CLASS FIELD TOWERS AND MINIMAL MODELS

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Abstract. We use the notion of an Etesi $C^*$-algebra to prove that the real class field towers are always finite.

1. Introduction

Let $k$ be a number field and let $\mathcal{H}(k)$ be the Hilbert class field of $k$, i.e. the maximal abelian unramified extension of $k$. The class field tower is a sequence of the field extensions:

$$k \subseteq \mathcal{H}(k) \subseteq \mathcal{H}^2(k) \subseteq \mathcal{H}^3(k) \subseteq \ldots,$$  \hspace{1cm} (1.1)

where $\mathcal{H}^2(k) = \mathcal{H}(\mathcal{H}(k))$, $\mathcal{H}^3(k) = \mathcal{H}(\mathcal{H}^2(k))$, etc. Whether there exists an integer $m \geq 1$ such that $\mathcal{H}^m(k) \cong \mathcal{H}^{m+1}(k) \cong \ldots$ is known as the class field tower problem [Furtwängler 1916] [4]. They say that the class field tower is finite if $m < \infty$ and infinite otherwise. It is easy to see, that the tower is finite if and only if the ring of integers of the field $\mathcal{H}^m(k)$ is the principal ideal domain, i.e. has the class number 1.

The Golod-Shafarevich Theorem says that (1.1) can be infinite for some fields $k$ [Golod & Shafarevich 1964] [5]. This result solves in negative the class field tower problem. On the other hand, many fields $k$ have finite class field towers. The sorting of $k$ by the finite and infinite towers (1.1) is a difficult open problem.

In this note we study the real class field towers, i.e. when all fields $\mathcal{H}^i(k)$ in (1.1) are real. It is shown that such towers are always finite, see corollary 1.2. To outline the idea, we prove that the blow-up map of an algebraic surface induces a Hilbert class field extension of a field coming from the $K_0$-group of the corresponding Etesi $C^*$-algebra [6, Section 7.5], see theorem 1.1. The Castelnuovo Theorem says that it takes a finite number of the blow-ups of an algebraic surface to get the minimal model, hence (1.1) is finite. To formalize our results, let us recall some definitions.

An algebraic surface $S$ is a variety of complex dimension 2. The rational map $\phi : S \dashrightarrow S'$ is called birational, if its inverse $\phi^{-1}$ is a rational map. A birational map $\phi : S \dashrightarrow S'$ is a blow-up, if it is defined everywhere except for a point $p \in S$ and a rational curve $C \subset S'$, such that $\phi^{-1}(C) = p$. Each birational map is composition of a finite number of the blow-ups. The surface $S$ is called a minimal model, if any birational map $S \dashrightarrow S'$ is an isomorphism. The Castelnuovo Theorem says that $S$ is a minimal model if and only if $S$ does not contain rational curves $C$ with the self-intersection index $-1$. In particular, the minimal model is obtained from $S$ by a finite number of the blow-ups along $C$.

Recall that $S$ can be identified with a smooth 4-dimensional manifold. We denote by $\text{Diff}(S)$ a group of the orientation-preserving diffeomorphisms of $S$ and...
by \( \text{Diff}_0(S) \) the connected component of \( \text{Diff}(S) \) containing the identity. The Etesi \( C^* \)-algebra \( \mathbb{E}_S \) is a group \( C^* \)-algebra of the locally compact group \( G := \text{Diff}(S)/\text{Diff}_0(S) \) [6, Definition 1.1]. Since \( G \) is a countable, discrete, amenable group acting on \( S \) and the action admits a faithful \( G \)-invariant Borel probability measure (e.g. by taking the Lebesgue measure of the orbit space \( S/G \)), we conclude that \( \mathbb{E}_S \cong C_0(S) \rtimes G \) embeds into a simple unital AF-algebra [Schafhauser 2020] [7, Theorem C]. Moreover, such an AF-algebra is stationary [6, Lemma 7.5.3] depending on a constant integer matrix \( A \in GL(n, \mathbb{Z}) \) [Blackadar 1986] [2, Section 7.2]. Let \( \lambda_A > 1 \) be the Perron-Frobenius eigenvalue of \( A \) and let \( k = \mathbb{Q}(\lambda_A) \) be a real number field generated by the algebraic number \( \lambda_A \). Our main result can be formulated as follows.

