Gluing minimal prime ideals in local rings

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ABSTRACT
Let \( B \) be a reduced local (Noetherian) ring with maximal ideal \( M \). Suppose that \( B \) contains the rationals, \( B/M \) is uncountable and \( |B| = |B/M| \). Let the minimal prime ideals of \( B \) be partitioned into \( m \geq 1 \) subcollections \( C_1, \ldots, C_m \). We show that there is a reduced local ring \( S \) of \( B \) with maximal ideal \( S \cap M \) such that the completion of \( S \) with respect to its maximal ideal is isomorphic to the completion of \( B \) with respect to its maximal ideal and such that, if \( P \) and \( Q \) are prime ideals of \( B \), then \( P \cap S = Q \cap S \) if and only if \( P \) and \( Q \) are in \( C_i \) for some \( i = 1, 2, \ldots, m \).

ARTICLE HISTORY
Received 17 March 2022
Revised 20 June 2022
Communicated by Scott Chapman

KEYWORDS
Spectra of Noetherian rings; minimal prime ideals; completions of local rings

2020 MATHEMATICS SUBJECT CLASSIFICATION
13E05; 13J10

1. Introduction

Given a Noetherian ring \( B \), it is often useful to find another Noetherian ring \( S \) such that the prime ideals of \( B \) and the prime ideals of \( S \) are related in some specific desired way. For example, if \( P \) is a prime ideal of \( B \), then, in many situations, passing to the localization \( S = B_P \) is incredibly useful, in part because there is a one-to-one (inclusion preserving) correspondence between the prime ideals of \( B \) contained in \( P \) and the prime ideals of \( B_P \). Similarly, it is a standard technique in many settings to study the domain \( B/P \) and, of course, the relationship between the prime ideals of \( B \) and the prime ideals of \( B/P \) is well understood. In this paper, we consider the following question. Let \( B \) be a local (Noetherian) ring and suppose that \( B \) has \( n \) minimal prime ideals. Let \( m \) be an integer such that \( 1 \leq m \leq n \). Is there a local subring \( S \) of \( B \) such that \( S \) and \( B \) have the same completion, and such that, when viewed as partially ordered sets (posets), \( \text{Spec}(B) \) and \( \text{Spec}(S) \) are the same except that \( \text{Spec}(B) \) has \( n \) minimal elements and \( \text{Spec}(S) \) has \( m \) minimal elements? Informally, in this setting, we think of obtaining the partially ordered set \( \text{Spec}(S) \) by taking the partially ordered set \( \text{Spec}(B) \) and “gluing” certain minimal nodes together while preserving everything else. We show that for a large class of local rings, such a subring does, in fact, exist.

We start with a reduced local ring \( B \) with maximal ideal \( M \) and we suppose that \( B \) contains the rationals, \( B/M \) is uncountable, and \( |B| = |B/M| \). Our goal is to construct a local ring \( S \) such that \( S \subseteq B \), the completion of \( S \) is isomorphic to the completion of \( B \), and, \( \text{Spec}(S) \) and \( \text{Spec}(B) \) when viewed as partially ordered sets, are the same except for their minimal elements. Specifically, the main result of this paper is the following theorem.

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Theorem 2.14. Let $B$ be a reduced local ring with maximal ideal $M$. Suppose that $B$ contains the rationals, $B/M$ is uncountable and $|B| = |B/M|$. Suppose also that the set of minimal prime ideals of $B$ is partitioned into $m \geq 1$ subcollections $C_1, ..., C_m$. Then there is a reduced local ring $S \subseteq B$ with maximal ideal $S \cap M$ such that

1. $S$ contains the rationals.
2. The completion of $S$ at its maximal ideal is isomorphic to the completion of $B$ at its maximal ideal.
3. $S/(S \cap M)$ is uncountable and $|S| = |S/(S \cap M)|$.
4. If $Q$ and $Q'$ are minimal prime ideals of $B$ then $Q \cap S = Q' \cap S$ if and only if there is an $i \in \{1, 2, ..., m\}$ with $Q \in C_i$ and $Q' \in C_i$.
5. The map $f : \text{Spec}(B) \to \text{Spec}(S)$ given by $f(P) = S \cap P$ is onto and, if $P$ is a prime ideal of $B$ with positive height, then $f(P)B = P$. In particular, if $P$ and $P'$ are prime ideals of $B$ with positive height, then $f(P)$ has positive height and $f(P) = f(P')$ implies that $P = P'$.

The properties of $f$ in Theorem 2.14 guarantee that it is an order-preserving onto map and, when $f$ is restricted to the prime ideals of $B$ with positive height, it is a poset isomorphism from the prime ideals of $B$ with positive height to the prime ideals of $S$ with positive height. In addition, $f$ maps all of the elements of a given $C_i$ to the same prime ideal of $S$. Hence, one could think of $f$ as gluing all the prime ideals in each respective $C_i$ together while totally preserving everything else about the spectrum. Moreover, there is no restriction on how the minimal prime ideals of $B$ are glued; that is, one can choose the sets $C_1, ..., C_m$ to be any partition of the set of minimal prime ideals of $B$. We refer to Theorem 2.14 as the gluing theorem.

This type of gluing is done in [1] where the ring $B$ contains the rationals and is required to be complete. We show that it is possible to do this type of gluing replacing the condition that $B$ is complete with the conditions that $B$ is reduced, $B/M$ is uncountable and $|B| = |B/M|$. In particular, whereas the gluing in [1] is done inside of a complete local ring, the gluing in this paper can be done inside a suitable localized polynomial ring which is not complete. To illustrate, we give an example for which our main result applies, but Theorem 3.12 in [1] does not.

