A Simple Proof of Maxwell Saturation for Coupled Scalar Recursions

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Abstract—Low-density parity-check (LDPC) convolutional codes (or spatially-coupled codes) were recently shown to approach capacity on the binary erasure channel (BEC) and binary-input memoryless symmetric channels. The mechanism behind this spectacular performance is now called threshold saturation via spatial coupling. This new phenomenon is characterized by the belief-propagation threshold of the spatially-coupled ensemble increasing to an intrinsic noise threshold defined by the uncoupled system.

In this paper, we present a simple proof of threshold saturation that applies to a wide class of coupled scalar recursions. Our approach is based on constructing potential functions for both the coupled and uncoupled recursions. Our results actually show that the fixed point of the coupled recursion is essentially determined by the minimum of the uncoupled potential function and we refer to this phenomenon as Maxwell saturation.

A variety of examples are considered including the density-evolution equations for: irregular LDPC codes on the BEC, irregular low-density generator matrix codes on the BEC, a class of generalized LDPC codes with BCH component codes, the joint iterative decoding of LDPC codes on intersymbol-interference channels with erasure noise, and the compressed sensing of random vectors with i.i.d. components.

Index Terms—convolutional LDPC codes, Maxwell conjecture, potential functions, spatial coupling, threshold saturation

I. INTRODUCTION

Convolutional low-density parity-check (LDPC) codes, or spatially-coupled (SC) LDPC codes, were introduced in [3] and shown to have excellent belief-propagation (BP) thresholds in [4], [5], [6]. Moreover, they have recently been observed to universally approach the capacity of various channels [6], [7], [8], [9], [10], [11], [12], [13].

The fundamental mechanism behind this is explained well in [14], where it is proven analytically for the binary erasure channel (BEC) that the BP threshold of a particular SC ensemble converges to the maximum-a-posteriori (MAP) threshold of the underlying ensemble. This phenomenon is now called threshold saturation. A similar result was also observed independently in [15] and stated as a conjecture. The same result for general binary memoryless symmetric (BMS) channels was first empirically observed [6], [7] and recently proven analytically [13]. This result implies that one can achieve the capacity universally over BMS channels because the MAP threshold of regular LDPC codes approaches the Shannon limit universally, as the node degrees are increased.

The underlying principle behind threshold saturation appears to be very general and spatial coupling has now been applied, with much success, to a variety of more general scenarios in information theory and coding. In [16], [17], the benefits of spatial coupling are described for K-satisfiability, graph coloring, and the Curie-Weiss model in statistical physics. SC codes are shown to achieve the entire rate-equivocation region for the BEC wiretap channel in [8]. The authors observe in [9] that the phenomenon of threshold saturation extends to multi-terminal problems (e.g., a noisy Slepian-Wolf problem) and can provide universality over unknown channel parameters.

Threshold saturation has also been observed for the binary-adder channel [18], for intersymbol-interference channels [10], [11], [12], for message-passing decoding of code-division multiple access (CDMA) [19], [20], [21], and for iterative hard-decision decoding of SC generalized LDPC codes [22]. For compressive sensing, SC measurement matrices were investigated with verification-based reconstruction in [23], shown to have excellent performance with BP decoding in [24], and proven to achieve the information-theoretic limit in [25].

In many of these papers it is conjectured, either implicitly or explicitly, that threshold saturation occurs for the studied problem. A general proof of threshold saturation (especially one where only a few details must be verified for each system) would allow one to settle all of these conjectures simultaneously. In this paper, we provide such a proof for the case where the system of interest is characterized by a coupled scalar recursion [1]. Our results actually go further and establish that the fixed point of the coupled recursion is essentially determined by the minimum of a potential function associated with the uncoupled recursion [2]. We call this phenomenon Maxwell saturation because the fixed point of the coupled recursion saturates to a value that is closely related to the Maxwell curve of the uncoupled system [26].

Our method is based on potential functions and was motivated mainly by the approach taken in [27]. It turns out that their approach is missing a few important elements and does not, as far as we know, lead to a general proof of threshold saturation. Still, it introduces the idea of using a
potential function defined by an integral of the DE recursion and this is an important element in our approach. Elements of this approach have also appeared in some previous proofs of threshold saturation (e.g., [23], [21]) where the authors relied on a continuum approach to DE. A little while after we posted the conference version of this article [1, Kudekar et al. posted their general proof of threshold saturation, which was derived independently based on a continuum approach and a slightly different potential function [23]. Later, we extended our approach to show Maxwell saturation in [2] and they also extended their approach in [29]. The approach used in this paper has also been extended to vector recursions in [30] and BMS channels in [31], [32]. The connections between these approaches is discussed more thoroughly in Section II-B.

A. Notation

The following notation is used throughout the paper. We define the closed real intervals \( \mathcal{X} = [0, x_{\text{max}}] \) with \( x_{\text{max}} \in (0, \infty) \), \( \mathcal{Y} = [0, y_{\text{max}}] \) with \( y_{\text{max}} \in (0, \infty) \), and \( \mathcal{E} = [0, \varepsilon_{\text{max}}] \) with \( \varepsilon_{\text{max}} \in (0, \infty) \). Intervals of natural numbers are denoted by \([n:n]\triangleq \{m, m+1, \ldots, n\}\).

Vectors are denoted in boldface (e.g. \( \mathbf{x} \in \mathcal{X}^n \)), assumed to be column vectors, and their elements are denoted by \( \mathbf{x}_1 \) or \( x_i \) for \( i \in [1 : n] \). For vectors (e.g. \( \mathbf{x}, \mathbf{z} \in \mathcal{X}^n \)), we use the partial order \( \mathbf{x} \preceq \mathbf{z} \) defined by \( x_i \leq z_i \) for \( i \in [1 : n] \). A vector mapping (e.g. \( \mathbf{f}(\mathbf{x}) \)) that is non-decreasing w.r.t. the partial order (i.e., \( \mathbf{x} \preceq \mathbf{z} \) implies \( \mathbf{f}(\mathbf{x}) \preceq \mathbf{f}(\mathbf{z}) \)) is called isotone. The symbols \( 0_n \) and \( \mathbf{1}_n \) denote the all-zeros and all-ones column vectors of length \( n \) and the subscript is sometimes dropped when the length is apparent.

We use standard-weight type (e.g. \( \mathbf{f}(\mathbf{x}) \)) and \( \mathbf{F}(\mathbf{x}) \) to denote scalar-valued functions and boldface (e.g. \( \mathbf{f}(\mathbf{x}) \)) and \( \mathbf{g}(\mathbf{x}) \) to denote vector-valued functions. We say a function \( f: \mathcal{Z} \rightarrow \mathbb{R} \) is \( C^k \) if its \( k \)-th derivative exists and is continuous on \( \mathcal{Z} \) and we denote the supremum of \( f \) over its domain by \( \|f\|_{\infty} \triangleq \sup_{z \in \mathcal{Z}} f(z) \).

The gradient vector of a scalar-valued function is defined by \( \nabla f(x) = \partial F(x)/\partial x_1, \ldots, \partial F(x)/\partial x_n \) and we use the convention that lowercase denotes the derivative (e.g., \( f(x) = F(x) \) and \( \mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \)) when both are defined. The Jacobian matrix of a vector-valued function is defined to be

\[
\mathbf{f}'(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \triangleq \begin{bmatrix}
\frac{\partial f_1(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n}
\end{bmatrix}.
\]

We note that the gradient and Jacobian definitions are transposed with respect to the standard convention.

B. Outline

Section II-A describes the problem of interest and states the main results for a fixed recursion. The recent history of this problem is discussed in Section II-B Section III details the proof of Theorem I and Section IV describes the proof of Theorem 2. Section V extends these results to the case where the recursion depends on a continuous parameter. Applications of these results are presented in Section VI.

II. Maxwell Saturation

A. Problem Statement and Main Results

Let \( f: \mathcal{Y} \rightarrow \mathcal{X} \) be a non-decreasing \( C^1 \) function, \( g: \mathcal{X} \rightarrow \mathcal{Y} \) be a strictly increasing \( C^1 \) function, and assume \( y_{\text{max}} = g(x_{\text{max}}) \). The main goal of this paper is to analyze coupled recursions of the form

\[
y_i^{(\ell+1)} = g \left( x_i^{(\ell)} \right)
\]

\[
x_i^{(\ell+1)} = \sum_{j=1}^{N} A_{j,i} f \left( \sum_{k=1}^{M} A_{j,k} y_k^{(\ell+1)} \right)
\]

for \( i \in [1 : M] \), where \( M \triangleq N + w - 1 \) and \( \mathbf{A} = \{A_{j,k}\} \) is the \( N \times M \) matrix defined by

\[
A_{j,k} = [\mathbf{A}]_{j,k} \triangleq \begin{cases}
1 & \text{if } 1 \leq k - j + 1 \leq w \\
0 & \text{otherwise}.
\end{cases}
\]

It is important to observe that, while the inner sum for \( x_i^{(\ell+1)} \) simply averages \( w \) adjacent \( y \)-values, the outer sum implicitly uses a boundary value of 0 because \( \sum_{j=1}^{N} A_{j,i} < 1 \) for \( i \in [1 : w-1] \cup [N+1 : N+w-1] \). Under the conditions described below, we show that the fixed point of the coupled recursion is intrinsically connected to the dynamics of the uncoupled recursion

\[
y_i^{(\ell+1)} = g \left( x_i^{(\ell)} \right)
\]

\[
x_i^{(\ell+1)} = f \left( y_i^{(\ell+1)} \right)
\]

This problem is motivated by the density evolution (DE) recursions that characterize the large-system performance of iterative demodulation and decoding schemes with and without spatial coupling [33], [34], [35], [14], [20], [12], [22].

The recursions (1) and (4) are initialized by choosing \( x_i^{(0)} = x_{\text{max}} \) for \( i \in [1 : M] \) and \( x_i^{(0)} = x_{\text{max}} \), respectively. Since \( f(g(X)) \subseteq \mathcal{X} \), this initialization implies that \( x_i^{(1)} \leq x_i^{(0)} \). Since \( f, g \) are non-decreasing, \( h(x) \triangleq f(g(x)) \) is non-decreasing and \( x_i^{(\ell)} \leq x_i^{(\ell-1)} \) implies

\[
x_i^{(\ell+1)} = f \left( g(x_i^{(\ell)}) \right) \leq f \left( g(x_i^{(\ell-1)}) \right) = x_i^{(\ell)}.
\]

Therefore, the sequence \( (x_i^{(\ell)}, y_i^{(\ell)}) \) converges to a limit \((x_i^{(\infty)}, y_i^{(\infty)})\) and that limit is a fixed point because \( f, g \) are continuous. The set of all fixed points will be important and is denoted by

\[
\mathcal{F} \triangleq \{ x \in \mathcal{X} | x = f(g(x)) \}.
\]

For the coupled recursion, it is easy to verify that \( A_{j,k} \geq 0 \) implies that

\[
x_i^{(\ell)} \leq x_i^{(\ell-1)} \quad \forall i \in [1 : M] \implies x_i^{(\ell+1)} \leq x_i^{(\ell)} \quad \forall i \in [1 : M].
\]

Therefore, \( x_i^{(\ell+1)} \leq x_i^{(\ell)} \) is given by induction from the base case \( x_i^{(1)} \leq x_i^{(0)} \), which follows from \( \sum_{k=1}^{M} A_{j,k} \leq 1 \) and \( \sum_{j=1}^{N} A_{j,k} \leq 1 \). Hence, for each \( i \in [1 : M] \), the sequence \( (x_i^{(\ell)}, y_i^{(\ell)}) \) converges to a limit \((x_i^{(\infty)}, y_i^{(\infty)})\) as \( \ell \rightarrow \infty \) and the limiting vector is a fixed point of the coupled recursion because the implied vector update is continuous.
Let the single-system (or uncoupled) potential function \( U_s : \mathcal{X} \to \mathbb{R} \) of the scalar recursion be
\[
U_s(x) \triangleq xg(x) - G(x) - F(g(x)),
\]
where \( F(x) = \int_0^x f(z) \, dz \) and \( G(x) = \int_0^x g(z) \, dz \). The main results of this paper are encapsulated in the following two theorems. The first theorem shows that the maximum element of the fixed-point vector for the coupled recursion is upper bounded by the largest minimizer of the uncoupled potential function.

**Theorem 1.** For any \( \delta > 0 \), there is \( w_0 < \infty \) such that, for all \( w > w_0 \) and all \( N \in [1: \infty] \), the fixed point of the coupled recursion satisfies the upper bound
\[
\max_{i \in [1:M]} x_i^{(\infty)} - \delta \leq \pi^* \triangleq \max \left( \min_{x \in \mathcal{X}} U_s(x) \right) .
\]
If \( \pi^* \) is not a limit point of \( \mathcal{F} \cap \{ \pi^*_i, x_{\text{max}} \} \), then \( w_0 < \infty \) for \( \delta = 0 \).

**Proof:** The proof is given in Section [III].

The second theorem shows that the maximum element of the fixed-point vector for the coupled recursion is lower bounded by the smallest minimizer of the uncoupled potential function.

**Theorem 2.** For any fixed \( w \geq 1 \) and \( \eta > 0 \), there is \( N_0 < \infty \) such that, for all \( N > N_0 \), the fixed point of the coupled recursion satisfies the lower bound
\[
\max_{i \in [1:M]} x_i^{(\infty)} + \eta \geq \pi^* \triangleq \min \left( \max_{x \in \mathcal{X}} U_s(x) \right) .
\]

**Proof:** The proof is given in Section [IV].

By combining Theorems 1 and 2, we see that the maximum element of the fixed-point vector for the coupled recursion is essentially determined by the minimizer of uncoupled potential. When the system also depends on a parameter (e.g., see Section [V]), this result is closely related to the Maxwell construction in [26]. For that reason, we refer to this phenomenon as Maxwell saturation. In contrast, threshold saturation refers to the slightly weaker statement that the iterative decoding threshold of the SC system (for reliable communication) converges to the MAP threshold of the single system for sufficiently large \( w \).

**Remark 3.** It is natural to wonder how the choice of a uniform fixed-width coupling (e.g., see [2]) affects our results. Suppose instead that \( \alpha \) is a symmetric non-negative vector that sums to 1. While it is possible to extend Theorem 1 to this case (e.g., see [25, 29]), the resulting bounds depend only on the maximum value \( \max_{i \in [1:w]} \alpha_i \). In terms of the bounds, the optimum \( \alpha \) vector is \( a_j = \frac{1}{w} \). Hence, for the sake of simplicity, we consider only uniform fixed-width coupling.

**Remark 4.** The statement of Theorem 1 is closely related to the definition of \( \lim_{w \to \infty} \mathcal{F} \) and this connection implies that
\[
\lim_{w \to \infty} \lim_{N \to \infty} \max_{i \in [1:N+w-1]} x_i^{(\infty)} \leq \pi^* .
\]

**Example 5.** Consider the recursion defined by \( x_{\text{max}} = y_{\text{max}} = 1 \), \( f(x) = \frac{97}{300} x^2 \) and \( g(x) = 1 - (1 - x)^2 \). This recursion can be seen as the DE recursion of a (3,3)-regular LDGM ensemble on a BEC with erasure probability \( \frac{97}{300} \). In this case, we have \( F(x) = \frac{97}{300} x^3 \), \( G(x) = x + \frac{1}{4} ((1 - x)^3 - 1) \), and
\[
U_s(x) = x(1 - (1 - x)^2) - \left( x + \frac{(1-x)^3-1}{3} \right) - \frac{97(1-(1-x)^2)^3}{300} .
\]
In Fig. 1 we see that the minimum of \( U_s(x) \) is uniquely achieved at \( \pi^* = 0 \). Since 0 is not a limit point of \( \mathcal{F} \) (e.g., see Proposition 10), there is a \( w_0 < \infty \) such that the coupled recursion converges to the 0 vector for all \( w > w_0 \). One can extract the upper bound, \( w_0 \leq \frac{1}{2\Delta} K_{f,g} x_{\text{max}}^2 \), from the proof of Theorem 1. For this example, we find that \( w_0 < 600 \) because straightforward computations show that \( \Delta = 0.01 \) and \( K_{f,g} < 12 \).

**Example 6.** Consider the recursion defined by \( x_{\text{max}} = y_{\text{max}} = 1 \), \( f(x) = x^5 \) and \( g(x) = 1 - \frac{1}{4} R'(1 - x)/R'(1) \) where \( R(x) = \frac{2}{15} x + \frac{1}{15} x^2 + \frac{7}{15} x^3 + \frac{2}{3} x^4 \). This recursion can be seen as the DE recursion of an irregular low-density generator matrix (LDGM) ensemble on a BEC with erasure probability \( \frac{1}{4} \). In this case, we have \( F(x) = \frac{1}{5} x^6 \), \( G(x) = x - \frac{1}{4} (1 - R(x)) / R'(1) \), and \( U_s(x) \) is given by (4). In Fig. 2 we see that the function \( \pi^* \) has a unique global minima at \( \pi^* \approx 0.05 \). Since \( \pi^* \) is not a limit point of \( \mathcal{F} \) (e.g., see Proposition 10), there is a \( w_0 < \infty \) such that the maximum element of the coupled fixed-point vector is upper bounded by \( \pi^* \) for all \( w > w_0 \). One can extract the upper bound, \( w_0 \leq \frac{1}{\Delta} K_{f,g} x_{\text{max}}^2 \), from the proof of Theorem 1. For this example, we find that \( w_0 < 2000 \) because straightforward computations show that \( \Delta \geq 0.0025 \) and \( K_{f,g} < 10 \).
Example 7. Consider the following pathological example defined by \( x_{\text{max}} = 1 \), \( g(x) = x \), and \( F(x) = \frac{1}{x}x^2 - \frac{1}{2}x^3\sin(\frac{x}{x}) - \frac{1}{3}x^6 \).

With a little work, one can verify (e.g., numerically) that this system satisfies the necessary conditions of Theorem 1 (i.e., \( F''(x) \) is continuous, bounded, and non-negative on \( x \)). The implied potential function,

\[
U_s(x) = \frac{x}{2}x^5\sin^4(\frac{x}{x}) + \frac{1}{3}x^6,
\]

has a countably infinite set of local minima (e.g., see Fig. 3) in the neighborhood of \( x = 0 \) whose values approach 0 as \( x \to 0 \).

