Abstract. We consider the structures formed by isogenies of abelian varieties with polarizations that are not necessarily principal, specifically with the $\ell$-polarizations we have previously defined. Our primary interest is in superspecial abelian varieties, where the isogenies are related to quaternionic hermitian forms. We first consider isogeny graphs. We show that these $\ell$-isogeny graphs are a generalized Brandt graph and construct them entirely in terms of definite quaternion algebras. We prove that they are connected and give examples to show that the regular graphs obtained are sometimes Ramanujan and sometimes not. Isogenies of $\ell$-polarized abelian varieties can be closed under composition, with the consequence that such isogenies naturally form semi-simplicial complexes as introduced by Eilenberg and Zilber in 1950 (later also called $\Delta$-complexes)—the higher-dimensional analogues of multigraphs. We show that these isogeny complexes can be constructed from the arithmetic of hermitian forms over definite quaternion algebras and that they are quotients of the Bruhat-Tits building of the symplectic group by the action of a quaternionic unitary group. Working with quaternions these isogeny graphs and complexes are amenable to machine computation and we include many examples, concluding with a detailed examination of the $[2]$-isogeny complexes of superspecial abelian surfaces in characteristic 7.

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1. Introduction

Fix primes $p$ and $\ell$ with $\ell \neq p$. Isogenies of degree $\ell$ (or $\ell$-isogenies) of supersingular elliptic curves in characteristic $p$ are naturally organized into isogeny graphs, which can in turn be described using definite rational quaternion algebras. The only subtlety is that there are variations depending on how polarizations and isogenies are identified, leading to three different isogeny graphs in Jordan-Zaytman [JZa]: the big isogeny graph $Gr_1(\ell, p)$, the little isogeny graph $gr_1(\ell, p)$, and the enhanced isogeny graph $\tilde{gr}_1(\ell, p)$. The literature seems to have only one isogeny graph: this ubiquitous graph is the big isogeny graph for us.

Distinguishing between these three makes many results clearer and more precise. For example, the little and enhanced isogeny graphs are uniformized by the Bruhat-Tits building $\Delta = \Delta_\ell$ of $SL_2(\mathbb{Q}_\ell)$, whereas the big isogeny graph is not [JZa] Sect. 9.1. The big isogeny graph is a regular graph, so it is natural to ask if it is Ramanujan, whereas the little and enhanced isogeny graphs are not. And it is the little and enhanced isogeny graphs which arise from the bad reduction of Shimura curves and not the familiar big isogeny graph [JZa] Sect. 9.2.

Fix a supersingular elliptic curve $E$ over $\overline{\mathbb{F}}_p$. Then $\mathcal{O} = \mathcal{O}_E = \text{End}(E)$ is a maximal order in the definite rational quaternion algebra $\mathbb{H}_p$ ramified at $p$. A superspecial abelian variety $A$ over $\overline{\mathbb{F}}_p$ of dimension $g$ is isomorphic to $E^g$; this is equivalent to requiring that $A$ is isomorphic to a product of supersingular elliptic curves. A polarization $\lambda$ of $A$ is a symmetric isogeny $\lambda : A \to \hat{A} = \text{Pic}^0(A)$ subject to the positivity condition involving the Poincaré line bundle given in Section 2.1. The degree $\text{deg}(\lambda)$ of a polarization $\lambda$ is the degree of $\lambda$ as an isogeny.
It is always a square and the reduced degree rdeg(λ) of λ is rdeg(λ) = \sqrt{\deg(λ)}. A principal polarization λ has deg(λ) = 1. Let \(\mathcal{A} = (A = E^g, \lambda)\) be a principally polarized superspecial abelian variety of dimension \(g\) over \(\overline{\mathbb{F}}_p\) with \(\mathbb{F}_p\)-isomorphism class \([\mathcal{A}]\). An \((\ell)^g\)-isogeny \(\psi : \mathcal{A} = (A = E^g, \lambda) \to \mathcal{A}' = (A' = E^g, \lambda')\) has kernel \(\ker(\psi)\) a maximal isotropic subgroup of \(A[\ell]\) and \(\psi^*(\lambda') = \ell \lambda\).

The theory of \(\ell\)-isogeny graphs for suppersingular elliptic curves in characteristic \(p\) extends to \((\ell)^g\)-isogenies of principally polarized superspecial abelian varieties of dimension \(g\), giving the big isogeny graph \(Gr_g(\ell, p)\), the little isogeny graph \(gr_g(\ell, p)\) and the enhanced isogeny graph \(\tilde{gr}_g(\ell, p)\); this was the focus of \([JZa]\). These \((\ell)^g\)-isogeny graphs are connected in higher dimensions \(g\) as they are for \(g = 1\) [\(JZa\), Thm. 43], but unlike the \(g = 1\) case the big isogeny graph is not in general Ramanujan for \(g > 1\) [\(JZa\), Sect. 10].

This paper further develops the category of \([\ell]\)-polarized abelian varieties introduced in \([JZa]\).

Isogenies in this category form cell complexes, whereas \((\ell)^g\)-isogenies of principally polarized abelian varieties only form graphs. A \(g\)-dimensional \([\ell]\)-polarized abelian variety \(\mathcal{X} = (X, \lambda)\) has a type \(t(\mathcal{X})\) with \(0 \leq t(\mathcal{X}) \leq g\) and a natural \([\ell]\)-dual \(\mathcal{X}^* = (\hat{X} = \text{Pic}^0(X), [\lambda])\), cf. Remark [\(\Box\]. If \(\mathcal{X}' = (X', \lambda')\) is an \([\ell]\)-polarized abelian variety, there is a notion \((\text{Section 2.1})\) of \([\ell]\)-isogeny \(f : \mathcal{X} \to \mathcal{X}'\) if \(t(\mathcal{X}) > t(\mathcal{X}')\). Compositions of \([\ell]\)-isogenies can again be \([\ell]\)-isogenies, resulting in Section [\(\Box\)] in an \([\ell]\)-isogeny complex \(\tilde{co}_g(A_0)\) once we fix a base principally polarized abelian variety \(A_0 = (A_0, \lambda_0)\). This enhanced isogeny complex \(\tilde{co}_g(A_0)\) is a semi-simplicial complex in the sense of Eilenberg and Zilber [\(EZ50\)] or \(\Delta\)-complex as in [\(Hat02\)]. Most importantly, in Theorem [\(\Box\)] we see that the connected component of \(\tilde{co}_g(A_0)\) containing the base \(A_0\) is a quotient of the Bruhat-Tits building \(B_g\) of the symplectic group \(\text{Sp}_{2g}(\mathbb{Q}_\ell)\).

Mapping an \([\ell]\)-polarized abelian variety \(\mathcal{A}\) to its dual \(\hat{\mathcal{A}}\) and \([\ell]\)-isogenies \(f : \mathcal{A} \to \mathcal{A}'\) to their duals \(\hat{f} : \hat{\mathcal{A}}' \to \hat{\mathcal{A}}\) gives an involution \(\iota\) on \(\tilde{co}_g(A_0)\); the quotient is the little isogeny complex \(co_g(A_0)\). Unfortunately, the action of \(\iota\) is not admissible as in Section [\(\Box\)] so \(co_g(A_0)\) does not inherit the structure of a \(\Delta\)-complex from \(\tilde{co}_g(A_0)\). We can avoid this difficulty if we barycentrically subdivide \(\tilde{co}_g(A_0)\) before taking the quotient by \(\iota\) (then the action is admissible), but this explodes the number of cells making it too difficult to compute examples. So we keep the picture we have and examine the quotient by \(\iota\) simplex by simplex. We show that if \(g \leq 4\), then the quotient \(co_g(A_0)\), although not a \(\Delta\)-complex, is a CW-complex. We explain the cell types we get in this case.

The real power of this approach to isogenies emerges when we restrict to the case that the base \(A_0\) is a superspecial abelian variety in characteristic \(p\). Let \(\mathbb{H} = \mathbb{H}_p\) be the definite quaternion algebra of reduced discriminant \(p\) with a maximal order \(\mathcal{O}_\mathbb{H}\). Then the isogeny complexes \(\tilde{co}_g(A_0)\) and \(co_g(A_0)\) are finite and connected, and they are the quotients of the building \(B_g\) of the symplectic group \(\text{Sp}_{2g}(\mathbb{Q}_\ell)\) by the quaternionic unitary group \(\text{U}_g(\mathcal{O}_\mathbb{H}[1/\ell])\) and the general quaternionic unitary group \(\text{GU}_g(\mathcal{O}_\mathbb{H}[1/\ell])\), respectively. Crucially, in this case it is possible to describe the isogeny complexes completely in terms of hermitian forms over definite quaternion algebras. Our notion of Brandt complexes generalizing Brandt matrices is developed in Section [\(\Box\)] Brandt complexes are amenable to machine computation and the paper contains many computational results.

In a forthcoming paper [\(JZb\)], we use the Brandt complex descriptions of

\[
\tilde{co}_g(A_0) = \text{U}_g(\mathcal{O}_\mathbb{H}[1/\ell]) \backslash B_g \quad \text{and} \quad co_g(A_0) = \text{GU}_g(\mathcal{O}_\mathbb{H}[1/\ell]) \backslash B_g
\]
to deduce results on the torsion in
$$H^*(U_g(\mathcal{O}_H[1/\ell]), \mathbb{Z}) \quad \text{and} \quad H^*(GU_g(\mathcal{O}_E[1/\ell]), \mathbb{Z})$$
for a range of examples. We conclude this paper with an extended example, giving in detail the with the $[2]$-isogeny complexes of superspecial abelian surfaces in characteristic $7$. We return to this example in [JZb] and compute cohomology.

2. Superspecial abelian varieties in characteristic $p$

2.1. $[\ell]$-polarizations on abelian varieties. The general reference for this section is [JZa, Sect. 6]. Proofs, details, and additional citations to the literature can be found there.

Let $X$ be an abelian variety over a field $k$ (not necessarily algebraically closed) with dual abelian variety $\hat{X} = \text{Pic}^0(X)$. The Poincaré line bundle on $X \times \hat{X}$ is denoted $\mathcal{P}$. A polarization of $X$ over $k$ is a symmetric isogeny $\lambda : X \rightarrow \hat{X}$ defined over $k$ such that the line bundle $(1, \lambda)^*\mathcal{P}$ is ample. The degree $\text{deg}(\lambda)$ of a polarization $\lambda : X \rightarrow \hat{X}$ is $\text{deg}(\lambda) = \# \ker(\lambda)$; this is always a square by the Riemann-Roch theorem (see [Mum08, Sect. 16]). The reduced degree of $\lambda$ is $\text{rdeg}(\lambda) = \sqrt{\text{deg}(\lambda)}$. A polarization of degree $1$ is a principal polarization. If $\lambda : X \rightarrow \hat{X}$ is a polarization and $\phi : X' \rightarrow X$ is an isogeny, then
$$\phi^* (\lambda) := \hat{\phi} \circ \lambda \circ \phi : X' \rightarrow \hat{X}'$$
is a polarization of $X'$ with
$$\text{deg}(\phi^*(\lambda)) = \text{deg}(\lambda) \text{deg}(\phi)^2 \quad \text{and} \quad \text{rdeg}(\phi^*(\lambda)) = \text{rdeg}(\lambda) \text{deg}(\phi). \quad (2)$$

Suppose $n \in \mathbb{N}$ and $(\text{char } k, n) = 1$ if $\text{char } k > 0$. A polarization $\lambda$ of $X$ gives rise to the nondegenerate Weil pairing
$$\langle \ , \rangle_\lambda := \langle \ , \rangle_{X, \lambda} : \ker(\lambda) \times \ker(\lambda) \rightarrow \mathfrak{g}_{\mu}.$$ \quad (3)

The more well known pairing
$$\langle \ , \rangle_{\lambda, n} := \langle \ , \rangle_{X, \lambda, n} : X[n] \times X[n] \rightarrow \mathfrak{g}_{\mu_n} \quad (4)$$
can in fact be given in terms of (3) by
$$\langle \ , \rangle_{X, \lambda, n} = \langle \ , \rangle_{X, n \circ \lambda} |_{X[n] \times X[n]}.$$

The notion of an $[\ell]$-polarization of an abelian variety, introduced in [JZa, Sect. 6], is crucial to what follows and is summarized below.

Remark 1. Let $\ell$ be a prime such that $\ell \neq \text{char } k$. Let $X$ be an abelian variety over the field $k$ with dim $X = g$.

(a) An $[\ell]$-polarization $\lambda$ on $X$ is a polarization such that $\ker(\lambda) \subseteq X[\ell]$. In particular, the degree of an $[\ell]$-polarization $\lambda$ is $\text{deg}(\lambda) = \ell^{2r}$ for $0 \leq r \leq g$. We say that $r = \log_{\ell}(\text{rdeg}(\lambda))$ is the type $t(\lambda)$ of the $[\ell]$-polarization $\lambda$. If $\lambda$ is an $[\ell]$-polarization on $X$ with $t(\lambda) = r$, we say that $\mathcal{X} = (X, \lambda)$ is an $[\ell]$-polarized abelian variety of type $r$ and write $t(\mathcal{X}) = r$.

(b) Let $\mathcal{X} = (X, \lambda)$ be an $[\ell]$-polarized abelian variety with type $t(\mathcal{X}) = r$. The $[\ell]$-dual of $\mathcal{X}$ is the $[\ell]$-polarized abelian variety $\mathcal{X}' = (\hat{X} = \text{Pic}^0(X), [\lambda])$ with the composition
$$X \xrightarrow{\lambda} \hat{X} \xrightarrow{[\lambda]} X$$
equal to multiplication by \(\ell\). We have \(t(\hat{X}) = \hat{r} := g - r\). If \(\mathcal{X'} = (X, \lambda)\) is principally polarized (so of type 0), then the \([\ell]\)-dual polarization \([\lambda]\) on \(X \cong \hat{X}\) is \(\ell\lambda\).

**Definition 2.** Let \(\ell\) be a prime such that \(\ell \neq \text{char} \, k\). Let \(X\) be an abelian variety over the field \(k\) with \(\dim X = g\).

(a) Let \(\mathcal{X} = (X, \lambda)\) and \(\mathcal{X}' = (X', \lambda')\) be polarized abelian varieties. An isogeny \(f : X \rightarrow X'\) is an isogeny from \(\mathcal{X}\) to \(\mathcal{X}'\), written \(f : \mathcal{X} \rightarrow \mathcal{X}'\), if \(\lambda = f^*(\lambda')\). Now suppose \(\mathcal{X}\) and \(\mathcal{X}'\) are \([\ell]\)-polarized abelian varieties of types \(r, s\), respectively. An isogeny \(f : \mathcal{X} \rightarrow \mathcal{X}'\) is strict if \(r \neq s\). If \(f : \mathcal{X} \rightarrow \mathcal{X}'\) is strict, then \(r > s\), \(\deg(f) = \ell^{(r-s)}\), and \(\lambda : X \rightarrow \hat{X}\) factors through \(f\).

**WARNING:** An \([\ell]\)-polarization \(\lambda : X \rightarrow \hat{X}\) does not give an isogeny from \(\mathcal{X}\) to \(\mathcal{X}'\).

(b) Let \(\mathcal{X} = (X, \lambda)\) be an \([\ell]\)-polarized abelian variety of type \(\ell\). A subgroup \(0 \neq C \subseteq X[\ell]\) of order \(\ell^d, 1 \leq d \leq r\), is an \([\ell]\)-subgroup of \(\mathcal{X}\) if \(C \subseteq \ker(\lambda)\) and \(C\) is isotropic with respect to the Weil pairing \(\langle \cdot, \cdot \rangle\).

**Proposition 3.** Suppose \(\mathcal{X} = (X, \lambda)\) and \(\mathcal{X}' = (X', \lambda')\) are \([\ell]\)-polarized abelian varieties of types \(r, s\), respectively. Let \(f : \mathcal{X} \rightarrow \mathcal{X}'\) be an isogeny. Then \(\hat{f} : \hat{\mathcal{X}}' \rightarrow \hat{\mathcal{X}}\) gives an isogeny \(\hat{f} : \hat{\mathcal{X}}' \rightarrow \hat{\mathcal{X}}\). If \(f : \mathcal{X} \rightarrow \mathcal{X}'\) is strict, then \(\hat{f} : \hat{\mathcal{X}}' \rightarrow \hat{\mathcal{X}}\) is strict.

