On the simplicity and conditioning of low rank semidefinite programs

Lijun Ding* and Madeleine Udell†

July 26, 2021

Abstract

Low rank matrix recovery problems appear widely in statistics, combinatorics, and imaging. One celebrated method for solving these problems is to formulate and solve a semidefinite program (SDP). It is often known that the exact solution to the SDP with perfect data recovers the solution to the original low rank matrix recovery problem. It is more challenging to show that an approximate solution to the SDP formulated with noisy problem data acceptably solves the original problem; arguments are usually ad hoc for each problem setting, and can be complex.

In this note, we identify a set of conditions that we call simplicity that limit the error due to noisy problem data or incomplete convergence. In this sense, simple SDPs are robust: simple SDPs can be (approximately) solved efficiently at scale; and the resulting approximate solutions, even with noisy data, can be trusted. Moreover, we show that simplicity holds generically, and also for many structured low rank matrix recovery problems, including the stochastic block model, $Z_2$ synchronization, and matrix completion. Formally, we call an SDP simple if it has a surjective constraint map, admits a unique primal and dual solution pair, and satisfies strong duality and strict complementarity.

However, simplicity is not a panacea: we show the Burer-Monteiro formulation of the SDP may have spurious second-order critical points, even for a simple SDP with a rank 1 solution.

1 Introduction

We consider a semidefinite program (SDP) in the standard form

$$\begin{align*}
\text{minimize} \quad & \langle C, X \rangle \\
\text{subject to} \quad & AX = b \quad \text{and} \quad X \succeq 0,
\end{align*}$$

where $\langle \cdot, \cdot \rangle$ denotes the matrix trace inner product. The primal variable is the symmetric positive semidefinite (PSD) matrix $X \in S_n^+ \subset \mathbb{R}^{n \times n}$. The problem data comprises a symmetric (but possibly indefinite) cost matrix $C \in S^n$, a righthand side $b \in \mathbb{R}^m$, and a linear constraint map $A : \mathbb{R}^{n \times n} \to \mathbb{R}^m$ with rank $m$ operating on any $H \in \mathbb{R}^{n \times n}$ by $[AH]_i = \langle A_i, H \rangle$, $i = 1, \ldots, m$ for some fixed symmetric $A_i \in S^n$. Denote an arbitrary solution of $(P)$ as $X^*$ and the optimal value as $p^*$.

The optimization problem $(P)$ appears in problems in statistics [SS05], combinatorics [GW95], and imaging [CMP10], among others. Due to the nature of these applications, practical instances of $(P)$ such as matrix completion [SS05, UT19] and MaxCut [GW95] are often expected to have low rank solutions. It is also notable that any instance of $(P)$ admits a solution with rank $r^*$ satisfying $r^* (r^* + 1) / 2 \leq m$ [Bar95, Pat08].

Simplicity Formally, we say an SDP is simple if it has a surjective constraint map, admits a unique primal and dual solution pair, and satisfies both strong duality and strict complementarity. (See Section 1.1 for more detail.) These conditions suffice to guarantee many useful properties about the resulting SDP.

*School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14850, USA; ld446@cornell.edu
†School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14850, USA; udell@cornell.edu
Simplicity was found by [AH097] to hold generically: for almost all $A$, $b$ and $C$, $P$ is simple so long as a primal and dual solution pair exists. A followup work [DIL16, Section 5] strengthens this result: for every surjective $A$, simplicity holds for almost all $b$ and $C$, again conditioning on the existence of a primal and dual solution pair.

However, realistic applications of semidefinite programming may place structural constraints on $A$, $b$, and $C$: for example, in matrix completion, the cost matrix $C = I$; in MaxCut type SDPs, the constraint map $A = \text{diag}$ and the right hand side $b$ is the vector of all ones. We will show in Section 2 and 4 that many of these SDPs, including $Z_2$ synchronization and the stochastic block model, are still simple. We also show in Section 5 that matrix completion is \textit{primal simplicity}: it satisfies all conditions for simplicity except for dual uniqueness.

\subsection*{Conditioning and simplicity}
Many authors have shown that instances of the primal SDP $P$ appearing statistical or signal processing problems [CR09, WdM15, Ban18], admit a unique low rank solution which coincides with (or is close to) the underlying true signal. However, this analysis does not fully solve the original statistical or signal processing problems; other-\textit{wise, inaccuracies in the problem data or incomplete convergence can lead to wildly different reconstructions of the underlying signal. Here we consider two different notions of problem conditioning:

1. \textit{Measurement error}. Suppose we obtain perturbed problem data $A + \Delta A$, $b + \Delta b$, and $C + \Delta C$ instead of the original problem data $A$, $b$, and $C$ due to noisy measurements. We solve $P$ with perturbed problem data and obtain a perturbed solution $X'$. To ensure that the perturbed solution $X'$ is meaningful for the original problem, we must ensure the error in the solution $X - X'$ is controlled by the size of the perturbation $(\Delta A, \Delta b, \Delta C)$ in the data.

We can describe the \textit{sensitivity of the solution} to measurement error by finding constants $\alpha, \beta > 0$ such that for all small $(\Delta A, \Delta b, \Delta C)$,

$$\|X - X'\|_F^\alpha \leq \beta (\|\Delta A\| + \|\Delta b\| + \|\Delta C\|).$$

2. \textit{Optimization error}. Most optimization algorithms offer guarantees on the suboptimality $\text{tr}(CX) - p_*$ of the putative solution $X$ they return, but many cannot guarantee bounds on the distance to the solution, $\|X - X_*\|$. However, the distance to the solution is usually the more important metric for statistical and signal processing applications. Hence it is important to understand how (and when) guarantees in suboptimality translate into guarantees on the distance to the solution.

We may seek to bound the distance to the solution, $\|X - X_*\|$, in terms of simpler metrics of optimization error: the infeasibility with respect to conic constraints, $(-\lambda_{\text{min}}(X))_+$, and linear constraints, $\|AX - b\|_2$, and the suboptimality, $\text{tr}(CX) - \text{tr}(CX_*)$. (Throughout the paper we define $(x)_+ = \max\{0, x\}$ for $x \in \mathbb{R}$.) We produce an \textit{error bound} on the solution by finding constants $\gamma, \rho > 0$ such that for all $X$ near $X_*$,

$$\|X - X_*\|_F^\rho \leq \gamma (\|AX - b\|_2 + (-\lambda_{\text{min}}(X))_+ + (\text{tr}(CX) - p_*)).$$

The exponents $\rho$ and $\alpha$ and the multiplicative factors $\gamma$ and $\beta$, can be interpreted as condition numbers of $P$.

Simple SDPs obey useful bounds on these condition numbers. In the literature, it has been found that if the SDP $P$ is simple, then $\rho = 1$ [NO99] and $\alpha = 2$ [Stu00]. We note that $\alpha = 2$ only requires primal simplicity. An upcoming work of ours [DU] shows that $\rho = 2$ under the weaker condition of primal simplicity. Estimates of $\beta$ and $\gamma$ for simple SDPs based on problem data and solutions are also available respectively in [NO99] and our upcoming work [DU]. When the SDP $P$ is not simple but only feasible, then the exponent of $\rho$ can become as large as $2^{n-1}$ which is shown to be tight [Stu00, Example 2]. In such cases, the SDP is very ill-conditioned. Thus if the SDP $P$ is simple or primal simple, neither measurement error nor optimization error impede signal recovery, as the distance to the solution (which is the true signal or close to it) grows at most quadratically in the measurement or optimization error.
Simplicity and algorithmic convergence  Simplicity also plays an important role in the convergence analysis of algorithms of SDP. For example:

- Simple SDP can be solved efficiently at scale: for example, the storage-optimal algorithm of \cite{DYC+19} requires simplicity to ensure the limit of the dual iterates produces a meaningful approximation of the primal solution \(X^\star\).
- For simple SDP, the central path of an interior point method (IPM) leads to the analytical center of the solution set \cite{HgKR02,LSZ98}.

Simplicity can also improve the convergence rate for many algorithms:

- For SDP that satisfy Slater’s condition, IPMs can only be shown to converge linearly \cite{Nes18}. But for primal simple SDP, IPMs achieve superlinear convergence \cite{LSZ98}; and for simple SDP, IPMs achieve quadratic convergence \cite{AHO98}.
- For the exact penalty formulation of the dual SDP \cite{DYC+19}, subgradient-type methods with constant or diminishing stepsize require \(O(1/\epsilon^2)\) iterations to reach an \(\epsilon\)-suboptimal dual solution. But for simple SDP, subgradient methods achieve faster sublinear rates \(O(1/\epsilon)\), using the quadratic error bound induced by simplicity for the analysis \cite{Stu00,JM17}.

Simplicity and the Burer-Monteiro method  The Burer-Monteiro (BM) \cite{BM03} approach solves the SDP \(\mathcal{P}\) by factoring the decision variable, building on earlier work by Homer and Peinado \cite{HP97} that introduced the approach for the MaxCut SDP. The BM approach factors the decision variable \(X = FF^\top\), with factor \(F \in \mathbb{R}^{n \times r}\), and solves the following (nonconvex) problem:

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(CFF^\top) \\
\text{subject to} & \quad \mathcal{A}(FF^\top) = b.
\end{align*}
\]

When \(r\) exceeds the rank of any solution to \(\mathcal{P}\), the BM and \(\mathcal{P}\) have the same solution set.

Usually, the BM is solved using a Riemannian gradient or trust region method \cite{BAC18}, which requires that the feasible set forms a smooth Riemannian manifold. Following \cite{BVB18}, we call such an SDP smooth: the feasible set \(\mathcal{A}(FF^\top) = b\) forms a smooth Riemannian manifold. In this paper we will consider many interesting smooth SDPs: including MaxCut, OrthogonalCut, and an SDP relaxation of a problem optimizing over a product of spheres; and statistical problems like \(Z_2\) synchronization and the stochastic block model. Notice that many interesting large scale SDPs, such as matrix completion \cite{CT10} and phase retrieval \cite{CSV13}, may not be smooth.

Since these Riemannian optimization methods are only guaranteed to find second order stationary points, we will say the BM method succeeds for a smooth SDP when all second order stationary points are globally optimal (and fails otherwise). A recent result \cite{BVB18} shows that for smooth SDP (and under a few more technical conditions), for almost all objectives \(C\), BM succeeds if \(r(r+1)/2 > m\).

Does the BM method succeed for every (smooth) simple SDP? Alas, no: we show the Burer-Monteiro approach (BM) can fail when \(r + r + 1 < m\), even if \(\mathcal{P}\) is simple. This result extends a recent counterexamples due to \cite{WW18} by showing uniqueness of the dual solution. Hence storage optimal algorithms for SDP, such as \cite{DYC+19}, that operate directly on the SDP (without factorization) have advantages over BM.

