EINSTEIN GRAVITY IN 2+1 DIMENSIONS
FROM A GAUGE MODEL WITH SYMMETRY BREAKING

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ABSTRACT

Einstein gravity in 2+1 dimensions arises as a consequence of the equations of motion of a gauge model in an external metric. Newton’s constant appears as an order parameter of a spontaneously broken discrete symmetry. Matter is coupled in a straightforward way.

Submitted to: Physical Review Letters
Introduction. The Chern-Simons $ISO(1, 2)$ gauge theory (CS) approach to gravity in 2+1 dimensions (see ref. [1,2]) has advanced the notion that the background metric is disparate from the field playing the role of the quantum metric. In this approach the two fields are present in the quantum (gauge fixed) theory and do not arise from a background-quantum splitting. Additionally, the fact that the spin-connection and dreibein appear as independent fields also played a crucial role in the development of the theory. As advocated in ref. [2], these points were key to the renormalizability of the theory. One would eventually like to couple the theory to matter but, thus far, attempts at this have proven intractable. In this paper, we will be less ambitious and simply investigate the issue of how to introduce Newton’s constant (or equivalently, the Planck length) into the gauge-theoretic formulation of (2+1)-dimensional gravity. This will lead us to a new model of (2 + 1)-dimensional gravity.

Superficially, the non-renormalizability of General Relativity (GR) with matter can be traced to the fact that the coupling constant is the Planck length, $l_P$, and is thus dimensionful. This is unlike QED, for example, where the coupling constant, $\alpha$, is dimensionless. If we are to find a conventional, perturbative quantum field theory of gravity, we must find some dimensionless constant to use as a parameter. However, we must also arrive at Einstein’s equations (with $G_N$, Newton’s constant) as the classical equations of motion from our model. This reminds us of the electroweak theory in which the Fermi coupling constant is constructed out of the parameters in the Higgs potential. In this paper, we propose a similar framework for (2 + 1)-dimensional gravity. In retrospect, our study is related to earlier works where $G_N$ is obtained via symmetry breaking [3]. It is not motivated by a popular approach to quantum gravity in which the metric is thought of as being a Goldstone boson [4].

Before proceeding with the construction of our model, we point out that we will be forced to abandon the CS approach to 2+1 quantum gravity in favor of a formulation which has similarities with Ashtekar’s construction [5,6,7]. This is due to the fact that in order to obtain $G_N$ via a $vev$, we must couple the order parameter to the curvature scalar. In the CS approach the Einstein-Hilbert action appears in the expansion of the action in terms of the components (dreibein and spin-connection) of the $ISO(1, 2)$ gauge field. However,
the Lagrangian is not gauge invariant and multiplying it by a field which is in the singlet
of the gauge group spoils the gauge invariance of the action. In contrast, the Lagrangian
in Ashtekar’s formulation is gauge invariant and so this problem does not arise. We will
ignore the question of whether or not this gauge invariance leads to counterterms in the
theory’s perturbative expansion and thus may endanger renormalizability. Our focus for
this paper will be on classical physics. With $SO(1, 2)$ as the gauge group our model is rich
enough to include the spectrum of fields in $(2 + 1)$-dimensional GR; it is not necessary
to use the full Poincaré group, $ISO(1, 2)$. The analysis below may be carried out with
$SO(1, 2)$ replaced by $SO(3)$ with suitable re-definitions of the metrics.

The Model. Our model consists of three fields, $E$, $A$ and $\phi$. Take $E \equiv dx^a E^a J_a$
a 1-form valued in the Lie algebra of $SO(1, 2)$ transforming in the adjoint representation.
The gauge group generators $J_a$ satisfy $[J_a, J_b] = \epsilon_{abc} J_c$. Furthermore, $A$ is a gauge field and
$\phi$ a $SO(1, 2)$ scalar field which plays the role of the order field. The peculiarity of the model
is that it depends, a priori, on a background metric, $g_{\alpha\beta}$. Note that there will be no kinetic
term for $g_{\alpha\beta}$ in our action, yet we will find a solution which imposes Einstein’s equations
on this background metric. $E$ transforms homogeneously under the Lorentz/gauge group
$SO(1, 2)$ as does the field strength $F_{\alpha\beta} \equiv \partial_{[\alpha} A_{\beta]} + [A_{\alpha}, A_{\beta}]$. Diffeomorphisms transform
$E$ as a 1-form, $F$ as a 2-form and $\phi$ as a scalar. We construct two gauge scalars$^\dagger$

$$X_{\alpha\beta} \equiv E^a E_{\alpha a}, \quad X \equiv X_{\alpha\beta} g^{\alpha\beta} = E^a E^a_a . \quad (1)$$

