Insurance, Reinsurance and Dividend Payment

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Abstract

The aim of this paper is to introduce an insurance model allowing reinsurance and dividend payment. Our model deals with several homogeneous contracts and takes into account the legislation regarding the provisions to be justified by the insurance companies. This translates into some restriction on the (maximal) number of contracts the company is allowed to cover. We deal with a controlled jump process in which one has free choice of retention level and dividend amount. The value function is given as the maximized expected discounted dividends. We prove that this value function is a viscosity solution of some first-order Hamilton-Jacobi-Bellman variational inequality. Moreover, a uniqueness result is provided.

Key words: Stochastic control, jump diffusion, viscosity solution, insurance, reinsurance
49L20, 60H30

1 Preliminaries

A common problem of the insurance companies is to find a strategy allowing to satisfy the claims appearing either from the insured parties as consequence to specified peril or from the shareholders in terms of dividends. To reduce their risks and protect themselves from very large losses, the companies usually choose to pay some of the premiums to a third party. This process is called reinsurance, and it commits the third party (the reinsurance company) to cover a certain part of the claims. It is obvious that the insurance company controls the contracts to be reinsured as well as the dividends to be paid to the shareholders. These elements justify the framework of stochastic control.
This paper considers a utility function given as the maximized expected discounted dividends. In the literature, this approach has been first used by Jeanblanc, and Shiryaev (1995). In their model, the capital of an insurance company is described with the help of a standard Brownian motion and the dividend payment strategy is understood as control process. More precisely, they deal with the following model

\[ dX_t = \mu dt + \sigma dW_t - dZ_t, \]

where \( \mu \) and \( \sigma \) are arbitrary constants, \( W \) is a 1-dimensional standard Brownian motion and \( Z \) is an adapted, non decreasing, right-continuous process which represents the dividend payment strategy.

In Asmussen et al. (2000), a model concerning excess-of-loss reinsurance and dividend payment has been studied. They use diffusion and proportional reinsurance for their model. More exactly, they take as model for the capital of the insurance company the process given by the following equation

\[ dX_t = a_t (\mu dt + \sigma dW_t) - dZ_t, \]

where \( 0 \leq a_t \leq 1 \) stands for the retention level. In the case where the rate of dividend pay-out is unrestricted, they characterize the value function as the (classical) solution of some associated Hamilton-Jacobi-Bellman equation.

The same problem is studied by Mnif, Sulem (2005), but the claims are represented by a compound Poisson process. In their collective risk model, a retention level is an adapted process \( \alpha_t \) which specifies that, for a claim \( y \), the direct insurer covers \( y \wedge \alpha_t \), while the reinsurance company covers the remaining \( (y - \alpha_t)^+ \). They consider a single insurance contract and the reserve of the insurance company satisfies

\[ dX_t = p(\alpha_t) dt - \int_B (y \wedge \alpha_t) \mu(dtdy) - dL_t, \]

where \( \mu \) is the random measure associated to the compound Poisson process. In the above equation, \( p(\alpha_t) \) is the actual premium of the insurance company given the retention level \( \alpha \). The process \( L \) describes the pay-out of dividends for shareholders and it is an adapted, càdlàg process such that \( L_t - L_{t^-} \leq X_{t^-} \) for all \( t \geq 0 \). The value function is defined as the maximized expected discounted dividends until the ruin time \( \tau \),

\[ V(x) = \sup_{(a,L)} E \left[ \int_0^\tau e^{-rs} dL_s \right]; \]

where \( r \) is some positive discount factor. The authors proved that, under the assumption that the value function satisfies the dynamic programming principle, \( V \) is a viscosity solution of the associated Hamilton-Jacobi-Bellman variational inequality.
In the present paper we consider the problem of optimal reinsurance and dividend pay-out with several insurance contracts. We will prove that in the framework of the collective risk model, even if the invested initial capital is arbitrarily small, one can expect a gain which exceeds an a priori fixed positive constant. Indeed, this comes from the fact that, independently of its initial capital, the model allows the insurance company to sell one contract. However, as it is precised in section 2, in the case of insurance companies, the codes of law impose that, at any time, these companies should be able to justify enough resources to cover the obligations contracted towards their clients. This condition imposes an upper limit for the number of contracts the company can have. In the work we present here, several contracts are considered. We obtain a stochastic differential equation with respect to a random measure and introduce the utility for the shareholders as in Mnif, Sulem (2005) to be the maximized discounted flow of dividends. We prove that the value function is regular enough (enjoys the Lipschitz property) and satisfies the associated Hamilton-Jacobi-Bellman Variational Inequality in the viscosity sense. We also provide an uniqueness result for the viscosity solution in the class of continuous functions of at most linear growth. We emphasize that the limitation of the number of contracts which comes from the codes of insurance, allows us to get the Lipschitz property of the value function $V$. This property insures that an initial capital close to 0 will induce a zero-expected gain (unlike the collective risk model). Moreover, in this case, the dynamic programming principle follows in a standard way, while it was only assumed by the authors of [9].

The paper is organized as follows. In the first section we present a simple example showing the limits of the collective risk model. The second section is concerned with the insurance problem with several contracts. We introduce the model, the basic assumptions and prove some elementary properties of the value function $V$. In the third section, we show that the value function is a viscosity solution of the associated Hamilton-Jacobi-Bellman variational inequality. The fourth section provides a comparison result which allows to obtain the uniqueness of the viscosity solution for the given variational inequality. A numerical example is given in the last section.

2 The limits of the collective risk model. A counter example

We consider the following special case of the collective risk model introduced by Mnif, Sulem (2005). We assume that the claims are generated by a Poisson process $N$ with intensity 1 on a complete probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by the random measure associated to $N$, completed by the family of $P$-null sets. Given an $\mathcal{F}_t$—adapted process
\( \alpha_t \in [0, 1] \) (retention level), the premium rate is

\[
p(\alpha_t) = k_1 - k_2 + (1 + k_2) \alpha_t, \text{ for all } t \geq 0,\]

where \( 0 \leq k_1 \leq k_2 \) are proportional factors. Moreover, if \( L \) denotes the \( F_t \)-adapted process of cumulative dividends, then the reserve of the insurance company satisfies the equation

\[
X_{x,\alpha,L}^t = x + \int_0^t p(\alpha_s)ds - N_t - \int_0^t dL_u.
\]

The process \( L \) should be right-continuous, non-decreasing and such that \( L_{0-} = 0 \) and \( L_t - L_{t-} \leq X_{t,x,\alpha,L} \) for all \( t \geq 0 \). We introduce the first jump time for the Poisson process \( N \)

\[
\tau_1 = \inf \{ t \geq 0 : N_t = 1 \}.
\]

Obviously, \( \tau_1 \) is of exponential law with intensity 1, and, in particular,

\[
P(\tau_1 > 1) = e^{-1}.
\]

If we consider the strategy \((\alpha, L)\) given by

\[
\begin{cases}
\alpha \equiv 1, \\
L_t(\omega) = I_{\{\tau_1 > 1\}}(\omega) I_{\{t \geq 1\}}(t),
\end{cases}
\]

then \((\alpha, L)\) is admissible and the ruin time

\[
\tau_{x,\alpha,L} > 1 \text{ on } \{\tau_1 > 1\}.
\]

Indeed,

\[
X_{x,\alpha,L}^t = x + (1 + k_1) t - N_t - \int_0^t dL_u,
\]

and on \( \{\tau_1 > 1\} \) we have that

\[
X_{x,\alpha,L}^t = x + (1 + k_1) t,
\]

for all \( t < 1 \).

It follows that

\[
V(x) \geq E \left[ \int_0^{\tau_{x,\alpha,L}} e^{-rt} dL_t \right] \geq E \left[ e^{-r} I_{\{\tau_1 > 1\}} \right] \geq e^{-(r+1)},
\]

for all \( x > 0 \). Obviously

\[
V(0+) \geq e^{-(r+1)} > 0.
\]
Therefore, investing an arbitrarily small capital in the insurance company, we expect to gain more than $e^{-(r+1)}$. This contradicts theorem 3.3 in [9]. This problem is due mainly to the fact that, independent of the initial capital, the insurance company is allowed to hold one contract.

