On the Tate spectrum of tmf at the prime 2

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Abstract
Computations involving the Mahowald invariant prompted Mahowald and Shick to develop the slogan: “the Mahowald invariant of $v_n$-periodic homotopy is $v_n$-torsion”. While neither a proof, nor a precise statement, of this slogan appears in the literature, numerous authors have offered computational evidence in support of its fundamental idea. The Mahowald invariant is closely related to his inverse limit description of the Tate spectrum, and computations have shown the Tate spectrum of $v_n$-periodic cohomology theories to be $v_n$-torsion. The purpose of this paper is to split the Tate spectrum of tmf as a wedge of suspensions of ko, providing yet another example in support of the slogan to the existing literature.

Keywords  Topological modular forms · Tate spectrum · Mahowald invariant

Mathematics Subject Classification  55P42 · 55T15

1 Introduction

Let $\lambda$ denote the canonical line bundle over $\mathbb{R}P^\infty = B(\mathbb{Z}/2\mathbb{Z})$. For $\ell \in \mathbb{Z}$, define $P_\ell$ to be the Thom spectrum of $\ell \lambda$. Induced maps on the level of Thom spectra give a naturally defined inverse system of projective spaces

$$
\cdots \rightarrow P_{n-1} \xrightarrow{j_{n-1}} P_n \xrightarrow{j_n} P_{n+1} \rightarrow \cdots
$$

Lin [8] demonstrated that the homotopy limit of (1) has the homotopy type of a desuspended 2-complete sphere, i.e.,

$$
\lim P_n \simeq \mathring{S}^{-1},
$$

where $(\mathring{\cdot})$ denotes 2-completion. Suppose $X$ is a finite complex and consider a cohomotopy class $\alpha \in [X, S^0]$. The equivalence (2) guarantees the existence of a largest $\ell \in \mathbb{Z}$ such
that the composite $\Sigma^{j-1} X \xrightarrow{\alpha} S^{-1} \to P_\ell$ is nontrivial. In particular, this induces a map $R(\alpha) : \Sigma^{j-1} X \to S^\ell$, the homotopy class of which is called the Mahowald invariant of $\alpha$. Computations inside the EHP sequence led Mahowald and Ravenel [14] to conjecture that the Mahowald invariant of a $v_n$-periodic element is $v_{n+1}$-periodic. This prompted Mahowald and Shick [13] to discuss the related slogan: “The Mahowald invariant of $v_n$-periodic homotopy is $v_n$-torsion”. They show, for a finite complex $X$ having a $v_n$-self map, that

$$\lim_{\leftarrow} \{ [X, P_\ell(v_n^{-1})] \} = 0. \tag{3}$$

In particular, Mahowald and Shick point out that if $\alpha \in [X, S^0]$ is $v_n$-periodic then its Mahowald invariant, at least when considered as an element of $[X, P_\ell]$, is $v_n$-torsion.

Let $X$ be a spectrum. Mahowald’s description of the Tate spectrum of $X$ is the homotopy inverse limit

$$t(X) = \Sigma \lim_{\leftarrow} (P_\ell \wedge X). \tag{4}$$

If $X$ is a finite spectrum, then (4) implies the Tate spectrum functor corresponds to completion at the prime 2. This is certainly not the case for all spectra since homotopy limits do not commute with the smash product.

While neither a proof, nor even a precise statement of the phenomenon suggested by the slogan has appeared in the literature, many authors have demonstrated the Tate spectrum functor sends $v_n$-periodic cohomology theories to $v_{n-1}$-periodic theories:

1984: Davis and Mahowald [5], for $p = 2$, show $t(\text{ko}) \simeq \bigvee_{j \in \mathbb{Z}} \Sigma^{4j} \tilde{H}\mathbb{Z}$;

1986: Davis et al. [4] demonstrate that, if $p$ is any prime and $q = 2(p-1)$, then there are equivalences of $p$-complete spectra $t(BP(2)) \simeq \prod_{j \in \mathbb{Z}} \Sigma^{4j} BP(1)$, and conjecture a similar splitting of $t(BP(n))$;

