A sufficient condition for $k$-contraction
of the series connection of two systems

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Abstract

The flow of contracting systems contracts 1-dimensional parallelotopes, i.e., line segments, at an exponential rate. One reason for the usefulness of contracting systems is that many interconnections of contracting sub-systems yield an overall contracting system.

A generalization of contracting systems is $k$-contracting systems, where $k \in \{1, \ldots, n\}$. The flow of such systems contracts the volume of $k$-dimensional parallelotopes at an exponential rate, and in particular they reduce to contracting systems when $k = 1$. It was shown by Muldowney and Li that time-invariant 2-contracting systems have a well-ordered asymptotic behaviour: all bounded trajectories converge to the set of equilibria.

Here, we derive a sufficient condition guaranteeing that the system obtained from the series interconnection of two sub-systems is $k$-contracting. This is based on a new formula for the $k$th multiplicative and additive compounds of a block-diagonal matrix, which may be of independent interest. As an application, we find conditions guaranteeing that 2-contracting systems with an exponentially decaying input retain the well-ordered behaviour of time-invariant 2-contracting systems.

Index Terms

Contracting systems, compound matrices, series connections, Thomas’ cyclically symmetric attractor.

I. INTRODUCTION

Contraction theory plays an important role in systems and control theory, with applications in robotics, systems biology, neuroscience, the design of observers and controllers, and more. This is due to several
reasons. First, contraction implies a highly-ordered asymptotic behavior: any two trajectories approach one another at an exponential rate [1]. Therefore, if an equilibrium point exists it is unique and globally exponentially stable. In a time-varying and $T$-periodic contracting system all trajectories converge to a unique and globally asymptotically stable limit cycle with period $T$, and the convergence occurs at an exponential rate [1], [2], [3]. Thus, the system *entrains* to the periodic excitation modeled by the $T$-periodic vector field. In fact, nonlinear contracting systems have a well-defined frequency response, as shown in [4] in the context of convergent systems [5]. Second, there exist sufficient conditions for contraction based on matrix measures [1], [6]. These conditions are easy to verify, and just like Lyapunov’s second theorem, do not require solving the differential equations. However, there is a growing awareness of the importance of finding the “correct metric” for establishing contraction [1], [7], [8], [9]. Third, various interconnections of contracting sub-systems, including parallel, series, and some feedback connections, yield an overall contracting system [1], [10].

Contraction theory is an active area of research, with recent contributions including both theoretical results and applications. Recent generalizations of contraction theory typically guarantee weaker conclusions on the asymptotic behaviour, but allow studying a larger class of systems. Some extensions include various notions of “weak contraction” (see, e.g. [11], [12]), contraction of piecewise-smooth dynamical systems [13], and the introduction of $\alpha$-contracting systems [14], with $\alpha \geq 1$ real, which is motivated in part by the seminal works of Douady and Oesterlé [15], and Leonov and his colleagues (see the recent monograph by Kuznetsov and Reitmann [16]) who developed powerful tools to bound the Hausdorff dimension of complex attractors in chaotic dynamical systems.

One generalization, which is the focus of this work, is called $k$-contraction, and was suggested in [17] (see also the note [18]). Let $n$ denote the dimension of the dynamical system, and fix $k \in \{1, \ldots, n\}$. The flow of a $k$-contracting system contracts the volume of $k$-dimensional paralleolopes at an exponential rate. In particular, for $k = 1$ these are just standard contracting systems. Ref. [17] also provides simple to check sufficient conditions for $k$-contraction based on the $k$th additive compound of the Jacobian of the vector field.

The notion of $k$-contraction is motivated in part by the seminal work of Muldowney and his colleagues [19], [20], on systems that, using the new terminology, are 2-contracting. One of the main results in this context is that every bounded solution of a time-invariant 2-contracting system converges to an equilibrium point. This is strictly weaker than the case of 1-contracting systems, as the equilibrium point
is not necessarily unique. Also, 2-contracting systems with a time-varying $T$-periodic vector fields do not necessarily entrain. The theory of 2-contracting systems proved useful in the analysis of epidemiological models (see, e.g. [21]). Indeed, such models typically have at least two equilibrium points, corresponding to the disease-free and the endemic steady-states, and thus are not 1-contracting with respect to (w.r.t.) any norm.

Since various interconnections of contracting sub-systems yield an overall contracting system, it is natural to ask if the same holds for $k$-contracting systems as well [18]. Ref. [22] considered the series interconnection of $k$-contracting systems with $k \in \{1, 2\}$, and showed that such interconnections are in general not 2-contracting. Furthermore, the asymptotic analysis of such systems is more delicate than in the case of 1-contracting systems because the well-ordered behaviour of 2-contracting systems only holds in the time-invariant case, while connecting two sub-systems implies that at least one sub-system has an input from the other sub-system and is thus time-varying.

Here, we derive a new sufficient condition guaranteeing that the series interconnection of two sub-systems is $k$-contracting for some $k \in \{1, \ldots, n\}$. Roughly speaking, this condition requires that both sub-systems are $k$-contracting and, in addition, they must satisfy “additive $j$-contraction conditions”, with $j < k$. We show using an example that in general this condition cannot be improved.

Our main result is based on a new closed-form expression for the $k$th multiplicative compound and the $k$th additive compound of a block-diagonal matrix. We believe that this result may find other applications as well. Matrix compounds have recently found many applications in systems and control theory, see the tutorial paper [23].

The remainder of this paper is organized as follows. The next section reviews known definitions and results that are used later on. Section III includes the main theoretical results. Section IV describes an application of these results, and the final section concludes.

We use standard notation. Small [capital] letters denote column vectors [matrices]. $I_n$ is the $n \times n$ identity matrix, and we drop the subscript if the dimension is clear from the context. For a matrix $A$, $A^T$ is the transpose of $A$. For a square matrix $A$, $\det(A)$ is the determinant of $A$, and $\text{trace}(A)$ is the trace of $A$.

II. Preliminaries

The sufficient condition for $k$-contraction in [17] is based on the $k$th additive compound of the Jacobian of the vector field. To make this paper more accessible, we briefly review matrix compounds, their
geometrical interpretation, and their role in the analysis of ODEs. For more details, see also [19]. For more recent applications of these compounds in systems and control theory, see [24], [25], [14], [26] and the tutorial paper [23].

A. Multiplicative compound

Let $C \in \mathbb{R}^{n \times m}$. Fix $k \in \{1, \ldots, \min\{n, m\}\}$. Recall that a $k$ minor of $C$ is the determinant of some $k \times k$ submatrix of $C$. The $k$th multiplicative compound of $C$, denoted $C^{(k)}$, is the \( \binom{n}{k} \times \binom{m}{k} \) matrix that contains all the $k \times k$ minors of $C$ in lexicographic order [19]. For example, for $n = m = 3$ and $k = 2$, $C^{(2)}$ is the $3 \times 3$ matrix:

$$C^{(2)} = \begin{bmatrix}
\det(\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}) & \det(\begin{pmatrix} c_{11} & c_{13} \\ c_{21} & c_{23} \end{pmatrix}) & \det(\begin{pmatrix} c_{12} & c_{13} \\ c_{22} & c_{23} \end{pmatrix}) \\
\det(\begin{pmatrix} c_{11} & c_{12} \\ c_{31} & c_{32} \end{pmatrix}) & \det(\begin{pmatrix} c_{11} & c_{13} \\ c_{31} & c_{33} \end{pmatrix}) & \det(\begin{pmatrix} c_{12} & c_{13} \\ c_{32} & c_{33} \end{pmatrix}) \\
\det(\begin{pmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}) & \det(\begin{pmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{pmatrix}) & \det(\begin{pmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{pmatrix})
\end{bmatrix}.$$  

This definition implies that $(C^{(k)})^T = (CT)^{(k)}$, $C^{(1)} = C$, and if $n = m$ then $C^{(n)} = \det(C)$. Also, $(I_n)^{(k)} = I_r$, where $r := \binom{n}{k}$.