**Theorem 1.1.** The birational map \( S \dasharrow S' \) is a blow-up if and only if \( k' \cong \mathcal{H}(k) \).

As explained above, theorem 1.1 can be used to study inclusions (1.1). Namely, in view of Castelnuovo’s theory of the minimal models, one gets the following application of theorem 1.1.

**Corollary 1.2.** The real class field towers are always finite.

**Remark 1.3.** The real class field towers and the class field towers over the real fields are not the same in general. The latter admit imaginary Hilbert class fields and can be infinite.

The article is organized as follows. The preliminary facts can be found in Section 2. Theorem 1.1 and corollary 1.2 are proved in Section 3.

2. **Preliminaries**

We briefly review the 4-dimensional topology, the Etesi \( C^* \)-algebras and the algebraic surfaces. We refer the reader to [Beauville 1996] [1], [Freedman & Quinn 1990] [3], and [6, Section 7.5] for a detailed account.

2.1. **Topology of 4-manifolds.** We denote by \( \mathcal{M} \) a compact 4-dimensional manifold. Unlike dimensions 2 and 3, the smooth structures are detached from the topology of \( \mathcal{M} \). Due to the works of Rokhlin, Freedman and Donaldson, it is known that \( \mathcal{M} \) can be non-smooth and if there exists a smooth structure, it need not be unique. In what follows, we assume \( \mathcal{M} \) to be a smooth 4-manifold endowed with the standard smooth structure, e.g. coming from a realization of \( \mathcal{M} \) as a complex algebraic surface \( S \).

Let \( \text{Diff}(\mathcal{M}) \) be a group of the orientation-preserving diffeomorphisms of \( \mathcal{M} \). Denote by \( \text{Diff}_0(\mathcal{M}) \) the connected component of \( \text{Diff}(\mathcal{M}) \) containing the identity. The group \( \text{Diff}(\mathcal{M})/\text{Diff}_0(\mathcal{M}) \) is discrete and therefore locally compact.

2.2. **Etesi \( C^* \)-algebras.**

2.2.1. **\( C^* \)-algebras.** The \( C^* \)-algebra is an algebra \( \mathcal{A} \) over \( \mathbb{C} \) with a norm \( a \mapsto ||a|| \) and an involution \( \{ a \mapsto a^* \mid a \in \mathcal{A} \} \) such that \( \mathcal{A} \) is complete with respect to the norm, and such that \( ||ab|| \leq ||a|| \cdot ||b|| \) and \( ||a^*a|| = ||a||^2 \) for every \( a, b \in \mathcal{A} \). Each commutative \( C^* \)-algebra is isomorphic to the algebra \( C_0(X) \) of continuous complex-valued functions on some locally compact Hausdorff space \( X \). Any other algebra \( \mathcal{A} \) can be thought of as a noncommutative topological space.

An AF-algebra (Approximately Finite-dimensional \( C^* \)-algebra) is the norm closure of an ascending sequence of finite dimensional \( C^* \)-algebras \( M_n \), where \( M_n \) is the
The $C^*$-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \cdots$, where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. If $\varphi_i = \text{Const}$, then the AF-algebra $\mathcal{A}$ is called stationary. In particular, such an AF-algebra is given by a constant positive integer matrix $A \in GL(n, \mathbb{Z})$.

