^2} - \frac{1}{1 + \varepsilon}\right) \varepsilon \leq \left(1 - \frac{1}{1 + \varepsilon}\right) \varepsilon = \frac{\varepsilon^2}{1 + \varepsilon},
\]

and thus, taking \( \eta := \varepsilon^2/(1 + \varepsilon) \) in (3.8) makes the inequality \( \Pi_0(s) < (1 + \eta)\Pi_\varepsilon(s) \) valid for every \( s \in \mathcal{R}_1 \).

• Step 2: \( \frac{\Pi_\varepsilon(s)}{(1 + 2\varepsilon)(1 + 2M\varepsilon(1 + \varepsilon))} < \Pi_0(s) \leq (1 + M\varepsilon)\Pi_\varepsilon(s) \) for \( s \in \mathcal{R}_2 \cup \mathcal{R}_3 \).

We start with the lower bound. The denominator of (3.6) is always positive, and so is the numerator, for every \( \varepsilon > 0 \). Thus,

\[
\Pi_0(s) - \frac{\Pi_\varepsilon(s)}{(1 + 2\varepsilon)(1 + g(\varepsilon))} \geq \frac{2\left(\tilde{\sigma} - \frac{\sigma_\varepsilon}{1 + g(\varepsilon)} + \left(1 - \frac{1}{1 + g(\varepsilon)}\right) + \left(\tilde{\sigma} - \frac{\sigma_\varepsilon}{1 + g(\varepsilon)}\right)\right) t^2 + \left(\tilde{\sigma} - \frac{\sigma_\varepsilon}{1 + g(\varepsilon)}\right) t^4}{\tilde{\sigma}^2 + (1 - 2\tilde{\sigma})t^2 + t^4} \frac{1}{1 + t^2},
\]

(3.9)

where \( g(\varepsilon) > 0 \) is some function of \( \varepsilon > 0 \). Therefore, since

\[
\left(\tilde{\sigma} - \frac{\sigma_\varepsilon}{1 + g(\varepsilon)}\right) + \left(1 - \frac{1}{1 + g(\varepsilon)}\right) + \left(\tilde{\sigma} - \frac{\sigma_\varepsilon}{1 + g(\varepsilon)}\right) t^2 = \frac{g(\varepsilon)}{1 + g(\varepsilon)} (1 + \tilde{\sigma}t^2 + t^2) - \frac{\varepsilon(1 + \varepsilon)}{1 + g(\varepsilon)} (1 + t^2),
\]

in order for (3.9) to be strictly positive it suffices to have

\[
g(\varepsilon) \geq \varepsilon(1 + \varepsilon) \frac{1 + t^2}{(1 + t^2)\tilde{\sigma} + t^2} = \varepsilon(1 + \varepsilon) \left(\tilde{\sigma} + \frac{t^2}{1 + t^2}\right)^{-1}.
\]

(3.10)
For $s \in R_2$, we have $\sigma = \sigma(1 - \sigma) \geq M^{-1}(1 - M^{-1}) \geq \frac{1}{2} M^{-1}$ (since $M \geq 2$), while for $s \in R_3$ we have $t^2/(1 + t^2) \geq \frac{1}{2} M^{-1}$. Thus, in both cases, it suffices to take $g(\varepsilon) := 2M \varepsilon (1 + \varepsilon)$ in order for $g$ to satisfy (3.10), giving us the desired lower bound.

For the upper bound, we have that

$$\left(\frac{\sigma - t^2/(\sigma + \varepsilon)}{\sigma^2 + t^2}\right) \varepsilon \leq \left(\frac{\varepsilon}{\sigma^2 + t^2}\right) \varepsilon \leq \begin{cases} \varepsilon/\sigma \leq M \varepsilon, & \text{if } s \in R_2, \\ \varepsilon/t^2 \leq M \varepsilon, & \text{if } s \in R_3; \end{cases}$$

implying that taking $\eta := M \varepsilon$ in (3.8) makes $\Pi_0(s) \leq (1 + \eta) \Pi_0(s)$ valid for every $s \in R_2 \cup R_3$, concluding step 2.

- **Step 3:** $|\Pi_0(s) - \Pi_0(s)| < 5M \varepsilon \Pi_0(s)$ for $s \in R_1 \cup R_2 \cup R_3$, $0 < \varepsilon < 1$.

Since $M \geq 2$ and $\varepsilon < 1$, we have from step 1 that, for $s \in R_1$,

$$|\Pi_0(s) - \Pi_0(s)| \leq \max \left\{ \frac{2 \varepsilon}{1 + 2 \varepsilon}, \frac{\varepsilon^2}{1 + \varepsilon} \right\} \Pi_0(s) \leq \max \{2 \varepsilon, \varepsilon^2\} \Pi_0(s)$$

$$\leq 2 \varepsilon \Pi_0(s) \quad (< 5M \varepsilon \Pi_0(s)),$$

and from step 2 we have, for $s \in R_2 \cup R_3$, that

$$|\Pi_0(s) - \Pi_0(s)| \leq \max \left\{ \frac{2 \varepsilon}{1 + 2 \varepsilon} \left(1 + \frac{M(1 + \varepsilon)}{1 + 2M \varepsilon(1 + \varepsilon)}\right), M \varepsilon \right\} \Pi_0(s) \leq \max \{2M \varepsilon \left(\frac{1}{M} + 1 + \varepsilon\right), M \varepsilon \} \Pi_0(s) \leq 5M \varepsilon \Pi_0(s).$$

thus proving part (ii). \(\square\)

We are now ready to prove the main propositions of this section.

### 3.2. Proofs of Propositions 3.1 and 3.2

**Proof of Proposition 3.1.** Write $\tilde{S}$ for the multiset consisting of $\varrho$, $1 - \varrho$ for each $\varrho \in S$, so that $|\tilde{S}| = 2|S|$, and write $G = \frac{1}{2}(\gamma + \log 2\pi + \chi(-1) \log 2)$. From the definition of the pairing-up function (3.4), we have

$$\frac{1}{2} \Re \left( \frac{1}{\varrho} \right), \frac{1}{2} \Re \left( \frac{1}{1 - \varrho} \right) \leq \frac{\Pi_0(\varrho)}{4}$$

and $\Pi_0(\varrho)/4 \leq \varepsilon^{-1}$, for $0 < \Re(\varrho) < 1$ and $\varepsilon > 0$. Thus, from (3.5) and Lemmas 3.3 (ii), 3.4 (i) we obtain:

$$\Re \left( \frac{L'}{L}(1, \chi) \right) = \sum_{\varrho \in \tilde{S}} \frac{\Pi_0(\varrho)}{4} + \left( \sum_{\varrho(\chi)} \frac{\Pi_0(\varrho)}{4} - \frac{1}{2} \log q + G - \sum_{\varrho \in \tilde{S}} \frac{\Pi_0(\varrho)}{4} \right)$$

$$> \Re \left( \sum_{\varrho \in \tilde{S}} \frac{1}{1 - \varrho} \right) + \frac{1}{2\varphi - 1} \left( \sum_{\varrho(\chi)} \frac{\Pi_{\varphi - 1}(\varrho)}{4} - \frac{1}{2} \log q + G - \sum_{\varrho \in \tilde{S}} \frac{\Pi_{\varphi - 1}(\varrho)}{4} \right)$$

$$+ \left(1 - \frac{1}{2\varphi - 1}\right) \left( - \frac{1}{2} \log q + G \right).$$
The proposition then follows by rearranging the terms. □

