Spontaneous supersymmetry breaking in the \(2d\ N=1\) Wess-Zumino model

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We study the phase diagram of the two-dimensional \(N=1\) Wess-Zumino model on the lattice using Wilson fermions and the fermion loop formulation. We give a complete nonperturbative determination of the ground state structure in the continuum and infinite volume limit. We also present a determination of the particle spectrum in the supersymmetric phase, in the supersymmetry broken phase and across the supersymmetry breaking phase transition. In the supersymmetry broken phase we observe the emergence of the Goldstino particle.

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INTRODUCTION

Understanding the spontaneous breakdown of supersymmetry is a generic nonperturbative problem which is relevant not only for particle physics but in fact for many physical systems beyond quantum field theories. The \(N=1\) Wess-Zumino model \([1, 2]\) in two dimensions is one of the simplest supersymmetric quantum field theories which allows for spontaneous supersymmetry breaking since it enjoys the necessary but not sufficient condition of a vanishing Witten index \([3]\). The model has been analysed employing various approaches such as Monte Carlo methods \([4, 5]\), Hamiltonian techniques \([6–8]\), or exact renormalisation group methods \([9]\). Wilson derivatives for fermions and bosons, guaranteeing a supersymmetric continuum limit \([10]\), were used in \([11]\) and a numerical analysis of the phase diagram using the SLAC derivative has been conducted in \([12]\). All approaches use various regulators which are more or less difficult to control. In this letter we report on our results for the two-dimensional \(N=1\) Wess-Zumino model regularized on a Euclidean spacetime lattice. The discretization using Wilson derivatives for the fermions and bosons \([10]\) together with the fermion loop formulation and a novel algorithm \([13]\) allows to systematically remove all effects from the IR and UV regulators by explicitly taking the necessary limits in a completely controlled way. One reason why this has not been achieved so far with other methods is the fact that all supersymmetric systems with spontaneously broken supersymmetry suffer from a fermion sign problem related to the vanishing of the Witten index \([14]\). However, that sign problem can be circumvented in our approach by using the exact reformulation of the lattice model in terms of fermion loops \([14]\). In this formulation the partition function is obtained as a sum over closed fermion loop configurations and separates naturally into its bosonic and fermionic parts for which the sign is perfectly under control. Efficient simulations with an open fermion string (or fermionic worm) algorithm \([13]\) are then possible even in the phase with spontaneously broken supersymmetry where the massless Goldstino mode is present.

The two-dimensional \(N=1\) Wess-Zumino model \([1, 2]\) contains a real two component Majorana spinor \(\psi\) and a real bosonic field \(\phi\) and is described in Euclidean spacetime by the on-shell continuum action

\[
S = \int d^2x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \bar{\psi} D \psi + \frac{[P'(\phi)]^2}{2} \right\}
\]

where \(D = [\phi + P''(\phi)]\) is the Majorana Dirac operator. Here, \(P(\phi)\) denotes a generic superpotential and \(P'\) and \(P''\) its first and second derivative with respect to \(\phi\), respectively. The action is invariant under a supersymmetry transformation \(\delta\) which transforms \(\phi, \psi\) and \(\bar{\psi}\) as

\[
\delta \phi = \bar{\psi} \epsilon, \quad \delta \psi = (\partial \phi - P') \epsilon, \quad \delta \bar{\psi} = 0
\]

where \(\epsilon\) is a constant Majorana spinor. In the following we will concentrate on the specific superpotential

\[
P(\phi) = \frac{1}{3} \phi^3 - \frac{m^2}{4g} \phi.
\]

With this potential, the action is also invariant under a discrete \(\mathbb{Z}_2/\text{chiral symmetry transformation}\n
\[
\phi \rightarrow -\phi, \quad \psi \rightarrow \sigma_3 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi} \sigma_3
\]

which in the following we denote by \(\mathbb{Z}_2^3\) symmetry. The potential yields a vanishing Witten index \(W = 0\) and hence allows for spontaneous supersymmetry breaking \([3]\). This can for example be derived from the transformation properties of the Pfaffian \(\text{Pf}(D)\) under the \(\mathbb{Z}_2\) symmetry \(\phi \rightarrow -\phi\) \([15]\).

