A PRACTICAL ANALYTIC METHOD FOR CALCULATING $\pi(x)$

II

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Abstract. In this paper we describe an improved version of the analytic method presented in the preceding paper [FKBJ], which we recently used to calculate the number of prime numbers $\leq 10^{25}$.

1. Introduction

Recently, two variants of the analytic method proposed in [LO87] have been implemented to evaluate the prime-counting function $\pi(x)$ at large height [Pla, FKBJ]. The new methods avoid the original idea of numerical integration by the use of explicit formulas. This way, $\pi(x)$ can be calculated by approximating a sum over the non-trivial zeros of the Riemann zeta function and calculating a correction term involving the prime powers in a neighbourhood of $x$.

The methods in [Pla] and [FKBJ] mainly differ with respect to the kernel functions which are used to speed up the convergence of the sum over zeros in the Riemann explicit formula. The method in [Pla] uses the Gaussian function, as suggested in [Gal04], while the methods in [FKBJ] use the Logan function [Log88].

In this paper we present an improved version of the methods in [FKBJ], which is more flexible than method I and simpler than method II. We also provide sharper bounds for calculations not assuming the Riemann hypothesis: for the truncation of the sum over zeros we make use of partial knowledge of the RH, and we use a bound from sieve theory to shorten the sieve interval about $x$.

We implemented the new method in cooperation with the authors of [FKBJ] and calculated the value

$$\pi(10^{25}) = 176,846,309,399,143,769,411,680.$$  

After the unconditional calculation in [Pla], we also confirmed the value

$$\pi(10^{24}) = 18,435,599,767,349,200,867,866$$

once more.

The paper is structured as follows. In section 2 we define a modified Riemann prime-counting function $\pi^*_c(x)$ and study the difference $\pi^*_c(x) - \pi^*(x)$. In section 3 we then prove an explicit formula for $\pi^*_c(x)$. This is followed by a study of the sum over zeros in section 4, and the sum over prime powers in section 5. Finally, we conclude this paper with some information about the actual calculations.

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2. A Modified Prime-Counting Function

In the preceding paper the Weil-Barner explicit formula [Wei52, Bar81] was used to obtain a correction term that, when added to the Riemann explicit formula [Rie59, vM95], would speed up the slow convergence of the sum over the zeros of the Riemann zeta function. In this paper we proceed in a different way: we introduce a modified Riemann prime-counting function $\pi_{c,\varepsilon}(x)$ for which an explicit formula will be derived directly from the Weil-Barner explicit formula. This way, the entire sum over zeros benefits from the sharp cutoff property of the Logan function [Log88].

First we introduce some notations, which will be used throughout this paper. We take $\ell_{c,\varepsilon}(t) = \ell_c(\varepsilon t)$ and

$$\eta_{c,\varepsilon}(\xi) = \frac{\hat{\ell}_{c,\varepsilon}(\xi)}{2\pi} = \chi_{(-\varepsilon,\varepsilon]}(\xi) I_0(c\sqrt{1-|\xi/\varepsilon|^2}),$$

where the Fourier transform is defined as in [FKBJ]. Furthermore, we use the notations $A_{c,\varepsilon} := -\ell_{c,\varepsilon}''(0)/2$ and $\lambda_{c,\varepsilon} := \ell_{c,\varepsilon}(i/2)$, and we define the auxiliary functions

$$f_k(t) = \frac{e^{t/2}}{t^k}.$$

For $\varepsilon < \log 2$ we may then define the modified Riemann prime-counting function

$$\pi_{c,\varepsilon}^*(x) = \sum_{p^m} \frac{1}{m} \frac{\log p}{p^m/2} \phi_{x,c,\varepsilon}(m \log p),$$

where

$$\phi_{x,c,\varepsilon}(t) = \lambda_{c,\varepsilon}^{-1} \left( \chi_{(-\infty, \log x]} \left( f_1 + A_{c,\varepsilon} \left( f_2 - 2f_3 \right) \right) \right) * \eta_{c,\varepsilon}(t)$$

for $|t| > \varepsilon$.

As opposed to the methods in [FKBJ], it is now no longer true that the difference $\pi^*(x) - \pi_{c,\varepsilon}^*(x)$ depends on the prime powers in the sieve interval $I = [e^{-\varepsilon}x, e^\varepsilon x]$ only. But the contribution of the prime powers in $[0, e^{-\varepsilon}x]$ turns out to be of size $O(\varepsilon^3 x)$. This imposes a mild lower bound of order $x^{1/3}$ on the minimal length of the zero range, which is less restrictive than the lower bound for method I in [FKBJ].

To simplify the error estimates, we will sometimes assume the parameters to satisfy

$$\frac{1}{2} \log x \leq c \leq 2 \log x, \quad (2.3)$$

and

$$x^{-2/3} \leq \varepsilon \leq x^{-1/3}, \quad (2.4)$$

These bounds should not impose any restrictions for practical applications.

The remaining part of this section will be devoted to the study of $\pi_{c,\varepsilon}^*(x) - \pi^*(x)$. To this end, we will be needing the auxiliary functions

$$\mu_{c,\varepsilon}(t) = \begin{cases} -\int_{-\infty}^t \eta_{c,\varepsilon}(\tau) \, d\tau & t < 0 \\ -\mu_{c,\varepsilon}(-t) & t > 0 \\ 0 & t = 0 \end{cases}$$
and
\[ \nu_{c,\varepsilon}(t) = \int_{-\infty}^{t} \mu_{c,\varepsilon}(\tau) \, d\tau. \]

These are connected to the functions \( \mu_1 \) and \( \mu_2 \) in [FKBJ] by
\[ \mu_{c,\varepsilon}(t) = \nu_1(t/\varepsilon) \quad \text{and} \quad \nu_{c,\varepsilon}(t) = \varepsilon \nu_2(t/\varepsilon). \]

The difference \( \pi_{c,\varepsilon}^*(x) - \pi^*(x) \) may then be described essentially in terms of the function
\[ (2.5) \quad M_{x,c,\varepsilon}(t) = \lambda_{c,\varepsilon}^{-1}\left[ \mu_{c,\varepsilon}\left(\log\frac{t}{\varepsilon}\right) + \left(\frac{1}{\log t} - \frac{1}{2}\right)\left(\mu_{c,\varepsilon}\left(\log\frac{1}{\varepsilon}\right) \log\frac{t}{\varepsilon} - \nu_{c,\varepsilon}\left(\log\frac{t}{\varepsilon}\right)\right)\right], \]

which is supported on the sieve interval.

**Theorem 2.1.** Let \( x > 10^{10} \), and let \( c \) and \( \varepsilon \) satisfy (2.3) and (2.4). Then we have
\[ (2.6) \quad \pi_{c,\varepsilon}^*(x) = \pi^*(x) + \sum_{e^{-x}<p^{m}<e^{x}} \frac{1}{m} M_{x,c,\varepsilon}(p^m) + \Theta\left(0.82 \cdot \frac{\varepsilon^3 x}{\log(\varepsilon x)^2}\right). \]

First, we investigate the difference \( f_1 - \phi_{\infty,c,\varepsilon} \).

**Lemma 2.2.** Let \( \varepsilon \leq 0.001 \), \( c \geq 1 \) and \( |t| \geq \log 2 \). Then we have
\[ (2.7) \quad \phi_{\infty,c,\varepsilon}(t) = f_1(t) + \Theta\left(39 \frac{\varepsilon^4}{c^2} f_2(t)\right). \]

**Proof.** Let \( t \in \mathbb{R} \) satisfy \( |t| \geq \log 2 \) and let
\[ g_k(\tau) = e^{-\tau/2}\left(\frac{t^2}{(t-\tau)^k} - t^{2-k}\right). \]

Then we have
\[ (2.8) \quad f_k * \eta_{c,\varepsilon}(t) = \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) \frac{e^{\tau \varepsilon}}{(t-\tau)^k} \, d\tau = \lambda_{c,\varepsilon} f_k(t) + f_2(t) \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) g_k(\tau) \, d\tau. \]

We intend to approximate the integral on the right hand side by combining Taylor approximation to the functions \( g_k \) and the well-known identity
\[ (2.9) \quad \ell_{c,\varepsilon}^{(n)}(0) = i^n \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) \tau^n \, d\tau. \]

Since \( \eta_{c,\varepsilon} \) is even, the derivatives \( \ell_{c,\varepsilon}^{(n)}(0) \) vanish for odd \( n \) and therefore the odd terms in the Taylor series approximation of \( g_k \) are negligible. Obviously, we have \( g_k(0) = 0 \) and the second derivative is given by
\[ g_k''(\tau) = e^{-\tau/2}\left[-\frac{t^{2-k}}{4} + \frac{1}{4} \frac{t^2}{(t-\tau)^k} - \frac{k t^{k+1}}{(t-\tau)^{k+2}} + \frac{k(k+1) t^{2-k}}{(t-\tau)^{k+2}}\right]. \]