However, the insurance law requires that, at any moment, the companies should be able to cover any liabilities that have been incurred on insurance contracts as far as can be reasonably foreseen. Experience of similar claim development trends is of particular relevance. Usually, the solvency margin is computed with respect to both the premium rates and the average claim. According to the current Solvency I prudence regime, "the life insurance capital requirements are arrived at by multiplying a factor of 4% to the mathematical reserves of participating business (for unit-linked business the factor is reduced to 1%) plus a factor of 0.3% to the sum-at-risk" (CEA and Mercer Oliver Wyman, Solvency Assessment Models Compared, http://www.cea.assur.org/cea/download/publ/article221.pdf).

The suitable formulae should take into account the specificities of life, non-life and reinsurance business. Various methods are, therefore, available. To give an example, according to the French legislation (Code des Assurances, R334-13) for the life insurance, the solvency margin (to be replaced by the Solvency Capital Requirement for Solvency II) should be superior to the result obtained by multiplying 0.3% of the capital under risk with the ratio between the capital under risk after reinsurance and the capital under risk before reinsurance computed for the previous exercise. The latter ratio cannot be inferior to 50%. To keep it simple, at time $t$ the result obtained by multiplying a constant $\zeta_0$ (depending on previous experience and the type of insurance business) by the average claim per contract and by the number of contracts $n_t$ should not exceed the fortune of the insurance company:

$$\zeta_0 \times n_t \times \text{average claim} \leq \text{fortune at time } t. \quad (1)$$

Corroborating these elements, it appears obvious that the simple collective risk model should be improved to a model involving several contracts. We emphasize the fact that only quantitative requirements are taken into consideration (therefore, the model covers only part of Solvency II Pillar 1 requirements).

### 3 The insurance problem with several contracts

We introduce a complete probability space $(\Omega, \mathcal{F}, P)$. In order to model the claims, as for Mnif, Sulem (2005), we use a compound Poisson process given by a random measure $\mu(dt\,dy)$ on $\mathbb{R}_+ \times B$, with $B \subset \mathbb{R}_+ \setminus \{0\}$. Moreover,
we assume that the compensator of $\mu$ takes the form $dt\pi(dy)$ and that the measure $\pi$ is finite $\pi(dy) = \beta G(dy)$ for some probability measure $G(dy)$ on $B$ and some positive constant $\beta$.

Throughout the section, we let $Y$ denote a generic random variable distributed according to $G(dy)$.

We consider the natural filtration $(F_t)_{t \geq 0}$ generated by the random measure $\mu$. We call retention level any $(F_t)$-adapted process $(u_t)_{t \geq 0}$ which specifies that, given a claim $y$ at time $t \geq 0$, the direct insurer covers $y \land u_t$ while the reinsurance company covers the excess of loss $(y - u_t)^+$. Since we are going to consider several insurance contracts, we introduce a function $f$ depending both on the number of insurance contracts and on the risk taken by the company to model the claims $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$. If the company chooses some retention level $u_t$, then the actual premium rate per contract is given as in Asmussen et al. (2000), or, again, in Mnif, Sulem (2005)

$$p(u_t) = (1 + k_1)\beta \nu - (1 + k_2)\beta E \left[ f(1, (Y - u_t)^+) \right] \text{ for all } t \geq 0,$$

where $k_i$ are real constants satisfying $0 \leq k_1 < k_2$ and

$$\nu = \int_B f(1, y)G(dy) = E[f(1, Y)].$$

The first term in (2) is the premium received from the client, while the second term is the quantity paid to the reinsurer.

Given the initial fortune $x \geq 0$ and the retention level $u$, if $L$ stands for the $(F_t)$–adapted process representing the cumulative dividends paid up to the time $t$, $n_t$ denotes the number of contracts of the insurance company at time $t$, and $X_t^{x,u,L}$ the fortune of the company , then we have

$$X_t^{x,u,L} = x + \int_0^t n_s p(u_s)ds - \int_0^{t+} \int_B f(n_s, y \land u_s)\mu(dyds) - \int_0^t dL_s.$$

If we denote by $a$ the quantity

$$a = \frac{1}{\zeta_0 \nu},$$

then, from (2) we get that the maximum number of insurance contracts is $n_t^{\max} = aX_t^{x,u,L}$. We have the following equation

$$X_t^{x,u,L} = x + a\int_0^t X_s^{x,u,L}p(u_s)ds - \int_0^{t+} \int_B f(aX_s^{x,u,L}, y \land u_s)\mu(dyds) - \int_0^t dL_s.$$

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and introduce the cost functional

\[ J(x, u, L) = E \left[ \int_0^\tau e^{-rs} dL_s \right], \]

where \( r \) is some discount factor and \( \tau \) is the ruin time

\[ \tau = \inf \{ t \geq 0 : X_t^{x, u, L} \leq 0 \}. \]

Our value function \( V \) will be defined as the maximum over some family of admissible couples \((u, L)\) of the cost functional \( J \).

In practice, whenever the solvency condition is not satisfied, one of the following two events may occur. In the first case, a capital infusion from the shareholders intervenes. In the second one, an external referee solves the problem: either by transferring some of the contracts to other insurance companies, or by dissolving the contracts in final phase. The Solvency II framework states that as soon as the Solvency Capital Requirement (SCR) is not satisfied, supervisory action will be triggered. However, if the Minimum Capital Requirement (MCR) is not satisfied, the control authority can invoke severe measures (including closure of the company). From the mathematical point of view, we do not allow capital infusions, these being obtained by taking a larger initial reserve. On the contrary, the latter events may appear and they allow the variation of the number of contracts.

Let us now return to the function \( f \) modelling the claims. It is natural to suppose that the claims increase with the number of contracts and are null if the company has no contract. Moreover, the claims should increase with the risks covered and should be 0 if dealing with no risk. If the number of contracts is positive and the risk covered by these contracts is not null, then the claims are expected to be strictly positive. An utility function is usually supposed to be concave. If we are given a concave function \( v \) such that \( v(0) = 0 \), then

\[ v(\lambda x) \geq \lambda v(x), \]

for any \( \lambda \leq 1 \). Since any nonlinearity in \( f \) may only come from \( f \), in order to obtain the previous property for our utility function \( V \), one should assume that \( f \) is convex in the first variable. These assumptions give

**Assumption 1 (A1)** Suppose that the function \( f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+ \) satisfies:

- \( f(\cdot, y) \) is convex, non decreasing and \( f(0, y) = 0 \) for all \( y \in \mathbb{R}_+ \);
- \( f(x, \cdot) \) is increasing and \( f(x, 0) = 0 \);
- \( f(x, y) > 0 \) if \( x > 0 \) and \( y > 0 \);
- \( f \) is uniformly continuous on \( \mathbb{R}_+ \times \mathbb{R} \);
f(x, y) is Lipschitz in x, uniformly in y ∈ ℝ+.

One expects to cover expenditures through the premium received

\[ p(u_t) \geq \beta E[f(1, Y \wedge u_t)]. \]

Recall that \( p(0) - \beta E[f(1, 0)] < 0 \) and that \( \lim_{u \to \infty} (p(u) - \beta E[f(1, Y \wedge u)]) > 0 \) (recall the definitions (2) and (3) of \( p \) and \( \nu \), respectively) and we obtain the existence of some \( u > 0 \) such that

\[ p(u) \geq \beta E[f(1, Y \wedge u)], \quad (7) \]

for all \( u \geq u_0 \). Thus, we are going to consider only the retention levels \( u_t \) satisfying

\[ u_t \geq u. \quad (8) \]

One should impose that the dividends paid at some time \( t \) do not exceed the reserve at the same time. Therefore, we call admissible strategy the couple of \((F_t)\)–adapted processes \((u, L)\) such that \( u \) satisfies \( (8) \) and \( L \) is càdlàg, non decreasing, \( L_{0-} = 0 \) and \( L_t - L_{t-} \leq X^{x,u,L}_{t-} \) for almost every \((t, \omega)\). We should first prove the existence of such admissible strategies.

**Remark 2** If \( l \) is an \((F_t)\)–adapted process which is càdlàg, non decreasing, \( l_{0-} = 0 \), then, for any initial condition \( x \geq 0 \), and any \((F_t)\)–adapted processes \( u \) which satisfies \( (8) \), there exists a unique \((F_t)\)–adapted right-continuous process \( X^{x,u,l}_t \) with left-hand limits which satisfies the equation

\[ X^{x,u,l}_t = x + a \int_0^t X^{x,u,l}_s p(u_s) ds - \int_0^t f(aX^{x,u,l}_s, y \wedge u_s) \mu(dsdy) - \int_0^t dl_s \quad (9) \]

(see also Ikeda, Watanabe (1989) IV, Theorem 9.1). We define the ruin time \( \tau = \inf \{ t \geq 0 : X^{x,u,l}_t \leq 0 \} \). Obviously, on \( \{ t < \tau \} \) we have \( \Delta l_t = l_t - l_{t-} \leq X^{x,u,l}_{t-} \). Let us define the process

\[ L_t = l_t 1_{\{ t < \tau \}} + \left( \Delta l_t \wedge X^{x,u,l}_{t-} \right) 1_{\{ t = \tau \}}. \]

We get an \((F_t)\)–adapted process which is càdlàg, non decreasing, and \( L_{0-} = 0 \). Let \( X^{x,u,L} \) denote the solution of (9) with \( L \) instead of \( l \). We notice that \((u, L)\) is admissible in the sense that \( L_t - L_{t-} \leq X^{x,u,L}_{t-} \) for almost every \((t, \omega)\).

