Improved State Feedback $H_{\infty}$ Control for Flexible Air-Breathing Hypersonic Vehicles on LMI Approach

Xue Zhang, Xiaogeng Liang

Luoyang Opto-Electro Technology Development Center, China

Abstract. Focusing on a nonlinear longitudinal dynamical model for Air-breathing Hypersonic Flight Vehicles (AHFV), a linearized model on a nominal trim condition is proposed. To stabilize the flight of an AHFV in the presence of external disturbances and actuator uncertainties, a state feedback $H_{\infty}$ control is designed. With bounds on the uncertainties, a feedback stabilization problem is converted to an optimal control problem and the cost function is minimized by solving a set of linear matrix inequalities. Since uncertainties in the design of AHFV are inevitable, to make a comparison, a general $H_{\infty}$ robust controller is constructed by only considering the disturbances firstly. Then the results are extended by incorporating the actual existing uncertainties as well as the external disturbances in the AHFV system. Numerical simulation shows that the controller, which takes both disturbances and uncertainties into account, can effectively stabilize the AHFV system.

1 Introduction

Considering the high demand for extremely fast global strike capability and development of defense systems, the research on hypersonic vehicles has been of a great importance for many researchers. Since the 1970s, the hypersonic speed has generally been defined as at least the value of Mach 5 (five times the speed of sound). High velocity may cause a hypersonic vehicle to be highly sensitive to the change of flight conditions [1], such as the Mach number, and the angle of attack. A comparison among subsonic, supersonic and hypersonic speeds shows that the drag and lift forces become non-linear functions of the angle of attack at hypersonic speeds [1]. The loss of linearity of some parameters may lead the system to be less robust and unstable. An Air-breathing Hypersonic Flight Vehicle (AHFV), such as X-43A, usually is equipped with an air breathing engine which can scoop oxygen from the air. Because of the highly integrated design between airframe and engine, the dynamic characteristics of hypersonic vehicles change drastically with the operating conditions and mass distribution. Meanwhile, the inaccuracy of a mathematical model or control law may also add some uncertainties. In addition, the disturbances from outside atmosphere and density variation at a high Mach number also add uncertainties in the performance of air-breathing propulsion systems [2]. Therefore, a design of a robust and performance oriented control system is the key issue in the design of an Air-breathing Hypersonic Flight Vehicle (AHFV).

Many strategies related to efficient controllers for the hypersonic vehicles have been presented, including adaptive control [3] [4], sliding mode control [5] [6], and $H_{\infty}$ control [7] [8]. Considering uncertainties due to both aerodynamic and center of gravity, adaptive controllers are suitable for command tracking [9]. However, for an unknown input matrix, the results of the adaptive control have only local stability. The sliding mode control can preferably solve the model uncertain problem [5], and it is insensitive to the out-side interference [6]. But more control force is needed in this method. Due to high parameter perturbation of controller, the air-breathing hypersonic flight model is extremely complex. Therefore, although the single $H_{\infty}$ controller or $\mu$ -synthesis methods may achieve the desired performance [10] [11] [12] [13] [14], it is not easy for these controllers to guarantee the stability and desired performance.

In this paper, the problem of stability is solved by designing an improved state feedback $H_{\infty}$ control law for a rigid body AHFV. Therefore, the stabilization design problem of a nonlinear system is converted into a problem of optimal control. Since uncertainties are inevitable, to make a comparison, a general controller is constructed by only considering the disturbances firstly. Then the results are extended by incorporating the actual existing uncertainties as well as the external disturbances. Ideally, the control model is MATLAB-based and the simulation results show that the control law is effective and can facilitate the development of the control system.

This paper is organized as follows: Section II presents a linearized AHFV dynamical model. In Section III, by
considering only disturbances, a general $H_\infty$ robust control-
troller is presented. Moreover, this control law is further
developed by taking both disturbances and uncertainties
into account. The simulation results are shown in Section
IV, which demonstrates the effectiveness of the proposed
control law. Section V draws conclusions.

Notation: The notation used in this paper is fairly
standard. The superscript ‘T’ indicates matrix
transposition; $\mathbb{R}$ denotes the $n$-dimensional Euclidean
space; sym(P) denotes $P + P^T$; $\ast$ denotes to represent a term
that is introduced by symmetry. For a real symmetric matrix $P$
and $P < 0$ means $P$ is positive definite and $P > 0$ means $P$
is negative definite.