1998: Ando et al. [1] prove the existence of a ring isomorphism $(tE(n)_*)_{I_{n-1}}^{\wedge} \simeq E(n-1)_{+}((x))_{I_{n-1}}^{\wedge}$, where $I_{n-1} = (p, v_1, \ldots, v_{n-2})$ and construct a map of spectra

$$\bigvee_{j \in \mathbb{Z}} \Sigma^{2j} E(n-1) \to tE(n)_{I_{n-1}}^{\wedge},$$

which, after completion at $I_{n-1}$ (or equivalently after localization with respect to the $(n-1)st$ Morava $K$-theory) induces the isomorphism of homotopy groups.

The purpose of this paper is to provide yet another example to the literature. Let $\text{tmf}$ denote the connective ring spectrum of topological modular forms (see [2,6,10]) at the prime 2. The main theorem is:

**Theorem 1.1** There is a 2-local weak equivalence of spectra

$$t(\text{tmf}) \simeq \prod_{i \in \mathbb{Z}} \Sigma^{8i} \text{ko}. \tag{5}$$

In the context of the above machinery, computations involving the homotopy of $t(\text{tmf})$ greatly benefit from Mahowald’s inverse limit description of the Tate spectrum. However, the Tate spectrum functor conserves other properties, such as ring structure, of the spectrum. This fact, however, is not immediately clear from the inverse limit point of view. On the other hand, such a structure is clear when placed in the framework established by Greenlees and May [7]. In their notation: let $G$ be a compact Lie group, $EG$ a free contractible $G$-space and $\tilde{E}G$ the cofiber of the map $EG_+ \to S^0$. If $k_G$ is a $G$-spectrum, then $t(k_G) = F(EG_+, k_G) \wedge \tilde{E}G$, where $F(EG_+, k_G)$ is the function $G$-spectrum of maps $EG_+ \to k_G$, is the Tate spectrum of $k_G$. Since $EG_+$ is equipped with a coproduct, if $k_G$ is a ring spectrum then $F(EG_+, k_G)$ is also a ring spectrum. Combining this with the product on $\tilde{E}G$, $t(k_G)$ is also a ring. Lewis-May fixed points give a lax monoidal functor, so $t(k_G)^G$ also has a ring structure.
The link to Mahowald’s inverse limit description is as follows [7]: If \( G \) is cyclic of order 2 and \( k_G \) is the equivariant \( G \)-spectrum associated to a non-equivariant spectrum \( k \), then there is a homotopy equivalence \( \mathfrak{t}(k_G)^G \simeq \Sigma \lim (P_t \wedge k) = \mathfrak{t}(k) \).

Before we are able to restate our main result, we recall the following notations and observations from [1]. Let \( E \) be a spectrum. For \( n > 0 \), we denote by \( E[x^n] \) the spectrum \( \sqrt{\sum_{i=0}^{\infty} \Sigma^{ni} E} \). Here \( x^n \) is just a placeholder. We also denote by \( E((x^n)) \) the limit \( \lim_{i} \sqrt{\sum_{i=-i}^{\infty} \Sigma^{kn} E} \). This notation is directly taken from [1]. Furthermore, when \( E \) is a connective spectrum, there is a weak equivalence

\[
\prod_{i=-\infty}^{\infty} \Sigma^{ni} E \cong E((x^n)).
\]

This is a well-known result (see [1]). We include its proof in Proposition 3.5 for completeness (the reader should be warned that \( n \geq 1 \) and \( E \) connective are necessary for this weak equivalence to hold).

Finally, whenever \( E \) is a ring, \( E((x^n)) \) is a ring spectrum as well (see Proposition 3.6 or [1]). However, there are possibly many different ring structures on \( E((x^n)) \), so it is unclear which one guarantees that the weak equivalence of Theorem 1.1 is a weak equivalence of ring spectra.

