On the BRST and finite field dependent BRST of a model where vector and axial vector interaction get mixed up with different weight

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The generalized version of a lower dimensional model where vector and axial vector interaction get mixed up with different weight is considered. The bosonized version of which does not posses the local gauge symmetry. An attempt has been made here to construct the BRST invariant reformulation of this model using Batalin Fradlin and Vilkovisky formalism. It is found that the extra field needed to make it gauge invariant turns into Wess-Zumino scalar with appropriate choice of gauge fixing. An application of finite field dependent BRST and anti-BRST transformation is also made here in order to show the transmutation between the BRST symmetric and the usual non-symmetric version of the model.

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I. INTRODUCTION

Dynamical equations of physical system cannot always be described in terms of observable physical degrees of freedom which pose problem to the straightforward physical interpretation of the solution of evaluation equations. In some cases, few solutions need to be excluded since they do not describe the real physical situation or it may be the case that certain class of apparently different solutions appears to be physically indistinguishable. The BRST-formalism has been developed precisely to deal with such systems. It is a technique to enlarge the phase space of a gauge theory and to restore the symmetry of the gauge fixed action in the extended phase space keeping the physical contents of the theory intact. To study the unitarity and renormalization it is instrumental. The unphysical ghost field acquires prominent status rendering its valuable service in bringing back the symmetry of the gauge fixed effective action with the preservation of the fundamental unitary property. Since this symmetry mixes all the fields (physical and ghost) in such a way that ghost field along with the other fields need to be treated on the same footing and that forces to regard the ghost field along with all the other field as a different component of a single geometrical object.

The role field dependent BRST (FFBRST) is almost similar to the BRST so far symmetry is concerned. It does protect nilpotency and reflects the symmetry of the gauge fixed action of a physically sensible theory. It can be considered as a generalization over the usual BRST formalism where transformation parameters becomes finite, field dependant and anti-commuting in nature. Unlike BRST transformation, it fails to keep the measure of the generating functional unchanged. However, the change appeared there renders several important services to make an equivalence between the different effective actions of a particular theory. In this context, the services obtained through the exploitation of the change entered into the measure of the generating functional to relate the different gauge fixed actions of a particular theory is remarkable. BRST and FFBRST are therefore equally important and interesting in their own right. So application of BRST as well as FFBRST formalism on any physically sensible theories would be of considerable interest and would certainly add a new contribution to the formal field theoretical regime.

In this context, we consider a (1+1) dimensional generalized version of Schwinger (GVSM) model where axial and vector interaction get mixed up with different weight. The most interning feature of the model is its ability to interpolate the two most important lower dimensional field theoretical models through its mixing weight factor: the models are well celebrated vector Schwinger model and its chiral generation, commonly known as chiral Schwinger mode. Schwinger model started a glorious journey for its potential of describing the mass generation along with its ability to describe the confinement aspect of fermion in lower dimension and has been extensively studied over the

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years \[14–21\]. Chiral generalization of this model too has been studied with great interest after the removal of its unitarity problem by Jackiw and Rajaraman \[13, 21–39\].

Recently, an attempts has been made by us to quantize both the gauge invariant and gauge non-invariant version of GVSM \[11\]. This model in its bosonized version does not possess the local gauge symmetry, since it becomes essential to take into account the anomaly to protect the unitarity of this model. Here mass generation takes place indeed, via a kind of dynamical symmetry breaking. However, unlike Schwinger model \[12\], here the fermions are found to get liberated which may be considered as de-confinement phase of fermions. We should mention here that the fermion are found to remain confined when the model turns into Schwinger model in absence of its axial interaction part. The model provides so many interesting and surprising insights into the phenomena related to mass generation and confinement aspect of fermion, charge shielding etc., that till now it remains as a subject of several interest. So naturally, the extension of the model coined in \[8\], which has the ability of combining these two model into a single structure would be of worth investigations. Besides, in order to protect unitarity, inclusion of anomaly becomes essential and it adds further interest in another direction, because one loop correction enters there holding the hand of anomaly. But it certainly breaks the local gauge symmetry. So the study related to the restoration of symmetry would be instructive which we have attempted in the present work. We, therefore, in our present work consider the GVSM and attempts is made here towards the reformulation of a BRST invariant effective action by the use of Batalin, Fradkin and Vilkovisky (BFV) formalism. The scheme developed by Batalin, Fradkin and Vilkovisky towards the conversion of a set second class constraint into first class set helps to get this symmetric transmuted form. It is known that for the above transmutation some extra fields are needed. These fields are known as auxiliary fields. These auxiliary fields turn into Wess-Zumino scalar with appropriate choice of gauge fixing conditions for some favorable situations. So at first an attempt has been made here towards the BRST invariant reformulation of this model using BFV formalism \[40–44\]. In fact, we have used the improved version presented by Fujiwara and Igarishi and Kubo (FIK) \[44\], since it is known that it generally helps to obtain the Wess-Zumino \[45\] action associated with the model in most of the cases \[46–50\].

Application of FFBRST formalism on this model would also be instructive like its ancestor BRST formalism and would add a new contribution to formal field theoretical regime. So an extension using FFBRST formulation is also made here to show how the contribution that enters into the measure of the generating functional under FFBRST transformation helps to convert the BRST invariant effective action into its original gauge non-invariant version to ensures that the physical contents of these two effective actions are identical. The recent works \[51–61\], indeed provides much insight intothe way of approach towards our recent attempt. It reminds the work of Falck and Kramer \[62\], where they explicitly showed that physical content of chiral Schwinger model \[13\] remains identical both in the usual gauge non invariant action and the gauge symmetric action of the extended phase space. But it has to be kept in mind that in that situation the symmetry that was handled was the local gauge symmetry.

