The Analogue of Regional Economic Models in Quantum Calculus

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Abstract. In this paper, we derive a new formulation for an optimal investment allocation in N-regional economic model using quantum calculus analogue. This model is described as an optimal control model and formulated in both primal and dual models using quantum calculus formulation. This formulation is an extension of regional economic models. Also, the new formulation provides an exact optimal investment allocation. In addition, the classical regional economic model is obtained by choosing q=1. Furthermore, we formulate the primal and the dual regional economic models in quantum calculus. Moreover, we present a new version of the duality theorems for quantum calculus case. Finally, example is provided and solved using MATLAB in order to show the given new results.

1. Introduction
Continuous optimization has a wide range of real world applications in the field of operation research. This problem was first studied by Bellman [1] as “bottleneck problem”. The duality theory of continuous optimization proposed by Tyndall [2, 3]. Also, solution methods presented by Drews [4] and Segers [5] based on a discrete version of this continuous –time problem.

On the other hand, dynamic models have been recently studied using quantum calculus theory. Advar and Bohner [6,7] presented spectral theory in quantum calculus. Bohner and Chieochan studied The beverton-holt q-difference equations[8-9]. Periodic averaging principle in quantum calculus introduced by Bohner and Mesquita[10]. Al-Salih et al [11,12] proposed quantum calculus formulation of lientef production model with linear objective function as well as quadratic objective function to determine the upper and lower the optimal production plan. Dynamic network flows in quantum calculus has been presented by Al-Salih[13].

Regional economics models presented by Rahman[14]. These models Also, studied by Takayam[15], Kendnck[16], and Tabata[17]. The regional economic model was personal by Tabata as a class of optimal control problem, In particular, continuous optimization problem.

In this paper, we derive a new formulation of the regional economic models which provides an exact optimal solution.

The paper is organized as follows. In Section 2, overview of quantum calculus is presented. In Section 3, we give a formulation for regional models as continuous-time problem. Quantum calculus analogue of regional models is described in Section 4. In Section 5, some of duality theorem is given. Example is provided in Section 6. In Section 7, conclusion is presented.

2. Quantum calculus
A quantum calculus is an analogue of calculus in which we represent derivatives as differences and integration as sums. This new theory has been received more attention recently. Now, we introduce an overview of the theory of quantum calculus. The material that we introduce can be found in [11,12,13,18,19,20].

Definition II-.1. “The q-derivation of a function \( f : q^{N_0} \to \mathbb{R}^n \) is defined as

\[
D_qf(t) = \frac{f(q(t))-f(t)}{(q-1)t}
\]
The q-derivative is also called Jackson derivative. See [6]:

Theorem II-2. "If \( f, g : q^{\mathbb{N}} \to \mathbb{R} \) are q-derivation, then we have the following:

1) \( D_q [af(t) + bg(t)] = aD_q f(t) + bD_q g(t) \), \( t \in q^{\mathbb{N}} \).

2) \( D_q \left( f(t)g(t) \right) = f(qt)D_q g(t) + g(t)D_q f(t) \), \( t \in q^{\mathbb{N}} \).

3) \( D_q \left( \frac{f(t)}{g(t)} \right) = \frac{g(t)D_q f(t) + f(t)D_q g(t)}{g(t)g(qt)} \), \( t \in q^{\mathbb{N}} \).

Theorem II-3. (Quantum Taylor's theorem). Let \( P_0, P_1, \ldots, P_N \) be polynomials and \( a \) be a number such that:

1) \( P_0(a) = 1 \) and \( P_n(a) = 0 \) for \( n \geq 1 \).

2) \( \deg(P_n) = n \).

Then any polynomial \( f \) of degree \( n \) can be written in the form:

\[
f(x) = \sum_{n=0}^{N} \begin{vmatrix} d^n f \end{vmatrix}_{x=a} P_n(x)
\]  

Definition II-4. Assume \( q^{\mathbb{N}} \to \mathbb{R} \) and \( a, b \in q^{\mathbb{N}} \) with \( a < b \). The definite integral of the function \( f \) is given by

\[
\int f(t) \, dq(t) = (q - 1) \sum_{i \in \{a,b\} \cap q^{\mathbb{N}}} t \cdot f(t)
\]

Definition II-5. If \( f : q^{\mathbb{N}} \to \mathbb{R} \) with \( q > 1 \), \( m, n \in \mathbb{N}_0 \) and \( m < n \), then

\[
\int_{q^n}^{q^m} f(t) \, dq(t) = \sum_{k=m}^{n-1} (q - 1) \cdot q^k \cdot f(q^k)
\]

3. Continuous-time regional economic model

Tabata in [28] described the mathematical formulation of regional economic model as continuous model as follows. "Consider the economic model with N-regional .This model gives a regional income \( w_i(t) \), \( i = 1, 2, \ldots, N \), with capital ratio \( b_i(t) \) at any time \( t \). Hence,

\[
w_i(t) = b_i(t) y_i(t)
\]

where \( y_i(t) \) represents the stock of capital in region \( i \).

