THE ORBIT SPACE AND BASIC FORMS OF A PROPER LIE GROUPOID

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Abstract. We show that the complex of basic differential forms on a proper Lie groupoid is isomorphic to the complex of differential forms on the orbit space equipped with the quotient diffeological structure.

1. Introduction

Given a compact Lie group \( G \) acting on a manifold \( M \), the complex of basic differential forms yields a cohomology that is shown by Koszul [7] to be isomorphic to the singular cohomology of the orbit space \( M/G \). If the action is free, then the orbit space is a manifold, and the basic differential forms on \( M \) are in bijection with the differential forms on the orbit space. Thus, in such a case, Koszul’s theorem combined with the de Rham theorem yields an isomorphism of de Rham cohomologies. Using the generalised slice theorem of Palais [9], the above results can be extended to proper Lie group actions.

In [12] (Chapter 3), it is shown for a compact Lie group \( G \) acting on a manifold \( M \) that if we equip the orbit space with the quotient diffeological smooth structure, then the corresponding differential forms on the orbit space are in bijection with the basic forms on \( M \). The corresponding de Rham cohomologies are therefore isomorphic. This, in particular, includes the non-free case. In [6], this result is generalised further to proper Lie group actions.

The purpose of this paper is to push the above result even further into the realm of Lie groupoids. In particular, we have the following result. (Throughout this paper, we make the assumption that the Lie groupoids we are working with are finite dimensional, paracompact, and Hausdorff.)

Main Theorem. Let \( G = (G_1 \rightrightarrows G_0) \) be a proper Lie groupoid. Then the de Rham complex of basic forms on \( G_0 \) is isomorphic to the de Rham complex of differential forms on the orbit space \( G_0/G_1 \).

It has been shown (2; see also [13], [14], [15]) that proper Lie groupoids are locally Morita equivalent to action groupoids of compact Lie group actions. Thus, locally, the Main Theorem is true for proper Lie groupoids, and we show that it in fact extends to a global result. In the language of groupoids, what the Main Theorem is saying is that there is a bijection between the basic forms with respect to the Lie groupoid structure, and the basic forms with respect to the structure of the relation groupoid \( G_0 \times_{\pi} G_0 \) (where \( \pi : G_0 \to G_0/G_1 \))

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is the quotient map), equipped with the diffeological structure induced by $G_0 \times G_0$. (See Example 2.8 and Remark 3.5.)

In [10] Section 8, a different but equivalent definition of basic differential form on a proper Lie groupoid is used to establish an isomorphism between the de Rham cohomology of the basic forms and the singular cohomology of the orbit space. Thus in conjunction with this paper we obtain a de Rham theorem for the diffeological differential forms on the orbit space; that is, an isomorphism between the de Rham cohomology on the orbit space, and the singular cohomology of the orbit space. What this tells us is that the de Rham cohomology of basic differential forms is an homotopy invariant of the orbit space. It turns out that this is even weaker as an invariant than the quotient diffeology on the orbit space itself, since there are examples of families of Lie group actions that yield homeomorphic but not diffeomorphic orbit spaces; the homeomorphisms imply that the corresponding de Rham cohomologies are isomorphic. (See, for example, Exercise 50 of [4] with solution at the end of the book.)

This paper is organised as follows. In Section 2, we review the basics of diffeology required for this paper. A more thorough source is [4]. In Section 3, we review basic differential forms in the Lie groupoid setting. In Section 4, we review bibundles and Morita equivalence of Lie groupoids, and the induced (diffeologically) smooth maps between orbit spaces. In Section 5, we apply the results of the previous section to basic differential forms. In Section 6, we review linearisations in the context of Lie groupoids. Finally, in Section 7, we prove the Main Theorem.

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2. Background: Diffeology

Definition 2.1 (Diffeology). Let $X$ be a set. A parametrisation of $X$ is a map of sets $p: U \to X$ where $U$ is an open subset of Euclidean space (no fixed dimension). A diffeology $D$ on $X$ is a set of parametrisations satisfying the following three conditions.

1. (Covering) For every $x \in X$ and every nonnegative integer $n \in \mathbb{N}$, the constant function $p: \mathbb{R}^n \to \{x\} \subseteq X$ is in $D$.
2. (Locality) Let $p: U \to X$ be a parametrisation such that for every $u \in U$ there exists an open neighbourhood $V \subseteq U$ of $u$ satisfying $p|_V \in D$. Then $p \in D$.
3. (Smooth Compatibility) Let $p: U \to X$ be a plot in $D$. Then for every $n \in \mathbb{N}$, every open subset $V \subseteq \mathbb{R}^n$, and every smooth map $F: V \to U$, we have $p \circ F \in D$.

A set $X$ equipped with a diffeology $D$ is called a diffeological space, and is denoted by $(X, D)$. When the diffeology is understood, we will drop the symbol $D$. The parametrisations $p \in D$ are called plots.

Definition 2.2 (Diffeologically Smooth Maps). Let $(X, D_X)$ and $(Y, D_Y)$ be two diffeological spaces, and let $F: X \to Y$ be a map. Then we say that $F$ is diffeologically smooth if for any plot $p \in D_X$,

\[ F \circ p \in D_Y. \]
Example 2.3. Let $M$ be a smooth manifold. Then the standard diffeology on $M$ is the set of all smooth maps $f : U \to M$ as $U$ runs over all open subsets of $\mathbb{R}^n$, and $n$ runs over all nonnegative integers.

