Global Solution of Atmospheric Circulation Models with Humidity Effect *

Hong Luo

College of Mathematics and Software Science, Sichuan Normal University,
Chengdu, Sichuan 610066, China

Abstract: The atmospheric circulation models are deduced from the very complex atmospheric circulation models based on the actual background and meteorological data. The models are able to show features of atmospheric circulation and are easy to be studied. It is proved that existence of global solutions to atmospheric circulation models with the use of the $T$-weakly continuous operator.

Key Words: Atmospheric Circulation Models; Humidity Effect; Global Solution

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1 Introduction

Mathematics is a summary and abstraction of the production practices and the natural sciences, and is a powerful tool to explain natural phenomena and reveal the laws of nature as well. Atmospheric circulation is one of the main factors affecting the global climate, so it is very necessary to understand and master its mysteries and laws. Atmospheric circulation is an important mechanism to complete the transports and balance of atmospheric heat and moisture and the conversion between various energies. On the contrary, it is also the important result of these physical transports, balance and conversion. Thus it is of necessity to study the characteristics, formation, preservation, change and effects of the atmospheric

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circulation and master its evolution law, which is not only the essential part of human’s understanding of nature, but also the helpful method of changing and improving the accuracy of weather forecasts, exploring global climate change, and making effective use of climate resources.

The atmosphere and ocean around the earth are rotating geophysical fluids, which are also two important components of the climate system. The phenomena of the atmosphere and ocean are extremely rich in their organization and complexity, and a lot of them cannot be produced by laboratory experiments. The atmosphere or the ocean or the coupled atmosphere and ocean can be viewed as an initial and boundary value problem [8] [10], or an infinite dimensional dynamical system [3][4][5]. These phenomena involve a broad range of temporal and spatial scales [2]. For example, according to [11], the motion of the atmosphere can be divided into three categories depending on the time scale of the prediction. They are motions corresponding respectively to the short time, medium range and long term behavior of the atmosphere. The understanding of these complicated and scientific issues necessitate a joint effort of scientists in many fields. Also, as [11] pointed out, this difficult problem involves a combination of modeling, mathematical theory and scientific computing.

Some authors have studied the atmospheric motions viewed as an infinite dimensional dynamical system. In [3], the authors study as a first step towards this long-range project which is widely considered as the basic equations of atmospheric dynamics in meteorology, namely the primitive equations of the atmosphere. The mathematical formulation and attractors of the primitive equations, with or without vertical viscosity, are studied. First of all, by integrating the diagnostic equations they present a mathematical setting, and obtain the existence and time analyticity of solutions to the equations. They then establish some physically relevant estimates for the Hausdorff and fractal dimensions of the attractors of the problems. In [12], based on the complete dynamical equations of the moist atmospheric motion, the qualitative theory of nonlinear atmosphere with dissipation and external forcing and its applications are systematically discussed by new theories and methods on the infinite dimensional dynamical system. In [13], by Lions theorem in the Hilbert space, the existence and uniqueness of the weak solution of water vapour-equation with the first boundary-value
problem are proven, and the scheme of the finite-element method according to the weak solution is proposed. In [14], the model of climate for weather forecast is studied, and the existence of the weak solution is proved by Galerkin method. The asymptotic behaviors of the weak solution is described by the trajectory attractors.

In this article, Atmospheric circulation equations with humidity effect is considered, which is different from the previous research. Previous studies are based on two kinds of equations, one is about the heat and humidity transfer([12],[13]), without considering the diffusion of heat and of humidity; the other is p-coordinates equation in([3],[14]), being used to consider the horizontal movement of the atmosphere, and the vertical direction of the velocity is transformed into pressure and topography. Atmospheric circulation equations with humidity effect, derivation from the original equations in [9], considering the diffusion of heat and of humidity, not be restricted by p-coordinates, can be deformed based on the different concerns. In the last part of the article, Existence of global solutions to the atmospheric circulation models is obtained, which implies that atmospheric circulation has its own running way as humidity source and heat source change, and confirms that the atmospheric circulation models are reasonable.

The paper is organized as follows. In Section 2 we present derivation of atmospheric circulation models. In Section 3, we prove that the atmospheric circulation models possess global solutions by using space sequence method.

2 Derivation of Atmospheric Circulation Models with Humidity Effect

2.1 Original Model

The hydrodynamical equations governing the atmospheric circulation are the Navier-Stokes equations with the Coriolis force generated by the earth's rotation, coupled with the first law of thermodynamics.

Let \((\varphi, \theta, r)\) be the spheric coordinates, where \(\varphi\) represents the longitude, \(\theta\) the latitude,
and $r$ the radial coordinate. The unknown functions include the velocity field $u = (u_\phi, u_\theta, u_r)$, the temperature function $T$, the humidity function $q$, the pressure $p$ and the density function $\rho$. Then the equations governing the motion and states of the atmosphere consist of the momentum equation, the continuity equation, the first law of thermodynamic, and the diffusion equation for humidity, and the equation of state (for ideal gas):

$$\rho \left[ \frac{\partial u}{\partial t} + \nabla u + 2\vec{\Omega} \times u \right] + \nabla p + \vec{r} \rho g = \mu \Delta u, \quad (2.1)$$

$$\frac{\partial p}{\partial t} + \text{div}(\rho u) = 0, \quad (2.2)$$

$$\rho c_v \left[ \frac{\partial T}{\partial t} + u \cdot \nabla T \right] + p \text{div} u = Q + \kappa_T \Delta T, \quad (2.3)$$

$$\rho \left[ \frac{\partial q}{\partial t} + u \cdot \nabla q \right] = G + \kappa_q \Delta q, \quad (2.4)$$

$$p = R_0 \rho T, \quad (2.5)$$

Where $-\infty < \varphi < +\infty$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $r_0 < r < r_0 + h$, $r_0$ is the radius of the earth, $h$ is the height of the troposphere, $\Omega$ is the earth’s rotating angular velocity, $g$ is the gravitative constant, $\mu, \kappa_T, \kappa_q, c_v, R_0$ are constants, $Q$ and $G$ are heat and humidity sources, and $\vec{r} = (0, 0, 1)$. The differential operators used are as follows:

1. The gradient and divergence operators are given by:

$$\nabla = \left( \frac{1}{r \cos \theta} \frac{\partial}{\partial \varphi}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial r} \right),$$

$$\text{div} u = \frac{1}{r^2 \partial r} (r^2 u_r) + \frac{1}{r \cos \theta} \frac{\partial (u_\theta \cos \theta)}{\partial \theta} + \frac{1}{r \cos \theta} \frac{\partial u_\varphi}{\partial \varphi},$$

2. In the spherical geometry, although the Laplacian for a scalar is different from the Laplacian for a vectorial function, we use the same notation $\Delta$ for both of them:

$$\Delta u = (\Delta u_\varphi + \frac{2}{r^2 \cos \theta} \frac{\partial u_r}{\partial \varphi} + \frac{2 \sin \theta}{r^2 \cos^2 \theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2 \cos^2 \theta}),$$

$$\Delta u_\theta = \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \cos^2 \theta} - \frac{2 \sin \theta}{r^2 \cos^2 \theta} \frac{u_\varphi}{\partial \varphi},$$

$$\Delta u_r + \frac{2u_r}{r^2} + \frac{2}{r^2 \cos \theta} \frac{\partial (u_\varphi \cos \theta)}{\partial \theta} - \frac{2}{r^2 \cos \theta} \frac{u_\varphi}{\partial \varphi},$$
\[
\Delta f = \frac{1}{r^2 \cos \theta} \frac{\partial f}{\partial \theta} (\cos \theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \cos^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{r^2 \partial r} (r^2 \frac{\partial}{\partial r}),
\]

3. The convection terms are given by

\[
\nabla u \cdot u = (u \cdot \nabla u_r + \frac{u_\varphi u_r}{r} - \frac{u_\varphi u_\theta}{r} \tan \theta),
\]

\[
u \cdot \nabla u_\theta + \frac{u_\varphi u_r}{r} + \frac{u_\varphi^2}{r} + \frac{u_\theta^2}{r} (\tan \theta, u \cdot \nabla u_r - \frac{u_\varphi^2 + u_\theta^2}{r}),
\]

4. The Coriolis term \(2\vec{\Omega} \times u\) is given by

\[
2\vec{\Omega} \times u = 2\Omega (\cos \theta u_r - \sin \theta u_\theta, \sin \theta u_\varphi, -\cos \theta u_\varphi),
\]

Here \(\vec{\Omega}\) is the angular velocity vector of the earth, and \(\Omega\) is the magnitude of the angular velocity.

They are supplemented with the following initial value conditions

\[
(u, T, q) = (\varphi_{10}, \varphi_{20}, \varphi_{30}) \quad \text{at} \quad t = 0. \quad (2.6)
\]

Boundary conditions are needed at the top and bottom boundaries \((r_0, r_0 + h)\). At the top and bottom boundaries \((r = r_0, r_0 + h)\), either the Dirichlet boundary condition or the free boundary condition is given

\[
\text{(Dirichlet)} \quad \begin{cases} 
(u, T, q) = (0, T_0, q_0), & r = r_0, \\
(u, T, q) = (0, T_1, q_1), & r = r_0 + h,
\end{cases} \quad (2.7)
\]

\[
\text{(free)} \quad \begin{cases} 
(u, T, q) = (0, T_0, q_0), & r = r_0, \\
(u_r, T, q) = (0, T_1, q_1), & \frac{\partial (u_\varphi, u_\theta)}{\partial r} = 0 \quad r = r_0 + h,
\end{cases} \quad (2.8)
\]

For \(\varphi\), periodic condition are usually used, for any integers \(k_1 \in \mathbb{Z}\)

\[
(u, T, q)(\varphi + 2k_1 \pi, \theta, r) = (u, T, q)(\varphi + 2k_1 \pi, \theta, r). \quad (2.9)
\]

Because \((\varphi, \theta, r)\) are in a circular field with \(C^\infty\) boundary, the space domain is taken as \((0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times (r_0, r_0 + h)\) and periodic condition is written as

\[
(u, T, q)(0, \theta, r) = (u, T, q)(2\pi, \theta, r). \quad (2.10)
\]
For simplicity, we study the problem with the Dirichlet boundary conditions, and all results hold true as well for other combinations of boundary conditions. Atmospheric convection equations can be read as (2.1)-(2.7), (2.10).

The above equations were basically the equations used by L. F. Richardson in his pioneering work [9]. However, they are in general too complicated to conduct theoretical analysis. As practiced by the earlier workers such as J. Charney, and from the lessons learned by the failure of Richardson’s pioneering work, one tries to be satisfied with simplified models approximating the actual motions to a greater or lesser degree instead of attempting to deal with the atmosphere in all its complexity. By starting with models incorporating only what are thought to be the most important of atmospheric influences, and by gradually bringing in others, one is able to proceed inductively and thereby to avoid the pitfalls inevitably encountered when a great many poorly understood factors are introduced all at once. The simplifications are usually done by taking into consideration of some main characterizations of the large-scale atmosphere. One such characterization is the small aspect ratio between the vertical and horizontal scales, leading to hydrostatic equation replacing the vertical momentum equation. The resulting system of equation are called the primitive equations (PEs); see among others [3]. The another characterization of the large scale motion is the fast rotation of the earth, leading to the celebrated quasi-geostrophic equations [1].