Example 1.1. Let $B = \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1x_2x_3x_4x_5x_6)$. Then $B$ satisfies the conditions of Theorem 2.14, and it has six minimal prime ideals. Let $C_1 = \{(x_1), (x_2), (x_3)\}$, $C_2 = \{(x_1), (x_3)\}$, and $C_3 = \{(x_6)\}$. Then there exists a local ring $S$ contained in $B$ such that the completion of $S$ is $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1x_2x_3x_4x_5x_6)$ and such that $S$ has exactly three minimal prime ideals. Moreover, the minimal prime ideals of $S$ are $(x_1) \cap S = (x_2) \cap S = (x_3) \cap S$, $(x_4) \cap S = (x_5) \cap S$, and $(x_6) \cap S$, and, if $P, Q \in \text{Spec}(B)$ are not minimal prime ideals of $B$, then $S \cap P = S \cap Q$ if and only if $P = Q$.

Since the ring $S$ in Theorem 2.14 is a reduced local ring that contains the rationals, $S/(S \cap M)$ is uncountable and $|S| = |S/(S \cap M)|$, we can apply the theorem multiple times to obtain a descending chain of rings where the number of minimal prime ideals of the rings in the chain decreases. We illustrate with an example.

Example 1.2. Let $B_1 = \mathbb{C}[[x, y, z]]/(xyz)$. By Theorem 2.14, there is a reduced local ring $B_2$ contained in $B_1$ such that the maximal ideal of $B_2$ is $B_2 \cap (x, y, z)$, $B_2$ contains the rationals, $B_2/(B_2 \cap (x, y, z))$ is uncountable, $|B_2| = |B_2/(B_2 \cap (x, y, z))|$, the completion of $B_2$ is $B_1$, $B_2$ has exactly two minimal prime ideals $B_2 \cap (x) = B_2 \cap (y)$ and $B_2 \cap (z)$, and, if $P, Q \in \text{Spec}(B_1)$ are not minimal prime ideals of $B_1$, then $B_2 \cap P = B_2 \cap Q$ if and only if $P = Q$. We now apply Theorem 2.14 to $B_2$ to obtain a local ring $B_3$ contained in $B_2$ such that the completion of $B_3$ is $B_1$ and such that $B_3$ has only one minimal prime ideal, namely $B_3 \cap (x) = B_3 \cap (y) = B_3 \cap (z)$. Moreover, if $P, Q \in \text{Spec}(B_1)$ are not minimal prime ideals of $B_1$, then $B_3 \cap P = B_3 \cap Q$ if and only if $P = Q$. 
If $R$ is a local ring and $T$ is the completion of $R$ with respect to its maximal ideal, then it is well known that minimal prime ideals of $R$ can “split” in $T$. That is, if $Q$ is a minimal prime ideal of $R$, then $QT$ need not be a prime ideal of $T$ and, when it is not, there are multiple minimal prime ideals of $T$ that lie over $Q$. The gluing in this paper is, in some sense, the inverted viewpoint to this kind of splitting; given a local ring $B$ satisfying the conditions of Theorem 2.14, we show that $B$ contains a local subring such that one can identify minimal prime ideals in $B$ while preserving everything else in the spectrum.

To prove our main result, we let $B$ be a reduced local ring with maximal ideal $M$. We assume $B$ contains the rationals, $B/M$ is uncountable, and $|B| = |B/M|$. The bulk of our proof is dedicated to showing that if $Q_1$ and $Q_2$ are distinct minimal prime ideals of $B$, then $B$ contains a subring $S$ such that $S$ is local with maximal ideal $S \cap M$, $S/S \cap M$ is uncountable, $|S| = |S/S \cap M|$, and $Q_1 \cap S = Q_2 \cap S$. We then induct on the number of minimal prime ideals of $B$ to get our main result.

All rings in this article are commutative with unity. If $R$ is a ring with exactly one maximal ideal and $R$ is not necessarily Noetherian, we say that $R$ is quasi-local. If $R$ is both quasi-local and Noetherian, we say that $R$ is local. We use $(R, M)$ to denote a local ring with maximal ideal $M$ and, if $(R, M)$ is a local ring, we use $\hat{R}$ to denote the $M$-adic completion of $R$. Finally, we use $\text{Min}(B)$ to denote the set of minimal prime ideals of $B$.

2. The gluing theorem

We are now ready to begin the proof of our main result, the gluing theorem. Much of the work in our proof is inspired by techniques from [1]. Throughout, $(B, M)$ will be a reduced local ring with $B/M$ uncountable. To prove the gluing theorem, we start by gluing two minimal prime ideals of $B$ together, and then we induct to get the final result. We begin our construction with the following useful definition.

**Definition 2.1.** Let $(B, M)$ be a reduced local ring with $B/M$ uncountable, and let $\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\}$ with $n \geq 2$. A quasi-local subring $(R, R \cap M)$ of $B$ is called a minimal-gluing subring of $B$, or an MG-subring of $B$, if $R$ is infinite, $|R| < |B/M|$, and $R \cap Q_1 = R \cap Q_2$.

The idea is that, if $R$ is an MG-subring of $B$, then two minimal prime ideals of $B$ are glued together, and we arrange it so that these two minimal prime ideals of $B$ are $Q_1$ and $Q_2$. So, even though $Q_1$ and $Q_2$ are two distinct minimal prime ideals of $B$, we have that $Q_1 \cap R = Q_2 \cap R$ in $R$.

Note that, if $B$ in Definition 2.1 contains the rationals, then $\bigcap \{Q\}$ is an MG-subring of $B$. To construct our final ring $S$ in the gluing theorem, we begin with $\bigcap \{Q\}$ and successively adjoin uncountably many elements while ensuring that our resulting rings remain MG-subrings of $B$. To do this, we make use of the following result which can be thought of as a generalization of the prime avoidance lemma.