Definition 2.4 (Quotient Diffeology). Let $(X, \mathcal{D})$ be a diffeological space, and let $\sim$ be an equivalence relation on $X$. Let $\pi : X \to X/\sim$ be the quotient map. Then $X/\sim$ comes equipped with the quotient diffeology, which is the set of all plots that locally factor through $\pi$. More precisely, a map $p : U \to X/\sim$ is a plot if for any $u \in U$ there exist an open neighbourhood $V \subseteq U$ of $u$ and a plot $q \in \mathcal{D}$ such that

$$p|_V = \pi \circ q.$$ 

Example 2.5. Let $G_1 \rightrightarrows G_0$ be a Lie groupoid. Then its orbit space $G_0/G_1$ comes equipped with the quotient diffeology induced by the standard manifold diffeology on $G_0$.

Definition 2.6 (Product Diffeology). Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be diffeological spaces. Then the product diffeology on $X \times Y$ contains a map $p : U \to X \times Y$ as a plot if $pr_X \circ p$ and $pr_Y \circ p$ are plots in $\mathcal{D}_X$ and $\mathcal{D}_Y$, respectively. Here, $pr_X$ and $pr_Y$ are the projection maps.

Definition 2.7 (Subset Diffeology). Let $(X, \mathcal{D})$ be a diffeological space, and let $Y \subseteq X$. Then $Y$ comes equipped with the subset diffeology, which is the set of all plots in $\mathcal{D}$ with image in $Y$.

Example 2.8. Let $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ be diffeological spaces and let $f : X \to Y$ be smooth. Then, the pullback of $X$ by $f$

$$X \times_f X = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$$

comes equipped with the subset diffeology induced by the product diffeology on $X \times X$. In particular, given a Lie groupoid $G_1 \rightrightarrows G_0$ with source $s$ and target $t$, the image of the map $(s, t) : G_1 \to G_0 \times G_0$ is exactly the relation groupoid

$$\{(x_1, x_2) \in G_0 \times G_0 \mid \exists g \in G_1 \text{ such that } s(g) = x_1 \text{ and } t(g) = x_2\}.$$ 

The reader can check that this is equal to the pullback of $G_0$ by the quotient map $\pi : G_0 \to G_0/G_1$. Since $s$ and $t$ are smooth maps into $G_0$, the map $(s, t) : G_1 \to G_0 \times \pi G_0$ is smooth with respect to the standard manifold diffeology on $G_1$ and the subset diffeology on $G_0 \times \pi G_0$. Note that while $G_0 \times \pi G_0$ is generally not a Lie groupoid, it is still a groupoid equipped with a smooth structure and smooth source and target maps.

Definition 2.9 (Differential Forms). Fix a diffeological space $(X, \mathcal{D})$. A differential k-form $\alpha$ on $X$ is an assignment to each plot $(p : U \to X) \in \mathcal{D}$ a differential form $\alpha_p \in \Omega^k(U)$ satisfying the following smooth compatibility condition: for any open subset $V$ of some Euclidean space and any smooth map $f : V \to U$, we have

$$\alpha_{pf} = f^* \alpha_p.$$ 

Denote the collection of k-forms on $X$ by $\Omega^k(X)$. Define the exterior derivative $d : \Omega^k(X) \to \Omega^{k+1}(X)$ plot-wise:

$$(d\alpha)_p = d(\alpha_p).$$

Remark 2.10. We have the following facts regarding differential forms.
(1) The 0-forms on a diffeological space are exactly the smooth functions \( f : X \to \mathbb{R} \) (see Article 6.31 of [4] for details).
(2) The collection of all differential forms on \( X \) forms a de Rham complex \( (\Omega^*(X), d) \) (see Articles 6.34 and 6.35 of [4]).
(3) If \( f : X \to Y \) is a smooth map between diffeological spaces, then for any differential form \( \alpha \) on \( Y \), the pullback map \( f^* : \alpha \mapsto f^* \alpha \) is well-defined, sending \( k \)-forms to \( k \)-forms. (See Article 6.32 of [4] for details.) In fact, given a diffeological space \( (X, \mathcal{D}) \) and plot \( p \in \mathcal{D} \), any differential form \( \alpha \) on \( X \) satisfies \( \alpha_p = p^* \alpha \). (See Article 6.33 of [4].)

3. Background: Basic Differential Forms

Reiterating what was stated in the introduction, henceforth, we will assume that all Lie groupoids are finite dimensional, paracompact, and Hausdorff.

Definition 3.1 (Basic Differential Forms). Let \( G_1 \xrightarrow{\Phi} G_0 \) be a Lie groupoid with source \( s \) and target \( t \). A differential form \( \alpha \) on \( G_0 \) is basic if \( s^* \alpha = t^* \alpha \).

Remark 3.2. Basic forms of a Lie groupoid form a subcomplex of the de Rham complex of the base manifold.

Lemma 3.3. Let \( K \) be a Lie group acting on a manifold \( M \). Then a differential form \( \alpha \) is basic with respect to the action if and only if it is basic with respect to the action groupoid \( K \ltimes M := (K \times M \rightrightarrows M) \).

Proof. Assume that \( \alpha \) is \( K \)-invariant and horizontal. Fix a vector \( v \in T_{(k,x)}(K \times M) \). Then,

\[
\begin{align*}
v \downarrow t^* \alpha & = t^*(t_* v \downarrow \alpha) \\
& = t^*(k_* s_* v \downarrow \alpha) \\
& = s^*(s_* v \downarrow \alpha) \quad \text{since } \alpha \text{ is } K \text{-invariant} \\
& = v \downarrow s^* \alpha.
\end{align*}
\]

This shows that \( \alpha \) is basic with respect to the action groupoid.

