Bose Einstein condensation of the classical axion field in cosmology?

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Abstract

The axion is a motivated cold dark matter candidate, which it would be interesting to distinguish from weakly interacting massive particles. Sikivie has suggested that axions could behave differently during non-linear galaxy evolution, if they form a Bose-Einstein condensate, and argues that “gravitational thermalisation” drives them to a Bose-Einstein condensate during the radiation dominated era. Using classical equations of motion during linear structure formation, we explore whether the gravitational interactions of axions can generate enough entropy. At linear order in $G_N$, we interpret that the principle activities of gravity are to expand the Universe and grow density fluctuations. To quantify the rate of entropy creation we use the anisotropic stress to estimate a short dissipation scale for axions, which does not confirm previous estimates of their gravitational thermalisation rate.

1 Introduction

Axions [1–3] are hypothetical light pseudoscalar bosons, with phenomenological and theoretical attractions. They could constitute a quarter of the mass density of the Universe today, being the cold dark matter(CDM) responsible for the growth of galaxies and large scale structure. The axion constitutes a minimal and theoretically attractive candidate, because it arises in models which solve the “strong CP problem” of QCD, and is accompanied by no other new particles [4, 5].

An interesting question is whether axions can be distinguished from more massive CDM candidates, such as weakly interacting massive particles(WIMPs) [6]. The axion $a$ is the Goldstone boson of the Peccei-Quinn $U_{PQ}(1)$ symmetry [2] that breaks at a high scale $f_{PQ} \sim 4 \times 10^8$ GeV. It acquires a small mass $m_a \lesssim 0.01$ eV by mixing with the pion. The very light axion can nonetheless constitute CDM if it is non-relativistic, which requires a non-thermal production mechanism. For instance, an oscillating classical axion field can be produced at the QCD phase transition. It is well-known that the energy density of a homogeneous and isotropic scalar field redshifts [7, 8] like CDM, and that the linear growth of density fluctuations is the same for axions and WIMPs [9–12]. Sikivie and collaborators [16–18] have extensively explored the differences between axions and WIMPs, in search of distinguishing observables. In an extended study [17], Erken, Sikivie, Tam and Yang (hereafter ESTY; see also the formalised analysis of [19]) argue that axion-CDM forms a Bose-Einstein (BE) condensate due to gravitational scattering at photon temperatures $\sim$ keV, and since the BE condensate can support vortices, this allows caustics in the dark matter distribution of our galaxy today. They conclude that axions behave differently from WIMPs during non-linear structure formation.

The results of [17, 19] are obtained using Quantum Field Theory and Newtonian gravity, and have various curious features, which are mentioned at the end of section 2. The aim of this paper is to study axion evolution in an alternative formalism. We use classical equations of motion for axions

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1The doubts in the appendix of [13] are addressed in [14] and [11, 15].
in an expanding Universe with metric perturbations. A classical analysis should give the lowest order solution, and gravity is a classical theory. We study axion evolution in the early Universe, after the QCD phase transition and in the regime where departures from the homogeneous and isotropic solutions can be treated in linear perturbation theory. (The Appendix recalls that the effect of the homogeneous and isotropic gravitational interactions is to redshift the axion momenta.) Even during this period, the equations of motion for the axion field are non-linear, so we study instead the fluctuations of the axion stress-energy tensor. This is the current to which gravity couples, so we anticipate that its components are appropriate variables for describing gravitational interactions of axions.

Section 2 contains a review of axion cosmology, and some basics of Bose-Einstein condensation. We wish to know if the gravitational interactions of axions generate entropy during linear structure formation, so we are looking for a dissipative process. In section 3, we focus on physics inside the horizon, and equate the axion stress-energy tensor of the perturbed Universe, with the stress-energy tensor of an imperfect fluid. This gives an estimate of the viscosity of the axion fluid due to metric/density fluctuations, and viscosity, in fluid dynamics, wipes out short distance modes. The damping scale this gives is very short, and does not indicate that the axion field forms a BE condensate. Section 4 discusses our result and its relation to the literature, and section 5 is a summary.

2 Review

After a brief introduction of the theory of axions, section 2.1 tells the story of their cosmological evolution before and after the epochs of interest in this paper. In section 2.2 we briefly recall what is Bose-Einstein condensation in equilibrium and non-equilibrium field theory.

2.1 Axions and their cosmology

The strong CP problem of QCD is that instantons should generate an $F\tilde{F}$ term in the Lagrangian, with coefficient $\theta \sim 1$. However, the non-observation of the neutron electric dipole moment [20] implies $\theta \lesssim 10^{-10}$ [21]. This discrepancy can be explained by making $\theta$ a massive dynamical field — the axion [2]. Axion models can be constructed by extending the particle content of the Standard Model Lagrangian to allow a global chiral U(1) symmetry, referred to as a Peccei-Quinn [2] symmetry, which is broken by colour anomalies [22]. The $U_{PQ}(1)$ is also spontaneously broken at a high scale, which ensures that the Goldstone boson is the only new particle at low energies, and that it has tiny interactions with the SM [4, 5]. The couplings to photons and gluons, are suppressed by $1/f_{PQ}$; other interactions of axions can be found in [23]. The colour anomalies give this axion a small mass from mixing with the pion:

$$m_a \simeq \frac{m_\pi f_\pi}{f_{PQ} \sqrt{m_u m_d}} \approx 6 \times 10^{-6} eV \frac{10^{12} GeV}{f_{PQ}}.$$ \hspace{1cm} (2.1)

Axions are searched for in various experiments [25, 26], and can be constrained by astrophysical observations [3]. The most stringent lower bound is $f_{PQ} \gtrsim 4 \times 10^8$ GeV, to ensure that axions do not carry to much energy out of stars [3, 24].

Due to the high scale $f_{PQ}$, it is unclear whether the Peccei Quinn (PQ) phase transition occurs before or after inflation. Both scenarios have been extensively studied (for reviews and references, see, e.g. [27–29]), with emphasis on large scale density fluctuations relevant to the CMB and galaxy formation. The axion fluctuations we will study in this paper are on distances at least a million times shorter.

If the PQ phase transition occurs sufficiently before inflation, a coherent patch of axion field will be inflated beyond the size of the visible Universe. During inflation, the axion field, like the inflaton,
will develop fluctuations on scales relevant to the CMB and large scale structure: \( \delta a/a \sim H_{\text{inf}}/(2\pi f_{\text{PQ}}) \) \cite{27}. Provided that \( H_{\text{inf}} \ll f_{\text{PQ}} \), these will be small fluctuations on a homogeneous and isotropic axion field (see \cite{28} for the case \( H_{\text{inf}} \rightarrow f_{\text{PQ}} \)). When the axion acquires a potential at the QCD phase transition, it inherits the adiabatic density perturbations that the inflaton imprinted on the plasma, and in addition, its own fluctuations become isocurvature density perturbations \cite{30}\(^3\). The non-observation in CMB data of isocurvature perturbations allows to constrain this axion scenario where the PQ transition is before inflation \cite{28}.

When the PQ phase transition occurs after inflation, which is the scenario of interest in this paper, the axion field will have random different values in different causally connected volumes of the Universe. The coherence scale of the field grows with the horizon \(^4\) until the QCD phase transition, and a network of cosmic strings develops. The strings disappear after the QCD phase transition \cite{32, 33}, radiating axions with momentum of order the Hubble expansion rate. This population of non-relativistic axions is difficult to calculate reliably \cite{32, 33}; a recent estimate of their contribution to the CDM density today \cite{32} is:

\[
\Omega_a \sim 0.2 \times \left( \frac{f_{\text{PQ}}}{10^{11} \text{ GeV}} \right)^{6/5} \quad \text{(from strings).} \tag{2.2}
\]

In this paper, for simplicity, we neglect this bath of cold axion particles. We also restrict to axion models which do not generate domain walls.

Until shortly before the QCD Phase Transition, the potential for the axion field was flat. Afterwards, massive pions appear, and the axion field develops a potential (we follow here \cite{1})

\[
V(a) \approx f_{\text{PQ}}^2 m_a^2 \left[ 1 - \cos(a/f_{\text{PQ}}) \right] \simeq \frac{1}{2} m_a^2 a^2 - \frac{1}{4!} m_a^2 a^4
\tag{2.3}
\]

The QCD phase transition is a cross-over in lattice simulations \cite{35}, suggesting that the turn-on of the axion mass is a smooth and homogeneous process. A few Hubble times later, the mass will have settled to its value today, and the axion field will oscillate around the minimum with a frequency \( \sim m_a \). The axions making up this field are non-relativistic, because their momenta \( \ll H_{\text{QCD}} \ll m_a \), where

\[
H_{\text{QCD}} = \frac{1}{2 t_{\text{QCD}}} = \frac{1.66 \sqrt{T_{\text{QCD}}^4}}{m_{\text{pl}}} \approx 2 \times 10^{-20} \text{ GeV} - \frac{T_{\text{QCD}}^4}{(200 \text{ MeV})^2} \tag{2.4}
\]

and \( t_{\text{QCD}} \approx 5 \text{ km} \) is the age of the Universe, or the horizon scale at the QCD Phase Transition. This comoving scale corresponds to \( \simeq 0.1 \) parsec today (recall the distance to the galactic centre is \( \simeq 8 \text{ kpc} \)).