If we consider \( v(t) \) as the national income of the country which equal to the sum of the N-regional incomes. We obtain

\[
v(t) = w_1(t) + w_2(t) + \ldots + w_N(t)
\]

Assuming the consumption depends on current and the increment of the stock of capital gives the investment on each region. Hence,

\[
\alpha_i(t) = r_i(t) w_i(t)
\]

\[
w_i(t + \Delta t) - w_i(t) = b_i(t) \beta_i(t)
\]

where \( \alpha_i(t) \) represents the consumption and \( \beta_i(t) \) represents the investment in region \( i \) at time \( t \).

The rate of the consumption in region \( i \) at time \( t \) is denoted by \( r_i(t) \). Now, using national income identity as in [17], we get

\[
v(t) = \sum_{i=1}^{N} w_i(t) = \sum_{i=1}^{N} \alpha_i(t) + \sum_{i=1}^{N} \beta_i(t)
\]

Therefore,
\[ \sum_{i=1}^{N} w_i(t) = \sum_{i=1}^{N} \eta_i(t) w_i(t) + \sum_{i=1}^{N} \frac{1}{b_i(t)} \frac{w_i(t + \Delta t) - w_i(t)}{\Delta t}. \]

Assuming \( 1 - \eta_i(t) = y_i(t) \) which represents the saving ratio in region \( i \) at time \( t \).

Thus,

\[ \sum_{i=1}^{N} y_i(t) w_i(t) = \sum_{i=1}^{N} \frac{1}{b_i(t)} \frac{w_i(t + \Delta t) - w_i(t)}{\Delta t}. \]

If \( \Delta t \) approach to zero, then

\[ \sum_{i=1}^{N} y_i(t) w_i(t) = \sum_{i=1}^{N} \frac{1}{b_i(t)} \frac{d(w_i(t))}{dt} \quad \text{(6)} \]

Using eq.(1), eq.(6) becomes:

\[ \sum_{i=1}^{N} \left[ b_i(t) y_i(t) - \frac{d(log b_i(t))}{dt} \right] y_i(t) = \sum_{i=1}^{N} \frac{d(y_i(t))}{dt} \quad \text{(7)} \]

Integrating both sides of (7) from 0 to \( t \), we get

\[ \sum_{i=1}^{N} y_i(t) = \sum_{i=1}^{N} y_i(0) + \sum_{i=1}^{N} \int_{0}^{t} M_i(s) y_i(s) ds \quad \text{(8)} \]

where \( M_i(s) = b_i(s) y_i(s) - \frac{d}{ds}(log b_i(s)) \).

Now, eq.(8) can be written as

\[ Iy(t) = Iy(0) + \int_{0}^{t} M(s)y(s)ds, \]

here \( I = (1,1,...,1) \), \( y(t) = (y_1(t),...,y_N(t)) \);
\( M(t) = (M_1(t),...,M_N(t)) \)

Our goal is to find \( y_i(t) \geq 0 \) and the investment \( \beta_i(t) \geq 0 \) to maximize the objective function value.

In this paper, we consider the Ramsay-type objective function as follows[17].

\[ \text{Max } \varphi(t) = \sum_{i=1}^{N} \int_{0}^{T} g_i(t) \alpha_i(t) \ dt \quad \text{(9)} \]

Eq.(9) “shows the total consumption over the planning horizon, where the weight attached to the consumption of each region

Now, using eq.(3), eq.(9) becomes

\[ \text{Max } \varphi(t) = \int_{0}^{T} \lambda(t)y(t) \ dt \quad \text{(10)} \]

where \( \lambda(t) = (\lambda_1(t),...,\lambda_N(t)) ; \lambda_i(t) = b_i(t) g_i(t) \eta_i(t) \).

