Geometric scaling in Mueller-Navelet jets

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Abstract

We argue that the production of Mueller–Navelet jets at the LHC represents a convenient environment to study gluon saturation and high–energy scattering in the presence of unitarity corrections. We show that, in a suitable range of transverse momenta for the produced jets, the cross–section for the partonic subprocess should exhibit geometric scaling. We point out that, in the presence of a running coupling, the cross–section for producing hard jets cannot be fully computed in perturbation theory, not even after taking into account the saturation effects: the non–perturbative physics affects the overall normalization of the cross–section, but not also its geometric scaling behavior.

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1 Introduction

Geometric scaling is one of the most important manifestations of the saturation physics in QCD at high energy [1–4], with striking consequences for the phenomenology. Identified first [1] in the HERA data, via a phenomenological analysis inspired by the idea of saturation [5], geometric scaling has been soon after understood [2–4] as a property of the BFKL evolution [6] in the presence of saturation. For problems with a single (transverse) resolution scale $Q^2$, so like deep inelastic scattering (DIS) or inclusive single–particle production in hadron–hadron collisions, ‘geometric scaling’ means that, within a wide kinematical window, the cross–section scales as a function of the ratio $Q^2/Q^2_s(Y)$. Here, $Q_s(Y)$ is the saturation momentum, i.e., the characteristic momentum scale for the onset of unitarity corrections in a collision in which the projectile and the target are separated by a rapidity gap $Y$. This scale grows rapidly with $Y$, according to the BFKL evolution [2, 3, 7].

Most importantly, geometric scaling is not restricted to the saturation region at $Q^2 \lesssim Q^2_s(Y)$, where the gluon occupation numbers are large and the scattering amplitudes are close to their unitarity limits, but it also extends over a relatively wide ‘geometric scaling window’ [2, 3] at $Q^2 \gg Q^2_s(Y)$, where the target is dilute and the scattering is weak, yet the scattering amplitudes ‘feel’ the effects of saturation, via the boundary condition at $Q^2 \sim Q^2_s(Y)$. This scaling window, which with increasing $Y$ is pushed towards larger and larger values of $Q^2$ (because of the corresponding rise in $Q^2_s(Y)$) and whose width is slowly increasing with $Y$ (via the BFKL diffusion), is essentially the same as the validity range for the BFKL approximation [6] at high energy. Hence, the large–$Q^2$ form of geometric scaling is a direct consequence of the BFKL dynamics precursory of saturation [2–4] and can be used to test the latter at the level of the phenomenology.

So far, the most compelling such tests have been performed on the HERA data at small values of Bjorken’s $x \simeq Q^2/s$, which are the data for which geometric scaling has been originally identified [1, 8] (see also Refs. [9, 10] for recent analyses, which include the diffractive data). Namely, one found that, within the whole small–$x$ domain at HERA, i.e., for $x \leq 0.01$ and $Q^2 \leq 450 \text{ GeV}^2$, the DIS cross–section can be well approximated by a scaling function: $\sigma(x, Q^2) \approx \sigma(Q^2/Q^2_s(Y))$ with $Q^2_s(Y) \sim e^{\lambda Y}$ and $Y = \ln(1/x)$. (For comparison, the proton saturation momentum at HERA is estimated in the ballpark of 1 GeV.) Remarkably, the value $\lambda \simeq 0.3$ for the ‘saturation exponent’ emerging from these analyses is in rough agreement with its perturbative calculation [7] from the next–to–leading order BFKL equation [11, 12]. More detailed analyses [13–16], combining BFKL dynamics and unitarity corrections within the framework of the dipole picture, have shown that the HERA data are consistent with some of the hallmarks of the BFKL evolution, like its characteristic ‘anomalous dimension’, or the violation of geometric scaling via BFKL diffusion. Similar parametrizations for the dipole cross–section, with the parameters fixed through fits to the HERA data, have been used [17–19] to describe particle production in deuteron–nucleus collisions at RHIC, with some success in explaining the ‘high–$p_T$ suppression’ in the nuclear modification factor at forward rapidities.

However, given the kinematical limitations inherent in the experiments at HERA and RHIC, the previous phenomenological studies of BFKL physics, geometric scaling, and saturation cannot be viewed as definitive. The situation should be more favorable in this respect at LHC, where the higher available energies and the experimental setup
should offer larger rapidity gaps to the BFKL evolution. For instance, in forward particle production at LHC one could measure values of $x$ as small as $x \sim 10^{-6}$ in the ‘target’ proton wavefunction for a produced jet with transverse momentum $k_\perp \sim 10$ GeV. With this kinematics, the jet should explore the geometric scaling window of the target proton, with interesting consequences, e.g., for the nuclear modification factor [20, 21].

Another interesting process that was proposed to test the BFKL dynamics and which could be measured at LHC under favorable conditions is the production of Mueller–Navelet jets [22] (see also Refs. [23–31] for various theoretical studies and [32, 33] for experimental searches at the Tevatron). This is a pair of jets separated by a large rapidity gap $Y$ which should favor the BFKL evolution of the cross-section for the partonic subprocess. For sufficiently large values of $Y$, saturation effects (in the form of unitarity corrections to the partonic scattering) should become important, as already emphasized by Mueller and Navelet in their original proposal [22]. However, with the exception of a few, preliminary, phenomenological studies [34, 35], such effects have been left out in previous studies of the Mueller–Navelet jets, which focused on the energies at the Tevatron. In particular, the modification of the BFKL dynamics by saturation and the phenomenon of geometric scaling have never been addressed in this context. These are the aspects that we would like to focus on in what follows.

Our first observation is that the Mueller–Navelet process is particularly favorable to study saturation physics. Unlike in DIS, where the gluon evolution inside the proton starts at the ‘soft’ scale $\Lambda_{QCD} \sim 250$ MeV and thus requires a relatively large rapidity evolution before it develops a hard saturation momentum, in the context of Mueller–Navelet jets this evolution starts with the ‘hard’ scale set by the transverse momentum $k_\perp \geq 10$ GeV of one of the two jets. Hence, the subsequent evolution with $Y$ should rapidly produce a system with very high gluon density around the position of the jet, i.e., with a very large local saturation momentum. The counterpart of that is that the dense region occupies only a small area $\sim 1/k_\perp^2$ in impact parameter space, and looks like a ‘dense spot’.

To be more specific, recall that one needs a rapidity evolution $Y_0 \simeq (1/\omega_P) \ln(1/\alpha_s)$, with $\omega_P$ the BFKL intercept, before a small hadronic system, so like a dipole or a high-momentum parton, reaches saturation on the resolution scale set by its own size, or transverse momentum [36]. A leading-order estimate for $\omega_P$ would yield $Y_0 \simeq 5$ (for $\alpha_s = 0.2$, as appropriate for a 10 GeV jet), but this is probably too optimistic. A more realistic estimate, using the NLO BFKL intercept [12], is $Y_0 \simeq 8$, which is fully within the reach of LHC. This means that, when producing a pair of Mueller–Navelet jets separated by a rapidity gap $Y = 8$ at the LHC, then one of the jets will ‘see’ the other one as a high-density gluonic system (a ‘color glass condensate’) with a saturation momentum $Q_s(Y) \sim 10$ GeV. Moreover, every additional unit of rapidity will make this saturation scale even harder, according to $Q_s^2(Y) \simeq Q_s^2 \exp[\lambda(Y - Y_0)]$. Such values for $Q_s$ are considerably higher than those that could ever be achieved in a proton, or nuclear, wavefunction at LHC energies, even for the most forward collisions.

Furthermore, the evolution towards saturation should favor the transverse momentum dissymmetry between the two jets. The typical transverse momentum of a gluon within

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3 We recall that in lowest-order perturbation theory the two jets come out with equal and opposite transverse momenta, because of momentum conservation. Hence, any momentum asymmetry
the wavefunction of an evolved jet is the respective saturation momentum $Q_s(Y)$, which for large $Y$ is as hard as, or even harder than, the original parton which comes out as a jet. Therefore, the other jet can easily be produced with a very different transverse momentum, because the momentum imbalance can be compensated by ‘inclusive’ gluons from the ‘evolved’ jet wavefunction. In fact, we expect a larger momentum asymmetry in the presence of saturation than from pure BFKL evolution: unlike the latter, which proceeds symmetrically towards soft and hard momenta, the evolution in the presence of saturation is biased towards large transverse momenta (larger than the saturation scale).

The most interesting kinematical situation for our subsequent analysis is precisely when the two jets are well separated in transverse momentum, say, $k_{1\perp} \gg k_{2\perp}$. More precisely, we shall be interested in configurations where the harder jet has a transverse momentum comparable to, or even larger than, the saturation momentum that would be generated by the evolution of the softer jet over the rapidity gap $Y$: $k_{1\perp} \gtrsim Q_s(Y)$, where $Q_s(Y)$ is implicitly a function of $k_{2\perp}$. Under these circumstances, we shall see that the partonic cross-section exhibits geometric scaling within a wide kinematical window. That is, for given $k_{2\perp}$, the cross-section scales as a function of the ratio $k_{1\perp}^2/Q_s^2(Y)$. On the other hand, the cross-section is small, of order $1/k_{2\perp}^2$, because of the small size of the dense spot, as alluded to above. While our conclusions may look natural, given the kinematics and the similarity with other problems like DIS, our analysis appears to reserve some difficulties and surprises.