The energy density in these coherent oscillations redshifts like matter, as \( 1/R(t)^3 \) where \( R(t) \) is the scale factor of the Universe, and contributes to the dark matter density today\cite{36}:

\[
\Omega_a \sim 0.7 \times \left( \frac{f_{\text{PQ}}}{10^{12} \text{ GeV}} \right)^{7/6} \left( \frac{a(t_{\text{QCD}})}{\pi f_{\text{PQ}}} \right)^2 \quad \text{(coherent oscillations)}, \tag{2.5}
\]

where \( a(t_{\text{QCD}}) \) is the value of the axion field averaged over the Universe at the QCD phase transition. Recall that \( a/f_{\text{PQ}} \) corresponds to the phase of a complex scalar field, so could have any value between \( -\pi \) and \( \pi \) in a causally connected volume. In the case of interest here, when the PQ phase transition is after inflation, if one supposes that all phases are equally probable with linear measure, then the value of the axion field, averaged over the Universe is

\[
a(t_{\text{QCD}}) \simeq \pi f_{\text{PQ}}/\sqrt{3} \quad . \tag{2.6}
\]

\(^3\)For a pedagogical introduction to adiabatic and isocurvature fluctuations, see e.g. \cite{31}

\(^4\)This follows from the equations of motion for a massless field in the Friedman-Roberson Walker Universe.
Requiring that the axions from topological defects (2.2) and the condensate (2.5) not over-contribute to $\Omega_{CDM} \lesssim 0.27$ gives $f_{PQ} \lesssim 10^{11}$ GeV [32] in this case of the PQ transition after inflation. On the other hand, if the PQ transition is before inflation, a($t_{QCD}$) can be tuned to be much smaller than $f_{PQ}$. This is referred to as the anthropic region of axion parameter space [37], and allows values of $f_{PQ}$ at the GUT scale.

Axions also have quartic interactions, as seen in eqn (2.3). The coupling $m_a/f_{PQ}$ is very small, but for $a \sim f_{PQ}$, the contributions to the potential from the quartic and quadratic terms can be comparable. We will neglect the quartic terms, because we are interested in axion evolution after the QCD, so the quartic term is suppressed by $[R(t_{QCD})/R(t)]^3$ with respect to the quadratic term (where $R(t)$ is the scale factor, see eqn (3.3)).

We review in section 3.2 some results of density fluctuation growth in a Universe whose CDM is axions.

Galactic halos made of BE condensed scalars, of diverse masses and self-interaction strengths (but not axions), have been studied in [38], who confirm the presence of vortices. Galaxy formation with axion-CDM has recently been studied by Banik and Sikivie [39].

2.2 Bose-Einstein condensation

This section discusses what is a BE condensate, and some approaches to calculating how to get there.

The notion of a BE condensate, in equilibrium, is familiar from the statistical mechanics of particles: in a thermal bath with a sufficiently large conserved charge density, the free energy is minimised if charge-carrying bosons migrate to the zero-momentum state.

Equilibrium BE condensation is also a familiar notion for scalar fields in cosmology. The classic papers of Kapusta [41] and Haber and Weldon [42], evaluate the partition function for an interacting complex scalar field $\Phi$ at finite temperature, in the presence of a net charge density. They show that the chemical potential $\mu$ associated to the conserved charge contributes a negative mass-squared to the effective potential. So a sufficient charge density can drive a phase transition, to a scalar vacuum expectation value, which carries the excess charge not stored in the equilibrium bath of particles:

$$n_Q = m(\Phi)^2 + \int \frac{d^3k}{(2\pi)^3} \frac{1}{(e(E-\mu)/T - 1) - \frac{1}{e(E+\mu)/T - 1}} \quad (2.7)$$

where $n_Q$ is the charge density of the plasma.

If these equilibrium estimates are applied to axions, after the QCD phase transition, it is clear that an axion number density $n_a \simeq m_a f_{PQ}^2 \gg T^3$, if in thermal equilibrium, must be in a BE condensate. However, axions are not thermally produced, and interact very feebly. ESTY [17] propose that they “thermalise gravitationally”. However [17, 19] do not show that the axion distribution approaches an equilibrium distribution, or a migration of axion modes towards the infrared.

The particle and scalar field descriptions of BE condensation make contact in coherent state notation [63], in second quantised field theory, where a coherent state is defined so that the expectation value of the field operator gives the classical field (see eqn (6.4)). This illustrates the observation of Bogoliubov [40], that BE condensation in non-relativistic systems can be described as a phase transition. In the coherent state perspective, the above two descriptions of BE condensation have two features:

1. a classical field is born from a state containing particles. This requires $\hbar$, because $\hbar$ should be distributed differently in the Lagrangian to obtain particles or fields in the classical limit [58] (the mass has dimension length$^{-1}$ for fields).

2. the particles move to a homogeneous and isotropic configuration where they are in their lowest energy state.
It is unclear to the authors which features define a BE condensate, or, more precisely, what are the characteristics required of the axion dark matter to allow caustic formation as envisaged by Sikivie and collaborators. Is it coherence — that is, a classical scalar field? Or is it a large population in the zero momentum lowest energy state?

If the crucial feature is coherence, then any (non-relativistic) classical field would be a BE condensate. For instance, the axion field made via the misalignment mechanism, which can be decomposed on Fourier modes, could correspond to a superposition of BE condensates (one for each three-momentum)\(^5\). This would be consistent with the detection of BE condensation in alkali gases [43], demonstrated by coherent collective behaviour of the atoms (the BE condensate is allowed velocity). Maybe the two simple equilibrium examples of BE condensates, introduced above, are homogeneous and isotropic because equilibrium is homogeneous and isotropic. If the classical axion field is by nature a BE condensate, then the “gravitational thermalisation” of misalignment axions is unnecessary and this paper is beside the point.

Experimentally, BE condensation occurs far from equilibrium [43–45]. Theoretically, a Closed-Time-Path [46] implementation of the 2 Particle Irreducible effective action [47] (see e.g. the chapter on this subject in [46]), allows to compute the out-of-equilibrium generation of a BE condensate. The 2PI effective action is a function of both the classical field and of the two point function (and the two point function in closed time path represents the number density and propagator). Analytic calculations have been performed in self-interacting scalar field theories [48], and show [49] that at NLO, an overpopulation of low momentum modes in the number density can institute an inverse cascade towards the infrared, without first establishing an equilibrium distribution. Recall, however, that a high density of low momentum modes is not a classical field (or a BE condensate); it lacks the required coherence.

In summary, we focus on the gravitational interactions of the misalignment axions, which are already a classical field. We look for dissipation in these interactions, because this increases entropy. In 2PI formalism, thermalisation does not occur at leading order in the coupling in \(\phi^4\) models; extrapolating naively, this suggests that gravitational thermalisation, or entropy generation, does not occur at leading order in \(G_N\). Instead of including \(O(G_N^2)\), we look for dissipation/thermalisation at order \(G_N|\vec{p}|^2/m_a^2\). This could be relevant to BE condensation, if the axion field produced by the misalignment mechanism is not already a BE condensate. (This assumption is consistent with [19]. References [16, 17] envisage, that in this case, the “gravitational thermalisation” of axions will drive them to a BE condensate.)

2.3 Axion Bose-Einstein condensation in cosmology

This project was motivated by the scenario envisaged in [16, 17], where axions form a Bose-Einstein condensate in the early Universe at photon temperatures \(T \sim \text{keV}\), due to gravitational scattering among the axions. The gravitational interaction rate of [16, 17] was confirmed by Saikawa and Yamaguchi (SY) [19], who calculate in Quantum Field Theory, the rate of change of the axion number operator \(-i[H, \hat{n}(k)]\). SY describe the axions as a coherent state (see eqn (6.4)) in Minkowski spacetime, interacting via Newtonian gravity. This significant calculation has various curious features: the Newtonian analysis is applied in the early Universe, without a distinction between the homogeneous energy density which drives expansion, and the fluctuations. Also, intuition and the equations of linear structure growth say that gravity grows inhomogeneities, which appears naively at odds with gravitational interactions driving axions to Bose-Einstein condense in the zero mode. Another curious feature is that, although gravity should be universal, the axions are found to “gravitationally thermalise” with themselves, but not with other particles. We discuss the interpretation of our estimates and these earlier calculations in section 4.

\(^5\)In the approximation of this paper where we neglect quartic axion self-interactions, these BE condensates have only gravitational interactions with each other.
Finally, we raise one more confusing issue. A BE condensate in statistical mechanics is a large number of particles in a \(\delta\)-function at zero kinetic energy. In the coherent state notation of eqn (6.4), these particles make up the first term of eqn (2.7). However, in cosmology, it is unclear how narrow is the energy range for the axions making up the “zero mode”, or BE condensate. At the QCD phase transition, the axions of mass \(m_a\) and momentum \(H_{QCD}\), have kinetic energy \(E_K = \frac{H^2_{QCD}}{2m_a} \ll H_{QCD}\). During radiation domination, the ratio

\[
\frac{E_K}{H} \simeq \frac{H_{QCD}}{2m_a} \ll 1
\]

remains constant; between matter-radiation equality and today, it increases by a factor \(\sqrt{T_{eq}/T_0}\), but does not attain one. Therefore, the Heisenberg uncertainty principle could imply that the age of the Universe is not long enough to distinguish that the axions are not in the zero mode. Does this imply that they are in a BE condensate? Notice that their three-momentum can be distinguished from zero, so if the condensate was defined as the zero-momentum state, then the axions are not in it.