Paper organization  We formally define simple SDPs in Section 1.1. Section 1.2 introduces the notation used in this paper. In Section 2 we show that every PSD matrix solves a simple SDP and that primal simplicity holds for almost all objectives \(C\) under Slater’s condition. In Section 3, we construct simple SDPs for which the Burer-Monteiro approach fails. In Section 4 we use simplicity to show that the SDPs corresponding to the stochastic block model and \(Z_2\) synchronization can recover the ground truth from noisy data. Notably, we show recovery is possible at higher noise thresholds than those for which the BM approach is known to succeed. Finally, in Section 5 we show that the celebrated matrix completion SDP is primal simple, but not simple.
1.1 Simplicity

To start, recall the dual problem of \(( \mathcal{P} )\) is

\[
\begin{align*}
\text{maximize} & \quad \langle b, y \rangle \\
\text{subject to} & \quad C - A^* y \succeq 0.
\end{align*}
\]

\(( \mathcal{D} )\)

Here \(\langle \cdot, \cdot \rangle\) is the dot product in \(\mathbb{R}^m\). The decision variable is the vector \(y \in \mathbb{R}^m\). The map \(A^* : \mathbb{R}^m \to \mathbb{R}^{n \times n}\) is the adjoint of the linear map \(A\), which satisfies \(\langle y, AX \rangle = \langle A^* y, X \rangle\). Explicitly, \(A^*(y) = C - \sum_{i=1}^{m} y_i A_i\) for \(y \in \mathbb{R}^m\).

We now formally state the conditions that define a simple SDP. The first two conditions, strong duality and surjective constraint map, are standard in the literature.

**Definition 1** (Strong Duality). \(( \mathcal{P} )\) and \(( \mathcal{D} )\) satisfy strong duality if there is a primal-dual solution pair and for any solution pair \((X_\star, y_\star)\) \(\in \mathbb{S}^n \times \mathbb{R}^m\) to \(( \mathcal{P} )\) and \(( \mathcal{D} )\),

\[
p_\star := \text{tr}(CX_\star) = b^\top y_\star =: d_\star.
\]

Notably, strong duality holds under primal and dual Slater’s condition: existence of feasible primal \(X_0 \succ 0\) and dual \(y_0\) with \(C - A^* y_0 \succ 0\).

Surjectivity of the linear constraint map \(A\) ensures that there are no redundant linear constraints.

**Definition 2** (Surjective constraint map). We say \(( \mathcal{P} )\) has a surjective constraint map if the linear constraint map \(A\) is surjective, or equivalently, the matrices \(A_i\) are linearly independent in \(\mathbb{S}^n\).

Simplicity also requires strict complementary slackness.

**Definition 3** (Strict complementarity). We say a solution pair \((X_\star, y_\star)\) \(\in \mathbb{S}^n \times \mathbb{R}^m\) to \(( \mathcal{P} )\) and \(( \mathcal{D} )\) is strictly complementary if

\[
\text{rank}(X_\star) + \text{rank}(C - A^* y_\star) = n.
\]

If the primal \(( \mathcal{P} )\) and dual \(( \mathcal{P} )\) SDP pair has one strictly complementary solution pair, we say the SDP pair satisfies strict complementarity, or simply that the primal SDP \(( \mathcal{P} )\) satisfies strict complementarity.

Linear programs always have some strictly complementary solution whenever they exist: there is always some primal optimal \(x_\star \in \mathbb{R}^n\) and dual optimal \(z_\star = c - A^\top y_\star \in \mathbb{R}^n\) such that

\[
nz(x_\star) + nz(z_\star) = n,
\]

where \(nz\) is the number of nonzeros \([\text{GT}56]\). In contrast, semidefinite programs may not satisfy strict complementarity \([\text{AHO}97]\, \text{Example on page } 117]\).

Finally, simplicity requires that both \(( \mathcal{P} )\) and \(( \mathcal{D} )\) have unique solutions.

**Definition 4** (Simple SDP). The SDP \(( \mathcal{P} )\) is simple if

1. \(( \mathcal{P} )\) satisfies strong duality;
2. \(( \mathcal{P} )\) has a surjective constraint map;
3. \(( \mathcal{P} )\) satisfies strict complementarity; and
4. \(( \mathcal{P} )\) and \(( \mathcal{D} )\) both have unique solutions.

Next, we introduce primal simplicity.

**Definition 5** (Primal simple SDP). The SDP \(( \mathcal{P} )\) is primal simple if it has a surjective constraint map, and satisfies strong duality and strict complementarity, and the solution to \(( \mathcal{P} )\) is unique.

The dual of a primal simple SDP may admit multiple solutions. Notice every simple SDP is primal simple. Primal simplicity is practically important: for example, the matrix completion SDP \([\text{CR}09]\), introduced in Section 5, is primal simple but not simple. Primal simple SDPs inherit some (but not all) of the nice properties of simple SDPs.
Simplicity under generic problem data As mentioned in Section 1 for almost all \( A, C, b \) (under the Lebesgue measure), if the SDP pair with problem data \( A, C, b \) has a primal and a dual solution then it is simple [AHO97, Theorem 11, 14 and 15]. In this paper, we also show in Theorem 2 that for fixed \( A \) and \( b \), the SDP pair is primal simple for almost all \( C \).

1.1.1 Relationship with conditions defined in the literature

Simple SDPs are related to many other ideas proposed earlier in the SDP literature. We review their connections here to clarify the terminology. In the following discussion, we assume that (\( P \)) satisfies strong duality and has a surjective constraint map.

First, nondegeneracy as defined in [AHO97] is well-known in the interior point methods community, and consists of primal nondegeneracy [AHO97, Definition 5] and dual nondegeneracy [AHO97, Definition 8]. These same two conditions are called primal and dual constraint nondegeneracy in the variational analysis community; see e.g. [Rob83, Definition 2.1] and [CS08, Definition 8]. Primal and dual nondegenerate SDPs are not necessarily simple, as these two conditions do not imply strict complementarity [AHO97, Example on page 117]. However, primal nondegeneracy does imply dual uniqueness [AHO97, Theorem 7], and conversely dual nondegeneracy implies primal uniqueness.

Second, the term strong regularity appears in the study of generalized equations [Rob80]. It has been shown in [CS08, Theorem 18] that strong regularity for the KKT equation of (\( P \)) is equivalent to primal and dual nondegeneracy. As noted before, primal and dual nondegeneracy together imply primal and dual uniqueness. Further, under the assumption of strict complementarity, they are equivalent [AHO97, Theorem 11]. A simple SDP satisfies strong regularity (of the KKT equation); the converse is false due to [AHO97, Example on page 117]. If (\( P \)) is primal simple but has multiple dual solutions, then strong regularity (for the KKT equation) fails: under strict complementarity, primal nondegeneracy is equivalent to dual uniqueness [AHO97, Theorem 7].

1.2 Notation

Norms and Eigenvalues For a matrix \( B \in \mathbb{R}^{n_1 \times n_2} \), we denote its Frobenius, operator two norm, and nuclear norm (sum of singular values) as \( \|B\|_F, \|B\|_{op}, \) and \( \|B\|_* \) respectively. The operator norm of a linear operator \( B : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{m_1 \times m_2} \) is defined as \( \|B\|_{op} = \max_{A \in \mathbb{R}^{n_1 \times n_2}} \|B(A)\|_F \). We write the eigenvalues of a symmetric matrix \( A \in \mathbb{S}^n \) in decreasing order as

\[
\lambda_1(A) \geq \lambda_2(A) \cdots \geq \lambda_n(A).
\]

We define the singular values \( \sigma_i : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R} \) similarly.

Inner product We use the Euclidean inner product for vectors: for \( y, z \in \mathbb{R}^n \), \( \langle y, z \rangle = \sum_{i=1}^n y_i z_i \). We use the trace inner product for matrices: for \( X \) and \( Y \in \mathbb{S}^n \) or \( X \) and \( Y \in \mathbb{R}^{n_1 \times n_2} \), \( \langle X, Y \rangle = \operatorname{tr}(X^T Y) = \sum_{i,j} X_{ij} Y_{ij} \).

Transposes and adjoints For a vector \( v \) or a matrix \( A \), \( v^T \) and \( A^\top \) denote the transpose. The adjoint map of a linear map \( A \) from \( \mathbb{R}^n \to \mathbb{R}^m \) is defined as the unique linear map \( A^* : \mathbb{R}^m \to \mathbb{R}^n \) such that for every \( X \in \mathbb{S}^n \), \( y \in \mathbb{R}^m \), \( \langle AX, y \rangle = \langle X, A^* y \rangle \).

SDP Optimization The notation \( X_\star \) denotes a primal solution to (\( P \)) and \( y_\star \) denotes a dual solution to (\( P \)). Define the slack operator \( Z : \mathbb{R}^n \to \mathbb{S}^n \) that maps a putative dual solution \( y \in \mathbb{R}^m \) to its associated slack matrix \( Z(y) := C - A^* y \). We omit the dependence on \( y \) if it is clear from the context.

2 Simple SDPs are generic

In this section, we first show that any psd matrix solves a simple SDP. We also demonstrate that for almost all \( C \), if SDP (\( P \)) satisfies primal Slater’s condition and has a surjective constraint map and has a primal solution, then it is primal simple. We then show that interesting SDPs, including MaxCut, OrthogonalCut, and
ProductSDP (introduced in Section 2.3), are simple for almost all $C$. Finally, we demonstrate numerically, MaxCut SDP of many graphs are indeed simple.

2.1 Any PSD matrix solves a simple SDP

Given any rank $r_*$ positive semidefinite matrix $X_*$, we can construct a simple SDP with $X_*$ as its unique solution.

Write the eigenvalue decomposition of $X_*$ as $X_* = \sum_{i=1}^n \lambda_i v_i v_i^T = V A V^\top$. Here the eigenvalues satisfy $\lambda_1 \geq \cdots \geq \lambda_{r_*} > \lambda_{r_*+1} = \cdots = \lambda_n = 0$, and we define the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{r_*}, 0, \ldots, 0)$ and the orthonormal matrix $V = [v_1, \ldots, v_n] \in \mathbb{R}^{n \times n}$.

We are now ready to construct the SDP and state our first theorem:

Theorem 1. For any rank $r_*$ positive semidefinite matrix $X_*$ with eigenvalue decomposition $X_* = \sum_{i=1}^n \lambda_i v_i v_i^T$, the SDP

$$\begin{align*}
\text{minimize} \quad & \text{tr}(X) \\
\text{subject to} \quad & \text{tr}(v_i v_i^T X) = \lambda_i, \quad i = 1, \ldots, r_* \\
& \text{tr}(v_j v_j^T X) = 0, \quad 1 \leq i < j \leq r_*, \\
& X \succeq 0,
\end{align*}$$

with variable $X \in \mathbb{S}^n$ is simple and has $X_*$ as its unique solution.

Proof. Let us first write down the dual, with variables $y_{ij}$, $1 \leq i \leq j \leq r_*$:

$$\begin{align*}
\text{maximize} \quad & \sum_{i=1}^{r_*} \lambda_i y_{ii} \\
\text{subject to} \quad & I - \sum_{i \leq j \leq r_*} \frac{v_i v_i^T + v_j v_j^T}{2} y_{ij} \succeq 0.
\end{align*}$$

(2)

We now verify each property required for simplicity.

Surjective constraint map The matrices $A_{ij} = \frac{v_i v_i^T + v_j v_j^T}{2}$, $1 \leq i \leq j \leq r_*$ are orthogonal and hence they are linearly independent. As a result, $A$ is surjective.