FERMION LOOP FORMULATION

When the model is regularized on a discrete spacetime lattice both the \(\mathbb{Z}_2^3\) and the supersymmetry are broken explicitly, but the discretization can be chosen such that the restoration of the symmetries is guaranteed in the continuum limit \([10]\). This can be achieved because the model is superrenormalisable and only one counterterm is necessary to renormalize the bare mass \(m\), while the coupling \(g\) is not renormalized and can hence be used to define the continuum limit \(ag \rightarrow 0\) where \(a\) is the lattice
The loop formulation is obtained by constructing an exact hopping expansion of the fermion action to all orders. When expanding the Boltzmann factor and subsequently performing the integration over the fermion fields, the nilpotency of the Grassmann elements ensures that only closed, nonoriented and selfavoiding fermion loops survive. The partition function then becomes a sum over all fermion loop configurations \( L \subset \mathcal{L} \),

\[
Z_L = \sum_{l \in \mathcal{L}} \prod_x w_l(x), \tag{5}
\]

where the weight for a given loop configuration is a product over site weights \( w_l(x) \) which depend only on the local geometry of the loop at the lattice site \( x \), if a fermion loop is present, or on an integral over an ultralocal function of the bosonic field \( \phi \). The configuration space of all loop configurations \( \mathcal{L} \) naturally separates into equivalence classes \( \mathcal{L}_{ij} \) characterized by the even or odd number of loops winding around the lattice in the spatial and temporal direction, respectively. The loop configurations in each equivalence class pick up a definite sign depending on the chosen boundary conditions (b.c.) for the Majorana fermion [16], so the partition function in eq.(5) represents a system with unspecified (or fluctuating) b.c., while the one with periodic b.c.,

\[
W \propto Z_{pp} = Z_{L_{00}} - Z_{L_{10}} - Z_{L_{01}} - Z_{L_{11}}, \tag{6}
\]

is proportional to the Witten index and the one with antiperiodic b.c. in time,

\[
Z_{ap} = Z_{L_{00}} + Z_{L_{10}} - Z_{L_{01}} + Z_{L_{11}}, \tag{7}
\]

describes the system at finite temperature. Note that the weight is not necessarily positive definite in each of the sectors, but sufficiently close to the continuum limit it turns out to be so. As described in [13] the system can most efficiently be simulated, essentially without critical slowing down, by introducing an open fermion string corresponding to the insertion of a Majorana fermion pair. By letting the ends of the string move around the lattice by a standard Metropolis update procedure, one samples the fermion 2-point function as well as the relative weights between \( Z_{L_{00}}, Z_{L_{10}}, Z_{L_{01}} \), and \( Z_{L_{11}} \), which allows precise determinations of eqs.(6) and (7) a posteriori. Finally, the bosonic fields are integrated over by standard Monte Carlo methods using a Metropolis algorithm.

VACUUM STRUCTURE

The vacuum structure of the system depends on the two bare parameters \( m \) and \( g \) and is hence a function of the dimensionless ratio \( f \equiv g/m \). The expected symmetry breaking pattern in the continuum [3] is characterized by a supersymmetric phase with spontaneously broken \( \mathbb{Z}_2 \) symmetry and a unique (bosonic or fermionic) vacuum (groundstate) at small \( f \), and a \( \mathbb{Z}_2 \) symmetric phase at large \( f \) with spontaneously broken supersymmetry accompanied by tunneling between the bosonic or fermionic vacua (groundstates). The two phases are separated by a phase transition at \( f_c = g/m \).

In FIG. 1 we show histograms of the partition functions \( Z_{L_{ij}} \), as a function of the vacuum expectation value \( \langle \phi \rangle \) of the bosonic field in both phases. The top panel shows data for \( f < f_c \) (\( \mathbb{Z}_2 \) broken/supersymmetric) where \( \langle \phi \rangle = \pm m/(2g) \) corresponds to the two classical minima of the potential with \( Z_{pp}/Z_{ap} = \pm 1 \) in the continuum. From the plots we infer that the groundstate at \(+m/2g\), where \( Z \simeq Z_{L_{00}} \) and \( Z_{L_{10}} \simeq Z_{L_{01}} \simeq Z_{L_{11}} = 0 \), and hence \( Z_{pp}/Z_{ap} \simeq +1 \), corresponds to the bosonic vacuum while the groundstate at \(-m/2g\), where \( Z_{L_{00}} \simeq Z_{L_{10}} \simeq Z_{L_{01}} \simeq Z_{L_{11}} \simeq 0 \), and hence \( Z_{pp} \simeq -Z_{ap} \), corresponds to the fermionic one [17]. In the infinite volume limit either the bosonic or fermionic groundstate is selected and consequently supersymmetry is intact (but the \( \mathbb{Z}_2 \) symmetry is spontaneously broken). For increasing \( f \) the tunneling between the groundstates is enhanced and eventually triggers the restoration of the \( \mathbb{Z}_2 \) symmetry accompanied by the spontaneous breakdown of the supersymmetry at \( f_c \). This is illustrated in the lower panel of FIG. 1 where the data for \( f > f_c \) clearly displays \( \langle \phi \rangle \simeq 0 \) and \( Z_{pp}/Z_{ap} \simeq 0 \), both of which become exactly zero in the continuum limit [17]. Note that the skewness of the distribution is due to the residual \( \mathbb{Z}_2 \) symmetry breaking at finite lattice spacing. Whether the spontaneous phase transition survives the continuum and infinite volume limit, i.e., whether \( f_c \) remains finite and nonzero, needs to be investigated by quantitatively determining \( f_c \) at various lattice spacings and volumes and carefully taking first the infinite volume limit followed by the continuum one.

