Generating robust entanglement via quantum feedback

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Abstract
Generating entangled states is one of the most important tasks in quantum information technology. However, in reality any entanglement generator must contain some characteristic uncertainty, and as a result the produced entangled state becomes an undesirable mixed state. This paper develops a coherent feedback control scheme that suppresses the characteristic uncertainty of a typical entanglement generator (the non-degenerate optical parametric oscillator) for producing robust Gaussian entangled states. In particular, we examine a two-mode squeezed state and Gaussian four-mode cluster states to demonstrate the effectiveness of the proposed control method.

Keywords: quantum control, cluster states, quantum feedback, entangled states, non-degenerate optical parametric oscillator

(Some figures may appear in colour only in the online journal)

1. Introduction

Entangled states are an essential resource in quantum information technology, such as the quantum cryptography [1] and the quantum teleportation [2–4]. In particular, continuous-variable (CV) systems are well-established platforms for demonstrating those quantum information processing [5]; for instance, Gaussian CV cluster states [6, 7] are an important class of entangled states that can be applied to the one-way quantum computation [8].

However, in practice there always exists a fragility issue in the process of generating entangled states, which as a result could largely degrade the performance of quantum information processing. To be specific, let us consider the non-degenerate optical parametric oscillator (NDPO) [9–12], which can be used for generating various types of entangled states. The NDPO is an optical cavity containing a nonlinear crystal; two photons entering the nonlinear crystal will be amplified by a strong electromagnetic wave (pump), and at the same time these photons become entangled; as a result the NDPO outputs an entangled light field. The fragility issue in this device is that the system parameters such as the pump gain and the cavity length easily change, which consequently induces fluctuation on the output entangled state. This means that the resulting entangled state must be a mixed state.

Therefore, it is important to devise a robust entanglement generator which is ideally free from the system’s characteristic uncertainties. The key technique that is generally used for suppressing such a system fluctuation is feedback control. In the classical (non-quantum) case, the basic configuration of the feedback control is shown in figure 1. Let us consider a system P (called the ‘plant’), which outputs the signal y. For example, think y as a voltage and ω as a frequency of the input. We want the voltage to be, say y = 5.0 V at ω = 0; however, due to the inevitable characteristic uncertainty contained in any electric device, the output voltage must vary from the target value. The general solution is to feed a portion of the output y back to the input by passing it through a robust system C (called the ‘controller’); then a suitably chosen controller may suppress the plant’s fluctuation, and as a result the total system generates a less-fluctuating output. This effect can be clearly seen especially when the plant is given by an amplifier. Let us now interpret P and C as the gain at ω = 0 of the plant and the controller, respectively; then an input signal to the process is feedforward.
total system, $u$, is transformed to the output

$$y = \frac{p}{1 + PC} u$$

which converges to $y = u/C$ in the limit $P \to \infty$. This does not depend on $P$, and thus the output is robust against the plant’s uncertainty involved in $P$.

Our idea is to apply the above idea to the problem of robust entanglement generation, where particularly the plant $P$ is given by a multi-mode Gaussian entanglement generator composed of a single NDPO and some beam splitters. In fact we show that, with the aid of feedback control, the controlled system becomes robust against the plant’s fluctuation and obtains the ability to selectively produce several types of cluster states in a robust way. Note that this is a non-trivial extension of the work [13], where a similar feedback scheme is applied to engineer a robust phase-insensitive quantum linear amplifier; here by the word ‘non-trivial’ we mean that how to configure the total feedback-controlled system composed of a multi-mode entangler and a controller is not clear, compared to the simple feedback amplification problem studied in [13]. Actually we show that, in the problem of generating four-mode cluster states, the effect of the feedback control differs depending on the structure of the feedback loop.

