ASYMPTOTIC $N_p$ PROPERTY OF RATIONAL SURFACES

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0. Introduction.

The pioneering work of Mumford ([M]), its amplifications by St. Donat ([SD]) and Fujita ([F]), the inspiring work of Green ([Gr]), followed by works of Green and Lasarsfeld ([G-L]) and Ein and Lasarsfeld ([E-L]), have captured the interest and have influenced a large number of researchers in the last fifteen years. Several authors have studied the defining equations of projective varieties and, more generally, the higher order syzygies among these equations. A significant algebraic property was introduced along this line of works ([Gr], [G-L]), the $N_p$ property. It says that a variety is generated by quadratics, and its minimal free resolution is linear up to the first $p$ steps. We recall in detail the definition of $N_p$ property from [G-L].

Definition. Let $Y$ be a smooth projective variety and let $\mathcal{L}$ be a very ample line bundle on $Y$ defining an embedding $\varphi_\mathcal{L} : Y \hookrightarrow \mathbb{P} = \mathbb{P}(H^0(Y, \mathcal{L})^*)$. Let $S = \text{Sym}^* H^0(Y, \mathcal{L})$, the homogenous coordinate ring of the projective space $\mathbb{P}$. Suppose $A$ is the homogeneous coordinate ring of $\varphi_\mathcal{L}(Y)$ in $\mathbb{P}(H^0(Y, \mathcal{L})^*)$, and

$$0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to A \to 0$$

is a minimal free resolution of $A$. The line bundle $\mathcal{L}$, or the embedding $\varphi_\mathcal{L}(Y)$ of $Y$, is said to have property $N_p$ (for $p \in \mathbb{N}$) if and only if $F_0 = S$ and $F_i = S(-i-1)^{\alpha_i}$ with $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq p$. One also says that $\mathcal{L}$, or the embedding $\varphi_\mathcal{L}(Y)$ of $Y$, satisfies property $N_0$ if $\varphi_\mathcal{L}(Y)$ is projectively normal, i.e. $\mathcal{L}$ is normally generated.

The question of having property $N_p$, for various values of $p$, has been studied on many different projective varieties, such as the Veronese varieties (cf. [O-P], [J-P-W]), and the Segre product of several projective spaces (cf. [G-P], [J-P-W], [Ru]). In this paper, we look at the same question for rational surfaces obtained by blowing up $\mathbb{P}^2$ at collections of points. Defining equations and the minimal free resolution of these surfaces have also been the objects of study for many authors, such as [C-S].

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Our main result is the following theorem.

**Theorem.** Suppose $X = \{P_1, \ldots, P_s\}$ is a set of $s$ distinct points in $\mathbb{P}^2$. $Z = m_1P_1 + \ldots + m_sP_s \subseteq \mathbb{P}^2$ is an arbitrary scheme of fat points with support $X$ ($m_i \in \mathbb{N}$ $\forall i$). Let $\mathbb{P}^2(X)$ be the blowup of $\mathbb{P}^2$ along the points in $X$, $D_t = tE_0 - \sum^s_{i=1} m_iE_i \in \text{Pic} (\mathbb{P}^2(X))$, and $\sigma = \sigma(Z)$. Then, for each $p \in \mathbb{Z}_{\geq 0}$, $O_{\mathbb{P}^2(X)}(D_t)$ has property $N_p$ for all $t \geq \max\{\sigma + 1, d, 1 + \frac{d+p}{3}\}$, where $d = \sum^s_{i=1} m_i$. In other words, the embedding of $\mathbb{P}^2(X)$ using $D_t$ satisfies property $N_p$ for all $t \geq \max\{\sigma + 1, d, 1 + \frac{d+p}{3}\}$.

This theorem gives a positive answer to a conjecture of Geramita and Gimigliano ([G-G]), where they considered the case when $Z$ is reduced (i.e. $m_i = 1$ for all $i = 1, \ldots, s$) and stated that the embedding of $\mathbb{P}^2(X)$ using $D_t$ should have property $N_p$ for $t$ big enough ($t \geq \sigma(Z) + p$). In this situation, even though our bound is not exactly the conjectural value $\sigma(Z) + p$, it will be easily seen that for $p \gg 0$, $\max\{\sigma + 1, d, 1 + \frac{d+p}{3}\} \ll \sigma + p$.