Conversely, assume that \( s^* \alpha = t^* \alpha \). Let \( k \in K \). For any \( v \in T_y M \), there exists \( w \in T_{(k,x)}(K \times M) \) such that \( s_* w = v \), since \( s \) is a surjective submersion. Thus,

\[
\begin{align*}
s^*(v \downarrow k^* \alpha) & = s^*(s_* w \downarrow k^* \alpha) \\
& = w \downarrow s^* k^* \alpha \\
& = w \downarrow t^* \alpha \\
& = w \downarrow s^* \alpha \\
& = s^*(v \downarrow \alpha).
\end{align*}
\]
Since \( s \) is a surjective submersion, \( s^* \) is injective on differential forms. This proves \( K \)-invariance.

Finally, assume that \( v \in T_x M \) is tangent to the orbit \( K \cdot x \). Identify \( T(K \times M) \) with \( K \times \mathfrak{k} \times TM \) using the left trivialisation of \( TK \), where \( \mathfrak{k} \) is the Lie algebra of \( K \). There exists \( \xi \in \mathfrak{k} \) such that

\[
v = \xi_M|_x = t^*((e, \xi), 0)
\]

where \( e \in K \) is the identity element, and \((e, \xi), 0) \in T_{(e,x)}(K \times M)\). Here, \( \xi_M \) is the vector field on \( M \) induced by \( \xi \). Checking:

\[
t^*((e, \xi), 0) = \left. \frac{d}{d\tau} \right|_{\tau=0} \exp(\tau \xi) \cdot x = \xi_M|_x.
\]

Note that a similar calculation shows that \( s^*((e, \xi), 0) = 0 \).

Thus,

\[
t^*(v \, \downarrow \alpha) = ((e, \xi), 0) \downarrow t^* \alpha = ((e, \xi), 0) \downarrow s^* \alpha = s^*(s^*((e, \xi), 0) \downarrow \alpha) = 0.
\]

Since \( t \) is a surjective submersion, \( t^* \) is injective on differential forms, and so \( v \downarrow \alpha = 0 \). This shows that \( \alpha \) is basic with respect to the action. \( \square \)

**Proposition 3.4 (Basic Forms of the Relation Groupoid).** Let \((X, D_X)\) be a diffeological space, let \( \sim \) be an equivalence relation on \( X \), and let \( Y = X/\sim \) be equipped with the quotient diffeology. Let \( \pi : X \to Y \) be the quotient map. Then a differential form \( \alpha \) on \( X \) is the pullback \( \pi^* \beta \) of a differential form \( \beta \) on \( Y \) if and only if for any plots \( p_1, p_2 : U \to X \) in \( D_X \) such that \( \pi \circ p_1 = \pi \circ p_2 \), we have the equality

\[
p_1^* \alpha = p_2^* \alpha.
\]

**Remark 3.5.** The above proposition can be reworded as follows. Let \( \text{pr}_1 : X \times \pi X \to X \) and \( \text{pr}_2 : X \times \pi X \to X \) be the canonical projection maps. A differential form \( \alpha \) on \( X \) is the pullback of a differential form \( \beta \) on \( Y \) if and only if \( \text{pr}_1^* \alpha = \text{pr}_2^* \alpha \).

**Proof.** The proof of the proposition and the remark can be found in Article 6.38 of \([4]\). \( \square \)

**Corollary 3.6 (Pullbacks from the Quotient are Basic).** Let \( G_1 \Rightarrow G_0 \) be a Lie groupoid, and let \( \pi : G_0 \to G_0/G_1 \) be the quotient map. Then if a differential form \( \alpha \) on \( G_0 \) is equal to the pullback \( \pi^* \beta \) of some differential form \( \beta \) on \( G_0/G_1 \), then \( \alpha \) is basic with respect to \( G_1 \Rightarrow G_0 \).

**Proof.** By Proposition 3.4 and Remark 3.5, we know that \( \text{pr}_1^* \alpha = \text{pr}_2^* \alpha \). Thus we know that \( (s, t)^* (\text{pr}_1^* \alpha - \text{pr}_2^* \alpha) = 0 \) (see Example 2.8), and so \( s^* \alpha = t^* \alpha \); that is, \( \alpha \) is basic with respect to the groupoid \( G_1 \Rightarrow G_0 \). \( \square \)
Theorem 3.7 (Proper Group Actions). Let $K$ be a Lie group, and let $K \ltimes M$ be a proper action groupoid with quotient map $\pi : M \to M/K$. Then $\pi^*$ is an isomorphism between the de Rham complexes of differential forms on $M/K$ and basic differential forms on $M$.

Proof. See [6], or [12] Theorem 3.20. The latter proves the compact group action case, but this is extended in [6].

4. Bibundles and Morita Equivalence

Some references on actions of groupoids, principal groupoid bundles, and bibundles, include [5] and [8].

Definition 4.1 (Right Action of a Groupoid). A right action of a Lie groupoid $H = (H_1 \Rightarrow H_0)$ on a manifold $P$ is a pair of smooth maps: the anchor map $a : P \to H_0$, and the action $\text{act} : P_a \times_t H_1 \to P$ sending $(p, h)$ to $p \cdot h$; along with a smooth functor of Lie groupoids making the following diagram commute:

In particular, $a(p \cdot h) = s(h)$, $a(p \cdot u(a(p))) = a(p)$ where $u : H_0 \to H_1$ is the unit map, and if $h$ acts on $p$, then $a(p) = t(h)$.

Definition 4.2 (Principal $H$-Bundles). Let $H = (H_1 \Rightarrow H_0)$ be a Lie groupoid. A principal (right) $H$-bundle $\rho : P \to B$ is a pair of manifolds $P$ and $B$ with a surjective submersion $\rho$ between them, along with a right $H$-action on $P$ with anchor map $a : P \to H_0$ such that $\rho$ is $H$-invariant, and the action of $H$ is free and transitive on fibres of $\rho$ (i.e. $P_a \times_t H_1$ is diffeomorphic to $P \times_B P$ via $(p, h) \mapsto (p, p \cdot h)$).