For all initial reserve \( x \geq 0 \), we denote by \( A(x) \) the set of admissible strategies described above. The value function is defined by

\[ V(x) = \sup_{(u, L) \in A(x)} J(x, u, L). \]

**Proposition 3** (Comparison for solutions of (9)) Given two \((F_t)\)–adapted processes \( u \) and \( l \) such that \( u \) satisfies \( (8) \) and \( L \) is càdlàg, non decreasing, and
l_0 = 0, and two initial states 0 ≤ x ≤ x', the solutions of (8) \( X^{x,u,l} \) and \( X^{x',u,l} \) starting from x (respectively \( x' \)) and associated with the pair \((u,l)\) satisfy

\[
X_t^{x,u,l} \leq X_t^{x',u,l}, \text{ for all } t, \ P - a.s.
\]

**PROOF.** Let us consider the sequence of functions \( \phi_n \in C^1(\mathbb{R}) \) such that

\[
\phi_n(x'') = 0 \text{ for all } x'' \leq 0, \ 0 \leq \phi'_n(x'') \leq 1, \text{ for all } x'' \in \mathbb{R}, \text{ and } \phi_n(x'') \uparrow (x'')^+ \text{ as } n \to \infty.
\]

A simple application of Itô’s formula yields

\[
\phi_n \left( X_{t-}^{x,u,l} - X_{t-}^{x',u,l} \right) = I_1 + I_2,
\]

where

\[
I_1 = \int_0^t a\,p(u_s) \left( X_{s-}^{x,u,l} - X_{s-}^{x',u,l} \right) \phi'_n \left( X_{s-}^{x,u,l} - X_{s-}^{x',u,l} \right) \, ds,
\]

\[
I_2 = \int_0^t \int_B \phi_n \left( X_{s-}^{x,u,l} - X_{s-}^{x',u,l} - f \left( aX_{s-}^{x,u,l}, y \wedge u_s \right) + f \left( aX_{s-}^{x',u,l}, y \wedge u_s \right) \right) \mu(\,ds\,dy) - \int_0^t \int_B \phi_n \left( X_{s-}^{x,u,l} - X_{s-}^{x',u,l} \right) \mu(\,ds\,dy).
\]

It is obvious that

\[
I_1 \leq C \int_0^t \left( X_{s-}^{x,u,l} - X_{s-}^{x',u,l} \right)^+ \, ds,
\]

where \( C \) is a constant independent of \( x \) and \( x' \). Since \( a \) can be chosen arbitrarily small (for that, it is enough to recall \( a = \frac{1}{\zeta \omega} \) and then choose an arbitrarily small monetary unit such that the quantity \( \nu \) becomes large), we may assume that \( aK_0 \leq 1 \) (here \( K_0 \) denotes the Lipschitz constant for \( f \)). Then the function \( x \mapsto x - f(ax, y) \) is increasing for all \( y \in \mathbb{R}_+ \). Therefore, we get

\[
I_2 \leq 0.
\]

Combining the two estimates for \( I_1 \) and \( I_2 \), we have

\[
E \left[ \phi_n \left( X_t^{x,u,l} - X_t^{x',u,l} \right) \right] \leq C \int_0^t E \left[ \left( X_s^{x,u,l} - X_s^{x',u,l} \right)^+ \right] \, ds.
\]

We allow \( n \to \infty \) to obtain

\[
E \left[ \left( X_t^{x,u,l} - X_t^{x',u,l} \right)^+ \right] \leq C \int_0^t E \left[ \left( X_s^{x,u,l} - X_s^{x',u,l} \right)^+ \right] \, ds.
\]

Finally, Gronwall’s inequality yields

\[
E \left[ \left( X_t^{x,u,l} - X_t^{x',u,l} \right)^+ \right] = 0.
\]

The proof of our Proposition is complete.
If the initial fortune is fixed, then the company has to make a choice over some family of admissible strategies. One may naturally wonder whether the same strategies are valid when dealing with a greater initial reserve or not. The answer is affirmative as proven by the following Proposition.

**Proposition 4** If $0 \leq x \leq x'$ are two initial capitals and if $(u, L)$ is an admissible strategy for $x$, then $(u, L)$ is also admissible for $x'$.

**PROOF.** Indeed, if $X^{x,u,L}_t$ (respectively $X^{x',u,L}_t$) denote the solutions of (9) starting from $x$ (respectively $x'$) associated with the control pair $(u, L)$, then the comparison result yields

$$X^{x,u,L}_t \leq X^{x',u,L}_t, \text{ dtdP - a.e. on } [0, \infty) \times \Omega.$$

Now, since $L$ is admissible for $x$, we have

$$L_t - L_{t-} \leq X^{x,u,L}_t - X^{x',u,L}_t, \text{ dtdP - a.e.,}$$

and $L$ is again admissible for $x'$. Moreover, if $\tau$ denotes the ruin time for $X^{x,u,L}_t$ and $\tau'$ denotes the ruin time for $X^{x',u,L}_t$, then, obviously

$$\tau \leq \tau', \text{ P - a.s.}$$

As one expects, using the previous results, we find that the utility function of the insurance company increases with the initial reserve. Since our strategy involves a dynamic programming approach, we would like to have finite value function. We suppose that the following assumption holds true

**Assumption 5 (A2)** The discount factor $r$ in (6) satisfies

$$r > \frac{2(1 + k_1)\beta}{\zeta_0}$$

Given an economic framework in which the discount factor $r$ is fixed, the above assumption says that the time between two claims is great enough to justify the demand for small solvency translated in the small constant $\zeta_0$.

Under this Assumption, we provide an upper bound estimate as well as Lipschitz regularity of the value function.

**Proposition 6** The value function $V$ is non decreasing, enjoys the Lipschitz property and satisfies

$$V(x) \leq Kx,$$

for some large enough positive constant $K$. 

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PROOF. The first assertion is straightforward from the previous Proposition. In order to establish the upper bound (11), we notice that

$$X_t^{x,u,L} \leq x + \frac{(1 + k_1)\beta}{\zeta_0} \int_0^t X_s^{x,u,L} ds,$$

for all $t \geq 0$. Gronwall’s inequality yields

$$X_t^{x,u,L} \leq xe^{\frac{(1+k_1)\beta t}{\zeta_0}}. \tag{12}$$

We write Itô’s formula for $e^{-rt}X_t^{x,u,L}$ and use (12) together with (A2) to obtain

$$J(x,u,L) \leq Cx.$$

Here $C$ is a constant which may change from line to line. Let us fix $x,x' \geq 0$. Suppose that $(u,L) \in A(x + x')$ and notice that, in this case, $(u,x+x'L) \in A(x)$. Indeed,

$$X_t^{x+x',u,L} = (x + x') + a \int_0^t X_s^{x+x',u,L} p(u_s) ds
- \int_0^{t+} \int_B f(aX_s^{x+x',u,L}, y \wedge u_s) \mu(dsdy) - \int_0^t dL_s,$$

and, by multiplying the latter equality by $\frac{x}{x + x'}$, we get

$$\frac{x}{x + x'} X_t^{x+x',u,L} = x + \int_0^t a p(u_s) \frac{x}{x + x'} X_s^{x+x',u,L} ds
- \int_0^{t+} \int_B \frac{x}{x + x'} f(aX_s^{x+x',u,L}, y \wedge u_s) \mu(dsdy)
- \int_0^t d \left( \frac{x}{x + x'} L_s \right).$$

On the other hand,

$$X_t^{x,u,\frac{x+x''}{x+x'}} = x + \int_0^t a p(u_s) X_s^{x,u,\frac{x+x''}{x+x'}} ds
- \int_0^{t+} \int_B f(aX_s^{x,\frac{x+x''}{x+x'}}, y \wedge u_s) \mu(dsdy)
- \int_0^t d \left( \frac{x}{x + x'} L_s \right).$$