2 Longitudinal dynamical model of
hypersonic flight

In this paper, the longitudinal model of an AHFV is
adopted from [15]. The aerodynamic coefficients are
the typical cruising flight coefficients with Mach number 15.
Flight velocity, altitude, path angle and the pitch rate are
respectively 15060 ft/s, 110000 ft, 0 deg, 0 deg/s
governed by the following equations:

$$
\dot{V} = (T \cos \alpha - D)/m - \mu \sin \theta / r^2
$$

$$
\dot{\theta} = (L + T \sin \alpha) / (mV) - ((\mu - V^2/r) \cos \theta) / (Vr^2)
$$

$$
\dot{\omega} = q - \dot{\theta}
$$

$$
\dot{q} = M_{yy} / I_{yy}
$$

Where $V$(ft/s), $\theta$(deg), $h$(ft), $\alpha$(deg), $q$(deg/s),
$m$(slug), $\mu$(ft/s$^2$), and $I_{yy}$ (slug·ft$^2$) denote velocity,
flight-path angle, altitude, angle of attack, pitch rate,
mass, gravitational constant, and moment of inertia,
respectively. The equations for lift $L$(1bf), drag $D$(1bf),
thrust $T$(1bf), pitching moment $M_{yy}$ (1bf·ft), and the
radial distance from earth’s center $r$(ft) are:

$$
L = \frac{1}{2} \rho V^2 S C_L
$$

$$
D = \frac{1}{2} \rho V S^2 C_D
$$

$$
T = \frac{1}{2} \rho V S^2 C_T
$$

$$
r = h + R_e
$$

$$
M_{yy} = \frac{1}{2} \rho V S [C_L(\alpha) + C_D(\delta) + C_M(q)]
$$

with the following lift coefficient $C_L$, drag coefficient
$C_D$, thrust coefficient $C_T$, and pitching moment
coefficients $C_a$, $C_b$, $C_q$:

$$
C_L = \alpha(0.493 + 1.91/M)
$$

$$
C_D = 0.0082(171\alpha^2 + 1.15\alpha + 1)
$$

$$
\times(0.0012M^2 - 0.054M + 1)
$$

$$
C_T = \frac{C_T(1 + 0.15)\beta}{1}
$$

$$
\frac{C_T(1 + 0.15)\beta}{1} \text{ if } \beta < 1
$$

$$
\frac{C_T(1 + 0.15)\beta}{1} \text{ if } \beta \geq 1
$$

$$
C_M(\alpha) = 10^{-4}(0.06 - e^{-4\alpha}) \times (-6565\alpha^2 + 6875\alpha + 1)
$$

$$
C_M(\alpha) = 0.0292(\delta - \alpha)
$$

$$
C_M(q) = \frac{q}{2\rho V} (-0.025M + 1.37) \times (-6.83\alpha^2 + 0.303\alpha - 0.23)
$$

where

$$
C_T = 0.0105[1 - 164(\alpha - \alpha_0)^2](1 + 17/M)
$$

$M$ is the Mach number,

$$
M = V/a \quad (a = 8.99 \times 10^{-9} h^2 - 9.16 \times 10^{-4} h + 996)
$$

and $\rho$ denotes the air density $\rho = 2.38 \times 10^{-3} (h/24000)$.

The longitudinal model of the hypersonic vehicle
can be transformed into a general MIMO nonlinear
system form [16]:

$$
\dot{X} = f(X) + \sum_{i=1}^{2} g_i(X)U_i = F(X, U)
$$

where the state and control vectors are $X = [V, \theta, h, \alpha, q]^T$
and $U = [\beta, \delta]^T$.