Example 4.3 (The Unit Principal $H$-Bundle). For any Lie groupoid $H = (H_1 \Rightarrow H_0)$, the target map $t : H_1 \to H_0$ is an example of a principal (right) $H$-bundle.

Example 4.4 (Pullback Bundles). Let $M$ be a manifold, $H = (H_1 \Rightarrow H_0)$ a Lie groupoid, $\rho : P \to B$ a principal $H$-bundle, and $f : M \to B$ a smooth map. Then, we can form the pullback bundle $f^*P = M \times_B P$, which also is a principal $H$-bundle.

Definition 4.5 (Left Action of a Groupoid). A left action of a Lie groupoid $G = (G_1 \Rightarrow G_0)$ on a manifold $P$ is a pair of smooth maps: the anchor map $a : P \to G_0$, and the action $\text{act} : G_1 \times_a P \to P$ sending $(g, p)$ to $g \cdot p$; along with a smooth functor of Lie groupoids making the following diagram commute:
In particular, \( a(g \cdot p) = t(g) \), \( a(u(a(p)) \cdot p) = a(p) \) where \( u : G_0 \to G_1 \) is the unit map, and if \( g \) acts on \( p \), then \( a(p) = s(g) \).

**Definition 4.6 (Bibundles).** Let \( G = (G_1 \rightrightarrows G_0) \) and \( H = (H_1 \rightrightarrows H_0) \) be Lie groupoids. Then a bibundle \( P : G \to H \) is a manifold \( P \) with smooth maps \( a_L : P \to G_0 \) and \( a_R : P \to H_0 \) such that the following are satisfied.

1. \( G \) acts on \( P \) on the left, with anchor map \( a_L \); and \( H \) acts on \( P \) on the right, with anchor map \( a_R \). Moreover, the two actions commute.
2. \( a_L : P \to G_0 \) is a principal \( H \)-bundle.
3. \( a_R \) is \( G \)-invariant.

**Example 4.7 (Smooth Functor).** Let \( G = (G_1 \rightrightarrows G_0) \) and \( H = (H_1 \rightrightarrows H_0) \) be Lie groupoids, and \( f : G \to H \) a smooth functor between them. Then the pullback of \( t : H_1 \to H_0 \) by \( f_0 : G_0 \to H_0 \) is a bibundle from \( G \) to \( H \).

**Definition 4.8 (Morita Equivalence).** Let \( G = (G_1 \rightrightarrows G_0) \) and \( H = (H_1 \rightrightarrows H_0) \) be Lie groupoids, and let \( P : G \to H \) be a bibundle between them. \( P \) is invertible if its right anchor map \( a_R : P \to H_0 \) makes \( P \) into a principal (left) \( G \)-bundle, defined similarly to a principal (right) bundle. In this case, we can construct a bibundle \( P^{-1} : H \to G \) by switching the anchor maps, inverting the left \( G \)-action into a right \( G \)-action, and doing the opposite for the \( H \)-action. Then, \( P \circ P^{-1} \) is isomorphic to the bibundle corresponding to the identity map on \( H \), and \( P^{-1} \circ P \) isomorphic to the bibundle representing the identity map on \( G \). In the case that \( G \) and \( H \) admit an invertible bibundle between them, they are called Morita equivalent groupoids.

**Example 4.9 (Saturation).** Let \( G = (G_1 \rightrightarrows G_0) \) be a Lie groupoid, and \( U \subseteq G_0 \). The saturation of \( U \), denoted \( U^G \), is the set of all points \( x \in G_0 \) such that there exists \( g \in G_1 \) with \( s(g) \in U \) and \( t(g) = x \). Equivalently, this is the smallest “invariant” open set containing \( U \). As shown in Example 3.2 of [2], the restriction of \( G \) to \( U \) is Morita equivalent to the restriction of \( G \) to \( U^G \). Indeed, take as a bibundle the submanifold \( t^{-1}(U) \) in \( G_1 \) with the appropriate restrictions of source and target maps as anchor maps.
Proposition 4.10 (Bibundles Descend to Smooth Maps). Let $G = (G_1 ightrightarrows G_0)$ and $H = (H_1 ightrightarrows H_0)$ be Lie groupoids and $P : G \to H$ a bibundle. Then there exists a unique smooth map $\Psi_P : G_0/G_1 \to H_0/H_1$ such that the diagram below commutes.

\[
\begin{array}{ccc}
G_1 & \xrightarrow{a_L} & P \\
\downarrow{\pi_G} & & \downarrow{\pi_H} \\
G_0 & \xleftarrow{a_R} & H_0 \\
\end{array}
\]

Proof. Define $\Psi_P$ as follows. Fix $x \in G_0$ and denote by $[x]$ the point $\pi_G(x)$. Then define

$$\Psi_P([x]) := \pi_H \circ a_R \circ \sigma(x)$$

for some smooth local section $\sigma$ of $a_L$ about $x$ (recall that $a_L$ is a surjective submersion since $P$ is a bibundle, so such smooth local sections always exist). We claim that $\Psi_P$ is well-defined; in particular, it is independent of representative of $[x]$ and local sections. If $y \in G_0$ is another representative of $[x]$, then there exists $g \in G_1$ such that $s(g) = x$ and $t(g) = y$, where $s$ and $t$ are the source and target maps of $G_1$, respectively. Then, $a_L(g \cdot \sigma(x)) = t(g) = y$. Thus, $g \cdot \sigma(x) \in a_L^{-1}(y)$. Let $\sigma'$ be a local section of $a_L$ about $y$. Since $a_L : P \to G_0$ is a principal $H$-bundle, there exists $h \in H_1$ such that $(g \cdot \sigma(x)) \cdot h = \sigma'(y)$. Thus,

$$a_R \circ \sigma'(y) = a_R((g \cdot \sigma(x)) \cdot h)$$

$$= a_R(g \cdot (\sigma(x) \cdot h))$$

$$= a_R(\sigma(x) \cdot h)$$

$$= s(h).$$

But, we have $a_R(\sigma(x)) = t(h)$, and hence

$$\pi_H(a_R(\sigma(x))) = \pi_H(a_R(\sigma'(y))).$$

Thus, $\Psi_P([x])$ is well-defined.