The paper is organized in the following manner. In Sec. II, we have given a brief introduction of the model. Sec. III is devoted to the BRST invariant reformulation of the model. In Sec. IV, FFBRST and anti-FFBRST formulation is applied to this BRST invariant effective action to get back the original gauge non-invariant form of the action through the incredible service of the field dependent parameter of FFBRST and anti-FFBRST

II. BRIEF REVIEW OF THE MODEL

The model where we find the mixing of both vector and axial vector interaction with different weight is given by the following generating functional

\[
Z(A) = \int d\psi d\bar{\psi} \exp[i \int d^2 x L_F],
\]

with \( L_F = \bar{\psi} \gamma^\mu (\partial_\mu + eA_\mu (1 - r\gamma_5)) \psi \). The integration over the fermionic degrees of freedom leads to a determinant and if that fermionic determinant is expressed in terms of auxiliary scalar field \( \phi \), we get

\[
Z(A) = \int d\phi \exp[i \int d^2 x L_B],
\]

where \( L_B = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + eA^\mu (\partial_\mu + r\partial_\mu) \phi + \frac{1}{2} ae^2 A_\mu A^\mu \). Here \( a \) is the regularization ambiguity emerged out during the process of regularization to remove the divergence of the fermionic determinant. If we now introduce the kinetic term of the back ground electromagnetic field we will get the total lagrange density:

\[
L_B = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + eA^\mu (\epsilon_{\mu\nu} \partial^\nu + rg_{\mu\nu} \partial^\nu) \phi + \frac{1}{2} ae^2 A_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.
\]
The Euler-Lagrange equations for the fields describing the lagrangian density (3) are
\[ \partial^\mu F_{\mu\nu} = -ae^2 A_\nu - e(\epsilon_{\mu\nu\rho} \partial^\rho \phi + \rho_{\mu\nu} \partial^\rho \phi), \]  
and
\[ \Box \phi = -e(\rho_{\mu\nu} \partial^\nu + \epsilon_{\mu\nu} \partial^\rho) A^\mu. \]

It is known that the most general solution for \( A_\mu \) is
\[ A_\mu = \frac{1}{ae^2} [r \partial_\mu \phi + (a - r^2) \tilde{\partial}_\mu \phi + (1 + a - r^2) \tilde{\partial}_\mu h], \]
and the theoretical spectra are given by
\[ (\Box + m^2) \sigma = 0, \]  
\[ \Box h = 0, \]
where
\[ \sigma = \phi + h, \]
and \( m^2 \) is given by
\[ m^2 = \frac{e^2 a(1 + a - r^2)}{(a - r^2)}. \]

So the physical subspace of the model is constituted with a massive boson with square of the mass \( m^2 = \frac{e^2 a(1 + a - r^2)}{(a - r^2)} \) and a massless boson. In short, this the physical content of the model.

**III. BRST INVARIANT REFORMULATION OF GVSM USING BFV FORMALISM**

To make the paper self contained let us start with the brief introduction of the BFV formalism. Consider a phase space of canonical variables \( q_i, p_i \) (\( i = 1, 2, \ldots, n \)) in terms of which the canonical Hamiltonian is \( H_c(q_i, p_i) \) and the constraints \( w_j(q_i, p_i) \) are embedded there in. The algebra between the constraints themselves and with the canonical hamiltonian respectively are
\[ [w_a, w_b] = iw_c U_{ab}^c, \]  
\[ [H_c, w_a] = iw_b V_{a}^b, \]  
where \( U_{ab}^c \) and \( V_{a}^b \) are the structure coefficients. To extract out the physical degrees of freedom, \( N \) number of additional conditions \( \phi^a = 0 \) are needed to be imposed. The constraints \( \phi^a = 0 \) and \( w_a = 0 \) together with the Hamiltonian equations may be obtained from the action
\[ S = \int [p_i q^i - H(p_i, q^i) - \lambda^a w_a + \pi^a \phi^a] dt, \]
where \( \lambda^a \) and \( \pi_a \) are lagrange multiplier having Poisson’s bracket \([\lambda^a, \pi^a] = i\delta^a_b\) and the gauge fixing conditions contain \( \lambda^a \) in the form \( \phi^a = \lambda^a + \chi^a \). In order to make the equivalence with the initial theory, we may introduce two sets of canonically conjugate anti-commuting ghost coordinates and momenta \((C^i, \bar{P}^i)\) and \((P^a, \bar{C}^a)\) having the algebra \([C^i, \bar{P}^i] = i\delta(x - y)\) and \([P^a, \bar{C}^i] = i\delta(x - y)\). The quantum theory, therefore, can be given by the generating functional
\[ Z_G = \int dq^i dp_1 d\lambda^a d\pi_a dC^a d\bar{P}_a dP^a d\bar{C}^a e^{i S_G}, \]
where the action \( S_G \) is
\[ S_G = \int [p_i q^i + \bar{P}^i C_i + \bar{C}^a P_a - H_m + \lambda^a \pi_a + i [Q, G]] dt. \]
Here $H_m$ is usually known as the minimal hamiltonian, $Q$ is the BRST charge and $G$ is the gauge fixing function and these are defined as follows

