Generalized translation operator and the Heat equation for
the canonical Fourier Bessel transform

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Abstract

The aim of this paper is to introduce a translation operator associated to the canonical Fourier Bessel transform $F_{\nu}^m$ and study some of the important properties. We derive a convolution product for this transform and as application we study the heat equation and the heat semigroup related to $\Delta_{\nu}^m$.

KEYWORDS: Canonical Fourier Bessel transform, Translation operator, Heat equation
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1 Introduction

The linear canonical transform (LCT) is a class of linear integral transform with matrix parameter $m \in SL(2, \mathbb{R})$ [1, 14, 20]. It includes many well-known transforms such as the Fourier transform, the fractional Fourier transform, the Fresnel transform etc. These integral transforms are of importance in several areas of physics and mathematics [1, 2, 16, 20]. As is well known, a wide number of papers have been successfully devoted to the extension of the theory of LCT to some other integral transforms such that the Hankel transform, the Dunkl transform [4, 7, 13] and the Fourier Bessel transform [17]. In [19], Wolf enlarged the concept of LCT to the Hankel setting. He built a class of linear integral transform with a kernel involving a Bessel function and matrix parameter $m \in SL(2, \mathbb{R})$. Special cases of this transformation are the Hankel transform and the fractional Hankel transform [13]. In [7], the authors introduced the Dunkl linear canonical transform (DLCT) which is a generalization of the LCT in the framework of Dunkl transform [4, 7]. DLCT includes many well-known transforms such as the Dunkl transform [4, 7, 13], the fractional Dunkl transform [6, 9, 10, 11] and the canonical Fourier Bessel transform [6, 9]. In [6], the authors developed a harmonic analysis related the canonical Fourier Bessel transform $F_{\nu}^m$. Several properties, such as a Riemann-Lebesgue lemma, inversion formula, operational formulas, Plancherel theorem and Babenko inequality and several uncertainty inequalities are established.

In the present work, we continue the analysis begun in [6] by studying the translation operator and the convolution product related to this transformation. Following the framework of Delsarte [3] and Levitan [15], we introduce a generalized translation operator $T_{\nu}^m f(y) = u(x, y) (x, y \geq 0, \ m \in SL(2, \mathbb{R}))$ of a function $f \in C^2([0, +\infty])$ as the solution to the following Cauchy problem

\[
\begin{cases}
\Delta_{\nu,x}^m u(x, y) = \Delta_{\nu,y}^m u(x, y), \\
u(x, 0) = f(x), \\
\frac{\partial}{\partial x} u(x, 0) = 0,
\end{cases}
\]

where $\Delta_{\nu,x}^m = \frac{d^2}{dx^2} + \left(\frac{2\nu+1}{x} - 2i \frac{\nu}{x^2}\right) \frac{d}{dx} - \left(t^2 x^2 + 2i (\nu + 1) \frac{d}{x}\right)$ and where $\Delta_{\nu,x}^m$ act in the $x$ variable. We prove that the solution of the above problem can be written explicitly as

\[
T_{\nu}^m f(y) = u(x, y) = e^{\frac{\nu}{2} (x^2 + y^2)} \int_0^{\infty} e^{-i s^2} f(s) \, ds
\]

where $T_{\nu}^m$ is the generalized translation operators associated with the Bessel operator $\Delta_{\nu}$. We also study some of the important properties of the translation operator $T_{\nu}^m$ such that the continuity property with respect to the norm $\|\cdot\|_{p,\nu}$. Next, we give a generalized convolution product $\ast_{\nu,m}$ on $(0, +\infty)$ tied to the differential operator $\Delta_{\nu}^m$, by putting

\[
\ast_{\nu,m} f(x) = \int_0^{\infty} \left(T_{\nu}^m f(y) \left[e^{-i \frac{\nu}{2} y^2} g(y)\right]^2\right) dy
\]

and study some of its very basic properties. As application, we conclude this paper by studying the heat equation associated to $\Delta_{\nu}^{m-1}$:

\[
\begin{cases}
\frac{\partial}{\partial t} u(t, x) = \sigma \Delta_{\nu}^{-1} u(t, x), \ (t, x) \in (0, +\infty) \times \mathbb{R} \\
u(0, x) = f(x)
\end{cases}
\]

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This paper is organized as follows. Section 2 is devoted to an overview of the canonical Fourier Bessel transform. In section 3 we introduce and study the translation operator and the convolution product related to the canonical Fourier Bessel transform $\mathcal{F}_\nu^m$ and we establish some of its properties. In section 4 we study the heat equation associated to $\Delta^{m-1}$.

## 2 A brief survey of the canonical Fourier Bessel transform

The aim of this section is to give a brief review of the theory of canonical Fourier Bessel transform that is relevant to the succeeding sections [6]. Throughout this paper, $\nu$ denotes a real number such that $\nu > -\frac{1}{2}$.

### 2.1 Notations and functional spaces

We denote by:

- $C_{*o}(\mathbb{R})$ the space of even continuous functions on $\mathbb{R}$ and vanishing at infinity. We provide $C_{*o}(\mathbb{R})$ with the topology of uniform convergence.
- $C_{*c}(\mathbb{R})$ the space of even continuous functions on $\mathbb{R}$ with compact support.
- $C_{*b}(\mathbb{R})$ the space of even and bounded continuous functions on $\mathbb{R}$. We provide $C_{*b}(\mathbb{R})$ with the topology of uniform convergence.
- $\mathcal{L}_{p,\nu}$ the Lebesgue space of measurable functions on $[0, +\infty[$ such that
  \[
  \|f\|_{p,\nu} = \left(\int_0^{+\infty} |f(y)|^p y^{2\nu+1} dy \right)^{\frac{1}{p}} < \infty, \text{ if } 1 \leq p < \infty, \\
  \|f\|_{\infty} = \text{ess sup}_{y\in[0,\infty]} |f(y)| < \infty, \text{ if } p = \infty.
  \]
  We provide $\mathcal{L}_{p,\nu}$ with the topology defined by the norm $\|\cdot\|_{p,\nu}$.
- $\mathcal{L}_{2,\nu}$ the Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\nu}$ given by
  \[
  \langle f, g \rangle_{\nu} = \int_0^{+\infty} f(y) \overline{g(y)} y^{2\nu+1} dy.
  \]
- $E_{*}(\mathbb{R})$ the space of even $C^\infty$-functions on $\mathbb{R}$. We provide it with the topology of uniform convergence on all compacts of $\mathbb{R}$, for functions and their derivatives.
- $S_{*}(\mathbb{R})$ the Schwartz space of even $C^\infty$-functions on $\mathbb{R}$ and rapidly decreasing together with their derivatives. We provide $S_{*}(\mathbb{R})$ with the topology defined by the seminorms:
  \[
  q_{n,m}(f) = \sup_{x \geq 0} (1 + x^2)^n \left| \frac{d^m}{dx^m} f(x) \right|; \quad f \in S_{*}(\mathbb{R}), \quad n, m \in \mathbb{N}.
  \]
- $D_{*}(\mathbb{R})$ the space of even $C^\infty$-functions on $\mathbb{R}$ with compact support. We have
  \[
  D_{*}(\mathbb{R}) = \bigcup_{a \geq 0} D_{*a}(\mathbb{R}),
  \]
  where $D_{*a}(\mathbb{R})$ is the space of even $C^\infty$-functions on $\mathbb{R}$ with support in the interval $[-a, a]$. We provide $D_{*a}(\mathbb{R})$ with the topology of uniform convergence of functions and their derivatives.
- $H_{*}(\mathbb{C})$ the space of even entire functions on $\mathbb{C}$ rapidly decreasing and of exponential type. We have
  \[
  H_{*}(\mathbb{C}) = \bigcup_{a \geq 0} H_{*a}(\mathbb{C}),
  \]
  where $H_{*a}(\mathbb{C})$ is the space of even entire functions $f : \mathbb{C} \to \mathbb{C}$ and satisfying for all $m \in \mathbb{N}$
  \[
  P_m(f) = \sup_{\lambda \in \mathbb{C}} (1 + |\lambda|)^m |f(\lambda)| e^{-a|\lambda|} < \infty.
  \]
  We provide $H_{*a}(\mathbb{C})$ with the topology defined by the seminorms $P_m, \ m \in \mathbb{N}$.

### 2.2 Canonical Fourier Bessel transform

Throughout this paper, we denote by $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitrary matrix in $SL(2, \mathbb{R})$. For $m \in SL(2, \mathbb{R})$ such that $b \neq 0$, the canonical Fourier Bessel transform of a function $f \in \mathcal{L}_{1,\nu}$ is defined by [6]

\[
\mathcal{F}_\nu^m f(x) = \frac{c_\nu}{(ib)^{\nu+1}} \int_0^{+\infty} K^m_\nu(x, y) y^{2\nu+1} dy,
\]
where $c_{\nu}^{-1} = 2^\nu \Gamma(\nu + 1)$ and

$$K^m_{\nu}(x, y) = e^{\frac{y}{b}(2x^2 + \frac{x}{y} y^2)} j_\nu \left( \frac{xy}{b} \right).$$

Here $j_\nu$ denotes the normalized Bessel function of order $\nu > -1/2$ and defined by [8] [18]

$$j_\nu(x) = 2^\nu \Gamma(\nu + 1) \frac{j_\nu(x)}{x^\nu} = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma(\nu + 1)}{n! \Gamma(n + \nu + 1)} \left( \frac{x}{2} \right)^{2n}; \ x \in \mathbb{C}.$$

It is well known that, for every $y \in \mathbb{C}$, the function $j_\nu(y \cdot) : x \mapsto j_\nu(yx)$ is the unique solution of

$$\begin{cases}
\Delta_\nu f = -y^2 f, \\
f(0) = 1, \ f'(0) = 0,
\end{cases}$$

where $\Delta_\nu$ is the Bessel operator given by

$$\Delta_\nu = \frac{d^2}{dx^2} + \frac{2\nu + 1}{x} \frac{d}{dx}.$$

For every $\nu > -\frac{1}{2}$, the normalized Bessel function $j_\nu$ has the Mehler integral representation

$$j_\nu(x) = \frac{2\Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos(\nu \theta) dt.$$

