Construction and exact solution of a nonlinear quantum field model in quasi-higher dimension

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Abstract

Nonperturbative exact solutions are allowed for quantum integrable models in one space-dimension. Going beyond this class we propose an alternative Lax matrix approach, exploiting the hidden multi-time concept in integrable systems and construct a novel quantum nonlinear Schrödinger model in quasi-two dimensions. An intriguing field commutator is discovered, confirming the integrability of the model and yielding its exact Bethe ansatz solution with rich scattering and bound-state properties. The universality of the scheme is expected to cover diverse models, opening up a new direction in the field.

Keywords: Nonlinear integrable quantum field theory; Quasi-higher dimensional exact models; Yang-Baxter equation; Algebraic Bethe ansatz; Bound states.

1. Introduction and Motivation

A large number of quantum models in one space-dimension (1D) admits exact nonperturbative solutions, in spite of their nonlinear interaction. This exclusive class of models, which also includes field models, constitute the family of quantum integrable (QI) systems [1, 2, 3, 4, 5] with extraordinary properties, like association with a quantum Lax and a quantum $\mathcal{R}$ matrix, possessing rich underlying algebraic structures to satisfy the quantum Yang-Baxter equation (QYBE), existence of a commuting set of conserved operators with an exact solution of their eigenvalue problem (EVP), etc. This exact method of solution, known as the Bethe ansatz (BA), was pioneered by Bethe way back in 1931 [6] and generalized later to algebraic BA [1, 2, 3, 4]. These QI systems defined in $1 + 1$-dimensions, include a wide variety of models, e.g. isotropic [6, 7] and anisotropic [8] quantum spin-$\frac{1}{2}$ chains, $\delta$ and $\delta'$-function Bose [9, 10] and anyon [11, 12] gases, nonlinear Schrödinger (NLS) field [1, 13] and lattice [3] model, relativistic [14] and nonrelativistic [15] Toda chain, t-J [16] and Hubbard [17] model, Gaudin model [18], derivative NLS [19], sine-Gordon [20] and Liouville [21] model, etc. The algebraic structures underlying these models are also rich and diverse, which include canonical, bosonic, fermionic, anyonic and spin algebras, quantum oscillator and quantum group algebras etc., having inherent Hopf algebra properties [22]. However, it is important to note, that among this diversity there is a deep unity, revealing that all known QI models, we are interested in, are realizable from a single ancestor Lax matrix or its q-deformation. At the same time, the diverse algebras underlying these integrable models are also reducible from the ancestor algebra or its quantum-deformation [23]. There is a separate class of models with long-range interactions [24], which although are solvable quantum many body systems, exhibit different properties than those listed above and will not be discussed here. The ancestor model scheme, though a significant achievement in unifying and generating integrable models, seems to be also an apparent disappointment, since it looks like a no-go theorem, allowing no construction of new integrable models beyond the known ancestor model. Moreover, since the ancestor model and hence all QI models as its descendants, are defined in 1D, it apparently excludes any construction of integrable quantum models in higher space dimensions. 2D Kitaev models [25], belonging to a different class, are possibly the only exception.