$$H_m = H_c + \bar{P}_a V^a_b C^b,$$

$$Q = C^a \omega_a - \frac{1}{2} C^b C^c U_{cb} \bar{P}_a + P^a \pi_a,$$

$$G = \bar{C}_a \chi_a + \bar{P}_a \chi_a.$$  

Let us now proceed to apply the above formalism to the model considered here for BRST invariant reformulation. The bosonized lagrangian density for this theory is

$$L_B = \frac{1}{2} (\dot{\phi}^2 - \phi''^2) + e(A_0 \phi' - A_1 \dot{\phi}) + er(A_0 \dot{\phi} - A_1 \phi')$$

$$+ \frac{e^2}{2} a(A_0^2 - A_1^2) + \frac{1}{2} (A_1 - A_0)^2.$$  

We are now in a state to proceed towards the BRST invariant reformulation of the lagrangian given in (18). In order to proceed to that end, we need to know the constraint structure of the theory described by the lagrangian (18). The momentum corresponding to the fields $\phi, A_0$ and $A_1$ respectively are

$$\pi_\phi = \dot{\phi} - eA_1 + erA_0,$$

$$\pi_0 = 0,$$

$$\pi_1 = \dot{A}_1 - A'_0.$$  

The equation $\omega_1 = \pi_0 = 0$, is identified as the primary constraint of the theory. By the Legendre transformation we obtain the following canonical Hamiltonian:

$$H_c = \int dx [\pi_\phi \dot{\phi} + \pi_1 \dot{A}_1 - \frac{1}{2} (\dot{\phi}^2 - \phi''^2) + e(A_0 \phi' - A_1 \dot{\phi})$$

$$+ er(A_0 \dot{\phi} - A_1 \phi') + \frac{e^2}{2} a(A_0^2 - A_1^2) + \frac{1}{2} (A_1 - A_0)^2].$$  

After a little algebra we find that (22) reduces to

$$H_c = \frac{1}{2} (\pi_\phi^2 + \pi_1^2 + \phi''^2) + \pi_1 A'_0 + er \pi_\phi (A_1 - rA_0)$$

$$- \frac{e^2}{2} a(A_0^2 - A_1^2) + \frac{1}{2} e^2 (A_1 - rA_0)^2 + e \phi' (rA_1 - A_0).$$  

The consistency of the primary constraint with respect to the time evolution leads to the secondary constraint

$$\omega_2 = [\pi_0, H_c],$$

$$= \pi_1' + e^2 (a - r^2) A_0 + e^2 r A_1 + e (r \pi_\phi + \phi') \approx 0.$$  

So the constraints that are embedded within the phase space of the theory are

$$\omega_1 = \pi_0 \approx 0,$$

$$\omega_2 = \pi_1' + e^2 (a - r^2) A_0 + e^2 r A_1 + e (r \pi_\phi + \phi') \approx 0.$$  

Therefore, the effective Hamiltonian in this situation reads

$$H_{eff} = H_c + u \omega_1 + v \omega_2.$$
where \( u \) and are two Lagrange multipliers. The preservation of \( \omega_2 \) with respect to the Hamiltonian determines the velocity \( u \) as follows.

\[
\dot{\omega}_2 = [\omega_2, H_{\text{eff}}] = e^2(a - r^2)u - e^2(a - r^2)A'_1 + e^2r\pi_1 = 0.
\]

Equation (28) now gives

\[
u = A'_1 - \frac{r}{(a - r^2)}\pi_1.
\]

Therefore, substituting \( u \) in (27) we get

\[
H_{\text{eff}} = \frac{1}{2}(\pi^2_1 + \pi'^2_\phi + \phi'^2) + \frac{e^2}{2}a(A^2_0 - A^2_1) + \frac{1}{2}e^2(A_1 - rA_0)^2 + e\phi'(rA_1 - A_0) + \pi_\phi(A_1 - rA_0) + \pi_0(A'_1 - \frac{r}{(a - r^2)}\pi_1).
\]

The closures of the constraints with respect to the Hamiltonian are

\[
\dot{\omega}_1 = [\omega_1, H_{\text{eff}}] = \omega_2,
\]

\[
\dot{\omega}_2 = [\omega_2, H_{\text{eff}}] = \omega''_1 - \frac{e^2r^2}{(a - r^2)}\omega_1.
\]

For BRST invariant reformulation the system with second class constraints (31) and (32) are needed to be converted into a first class set. In this respect, we introduce the auxiliary field \( \theta \) and \( \pi_\theta \). This set of auxiliary fields satisfy the following canonical relation.

\[
[\theta(x), \pi_\theta(y)] = i\delta(x - y).
\]

The auxiliary fields are known as BF fields. With some suitable linear combinations of BF fields the second class constraints get convert into first class constraints in the following way.

\[
\bar{\omega}_1 = \omega_1 + e(a - r^2)\theta,
\]

\[
\bar{\omega}_2 = \omega_2 + e\pi_\theta.
\]

For preservation of the constraints (34) and (35), \( \bar{\omega}_1 \) and \( \bar{\omega}_2 \) need to satisfy the same closures (31) and (32) as satisfied by \( \omega_1 \) and \( \omega_2 \):

\[
[H_c, \bar{\omega}_1] = \bar{\omega}_2,
\]

\[
[H_c, \bar{\omega}_2] = \bar{\omega}''_1 - \frac{e^2r^2}{a - r^2}\bar{\omega}_1.
\]

We may expect to get first class Hamiltonian by appropriate insertion of BF fields within the Hamiltonian (30). A little algebra shows that the first class hamiltonian reads

\[
\bar{H} = H_R + H_{\text{BF}},
\]

where \( H_{\text{BF}} \) for this theory is found out to be

\[
H_{\text{BF}} = \frac{1}{2(a - r^2)}\pi'^2_\theta + \frac{1}{2}(a - r^2)\theta'^2 + \frac{1}{2}e^2r^2\theta'^2.
\]