Consequently, we have

$$\forall x \in \mathbb{C}, \ |j_\nu(x)| \leq e^{\|\nu\| \text{Re}(x)}.$$

The normalized Bessel function $j_\nu$ satisfies, for all $x, z \in [0, +\infty[$, the following product formula [18]

$$j_\nu(x)j_\nu(y) = \frac{\Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_0^\pi j_\nu \left( y \sqrt{x^2 + z^2 - 2xz \cos(\theta)} \right) \sin^{2\nu}(\theta) d\theta.$$

It is easy to see that, for each $y \in \mathbb{R}$, the kernel $K^m_{\nu}(\cdot, y)$ of the canonical Fourier Bessel transform $F^m_\nu$ is the unique solution of [6]:

$$\begin{cases}
\Delta^m_{\nu} K^m_{\nu}(\cdot, y) = -y^2 K^m_{\nu}(\cdot, y), \\
K^m_{\nu}(0, y) = e^{\frac{y}{b} y^2}, \ \frac{d}{dx} K^m_{\nu}(0, y) = 0,
\end{cases}$$

where for each $m \in SL(2, \mathbb{R})$ such that $b \neq 0$, $\Delta^m_{\nu}$ denotes the differential operator

$$\Delta^m_{\nu} = \frac{d^2}{dx^2} + \left( \frac{2\nu + 1}{x} - 2i \frac{d}{dx} \right) \frac{d}{dx} - \left( \frac{d^2}{b^2 x^2} + 2i (\nu + 1) \frac{d}{b} \right).$$

(2.1)

Many properties of the Fourier Bessel transform carry over to the canonical Fourier Bessel transform. In particular:

**Theorem 2.1** [6]

(1) Let $m \in SL(2, \mathbb{R}).$

(a) **The reversibility property:** For all $f \in L_{1,\nu}$ with $F^m_\nu f \in L_{1,\nu},$

$$\left( F^m_\nu \circ F^{-1}_{\nu} \right) f = \left( F^{-1}_{\nu} \circ F^m_\nu \right) f = f, \ a.e.$$

(b) The canonical Fourier Bessel transform $F^m_\nu$ is a topological isomorphism from $S_s(\mathbb{R})$ into itself. More precisely, for each $(n, m) \in \mathbb{N} \times \mathbb{N}$ there exist $1_{n_1}, \ldots, 1_{m_k}, m_1, \ldots, m_k \in \mathbb{N}$ and $c > 0$ such that

$$q_{n, m} (F^m_\nu f) \leq c \sum_{i=1}^{k} q_{n_i, m_i}(f).$$

(2.2)

(2) **Operational formulas:** Let $m \in SL(2, \mathbb{R})$ such that $b \neq 0$ and $f \in S_s(\mathbb{R}).$ Then

(a) $F^m_\nu \left[ y^2 f(y) \right](x) = -b^2 \Delta^m_\nu \left[ F^m_\nu f \right](x).$

(b) $x^2 F^m_\nu(f)(x) = -b^2 F^m_\nu \left[ \Delta^{-1} \nu m \right](f).$

(2.3)

(3) **Babenko inequality:** Let $m \in SL(2, \mathbb{R})$ such that $b \neq 0.$ Let $p$ and $q$ be real numbers such that $1 < p \leq 2$ and $1 + \frac{1}{p} + \frac{1}{q} = 1.$ Then $F^m_\nu$ extends to a bounded linear operator on $L_{p,\nu}$ and we have

$$\|F^m_\nu f\|_{q,\nu} \leq |b|^\nu \left( \frac{c_{\nu} p}{c_{\nu} q} \right)^{\nu + 1} \|f\|_{p,\nu}.$$
3 Translation operators associated with $\Delta^m_\nu$

We shall begin the section by recalling some well-known facts about the generalized translation operators $T^\nu_x$ associated with the Bessel operator $\Delta_\nu$. Next we introduce and study a translation operators associated with $\Delta^m_\nu$.

3.1 Generalized translation operators associated with $\Delta_\nu$

Levitan [15] defined the generalized translation operators $T^\nu_x$ associated with $\Delta_\nu$ by:

$$T^\nu_x f(y) = u(x, y)$$

where $u(x, y)$ is the unique solution of the Cauchy problem [15]:

$$\begin{cases} \left( \frac{\partial^2}{\partial x^2} + \frac{\nu + 1}{x} \frac{\partial}{\partial x} \right) u(x, y) = \left( \frac{\partial^2}{\partial y^2} + \frac{2\nu + 1}{y} \frac{\partial}{\partial y} \right) u(x, y), \\
u(x, 0) = f(x), f \in \mathcal{E}_s(\mathbb{R}), \\
\frac{\partial}{\partial x} u(x, 0) = 0. \end{cases} \tag{3.1}$$

The solution of (3.1) can be represented in the following form [15]:

$$T^\nu_x f(y) = \frac{\Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_0^\pi f \left( \sqrt{x^2 + y^2 - 2xy \cos(\theta)} \right) (\sin(\theta))^{2\nu} d\theta. \tag{3.2}$$

It is interesting to note that $T^\nu_x f(y)$ is an even function with respect to both variables and then we can assume that $x \geq 0$ and $y \geq 0$. By a change of variables we get

$$T^\nu_x f(y) = \int_0^{+\infty} f(z) \mathcal{W}_\nu(x, y, z) z^{2\nu+1} dz; \ x, y \geq 0. \tag{3.3}$$

Here, the kernel $\mathcal{W}_\nu(x, y, z)$ is given by

$$\mathcal{W}_\nu(x, y, z) = \frac{2^{1-2\nu} \Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + 1/2)} \frac{\Delta^{2\nu-1}}{(xyz)^{2\nu} |x-y|, x+y}$$

where

$$\Delta = \frac{1}{4} \sqrt{(x+y+z)(x+y-z)(x-y+z)(y+z-x)}$$

denotes the area of the triangle with sides $x, y, z > 0$ and $|A|$ is the indicator function of $A$. We list some standard properties of the generalized translation operators $T^\nu_x$ that we shall use in this article.

**Theorem 3.1** [15] The operators $T^\nu_x$, $x \in \mathbb{R}$, satisfy:

1) **Symmetry:** $T^\nu_0 = \text{identity}$ and $T^\nu_x f(y) = T^\nu_y f(x)$.
2) **Linearity:** $T^\nu_x [af + bg] = aT^\nu_x f + bT^\nu_x g(y)$.
3) **Positivity:** If $f(y) \geq 0$ then $T^\nu_x f(y) \geq 0$.
4) **Compact support:** If $f(y) = 0$ for $y \geq a$ then $T^\nu_x f(y) = 0$ for $|x-y| \geq a$.
5) **Product formula:** For all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we have the product formula

$$T^\nu_x [s \mapsto j_\nu(\lambda s)] y) = j_\nu(\lambda x)j_\nu(\lambda y). \tag{3.4}$$

5) **Continuity of translation operators $T^\nu_x$:** The operator $T^\nu_x$ is continuous from:

- $C(\mathbb{R})$ into itself. (The space $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ is equipped with the topology of uniform convergence on compact sets)
- $\mathcal{E}_s(\mathbb{R})$ into itself.
- $L_{p,\nu}$ into itself. More precisely, for all $f \in L_{p,\nu}$, $p \in [1, +\infty]$, and $x \geq 0$, the function $T^\nu_x f$ is defined almost everywhere on $[0, +\infty[$, belongs to $L_{p,\nu}$ and we have

$$\|T^\nu_x f\|_{p,\nu} \leq \|f\|_{p,\nu}. \tag{3.5}$$

6) **Commutativity:**

- The following equality hold in $C(\mathbb{R})$: $T^\nu_x \circ T^\nu_y = T^\nu_y \circ T^\nu_x$.
- The following equality hold in $\mathcal{E}_s(\mathbb{R})$: $\Delta_\nu \circ T^\nu_x = T^\nu_x \circ \Delta_\nu$.

7) **Self-adjointness of $T^\nu_x$:** Let $f \in L_{1,\nu}$ and $g \in C_{s,b}(\mathbb{R})$. Then

$$\int_0^{+\infty} T^\nu_x f(y)g(y)y^{2\nu+1} dy = \int_0^{+\infty} f(y)T^\nu_x g(y)y^{2\nu+1} dy. \tag{3.6}$$

In particular:

- For $g(y) = 1$,

$$\int_0^{+\infty} T^\nu_x f(y)y^{2\nu+1} dy = \int_0^{+\infty} f(y)y^{2\nu+1} dy. \tag{3.7}$$

- For $g(y) = j_\nu(\lambda y)$, $\lambda \in \mathbb{R}$,

$$\mathcal{F}_\nu [T^\nu_x f](\lambda) = j_\nu(\lambda x) \mathcal{F}_\nu f(\lambda). \tag{3.8}$$
3.2 Generalized translation operators associated with $\Delta^m_{\nu}$

We begin with the following Proposition

**Proposition 3.1** Let $m \in SL(2, \mathbb{R})$ such that $b \neq 0$. For each $f \in C_*(\mathbb{R})$ the problem

$$
\begin{align*}
\begin{cases}
\Delta^m_{\nu,x} u(x,y) = \Delta^m_{\nu,y} u(x,y), \\
u(x,0) = f(x), \\
\frac{\partial}{\partial y}u(x,0) = 0,
\end{cases}
\end{align*}
$$

(3.3)

has a unique solution given by:

$$
u(x,y) = e^{\frac{i}{2}x^2 + y^2} T_x^\nu \left[ e^{-\frac{i}{2}x^2} f(s) \right](y).
$$

Here $\Delta^m_{\nu,x}$ act in the $x$ variable and given by (2.1).