The first difficulty refers to the inclusion of saturation effects and unitarity corrections in the cross-section for Mueller–Navelet jets. This in turn requires two steps: (i) a factorization formula which is general enough to allow for unitarity corrections, and (ii) the calculation of the ingredients which enter this factorization formula within the framework of high–density QCD (i.e., from the solutions to the non–linear evolution equations which generalize the BFKL equation to the region of high gluon density).

Concerning step (i), we shall proceed in a heuristic way, by generalizing, in Sect. 2, a known formula for single–jet production in the presence of unitarity corrections [37–41]. This leads us to a generalization of the standard $k_T$–factorization for Mueller–Navelet jets, which has been already presented in Ref. [34,35,41], and in which the BFKL Green’s function is replaced by the total cross-section for the scattering between two effective color dipoles. Unlike the quark–antiquark dipole familiar in the context of DIS (see, e.g., [5,13–16]), which is a physical fluctuation of the virtual photon, the dipoles that enter our factorization for Mueller–Navelet jets are merely mathematical constructions, which appear in the calculation of the cross-section and are built with one parton in the direct amplitude and another parton in the complex conjugate amplitude. The dipole–dipole cross-section is written in coordinate space, as appropriate for the inclusion of unitarity corrections (multiple scattering) in the eikonal approximation. The transverse momenta $k_{1\perp}$ and $k_{2\perp}$ of the produced jets are then fixed via a double Fourier transform from the dipole sizes.

The second step, i.e., the calculation of the dipole–dipole cross-section within high–density QCD, turns out to be particularly subtle. Since the scattering involves two systems
(dipoles) which start by being dilute at low energy, it seems that we cannot rely on the standard Balitsky–JIMWLK, or BK, equations [42–45], which apply only to dense–dilute scattering. Instead, one should use the more general, ‘Pomeron loop’, equations [46–48], which also allow for particle number fluctuations in the course of the evolution. From the correspondence with statistical physics [49], and also from the numerical simulations of simple models inspired by QCD [50, 51], we know that, with a fixed coupling, the effects of the fluctuations are truly crucial: with increasing energy, they rapidly wash out both the BFKL approximation and the ‘geometric scaling’ behavior predicted by the BK equation [2–4]. However, a very recent numerical analysis [52] has shown that the fluctuations are strongly suppressed by the running of the coupling, in such a way that their effects remain negligible for all energies of practical interest.

With the philosophy that the running–coupling case is the only one of fundamental interest for real QCD, in what follows we shall perform a ‘mean field’ type of analysis, based on BK equation, for both fixed and running coupling. The fixed–coupling analysis, as developed in Sects. 3 and 4, turns out to be rather straightforward: The dipole–dipole cross–section, as obtained from approximate solutions to BK equation [2–4], exhibits geometric scaling for suitably chosen dipole sizes. After a Fourier transform, this scaling property gets transmitted to the partonic core of the Mueller–Navelet cross–section, for appropriate transverse momenta of the two jets.

The running–coupling case, that we shall treat in Sect. 5, is both more interesting and more subtle. First, it might look inconsistent to include the running of the coupling, but at the same time ignore other next–to–leading order corrections in perturbative QCD. But it turns that the running of the coupling plays a special role in the context of the high–energy dynamics: because of it, all the other perturbative corrections die away in the high–energy limit [7, 53]. Indeed, the evolution towards saturation is controlled by momenta $k_\perp \sim Q_s(Y)$; so, the relevant value of the coupling is $\alpha_s(Q^2_s(Y))$, which decreases with $Y$, and therefore so do the perturbative corrections, whose strength is proportional to $\alpha_s(Q^2_s(Y))$. Moreover, the running of $\alpha_s$ has qualitative consequences which modify the high–energy evolution in depth. We have already mentioned its role in suppressing the particle–number fluctuations. This is related to a more general property of the running of the coupling, which is to slow down the evolution towards saturation [2, 3, 7]. Another manifestation of this property is in the growth of the saturation momentum with $Y$: for sufficiently large $Y$, and with a running coupling, $\ln Q^2_s$ grows like $\sqrt{Y}$, and not like $Y$.

An important consequence of the running of the coupling, which is not specific to the high–energy problem, but has dramatic consequences for it, is the fact that it introduces an intrinsic scale in the problem — the ‘soft’ scale $\Lambda_{QCD}$ —, thus breaking down the conformal invariance of the leading–order formalism. With increasing energy, this scale progressively replaces within the saturation momentum any other scale introduced by the initial conditions at low energy, so like the target dipole size. Accordingly, for sufficiently high energy, the saturation momentum becomes independent of the target size [54]. This has important consequences for the high–energy evolution in general (e.g., it implies that a large nucleus is not more dense than a proton at very high energies), and for the

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4 Of course, these corrections may be numerically important for the phenomenology at LHC, but here we focus on the dominant asymptotic behavior, for simplicity.
Mueller–Navelet process in particular: it implies that the perturbative calculation of the Mueller–Navelet cross–section breaks down, even if the jet transverse momenta are restricted to be hard. The precise argument in that sense will be developed in Sect. 4, but here we would like to emphasize that this argument is in fact very general (and hence also very robust): it reflects the fact that, with running coupling, the dipole–dipole scattering amplitude at a fixed impact parameter is independent of the target dipole size $R$ (rather than dying away as an inverse power of $R$, as it would happen in the fixed–coupling formalism, by conformal invariance). Accordingly, the dipole–dipole cross–section, which is obtained by integrating the amplitude over all impact parameters, is proportional to $R^2$, and hence it strongly favors large dipole fluctuations. Without the non–perturbative cut–off introduced by confinement, the partonic cross–section would be controlled by dipole fluctuations with arbitrarily large size. As we shall argue in Sect. 4, the ad–hoc introduction of a non–perturbative cutoff on the dipole sizes affects the normalization of the total cross–section, but not also its property of geometric scaling (which merely refers to the functional dependencies of the cross–section upon the rapidity gap $Y$ and upon the transverse momentum $k_{1\perp}$ of the hardest jet).

2 Forward jets with unitarity corrections

Although our main interest here is in the production of a pair of (Mueller–Navelet) jets, it is instructive to start our presentation with the case of a single jet, for which the high–energy factorization in the presence of unitarity corrections is more firmly established. This will also allow us to introduce the physics and the theoretical description of the unitarity corrections in a simpler setting. The two–jet problem will then be easier to explain, by analogy.

For definiteness, we focus on jets initiated by gluons (quarks will be added later on), and thus consider the cross–section for inclusive gluon production at forward rapidity in a hadron–hadron collision at high energy. By ‘forward rapidity’ we mean that the produced gluon carries a sizeable fraction $x \sim \mathcal{O}(1)$ of the longitudinal momentum of one of the incoming hadrons (the ‘projectile’), so that there is a large rapidity gap $Y = Y_0 - y$ between this produced gluon and the other hadron (the ‘target’). Here, $Y_0 = \ln(s/M_1 M_2)$, with $s$ the invariant energy squared and $M_1, 2$ the masses of the participating hadrons, is the rapidity gap between the projectile and the target, and $y = \ln(1/x) + \ln(k_{\perp}/M_1)$ is the (relatively small) rapidity separation between the produced gluon, which has transverse momentum $k_{\perp}$, and the projectile. Alternatively, one could trade the rapidity gap $y$ for the pseudo–rapidity $\eta$ of the produced jet in the laboratory frame; e.g., if the lab frame coincides with the hadron center–of–mass frame, so like at LHC, then $x = (k_{\perp}/\sqrt{s}) e^\eta$, with $\eta > 0$ for forward jets (see also Fig. 1.a).

Under these circumstances, and in the leading–order formalism of perturbative QCD (meaning, in particular, that the coupling is fixed), the cross–section for gluon production can be expressed in a ‘$k_T$–factorized’ form, which is formally similar to, but more general than, the corresponding factorization used in the context of the BFKL physics [6]. Namely, the $k_T$–factorization is now extended towards the high–energy regime where unitarity corrections (multiple scattering, gluon saturation) become important. The general respective
formula can be found in the literature (see, e.g., Refs. [37–41, 55]). Here, we shall need only a simpler form of it, valid when the transverse momentum $k_\perp$ of the produced gluon is large enough — much larger than the typical momentum transferred from the projectile to this gluon (see below for a more precise condition). We then have

$$
\frac{d\sigma^{P_T\rightarrow JX}}{d\eta \, d^2k_\perp} = \frac{1}{8\pi^2 k_\perp^2} x G_P(x, k_\perp^2) \int d^2r \, e^{-i k \cdot r} \nabla_r^2 \sigma(gg)_T(r, Y),
$$

(2.1)

where $x G_P(x, k_\perp^2)$ is the gluon distribution in the projectile ($P$) on the resolution scale of the jet (i.e., the number of gluons with longitudinal momentum fraction $x$ equal to that of the jet, and with transverse momenta $p_\perp^2 \leq k_\perp^2$). Furthermore, $\sigma(gg)_T(r, Y)$ is the total cross-section for the scattering between a gluonic dipole ($gg$ pair in a color singlet state) with transverse size $r$ and the hadronic target ($T$), for a rapidity separation $Y$. The $gg$ dipole here is the effective dipole made with the produced gluon in the direct amplitude (located at transverse coordinate $x$) and the corresponding gluon in the complex conjugate amplitude (located at $y$). The gluon transverse momentum $k_\perp = |k|$ in the final state is then fixed via the Fourier transform from $r = x - y$ to $k$.