**Lemma 2.2** ([2, Lemma 3]). Let $(B, M)$ be a local ring. Let $C \subseteq \text{Spec}(B)$, let $I$ be an ideal of $B$ such that $I \not\subseteq P$ for every $P \in C$, and let $D$ be a subset of $B$. Suppose $|C \times D| < |B/M|$. Then $I \not\subseteq \bigcup\{P + r \mid P \in C, r \in D\}$.

If $R$ is a subring of the ring $B$, and if $Q$ is a prime ideal of $B$, then the map $R/(Q \cap R) \rightarrow B/Q$ is an injection, and so we can think of $R/(Q \cap R)$ as a subring of $B/Q$. Suppose $(R, R \cap M)$ is an MG-subring of $B$. The next lemma gives sufficient conditions on an element $x \in B$ for $R[x]/(R[x] \cap M)$ to also be an MG-subring of $B$.

**Lemma 2.3.** Let $(B, M)$ be a reduced local ring with $B/M$ uncountable, and let $\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\}$ with $n \geq 2$. Suppose $(R, R \cap M)$ is an MG-subring of $B$. If $x \in B$ satisfies the
condition that \( x + Q_i \in B/Q_i \) is transcendental over \( R/(Q_i \cap R) \) for \( i \in \{1, 2\} \), then \( S = R[x/(R[x] \cap M)] \) is an MG-subring of \( B \) with \( |S| = |R| \).

**Proof.** Since \( R \) is infinite, \( |S| = |R| \), and we have \( |S| < |B/M| \). Now suppose \( f \in R[x] \cap Q_i \). Then \( f = r_m x^m + \cdots + r_1 x + r_0 \in Q_i \) where \( r_j \in R \) for \( 0 \leq j \leq m \). Since \( x + Q_i \) is transcendental over \( R/(R \cap Q_i) \), we have \( r_j \in R \cap Q_i \). Hence, \( f \in Q_i \), and so \( R[x] \cap Q_i \subseteq R[x] \cap Q_i \). Similarly, \( R[x] \cap Q_2 \subseteq R[x] \cap Q_2 \), and therefore \( R[x] \cap Q_1 = R[x] \cap Q_2 \). It follows that \( S \cap Q_i = S \cap Q_2 \), and so \( S \) is an MG-subring of \( B \). \( \square \)

We now use Lemma 2.3 to show that we can adjoin very specific elements to an MG-subring of \( B \) to obtain a larger MG-subring of \( B \). Lemma 2.4 is very useful and will be employed several times.

**Lemma 2.4.** Let \( (B, M) \) be a reduced local ring with \( B/M \) uncountable, and let \( (\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\} \) with \( n \geq 2 \). Suppose \( (R, R \cap M) \) is an MG-subring of \( B \). Let \( B \in B \) and let \( z \in B \) such that \( z \not\in Q_1 \) and \( z \not\in Q_2 \). Let \( J \) be an ideal of \( B \) such that \( J \not\subseteq Q_1 \) and \( J \not\subseteq Q_2 \). Then there is an element \( w \in J \) such that \( S = R[\bar{b} + zw]/(R[\bar{b} + zw] \cap M) \) is an MG-subring of \( B \) with \( |S| = |R| \).

**Proof.** Let \( i \in \{1, 2\} \), and suppose \( b + tz + Q_i = b + t'z + Q_i \) with \( t, t' \in B \). Then \( z(t - t') \in Q_i \) and \( z \not\in Q_i \). Therefore, \( b + tz + Q_i = b + t'z + Q_i \) if and only if \( t + Q_i = t' + Q_i \). Let \( D_i \) be a full set of coset representatives for the cosets \( t + Q_i \in B/Q_i \), that make \( b + zt + Q_i \) algebraic over \( R/(R \cap Q_i) \). Note that \( |D_i| \leq |R| \). Define \( D = D_1 \cup D_2 \) and \( C = \{Q_1, Q_2\} \). Then \( |C \times D| \leq |R| < |B/M| \). By Lemma 2.2 using \( I = J \), there is an element \( w \in J \) such that \( w \not\in (\bar{p} + r \mid p \in C, r \in D) \). Then \( b + zw + Q_i \) is transcendental over \( R/(R \cap Q_i) \). By Lemma 2.3, \( S = R[\bar{b} + zw]/(R[\bar{b} + zw] \cap M) \) is an MG-subring of \( B \) and \( |S| = |R| \). \( \square \)

Recall that we want our final ring to have the same completion as \( B \). To achieve this, we use the following two propositions.

**Proposition 2.5** ([3, Proposition 1]). If \( (R, R \cap M) \) is a quasi-local subring of a complete local ring \( (T, M) \) such that the map \( R \to T/M^2 \) is onto and \( IT \cap R = IR \) for every finitely generated ideal \( I \) of \( R \), then \( R \) is Noetherian and the natural homomorphism \( \bar{R} \to T \) is an isomorphism.

The converse of Proposition 2.5 also holds (for a proof of this, see, for example, Proposition 2.4 in [4]). That is, if \( R \) is a local ring with completion \( (T, M) \), then the map \( R \to T/M^2 \) is onto and \( IT \cap R = I \) for every finitely generated ideal \( I \) of \( R \).

**Proposition 2.6.** Let \( (B, M) \) be a local ring and let \( T = \bar{B} \). Suppose \( (S, S \cap M) \) is a quasi-local subring of \( B \) such that the map \( S \to B/M^2 \) is onto and \( IB \cap S = I \) for every finitely generated ideal \( I \) of \( S \). Then \( S \) is Noetherian and \( \bar{S} = T \). Moreover, if \( B/M \) is uncountable and \( |B| = |B/M| \) then \( S/(S \cap M) \) is uncountable and \( |S| = |S/(S \cap M)| \).