We are now in a position to introduce the two pairs of ghost and anti-ghost fields \((C^i, \bar{P}_i)\) and \((P^i, \bar{C}_i)\). We also need a pair of multiplier fields \((N_i, B_i)\). The fields satisfy the following canonical Poisson’s Bracket
\[ [C^i, \bar{P}_i] = [P_i, \tilde{C}_i] = [N^i, B_j] = i\delta^i_j \delta(x - y). \]

From the definition (15), we can write the BRST invariant Hamiltonian for the theory under the present situation is

\[ H_m = H_{\text{eff}} + H_{\text{BF}} + \int [Q, G] dx + \bar{P}_a V^a_b C^b. \] (40)

Here BRST charge \( Q \) and the fermionic gauge fixing function \( G \) are defined by

\[ Q = \int dx (C^1 \bar{\omega}_1 + C^2 \bar{\omega}_2 + P^1 B_1 + P^2 B_2), \] (41)

\[ G = \int dx (\bar{C}_1 \chi^1 + \bar{C}_2 \chi^2 + \bar{P}_1 N^1 + \bar{P}_2 N^2). \] (42)

Right now we have to fix up the gauge condition which is very crucial for getting appropriate Wess-Zumino term. It is found that these two very conditions only meet our need successfully.

\[ \chi_1 = A_0, \] (43)

\[ \chi_2 = A'_1 + \frac{\alpha}{2} B_2. \] (44)

Let us now calculate the commutation relation in between BRST charge, and gauge fixing function

\[ [Q, G] = [B_i P^i + C^i \bar{\omega}_i, \bar{C}_j \chi^j + \bar{P}_j N^j] \]

\[ = B_1 \chi^1 + B_2 \chi^2 - P^1 B_1 - P^2 B_2 - C^1 \bar{C}_1 + C^2 \bar{C}_2'' + \bar{\omega}_1 N^1 + \bar{\omega}_2 N^2. \] (45)

Using equation (40), BRST invariant Hamiltonian is obtained which is given by

\[ H_m = H_R + H_{BF} + \bar{P}_2 C_1 + \bar{P}_1'' C_2 - \frac{e^2 r^2}{a - r^2} \bar{P}_1 C_2 + \int [Q, G] dx. \] (46)

The generating functional for this system can now be written down as

\[ Z = \int [D\mu]\exp^{iS}, \] (47)

where \([D\mu]\) is the Liouville measure in the extended phase space.

\[ [D\mu] = d\phi d\pi_0 dA_0 d\pi_0 dA_1 d\pi_1 d\theta d\pi_0 \prod_{i=1}^{2} dN^i dB_i dC^i \] d\bar{C}_i dP^i d\bar{P}_i. \] (48)

The action \( S \) in equation (47) reads

\[ S = \int d^2 x [\pi_0 \dot{\phi} + \pi_0 \dot{A}_0 + \pi_1 \dot{A}_1 + \dot{\theta} \pi_0 + \dot{N}^1 B_1 \]

\[ + \dot{N}^2 B_2 + \dot{C}_1 \bar{B}^1 + \dot{C}_2 \bar{B}^2 + \bar{P}_1 C_1 + \bar{P}_2 C_2 - H_m]. \] (49)

The explicit form of \( H_m \) lying in equation (47) is

\[ H_m = H_R + H_{BF} - P^1 \bar{P}_1 - P^2 \bar{P}_2 - C^1 \bar{C}_1 + C^2 \bar{C}_2'' \]

\[ + \bar{\Omega}_1 N^1 + \bar{\Omega}_2 N^2 + \bar{P}_2 C_1 + \bar{P}_1'' C_2 \]

\[ - \frac{e^2 r^2}{a - r^2} \bar{P}_1 C_2 + B_1 A_0 + B_2 (A'_1 + \frac{\alpha}{2} B_2). \] (50)
To get the effective action in the desired form it is necessary to integrate out the fields $B_1, N^1, \pi_0, \pi_1, \pi_\phi, \bar{P}_1, C_1$ and $C_1$. After integrating out of these fields we obtain a simplified form of the generating functional which is constituted with the following action:

$$S = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \epsilon \epsilon_{\mu \nu} A^\mu \partial^\nu \phi + e r g_{\mu \nu} A^\mu \partial^\nu \phi + \frac{1}{2} \epsilon e^2 A_\mu A^\mu - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}$$

$$+ \frac{1}{2} (a - r^2) \partial_\mu \theta \partial^\mu \theta + e (a - r^2) g_{\mu \nu} A^\mu \partial^\nu \theta - e r \epsilon_{\mu \nu} A^\nu \partial^\mu \theta$$

$$+ \partial^\mu C \partial_\mu \bar{C} + \frac{\alpha}{2} B^2 + B \partial_\mu A^\mu.$$

(51)

We have to choose $C^2 = C, N^2 = A_0, B_2 = B$ to reach equation (51) from equation (47). It is interesting to see that the action (51) is invariant under the transformation

$$\delta \phi = e r \lambda C, \delta N_0 = - \lambda \bar{C}, \delta A_1 = - \lambda C^\alpha, \delta \theta = - e r C, \delta C = 0, \delta \bar{C} = - \lambda B.$$

(52)

These are the very BRST transformations corresponding to the fields that describe the system under consideration. It would be of worth to reiterate that the choice of gauge fixing is very crucial here. The choice of gauge fixing which we have considered here renders a great service to obtain the appropriate Wess-Zumino term. The Wess-Zumino term $L_{wz}$ can easily be identified as

$$L_{wz} = \frac{1}{2} (a - r^2) \partial_\mu \theta \partial^\mu \theta + e (a - r^2) g_{\mu \nu} A^\mu \partial^\nu \theta - e r \epsilon_{\mu \nu} A^\nu \partial^\mu \theta.$$

(53)

This completes the BRST invariant reformulation of the theory. we will now proceed to the application of FFBRST and anti-FFBRST formalism on the same model in the next section.

