BINOMIAL APPROXIMATIONS FOR BARRIER OPTIONS OF ISRAELI STYLE.

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ABSTRACT. We show that prices and shortfall risks of game (Israeli) barrier options in a sequence of binomial approximations of the Black–Scholes (BS) market converge to the corresponding quantities for similar game barrier options in the BS market with path dependent payoffs and the speed of convergence is estimated, as well. The results are new also for usual American style options and they are interesting from the computational point of view, as well, since in binomial markets these quantities can be obtained via dynamical programming algorithms. The paper continues the study of [11] and [7] but requires substantial additional arguments in view of peculiarities of barrier options which, in particular, destroy the regularity of payoffs needed in the above papers.

1. INTRODUCTION

This paper deals with knock--out and knock--in double barrier options of the game (Israeli) type sold in a standard securities market consisting of a nonrandom component \( b_t \) representing the value of a savings account at time \( t \) with an interest rate \( r \) and of a random component \( S_t \) representing the stock price at time \( t \). As usual, we view \( S_t, t > 0 \) as a stochastic process on a probability space \( (\Omega, \mathcal{F}, P) \) and we assume that it generates a right continuous filtration \( \{\mathcal{F}_t\} \). The setup includes also two right continuous with left limits (caldlag) stochastic payoff processes \( X_t \geq Y_t \geq 0 \) adapted to the above filtration. Recall, that a game contingent claim (GCC) or a game option was defined in [10] as a contract between the seller and the buyer of the option such that both have the right to exercise it at any time up to a maturity date (horizon) \( T \) which in this paper assumed to be finite. If the buyer exercises the contract at time \( t \) then he receives the payment \( Y_t \), but if the seller exercises (cancels) the contract before the buyer then the latter receives \( X_t \). The difference \( \Delta_t = X_t - Y_t \) is the penalty which the seller pays to the buyer for the contract cancellation. In short, if the seller will exercise at a stopping time \( \sigma \leq T \) and the buyer at a stopping time \( \tau \leq T \) then the former pays to the latter the amount \( H(\sigma, \tau) = X_\sigma I_{\sigma < \tau} + Y_\tau I_{\tau < \sigma} \) where we set \( I_A = 1 \) if an event \( A \) occurs and \( I_A = 1 \) if not.

A hedge (for the seller) against a GCC is defined here as a pair \( (\pi, \sigma) \) which consists of a self financing strategy \( \pi \) (i.e. a trading strategy with no consumption and no infusion of capital) and a stopping time \( \sigma \) which is the cancellation time for the seller. A hedge is

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that if no matter what exercise time the buyer chooses, the seller can cover his liability to the buyer (with probability one). The option price $Y^*$ is defined as the minimal initial capital which is required for a perfect hedge, i.e. for any $x > Y^*$ there is a perfect hedge with an initial capital $x$. Recall, (see [10]) that pricing a GCC in a complete market leads to the value of a zero sum optimal stopping (Dynkin’s) game with discounted payoffs $\tilde{X}_t = b_0 \frac{x t}{n}$, $\tilde{Y}_t = b_0 \frac{R t}{n}$ considered under the unique martingale measure $\tilde{P} \sim P$.

We consider a double knock–out barrier option with a two constant barriers $L, R$ such that $0 \leq L < S_0 < R \leq \infty$ which means that the option is worthless to its holder (buyer) at the first time $\tau$ the stock price $S_t$ exits the open interval $I = (L, R)$. Thus for $t \geq \tau_{L,R}$ the payoff is $X_t = Y_t = 0$. For $t < \tau_{L,R}$ we consider path dependent payoffs. Such a contract is of potential value to a buyer who believes that the stock price will not exit the interval $I$ up to a maturity date and to a seller who believes otherwise and does not want to have to worry about hedging if the stock price will reach one of the barriers $L, R$. Double knock–in barrier options which start when $S_t$ exits an interval $I$ will be considered, as well. Observe, that we view barrier game options as a generalization of regular game options where the supremum is taken over all stopping times not exceeding a horizon $T$, the infimum is taken over all hedges with an initial capital $x$. $Q(\sigma, \tau) = X_t 1_{t < \tau} + Y_t 1_{t \geq \tau}$ is the discounted payoff, $V_t^{\pi, \tau}$ is the portfolio value of $\pi$ at time $t$ and $E$ denotes the expectation with respect to the objective probability measure $P$. An investor (seller) whose initial capital $x$ is less than the option price still wants to compute the minimal possible shortfall risk.

The Cox, Ross and Rubinstein (CRR) binomial model which was introduced in [4] is an efficient tool to approximate derivative securities in a Black–Scholes (BS) market. We will show that for a double barrier options in the BS model the option price can be approximated by a sequence of option prices of a barrier options (with the same barriers) in appropriate CRR $n$–step models with errors bounded by $Cn^{-1/4}(\ln n)^{3/4}$ where $C$ is a constant which does not depend on the value of the barriers. These both generalize the results from [11] which were obtained for regular (without barriers) game options with path dependent payoffs and provide an algorithm for computation of this important class of derivative securities since pricing of game options in CRR markets can be done by dynamical programming (see [10]).

Pricing of European and American type barrier options was studied in several papers (see, for instance, [2] and [14]) and a number of papers dealt with error estimates for discrete approximations of barrier European options (see, for instance, [2], [3], [20] and references there). On the other hand, binomial approximations and their error estimates for look back American style, let alone for Israeli style, barrier options were not studied rigorously before.

We also deal with partial hedging (under the same assumption on the payoffs) which becomes relevant if for instance, an investor (seller) is not willing for various reasons to tie in a hedging portfolio the full initial capital required for a perfect hedge. In this case the seller is ready to accept a risk that his portfolio value at an exercise time may be less than his obligation to pay and he will need additional funds to fulfill the contract. Thus a portfolio shortfall comes into the picture and by this reason we distinguish here between hedges and perfect hedges.

In this paper, we deal with certain type of risk called the shortfall risk (cf. for instance, [5], [6], [8], [17]) which was defined for game options in [6] by the formulas

$$R(\pi, \sigma) = \sup_{\tau} E(Q(\sigma, \tau) - b_0 \frac{V^{\pi, \tau} - x}{b_0 \sigma \tau})^+$$

and

$$R(x) = \inf_{(\pi, \sigma)} R(\pi, \sigma)$$

where the supremum is taken over all stopping times not exceeding a horizon $T$, the infimum is taken over all hedges with an initial capital $x$. $Q(\sigma, \tau) = X_t 1_{t < \tau} + Y_t 1_{t \geq \tau}$ is the discounted payoff, $V_t^{\pi, \tau}$ is the portfolio value of $\pi$ at time $t$ and $E$ denotes the expectation with respect to the objective probability measure $P$. An investor (seller) whose initial capital $x$ is less than the option price still wants to compute the minimal possible shortfall risk.
and to find a hedge with the initial capital \( x \) which minimizes or "almost" minimizes the shortfall risk.

In [6] we proved that for a game option in the multinomial model with general payoffs there exists a hedge which minimizes the shortfall risk under constraint on the initial capital, and the above hedge together with the corresponding shortfall risk can be computed via a dynamical programming procedure. For game option in the BS model the problem of finding an optimal hedge is more complicated and for now remains open even for regular payoffs. We will prove that in the BS model the shortfall risk \( R(x) \) of a seller with initial capital \( x \) for double barrier options is a limit of the shortfall risks \( R_n(x) \) for double barrier options in the CRR markets with the same barriers and initial capital as in the BS model. Here we are able to provide only a one sided error estimate \( R(x) - R_n(x) \leq \tilde{C} n^{-1/4} (\ln n)^{-3/4} \) where \( \tilde{C} > 0 \) is a constant which does not depend on the value of the barriers. These results generalize the ones which were obtained in [7] for regular game options with path dependent payoffs and again provide a way of computation of the shortfall risk for barrier game options. Binomial approximations of shortfall risks for barrier options were not studied before even for European options.

For a given initial capital \( x \) we will use hedges which minimize the shortfall risk in CRR markets under the above constraint on the initial capital, in order to construct hedges which "almost" minimize the shortfall risk in the BS model under the same constraint on the initial capital. Furthermore we will see that the corresponding portfolios are managed on a finite set of random times as it was done in [7] for regular game options. We consider also another situation where the seller of a game option in the BS model has an initial capital which is a little bit larger than the option price. In this case we use perfect hedges in CRR markets in order to build explicitly hedges with small shortfall risks in the BS model where the corresponding portfolios are managed on a finite set of random times as it was done in [11] for regular game options.

Our main tool is the Skorohod type embedding of sums of i.i.d. random variables into a Brownian motion with a constant drift. This tool was employed for a regular options in [7] and [11] in order to obtain error estimates for approximation of shortfall risks and for approximation of option prices, respectively. However, in the barrier options case the payoffs lose their Lipschitz continuity which was crucial in [11] and [7], and so this case requires substantial additional arguments and estimates leading to a generalization of our previous results. Moreover, observe that discontinuities of payoffs occur at random times since they depend on the stock behavior. Since the discretization does not necessarily adjusted to the barrier value where discontinuities occur we have to estimate the deviation of the option price as the barrier value changes a bit which is the key additional part of the proof in comparison to [7] and [11] (see Lemmas 3.3, 5.4 and 5.2).

Main results of this paper are formulated in the next section where we discuss also the Skorohod type embedding. In Section 3 we introduce recursive formulas which enable us to compare various option prices and risks. In this section we also derive auxiliary estimates for option prices and risks. In Section 4 we complete the proof of main results of the paper for knock–out options while in Section 5 we deal with the knock–in case which requires a somewhat different definitions and a separate treatment yielding a bit worse error estimates. Some definitions and estimates in this paper are similar to [7] and [11] but for the sake of the reader and in order to keep the paper relatively self-contained we repeat them here with needed modifications. On the other hand, the reader may benefit reading this paper consulting occasionally for more details also [7] and [11].
2. PRELIMINARIES AND MAIN RESULTS

First, we describe the setup. Denote by $M(0,t]$ the space of Borel measurable functions on $[0,t]$ with the uniform metric $d_{0}(v,\tilde{v}) = \sup_{0\leq s\leq t}\|v_{s} - \tilde{v}_{s}\|$. For each $t > 0$ let $F_{t}$ and $\Delta_{t}$ be nonnegative functions on $M(0,t]$ such that for some constant $\mathcal{L} \geq 1$ and for any $t \geq s \geq 0$ and $v, \tilde{v} \in M(0,t]$,

$$(2.1) \quad |F_{s}(v) - F_{s}(\tilde{v})| + |\Delta_{s}(v) - \Delta_{s}(\tilde{v})| \leq \mathcal{L}(s+1)d_{0}(v, \tilde{v}),$$

and

$$(2.2) \quad |F_{t}(v) - F_{t}(\tilde{v})| + |\Delta_{t}(v) - \Delta_{t}(\tilde{v})| \leq \mathcal{L}(t-s)(1 + \sup_{u\in[0,t]}|v_{u}|) + \sup_{u\in[0,t]}|v_{u} - \tilde{v}_{u}|).$$

By (2.1), $F_{0}(v) = F_{0}(\tilde{v})$ and $\Delta_{0}(v) = \Delta_{0}(\tilde{v})$ are functions of $v_{0}$ only and by (2.2),

$$(2.3) \quad F_{t}(v) + \Delta_{t}(v) \leq F_{0}(v_{0}) + \Delta_{0}(v_{0}) + \mathcal{L}(t)(1 + \sup_{0 \leq s \leq t}|v_{s}|).$$

Next we consider a complete probability space $(\Omega_{B}, \mathcal{F}_{B}, P^{B})$ together with a standard one-dimensional continuous in time Brownian motion $\{B_{t}\}_{t=0}^{\infty}$, and the filtration $\mathcal{F}_{B} = \sigma\{B_{s}|s \leq t\}$. A BS financial market consists of a savings account and a stock whose prices $b_{t}$ and $S_{t}^{B}$ at time $t$, respectively, are given by the formulas

$$(2.4) \quad b_{t} = b_{0}e^{rt} \quad \text{and} \quad S_{t}^{B} = S_{0}e^{rt + \kappa B_{t}}, \quad b_{0}, S_{0} > 0,$$

where

$$(2.5) \quad B_{t}^{*} = (\frac{\mu}{\kappa} - \frac{1}{2})t + B_{t}, \quad t \geq 0,$$

$r$ is the interest rate, $\kappa > 0$ is called volatility and $\mu$ is another parameter. Denote by $\bar{S}_{t}^{B} = e^{-rt}S_{t}^{B}$ the discounted stock price.

For any open interval $I = (L,R)$ such that $0 \leq L < S_{0} < R \leq \infty$ let

$$(2.6) \quad \tau_{I} = \inf\{t \geq 0|S_{t}^{B} \notin I\}$$

be the first time the stock price exit from the interval $I$. Clearly $\tau_{I}$ is a stopping time (not necessary finite since we allow the cases $L = 0$ and $R = \infty$). In this paper we assume that either $L > 0$ or $R < \infty$ while the case $L = 0$ and $R = \infty$ of regular options is treated in [11] and [17]. Consider a game option with the payoffs

$$(2.7) \quad Y_{I}^{t} = F_{t}(S_{t}^{B})1_{X_{t}^{I} < \tau_{I}} \quad \text{and} \quad X_{I}^{t} = G_{t}(S_{t}^{B})1_{X_{t}^{I} < \tau_{I}}, \quad t \geq 0,$$

where $G_{t} = F_{t} + \Delta_{t}$ with $F$ and $\Delta$ satisfying (2.1) and (2.2), $S_{t}^{B} = S_{t}^{B}(\omega) \in M(0,\infty)$ is a random function taking the value $S_{t}^{B} = S_{t}^{B}(\omega)$ at $t \in [0,\infty)$. When considering $F_{t}(S_{t}^{B}), G_{t}(S_{t}^{B})$ for $t < \infty$ we take the restriction of $S_{t}^{B}$ to the interval $[0,t]$. Denote by $T$ the horizon of our game option assuming that $T < \infty$. Observe that the contract is "knocked–out" (i.e. becomes worthless to the buyer) at the first time that the stock price exit from the interval $I$. The case of knock–in options will be considered in Section 5. The discounted payoff function is given by

$$(2.8) \quad Q^{B,I}(s,t) = X_{I}^{t}1_{s < t} + Y_{I}^{t}1_{s < \tau_{I}},$$

where $Y_{I}^{t} = e^{-rt}Y_{I}^{t}$ and $X_{I}^{t} = e^{-rt}X_{I}^{t}$ are the discounted payoffs.

Among examples of barrier options which fit our setup are put or call barrier options given by

$$(\Delta \equiv \delta, \quad F_{t}(v) = (K - v_{t})^{+} \quad \text{or} \quad F_{t}(v) = (v_{t} - K)^{+},$$

By (2.1), $F_{0}(v) = F_{0}(\tilde{v})$ and $\Delta_{0}(v) = \Delta_{0}(\tilde{v})$ are functions of $v_{0}$ only and by (2.2),
respectively, Russian type barrier options given by

\[ F_t(v) = \max(m, \sup_{u \in [0,t]} u_t) \quad \text{and} \quad \Delta_t(v) = \delta u_t, \]

and integral put or call barrier options given by

\[ \Delta_t(u) = \int_0^t \delta(u_u)du, \quad F_t(u) = (K - \int_0^t f_u(u_u)du)^+ \quad \text{or} \quad F_t(u) = (\int_0^t f_u(u_u)du - K)^+, \]

respectively, where we assume that for all \( x, y, u \geq 0,\)

\[ |f_u(x) - f_u(y)| + |\delta_u(x) - \delta_u(y)| \leq \mathcal{L}|x - y| \quad \text{and} \quad f_u(x) + \delta_u(x) \leq \mathcal{L}x \]

where \( \mathcal{L} \) is the same constant as in (2.1) and (2.2).

Denote by \( \mathcal{P}_B \) the unique martingale measure for the BS model. Using standard arguments it follows that the restriction of the probability measure \( \mathcal{P}_B \) to the \( \sigma \)-algebra \( \mathcal{F}_t^B \)

\[(2.9) \quad \mathcal{Z}_t = \frac{d\mathcal{P}_B}{d\mathcal{P}}|_{\mathcal{F}_t^B} = e^{\frac{1}{2} \sigma_t^2 + \frac{1}{2} \mathcal{B}_t^B}. \]

Denote by \( \mathcal{F}_t^B \) the set of all stopping times with respect to the Brownian filtration \( \mathcal{F}_t^B, t \geq 0 \) and let \( \mathcal{T}_{0t}^B \) be the set of all stopping times with values in \([0, T]\). From Theorem 3.1 in [10] we obtain the fair price of a game option in the BS model by

\[(2.10) \quad \gamma^I = \inf_{\sigma \in \mathcal{T}_{0t}^B} \sup_{\tau \in \mathcal{T}_{0t}^B} \mathcal{E}BQ^{\mathcal{J}, I}(\sigma, \tau) \]

where \( \mathcal{E}B \) is the expectation with respect to \( \mathcal{P}_B \).

Recall, (see, for instance, (21), Section 7.1) that a self financing strategy \( \pi \) with a (finite) horizon \( T \) and an initial capital \( x \) is a process \( \pi = \{ (\beta_t, \gamma_t) \}_{t=0}^T \) of pairs where \( \beta_t \) and \( \gamma_t \) are progressively measurable with respect to the filtration \( \mathcal{F}_t^B, t \geq 0 \) and satisfy

\[(2.11) \quad \int_0^T e^{\gamma_t} |\beta_t| dt < \infty \quad \text{and} \quad \int_0^T (\gamma_tS_t^B)^2 dt < \infty. \]

The portfolio value \( V^\pi_t \) for a strategy \( \pi \) at time \( t \in [0, T] \) is given by

\[(2.12) \quad V^\pi_t = \beta_t b_t + \gamma_t S_t^B = x + \int_0^t \beta_u db_u + \int_0^t \gamma_u dS_u^B. \]

Denote by \( \bar{V}^\pi_t = e^{-\eta} V^\pi_t \) the discounted portfolio value at time \( t \). Then it is easy to see that (see, for instance, (21)),

\[(2.13) \quad \bar{V}^\pi_t = x + \gamma_t dS^B_t \quad \text{and} \quad \beta_t = (x + \int_0^t \gamma_u dS^B_u - \gamma_t S^B_t)/b_0. \]

Observe that the discounted portfolio value depends only on the process \( \{ \gamma_t \}_{t=0}^T \). Thus in order to determine a self financing strategy it suffices to fix a process \( \{ \gamma_t \}_{t=0}^T \) and to obtain the process \( \{ \beta_t \}_{t=0}^T \) by (2.13). A self financing strategy \( \pi \) is called admissible if \( V^\pi_t \geq 0 \) for all \( t \in [0, T] \) and the set of such strategies with an initial capital \( x \) will be denoted by \( \mathcal{A}^B(x) \). Set also \( \mathcal{A}^B = \bigcup_{x \geq 0} \mathcal{A}^B(x) \). A pair \( (\pi, \sigma) \in \mathcal{A}^B \times \mathcal{T}_{0t}^B \) of an admissible self financing strategy \( \pi \) and of a stopping time \( \sigma \) will be called a hedge. For a hedge \( (\pi, \sigma) \) the shortfall risk is given by (see [6]),

\[(2.14) \quad R^I(\pi, \sigma) = \sup_{\tau \in \mathcal{T}_{0t}^B} \mathcal{E}B[\frac{1}{(Q^{\mathcal{J}, I}(\sigma, \tau) - \bar{V}^\pi_{\sigma \wedge \tau})^+}], \]
which is the maximal possible expectation with respect to the probability measure \( P^B \) of the discounted shortfall. The shortfall risks for a portfolio \( \pi \in \mathcal{A}^B \) and for an initial capital \( x \) are given by

\[
R^f(\pi) = \inf_{\sigma \in \mathcal{F}^B_0} R^f(\pi, \sigma) \quad \text{and} \quad R^f(x) = \inf_{\pi \in \mathcal{A}^B(x)} R^f(\pi),
\]

respectively.

