Supercharacter Table of Certain Finite Groups

H. Saydi$^{1,a}$, M. R. Darafsheh$^{2,b}$ and A. Iranmanesh$^{1,c}$

(1) College of Mathematical Science, Tarbiat Modares University, Tehran, Iran
(2) School of mathematics, statistics and computer science, College of science, University of Tehran, Tehran, Iran

(a) e-mail: h.seydi@modares.ac.ir
(b) e-mail: darafsheh@ut.ac.ir
(c) e-mail: iranmana@modares.ac.ir

Abstract

Supercharacter theory is developed by P. Diaconis and I. M. Isaacs as a natural generalization of the classical ordinary character theory. Some classical sums of number theory appear as supercharacters which are obtained by the action of certain subgroups of $GL_d(\mathbb{Z}_n)$ on $\mathbb{Z}_n^d$.

In this paper we take $\mathbb{Z}_p^d$, $p$ prime, and by the action of certain subgroups of $GL_d(\mathbb{Z}_p)$ we find supercharacter table of $\mathbb{Z}_p^d$.

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1 Introduction

Let $Irr(G)$ denote the set of all the irreducible complex characters of a finite group $G$, and let $Con(G)$ denote the set of all the conjugacy classes of $G$. The identity element of $G$ is denoted by 1 and the trivial character is denoted by $1_G$. By definition a supercharacter theory for $G$ is a pair $(\mathcal{X}, \mathcal{K})$ where $\mathcal{X}$ and $\mathcal{K}$ are partitions of $Irr(G)$ and $G$ respectively, $|\mathcal{X}| = |\mathcal{K}|$, $\{1\} \in \mathcal{K}$, and for each $X \in \mathcal{X}$ there is a character $\sigma_X$ such that $\sigma_X(x) = \sigma_X(y)$ for all $x, y \in K$, $K \in \mathcal{K}$. We call $\sigma_X$ supercharacter and each member of $\mathcal{K}$ a superclass. We write $Sup(G)$ for the set of all the supercharacter theories of $G$.

Supercharacter theory of a finite group were defined by Diaconis and Isaacs as a general case of the ordinary character theory. In fact, in a supercharacter theory characters play the role of irreducible ordinary characters and union of conjugacy classes play the role of conjugacy classes.
In [3] it is shown that \( \{1_G\} \in \mathcal{X} \) and if \( X \in \mathcal{X} \) then \( \sigma_X \) is a constant multiple of \( \sum_{\chi \in X} \chi(1)\chi \), and that we may assume that \( \sigma_X = \sum_{\chi \in X} \chi(1)\chi \).

For any finite group there are two trivial supercharacter theories as follows. In the first case, \( \mathcal{X} = \bigcup_{\chi \in \operatorname{Irr}(G)} \{\chi\} \) and \( \mathcal{K} \) is the set of all conjugacy classes of \( G \). In the second case, \( \mathcal{X} = \{1_G\} \cup \{\operatorname{Irr}(G) - \{1_G\}\} \)

and \( \mathcal{K} = \{1\} \cup \{G - \{1\}\} \). In the first case, supercharacters are just irreducible characters and superclasses are conjugacy classes. In the second case, the non-trivial supercharacter is \( \rho_G - 1_G \), where \( \rho_G \) denotes the regular character of \( G \). These two supercharacter theories of \( G \) are denoted by \( m(G) \) and \( M(G) \) respectively.

It is mentioned in [6] that the set of supercharacter theories of a group form a Lattice in the following natural way. \( \operatorname{Sup}(G) \) can be made to a poset by defining \( (\mathcal{X}, \mathcal{K}) \leq (\mathcal{Y}, \mathcal{L}) \) if \( \mathcal{X} \leq \mathcal{Y} \) in the sense that every part of \( \mathcal{X} \) is a subset of some part of \( \mathcal{Y} \). In [6] it is shown that this definition is equivalent to \( (\mathcal{X}, \mathcal{K}) \leq (\mathcal{Y}, \mathcal{L}) \) if \( \mathcal{K} \leq \mathcal{L} \). By this definition \( m(G) \) is the least and \( M(G) \) is the largest element of \( \operatorname{Sup}(G) \).