**Outline of the proof** The approach we use here is similar to [1] with a twist. The main difference with the latter is that we want to get an explicit description of the stages \( \Sigma^{-1} \text{tmf} \wedge P_t \). This is a complicated problem as stated, but we introduce in Definition 2.3 a small spectrum \( L_0 \subset P_{-1} \) which plays a crucial role in our proof. We first show that the computation of the Tate spectrum of \( \text{tmf} \) can be done by neglecting the part coming from \( L_0 \) (see Lemma 3.3). Then, we identify \( \Sigma^{-1} \text{tmf} \wedge P_{-1}/L_0 \cong \text{ko}[x^8] \) (see Lemma 3.4) by an Adams spectral sequence computation. The reduction to the analysis of the connective spectrum \( \Sigma^{-1} \text{tmf} \wedge P_{-1}/L_0 \) is necessary for the identification, since the Adams spectral sequence converges for connective spectra. This reduces the proof of our main result to the computation of \( \lim \Sigma^{-8i} \text{ko}[x^8] \), where the limit is taken along the projections (see Lemma 3.4). Note that this is also the reason why we only obtain a 2-local weak equivalence, since the Adams spectral sequence only captures 2-local information.

### 2 Some particular \( \mathcal{A}(2) \)-modules

Let \( \mathcal{A} \) denote the mod-2 Steenrod algebra generated by the squaring operations \( \{Sq^{2^i}\}_{i \geq 0} \). Let \( M \) be an \( \mathcal{A} \)-module, and consider the \( \mathcal{A} \)-modules \( \mathcal{A}/\mathcal{A}(n) \otimes M \) via the diagonal action and \( \mathcal{A} \otimes_{\mathcal{A}(n)} M \) via left action on \( \mathcal{A} \). There is a shearing isomorphism

\[
\Phi : \mathcal{A}/\mathcal{A}(n) \otimes M \rightarrow \mathcal{A} \otimes_{\mathcal{A}(n)} M
\]

defined by \( \Phi(a \otimes m) = \sum a' \otimes a'' m \), where \( \psi(a) = \sum a' \otimes a'' \) is the coproduct on \( \mathcal{A} \). This isomorphism induces a change-of-rings isomorphism at the level of Ext-groups

\[
\text{Ext}^{i,i}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}(n) \otimes M, N) \cong \text{Ext}^{i,i}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(n)} M, N) \cong \text{Ext}^{i,i}_{\mathcal{A}(n)}(M, N).
\]

Note that this isomorphism of Ext-groups can be used to simplify computations within the Adams spectral sequence \( E_2 \)-term, reducing the computation from the whole Steenrod algebra to its subalgebra \( \mathcal{A}(n) \). For instance, since \( H^*(\text{tmf}) \cong \mathcal{A}/\mathcal{A}(2) [12,15] \), to compute the homotopy groups of \( P_t \wedge \text{tmf} \) it suffices to understand the left \( \mathcal{A}(2) \)-module structure of \( H^*P_t \).
The following two propositions are results of Lin et al. [9, p. 459].

**Proposition 2.1** As a $\mathbb{F}_2$-vector space, $H^* P_i = x^i \mathbb{F}_2[x]$, where $x$ is in degree 1. The action of $A$ on $H^* P_i$ is determined by

$$Sq^j x^k = \binom{k}{j} x^{j+k}. \quad (8)$$

Denote by $\mathbb{F}_2[x^{\pm 1}]$ the colimit of these $A$-modules. Note that this is certainly not the cohomology of the limit given by (2). A consequence is the following interpretation of the $A(2)$-module structure of some quotients of $\mathbb{F}_2[x^{\pm 1}]$. Let $F_\ell$ be the sub-$A$-$A(2)$-module of $\mathbb{F}_2[x^{\pm 1}]$ generated by the classes in degrees less than $\ell$. The following result is one of the key steps in the analysis of $\mathbb{F}[x^{\pm 1}]$ in [9].

**Proposition 2.2** [9, Lemma 2.2] There is an isomorphism of $A(2)$-module

$$Gr_{(\mathbb{F}_2[x^{\pm 1}])} \cong \bigoplus_{j \in \mathbb{Z}} \Sigma^{8j-1} A(2)/A(1), \quad (9)$$

where $Gr_{(\mathbb{F}_2[x^{\pm 1}])}$ denotes the associated graded of $\mathbb{F}_2[x^{\pm 1}]$ under the increasing filtration $\mathcal{F} = \{ F_\ell : \ell \equiv -1 \mod 8 \}$.