1. Preliminaries and Notations.

Suppose $X = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^2 = \mathbb{P}_k^2$ is a set of $s$ distinct points ($k$ is an algebraic closed field of characteristic $0$) and let $\varphi_i \subseteq R = k[w_0, w_1, w_2]$ be the defining ideal of $P_i$ for all $i = 1, \ldots, s$. We also let $\mathbb{P}^2(X)$ be the blowup of $\mathbb{P}^2$ along the points in $X$. It is well-known (see [Hart]) that $\text{Pic} (\mathbb{P}^2(X)) \simeq \mathbb{Z}^{s+1} \simeq < E_0, E_1, \ldots, E_s >$, where $E_1, \ldots, E_s$ are the classes of the exceptional divisors and $E_0$ denotes the class of the pull-back of a general line in $\mathbb{P}^2$.

An approach to the study of the embeddings of $\mathbb{P}^2(X)$ that has been investigated is through “fat points” in $\mathbb{P}^2$ with support $X$. To be more precise, suppose $m_1, \ldots, m_s$ are positive integers, and let $I = \varphi_1^{m_1} \cap \ldots \cap \varphi_s^{m_s} \subseteq R$. Then, the scheme of fat points associated to $I$ is the subscheme $Z = m_1P_1 + \ldots + m_sP_s$ of $\mathbb{P}^2$ defined by the ideal $I$. Let $\sigma = \sigma(I) = \sigma(Z)$ be the least integer start from which the difference function of the Hilbert function of $I$ vanishes (cf. [C-G-Q]). For each $t$, we have a divisor $D_t = tE_0 - \sum^s_{i=1} m_iE_i$ on $\mathbb{P}^2(X)$. By a general result of Coppens (cf. [G-G-P]) or a stronger version in $\mathbb{P}^2$ of Davis and Geramita ([D-G]), $D_t$ is very ample on $\mathbb{P}^2(X)$ for all $t \geq \sigma + 1$, and $D_\sigma$ is also very ample provided any line in $\mathbb{P}^2$ meets $Z$ at least
than $\sigma$ points (counted properly). It was shown in [G-G-P] that the embedding of $\mathbb{P}^2(X)$ using $D_t$, for $t \geq \sigma$ and $D_t$ is very ample, is projectively normal and arithmetic Cohen-Macaulay (a.CM), i.e. the embedding of $\mathbb{P}^2(X)$ using $D_t$ satisfies property $N_0$. When the scheme $Z$ is reduced (i.e. $m_i = 1$ for all $i$) it was shown in [Gi-2] and [G-G] that the embedding of $\mathbb{P}^2(X)$ using $D_t$, for $t \geq \sigma + 1$, not only is projectively normal and a.CM, but also is generated by quadratics (i.e. satisfying properties $N_0$ and $N_1$). In [G-G], the authors further believed and conjectured that for $t \geq \sigma + p$, the embedding of $\mathbb{P}^2(X)$ using $D_t$ should have property $N_p$.

We shall now recall some notation and result introduced by Green ([Gr]) that will be used for our proof of the main theorem.

Let $Y$ be a projective scheme. Let $L$ be a very ample line bundle and $F$ a coherent sheaf on $Y$. Let $W = H^0(Y, L)$ and $S = \text{Sym}^* W$. Then, $S$ is the homogeneous coordinate ring of $\mathbb{P}(W)$, the projective space into which $Y$ is embedded using $L$. Let $B = B(L, F) = \bigoplus_{q \in \mathbb{Z}} H^0(Y, F \otimes qL) = \bigoplus_{q \in \mathbb{Z}} B_q$ a $S$-graded module.

**Definition.** The Koszul complex of $B$ is defined to be

$$\ldots \rightarrow \bigwedge^{p+1} W \otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^p W \otimes B_q \xrightarrow{d_{p,q}} \bigwedge^{p-1} W \otimes B_{q+1} \rightarrow \ldots$$

and the Koszul cohomology groups of $B$ are defined to be

$$K_{p,q}(L, F) = \frac{\ker d_{p,q}}{\text{im } d_{p+1,q-1}}, \ p, q \in \mathbb{Z}.$$ 

The following theorem relates the Koszul cohomology groups of $B$ and its minimal free resolution over $S$.

**Theorem A.** (Green’s syzygy theorem - [Gr, 1.b.4]) Suppose

$$\ldots \rightarrow \bigoplus_{q \geq q_0} M_{1,q} \otimes S(-q) \rightarrow \bigoplus_{q \geq q_0} M_{0,q} \otimes S(-q) \rightarrow B \rightarrow 0$$

is a minimal free resolution of $B$ over $S$, then

$$M_{p,p+q}(L, F) = K_{p,q}(L, F) \ \forall p, q.$$ 

Here, we write $M_{p,p+q}$ and $K_{p,q}$ as functions of $L$ and $F$ because $B$ itself depends on $L$ and $F$.