2. Preliminaries

2.1. The NDPO

Here we review the dynamics of the NDPO (see figure 2). The NDPO has two internal modes (annihilation operators) $a_1$ and $a_2$, which are called the signal mode and the idler mode, respectively. The signal mode with frequency $\omega_1$ and the idler mode with frequency $\omega_2$ couple at the nonlinear crystal driven by the classical pump mode with frequency $2\omega_p$. The interaction Hamiltonian is given by

$$H = \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2 + \hbar \lambda (a_1^\dagger a_2^\dagger e^{-2i\omega_p t} - a_1 a_2 e^{2i\omega_p t})$$

where $\lambda \in \mathbb{R}$ is a coupling constant. The two modes $a_1$ and $a_2$ become entangled through this interaction; they leave the cavity and are transformed to an entangled traveling field between the field annihilation operators $b_1^{\text{out}}$ and $b_2^{\text{out}}$. Now let $b_1^{\text{in}}$ and $b_2^{\text{in}}$ be the input modes entering the cavity at the mirrors. Then, the quantum Langevin equations [14–16] of the cavity modes in the rotating frame at $\omega_p$ are given by

$$\frac{da_1}{dt} = -i\Delta_1 + \frac{\kappa}{2} a_1 + \lambda a_2^\dagger - \sqrt{\kappa} b_1^{\text{in}}$$

$$\frac{da_2}{dt} = -i\Delta_2 + \frac{\kappa}{2} a_2^\dagger + \lambda a_1 - \sqrt{\kappa} b_2^{\text{in}}$$

where $\kappa$ is a damping rate and $\Delta_i := \omega_i - \omega_p$ are detunings of the cavity modes from the pump mode ($\hbar = 1$ in this paper). Also, the input-output relations are

$$b_1^{\text{out}} = \sqrt{\kappa} a_1 + b_1^{\text{in}}, \quad b_2^{\text{out}} = \sqrt{\kappa} a_2^\dagger + b_2^{\text{in}}$$

By Laplace transforming equations (1)–(3), followed by eliminating $a_1$ and $a_2$, one obtains

$$\begin{bmatrix} b_1(s) \\ b_2(s) \end{bmatrix} = G(s) \begin{bmatrix} b_1^{\text{in}}(s) \\ b_2^{\text{in}}(s) \end{bmatrix}$$

$$G(s) = \frac{G_1(s)}{G_2(s)}$$

where $s \in \mathbb{C}$ is the Laplace variable and $G_i(s)$ are

$$G_1(s) = \frac{(s - \kappa/2 + i\Delta_1)(s + \kappa/2 - i\Delta_2) - \lambda^2}{D(s)}$$

$$G_2(s) = \frac{-\lambda \kappa}{D(s)}$$

with

$$D(s) = \left(s + \frac{\kappa}{2} + i\Delta_2\right)\left(s + \frac{\kappa}{2} - i\Delta_2\right) - \lambda^2$$

They satisfy $|G_1(i\omega)| = |G_2(i\omega)|$, $|G_1(\omega)| = |G_2(\omega)|$, and $G_1(\omega)G_2(\omega) - G_1(i\omega)G_2(i\omega) = G_2(\omega)/G_1(\omega)$ for all $\omega$. 

---

Figure 1. Configuration of the general classical feedback control. In the absence of feedback, the output $y$ fluctuates (see the right figure). By adding a feedback loop with controller $C$, the fluctuation of $y$ can be suppressed.

Figure 2. Schematic of NDPO. The two input modes $b_1^{\text{in}}, b_2^{\text{in}}$ enter the system, and two outputs $b_1^{\text{out}}, b_2^{\text{out}}$ come out.
2.2. The quantum feedback amplification method

Here we review the coherent feedback method for engineering a robust quantum amplifier [13]. The plant $G$ is a general 2-inputs and 2-outputs linear phase-insensitive amplifier shown in figure 3(a); the elements $G_i(\omega)$ satisfy the relations below equation (4). The feedback structure is shown in figure 3(b); the idler output of $G$ is connected to the idler input of $G$, through a 2-inputs and 2-outputs linear passive quantum system $K$ such as an empty optical cavity, without involving any measurement process [17–22]. We express the controller’s transfer function matrix $K(s)$ in the Laplace domain as

$$K(s) = \begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix}. $$

Note that, from the passivity property, $K(\omega)$ is a unitary matrix satisfying $K(\omega)K'(\omega) = I$, where $I = \text{diag}[1, 1]$. Then, the input–output relation of the total system depicted in figure 3(b) is given by