For \( k = 1 \) we will also be needing the fourth derivative, which is given by
\[ g_1^{(4)}(\tau) = e^{-\tau/2}\left[\frac{1}{16} \frac{t}{t-\tau} - \frac{1}{2} \frac{t^2}{(t-\tau)^2} + \frac{3 t^2}{(t-\tau)^3} - \frac{12 t^2}{(t-\tau)^4} + \frac{24 t^2}{(t-\tau)^5}\right]. \]

Now for \( |\tau| \leq \varepsilon \leq 0.001 \) we have
\[ \left|\frac{t}{t-\tau}\right| \leq \frac{\log 2}{\log 2 - 0.001} \leq 1.002, \]
and consequently we get
\[ |g''(\tau)| \leq 1.002^{k+2}e^{0.0005} \left[ \frac{1}{2\log(2)^{k-2}} + \frac{k}{\log(2)^{k-1}} + \frac{k(k+1)}{\log(2)^k} \right] \leq \begin{cases} 16.1 & k = 2, \\ 43.5 & k = 3, \end{cases} \]
and
\[ |g^{(4)}(\tau)| \leq 1.002^5 \cdot e^{0.0005} \left[ \frac{1}{16} \cdot 0.001 + \frac{1}{2} + \frac{3}{\log(2)^2} + \frac{12}{\log(2)^3} + \frac{24}{\log(2)^4} \right] \leq 103. \]
We therefore have
\[ g_1(\tau) = \alpha_1 \tau + \beta_1 \tau^3 + \Theta(4.3\tau^4), \]
\[ g_2(\tau) = \alpha_2 \tau + \Theta(8.1\tau^2), \]
and
\[ g_3(\tau) = \alpha_3 \tau + \Theta(21.8\tau^2) \]
for suitable \( \alpha_k \) and \( \beta_1 \). We thus get
\[ (2.10) \quad \left| \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) g_k(\tau) d\tau \right| \leq \begin{cases} 8.1 |\eta_{c,\varepsilon}'(0)| & k = 2, \\ 21.8 |\eta_{c,\varepsilon}'(0)| & k = 3, \end{cases} \]
and
\[ (2.11) \quad \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) g_1(\tau) d\tau = -\frac{\eta_{c,\varepsilon}'(0)}{2} \left( \frac{2}{t^2} - \frac{1}{t} \right) + \Theta(4.3\ell_{c,\varepsilon}'(0)). \]
from (2.9).
Next, we estimate the derivatives of \( \ell_{c,\varepsilon} \) at 0. For \( c \geq 1 \) we have
\[ 0 \leq \frac{\cosh(c)}{\sinh(c)} - \frac{1}{c} = 1 + \frac{e^{-c}}{\sinh(c)} - \frac{1}{c} \leq 1 + \frac{1}{c} \left( \frac{e^{-1}}{\sinh(1)} - 1 \right) \leq 1 \]
and from (2.9) we get \((-1)^n\ell_{c,\varepsilon}^{(2n)}(0) > 0\). We therefore have
\[ (2.12) \quad 0 < -\ell_{c,\varepsilon}''(0) = \frac{\varepsilon^2}{c} \left( \frac{\cosh(c)}{\sinh(c)} - \frac{1}{c} \right) \leq \frac{\varepsilon^2}{c} \]
and
\[ 0 < \ell_{c,\varepsilon}^{(4)}(0) = \frac{9\varepsilon^4}{c^2} \left( \frac{1}{c^2} - \frac{\cosh(c)}{c \sinh(c)} + \frac{1}{3} \right) \leq \frac{3\varepsilon^4}{c^2}. \]
If we use these bounds in (2.10) and (2.11) and take into account that \( \lambda_{c,\varepsilon} > 1 \) holds (see also (3.18)), we get
\[ \phi_{\infty,c,\varepsilon}(t) = \lambda_{c,\varepsilon}^{-1} \left( f_1 + A\varepsilon,\varepsilon(f_2 - 2f_3) \right) * \eta_{c,\varepsilon}(t) \]
\[ = f_1(t) \left[ 1 + \Theta \left( 4.3\frac{\varepsilon^4}{t^2} \right) + A\varepsilon,\varepsilon \Theta \left( 8.1 + 43.6\frac{\varepsilon^2}{t^2} \right) \right] \]
\[ = f_1(t) \left[ 1 + \Theta \left( 39\frac{\varepsilon^4}{t^2} \right) \right] \]
from (2.8).
\[ \square \]
Next, we investigate the difference \( \chi^*_{[\log 2, \log x]} \phi_{\infty,c,\varepsilon} - \phi_{x,c,\varepsilon} \) in \( B_\varepsilon(\log x) \).
Lemma 2.3. Let \( x \geq 10^{10}, \varepsilon \leq 0.001, c \geq 1, \) and let
\[
(2.13) \quad m_{x,c,\varepsilon}(t) = \frac{e^{t/2}}{\lambda_{c,\varepsilon}} \left[ \mu_{c,\varepsilon}(y) + \left( \frac{1}{2} - \frac{1}{t} \right) (y \mu_{c,\varepsilon}(y) - \nu_{c,\varepsilon}(y)) \right],
\]
where \( y = t - \log(x) \). Then we have
\[
(2.14) \quad \phi_{x,c,\varepsilon}(t) = \chi_{[2z, \log x]}(t) \phi_{\infty,c,\varepsilon}(t) + m_{x,c,\varepsilon}(t) + \Theta \left( 0.1 e^{-\varepsilon/2} \frac{\varepsilon^2 \sqrt{x}}{c \log(e^\varepsilon x)} \right)
\]
for \( |t - \log x| \leq \varepsilon \).

Proof. Let \( t, \varepsilon \) and \( y \) satisfy the conditions of the lemma. Then we have
\[
(2.15) \quad \chi_{[2z, \log x]} f_k \ast \eta_{c,\varepsilon}(t) = \chi_{[2z, \log x]}(t) f_k(t - \tau) d\tau + \chi_{[2z, \log x]}(t) \left[ f_k \ast \eta_{c,\varepsilon}(t) - \int_{-\varepsilon}^{y} \eta_{c,\varepsilon}(\tau) f_k(t - \tau) d\tau \right]
\]
Since
\[
0 < f_k(t) \leq f_k(\log(x) + \varepsilon) \leq e^{\varepsilon/2} \frac{\sqrt{\varepsilon}}{\log(x)^{2/3}}
\]
holds for \( t \in B_{\varepsilon}(\log x) \), this gives
\[
(\chi_{[2z, \log x]} f_k) \ast \eta_{c,\varepsilon}(t) = \chi_{[2z, \log x]}(t) f_k(t) + \Theta \left( e^{\varepsilon/2} \frac{\sqrt{\varepsilon}}{2 \log(x)^{2}} \right),
\]
where the 2 in the denominator of the \( \Theta \)-term results from
\[
\int_{0}^{\varepsilon} \eta_{c,\varepsilon}(\tau) d\tau = \int_{-\varepsilon}^{0} \eta_{c,\varepsilon}(\tau) d\tau = \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau) d\tau = \frac{1}{2} \ell_{c,\varepsilon}(0) = \frac{1}{2}
\]
For \( k = 1 \) we further evaluate the integrals in (2.15). Since we have \( t > 23 \), a simple computation shows
\[
\frac{e^{t/2}}{t - \tau} = f_1(t) \left( 1 + \tau \left( 1 - \frac{1}{t} \right) + \Theta(0.15 \tau^2) \right).
\]
Now we first assume \( y > 0 \). Then we get
\[
\int_{y}^{\varepsilon} \eta_{c,\varepsilon}(\tau) f_1(t - \tau) d\tau = \frac{e^{t/2}}{t} \int_{y}^{\varepsilon} \eta_{c,\varepsilon}(\tau) (1 + \tau (1 - \frac{1}{t}) + \Theta(0.15 \tau^2)) d\tau
\]
\[
= \frac{e^{t/2}}{t} \left( \mu_{c,\varepsilon}(y) + (1 - \frac{1}{t}) \int_{y}^{\varepsilon} \eta_{c,\varepsilon}(\tau) \tau d\tau + \Theta \left( 0.15 \frac{\varepsilon^2}{c} \right) \right)
\]
\[
= \frac{e^{t/2}}{t} \left( \mu_{c,\varepsilon}(y) + (1 - \frac{1}{t}) (y \mu_{c,\varepsilon}(y) - \nu_{c,\varepsilon}(y)) + \Theta \left( 0.075 \frac{\varepsilon^2}{c} \right) \right)
\]
for the first integral in (2.15), where we used (2.9) and (2.12) again. A similar computation shows that
\[
\int_{-\varepsilon}^{y} \eta_{c,\varepsilon}(\tau) f_1(t - \tau) d\tau = -f_1(t) \left( \mu_{c,\varepsilon}(y) + (1 - \frac{1}{t}) (y \mu_{c,\varepsilon}(y) - \nu_{c,\varepsilon}(y)) + \Theta \left( 0.075 \frac{\varepsilon^2}{c} \right) \right)
\]
holds for \( y < 0 \). This also includes the case \( y = 0 \) since \( \mu_{c,\varepsilon} \) and \( \nu_{c,\varepsilon} \) are normalized. Since we have
\[
e^{\varepsilon/2} \frac{\sqrt{\varepsilon}}{\log x} \left[ 0.075 \frac{\varepsilon^2}{c} + A_{c,\varepsilon} \left( \frac{1}{\log(x)^{2}} + \frac{2}{\log(x)^{2}} \right) \right] \leq 0.1 e^{-\varepsilon/2} \frac{\varepsilon^2 \sqrt{\varepsilon}}{c \log(e^\varepsilon x)},
\]
the assertion follows from (2.15) and (2.2).