Now, let the functions $\phi_n \in C^1(\mathbb{R})$ be such that $\phi_n(x'') = 0$ for all $x'' \leq 0$, and $0 \leq \phi_n'(x'') \leq 1$, for all $x'' \in \mathbb{R}$, and $\phi_n(x'') \uparrow (x'')^+$ as $n \to \infty$. We make the following notation

$$\frac{x}{x + x'} X_t^{x+x',u,L} = Y.$$

We apply Itô’s formula to have

$$\phi_n \left( Y_t - X_t^{x,u,\frac{x+x''}{x+x'}} \right) = I_1 + I_2, \tag{13}$$
where
\[
I_1 = \int_0^t a p(u_s) \left( Y_s - X_s^x \frac{x}{x + x'} L \right) \phi' \left( Y_s - X_s^x \frac{x}{x + x'} L \right) ds,
\]
\[
I_2 = \int_{t+}^t \int_B \phi_n \left( Y_s - X_s^x \frac{x}{x + x'} L - \frac{x}{x + x'} (aX_s^{x,x',u, L}_t, y \wedge u_s) + f \left( aX_s^{x,x',u, L}_s, y \wedge u_s \right) \right) \mu(dsdy)
- \int_0^{t+} \int_B \phi_n \left( Y_s - X_s^x \frac{x}{x + x'} L \right) \mu(dsdy).
\]

It is obvious that
\[
I_1 \leq C \int_0^t \left( Y_s - X_s^x \frac{x}{x + x'} L \right)^+ ds,
\]
where \( C \) is a constant independent of \( x \) and \( x' \), and we use the convexity of \( f \) in the first variable and \( f(0, \cdot) = 0 \), together with the monotonicity of \( \phi_n \) to get (as in the proof of the comparison result),
\[
I_2 \leq 0.
\]

Thus we obtain, as in the comparison result,
\[
\frac{x}{x + x'} X_t^{x+x',u,L} \leq X_t^{x,x',\frac{x}{x + x'} L} dtdP - a.e. \text{ on } [0, \infty) \times \Omega.
\]

Obviously, \( (u, \frac{x}{x + x'} L) \) is an admissible strategy for the initial reserve \( x \). If \( \tau \) is the ruin time for the strategy \( (u, L) \) for the initial reserve \( x + x' \), then the above inequality states that the ruin time for the strategy \( (u, \frac{x}{x + x'} L) \) when the initial reserve is \( x \) is greater than or equal to \( \tau \). Therefore, we have
\[
V(x + x') = \frac{x + x'}{x} \sup_{(u, L) \in A(x+x')} E \left[ \int_0^\tau e^{-rs} d \left( \frac{x}{x + x'} L_s \right) \right] \leq \frac{x + x'}{x} V(x),
\]

and (11) gives the Lipschitz property of \( V \). The proof of the Proposition is complete.

4 Hamilton Jacobi Bellman Variational Inequality

We have already seen that our value function \( V \) is increasing and Lipschitz continuous. These properties allow us to prove in a standard way that \( V \) satisfies the following Dynamic Programming Principle

Principle 7 (DPP)

\[
V(x) = \sup_{(u, L) \in A(x)} E \left[ e^{-r(t \wedge \tau)} V(x^{x+u,L}_{t \wedge \tau}) + \int_0^{t \wedge \tau} e^{-rs} dL_s \right],
\]

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for all \( t \geq 0, \ x \geq 0 \).

For further literature on the subject, the reader is referred to Fleming, Soner (1993), Krylov (1980), or Yong, Zhou (1999) (Theorem 4.3.3), for diffusion state processes or to Pham (1998) in the case of jump diffusion processes.

We consider at this point the following HJB variational inequality:

\[
\begin{cases}
\max\{H(x, V, V'(x)), 1 - V'(x)\} = 0 \text{ in } \mathbb{R}^*_+, \\
V(0) = 0.
\end{cases}
\] (14)

where

\[
H(x, V, q) = \sup_{u \geq 0} \left\{ -rV(x) + a x p(u)q + \int_B [V(x - f(ax, y)) - V(x)] \pi(dy) \right\}.
\] (15)

Let us recall that \( C^{1,1}(\mathbb{R}_+) \) stands for the class of all real-valued, differentiable functions on \( \mathbb{R}_+ \) such that the derivative is locally Lipschitz.

We also recall the definition of the viscosity supersolution, respectively viscosity subsolution.

**Definition 8** (i) Any lower semi-continuous (respectively upper semi-continuous) function \( v \) is a viscosity supersolution (subsolution) of (14) if \( v(0) \geq 0 \) (\( \leq 0 \)) and

\[
\max \{H(x, \varphi, \varphi'(x)), 1 - \varphi'(x)\} \leq 0,
\]

(respectively \( \geq 0 \)) whenever \( \varphi \in C^{1,1}(\mathbb{R}_+) \) is such that \( v - \varphi \) has a global minimum (maximum) at \( x > 0 \).

(ii) A function \( v \) is a viscosity solution of (14) if it is both super and subsolution.

**Theorem 9** The value function \( V \) is a viscosity solution for the associated Hamilton-Jacobi-Bellman Variational Inequality (14).

**Proof.** First, we prove that \( V \) is a viscosity supersolution for (14). In order to do this, let us consider \( x \in \mathbb{R}_+^* \) and a \( C^{1,1} \) test function \( \varphi \) such that \( V(x') - \varphi(x') \geq V(x) - \varphi(x) = 0 \), for all \( x' \in \mathbb{R}_+^* \). Moreover, consider \( 0 < h < x \) and the admissible strategy \((u, L) \in \mathcal{A}(x)\) where \( L_s = h \), for all \( s \geq 0 \) and \( u \).
is admissible and arbitrarily chosen. We have
\[ \varphi(x) = V(x) \geq E \left[ \int_0^{t \wedge \tau} e^{-rs} dL_s + e^{-r(t \wedge \tau)} V(X_{t \wedge \tau}^{x,u,L}) \right] \]
\[ \geq h + E \left[ e^{-r(t \wedge \tau)} \varphi(X_{t \wedge \tau}^{x,u,L}) \right] , \]
for all \( t \geq 0 \). We take the limit as \( t \to 0^+ \) and get
\[ \varphi(x) \geq h + \varphi(x - h) . \]
This latter inequality yields
\[ 1 - \varphi'(x) \leq 0 . \]

In order to prove \( H(x, \varphi, \varphi'(x)) \leq 0 \), we consider the admissible pair \( L_s = 0, u_s = u_0 \), for all \( s \geq 0 \) (here \( u_0 \geq u \) is arbitrarily chosen). We apply Itô’s formula to \( e^{-r(t \wedge \tau)} \varphi(X_{t \wedge \tau}^{x,u,L}) \) to obtain
\[ E \left[ e^{-r(t \wedge \tau)} \varphi(X_{t \wedge \tau}^{x,u,L}) \right] - \varphi(x) \]
\[ = E \left[ \int_0^{t \wedge \tau} \left( -r e^{-rs} \varphi(X_{s}^{x,u,L}) + e^{-rs} a X_{s}^{x,u,L} p(u_0) \varphi'(X_{s}^{x,u,L}) \right) ds \right] \]
\[ + E \left[ \int_0^{t \wedge \tau} \int_B e^{-rs} \left( \varphi \left( X_{s}^{x,u,L} - f \left( a X_{s}^{x,u,L} , y \wedge u_0 \right) \right) - \varphi \left( X_{s}^{x,u,L} \right) \right) \mu(dsdy) \right] . \]
Recalling that \( \varphi(x) \geq E \left[ e^{-r(t \wedge \tau)} V(X_{t \wedge \tau}^{x,u,L}) \right] \), and dividing by \( t > 0 \), we have
\[ 0 \geq E \left[ \frac{1}{t} \int_0^{t \wedge \tau} \left( -r e^{-rs} \varphi(X_{s}^{x,u,L}) + e^{-rs} a X_{s}^{x,u,L} p(u_0) \varphi'(X_{s}^{x,u,L}) \right) ds \right] \]
\[ + E \left[ \frac{1}{t} \int_0^{t \wedge \tau} \int_B \left( e^{-rs} \left( \varphi \left( X_{s}^{x,u,L} - f \left( a X_{s}^{x,u,L} , y \wedge u_0 \right) \right) - \varphi \left( X_{s}^{x,u,L} \right) \right) \mu(dsdy) \right) \right] \]
\[ \geq E \left[ \frac{1}{t} \int_0^{t \wedge \tau} \left( -r \varphi(x) + e^{-rt} a x p(u_0) \varphi'(x) \right) ds \right] \]
\[ + E \left[ \frac{1}{t} \int_0^{t \wedge \tau} ds \int_B \left( e^{-rt} \varphi \left( x - f \left( a x , y \wedge u_0 \right) \right) - \varphi \left( x \right) \right) \pi(dy) \right] \]
\[ - O \left( E \left[ \sup_{s \leq t \wedge \tau} e^{-rs} \left( X_{s}^{x,u,L} - x \right) \right] \right) , \]
where \( O(\delta) \to 0 \) whenever \( \delta \to 0 \).