In the single–scattering approximation to the dipole–target cross–section, the Fourier transform in Eq. (2.1) yields the usual ‘unintegrated’ gluon distribution in the target wavefunction (evaluated in the BFKL approximation) times a constant of order $\alpha_s$. We then recover from Eq. (2.1) the traditional $k_T$–factorization. But Eq. (2.1) remains valid also very large values of $Y$, where the unitarity corrections to the dipole scattering become important and the BFKL approximation ceases to apply.

The physical interpretation of the unitarity corrections is most transparent in the ‘target infinite momentum frame’, where the dipole has relatively low energy while the target carries most of the total rapidity $Y$. Then the target wavefunction has evolved into a ‘color glass condensate’ (CGC) — a system with high gluon density characterized by a hard intrinsic scale, the saturation momentum $Q_s(Y)$, which grows rapidly with $Y$ and separates between

- a high density region at low momenta $p_\perp \lesssim Q_s(Y)$, where the gluon occupation numbers are large, $\sim \mathcal{O}(1/\alpha_s)$, but ‘saturated’ (they do not rise with the energy anymore), and
- a low density region at high momenta $p_\perp \gtrsim Q_s(Y)$, where the occupation numbers are low, but rapidly growing with $Y$, via the BFKL evolution.

For sufficiently high energy and/or large dipole sizes, such that $r \gtrsim 1/Q_s(Y)$, the dipole will undergo multiple scattering off the CGC. The ‘unitarity corrections’ refer to this multiple scattering, as well as to the saturation effects in the target gluon distribution, i.e., to all the non–linear effects which reduce the gluon density and enforce the unitarity bound on the scattering process. Such effects are resummed — within the eikonal approximation, and within the limits of the LO formalism — by the Balitsky–JIMWLK equations, which in particular determine the dipole scattering amplitude in this high energy regime. The corresponding solution will be further described in the next sections. Here, it suffices to

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5 As mentioned in the Introduction, these equations neglect the particle number fluctuations, which would be important in the context of the fixed–coupling evolution [46,49], but which are suppressed by the running of the coupling [52].
Fig. 1. The inclusive production of a single, forward, jet (left) and of a pair of Mueller–Navelet jets (right) in the presence of unitarity corrections.

We are now prepared to present the corresponding formulæ for the Mueller–Navelet jets. This is a pair of jets produced in a high–energy hadron–hadron collision such that each jet carries a relatively large fraction \( x_i, i = 1, 2 \) of the longitudinal momentum of its parent hadron. Accordingly, each jet is relatively close in rapidity to its respective parent, so there is a large rapidity gap \( Y \gg 1 \) between the jets: \( Y = \ln(x_1 x_2 s/k_{1\perp} k_{2\perp}) \), where \( k_{1\perp} \) and \( k_{2\perp} \) are the jet transverse momenta (see Fig. 1(b)). When the scattering is viewed in the hadron center–of–mass frame, the momentum fractions \( x_i \) and the rapidity gap are determined by the pseudo–rapidities \( \eta_i \) of the two jets, according to

\[
x_1 \simeq \frac{k_{1\perp}}{\sqrt{s}} e^{\eta_1}, \quad x_2 \simeq \frac{k_{2\perp}}{\sqrt{s}} e^{-\eta_2}, \quad Y = \eta_1 - \eta_2.
\]

The typical kinematics for Mueller–Navelet jets is such that \( \eta_1 \) is large and positive, while \( \eta_2 \) is large and negative.

The differential cross–section for Mueller–Navelet jets has been rigorously computed [22] (within the LO formalism, once again) only at the level of the BFKL approximation, which ignores unitarity corrections. The corresponding result is the expected generaliza-
tion of the corresponding single–jet cross–section — the BFKL version of Eq. \((2.1)\) — which is symmetric w.r.t. the two jets. In view of this, and of the symmetry of the problem, it has been conjectured \([34, 35, 41]\) that, after including the unitarity corrections, the cross–section for Mueller–Navelet jets should be given by the properly symmetrized version of Eq. \((2.1)\), that is

\[
\frac{d\sigma^{pp\rightarrow JXJ}}{dx_1 dx_2 d^2 k_{1\perp} d^2 k_{2\perp}} = \frac{1}{64\pi^4} G(x_1, k_{1\perp}^2) G(x_2, k_{2\perp}^2) \frac{1}{k_{1\perp}^2 k_{2\perp}^2} \times \int d^2 r_1 e^{-i\mathbf{k}_1 \cdot \mathbf{r}_1} \int d^2 r_2 e^{-i\mathbf{k}_2 \cdot \mathbf{r}_2} \nabla_{r_1}^2 \nabla_{r_2}^2 \sigma_{(gg)(gg)}(r_1, r_2, Y), \tag{2.3}
\]

where, for definiteness, we have chosen the incoming hadrons to be protons, so like at LHC. \(\sigma_{(gg)(gg)}(r_1, r_2, Y)\) is the total cross–section for the scattering between two (effective) gluonic dipoles with the indicated transverse sizes and separated by a rapidity gap \(Y\).

The cross–section \((2.3)\) is generally a function of \(k_{1\perp}, k_{2\perp}\), and the relative angle \(\phi\) between the vectors \(\mathbf{k}_1\) and \(\mathbf{k}_2\) (the azimuthal angle between the two jets). Although interesting in view of the phenomenology (and largely studied in the context of the BFKL approximation \([23–31]\)), the azimuthal correlations represent a subleading effect at high energies, and thus are irrelevant for our present study of the unitarity corrections. So, in what follows we shall average over \(\phi\), which is tantamount to replacing the dipole cross–section in \((2.3)\) by \(\sigma_{(gg)(gg)}(r_1, r_2, Y)\) — the corresponding cross–section averaged over the relative orientations of the two dipoles.

It is furthermore convenient to introduce some hard momentum cutoffs \(Q_1\) and \(Q_2\) (corresponding to the experimental \(k_T\)–cuts) and compute the cross–section for producing two jets with transverse momenta \(k_{1\perp} > Q_1\) and \(k_{2\perp} > Q_2\) and with given longitudinal momentum fractions \(x_1\) and \(x_2\). That is,

\[
\frac{d\sigma^{pp\rightarrow JXJ}}{dx_1 dx_2} \equiv \int d^2 k_1 \int d^2 k_2 \frac{d\sigma^{pp\rightarrow JXJ}}{dx_1 dx_2 d^2 k_{1\perp} d^2 k_{2\perp}} \Theta(k_{1\perp} - Q_1) \Theta(k_{2\perp} - Q_2). \tag{2.4}
\]

(These integrations automatically implement the average over \(\phi\).) A priori, the integrations are complicated by the \(k_{\perp}\)–dependencies of the gluon distributions in \((2.3)\), and by the one implicit in the dipole–dipole cross–section, via the rapidity gap \(Y = \ln(x_1 x_2 s/k_{1\perp} k_{2\perp})\). Note however that for given \(x_1, x_2\) and \(s\), the maximal rapidity gap \(Y_{\text{max}} = \ln(x_1 x_2 s/Q_1 Q_2)\) is attained for the threshold momenta \(k_{1\perp} = Q_1\) and \(k_{2\perp} = Q_2\). Hence the dominant contribution, in the sense of the leading–logarithmic approximation, is obtained by replacing \(Y \rightarrow Y_{\text{max}}\) within \(\sigma_{(gg)(gg)}\). Moreover, the integrand in \((2.4)\) is rapidly decreasing at very large values for \(k_{1\perp}\) and \(k_{2\perp}\) — this can be checked, e.g., by using the BFKL approximation for the dipole–dipole cross–section \([22]\), which enables us to replace \(G(x_i, k_{i\perp}^2) \rightarrow G(x_i, Q_i^2)\) in the slowly varying gluon distributions. The remaining integrations can be easily performed, with the final result

\[
\frac{d\sigma^{pp\rightarrow JXJ}}{dx_1 dx_2} \simeq \frac{1}{64\pi^4} G(x_1, Q_1^2) G(x_2, Q_2^2) \times \int_0^\infty dr_1 \int_0^\infty dr_2 Q_1 J_1(Q_1 r_1) Q_2 J_1(Q_2 r_2) \sigma_{(gg)(gg)}(r_1, r_2, Y). \tag{2.5}
\]
Here and from now on it is understood that \( Y = \ln(x_1 x_2 s/Q_1 Q_2) \). So far, we have considered only gluon jets, but quarks or antiquarks jets can be similarly included: when the jet \( i \), with \( i = 1, 2 \), is initiated by a quark with flavor \( f \), we have a formula similar to Eq. (2.5) in which the gluon distribution \( G(x_i, Q_i^2) \) is replaced by the quark distribution \( q_f(x_i, Q_i^2) \) (or \( \bar{q}_f(x_i, Q_i^2) \) for an antiquark), and the corresponding dipole within \( \sigma_{(gg)(gg)} \) is replaced by a dipole made with a quark and an antiquark. We thus encounter three types of dipole–dipole processes: \((gg)(gg)\), \((gg)(q\bar{q})\), and \((q\bar{q})(q\bar{q})\), whose cross–sections differ at most through color factors (see below).