Proof of Theorem 2.1. First we note that under the conditions imposed on \(x, c\) and \(\varepsilon\) we have \(\varepsilon < 0.0005\) and \(c > 10\), so the conditions of Lemma 2.2 and Lemma 2.3 are satisfied. Now let \(I = [\exp(-\varepsilon)x, \exp(\varepsilon)x]\) denote the sieve interval.

By Lemma 2.3 we have

\[
\pi^*_c(x) = \sum_{p \leq x} \log p \frac{\phi_{x,c,\varepsilon}(m \log x)}{p^{m/2}} = \sum_{p \leq x} \log p \frac{\chi_{[2\varepsilon, \log x]}(\phi_{\infty, c, \varepsilon})(m \log p)}{p^{m/2}} \frac{\phi_{x,c,\varepsilon}(m \log p)}{p^{m/2}} + \sum_{p \leq x} \log p \frac{m_{x,c,\varepsilon}(m \log p) + \Theta\left(0.1 \frac{e^{-\varepsilon/2} z \sqrt{x}}{c \log(e^\varepsilon x)}\right)}{p^{m/2}}.
\]

The sum on the second line of (2.16) equals

\[
\pi^*(x) + \sum_{p \leq x} \log p \frac{\chi_{[2\varepsilon, \log x]}(\phi_{\infty, c, \varepsilon} - f_1)(m \log p)}{p^{m/2}}.
\]

We may use the bound from Lemma 2.2 and \(c \geq \log(e^\varepsilon x)/2\), which gives

\[
\sum_{p \leq x} \frac{\log p}{p^{m/2}} |(\phi_{\infty, c, \varepsilon} - f_1)(m \log p)| \leq 39 \frac{e^4}{c^2} \sum_{p \leq x} \frac{1}{m \log(p)} \leq 39 \frac{e^4}{c^2} \leq 0.08 \frac{e^{3} x}{\log(e^{3} x)^2}.
\]

For the sum on the third line of (2.16) we use the bound

\[
\log p \frac{1}{p^{m}} \leq e^{c/2} \frac{\log(e^\varepsilon x)}{\sqrt{x}},
\]

which holds for all \(p^m \in I\), so this line takes the form

\[
\sum_{p \leq x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) + \Theta\left(0.1 \frac{e^2}{c} \sum_{p \leq x} \frac{1}{m}\right).
\]

It remains to treat the sum over the prime powers in \(I\). We begin by estimating the number of prime numbers. By the Brun-Titchmarsh inequality, as stated in [MV73], we have

\[
0.1 \frac{e^2}{c} \sum_{p \leq x} 1 \leq 0.1 \frac{e^2}{c} \frac{4.01 e x}{\log(e^\varepsilon x)} \leq 0.802 \frac{e^3 x}{\log(e^{3} x)^2}.
\]

For the higher prime powers we use the following lemma.

Lemma 2.4. Let \(x \geq 100, \varepsilon \leq \frac{1}{100}\) and let \(I = [e^{-\varepsilon} x, e^{\varepsilon} x]\). Then we have

\[
\sum_{p \leq x} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \leq 4.01 e \sqrt{x} + \log \log(2x^2).
\]
Proof. Let \(0 < 2Y < X\). Then we have

\[
(X - Y)^{1/m} \geq X^{1/m} - \frac{Y}{m}X^{1/m-1} - \frac{Y^2}{2m}(1 - \frac{1}{m})(X - Y)^{1/m-2}
\]

\[
\geq X^{1/m} - 2\frac{Y}{m}X^{1/m-1}
\]

for \(m > 1\), and consequently we get

\[
\# \{p | p^m \in [X, X - Y]\} \leq X^{1/m} - (X - Y)^{1/m} + 1
\]

\[
\leq \frac{2}{m}YX^{1/m-1} + 1.
\]

For the choice \(X = e^\varepsilon x\), \(Y = 2 \sinh(\varepsilon)x\) and \(m \geq 2\) this is bounded by

\[
\frac{4.01}{m}\sqrt{x} + 1,
\]

so we get

\[
\sum_{p^m \in I} \frac{1}{m} \leq \int_1^{2\log(2x)} \frac{4.01\varepsilon\sqrt{x}}{t^2} + \frac{1}{t} \, dt \leq 4.01\varepsilon\sqrt{x} + \log(2x^2).
\]

Since we assume \(x^{1/3} \leq \varepsilon x \leq x^{2/3}\), Lemma 2.4 gives the bound

\[
\sum_{p^m \in I} \frac{1}{m} \leq \frac{\varepsilon x}{\log(\varepsilon x)} \left( 4.01 \frac{\log(\varepsilon x)}{\sqrt{x}} + \frac{\log(2x^2)\log(\varepsilon x)}{\varepsilon x} \right) \leq 0.03 \frac{\varepsilon x}{\log(\varepsilon x)}.
\]

This in combination with (2.19) gives the bound

\[
0.81 \frac{\varepsilon^3 x}{\log(\varepsilon x)^2}
\]

for the \(\Theta\)-term in (2.18), where we used \(2c \geq \log(\varepsilon x)\) again. Since we have \(0.08 + 0.81 < 0.82\), the assertion of the theorem thus follows from (2.16) and (2.17).

\(\square\)

3. The explicit formula for \(\pi^*_c(x)\)

The function \(\pi^*_c(x)\) satisfies an explicit formula similar to the Riemann explicit formula, but where the cutoff point and the size of the can be controlled by the Logan function.

First, we introduce some auxiliary exponential integrals.

**Definition 3.1.** For \(\text{Re}(z) > 0\) let \(\gamma^+(z)\) denote the polygonal chain \((-\infty + i, i, z)\) and let \(\gamma^-(z)\) denote the polygonal chain \((-\infty - i, -i, z)\). Then we define

\[
Ei_k(z) = \frac{1}{2} \left( \int_{\gamma^+(z)} + \int_{\gamma^-(z)} \right) \frac{e^u}{u^k} \, du.
\]

Our main result is

**Theorem 3.2.** Let \(x > 30000, c \geq 1, 0 < \varepsilon \leq 0.01\), and let

\[
\Psi_{x,c,\varepsilon}(\rho) = \lambda_{c,\varepsilon}^{-1} \left( Ei_1(\rho \log x) + A_{c,\varepsilon}(\rho Ei_2(\rho \log x) - 2\rho^2 Ei_3(\rho \log x)) \right) \varepsilon_{c,\varepsilon} \left( \frac{\rho}{4} - \frac{1}{4} \right).
\]
Then we have
\[ \pi_{c,\varepsilon}(x) = \text{li}(x) + \frac{A_{c,\varepsilon} x}{\log(x)} + \sum_{\rho}^* \Psi_{x,c,\varepsilon}(\rho) - \log(2) + \int_{x}^{\infty} \frac{dt}{t \log t(t^2 - 1)} + \Theta(50\varepsilon). \]

Proof. The proof is based on the Weil-Barner formula [Bar81], for which we use the notation
\[ w_s(\hat{f}) = w_f(f) + w_\infty(f). \]
from [FKBJ].