For convenience of research, the approximations we adopt involves the following components:

First, we often use Boussinesq assumption, where the density is treated as a constant except in the buoyancy term and in the equation of state.

Second, because the air is generally compressible, we do not use the equation of state for ideal gas, rather, we use the following empirical formula, which can by considered as the linear approximation of

\[ \rho = \rho_0 [1 - \alpha_T (T - T_0) + \alpha_q (q - q_0)], \]  

(2.11)

where \( \rho_0 \) is the density at \( T = T_0 \) and \( q = q_0 \), and \( \alpha_T \) and \( \alpha_q \) are the coefficients of thermal and humidity expansion.
Third, since the aspect ratio between the vertical scale and the horizontal scale is small, the spheric shell the air occupies is treated as a product space $S^2_0 \times (r_0, r_0 + h)$. This approximation is extensively adopted in geophysical fluid dynamics. Under the above simplification, we have the following equations governing the motion and states of large scale atmospheric circulations:

$$\frac{\partial u}{\partial t} + \nabla u = \nu \Delta u - 2\Omega \times u - \frac{1}{\rho_0} \nabla p - \left[1 - \alpha_T(T - T_0) + \alpha_q(q - q_0)\right]g e_z, \tag{2.12}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T = Q + \kappa_T \Delta T, \tag{2.13}$$

$$\frac{\partial q}{\partial t} + (u \cdot \nabla)q = G + \kappa_q \Delta q, \tag{2.14}$$

$$\text{div} u = 0, \tag{2.15}$$

where $(\phi, \theta, z) \in M = S^2_0 \times (r_0, r_0 + h)$, $\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity, $u = u_{\phi}e_{\phi} + u_{\theta}e_{\theta} + u_r e_r$, $(e_{\phi}, e_{\theta}, e_r)$ the local normal basis in the sphereric coordinates, and

$$\nabla u = \left((u \cdot \nabla)u_{\phi} + \frac{u_{\phi}u_{z}}{r_0} - \frac{u_{\phi}u_{\theta}}{r_0} \tan \theta\right)e_{\phi} +$$

$$\left((u \cdot \nabla)u_{\theta} + \frac{u_{\theta}u_{z}}{r_0} + \frac{u_{\theta}^2}{r_0} \tan \theta\right)e_{\theta} + \left((u \cdot \nabla)u_{z} - \frac{u_{\phi}^2 + u_{\theta}^2}{r_0}\right)e_z,$$

$$\Delta u = \left(\Delta u_{\phi} + \frac{2}{r_0^2} \cos \theta \frac{\partial u_{\phi}}{\partial \phi} + \frac{2 \sin \theta}{r_0^2 \cos^2 \theta} \frac{\partial u_{\phi}}{\partial \theta} - \frac{u_{\phi}}{r_0^2 \cos^2 \theta}\right)e_{\phi} +$$

$$\left(\Delta u_{\theta} + \frac{2}{r_0^2} \frac{\partial u_{\phi}}{\partial \theta} - \frac{u_{\theta}}{r_0^2 \cos^2 \theta} - \frac{2 \sin \theta}{r_0^2 \cos^2 \theta} \frac{u_{\phi}}{\partial \phi}\right)e_{\theta} +$$

$$\left(\Delta u_{z} + \frac{2u_{\phi}}{r_0^2} + \frac{2}{r_0^2} \cos \theta \frac{\partial (u_{\theta} \cos \theta)}{\partial \theta} - \frac{2}{r_0^2 \cos \theta} \frac{u_{\phi}}{\partial \phi}\right)e_z,$$

$$\nabla p = \frac{1}{r_0 \cos \theta} \frac{\partial p}{\partial \theta} e_{\phi} + \frac{1}{r} \frac{\partial p}{\partial \theta} e_{\theta} + \frac{\partial p}{\partial z} e_z,$$

$$\text{div} u = \frac{1}{r_0 \cos \theta} \frac{\partial u_{\phi}}{\partial \phi} + \frac{1}{r_0 \cos \theta} \frac{\partial (u_{\theta} \cos \theta)}{\partial \theta} + \frac{\partial u_{z}}{\partial z},$$

and the differential operators $(u \cdot \nabla)$ and $\Delta$ are expressed as

$$(u \cdot \nabla) = \frac{u_{\phi}}{r_0 \cos \theta} \frac{\partial}{\partial \phi} + \frac{u_{\theta}}{r_0} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z},$$

$$\Delta = \frac{1}{r_0^2 \cos \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r_0^2 \cos \theta} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial}{\partial \theta}\right) + \frac{\partial^2}{\partial z^2}.$$
2.2 Simplification of Model

Atmospheric circulation is the large-scale motion of the air, which is essentially a thermal convection process caused by the temperature and humidity difference between the earth's surface and the tropopause. It is a crucial means by which heat and humidity are distributed on the surface of the earth. Air circulates within the troposphere, limited vertically by the tropopause at about 8-10km. Atmospheric motion in the troposphere, together with the oceanic circulation, plays a crucial role in leading to the global climate changes and evolution on the earth. There are two types of circulation cells: the latitudinal circulation and the longitudinal circulation. The latitudinal circulation is characterized by the Polar cell, the Ferrel cell, and the Hadley cell, which are major players in global heat and humidity transport, and do not act alone. The zonal circulation consists of six circulation cells over the equator. The overall atmospheric motion is known as the zonal overturning circulation, and also called the Walker circulation. The most remarkable feature of the global atmospheric circulation is that the equatorial Walker circulation divides the whole earth into three invariant regions of atmospheric flow: the northern hemisphere, the southern hemisphere, and the equatorial zone. We also note the important fact that the large-scale structure of the zonal overturning circulation varies from year to year, but its basic structure remains fairly stable, it never vanishes. Based on these natural phenomena, we here present the Zone Hypotheses for atmospheric dynamics, which amounts to saying that the global atmospheric system can be divided into three sub-systems: the North-Hemispheric System, the South-Hemispheric System, and the Tropical Zone System, which are relatively independent, and have less influence on each other in their basic structure. More precisely, the Atmospheric Zone Hypothesis is stated in the following form[6].