**Proof.** Since \(f\) is an isogeny of polarized abelian varieties, by (1) that means \(\lambda = \hat{f} \circ \lambda' \circ f\). Now by Remark \(\square\) that means \(\lambda \circ \hat{f} \circ \lambda' \circ f = \ell\). Hence, \(f \circ [\lambda] \circ \hat{f} \circ \lambda' = \ell\), and thus \(\lambda([\lambda]) = f \circ [\lambda] \circ f = [\lambda]\), which implies \(\hat{f} : \hat{\mathcal{X}}' \rightarrow \hat{\mathcal{X}}\) is an isogeny.

Since \(t(\hat{\mathcal{X}}') = \hat{s} = g - s\) and \(t(\hat{\mathcal{X}}) = \hat{r} = g - r\), \(\hat{f}\) is strict if and only if \(f\) is. \(\square\)

**Proposition 4.** Suppose \(\mathcal{X} = (X, \lambda)\) is an \([\ell]\)-polarized abelian variety. Let \(C \subseteq X[\ell]\) be an \([\ell]\)-subgroup of \(\mathcal{X}\) of order \(\ell^d, 1 \leq d \leq r\), with \(f_C : X \rightarrow X_C := X/C\) the isogeny taking the quotient by \(C\). Then \(X_C = X/C\) has a canonical \([\ell]\)-polarization \(\lambda_C\) such that the composition

\[
X \xrightarrow{f_C} X_C \xrightarrow{\lambda_C} \hat{X}_C \xrightarrow{\hat{f}_C} \hat{X}
\]

is \(\lambda\). The \([\ell]\)-polarized abelian variety \(\mathcal{X}_C = (X_C, \lambda_C)\) has type \(t(\mathcal{X}_C) = s = r - d\) and the isogeny \(f_C : \mathcal{X} \rightarrow \mathcal{X}_C\) is strict as in Definition \(\square\). Moreover a strict isogeny \(f : \mathcal{X} \rightarrow \mathcal{X}'\) of \([\ell]\)-polarized abelian varieties is of the form \(f_C\) with \(\mathcal{X}' \cong \mathcal{X}_C\) for \(C\) an \([\ell]\)-subgroup of \(\mathcal{X}\).

**Proof.** First suppose that \(C \subseteq X[\ell]\) is an \([\ell]\)-subgroup of \(\mathcal{X}\). Then since \(C\) is isotropic, we have \(C \subseteq C^\perp \subseteq \ker \lambda\). Consider the corresponding decomposition

\[
\lambda : X \xrightarrow{\lambda_C} X_C \xrightarrow{\lambda_C} X/C^\perp \rightarrow \hat{C},
\]

By properties of duals, there is an identification \(X/C^\perp = \hat{X}_C\) such that the map \(X/C^\perp = \hat{X}_C \rightarrow \hat{C}\) is isomorphic to \(\hat{f}_C\). Since \(\lambda = \hat{f}_C \circ \lambda_C \circ f_C\) and \(\lambda\) is a polarization, its not hard to see that so is \(\lambda_C\). The formula for the type follows trivially by counting dimensions.

Now suppose, \(f : X \rightarrow X'\) be a strict isogeny. Let \(C = \ker f\). Then \(\hat{f} : \hat{X}' \rightarrow \hat{X}\) corresponds to \(C^\perp \subseteq \ker (\lambda)\). Thus \(C \subseteq C^\perp\), hence \(C\) is isotropic and thus an \([\ell]\)-subgroup of \(\mathcal{X}\).

A strict isogeny \(f : \mathcal{X} \rightarrow \mathcal{X}'\) between \([\ell]\)-polarized abelian varieties will be called an \([\ell]\)-isogeny.
2.2. Superspecial abelian varieties. A superspecial abelian variety $A/\mathbb{F}_p$ with $\dim A = g$ is isomorphic to the product of $g$ supersingular elliptic curves. Fix a supersingular elliptic curve $E/\mathbb{F}_p$ with $O = O_E = \text{End}(E)$ a maximal order in the rational definite quaternion algebra $\mathbb{H}_p$ ramified at $p$. It is a theorem of Deligne, Ogus, and Shioda that such a superspecial $A/\mathbb{F}_p$ is isomorphic to $E^g$.

Fix a prime $\ell$ with $\ell \neq p$.

**Definition 5.** For a dimension $g \geq 1$, let $\text{SP}_g(\ell, p)_r$ be the set of $\mathbb{F}_p$-isomorphism classes $[\mathcal{A}]$ of $g$-dimensional $[\ell]$-polarized superspecial abelian varieties $\mathcal{A} = (A, \lambda)$ over $\mathbb{F}_p$ of type $r$, $0 \leq r \leq g$. In case $r = 0$, $\text{SP}_g(p)_0 := \text{SP}_g(\ell, p)_0$ is the set of $\mathbb{F}_p$-isomorphism classes of principally polarized superspecial abelian varieties. The set $\text{SP}_g(\ell, p)_r$ is finite; set

$$h_g(\ell, p)_r = \#\text{SP}_g(\ell, p)_r, \quad h_g(\ell, p)_r = \sum_{r=0}^{g} h_g(\ell, p)_r, \quad \text{and } h_g(p) = h_g(\ell, p)_0 = \#\text{SP}_g(p)_0.$$  \hspace{1cm} (5)

For $g$ fixed, it is convenient to define for $0 \leq r \leq g$:

$$\tau = \tilde{\tau} = \{r, \tilde{r}\}, \text{ viewed as a multiset.}$$ \hspace{1cm} (6)

Taking the $[\ell]$-dual gives a canonical bijection

$$\iota : \text{SP}_g(\ell, p)_r \xrightarrow{\sim} \text{SP}_g(\ell, p)_{\tilde{r}}, \quad 0 \leq r \leq g, \text{ with } \iota([\mathcal{A}] = [\mathcal{A}^\ell];$$  \hspace{1cm} (7)

see [JZa, Defn. 28]. Hence $h_g(\ell, p)_r = h_g(\ell, p)_{\tilde{r}}$. If $g$ and $p$ are fixed, set

$$h(\ell; r) := h_g(\ell, p)_r, \quad h := h(\ell; 0), \quad \text{and } h(\ell) := h_g(\ell, p).$$ \hspace{1cm} (8)

Table 1 gives the $h_g(\ell, p)_r$ which we need for subsequent examples.

| $g$ | $\ell$ | $p$ | $h_g(\ell, p)_0$ | $h_g(\ell, p)_1$ |
|-----|-----|-----|----------------|----------------|
| 2   | 2   | 7   | 2              | 4              |
| 2   | 2   | 11  | 5              | 10             |
| 3   | 2   | 3   | 2              | 3              |

**Table 1.** $h_g(\ell, p)_r = \#\text{SP}_g(\ell, p)_r = h_g(\ell, p)_{\tilde{r}} = \#\text{SP}_g(\ell, p)_{\tilde{r}}$

Using the notation of (6) and (7), put

$$\text{SP}_g(\ell, p)_r = \{[\mathcal{A}], \ldots, [\mathcal{A}_{h(\ell, r)}]\} \text{ with } \mathcal{A}_i = (A_i, \lambda_i) \text{ and } \text{SP}_g(p)_0 = \{[\mathcal{A}], \ldots, [\mathcal{A}_h]\},$$

$$\text{SP}_g(\ell, p) = \prod_{r=0}^{g} \text{SP}_g(\ell, p)_r = \{[\mathcal{A}], \ldots, [\mathcal{A}_{h(\ell)}]\},$$

$$\text{SP}_g(\ell, p)_{\tilde{r}} = \begin{cases} \prod \text{SP}_g(\ell, p)_r / \iota & \text{if } r \neq \tilde{r}, \\ \text{SP}_g(\ell, p)_g / \iota & \text{if } r = g/2. \end{cases}$$ \hspace{1cm} (9)

$$\overline{\text{SP}}_g(\ell, p) = \prod_{r=0}^{\lfloor g/2 \rfloor} \text{SP}_g(\ell, p)_r.$$
Remark 6. We can identify $\mathcal{SP}_g(\ell, p)_\tau$ with the set of multisets $[\mathcal{A}] := \{[\mathcal{A}], [\mathcal{A}']\}$ with $[\mathcal{A}] \in \mathcal{SP}_g(\ell, p)_r$. If $r \neq \hat{r}$, i.e., if $r \neq g/2$, then $\mathcal{SP}_g(\ell, p)_\tau \cong \mathcal{SP}_g(\ell, p)_r \cong \mathcal{SP}_g(\ell, p)_{\hat{r}}$. Elements of $\mathcal{SP}_g(\ell, p)_\tau$ have type $\tau = \{r, \hat{r}\}$. Let

$$h_g(\ell, p)_\tau = \#\mathcal{SP}_g(\ell, p)_\tau \quad \text{and} \quad \overline{h}_g(\ell, p)_\tau = \#\overline{\mathcal{SP}}_g(\ell, p)_\tau = \sum_{r=0}^{[g/2]} h_g(\ell, p)_\tau. \quad (10)$$

Then $h_g(\ell, p)_\tau = h_g(\ell, p)_r$ if $r \neq g/2$. In Table 2 we augment Table 1 by giving the $h_g(\ell, p)_\tau$.

| $g$ | $\ell$ | $p$ | $h_g(\ell, p)_{\overline{\tau}}$ | $h_g(\ell, p)_\tau$ |
|-----|-------|-----|-------------------------------|-------------------|
| 2   | 2     | 7   | 2                            | 4                 |
| 2   | 2     | 11  | 5                            | 8                 |
| 3   | 2     | 3   | 2                            | 3                 |

Table 2. $h_g(\ell, p)_\tau = \#\mathcal{SP}_g(\ell, p)_\tau$

It is key that the image of an $[\ell]$-polarized superspecial abelian variety under a strict isogeny is superspecial:

Proposition 7. Suppose $\mathcal{A} = (A, \lambda)$ is an $[\ell]$-polarized superspecial abelian variety with type $t(\mathcal{A}) = r$ and $C \leq A[\ell]$ is an $[\ell]$-subgroup of $\mathcal{A}$ of order $\ell^d$, $1 \leq d \leq r$, as in Definition 2(b). Then $A/C$ is a superspecial abelian variety and hence $\mathcal{A}_C = (A/C, \lambda_C)$ as in Proposition 4 is an $[\ell]$-polarized superspecial abelian variety with $t(\mathcal{A}_C) = s = r - d$.

Proof. This follows from Proposition 4 and the fact the image under a separable isogeny of a superspecial abelian variety is superspecial—see [Oor75, p. 36]. \hfill \square

We now define two equivalence relations on strict isogenies of $[\ell]$-polarized superspecial abelian varieties.

Definition 8. Suppose $[\mathcal{A}] = (A, \lambda) \in \mathcal{SP}_g(\ell, p)_r$ and $[\mathcal{A}]' = (A', \lambda') \in \mathcal{SP}_g(\ell, p)_s$ with $0 \leq s < r \leq g$. Suppose $f, g : \mathcal{A} \to \mathcal{A}'$ are strict isogenies.

(a) Say $f \sim_b g$ if there exists $\alpha_2 \in \text{Aut}(\mathcal{A}')$ such that $f = \alpha_2 g$. Write $[f]_b$ for the equivalence class with respect to $\sim_b$ containing $f$. Put $\text{Iso}_b(\mathcal{A}, \mathcal{A}') = \{[f]_b \mid f : \mathcal{A} \to \mathcal{A}' \text{ is a strict isogeny}\}$.

Set $\text{Iso}_b(\mathcal{A})_s = \{[f]_b \in \text{Iso}_b(\mathcal{A}, \mathcal{A}'') \text{ for some } \mathcal{A}'' \in \mathcal{SP}_g(\ell, p)_s\}$.

(b) Say $f \sim_l g$ if there exist $\alpha_1 \in \text{Aut}(\mathcal{A})$ and $\alpha_2 \in \text{Aut}(\mathcal{A}')$ such that $f = \alpha_2 g \alpha_1$. Write $[f]_l$ for the equivalence class with respect to $\sim_l$ containing $f$. Put $\text{iso}_l(\mathcal{A}, \mathcal{A}') = \{[f]_l \mid f \text{ is a strict isogeny from } \mathcal{A} \text{ to } \mathcal{A}'\}$.

Set $\text{iso}_l(\mathcal{A})_s = \{[f]_l \in \text{iso}_l(\mathcal{A}, \mathcal{A}'') \text{ for some } \mathcal{A}'' \in \mathcal{SP}_g(\ell, p)_s\}$. 

Remark 9. In light of Proposition 4, we could equivalently give Definition 8 in terms of the $[\ell]$-subgroups of Definition 2(b). Suppose $[\mathcal{A}] \in \text{SP}_g(\ell, p)_r$ for $0 \leq r \leq g$. For $0 \leq s < r$, set

$$\text{Iso}_\ell([\mathcal{A}])_s = \{ C \mid C \text{ is an } [\ell]\text{-subgroup of } \mathcal{A} \text{ of order } \ell^{r-s} \}. \quad (11)$$

Define an equivalence relation $\sim$ on $[\ell]$-subgroups of $\mathcal{A}$ by $C \sim C'$ if there exists $\alpha \in \text{Aut}(\mathcal{A})$ such that $\alpha(C) = C'$. Let $[C]$ denote the equivalence class with respect to $\sim$ containing $C$. Set

$$\text{iso}_\ell([\mathcal{A}])_s = \{ [C] \mid C \in \text{Iso}_\ell([\mathcal{A}])_s \}. \quad (12)$$

**Proposition 10.** Suppose $[\mathcal{A}] \in \text{SP}_g(\ell, p)_r$. Then $\# \text{Iso}_\ell([\mathcal{A}])_s$ neither depends on $p$ nor $g$ nor on the choice of $[\mathcal{A}] \in \text{SP}_g(\ell, p)_r$.

**Proof.** Let $\mathcal{A} = (A, \lambda)$. By Definition 2(b) and (11)

$$\text{Iso}_\ell([\mathcal{A}])_s = \{ C \mid C \subset \ker(\lambda) \text{ is an isotropic subgroup of order } \ell^{r-s} \},$$

where $\ker(\lambda)$ is an $\ell$-group of rank $2r$ with a nondegenerate symplectic pairing. Thus $\# \text{Iso}_\ell([\mathcal{A}])_s$ depends only on $\ell$, $r$, and $s$. □

For $0 \leq s < r \leq g$ and $[\mathcal{A}] \in \text{SP}_g(\ell, p)_r$, set

$$N(\ell)_{r,s} = \# \text{Iso}_\ell([\mathcal{A}])_s. \quad (13)$$

We saw in [JZa, (30)] that

$$N(\ell)_{r,0} = \prod_{k=1}^{r} (\ell^k + 1). \quad (14)$$

The general formula for $N(\ell)_{r,s}$ is given below; it reduces to (14) when $s = 0$. We define $N(\ell)_{r,r} = 1$ for all $r \geq 0$.

**Proposition 11.** We have

$$N(\ell)_{r,s} = \prod_{k=0}^{r-s-1} \frac{\ell^{2(r-k)} - 1}{\ell^{k+1} - 1}. \quad (15)$$

**Proof.** By the proof of Proposition 10, $N(\ell)_{r,s}$ is equal to the number of isotropic rank $r - s$ subgroups $C$ of an $\ell$-group $G$ of rank $2r$ with a nondegenerate symplectic pairing.

If we pick a basis $(v_0, \ldots, v_{r-s-1})$ for $C$, we have $\ell^{2r} - 1$ choices for $v_0 \in G \setminus \{0\}$. Now $v_1$ must be orthogonal to $v_0$ and independent of $v_0$ so we have $\ell^{2r-1} - \ell = \ell(\ell^{2(r-1)} - 1)$ choices for $v_1$, and in general we have $\ell^{k(\ell^{2(r-k)} - 1)}$ choices for $v_k$. Thus there are

$$\prod_{k=0}^{r-s-1} \ell^{k(\ell^{2(r-k)} - 1)}$$

isotropic rank $r - s$ subspaces $C$ with basis specified.