**Proof** The asserted result follows from [9, Lemma 2.2] which states that there is an exact sequence of $A(2)$-modules

$$\Sigma^{-1} A(2)/A(1) \to \mathbb{F}[x^{\pm 1}]/F_{-1} \to \mathbb{F}[x^{\pm 1}]/F_1,$$

together with the fact that the modules $\mathbb{F}[x^{\pm 1}]/F_{8j-1}$ for $j \in \mathbb{Z}$ are isomorphic up to a suspension, as observed in [9, p. 461].

**Definition 2.3** Let $L_0$ denote the spectrum $(S^1 \cup_2 e^2 \cup e^4 \cup_3 e^8)_+$, which exists since the Toda bracket $\langle 2, \eta, \nu \rangle$ contains zero. By construction, $H^* L_0 = \mathbb{F}_2[1, x, Sq^1 x, Sq^2 Sq^1 x, Sq^4 Sq^2 Sq^1 x]$, with the action of $A$ indicated by the names of the elements.

**Proposition 2.4** There is a filtration of $A(2)$-modules of $H^* P_{-1}$ with associated graded $H^* L_0 \oplus \bigoplus_{j \geq 0} \Sigma^{8j-1} A(2)/A(1)$.

**Proof** Define $\mathcal{F}' = \{ F_\ell \cap H^* P_{-1} : \ell \equiv -1 \mod 8 \}$, to be the restriction of $\mathcal{F}$ to the sub-$A$-$A(2)$-module $H^* P_{-1}$. By construction, $H^* L_0 \cong F_{-1} \cap H^* P_{-1}$, hence the $A(2)$-module isomorphism (9) yields the result.

**Proposition 2.5** There is a tmf-module map $\iota : \text{tmf} \wedge P_{-1} \to \text{tmf} \wedge L_0$ realizing the inclusion $\iota : H^* L_0 \to H^* P_{-1}$ of $A(2)$-modules. Explicitly,

$$H^* \iota : A/\langle A(2) \otimes H^* L_0 \to A/\langle A(2) \otimes H^* P_{-1}$$

is $A/\langle A(2) \otimes \iota$.

**Proof** By adjunction, it suffices to show that the corresponding map of spectra $P_{-1} \to \text{tmf} \wedge L_0$ survives the Adams spectral sequence. By the change-of-rings isomorphism (7) the Adams $E_2$-page computing the desired homotopy classes is

$$E_2 \cong Ext_{A(2)}(H^* L_0, x^{-1} \mathbb{F}_2[x]). \quad (10)$$
One concludes the argument by observing that the relevant extension group vanishes whenever $t - s = -1$. Indeed, there is a spectral sequence

$$E_1 \cong \text{Ext}^s_{\mathcal{A}(2)}(H^*L_0, H^*L_0) \oplus \bigoplus_{j \geq 0} \text{Ext}^{s+j}_{\mathcal{A}(2)}(H^*L_0, \Sigma^{8j-1} \mathcal{A}(2)/\mathcal{A}(1))$$

which abuts to the $E_2$-term (10). It then suffices to check there is nothing in degree $t - s = -1$ in (11).

The first summand, $\text{Ext}^s_{\mathcal{A}(2)}(H^*L_0, H^*L_0)$, requires a direct computation. The relevant Ext chart is displayed in Fig. 1 clearly has no classes in stem $t - s = -1$.