Strong duality and strict complementarity Define $y_*$ with $(y_*)_{ii} = 1$ for $i = 1, \ldots, r_*$ and $(y_*)_{ij} = 0$ for $i \neq j \leq r_*$. To verify strong duality and the strict complementarity, we claim $X_*$ and $y_*$ are solutions to the primal SDP, Eq. (1), and dual SDP, Eq. (2), respectively. Indeed, it is easy to verify that $X_*$ is primal feasible. Furthermore, by writing the slack matrix

$$Z(y_*) = I - \sum_{i \leq j \leq r_*} \frac{v_i v_i^T + v_j v_j^T}{2} (y_*)_{ij} = I - \sum_{i=1}^{r_*} v_i v_i^T \succeq 0,$$

we see $y_*$ is dual feasible and $Z(y_*)$ has rank $n - r_*$. Since the primal and dual objective match, $\text{tr}(X) = \sum_{i=1}^{r_*} \lambda_i = \sum_{i=1}^{r_*} \lambda_i (y_*)_{ii}$, we see $X_*$ and $y_*$ are a primal-dual optimal solution pair. Since $Z(y_*)$ has rank $n - r_*$ and $\text{rank}(X_*) = r_*$, we see strict complementarity holds.

Uniqueness Suppose that $y'_*$ solves the dual problem (2). We will show that $y'_* = y_*$, and hence the dual has a unique solution. Using strong duality, we know $Z(y'_*) X_* = 0$. Moreover, $Z(y'_*)$ and $X_*$ are psd. Hence $Z(y'_*)$ has rank at most $n - r_*$. By the definition of $Z(y'_*)$, we see

$$Z(y'_*) = I - \sum_{i \leq j \leq r_*} \frac{v_i v_i^T + v_j v_j^T}{2} y'_{ij}$$

$$= V \begin{bmatrix}
1 - (y'_*)_{11} & -\frac{(y'_*)_{12}}{2} & \cdots & -\frac{(y'_*)_{1r_*}}{2} & 0 \\
\vdots & \ddots & \vdots & \vdots & 0 \\
-\frac{(y'_*)_{r_1}}{2} & \cdots & 1 - (y'_*)_{r_*r_*} & 0 \\
0 & \cdots & 0 & I_{n-r_*}
\end{bmatrix} V^\top.$$

(3)
The lower right block of the inner matrix above is the identity \( I_{n-r_*} \in \mathbb{S}^{n-r_*} \). Hence we see \( Z(y'_r) \) has rank at least \( n-r_* \). Thus \( Z(y'_r) \) must have rank exactly \( n-r_* \). This fact forces the upper left block of \( Z(y'_r) \) in (3) to be 0. Hence, we must have \( y'_r = y_* \).

To show the primal solution is unique, introduce the new variable \( S \in \mathbb{S}^n \) so that \( V S V^T = X \). Using this change of variables in (1), we see \( X_* \) uniquely solves (1) if and only if \( \Lambda \in \mathbb{S}^n \) uniquely solves

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(S) \\
\text{subject to} & \quad S_{ii} = \lambda_i, \quad i = 1, \ldots, r_* \\
& \quad S_{ij} = 0, \quad 1 \leq i < j \leq r_*, \\
& \quad S \succeq 0. \\
\end{align*}
\]

(Notice that \( S = \Lambda \) is optimal for Eq. (4), using the same argument we used to show the optimality of \( X_* \) for Eq. (1) above.) Since the optimal value of Eq. (4) is \( \text{tr}(\Lambda) = \sum_{i=1}^{r_*} \lambda_i \), from the constraints \( S_{ii} = \lambda_i \) of (3), we see that any feasible \( S \succeq 0 \) of (4) has objective value \( \geq \text{tr}(\Lambda) \). To achieve optimality, we must have \( S_{rr} = 0 \) for \( r_* < r \leq n \). Now use the fact that \( S \succeq 0 \) to see \( \Lambda \) is the unique solution.

\[ \square \]

### 2.2 Almost all cost matrices yield a primal simple SDP

We establish the fact that \((P)\) is primal simple for almost all cost matrices \( C \), whenever the primal solution exists.

**Theorem 2.** Suppose \((P)\) satisfies the surjective constraint map condition and the primal Slater’s condition: there is some \( X_0 \in \mathbb{S}^n \) such that \( X_0 > 0 \) and \( A(X_0) = b \). Then for almost all \( C \in \mathbb{S}^n \), \((P)\) is primal simple as long as the primal solution exists.

**Proof.** We utilize [DL11 Corollary 3.5]: for a convex extended value function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \), for almost all \( v \in \mathbb{R}^n \), the perturbed function \( f_v(x) = f(x) - v^T x \) admits at most one minimizer \( x_v \) and satisfies \( v \in \text{ri} \left( \partial f(x_v) \right) \), the relative interior of \( \partial f(x_v) \).

To exploit this theorem, we set \( v = -C \in \mathbb{S}^n \) and take \( f \) to be the function

\[
\chi_{\{AX=b\}} + \chi_{\{X \succeq 0\}},
\]

where \( \chi_C(x) \) is the indicator function of a convex set \( C \): 0 if \( X \in C \) and \( +\infty \) otherwise. Using [DL11 Corollary 3.5], we see that for almost all \( C \), the problem \( \min(\chi_{\{AX=b\}} + \chi_{\{X \succeq 0\}})(X) + \text{tr}(CX) \) has at most one solution \( X_C \), and

\[
\begin{align*}
-C & \in \text{ri} \left( \partial \left( \chi_{\{AX=b\}} + \chi_{\{X \succeq 0\}} \right) (X_C) \right) \\
& \overset{(a)}{=} \text{ri} \left( \partial \left( \chi_{\{AX=b\}} \right) (X_C) + \partial \left( \chi_{\{X \succeq 0\}} \right) (X_C) \right) \\
& \overset{(b)}{=} \text{ri} \left( \partial \left( \chi_{\{AX=b\}} \right) \right) (X_C) + \text{ri} \left( \partial \left( \chi_{\{X \succeq 0\}} \right) \right) (X_C) \\
& \overset{(c)}{=} -\{A^*y \mid y \in \mathbb{R}^m\} - \{Z \mid Z \succeq 0, \ker(Z) = \text{range}(X_C)\},
\end{align*}
\]

which implies that \( C = A^*y - Z \) for some slack matrix \( Z \succeq 0 \) satisfying \( \ker(Z) = \text{range}(X_C) \). Here step \((a)\) uses Slater’s condition to apply the sum rule of the subdifferential. Step \((b)\) uses [Ber09 Proposition 1.3.6]: the sum rule for the relative interior. Step \((c)\) uses basic sub-differential calculus. Hence, there is some \( y \) and \( Z \) such that \( Z = C - A^*y \geq 0 \), \( \ker(Z) = \text{range}(X_C) \), and

\[
\text{rank}(Z) + \text{rank}(X_C) = n \quad \text{and} \quad \text{tr}(ZX_C) = 0.
\]

Hence \( y \) is dual optimal and strict complementarity holds. \[ \square \]

### 2.3 MaxCut-type SDP are simple for almost all \( C \)

In this section, we introduce three classes of SDPs that generalize the SDP relaxation of the MaxCut problem [GW95], with applications in statistical signal recovery, optics, and subproblems of important algorithms. We show in Corollary [1] that they are simple for almost all \( C \) based on Theorem 2.
MaxCut We call an SDP a MaxCut-type SDP if it is of the form
\[
\begin{align*}
\text{minimize} & \quad \text{tr}(CX) \\
\text{subject to} & \quad \text{diag}(X) = 1 \quad \text{and} \quad X \succeq 0.
\end{align*}
\]
(MaxCut)

Here we do not require the cost matrix $C$ to be a negative Laplacian matrix.

MaxCut-type SDP can be used to find approximations of the maximum weight cut in a graph [GW95], to recover an object of interest from optical measurements [WdM15], and to identify the cluster corresponding to each node in the stochastic block model [Ban18].

Orthogonal cut For any $M \in \mathbb{R}^{Sd \times Sd}$, $s \leq S$, we denote by $\text{Block}_s(M)$ the $s$-th diagonal $d \times d$ block of $M$. An OrthogonalCut-type problem has decision variable $X \in \mathbb{S}^D$ for some integer $S > 0$ and $d = 1, 2, \text{or} 3$, and is of the form
\[
\begin{align*}
\text{minimize} & \quad \text{tr}(CX) \\
\text{subject to} & \quad \text{Block}_s(X) = I, \quad s = 1, \ldots, S, \\
& \quad X \succeq 0.
\end{align*}
\]
(OrthogonalCut)

Note that when $d = 1$, (OrthogonalCut) reduces to (MaxCut), with $m = \frac{Sd(d+1)}{2}$ constraints.

The OrthogonalCut-type SDP generalizes the MaxCut-type SDP, and appears in sensor network localization [CLS12] and ranking problems [Cuc16].

ProductSDP: optimization over a product of spheres Finally, we introduce (ProductSDP), an SDP relaxation of a quadratic program over a product of spheres. Let $D$ be a positive integer and let $S_1, \ldots, S_m$ be a partition of the set $[D] := \{1, \ldots, D\}$: $S_i \cap S_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^m S_i = [D]$. A ProductSDP-type problem, with decision variable $X \in \mathbb{S}^D$, takes the form
\[
\begin{align*}
\text{minimize} & \quad \text{tr}(CX) \\
\text{subject to} & \quad \sum_{k \in S_i} X_{kk} = 1, \quad i = 1, \ldots, m, \\
& \quad X \succeq 0.
\end{align*}
\]
(ProductSDP)

Note that when the cardinality of each $S_i$ is one, (ProductSDP) reduces to (MaxCut).

To explain the name of this SDP, suppose $x_i \in \mathbb{R}^{|S_i|}$ for each $i = 1, \ldots, m$. The constraint $\langle x_i, x_i \rangle = 1$ ensures that $x_i$ is on the sphere $S_i^{S_i|-1}$ in $\mathbb{R}^{|S_i|}$. Now stack the variables $x_i$ for $i = 1, \ldots, m$ as a vector $x \in \mathbb{R}^n$. The SDP (ProductSDP) is a relaxation of the quadratic program
\[
\begin{align*}
\text{minimize} & \quad \text{tr}(Cxx^\top) \\
\text{subject to} & \quad x \in \prod_{i=1}^m S_i^{|S_i|-1}
\end{align*}
\]
(7)

with $xx^\top$ replaced by $X$. Problems of this form can appear as trust-region subproblems, e.g., [BVB18, Section 5.3].

Having defined these three classes of SDP, we show all of these problems are almost always simple.

**Corollary 1.** The MaxCut-type, OrthogonalCut-type, and ProductSDP-type SDPs are simple for almost any cost matrix $C$.

**Proof.** We first check dual uniqueness and verify the constraint map is surjective. Then we verify primal simplicity to conclude that these three classes of SDP are simple.

**Dual uniqueness and surjective constraint map** First, note the property surjective constraint map follows directly from the uniqueness of the dual solution. We show dual uniqueness by contradiction: if the dual is not unique, there is some $\Delta y$ such that $A^\ast(\Delta y) = 0$ and $y_\ast + \alpha \Delta y$ for some $\alpha \in \mathbb{R}$ is still optimal. Using [WW18, Proposition 9], we know there is no nonzero $y$ such that $A^\ast(y)X_\ast = 0$.