Therefore for a breakthrough, we look for new ideas and observe, that the rational ancestor Lax matrix dependents on the spectral parameter $\lambda$ only linearly, while its q-deformation depends on its trigonometric...
functions \[23\]. Consequently, all quantum Lax matrices of known integrable models, since realized from the ancestor model, depend also linearly (for the rational class), or trigonometrically on \(\lambda\) (for q-deformed class). For going beyond the prescribed form of the ancestor model, we search for Lax matrices with higher scaling dimension, for which we exploit the concept of multi-time \(t_n, n = 1, 2, 3, \ldots \) \[26\], hidden in integrable systems. We propose an alternative Lax matrix approach with \(\lambda^2\) dependence, confining to the case \(n = 2\) and focus on the NLS field model as a concrete example. It is quite surprising, that although such higher order Lax matrices are known in the context of classical integrable systems, they have been ignored completely, as far as we know, in the construction of quantum models. Denoting \(t_1 = x, t_2 = y, t_3 = t\), this would result to a novel quasi-(2 + 1) dimensional NLS quantum field model, involving the scalar field \(q(x, y, t)\) and its conjugate \(q^\dagger\). For confirming the complete integrability of the model, one needs to show the mutual commutativity of all its conserved operators, which is guaranteed when the associated Lax matrix satisfies the QYBE. However, this task for the present Lax matrix turns out to be the most difficult one, since the commutation relations (CR) for the basic fields, known for the existing QI models fail here, due to significantly different structure of our Lax matrix and its higher \(\lambda\) dependence. Moreover, we are unable to seek the guidance of the ancestor algebra \[23\], since we have gone beyond the known ancestor model. Fortunately, we could discover intriguingly new algebraic relations for our basic quantum fields, which solve the required QYBE with the known rational L-matrix and its higher \(\lambda\) dependence. Content, nature of the basic fields and underlying algebras, whereas the quantum \(\lambda\) matrix and its higher \(\lambda\) dependence. Moreover, we are unable to seek the guidance of the ancestor algebra \[23\], since we have gone beyond the known ancestor model. Fortunately, we could discover intriguingly new algebraic relations for our basic quantum fields, which solve the required QYBE with the known rational L-matrix and its higher \(\lambda\) dependence. Moreover, we are unable to seek the guidance of the ancestor algebra \[23\], since we have gone beyond the known ancestor model. Fortunately, we could discover intriguingly new algebraic relations for our basic quantum fields, which solve the required QYBE with the known rational L-matrix and its higher \(\lambda\) dependence. 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2. Quantum integrable models as descendants

QI models are associated with a quantum Lax matrix \(U(\lambda)\), the operator elements of which, for ensuring the integrability of the model, must satisfy certain algebraic relations, which are expressed in a compact matrix form through the QYBE

\[ R(\lambda - \mu) U_j(\lambda) \otimes U_j(\mu) = U_j(\mu) \otimes U_j(\lambda) R(\lambda - \mu), \]

at each lattice site \(j = 1, 2, \ldots N\), together with an ultralocality condition

\[ [U_j(\lambda) \otimes U_k(\mu)] = 0, \quad j \neq k. \]

Individual Lax matrices, each representing a particular integrable model, differ substantially in their structure, content, nature of the basic fields and underlying algebras, whereas the quantum \(R\)-matrix, appearing in the QYBE as structure constants, remains the same for all models belonging to the same class and therefore can be of only three types: rational, trigonometric and elliptic. However, in spite of widely different Lax matrices linked to the rich variety of known QI models, they are in fact realizable from a single rational ancestor Lax matrix or its q-deformed trigonometric form \[23\]. We will not be concerned here with the elliptic models, which are anyway few in number. The rational ancestor Lax matrix taken in the form

\[ U_{rAnc}(\lambda) = \left( \begin{array}{c} c_1(\lambda + s^3) + c_2, c_3(\lambda - s^3) - c_4 \\ s^+, s^- \end{array} \right), \]

satisfies the QYBE with the rational \(R\)-matrix, due to its underlying generalized spin algebra

\[ [s^-, s^+] = 2m^+ s^3 + m^-, \quad [s^3, \hat{s}^\pm] = \pm \hat{s}^\pm \]

where \(m^+ = c_1c_3, m^- = c_2c_3 + c_1c_4\), with \(c, s\) as constant parameters admitting zero values, and is capable of generating the known quantum integrable models of the rational class. The rational quantum \(R(\lambda - \mu)\) matrix in its \(4 \times 4\) matrix representation may be defined through its nontrivial elements as

\[ R^{11}_{11} = R^{22}_{22} \equiv a(\lambda - \mu) = \lambda - \mu + i\alpha, \quad R^{12}_{21} = R^{21}_{12} \equiv c = i\alpha, \]

\[ R^{11}_{22} = R^{22}_{11} \equiv b(\lambda - \mu) = \lambda - \mu. \]
while the trigonometric case has q-deformed elements: \( a = \sinh(\lambda - \mu + i\alpha) \), \( b = \sinh(\lambda - \mu) \), \( c = \sinh(i\alpha) \). The representative Lax matrices of known QI models of the rational class can be recovered from the rational ancestor model. We present below a few of such examples to illuminate the situation.