The observant reader will note, as an $\mathcal{A}(2)$-module, the vector space dual of $\mathcal{A}(2)/\mathcal{A}(1)$ is $\Sigma^{-17} \mathcal{A}(2)/\mathcal{A}(1)$. In particular, by adjunction and change-of-rings, the second summand is isomorphic to $\bigoplus_{j \geq 0} \text{Ext}^{s+j}_{\mathcal{A}(1)}(\Sigma^{-8(j+2)}H^*L_0, \mathbb{F}_2)$. This is a straightforward computation in $\mathcal{A}(1)$-modules, since $H^*L_0 = \mathbb{F}_2 \oplus \Sigma QM \oplus \Sigma^8 \mathbb{F}_2$ where $QM$ is the question mark complex, yielding the $E_2$-term of $ko \cup ko(1) \cup \Sigma^8 ko$ which are all 8-periodic. Figure 2 displays $\text{Ext}^s_{\mathcal{A}(1)}(H^*L_0, \mathbb{F}_2)$, and shows there cannot be a class in stem $t - s = -1$ for degree reasons.

**Definition 2.6** Choose a tmf-module map $\pi$ satisfying the hypothesis of Proposition 2.5. Let $t(tmf)_{-1}$ be the fiber of $\pi$.

Together with the isomorphism $\Sigma^8 tmf \wedge P_n \cong tmf \wedge P_{n+8}$ for all $n \in \mathbb{Z}$ [3], we have the following:

**Proposition 2.7** There is a family of cofiber sequences

$$\Sigma^{-8k} t(tmf)_{-1} \to tmf \wedge P_{-1-8k} \to \Sigma^{-8k} tmf \wedge L_0$$

for all $k \in \mathbb{Z}$.

### 3 The Tate spectrum of tmf

The proof of Theorem 1.1 is decomposed into a series of lemmas. The argument presented here is similar to that given for the splitting of $t(ko)$ [5]. In what follows, we denote $\Sigma t(tmf)_{-1}$ by $\widetilde{t}(tmf)$. We will show that $\widetilde{t}(tmf)$ splits as a wedge of copies of $ko$. This is done in two steps: first, we show that the cohomology of $\widetilde{t}(tmf)$ splits as a module over $\mathcal{A}$ in Lemma 3.2; second, we compute the $E_2$-page of the Adams spectral sequence converging to $[\widetilde{t}(tmf), ko]$ to show that there are classes in $[\widetilde{t}(tmf), \Sigma^j ko]$ for all $j \in \mathbb{Z}$ that realize the splitting (this is done in Lemma 3.4). The main result we need is the following computation of Mahowald [11, Theorem 2.8]:

**Lemma 3.1** If $s > 0$ then $\text{Ext}^s_{\mathcal{A}}(H^*(ko \wedge ko), \mathbb{F}_2))$ is isomorphic to

$$\bigoplus_{\ell \geq 0} \left( \text{Ext}^{2+4\ell - \alpha(\ell),-4\ell - \alpha(\ell)}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2) \oplus \text{Ext}^{2+4\ell - \alpha(\ell),-4\ell - \alpha(\ell)-4}_{\mathcal{A}(1)}(N, \mathbb{F}_2) \right)$$

where $N = H^* bsp$ and $\alpha(\ell)$ is the number of ones in the dyadic expansion of $\ell$.

Note that, in particular, for all $s \geq 1$, the group $\text{Ext}^s_{\mathcal{A}(1)}(\mathbb{F}_2, \mathcal{A}/\mathcal{A}(1))$ is zero as soon as $t - s \equiv 3 \mod 4$. 

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Lemma 3.2 There is an isomorphism of $A$-modules $H^*t(\text{tmf}) \cong A//A(1)[x^8]$ where $x^8$ is in degree 8.

Proof By Propositions 2.4, 2.5, and Definition 2.6 there is a filtration of $H^*t(\text{tmf})$ whose associated graded is $\bigoplus_{j \geq 0} \Sigma^{8j-1} A(2) // A(1)$. To conclude, we show inductively that there are no non-trivial extensions.
\[ \text{Ext}^{1,0}_A \left( \bigoplus_{j=0}^n \Sigma^{8j}A/\langle A(1), A/\langle A(1) \rangle \right) = \text{Ext}^{1,8}_A(F_2, A/\langle A(1)[x^8]/(x^n) \rangle). \]

By Lemma 3.1, this Ext vanishes whenever \( t - s \equiv -1 \mod 4 \). The degrees in which we are looking for non-trivial extensions are of the form \((1, 8j)\). The result follows.