To show uniqueness, fix $p \in P$. Then, $\pi_G(a_L(p)) = [a_L(p)]$ and $\pi_H(a_R(p)) = [a_R(p)]$, so any map between $G_0/G_1$ and $H_0/H_1$ would need to send $[a_L(p)]$ to $[a_R(p)]$. This defines such a map uniquely, and $\Psi_P$ satisfies this.

Next, we show that $\Psi_P$ is diffeologically smooth. Fix a plot $p : U \to G_0/G_1$ and $u \in U$. By definition of the quotient diffeology, there exist an open neighbourhood $V \subseteq U$ of $u$ and a plot $q : V \to G_0$ such that $p|_V = \pi_G \circ q$. Since $a_L$ is a surjective submersion, there exist an open neighbourhood $W \subseteq G_0$ of $q(u)$ and a smooth section $\sigma : W \to P$ of $a_L|_{a_L^{-1}(W)}$. Since $a_R \circ \sigma$ is a smooth map of manifolds, we have that $a_R \circ \sigma \circ q|_{q^{-1}(W)}$ is a plot of $H_0$, and so $\pi_H \circ a_R \circ \sigma \circ q|_{q^{-1}(W)}$ is a plot of $H_0/H_1$. That is to say, $\Psi_P \circ p|_{q^{-1}(W)}$ is a plot of $H_0/H_1$. Since $u \in U$ is arbitrary, the locality axiom of diffeology guarantees that $\Psi_P$ is a smooth map. □
**Definition 4.11 (Composition of Bibundles).** Let $G = (G_1 \rightrightarrows G_0)$, $H = (H_1 \rightrightarrows H_0)$, and $K = (K_1 \rightrightarrows K_0)$ be Lie groupoids, and let $P : G \to H$ and $Q : H \to K$ be bibundles. Define the composition $Q \circ P : G \to K$ to be the bibundle $(P \times_{H_0} Q)/H$, where $H$ acts on the fibred product via the diagonal action. (See [5], Remark 3.30 for a proof that this in fact is a bibundle between $G$ and $K$.)

**Lemma 4.12.** $\Psi_{Q\circ P} = \Psi_Q \circ \Psi_P$.

**Proof.** We have the following diagrams, where $\pi_{Q\circ P}$ is the quotient map induced by the diagonal $H$-action on $P \times_{H_0} Q$, and $a_j^i, i = P, Q, Q \circ P$ and $j = L, R$ are anchor maps.

![Diagram](attachment:diagram.png)

The diagrams above commute, except *a priori* for $\Psi_{Q\circ P} = \Psi_Q \circ \Psi_P$, which we intend to show commutes. Fix $x \in G_0$. Then, $\Psi_{Q\circ P}([x]) = \pi_K \circ a_{Q\circ P}^Q \circ \gamma(x)$ for some local section $\gamma$ of $a_{Q\circ P}^Q$ about $x$. Note: $\pi_{Q\circ P}$ is a surjective submersion since $H$ acts properly and freely on $P \times_{H_0} Q$. Thus, there exists a local section $\nu$ of $\pi_{Q\circ P}$ about $\gamma(x)$ such that $\nu \circ \gamma$ is a local section of $a_{Q\circ P}^Q \circ \pi_{Q\circ P}$ about $x$. Since $a_{Q}^L \circ \text{pr}_1 = a_{Q}^L \circ \pi_{Q\circ P}$, we conclude that $\sigma := \text{pr}_1 \circ \nu \circ \gamma$ is a local section of $a_{Q}^L$ about $x$.

Next, let $\tau$ be a local section of $a_{Q}^L$ about $\tau a_{Q}^R \circ \sigma(x)$ (shrinking the domain of $\sigma$ so that $\tau a_{Q}^R \circ \sigma$ is well-defined). Note that for any $q \in Q$, we have $\tau a_{Q}^L(q) = h \cdot q$ for some $h \in H_1$. However, $a_{Q}^R(h \cdot q) = a_{Q}^R(q)$ due to the $H$-invariance of $a_{Q}^R$. Thus,

$$\pi_K \circ a_{Q}^R \circ \tau \circ a_{Q}^L = \pi_K \circ a_{Q}^R.$$  

(2)

We also have

$$a_{Q}^R \circ \text{pr}_2 = a_{Q}^R \circ \pi_{Q\circ P}$$  

(3)

and

$$a_{Q}^R \circ \text{pr}_1 = a_{Q}^L \circ \text{pr}_2.$$  

(4)
Putting all of these facts together, we have

\[
\Psi_{Q \circ P}([x]) = \pi_K \circ a_{R}^{Q \circ P} \circ \gamma(x)
\]

\[
= \pi_K \circ a_{R}^{Q \circ P} \circ (\pi_{Q \circ P} \circ \nu) \circ \gamma(x)
\]

\[
= \pi_K \circ (a_{R}^{Q} \circ \text{pr}_2) \circ \nu \circ \gamma(x)
\]

\[
= \pi_K \circ a_{R}^{Q} \circ (\tau \circ a_{L}^{Q}) \circ \text{pr}_2 \circ \nu \circ \gamma(x)
\]

\[
= \pi_K \circ a_{R}^{Q} \circ \tau \circ (a_{R}^{P} \circ \text{pr}_1) \circ \nu \circ \gamma(x)
\]

\[
= (\pi_K \circ a_{R}^{Q} \circ \tau) \circ (a_{R}^{P} \circ \sigma)(x)
\]

\[
= \Psi_Q \circ \Psi_P([x]).
\]

\[
\square
\]