Since all local minima are fixed points (e.g., see Lemma 18), this implies that \( F \) has a limit point at \( x = 0 \). One also finds that \( \delta = 0 \) implies \( w_0 = \infty \). Thus, one cannot guarantee that \( \max_i x_i^{(\infty)} = 0 \) occurs for any finite \( w \). Still, Remark 4 implies that \( \max_i x_i^{(\infty)} \to 0 \) as \( w \to \infty \).

Remark 8. While the stated result for Example 5 is an instance of threshold saturation (e.g., see [14], [11], [28]), the result in Example 6 is an instance of Maxwell saturation and does not follow from previously published proofs of threshold saturation. Also, the bounds on \( w_0 \) computed in these examples are quite loose and shown only for completeness. There is some evidence that the necessary \( w_0 \) for Theorem 1 could be as small as \( O(\ln \frac{1}{\lambda}) \) if \( f, g \) are analytic [36].

B. History and Motivation

The history of spatial coupling and threshold saturation starts with the introduction of LDPC convolutional codes by Felstrom and Zigangirov in 1999 [3]. After a few years, it became apparent that these codes had significantly better performance (i.e., noise thresholds close to capacity) when terminated [5]. Without termination, however, these codes perform very similarly to standard LDPC codes. The next big advance came when researchers realized [14], [6] that the noise threshold of the \((j,k)\)-regular convolutional LDPC ensemble was very close to the MAP threshold of the standard \((j,k)\)-regular LDPC ensemble and then were able to prove it [14]. It is worth noting that this connection would not have been possible without the calculation of the MAP thresholds, via (G)EXIT functions, for standard LDPC ensembles [26], [37].

Kudekar, Richardson, and Urbanke showed that this observation was part of a very general phenomenon which they named threshold saturation via spatial coupling [14]. The term spatial coupling refers to the fact that LDPC convolutional codes can be seen as a spatial arrangement of standard LDPC ensembles that are locally connected to each other along the spatial dimension. Their proof shows that the BP noise threshold of the coupled ensemble must equal the MAP-decoding noise threshold of the underlying LDPC ensemble by using the coupled BP EXIT function to construct a spatial integration of the standard BP EXIT function. Their result represents a major breakthrough in coding theory and their paper introduces many ideas and techniques (e.g., the \((1, r, L, w)\) ensemble and the modified DE recursion) used in this paper. Still, the result was proven only for regular LDPC ensembles on the BEC and the proof technique does not generalize easily to other cases. For example, their extension to BMS channels is a tour de force in analysis [13]. Hassani, Macris, and Urbanke have also presented results for problems in statistical physics including a rigorous analysis of the coupled Curie-Weiss model [16]. Since then, there has been mounting evidence that the phenomenon of threshold saturation is indeed very general [4].

In [27], Takeuchi, Tanaka, and Kawabata present a phenomenological description of threshold saturation that is based on the single-system potential function

\[
\int_0^x (z - f(g(z))) dz.
\]

Our initial goal was to construct a rigorous proof based on the outline presented in [27]. After some time, we concluded that this is not possible without two significant modifications. First, their uncoupled potential does not correctly predict the MAP threshold for regular LDPC codes on the BEC. Therefore, we use instead [4] the uncoupled potential function [4] whose integral form is given by

\[
U_s(x) = \int_0^x (z - f(g(z))) g'(z) dz + \text{const}.
\]

For the special case of irregular LDPC codes, \( U_s(x) \) is also a scalar multiple of the trial entropy \( P_\varepsilon(x, y(x)) \) in [26, Lem. 4], which defines the conjectured MAP (or Maxwell) threshold of these codes on the BEC [20]. Second, we were unable to rigorously connect the coupled recursion [1] to their proposed free energy functional for the coupled system [27]. During this process, however, we realized that the coupled system also has a simple closed-form potential function [11]. We observed later that, for the special case of the coupled Curie-Weiss model, a similar potential function was defined earlier in [36] (18). Still, [27] introduced us to the idea of defining a potential function for a discrete-time recursion as the integral of the current value minus the updated value.

In 2011, rigorous proofs based on coupled potential functions in the continuum limit were posted to arXiv by Truhachev, for iterative demodulation of packet-based CDMA with outer codes, and by Donoho, Montanari, and Javanmard, for compressed sensing [21], [25]. Our conference papers on this subject were submitted a few months later [1], [30].

This paper is based on two conference papers by the same authors [1], [30]. These papers use coupled-system potentials.
for the discrete system along with a Taylor series expansion to prove threshold saturation for both coupled scalar and coupled vector recursions. For some coding problems, these results imply that the iterative-decoding noise threshold of the coupled system equals the conjectured MAP-decoding noise threshold. Independently in [28], Kudekar, Richardson, and Urbanke develop a different approach based on potential functions that proves threshold saturation for general scalar recursions. Their approach is somewhat more complicated but also produces stronger performance guarantees and requires only that $f$ and $g$ are non-decreasing. In this paper, the techniques from [11, 30] are extended to show the coupled recursion essentially converges to a point determined by the minimum of the single-system potential (e.g., the point given by the Maxwell construction in [26]). On the other hand, threshold saturation is only informative if this point is 0. The approach taken in [11, 30] was also extended to irregular LDPC codes on BMS channels in [31, 32]. In fact, this generalization highlights both the simplicity and generality of our approach.

Some of the aforementioned proofs (e.g., [21, 25, 1], [30]) can also be seen as using Lyapunov techniques to prove the stability of discrete-time dynamical systems (e.g., the scalar and coupled recursions). For the single-system potential of LDPC codes on the BEC (i.e., the Bethe free energy), this connection was observed in 2009 by Vontobel [49]. More recently, it was discussed in [50]. However, we note that something much more interesting is at work here. The single-system potential function that we use is something more than just a Lyapunov function for the single system. It is the thermodynamic potential associated with some implicit thermodynamic system and therefore the global minimum is intrinsically connected to some notion of a minimum energy state. This is exactly why we can establish the lower bound in Theorem 2. While many functions can be used as a Lyapunov function for the single system, most of them will not determine the fixed point of the coupled system. For example, the choice of Takeuchi et al. in [7] is a Lyapunov function for the single system whose minimum does not correctly predict the fixed point of the coupled system.

**C. Some Important Details**

Consider the integral form (8) of the potential (4). Taking the derivative of (4) gives

$$U_s(x) = xg'(x) + g(x) - g(x) - f(g(x))g'(x)$$

$$= (x - f(g(x))) g'(x)$$

and shows that (4) is consistent with the integral form (8). Calculating $U_s(0)$ in both equations shows that the constant in (3) is given by $-F(g(0))$. In our previous work [11, 30], this constant was 0 due to the assumption that $g(0) = 0$.

In this work, this choice simplifies some parts of Section V. Moreover, we have the following proposition.

**Proposition 9.** If 0 is a fixed point of the uncoupled recursion (i.e., $f(g(0)) = 0$), then $F(g(0)) = 0$ and $U_s(0) = 0$.

**Proof:** Under this condition, we find $F(g(0)) = 0$ because $F$ is non-negative and

$$F(g(0)) = \int_0^{g(0)} f(x) dx \leq g(0)f(g(0)) = 0,$$

where the inequality holds because $f(x)$ is non-decreasing. Since (4) implies $U_s(0) = -F(g(0))$, the result follows.

Another important detail has to do with the case of $\delta = 0$ in Theorem 1. Under the following conditions, Theorem 1 predicts Maxwell saturation with $w_0 < \infty$ even for the case of $\delta = 0$.

**Proposition 10.** In Theorem 7, $w_0 < \infty$ for $\delta = 0$ if any of the following hold:

(i) the fixed-point set $\mathcal{F}$ is finite,

(ii) $f(x)$ and $g(x)$ are real analytic functions on $\mathcal{X}$ and $x - f(g(x))$ is not identically zero,

(iii) there exists a $\gamma > 0$ such that $f(g(x)) < x$ for all $x \in (\pi^*, \pi^* + \gamma)$, or

(iv) $f'(\pi^*)g'(\pi^*) < 1$.

**Proof:** See Appendix A.

**D. A Half-Iteration Shift**

One can also shift the uncoupled recursion by half an iteration by swapping $f$ and $g$. In this case, the recursion of interest becomes $y^{(E+1)} = g(y^{(E)})$, starting from $y^{(0)} = g(x_{\text{max}})$, and the associated uncoupled potential function becomes

$$V_s(y) \triangleq g(y) - F(y) - G(f(y)).$$

While the uncoupled recursion only experiences a time shift, the change in the coupled recursion is more complicated because of the 0-boundary condition. Still, the coupled system associated with the shifted recursion behaves very similar to the original system due to the following proposition.

**Lemma 11.** If $f$ and $g$ are both strictly increasing $C^2$ functions, then

(i) the fixed points satisfy $F' = g(F)$ and $F = f(F')$ with $\mathcal{F}' = \{y \in \mathcal{Y} \mid y = g(f(y))\}$.

(ii) if $x \in \mathcal{F}$ (resp. $y \in \mathcal{F}'$), then $U_s(x) = V_s(g(x))$ (resp. $V_s(y) = U_s(f(y))$), and

(iii) the minimizers satisfy $\mathcal{M}' = g(\mathcal{M})$ and $\mathcal{M} = f(\mathcal{M}')$ with

$$\mathcal{M} = \arg \min_{x \in \mathcal{X}} U_s(x), \quad \mathcal{M}' = \arg \min_{y \in \mathcal{Y}} V_s(y).$$

**Proof:** See Appendix A.

**Remark 12.** The above proposition implies that all quantities computed from $U_s(x)$ in Theorem 1 can alternatively be computed from $V_s(y)$ (e.g., if it has a simpler functional form). Moreover, the coupled systems based on the original and shifted recursions receive the same guarantees (after the appropriate variable change) from Theorem 1.
E. The Coupled Potential Function

First, we observe that, using the vector notation \( x^{(t)} = (x_1^{(t)}, \ldots, x_M^{(t)}) \), the recursion (1) can be written compactly as

\[
x^{(t+1)} = h \left( x^{(t)} \right) \triangleq A^T f \left( Ag(x^{(t)}) \right),
\]

where \( f : \mathbb{R}^N \to \mathbb{R}^N \) and \( g : \mathcal{X}^M \to \mathcal{Y}^M \) are defined by \( f(x)_i = f(x_i) \), and \( g(x)_i = g(x_i) \). The recursion starts from \( x^{(0)} = x_{\text{max}} \triangleq x_{\text{max}} \cdot 1_M \) and it is easy to verify that the vector mappings \( f, g \) are isotone, which means that \( x \preceq z \) implies \( f(x) \preceq f(z) \) and \( g(x) \preceq g(z) \). Moreover, the linear transformations defined by \( A, A^T \) are also isotone and \( x \preceq z \) implies \( Ax \preceq Az \). The mapping \( h \) is also isotone because it is the composition of four isotone mappings. Finally, we say a length-\( M \) vector \( x \) is symmetric if \( [x]_i = [x]_{M-i+1} \) and a symmetric vector is unimodal if \( [x]_{i+1} \preceq [x]_i \) for \( i < \lceil M/2 \rceil \).

We will see that each \( x^{(t)} \) is symmetric and unimodal.

Next, we extend the definition of the potential function to general coupled recursions of the form (1).

**Definition 13.** Let the potential function of the coupled system be defined by

\[
U_c(x) \triangleq \sum_{i=1}^M (g(x_i)x_i - G(x_i)) - \sum_{i=1}^N F \left( \sum_{j=1}^M A_{i,j}g(x_j) \right),
\]

\[
= g(x)^T x - G(x) - F(Ag(x)),
\]

where \( G(x) = \sum_{i=1}^M G(x_i) \) and \( F(y) = \sum_{i=1}^N F(y_i) \). One can verify that this is equivalent to

\[
U_c(x) = \int_C g'(z)(z - A^T f(Ag(z))) \cdot dz - F(Ag(x)),
\]

where \( C \) is a smooth curve in \( \mathcal{X}^M \) from 0 to \( x \), \( G(x) = \int_C g(z) \cdot dz \), and \( F(y) = \int_C f(z) \cdot dz \). We note that the constant term in the integral \( U_c(x) \) formula is chosen to be consistent with the scalar potential.

**Remark 14.** An important property of the potential function \( U_c(x) \) is that its derivative is closely related to the step taken by the recursion, \( x - f(g(x)) \). A similar result holds for the coupled potential \( U_c(x) \) and computing the derivative of (11) shows that

\[
\frac{d}{dx_k} U_c(x) = \frac{d}{dx_k} \sum_{i=1}^M (g(x_i)x_i - G(x_i))
\]

\[= \sum_{i=1}^M \frac{d}{dx_k} \left( \sum_{j=1}^M A_{i,j}g(x_j) \right)
\]

\[= g'(x_k)x_k + g(x_k) - g(x_k)
\]

\[= \sum_{j=1}^M A_{i,j} \frac{d}{dx_k} \left( \sum_{i=1}^M A_{i,j}g(x_j) \right)
\]

\[= [g'(x_k)(x - A^T f(Ag(x)))]^T_k.
\]

A key observation in this work is that a potential function for general coupled systems can be written in the simple form given by Def. (13). Remarkably, this holds for general coupling coefficients because of the reciprocity between \( A \) and \( A^T \) that appears naturally in spatial coupling.

**Remark 15.** It is natural to wonder why the integral form of the single-system potential needs a factor of \( g'(x) \). Focusing instead on the coupled potential function, one notices that the vector field \( x - A^T f(Ag(x)) \) is not conservative in general. Thus, there is no scalar function whose gradient equals \( x - A^T f(Ag(x)) \). But, if one multiplies by \( g'(x) \) on the left, then \( g'(x)(x - A^T f(Ag(x))) \) is conservative for all \( A, f, g \) (e.g., it is the gradient of the coupled potential). Hence, we need the factor of \( g'(x) \) in the derivative of the coupled potential just to make it the gradient of a scalar function and we need the \( g'(x) \) in the derivative of the single-system potential so that it matches the coupled potential.

F. A Convenient Change of Variables

Now, we introduce a change of variables for the uncoupled recursion that allows one to translate any fixed point to \( x = 0 \). This result is used in the proof of Theorem 1.

For any fixed point \( \hat{x} \in \mathcal{F} \) of the uncoupled recursion, the functions \( \hat{f}(y) \triangleq f(y + \hat{x}) - \hat{x} \) and \( \hat{g}(x) \triangleq g(x + \hat{x}) - g(\hat{x}) \) define a translated uncoupled recursion that satisfies

\[
\hat{f}(\hat{g}(x)) = f(g(x + \hat{x}) - \hat{x}.
\]

This translation is given by the change of variable \( x \mapsto x - \hat{x} \) and, hence, the recursion operates on the space \( X \triangleq [0, \hat{x}_{\text{max}}] \), where \( \hat{x}_{\text{max}} \triangleq \hat{x}_{\text{max}} - \hat{x} \). For the uncoupled system, this change of variables has no effect other than mapping the value \( \hat{x} \) in the original uncoupled recursion to the value 0 in the translated uncoupled recursion (e.g., \( f(0) = \hat{g}(0) = 0 \)). For the translated coupled system, \( \hat{f} \) and \( \hat{g} \) are defined by the pointwise application of \( f \) and \( g \) to vectors of length \( N \) and \( M \). This implies that \( \hat{f}(y) = f(y + \hat{x}1_N) - \hat{x}1_N \) and \( \hat{g}(x) = g(x + \hat{x}1_M) - g(\hat{x})1_M \). It is important to note that the translation also affects implicitly the boundary conditions introduced by coupling.

**Lemma 16.** The translated scalar system is well defined and the overall update of the translated coupled system, \( \hat{h} \), is

\[
\hat{h}(x) = A^T f \left( Ag(x + \hat{x}1_M) - \hat{x}1_M + \hat{x} \right) \left( 1_M - A^T \right) 1_N.
\]

This differs from \( h(x + 1_M - \hat{x}1_M) \) (i.e., translating the coupled recursion) by the boundary condition term \( \hat{x}(1_M - A^T 1_N) \). The potentials for the translated recursion are given by

(i) \( \hat{F}(y) = F(y + \hat{x}) - y\hat{x} - F(g(\hat{x})) \)

(ii) \( \hat{G}(x) = G(x + \hat{x}) - x\hat{x} - G(\hat{x}) \)

(iii) \( \hat{U_s}(x) = x\hat{g}(x) - \hat{G}(x) - \hat{F}(\hat{g}(x)) = U_s(x + \hat{x}) - U_\hat{s}(\hat{x}) \)

Proof: See Appendix A

**Remark 17.** The term \( \hat{x}(1_M - A^T 1_N) \) in Lemma 16 effectively changes the boundary values from 0 in original coupled recursion to \( \hat{x} \) in the translated coupled recursion. However, this is done by shifting the recursion down and keeping
the same boundary value at 0 rather than keeping the same recursion and changing the boundary value. Therefore, if the translated coupled recursion converges to the zero vector, then the original coupled recursion (with a modified boundary value of \( \hat{\alpha} \)) converges to the all-\( \hat{\alpha} \) vector.

### III. Proof of Theorem 1

Before we present the proof of Theorem 1, we introduce some necessary definitions and lemmas. Proofs are relegated to Appendix B unless they reveal ideas central to our approach.

**Lemma 18.** The potential value is non-increasing with iteration (i.e., \( U_n(f(g(x))) \leq U_n(x) \)) and strictly decreasing if \( x \) is not a fixed point (i.e., \( f(g(x)) \neq x \)). Furthermore, all local minima of \( U_n(x) \) occur at fixed points.

**Proof:** See Appendix B

**Remark 19.** It turns out that the second part of Lemma 18 depends crucially on the assumption that \( g(x) \) is strictly increasing. If \( g(x) \) is allowed to be constant on an interval, then one can construct recursions where a minimizer of the potential is not a fixed point.

Throughout, we will use \( i_0 \triangleq \lceil M/2 \rceil \) to denote the spatial midpoint of the coupled system. The following lemma shows that \( x(t) \) is a non-increasing sequence of symmetric unimodal vectors.

**Lemma 20.** The coupled recursion satisfies: (i) \( x(t+1) \leq x(t) \), (ii) \( x(t)_i = x_M - t + 1 \) and (iii) \( x(t+1)_i \geq x(t)_i \) for \( i < i_0 \).

**Proof:** See Appendix B

**Definition 21.** The modified coupled recursion is defined to be \( x(t) = q(h(x(t))) \), starting from \( x(0) = x_{\text{max}} \), where

\[
[q(x)]_i = \begin{cases} 
\lceil x \rceil_{i_0} & \text{if } i \in [i_0 + 1 : M] \\
\lfloor x \rfloor_i & \text{otherwise}
\end{cases}
\]

**Remark 22.** The modified coupled recursion provides an upper bound on the coupled recursion that has a few important properties which will be used in the proof of Theorem 1.