But each $C$ has

$$\prod_{k=0}^{r-s-1} \ell^{k(\ell^{k+1} - 1)}$$

choices of basis. Dividing these two values gives the result. □

We use Proposition 11 to tabulate $N(\ell)_{r,s}$ for $0 \leq s < r < 3$ and $\ell = 2$ in Table 3. The reader can verify that $N(\ell)_{3m-1,m-1} = N(\ell)_{3m-1,m}$ for $m \geq 1$, a special case of which is the equality $N(2)_{2,0} = N(2)_{2,1}$ in Table 3.
Table 3. \( N(2)_{r,s} \) for \( 0 \leq s < r \leq 3 \)

| \( N(2)_{1,0} \) | \( N(2)_{2,0} \) | \( N(2)_{2,1} \) | \( N(2)_{3,0} \) | \( N(2)_{3,1} \) | \( N(2)_{3,2} \) |
|---|---|---|---|---|---|
| 3  | 15 | 15 | 135 | 315 | 63 |

3. Isogeny graphs of [\( \ell \)]-polarized superspecial abelian varieties

Let \( \ell \) and \( p \) be primes with \( p \neq \ell \). Recall that an \( (\ell)^g \)-isogeny of a \( g \)-dimensional principally polarized abelian variety is taking the quotient by a maximal isotropic subgroup of its \( \ell \)-torsion. In [JZa] we defined big, little and enhanced \( (\ell)^g \)-isogeny graphs of \( g \)-dimensional principally polarized abelian varieties. In the case of principally polarized superspecial abelian varieties over \( \mathbb{F}_p \), it is possible to construct these graphs from definite quaternion algebras and hermitian forms over their maximal orders. Such a construction gives the Brandt graphs of [JZa, Sect. 5].

In this section we relax the requirement of principal polarizations to that of \([\ell]\)-polarizations and consider \([\ell]\)-isogeny graphs; see Section 2.1 for definitions. The definitions of the big, little, and enhanced \( (\ell)^g \)-isogeny graph carry over to this more general setting. The major results of [JZa] carry over to \([\ell]\)-isogeny graphs: We prove in Theorem 17 that they are connected; we show in Section 4.2.1 that they are sometimes (albeit rarely) Ramanujan and sometimes non-Ramanujan; and we show that the little and enhanced graphs are \( \ell \)-adically uniformized by the 1-skeleton of the Bruhat-Tits building for \( \text{Sp}_{2g}(\mathbb{Q}_\ell) \) in Section 5.

3.1. Graphs.

**Definition 12.** A (finite) graph \( \text{Gr} \) has a finite set of vertices \( \text{Ver}(\text{Gr}) = \{v_1, \ldots, v_s\} \) and a finite set of (directed) edges \( \text{Ed}(\text{Gr}) \). And edge \( e \in \text{Ed}(\text{Gr}) \) has initial vertex \( o(e) \) and terminal vertex \( t(e) \). For vertices \( v_i, v_j \in \text{Ver}(\text{Gr}) \), put

\[
\text{Ed}(\text{Gr})_{ij} = \{e \in \text{Ed}(\text{Gr}) \mid o(e) = v_i \text{ and } t(e) = v_j\}.
\]

The adjacency matrix \( \text{Ad}(\text{Gr}) \in \text{Mat}_{s \times s}(\mathbb{Z}) \) is the matrix with

\[
\text{Ad}(\text{Gr})_{ij} = \# \text{Ed}(\text{Gr})_{ij}.
\]

We place no further restrictions on our definition of a graph. Serre [Ser03] requires graphs to be graphs with opposites: every directed edge \( e \in \text{Ed}(\text{Gr}) \) has an opposite edge \( \overline{e} \in \text{Ed}(\text{Gr}) \). An edge \( e \) with \( \overline{e} = e \) is called a half-edge. Serre forbids half-edges; we will call a graph with opposites satisfying his requirements a graph without half-edges. Kurihara [Kur79] relaxes Serre’s definition to allow half-edges on a graph with opposites giving the notion of a graph with half-edges, also called an h-graph as in [JK20] and [JK21]). A graph with half-edges may have \( \emptyset \) as its set of half-edges, so every graph without half-edges is a graph with half-edges.

**Definition 13.** (a) A graph with weights is a graph with opposites together with a weight function \( w \) mapping vertices and edges to positive integers that agrees on edges and their opposites and such that for each edge \( e \) we have \( w(e) \mid w(o(e)) \) (which implies \( w(e) \mid w(t(e)) \)) since \( w(e) = w(\overline{e}) \).

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(b) The weighted adjacency matrix $A_w := \text{Ad}_w(Gr)$ of a graph with weights $Gr$ with vertices $\text{Ver}(Gr) = \{v_1, \ldots, v_s\}$ is

\[
(A_w)_{ij} = \sum_{e \in \text{Ed}(Gr)_{ij}} \frac{w(v_i)}{w(e)}.
\]

3.2. The enhanced $[\ell]$-isogeny graph $\tilde{g}_g([\ell], p)$ and its subgraphs $\tilde{g}_g([\ell], p)_{r,s}$. The enhanced $[\ell]$-isogeny graph $\tilde{g}_g([\ell], p)$ has vertices $\text{Ver}(\tilde{g}_g([\ell], p)) = \text{SP}_g([\ell], p) = \prod_{r=0}^g \text{SP}_g([\ell], p)_r$ in the notation (9). In particular, each vertex of $\tilde{g}_g([\ell], p)$ has a type $r$ with $0 \leq r \leq g$. The vertices $\text{SP}_g([\ell], p)_0$ of type 0 and $\text{SP}_g([\ell], p)_g$ of type $g$ are the special vertices.

The edges of $\tilde{g}_g([\ell], p)$ from the vertex $[\mathcal{A}_i] \in \text{SP}_g([\ell], p)_r \subseteq \text{SP}_g([\ell], p)$ to the vertex $[\mathcal{A}_j] \in \text{SP}_g([\ell], p)_s \subseteq \text{SP}_g([\ell], p)$ are

\[
\text{Ed}(\tilde{g}_g([\ell], p))_{i,j} = \text{iso}_\ell([\mathcal{A}_i], [\mathcal{A}_j]) \cup \text{iso}_\ell([\mathcal{A}_i], [\mathcal{A}_j]).
\]

Note that at most one of the sets $\text{iso}_\ell([\mathcal{A}_i], [\mathcal{A}_j])$, $\text{iso}_\ell([\mathcal{A}_i], [\mathcal{A}_j])$ is nonempty. We denote by $\tilde{g}_g([\ell], p)_{r,s} = \tilde{g}_g([\ell], p)_{s,r}$ the subgraph of $\tilde{g}_g([\ell], p)$ induced by the vertex subset

\[
\text{SP}_g([\ell], p)_r \cup \text{SP}_g([\ell], p)_s \subseteq \text{Ver}(\tilde{g}_g([\ell], p)) = \text{SP}_g([\ell], p).
\]

In particular $\text{Ed}(\tilde{g}_g([\ell], p)_{r,s}) = \emptyset$ for $0 \leq r \leq g$.

Remark 14. The graphs $\tilde{g}_g([\ell], p)$ and $\tilde{g}_g([\ell], p)_{r,s}$, $0 \leq r, s \leq g$, are graphs with opposites: if $e \in \text{Ed}(\tilde{g}_g([\ell], p)_{r,s})$ corresponds to an isogeny $f$, then $\bar{e}$ corresponds to the dual isogeny $\bar{f}$.

Define a weight function $w$ on $\tilde{g}_g([\ell], p)$ by $w([\mathcal{A}]) = \# \text{Aut}(\mathcal{A}, \lambda)$ for $\mathcal{A} = (\mathcal{A}, \lambda) \in \text{SP}_g([\ell], p) = \text{Ver}(\tilde{g}_g)$. On edges put $w(e) = \# \text{Aut}(f : \mathcal{A} \rightarrow \mathcal{A})$ if $e$ corresponds to the equivalence class of the strict isogeny $f : \mathcal{A} \rightarrow \mathcal{A}'$. The subgraphs $\tilde{g}_g([\ell], p)_{r,s}$ inherit weights $w$ from $\tilde{g}_g([\ell], p)$.

The enhanced isogeny graph $\tilde{g}_g([\ell], p)$ has a natural involution. Let $\iota : \tilde{g}_g([\ell], p) \rightarrow \tilde{g}_g([\ell], p)$ and by restriction $\iota : \tilde{g}_g([\ell], p)_{r,s} \rightarrow \tilde{g}_g([\ell], p)_{s,r}$ be the map with $\iota^2$ the identity defined on vertices by $\iota([\mathcal{A}]) = [\mathcal{A}]$ and on edges such that if $e \in \text{Ed}(\tilde{g}_g([\ell], p)_{r,s})$ corresponds to the class $[C]$, then $\iota(e) \in \text{Ed}(\tilde{g}_g([\ell], p)_{r,s})$ corresponds to the same class $[C]$. This is consistent with the definition of $\iota : \text{SP}_g([\ell], p) \rightarrow \text{SP}_g([\ell], p)$ in Remark 6. The involution $\iota$ can fix vertices of $\tilde{g}_g([\ell], p)$. It does not fix any edge of $\tilde{g}_g([\ell], p)$, but can map an edge to its opposite.

The weighted adjacency matrix $A := \text{Ad}_w(\tilde{g}_g([\ell], p))$ of Definition 3 is a block matrix $A = [A_{r,s}]_{0 \leq r, s \leq g}$ with $A_{r,s}$ arising from the edges going from the vertices $\text{SP}_g([\ell], p)_r \subseteq \text{Ver}(\tilde{g}_g([\ell], p))$ to the vertices $\text{SP}_g([\ell], p)_s$. It is convenient to establish the notation that $A_{0,0}$ is a matrix having all entries 0 with size determined by the context. We have that $A_{r,s} = A_{s,r}$; in particular $A_{r,\bar{r}} = A_{\bar{r},r}$. Moreover $A_{r,r} = [\cdot \cdot \cdot]$ since $\text{Ed}(\tilde{g}_g([\ell], p)_{r,r}) = \emptyset$. The matrix $A_{r,s}$ has size $h_g([\ell], p)_r \times h_g([\ell], p)_s$. In particular the matrix $A_{r,\bar{r}}$ is square since $h_g([\ell], p)_r = h_g([\ell], p)_\bar{r}$. The matrix $A_{r,\bar{r}}$ is a constant row-sum matrix with all the rows adding up to $N([\ell])_{r,s}$ if $r > s$, $N([\ell])_{r,\bar{r}}$ if $s > r$, and 0 if $s = r$. The matrix $\text{Ad}_w(\tilde{g}_g([\ell], p))_{r,s}$ is of size $(h_g([\ell], p)_r + h_g([\ell], p)_s) \times (h_g([\ell], p)_r + h_g([\ell], p)_s)$ if $r \neq s$, and of size $h_g([\ell], p)_r \times h_g([\ell], p)_r$ if $r = s$. 10
It is a block matrix of the form

\[
\text{Ad}_w(\tilde{g}_r([\ell], p), r, s) = \begin{cases} 
\begin{bmatrix} A_{r,r} & A_{r,s} \\
A_{s,r} & A_{s,s} \end{bmatrix} = \begin{bmatrix} -0. & A_{r,s} \\
A_{s,r} & -0. \end{bmatrix} & \text{if } r \neq s, \\
-0. & -0. \end{cases} \quad \text{if } r = s.
\]

Hence we have the following proposition.

**Proposition 15.** (a) The graph \(\tilde{g}_r([\ell], p), r, s\) is bipartite for \(0 \leq r, s \leq g\).

(b) The graph \(\tilde{g}_r([\ell], p)_{r,\tilde{r}}\) is a regular graph with regularity \(N(\ell)_{r,\tilde{r}}\).

### 3.3. Two examples of \(\text{Ad}_w(\tilde{g}_r([\ell], p))\).

**3.3.1. \(\text{Ad}_w(\tilde{g}_r([2], 11))\).** Let \(A = \text{Ad}_w(\tilde{g}_r([2], 11)) = [A_{r,s}]_{0 \leq r, s \leq 2}\). We have

\[
A = \begin{bmatrix} 
A_{0,0} & A_{0,1} & A_{0,2} \\
A_{1,0} & A_{1,1} & A_{1,2} \\
A_{2,0} & A_{2,1} & A_{2,2} 
\end{bmatrix} = 
\begin{bmatrix} 
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 1 & 0 & 0 & 0 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 3 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 6 & 0 & 0 & 0 & 0 & 3 & 1 & 3 & 0 & 8 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 6 & 0 & 0 & 1 & 3 & 3 & 0 & 0 & 6 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 3 & 0 & 0 & 4 & 3 & 8 \\
1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\
3 & 4 & 4 & 4 & 0 & 2 & 4 & 1 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 3 & 3 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & 3 & 0 & 8 & 0 & 3 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 6 & 6 & 3 & 0 & 0 & 0 & 2 & 6 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 3 & 8 & 0 & 0 & 0 & 6 & 0 & 0 & 6 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
3 & 4 & 4 & 4 & 0 & 2 & 4 & 1 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
9 & 3 & 3 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & 3 & 0 & 8 & 0 & 3 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 6 & 6 & 3 & 0 & 0 & 0 & 2 & 6 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 3 & 8 & 0 & 0 & 0 & 6 & 0 & 0 & 6 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]

**3.3.2. \(\text{Ad}_w(\tilde{g}_r([2], 3))\).** With \(g = 3\), there are four possible types of vertices: 0, 1, 2, 3. We will use the notation of Definition 5. The matrix \(A = \text{Ad}_w(\tilde{g}_r([\ell], p))\) is size \(h_3(\ell, p) \times h_3(\ell, p)\). It is a \(4 \times 4\) block matrix \(A = [A_{r,s}]_{0 \leq r, s \leq 3}\). We have \(A_{r,r} = -0.\) for \(0 \leq r \leq 3\) and \(A_{r,s} = A_{s,r}\).

From Table [I] we have \(h_3(2, 3)_0 = h_3(2, 3)_3 = 2\) and \(h_3(2, 3)_1 = h_3(2, 3)_2 = 3\). Let \(A = \text{Ad}_w(\tilde{g}_r([2], 3)) = [A_{r,s}]_{0 \leq r, s \leq 3}\). We have...


Indeed the rows of $A_{\tau,s}$ all add up to $N(3)_{r,s}$ with $N(3)_{1,0} = N(3)_{2,3} = 3$, $N(3)_{2,0} = N(3)_{1,3} = 15$, $N(3)_{3,0} = N(3)_{3,3} = 135$, $N(3)_{3,1} = N(3)_{0,2} = 315$, $N(3)_{3,2} = N(3)_{0,1} = 63$, and $N(3)_{0,0} = N(3)_{1,1} = N(3)_{2,2} = N(3)_{3,3} = 0$ from Table 3.