As in [17] and [11] we consider a sequence of CRR markets on a complete probability space such that for each \( n = 1, 2, \ldots \) the bond prices \( b^{(n)}_t \) at time \( t \) are

\[
b^{(n)}_t = b_0 e^{r[n/T]T/n} = b_0 (1 + r_n)[n/T], \quad r_n = e^{r/T/n} - 1
\]

and stock prices \( S^{(n)}_t \) at time \( t \) are given by the formulas \( S^{(n)}_t = S_0 \) for \( t \in [0, T/n] \) and

\[
S^{(n)}_t = S_0 \exp\left( \sum_{k=1}^{[n/T]} \left( r T/k + \kappa (T/k)^{1/2} \xi_k \right) \right) = S_0 \prod_{k=1}^{[n/T]} (1 + \rho^{(n)}_k) \text{ if } t \geq T/n
\]

where \( \rho^{(n)}_k = \exp(\frac{r T}{n} + \kappa (T/k)^{1/2} \xi_k) - 1 \) and \( \xi_1, \xi_2, \ldots \) are i.i.d. random variables taking values 1 and -1 with probabilities \( p^{(n)} = (\exp(\kappa - \frac{2r}{k}) \sqrt{\frac{T}{n}} + 1)^{-1} \) and \( 1 - p^{(n)} = (\exp(\frac{2r}{k} - \kappa) \sqrt{\frac{T}{n}} + 1)^{-1} \), respectively. Let \( P^{\mathbb{F}}_n = \{ p^{(n)}, 1 - p^{(n)} \}^\infty \) be the corresponding product probability measure on the space of sequences \( \Omega_\mathbb{F} = \{-1, 1\}^\infty \). Namely, for each \( n \) we consider a CRR market with horizon \( n \) on the probability space \( (\Omega_\mathbb{F}, P^{\mathbb{F}}_n) \) with bond prices \( b_m = b^{(n)}_m \) and stock prices \( S_m = S^{(n)}_m \). We view \( S^{(n)}(\omega) \) as a random function on \([0, T]\), so that \( S^{(n)}(\omega) \in \mathcal{M}[0, T] \) takes the value \( S^{(n)}_t = S^{(n)}_t(\omega) \) at \( t \in [0, T] \). For \( k \leq n \) denote the discounted stock price at the moment \( kT/n \) by \( \tilde{S}^{(n)}_{kT/n} = (1 + r_n)^{-k} S^{(n)}_{kT/n} \). Let \( \mathcal{F}^{\mathbb{F}}_k = \sigma(\xi_1, \xi_2, \ldots) \) and \( \mathcal{F}^{\mathbb{F}} = \bigcup_{k=1}^{[n/T]} \mathcal{F}^{\mathbb{F}}_k \). Denote by \( \mathcal{F}^{\mathbb{F}}_k \) the set of all stopping times with respect to the filtration \( \mathcal{F}^{\mathbb{F}}_k \) and let \( \mathcal{T}_k \) be the set of all stopping times with values in \( \{0, 1, \ldots, n\} \). Similarly to \( \mathcal{F}^{\mathbb{F}}_k \), given an open interval \( I \) introduce a stopping time (with respect to the filtration \( \{ \mathcal{F}^{\mathbb{F}}_k \}_{k=0}^{\infty} \))

\[
\tau^{(n)}_I = \min\{ k \geq 0 | S^{(n)}_{kT/n} \notin I \}
\]

together with barrier options having the payoffs

\[
Y^{L,n}_k = F^{\mathbb{F}}_k(S^{(n)}) 1_{k < \tau^{(n)}_I} \quad \text{and} \quad X^{L,n}_k = G^{\mathbb{F}}_k(S^{(n)}) 1_{k < \tau^{(n)}_I}.
\]

The corresponding discounted payoff function is given by

\[
Q^{L,n}(s, k) = X^{L,n}_s 1_{s < k} + Y^{L,n}_s 1_{s \leq k}, \quad k, s \leq n
\]

where \( X^{L,n}_s = (1 + r_n)^{-k} X^{L,n}_s \) and \( Y^{L,n}_s = (1 + r_n)^{-k} Y^{L,n}_s \) are the discounted payoffs. Let \( \mathbb{P}^{\mathbb{F}}_n \) be a probability measure on the \( \Omega_\mathbb{F} \) such that \( \xi_1, \xi_2, \ldots \) is a sequence of i.i.d. random variables taking on the values 1 and -1 with probabilities \( \tilde{p}^{(n)} = (\exp(\kappa \sqrt{T/n} + 1)^{-1} \) and \( 1 - \tilde{p}^{(n)} = (\exp(-\kappa \sqrt{T/n} + 1)^{-1} \), respectively (with respect to \( \mathbb{P}^{\mathbb{F}}_n \)). Observe that for any \( n \) the process \( \{ S^{(n)}_{mT/n} \}_{m=0}^{\infty} \) is a martingale with respect to \( \mathbb{P}^{\mathbb{F}}_n \), and so we conclude that \( \mathbb{P}^{\mathbb{F}}_n \) is the
unique martingale measure for the above CRR markets. Thus from Theorem 2.1 in [10] it follows that the fair price of the game option in the \( n \)-step CRR market is given by

\[
(2.21) \quad \gamma^I_n = \min_{\xi \in \mathcal{P}_n^a} \max_{\eta \in \mathcal{P}_n^b} E^\xi_n Q^{\pi} (\xi, \eta),
\]

where \( E^\xi_n \) is the expectation with respect to \( \mathcal{P}_n^\xi \). The following theorem provides an estimate for the error term in approximations of the fair price of a knock-out game option in the BS model by fair prices of the sequence of knock out game options in the CRR markets defined above. This result is a generalization of Theorem 2.1 in [11] which deals with regular game options.

**Theorem 2.1.** There exists a constant \( C_1 \) such that for any open interval \( I \) and \( n \in \mathbb{N} \),

\[
(2.22) \quad | \gamma^I_n - \gamma^I_n | \leq C_1 n^{-\frac{1}{2}} (\ln n)^{\frac{1}{2}}.
\]

Denote by \( \mathcal{A}^{\xi,n}(x) \) the set of all admissible self financing strategies with an initial capital \( x \) and set \( \mathcal{A}^{\xi,n} = \cup_{\geq 0} \mathcal{A}^{\xi,n}(x) \). Recall (see (2.22)) that a self financing strategy \( \pi \) with an initial capital \( x \) and a horizon \( n \) is a sequence \( (\pi_1, \ldots, \pi_n) \) of pairs \( \pi_k = (b_k, \gamma_k) \) where \( b_k, \gamma_k \) are \( \mathcal{F}^\xi_{k-1} \)-measurable random variables representing the number of bond and stock units, respectively, at time \( k \). Thus the portfolio value \( V_k^\pi, k = 0, 1, \ldots, n \) is given by

\[
(2.23) \quad V_0^\pi = x, V_k^\pi = \beta_k b_{k+1}^\pi + \gamma_k S_{k+1}^\pi, \quad 1 \leq k \leq n.
\]

Denote by \( \tilde{V}_k^\pi = (1 + r_n)^{-k} V_k^\pi \) the discounted portfolio value at time \( k \). Since \( \pi \) is self financing then

\[
(2.24) \quad \beta_k b_{k+1}^\pi + \gamma_k S_{k+1}^\pi = \beta_{k+1} b_k^\pi + \gamma_{k+1} S_k^\pi,
\]

and so (see (2.21) and (2.22)),

\[
(2.25) \quad \tilde{V}_k^\pi = x + \sum_{i=0}^{k-1} \gamma_{i+1} (S_i^\pi - S_{i+1}^\pi) \text{ and } \beta_k = \frac{x + \sum_{i=0}^{k-1} \gamma_{i+1} (S_i^\pi - S_{i+1}^\pi)}{\tilde{V}_k^\pi / \gamma_{k+1}}.
\]

Hence, as before, in order to determine a self financing strategy it suffices to introduce a process \( \{ \gamma_k \}_{k=0}^n \) and to obtain the process \( \{ \beta_k \}_{k=0}^n \) by (2.25). We call a self financing strategy \( \pi \) admissible if \( V_k^\pi \geq 0 \) for any \( k \leq n \). A hedge with an initial capital \( x \) is an element in the set \( \mathcal{A}^{\xi,n}(x) \times \mathcal{F}^\xi_{0n} \). The definitions for the shortfall risks in the CRR markets are similar to the definitions in the BS model. Thus for the \( n \)-step CRR market the shortfall risks are given by

\[
(2.26) \quad R_n^I (\pi, \sigma) = \max_{(\tau \in \mathcal{A}^\pi_{0n})} E^\pi_n (Q^{\pi \sigma}(\tau, \tau) - \tilde{V}_\tau^\pi)^+,
\]

\[
R_n^I (\pi) = \min_{\sigma \in \mathcal{F}^\pi_{0n}} R_n^I (\pi, \sigma) \quad \text{and} \quad R_n^I (x) = \inf_{\pi \in \mathcal{A}^{x,n}(x)} R_n^I (\pi),
\]

where \( E^\pi_n \) is the expectation with respect to \( \mathcal{P}_n^\pi \).

**Theorem 2.2.** For any open interval \( I \)

\[
(2.27) \quad \lim_{n \to \infty} R_n^I (x) = R^I (x).
\]

Furthermore, there exists a constant \( C_2 \) (which does not depend on the interval \( I \)) such that for any \( n \in \mathbb{N} \)

\[
(2.28) \quad R^I (x) \leq R_n^I (x) + C_2 n^{-\frac{1}{2}} (\ln n)^{\frac{1}{3}}.
\]
The above result says that the shortfall risk $R^0(x)$ for double barrier options in the BS model can be approximated by a sequence of shortfall risks with an initial capital $x$ for a similar options in the CRR markets and it provides also a one sided error estimate of the approximation. This result is a generalization of Theorem 2.1 in [7] which deals with regular game options.

In order to compare the option prices and the shortfall risks in the BS model with the corresponding quantities in the CRR markets, we will use a trivial form of the Skorohod type embedding (see [11]) which allows us to consider the above objects on the same probability space. Thus, define recursively

$$\theta_0^{(n)} = 0, \quad \theta_k^{(n)}(\omega) = \inf \{ t > \theta_{k-1}^{(n)}(\omega) : |B_t - B_{\theta_k^{(n)}}| = \sqrt{T/n} \},$$

where, recall, $B_t^{*} = (\frac{B_t}{\sqrt{T}} - \xi_{\frac{T}{2}})B_t + B_t$. Using the same arguments as in [11] we obtain that for each of the measures $\mathbb{P}^B, \mathbb{P}^B$, the sequence $\theta_k^{(n)} - \theta_k^{(n)}$, $k = 1, 2, \ldots$ is a sequence of i.i.d. random variables such that $(\theta_k^{(n)} - \theta_k^{(n)}, B_t^{*} - B_{\theta_k^{(n)}}^{*})$ are independent of $\mathcal{F}_{\theta_k^{(n)}}^B$.

Employing the exponential martingale $\exp((\kappa - \frac{2\mu}{\kappa})B_t^*)$ for the probability $\mathbb{P}^B$ we obtain that $E^B\exp((\kappa - \frac{2\mu}{\kappa})B_T^*) = 1$ concluding that $B_{\theta_k^{(n)}}^* = \sqrt{T/n}$ or $-\sqrt{T/n}$ with probability $\rho^{(n)}$ or $1 - \rho^{(n)}$, respectively. Using the martingale $\tilde{S}_T = S_0\exp(\kappa B_T^*)$ for the probability $\tilde{P}^B$ we obtain $E^B \exp(\kappa B_{\theta_k^{(n)}}^*) = 1$, and so $B_{\theta_k^{(n)}}^* = \sqrt{T/n}$ or $-\sqrt{T/n}$ with probability $\tilde{\rho}^{(n)}$ or $1 - \tilde{\rho}^{(n)}$ respectively.

The Skorohod embedding also allows us to define mappings (introduced in [7] and [11]) which map hedges in CRR markets to hedges in the BS model and which will play a decisive role in Theorems 2.3 and 2.4 below. For readers convenience we review the definitions. For any $n \in \mathbb{N}$ set $b_i^{(n)} = B_i - B_{\theta_i^{(n)}}^{*}, i = 1, 2, \ldots$ and following [11] introduce for each $k = 1, 2, \ldots$ the finite $\sigma$–algebra $\mathcal{G}_k^{B,n} = \sigma \{b_1^{(n)}, \ldots, b_k^{(n)} \}$ with $\mathcal{G}_0^{B,n} = \{\emptyset, \Omega_B \}$. Let $\mathcal{G}_0^{B,n}$ be the set of all stopping times with respect to the filtration $\mathcal{G}_k^{B,n}, k = 0, 1, 2, \ldots$ with values in $\{0, 1, \ldots, n \}$. Observe that for any $n$ we have a natural bijection $\Pi_n : L^\infty(\mathcal{F}_n^B, \mathbb{P}_n^B) \rightarrow L^\infty(\mathcal{G}_n^{B,n}, \mathbb{P}_n^B)$ which is given by $\Pi_n(Z) = Z$ so that if $Z = f(\xi_1, \ldots, \xi_n)$ for a function $f$ on $\{-1, 1\}^n$ then $\tilde{Z} = f(\sqrt{T/n} \xi_1, \ldots, \sqrt{T/n} \xi_n)$. Notice that if we restrict $\Pi_n$ to $L^\infty(\mathcal{F}_k^B, \mathbb{P}_n^B)$ we obtain a bijection $\Pi_{n,k} : L^\infty(\mathcal{F}_k^B, \mathbb{P}_n^B) \rightarrow L^\infty(\mathcal{G}_k^{B,n}, \mathbb{P}_n^B)$ and if we restrict $\Pi_n$ to $\mathcal{G}_n^{B,n}$ we get a bijection $\Pi_n : \mathcal{G}_n^{B,n} \rightarrow \mathcal{G}_n^{B,n}$. In addition to the set $\mathcal{G}_0^{B,n}$ consider also the set $\mathcal{G}_n^{B,n}$ of stopping times with respect to the filtration $\{\mathcal{G}_k^{B,n}\}_{k=0}^n$ with values in $\{0, 1, \ldots, n \}$. Clearly $\mathcal{G}_n^{B,n} \subset \mathcal{G}_n^{B,n}$. Next, we define a function $\phi_n : \mathcal{G}_0^{B,n} \rightarrow \mathcal{G}_0^{B,n}$ which maps stopping times in CRR markets to stopping times in the BS model by

$$\phi_n(\sigma) = T \wedge \theta_n^{(n)} \text{ if } \Pi_n(\sigma) < n \text{ and } \phi_n(\sigma) = T \text{ if } \Pi_n(\sigma) = n.$$  

It is easy to see that $\phi_n(\sigma) \in \mathcal{G}_0^{B,n}$ (see (2.28) in [7]). For each $n$ and $x > 0$ let $\mathcal{A}^{B,n}(x)$ be the set of all admissible self financing strategies with an initial capital $x$ in the BS model which can be managed only on the set $\{0, \theta_1^{(n)}, \ldots, \theta_n^{(n)} \}$, such that the discounted portfolio value remains constant after the moment $\theta_n^{(n)}$ and set $\mathcal{A}^{B,n} = \bigcup_{x \in 0} \mathcal{A}^{B,n}(x)$. Thus if $\pi = \{ (\beta_t, \gamma_t) \}_{t=0}^{\infty} \in \mathcal{A}^{B,n}$ then $\beta_t = \beta_{\theta_k^{(n)}}$ and $\gamma_t = \gamma_{\theta_k^{(n)}}$ for any $k < n$ and $t \in [\theta_k^{(n)}, \theta_{k+1}^{(n)}]$. 

(2.29) 

$$\phi_n(\sigma) = T \wedge \theta_n^{(n)} \text{ if } \Pi_n(\sigma) < n \text{ and } \phi_n(\sigma) = T \text{ if } \Pi_n(\sigma) = n.$$
Furthermore, in order to keep the discounted portfolio constant after $\theta^*_n$, the investor should sell all his stocks at the moment $\theta^*_n$ and buy bonds for all money, and so $\gamma = 0$ for $t \geq \theta^*_n$. From (2.13) it follows that for $\pi = ((\beta_k, \gamma_k))_{k=0}^\infty \in \mathcal{A}^{B,n}$ the corresponding discounted portfolio value is given by

$$V_t^\pi = V_t^{\pi | \theta_k} + \gamma_k (S^B_k - S^B_l^\theta_k), \quad t \in [\theta_k, \theta_{k+1}], \quad t > \theta^*_n.$$  

Finally, we define a function $\psi_n : \mathcal{A}^{\xi,n}(x) \rightarrow \mathcal{A}^{B,n}(x)$ which maps admissible self financing strategies in the CRR $n$-step model to the set of the above self financing strategies in the BS model. For $\pi = ((\beta_k, \gamma_k))_{k=0}^n \in \mathcal{A}^{\xi,n}(x)$ define $\psi_n(\pi) \in \mathcal{A}^{B,n}(x)$ by

$$V_t^{\psi_n(\pi)} = V_t^{\psi_n(\pi) | \theta_k} + \Pi_n(\gamma_k+1) (S^B_k - S^B_l^\theta_k), \quad t \in [\theta_k, \theta_{k+1}],$$

and

$$V_t^{\psi_n(\pi)} = V_t^{\psi_n(\pi) | \theta_k} + \Pi_n(\gamma_k+1) (S^B_k - S^B_l^\theta_k), \quad t = \theta^*_n.$$  

Observe that $\Pi_n(\xi^{(n)}_n) = S^B_k$ for any $k \leq n$, and so we obtain from (2.25) and (2.30) that

$$V_t^{\psi_n(\pi) | \theta_k} \geq 0 \text{ for any } k \leq n.$$  

Since the process $V_t^{\psi_n(\pi)}$, $t \geq 0$ is a martingale with respect to the martingale measure $\tilde{P}$ and it remains constant for $t \geq \theta^*_n$, we get that the portfolio $\psi_n(\pi)$ is admissible concluding that $\psi_n(\pi) \in \mathcal{A}^{B,n}(x)$, as required. Clearly, if we restrict the portfolio $\psi_n(\pi)$ to the interval $[0, T]$ we can consider $\psi_n(\pi)$ as an element in $\mathcal{A}^B(x)$.

Let $I = (L, R)$ be an open interval and set $L_n = L \exp(-n^{-\frac{1}{2}})$, $R_n = R \exp(n^{-\frac{1}{2}})$ (with $R_n = \infty$ if $R = \infty$) and $I_n = (L_n, R_n)$. Let $(\pi, \sigma) \in \mathcal{A}^{\xi,n}(\gamma_n) \times \mathcal{T}^\xi_{\infty}$ be a perfect hedge for a double barrier option in the $n$-step CRR market with the barriers $L_n, R_n$, i.e. a hedge which satisfies $V_t^{\pi, \sigma} \geq Q^{\xi,n}(\sigma, k)$ for any $k \leq n$. In general the construction of perfect hedges for game options in CRR markets can be done explicitly (see [10], Theorem 2.1). The following result shows that if we embed the perfect hedge $(\pi, \sigma)$ into the BS model we obtain a hedge with small shortfall risk for the barrier option with barriers $L, R$.

**Theorem 2.3.** Let $I = (L, R)$ be an open interval. For any $n$ let $(\pi_n^b, \sigma_n^b) \in \mathcal{A}^{\xi,n}(\gamma_n) \times \mathcal{T}^\xi_{\infty}$ be a perfect hedge for a double barrier option in the $n$-step CRR market with the barriers $L_n, R_n$. Define $(\pi_n^b, \sigma_n^b) \in \mathcal{A}^{\xi,n}(\gamma_n) \times \mathcal{T}^\xi_{\infty}$ by $\pi_n^b = \psi_n(\pi_n^b)$ and $\sigma_n^b = \phi_n(\sigma_n^b)$. There exists a constant $C_3$ (which does not depend on the interval $I$) such that for any $n$,

$$R^i(\pi_n^b, \sigma_n^b) \leq C_3 n^{-\frac{1}{2}} (\ln n)^{\frac{1}{2}}.$$  

We will see (as a conclusion of (3.12) and Theorem 2.1) that there exists a constant $\tilde{C}$ (which does not depend on the interval $I$) such that $|\gamma^I - \gamma^I_n| \leq \tilde{C} n^{-\frac{1}{2}} (\ln n)^{\frac{1}{2}}$ for any $n$. Since the above term is small then in practice a seller of a double barrier game option with the barriers $L, R$ can invest the amount $\gamma^I_n$ in the portfolio and use the above hedges facing only small shortfall risk.