It is then immediate the dual is unique by noting $Z(y)X_\ast = 0$ for any dual optimal $y$.\[1\]

\[1\] In fact, primal nondegeneracy [Definition 5][AHO97], a stronger condition than dual uniqueness, always holds for MaxCut even when strict complementarity fails. To prove this fact, one can check the definition directly, or check [BVB18, Assumption 1.1a] and use [BYB18, Proposition 6.6].
Primal simplicity  The primal solution exists because the feasible region of each class is compact and nonempty. Slater’s condition for these three classes of SDP can be easily verified using a well-chosen diagonal matrix. Hence Theorem 2 asserts these three classes are primal simple for almost all C.

2.4 Numerical verification for real-world SDP

In this section, we numerically verify that the MaxCut problems (MaxCut) corresponding to several graphs are simple. In particular, we use the Gset graphs G1 to G20 [Gse]. In the MaxCut relaxation, the cost matrix $C$ is the negative graph Laplacian. Each graph has $n = 800$ vertices, so the MaxCut SDP (MaxCut) has a decision variable $X$ of size $800 \times 800$.

To verify strict complementarity, we must compute the rank of the primal and dual solution $X_*$ and $Z(y_*)$, $r_p$ and $r_d$, and see whether $r_p + r_d = n$.

To verify uniqueness of the primal solution, define a matrix $U \in \mathbb{R}^{n \times (n - r_d)}$ whose columns form an orthonormal basis for the null space of $Z$. Define the linear operator $A_Z: \mathbb{R}^{n - r_d} \rightarrow \mathbb{R}^n$, $A(U(S)) = USU^\top$, where $\mathcal{A} = \text{diag}$. According to [AHO97] (Theorem 9 & 10), the primal solution $X_*$ is unique if the smallest singular value $\sigma_{\min}(A_{Z_*}) := \min_{\|S\|_p = 1} \|A_Z(S)\|_2$ is nonzero.

To verify uniqueness of the dual solution, define a matrix $V_1 \in \mathbb{R}^{n \times r_p}$ whose columns form an orthonormal basis for the column space of $X$ and $V_2 \in \mathbb{R}^{n \times (n - r_p)}$ whose columns form a basis for the null space of $X$. Define the matrix $A_{X_*}^* \in \mathbb{R}^{n \times n}$ where the $k$-th column of $A_{X_*}^*$ is $[A_{X_*}]_k = \text{vec}\left(\begin{bmatrix} V_1^T e_k e_k^T V_1 \\ V_2^T e_k e_k^T V_1 \end{bmatrix} \right)$ for $k = 1, \ldots, n$. Then according to [AHO97] (Theorem 6 & 7), the dual is unique if the smallest singular value of $A_{X_*}^*$ is nonzero.

Numerically, we obtain $X_*$ and $Z(y_*)$ using the MOSEK solver [Mos10]. We estimate the rank by the number of eigenvalues larger than $10^{-6}$, and denote the smallest eigenvalue larger than $10^{-6}$ as $\lambda_{\min > 0}(X_*)$ and $\lambda_{\min > 0}(Z(y_*))$ respectively. We compute their condition numbers defined as $\kappa_{X_*} := \frac{\lambda_1(X_*)}{\lambda_{\min > 0}(X_*)}$ and $\kappa_{Z_*} := \frac{\lambda_1(Z_*)}{\lambda_{\min > 0}(Z_*)}$. We compute the condition numbers of $A_{X_*}$ and $A_{Z_*}$ defined as $\kappa(A_{Z_*}) := \frac{\|A_{Z_*}\|_{\infty}}{\sigma_{\min}(A_{Z_*})}$ and $\kappa(A_{X_*}) := \frac{\sigma_1(A_{X_*})}{\sigma_n(A_{X_*})}$. The results are reported in Table 1. As can be seen, simplicity is indeed satisfied for every MaxCut problem from G1 to G20. For graph G11, the condition number $Z_*$ is actually only $10^{-5}$ (not shown here) meaning that strict complementarity holds in a very weak sense.

3 Burer-Monteiro may fail for simple SDPs

In this section, we show that the [BM] formulation of (P) admits second order stationary points that are not globally optimal even for simple SDPs with low rank (1 or 2 or 3) solutions.

Recall from the introduction the Burer and Monteiro approach (BM approach) to semidefinite programming, which replaces the SDP (P) by the following nonlinear optimization problem with decision variable $F \in \mathbb{R}^{n \times r}$:

$$
\begin{align*}
\text{minimize} & \quad \text{tr}(CFF^\top) =: f(F) \\
\text{subject to} & \quad \mathcal{A}(FF^*) = b.
\end{align*}
$$

(BM)

This problem is in general nonconvex.

Nonlinear optimization solvers such as Riemannian trust regions [BAC18] can guarantee that they find a second order stationary point (SOSP) of such a problem, but cannot guarantee (or even check) that they have found a global solution. When the constraint set is a manifold, as it is for all the examples discussed in the previous section, a putative solution $F$ is second order stationary if its Riemannian gradient is 0 and its Riemannian Hessian is positive semidefinite. See Appendix A for further discussion.

Hence we can guarantee that the BM approach finds the global optimum if we can prove that all SOSPs are globally optimal. The following definition serves as a useful shorthand as we understand when this condition holds.

---

2Here $e_i$ is the $i$-th standard basis vector in $\mathbb{R}^n$ and $\text{vec}$ stacks the columns of a matrix.
Table 1: Summary statistics of MaxCut problem on Gset graphs verify simplicity. Strict complementarity holds, as numerically rank(X) + rank(Z) is equal to the dimension \( n = 800 \) for every problem. The small condition numbers of the linear maps \( A_Z \) and \( A_X \), verify primal and dual uniqueness, respectively.

**Definition 6.** We say the BM approach succeeds for an SDP \([P]\) if every SOSP \( F \) of (BM) is globally optimal, and hence \( X = FF^\top \) is optimal for \([P]\). Conversely, we say the BM approach fails if (BM) has any SOSP that is not global optimal.

Note that as a practical matter, a nonlinear solver for (BM) might produce a globally optimal SOSP even for a problem that admits non-optimal SOSPs.

Recall from the introduction that for almost all \( C \), when \( \frac{r(r+1)}{2} > m \), any SOSP of (BM) is globally optimal [BVB18]. On the other hand, building on results by [WW18], we will demonstrate a positive measure set of simple SDP of each of the three classes described in Section 2.3 for which BM fails whenever \( \frac{r(r+1)}{2} + r \leq m \).

### 3.1 Examples: MaxCut, OrthogonalCut, and ProductSDP

Let us first recall the (MaxCut) SDP we described in Section 2.3:

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(CX) \\
\text{subject to} & \quad \text{diag}(X) = 1 \quad \text{and} \quad X \succeq 0.
\end{align*}
\]

As demonstrated in [WW18, Corollary 1], if

\[
\frac{r(r+1)}{2} + r > n,
\]

then for almost all \( C \), any SOSP \( F \) of the BM formulation (BM) of (MaxCut) is global optimal. Hence the matrix \( FF^\top \) solves (MaxCut). However in [WW18, Corollary 1], the authors show that if

\[
\frac{r(r+1)}{2} + r \leq n,
\]

then there is a positive measure set of the cost matrix \( C \) for which (MaxCut) has a unique rank 1 solution but the BM approach fails.
Are these SDP particularly nasty? On the contrary! Our contribution, stated in the following theorem, is to show that these SDPs are simple. We also generalize these results to (OrthogonalCut) and (ProductSDP).

**Theorem 3.** Fix a positive integer $r$. If

$$\frac{r(r+1)}{2} + r \leq n,$$

then there is a set of cost matrices $C$ with positive measure for which (MaxCut) admits a unique rank 1 solution and is simple, but the BM approach fails.

The same result holds for (ProductSDP) if

$$\frac{r(r+1)}{2} + r \leq m.$$

For (OrthogonalCut), the same result holds, except that the solution has rank $d$, if

$$\frac{r(r+1)}{2} + rd \leq m = \frac{Sd(d+1)}{2}.$$

**Proof.** The proofs of dual uniqueness and the surjective constraint map property are the same as in the proof of Corollary 1. We next verify the failure of BM, and primal simplicity.

**Failure of BM, and Primal simplicity** Waldspurger and Waters show that there is a positive measure set of cost matrices $C$ for which (MaxCut) satisfies: (1) strong duality [WW18, Proposition 4], (2) uniqueness of a primal solution $X_\star$ with rank 1 [WW18, Corollary 2], (3) strict complementarity for a dual solution $y_\star$, [WW18, Lemma 2, Lemma 9 and $X_\star Z(y_\star) = 0$], (4) the BM approach fails [WW18, Corollary 1]. Together with dual uniqueness and the surjective constraint map property, these results verify the theorem statement for (MaxCut).

**OrthogonalCut and ProductSDP** The proof for the other two SDPs follows exactly the same argument as above, using [WW18, Corollary 2] for (OrthogonalCut) and [WW18, Corollary 3] for (ProductSDP). 

---

### 4 Noisy SDPs are simple

In section 2, we saw that many interesting SDPs are simple for almost any cost matrix $C$. In this section, we show that the (very structured) cost matrices that appear in certain statistical problems also yield simple SDPs. In these problems, the objective measures agreement with observations of a ground-truth object, while the constraints restrict the complexity of the solution. Importantly, simplicity of these problems guarantees that the solution of the SDP recovers the ground truth.

More precisely, we consider the SDP relaxations of the following statistical problems:

- **$Z_2$ Synchronization**
- **Stochastic Block Model**

We show that these SDP relaxations are simple with high probability.

We also demonstrate a strong advantage to solving the original SDP rather than using the BM approach (when applicable): these SDPs can provably recover the ground truth under much higher noise than the noise level (provably) allowable using the BM approach.

#### 4.1 $Z_2$ Synchronization

Consider a binary vector $z \in \{\pm 1\}^n$. The $Z_2$ synchronization problem is to to recover the vector $z$ up to a sign from the observations $Y = zz^\top + \gamma W$, where $W$ is symmetric with iid standard normal upper diagonal
entries, and 0 diagonal entries. The value $\gamma$ is the noise level. The SDP proposed in the literature with decision variable $X \in S^n$ is

$$\begin{array}{ll}
\text{minimize} & \text{tr}(-YX) \\
\text{subject to} & \text{diag}(X) = 1 \quad \text{and} \quad X \succeq 0.
\end{array} \quad (Z_2 \text{ Sync})$$

The corresponding Burer-Monteiro formulation with variable $F \in \mathbb{R}^{n \times r}$ is

$$\begin{array}{ll}
\text{minimize} & \text{tr}(-YFF^\top) \\
\text{subject to} & \text{diag}(FF^\top) = 1.
\end{array} \quad (BM \ Z_2 \text{ Sync})$$

It is intuitive that the problem is more challenging as the noise level $\gamma$ increases. For (Z_2 Sync), if the noise level satisfies $\gamma \leq \sqrt{\frac{n}{(2+\epsilon) \log n}}$ for some numerical constant $\epsilon > 0$, it admits $zz^\top$ as its unique solution with high probability [Ban18 Proposition 3.6]. But for (BM Z_2 Sync) with $r = 2$, the best known theoretical results state that the noise level $\gamma$ must be less than $c n^{\frac{2}{3}}$ for some small numerical constant $c > 0$ to ensure the BM formulation succeeds, i.e., all second order stationary points $F$ satisfy $FF^\top = zz^\top$ [BBV16]. The gap between $\gamma = \sqrt{\frac{n}{(2+\epsilon) \log n}}$ and $O(n^{\frac{2}{3}})$ is polynomially large.

