Symmetric functions of binary products of bivariate complex Lucas polynomials and some special numbers and polynomials

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Abstract In this paper, we introduce an operator in order to derive some new symmetric properties of bivariate complex Lucas polynomials, then we give some new generating functions of the products of bivariate complex Lucas polynomials with Chebyshev polynomials of the first, second, third and fourth kinds, \(k\)-Fibonacci numbers, \(k\)-Lucas numbers, \(k\)-Pell numbers, \(k\)-Pell Lucas numbers, \(k\)-Jacobsthal numbers and \(k\)-Jacobsthal Lucas numbers. By making use of the operator defined in this paper, we give some new generating functions of the products of bivariate complex Lucas polynomials with Fibonacci polynomials, Pell polynomials and Jacobsthal polynomials.

Keywords symmetric functions · generating functions · bivariate complex Lucas polynomials · \(k\)-Fibonacci numbers · third and fourth kinds Chebyshev polynomials.

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1 Introduction

Falcón and Plaza in [15] introduced the \(k\)-Fibonacci numbers \(\{F_{k,n}\}_{n \in \mathbb{N}}\) which is defined by

\[
F_{k,n} = kF_{k,n-1} + F_{k,n-2} \quad \text{for} \quad k \geq 1 \text{ and } n \geq 2,
\]

with the initial values \(F_{k,0} = 1, F_{k,1} = k\). After that, Falcón in [16] showed some results of the \(k\)-Lucas numbers \(\{L_{k,n}\}_{n \in \mathbb{N}}\) which is defined as

\[
L_{k,n} = kL_{k,n-1} + L_{k,n-2} \quad \text{for} \quad k \geq 1 \text{ and } n \geq 2,
\]

with initial conditions \(L_{k,0} = 2, L_{k,1} = k\). The Binet’s formulas for \(k\)-Fibonacci and \(k\)-Lucas numbers are given by

\[
F_{k,n} = \frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2} \quad \text{and} \quad L_{k,n} = x_1^n + x_2^n,
\]
respectively, where \( x_1 = \frac{k + \sqrt{k^2 + 4}}{2} \) and \( x_2 = \frac{k - \sqrt{k^2 + 4}}{2} \) are roots of the characteristic equation \( x^2 - kx - 1 = 0 \). We note that

\[
x_1 + x_2 = k, \quad x_1 x_2 = -1 \quad \text{and} \quad x_1 - x_2 = \sqrt{k^2 + 4}.
\]

M. Asci and E. Gurel defined and studied the bivariate complex Fibonacci and bivariate complex Lucas polynomials in [3]. They gave generating function, Binet’s formula, explicit formula and partial derivation of these polynomials.

The bivariate complex Lucas polynomials \( \{L_n(x, y)\}_{n \in \mathbb{N}} \) is defined by

\[
\begin{align*}
L_{n+1}(x, y) &= ixL_n(x, y) + yL_{n-1}(x, y), \quad \text{for} \quad n \geq 1, \\
L_0(x, y) &= 2, \quad L_1(x, y) = ix.
\end{align*}
\]

The first few terms of bivariate complex Lucas polynomials are listed in the Table below:

| \( n \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) |
| --- | --- | --- | --- | --- | --- | --- |
| \( L_n(x, y) \) | \( 2 \) | \( ix \) | \( -x^2 + 2y \) | \( -x^3 i + 3xyi \) | \( x^4 - 4x^2 y + 2y^2 \) | \( x^5 i - 5x^3 y i + 5x y^2 i \) |

**Table 1.** The bivariate complex Lucas polynomials for \( 0 \leq n \leq 5 \).

The explicit formula of bivariate complex Lucas polynomials is given by

\[
L_n(x, y) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{n-j} \left( \frac{ix}{j} \right)^{n-j} y^j,
\]

(see [3]).

For any integer \( n \geq 2 \), the Chebyshev polynomials of the first, second, third and fourth kinds \( T_n(x) \), \( U_n(x) \), \( V_n(x) \) and \( W_n(x) \) are respectively defined as follows

\[
T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),
\]

with initial conditions \( T_0(x) = 1 \) and \( T_1(x) = x \), and

\[
U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),
\]

with initial conditions \( U_0(x) = 1 \) and \( U_1(x) = 2x \). For more information, please see the paper [4].

\[
V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x),
\]

with the initial values \( V_0(x) = 1 \) and \( V_1(x) = 2x - 1 \), and

\[
W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x),
\]

with the initial values \( W_0(x) = 1 \) and \( W_1(x) = 2x + 1 \). (see [14]).

Catarino in [11] studied the k-Pell sequence \( \{P_{k,n}\}_{n \in \mathbb{N}} \) which is defined by

\[
P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \quad \text{for} \quad k \geq 1 \quad \text{and} \quad n \geq 2,
\]
with the initial terms $P_{k,0} = 0$, $P_{k,1} = 1$. After that Catarino and Vasco in [12] found some results of the $k$-Pell Lucas numbers $\{Q_{k,n}\}_{n \in \mathbb{N}}$ which is defined as

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \text{ for } k \geq 1 \text{ and } n \geq 2,$$

with the initial terms $Q_{k,0} = 2$, $Q_{k,1} = 2$.

In this part, we define $k$-Jacobsthal and $k$-Jacobsthal Lucas numbers, Fibonacci, Pell and Jacobsthal polynomials.