As a consistency check of our above factorization of the Mueller–Navelet cross–section, cf. Eq. (2.3) or (2.5), let us now verify its BFKL limit. For two gluonic dipoles, the BFKL cross–section (averaged over angle) reads

\[
\sigma_{(gg)(gg)}(r_1, r_2, Y)_{\text{BFKL}} = 2\pi \alpha_s^{2} \frac{N_c}{C_F} \frac{r_2}{r_1} \int \frac{d\gamma}{2\pi i} \frac{(r_2/r_1)^{2\gamma}}{\gamma^2(1-\gamma)^2} \exp \left[ \frac{\alpha_s N_c}{\pi} \chi(\gamma) Y \right], \quad (2.6)
\]

where we use the standard representation for the BFKL solution in Mellin space (see, e.g., [2, 3]). This result is symmetric under the exchange \( r_1 \leftrightarrow r_2 \) of the two dipoles, as it can be checked by using the property \( \chi(\gamma) = \chi(1-\gamma) \) of the BFKL characteristic function. When one or both of the gluonic dipoles are replaced by fermionic \((q\bar{q})\) ones, the expression in Eq. (2.6) must be multiplied by a factor \( C_F / N_c \) for each such a replacement.

Substituting (2.6) into (2.5), making use of the Bessel function integration formula

\[
\int_0^\infty dx \, x^\beta J_n(x) = 2^\beta \frac{\Gamma\left(n + 1 + \beta \right)}{\Gamma\left(n + 1 - \beta \right)}, \quad (2.7)
\]

and summing up over all types of partons, we obtain (with \( \bar{\alpha} \equiv \alpha_s N_c / \pi \))

\[
\frac{d\sigma_{pp \to JXJ}}{dx_1 dx_2}_{\text{BFKL}} = F_{\text{eff}} \frac{8\pi N_c}{C_F} \frac{\alpha_s^2}{Q_1^2} \int \frac{d\gamma}{2\pi i} \frac{(Q_1^2/Q_2^2)^\gamma}{\gamma(1-\gamma)} \exp \left[ \bar{\alpha} \chi(\gamma) Y \right], \quad (2.8)
\]

where \( F_{\text{eff}} \) involves contributions from quarks, antiquarks, and gluons, with appropriate color factors (the sum over the quark flavors is kept implicit):

\[
F_{\text{eff}} = \frac{1}{64\pi^4} f_{\text{eff}}(x_1, Q_1^2) f_{\text{eff}}(x_2, Q_2^2) 
\]

\[
f_{\text{eff}}(x, Q^2) \equiv G(x, Q^2) + \frac{C_F}{N_c} \left[ q(x, Q^2) + \bar{q}(x, Q^2) \right]. \quad (2.9)
\]

As anticipated, Eq. (2.8) is in precise agreement (including the normalization) with the corresponding result in Ref. [22].

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6 The appearance of the Casimir \( C_F \) for the fundamental representation in a cross–section pertinent to gluons alone may look surprising. In reality, this has been generated via the identity \( N_c^2 / (N_c^2 - 1) = N_c / 2C_F \), where all the \( N_c \) factors arise from the gluon color algebra.
3 Fixed coupling case: the dipolar cross-section

In addition to taming the BFKL growth of the dipole scattering amplitude, in compliance with the unitarity bound, the non-linear effects encoded in the Balitsky–JIMWLK (or BK) equations have also an interesting consequence for the functional form of the amplitude in the transition region from weak to strong scattering: within a rather wide kinematical region, whose width is increasing with $Y$, this amplitude shows geometric scaling \cite{2, 3}, i.e., it depends upon the size $r$ of the projectile dipole and upon $Y$ only via the dimensionless variable $\tau \equiv r^2 Q_s^2(Y)$, with $Q_s(Y)$ the saturation momentum of the target evolved up to rapidity $Y$. It is then tempting to conjecture that this scaling property should transmit from the dipole–dipole amplitude to the cross-section for Mueller–Navelet jets, via the convolutions in Eq. (2.5) (within a suitable range of values for the variables $Q_1$, $Q_2$, and $Y$). This is a simple argument in that sense: due to the presence of the rapidly oscillating Bessel functions in the integrand of Eq. (2.5), one expects the integrals there to be dominated by values $r_1 \sim 1/Q_1$ and $r_2 \sim 1/Q_2$. If this is true, then one can choose $Q_1$ and $Q_2$ (for a given $Y$) in such a way that the dipole–dipole cross-section, and hence the dijet cross-section, are in the geometric scaling window. As we shall later discover, via explicit calculations, this simple argument is indeed correct in the case of a fixed coupling, but not also for a running coupling.

We start with the fixed–coupling case, i.e., the LO formalism. Within the eikonal approximation, the dipole–dipole cross-section is computed as (for dipoles made with partons in a generic color representation)

$$
\sigma_{dd}(r_1, r_2, Y) = 2 \int d^2b \ T_{dd}(r_1, r_2, b, Y),
$$

(3.1)

where $T_{dd}(r_1, r_2, b, Y)$ is the scattering amplitude for two dipoles with transverse sizes $r_1$ and $r_2$, relative impact parameter $b$, and rapidity separation $Y$. (The average over the relative angle between $r_1$ and $r_2$ is implicit here and from now on.) We use conventions in which the $S$–matrix is written as $S = 1 - T$, where $T$ is taken to be real, as appropriate for the dominant behavior at high energy. In general, the dipole–dipole amplitude and cross-section are, of course, symmetric under the exchange of the two dipoles, but their approximate forms that we shall derive below are valid only when one of the dipoles is much smaller than the other one. It is then convenient to introduce the notations $r \equiv \min(r_1, r_2)$ and $R \equiv \max(r_1, r_2)$, and refer to the small (large) dipole as the ‘projectile’ (respectively, the ‘target’). Also, as explained in Sect. 2, it is convenient to visualize the evolution with increasing energy as gluon evolution in the ‘target’ (the larger dipole).

The unitarity of the $S$–matrix implies $T \leq 1$, with the upper bound $T = 1$ (the ‘black disk limit’) describing a situation where the scattering occurs with probability one. This constraint is indeed obeyed by the solution $T$ to the Balitsky–JIMWLK (or BK) equations, which, moreover, appears to saturate the black disk limit $T = 1$ for sufficiently large $Y$. But the cross-section (3.1) can rise indefinitely with $Y$, even after the ‘black disk’ limit has been reached at central impact parameters, because the gluon distribution in the target keeps expanding towards larger impact parameters, due to the non–locality of the BFKL evolution. This radial expansion of the gluon distribution is however much slower than its evolution towards the black disk limit at a fixed value of $b$. In particular, we do
not expect this expansion to be essential at the LHC energies. Therefore, a target which at $Y = 0$ starts as a single dipole of size $R$, evolves with $Y$ towards blackness, first, on its own scale $R$, then, on smaller and smaller scales, without significantly expanding towards larger sizes\footnote{This property is not correctly encoded in the Balitsky–JIMWLK, or BK, equations, which rather predict a rapid radial expansion of the black disk, because of the long–range tails (in $b$) of the perturbative gluon distribution \cite{56}. However, in real QCD we expect this tails to be cut–off by confinement, with the effect that the radial expansion is drastically slowed down, in compliance with Froissart bound \cite{57}.}. Accordingly, when this evolved target is probed by a projectile dipole with size $r \ll R$, the amplitude $T_{dd}(r, R, b, Y)$ is negligibly small when the two dipoles have no overlap with each other ($b \gg R$). The typical impact parameters which contribute to the cross–section are such that $b \ll R$, and for them the amplitude is roughly independent of $b$. These considerations motivate the following approximation to the dipole–dipole cross–section \cite{3.1}, valid when $r \ll R$

$$\sigma_{dd}(r, R, Y) \simeq 2\pi R^2 T_{dd}(r, R, Y), \quad (3.2)$$

where $T_{dd}(r, R, Y)$ is independent of $b$ and satisfies the unitarity bound $T_{dd} \leq 1$. For this amplitude, we shall use approximate solutions to the BK equation with fixed coupling.