Next, consider an investor in the BS market whose initial capital $x$ which is less than the option price $\gamma^I_n$. A hedge $(\pi, \sigma) \in \mathcal{A}^{B}(x) \times \mathcal{T}^B_{\infty}$ will be called $\varepsilon$-optimal if $R^i(\pi, \sigma) \leq R^i(x) + \varepsilon$. For $\varepsilon = 0$ the above hedge is called an optimal hedge. For the CRR markets we have analogous definitions. In the next section we will follow [6] and construct optimal hedges $(\pi_n, \sigma_n) \in \mathcal{A}^{\xi,n}(x) \times \mathcal{T}^\xi_{\infty}$ for double barrier options in the $n$-step CRR markets with barriers $L_n, R_n$. By embedding this hedges into the BS model we obtain a simple representation of $\varepsilon$-optimal hedges for the BS model.
Theorem 2.4. For any $n$ let $(\pi_n, \sigma_n) \in \mathcal{A}^{\xi, n}(\pi) \times \mathcal{F}^{\xi}_0$ be the optimal hedge which is given by (2.25) with $H = I_{\pi}$. Then

$$\lim_{n \to \infty} R^I(\psi_{\pi_n}(\pi_n), \phi_{\pi_n}(\sigma_n)) = R^I(\pi).$$

In Section 3 we formulate and prove corresponding results for knock–in Israely style barrier options.

3. Auxiliary Lemmas

First we introduce the machinery which enables us to reduce optimization of the shortfall risk to optimal stopping problems for Dynkin’s games with appropriately chosen payoff processes so that on the next stage we will be able to employ the Skorohod embedding in order to compare values of the corresponding discrete and continuous time games. This machinery was used in [7] for similar purposes in the case of regular game options. For any $n$ set $a_1^{(n)} = e^{\kappa \sqrt{T/n}} - 1$, $a_2^{(n)} = e^{-\kappa \sqrt{T/n}} - 1$ and observe that for any $m \leq n$ the random variable

$$\frac{S^{(n)}_m}{(m-1)T/n} - 1 = \exp(\kappa(T/n)^{1/2} \xi_m) - 1$$

takes on only the values $a_1^{(n)}, a_2^{(n)}$. For each $y > 0$ and $n \in \mathbb{N}$ introduce the closed interval $K_n(y) = \left[ -\frac{y}{\alpha_1}, \frac{y}{\alpha_2} \right]$ and for $0 \leq k < n$ and a given positive $\mathcal{F}_{k}^{\xi}$-measurable random variable $X$ define

$$\mathcal{A}^{\xi, n}_k(X) = \{ Y | Y = X + \alpha(\exp(\kappa(T/n)^{1/2} \xi_{k+1}) - 1) \}$$

for some $\mathcal{F}_{k}^{\xi}$-measurable $\alpha \in K_n(X)$.

Notice that if for $\pi = \{ (\beta_k, \gamma_k) \}_{k=1}^{n}$, $V^{\pi}_k = X$ and $V_{k+1}^{\pi} = Y$ then by (2.25), $Y = X + \alpha(\exp(\kappa(T/n)^{1/2} \xi_{k+1}) - 1)$ where $\alpha = \gamma_{k+1}S^{(n)}_{k+1}$ is $\mathcal{F}_{k}^{\xi}$-measurable. Since we allow only nonnegative portfolio values, and so $Y \geq 0$ which must be satisfied for all possible values of $\exp(\kappa(T/n)^{1/2} \xi_{k+1}) - 1$ we conclude in view of independency of $\alpha$ and $\xi_{k+1}$ that $\mathcal{A}^{\xi, n}_k(X)$ is the set of all possible discounted portfolio values at the time $k+1$ provided that the discounted portfolio value at the time $k$ is $X$.

Let $H$ be an open interval. For any $\pi \in \mathcal{A}^{\xi, n}$ define a sequence of random variables $\{ W^H_{k, \pi} \}_{k=0}^{n}$

\[
W^H_{n, \pi} = (\tilde{V}^H_{n, \pi} - \tilde{V}^\pi_{n, \pi})^+, \quad W^H_{k, \pi} = \min \left( (\tilde{X}^H_{k+1} - \tilde{V}^\pi_{k+1})^+, \max \left( (\tilde{X}^H_{k+1} - \tilde{V}^\pi_{k+1})^+, E^\pi_{k+1}(W^H_{k+1} | \mathcal{F}^{\xi}_k) \right) \right), \quad k < n.
\]

Applying the results for Dynkin’s games from [18] for the processes

$$\{ (\tilde{X}^H_{k} - \tilde{V}^\pi_{k})^+ \}_{k=0}^{n}, \{ (\tilde{X}^H_{k} - \tilde{V}^\pi_{k})^+ \}_{k=0}^{n}$$

we obtain

\[
W^H_{0, \pi} = \min_{\sigma \in \mathcal{A}^{\xi, n}_0, \tau \in \mathcal{F}^{\xi}_0} \max_{\sigma \in \mathcal{A}^{\xi, n}_0, \tau \in \mathcal{F}^{\xi}_0} E^\pi_{n}(Q^H_{n}(\sigma, \tau) - \tilde{V}^\pi_{\sigma \land \tau})^+ = R^H_{n}(\pi) = R^H_{n}(\pi, \sigma(H, \pi))
\]

where

\[
\sigma(H, \pi) = \min \{ k | (\tilde{X}^H_{k} - \tilde{V}^\pi_{k})^+ = W^H_{k, \pi} \} \land n.
\]
On the Brownian probability space set

\[(3.5)\quad S_t^{B,n} = S_0, \quad t \in [0,T/n] \quad \text{and} \quad S_t^{B,n} = S_0 \exp\left( \sum_{k=1}^{\lfloor nT \rfloor} \left( \frac{\Theta^k}{n} + \kappa b_k^{(n)} \right) \right), \quad t \in [T/n,T].\]

Define

\[(3.6)\quad \tau_H^{B,n} = \min\{k \geq 0|S_k^{B,n} \notin H\}.\]

Clearly \(\tau_H^{B,n}\) is a stopping time with respect to the filtration \(\mathscr{F}_k^{B,n}\), \(k \geq 0\). Consider the new payoffs \(Y_k^{B.H,n} = \mathbb{E}(S_{\tau_H^{B,n}}^{B,n})_{k < \tau_H^{B,n}}\) and \(X_k^{B.H,n} = \mathbb{E}(S_{\tau_H^{B,n}}^{B,n})_{k < \tau_H^{B,n}}\), \(k \leq n\). The corresponding payoff function is given by

\[(3.7)\quad Q_{k,l}^{B.H,n}(k,l) = X_k^{B.H,n} + \bar{Y}_l^{B.H,n}, \quad k,l \leq n\]

where \(\bar{Y}_k^{B.H,n} = (1+r_n)^{-k}Y_k^{B.H,n}\) and \(\bar{X}_k^{B.H,n} = (1+r_n)^{-k}X_k^{B.H,n}\) are the discounted payoffs. For any \(n\) we consider now hedges which are elements in \(\mathscr{A}_k^{B,n} \times \mathscr{B}_k^{B,n}\). Given a positive \(\mathscr{F}_k^{B,n}\)-measurable random variable \(X\) define \(\mathscr{A}_k^{B,n}(X)\) by \(3.1\) with \(\sqrt{\frac{T}{n}}\mathbb{E}_k^{B,n}\) and \(\mathscr{F}_k^{B,n}\) replaced by \(b_k^{(n)}\) and \(\mathscr{F}_k^{B,n}\), respectively. By \(2.3\) we conclude similarly to the above that \(\mathscr{A}_k^{B,n}(X)\) consists of all possible discounted values at the time \(\theta_k^{(n)}\) of portfolios managed only at embedding times \(\{\theta_k^{(n)}\}\) with the discounted stock evolution \(\tilde{S}_t^{B}\), provided the discounted portfolio value at the time \(\theta_k^{(n)}\) is \(X\).

Next, define the shortfall risk by

\[(3.8)\quad R_{k,n}^{H} = \sup_{\eta \in \mathcal{B}_k^{B,n}} \mathbb{E}B(Q_{k,n}^{B.H,n}(\zeta,\eta) - \bar{V}_k^{\pi \eta})^+,\]

\[(3.9)\quad R_{k,n}^{H}(\pi) = \inf_{\zeta \in \mathcal{F}_k^{B,n}} R_{k,n}^{H}(\pi,\zeta) \quad \text{and} \quad R_{n,n}^{H}(\pi) = \inf_{\eta \in \mathcal{F}_n^{B,n}(\pi)} R_{n,n}^{B.H}(\pi).\]

For any \(\pi \in \mathcal{A}_k^{B,n}\) define a sequence of random variables \(\{U_k^{H,\pi}\}_{k=0}\).

\[(3.10)\quad U_k^{H,\pi} = \min\{k|\langle \bar{X}_k^{B.H,n} - \bar{V}_k^{\pi \eta} \rangle^+ = U_k^{H,\pi}\} \wedge n.\]

Again, using the results on Dynkin’s games from [18] for the adapted (with respect to the filtration \(\mathcal{F}_k^{B,n}\), \(k \geq 0\)) payoff processes \(\{\langle \bar{X}_k^{B.H,n} - \bar{V}_k^{\pi \eta} \rangle^+\}_{k=0}^n\), \(\{\langle \bar{X}_k^{B.H,n} - \bar{V}_k^{\pi \eta} \rangle^+\}_{k=0}^n\) we obtain that

\[(3.11)\quad U_0^{H,\pi} = \inf_{\zeta \in \mathcal{F}_0^{B,n}} \sup_{\eta \in \mathcal{B}_0^{B,n}} \mathbb{E}B(Q_{0,n}^{B.H,n}(\zeta,\eta) - \bar{V}_0^{\pi \eta})^+ = R_{n,n}^{B.H}(\pi).\]

For \(k \leq n\) and \(x_1,\ldots,x_k \in \mathbb{R}\), consider the function \(\psi^{x_1,\ldots,x_k} \in \mathcal{M}(0,L_T/\pi)\) given by

\[\psi^{x_1,\ldots,x_k}(t) = S_0 \exp\left( \frac{\Theta^k}{n} + \kappa \sum_{i=1}^j x_i \right), \quad t \in [jt/n, (j+1)T/n], \quad 1 \leq j \leq k\]

and \(\psi^{x_1,\ldots,x_k}(0) = S_0, \quad t \in [0, T/n],\)
there exist \( f^n_k, g^n_k : \mathbb{R}^k \to \mathbb{R} \) such that for any \( x_1, \ldots, x_k \in \mathbb{R} \),
\[
 f^n_k(x_1, \ldots, x_k) = (1 + \rho_k^n)^{-1} F_{\frac{n}{k}}(\psi^{x_1} \cdots \psi^{x_k}) = e^{-rKT/n} F_{\frac{n}{k}}(\psi^{x_1} \cdots \psi^{x_k}),
\]
and \( g^n_k(x_1, \ldots, x_k) = (1 + \rho_k^n)^{-1} G_{\frac{n}{k}}(\psi^{x_1} \cdots \psi^{x_k}) = e^{-rKT/n} G_{\frac{n}{k}}(\psi^{x_1} \cdots \psi^{x_k}) \).

Set
\[
 q^n_{k,H}(x_1, \ldots, x_k) = \frac{\min_{0 \leq s \leq k} \psi^{x_1+\cdots+x_{k-s}}} {\max_{0 \leq s \leq k} \psi^{x_1+\cdots+x_{k-s}}} |H| < H.
\]

Observe that for the above functions,
\[
(3.12) \quad V^{B,H,n}_{k}(b^{(1)}_1, \ldots, b^{(n)}_k) = H^n_k(b^{(1)}_1, \ldots, b^{(n)}_k),
\]
\[
X^{B,H,n}_{k}(b^{(1)}_1, \ldots, b^{(n)}_k) = \left( \begin{array}{c} \frac{1}{n} \sum_{1}^{k} b_i^{(1)} \end{array} \right), \ldots, \left( \begin{array}{c} \frac{1}{n} \sum_{1}^{k} b_i^{(n)} \end{array} \right)
\]
\[
\text{and } X^{B,H,n}_{k}(b^{(1)}_1, \ldots, b^{(n)}_k) = Y^{B,H,n}_{k}(b^{(1)}_1, \ldots, b^{(n)}_k).
\]

Finally, define a sequence \( \{J^n_k\}_{k=0}^{\infty} \) of functions \( J^n_k : [0, \infty) \times \mathbb{R}^k \to \mathbb{R} \) by the following backward recursion
\[
(3.13) \quad J^n_k(y, u_1, u_2, \ldots, u_n) = \min \left( \left( f^n_k(u_1, \ldots, u_k)q^n_{k,H}(u_1, \ldots, u_k) - y \right)^+, \max \left( \left( f^n_k(u_1, \ldots, u_k) \right)^+ \right) \right)
\]
\[
\times q^n_{k,H}(u_1, \ldots, u_k) - y \right)^+, \max \left( \left( f^n_k(u_1, \ldots, u_k) \right)^+ \right) + \min_{w \in K_{\beta}} \|f(w, \beta)\| \left( 1 - p^n_k \right) J^n_{k+1}(y + u_{k+1}^{(n)}, u_1, \ldots, u_k, \sqrt{T/n}) \right)
\]
\[
\text{for } k = n - 1, n - 2, \ldots, 0.
\]

Similarly to [7] this dynamical programming relations will enable us to compute shortfall risks defined in (2.28) and (3.8).

**Lemma 3.1.** The function \( J^n_k(y, u_1, \ldots, u_k) \) is continuous and decreasing with respect to \( y \) for any \( n, k \leq n \) and an open interval \( H \).

**Proof.** The proof is the same as the proof of Lemma 3.2 in [7], just replace \( J^n_k \) by \( J^n_k \). \( \square \)

For a given closed interval \( K = [a, b] \) and a function \( f : K \times \mathbb{R}^k \to \mathbb{R} \) such that \( f(\cdot, v) \) is continuous for all \( v \in \mathbb{R}^k \) define \( \arg min_{w \in K} f(u, v) = \min \{ w \in K | f(w, v) = \min_{\beta \in K} f(\beta, v) \} \).

**Lemma 3.1** enables us to define the following functions
\[
(3.14) \quad h^n_k(y, x_1, \ldots, x_k) = \arg \min_{w \in K_{\beta}} \left( p^n_k \right) J^n_k(y + u_{k+1}^{(n)}, x_1, \ldots, x_k, \sqrt{T/n}), k < n.
\]

Let \( x \) be an initial capital. For any \( n \) and an open interval \( H \) there exists a hedge \( (\pi^n_H, \sigma^n_H) \in \mathcal{A}^n_{\mathcal{P}^n}(x) \times \mathcal{B}^n_{\mathcal{Q}^n} \) such that
\[
(3.15) \quad \Pi^n_{0,H} = x \text{ and } \Pi^n_{k+1} = \Pi^n_k + h^n_k(V^n_k, \sigma^n_k, e^{K\sqrt{T/n}}, \ldots, e^{K\sqrt{T/n}}).
\]
\[
\text{for } k > 0 \text{ and } \sigma^n_H = \sigma(H, \pi^n_H).
\]

From the arguments concerning \( \mathcal{A}^n_{\mathcal{P}^n}(X) \) at the beginning of this section it follows that \( \pi^n_H \) is an admissible strategy. Let \( \{\pi^n_{B,H}, \sigma^n_{B,H}\} \in \mathcal{B}^n_{\mathcal{R}^n} \times \mathcal{B}^n_{\mathcal{Q}^n} \) be a hedge which is given by
\( \pi_n^{B,H} = \psi_n(\pi_n^{H}) \) and \( \zeta_n^{H} = \Pi_n(\sigma_n^{H}) \) where, recall, the maps \( \psi_n, \Pi_n \) were defined in Section 2. Namely, we consider a hedge which is determined by

\begin{equation}
V_0^{\pi_n^{B,H}} = x \quad \text{and} \quad V_k^{\pi_n^{B,H}} = V_{k-1}^{\pi_n^{B,H}} + R_n^{H,n}(\pi_n^{B,H}, \zeta_n^{H}) \times (e^{(k+1)/2} - 1) \quad \text{for} \quad k > 0 \quad \text{and} \quad \zeta_n^{H} = \zeta(H, \pi_n^{B,H}).
\end{equation}

The following lemma enables us to consider all relevant processes on the Brownian probability space and to deal with stopping times with respect to the same filtration.

**Lemma 3.2.** For any initial capital \( x, n \in \mathbb{N} \) and an open interval \( H \),

\begin{equation}
R_n^{H}(x) = R_n^{H}(\pi_n^{H}, \sigma_n^{H}) = J_0^{H,n}(x) = R_n^{B,H}(\pi_n^{B,H}, \zeta_n^{H}) = R_n^{B,H}(x).
\end{equation}

**Proof.** The proof is the same as in Lemma 3.3 of \([7]\), just replace \( J_k^{H,n}, R_n^{H,n}, (\pi_n, \sigma_n) \) and \( (\pi_n^{B,H}, \zeta_n^{H}) \) by \( J_k^{H,n}, R_n^{H,n}, (\pi_n^{B,H}, \sigma_n^{H}) \) and \( (\pi_n^{B,H}, \zeta_n^{H}) \), respectively. \( \square \)

Observe that if the initial capital \( x \) is no less than \( \gamma^{H} \) then the hedge which is given by \([3.15]\) satisfy \( R_n^{H}(\pi_n^{H}, \sigma_n^{H}) = R_n^{H}(x) = 0 \). Namely, \( (\pi_n^{H}, \sigma_n^{H}) \) is a perfect hedge for a game option with the payoffs \( Y_k^{H,n}X_k^{H,n} \), \( k \geq 0 \). Thus, the dynamical algorithm which is given by \([3.13]\) provides a way to find a perfect hedge (when the initial capital is no less than the option price) for CRR markets. Of course, in general a perfect hedge should not be unique taking different versions of the term \( \text{argmin} \) which was defined before \([3.14]\) we will obtain other perfect hedges. However, a more efficient way to find a perfect hedge is via the Doob decomposition exactly as in Theorem 2.1 of \([10]\).

Next we deal with estimates for the BS model. Let \( H = (L, R) \) be an open interval. For any \( \varepsilon > 0 \) set \( H_\varepsilon = (L e^{-\varepsilon}, R e^\varepsilon) \). Clearly, \( \gamma^{H_\varepsilon} \geq \gamma^{H} \) for any \( \varepsilon > 0 \) and \( R^{H_\varepsilon}(x) \geq R^{H}(x) \) for any initial capital \( x \). The following result provides an estimate from above of the term \( R^{H_\varepsilon}(x) - R^{H}(x) \).

**Lemma 3.3.** There exists a constant \( A_1 \) such that for any initial capital \( x, \varepsilon > 0 \) and an open interval \( H \),

\begin{equation}
R^{H_\varepsilon}(x) - R^{H}(x) \leq A_1 \varepsilon^{3/4}.
\end{equation}

**Proof.** Before proving the lemma observe that if \( P = \hat{P} \) then the option price can be represented as the shortfall risk for an initial capital \( x = 0 \), i.e. if \( \mu = 0 \) then \( \gamma^{I} = R^I(0) \) for any open interval \( I \). Hence, by \([3.18]\) there exists a constant \( A_2 \) (which is equal to \( A_1 \) for the case \( \mu = 0 \)) such that for any open interval \( H \) and \( \varepsilon > 0 \),

\begin{equation}
\gamma^{H_\varepsilon} - \gamma^{H}(x) \leq A_2 \varepsilon^{3/4}.
\end{equation}

Next we turn to the proof of the lemma. Choose an initial capital \( x \), an open interval \( H = (L, R) \), some \( \varepsilon > 0 \) and fix \( \delta > 0 \). There exists a \( \pi_1 \in \omega^B(x) \) such that \( R^H(\pi_1) < R^H(x) + \delta \).