Proof of Proposition 3.3 Writing $\mathcal{B} = \mathcal{B}_M$, we have from (3.5) that

$$\Re \left( \frac{L'}{L}(1, \chi) - \sum_{\varrho(\chi) \cap \mathcal{B}} \frac{1}{1 - \varrho} \right) = \sum_{\varrho(\chi) \cap \mathcal{B}} \frac{\Pi_\varrho(\chi) - \Pi_\varrho(\chi)}{4} =: S_2$$

$$- \left( \sum_{\varrho(\chi) \cap \mathcal{B}} \frac{\Pi_\varrho(\chi)}{4} - \sum_{\varrho(\chi) \cap \mathcal{B}} \Re \left( \frac{1}{\varrho} \right) \right) =: S_2$$

$$+ \left( \sum_{\varrho(\chi)} \frac{\Pi_\varrho(\chi)}{4} - \frac{1}{2} \log q + \frac{1}{2} \left( \gamma + \log 2\pi + \chi(-1) \log 2 \right) \right).$$

From Lemma 3.4 (ii), we have $|S_1| < 5M\varepsilon \sum_{\varrho(\chi)} \Pi_\varrho(\chi)/4$, and thus, from Lemma 3.3 (i), it follows that $|S_1| < \frac{3}{2} M\varepsilon \log q + 5M$. Next, since $\Pi_\varrho(\chi)/4 < \varepsilon^{-1}$ for $0 < \Re(\varrho) < 1$, and $\Re(1/\varrho) \leq M^{-1}$ for $\varrho \in \mathcal{B}$, we have $|S_2| < 2|\varrho(\chi) \cap \mathcal{B}|/\varepsilon$. Finally, from Lemma 3.3 (ii), we have $|S_3| \leq 1 + \varepsilon^{-1}$. Then, putting everything together and substituting $\varepsilon = 1/\sqrt{M\log q}$ yields

$$\left| \Re \left( \frac{L'}{L}(1, \chi) - \sum_{\varrho(\chi) \cap \mathcal{B}} \frac{1}{1 - \varrho} \right) \right| < \left( \frac{7}{2} + 2|\varrho(\chi) \cap \mathcal{B}| \right)s + M\log q + (5M + 1).$$

Note that $5M + 1 \leq 11M/2$. By taking $M := f(q) (\leq 4\log q)$, we have $f(q) \leq 2\sqrt{f(q)\log q}$, and the estimate from Proposition 3.2 follows. □

3.3. Chang’s zero-free regions: Theorem 1.4 For $q \geq 2$, write $q' := \prod_{p \mid q} p$, $K_q := \log q'/\log q'$, and $\mathcal{P}(q)$ for the largest prime divisor of $q$. Then, there is an effectively computable constant $c > 0$ such that, for every $T \geq 1$, the region

$$\left\{ s = \sigma + it \mid \sigma \geq 1 - c \min \left\{ \frac{1}{\log \mathcal{P}(q)}, \frac{\log \log q'}{(\log q')(\log 2K_q)}, \frac{1}{(\log q)(\log q')^{9/10}} \right\} \right\}$$

is zero-free for $|t| < T$, with the possible exception of a simple, real zero in this region in case $\chi$ is real (Theorem 10 of Chang [11]). In particular, it follows that we can take

$$f(q) := \frac{1}{c} \max \left\{ \log \mathcal{P}(q), \frac{(\log q')(\log 2K_q)}{\log q'}, \frac{1}{(\log q)^{9/10}} \right\}$$
in (3.1), so that the only possible element in
\[ \varrho(\chi) \cap \left\{ s \in \mathbb{C} \mid \sigma \geq 1 - \frac{1}{f(q)}, \ |t| \leq 1 \right\} \quad (\supset \varrho(\chi) \cap B_f) \]
is the potential Siegel zero, and hence the only potential term in the summation over zeros in Proposition 3.2. Writing \( L(q) := \log \log q / \log \log q' \), it becomes clear that
\[ \frac{(\log q')(\log 2K_q)}{\log \log q'} = \left( \frac{(\log 2)L(q)}{\log q} + L(q) - 1 \right) \left( \log q \right)^{1/L(q)} = o(\log q), \]
and thus, for sufficiently smooth moduli, Chang’s zero-free region is much wider than the classical one (which yields \( O(\log q) \)). More precisely, if \( \delta > 0 \) and \( q \) is \( q^2 \)-smooth, then there is \( N_\delta \in \mathbb{R}_{>0} \) such that \( f(q) \leq c^{-1} \delta \log q \) for all \( q > N_\delta \). This combined with Proposition 3.2 proves Theorem 1.4. \( \square \)

3.4. Proof of Corollary 1.5. Using that \( \Re(\frac{1}{1-s}) = ((1-s) + \frac{t^2}{1-s})^{-1} \), we can apply Proposition 3.1 to deduce that a number \( s = \sigma + it \) in the critical strip cannot be a zero of \( L(s, \chi_D) \) if
\[ 1 > \left( (1 - \sigma) + \frac{t^2}{1 - \sigma} \right) \left( 1 - \frac{1}{\sqrt{5}} \right) \frac{1}{2} \log |D| + \frac{L'(1, \chi_D)}{L}(1, \chi_D) + O(1) \]equals \( X \).

Multiplying by \( 1 - \sigma \) and rearranging we get the equation
\[ (1 - \sigma)^2 X - (1 - \sigma) + t^2 X < 0, \]
which, setting \( t = 0 \) and solving in \( 1 - \sigma \) yields \( 0 < 1 - \sigma < X^{-1} \). As \( (1 - \frac{1}{\sqrt{5}})^{-1} = \sqrt{5}\varphi \) for \( \varphi = \frac{1 + \sqrt{5}}{2} \), it follows from Theorem 1.2 that, under weak uniform \( abc \), we have \( X^{-1} > (\sqrt{5}\varphi + o(1))(\log |D|)^{-1} \), thus proving the first assertion.