$$\begin{bmatrix} b_1(s) \\ b_2(s) \end{bmatrix}_{\text{out}} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} b_1(s) \\ b_2(s) \end{bmatrix}_{\text{in}},$$

where

$$G_{11}(s) = \frac{G_{11} - K_{21}(G_{12}G_{22} - G_{12}G_{21})}{1 - K_{21}G_{22}},$$

$$G_{12}(s) = \frac{G_{12}K_{22}}{1 - K_{21}G_{22}},$$

$$G_{21}(s) = \frac{G_{21}K_{12}}{1 - K_{21}G_{22}},$$

$$G_{22}(s) = \frac{K_{21} + K_{22}\det K}{1 - K_{21}G_{22}}.$$

(We often omit the variable $s$ or $i\omega$ for short, like the above equations.) From the equations below equation (4), one can obtain the following result:

$$|G_{11}^{(b)}(i\omega)| = \frac{|G_{11}/G_{11} - K_{21}(G_{11}G_{22} - G_{12}G_{21})/|G_{11}|}{1/|G_{11}| - K_{21}G_{22}/|G_{11}|} \rightarrow \frac{1}{|K_{21}(i\omega)|}, \quad (|G_{11}| \rightarrow \infty).$$

Therefore, if $G(\omega)$ is a large-gain amplifier, the gain of the whole controlled system depends only on the passive (and thus robust) component $|K_{11}(i\omega)|$. That is, the controlled system functions as a robust amplifier with gain $1/|K_{21}(i\omega)| > 1$.

3. Robust two-mode squeezed (TMS) state

In this section we show that the feedback amplification technique discussed in section 2.2 can be applied to generate a robust TMS state. In our scenario this is a Gaussian entangled state between the signal and idler output fields of the NDPO [12]. To quantify the entanglement of TMS state, we apply the entanglement entropy, which can be explicitly calculated in terms of the covariance matrix (CM) (see appendix A). The CM of the output state of the general (non-controlled) linear amplifier $G$ is given, in the frequency domain, by

$$\gamma(\omega) = \frac{1}{2} \begin{bmatrix} |G_{11}|^2 + |G_{12}|^2 & 2G_{11}G_{12}I_{11} \\ 2G_{11}G_{12}I_{11} & |G_{11}|^2 + |G_{12}|^2 \end{bmatrix},$$

where $I_{11} = \text{diag}[1, -1]$. Also $|G_{11}| = |G_{22}|$ and $|G_{12}| = |G_{21}|$ are used. From this CM, one obtains the entanglement entropy $S(\omega)$ of the TMS state as

$$S(\omega) = |G_{11}(i\omega)|^2 \ln |G_{11}(i\omega)|^2 - |G_{12}(i\omega)|^2 \times \ln |G_{12}(i\omega)|^2.$$ 

Note that the output field state of the feedback-controlled amplifier is also a TMS state, and its CM $\gamma^{fb}(\omega)$ and the entanglement entropy $S^{fb}(\omega)$ can be obtained simply by replacing $G(\omega)$ by $G^{fb}(\omega)$ in the above two formulas $\gamma(\omega)$ and $S(\omega)$. Then, because $|G^{fb}(\omega)|$ are free from the characteristic uncertainty of the original amplifier $G$ in the limit $|G_{11}(i\omega)| \gg 1$, the entanglement property of the output state of the feedback-controlled system is robust against those uncertainty in such a large amplification limit. This is the central idea of robust entanglement generation via feedback amplification.

We consider the NDPO discussed in section 2.1. The pole of this linear system is $s_{\pm} = -\kappa/2 \pm \sqrt{\lambda^2 - \Delta^2}$, and the gain at $s = 0$ is

$$|G_{11}(0)|^2 = \frac{\kappa^4 + 16(\Delta^2 - \lambda^2)^2 + 8\kappa^2(\Delta^2 + \lambda^2)}{(\kappa^2 - 4\lambda^2 + 4\Delta^2)^2},$$