We wish to prove that \( E \left[ \sup_{s \leq t \wedge \tau} e^{-rs} \left( X_{s}^{x,u,L} - x \right) \right] \to 0 \), when \( t \to 0 \). In order to do this, we use
\[ \left| X_{s}^{x,u,L} - x \right| \leq \int_0^s a p(u_0) X_{s'}^{x,u,L} ds' \]
\[ + \int_0^{s'} \int_B f \left( a X_{s'}^{x,u,L} , y \wedge u_0 \right) \mu(ds'dy) . \]
Therefore, with the notation $C_0 = \frac{(1+k_1)^\beta}{\delta \phi}$, we have, for some constant $C,$

$$|X^{x,u,L}_s - x| \leq x \left( e^{C_0 s} - 1 \right) + C x \int_0^{s+} \int_B e^{C_0 s'} \mu(ds'dy)$$

for all $0 \leq s \leq t \land \tau$ (we use the Lipschitz property of $f$ in $x$ uniformly in $y$, $f(0, \cdot) = 0$ and the upper bound for $X^{x,u,L}_s$ given by (12)). We multiply the last inequality by $e^{-rs}$, take the supremum over all $0 \leq s \leq t \land \tau$, then the expectation with respect to $P$ to obtain

$$\lim_{t \to 0^+} E \left[ \sup_{s \leq t \land \tau} e^{-rs} |X^{x,u,L}_s - x| \right] = 0.$$  

(18)

Notice that

$$\frac{E[t \land \tau]}{t} \geq 1 - P(\tau \leq t) \geq 1 - P(\eta_1 \leq t),$$

where $\eta_1$ is the first time a claim occurs (it follows the exponential law). Consequently,

$$\lim_{t \to 0^+} \frac{E[t \land \tau]}{t} = 1.$$  

(19)

Returning to (17) we let $t \to 0^+$ and use (18) and (19) to get

$$0 \geq \left( -r \varphi(x) + axp(u_0) \varphi'(x) \right)$$

$$+ \int_B \{ \varphi(x - f(ax, y \land u_0)) - \varphi(x) \} \pi(dy)$$

Combining (20) and (16), we prove that $V$ is a viscosity supersolution for (14).

In order to prove that the value function is a viscosity subsolution for (14), we fix $x > 0$ and consider an arbitrary test function $\varphi \in C^{1,1}$ such that $V(x') - \varphi(x') \leq V(x) - \varphi(x) = 0$, for all $x' \in \mathbb{R}_+$. Let us suppose that the subsolution inequality does not hold. Therefore, there exists $\delta > 0$ such that

$$\max \{ H(x, \varphi, \varphi'(x)), 1 - \varphi'(x) \} < -\delta.$$  

We use the continuity of $H$ and of $\varphi'$ to obtain the existence of some $\eta \in \left( 0, x \land \frac{\delta}{4K_\varphi} \right)$, where $K_\varphi$ denotes the Lipschitz constant for $\varphi$ on $[0, e^r x]$, such that

$$\max \{ H(x', \varphi, \varphi'(x')), 1 - \varphi'(x') \} < -\delta, \text{ if } x' \in B(x, \eta).$$  

(21)

Let us consider an arbitrary strategy $(u, L) \in \mathcal{A}(x)$ and let $X^{x,u,L}$ denote the solution of (14) for $(u, L)$ instead of $(u, l)$. We define the stopping time

$$\sigma = \inf\{ t \geq 0 : X^{x,u,L}_t \notin B(x, \eta) \}.$$  

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Obviously \( \sigma \leq \tau \) (the ruin time). We apply Itô’s formula to \( e^{-r(t\wedge \sigma)}\phi(X_{t\wedge \sigma}^{x,u,L}) \) and write

\[
E[e^{-r(t\wedge \sigma)}\phi(X_{t\wedge \sigma}^{x,u,L})] - \phi(x) =
\]

(22)

\[
E\left[\int_0^{t\wedge \sigma} \left(-re^{-rs}\phi(X_{s-}^{x,u,L}) + e^{-rs}aX_{s-}^{x,u,L}p(u_0)\phi'(X_{s-}^{x,u,L})\right) ds\right]
\]

+ \( E\left[\int_0^{t\wedge \sigma} \left(\varphi(X_{s-}^{x,u,L} - f(aX_{s-}^{x,u,L}, y \wedge u_0)) - \varphi(X_{s-}^{x,u,L})\right) \mu(dsdy)\right]\)

- \( E\left[\int_0^{t\wedge \sigma} e^{-rs}\varphi'(X_{s-}^{x,u,L}) dL_s\right]\)

+ \( E\left[\sum_{s\leq t\wedge \sigma} e^{-rs} \left(\varphi(X_{s-}^{x,u,L} - \triangle L_s) - \varphi(X_{s-}^{x,u,L})\right)\right]\)

For \( s < t \wedge \sigma \) we have, from (21)

\[
-re^{-rs}\varphi(X_{s-}^{x,u,L}) + e^{-rs}aX_{s-}^{x,u,L}p(u_0)\varphi'(X_{s-}^{x,u,L})
\]

(23)

\[+e^{-rs} \int_B \left(\varphi(X_{s-}^{x,u,L} - f(aX_{s-}^{x,u,L}, y \wedge u_0)) - \varphi(X_{s-}^{x,u,L})\right) \pi(dy) < -\delta e^{-rs},\]

and, again from (21),

\( \varphi'(X_{s-}^{x,u,L}) > 1.\)

It follows that

\[
\varphi(X_{s-}^{x,u,L} - \triangle L_s) - \varphi(X_{s-}^{x,u,L}) \leq -\triangle L_s.
\]

(24)

Using (23) we get

\[
E\left[\int_0^{t\wedge \sigma} \left(-re^{-rs}\varphi(X_{s-}^{x,u,L}) + e^{-rs}aX_{s-}^{x,u,L}p(u_0)\phi'(X_{s-}^{x,u,L})\right) ds\right]
\]

\[
+ E\left[\int_0^{t\wedge \sigma} \int_B e^{-rs} \left(\varphi(X_{s-}^{x,u,L} - f(aX_{s-}^{x,u,L}, y \wedge u_0)) - \varphi(X_{s-}^{x,u,L})\right) \mu(dsdy)\right]\]

\[
\leq \delta E\left[\frac{e^{-r(t\wedge \sigma)} - 1}{r}\right]
\]

\[-E\left[\int_0^{t\wedge \sigma} \int_B e^{-rs} \left(\varphi(X_{s-}^{x,u,L} - f(aX_{s-}^{x,u,L}, y \wedge u_0)) - \varphi(X_{s-}^{x,u,L})\right) \pi(dy) ds\right]\]

\[+ \int_0^{t\wedge \sigma} ds \int_B e^{-rs} \left(\varphi(x - f(ax, y \wedge u_0)) - \varphi(x)\right) \pi(dy)\]

\[-E\left[\int_0^{t\wedge \sigma} \int_B e^{-rs} \left(\varphi(x - f(ax, y \wedge u_0)) - \varphi(x)\right) \mu(dsdy)\right]\]

\[+ E\left[\int_0^{t\wedge \sigma} \int_B e^{-rs} \left(\varphi(X_{s-}^{x,u,L} - f(aX_{s-}^{x,u,L}, y \wedge u_0)) - \varphi(X_{s-}^{x,u,L})\right) \mu(dsdy)\right]\]

\[
\leq \delta E\left[\frac{e^{-r(t\wedge \sigma)} - 1}{r}\right] + 4K\phi\eta E\left[\frac{1 - e^{-r(t\wedge \sigma)}}{r}\right].
\]