We now prove [BBV16]. The dual optimal solution proposed in [Ban18 Proposition 3.6] is

$$y_\star = -\text{ddiag}(Yzz^\top), \quad \text{and} \quad Z_\star = -Y - (\text{diag}(y_\star)) = \text{ddiag}(Yzz^\top) - Y,$$

where $\text{ddiag} : S^n \rightarrow \mathbb{R}^n$ is the adjoint operator of $\text{diag} : \mathbb{R}^n \rightarrow S^n$. Note that $Z_\star z = \text{ddiag}(Yzz^\top)z - Yz = 0$ using the fact that $z \in \{\pm 1\}^n$. Using the proof of [Ban18 Proposition 3.6, proof on pp356] and $\gamma \leq \sqrt{\frac{n}{(2+\epsilon) \log n}}$, we find that with high probability, there is a numerical constant $c \in (0,1)$ such that

$$\lambda_{n-1}(Z_\star) \geq cn. \quad (8)$$

We see $Z_\star \succeq 0$ and is optimal as $Z_\star zz^\top = 0$. Moreover, strict complementarity is satisfied, as $\lambda_{n-1}(Z_\star) > 0$. Surjectivity of the constraint map and the uniqueness of the dual can be verified in the same way as in the proof of Theorem 3. We summarize our findings as the following theorem.

**Theorem 4.1.** For the $Z_2$ synchronization problem, if the noise level $\gamma < \sqrt{\frac{n}{(2+\epsilon) \log n}}$ for some numerical constant $\epsilon > 0$, then with high probability the SDP (Z_2 Sync) is simple with primal solution $zz^\top$. Moreover, the dual solution satisfies $\lambda_{n-1}(Z_\star) > cn$ for some numerical constant $c \in (0,1)$.

### 4.2 Stochastic Block Model

The stochastic block model (SBM) is structurally quite similar to $Z_2$ synchronization. The SBM posits that we observe the edges and vertices of a graph $G$ with $n$ vertices that are split into two clusters according to a binary membership vector $z \in \{-1,1\}^n$. For each pair of vertices $(i,j) \in [n] \times [n]$ with $i \neq j$, the undirected edge $(i,j)$ is formed with probability $p \in [0,1]$ if vertices $i$ and $j$ are in the same cluster ($z_i = z_j$) and with probability $0 \leq q < p$ otherwise. The goal is to recover the cluster membership vector $z$. For simplicity, we further assume that $n$ is even and that the clusters are balanced: $n/2$ entries of $z$ are +1 and $n/2$ are −1. Let $A$ be the adjacency matrix of $G$ with diagonal entries set to be $\frac{q+p}{2}$. The SDP proposed to recover $z$ by [BBV16], with variable $X$, is

$$\begin{array}{ll}
\text{maximize} & \text{tr}((A - \frac{q+p}{2}J)X) \\
\text{subject to} & \text{diag}(X) = 1 \quad \text{and} \quad X \succeq 0.
\end{array} \quad (SBM)$$

where the matrix $J = 11^\top - I \in S^n$. The corresponding Burer-Monteiro formulation with variable $F \in \mathbb{R}^{n \times r}$ is

$$\begin{array}{ll}
\text{minimize} & \text{tr}((A - \frac{q+p}{2}J)FF^\top) \\
\text{subject to} & \text{diag}(FF^\top) = 1.
\end{array} \quad (BM \ SBM)$$
(There are other SDP formulations for SBM which make weaker assumptions; see [Ban18]. However, there are no guarantees for the corresponding Burer-Monteiro relaxations.)

To see the relation between \( Z_2 \text{Sync} \) and \( \text{SBM} \), we note the cost matrix \( A - \frac{p+q}{2} J \) can be decomposed as

\[
A - \frac{p+q}{2} J = \frac{p-q}{2} zz^\top + E,
\]

where the error matrix \( E \) has zero diagonal, expectation 0, and satisfies that for \( z_i = z_j, i \neq j \),

\[
E_{ij} = \begin{cases} 
1 - p & \text{with probability } p \\
-p & \text{with probability } 1 - p
\end{cases}
\]

and for \( z_i \neq z_j \),

\[
E_{ij} = \begin{cases} 
1 - q & \text{with probability } q \\
-q & \text{with probability } 1 - q.
\end{cases}
\]

We may rescale the cost matrix \( A - \frac{p+q}{2} J \) by \( \frac{2}{p-q} \) to form

\[
\tilde{A} = \frac{2}{p-q} (A - \frac{p+q}{2} J) = zz^\top + \frac{2}{p-q} E,
\]

which has the same form as the observation matrix \( Y = zz^\top + \gamma W \) in Section 4.1.

To establish the fact that \( \text{SBM} \) is simple, let us work with the following form of SDP whose cost matrix is the rescaled version \( \tilde{A} \):

\[
\begin{align*}
\text{maximize} & \quad \text{tr}(\tilde{A}X) \\
\text{subject to} & \quad \text{diag}(X) = 1 \quad \text{and} \quad X \succeq 0.
\end{align*}
\]

Clearly \( \text{SBM} \) is simple if and only if \( \text{(A-SBM)} \) is simple. The dual certificate we construct here is

\[
y_\star = -\text{ddiag}(\tilde{A}zz^\top), \quad \text{and} \quad Z_\star = -\tilde{A} - \text{diag}(y_\star) = \text{ddiag}(\tilde{A}zz^\top) - \tilde{A},
\]

Using [BBV16, Lemma 11 and its proof], \( Z(y_\star) \) is dual optimal, certifies \( zz^\top \) as the unique solution of \( \text{(A-SBM)} \) and satisfies \( \lambda_2(Z(y_\star)) > cn \) (for some small but universal constant \( c \)) if (for some large but universal constant \( C \))

\[
\left\| \frac{2}{\sqrt{n(p-q)} E} \right\|_\infty \leq \frac{n}{C \log n}, \quad \text{and} \quad \left\| \frac{2}{\sqrt{n(p-q)} E z} \right\|_\infty \leq \sqrt{\frac{n}{C}}.
\]

Here \( \left\| \frac{2}{\sqrt{n(p-q)} E z} \right\|_\infty \) is the largest entry in absolute value of \( \frac{2}{\sqrt{n(p-q)} E z} \). Using [BBV16, Lemma 18, 19], the inequality of the previous display holds if the signal strength satisfies

\[
\lambda(p,q) = \frac{p-q}{\sqrt{2(p+q)}} \sqrt{n} \geq C \sqrt{\log n}
\]

for some large universal constant \( C \).

[BBV16 Theorem 6] also states conditions under which \( \text{BM SBM} \) provably succeeds. These conditions require \( \lambda(p,q) \geq Cn^{1/3} \). This requirement is polynomially larger than the \( \lambda(p,q) \) that guarantees simplicity of \( \text{SBM} \). We summarize our findings as the following theorem.

**Theorem 4.2.** For the SBM problem, if the signal strength satisfies that

\[
\lambda(p,q) = \frac{p-q}{\sqrt{2(p+q)}} \sqrt{n} \geq C \sqrt{\log n}
\]

for some numerical constant \( C > 0 \), then with high probability, the SDP \( \text{SBM} \) is simple with primal solution \( zz^\top \) and the dual solution of \( \text{(A-SBM)} \) satisfies \( \lambda_{n-1}(Z_\star) > cn \) for some numerical constant \( c \in (0,1) \).
5 Primal simple SDP: Matrix Completion

We have seen many simple SDPs in previous sections. In this section, we demonstrate that the matrix completion SDP, a celebrated method for data imputation, is not simple but only primal simple.

The matrix completion problem seeks to recover a rank $r_2$ matrix $X_2 \in \mathbb{R}^{n_1 \times n_2}$ from a few entrywise observations $(X_2)_{ij}, (i,j) \in \Omega$, where $\Omega \subset [n_1] \times [n_2]$ is an index set of the observed entries. Define the projection operator $\Pi_\Omega: \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{n_1 \times n_2}$ as $[\Pi_\Omega(A)]_{ij} = A_{ij}$ if $(i,j) \in \Omega$ and 0 otherwise.

One popular recovery method for matrix completion, Nuclear Norm Minimization (NNM) [CR09], imputes the missing entries by solving the SDP

$$\text{minimize} \quad \|X\|_*$$
$$\text{subject to} \quad \Pi_\Omega(X) = \Pi_\Omega(X_2).$$

(Matrix-Completion)

A standard result in this literature [Faz02] Lemma 2 represents the nuclear norm by semidefinite-representable constraints on a lifted matrix $\begin{bmatrix} X_1 & X \\ X^\top & X_2 \end{bmatrix}$:

$$\|X\|_* \leq t \iff \exists X_1, X_2 \text{ such that } \begin{bmatrix} X_1 & X \\ X^\top & X_2 \end{bmatrix} \succeq 0, \text{tr}(X_1) + \text{tr}(X_2) \leq 2t. \quad (9)$$

Hence (Matrix-Completion) can be reformulated as

$$\text{minimize} \quad \text{tr}(W_1) + \text{tr}(W_2)$$
$$\text{subject to} \quad W_1 \begin{bmatrix} X_1 & X \\ X^\top & X_2 \end{bmatrix} W_2 \succeq 0,$$

(SDP Matrix-Completion)

where $W_1, W_2, X$ are the decision variables. In particular, if $X_* = \begin{bmatrix} W_1 & X \\ X^\top & W_2 \end{bmatrix}$ solves (SDP Matrix-Completion), then $X$ solves (Matrix-Completion) using (9).

As is standard in this literature, we measure the difficulty of the matrix completion problem by the incoherence $\mu$, which we now define. Let $X_2 = U \Sigma V^\top$ be the SVD of $X_2$ with $U \in \mathbb{R}^{n_1 \times r_2}, V \in \mathbb{R}^{n_2 \times r_2}$ having orthonormal columns and the diagonal matrix $\Sigma \in \mathbb{R}^{r_2 \times r_2}$ having positive entries on the diagonal. The incoherence $\mu$ is the smallest number that satisfies

$$\max_{1 \leq i \leq n_1} \|e_i^\top U\|_2 \leq \sqrt{\frac{\mu r_2}{n_1}} \quad \text{and} \quad \max_{1 \leq i \leq n_2} \|e_i^\top V\|_2 \leq \sqrt{\frac{\mu r_2}{n_2}}. \quad (10)$$

If each entry of $X_2$ is observed independently with probability $p$ and $X_2$ is $\mu$-incoherent, Problem (Matrix-Completion) has $X_2$ as its unique solution with high probability when the observation probability $p$ exceeds a certain threshold. A string of celebrated results have placed bounds on this threshold [CT10, Gro11, Rec11, Che15]. The tightest bound available is $p > \frac{C(\mu r_2 \log(\mu r_2) \log(\max(n_1, n_2)))}{\min(n_1, n_2)}$ for some large enough constant $C$ [DC18].