**Lemma 4.13.** Let \( G \) and \( H \) be Lie groupoids, and let \( P : G \to H \) and \( Q : G \to H \) be bibundles between them. Assume there exists a \((G-H)\)-equivariant diffeomorphism \( \alpha : P \to Q \). Then, \( \Psi_P = \Psi_Q \).

**Proof.** We have that \( a_L^Q \circ \alpha = a_L^P \) and \( a_R^Q \circ \alpha = a_R^P \). Thus, fixing \( x \in G_0 \), since \( \Psi_P([x]) = \pi_H \circ a_R^P \circ \sigma(x) \) for any local section \( \sigma \) of \( a_R^P \) about \( x \), and noting that \( \alpha \circ \sigma \) is a local section of \( a_L^Q \), we conclude

\[
\Psi_P([x]) = \pi_H \circ a_R^Q \circ (\alpha \circ \sigma)(x) = \Psi_Q([x]).
\]

\[
\square
\]

**Proposition 4.14 (Morita Equivalence Descends to a Diffeomorphism).** Let \( G \) and \( H \) be Morita equivalent Lie groupoids, and let \( P \) be an invertible bibundle representing this equivalence. Then \( \Psi_P \) is a diffeomorphism between \( G_0/G_1 \) and \( H_0/H_1 \).

**Proof.** Since \( G \) and \( H \) are Morita equivalent, there exists an invertible bibundle \( P : G \to H \) in which \( P^{-1} \circ P \) is equivariantly diffeomorphic to the bibundle induced by the identity functor on \( G \). \( P \circ P^{-1} \) has a similar relation with the identity functor on \( H \). The result follows from Lemma 4.12 and Lemma 4.13.

\[
\square
\]

### 5. Bibundles and Differential Forms

**Proposition 5.1 (Pullbacks of Basic Forms by Bibundles).** Let \( G = (G_1 \rightrightarrows G_0) \) and \( H = (H_1 \rightrightarrows H_0) \) be Lie groupoids, and let \( P : G \to H \) be a bibundle between them, with anchor maps \( a_L : P \to G_0 \) and \( a_R : P \to H_0 \). Then for any \( H \)-basic form \( \beta \in \Omega_k^{\text{basic}}(H_0) \), there exists a unique \( G \)-basic form \( \alpha \in \Omega_k^{\text{basic}}(G_0) \) such that \( a_L^* \alpha = a_R^* \beta \).
Proof.

Fix an $H$-basic $k$-form $\beta$ on $H_0$. Consider the pullback $a^*_R \beta$. Recall that by Definition 4.1, since $H$ acts on $P$ on the right, we have a Lie groupoid $P \times_a H_1 \Rightarrow P$ with source $\text{act}_H$ and target $\text{pr}_1$. We claim that $a^*_R \beta$ is basic with respect to this Lie groupoid structure. Indeed, since $\beta$ is $H$-basic, we have

\[ \text{act}_H^* a^*_R \beta = \text{pr}_2^* s^* \beta = \text{pr}_2^* t^* \beta = \text{pr}_1^* a^*_R \beta. \]

Now, since $H$ acts on $P$ freely and properly, with quotient manifold $G_0$ and quotient map $a_L$, we have that $a^*_R \beta$ descends uniquely to a form on $G_0$, which we denote by $\alpha$. We claim that $\alpha$ is $G$-basic. Indeed,

\[ \text{pr}_1^* s^* \alpha = \text{pr}_2^* a^*_L \alpha = \text{pr}_2^* a^*_R \beta = \text{act}_G^* a^*_R \beta = \text{act}_G^* a^*_L \alpha = \text{pr}_1^* t^* \alpha. \]

Since $a_L$ is a surjective submersion, so is $\text{pr}_1 : G_1 \times_{G_0} P \to G_1$. Thus, $\text{pr}_1^*$ is an injection on differential forms, and we conclude that $s^* \alpha = t^* \alpha$. This completes the proof. \hfill \Box

Remark 5.2. In the proposition above, we can think of $\alpha$ as the pullback of $\beta$ by $P$, and so we will denote $\alpha$ by $P^* \beta$.

Corollary 5.3. Let $G$ and $H$ be Morita equivalent Lie groupoids. If $P : G \to H$ is an invertible bibundle representing this equivalence, then $P^*$ is an isomorphism of de Rham complexes between $H$-basic forms and $G$-basic forms.

Proof. Since $P$ is invertible, the right anchor map $a_R : P \to G_0$ is a surjective submersion, and we can use the arguments in the proof of Proposition 5.1 to obtain a bijection between $H$-basic forms and $G$-basic forms. \hfill \Box

Remark 5.4. The Morita invariance of basic differential forms described in Corollary 5.3 can be seen quite easily in the language of stacks. Let $G$ be a Lie groupoid, and $BG$ its corresponding geometric stack. Then a basic differential $k$-form $\alpha$ on $G_0$ yields a map of stacks $\Omega : BG \to \Omega^k(\cdot)$. Conversely, any map of stacks $\beta : BG \to \Omega^k(\cdot)$ pulls back to a map $G_0 \to \Omega^k(\cdot)$, which pulls back further to $G_0 \times_{BG} G_0 \simeq G_1$ via two isomorphic maps of stacks. 


induced by $s : G_1 \to G_0$ and $t : G_1 \to G_0$. Since $\Omega^k(\cdot)$ is a discrete stack, the two pullbacks are in fact equal, coinciding with the definition of a basic form.