**IV. APPLICATION OF FFBRST AND ANTI-FFBRST FORMALISM IN THE GVSM**

An ingenious attempt was made in [7] to generalize the well celebrated BRST formulation. It was shown there that even making the BRST transformation field dependent the nilpotency can be protected and it is equally effective for anti BRST formalism. Under finite field dependent transformation the path integral measure acquires a nontrivial change that though leads to a different effective theory, the physical contents of the theory remains unaffected. This generalization however is advantageous since that renders several important services. One of such advantage is that it helps to correlate the different gauge fixed versions of a particular theory [7]. The ability to relate a theory embedded with a set of first class constraint to an equivalent theory embedded with a set of second class constraint through appropriate choice of gauge fixing parameter is also an interesting extension of the field dependent BRST (FFBRST) formalism. [51-61].

An illustration related to the calculation of jacobian for this field dependent transformation is available in [7]. To make this paper a self contained one let us now proceed with the brief introduction concerning how FFBRST transformation brings a non trivial change in the interactional measure of the generating functional and how this change adds a contribution to the effective action.

If the fields that describe a physically sensible theory are function of a parameter $\eta$ such that $\phi(x, \eta) = 0$ is $\phi(x)$ and $\phi(x, \eta = 1) = \phi(x)$ the infinitesimal BRST transformation is given by

$$\frac{d}{d\eta} \phi(x) = \delta_{BRS}[\phi(\eta)] \Theta'[\phi(\eta)].$$

(54)

The finite field dependence can be obtained through the integration over the infinitesimal transformation within the limit $\eta = 0$ to $\eta = 1$

$$\tilde{\phi}(x) = \phi(x, \eta = 1) = \phi(x, \eta = 0) + \delta_{BRS}[\phi(x)] \Theta[\phi(x)],$$

(55)

where

$$\Theta[\phi(x)] = \int_0^1 d\eta' \Theta'[\phi(x, \eta')].$$

(56)

It should be mentioned here that the condition $\Theta^2 = 0$ is to be maintained in order to protect nilpotency. The jacobian for the transformation can be evaluated from the field dependent function $\Theta[\phi(x)]$ by

$$\prod d\phi = J(\eta) \prod d\phi(\eta) = J(\eta + d\eta) \prod d\phi(\eta + d\eta).$$

(57)
The infinitesimal nature of transformation from $\phi(\eta)$ to $\phi(\eta + d\eta)$ leads to the following relations of the jacobian $J(\eta)$:

$$\frac{J(\eta)}{J(\eta + d\eta)} = \sum \pm \frac{\delta \phi(x, \eta + d\eta)}{\delta \phi}.$$  \hspace{1cm} (58)

Here $\prod$ and $\sum$ represents the product and sum over all the fields involved within the theory respectively. In equation (58) $(\pm)$ signs are to be used for boson and fermion field respectively. Equation (58) renders the following in infinitesimal change in the jacobian $J(\eta)$.

$$\frac{1}{J} \frac{dJ}{d\eta} = - \int d^2x [\pm \delta \phi(x, \eta) \frac{\partial \Theta'}{\partial \phi}].$$  \hspace{1cm} (59)

The incredible characteristic of this extension is that within the functional integration $J(\eta)$ can be expressed as

$$J(\eta) = e^{iS_c(x, \eta)},$$  \hspace{1cm} (60)

if and only if the condition

$$\int \prod \delta \phi(x) [\frac{1}{J} \frac{dJ}{d\eta} - i \frac{dS_c(x, \eta)}{d\eta}] e^{i(S_{eff} + S_C)} = 0,$$  \hspace{1cm} (61)

is maintained within the phase space of the theory. The role of $\Theta$ though surprising, nevertheless plays a very crucial as well as intriguing role since the appropriate choice $\Theta$ leads to another equivalent effective action corresponding to the starting theory which is given by the generating functional.

$$\bar{Z} = \int \prod \delta \phi(x) \exp[i(S_{eff} + S_C)].$$  \hspace{1cm} (62)

It indeed keeps the physical contents of the theory unchanged.

To see the transmutation between the gauge invariant and gauge non-invariant effective theory of GVSM we begin our analysis starting from the BRST invariant effective action of the theory which reads

$$S_{eff} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + e \epsilon_{\mu \nu} A^\mu \partial^\nu \phi + e r g_{\mu \nu} A^\mu \partial^\nu \phi + \frac{1}{2} \alpha e^2 A_\mu A^\mu - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}$$

$$+ \frac{1}{2} (a - r^2) \partial_\mu \theta \partial^\mu \theta + e (a - r^2) g_{\mu \nu} A^\mu \partial^\nu \theta - e r \epsilon_{\mu \nu} A^\mu \partial^\nu \theta$$

$$+ \partial^\mu C \partial_\mu \bar{C} + \frac{\alpha}{2} B^2 + B \partial_\mu A^\mu$$

$$= S_{ORI} + S_{WZ} + S_{GHOST} + S_{GF}.$$  \hspace{1cm} (63)