**Lemma 23.** The sequence generated by the modified coupled recursion satisfies: (i) \( \hat{x}(t+1) \leq \hat{x}(t) \), (ii) \( \hat{x}(t) \geq x(t) \), and (iii) \( \hat{x}(t+1) \geq \hat{x}(t) \) for \( i \in [1 : M-1] \).

**Proof:** See Appendix B

**Lemma 24.** Consider the coupled system potential \( U_c(x) \) for any \( x \in \mathcal{X}^M \) satisfying \( x_i = x_{i_0} \) for \( i \in [i_0:M] \) and \( x_i \leq x_0 \) for \( i \in [0:i_0] \). Let the shift operator \( S \) be defined by \( Sx_i = x_{i-1} \) for \( i \in [1:M] \) with \( x_0 = 0 \). Then, applying the shift to the vector changes the potential by \( U_c(Sx) - U_c(x) \leq U_c(0) - U_c(x_{i_0}) \).

**Proof:** We compute \( U_c(Sx - \alpha) - U_c(x - \alpha) \) by treating the three terms in \( [12] \) separately. The first term gives

\[
g(Sx)^\top (Sx) - g(x)^\top x = \sum_{i=0}^{M-1} g(x_i) x_i - \sum_{i=1}^M g(x_i) x_i = -[g(x_0) x_0 - g(x_M) x_M] = -\sum_{i=0}^{i_0} g(x_i) x_{i_0}.
\]

Similarly, the second term gives

\[
-G(Sx) + G(x) = -\sum_{i=0}^{M-1} G(x_i) + \sum_{i=0}^M G(x_i) = -G(x_0) + G(x_M) = G(x_{i_0}).
\]

For the third term, we consider \(-F(Ag(Sx)) \neq -F(g(x))\) and observe that \([Ag(Sx)]_i = g(0)\) and \([Ag(Sx)]_i = [Sg(x)]_i \) for \( i \in [2 : M] \). Since \( g \) is monotonic, this implies that \(-F(Ag(Sx))_1 \leq -F(g(0))\) and \(-F(Ag(Sx))_i = -F(Ag(x)|_{i-1}) \) for \( i \in [2 : M] \). Therefore, we have the upper bound

\[
-G(Sx) + G(x) \leq -F(g(0)) \leq \sum_{i=0}^{N} F([Ag(x)]_{i-1}) + \sum_{i=1}^{N} F([Ag(x)]_i).
\]

**Lemma 25.** Let \( \hat{x} = \hat{x}(\infty) \) be the limit of the modified coupled recursion in Def. 27 and let \( S \) be the shift operator defined in Lemma 24. Then, the directional derivative of the coupled potential in the direction \( S \hat{x} - \hat{x} \) satisfies

\[
U'(x)^\top (S \hat{x} - \hat{x}) = 0.
\]

**Proof:** By Lemma 23 the modified coupled recursion generates a decreasing sequence that converges to a limit, which is a fixed point because the update function is continuous. From the definition of \( g(\cdot) \) and the fact that \( \hat{x} \) is non-decreasing, it follows that \( \hat{x}_{i+1} = h(\hat{x}_i) \) for \( i \in [1 : i_0] \), and \( \hat{x}_{i_0} = x_{i_0} \) for \( i \in [i_0 + 1 : M] \). Since the last half of \( \hat{x} \) is constant, we find that \( S \hat{x} - \hat{x} = 0 \) for \( i \in [i_0 + 1 : M] \). For the first half of the vector, the recursion is unaffected by \( g(\cdot) \) and this implies that \( U'(\hat{x})_i = [g'(\hat{x})(\hat{x} - h(\hat{x}))]_i = 0 \) for \( i \in [1 : i_0] \). Therefore, we have \( U'(\hat{x})^\top (S \hat{x} - \hat{x}) = 0 \) because, in each position, one of the two vectors is zero.

**Lemma 26.** The Hessian matrix \( U''(x) \) of \( U_c(x) \) satisfies

\[
\|U''(x)\|_{\infty} \leq K_{f,g}, \quad \text{where}
\]

\[
K_{f,g} \triangleq \|g''\|_{\infty} x_{\text{max}} + \|g'\|_{\infty} + \|f'\|_{\infty} \|g''\|_{\infty}^2
\]

for all \( x \in \mathcal{X}^M \) and \( p \in \{1, 2, \infty\} \).

**Proof:** See Appendix B

**Lemma 27.** Let \( x = g(A^T z) \) for some \( z \in \mathcal{X}^N \) satisfying \( z_{i+1} \geq z_i \) for \( i \in [1 : M - 1] \). Let the down-shift operator \( S : \mathcal{X}^M \rightarrow \mathcal{X}^M \) be defined by \( Sx_i = x_i-1 \) for \( i \in [1 : M] \) with \( x_0 = 0 \). Then, we have the bounds \( \|Sx - x\|_{\infty} \leq \frac{1}{\|x\|_{\infty}} z_{i_0} \leq \frac{1}{\|x\|_{\infty}} x_{\text{max}} \), and \( \|Sx - x\|_{1} = x_{i_0} - x_0 \leq x_{\text{max}} \).
Proof: For \( i > i_0 \), \( x_i = x_{i-1} \) due to \( q(\cdot) \) and \( |x_i - x_{i-1}| = 0 \). For \( i \leq i_0 \), we have
\[
|x_i - x_{i-1}| = \left| \frac{1}{w} \sum_{k=0}^{w-1} z_i - k - \frac{1}{w} \sum_{k=0}^{w-1} z_i - k \right| \\
\leq \frac{1}{w} \delta_{i_0} \leq \frac{1}{w} x_{\max}.
\]
where we assume \( z_i = 0 \) for \( i \notin [1 : N] \). Hence, \( \|Sx - x\|_\infty \leq \frac{1}{w} x_{\max} \). By Lemma 23.(iii), \( x \) is non-decreasing because \( z \) is non-decreasing. Therefore, we find that \( \|Sx - x\|_1 = x_{i-1} - x_i \leq 0 \) and the 1-norm sum telescopes to give
\[
\|Sx - x\|_1 = \sum_{i=1}^{M} |x_i - x_{i-1}| = \sum_{i=1}^{i_0} (x_i - x_{i-1}) \\
= x_{i_0} - x_0 \leq x_{\max}.
\]

Remark 28. It is natural to wonder, “Where in the proof of Theorem 1 is the 0-boundary of the SC system used?”. The best answer to this question is probably Lemma 27. This is because the bound \( \|Sx - x\|_\infty \leq \frac{1}{w} x_{\max} \) depends on the fact that \( |x_0 - x_1| = |x_0| \leq \frac{1}{w} x_{\max} \) and this holds only because the 0-boundary is implicit in the equation \( x_1 = [A^T \tilde{z}]_1 = \frac{1}{w} x_1 \).

Now, we are ready to combine the previous results into a single lemma.

Lemma 29. Consider an uncoupled recursion satisfying \( f(0) = g(0) = 0 \). For any \( \delta \geq 0 \), if \( \Delta_0 \triangleq \inf \{U_b(x) \mid x \in \mathcal{F} \cap (\delta, x_{\max}) \} > 0 \) and \( w > \frac{1}{2\Delta_0} K_{f,g} x_{\max}^2 \) (where \( K_{f,g} \) is defined in (13)), then the fixed point of the coupled recursion must satisfy \( x^{(\infty)}_{i} \leq \delta \) for \( i \in [1 : M] \).

Proof: First, we use Lemma 23 to see that the modified coupled recursion converges to a fixed-point limit denoted by \( \tilde{x} \). If \( \tilde{x}_{i_0} \leq \delta \), then the proof is complete because \( \tilde{x}_{i_0} \) is the maximum value of \( x \). Hence, we can assume \( \delta \in [0, x_{\max}] \). Now, suppose that \( \tilde{x}_{i_0} > \delta \). The remainder of the proof hinges on calculating \( U_c(S\tilde{x}) - U_c(\tilde{x}) \) using two different approaches.

The first approach is by applying Lemma 24 and observing that
\[
U_c(S\tilde{x}) - U_c(\tilde{x}) \leq U_c(0) - U_c(\tilde{x}_{i_0}) = -U_c(\tilde{x}_{i_0})
\]
because \( f(0) = g(0) = 0 \) implies \( U_c(0) = 0 \). In addition, we observe that
\[
\tilde{x}_{i_0} = [q(h(\tilde{x}))]_{i_0} = [h(\tilde{x})]_{i_0} \leq [h(\tilde{x}_{i_0})]_{i_0} = h(\tilde{x}_{i_0})
\]
holds because \( \tilde{x} \) is a fixed point of \( q(h(\cdot)) \) and \( \tilde{x} \leq \tilde{x}_{i_0} \).}

1. This implies that the recursion \( z^{(t+1)} = f(g(z^{(t)})) \) from \( z^{(0)} = \tilde{x}_{i_0} \) satisfies \( z^{(t+1)} \geq z^{(t)} \) and \( U_c(z^{(t)}) \leq U_c(z^{(t+1)}) \) (by Lemma 18). Therefore, \( z^{(t)} \) increases to a stable fixed point \( z^{(\infty)} \geq \tilde{x}_{i_0} > \delta \) that satisfies \( U_c(\tilde{x}_{i_0}) \geq U_c(z^{(\infty)}) \) and
\[
U_c(z^{(\infty)}) \geq \inf \{U_c(x) \mid x \in (\delta, x_{\max}) \},
\]
\( x = f(g(x)) \).

Thus, \( -U_c(\tilde{x}_{i_0}) \leq -\Delta_0 \) and, if \( \Delta_0 > 0 \), then this implies that \( U_c(S\tilde{x}) - U_c(\tilde{x}) \geq \Delta_0 \).

For the second approach, we evaluate \( U_c(S\tilde{x}) \) using a 2nd-order Taylor series expansion. Let \( \tilde{z}(t) = \tilde{x} + t(S\tilde{x} - \tilde{x}) \) and \( \phi(t) = U_c(\tilde{z}(t)) \). Then, Taylor’s theorem says that, for some \( t_0 \in [0, 1] \), one has
\[
\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2} \phi''(t_0).
\]

Using the chain rule to calculate the derivatives shows that
\[
U_c(S\tilde{x}) - U_c(\tilde{x}) = U_c'(\tilde{x})^T (S\tilde{x} - \tilde{x}) + \frac{1}{2} \|S\tilde{x} - \tilde{x}\|_2^2 U_c''(\tilde{x})(S\tilde{x} - \tilde{x})
\]

Using Lemmas 25 and 26 this can be rewritten as
\[
U_c(S\tilde{x}) - U_c(\tilde{x}) = \frac{1}{2} \|S\tilde{x} - \tilde{x}\|_2^2 U_c''(\tilde{x})(S\tilde{x} - \tilde{x})
\]

where the norm bounds in the last step are computed in Lemma 27. This bound contradicts the result of the first approach if
\[
w > \frac{1}{2\Delta_0} K_{f,g} x_{\max}^2.
\]

Based on this contradiction, we conclude that (14) implies \( \tilde{x}_{i_0} \leq \delta \).

Now, we are ready to prove Theorem 1. This theorem says that, for any \( \delta > 0 \), there is \( w_0 < \infty \) such that, for all \( w > w_0 \) and all \( N \in [1 : \infty] \), we have \( \max_{i \in [1 : M]} x^{(\infty)}_{i} - \delta \leq \bar{x} \), where \( \bar{x} \triangleq \max (\arg \min x \in \mathcal{X} U_c(x)) \). The primary tools are the change of variables defined in Lemma 16 and threshold saturation result in Lemma 29.

Proof of Theorem 7: This theorem now follows from combining Lemmas 29 and 16. The idea is to first translate the uncoupled recursion by applying Lemma 16 with \( \tilde{x} = \bar{x} \) and then apply Lemma 29 to show that coupled translated recursion converges to the zero vector.

First, we handle some special cases. If the uncoupled recursion has no fixed points in \( [\bar{x} + \delta, x_{\max}] \), then the promised upper bound follows trivially because the coupled system is upper bounded by the uncoupled system (i.e., \( x^{(\infty)} \leq x^{(\infty)} \) for \( i \in [1 : M] \)) and the uncoupled system satisfies \( x^{(\infty)} \leq \bar{x} + \delta \). Therefore, we consider only the case where the uncoupled system has a fixed point in \( [\bar{x} + \delta, x_{\max}] \) and, hence, \( \delta \in [0, x_{\max} - \bar{x}] \).

Next, we define \( w_0 \triangleq \frac{1}{2\Delta} K_{f,g} x_{\max}^2 \) where \( K_{f,g} \) is defined in (13), and
\[
\Delta \triangleq \inf \{U_c(x) - U_c(\bar{x}) \mid x \in \mathcal{F} \cap (\bar{x} + \delta, x_{\max}) \}
\]
To show that \( w_0 \leq \infty \), we will show that \( \Delta > 0 \) for any \( \delta \in (0, x_{\max} - \bar{x}) \). Observe that the set of fixed points \( \mathcal{F} = \{x \in \mathcal{X} \mid x = f(g(x)) \} \) is compact because it is closed subset (i.e., it contains all its limit points because it is defined by a continuous equality) of a compact set. From this, we see that the set \( \mathcal{F} \cap (\bar{x} + \delta, x_{\max}) \) is compact (i.e., it is intersection
of compact sets) and non-empty (i.e., there is a fixed point in \([\bar{x}^* + \delta, x_{\text{max}}]\)). Since \(\bar{x}^*\) is the largest value that minimizes \(U_s(x)\), it follows that \(U_s(x) > U_s(\bar{x}^*)\) for all \(x \in (\bar{x}^*, x_{\text{max}}]\).

Hence, for any \(\delta \in (0, x_{\text{max}} - \bar{x}^*)\), we have
\[
\Delta \geq \min \{U_s(x) - U_s(\bar{x}^*) \mid x \in \mathcal{F} \cap [\bar{x}^* + \delta, x_{\text{max}}]\} > 0.
\]

Except for pathological cases (e.g., where \(\bar{x}^*\) is a limit point of \(\mathcal{F} \cap [\bar{x}^*, x_{\text{max}}]\)), the infimum must equal the minimum. This implies that, if \(\bar{x}^*\) is not a limit point of \(\mathcal{F} \cap [\bar{x}^*, x_{\text{max}}]\), then \(\Delta > 0\) and \(w_0 < \infty\) for all \(\delta \in (0, x_{\text{max}} - \bar{x}^*)\).

Now, we focus on the translated uncoupled recursion defined by Lemma 16 with \(\bar{x} = \bar{x}^*\). For the translated system, one finds that \(f(0) = \tilde{g}(0) = 0\), \(x_{\text{max}} = x_{\text{max}} - \bar{x}^*\), and \(U_s(x) = U_s(x + \bar{x}^*) - U_s(\bar{x}^*)\) for all \(x \in (0, x_{\text{max}}]\). To apply Lemma 16 to the translated system, we start by computing
\[
\Delta = \inf \{U_s(x) \mid x \in (0, x_{\text{max}}], x = \bar{f}(\tilde{g}(x))\}
\]
\[
= \inf \{U_s(x + \bar{x}^*) - U_s(\bar{x}^*) \mid x \in (0, x_{\text{max}}], x = \bar{f}(\tilde{g}(x))\}
\]
\[
= \inf \{U_s(x) - U_s(\bar{x}^*) \mid x \in (\bar{x}^* + \delta, x_{\text{max}}], x = f(g(x))\}
\]
\[
= \Delta > 0.
\]

Therefore, Lemma 29 implies that the fixed point of the translated coupled system, \(\bar{x}^{(\infty)}_i\), must satisfy \(\bar{x}^{(\infty)}_i \leq \delta\) for \(i \in [1: M]\). The proof is completed by noting that the fixed point of the original coupled system, \(x^{(\infty)}_i\), is upper bounded by \(\bar{x}^{(\infty)}_i + \bar{x}^*\) because the two coupled recursions are identical except for the translation and the fact that the translated system uses the larger boundary value \(\bar{x}^* \geq 0\).

IV. PROOF OF THEOREM 2

Before we present the proof of Theorem 2, we introduce the following lemma.

Lemma 30. The functions \(U_s(x)\) and \(U_c(x)\) satisfy

(i) \(U_s(x) > U_s(\bar{x}^*)\) for all \(x \in [0, \bar{x}^*)\)

(ii) If \(x \geq A^T f(Ag(x))\), then \(U_c(A^T f(Ag(x))) \leq U_c(x)\)

(iii) For an arbitrary vector \(x\), we have
\[
U_c(x) \geq \sum_{i=1}^M U_s(x_i)
\]

(iv) For a constant vector \(x = (x, \ldots, x)\), we have
\[
U_c(x) = M U_s(x) + (w - 1) F(g(x))
\]

Proof: For (i), we observe that \(\bar{x}^*\) is precisely the smallest value of \(x\) that achieves the minimum value of \(U_s(x)\). It follows easily that \(U_s(x) > U_s(\bar{x}^*)\) for \(x \in [0, \bar{x}^*)\). For (ii), we let \(z(t) = x + t(A^T f(Ag(x)) - x)\) and observe that \(x \geq z(t) \geq A^T f(Ag(x))\) holds because \(z(t)\) is a convex combination of the endpoints \(x \geq A^T f(Ag(x))\). Also, \(A^T f(Ag(x)) \geq A^T f(Ag(z(t)))\) is implied by \(x \geq z(t)\) because \(h(x) = A^T f(Ag(z(t)))\) is isometric. Next, we use (12) to write \(B = U_c(A^T f(Ag(x))) - U_c(x)\) as
\[
B = \int_{\bar{x}^*}^{\bar{x}^*} g'(z(t)) (z(t) - A^T f(Ag(z(t))))_{t \geq 0} (z'(t)) dt.
\]

Since \(g'(z(t))\) is an \(M \times M\) non-negative matrix (e.g., \(g\) is non-decreasing), it follows that \(B \leq 0\).