According to \([2.13]\) the discounted portfolio process \( \{V_t^{\pi_1}\}_{t=0}^T \) is given by a stochastic integral whose integrand in view of \([2.11]\) satisfies the standard conditions assumed in the construction of stochastic integrals, and so \( \{V_t^{\pi_1}\}_{t=0}^T \) has a continuous modification (see, for instance, Ch.2 in \([10]\) or Ch.4 in \([15]\) which we take as the portfolio process. Observe that \( (R^B(\sigma, \tau) - V_{\tau_\varepsilon})^+ = (V_{\tau_\varepsilon} - V_{\tau_\varepsilon}^-)^+ \) for all stopping times \( \sigma, \tau \in \mathcal{B}_T \). Thus, there exists a hedge \( (\pi_1, \sigma_1) \in \omega^B(x) \times \mathcal{B}_0^T \) such that

\begin{equation}
R^H(\pi_1, \sigma_1) < R^H(x) + \delta \quad \text{and} \quad \sigma_1 \leq \tau_\varepsilon.
\end{equation}

Set \( \sigma_2 = \sigma_1 1_{\tau_\varepsilon < \tau} + T 1_{\tau_\varepsilon \geq \tau} \). Clearly, \( \{\sigma_2 \leq t\} = \{\sigma_1 \leq t\} \cap \{\sigma_1 < \tau_\varepsilon\} \in \mathcal{B}_t^T \) for any \( t < T \), and so we conclude that \( \sigma_2 \in \mathcal{B}_T^T \). Observe that if \( \pi_1 = \{\langle \beta_t, \gamma_t \rangle\}_{t=0}^T \) and \( \pi_2 =
In order to estimate \((\beta_t, \tilde{\xi})\) let \(\gamma = \gamma_{1, \sigma_1, \tau}\) and \(\tilde{\beta}_t = (x + \int_0^t \tilde{\gamma}_u dS_u^B - \tilde{\gamma}_0 S_0^B) / b_0\) then \(\pi_2\) is an admissible self financing strategy and \(\tilde{V}_{\pi_2} = \tilde{V}_{\pi_2}^B\). Consider the hedge \((\pi_1, \pi_2) \in \mathcal{L}_0^B\) then

\[
Q^{B,H}(\pi_2, \pi_1, \tau) = (Q^{B,H}(\sigma_2, \tau) - \tilde{V}_{\sigma_2, \tau}^+) + \delta \quad \text{and} \quad \tau \leq \tau_H.
\]

For any \(\alpha > 0\) denote \(J_\alpha = (Le^\alpha, Re^{-\alpha})\). Set \(U_\alpha = (Q^{B,H}(\sigma_1, \tau) - \tilde{V}_{\sigma_1, \tau}^+)\). Clearly, \(\tau \land \tau_I \leq \tau \land \tau_H\) for any \(\alpha > 0\) and \(\tau \land \tau_I \uparrow \tau \land \tau_H\) as \(\alpha \to 0\). This together with \ref{26} yields that

\[
\lim_{\alpha \to 0} Q^{B,H}(\sigma_1, \tau \land \tau_I) = \tilde{X}_{\sigma_1}^H \lim_{\alpha \to 0} \tilde{Y}_{\tau \land \tau_I}^H \|\sigma_I \geq \tau \land \tau_I \| = e^{-r\sigma_1} G_{\sigma_1}(S_T^B) \|_{\sigma_I \geq \tau \land \tau_I} \| + \lim_{\alpha \to 0} \tilde{Y}_{\tau \land \tau_I}^H \|_{\sigma_1 \geq \tau \land \tau_I} = e^{-r\sigma_1} G_{\sigma_1}(S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} \| + e^{-r(\tau \land \tau_I)} F_{\tau \land \tau_I} (S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} \|
\]

By the choice of \(\tilde{\xi}\) we obtain by the choice of \(\pi_2\) that

\[
\lim_{\alpha \to 0} U_\alpha = (e^{-r\sigma_1} G_{\sigma_1}(S_T^B) \|_{\sigma_I \geq \tau \land \tau_I} + e^{-r(\tau \land \tau_I)} F_{\tau \land \tau_I} (S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} - \tilde{V}_{\sigma_1, \tau}^+) = (e^{-r\sigma_1} G_{\sigma_1}(S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} + e^{-r(\tau \land \tau_I)} F_{\tau \land \tau_I} (S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} - \tilde{V}_{\sigma_2, \tau}^+).\]

Observe that \(R^H(\pi_1, \sigma_1) \geq E^B U_\alpha\) for any \(\alpha\). Thus from \ref{22} and the Fatou's lemma we obtain

\[
R^H(\pi_1, \sigma_1) \geq E^B \lim_{\alpha \to 0} U_\alpha = E^B (e^{-r\sigma_1} G_{\sigma_1}(S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} + e^{-r(\tau \land \tau_I)} F_{\tau \land \tau_I} (S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} - \tilde{V}_{\sigma_1, \tau}^+).\]

Since \(\sigma_2 \geq \sigma_1\) a.s. then from the definition of \(\pi_2\) it follows that \(\tilde{V}_{\sigma_1, \tau} = \tilde{V}_{\sigma_2, \tau} \leq \tilde{V}_{\sigma_1, \tau} \leq \tilde{V}_{\sigma_1, \tau}^+\). Thus \(\tilde{V}_{\sigma_1, \tau} \leq \tilde{V}_{\sigma_2, \tau} \leq \tilde{V}_{\sigma_1, \tau}^+\).

Observe that if \(\sigma_2 < \tau\) then \(\sigma_2 = \sigma_1 < \tau \land \tau_H\) and if \(\sigma_2 \geq \tau\) then \(\sigma_1 \geq \tau \land \tau_H\). And so from \ref{20}, \ref{23} and \ref{24} we obtain that

\[
R^H(\pi_2, \sigma_2) \leq E^B (e^{-r\sigma_1} G_{\sigma_1}(S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} + e^{-r(\tau \land \tau_I)} F_{\tau \land \tau_I} (S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} - \tilde{V}_{\sigma_2, \tau}^+ + \delta).\]

Observe that if \(\sigma_2 < \tau\) then \(\sigma_2 = \sigma_1 < \tau \land \tau_H\) and if \(\sigma_2 \geq \tau\) then \(\sigma_1 \geq \tau \land \tau_H\). And so from \ref{20}, \ref{23} and \ref{24} we obtain that

\[
R^H(\pi_2, \sigma_2) \leq E^B (e^{-r\sigma_1} G_{\sigma_1}(S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} + e^{-r(\tau \land \tau_I)} F_{\tau \land \tau_I} (S_T^B) \|_{\sigma_1 \geq \tau \land \tau_I} - \tilde{V}_{\sigma_2, \tau}^+ + \delta).\]

In order to estimate \(E^B \Gamma_1\) and \(E^B \Gamma_2\) introduce the process \(W_t = \frac{\ln S_t^B - \ln S_0^B}{\sqrt{t}} - B_t + \left(\frac{r + \mu}{\kappa}\right)t, t \geq 0\). From Girsanov's theorem (see \ref{13}) it follows that \(\{W_t\}_{t=0}^T\) is a Brownian motion with respect to the measure \(P^B\) whose restriction to the \(\sigma\)-algebra \(\mathcal{F}_t^B\) satisfies

\[
D_t = \frac{dP^B}{dP} |_{\mathcal{F}_t^B} = \exp \left(\left(\frac{r + \mu}{\kappa} - \frac{\kappa}{2}\right)B_t + \left(\frac{r + \mu}{\kappa} - \frac{\kappa}{2}\right)^2 t\right).\]

Denote the expectation with respect to \(P^B\) by \(E^B\) then by \ref{23} and the Hölder inequality,

\[
E^B \Gamma_1 \leq E^B (r(\tau - \tau \land \tau_H)(F_0(S_0) + \mathcal{L}(T + 2)(1 + \sup_{0 \leq t \leq T} S_t^B)))D_T \leq c_1 (E^B (\tau - \tau \land \tau_H)^{4/3})^{3/4}.
\]
for some constant $c_1$. From (2.2) it follows that $\Gamma_2 \leq \Gamma_3 + \Gamma_4$ where
\[ \Gamma_3 = \mathcal{L}((\tau - \tau \wedge \tau_H) (1 + \sup_{0 \leq t \leq T} S^B_t))\] and $\Gamma_4 = \sup_{\tau \wedge \tau_H \leq t} \mathcal{L}(|S^B_{\tau \wedge \tau_H}|)$. 

By the Hölder inequality,
\[ E^B \Gamma_3 = E_W(\mathcal{L}((\tau - \tau \wedge \tau_H) (1 + \sup_{0 \leq t \leq T} S^B_t))) \leq c_2(E_W(\tau - \tau \wedge \tau_H)^{4/3})^{3/4} \]
for some constant $c_2$. Set $\Gamma_5 = \sup_{\tau \wedge \tau_H \leq t} |W_t - W_{\tau \wedge \tau_H}|$. Employing the inequality $|e^x - 1| \leq x$ for $0 \leq x \leq 1$ it follows that $\Gamma_4 \leq \mathcal{L}(\sup_{0 \leq t \leq T} S^B_t(\|\tau_H\| + \Gamma_5))$ and together with the Markov and Hölder inequalities we obtain that there exists a constant $c_3$ such that
\[ E^B \Gamma_4 \leq E_W(D_T \mathcal{L}(\sup_{0 \leq t \leq T} S^B_t(\|\tau_H\| + \Gamma_5))) \leq c_3(P_W(\|\tau_H\| > 1))^{3/4} + c_3(E_W(\Gamma_5))^{3/4} \leq 2c_3(E_W(\Gamma_5))^{3/4}. \]

Using the Burkholder–Davis–Gundy inequality (see [13]) for the martingale $W_t - W_{\tau \wedge \tau_H}$, $t \geq \tau \wedge \tau_H$ we obtain that there exists a constant $c_4$ such that
\[ E_W(\Gamma_5) \leq c_4E_W(\tau - \tau \wedge \tau_H)^{2/3}. \]

Since $\tau - \tau \wedge \tau_H \leq T$ then from (3.27)–(3.30) we obtain
\[ E^B (\Gamma_1 + \Gamma_2) \leq c_5(E_W(\tau - \tau \wedge \tau_H)^{2/3})^{3/4} \]
for some constant $c_5$. Finally, we estimate the term $E_W(\tau - \tau \wedge \tau_H)^{2/3}$. First assume that $L > 0$ and $R < \infty$. Set $x_1 = (\ln L - \ln S_0)/\kappa$, $x_2 = (\ln R - \ln S_0)/\kappa$, $y_1 = x_1 - \frac{e}{\kappa}$ and $y_2 = x_2 + \frac{e}{\kappa}$. For any $x \in \mathbb{R}$ let $\tau^{(x)} = \inf\{t \geq 0 | W_t = x\}$ be the first time the process $\{W_t\}_{t=0}^\infty$ hits the level $x$. Clearly $\tau^{(x)}$ is a finite stopping time with respect to $P_W$. By (3.21) we obtain that
\[ \tau - \tau \wedge \tau_H \leq T \wedge (\tau_{H_1} - \tau_{H_2}) = T \wedge (\tau^{(y_1)} \wedge \tau^{(y_2)} - \tau^{(x_1)} \wedge \tau^{(x_2)}) \leq T \wedge (\tau^{(y_1)} - \tau^{(x_1)}) + T \wedge (\tau^{(y_2)} - \tau^{(x_2)}). \]

From the strong Markov property of the Brownian motion it follows that under $P_W$ the random variable $\tau^{(y_1)} - \tau^{(x_1)}$ has the same distribution as $\tau^{(y_1 - x_1)} = \tau^{(-\hat{\mu})}$ and the random variable $\tau^{(y_2)} - \tau^{(x_2)}$ has the same distribution as $\tau^{(y_2 - x_2)} = \tau^{\hat{\mu}}$. Recall, (see [14]) that for any $z \in \mathbb{R}$ the probability density function of $\tau^{(z)}$ (with respect to $P_W$) is $f_{\tau^{(z)}}(t) = \frac{|z|}{\sqrt{2\pi t^3}} \exp(-\frac{|z|^2}{2t})$. Hence, using the inequality $(a + b)^{2/3} \leq a^{2/3} + b^{2/3}$ together with (3.32) we obtain that
\[ E_W(\tau - \tau \wedge \tau_H)^{2/3} \leq E_W(T \wedge (\tau^{(-\hat{\mu})}))^{2/3} + E_W(T \wedge \tau^{\hat{\mu}})^{2/3} \leq \frac{2\hat{\mu}}{\sqrt{2\pi R}} \left( \int_0^{1/R} \frac{1}{t^{2/3}} dt + T^{2/3} \int_0^{T^{2/3}} \frac{1}{t^{1/2}} dt \right) = \frac{16\hat{\mu}}{\sqrt{2\pi R}} T^{1/6}. \]

Observe that when either $L = 0$ or $R = \infty$ (but not both) we obtain either $\tau - \tau \wedge \tau_H \leq T \wedge (\tau^{(y_2)} - \tau^{(x_2)})$ or $\tau - \tau \wedge \tau_H \leq T \wedge (\tau^{(y_1)} - \tau^{(x_1)})$, respectively. Thus for these cases (3.33) holds true, as well. From (3.23), (3.31) and (3.33) we see that there exists a constant $A_1$ such that
\[ R^H(x) - R^H(x) \leq 2\delta + A_1 e^{3/4} \]
and since $\delta > 0$ is arbitrary we complete the proof.

The next result provides an estimate from above of the shortfall risk when one of the barriers is close to the initial stock price $S_0$. 

\]
**Lemma 3.4.** Let $I = (L, R)$ be an open interval which satisfy $\min(\frac{R}{S_0}, \frac{S_0}{L}) \leq e^\epsilon$, where we set $\frac{S_0}{R} = \frac{\infty}{S_0} = \infty$. There exists a constant $A_3$ independent of $L, R$ such that for any $\epsilon > 0$ and an initial capital $x$
\[(3.34)\] 
\[R^l(x) \leq (F_0(S_0) - x)^+ + A_3 \epsilon^{3/4}.\]

**Proof.** Let $x$ be an initial capital. Consider the constant portfolio $\pi \in \mathcal{A}^B(x)$ which satisfy $\tilde{V}_t^\pi = x$ for all $t$. Using the same notations as in Lemma 3.3 set $\sigma = (\tau(\xi^\pi) \vee \tau(-\xi^\pi)) \wedge T$. Since $\tau(\xi^\pi) \vee \tau(-\xi^\pi) \geq \tau$ we obtain that
\[(3.35)\] 
\[R^l(x) \leq R^l(\pi, \sigma) \leq \sup_{\tau \in \mathcal{B}_{0T}^R} E^B(e^{-r(\tau(\xi^\pi) \vee \tau(-\xi^\pi))}) F_{\tau(\xi^\pi) \vee \tau(-\xi^\pi)}(S^B) - x^+.\]

Similarly to (3.31) (by letting $\tau_T = 0$) we obtain that
\[(3.36)\] 
\[\sup_{\tau \in \mathcal{B}_{0T}^R} E^B(e^{-r(\tau(\xi^\pi) \vee \tau(-\xi^\pi))}) F_{\tau(\xi^\pi) \vee \tau(-\xi^\pi)}(S^B) - F_0(S_0) \leq c_s(E_W(T \wedge (\tau(\xi^\pi) \vee \tau(-\xi^\pi))^2/3)^{3/4}.\]

In the same way as in (3.35) we derive that
\[(3.37)\] 
\[E_W(T \wedge (\tau(\xi^\pi) \vee \tau(-\xi^\pi))^2/3 \leq \frac{16 \epsilon}{\sqrt{2\pi k}} T^{1/6}.\]

and combining (3.35), (3.37) we complete the proof. \hfill \square

4. Proving the main results

In this section we complete the proof of Theorems 2.2, 2.3. We start with the proof of Theorem 2.2. Though Theorem 2.2 provides only one sided estimates for shortfall risks we will see that Theorem 2.3 which provide two sided estimates for option prices follows from the proof of Theorem 2.2. In order to provide second side estimates in Theorem 2.2 we should have more precise information on optimal portfolios of shortfall risk in the BS model. However, this problem does not arise when we are dealing with option prices. Theorem 2.4 will also follow from the proof of Theorem 2.2. At the end of this section we prove Theorem 2.3. The proof of (2.27) and (2.28) is necessarily rather technical and it is marked by various risk comparisons via the formulas (4.1), (4.7), (4.8), (4.11), then estimates of terms in the right hand side of (4.11), then (4.25)–(4.30), then (4.34) and estimates of its right hand side and, finally, (4.45) and (4.46) so that these formulas may serve as road posts for the reader going through all these details.

Let $x > 0$ be an initial capital and let $I = (L, R)$ be an open interval as before. Fix $\epsilon > 0$ and denote $I_\epsilon = (e^{-\epsilon}, e^\epsilon)$. Choose $\delta > 0$. For any $z \in \mathcal{A}^B(z) \subset \mathcal{A}^B(z)$ be the subset consisting of all $\pi \in \mathcal{A}^B(z)$ such that the discounted portfolio process $\{\tilde{V}_t^\pi\}_{t=0}^T$ is a right continuous martingale with respect to the martingale measure $\tilde{P}^B$ and $\tilde{V}_T^\pi = f(B_{t_1}^\pi, ..., B_{t_k}^\pi)$ for some smooth function $f \in C^0_0(\mathbb{R}^k)$ with a compact support and $t_1, ..., t_k \in [0, T]$. Using the same arguments as in Lemmas 4.1–4.3 in [17] we obtain that there exists $z < x$ and $\pi \in \mathcal{A}^B(z)$ such that $R^k(\pi) < R^k(x) + \delta$. Thus there exist $k, 0 < t_1 < t_2 < ... < t_k \leq T$ and $0 \leq f_\delta \in C^0_0(\mathbb{R}^k)$ such that the portfolio $\pi \in \mathcal{A}^B$ with $\tilde{V}_0^\pi = \tilde{E}(f_\delta(B_{t_1}^\pi, ..., B_{t_k}^\pi)|\mathcal{F}_T^B)$ satisfies
\[(4.1)\] 
\[R^k(\pi) < R^k(x) + \delta \quad \text{and} \quad V_0^\pi < x.\]

Set
\[(4.2)\] 
\[\Psi_n = f_\delta(B_{\theta_n/T}^*, ..., B_{\theta_n/T}^*),\]
markets are complete we can find a portfolio's professional distributions of the sequence $R$

There exists a stopping time $\tau$ such that for all $n$,

\begin{equation}
E^B \pi \leq K(n)m - m \quad \text{and} \quad E^B w_n^{2m} \leq K(n) m^{-m}.
\end{equation}

From the exponential moment estimates (4.8) and (4.25) of Barlow and Lehoczky, it follows that there exists a constant $K_1$ such that for any natural $n$ and a real $a$,

\begin{equation}
E^B \exp(a\theta^{(n)}_T) \leq e^{aK_1T} \quad \text{and} \quad E^B \sup_{0 \leq t \leq \theta^{(n)}_T} \exp(a\theta^{(n)}_t) \leq 2e^{aK_1T}.
\end{equation}

Clearly $(B_t - B_{\theta^{(n)}_T})^2 \leq 2(B_t - B\theta^{(n)}_T)^2 + 2((\frac{\theta^{(n)}_T - \theta^{(n)}_T}{m/a_T}))^2$ and $|t - \theta^{(n)}_T| \leq \frac{T}{n} + u_n$.

Hence, from (4.3) and Itô's isometry for the Brownian motion it follows that there exists a constant $C(1)$ such that $E^B \|B^*_t - B_{\theta^{(n)}_T}\|^2 \leq C(1)n^{-1/2}$ for all $t$. Let $L^2(f_b) = \max_{0 \leq t \leq T} |\frac{\partial f_b}{\partial \theta^{(n)}_T}(x_1, ..., x_k)|$.