**Proof.** From the transmutation property

$$e^{-\frac{i}{2}x^2} \circ \Delta^m_{\nu} \circ e^{\frac{i}{2}x^2} = \Delta^m_{\nu},$$

the system (3.3) is equivalent to

$$
\begin{align*}
\begin{cases}
\left( \frac{\partial^2}{\partial x^2} + \frac{2\nu+1}{x} \frac{\partial}{\partial x} \right) \tilde{u}(x,y) = \left( \frac{\partial^2}{\partial y^2} + \frac{2\nu+1}{y} \frac{\partial}{\partial y} \right) \tilde{u}(x,y), \\
\tilde{u}(x,0) = e^{-\frac{i}{2}x^2} f(x), \\
\frac{\partial}{\partial y}\tilde{u}(x,0) = 0,
\end{cases}
\end{align*}
$$

where $\tilde{u}(x,y) = e^{-\frac{i}{2}x^2 + y^2} u(x,y)$. By (3.1) we obtain

$$
u(x,y) = T_x^\nu \left[ e^{-\frac{i}{2}x^2} f(s) \right](y)
$$

and then

$$
u(x,y) = e^{\frac{i}{2}x^2 + y^2} T_x^\nu \left[ e^{-\frac{i}{2}x^2} f(s) \right](y).
$$

\[\blacksquare\]

**Definition 3.1** Let $m \in SL(2, \mathbb{R})$ such that $b \neq 0$. For $f \in C(\mathbb{R})$, we define the generalized translation operators associated with the operator $\Delta^m_{\nu}$ by:

$$
T_x^\nu m f(y) = e^{\frac{i}{2}x^2 + y^2} T_x^\nu \left[ e^{-\frac{i}{2}x^2} f(s) \right](y),
$$

where $T_x^\nu$ is the generalized translation operator associated with $\Delta^m_{\nu}$.

The following definition will be important in the sequel:

**Definition 3.2** The chirp multiplication operator $L_a$ and the dilatation operator $D_a$ are defined, respectively, by

$$
\begin{align*}
L_a f(x) &= e^{\frac{i}{2}x^2} f(x); \ a \in \mathbb{R}, \\
D_a f(x) &= \frac{1}{|a|^{\nu+1}} f \left( \frac{x}{a} \right); \ a \in \mathbb{R}\{0\}.
\end{align*}
$$

In the case where $f$ is an even function, we have the following result:

$$
D_a f(x) = D_{|a|} f(x); \ a \in \mathbb{R}\{0\}.
$$

The inverse of $L_a$ and the inverse of $D_a$ are given, respectively, by

$$
(L_a)^{-1} = L_{-a}, \ (D_a)^{-1} = D_{\frac{1}{a}}.
$$
We give in the following proposition some useful properties of L_a and D_a.

**Proposition 3.2 [6]** Let 1 \leq p < \infty and \alpha \in \mathbb{R}.
1) The operator L_a is an isometric isomorphism from:
- L_{p,\nu} into itself.
- C_{*,\nu}(\mathbb{R}) into itself.
2) L_a is a unitary operator from L_{2,\nu} into itself.
3) The operator L_a is a topological isomorphism from:
- S_0(\mathbb{R}) into itself.
- D_{*,\nu}(\mathbb{R}) into itself.

2) For each \alpha > 0, the operator D_a is a topological isomorphism from L_{p,\nu} into itself and we have
\[ \|D_a f\|_{p,\nu} = a^{-(\nu+1)} \|f\|_{p,\nu} ; f \in L_{p,\nu}. \]
In particular, D_a is a unitary operator from L_{2,\nu} into itself.
3) For each \alpha \neq 0, the operator D_a is a topological isomorphism from C_{*,\nu}(\mathbb{R}) into itself and we have
\[ \|D_a f\|_\infty = |a|^{-(\nu+1)} \|f\|_\infty ; f \in C_{*,\nu}(\mathbb{R}). \]
iii) The operator D_a is a topological isomorphism from:
- S_0(\mathbb{R}) into itself.
- D_{*,\nu}(\mathbb{R}) into D_{*,\nu}(\mathbb{R}).

Now we are in a position to give some properties of the \( T_x^{\nu,m} \).

**Proposition 3.3** Let \( m \in SL(2, \mathbb{R}) \) such that \( b \neq 0 \). The operators \( T_x^{\nu,m} \), \( x \in \mathbb{R} \), satisfy:
1) **Symmetry:** \( T_0^{\nu,m} = \) identity and \( T_x^{\nu,m} f(y) = T_{y}^{\nu,m} f(x). \)
2) **Linearity:** \( T_x^{\nu,m} [f + g](y) = \lambda T_x^{\nu,m} f(y) + T_x^{\nu,m} g(y). \)
3) **Compact support:** If \( f(y) = 0 \) for \( y \geq r \) then \( T_x^{\nu,m} f(y) = 0 \) for \( |x - y| \geq r \).
4) **Product formula:** For all \( x, y, z \in \mathbb{R} \), we have the product formula
\[ T_x^{\nu,m} [K^{\nu}_{\nu} (., y)] (z) = e^{-4\pi^2 y^2} K^{\nu}_{\nu} (x, y) K^{\nu}_{\nu} (z, y). \]
5) For all \( x, y > 0 \) we have
\[ T_x^{\nu,m} f(y) = \int_0^{+\infty} e^{-i\theta z} f(z) \mathcal{W}_\nu(x, y, z) z^{2\nu+1} dz, \]
where
\[ \mathcal{W}_\nu(x, y, z) = e^{\frac{4\pi^2}{\nu}(x^2 + y^2 + z^2)} \mathcal{W}_\nu(x, y, z). \]
6) **Continuity of translation operators** \( T_x^{\nu,m} \): The operator \( T_x^{\nu,m} \) is continuous from:
- \( C(\mathbb{R}) \) into itself. (The space \( C(\mathbb{R}) \) of all continuous functions on \( \mathbb{R} \) is equipped with the topology of uniform convergence on compact sets)
- \( L_{p,\nu} \) into itself.
- \( L_{p,\nu} \) into itself. More precisely, for all \( f \in L_{p,\nu}, p \in [1, +\infty], \) and \( x \geq 0 \), the function \( T_x^{\nu,m} f \) is defined almost everywhere on \([0, +\infty[\), belongs to \( L_{p,\nu} \) and we have
\[ \|T_x^{\nu,m} f\|_{p,\nu} \leq \|f\|_{p,\nu}. \] (3.4)
7) **Commutativity:**
- The following equality hold in \( C(\mathbb{R}) \) : \( T_x^{\nu,m} \circ T_y^{\nu,m} = T_y^{\nu,m} \circ T_x^{\nu,m} \).
- The following equality hold in \( L_{p,\nu} \) : \( \Delta^{\nu} \circ T_x^{\nu,m} = T_x^{\nu,m} \circ \Delta^{\nu} \).
8) Let \( f \in L_{1,\nu} \) and \( g \in C(\mathbb{R}) \). Then
\[ \int_0^{+\infty} [T_x^{\nu,m} f(y)] e^{-4\pi^2 y^2} g(y) y^{2\nu+1} dy = \int_0^{+\infty} e^{-4\pi^2 y^2} f(y) [T_x^{\nu,m} g(y)] y^{2\nu+1} dy. \] (3.5)
9) For all \( f \in L_{1,\nu}, \) we have
\[ F_{\nu}^{\nu} [T_x^{\nu,m} f] (\lambda) = e^{\frac{4\pi^2 \lambda^2}{\nu}} K_{\nu}^{-1} (x, \lambda) F_{\nu}^{\nu} f(\lambda). \] (3.6)
10) For all \( f \in L_{p,\nu}, p \in [1, 2], \) we have
\[ F_{\nu}^{\nu} [T_x^{\nu,m} f] (\lambda) = e^{\frac{4\pi^2 \lambda^2}{\nu}} K_{\nu}^{-1} (x, \lambda) F_{\nu}^{\nu} f(\lambda), \text{ a.e.} \]
Proof. Statements 1), 2) and 3) are obvious.
4) From the definition of $T_x^{\nu,m}$ it is easy to see that
\[
T_x^{\nu,m}[K_{\nu}(\cdot,y)](z) = e^{\frac{\nu}{2}(x^2+y^2)}T_x^{\nu}[s \mapsto e^{\frac{\nu}{2}y^2}j_\nu(sy/b)](z)
\]
\[
= e^{\frac{\nu}{2}(x^2+y^2)}e^{\frac{\nu}{2}y^2}T_x^{\nu}[s \mapsto j_\nu(sy/b)](z)
\]
\[
= e^{\frac{\nu}{2}(x^2+y^2)}e^{\frac{\nu}{2}y^2}j_\nu(xy/b)j_\nu(zy/b)
\]
\[
= e^{-\frac{\nu}{2}y^2}K_{\nu}(x,y)K_{\nu}(z,y).
\]
5) This is a direct consequence of (3.2).
6) This follows from
\[
T_x^{\nu,m}f(y) = \left[ L_{\nu} \circ T_x f \right] \left( L_{-\frac{\nu}{2}} f \right)
\]
and the fact that $L_{\nu}$, $L_{-\frac{\nu}{2}}$, $T_x^{\nu}$ are continuous from $\mathcal{C}(\mathbb{R})$ into itself, $\mathcal{E}_*(\mathbb{R})$ into itself and $\mathcal{L}_{p,\nu}$ into itself respectively.
Now let $f \in \mathcal{L}_{p,\nu}$. The function $T_x^{\nu,m}f$ belongs to $\mathcal{L}_{p,\nu}$ and we have
\[
\|T_x^{\nu,m}f\|_{p,\nu} = \|T_x^{\nu} f\|_{p,\nu}
\]
\[
\leq \left\| L_{-\frac{\nu}{2}} f \right\|_{p,\nu} = \|f\|_{p,\nu}.
\]
7) • Let $f \in \mathcal{C}(\mathbb{R})$. Then
\[
[T_x^{\nu,m} \circ T_y^{\nu,m}] f(z) = e^{\frac{\nu}{2}(x^2+y^2+z^2)} \left[ T_x^{\nu} \circ T_y^{\nu} \right] e^{-\frac{\nu}{2}x^2}f(s)(z)
\]
\[
= e^{\frac{\nu}{2}(x^2+y^2+z^2)} \left[ T_y^{\nu} \circ T_x^{\nu} \right] e^{-\frac{\nu}{2}x^2}f(s)(z)
\]
\[
= \left[ T_y^{\nu,m} \circ T_x^{\nu,m} \right] f(z).
\]
• Let $f \in \mathcal{E}_*(\mathbb{R})$. Then
\[
[\Delta_{\nu}^{m} \circ T_x^{\nu,m}] f(y) = e^{\frac{\nu}{2}(x^2+y^2)} \left[ \Delta_{\nu} \circ T_x^{\nu} \right] e^{-\frac{\nu}{2}x^2}f(s)(y)
\]
\[
= e^{\frac{\nu}{2}(x^2+y^2)} \left[ T_x^{\nu} \circ \Delta_{\nu} \right] e^{-\frac{\nu}{2}x^2}f(s)(y)
\]
\[
= \left[ T_x^{\nu,m} \circ \Delta_{\nu}^{m} \right] f(y).
\]
8)Let $f \in \mathcal{L}_{1,\nu}$ and $g \in \mathcal{C}_{b}(\mathbb{R})$. Then
\[
\int_0^{+\infty} \left[ T_x^{\nu,m} f(y) \right] e^{i\frac{\nu}{2}y^2} g(y) y^{2\nu+1} dy = e^{\frac{\nu}{2}x^2} \int_0^{+\infty} T_x^{\nu} f(s) e^{-\frac{\nu}{2}s^2} g(y) y^{2\nu+1} dy
\]
\[
= e^{\frac{\nu}{2}x^2} \int_0^{+\infty} e^{-\frac{\nu}{2}s^2} f(y) T_x^{\nu} e^{-\frac{\nu}{2}y^2} g(s) y^{2\nu+1} dy
\]
\[
= \int_0^{+\infty} e^{-\frac{\nu}{2}y^2} f(y) \left[ T_x^{\nu,m} g(y) \right] y^{2\nu+1} dy.
\]
9)Let $f \in \mathcal{L}_{1,\nu}$. Then
\[
\left[ (ib)^{\nu+1}/c_{\nu} \right] \mathcal{F}_{\nu}^{-1} \left[ T_x^{\nu,m-1} f \right](\lambda) = e^{i\frac{\nu}{2}(\lambda^2-\frac{\pi^2}{4})} \int_0^{+\infty} T_x^{\nu} f(s) \left( y j_\nu(\lambda y/b) y^{2\nu+1} dy
\]
\[
= e^{i\frac{\nu}{2}(\lambda^2-\frac{\pi^2}{4})} \int_0^{+\infty} e^{\frac{\nu}{2}y^2} f(y) T_x^{\nu} \left( y j_\nu(\lambda y/b) y^{2\nu+1} dy
\]
\[
= e^{i\frac{\nu}{2}(\lambda^2-\frac{\pi^2}{4})} j_\nu(\lambda y/b) \int_0^{+\infty} e^{\frac{\nu}{2}y^2} f(y) j_\nu(\lambda y/b) y^{2\nu+1} dy
\]
\[
= \left[ (ib)^{\nu+1}/c_{\nu} \right] e^{\frac{\nu}{2}y^2} K_{\nu}^{-1}(x,\lambda) \mathcal{F}_{\nu} f(\lambda).
\]
10) From 9) the result is true for $f \in \mathcal{L}_{1,\nu} \cap \mathcal{L}_{p,\nu}$. On the other hand, Babenko inequality (2.4) and the relation (3.3) show that the mappings $f \mapsto \mathcal{F}_{\nu}^{-1} \left[ T_x^{\nu,m-1} f \right]$ and $f \mapsto \mathcal{F}_{\nu} f$ are continuous from $\mathcal{L}_{p,\nu}$ into $\mathcal{L}_{q,\nu}$ ($1/p + 1/q = 1$). We obtain the result from density of $\mathcal{L}_{1,\nu} \cup \mathcal{L}_{p,\nu}$ in $\mathcal{L}_{p,\nu}$. \textbullet