In order to simplify the design of controller for the
longitudinal model, the model is linearized which is
represented by:

$$
\dot{X}(t) = AX(t) + Bu(t)
$$

$$
z(t) = Cx(t)
$$

$$
A = \frac{\partial F(X, U)}{\partial X}|_{x=x_0, u=0} = \begin{bmatrix}
\begin{array}{cccc}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    -a_{31} & -a_{32} & -a_{33} & -a_{34} \\
    a_{31} & 0 & a_{33} & a_{34}
\end{array}
\end{bmatrix}
$$

$$
B = \frac{\partial F(X, U)}{\partial U}|_{x=x_0, u=0} = \begin{bmatrix}
\begin{array}{cccc}
    b_{11} & 0 \\
    b_{21} & 0 \\
    0 & 0 \\
    0 & b_{32}
\end{array}
\end{bmatrix}
$$

and

$$
a_{11} = \frac{\partial S}{2m} \left[ 2VC_T + V^2 \frac{\partial C_T}{\partial V} \cos \alpha - 2VC_D + V^2 \frac{\partial C_D}{\partial V} \right]
$$

$$
a_{12} = -\frac{\mu \cos \theta}{r^2}
$$

$$
a_{13} = -\frac{\rho V^2S}{48000m} (C_T \cos \alpha - C_D) + \frac{2 \mu \sin \theta}{r^2}
$$

$$
a_{14} = \frac{\rho V^2S}{2m} \left[ (1 + 17/M)(1 + 1.15)\beta \times 0.0105[1 - 164(\alpha - \alpha_0)^2] \sin \alpha \right] - \frac{\rho V^2S}{2m} \times 0.0082(0.0012M^2 - 0.054M + 1)(342\alpha + 1.15)
$$

$$
\times \cos \alpha - [1 - 164(\alpha - \alpha_0)^2] \sin \alpha
$$

$$
\times 0.0082(0.0012M^2 - 0.054M + 1)(342\alpha + 1.15)
$$
The conventional LQR control does not consider disturbances which may significantly compromise the stability of system. In order to increase the robustness, the external disturbances $\epsilon(t)$ should be taken into account. For a guaranteed cost controller $u(t) = Kx(t)$, a system with disturbances can be expressed as follows:

$$
\begin{align*}
\dot{x}(t) &= (A + B_1 K)x(t) + B_2 \epsilon(t), \\
\epsilon(t) &= C_x x(t), \\
\end{align*}
$$

(1)

where $B_2$ is a constant matrix and $z(t)$ is the controlled output. The objective is to find a desired controller $u(t) = Kx(t)$ to make the system (1) asymptotically stable and minimize the cost function. Hence, the analysis of stability performance is required.

**Theorem 1** Consider the AHFV dynamics (1) with given parameters. The system is asymptotically stable with a given $H_\infty$ performance if there exists a positive definite symmetric matrix $P = P^T > 0$ such that the following condition holds:

$$
\begin{bmatrix}
\text{sym}(P(A + B_K)) & PB_2 \\
PB_2^T & [Q - K^T 0 R] - I
\end{bmatrix}
< 0
$$

(2)

**Proof.** Define a Lyapunov function as

$$
V(t) = x^T(t)Px(t),
$$

(3)

where $P$ is a positive-definite symmetric matrix. The $H_\infty$ cost function is as follows:

$$
J_H = \int_0^T \left[ z^T(t)z(t) - \gamma^2 \dot{w}^T(t) \dot{w}(t) + \dot{V} \right] dt,
$$

where $\dot{V}$ is the derivative of $V(t)$.

In order to connect the $H_\infty$ performance index with LQR control, a new vector is defined:

$$
z(t) = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R \end{bmatrix} x(t),
$$

$$
\dot{z}(t) = \begin{bmatrix} \dot{x} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} PB_2 \\ 0 \end{bmatrix} z(t),
$$

Thus, by considering the disturbances under the zero initial condition, the improved LQR cost function with a prescribed $H_\infty$ performance can be rewritten as

$$
J = \int_0^T \left[ u^T(t)R u(t) + x^T(t)Q x(t) - \gamma^2 \dot{w}^T(t) \dot{w}(t) + \dot{V} \right] dt.
$$

The integrand of cost function is denoted as $J_r$.

$$
J_r = z^T(t)z(t) - \gamma^2 \dot{w}^T(t) \dot{w}(t) + \dot{V},
$$

where

$$
C_1 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}^T + K^T \begin{bmatrix} 0 & R \end{bmatrix}^T.
$$

According to the Schur complement, the conditional in equality can be replaced by (23):