The Cauchy–Binet formula [27, Chapter 0], asserts that for any $B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times p}$ and any $k \in \{1, \ldots, \min\{n, m, p\}\}$, we have

$$(BC)^{(k)} = B^{(k)}C^{(k)}. \quad (1)$$

This justifies the term multiplicative compound. In particular, (1) implies that if $n = m = p$ then $\det(BC) = \det(B)\det(C)$, and that if $A \in \mathbb{R}^{n \times n}$ is non-singular then

$$(I_n)^{(k)} = (AA^{-1})^{(k)} = A^{(k)}(A^{-1})^{(k)},$$

so $(A^{(k)})^{-1} = (A^{-1})^{(k)}$.

One reason for the importance of the multiplicative compound is that it has a useful geometric interpretation. Fix $k \in \{1, \ldots, n\}$ and vectors $x^1, \ldots, x^k \in \mathbb{R}^n$. The parallelotope generated by these vectors (and the zero vertex) is

$$P(x^1, \ldots, x^k) := \sum_{i=1}^{k} r_i x^i : r_i \in [0, 1] \}.$$  

Note that this implies that $0 \in \mathbb{R}^n$ is a vertex of $P$. This can always be assured by a simple translation. Collect the vectors in the matrix $X := \begin{bmatrix} x^1 & \ldots & x^k \end{bmatrix} \in \mathbb{R}^{n \times k}$. The Gram matrix associated with $x^1, \ldots, x^k$
is the $k \times k$ symmetric matrix:

$$G(x^1, \ldots, x^k) : = X^T X$$

$$= \begin{bmatrix}
(x^1)^T x^1 & (x^1)^T x^2 & \ldots & (x^1)^T x^k \\
(x^2)^T x^1 & (x^2)^T x^2 & \ldots & (x^2)^T x^k \\
\vdots & \vdots & \ddots & \vdots \\
(x^k)^T x^1 & (x^k)^T x^2 & \ldots & (x^k)^T x^k 
\end{bmatrix}. \quad (2)$$

It follows from (2) that for any $r \in \mathbb{R}^k$ we have $|\sum_{i=1}^k r_i x^i|^2 = r^T G(x^1, \ldots, x^k) r$, so $G(x^1, \ldots, x^k)$ is non-negative definite, and it is positive-definite if and only if $x^1, \ldots, x^k$ are linearly independent. The volume of the parallelotope $P(x^1, \ldots, x^k)$ is

$$\text{vol}(P(x^1, \ldots, x^k)) = \sqrt{\det(G(x^1, \ldots, x^k))}, \quad (3)$$

(see, e.g. [28, Chapter IX]). For example, suppose that $x^i = e^i$, $i = 1, \ldots, k$, where $e^i$ is the $i$th canonical vector in $\mathbb{R}^n$. Then (3) gives

$$\left(\text{vol}(P(e^1, \ldots, e^k))\right)^2 = \det(G(e^1, \ldots, e^k))$$

$$= \det\left(\begin{bmatrix}
(e^1)^T e^1 & (e^1)^T e^2 & \ldots & (e^1)^T e^k \\
(e^2)^T e^1 & (e^2)^T e^2 & \ldots & (e^2)^T e^k \\
\vdots & \vdots & \ddots & \vdots \\
(e^k)^T e^1 & (e^k)^T e^2 & \ldots & (e^k)^T e^k 
\end{bmatrix}\right)$$

$$= 1.$$

The volume of $P(x^1, \ldots, x^k)$ can be expressed using the $k$th multiplicative compound of $X$. To see this, note that combining (2) and the Cauchy-Binet formula yields

$$\det(G(x^1, \ldots, x^k)) = \det(X^T X)$$

$$= (X^T X)^{(k)}$$

$$= (X^T)^{(k)} X^{(k)}$$

$$= (X^{(k)})^T X^{(k)}.$$
$$A^{(3)} = \begin{bmatrix} a_{11}a_{22}a_{33} & a_{11}a_{22}a_{34} & a_{11}(a_{23}a_{34} - a_{24}a_{33}) & a_{14}a_{22}a_{33} - a_{12}a_{24}a_{33} - a_{13}a_{22}a_{34} + a_{12}a_{23}a_{34} \\ 0 & a_{11}a_{22}a_{44} & a_{11}a_{23}a_{44} & a_{12}a_{23}a_{44} - a_{13}a_{22}a_{44} \\ 0 & 0 & a_{11}a_{33}a_{44} & a_{12}a_{33}a_{44} \\ 0 & 0 & 0 & a_{22}a_{33}a_{44} \end{bmatrix}$$

Fig. 1: The matrix $A^{(3)}$.

Since $X \in \mathbb{R}^{n \times k}$, $X^{(k)}$ is an \binom{n}{k} column vector, so $\det(G(x^1, \ldots, x^k)) = |X^{(k)}|^2$. Thus,

$$\text{vol}(P(x^1, \ldots, x^k)) = |X^{(k)}|. \quad (4)$$

We emphasize again that here $X^{(k)}$ is a column vector. The norm $| \cdot |$ here is the $L_2$ norm, but using the equivalence of norms in $\mathbb{R}^n$ other norms can be used when studying asymptotic properties like convergence to zero of the volume of a parallelotope when the vectors $x^1, \ldots, x^k$ evolve in time.

In the special case where $k = n$, $X$ is a square matrix, and (4) reduces to the well-known formula

$$\text{vol}(P(x^1, \ldots, x^k)) = | \det(X) | = | \det([x^1 \ldots x^n]) |.$$ 

The multiplicative compound has a useful spectral property. Let $A \in \mathbb{R}^{n \times n}$ with eigenvalues $\lambda_i$, $i \in \{1, \ldots, n\}$. The eigenvalues of $A^{(k)}$ are all the products $\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

The next example demonstrates this.

**Example 1.** Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$ 

A calculation gives that $A^{(3)}$ is the matrix depicted in Fig. 1 and thus the eigenvalues of $A^{(3)}$ are the products of three eigenvalues of $A$. □
Suppose now that the vectors $x^1, \ldots, x^k$ vary with time via $\dot{x}^i(t) = Ax^i(t)$, and let

$$X(t) := \begin{bmatrix} x^1(t) & \cdots & x^k(t) \end{bmatrix} = \exp(At) \begin{bmatrix} x^1(0) & \cdots & x^k(0) \end{bmatrix}.$$  

Then the norm of $X^{(k)}(t) = (\exp(At))^{(k)} \begin{bmatrix} x^1(0) & \cdots & x^k(0) \end{bmatrix}^{(k)}$ is the volume of the parallelotope generated by these vectors at time $t$. This naturally leads to the following question: how does $(\exp(At))^{(k)}$ evolve in time? To address this, we require the $k$th additive compound of $A$.

