Robust pricing and hedging of double no-touch options

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Abstract Double no-touch options are contracts which pay out a fixed amount provided an underlying asset remains within a given interval. In this work, we establish model-independent bounds on the price of these options based on the prices of more liquidly traded options (call and digital call options). Key steps are the construction of super- and sub-hedging strategies to establish the bounds, and the use of Skorokhod embedding techniques to show the bounds are the best possible.

In addition to establishing rigorous bounds, we consider carefully what is meant by arbitrage in settings where there is no a priori known probability measure. We discuss two natural extensions of the notion of arbitrage, weak arbitrage and weak free lunch with vanishing risk, which are needed to establish equivalence between the lack of arbitrage and the existence of a market model.

Keywords Double no-touch option · Robust pricing and hedging · Skorokhod embedding problem · Weak arbitrage · Weak free lunch with vanishing risk · Model-independent arbitrage

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JEL Classification C60 · G13
1 Introduction

It is classical in the mathematical finance literature to begin by assuming the existence of a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) on which an underlying price process is defined. In this work, we do not assume any given model. Instead we are given the observed prices of vanilla options and our aim is to derive information concerning the arbitrage-free price of an exotic option, while assuming as little as possible about the underlying asset’s behaviour.

More precisely, we consider a double no-touch option, which is a contract that pays a fixed amount at maturity (which we assume always to be 1 unit), provided the asset remains (strictly) between two fixed barriers. These options appear most commonly in FX markets, and (in different contexts to the one we consider) have been considered recently by e.g. [6] and [27]. Our starting point is the following question: Suppose we know the call prices on a fixed underlying at a given maturity date, what can we deduce about the price of a double no-touch option, written on the same underlying and settled at the same maturity as the call options?

Such robust pricing questions have important practical implications. The discrepancy between the lower and upper bounds is sometimes used as a crude measure of model risk, see Cont [11], and may influence the bid–ask spreads on a given product set by the traders. Further, we also derive robust hedging strategies which enforce the price bounds and which can be seen as a viable alternative to model-specific hedging in the presence of model ambiguity.

The hedging strategies we derive are semi-static; they involve taking an initial position in co-maturing European calls or puts and further forward transactions upon hitting the barriers. We note that there is ample literature on static hedging of barrier options; see [26] and the references therein. However, this stream of literature asks different questions and uses linear programming and optimisation methods. The approach we take to the problem is very different and is based on the approach which was initially established in [22], and later used in different settings in [4, 16, 17]. The basic principle is to use constructions from the theory of Skorokhod embeddings to identify extremal processes, which may then give intuition to identify optimal super- and sub-hedges. A model based on the extremal solution to the Skorokhod embedding allows one to deduce that the price bounds implied by the hedges are tight. In the setting considered here, as we shall show shortly, the relevant constructions already exist in the Skorokhod embedding literature (due to [15, 23, 30]). However, the hedging strategies have not been explicitly derived. One of the goals of this paper is to open up these results to the finance community.

A second aspect of our discussion concerns a careful consideration of the technical framework in which our results are valid: We let the ‘market’ determine a set of asset prices, and we assume that these prices satisfy standard linearity assumptions. In particular, our starting point is a linear operator on a set of functions from a path space (our asset histories) to the real line (the payoff of the option). Since there is no specified probability measure, a suitable notion of arbitrage has to be introduced; the simplest no-arbitrage concept here is that any non-negative payoff must be assigned a non-negative price, which we call absence of a ‘model-independent arbitrage’. However, as noted in [18], this definition is insufficient to exclude some undesirable cases.