If the (SDP Matrix-Completion) is simple, then it is computationally tractable and gives a statistical robust estimator as argued in the introduction. The strong duality, and the condition of surjective constraint map (SDP Matrix-Completion) can be easily verified. Previous work has established primal uniqueness, but not strict complementarity because the dual certificate is only approximate. In this paper, we show that it also satisfies strict complementarity but has multiple dual solution. Hence (SDP Matrix-Completion) is only primal simple. Still, as discussed in the introduction, primal simplicity guarantees that the recovery of $X_*$ is not stymied by the optimization and measurement error.

**Theorem 5.1.** Let $n_{\min} = \min\{n_1, n_2\}$ and $n_{\max} = \max\{n_1, n_2\}$. Suppose the ground truth rank $r_2$ matrix $X_2$ is $\mu$-incoherent, and each entry of it is observed with probability $p$ independently. If $p \geq C \frac{\log(\mu r_2)}{n_{\min}} \frac{\log(n_{\max})}{n_{\max}}$ for some large enough numerical constant $C > 1$, then with probability at least $1 - n_{\min}^{-\epsilon}$ for some numeric constant $c > 0$, every item of the following holds
1. Problem (SDP Matrix-Completion) is primal simple and has a unique solution \( \tilde{X}_* = \begin{bmatrix} UΣUT & X_2 \\ X_2^T & VΣV^T \end{bmatrix} \) with rank \( r_2 \).

2. It admits multiple dual solutions.

3. It has a dual optimal solution \( \tilde{Y}_0 \) strictly complementary to \( \tilde{X}_* \), satisfying \( \lambda_{n-r_1}(\tilde{Y}_0) \geq \frac{3}{8} \).

The rest of the section is devoted to the proof of this theorem. We assume \( n_1 = n_2 = n \) to simplify the presentation. The case for rectangular matrices \( n_1 \neq n_2 \) follows exactly the same reasoning. \( ^3 \)

### 5.1 Surjective constraint map and uniqueness of primal

The surjective constraint map property is satisfied because the constraint map \( A \) has

\[
A_{i,j} = \frac{e_i e_{n_2+j}^T + (e_i e_{n_2+j}^T)^T}{2}, (i,j) \in \Omega,
\]

which are orthogonal and hence linearly independent.

It has been proved that with high probability \( X_2 \) is the unique solution to (Matrix-Completion) [DC18, Theorem 2]. Hence the matrix

\[
\tilde{X}_* = \begin{bmatrix} UΣUT & UΣV^T \\ VΣU^T & VΣV^T \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} Σ \begin{bmatrix} U^T & V^T \end{bmatrix} \geq 0
\]

(11)

is a solution to (SDP Matrix-Completion) and has rank equal to \( r_* \). Moreover, for any solution \( \tilde{X} \) of (SDP Matrix-Completion), because \( X_2 \) is the unique solution of (Matrix-Completion), it must be of the form

\[
\tilde{X} = \begin{bmatrix} W_1 & X_2 \\ X_2^T & W_2 \end{bmatrix}.
\]

Since the objective value should be equal for \( \tilde{X} \) and \( \tilde{X}_* \):

\[
\text{tr}(\tilde{X}) = \text{tr}(\tilde{X}_*) = 2\|X_2\|_*,
\]

we must have \( \tilde{X} = \tilde{X}_* \). We prove this formally in Lemma 3 using complementarity. Hence the primal solution to (SDP Matrix-Completion) is unique.

### 5.2 Strong duality and strict complementarity

In this section, we will construct a dual optimal solution to assert strong duality and strict complementarity by using the following lemma (proved in [B.1]).

**Lemma 1.** Under the setting of Theorem 5.1 with probability at least \( 1 - n_{\min}^{-c} \), there exists a \( Y_0 \in \mathbb{R}^{n \times n} \) such that (1) \( Π(\Omega)(Y_0) = Y_0 \), (2) \( Π_τ(Y_0) = UV^T \), and (3) \( Π_τ(Y_0) \|_{op} \leq \frac{5}{8} \).

The operator \( Π_τ \) is the projection to the linear space \( \mathcal{T} \subset \mathbb{R}^{n^2} \) consisting of matrices with columns in \( \text{range}(U) \) or rows in \( \text{range}(V) \). The projection \( Π_τ \) can be written explicitly as

\[
Π_τ Z = UU^T Z + ZV V^T - UU^T ZV V^T
\]

for any \( Z \in \mathbb{R}^{n^2} \). The projection \( Π_{τ^*}(Z) = Z - Π_τ(Z) \) is the projection on to the subspace orthogonal to \( \mathcal{T} \).

Let us write down the dual of (SDP Matrix-Completion) with variable \( y_{ij}, (i,j) \in \Omega \) for the purpose of constructing a dual solution:

\[
\begin{align*}
\text{maximize} & \quad \sum_{(i,j) \in \Omega} y_{ij}(X_2)_{ij} \\
\text{subject to} & \quad I - \sum_{(i,j) \in \Omega} c_{ij} e_i e_j^T y_{ij} \geq 0.
\end{align*}
\]

\( ^3 \)Primal and dual uniqueness and strict complementarity can be defined for (Matrix-Completion) directly instead of for the lifted version (SDP Matrix-Completion). However, the conclusions are the same for the lifted or standard versions: in Section B.2, we show that primal uniqueness and strict complementarity still hold for (Matrix-Completion), and (Matrix-Completion) has multiple dual solution with the same probability and assumptions as Theorem 5.1.
By introducing a variable \( \tilde{Y} = \begin{bmatrix} 0 & Y \\ Y^T & 0 \end{bmatrix} = \sum_{(i,j) \in \Omega} \frac{e_i e_{n+j}^T + e_{n+j} e_i^T}{2} y_{ij} \in \mathbb{R}^{2n \times 2n} \) and \( Y \in \mathbb{R}^{n^2} \), the dual problem (12) is equivalent to

\[
\begin{align*}
\text{maximize} & \quad 2 \langle X, Y \rangle \\
\text{subject to} & \quad I - \tilde{Y} \succeq 0 \\
& \quad \Pi_\Omega(Y) = Y.
\end{align*}
\]

with decision variable \( \tilde{Y} = \begin{bmatrix} 0 & Y \\ Y^T & 0 \end{bmatrix} \). We work with \( \tilde{Y} \) instead of \([y_{ij}]_{(i,j) \in \Omega}\) because working with matrices is more convenient. We claim the dual matrix

\[
\tilde{Y}_0 = \begin{bmatrix} 0 & Y_0 \\ Y_0^T & 0 \end{bmatrix}
\]

solves the dual problem (13). Our derivation will also show strong duality and strict complementarity of (19). We first verify that \( \tilde{Y}_0 \) is optimal to (12) as shown in the next paragraph:

**Linear feasibility:** This is due to \( \Pi_\Omega(Y_0) = Y_0 \) by assumption on \( Y_0 \).

**PSD feasibility and strict complementarity:** Take any \( w = \begin{bmatrix} u \\ v \end{bmatrix} \in \text{range} \left( \begin{bmatrix} U \\ V \end{bmatrix} \right) \). As \( \frac{1}{2} \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^T, V^T \end{bmatrix} \) is the projection matrix to \( \text{range} \left( \begin{bmatrix} U \\ V \end{bmatrix} \right) \), we have

\[
w = \left( \frac{1}{2} \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^T, V^T \end{bmatrix} \right) w.
\]

By expanding the righthand side of the above equality, we reach \( w = \begin{bmatrix} UV^T u \\ VU^T u \end{bmatrix} \). Using this fact and the definition of \( T^\perp \), we have

\[
(I - \tilde{Y}_0)w = w - \begin{bmatrix} UV^T u \\ VU^T u \end{bmatrix} - \left[ \Pi_{T^\perp}(Y_0) \right] u = 0.
\]

Thus the null space of \( I - \tilde{Y}_0 \) contains \( \text{range} \left( \begin{bmatrix} U \\ V \end{bmatrix} \right) \). Now take any \( z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \perp \text{range} \left( \begin{bmatrix} U \\ V \end{bmatrix} \right) \), then \( U^T z_1 + V^T z_2 = 0 \). Thus the quadratic form \( z^T (I - \tilde{Y}_0)z \) satisfies

\[
\begin{align*}
& z^T (I - \tilde{Y}_0)z = \| z_1 \|^2 + \| z_2 \|^2 - 2 z_1^T V U^T z_1 - 2 z_2^T V U^T z_2 \\
& \quad \geq \| z_1 \|^2 + \| z_2 \|^2 - 2 z_1^T V V^T z_2 - 2 \| z_1 \|_2 \| \Pi_{T^\perp}(Y_0) \|_{op} \| z_2 \|_2 \\
& \quad \geq \| z_1 \|^2 + \| z_2 \|^2 - \frac{5}{4} \| z_1 \|_2 \| z_2 \|_2 \geq \frac{3}{8} (\| z_1 \|^2 + \| z_2 \|^2),
\end{align*}
\]

where step (a) is due to the fact \( U^T z_1 + V^T z_2 = 0 \) and step (b) is because of \( \| \Pi_{T^\perp}(Y_0) \|_{op} \leq \frac{5}{8} \). We hence have shown that \( I - \tilde{Y}_0 \) is PSD when restricted to the space orthogonal to \( \text{range} \left( \begin{bmatrix} U^T \\ V \end{bmatrix} \right) \). To conclude \( I - \tilde{Y}_0 \) is PSD, we recall the null space of \( I - \tilde{Y}_0 \) contains \( \text{range} \left( \begin{bmatrix} U^T \\ V \end{bmatrix} \right) \). Note that strict complementarity is satisfied as \( \tilde{Y} \) is optimal to (12) as shown in the next paragraph:

\[
\text{rank}(I - \tilde{Y}_0) = n - r_2, \quad \text{and} \quad \lambda_{n-r_2} (I - \tilde{Y}_0) \geq \frac{3}{8}.
\]
Optimality and strong duality: the objective value of $\hat{Y}_0$ and $\hat{X}_*$ satisfy
\[ 2\tr(X_0Y_0) \overset{(a)}{=} 2\tr(X_0\Pi_T(Y_0)) \overset{(b)}{=} 2\tr(V\Sigma U^TU^T) = 2\tr(\Sigma) \overset{(c)}{=} \tr(\hat{X}_*). \quad (16) \]
Here step (a) uses the fact that $X_0 \in T$. In step (b), we use the fact $\Pi_T(Y_0) = UV$ in Lemma 1. Step (c) uses the form of $\hat{X}_*$ in (11). Thus we see $I - \hat{Y}_0$ is indeed dual optimal and satisfies strong duality.

5.3 Multiple dual solutions
To establish the fact that the dual has multiple solutions, let us introduce the following lemma concerning the uniqueness of the dual.