For \( 0 < \delta < 1/x \) let
\[ f_{\delta,x} = \chi_{(\log \delta, \log x)} f_1, \quad \text{and} \quad g_{\delta,x} = \chi_{(\log \delta, \log x)} (f_2 - 2f_3), \]
and let
\[ F_{\delta,x}(t) = \frac{1}{2} (f_{\delta,x}(t) + f_{\delta,x}(-t)), \quad \text{and} \quad G_{\delta,x}(t) = \frac{1}{2} (g_{\delta,x}(t) + g_{\delta,x}(-t)). \]

We will prove the theorem by applying the Weil-Barner formula to the function
\[ H_{\delta,x,c,\varepsilon} = \lambda_{c,\varepsilon}^{-1}(F_{\delta,x} + A_{c,\varepsilon} G_{\delta,x}) * \eta_{c,\varepsilon} \]
and taking the limit \( \delta \to 0 \). For the function \( F_{\delta,x} \) this gives the original Riemann explicit formula [BFJK13]. We will be needing the assertions of lemmas 3.3, 3.4 and 3.5 in [BFJK13], which we restate for reference:
\[
\begin{aligned}
(3.2) \quad & \lim_{\delta \to 0} (w_s(F_{\delta,x}) - \log |\log \delta|) = \sum_{\rho}^* \text{Ei}(\rho \log x) - \text{li}(x) - \log x \\
(3.3) \quad & \lim_{\delta \to 0} (w_f(F_{\delta,x}) - \log |\log \delta|) = \gamma - \pi^*(x) \\
(3.4) \quad & \lim_{\delta \to 0} w_\infty(F_{\delta,x}) = \int_{x}^{\infty} \frac{dt}{t \log t(t^2 - 1)} - \gamma - \log 2 
\end{aligned}
\]

We investigate the functionals \( w_s \) separately, beginning with \( w_s(H_{\delta,x,c,\varepsilon}) \).

**Lemma 3.3.** Let \( x > 30000 \). Then we have
\[
\begin{aligned}
& \lim_{\delta \to 0} (w_s(H_{\delta,x,c,\varepsilon}) - \log |\log \delta|) = -\text{li}(x) - \log \log(x) - A_{c,\varepsilon} \frac{x}{\log(x)^2} \\
& \quad + \sum_{\rho}^* \Psi_{x,c,\varepsilon}(\rho) + \Theta\left(\frac{3.2 A_{c,\varepsilon}}{\log(x)}\right).
\end{aligned}
\]

Proof. We recall the definition
\[ w_s(\hat{f}) = \sum_{\rho}^* \hat{f}(\frac{\rho}{t}) - \frac{1}{2t} - \hat{f}(i/2) - \hat{f}(-i/2). \]

For \(|t| \leq \log x\) we have
\[ F_{\delta,x}(t) = \frac{\sinh(t/2)}{t} \quad \text{and} \quad G_{\delta,x}(t) = \frac{1}{t^2} \left( \cosh(t/2) - \frac{2}{t} \sinh(t/2) \right), \]
so \( F_{\delta,x} \) and \( G_{\delta,x} \) are bounded on \( \mathbb{R} \), and \( \hat{F}_{\delta,x} \) (resp. \( \hat{G}_{\delta,x} \)) converges pointwise to \( \hat{F}_{0,x} \) (resp. \( \hat{G}_{0,x} \)) for \( |\text{Im}(\xi)| < \frac{1}{2} \) and \( \delta \to 0 \).

Since we have
\[
\begin{aligned}
(3.5) \quad & w_s(H_{\delta,x,c,\varepsilon}) = \lambda_{c,\varepsilon}^{-1}(w_s(F_{\delta,x} * \eta_{c,\varepsilon}) + A_{c,\varepsilon} w_s(G_{\delta,x} * \eta_{c,\varepsilon})),
\end{aligned}
\]
we can treat the functions \( F_{\delta,x} \ast \eta_{c,\varepsilon} \) and \( G_{\delta,x} \ast \eta_{c,\varepsilon} \) separately. First we have to show that the limit \( \delta \to 0 \) may be interchanged with the sum over zeros, which we illustrate for the function \( \widetilde{F}_{\delta,x} \ast \eta_{c,\varepsilon} \).

By well-known properties of the zeros of \( \zeta(s) \), it is sufficient to investigate the function \( \widetilde{F}_{\delta,x} \ast \eta_{c,\varepsilon} \) for \( \text{Re}(\xi) > 0 \) and \( |\text{Im}(\xi)| < \frac{1}{2} \).

We have
\[
\widetilde{F}_{\delta,x} \ast \eta_{c,\varepsilon} = \widetilde{F}_{\delta,x} \cdot \eta_{c,\varepsilon} = \widetilde{F}_{\delta,x} \cdot \ell_{c,\varepsilon},
\]
the Logan function \( \ell_{c,\varepsilon} \) obviously satisfies the bound
\[
|\ell_{c,\varepsilon}(\xi)| \ll \frac{1}{\text{Re}(\xi)}
\]
in this region, and the Fourier transform of \( \widetilde{F}_{\delta,x} \) is given by
\[
\widehat{\widetilde{F}}_{\delta,x}(\xi) = \int_{-\log x}^{\log x} e^{i\xi t} \frac{\sinh(t/2)}{t} dt - \int_{-\log x}^{\log x} \frac{e^{(-1/2+i\xi)t} + e^{(-1/2-i\xi)t}}{t} dt.
\]
We bound the integrals on the right hand side separately. For the first integral we get
\[
\int_{-\log x}^{\log x} e^{i\xi t} \frac{\sinh(t/2)}{t} dt = \left[ e^{i\xi t} \frac{\sinh(t/2)}{i t} \right]_{-\log x}^{\log x} - \frac{1}{i \xi} \int_{-\log x}^{\log x} e^{i\xi t} \left( \frac{\sinh(t/2)}{t} \right) \prime dt = O\left( \frac{1}{\text{Re}(\xi)} \right),
\]
and since
\[
\int_{-\log x}^{\log x} e^{(\pm i\xi - 1/2)t} \frac{\pm i\xi - 1/2}{t} dt = \left[ \frac{e^{(\pm i\xi - 1/2)t}}{(\pm i\xi - 1/2)t} \right]_{-\log x}^{\log x} + \frac{1}{\pm i\xi - 1/2} \int_{-\log x}^{\log x} \frac{e^{(\pm i\xi - 1/2)t}}{t^2} dt = O\left( \frac{1}{\text{Re}(\xi)} \right)
\]
holds uniformly in \( \delta \), the second integral in (3.6) is also \( O(1/\text{Re}(\xi)) \).

Therefore, we get
\[
\left| (\widetilde{F}_{\delta,x} \cdot \ell_{c,\varepsilon})(\xi) \right| \ll \frac{1}{\text{Re}(\xi)^2}
\]
uniformly for \( |\text{Im}(\xi)| < 1/2 \) and \( 0 < \delta < 1/x \). Since the sum
\[
\sum_{\rho}^\ast \frac{1}{|\text{Im}(\rho)|^\alpha}
\]
converges for every \( \alpha > 1 \), we thus get
\[
\sum_{\rho}^\ast (\widehat{\widetilde{F}}_{\delta,x} \cdot \ell_{c,\varepsilon})(\rho - \frac{1}{2i}) = \sum_{\rho}^\ast (\widehat{F}_{0,x} \cdot \ell_{c,\varepsilon})(\rho - \frac{1}{2i})
\]
for \( \delta \to 0 \).

Next, we will show that
\[
(3.7) \quad \widehat{F}_{0,x}(\xi) = \frac{1}{2}(\text{Ei}_1(z \log x) + \text{Ei}_1(\bar{z} \log x))
\]
and
\[
(3.8) \quad \widehat{G}_{0,x}(\xi) = \frac{z}{2} \text{Ei}_2(z \log x) - z^2 \text{Ei}_3(z \log x) + \frac{\bar{z}}{2} \text{Ei}_2(\bar{z} \log x) - \bar{z}^2 \text{Ei}_3(\bar{z} \log x)
\]
hold for $|\text{Im}(\xi)| < 1/2$, where we used the abbreviations $z = \frac{1}{2} + i\xi$ and $\tilde{z} = 1 - z$. It suffices to show this for purely imaginary $\xi$. Then we have $z, \tilde{z} \in (0, 1)$ and the substitution $u = zs$ in (3.1) yields

$$z^{1-k} \mathrm{Ei}_k(z \log x) = \frac{1}{2} \int_{-\infty}^{\log x} \frac{e^{z(ir+t)}}{(ir+t)^k} + \frac{e^{z(-ir+t)}}{(-ir+t)^k} \, dt + O(r).$$

Similarly, the substitution $u = -\tilde{z}s$ yields

$$\tilde{z}^{1-k} \mathrm{Ei}_k(\tilde{z} \log x) = \frac{1}{2} \int_{-\infty}^{\log x} \frac{e^{\tilde{z}(-ir-t)}}{(-ir-t)^k} + \frac{e^{\tilde{z}(ir-t)}}{(ir-t)^k} \, dt + O(r).$$

If we use this in (3.7) and (3.8), the sum of the integrands is bounded in a neighbourhood of 0. We may therefore interchange the limit $r \to 0$ with the integral, which gives the Fourier integrals of $F_{0,x}$ and $G_{0,x}$.