Specifically, the saturation momentum $Q_s(R, Y)$ is defined by the condition

$$T_{dd}(r, R, Y) = \kappa \quad \text{for} \quad r = 1/Q_s(R, Y), \quad (3.3)$$

where $\kappa < 1$ is a number of order one (its precise value is irrelevant to the accuracy of interest). For $r \gtrsim 1/Q_s$ we have $T_{dd} \sim \mathcal{O}(1)$, whereas for $r \ll 1/Q_s$ the amplitude is small, $T_{dd} \ll 1$, and approximately given by the following, universal, function

$$T_{dd}(r, R, Y) \simeq \left(\ln \frac{1}{r^2 Q_s^2}\right) (r^2 Q_s^2)^{\gamma_s} \exp \left\{-\frac{\ln^2(r^2 Q_s^2)}{4D_s \bar{\alpha} Y}\right\}, \quad (3.4)$$

(up to a normalization factor), with the saturation momentum

$$Q_s^2(R, Y) = (C \alpha_s^2)^{1/\gamma_s} \frac{1}{R^2} e^{\lambda_s \bar{\alpha} Y}. \quad (3.5)$$

The various exponents which appear in these formulae are pure numbers determined by the BFKL characteristic function (see Refs. [2, 3] for details). Specifically, $\gamma_s \approx 0.63$ ($1 - \gamma_s \approx 0.37$ is the BFKL anomalous dimension at saturation), $D_s \approx 48.5$ plays the role of a diffusion coefficient, and $\lambda_s \approx 4.88$ is the saturation exponent. Note that rapid growth of the saturation momentum with $Y$. Its dependence upon the target size $R$ could have been anticipated from dimensional arguments.

Eq. \cite{3.4} is universal in the sense that the dependence upon the initial conditions at low energy is fully encoded in the value of the saturation momentum. But the latter is, of course, process–dependent: it depends upon the size $R$ of the target, and also upon the color representations of the partons making up the two dipoles participating in the collision (via the coefficient $C$ which here is left unspecified). For instance, the value of
$C$ corresponding to the process $(gg)(q\bar{q})$ is larger by a factor $N_c/C_F$ than that for the process $(q\bar{q})(q\bar{q})$.

The prefactor involving $\alpha_s^2$ in the expression (3.5) for the saturation momentum reflects the fact that the scattering amplitude starts at order $\alpha_s^2$ in perturbation theory, hence one needs some non–trivial rapidity evolution $Y_0$ before the dipole becomes ‘black’ on the resolution scale fixed by its own size: $T_{dd}(r = R, R, Y_0) = \kappa$. This condition yields $Y_0 \simeq (1/\gamma_s \lambda_s \bar{\alpha}) \ln(\kappa/\alpha_s^2)$, and then Eq. (3.5) can be rewritten in such a way to exhibit the rapidity excess beyond $Y_0$:

$$Q_s^2(R, Y) = \frac{e^{\lambda_s \bar{\alpha}(Y - Y_0)}}{R^2}.$$  \hspace{1cm} (3.6)

Eq. (3.4) is valid when $\bar{\alpha} Y \gg 1$ and in a rather wide range of values for $r$, namely for $1 \ll \ln(1/r^2 Q_s^2) \lesssim c \bar{\alpha} Y$, with $c \sim O(1)$. In particular, within the more restricted window

$$1 \ll \ln \frac{1}{r^2 Q_s^2} \ll \sqrt{4D_s \bar{\alpha} Y},$$  \hspace{1cm} (3.7)

the last, Gaussian, factor in Eq. (3.4) can be ignored, and then the amplitude shows geometric scaling, as anticipated. Note that, when $\bar{\alpha} Y \gg 1$, this scaling window is quite wide, especially since the parameter $D_s$ is numerically large.

4 Fixed coupling case: Mueller–Navelet jets

We now have all the ingredients to compute the cross–section for Mueller–Navelet jets in the presence of unitarity corrections and for fixed coupling. We shall focus on the range of values for the momentum cutoffs $Q_1$ and $Q_2$ in which we expect geometric scaling. Namely we choose $Q_1 \gg Q_2$ in such a way that $Q_1^2 \gtrsim Q_2^2 e^{\lambda y}$, with the compact notations $\lambda \equiv \lambda_s \bar{\alpha}$ and $y \equiv Y - Y_0$. That is, $y$ is the rapidity excess introduced in Eq. (3.6). (Throughout this paper, we assume $Y > Y_0$.) Therefore, $Q_1$ is larger, but not much larger, than the saturation scale that would generated by a target dipole with size $\sim 1/Q_2$ after a rapidity evolution $Y$. As we shall shortly check, under this condition the convolutions in Eq. (2.5) are indeed dominated by values for $r_1$ and $r_2$ within the scaling window (3.7).

To simplify the calculation, we therefore keep only the scaling piece in the dipole–dipole amplitude (3.4) in the weak scattering regime. We therefore replace Eqs. (3.2)–(3.4) by the following, piecewise, approximation to the dipole–dipole cross–section (we recall the notations $r \equiv \min(r_1, r_2)$ and $R \equiv \max(r_1, r_2)$)

$$\sigma_{dd}(r, R, Y) = 2\pi R^2 \begin{cases} \left( \frac{r^2}{R^2} e^{\lambda y} \right)^{\gamma_s} & \text{for } r^2 < R^2 e^{-\lambda y} \\ 1 & \text{for } r^2 > R^2 e^{-\lambda y} \end{cases}$$  \hspace{1cm} (4.1)

In the Introduction, this was written as $Y_0 \simeq (1/\omega_P) \ln(1/\alpha_s^2)$, which is essentially the same, since $\gamma_s \lambda_s \bar{\alpha}$ plays the role of the intercept at the saturation saddle point, as manifest in Eq. (3.4).
where we have ignored the slowly varying logarithm in Eq. (3.4). The precise normalization of the cross-section is not an issue here, as we are merely interested in its functional dependencies.

Because of the symmetry of the above cross-section under \( r_1 \leftrightarrow r_2 \), when evaluating the double integral in Eq. (2.5) it is enough to consider the case \( r_1 < r_2 \). Then the contribution from the other region \( r_1 > r_2 \) can be simply obtained by letting \( Q_1 \leftrightarrow Q_2 \) in the result of the first case\(^9\). Thus, substitution of (4.1) into (2.5) gives

\[
\frac{d\sigma_{pp \rightarrow JXJ}}{d x_1 d x_2} = F e^{\gamma_s \lambda y} \int_0^\infty dr_1 Q_1 J_1(Q_1 r_1) r_1^{2\gamma_s} \int_{r_1 e^{\lambda y/2}}^\infty dr_2 Q_2 J_1(Q_2 r_2) r_2^{2-2\gamma_s} + F \int_0^\infty dr_1 Q_1 J_1(Q_1 r_1) \int_{r_1}^\infty dr_2 Q_2 J_1(Q_2 r_2) r_2^2 + \{Q_1 \leftrightarrow Q_2\},
\]

where for the time being we consider the gluon jets alone (hence, the overall factor \( F \) includes the gluon distributions, together with other numerical factors); the quark jets will be added later on. Note that the exchange \( Q_1 \leftrightarrow Q_2 \) should not be done inside \( F \), but only in the result of the integration. Clearly, the two explicit terms in the r.h.s. of the above equation arise from the corresponding pieces in (1.1). It is convenient to change variables by letting \( u_1 = Q_1 r_1 \) and \( u_2 = Q_2 r_2 \). Then the above equation becomes

\[
\frac{d\sigma_{pp \rightarrow JXJ}}{d x_1 d x_2} = F \frac{1}{Q_2} \left( Q_2^2 e^{\lambda y} \right)^{\gamma_s} \int_0^\infty du_1 u_1^{2\gamma_s} J_1(u_1) \int_{a u_1}^\infty du_2 u_2^{2-2\gamma_s} J_1(u_2) + F \frac{1}{Q_2} \int_0^\infty du_1 J_1(u_1) \int_{b u_1}^{a u_1} du_2 u_2^2 J_1(u_2) + \{Q_1 \leftrightarrow Q_2\},
\]

where we set \( a = (Q_2/Q_1) \exp(\lambda y/2) \) and \( b = Q_2/Q_1 \). In what follows we shall show that the only non-zero term in the r.h.s. of Eq. (4.3) is the first one, and moreover this term shows geometric scaling.

The fact that the second term in (4.3) vanishes could have been anticipated, since this term arises from the saturation piece in Eq. (1.1), which in turn depends only on one of the two variables \( r_1 \) and \( r_2 \). Hence, if we return to the unintegrated version of the cross-section, Eq. (2.3), it becomes obvious that this saturation piece goes away by the successive action of the two Laplacians \( \nabla_{r_1}^2 \nabla_{r_2}^2 \). It is a little bit more difficult to see the corresponding cancelation in (4.3). Performing the integration over \( u_2 \) we obtain for this second term under consideration

\[
F \left( \frac{1}{Q_2} \int_0^\infty du_1 u_1^2 J_1(u_1) \left[ a^2 J_2(a u_1) - b^2 J_2(b u_1) \right] \right).
\]

To evaluate this integral, let us choose \( n = 1 \) in the completeness formula.