3 Estimating axion viscosity

This section aims to address whether classical gravitational interactions among axions can dissipate fluctuations and generate entropy. The Appendix suggests that in a homogeneous and isotropic Universe, the answer is “no” (gravity merely redshifts the axion momenta, in a Friedman-Robertson-Walker Universe). So we study a cosmological scenario with density fluctuations, whose gravitational effects can be treated in the linear approximation. The question would be more difficult, in the case where gravity is non-linear.

We take an initial condensate made of axions with comoving momenta of order \(H_{QCD}\). We compute their stress-energy tensor in an almost homogeneous and isotropic Universe, including scalar metric perturbations in Newtonian gauge [27, 50]. In particular, the metric perturbations will give spatial off-diagonal components \(T_{ij}\), for \(i \neq j\). We equate the term involving the gravitational potential with the off-diagonal \(T_{ij}\) of an imperfect fluid in a homogeneous and isotropic Universe. In an imperfect fluid, the \(T_{ij}\) elements, for \(i \neq j\), are proportional to the viscosity, which damps short-scale fluctuations. The only interactions of the axions are gravitational, so implicitly, the “bath” responsible for dissipation in the fluid contains metric and density fluctuations. Equation (3.23) is an estimate of the damping scale of density fluctuations, due to the gravitational self-interactions of axions.

3.1 Axion initial conditions after the QCD phase transition

We take our initial conditions a few Hubble times after the QCD phase transition, when the axion mass is settled to its value today. We focus on the classical axion field produced by misalignment, this does not include the axions from strings. Classical field means in principle the variable in the 1PI action, and in practise the expectation value of the field operator in the ground state. It can be expressed as a coherent state, see eqn (6.4). We suppose that at the QCD phase transition, the axion field was approximately constant within a horizon volume, and randomly distributed between \(-\pi\) and \(\pi\) from one horizon volume to another. So the initial axion field oscillates rapidly in time with frequency \(m_a\), and more slowly in space with co-moving momentum \(\sim H_{QCD}\). It can be expanded on Fourier modes of a homogeneous and isotropic Universe:

\[
a(\vec{x}, t) = \frac{1}{\sqrt{2m_a V R^3(t)}} \sum_p \tilde{a}(\vec{p}, t) \exp\{i(\vec{p} \cdot \vec{x} - \omega t)\} + \tilde{a}^*(\vec{p}, t) \exp\{-i(\vec{p} \cdot \vec{x} - \omega t)\}
\]

where \(\vec{p}\) is the comoving three-momentum, and the field is normalised in a comoving box of volume \(V\). Recall that \(a(\vec{x}, t)\) has mass dimension one, so the \(\tilde{a}(\vec{p}, t)\) are dimensionless, and \(|\tilde{a}(\vec{p}, t)|^2 / 2\) is the number of axions of momentum \(\vec{p}\) in the volume \(V\) (see eqn (3.7)). The fast time dependence \(e^{-i\omega t}\),
which we approximate as $e^{-imt}$ can be averaged [8, 9, 11] on the longer evolution timescale of the spatial variations. The axion is a real field, so the Fourier coefficients satisfy $\tilde{a}(\vec{p}, t) = \tilde{a}^*(\vec{p}, t)$. Fourier transforms are performed in a comoving box $V = L^3$, and defined to be “dimensionless” to simplify dimensional analysis; we write

$$\frac{d^3x}{V}, \text{ and } \sum_p = V \int \frac{d^3p}{(2\pi)^3}.$$

Notice that the density fluctuations are of order one on the co-moving scale

$$\delta \rho(\vec{x}, t) = \frac{1}{\sqrt{2mV\langle R^3(t) \rangle}} \left( \tilde{a}_0(t_{QCD}) \cos(mt) + \sum_i [\delta \tilde{a}(\vec{p}, t)e^{i(\vec{p} \cdot \vec{x} - mt)} + \delta \tilde{a}^*(\vec{p}, t)e^{-i(\vec{p} \cdot \vec{x} - mt)}] \right)$$

where $\tilde{a}_0(t_{QCD})/\sqrt{2mV} = a(t_{QCD})$ is the averaged-over-the-Universe value of the field at the QCD phase transition, and the small fluctuations considered are on large scale structure scales. As discussed in [28], the difference between these two forms for the growth of linear density perturbations is negligible: the kinetic energy density in the horizon-scale fluctuations is negligible (compared to the potential and the time oscillations), and on structure formation scales, $(a(\vec{x}, t))(t_{QCD}) \simeq \pi f_{PQ}/\sqrt{3}$.

### 3.2 The stress-energy tensor with perturbed metric

This section reviews the stress-energy tensor and equations of motion for scalar perturbations in an almost homogeneous and isotropic Universe.

The metric in Newtonian gauge can be written

$$ds^2 = (1 + 2\phi)dt^2 - R^2(t)(1 - 2\phi)\delta_{ij}dx^i dx^j$$

where $\phi \simeq \psi$ will be the Newtonian potential inside the horizon, and we take the scale factor $R(t)$ dimensionless and equal to 1 at the QCD phase transition.

The stress-energy tensor for a homogeneous and isotropic Universe is $T^\mu_\nu = \text{diag}(\bar{\rho}, -\bar{P}, -\bar{P}, -\bar{P})$, where $\bar{\rho}$ and $\bar{P}$ are the (homogeneous and isotropic) energy density and pressure. In the presence of scalar fluctuations, $T^\mu_\nu$ can be described with four additional parameters, written in Fourier space as [50]

$$\bar{\rho}(t) \rightarrow \bar{\rho}(t) + \delta \tilde{\rho}(\vec{k}, t), \quad \bar{P}(t) \rightarrow \bar{P}(t) + \delta P(\vec{k}, t)$$

$$ik_j \delta T^i_0 = (\bar{\rho} + \bar{P})\delta \theta(\vec{k}, t), \quad (\vec{k}, \vec{k} \cdot \vec{k} - \frac{1}{3} \delta_{ij}) \delta T^i_j = -(\bar{\rho} + \bar{P})\bar{\sigma}(\vec{k}, t)$$

where $\theta$ parametrises a fluid velocity, and $\sigma$ is the anisotropic stress.

For a massive non-interacting real scalar field, such as the axion, the stress-energy tensor has the form

$$T^\mu_\nu = a^{\mu \nu}a_{\mu \nu} - \frac{1}{2} (a^{\mu \alpha}a_{\alpha \nu} - m^2 a^2) \delta^\mu_\nu.$$

Equating (3.5) and (3.4) allows to determine the density fluctuations and other fluid parameters of the classical axion field.

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6It can be removed more elegantly by studying the non-relativistic field [12].

7This means that fluctuations get smaller on larger distances $L$: $\int_L \delta \rho \sim |\bar{\rho}|/L^3 H_{QCD}^{-1}$.
For axions of the form given in eqn (3.1), \( \overline{\rho}_a(t) \) is the \( \overline{\rho} = 0 \) Fourier mode of the density \( \overline{\rho}(\overline{p},t) \):

\[
\overline{\rho}_a(t) = \int \frac{d^3 x}{V} T^0_a(\overline{x},t) = \frac{m^2}{[R(t)]^3} \sum_q \frac{|\tilde{a}(\overline{q},t)|^2}{mV} \left( 1 + \frac{q^2}{m^2 R(t)^2} \right) + \ldots \tag{3.6}
\]

where the \(+\ldots\) contains subdominant terms involving \( H, \phi \) and \( \psi \), and the \( q^2/m^2 \) term will be neglected. The (volume-averaged) number density of axions in the classical field can similarly be expressed as

\[
n_a(t) = \frac{m}{[R(t)]^3} \sum_q \frac{|\tilde{a}(\overline{q},t)|^2}{mV} + \ldots \tag{3.7}
\]

If \( n_a(t_{QCD}) \simeq m\pi^2 f_{QCD}^3/3 \), and \( \tilde{a}(\overline{q},t) \) is approximately constant for \( |\overline{q}| \lesssim \sqrt{3}H_{QCD} \), then \( |\tilde{a}(\overline{q},t)|^2 \simeq \frac{\pi^4 m_a f_a f_{QCD}}{H_{QCD}^2} \).

The Fourier transform of the density fluctuations in the classical axion field is

\[
\delta\tilde{\rho}_a(\overline{k},t) = \int \frac{d^3 x}{V} e^{-i\overline{k}\cdot\overline{x}} [\rho_a(\overline{x},t) - \overline{\rho}_a(t)] = \frac{m^2}{mV[R(t)]^3} \sum_q \tilde{a}(\overline{q} + \overline{k}/2, t) \tilde{a}^*(\overline{q} - \overline{k}/2, t) \quad k \neq 0 \tag{3.8}
\]

where we have dropped terms proportional to \( H, \phi \) and \( \psi \), and \( (q^2,k^2)/m^2 \). Notice that this formula is different (less intuitive) from the case usually studied in structure formation, where most axions are in a zero-momentum condensate. If most axions are in the zero mode, the density fluctuations on scale \( k^{-1} \) are linear in the field fluctuations, \( \delta \tilde{\rho}(\overline{k},t) \sim \tilde{a}_0 \delta \tilde{a}(\overline{k},t) \) so are made up of axions of momentum \( k \) (in the coherent state formalism of eqn (6.4)).