2.1. Generation of rational models

**XXX-spin chain**: The Lax matrix may be reduced from the ancestor matrix at \( c_1 = c_3 = 1 \), \( c_2 = c_4 = 0 \), giving \( m^+ = 1 \), \( m^- = 0 \), which transforms ancestor algebra to the spin algebra for Pauli matrices.

**Lattice NLS model**: The Lax matrix may be obtained from at the above parameter values, by mapping spin operators through the Holstein-Primakov transformation to the bosonic operators: \([q_j, q_k^\dagger] = \delta_{jk}\).

**NLS field model**: The Lax matrix may be recovered from its lattice version at the field limit, giving the simple familiar form

\[
U = i \begin{pmatrix} \lambda & q \\ q^\dagger & -\lambda \end{pmatrix}
\]

with bosonic field CR: \([q(x), q^\dagger(x')] = \delta(x - x')\).

**Toda chain**: Lax matrix may be obtained from at the parameter choice \( c_1 = c_3 = 1 \), \( c_2 = c_4 = 0 \), resulting both \( m^\pm = 0 \), with generators of the reduced algebra realized through canonical variables \([q_j, p_k] = \delta_{jk}\).

The rest of the QI models of the rational class, like XXX Gaudin model, tJ and Hubbard model etc. can be covered also from the rational ancestor model, employing limiting procedures, higher rank representations, fermionic realizations etc., details of which we omit.

2.2. Trigonometric models

Similarly QI models belonging to the trigonometric class, e.g. **XXZ** spin chain, relativistic Toda chain, sine-Gordon model, Liouville model, derivative NLS model etc., are derivable from their representative Lax matrices, which in turn can be generated from a single trigonometric ancestor Lax matrix. This ancestor matrix is a q-deformation of and satisfies the QYBE with the trigonometric \( R \)-matrix, thanks to its underlying generalized quantum group algebra. The details, which we skip here, can be found in [23].

3. Novel quasi-2D NLS model

Since the known quantum Lax matrices including as discussed above, inherit their properties from the ancestor models, all of them depend on the spectral parameter \( \lambda \) linearly (for rational models) or on \( \sin \lambda, \cos \lambda \) functions (for trigonometric models). Therefore for going beyond the known models, we look for a higher order Lax matrix \( U_n \) with scaling dimension \( n \), defined as generators of infinitesimal transformations: \( \Phi_{\lambda_n} = U_n(\lambda)\Phi \), \( n = 1, 2, 3, \ldots \), along the time directions \( t_n \) in a multi-time space, involving a multi-dimensional field \( q(t_1, t_2, t_3, \ldots ) \). (Here and in what follows we denote partial derivatives as subscripts, as a short-hand notation.) We intend to use this concept of the hierarchy of multiple times, hidden in the theory of integrable systems.

3.1. Alternative Lax matrix

For a concrete application, we consider the NLS hierarchy, which belongs to the rational class and choose its quantum Lax matrix as

\[
U_2(\lambda) = i \begin{pmatrix} 2\lambda^2 - q^\dagger q & 2\lambda q - iq_x \\ -2\lambda q^\dagger - iq_x^\dagger & -2\lambda^2 + q^\dagger q \end{pmatrix}
\]

It is interesting to compare the structure of Lax matrix with that of the well known NLS model, to note the crucial differences, that the matrix elements of depend on the spectral parameter up to \( \lambda^2 \) and involve field operators \( q, q^\dagger, q_x, q_x^\dagger \) defined in quasi-(2 + 1) dimensions: \((x, y, t)\). It seems, that the Lax matrix, having scaling dimension 2, has never been used in the context of quantum models.
3.2. Quantum integrability through Yang-Baxter equation