**Definition 1.1** [19] We define $k$-Jacobsthal numbers $\{J_{k,n}\}_{n \in \mathbb{N}}$ by the following recurrence relation

$$\begin{cases} J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2}, & \text{for } n \geq 2 \\ J_{k,0} = 0, J_{k,1} = 1 \end{cases}.$$

**Definition 1.2** [20] For $n \in \mathbb{N}$, the $k$-Jacobsthal Lucas numbers, denoted by $\{j_{k,n}\}_{n \in \mathbb{N}}$ is defined recursively by

$$\begin{cases} j_{k,n} = kj_{k,n-1} + 2j_{k,n-2}, & \text{for } n \geq 2 \\ j_{k,0} = 2, j_{k,1} = k \end{cases}.$$

**Definition 1.3** [17] We define Pell polynomials $\{P_n(x)\}_{n \in \mathbb{N}}$ by the following recurrence relation

$$\begin{cases} P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), & \text{for } n \geq 2 \\ P_0(x) = 0, P_1(x) = 1 \end{cases}.$$

**Definition 1.4** [13] For $n \in \mathbb{N}$, the Fibonacci polynomials, denoted by $\{F_n(x)\}_{n \in \mathbb{N}}$ is defined recursively by

$$\begin{cases} F_n(x) = xF_{n-1}(x) + F_{n-2}(x), & \text{for } n \geq 2 \\ F_0(x) = 1, F_1(x) = x \end{cases}.$$

**Definition 1.5** [18] We define Jacobsthal polynomials $\{J_n(x)\}_{n \in \mathbb{N}}$ by the following recurrence relation

$$\begin{cases} J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x), & \text{for } n \geq 2 \\ J_0(x) = 0, J_1(x) = 1 \end{cases}.$$

**2 Definitions and some properties**

In this section, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the
A symmetric function of an alphabet $A$ is a function of the letters which is invariant under permutation of the letters of $A$. Taking an extra indeterminate $z$, one has two fundamental series (see [10]).

$$\lambda_z(A) = \prod_{a \in A} (1 + za), \quad \sigma_z(A) = \frac{1}{\prod_{a \in A} (1 - za)}.$$  

The expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{+\infty} \Lambda_n(A) z^n, \quad \sigma_z(A) = \sum_{n=0}^{+\infty} S_n(A) z^n.$$  

Let us now start at the following definition.

**Definition 2.1** [1] Let $A$ and $B$ be any two alphabets, then we give $S_n(A - B)$ by the following form:

$$\prod_{b \in B} (1 - zb) = \sum_{n=0}^{+\infty} S_n(A - B) z^n = \sigma_z(A - B), \quad (2.1)$$

with the condition $S_n(A - B) = 0$ for $n < 0$.

**Remark 2.1** Taking $A = \{0\}$ in (2.1) gives

$$\prod_{b \in B} (1 - zb) = \sum_{n=0}^{+\infty} S_n(-B) z^n = \lambda_z(-B). \quad (2.2)$$

Further, in the case $A = \{0\}$ or $B = \{0\}$, we have

$$\sum_{n=0}^{+\infty} S_n(A - B) z^n = \sigma_z(A) \times \lambda_z(-B). \quad (2.3)$$

Thus, (see [1])

$$S_n(A - B) = \sum_{k=0}^{n} S_{n-k}(A) S_k(-B).$$

**Definition 2.2** [21] Let $g$ be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form

$$\partial_{x_i, x_{i+1}} (g) = \frac{g(x_1, \ldots, x_i, x_{i+1}, \ldots x_n) - g'(x_1, \ldots, x_i, x_{i+1}, \ldots x_n)}{x_i - x_{i+1}},$$

where $g'$ is given by

$$g'(x_1, \ldots, x_i, x_{i+1}, \ldots x_n) = g(x_1, \ldots x_{i-1}, x_{i+1}, x_i, x_{i+2} \ldots x_n).$$
Definition 2.3 Let \( n \) be positive integer and \( A = \{a_1, a_2\} \) are set of given variables, then, the \( n \)-th symmetric function \( S_n(a_1 + a_2) \) is defined by
\[
S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},
\]
with
\[
S_0(A) = S_0(a_1 + a_2) = 1,
S_1(A) = S_1(a_1 + a_2) = a_1 + a_2,
S_2(A) = S_2(a_1 + a_2) = a_1^2 + a_1 a_2 + a_2^2,
\]

Definition 2.4 [7] Given an alphabet \( A = \{a_1, a_2\} \), the symmetrizing operator \( \delta_{a_1 a_2}^k \) is defined by
\[
\delta_{a_1 a_2}^k f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \quad k \in \mathbb{N}_0. \tag{2.4}
\]
If \( f(a_1) = a_1^n \), the operator (2.4) gives us
\[
\delta_{a_1 a_2}^k (a_1^n) = \frac{a_1^{k+n} - a_2^{k+n}}{a_1 - a_2} = S_{k+n-1}(a_1 + a_2).
\]

3 Generating functions of the products of bivariate complex Lucas polynomials with some well-known numbers and polynomials

The following propositions are key tools of the proof of our main result. They have been proved in [6] and [7].

Proposition 3.1 Given two alphabets \( E = \{e_1, e_2\} \) and \( A = \{a_1, a_2\} \), then
\[
\sum_{n=0}^{+\infty} S_n(A) S_n(E) z^n = \frac{1 - a_1 a_2 e_1 e_2 z^2}{\left( \sum_{n=0}^{+\infty} S_n(-A) e_1^n z^n \right) \left( \sum_{n=0}^{+\infty} S_n(-A) e_2^n z^n \right)}. \tag{3.1}
\]

Based on the relationship (3.1) we get
\[
\sum_{n=0}^{+\infty} S_{n-1}(A) S_{n-1}(E) z^n = \frac{z - a_1 a_2 e_1 e_2 z^3}{\left( \sum_{n=0}^{+\infty} S_n(-A) e_1^n z^n \right) \left( \sum_{n=0}^{+\infty} S_n(-A) e_2^n z^n \right)}. \tag{3.2}
\]
Proposition 3.2 Given two alphabets \( E = \{e_1, e_2\} \) and \( A = \{a_1, a_2\} \), then
\[
\sum_{n=0}^{+\infty} S_{n-1}(A)S_n(E)z^n = \frac{(e_1 + e_2)z - e_1e_2(a_1 + a_2)z^2}{\left( \sum_{n=0}^{+\infty} S_n(-A)e_1^n z^n \right) \left( \sum_{n=0}^{+\infty} S_n(-A)e_2^n z^n \right)}.
\] (3.3)

In this part, we now derive the new generating functions of the products of bivariate complex Lucas polynomials with \( k \)-Fibonacci numbers, \( k \)-Lucas numbers and \( k \)-Pell numbers, \( k \)-Pell Lucas numbers and \( k \)-Jacobsthal numbers, \( k \)-Jacobsthal Lucas numbers and Fibonacci polynomials, Pell polynomials and Jacobsthal polynomials. For the case \( A = \{a_1, -a_2\} \), \( E = \{e_1, -e_2\} \) with replacing \( a_2 \) by \((-a_2)\) and \( e_2 \) by \((-e_2)\) in (3.1), (3.2) and (3.3), we have
\[
\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{1 - a_1a_2e_1e_2z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}.
\] (3.4)
\[
\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{z - a_1a_2e_1e_2z^3}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}.
\] (3.5)
\[
\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{(e_1 - e_2)z + e_1e_2(a_1 - a_2)z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}.
\] (3.6)

respectively.