The dynamics is controlled by \( T^{\alpha\beta}_{a;\beta} = 0 \), and by Einstein’s Equations \( G_{\alpha\beta} = 8\pi G_N T_{\alpha\beta} \). In the absence of perturbations, these give the Hubble expansion rate

\[
\left( \frac{\dot{R}}{R(t)} \right)^2 \equiv H^2(t) = \frac{8\pi G_N}{3} (\overline{\rho}_a(t) + \overline{\rho}_{\text{strings}}(t) + \overline{\rho}_{\text{rad}}(t)) \tag{3.9}
\]

where \( \overline{\rho}_a(t) + \overline{\rho}_{\text{strings}}(t) + \overline{\rho}_{\text{rad}}(t) \) is the Universe-averaged density in the axion field, in the axions from strings, and in radiation.

The equations for the scalar metric and density fluctuations can be found in [13, 27, 50]. For the stress-energy tensor of the axion field, eqn (3.4), the condition \( T^{\alpha\beta}_{a;\beta} = 0 \), gives the scalar equation of motion in the perturbed Universe. Whereas for a perturbed fluid, eqn (3.5), \( T^{\alpha\beta}_{\gamma;\beta} = 0 \), gives two equations for the four parameters. If the speed of sound \( c_s^2 = \delta P/\delta \rho \) can be calculated, and \( \sigma \) neglected, then to determine the dynamics of the fluid (like those of the field), requires only one additional equation from Einsteins Equations. We will be interested in fluctuations inside the horizon, so neglecting terms of order \( \partial^a \phi, H(t) \), the Einsteins Equations give, in Fourier space, the Poisson equation for \( \phi \):

\[
-\frac{\overline{\rho}^2}{R^2(t)} \tilde{\phi}(\overline{p},t) \simeq 4\pi G_N \delta \overline{\rho}(\overline{p},t) \tag{3.10}
\]

where \( \delta \overline{\rho}(\overline{p},t) \) is the Fourier transform of the density fluctuation (in radiation and axions). Notice that this can be interpreted as the potential due to single graviton exchange [51], which we will allude to in the discussion.
Combining the various equations gives the well-known equation \([9-11]\) for the evolution of adiabatic scalar density fluctuations \(\delta \equiv \delta \rho(\vec{k}, t)/\rho(t)\):

\[
\ddot{\delta} + 2H \dot{\delta} - 4\pi G_N \delta + \frac{c_s^2 k^2}{R^2(t)} \delta = 0
\]  
(3.11)

The equations for isocurvature fluctuations are different \([52]\), but they share with this equation the property that density fluctuations are frozen within the horizon during radiation domination\([52]\), and can grow during matter domination. For a homogeneous axion field with small fluctuations, as would arise if the PQ phase transition was before inflation (see eqn (3.2)), the equation (3.11) is elegantly obtained in \([10]\). In this case, it is shown that \(\sigma = 0\) and \(c_s^2 \approx k^2/(4m^2R^2(t))\). The (physical) axion Jeans length is therefore

\[
\lambda_j(t) \approx \frac{2\pi}{[16\pi G_N \rho(t)m^2]^{1/4}} \sim \frac{6}{\sqrt{H(t)m}}
\]  
(3.12)

on shorter distances, the fluctuations oscillate due to axion pressure, on larger distances, they can grow in a matter-dominated Universe. It can be checked that \(\lambda_j \sim \sqrt{H_{\text{QCD}}/m} \times H_{\text{QCD}}^{-1} R(t)\), suggesting that axions behave like dust on the comoving distance of the QCD horizon.

A caveat is that \(\sigma\) and \(c_s\) may be different for the axion field configuration arising when the PQ transition is after inflation (see eqn (3.1)). However, by dimensional analysis, \(\sigma \lesssim H_{\text{QCD}}^{-1}/m^2\) and \(\overline{\rho} \sim \delta \rho \lesssim \rho R_{\text{QCD}}^2/m^2\), so they naively appear insignificant to fluctuation evolution on the comoving scale \(H_{\text{QCD}}^{-1}\).

Recall that the fluctuations in the density of the axion field on the scale \(H_{\text{QCD}}^{-1}\) are of \(O(1)\). After matter-radiation equality, these short-distance isocurvature fluctuations can grow and promptly decouple from the Hubble flow, to form gravitationally bound axion configurations called “miniclusters” \([53]\). The miniclusters can further cool and contract due to gravitational interactions \([54]\). The position-space perspective on these \(O(1)\) inhomogeneities is instructive. One can estimate that an axion (particle?) with comoving momentum \(H_{\text{QCD}}\) cannot escape from a fluctuation of comoving size \(H_{\text{QCD}}^{-1}\) prior to matter-radiation equality. That is, the fluctuations are not damped by free-streaming. If an axion BE condensate should be approximately homogeneous, then it is unclear to the authors how the axions making up \(O(1)\) density fluctuations on scales \(H_{\text{QCD}}^{-1}\) can migrate to the zero-momentum mode, because they do not seem to move fast enough to homogenise in position space.

### 3.3 Anisotropic stress

The previous section showed that the small \(T_{ij}^k\) elements of the stress energy tensor of the classical axion field where unimportant for fluctuation growth. This section calculates these off-diagonal elements, with the aim of identifying in them some gravitational dissipation.

The off-diagonal spatial elements \(T_{ij}^k\) are interesting for two reasons: they are gauge invariant, and in the fluid approximation, they are proportional to the viscosity. Viscosity damps fluctuations on small scales \([55]\), so we hope that an estimate of the viscosity will give some notion of gravity’s ability to generate entropy. The first step is to compute \(T_{ij}^k(\vec{x}, t)\), for \(i \neq j\):

\[
T_{ij}^k(\vec{x}, t) = -\frac{(1 + 2\phi)}{R^2(t)} \partial_i \partial_j \partial_k a
\]  
(3.13)

so in Fourier space:

\[
T_{ij}^k(\vec{k}, t) = -\frac{1}{m VR^2(t)} \left[ \sum_q (q + k/2)_i(q - k/2)_j \tilde{a}(\vec{q} + \vec{k}/2, t) \tilde{a}^*(\vec{q} - \vec{k}/2, t) \\
+ 2 \sum_{pq} (q + k/2)_i(q + p - k/2)_j \tilde{\phi}(\vec{p}, t) \tilde{a}(\vec{q} + \vec{k}/2, t) \tilde{a}^*(\vec{q} + \vec{p} - \vec{k}/2, t) \right].
\]  
(3.14)
We drop the first term (not involving the metric fluctuation) in this expression, which arises because the condensate of axions with finite momentum is not perfectly homogeneous and isotropic. This term is naively of order the axion pressure \( \sim \rho_a(T) \), which we also neglect. This first term in principle contributes to distinguishing \( \phi \) from \( \psi \) (see the metric of eqn (3.3)) in the perturbed Einstein Equations:

\[
k_i k_j \left( \phi(k, t) - \psi(k, t) \right) = 12\pi G_N T_i^j \quad (i \neq j). \quad (3.15)
\]

However, we neglect this effect and take \( \phi = \psi \), because the axions constitute initially a tiny fraction of the energy density, and this contribution to \( T_i^j \) decreases as \( 1/R(t) \) compared to the total density (second order radiation perturbations could be more significant). The second term of eqn (3.14), which contains the gravitational potential of density perturbations, is the piece from which we wish to extract axion viscosity.

### 3.4 Matching to an imperfect fluid

We aim to obtain a viscosity coefficient for our axion fluid, despite that we do a classical field analysis with coherent initial conditions. We need fluctuations and dissipation, and since we approximate the axions to have only gravitational interactions, these must involve gravity. We therefore map the stress tensor of the Universe with metric fluctuations, onto the imperfect fluid stress tensor of a homogeneous and isotropic Universe. One can imagine that the density/metric fluctuations generate the viscosity.

The stress-energy tensor for an imperfect fluid in a homogeneous and isotropic expanding Universe is given in [55]. For \( i \neq j \):

\[
T_i^j(\vec{x}, t) = -\eta(t)(\partial_j U^i(\vec{x}, t) + \partial^i U_j(\vec{x}, t))
\]

(3.16)

where \( U^\alpha \) is the fluid four-velocity, which Weinberg defines from the conserved number current \( N^\alpha = nU^\alpha \). Since axions are a real field, it is convenient to use an alternative definition, so we define \( U^\alpha \) from the energy flux: \( T_i^0 = \rho U^0 U_i \). This has the added interest of giving a \( k_i k_j \) term in eqn (3.18). As discussed with care in Weinberg’s paper [55], it is important to use a self-consistent formalism, so we anticipate that our estimate will not have the correct constant factors. We hope that the dependence on physical parameters will nonetheless be correct. For non-relativistic axions, eqn (3.5) in a homogeneous and isotropic Universe gives

\[
U_0 U^i(\vec{x}, t) \simeq -\frac{1}{R^2(t) p(t)} \partial_i a(\vec{x}, t) \partial a(\vec{x}, t)
\]

(3.17)

which gives

\[
T_i^0(\vec{k}, t) = -\frac{\eta(t)}{m V R^3(t) n_a(t)} \sum_q [q_i k_j + k_i q_j - k_i k_j] \bar{a}(\vec{q} + \vec{k}/2, t) \bar{a}^*(\vec{q} - \vec{k}/2, t)
\]

(3.18)

with \( n_a(t) \) from eqn (3.7).