For the second assertion, we have from Theorem 1.4 that, for any fixed \( \delta > 0 \), there is \( N_\delta \in \mathbb{R}_{>0} \) such that, if \( |D| > N_\delta \) is \( |D|^{\delta} \)-smooth, then any real zero \( 0 < \beta < 1 \) of \( L(s, \chi_D) \) must satisfy
\[ 0 < \frac{1}{1 - \beta} < \frac{L'(1, \chi_D)}{L}(1, \chi_D) + M\delta^{\frac{1}{2}} \log |D|, \]
where \( M > 0 \) is some absolute constant. It follows from Theorem 1.2 that, under weak uniform \( abc \), we must have \( \beta < 1 - \frac{m\delta^{-1/2} + o(1)}{\log |D|} \), where \( m = M^{-1} \). Changing \( o(1) \) to \( o_\delta(1) \) allows us to drop the condition \( |D| > N_\delta \), and since \( m\delta^{-\frac{1}{2}} = o_{\delta \to 0}(\delta^{-1}) \), Theorem 1.4 allows us to extend vertically the zero-free region obtained to the height 1 box \( \{ s \mid |t| \leq 1 \} \), concluding the proof. \( \square \)

4. The Bridge from \( \frac{L'}{L}(1, \chi_D) \) to \( \text{ht}(j(\tau_D)) \)

We will now prove Theorem 1.1, which is how we connect Siegel zeros with \( abc \). The proof essentially gets reduced down to a calculation once three concepts are introduced: the framework of Euler–Kronecker constants due to Ihara [2], Kronecker’s limit formula (KLF), and the uniform distribution of Heegner points due to Duke [6]. After briefly describing each of these, we compute the constant \( C \) from Theorem 1.1 with Lemmas 4.3, 4.5 and finish the proof in subsection 4.5.
4.1. Euler–Kronecker constants. Let $K/\mathbb{Q}$ be a number field, $\mathcal{O}_K$ its ring of integers, and $\zeta_K(s) = \sum_{a \in \mathcal{O}_K} [\mathcal{O}_K : a]^{-s}$ its Dedekind zeta function. In 2006, Ihara [9] introduced and studied the Euler–Kronecker constants $\gamma_K \in \mathbb{R}$, which are defined as the constant term in the Laurent expansion of $\zeta_K'/\zeta_K$ at $s = 1$:

$$\frac{\zeta_K'}{\zeta_K}(s) = -\frac{1}{s-1} + \gamma_K + O(s-1) \quad (s \to 1).$$

Write $\mathcal{C}_K$ for the ideal class group of $K$, and $h_K(=|\mathcal{C}_K|)$ for its class number. For each $A \in \mathcal{C}_K$, the partial zeta function $\zeta_K(s, A)$ is given by

$$\zeta_K(s, A) := \sum_{a \in A, a \text{ integral}} [\mathcal{O}_K : a]^s,$$

and so we have $\zeta_K(s) = \sum_{A \in \mathcal{C}_K} \zeta_K(s, A)$. Roughly following Ihara’s naming scheme, we define the Kronecker limits $\mathcal{R}(A)$ as the constant term in the Laurent expansion of $\zeta_K'/\zeta_K(s, A)$ at $s = 1$. The expansion of $\zeta_K(s, A)$ at $s = 1$ is given by

$$\zeta_K(s, A) = \frac{\mathcal{R}_K}{s-1} + \mathcal{R}_K \mathcal{R}(A) + O(s-1) \quad (s \to 1),$$

where $\mathcal{R}_K$, which is independent of $A$, is determined by the analytic class number formula. Hence, the Euler–Kronecker constant of $K$ is the average of its Kronecker limits:

$$\gamma_K = \frac{1}{h_K} \sum_{A \in \mathcal{C}_K} \mathcal{R}(A).$$

Using the classical limit formulas of Kronecker (for imaginary quadratic fields) and Hecke (for real quadratic fields – see Zagier [20]), together with the equidistribution results in Theorem 1 of Duke [6], one can obtain estimates for $\gamma_K$ of quadratic fields in terms of cycle integrals with good error terms. We will do this in detail for the imaginary quadratic case.

4.2. Kronecker’s limit formula. The classical real analytic Eisenstein series is given by

$$E(\tau, s) = \sum_{(m,n) \in \mathbb{Z}^2 \atop (m,n) \neq 0} \frac{\Im(\tau)^s}{|m\tau + n|^{2s}},$$

for $\tau \in \mathfrak{h}$ and $\Re(s) > 1$[5]. This is the simplest example of a “non-holomorphic modular function”, meaning it is invariant under modular transformations $\tau \mapsto \tau' = (\alpha \tau + \beta) / (\gamma \tau + \delta)$ for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$.

In the notation of (2.2), we have the identity

$$\zeta_{\mathbb{Q}(\sqrt{D})}(s, A) = \frac{1}{w_D} \left( \frac{2}{\sqrt{|D|}} \right)^s E(\tau_A, s).$$

[5] Some authors consider slightly modified versions of this function, such as divided by 2, or with the pair $(m, n)$ running through relatively prime integers (equivalent to considering $\zeta(2s)^{-1} E(\tau, s)$), or multiplied by $\pi^{-s} \Gamma(s)$. As we are following Siegel [16], we stick to his convention.
where \( w_D \) is the number of roots of unity in \( \mathbb{Q}(\sqrt{D}) \). The residue and constant term in the Laurent expansion of \( E(\tau, s) \) at \( s = 1 \) are given by

\[
E(\tau, s) = \frac{\pi}{s-1} + \pi \left( \frac{\pi}{3} \Im(\tau) - \log \Im(\tau) + \mathcal{U}(\tau) \right) + 2\pi(\gamma - \log 2) + O(s - 1),
\]

where

\[
\mathcal{U}(\tau) := 4 \sum_{n \geq 1} \left( \sum_{d \mid n} \frac{1}{d} \right) \frac{\cos(2\pi n \Re(\tau))}{e^{2\pi n \Im(\tau)}} = -2 \log(|\eta(\tau)|^2) - \frac{\pi}{3} \Im(\tau);
\]

this expansion is also sometimes called “Kronecker’s limit formula”. Here, \( \eta \) is Dedekind’s \( \eta \)-function, defined as

\[
\eta(z) := q^{1/24} \prod_{k \geq 1} (1 - q^k)
\]

for \( z = x + iy \in \mathfrak{h} \), where \( q := e^{2\pi i z} \) and \( q^{1/24} = e^{\pi iz/12} \). One checks that

\[
\mathcal{U}(z) = 4 \sum_{n \geq 1} \left( \sum_{d \mid n} \frac{1}{d} \right) \cos(2\pi n x) e^{2\pi ny} = 4 \Re \left( \sum_{n \geq 1} \left( \sum_{d \mid n} \frac{1}{d} \right) q^n \right)
\]

\[
= 4 \Re \left( \sum_{d \geq 1} \sum_{k \geq 1} \frac{q^{dk}}{d} \right) = -4 \Re \left( \sum_{k \geq 1} \log(1 - q^k) \right)
\]

\[
= -4 \sum_{k \geq 1} \log(1 - q^k) = -2 \log(|\eta(z)|^2) - \frac{\pi}{3} y,
\]

so the formula in (4.3) is indeed equivalent to the usual formulation.

Remark. The function \( \mathcal{U} \) in (4.4) is based on Stopple’s notation in p. 867 of [17]. Although (4.3) is usually given in terms of \( \eta \), this form of the statement emphasizes the dominant part \( \pi \Im(\tau) \) in the constant term of \( E(\tau, s) \).