Recall the dreibein is related to the metric by the formula

$$g_{\alpha\beta} = e_\alpha^a e_{\beta a} . \quad (2)$$

Although the gauge and Lorentz groups are identified, we have two distinct covariant derivatives

$$\nabla_\alpha V_\beta^a = \partial_\alpha V_\beta^a - \Gamma_{\alpha\beta}^\gamma V_\gamma^a + [A_\alpha, V_\beta]^a ,$$

$$\nabla_\alpha^B V_\beta^a = \partial_\alpha V_\beta^a - \Gamma_{\alpha\beta}^\gamma V_\gamma^a + [\omega_\alpha, V_\beta]^a , \quad (3)$$

$^\dagger$ The latin indices are raised and lowered with the Minkowski metric. The greek indices
are raised and lowered with the background (curved) metric, $g_{\alpha\beta}$.
written here acting on a representative field \( V \). These derivatives are both \( SO(1, 2) \) and general-coordinate covariant. \( \nabla^B_\mu \) is the covariant derivative with respect to the background metric \( g_{\alpha\beta} \) of the manifold. Thus \( \Gamma_{\alpha\beta}^{\gamma} \) is the Christoffel symbol and \( \omega_\alpha \) is the Lorentz spin-connection (defined by \( \nabla^B_\alpha e_\beta^a = 0 \)). Note that \( A \) and \( E \) are independent fields while the background spin-connection \( \omega \) is constrained to be a function of the dreibein.

Our model is defined by the classical action

\[
S^G = \int d^3x \sqrt{-g} \left\{ \phi^2 E_\alpha^a F_{\beta\gamma}^{\phantom{\beta\gamma}a} e^{\alpha\beta\gamma} \frac{1}{\sqrt{-g}} + \frac{1}{2} g^{\alpha\beta} \nabla^B_\alpha \phi \nabla^B_\beta \phi - V(\phi, X) \right\} ,
\]

where the mass dimensions are \( \{ E, A, \phi \} = \{ 0, 1, \frac{1}{2} \} \). The first term looks like a Chern-Simons density since it is topological, but it differs in two respects. It is truly gauge invariant and \( F \) is not the curvature associated with \( E \) but with the gauge field \( A \). It is modelled after the three dimensional \( BF \) (Schwarz type) topological gauge theory [8]. Besides the kinetic term for \( \phi \) there is also a potential (bounded from below) in the scalars \( \phi \) and \( X \) which contains the only dimensionful constants of our model. They are necessary to build Newton’s constant \( G_N \). Although \( E_\alpha^a \) appears as an auxiliary field in our action, we do not integrate it out of the action. Removing it in this way would obscure much of the subsequent analysis. The equations of motion we obtain by varying \( S^G \) with respect to \( A, E \) and \( \phi \) are

\[
\nabla_{[\alpha} \phi^2 E_{\beta]}^{\phantom{\beta}a} = 0 ,
\]

\[
\phi^2 \tilde{F}_\alpha^a = 2 \frac{\partial V}{\partial X} E_\alpha^a ,
\]

\[
g^{\alpha\beta} \nabla^B_\alpha \nabla^B_\beta \phi = - \frac{\partial V}{\partial \phi} + 2 \phi E_\alpha^a \tilde{F}_\alpha^a .
\]

where \( \tilde{F}_\alpha^a \equiv F_{\beta\gamma} \left( e^{\alpha\beta\gamma} / \sqrt{-g} \right) \). The metric appears in our action as an external field. Thus the full energy-momentum tensor of the model has no “gravitational” contribution. In fact, varying the action with respect to \( g^{\alpha\beta} \) we find the stress-energy tensor

\[
T^B_{\alpha\beta} = \nabla^B_\alpha \phi \nabla^B_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\nabla^B \phi)^2 + g_{\alpha\beta} V(\phi, X) - 2 \frac{\partial V}{\partial X} X_{\alpha\beta} .
\]