**Atmospheric Zone Hypothesis.** The atmospheric circulation has three invariant regions: the northern hemisphere domain \((0 < \theta \leq \frac{\pi}{2})\), the southern hemisphere domain \((-\frac{\pi}{2} \leq \theta < 0)\), and the equatorial zone \((\theta = 0)\). Namely, the large-scale circulations in their invariant regions can act alone with less influence on the others. In particular, the velocity field \(u = (u_\varphi, u_\theta, u_z)\) of the atmospheric circulation has a vanished latitudinal component in
a narrow equatorial zone, i.e., \( u_\theta = 0 \), for \(-\varepsilon < \theta < \varepsilon\), where \( \varepsilon > 0 \) is a small number.

Atmospheric Zone Hypothesis is based on the following several evidences from theory and practice:

1. The global atmospheric motion equations (2.12)-(2.15) are of \( \theta \)--reflexive symmetry, i.e. under the \( \theta \)--reflexive transformation \((\varphi, \theta, z) \rightarrow (\varphi, -\theta, z)\), the velocity field \( u \) becomes \((u_\varphi, u_\theta, u_z) \rightarrow (u_\varphi, -u_\theta, u_z)\), and equations (2.12)-(2.15) are invariant, which implies that this system is compatible with the Atmospheric Zone Hypothesis.

2. Climatic observation data show that when the El Niño-Southern Oscillation (the behavior that the Walker circulation cell in the Western Pacific stops or reverses its direction) takes place, no oscillation occurs in the latitudinal cells. It demonstrates the relative independence of these circulations in their invariant domain.

3. When a cold current moves southward from Siberia, or a violent typhoon sweeps northward from the tropics, the weather in Southern Hemisphere has no response. Atmospheric Zone Hypothesis provides a theoretic basis for the study of atmospheric dynamics, by which we can establish locally simplified models to treat many difficult problems in atmospheric science.

4. For the three-dimensional atmospheric circulation equation, it is too difficult to study. So we study the equatorial atmospheric circulation.

The atmospheric motion equations over the tropics are the equations restricted on \( \theta = 0 \), where the meridional velocity component \( u_\theta \) is set to zero, and the effect of the turbulent friction is taking into considering

\[
\begin{align*}
\frac{\partial u_\varphi}{\partial t} &= -(u \cdot \nabla) u_\varphi - \frac{u_\varphi u_z}{r_0} + \nu(\Delta u_\varphi + \frac{2}{r_0^2} \frac{\partial u_\varphi}{\partial \varphi} - \frac{2u_\varphi}{r_0^3}) - \sigma_0 u_\varphi - 2\Omega u_z - \frac{1}{\rho_0 r_0} \frac{\partial p}{\partial \varphi}, \\
\frac{\partial u_z}{\partial t} &= -(u \cdot \nabla) u_z + \frac{u_\varphi^2}{r_0} + \nu(\Delta u_z + \frac{2}{r_0^2} \frac{\partial u_\varphi}{\partial \varphi} - \frac{2u_z}{r_0^3}) - \sigma_1 u_z - 2\Omega u_\varphi \\
&\quad - \frac{1}{\rho_0} \frac{\partial p}{\partial z} - [1 - \alpha_T(T - T_0) + \alpha_q(q - q_0)]g, \\
\frac{\partial T}{\partial t} &= -(u \cdot \nabla) T + \kappa_T \Delta T + Q,
\end{align*}
\]
\[ \frac{\partial q}{\partial t} = -(u \cdot \nabla)q + \kappa q \Delta q + G, \quad (2.19) \]

\[ \frac{1}{r_0} \frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} = 0, \quad (2.20) \]

Here \( \sigma_i = C_i h^2 (i = 0, 1) \) represent the turbulent friction, \( r_0 \) is the radius of the earth, the space domain is taken as \( M = S^1_{r_0} \times (r_0, r_0 + h) \) with \( S^1_{r_0} \) being the one-dimensional circle with radius \( r_0 \), and

\[ (u \cdot \nabla) = \frac{u_\varphi}{r_0} \frac{\partial}{\partial \varphi} + u_z \frac{\partial}{\partial z}, \quad \Delta = \frac{1}{r_0^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \]

For simplicity, we denote

\[ (x_1, x_2) = (r_0 \varphi, z), \quad (u_1, u_2) = (u_\varphi, u_z). \]