Lemma 2. [AHO97, Theorem 6, 7, and 11] Suppose the primal SDP $[P]$ is primal simple. A necessary condition for the dual to be unique is $\frac{[n-r,j(n-r_j+1)]}{2} \leq \frac{n(n+1)}{2} - m$. Using the necessary condition from Lemma 2, we have the dual is not unique unless
\[ \frac{(2n - r_s)(2n - r_s + 1)}{2} \leq \frac{2n(2n + 1)}{2} - m \iff \frac{2nr_s - (r_s)^2 + r_s}{2} \geq m. \quad (17) \]
However, since $m \geq \frac{1}{2}pm^2 \geq Cnr_s\mu \log(r_s\mu) \log n$ with probability at least $1 - n^{-2}$ for some large constant $C > 1$ and $r_s = r_2$ (as $\hat{X}_*$ has rank $r_2$), we see (17) cannot be satisfied for any $n \geq 1$ and hence the dual is not unique.

5.4 Numerical verification of multiple dual solution
In this section, we demonstrate numerically that the matrix completion problem (SDP Matrix-Completion) indeed admits multiple dual solutions by constructing two dual solutions. The problem instance we consider is the matrix completion problem with $n = n_1 = n_2 = 50$. The original matrix $X_*$ is generated randomly with rank $r_2 = 2$. We set the observation probability $p$ to be $p = 3r_2 \log(n)/n$. We compute the primal solution $X_*$ of (SDP Matrix-Completion) using the SDPT3 solver [TTT99] and find it is exactly $\hat{X}_*$.

Now we demonstrate the multiplicity of the dual solution by constructing several dual solutions. Let $U \in \mathbb{R}^{2n \times (2n-r_s)}$ have columns that form an orthonormal basis for $\null(\hat{X}_*)$. Recall $X_*$ has rank $r_s = \rank(\hat{X}_*) = r_2$. For a given cost matrix $C \in \mathbb{S}^{2n-r_s}$, we solve the problem
\[ \begin{array}{cl}
\text{maximize} & \langle C, Z_s \rangle \\
\text{subject to} & Z_s \succeq 0 \\
& UZ_sU^T = I - \sum_{(i,j) \in \Omega} \frac{r_ir_j^T + r_{i+n}r_{j+n}^T}{2} y_{ij} 
\end{array} \quad (18) \]
with decision variable $Z_s \in \mathbb{S}^{2n-r_s}$ and $y \in \mathbb{R}^m$. We solve this problem twice, with (i) $C = I$ and (ii) $C$ having iid standard Gaussian entries.

Denote the solution of (18) as $Z_{s,C}$ for each different $C$. By construction, the matrix $Z_C = UZ_{s,C}U^T$ is dual optimal as it is feasible and $\tr(ZCX_*) = 0$.

We plot the spectrum of $Z_C$ in Figure 1. We see the spectra are quite different; clearly these two dual solutions are not the same!

6 Discussion and conclusion
In this note, we have shown that generic SDPs are simple, and that many structured low rank matrix recovery problems are also simple. Building on the framework established here, an important future direction is to understand whether most SDPs with linear inequality constraints are simple. These inequality constraints

\footnote{We note that the results in [AHO97] require either the primal ([P]) or the dual ([D]) satisfies the Slater’s condition. Since this condition is only used to ensure $p_* = d_*$, which is covered by our definition of strong duality, Slater’s condition is no longer required.}
Figure 1: The eigenvalues of different dual solution $Z_C$. 
can be embedded into the standard form SDP presented here, but this embedding can often lead to a problematic increase in the dimensionality of the problem and give too much degree of freedom in the dual space. For example, one important special case concerns entrywise constraints of $X$ such as the nonnegativity constraint $X \geq 0$.

We conjecture that primal simplicity continues to hold for SDP applications in statistics and signal processing, even in the presence of nonnegativity constraints. Intuitively, in this applications, we expect the optimal solution to coincide with the underlying signal, which means the optimal solution is likely to be unique. One common way to prove primal uniqueness for these problems is to first prove strict complementarity holds; see e.g. [LCX18, Equation (1.5) and Section 2].

On the other hand, we expect the dual solution of these problems not to be unique, as the number of constraints, which is the number of measurements, is usually a bit larger than the intrinsic dimension. (For example, in matrix completion, the intrinsic dimension of a rank $r$ matrix of size $n \times n$ is $\mathcal{O}(nr)$ but the number of measurements needs to be greater by a log factors as shown in Theorem 5.1 which is $pn^2 = \mathcal{O}(nr\mu \log(\mu r) \log n))$. The excess of constraints seems necessary to ensure successful recovery with high probability, but destroys dual uniqueness as there is more freedom in the dual.

As a first step towards handling problems with more constraints, consider a simple SDP arising from community detection [LCX18, Equation (1.5)], which is a Max-Cut SDP with an additional nonnegativity constraint $X \geq 0$ and admits a unique completely positive primal solution. As shown in [LCX18, section 2], it is primal simple but not simple (with the appropriate generalization of simplicity and primal simplicity to problems with inequality constraints). We leave the exact details to future work.

\section*{A Definition of stationary points for \textbf{(BM)}}

We define second order stationary points formally below:

**Definition 7.** Suppose $\mathcal{M}_r = \{ F \mid \mathcal{A}(FF^\top) = b \}$ is a Riemannian manifold equipped with the trace inner product of $\mathbb{R}^{n \times p}$. Denote the tangent space of any $F \in \mathcal{M}_r$ as $T_F \mathcal{M}_r \subset \mathbb{R}^{n \times r}$. A point $F \in \mathbb{R}^{n \times r}$ is a second order stationary point if the following two conditions are satisfied:

- the Riemannian gradient $\nabla f(F) \in T_F \mathcal{M}_r$ satisfies $\nabla f(F) = 0$
- the Riemannian Hessian $\text{Hess} f(F)$ is a positive semidefinite symmetric linear map from $T_F \mathcal{M}_r$ to $T_F \mathcal{M}_r$.

Informally, we can see the conditions required of second order stationary points for \textbf{(BM)} match the conditions required in the unconstrained case. Ideas from Riemannian optimization makes the gradient and Hessian in the constrained setting precise and rigorous. We refer the reader to [AMS09] for the general definition of Riemannian gradient and Hessian for smooth functions defined on Riemannian manifold.

For the condition guaranteeing $\mathcal{M}_r$ being a Riemannian manifold, the detailed formula of the tangent space $T_F \mathcal{M}_r$, the $\nabla f(F)$, and $\text{Hess} f(F)$, see [BYBH18] Assumption 1.1, Equations (3), (7), and (10) respectively.

\section*{B Lemma for Section \textbf{5}}

**Lemma 3.** Fix a rank $r_2$ matrix $X_2 \in \mathbb{R}^{n_1 \times n_2}$ with singular value decomposition $X = U\Sigma V^\top$. Here $U \in \mathbb{R}^{n_1 \times r_2}$, $V \in \mathbb{R}^{n_2 \times r_2}$ have orthonormal columns and $\Sigma \in \mathbb{S}^{r_2}$ is diagonal. The optimization problem with decision variable $W_1 \in \mathbb{S}^{n_1}$ and $W_2 \in \mathbb{S}^{n_2}$

\begin{equation}
\begin{aligned}
\text{minimize} & \quad \text{tr}(W_1) + \text{tr}(W_2) \\
\text{subject to} & \quad \begin{bmatrix} W_1 & X_2 \\ X_2^\top & W_2 \end{bmatrix} \succeq 0
\end{aligned}
\end{equation}

has a unique solution $W_1 = U\Sigma U^\top$ and $W_2 = V\Sigma V^\top$ with optimal value $2\|X_2\|_*$.\footnote{Do note that the term of strict complementarity is not explicitly specified.}
Proof. The dual problem of Problem (19) is simply
\[
\begin{aligned}
\text{maximize} & \quad -2 \, \text{tr}(X_0^T Z) \\
\text{subject to} & \quad \begin{bmatrix} I & Z^\top \\ Z & I \end{bmatrix} \succeq 0,
\end{aligned}
\]
with decision variable \( Z \in \mathbb{R}^{n_1 \times n_2} \). First take \( Z_* = -UV^\top \) and \( W_1^* = USU^\top \) and \( W_2^* = VSV^\top \). It can be easily verified that \( Z_*, W_1^* \) and \( W_2^* \) are feasible. We also find that
\[
\text{tr}(W_1^*) + \text{tr}(W_2^*) - (2 \, \text{tr}(X_0^T Z_*)) = 2 \, \text{tr}(\Sigma) - 2 \, \text{tr}(\Sigma) = 0.
\]
Hence both \( Z_* \) and \( W_1^* = USU^\top, W_2^* = VSV^\top \) are optimal, and the optimal value of (19) is \( 2 \, \text{tr}(\Sigma) = 2 \| X_0 \|. \) Now take any \( \bar{W}_1 \) and \( \bar{W}_2 \) that is optimal to (19). Using the optimality of \( \begin{bmatrix} I & \bar{Z}_* \\ \bar{Z}_* & I \end{bmatrix} \succeq 0 \), we have
\[
0 = \begin{bmatrix} I & \bar{Z}_* \\ \bar{Z}_* & I \end{bmatrix} \begin{bmatrix} \bar{W}_1 & X_0 \\ X_0^\top & \bar{W}_2 \end{bmatrix} = \begin{bmatrix} \bar{W}_1 - USU^\top & \bar{W}_1 - USU^\top X_0 \\ X_0^\top \bar{W}_1 + X_0^\top & X_0^\top \bar{W}_2 \end{bmatrix}.
\]
Thus we must have \( \bar{W}_1 = USU^\top \) and \( \bar{W}_2 = VSV^\top \). \( \Box \)

B.1 Proof of Lemma 1

The matrix \( Y_0 \) is actually a dual certificate for \( X_0 \) for [Matrix-Completion]. We follow the construction procedure in [DC18].

First set \( k_0 := C_0 \log(\mu r_q) \) for some large enough numerical constant \( C_0 \). We can suppose (without loss of generality) that the set \( \Omega \) of observed entries is generated from \( \Omega = \bigcup_{t=1}^{k_0} \Omega_t \), where for each \( t \) and matrix index \((i,j)\), \( \text{Prob}(\{(i,j) \in \Omega_t\}) = q := 1 - (1 - p) \frac{m}{n} \), and the event \( \{(i,j) \in \Omega_t\} \) is independent of all others. Denote the projection \( \Pi_{\Omega_t} \) by \( [\Pi_{\Omega_t}(Z)]_{ij} = Z_{ij} \mathbb{1}\{\{(i,j) \in \Omega_t\} \} \), where \( \mathbb{1}\{\{(i,j) \in \Omega_t\} \} = 1 \) if \( (i,j) \in \Omega_t \) and 0 otherwise. We also denote \( \bar{R}_{\Omega_t} := \frac{1}{q} \Pi_{\Omega_t} \).

We use independent samples in constructing \( k_0 \) building blocks of the first piece of the dual certificate: set \( W^0 := UV^\top \) and
\[
W^t := \Pi_T \mathcal{H}_{\Omega_t} (W^{t-1}), \quad t = 1, 2, \ldots, k_0 - 1,
\]
where \( \mathcal{H}_{\Omega_t} = I - \frac{1}{\bar{q}} \Pi_{\Omega_t} \), and \( I \) is the identity map on \( \mathbb{R}^{n \times n} \).