**B. Additive compound**

Let $A \in \mathbb{R}^{n \times n}$. For $k \in \{1, \ldots, n\}$, the $k$th additive compound of $A$ is the $\binom{n}{k} \times \binom{n}{k}$ matrix defined by

$$A^{[k]} := \frac{d}{dt}(\exp(At))^{(k)}|_{t=0}.$$  

This implies that $(I + tA)^{(k)} = I + tA^{[k]} + o(t)$. In particular, $A^{[1]} = A$, and $A^{[n]} = \text{trace}(A)$.

It can be shown using the definition of the additive compound and the properties of the multiplicative compound that if $A, B \in \mathbb{R}^{n \times n}$ then

$$(A + B)^{[k]} = A^{[k]} + B^{[k]}.$$  

This justifies the term additive compound.

If $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $A^{[k]}$ are all the sums $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ [19].

A useful relation between the multiplicative and additive compounds under the matrix exponential is [19]

$$\exp(A)^{(k)} = \exp(A^{[k]}).$$  

(5)

**C. Compounds and ODEs**

In the context of dynamical systems, the importance of these compounds is due to following fact. If $\Phi : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ is the solution of the matrix differential equation

$$\frac{d}{dt} \Phi(t) = A(t)\Phi(t), \quad \Phi(0) = I,$$  

then...
where $t \to A(t)$ is continuous, then

$$
\frac{d}{dt} (\Phi(t))^{(k)} = (A(t))^{[k]}(\Phi(t))^{(k)}.
$$

(6)

In other words, $(\Phi(t))^{(k)}$ also evolves according to a linear dynamics, with the matrix $(A(t))^{[k]}$. Roughly speaking, $A^{[k]}$ determines the evolution of $k$-dimensional parallelotopes under the LTI dynamics $\dot{x} = Ax$ [29].

A time-varying nonlinear system is called $k$-contracting if its variational equation along any solution (which is an LTV) contracts $k$-dimensional parallelograms; see [17] for the exact definition. For our purposes, it is enough to review a sufficient condition for $k$-contraction. Recall that a vector norm $| \cdot | : \mathbb{R}^n \to \mathbb{R}_+$ induces a matrix norm $||A|| := \max_{|x|=1} |Ax|$, and a matrix measure $\mu(A) := \lim_{\varepsilon \to 0^+} (||I + \varepsilon A|| - 1)/\varepsilon$. If $\mu((A(t))^{[k]}) \leq -\eta < 0$ all $t \geq 0$ then applying Coppel’s inequality [30] to (6) yields $||(\Phi(t))^{(k)}|| \leq \exp(-\eta t)|||\Phi(0))^{(k)}||$ for all $t \geq 0$. This leads to the following.

**Proposition 1.** [17] Consider the time-varying nonlinear system $\dot{x}(t) = f(t, x(t))$, with $f$ a $C^1$ mapping, and suppose that its trajectories evolve on a convex set $\Omega \subseteq \mathbb{R}^n$. Let $J(t, x) := \frac{\partial}{\partial x} f(t, x)$ denote the Jacobian of $f$ with respect to $x$. If

$$
\mu \left( (J(t, z))^{[k]} \right) \leq -\eta < 0, \text{ for all } t \geq 0, z \in \Omega.
$$

(7)

then the system is $k$-contracting.

Note that for $k = 1$ this reduces to the standard infinitesimal contraction condition [6], as $J^{[1]} = J$. Thus, a 1-contracting system is a contracting system.

Note also that condition (7) is robust in the sense that if it holds for $f$ then it also holds for small perturbations of $f$ (but perhaps with a different value $\eta$).

For $p \in \{1, 2, \infty\}$, let $\mu_p$ denote the matrix measure induced by the $L_p$ vector norm $| \cdot |_p$. An important advantage of contraction theory is that there exist easy to verify sufficient conditions for contraction in terms of matrix measures. It is useful to provide similar conditions for $k$-contraction. These can be easily derived using the following result. Let $Q(k, n)$ denote the set of all increasing sequences of $k$ numbers from $\{1, \ldots, n\}$ ordered lexicographically. For example,

$$
Q(3, 4) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.
$$
Proposition 2. (see, e.g. [19]) Let $A \in \mathbb{R}^{n \times n}$. Fix $k \in \{1, \ldots, n\}$. Then the $L_1$, $L_2$, and $L_\infty$ matrix measures of $A[k]$ are:

$$
\mu_1(A[k]) = \max_{\alpha \in \mathcal{Q}(k,n)} \left( \sum_{p=1}^{k} a_{\alpha p, \alpha p} + \sum_{j \notin \alpha} (|a_{j,\alpha_1}| + \cdots + |a_{j,\alpha_k}|) \right),
$$

$$
\mu_2(A[k]) = \sum_{i=1}^{k} \lambda_i \left( \frac{A + A^T}{2} \right),
$$

$$
\mu_\infty(A[k]) = \max_{\alpha \in \mathcal{Q}(k,n)} \left( \sum_{p=1}^{k} a_{\alpha p, \alpha p} + \sum_{j \notin \alpha} (|a_{\alpha_1,j}| + \cdots + |a_{\alpha_k,j}|) \right),
$$

where for a symmetric matrix $S \in \mathbb{R}^{n \times n}$, $\lambda_1(S) \geq \cdots \geq \lambda_n(S)$, are the eigenvalues of $S$.

D. Contraction with a hierarchic norm

To analyze contraction for block-diagonal matrices, we use an interesting result of Ström [31] (see also [10]). For the sake of completeness, we briefly review this result. Given $x \in \mathbb{R}^s$, decompose it as

$$
x = \begin{bmatrix} x^1 \\ \vdots \\ x^r \end{bmatrix},
$$

where $x^i \in \mathbb{R}^{s_i}$, and $\sum_{i=1}^{r} s_i = s$. Let $| \cdot |_i$ denote a norm on $\mathbb{R}^{s_i}$, and let $| \cdot |_0$ denote a monotonic norm on $\mathbb{R}^r$ (see [32] for the definition and properties of monotonic norms). Define a norm $| \cdot | : \mathbb{R}^s \to \mathbb{R}_+$ by

$$
|x| := \begin{bmatrix} |x^1|_1 \\ \vdots \\ |x^r|_0 \end{bmatrix}.
$$

This may be interpreted as a “hierarchic norm”, as we first decompose $x$ into sub-vectors and then combine the norms of all these sub-vectors to form $|x|$.