3.4. The little $[\ell]$-isogeny graph $gr_g([\ell], p)$ and its subgraphs $gr_g([\ell], p)_{\tau,\overline{\tau}}$. The little $[\ell]$-isogeny graph $gr_g([\ell], p)$ is the quotient of the enhanced $[\ell]$-isogeny graph $\tilde{g}_g([\ell], p)$ by the involution $\iota$ in Remark 14. Let $\pi : \tilde{g}_g([\ell], p) \rightarrow gr_g([\ell], p)$ be the double covering. In particular

\[
Ver(gr_g([\ell], p)) = Ver(\tilde{g}_g([\ell], p)) / \iota = \frac{\overline{SP}_g(\ell, p)}{\iota} = \prod_{r=0}^{\lfloor g/2 \rfloor} SP_g(\ell, p)_{\tau}
\]  

(16)

using the notation (9). The vertices of $gr_g([\ell], p)$ have type given by the multiset $\bar{\tau} = \{r, \hat{r}\}$ for $0 \leq r \leq \lfloor g/2 \rfloor$. For $[\mathcal{A}] \in SP_g(\ell, p)$, let $[\mathcal{A}] := \{[\mathcal{A}], [\mathcal{A}]^{\hat{}}\}$, viewed as a multiset. Then

\[
\{[\mathcal{A}] | [\mathcal{A}] \in SP_g(\ell, p)\} = \overline{SP}_g(\ell, p) = Ver(gr_g([\ell], p)).
\]

For edges we have $Ed(gr_g([\ell], p)) = Ed(\tilde{g}_g([\ell], p)) / \iota$. The edges of $gr_g([\ell], p)$ going from the vertex $[\mathcal{A}]_i \in SP_g(\ell, p)_{\tau}$ to the vertex $[\mathcal{A}]_j \in SP_g(\ell, p)_{\bar{\tau}}$ are

\[
Ed(gr_g([\ell], p))_{ij} = \text{iso}_\iota(\mathcal{A}_i, \mathcal{A}_j) \cup \text{iso}_\iota(\mathcal{A}_j, \mathcal{A}_i^{\hat{}}) \cup \text{iso}_\iota(\mathcal{A}_i^{\hat{}}), \mathcal{A}_j) \cup \text{iso}_\iota(\mathcal{A}_i, \mathcal{A}_j^{\hat{}}).
\]

Note that at most two of the above sets are nonempty. If $e \in Ed(gr_\ell)_{ij}$, the opposite edge

\[
\bar{e} \in Ed(gr_g([\ell], p))_{ji} = \text{iso}_\iota(\mathcal{A}_j, \mathcal{A}_i) \cup \text{iso}_\iota(\mathcal{A}_i, \mathcal{A}_j^{\hat{}}) \cup \text{iso}_\iota(\mathcal{A}_j^{\hat{}}, \mathcal{A}_i) \cup \text{iso}_\iota(\mathcal{A}_j^{\hat{}}), \mathcal{A}_i^{\hat{}})
\]

is the equivalence class of the dual isogeny.

Define a weight function $w$ on $gr$ by descending the weight function $w$ on $\tilde{g}_g([\ell], p)$: If $\tilde{e} \in Ed(\tilde{g}_g([\ell], p))$ with $\pi(\tilde{e}) = e \in Ed(gr_g([\ell], p))$, set $w(e) = w(\tilde{e})$, unless $\iota(e) = e$, in which case $w(e) = 2w(\tilde{e})$. Suppose $\tilde{v} \in Ver(\tilde{g}_g([\ell], p))$ and $\pi(\tilde{v}) = v$. If $\pi$ is ramified at $v$, set $w(v) = w(\tilde{v})$. If $\tau$ is ramified at $v$, set $w(v) = 2w(\tilde{v})$.

The weighted adjacency matrix $Ad_w(gr_g([\ell], p))$ is of size $\overline{H}_g(\ell, p) \times \overline{H}_g(\ell, p)$. It is a block matrix of the form

\[
Ad_w(gr_g([\ell], p)) = \begin{bmatrix}
    A_{0,0} & A_{0,1} & \cdots & A_{0,\overline{g/2}} \\
    A_{1,0} & A_{1,1} & \cdots & A_{1,\overline{g/2}} \\
    \vdots & \vdots & \ddots & \vdots \\
    A_{\overline{g/2},0} & A_{\overline{g/2},1} & \cdots & A_{\overline{g/2},\overline{g/2}}
\end{bmatrix}
\]  

(17)
The subgraph \( g_r([\ell], p)_{\tau, \bar{\tau}} \) of \( g_r([\ell], p) \) is induced by the vertex subset
\[
\mathcal{SP}_g(\ell, p)_\tau \cup \mathcal{SP}_g(\ell, p)_{\bar{\tau}} \subseteq \text{Ver}(g_r([\ell], p)) = \mathcal{SP}_g(\ell, p).
\]
We have
\[
\# \text{Ver}(g_r([\ell], p)_{\tau, \bar{\tau}}) = \begin{cases} \overline{h}_g(\ell, p)_{\tau, \bar{\tau}} + \overline{h}_g(\ell, p)_{\tau, \bar{\tau}} & \text{if } \tau \neq \bar{\tau}, \\ \overline{h}_g(\ell, p)_{\tau, \bar{\tau}} & \text{if } \tau = \bar{\tau}. \end{cases} \tag{18}
\]
The weighted adjacency matrix \( \text{Ad}_w(g_r([\ell], p)_{\tau, \bar{\tau}}) \) is a block matrix
\[
\text{Ad}_w(g_r([\ell], p)_{\tau, \bar{\tau}}) = \begin{bmatrix} A_{\tau, \tau} & A_{\tau, \bar{\tau}} \\ A_{\tau, \bar{\tau}} & A_{\bar{\tau}, \bar{\tau}} \end{bmatrix}, \quad \text{if } \tau \neq \bar{\tau},
\]
of size \((\overline{h}_g(\ell, p)_{\tau, \bar{\tau}} + \overline{h}_g(\ell, p)_{\tau, \bar{\tau}}) \times (\overline{h}_g(\ell, p)_{\tau, \bar{\tau}} + \overline{h}_g(\ell, p)_{\tau, \bar{\tau}})\) if \( \tau \neq \bar{\tau} \), \( \overline{h}_g(\ell, p)_{\tau, \bar{\tau}} \times \overline{h}_g(\ell, p)_{\tau, \bar{\tau}} \) if \( \tau = \bar{\tau} \).
We can give the blocks \( A_{\tau, \bar{\tau}} \) in terms of the blocks \( A_{\tau, s} \) as in (15). Let us first suppose that we are in the generic case that neither of \( \tau, \bar{\tau} \) is \( \{g/2, g/2\} \). Then \( \overline{h}_g(\ell, p)_{\tau, \bar{\tau}} = \overline{h}_g(\ell, p)_{\tau, r} \) and \( \overline{h}_g(\ell, p)_{\tau, \bar{\tau}} = \overline{h}_g(\ell, p)_{r, s} \). We have
\[
A_{\tau, \tau} = A_{r, r}, \quad A_{\tau, \bar{\tau}} = A_{r, s} + A_{s, \bar{\tau}} \quad \text{if } \tau \neq \bar{\tau}. \tag{19}
\]
It is convenient to establish notation for the vertices \([\mathcal{A}] \in \mathcal{SP}_g(\ell, p)_{g/2} \) ramified, respectively étale, in the double cover \( \mathcal{SP}_g(\ell, p)_{g/2} \to \mathcal{SP}_g(\ell, p)_{g/2} \) which is quotienting by the involution \( t \).
Set
\[
\mathcal{Ram}_g(\ell, p)_{g/2} = \{ [\mathcal{A}] \in \mathcal{SP}_g(\ell, p)_{g/2} \mid [\mathcal{A}] = [\mathcal{A}'] \} \tag{20}
\]
and put
\[
r_g(\ell, p)_{g/2} = \# \mathcal{Ram}_g(\ell, p)_{g/2} \quad \text{and} \quad e_g(\ell, p)_{g/2} = \# \mathcal{Et}_g(\ell, p)_{g/2}. \tag{21}
\]
Then
\[
r_g(\ell, p)_{g/2} + e_g(\ell, p)_{g/2} = h_g(\ell, p)_{g/2},
\]
\[
r_g(\ell, p)_{g/2} + (1/2)e_g(\ell, p)_{g/2} = h_g(\ell, p)_{g/2} \tag{22}
\]
Suppose \( \tau = \{g/2, g/2\} \) and \( \bar{\tau} \neq \tau \).
(a) \( A_{\tau, \tau} \) is the matrix \([0\cdot]\) of size \( h_g(\ell, p)_{g/2} \times h_g(\ell, p)_{g/2} \).
(b) Suppose \([\mathcal{A}] \in \mathcal{SP}_g(\ell, p)_{g/2} \):
   (i) If \([\mathcal{A}] \in \mathcal{Ram}_g(\ell, p)_{g/2} \), then the \([\mathcal{A}]\)-row of \( A_{\tau, \bar{\tau}} \) is double the \([\mathcal{A}]\)-row of \( A_{r, s} \).
   (ii) If \([\mathcal{A}] \in \mathcal{Et}_g(\ell, p)_{g/2} \), then the \([\mathcal{A}]\)-row of \( A_{\tau, \bar{\tau}} \) is the sum of the \([\mathcal{A}]\)-row of \( A_{r, s} \) and the \([\mathcal{A}]\)-row of \( A_{s, \bar{\tau}} \).
(c) Suppose \([\mathcal{A}] \in \mathcal{SP}_g(\ell, p)_{g/2} \):
   (i) If \([\mathcal{A}] \in \mathcal{Ram}_g(\ell, p)_{g/2} \), then the \([\mathcal{A}]\)-column of \( A_{\tau, \tau} \) is the \([\mathcal{A}]\)-column of \( A_{s, r} \).
   (ii) If \([\mathcal{A}] \in \mathcal{Et}_g(\ell, p)_{g/2} \), then the \([\mathcal{A}]\)-column of \( A_{\tau, \tau} \) is the sum of the \([\mathcal{A}]\)-column of \( A_{s, r} \) and the \([\mathcal{A}]\)-column of \( A_{s, r} \).
3.5. The big $[\ell]$-isogeny graph $Gr_g([\ell], p)$ and its subgraphs $Gr_g([\ell], p)_{r, \overline{r}}$. Retain the notation of Section 3.4. The vertices of the big $[\ell]$-isogeny graph $Gr_g([\ell], p)$ are

$$\text{Ver}(Gr_g([\ell], p)) = \text{Ver}(gr_g([\ell], p)) = \overline{\text{SP}}_g(\ell, p) = \prod_{r=0}^{[g/2]} \text{SP}_g(\ell, p)_r.$$  \hfill (23)

The edges of $Gr_g([\ell], p)$ going from the vertex $[\overline{\mathcal{A}}_i] \in \text{SP}_g(\ell, p)_r$ to the vertex $[\overline{\mathcal{A}}_j] \in \text{SP}_g(\ell, p)_{\overline{r}}$ are

$$\text{Ed}(Gr_g([\ell], p)) = \text{Iso}(\mathcal{A}_i, \mathcal{A}_j) \cup \text{Iso}(\mathcal{A}_i, \mathcal{A}'_j),$$

with definitions as in Definition 8 and Remark 9. Note that at most two of the above sets are nonempty.

The adjacency matrix $\text{Ad}(Gr_g([\ell], p))$ is of size $\overline{h}_g(\ell, p) \times \overline{h}_g(\ell, p)$. It is a block matrix of the form

$$\text{Ad}(Gr_g([\ell], p)) = \begin{bmatrix}
A_{0, 0} & A_{0, 1} & \cdots & A_{0, [g/2]} \\
A_{1, 0} & A_{1, 1} & \cdots & A_{1, [g/2]} \\
\vdots & \vdots & \ddots & \vdots \\
A_{[g/2], 0} & A_{[g/2], 1} & \cdots & A_{[g/2], [g/2]} \\
\end{bmatrix}.$$  \hfill (24)

The subgraph $Gr_g([\ell], p)_{r, \overline{r}}$ of $Gr_g([\ell], p)$ is induced by the vertex subset

$$\overline{\text{SP}}_g(\ell, p)_r \cup \overline{\text{SP}}_g(\ell, p)_{\overline{r}} \subseteq \text{Ver}(gr_g([\ell], p)) = \overline{\text{SP}}_g(\ell, p).$$

The adjacency matrix $\text{Ad}(Gr_g([\ell], p)_{r, \overline{r}})$ is a block matrix

$$\text{Ad}(Gr_g([\ell], p)_{r, \overline{r}}) = \begin{cases}
A_{r, r} & A_{r, \overline{r}} \\
A_{\overline{r}, r} & A_{\overline{r}, \overline{r}}.
\end{cases}$$

For $r \neq g/2$ set $Gr_g([\ell], p)_r = Gr_g([\ell], p)_{r, \overline{r}}$. The graph $Gr_g([\ell], p)_r$ is a regular graph with valence $N(\ell)_{r, \overline{r}}$ if $r > \hat{r}$ and $N(\ell)_{\overline{r}, r}$ if $\hat{r} > r$. In case $r = 0$, the graph $Gr_g([\ell], p)_0 = Gr_g([\ell], p)_g$ is the big isogeny graph $Gr_g(\ell, p)$ of \cite{12}. Sect. 7.1.

**Theorem 16.** Let $A_{\overline{r}, \overline{r}}$ be as in (17) and $A_{r, \overline{r}}$ be as in (24). Then $A_{r, \overline{r}} = A_{\overline{r}, \overline{r}}$. In particular, $\text{Ad}(Gr_g([\ell], p)) = \text{Ad}_w(gr_g([\ell], p))$ and $\text{Ad}(Gr_g([\ell], p)_{r, \overline{r}}) = \text{Ad}_w(gr_g([\ell], p)_{r, \overline{r}})$.

3.6. Two examples of $\text{Ad}_w(gr_g([\ell], p)) = \text{Ad}(Gr_g([\ell], p))$. In this section we give the weighted adjacency matrix for the little isogeny graphs arising from the examples given in Section 3.3. In particular, we have to apply the discussion in Section 3.4 to calculate the $A_{r, \overline{r}}$ from the $A_{r, s}$ given in Sections 3.3.2 and 3.3.1.

3.6.1. $\text{Ad}_w(gr_2([2], 11)) = \text{Ad}(Gr_2([2], 11))$. From Table 2, $h_2(2, 11)_{\overline{r}} = 3$ and $h_2(2, 11)_r = 8$. We have $e_2(2, 11)_1 = 4$ and $r_2(2, 11)_1 = 6$; observe that the relations (22) are satisfied. In the matrix $\text{Ad}_w(\overline{gr}_2([2], 11))$ given in Section 3.3.1, the block $A_{0, 1}$ has its first six columns corresponding to $[\mathcal{A}] \in \text{SP}_2([2], 11)$ with $[\mathcal{A}'] = [\mathcal{A}]$. The $[\mathcal{A}]$ is in $\text{SP}_2(2, 11)_1$ corresponding to Column 7 in $A_{0, 1}$ has $[\mathcal{A}] \in \text{SP}_2(2, 11)_1$ corresponding to Column 9. Similarly the $[\mathcal{A}]$ corresponding to Column 8 in $A_{0, 1}$ has $[\mathcal{A}]$ corresponding to Column 10. There is the analogous situation in the rows of $A_{1, 0}$: Rows 1-6 each correspond to a vertex fixed by $\ell$, the vertex of Row 7 is mapped under $\ell$ to the vertex of Row 9, and the vertex of Row 8 is mapped...
under \( \iota \) to the vertex of Row 10. This information is sufficient to calculate \( \text{Ad}_w(\text{gr}_2([2], 11)) \) using the recipe in Section 3.4. Let \( A = \text{Ad}(\text{Gr}_2([2], 11)) = \text{Ad}_w(\text{gr}_2([2], 11)) = [A_{r,s}]_{0 \leq r, s \leq 1}, \) cf. Theorem 16. Then

\[
A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} 3 & 4 & 4 & 4 & 0 & 2 & 4 & 1 & 0 & 0 & 0 & 4 & 4 \\ 9 & 3 & 3 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 6 & 0 \\ 3 & 1 & 3 & 0 & 8 & 0 & 3 & 0 & 6 & 0 & 0 & 0 & 6 \\ 3 & 0 & 0 & 6 & 6 & 3 & 0 & 0 & 2 & 6 & 1 & 3 \\ 0 & 0 & 4 & 3 & 8 & 0 & 0 & 0 & 6 & 0 & 6 & 0 & 3 \end{bmatrix}.
\]

(25)

3.6.2. \( \text{Ad}_w(\text{gr}_3([2], 3)) = \text{Ad}(\text{Gr}_3([2], 3)) \). From Table 2, \( h_3(2, 3)_\Pi = 2 \) and \( h_3(2, 3)_1 = 3 \). Let \( A = \text{Ad}(\text{Gr}_3([2], 3)) = \text{Ad}_w(\text{gr}_3([2], 3)) = [A_{r,s}]_{0 \leq r, s \leq 1}, \) see Theorem 16. Then

\[
A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} 81 & 54 & 54 & 108 & 216 \\ 63 & 72 & 0 & 63 & 315 \\ 18 & 0 & 6 & 9 & 0 \\ 12 & 6 & 3 & 3 & 9 \\ 8 & 10 & 0 & 3 & 12 \end{bmatrix}.
\]

(26)

4. First results on the graphs \( \tilde{\text{gr}}_g([\ell], p) \), \( \text{gr}_g([\ell], p) \), and \( \text{Gr}_g([\ell], p) \)

For \( r \neq g/2 \) let \( \tilde{\text{gr}}_g([\ell], p)_r := \tilde{\text{gr}}_g([\ell], p)_{r, \iota} \) and \( \text{gr}_g([\ell], p)_r := \text{gr}_g([\ell], p)_{r, \iota} \).