Upon applying the equations of motion (5), we verify \( \nabla^B_\alpha T^B_{\alpha\beta} = 0 \).
We now study the solutions of the equations of motion (5) which minimize the total energy. The latter is defined for a manifold of the form $\mathbb{R} \times \Sigma$ with a time independent metric
\[ ds^2 = dt^2 - \gamma_{ij} dx^i dx^j , \] (7)
where $\gamma_{ij}$ is an Euclidean metric on $\Sigma$. Let $\Pi$ be the canonical momentum conjugate to $\phi$. The Hamiltonian density is then
\[ H = \frac{1}{2} \Pi^2 + \frac{1}{2} \gamma^{ij} \partial_i \phi \partial_j \phi + V(\phi, \mathbf{X}) - 2 \frac{\partial V}{\partial \mathbf{X}} \mathbf{X}_{00} . \] (8)
As the first two terms are positive semi-definite, we take $\phi$ and $\mathbf{X}$ to be the constants which minimize the last two (potential) terms. This yields
\[ \begin{align*}
\delta \phi : & \quad \frac{\partial V}{\partial \phi} - 2 \frac{\partial^2 V}{\partial \phi \partial \mathbf{X}} \mathbf{X}_{00} = 0 , \\
\delta E^i : & \quad \left( \frac{\partial V}{\partial \mathbf{X}} - 2 \frac{\partial^2 V}{\partial \mathbf{X}^2} \mathbf{X}_{00} \right) E_i = 0 , \\
\delta E^0 : & \quad \left( \frac{\partial V}{\partial \mathbf{X}} + 2 \frac{\partial^2 V}{\partial \mathbf{X}^2} \mathbf{X}_{00} \right) E_0 = 0 .
\end{align*} \] (9)

In general, given a potential, there may be many solutions. However, they need not all be degenerate. Two classes of potentials interest us. The first class is composed of those with $\mathbf{X}_{00} \neq 0$ and $\mathbf{X}_{ij} \neq 0$. The second class has $E_{\alpha a} = 0$. For the first class, then the last two equations of (9) imply
\[ \begin{align*}
\frac{\partial V}{\partial \mathbf{X}} & = 0 , \\
\frac{\partial^2 V}{\partial \mathbf{X}^2} & = 0 .
\end{align*} \] (10)
Using these, the equations of motion (5) lead to
\[ \begin{align*}
\frac{\partial V}{\partial \phi} & = 0 \quad \text{and} \quad \frac{\partial^2 V}{\partial \phi \partial \mathbf{X}} = 0 ,
\end{align*} \] (11)
then if $\phi$ is non-zero
\[ \begin{align*}
\nabla_{[\alpha} E_{\beta]} & = 0 \quad \text{and} \quad E_{\alpha a} \tilde{F}_{a}^{\alpha} = 0 .
\end{align*} \] (12)
\[ \dagger \] The vertical bar signifies evaluation on a minimum energy solution.
To focus ideas further, we take the potential to be

\[ V(\phi, X) = -\frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \lambda_4 \phi^4 + \frac{1}{6} \lambda_6 \phi^6 + \lambda_X (X - X_0)^4 , \]  

where \( \lambda_X, \mu^2, \lambda_4, \lambda_6 \) and \( X_0 \) are positive constants with mass dimensions three, two, one, zero and zero, respectively. For this potential, the minimal-energy solution for \( X \) is \( X_0 \) and for \( \phi \) it is \( \phi = \nu \) where \( \nu^2 = (-\lambda_4 + \sqrt{\lambda_4^2 + 4\mu^2\lambda_6})/2\lambda_6 \). As is well known, the \( \mathbb{Z}_2 \) symmetry \( \phi \rightarrow -\phi \) is spontaneously broken and as it is a discrete symmetry there is no massless Goldstone boson. The conditions (10) are satisfied by \( E_\alpha^a = \sqrt{X_0} \epsilon_\alpha^a \) so that \( X = e_\alpha^ae_\beta_ag^{\alpha\beta} = X_0 \). Observe that this directly links the solutions for \( E_\alpha^a \) to the metric on our manifold and hence in our action. Such a link is a mystery in the Chern-Simons approach. It is for this reason that the potential in \( X \) is important. Henceforth, we normalize such that \( X_0 = 3 \). The first equation in (12) is solved by \( A_\alpha = \omega_\alpha \). We summarize our minimal-energy solutions by the equations

\[ \phi = \nu , \quad E_\alpha^a = e_\alpha^a , \quad A_\alpha^{ab} = \omega_\alpha^{ab} . \]  