Theorem 1
\[
\begin{bmatrix}
\text{sym}(P(A + B_K K)) & PB_2 \\
B_2^T P \\
\end{bmatrix}
- \begin{bmatrix}
C_i \\
0 \\
\end{bmatrix}^T \begin{bmatrix}
-C_i \\
-I \\
\end{bmatrix} < 0.
\]

Then
\[
\begin{bmatrix}
\text{sym}(P(A + B_K K)) + C_i^T C_i & PB_2 \\
B_2^T P \\
\end{bmatrix} < 0
\]

Therefore:
\[
J_r < 0,
\]

and
\[
\text{sym}(P(A + B_K K)) + C_i^T C_i < 0,
\]

\[
\text{sym}(P(A + B_K K)) < 0.
\]

The quadratic Lyapunov function candidate is shown as (3) and taking the time derivative of V (t) will result:
\[
\dot{V}(t) = x^T(t) \text{sym}(P(A + B_K K)) x(t).
\]

Thus \( \dot{V} < 0 \), which means the system is stable.

On the other hand, since \( J_r < 0 \), the following inequality holds:
\[
\int_0^t \dot{V}(t) dt + \int_0^t (z(t) x(t) - \gamma^2 \omega^T(t) \omega(t)) dt < 0,
\]

Assuming zero condition, integration from 0 to \( \infty \) yields:
\[
\| z(t) \| < \gamma^2 \| \omega(t) \| \|
\]

Therefore, the \( H_\infty \) performance is guaranteed. The proof is completed.

Now, the minimum optimal problem is converted into a problem of solving a prescribed \( H_\infty \) control, which will significantly enhance the robustness of the system. In this control problem, transfer function from disturbances \( \omega(t) \) to output \( z(t) \) would satisfy the \( H_\infty \) norm bound restriction. At the same time, by substituting the control input \( u(t) \) into (1), the control system is turned into:
\[
\begin{align*}
\dot{x}(t) &= (A + B_K K)x(t) + B_o \omega(t), \\
z(t) &= \begin{bmatrix}
\begin{bmatrix}
\hat{Q}^2 & 0 \\
0 & \begin{bmatrix}
K^T & 0 \\
R^2 & 0 \\
\end{bmatrix}
\end{bmatrix}^T \\
x(t) + K \begin{bmatrix}
0 & R^2 \\
0 & 0 \\
\end{bmatrix} x(t).
\end{align*}
\]

Furthermore, in order to obtain desired dynamics of the closed-loop systems, usually some pole placement constraints are considered. In this paper, a disc pole placement region is chosen [17].

**Lemma 1 (Disc region)** Let \( \mathcal{U}(\eta, r) \) denotes any disc region centered in \( \eta \) with radius \( r \) in a complex plane \( (\eta, r \in \mathbb{R}) \). Then all the eigenvalues of \( A \) in (1) lie in the region \( \mathcal{U}(\eta, r) \) if and only if there exists a matrix \( P > 0 \) satisfying
\[
\begin{bmatrix}
-P & P(A - \eta I) \\
0 & -\gamma^2 P
\end{bmatrix} < 0,
\]

where \( A = A + B_K K \).

Now, a multi-objective controller can be obtained by setting \( P = X^{-1} \).

**Theorem 2** Consider the system in (1). If the scalars \( r \) and \( \eta \) are known and there exist matrices \( X > 0 \) and \( Y \) with appropriate dimensions and also the following LMIs are satisfied
\[
\begin{bmatrix}
AX + XA^T + BKY + Y^TB_i^T & B_i \\
\ast & -\gamma^2 I
\end{bmatrix} < 0
\]

and
\[
\begin{bmatrix}
-X & AX + BKY - \eta X \\
\ast & -r^2 X
\end{bmatrix} < 0,
\]

then there is a proper controller \( K \) which can make the system described by (1) asymptotically stable and the poles lie in a disc region. The nominal controller gain matrix is given by
\[
K = YX^{-1}.
\]

**Proof.** In order to get the state feedback controller \( K \), congruence transformation is performed on (4) by \( P^{-1} \)
\[
P^{-1} (P(A + B_K K) + (A + B_K K) P^{-1}) P^{-1} < 0.
\]

Accordingly,
\[
(A + B_K K) P^{-1} + P^{-1} (A + B_K K) < 0.
\]

Let \( X = P^{-1} (X = X^T > 0) \), therefore:
\[
AX + B_K KX + XA^T + XK^T B_i^T < 0.
\]

To calculate the inequality, it is denoted that \( Y = KX \), thus
\[
AX + B_K Y + XA^T + Y^TB_i^T < 0,
\]

and the controller is:
\[
K = YX^{-1}.
\]

When \( K = YX^{-1} \), performing the congruence transformation to (7) by \( \text{diag} \{P,P^T,1\} \), condition (2) is obtained. On the other hand, performing congruence transformation to (8) by \( \text{diag} \{P,P\} \), one can get (6). The proof is finished.