Then by (4.2) and the inequality $(\sum_{i=1}^k a_i^2)^{1/2} \leq k \sum_{i=1}^k a_i^2$ we obtain

\begin{equation}
E^B |\Psi_n - \tilde{\Psi}_T|^2 \leq L^2(f_b)^2 E^B (\sum_{i=1}^k |B^*_i - B_{\theta^{(n)}_T}|)^2 \leq k^2 L^2(f_b)^2 C(1)n^{-1/2}.
\end{equation}

By (4.4) and the Cauchy-Schwarz inequality,

\begin{equation}
lm_{n \to \infty} E^B |\Psi_n - \tilde{\Psi}_T| = \lim_{n \to \infty} \left( E^B |\Psi_n - \tilde{\Psi}_T|^2 \right)^{1/2} \left( E^B \tilde{\Psi}_T \right)^{1/2} \Psi_n = 0
\end{equation}

where $Z_\delta$ is the Radon-Nikodim derivative given by (2.29). Since $E^B \tilde{\Psi}_T < x$ then for sufficiently large $n$ we can assume that $\nu_n = E(\tilde{\Psi}_n) < x$. Observe that the finite dimensional distributions of the sequence $\sqrt{\frac{T}{n}} \xi_1, ..., \sqrt{\frac{T}{n}} \xi_n$ with respect to $\tilde{P}^\delta$ and the finite dimensional distributions of the sequence $\tilde{b}^{(n)}_1, ..., \tilde{b}^{(n)}_n$ with respect to $P^\delta$ are the same, and so

\begin{equation}
\nu_n = E \tilde{f}_n \left( \sqrt{\frac{T}{n}} \sum_{i=1}^{[m/a_T]} \xi_i, ..., \sqrt{\frac{T}{n}} \sum_{i=1}^{[n/a_T]} \xi_i \right) < x \quad \text{(for sufficiently large $n$)}.
\end{equation}

Since CRR markets are complete we can find a portfolio $\pi(n) \in \mathcal{A}^\delta, n(\nu_n)$ such that

\begin{equation}
\nu^\pi_n = f_b \left( \sqrt{\frac{T}{n}} \sum_{i=1}^{[m/a_T]} \xi_i, ..., \sqrt{\frac{T}{n}} \sum_{i=1}^{[n/a_T]} \xi_i \right).
\end{equation}

For a fixed $n$ let $\pi = \pi(n) \in \mathcal{A}^\delta, n(\nu_n)$. From (2.31) it follows that $\tilde{\pi}^{\pi(n)} = \Psi_n$. Since $R^\delta_n(\cdot)$ is a non increasing function then by (4.4) and Lemma 3.2,

\begin{equation}
R^\delta_n(x) - R^\delta(x) \leq R^\delta_n(\nu_n) - R^\delta(x) \leq \delta + R^\delta_n(\pi) - R^\delta(\pi).
\end{equation}

There exists a stopping time $\sigma \in \mathcal{T}_{\mathcal{Q}_0}$ such that

\begin{equation}
R^\delta(\pi) > \sup_{\tau \in \mathcal{T}_{\mathcal{Q}_0}} E^B (Q^\delta(\pi, \sigma - \tau) - \tilde{\pi}^{\pi}_\sigma)^+ - \delta.
\end{equation}

Set

\begin{equation}
\zeta = (n \wedge \min \{ \varphi^{(n)}_i \geq \sigma \}) \mathbb{I}_{\sigma \in T} + n \mathbb{I}_{\sigma = T}.
\end{equation}
Clearly, $\zeta \leq n$ a.s. and $\{\zeta \leq i\} = \{\sigma \leq \theta_i^{(n)}\} \cap \{\sigma < T\} \in \mathcal{F}_{\theta_i^{(n)}}$ for any $i < n$ implying that $\zeta \in \mathcal{F}_{\theta_i^{(n)}}$. There exists a stopping time $\eta \in \mathcal{F}_{\theta_i^{(n)}}$ such that

$$E^B(Q^{B,I,n}(\frac{\zeta^T}{n}, \frac{\eta^T}{n} - \bar{V}_{\theta_i^{(n)}}^{(\eta)}) + > \sup_{\theta \in \mathcal{F}_{\theta_i^{(n)}}} E^B(Q^{B,I,n}(\frac{\zeta^T}{n}, \frac{\eta^T}{n} - \bar{V}_{\theta_i^{(n)}}^{(\eta)}) + \bar{V}_{\theta_i^{(n)}}^{(\eta)} + \delta \geq R_{n,I}(\pi') - \delta. \tag{4.10}$$

From (4.8) and (4.10) we obtain that

$$R_{n,I}(\pi') - R_\epsilon(\pi) < 2\delta + E^B(Q^{B,I,n}(\frac{\zeta^T}{n}, \frac{\eta^T}{n} - \bar{V}_{\theta_i^{(n)}}^{(\eta)}) + \bar{V}_{\theta_i^{(n)}}^{(\eta)} + \delta \leq 2\delta + E^B(\Lambda_1 + \Lambda_2 + \Lambda_3) \tag{4.11}$$

where

$$\Lambda_1 = \left| \bar{V}_{\theta_i^{(n)}}^{(\eta)} - \bar{V}_{\theta_i^{(n)} \wedge T} \right|, \quad \Lambda_2 = \left| \bar{V}_{\theta_i^{(n)} \wedge T} - \bar{V}_{\theta_i^{(n)} \wedge \sigma^\eta} \right|$$

and $\Lambda_3 = (Q^{B,I,n}(\frac{\zeta^T}{n}, \frac{\eta^T}{n} - Q^{B,I}(\sigma, \theta_i^{(n)} \wedge T))$.\footnote{\textcolor{red}{Redacted}}

Since the processes $\{\bar{V}_{\theta_i^{(n)}}\}$, $t \geq 0$ is a martingale then $\bar{V}_{\theta_i^{(n)} \wedge T} = E^B(\bar{V}_{\theta_i^{(n)} \wedge T}| \mathcal{F}_{\theta_i^{(n)} \wedge T}) = E^B(\bar{V}_{\theta_i^{(n)} \wedge T}| \mathcal{F}_{\theta_i^{(n)} \wedge T})$.\footnote{\textcolor{red}{Redacted}}

Taking into account that $\bar{V}_{\theta_i^{(n)}}$ is $\mathcal{F}_{\theta_i^{(n)} \wedge T}$ measurable, since the processes $\{\bar{V}_{\theta_i^{(n)}}\}$, $t \geq 0$ is a martingale and $\Psi_n = \bar{V}_{\theta_i^{(n)} \wedge T}$ then $\bar{V}_{\theta_i^{(n)} \wedge T} = E^B(\Psi_n| \mathcal{F}_{\theta_i^{(n)} \wedge T})$. Thus

$$\bar{V}_{\theta_i^{(n)} \wedge T} - \bar{V}_{\theta_i^{(n)} \wedge T} = E^B(\Psi_n - \bar{V}_{\theta_i^{(n)} \wedge T}| \mathcal{F}_{\theta_i^{(n)} \wedge T}) = E^B\left(\frac{Z_{\theta_i^{(n)}}(\Psi_n - \bar{V}_{\theta_i^{(n)} \wedge T})}{Z_{\theta_i^{(n)}}}\right). \tag{4.12}$$

By (4.12), (4.13), the Cauchy-Schwarz and Jensen inequalities,

$$E^B \Lambda_1 \leq \left( E^B\left(\frac{Z_{\theta_i^{(n)}}(\Psi_n - \bar{V}_{\theta_i^{(n)} \wedge T})}{Z_{\theta_i^{(n)}}}\right)^2 \right)^{1/2} (E^B(\Psi_n - \bar{V}_{\theta_i^{(n)} \wedge T})^2)^{1/2} \leq C(f_\delta)n^{-1/4} \tag{4.14}$$

where $C(f_\delta)$ is a constant which depends only on $f_\delta$. By using the same arguments as in (5.14)-(5.17) of \textcolor{red}{[1]} we obtain that

$$E^B \Lambda_2 \leq C(f_\delta)n^{-1/2} \tag{4.15}$$

for some constant $C(f_\delta)$ which depends only on $f_\delta$. Next, we estimate $\Lambda_3$. Set

$$Q^B(s,t) = e^{-rt}G_1(S^B)_{s \leq \xi} + e^{-rs}F_1(S^B)_{\xi \leq t}, \quad s,t \geq 0 \quad \text{and}$$

$$Q^{B,n}(k,l) = \prod_{k \leq \xi}^{-l}(1 + r_n)^{-k} G_{\mathcal{F}_{B,n}}(S^B)_{\xi \leq l} + \prod_{l \leq \xi}^{-k}(1 + r_n)^{-l} F_{\mathcal{F}_{B,n}}(S^B)_{l \leq \xi}. \tag{4.16}$$

From (2.3) and (4.12) we get

$$\Lambda_3 \leq (Q^{B,n}(\frac{\zeta^T}{n}, \frac{\eta^T}{n} - Q^{B}(\sigma, \theta_i^{(n)} \wedge T)) + I_{\Theta}(G_0(S_0) + \mathcal{L}(T + 2)(1 + \max_{0 \leq k \leq n} \mathcal{S}_{r, n})) \tag{4.17}$$

where $\Theta = \{\zeta < \tau_i^{(n)}\} \cap \{\sigma < \theta_i^{(n)}\}$. Similarly to Lemmas 3.2 and 3.3 in \textcolor{red}{[1]} it follows that there exists a constant $C(2)$ such that

$$\sup_{\zeta \in \mathcal{F}_{B,n}} \sup_{\eta \in \mathcal{F}_{B,n}} E^B(Q^{B}(\theta_i^{(n)} \wedge T) - Q^{B,n}(\frac{\zeta^T}{n}, \frac{\eta^T}{n})) \leq C(2)n^{-1/4}(\ln n)^{3/4}. \tag{4.18}$$
From (4.4) and the Cauchy-Schwarz inequality it follows that
\begin{equation}
E^B \left( I_{0}(G_0(S_0) + \mathcal{L}(T + 2)(1 + \max_{0 \leq k \leq n} \frac{B_k}{\mathcal{X}})) \right) \leq C^3 P(\Theta)^{1/2}
\end{equation}
for some constant $C^3$. By (4.9) we see that $\sigma < \theta^{(n)}_{\xi \eta} \wedge T$ provided $\zeta < \eta$. This together with (4.16) - (4.18) gives
\begin{equation}
E^B \Lambda_1 \leq C^3 P(\Theta)^{1/2} + C^2 n^{-1/4}(\ln n)^{3/4} + E^B \left( Q^B(\sigma, \theta^{(n)}_{\xi \eta} \wedge T) \right) \leq C^3 P(\Theta)^{1/2} + C^2 n^{-1/4}(\ln n)^{3/4} + \alpha_1 + \alpha_2
\end{equation}
where
\begin{equation}
\alpha_1 = E^B \left| e^{-\theta^{(n)}_{\xi \eta}} G_{\theta^{(n)}_{\xi \eta}}(S^B) - e^{-\sigma \wedge \theta^{(n)}_{\xi \eta}} G_{\sigma \wedge \theta^{(n)}_{\xi \eta}}(S^B) \right|
\end{equation}
and
\begin{equation}
\alpha_2 = E^B \left| e^{-\theta^{(n)}_{\xi \eta}} F_{\theta^{(n)}_{\xi \eta}}(S^B) - e^{-\sigma \wedge \theta^{(n)}_{\xi \eta}} F_{\sigma \wedge \theta^{(n)}_{\xi \eta}}(S^B) \right|
\end{equation}
From Lemma 4.4 in [7] it follows that there exists a constants $C^4, C^5$ such that
\begin{equation}
\alpha_1 + \alpha_2 \leq C^4 \left( E^B(\theta^{(n)}_{\xi \eta} - \theta^{(n)}_{\xi \eta} \wedge \sigma)^2 \right)^{1/2} + C^5 \left( E^B(\theta^{(n)}_{\xi \eta} - \theta^{(n)}_{\xi \eta} \wedge \sigma)^2 \right)^{1/4}.
\end{equation}
By (4.9) we obtain that $|\theta^{(n)}_{\xi \eta} - \theta^{(n)}_{\xi \eta} \wedge \sigma| \leq |\theta^{(n)}_{\xi \eta} - \sigma| \leq |T - \theta^{(n)}_{\xi \eta}| \leq u_n$. Thus by (4.3),
\begin{equation}
\alpha_1 + \alpha_2 \leq C^6 n^{-1/4}
\end{equation}
for some constant $C^6$. Finally, we estimate $P(\Theta)$. Observe that $\sigma \wedge \theta^{(n)}_{\xi \eta} \leq \theta^{(n)}_{\xi \eta}$, and so
\begin{equation}
\Theta \subseteq \left\{ \sup_{0 \leq k \leq n} \frac{s^B_k}{\theta^{(n)}_{\xi \eta}} > e^{-\epsilon} \right\} \cup \left\{ \inf_{0 \leq k \leq n} \frac{s^B_k}{\theta^{(n)}_{\xi \eta}} \leq e^{-\epsilon} \right\} \subseteq \left\{ \sup_{0 \leq k \leq n} \frac{s^B_k}{\theta^{(n)}_{\xi \eta}} > e^{-\epsilon} \right\} \cup \left\{ \inf_{0 \leq k \leq n} \frac{s^B_k}{\theta^{(n)}_{\xi \eta}} \leq e^{-\epsilon} \right\}
\end{equation}
\begin{equation}
\left\{ \max_{0 \leq k \leq n-1} \sup_{0 \leq s \leq \theta^{(n)}_{\xi \eta} t \leq \theta^{(n)}_{\xi \eta} \wedge \theta^{(n)}_{\xi \eta} + 1} \max_{0 \leq k \leq n-1} \sup_{0 \leq s \leq \theta^{(n)}_{\xi \eta} t \leq \theta^{(n)}_{\xi \eta} \wedge \theta^{(n)}_{\xi \eta} + 1} \max_{0 \leq k \leq n-1} \sup_{0 \leq s \leq \theta^{(n)}_{\xi \eta} t \leq \theta^{(n)}_{\xi \eta} \wedge \theta^{(n)}_{\xi \eta} + 1} \frac{r|t - \frac{kT}{n}| + \kappa|B^{*}_{t} - B^{*}_{\theta^{(n)}_{\xi \eta}}|}{\theta^{(n)}_{\xi \eta}} > \epsilon \right\}
\end{equation}
Since $|B^{*}_t - B^{*}_{\theta^{(n)}_{\xi \eta}}| \leq \frac{1}{n}$ and $|t - \frac{kT}{n}| \leq u_n + \frac{1}{n}$ for any $k < n$ and $t \in \left[ \theta^{(n)}_{k}, \theta^{(n)}_{k+1} \right]$ (where $u_n$ was defined after (4.2)) then using the inequality $(a + b)^3 \leq 4(a^3 + b^3)$ for $a, b \geq 0$ we obtain by (4.4) that
\begin{equation}
E^B(\max_{0 \leq k \leq n-1} \sup_{0 \leq s \leq \theta^{(n)}_{\xi \eta} t \leq \theta^{(n)}_{\xi \eta} \wedge \theta^{(n)}_{\xi \eta} + 1} \max_{0 \leq k \leq n-1} \sup_{0 \leq s \leq \theta^{(n)}_{\xi \eta} t \leq \theta^{(n)}_{\xi \eta} \wedge \theta^{(n)}_{\xi \eta} + 1} \max_{0 \leq k \leq n-1} \sup_{0 \leq s \leq \theta^{(n)}_{\xi \eta} t \leq \theta^{(n)}_{\xi \eta} \wedge \theta^{(n)}_{\xi \eta} + 1} \frac{r|t - \frac{kT}{n}| + \kappa|B^{*}_{t} - B^{*}_{\theta^{(n)}_{\xi \eta}}|}{\theta^{(n)}_{\xi \eta}})^3 \leq C^7 n^{-3/2}
\end{equation}
for some constant $C^7$. From (4.23) and the Markov inequality it follows that $P(\Theta) \leq C^7 n^{-3/2}$ and together with (4.7), (4.11), (4.13), (4.15), (4.19) and (4.22) we conclude that
\begin{equation}
R^B_n(x) - R^C_n(x) \leq 3\delta + (C^6 + C(f_{\delta}))n^{-1/4} + \tilde{C}(f_{\delta})n^{-1/2} + C^2 n^{-1/4}(\ln n)^{3/4} + C^3 \sqrt{C^7 \frac{n^{-3/2}}{\epsilon^3}}.
\end{equation}
Since the above constants do not depend on \( n \) then \( R^L(x) \geq \limsup_{n \to \infty} R_n^l(x) - 3\delta \). Letting \( \delta \downarrow 0 \) we obtain that \( R^L(x) \geq \limsup_{n \to \infty} R_n^l(x) \) and by Lemma 3.3

\[
R^l(x) = \lim_{\varepsilon \to 0} R^L(x) \geq \limsup_{n \to \infty} R_n^l(x). 
\]

In order to complete the proof of Theorem 2.2 we should prove (2.28). Fix an initial capital \( x \), an open interval \( I = (L, R) \) and a natural number \( n \). If \( \min(\frac{R}{2}, \frac{S_0}{2}) \leq e^{n^{-1/3}} \) then from Lemma 3.4 and the inequality \( R_n^l(x) \geq (F_0(S_0) - x)^+ \) it follows

\[
R^l(x) - R_n^l(x) \leq R^l(x) - (F_0(S_0) - x)^+ \leq A_3 n^{-1/4}. 
\]

Next, we deal with the case where \( \min(\frac{R}{2}, \frac{S_0}{2}) > e^{n^{-1/3}} \) (which is true for sufficiently large \( n \)). Introduce the open interval \( J_n = (L \exp(n^{-1/3}), R \exp(-n^{-1/3})) \). Set \( (\pi, \sigma) = (\psi_n(\pi_n^l), \phi_n(\sigma_n^l)) \) where \( \pi_n^l, \sigma_n^l \) is the optimal hedge given by (5.15) and the functions \( \psi_n, \phi_n \) were defined in Section 2. We can consider the portfolio \( \pi = \psi_n(\pi_n) \) not only as an element in \( \mathcal{B}_n^B(x) \) but also as an element in \( \mathcal{B}_n^L(x) \) if we restrict the above portfolio to the interval \([0, T]\). From Lemma 3.2 we obtain that

\[
R^L(\pi, \sigma) - R_n^l(x) = R^L(\pi, \sigma) - R_n^J(\pi, \zeta_n^J) 
\]

where, recall, \( \zeta_n^J \) was defined in (3.16). Since \( I \) and \( n \) are fixed we denote \( \zeta = \zeta_n^J \). Recall that \( \Pi_n(\zeta_n^J) = \zeta \) and so from (2.29) we get \( \sigma = (T \land \theta_{\zeta}^{(n)}) \mathbb{1}_{\zeta < n} + T \mathbb{1}_{\zeta = n} \). For a fixed \( \delta > 0 \) choose a stopping time \( \tau \) such that

\[
R^L(\pi, \sigma) < \delta + E^B[(Q^{B,J_n}(\sigma, \tau) - \bar{V}_{\sigma \land \tau})^+].
\]

Observe that \( \min\{k|\theta_k^{(n)} \geq \tau\} \in \mathcal{B}_n^B \) since \( \min\{k|\theta_k^{(n)} \geq \tau\} = \{\theta_j^{(n)} \geq \tau\} \in \mathcal{B}_n^B \) and set \( \eta = n \land \min\{k|\theta_k^{(n)} \geq \tau\} \in \mathcal{B}_n^B \). Denote

\[
\Gamma_1 = (Q^{B,J_n}(\sigma, \tau) - Q^{B,J_n}(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}))^+ \\
\text{and} \quad \Gamma_2 = (Q^{B,J_n}(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - Q^{B,J_n}(\zeta_n^J, T_n))^+.
\]

From (2.29) it follows that

\[
R^L(\pi, \sigma) < \delta + E^B(Q^{B,J_n}(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - \bar{V}_{\sigma \land \tau})^+ + E^B \Gamma_1 \\
\text{and} \quad R_n^J(\pi, \zeta) \geq E^B(Q^{B,J_n}(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - \bar{V}_{\theta_n^{(n)} \land \eta})^+ - E^B \Gamma_2.
\]

Hence,

\[
R^L(\pi, \sigma) - R_n^J(\pi, \zeta) < E^B(Q^{B,J_n}(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - \bar{V}_{\sigma \land \tau})^+ \\
- E^B(Q^{B,J_n}(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - \bar{V}_{\theta_n^{(n)} \land \eta})^+ + \delta + E^B(\Gamma_1 + \Gamma_2).
\]