Note that, in particular, the Adams spectral sequence computing \( \pi_*(\overline{t}(\text{tmf})) \) gives homotopy classes \( x^{8j} \in \pi_{8j}(\overline{t}(\text{tmf})) \). Indeed, the \( E_2 \)-page of this spectral sequence is given by

\[ E_2 = \text{Ext}_A(H^*(\overline{t}(\text{tmf})), F_2) \cong \text{Ext}_{A(1)}(F_2[x^8], F_2) \tag{14} \]

and Lemma 3.1 ensures that the elements of \( x^{8j} \) survive the Adams spectral sequence for degree reasons.

**Lemma 3.3** We can arrange the cofiber sequences (12) in a commutative diagram

\[
\begin{array}{ccc}
\Sigma^{-8k}t(\text{tmf}) & \longrightarrow & \Sigma \text{tmf} \wedge P_{-8k} \\
\downarrow & & \downarrow \\
\Sigma^{-8k+8}t(\text{tmf}) & \longrightarrow & \Sigma \text{tmf} \wedge P_{7-8k} \\
\end{array}
\]

Moreover, \( t(\text{tmf}) \) is the limit of both the leftmost and middle terms.

**Proof** The existence of the left commutative square in (15) comes from the identification \( \Sigma^8 \text{tmf} \wedge P_n \simeq \text{tmf} \wedge P_{n+8} \) for all \( n \in \mathbb{Z} \). This gives the asserted compatibility between the cofiber sequences.

We now take the inverse limit on the three terms. By definition, the limit of the middle term is \( \Sigma^{-1}t(\text{tmf}) \). Note that the composite of the right-most vertical maps in (15) is zero, since it belongs to \([L_0, \text{tmf} \wedge L_0]_{-16} = 0\), by adjunction. Thus the limit is contractible, and the induced map \( \varinjlim \Sigma^{-8k}t(\text{tmf}) \rightarrow t(\text{tmf}) \) is a weak equivalence.

In light of Lemma 3.3, we reduce our analysis of \( t(\text{tmf}) \) to \( \overline{t}(\text{tmf}) \). This, in turn, reduces to a simple Adams spectral sequence computation.

**Lemma 3.4** There is a weak equivalence \( \overline{t}(\text{tmf}) \simeq \mathbb{ko}[x^8] \). Moreover, the maps \( \Sigma^{-8k}t(\text{tmf}) \rightarrow \Sigma^{-8k+8}t(\text{tmf}) \) coincide with the projection \( x^{-8k} \mathbb{ko}[x^8] \rightarrow x^{-8k+8} \mathbb{ko}[x^8] \).

**Proof** We argue as follows: we build a map \( \phi_j : \overline{t}(\text{tmf}) \rightarrow \Sigma^8 \mathbb{ko} \) for all \( j \geq 0 \), which realizes the injection \( \phi_j^* : \Sigma^8 A/\langle A(1), A/\langle A(1) \rangle \} \{x^{\pm 8}\} \). Then, we consider the coproduct of the \( \phi_j \), for \( j \geq 0 \), and show that it is the desired weak equivalence.

First, fix \( j \geq 0 \). To build the map \( \phi_j \), we compute the Adams spectral sequence converging to \([t(\text{tmf}), \Sigma^8 \mathbb{ko}]\). The \( E_2 \)-page of this spectral sequence is \( \text{Ext}_A^{s, t+8j}(A/\langle A(1), A/\langle A(1) \rangle) \} \{x^{\pm 8}\} \). By Lemma 3.1, any element in \((s, t + 8j) = (0, 8j)\) cannot be the source of a differential for degree reasons. Consequently, any \( A \)-module map \( \Sigma^8 H^* \mathbb{ko} \rightarrow H^*\overline{t}(\text{tmf}) \) comes from a map \( \overline{t}(\text{tmf}) \rightarrow \Sigma^8 \mathbb{ko} \). Choose one map corresponding to the inclusion of \( \Sigma^8 A/\langle A(1) \rangle \) into \( A/\langle A(1) \rangle \{x^{\pm 8}\} \) and call it \( \phi_j \).