For (iii), consider any vector \(x\) and observe that
\[
U_c(x) = x^T g(x) - G(x) - F(Ag(x))
\]
\[
= \sum_{i=1}^M x_i g(x_i) - \sum_{i=1}^M G(x_i) - \sum_{j=1}^N F \left( \sum_{i=1}^M A_{j,i} g(x_i) \right)
\]
\[
\geq \sum_{i=1}^M x_i g(x_i) - \sum_{i=1}^M G(x_i) - \sum_{j=1}^N \sum_{i=1}^M A_{j,i} F(g(x_i))
\]
\[
\geq \sum_{i=1}^M x_i g(x_i) - \sum_{i=1}^M G(x_i) - \sum_{i=1}^M F(g(x_i))
\]
\[
= \sum_{i=1}^M U_s(x_i),
\]

where \((a)\) \(F \left( \sum_{i=1}^M A_{j,i} g(x_i) \right) \leq \sum_{i=1}^M A_{j,i} F(g(x_i))\) holds because \(F\) is convex and \(\sum_{i=1}^M A_{j,i} = 1\) and \((b)\) follows from \(\sum_{j=1}^N A_{j,i} \leq 1\).

For (iv), \(x = (x, \ldots, x)^T\) is a constant vector and we have
\[
U_c(x) = x^T g(x) - G(x) - F(Ag(x))
\]
\[
= \sum_{i=1}^M x g(x) - \sum_{i=1}^M G(x) - \sum_{i=1}^M F(\sum_{i=1}^M A_{j,i} g(x_i))
\]
\[
= \sum_{i=1}^M x g(x) - \sum_{i=1}^M G(x) - \sum_{i=1}^M F(g(x))
\]
\[
= \sum_{i=1}^M U_s(x) + (M - N) F(g(x))
\]
\[
= M U_s(x) + (w - 1) F(g(x)).
\]

Now, we are ready to prove Theorem 2. This theorem says that, for any fixed \(w \geq 0\) and \(\eta > 0\), there is a \(N_0 < \infty\) such that, for all \(N \geq N_0\), we have \(\max_{x \in [1: M]} z_i^{(\infty)} \geq \bar{x}^* - \eta\), where \(\bar{x}^* \triangleq \min (\arg \min_{x \in X} U_s(x))\). [Proof of Theorem 2] The fixed point of the vector recursion is given by \(z_i^{(\infty)}\). The idea is to lower bound this fixed point by starting the vector recursion \(z_i^{(t+1)} = A^T f(Ag(z_i^{(t)}))\) from \(z_i^{(0)} = \bar{x}^*1_M\). Since \(\bar{x}^*\) is a fixed point of the scalar recursion and the coupled recursion has 0’s at the boundary, \(z_i^{(t)} \geq z_i^{(t)}\) and the vector sequence \(z_i^{(0)} \geq z_i^{(1)} \geq \cdots\) is non-increasing. This implies that \(z_i^{(t)}\) converges to a limit \(z_i^{(\infty)}\) and, using the fact that \(x_i^{(t)} \geq x_i^{(0)}\), one finds that \(z_i^{(\infty)} \leq z_i^{(\infty)}\). From Lemma 30, it follows that
\[
\sum_{i=1}^M U_s \left( z_i^{(\infty)} \right) \quad \text{part (iii)} \quad \leq \quad U_c(z_i^{(\infty)}) \quad \text{part (ii)} \quad \leq \quad U_c(z_i^{(0)})
\]
\[
= U_c(\bar{x}^*1_M) \quad \text{part (iv)} \quad = \quad M U_s(\bar{x}^*) + (w - 1) F(g(\bar{x}^*)).
\]

The proof continues by contradiction. First, we define \(\gamma = \min_{x \in [0, \bar{x}^* - \eta]} U_s(x) - U_s(\bar{x}^*)\) and observe that \(\gamma > 0\) is implied by Lemma 30. Next, we choose
\[
N > N_0 \triangleq \frac{(w - 1) F(g(\bar{x}^*))}{\gamma} - (w - 1)
\]
and suppose that \(\max_{x \in [1: M]} z_i^{(\infty)} < \bar{x}^* - \eta\). This implies
\[ z_i^{(\infty)} < x^* - \eta \] for all \( i \in [1 : M] \) and, hence, that
\[ \sum_{i=1}^M U_s(z_i^{(\infty)}) \geq \sum_{i=1}^M \min_{x \in [0, x^* - \eta]} U_s(x) = M \left( U_s(x^*) + \gamma \right). \]

Next, we observe that \( N > N_0 \) implies
\[ M > N_0 + (w - 1) > \frac{(w - 1)F(g(x^*))}{\gamma} \]
and
\[ M \left( U_s(x^*) + \gamma \right) > M U_s(x^*) + (w - 1)F(g(x^*)). \]

But, this contradicts \( \text{(15)} \) and implies that \( \max_{i \in [1 : M]} z_i^{(\infty)} \geq x^* - \eta. \) The stated result follows from \( x^{(\infty)} \geq z^{(\infty)}. \) \( \blacksquare \)

V. Dependence on a Parameter

In many applications, the recursion of interest also depends on an additional parameter. If the uncoupled recursion has a stable fixed point at 0 when this parameter is sufficiently small, then one is often interested in the largest parameter value such that the coupled recursion converges to the zero vector. In this section, we characterize this threshold and show that, under some conditions, it is equal to the natural generalization of the threshold associated with the Maxwell conjecture for LDPC codes [26, Conj. 1].

A. Admissible Systems

For \( \varepsilon_{\max} \in (0, \infty) \), let \( \mathcal{E} \triangleq [0, \varepsilon_{\max}] \) and suppose that the system of interest depends on a parameter \( \varepsilon \in \mathcal{E} \). In this case, the recursion is defined by the bivariate functions
\[ f : \mathcal{Y} \times \mathcal{E} \to \mathcal{X} \quad \text{and} \quad g : \mathcal{X} \times \mathcal{E} \to \mathcal{Y}. \]

For bivariate functions with dependence on \( x \) and \( \varepsilon \), we use the notation \( f(x; \varepsilon) \) and the first two \( x \)-derivatives are denoted by \( f'(x; \varepsilon) \) and \( f''(x; \varepsilon) \). Derivatives in \( \varepsilon \) (or mixed partial derivatives) are denoted by \( f'(m,n)(x; \varepsilon) \triangleq \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial \varepsilon^n} f(x; \varepsilon) \). For convenience, we also define \( h(x; \varepsilon) \triangleq f(g(x; \varepsilon); \varepsilon) \) and \( \mathcal{X}_0 \triangleq \{0, x_{\max}\}. \)

**Definition 31.** An admissible system is a system where the functions \( f(x; \varepsilon) \) and \( g(x; \varepsilon) \) satisfy:

(i) \( f(x; \varepsilon) \) and \( g(x; \varepsilon) \) are \( C^1 \) functions on \( \mathcal{X} \times \mathcal{E} \);
(ii) \( f'(x; \varepsilon) \) and \( g'(x; \varepsilon) \) are non-decreasing in both \( x \) and \( \varepsilon \);
(iii) \( g(x; \varepsilon) \) is strictly increasing in \( x \), and
(iv) \( g''(x; \varepsilon) \) exists and is jointly continuous on \( \mathcal{X} \times \mathcal{E} \).

In addition, we say that an admissible system is proper if \( h^{(0,1)}(x; \varepsilon) > 0 \) for all \( (x, \varepsilon) \in \mathcal{X}_0 \times \mathcal{E} \).

Based on the above conditions, it is easy to verify that the results of Theorem 1 and Theorem 2 hold for each \( \varepsilon \in \mathcal{E} \). For admissible systems, the uncoupled potential function, \( U_s : \mathcal{X} \times \mathcal{E} \to \mathbb{R} \), is defined to be

\[ U_s(x; \varepsilon) \triangleq xg(x; \varepsilon) - G(x; \varepsilon) - F(g(x; \varepsilon); \varepsilon), \]

where \( F(x; \varepsilon) = \int_0^x f(z; \varepsilon)dz \) and \( G(x; \varepsilon) = \int_0^x g(z; \varepsilon)dz \).

**Definition 32.** Applying Theorem 1 to a system that depends on a parameter naturally leads to the following definitions:

\[ \Psi(\varepsilon) \triangleq \min_{x \in \mathcal{X}} U_s(x; \varepsilon), \]
\[ X^*(\varepsilon) \triangleq \{ x \in \mathcal{X} : U_s(x; \varepsilon) = \Psi(\varepsilon) \}, \quad \text{and} \]
\[ \pi^*(\varepsilon) \triangleq \max X^*(\varepsilon). \]

In many cases, one is interested in the \( \varepsilon \)-threshold below which the uncoupled (resp. coupled) recursion converges to 0 (resp. the 0 vector). This leads us to the following definitions.

**Definition 33.** Let the single-system threshold \( \varepsilon^*_s \) be

\[ \varepsilon^*_s \triangleq \sup \{ \varepsilon \in \mathcal{E} \mid h(x; \varepsilon) < x, \ x \in \mathcal{X}_0 \}, \quad (17) \]

which is well-defined as long as \( h(x; 0) < x \) for \( x \in \mathcal{X}_0 \).

**Definition 34.** Let the stability threshold \( \varepsilon^*_{\text{stab}} \) be

\[ \varepsilon^*_{\text{stab}} \triangleq \sup \{ \varepsilon \in \mathcal{E} \mid \exists \delta > 0, \ \forall x \in (0, \delta], h(x; \varepsilon) < x \}, \quad (18) \]

which is well-defined as long as 0 is a stable fixed point of the uncoupled recursion for \( \varepsilon = 0 \). We say that a system has a strict stability threshold if, for all \( \varepsilon \in (\varepsilon^*_{\text{stab}}, \varepsilon_{\max}] \), there is a \( \delta > 0 \) such that \( h(x; \varepsilon) > x \) for all \( x \in (0, \delta] \). Likewise, a system is called unconditionally stable if there is a \( \delta > 0 \) such that \( h(x; \varepsilon_{\max}) < x \) for all \( x \in (0, \delta] \).

For the coupled recursion, Theorem 1 implies a simple threshold for convergence to 0 as \( w \to \infty \). This expression is stated precisely below. Lemma 46 also gives a number of equivalent expressions.

**Definition 35.** Let the coupled threshold (or potential threshold) \( \varepsilon^*_c \) be defined by

\[ \varepsilon^*_c \triangleq \sup \{ \varepsilon \in \mathcal{E} \mid \pi^*(\varepsilon) = 0 \}, \quad (19) \]

which is well-defined as long as \( \pi^*(0) = 0 \).

**Lemma 36.** If the above thresholds are well-defined, then we have the following:

(i) If \( \varepsilon < \varepsilon^*_s \), then the uncoupled recursion converges to a fixed point at 0. If \( \varepsilon > \varepsilon^*_s \), then it converges to a non-zero fixed point.
(ii) If \( \varepsilon < \varepsilon^*_{\text{stab}} \), then 0 is a stable fixed point of the uncoupled recursion. If \( \varepsilon > \varepsilon^*_{\text{stab}} \), then 0 is not a stable fixed point.
(iii) If \( \varepsilon < \varepsilon^*_c \), then \( \pi^*(\varepsilon) = 0 \) and the fixed point of the coupled recursion converges to 0 as \( w \to \infty \).
(iv) If \( \varepsilon < \varepsilon^*_c \leq \varepsilon^*_{\text{stab}} \), then there is a \( w_0 < \infty \) such that the coupled recursion converges to 0 for all \( w > w_0 \).

**Proof:** The statement of (i) follows from well-known properties of continuous scalar recursions [51, p. 96]. A fixed point of a recursion is called stable if all sufficiently small perturbations return to that fixed point. Using this definition, we see that (ii) is true by definition. Statement (iii) follows from the definition of \( \varepsilon^*_c \) and Theorem 1.

For (iv), we note that \( \varepsilon < \varepsilon^*_c \leq \varepsilon^*_{\text{stab}} \) implies that \( \pi^*(\varepsilon) = 0 \) and that there is a \( \delta > 0 \) such that \( h(x; \varepsilon) < x \) for all \( x \in (0, \delta] \). Therefore, Proposition 10(iii) implies that there is
a \( w_0 < \infty \) such that the coupled recursion converges to 0 for all \( w > w_0 \).

B. Connections with EXIT Functions

Recursions that depend on a parameter also have an extra degree of freedom that allows some quantities to be computed in creative ways. For example, the DE equations for LDPC codes on a BEC have a deep structure and the Maxwell construction in statistical physics can be applied to their EXIT functions to bound the performance of MAP detectors \([29]\).

Some elements of that theory (e.g., parts that depend only on the mathematical structure of the problem) can be extended to general recursions. The remainder of this section describes these extensions.

Example 37. For the case of irregular LDPC codes on the BEC \([51]\), we have \( \varepsilon_{\text{max}} = x_{\text{max}} = y_{\text{max}} = 1 \) and

\[
\begin{align*}
    f(x; \varepsilon) &= \varepsilon L(x) \\
    g(x; \varepsilon) &= 1 - \rho(1 - x).
\end{align*}
\]

The polynomials \( L(x) \) (resp. \( R(x) \)) define the bit (resp. check) node degree distributions and we use the standard convention that \( \lambda(x) = L'(x)/L'(1) \) (resp. \( \rho(x) = R'(x)/R'(1) \)) \([51]\). In terms of these functions, we find that

\[
\begin{align*}
    F(x; \varepsilon) &= \varepsilon L(x)/L'(1) \\
    G(x; \varepsilon) &= x - (1 - R(1 - x))/R'(1). \tag{20}
\end{align*}
\]

The trial entropy\(^6\) for this problem is defined by \( \phi(x, 1 - \rho(1 - x)) \) in \([52] (9.10)\) and \( P_\varepsilon(x, y(x)) \) in \([26] \text{Lem. 4}\). Translating notation between these papers shows that the uncoupled potential \( U_s(x; \varepsilon) \) is exactly equal to \(-1/L'(1)\) times the trial entropy. Conjectured values for the conditional entropy and the MAP erasure rate are also given explicitly in \([52] \text{Sec. X}\). The Maxwell construction in \([26] \) also implies conjectured values for conditional entropy, the MAP EXIT function, and erasure rate of the MAP decoder. The conjectured value of the conditional entropy of the code bits given the observations is given by \( \max_{x \in [0,1]} \phi(x, 1 - \rho(1 - x)) = -L'(1)\Psi(\varepsilon) \) and this was first proven to be an upper bound (under some restrictions) in \([52] (9.10)\). The conjectured value of the MAP EXIT function is given by \(-L'(1)\Psi(\varepsilon)\). The conjectured value of the MAP erasure rate is given by \( \varepsilon L(\tau^*(\varepsilon)) = L'(1)F(\tau^*(\varepsilon); \varepsilon) \), where \( \tau^*(\varepsilon) \) is a DE fixed point because Lemma \([18]\) shows that all minima of \( U_s(x; \varepsilon) \) occur at fixed points. We note that the results of this paper show that SC ensembles achieve these conjectured values.

The perspective in this section provides generalizations of the quantities in Example \([37]\) for general \( f(x; \varepsilon) \) and \( g(x; \varepsilon) \).

An important part of this is the result that \( \Psi(\varepsilon) \) has a simple integral expression. This follows from the fact that \( x \in X^\varepsilon(\varepsilon) \)

implies that \( x \) is a fixed point (i.e., \( h(x; \varepsilon) = x \)) and, hence, \( U_s^{(1,0)}(x; \varepsilon) = 0 \) and \( U_s^{(0,1)}(x; \varepsilon) \) has a simple expression (e.g., see \([23]\)). If the system satisfies a few additional constraints, then this simple expression can also be used to compute \( U_s(x; \varepsilon) \) at any fixed point. The following theorem forms a basis for these connections.

**Theorem 38.** The function \( \Psi(\varepsilon) \) is non-increasing, and satisfies \( \Psi(\varepsilon) = \int_0^\varepsilon \psi(t) dt \) with

\[
\psi(t) \triangleq -G^{(0,1)}(\tau^*(t); t) - F^{(0,1)}(g(\tau^*(t); t); t). \tag{21}
\]

For a proper admissible system, \( \Psi(\varepsilon) \) is strictly decreasing on \( \varepsilon \in [\varepsilon_0, \varepsilon_{\text{max}}] \) or, similarly, if \( \tau^*(\varepsilon) > 0 \).

The following Lemma, whose proof appears in Appendix \([C]\) is used in the proof of Theorem \([38]\).

**Lemma 39.** For admissible systems, the function \( U_s(x; \varepsilon) \) (resp. \( \Psi(\varepsilon) \)) is (resp. uniformly) Lipschitz continuous in \( \varepsilon \). Moreover, \( F^{(0,1)}(x; \varepsilon) \) and \( G^{(0,1)}(x; \varepsilon) \) are non-negative and non-decreasing in \( x \).

**Proof of Theorem 38.** Consider a continuously differentiable bivariate function (e.g., \( U_s(x; \varepsilon) \)) and its \( x \)-minimum over a compact set (e.g., \( \Psi(\varepsilon) = \min_{x \in X} U_s(x; \varepsilon) \)). The standard *envelope theorem* states that, if the minimum is achieved uniquely, then the derivative of \( \Psi \) equals \( U_s^{(0,1)}(x; \varepsilon) \) evaluated at the \( x \)-minimizer (e.g., \( x = \tau^*(\varepsilon) \)). In this proof, we make use of an elegant generalization of the envelope theorem to the case where the minimum is not necessarily unique \([56]\).

Since \( \Psi(\varepsilon) \) is Lipschitz continuous by Lemma \([39]\) it follows that \( \Psi(\varepsilon) \) is differentiable almost everywhere and satisfies the fundamental theorem of calculus \([57] \text{pp. 106-108}\). Therefore, there is a Lebesgue integrable function \( \psi(t) = \Psi'(t) \) such that

\[
\Psi(\varepsilon) = \int_0^\varepsilon \psi(t) dt.
\]

Based on this, the envelope theorem in \([56]\) shows that \( \psi(t) = U_s^{(0,1)}(\tau^*(t); t) \) almost everywhere. Using \([16]\), we can compute

\[
\begin{align*}
    U_s^{(0,1)}(x; \varepsilon) &= \frac{d}{d\varepsilon} U_s(x; \varepsilon) \\
    &= \frac{d}{d\varepsilon} \left[ (xg(x; \varepsilon) - G(x; \varepsilon) - F(g(x; \varepsilon); \varepsilon)) \right] \\
    &= (x - f(g(x; \varepsilon); \varepsilon)) g^{(0,1)}(x; \varepsilon) - G^{(0,1)}(x; \varepsilon) \\
    &= F^{(0,1)}(g(x; \varepsilon); \varepsilon). \tag{22}
\end{align*}
\]

Since Lemma \([18]\) shows that all minima of \( U_s(x; \varepsilon) \) occur at fixed points, we see that

\[
U_s^{(0,1)}(\tau^*(\varepsilon); \varepsilon) = -G^{(0,1)}(\tau^*(\varepsilon); \varepsilon) - F^{(0,1)}(g(\tau^*(\varepsilon); \varepsilon)).
\]

Finally, we note that Lemma \([39]\) also shows that \( F^{(0,1)}(x; \varepsilon) \) and \( G^{(0,1)}(x; \varepsilon) \) are non-negative on \( \mathcal{X} \times \mathcal{E} \). From this, we see that \( \psi(\varepsilon) \) is almost everywhere non-positive and conclude that \( \Psi(\varepsilon) \) is non-increasing.