where $\Delta_1 = \Delta_2 = \Delta$ is assumed. In the usual setting with cavity-locked NDPO ($\Delta = 0$), the parameter is chosen as $\kappa = 2\Delta + 0$ to realize a high-gain amplification. However, this induces $s_+ \rightarrow 0$, meaning that the amplifier becomes nearly unstable; that is, there is a trade-off between the gain and stability of the system. To circumvent this issue, we take a special type of NDPO satisfying $\Delta = \lambda$ [13]; then, because $s_{\pm} = -\kappa/2$, such a trade-off does not appear. In this case, from $|G_{11}(0)| = \sqrt{1 + 16\lambda^2/\kappa^2}$, the gain of the NDPO increases monotonically with $\lambda$. Here the controller is set to a

\[ \text{Figure 3. (a) Diagram of the NDPO without control. (b) The coherent feedback configuration; the idler mode is used for feedback while the signal mode is not touched.} \]
beamsplitter, which is independent of the frequency:
\[ K(i\omega) = K = \begin{bmatrix} \tau & -\rho \\ \rho & \tau \end{bmatrix}, \quad \tau^2 + \rho^2 = 1, \]
(5)
where \(\tau\) is the transmissivity and \(\rho\) is the reflectivity. In particular here we take \(\rho = 0.04\), which satisfies the stability condition \(|\rho| < \kappa/2\lambda\) for the feedback-controlled system [13]. Also \(\lambda = 10\kappa\). The effect of feedback can be clearly seen by examining the sensitivity of the entanglement entropy, \(\partial S(\omega)/\partial \lambda\); here, for simplicity, only the coupling strength \(\lambda\) is assumed to change in the vicinity of \(\lambda = 10\) and \(\kappa = 1\). Figure 4(a) shows the sensitivity of the non-controlled system \(\partial S(0)/\partial \lambda\big|_{\lambda=0}\) and that of the controlled one \(\partial S^b(0)/\partial \lambda\big|_{\lambda=10}\). It is clear that the feedback control drastically lowers the sensitivity, meaning that the entangled state is robust against an unexpected change of \(\lambda\).

Let us now see the robustness of the entanglement entropy in the frequency domain. The two parameters \(\lambda\) and \(\kappa\) can vary up to 10\% from their nominal values, i.e., \(\lambda = (1 + \delta_j)\lambda_0\) and \(\kappa = (1 + \delta_j)\kappa_0\), where \(\lambda_0\) and \(\kappa_0\) are the nominal values satisfying \(\lambda_0 = 10\kappa_0\). Figure 4(b) shows 90 samples of entanglement entropy, where the red and blue lines correspond to the case with and without feedback, respectively. Also \(\delta_1\) and \(\delta_2\) deterministically and linearly change from −0.1 to 0.1. Clearly, in the low-frequency regime, the variation of \(S^b(\omega)\) due to the fluctuation of \((\lambda, \kappa)\) is smaller than that of \(S(\omega)\), at the price of decreasing the degree of entanglement. Also, it is true that lowering the NDPO’s gain causes the entanglement less-fluctuate, which brings the question of whether the robustness shown in figure 4(b) intrinsically stems from the feedback control. To demonstrate that the robustness comes from the feedback itself, we compare entanglement entropy of the low-gain non-controlled TMS states (figure 5(a)) with that of the feedback-controlled TMS states (figure 5(b)). Note that figure 5(b) is exactly the same as the red lines in figure 4(b). To compare the effect of ‘the low-gain prescription’ with the feedback control, we set the nominal values of their entanglement entropy the same. This requires \(\lambda_0 = 5\kappa_0\) for the low-gain prescription and \(\lambda_0 = 10\kappa_0\) for the feedback-controlled system. From this comparison, we can conclude that the robustness feature represented in figure 4(b) stems from the proposed feedback control since the fluctuation cannot be more suppressed than the feedback-controlled one.

We note that it is shown in [13] that the proposed feedback control has robustness against losses in the feedback channel. On the other hand, it is true that the whole feedback controlled system may be largely affected by losses in the controller \(K\). However, as long as the controller is a static device, such as a beamsplitter as in our case, the whole feedback-controlled system is insensitive to such losses. There are other types of methods [23, 24] to generate Gaussian TMS states from the NDPOs, which is robust to losses in connection channels.