The (pseudo-)critical point \( am_c \) of the spontaneous \( \mathbb{Z}_2 \) symmetry breaking phase transition is determined at fixed lattice spacing \( ag \) by considering the intersection
point of the Binder cumulant $U = 1 - \langle \phi^4 \rangle / 3 \langle \phi^2 \rangle^2$ obtained from different volumes using $Z_{ap}$ [17]. This can be compared with the determination from the peak of the susceptibility $\chi$ of the average sign of the bosonic field. In the inset of FIG. 2 we show data exemplarily for $ag = 0.125$ extrapolated to the infinite volume limit $gL \to \infty$ using linear and quadratic terms in $1/gL$. Both observables yield values which agree in the thermodynamic limit. Similarly, the (pseudo-)critical point of the spontaneous supersymmetry breaking phase transition can be determined from the supersymmetric Ward identity $\langle P'/m \rangle$ yielding results which in the infinite volume limit are in agreement with the determinations from the $Z_2^\chi$ transition already at finite lattice spacing. In order to take the continuum limit this procedure is repeated for a range of lattice spacings and the resulting bare critical couplings $f_c = g/m_c(0)g$ are renormalised by subtracting the logarithmically divergent one-loop self energy from the bare mass $m^2$ and computing the renormalised critical coupling $f_c^{R} = g/m_c^{R}(ag)$. In FIG. 2 we show $1/f_c^{R}$ in the infinite volume limit as a function of the lattice spacing together with an extrapolation to the continuum using corrections linear plus quadratic in $a$. The average of this extrapolation with one using only a linear correction yields

$$1/f_c^{R} = 2.286(28)(36)$$

where the first error is statistical and the second comes from the difference of the two extrapolations. The result demonstrates that the supersymmetry breaking phase transition coinciding with the $Z_2^\chi$ symmetry restoration survives the infinite volume and the continuum limit, and it provides a precise nonperturbative determination of the perturbatively renormalised critical coupling. Finally we note that the determination in [12] using SLAC fermions is fully compatible with our result once the exact same renormalization procedure is applied.