Equating the coefficients of \( \bar{a} \bar{a} \) in eqns (3.18) and the second line of eqn (3.14), gives

\[
\frac{\eta(t)}{n_a(t)} \sim -2\pi G_N \sum_p \frac{\delta \rho(p, t) R^2(t)}{|p|^2}
\]

(3.19)

where we suppose the \( p \) in the sum on \( q \) of \(|\bar{a}|^2\) makes little difference.

This estimate used a description of imperfect fluids [55] which can suffer from non-causal information propagation. Such difficulties are avoided with the causal thermodynamics of [56], which adds approximately a factor \( \left(1 + H/\Gamma_g \right) \) to the right side of (3.19), where \( \Gamma_g \sim 8\pi G \rho_a R(t)/H_{QCD}^2 \) is the gravitational interaction rate of axions (see eqn 4.1). The correction factor exceeds 2 for \( T > 2 \text{keV} \) (for \( f_{PQ} \sim 10^{12} \text{GeV} \) fixed) and grows linearly with \( T \). However, we neglect this effect, because it never
allows the time or length scale of dissipation to reach the horizon, and because axions gravitationally thermalise after $T \sim \text{keV}$ in the scenario of Sikivie and collaborators.

There are two simple limits for the estimate of eqn (3.19). First, if the dominant density fluctuations are the no-scale adiabatic fluctuations in the radiation, then the sum is infrared divergent and dominated by horizon-scale fluctuations:

$$\frac{\eta(t)}{n_a(t)} \sim 2\pi G_N \frac{\delta \tilde{\rho}(H(t)T_{QCD}, t)/H^2(t)}{H_{QCD}^2} \approx 3 \frac{\delta \tilde{\rho}(H(t)T_{QCD}, t)}{\rho(t)}$$  \hspace{1cm} (3.20)

Before and during linear fluctuation growth, this gives $\eta(t) < n_a(t)$.

The second case is when the dominant density fluctuations are the axion inhomogeneities on the co-moving scale $H_{QCD}^{-1}$. Then the sum $d^3p \delta \tilde{\rho}_a/|\vec{p}|^2$ in eqn (3.19) is dominated by $p \sim H_{QCD}$, giving

$$\frac{\eta(t)}{n_a(t)} \sim 2\pi G_N \frac{\delta \tilde{\rho}(H_{QCD}, t)R^2(t)}{H_{QCD}^2} \approx 8\pi G_N \rho_a(t_{QCD}) \frac{T}{T_{QCD}} = \frac{\rho_a(t_{QCD})}{\rho_{\text{rad}}(t_{QCD})} \left( \frac{T}{T_{QCD}} \right) = \frac{T_{\text{eq}}T}{T_{QCD}}$$ \hspace{1cm} (3.21)

where $T_{\text{eq}}$ is the photon temperature at matter-radiation equality.

Weinberg gives [55] that modes of comoving wavenumber $\vec{p}$ decay at a rate

$$\Gamma \sim \frac{\eta(t)|\vec{p}|^2}{R^2(t)|\vec{p}(t)|}$$ \hspace{1cm} (3.22)

So the physical distance $\ell_{\text{damp}}(t)$ on which fluctuations could disappear grows as the square root of the time available. This makes intuitive sense when fluctuations are damped by particles random-walking out$^9$. In the axion case studied here, in the lifetime of the Universe $\sim 1/H$, fluctuations on physical distances less than $\ell_{\text{damp}}$ could dissipate, where

$$\ell_{\text{damp}}^2(t = 1/H) \sim \frac{1}{H(t)m_a} \frac{\eta(t)}{n_a(t)} \frac{\rho_a(t)}{\rho(t)}$$ \hspace{1cm} (3.23)

It is clear that these estimates give a damping distance much shorter than the comoving scale $H_{QCD}$. The Jeans distance for axions is $1/\sqrt{H(t)m}$ [9]: at shorter distances, density fluctuations in axions oscillate due to pressure, and at larger distances the fluctuations can grow (during matter domination). It is reassuring that the damping distance due to viscosity (3.23) is shorter than the Jeans length.

4 Discussion and comparison to previous results

The estimated damping scale (3.23) for axion density fluctuations prior to the period of non-linear structure formation, is always shorter than the QCD horizon scale $T_{QCD}/(TH_{QCD})$. It does not confirm that “gravitational thermalisation” erases axion fluctuations on the QCD horizon scale at $T \sim \text{keV}$. We first comment on our estimate, then compare to the calculation of Saikawa and Yamaguchi [19].

$^8$If $k_H$ is the comoving scale at the horizon, we can write $\delta \tilde{\rho}(k,t)/|\vec{p}(t)| = A(k_H/k)^{3/2}$, to obtain a no-scale power spectrum such that $V \int d^3k P(k) = 4\pi V^{k_H}|A|^2 \ln k_{\text{max}}/k_{\text{min}}$. Then $V \int_{k_H} \int d^3p \delta \tilde{\rho}(p,t)/|\vec{p}|^2 = 4\pi V^{k_H}A \sim \delta \tilde{\rho}(k_H,t)/(|\vec{p}(t)|k_H^2)$.

$^9$Recall, however, that this picture corresponds to perturbation theory in the mean free path $\sim 1/\sigma(n)$. So weakly interacting particles diffuse more easily out of perturbations, and the interpretation is unclear for particles whose mean free path is the size of the perturbation.
4.1 Our estimate

A first simplifying approximation made in this paper, is that we only work to linear order in $G_N$. This could appear curious compared to thermalisation rates associated to Boltzmann Equations, where the rates are proportional to couplings-squared. But classical fields, expressed as coherent states, correspond to the coherent superposition of amplitudes, so classical gravitational effects appear at linear order in $G_N$ (as is well known).

We focus on the axion energy density, and fluctuations therein, rather than on the axion field. It is clear that in general, the field carries more information than the energy density, since it allows to compute a wider variety of correlation functions. However, at the classical level used in this paper, the equations of motion for both the axion field and the density fluctuations are obtained from $T^{\mu\nu} = 0$ and the Poisson Equation (3.10), which suggests that it is merely two different parametrisations of the same physics. The equations for the density fluctuations have the advantage that they are linear and can be solved. They say that gravity grows inhomogeneities in axions. Whereas the equations for the field are non-linear; a gravitational interaction rate for axions can be calculated without solving the equations, but that does not say what gravity does with the axions. So we do not disagree with the gravitational interaction rate of axions obtained by [17, 19] (we can reproduce it, see eqn (4.1)); however, we disagree with its interpretation as a thermalisation rate. We suspect that it is the rate associated with the gravitational growth of density fluctuations (which is compensated by the expansion of the universe during radiation domination).

We supposed that BE condensation requires dissipation, and furthermore, that leading order solutions of classical equations of motion do not exhibit dissipation or thermalisation. This is a usual perspective in non-equilibrium field theory — to obtain dissipation from time-reversal invariant equations requires summing over a bath of fluctuations. We are unclear on how to separate gravitational interactions in the early Universe into a leading order solution plus fluctuations that we can integrate. Therefore, we hesitate to discuss a “gravitational thermalisation” rate, because its definition seems to require this separation of gravitational interactions into “leading order” and “dissipative”. It may be unwise to identify the homogeneous and isotropic component of the Universe as the leading order solution, and the fluctuations as the bath, because density fluctuation growth is an important part of the classical solution. However, the $O(|p|^2/m_a^2)$ terms are usually neglected in these equations, so we attempt to associate dissipation with them: in the perturbed, expanding Universe, the off-diagonal spatial elements of the stress-energy tensor are gauge invariant, of $O(|p|^2/m_a^2)$, and unimportant for fluctuation growth. We identify them with the off-diagonal elements of the stress energy tensor of an imperfect fluid. An imperfect fluid can grow density fluctuations, but contains dissipation, so we hope, by this identification, to be summing over the gravitational fluctuations that are not an important part of the classical solution. Fortunately, the damping scale we obtain is irrelevantly short, so whether this trick is credible is of minor importance.

It can be useful to compare to classical thermalisation studies in $\phi^4$ models, using the 2PI action in closed time path formalism. In this formalism, the dynamical variables are the classical field and the two point function (which describes the density of incoherent modes and the propagator). Intuitively, one could anticipate that the classical field could dissipate by interacting with the bath of incoherent fluctuations. In this paper, we neglected the cold axion particles produced by strings, which could be the bath thermalising the axion field (because at linear order in $G_N$, they should just constitute an additional contribution to density fluctuations, with which the fluctuations in the density of the axion field could interact). Studies of thermalisation in $\phi^4$ find that the incoherent modes thermalise at NLO. So perhaps it would be interesting to study the evolution of axions.