**Remark 1** In the problem formulation above, only external disturbances are considered. However, the actuator uncertainties are inevitable in the practical control system. Therefore, it is quite necessary to incorporate both external disturbances and actuator uncertainties factors to the system model.

A controller is designed to make AHFV asymptotically stable with a given \( H_\infty \) attenuation level \( \gamma \) considering external disturbances as well as actuator uncertainties. Assuming the actuator uncertainties can be denoted by \( \Delta A \) and actual system matrix \( \tilde{A} = A + \Delta A \), the AHFV dynamical system can be written as
\[
\begin{bmatrix}
\dot{x}(t) &= (A + \Delta A)x(t) + Bu(t) + B_o \omega(t), \\
z(t) &= C_i x(t).
\end{bmatrix}
\]

The uncertain matrix \( \Delta A \) is practically assumed to be norm bounded, which means
\[
\Delta A = LN(t) E,
\]

where \( N(t) \) and \( E \) are bounded matrices.
where \( L \) and \( E \) are known constant matrices of certain appropriate dimensions. \( N(t) \) is an unknown time-varying variable which satisfies \( N^T(t)N(t) \leq I \).

In order to deal with the uncertainties, the following lemma is used [18]:

**Lemma 2** Let \( \Theta = \Theta^T, \overline{E} \) and \( \overline{E} \) be real matrices with compatible dimensions and \( N(t) \) be time-varying variable which satisfies (11). Then the following condition

\[
\Theta + \overline{N}(t)\overline{E} + \overline{E}^T N^T(t)\overline{E}^T < 0, \tag{12}
\]

holds if and only if there exists a positive scalar \( \varepsilon > 0 \) such that

\[
\begin{bmatrix}
\Theta & \varepsilon \overline{E}^T \\
* & -\varepsilon I
\end{bmatrix} < 0. \tag{13}
\]

The following theorem is derived to eliminate the influence from the actuator uncertainties.

**Theorem 3** Consider the AHFV dynamics in (10) with given parameters. The system is asymptotically stable with a given \( H_\infty \) performance \( \gamma \) if there exists a positive-definite symmetric matrix \( P = P^T \) such that the following inequality holds

\[
\begin{bmatrix}
sym(P(A + B, K)) & PB_2 \\
* & -\gamma^2 I
\end{bmatrix} < 0.
\]

\[
\begin{bmatrix}
\frac{1}{2} I & K^T \begin{bmatrix} 0 & R^T \end{bmatrix} \\
0 & 0
\end{bmatrix} PL \varepsilon E^T < 0 \quad \tag{14}
\]

**Proof.** According to Lemma 2, the system described by (10) is asymptotically stable with a given \( H_\infty \) performance \( \gamma \) if the condition in (13) holds. Therefore, the condition (14) can be changed into

\[
\begin{bmatrix}
sym(P(A + B, K)) & PB_2 \\
* & -\gamma^2 I
\end{bmatrix} + \begin{bmatrix}
\frac{1}{2} I & K^T \begin{bmatrix} 0 & R^T \end{bmatrix} \\
0 & 0
\end{bmatrix} PL \varepsilon E^T < 0 \quad \tag{15}
\]

By means of Lemma 2, the matrix inequality (14) can be obtained. This completes the proof.