4.1. Connectedness results. We proved in [Za, Sect. 8] that the graphs \( \tilde{\text{gr}}_g([\ell], p)_0 \) and \( \text{gr}_g([\ell], p)_\Pi \) are connected. Additionally we proved that \( \text{gr}_g([\ell], p)_{\overline{r}, \overline{s}} \) is not bipartite. The analogous statements are true for our more general \( [\ell] \)-isogeny graphs.

Theorem 17. 1. The graphs \( \tilde{\text{gr}}_g([\ell], p) \) and \( \text{gr}_g([\ell], p) \) are connected.

2. If \( r \neq s \), the graph \( \tilde{\text{gr}}_g([\ell], p)_{r, s} \) is connected. If \( (\overline{r}, \overline{s}) \neq (g/2, g/2) \), the graph \( \text{gr}_g([\ell], p)_{\overline{r}, \overline{s}} \) is connected.

We will need the following lemma.

Lemma 18. Consider the lattice of isotropic subgroups of a symplectic space of dimension \( 2g \). Now consider the subgraph consisting of subspaces of dimension \( r \) and \( s \) with \( 0 \leq r < s \leq g \), i.e., there is an edge between two spaces if one is contained in the other. That subgraph is connected.

Proof. Let \( V \) and \( V' \) be any two isotropic subspaces of dimension \( s \), it suffices to prove there is a path between them, so suppose that there isn’t. Let \( W = V \cap V' \) with \( t = \dim W \) and
suppose \( t \) is maximal among all \( s \)-dimensional subspaces that don’t have a path between them. We must clearly have \( t < r \leq s - 1 \).

Now let \( x \in V \setminus W \) and consider \( U = x^\perp \cap V' \). Since \( V \) is isotropic, \( W \subset U \). And \( \dim U \) is either \( s \) or \( s + 1 \), in the latter case replace \( U \) by an \( s \)-dimensional subset containing \( W \). Set \( V'' = U \oplus \langle x \rangle \). Now \( \dim V \cap V'' = t + 1 \) and \( \dim V' \cap V'' = s - 1 \), so by assumption \( V'' \) is connected to both \( V \) and \( V' \) and we are done. \( \square \)

**Proof of Theorem 17.** Part 1 follows trivially from [JZa] [Theorem 43] since by Proposition 4 every superspecial abelian variety with an \([\ell]\)-polarization has an isogeny to a type 0 abelian variety.

We will prove part 2 is stages.

The connectedness of \( \tilde{g}_r([\ell], p)_{g,0} \) is just [JZa] [Theorem 43].

Now to prove the connectedness of \( \tilde{g}_r([\ell], p)_{r,0} \) it suffices to show that if \([\mathscr{A}'] - [\mathscr{A}] - [\mathscr{A}''] \) is a subgraph of \( \tilde{g}_r([\ell], p)_{g,0} \) with \( \mathscr{A} \) of type \( g \) and \( \mathscr{A}' \), \( \mathscr{A}'' \) of type 0 we can find a path in \( \tilde{g}_r([\ell], p)_{r,0} \) between \([\mathscr{A}']\) and \([\mathscr{A}'']\). Let \( \mathscr{A} = (A, \lambda) \), by Proposition 4 \( \mathscr{A}' \) and \( \mathscr{A}'' \) correspond to isotropic rank-\( g \) subgroups of \( \ker(\lambda) \) and rank-(\( g - r \)) subgroups of \( \ker(\lambda) \) give type \( r \) vertices of \( \tilde{g}_r([\ell], p)_{r,0} \). Since the graph of isotropic subspaces of \( \ker(\lambda) \) of ranks \( g \) and \( g - r \) is connected, we have a path from \([\mathscr{A}']\) to \([\mathscr{A}'']\) in \( \tilde{g}_r([\ell], p)_{r,0} \).

Next apply the involution we see that the graphs \( \tilde{g}_r([\ell], p)_{g,s} \) are connected. And by the same logic as above we get the connectedness of \( \tilde{g}_r([\ell], p)_{r,s} \) for general \( r \) and \( s \).

Finally, \( gr_r([\ell], p)_{r,s} \) connected since it’s the quotient of the connected graph \( \tilde{g}_r([\ell], p)_{r,s} \). \( \square \)

### 4.2. The Ramanujan condition.

Let \( A = \text{Ad}_w(\tilde{g}_r([\ell], p)) \); it is a block matrix with \( A = [A_{r,s}]_{0 \leq r, s \leq g} \). The matrices \( A_{r,s} \) are symmetric with constant row sum equal to \( N(\ell)_{r,s} \) if \( r > s \), \( N(\ell)_{r,s} \) if \( r > s \), and 0 if \( r = s \). Let \( Gr_r([\ell], p) \) be the regular graph with adjacency matrix \( A_{r,s} \) for \( r \neq g/2 \). The graph \( Gr_r([\ell], p)_{0} = Gr_r([\ell], p)_{g} \) is the big isogeny graph \( Gr_r([\ell], p) \) for principally polarized superspecial abelian varieties defined in [JZa] Sect. 7.1.

We saw in [JZa] Sect. 10 that this graph is sometimes, but rarely, Ramanujan and usually non-Ramanujan. We get the same behavior for the graphs \( Gr_r([\ell], p)_r \) for \( 0 < r < g \). We give a Ramanujan and a non-Ramanujan example below.

#### 4.2.1. A non-Ramanujan example: \( Gr_3([3], 2)_1 \).

We have

\[
\text{Ad}(Gr_3([3], 2)_1) = \text{Ad}_w(\tilde{g}_r([3], 2)_{1,2}) = \begin{bmatrix} 8 & 32 & 0 \\ 4 & 4 & 32 \\ 0 & 12 & 28 \end{bmatrix}.
\]

This matrix has eigenvalues 40, ±12. As \( 12 < 2\sqrt{39} \), the graph \( Gr_3([3], 2)_1 \) is Ramanujan.

#### 4.2.2. A Ramanujan example: \( Gr_3([2], 3)_1 \).

The graph \( \tilde{g}_3([2], 3) \) was considered in Section 3.3.2. We have

\[
\text{Ad}(Gr_3([2], 3)_1) = \text{Ad}_w(g_r([2], 3)_{2,1,2}) = \begin{bmatrix} 6 & 9 & 0 \\ 3 & 3 & 9 \\ 0 & 3 & 12 \end{bmatrix}.
\]

The eigenvalues of this matrix are 15 and \( 3 \pm 3\sqrt{3} \). Since \( 3 + 3\sqrt{3} > 2\sqrt{14} \), the graph \( Gr_3([2], 3)_1 \) is not Ramanujan.
5. The $\ell$-adic uniformization of $\tilde{g}_g([\ell], p)$ and $g_r([\ell], p)$

Let $B_{2g}$ be the Bruhat-Tits building for $GSp_{2g}$ over $\mathbb{Q}_\ell$. Let $Sk = Sk_{2g}$ be the 1-skeleton of $B_{2g}$. Vertices $v \in Sk \subseteq B_{2g}$ have a type $t(v) \in \mathbb{Z}$ with $0 \leq t(v) \leq g$. The special vertices are the vertices $v$ with $t(v) = 0$ or $t(v) = g$. To handle types of positive-dimensional cells, set

$$\mathcal{T}_s := \{(r_0, \ldots, r_s) \in \mathbb{Z}^{s+1} | g \geq r_0 > r_1 > \cdots > r_g \geq 0\} \quad (27)$$

for $0 \leq s \leq g$. Edges $e \in Sk \subseteq B_{2g}$ have a type $t(e) = r \in \mathcal{T}_1$. For $r,s \in \mathcal{T}_0$, $r \neq s$, let $Ver(Sk)_{r,s} = \{v \in Ver(Sk) | t(v) = r \text{ or } t(v) = s\}$ with $Sk_{r,s}$ the induced subgraph of $Sk$ on the vertex set $Ver(Sk)_{r,s}$.

Let $R$ be a commutative ring and $M$ be an $R$-algebra with an anti-involution $x \mapsto x^\dagger$. Define the unitary group $U(M)$ and general unitary group $GU_R(M)$ by

$$U(M) = \{x \in M | x^\dagger x = 1\} \quad (28)$$

$$GU_R(M) = \{x \in M | x^\dagger x \in R^x\}.$$

Recall that $E/\overline{F}_p$ is a supersingular elliptic curve with $\mathcal{O} = \mathcal{O}_E = \text{End}(E)$. We apply (28) with $R = \mathbb{Z}[1/\ell]$ and $M = \text{Mat}_{g \times g}(\mathcal{O}_E[1/\ell])$ with anti-involution $x^\dagger = \overline{x}^\dagger$ for $x \in M$. Set

$$U_g(\mathcal{O}[1/\ell]) = U(\text{Mat}_{g \times g}(\mathcal{O}[1/\ell])) \quad \text{and} \quad (29)$$

$$GU_g(\mathcal{O}[1/\ell]) = GU_{\mathbb{Z}[1/\ell]}(\text{Mat}_{g \times g}(\mathcal{O}[1/\ell])).$$

The groups $U_g(\mathcal{O}[1/\ell])$ and $GU_g(\mathcal{O}[1/\ell])$ act on $B_{2g}$ with finite quotients.

Theorem 19. (a) $U_g(\mathcal{O}[1/\ell]) \backslash Sk_{2g} \cong g_g([\ell], p)$ as graphs with weights.

(b) The group $U_g(\mathcal{O}[1/\ell])$ acting on $Sk = Sk_{2g}$ stabilizes the subgraph $Sk_{r,s}$ for $r,s \in \mathcal{T}_0$, $r \neq s$, and $g_g([\ell], p)_{r,s} \cong U_g(\mathcal{O}[1/\ell]) \backslash Sk_{r,s}$ as graphs with weights.

Proof. This will follow as a special case of Theorem 26(30) and the fact that the action of $U_g(\mathcal{O}[1/\ell])$ preserves types.

Theorem 19(a) implies that the graph $g_g([\ell], p)$ is connected, which we have already established in Theorem 17(1). Likewise Theorem 19(1) would imply that $g_g([\ell], p)_{r,s} \cong U_g(\mathcal{O}[1/\ell]) \backslash Sk_{r,s}$ is connected if we knew that $Sk_{r,s}$ was connected—but we already know that $g_g([\ell], p)_{r,s} \cong U_g(\mathcal{O}[1/\ell]) \backslash Sk_{r,s}$, $r \neq s$, is connected from Theorem 17(2).

Theorem 20. $GU_g(\mathcal{O}[1/\ell]) \backslash Sk_{2g} \cong g_g([\ell], p)$ as graphs with weights.

Proof. This will follow as a special case of Theorem 26(31).

In particular, Theorem 20 implies that the graphs $g_g([\ell], p)$ and $g_g([\ell], p)_{r,s}$ are connected. However, as above, connectedness was already shown in Theorem 17(1).

6. ISOGONY COMPLEXES OF [\ell]-POLARIZED ABELIAN VARIETIES

Fix a prime $\ell$. In this Section 6 and Section 7, we work with isogenies of degree $\ell^n$ of general polarized abelian varieties $A = (A, \lambda)$ over an algebraically closed field $k$ with $(\text{char } k, \ell) = 1$. We then return to superspecial abelian varieties in Section 8.

We will define $[\ell]$-isogeny complexes of $[\ell]$-polarized abelian varieties, generalizing the $[\ell]$-isogeny graphs of Section 3.
6.1. \(\Delta\)-complexes and \(\Delta\)-complexes with half-faces. The complexes \([\ell]\)-isogenies naturally form the higher-dimensional analogues of multigraphs, and quotients of these complexes by an involution. They are a mild generalization of simplicial complexes introduced by Eilenberg and Zilber [EZ50] in 1950 as semi-simplicial complexes. Hatcher [Hat02] Sect. 2.1 calls them \(\Delta\)-complexes and we follow this in the interest of brevity and disambiguation. However, we use the purely combinatorial definition of Eilenberg-Zilber [EZ50] Sect. 1. Taking the quotient of a \(\Delta\)-complex by an involution gives a \(\Delta\)-complex with half-faces or an \(h\)-\(\Delta\)-complex. A one-dimensional \(\Delta\)-complex is a multigraph, called simply a graph in Section 3.1. A one-dimensional \(\Delta\)-complex with half-faces or a one-dimensional \(h\)-\(\Delta\)-complex is a graph with half-edges or an \(h\)-graph as in Section 3.1.

6.1.1. \(\Delta\)-complexes.

**Definition 21.** (a) ([EZ50] Sect. 1) A \(\Delta\)-complex \(K\) of dimension \(g\) consists of a set \(\Sigma = \Sigma(K) = \{\sigma\}\) of simplexes together with two functions on \(\Sigma\). The first assigns to any \(\sigma \in \Sigma\) an integer \(j = \text{dim}(\sigma)\) with \(g \geq j \geq 0\) called the dimension of \(\sigma\). Put \(\Sigma_j = \Sigma_j(K) = \{\sigma \mid \text{dim}(\sigma) = j\} \subseteq \Sigma\) and say that \(\sigma \in \Sigma_j\) is a \(j\)-simplex. The 0-simplexes \(\Sigma_0(K)\) are called the vertices \(\text{Ver}(K)\) of \(K\). Likewise the 1-simplexes \(\Sigma_1(K)\) are called the edges \(\text{Ed}(K)\) of \(K\). We assume \(\Sigma_0 \neq \emptyset\); the simplexes \(\sigma \in \Sigma_0(K)\) are called facets.

The second function on \(\Sigma\) associates to each \(j\)-simplex \(\sigma\) and to each integer \(0 \leq i \leq j\) a \((j-1)\)-simplex \(\sigma_i(h)\) called the \(i\)th face of \(\sigma\) such that \(\sigma_i(h(h_i)\sigma) = \sigma_{i-1}(h_i(h)(\sigma))\) for \(i < j\) and \(j > 1\). Iteration of this map gives lower-dimensional faces. For \(0 \leq i_1 < \cdots < i_n \leq j\) define inductively

\[
\sigma_{i_0, \ldots, i_n}(h) = \sigma_{i_1} h_{i_2, \ldots, i_n}(h)
\]

which is a \((j-n)\)-simplex. If \(0 \leq k_0 < \cdots < k_{j-n} \leq j\) is the set complementary to \(\{i_1, \ldots, i_n\} \subseteq \{1, 2, \ldots, j\}\), then we also write \(\sigma_{i_0, \ldots, i_n}(h) = \sigma(k_{j-n}, \ldots, k_0)\). In particular for \(\sigma \in \Sigma_j\) we have \(\sigma(i) \in \Sigma_0\) for \(0 \leq i \leq j\) called the \(i\)th vertex of \(\sigma\). The definition of the \(i\)th face does not exclude the possibility of \(\sigma, \tau \in \Sigma_j, \sigma \neq \tau, \) with \(\sigma_i(h(h_i)\tau) = \tau_{i-1}(h_i(h)(\tau))\) for \(0 \leq i \leq j\) (which would not be allowed in a simplicial complex). Hence a \(\Delta\)-complex is analogous to a multigraph.

(b) A **weighted \(\Delta\)-complex** is a \(\Delta\)-complex together with a **weight function** \(w\) mapping simplexes to positive integers such that if \(\sigma\) is a simplex and \(\sigma'\) is a face of \(\sigma\) then \(w(\sigma') = w(\sigma)\).