### 4. Robust Gaussian cluster states

In the previous section, we showed how much the feedback control suppresses the fluctuation of the two-mode entangled state. Here we expand this result to a multi-mode case, in particular four-mode Gaussian cluster states with linear, T-shape, and square structures [7, 25, 26]. Although there are several ways to create cluster states [27], we take the method using only a single multi-mode NDPO [26, 28–31].

#### 4.1. Linear cluster state

We begin with a linear cluster state depicted in figure 7(a); the label \(j = (1, 2, 3, 4)\) in the figure corresponds to \(b_{j,\text{out}}\), which is the \(j\)th output field mode of a single multi-mode NDPO. The input–output relation of the NDPO in the Laplace domain is written as

\[
\begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    b_4^{\text{out}}
\end{bmatrix} = G(s) \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    b_4^{\text{in}}
\end{bmatrix},
\]
(6)
where \(G(s)\) is the \(4 \times 4\) transfer function matrix (see appendix B). Each matrix element contains the coupling constants \(\lambda_i (i = 1, 2, 3)\), the damping rates \(\kappa_j (j = 1, 2, 3, 4)\), and the detunings \(\Delta_j (j = 1, 2, 3, 4)\). Here, for simplicity, we assume \(\lambda = \lambda_i\), \(\kappa = \kappa_j\), and \(\Delta = \Delta_j\) for all \(i, j\). The poles...
of the NDPO is given by
\[ s = \frac{-\kappa}{2} \pm \sqrt{\frac{3 - \sqrt{5}}{2} \lambda^2 - \Delta^2}, \]
\[ = \frac{-\kappa}{2} \pm \sqrt{\frac{3 + \sqrt{5}}{2} \lambda^2 - \Delta^2}. \]

As discussed in section 3, we take \( \Delta = \sqrt{(3 + \sqrt{5})/2 \lambda} \), leading that all the poles strictly locate in the left-side of the complex plane for arbitrary large gain of the amplifier; that is, the gain-stability trade-off does not appear.

Let us apply the feedback control to the above 4-inputs and 4-outputs NDPO. Unlike figure 3(b), the feedback configuration is non-trivial to design; we need to choose pairs of \((b_{j,\text{out}}, b_{k,\text{in}})\) and connect them via coherent feedback through a controller \(K(s)\). In this paper we particularly consider the mode-\(j\) feedback (FB), meaning that \(b_{j,\text{out}}\) and \(b_{j,\text{in}}\) are connected for an index \(j\) (figure 6 is the case of \(j = 3\)). Then the transfer function matrix \(G_{ij}(s)\) for the feedback-controlled system is composed of the following elements:

\[ G_{ij}^{\text{fb}} = \frac{K_{i2} + G_{ij} \det K}{1 - G_{ij} K_{21}}, \]
\[ G_{ik}^{\text{fb}} = \frac{G_{ik} K_{12}}{1 - G_{ij} K_{21}}, \quad G_{ij}^{\text{fb}} = \frac{G_{ij} K_{22}}{1 - G_{ij} K_{21}}, \quad (k, l \neq j), \]
\[ G_{ik}^{\text{fb}} = \frac{G_{ik} + K_{21}(G_{ij} G_{ik} - G_{ij} G_{ik})}{1 - G_{ij} K_{21}}, \quad (k, l \neq j). \]

Because \(G_{ij}\) shows up in the denominators of \(G_{ij}^{\text{fb}}\), the effect of feedback control appears when \(|G_{ij}| \to \infty\).

We now see the robustness property of the feedback-controlled system, by examining the entanglement entropy and its sensitivity; here particularly we focus on the entanglement entropy between the 1th mode and the other modes, with (the red and green lines) and without (the blue lines) feedback.

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robust than the mode-4 FB. This is simply because the mode-2 has two links while the mode-4 has only one; hence, the use of the former as feedback would be more effective to suppress the fluctuation added on all nodes. This difference can be observed in the frequency domain as well; figure 7(c) shows 90 sample paths of $S_I(i\omega)$, where, as in the TMS case, $(\lambda, \kappa)$ deterministically and linearly change up to 10% from their nominal values satisfying $\lambda_0 = 10\kappa_0$.