**Theorem 4 (Disc region)** Let \( \mathcal{U}(\eta, r) \) denotes any disc region centered in \( \eta \) with radius \( r \) in a complex plane \( (\eta, r) \in \mathbb{C} \) and \( r > 0 \). Then all the eigenvalues of \( \Delta A + \Delta A \) in (10) lie in the region \( \mathcal{U}(\eta, r) \) if and only if there exists a matrix \( P > 0 \) satisfying

\[
\begin{bmatrix}
-P & P(A - \eta I) \\
* & -\gamma^2 P
\end{bmatrix} + \begin{bmatrix}
0 & N(t)
\end{bmatrix} \begin{bmatrix} 0 & E \end{bmatrix} < 0. \tag{16}
\]

**Proof.** When taking the uncertain matrix \( \Delta A \) into account, Lemma 1 can be changed into

\[
\begin{bmatrix}
-P & P(A + \Delta A - \eta I) \\
* & -\gamma^2 P
\end{bmatrix} + \begin{bmatrix}
0 & N(t)
\end{bmatrix} \begin{bmatrix} 0 & E \end{bmatrix} < 0. \tag{17}
\]

since \( \Delta A = LN(t)E \), (17) can be rewritten as

\[
\begin{bmatrix}
-P & P(A - \eta I) \\
* & -\gamma^2 P
\end{bmatrix} + \begin{bmatrix}
0 & N(t)
\end{bmatrix} \begin{bmatrix} 0 & E \end{bmatrix} < 0. \tag{18}
\]

By means of Lemma 2, the matrix inequality (16) can be obtained. This completes the proof.

Under an uncertain precondition, the \( H_\infty \) based LQR control design with pole placement can be presented in the following theorem.

**Theorem 5** Consider the system in (10). If the scalars \( r \) and \( \eta \) are known and there exist matrices \( X > 0 \) and \( Y \) with appropriate dimensions, the following LMIs are satisfied

\[
\begin{bmatrix}
AX + XA^T + B_2 Y + Y^T B_1^T & B_2 \\
* & -\gamma^2 I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
X \begin{bmatrix} \frac{1}{2} I & 0 \end{bmatrix} + Y \begin{bmatrix} 0 & R \end{bmatrix} & \varepsilon L \\
0 & X E^T
\end{bmatrix} < 0,
\]

and

\[
\begin{bmatrix}
-X & AX + B_2 Y - \eta X \\
* & -\gamma^2 X
\end{bmatrix} + \begin{bmatrix}
0 & X E^T
\end{bmatrix} < 0. \tag{20}
\]

Then there is a proper controller \( K \) can make system (10) asymptotically stable with poles lying in a disc region. The nominal controller gain matrix is given by:

\[
K = YX^{-1}.
\]
Proof. Applying congruence transformation to (14) by 
\( \text{diag} \{ P^{-1}, I, I, I \} \), and set 
\( X = P^{-1}Y = KP^{-1} \), will result:
\[
AX + XA^T + BY + Y^TB_B^T B_B = 0
\]
\[
\begin{bmatrix}
X 
\end{bmatrix} \begin{bmatrix}
Q^T \\
0
\end{bmatrix} + Y \begin{bmatrix}
0 \\
R^T
\end{bmatrix} + L X E E^T < 0,
\]
(22)

Then, left-multiply (22) by \( \text{diag} \{ I, I, I, \epsilon, \epsilon \} \) and right-multiply (22) by the transpose of diag \( \{ I, I, I, \epsilon, \epsilon \} \). Next, replace \( \epsilon^{-1} \) by \( \epsilon \), and multiply the resulting LMI by diag \( \{ I, I, I, \epsilon, \epsilon \} \) on the left and its transpose on the right to obtain (19). Additionally, by application of congruence transformation on (20) using diag \( \{ P, P, I, I \} \), the condition (16) can be obtained.

Theorem 5 indicates that even in the presence of unavoidable actuator uncertainties, the robustness of the real system can be guaranteed by computing the inequalities.

4 Simulation results
To illustrate the performance of improved state feedback \( H_2 \) controller, some numerical simulations are done using MATLAB. The parameters used in this study are adopted from [15] as listed in Table 1.

| Symbol | Value | Symbol | Value |
|--------|-------|--------|-------|
| \( h \) | 110000ft | \( \theta \) | 0rad |
| \( S \) | 3603ft² | \( \beta \) | 0.183 |
| \( \tau \) | 80ft | \( \delta \) | -0.0066 |
| \( m \) | 9375slug | \( V \) | 15060ft/s |
| \( I_w \) | \( 7 \times 10^6 \)slug·ft² | \( \alpha \) | 0.0315rad |
| \( R_e \) | 20903500ft | \( M \) | 15 |
| \( \mu \) | \( 1.39 \times 10^{-6} \) ft³/s | \( q \) | 0rad/s |
| \( \alpha_0 \) | 0.0315rad |

Table 1.Trim Condition States Value.