In dealing with quantum field models one has to lattice regularize the Lax operators first to avoid short-distance singularities [1]. Therefore, our intention is to show, that the discretized Lax matrix

\[ U^3 = I + \Delta U_2(\lambda, q_j), \]

where \( q_j = q(x, y = j, t) \), with lattice constant \( \Delta \to 0 \), does satisfy the QYBE [9] with the rational \( R \)-matrix [2]. However, this becomes a highly involved problem, since due to more complicated structure of the present Lax matrix [7], ten out of total 16 relations of the 4×4 matrix QYBE remain nontrivial, all of which are to be satisfied with a suitable field CR. Compare this situation with the known 1D NLS case [1,13], where due to much simpler form of the Lax matrix (see [8]) only two nontrivial relations in the QYBE survive, which can be solved successfully using the bosonic field CR. However, we realize that, no algebraic relations, including the bosonic CR, appearing in the existing integrable models would work here, since the choice of \( U_2 \) has taken us beyond the scope of the known ancestor models and the associated algebras, and therefore we have to look for some innovative commutation relations for the basic quantum fields. Fortunately, we find a new set of such relations for our quasi-2D fields as

\[ [q(x, y, t), q^\dagger(x, y', t)] = -2i\alpha \delta(y - y'), \]

(8)

together with its hermitian conjugate and with all other relations being trivial, like \([q(x, y, t), q^\dagger(x, y', t)] = 0\). Note that CR [3] exhibiting an asymmetry in space are fundamentally new relations, application of which satisfies miraculously all ten nontrivial relations in the QYBE, involving the discretized Lax matrix \( U^3(\Delta) \), up to order \( O(\Delta) \) (See Appendix for details). This is however enough for proving the integrability of the field models, obtained at the limit \( \Delta \to 0 \). It is remarkable, that in spite of the presence of a \( x \)-derivative term, the new CR [6] satisfies the necessary ultralocality condition [2], i.e. the fields commute at space separated points along \( j \to y \), which is the only relevant direction here, reflecting the quasi-2D nature of our model.

Therefore, since the lattice regularized quantum Lax operator \( U^3(\lambda) \) constructed from [7] satisfies the QYBE [1] for the rational \( R \)-matrix, together with the ultralocality condition, the transition matrix for our model, defined for \( N \)-lattice sites: \( T(\lambda) = \prod_{j=1}^{N} U^3(\lambda) \), must also satisfy the QYBE

\[ R(\lambda - \mu) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) R(\lambda - \mu), \]

(9)

with the same \( R(\lambda - \mu) \)-matrix. This happens due to the coproduct property of the underlying Hopf algebra [22], which keeps an algebra invariant under its tensor product. This global QYBE [2] serves two important purposes. First, it proves the quantum integrability of the model by showing the mutual commutativity of all conserved operators. Second, it derives the commutation relations between the operator elements of \( T(\lambda) \), which are used for the exact algebraic Bethe ansatz solution of the EVP.

In more details: multiplying QYBE [2] from left by \( R^{-1} \), taking the trace from both sides and using the property of cyclic rotation of matrices under the trace, one can show that \( \tau(\lambda) = \text{trace} T(\lambda) \) commutes: \( [\tau(\lambda), \tau(\mu)] = 0 \). This in turn leads to the Liuoville integrability condition: \( [C_j, C_k] = 0 \), \( j, k = 1, 2, \ldots \), since the conserved set of local operators are generated from ln \( \tau(\lambda) = \sum_j C_j \lambda^{-j} \), through expansion in the spectral parameter \( \lambda \). Following this construction and exploiting the explicit form of the Lax matrix [7], we can derive, in principle, all conserved operators \( C_j, j = 1, 2, \ldots \) for our model. Skipping the details, which can be found for the classical case in [27], we present only the Hamiltonian of our model as \( H = C_4 \) in the explicit form

\[ H = \int dy : [iq^4_y q_x + q^4_y q_x - i(q^4_y q_x - q^4_y q)] - 2(q^4_y q_x + q^2_y q_x^2 + q^4_y q_x^2). \]

(10)

Notice the quasi \((2 + 1)\) dimensional nature of the the Hamiltonian, since though [10] involves both \( x \) and \( y \) derivatives of the field, the volume integral is taken only along \( y \).