Observe that \( \sigma \land \theta_n^{(n)} \leq \theta_n^{(n)} \) and \( \tau \land \theta_n^{(n)} \leq \theta_n^{(n)} \), thus

\[
\sigma \land \tau \land \theta_n^{(n)} \leq \theta_n^{(n)}. 
\]
Since $\pi \in \mathcal{A}^B(x)$ then by (2.30), $\nabla^{\pi}_{\sigma \wedge \tau} = \nabla^{\pi}_{\sigma \wedge \theta_n^{(n)}} = E^B(\nabla^{\pi}_{\theta_n^{(n)}} | \mathcal{F}^B_{\sigma \wedge \theta_n^{(n)}})$. This together with the Jensen inequality yields that

$$
(4.32) \quad E^B((Q^{B,J_n}(\sigma \wedge \theta_n^{(n)}, \tau \wedge \theta_n^{(n)}) - \nabla^{\pi}_{\sigma \wedge \tau})^+) \leq E^B(Q^{B,J_n}(\sigma \wedge \theta_n^{(n)}, \tau \wedge \theta_n^{(n)}) - \nabla^{\pi}_{\sigma \wedge \tau})^+ | \mathcal{F}^B_{\sigma \wedge \theta_n^{(n)}}).
$$

Thus,

$$
(4.33) \quad E^B(Q^{B,J_n}(\sigma \wedge \theta_n^{(n)}, \tau \wedge \theta_n^{(n)}) - \nabla^{\pi}_{\sigma \wedge \tau})^+ \leq E^B(Q^{B,J_n}(\sigma \wedge \theta_n^{(n)}, \tau \wedge \theta_n^{(n)}) - \nabla^{\pi}_{\sigma \wedge \tau})^+ | \mathcal{F}^B_{\sigma \wedge \theta_n^{(n)}}).
$$

By (4.30) and (4.33) we obtain that

$$
(4.34) \quad R^{B,J_n}(\pi, \sigma) - R^{B,J_n}(\pi, \xi) < \delta + E^B(\Gamma_1 + \Gamma_2) + \alpha_3
$$

where

$$
\alpha_3 = E^B\left(\frac{Z_{\sigma \wedge \theta_n^{(n)}} - Z_{\theta_n^{(n)}}}{Z_{\theta_n^{(n)}}} Q^{B,J_n}(\sigma \wedge \theta_n^{(n)}, \tau \wedge \theta_n^{(n)})\right).
$$

Notice that $|\sigma - \theta_n^{(n)}| \leq w_n$ and $|\tau - \theta_n^{(n)}| \leq w_n$ (where $w_n$ was defined after (4.2)). Thus by (4.31) we obtain that

$$
(4.35) \quad 0 \leq \theta_n^{(n)} - \sigma \wedge \tau \wedge \theta_n^{(n)} \leq \max(|\sigma - \theta_n^{(n)}|, |\tau - \theta_n^{(n)}|) \leq w_n.
$$

From Ito’s formula it follows that $dZ_t = \frac{\mu}{\kappa}Z_t dB_t + \left(\frac{\kappa}{\nu} \right)^2 Z_t dt$, and so

$$
Z_{\theta_n^{(n)}} - Z_{\sigma \wedge \theta_n^{(n)}} = \frac{\mu}{\kappa} \int_{\theta_n^{(n)}}^{\sigma \wedge \theta_n^{(n)}} Z_t dB_t + \left(\frac{\kappa}{\nu} \right)^2 \int_{\theta_n^{(n)}}^{\sigma \wedge \theta_n^{(n)}} Z_t dt.
$$

Set $E_n = \sup_{0 \leq t \leq \theta_n^{(n)}} Z_t$. From (4.33), (4.4), the Cauchy-Schwarz inequality and Ito’s isometry we obtain that

$$
(4.36) \quad E^B(Z_{\theta_n^{(n)}} - Z_{\sigma \wedge \theta_n^{(n)}})^2 \leq 2\left(\frac{\mu}{\kappa} \right)^2 E^B \int_{\theta_n^{(n)}}^{\sigma \wedge \theta_n^{(n)}} Z_t^2 dt + 2\left(\frac{\kappa}{\nu} \right)^4 E^B(w_n E_n)^2 \leq 2\left(\frac{\mu}{\kappa} \right)^2 E^B(w_n E_n)^2 + 2\left(\frac{\kappa}{\nu} \right)^4 E^B(w_n E_n)^2 \leq C(8)n^{-1/2}
$$

for some constant $C(8)$. By (2.3) it follows that $Q^{B,J_n}(\sigma \wedge \theta_n^{(n)}, \tau \wedge \theta_n^{(n)}) \leq G_0(S_0) + \mathcal{D}(T + 2)(1 + \sup_{0 \leq t \leq T} \mathcal{D}_n^2)$, and so (4.34) and the Cauchy-Schwarz inequality yields that

$$
(4.37) \quad \alpha \leq C(9)n^{-1/4}
$$

for some $C(9) > 0$ independent of $n$. Now we estimate $E^B\Gamma_1$. Clearly $\Gamma_1 \leq (Q^B(\sigma, \tau) - Q^B(\sigma \wedge \theta_n^{(n)}, \tau \wedge \theta_n^{(n)}))^+$. From the definitions it follows easily that $\sigma < \tau$ is equivalent to $\sigma \wedge \theta_n^{(n)} < \tau \wedge \theta_n^{(n)}$. Furthermore, $\sigma \wedge \tau - \sigma \wedge \theta_n^{(n)} \leq |T - \theta_n^{(n)}| \leq u_n$ (with $u_n$ defined
after (4.2). Thus from (4.3) and Lemma 4.4 in [7] we obtain that there exists a constant $C^{(10)}$ such that for all $n \in \mathbb{N}$,

\begin{align}
E^B \Gamma_1 & \leq E^B |e^{-r\sigma \land \tau} G_{\sigma \land \tau}(S^B) - e^{-r\theta_n^{(n)} \land \tau} G_{\theta_n^{(n)} \land \tau}(S^B)| + e^{-r\theta_n^{(n)} \land \tau} F_{\theta_n^{(n)} \land \tau}(S^B) - e^{-r\theta_n^{(n)} \land \tau} F_{\theta_n^{(n)} \land \tau}(S^B)| \leq \\
& \leq C^{(4)} (E^B(\eta_n))^2/2 + C^{(5)} (E^B(\eta_n))^2/4 \leq C^{(10)} n^{-1/4}
\end{align}

where the last inequality follows from (4.3). Next, we estimate $E^B \Gamma_2$. From (2.3) it follows that

\begin{align}
\Gamma_2 & \leq (Q^B(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - Q^B_n(\tilde{\zeta}, nT/\pi))^+ + + \sup_{0 \leq t \leq T} (S^B_t) \setminus \{\eta \land \zeta \geq \tilde{\zeta}_n^B\} \cap \{\sigma \land \tau \land \theta_n^{(n)}} < \tau_n\}.
\end{align}

By the Cauchy-Schwarz inequality,

\begin{align}
E^B \Gamma_2 & \leq (Q^B(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - Q^B_n(\tilde{\zeta}, nT/\pi))^+ + \sup_{0 \leq t \leq T} (S^B_t) \setminus \{\eta \land \zeta \geq \tilde{\zeta}_n^B\} \cap \{\sigma \land \tau \land \theta_n^{(n)}} < \tau_n\}.
\end{align}

From the definitions it follows easily that if $\sigma \land \theta_n^{(n)} < \tau \land \theta_n^{(n)}$ then $\zeta < \eta$. Hence, from (4.3), (4.35) and Lemma 4.4 in [7] we obtain that

\begin{align}
E^B \Gamma_2 & \leq (Q^B(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - Q^B_n(\tilde{\zeta}, nT/\pi))^+ + \sup_{0 \leq t \leq T} (S^B_t) \setminus \{\eta \land \zeta \geq \tilde{\zeta}_n^B\} \cap \{\sigma \land \tau \land \theta_n^{(n)}} < \tau_n\}.
\end{align}

for some constant $C^{(11)}$ independent of $n$. From (4.17), (4.39) and (4.40),

\begin{align}
E^B \Gamma_2 & \leq (Q^B(\sigma \land \theta_n^{(n)}, \tau \land \theta_n^{(n)}) - Q^B_n(\tilde{\zeta}, nT/\pi))^+ + \sup_{0 \leq t \leq T} (S^B_t) \setminus \{\eta \land \zeta \geq \tilde{\zeta}_n^B\} \cap \{\sigma \land \tau \land \theta_n^{(n)}} < \tau_n\}.
\end{align}

for some constant $C^{(12)}$. Finally, we estimate $P(\tilde{\Theta})$. Observe that $\sigma \land \tau \land \theta_n^{(n)} \geq \theta_n^{(n)}(\tilde{\eta} \land \tilde{\eta})$. Indeed, from the definitions it follows that $\tau \geq \theta_n^{(n)}(\tilde{\eta} \land \tilde{\eta})$. If $\sigma = T$ then $\sigma \land \tau \land \theta_n^{(n)} = \tau \land \theta_n^{(n)} \geq \theta_n^{(n)}(\tilde{\eta} \land \tilde{\eta})$. If $\sigma < T$ then $\sigma = \theta_n^{(n)}(\tilde{\eta} \land \tilde{\eta})$, and so $\sigma \land \tau \land \theta_n^{(n)} \geq \theta_n^{(n)}(\tilde{\eta} \land \tilde{\eta})
\( \theta^{(n)}_{(\xi, \eta-1)^j} \). Thus

\[
\Phi \subseteq \left\{ \max_{0 \leq k \leq n} \frac{\theta^n_{k+1}}{\theta^n_k} > e^{\alpha/3} \right\} \cup \left\{ \min_{0 \leq k \leq n} \frac{\theta^n_{k+1}}{\theta^n_k} < e^{-\alpha/3} \right\}
\]

\[
\subseteq \left\{ \max_{0 \leq k \leq n} \frac{\theta^n_{k+1}}{\theta^n_k} > e^{\alpha/3} \right\} \cup \left\{ \min_{0 \leq k \leq n} \frac{\theta^n_{k+1}}{\theta^n_k} < e^{-\alpha/3} \right\} \subseteq \left\{ \max_{0 \leq k \leq n} \left( r \left| \theta^{(n)}_{k+1} - \frac{\tau_k}{n} \right| \right) > 1/n \right\}
\]

From (4.3), (4.43) and the Markov inequality it follows that

\[
P (\hat{\Phi}) \leq n E \left( r(u_n + w_n) + \kappa \sqrt{\frac{T}{n}} \right)^3 \leq C (13) n^{-1/2}
\]

for some constant \( C (13) \) independent of \( n \). Since \( \delta \) is arbitrary then combining (4.28), (4.34), (4.37), (4.38), (4.41) and (4.44) we conclude that there exists a constant \( C (14) \) such that

\[
R_{\delta}^I (\pi, \sigma) - R_{\delta}^I (x) = R_{\delta}^I (\pi, \sigma) - R_{\delta}^I (\pi, \xi) \leq C (14) n^{-1/4} (\ln n)^{3/4}.
\]

By (4.43) and Lemma 3.3 it follows that for \( n \) which satisfy \( \min \left( \frac{n}{3}, \frac{\ln n}{n} \right) > e^{\alpha/3} \) we have

\[
R_{\delta}^I (x) - R_{\delta}^I (x) \leq R_{\delta}^I (x) - R_{\delta}^I (x) + R_{\delta}^I (\pi, \sigma) - R_{\delta}^I (x) \leq A_1 n^{-1/4} + C (14) n^{-1/4} (\ln n)^{3/4}.
\]

From (4.27) and (4.46) we derive (4.28) and complete the proof of Theorem 2.2.

Next, we prove Theorem 2.4. Let \( H = (L, R) \) be an open interval as before and for any \( n \) set \( H_n = (L \exp(-n^{-1/3}), R \exp(n^{-1/3})) \). Fix \( n \) and let \( (\pi_n^{H_n}, \sigma_n^{H_n}) \in \mathcal{A}^{\Xi} (n) \times \mathcal{P}^{\Xi} \) be the optimal hedge given by (3.15). Using (4.43) for \( I = H_n \) we obtain that

\[
R_{\delta}^H (\psi_n^{H_n}, \phi_n (\sigma_n^{H_n})) \leq R_{\delta}^H (x) + A_1 n^{-1/4} + C (14) n^{-1/4} (\ln n)^{3/4}.
\]

Thus

\[
\limsup_{n \to \infty} R_{\delta}^H (\psi_n^{H_n}, \phi_n (\sigma_n^{H_n})) \leq \limsup_{n \to \infty} R_{\delta}^H (x) + A_1 n^{-1/4} + C (14) n^{-1/4} (\ln n)^{3/4}.
\]

For any \( \epsilon > 0 \) denote \( J_\epsilon = (L \epsilon^{-1}, R \epsilon^{-1}) \). Since \( H_n \subseteq J_\epsilon \) for sufficiently large \( n \) then from (4.48) and Theorem 2.2 we obtain that for any \( \epsilon > 0 \),

\[
\limsup_{n \to \infty} R_{\delta}^H (\psi_n^{H_n}, \phi_n (\sigma_n^{H_n})) \leq \limsup_{n \to \infty} R_{\delta}^H (x) = R_{\delta}^H (x).
\]

By (4.49) and Lemma 3.3,

\[
\limsup_{n \to \infty} R_{\delta}^H (\psi_n^{H_n}, \phi_n (\sigma_n^{H_n})) \leq \limsup_{n \to \infty} R_{\delta}^H (x) = R_{\delta}^H (x)
\]

which completes the proof of Theorem 2.4.

Next, we prove Theorem 2.1. Let \( I = (L, R) \) be an open interval as before. Assume that \( \mu = 0 \). In this case \( \bar{\mu} = \bar{\mu}^g \) and \( \bar{p}_n^g = \bar{p}_n^g \) for any \( n \). Thus \( \gamma^I = \gamma^I + \gamma^I \). Hence, using the same procedure as in first part of the proof of Theorem 2.2 and taking into account that the value of the portfolios \( \pi, \pi' \) is zero (which means that \( C (f) = \tilde{C} (f) = 0 \)
and so we can let \( \delta \downarrow 0 \) in \((4.25)\) we obtain that there exist constants \( C^{(15)} \) and \( C^{(16)} \) such that for any \( \varepsilon > 0 \),

\[
(4.51) \quad \gamma_n^I - \gamma^I \leq C^{(15)} n^{-1/4} (\ln n)^{3/4} + C^{(16)} n^{-3/2} \sqrt{\frac{n-3/2}{\varepsilon^3}}
\]

where \( I_\varepsilon = (Le^{-\varepsilon}, Re^{\varepsilon}) \). Taking \( \varepsilon = n^{-1/3} \) we obtain by \((3.19)\) and \((4.51)\) that

\[
(4.52) \quad \gamma_n^I - \gamma^I \leq C^{(15)} n^{-1/4} (\ln n)^{3/4} + (C^{(16)} + A_2) n^{-1/4}.
\]

From Theorem 2.2 it follows that there exists a constant \( C^{(17)} \) such that

\[
\gamma_n^I - \gamma^I = R^I(0) - R^I_n(0) \leq C^{(17)} n^{-1/4} (\ln n)^{3/4}.
\]

This together with \((4.52)\) completes the proof of Theorem 2.1

Finally, we prove Theorem 2.3 Let \( H = (L, R) \) be an open interval and \( n \) be a natural number. Set \( H_n = (\text{Exp}(-n^{-1/3}), R \text{Exp}(n^{-1/3})) \) and let \((\pi_n, \sigma_n) \in \mathcal{A}^B_n(\gamma^H_n) \times \mathcal{F}^B_n \) be a perfect hedge for a double barrier option in the \( n \)-step CRR market with the barriers \( \text{Exp}(-n^{-1/3}), R \text{Exp}(n^{-1/3}) \), i.e. for any \( k \leq n \),

\[
(4.53) \quad \tilde{V}^\pi_{\sigma_n/k} \geq Q^{H, n}(\sigma_n, k).
\]

Set \((\pi, \zeta) = (\psi_n(\pi_n), \Pi_n(\sigma_n)) \in \mathcal{A}^B_n(\gamma^H_n) \times \mathcal{F}^B_n \). From \((4.53)\) and the definition of \( \Pi_n \) we obtain that for any \( k \leq n \),

\[
(4.54) \quad \tilde{V}^\pi_\zeta \geq \Pi_n(\tilde{V}^\pi_{\sigma_n/k}) \geq \Pi_n(Q^{H, n}(\sigma_n, k)) = Q^{R_H, n}(\zeta, k)
\]

implying that \( R^H_n(\pi, \zeta) = 0 \). Set \( \sigma = \phi_n(\sigma_n) \in \mathcal{F}^B \) then \( \sigma = (T \land \theta^n(\zeta)) \tilde{\xi} < n + T \tilde{\xi} = n. \)

Hence, using \((4.45)\) for \( I = H \) we obtain that

\[
(4.55) \quad R^H(\pi, \sigma) \leq R^H_n(\pi, \zeta) + C^{(14)} n^{-1/4} (\ln n)^{3/4} = C^{(14)} n^{-1/4} (\ln n)^{3/4}
\]

completing the proof.

**Remark 4.1.** Consider another definition of the discounted payoff function where in place of \((2.8)\) we set

\[
(4.56) \quad Q_1^{B, I}(t, x) = e^{-r(t/x)} (G_t(S^B_t)^{\Pi}_{s \leq t} + \tilde{V}^I_{s \leq t})
\]

which means that the seller pays for cancellation an amount which does not depend on the barriers. For such discounted payoff function the option price will be equal to the original option price \( \gamma^I \) given by \((2.7)\) and for any initial capital \( x \) the shortfall risk will be equal to \( R^I(x) \) given by \((2.15)\). Indeed, the terms in the formula \((4.56)\) for the discounted payoff function are not less than the corresponding terms for the payoff function given by \((2.8)\). On the other hand, for any \( \pi \in \mathcal{A}^B \) and \( \sigma \in \mathcal{F}^B \),

\[
(4.57) \quad K^I(\pi) = \inf_{\sigma \in \mathcal{F}^B} \sup_{\tau \in \mathcal{F}^B} E^B(Q^{B, I}(\sigma, \tau) - \tilde{V}^\pi_{\sigma \land \tau})^+ \geq \inf_{\sigma \in \mathcal{F}^B} \sup_{\tau \in \mathcal{F}^B} E^B(Q^{B, I}_1(\sigma, \tau) - \tilde{V}^\pi_{\sigma \land \tau})^+
\]

and we conclude that \((4.37)\) is, in fact, an equality which proves that for a given portfolio the shortfall risk remains as before. Since option prices can be represented as shortfall risk for the case where \( P = \tilde{P} \) and the initial capital is \( 0 \) then it follows that the option price remains as before, as well. The same holds true for CRR markets. We note that the proof of
our main results for the discounted payoff function given by (4.30) becomes a bit simpler than for the original definitions but the latter seem to more natural.

5. The knock–in case

In this section we present results similar to Theorems 2.1, 2.4 (with a little bit different estimates) for knock–in barrier options. For a given open interval \(I = (L, R)\) the payoff processes in the BS model and the \(n\)-step CRR market are defined in this case by

\[
\mathcal{X}_t = G_t(S^B), \quad \mathcal{Y}_t^I = F_t(S^B)\mathbb{I}_{t \geq \tau_I} \quad \text{and} \quad \mathcal{X}_t^{(n)} = G_t^{(n)}(S^B), \quad \mathcal{Y}_t^{(n)} = F_t^{(n)}(S^B)\mathbb{I}_{k \geq \tau^{(n)}_I},
\]

respectively. Notice that the seller will pay for cancellation an amount which does not depend on the barriers. If we would define the high payoff process \(\mathcal{X}_t^I, t \geq 0\) in a way similar to the low payoff process \(\mathcal{Y}_t^I, t \geq 0\), namely, \(\mathcal{X}_t^I = G_t(S^B)\mathbb{I}_{t \geq \tau_I}\), then the seller could cancel the contract at the moment \(t = 0\) without paying anything to the buyer which would make such contract worthless.