The benefit of this approach is that defining a basic form as a map of stacks between $BG$ and $\Omega^k(\cdot)$ automatically is independent of an atlas; in particular, since Morita equivalent Lie groupoids yield isomorphic stacks, their basic differential forms seen as the set of maps of stacks to $\Omega^k(\cdot)$ are isomorphic as well. For details on stacks and their relation to Lie groupoids, see for example [3] and [5].

6. Linearisations

We begin this section by reviewing linearisations in the context of Lie groupoids. We mostly adopt the notation used in [2] (see Section 1.2). See also [13], [14], and [15].

Let $G = (G_1 \rightrightarrows G_0)$ be a Lie groupoid, and let $\mathcal{O} \subseteq G_0$ be an orbit. Then, $G$ restricts to the Lie groupoid $G|_{\mathcal{O}} = s^{-1}(\mathcal{O})$.

Let $\mathcal{N}_\mathcal{O}$ be the normal bundle to $\mathcal{O}$. Then the normal bundle to $G|_{\mathcal{O}}$ in $G_1$, denoted $\mathcal{N}_\mathcal{O}(G)$, is a Lie groupoid over $\mathcal{N}_\mathcal{O}$ with source and target the maps induced by $ds$ and $dt$, where $s$ and $t$ are the source and target of $G_1$.

**Definition 6.1 (Linearisations).** Let $G = (G_1 \rightrightarrows G_0)$ be a Lie groupoid, and let $\mathcal{O}$ be an orbit in $G_0$. Then the Lie groupoid $\mathcal{N}_\mathcal{O}(G) \rightrightarrows \mathcal{N}_\mathcal{O}$ is called a linearisation of $G$ at $\mathcal{O}$. We say that $G$ is linearisable at $\mathcal{O}$ if there exist open neighbourhoods $U \subseteq G_0$ and $V \subseteq \mathcal{N}_\mathcal{O}$ of $\mathcal{O}$ and an isomorphism of groupoids $G|_U \cong \mathcal{N}_\mathcal{O}(G)|_V$ which is equal to the identity on $G|_{\mathcal{O}}$. Finally, we say that $G$ is linearisable if it is linearisable at each of its orbits.

**Remark 6.2.** Note that we did not require $U$ and $V$ to be invariant above. The reader should note that arbitrarily small invariant neighbourhoods may not always exist. See Example 6.4 below.

**Lemma 6.3.** Let $G = (G_1 \rightrightarrows G_0)$ be a Lie groupoid, and let $\mathcal{O}$ be an orbit in $G_0$. Fix $x \in \mathcal{O}$. Denote by $G_x$ the stabiliser of $x$ and $\mathcal{N}_x$ the normal space to $\mathcal{O}$ at $x$. Then, $\mathcal{N}_\mathcal{O}(G)$ is Morita equivalent to the action groupoid $G_x \ltimes \mathcal{N}_x$.

**Proof.** This is Example 3.3 of [2]. Take as a bibundle $s^{-1}(x) \times \mathcal{N}_x$. □

**Example 6.4 (No Arbitrarily Small Invariant Neighbourhoods).** This example is from [1]. Consider the vector field $X$ on $\mathbb{R}^2$ defined as

$$X|_{(x,y)} = \sin(x)\partial_x|_{(x,y)} + \cos(x)\partial_y|_{(x,y)}.$$  

$X$ is $2\pi$-periodic in both the $x$ and $y$ directions, and is bounded, and so descends to a vector field $\bar{X}$ on the torus $\mathbb{R}^2/(2\pi\mathbb{Z})^2$. The flow of $\bar{X}$ induces an $\mathbb{R}$-action that is free everywhere except on two disjoint circles. The stabiliser at any point on either of these circles is equal to $2\pi\mathbb{Z} \subseteq \mathbb{R}$. Thus, the corresponding action groupoid is not proper. Also, the circles do not admit arbitrarily small invariant open neighbourhoods. By Lemma 6.3 a linearisation of one of the circles is Morita equivalent to an action of $2\pi\mathbb{Z}$ on $\mathbb{R}$.
Theorem 6.5 (Proper Lie Groupoids are Linearisable). Let $G = (G_1 \rightrightarrows G_0)$ be a proper Lie groupoid, and let $O$ be an orbit in $G_0$. Then there exist open neighbourhoods $U \subseteq G_0$ and $V \subseteq N_O$ of $O$ such that $G|_U$ is isomorphic to $N_O(G)|_V$.

Proof. See Theorem 1 of [2].

Corollary 6.6. Let $G = (G_1 \rightrightarrows G_0)$ be a proper Lie groupoid, and let $O$ be an orbit in $G_0$. Fix $x \in O$. Then there exists an open neighbourhood $U \subseteq G_0$ of $O$ such that $G|_U$ is Morita equivalent to the action groupoid $G_x \ltimes N_x$. In particular, $G|_U$ is Morita equivalent to the action groupoid of a compact Lie group action.

Proof. This is just the combination of Theorem 6.5 and Lemma 6.3 along with the fact that stabilisers of proper Lie groupoids are compact.

7. PROOF OF MAIN THEOREM AND CONCLUDING REMARKS

Proof of Main Theorem. Let $G = (G_1 \rightrightarrows G_0)$ be a proper Lie groupoid. Let $\pi_G : G_0 \to G_0/G_1$ be the quotient map. By Corollary 3.6, the pullback by $\pi_G$ of any differential form on $G_0/G_1$ is basic. We wish to show the other direction.