6.1.2. \(\Delta\)-complexes with half-faces. Suppose \(K\) is a \(\Delta\)-complex and \(\iota : K \to K\) is an involution. As in [Bro94] Sect. IX.10, say that \(\iota\) is **admissible** if \(\iota\) fixing a simplex \(\sigma\) of \(K\) is equivalent to \(\iota\) fixing \(\sigma\) pointwise. If the involution \(\iota\) of the \(\Delta\)-complex \(K\) is admissible, then the quotient \(K/\iota\) is naturally again a \(\Delta\)-complex. However, if the action is not admissible, then the quotient \(K/\iota\) will in general have half-faces and will not be a \(\Delta\)-complex. We refer to the images of simplexes in \(K\) under the natural projection \(K \to K/\iota\) as cells. Many references only consider admissible actions since this is obtainable by passing to a subdivision: Let \(K'\) be the first barycentric subdivision of \(K\). Then \(\iota\) induces an involution \(\iota' : K' \to K'\), which is admissible. However, passing to the barycentric subdivision explodes the number of cells, making computing examples too difficult. Hence we do not pass to the barycentric subdivision, and so we have to deal directly with half-faces. We draw some of the half-faces in dimensions 1, 2, and 3 which occur in Section 6.4.
6.2. The enhanced \([\ell]-\)isogeny complex \(\tilde{\text{co}}_\ell(\mathcal{A}_0)\). The isogenies we will consider are of the following type:

**Definition 22.** Two polarized abelian varieties \(\mathcal{X} = (X, \lambda)\) and \(\mathcal{X}' = (X', \lambda')\) are said to be \(\ell\)-power isogenous if there exists an isogeny in the sense of Definition 2(a) from \(\ell^n \mathcal{X} = (X, \ell^n \lambda)\) to \(\mathcal{X}'\) for some \(n \in \mathbb{N}\). We will say that they’re **evenly** (respectively, **oddly**) isogenous if \(n\) is even (respectively, odd). Note that a principally polarized abelian variety is oddly isogenous to its \([\ell]-\)dual by Remark 1(b) since 1 is odd.

Pick a base \(g\)-dimensional principally polarized abelian variety \(\mathcal{A}_0 = (\mathcal{A}_0, \lambda_0)\) over an algebraically closed field \(k\) with \(\ell\) coprime to \(\text{char} \ k\); \(\mathcal{A}_0\) is not required to be superspecial. All polarized abelian varieties discussed in this section and the next will be polarized and \(\ell\)-power isogenous to \(\mathcal{A}_0\).

We will first define the enhanced isogeny complex \(\tilde{\text{co}}_\ell(\mathcal{A}_0)\).

**Definition 23.** (a) The vertices \(\text{Ver}(\tilde{\text{co}}_\ell(\mathcal{A}_0))\) or 0-simplexes \(\Sigma_0(\tilde{\text{co}}_\ell(\mathcal{A}_0))\) correspond to isomorphism classes \([\mathcal{A} = (A, \lambda)]\) of abelian varieties \(A\) with a polarization \(\lambda\) of type \(r\) with \(0 \leq r \leq g\) as in Remark 1(a) such that \(\mathcal{A}\) is \(\ell\)-power isogenous to \(\mathcal{A}_0\). Recall that a polarization of type 0 is simply a principal polarization. If \(\mathcal{A} = (A, \lambda)\) with \(\lambda\) a polarization of type \(g\), then there is a principal polarization \(\lambda'\) of \(A\) with \([\mathcal{A} = \ell \mathcal{A}' = (A, \ell \lambda')]\) for \(\mathcal{A}' = (A, \lambda')\) as in Remark 1(b). Thus there is a natural bijection between polarizations of type \(g\) on \(A\) and those of type 0. Nevertheless, \([\mathcal{A}]\) and \([\mathcal{A}']\) correspond to distinct 0-cells of the enhanced complex. The weight of a vertex \([\mathcal{A}]\) in \(\Sigma_0(\tilde{\text{co}}_\ell(\mathcal{A}_0))\) is \(\# \text{Aut}(\mathcal{A})\).

Suppose \(\mathcal{A}_i = (A_i, \lambda_i)\) with \(\lambda_i\) a polarization of type \(r_i\). For \(r_0 > r_1\) the (unoriented) edges \(\Sigma_1(\tilde{\text{co}}_\ell(\mathcal{A}_0))\) are isomorphism classes of strict maps \(f : \mathcal{A}_0 \to \mathcal{A}_1\) as in Definition 2(a), where an isomorphism between two strict maps is defined in the natural manner. Such an edge will be said to be of **type** \((r_0, r_1)\), if \(e \in \Sigma_1(\tilde{\text{co}}_\ell(\mathcal{A}_0))\) is the edge corresponding to \(f\), then the faces of \(e\) are the vertices \(\text{Fa}_0(e) = [\mathcal{A}_1] = e_{(1)} \in \Sigma_0(\tilde{\text{co}}_\ell(\mathcal{A}_0))\) and \(\text{Fa}_1(e) = [\mathcal{A}_0] = e_{(0)} \in \Sigma_0(\tilde{\text{co}}_\ell(\mathcal{A}_0))\). The weight \(w(e)\) of the edge \(e\) is the cardinality of the automorphism group of \(f : \mathcal{A}_1 \to \mathcal{A}_2\).

At height \(k\), a \(k\)-simplex \(\sigma\) is an isomorphism class of sequences of strict maps

\[
\mathcal{A}_0 \xrightarrow{f_1} \mathcal{A}_1 \xrightarrow{f_2} \mathcal{A}_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{k-1}} \mathcal{A}_{k-1} \xrightarrow{f_k} \mathcal{A}_k.
\]

Necessarily \(g \geq r_0 > r_1 > \cdots > r_{k-1} > r_k \geq 0\). Such a \(k\)-simplex will be said to be of **type** \(\mathbf{r} = (r_0, \ldots, r_k)\). We denote by \(\Sigma_r \subseteq \Sigma_k\) the set of \(k\)-cells of type \(\mathbf{r} = (r_0, \ldots, r_k)\), \(g \geq r_0 > r_1 > \cdots > r_{k-1} > r_k \geq 0\). The \((k+1)\) faces of \(\sigma \in \Sigma_r\) are the \((k-1)\)-simplexes as follows:

(a) The face \(\text{Fa}_0(\sigma)\) of type \((r_1, r_2, \ldots, r_k)\) is the isomorphism class of the sequence of strict maps

\[
\mathcal{A}_1 \xrightarrow{f_2} \mathcal{A}_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{k-1}} \mathcal{A}_{k-1} \xrightarrow{f_k} \mathcal{A}_k.
\]

(b) The face \(\text{Fa}_i(\sigma)\) for \(1 \leq i \leq k-1\) of type \((r_0, \ldots, \hat{r}_i, \ldots, r_k)\) (here \(\hat{r}_i\) means omit \(r_i\)) is the isomorphism class of the sequence of strict maps

\[
\mathcal{A}_0 \xrightarrow{f_{i-1}} \mathcal{A}_1 \xrightarrow{f_2} \mathcal{A}_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{i-1}} \mathcal{A}_{i-1} \xrightarrow{f_{i+1} \circ f_i} \mathcal{A}_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_{k-1}} \mathcal{A}_{k-1} \xrightarrow{f_k} \mathcal{A}_k.
\]

(c) The face \(\text{Fa}_k(\sigma)\) of type \((r_0, \ldots, r_{k-1})\) is the isomorphism class of the sequence of strict maps

\[
\mathcal{A}_0 \xrightarrow{f_1} \mathcal{A}_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} \mathcal{A}_{k-1}.
\]
The weight \( w(\sigma) \) of the \( k \)-simplex \( \sigma \) is the cardinality of the automorphism group of the sequence of maps. In particular that there are \( \binom{n+1}{k+1} \) types of \( k \)-simplexes.

**NOTE:** The \( \Delta \)-complex \( \tilde{\mathcal{C}}_{\ell}(A_0) \) is connected if and only if \( A_0 = (A_0, \lambda_0) \) is oddly \( \ell \)-isogenous to itself, otherwise it has two connected components consisting of those polarized abelian varieties that are evenly and oddly \( \ell \)-isogenous to \( A_0 \). Let \( \tilde{\mathcal{C}}_{\ell,0}(A_0) \) be the connected component of \( \tilde{\mathcal{C}}_{\ell}(A_0) \) containing the vertex \([A_0]\). The vertices of \( \tilde{\mathcal{C}}_{\ell,0}(A_0) \) are all isomorphism classes of polarized abelian varieties evenly \( \ell \)-power isogenous to \( A_0 \).

(b) The **dual map** \( \iota : \tilde{\mathcal{C}}_{\ell}(A_0) \to \tilde{\mathcal{C}}_{\ell}(A_0) \) is the involution sending each simplex to its dual: a vertex \([\mathcal{A}]\) is sent to its dual \([\mathcal{A}]^\circ\); observe that if \( \mathcal{A} \) was of type \( r \) then \( \mathcal{A}^\circ \) will be of type \( \tilde{r} = g-r \). Each higher simplex \( \mathcal{A}_0 \xrightarrow{1} \mathcal{A}_1 \xrightarrow{2} \cdots \xrightarrow{k-1} \mathcal{A}_{k-1} \xrightarrow{k} \mathcal{A}_k \) of type \( r = (r_0, \ldots, r_k) \) gets sent to \( \mathcal{A}_0^\circ \xrightarrow{k} \mathcal{A}_1^\circ \xrightarrow{k-1} \cdots \xrightarrow{2} \mathcal{A}_k^\circ \xrightarrow{1} \mathcal{A}_0^\circ \) of type \( \tilde{r} := (g-r_k, \ldots, g-r_0) \). Observe that this reverses any orientation of the simplex precisely when \( k \equiv 1, 2 \pmod{4} \).

**NOTE:** If \( \tilde{\mathcal{C}}_{\ell}(A_0) \) is disconnected, then \( \iota \) swaps the two components.

### 6.3. The little \([\ell]\)-isogeny graph \( \mathcal{C}_{\ell}(A_0) \)

We define the little \([\ell]\)-isogeny complex as the quotient \( \mathcal{C}_{\ell}(A_0) := \tilde{\mathcal{C}}_{\ell}(A_0) / \iota \). Every \( k \)-cell of \( \mathcal{C}_{\ell}(A_0) \) has a type \( \overline{r} = \{r, \tilde{r}\} \) with \( \overline{r} = \tilde{r} \). Unfortunately, under this map some faces are self-dual without every face of their boundary being self-dual, resulting in what we call **half-faces**. Because of this, \( \mathcal{C}_{\ell}(A_0) \) does not inherit the structure of a \( \Delta \)-complex from \( \tilde{\mathcal{C}}_{\ell}(A_0) - \) it is a \( \Delta \)-complex with half-faces, or an \( h-\Delta \)-complex.

In a future paper \[JZb\] we consider the cohomology of \( \tilde{\mathcal{C}}_{\ell}(A_0) \) and \( \mathcal{C}_{\ell}(A_0) \) when \( A_0 \) is a principally polarized superspecial abelian variety in characteristic \( p \neq \ell \). The cohomology of \( \tilde{\mathcal{C}}_{\ell}(A_0) \) is straightforward – Eilenberg and Zilber \[EZ50\] proved that the canonical cochain complex constructed from the simplexes computes the cohomology of a \( \Delta \)-complex. But the cohomology of \( \mathcal{C}_{\ell}(A_0) \) is much trickier. Topologically the right thing to do is clearly to barycentrically subdivide all the simplexes of \( \tilde{\mathcal{C}}_{\ell}(A_0) \) before taking the quotient by the dual map \( \iota \). But this greatly increases the number of cells, making computing examples out of reach. So we will deal with the half-faces directly. We examine the low-dimensional cells in \( \mathcal{C}_{\ell}(A_0) \) in the next Section \[6.4\] and find that the \( k \)-cells in \( \mathcal{C}_{\ell}(A_0) \) are balls for \( k \leq 4 \), so that \( \mathcal{C}_{\ell}(A_0) \) is naturally a CW-complex if the dimension \( g \leq 4 \). We can therefore compute its cohomology in this case from a canonical cochain complex constructed from its cells.

### 6.4. Half-faces of low dimension

In this section we describe what half-faces look like in low dimension. For each example, we consider the involution \( \iota \) which reverses the order of the vertices in a standard simplex with the induced action on each simplex in the picture.

A 0-dimensional face is a vertex or point, a 0-dimensional half-face is also functionally a point.