Given $B \in \mathbb{R}^{s \times s}$, partition it into $r \times r$ blocks $B^{ij} \in \mathbb{R}^{s_i \times s_j}$, and define their matrix norms by

$$
||B^{ij}||_{ij} := \sup_{z \in \mathbb{R}^s \setminus \{0\}} \frac{|B^{ij} z|_i}{|z|_j}.
$$

Theorem 1. [31, Thm. 9] Let $\mu$ denote the matrix measure induced by the norm $| \cdot |$ defined in (10).
Let \( \mu_i \) denote the matrix measure induced by \( | \cdot | \), \( i = 0, \ldots, r \). Define \( C \in \mathbb{R}^{r \times r} \) by

\[
c_{ij} := \begin{cases} 
\mu_i(B^{ii}), & i = j, \\
||B^{ij}||_{ij}, & i \neq j.
\end{cases}
\] (12)

Then

\[
\max_i \mu_i(B^{ii}) \leq \mu(B) \leq \mu_0(C).
\] (13)

This provides in particular an upper bound on \( \mu(B) \), for the “big” matrix \( B \in \mathbb{R}^{s \times s} \), using \( \mu_0(C) \), with the smaller matrix \( C \in \mathbb{R}^{r \times r} \). If \( \mu_0(C) \leq -\eta < 0 \) then (13) implies that \( \dot{x} = Bx \) is contracting.

Note that if \( B \) is block-diagonal, that is, \( B^{ij} = 0 \) for all \( i \neq j \), then \( C \) is diagonal and the lower and upper bounds in (13) are equal, since for any matrix measure \( \mu \) induced by a monotonic norm and any square diagonal matrix \( E \),

\[
\mu(E) = \max_i \lambda_i,
\] (14)

where \( \lambda_i \) is the \( i \)th eigenvalue of \( E \). Therefore, for block-diagonal matrices Thm. 1 yields

\[
\mu(B) = \max_i \mu_i(B^{ii}).
\] (15)

The next section describes our main results.

III. MAIN RESULTS

Consider the series interconnection of two time-varying nonlinear sub-systems

\[
\dot{x}^1 = f^1(t, x^1),
\]

\[
\dot{x}^2 = f^2(t, x^2, x^1),
\] (16)

with \( x^1 \in \mathbb{R}^n \) and \( x^2 \in \mathbb{R}^m \). We assume that the trajectories of this system evolve in a convex state-space \( \Omega^1 \times \Omega^2 \subseteq \mathbb{R}^n \times \mathbb{R}^m \). We also assume that \( f^1, f^2 \) are \( C^1 \). The Jacobian of (16) is

\[
J(t, x) = \begin{bmatrix}
J^{11}(t, x^1) & 0 \\
J^{21}(t, x) & J^{22}(t, x)
\end{bmatrix},
\] (17)

where \( x := \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \) and \( J^{ij} := \frac{\partial}{\partial x^j} f^i \). From here on, we always assume that the initial time is \( t = 0 \).
A. Sufficient condition for \( k \)-contraction of a series interconnection

We can now state our first main result. For a square matrix \( M \), we define the 0th compounds by \( M^{(0)} := 1 \) and \( M^{[0]} := 0 \).

**Theorem 2.** Assume that \( J^{21}(t, x) \) is uniformly bounded for all \( x \in \Omega^1 \times \Omega^2 \) and \( t \geq 0 \). Fix \( k \in \{1, \ldots, n + m\} \), and let \( i_1 := \max\{0, k - n\} \), \( i_2 := \min\{m, k\} \). If there exist matrix measures \( \mu_i \) induced by (possibly different) \( L_p \) norms and constants \( \eta_i > 0 \) such that

\[
\mu_i((J^{11}(t, x^1))^{[k-i]}) + \mu_i((J^{22}(t, x))^{[i]}) \leq -\eta_i < 0.
\]  

(18)

for all \( i \in \{i_1, \ldots, i_2\} \), \( x \in \Omega^1 \times \Omega^2 \) and \( t \geq 0 \), then (16) is \( k \)-contracting.

**Remark 1.** For \( k = 1 \), \( i_1 = 0 \) and \( i_2 = 1 \), so (18) becomes

\[
\mu((J^{11}(t, x^1))^{[1]}) \leq -\eta_1,
\]
\[
\mu((J^{22}(t, x))^{[1]}) \leq -\eta_2,
\]

(where for simplicity we use the same matrix measure). Thus, the sufficient condition for contraction is that both sub-systems are contracting. This is a well-known result (see for example [1] or [9]).

For \( k = 2 \) (and assuming that \( n, m \geq 2 \)), \( i_1 = 0 \) and \( i_2 = 2 \), so (18) becomes

\[
\mu((J^{11}(t, x^1))^{[2]}) \leq -\eta_1,
\]
\[
\mu((J^{11}(t, x^1))^{[1]}) + \mu((J^{22}(t, x))^{[1]}) \leq -\eta_2,
\]
\[
\mu((J^{22}(t, x))^{[2]}) \leq -\eta_3.
\]  

(19)

Thus, the sufficient condition for 2-contraction of the series interconnection (16) is that both sub-systems are 2-contracting, and also the “additive 1-contraction condition” (19). Note that this condition implies that for any \((t, x)\) at least one of the sub-systems is 1-contracting.

A fundamental issue in systems theory is the complexity vs. stability problem: “Does an increase of complexity lead to an improvement of system stability, or is it the other way around?” (see, e.g. [33], [34]). Thm. 2 shows that \( k \)-contraction of the series connection requires more than \( k \)-contraction of each sub-system. In this respect, an increase in the complexity of the system can only destroy the \( k \)-contraction property, for any \( k > 1 \).
Example 2. Consider the series connection

\[ \begin{align*}
\dot{x}_1 &= Ax_1, \\
\dot{x}_2 &= Bx_1 + Cx_2,
\end{align*} \tag{20} \]

with \( x_1, x_2 \in \mathbb{R}^2, \) \( A = \text{diag}(1, -2), \) \( B = 0, \) and \( C = \text{diag}(\zeta_1, \zeta_2), \) with \( \zeta_1 \geq \zeta_2 \) and \( \zeta_1 + \zeta_2 < 0. \) Both sub-systems are 2-contracting, as \( \text{trace}(A) = -1 \) and \( \text{trace}(C) = \zeta_1 + \zeta_2. \) Let \( e^i \) denote the \( i \)-th canonical vector in \( \mathbb{R}^4, \) and consider the 2D time-varying paralleloptope

\[ P(x(t, e^1), x(t, e^3)) = \{ r_1 x(t, e^1) + r_3 x(t, e^3) : r_i \in [0, 1] \}. \]

Since \( x(t, e^1) = \exp(t)e^1 \) and \( x(t, e^3) = \exp(\zeta_1 t)e^3, \) the volume of \( P(x(t, e^1), x(t, e^3)) \) evolves like \( \exp((1 + \zeta_1)t). \) Thus, a necessary condition for 2-contraction is that \( 1 + \zeta_1 < 0 \) and this is exactly the additive 1-contraction condition. This shows that the sufficient condition in Theorem 2 cannot be improved in the general case.

Remark 2. It is important to note however that the additive 1-contraction condition in (19) does not require that either of the two sub-systems is contracting on the entire state-space. To illustrate this, consider the series interconnection

\[ \begin{align*}
\dot{x}_1 &= -\frac{1}{2} x_1^2 - x_1, \\
\dot{x}_2 &= x_2 x_1,
\end{align*} \tag{21} \]

with \( \Omega = \mathbb{R}_+^2. \) The Jacobian of this system is \( J(x) = \begin{bmatrix} -x_1 - 1 & 0 \\ x_2 & x_1 \end{bmatrix}, \) and \( J^{[2]} = \text{trace}(J) = -1, \) so the system is 2-contracting. Yet, the dynamics of \( x_1 \) is not contracting on the entire state-space, and the same holds for the dynamics of \( x_2. \) In this case, the additive 1-contraction condition is

\[ \mu(-x_1 - 1) + \mu(x_1) \leq -\eta < 0, \]

and for \( \eta = 1 \) this indeed holds on the entire state-space.