(25)
We return to (22) and use (24) and (25) to get
\[
E \left[ e^{-r(t \wedge \sigma)} \varphi(X_{t \wedge \sigma}^x) \right] - \varphi(x) \leq \delta E \left[ \frac{e^{-r(t \wedge \sigma)} - 1}{r} \right] + 4K \varphi \eta E \left[ \frac{1 - e^{-r(t \wedge \sigma)}}{r} \right] - E \left[ \int_0^{t \wedge \sigma} e^{-rs} dL_s \right],
\]
and, from this,
\[
V(x) = \varphi(x)
\]
\[
\geq E \left[ e^{-r(t \wedge \sigma)} \varphi(X_{t \wedge \sigma}^x) + \int_0^{t \wedge \sigma} e^{-rs} dL_s \right] + (\delta - 4K \varphi \eta) E \left[ \frac{1 - e^{-r(t \wedge \sigma)}}{r} \right].
\]  
(26)

for \( t \) small enough. We can suppose that \( x \) is a strict global maximum point. Then there exists \( \lambda > 0 \) such that
\[
\sup_{x' \notin B^\varphi(x, \eta)} (V(x') - \varphi(x')) = -\lambda.
\]

We use (26) and write
\[
V(x) \geq E \left[ e^{-r(t \wedge \sigma)} V(X_{t \wedge \sigma}^x) + \int_0^{t \wedge \sigma} e^{-rs} dL_s \right] + \lambda E \left[ e^{-r(t \wedge \sigma)} 1_{\sigma \leq t} \right] + \frac{\delta - 4K \varphi \eta}{2} t P(\sigma > t)
\]
\[
\geq E \left[ e^{-r(t \wedge \sigma)} V(X_{t \wedge \sigma}^x) + \int_0^{t \wedge \sigma} e^{-rs} dL_s \right] + \left( \lambda e^{-rt} \right) \wedge \left( \frac{\delta - 4K \varphi \eta}{2} t \right)
\]  
(27)

The dynamic programming principle yields
\[
V(x) \leq \sup_{(u, L)} E \left[ e^{-r(t \wedge \sigma)} V(X_{t \wedge \sigma}^x) + \int_0^{t \wedge \sigma} e^{-rs} dL_s \right].
\]  
(28)

Therefore, by the choice of \( \eta < \frac{\delta}{4K \varphi} \) and \( \lambda > 0 \), (27) contradicts (28). This proves that \( V \) is a viscosity subsolution for (14). Our Theorem is now complete.

5 The Comparison Theorem

The following Lemma provides an equivalent definition for the notions of viscosity super and subsolution.
Lemma 10  

(i) A continuous function $U$ is a viscosity supersolution for (14) in $\mathbb{R}^*_+$ if and only if, $U(0) \geq 0$ and, for any $x \in \mathbb{R}^*_+$ and any test function $\varphi \in C^{1,1}$ such that $U - \varphi$ has a global strict minimum at $x$, we have

$$\max \{H(x,U,\varphi'(x)), 1 - \varphi'(x)\} \leq 0. \quad (29)$$

(ii) A continuous function $U$ is a viscosity subsolution for (14) in $\mathbb{R}^*_+$ if and only if, $U(0) \leq 0$ and, for any $x \in \mathbb{R}^*_+$ and any test function $\varphi \in C^{1,1}$ such that $U - \varphi$ has a global strict maximum at $x$, we have

$$\max \{H(x,U,\varphi'(x)), 1 - \varphi'(x)\} \geq 0. \quad (30)$$

PROOF. We only prove the assertion for viscosity supersolution, the proof for subsolution being similar.

Suppose that (i) holds true. For any test function $\varphi \in C^{1,1}$ such that $U(x) = \varphi(x)$ and $U - \varphi$ has a global minimum at $x$, and all $\delta > 0$, we define

$$\varphi_\delta(x') = \varphi(x') - \delta |x' - x|^2, \text{ for all } x' \in \mathbb{R}^*_+. \quad (31)$$

Then $\varphi_\delta \in C^{1,1}$ and $U - \varphi_\delta$ has a global strict minimum at $x$. The assumption implies that

$$\max \{H(x,U,\varphi_\delta(x)), 1 - \varphi'_\delta(x)\} \leq 0.$$

Obviously, $U(x') - U(x) > \varphi_\delta(x') - \varphi_\delta(x)$, for all $x' \in \mathbb{R}^*_+ \setminus \{x\}$. The definition of $H$, together with the last inequality, yields

$$\max \{H(x,\varphi_\delta,\varphi'(x)), 1 - \varphi'(x)\} \leq 0. \quad (31)$$

Moreover, again from the definition of $H$,

$$H(x,\varphi,\varphi'(x)) \leq H(x,\varphi_\delta,\varphi'(x))$$

$$+ \sup_{u \geq u_0} \left( \int_B (\varphi(x - f(ax,y \wedge u)) - \varphi_\delta(x - f(ax,y \wedge u)) \pi \, dy \right)$$

$$\leq H(x,\varphi_\delta,\varphi'(x)) + C\delta,$$

where $C$ is a generic constant independent of $\delta$. We get, using (31) then taking the limit as $\delta$ \to 0,

$$\max \{H(x,\varphi,\varphi'(x)), 1 - \varphi'(x)\} \leq 0.$$

For the converse, consider an arbitrary test function $\varphi \in C^{1,1}$ and $x \in \mathbb{R}^*_+$ such that

$$0 = U(x) - \varphi(x) < U(x') - \varphi(x'),$$

but $\varphi_\delta(x') = \varphi(x') - \delta |x' - x|^2$ would be a global strict minimum. Hence, $U - \varphi$ has a strict minimum at $x$ and $\varphi_\delta(x)$ is a test function that satisfies $U - \varphi_\delta$ has a global strict minimum at $x$. Therefore, $H(x,U,\varphi'(x)) \leq 0$. By taking the limit as $\delta$ \to 0, we get $H(x,U,\varphi'(x)) \leq 0$, and hence $U$ is a viscosity supersolution for (14) in $\mathbb{R}^*_+$. The proof for subsolution is similar.
for all \( x' \in \mathbb{R}_+^* \setminus \{x\} \). For \( \varepsilon > 0 \) such that \( \varepsilon < \frac{x}{2} \), we define

\[
\delta_\varepsilon = \sup_{x' \in B(x,4\varepsilon)} (U(x') - \varphi(x')) > 0.
\]

It is obvious that \( \lim_{\varepsilon \to 0} \setminus \delta_\varepsilon = 0 \). We introduce

\[
\varphi_\varepsilon = (U - \varphi - \delta_\varepsilon) 1_{[0,x-2\varepsilon]} + (U(0) - \varphi(0) - \delta_\varepsilon) 1_{\mathbb{R}_-}.
\]

We consider some sequence of mollifiers \( \rho_n \in C^\infty_c (\mathbb{R}; \mathbb{R}_+) \), \( \text{Supp}\ \rho_n \subset B \left( 0, \frac{1}{n} \right) \) and \( \int_{\mathbb{R}} \rho_n(t) dt = 1 \). Since \( U - \varphi \) is continuous, the sequence \( \{\rho_n * \varphi_\varepsilon\}_n \) converges uniformly on \( [0, x-3\varepsilon] \) to \( \varphi_\varepsilon \). Then there exists a subsequence (denoted by \( (\rho_\varepsilon) \)) such that \( \text{Supp}\ \rho_\varepsilon \subset B (0, \varepsilon) \) and

\[
U(x') - \varphi(x') - 2\delta_\varepsilon \leq (\rho_\varepsilon * \varphi_\varepsilon)(x') < U(x') - \varphi(x')
\]

for all \( 0 \leq x' \leq x - 3\varepsilon \) and all \( \varepsilon > 0 \). Finally, we define the function

\[
F_\varepsilon(x') = \varphi(x') + (\rho_\varepsilon * \varphi_\varepsilon)(x').
\]

It is obvious that \( F_\varepsilon \in C^{1,1} \) has the following properties:

\[
\begin{aligned}
F_\varepsilon(x') &= \varphi(x'), \quad \text{if } x' \geq x - \varepsilon, \\
U(x') - 2\delta_\varepsilon &\leq F_\varepsilon(x'), \quad \text{if } 0 \leq x' \leq x - 3\varepsilon, \\
F_\varepsilon(x') &< U(x'), \quad \text{if } x' \neq x.
\end{aligned}
\]

The assumptions give

\[
\max \{ H(x, F_\varepsilon, F'_\varepsilon(x)), 1 - F'_\varepsilon(x) \} \leq 0.
\]