4. Algebraic Bethe ansatz for the eigenvalue problem

Since \( T(\lambda) \) satisfies the QYBE with the rational \( R \)-matrix, we can follow the procedure for the algebraic BA, close to the formulation for the 1D quantum NLS model [1,13]. As we have discussed above, \( \tau(\lambda) = \text{trace} T(\lambda) \) commutes:
trace$T(\lambda) = A(\lambda) + A^\dagger(\lambda)$ is linked to the generator of the conserved operators $C_j$, $j = 1, 2, \ldots$, including the Hamiltonian $[10]$. The off-diagonal elements of $T^{kl}(\lambda) = B(\lambda)$ and $T^{kj}(\lambda) = B^\dagger(\lambda)$, on the other hand, can be considered as generalized creation and annihilation operators, respectively. For solving the EVP for all conserved operators: $C_j |M> = c^M_j |M>$, $j = 1, 2, \ldots$, simultaneously we construct exact M-particle Bethe state $|M>= B(\mu_1)B(\mu_2)\cdots B(\mu_M)|0>$, on a pseudo-vacuum $|0>$ with the property $B^\dagger(\mu_j)|0>=0$, $A(\lambda)|0>=|0>$, and aim to solve the EVP: $\tau(\lambda)|M>=\Lambda_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M)|M>$, with exact eigenvalues $\ln \Lambda(\lambda, \{\mu_i\}) = \sum c^M_j (\{\mu_i\}) \lambda^j$.

4.1. Exact solution for quasi 2D quantum field model

For obtaining the final result for our quantum NLS field model on infinite space interval, we have to switch over to the field limit: $\Delta \to 0$ with total lattice site $N \to \infty$ and then take the interval $L = N\Delta \to \infty$, assuming vanishing of the field $q_j \to 0$, at $j \to \infty$, compatible with the natural condition of having the vacuum state at space infinities, yielding the asymptotic Lax matrix $U^j(\lambda)|_{j\to\infty} = U_0(\lambda)$. Therefore, we have to shift over to the field transition matrix defined as

$$T_f(\lambda) = U_0^{-N} T(\lambda) U_0^{-N}, \quad N \to \infty,$$

and for further construction introduce $V(\lambda, \mu) \equiv U_0(\lambda) \otimes U_0(\mu)$, $W(\lambda, \mu) = (U^j(\lambda) \otimes U^j(\mu))_{j\to\infty}$. We may check from the QYBE [1] that $W$ satisfies the relation $R(\lambda - \mu)W(\lambda, \mu) = W(\mu, \lambda)R(\lambda - \mu)$, using which we can derive from QYBE [5], that the field transition matrix (11) also satisfies the QYBE

$$R_0(\lambda, \mu) T_f(\lambda) \otimes T_f(\mu) = T_f(\mu) \otimes T_f(\lambda) R_0(\lambda, \mu),$$

but with a transformed $R$-matrix:

$$R_0 = S(\mu, \lambda) R(\lambda - \mu) S(\lambda, \mu), \quad S(\lambda, \mu) = W^{-N} V^N, \quad N \to \infty,$$

where a $R(\lambda - \mu)$ is the original rational $R$–matrix [5] (see [5] for similar details on 1D NLS model).

Based on the above formulation, using the field operator products: $q_j q^\dagger_{j,x} = -2i\Delta$, $q^\dagger_{j,x} q_j = 0$, at $j \to \infty$, we can calculate explicitly the relevant objects needed for our field model. In particular, the central $2 \times 2$ block $W_c$ for matrix $W$ turns out to be

$$W_c(\lambda, \mu) = I + \Delta M(\lambda, \mu) \begin{pmatrix} (\lambda - \mu) & 0 \\ -2\alpha & -(\lambda - \mu) \end{pmatrix},$$

with an intriguing factorization of its spectral dependence by a prefactor $M(\lambda, \mu) = 2(\lambda + \mu)$, which is the key reason behind the success of the exact algebraic Bethe ansatz solution for our field model, inspite of the more complicated form of its Lax operator. Note, that since our model shares the same rational $R$-matrix with the known NLS case, which is independent of any specific Lax operator, the present result coincides in part with that of the known NLS model [1, 13], though only formally. Note on the other hand, that the transformed $R_0$ matrix, relevant for the field model, depends on the corresponding asymptotic Lax matrix and its product through matrix $S(\lambda, \mu)$.

Therefore, since Lax matrix [7] for our model is more complicated, compared to [5] for the 1D NLS model, our final result shows intriguing differences from the known NLS result, which we highlight below.