**Proof.** Since \( T \) is the completion of \( B \), the map \( B \to T/(MT)^2 \) is onto. Since \( M^2 \subseteq (MT)^2 \cap B \), the map \( B/M^2 \to T/(MT)^2 \) is well defined and onto. By hypothesis, the map \( S \to B/M^2 \) is onto, and so the map \( S \to B/M^2 \to T/(MT)^2 \) is onto. Let \( I \) be a finitely generated ideal of \( S \). Then, since \( T \) is the completion of \( B \), \( IT \cap B = IB \). It follows that \( IT \cap S = (IT \cap B) \cap S = IB \cap S = I \). By Proposition 2.5, \( S \) is Noetherian and \( \bar{S} = T \).

Now suppose \( B/M \) is uncountable and \( |B| = |B/M| \). Since \( T \) is the completion of both \( S \) and \( B \), we have \( S/(S \cap M) \cong B/M \cong T/MT \). Hence, \( S/(S \cap M) \) is uncountable and \( |S/(S \cap M)| = |B/M| \). Now \( |S| \leq |B| = |B/M| = |S/(S \cap M)| \), and it follows that \( |S| = |S/(S \cap M)| \). \( \square \)
Because we will use Proposition 2.6 to show that our final ring has the same completion as $B$, we want our final ring to contain an element of every coset in $B/M^2$. The next lemma shows that we can adjoin an element of a specific coset $b + M^2$ to an MG-subring of $B$ that will result in another MG-subring of $B$. Later in this section (Theorem 2.13), we will adjoin elements from every coset in $B/M^2$.

**Lemma 2.7.** Let $(B, M)$ be a reduced local ring with $B/M$ uncountable, and let $\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\}$ with $n \geq 2$. Let $b \in B$ and suppose $(R, R \cap M)$ is an MG-subring of $B$. Then there exists an MG-subring $(S, S \cap M)$ of $B$ such that $R \subseteq S$, $|S| = |R|$, and $S$ contains an element of the coset $b + M^2$.

**Proof.** Since $n \geq 2$ and $B$ is local, $M \not\subseteq Q_i$ for $i \in \{1, 2\}$, and so $M^2 \not\subseteq Q_i$ for $i \in \{1, 2\}$. Use Lemma 2.4 with $J = M^2$ and $z = 1$ to find $m \in M^2$ such that $S = R[b + m]_{(R[b + m] \cap M)}$ is an MG-subring of $B$ with $|S| = |R|$. Note that $R \subseteq S$ and $S$ contains $b + m$, an element of the coset $b + M^2$.

In light of Proposition 2.6, we want to make sure that, if $S$ is our final ring, $IB \cap S = I$ for every finitely generated ideal $I$ of $S$. Lemma 2.8 will help us do this.

**Lemma 2.8.** Let $(B, M)$ be a reduced local ring with $B/M$ uncountable, and let $\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\}$ with $n \geq 2$. Let $(R, R \cap M)$ be an MG-subring of $B$. Then, for any finitely generated ideal $I$ of $R$ and for any $c \in IB \cap R$, there is an MG-subring $(S, S \cap M)$ of $B$ such that $R \subseteq S$, $|S| = |R|$, and $c \in IS$.

**Proof.** Let $I = \langle y_1, \ldots, y_m \rangle$. We induct on $m$. If $m = 1$ then $I = aR$ for $a \in R$, and $c = au$ for some $u \in B$. If $a = 0$, then $S = R$ works. So assume $a \neq 0$.

First suppose $a \not\in Q_1$. Then $a \not\in Q_2$. We claim that $S = R[u]_{(R[u] \cap M)}$ is the desired subring of $B$. Suppose $f \in R[u] \cap Q_1$. Then $f = r_m u^m + \cdots + r_1 u + r_0$ where $r_i \in R$. Hence, $a^m f = r_m c^m + \cdots + r_1 c a^{m-1} + r_0 a^m \in R \subseteq Q_1 = R \cap Q_2$. Since $a \not\in Q_2$, we have $f \in Q_2$, and so $R[u] \cap Q_1 \subseteq R[u] \cap Q_2$. Similarly, $R[u] \cap Q_2 \subseteq R[u] \cap Q_1$, and so $R[u] \cap Q_1 = R[u] \cap Q_2$. It follows that $S \subseteq R$ and $|S| = |R|$, and $c \in IS$. This completes the base case of the induction.

Now assume that $a \in Q_1$. Then $a \in Q_2$. Since $B$ is reduced, $B_{Q_1}$ is a field, and so $\text{ann}_B(a) \subseteq Q_1$. Similarly, $\text{ann}_B(a) \subseteq Q_2$. Using Lemma 2.4 with $z = 1$, there exists $w \in \text{ann}_B(a)$ such that $S = R[u + w]_{(R[u + w] \cap M)}$ is an MG-subring of $B$ with $|S| = |R|$. Now, $u + w \in S$ and $a(u + w) = au = c$, and so $c \in IS$. This completes the base case of the induction.

Suppose that $m > 1$ and that the lemma holds for all ideals generated by fewer than $m$ elements. We have $c = y_1 b_1 + y_2 b_2 + \cdots + y_m b_m$ for some $b_i \in B$.