(14)

With the gauge field equated to the spin-connection we have

\[ F_{\alpha\beta}^{ab} = \partial_{[\alpha} \omega_{\beta]}^{ab} + [\omega_\alpha, \omega_\beta]^{ab} = R_{\alpha\beta}^{ab} , \]  

(15)

where \( R_{\alpha\beta}^{ab} \) is the Riemann curvature tensor in the background metric \( g_{\alpha\beta} \). The second equation in (12) (which was obtained from the equations of motion (5c) evaluated on \( \phi \)) now reads

\[ R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0 , \]  

(16)

namely, Einstein’s vacuum equations. Of course, any background metric may appear in our action, but only those which solve Einstein’s equations will be consistent with our ansatz. For example flat Minkowski space for which our solution (14) defines a translation invariant vacuum.

The second interesting class of potentials, those for which \( E_\alpha^a = 0 \) also require \( \frac{\partial V}{\partial \phi} = 0 \). Then \( \phi = \nu \) and \( F_\alpha^a = 0 \) are minimal energy solutions. We interpret the solution with \( A_\alpha^a = 0 \) as the generally covariant one. However, here we are unable to identify \( E_\alpha^a \) solutions with the background dreibein.
As an aside, we compare our model, $S^G$, with the two dominant gauge-theoretic models appearing in the literature. In this case, we take the coupling constants in our potential to be such that $\phi = \nu$ dominates the path integral. Furthermore, we restrict the potential to be $X$ independent. Then the action reduces to

$$S' = \nu^2 \int d^3x E^a \hat{F}^{a}_{\beta\gamma} \epsilon^{\alpha\beta\gamma}.$$ \hspace{1cm} (17)

This is the Ashtekar formulation of (2 + 1)-dimensional gravity as given in ref. [6,7]. As shown in those references, the constraints obtained are the same as those of the CS approach [2] when the spatial metric is non-degenerate; they satisfy the Poincaré algebra.

**Matter Coupling.** We minimally couple matter to our model such that we get Einstein’s equations, $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = -8\pi G_N T^M_{\alpha\beta}$. $T^M_{\alpha\beta}$ is the energy-momentum tensor one would obtain from the conventional minimal coupling of gravity to matter. It should not be confused with the stress-energy tensor which is the response of our action to a change in the background metric, $g_{\alpha\beta}$.

Consider the following addition to our action (4):

$$S^M = -\frac{1}{2} \int d^3x \sqrt{-g} \left[ X^{\alpha\beta} f(X) Y_{\alpha\beta} + h(X) J \right].$$ \hspace{1cm} (18)

$Y_{\alpha\beta}$ (symmetric tensor) and $J$ depend on the matter and the metric $g_{\alpha\beta}$ but not on $E$, $\phi$ or $A$. $f$ and $h$ are dimensionless functions of $X$. As $A$ does not couple in $S^M$, its equation of motion is also as before (5a) and is satisfied by a constant $\phi$ and $\nabla \times E = 0$. Since we choose not to couple $\phi$ to matter, its equation of motion is as before, namely (5c)

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = (\mu^2 + 2E^a \hat{F}^{a}_{\alpha}) \phi - \lambda_4 \phi^3 - \lambda_6 \phi^5.$$ \hspace{1cm} (19)