Now, we can describe regional economic model as:

\[
\begin{cases}
\text{Max } \varphi(t) = \int_{0}^{T} \lambda(t)y(t) \ dt \\
S.t. \ Iy(t) = Iy(0) + \int_{0}^{T} M(s)y(s)ds, \\
y(t) \geq 0, \text{ for each } t \in [0,T].
\end{cases}
\]
4. Regional models in quantum calculus
We describe the mathematical formulation of regional economic model using quantum calculus analogue.
We use \( J \) to represent the quantum calculus interval
\( J = [1,T] \cap q^{[0]} \).
And by \( U_k \), to represent the space of all rd-continuous functions from \( J \) into \( \mathbb{R}^k \).
The primal quantum regional economic model (PQREM) is given as
\[
\begin{aligned}
\text{(PQREM)} \quad & \text{Max } \phi(y) = \int_1^q \lambda(t)y(t)d_q(t) \\
& \text{S.t. } l y(t) = l y(0) + \int_1^q M y(s)d_q(s) , \quad q^n \in J \\
& \text{and } y \in U_k , \quad y(t) > 0 , \quad t \in J
\end{aligned}
\]
where \( \lambda \in U_n \) and \( M \) is an \( n \) by \( n \) constant matrix.
The dual quantum regional economic model (DQREM) is
\[
\begin{aligned}
\text{(DQREM)} \quad & \text{Min } \Psi(z) = \int_1^{q^n+1} l y(0) z(t)d_q(t) \\
& \text{S.t. } z(q^n) \leq \hat{\lambda}(t) + \int_1^{q^n+1} z(s) M(t)d_q(s) , \quad q^n \in J \\
& \text{and } z \in U_m.
\end{aligned}
\]
5. Quantum calculus version of the duality theorems
We present a quantum calculus analogue of duality theorems for regional economic model and the proof of these theorems are immediate from the proof of standard duality theorems as in [7].

Theorem V-1 (Weak duality theorem)
If \( y \) and \( z \) are arbitrary feasible solution of (PQREM) and (DQREM), respectively, then \( \phi(y) \leq \Psi(z) \).

Theorem V-2 (Strong duality theorem)
If (PQREM) has an optimal solution \( y^* \), then (DQREM) also has an optimal solution \( z^* \) with \( \phi(y^*) = \Psi(z^*) \).

6. Example: two regional model
We present an example of a two regional model in quantum calculus.
Let \( T = q^3 \) and \( J = \{1,2,4\} \) with \( q = 2 \). Then we consider the following regional model in quantum calculus.
\[
\begin{aligned}
\text{(PQREM)} \quad & \text{Max } \phi(y = (y_1,y_2)) = \int_1^2 [2y_1(t) + 5y_2(t)]d_q(t) \\
& \text{S.t. } y_1(t) + y_2(t) = y_1(1) + y_2(1) + \int_1^t [10y_1(s) + y_2(s)]d_q(s) \\
& \text{and } y_1(t) \geq 0 , y_2(t) \geq 0 , \quad y_1(1) > 0 , y_2(1) > 0.
\end{aligned}
\]
Using Definition 2.4, the regional model becomes
\[
\begin{aligned}
\text{(PQREM)} \quad & \text{Max } \phi(y = (y_1,y_2)) = \sum_{t \in \mathbb{Z}} 2^t[2y_1(2^t) + 5y_2(2^t)] \\
& \text{S.t. } y_1(1) + y_2(1) + \sum_{t \in \mathbb{Z}} 2^t[10y_1(2^t) + y_2(2^t)] \\
& \text{and } y_1(t) \geq 0 , y_2(t) \geq 0 , \quad y_1(1) > 0 , y_2(1) > 0.
\end{aligned}
\]

Now, the optimal solution of this problem is obtained by MATLAB as follows.
\( y_1(1) = 5 , y_2(1) = 3 , y_1(2) = 61 , y_2(2) = 0 , \quad y_1(4) = 0 , y_2(4) = 1281 \)
and the optimal value is \( \phi((y_1,y_2)) = 25889 \).
Now the dual quantum calculus regional economic model is
Using Definition 2.4 and MATLAB, the optimal solution of the dual model is
\[ \Psi(z) = 25889, \]
and this shows the duality theorems hold, i.e., \( \Psi(z) = \phi(y = (y_1, y_2)) \).

7. Conclusions
In this paper, the regional economic model has been described using quantum calculus formulation. This model is presented as an optimal control model and formulated in both primal and dual models using quantum calculus formulation. Also, we present a new version of the duality theorems for this economics model. This approach yields an exact optimal solution of regional allocation of investment. This optimal solution is obtained using MATLAB. Furthermore, the upper and the lower bounds for any investment plan can be calculated based on the objective function values for both primal and dual regional models.

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