This case consists of six related parts. Firstly, the substitutions
\[
\begin{cases}
    a_1 - a_2 = ix & \text{and} \\
    a_1a_2 = y & \text{and} \\
    e_1 - e_2 = k & \text{and} \\
    e_1e_2 = 1
\end{cases}
\]
in (3.4), (3.5) and (3.6), we obtain
\[
\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{1 - yz^2}{1-ikxz - (y(k^2 + 2) - x^2)z^2 - ixyz^3 + y^2z^4},
\] (3.7)
\[
\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{z - yz^3}{1-ikxz - (y(k^2 + 2) - x^2)z^2 - ixyz^3 + y^2z^4},
\] (3.8)
\[
\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{yzz^2}{1-ikxz - (y(k^2 + 2) - x^2)z^2 - ixyz^3 + y^2z^4},
\] (3.9)
respectively, we have the following theorems.

Theorem 3.3 For \( n \in \mathbb{N} \), the new generating function of the product of bivariate complex Lucas polynomials with \( k \)-Fibonacci numbers is given by
\[
\sum_{n=0}^{+\infty} L_n(x, y)F_{k,n}z^n = \frac{2 - ikxz + (x^2 - 2y)z^2}{1-ikxz - (y(k^2 + 2) - x^2)z^2 - ixyz^3 + y^2z^4}.
\] (3.10)
Proof. Recall that, we have $F_{k,n} = S_n(e_1 + [-e_2])$ (see [5]). We see that

$$
\sum_{n=0}^{+\infty} L_n(x,y) F_{k,n} z^n = \sum_{n=0}^{+\infty} (2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])) S_n(e_1+[-e_2]) z^n
$$

$$
= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2]) z^n
$$

$$
- ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2]) z^n.
$$

Using the relationships (3.7) and (3.9), we obtain

$$
\sum_{n=0}^{+\infty} L_n(x,y) F_{k,n} z^n = \frac{2(1-yz^2)}{1-ikxz - (y(k^2 + 2) - x^2)z^2 - i kxyz^3 + y^2z^4}
$$

$$
- \frac{ix(kz + ixz^2)}{1-ikxz - (y(k^2 + 2) - x^2)z^2 - i kxyz^3 + y^2z^4}
$$

$$
= \frac{2 - ikxz + (x^2 - 2y)z^2}{1-ikxz - (y(k^2 + 2) - x^2)z^2 - i kxyz^3 + y^2z^4}.
$$

This completes the proof.

\[\square\]

**Theorem 3.4** For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Lucas polynomials with $k$-Lucas numbers is given by

$$
\sum_{n=0}^{+\infty} L_n(x,y) L_{k,n} z^n = \frac{4 - 3ikxz + (2x^2 - 2y(k^2 + 2))z^2 - i kxyz^3}{1-ikxz - (y(k^2 + 2) - x^2)z^2 - i kxyz^3 + y^2z^4}.
$$

(3.11)
Proof. By [5], we have \( L_{k,n} = 2S_n(e_1 + [-e_2]) - kS_{n-1}(e_1 + [-e_2]) \). Then, we can see that
\[
\sum_{n=0}^{+\infty} L_n(x, y)L_{k,n}z^n = \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])) \times (2S_n(e_1 + [-e_2]) - kS_{n-1}(e_1 + [-e_2]))z^n
\]
\[
= 4 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n - 2k \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n
\]
\[
- 2ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n + ikx \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n.
\]
Using the relationships (3.7), (3.8) and (3.9), we obtain
\[
\sum_{n=0}^{+\infty} L_n(x, y)L_{k,n}z^n = \frac{4 (1 - yz^2)}{1 - ikxz - (y(k^2 + 2) - x^2)z^2 - ikxyz^3 + y^2z^4} - \frac{2k (ixz + kyz^2)}{1 - ikxz - (y(k^2 + 2) - x^2)z^2 - ikxyz^3 + y^2z^4} - \frac{2ix (kz + ixz^2)}{1 - ikxz - (y(k^2 + 2) - x^2)z^2 - ikxyz^3 + y^2z^4} + \frac{ikx (z - yz^3)}{1 - ikxz - (y(k^2 + 2) - x^2)z^2 - ikxyz^3 + y^2z^4} + \frac{4 - 3ikxz + (2x^2 - 2y(k^2 + 2))z^2 - ikxyz^3}{1 - ikxz - (y(k^2 + 2) - x^2)z^2 - ikxyz^3 + y^2z^4}.
\]
This completes the proof. \(\square\)

- Put \(k = 1\) in the relationships (3.10) and (3.11), we obtain the following corollaries.

**Corollary 3.5** For \(n \in \mathbb{N}\), the new generating function of the product of bivariate complex Lucas polynomials with Fibonacci numbers is given by
\[
\sum_{n=0}^{+\infty} L_n(x, y)F_nz^n = \frac{2 - ixz + (x^2 - 2y)z^2}{1 - ixz - (3y - x^2)z^2 - ixyz^3 + y^2z^4}.
\]
Corollary 3.6 For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Lucas polynomials with Lucas numbers is given by

$$\sum_{n=0}^{+\infty} L_n(x, y)L_n z^n = \frac{4 - 3ixz + 2(x^2 - 3y)z^2 - ixyz^3}{1 - ixz - (3y - x^2)z^2 - ixyz^3 + y^2z^4}.$$  

Secondly, the substitutions

$$\begin{cases} a_1 - a_2 = ix \\ a_1a_2 = y \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = 2 \\ e_1e_2 = k \end{cases},$$

in (3.4), (3.5) and (3.6), we give

$$\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{1 - kyz^2}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4},$$  \hspace{1cm} (3.12)

$$\sum_{n=0}^{+\infty} S_n-1(a_1 + [-a_2])S_n-1(e_1 + [-e_2])z^n = \frac{z - kyz^3}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4},$$  \hspace{1cm} (3.13)

$$\sum_{n=0}^{+\infty} S_n-1(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{2z + ikxz^2}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4},$$  \hspace{1cm} (3.14)

respectively, we have the following results.