For a proper admissible system,

\[
l^{(0,1)}(x; \varepsilon) = f^{(1,0)}(g(x; \varepsilon); \varepsilon) g^{(0,1)}(x; \varepsilon) + f^{(0,1)}(g(x; \varepsilon); \varepsilon).
\]
is positive on $X_c \times E$ and this implies either $g^{(0,1)}(x; \varepsilon) > 0$ or $f^{(0,1)}(g(x; \varepsilon); \varepsilon) > 0$ for each $(x, \varepsilon) \in X_c \times E$. By integrating, we see that, if $\Xi^* (\varepsilon) > 0$, then either $G^{(0,1)}(\Xi^* (\varepsilon); \varepsilon) > 0$ or $F^{(0,1)}(g(\Xi^* (\varepsilon); \varepsilon); \varepsilon) > 0$. Since the definition of $\varepsilon^* \varepsilon$ implies that $\Xi^* (\varepsilon) > 0$ for $\varepsilon > \varepsilon^*$, it follows that $\Psi (\varepsilon)$ is strictly decreasing on $\varepsilon \in [\varepsilon^*, \varepsilon_{\text{max}}]$.

**Remark 40.** The function $\psi (t)$ is analogous to the MAP EXIT function and is computed by evaluating a function, which is closely related to the recursion, along a fixed-point curve (e.g., $\Xi^* (t)$) of the recursion. The difference between the MAP EXIT function and the EBP EXIT function is the fixed-point curve along which the integral is taken. For the DE recursion of irregular LDPC codes with degree profile $(\lambda, \rho)$, the EBP EXIT function associated with a fixed point $\Xi^* (t)$ is $L(1 - \rho(1 - \Xi^* (t)))$. Likewise, we find that $G^{(0,1)}(x; \varepsilon) = 0$ and $F^{(0,1)}(x; \varepsilon) = L(1 - \rho(1 - x))/L'(1)$ implies $\psi (t) = -L(1 - \rho(1 - \Xi^* (t))) /L'(1)$.

**Definition 41.** For admissible systems, the following observations and definitions will be useful:

(i) Since $h(x; \varepsilon)$ is continuous and non-decreasing in $\varepsilon$, the subset of $X_c$ that supports a fixed point is

$$X_f = \{ x \in X_c \mid h(x; 0) \leq x, h(x; \varepsilon_{\text{max}}) \geq x \} .$$

(ii) For each $x \in X_f$, let $\varepsilon (x)$ be the smallest $\varepsilon$ that supports a fixed point at $x$.

(iii) The fixed-point potential, $Q: X_f \rightarrow \mathbb{R}$, is defined by $Q(x) = U_s(x; \varepsilon (x))$.

**Lemma 42.** For a proper admissible system, $\varepsilon (x)$ maps $x \in X_f$ to the unique $\varepsilon \in E$ such that $h(x; \varepsilon) = x$. Moreover, each $x_0 \in X_f$ with $\varepsilon (x_0) \in (0, \varepsilon_{\text{max}}]$ lies in the interior of $X_f$ and $\varepsilon (x)$ is $C^1$ on any interval in $X_f$.

**Proof:** See Appendix C.

The following lemma is the natural generalization of [26, Lem. 4] to the more general setup considered in this paper.

**Lemma 43.** For a proper admissible system, if the interval $[x_1, x_2] \subset X_f$, then $Q(x_2) - Q(x_1)$ is given by

$$- \int_{x_1}^{x_2} \left( G^{(0,1)}(x; \varepsilon (x)) + F^{(0,1)}(x; \varepsilon (x)) \right) \varepsilon' (x) dx .$$

**Proof:** See Appendix C.

**Remark 44.** It is very likely that the precise connection between the Maxwell construction and the $\Psi (\varepsilon)$ function is still rather opaque. The key to understanding this connection is to note that $\Xi^* (\varepsilon)$ and $\Psi (\varepsilon)$ can only jump when the minimum of $U_s(x; \varepsilon)$ is achieved at multiple $x$-values. In this case, there exist $x_1, x_2 \in X^* (\varepsilon)$ such that $U_s(x_1; \varepsilon) = U_s(x_2; \varepsilon) = \Psi (\varepsilon)$ and $\varepsilon (x_1) = \varepsilon (x_2) = \varepsilon$. Therefore, if $[x_1, x_2] \subset X_f$, then Lemma 43 shows that the natural analogue of the EBP EXIT area integral is given by

$$0 = U_s(x_2; \varepsilon) - U_s(x_1; \varepsilon) = Q(x_2) - Q(x_1) = \int_{x_1}^{x_2} U_s^{(0,1)}(x; \varepsilon (x)) \varepsilon' (x) dx .$$

Hence, the positive and negative portions of the parametric area computation (e.g., EBP EXIT integral) must balance.

We note that this connection between the Maxwell construction and minimizers of thermodynamic potential functions is certainly known by many physicists. It can be seen implicitly in the statistical physics literature (via Legendre transforms) but is rarely discussed explicitly.

**Lemma 45.** The following properties of $X_f$ and $\varepsilon (x)$ will be useful:

(i) The set $X_f$ is either closed or it is missing a single limit point at $x = 0$.

(ii) If $\varepsilon^*_0 \in [0, \varepsilon_{\text{max}}]$, then there exists a sequence $x_n \in X_f$ such that $x_n \rightarrow 0$ and $\lim_{n} \varepsilon (x_n) = \varepsilon^*_0$.

(iii) If the system is unconditionally stable, then $X_f$ is closed and $\varepsilon^*_0 \leq \varepsilon^*_0$.

(iv) If the system has a strict stability threshold $\varepsilon^*_0 \in [0, \varepsilon_{\text{max}}]$, then $\varepsilon^*_0 \leq \varepsilon^*_0$ and $\lim_{n} \varepsilon (x_n) = \varepsilon^*_0$ for any sequence $x_n \in X_f$ such that $x_n \rightarrow 0$.

**Proof:** See Appendix C.

**Lemma 46.** The following expressions will also be useful:

(i) For any admissible system, $\Psi (\varepsilon) = 0$ for $\varepsilon \leq \varepsilon^*$ and $\Psi (\varepsilon) > 0$ for $\varepsilon > \varepsilon^*$.

(ii) For a proper admissible system,

$$\varepsilon^* = \sup \{ \varepsilon \in E \mid \min_{x \in X} U_s(x; \varepsilon) \geq 0 \} .$$

Also, if $Q(x) < 0$ for $x \in X_f$, then $\varepsilon^* < \varepsilon (x)$.

(iii) Consider a proper admissible system where either $X_f$ is closed or the system has a strict stability threshold $\varepsilon^*_0 \in [0, \varepsilon_{\text{max}}]$. If $\varepsilon^*_0 \in [0, \varepsilon_{\text{max}}]$ then

$$\varepsilon^*_0 = \min \{ \varepsilon (x) \mid x \in X_f, Q(x) = 0 \} .$$

**Proof:** See Appendix C.

**Remark 47.** We note that the definition of the potential threshold in (23) is exactly the same as the definition given in [22] but there is one caveat. In [22], the energy gap is defined as the minimum of the potential over all $x$-values but less than the first unstable fixed point. It is stated incorrectly in that this value will always be positive. Instead, this value is strictly positive only if there are no unstable fixed points arbitrarily close to $x = 0$. For some pathological cases (e.g., see Example 7), the energy gap can be zero. Remark 44 shows that threshold of the SC system still converges to $\varepsilon^*_0$, but only as $w \rightarrow \infty$.

VI. APPLICATIONS

A. Application to Irregular LDPC Codes

Consider the ensemble LDPC($\lambda, \rho$) of irregular LDPC codes and assume transmission takes place over an erasure channel with parameter $\varepsilon$ [51]. For this ensemble, the node degree distributions are given by $L(z) = \sum_{i=2}^{d_z} L_i z^i$ (resp. $R(z) = \sum_{i=1}^{d_z} R_i z^i$) where $L_i$ (resp. $R_i$) is a rational number that defines the fraction of variable (resp. check) nodes with degree $i$. The edge degree distributions are given by $\Lambda (z) = L'(z)/L'(1) = \sum_{i=2}^{d_u} \lambda_i z^{i-1}$ (resp. $\rho (z) = R'(z)/R'(1) = \sum_{i=1}^{d_u} \rho_i z^{i-1}$).
Definition 48. The ensemble of SC irregular LDPC($\lambda, \rho$) codes with length $kN$ is defined as follows. Let $\kappa$ be chosen such that $\kappa L_i, \kappa R_j \ell(1)/R(1)$, and $\kappa L'(1)/w$ are integers. A collection of $N$ variable-node groups are placed at positions labeled $\{1, 2, \ldots, N\}$ and a collection of $M = N + w - 1$ check-node groups are placed at positions labeled $\{1, 2, \ldots, M\}$. In each variable-node group, $\kappa L_i$ nodes of degree $i$ are placed for $i \in [2 : d_v]$. Similarly, in each check-node group, $\kappa R_j \ell(1)/R(1)$ nodes of degree $j$ are placed for $j \in [1 : d_c]$. Next, the $\kappa L'(1)$ edge sockets in each group of variable and check nodes are partitioned in $w$ groups using a uniform random permutation. The set of variable-node (resp. check-node) sockets in the $k$-th group of position $i$ is denoted by $\varphi_{i,k}^\ell$ (resp. $\varphi_{i,k}^R$) for $k = 0, 1, \ldots, w - 1$. The SC code is constructed by defining edges that connect the sockets in $\varphi_{i,k}^\ell$ to the sockets in $\varphi_{i+k-1,k}^\ell$ in some fixed manner. This construction leaves some sockets of the check-node groups at the boundaries unconnected and these sockets are removed (i.e., the bit value is assumed to 0). It is easy to verify that the DE analysis equations for this ensemble are given by [1] as $\kappa \to \infty$. This construction can be seen as a variation on the original $(1, x, L, w)$ SC ensemble defined in [14].

The results of this paper allow us to analyze the performance of the above SC LDPC ensemble. In particular, the uncoupled potential function can be computed easily from [20] and we note that it equals $-\phi(x, 1 - \rho(1 - x))/L'(1)$, where $\phi$ is given by [52] (9.10). For the uncoupled ensemble, this implies that, under mild conditions on $R(x)$, the conditional entropy of the code bits given the received sequence is upper bounded by $-L'(1)\psi(z)$ [52] Thm. 2.

For this system, the unique $\varepsilon$ associated with a fixed point $x$ is well-known and given by $\varepsilon(x) = x/\lambda(1 - \rho(1 - x))$. For consistency with Definition 41 we use $\tilde{\varepsilon}(x)$ to denote the restriction of $\varepsilon(x)$ to the domain $X_f = \{x \in [0, 1] | \varepsilon(x) \in [0, 1]\}$. For this setup, the conjectured MAP decoding threshold is called the Maxwell threshold [26] Conj. 1) and is defined in terms of the trial entropy

$$P(x) = \int_0^x L \left(1 - \rho(1 - z)\right) \tilde{\varepsilon}'(z) dz$$

$$\equiv -L'(1)x(1 - \rho(1 - x)) + L'(1) \left(\frac{x + 1 - R(1 - x)}{R'(1)}\right) + \varepsilon(x)L(1 - \rho(1 - x))$$

$$= -L'(1)Q(x)$$

for $x \in X_f$,

where $(a)$ comes from [51] p. 124 and the relationship to $Q(x)$ can be verified directly by computing $Q(x)$ according to Definition 41. In particular, the Maxwell threshold [26] Conj. 1] is given by

$$\varepsilon_{\text{Max}} = \min \left\{ \frac{1}{x(0)\rho'(1)} \min \{\varepsilon(x) | P(x) = 0, x \in [0, 1]\} \right\}$$

$$= \min \{\varepsilon(x) | P(x) = 0, x \in [0, 1]\},$$

where $P(0) \equiv \lim_{x \to 0} P(x) = 0$ and

$$\varepsilon(0) \equiv \lim_{x \to 0} \varepsilon(x) = \frac{1}{x(0)\rho'(1)}$$

for $R(0) > 0$ otherwise.

In many cases, the results of [52], [20], [58] can be used to show that $\varepsilon_{\text{MAP}} \leq \varepsilon_{\text{Max}}$, where $\varepsilon_{\text{MAP}}$ is noise threshold of the MAP decoder. We also note that the potential $U_s(x; \varepsilon)$ is the same as the pseudo-dual of the average Bethe variational entropy (e.g., see [59], [49] Part 2, pp. 62-65).

Lemma 49. For the ensemble LDPC($\lambda, \rho$), the potential threshold [19] equals the Maxwell threshold if the code rate $r = 1 - L'(1)/R'(1)$ satisfies $r > 0$.

Proof: Based on well-known properties of LDPC DE equations, it is easy to verify that they define an admissible system. It is a proper admissible system because $h^{(0,1)}(x; \varepsilon) = \lambda(1 - \rho(1 - x)) > 0$ for $x \in (0, 1)$. Also, the expression $h(x; \varepsilon) = \varepsilon(0)\rho'(1)x + O(x^2)$ shows that the system has a strict stability threshold $\varepsilon_{\text{stab}}(0) \in (0, 1)$. If $r > 0$, then we can compute directly $L'(1)U_s(1; 1) = L'(1)/R'(1) - 1 < 0$. Since $U_s(0, 1) = 0$, this implies that $\varepsilon^*(1) > 0$ and, hence, that $\varepsilon^*(0) \in [0, 1]$. Along with the strict stability threshold, this allows us to apply Lemma 46(iii) to see that (24) holds. This also implies that there is an $x^* \in X_f$ such that $\varepsilon(x^*) < 1$ and $Q(x^*) = 0$. From this, we see that $\varepsilon_{\text{Max}} < 1$ and that

$$\varepsilon^* = \inf \{\tilde{\varepsilon}(x) | x \in X_f, Q(x) = 0\}$$

$$= \inf \{\varepsilon(x) | x \in [0, 1], P(x) = 0\} = \varepsilon_{\text{Max}}$$

because $P(x) = -L'(1)Q(x)$ for $x \in X_f$ and $\varepsilon(x) > 1$ for $x \in [0, 1] \backslash X_f$.

Corollary 50. If the conditions of Lemma 49 hold and $\varepsilon < \varepsilon_{\text{Max}}$, then there is a $w_0 < \infty$ such that the SC DE recursion converges to the zero vector for all $w > w_0$.

Proof: Since $\varepsilon_{\text{Max}} = \varepsilon_{\text{ stabilization}}^* \leq \varepsilon_{\text{ stab}}^*$ by Lemma 49 and $\varepsilon < \varepsilon_{\text{Max}}$, we can apply Lemma [36](w) to see that there is a $w_0 < \infty$ such that, for all $w < w_0$, the SC DE recursion will converge to the zero vector.

Example 51. Consider the irregular LDPC ensemble with degree distribution

$$\lambda(x) = \frac{4}{20}x + \frac{5}{20}x^2 + \frac{7}{20}x^6 + \frac{9}{20}x^{20}$$

$$\rho(x) = \frac{6}{10}x^4 + \frac{4}{10}x^{12}$$

In this case, $\varepsilon^*$ equals the conjectured MAP threshold and we can compute $\varepsilon^* = 0.625$.

Recall that the parametric EBP EXIT curve [51] Ch. 3] is given by $\varepsilon(x; L(1 - \rho(1 - x)))$ and notice that, up to a scale factor, this equals $U_s(L(1; x; \varepsilon)(x))$. Likewise, the conjectured

8Their statement actually neglects the stability condition and is incorrect if the threshold is determined by stability. Also, there are pathological cases where this definition implies $\varepsilon_{\text{ Max}} = \infty$ (e.g., when the design rate $1 - L'(1)/R'(1)$ is negative).
MAP EXIT curve is given by $L(1 - \rho(1 - \pi^*(\varepsilon)))$ and, up to a scale factor, this equals $\psi(\varepsilon)$. We can also compute $L(1 - \rho(1 - x_{406}(\varepsilon)))$, where $x_{406}(\varepsilon)$ is the fixed-point erasure rate for the worst-case position of a finite SC system with $w = 11$ and $N = 800$. Fig. 4 shows the EBP EXIT curve, the conjectured MAP EXIT curve, and the performance of the finite SC system. Notice that the finite SC system matches the asymptotic prediction almost exactly.

**B. Application to Irregular LDGM Codes**

Any code in the irregular LDGM ensemble, LDGC($\lambda, \rho$), can be converted into an LDGM code by adding a degree-1 variable node to each check node. During transmission, all degree-1 variable nodes are transmitted and all other variable nodes are punctured. Let LDGC($\lambda, \rho$) denote the standard irregular LDGM ensemble formed by converting each code in LDGC($\lambda, \rho$) to an LDGM code. The degree distributions are defined identically to the LDGC case in Section VI-B except that the new degree-1 variable nodes are not counted.

Consider the iterative decoding of a code in LDGC($\lambda, \rho$) assuming transmission takes place over an erasure channel with parameter $\varepsilon$ [51]. Let $x(\ell)$ be the fraction of erasure messages sent from variable to check nodes during iteration $\ell$. Then, the DE equation can be written in the form of (3), where $f(x; \varepsilon) = \lambda(x)$ and $g(x; \varepsilon) = 1 - (1 - \varepsilon)\rho(1 - x)$ [51]. This example was discussed in Example 6 and it is easy to verify that $f$ and $g$ describe an proper admissible system with $\varepsilon_{\text{max}} = x_{\text{max}} = y_{\text{max}} = 1$. The SC LDGC ensemble is essentially identical to the SC LDGC ensemble in Definition 48 except that degree-1 variable nodes are added to each check node.