### 4.2. T-shape cluster state

The next example is the T-shape cluster state whose structure is shown in figure 8(a). The system Hamiltonian is given in appendix B, where for simplicity $\lambda = \lambda_j$, $\kappa = \kappa_j$, and $\Delta = \Delta_j$ are assumed for all $l, j$. The poles of the non-feedback NDPO are 

$$s = -\kappa/2 + i\Delta$$

and 

$$s = \kappa/2 \pm \sqrt{3\lambda^2 - \Delta^2}.$$ 

Similar to the previous cases, we take $\Delta = \sqrt{3}\lambda$ to avoid the gain-stability trade-off. As for the controller, the beam splitter with $\varrho = 0.024$ and $\varphi = 0.024$ are chosen for the mode-1 FB and the mode-2 FB, respectively. Under this condition, the sensitivity $\delta S_I(0)/\partial \lambda$ and the entanglement entropy $S_I(i\omega)$ are depicted in figures 8(b) and (c) respectively, where $\lambda_0 = 10\kappa_0$ and up to 10% fluctuation are added to $(\lambda, \kappa)$. Likewise the linear cluster case, the mode-1 FB scheme realizes the better suppression than the mode-2 FB, presumably because controlling the mode-1 can affect on all the other modes through the direct links while the mode-2 can do that only via an indirect way.

### 4.3. Square cluster state

Finally, we examine the square cluster state shown in figure 9(a). Again we assume $\lambda = \lambda_j$, $\kappa = \kappa_j$, and $\Delta = \Delta_j$. The poles are 

$$s = -\kappa/2 + i\Delta,$$ 

and 

$$s = -\kappa/2 \pm \sqrt{4\lambda^2 - \Delta^2},$$

leading to $\Delta = 2\lambda$. The controller is a beam splitter with $\varrho = 0.04$. Then with the same parameters choice as in the previous case, the sensitivities and the entanglement entropies are depicted in figures 9(b) and (c). As expected, the mode-1 FB and the mode-2 FB schemes have the same effect on the robustness, due to the symmetric structure.

### 5. Conclusion

This paper has demonstrated that the feedback amplification technique proposed in [13] is effective for generating robust Gaussian entangled states. In particular, we have seen that the degree of robustness depends on the structure of feedback control; for the four-mode cluster states examined in this paper, our conclusion was that we should choose the mode having the biggest number of connection to the others, to construct the feedback loop. However, determining the most effective feedback for the general case is not a trivial problem and needs extensive investigation. Considering the fact that a feedback amplification architecture is involved in almost all electric circuits to generate robust functionalities, therefore, the result shown in this paper would be a first step toward
developing a quantum circuit theory for robust quantum functionalities such as teleportation.

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Appendix A. Entanglement measure

Let us consider a density operator \( \rho \) of a total system \( \mathcal{H}_{\text{tot}} \), which we can divide into two Hilbert spaces: \( \mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B \). Then, the entanglement entropy \( S_A \) of the Hilbert space \( \mathcal{H}_A \) is defined as \( S_A = - \text{Tr} (\rho_A \ln \rho_A) \), where \( \rho_A \) is the reduced density operator \( \rho_A = \text{Tr}_B (\rho) \). It is worth noting that the entropy always satisfies \( S_A \geq 0 \) where \( S_A = 0 \) means that \( \mathcal{H}_A \) and \( \mathcal{H}_B \) are not entangled. Also, it satisfies \( S_A = S_B \) if the state in the total system \( \mathcal{H}_{\text{tot}} \) is in a pure state.