Among construction of supercharacter theories of a finite group \( G \) the following is of great importance which is a lemma by Brauer on character tables of groups. Let \( A \) be a subgroup of \( \operatorname{Aut}(G) \) and \( \operatorname{Irr}(G) = \{\chi_1 = 1_G, \ldots, \chi_h\} \), \( \operatorname{Con}(G) = \{C_1 = \{1\}, \ldots, C_h\} \). Suppose for each \( \alpha \in A \), \( C_\alpha^i = C_j \), \( 1 \leq i \leq h \), and \( \chi_i(\alpha)(g) = \chi_i(g^\alpha) \) for all \( g \in G \), \( \alpha \in A \), then the number of conjugacy classes fixed by \( \alpha \) equals the number of irreducible characters fixed by \( \alpha \), and moreover the number of orbits of \( A \) on \( \operatorname{Con}(G) \) equals the number of orbits of \( A \) on \( \operatorname{Irr}(G) \), [4]. It is easy to see that the orbits of \( A \) on \( \operatorname{Irr}(G) \) and \( \operatorname{Con}(G) \) yield a supercharacter theory for \( G \). This supercharacter theory of \( G \) is called automorphic. In [7] it is shown that all the supercharacter theories of the cyclic group of order \( p \), \( p \) prime, are automorphic.

Another aspect of the supercharacter theory of finite groups is to employ the theory to the group \( U_n(F) \), the group of \( n \times n \) unimodular upper triangular matrices over the Galois field \( GF(p^m) \), \( p \) prime. Computation of the conjugacy classes and irreducible characters of \( U_n(F) \) is still open, but in [1] the author has developed an applicable supercharacter theory for \( U_n(F) \). This result is reviewed in [3].

## 2 Supercharacter table

Let \( G \) be a finite group and \( (\mathcal{X}, \mathcal{K}) \) be a supercharacter theory for \( G \). Suppose \( \mathcal{X} = \{X_1, X_2, \ldots, X_h\} \) be a partition for \( \operatorname{Irr}(G) \) with the corresponding supercharacter \( \sigma_i = \sum_{\chi \in X_i} \chi(1)\chi \).

Let \( \mathcal{K} = \{K_1, K_2, \ldots, K_h\} \) be the partition of \( G \) into superclasses. In fact \( X_1 = \{1_G\} \), \( K_1 = \{1\} \) and \( K_i \)'s are conjugacy classes of \( G \). The supercharacter table of \( G \) corresponding to \( (\mathcal{X}, \mathcal{K}) \) is the following \( h \times h \) array:
Let us set \( S = (\sigma_i(K_j))_{i,j=1}^h \), and call it the supercharacter table of \( G \).

Recall that a class function on \( G \) is a function \( f : G \rightarrow \mathbb{C} \) which is constant on conjugacy classes of \( G \). The set of all the class functions on \( G \), \( Cf(G) \) has the structure of a vector space over \( \mathbb{C} \) with an orthonormal basis \( \text{Irr}(G) \) with respect the inner product \( \langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)} \).

Since supercharacters are constant on superclasses, it is natural to call them superclass functions. We have:

\[
\langle \sigma_i, \sigma_j \rangle = \frac{1}{|G|} \sum_{k=1}^h |K_k| \sigma_i(K_k) \sigma_j(K_k)
\]

But using the orthogonality of \( \text{Irr}(G) \) we also can write:

\[
\langle \sigma_i, \sigma_j \rangle = \langle \sum_{\chi \in X_i} \chi(1) \chi, \sum_{\varphi \in X_j} \varphi(1) \varphi \rangle = \delta_{ij} \sum_{\chi \in X_i} \chi(1)^2
\]

Therefore

\[
\frac{1}{|G|} \sum_{k=1}^h |K_k| \sigma_i(K_k) \sigma_j(K_k) = \delta_{ij} \sum_{\chi \in X_i} \chi(1)^2.
\]

If we set the matrix

\[
U = \frac{1}{\sqrt{|G|}} \left[ \sum_{\chi \in X_i} \chi(1)^2 \right]^{h}_{i,j=1}
\]

We see that \( U \) is a unitary matrix with the following properties, which are proved in [2]. We have \( U = U^t \), \( U^2 = P \) where \( P \) is a permutation matrix and \( U^4 = I \).

In the course of studying the supercharacter theory of a group \( G \) finding the supercharacter table of \( G \) and the matrix \( U \) is of great importance. In this paper, we will do this task for certain groups acting on certain sets.