\(^9\) Given our choice that \( Q_1 \gg Q_2 \), one could anticipate that the dominant contribution comes from the region \( r_1 \ll r_2 \). For completeness, we shall nevertheless consider the region \( r_1 > r_2 \) too.
\[ \int_0^\infty du u J_n(u) J_n(au) = \delta(a - 1), \quad (4.5) \]

then differentiate this identity with respect to \( a \) and use the fact that \( J'_1(x) = J_1(x)/x - J_2(x) \). We thus obtain

\[ \int_0^\infty du u^2 J_1(u) J_2(au) = \delta(a - 1) - \delta'(a - 1), \quad (4.6) \]

which provides the result for the integrals in Eq. (4.4). Namely, since we are interested in momenta such that \( Q_1^2 > Q_2^2 \exp(\lambda y) \), we see that \( a < 1 \) and \( b < 1 \), hence both terms in Eq. (4.4) vanish, as anticipated.

Let us now turn to the calculation of the first term in (4.3). Putting aside the prefactor, the remaining double integration, let us call it \( h(a) \), can be simplified by differentiating with respect to \( a \). We have

\[ h'(a) = -a^{2-2\gamma_s} \int_0^\infty du u^3 J_1(u) J_1(au). \quad (4.7) \]

The above integral can be again recognized as the derivative of a known integral: by differentiating (4.6) with respect to \( a \) and using \( J'_2(x) = J_1(x) - 2J_2(x)/x \), we find

\[ \int_0^\infty du u^3 J_1(u) J_1(au) = -\delta'(a - 1) - \delta''(a - 1). \quad (4.8) \]

It is now straightforward to obtain \( h(a) \) by integrating over \( a \). Integrating by parts to get rid of the derivatives acting on the \( \delta \)–function and using \( h(\infty) = 0 \), we obtain

\[ h(a) = 4\gamma_s (1 - \gamma_s) \Theta(1 - a). \quad (4.9) \]

Recalling that \( a < 1 \) in the kinematical region of interest, we see that the step function is equal to 1. That is, the first (double) integral in the r.h.s. of (4.3) is independent of \( a \) so long as \( a < 1 \).

By a similar argument, it is now easy to see that the term obtained by exchanging \( Q_1 \leftrightarrow Q_2 \) (that is, the term coming from the region \( r_1 > r_2 \)) is equal to zero: indeed, the corresponding contribution would be proportional to \( \Theta(1 - 1/a) \).

Now, whereas the above, exact, results are of course attributed to the precise form of the interpolation chosen in Eq. (4.1) for the dipole–dipole cross–section, it is clear that these results will approximately hold for any smooth interpolation (in between the shown limiting expressions) provided we impose the strong inequality \( Q_1^2 \gg Q_2^2 \exp(\lambda y) \). Therefore, putting everything together, we arrive at (recall that \( y = Y - Y_0 \))

\[ \frac{d\sigma^{pp \rightarrow J \times J}}{dx_1 dx_2} \simeq F_{\text{eff}} \frac{1}{Q_2^2} \left( \frac{Q_2^2 \exp(\lambda y)}{Q_1^2} \right)^{\gamma_s} \quad \text{for} \quad Q_1^2 \gg Q_2^2 \exp(\lambda y), \quad (4.10) \]
valid up to an overall, numerical, factor which is not under control. The effective parton distribution $F_{\text{eff}}$ of Eq. (2.9) has been generated because the saturation momenta are, strictly speaking, different for different types of dipoles, as explained below Eq. (3.5), and these differences can be absorbed in the normalization of the parton distributions, as shown in Eq. (2.9).

Apart the prefactor $F_{\text{eff}}$, the above expression can be obtained from the weak–scattering piece in Eq. (4.1) via the replacements $r \to 1/Q_1$ and $R \to 1/Q_2$. The dimensionfull factor $1/Q_2^2$ plays the role of the ‘area of the larger (target) dipole’, whereas the dimensionless ratio $Q_2^2 e^{\lambda y}/Q_1^2$ is recognized as the scaling variable $\tau \equiv Q_2^2(Y)/Q_1^2$, with $Q_s(Y)$ the saturation momentum of this ‘target dipole’. Accordingly, Eq. (4.10) exhibits geometric scaling, as anticipated: it depends upon the resolution $Q_1^2$ of the ‘small dipole’ and the rapidity $y$ only via the scaling variable $\tau$. The validity region for this behavior should be clear too from the previous manipulations: since the effect of the Fourier transforms is to select $r_1 \sim 1/Q_1$ and $r_2 \sim 1/Q_2$ (at least, so long as the external momenta $Q_1$ and $Q_2$ are well separated from each other), it is quite clear that the geometric scaling in the cross–section for Mueller–Navelet jets at fixed coupling holds in the same kinematical window as for the dipole–dipole scattering amplitude, that is,

$$1 \ll \ln \frac{Q_1^2}{Q_2^2(Q_2, y)} \ll \sqrt{4D s\alpha Y}. \quad (4.11)$$

By the same argument, we also expect geometric scaling behavior in the ‘unintegrated’ cross–section (2.3), for transverse momenta $k_{1\perp}$ and $k_{2\perp}$ replacing $Q_1$ and $Q_2$ in the above formulæ.

5 Running coupling

With a running coupling, the theoretical situation is less firmly under control, since the NLO formalism is not yet fully developed for the unitarity corrections. (The running–coupling version of the BK equation became available only recently [58–60].) Still, for the specific problem at hand, we need only some limited information about the NLO effects, that we believe to be reliably described by the present formalism. Indeed, to study geometric scaling in the Mueller–Navelet jets, we need the dipole–dipole cross–section in the weak scattering regime, where the BFKL approximation is expected to apply, and for which the NLO formalism is by now well established [11, 12] (including the approach towards saturation [3, 7]). As mentioned in the Introduction, the use of the BFKL approximation is better justified in the running–coupling scenario than in the fixed–coupling one, since the effects of gluon–number fluctuations (which tend to invalidate this approximation) are drastically suppressed by the running of the coupling [52]. In fact, as we shall shortly discover, the main obstruction to our calculation does not come from our limited knowledge of the NLO perturbative formalism, but rather from a drawback of perturbation theory itself, ultimately associated with the running of the coupling. To identify this difficulty, we start by assuming perturbation theory to apply.

As before, we choose hard cut–off momenta, $Q_1, Q_2 \gg \Lambda_{\text{QCD}}$, with moreover $Q_1 \gg Q_2$. In the fixed–coupling case, this condition was enough to ensure that the relevant dipole
sizes are sufficiently small, $r_i \sim 1/Q_i \ll 1/\Lambda_{\text{QCD}}$, for perturbation theory to apply. With a running coupling, this strong correlation between $Q_i$ and $r_i$ is lost, as we shall see, but for the time being let us simply assume that the (effective) dipoles are perturbatively small. The typical situation is such that one dipole is much larger than the other, $R \gg r$, and we shall assume, once again, that the larger dipole (the ‘target’) evolves towards high gluon density and blackness on transverse sizes $r \lesssim R$ much faster than it expands in impact parameter space. Under these assumptions, the dipole–dipole cross–section is again given by Eq. (3.2), but with the scattering amplitude $T_{dd}(r, R, Y)$ now computed for a running coupling. This calculation has been described somewhere else [3, 7, 20], and here we present only the relevant results.

(i) The saturation momentum is now estimated as

$$Q_s^2(R, Y) = \Lambda_{\text{QCD}}^2 \exp \left[ 2c(Y - Y_0) + \rho_R^2 \right], \quad \rho_R \equiv \ln \frac{1}{R^2 \Lambda_{\text{QCD}}^2}$$

(5.1)

where the QCD scale $\Lambda_{\text{QCD}}$ has been introduced via the running of the coupling, for which we used the one–loop result $\alpha_s(Q^2) = b_0 / \ln(Q^2/\Lambda_{\text{QCD}}^2)$. We have denoted $c \equiv b_0 N_c \lambda_s / \pi$, with the saturation exponent $\lambda_s \approx 4.88$ (the same as in Eq. (3.5)). As before, $Y_0$ denotes the rapidity evolution necessary to build a saturation scale $Q_s$ equal to $1/R$.