The atmospheric motion equations (2.16)-(2.20) can be written as

\[ \frac{\partial u_1}{\partial t} = -(u \cdot \nabla)u_1 - \frac{u_1 u_2}{r_0} + \nu (\Delta u_1 + \frac{2}{r_0} \frac{\partial u_2}{\partial x_1} - \frac{2 u_1}{r_0^2}) - \sigma_0 u_1 - 2 \Omega u_2 - \frac{1}{\rho_0} \frac{\partial p}{\partial x_1}, \quad (2.21) \]

\[ \frac{\partial u_2}{\partial t} = -(u \cdot \nabla)u_2 + \frac{u_2^2}{r_0} + \nu (\Delta u_2 + \frac{2}{r_0} \frac{\partial u_1}{\partial x_1} - \frac{2 u_2}{r_0^2}) - \sigma_1 u_2 - 2 \Omega u_1 \]

\[ -\frac{1}{\rho_0} \frac{\partial p}{\partial x_2} - [1 - \alpha_T (T - T_0) + \alpha_q (q - q_0)] g, \quad (2.22) \]

\[ \frac{\partial T}{\partial t} = -(u \cdot \nabla)T + \kappa_T \Delta T + Q, \quad (2.23) \]

\[ \frac{\partial q}{\partial t} = -(u \cdot \nabla)q + \kappa q \Delta q + G, \quad (2.24) \]

\[ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0. \quad (2.25) \]

To make the nondimensional form, let

\[ x = h x', \quad t = \frac{h^2}{\kappa_T} t', \quad u = \frac{\kappa_T}{h} u', \]

\[ T = T_0 - (T_0 - T_1) \frac{x_2}{h} + (T_0 - T_1) T', \]

\[ q = q_0 - (q_0 - q_1) \frac{x_2}{h} + (q_0 - q_1) q'. \]
The nondimensional form of (2.21)-(2.25) reads

\[
\frac{\partial u_1'}{\partial t'} = -(u' \cdot \nabla)u_1' + \frac{\nu}{\kappa_T} \Delta u_1' - \frac{\sigma_0 h^2}{\kappa_T} u_1' - \frac{2\Omega h^2}{\kappa_T} u_2' - \frac{\nu}{\kappa_T} \frac{\partial p'}{\partial x_1'}, \tag{2.26}
\]

\[
- \frac{h}{r_0} u_1 u_2 + \frac{2h}{r_0 \kappa_T} \frac{\partial u_2'}{\partial x_1} - \frac{2h^2}{r_0 \kappa_T} u_1',
\]

\[
\frac{\partial u_2'}{\partial t'} = -(u' \cdot \nabla)u_2' + \frac{\nu}{\kappa_T} \Delta u_2' - \frac{\sigma_1 h^2}{\kappa_T} u_2' - \frac{2\Omega h^2}{\kappa_T} u_1' - \frac{h}{r_0} u_1^2 + \frac{h}{r_0 \kappa_T} \frac{\partial u_1'}{\partial x_2'} - \frac{2h^2}{r_0 \kappa_T} u_2',
\]

\[
- \frac{\nu}{\kappa_T} \frac{\partial p'}{\partial x_2'} + \frac{gh^3}{\kappa_T^2} [\alpha_T(T_0 - T_1)T' - \alpha_q(q_0 - q_1)q'], \tag{2.27}
\]

\[
\frac{\partial T'}{\partial t'} = -(u' \cdot \nabla)T' + \Delta T' + u_2' + \frac{h^2}{(T_0 - T_1)\kappa_T} Q, \tag{2.28}
\]

\[
\frac{\partial q'}{\partial t'} = -(u' \cdot \nabla)q' + \frac{\kappa_q}{\kappa_T} \Delta q' + u_2' + \frac{h^2}{(T_0 - T_1)\kappa_T} G, \tag{2.29}
\]

\[
div u' = 0. \tag{2.30}
\]

Because \(r_0\) is far larger than \(u_1, u_2\), the atmospheric motion equations (2.26)-(2.30) can be read

\[
\frac{\partial u_1'}{\partial t'} = -(u' \cdot \nabla)u_1' + \frac{\nu}{\kappa_T} \Delta u_1' - \frac{\sigma_0 h^2}{\kappa_T} u_1' - \frac{2\Omega h^2}{\kappa_T} u_2' - \frac{\nu}{\kappa_T} \frac{\partial p'}{\partial x_1'}, \tag{2.31}
\]

\[
\frac{\partial u_2'}{\partial t'} = -(u' \cdot \nabla)u_2' + \frac{\nu}{\kappa_T} \Delta u_2' - \frac{\sigma_1 h^2}{\kappa_T} u_2' - \frac{2\Omega h^2}{\kappa_T} u_1' - \frac{\nu}{\kappa_T} \frac{\partial p'}{\partial x_2'},
\]

\[
- \frac{h}{r_0} u_1 u_2 + \frac{2h}{r_0 \kappa_T} \frac{\partial u_2'}{\partial x_1} - \frac{2h^2}{r_0 \kappa_T} u_1',
\]

\[
\frac{\partial T'}{\partial t'} = -(u' \cdot \nabla)T' + \Delta T' + u_2' + \frac{h^2}{(T_0 - T_1)\kappa_T} Q, \tag{2.33}
\]

\[
\frac{\partial q'}{\partial t'} = -(u' \cdot \nabla)q' + \frac{\kappa_q}{\kappa_T} \Delta q' + u_2' + \frac{h^2}{(T_0 - T_1)\kappa_T} G, \tag{2.34}
\]
\[
\text{div} u' = 0. \quad (2.35)
\]

Let \( P_r = \frac{\nu}{\kappa T}, \quad L_e = \frac{\sigma_0}{\kappa T}, \quad R = \frac{g\alpha (q_0 - q)h^3}{\kappa T^2}, \quad \tilde{R} = \frac{g\alpha (q_0 - q)h^3}{\kappa T^2}, \quad \sigma'_i = \frac{\sigma_i h^2}{\nu}, \quad \omega = \frac{2\Omega h^2}{\nu}, \)

\[
Q' = \frac{h^2}{(T_0 - T_1)\kappa T} Q, \quad G' = \frac{h^2}{(T_0 - T_1)\kappa T} G, \]

omitting the primes, the nondimensional form of (2.31)-(2.35) reads

\[
\frac{\partial u}{\partial t} = P_r (\Delta u - \nabla p - \sigma u) + P_r (RT - \tilde{R}q)\kappa - (\nabla \cdot u)u, \quad (2.36)
\]

\[
\frac{\partial T}{\partial t} = \Delta T + u_2 - (u \cdot \nabla)T + Q, \quad (2.37)
\]

\[
\frac{\partial q}{\partial t} = L_e \Delta q + u_2 - (u \cdot \nabla)q + G, \quad (2.38)
\]

\[
\text{div} u = 0, \quad (2.39)
\]

where \( \sigma \) is constant matrix

\[
\sigma = \begin{pmatrix}
\sigma_0 & \omega \\
\omega & \sigma_1
\end{pmatrix}.
\]