We first consider the case where $y_i \not\in Q_1$ for some $i$. Without loss of generality, suppose $y_2 \not\in Q_1$. Then $y_2 \not\in Q_2$. Use Lemma 2.4 with $J = B$ to find $w \in B$ such that $S' = R[b_1 + y_2 w]_{(R[b_1 + y_2 w] \cap M)}$ is an MG-subring of $B$ with $|S'| = |R|$. Note that

$$c = y_1 b_1 + y_1 y_2 w + y_2 b_2 + \cdots + y_m b_m = y_1 (b_1 + y_2 w) + y_2 (b_2 - y_1 w) + \cdots + y_m b_m.$$ 

Now consider the ideal $(y_2, \ldots, y_m)$ of $S'$ and let $c^* = c - y_1 (b_1 + y_2 w)$. Then, $c^* \in (y_2, \ldots, y_m) B \cap S'$. By our induction assumption, there is an MG-subring $(S, S \cap M)$ of $B$ such that $S' \subseteq S$, $|S| = |S'|$, and $c^* \in (y_2, \ldots, y_m) S$. So we have $c^* = y_2 s_2 + \cdots + y_m s_m$ for some $s_i \in S$. Hence, $c = c^* + y_1 (b_1 + y_2 w) \in (y_1, \ldots, y_m) S = IS$, and it follows that $S$ is the desired MG-subring of $B$.

We now consider the case where $y_i \in Q_1$ for all $i = 1, 2, \ldots, m$. Then $y_i \in Q_2$ for all $i = 1, 2, \ldots, m$. As before, $\text{ann}_B(y_1) \not\subseteq Q_1$ and $\text{ann}_B(y_1) \not\subseteq Q_2$. Use Lemma 2.4 with $J = \text{ann}_B(y_1)$ and $z = 1$ to find $w \in \text{ann}_B(y_1)$ such that $S' = R[b_1 + w]_{(R[b_1 + w] \cap M)}$ is an MG-subring of $B$ with $|S'| = |R|$. Consider the ideal $(y_2, \ldots, y_m)$ of $S'$ and let $c^* = c - y_1 (b_1 + w)$. Then $c^* \in (y_2, \ldots, y_m) B \cap S'$.
so by our induction assumption there is an MG-subring \((S, S \cap M)\) of \(B\) such that \(S' \subseteq S\), \(|S'| = |S|\), and \(c^* \in (y_2, \ldots, y_m)S\). So we have \(c^* = y_2s_2 + \cdots + y_ms_m\) for some \(s_i \in S\). Since \(c = c^* + y_1(b_1 + w)\), we have \(c \in (y_1, y_2, \ldots, y_m)S = IS\), and it follows that \(S\) is the desired MG-subring of \(B\).

To ensure that \(B\) and our final ring \(S\) have the same spectrum except at the minimal prime ideals, we guarantee that, if \(J\) is an ideal of \(B\) of positive height, then \(S\) contains a generating set for \(J\). In Lemma 2.9, we show that, for a particular ideal \(J\) of \(B\), we can start with an MG-subring, and adjoin appropriate elements so that the resulting ring is not only an MG-subring of \(B\), but it also contains a generating set for \(J\).

**Lemma 2.9.** Let \((B, M)\) be a reduced local ring with \(B/M\) uncountable, and let \(\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\}\) with \(n \geq 2\). Suppose \(J\) is an ideal of \(B\) with \(J \nsubseteq Q_1\) and \(J \nsubseteq Q_2\). Let \((R, R \cap M)\) be an MG-subring of \(B\). Then there exists an MG-subring \((S, S \cap M)\) of \(B\) such that \(R \subseteq S\), \(|R| = |S|\), and \(S\) contains a generating set for \(J\).

**Proof.** Let \(J = (x_1, x_2, \ldots, x_k)\). By the prime avoidance theorem, there exists \(z \in J\) such that \(z \notin Q_1\) and \(z \notin Q_2\). Note that \(M \nsubseteq Q_1\) and \(M \nsubseteq Q_2\). By Lemma 2.4, there is an \(m_1 \in M\) such that \(R_1 = R[x_1 + m_1z, x_2, \ldots, x_k]\) is an MG-subring of \(B\) and \(|R_1| = |R|\). Note that \((x_1 + m_1z, x_2, \ldots, x_k) + M = J\), and so by Nakayama’s lemma, \((x_1 + m_1z, x_2, \ldots, x_k) = J\). Now repeat this procedure replacing \(x_1\) with \(x_2\), and \(R\) with \(R_1\) to find \(m_2 \in M\) such that \(R_2 = R_1[x_2 + m_2z, x_3, \ldots, x_k]\) is an MG-subring of \(B\), \(|R_2| = |R_1|\), and \(J = (x_1 + m_1z, x_2 + m_2z, x_3, \ldots, x_k)\). Continue the procedure to find an MG-subring \(R_k\) of \(B\) such that \(R \subseteq R_k\), \(|R_k| = |R|\), \(J = (x_1 + m_1z, x_2 + m_2z, \ldots, x_k + m_kz)\), and \(x_1 + m_1z \in R_k\) for all \(j = 1, 2, \ldots, k\). Then \(S = R_k\) is the desired MG-subring of \(B\).

Our strategy is to start with \(\emptyset\) and successively adjoin uncountably many carefully chosen elements of \(B\) to get our final ring. In the process, we construct increasing chains of MG-subrings. The next lemma ensures that the union of these increasing chains satisfies most properties of MG-subrings.

**Lemma 2.10.** Let \((B, M)\) be a reduced local ring with \(B/M\) uncountable, and let \(\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\}\) with \(n \geq 2\). Let \(\Omega\) be a well-ordered index set and suppose that \((R_\alpha, R_\alpha \cap M)\) for \(\alpha \in \Omega\) is a family of MG-subrings of \(B\) such that, if \(\alpha, \mu \in \Omega\) with \(\alpha < \mu\), then \(R_\alpha \subseteq R_\mu\). Then \(S = \bigcup_{\alpha \in \Omega} R_\alpha\) is an infinite subring of \(B\) such that \(S \cap Q_1 = S \cap Q_2\). Furthermore, if there is some cardinal \(\lambda < |B/M|\) such that \(|R_\alpha| = \lambda\) for all \(\alpha \in \Omega\), and if \(|\Omega| < |B/M|\), then \(|S| \leq \max\{\lambda, |\Omega|\}\), and \(S\) is an MG-subring of \(B\).