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10 Whether this formulations are equivalent is important, because BE condensation corresponds to suppressing the field fluctuations. One can wonder if gravitational interactions could homogenize the field configuration without changing the stress-energy tensor. The linearised Einstein Equations might suggest not: the stress-energy fluctuations induce the Newtonian potential.

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from strings and misalignment using the 2PI effective action, in the Closed Time Path formalism used by Saikawa and Yamaguchi.

### 4.2 Making contact with previous calculations

We now address the differences between our estimate and the calculation of Saikawa and Yamaguchi (SY) [19]. We focus on this impressive analytic calculation, because they introduce very clearly the used formalism and obtain the same result as [17]. SY calculate the time evolution of the axion number operator, using a closed time path formalism of Quantum Field Theory, in flat space-time with Newtonian gravity. They evaluate $dn(q)/dt$ (which is the rate of change of the number density of axions of momentum $q$ due to gravitational interactions), in a coherent state representing highly populated low-momentum axion states. This rate is interpreted as an axion thermalisation rate, and it is larger than $H$ for photon temperatures $\lesssim 1$ keV.

An unimportant difference is that the redshifting due to Universe expansion does not appear in the SY calculation. The gravitational effect of the homogeneous and isotropic axion density is to drive expansion, but since SY calculate with Newtonian gravity in a non-expanding space-time, all the gravitational effects of the axions are included in the “thermalisation” process. This is a relatively minor issue; scale factors can be judiciously distributed in their formulae, and within the horizon, density fluctuations can be described by Newtonian gravity. If the density $\rho(x,t)$ in the SY formulae is replaced by the density fluctuation $\delta \rho$, then their equations are consistent with the classical linearised Einsteins Equations in Newtonian gauge.

An obvious difference from our classical discussion is that SY calculate in quantum field theory. This seems also to be unimportant, because using classical equations of motion we can obtain a similar result:

$$i \frac{\partial}{\partial t} |\tilde{a}(\vec{q},t)|^2 \approx 4\pi m G_N \sum_{\vec{k}} \frac{R^2(t)}{|\vec{k}|^2} \delta \rho(\vec{k},t) \left\{ \tilde{a}^\ast(\vec{q} + \vec{k},t)\tilde{a}(\vec{q},t) - \tilde{a}^\ast(\vec{q},t)\tilde{a}(\vec{q} - \vec{k},t) \right\} . \quad (4.1)$$

SY describe the axions as a coherent state, so it is unsurprising that their calculation gives the same result as the classical field equations, because coherent states are constructed for that purpose. In the understanding of the authors, the quantum aspect of the SY result is to identify $|\tilde{a}(\vec{q},t)|^2 / \sum_p |\tilde{a}(\vec{p},t)|^2$ as a number density of axion particles.

An important difference is that the axion number density $n(\vec{q},t)$ studied by SY is labelled by the axion momentum, whereas density fluctuations $\delta \rho(\vec{k},t)$ are labelled by the momentum of the graviton they exchange. Notice that the same dynamics should be included in $\sum_{\vec{p}} n(\vec{p},t)$ of SY and the equation of fluctuation growth (eqn 3.11), because they are both the result of $T_{\mu\nu} = 0$ and the Poisson equation. To the understanding of the authors, the SY calculation shows that the rate for an axion to emit a graviton of any wavelength is large (compared to $H$). However, what the gravitons then do is unknown. Whereas the solution of the equations of motion for density fluctuations say that the gravitons cause the density fluctuations to grow.

Finally, SY show that axions do not have coherent gravitational interactions with other particles, such as photons, in the early Universe plasma. This is good, because it means that axions are not heated to the photon temperature. However, the classical Einsteins Equations say that gravity is universal, so that density fluctuations in the axions are subject to the gravitational attraction of other fluids (at linear order in $G_N$). Indeed, our estimate of the gravitational damping scale involves the density fluctuation, irrespective of whether it is made of axions or other particles. How can these two perspectives be consistent? At order $G_N$, the axions should have gravitational interactions with

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\[11\] From eqn (3.7), $|\tilde{a}(\vec{q},t)|^2 / (\sum_p |\tilde{a}(\vec{p},t)|^2)$ is the fractional number density of axions of momentum $\vec{q}$. This equation can be obtained from the equations of motion for the fourier-transformed field, multiplied by $\tilde{a}(\vec{q},t)$. The equations of motion for the field, like those for $\delta \rho$, are obtained from eqns (3.10) and $T_{\mu\nu} = 0$.

\[12\] This is because the distribution of $\delta s$ in the Lagrangian is different, depending on whether the classical limit should be fields or particles [58]. So to define the particle number of a classical field configuration requires $\hbar$. 

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the fluctuations in the density of other particles, rather than with the individual particles. That is, to find the universal gravitational attraction between hot other particles and the cold axions, one should describe the other particles with an “effective Lagrangian” at the scale of the graviton momentum. For instance, in the Closed Time Path formalism of SY, the radiation plasma in the early Universe can be described (in 2PI formalism) by its two-point function. Averaged over short distances $\sim T^{-1}$, the two point function becomes a Wigner function, which can be approximated as a Boltzmann phase space distribution on the scale of axion momenta. The density fluctuations encoded in the temperature variations of this Boltzmann distribution are the density fluctuations which interact gravitationally with the axions at order $G_N$. We interpret that the axions do not interact with individual hot photons, which could destroy the axion condensate, but rather, that the axion interactions with the long range density fluctuations in other particles will grow density perturbations, and could contribute to the axion dissipation.

5 Summary

The question of interest for this paper is whether gravitational interactions can “thermalise” the axions produced via the misalignment mechanism. We reproduce earlier estimates of the gravitational interaction rate of these axions, but do not confirm that it is a thermalisation rate.

We discuss this issue in cosmology, prior to the epoch of non-linear structure formation, because gravitational interactions can be treated in the linear approximation. We suppose that the Peccei-Quinn phase transition occurred after inflation, so when the axion mass turns on at the QCD phase transition and the axion field starts to oscillate, the coherence length of the field is of order the horizon. Equivalently, the comoving momentum of the field (or of the axion particles in the coherent state that makes it up) is of order the expansion rate $H_{QCD}$. The axions should form CDM, therefore the gravitational interactions of the homogeneous component must drive expansion, and the gravitational interactions of density fluctuations should cause them to grow. The question is whether, in addition, gravity can “thermalise” these axions, and cause them to form a Bose Einstein condensate as anticipated by Sikivie and collaborators [16, 17]. This question is only relevant, if the misalignment axions are not already a Bose Einstein condensate, as discussed in section 2.2.

The field theory literature indicates that Bose Einstein condensation can arise in non-equilibrium situations, as well as in thermal equilibrium – but that entropy is not generated in the leading order classical solution of time-reversal-invariant equations. Instead, some fluctuations must be resummed to obtain a Bose Einstein condensate in a calculation. So in this paper, we attempt to identify and “resum” some gravitational interactions which are not those driving the expansion or growing density perturbations. In section 3.3, we estimate the contribution of metric fluctuations to the off-diagonal elements of the stress-energy tensor $T^i_j$. These elements are commonly neglected in calculating the evolution of axion density perturbations, so we imagine that we can resum these fluctuations. We do this in section 3.4, by equating the $T^i_j$ of section 3.3 to the $T^i_j$ of an imperfect fluid in a homogeneous and isotropic Universe. This gives an estimate for the “gravitational viscosity” of the axion fluid. We find that this viscosity damps fluctuations on distances smaller than the axion Jeans length $\sqrt{1/m_a H}$. The damping scale is given in eqn (3.23). In particular, fluctuations on the comoving scale $H^{-1}_{QCD}$ are not damped during the cosmological periods we consider. So we do not confirm the interpretation of [17, 19] that axions migrate to the zero mode (form a Bose Einstein condensate) at a photon temperature $T_\gamma \sim \text{keV}$, due to “gravitational thermalisation”. We can reproduce the gravitational interaction rate obtained by [17, 19], but it is unclear to us that this is a thermalisation rate: some of the gravitons should be contributing to the growth of density fluctuations. Section 4 discusses our estimates and compares to the calculation of [19].
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Note added: When this paper was nearing completion, appeared an interesting discussion [60] of the connection between the field theory and fluid descriptions of a BE condensate.

6 Appendix

In this Appendix, we study whether gravity can redistribute the momenta of CDM axions in a homogeneous and isotropic Universe described by Einsteins Equations. We consider a classical free scalar field, that is, a coherent state, evolving in a Friedman-Robertson-Walker universe. We suppose this to be an adequate description of dark matter axions after the QCD phase transition, during linear structure formation. So from an S-matrix perspective, we take the “in” states to be axion modes shortly after the QCD phase transition, and the “out” states prior to $z \sim 10$. We describe the CDM axions as a coherent state of “in-particles”. We then evaluate, in that state, the expectation value of the number operator of “out-particles”, thereby obtaining their momentum distribution. As expected, the physical momentum of the modes redshifts, and the comoving momentum distribution does not change at leading order.