Although the entanglement entropy is generally hard to calculate, it is straightforward in the Gaussian case [32–35]. Let us consider an \( n \)-mode Gaussian system with vector of quadratures \( \mathbf{x} = [q_1, p_1, q_2, p_2, \ldots, q_n, p_n] \), with \( q = (b + b^\dagger)/\sqrt{2} \) and \( p = (b - b^\dagger)/\sqrt{2}i \). Then the CM \( \gamma \) is defined as

\[
\gamma = \mathfrak{N} \{ \mathbf{x} \mathbf{x}^\dagger \},
\]

where \( \mathfrak{N} \) is an expected value. Let us denote \( \mathbf{x}_{\text{out}} \) and \( \mathbf{x}_{\text{in}} \) as vectors of the output and the input modes respectively. These two vectors are related by \( \mathbf{x}_{\text{out}} = Y \mathbf{x}_{\text{in}} \), where \( Y \) is a \( 2n \times 2n \) matrix. In the vacuum input case, where \( \langle q^2 \rangle = \langle p^2 \rangle = 1/2 \), \( \langle qp \rangle = i/2 \), and \( \langle pq \rangle = -i/2 \) hold, the CM of the output state is given by

\[
\gamma = \mathfrak{N} \{ Y \mathbf{x}_{\text{in}} \mathbf{x}_{\text{in}}^\dagger \} = \frac{1}{2} \mathfrak{N} \{ Y Y^\dagger \},
\]

where

\[ R = \oplus_{k=1}^n \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \]

Now, the entanglement entropy between the \( j \)th mode and all the other modes is defined by

\[
S_j = \left( \sigma_j + \frac{1}{2} \right) \ln \left( \sigma_j + \frac{1}{2} \right) - \left( \sigma_j - \frac{1}{2} \right) \ln \left( \sigma_j - \frac{1}{2} \right),
\]

where \( \sigma_j \) is the \( j \)th symplectic eigenvalue of the CM. More specifically, \( \pm i \sigma_j \) is the eigenvalue of \( \gamma^j / \Omega \), where

\[
\begin{bmatrix}
\langle q_j^2 \rangle \\
\langle qp_j + p_j q_j \rangle / 2 \\
\langle q_j p_j + p_j q_j \rangle / 2 \\
\langle p_j^2 \rangle
\end{bmatrix}
\]

\[
\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

Note that if \( \sigma_j = 1/2 \) then \( S_j = 0 \), meaning that there is no entanglement between the mode-\( j \) and the others.

Appendix B. NDPO dynamics for the four mode Gaussian cluster state

The linear cluster state is generated, via the use of two pump beams entering a nonlinear crystal (see figure B1). The pump with frequency \( 2\omega_{p1} \) generates a pair of the modes \( \{1, 2\} \), and the pump with frequency \( 2\omega_{p2} \) generates two pairs of the modes \( \{2, 3\} \) and \( \{1, 4\} \). The system Hamiltonian representing this couplings is given by

\[
H_{\text{sys}} = \sum_{j=1}^4 \omega_j a_j^\dagger a_j + [i \eta_1 a_1^\dagger a_2^\dagger a_3 a_4 + \eta_2 a_1^\dagger a_2^\dagger a_3 a_4 + \eta_3 a_1^\dagger a_2^\dagger a_4 + \eta_4 a_1^\dagger a_2^\dagger a_3 + \text{h.c.}],
\]

where \( a_j \) is the annihilation operator of mode-\( j \). Also \( \omega_j \) is the frequency of mode-\( j \), and \( \eta_j \) is the coupling constant between the cavity modes. We now use the undepleted pump approximation [30, 31, 36] to make the interactions linear; that is, \( a_{p1} \) and \( a_{p2} \) are treated as c-numbers and are absorbed into \( \eta_j \) which redefines the coupling constants as \( \eta_j \rightarrow \lambda_j \). The resulting quantum Langevin equations for the cavity modes \( a_j \) are

\[
\dot{a}_1 = -i \Delta_1 + \frac{\kappa_1}{2} a_1 + \lambda_1 a_3 a_2 + \sqrt{\kappa_1} b_{1,\text{in}},
\]

\[
\dot{a}_2 = -i \Delta_2 + \frac{\kappa_2}{2} a_2^\dagger + \lambda_1 a_1 + \lambda_3 a_3 - \sqrt{\kappa_2} b_{2,\text{in}},
\]

\[
\dot{a}_3 = -i \Delta_3 + \frac{\kappa_3}{2} a_3 + \lambda_2 a_2 - \sqrt{\kappa_3} b_{3,\text{in}},
\]

\[
\dot{a}_4 = -i \Delta_4 + \frac{\kappa_4}{2} a_4^\dagger + \lambda_2 a_1 - \sqrt{\kappa_4} b_{4,\text{in}},
\]