We then use the same sample set \( \Omega_{k_0} \) in the next \( t_0 := 2 \log n + 2 \) building blocks of the second piece of the dual certificate: set \( Z^0 := W^{k_0-1} \) and
\[
Z^t := \Pi_T \mathcal{H}_{\Omega_{k_0}} (Z^{t-1}) = (\Pi_T \mathcal{H}_{\Omega_{k_0}})^t(W^{k_0-1}), \quad t = 1, 2, \ldots, t_0 - 1.
\]

The building block of the last piece is simply running (22) for all \( t \geq t_0 \). The final dual certificate \( Y \) is constructed by summing up the above iterates: set
\[
Y_1 := \sum_{t=1}^{k_0-1} \bar{R}_{\Omega_t} \Pi_T (W^{t-1}), \quad Y_2 := \sum_{t=1}^{t_0} \bar{R}_{\Omega_{k_0}} \Pi_T (Z^{t-1}), \quad Y_3 := \sum_{t=t_0+1}^{\infty} \bar{R}_{\Omega_{k_0}} \Pi_T (Z^{t-1})
\]
and our desired \( Y \) is simply
\[
Y := Y_1 + Y_2 + Y_3.
\]

Convergence of \( Y_3 \) We first verify the infinite series \( Y_3 \) indeed converges. Denote the Frobenius norm as \( \| \cdot \|_F \). Using [CR09] Theorem 4.1, we have \( \| \Pi_T \mathcal{H}_{\Omega_{k_0}} \Pi_T \| \leq \frac{1}{4} \) for all \( t \geq 1 \)
\[
\| Z^t \|_F \leq \| \Pi_T \mathcal{H}_{\Omega_{k_0}} \Pi_T \| \| Z^{t-1} \|_F \leq \frac{1}{4} \| Z^{t-1} \|_F.
\]
Hence the series \( \| Y_3 \|_F \leq \frac{1}{4} \| Z^{t_0} \|_F \sum_{t=t_0+1}^{\infty} \frac{1}{4^t} \), and the infinite series in \( Y_3 \) indeed converges.
The condition $\Pi_{\Omega}(Y) = Y$ Note that $\Pi_{\Omega}\mathcal{R}_{\Omega} = \mathcal{R}_{\Omega}$ for any $t$ by construction. Hence using the convergence of series in $Y_\alpha$ and $\Pi_{\Omega}$ is a continuous map, we reach $\Pi_{\Omega}(Y) = Y$.

The condition $\Pi_{\tau}(Y) = UV^T$ Using the construction of $Y$, we find that $\Pi_{\tau}(Y_1 + Y_2 + \sum_{\tau=t_0+1}^{t} \mathcal{R}_{\Omega_{k_{\tau}}} \Pi_{\tau}(Z^\tau_{-1})) - UV^T = -Z^t$. Hence, we have

$$\|\Pi_{\tau}(Y) - UV^T\|_F = \lim_{t \to \infty} \|\Pi_{\tau}(Y_1 + Y_2 + \sum_{\tau=t_0+1}^{t} \mathcal{R}_{\Omega_{k_{\tau}}} \Pi_{\tau}(Z^\tau_{-1})) - UV^T\|_F$$

$$= \lim_{t \to \infty} \|Z^t\|_F. \tag{25}$$

Now using (24), we see the above is actually 0 and hence $\Pi_{\tau}(Y) = UV^T$.

The condition $\|\Pi_{\tau}\|_{op} \leq \frac{5}{8}$ In [DC18] Section 6, “Validating Condition 2(a)”, pp 30-31, it has been shown that $\|\Pi_{\tau}(Y_1 + Y_2)\|_{op} \leq \frac{1}{2}$. Using [DC18] Inequality (92), we have $\|Z_0\|_F \leq \frac{1}{4\pi} < \frac{1}{8}$. Hence $\|\Pi_{\tau}(Y_3)\|_{op} \leq \|\Pi_{\tau}(Y_3)\|_F \leq \|Y_3\|_F \leq \frac{1}{8}$. Hence the operator norm of $\Pi_{\tau}(Y)$ satisfies $\|\Pi_{\tau}(Y)\|_{op} \leq \|\Pi_{\tau}(Y_1 + Y_2)\|_{op} + \|\Pi_{\tau}(Y_3)\|_{op} \leq \frac{5}{8}$.

B.2 Primal simplicity for (Matrix-Completion)

Here we show that primal uniqueness and strict complementarity hold for (Matrix-Completion), yet the problem has multiple dual solutions, exactly like the lifted version (SDP Matrix-Completion).

To show the problem has multiple dual solutions, note the Lagrangian dual of (Matrix-Completion) is

$$\text{maximize} \quad \langle X_\Omega, \Pi_{\Omega}(Y) \rangle$$

$$\text{subject to} \quad \|\Pi_{\Omega}(Y)\|_{op} \leq 1. \tag{26}$$

Equation (26) is equivalent to equation (13) in the following sense: if $\tilde{Y} = \begin{bmatrix} 0 & Y^T \\ Y & 0 \end{bmatrix}$ is optimal for (13), then $Y$ is optimal for (26), and vice versa. This is a simple consequence of $\|\tilde{Y}\|_{op} = \|Y\|_{op} \leq 1 \iff I \succeq \tilde{Y}$, and the constraint $\Pi_{\Omega}(Y) = Y$. Hence multiple dual solutions to (13) implies multiple solutions of (26), and hence (Matrix-Completion) has multiple dual solutions by Theorem 5.1.

A proof of primal uniqueness appears in [DC18] Theorem 2.

Finally, we show that strict complementarity holds. Strict complementarity in this context means there exists $Y = \Pi_{\Omega}(Y)$ such that $Y \in \text{ri} (\partial \|X_\Omega\|_\cdot)$, where $\text{ri}(\cdot)$ extracts the relative interior of its argument. The existence of such $Y$ is ensured by Lemma 4.

Acknowledgments

We would like to thank Yudong Chen, James Renegar, Adrian Lewis, and Michael L. Overton for helpful discussions. We would also like to thank the editor and anonymous reviewers for their feedback.

References

[AH97] Farid Alizadeh, Jean-Pierre A Haeberly, and Michael L Overton. Complementarity and nondegeneracy in semidefinite programming. *Mathematical programming*, 77(1):111–128, 1997.

[AO98] Farid Alizadeh, Jean-Pierre A Haeberly, and Michael L Overton. Primal-dual interior-point methods for semidefinite programming: convergence rates, stability and numerical results. *SIAM Journal on Optimization*, 8(3):746–768, 1998.

[1] The definition of strict complementarity is inspired from the dual strict complementarity defined in [DL18] Section 4 and the equality in [ZS17] Equation (49) and (50).
[AMS09] P-A Absil, Robert Mahony, and Rodolphe Sepulchre. *Optimization algorithms on matrix manifolds*. Princeton University Press, 2009.

[BAC18] Nicolas Boumal, Pierre-Antoine Absil, and Coralia Cartis. Global rates of convergence for non-convex optimization on manifolds. *IMA Journal of Numerical Analysis*, 39(1):1–33, 2018.

[Ban18] Afonso S Bandeira. Random laplacian matrices and convex relaxations. *Foundations of Computational Mathematics*, 18(2):345–379, 2018.

[Bar95] Alexander I. Barvinok. Problems of distance geometry and convex properties of quadratic maps. *Discrete & Computational Geometry*, 13(2):189–202, 1995.

[BBV16] Afonso S Bandeira, Nicolas Boumal, and Vladislav Voroninski. On the low-rank approach for semidefinite programs arising in synchronization and community detection. In *Conference on learning theory*, pages 361–382, 2016.

[Ber09] Dimitri P Bertsekas. *Convex optimization theory*. Athena Scientific Belmont, 2009.

[BM03] Samuel Burer and Renato DC Monteiro. A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization. *Mathematical Programming*, 95(2):329–357, 2003.

[BVB18] Nicolas Boumal, Vladislav Voroninski, and Afonso S Bandeira. Deterministic guarantees for burer-monteiro factorizations of smooth semidefinite programs. *arXiv preprint arXiv:1804.02008*, 2018.

[Che15] Yudong Chen. Incoherence-optimal matrix completion. *IEEE Transactions on Information Theory*, 61(5):2909–2923, 2015.

[CLS12] Mihai Cucuringu, Yaron Lipman, and Amit Singer. Sensor network localization by eigenvector synchronization over the euclidean group. *ACM Transactions on Sensor Networks (TOSN)*, 8(3):19, 2012.

[CMP10] Anwei Chai, Miguel Moscoso, and George Papanicolaou. Array imaging using intensity-only measurements. *Inverse Problems*, 27(1):015005, 2010.

[CR09] Emmanuel J Candès and Benjamin Recht. Exact matrix completion via convex optimization. *Foundations of Computational mathematics*, 9(6):717, 2009.

[CS08] Zi Xian Chan and Defeng Sun. Constraint nondegeneracy, strong regularity, and nonsingularity in semidefinite programming. *SIAM Journal on optimization*, 19(1):370–396, 2008.

[CSV13] Emmanuel J Candes, Thomas Strohmer, and Vladislav Voroninski. Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming. *Communications on Pure and Applied Mathematics*, 66(8):1241–1274, 2013.

[CT10] EJ Candes and T Tao. The power of matrix completion: near-optimal convex relaxation. *IEEE Trans. Information Theory*, 56(5):2053–2080, 2010.

[Cuc16] Mihai Cucuringu. Sync-rank: Robust ranking, constrained ranking and rank aggregation via eigenvector and sdp synchronization. *IEEE Transactions on Network Science and Engineering*, 3(1):58–79, 2016.

[DC18] Lijun Ding and Yudong Chen. The leave-one-out approach for matrix completion: Primal and dual analysis. *arXiv preprint arXiv:1803.07554*, 2018.

[DIL16] Dmitriy Drusvyatskiy, Alexander D Ioffe, and Adrian S Lewis. Generic minimizing behavior in semialgebraic optimization. *SIAM Journal on Optimization*, 26(1):513–534, 2016.

[DL11] Dmitriy Drusvyatskiy and Adrian S Lewis. Generic nondegeneracy in convex optimization. *Proceedings of the American Mathematical Society*, pages 2519–2527, 2011.
[SS05] Nathan Srebro and Adi Shraibman. Rank, trace-norm and max-norm. In International Conference on Computational Learning Theory, pages 545–560. Springer, 2005.

[Stu00] Jos F Sturm. Error bounds for linear matrix inequalities. SIAM Journal on Optimization, 10(4):1228–1248, 2000.

[TTT99] Kim-Chuan Toh, Michael J Todd, and Reha H Tütüncü. Sdpt3—a matlab software package for semidefinite programming, version 1.3. Optimization methods and software, 11(1-4):545–581, 1999.

[UT19] Madeleine Udell and Alex Townsend. Why are big data matrices approximately low rank? SIAM Journal on Mathematics of Data Science, 1(1):144–160, 2019.

[WdM15] Irène Waldspurger, Alexandre d’Aspremont, and Stéphane Mallat. Phase recovery, maxcut and complex semidefinite programming. Mathematical Programming, 149(1-2):47–81, 2015.

[WW18] Irène Waldspurger and Alden Waters. Rank optimality for the burer-monteiro factorization. arXiv preprint arXiv:1812.03046, 2018.

[ZS17] Zirui Zhou and Anthony Man-Cho So. A unified approach to error bounds for structured convex optimization problems. Mathematical Programming, 165(2):689–728, 2017.