The infinitesimal BRST transformations of the fields under which the above action (63) is found to remain invariant are

$$\delta A_\mu = - \frac{1}{e} \lambda \partial_\mu C, \delta \phi = r \lambda C, \delta \theta = \lambda C, \delta C = 0, \delta \bar{C} = - B, \delta B = 0,$$  \hspace{1cm} (64)

and the FFBRST transformations of those fields describing the theory under consideration are

$$\delta A_\mu = - \frac{1}{e} \partial_\mu C \Theta, \delta \phi = r C \Theta, \delta \theta = - \Theta C, \delta C = 0, \delta \bar{C} = - B \Theta, \delta B = 0,$$  \hspace{1cm} (65)

where $\Theta$ is an arbitrary finite field dependent function serving the role of transformation parameter corresponding to the FFBRST transformation. Our objective is to connect this BRST invariant effective action of the extended phase space to the original effective action of the usual physical phase space. To relate this we make a choice over the $\Theta$ in such a way that the change that would enter into the measure of the generating functional can be exploited to serve the desired purpose. To this end we define $\Theta$ as follows:

$$\Theta' = i \gamma \int d^2x [\bar{C} (\partial_\mu A^\mu + \frac{\alpha}{2} B)].$$  \hspace{1cm} (66)

Here $\gamma$ is an arbitrary parameter that would be fixed later. For the finite field dependent parameter the nontrivial infinitesimal change that would enter in the jacobian can be be computed using equation (59)

$$\frac{1}{J} \frac{dJ}{d\eta} = i \gamma \int d^2x [\delta \bar{C} \frac{d\Theta'}{dC} + \delta A_\mu \frac{d\Theta'}{dA_\mu} + \delta B \frac{d\Theta'}{dB}].$$  \hspace{1cm} (67)
\[
= i\gamma \int d^2x [B(\partial_\mu A^\mu + \frac{\alpha}{2} B) + \frac{\lambda}{\epsilon} \partial_\mu C \partial^\mu \tilde{C}] .
\]

The Euler-Lagrange equation of motion for the ghost field simplifies the above equation to the following form
\[
\frac{1}{J} \frac{dJ}{d\eta} = i\gamma \int d^2x [-B(\partial_\mu A^\mu + \frac{\alpha}{2} B)].
\]

We are now in a state to choose an ansatz for \( S_C \). The following ansatz for \( S_C \) suffices our need without violating any physical principle
\[
S_C = \int d^2x [\zeta_1(k)B^2 + \zeta_2(k)B\partial_\mu A^\mu].
\]

Here \( \zeta_1(\eta) \) and \( \zeta_2(\eta) \) are some function of the parameter \( \eta \). The differentiation of the action \( S_C \) with respect to \( \eta \) yields
\[
\frac{\partial S_C}{\partial \eta} = \int d^2x (B^2 \zeta_1'(\eta) + B\partial_\mu A^\mu \zeta_2'(\eta)).
\]

The over prime denotes here the differentiation with respect to the parameter \( \eta \). The contribution that enters into the measure of the generating function through the Jacobian under FFBRST transformation can be written down in the form of \( e^{i S_C} \), provided the following very equation
\[
\int d^2x e^{i(S_{eff} + S_C)} [i(\zeta_1' - \frac{\gamma \alpha}{2} B^2) + iB(\partial_\mu A^\mu(\zeta_2' - \gamma))] = 0,
\]

is satisfied. It fixes \( \zeta_1 \) and \( \zeta_2 \) and make it expressible in terms of \( \gamma \) and \( \alpha \):
\[
\zeta_1 = \frac{\gamma \alpha}{2} \eta \\
\zeta_2 = \gamma \eta.
\]

Setting \( \eta = 1 \), we get
\[
S_C = \int d^2x [\frac{\gamma \alpha}{2} B^2 + \gamma B\partial_\mu A^\mu],
\]

and for \( \gamma = -1 \), \( S_C \) turns into
\[
S_C = \int d^2x [-\frac{\alpha}{2} B^2 + B\partial_\mu A^\mu].
\]

So through the exploitation of the change entered in to the measure of the generating functional through FFBRST transformation enables us to eliminate the gauge fixing part \( S_{GF} \) from the \( S_{eff} \) for the above setting of \( \gamma \) and \( \eta \). It is the first step to proceed towards the effective action defined in the usual phase space. So the remaining part in the \( S_{eff} \) are
\[
S_{ST} = S_{ORI} + S_{WZ} + S_{GHOST}.
\]

Precisely, the parameter of FFBRST transformation (being field dependent) renders here a great job which is the elimination of the gauge fixing term through the contribution entered into the path integral measure of generating functional due to the finite field dependent nature of the transformation parameter of the FFBRST transformation. Our next task is to eliminate the ghost and the Wess-Zumino part one by one. The elimination of the ghost part is trivial because under integration the contribution that evolve out from this part can be absorbed within the normalization. However, the elimination of the Wess-Zumino term is not so trivial. It certainly needs the integrating out of the Wess-Zumino field but one has to keep it in mind that the theory has now converted into gauge invariant one and the constraints that embedded in its phase space are first class in nature. So proper gauge fixing is needed to land onto the theory of the usual phase space \[62\]. This can be done in different ways. In \[62\], the authors did not use the path integral approach. However, since in the full body of the paper path integral approach is followed we use gauge fixing with the path integral formulation to which we now turn.
From the Hamiltonian analysis which is available in [11], it is known that the phase space of the theory contains two first class constraint. From [11], we find that the the original action along with the Wess-Zumino part of the $S_{eff}$ [33], leads to the following Hamiltonian.