Under a straight-and-level flight trim condition, the values for state vectors and input control are:
\[
x_0 = [15060 0 110000 0.0315 0]^T,
\]
\[
u_0 = [0.183 -0.0066]^T.
\]

The values of \( A, B_1 \) are shown as follows:
\[
A = \begin{bmatrix}
-3.4700 \times 10^{-5} & -31.478 & -7.2040 \times 10^{-6} \\
2.7135 \times 10^{-7} & 0 & -5.3151 \times 10^{-4} \\
5.5375 \times 10^{-6} & 0 & -1.4010 \times 10^{-5} \\
-47.8373 & 0.0406 & 0 \\
0.0406 & 0 & 0 \\
0 & 0 & 0.5923 & -0.0682
\end{bmatrix}
\]

When there are only disturbances, the gain matrices \( K_i \) can be computed by (9). The external disturbances are modeled by Dryden turbulence model, and the turbulence spectrum is defined by the weight [13]:
\[
\sigma_v = 5.8 \, \text{ft/s}, L_v = 65574 \, \text{ft/s}.
\]

The matrix of external disturbances is \( B_2 = [0 0.5 0 0.5 0]^T \). Within a desired disc region, assuming \( Q = 0.0001I \), \( R = 0.01I \), controller \( u(t) = Kx(t) \) is designed to make the system asymptotically stable. It is assumed that the disc region is \( \mathfrak{D}(8,10) \) and \( Q = 0.0001I \), \( R = 0.01I \). The gain matrices \( K_i \) is
\[
K_i = \begin{bmatrix}
-0.5636 & -32.7615 & -0.0089 \\
0.0454 & -2.1398 \times 10^4 & -4.7814 \\
1.6366 & -0.0223 \\
-132.1430 & -9.7263
\end{bmatrix}
\]

Minimum disturbance attenuation level is \( \gamma_i = 3.4652 \).

The uncertainties inherently exist in the system and will inevitably degrade the performance of AHFV. Hence, it is important to consider these uncertainties which may significantly affect the stability of system.

By using the approach developed in this paper, the gain matrix \( K_2 \) can be computed by (21).
The minimum disturbance attenuation level is $\gamma_2 = 5.4144$.

Figures 1 and 2 show the simulation results when uncertainties exist. As shown, the controller developed in [19] cannot stabilize the system while the controller developed in this paper is able to attenuate the uncertainties and can quickly enter the stable state when uncertainties happen.

**Remark 2** This paper is motivated by [19]. Gao et al. presenting an LQR controller but the actual existing disturbances $\alpha(t)$ and uncertainties $\Delta A$ are not considered. In this paper, firstly, when there are only disturbances, an improved state feedback $H_\infty$ controller is developed by considering the $H_\infty$ and LQR indexes, which can increase the robustness of system. Secondly, since the actual existing disturbances is inevitable, it is critical to take the actual existing disturbances into account in terms of both theoretical merits and practical application perspective. When there are both disturbances and uncertainties, the output responses in the control mode of [19] cannot converge to an equilibrium state, however, using the control law in this paper, the output responses can rapidly converge to equilibrium states, which means the stability performance of AHFV system is significantly improved.

**5 Conclusions**

In this paper, a technique of designing improved state feedback $H_\infty$ controller has been developed. Indexes for $H_\infty$ performance, quadratic performance and pole placement are concerned together. By converting the mixed cost function, the multi-objective controller can be deduced by LMIs. Moreover, in terms of a more complete consideration on both external disturbances and actuator uncertainties, the simulation results in [19] and in this paper are presented to make a clear comparison. The results show that, the controller developed in this paper, which takes the actual existing actuator uncertainties into account, can stabilize the flight of AHFV successfully. Performances of flexible air-breathing hypersonic vehicles demonstrate the capability of the algorithm and the feasibility of the controller.

