Astrophysical factors: Zero energy vs. Most effective energy

Theodore E. Liolios $^{1,2,3}$ *

$^1$Institute of Physics and Astronomy, University of Aarhus, DK-8000 Aarhus C, Denmark
$^2$University of Thessaloniki, Department of Theoretical Physics, Thessaloniki 54006, Greece
$^3$Hellenic War College, BST 903, Greece

Abstract

Effective astrophysical factors for non-resonant astrophysical nuclear reaction are usually calculated with respect to a zero energy limit. In the present work that limit is shown to be very disadvantageous compared to the more natural effective energy limit. The latter is used in order to modify the thermonuclear reaction rate formula in stellar evolution codes so that it takes into account both plasma and laboratory screening effects.

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Nuclear astrophysics experiments are characterized by their continuous effort towards the measurement of the zero energy astrophysical factor \( AF \) which is used in the thermonuclear reaction rate formulas. Recent and past experiments \[1,6\] have all attempted to lower the energy of the experiment in order to avoid the inaccuracies incurred by the inevitable extrapolation to zero energies. That action, however, gives rise to an undesirable electron screening effect which enhances low energy data (\[7\] and references therein). Moreover, experimentalists often fit their \( AF \) data with a zero-energy Taylor polynomial which will be shown to be less suitable than the more natural effective-energy one. The results of this paper should be viewed as an improvement (for some cases) in the theoretical extraction of the astrophysical factor and not as a correction of the way experimentalists present their results. In the formulas below we follow the formalism and notation of Ref. \[8\].

The effective astrophysical factor \( EAF \) \( S_{\text{eff}} \) which appears in the non-resonant thermonuclear reaction rate

\[
\begin{align*}
\left[ 1 \right] & = \frac{7.20 \times 10^{-10} N_1 N_2}{1 + \delta_{12}} A Z_1 Z_2 f S \tau^2 e^{-\tau} S_{\text{eff}} \\
\end{align*}
\]

is defined as

\[
\begin{align*}
S_{\text{eff}} &= \sqrt{\frac{\tau}{4\pi E_0}} \int_0^\infty S (E) \exp \left[ - \left( \frac{E}{kT} + 2\pi n \right) \right] dE \\
\end{align*}
\]

and can be calculated by expanding the astrophysical factor \( S (E) \) either around zero or around the most effective energy of interaction \( E_0 \). In the first case the expansion

\[
S (E) = \sum_{n=0}^\infty \frac{1}{n!} S^{(n)} (0) E^n
\]

yields \([9], [10]\)

\[
S_{\text{eff}}^{(0)} = \sum_{n=0}^{n_M} \frac{1}{n!} S^{(n)} (0) E_0^n \sum_{k=0}^{k_M} \frac{P_{2k} (n)}{k! (12)^k \tau^k}
\]

while an expansion around the Gamow peak energy \( E_0 \)

\[
S (E) = \sum_{n=0}^\infty \frac{1}{n!} S^{(n)} (E_0) (E - E_0)^n
\]

yields \([9], [10]\)

\[
S_{\text{eff}}^{(G)} = \sum_{n=0}^{n_M} \frac{1}{n!} S^{(n)} (E_0) E_0^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} \sum_{k=0}^{k_M} \frac{P_{2k} (n - r)}{k! (12)^k \tau^k}
\]

where the polynomials are given in Ref. \[10\]. For example, the first two are

\[
P_0 (n) = 1, \quad P_2 (n) = 12n^2 + 18n + 5
\]

In most nuclear astrophysics experiments of non-resonant reaction experimentalist \[11\] use a second order truncation of Eq. \( 3 \) as a fitting formula. They obtain the values \( S (0) , S^{(1)} (0) , S^{(2)} (0) \) which are then inserted into a second order truncation of Eq. \( 4 \).
However, zero energy proximity has serious disadvantages since at such very low energies, which are extremely difficult to attain, the experiment is hampered by such effects as beam instabilities, impurities, electron screening effects, very small cross sections etc. On the other hand the Gamow peak polynomial of Eq. (3) is hardly ever used although it leads to a more accurate calculation of the EAF. In Ref. [10] it was noted that the accuracy obtained by the first term of Eq. (3), that is $S_{\text{eff}}^{(G)} \simeq S(E_0)$, is equivalent to the accuracy achieved with knowledge of $S(0)$ and its derivatives. That, of course, is also obvious from Eq. (3) which gives

$$S(E_0) = \sum_{n=0}^{\infty} \frac{1}{n!} S^{(n)}(0) E_0^n$$

(8)

We can further elaborate on that competition of Eq. (3) versus Eq. (4) by disregarding the associated derivatives of the AFs, an approximation which doesn’t cause any significant error [12]. In that case we can write

$$S_{\text{eff}}^{(G)} = S_{\text{eff}}^{(0)} + \left(1 + \frac{5}{12 \tau}\right) \sum_{n=1}^{\infty} \frac{1}{n!} S^{(n)}(0) E_0^n$$

(9)

where it is now obvious that $S_{\text{eff}}^{(G)}$ incorporates $S_{\text{eff}}^{(0)}$ along with an infinite number of corrective terms. It is again obvious that a simple first order correction to the constant Gamow peak AF $S(E_0)$ is equivalent to knowing an infinite number of zero energy derivatives.