Now, for the BS model we define the option price and the shortfall risks by

\[
\tilde{\mathcal{P}}(t, x) = e^{-r(t, \lambda)}(\mathcal{X}_{t \wedge \lambda} + \mathcal{Y}_{t \wedge \lambda}^I)\] is the discounted payoff function. For the \(n\)-step CRR market the corresponding definitions are

\[
\tilde{\mathcal{P}}^n(t, x) = e^{-r(t, \lambda)}(\mathcal{X}_{t \wedge \lambda}^n + \mathcal{Y}_{t \wedge \lambda}^I).
\]

Let \(\mathcal{Y} = \min_{I \in \mathcal{I}_0^T} \mathcal{Y}_t^I\) be a perfect hedge for a double barrier knock–in option as above in the \(n\)-step CRR market with the barriers \(L, R\).

**Theorem 5.1.** Let \(I = (L, R)\) be an open interval.

(i) For each \(\varepsilon > 0\) there exists a constant \(\tilde{C}_{1, \varepsilon}\) such that for any \(n \in \mathbb{N}\),

\[
|\tilde{\mathcal{P}}(t, x) - \tilde{\mathcal{P}}^n(t, x)| \leq \tilde{C}_{1, \varepsilon} n^{-\frac{1}{2} + \varepsilon}.
\]

(ii) For each initial capital \(x\),

\[
l_{n \to \infty} \tilde{\mathcal{P}}^n(x) = \tilde{\mathcal{P}}(x).
\]

Furthermore, for each \(\varepsilon > 0\) there exists a constant \(\tilde{C}_{2, \varepsilon}\) such that for any \(x, n \in \mathbb{N}\),

\[
\tilde{\mathcal{P}}^n(x) \leq \tilde{\mathcal{P}}(x) + \tilde{C}_{2, \varepsilon} n^{-\frac{1}{2} + \varepsilon}.
\]

(iii) For each \(n \in \mathbb{N}\) let \((\pi^n, \sigma^n) \in \mathcal{A}^\varepsilon \times \mathcal{F}^\varepsilon_{\mathcal{I}_0^T}\) be a perfect hedge for a double barrier knock–in option as above in the \(n\)-step CRR market with the barriers \(L, R\). Then for any \(\varepsilon > 0\) and \(n \in \mathbb{N}\),

\[
\tilde{\mathcal{P}}^n(\psi_n(\pi^n), \phi_n(\sigma^n)) \leq \tilde{C}_{2, \varepsilon} n^{-\frac{1}{2} + \varepsilon}.
\]

(iv) For any \(n \in \mathbb{N}\) let \((\tilde{\pi}^n, \tilde{\sigma}^n) \in \mathcal{A}^\varepsilon \times \mathcal{F}^\varepsilon_{\mathcal{I}_0^T}\) be the optimal hedge which is given by (5.13) below. Then

\[
l_{n \to \infty} \tilde{\mathcal{P}}^n(\psi_n(\tilde{\pi}^n), \phi_n(\tilde{\sigma}^n)) = \tilde{\mathcal{P}}(x).
\]
All the constants above are not depend on the interval $I$.

In order to prove Theorem 5.1 we should establish a result similar to Lemma 3.2. For each open interval $H$ set $\gamma^H_k = F_k(\mathcal{S}^B(n))$ and $\gamma^H_{k}\pi^B(n) = \gamma^H_k(\mathcal{S}^B(n))$ and $\mathcal{Q}^B(n,k,l) = (1 + r_n)^{-k/l}(\xi^{B,n}_{k,l} + \gamma^H_{l}\pi^B(n))$, $k, l \leq n$. Similarly to (3.8)-(3.10) define the shortfall risk by

$$R^H_k(\pi, \zeta) = \sup_{\eta \in \mathcal{A}^{B,n}} E^B(\mathcal{Q}^B(\zeta, \eta) - V_{\theta^{(n)}}^{(\pi)})^+, \quad R^H_k(\pi) = \inf_{\zeta \in \mathcal{A}^{B,n}} R^H_k(\pi, \zeta) \text{ and } R^H_n(x) = \inf_{\zeta \in \mathcal{A}^{B,n}} R^H_k(\pi).$$

Similarly to (3.9) and (3.10) for any $\pi \in \mathcal{A}^{B,n}$ set

$$U^H_k \pi = ((1 + r_n)^{-n} \gamma^H_{k,n} - V_{\theta^{(n)}}^{(\pi)})^+, \quad U^H_k = \min \left( (1 + r_n)^{-k} \xi^{H,n}_k - V_{\theta^{(n)}}^{(\pi)}, \max \left( (1 + r_n)^{-k} \gamma^B_{H,n} - V_{\theta^{(n)}}^{(\pi)} + E^B(\mathcal{Q}^B_{k+1}\mathcal{S}^B_{k+1} | \mathcal{F}^{H}_k) \right) \right), \quad k < \mu.$$}

and $\bar{\zeta}(H, \pi) = \min \{ k \left( (1 + r_n)^{-k} \xi^{H,n}_k - V_{\theta^{(n)}}^{(\pi)} \right) \}$ for $k \leq n$ and $x_1, \ldots, x_k$ set

$$d^H_k(x_1, \ldots, x_k) = 1 - \prod_{\gamma \in \gamma_k} \max_{\eta \in \mathcal{A}^{H,n}} \mathcal{Q}^B(\gamma, \eta) \left( \frac{\gamma}{\eta} \right) \in \mathcal{A}^{B,n} \times \mathcal{A}^{B,n}$$

with the functions $\psi_{x_1}^{\gamma_1} \cdots \psi_{x_k}^{\gamma_k}$ introduced after (3.11). Similarly to (3.13) define a sequence $\{H_{k,n}\}_{k=0}^n$ of functions $H_{k,n} : [0, \infty) \times \mathbb{R}^k \rightarrow \mathbb{R}$ by the following backward recursion

$$f^H_{k,n}(y, u_1, \ldots, u_n) = (f^H_n(u_1, \ldots, u_n) \gamma_{H,n}^{H,n}(u_1, \ldots, u_n) - y)^+ \text{ and }$$

$$f^H_{k,n}(y, u_1, \ldots, u_k) = \min \left( (f^H_n(u_1, \ldots, u_k) - y)^+, \max \left( (f^H_n(u_1, \ldots, u_k) \gamma_{H,n}^{H,n}(u_1, \ldots, u_k) - y)^+, \inf_{\pi \in \mathcal{A}^{B,n}} \mathcal{Q}^B(x_1, \ldots, x_k, \mathcal{F}^{H}_{k+1}) \right) \right), \quad k = n - 1, n - 2, \ldots, 0.$$
Using the same arguments as in Section 3 we obtain

\[ R^H_n(x) = \Gamma_n^H(\pi_n^H, \sigma_n^H) = J_0^{H_n}(x) = R_{n}^{B,H}(\pi_n^H, \sigma_n^H) = \tilde{R}_{n}^{B,H}(x). \]

Next we derive estimates in the spirit of Lemmas 3.3 and 3.4

**Lemma 5.2.** For any \( \gamma > 1 \) there exists a constant \( A_\gamma \) such that for any open interval \( H = (L, R), \varepsilon > 0 \) and a hedge \((\pi, \sigma) \in \mathcal{A}^B \times \mathcal{S}_{01}^B\)

\[ \tilde{R}^H(\pi, \sigma) - \tilde{R}^{He}(\pi, \sigma) \leq A_\gamma e^{1/\gamma} \]

where \( H_e = (Le^{-\varepsilon}, Re^{\varepsilon}) \).

**Proof.** Choose an open interval \( H = (L, R), \varepsilon > 0 \) and a hedge \((\pi, \sigma) \in \mathcal{A}^B \times \mathcal{S}_{01}^B\).

Since \((\tilde{Q}^H(\sigma, \tau) - \tilde{V}_{\sigma\wedge \tau})^+ \leq (\tilde{Q}^H(\sigma, \tau \wedge (\tau H \vee T))) - \tilde{V}_{\sigma\wedge \tau}^+(\tau H \vee T)) \) for any \( \tau \in \mathcal{S}_{01}^B \) then for each \( \delta > 0 \) there exists a stopping time \( \tau_1 \in \mathcal{S}_{01}^B \) such that

\[ \tilde{R}^H(\pi, \sigma) < E^B(\tilde{Q}^H(\sigma, \tau_1) - \tilde{V}_{\sigma\wedge \tau_1}^+) + \delta \text{ and } \tau_1 \geq \tau H \vee T. \]

Set \( \tau_2 = \tau_1 \wedge (\tau H \vee T) \) and \( \Gamma = (\tilde{Q}^H(\sigma, \tau_1) - \tilde{Q}^H(\sigma, \tau_2))^+ \). Since \( \{\tilde{V}_\tau\}_{\tau=0}^T \) is a supermartingale (with respect to the martingale measure) then by Jensen’s inequality,

\[ (\tilde{Q}^H(\sigma, \tau_1) - \tilde{V}_{\sigma\wedge \tau_1}^+) \leq E^B((\tilde{Q}^H(\sigma, \tau_1) - \tilde{V}_{\sigma\wedge \tau_1}^+)\mathcal{F}_{\sigma\wedge \tau_1}^B) = E^B\left(\frac{Z_{\sigma\wedge \tau_1}}{Z_{\sigma\wedge \tau_2}}(\tilde{Q}^H(\sigma, \tau_1) - \tilde{V}_{\sigma\wedge \tau_2}^+)\mathcal{F}_{\sigma\wedge \tau_1}^B\right). \]

Thus, from (2.3), 5.16 and the Hölder inequality it follows that for any \( \beta > 1 \) there exists a constant \( c_\beta^{(1)} \) such that

\[ \tilde{R}^H(\pi, \sigma) \leq \delta + E^B\left(\frac{|Z_{\sigma\wedge \tau_1} - Z_{\sigma\wedge \tau_2}|}{Z_{\sigma\wedge \tau_2}}(\tilde{Q}^H(\sigma, \tau_1) - \tilde{V}_{\sigma\wedge \tau_2}^+)\right) + E^B(\tilde{Q}^H(\sigma, \tau_1)) - \tilde{V}_{\sigma\wedge \tau_2}^+ + E^B(\Gamma_1 \leq \delta + c_\beta^{(1)} E^B|Z_{\sigma\wedge \tau_1} - Z_{\sigma\wedge \tau_2}|^\beta) + \tilde{R}^{He}(\pi, \sigma) + E^B(\Gamma_3). \]

Observe that

\[ \Gamma_3 = \Gamma_1 + \Gamma_2 + \Gamma_3 \]

where

\[ \Gamma_1 = |e^{-r(\tau_1 \wedge \sigma)} - e^{-r(\tau_2 \wedge \sigma)}|F_{\tau_1 \wedge \sigma}(S^B), \quad \Gamma_2 = |F_{\tau_1 \wedge \sigma}(S^B) - F_{\tau_2 \wedge \sigma}(S^B)| \quad \text{and} \quad \Gamma_3 = \sup_{0 \leq T \leq T} F_I(S^B). \]

In the same way as in (3.27)–(3.31) for any \( \beta > \frac{1}{2} \) (and not necessarily \( \beta = \frac{1}{2} \) as there) there exists a constant \( c_\beta^{(0)} \) such that \( E^B(\Gamma_1 + \Gamma_2) \leq c_\beta^{(0)} E^B(\sup_{0 \leq T \leq T} F_I(S^B)). \) Since \( \tau_2 \wedge \sigma - \tau_1 \wedge \sigma \leq T \wedge (\tau H - \tau H) \) then similarly to (3.32)–(3.33) it follows that \( E^B(\tau_2 \wedge \sigma - \tau_1 \wedge \sigma)^\beta \leq c_\beta^{(0)} \varepsilon \) for some constant \( c_\beta^{(1)} \). We conclude that for any \( \beta > 1 \) there exists a constant \( c_\beta^{(4)} \) such that

\[ E^B(\Gamma_1 + \Gamma_2) \leq c_\beta^{(4)} e^{1/\beta}. \]

Next, we estimate \( E^B(\Gamma_3) \). First assume that \( L > 0 \) and \( R < \infty \). Set \( x_1 = (\ln L - \ln S_0)/\kappa \), \( x_2 = (\ln R - \ln S_0)/\kappa \), \( y_1 = x_1 - \frac{\varepsilon}{2\kappa} \) and \( y_2 = x_2 + \frac{\varepsilon}{2\kappa} \) where we set \( \ln 0 = -\infty \) and \( \ln \infty = \infty \).
Using the stopping times $\tau^{(i)}$ and the probabilities $P_W$ introduced in the proof of Lemma 3.3 we observe that \( \{\tau_H \leq T < \tau_{H_c}\} \subseteq \{\tau^{(i)}_1 \leq T < \tau^{(i)}_1\} \cup \{\tau^{(i)}_2 \leq T < \tau^{(i)}_2\} \), and so

\[
(5.20) \quad P_W(\{\tau_H \leq T < \tau_{H_c}\}) \leq P_W(\{\tau^{(i)}_1 > T\} - P_W(\{\tau^{(i)}_1 > T\} + P_W(\{\tau^{(i)}_2 > T\})
\]

\[
- P_W(\{\tau^{(i)}_2 > T\}) = \int_0^\infty \frac{1}{\sqrt{2\pi t}} \sum_{i=1}^2 (|y_i| \exp(-\frac{y_i^2}{2}) - |x_i| \exp(-\frac{x_i^2}{2})) dt.
\]

Since \( \frac{d}{dt}(x \exp(-\frac{x^2}{2})) = (1 - \frac{x^2}{2}) \exp(-\frac{x^2}{2}) \leq 1 \) then it follows from the mean value theorem that \( |y_i| \exp(-\frac{y_i^2}{2}) - |x_i| \exp(-\frac{x_i^2}{2}) \leq |y_i| - |x_i| = \frac{\pi}{a} \) for any \( i \) which together with (5.20) gives

\[
(5.21) \quad P_W(\{\tau_H \leq T < \tau_{H_c}\}) \leq \frac{2\sqrt{x}}{\sqrt{\pi T}}.
\]

For the cases \( L = 0 \) and \( R = \infty \), \( P_W(\{\tau_H \leq T < \tau_{H_c}\}) \leq P_W(\{\tau^{(i)}_2 > T\}) \) and \( P_W(\{\tau_H \leq T < \tau_{H_c}\}) \leq P_W(\{\tau^{(i)}_2 > T\}) - P_W(\{\tau^{(i)}_1 > T\}) \), respectively. Thus for the above cases (5.21) holds true. By (5.21) and the Hölder inequality we see that for any \( \beta > 1 \) there exists a constant \( c_\beta^{(8)} \) such that

\[
(5.22) \quad E^B\Gamma_3 = E_W(\sup_{0 \leq x \leq T} F_1(S^B)) \leq c_\beta^{(8)} e^{1/\beta}.
\]

Finally, we estimate \( E_W|Z_{\sigma \wedge \tau_2} - Z_{\sigma \wedge \tau_1}|^\beta \). Set \( \Gamma_4 = I_{\{W_{\sigma \wedge \tau_2} - W_{\sigma \wedge \tau_1}\}} + (\frac{\beta}{2} - \frac{\beta^2}{2\kappa}) (\sigma \wedge \tau_2 - \sigma \wedge \tau_1) \). From the Burkholder-Davis-Gundy inequality it follows that there exists a constant \( c_\beta^{(7)} \) such that \( E_W\Gamma_4 \leq c_\beta^{(7)} E_w(\sigma \wedge \tau_2 - \sigma \wedge \tau_1)^{\beta/2} \). By the mean value theorem we obtain that \( (e^{\tau} - 1)^\beta \leq \beta e^{\tau} x \) provided \( 0 \leq x \leq 1 \) and since \( Z_t = \exp(\frac{t}{\kappa} W_t + (\frac{\beta}{2} - \frac{\beta^2}{2\kappa}) t) \) it follows from the Markov and Hölder inequalities that for any \( \beta > 1 \) there exists a constant \( c_\beta^{(8)} \) such that

\[
(5.23) \quad E_W|Z_{\sigma \wedge \tau_2} - Z_{\sigma \wedge \tau_1}|^\beta \leq E_W(\sup_{0 \leq x \leq T} Z_t^\beta I_{\{\Gamma_4 > 1\}} + \beta e^{\tau} E_W(\sup_{0 \leq x \leq T} Z_t^\beta I_{\{\Gamma_4 > 1\}}) + \beta e^{\tau} E_W(\sup_{0 \leq x \leq T} Z_t^\beta I_{\{\Gamma_4 > 1\}})
\]

\[
\leq c_\beta^{(8)} (P_w(\{\Gamma_4 > 1\})^{1/\beta} + \beta e^{\tau} c_\beta^{(8)} (E_W\Gamma_4)^{1/\beta}) \leq (1 + \beta e^{\tau}) c_\beta^{(8)} (E_W\Gamma_4)^{1/\beta}
\]

\[
\leq (1 + \beta e^{\tau}) c_\beta^{(8)} (c_\beta^{(7)} c_\beta^{(3)} e^{1/\beta}.
\]

Letting \( \delta \to 0 \) we complete the proof by (5.17), (5.19), (5.22) and (5.23).

Repeating the proof of the last lemma with \( \tau_H = 0 \) and a portfolio \( \pi \) satisfying \( V^\pi \equiv 0 \) we arrive at the following result.

**Corollary 5.3.** Let \( \bar{H} = (L, R) \) be an open interval satisfying \( \min(\frac{L}{R}, \frac{S_0}{R}) \leq e^\gamma \). For any \( \gamma > 1 \) there exists a constant \( \bar{A}_\gamma \) such that

\[
(5.24) \quad \gamma - \gamma^\beta \bar{A}_\gamma e^{1/\gamma}
\]

where \( \gamma^\prime = \inf_{\sigma \in [\bar{H}]} \sup_{\tau \in [\bar{H}]} E^B Q^\beta(\sigma, \tau) \) is the option price for the regular payoff function \( Q^\beta(k, l) \).

Now we are ready to prove Theorem 5.1. Let \( I = (L, R) \) be an open interval. We start with the proof of the second statement in the above theorem. Let \( x > 0 \) be an initial capital and choose \( \delta > 0 \). As before there exists \( k, 0 \leq t_1 < t_2 \ldots < t_k \leq T \) and \( 0 \leq f_\delta \in C^0(\mathbb{R}^k) \) such that the portfolio \( \pi \in \omega^B \) with \( V^\pi = E(f_\delta(B_{\delta}^{t_1}, \ldots, B_{\delta}^{t_k})|\mathcal{F}_T^\delta) \) satisfies

\[
(5.25) \quad R^\prime(\pi) < R^\prime(x) + \delta \quad \text{and} \quad V^\pi < x.
\]
For any \( n \) set

\[
\Psi_n = f_\delta \left( B_{\theta^{(n)}_{|T|}}, \ldots, B_{\theta^{(n)}_{|T|}} \right).
\]

Using the same arguments as after the formula (4.53) it follows that for sufficiently large \( n \) there exists a portfolio \( \pi' (n) \in \mathcal{A}_{3, n} \) with an initial capital less than \( x \) satisfying \( \tilde{V}^{\pi'}_{\theta^{(n)}} = \Psi_n \).

For any \( \beta > 0 \) which satisfy \( e^{\beta} < \min \left( \frac{\tilde{S}_1}{\tilde{S}_0}, \frac{\tilde{S}_2}{\tilde{S}_1} \right) \) introduce the open interval \( \tilde{I}_\beta = ( L e^{\beta}, R e^{-\beta} ) \).