More precisely, Eq. (5.1) has been obtained by interpolating between the asymptotic behavior at large $Y$, where the calculation is better under control, and the expected behavior at low $Y$, where one should recover the fixed–coupling result (3.6). Indeed, for $2cy \ll \rho_R^2$, with $y \equiv Y - Y_0$, Eq. (5.1) reduces to $Q_s^2 \simeq (1/R^2) \exp(\lambda_s \bar{\alpha}y)$, with $\bar{\alpha}$ evaluated at $Q^2 = 1/R^2$. On the other hand, for energies high enough such that $2cy \gg \rho_R^2$, the saturation momentum loses any dependence upon the target size [54]:

$$Q_s^2(R, Y) \simeq Q_c^2(Y) \equiv \Lambda_{\text{QCD}}^2 e^{2cy} \quad \text{when} \quad 2cy \gg \rho_R^2.$$  

(5.2)

(ii) Consider the scattering amplitude $T_{dd}(r, R, Y)$ for a projectile dipole with size $r$. Within the relatively wide region at

$$1 \ll \ln \frac{1}{r^2 Q_s^2} \lesssim (2cy + \rho_R^2)^{1/3},$$

(5.3)

where the scattering is weak ($T_{dd} \ll 1$), this amplitude is a universal function of the ‘scaling’ variable $\tau \equiv r^2 Q_s^2(R, Y)$ and of $Y$, whose structure is quite similar to that at fixed coupling, cf. Eq. (3.4): namely, it involves the power $\tau^{\gamma_s}$ times a function of $\tau$ and $Y$ which violates geometric scaling via a diffusive pattern.

(iii) Within the more restricted window at

$$1 \ll \ln \frac{1}{r^2 Q_s^2} \lesssim (2cy + \rho_R^2)^{1/6},$$

(5.4)

the scaling violations can be neglected, and the amplitude takes the same scaling form as in the fixed–coupling case, that is,

\[10^{\text{See Eq. (3.27) in Ref. [20] for the specific function in the case of a running coupling.}}

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Two important observations about the above results are here in order: (i) The running of the coupling considerably slows down the evolution, as clear from the fact that both the saturation momentum (5.1) and the width of the scaling region (5.4) rise much slower with $Y$ than at fixed coupling. (ii) In the high–energy regime where Eq. (5.2) applies, the dipole–dipole amplitude is insensitive to the target dipole size within the whole validity range for the BFKL approximation (cf. Eq. (5.3)). This second observation has dramatic consequences for the Mueller–Navelet process, to which we now return.

For simplicity, we keep only the scaling piece (5.5) in the dipole–dipole amplitude (as we shall later argue, our main results are independent of this approximation), and thus write (compare to Eq. (4.1) at fixed coupling)

$$\sigma_{dd}(r_1, r_2, Y) \approx 2\pi R^2 \begin{cases} \left( \frac{r_2^2}{r_s^2} \right)^{\gamma_s} & \text{for } r < r_s, \\ 1 & \text{for } r > r_s. \end{cases}$$

(5.6)

where $r_s \equiv 1/Q_s(R, Y)$ encodes the whole dependence upon both the rapidity and the target size. Below, we shall also use $r_c \equiv 1/Q_c(Y)$, cf. Eq. (5.2).

Let us calculate the contribution of the weak–scattering piece in Eq. (5.6) (the first line there) to the Mueller–Navelet dijet cross section. (As in the fixed–coupling case, one can show that the respective contribution coming from the saturation piece is equal to zero.) We focus on the case $r_2 > r_1$, which we expect to be the most interesting one, in view of our condition that $Q_1 \gg Q_2$. Hence, from now on, $r = r_1$ and $R = r_2$, and we have

$$\frac{d\sigma_{pp\rightarrow JXJ}}{d\chi_1 d\chi_2} = F \frac{r_c}{r_1 \leq r_c} \int dr_1 dr_2 Q_1 J_1(Q_1 r_1) Q_2 J_1(Q_2 r_2) r_2^2 \left( \frac{r_1^2}{r_s^2} \right)^{\gamma_s}.$$  

(5.7)

It is convenient to distinguish between two regions of integration over $r_2$: (a) $r_2 > r_c$ and (b) $r_2 < r_c$. Given that $r_2^2 \propto \exp\left(-\sqrt{2c y}\right)$ is rapidly decreasing with $y$, it is quite clear (and easy to check) that the parametrically dominant contribution at large $y$ is the one coming from region (a). In this region, we typically have $r_2 \gg r_c$, so that the saturation scale $r_s \approx r_c$ is independent of the size $r_2$ of the target. (Indeed, the condition $r_2 \gg r_c$ is the same as $\rho_R^2 \ll 2c y$, cf. Eq. (5.2), applied to $R = r_2$.) Then Eq. (5.7) becomes

$$\frac{d\sigma_{pp\rightarrow JXJ}}{d\chi_1 d\chi_2} \approx F \frac{1}{r_c^{2\gamma_s}} \int_0^{r_c} dr_1 Q_1 J_1(Q_1 r_1) r_1^{2\gamma_s} \int_{r_c}^{\infty} dr_2 Q_2 J_1(Q_2 r_2) r_2^2 + \ldots,$$  

(5.8)

with the dots standing for the contributions coming from dipoles of size $r_2 \lesssim r_c$. We do our standard change of variables $u_1 = Q_1 r_1$ and $u_2 = Q_2 r_2$ to obtain

$$\frac{d\sigma_{pp\rightarrow JXJ}}{d\chi_1 d\chi_2} \approx F \frac{1}{Q_c^{2\gamma_s}} \left( \frac{Q_c^2}{Q_1^2} \right)^{\gamma_s} \int_0^{Q_1/Q_c} du_1 u_1^{2\gamma_s} J_1(u_1) \int_{Q_2/Q_c}^{\infty} du_2 u_2^{2\gamma_s} J_1(u_2) + \ldots,$$  

(5.9)
Naturally, the regime that we are interested in is \( Q_1 \gg Q_c(Y) \gg Q_2 \). In this regime, \( Q_c(Y) \) is essentially the same as the saturation scale for a target dipole with size \( 1/Q_2 \). Therefore, the prefactor appearing outside the integrations in Eq. (5.9) has the right structure to exhibit geometric scaling. There are, of course, additional functional dependencies in the limits of the remaining integrations, but at a first sight it seems that these dependencies are rather weak and therefore negligible: Since \( Q_1 \gg Q_c \), we can extend the upper limit of the \( u_1 \) integration to \( \infty \); then, by also making use of (2.7), one sees that this integration yields a positive number of \( \mathcal{O}(1) \). Similarly, since \( Q_2 \ll Q_c \), we can extend the lower limit of the \( u_2 \) integration to 0. But the problem that we are facing then is that, according to Eq. (2.7), the result of the ensuing integration over \( u_2 \) is exactly zero.

Thus, by making approximations aiming at preserving the dominant contributions to the Mueller–Navelet cross–section at running coupling, we have found a result which is identically zero. Of course, a non–zero result could be instead obtained by keeping the formerly discarded (since formally subleading) contributions. But would that result be correct indeed? We do not believe so since, first, that result would be generated by physically implausible corners of the phase–space and, second, it would be strongly sensitive to the fine details of our approximations — it could even oscillate between positive and negative values, an unacceptable feature for a cross–section.

This invites us to critically reexamine the above calculation, in order to better understand why we obtained this vanishing result. At a mathematical level, this is related to the oscillatory behavior of the Bessel functions. The relevant integral, that is \( \int_0^\infty \, du_2 \, u_2^2 \, J_1(u_2) \), is quite peculiar: for large values of the variable \( u_2 \), the oscillations of \( J_1(u_2) \) are strongly amplified by the factor \( u_2^2 \). Hence, if the overall result turns out to be zero, it is because of exact cancelations between large contributions with opposite signs. Then, clearly, the result of this integration is controlled by the behavior of its integrand at large \( u_2 \gg 1 \). If one sharply cuts off the integral at some value \( u_2^{\text{max}} \gg 1 \), then the result is an oscillating function of \( u_2^{\text{max}} \), which can vary from large positive values to large negative ones.

We see that, even though we have tried to set up a perturbative calculation, the final, vanishing, result that we have obtained is in fact controlled by large dipoles — in fact, arbitrarily large —, for which the perturbative approach is not justified anymore. At this point, one may wonder about the real significance of this result: does it signal a true failure of perturbation theory, or is this merely an artifact of our specific approximations? The following argument, based on a comparison with the situation at fixed coupling, suggests that the first answer should be the correct one.

The corresponding integral in the fixed–coupling case, namely \( \int_0^\infty \, du_2 \, u_2^{2-2\gamma_s} \, J_1(u_2) \) (see Eq. (1.3)), was convergent and dominated by \( u_2 \sim 1 \) (i.e., \( r_2 \sim 1/Q_2 \)) because, in the dipole–dipole cross–section (3.2), the rapid growth \( \propto r_2^2 \) of the target area was partially compensated by the decay \( \propto (1/r_2)^{2\gamma_s} \) of the scattering amplitude \( T_{dd} \) at very large \( r_2 \) (cf. Eqs. (3.4)–(3.5) with \( r \to r_1 \) and \( R \to r_2 \)). In that case, the dipole–dipole amplitude at a given impact parameter can be made arbitrarily small be increasing the overall size of the target. This is a very peculiar feature of the leading–order formalism, ultimately related to its conformal invariance: the target size is the only dimensionfull parameter in the problem, so the gluon density in the target, as measured by the (local) saturation momentum (3.5), must be proportional to an appropriate power of \( 1/r_2 \).