2.2.2. K-theory. By $M_\infty(\mathcal{A})$ one understands the algebraic direct limit of the $C^*$-algebras $M_n(\mathcal{A})$ under the embeddings $a \mapsto \text{diag}(a, 0)$. The direct limit $M_\infty(\mathcal{A})$ can be thought of as the $C^*$-algebra of infinite-dimensional matrices whose entries are all zero except for a finite number of the non-zero entries taken from the $C^*$-algebra $\mathcal{A}$. Two projections $p, q \in M_\infty(\mathcal{A})$ are equivalent, if there exists an element $v \in M_\infty(\mathcal{A})$, such that $p = vv^*v$ and $q = vv^*$. The equivalence class of projection $p$ is denoted by $[p]$. We write $V(\mathcal{A})$ to denote all equivalence classes of projections in the $C^*$-algebra $M_\infty(\mathcal{A})$, i.e. $V(\mathcal{A}) := \{ [p] : p = p^* = p^2 \in M_\infty(\mathcal{A}) \}$. The set $V(\mathcal{A})$ has the natural structure of an abelian semi-group with the addition operation defined by the formula $[p] + [q] := \text{diag}(p, q) = [p' \oplus q']$, where $p' \sim p$, $q' \sim q$ and $p' \perp q'$. The identity of the semi-group $V(\mathcal{A})$ is given by $[0]$, where $0$ is the zero projection. By the $K_0$-group $K_0(\mathcal{A})$ of the unital $C^*$-algebra $\mathcal{A}$ one understands the Grothendieck group of the abelian semi-group $V(\mathcal{A})$, i.e. a completion of $V(\mathcal{A})$ by the formal elements $[p] - [q]$.

The canonical trace $\tau$ on the AF-algebra $\mathcal{A}$. induces a homomorphism $\tau_* : K_0(\mathcal{A}) \to \mathbb{R}$. We let $\mathfrak{m} := \tau_*(K_0(\mathcal{A})) \subset \mathbb{R}$. If $\mathcal{A}$ is the stationary AF-algebra given by a matrix $A \in GL(n, \mathbb{Z})$, then $\mathfrak{m}$ is a $\mathbb{Z}$-module in the number field $k = \mathbb{Q}(\lambda_A)$, where $\lambda_A > 1$ is the Perron-Frobenius eigenvalue of the matrix $A$. The endomorphism ring of $\mathfrak{m}$ is denoted by $\Lambda$ and the ideal class of $\mathfrak{m}$ is denoted by $[\mathfrak{m}]$. The $(\Lambda, [\mathfrak{m}], k)$ is called a Handelman triple of the stationary AF-algebra $\mathcal{A}$.

2.2.3. Etesi $C^*$-algebra. Let $\mathcal{M}$ be a smooth 4-dimensional manifold. The group $C^*$-algebra $\mathbb{E}_\mathcal{M}$ of the locally compact group $\text{Diff}(\mathcal{M})/\text{Diff}_0(\mathcal{M})$ is called the Etesi $C^*$-algebra of $\mathcal{M}$. Some properties of the $\mathbb{E}_\mathcal{M}$ are described in below.

**Theorem 2.1.** ([6, Section 7.5]) The following is true:

(i) the $\mathbb{E}_\mathcal{M}$ is a stationary AF-algebra;

(ii) the Handelman triple $(\Lambda, [\mathfrak{m}], k)$ is a topological invariant of $\mathcal{M}$.

**Remark 2.2.** The map $F$ acting by the formula:

$$\mathcal{M} \mapsto (\Lambda, [\mathfrak{m}], k)$$

is a covariant functor on the category of all 4-dimensional manifolds with values in a category of the Handelman triples. In particular, the number field $k$ is a topological invariant of $\mathcal{M}$. It is not hard to see, that (2.1) has an inverse $(\Lambda, [\mathfrak{m}], k) \mapsto \mathcal{M}$. Indeed, the group algebra $\mathbb{E}_\mathcal{M}$ can be recovered from the triple $(\Lambda, [\mathfrak{m}], k)$, while the manifold $\mathcal{M}$ is defined by the group $\text{Diff}(\mathcal{M})/\text{Diff}_0(\mathcal{M})$.