**MASS SPECTRUM**

Next we consider the mass spectrum of the system below, above and across the phase transition. The lowest masses are obtained from the exponential temporal decay of 2-point correlation functions $\langle O(t)O(0) \rangle$ of appropriate fermionic or bosonic operators $O$ projected to zero spatial momentum [18]. Due to the open fermion string algorithm the correlation functions can be determined to very high accuracy even in the massless phase or when the signal falls off by many orders of magnitude [13]. In FIG. 3 we show the lowest boson and fermion masses $m_\psi^{(0)}, m_\psi^{(1)}$ and $m_\phi^{(0)}, m_\phi^{(1)}$, respectively, in the bosonic sector $Z_{c_{00}}$ at $ag = 0.25$ on a lattice with extent $128 \times 48$. For bare masses $m > m_c$ the system is in the supersymmetric phase and we observe perfect mass degeneracy between the lowest fermion and boson mass already at finite lattice spacing. In the supersymmetry broken phase $m < m_c$ the masses split up and the degeneracy is lifted. In this phase the masses can also be determined in the fermionic sector $Z_{c_{10}} + Z_{c_{11}}$ and we find the same values within our numerical accuracy. Further excited states can be obtained by employing the operators $O = \psi\phi$ and $\phi^2$ which in the $Z_2^\chi$ symmetric phase do not mix with the above operators $O = \psi$ and $\phi$, respectively. The result for $O = \phi^2$ is also displayed in FIG. 3. In the inset we show a zoom of the mass $m_\psi^{(0)}$ and the amplitude $A_\psi^{(0)}$ of the lowest fermionic state. When approaching the phase transition in the supersymmetry broken phase, the amplitude decreases and vanishes at the critical point $m_c$, i.e., the particle decouples from the system when entering the supersymmetric phase. The mass is by an order of magnitude smaller than the next-to-lowest mass and the investigation of the finite volume corrections shows that the data for $m_\psi^{(0)}$ is compatible with zero in the infinite volume limit, even at finite lat-
phase for three different bare masses am. The data clearly indicates a finite boson mass which in the lowest boson mass allows for a universality class different from the Ising one. Though, that the Goldstino mode materializes so clearly accompanying the vanishing fermion Witten index. We clearly observe a \( \mathbb{Z}_2 \) symmetric, supersymmetry broken phase where the bosonic and fermionic vacua (groundstates) are degenerate and a \( \mathbb{Z}_2 \) broken, supersymmetric phase where one of the two groundstates is spontaneously selected in the infinite volume limit. This confirms the expected symmetry breaking pattern and the corresponding vacuum structure. The phase transition separating those two phases can be analysed using different observables in the infinite volume limit and our calculations at several lattice spacings provides a precise nonperturbative determination of the renormalised critical coupling in the continuum limit. Concerning the mass spectrum we observe degenerate boson and fermion masses in the supersymmetric \( \mathbb{Z}_2 \) broken phase, surprisingly even at finite and rather coarse lattice spacing. In the \( \mathbb{Z}_2 \) symmetric, supersymmetry broken phase the nondegeneracy of the lowest few bosonic and fermionic masses can also be accurately resolved due to the efficient algorithm employed. The mass of the lowest fermionic state is compatible with zero in the thermodynamic and continuum limit, allowing us to identify it with the expected massless Goldstino mode.

\[ \text{FIG. 4: Left: Infinite volume extrapolation of } m_\psi^{(0)} \text{ at fixed lattice spacing } ag = 0.25 \text{ for three bare masses } am = 0.30, 0.18 \text{ and } 0.02 \text{ (from bottom up) in the supersymmetry broken phase and } m_\phi^{(0)}, m_\psi^{(0)} \text{ for one bare mass } am = 0.70 \text{ in the supersymmetric phase (top). Right: Continuum extrapolation of } m_\psi^{(0)}/g \text{ and } m_\phi^{(0)}/g \text{ at fixed volume } gL = 8 \text{ and renormalised coupling } 1/f^R = 3 \text{ in the supersymmetric phase.} \]

\[ \text{FIG. 5: Infinite volume extrapolation of } m_\psi^{(0)} \text{ and } m_\phi^{(0)} \text{ in the supersymmetry broken phase at } 1/f^R = 1.2. \]

CONCLUSION

We have established the fermion loop formulation for the two-dimensional \( \mathcal{N} = 1 \) Wess-Zumino model which allows efficient simulations with a worm algorithm by avoiding the fermion sign problem generically appearing in the phase with spontaneously broken supersymmetry due to the vanishing Witten index. We clearly observe a \( \mathbb{Z}_2 \) symmetric, supersymmetry broken phase where the bosonic and fermionic vacua (groundstates) are degenerate and a \( \mathbb{Z}_2 \) broken, supersymmetric phase where one of the two groundstates is spontaneously selected in the infinite volume limit. This confirms the expected symmetry breaking pattern and the corresponding vacuum structure. The phase transition separating those two phases can be analysed using different observables in the infinite volume limit and our calculations at several lattice spacings provides a precise nonperturbative determination of the renormalised critical coupling in the continuum limit. Concerning the mass spectrum we observe degenerate boson and fermion masses in the supersymmetric \( \mathbb{Z}_2 \) broken phase, surprisingly even at finite and rather coarse lattice spacing. In the \( \mathbb{Z}_2 \) symmetric, supersymmetry broken phase the nondegeneracy of the lowest few bosonic and fermionic masses can also be accurately resolved due to the efficient algorithm employed. The mass of the lowest fermionic state is compatible with zero in the thermodynamic and continuum limit, allowing us to identify it with the expected massless Goldstino mode.

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