6.1 The Calculation

We take the starting time $t = 0$ for our study to be a few Hubble times after the QCD phase transition, when the axion dark matter can be described as a real free scalar field in a Friedmann-Robertson-Walker background, with metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R^2(t)[dx^2 + dy^2 + dz^2]$$  \hspace{1cm} (6.1)

We follow the evolution of axion dark matter until this description breaks down, when structure formation becomes non-linear. We approximate this time as $t \to \infty$. During this period, the axion field $a(x)$ satisfies the equations of motion:

$$\ddot{a} + 3H\dot{a} - \frac{1}{R^2(t)}\partial_i\partial^i a + m^2 a = 0$$  \hspace{1cm} (6.2)

where $\dot{a} = \frac{\partial a}{\partial t}$. Following [61, 62], the field can be expanded on a complete set of orthogonal eigen-modes $\{u_k(x)\}$, which are solutions of eqn (6.2), and which correspond to axion particles at $t = 0$. We follow the conventions of [62], but with the metric of eqn (6.1). It is convenient to normalise the eigenmodes in a box of physical volume $R^3(t)L^3$:

$$u^\text{in}_k(t, x) = \frac{1}{[R(t)L]^3/2}\chi^\text{in}(t)e^{i\vec{k} \cdot \vec{x}}$$  \hspace{1cm} (6.3)

Notice that solutions of the Klein Gordon equation (6.2) are separable, due to the homogeneity of FRW spacetime.

In a second-quantised formalism, the axion field operator $\hat{a}(x)$ can be expanded in the usual way on (time-dependant) annihilation and creation operators which satisfy $[\hat{b}_k(t), \hat{b}_q(t)] = \delta_{k,q}$ [61], and
which multiply the modes \( \{ u_k \} \). These annihilation operators define the “in” vacuum. The classical axion field can therefore be written as a coherent state \([63]\) of (non-relativistic) axion particles:

\[
|a(\vec{x},t)\rangle = \frac{1}{N} \exp \left\{ \sum_\vec{p} a(\vec{p},t) b_\vec{p}^\dagger \right\} |0_m\rangle .
\]

(6.4)

where \( N \) is a normalisation factor to ensure \( \langle a|a \rangle = 1 \). This state describes the classical axion field: \( \langle a|\hat{a}(x)|a \rangle = a(x) \).

We wish to know the spectrum of axions that this state describes at \( t \to \infty \). It is well-known that gravity can change momenta and create particles. Canonical examples are momentum red-shifting in FRW cosmologies, and black hole radiation: in a the curved space-time outside a black hole, the state with no particles at \( t \to -\infty \) will contain particles at \( t \to +\infty \). This can be described \([62]\) by writing the in- vacuum creation operators (or equivalently, eigenmodes) in terms of the out-vacuum operators using Bogoliubov coefficients:

\[
u^\text{out}_k = \sum_\vec{q} \alpha_{\vec{k},\vec{q}} u^\text{in}_{\vec{q}} + \beta_{\vec{k},\vec{q}} u^\text{in,*}_{\vec{q}}
\]

(6.5)

Recall that the \( \beta \) coefficients, which parametrise the overlap between positive and negative frequency modes, describe particle creation by gravity. In the axion case, we wish to know the momentum distribution of “out-state” axions — that is, axion particles at the end of linear structure formation —in the coherent state of eqn (6.4). This can be evaluated if we know the Bogoliubov transformation between the “in” and “out” creation and annihilation operators, or equivalently, if one can express the “out” eigenmodes in terms of the “in” eigenmodes, as in eqn (6.5).

The out eigenmodes can be written

\[
u^\text{out}_k(t,\vec{x}) = \frac{1}{|R(t)L|^{3/2}} \chi^\text{out}(t)e^{i\vec{k} \cdot \vec{x}}
\]

(6.6)

where \( \chi^\text{out}(t) \) is a solution of

\[
\frac{\partial^2}{\partial t^2} \chi^\text{out} + \frac{|\vec{k}|^2}{R^2(t)} \chi^\text{out} + m^2 \chi^\text{out} = 0
\]

and chosen to describe axion particles at \( t \to \infty \) (\( \chi^\text{in} \) is a solution of the same equation). The homogeneity of FRW spacetimes means that co-moving momentum \( \vec{k} \) is conserved, or equivalently, the Bogoliubov coefficients are diagonal in momentum space \([62]\):

\[
\alpha_{\vec{k},\vec{q}} = (u^\text{out}_k, u^\text{in}_q) \propto \delta_{\vec{k},\vec{q}} , \quad \beta_{\vec{k},\vec{q}} = -(u^\text{out}_k, u^\text{in,*}_q) \propto \delta_{\vec{k},-\vec{q}}
\]

(6.7)

so the number operator for axion particles at \( t \to \infty \) is

\[
\hat{b}^\text{out}_k \hat{b}^\dagger \text{out}_k = (\alpha_{\vec{k},\vec{k}} \delta_{\hat{\text{in}}^\dagger_{\vec{k}}, \hat{\text{in}}_{\vec{k}}} - \beta_{\vec{k},-\vec{k}} \delta_{\hat{\text{in}}^\dagger_{\vec{k}}, -\hat{\text{in}}_{\vec{k}}} \epsilon_{\hat{\text{in}}^\dagger_{\vec{k}}} \hat{\text{in}}_{\vec{k}} - \beta^*_{\vec{k},-\vec{k}} \delta_{\hat{\text{in}}^\dagger_{\vec{k}}, -\hat{\text{in}}_{\vec{k}}} \epsilon_{\hat{\text{in}}^\dagger_{\vec{k}}} \hat{\text{in}}_{\vec{k}})
\]

(6.8)

This shows that the effect of gravity on axions, in an expanding FRW Universe, is to redshift their momenta (and possibly create particles). There is no indication, from this calculation, that gravity modifies the co-moving momentum distribution of the axions.

The axion creation by gravity, encoded in the coefficients \( \beta \), is expected to be negligible because \( H \ll m_a \). It can be estimated, following \([61]\), by taking the lowest order adiabatic approximation

\[
\chi(t) = \frac{1}{\sqrt{2\omega}} e^{i\int t \omega dt'}
\]

(6.9)
with $\omega^2 = |\vec{k}|^2 / R^2 + m^2$. For $|\vec{k}|^2 \ll m^2$, we obtain \(^{13}\)

$$|\beta_{\vec{k}-\vec{k}}| \ll \frac{H(t = 0)}{m_a}, \quad \alpha_{\vec{k}, \vec{k}} \simeq 1$$  \hspace{1cm} (6.10)$$

so gravitational particle production can be neglected, as expected, because $H(t = 0) \simeq H_{QCD} \ll m_a$. Setting $\beta_{\vec{k}-\vec{k}} \rightarrow 0$ in eqn (6.8) implies that the number of axion particles making up the classical field, and their co-moving momentum distribution, are unchanged in the expanding FRW Universe.

References

[1] for a review, see e.g. J. E. Kim, “Light Pseudoscalars, Particle Physics and Cosmology,” Phys. Rept. 150 (1987) 1.

[2] R. D. Peccei and H. R. Quinn, “CP Conservation in the Presence of Instantons,” Phys. Rev. Lett. 38 (1977) 1440.

[3] G. G. Raffelt, “Astrophysical methods to constrain axions and other novel particle phenomena,” Phys. Rept. 198 (1990) 1.

[4] M. Dine, W. Fischler and M. Srednicki, “A Simple Solution to the Strong CP Problem with a Harmless Axion,” Phys. Lett. B 104 (1981) 199.

[5] G. Bertone, D. Hooper and J. Silk, “Particle dark matter: Evidence, candidates and constraints,” Phys. Rept. 405 (2005) 279 [hep-ph/0404175].

[6] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, “Can Confinement Ensure Natural CP Invariance of Strong Interactions?,” Nucl. Phys. B 166 (1980) 493.

[7] M. Dine and W. Fischler, “The Not So Harmless Axion,” Phys. Lett. B 120 (1983) 137.

[8] M. Khlopov, B. A. Malomed and I. B. Zeldovich, “Gravitational instability of scalar fields and formation of primordial black holes,” Mon. Not. Roy. Astron. Soc. 215 (1985) 575.

[9] Y. Nambu and M. Sasaki, “Quantum Treatment Of Cosmological Axion Perturbations,” Phys. Rev. D 42 (1990) 3918.

\(^{13}\)Specifically, by substituting (6.9) into eqn (22) of [61], then integrating eqn (26) of [61] neglecting $\beta$ under the integral.
[13] H. Kodama and M. Sasaki, “Cosmological Perturbation Theory,” Prog. Theor. Phys. Suppl. 78 (1984) 1.

[14] E. W. Kolb, A. Singh and M. Srednicki, “Quantum fluctuations of axions,” Phys. Rev. D 58 (1998) 105004 [hep-ph/9709285].

[15] B. Ratra and P. J. E. Peebles, “Cosmological Consequences of a Rolling Homogeneous Scalar Field,” Phys. Rev. D 37 (1988) 3406.

[16] P. Sikivie and Q. Yang, “Bose-Einstein Condensation of Dark Matter Axions,” Phys. Rev. Lett. 103 (2009) 111301 [arXiv:0901.1106 [hep-ph]].

[17] O. Erken, P. Sikivie, H. Tam and Q. Yang, “Cosmic axion thermalization,” Phys. Rev. D 85 (2012) 063520 [arXiv:1111.1157 [astro-ph.CO]].

[18] O. Erken, P. Sikivie, H. Tam and Q. Yang, “Axion BEC DM” [arXiv:1111.3976 [astro-ph.CO]].