The results of this paper also allow us to analyze the performance of the SC LDGC ensemble. This is in contrast with our previous results in [11, 50] because $g(0; \varepsilon) = 1 - (1 - \varepsilon)\rho(1) > \varepsilon$ and $h(0; \varepsilon) > \lambda(\varepsilon)$ do not satisfy the necessary conditions in those papers. This is related to the well-known fact that LDGC codes do not have a perfect-decoding fixed point. Still, the uncoupled potential function can be computed easily from (20) using the fact that $F(x; \varepsilon) = L(x)/L'(1)$ and $G(x; \varepsilon) = x - (1 - \varepsilon)(1 - R(1 - x))/R'(1)$. In this case, $U_s(\varepsilon)$ is the same as $-\phi(x, 1 - (1 - \varepsilon)\rho(1 - x))/L'(1)$, where $\phi$ is given by [52] (9,16). For the uncoupled ensemble, [52] Thm. 2] states that, under mild conditions on $R(x)$, the conditional entropy per information bit is upper bounded by $\max_{\varepsilon \in [0,1]} \phi(x, 1 - (1 - \varepsilon)\rho(1 - x)) = -L'(1)\Psi(\varepsilon)$. Normalizing instead by the number of code bits, we find that the conditional entropy per code bit is upper bounded by $-R'(1)\Psi(\varepsilon)$.

Likewise, one can solve for $\varepsilon$ in the fixed point equation to see that

$$
\varepsilon(x) = 1 - \frac{1 - \lambda^{-1}(x)}{\rho(1 - x)}
$$

and $X_f = \{x \in (0, 1) | \varepsilon(x) \in [0, 1]\}$. If there are no degree-1 checks (i.e., $\rho(0) = 0$), then $x = 1$ is a fixed point for all $\varepsilon \in \mathcal{E}$ and the formula for $\varepsilon(x)$ must be treated carefully at this point. Regardless, we can write the fixed-point potential as

$$
Q(x) = x(1 - (1 - \varepsilon(x))\rho(1 - x)) - \left( x - (1 - \varepsilon(x)) \frac{1 - R(1 - x)}{R'(1)} \right) - \frac{1}{L'(1)} L(1 - (1 - \varepsilon(x))\rho(1 - x)).
$$

We note that, since the DE recursion for LDGC codes does not converge to 0 for any $\varepsilon > 0$, the previous threshold results do not apply.

**Remark 52.** Although we cannot use the previous threshold results for this system, the following observations provide something similar. Notice that $\Psi(\varepsilon(\pi^*(\varepsilon))) = Q(\pi^*(\varepsilon))$ implies $\varepsilon(\pi^*(\varepsilon)) = \Psi^{-1}(Q(\pi^*(\varepsilon)))$, where the inverse exists because $h(0,1)(x; \varepsilon) > 0$ for $\varepsilon \in [0,1]$ implies $\Psi$ is strictly decreasing for $\varepsilon \in [0,1]$. From this, we see that, for any $x \in X_f$, $\Psi^{-1}(Q(x))$ can be seen as the $\varepsilon$-threshold below which $\pi^*(\varepsilon) \leq x$.

**Example 53.** Consider the irregular LDGC ensemble with degree distribution

$$
\lambda(x) = x^5, \quad \rho(x) = \frac{2}{45} + \frac{7}{15}x^2 + \frac{4}{9}x^3.
$$

In this case, the Maxwell curve has a single discontinuity at $\varepsilon_0 \approx 0.508$.

Applying the techniques in [51] Ch. 3] to the LDGC ensemble (i.e., differentiating the conditional entropy) shows that the parametric EBP EXIT curve is given by

$$
(\varepsilon(x); 1 - R(1 - x)) = \left( \varepsilon(x), R'(1)G^{(0,1)}(x; \varepsilon(x)) \right)
$$

and notice that, up to a scale factor, this equals $U_s^{(0,1)}(x; \varepsilon(x))$. Likewise, the conjectured MAP EXIT curve is given by

$$
1 - R(1 - \pi^*(\varepsilon)) = R'(1)G^{(0,1)}(\pi^*(\varepsilon); \varepsilon)
$$

and, up to a scale factor, this equals $\psi(\varepsilon)$. We can also compute $1 - R(1 - x_{406}(\varepsilon))$, where $x_{406}(\varepsilon)$ is the fixed-point erasure rate for the worst-case position of a finite SC system with $w = 11$ and $N = 800$. Fig. 5 shows the parametric EBP
EXIT, the conjectured MAP EXIT curve, and the performance of the finite SC system. Notice that the finite SC system has a slight overhang at $\varepsilon_0$ due to finite $w$ but otherwise matches the asymptotic prediction very well.

C. Application to Generalized LDPC Codes

Consider a generalized LDPC (GLDPC) code with degree-2 bits and generalized check constraints given by a primitive BCH code of block-length $n$. For an iterative decoder based on bounded-distance decoding of the BCH code, the DE recursions can be derived for both the BEC and binary symmetric channel (BSC) [22]. On the BEC, the code is chosen to correct all patterns of at most $t$ erasures. On the BSC, the code is chosen to correct all error patterns of weight at most $t$ and it is assumed that miscorrections can be ignored.

For fixed $t$ and large $n$, the rate of a $t$ error-correcting primitive BCH code is given by $r_{BCH} = 1 - t \log_2(n+1)/n$ and the same code corrects $2t$ erasures. For the GLDPC code, the overall design rate is $r_{GLDPC} = 1 - 2(1 - r_{BCH})$ [22]. Therefore, the GLDPC rates for the BSC and BEC constructions are $r_{BSC} = 1 - 2t \log_2(n+1)/n$ and $r_{BEC} = 1 - t \log_2(n+1)/n$.

For both cases, the iterative decoding performance of this ensemble is characterized by a DE recursion of the form (3), where $\varepsilon$ denotes the channel parameter. In this case, the recursion is defined by $\varepsilon_{\max} = x_{\max} = y_{\max} = 1$, $f(x; \varepsilon) \triangleq \varepsilon x$, and $g(x; \varepsilon) \triangleq \sum_{i=1}^{n-1} \binom{n-1}{i} x^i (1-x)^{n-1-i}$ [22]. Here, $x$ denotes the erasure (resp. error) probability of bit-to-check messages for the BEC (resp. BSC) case. We note that $g(0; \varepsilon) = 0$ and $g(1; \varepsilon) = 1$. To highlight the fact that $g(x; \varepsilon)$ is independent of $\varepsilon$, we also write $g(x)$ instead of $g(x; \varepsilon)$.

Let $B(a, b) \triangleq \binom{(a-1)(b-1)}{(a+b-1)}$ denote the Beta function and

$$I_x(t, n - t) \triangleq \frac{1}{B(t, n - t)} \int_0^x z^{t-1} (1-z)^{n-t-1} dz$$

denote the regularized incomplete Beta function [60]. Using the fact that $g(x) = I_x(t, n - t)$ [60] Sec. 8.17], we can verify that, for $x \in (0, 1)$,

$$g'(x) = \frac{d}{dx} I_x(t, n - t) = \frac{x^{t-1} (1-x)^{n-t-1}}{B(t, n - t)} > 0.$$  

Therefore, the functions $f, g$ define an admissible system because: map $[0, 1]$ to $[0, 1]$, are polynomial in $x, \varepsilon$, are strictly increasing in $x$ for fixed $\varepsilon \neq 0$, and are non-decreasing in $\varepsilon$. It is a proper admissible system because $h(0, 1)(x; \varepsilon) = g(x) > 0$ for $x \in (0, 1]$. Since $h(x; \varepsilon) = \varepsilon g(x)$, we also see that $\varepsilon(x) = x/g(x)$. For consistency with Definition [47], we use $\varepsilon(x)$ to denote the restriction of $\varepsilon(x)$ to the domain $\tilde{\varepsilon} = \{x \in (0, 1] | \varepsilon(x) \in [0, 1]\}$.

Using [4], we find that

$$U_\varepsilon(x; \varepsilon) = x g(x) - \int_0^x g(z) \, dz - \varepsilon \frac{1}{2} g(x)^2.$$  

Likewise, the fixed-point potential is given by

$$Q(x) = x g(x) - \int_0^x g(z) \, dz - \frac{1}{2} \frac{g(x)^2}{g(x)} = \frac{1}{2} x g(x) - \int_0^x g(z) \, dz,$$

where the last expression is stated without derivation and can be verified using well-known properties of $Q(x) = x$ [60] Sec. 8.17]. Since $I_0(a, b) = 0$ and $I_1(a, b) = 1$, we find that $Q(0) = 0$ and $Q(1) = -(1 - 2t/n)/2$. For the BEC case, the EBP EXIT function is $h_{EBP}(x) = x^2$ and the “trial entropy” is given by

$$P(x) = \int_0^x h_{EBP}(g(z)) \varepsilon'(z) \, dz = \int_0^x g(z)^2 \varepsilon'(z) \, dz = -rg(x) + 2I_x(t, n - t) = -2Q(x).$$

Therefore, we have $P(0) = 0$ and $P(1) = 1 - 2t/n$.

It also turns out that $P(x)$ has a unique root $x^*$ in $(0, 1]$ (e.g., see Lemma [55]). For LDPC codes, this typically means that $\varepsilon(x^*)$ is a (tight) upper bound on the MAP threshold. But, the bounded-distance decoder associated with $g(x)$ is suboptimal for BCH codes. Therefore, we find that $P(1) > r_{BEC}$ and, hence, $\varepsilon(x^*)$ is not an upper bound on the MAP threshold of the system. Instead, we will find that $\varepsilon^*_R = \varepsilon(x^*)$ is the threshold of the SC system defined in [22].

Lemma 54. For a fixed $t \geq 2$, $P^R(x) < 0$ for all $x \in (0, 1 - \frac{1}{n-2})$, and $P^R(x) \geq 0$ for all $x \in (1 - \frac{1}{n-2}, 1)$.

Proof: See Appendix [10].

Lemma 55. For any $2 \leq \ell \leq \lfloor \frac{n-1}{\varepsilon} \rfloor$, $P(x)$ has an unique root $x^*$ in $(0, 1]$ and $\varepsilon^*_R = \varepsilon(x^*) < 1$.

Proof: First, we observe that $P(0) = 0$. Next, we see that $P(x) < 0$ for all $x \in (0, \frac{1}{n-2})$ because Lemma 54 shows
that $P'(x) < 0$ in this range. Since $P(1) > 0$ for $t$ in the stated range, one can see that $P(x)$ has at least one root in $[\frac{1}{n+1}, 1]$. Finally, we see that $P(x)$ is convex on $[\frac{1}{n+1}, 1]$ because Lemma 43 shows that $P''(x) \geq 0$ for all $x \in [\frac{1}{n+1}, 1]$. Therefore, $P(x)$ has exactly one root in $[\frac{1}{n+1}, 1]$. Since $Q(1) = -(1 - 2t/n)/2 < 0$ for $t \leq \frac{n-1}{n+1}$, we can apply Lemma 46(ii) to see that $\varepsilon^* < \varepsilon(1) = 1/g(1) = 1$. Also, $h'(0; 1) = 0$ implies that the system is unconditionally stable and, hence, Lemma 43 shows that $X_f$ is closed. Therefore, we can use Lemma 46(iii) to see that

$$\varepsilon^*_c = \min\{\varepsilon(x) : x \in X_f, P(x) = 0\} = \varepsilon(x^*)$$

because $P(x) = -2Q(x)$ and $x^* \in X_f$.

Corollary 56. If $\varepsilon < \varepsilon^*_c$, then there is a $w_0 < \infty$ such that the DE recursion for the SC GLDPC in ensemble defined in [22] converges to the zero vector for all $w > w_0$.

Proof: Since $\varepsilon < \varepsilon^*_c < \varepsilon^*_\text{stab} = 1$, we can apply Lemma 56(v) to see that there is a $w_0 < \infty$ such that, for all $w < w_0$, the SC DE recursion will converge to the zero vector.

D. Application to ISI Channels with Erasure Noise

In [35], a family of intersymbol-interference (ISI) channels with erasure noise is investigated as an analytically tractable model of joint iterative decoding of LDPC codes and channels with memory. Let $\phi(x; \varepsilon)$ be the function that maps the a priori erasure rate $x$ from the code and the channel erasure rate $\varepsilon$ to the erasure rate of extrinsic messages from the channel detector to the bit nodes [10]. Then, the resulting DE update equation for the erasure rate, $x^{(t)}$, of bit-to-check messages can be written in the form of (3), where $f(x; \varepsilon) = \phi(L(x); \varepsilon)\lambda(x)$ and $g(x; \varepsilon) = 1 - \rho(1 - x)$ [35]. If $\phi(x; \varepsilon)$ is continuously differentiable on $X \times \mathcal{E}$, then these equations define an admissible system with $\varepsilon_{\text{max}} = x_{\text{max}} = y_{\text{max}} = 1$ and

$$U_n(x; \varepsilon) = x(1 - \rho(1 - x)) - \left(1 - \frac{R(1 - x)}{R'(1)}\right)$$

where $\Phi(x; \varepsilon) = \int_0^x \phi(x; \varepsilon)dz$. If the DE equations define a proper admissible system, then we can also compute the fixed-point potential $Q(x)$ using Lemma 43. The main benefit that the calculation is the same (up to a scale factor) as the calculation of the “trial entropy” $P(x)$ in [11], [61].

Assuming that $h(x; \varepsilon)$ is defined and strictly increasing in $\varepsilon$ for $\varepsilon \in [0, \infty]$, we use $\varepsilon(x)$ to denote the unique $\varepsilon \in [0, \infty]$ associated with a fixed point $x \in [0, 1]$. For consistency with Definition 44 and Definition 43, we use $\varepsilon(x)$ to denote the restriction of $\varepsilon(x)$ to the domain $X_f = \{x \in [0, 1] : \varepsilon(x) \in [0, 1]\}$. Adjusting notation from [11], [61], we find that $P(x) = -L'(1)Q(x)$ for $x \in X_f$, where $Q(x) = U_n(x; \varepsilon(x))$. Using this, the natural generalization of the Maxwell threshold is given by

$$\varepsilon^* \equiv \varepsilon(x^*)$$

where $\epsilon(0) \equiv \varepsilon^*_\text{stab}$. If $P(x)$ has a unique root $x^* \in (0, 1]$ and the threshold is not determined by stability, then the EXIT area theorem shows that $\varepsilon^* \text{MAP} \leq \varepsilon^* \text{Max} = \varepsilon(x^*)$ because $P(1) = 1 - L'(1)/R'(1)$ equals the code rate [11], [64]. For the dicode erasure channel (DEC), the counting argument in [26] has also been extended to prove the tightness of this upper bound [11], [61].

The analysis of irregular LDPC codes on ISI channels with erasure noise was extended to SC ensembles in [10], [11], [12], [61] and the resulting DE equations depend on bit ordering during transmission. If the bits in each spatial group are clustered together in the channel input (rather than interleaved across the entire input), then the SC DE equations are given by (1) and we can apply our analysis to the SC system. If the channel is the DEC, precoded DEC, or a class-2 partial-response channel with erasure noise (PR2EC), then there are closed-form expressions for $\phi(x; \varepsilon)$ and it is easy to verify that the DE equations define proper admissible systems with strict stability thresholds [35], [41], [11], [12].

Lemma 57. Consider the ensemble LDPC($\lambda, \rho$) on a channel defined by $\phi(x; \varepsilon)$ and assume the code rate $r = 1 - L'(1)/R'(1)$ satisfies $r > 0$. If $\phi(x; \varepsilon)$ is continuously differentiable on $X \times \mathcal{E}$, then $\varepsilon^*_\text{max}$ given by (19) equals the Maxwell threshold $\varepsilon^*_\text{Max}$.

Proof: Under the given conditions, it is easy to verify that the DE equations define a proper admissible system. From $\phi(1; 1) = 1$, we see that $h(1; 1) = 1$ and $\varepsilon(1) = 1$. Using $Q(1) = -P(1)/L'(1) = -r/L'(1) < 0$, we can apply Lemma 46(ii) to see that $\varepsilon^*_c < \varepsilon(1) = 1$. Since $\phi'(0; \varepsilon)$ is strictly increasing in $\varepsilon$, the expansion $h(x; \varepsilon) = \phi'(0, \varepsilon)\lambda(0)\rho'(0)x + o(x)$ shows that the system has a strict stability threshold. Therefore, we can apply Lemma 46(iii) to see that

$$\varepsilon^*_c = \min\{\varepsilon(x) : x \in X_f, P(x) = 0, x \in [0, 1]\} = \varepsilon(x^*)$$

because $P(x) = -L'(1)Q(x)$ for $x \in X_f$ and $\varepsilon(x) > 1$ for $x \in (0, 1]\setminus X_f$.

Corollary 58. If the conditions of Lemma 57 hold and $\varepsilon < \varepsilon^*_\text{Max}$, then there is a $w_0 < \infty$ such that the SC DE recursion converges to the zero vector for all $w > w_0$.

Proof: Since $\varepsilon^*_\text{Max} = \varepsilon^*_c \leq \varepsilon^*_\text{stab}$ by Lemma 57 and $\varepsilon < \varepsilon^*_\text{Max}$, we can apply Lemma 56(v) to see that there is a $w_0 < \infty$ such that, for all $w < w_0$, the SC DE recursion will converge to the zero vector.

E. Application to Compressed Sensing

Spatial coupling can also be used to improve the performance of compressed sensing [23], [24], [25], [62]. Under some conditions, one can analyze belief-propagation reconstruction for compressed sensing using a Gaussian approximation [63], [64], [65], [23], [24], [62]. This leads to a simple scalar recursion for the mean-square error (MSE) in the estimate of each signal component. In particular, we
consider the reconstruction of a length-$n$ signal vector, whose entries are i.i.d. copies of a random variable $X$, from $\delta n$ linear measurements in the limit as $n \to \infty$. We assume that $\mathbb{E}[X^2] < \infty$ and that each measurement is corrupted by independent Gaussian noise with variance $\sigma^2$.