$$H_{ce} = \int dx \left[ \frac{1}{2} \left( \pi_\phi^2 + \pi_1^2 + \phi'^2 \right) + \pi_1 A_0' + \frac{e^2 a (1 + a - r^2)}{2(a - r^2)} A_1^2 + e \phi' (r A_1 - A_0) \right.$$ 

$$+ e \phi (A_1 - r A_0) + \frac{1}{2} (a - r^2) \theta'^2 + e \theta' ((a - r^2) A_1 + A_0) + \frac{1}{2(a - r^2)} \pi_0^2$$

$$+ \frac{e}{(a - r^2)} (r A_1 + (a - r^2) A_0) \pi_\theta \right], \quad (77)$$

and there embeds the following two first class constraints in the phase space of the theory:

$$\tilde{\omega}_1 = \pi_0 \approx 0, \quad (78)$$

$$\tilde{\omega}_2 = \pi_1' + e (r \pi_\phi + \phi') + e (\pi_\theta - r \theta') \approx 0. \quad (79)$$

Therefore, two gauge fixing conditions are needed at this stage to get back the gauge non-invariant theory of the usual phase space. These two gauge fixing conditions that are chosen here are

$$\tilde{\omega}_3 = \theta' \approx 0, \quad (80)$$

$$\tilde{\omega}_4 = \pi_\theta + e ((a - r^2) A_0 + r A_1) \approx 0. \quad (81)$$

With these inputs the generating functional now can be written down as

$$Z = \int |D\mu| \det[\tilde{\omega}_i, \tilde{\omega}_j] i^{\frac{1}{4}} e^i d^2 x [\pi_1 A_1 + \pi_0 A_0 + \pi_\phi + \pi_\theta - H_{ce}] \delta(\tilde{\omega}_1) (\tilde{\omega}_2) \delta(\tilde{\omega}_3) \delta(\tilde{\omega}_4), \quad (82)$$

where the Liouville measure $|D\mu| = d\pi_\phi d\phi d\pi_1 dA_1 d\pi_0 dA_0 d\pi_\theta d\theta$, and $i$ and $j$ runs from 1 to 4. After integrating out of the field $\theta$ and $\pi_\theta$ we find that equation (82) turns into

$$Z = N \int |d\mu| e^i d^2 x [\pi_1 A_1 + \pi_0 A_0 + \pi_\phi + \pi_\theta - H_{ce}] \delta(\pi_0) \delta(\pi_1' + e (r \pi_\phi + \phi') + e^2 ((a - r^2) A_0 + r A_1), \quad (83)$$

where $|d\mu| = d\pi_\phi d\phi d\pi_1 dA_1 d\pi_0 dA_0$ and $N$ is a normalization constant having no significant physical importance and $H_{ce}$ is

$$\tilde{H}_{ce} = \frac{1}{2} (\pi_1^2 + \phi'^2) + \pi_1 A_0' + \frac{1}{2} a e^2 (A_1^2 - A_0)^2 + e \phi' (r A_1 - A_0) + \frac{1}{2} \pi_0^2 + e (A_1 - r A_0)^2 \right]. \quad (84)$$

Now after integrating out of the the field $\pi_\phi$, $\pi_1$ and $\pi_0$ we land on to the required result

$$Z = \int dA_1 dA_0 d\phi e^{i S_{BRST}}. \quad (85)$$

Like BRST symmetry anti-BRST is also a symmetry of the effective action of a given theory and like the BRST transformations anti-BRST transformations do generate from a nilpotent charge. In the anti-BRST formulation the role of ghost and anti ghost fields interchanges. In addition to that there may be change in the coefficient depending upon the system. Therefore, study with anti-FFBRST is equally important like FFBRST. So we are intended to examine whether the anti-FFBRST formalism can be brought into the same service as it has been served by the FFBRST formalism. The ant-BRST transformations for the fields describing the theory are given by

$$\delta A_\mu = -\frac{1}{e} \lambda \partial_\mu \tilde{C}, \delta \phi = r \lambda \tilde{C}, \delta \theta = -\lambda \tilde{C}, \delta \tilde{C} = B \lambda \delta \tilde{C} = 0, \delta B = 0, \quad (86)$$

and the corresponding anti-FFBRST transformations of the fields are

$$\delta A_\mu = -\frac{1}{e} \partial_\mu \tilde{C} \Theta_A, \delta \phi = r \tilde{C} \Theta_A, \delta \theta = -\tilde{C} \Theta_A, \delta \tilde{C} = 0, \delta C = -B \Theta_A, \delta B = 0 \quad (87)$$
where $\Theta_A$ is an arbitrary finite field dependent function serving the role of transformation parameter corresponding to the anti-FFBRST transformation. Our objective is the same as we have done for the FFBRST. For that purpose here also we need to choose a $\Theta_A$ in such a way that the change that would enter into the generating functional can be exploited to serve the same purpose as it has been found to serve in the earlier situation. In an analogous manner let us define $\Theta_A$ as

$$\Theta_A' = i\tilde{\gamma} \int d^2 x[C(\partial_\mu A^\mu + \frac{\beta}{2} B)], \quad (88)$$

here $\tilde{\gamma}$ is an arbitrary parameter that would be fixed later like the previous situation. For anti-FFBRST transformation also the nontrivial infinitesimal change in the jacobian that would enter can be calculated using equation (59).