The problem (2.36)-(2.39) are supplemented with the following Dirichlet boundary condition at \( x_2 = 0, 1 \) and periodic condition for \( x_1 \):

\[
(u, T, q) = 0, \quad x_2 = 0, 1, \quad (2.40)
\]

\[
(u, T, q)(0, x_2) = (u, T, q)(2\pi, x_2), \quad (2.41)
\]

and initial value conditions

\[
(u, T, q) = (u_0, T_0, q_0), \quad t = 0. \quad (2.42)
\]

3 Global Solution of Atmospheric Circulation Equations with Humidity Effect

3.1 Preliminaries

Let \( X \) be a linear space, \( X_1, X_2 \) two separable reflexive Banach spaces, \( H \) a Hilbert space. \( X_1, X_2 \) and \( H \) are completion space of \( X \) under the respective norms. \( X_1, X_2 \subset H \) are
dense embedding. $F : X_2 \times (0, \infty) \to X_1^*$ is a continuous mapping. We consider the abstract equation

$$
\begin{cases}
\frac{du}{dt} = Fu, & 0 < t < \infty, \\
u(0) = \varphi,
\end{cases}
$$

(3.1)

where $\varphi \in H$, $u : [0, +\infty) \to H$ is unknown.

**Definition 3.1** Let $\varphi \in H$ be a given initial value. $u \in L^p((0, T), X_2) \cap L^\infty((0, T), H)$, $(0 < T < \infty)$ is called a global solution of Eq(3.1), if $u$ satisfies

$$(u(t), v)_H = \int_0^t < Fu, v > dt + (\varphi, v)_H, \quad \forall v \in X_1 \subset H.$$  

**Definition 3.2** Let $u_n, u_0 \in L^p((0, T), X_2)$. $u_n \to u_0$ is called uniformly weak convergence in $L^p((0, T), X_2)$, if $\{u_n\} \subset L^\infty((0, T), H)$ is bounded, and

$$
\begin{cases}
u_n \rightharpoonup u_0, & \text{in } L^p((0, T), X_2), \\
\lim_{n \to \infty} \int_0^T |u_n - u_0, v >_H|^2 dt = 0, & \forall v \in H.
\end{cases}
$$  

(3.2)

**Definition 3.3** A mapping $F : X_2 \times (0, \infty) \to X_1^*$ is called $T$-weakly continuous, if for $p = (p_1, \ldots, p_m)$, $0 < T < \infty$, and $u_n$ uniformly weakly converge to $u_0$ under Eq(3.2), we have

$$
\lim_{n \to \infty} \int_0^T < Fu_n, v > dt = \int_0^T < Fu_0, v > dt, \quad \forall v \in X_1.
$$

**Lemma 3.4** Let $\{u_n\} \subset L^p((0, T), W^{m,p})(m \geq 1)$ be bounded sequence, and $\{u_n\}$ uniformly weakly converge to $u_0 \subset L^p((0, T), W^{m,p})$, i.e.

$$
\begin{cases}
u_n \rightharpoonup u_0 & \text{in } L^p((0, T), W^{m,p}), \quad p \geq 2, \\
\lim_{n \to \infty} \int_0^T [\int_\Omega (u_n - u_0)v dx]^2 dt = 0, & \forall v \in C^\infty_0(\Omega).
\end{cases}
$$

(3.3)

Then for all $|\alpha| \leq m - 1$, we have

$$
D^\alpha u_n \to D^\alpha u_0 \quad \text{in } L^2((0, T) \times \Omega).
$$

**Lemma 3.5** Assume $F : X_2 \times (0, \infty) \to X_1^*$ is $T$-weakly continuous, and satisfies
(A1) there exists $p = (p_1, \cdots, p_m)$, $p_i > 1 (1 \leq i \leq m)$, such that
\[
<Fu, u> \leq -C_1\|u\|_{X_2}^p + C_2\|u\|_H^2 + f(t), \; \forall u \in X,
\]
where $C_1, C_2$ are constants, $f \in L^1(0, T)(0 < T < \infty)$, $\| \cdot \|_{X_2}^p = \sum_{i=1}^m | \cdot |_{i}^p$, $| \cdot |_i$ is seminorm of $X_2$.

(A2) there exists $0 < \alpha < 1$, for all $0 < h < 1$ and $u \in C^1([0, \infty), X)$,
\[
| \int_t^{t+h} < Fu, v > dt | \leq Ch^\alpha, \; \forall v \in X \; \text{and} \; 0 \leq t < T,
\]
where $C > 0$ depends only on $T$, $\|v\|_{X_1}$, $\int_0^t \|u\|_{X_2}^p dt$ and $\sup_{0 \leq t \leq T} \|u\|_H$.