On the other hand the crucial region of energies is not close to zero but the Gamow window $[-\Delta/2 + E_0, E_0 + \Delta/2]$, with $\Delta$ being as usual the full width at 1/e. Consequently, while in the case of $S_{\text{eff}}^{(0)}$ we attempt to lower the energy as close to zero as possible, paying a great price, when $S_{\text{eff}}^{(G)}$ is used the situation improves considerably. First, we only have to perform measurements inside the Gamow window, thus avoiding the dangerous zero energy region. Then we obtain accurate results by including a limited number of terms while in the zero-energy formulas a second order expansion is not adequate as measurements are far away from the origin. It is surprising that some experimentalist have been satisfied with fitting a second order expansion of the AF given by Eq. (3) to their data. It is easy to show that including some more terms will result in different zero energy AFs. Fortunately the results that are used in stellar calculations are often based on independent analyses of the experiments (often with theoretical guidance rather than the naive Taylor expansion).

Another positive aspect of adopting Eq. (4) instead of Eq. (3) is the electron screening effect, which enhances all very low energy data. The lower the energy, the more pronounced the effect which makes Eq. (4) seem all the more attractive. In fact, in the above formulas, the quantities $S^{(n)}(0)$ and $S^{(n)}(E_0)$ are referring to bare nuclei. However, in the laboratory (L) the experimental (ex) values $S_{\text{ex}}^{(n)}(0)$ and $S_{\text{ex}}^{(n)}(E_0)$ are enhanced as they are multiplied by the screening enhancement factor (SEF) given by [13]

$$f_L(E) = \exp \left( \frac{\pi n U_e^{L \text{e}}}{E} \right)$$

(10)
where \( n \) is the Sommerfeld parameter, and \( U_e^L \) is the screening energy obtained via theoretical models. For nuclei involved in the CNO bi-cycle or more advanced burning stages \( U_e^L \) can be given by \[14\]

\[
U_{TF}^{SL} = -32.9 Z_1^{4/3} Z_2 eV
\] (11)

where \( Z_1, Z_2 \) are the atomic numbers of the target and the projectile respectively. Alternatively, if one wants to fully explore the thermal, ionization and relativistic effects on screening more elaborate formulas can be used \[13\]. The bare-nucleus AF \( S(E) \) is related to the corresponding experimental value \( S_{ex}(E) \) by the formula:

\[
S(E) = S_{ex}(E) f_L^{-1}(E)
\] (12)

Another disadvantage of the zero-energy AF is that we cannot obtain \( S(0) \) from the above formula, whereas the Gamow-peak AF is readily given by \( S(E_0) = S_{ex}(E_0) f_L^{-1}(E_0) \). Disregarding second order derivatives \( S^{(2)}(E_0) \), which play a minor role and their correction (or their inclusion according to Ref. \[12\]) would be pointless we have

\[
S_{eff}^{(G)} \approx f_L(E_0) S_{ex}(E_0) \left[ 1 + \frac{5}{12\tau} + \frac{5}{2\tau} \frac{S_{ex}^{(1)}(E_0)}{S_{ex}(E_0)} E_0 + \frac{15}{12\tau} \ln f_L(E_0) \right]
\] (13)

Actually, neglecting the screening effect is equivalent to setting \( f_L(E) = 1 \).

The above formula vividly depicts that the value of the EAF used in Eq. \[1\] actually depends on the SEF and the model used to describe the screening effect. This of course is known empirically to experimentalists who try to correct their data in the laboratory. What is derived here for the first time is the analytic dependence of the EAF on the screening models selected each time. And there are many, indeed. Another novelty of the above formula is that it eliminates all the tedious corrections that are necessary during nuclear astrophysics experiments. Moreover, it corrects all nuclear astrophysics experiments performed in the past, which have disregarded the screening effect.

There is another major advantage of Eq. \[13\] that needs to be underlined. By now it is obvious that, if we disregard screening effects, the EAF is approximately equal to either \( S(E_0) \) or \( S(0) \). Therefore, if the latter is selected by means of Eq. \[1\], all temperature effects of the EAF are actually neglected, since \( S(0) \) will be the same along the whole profile of the star. On the contrary, if the former is selected then the EAF takes into account the region of the star where the reaction takes place via the temperature dependence of the most effective energy of interaction \( E_0(T) \). For example, if we adopt Eq. \[1\] in our solar codes using the bare-nucleus value \[1\] \( S(0) = 5.32 \) for the \( He^3(He^3, 2p)He^4 \) reaction then that will practically be the value of the EAF at the center of the sun, throughout its energy production core as well as in any other stellar interior no matter how hot it is. That is indeed an unnatural result. The only way Eq. \[1\] could counterbalance the temperature dependence of the leading term \( S(E_0) \) of Eq. \[1\] is to include an infinite number of temperature dependent terms, which is of course futile since that can be accomplished by merely setting \( S_{eff} \approx S(E_0) \) in the reaction rate.