A 1-dimensional face is an edge. A 1-dimensional half-face is a half-edge as described and drawn in \[Kur79\] and subsequently \[IJK+, IJK+21, JZa\]:

```
\[ \begin{array}{c}
\bullet \\
\cdot \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
\bullet \\
\cdot \\
\end{array} \]```

20
Here the original complex has 2 vertices and 1 edge. The quotient complex by \( \iota \) has 1 vertex and 1 half-edge. Note that the half-edge is contractible to its vertex.

A 2-dimensional face is a triangle; it has 1 2-simplex, 3 edges, and 3 vertices. The quotient by \( \iota \) in turn has 1 half-2-simplex, 1 edge, 1 half-edge, and 2 vertices. A 2-dimensional half-face is a half-triangle as seen below:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{triangle}\quad\rightarrow\quad\includegraphics[width=0.5\textwidth]{half-triangle}
\end{array}
\]

Note that the half-face and its associated half-edge are contractible to its full edge.

It’s also possible for a full 2-face to have a half-edge as shown below:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{2-face}\quad\rightarrow\quad\includegraphics[width=0.5\textwidth]{2-face-half-edge}
\end{array}
\]

The quotient has 1 2-face, 2 edges, 1 half-edge, and 2 vertices. Note that the half-edge is contractible leaving the triangle a digon.

A 3-dimensional face is a tetrahedron with the involution acting as rotation about the indicated axis which passes through the midpoint \( A \) of an edge and the midpoint \( B \) of the opposing edge. We label where these midpoints go in the quotient by \( \iota \) using the same letter. A 3-dimensional half-face is a half-tetrahedron as seen below:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{tetrahedron}\quad\rightarrow\quad\includegraphics[width=0.5\textwidth]{half-tetrahedron}
\end{array}
\]
The original simplex has 1 3-simplex, 4 2-simplexes, 6 edges, and 4 vertices. The quotient has 1 half-3-simplex, 2 half-simplexes, 2 edges, 2 half-edges, and 2 vertices. Note that the boundary of the half tetrahedron contains half-edges and faces that turn into digons when the half-edges are contracted as described above. The half-tetrahedron itself is still a topological 3-ball $B^3$, and it is not contractible to its boundary.

Once we hit faces of dimension 4 and higher, it is no longer really possible to draw pictures. The standard $k$-simplex $\Delta_k$ in $\mathbb{R}^{k+1}$ is

$$\Delta_k = \{(x_1, \ldots, x_{k+1}) \mid \sum_{i=1}^{k+1} x_i = 1, x_i \geq 0 \text{ for } 1 \leq i \leq k+1\}.$$ 

In this standard $k$-simplex, the involution $\iota$ reversing the order of the vertices is induced by the linear involution of $\mathbb{R}^{k+1}$ given in the standard basis by the matrix $M = M_{k+1}$ with 1’s on the antidiagonal and 0’s elsewhere. The eigenvalues of the matrix $M_{k+1}$ are $(k+1)/2$ +1’s, $(k+1)/2$ −1’s if $k$ is odd and $k/2 + 1$ +1’s, $k/2$ −1’s if $k$ is even. The vector $\vec{v} \in \mathbb{R}^{k+1}$ with all entries 1 is an eigenvector of $M_{k+1}$ with eigenvalue 1 and $\vec{v}$ is perpendicular to the affine hyperplane $\sum_{i=1}^{k+1} i = 1$. Hence the eigenvalues of $M_{k+1}$ acting on $\Delta_k$ are $(k-1)/2$ +1’s, $(k+1)/2$ −1’s if $k$ is odd and $k/2$ +1’s, $k/2$ −1’s if $k$ is even.

In particular on $\Delta_4$ the involution acts on $\Delta_4$ with 2 +1 eigenvalues and 2 −1 eigenvalues. Hence the quotient of $\Delta_4$ by $\iota$ is (the cone on $\mathbb{R}P^4$) $\times B^2$, the cone on $\mathbb{R}P^4$ coming from the 2 −1 eigenvalues and the 2-ball $B^2$ coming from the 2 +1-eigenvalues. But $\mathbb{R}P^4$ is just the circle $S^1$, so the half 4-simplex $\Delta_4/\iota$ is homeomorphic to (the cone on $S^1$) $\times B^2$, which is just $B^2 \times B^2 \cong B^4$. Hence a half 4-simplex is still a topological ball $B^4$ and it not contractible to its boundary.

Now on the standard 5-simplex $\Delta_5$ the involution acts with eigenvalues 3 −1’s and 2 +1’s. As above this gives us that a half 5-simplex is homeomorphic to $S := (\text{the cone on } \mathbb{R}P^2) \times B^2$. However, we claim that now in dimension 5 this is not the ball $B^5$; here is one way to see this:

**Proposition 24.** Let $S = (\text{the cone on } \mathbb{R}P^2) \times B^2$. Then $S$ is not homeomorphic to $B^5$, and in fact $\partial S$ is not homotopic to $\partial B^5$.

**Proof.** The first statement clearly follows from the second. We claim that

$$H_3(\partial S, \mathbb{Z}) \neq 0 \text{, whereas } H_3(\partial B^5, \mathbb{Z}) = 0.$$ 

Firstly, recall that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$. But then $\partial S$ is homotopic to the double suspension $\Sigma^2(\mathbb{R}P^2)$. Since each suspension moves homology up by 1 we have that $H_3(\Sigma^2(\mathbb{R}P^2), \mathbb{Z}) \neq 0$ since $H_1(\mathbb{R}P^2, \mathbb{Z}) \neq 0$. But $H_3(\partial B^5, \mathbb{Z}) = 0$ since $\partial B^5 = S^4$. \qed

We summarize our discussion above in the following theorem.

**Theorem 25.** Let $A_0$ be a $g$-dimensional principally polarized abelian variety over an algebraically closed field $k$ with char $k \neq \ell$. Let $K$ be the $\Delta$-complex $\widetilde{\mathfrak{D}}(A_0)$ and $\iota$ the involution of $K$ given by taking isogenies to their duals and taking $[\ell]$-polarized abelian varieties to their $[\ell]$-duals. Set $K/\iota = \mathfrak{D}(A_0) = \widetilde{\mathfrak{D}}(A_0)/\iota$.

(a) The half-faces of $K/\iota$ of dimensions 1 and 2 are homeomorphic to the standard cells $B^1$ and $B^2$, and are contractible to their boundary.

(b) The half-faces of $K/\iota$ of dimensions 3 and 4 are homeomorphic to the standard cells $B^3$ and $B^4$, and are not contractible to their boundary.
(c) The half-faces of $K/\iota$ of dimension 5 and higher are not homeomorphic to the standard cell (i.e., the ball of the same dimension).

(d) Suppose the dimension $g \leq 4$. Then $\co_\ell(A_0) = K/\iota$ can be given the structure of a CW-complex $K^*$, after homotopically collapsing the h-1-cells and h-2-cells. The CW-complex $K^*$ with 0-cells the 0-cells of $K/\iota$, 1-cells the regular edges of $K/\iota$, 2-cells the regular 2-simplexes of $K/\iota$, 3-cells equal to the regular 3-simplexes and half 3-simplexes of $K/\iota$, and 4-cells equal to the regular 4-simplexes and half 4-simplexes of $K/\iota$.

7. Identifying isogeny complexes for $\ell$-polarized abelian varieties as quotients of the building for $\text{Sp}_{2g}(\mathbb{Q}_\ell)$

Let $(V, \langle \ , \rangle)$ be a $2g$-dimensional symplectic space over $\mathbb{Q}_\ell$. Let $W \subset V$ be a $\mathbb{Z}_\ell$-sublattice. Recall that there exists a basis of $W$ such that the symplectic form decomposes as the sum of $g$ forms of the form $\begin{pmatrix} 0 & \ell^m \\ -\ell^m & 0 \end{pmatrix}$ with the $g$-tuple $(m_i) = (m_1, m_2, \ldots, m_g)$ of $m$'s uniquely determined by $W$ (up to permutation). We will call $W$ an $(m_i)$-lattice. Also recall that the dual lattice $W^*$ is defined as $\{ x \in V \mid \langle W, x \rangle \subset \mathbb{Z}_\ell \}$. If $W$ is an $(m_i)$-lattice then $\ell^a W$ is a $(2a + m_i)$-lattice and $W^*$ is a $(-m_i)$ lattice. A $((m + 1)^n, (m)^{-n})$-lattice will be called special and a $((1)^n, (0)^{-n})$-lattice will be called extra special of type $n$.

Recall that the vertices of the building $B_g$ for $\text{Sp}_{2g}(\mathbb{Q}_\ell) = \text{Sp}(V)$ are given by the extra special lattices. We shall say a vertex corresponding to a $((1)^r, (0)^{g-r})$-lattice $W$ has type $r$. Recall also that the $k$-faces of the building are given by chains of vertices with proper inclusions $W_0 \subset W_1 \subset \cdots \subset W_k$ where we call $r = (r_0, r_1, \ldots, r_k)$ the type of the face if $r_i$ is the type of $W_i$. Note that $r_0 > r_1 > \cdots > r_k$.

Recall that $\text{Sp}(V)$ acts on all these faces and hence the building in the obvious way preserving the types of faces. We shall take this action to be on the left.

Finally recall that we can extend this action to an action of $\text{GSp}(V)$. First identify $W$ with the generalized homothety set $\{ \ell^a W \} \cup \{ \ell^a W^* \}$. Notice that: this operation partitions the lattices into equivalence classes; if $W$ is special, all the elements of the set are; and each special set has precisely one extra special element. Also the action of $\text{GSp}(V)$ preserves generalized homothety. This induces an action of $\text{GSp}(V)$ on the building. However, unlike the $\text{Sp}(V)$ action which preserves types, the elements of $\text{GSp}(V)$ which scale by an odd power of $\ell$ send faces of type $r = (r_0, r_1, \ldots, r_k)$ to faces of type $\ell^{-1} r := (g - r_k, g - r_{k-1}, \ldots, g - r_0)$. Also since scalar multiplication fixes generalized homothety, this action factors through $\text{PGSp}(V)$.

**Theorem 26.** (cf. JZa Theorem 21.) Let $A_0 = (A_0, \lambda_0)$ be a principally polarized abelian variety of dimension $g$ over an algebraically closed field $k$ of characteristic $\text{char } k$ and $\ell \neq \text{char } k$ be a prime. In the notation of (28) and Definition 23(a) we have

\[
\co_\ell^0(A_0) \cong \text{U(End}(A_0)[1/\ell]) \backslash B_g
\]

and

\[
\co_\ell(A_0) \cong \text{GU(End}(A_0)[1/\ell]) \backslash B_g.
\]

**Proof.** Let $W = \text{Ta}_\ell(A_0)$ be the Tate module of $A_0$, and let $V = \text{Ta}_\ell(A_0) \otimes \mathbb{Q}_\ell$, both equipped with the symplectic Weil pairing. Note that we have an exact sequence

\[0 \to W \to V \xrightarrow{\phi} A_0[\ell^\infty] \to 0.\]
This gives us induced injections $U(\text{End}(A_0)[1/\ell]) \to \text{Sp}(V)$ and $\text{GU}(\text{End}(A_0)[1/\ell]) \to \text{GSp}(V)$.

Let $\mathcal{A}_1$ be evenly $\ell$-isogenous to $A_0$ via $\phi : \ell^{2m}A_0 \to \mathcal{A}_1$. We will associate to the pair $(\phi, \mathcal{A}_1)$ the lattice $W_1 = \ell^m\pi^{-1}(\ker \phi) \subset V$. Since $\mathcal{A}_1$ is $[\ell]$-polarized, we can see that $W_1$ is extra special of the same type as $\mathcal{A}_1$ by chasing through the matrices. Conversely, if $W_1$ is extra special and we can pick an $m \in \mathbb{N}$ such that $\ell^{-m}W_1 \supset W$, then $\pi(\ell^{-m}W_1)$ is the kernel of an $\ell$-power isogeny $\phi : \ell^{2m}A_0 \to \mathcal{A}_1$ whose image is $[\ell]$-polarized of the same type as $W_1$. Furthermore, picking a different representative corresponds to composing $\phi$ with multiplication by a scalar power of $\ell$.

Note that if $\psi : \mathcal{A}_0 \to \mathcal{A}_1$ is an isogeny of type $(\ell_0, \ell_1)$, $\ell_0 > \ell_1$, then

$$W_1 = \ell^m\pi^{-1}(\ker \psi \circ \phi) \supseteq W_0$$

is an extra-special lattice of type $\ell_1$. Hence the chain $W_0 \subset W_1$ is an edge of type $(\ell_0, \ell_1)$. The preceding can clearly be applied to each isogeny in a chain to obtain higher faces. Conversely, if the chain $W_0 \subset W_1$ is an edge of type $(\ell_0, \ell_1)$ with $\ell^{-m}W_0 \supset W$ and $\pi(\ell^{-m}W_0)$ the kernel of $\mathcal{A}_1 : \ell^{2m}A_0 \to \mathcal{A}_1$, we can decompose $\phi_1 = \psi \circ \phi_0$ with $\psi : \mathcal{A}_0 \to \mathcal{A}_1$ an isogeny of type $(\ell_0, \ell_1)$. Again the preceding can be applied to higher faces to obtain a chain of isogenies.

Now we show that $W'$ and $W''$ correspond to isomorphic $[\ell]$-polarized abelian varieties if and only if $W' = \psi W''$ for some $\psi \in U(\text{End}(A_0)[1/\ell])$. First assume we are given $\psi$. Pick an $m \in \mathbb{N}$ such that $\ell^m\psi \in \text{End}(A_0)$, and an $n$ such that $\ell^{-n}W', \ell^{-n}W'' \supset W$. Therefore $\pi(\ell^{-n}W') = \ell^m\psi(\pi(\ell^{-n}W''))$. Hence if $\ker \psi' = \pi(\ell^{-n}W')$ and $\ker \psi'' = \pi(\ell^{-n}W'')$, then $\psi' \circ \ell^m\psi = \psi''$ and both have the same codomain.

Conversely, suppose $\psi'$ and $\psi''$ (using scalings $m'$ and $m''$) induced from $W'$ and $W''$ have the same codomain. Let $\phi$ be such that $\phi \circ \phi' = \ell^{2m'}$ let $\psi = \ell^{-m'-m''}\phi \circ \phi''$. Note that $\psi \in U(\text{End}(A_0)[1/\ell])$ since it preserves the polarization up to scaling, and chasing through we see that the scale factor is 1. Chasing through the equations we furthermore see that $W' = \psi W''$.

The same logic can be applied to the higher faces thus showing (31).

We will now show (31). First suppose we are given $\psi \in \text{GU}(\text{End}(A_0)[1/\ell])$ such that $[W'] = [\psi W'']$ as generalized homothety classes. If $\ell^aW' = \psi W''$, then $\psi$ is a scalar multiple of an element of $U(\text{End}(\mathcal{A})[1/\ell])$. Since that case was dealt with above, we may assume that $\ell^aW' = \psi W'$. After possible scaling $W', W''$, and $\psi$ by powers of $\ell$ we may assume that $\psi \in \text{End}(\mathcal{A}), W'^* = \psi W''$, and $W'^*, \ell^bW', W'' \supset W$. Therefore $\pi(W'^*) = \psi(\pi(W''))$. Hence if $\ker \phi = \pi(W'^*)$, $\ker \phi' = \pi(\ell^bW')$, and $\ker \phi'' = \pi(W'')$, then $\psi = \phi''$ and both have the same codomain which is the dual of the codomain of $\phi'$.

Conversely, if $\phi'$ and $\phi''$ induced from $W'$ and $W''$ have dual codomains, let $\psi = \phi'' \circ \phi'$. Note that $\psi \in \text{GU}(\text{End}(A_0)[1/\ell])$ since it preserves the polarization up to scaling, and chasing through we see that the scale factor is 1. And chasing through the equations we see that $\ell^aW'^* = \psi W''$ for some $a \in \mathbb{Z}$.

As before, the same logic can be applied to the higher faces thus showing (31).

\[\Box\]

8. Mass formulas

We now return to the special case of superspecial abelian varieties. If the principally polarized base abelian variety $A_0 = (A_0, \lambda_0)$ has $A_0$ superspecial in characteristic $p$, then
for primes $\ell \neq p$ the isogeny complexes $\tilde{c}_A(A_0)$ and $c_A(A_0)$ enjoy important properties not necessarily present in the general case:

(a) In the superspecial case, the isogeny complexes are finite.
(b) In the superspecial case, the isogeny complexes are connected.
(c) In the superspecial case, the isogeny complexes can be described in terms of hermitian forms over definite quaternion algebras – this is developed in our notion of Brandt complexes in Section 9. In particular, there are mass formulas for the isogeny complexes arising from the definite quaternion algebra description; this is the subject of the present Section 8.

Computing the number of simplexes of a given type in the isogeny complex of a superspecial abelian variety is in general hard; however, it’s relatively easy to compute their mass.

**Definition 27.** The mass of a set $S$ of $k$-simplexes in a weighted simplicial complex is equal to

$$\sum_{\sigma \in S} \frac{1}{w(\sigma)}.$$