Theorem 3.7 For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Lucas polynomials with $k$-Pell numbers is given by

$$\sum_{n=0}^{+\infty} L_n(x, y)P_{k,n} z^n = \frac{ixz + 4yz^2 + ikxyz^3}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4}.$$  \hspace{1cm} (3.15)

Proof. By referred to [5], we have

$$P_{k,n} = S_{n-1}(e_1 + [-e_2]).$$

We see that

$$\sum_{n=0}^{+\infty} L_n(x, y)P_{k,n} z^n = \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_n-1(a_1 + [-a_2] - S_{n-1}(e_1 + [-e_2])z^n$$

$$= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n-1(e_1 + [-e_2])z^n$$

$$- ix \sum_{n=0}^{+\infty} S_n-1(a_1 + [-a_2])S_n-1(e_1 + [-e_2])z^n.$$
then, we get the following equality:

\[
\sum_{n=0}^{\infty} L_n(x, y) P_{k,n} z^n = \frac{2 (ixz + 2yz^2)}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4}
\]

\[
\frac{ix(z - kyz^3)}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4}
\]

\[
= 1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4 .
\]

This completes the proof.

\[\square\]

**Theorem 3.8** For \( n \in \mathbb{N} \), the new generating function of the product of bivariate complex Lucas polynomials with \( k \)-Pell Lucas numbers is given by

\[
\sum_{n=0}^{\infty} L_n(x, y) Q_{k,n} z^n = \frac{4 - 6ixz + (2kx^2 - 4y(k + 2))z^2 - 2ikxyz^3}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4} .
\]

**Proof.** Recall that, we have [5].

\[Q_{k,n} = 2S_n(e_1 + [-e_2]) - 2S_{n-1}(e_1 + [-e_2]),\]
by using the relationships (3.12), (3.13) and (3.14), we get

\[
\sum_{n=0}^{+\infty} L_n(x,y)Q_{k,n}z^n = \sum_{n=0}^{+\infty} \left( 2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2]) \right) \\
\times (2S_{n}(e_1 + [-e_2]) - 2S_{n-1}(e_1 + [-e_2]))z^n \\
= 4 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
- 4 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
- 2ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
+ 2ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
= \frac{4 (1 - kyz^2)}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4} \\
- \frac{4 (ixz + 2yz^2)}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4} \\
- \frac{2ix (2z + ikxz^2)}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4} \\
+ \frac{2ix (z - kyz^3)}{1 - 2ixz - (2y(k + 2) - kx^2)z^2 - 2ikxyz^3 + k^2y^2z^4} \\
= \frac{4 - 6ixz + (2kx^2 - 4y(k + 2))z^2 - 2ikxyz^3}{1 - 2ixz - (6y - kx^2)z^2 - 2ikxyz^3 + y^2z^4}.
\]

This completes the proof.

- Put \(k = 1\) in the relationships (3.15) and (3.16), we deduce the following corollaries.

**Corollary 3.9** For \(n \in \mathbb{N}\), the new generating function of the product of bivariate complex Lucas polynomials with Pell numbers is given by

\[
\sum_{n=0}^{+\infty} L_n(x,y)P_{n}z^n = \frac{ixz + 4yz^2 + ixyz^3}{1 - 2ixz - (6y - x^2)z^2 - 2ikxyz^3 + y^2z^4}.
\]
Corollary 3.10 For \( n \in \mathbb{N} \), the new generating function of the product of bivariate complex Lucas polynomials with Pell Lucas numbers is given by

\[
\sum_{n=0}^{+\infty} L_n(x, y)Q_nz^n = \frac{4 - 6ixz + 2(x^2 - 6y)z^2 - 2ixyz^3}{1 - 2ixz - (6y - x^2)z^2 - 2ixyz^3 + y^2z^4}.
\]

Thirdly, the substitutions

\[
\begin{aligned}
\{ a_1 - a_2 = ix \\
\{ a_1a_2 = y
\end{aligned}
\text{ and } \begin{aligned}
\{ e_1 - e_2 = k \\
e_1e_2 = 2
\end{aligned}
\]

in (3.4), (3.5) and (3.6), we give

\[
\begin{align*}
\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n &= \frac{1 - 2yz^2}{1 - ikxz - ((k^2 + 4) y - 2x^2)z^2 - 2ikxyz^3 + 4y^2z^4}, \\
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n &= \frac{z - 2yz^3}{1 - ikxz - ((k^2 + 4) y - 2x^2)z^2 - 2ikxyz^3 + 4y^2z^4}, \\
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n &= \frac{kz + 2ixz^2}{1 - ikxz - ((k^2 + 4) y - 2x^2)z^2 - 2ikxyz^3 + 4y^2z^4},
\end{align*}
\]

respectively, thus we get the following theorems.