From (5.14), (5.23) and Lemma 5.2 it follows that for any \( \gamma > 1 \),

\[
\hat{R}^l_n (x) - \hat{R}^l (x) \leq \delta + \hat{R}^B_n (\pi') - \hat{R}^l (\pi') \leq \delta + A_1 \beta^{1/\gamma} + \hat{R}^B_n (\pi') - \hat{R}^l (\pi').
\]

Let \( \sigma \in \mathcal{A}_{3, n} \) and \( \eta \in \mathcal{A}_{3, n} \) be such that

\[
\hat{R}^B_n (\pi') - \hat{R}^l (\pi') < 2 \delta + E^B (\tilde{Q}^{B, l, n} (\sigma, \eta, \tau_{\pi'}) - \tilde{V}^{\pi'}_{\theta^{(n)}_{\bar{\theta}^{(n)}}})^+ - \tilde{V}^{\pi'}_{\theta^{(n)}_{\bar{\theta}^{(n)}}}
\]

where \( \hat{a} = (n \wedge \min \{ i : \theta_i^{(n)} \geq \sigma \}) \| \sigma_{\bar{T}} \sigma + n \|_{\bar{T}} \). From (5.28) we obtain that

\[
\hat{R}^B_n (\pi') - \hat{R}^l (\pi') < 2 \delta + E^B (\tilde{Q}^{B, l, n} (\sigma, \eta, \tau_{\pi'}) - \tilde{V}^{\pi'}_{\theta^{(n)}_{\bar{\theta}^{(n)}}})^+ - \tilde{V}^{\pi'}_{\theta^{(n)}_{\bar{\theta}^{(n)}}},
\]

where

\[
\hat{A}_1 = |\tilde{V}^{\pi'}_{\theta^{(n)}_{\bar{\theta}^{(n)}}} - \tilde{V}^{\pi'}_{\theta^{(n)}_{\bar{\theta}^{(n)}}} |, \quad \hat{A}_2 = |\tilde{V}^{\pi'}_{\theta^{(n)}_{\bar{\theta}^{(n)}}} - \tilde{V}^{\pi'}_{\theta^{(n)}_{\bar{\theta}^{(n)}}} |, \quad \hat{A}_3 = (\tilde{Q}^{B, l, n} (\sigma, \eta, \tau_{\pi'}) - \tilde{Q}^{B, l, \eta, \tau_{\pi'}} (\sigma, \eta, \tau_{\pi'})^+).
\]

The quantities \( \hat{A}_1 \) and \( \hat{A}_2 \) can be estimated exactly as \( A_1 \) and \( A_2 \) in the formulas (4.13)–(4.14), i.e. for some constant \( C (f_\delta) \) depending only on \( f_\delta \),

\[
E^B (\hat{A}_1 + \hat{A}_2) \leq C (f_\delta) n^{-1/4}.
\]

Using the quantities \( Q^{B, (s, t)} \) and \( Q^{B, (n, k, l)} \) (introduced before the formula (4.16)) and observing that \( \sigma < \theta_{\tau_{\pi'}} \wedge T \) if \( \xi < \eta \) we obtain from (2.3) and (5.30) that

\[
A_3 \leq (Q^{B, n} (\xi_{T_n}, \eta_{T_n}) - Q^B (\sigma, \eta_{\tau_{\pi'}}))^+ + \| \mathbb{E} (G_0 (S_0) + \mathcal{L} (T + 2) (1 + \max_{0 \leq k \leq n} S_{T_k})) \|
\]

where \( \mathbb{E} = \{ \eta \geq \xi_{\tau_{\pi'}} \} \cap \{ \theta_{\tau_{\pi'}} \wedge T < \tau_\delta \} \). Similarly to (4.17)–(4.22) we see that there exists a constant \( C (1) \) such that

\[
E^B A_3 \leq C (2) n^{-1/4} (\ln n)^{3/4} + C (6) n^{-1/4} + C (1) P (\Xi)^{1/2}.
\]
Similarly to (4.23) we observe that

$$
\Xi \subseteq \left\{ \max_{0 \leq k \leq n} S^{\beta_n}_{t \wedge \theta_k^m} > e^\beta \right\} \bigcup \left\{ \min_{0 \leq k \leq n} S^{\beta_n}_{t \wedge \theta_k^m} < e^{-\beta} \right\} \subseteq \left\{ \max_{0 \leq k \leq n} \right\}
$$

$$
\left( \frac{S^{\beta_n}_{t \wedge \theta_k^m}}{S^0_{t \wedge \theta_k^m}}, \frac{S^{\beta_n}_{t \wedge \theta_k^m}}{S^0_{t \wedge \theta_k^m}} \right) > e^\beta \right\} \subset \{ |r + \mu - \frac{\kappa^2}{2} |\eta_n + \kappa \sup_{T \wedge \theta_k^m \leq t \leq \theta_k^m} |B_t - B_{\theta_k^m}^{(n)}| T \wedge \theta_k^m > \beta \}
$$

where the term $u_n$ was defined before formula (4.3). Using the Burkholder-Davis-Gandy inequality for the martingale $B_t - B_T$, $t \geq T$ it follows that for any $m > 1$ there exists a constant $\lambda_m$ such that $E^B(\sup_{T \wedge \theta_k^m \leq t \leq \theta_k^m} |B_t - B_{\theta_k^m}^{(n)}| T \wedge \theta_k^m)^m \leq \lambda_m E^B(\theta_k^m - T)^m / 2$. Thus from (4.3), (5.34) and the Markov inequality we derive that for any $m > 1$ there exists a constant $K(m)$ such that $P(\Xi) \leq \frac{K(m)n^{-m/4}}{\beta^m}$. This together with (5.27), (5.29), (5.31) and (5.23) yields that for any $\gamma, m > 1$,

$$
\bar{R}^I_n(x) - \bar{R}^I(x) \leq 3\delta + A_1 \beta^{1/\gamma} + (C(\delta) + C(\delta)) n^{-1/4} + C(\delta) n^{-1/4} |\ln n|^{-3/4} + C(\delta) \sqrt{n} \eta_n^{-1/4} \beta^{-m/4}.
$$

Thus $\bar{R}^I(x) \geq \limsup_{n \to \infty} \bar{R}^I_n(x) - 3\delta - A_1 \beta^{1/\gamma}$ and by letting $\beta, \delta \to 0$ we get that

$$
\bar{R}^I(x) \geq \limsup_{n \to \infty} \bar{R}^I_n(x).
$$

In order to compete the proof of the second statement in Theorem 5.1 we should prove (5.6). Fix $\beta > 0$ and $n \in \mathbb{N}$. Set $J^{(n, \beta)} = (\exp(-2n^{-1/4+\beta}), \exp(2n^{-1/4+\beta}))$ and let $(\pi, \sigma) = (\psi_0(\pi_1^m), \phi_0(\sigma_1^m))$ where $(\pi_1^m, \sigma_1^m)$ is the optimal hedge given by (5.13). Once again we consider the portfolio $\pi = \psi_0(\pi_1^m)$ not only as an element in $\mathcal{A}^B(x)$ but also as an element in $\mathcal{A}^B, n(x)$. From (5.14) we obtain that

$$
\bar{R}^{I,(n, \beta)}(\pi, \sigma) - \bar{R}^I_n(x) = \bar{R}^{I,(n, \beta)}(\pi, \sigma) - \bar{R}^B_n(\pi, \tilde{\pi}_n^m)
$$

where, recall, $\tilde{\pi}_n^m$ was defined in (5.13). Set $\xi = \tilde{\pi}_n^m$ then from (5.13) it follows that $\sigma = (T \wedge \theta_k^m)_{1 \leq n} + T|\xi = n$. Fix $\delta > 0$ and let $\tau \in \mathcal{A}^B, n$ be such that

$$
\bar{R}^{I,(n, \beta)}(\pi, \sigma) < \delta + E^B (\bar{Q}^{B, I, J^{(n, \beta)}}(\sigma, \tau) - \bar{V}_n^{\pi, \sigma, \tau} + E^B (\bar{Q}^{B; I, J^{(n, \beta)}}(\sigma, \tau) - \bar{V}_n^{\pi, \sigma, \tau})^+.
$$

Set $\eta = n \wedge \min\{k | \theta_k^m \geq \tau \} \in \mathcal{A}^B, n$ and let $J^{(n, \beta)} = (\exp(-n^{-1/4+\beta}), \exp(n^{-1/4+\beta}))$. Denote

$$
\tilde{\Gamma}_1 = (\bar{Q}^{B, I, J^{(n, \beta)}}(\sigma, \tau) - \bar{Q}^{B; I, J^{(n, \beta)}}(\sigma \wedge \theta_k^m, \tau \wedge \theta_k^m)^+, \text{ and } \tilde{\Gamma}_1 = (\bar{Q}^{B, I, J^{(n, \beta)}}(\sigma \wedge \theta_k^m, \tau \wedge \theta_k^m) - \bar{Q}^{B; I, J^{(n, \beta)}}(\sigma \wedge \theta_k^m, \tau \wedge \theta_k^m))^+.
$$

From (5.33) it follows that

$$
\bar{R}^{I, J^{(n, \beta)}}(\pi, \sigma) - \bar{R}^B_n(\pi, \xi) < E^B (\bar{Q}^{B; I, J^{(n, \beta)}}(\sigma \wedge \theta_k^m, \tau \wedge \theta_k^m) - \bar{V}_n^{\pi, \sigma, \tau})^+ + \delta + E^B (\tilde{\Gamma}_1 + \tilde{\Gamma}_2).
$$
In the same way as in the formulas \((4.31) - (4.37)\) we derive that
\[
E^B(\tilde{Q}^{B,1(n,\beta)}(\sigma \land \theta_{n}^{(\beta)}, \tau \land \theta_{n}^{(\beta)}) - \bar{V}_{\sigma \land \tau}^\pi)^+ - E^B(\tilde{Q}^{B,1(n,\beta)}(\sigma \land \theta_{n}^{(\beta)}, \tau \land \theta_{n}^{(\beta)}) - \bar{V}_{\theta_{n}^{(\beta),n}}^\pi)^+ \leq C^{(9)} n^{-1/4}
\]
where \(C^{(9)}\) is the same constant as in formula \((4.37)\).

Next, we estimate \(E^B \Gamma_1\). Since in our case \(\sigma < \tau\) is equivalent to \(\sigma \land \theta_{n}^{(\beta)} < \tau \land \theta_{n}^{(\beta)}\) then from \((4.3)\) it follows that
\[
\Gamma_1 \leq (Q^B(\sigma, \tau) - Q^B(\sigma \land \theta_{n}^{(\beta)}, \tau \land \theta_{n}^{(\beta)}))^+ + \mathbb{E}(G_0(\delta^B_0) + \mathcal{L}(T + 2)(1 + \sup_{0 \leq t \leq T} S^B_t))
\]
where \(\Xi_1 = \{ \tau \geq \tau_{f(n,\beta)} \} \cup \{ \tau \land \theta_{n}^{(\beta)} < \tau_{f(n,\beta)} \}\). The term \(E^B(\tilde{Q}^B(\sigma, \tau) - Q^B(\sigma \land \theta_{n}^{(\beta)}, \tau \land \theta_{n}^{(\beta)})^+\) can be estimated by the right hand side of \((4.38)\). Hence, by \((4.38)\) and the Cauchy–Schwarz inequality we obtain that
\[
E^B \Gamma_1 \leq C^{(10)} n^{-1/4} + C^{(11)} (P(\Xi_1))^{1/2}
\]
where \(C^{(10)}\) and \(C^{(11)}\) are the same constants as in the formulas \((4.38)\) and \((4.40)\), respectively. Similarly to \((4.43)\) we see that
\[
\Xi_1 \subseteq \left\{ \sup_{0 \leq t \leq T \land \theta_{n}^{(\beta)}} S^B_t > e^{-n^{-1/4} \beta} \right\} \cup \left\{ \inf_{0 \leq t \leq T \land \theta_{n}^{(\beta)}} S^B_t < e^{-n^{-1/4} \beta} \right\} \subseteq \left\{ \sup_{0 \leq t \leq T \land \theta_{n}^{(\beta)}} \max\left(\frac{S^B_t}{\theta_{n}^{(\beta)}}\right) > e^{-n^{-1/4} \beta} \right\} \subseteq \{ r + \mu - \frac{S^B_t}{\tau - \theta_{n}^{(\beta)}} + \kappa \sup_{0 \leq t \leq T} \left\{ B_t - B_{\theta_{n}^{(\beta)}} \right\} > n^{-1/4} + \beta \}.
\]
Employing the Burkholder-Davis-Gundy inequality for the martingale \(B_t - B_T \land \theta_{n}^{(\beta)}, t \geq T \land \theta_{n}^{(\beta)}\) we obtain that \(E^B(\sup_{T \land \theta_{n}^{(\beta)} \leq t \leq T} |B_t - B_{\theta_{n}^{(\beta)}}|)^m \leq \lambda_m E^B(\theta_{n}^{(\beta)} - T)^{m/2} \) for any \(m > 1\). Thus, by \((4.3), \(5.43)\) and the Markov inequality it follows that \(P(\Xi_1) \leq \frac{K^{(m-1)/m}}{n^{m/(m-1)+\beta}}\) for any \(m > 1\). This together with \((5.42)\) gives that
\[
E^B \Gamma_1 \leq C^{(10)} n^{-1/4} + C^{(11)} \sqrt{K^{(m-1)/m}} n^{-m \beta}.
\]
Finally, we estimate \(E^B \Gamma_2\). Since \(\zeta < \eta\) provided \(\sigma \land \theta_{n}^{(\beta)} < \tau \land \theta_{n}^{(\beta)}\) then by \((2.3)\),
\[
\Gamma_2 \leq (Q^B(\sigma \land \theta_{n}^{(\beta)}, \tau \land \theta_{n}^{(\beta)}) - Q^B(\bar{\tau}_{\eta}, \bar{\eta}))^+ + \mathbb{E}(G_0(\delta^B_0) + \mathcal{L}(T + 2)(1 + \sup_{0 \leq t \leq T} S^B_t))
\]
where \(\Xi_2 = \{ \tau \geq \tau_{f(n,\beta)} \} \cup \{ \eta < \tau_{f(n,\beta)} \}\). The term \(E^B(\tilde{Q}^B(\sigma \land \theta_{n}^{(\beta)}, \tau \land \theta_{n}^{(\beta)}) - Q^B(\bar{\tau}_{\eta}, \bar{\eta})^+\) can be estimated applying \((4.17)\) and \((4.43)\) which gives
\[
E^B \Gamma_2 \leq C^{(2)} n^{-1/4} (\ln n)^{3/4} + C^{(12)} n^{-1/4} + E^B(\mathbb{E}(G_0(\delta^B_0) + \mathcal{L}(T + 2)(1 + \sup_{0 \leq t \leq T} S^B_t)))).
\]
This together with the Cauchy–Schwarz inequality yields that
\[
E^B \Gamma_2 \leq C^{(2)} n^{-1/4} (\ln n)^{3/4} + C^{(12)} n^{-1/4} + C^{(11)} (P(\Xi_2))^{1/2}.
\]
Since \(\tau \land \theta_{n}^{(\beta)} \geq \theta_{(\eta-1)}^{(\beta)}\), then similarly to \((4.43)\) we obtain that \(\Xi_2 \subseteq \{ r(u_n + w_n) + \kappa \sqrt{n} > n^{-1/4} + \beta \}\). Thus \(P(\Xi_2)\) can be estimated by the right hand side of \((4.44)\) for \(\beta > \frac{1}{12}\), and
so

\[ P(\mathcal{Z}_k) \leq C(13)n^{-1/2}. \]

Since \( \delta \) is arbitrary then combining (5.37), (5.39), (5.40), (5.44), (5.46) and (5.47) we conclude that there exists a constant \( \tilde{C}^{(2)} \) such that

\[
R^{(n,\beta)}(\pi, \sigma) - R_{n}^{B,I}(\pi, \zeta) = R^{(n,\beta)}(\pi, \sigma) - R_{n}^{a}(x) \leq \tilde{C}^{(2)}n^{-1/4}(\ln n)^{3/4} + C^{(11)}\sqrt{K(m)n^{-m\beta}}.
\]

From (5.48) and Lemma [5.2] it follows that for any \( \gamma > 1 \),

\[
R^{(n,\beta)}(\pi, \sigma) - R_{n}^{B,I}(\pi, \zeta) = R^{(n,\beta)}(\pi, \sigma) - R_{n}(x) \leq \tilde{C}^{(2)}n^{-1/4}(\ln n)^{3/4} + C^{(11)}\sqrt{K(m)n^{-m\beta}} + A_1\gamma^{1/2}n^{-1/2}\beta.
\]

Let \( 0 < \varepsilon < \frac{1}{4} \) and set \( \beta = \frac{\varepsilon}{\gamma}, \gamma = \frac{1/4-\varepsilon/2}{1/4-\varepsilon} > 1 \) and \( m = \frac{1}{\varepsilon} \). From (5.49) we obtain that there exists a constant \( \tilde{C}_2 \) such that

\[
R^{(n,\beta)}(\pi, \sigma) - R_{n}(x) = R^{(n,\beta)}(\pi, \sigma) - R_{n}^{a}(x) \leq \tilde{C}_2\varepsilon n^{-\frac{1}{4}+\varepsilon}.
\]

Combining (5.36) and (5.50) we complete the proof of the second and the fourth statements in Theorem 5.1.

Next, we prove the first statement in Theorem 5.1. Assume that \( \mu = 0 \). In this case \( \tilde{\gamma}^{I} = R^{(0)}(0) \) and \( \hat{\gamma}^{I} = R_{n}^{a}(0) \). Let \( 0 < \varepsilon < \frac{1}{4} \) and fix \( n \) assuming, first, that \( \exp(n^{-1/4+\varepsilon/2}) \geq \min\left(\frac{R_{0,n}}{\varepsilon}, \frac{\Delta}{\varepsilon}\right) \). Using Corollary 5.3 for \( \gamma = \frac{1/4-\varepsilon/2}{1/4-\varepsilon} > 1 \) we get that

\[
\gamma - \tilde{\gamma}^{I} \leq \tilde{\lambda} n^{-1/4+\varepsilon}.
\]

From Theorem 2.1 in [1] it follows that there exists a constant \( C \) such that \( |\gamma_n - \gamma| \leq Cn^{-1/4}(\ln n)^{3/4} \). This together with (5.51) yields that for \( n \) as above,

\[
\hat{\gamma}^{I} - \tilde{\gamma}^{I} \leq \gamma_n - \gamma \leq Cn^{-1/4}(\ln n)^{3/4} + \tilde{\lambda} n^{-1/4+\varepsilon}.
\]

Next, assume that \( \exp(n^{-1/4+\varepsilon/2}) < \min\left(\frac{R_{0,n}}{\varepsilon}, \frac{\Delta}{\varepsilon}\right) \). In this case we can apply (5.55) for \( \beta = n^{-1/4+\varepsilon/2}, \gamma = \frac{1/4-\varepsilon/2}{1/4-\varepsilon} > 1 \) and \( m = \frac{1}{\varepsilon} \), with \( C'(f_\delta) = 0 \) since portfolios with zero initial capital will preserve zero value, and so the left hand side of (5.51) is zero. Thus we can let \( \delta \downarrow 0 \) in (5.53) and obtain that for some constant \( C(\varepsilon) \)

\[
\hat{\gamma}^{I} - \tilde{\gamma}^{I} \leq C(\varepsilon) n^{-1/4+\varepsilon}.
\]

From (5.6) we obtain that there exists a constant \( \tilde{C}^{\varepsilon} \) such that for any \( n \),

\[
\hat{\gamma}^{I} - \tilde{\gamma}^{I} \leq \tilde{C}^{\varepsilon} n^{-1/4+\varepsilon}.
\]

Combining (5.52), (5.53) and (5.54) we complete the proof of the first statement in Theorem 5.1.

Finally, we prove the third statement in Theorem 5.1. Fix \( n > 0 \). Clearly, \( V_{\pi_n}(\sigma_n^{B,k}) \leq Q_{\pi_n}(\sigma_n^{B,k}) \) for any \( k \), and so

\[
V_{\pi_n}^{B,k} = \Pi_n(\psi_n^{B,n}(\sigma_n^{B,k})) \geq \Pi_n(\tilde{Q}^{B,n}(\sigma_n^{B,k})) = \tilde{Q}^{B,n}(\xi, k).
\]

where \( (\pi, \zeta) = (\psi_n(\pi_n^{B,k}), \Pi_n(\sigma_n^{B,k})) \in \mathcal{Q}^{B,n}(\gamma_n^{B,k}) \times \mathcal{Q}^{B,n}_{0,n} \). Thus, \( R^{B,I}(\pi, \zeta) = 0 \). Set \( \sigma = \phi_n(\sigma_n) \in \mathcal{Q}^{B,n}_{0,n} \) then \( \sigma = (T \wedge \Theta_n(\xi))\|_{\xi \in n} + T\|_{\xi \in n} \) and applying (5.50) we obtain that

\[
R^{(n,\beta)}(\pi, \sigma) \leq R^{B,I}(\pi, \zeta) + \tilde{C}_2\varepsilon n^{-\frac{1}{4}+\varepsilon} = \tilde{C}_2\varepsilon n^{-\frac{1}{4}+\varepsilon}.
\]
completing the proof.

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