**Proof.** It is clear that \(S\) is infinite and \(S \cap Q_1 = S \cap Q_2\). Now, suppose there is some cardinal \(\lambda < |B/M|\) such that \(|R_\alpha| \leq \lambda\) for all \(\alpha \in \Omega\), and \(|\Omega| < |B/M|\). Then \(S \leq |\lambda| |\Omega| = \max\{\lambda, |\Omega|\}\). So, \(|S| < |B/M|\), and it follows that \((S, S \cap M)\) is an MG-subring of \(B\).

The next two results show that we can construct a subring of \(B\) that satisfies several of our desired properties simultaneously. Before we state and prove the results, we state a technical definition.

**Definition 2.11.** Let \(\Psi\) be a well-ordered set and let \(z \in \Psi\). Define \(\gamma(z) = \sup\{\beta \in \Psi \mid \beta < z\}\).

**Lemma 2.12.** Let \((B, M)\) be a reduced local ring with \(B/M\) uncountable, and let \(\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\}\) with \(n \geq 2\). Let \(J\) be an ideal of \(B\) with \(J \nsubseteq Q_1\) and \(J \nsubseteq Q_2\), and let \(b \in B\). Suppose \((R, R \cap M)\) is an MG-subring of \(B\). Then there exists an MG-subring \((S, S \cap M)\) of \(B\) such that \(R \subseteq S\), \(|S| = |R|\), \(b + M^2\) is in the image of the map \(S \to B/M^2\), \(S\) contains a generating set for \(J\), and \(IB \cap S = I\) for every finitely generated ideal \(I\) of \(S\).
Proof. First use Lemma 2.7 to obtain an MG-subring \((R', R' \cap M)\) of \(B\) such that \(R \subseteq R'\), \(|R'| = |R|\), and \(R'\) contains an element of \(b + M^2\). Next, use Lemma 2.9 to get an MG-subring \((R'', R'' \cap M)\) of \(B\) such that \(R' \subseteq R''\), \(|R''| = |R'|\), and \(R''\) contains a generating set for \(I\). Define

\[ \Psi = \{(I, c) \mid I \text{ is a finitely generated ideal of } R'' \text{ and } c \in IB \cap R''\}. \]

Well-order \(\Psi\) so that it has no maximal element, and let 0 denote its first element. Note that \(|\Psi| \leq |R''| = |R|\). We proceed by using transfinite induction. Recursively define a family of MG-subrings \((R_\mu, R_\mu \cap M)\) of \(B\) for each \(\mu \in \Psi\) such that \(|R_\mu| = |R|\) and, if \(a, \rho \in \Psi\) with \(a < \rho \leq \mu\), then \(R_a \subseteq R_\rho\). Define \(R_0 = R''\). Now, for \(\mu \in \Psi\) assume that \(R_{\beta}\) has been defined for all \(\beta < \mu\) such that \((R_{\beta}, R_\beta \cap M)\) is an MG-subring of \(B\), \(|R_\beta| = |R|\) and if \(a, \rho \leq \beta\) with \(a < \rho\), then \(R_a \subseteq R_\rho\). Suppose \(\gamma(\mu) < \mu\), and let \(\gamma(\mu) = (I, c)\). Then define \((R_\mu, R_\mu \cap M)\) to be the MG-subring obtained from Lemma 2.8 such that \(R_{\gamma(\mu)} \subseteq R_\mu\), \(|R_{\gamma(\mu)}| = |R_\mu|\), and \(c \in IR_\mu\). On the other hand, if \(\gamma(\mu) = \mu\), define \(R_\mu = \cup_{\beta < \mu} R_\beta\). In this case, by Lemma 2.10, \((R_\mu, R_\mu \cap M)\) is an MG-subring of \(B\) with \(|R_\mu| = |R|\). In either case, we have that \((R_\mu, R_\mu \cap M)\) is an MG-subring of \(B\), \(|R_\mu| = |R|\), and if \(a, \rho \leq \mu\) with \(a < \rho\), then \(R_a \subseteq R_\rho\).

Let \(S_1 = \cup_{\mu \in \Psi} R_\mu\). By Lemma 2.10, \((S_1, S_1 \cap M)\) is an MG-subring of \(B\) and \(|S_1| = |R|\). Let \(I\) be a finitely generated ideal of \(R''\) and let \(c \in IB \cap R''\). Then \((I, c) = \gamma(\mu)\) for some \(\gamma(\mu) \in \Psi\) with \(\gamma(\mu) < \mu\). By construction, \(c \in IR_\mu \subseteq IS_1\). It follows that \(IB \cap R'' \subseteq IS_1\) for every finitely generated ideal \(I\) of \(R''\).

Repeat this process with \(R''\) replaced by \(S_1\) to obtain an MG-subring \((S_2, S_2 \cap M)\) of \(B\) with \(S_1 \subseteq S_2\), \(|S_2| = |S_1|\), and \(IB \cap S_1 \subseteq IS_2\) for every finitely generated ideal \(I\) of \(S_1\). Continue to obtain a chain of MG-subrings \(R'' \subseteq S_1 \subseteq S_2 \subseteq \cdots\) with \(S_i \subseteq S_{i+1}\), \(|S_{i+1}| = |S_i|\) and \(IB \cap S_i \subseteq IS_{i+1}\) for every finitely generated ideal \(I\) of \(S_i\).

Let \(S = \cup_{i=1}^{\infty} S_i\). By Lemma 2.10, \((S, S \cap M)\) is an MG-subring of \(B\) with \(|S| = |R|\). Now suppose \(I\) is a finitely generated ideal of \(S\), and \(c \in IB \cap S\). Then \(I = (s_1, \ldots, s_k)\) for \(s_i \in S\). Choose \(N\) such that \(c, s_1, \ldots, s_k \in S_N\). Then \(c \in IB \cap S_N \subseteq IS_{N+1} \subseteq IS\). It follows that \(IB \cap S = I\), and so \(S\) is the desired MG-subring of \(B\).