For the pole contributions we find

$$F_{\delta, x}(i/2) + F_{\delta, x}(-i/2) = \text{li}(x) + \log \log(x) - \log |\log \delta| + O(\delta)$$

and

$$(3.9) \quad G_{\delta, x}(i/2) + G_{\delta, x}(-i/2) = \int_{-\log x}^{\log x} \frac{e^t - 1}{t^2} - 2\frac{e^t - 1}{t^3} \, dt$$

$$+ \Theta \left( \int_{\log x}^{\infty} \left( \frac{1}{t^2} + \frac{2}{t^3} \right) (1 + e^{-t}) \right) + O(|\log \delta|^{-1}).$$

Here, the first integral on the right hand side equals

$$\left[ \frac{e^t - t - 1}{t^2} \right]_{-\log x}^{\log x} = \frac{x}{\log(x)^2} + \Theta \left( \frac{2.1}{\log x} \right)$$

and the integral in the $\Theta$-term is bounded by $1.1/\log(x)$. We therefore get

$$\lim_{\delta \searrow 0} \left( G_{\delta, x}(i/2) + G_{\delta, x}(-i/2) \right) = \frac{x}{\log(x)^2} + \Theta \left( \frac{3.2}{\log x} \right).$$

Since $\ell_{c,\varepsilon}(\xi) = \ell_{c,\varepsilon}(-\xi)$ holds and since $\rho \mapsto 1 - \rho$ is a bijection of the non-trivial zeros of $\zeta(s)$, we have

$$\sum_{\rho}^* \mathrm{Ei}_k((1 - \rho) \log x) \ell_{c,\varepsilon}(\rho - \frac{1}{2i}) = \sum_{\rho}^* \mathrm{Ei}_k(\rho \log x) \ell_{c,\varepsilon}(\rho - \frac{1}{2i})$$

for $k \geq 1$. We may therefore exchange $z$ for $\tilde{z}$ in (3.8) and (3.8). Adding up the pole contributions and the zero contributions as they occur in (3.3) then gives the desired result.

**Lemma 3.4.** Let $x > 1$, $\varepsilon \leq 0.01$ and $c \geq 1$. Then we have

$$\lim_{\delta \searrow 0} \left( w_f(H_{\delta, x, c, \varepsilon}) - \log |\log \delta| \right) = \gamma - \pi_{c,\varepsilon}^*(x) + O\left( \frac{440 \varepsilon^4 \chi(x)}{c^7} \right).$$

**Proof.** We recall the definition

$$w_f(f) = -\sum_{p^m \leq x} \log p \left( f(m \log p) + f(-m \log p) \right).$$

The result follows by comparison of $w_f(H_{\delta, x, c, \varepsilon})$ and $w_f(F_{\delta, x})$.

Let

$$h(t) = \lambda_{c,\varepsilon}^{-1}(\chi(\log \delta \log x)(f_1 + A_{c,\varepsilon}(f_2 - 2f_3)) \ast \eta_{c,\varepsilon}(t)).$$
Then we have
\[ H_{\delta,x,c,\varepsilon}(t) = \frac{1}{2}(h(t) + h(-t)) \]
for \(|t| > \varepsilon\), and since we assume \(\varepsilon \leq 0.01 < \log 2\) we have
\[ w_f(H_{\delta,x,c,\varepsilon}) = w_f(h). \]

Furthermore, we have
\[ w_f(H_{\delta,x,c,\varepsilon} - F_{\delta,x}) = \pi^+(x) - \pi_{c,\varepsilon}^+(x) - \sum_{p^m} \log \frac{p}{p^{m/2}} (h - f_{\delta,x})(-m \log p). \]

The assertion thus follows from (3.3), if we show that the sum over prime powers is bounded by \(440 \varepsilon^2/c^2\) for \(\delta \to 0\).

Let \(g(t) = (h - f_{\delta,x})(-t)\) and \(y = \delta^{-1}\). Then \(g\) vanishes for \(t > y + \varepsilon\). We first consider the contribution of the prime powers with \(m \log p \in B_{\varepsilon}(y)\). By the Brun-Titchmarsh inequality, their number is \(\ll \varepsilon y \log y + \sqrt{y}\).

Since we have \(|g(t)| \ll \varepsilon \frac{1}{\sqrt{y} \log y}\) for \(t \in B_{\varepsilon}(y)\), their contribution is \(O(1/\log(y))\) and thus vanishes for \(\delta \to 0\).

For the remaining prime powers with \(m \log p \leq y - \varepsilon\) we use the bound
\[ |g(t)| = |(\phi_{\infty,c,\varepsilon} - f_1)(-t)| \leq 39 \frac{\varepsilon^4}{c^2} e^{-t/2} \]
for \(t \geq \log 2\) from Lemma 2.2. This gives
\[ \sum_{p^m \leq y - \varepsilon} \log \frac{p}{p^{m/2}} |g(m \log p)| \leq 39 \frac{\varepsilon^4}{c^2} \sum_{p^m} \frac{1}{m^2 p^m \log p} \leq 39 \frac{\varepsilon^4}{c^2} \zeta(2) \sum_{p^m} \frac{1}{p \log p}. \]

By the Brun-Titchmarsh inequality we have
\[ \sum_{2^k \leq p < 2^{k+1}} \frac{1}{p \log p} < \frac{2^{k+1}}{k \log 2} \frac{2^{-k}}{k \log 2} < \frac{2}{\log(2)^2} \frac{1}{k^2}, \]
so we obtain the desired bound
\[ 39 \zeta(2)^2 \frac{2}{\log(2)^2} \frac{\varepsilon^4}{c^2} < 440 \frac{\varepsilon^4}{c^2} \]
for the right hand side of (3.11).

**Lemma 3.5.** Let \(x > 30000\), \(0 < \varepsilon < 0.01\) and let \(c \geq 1\). Then we have
\[ \lim_{\delta \to 0} w_{\infty}(H_{\delta,x,c,\varepsilon}) = \int_x^{\infty} \frac{dt}{t \log(t^2 - 1)} - \gamma - \log \log x - \log 2 + \Theta(49.4 \varepsilon). \]

**Proof.** We recall the definition
\[ w_{\infty}(f) = f(0) \left( \frac{\Gamma'(1/4)}{\Gamma(1/4)} - \log \pi \right) - \int_0^\infty \frac{f(t) + f(-t) - 2f(0)}{1 - e^{-2t}} e^{-t/2} dt. \]

The functional \(w_{\infty}\) may be applied directly to the function \(H_{0,x,c,\varepsilon}\). Let \(\Delta = H_{0,x,c,\varepsilon} - F_{0,x}\). Then we have
\[ w_{\infty}(H_{0,x,c,\varepsilon}) = w_{\infty}(F_{0,x}) + w_{\infty}(\Delta). \]
It is therefore sufficient to show $w_\infty(\Delta) = \Theta(49.4 \varepsilon)$. This will be done by proving the following bounds.

\begin{equation}
-2 \int_0^{\log 2} \frac{\Delta(t) - \Delta(0)}{1 - e^{-2t}} e^{-t/2} \, dt = \Theta(14.1 \varepsilon) \tag{3.12}
\end{equation}

\begin{equation}
2\Delta(0) \int_{\log 2}^\infty \frac{e^{-t/2}}{1 - e^{-2t}} \, dt = \Theta(14.5 \varepsilon) \tag{3.13}
\end{equation}

\begin{equation}
-\int_{\log 2}^{\log x} \frac{(\phi_{\infty,c,\varepsilon} - f_1)(t)}{1 - e^{-2t}} e^{-t/2} \, dt = \Theta(76 \varepsilon^4) \tag{3.14}
\end{equation}

\begin{equation}
-\int_{\log x - \varepsilon}^{\log x + \varepsilon} \frac{(\phi_{x,c,\varepsilon} - \chi_{x,\log x})(\phi_{\infty,c,\varepsilon})(t)}{1 - e^{-2t}} e^{-t/2} \, dt = \Theta(0.12 \varepsilon) \tag{3.15}
\end{equation}

\begin{equation}
-\int_{\log 2}^\infty \frac{(\phi_{x,c,\varepsilon} - f_1)(-t)}{1 - e^{-2t}} e^{-t/2} \, dt = \Theta(38 \varepsilon^4) \tag{3.16}
\end{equation}

\begin{equation}
\Delta(0) \left( \frac{\Gamma'}{\Gamma}(1/4) - \log \pi \right) = \Theta(20.6 \varepsilon) \tag{3.17}
\end{equation}