Note that for the special case \( k = 2 \) and the matrix measure \( \mu_2 \) induced by the \( L_2 \) norm, Theorem 2 reduces to [18, Thm. 3].

The remainder of this section is devoted to the proof of Theorem 2. This is based on an auxiliary result that describes the \( k \)th compounds of a block-diagonal matrix.
B. Compounds of a block-diagonal matrix

For $A_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, s$, let

$$\text{diag}(A_1, \ldots, A_s) := \begin{bmatrix} A_1 & 0 & 0 & \ldots & 0 \\ 0 & A_2 & 0 & \ldots & 0 \\ \vdots \\ 0 & 0 & \ldots & 0 & A_s \end{bmatrix}.$$ 

We use $\otimes$ to denote the Kronecker product. The Kronecker sum [35, Ch. 4] of $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$ is

$$A \oplus B := A \otimes I_m + I_n \otimes B. \quad (22)$$

We can now state our second main result. This relates the multiplicative [additive] compound matrix of block-diagonal matrices to Kronecker products [sums] of the compound matrices of the individual blocks.

**Theorem 3.** Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$. Define $C := \text{diag}(A, B)$. Pick $k \in \{1, \ldots, n + m\}$, and let $r := \binom{n + m}{k}$. Let $i_1 := \max\{0, k - n\}$, and $i_2 := \min\{m, k\}$. There exists a permutation matrix $P \in \mathbb{R}^{r \times r}$ such that

$$C^{(k)} = P \left( \text{diag}_{i \in \{i_1, \ldots, i_2\}} (A^{(k-i)} \otimes B^{(i)}) \right) P^{-1}, \quad (23)$$

and

$$C^{[k]} = P \left( \text{diag}_{i \in \{i_1, \ldots, i_2\}} (A^{[k-i]} \oplus B^{[i]}) \right) P^{-1}, \quad (24)$$

where for a square matrix $M$, we define $M^{(0)} := 1$ and $M^{[0]} := 0$.

**Remark 3.** It is useful to verify that the dimensions of the matrices in (23) agree. Clearly, $C^{(k)}$ has dimensions $\binom{n + m}{k} \times \binom{n + m}{k}$. The matrix on the right-hand side in (23) has dimensions $\ell \times \ell$, where

$$\ell := \sum_{i=\max\{0, k-n\}}^{\min\{m,k\}} \binom{n}{k-i} \binom{m}{i}$$

$$= \sum_{i=0}^{k} \binom{n}{k-i} \binom{m}{i}$$

$$= \binom{n + m}{k},$$

where
where the second equality holds since any potentially added term is zero, and the third equality is Vandermonde’s identity.

**Example 3.** Note that for \( k = 1 \) we have \( i_1 = 0 \) and \( i_2 = 1 \) so both (23) and (24) give \( C = P \text{diag}(A, B)P^{-1} \) which indeed holds for \( P = I \), whereas for \( k = n + m \) we get

\[
C^{(n+m)} = P \left( A^{(n)} \otimes B^{(m)} \right) P^{-1} = P \text{det}(A) \text{det}(B)P^{-1} = \text{det}(A) \text{det}(B),
\]

as the only \( 1 \times 1 \) permutation matrix is the scalar one, and similarly

\[
C^{[n+m]} = P \left( A^{[n]} \oplus B^{[m]} \right) P^{-1} = \text{trace}(A) + \text{trace}(B).
\]

\[\blacksquare\]

**Proof of Theorem 3:** We begin by proving (23). Let \( Q(k, n, m) \) denote the set of increasing sequences of \( k \) integers in \( \{1, \ldots, n+m\} \). (We write \( Q(k, n, m) \) rather than just \( Q(k, n+m) \) because we define below a special lexicographic ordering of the elements in \( Q(k, n, m) \) that depends on the parameter \( n \)).
The cardinality of $Q(k, n, m)$ is $\binom{n+m}{k}$. For a sequence $\alpha = \{\alpha_1, \ldots, \alpha_k\} \in Q(k, n, m)$, let $s_\alpha$ denote the minimal $i \in \{1, \ldots, k\}$ such that $\alpha_i > n$, or $k + 1$ if no such $i$ exists. In other words, $s_\alpha < k + 1$ implies that $\alpha$ includes at least one index $\alpha_\ell > n$ and since $A \in \mathbb{R}^{n \times n}$, this index corresponds to a row in $C$ that is a row from $B$. Note that $i_i + 1 \leq s_\alpha \leq i_2 + 1$.

We define a “block-lexicographic ordering” by $\alpha \in Q(k, n, m)$ precedes $\beta \in Q(k, n, m)$ if

1) $s_\alpha > s_\beta$, or

2) $s_\alpha = s_\beta$, and $\alpha$ precedes $\beta$ in the standard lexicographic ordering.

For example, if $n = 3$, $m = 2$, and $k = 2$ then the ordering is

$$Q(2, 3, 2) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\},$$

whereas, for $n = 2$, $m = 3$, and $k = 2$ the ordering is

$$Q(2, 2, 3) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\},$$

which in this case is just the lexicographic ordering on $Q(2, 5)$.

Let $D^{(k)} := P^{-1} C^{(k)} P$, where $P \in \mathbb{R}^{r \times r}$ is the permutation matrix such that the entries of $D^{(k)}$ are those of $C^{(k)}$, but now organized in the block-lexicographic ordering. We first show that each entry of $D^{(k)}$ is the determinant of a block-triangular matrix, or equivalently that for any $\alpha, \beta \in Q(k, n, m)$, the submatrix $C[\alpha|\beta]$ is a block-triangular matrix. Fix $\alpha, \beta \in Q(k, n, m)$ such that $s_\alpha \geq s_\beta$. Decompose the $k \times k$ matrix $C[\alpha|\beta]$ as the block matrix

$$C[\alpha|\beta] = \begin{bmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{bmatrix},$$

where

$$C^{11} := C[\{\alpha_1, \ldots, \alpha_{s_\alpha-1}\} | \{\beta_1, \ldots, \beta_{s_\beta-1}\}],$$

$$C^{12} := C[\{\alpha_1, \ldots, \alpha_{s_\alpha-1}\} | \{\beta_{s_\beta}, \ldots, \beta_k\}].$$

Note that $C^{11} \in \mathbb{R}^{(s_\alpha-1) \times (s_\beta-1)}$, and $C^{12} \in \mathbb{R}^{(s_\alpha-1) \times (k-s_\beta+1)}$.

By definition, $s_\alpha$ is the minimal index $i$ such that $\alpha_i > n$, so the block $C^{11}$ is a submatrix of $A$
with $(s_\alpha - s_\beta)$ additional columns of zeros added to the right, and the block $C_{22}^{22}$ is a submatrix of $B$. Furthermore, every entry of the block $C_{12}^{12}$ is zero, since $\alpha_{s_\alpha - 1} \leq n$ and $\beta_{s_\alpha} > n$.