Taking logarithmic derivatives in (4.2) and (4.3) immediately yields:

**Lemma 4.1 (KLF).** For each \( \mathcal{A} \in \mathcal{C}(D) \), we have

\[
\Re(\mathcal{A}) = \frac{\pi}{3} \Im(\tau_{\mathcal{A}}) - \log \Im(\tau_{\mathcal{A}}) + \mathcal{U}(\tau_{\mathcal{A}}) - \frac{1}{2} \log|D| + 2\gamma - \log 2.
\]

4.3. Duke’s equidistribution theorem. In 1988, W. Duke showed that the set \( \Lambda_D \) of Heegner points is uniformly distributed in \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h} \). To state his result, consider the probability space \( (\mathcal{F}, \Sigma, \mu) \), where:

- \( \mathcal{F} \subseteq \mathfrak{h} \) is the usual fundamental domain of \( \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h} \);
- \( \Sigma \) is the usual \( \sigma \)-algebra of Lebesgue measurable sets inherited from \( \mathbb{R}^2 \supseteq \mathfrak{h} \);
- \( d\mu := \frac{3}{\pi} dx dy/y^2 \) for \( z = x + iy \in \mathcal{F} \), so that \( \mu(\mathcal{F}) = 1 \).

Endowing \( \mathfrak{h} \) with its usual hyperbolic structure, we have the following:

\[\text{Cf. Chapter 20, §4 of Lang [13].}\]
Duke’s Theorem (Theorem 1 i) in [6]. Let \( \Omega \subseteq F \) be a convex set (in the hyperbolic sense) with piecewise smooth boundary. Then, there is a real number \( \delta = \delta(\Omega) > 0 \) such that
\[
\frac{|\Lambda_D \cap \Omega|}{|\Lambda_D|} = \mu(\Omega) + O_{\Omega, \delta}(|D|^{-\delta}),
\]
where the implied constant is ineffective.

Since hyperbolic convex subsets of \( \mathbb{H} \) constitute a basis for the usual topology inherited from \( \mathbb{R}^2 \supseteq \mathbb{H} \), the following is a direct corollary:

Lemma 4.2. Let \( f : F \to \mathbb{C} \) be a Riemann-integrable function in \( (F, \Sigma, \mu) \). Then:
\[
\lim_{D \to -\infty} \left( \frac{1}{h(D)} \sum_{\mathcal{O} \in C(D)} f(\tau_{\mathcal{O}}) \right) = \frac{3}{\pi} \int_F f(x + iy) \frac{dx dy}{y^2}.
\]

4.4. Computing \( C \). With Lemma 4.2, we are able to not only bound but also compute the average over \( \mathcal{O} \in C(D) \) of the non-dominant terms in Kronecker’s limit formula (4.6). While \( \kappa_1 \) and \( \kappa_3 \) are more easily shown to be bounded without Duke’s theorem, the main point of the next three lemmas is the computation of \( C \) in Theorem 1.1.

Lemma 4.3. \( \kappa_1 := -\frac{1}{6} \int_F \left( \log^+ |j(z)| - 2\pi y \right) d\mu = 0.011448\ldots \)

Proof. Writing \( q = e^{2\pi i z} \), we have \( |q| = e^{-2\pi y} \), so we can rewrite \( \kappa_1 \) as
\[
\kappa_1 = -\frac{1}{2\pi} \int_F \log \max \{|j(z)| \cdot |q|, |q|\} \frac{dx dy}{y^2}.
\]

We will prove the convergence of (4.8) in three steps, and then we will estimate its value numerically. Writing \( c(n) \) for the \( n \)-th coefficient in the \( q \)-expansion of the \( j \)-invariant in (2.3), we have the following:

- Step 1: For every \( n \geq 1 \), we have \( 0 \leq c(n) < e^{4\pi \sqrt{n}} \).
  Since \( (1 - q^n)^{-1} = \sum_{k \geq 0} q^{nk} \), it is clear from the \( q \)-expansion of \( j(z) \) in (2.3) that the \( c(n) \) are nonnegative. To show the upper bound, we use the fact that \( j \) is a modular function of weight 0 for \( \text{SL}_2(\mathbb{Z}) \). For every \( 0 < t < 1 \), we have \( j(it) = j(it) \).
  Thus, in terms of the \( q \)-expansion, rearranging this equality yields:
  \[
  \sum_{n \geq 0} c(n) \left( \frac{e^{2\pi n/t} - e^{2\pi nt}}{e^{2\pi n(1+t^{-1})}} \right) = e^{2\pi /t} - e^{2\pi t}.
  \]
  For \( n = 1 \) we have \( c(1) = 196884 < e^{4\pi} \). For \( n \geq 2 \), since the \( c(n) \) are nonnegative, we have
  \[
  c(n) \leq \left( \frac{e^{2\pi n/t} - e^{2\pi t}}{e^{2\pi n/t} - e^{2\pi nt}} \right) e^{2\pi nt(1+t^{-2})},
  \]
  This comes from the use of Siegel’s theorem in the proof. This ineffectiveness is hence passed down to all subsequent calculations.
so taking $t = t(n) := 1/\sqrt{n}$, it follows that

$$c(n) \leq \left( \frac{e^{2\pi \sqrt{n}} - e^{\pi \sqrt{n}}}{e^{\pi \sqrt{n}} - e^{2\pi \sqrt{n}}} \right) e^{2\pi \sqrt{n}(1+n)} = \left( \frac{1 - e^{-2\pi \sqrt{n}(1-n^{-1})}}{1 - e^{-2\pi \sqrt{n}(n-1)}} \right) e^{4\pi \sqrt{n}} < e^{4\pi \sqrt{n}},$$

as desired.

- **Step 2:** \( \frac{1}{2\pi} \int_{\mathcal{F} \cap \{\Im(z) \geq 16\}} \log \max \{|j(z)|, |q|, |q|\} \frac{dy}{y^2} \) \( \leq 10^{-21} \).
  For \( y \geq 4 \), we have \( 4\pi \sqrt{k} - 2 \pi ky \leq -\pi ky \) for every \( k \geq 1 \). Thus, by step 1, for \( y \geq 4 \), it follows that

$$|j(z)| \cdot |q| \leq 1 + \sum_{n \geq 0} c(n)|q|^{n+1} \leq 1 + \sum_{n \geq 0} e^{4\pi \sqrt{n} - 2\pi(n+1)y} \leq 1 + \sum_{n \geq 1} e^{-\pi ny}.$$

By partial summation,

$$\sum_{n \geq 1} e^{-\pi ny} = \pi y \int_{1}^{+\infty} [t] e^{-\pi yt} dt \leq \pi y \int_{1}^{+\infty} t e^{-\pi yt} dt = \left( 1 + \frac{1}{\pi y} \right) e^{-\pi y}.$$  

Hence, as \( \log(1 + t) \leq t \) for all \( t \geq 0 \), we have

$$\left| \frac{1}{2\pi} \int_{\mathcal{F} \cap \{\Im(z) \geq 16\}} \log \max \{|j(z)|, |q|, |q|\} \frac{dy}{y^2} \right| \leq \frac{1}{2\pi} \int_{16}^{+\infty} \frac{1 + (\pi y)^{-1}}{y^2 e^{\pi y}} dy \leq \frac{1}{2\pi} \int_{16}^{+\infty} \frac{1 + \pi y}{y^2 e^{\pi y}} dy \leq \frac{1}{16 e^{16\pi}}.$$  

Then, since \( e^{4\pi} > 10^5 \), we have \( e^{16\pi} > 10^{20} \) and \( 32\pi > 10 \), yielding step 2.