The situation changes at NLO, where the running of the coupling introduces an addi-
tional mass scale in the problem, the ‘soft’ scale \( \Lambda_{\text{QCD}} \). Then, for sufficiently high energy, the local saturation momentum becomes insensitive to the overall target size, as manifest on Eqs. (5.1)–(5.2). This result is quite natural: the local gluon distribution is determined by the physics on the distance scale \( 1/Q_s(Y) \), which decreases with increasing \( Y \), and is in any case much smaller than the target size \( r_2 \). Similarly, the amplitude \( T_{3d}(r_1, r_2, Y) \) for a small dipole \( (r_1 < 1/Q_s \ll r_2) \) and for large enough \( Y \) is insensitive to \( r_2 \), as emphasized after Eq. (5.5). This property holds within the validity range (5.3) of the BFKL approximation, and not only within the narrower window (5.4) for geometric scaling. Accordingly, the above conclusion about Eq. (5.9) is more general than the geometric–scaling ansatz in Eq. (5.6): within the whole range in which the BFKL approximation is expected to be valid, the (perturbative) dipole–dipole cross–section for two dipoles with very disparate sizes is expected to grow like the area \( \sim r_2^2 \) of the larger dipole. The growth is so fast that the cross–section for Mueller–Navelet jets is ineluctably dominated by the largest possible ‘target’ dipoles, whose treatment goes beyond the scope of perturbation theory.

Of course, in QCD dipoles cannot become arbitrarily large, because of confinement. In what follows we propose a heuristic modification of the previous calculation which limits the dipole sizes to a value \( \sim 1/\Lambda_{\text{QCD}} \) and thus yields a finite result for the Mueller–Navelet dijet cross–section. Clearly, the precise value of this result will be sensitive to our specific prescription for introducing confinement, and we shall try to motivate this prescription on physical grounds. But before we proceed, it is important to emphasize that this prescription will affect only the \( u_2 \)–integration in Eq. (5.9), but not also the functional dependencies upon \( Q_1 \) and \( Y \), as encoded in the prefactor there. In other terms, whatever prescription we choose to eliminate the large target dipoles, this will not change the geometric scaling behavior of the cross–section, as determined by the prefactor.

To motivate our prescription for introducing confinement, let us recall that the ‘dipoles’ under consideration are effective dipoles, built with one gluon at \( x \) in the amplitude and another gluon at \( y \) in the complex conjugate amplitude, and such that \( R = |x - y| \). Hence, the maximal possible value for \( R \) is the same as the maximal dispersion between the positions in impact parameter space at which a gluon can be produced in the proton wavefunction. This distance is of the order of the proton size \( \sim 1/\Lambda_{\text{QCD}} \). Moreover, larger impact parameters \( \gtrsim 1/\Lambda_{\text{QCD}} \) lie in the tail of the proton wavefunction, where the gluon distribution must decay exponentially, so as it happens for any quantum–mechanical system with a mass gap. Thus we conjecture that the probability to produce an effective dipole with large size \( R \gtrsim 1/\Lambda_{\text{QCD}} \) should fall exponentially, according to \( \exp(-R\Lambda_{\text{QCD}}) \).

We can implement this prescription in our calculation via the following replacement for the area factor in Eq. (5.6) (with \( \Lambda \equiv \Lambda_{\text{QCD}} \) from now on):

\[
R^2 \to R^2 \exp(-R\Lambda). \tag{5.10}
\]

Then it is straightforward to see that the last integration in (5.9) becomes

\[
\int_{Q_2/Q_c}^\infty du_2 u_2^2 J_1(u_2) \exp(-u_2\Lambda/Q_2) \simeq \frac{3\Lambda}{Q_2} \frac{1}{(1 + \Lambda^2/Q_2^2)^{5/2}} - \frac{1}{8} \frac{Q_4^4}{Q_c^4} \simeq \frac{3\Lambda}{Q_2}, \tag{5.11}
\]

where the approximate equality holds when \( \Lambda/Q_2 \gg Q_4^4/Q_c^4(Y) \) (with \( \Lambda/Q_2 \ll 1 \), though),
a situation which is eventually reached with increasing energy at fixed $Q_2$. Under these assumptions, the high–energy behavior of the Mueller–Navelet cross–section reads

$$
\frac{d\sigma^{pp\rightarrow JXJ}}{dx_1 dx_2} \simeq F_{\text{eff}} \frac{\Lambda}{Q_2^2} \left( \frac{Q_c^2(Y)}{Q_1^2} \right)^{\gamma_s},
$$

(5.12)
a formula which should be valid for sufficiently large $Y$ and for $Q_1 \gg Q_c \gg Q_2 \gg \Lambda$. The overall normalization factor $\Lambda/Q_2^2$ in this equation depends, of course, upon our specific model for introducing confinement, but the scaling behavior w.r.t. $\tau \equiv Q_c^2(Y)/Q_1^2$ does not. We therefore consider this scaling behavior as a robust prediction of our analysis. More precisely, this behavior should hold within the window

$$
1 \ll \ln \frac{Q_1^2}{Q_c^2(Y)} \lesssim (2cy)^{1/6},
$$

(5.13)
which is obtained after replacing $r \equiv r_1 \rightarrow 1/Q_1$ and $Q_s \rightarrow Q_c$ in Eq. (5.4). (Such a replacement is legitimate, since the integration over $r_1$ in Eq. (5.8) is indeed dominated by $r_1 \sim 1/Q_1$.)

6 Conclusion and perspectives

Our main result in this paper is that, under specific kinematical conditions — namely, for a sufficiently large rapidity gap $Y$ and for a sufficiently pronounced asymmetry between the transverse momenta of the two jets — the partonic core of the cross–section for Mueller–Navelet jets should exhibit geometric scaling.

In the leading–order formalism, where the coupling is fixed, this result is a rather straightforward consequence of the factorization (2.3) for the dijet cross–section together with known results about the dipole–dipole scattering within the framework of the BK (or Balitsky–JIMWLK) equation. Although the factorization (2.3) has not been established here in full rigor, this should not affect the generality of our analysis, which employed Eq. (2.3) only in the weak scattering regime where the $k_T$–factorization is firmly established (at LO). However, as explained in the Introduction, this fixed–coupling analysis is a bit academic since, first, within a complete LO calculation its conclusions would be affected by particle–number fluctuations (at least, for sufficiently high energy) and, second, in real QCD the coupling is running anyway.

With a running coupling, on the other hand, our analysis lacks rigor at several points — it neglects other NLO corrections except for the running of the coupling and, especially, it turns out to transcend the framework of perturbation theory — and hence should be viewed as merely exploratory. Yet, we believe that our main conclusion (concerning the emergence of geometric scaling) is rather robust even in that context, because it is mainly based on the analysis of the harder jet, for which perturbation theory appears to be reliable. As explained in Sect. 5, the failure of perturbation theory refers merely to the softer jet, and, more precisely, to the connection between the transverse momentum $Q_2$ of that jet and the size $r_2$ of the associated, effective, dipole: even when $Q_2$ is relatively hard, $Q_2^2 \gg \Lambda_{\text{QCD}}^2$, the cross–section for dijet production is still dominated by very large ‘target’
dipoles, with \( r_2 \sim 1/\Lambda_{\text{QCD}} \), because the perturbative dipole–dipole cross-section grows very fast with \( r_2 \). This growth is faster with a running coupling since the corresponding saturation momentum is independent of the target size.

This failure of perturbation theory is perhaps a bit surprising, as this is not the usual failure associated with the BFKL ‘infrared diffusion’ in the presence of a running coupling: as expected, gluon saturation eliminates the IR diffusion and sets the argument of the coupling to a relatively hard scale \( \sim Q_s(Y) \), so that the Landau pole in the coupling is not an issue any more. In spite of that, an infrared problem remains, as alluded to above, and its identification can be viewed as our second main result.

Since based on asymptotic expansions, our results can be trusted, strictly speaking, only for sufficiently high energies, so their applicability to LHC may be questionable. This being said, and in view of the rather successful phenomenology at HERA, it would be nevertheless interesting to look for traces of this geometric scaling behavior in the forthcoming data at LHC. For instance, while keeping fixed the kinematics \((Q_2, \eta_2)\) of the softest jet, one could vary the the transverse momentum cutoff \( Q_1 \) (with \( Q_1 > Q_2 \)) and the pseudo–rapidity \( \eta_1 \) of the hardest jet, in such a way to preserve a constant value for the respective longitudinal momentum fraction \( x_1 \sim (Q_1 / \sqrt{s}) e^{\eta_1} \). In this way, \( Q_1 \) and the rapidity gap \( Y \) would be simultaneously changing, and then one could check whether the ensuing variation in the measured dijet cross–section follows indeed a geometric scaling pattern, cf. Eq. (5.12), at least approximately. Of course, such a scaling should be partially violated by the \( Q_1 \)–dependence of the parton distributions within \( F_{\text{eff}} \), but this dependence should be rather weak and, in any case, controllable within perturbation theory.

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