Green’s notations and results are applicable to our situation when the scheme $Y$ is the blowup $\mathbb{P}^2(X)$ of $\mathbb{P}^2$, the line bundle $L$ is the invertible sheaf $L(D_t)$ corresponding to the divisor $D_t$ on $\mathbb{P}^2(X)$. In this case, we write $M_{p,p+q}(D_t, F)$ and $K_{p,q}(D_t, F)$ for
When the coherent sheaf $\mathcal{F}$ is the structure sheaf $\mathcal{O}_{\mathbb{P}^2(X)}$ of $\mathbb{P}^2(X)$, we write $K_{p,q}(D_t)$ for $K_{p,q}(D_t, \mathcal{F})$.

2. Proof of the main theorem.

For $p = 0$, the result was already proved by [G-G-P]. Suppose $p \geq 1$. Let

$$d_p = \max\{\sigma + 1, d, 1 + \frac{d + p}{3}\},$$

where $d = \sum_{i=1}^{s} m_i$.

Let $t$ be an arbitrary integer such that $t \geq d_p$. Let $D_t = tE_0 - \sum_{i=1}^{s} m_i E_i$ and $\mathcal{L} = \mathcal{L}(D_t)$ the invertible sheaf corresponding to $D_t$. Let $W = H^0(\mathbb{P}^2(X), \mathcal{L})$ and $S = \text{Sym}^t W$. Since $t \geq d_p \geq \sigma + 1$, $D_t$ is very ample on $\mathbb{P}^2(X)$. Let $\mathcal{V}$ be the embedding of $\mathbb{P}^2(X)$ into $\mathbb{P}(W)$ using the divisor $D_t$. We need to show that $\mathcal{V}$ possesses property $N_p$.

Let $I_V \subseteq S$ be the defining ideal of $\mathcal{V}$, and let $\mathcal{I}_V$ be the ideal sheaf of $\mathcal{V}$ in $\mathbb{P}(W)$. Since $D_t$ is very ample, we have an exact sequence

$$0 \to I_V \to S \to \bigoplus_{q \in \mathbb{Z}} H^0(\mathbb{P}^2(X), qD_t) \to \bigoplus_{m \in \mathbb{Z}} H^1(\mathbb{P}(W), \mathcal{I}_V(m)) \to 0.$$

Moreover, it was proved in [G-G-P] that $\mathcal{V}$ is projectively normal, i.e.

$$\bigoplus_{m \in \mathbb{Z}} H^1(\mathbb{P}(W), \mathcal{I}_V(m)) = 0;$$

or in other words,

$$S/I_V \simeq \bigoplus_{q \in \mathbb{Z}} H^0(\mathbb{P}^2(X), qD_t).$$

Thus, the minimal free resolution of the homogeneous coordinate ring of $\mathcal{V}$ is given by that of $\bigoplus_{q \in \mathbb{Z}} H^0(\mathbb{P}^2(X), qD_t)$. This, by Green’s syzygy theorem (Theorem A), is given by the Koszul cohomology groups $K_{r,n}(D_t)$, $r, n \in \mathbb{Z}$. More precisely, let $N = \dim W - 1$, then the minimal free resolution

$$0 \to F_{N-2} \to \ldots \to F_1 \to F_0 = S \to S/I_V \to 0$$

of the homogeneous coordinate ring of $\mathcal{V}$ (since $\mathcal{V}$ is projectively normal and a.CM as proved by [G-G-P], and the codimension of $\mathcal{V}$ is 2, the length of its minimal free resolution must be $N - 2$) is given by

$$F_i = \bigoplus_{q \geq 1} K_{i,q}(D_t) \otimes S(-i - q)$$

for $i = 1, \ldots, N - 2$. 
From [G-G-P, Corollary 2.6], we know that the Castelnuovo-Mumford regularity of $I_V$ is at most 3, so

$$K_{i,q}(D_t) = 0 \text{ for all } i = 1, \ldots, N - 2, \text{ and } q \geq 3.$$ 

Observe further that if we can show $K_{p,2}(D_t) = 0$ for all $p \in \mathbb{N}$ and $t \geq d_p$, then the theorem is proved. This is because, by then, since $d_1 \leq d_2 \leq \ldots \leq d_p$, we can use induction to show that $V$ has property $N_i$ for all $i = 1, \ldots, p$; in particular, it has property $N_p$.