- **Step 3:** \( \frac{1}{2\pi} \int_{\mathcal{F}} \log \max \{|j(z)|, |q|, |q|\} \frac{dy}{y^2} = -0.011448\ldots \)
  Since \( \log \max \{|j(z)|, |q|, |q|\} \) is continuous in the closure of \( \mathcal{F} \cap \{\Im(z) < 16\} \), which is compact, the integral

$$\int_{\mathcal{F} \cap \{\Im(z) < 16\}} \log \max \{|j(z)|, |q|, |q|\} \frac{dy}{y^2}$$

converges. This, together with step 2, implies that \( (4.8) \) converges. Furthermore, by step 2, in order to obtain a computational estimate of \( (4.8) \) with 20 decimal places of accuracy, it suffices to estimate \( (4.9) \) to 20 decimal places of accuracy. Using Python’s mpmath library, the first 6 decimal places are \( 0.011448\ldots \). \( \square \)

**Lemma 4.4.** We have:

$$\kappa_2 := \int_{\mathcal{F}} \log(y) \, d\mu = 1 - \log 2 + \frac{3}{\pi} \sum_{n \geq 1} \frac{\sin(2\pi n / 3)}{n^2} = (0.952984\ldots).$$

**Proof.** Using that \( \int \cot(\vartheta) \, d\vartheta = \log(\sin(\vartheta)) \) and \( \int \cot(\vartheta)^2 \, d\vartheta = -\vartheta - \cot(\vartheta) \) (omitting the integration constants), we have:

$$\frac{3}{\pi} \int_{\mathcal{F}} \log(y) \frac{dy}{y^2} = 6 \int_{\frac{\pi}{2}}^{+\infty} \left( \int_{0}^{\frac{1}{2\sqrt{y}}} \log(y) \, dx \right) dy - 6 \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \left( \int_{0}^{\sqrt{4-y^2}} \log(y) \, dx \right) dy.$$  

\[ = \frac{2\sqrt{3}}{\pi} \left( \log \left( \frac{\sqrt{3}}{2} \right) + 1 \right) - \frac{6}{\pi} \int_{1}^{1} \frac{\log(y) \sqrt{1 - y^2}}{y^2} \, dy \]

\[ = \frac{2\sqrt{3}}{\pi} \left( \log \left( \frac{\sqrt{3}}{2} \right) + 1 \right) - \frac{6}{\pi} \int_{\frac{\pi}{3}}^{\pi} \log(\vartheta) \cot(\vartheta)^2 \, d\vartheta \]

(4.10)

\[ = 1 + \frac{3}{\pi} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \log(\sin(\vartheta)) \, d\vartheta. \]

Since \( \sin(-i \log(\xi)) = i (1 - \xi^2)/2\xi \), making the substitution \( \xi := e^{i\vartheta} \) yields:

\[ \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \log(\sin(\vartheta)) \, d\vartheta = -i \int_{r} \left( \log \left( \frac{1 - \xi^2}{2} \right) + \log \left( \frac{i}{\xi} \right) \right) \frac{d\xi}{\xi} \]

\[ = i \log(2) \log(e^{\pi i/3}) + \frac{1}{2i} \left( \text{Li}_2(e^{2\pi i/3}) - \text{Li}_2(e^{4\pi i/3}) \right) \]

\[ = -\frac{\pi}{3} \log(2) + \mathfrak{I} \left( \text{Li}_2(e^{i\vartheta}) \right), \]

where \( \omega := e^{2\pi i/3}, \Gamma := \{ e^{i\vartheta} \mid \vartheta \in [\pi/3, 2\pi/3] \} \) is oriented counterclockwise, and \( \text{Li}_2(z) := \sum_{n \geq 1} \frac{z^n}{n^2} \) is the dilogarithm function. Putting this together with (4.10), we deduce the lemma by applying Lemma 4.2 and by observing that for every \( \vartheta \in (-\pi, \pi) \) we have \( \mathfrak{I}(\text{Li}_2(e^{i\vartheta})) = \sum_{n \geq 1} \sin(n\vartheta)/n^2 \).

□

**Lemma 4.5.** Writing \( \omega := e^{2\pi i/3} \), we have:

\[ \kappa_3 := \int_{\mathfrak{U}} U(z) \, d\mu = \frac{24}{\pi} \Re \left( \sum_{n \geq 1} \left( \sum_{d \mid n} \frac{1}{d} \right) \int_{\omega} e^{2\pi inz} \frac{dz}{z} \right) \]

\[ (= -0.000303 \ldots). \]

**Proof.** From Equation (4.5), for \( z = x + iy \in \mathfrak{h} \) we have

(4.11)

\[ U(z) = 4 \Re \left( \sum_{n \geq 1} \left( \sum_{d \mid n} \frac{1}{d} \right) e^{2\pi inz} \right) \]

Writing \( \xi = \cos(\vartheta) + i \sin(\vartheta) \), we have

\[ \int_{\mathfrak{U}} e^{2\pi inz} \frac{dxdy}{y^2} = -\frac{1}{2\pi in} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} e^{2\pi in\xi} \cos(\vartheta) \sin(\vartheta)^2 \, d\vartheta \]

\[ = -\frac{1}{2\pi in} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} e^{2\pi in\xi} \cos(\vartheta) \, d\vartheta \]

\[ = -i \int_{r} e^{2\pi in\xi} \left( \frac{2\xi}{\xi^2 - 1} \right) \, d\xi, \]

where \( \Gamma := \{ e^{i\vartheta} \mid \vartheta \in [\pi/3, 2\pi/3] \} \) is oriented counterclockwise. Since \( 2\xi/(\xi^2 - 1) = (\xi - 1)^{-1} + (\xi + 1)^{-1} \), we have

\[ -i \int_{r} e^{2\pi in\xi} \left( \frac{2\xi}{\xi^2 - 1} \right) \, d\xi = -i \left( \int_{r-1} e^{2\pi in\xi} \frac{d\xi}{\xi} + \int_{r+1} e^{2\pi in\xi} \frac{d\xi}{\xi} \right) \]
\[
2 \Im \left( \int_{\omega}^{\omega^{-1}} e^{2\pi i n \xi} \frac{d\xi}{\xi} \right),
\]
which is a real number. Thus, putting this together with (4.11), we get the series in the statement of the lemma. To see that \( \kappa_3 \) converges, note that
\[
\left| \int_{\omega}^{\omega^{-1}} e^{2\pi i n \xi} \frac{d\xi}{\xi} \right| \leq e^{-2\pi n \Im(\omega)} \frac{1}{|\omega|},
\]
and hence, \( |\kappa_3| \leq 24(\pi |\omega|)^{-1} \sum_{n \geq 1} \left( \sum_{d|n} d^{-1} \right) e^{-\pi n \sqrt{3}} \), which converges. \( \square \)

4.5. **Proof of Theorem 1.1.** Since \( j(\tau_D) \) is integral, (2.4) gives us
\[
ht(j(\tau_D)) = \frac{1}{h(D)} \sum_{\omega \in \text{Cl}(D)} \log^+ |j(\tau_{\omega})|.
\]
Hence, from (4.1), we deduce that
\[
\gamma_{Q(\sqrt{D})} = \frac{1}{6} \text{ht}(j(\tau_D)) - \frac{1}{2} \log |D| + (\kappa_1 - \kappa_2 + \kappa_3 + 2\gamma - \log 2) + o(1)
\]
by Lemmas 4.1, 4.2, and 4.3–4.5. From the identity \( \zeta_{Q(\sqrt{D})}(s) = \zeta(s)L(s, \chi_D) \) we have \( \gamma_{Q(\sqrt{D})} = \gamma + \frac{L'}{L}(1, \chi_D) \), concluding the first part of Theorem 1.1.