Corollary 3.1 Let $m \in \text{SL}(2,\mathbb{R})$ such that $b \neq 0$ and $x \in \mathbb{R}$. Then the operator $T_x^{\nu,m-1}$ leaves $\mathcal{S}_*(\mathbb{R})$ invariant and for each $f \in \mathcal{S}_*(\mathbb{R})$ we have:
\[
T_x^{\nu,m-1} f(y) = \frac{c_{\nu}}{(-ib)^{\nu+1} e^{-\frac{\nu}{2}y^2}} \int_0^{+\infty} j_\nu(\lambda y/b)K_{\nu}^{-1}(x,\lambda) \mathcal{F}_{\nu} f(\lambda) \lambda^{2\nu+1} d\lambda.
\]
Proof. To prove this, we first observe that if \( f \in \mathcal{S}_s(\mathbb{R}) \) then the inequality \( \| f \|_{C_{\nu}} \) shows that \( y \mapsto \| T_{\nu}^{y,m} f(y) \) is a continuous function of \( L_{1,\nu} \). The result is then a consequence of \( \mathcal{H},(3.0) \) and the reversibility property of the canonical Fourier Bessel transform (see Theorem \( \mathcal{H},(2.4) \)).

We conclude this subsection with the following two Theorems.

**Theorem 3.2** Let \( m \in SL(2, \mathbb{R}) \) such that \( b \neq 0 \).

(i) For all \( f \in \mathcal{C}_{s,0}(\mathbb{R}) \), we have \( \lim_{y \to 0} \| T_{\nu}^{y,m} f - f \|_{\infty} = 0 \).

(ii) For all \( f \in L_{p,\nu} \) with \( 1 \leq p < +\infty \), we have \( \lim_{y \to 0} \| T_{\nu}^{y,m} f - f \|_{p,\nu} = 0 \).

**Proof.** (i) First consider the case where \( f \in \mathcal{C}_{s,c}(\mathbb{R}) \). From the definition of \( T_{\nu}^{y,m} \) and the fact that \( \frac{\Gamma(\nu+1)}{\sqrt{\pi}(\nu+1/2)} \int_0^\infty \sin(\theta)^{2\nu} d\theta = 1 \), we can write

\[
T_{\nu}^{y,m} f(x) - f(x) = T_{\nu}^{y,m} f(y) - f(x) = a_y(x) + b_y(x)
\]

where

\[
a_y(x) = f(x) \frac{\Gamma(\nu+1)}{\sqrt{\pi(\nu+1/2)}} \int_0^\pi \left[ e^{iy\theta} \cos(\theta) - 1 \right] \sin(\theta)^{2\nu} d\theta,
\]

\[
b_y(x) = \frac{\Gamma(\nu+1)}{\sqrt{\pi(\nu+1/2)}} \int_0^\pi e^{iy\theta} \cos(\theta) \left[ f\left( \sqrt{x^2 + y^2 - 2xy \cos(\theta)} \right) - f(x) \right] \sin(\theta)^{2\nu} d\theta.
\]

Clearly

\[
a_y(x) = f(x) \frac{\Gamma(\nu+1)}{\sqrt{\pi(\nu+1/2)}} \sum_{n=1}^{+\infty} \left( \frac{i^n y^n}{n!} \right) \int_0^\pi \cos^n(\theta) (\sin(\theta))^{2\nu} d\theta,
\]

and since

\[
\int_0^\pi \cos^n(\theta) (\sin(\theta))^{2\nu} d\theta = \frac{1 + (-1)^n}{2} \int_0^\pi \cos^n(\theta) (\sin(\theta))^{2\nu} d\theta = \frac{\Gamma(\nu+1)}{\sqrt{\pi(\nu+1/2)}} \frac{\Gamma(\nu+1/2)}{\Gamma(\nu+1)}
\]

we conclude

\[
a_y(x) = \left[ j_\nu \left( \frac{dy}{b} \right) - 1 \right] f(x).
\]

Let \( R > 0 \) such that \( \text{supp} f \subset [0,R] \). Then

\[
\|a_y\|_{\infty} \leq \|f\|_{\infty} \left[ j_\nu \left( \frac{dy}{b} \right) - 1 \right] \to 0 \quad \text{as} \quad y \to 0.
\]

By the triangle inequality, we have

\[
\left| \sqrt{x^2 + y^2 - 2xy \cos(\theta)} - x \right| = \left| \sqrt{(x-y \cos(\theta))^2 + y^2 \sin^2(\theta)} - \sqrt{x^2} \right| \leq |y|.
\]

Now given \( \epsilon > 0 \), by the uniform continuity of \( f \) there exists \( \delta > 0 \) such that \( |f(u) - f(v)| \leq \epsilon \) whenever \( |u-v| < \delta \). Thus we get for \( |y| < \delta \) and \( x \in \mathbb{R} \),

\[
|b_y(x)| \leq \frac{\Gamma(\nu+1)}{\sqrt{\pi(\nu+1/2)}} \int_0^\pi \left| f\left( \sqrt{x^2 + y^2 - 2xy \cos(\theta)} \right) - f(x) \right| \sin(\theta)^{2\nu} d\theta \leq \epsilon.
\]

Hence

\[
\lim_{y \to 0} \| T_{\nu}^{y,m} f - f \|_{\infty} = 0.
\]

Now suppose \( f \in \mathcal{C}_{s,0}(\mathbb{R}) \). If \( \epsilon > 0 \) and since \( \mathcal{C}_{s,c}(\mathbb{R}) \) is dense in \( \mathcal{C}_{s,0}(\mathbb{R}) \), there exists \( g \in \mathcal{C}_{s,c}(\mathbb{R}) \) such that \( \|f-g\|_{\infty} \leq \frac{\epsilon}{3} \), so

\[
\| T_{\nu}^{y,m} f - f \|_{\infty} \leq \| T_{\nu}^{y,m} (f-g) \|_{\infty} + \| T_{\nu}^{y,m} g - g \|_{\infty} + \| g - f \|_{p,\nu} \leq 2 \| g - f \|_{\infty} + \| T_{\nu}^{y,m} g - g \|_{\infty} \leq \frac{2\epsilon}{3} + \| T_{\nu}^{y,m} g - g \|_{\infty}
\]

and \( \| T_{\nu}^{y,m} g - g \|_{\infty} \leq \frac{\epsilon}{3} \) if \( y \) is sufficiently small.