### 3 Automorphic supercharacter table

In this section we follow the method used in [2] considering the group \( G = \mathbb{Z}_n^d \) which abelian of order \( n^d \). The automorphism group of \( G \) is \( GL_d(\mathbb{Z}_n) \), the group of \( d \times d \) invertible matrices with entries in \( \mathbb{Z}_n \). We write elements of \( G \) as row vectors \( y = (y_1, \ldots, y_d) \) and let the action of \( GL_d(\mathbb{Z}_n) \) on \( G \) be as follows:
Irreducible characters of $G$ are of degree 1 and the number of them is equal to $|G|$. For $x \in G$, let us define $\psi_x : G \rightarrow \mathbb{C}^\times$, by $\psi_x(\zeta) = e(x \cdot \zeta)$, where $e(t) = e^{2\pi i t}$ and $x \cdot \zeta$ is the inner product of two elements $x$ and $\zeta$ of $G$ as row vectors in $G = \mathbb{Z}_n^d$. Therefore $Irr(G) = \{ \psi_x | x \in G \}$ and the action of $GL_d(\mathbb{Z}_n)$ on $Irr(G)$ is as follows:

$$\psi_x^A = \psi_x_{|A^{-1}}$$ where $A \in GL_d(\mathbb{Z}_n)$, $x \in G$.

Now let $\Gamma$ be a symmetric subgroup of $GL_d(\mathbb{Z}_n)$, i.e. $\Gamma^t = \Gamma$. Then $\Gamma$ acts on $G$ and $Irr(G)$ as above. Let $\mathcal{X}$ be the set of orbits of $\Gamma$ on $Irr(G)$ and $\mathcal{K}$ be the set of orbits of $GL_d(\mathbb{Z}_n)$ on $Irr(G)$.

It is shown in [2] that $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of $G$. Following the notations used in [2] we identity $\psi_x$ with $x$ and $\psi_x^A = \psi_{xA^{-1}} = xA^{-t}$. Therefore $\mathcal{X}$ is identified with the set of orbits of $GL_d(\mathbb{Z}_n)$ on $G$, by $x \mapsto xA^{-t}$, and $\mathcal{K}$ is identified with the orbits of the action of $GL_d(\mathbb{Z}_n)$ on $G$ by $y \mapsto yA$.

In [2] using different subgroups of $GL_d(\mathbb{Z}_n)$ the authors provide supercharacter tables for $G$. For example the discrete Fourier transform in the case of $\Gamma = \{1\}$, or $\Gamma = \{\pm 1\}$ a group of order 2. The Gauss sums is obtained in the case of $G = \mathbb{Z}_p$, $p$ an odd prime, $\Gamma = \langle g^2 \rangle$ where $g$ is a primitive root modulo $p$. Kloosterman sums in the case $G = \mathbb{Z}_p^2$, $p$ an odd prime and $\Gamma = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid 0 \neq a \in \mathbb{Z}_p \right\}$. Helibronn sums, in the case of $G = \mathbb{Z}_p$ and $\Gamma = \{ x^p \mid 0 \neq x \in \mathbb{Z}_p \}$. The Ramanujan sums in the case of $G = \mathbb{Z}_n$ and $\Gamma = \mathbb{Z}_n^\times$. It is worth mentioning that all the above sums appear as supercharacters.

As a generalization of the group $\Gamma$ in Kloosterman sum we let

$$\Gamma = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z}_p^\times \right\}$$

a group of order $(p-1)^2$. Here $G = \mathbb{Z}_p \times \mathbb{Z}_p$ and orbits of $\Gamma$ on $G$ are:

- $Y_1 = \{(0,0)\}$ of size 1
- $Y_2 = (1,0)\Gamma = \{(a,0) \mid a \in \mathbb{Z}_p^\times \}$ of size $p-1$
- $Y_3 = (0,1)\Gamma = \{(0,b) \mid b \in \mathbb{Z}_p^\times \}$ of size $p-1$
- $Y_4 = (1,1)\Gamma = \{(a,b) \mid a, b \in \mathbb{Z}_p^\times \}$ of size $(p-1)^2$

Orbits of $\Gamma$ on $Irr(G)$ are as follows:

- $X_1 = \{(0,0)\}$ of size 1
- $X_2 = (1,0)\Gamma = \{(a,0) \mid a \in \mathbb{Z}_p^\times \}$ of size $p-1$
- $X_3 = (0,1)\Gamma = \{(0,b) \mid b \in \mathbb{Z}_p^\times \}$ of size $p-1$
- $X_4 = (1,1)\Gamma = \{(a,b) \mid a, b \in \mathbb{Z}_p^\times \}$ of size $(p-1)^2$

Now we form the supercharacter table of $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Let $\sigma_i$ be the supercharacter associated with $X_i$, with $\sigma_1 = 1$. 
We know \( \sigma_i = \sum_{\psi_x \in X_i} \psi_x e^{\frac{x_i \cdot y}{p}}, 1 \leq i \leq 4 \). Therefore, the following table is calculated:

| \( \mathbb{Z}_p^2 \) | \( Y_1 \) | \( Y_2 \) | \( Y_3 \) | \( Y_4 \) |
|----------------|-------|-------|-------|-------|
| superclass size | 1     | \( p-1 \) | \( p-1 \) | \( (p-1)^2 \) |
| \( \sigma_1 \)   | 1     | 1     | 1     | 1     |
| \( \sigma_2 \)   | \( p-1 \) | -1    | \( p-1 \) | -1    |
| \( \sigma_3 \)   | \( p-1 \) | \( p-1 \) | -1    | -1    |
| \( \sigma_4 \)   | \( (p-1)^2 \) | \( -(p-1) \) | \( -(p-1) \) | \( -(p-1) \) |

To find the unitary matrix \( U \) we use the formula written down in section 2 to obtain the \( 4 \times 4 \) matrix \( U \) as follows:

\[
U = \frac{1}{p} \begin{bmatrix}
1 & \sqrt{p-1} & \sqrt{p-1} & p-1 \\
\sqrt{p-1} & -1 & p-1 & -\sqrt{p-1} \\
\sqrt{p-1} & p-1 & -1 & -\sqrt{p-1} \\
p-1 & -\sqrt{p-1} & -\sqrt{p-1} & 1
\end{bmatrix}
\]

At this point it is convenient to consider the general case of \( G = \mathbb{Z}_p^d \), the diagonal subgroup of order \( (p-1)^d \) of \( GL_d(\mathbb{Z}_p) \).

Orbits of \( \Gamma \) on \( G \) are as follows:

\( Y_1 = \{(0,0,\ldots,0)\} \) is one orbit. Let \( y^{(k)} = (1^k,0^{d-k}) \) be a vector of \( G \) with \( k \) one's in different positions. Then \( y^{(k)} \Gamma \) consists of vectors with non-zero entries in exactly \( k \) different positions. Therefore, the orbit \( y^{(k)} \) has size \( (p-1)^k \). Since this \( k \) positions is taken out of \( d \) positions, therefore we have \( \binom{d}{k} \) orbits of this shapes each of size \( (p-1)^k \). Hence we have \( \sum_{k=0}^{d} \binom{d}{k} = 2^d \) orbits of \( \Gamma \) on \( G \). Each orbit has size \( (p-1)^k \). Since \( \sum_{k=0}^{d} \binom{d}{k} (p-1)^k = p^d = |G| \), all the orbits are counted.

Orbits of \( \Gamma \) on \( \text{Irr}(G) \) have the same setting as above. In this case if \( \psi_x \) is a representative of the orbit \( X \) of \( \Gamma \) on \( \text{Irr}(G) \), then we may assume \( x = x^{(l)} = (1^l,0^{d-l}) \) is a vector with \( l \) ones in different positions, hence.

\[
\sigma_X(y) = \sum_{x \in X} \psi_x(y) = \sum_{x \in X} e^{\frac{x \cdot y}{p}}
\]

And it is computable if the inner product \( x \cdot y \) is known.
4 J-Symmetric groups

Let $G = \mathbb{Z}_p^d$ and $\Gamma$ be a subgroup of $GL_d(\mathbb{Z}_p)$. By \cite{2} we have to assume that $\Gamma$ is symmetric, i.e. $\Gamma = \Gamma^t$, in order to conclude that the action of $\Gamma$ on $G$ and on $\text{Irr}(G)$ generate the same orbits. Most of the results on supercharacter theory of $G$ holds if we assume $\Gamma$ is J-Symmetric. Suppose there is a fixed symmetric invertible matrix $J \in GL_d(\mathbb{Z}_p)$ such that $J\Gamma = \Gamma^t J$. As before the action of $\Gamma$ on $G$ is by $y \mapsto yA$ and by identifying $\psi_x \in \text{Irr}(G)$ with $x$, the action of $\Gamma$ on $\text{Irr}(G)$ is by $x \mapsto xA^{-t}$ for $A \in \Gamma$.