$$\frac{1}{J} \frac{dJ}{d\eta} = i\tilde{\gamma} \int d^2 x[B(\partial_\mu A^\mu + \frac{\beta}{2} B) + \lambda \epsilon^{\mu\nu} \partial_\mu C \partial_\nu \bar{C}]. \quad (89)$$

By the use of Euler-lagrange equation of motion for the anti ghost field the above equation reduces to

$$\frac{1}{J} \frac{dJ}{d\eta} = i\tilde{\gamma} \int d^2 x[-B(\partial_\mu A^\mu + \frac{\beta}{2} B)]. \quad (90)$$

In order to express the above in the form of $e^{i\tilde{S}_C}$, the following ansatz can be chosen for $\tilde{S}_C$ without any loss of generic condition, and of course, without violating any physical principle:

$$\tilde{S}_C = \int d^2 x[\xi_1(k)B^2 + \xi_2(k)B\partial_\mu A^\mu]. \quad (91)$$

Here $\xi_1(\eta)$ and $\xi_2(\eta)$ are some function of the parameter $\eta$. If we now take the derivative of the action $\tilde{S}_C$ with respect to $\eta$ we get

$$\frac{\partial S_C}{\partial \eta} = \int d^2 x[B^2 \xi'_1(\eta) + B\partial_\mu A^\mu \xi'_2(\eta)]. \quad (92)$$

The symbol over prime denotes here the differentiation with respect to the parameter $\eta$ as usual. The contribution that the path integral measure of the generating functional acquires under anti-FFBRST transformation can be written down in the form of $e^{i\tilde{S}_C}$, iff the following relation

$$\int d^2 x e^{i(S_{eff} + \tilde{S}_C)}[i(\xi'_1 - \frac{\gamma}{2} \beta B^2) + iB(\partial_\mu A^\mu (\xi'_2 - \gamma))] = 0, \quad (93)$$

holds. The equation (93) fixes $\xi_1$ and $\xi_2$ so one can express these in terms of $\gamma$ and $\beta$:

$$\xi_1 = \frac{\tilde{\gamma} \beta}{2}\eta, \quad \xi_2 = \tilde{\gamma}\eta. \quad (94)$$

Thus setting the parameter $\eta = 1$ we get

$$\tilde{S}_C = \int d^2 x[\frac{\tilde{\gamma} \beta}{2} B^2 + \tilde{\gamma} B\partial_\mu A^\mu], \quad (95)$$

and finally putting $\gamma = -1$, we get the appropriate $\tilde{S}_C$ in his situation:

$$\tilde{S}_C = \int d^2 x[-(\frac{\beta}{2} B^2 + B\partial_\mu A^\mu)]. \quad (96)$$

Therefore, we find that the exploitation of the change entered into path integral measure of the generating functional due to the anti-FFBRST transformation with the above choice of $\gamma$ and $\eta$ enables us to eliminate the gauge fixing term from our starting $S_{eff}$ and it now reduces to

$$S_{eff} = S_{ORI} + S_{WZ} + S_{GHOST}. \quad (97)$$

So the first step to reach towards the effective action defined in the usual phase space is successfully made in case of anti-FFBRST transformation too. Note that for this system the calculation may look similar to the FFBRT.
However, for the other situations this task may not be so straightforward. For instance, the BRST and ant-BRST transformation for the gauge fixed Yang-Mills theory differs in coefficient \[7\], and as a result the calculations in this regard takes a different shape.

After the first step, the task that is yet to be done is to make the \( S_{\text{eff}} \) part free from ghost as well as the Wess-Zumino part. The elimination of the ghost part is trivial like the previous case since under integration of the anti-ghost field the contribution that evolve out can again be absorbed within the normalization. However, the elimination of the Wess-Zumino term is not trivial but it is identical to the previous case as it has already been made for the FFBRST transformation. Explicit calculation in this situation therefore does not carry any new information. So, it is not shown here.

V. CONCLUSION

The present work is an application of the well celebrated BRST formulation and its incredible generalization, so called FFBRST and anti-FFBRST formulation on a lower dimensional model where vector and axial vector coupling gets mixed up with different weight \[8, 9\]. Initially, we have attempted to construct the BRST invariant reformulation using the improved version of BFV formulation due to Fujiwara, Igarashi and Kubo \[44\]. This improved version has helped us to obtain a BRST invariant reformulation along with the emergence of appropriate Wess-Zumino term. We would like to emphasize the fact that the role of gauge fixing is very crucial to get the appropriate Wess-Zumino term.

Application of FFBRST and anti-FFBRST are made for another interesting as well as important purpose. In fact, it has been used here to make the equivalence between the physical content of the model in the usual and in the extended phase space. Here extended phase space implies the presence of not only the Wess-Zumino field, but also the presence of ghost and the auxiliary \( B \) fields too. It has been found that both the FFBRST and anti-FFBRS T formulation have successfully render their great services to show the equivalence.

In both the cases FFBRST and ant-FFBRST it has been found that the gauge fixing part, i.e., the part of the effective action involving the auxiliary \( B \) gets eliminated by the contribution entered into the effective action through the acquired contribution of the measure of the generating functional under FFBRST and anti-FFBRST transformations respectively. To eliminate rest of the part we have adopted here the formalism developed by Falck and Kramer \[62\]. So, the joint action of the two formalisms developed in \[7\] and \[62\] have done their novel services to show the equivalence. We can conclude that the joint action of these two formalisms would be instrumental to show the equivalence between the different effective actions of any field theoretical model if these two are employed in appropriate manner.

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