(ii) First, if \( f \in \mathcal{C}_{s,c}(\mathbb{R}) \), for \( |y| \leq 1 \) the functions \( T_{\nu}^{y,m} f \) are all supported in a common compact set \( [0,R] \), so

\[
\| T_{\nu}^{y,m} f - f \|_{p,\nu} \leq \left( \int_0^R x^{2\nu+1} dx \right) \| T_{\nu}^{y,m} f - f \|_{\infty} \to 0 \quad \text{as} \quad y \to 0.
\]

The general case then follows from density of \( \mathcal{C}_{s,c}(\mathbb{R}) \) in \( L_{p,\nu} \).
Theorem 3.3 Let \( m \in \text{SL}(2, \mathbb{R}) \) such that \( b \neq 0 \). Let \( x, y \in \mathbb{R} \) and \( \sigma > 0 \). For all \( f \in S_\sigma(\mathbb{R}) \), we have:

(i) The function \( t \mapsto e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}] \) is \( C^\infty \) on \([0, +\infty)\) with

\[
\frac{d^n}{dt^n}\left[e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}]\right] = \frac{(-\sigma)^n}{2^\nu(b)^{\nu+1}(n+\nu+1)} \times y^{2n} \int_0^{+\infty} j_{\nu+n}\left(\lambda\sqrt{4\sigma ty}/b\right) \lambda^{2n} K^m_\nu(x, \lambda) \mathcal{F}^{m-1}_{\nu} f(\lambda) \lambda^{2\nu+1} d\lambda.
\]

(ii) For all non-negative integers \( n \),

\[
\frac{d^n}{dt^n}\left(e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}]\right)|_{t=0} = \frac{\Gamma(\nu+1)}{\Gamma(n+\nu+1)} \sigma^n y^{2n} (\Delta^m_{\nu})^n f(x).
\]

(iii) For all non-negative integers \( n \) and \( t > 0 \),

\[
e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}] = \sum_{k=0}^{n} \frac{\Gamma(\nu+1)}{\Gamma(k+\nu+1)k!} t^k \sigma^k y^{2k} (\Delta^m_{\nu})^k f(x)
\]

\[+ \frac{n+1}{n!} \int_0^1 (1-s)^n \left(\frac{d^{n+1}}{dt^{n+1}}\right) \left[e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}]\right] (st) d\lambda.
\]

(iv) For all non-negative integers \( n \), there is a constant \( c_n > 0 \) and \( n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N} \) such that

\[
\left|\frac{d^n}{dt^n}e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}]\right|_t \leq c_n y^{2n} \sum_{i=1}^k q_{n_i, m_i}(f).
\]

Proof. (i) By Corollary 3.1 we have

\[
e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}] = \frac{e_\nu}{(ib)^{\nu+1}} \int_0^{+\infty} j_\nu\left(\lambda\sqrt{4\sigma ty}/b\right) K^m_\nu(x, \lambda) \mathcal{F}^{m-1}_{\nu} f(\lambda) \lambda^{2\nu+1} d\lambda.
\]

The desired result is therefore a consequence of Theorem of differentiation under the integral sign; since

\[
\frac{d^n}{dt^n} j_\nu\left(\lambda\sqrt{4\sigma ty}/b\right) = (-1)^n \frac{\Gamma(\nu+1)}{\Gamma(n+\nu+1)} \frac{\lambda^{2n} y^{2n} \sigma^n}{b^{2n}} j_{n+\nu}\left(\lambda\sqrt{4\sigma ty}/b\right),
\]

\[
\left|j_{n+\nu}\left(\lambda\sqrt{4\sigma ty}/b\right) K^m_\nu(x, \lambda)\right| \leq 1 \text{ and } \lambda \mapsto \lambda^{2n} \mathcal{F}^{m-1}_{\nu} f \in S_\sigma(\mathbb{R}).
\]

(ii) Clearly

\[
\frac{d^n}{dt^n}\left(e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}]\right)|_{t=0} = \frac{(-\sigma)^n}{2^\nu(b)^{\nu+1}(n+\nu+1)} \int_0^{+\infty} K^m_\nu(x, \lambda) \lambda^{2n} \mathcal{F}^{m-1}_{\nu} f(\lambda) \lambda^{2\nu+1} d\lambda.
\]

By (2.3)

\[
\lambda^{2n} \mathcal{F}^{m-1}_{\nu} f(\lambda) = (-b^2)^n \mathcal{F}^{m-1}_{\nu} (\Delta^m_{\nu})^n f(\lambda).
\]

Then

\[
\frac{d^n}{dt^n}\left(e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}]\right)|_{t=0} = \frac{\Gamma(\nu+1)}{\Gamma(n+\nu+1)} \sigma^n y^{2n} \mathcal{F}^{m-1}_{\nu} \left(\Delta^m_{\nu}\right)^n f(x).
\]

(iii) This follows from Taylor’s formula and (ii).

(iv) From (i) it is clear that

\[
\left|\frac{d^n}{dt^n}e^{-2it^2xy}\left[T_{x^v}^\nu f\right][\sqrt{4\sigma ty}]\right| \leq \left|\frac{(-\sigma)^n}{2^\nu(b)^{\nu+1}(n+\nu+1)} \times y^{2n} \int_0^{+\infty} \lambda^{2n} \left|\mathcal{F}^{m-1}_{\nu} f(\lambda)\right| \lambda^{2\nu+1} d\lambda.
\]

Now choose \( m \in \mathbb{N} \) and \( m > \nu + 1 \). Then

\[
\int_0^{+\infty} \lambda^{2n} \left|\mathcal{F}^{m-1}_{\nu} f(\lambda)\right| \lambda^{2\nu+1} d\lambda \leq \left(\int_0^{+\infty} \lambda^{2\nu+1} d\lambda\right) \sup_{\lambda \geq 0} \left(1 + \lambda^{2m}\right) \lambda^{2n} \left|\mathcal{F}^{m-1}_{\nu} f(\lambda)\right| \leq \left(\int_0^{+\infty} \lambda^{2\nu+1} d\lambda\right) \left(q_{n+m, 0}\left|\mathcal{F}^{m-1}_{\nu} f\right| + q_{n, 0}\left|\mathcal{F}^{m-1}_{\nu} f\right|\right).
\]

The desired result is therefore a consequence of (2.2).
3.3 Generalized convolution product

Let $m \in SL(2, \mathbb{R})$ such that $b \neq 0$ and let $f$ and $g$ be measurable functions on $[0, +\infty[$. The generalized convolution of $f$ and $g$ is the function $f \ast_{m} g$ defined by

$$f \ast_{m} g(x) = \int_{0}^{+\infty} [T_{x}^{\nu,m} f(y)] e^{-i\frac{\nu}{2} y^2} g(y) dy$$

(3.8)

for all $x$ such that the integral exists. The elementary properties of convolutions are summarized in the following proposition.

**Proposition 3.4** Assuming that all integrals in question exist, we have

1) $f \ast_{m} g = g \ast_{m} f$.

2) $T_{x}^{\nu,m} (f \ast_{m} g) = [T_{x}^{\nu,m} f] \ast_{m} g = f \ast_{m} [T_{x}^{\nu,m} g]$.

3) $(f \ast_{m} g) \ast_{m} h = f \ast_{m} (g \ast_{m} h)$.

**Proof.**

1) This is [56].

2) By Fubini’s theorem.

3) follows from (1), (2) and Fubini’s theorem: ■

The following proposition contain the basic facts about convolutions of $L_{p,\nu}$ functions.

**Proposition 3.5** *(Young’s Inequality)* Let $m \in SL(2, \mathbb{R})$ such that $b \neq 0$. Suppose $1 \leq p, q, r \leq \infty$ and $p^{-1} + q^{-1} = r^{-1} + 1$. If $f \in L_{p,\nu}$ and $g \in L_{q,\nu}$, then $f \ast_{m} g \in L_{r,\nu}$ and

$$\|f \ast_{m} g\|_{r,\nu} \leq \|f\|_{p,\nu} \|g\|_{q,\nu}.$$  

(3.9)

**Proof.** By applying Hölder’s inequality to the product

$$\left| T_{x}^{\nu,m} f(y) e^{-i\frac{\nu}{2} y^2} g(y) \right| = \left| (T_{x}^{\nu,m} f(y))^p |g(y)|^q \right|^{1/r} \left| (T_{x}^{\nu,m} f(y))^p |g(y)|^q \right|^{1/q - 1/r},$$

we have

$$\int_{0}^{+\infty} T_{x}^{\nu,m} f(y) e^{-i\frac{\nu}{2} y^2} g(y) y^{2r+1} dy \leq \left( \int_{0}^{+\infty} |T_{x}^{\nu,m} f(y)|^p |g(y)|^q y^{2r+1} dy \right)^{1/r} \left( \int_{0}^{+\infty} |T_{x}^{\nu,m} f(y)|^p y^{2q+1} dy \right)^{1/q - 1/r} \times \left( \int_{0}^{+\infty} |g(y)|^q y^{2r+1} dy \right)^{1/q - 1/r}.