If $(\mathcal{X}, \mathcal{Y})$ is the supercharacter theory obtained in this way, then we set $\mathcal{X} = \{X_1, X_2, \ldots, X_h\}$ and $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_h\}$ and $\sigma_i = \sigma_{X_i}$, $1 \leq i \leq r$, then the unitary matrix $U$ is replaced by

$$U = \frac{1}{\sqrt{n^d}} \left[ \frac{\sigma_i(Y_j)\sqrt{|J_j|}}{\sqrt{|X_i|}} \right]_{i,j=1}^h.$$

In this section we consider $G = \mathbb{Z}_p^3$, $p$ a prime, and $J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

$$\Gamma = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & b \\ 0 & 0 & a \end{bmatrix} \mid a, b, c \in \mathbb{Z}_p^\times, b, c \in \mathbb{Z}_p \right\}$$

is a subgroup of $GL_3(\mathbb{Z}_p)$ of order $p^2(p-1)^2$. It is obvious that $\Gamma$ is a J-Symmetric group.

Orbits of $\Gamma$ on $G$ are as follows:

$Y_1 = \{(0,0,0)\}$

$Y_2 = (0,0,1)\Gamma = \{(0,0,a) \mid a \in \mathbb{Z}_p^\times\}$

$Y_3 = (0,1,0)\Gamma = \{(0,d,b) \mid d \in \mathbb{Z}_p^\times, b \in \mathbb{Z}_p\}$

$Y_4 = (1,0,0)\Gamma = \{(a,b,c) \mid a \in \mathbb{Z}_p^\times, b, c \in \mathbb{Z}_p\}$.

We have

$|Y_1| = 1$

$|Y_2| = p - 1$

$|Y_3| = p(p-1)$

$|Y_4| = p^2(p-1)$.

Since $|Y_1| + |Y_2| + |Y_3| + |Y_4| = p^3$, we deduce that $Y_1, Y_2, Y_3$ and $Y_4$ are orbits of $\Gamma$ on $G$. It is easy to see that the orbits of $\Gamma$ on $\text{Irr}(G)$ are as follows:

$X_1 = \{(0,0,0)\}$

$X_2 = (1,0,0)\Gamma = \{(a,0,0) \mid a \in \mathbb{Z}_p^\times\}$

$X_3 = (0,1,0)\Gamma = \{(a,b,0) \mid a \in \mathbb{Z}_p, b \in \mathbb{Z}_p^\times\}$

$X_4 = (1,0,0)\Gamma = \{(a,b,c) \mid a, b \in \mathbb{Z}_p, c \in \mathbb{Z}_p^\times\}$. 

6
We have

\[ |X_1| = 1 \]
\[ |X_2| = p - 1 \]
\[ |X_3| = p(p - 1) \]
\[ |X_4| = p^2(p - 1). \]

Let the supercharacter associated to \( X_i \) be \( \sigma_i \). The following supercharacter table for the group \( G \) is constructed:

| \( \mathbb{Z}_p^3 \) | \( Y_1 \) | \( Y_2 \) | \( Y_3 \) | \( Y_4 \) |
|-----------------|--------|--------|--------|--------|
| superclass size | 1      | \( p-1 \) | \( p(p-1) \) | \( p^2(p-1) \) |
| \( \sigma_1 \)  | 1      | 1      | 1      | 1      |
| \( \sigma_2 \)  | \( p-1 \) | \( p-1 \) | \( p-1 \) | -1     |
| \( \sigma_3 \)  | \( p(p-1) \) | \( p(p-1) \) | -p     | 0      |
| \( \sigma_4 \)  | \( p^2(p-1) \) | -p^2   | 0      | 0      |

The unitary table associated with the above table is:

\[
U = \frac{1}{p \sqrt{p}} \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\sqrt{p-1} & 1 & \cdots & 0 \\
\sqrt{p(p-1)} & \sqrt{p(p-1)} & \cdots & 0 \\
\sqrt{p^2(p-1)} & \sqrt{p^2(p-1)} & \cdots & 0 \\
\end{bmatrix}
\]

As general case let us consider \( G = \mathbb{Z}_p^d \),

\[
\Gamma = \begin{bmatrix}
1 & a_2 & a_3 & \cdots & a_d \\
0 & 1 & a_2 & \cdots & a_{d-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 1 & 1 \\
\end{bmatrix}
\]

where \( a_i \in \mathbb{Z}_p, 2 \leq i \leq d \}

which is J-symmetric with respect to the \( d \times d \) matrix \( J = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\vdots & \vdots \\
0 & 0 \\
1 & 0 \\
\end{bmatrix} \). We have \(|\Gamma| = p^d - 1 \)

and it is a \( p \)-group.