Let us put

\[
G(x') = \sup_{u \geq u_0} \left\{ -r (U(x') - F_\varepsilon(x')) + ap(u)x' (\varphi'(x') - F'_\varepsilon(x')) \\
+ \int_B (U(x' - f(ax', y \wedge u)) - F_\varepsilon(x' - f(ax', y \wedge u))) \pi(dy) \\
- \int_B (U(x') - F_\varepsilon(x')) \pi(dy) \right\},
\]

for any \( x' \in \mathbb{R}_+^* \). Then

\[
H(x, U, \varphi'(x)) - H(x, F_\varepsilon, F'_\varepsilon(x)) \leq G(x),
\]

where

\[
G(x) \leq \sup_{u \geq u_0} \left\{ \int_B (U(x - f(ax, y \wedge u)) - F_\varepsilon(x - f(ax, y \wedge u))) \pi(dy) \\
- \int_B (U(x) - F_\varepsilon(x)) \pi(dy) \right\}.
\]

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We then consider the sets $B^u = \{ y \in B : x - f(ax, y \wedge u) \in \overline{B}(x, 3\varepsilon) \}$ and get

$$
\int_B (U(x - f(ax, y \wedge u)) - F_\varepsilon(x - f(ax, y \wedge u))) \pi(dy)
\leq \int_{B \setminus B^u} 2\delta_\varepsilon \pi(dy) + C \pi(B^u)
\leq 2\beta\delta_\varepsilon + C \pi(B^u),
$$

(33)

where $C > 0$ is a generic constant independent of $\varepsilon$. Moreover, if $y \in B^u$, then

$$
x - f(ax, y \wedge u) \geq x - 3\varepsilon.
$$

Therefore,

$$
f(ax, y \wedge u) \leq f(ax, y \wedge u) \leq 3\varepsilon.
$$

Since $f(ax, y \wedge u) > 0$ for $xy > 0$ and $f(ax, \cdot)$ is nondecreasing, we deduce the existence of some $\eta_\varepsilon > 0$ such that $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$ and $y \in B^u$ only if $y \leq \eta_\varepsilon$. Thus, returning to (33), we get

$$
\int_B (U(x - f(ax, y \wedge u)) - F_\varepsilon(x - f(ax, y \wedge u))) \pi(dy)
\leq C \delta_\varepsilon + C \pi(B \cap [0, \eta_\varepsilon]).
$$

Consequently,

$$
G(x) \leq C \delta_\varepsilon + C \pi(B \cap [0, \eta_\varepsilon]).
$$

(34)

Recall that $0 \notin B$. Thus, using (34) in (32) and taking the limit as $\varepsilon \to 0$, we obtain

$$
H(x, U, \varphi') \leq 0,
$$

and (i) follows.

The assertion (ii) follows in the same way.

Under the assumption (A2) we are able to prove the following result on the comparison of viscosity solutions for (14).

**Theorem 11** Let $U$ and $V$ be respectively a continuous viscosity subsolution and a continuous viscosity supersolution for (14) both of at most linear growth. Then, if (A2) holds true, we have

$$
U(x) \leq V(x), \text{ for all } x \in \mathbb{R}_+^*.
$$

**Proof.** For $\delta > 0$ and $\varepsilon > 0$, we denote by $\Phi_{\varepsilon, \delta}$ the function $\Phi_{\varepsilon, \delta} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ given by

$$
\Phi_{\varepsilon, \delta}(x, x') = U(x) - V(x') - \frac{1}{2\varepsilon}(x - x')^2 - \delta \left(x^2 + (x')^2\right),
$$

(35)
for all \( x, x' \geq 0 \). Suppose that for some \( x_0 \in \mathbb{R}_+^\ast \) and some \( \theta > 0 \) we have
\[
U(x_0) - V(x_0) \geq \theta.
\]

Since \( \Phi_{\varepsilon, \delta} \) is upper semi-continuous and \( U \) and \( V \) are of linear growth, there exists a global maximum point of \( \Phi_{\varepsilon, \delta} \), denoted by \( (x_{\varepsilon, \delta}, x'_{\varepsilon, \delta}) \in \mathbb{R}_+ \times \mathbb{R}_+ \).

Obviously, since \( \Phi_{\varepsilon, \delta}(0, x') \leq 0 \) for all \( x' \in \mathbb{R}_+ \), it holds that \( x_{\varepsilon, \delta} \geq 0 \). Moreover,
\[
\gamma_{\varepsilon, \delta} = \Phi_{\varepsilon, \delta}(x_{\varepsilon, \delta}, x'_{\varepsilon, \delta}) \geq \Phi_{\varepsilon, \delta}(x_0, x_0) \geq \theta - 2\delta x_0^2 \geq \frac{\theta}{2},
\]
for any \( \delta \leq \delta_0 = \frac{\theta}{4x_0^2} \). Obviously, for \( \delta \leq \delta_0 \) fixed, \( (\gamma_{\varepsilon, \delta})_{\varepsilon} \) is increasing and
\[
\gamma_{2\varepsilon, \delta} \geq \gamma_{\varepsilon, \delta} + \frac{1}{4\varepsilon}(x_{\varepsilon, \delta} - x'_{\varepsilon, \delta})^2.
\]
Therefore,
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (x_{\varepsilon, \delta} - x'_{\varepsilon, \delta})^2 = 0.
\]
If, for all \( \varepsilon > 0 \) (or, at least for some arbitrary sequence \( \varepsilon_n \) such that \( \varepsilon_n \to 0 \) when \( n \to \infty \)) \( x'_{\varepsilon, \delta} = 0 \), then \( \lim_{\varepsilon \to 0} x_{\varepsilon, \delta} = 0 \), and, by taking the upper limit when \( \varepsilon \to 0 \) in (35), we get
\[
\frac{\theta}{2} \leq U(0) - V(0) \leq 0,
\]
which contradicts the assumption \( \theta > 0 \). We deduce that, for \( \varepsilon > 0 \) small enough, \( x_{\varepsilon, \delta} \) and \( x'_{\varepsilon, \delta} \) are strictly positive. We consider the test function
\[
\varphi(x) = V(x'_{\varepsilon, \delta}) + \frac{1}{2\varepsilon}(x - x'_{\varepsilon, \delta})^2 + \frac{1}{\varepsilon}(x^2 + (x'_{\varepsilon, \delta})^2), \text{ for } x \in \mathbb{R}_+^\ast,
\]
such that \( U - \varphi \) has a maximum point at \( x_{\varepsilon, \delta} \). We write the variational inequality and use the previous Lemma to get
\[
\max \{H(x_{\varepsilon, \delta}, U, \varphi'(x_{\varepsilon, \delta})), 1 - \varphi'(x_{\varepsilon, \delta})\} \geq 0.
\]
In a similar way we have
\[
\max \{H(x'_{\varepsilon, \delta}, V, \psi'(x'_{\varepsilon, \delta})), 1 - \psi'(x'_{\varepsilon, \delta})\} \leq 0,
\]
where
\[
\psi(x') = U(x_{\varepsilon, \delta}) - \frac{1}{2\varepsilon}(x_{\varepsilon, \delta} - x')^2 - \delta(x_{\varepsilon, \delta}^2 + (x')^2), \text{ for all } x' \in \mathbb{R}_+^\ast.
\]

(a) We suppose that
\[
H(x_{\varepsilon, \delta}, U, \varphi'(x_{\varepsilon, \delta})) \geq H(x'_{\varepsilon, \delta}, V, \psi'(x'_{\varepsilon, \delta})).
\]
Then
\[
0 \leq \sup_{u \geq u} \left\{ -r \left( U(x, \delta) - V(x', \delta) \right) + ap(u) \left[ x_{\varepsilon, \delta} - x'_{\varepsilon, \delta} \right] + \int_{B} \left( U(x_{\varepsilon, \delta} - f(ax_{\varepsilon, \delta}, y \land u)) - V(x', \delta - f(ax', \delta, y \land u)) \right) \pi(dy) + \int_{B} \left( V(x_{\varepsilon, \delta} - U(x_{\varepsilon, \delta})) \right) \pi(dy) \right\}. \tag{40}
\]