We consider three cases. First, assume that $s_\alpha = 1$, then $C[\alpha|\beta] = C_{22}^{22}$ and $C(\alpha|\beta) = \det(C_{22}^{22})$. Second, assume that $s_\alpha = k + 1$, then $C[\alpha|\beta] = C_{11}^{11}$ and $C(\alpha|\beta) = \det(C_{11}^{11})$. Finally, assume that $1 < s_\alpha < k + 1$. In this case, all blocks are defined, but since all entries of $C_{12}^{12}$ are zero, we have that $C(\alpha|\beta) = \det(C_{11}^{11}) \det(C_{22}^{22})$.

We now show that $D^{(k)}$ has a block-diagonal structure. This is equivalent to showing that $C(\alpha|\beta) = 0$ whenever $s_\alpha > s_\beta$ or $s_\alpha < s_\beta$. Consider the case $s_\alpha > s_\beta$, then the block $C_{11}^{11}$ has at least one column of zeros, so $C(\alpha|\beta) = 0$. The proof that $C(\alpha|\beta) = 0$ when $s_\alpha < s_\beta$ is similar.

Consider the diagonal blocks of $D^{(k)}$. The set

$$\{D(\alpha|\beta) : s_\alpha = s_\beta = \ell\}$$

is the $\ell$th diagonal block of $D^{(k)}$, and its dimensions are $\left(\binom{n}{\ell}, \binom{m}{k-\ell+1}\right) \times \left(\binom{n}{\ell}, \binom{m}{k-\ell+1}\right)$. Consider an entry of the $\ell$th diagonal block for some $\ell \in \{i_1 + 1, \ldots, i_2 + 1\}$, and fix $\alpha, \beta$ such that $s_\alpha = s_\beta = \ell$. Since $s_\alpha = s_\beta$, the block $C_{11}^{11}$ is a submatrix of $A$ and $C(\alpha|\beta) = \det(C_{11}^{11}) \det(C_{22}^{22})$ is a product of an $(\ell - 1)$ minor of $A$ and a $(k - \ell + 1)$ minor of $B$. Therefore, $C(\alpha|\beta)$ is an entry of $A^{(\ell-1)} \otimes B^{(k-\ell+1)}$.

We now show that the $\ell$th diagonal block of $D^{(k)}$ is just $A^{(\ell-1)} \otimes B^{(k-\ell+1)}$. Let $\alpha \in Q(k, n, m)$ such that $s_\alpha = \ell$. Then $\alpha = \{\alpha^A, \alpha^B\}$ where $\alpha^A = \{\alpha_1, \ldots, \alpha_{\ell-1}\}$ is a sequence of indices “pointing” to entries of $A$, and $\alpha^B = \{\alpha_\ell, \ldots, \alpha_k\}$ is a sequence of indices “pointing” to entries of $B$. The sequence $\beta = \{\beta^A, \beta^B\} \in Q(k, n, m)$ that immediately succeeds $\alpha$ according to the block-lexicographic ordering may be found as follows: if $\alpha^B$ is not the last element in $Q(k - \ell + 1, m)$ according to the lexicographic ordering, i.e. $\alpha^B \neq \{m - k + \ell - 1, \ldots, m\}$, choose $\beta^B$ to be the element following $\alpha^B$ in $Q(k - \ell + 1, m)$ and $\beta^A = \alpha^A$. Otherwise, choose $\beta^B$ to be the first element in $Q(k - \ell + 1, m)$ and $\beta^A$ to be the element succeeding $\alpha^A$ in $Q(\ell, n)$. Following this, since the cardinality of $Q(k - \ell + 1, m)$ is $\binom{m}{k-\ell+1}$, we conclude that if $\alpha$ is the $i$th element of $Q(k, n, m)$ in the block-lexicographic ordering such that $s_\alpha = \ell$, then $\alpha_A$ is element number $\left[i / \binom{m}{k-\ell+1}\right]$ in $Q(\ell, n)$ and $\alpha^B$ is element number $(i - 1)\binom{m}{k-\ell+1} + 1$ in $Q(k - \ell + 1, m)$, where $a \% b$ denotes the remainder of the integer division $a / b$. Recalling that

$$(A \otimes B)_{i,j} = A_{\left[i/m\right], \left[j/m\right]}(B)_{(i-1)\%m+1, (j-1)\%m+1}, \quad (26)$$

we conclude that the $\{i, j\}$ entry of the $\ell$th diagonal block of $D^{(k)}$ is equal to the $\{i, j\}$ entry of $A^{(\ell-1)} \otimes B^{(k-\ell+1)}$.\]
This completes the proof of (23).

To prove (24), let \( H_i := \exp(At)^{(k-i)} \otimes \exp(Bt)^{(i)} \) for \( i \in \{i_1, \ldots, i_2\} \). Then, by (5) and the properties of the matrix exponential,

\[
H_i = \exp(A^{k-i}t) \otimes \exp(B^it) = \exp((A^{k-i} \oplus B^it)t).
\]

(27)

Thus, \( \frac{d}{dt} H_i|_{t=0} = A^{k-i} \oplus B^it \), and combining this with (23) proves (24).

Theorem 3 allows to derive a lower and an upper bound on the matrix measure of the \( k \)th additive compound of a block-diagonal matrix.

**Corollary 1.** Let \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{m \times m} \) and define \( C := \text{diag}(A, B) \). Fix \( k \in \{1, \ldots, n + m\} \), and let \( r := \binom{n+m}{k} \). Let \( i_1 := \max\{0, k - n\} \) and \( i_2 := \min\{m, k\} \). Let \( \mu_i, i \in \{i_1, \ldots, i_2\} \), be matrix measures induced by (possibly different) \( L_p \) vector norms. Let \( |\cdot|_0 \) denote some monotonic norm, define a norm \( |\cdot| \) as in (10) and let \( \mu \) denote the matrix measure induced by the norm \( |\cdot| \). Let \( P \in \mathbb{R}^{r \times r} \) denote the permutation matrix in Thm. 3, and define the norm \( |x|_P := |P^{-1}x| \). Let \( \mu_P \) denote the matrix measure induced by \( |\cdot|_P \). Then

\[
\mu_P(C^{[k]}) = \max_{i \in \{i_1, \ldots, i_2\}} \{\mu_i(A^{k-i}) + \mu_i(B^i)\},
\]

and

\[
\min_{i \in \{i_1, \ldots, i_2\}} \{-\mu_i(-A^{k-i}) - \mu_i(-B^i)\} \leq \mu_P(C^{[k]}) \tag{29}
\]

**Proof:** By Thm. 3

\[
\mu_P(C^{[k]}) = \mu(\text{diag}_{i \in \{i_1, \ldots, i_2\}}(A^{k-i} \oplus B^i)) = \max_i \mu_i(A^{k-i} \oplus B^i),
\]

where we used Thm. 1 and Eq. (15) for the second equality. Since \( \mu_i \) are induced by \( L_p \) norms, we have that (see for example Prop. 5 and Thm. 6 in [14])

\[
\mu_i(A^{k-i} \oplus B^i) = \mu_i(A^{k-i}) + \mu_i(B^i),
\]

(30)

and this proves the equality in (28). The lower bound in (29) may be derived by noting that for any matrix measure and any matrix \( M \)

\[
-\mu(-M) \leq \mu(M).
\]

(31)
We may now prove Thm. 2.