This leads to:

$$\left| (f \ast_{m} g)(x) \right|^{r} \leq \|T_{x}^{\nu,m} f\|_{p,\nu}^{r-p} \|g\|_{q,\nu}^{r-q} \int_{0}^{+\infty} |T_{x}^{\nu,m} f(y)|^p |g(y)|^q y^{2r+1} dy$$

and using [5.4] we obtain

$$\left| (f \ast_{m} g)(x) \right|^{r} \leq \|f\|_{p,\nu}^{r-p} \|g\|_{q,\nu}^{r-q} \int_{0}^{+\infty} |T_{x}^{\nu,m} f(y)|^p |g(y)|^q y^{2r+1} dy.$$ 

Multiply both sides by $x^{2r+1}$ and integrate from 0 to $+\infty$, we get

$$\|f \ast_{m} g\|_{r,\nu}^{r} \leq \|f\|_{p,\nu}^{r-p} \|g\|_{q,\nu}^{r-q} \int_{0}^{+\infty} \left[ \int_{0}^{+\infty} |T_{x}^{\nu,m} f(y)|^p |g(y)|^q y^{2r+1} dy \right] x^{2r+1} dx \times \left[ \int_{0}^{+\infty} |T_{y}^{\nu,m} f(x)|^p x^{2r+1} dx \right]$$

$$\leq \|f\|_{p,\nu} \|g\|_{q,\nu}^{r}.$$
Proposition 3.6 Let $m \in SL(2, \mathbb{R})$ such that $b \neq 0$.
1) Let $f$ and $g$ be two function in $L_{1, \nu}$. We have
\[
\forall x \in \mathbb{R}, \quad (c_{\nu}/(ib)^{\nu+1}) \mathcal{F}^{m}_{\nu} \left( f_{\nu, m^{-1}} \ast g \right)(x) = e^{-\frac{\beta}{2} x^2} \mathcal{F}^{m}_{\nu} f(x) \mathcal{F}^{m}_{\nu} g(x).
\]
2) Let $f \in L_{1, \nu}$ and $g \in L_{p, \nu}$ ($p \in [1, 2]$). We have
\[
(c_{\nu}/(ib)^{\nu+1}) \mathcal{F}^{m}_{\nu} \left( f_{\nu, m^{-1}} \ast g \right)(x) = e^{-\frac{\beta}{2} x^2} \mathcal{F}^{m}_{\nu} f(x) \mathcal{F}^{m}_{\nu} g(x), \ a.e.
\]

**Proof.** 1) According to the previous proposition, one has $(f_{\nu, m^{-1}} \ast g) \in L_{1, \nu}$. From the definition of $\mathcal{F}^{m}_{\nu}$ and (3.6) it is easy to see that
\[
\mathcal{F}^{m}_{\nu} \left( f_{\nu, m^{-1}} \ast g \right)(x) = \frac{c_{\nu}}{(ib)^{\nu+1}} \int_{0}^{\infty} K^{m}_{\nu}(x, y) \left[ \int_{0}^{\infty} T^{\nu, m^{-1}}_{y} f(z) e^{i \frac{\beta}{2} x^2} g(z) \right] y^{2\nu+1} dy dz,
\]
where $F_{\nu}$ and we require that $F_{\nu} \in L_{p, \nu}$.

The Fubini theorem was used in the second line, since
\[
\int_{0}^{\infty} \int_{0}^{\infty} |K^{m}_{\nu}(x, y) T^{\nu, m^{-1}}_{y} f(z) e^{i \frac{\beta}{2} x^2} g(z)| y^{2\nu+1} dy dz \leq \int_{0}^{\infty} \int_{0}^{\infty} |T^{\nu, m^{-1}}_{y} f(z)| |g(z)| y^{2\nu+1} z^{2\nu+1} dy dz = \|f\|_{L_{1, \nu}} \|g\|_{L_{1, \nu}} < \infty.
\]

2) From 1) the result is true for $g \in L_{1, \nu} \cap L_{p, \nu}$. On the other hand, the Babenko inequality (2.4) and Proposition 3.5 show that the mappings $g \mapsto \mathcal{F}^{m}_{\nu} \left( f_{\nu, m^{-1}} \ast g \right)$ and $g \mapsto \mathcal{F}^{m}_{\nu} f \mathcal{F}^{m}_{\nu} g$ are continuous from $L_{p, \nu}$ into $L_{q, \nu}$ (1/p + 1/q = 1).

We obtain the result from density of $L_{1, \nu} \cap L_{p, \nu}$ in $L_{p, \nu}$. 

### 4 Heat equations and heat semigroups related to $\Delta^{m^{-1}}_{\nu}$.

In this section, we shall illustrate our theory by studying the following heat equations associated to the operator $\Delta^{m^{-1}}_{\nu}$:
\[
\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} u(t, x) = \sigma \Delta^{m^{-1}}_{\nu} u(t, x), & (t, x) \in (0, +\infty) \times \mathbb{R} \\
u(0, x) = f(x),
\end{array} \right.
\]
where $f$ is a given function in $L_{p, \nu}$ (1 $\leq p \leq \infty$) and $\sigma > 0$ is the coefficient of heat conductivity. Here the initial data $u(0, x) = f(x)$ means that $u(t, x) \to f(x)$ as $t \to 0$ in the norm of $L_{p, \nu}$.

In order to solve the Cauchy problem (4.1), it suggests itself to apply the canonical Fourier Bessel transform $\mathcal{F}^{m}_{\nu}$, thus converting the partial differential equation $(\partial_{t} - \sigma \Delta^{m^{-1}}_{\nu}) u = 0$ into the simple ordinary differential equation $[\partial_{t} + (\sigma x^2/b^2)] \mathcal{F}^{m}_{\nu} [s \mapsto u(t, s)](x) = 0$. The general solution of this equation is
\[
\mathcal{F}^{m}_{\nu} [s \mapsto u(t, s)](x) = c(x) e^{-\frac{\beta}{2} x^2},
\]
and we require that $\mathcal{F}^{m}_{\nu} [s \mapsto u(0, s)](x) = \mathcal{F}^{m}_{\nu} f(x)$. We therefore obtain a solution to our problem by taking $c(x) = \mathcal{F}^{m}_{\nu} f(x)$; this gives $\mathcal{F}^{m}_{\nu} [s \mapsto u(t, s)](x) = e^{-\frac{\beta}{2} x^2} \mathcal{F}^{m}_{\nu} f(x)$.

By
\[
\int_{0}^{\infty} e^{-\beta y^2} j_{\nu} \left( \frac{xy}{b} \right) y^{2\nu+1} dy = \Gamma(\nu+1) 2^{\nu+1} \beta^{-\nu-1} \frac{e^{-\frac{x^2}{4\beta}}}{\sqrt{\pi}}, \quad \beta > 0, \ b \neq 0,
\]
we can write
\[
e^{-\frac{\beta}{2} x^2} = \left( (ib)^{\nu+1}/c_{\nu} \right) e^{-i \frac{\beta}{2} x^2} \mathcal{F}^{m}_{\nu} \left[ \mathcal{F}^{m}_{\nu}^{-1} \right](x),
\]
(4.2)
We begin with the following Lemma.

The heat equation

After applying the Proposition 3.6, we obtain

\[ \mathcal{F}_\nu [u(t,s)](x) = \mathcal{F}_\nu [\mathcal{P}_t^{m-1}(x), f(x)] = \frac{e^{-\frac{(y^2)}{4t}}}{\sqrt{\pi t}} \frac{(y^2)}{4t} f(y). \]

So we have a candidate for a solution:

\[ u(t,x) = \left[ \mathcal{P}_t^{m-1}(x) \ast f \right](x) = \int_0^{+\infty} T_x^{(\nu,m-1)} \left[ \mathcal{P}_t^{m-1}(y) \right] e^{\frac{4t}{y^2} f(y)} y^{2\nu+1} dy. \]

So far this is all formal, since we have not specified conditions on \( f \) to ensure that these manipulations are justified. Our object is to show that the function \( u(t,x) = \left[ \mathcal{P}_t^{m-1}(x) \ast f \right](x) \) with \( (t,x) \in (0,+) \times \mathbb{R} \) and \( f \in \mathcal{L}_{\nu,\nu} \) solves the heat equation \((\partial_t - \sigma_\nu^{\Delta^{m-1}}) u = 0\) and we study the problem of uniqueness of solutions of the Cauchy problem \((4.1)\).

We begin with the following Lemma.

**Lemma 4.1** Let \( \delta, r, s \in \mathbb{C} \) such that \( \text{Re}(\delta) > 0 \). Then

\[ \int_0^{+\infty} e^{-\delta x^2} j_{\nu}(2rx) j_{\nu}(2sx) x^{2\nu+1} dy = \frac{\Gamma(\nu + 1)}{2^\nu \nu + 1} e^{-(1/\delta)(s^2 + r^2)} j_{\nu}(2irs/\delta). \]

**Proof.** We need the following formulas (see 6.615 in [12]):

\[ \int_0^{+\infty} e^{-\delta y} j_{\nu}(2r \sqrt{y}) j_{\nu}(2s \sqrt{y}) dy = \frac{e^{-\frac{(1/\delta)(s^2 + r^2)}{\delta}}}{\delta} I_{\nu}(2rs/\delta); \quad |\arg(\delta)| < \pi/2, \quad \nu \geq 0, \]

where

\[ I_{\nu}(z) = \sum_{n=0}^{+\infty} \frac{(z/2)^{2n+\nu}}{n! \Gamma(n + \nu + 1)}. \]

In view of the relationship

\[ j_{\mu}(x) = 2^\mu \Gamma(\mu + 1) j_{\mu}(x) x^\mu \]

between the normalized Bessel function \( j_{\mu} \) and the classical Bessel function \( J_{\mu} \), the equation \((4.3)\) takes when \( \delta, r, s > 0 \) the following form:

\[ \int_0^{+\infty} e^{-\delta x^2} j_{\nu}(2rx) j_{\nu}(2sx) x^{2\nu+1} dy = \frac{\Gamma(\nu + 1)}{2^\nu \nu + 1} e^{-(1/\delta)(s^2 + r^2)} j_{\nu}(2irs/\delta). \]

After the change of variables \( x = \sqrt{y} \) and the analytic continuation of holomorphic functions, the equation \((4.3)\) becomes

\[ \int_0^{+\infty} e^{-\delta x^2} j_{\nu}(2rx) j_{\nu}(2sx) x^{2\nu+1} dy = \frac{\Gamma(\nu + 1)}{2^\nu \nu + 1} e^{-(1/\delta)(s^2 + r^2)} j_{\nu}(2irs/\delta); \quad \delta, r, s \in \mathbb{C}, \ \text{Re}(\delta) > 0. \]

\[ \square \]

**Definition 4.1** Let \( \mathbf{m} \in SL(2, \mathbb{R}) \) such that \( b \neq 0 \). The generalized heat kernel \( G_t^{m-1} \) associated to \( \Delta^{m-1} \) is defined by

\[ G_t^{m-1}(x,y) = T_x^{(\nu,m-1)} \left[ \mathcal{P}_t^{m-1} \right](y), \quad x, y \in \mathbb{R}, \quad t > 0. \]

We collect some basic properties of the generalized heat kernel \( G_t^{m-1} \).