We use \( \Phi(x_{\varepsilon, \delta}; x'_{\varepsilon, \delta}) \geq \Phi(x_{\varepsilon, \delta} - f(x_{\varepsilon, \delta}, y \land u), x'_{\varepsilon, \delta} - f(x'_{\varepsilon, \delta}, y \land u)) \) to get
\[
U(x_{\varepsilon, \delta} - f(ax_{\varepsilon, \delta}, y \land u)) - V(x', \delta - f(ax', \delta, y \land u)) \leq U(x_{\varepsilon, \delta}) - V(x', \delta) - \frac{1}{2\varepsilon} \left( x_{\varepsilon, \delta} - f(ax_{\varepsilon, \delta}, y \land u) - x'_{\varepsilon, \delta} + f(ax', \delta, y \land u) \right)^2 \]
\[
+ \delta \left( (x_{\varepsilon, \delta} - f(ax_{\varepsilon, \delta}, y \land u))^2 - x_{\varepsilon, \delta}^2 + (x', \delta - f(ax', \delta, y \land u))^2 - (x_{\varepsilon, \delta})^2 \right) \]
\[
\leq U(x_{\varepsilon, \delta}) - V(x', \delta),
\]
and, returning to (40), we have
\[
0 \leq \sup_{u \geq u} \left\{ -r \left( U(x_{\varepsilon, \delta}) - V(x', \delta) \right) + 2ap(u) \left( \frac{x_{\varepsilon, \delta} - x', \delta^2}{2\varepsilon} + \delta \left( x_{\varepsilon, \delta}^2 + (x_{\varepsilon, \delta}^2) \right) \right) \right\}
\leq \sup_{u \geq u} \left( 2ap(u) - r \right) \times \left( U(x_{\varepsilon, \delta}) - V(x', \delta) \right).
\]

Recall that \( \sup_{u \geq u} 2ap(u) \leq \frac{2(1+k_1)}{c_0} < r \) (see (A2)). Thus, it follows that \( \gamma_{\varepsilon, \delta} < 0 \) which contradicts (36).

(b) If (39) does not hold, we use (37) and (38) and we must have
\[
1 - \varphi'(x_{\varepsilon, \delta}) \geq 0 \geq 1 - \psi'(x_{\varepsilon, \delta}), \tag{41}
\]
thus
\[
\frac{1}{\varepsilon} \left( x_{\varepsilon, \delta} - x'_{\varepsilon, \delta} \right) - 2\delta x_{\varepsilon, \delta} \geq \frac{1}{\varepsilon} \left( x_{\varepsilon, \delta} - x'_{\varepsilon, \delta} \right) + 2\delta x_{\varepsilon, \delta}.
\]

We deduce that \( x_{\varepsilon, \delta} = x_{\varepsilon, \delta} = 0 \) and get a contradiction. The proof of the comparison result is now complete.

6 Numerical results

We now turn our attention to some particular case and observe the optimal retention process by means of numerical simulation. We have seen that, for
the collective risk model introduced in [9], a single insurance contract is considered and, of course, the risk is given for this one contract. A possible way to extend this model is to suppose that the risk concerns all contracts (or at least a percentage). We assume that the claims have constant intensity $\delta$ and the random measure $\mu$ is associated with some Poisson process of constant intensity $\pi(dy) = \beta G_{\delta}(dy)$, where $G_{\delta}$ corresponds to the Dirac mass. Moreover, the function $f$ is given by $f(x, y) = \rho xy$, with $0 < \rho \leq 1$ (that is only some $\rho$ part of the total contracts is subject to claims). In this case, the minimal retention level needed to cover expenditures is given explicitly by

$$u = \frac{(k_2 - k_1)\delta}{k_2}$$

and $p(u) = (k_1 - k_2)\beta + (1 + k_2)\beta pu$, for all $\frac{(k_2 - k_1)\delta}{k_2} \leq u \leq \delta$.

Under the above assumptions, Eq. (5) reads

$$X_t^{x,u,L} = x + a \int_0^t X_s^{x,u,L} p(u_s)ds - \rho a \int_0^t X_s^{x,u,L} u_s dN_s - \int_0^t dL_s. \tag{42}$$

Theorem 10 states that the maximized expected discounted dividends is the unique viscosity solution for the Hamilton-Jacobi-Bellman variational inequality

$$\begin{cases}
\max \{H(x, V, V'(x)), 1 - V'(x)\} = 0 \text{ in } \mathbb{R}^*_+,
V(0) = 0,
\end{cases} \tag{43}$$

where

$$H(x, V, q) = \sup_{\frac{(k_2 - k_1)\delta}{k_2} \leq u \leq \delta} \{-rV(x) + aq(x) + \beta [V(x - a\rho xu) - V(x)]\}.$$

The standard procedure in order to apply numerical arguments is to obtain a bounded space. Thus, we write the previous equation on $[0, 1)$ by taking $y = \frac{x}{x+1}$ and $\psi(y) = V(x)$. This leads to the following HJB equation

$$\begin{cases}
\max \{G(y, \psi, \psi'(y)), 1 - (1 - y)^2 \psi'(y)\} = 0 \text{ in } [0, 1),
\psi(0) = 0,
\end{cases} \tag{44}$$

where

$$G(y, \psi, q) = \sup_{\frac{(k_2 - k_1)\delta}{k_2} \leq u \leq \delta} \left\{-r\psi(y) + ap(x)q(1 - y) + \beta \left[\psi(y - a\rho u) - \psi(y)\right] + \beta \left[\psi(y(1 - a\rho u)) - \psi(y)\right] \right\}.$$

As in Mnif, Sulem (2005), the approximate solution of Eq. (44) is computed with the help of finite difference approximations and the policy iteration algorithm.
We consider two particular cases: the first one illustrates the natural framework in which the reinsurance company perceives a relative safety loading greater than that of the insurer, while the second example assumes the opposite. The data set we use is given in the following table:

|     | $k_1$ | $k_2$ | $\delta$ | $r$  | $\beta$ | $\rho$ |
|-----|-------|-------|----------|------|---------|--------|
| Fig 1 | 0.2   | 0.25  | 1        | 0.07 | 0.0011  | 10%    |
| Fig 2 | 0.2   | 0.19  | 1        | 0.07 | 0.0011  | 10%    |

For the first framework, the optimal retention level turns out to be maximal as shown by Fig 1.

As can be expected in the second case, if the initial reserve is great enough, then the direct insurer should play the safety card in order to maximize expected discounted dividends. Indeed, since the relative safety loadings guarantee a proportional steady income to the insurer, the optimal retention level is null (see Fig 2).
Fig 2. Optimal retention level for $\delta = 1$, $k_1 = 0.2$, $k_2 = 0.19$

References

[1] Asmussen, Højgaard, B., Taksar, M., 2000. Optimal risk control and dividend distribution policies. Example of excess-of-loss reinsurance for an insurance corporation. Finance and Stochastics, 4, 299-324.

[2] G. Barles, R. Buckdahn, E. Pardoux, 1997. Backward stochastic differential equations and integral-partial differential equations, Stochastic & Stochastic Reports 60 (1–2), 57–83.

[3] Bensoussan, A., Lions, J.L.,1978. Applications des Inéquations Variationnelles en Contrôle Stochastique, Paris: Dunod.

[4] Crandal, M.G., Ishii, H., Lions, P.L., 1992. User’s guide to viscosity solutions of second order partial differential equations, Bulletin des Sciences Mathématiques Soc., 27, 1-67.

[5] Fleming, W., Soner, H.M., 1993. Controlled Markov Processes and Viscosity Solutions, New York: Springer Verlag.

[6] Ikeda, N, Watanbe, S, 1989. Stochastic differential equations and diffusion processes. 2nd ed., Amsterdam: North-Holland.

[7] Jeanblanc-Piqué, M., Shiryaev, A.N., 1995. Optimization of the flow of dividends, Russian Mathematical Surveys 50:2, 257-277.
[8] Krylov, N.V., 1980. *Controlled Diffusion Processes*, Berlin: Springer Verlag.

[9] Mnif, M., Sulem, A., 2005. *Optimal risk control and dividend policies under excess of loss reinsurance*, Stochastics 77, No.5, 455-476.

[10] Pham, H., 1998. *Optimal Stopping of Controlled Jump Diffusion Processes: A Viscosity Solution Approach*, Journal of mathematical systems, estimation, and control 8, No.1, 127-130.

[11] Sayah, A., 1991. *Equations d’Hamilton-Jacobi du premier ordre avec termes intégro-différentiels: Parties I at II*, Communications in Partial Differential Equations, 10, 1057-1093.

[12] Soner, H.M., 1986. *Optimal control with state-space constraint II*, SIAM J. Control and Optimization 24, 1110-1122.

[13] Yong, J., Zhou, X. Y., 1999. *Stochastic Controls (Hamiltonian Systems and HJB Equations)*, Springer Verlag, Berlin.