Theorem 2.13. Let \((B, M)\) be a reduced local ring with \(B/M\) uncountable and \(|B| = |B/M|\). Suppose \(B\) contains the rationals and \(\text{Min}(B) = \{Q_1, Q_2, \ldots, Q_n\}\) with \(n \geq 2\). Then there is a quasi-local ring \(S \subseteq B\) with maximal ideal \(S \cap M\) such that the map \(S \rightarrow B/M^2\) is onto, \(IB \cap S = I\) for every finitely generated ideal \(I\) of \(S\), \(Q_1 \cap S = Q_2 \cap S\), and, if \(J\) is an ideal of \(B\) with \(J \not\subseteq Q_1\) and \(J \not\subseteq Q_2\), then \(S\) contains a generating set for \(J\).

Proof. First note that if

\[ \Psi = \{J \text{ an ideal of } B \mid J \not\subseteq Q_1 \text{ and } J \not\subseteq Q_2\} \]

then \(|\Psi| \leq |B|\) and \(|B/M^2| = |B/M| = |B|\). Well-order \(B/M^2\) using an index set \(\Omega\) such that 0 is the initial element of \(\Omega\) and every element of \(\Omega\) has fewer than \(|\Omega|\) predecessors. Let \(b_x + M^2\) be the element of \(B/M^2\) corresponding to \(x \in \Omega\). Fix a surjective map \(f\) from \(\Omega\) to \(\Psi\), and, for \(x \in \Omega\), define \(J_x = f(x)\).

We recursively define \(R_x\) for each \(x \in \Omega\). First, define \(R_0 = \emptyset\) and note that \(\emptyset\) is an MG-subring of \(B\). Let \(x \in \Omega\) and assume \(R_\beta\) has been defined for all \(\beta < x\) such that \((R_\beta, R_\beta \cap M)\) is an MG-subring of \(B\) and \(|R_\beta| \leq \{|\mu \in \Omega \mid \mu < \beta\}|R_0|\). If \(\gamma(x) < x\), define \((R_x, R_x \cap M)\) to be the MG-subring of \(B\) obtained from Lemma 2.12 such that \(R_{\gamma(x)} \subseteq R_x\), \(|R_{\gamma(x)}| = |R_x|\), \(b_{\gamma(x)} + M^2\) is in the image of the map \(R_x \rightarrow B/M^2\), \(R_x\) contains a generating set for \(J_{\gamma(x)}\), and \(IB \cap R_x = IR_x\) for every finitely generated ideal \(I\) of \(R_x\). Then \(|R_x| = |R_{\gamma(x)}| \leq \{|\mu \in \Omega \mid \mu < \gamma(x)\}|R_0| = \{|\mu \in \Omega \mid \mu < x\}|R_0|\) for each \(x \in \Omega\).
\( \Omega \mid \mu < x \} \{ R_0 \}. \) If \( \gamma(x) = x \), define \( R_x = \bigcup_{\beta \leq x} R_\beta \). In this case, \(| \{ \mu \in \Omega \mid \mu < x \} \{ R_0 | = B/M | \), and \( |R_x| \leq | \{ \mu \in \Omega \mid \mu < x \} \{ R_0 | \). By Lemma 2.10, \( (R_x, R_x \cap M) \) is an MG-subring of \( B \).

Define \( S = \bigcup_{x \in \Omega} R_x \). By Lemma 2.10, \( Q_1 \cap S = Q_2 \cap S \). By construction, \( S \cap M \) is the maximal ideal of \( S \), the map \( S \to B/M^2 \) is onto, and, if \( J \) is an ideal of \( B \) with \( J \not\subseteq Q_1 \) and \( J \not\subseteq Q_2 \), then \( S \) contains a generating set for \( J \). Let \( I = (s_1, \ldots, s_m) \) be a finitely generated ideal of \( S \) and let \( c \in IB \cap S \). Then, for some \( \mu \in \Omega \) with \( I'B \cap R_\mu = I' \), for every finitely generated ideal \( I' \) of \( R_\mu \) we have \( c, s_1, \ldots, s_m \in R_\mu \). It follows that \( c (s_1, \ldots, s_m)B \cap R_\mu = (s_1, \ldots, s_m)R_\mu \subseteq (s_1, \ldots, s_m)S = I \). Hence, we have that \( IB \cap S = I \) for every finitely generated ideal \( I \) of \( S \).

Before we state and prove the gluing theorem, we make two observations. First, suppose that \( (B, M) \) is a local ring and \( (S, S \cap M) \) is a subring of \( B \) with the same completion as \( B \). Let \( Q \) be a minimal prime ideal of \( B \). Then there is a minimal prime ideal \( Q \) of \( B \) such that \( B \cap \hat{Q} = Q \). Note that \( S \cap \hat{Q} \) is a minimal prime ideal of \( S \). It follows that \( S \cap Q = S \cap (B \cap \hat{Q}) = S \cap \hat{Q} \) is a minimal prime ideal of \( S \). Therefore, if \( Q \) is a minimal prime ideal of \( B \) then \( S \cap Q \) is a minimal prime ideal of \( S \).

Second, suppose \( S \) is a subring of the ring \( B \) and \( P \) is a prime ideal of \( B \) with positive height satisfying \( (S \cap P)B = P \). Then \( S \cap P \) is not a minimal prime ideal of \( S \). To see this, observe that since \( P \) has positive height, it strictly contains a minimal prime ideal \( Q \) of \( B \). If \( S \cap P \) is a minimal prime ideal of \( S \), then \( S \cap P = S \cap Q \) and so \( P = (S \cap P)B = (S \cap Q)B \subseteq Q \), a contradiction.