where \( \kappa_j \) is the damping rates, \( \Delta_j \) is the detuning, and \( b_{j,\text{in}} \) is the input field mode interacting with \( a_j \). Also, the input–output relations are given by \( b_{j,\text{out}} = b_{j,\text{in}} + \sqrt{\kappa_j} a_j \). After the Laplace transform is applied to the above quantum Langevin equations, the input fields \( b_{j,\text{in}} \) and the output fields \( b_{j,\text{out}} \) are related by the following symmetric transfer function matrix \( G(s) = G(s)^T \):

\[
\begin{bmatrix}
\tilde{b}_1 \\
\tilde{b}_2 \\
\tilde{b}_3 \\
\tilde{b}_4
\end{bmatrix}_{\text{out}} =
\begin{bmatrix}
G_{11} & G_{12} & G_{13} & G_{14} \\
G_{22} & G_{23} & G_{24} & b_1 \\
G_{34} & b_3 & G_{33} & b_2 \\
G_{44} & b_4 & b_3 & G_{44}
\end{bmatrix}_{\text{in}},
\]
where * denote the symmetric elements and
\[
G_{11} = 1 + \kappa_1 A_1 A_3 - \lambda_1^2 / D,
\]
\[
G_{12} = \lambda_1 \sqrt{\kappa_2 / \kappa_3} A_2 A_4 / D,
\]
\[
G_{13} = \lambda_1 \sqrt{\kappa_3 / \kappa_2} A_4 / D,
\]
\[
G_{14} = \lambda_1 \sqrt{\kappa_3 / \kappa_2} (A_2 A_4 - \lambda_1^2) / D,
\]
\[
G_{22} = 1 + \kappa_2 A_3 (A_1 A_4 - \lambda_2^2 / D),
\]
\[
G_{23} = \lambda_2 \sqrt{\kappa_1 / \kappa_3} (A_1 A_4 - \lambda_2^2) / D,
\]
\[
G_{24} = \lambda_2 \sqrt{\kappa_1 / \kappa_3} (A_1 A_4 - \lambda_2^2 A_2) / D,
\]
\[
G_{33} = 1 + \kappa_3 (A_1 A_4 - \lambda_3^2 A_4 - \lambda_2^2 A_2) / D,
\]
\[
G_{34} = \lambda_3 \sqrt{\kappa_2 / \kappa_1} A_2 A_4 / D,
\]
\[
G_{44} = 1 + \kappa_4 (A_1 A_3 A_4 - \lambda_4^2 A_3 - \lambda_2^2 A_1) / D,
\]
where
\[
D = \lambda_1^2 A_4 - (A_1 A_4 - \lambda_2^2) (A_2 A_3 - \lambda_3^2),
\]
\begin{align*}
A_1 &= s + i \Delta_1 + \kappa_1 / 2, \\
A_2 &= s - i \Delta_2 + \kappa_2 / 2, \\
A_3 &= s + i \Delta_4 + \kappa_4 / 2.
\end{align*}

The Y matrix appearing in appendix A is readily obtained from G, and as a result the entanglement entropy of the linear cluster state can also be calculated easily.

As for the T-shape and square cluster states, we just give the system Hamiltonians. In these cases three pump modes are used, as seen in figure B2. For the T-shape case, the system Hamiltonian within the undepleted pump approximation is given by
\[
H_{\text{sys}} = \sum_{j=1}^{4} \omega_j a_j^\dagger a_j + i [\lambda_j a_j^\dagger a_j^2 + \lambda_2 a_j^\dagger a_j^3 + \lambda_3 a_j^\dagger a_j^4 + \lambda_4 a_j^\dagger a_j^4] + \text{h.c.}.
\]

Also for the square cluster state, it is given by
\[
H_{\text{sys}} = \sum_{j=1}^{4} \omega_j a_j^\dagger a_j + i [\lambda_j a_j^\dagger a_j^2 + \lambda_2 a_j^\dagger a_j^3 + \lambda_3 a_j^\dagger a_j^4 + \lambda_4 a_j^\dagger a_j^4] + \text{h.c.}.
\]

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