To gain an idea of the screening corrections to the EAF let us consider the break-up reaction \( He^3(He^3, 2p)He^4 \) whose SEF according to a recent model \[14\] is \( f_L(E_0) = 1.23 \).
The screening correction inside the brackets in Eq. (13) is roughly \((15/12\tau) \ln f_L (E_0) = 4 \times 10^{-3}\) while the first significant term is \((5/12\tau) = 7.7 \times 10^{-3}\). Obviously, the screening correction is comparable to the first corrective term which is usually retained in the formula of the EAF. Of course they are both negligible and the only significant correction in Eq. (13) is that of the SEF multiplied by \(S_{ex} (E_0)\). As for reactions involved in more advanced burning stages than the \(pp\) one we can consider the most important reaction of the \(CNO\) bi-cycle namely \(N^{14} (p, \gamma) O^{15}\). For that reaction the two previously mentioned corrections are roughly the same \((\sim 6. \times 10^{-3})\). Note that including any of the corrective terms inside the brackets would be meaningless for another reason. As we observed their contribution is of order 1\% while the experimental error of the leading terms \(S (E_0)\) (or even \(S (0)\) when Eq. (3) is adopted) is much larger [4].

We can now safely argue that all screening effects on the EAF can be taken into account by multiplying the (uncorrected) EAF with the SEF given by \(f_L (E_0)\). Therefore the reaction rate itself is now multiplied by a SEF which is the combination of laboratory and plasma screening effects. The reaction rate is now written

\[
r_{12} = \frac{7.20 \times 10^{-19}}{1 + \delta_{12}} \frac{N_1 N_2}{AZ_1 Z_2} f_{SL} \tau^2 e^{-\tau S_{eff}}
\]

where \(S_{eff}^{(G)}\) is merely a single experimental measurement, that is \(S_{eff}^{(G)} \approx S_{ex} (E_0)\), and the combined SEF \(f_{SL}\) is given by

\[
f_{SL} (E_0) = \exp \left[ \frac{\pi n (E_0)}{E_0} (U_e^S - U_e^L) \right]
\]

Note that the values of the screening energies \((U_e^S, U_e^L)\) in the above formula are absolute ones. At first sight it seems that Eq. (14) and Eq. (13) indicate that \(r_{12} \neq r_{21}\), since \(U_e^L (Z_1, Z_2) \neq U_e^L (Z_2, Z_1)\), an error also committed in the plasma SEF derived in Ref. [16]. This is not the case here. The combined SEF given by \(f_{SL}\) must always be coupled with the appropriate experimental value \(S_{ex} (E_0)\) so that it is always \(r_{12} = r_{21}\).

According to Eq. (13), plasma and laboratory screening have opposite effects on the reaction rate. The theoretical value of \(U_e^L\) has already been discussed. On the other hand the value of the plasma screening energy can be derived using various theoretical models such as Salpeter’s [17] according to which \(U_e^S = Z_1 Z_2 e^2 r_D^{-1}\) where \(r_D\) the Debye radius, or even Shaviv’s [18] according to which \(U_e^S = 1.5 Z_1 Z_2 e^2 r_D^{-1}\).

Regarding these two plasma screening prescriptions a comment is imperative. In a recent work [19], Shaviv’s prescriptions has been argued against. However, the relation between Salpeter’s approach and Shaviv’s is as it should be. Actually they provide upper and lower limits for the screening energies, as has often been done in laboratory screening models [14] where a sudden (Salpeter’s) and an adiabatic (Shaviv’s) limit is defined. It is important to realize that the most consistent way of deriving a SEF is to provide a sudden and an adiabatic limit with a minimum discrepancy between them. The two formulas in question fulfill that requirement and constitute the most reliable screening constraints at the moment.

In conclusion, the competition between Eq. (3) and Eq. (4) is an unequal one. Just a single term of the former yields a more accurate EAF than a great number of terms of the
latter. Moreover, Eq. (6) takes into account temperature effects in a consistent and accurate way, whereas Eq. (4) practically disregards them. Finally, using Eq. (6) we were able to modify the reaction rate formula so that it automatically takes into account the laboratory electron screening enhancement. The combination of the plasma and laboratory screening led to the derivation of a combined screening factor which allows theoreticians to have full control over the theoretical models used for the description of the electron screening effect.

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