For constructing $R_0$ using definition [13], we have to construction first matrix $S(\lambda, \mu)$, taking proper limit of $W^{-N}$ at $L \to \infty$ using (14). Through some algebraic manipulations, which are skipped here, we finally arrive at the field limit: to a simple expression for $R_0$ matrix, expressed through its nontrivial elements as

$$R_{11}^{11} = R_{22}^{22} = a(\lambda - \mu), \quad R_{12}^{11} = b(\lambda - \mu), \quad R_{22}^{11} = R_{11}^{22} = 0,$$

$$R_{12}^{21} = b(\lambda - \mu) - \frac{a^2}{\lambda - \mu} + \frac{a^2 \pi}{M(\lambda, \mu)} \delta(\lambda - \mu),$$

where $M(\lambda, \mu) = 2(\lambda + \mu)$, $a(\lambda - \mu), b(\lambda - \mu)$ as in [5] and the $\delta(\lambda - \mu)$ term vanishes at $\lambda \neq \mu$. It is interesting to compare [15] with the original $R$-matrix [5]. Now from QYBE [12] relevant for the field models, we can derive
For the known NLS model the binding energy \[1, 13\]
\[E\]
negative values of the binding energy: when its energy is lower than the sum of the individual free-particle energies, which in turn is ensured by the result of the 1D NLS model.

Going beyond the known form of the existing integrable quantum models in 1D, we propose an alternative higher order Lax matrix approach, exploiting the concept of multi-time dimension hidden in integrable...
systems, and applied it for constructing and solving a novel quasi-2D quantum NLS field model. Due to quantum integrability of the model the eigenvalue problem can be solved exactly for the commuting set of all its conserved operators, with intriguing result for the many particle scattering and bound-states. The key to our success in proving the crucial quantum Yang-Baxter equation, which guarantees the quantum integrability of the model, is the discovery of a new type of field operator algebra, not covered by the existing rules. The present approach, general enough for applying to other quasi higher dimensional quantum models, could open up a new direction in the theory of quantum integrable systems. It is a challenge to find a possible q-deformation of the algebra used here, which could lead to a novel class of quantum algebra, while an exact lattice version of the present Lax matrix could unravel an ancestor matrix for generating a new family of integrable quantum models.

6. Appendix

In QYBE \( (1) \) with \( R \)-matrix \( (5) \) and discretized version \( U^j \) of the quantum Lax matrix\( (7) \), out of total 16 matrix operator relations, except 4 diagonal and 2 extreme off-diagonal terms, all other 10 relations \( Q_{ki}^j \) stand nontrivial and their validity need to be proved using the CR, discretized from \( (8) \):

\[
[q_j, q_{j,x}] = -2i \frac{\alpha}{\Delta}, \quad [q_j, q_k] = 0 \tag{19}
\]

and their conjugates.

6.1. QYBE relation for matrix elements

\[
Q_{12}^{12} = aU^{j}_{11}(\lambda)U^{j}_{12}(\mu) - bU^{j}_{12}(\mu)U^{j}_{11}(\lambda) - cU^{j}_{11}(\mu)U^{j}_{12}(\lambda) \\
= i\Delta(\lambda - \mu)q(-\Delta[q_j, q_{j,x}] + 2c) + O(\Delta^2) = 0,
\]

valid up to \( O(\Delta^2) \), using expressions for \( a(\lambda - \mu), b(\lambda - \mu), c \) and CR \( (19) \). Similarly one proves the conjugate relations \( Q_{11}^{11}, Q_{12}^{12}, Q_{11}^{12} \) and similar relations \( Q_{12}^{11}, Q_{21}^{11}, Q_{11}^{12}, Q_{12}^{11} \). The remaining

\[
Q_{21}^{12} = b [U^{j}_{12}(\lambda), U^{j}_{21}(\mu)] + c(U^{j}_{22}(\lambda)U^{j}_{11}(\mu) - U^{j}_{11}(\lambda)U^{j}_{22}(\mu)) =
\]
\[2i\Delta^2(\lambda - \mu)(\mu[q_j,x, q_j^\dagger] + \lambda[q_j^\dagger,x, q_j]) + 4i\Delta c(\mu^2 - \lambda^2) = 0,\]
valid exactly with the use of the CR (19). Similarly the conjugate relation \(Q_{12}^{i1}\) is proved, showing thus the validity of all QYBE relations for the quantum quasi-2D NLS model.

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