Fix a basic differential form $\alpha$. Since $G$ is proper, by Corollary 6.6 it is linearisable and has compact stabilisers. So there exists an open covering $\{V_\nu\}$ of $G_0$ such that for each $\nu$, the restriction $G|_{V_\nu}$ is Morita equivalent to an action groupoid $H_\nu \ltimes N_\nu$ of a compact Lie group $H_\nu$ acting on a manifold $N_\nu$. Moreover, by Example 4.9 for each $\nu$ we have that $G|_{V_\nu}$ is Morita equivalent to $G|_{V_\nu^G}$, where $V_\nu^G$ is the saturation of $V_\nu$. Thus, for each $\nu$ there exists an invertible bibundle $P_\nu : H_\nu \ltimes N_\nu \to G|_{V_\nu^G}$.

Let $i_\nu : V_\nu^G \to G_0$ be the inclusion map. Then, by Corollary 5.3, there exists an $H_\nu$-basic form $\alpha_\nu$ on $N_\nu$ such that

$$P_\nu^* i_\nu^* \alpha = \alpha_\nu. \quad (5)$$

Let $\pi_\nu : N_\nu \to N_\nu/H_\nu$ be the quotient map. By Theorem 3.7, there exists a differential form $\beta_\nu$ on $N_\nu/H_\nu$ such that $\pi_\nu^* \beta_\nu = \alpha_\nu$. Let $\Psi_{P_\nu^{-1}}$ be the smooth map between orbit spaces induced by the inverse bibundle $P_\nu^{-1}$. By Proposition 4.14 and Equation 5

$$\pi_G|_{V_\nu^G} \Psi_{P_\nu^{-1}}^* \beta = i_\nu^* \alpha. \quad (6)$$

Let $p_i : U \to G_0$ ($i = 1, 2$) be plots satisfying $\pi_G \circ p_1 = \pi_G \circ p_2$. By Proposition 3.4 to finish the proof it is enough to show that $p_1^* \alpha = p_2^* \alpha$. Since $\{V_\nu^G\}$ cover $G_0$, it is enough to show this for $p_1|_{W_\nu}$ and $p_2|_{W_\nu}$, where for each $\nu$ we define $W_\nu := p_1^{-1}(V_\nu^G) = p_2^{-1}(V_\nu^G)$. To this end:

$$p_1|_{W_\nu}^* \alpha = p_1|_{W_\nu}^* i_\nu^* \alpha = p_1|_{W_\nu}^* \pi_G|_{V_\nu^G} \Psi_{P_\nu^{-1}}^* \beta = p_2|_{W_\nu}^* \pi_G|_{V_\nu^G} \Psi_{P_\nu^{-1}}^* \beta = p_2|_{W_\nu}^* \alpha$$

by Equation 6 and abuse of notation.
Since this holds for each $\nu$, the proof is complete. □

Under the assumption that our Lie groupoids are finite dimensional, paracompact, and Hausdorff, it is worth noting that the proof of the Main Theorem above is designed exactly for proper Lie groupoids, and nothing more general, as the following proposition indicates. It remains an open problem how exactly to generalise from here, and there are examples of non-proper Lie groupoids in which the Main Theorem holds. For example, any $\mathbb{R}$-action that covers a circle action is not proper, but certainly the basic forms and forms on the orbit space are isomorphic, since this is true for the circle action. Clearly we could resolve this issue by taking the quotient of the group by the kernel of the corresponding group representation, and perhaps this is not so interesting. However, there are extreme examples where the theorem continues to hold: for instance, the 1-dimensional irrational torus (see Exercises 4 and 105 of [4], with solutions at the back of the book, along with many other related exercises).

Proposition 7.1 (Characterisation of Proper Lie Groupoids). Let $G = (G_1 \rightarrow G_0)$ be a Lie groupoid. Then $G$ is proper if and only if it is linearisable and its stabilisers are compact.

Proof. Corollary 6.6 provides one direction of the proof. To prove the other direction, assume that the stabilisers of $G$ are compact. Fix a compact subset $K$ of $G_0 \times G_0$, and let $C := (s, t)^{-1}(K)$. Our goal is to show that $C$ is compact. Since $G_1$ is a finite dimensional, paracompact, and Hausdorff manifold, it is metrisable. Thus, it is sufficient to show that $C$ is sequentially compact.

Let $\{g_i\}$ be a sequence in $C$. We wish to find a convergent subsequence. Since $K$ is compact, $\{(s, t)(g_i)\}$ has a convergent subsequence $\{(s, t)(g_{ij})\}$, whose limit we denote by $(x, y)$. Note that $x$ and $y$ are in the same orbit $O$. Let $U$ be a linearisation about $O$. Then infinitely many elements of $\{g_{ij}\}$ are contained in $G|_U$. In fact, if we let $B_x$ and $B_y$ be compact neighbourhoods of $x$ and $y$, respectively, both contained in $U$, then there are infinitely many elements of $\{g_{ij}\}$ contained in $(s, t)^{-1}(B_x \times B_y) \cap C$.

Now, the linearisations of $G$ are Morita equivalent to action groupoids of compact Lie groups. Since properness is a Morita invariant property (see Proposition 5.26 of [8]), and compact Lie group actions are proper, we have that $(s, t)|_{G|_U}$ is a proper map. Since $B_x \times B_y \subset U \times U$ is a compact set, $(s, t)^{-1}(B_x \times B_y) \cap C$ is compact, and so there must be a convergent subsequence of $\{g_{ij}\}$, and hence of $\{g_i\}$. □

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