2.3. Algebraic surfaces. An algebraic surface is a variety $S$ of the complex dimension 2. One can identify the non-singular variety $S$ with a complex surface and therefore with a smooth 4-dimensional manifold $\mathcal{M}$. In what follows, we denote by $\mathbb{E}_S$ the Etesi $C^*$-algebra of $\mathcal{M}$ corresponding to the surface $S$.

The map $\phi : S \dashrightarrow S'$ is called rational, if it is given by a rational function. The rational maps cannot be composed unless they are dominant, i.e. the image
of $\phi$ is Zariski dense in $S'$. The map $\phi$ is birational, if the inverse $\phi^{-1}$ is a rational map. A birational map $\epsilon : S \dashrightarrow S'$ is called a blow-up, if it is defined everywhere except for a point $p \in S$ and a rational curve $C \subset S'$, such that $\epsilon^{-1}(C) = p$. Every birational map $\phi : S \dashrightarrow S'$ is composition of a finite number of the blow-ups, i.e. $\phi = \epsilon_1 \circ \cdots \circ \epsilon_k$.

The surface $S$ is called a minimal model, if any birational map $S \dashrightarrow S'$ is an isomorphism. The minimal models exist and are unique unless $S$ is a ruled surface. By the Castelnuovo Theorem, the smooth projective surface $S$ is a minimal model if and only if $S$ does not contain a rational curves $C$ with the self-intersection index $-1$.

3. Proofs

3.1. Proof of theorem 1.1. For the sake of clarity, let us outline the main ideas. Let $S \to S'$ be a regular (polynomial) map between the surfaces $S$ and $S'$. It is known that such a map induces a homomorphism $E_S \to E_{S'}$ of the Etesi $C^*$-algebras and an extension of the fields $k \subseteq k'$ in the corresponding Handelman triples $(\Lambda, [m], k) \subseteq (\Lambda', [m'], k')$, see Section 2.2.3. Recall that the rational map $S \dashrightarrow S'$ is regular only on an open subset $U \subset S$, such that the Zariski closure of $U$ coincides with $S$. Roughly speaking, we prove that an operation corresponding to the Zariski closure of $U$ consists in passing from the field $k$ to its Hilbert class field $H(k)$ (lemma 3.1). The rest of the proof follows from the inclusion of fields $H(k) \subseteq H(k')$ induced by the rational map $S \dashrightarrow S'$.

We shall split the proof in a series of lemmas.

Lemma 3.1. If $S \dashrightarrow S'$ is a rational map, then $H(k) \subseteq H(k')$.

Proof. In outline, an open set $U \subset S$ is an open 4-dimensional manifold with boundary. Taking a connected sum with the copies of $S^4$, one gets a compact smooth manifold $S_0$ and a regular map $S_0 \to S$. Such a map defines a field extension $k_0 \subseteq k$. Since $U$ is Zariski dense in $S$, we conclude that the surface $S_0$ determines $S$ up to an isomorphism. Therefore the field extension $k_0 \subseteq k$ must depend solely on the arithmetic of the field $k_0$. In other words, the intrinsic invariants of $k_0$ control the Galois group $Gal (k|k_0)$. This can happen if and only if $k \cong H(k_0)$, so that $Gal (k|k_0) \cong Cl (k_0)$, where $Cl (k_0)$ is the ideal class group of $k_0$. We pass to a detailed argument.
Figure 2.

(i) Let $\phi : S \dashrightarrow S'$ be a rational map. Then there exist the open sets $U \subset S$ and $U' \subset S'$, such that

$$\phi : U \rightarrow U'.$$  

(3.1)

is a regular map.

(ii) Since $U \subset S$ is an open set, it is Zariski dense in $S$. The set $U$ is a 4-dimensional manifold with a boundary corresponding to the poles of the rational map $S \dashrightarrow S'$. Let $n$ be the total number of the boundary components of $U$. Consider a compact smooth 4-dimensional manifold

$$S_0 := U \#_n S^4.$$  

(3.2)

coming from the connected sum of $U$ with the $n$ copies of the 4-dimensional sphere $S^4$ equipped with the standard smooth structure.