Since we have

\[ \Delta(t) + \Delta(-t) = (\phi_{x,c,\varepsilon} - f_{0,x})(t) + (\phi_{x,c,\varepsilon} - f_1)(-t) \]

for $t > \varepsilon$, the left hand sides of these equations sum to $w_\infty(\Delta)$ and since we assume $\varepsilon < 0.01$, this implies indeed

\[ |w_\infty(\Delta)| \leq (14.1 + 14.5 + 76 \times 10^{-6} + 0.12 + 38 \times 10^{-6} + 20.6) \varepsilon < 49.4 \varepsilon. \]

First, we estimate $\Delta$. We define the entire auxiliary functions

\[ F(z) = \frac{\sinh(z/2)}{z}, \quad G(z) = \frac{\cosh(z/2)}{z^2} - 2\frac{\sinh(z/2)}{z^3}. \]

Now let $U_1 = \{ z \in \mathbb{C} \mid |\text{Re}(z)| < 3/2, |\text{Im}(z)| < 3/2 \}$. Then $F_{0,x}$ (resp. $G_{0,x}$) coincides with $F$ (resp. $G$) on $U_1 \cap \mathbb{R}$, and we have

\[ \max_{z \in \partial U_1} \{|\sinh(z/2)|, |\cosh(z/2)|\} \leq e^{3/4} < 2.2. \]

So by the maximum-principle we have

\[ |F(z)| \leq 2.2 \frac{2}{3} < 1.5 \quad \text{and} \quad |G(z)| \leq 2.2 \left( \frac{4}{9} + \frac{16}{27} \right) < 2.3 \]

for $z \not\in U_1$. If we shrink $U_1$ to $U_2 = \{ z \in \mathbb{C} \mid |\text{Re}(z)| < 1, |\text{Im}(z)| < 1 \}$, we get

\[ F'(z) = \frac{1}{2\pi i} \int_{|z-\xi|=\frac{1}{2}} \frac{F(\xi)}{(\xi-z)^2} \, d\xi = \Theta(3), \]

for $z \in U_2$ and similarly we get $G'(z) = \Theta(4.6)$ in $U_2$.

Next, we need to bound $\lambda_{c,\varepsilon}^{-1}$. The mapping $t \mapsto \sinh(t)/t$ is monotonously increasing for $t \in (0, \infty)$, so we have

\[ \frac{\sinh(\sqrt{c^2 + \varepsilon^2/4})}{\sqrt{c^2 + \varepsilon^2/4}} \leq \frac{\sinh(c + \varepsilon/2)}{c + \varepsilon/2} \leq e^{\varepsilon/2} \frac{\sinh c}{c} \]

and consequently we get

\begin{equation}
1 > \lambda_{c,\varepsilon}^{-1} \geq e^{-\varepsilon/2} \geq 1 - \frac{\varepsilon}{2} \tag{3.18}
\end{equation}
Now let \( f \) be a holomorphic function in \( U \). Then we have

\[
\frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau)f(z - \tau) d\tau = \frac{1}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{\varepsilon} \eta_{c,\varepsilon}(\tau)(f(z) + \Theta(\varepsilon \| f' \|_{\infty,U})) d\tau
\]

\[
= \lambda_{c,\varepsilon}^{-1} f(z) + \Theta(\varepsilon \| f' \|_{\infty,U}) = f(z) + \Theta(\varepsilon \| f' \|_{\infty,U} + 0.5 \| f \|_{\infty,U})
\]

for all \( z \) satisfying \( B_\varepsilon(z) \subset U \). So if we shrink \( U_2 \) to

\[
U_3 = \{ z \in \mathbb{C} \mid |\text{Re}(z)| < \log 2, |\text{Im}(z)| < \pi/4 \},
\]

we get

\[
\lambda_{c,\varepsilon}^{-1} f * \eta_{c,\varepsilon}(z) = F(z) + \Theta(3.75 \varepsilon) \quad \text{and} \quad \lambda_{c,\varepsilon}^{-1} G * \eta_{c,\varepsilon}(z) = G(z) + \Theta(5.75 \varepsilon)
\]

for \( z \in U_3 \). This gives the bound

\[(3.19) \quad \Delta(z) = \Theta(3.8 \varepsilon) \quad \text{for} \quad z \in U_3.\]

We proceed with the proofs of (3.14) - (3.17), starting with (3.12). For \( z \in \partial U_3 \), we have

\[|1 - e^{-2z}| \geq 1 - e^{-2\log 2} = \frac{3}{4}.\]

So by applying the maximum principle to \( \frac{\Delta(z) - \Delta(0)}{1 - \exp(-2z)} \) and using (3.19), we get

\[2 \int_0^{\log 2} \left| \Delta(t) - \Delta(0) \right| e^{-t/2} dt \leq 2 \cdot \log 2 \cdot \left( \frac{4}{3} \cdot 2 \cdot 3.8 \varepsilon \right) < 14.1 \varepsilon.\]

For (3.13) we use

\[
\int_0^{\log x} e^{-t/2} dt \leq \frac{4}{3} \int_0^{\log x} e^{-t/2} dt = \frac{4}{3} \sqrt{2} < 1.9,
\]

and \( 2 \cdot 3.8 \varepsilon \cdot 1.9 < 14.5 \varepsilon \).

For (3.14) we use (2.7), which gives

\[\int_{\log 2}^{\log x} \left| \phi_{\infty,c,\varepsilon}(t) - f_1(t) \right| e^{-t/2} dt \leq 39 \varepsilon^4 \frac{4}{3} \int_{\log 2}^{\log x} dt < 39 \cdot 4 \varepsilon^4 < 76 \varepsilon^4.\]

Next, we treat (3.15). Using the obvious bounds

\[(3.20) \quad |\mu_{c,\varepsilon}(t)| \leq \frac{1}{2}, \quad |\nu_{c,\varepsilon}(t)| \leq \varepsilon \quad \text{and} \quad |y| \leq \varepsilon\]

in (2.13) gives

\[|m_{c,\varepsilon}(t)| \leq \frac{e^{t/2}}{t} \left( \frac{1}{2} + \varepsilon \right).\]

So by Lemma 2.3 the integrand in (3.15) is bounded by

\[\left| (\phi_{c,\varepsilon} - \chi_{[2x,\log x]})(t) \right| e^{-t/2} \leq 1.001 \left( \frac{1}{2} + \varepsilon \right) < 0.06.\]

For (3.16) we use (2.7) again, which gives

\[\int_{\log 2}^{\log x} \left| (\phi_{c,\varepsilon} - f_1)(-t) \right| e^{-t/2} dt \leq 39 \varepsilon^4 \frac{4}{3} \int_{\log 2}^{\log x} dt < 38 \varepsilon^4.\]

Finally, for (3.17) we use \( \Gamma'/\Gamma(1/4) - \log \pi = \Theta(5.4) \) and (3.19).
To conclude the proof of Theorem 3.2, we note that we have

\[(3.21) \quad w_s(H_{\delta,x,c,\varepsilon}) - \log |\log \delta| = w_f(H_{\delta,x,c,\varepsilon}) - \log |\log \delta| + w_\infty(H_{\delta,x,c,\varepsilon}),\]

by the Weil-Barner explicit formula. Using the bounds \(\log x > 10\), \(c \geq 1\), \(\varepsilon \leq 0.01\) and \(A_{c,\varepsilon} \leq 0.005\), we obtain the bound

\[A_{c,\varepsilon} \frac{3.2}{\log x} + \frac{440 \varepsilon^4}{c^2} + 49.4 \varepsilon < 50 \varepsilon,\]

for the sum of the remainder terms in lemmas 3.3, 3.4 and 3.5. The limit \(\delta \to 0\) thus yields the assertion. \(\square\)

4. Evaluation of the sum over zeros

The natural point for the truncation of the sum over zeros is \(|\text{Im}(\rho)| \approx c/\varepsilon\), since the Logan function \(\ell_c\) decreases fast in \([0, c]\) and only slowly after this point. However, in the case of unconditional computations we can make use of the larger choice of \(c\) and decrease the truncation point. This way, the length of the zero range can always be reduced by a factor \(\sqrt{3/4}\).

The results are summarized in the following theorem.