Some results from [60] are required to understand the next few equations. Let $Y = \sqrt{\text{snr}} X + Z$ be a scaled observation of $X$ with standard Gaussian noise $Z$. If $\mathbb{E}[X^2] = 1$, then the variable $\text{snr}$ corresponds to the standard notion of signal-to-noise ratio for $Y$. For general $X$, $\text{snr}$ can be seen simply as a scaling factor. Regardless, the scalar estimate, $\hat{X} = \mathbb{E}[X|Y]$, minimizes the MSE, $\mathbb{E}[(X - \hat{X})^2]$, and the resulting MSE is denoted $\text{mmse}(\text{snr})$. This function is bounded because $0 \leq \text{mmse}(\text{snr}) \leq \text{mmse}(0) \leq \mathbb{E}[X^2] < \infty$. It also satisfies the differential relationship, with the mutual information, given by

$$\frac{1}{2} \text{mmse}(\text{snr}) = \frac{d}{d\text{snr}} \mathcal{I} (X; \sqrt{\text{snr}} X + Z). \tag{26}$$

Using this, the recursion given by [25, (33)] for the MSE of SC compressed sensing is essentially equal to

$$x_i^{(\ell+1)} = \sum_{j=1}^{N} A_{j,i} \text{mmse} \left( \sum_{k} A_{j,k} \frac{1}{\sigma^2 + \frac{1}{\delta} x_k^{(\ell)}} \right). \tag{27}$$

Now, we will construct an uncoupled recursion that results in the same coupled recursion. Since the $\text{mmse}$ function is non-increasing and $(\sigma^2 + \frac{1}{\delta} x)^{-1} \in (0, 1/\sigma^2]$ for $x \in [0, \infty)$, we introduce a slight twist and define

$$f(y) = \text{mmse} \left( \frac{1}{\sigma^2} y \right).$$

This implies that $f(y)$ is non-decreasing in $y$ and satisfies $f(y) \in [0, \text{mmse}(0)]$ for $y \in [0, 1/\sigma^2]$. Likewise, to undo this small twist, we define

$$g(x) = \frac{1}{\sigma^2} - \frac{1}{\sigma^2 + \frac{1}{\delta} x},$$

where $g(x) \in (0, 1/\sigma^2]$ for $x \in [0, \infty)$. The given bounds on $f$ and $g$ also allow us to choose $x_{\text{max}} = \text{mmse}(0)$ and $y_{\text{max}} = g(x_{\text{max}}) < 1/\sigma^2$. From this, we see that the uncoupled recursion is given by

$$f(g(x)) = \text{mmse} \left( \frac{1}{\sigma^2 + \frac{1}{\delta} x} \right). \tag{28}$$

For Theorems 1 and 2, we must also verify that $f$ and $g$ satisfy some technical conditions. First, it is easy to verify that $g(x)$ is strictly increasing in $x$ and $C^2$ on $X$. One can also show that $f(y)$ is $C^1$ on $Y$ (e.g., see [67]) and, hence, the recursion satisfies the necessary conditions of the theorems for $\sigma^2 \in (0, \infty)$. Since $\sum_{k} A_{j,k} = 1$, one can also verify that these definitions make the coupled recursion in (1) identical to (27). Lastly, we note that the boundary conditions assumed by (27) are correct if we have perfect estimates from signal nodes beyond the boundaries. This is easily achieved by choosing the signal values to be zero outside the range of the SC system.

Now, we can use (4) to construct the potential function. For $f$, we note that

$$F(y) = 2I \left( X; \sqrt{1/\sigma^2} X + Z \right) - 2I \left( X; \sqrt{1/\sigma^2} - y X + Z \right)$$

satisfies $F(0) = 0$ and it is easy to verify that $F'(y) = f(y)$ follows from (26). For $g$, we observe that

$$G(x) = \frac{1}{\sigma^2} x - \delta \ln \left( 1 + \frac{x}{\delta \sigma^2} \right)$$

satisfies $G(0) = 0$ and it is easy to check that $G'(x) = g(x)$. Combining these, we see that

$$U_n(x) = -\frac{x}{\sigma^2 + \frac{1}{\delta} x} + \delta \ln \left( 1 + \frac{x}{\delta \sigma^2} \right) - 2I \left( X; \sqrt{1/\sigma^2} X + Z \right) + 2I \left( X; \sqrt{1/\sigma^2 + x/\delta} X + Z \right). \tag{29}$$

Corollary 59. Consider the MSE achieved by SC compressed sensing with BP reconstruction (e.g., see [24], [25], [62]). For any $\delta > 0$, there is a $w_0 < \infty$ such that this MSE is upper bounded by $\mathcal{F}^* + \delta$ for all $w > w_0$, where

$$\mathcal{F}^* = \max \left( \arg \min_{x \in X} U_n(x) \right)$$

and $U_n$ is given by (29).

Proof: This follows directly from the above definitions and Theorem 1.

Remark 60. After we developed this example, we discovered that Kudekar et al. also added a very similar example to the recent update of their paper [29].

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we study a class of coupled scalar recursions and provide a tight characterization of their behavior based on an underlying uncoupled recursion. This result enables one to easily compute the asymptotic decoding thresholds for a variety of iterative decoding systems. We also demonstrate a precise connection between the threshold, $\mathcal{C}_*, \epsilon_{\text{Max}}$, below which the coupled system converges to the zero vector and the Maxwell threshold, $\epsilon^*$, which is the conjectured MAP threshold for LDPC ensembles.

The proof techniques used in this paper can also be extended, with some complications, to coupled recursions on vectors (e.g., see [30]) and log-likelihood-ratio densities (e.g., see [31], [32]). Since the cited works only consider threshold saturation, we are currently working to prove Maxwell saturation for these more general systems.

APPENDIX A

PROOFS FROM SECTION II

A. Proof of Proposition 1

For (i), we note that a bounded subset of real numbers has a limit point iff it contains infinitely many points. Therefore, $\mathcal{F} \cap [\mathcal{F}^*, x_{\text{max}}]$ cannot have a limit point if $\mathcal{F}$ is a finite set. From Theorem 1 we see that $w_0 < \infty$ whenever $\mathcal{F} \cap [\mathcal{F}^*, x_{\text{max}}]$ does not have a limit point. For (ii), recall that the composition of real analytic functions is real analytic.
and that a real analytic function is identically zero if its set of zeros has a limit point \[68\]. Since \(\cal F\) is the zero set of the real analytic function \(x - f(g(x))\) and \(x - f(g(x)) \neq 0\) for some \(x \in \cal X\), it follows that \(\cal F\) cannot have a limit point and must be finite. For (iii), the condition implies that \(\cal F \cap [\tau^*, \tau^* + \gamma] = \{\tau^*\}\) and therefore \(\tau^*\) is an isolated point in \(\cal F \cap [\tau^*, x_{\text{max}}]\). Hence, \(\tau^*\) is not a limit point of \(\cal F\). For (iv), the Taylor expansion of \(f(g(x))\) about \(x = \tau^*\) shows that there exists a \(\gamma > 0\) such that \(f(g(x)) < x\) for all \(x \in [\tau^*, \tau^* + \gamma]\) and, thus, we can apply (iii).

**B. Proof of Proposition \[77\]**

For (i), we note that \(g(x)\) is invertible and write

\[
y \in g(\cal F) \Leftrightarrow g^{-1}(y) \in \cal F \Leftrightarrow f(g^{-1}(y))) = g^{-1}(y) \Leftrightarrow f(y) = g^{-1}(y) \Leftrightarrow g(f(y)) = y \Leftrightarrow y \in \cal F'.
\]

A very similar argument shows \(x \in f(\cal F') \Leftrightarrow x \in \cal F\) because \(f(y)\) is invertible. For (ii), we use \(x = f(g(x))\) to simplify \(V_s(g(x))\) and this gives

\[
V_s(g(x)) = g(x)f(g(x)) - F(g(x)) - G(f(g(x))) = xg(x) - G(x) - F(g(x)).
\]

Similarly, \(y = g(f(y))\) implies that \(U_s(f(y))\) simplifies to \(V_s(g(x))\). For (iii), we note that all minimizers of \(U_s(x)\) must satisfy \(x = f(g(x))\) (see Lemma \[18\] and, hence, (ii) implies \(U_s(x) = V_s(g(x))\). Likewise, all minimizers of \(V_s(x)\) must satisfy \(x = g(f(x))\) and \(V_s(x) = U_s(f(x))\). Therefore, one gets a contradiction if \(U_s(x)\) and \(V_s(x)\) have different minimum values. Next, we let \(m = \min_{x \in \cal X} U_s(x) = \min_{g \in \cal Y} V_s(y)\) be that minimum value and write

\[
y \in g(\cal M) \Leftrightarrow g^{-1}(y) \in \cal M, U_s(g^{-1}(y)) = m \Leftrightarrow f(g^{-1}(y))) = g^{-1}(y), V_s(y) = m \Leftrightarrow f(y) = g^{-1}(y), V_s(y) = m \Leftrightarrow g(f(y)) = y, V_s(y) = m \Leftrightarrow y \in \cal M'.
\]

A very similar argument shows \(x \in f(\cal M') \Leftrightarrow x \in \cal M\).

**C. Proof of Lemma \[76\]**

First, we verify that the translated scalar system is well defined. In particular, we define \(\cal Y = [0, y_{\text{max}} - g(\tilde{x})]\) and observe that \(\tilde{f} : \cal Y \to \cal X\) because \(\tilde{f}(0) = f(0 + g(\tilde{x})) = f(g(\tilde{x})) - \tilde{x} = 0\) and \(\tilde{f}(y_{\text{max}} - g(\tilde{x})) = f(y_{\text{max}} - g(\tilde{x})) - \tilde{x} = f(y_{\text{max}}) - \tilde{x} = x_{\text{max}} - \tilde{x}\). Likewise, we observe that \(\tilde{g} : \cal X \to \cal Y\) because \(\tilde{g}(0) = g(0 + \tilde{x}) - g(\tilde{x}) = 0\) and \(\tilde{g}(x_{\text{max}} - \tilde{x}) = g(x_{\text{max}} - \tilde{x} + \tilde{x}) - g(\tilde{x}) = g(x_{\text{max}}) - g(\tilde{x}) \leq y_{\text{max}} - g(\tilde{x})\). Moreover, it is easy to verify that \(f\) and \(g\) inherit their monotonicity and differentiability directly from \(f\) and \(g\).

For translated coupled system, we derive

\[
\hat{h}(x) \triangleq A^T \hat{f}(A\hat{g}(x)) = A^T (f(A(g(x)+\tilde{x}1_M)-g(x)1_M)+g(\tilde{x})1_N) - \tilde{x}1_N.
\]

For translated coupled system, we derive

\[
\hat{h}(x) \triangleq A^T \hat{f}(A\hat{g}(x)) = A^T (f(A(g(x)+\tilde{x}1_M)-g(x)1_M)+g(\tilde{x})1_N) - \tilde{x}1_N.
\]

**Appendix B**

**Proofs from Section \[III\]**

**A. Proof of Lemma \[78\]**

First, we note that \(F(x)\) is convex because \(f(x)\) is non-decreasing and \(G(x)\) is strictly convex because \(g(x)\) is strictly increasing. For any \(x_0 \in \cal X\), this implies that \(-F(g(x_0)) \leq -F(g(x_0))-f(g(x_0))(x-x_0)\) and \(-G(x) \leq -G(x_0)-g(x_0)(x-x_0)\) with equality iff \(x = x_0\). Using these to upper bound \(U_s(x)\) gives

\[
U_s(x) = xg(x) - G(x) - F(g(x)) \leq xg(x) - G(x_0) - g(x_0)(x-x_0) - F(g(x_0))-f(g(x_0))(x-x_0)\]

\[
= U_s(x_0) + (x-x_0)(g(x_0)-g(x_0)),
\]

with equality iff \(x = x_0\). Now, choosing \(x = f(g(x_0))\) shows that \(U_s(f(g(x_0))) \leq U_s(x_0)\) with equality iff \(f(g(x_0)) = x_0\). Hence, one step of the recursion from \(x_0\) must strictly decrease the potential if \(x_0\) is not a fixed point.

Next, we prove that, if \(x_0\) is a local minimum of \(U_s(x)\) on \(\cal X\), then \(x_0\) is a fixed point. We do this by showing the contrapositive: if \(x_0\) is not a fixed point, then \(x_0\) is not a local minimum of \(U_s(x)\) on \(\cal X\). If \(x_0\) is not a fixed point, then \(x_0 - f(g(x_0)) \neq 0\) and there are two cases. If \(x_0 - f(g(x_0)) < 0\), then \(x_0 < f(g(x_0)) \leq x_{\text{max}}\) and the continuity of \(x - f(g(x))\) implies that there is a \(\delta > 0\) and \(\eta > 0\) such that \(x - f(g(x)) < -\eta\) for all \(x \in [x_0, x_0 + \delta]\). This implies that, for any \(x \in (x_0, x_0 + \delta]\), we have

\[
U_s(x) - U_s(x_0) = \int_{x_0}^{x} (x - f(g(x)))g'(x)dx < \eta \int_{x_0}^{x} g'(x)dx < -\eta (g(x) - g(x_0)) < 0,
\]

Therefore, \(x_0\) is not a local minimum of \(U_s(x)\) on \(\cal X\).
where the last step holds because \( q \) is strictly increasing. This shows that \( x_0 \) is not a local minimum. If \( x_0 - f(g(x_0)) > 0 \), then a very similar argument shows that, for some \( \delta > 0 \) and any \( x \in [x_0 - \delta, x_0) \), we have \( U_q(x) < U_q(x_0) \). Thus, \( x_0 \) is not a local minimum and the proof is complete.

**B. Proof of Lemma 20**

For \( \ell = 0 \), property (i) follows from

\[
x^{(1)} = A^T f(Ag(x_{\text{max}})) \leq A^T f(A1_N \cdot y_{\text{max}}) = A^T f(1_N \cdot y_{\text{max}}) \leq A^T 1_N \cdot x_{\text{max}} \leq x_{\text{max}},
\]

where \( A1_M = 1_N \) and \( A^T 1_N \leq 1_M \). The inductive step for \( \ell > 1 \) follows from the fact that \( h(x) = A^T f(Ag(x)) \) is isotone.

Let \( T_M \) denote the matrix, defined by \([T_M x]_{i} = x_{M-i+1}\), that reverses the order of elements in a vector. This matrix can be represented by \([T_M]_{i,j} = \delta_{j,M-i+1} \) because

\[
\sum_{j=1}^{M} [T_M]_{i,j} x_j = \sum_{j=1}^{M} \delta_{j,M-i+1} x_j = x_{M-i+1}.
\]

Since \( j = M - i + 1 \) implies \( i = M - j + 1 \), it follows that \([T_M]_{i,j} = \delta_{i,M-j+1} \) and \( T_M^T = T_M \). Also, \( A \) is symmetric under simultaneous row-column reversal (i.e., \( A_{j,k} = A_{N-j+1,N-k+1} \)) and this implies that \( AT_M = T_N A \).

Since \( x^{(0)} = T_M x^{(0)} \) by definition and property (ii) is equivalent to \( x^{(l)} = T_M x^{(l)} \), this property follows by induction using

\[
x^{(l+1)} = A^T f(Ag(T_M x^{(l)})) = A^T f(A^T f(T_N Ag(x^{(l)})))
\]

\[
= A^T f(T_N Ag(x^{(l)})) = A^T T_N f(Ag(x^{(l)}))
\]

\[
= T_M A^T f(Ag(x^{(l)})) = T_M x^{(l+1)}.
\]

For property (iii), we note that each mapping \( f, g, A, A^T \) is closed on the set of symmetric unimodal vectors. For \( f, g \), this holds because \( f, g \) are monotone and operate on each element of the vector. For \( A, A^T \), symmetry follows from \( AT_M = T_N A \) above. For unimodality, we have

\[
[Ax]_{i+1} - [Ax]_i = \frac{1}{w} \sum_{j=0}^{w-1} x_{i+1+j} - \frac{1}{w} \sum_{j=0}^{w-1} x_{i+j}
\]

\[
= \frac{1}{w} (x_{i+w} - x_i) \geq 0
\]

if \( i \leq M - (i + w) + 1 \) (due to symmetry) and this is equivalent to \( i \leq N - i \). Therefore, if the length-\( M \) vector \( Ax \) is symmetric and unimodal, then the length-\( N \) vector \( A^T x \) is symmetric and unimodal. The proof for \( A^T \) is very similar and, hence, omitted.

**C. Proof of Lemma 21**

First, we note that \( x \leq z \) implies \( q(x) \leq q(z) \) and \( q(h(x)) \leq q(h(z)) \). One can verify this by observing that, for each \( j \in [1 : M] \), \([q(x)]_j \) is a non-decreasing function of each \( x_j \). Now, we consider property (i). Since \( \hat{x}^{(1)} = q(x^{(1)}) \) and \( x^{(1)} \preceq x^{(0)} \), we have

\[
\hat{x}^{(1)} = q(h(x^{(0)})) = q(x^{(1)}) \leq q(x^{(0)}) = \hat{x}^{(0)} = \hat{x}^{(0)}.
\]

Assuming \( \hat{x}^{(l)} \preceq \hat{x}^{(l-1)} \), we proceed by induction and observe that

\[
\hat{x}^{(l+1)} = q(h(\hat{x}^{(l)})) \preceq q(h(\hat{x}^{(l-1)})) = \hat{x}^{(l)}.
\]

Next, we consider property (ii). Since \( \hat{x}^{(0)} = x^{(0)} \), induction on \( x^{(l)} \geq x^{(l)} \) shows that

\[
\hat{x}^{(l+1)} = q(h(\hat{x}^{(l)})) \geq q(h(x^{(l)})) = x^{(l+1)}.
\]

To show property (iii), we note that \( \hat{x}^{(0)} \) is non-decreasing and proceed by induction. Informally, the vector update is symmetric about \( y_0 \) but the argument vector is asymmetric and flat after \( y_0 \). Due to monotonicity, it is not too hard to see that \( z = A^T f(Ag(x)) \) will increase to a maximum and then decrease due to the 0-boundary. The key observation is that the maximum will occur at \( i \geq i_0 \) so that \( q(z) \) is non-decreasing.

Mathematically, we observe that, for a non-decreasing \( z \) (i.e., \( [x]_{i+1} \geq [x]_i \) for \( i \in [1 : M - 1] \)), it follows that \( q(x) \), and \( Ag(x) \), \( y = f(Ag(x)) \) are non-decreasing. For \( z = A^T y \), the sublty is treating the 0-boundary properly and we find

\[
[A^T y]_i - [A^T y]_{i-1} = \frac{1}{w} \sum_{j=0}^{w-1} y_{i-j} - \frac{1}{w} \sum_{j=0}^{w-1} y_{i-1-j} = \frac{1}{w} (y_i - y_{i-w}).
\]

Therefore, we need to show that \( y_i \geq y_{i-w} \) for \( i \in [1 : i_0] \). Since \( y \) is a length-\( N \) non-decreasing vector, this holds unless \( y_i = 0 \) due to the boundary because \( i \notin [1 : N] \). The key observation is that \( y_i = 0 \) if \( i > N \) and this can cause a problem if \( N < i \leq i_0 \). Since \( i_0 = [(N + w - 1)/2] \), this requires \( w \geq 2i - N + 1 \). But, \( i - w \leq N - i - 1 \leq 0 \) if \( i \leq N + 1 \) and, hence, there is no problem because \( y_i = 0 \) due to the boundary only if we also have \( y_{i-w} = 0 \) due to the boundary.