(iii) Notice that $S_0$ can be endowed with a complex structure and can be identified with an algebraic surface. It is not hard to see, that there exists a regular map $S_0 \rightarrow S$ and the corresponding field extension $k_0 \subseteq k$ given by the commutative diagram in Figure 1. Since $k_0$ and $k$ are totally real number fields, the $k_0 \subseteq k$ is a Galois extension. In particular, the Galois group $Gal (k|k_0)$ is correctly defined.

(iv) Recall that the Zariski closure of $U$ coincides with the surface $S$. Since $S_0$ contains $U$, the Zariski closure of $S_0$ will coincide with $S$ as well. Using the diagram in Figure 1, we conclude that the Galois extension $k_0 \subseteq k$ depends only on the arithmetic of the ground field $k_0$. This means that the Galois group $Gal (k|k_0)$ must be an invariant of the field $k_0$. The only extension with such a property is the Hilbert class field $\mathcal{H}(k_0)$, i.e. $Gal (k|k_0) \cong Cl (k_0)$, where $Cl (k_0)$ is the ideal class group of $k_0$. Thus $k \cong \mathcal{H}(k_0)$.

(v) Using the regular map (3.1), one gets an inclusion of the number fields $k_0 \subseteq k'_0$ and therefore an inclusion $\mathcal{H}(k_0) \subseteq \mathcal{H}(k'_0)$. In other words, the diagram in Figure 1 implies a commutative diagram in Figure 2.

(vi) Lemma 3.1 follows from Figure 2 after an adjustment of the notation, i.e. dropping the subscript zero for the number field $k_0$. □

Lemma 3.2. If $S \dashrightarrow S'$ is a birational map, then $\mathcal{H}(k) \cong \mathcal{H}(k')$. 
Proof. In view of lemma 3.1, the rational map $S \dashrightarrow S'$ implies an inclusion of the number fields $\mathcal{H}(k) \subseteq \mathcal{H}(k')$. Since $S \dashrightarrow S'$ is birational, the inverse rational map $S' \dashrightarrow S$ gives an inclusion of the number fields $\mathcal{H}(k') \subseteq \mathcal{H}(k)$. Clearly, the above inclusions are compatible if and only if $\mathcal{H}(k') \cong \mathcal{H}(k)$. Lemma 3.2 is proved.

Lemma 3.3. If $S \dashrightarrow S'$ is a blow-up, then $k' \cong \mathcal{H}(k)$.

Proof. Roughly speaking, we use the same argument as in lemma 3.1. Since the blow-up is a dominant rational map, the image of surface $S$ must be Zariski dense in $S'$. As it was shown earlier, one gets an isomorphism between the field $k'$ and the Hilbert class field of $k$. We pass to a detailed argument.

(i) Recall that any birational map $\phi : S \dashrightarrow S'$ is a composition

$$\phi = \epsilon_1 \circ \cdots \circ \epsilon_m,$$

where $\epsilon_i$ is a blow-up and $m < \infty$. In particular, the $\epsilon_i$ must be dominant rational maps. The latter means that the image $\epsilon_i(S)$ is Zariski dense in $S'$.

(ii) Consider a dominant rational map $\phi : S \dashrightarrow S'$ and the corresponding Galois extension of the number fields $k \subseteq k'$ shown in Figure 3. Since $S'$ is the closure of a Zariski dense subset $\phi(S)$, we conclude that the extension $k \subseteq k'$ depends only on the arithmetic of the ground field $k$. In particular, the Galois group $Gal \left( k'|k \right)$ is an invariant of the field $k$. The only extension with such a property is the Hilbert class field $\mathcal{H}(k)$ for which $Gal \left( k'|k_0 \right) \cong Cl \left( k_0 \right)$, where $Cl \left( k_0 \right)$ is the ideal class group of $k_0$. Thus $k' \cong \mathcal{H}(k)$. Lemma 3.3 is proved.