To prove (1.4), one simply expands the definition of \( \text{ht}(j(\tau_D)) \) and uses the \( q \)-expansion of the \( j \)-invariant, thus obtaining
\[
\text{ht}(j(\tau_D)) = \frac{1}{h(D)} \sum_{(a,b,c)} \frac{\pi \sqrt{|D|}}{a} + O(1).
\]
Rearranging the expression of the first part of Theorem 1.1 yields (1.4). \( \square \)

**Remark 4.6.** Theorem 1.1 is reminiscent of a result by Colmez [2, 3]. Writing \( E_D/\mathbb{C} \) for an elliptic curve with complex multiplication by \( \mathbb{Z}[\tau_D] \), Colmez showed that, writing \( \text{ht}_{\text{Fal}} \) for the Faltings height\(^8\)
\[
-2 \text{ht}_{\text{Fal}}(E_D) = \frac{1}{2} \log |D| + \frac{L'}{L}(0, \chi_D) + \log 2\pi.
\]
Together with Theorem 1.1 this yields
\[
\text{ht}_{\text{Fal}}(E_D) = \frac{1}{12} \text{ht}(j(\tau_D)) + \kappa + o_{D \to \infty}(1),
\]
where \( \kappa = \frac{C}{2} - \frac{7}{2} - \log 2\pi = -2.655370 \ldots \). It is interesting to note that in Remarque (ii), p. 364 of [3], Colmez hinted at a possible connection between \( abc \) and Siegel zeros, two years before the work of Granville and Stark.

\(^8\)See the diagram in subsection 0.6, p. 663 of [2] – note that there is a typo: the upper right corner should read “\(-2\text{ht}_{\text{Fal}}(X) - \frac{1}{2} \log D\)” instead.

\(^9\)Cf. the standard “\( O(\log(\text{ht})) \)” in Chapter X of Cornell–Silverman [4].
5. Estimating $\text{ht}(j(\tau_D))$ with $abc$

We are now going to prove Theorem 1.2. Already from Theorems 1.4 and 1.1 we have

\begin{equation}
\liminf_{D \to -\infty} \frac{1}{|D|^{|\epsilon|} \text{-smooth}} \frac{L'}{L}(1, \chi_D) \geq 0,
\end{equation}

so all we need is to work out the upper bounds. That will be done using the two uniform formulations of the $abc$-conjecture mentioned in the introduction, following Granville–Stark’s method.

5.1. Uniform $abc$-conjecture. We start with the usual $abc$-conjecture for number fields. Let $\text{rd}_K = |\Delta_K|^{1/[K:\mathbb{Q}]}$ be the root-discriminant of $K/\mathbb{Q}$.

**Conjecture 5.1** (uniform $abc$-conjecture for number fields – see Vojta [18]). Let $K/\mathbb{Q}$ be a number field. For every $\epsilon > 0$, there is a constant $C = C(K, \epsilon) \in \mathbb{R} \geq 0$ such that, for any $a, b, c \in K$ with $a + b + c = 0$, the following holds:

\begin{equation}
\text{ht}([a : b : c]) < (1 + \epsilon) \left( N_K([a : b : c]) + \log(\text{rd}_K) \right) + C(K, \epsilon).
\end{equation}

**Remark.** For $a, b, c \in \mathbb{Z}$ coprime, we have $\text{ht}([a : b : c]) = \log \max \{|a|, |b|, |c|\}$ and $N_\mathbb{Q}([a : b : c]) = \log \left( \prod_{p \mid abc} p \right)$, recovering the $abc$-conjecture for $\mathbb{Z}$.

For the method to work, we also need information about the growth of $C(K, \epsilon)$ as a function of $K$. The approach we take is to assume that $C(K, \epsilon)$ is dominated by the contribution "log(\text{rd}_K)" of ramified primes in the inequality, in two senses of the concept of domination: either big-$O$ or small-$o$. Both have implications to the Siegel zeros problem, and we state the usual uniform $abc$-conjecture for comparison:

**Conjecture 5.2** (Uniformity). Let $K/\mathbb{Q}$ be a number field and $\epsilon > 0$. Then:

(i) (O-weak uniform $abc$) $C(K, \epsilon) = O_\epsilon(\log(\text{rd}_K))$.

(ii) (Weak uniform $abc$) $C(K, \epsilon) = o_\epsilon(\log(\text{rd}_K))$ as $\text{rd}_K \to +\infty$.

(iii) (Uniform $abc$) $C(K, \epsilon) = O_{\epsilon}(1)$.

It is remarked in p. 510 of Granville–Stark [7] that Conjecture 5.2 (iii) follows from Vojta’s General Conjecture under the assumption that $[K : \mathbb{Q}]$ is bounded; consequently, so does (i) and (ii).\[10\]

**Remark 5.3.** See also Remark 2.2.3 of Mochizuki’s “IUT IV” [14], in which it is explained that the calculations of Corollary 2.2 (ii), (iii) of IUT IV can be regarded as a sort of “weak” version of uniform $abc$. Such version, however, is much weaker than the O-weak uniform $abc$ in Conjecture 5.2 (i), and thus, in principle, one is not able to deduce “no Siegel zeros” from Corollary 2.2 of [14] by using the methods we are employing here.

\[10\]Writing $n = [K : \mathbb{Q}]$, the Hermite–Minkowski theorem says that $\text{rd}_K \geq \frac{\pi}{4} (n/\sqrt{n})^2$. Hence, from Stirling’s formula, it follows that $\log(\text{rd}_K) \geq 2 + \log(\pi/4) + O(\log n/n) \gg 1$ uniformly for $n \geq 2$, and so (iii) $\implies$ (ii) $\implies$ (i).
5.2. Proof of Theorem 1.2. Consider the Weber modular functions $\gamma_2, \gamma_3$, related to the $j$-invariant via the identities
\[(5.3)\quad j(\tau) = \gamma_2(\tau)^3 = \gamma_3(\tau)^2 + 1728.\]
For each $D < 0$, the pair $(\gamma_2(\tau_D), \gamma_3(\tau_D))$ provides a solution $(x, y)$ to the Diophantine equation $x^3 - y^2 = 1728$. This solution lies in the extension $\tilde{H}_D := H_D(\gamma_2(\tau_D), \gamma_3(\tau_D))$, where $H_D = \mathbb{Q}(\sqrt{-D}, j(\tau_D))$ is the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$. If $\gcd(D, 6) = 1$, then we actually have $\tilde{H}_D = H_D$.