**References**

1. C. C. Coleman and F. A. Faruqi, On stability and control of hypersonic vehicles, Australian Government Department of Defence, PO Box 1500 Edinburgh South Australia 5111 Australia, Technical Report, (2009).
2. I. M. Gregory, R. S. Chowdhry, J. D. McMinn, and J. D. Shlaughnessy, Hypersonic vehicle model and control law development using $H_\infty$ and synthesis, National Aeronautics and Space Administration, Hampton, Virginia, Technical Memorandum, (1992)
3. T. E. Gibson, L. G. Crespo, and A. M. Annaswamy, Adaptive control of hypersonic vehicles in the presence of modeling uncertainties, in American Control Conference, pp. 3178–3183 (2009)
4. H. Xu, M. D. Mirmirani, and P. A. Ioannou, Adaptive sliding mode control design for a hypersonic flight vehicle, Journal of Guidance, Control, and Dynamics, vol. 27, no. 5, pp. 829–838 (2004)
5. J. Liu and F. Sun, Research and development on theory and algorithms of sliding mode control, Control Theory & Applications, vol. 24, no. 3, pp. 407–418 (2007)
6. H. Li, W. Sun, and Z. Li, Index approach law based sliding control for a hypersonic aircraft, in Systems and Control in Aerospace and Astronautics, ISSCAA 2008. 2nd International Symposium on, pp. 1–5 (2008)
7. H. Buschek and A. J. Calise., Fixed order robust
control design for hypersonic vehicles, in *AIAA Guidance, Navigation and Control Conference*, pp. 1094–1103(1994)

8. P. Lohsoonthorn, E. Jonckheere, and S. Dalzell, Eigenstructure vs constrained $H_\infty$ design for hypersonic winged cone, *Journal of Guidance, Control, and Dynamics*, vol. 24, no. 4, pp. 648–658 (2001)

9. H. Zhang, Y. Shi, and A. Saadat Mehr, Robust static output feedback control and remote PID design for networked motor systems, *IEEE Transactions on Industrial Electronics*, vol. 58, no. 12, pp. 5396–5405 (2011)

10. ———, Robust $H_\infty$ PID control for multivariable networked control systems with disturbance/noise attenuation, *International Journal of Robust and Nonlinear Control*, vol. 22, no. 2, pp. 183–204, 2012.

11. H. Zhang, Y. Shi, A. Saadat Mehr, and H. Huang, Robust energy-to-peak FIR equalization for time-varying communication channels with intermittent observations, *Signal Processing*, vol. 91, no. 7, pp. 1651–1658, 2011.

12. H. Zhang, Y. Shi, and A. Saadat Mehr, On $H_\infty$ filtering for discrete-time takagi-sugeno fuzzy systems, *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 2, pp. 396–401 (2012)

13. C. Dong, Y. Hou, Y. Zhang, and Q. Wang, Model reference adaptive switching control of a linearized hypersonic flight vehicle model with actuator saturation, *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, vol. 224, pp. 289–303 (2010)

14. H. Zhang, Y. Shi, and A. Saadat Mehr, Robust weighted H1 filtering for networked systems with interleaved measurements of multiple sensors, *International Journal of Adaptive Control and Signal Processing*, vol. 25, no. 4, pp. 313–330 (2011)

15. Q. Wang and R. F. Stengel, Robust nonlinear control hypersonic aircraft, *Journal of Guidance, Control, and Dynamics*, vol. 23, no. 4, pp. 577–585 (2000)

16. Z. Gao, B. Jiang, P. Shi, J. Liu, and Y. Xu, Passive fault-tolerant control design for near-space hypersonic vehicle dynamical system, *Circuits, Systems and Signal Processing*, vol. 31, no. 4, pp. 565–581 (2012)

17. H. Li, Y. Si, L. Wu, X. Hu, and H. Gao, Guaranteed cost control with poles assignment for a flexible air-breathing hypersonic vehicle, *International Journal of Systems Science*, vol. 42, no. 5, pp. 863–876 (2011)

18. H. Zhang, Y. Shi, and A. S. Mehr, Robust non-fragile dynamic vibration absorbers with uncertain factors, *Journal of Sound and Vibration*, vol. 330, no. 4, pp. 559–566 (2010)

19. H. Gao, Y. Si, H. Li, X. Hu, and C. Wang, Modeling and control of an air-breathing hypersonic vehicle, in *Proceedings of the 7th Asian Control Conference*, pp. 1114–1119 (2009)