Let \( \phi : \overline{t}(\text{tmf}) \rightarrow \mathbb{ko}[x^8] \) be the wedge of the above \( \phi_j \) for \( j \geq 0 \). By construction, it induces an isomorphism in cohomology between connective spectra, and thus it is a 2-local equivalence.

To conclude the proof, we need to show the assertion about the maps \( \pi : \Sigma^{-8k}t(\text{tmf}) \rightarrow \Sigma^{-8k+8}t(\text{tmf}) \). By 8-periodicity, we only have to check the case \( k = 0 \). Recall that, since...
\[ H^* \pi \text{ realizes the inclusion } \mathcal{A}^8 / \mathcal{A}(1)[x^8] \rightarrow \mathcal{A}^8 / \mathcal{A}(1)[x^8], \] we automatically know that the cohomology of its fiber is \( \mathcal{A}^8 / \mathcal{A}(1) \), i.e. is a copy of \( \mathbb{K} \) (more precisely, it is weakly equivalent to a suspension of \( \mathbb{K} \), by an immediate Adams spectral sequence computation). Now, there is no non-trivial map \( \Sigma^8 \mathbb{K}[x^8] \rightarrow \Sigma \mathbb{K} \) (see [11]). The morphism \( \pi \) has to be the desired one.

**Proof** (Proof of Theorem 1.1) By Lemma 3.3, \( t(\text{tmf}) \sim \lim_{\leftarrow} \Sigma^{-8k} \mathbb{K} \) which, by Lemma 3.4, is weakly equivalent to \( \lim_{\leftarrow} \Sigma^{-8k} \mathbb{K}[x^8] \sim \mathbb{K}((x^8)) \).

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**Appendix: Elementary properties of the spectrum \( E((x)) \)**

The aim of this appendix is to provide a proof of the statements about \( E((x^n)) \) asserted in the introduction.

Let \( E \) be a connective commutative ring spectrum and \( n \) a positive integer. Recall that \( E((x^n)) \) stands for the limit

\[ \lim_{i} \bigvee_{k=-i}^{\infty} \Sigma^{kn} E. \]

This should be thought of as a topological version of the ring of Laurent series in \( x^n \) which are not bounded below, but bounded above (i.e. of the form \( \sum_{i=-\infty}^{f_0} a_i x^{ni} \)).

**Proposition 3.5** There is a weak equivalence

\[ \prod_{i=-\infty}^{\infty} \Sigma^{ni} E \simeq E((x^n)). \]

**Proof** First, note that for all \( j \in \mathbb{Z} \), the natural map

\[ \prod_{i=j}^{\infty} \Sigma^{ni} E \rightarrow \bigvee_{i=j}^{\infty} \Sigma^{ni} E \]

is an isomorphism in homotopy groups since \( E \) is connective and \( n > 0 \). Thus, the previous map is a weak equivalence. Moreover, it makes the square

\[
\begin{array}{ccc}
\prod_{i=j}^{\infty} \Sigma^{ni} E & \rightarrow & \bigvee_{i=j}^{\infty} \Sigma^{ni} E \\
\downarrow & & \downarrow \\
\prod_{i=j-1}^{\infty} \Sigma^{ni} E & \rightarrow & \bigvee_{i=j-1}^{\infty} \Sigma^{ni} E
\end{array}
\]

commute. Taking the limit over \( i \) on both sides gives the result.

**Proposition 3.6** The natural maps

\[
\left( \bigvee_{k=-i}^{\infty} \Sigma^{kn} E \right) \wedge \left( \bigvee_{k=-j}^{\infty} \Sigma^{kn} E \right) \rightarrow \left( \bigvee_{k=-i-j}^{\infty} \Sigma^{kn} E \right)
\]

induces a commutative and associative \( E \)-algebra structure up to homotopy on \( E((x^n)) \).
Proof. Note that the natural maps described in the statement of the Proposition induces a map of functors indexed by \( \mathbb{N} \times \mathbb{N} \) (the appropriate squares commute on the nose). In particular, functoriality of the Holim in this diagram category finishes the proof of the existence of such a product. \( \square \)

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