Then for all $\varphi \in H$, Eq(3.1) has a global weak solution
\[u \in L^\infty((0, T), H) \cap L^p((0, T), X_2), \; 0 < T < \infty, \; p \; \text{in} \; (A1).\]

**Remark 3.6** $\| \cdot \|_X$ denotes norm of $X$, and $C_i$ are variable constants.

### 3.2 Existence of Global Solution

We introduce spatial sequences
\[
X = \{ \phi = (u, T, q) \in C^\infty(\Omega, R^4)|(u, T, q) \; \text{satisfy} \; (2.39) - (2.41) \},
\]
\[
H = \{ \phi = (u, T, q) \in L^2(\Omega, R^4)|(u, T, q) \; \text{satisfy} \; (2.39) - (2.41) \},
\]
\[
H_1 = \{ \phi = (u, T, q) \in H^1(\Omega, R^4)|(u, T, q) \; \text{satisfy} \; (2.39) - (2.41) \},
\]

**Theorem 3.7** If $\phi_0 = (u_0, T_0, q_0) \in H$, $Q, G \in L^2(\Omega)$, then Eq(2.36)-(2.42) there exists a global solution
\[(u, T, q) \in L^\infty((0, T), H) \cap L^2((0, T), H_1), \; 0 < T < \infty.\]

**Proof.** Define $F : H_1 \to H_1^*$ as
\[
<F\phi, \psi> = \int_{\Omega}[-P_r\nabla u \nabla v - P_r\sigma u \cdot v + P_r(RT - \tilde{R})v_2 - (u \cdot \nabla)u \cdot v
\]
\[
-\nabla T \nabla S + u_2 S - (u \cdot \nabla)TS + QS - L_e\nabla q \nabla z + u_2 z
\]
\[
-(u \cdot \nabla)qz + Gz]dx, \; \forall \psi = (v, S, z) \in H_1.
Let $\psi = \phi$. Then

$$<F\phi, \phi> = \int_{\Omega} \left[ -P_r|\nabla u|^2 - P_r\sigma u \cdot v + P_r(RT - \tilde{R}q)u_2 - (u \cdot \nabla)u \cdot u \

-|\nabla T|^2 + u_2 T - (u \cdot \nabla)TT + QT - L_e|\nabla q|^2 \\

+u_2q - (u \cdot \nabla)qq + Gq \right] \ dx$$

$$= \int_{\Omega} \left[ -P_r|\nabla u|^2 - P_r\sigma u \cdot u + (P_rR + 1)u_2 T - (P_r\tilde{R} - 1)qu_2 \\

-|\nabla T|^2 + QT - L_e|\nabla q|^2 + Gq \right] \ dx$$

$$\leq -C_1 \int_{\Omega} [||u||^2 + ||\nabla T||^2 + ||\nabla q||^2] \ dx + C_2 \int_{\Omega} [||u||^2 + ||u||T + ||q||u] \\

+||Q||T + ||G||q] \ dx$$

$$\leq -C_1 \int_{\Omega} [||u||^2 + ||\nabla T||^2 + ||\nabla q||^2] \ dx + C_2 \int_{\Omega} [||u||^2 + ||T||^2 + ||q||^2] \ dx \\

+C_3 \int_{\Omega} [||Q^2 + ||G^2||] \ dx$$

$$\leq -C_1 \|\phi\|_{H_1}^2 + C_2 \|\phi\|_{H}^2 + C_4,$$

which implies $(A_1)$.

For $\forall \phi \in L^2(0,T), H_1) \cap L^\infty((0,T), H)$ and $\psi \in X$, $h(0 < h < 1)$, we have

$$\left| \int_{t^h}^{t^+} <F\phi, \psi> \ dt \right|$$

$$= \left| \int_{t^h}^{t^+} \int_{\Omega} \left[ -P_r \nabla u \nabla v - P_r\sigma u \cdot v + P_r(RT - \tilde{R}q)v_2 - (u \cdot \nabla)u \cdot v - \nabla T \nabla S \\

+u_2 S - (u \cdot \nabla)TS + QS - L_e \nabla q \nabla z + u_2 z - (u \cdot \nabla)qz + Gz \right] \ dx \ dt \right|$$
\begin{align*}
\leq & \; C \int_t^{t+h} \int_\Omega \left[ |\nabla u||\nabla v| + |u||v| + |T||v| + |q||v| + |\nabla T||\nabla S| + |u||S| \\
& + |Q||S| + |\nabla q||\nabla z| + |u||z| + |G||z| \right] dx \, dt + C \int_t^{t+h} \int_\Omega (u \cdot \nabla)u \cdot v dx \\
& + \int_\Omega (u \cdot \nabla)TS dx + |f_\Omega (u \cdot \nabla)q z dx| \\
\leq & \; C \int_t^{t+h} \left[ (f_\Omega |u|^2 dx)^{\frac{1}{2}} (f_\Omega |\nabla v|^2 dx)^{\frac{1}{2}} + (f_\Omega |T|^2 dx)^{\frac{1}{2}} \right] \\
& + (f_\Omega |q|^2 dx)^{\frac{1}{2}} (f_\Omega |\nabla S|^2 dx)^{\frac{1}{2}} + (f_\Omega |Q|^2 dx)^{\frac{1}{2}} (f_\Omega |u|^2 dx)^{\frac{1}{2}} \\
& + (f_\Omega |\nabla q|^2 dx)^{\frac{1}{2}} (f_\Omega |\nabla z|^2 dx)^{\frac{1}{2}} + (f_\Omega |u|^2 dx)^{\frac{1}{2}} (f_\Omega |z|^2 dx)^{\frac{1}{2}} \\
& + (f_\Omega |G|^2 dx)^{\frac{1}{2}} (f_\Omega |z|^2 dx)^{\frac{1}{2}} |dt + C \int_t^{t+h} \left| \sum_{i,j=1}^{2} f_\Omega (u_i u_j \frac{\partial \Omega}{\partial x_i}) dx \right| \\
& + |\sum_{i=1}^{2} f_\Omega u_i T \frac{\partial S}{\partial x_i} dx| + |\sum_{i=1}^{2} f_\Omega u_i q \frac{\partial S}{\partial x_i} dx| |dt \\
\leq & \; C \left( \|u\|_{L^2(0,T;H^1)} \right) \left( \|Du\|_{L^2} \right)^{\frac{1}{2}} + (\|T\|_{L^2(0,T;L^2)} \|v\|_{L^2} h^{\frac{1}{2}} + \|q\|_{L^2(0,T;L^2)} \|v\|_{L^2} h^{\frac{1}{2}} \\
& + \|T\|_{L^2(0,T;H^1)} \|DS\|_{L^2} h^{\frac{1}{2}} + \|u\|_{L^\infty(0,T;L^2)} \|S\|_{L^2} h^{\frac{1}{2}} + \|Q\|_{L^2} \|S\|_{L^2} h \\
& + \|q\|_{L^2(0,T;H^1)} \|Dz\|_{L^2} h^{\frac{1}{2}} + \|u\|_{L^\infty(0,T;H^1)} \|z\|_{L^2} h + \|G\|_{L^2} \|z\|_{L^2} h \\
& + \|v\|_{C^1} \|u\|_{L^\infty(0,T,L^2)} h + \|S\|_{C^1} \|u\|_{L^\infty(0,T;L^2)} h \|T\|_{L^\infty(0,T;L^2)} h \\
& + \|z\|_{C^1} \|u\|_{L^\infty(0,T,L^2)} h \|q\|_{L^\infty(0,T;L^2)} h \right) \\
\leq & \; Ch^{\frac{1}{2}},
\end{align*}
which implies $(A_2)$.