Theorem 3.11 For \( n \in \mathbb{N} \), the new generating function of the product of bivariate complex Lucas polynomials with \( k \)-Jacobsthal numbers is given by

\[
\sum_{n=0}^{+\infty} L_n(x, y)J_{k,n}z^n = \frac{ixz + 2kyz^2 + 2ixyz^3}{1 - ikxz - ((k^2 + 4) y - 2x^2)z^2 - 2ikxyz^3 + 4y^2z^4}.
\]

Proof. By [5], we have \( J_{k,n} = S_{n-1}(e_1 + [-e_2]). \) Then, we can see that

\[
\begin{align*}
\sum_{n=0}^{+\infty} L_n(x, y)J_{k,n}z^n &= \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2]))S_{n-1}(e_1 + [-e_2])z^n \\
&= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&\quad - ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n,
\end{align*}
\]
then, we get the following equality:

\[
\sum_{n=0}^{+\infty} L_n(x, y) J_{k,n} z^n = \frac{2 (ixz + kyz^2)}{1 - ikxz - ((k^2 + 4) y - 2x^2)z^2 - 2ikxyz^3 + 4y^2 z^4} \\
- \frac{ix(z - 2y^3)}{1 - ikxz - ((k^2 + 4) y - 2x^2)z^2 - 2ikxyz^3 + 4y^2 z^4} \\
= \frac{ixz + 2kxy^2 + 2ixyz^3}{1 - ikxz - ((k^2 + 4) y - 2x^2)z^2 - 2ikxyz^3 + 4y^2 z^4}.
\]

This completes the proof.

\[\square\]

**Theorem 3.12** For \( n \in \mathbb{N} \), the new generating function of the product of bivariate complex Lucas polynomials with \( k \)-Jacobsthal Lucas numbers is given by

\[
\sum_{n=0}^{+\infty} L_n(x, y) J_{k,n} z^n = \frac{4 - 3ikxz + (4x^2 - 2y(k^2 + 4))z^2 - 2ikxyz^3}{1 - ikxz - ((k^2 + 4)y - 2x^2)z^2 - 2ikxyz^3 + 4y^2 z^4}.
\]

**Proof.** We know that

\[
j_{k,n} = 2S_n(e_1 + [-e_2]) - kS_{n-1}(e_1 + [-e_2]), \text{ (see [9]).}
\]
We see that
\[ \sum_{n=0}^{+\infty} L_n(x,y) j_{k,n} z^n = \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_n-1(a_1 + [-a_2])) \]
\[ \times (2S_n(e_1 + [-e_2]) - kS_n-1(e_1 + [-e_2])) z^n \]
\[ = 4 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n \]
\[ - 2k \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n-1(e_1 + [-e_2]) z^n \]
\[ - 2ix \sum_{n=0}^{+\infty} S_n-1(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n \]
\[ + ikz \sum_{n=0}^{+\infty} S_n-1(a_1 + [-a_2]) S_n-1(e_1 + [-e_2]) z^n \]
\[ = \frac{4 (1 - 2y^2 z^2)}{1 - ikxz - ((k^2 + 4) y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4} \]
\[ - \frac{2k (ixz + kyz^2)}{1 - ikxz - ((k^2 + 4) y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4} \]
\[ - \frac{2iz (kz + 2ixz^2)}{1 - ikxz - ((k^2 + 4) y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4} \]
\[ + \frac{ikx (z - 2y^3)}{1 - ikxz - ((k^2 + 4) y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4} \]

after a simple calculation, we have
\[ \sum_{n=0}^{+\infty} L_n(x,y) j_{k,n} z^n = \frac{4 - 3ikxz + (4x^2 - 2y (k^2 + 4)) z^2 - 2ikxyz^3}{1 - ikxz - ((k^2 + 4) y - 2x^2) z^2 - 2ikxyz^3 + 4y^2 z^4} . \]

This completes the proof. \( \square \)

- Put \( k = 1 \) in the relationships (3.20) and (3.21), we have the following corollaries.

**Corollary 3.13** For \( n \in N \), the new generating function of the product of bivariate complex Lucas polynomials with Jacobsthal numbers is given by
\[ \sum_{n=0}^{+\infty} L_n(x,y) J_n z^n = \frac{ixz + 2yz^2 + 2ixyz^3}{1 - ixz - (5y - 2x^2) z^2 - 2ixyz^3 + 4y^2 z^4} . \]
Corollary 3.14 For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Lucas polynomials with Jacobsthal Lucas numbers is given by

$$
\sum_{n=0}^{+\infty} L_n(x,y) j_n z^n = \frac{4 - 3 i x z + 2 (2 x^2 - 5 y) z^2 - 2 i x y z^3}{1 - i x z - (5 y - 2 x^2) z^2 - 2 i x y z^3 + 4 y^2 z^4}.
$$

Fourthly, the substitutions

$$
\begin{align*}
&\begin{cases}
  a_1 - a_2 = ix \\
  a_1 a_2 = y
\end{cases} \quad \text{and} \quad \begin{cases}
  e_1 - e_2 = t \\
  e_1 e_2 = 1
\end{cases},
\end{align*}
$$

in (3.4) and (3.6), we obtain

$$
\begin{align*}
\sum_{n=0}^{+\infty} S_n (a_1 + [-a_2]) S_n (e_1 + [-e_2]) z^n &= \frac{1 - y z^2}{1 - i x t z - (t^2 y + 2y - x^2) z^2 - i x y t z^3 + y^2 z^4}, \\
\sum_{n=0}^{+\infty} S_{n-1} (a_1 + [-a_2]) S_n (e_1 + [-e_2]) z^n &= \frac{t z + i x z^2}{1 - i x t z - (t^2 y + 2y - x^2) z^2 - i x y t z^3 + y^2 z^4},
\end{align*}
$$

respectively, we obtain the following theorem.