In general, we have the following:

**Lemma 5.4** (Lemma 1 of Granville–Stark [7]).
\[\rd_{\tilde{H}_D} \leq 6 \sqrt{|D|}.\]
This lemma quantifies the fact that $\tilde{H}_D/H_D$ has very little ramification, which is a key point. We remark that the particular factor of “6” plays little role here, since it would be enough to have $\rd_{\tilde{H}_D} \ll \varepsilon \sqrt{|D|}$.

**Lemma 5.5.** Assume the abc-conjecture (5.2). Then, for every fundamental discriminant $D < 0$ and $0 < \varepsilon < 1/5.01$, we have
\[\ht(j(\tau_D)) < \frac{1}{1 - 5\varepsilon} ((1 + \varepsilon) 3 \log |D| + 6 C(\tilde{H}_D, \varepsilon)) + O(1).\]

**Proof.** Writing $M := (1 + \varepsilon) \log(\rd_{\tilde{H}_D}) + C(\tilde{H}_D, \varepsilon)$ and applying abc to the equation $\gamma_2(\tau_D)^3 - \gamma_3(\tau_D)^2 - 1728 = 0$, we get
\[\ht([\gamma_2^3 : \gamma_3^2 : 1728]) \leq (1 + \varepsilon) (\frac{1}{3} \ht(\gamma_2^3) + \frac{1}{2} \ht(\gamma_3^2)) + M + O(1)\]
\[\leq (1 + \varepsilon) \frac{2}{3} \ht([\gamma_2^3 : \gamma_3^2 : 1728]) + M + O(1).\]
Since $\ht(j(\tau_D)) \leq \ht([j(\tau_D) : j(\tau_D) - 1728 : 1728])$, the claim of the lemma follows by rearranging the expression above and using Lemma 5.4. \(\square\)

**Proof of Theorem 1.2.** From Lemma 5.4 we have $\log(\rd_{\tilde{H}_D}) = \frac{1}{2} \log |D| + O(1)$, so it follows from Lemma 5.5 together with (5.1) that
\[O\text{-weak uniform } abc \text{ (Conjecture 5.2 (i)) } \implies \limsup_{D \to -\infty} \frac{\ht(j(\tau_D))}{\log |D|} < +\infty,\]
weak uniform abc (Conjecture 5.2 (ii)) \(\implies \limsup_{D \to -\infty} \frac{\ht(j(\tau_D))}{\log |D|} = 3;\)
hence, Theorem 1.2 follows directly from Theorem 1.1. \(\square\)

**Appendix A.** On “$B + \sum 1/\varrho = 0$”
Let $\chi \pmod{q}$ be a primitive Dirichlet character, $q \geq 2$. In section [3] we stated an expression for $L'(s, \chi)$ in (3.2) that is different from what usually appears in
\[\text{Cf. §72 of Weber [19].}\]
textbooks. Many investigations on Dirichlet \( L \)-functions have as a starting point the following classical formula:

\[
\frac{L'}{L}(s, \chi) = \left( \sum_{\varrho(\chi)} \frac{1}{s - \varrho} \right) - \frac{1}{2} \log \left( \frac{q}{\pi} \right) - \frac{1}{2} \frac{\Gamma'(s + a)}{\Gamma(s + a)} + \left( B_\chi + \sum_{\varrho(\chi)} \frac{1}{\varrho} \right).
\]

where \( a_\chi := \frac{1}{2}(1 - \chi(-1)) \), “\( \sum_{\varrho(\chi)} \)” denotes a sum over the non-trivial zeros of \( L(s, \chi) \), and \( B_\chi \in \mathbb{C} \) is a constant not depending on \( s \). While the fact that \( \mathfrak{R}(B_\chi + \sum_{\varrho(\chi)} 1/\varrho) = 0 \) is well-known, the more precise

\[
(A.1) \quad B_\chi + \sum_{\varrho(\chi)} \frac{1}{\varrho} = 0
\]

does not appear to be as much so, despite holding true in similar generality. It follows from Theorem 2 (p. 257) of Ihara–Murty–Shimura [10], for example, that the equivalent of \( (A.1) \) holds for finite-order Hecke characters in number fields (thus, in particular, for Dirichlet characters).

While the proof in [10] uses Weil’s “explicit formula”, it is possible to prove \( (A.1) \) with a simpler, more direct analytic argument, as shown by Lucia [8]. We will reproduce Lucia’s argument here in some generality, given the conspicuous absence of this fact in the literature, showing \( (A.1) \) for the extended Selberg class of \( L \)-functions, thus highlighting that it is a relatively simple consequence of the functional equation.

A.1. The extended Selberg class \( S^\# \). Following Kaczorowski [12], a Dirichlet series \( L(s) = \sum_{n \geq 1} a_n n^{-s} \) is said to be an \( L \)-function in the extended Selberg class (denoted \( S^\# \)) if it satisfies the following properties:

(S1) \( \sum_{n \geq 1} a_n n^{-s} \) is absolutely convergent for \( \mathfrak{R}(s) > 1 \).

(S2) (Analytic continuation) There exists \( k \in \mathbb{Z}_{\geq 0} \) such that \( (s - 1)^k L(s) \) is entire of finite order. Write \( k(L) \) for the smallest such \( k \).

(S3) (Functional equation) There exists a triple \( (c, Q, \gamma_L) \), where \( c, Q \) are positive real numbers, and \( \gamma_L \) is a function of the form

\[
\gamma_L(s) := \prod_{j=1}^f \Gamma(\lambda_j s + \mu_j)
\]

(called a Gamma factor), with \( f \in \mathbb{Z}_{\geq 1}, \) positive real numbers \( \lambda_j, 1 \leq j \leq f \), and complex numbers \( W, \mu_j, 1 \leq j \leq f \) with \( |W| = 1 \) and \( \mathfrak{R}(\mu_j) \geq 0 \), for which the completed \( L \)-function

\[
(A.2) \quad \Lambda_L(s) := c Q^s \gamma_L(s) L(s)
\]

satisfies a functional equation: \( \Lambda_L(s) = W \Lambda_L(1 - s) \).