We will prove that $F : H_1 \to H_1^*$ is $T$-weakly continuous. Let $\phi_n = (u^n, T^n, q^n) \to \phi_0 = (u^0, T^0, q^0)$ be uniformly convergence, i.e., $\{\phi_n\} \subset L^\infty((0, T); H)$ is bounded, and

$$
\begin{cases}
\phi_n \rightharpoonup \phi_0 \quad &\text{in } L^p((0, T); H),
\lim_{n \to \infty} \int_0^T |\phi_n - \phi_0, \psi >_H|^2 dt = 0, &\forall \psi \in H.
\end{cases}
$$

From Lemma 3.4, we known that $\phi_n \to \phi_0$ in $L^2(\Omega \times (0, T))$.

Then $\forall \psi \in X \subset C^\infty(\Omega, R^4) \cap H_1$, we have

$$
\lim_{n \to \infty} \int_0^T \int_\Omega (u^n \cdot \nabla) u^n \cdot v dx dt = \lim_{n \to \infty} \int_0^T \int_\Omega \sum_{i,j=1}^2 u^n_i \frac{\partial u^n_j}{\partial x_i} v_j dx dt
$$

$$
= - \lim_{n \to \infty} \int_0^T \int_\Omega \sum_{i,j=1}^2 u^n_i u^n_j \frac{\partial v_j}{\partial x_i} dx dt
$$

$$
= - \int_0^T \int_\Omega (u^0 \cdot \nabla)v \cdot u^0 dx dt
$$

$$
= \int_0^T \int_\Omega (u^0 \cdot \nabla)u^0 \cdot v dx dt,
$$

and

$$
\lim_{n \to \infty} \int_0^T \int_\Omega (u^n \cdot \nabla) T^n S dx dt = \lim_{n \to \infty} \int_0^T \int_\Omega \sum_{i=1}^2 u^n_i \frac{\partial T^n}{\partial x_i} S dx dt
$$

$$
= - \lim_{n \to \infty} \int_0^T \int_\Omega \sum_{i=1}^2 u^n_i T^n \frac{\partial S}{\partial x_i} dx dt
$$

$$
= - \int_0^T \int_\Omega (u^0 \cdot \nabla) ST^0 dx dt
$$

$$
= \int_0^T \int_\Omega (u^0 \cdot \nabla)T^0 S dx dt,
$$

and

$$
\lim_{n \to \infty} \int_0^T \int_\Omega (u^n \cdot \nabla) q^n z dx dt = \lim_{n \to \infty} \int_0^T \int_\Omega \sum_{i=1}^2 u^n_i \frac{\partial q^n}{\partial x_i} z dx dt
$$

$$
= - \lim_{n \to \infty} \int_0^T \int_\Omega \sum_{i=1}^2 u^n_i q^n \frac{\partial z}{\partial x_i} dx dt
$$

$$
= - \int_0^T \int_\Omega (u^0 \cdot \nabla) z q^0 dx dt
$$

$$
= \int_0^T \int_\Omega (u^0 \cdot \nabla) q^0 z dx dt,
$$

and
Thus,

$$\lim_{n \to \infty} \int_0^t \langle F\phi_n, \psi \rangle \, dt = \int_0^t \langle F\phi_0, \psi \rangle \, dt. \quad (3.4)$$

Because $X$ is dense in $H_1$, Eq(3.4) holds for $\psi \in H_1$. In other words, the mapping $F : H_1 \to H_1^*$ is $T$-weakly continuous.

From Lemma 3.5, Eq(2.36)-(2.42) has a global weak solution

$$(u, T, q) \in L^\infty((0,T), H) \cap L^2((0,T), H_1), \quad 0 < T < \infty.$$

$\square$

Remark 3.8 Existence of global solutions to the atmospheric circulation models implies that atmospheric circulation has its own running way as humidity source and heat source change, and confirms that the atmospheric circulation models are reasonable.

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