It follows from [G-G-P] that $H^1(\mathbb{P}^2(\mathbb{X}), mD_t) = 0$ for all $m \in \mathbb{Z}$. Let $K_{\mathbb{P}^2(\mathbb{X})}$ be the canonical divisor on $\mathbb{P}^2(\mathbb{X})$. By Green’s Duality theorem ([Gr, 2.c.6]), we have

$$K_{p,2}(D_t)^* = K_{N-2-p,1}(D_t, K_{\mathbb{P}^2(\mathbb{X})}).$$

Moreover, by Green’s Vanishing theorem ([Gr, 3.a.1]), we have

$$K_{N-2-p,1}(D_t, K_{\mathbb{P}^2(\mathbb{X})}) = 0 \text{ when } h^0(\mathbb{P}^2(\mathbb{X}), K_{\mathbb{P}^2(\mathbb{X})} + D_t) \leq N - 2 - p.$$ 

Therefore, it remains to check that

$$h^0(\mathbb{P}^2(\mathbb{X}), K_{\mathbb{P}^2(\mathbb{X})} + D_t) \leq N - 2 - p \text{ for all } t \geq d_p.$$ (2.1)

It is well-known that $K_{\mathbb{P}^2(\mathbb{X})} = -3E_0 + \sum_{i=1}^{s} E_i$. Thus,

$$K_{\mathbb{P}^2(\mathbb{X})} + D_t = (t - 3)E_0 - \sum_{i=1}^{s} (m_i - 1)E_i.$$ 

We have $D_t^2 = t^2 - \sum_{i=1}^{s} m_i^2 > t^2 - (\sum_{i=1}^{s} m_i)^2 \geq 0$ since $t \geq d_p \geq d = \sum_{i=1}^{s} m_i$, so

$$(K_{\mathbb{P}^2(\mathbb{X})} + D_t).D_t = K_{\mathbb{P}^2(\mathbb{X})}.D_t + D_t^2 > K_{\mathbb{P}^2(\mathbb{X})}.D_t.$$ 

Therefore, $H^2(K_{\mathbb{P}^2(\mathbb{X})} + D_t) = 0$ ([Hart, Lemma V.1.7]).

It follows from Serre’s Duality theorem that $H^1(K_{\mathbb{P}^2(\mathbb{X})} + D_t) = H^1(-D_t)$. By Kodaira’s theorem (cf. [Hart, p. 248]), since $D_t$ is very ample, we have $H^1(-D_t) = 0$. Thus, $H^1(K_{\mathbb{P}^2(\mathbb{X})} + D_t) = 0$.

Now, using the Riemann-Roch theorem, we obtain

$$h^0(\mathbb{P}^2(\mathbb{X}), K_{\mathbb{P}^2(\mathbb{X})} + D_t) = \frac{1}{2}(K_{\mathbb{P}^2(\mathbb{X})} + D_t).D_t + 1$$

$$= \frac{1}{2}(t(t - 3) - \sum_{i=1}^{s} m_i(m_i - 1)) + 1$$

$$= \frac{1}{2}t(t - 3) - \sum_{i=1}^{s} \left(\frac{m_i}{2}\right) + 1.$$
Moreover, the Hilbert function of $Z$ is increasing and stabilizes at $\deg(Z) = \sum_{i=1}^{s} (m_i + 1)$. Therefore,

$$N \geq \left(\frac{t + 2}{2}\right) - \deg(Z) - 1 = \frac{1}{2} (t + 1)(t + 2) - \sum_{i=1}^{s} \left(\frac{m_i + 1}{2}\right) - 1.$$ 

Hence,

$$N - 2 - p - h^0(\mathbb{P}^2(X), K_{\mathbb{P}^2(X)} + D_t) \geq 3t - 3 - p - \sum_{i=1}^{s} \left(\frac{m_i + 1}{2}\right) + \sum_{i=1}^{s} \left(\frac{m_i}{2}\right) = 3t - 3 - p - \sum_{i=1}^{s} m_i = 3t - 3 - p - d \geq 0.$$ 

The inequality (2.1) is verified, and the theorem is proved. \(\square\)

**Remark:** When $p \gg 0$, $1 + \frac{d + p}{3}$ is clearly the larger value compared to $\sigma + 1$ and $d$. Thus, for $p \gg 0$, $\max\{\sigma + 1, d, 1 + \frac{d + p}{3}\} = 1 + \frac{d + p}{3}$, which is essentially smaller than $\sigma + p$.

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