The orbits of \( \Gamma \) on \( G = \mathbb{Z}_p^d \) are grouped as follows:

\[ Y_1 = \{(0, 0, \ldots, 0)\} \]
\[ Y_2 = (\alpha, 0, \ldots, 0) \Gamma = \{(\alpha, a_2, a_3, \ldots, a_d) \mid a_i \in \mathbb{Z}_p, \alpha \in \mathbb{Z}_p^x \} \]
hence \( Y_2 \) is the union of \( p - 1 \) orbits each of size \( p^{d-1} \).

\[ Y_3 = (0, \alpha, 0, \ldots, 0)\Gamma = \{(0, \alpha, \alpha_2, \ldots, \alpha_{d-1}) \mid \alpha_i \in \mathbb{Z}_p \}, \alpha \in \mathbb{Z}_p^d \]

hence \( Y_3 \) is the union of \( p - 1 \) orbits each of size \( p^{d-2} \). If we continue in this way we obtain

\[ Y_d = (0, 0, \ldots, \alpha, 0)\Gamma = \{(0, 0, \ldots, \alpha, a_2) \mid a_2 \in \mathbb{Z}_p \} \]

has size \( p \) and is the union of \( p - 1 \) orbits,

\[ Y_{d+1} = (0, 0, \ldots, \alpha)\Gamma = \{(0, 0, \ldots, \alpha)\} \]

is the union of \( p - 1 \) orbits each of size 1.

Since \( 1 + (p-1)(p^{d-1} + p^{d-2} + \ldots + 1) = p^d = |G| \), all the orbits of \( \Gamma \) on \( G \) are counted. Therefore there are \( 1 + (p-1)d \) orbits.

To find the shapes of the orbits of \( \Gamma \) on \( \text{Irr}(G) \), we mention that each irreducible character of \( G \) has degree 1. We have \( \text{Irr}(G) = \{\psi_x \mid x \in G\} \) which may be represented by elements \( x \) of \( G \) under the action \( x \mapsto xA^{-t} \) where \( A \in \Gamma \). Therefore we obtain the following orbits:

\[ X_1 = \{(0,0,\ldots,0)\} \]
\[ X_2 = (\alpha,0,\ldots,0)\Gamma = \{(\alpha,0,\ldots,0)\} \]
\[ X_3 = (0,\alpha,0,\ldots,0)\Gamma = \{(a_2,\alpha,0,\ldots,0) \mid a_2 \in \mathbb{Z}_p\} \]

\( \vdots \)
\[ X_{d+1} = (0,0,\ldots,\alpha)\Gamma = \{(a_2,a_3,\ldots,a_d,\alpha) \mid \alpha_i \in \mathbb{Z}_p\}. \]

Each set \( X_i \), \( 2 \leq i \leq d+1 \) is the union of \( p - 1 \) orbit each of size \( p^{i-2} \).

Now if \( \psi_x \) is a representative of the orbit \( X \) of \( \Gamma \) on \( \text{Irr}(G) \), we may choose \( x = (0,0,\ldots,0) \) \( \in X_i \)

hence if \( \sigma_X \) is the supercharacter associated to \( X \), then for \( y \in Y_j \) we have

\[ \sigma_X(y) = \sum_{x \in X} \psi_x(y) = \sum_{x \in X} e^{\frac{x \cdot y}{p}} \]

if \( x \) and \( y \) are taken from orbits such that \( x \cdot y = 0 \), then \( \sum_{x \in X} e^{\frac{x \cdot y}{p}} = |X| = p^{i-2} \), provided \( X = X_i \). Otherwise if \( x \cdot y \neq 0 \) we obtain \( \sum_{x \in X} e^{\frac{x \cdot y}{p}} = 0 \). In this way the supercharacter table of \( G \) is computed.

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