This class and the more restrictive Selberg class (with the additional Euler product and Ramanujan hypothesis properties) contain most of the \( L \)-functions appearing in number theory, including \( L \)-functions of primitive Dirichlet characters and finite-order primitive Hecke characters in number fields. From now on, let \( L \) be a fixed \( L \)-function in \( S^\# \). See section 2 of Kaczorowski [12] for the facts used below.

\[\text{Cf. Eqs. (17), (18), Chapter 12, p. 83 of Davenport [5].}\]
Denote by \( \varrho(\mathcal{L}) \) the multiset of non-trivial zeros of \( \mathcal{L} \) (i.e., zeros of \( \Lambda_{\mathcal{L}}(s) \)). There is a real number \( \sigma_{\mathcal{L}} \geq 1 \) such that \( 1 - \sigma_{\mathcal{L}} \leq \Re(\varrho) \leq \sigma_{\mathcal{L}} \) for every non-trivial zero \( \varrho \) of \( \mathcal{L} \). Writing \( N_{\mathcal{L}}^T(T) = \#\{\varrho(\mathcal{L}) \mid 0 \leq \pm \Im(\varrho) < T\} \), by standard techniques based on the argument principle one shows that

\[
N_{\mathcal{L}}^{-}(T), \quad N_{\mathcal{L}}^{+}(T) = \frac{d_{\mathcal{L}}}{2\pi} T \log T + c_{\mathcal{L}} T + O(\log T),
\]

where \( d_{\mathcal{L}} = 2\sum_{j=1}^{f} \lambda_j \) is the degree of \( \mathcal{L} \), and \( c_{\mathcal{L}} \in \mathbb{R} \) is some constant depending only on \( \mathcal{L} \). This implies, in particular, that \( \Lambda_{\mathcal{L}} \) has order 1 as an entire function, and that \( \sum_{\varrho(\mathcal{L}) \neq 0} 1/\varrho \) converges in the principal-value sense \( \lim_{T \to +\infty} \sum_{\varrho(\mathcal{L}) \neq 0, |\Im(\varrho)| \leq T} \).

Thus, from \([S2]\) \([S3]\) by Hadamard’s factorization theorem we have

\[
(s(1 - s))^{k(\mathcal{L})} \Lambda_{\mathcal{L}}(s) = s^{m_0}(1 - s)^{m_1} e^{A+Bs} \prod_{\varrho(\mathcal{L}) \neq 0,1} \left(1 - \frac{s}{\varrho}\right) e^{s/\varrho},
\]

where \( m_0 - k(\mathcal{L}) \), \( m_1 - k(\mathcal{L}) \) are the order of \( s = 0, 1 \) as zeros of \( \Lambda_{\mathcal{L}}(s) \) (hence negative if it is a pole), respectively, and \( A, B \in \mathbb{C} \) are constants. Since \( \Lambda_{\mathcal{L}}(0) = \Lambda_{\mathcal{L}}(1) \), it follows that \( m_0 - k(\mathcal{L}) = m_1 - k(\mathcal{L}) \); thus, by taking logarithmic derivatives, we derive:

\[
\frac{\Lambda'_{\mathcal{L}}(s)}{\Lambda_{\mathcal{L}}(s)} + \frac{k(\mathcal{L})}{s} + \frac{k(\mathcal{L})}{s - 1} = \sum_{\varrho(\mathcal{L})} \frac{1}{s - \varrho} + \left(B + \sum_{\varrho(\mathcal{L}) \neq 0,1} \frac{1}{\varrho}\right).
\]

This is our starting point.

### A.2. Proof of “\( B + \sum 1/\varrho = 0 \)”.

**Theorem A.1.** For \( \mathcal{L} \in S' \), in the notation of \([S2]\) \([S3]\) we have

\[
\frac{\mathcal{L}'}{\mathcal{L}}(s) + \frac{k(\mathcal{L})}{s} + \frac{k(\mathcal{L})}{s - 1} = \left(\sum_{\varrho(\mathcal{L})} \frac{1}{s - \varrho}\right) - \log Q - \frac{\gamma'_{\mathcal{L}}(s)}{\gamma_{\mathcal{L}}(s)}.
\]

**Proof.** From \([S3]\) we have that, if \( \varrho \) is a zero of \( \mathcal{L} \), then so is \( 1 - \overline{\varrho} \). Moreover, \( \varrho(\overline{\mathcal{L}}) = \{1 - \varrho \mid \varrho \in \varrho(\mathcal{L})\} \), where \( \overline{\mathcal{L}}(s) = \overline{\mathcal{L}(s)} \) is the dual of \( \mathcal{L} \). Therefore, by \([A.2]\) and \([A.4]\),

\[
0 = \lim_{s \to 1} \left(\frac{\Lambda'_{\mathcal{L}}}{\Lambda_{\mathcal{L}}}(s) + \frac{\Lambda'_{\overline{\mathcal{L}}}}{\Lambda_{\overline{\mathcal{L}}}}(1 - s)\right) = 2\Re\left(B + \sum_{\varrho(\mathcal{L}) \neq 0,1} \frac{1}{\varrho}\right).
\]

Thus, in view of \([A.2]\), to prove the theorem we just need to show that \( \Im\left(B + \sum_{\varrho(\mathcal{L}) \neq 0,1} 1/\varrho\right) = 0 \). Since both \( \Im\left(\frac{\mathcal{L}'}{\mathcal{L}}(R)\right) \), \( \Im\left(\frac{\overline{\mathcal{L}}'}{\overline{\mathcal{L}}}(R)\right) \to 0 \) as \( R \to +\infty \) through the real numbers, by \([A.2]\) and \([A.4]\) it suffices to show that

\[
\Im\left(\sum_{\varrho(\mathcal{L})} \frac{1}{R - \varrho}\right) \ll \frac{\log R}{R}.
\]
Let $R > 0$ be fixed and consider zeros up to height $T > 0$. Writing $\rho = \beta + i\gamma$ for a generic non-trivial zero, since $|\beta| \ll 1$, we have

$$\sum_{\rho \in \mathcal{L}} \frac{(R - \beta) + i\gamma}{(R - \beta)^2 + \gamma^2} = \sum_{\rho \in \mathcal{L}, |\gamma| < T} \left( \frac{R - \beta + i\gamma}{R^2 + \gamma^2} + O \left( \frac{R}{(R^2 + \gamma^2)^{3/2}} \right) \right).$$

For the error term, we have

$$\sum_{\rho \in \mathcal{L}} \frac{R}{(R^2 + \gamma^2)^{3/2}} \leq \sum_{\rho \in \mathcal{L}, |\gamma| \leq R} \frac{1}{R^2} + \sum_{\rho \in \mathcal{L}, |\gamma| > R} \frac{R}{|\gamma|^3} \ll \frac{\log R}{R},$$

while, for the imaginary part of the main term, since from (A.3) it follows that $|N_{\mathcal{L}}^+(T) - N_{\mathcal{L}}^-(T)| \ll \log T$,

$$\sum_{\rho \in \mathcal{L}, |\gamma| < T} \frac{\gamma}{R^2 + \gamma^2} = \int_0^T \frac{t}{R^2 + t^2} dN_{\mathcal{L}}^+(t) - \int_0^T \frac{t}{R^2 + t^2} dN_{\mathcal{L}}^-(t)$$

$$= \frac{T}{R^2 + T^2} (N_{\mathcal{L}}^+(T) - N_{\mathcal{L}}^-(T)) - \int_0^T \frac{R^2 + 3t^2}{(R^2 + t^2)^2} (N_{\mathcal{L}}^+(t) - N_{\mathcal{L}}^-(t)) dt$$

$$\ll \frac{T \log T}{R^2 + T^2} + \int_0^T \frac{\log t}{R^2 + t^2} dt \ll \frac{\log R}{R}. \quad \Box$$

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