The ‘if’ part of theorem 1.1 follows from lemma 3.3. Let us show that if $k' \cong \mathcal{H}(k)$, then the corresponding birational map $\phi : S \dashrightarrow S'$ is a blow-up. Indeed, since $\phi$ is a birational map, one can apply lemma 3.2 to obtain an isomorphism of the number fields

$$\mathcal{H}(k) \cong \mathcal{H}(k').$$

The substitution $k' \cong \mathcal{H}(k)$ into (3.4) will imply that $\mathcal{H}(k) \cong \mathcal{H}^2(k)$. In other words, the class field tower of $k$ is stable after the first step. In particular, such a tower cannot be decomposed into the sub-towers, i.e. the birational map $\phi$ cannot be decomposed into a composition of the blow-ups $\epsilon_i$. The latter means that $\phi$ is a blow-up itself, see (3.3). The ‘only if’ part of theorem 1.1 follows.
This argument finishes the proof of theorem 1.1.

3.2. Proof of corollary 1.2. Our proof is based on the Castelnuovo theory of the minimal models for algebraic surfaces. Namely, such a theory says that it takes a finite number of the blow-ups to get the minimal model of $S$. The rest of the proof follows from theorem 1.1 applied to the corresponding class field tower. We pass to a detailed argument.

(i) Let us prove that if $S$ is an algebraic surface, then its minimal model gives rise to a finite real class field tower. We denote by $S^{(m)}$ the minimal model obtained from $S$ by composition of the blow-ups

$$
\epsilon_i : S^{(i-1)} \to S^{(i)},
$$

where $1 \leq i \leq m$. In view of theorem 1.1, each blow-up $\epsilon_i$ defines a Hilbert class field extension of the $k_{i-1}$, i.e.

$$
k_i = \mathcal{H}(k_{i-1}),
$$

where all $\{k_i \mid 1 \leq i \leq m\}$ are real number fields.

(ii) After a finite number of the blow-ups (3.5), one gets a commutative diagram in Figure 4. Thus the finite real class field tower corresponding to the minimal model $S^{(m)}$ has the form:

$$
k \subset \mathcal{H}(k) \subset \mathcal{H}^2(k) \subset \cdots \subset \mathcal{H}^m(k) \cong \mathcal{H}^{m+1}(k) \cong \ldots
$$

(iii) Let us show that if

$$
k := \mathcal{H}^0(k) \subset \mathcal{H}^1(k) \subset \mathcal{H}^2(k) \subset \ldots
$$

is a real class field tower, then it is finite, i.e. there exists an integer $m < \infty$ such that $\mathcal{H}^m(k) \cong \mathcal{H}^{m+1}(k) \cong \ldots$. Indeed, since $k_i := \{\mathcal{H}^i(k) \mid i \geq 0\}$ is a real number field, one can construct a Handelman triple $(\Lambda_i, [m], k_i)$ and a smooth 4-dimensional manifold $\mathcal{M}_i$, see remark 2.2. Since the triple $(\Lambda_i, [m], k_i)$ is a topological invariant of $\mathcal{M}_i$, we can always assume that $\mathcal{M}_i$ is a complex surface by choosing a proper smoothing of $\mathcal{M}_i$ if necessary. Notice that one can identify $\mathcal{M}_i$ with an algebraic surface $S^{(i)}$.

(iv) Since $k_i = \mathcal{H}(k_{i-1})$, one can apply theorem 1.1 saying that the $S^{(i)}$ is a blow-up of the surface $S^{(i-1)}$. Using the Castelnuovo theorem, one concludes that
the class field tower (3.8) must stabilize for an integer $m < \infty$, i.e. such a the tower is always finite.

This argument finishes the proof of corollary 1.2.

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