Let $A_0$ be a principally polarized superspecial abelian variety of dimension $g$ in characteristic $p$. We will let $m_g([\ell],p)$ be the mass of the set of $k$-simplexes of type $r = (r_0, r_1, \ldots, r_k)$ in the enhanced $[\ell]$-isogeny complex $\tilde{c}_A^0(A_0)$.

**Theorem 28.** (Mass formula of Ekedahl and Hashimoto/Ibukiyama)

$$m_g([\ell],p) = M_g(p) := \frac{(-1)^{g(g+1)/2}}{2^g} \left\{ \prod_{k=1}^{g} \zeta(1-2k) \right\} \prod_{k=1}^{g} \{p^k + (-1)^k\}.$$  

*Proof.* See [Eke87, p. 159] and [HI80, Prop. 9], cf. [Yu06, Thm. 3.1].

**Proposition 29.**

$$m_g([\ell],p)_{[\ell],p} = m_g([\ell],p)_{[\ell],p} N([\ell],p)_{[\ell],p}$$

where $N([\ell],p)$ is as given in [13] and Proposition 11.

*Proof.* Let $\sigma$ be the $(k-1)$-simplex of type $(r_0, r_1, \ldots, r_{k-1})$ corresponding to the sequence

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} A_{k-1}.$$  

Let $G(\sigma)$ be the automorphism group of this sequence, so $w(\sigma) = \#G(\sigma)$.

By Definition (13), there are $N([\ell],p)$ maps $f_k \in \text{Iso}(A_{k-1})$ from $A_{k-1}$ to some $A_k$ of type $r_k$. Two such $f_k$’s extend $\sigma$ to the same $k$-simplex $\sigma[f_k]$ if and only if there exists an element $g \in G(\sigma)$ sending one to the other. Let $\text{Orb}_{G(\sigma)}(f_k)$ be the orbit under $G(\sigma)$ of $f_k$. We have

$$\frac{N([\ell],p)_{[\ell],p}}{w(\sigma)} = \sum_{f_k \in \text{Iso}(A_{k-1})_{[\ell],p}} \frac{1}{\#G(\sigma)} = \sum_{f_k \in \text{Iso}(A_{k-1})_{[\ell],p}} \frac{\#\text{Orb}_{G(\sigma)}(f_k)}{\#G(\sigma)} = \sum_{f_k \in \text{Iso}(A_{k-1})_{[\ell],p}} \frac{1}{\#G(\sigma[f_k])} = \sum_{f_k \in \text{Iso}(A_{k-1})_{[\ell],p}} \frac{1}{w(\sigma[f_k])}.$$  

Summing the above over all $(k-1)$-simplexes of type $(r_0, r_1, \ldots, r_{k-1})$ gives the result.

**Proposition 30.**

$$m_g([\ell],p)_{(r_0,r_1,\ldots,r_k)} = m_g([\ell],p)_{(\hat{r}_k,\hat{r}_1,\hat{r}_0)}.$$  

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Proof. The dual map $\iota$ send $k$-simplexes of type $r = (r_0, r_1, \ldots, r_k)$ to $k$-simplexes of type $\hat{r} = (\hat{r}_k, \ldots, \hat{r}_1, \hat{r}_0)$ and preserves weights. $\square$

Corollary 31.

$$m_g([\ell], p)_{(r)} = M_g(p) \frac{N(\ell)_{g, \hat{r}}}{N(\ell)_{r, 0}}.$$ 

Proof. Using Propositions 29 and 30 we compute

$$m_g([\ell], p)_{(0)} N(\ell)_{g, \hat{r}} = m_g([\ell], p)_{(g)} N(\ell)_{g, \hat{r}} = m_g([\ell], p)_{(g, \hat{r})} = m_g([\ell], p)_{(r, 0)} N(\ell)_{r, 0}.$$ $\square$

Combining the above results we get:

Theorem 32.

$$m_g([\ell], p)_{(r_0, r_1, \ldots, r_k)} = M_g(p) \frac{N(\ell)_{g, \hat{r}_0}}{N(\ell)_{r_0, 0}} \prod_{i=1}^{k} N(\ell)_{r_{i-1}, r_i}.$$ 

9. The Brandt complex of a definite rational quaternion algebra

There are several ways to define the Brandt complex of a definite rational quaternion algebra. Here we present the one we used in our computations.

Let $\mathbb{H}$ be a definite rational quaternion algebra and $\mathcal{O}_\mathbb{H} \subset \mathbb{H}$ a maximal order. Recall that a positive definite hermitian form over $\mathcal{O}_\mathbb{H}$ is a projective left $\mathcal{O}_\mathbb{H}$-module $M$ of rank $g$ with a $\mathbb{Z}$-linear pairing $(\cdot, \cdot)$ into $\mathcal{O}_\mathbb{H}$ satisfying

1. $(\alpha x, y) = \alpha (x, y),$
2. $(x, y) = \overline{(y, x)},$
3. $(x, x) \geq 0$ with equality only if $x = 0$

for all $\alpha \in \mathcal{O}_\mathbb{H}$ and $x, y \in M$.

Given such a form recall that its dual $M^*$ is defined as $\{x \in M \otimes \mathbb{H} \mid (M, x) \subset \mathcal{O}_\mathbb{H}\}$.

Definition 33. For a prime $\ell$ unramified in $\mathbb{H}$, a positive definite hermitian form $M$ is $\ell$-bounded if

$$\ell M^* \subset M.$$ (32)

Observe that by comparing the reduced discriminant of both sides of equation (32) we conclude that the reduced discriminant $\text{disc}(M) = \ell^n$ for some integer $0 \leq n \leq g$. Given $M, (\cdot, \cdot)$ an $\ell$-bounded positive definite hermitian form, we define the scaled dual form (denoted $\hat{M}$) to be $M^*, \ell(\cdot, \cdot)$.

Remark 34. There are two ways to think about $\ell$-bounded forms.

(a) Recall that all positive definite $g$-dimensional hermitian forms over $\mathbb{H}$ are isomorphic. Thus we can fix the form $(\cdot, \cdot)$ on $\mathbb{H}^g \cong M \otimes \mathbb{H}$ and vary $M \subset \mathbb{H}^g$. In fact we can simultaneously fix $M \otimes \mathbb{Z}[1/\ell] = \mathcal{O}_\mathbb{H}[1/\ell]^g$.

(b) Also recall that for $g > 1$ all projective rank $g$ $\mathcal{O}_\mathbb{H}$-modules are isomorphic. Thus for $g > 1$ we can alternatively fix $M \cong \mathcal{O}_\mathbb{H}^g$ and vary the $\ell$-bounded hermitian form $(\cdot, \cdot)$, given by $(x, y) = y^\dagger H x$ for a hermitian matrix $H$. 

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We will now define the enhanced Brandt complex \( \widetilde{brc}_g(\ell, O_{\mathbb{H}}) \) of \( O_{\mathbb{H}} \) with respect to the prime \( \ell \) of dimension \( g \). The vertices of \( \widetilde{brc}_g(\ell, O_{\mathbb{H}}) \) correspond to the isomorphism classes of \( \ell \)-bounded positive definite hermitian forms \( M \) over \( O_{\mathbb{H}} \) of rank \( g \). If the reduced discriminant \( \text{disc}(M) = \ell^n \) we will say the corresponding vertex has type \( n \).

The \( k \)-faces of the enhanced Brandt complex correspond to chains of \( \ell \)-bounded positive definite hermitian forms \( M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_k \) with the same pairing \((\ , \)\). Such a face will be said to be of type \( n := (n_0, n_1, \ldots, n_k) \) if \( \text{disc}(M_i) = \ell^{n_i} \).

The little Brandt complex \( brc_g(\ell, O_{\mathbb{H}}) \) is obtained from the enhanced Brandt complex by quotienting by the involution corresponding to taking scaled duals.

As preparation for the proof of the theorem below, the adjugate (or classical adjoint) \( \text{Adj}(M) \) of an invertible \( n \times n \)-matrix \( M \) is the \( n \times n \) matrix with \( M \text{ Adj}(M) = \det M \text{ Id}_{n \times n} \).

**Theorem 35.** The enhanced Brandt complex \( \widetilde{brc}_g(\ell, O_{\mathbb{H}}) \) is isomorphic to

\[
U_g(O_{\mathbb{H}}[1/\ell])\backslash \mathcal{B}_g.
\]

and the little Brandt complex \( brc_g(\ell, O_{\mathbb{H}}) \) is isomorphic to

\[
GU_g(O_{\mathbb{H}}[1/\ell])\backslash \mathcal{B}_g.
\]

**Proof.** Since \( \mathbb{H} \) is unramified at \( \ell \), we can fix an isomorphism \( \mathbb{H} \otimes \mathbb{Q}_\ell \cong \text{Mat}_{2 \times 2}(\mathbb{Q}_\ell) \) such that \( O_{\mathbb{H}} \otimes \mathbb{Z}_\ell \) corresponds to \( \text{Mat}_{2 \times 2}(\mathbb{Z}_\ell) \). Under this isomorphism conjugating a quaternion corresponds to taking the adjugate matrix. For an invertible \( 2 \times 2 \) matrix \( m \) the adjugate is given by \( \text{Adj}(m) = sm^Ts^{-1} \), where \( s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) is the matrix of a symplectic form.

Interpreting \( \mathbb{H}^g \) as row vectors, we get an isomorphism \( \mathbb{H}^g \otimes \mathbb{Q}_\ell \cong \text{Mat}_{2 \times 2g}(\mathbb{Q}_\ell) \cong V \oplus V \) where \( V = \mathbb{Q}_\ell^{2g} \) as a row vector space. Since \( \mathbb{H} \otimes \mathbb{Q}_\ell \cong \text{Mat}_{2 \times 2}(\mathbb{Z}_\ell) \) acts on \( \text{Mat}_{2 \times 2g}(\mathbb{Q}_\ell) \) on the left, any \( \text{Mat}_{2 \times 2}(\mathbb{Z}_\ell) \)-lattice \( W \subset \text{Mat}_{2 \times 2g}(\mathbb{Q}_\ell) \) is of the form \( W \oplus W \) for \( W \subset V \). Also the hermitian form \((x, y)\) on \( \text{Mat}_{2 \times 2g}(\mathbb{Q}_\ell) \) is given by \( xs^{\otimes g}y^T = s^{-1} \), where \( s^{\otimes g} \) is the block diagonal matrix consisting of \( g \) copies of \( s \) which gives \( V \) the structure of a \( 2g \)-dimensional symplectic space over \( \mathbb{Q}_\ell \) as used in Section 7 to define the building.

Define a map \( \nu : \mathcal{B}_g \rightarrow \widetilde{brc}_g(\ell, O_{\mathbb{H}}) \) that maps the vertex corresponding to the extra special \((1)^g, (0)^{g-r})\)-lattice \( W \subset V \) to the hermitian form given by \( M = (W \oplus W) \cap O_{\mathbb{H}}[1/\ell]^g \). Note that \( M \) will be \( \ell \)-bounded of type \( r \) since \( W \) was extra special. Extend \( \nu \) to higher faces in the natural manner, namely by mapping the face corresponding to the chain \( W_0 \subset W_1 \subset \cdots \subset W_k \) to \((W_0 \oplus W_0) \cap O_{\mathbb{H}}[1/\ell]^g \subset (W_1 \oplus W_1) \cap O_{\mathbb{H}}[1/\ell]^g \subset \ldots \subset (W_k \oplus W_k) \cap O_{\mathbb{H}}[1/\ell]^g \). By Remark 34 a\) every \( \ell \)-bounded \( M \) can be embedded into \( \mathcal{B}_g \) with \( M \otimes \mathbb{Z}[1/\ell] = \mathbb{O}_{\mathbb{H}}[1/\ell]^g \) so comes from the extra special lattice \( W \subset W \) \( M \otimes \mathbb{Z} \subset V \). Hence \( \nu \) is surjective.

Now suppose we have two modules \( M, M' \subset \mathbb{H}^g \) with \( M \otimes \mathbb{Z}[1/\ell] = M' \otimes \mathbb{Z}[1/\ell] = \mathbb{O}_{\mathbb{H}}[1/\ell]^g \). Any isomorphism between them extends to an automorphism of \( \mathbb{O}_{\mathbb{H}}[1/\ell]^g \), i.e., an element of \( U_g(O_{\mathbb{H}}[1/\ell]) \). The same logic applies to chains. Thus the image of two faces under \( \nu \) correspond to the same element of the enhanced Brandt complex if and only if they’re related by an element of \( U_g(O_{\mathbb{H}}[1/\ell]) \). This proves 33.

Now pick a module \( M \subset \mathbb{H}^g \) with \( M \otimes \mathbb{Z}[1/\ell] = \mathbb{O}_{\mathbb{H}}[1/\ell]^g \) and consider an embedding of the scaled dual \( \tilde{M} \cong M' \subset \mathbb{H}^g \) also with \( M' \otimes \mathbb{Z}[1/\ell] = \mathbb{O}_{\mathbb{H}}[1/\ell]^g \). This gives us a scaling from \( M' \) to \( M' \) which extends to an element of \( GU_g(O_{\mathbb{H}}[1/\ell]) \). Since \( M \otimes \mathbb{Z}_\ell \) and \( M' \otimes \mathbb{Z}_\ell \) give elements of the same generalized homothety, we see that the preimages of \( M \) and \( \tilde{M} \)
under ν are related by an element of GU_2(O_H[1/ℓ]). Extending the same logic to higher faces and using that both sides are quotients of order 2 gives (34).

\[\square\]

**Corollary 36.** The enhanced and little Brandt complexes are independent of the choice of maximal order \(O_H\) and depend only on the algebra \(H\).

**Proof.** This follows from the fact that for \(ℓ\) unramified in \(H\), the algebra \(O_H[1/ℓ]\) is independent of the maximal order \(O_H\) and depends only on \(H\). \[\square\]

**Corollary 37.** If \(A_0 = E^g\) is a principally polarized superspecial abelian variety of dimension \(g\) with \(O = \text{End}(E)\), then the enhanced (respectively, little) \(ℓ\)-complex of \(\mathcal{A}_0\) is isomorphic to the enhanced Brandt complex \(\tilde{brc}_g(ℓ, O)\) (respectively, the little Brandt complex \(brc_g(ℓ, O)\)).

**Proof.** Combine Theorem 35 with Theorem 26. \[\square\]

### 9.1. Computations of Brandt complexes

Let \(N\) be a square-free positive integer with an odd number of prime factors. Let \(O_H\) be a maximal order in the definite rational quaternion algebra \(H = \mathbb{H}(N)\) of reduced discriminant \(\text{Disc}(H) = N\). We computed the Brandt complexes \(\tilde{brc}_g(ℓ, \mathcal{O}_H)\) and \(brc_g(ℓ, \mathcal{O}_H)\) for the values of \(N\) with \((N, ℓ) = 1\) in Table 4.

| \(g\) | \(ℓ\) | \(N = \text{Disc}(\mathbb{H})\) |
|-------|-------|-------------------------------|
| 2     | 2     | \(N \leq 347\)               |
| 3     |       | \(N \leq 227\)               |
| 5     |       | \(N \leq 157\)               |
| 7     |       | \(N \leq 107\)               |
| 3     | 2     | \(N \leq 23\)                |
| 3     | 3     | \(N \leq 13\)                |

Table 4. The range of our computations of \(\tilde{brc}_g(ℓ, \mathcal{O}_H)\) and \(brc_g(ℓ, \mathcal{O}_H)\)

### 10. \(\tilde{co}_2(2, 7)\) and \(co_2(2, 7)\)

In this section we examine the isogeny complexes \(\tilde{co}_2(2, 7)\) and \(co_2(2, 7)\) in detail, giving their faces and half-faces in each dimension. In [JZ2] we build on this, using these examples to illustrate how we compute the cohomology of isogeny complexes. We give the standard cochain complexes with their differentials for \(\tilde{co}_2(2, 7)\) and \(co_2(2, 7)\) and then compute the cohomology \(H^*(\tilde{co}_2(2, 7), \mathbb{Z})\) and \(H^*(co_2(2, 7), \mathbb{Z})\).

10.1. \(\tilde{co}_2(2, 7) = \tilde{brc}_2(2, \mathcal{O}_{\mathbb{H}(7)})\). The cell complex \(\tilde{co}_2(2, 7) = \tilde{brc}_2(2, \mathcal{O}_{\mathbb{H}(7)})\) has 8 0-cells, 23 1-cells, and 16 2-cells. They are divided among the possible types as shown in Table 5.

The weights and masses of the cells of the various types are given in Table 6.
Table 5. Number of cells in $\tilde{c}o_2(2,7)$ by type

| Type | Number of cells |
|------|----------------|
| (0)  | 2              |
| (1)  | 4              |
| (2)  | 2              |
| (1,0)| 7              |
| (2,0)| 9              |
| (2,1)| 7              |
| (2,1,0)| 16         |

Table 6. Weights of the cells in $\tilde{c}o_2(2,7)$ by type

| Type $t$ | Weights | Mass $m_2(2,7)_t$ |
|----------|---------|-------------------|
| (0)      | 32, 48  | $1/32 + 1/48 = 5/96$ |
| (1)      | 16, 16, 8, 96 | 25/96 |
| (2)      | 32, 48  | 5/96               |
| (1,0)    | 8, 8, 32, 16, 8, 12, 12, 48 | 25/32 |
| (2,0)    | 8, 16, 8, 8, 32, 4, 16 | 25/32 |
| (2,1)    | 8, 16, 8, 16, 8, 4, 32 | 25/32 |
| (2,1,0)  | 4, 8, 8, 16, 8, 8, 8, 4, 32, 16, 4, 4, 8, 16 | 75/32 |

From Theorem 28 we have

$$m_2(2,7)_{(0)} = \frac{(-1)^3}{4} \{\zeta(-1)\zeta(-3)\} \cdot (7 - 1)(7^2 + 1) = \frac{-1}{4} \left\{\frac{-1}{12} \cdot \frac{1}{120}\right\} \cdot (300) = \frac{5}{96},$$

agreeing with the computation of $m_2(2,7)_{(0)}$ in Table 6. The other $m_2(2,7)_t$ can now be computed from Theorem 32 using Table 3 for the $N(2)_{r,s}$ – reassuringly they all agree with Table 6.

10.2. $co_2(2,7) = brc_2(2, \mathcal{O}_{H(7)})$. The cell complex $co_2(2,7) = brc_2(2, \mathcal{O}_{H(7)})$ has 6 0-cells, 14 1-cells, and 12 2-cells. They are divided among the possible types as in Table 7.

| Type | # cells |
|------|---------|
| (0)  | (2)     |
| (1)  | (1)     |
| (2)  | (2,1)   |
| (2,0)| regular edges | 2 |
| (2,1)| half-edges | 5 |
| (2,0)| regular facets | 4 |
| (2,1,0)| half-facets | 8 |

Table 7. Number of cells in $co_2(2,7)$ by type

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Department of Mathematics, Baruch College, The City University of New York, One Bernard Baruch Way, New York, NY 10010-5526, USA

Email address: bruce.jordan@baruch.cuny.edu

Independent Mathematician, Newton, MA 02465, USA

Email address: gzaytman@alum.mit.edu