**Theorem 4.1.** Let \(c \geq 10, \varepsilon \leq 10^{-5}\), and \(x \geq 10^{10}\). Let \(h = \frac{1}{2}\) if the Riemann hypothesis is assumed and \(h = 1\) otherwise. Then we have

\[(4.1) \quad \sum_{\rho : |\text{Im}(\rho)| > c/\varepsilon} \Psi_{x,c,\varepsilon}(\rho) \leq 0.66 e^{c(\sqrt{e}/4 - 1)} \log(3c) \log\left(\frac{c}{\varepsilon}\right) \frac{x^h + 1}{2h \log x}.\]

If in addition \(a \in (0, 1)\) satisfies \(ac/\varepsilon \geq 10^3\), and if the Riemann hypothesis holds for \(|\text{Im}(\rho)| \leq \frac{c}{\varepsilon}\), then we have

\[(4.2) \quad \sum_{\rho : \text{Re}(\rho) < |\text{Im}(\rho)| \leq \frac{c}{\varepsilon}} \Psi_{x,c,\varepsilon}(\rho) \leq 0.33 + 3.6 \frac{c \varepsilon}{ca^2} \log\left(\frac{c}{\varepsilon}\right) \frac{\cosh(c \sqrt{1 - a^2})}{\sinh(c)} \sqrt{x}.\]

**Remark 4.2.** If we additionally impose the restrictions (2.3) and (2.4) on the parameters \(c\) and \(\varepsilon\), (4.1) can be simplified to

\[\sum_{|\text{Im}(\rho)| > c/\varepsilon} \Psi_{x,c,\varepsilon}(\rho) \leq e^{-c} \frac{x^h}{2h} \log \log(x).\]

We need some preparation in order to prove the theorem. First we give an asymptotic expansion for the functions \(E_{i,k}\).

**Lemma 4.3.** Let \(E_{i,k}\) be as in Definition 3.7. Then we have

\[(4.3) \quad E_{i,k}(z) = \sgn(\text{Im}(z)) \frac{\pi i}{(k-1)!} + \sum_{i=0}^{n-1} \frac{(k + l - 1)!}{(k-1)!} e^{\varepsilon} \frac{\varepsilon^i}{z^{l+k}} + O\left(\frac{2^{(k + n - 1)!} e^{\text{Re}(z)}}{|\text{Im}(z)|^{k+n}}\right).\]
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**Proof.** Let $z = x + iy$. Since both $E_i$ and the terms on the right hand side of (4.4) interchange with complex conjugation, we may assume $y < 0$. Then, by shifting $\gamma^+(z)$ towards $i\infty$ and $\gamma^-(z)$ towards $-i\infty$ in (4.3), we get

$$E_i^k(z) = \frac{1}{2} \int_{x-i\infty}^{x+i\infty} \frac{e^\xi}{\xi^k} d\xi + \frac{1}{2} \int_{-\infty}^{x-i\infty} \frac{e^\xi}{\xi^k} d\xi.$$  

Since

$$\int_{x-i\infty}^{x+i\infty} \frac{e^\xi}{\xi^k} d\xi = 2\pi i \text{Res}_{\xi=0} \frac{e^\xi}{\xi^k} = \frac{2\pi i}{(k-1)!}$$

we therefore find

$$E_i^k(z) = -\frac{\pi i}{(k-1)!} + \int_{x-i\infty}^{x+i\infty} \frac{e^\xi}{\xi^k} d\xi.$$  

Now, by integrating by parts repeatedly, we get

$$\int_{x-i\infty}^{x+i\infty} \frac{e^\xi}{\xi^k} d\xi = \sum_{j=0}^{n} \frac{(j + k - 1)!}{(k - 1)!} \frac{e^x}{z^{j+k}} + \frac{(k + n)!}{(k - 1)!} \int_{x-i\infty}^{x+i\infty} \frac{e^\xi}{\xi^{k+n+1}} d\xi,$$

where the integral is bounded by

$$\left| \frac{(k + n)!}{(k - 1)!} \int_{x-i\infty}^{x+i\infty} \frac{e^\xi}{\xi^{k+n+1}} d\xi \right| \leq \frac{(k + n)!}{(k - 1)!} e^x \int_{-\infty}^{y} \frac{d\xi}{|\xi|^{k+n+1}} \leq \frac{(k + n - 1)!}{(k - 1)!} \frac{e^x}{|y|^{k+n}}.$$  

Next, we define the auxiliary function

$$(4.4) \quad \psi_{x,c,\varepsilon}(\rho) = E_i(\rho \log x) + A_{c,\varepsilon} (\rho E_i(\rho \log x) - 2\rho^2 E_i(\rho \log x)),$$

so that we have

$$\Psi_{x,c,\varepsilon}(\rho) = \psi_{x,c,\varepsilon}(\rho) \ell_{c,\varepsilon}\left(\frac{\rho}{i} - \frac{1}{2i}\right).$$

The previous lemma gives rise to an asymptotic expansion of the function $\psi_{x,c,\varepsilon}$, which is used to evaluate $\Psi_{x,c,\varepsilon}(\rho)$.

**Lemma 4.4.** Let $c \geq 10$ and $\varepsilon \leq 0.01$. For $l \geq 1$, let

$$\alpha_l(x, c, \varepsilon) = \frac{(l - 1)!}{\log(x)^l} + A_{c,\varepsilon}\left(\frac{l!}{\log(x)^{l+1}} - \frac{(l+1)!}{\log(x)^{l+2}}\right).$$

and let

$$\alpha_0(\rho, c, \varepsilon) = \text{sgn}(\text{Im}(\rho)) \pi i \left(1 + A_{c,\varepsilon}(\rho - \rho^2)\right).$$

Furthermore, let $|\text{Im}(\rho)| > 14$, $\text{Re}(\rho) \in (0, 1)$ and $n + 1 \leq 10 \log x$. Then we have

$$\psi_{x,c,\varepsilon}(\rho) = \alpha_0(\rho, c, \varepsilon) + \sum_{j=1}^{n-1} \alpha_l(x, c, \varepsilon) x^\rho + \Theta\left(2.01(n-1)! \frac{x^{\text{Re}(\rho)}}{|\text{Im}(\rho)| \log x^n}\right).$$

We note that the term $\alpha_0(\rho, c, \varepsilon)$ plays no further role in the computation of the sum over zeros, since we have $\alpha_0(\rho, c, \varepsilon) + \alpha_0(1 - \rho, c, \varepsilon) = 0$.

**Proof.** We get the main term by adding up the asymptotic expansions of $E_i(\rho \log x)$ as they occur in (4.3). The sum of the $\Theta$-terms is then bounded by

$$(4.5) \quad \frac{n \rho}{|\text{Im}(\rho)\log x|} \left[1 + A_{c,\varepsilon}\left(\frac{n \rho}{|\text{Im}(\rho)\log x|} + \frac{n(n + 1)|\rho|^2}{|\text{Im}(\rho)\log x|^2}\right)\right].$$
Under the conditions of the lemma we have

\[ |\rho| / |\text{Im}(\rho)| \leq \frac{1 + 14}{14} < 1.08, \quad \frac{n + 1}{\log(x)} \leq 10, \]

and \( A_{c,\varepsilon} \leq 5 \times 10^{-6} \), so the inner bracket in (4.5) is bounded by 0.0007, from which the assertion follows.

For the second part of the theorem we will also be needing the following lemma.

**Lemma 4.5.** Let \( c, \varepsilon > 0 \), and \( a \in (0, 1) \) satisfy \( \frac{ac}{\varepsilon} \geq 10^3 \). Then we have

\[
\sum_{\frac{c}{\varepsilon} \leq |\text{Im}(\rho)|} |\ell_c(\varepsilon \text{Im}(\rho)))| \leq 1 + 11 \frac{c \varepsilon}{\pi c a^2} \log \left( \frac{c}{2\pi \varepsilon} \right) \cosh \left( c \sqrt{1 - a^2} \right) \sinh(c).
\]

**Proof.** We use the notation \( \rho = \beta + i\gamma \) for the zeros \( \rho \) of the zeta function, where \( \beta \) and \( \gamma \) are real. Let

\[
\tilde{N}(t) = \frac{t}{2\pi} \log \frac{t}{2\pi e} \quad \text{and} \quad R(t) = N(t) - \tilde{N}(t),
\]

where \( N \) denotes the zero-counting function. Then for \( t \geq 10^3 \) we have

\[
R(t) = \Theta(0.5 \log t)
\]

by Rosser’s estimate [Ros41, p. 223]. By symmetry of the zeros, it suffices to treat the sum over \( \gamma > 0 \).