**Proof of Theorem 2:** Recall that the Jacobian $J$ of the series connection has the form (17). For $\epsilon > 0$, let $T(\epsilon) = \begin{bmatrix} I_n & 0 \\ 0 & \epsilon I_m \end{bmatrix}$. Then

$$T(\epsilon)J^{-1}(\epsilon) = \begin{bmatrix} J^{11} & 0 \\ \epsilon J^{21} & J^{22} \end{bmatrix}.$$ 

Since $J^{21}(t, x)$ is uniformly bounded, we can make the term $\epsilon J^{21}$ arbitrarily small. Let $\tilde{J} := \begin{bmatrix} J^{11} & 0 \\ 0 & J^{22} \end{bmatrix}$. By Corollary 1 and (18), there exists a matrix measure $\mu$ such that

$$\mu(\tilde{J}) \leq -\min_i \eta_i < 0.$$ 

Let $| \cdot |$ denote the vector norm corresponding to $\mu$, and define a scaled vector norm by $|y|_{\epsilon} := |T(\epsilon)y|$. Let $\mu_{\epsilon}$ denote the matrix measure induced by this scaled norm. Then for all sufficiently small $\epsilon > 0$,

$$\mu_{\epsilon}(J) \leq -\min_i \eta_i/2 < 0$$

for all $x^1 \in \Omega^1, x^2 \in \Omega^2$, and $t \geq 0$, and this completes the proof.

Up to this point we considered only series interconnections. However, for the special case of $k$-contraction w.r.t. the $L_2$ norm it is possible to study also another form of interconnection, namely, a skew-symmetric feedback connection.

C. $k$-contraction of a skew-symmetric feedback interconnection

Consider the system

$$\begin{align*}
\dot{x}^1 &= f^1(t, x^1, x^2), \\
\dot{x}^2 &= f^2(t, x^1, x^2),
\end{align*}$$

(32)

where we assume that $x^1$ and $x^2$ evolve on convex sets $\Omega^1 \subseteq \mathbb{R}^n$ and $\Omega^2 \subseteq \mathbb{R}^m$ respectively, and for every initial condition $a \in \Omega^1 \times \Omega^2$ a unique solution exists. Let $J^{ij} := \frac{\partial}{\partial x^j} f^i$. In this section, we assume that there exists $c > 0$ such that

$$J^{21}(t, x^1, x^2) = -c(J^{12}(t, x^1, x^2))^T$$

(33)
for all \( x^i \in \Omega^i \) and all \( t \geq 0 \). Let \( \mu_2 \) denote the matrix measure induced by the \( L_2 \) norm.

**Proposition 3.** Fix \( k \in \{1, \ldots, n + m\} \), and let \( i_1 := \max\{0, k - n\} \) and \( i_2 := \min\{m, k\} \). Suppose that (33) holds. If

\[
\mu_2((J^{11}(t, x^1, x^2))^{[k-i]}) + \mu_2((J^{22}(t, x^1, x^2))^{[i]}) \leq -\eta < 0
\]

(34)

for all \( i \in \{i_1, \ldots, i_2\} \), \( x^1 \in \Omega^1 \), \( x^2 \in \Omega^2 \) and \( t \geq 0 \) then (32) is \( k \)-contracting.

**Proof:** The Jacobian of (32) is

\[
J(t, x^1, x^2) = \begin{bmatrix}
J^{11}(t, x^1, x^2) & J^{12}(t, x^1, x^2) \\
-c(J^{12}(t, x^1, x^2))^T & J^{22}(t, x^1, x^2)
\end{bmatrix}.
\]

Define \( T := \begin{bmatrix} \sqrt{c}I_n & 0 \\ 0 & I_m \end{bmatrix} \). Then

\[
TJT^{-1} = \begin{bmatrix}
J^{11} & \sqrt{c}J^{12} \\
-c(J^{12})^T & J^{22}
\end{bmatrix},
\]

and

\[
(TJT^{-1})^{[k]} = \begin{bmatrix} J^{11} & 0 \\ 0 & J^{22} \end{bmatrix}^{[k]} + \begin{bmatrix} 0 & \sqrt{c}J^{12} \\ -\sqrt{c}(J^{12})^T & 0 \end{bmatrix}^{[k]}.
\]

It follows from (8) that \( \mu_2((TJT^{-1})^{[k]}) = \mu_2 \left( \begin{bmatrix} J^{11} & 0 \\ 0 & J^{22} \end{bmatrix}^{[k]} \right) \), so effectively we have a block-diagonal Jacobian. Applying Corollary 1 completes the proof.

IV. AN APPLICATION

Desoer and Haneda [36] have shown that 1-contracting systems with an additive input satisfy what is now known as an input-to-state stability (ISS) property. The ISS property has become a fundamental topic in systems and control theory [37]. More recently, contracting (i.e. 1-contracting) systems with inputs have been studied in [9].

In this section, we consider a closely related question, namely, under what conditions a control system, with a time-varying exponential input, is equivalent to a time-invariant \( k \)-contracting system. This can be
studied using the results derived above after expressing the control system as the series interconnection of two sub-systems.

Consider the control system

\[ \dot{x} = f(x) + g(u), \]  
\[ (35) \]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( g : \mathbb{R} \rightarrow \mathbb{R}^n \) are \( C^1 \). Consider the input

\[ u(t) = \exp(\alpha t), \]  
\[ (36) \]

where \( \alpha \in \mathbb{R} \).

Write the closed-loop system as the time-invariant \((n + 1)\)-dimensional system

\[ \begin{align*}
\dot{x} &= f(x) + g(y), \\
\dot{y} &= \alpha y,
\end{align*} \]
\[ (37) \]

with \( y(0) = 1 \). The Jacobian of (37)

\[ J(x, y) = \begin{bmatrix}
\frac{\partial}{\partial x} f(x) & \frac{\partial}{\partial y} g(y) \\
0 & \alpha
\end{bmatrix}. \]

Applying Theorem 2 yields the following result.

**Corollary 2.** Suppose that the trajectories of (37) evolve on a convex set \( \Omega \subset \mathbb{R}^{n+1} \), and that \( \frac{\partial}{\partial u} g(u) \) is uniformly bounded. Fix \( k \in \{1, \ldots, n\} \). If there exist \( \eta > 0 \) and matrix measures \( \mu_i, \mu_j \) induced by (possibly different) \( L_p \) norms such that

\[ \begin{align*}
\mu_i(\frac{\partial}{\partial x} f(x))^{[k]} &\leq -\eta, \\
\mu_j(\frac{\partial}{\partial x} f(x))^{[k-1]} + \alpha &\leq -\eta,
\end{align*} \]
\[ (38) \]

for all \( x \in \Omega \) then the time-invariant system (37) is \( k \)-contracting.