Theorem 3.15 For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Lucas polynomials with Fibonacci polynomials is given by

$$
\sum_{n=0}^{+\infty} L_n(x,y) F_n (t) z^n = \frac{2 - i x t z + (x^2 - 2 y) z^2}{1 - i x t z - (t^2 y + 2y - x^2) z^2 - i x y t z^3 + y^2 z^4}.
$$

Proof. Recall that, we have [22].

$$
F_n (t) = S_n(e_1 + [-e_2]),
$$
by using the relationships (3.22) and (3.23), we get
\[
\sum_{n=0}^{+\infty} L_n(x,y) F_n(t) z^n = \sum_{n=0}^{+\infty} \left(2S_n(a_1 + [-a_2]) - ixS_n-1(a_1 + [-a_2])\right) S_n(e_1 + [-e_2]) z^n 
\]
\[
= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n 
\]
\[-ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n 
\]
\[
= \frac{2}{1 - ixtz - (t^2y + 2y - x^2)z^2 - ixytz^3 + y^2z^4}
\]
\[
1 - ixtz + (t^2 + 2y) z^2
\]
\[
1 - ixtz - (t^2 + 2y) z^2 - ixytz^3 + y^2z^4
\]

This completes the proof.

\[\square\]

**Fifthly**, the substitutions
\[
\begin{align*}
\{ & a_1 - a_2 = ix \\
& a_1a_2 = y \\
\} \quad \text{and} \quad \begin{align*}
\{ & e_1 - e_2 = 2t \\
& e_1e_2 = 1 \\
\}
\end{align*}
\]
in (3.5) and (3.6), we have

\[
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{z - yz^3}{1 - 2ixtz - (4yt^2 - x^2 + 2y)z^2 - 2ixyz^3 + y^2z^4},
\]

(3.25)

\[
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{2tz + ixz^2}{1 - 2ixtz - (4yt^2 - x^2 + 2y)z^2 - 2ixyz^3 + y^2z^4},
\]

(3.26)

respectively, thus we get the following theorem.

**Theorem 3.16** For \(n \in N\), the new generating function of the product of bivariate complex Lucas polynomials with Pell polynomials is given by

\[
\sum_{n=0}^{+\infty} L_n(x,y) P_n(t) z^n = \frac{ixz + 4ytz^2 + ixyz^3}{1 - 2ixtz - (4yt^2 - x^2 + 2y)z^2 - 2ixyz^3 + y^2z^4}.
\]

(3.27)
Symmetric functions of binary products of bivariate complex

Proof. We know that
\[ P_n(t) = S_{n-1}(e_1 + [-e_2]), \] (see [22]).

We see that
\[
\sum_{n=0}^{+\infty} L_n(x,y)P_n(t) z^n = \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2]))S_{n-1}(e_1 + [-e_2])z^n
\]
\[= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n\]
\[= -ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n\]
\[= \frac{2(ixz + 2yzt^3)}{1 - 2ixz - (4yt^2 - x^2 + 2y)z^2 - 2ixyz^3 + y^2z^4},\]
after a simple calculation, we have
\[
\sum_{n=0}^{+\infty} L_n(x,y)P_n(t) z^n = \frac{ixz + 4ytz^2 + ixyz^3}{1 - 2ixz - (4yt^2 - x^2 + 2y)z^2 - 2ixyz^3 + y^2z^4} .
\]

This completes the proof. \(\square\)

Sixthly, the substitutions
\[
\begin{cases}
a_1 - a_2 = ix \\ a_1a_2 = y
\end{cases}
\text{and}
\begin{cases}
e_1 - e_2 = 1 \\ e_1e_2 = 2t
\end{cases}
\]
in (3.5) and (3.6), we give
\[
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n = \frac{z - 2ytz^3}{1 - ixz - (y - 2tx^2 + 4yt)z^2 - 2ixyz^3 + 4y^2t^2z^4},
\]
(3.28)
\[
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n}(e_1 + [-e_2])z^n = \frac{z + 2itz^2}{1 - ixz - (y - 2tx^2 + 4yt)z^2 - 2ixyz^3 + 4y^2t^2z^4},
\]
(3.29)
respectively, we have the following theorem.

Theorem 3.17 For \(n \in \mathbb{N}\), the new generating function of the product of bivariate complex Lucas polynomials with Jacobsthal polynomials is given by
\[
\sum_{n=0}^{+\infty} L_n(x,y)J_n(t) z^n = \frac{ixz + 2yz^2 + 2ixyz^3}{1 - ixz - (y - 2tx^2 + 4yt)z^2 - 2ixyz^3 + 4y^2t^2z^4},
\]
(3.30)
Proof. By [22], we have $J_n(t) = S_{n-1}(e_1 + [-e_2])$. Then, we can see that

$$
\sum_{n=0}^{+\infty} L_n(x, y) J_n(t) z^n = \sum_{n=0}^{+\infty} (2S_n(a_1+[-a_2]) - ixS_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n
$$

$$
= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n
$$

$$
- ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n,
$$

then, we get the following equality:

$$
\sum_{n=0}^{+\infty} L_n(x, y) J_n(t) z^n = \frac{2(iz + yz^2)}{1 - iz - (y - 2tx^2 + 4yt)z^2 - 2ixyz^3 + 4y^2t^2z^4} - \frac{ix(z - 2ytz^3)}{1 - iz - (y - 2tx^2 + 4yt)z^2 - 2ixyz^3 + 4y^2t^2z^4}
$$

$$
\sum_{n=0}^{+\infty} L_n(x, y) J_n(t) z^n = \frac{ixz + 2yz^2 + 2ixyz^3}{1 - iz - (y - 2tx^2 + 4yt)z^2 - 2ixyz^3 + 4y^2t^2z^4}.
$$

This completes the proof.

\[\square\]

4 A new class of ordinary generating functions of binary products of bivariate complex Lucas polynomials with Chebyshev polynomials

In this section, we now derive the new generating functions of the products of bivariate complex Lucas polynomials with Chebyshev polynomials of the first, second, third and fourth kinds.

For the case $A = \{a_1, -a_2\}, E = \{2e_1, -2e_2\}$ with replacing $a_2$ by $(-a_2)$, $e_1$ by $2e_1$ and $e_2$ by $(-2e_2)$ in (3.1), (3.2) and (3.3), we have

$$
\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{1 - 4a_1a_2e_1e_2z^2}{(1 - 2a_1e_1z)(1 + 2a_2e_1z)(1 + 2a_1e_2z)(1 - 2a_2e_2z)},
$$

(4.1)

$$
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n = \frac{z - 4a_1a_2e_1e_2z^3}{(1 - 2a_1e_1z)(1 + 2a_2e_1z)(1 + 2a_1e_2z)(1 - 2a_2e_2z)},
$$

(4.2)

$$
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{2(e_1 - e_2)z + 4e_1e_2(a_1 - a_2)z^2}{(1 - 2a_1e_1z)(1 + 2a_2e_1z)(1 + 2a_1e_2z)(1 - 2a_2e_2z)},
$$

(4.3)
respectively, the substitutions
\[
\begin{align*}
&\left\{ \begin{array}{l}
a_1 - a_2 = ix \\
a_1a_2 = y \\
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l}
e_1 - e_2 = t \\
e_1e_2 = -1 ,
\end{array} \right.
\end{align*}
\]
in (4.1), (4.2) and (4.3), we obtain
\[
\begin{align*}
\sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n &= \frac{1 + yz^2}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}, \\
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n &= \frac{z + yz^3}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}, \\
\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n &= \frac{2tz - ixz^2}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}.
\end{align*}
\]
respectively, we get the following theorems.