We are now ready to state and prove the gluing theorem. In our proof, we use Theorem 2.13 and induct on the number of minimal prime ideals of \( B \).

**Theorem 2.14** (The gluing theorem). Let \( (B, M) \) be a reduced local ring containing the rationals with \( B/M \) uncountable and \( |B| = |B/M| \). Suppose \( \text{Min}(B) \) is partitioned into \( m \geq 1 \) subcollections \( C_1, \ldots, C_m \). Then there is a reduced local ring \( S \subseteq B \) with maximal ideal \( S \cap M \) such that

1. \( S \) contains the rationals.
2. \( \hat{S} = B \).
3. \( S/(S \cap M) \) is uncountable and \( |S| = |S/(S \cap M)| \).
4. For \( Q, Q' \in \text{Min}(B) \), \( Q \cap S = Q' \cap S \) if and only if there is an \( i \in \{1, 2, \ldots, m\} \) with \( Q \in C_i \) and \( Q' \in C_i \).
5. The map \( f : \text{Spec}(B) \to \text{Spec}(S) \) given by \( f(P) = S \cap P \) is onto and, if \( P \) is a prime ideal of \( B \) with positive height, then \( f(P)B = P \). In particular, if \( P \) and \( P' \) are prime ideals of \( B \) with positive height, then \( f(P) \) has positive height and \( f(P) = f(P') \) implies that \( P = P' \).

**Proof.** Let \( |\text{Min}(B)| = k \). We proceed by induction on \( k \). If \( k = 1 \), then \( S = B \) works. So let \( k > 1 \) and assume that the result holds for rings with fewer than \( k \) minimal prime ideals. If \( |C_i| = 1 \) for every \( i = 1, 2, \ldots, m \), then \( S = B \) works, so assume that \( |C_i| \geq 2 \) for some \( i \). Without loss of generality, assume \( |C_1| \geq 2 \). Let \( Q_1, Q_2 \) be distinct elements of \( C_1 \). By Theorem 2.13, there is a quasi-local ring \( S' \subseteq B \) with maximal ideal \( S' \cap M \) such that the map \( S' \to B/M^2 \) is onto, \( IB \cap S' = I \) for every finitely generated ideal \( I \) of \( S' \), \( Q_1 \cap S' = Q_2 \cap S' \), and, if \( J \) is an ideal of \( B \) with \( J \not\subseteq Q_1 \) and \( J \not\subseteq Q_2 \), then \( S' \) contains a generating set for \( J \). Since \( B \) is reduced and contains the rationals, \( S' \) also satisfies these properties. By Proposition 2.6, \( S' \) is Noetherian, \( \hat{S'} = \hat{B}, \hat{S'}/(\hat{S'} \cap M) \) is uncountable and \( |\hat{S'}| = |\hat{S'}/(\hat{S'} \cap M)| \). Consider the map \( f' : \text{Spec}(B) \to \text{Spec}(S') \) given by \( f'(P) = S' \cap P \). Let \( J \in \text{Spec}(S') \). Then there is a \( P \in \text{Spec}(\hat{S'}) \) such that \( P \cap \hat{S'} = J \). Hence, \( (P \cap B) \cap S' = P \cap S' = J \), and since \( P \cap B \in \text{Spec}B \), we have that \( f' \) is onto. If \( P \) is a prime ideal of \( B \) with \( P \not\subseteq Q_1 \) and \( P \not\subseteq Q_2 \), then \( S' \) contains a generating set for \( P \) and so \( f'(P)B = (S' \cap P)B = P \). It follows that \( S' \) has \( k - 1 \) minimal prime ideals.

Consider the partition \( C_1, C_2, \ldots, C_m \) on \( \text{Min}(S') \) given by \( C_i = \{Q \cap S' \mid Q \in C_i\} \). Note that \( |C_i| = |C_1| - 1 \) and \( |C_i| = |C_i| \) for \( i = 2, 3, \ldots, m \). By induction there is a reduced local ring \( S \subseteq \)}
$S' \subseteq B$ with maximal ideal $S \cap M$ such that $S$ contains the rationals, $\hat{S} = \hat{S}' = \hat{B}, S/(S \cap M)$ is uncountable, $|S| = |S/(S \cap M)|$, for $Q, Q' \in \text{Min}(S')$, $Q \cap S = Q' \cap S$ if and only if there is an $i$ with $Q \in C_i'$ and $Q' \in C_i'$, the map $f'' : \text{Spec}(S') \rightarrow \text{Spec}(S)$ given by $f''(P) = S \cap P$ is onto and, if $P$ is a prime ideal of $S'$ with positive height, then $f''(P)S' = P$.

Note that $f : \text{Spec}(B) \rightarrow \text{Spec}(S)$ given by $f(P) = S \cap P$ satisfies $f = f'' \circ f'$. Since $f'$ and $f''$ are onto, so is $f$. Let $P$ be a prime ideal of $B$ with positive height. Then $P \not\subseteq Q_1$ and $P \not\subseteq Q_2$ and so $f'(P)B = P$. Now $f'(P)$ is a prime ideal of $S'$ of positive height, and so $f''((f'(P))S') = f'(P)$.

Therefore, $P = f'(P)B = (f''(f'(P))S')B = f(P)B$.

Let $Q, Q' \in \text{Min}(B)$. Then $Q \cap S = Q' \cap S$ if and only if $(Q \cap S') \cap S = (Q' \cap S') \cap S$ if and only if there is an $i \in \{1, 2, ..., m\}$ such that $Q \cap S' \in C_i'$ and $Q' \cap S' \in C_i'$ if and only if $Q, Q' \in C_i$. \qed

**Funding**

Cory H. Colbert was partially supported by a Lenfest grant from Washington and Lee University.

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