We have

\[
\sum_{\frac{c}{\varepsilon} \leq |\text{Im}(\rho)|} \frac{\ell_c(\varepsilon \text{Im}(\rho)))}{\gamma} \leq \int_{c/\varepsilon}^{\infty} \frac{\ell_c(t)}{t} d(\tilde{N}(t) + R(t))
\]

(4.8)

First, we estimate the first integral on the right hand side of (4.8). Using the inequality

\[
0 < \frac{1}{t} \log \frac{t}{2\pi} \leq \frac{\varepsilon^2}{(ac)^2} \log \left( \frac{c}{2\pi \varepsilon} \right) t
\]

and applying the substitution \( u = \sqrt{c^2 - (\varepsilon t)^2} \) gives the bound

\[
0 < \frac{1}{2\pi} \int_{c/\varepsilon}^{\infty} \ell_c(t) \log \frac{t}{2\pi} \frac{dt}{t} \leq \frac{1}{2\pi(ac)^2} \log \left( \frac{c}{2\pi \varepsilon} \right) \frac{c}{\sinh(c)} \int_{0}^{c\sqrt{1-a^2}} \sinh(u) du
\]

(4.9)

For the second integral in (4.8) we use partial integration, the bound in (4.7) and the negativity of the derivative of \( \ell_c(t)/t \) in the range of integration, which
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\[\int_{a}^{c} \frac{\ell_{c,\epsilon}(t)}{t} dR(t) = \Theta \left[ \int_{a}^{c} \frac{\ell_{c,\epsilon}(t)}{t} R(t) \right]_{a}^{c} \leq \epsilon \ell(c) \log \left( \frac{c}{\epsilon} \right) - 0.5 \int_{a}^{c} \frac{\ell_{c,\epsilon}(t)}{t} \log(t) dt \]

(4.10)

\[\leq 1.5 \frac{\epsilon}{ac} \ell(c) \log \left( \frac{c}{\epsilon} \right) + 0.5 \int_{a}^{c} \frac{\ell_{c,\epsilon}(t)}{t^2} dt \leq 1.6 \frac{\epsilon}{ac} \ell(c) \log \left( \frac{c}{\epsilon} \right),\]

where we also used $\log(c/\epsilon) > 6$ on the last line. Since the function $t \mapsto t \cosh(1 - a^2)$ is monotonically increasing in $[0, \infty)$, we have

\[c \sqrt{1 - a^2} \cosh(c \sqrt{1 - a^2}) \sinh \left( c \sqrt{1 - a^2} \right) \geq 1\]

and therefore get

\[1.6 \frac{\epsilon}{ac} \log \left( \frac{c}{\epsilon} \right) \ell(c) \leq 1.6 \frac{\epsilon c}{ac} \log \left( \frac{c}{\epsilon} \right) \cosh(c \sqrt{1 - a^2}) \sinh(c) \]

This and (4.9) yields the assertion, since we have

\[\frac{1}{2 \pi \epsilon} \log \left( \frac{c}{2 \pi \epsilon} \right) + 1.6 \frac{\epsilon c}{ac} \log \left( \frac{c}{\epsilon} \right) \leq \frac{1}{2 \pi \epsilon} \log \left( \frac{c}{\epsilon} \right).\]

Proof of Theorem 4.1. We recall that we have $|\text{Im}(\rho)| > 14$ for all non-trivial zeros of the zeta function by well-known numerical results (see e.g. [Leh70]). If we take $n = 2$ in Lemma 4.4 and use the bound

\[|\alpha_1(x, c, \epsilon)| \leq \frac{1}{\log x} + \frac{\epsilon^2}{2c} \left( \frac{1}{\log(x)^2} + \frac{2}{\log(x)^3} \right) \leq \frac{1.001}{\log x},\]

we thus get

(4.11)

\[|\psi_{x,c,\epsilon}(\rho) - \alpha_0(\rho, c, \epsilon)| \leq 1.001 \frac{x \text{Re}(\rho)}{|\rho| \log x} + 2.01 \frac{x \text{Re}(\rho)}{|\text{Im}(\rho)| \log x} \leq \frac{1.01 x \text{Re}(\rho)}{|\text{Im}(\rho)| \log x}.\]

Now [FKBJ, Lemma 2.4] together with the pairing argument from [Pla] for zeros off the line gives (4.1) and Lemma 4.5 gives (4.2).

\[\Box\]

5. EVALUATION OF THE SUM OVER PRIME POWERS

The sum over prime powers is evaluated in the same way as in [FKBJ]. But since the functions $\eta_{c,\epsilon}$ also form a Dirac sequence for $c \to \infty$, the function $M_{x,c,\epsilon}$ tends to be very small near the boundary of the sieve interval if $c$ is large. For calculations not assuming the RH this can be used to reduce the length of the sieve interval by bounding the contribution of the prime powers near the boundary. To this end, we apply a weighted version of the Brun-Titchmarsh inequality. For applications with $x \approx 10^{24}$ this reduces the length of the sieve interval up to 20%.
Proposition 5.1. Let $x \geq 100$, $\varepsilon \leq 0.01$, $c \geq 1$, and let $\alpha \in (0, 1)$, such that

$$B := \frac{\varepsilon xe^{-\varepsilon} |\nu_c(\alpha)|}{2\mu_c(\alpha)} > 1$$

holds. Furthermore, let $I_+^\alpha = [e^{\alpha \varepsilon} x, e^{\varepsilon} x]$ and $I_-^\alpha = [e^{-\varepsilon} x, e^{-\alpha \varepsilon} x]$. Then we have

$$\left| \sum_{p^m \in I_+^\alpha} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \right| \leq \frac{2\varepsilon x e^{2\varepsilon} |\nu_c(\alpha)|}{\log B} + \varepsilon \mu_c(\alpha) \left( 4.01 \varepsilon \sqrt{x} + \log \log(2x^2) \right).$$

We start with the following lemma.

Lemma 5.2. Let $x > 1, \varepsilon < 1$ and $\alpha \in (0, 1)$, such that

$$B := \frac{\varepsilon xe^{-\varepsilon} |\nu_c(\alpha)|}{2\mu_c(\alpha)} > 1$$

holds. Then we have

$$\sum_{p \in I_+^\alpha} \left| \mu_{c,\varepsilon} \left( \log \frac{p}{x} \right) \right| \leq 2 \frac{\varepsilon x e^{\varepsilon} |\nu_c(\alpha)|}{\log B}.$$

Proof. We give the proof for $I = I_+^\alpha$. For $t \in I$ let

$$f(t) = \mu_c \left( \frac{1}{\varepsilon} \log \frac{t}{x} \right).$$

Then $f$ satisfies the conditions of [Büt, Theorem 4.3] and we therefore have

$$\sum_{p \in I} f(p) \leq 2 \left\| f \right\|_{1,I} \left( \log \frac{\left\| f \right\|_{1,I}}{\left\| f \right\|_{\infty,I} + \left\| f' \right\|_{1,I}} \right)^{-1}.$$

Since $f$ is monotonously decreasing on $I$, we have $\left\| f \right\|_{\infty,I} = \left\| f' \right\|_{1,I} = \mu_c(\alpha)$, and the substitution $u = \frac{1}{\varepsilon} \log \frac{t}{x}$ yields

$$\left\| f \right\|_{1,I} = \int_{e^{\alpha \varepsilon} x}^{e^{\varepsilon} x} f(t) \, dt = \varepsilon x \int_{\alpha}^{\varepsilon} e^{\varepsilon u} \mu_c(\alpha) \, du,$$

so we get (5.2). The interval $I_-^\alpha$ can be treated similarly. \qed

Proof of Proposition 5.1. From (3.20) we obtain the bound

$$|m_{x,c,\varepsilon}(t)| \leq \frac{e^{t/2}}{t} \left| \mu_{c,\varepsilon}(t - \log(x)) \right| (1 + \varepsilon)$$

for $t \neq \log(x)$. Therefore, we have

$$\left| \sum_{p^m \in I_+^\alpha} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \right| \leq \varepsilon \mu_c(\alpha) \left( \log \frac{p^m}{x} \right).$$

Using Lemma 5.2 to bound the contribution of prime numbers and Lemma 2.4 to bound the contribution of prime powers we get (5.1). \qed
6. Calculations

For the unconditional calculations of $\pi(10^{24})$ and $\pi(10^{25})$ we both took $c = 62$ and $\varepsilon = 6.2 \times 10^{-10}$ and computed the sum for $|\operatorname{Im}(\rho)|$ up to $10^{11}$. The calculation of $\pi(10^{24})$ took less than 3,900 CPU hours and the calculation of $\pi(10^{25})$ took less than 40,000 CPU hours this way. In both cases almost all time was spent generating the prime numbers in the sieve interval. The run time for both calculations could have been reduced by the use of additional zeros of the Riemann zeta function. For the calculation of $\pi(10^{24})$ it would have been optimal to take all zeros with imaginary part up to $4 \times 10^{11}$, which projects to a run time of less than 900 CPU hours, i.e. about 5 weeks on a single CPU.

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