**Theorem 4.1** For \( n \in \mathbb{N} \), the new generating function of the product of bivariate complex Lucas polynomials with Chebyshev polynomials of the first kind is given by
\[
\sum_{n=0}^{+\infty} L_n(x, y)T_n(t)z^n = \frac{2 - 3ixtz + (2y - 4t^2y - x^2)z^2 + ixyt^3z^3}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}.
\]

**Proof.** Recall that, we have [8]
\[
T_n(t) = S_n(2e_1 + [-2e_2]) - tS_{n-1}(2e_1 + [-2e_2]).
\]
Then, we can see that
\[
\begin{align*}
\sum_{n=0}^{+\infty} L_n(x, y)T_n(t)z^n &= \sum_{n=0}^{+\infty} \left( 2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2]) \right) \\
&\quad \times \left( S_n(2e_1 + [-2e_2]) - tS_{n-1}(2e_1 + [-2e_2]) \right)z^n \\
&= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\
&\quad - 2t \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n \\
&\quad - ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \\
&\quad + ixt \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n.
\end{align*}
\]
By using the relationships (4.4), (4.5) and (4.6), we get
\[
\sum_{n=0}^{+\infty} L_n(x, y)T_n(t)z^n = \frac{2(1 + yz^2)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}
\]
\[
- 2t(ixz + 2yzt^2)
\]
\[
- \frac{ix(2tz - ixz^2)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}
\]
\[
+ \frac{ixt(z + yz^3)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}
\]
\[
- \frac{2 - 3ixtz + (2y - 4t^2y - x^2)z^2 + iyzt^3}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}.
\]

This completes the proof.

\[\square\]

**Theorem 4.2** For \( n \in N \), the new generating function of the product of bivariate complex Lucas polynomials with Chebyshev polynomials of the second kind is given by
\[
\sum_{n=0}^{+\infty} L_n(x, y)U_n(t)z^n = \frac{2 - 2ixtz + (2y - x^2)z^2}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}.
\]

**Proof.** By [8], we have \( U_n(t) = S_n(2e_1 + [-2e_2]) \). Then, we can see that
\[
\sum_{n=0}^{+\infty} L_n(x, y)U_n(t)z^n = \sum_{n=0}^{+\infty} \left(2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])\right)S_n(2e_1 + [-2e_2])z^n
\]
\[
= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n
\]
\[
- ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n
\]
\[
= \frac{2(1 + yz^2)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}
\]
\[
- \frac{ix(2tz - ixz^2)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4},
\]

after a simple calculation, we have
\[
\sum_{n=0}^{+\infty} L_n(x, y)U_n(t)z^n = \frac{2 - 2ixtz + (2y - x^2)z^2}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}.
\]

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This completes the proof.

\[ \sum_{n=0}^{+\infty} L_n(x, y)V_n(t)z^n = \frac{2 - i x (2t + 1) z + (2y - 4yt - x^2) z^2 + ixyz^3}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4}. \]

(4.9)

**Theorem 4.3** For \( n \in \mathbb{N} \), the new generating function of the product of bivariate complex Lucas polynomials with Chebyshev polynomials of the third kind is given by

\[ \sum_{n=0}^{+\infty} L_n(x, y)V_n(t)z^n = \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - i a_1 S_{n-1} + [-a_2]) \times (S_n(2e_1 + [-2e_2]) - S_{n-1}(2e_1 + [-2e_2]))z^n \]

\[ = 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \]

\[ - 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n \]

\[ - ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \]

\[ + ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n. \]

Proof. By [2], we have \( V_n(t) = S_n(2e_1 + [-2e_2]) - S_{n-1}(2e_1 + [-2e_2]) \). Then, we can see that

According to relationships (4.4), (4.5) and (4.6) this gives the following equality:

\[ \sum_{n=0}^{+\infty} L_n(x, y)V_n(t)z^n = \frac{2 (1 + yz^2)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4} \]

\[ = \frac{2 (1 + yz^2)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4} \]

\[ - 2 (ixz + 2ytz^2) \]

\[ - \frac{2 (2tz - ixz^2)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4} \]

\[ - \frac{ix (z + yz^3)}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4} \]

\[ + \frac{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4} \]

\[ = \frac{2 - ix (2t + 1) z + (2y - 4yt - x^2) z^2 + ixyz^3}{1 - 2ixtz - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4}. \]

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This completes the proof.

\[\sum_{n=0}^{+\infty} L_n(x,y)W_n(t)z^n = 2 + ix (1 - 2t) z + (2y + 4yt - x^2) z^2 - ixyz^3 \]

\[\frac{1 - 2ixt z - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4}{1 - 2ixt z - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4}.\]

(4.10)

\textbf{Theorem 4.4} For \(n \in \mathbb{N}\), the new generating function of the product of bivariate complex Lucas polynomials with Chebyshev polynomials of the fourth kind is given by

\[\sum_{n=0}^{+\infty} L_n(x,y)W_n(t)z^n = 2 + ix (1 - 2t) z + (2y + 4yt - x^2) z^2 - ixyz^3 \]

\[\frac{1 - 2ixt z - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4}{1 - 2ixt z - (x^2 + 4yt^2 - 2y)z^2 + 2ixytz^3 + y^2z^4}.\]

Proof. By referred to [2], we have

\[W_n(t) = S_n(2e_1 + [-2e_2]) + S_{n-1}(2e_1 + [-2e_2]).\]