**Proposition 4.1** Let \( \mathbf{m} \in SL(2, \mathbb{R}) \) such that \( b \neq 0 \). The generalized heat kernel \( G_t^{m-1} \) has the following properties:

1) \[ G_t^{m-1}(x,y) = \frac{2 \Gamma(\nu + 1) (4\pi t)^{-\nu} e^{-\frac{y^2}{4t}} e^{-\frac{x^2}{4t}}}{\Gamma(\nu + 1) (4\pi t)^{\nu}} j_{\nu} \left( \frac{xy}{2\sqrt{t}} \right). \]

2) \[ |G_t^{m-1}(x,y)| \leq \frac{2e^{-\frac{y^2}{4t}}}{\Gamma(\nu + 1) (4\pi t)^{\nu+1}}. \]

12
3) \( \int_0^{+\infty} e^{\frac{i}{2}(x^2+y^2)} G_t^{m^{-1}}(x,y)y^{2\nu+1}dy = 1. \)

4) \( G^{m^{-1}}_{t+s}(x,y) = \int_0^{+\infty} G_t^{m^{-1}}(x,z)G_s^{m^{-1}}(y,z) e^{\frac{i}{2}z^2}z^{2\nu+1}dz. \)

5) For fixed \( y \in \mathbb{R} \), the function \( u(t, x) = G^{m^{-1}}_t(x,y) \) solves the heat equation \( \frac{\partial}{\partial t} u(t,x) = \sigma \Delta^{m^{-1}} u(t,x) \) on \( (0, +\infty) \times \mathbb{R} \). (\( G^{m^{-1}}_t \) defines a fundamental solution of the heat equation \((4.4)\).)

**Proof.** 1) From the definition of \( T_x^{\nu, m^{-1}} \) it follows that

\[
G^{m^{-1}}_t(x,y) = \frac{2}{\Gamma(\nu+1)(4\sigma t)^{\nu+1}} e^{-\frac{2}{4\sigma t}(x^2+y^2)} T_x^{\nu} e^{-\frac{2}{4\sigma t}}(y).
\]

Now a simple calculation shows that

\[
T_x^{\nu} e^{-\frac{2}{4\sigma t}}(y) = \frac{\Gamma(\nu+1)}{\sqrt{\pi} \Gamma(\nu+1/2)} e^{-\frac{x^2+y^2}{4\sigma t}} \int_0^{\pi} e^{\frac{xy}{2\sigma t}} \cos(\theta) (\sin(\theta))^{2\nu} d\theta
\]

By \((3.7)\) we conclude

\[
T_x^{\nu} e^{-\frac{2}{4\sigma t}}(y) = e^{-\frac{x^2+y^2}{4\sigma t}} j_{\nu} \left( \frac{ixy}{2\sigma t} \right).
\]

3) From Lemma \(4.1\) we know that

\[
\int_0^{+\infty} e^{\frac{1}{2}(x^2+y^2)} G_t^{m^{-1}}(x,y)y^{2\nu+1}dy = \frac{2e^{-\frac{2}{4\sigma t}}}{\Gamma(\nu+1)(4\sigma t)^{\nu+1}} \int_0^{+\infty} e^{-\frac{2}{4\sigma t}} j_{\nu} \left( \frac{ixy}{2\sigma t} \right) y^{2\nu+1}dy = 1.
\]

4) We have

\[
\int_0^{+\infty} G_t^{m^{-1}}(x,z)G_s^{m^{-1}}(y,z) e^{\frac{i}{2}z^2}z^{2\nu+1}dz = \frac{4e^{-\frac{2}{4\sigma t}(x^2+y^2)} e^{-\frac{2}{4\sigma t}} e^{-\frac{2}{4\sigma t}}}{\Gamma^2(\nu+1)(4\sigma t)^{\nu+1}(4\sigma s)^{\nu+1}}
\]

\[
\times \int_0^{+\infty} e^{-\frac{2}{4\sigma t}z} j_{\nu} \left( \frac{ixz}{2\sigma t} \right) j_{\nu} \left( \frac{iys}{2\sigma s} \right) z^{2\nu+1}dz.
\]

From Lemma \(4.1\),

\[
\int_0^{+\infty} e^{-\frac{2}{4\sigma t}z} j_{\nu} \left( \frac{ixz}{2\sigma t} \right) j_{\nu} \left( \frac{iys}{2\sigma s} \right) z^{2\nu+1}dz = \frac{(4\sigma ts)^{\nu+1} \Gamma(\nu+1)}{2(t+s)^{\nu+1}} e^{\frac{iz^2(\nu+1)}{2\sigma t} + \frac{iys^2(\nu+1)}{2\sigma s}} j_{\nu} \left( \frac{ixy}{2\sigma(t+s)} \right).
\]

These give the desired result.

5) We simply compute for \( y \in \mathbb{R} \) and \( t > 0 \),

\[
\frac{\partial}{\partial t} G_t^{m^{-1}}(x,y) = \left[ \frac{x^2+y^2}{4\sigma t^2} - \frac{\nu+1}{t} \right] G_t^{m^{-1}}(x,y) + \frac{2e^{-\frac{2}{4\sigma t}(x^2+y^2)} e^{-\frac{2}{4\sigma t}}}{\Gamma(\nu+1)(4\sigma t)^{\nu+1}} \left( - \frac{ixy}{2\sigma t} \right) j_{\nu} \left( \frac{ixy}{2\sigma t} \right).
\]

By the transmutation property

\[
e^{-\frac{1}{2}x^2} \sigma^{m^{-1}} e^{-\frac{1}{2}x^2} = \Delta_{\nu},
\]

we have

\[
\sigma \Delta^{m^{-1}} \left[ G_t^{m^{-1}}(x,y) \right] = \sigma e^{-\frac{1}{2}x^2} \Delta_{\nu} \left[ e^{\frac{1}{2}x^2} G_t^{m^{-1}}(x,y) \right]
\]

\[
= \frac{2\sigma e^{-\frac{1}{2}y^2(x^2+y^2)} e^{-\frac{2}{4\sigma t}}}{\Gamma(\nu+1)(4\sigma t)^{\nu+1}} \Delta_{\nu} \left[ e^{-\frac{1}{2}x^2} j_{\nu} \left( \frac{ixy}{2\sigma t} \right) \right].
\]

An easy calculation shows that

\[
\Delta_{\nu} \left[ e^{-\frac{1}{2}x^2} j_{\nu} \left( \frac{ixy}{2\sigma t} \right) \right] = e^{-\frac{1}{2}x^2} \left[ \frac{i}{2\sigma t} \right] j_{\nu} \left( \frac{ixy}{2\sigma t} \right) + \left( \frac{2\nu+1}{4\sigma t^2} - \frac{ixy}{2\sigma t} \right) j_{\nu} \left( \frac{ixy}{2\sigma t} \right) + \left( \frac{x^2}{4\sigma t^2} - \frac{\nu+1}{\sigma t} \right) j_{\nu} \left( \frac{ixy}{2\sigma t} \right).
\]
Since \( j'_\nu(z) + \frac{2\nu+1}{z} j'_\nu(z) = -j_\nu(z), \) \( z \in \mathbb{C}, \) we deduce
\[
\Delta_\nu \left[ e^{-\frac{x^2 + y^2}{4\sigma^2 t^2}} j_\nu \left( \frac{ixy}{2\sigma t} \right) \right] = e^{-\frac{x^2}{4\sigma^2 t^2}} \left[ \left( \frac{x^2 + y^2}{4\sigma^2 t^2} - \frac{\nu + 1}{t} \right) j_\nu \left( \frac{ixy}{2\sigma t} \right) - \frac{ixy}{2\sigma^2 t} j_\nu \left( \frac{ixy}{2\sigma t} \right) \right].
\]
Then
\[
\left[ \frac{\partial}{\partial t} - \sigma \Delta_\nu^{m-1} \right] G_t^{m-1}(x, y) = 0.
\]

**Definition 4.2** Let \( \mathbf{m} \in SL(2, \mathbb{R}) \) such that \( b \neq 0. \) For \( f \in L_{p,\nu} \) and \( t \geq 0, \) set
\[
S_{\nu}^{m-1}(t)f = \left\{ \begin{array}{ll}
|p_t^{m-1}_{\nu, \nu, m-1} f | & \text{if } t > 0, \\
f & \text{if } t = 0.
\end{array} \right.
\]

**Theorem 4.1** For each \( f \in L_{p,\nu} \) \( (1 \leq p \leq \infty) \) set \( u(t, x) = S_{\nu}^{m-1}(t)f(x) \) for any \( (t, x) \in (0, +\infty) \times \mathbb{R}. \) Then:
1. The function \( u \) satisfies the heat equation \( (\partial_t - \sigma \Delta_\nu^{m-1})u = 0 \) in \( (0, +\infty) \times \mathbb{R}. \)
2. The solution \( u(t, x) \rightarrow f(x) \) as \( t \rightarrow 0 \) in the norm of \( L_{p,\nu}. \)
3. \( \| u(t, \cdot) \|_{r, \nu} \leq \left[ (4\sigma t)^{\nu+1} \Gamma(\nu + 1) / 2 \right]^{1/2} q^{(\nu + 1)/q} \| f \|_{p, \nu} \) with \( p, q, r \in [1, +\infty] \) be such that \( 1/p + 1/q = 1 + 1/r \) and \( t > 0. \)
4. \( u(t, \cdot) \) is the unique solution of \( \{4.7\} \) belonging in \( L_{r,\nu} \) with \( r \in [1, +\infty]. \)
5. If \( f \in S_{\nu}(\mathbb{R}) \) then \( u(t, \cdot) \in S_{\nu}(\mathbb{R}) \) and for all non-negative integers \( n, \) there is a constant \( c_{n+1} > 0 \) and \( n_1, \ldots, n_{k_{n+1}}, \)
\( m_1, \ldots, m_{k_{n+1}} \in \mathbb{N} \) such that
\[
\left\| u(t, \cdot) - \sum_{k=0}^{n} \frac{t^k}{k!} \left( \sigma \Delta_\nu^{m-1} \right)^k f \right\|_{\infty} \leq \left[ \frac{t^{n+1}}{(n+1)!} \right] c_{n+1} \sum_{i=1}^{k_{n+1}} q_{n,m_i}(f).
\]