Note that for \( k = 1 \), i.e. when we require 1-contractivity, this reduces to the requirement that the open-loop system is 1-contracting and that \( \alpha < 0 \), that is, an exponentially converging input. For \( k = 2 \), the first condition in (38) is that the open-loop system is 2-contracting. If \( \alpha \geq 0 \) then the second condition in (38) implies that the open-loop system is also 1-contracting. But, for \( \alpha < 0 \), the second condition may
hold even if the open-loop system is not 1-contracting.

Since the system (37) is time-invariant, specializing Corollary 2 to the case \( k = 2 \) gives the following result.

**Corollary 3.** Consider the control system described by (35) and (36), and suppose that its trajectories evolve on a convex set \( \Omega \subseteq \mathbb{R}^n \), and that \( \frac{\partial}{\partial u} g(u) \) is uniformly bounded. If there exist \( \eta > 0 \) and matrix measures \( \mu_i, \mu_j \) induced by (possibly different) \( L_p \) norms such that

\[
\mu_i((\frac{\partial}{\partial x} f(x))^{[2]}) \leq -\eta, \\
\mu_j(\frac{\partial}{\partial x} f(x)) + \alpha \leq -\eta,
\]

(39)

for all \( x \in \Omega \) then the system (37) is 2-contracting, and thus any bounded trajectory of the closed-loop system converges to the set of equilibria.

Corollary 3 is a kind of analogue of the result of Desoer and Haneda [36] for 1-contracting systems, as it provides conditions guaranteeing that a 2-contracting system with an exponential input retains the useful properties of time-invariant 2-contracting systems. The next example demonstrates how these theoretical results can be used to study the robustness of a nonlinear control system.

**Example 5.** Ref. [14] designed a feedback controller for a popular chaotic system, introduced by Thomas [38] (see also the recent review [39]), so that the closed-loop system is 2-contracting. Here we show how our results can be used to analyse the closed-loop system with an added perturbation modeled as a decaying exponential.

Thomas’ cyclically symmetric attractor is given by:

\[
\begin{align*}
\dot{x}_1 &= \sin(x_2) - dx_1, \\
\dot{x}_2 &= \sin(x_3) - dx_2, \\
\dot{x}_3 &= \sin(x_1) - dx_3,
\end{align*}
\]

(40)

where \( d > 0 \) is the dissipation constant. Note that the convex and compact set \( D := \{ x \in \mathbb{R}^3 : d|x|_\infty \leq 1 \} \) is an invariant set of the dynamics. For \( d > 1 \) the origin is the single equilibrium of (40). The system undergoes a series of bifurcations as \( d \) decreases, and becomes chaotic at \( d \approx 0.208186 \). Fig. 2 depicts the trajectories emanating from several initial conditions for \( d = 0.193186 \). It may be seen that they all...
converge to a strange attractor.

Let \( f(x) \) denote the vector field in (40). Ref. [14] considered the closed-loop system

\[
\dot{x} = f(x) + g(x),
\]

where \( g(x) \) is the linear partial-state controller

\[
g(x) = -\text{diag}(c, c, 0)x,
\]

with \( c > 0 \). The Jacobian of the closed-loop system is

\[
J(x) = \begin{bmatrix}
-d - c & \cos(x_2) & 0 \\
0 & -d - c & \cos(x_3) \\
\cos(x_1) & 0 & -d
\end{bmatrix},
\]

and

\[
J^{[2]}(x) = \begin{bmatrix}
-2d - 2c & \cos(x_3) & 0 \\
0 & -2d - c & \cos(x_2) \\
-\cos(x_1) & 0 & -2d - c
\end{bmatrix}.
\]

Thus,

\[
\mu_1(J) = \max\{-d - c + |\cos(x_1)|, -d - c + |\cos(x_2)|, -d + |\cos(x_3)|\}
\]

\[
\leq -d + 1,
\]

and

\[
\mu_1(J^{[2]}) = \max\{-2(d + c) + |\cos(x_1)|, -2d - c + |\cos(x_3)|, -2d - c + |\cos(x_2)|\}
\]

\[
\leq -2d - c + 1.
\]

We conclude that the closed-loop system is 1-contracting if \( d > 1 \), and 2-contracting if \( c > 1 - 2d \). Note that if the closed-loop system is 2-contracting then in particular it admits a well-ordered behaviour, so the controller “de-chaotifies” the original dynamics.
Assume that the closed-loop system is perturbed by an additive exponentially decaying noise, so that the dynamics becomes

\[ \dot{x} = f(x) + g(x) + b \exp(\alpha t), \tag{42} \]

with \( b \in \mathbb{R}^3 \) and \( \alpha < 0 \). Since the uncontrolled system is chaotic, the perturbation may have a strong effect on the dynamical behaviour.

By Corollary 3, the perturbed system will be equivalent to a time-invariant 2-contracting system if

\[ c > 1 - 2d, \quad \alpha < 1 - d. \tag{43} \]

Fig. 3 depicts trajectories of the system (42) with \( b = (1/8) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \), \( d = 0.193186 \), \( c = 1.1 - 2d \), and \( \alpha = -0.1 \), so (43) holds. It may be seen that every trajectory converges to an equilibrium point, as expected. Note that there are several equilibrium points, so the system is not 1-contracting w.r.t. any norm.

V. Discussion

A fundamental topic in systems theory is the analysis of an interconnected system based on the properties of its sub-systems and the interconnection network. In this context, an important advantage of contracting systems is that various interconnections of contracting sub-systems yield a contracting system.
We derived a new sufficient condition guaranteeing that the series connection of two sub-systems is $k$-contracting. This is based on a new formula for the $k$th compounds of a block-diagonal matrix. The latter result may find more applications. For example, recall that an LTV system is called $k$-positive if it maps the set of vectors with up to $k - 1$ sign variations to itself [40], [41], [42], [43]. In particular, 1-positive systems are just positive systems. The conditions for $k$-positivity are based on the structure of the $k$ compounds of the system, so Thm. 3 may perhaps be used to determine when a series connection of two sub-systems generates a $k$-positive system.

Thms. 2 and 3 suggest that it might not be possible to improve the contraction order nor the contraction rate using a series or skew-symmetric feedback interconnection. However, these ideas may still be useful in designing a controller which retains the $k$-contraction of a system, while adding other desirable properties. For example, it might be possible to stabilize a 2-contracting system by adding a controller which guarantees that all solutions are bounded while at the same time retaining 2-contraction.

Thm. 2 may also be used to study $k$-contraction in a system whose output is fed to an integrator. Indeed, given $\dot{x} = f(t, x)$, let $g(x)$ denote the output of the system, and consider the augmented system

$$
\dot{x} = f(t, x), \\
\dot{y} = g(x).
$$

Thm. 2 can be applied to study $k$-contraction of this serial interconnection.

As another topic for further research, recall that the theory of asymptotic autonomous systems considers
the system $\dot{x} = f(t, x)$, where $f(t, x)$ converges (in some technical sense) to the time-invariant vector field $g(x)$ as $t \to \infty$. The question is what is the relation between the solutions $x(t, t_0, x_0)$ and the solutions $y(t, y_0)$ of the time-invariant system $\dot{y} = g(y)$ (see e.g. [44], [45] and the references therein). This is related to the question of retaining $k$-contraction in a system perturbed by a decaying exponential, and it might be of interest to further explore this connection.

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