We see that

\[\sum_{n=0}^{+\infty} L_n(x,y)W_n(t)z^n = \sum_{n=0}^{+\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])) \]

\[\times (S_n(2e_1 + [-2e_2]) + S_{n-1}(2e_1 + [-2e_2]))z^n \]

\[= 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \]

\[+ 2 \sum_{n=0}^{+\infty} S_n(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n \]

\[- ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n \]

\[- ix \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n.\]
According to relationships (4.4), (4.5) and (4.6) this gives the following equality:

\[
\sum_{n=0}^{\infty} L_n(x, y)W_n(t)z^n = \frac{2\left(1 + yz^2\right)}{1 - 2ixt - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4} \\
+ \frac{2\left(ixz + 2ytz^2\right)}{ix \left(2tz - ixz^2\right)} \\
- \frac{1 - 2ixt - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}{ix \left(z + yz^3\right)} \\
- \frac{1 - 2ixt - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}{ix \left(z + yz^3\right)} \\
+ \frac{2 + ix(1 - 2t)z + (2y + 4yt - x^2)z^2 - ixyz^3}{1 - 2ixt - (x^2 + 4yt^2 - 2y)z^2 + 2ixyz^3 + y^2z^4}.
\]

This completes the proof.

\[
\Box
\]

5 Conclusion

In this paper, by making use of Eqs. (3.1) and (3.3), we have derived some new generating functions of the products of bivariate complex Lucas polynomials with $k$-Fibonacci numbers, $k$-Lucas numbers, $k$-Jacobsthal numbers, $k$-Jacobsthal Lucas numbers, $k$-Pell numbers and $k$-Pell Lucas numbers, and the products of bivariate complex Lucas polynomials with Fibonacci polynomials, Jacobsthal polynomials, Pell polynomials and Chebyshev polynomials of the first, second, third and fourth kinds. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

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References

1. Abderrazak, A. – Généralisation de la transformation d’Euler d’une série formelle, Adv. Math., 103 (1994), no. 2, 180-195.
2. Aloui, B.; Boussayoud, A. – Generating function of the product of the $k$-Fibonacci and $k$-Pell numbers and Chebyshev polynomials of the third and fourth kind, (Submitted).
3. Asci, M.; Gurel, E. – On bivariate complex Fibonacci and Lucas polynomials, Conference on Mathematical Sciences ICM 2012, March 11-14 (2012).
4. Benoit, A. – Algorithmique semi-numérique rapide des séries de Tchebychev, Thèse Doctorat, École doctorale de Mathématiques et Informatiques de Bordeaux, (2012).
5. **Boughaba, S.; Boussayoud, A.; Kerada, M.** – Construction of symmetric functions of generalized Fibonacci numbers, Tamap Journal of Mathematics and Statistics, **3** (2019), no. 1, 1-7.

6. **Boussayoud, A.; Kerada, M.; Sahai, R.; Rouibah, W.** – Some applications on generating functions, J. Concr. Appl. Math., **12** (2014), no. 3-4, 321-330.

7. **Boussayoud, A.; Kerada, M.** – Symmetric and generating functions, Int. Electron. J. Pure Appl. Math., **7** (2014), no. 4, 195-203.

8. **Boussayoud, A.; Abderrezzak, A.; Kerada, M.** – Some applications of symmetric functions. Integers., **15** (2015), no. A48, 1-7.

9. **Boussayoud, A.; Boughaba, S.** – On some identities and generating functions for k-Pell sequences and Chebyshev polynomials, Online J. Anal. Comb., **14** (2019), no. 3, 1-13.

10. **Boussayoud, A.; Boughaba, S.; Kerada, M.; Araci, S.; Acikgoz, M.** – Generating functions of binary products of k-Fibonacci and orthogonal polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM., **113** (2019), no. 3, 2575-2586.

11. **Catarino, P.** – On some identities and generating functions for k-Pell numbers, Int. J. Math. Anal. (Ruse), **7** (2013), no. 37-40, 1877-1884.

12. **Catarino, P.** – On generating matrices of the k-Pell, k-Pell-Lucas and modified k-Pell sequences, Pure Math. Sci., **3** (2014), no. 2, 71-77.

13. **Doman, B.G.S.; Williams, J.K.** – Fibonacci and Lucas polynomials, Math. Proc. Cambridge Philos. Soc., **90** (1981), no. 3, 385-387.

14. **Eslahchi, M.R.; Dehghan, M.; Amani, S.** – The third and fourth kinds of Chebyshev polynomials and best uniform approximation, Math. Comput. Modelling., **55** (2012), no. 5-6, 1746-1762.

15. **Falcón, S.; Plaza, Á.** – The k-Fibonacci sequence and the Pascal 2-triangle, Chaos Solitons Fractals., **33** (2007), no. 1, 38-49.

16. **Falcón, S.** – On the k-Lucas numbers of arithmetic indexes, Appl. Math., **3** (2012), no. 10, 1202-1206.

17. **Horadam, A.F.; Mahon, J.M.** – Pell and Pell-Lucas polynomials, Fibonacci Quart., **23** (1985), no. 1, 7-20.

18. **Horadam, A.F.; Filipponi, P.** – Derivative sequences of Jacobsthal and Jacobsthal-Lucas polynomials, Fibonacci Quart., **35** (1997), no. 4, 352-357.

19. **Jhala, D.; Sisodiya, K.; Rathore, G.P.S.** – On some identities for k-Jacobsthal numbers, Int. J. Math. Anal. (Ruse), **7** (2013), no. 9-12, 551-556.

20. **Jhala, D.** – Some properties of the k-Jacobsthal Lucas sequence, International Journal of Modern Sciences and Engineering Technology, **1** (2014), no. 3, 87-92.

21. **Macdonald, I.G.** – Symmetric functions and Hall polynomials, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, (1979).

22. **Merzouk, H.; Boussayoud, A.; Chelgham, M.** – Symmetric functions of generalized polynomials of second order, Turkish Journal of Analysis and Number Theory, **7** (2019), no. 5, 135-139.

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