[19] K. ’i. Saikawa and M. Yamaguchi, “Evolution and thermalization of dark matter axions in the condensed regime,” Phys. Rev. D 87 (2013) 085010 [arXiv:1210.7080 [hep-ph]].

[20] C. A. Baker, D. D. Doyle, P. Geltenbort, K. Green, M. G. D. van der Grinten, P. G. Harris, P. Iaydjiev and S. N. Ivanov et al., “An Improved experimental limit on the electric dipole moment of the neutron,” Phys. Rev. Lett. 97 (2006) 131801 [hep-ex/0602020].

[21] A. Pich and E. de Rafael, “Strong CP violation in an effective chiral Lagrangian approach,” Nucl. Phys. B 367 (1991) 313. M. Pospelov and A. Ritz, “Theta induced electric dipole moment of the neutron via QCD sum rules,” Phys. Rev. Lett. 83 (1999) 2526 [hep-ph/9904483].

[22] G. ’t Hooft, “Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle,” Phys. Rev. D 14 (1976) 3432 [Erratum-ibid. D 18 (1978) 2199].

[23] M. Srednicki, “Axion Couplings to Matter. 1. CP Conserving Parts,” Nucl. Phys. B 260 (1985) 689.

[24] N. Viaux, M. Catelan, P. B. Stetson, G. Raffelt, J. Redondo, A. A. R. Valcarce and A. Weiss, “Neutrino and axion bounds from the globular cluster M5 (NGC 5904),” arXiv:1311.1669 [astro-ph.SR].

[25] G. Carosi [ADMX Collaboration], “Searching for old (and new) light bosons with the axion dark matter experiment (ADMX),” AIP Conf. Proc. 1441 (2012) 494.

[26] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, “Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions,” Phys. Rept. 215 (1992) 203.

[27] M. Beltran, J. Garcia-Bellido and J. Lesgourgues, “Isocurvature bounds on axions revisited,” Phys. Rev. D 75 (2007) 103507 [hep-ph/0606107].

[28] G. Efstathiou, “Cosmological perturbations,” In *Edinburgh 1989, Proceedings, Physics of the early universe* 361-463.

[29] G. Efstathiou and J. R. Bond, “Isocurvature cold dark matter fluctuations”, Mon. Not. Roy. Astron. Soc. 218 (1986) 103.

[30] D. Langlois, “Isocurvature cosmological perturbations and the CMB,” Comptes Rendus Physique 4 (2003) 953.
[33] R. L. Davis and E. P. S. Shellard, “Do Axions Need Inflation?,” Nucl. Phys. B 324 (1989) 167.
[34] O. Wantz and E. P. S. Shellard, “Axion Cosmology Revisited,” Phys. Rev. D 82 (2010) 123508 [arXiv:0910.1066 [astro-ph.CO]].
[35] Y. Aoki, S. Borsanyi, S. Durr, Z. Fodor, S. D. Katz, S. Krieg and K. K. Szabo, “The QCD transition temperature: results with physical masses in the continuum limit II.,” JHEP 0906 (2009) 088 [arXiv:0903.4155 [hep-lat]].
[36] J. Beringer et al. [Particle Data Group Collaboration], “Review of Particle Physics (RPP),” Phys. Rev. D 86 (2012) 010001.
[37] F. Wilczek, “A Model of anthropic reasoning, addressing the dark to ordinary matter coincidence,” In *Carr, Bernard (ed.): Universe or multiverse* 151-162 [hep-ph/0408167].
[38] T. Rindler-Daller and P. R. Shapiro, “Finding new signature effects on galactic dynamics to constrain Bose-Einstein-condensed cold dark matter,” arXiv:1209.1835 [astro-ph.CO]. ibid. “Angular Momentum and Vortex Formation in Bose-Einstein-Condensed Cold Dark Matter Haloes,” arXiv:1106.1256 [astro-ph.CO]. ibid. “Vortices and Angular Momentum in Bose-Einstein-Condensed Cold Dark Matter Halos,” arXiv:0912.2897 [astro-ph.CO].
[39] N. Banik and P. Sikivie, “Axions and the Galactic Angular Momentum Distribution,” arXiv:1307.3547 [astro-ph.GA].
[40] N. N. Bogoliubov, J. Phys. (Moscow) 11 (1947) 23.
[41] J. I. Kapusta, “Bose-Einstein Condensation, Spontaneous Symmetry Breaking, and Gauge Theories,” Phys. Rev. D 24 (1981) 426.
[42] H. E. Haber and H. A. Weldon, “Finite Temperature Symmetry Breaking as Bose-Einstein Condensation,” Phys. Rev. D 25 (1982) 502.
[43] A. J. Leggett, “Bose-Einstein condensation in the alkali gases: Some fundamental concepts,” Rev. Mod. Phys. 73 (2001) 307 [Erratum-ibid. 75 (2003) 1083].
[44] “Coherent versus Incoherent Dynamics during Bose-Einstein Condensation in Atomic Gases”, Stoof, H. T. C., cond-mat/9805393,
[45] Bose Einstein Condensation, editors Griffin, Snoke, Stringari, Cambridge University Press, 1996.
[46] E Calzetta, B-L Hu, “Non Equilibrium Quantum Field Theory”, Cambridge University Press, Cambridge, UK.
[47] J. M. Cornwall, R. Jackiw and E. Tomboulis, “Effective Action for Composite Operators,” Phys. Rev. D 10 (1974) 2428.
[48] J. Berges and J. Serreau, “Progress in nonequilibrium quantum field theory,” hep-ph/0302210.
J. Berges and J. Serreau, “Progress in nonequilibrium quantum field theory II,” hep-ph/0410330.
J. Berges and S. Borsanyi, “Progress in nonequilibrium quantum field theory. III.,” Nucl. Phys. A 785 (2007) 58 [hep-ph/0610015].
[49] J. Berges and D. Sexty, “Bose condensation far from equilibrium,” Phys. Rev. Lett. 108 (2012) 161601 [arXiv:1201.0687 [hep-ph]].
[50] C. -P. Ma and E. Bertschinger, “Cosmological perturbation theory in the synchronous and conformal Newtonian gauges,” Astrophys. J. 455 (1995) 7 [astro-ph/9506072].
[51] M. E. Peskin and D. V. Schroeder, “An Introduction to quantum field theory,” Reading, USA: Addison-Wesley (1995) 842 p. page 125.
[52] G. Efstathiou, J.R. Bond, “Isocurvature cold dark matter fluctuations”, *Mon. Not. R. astr. Soc.* (1986) 218, 103-121.
[53] C. J. Hogan and M. J. Rees, “Axion Miniclusters,” Phys. Lett. B 205 (1988) 228.
[54] E. Seidel and W. M. Suen, “Oscillating soliton stars,” Phys. Rev. Lett. 66 (1991) 1659.
E. Seidel and W. -M. Suen, “Formation of solitonic stars through gravitational cooling,” Phys. Rev. Lett. 72 (1994) 2516 [gr-qc/9309015].
[55] S. Weinberg, “Entropy generation and the survival of protogalaxies in an expanding universe,”
Astrophys. J. 168 (1971) 175.

[56] R. Maartens, “Causal thermodynamics in relativity,” [astro-ph/9609119].

[57] G. Aarts, G. F. Bonini and C. Wetterich, “On Thermalization in classical scalar field theory,” Nucl.
Phys. B 587 (2000) 403 [hep-ph/0003262]. G. Aarts and J. Berges, “Classical aspects of quantum
fields far from equilibrium,” Phys. Rev. Lett. 88 (2002) 041603 [hep-ph/0107129].

[58] S. J. Brodsky and P. Hoyer, “The $\hbar$ Expansion in Quantum Field Theory,” Phys. Rev. D 83 (2011)
045026 [arXiv:1009.2313 [hep-ph]].
B. R. Holstein and J. F. Donoghue, “Classical physics and quantum loops,” Phys. Rev. Lett. 93
(2004) 201602 [hep-th/0405239].
N. E. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, “Quantum gravitational corrections to the
nonrelativistic scattering potential of two masses,” Phys. Rev. D 67 (2003) 084033 [Erratum-ibid. D
71 (2005) 069903] [hep-th/0211072].
C. Montonen and D. I. Olive, “Magnetic Monopoles as Gauge Particles?,” Phys. Lett. B 72 (1977)
117.

[59] T. Noumi, K. 'i. Saikawa, R. Sato and M. Yamaguchi, “Effective gravitational interactions of dark
matter axions,” arXiv:1310.0167 [hep-ph].

[60] M. G. Alford, S. K. Mallavarapu, A. Schmitt and S. Stetina, “From field theory to superfluid
hydrodynamics of dense quark matter,” arXiv:1304.7102 [hep-ph].

[61] L. Parker, “Quantized fields and particle creation in expanding universes. 1.,” Phys. Rev. 183
(1969) 1057.

[62] N. D. Birrell and P. C. W. Davies, “Quantum Fields In Curved Space,” Cambridge, Uk: Univ. Pr.
(1982).

[63] C Cohen-Tannoudji, B Diu, F Laloé, Mécanique Quantique, vol 1, chapter 5.G, Hermann, 1977.
C Itzykson, J.B. Zuber, “Quantum Field Theory”, McGraw-Hill, New York, USA.