**Proof.** 1) Let \( f \in L_{p,\nu} \) \( (1 \leq p \leq \infty) \) In view of \( 4.8 \) and \( 4.6 \) we can write
\[
u(t, x) = |p_t^{m-1}_{\nu, \nu, m-1} f |(x) = \int_0^{+\infty} G_t^{m-1}(x, y)e^{ixy}f(y)y^{2\nu+1} dy.
\]
For any \( (t, x) \in (0, +\infty) \times \mathbb{R} \) the function \( y \rightarrow G_t^{m-1}(x, y) \) is in \( L_{q,\nu} \) for \( 1 \leq q \leq \infty. \) Then \( |p_t^{m-1}_{\nu, \nu, m-1} f | \) is well defined and by \( 4.7, \) we can show that
\[
\frac{\partial}{\partial t} u(t, x) = \int_0^{+\infty} \frac{\partial}{\partial t} G_t^{m-1}(x, y)e^{ixy}f(y)y^{2\nu+1} dy.
\]
Now a rough calculation shows that
\[
\left| \sigma \Delta_\nu^{m-1} G_t^{m-1}(x, y) \right| \leq \frac{2}{\Gamma(\nu + 1)(4\sigma t)^{\nu+1}} \left( \frac{r^2 + y^2}{4\sigma t^2} + \frac{\nu + 1}{t} + \frac{ry}{2\sigma t^2} \right) e^{-\frac{(r-y)^2}{4\sigma t}}
\]
for \( |x| \leq r \) and \( y \geq r. \) Hence
\[
\sigma \Delta_\nu^{m-1} u(t, x) = \int_0^{+\infty} \sigma \Delta_\nu^{m-1} G_t^{m-1}(x, y) e^{ixy}f(y)y^{2\nu+1} dy
\]
which yields the desired result by virtue of \( \frac{\partial}{\partial t} G_t^{m-1}(x, y) = \sigma \Delta_\nu^{m-1} G_t^{m-1}(x, y). \)
2) We begin by showing that if \( f \in C_{r,c}(\mathbb{R}) \), then
\[
\lim_{t \rightarrow 0} \| S_{\nu}^{m-1}(t)f - f \|_{p,\nu} = 0.
\]
By \( 3.5, \) we can write
\[
S_{\nu}^{m-1}(t)f(x) = \int_0^{+\infty} |T_x^{r, m-1} | e^{ixy}f(y)y^{2\nu+1} dy
\]
\[
= \int_0^{+\infty} e^{ixy}f(T_x^{r, m-1})|y|y^{2\nu+1} dy
\]
\[
= \frac{2}{\Gamma(\nu + 1)(4\sigma t)^{\nu+1}} \int_0^{+\infty} e^{-\frac{r^2+y^2}{4\sigma t}} T_x^{r, m-1} f(y)y^{2\nu+1} dy
\]
\[
= \frac{2}{\Gamma(\nu + 1)} \int_0^{+\infty} e^{2i\sigma ty} e^{-\frac{r^2+y^2}{4\sigma t}} T_x^{r, m-1} f(y)y^{2\nu+1} dy.
\]
Since \( \frac{2}{\Gamma(\nu+1)} \int_0^{+\infty} e^{-y^2} y^{2\nu+1} dy = 1 \), the formula (4.8) implies
\[
S_{\nu}^{-1}(t)f(x) - f(x) = a_t(x) + b_t(x) \tag{4.9}
\]
where
\[
a_t(x) = \frac{2}{\Gamma(\nu+1)} \int_0^{+\infty} e^{-y^2} \left(e^{2i\delta t y^2} - 1\right) f(x) y^{2\nu+1} dy,
\]
\[
b_t(x) = \frac{2}{\Gamma(\nu+1)} \int_0^{+\infty} e^{2i\delta t y^2} e^{-y^2} \left([T_x^{\nu,m-1} f] \left(\sqrt{4\sigma t y}\right) - f(x)\right) y^{2\nu+1} dy.
\]
The relation (4.9) implies that
\[
\|S_{\nu}^{-1}(t)f - f\|_{p,\nu} \leq \|a_t\|_{p,\nu} + \|b_t\|_{p,\nu}
\]
with
\[
\|a_t\|_{p,\nu} \leq \frac{2}{\Gamma(\nu+1)} \int_0^{+\infty} e^{-y^2} \left|e^{2i\delta t y^2} - 1\right| y^{2\nu+1} dy \|f\|_{p,\nu} \rightarrow 0 \text{ as } t \rightarrow 0,
\]
and by Minkowski’s inequality for integrals
\[
\|b_t\|_{p,\nu} \leq \frac{2}{\Gamma(\nu+1)} \int_0^{+\infty} e^{-y^2} \|T_x^{\nu,m-1} f - f\|_{p,\nu} y^{2\nu+1} dy \rightarrow 0 \text{ as } t \rightarrow 0.
\]
This is clear from the dominated convergence theorem since:
\[
\lim_{t \to 0} \|T_x^{\nu,m-1} f - f\|_{p,\nu} = 0 \quad \text{(see Theorem 3.2)}.
\]
Now suppose \( f \in \mathcal{L}_{p,\nu} \). If \( \epsilon > 0 \) and since \( C_{*,c}(\mathbb{R}) \) is dense in \( \mathcal{L}_{p,\nu} \), there exists \( g \in C_{*,c}(\mathbb{R}) \) such that \( \|f - g\|_{p,\nu} \leq \frac{\epsilon}{3} \), so
\[
\|S_{\nu}^{-1}(t)f - f\|_{p,\nu} \leq \|S_{\nu}^{-1}(t)(f - g)\|_{p,\nu} + \|S_{\nu}^{-1}(t)g - g\|_{p,\nu} + \|g - f\|_{p,\nu}
\]
\[
\leq 2\|g - f\|_{p,\nu} + \|S_{\nu}^{-1}(t)g - g\|_{p,\nu} \leq \frac{2\epsilon}{3} + \|S_{\nu}^{-1}(t)g - g\|_{p,\nu}
\]
and \( \|S_{\nu}^{-1}(t)g - g\|_{p,\nu} \leq \frac{\epsilon}{3} \) if \( t \) is sufficiently small.
3) By Young’s inequality (4.9)
\[
\|u(t,.)\|_{r,\nu} = \|P_{r,m-1}^{-1} v\|_{r,\nu} \leq \|P_{r,m-1}^{-1}\|_{q,\nu} \|f\|_{p,\nu}.
\]
The desired result is therefore a consequence of
\[
\|P_{r,m-1}^{-1}\|_{q,\nu} = \frac{2}{\Gamma(\nu+1)(4\sigma t)^{(\nu+1)}} \int_0^{+\infty} e^{-\frac{y^2}{4\sigma t}} y^{2\nu+1} dy^{1/q}
\]
\[
= \left(\frac{\Gamma(\nu+1)(4\sigma t)^{(\nu+1)}}{2}\right)^{1/q} \left(\frac{1}{q}\right)^{\frac{\nu+1}{q}}.
\]
4) Let \( v(t,.) \) be another solution of (4.1) belonging in \( \mathcal{L}_{r,\nu} \) with \( r \in [1, +\infty) \). For \( 0 \leq s < t < +\infty \)
\[
\frac{d}{ds} \left[ S_{\nu}^{-1}(t-s)v(s,) \right] = S_{\nu}^{-1}(t-s)\Delta_{\nu}^{-1}v(s,.) - S_{\nu}^{-1}(t-s)\sigma \Delta_{\nu}^{-1}v(s,.) = 0.
\]
Then \( S_{\nu}^{-1}(t-s)v(s,.) \) is independent of \( s \); setting \( s = 0, s = t \) yeild
\[
v(t,.) = S_{\nu}^{-1}(t)v(0,.) = S_{\nu}^{-1}(t)f = u(t,.).
\]
4) For \( f \in \mathcal{S}_{*}(\mathbb{R}) \) we have
\[
u(t, x) = S_{\nu}^{-1}(t)f(x) = \frac{2}{\Gamma(\nu+1)} \int_0^{+\infty} e^{-y^2} e^{2i\delta t y^2} [T_x^{\nu,m-1} f] \left(\sqrt{4\sigma t y}\right) y^{2\nu+1} dy.
\]
According to Theorem \ref{thm:main}, we can write:

\[ e^{2i\pi \sigma uy^2} [T^\nu_m \cdot f][\sqrt{4\sigma uy}] = \sum_{k=0}^{n} \frac{\Gamma(\nu + 1)}{\Gamma(k + \nu + 1)k!} t^k \sigma^k y^{2k} \left( \Delta^m_{\nu} \right)^k f(x) + \frac{t^{n+1}}{n!} \int_0^1 (1-s)^n \frac{d^{n+1}}{du^{n+1}} \left[ e^{2i\pi \sigma uy^2} [T^\nu_m \cdot f][\sqrt{4\sigma uy}] \right] (s) \ ds. \]

Multiply both sides by \( \frac{2}{\Gamma(\nu + 1)} e^{-y^2} y^{2\nu+1} \) and integrate from 0 to \(+\infty\), we get

\[ u(t, x) = \sum_{k=0}^{n} \frac{t^k}{k!} \sigma^k \left( \Delta^m_{\nu} \right)^k f(x) \left( \frac{2}{\Gamma(\nu + 1)} \int_0^{+\infty} e^{-y^2} y^{2(\nu+k+1)} \ dy \right) + \frac{t^{n+1}}{n!} \left( \frac{2}{\Gamma(\nu + 1)} \int_0^{+\infty} (1-s)^n \frac{d^{n+1}}{du^{n+1}} \left[ e^{2i\pi \sigma uy^2} [T^\nu_m \cdot f][\sqrt{4\sigma uy}] \right] (s) \ ds \right) y^{2\nu+1} \ dy. \]

The desired result is therefore a consequence of \( \frac{2}{(x^2+y^2+1)} \int_0^{+\infty} e^{-y^2} y^{2(\nu+k+1)} \ dy = 1. \]

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