**D. Proof of Lemma 22**

Let \( u_{m,i}(x) = [U_m^c(x)]_{m,i} \), and recall that \( U_m^c(x) = g'(x) (x - A^T f(Ag(x))) \). This implies that

\[
u_{m,i}(x) = \frac{d}{dx_m} [g'(x) (x - A^T f(Ag(x)))]_i
\]

\[
= \frac{d}{dx_m} \left[ \left( x_i - \sum_{j=1}^{N} A_{j,i} f \left( \sum_{k=1}^{M} A_{j,k} g(x_k) \right) \right) g'(x_i) \right]
\]

\[
= \delta_{i,m} g'(x_i) \left( x_i - \sum_{j=1}^{N} A_{j,i} f \left( \sum_{k=1}^{M} A_{j,k} g(x_k) \right) \right) + g'(x_i) \left( \delta_{i,m} - \sum_{j=1}^{N} A_{j,i} f' \left( \sum_{k=1}^{M} A_{j,k} g(x_k) \right) A_{j,m} g'(x_m) \right).
\]

Hence, we have the upper bound

\[
|u_{m,i}(x)| \leq \delta_{i,m} \|g''\|_{\infty} x_{\text{max}} + \|g''\|_{\infty} \left( \delta_{i,m} + \|f'\|_{\infty} \|g''\|_{\infty} \sum_{j=1}^{N} A_{j,i} A_{j,m} \right).
\]
Since \(\|U''_c(x)\|_1\) equals the maximum absolute column sum,
\[
\|U''_c(x)\|_1 = \max_{i \in [1:M]} \sum_{m=1}^{M} |u_{m,i}(x)|
\]
\[
\leq \max_{i \in [1:M]} \left( \sum_{m=1}^{M} \delta_{i,m} \|g''\|_{\infty} \right)
\]
\[
+ \sum_{m=1}^{M} \left( \|g''\|_{\infty} \delta_{i,m} + \|f''\|_{\infty} \|g''\|_{\infty} \delta_{i,m} \sum_{j=1}^{N} A_{j,i} A_{j,m} \right)
\]
\[
\leq 2 \|g''\|_{\infty} \delta + \|\varepsilon\|_{\infty} + \|f''\|_{\infty} \|g''\|_{\infty} \delta \sum_{j=1}^{N} A_{j,i} A_{j,m}.
\]
Since the Hessian \(U''_c(x)\) is symmetric, it follows that
\[
\|U''_c(x)\|_{\infty} = \|U''_c(x)\|_1.\]
Hence, the bound on \(\|U''_c(x)\|_2\) follows from the standard inequality
\[
\|U''_c(x)\|_2 \leq \sqrt{\|U''_c(x)\|_1 \|U''_c(x)\|_{\infty}}.
\]

**APPENDIX C**

**PROOFS FROM SECTION VII**

**A. Proof of Lemma 39**

Since \(\frac{d}{dx} U_s(x; \varepsilon)\) exists and is continuous on \(X \times \mathcal{E}\), there exists a \(\beta < \infty\) such that \(|U_s(x; t_0) - U_s(x; t_1)| \leq \beta |t_0 - t_1|\) (i.e., \(U_s\) is uniformly Lipschitz continuous in \(\varepsilon\)). For arbitrary \(t_0, t_1 \in \mathcal{E}\), we can assume without loss of generality that \(\Psi(t_0) \geq \Psi(t_1)\). Therefore, the Lipschitz continuity of \(\Psi(\varepsilon)\) follows from
\[
|\Psi(t_0) - \Psi(t_1)| = \left| \min_{x \in X} U_s(x; t_0) - \min_{x \in X} U_s(x; t_1) \right|
\]
\[
= U_s(x_0^{*}; t_0) - U_s(x_1^{*}; t_1)
\]
\[
\leq \beta |t_0 - t_1|,
\]
where the respective minima are achieved (at points \(x_0^{*}, x_1^{*}\)) because \(U_s(x; \varepsilon)\) is continuous in \(x\) and \(X\) is compact. Next, we show that \(F^{(0,1)}(x; \varepsilon)\) is non-negative and non-decreasing in \(x\). This holds because \(f^{(0,1)}(x; \varepsilon)\) is continuous and non-negative on \(X \times \mathcal{E}\) and therefore, we can write
\[
F^{(0,1)}(x; \varepsilon) \triangleq \frac{d}{dx} \int_{0}^{x} \int_{0}^{t} f^{(0,1)}(s; t) dt ds
\]
\[
= \int_{0}^{x} f^{(0,1)}(s; \varepsilon) ds \geq 0.
\]
The argument for \(G^{(0,1)}(x; \varepsilon)\) is identical.

**B. Proof of Lemma 42**

The definition of \(X_f\) shows that there exists an \(\varepsilon \in \mathcal{E}\) such that \(h(x; \varepsilon) - x = 0\) for each \(x \in X_f\). This \(\varepsilon\) is unique and denoted by \(\hat{\varepsilon}(x)\) because \(h(x; \varepsilon)\) is strictly increasing in \(\varepsilon\) for \(x \in X_f \subset X_0\) (e.g., \(h^{(0,1)}(x; \varepsilon) > 0\)). Now, consider any \(x_0 \in X_f\) satisfying \(\hat{\varepsilon}(x_0) \in (0, \varepsilon_{\max})\). Since \(h(x; \varepsilon)\) is strictly increasing in \(\varepsilon\) for \(x \in X_f\), it follows that \(h(x_0; 0) - x_0 < h(x_0; \hat{\varepsilon}(x_0)) - x_0 = 0\) and, likewise, \(h(x_0; \varepsilon_{\max}) - x_0 > 0\).

Using the continuity of \(h\), we see that there must be a \(\delta > 0\) such that \(h(x; 0) \leq x \leq h(x; \varepsilon_{\max})\) for all \(x \in I = [x_0 - \delta, x_0 + \delta]\). Therefore, the definition of \(X_f\) shows that \(I \subseteq X_f\) and, hence, \(x_0\) lies in the interior of \(X_f\).

For \(x \in X_f\), the positive derivative condition also allows us to apply the implicit function theorem for continuously differentiable functions to \(h(x; \varepsilon) - x = 0\). The result is that \(\hat{\varepsilon}(x)\) is continuously differentiable on \(X_f\) with derivative
\[
\hat{\varepsilon}'(x) = \frac{1 - h^{(1,0)}(x; \hat{\varepsilon}(x))}{h^{(0,1)}(x; \hat{\varepsilon}(x))}.
\]

**C. Proof of Lemma 43**

Under the stated assumptions, Lemma 42 shows that \(\hat{\varepsilon}(x)\) is a continuously differentiable function satisfying the fixed-point equation \(h(x; \hat{\varepsilon}(x)) = x\) for all \([x_1, x_2] \in X_f\). Since \(U_s(x; \hat{\varepsilon}(x))\) is differentiable, we can write
\[
\begin{align*}
Q(x_2) - Q(x_1) &= U_s(x_2; \hat{\varepsilon}(x_2)) - U_s(x_1; \hat{\varepsilon}(x_1)) \\
&= \int_{x_1}^{x_2} \left( \frac{d}{dx} U_s(x; \hat{\varepsilon}(x)) \right) dx \\
&= \int_{x_1}^{x_2} \left( U_s^{(1,0)}(x; \hat{\varepsilon}(x)) + U_s^{(0,1)}(x; \hat{\varepsilon}(x)) \hat{\varepsilon}'(x) \right) dx \\
&\quad \text{subject to:} \int_{x_1}^{x_2} \left( G^{(0,1)}(x; \hat{\varepsilon}(x)) + F^{(0,1)}(x; \hat{\varepsilon}(x)) \right) \hat{\varepsilon}'(x) dx, \\
&\text{where (a) follows from } U_s^{(1,0)}(x; \hat{\varepsilon}(x)) = 0 \text{ because } h(x; \hat{\varepsilon}(x)) = x \\
&\text{and (b) follows from using (22) to expand } U_s^{(0,1)}(x; \hat{\varepsilon}(x)) \text{ under the condition } h(x; \hat{\varepsilon}(x)) = x.
\end{align*}
\]

**D. Proof of Lemma 45**

To see (i), we observe that \(A = \{x, \varepsilon \in X \times \mathcal{E} \mid h(x; \varepsilon) = x\}\) is compact because \(X \times \mathcal{E}\) is compact and \(h(x; \varepsilon) - x\) is jointly continuous. Since the projection \(\pi: X \times \mathcal{E} \to \mathcal{E}\) defined by \((x, \varepsilon) \mapsto x\) is continuous, we also find that \(\pi(A) = \{x \in X \mid \exists \varepsilon \in \mathcal{E}, h(x; \varepsilon) = x\}\) is compact. Therefore, the set
\[
X_f = \{x \in X \mid \exists \varepsilon \in \mathcal{E}, h(x; \varepsilon) = x\} \setminus \{0\}
\]
is either closed or missing a single limit point at \(x = 0\).

Consider (ii). Consider the sequences \(\xi_n = (1 - \frac{1}{n}) \varepsilon_{\text{stab}}\) and \(\tau_n = \varepsilon_{\text{stab}} + \frac{\varepsilon_{\max} - \varepsilon_{\text{stab}}}{n}\) for \(n \in \mathbb{N}\). For each \(n \in \mathbb{N}\), the condition \(0 \leq \xi_n < \varepsilon_{\text{stab}}\) (or \(\xi_n = \varepsilon_{\text{stab}} = 0\) if \(\varepsilon_{\text{stab}} = 0\)) implies that there is a \(\delta_n > 0\) such that \(h(x; \xi_n) < x\) for all \(x \in (0, \delta_n]\). Likewise, the condition \(\varepsilon_{\text{stab}} < \tau_n \leq \varepsilon_{\max}\) implies that, for any \(n > 0\), there is an \(x_n \in (0, \gamma_n)\) such that \(h(x_n; \tau_n) = x_n\). Starting from any \(x_1 \in \delta_1\), we can generate a decreasing sequence \(x_n \in (0, x_{n-1})\) by choosing \(x_n\) in the second step based on the parameter \(\gamma_n = \min\{x_{n-1}/2, \delta_n\}\) given by the first step. By construction, the \(x_n\) sequence satisfies \(h(x_n; \xi_n) < x_n \leq h(x_n; \tau_n)\) and, hence, the continuity of \(h\) shows that there is a sequence \(\varepsilon_n \in [\xi_n, \tau_n]\) such that \(h(x_n; \varepsilon_n) = x_n\). Finally, we observe that \(\varepsilon_n \to \varepsilon_{\text{stab}}\) and \(x_n \to 0\).
fixed point and $X_f \cap [0, \delta] = \emptyset$. Along with (i), we see that $X_f$ is closed.

Consider (iv). If a system has a strict stability threshold $\varepsilon_{\text{stab}}^* \in [0, \varepsilon_{\text{max}}]$, then, for $\varepsilon \in (\varepsilon_{\text{stab}}^*, \varepsilon_{\text{max}}]$, there is a $\delta > 0$ such that $h(x; \varepsilon) > x$ for all $x \in (0, \delta]$. Since $g(q; \varepsilon)$ is strictly increasing in $x$, it follows that $g(\delta; \varepsilon) - g(0; \varepsilon) = \int_0^\delta g'(z; \varepsilon)dz > 0$. Using this and $x - h(x; \varepsilon) < 0$ for all $x \in (0, \delta]$, we find that

$$U_s(\delta; \varepsilon) - U_s(0; \varepsilon) = \int_0^\delta (x - h(x; \varepsilon))g'(x; \varepsilon)dz < 0.$$ 

This implies that $\varepsilon \geq \varepsilon_c^*$ because 0 is not an $x$-minimizer of $U_s(x; \varepsilon)$ and, hence, $\pi(x; \varepsilon) > 0$. As $\varepsilon \in (\varepsilon_{\text{stab}}^*, \varepsilon_{\text{max}}]$ was arbitrary, we conclude that $\varepsilon_{\text{stab}}^* \geq \varepsilon_c^*$. For the second part, consider any sequence $x_n \in X_f$ such that $x_n \to 0$. Such as sequence exists by (ii). If $\liminf_n \varepsilon(x_n) < \varepsilon_{\text{stab}}^*$, then there is a subsequence $x_{n_k} \in X_f$ and an $\eta > 0$ such that $x_{n_k} \to 0$ and $h(x_{n_k}; \varepsilon_{\text{stab}}^* - \eta) \geq x_{n_k}$. But, this contradicts the definition of the stability threshold. A similar argument shows that $\limsup_n \varepsilon(x_n) > \varepsilon_{\text{stab}}^*$ contradicts the strict stability threshold condition. Therefore, we find that $\lim_n \varepsilon(x_n) = \varepsilon_{\text{stab}}^*$.

E. Proof of Lemma 46

Consider (i). For $\varepsilon < \varepsilon_c^*$, Lemma 36(iii) shows that $\pi^*(\varepsilon) = 0$ and this implies that $0$ is a fixed point of the uncoupled recursion. For $\varepsilon < \varepsilon_c^*$, Proposition 9 shows that $U_s(0; \varepsilon) = 0$ and, hence, $\Psi(\varepsilon) = U_s(\pi^*(\varepsilon), \varepsilon) = 0$. Moreover, $\varepsilon(\varepsilon_c^*) = 0$ by the continuity of $\Psi$. Finally, $\Psi(\varepsilon) < 0$ implies $\varepsilon > \varepsilon_c^*$ and the definition of $\varepsilon_c^*$ implies that $\pi^*(\varepsilon) > 0$.

Consider (ii). For a proper admissible system, Theorem 38 shows that $\Psi(\varepsilon)$ is strictly decreasing for $\varepsilon \in [\varepsilon_c^*, \varepsilon_{\text{max}}]$. On the other hand, (i) shows that $\Psi(\varepsilon) = 0$ for $\varepsilon \leq \varepsilon_c^*$. From this, it also easy to verify that

$$\varepsilon_c^* = \sup \{ \varepsilon \in \mathcal{E} \mid \Psi(\varepsilon) \geq 0 \} = \sup \{ \varepsilon \in \mathcal{E} \mid \Psi(\varepsilon) = 0 \} = \min \{ \inf \{ \varepsilon \in \mathcal{E} \mid \Psi(\varepsilon) < 0 \} ; \varepsilon_{\text{max}} \},$$

(32)

where the last expression requires the outer maximum because $\inf \{ \varepsilon \in \mathcal{E} \mid \Psi(\varepsilon) < 0 \} = \infty$ if $\Psi(\varepsilon) = 0$ for all $\varepsilon \in \mathcal{E}$. Since $\Psi(\varepsilon) = \min_{x \in X_f} U_s(x; \varepsilon)$, the first of these proves the first stated result. For the second statement, if $Q(x) < 0$ for $x \in X_f$, then $\Psi(\varepsilon(x)) = \min_{x \in X_f} U_s(x' ; \varepsilon(x)) \leq Q(x)$ implies that $\Psi(\varepsilon(x)) \leq Q(x) < 0$ and, hence, $\varepsilon(x) > \varepsilon_c^*$.

Consider (iii). Let $A = \{ \varepsilon \in \mathcal{E} \mid \Psi(\varepsilon) < 0 \}$ and observe that $\varepsilon_c^* = \varepsilon_c = \inf A$. Now, observe that $\varepsilon < \varepsilon_c^*$ implies $\varepsilon_c^* = \varepsilon_{\text{max}}$. From this, we see that, if $\varepsilon_c^* < \varepsilon_{\text{max}}$, then $\varepsilon_c^* = \inf A$. We, now, will show that $B = \{ \varepsilon \in \mathcal{E} \mid \exists x \in X_f, \varepsilon(x) = \varepsilon, Q(x) < 0 \}$ satisfies $A = B$. To see this, we note that $\Psi(\varepsilon) < 0$ (i.e., $\varepsilon \in A$) implies $\pi^*(\varepsilon) > 0$ and, hence, $\pi^*(\varepsilon) \in X_f$. Since $\varepsilon(\pi^*(\varepsilon)) = \varepsilon$ implies $Q(\pi^*(\varepsilon)) = \Psi(\varepsilon) < 0$, we find that $\varepsilon \in B$. For the other direction, consider the following. If there is an $\varepsilon \in \mathcal{E}$ such that $\varepsilon(x) = \varepsilon$ for some $x \in X_f$ with $Q(x) < 0$ (i.e., $\varepsilon \in B$), then

$$\Psi(\varepsilon) = \min_{x \in X_f} U_s(x' ; \varepsilon(x)) \leq Q(x) < 0.$$  

(33)

Hence, $\varepsilon \in A$. Therefore, $A = B$ and $\inf A = \inf B = \varepsilon_c^*$. Since $A \neq \emptyset$, there is a sequence $\varepsilon_n \in A$ such that $\lim_n \varepsilon_n = \inf A = \varepsilon_c^*$.

APPENDIX D

PROOFS FROM SECTION VI

A. Proof of Lemma 54

From $P(x) = -2Q(x)$ and (25), we can see that

$$P'(x) = g(x) - xg'(x) = \frac{1}{B(t, n - t)} \left( \int_0^x z^{t-1} (1 - z)^{n-t-1}dz - x^t(1-x)^{n-t-1} \right).$$

Since the derivative $\frac{d}{dx}x^t(1-x)^{n-t-1}$ is given by

$$= tx^{t-1}(1-x)^{n-t-1} - (n-t-1)x^{t-1}(1-x)^{n-t-2}$$

$$= (t(1-x) - (n-t-1)x)x^{t-1}(1-x)^{n-t-2}$$

and the expression $x^t(1-x)^{n-t-1}$ can be written as

$$\int_0^x (t - (n-1)z)z^{t-1}(1-z)^{n-t-2}dz,$$
we find that $P'(x)$ can be expressed as

$$P'(x) = \frac{1}{B(t, n-t)} \int_0^\infty \left( z^{t-1}(1-z)^{n-t-1} + (n-1)z z^{t-1}(1-z)^{n-t-2} \right) dz$$

Using $z^{t-1}(1-z)^{n-t-2} > 0$, it can be shown $P'(x) < 0$ for all $0 < x < \frac{t-1}{n-2}$. By applying fundamental theorem of calculus to (35), we find that

$$P''(x) = \frac{1}{B(t, n-t)} \left( (1-t) + (n-2)x z^{t-1}(1-z)^{n-t-2}, \right.$$  

and, hence, $P''(x) > 0$ for all $\frac{t-1}{n-2} < x < 1$.

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