If $|E^a \hat{F}^{a}_{\alpha}| \ll \mu^2$ at each point on the manifold (we will soon see what this means geometrically), then $g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi \approx -\frac{\partial V}{\partial \phi}$ and $\phi = \nu$ is an approximate solution. When we compute the variation of our new action $S = S^G + S^M$, with respect to $E$ and use $\phi = \nu$ we find that (5b) becomes

$$\nu^2 \hat{F}^{a}_{\alpha} = \left[ 2 \frac{\partial V}{\partial X} E^a_{\alpha} \right] + f(X) Y^{\alpha\beta} E_{\beta a} + \frac{\partial f}{\partial X} Y^{\beta\gamma} X_{\beta\gamma} E^a_{\alpha} + \frac{\partial h}{\partial X} J E^a_{\alpha}.$$ \hspace{1cm} (20)
Now, as before, $\nabla \times E = 0$ is solved by
\[ E_\alpha^a = e_\alpha^a \quad \text{and} \quad A_\alpha^{ab} = \omega_\alpha^{ab}. \tag{21} \]

Using these, Eq. (20) reads
\[ R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi G_N T^M_{\alpha\beta}, \tag{22} \]
where
\[ T^M_{\alpha\beta} = f(X_0)\Upsilon_{\alpha\beta} + g_{\alpha\beta} \frac{\partial f}{\partial X}(X_0)\Upsilon_{\gamma\gamma} + \frac{\partial h}{\partial X}(X_0) Jg_{\alpha\beta}, \tag{23} \]
\[ \nu^2 = \frac{1}{32\pi G_N}. \]

As $\nu^2$ is now given by the inverse Planck length, $l_P^{-1}$, it is natural that $\mu^2 = \mathcal{O}(l_P^{-2})$ (note that $\mu = \frac{1}{32\pi G_N}$ for $\lambda_4 = 0$). Our condition that $\phi = \nu$ is an approximate solution now translates into a condition on the scalar curvature of the manifold: $|R| \ll l_P^{-2}$ at each point. When the energy-momentum tensor of the matter is such that $|T^M| = \mathcal{O}(l_P^{-3})$ (or equivalently, when we start to probe the Planck length), the excitations of $\phi$ become important.

We now give examples. The addition to the action of the term
\[ S^M(\Lambda) = -\frac{\Lambda}{16\pi G_N} \int d^3x \sqrt{-g} h(X), \tag{24} \]
with $\left| \frac{\partial h}{\partial X} \right| = 1$ yields Einstein’s equations with a cosmological constant $\Lambda$. This gives a contribution to the potential for $E$. Now consider a massive scalar field, $\Phi$. We find that a minimally coupled action is given by (18) with $f(X) = h(X) = \frac{1}{2}(X - 5)$ so that
\[ S^M(\Phi) = \frac{1}{4} \int d^3x \sqrt{-g} (X - 5) \left[ X^{\alpha\beta} \nabla^B \Phi \nabla^B \Phi - m^2 \Phi^2 \right]. \tag{25} \]
Evaluated on our solution (14), the equations of motion obtained from $S_0 + S^M(\Phi)$ are Einstein’s equations (22) and the scalar field equation $g^{\alpha\beta} \nabla^B \nabla^B \Phi = -m^2 \Phi$. Similarly, the minimally coupled action for a massive fermion is
\[ S^M(\psi) = -\int d^3x \sqrt{-g} \left\{ \frac{i}{2} X^{\alpha\beta} \left[ \bar{\psi}_{\gamma\alpha} \nabla^B \psi - \nabla^B \bar{\psi}_{\gamma\beta} \psi \right] - m\bar{\psi}\psi \right\}. \tag{26} \]

This discussion may be generalized to other types of matter.
Comments and Conclusions. In order to do quantum calculations with our action (4), we must expand around our vacuum configurations given by (14) with \( e_\alpha^a = \delta_\alpha^a \).

In particular, \( \phi = \nu + \hat{\phi}, E_\alpha^a = \delta_\alpha^a + \hat{E}_\alpha^a, A_\alpha^{ab} = \hat{A}_\alpha^{ab} \), where the hatted fields are quantum. Using these expressions in the action \( S = S^G + S^M \), we can read off propagators and vertices for the quantum fields in a Minkowski background.

\( E_\alpha^a \neq 0 \) breaks general covariance. Alternatively, we could have taken a potential for which \( E_\alpha^a | = 0 \), thereby not breaking general covariance. For this potential, coupling to matter still leads to Einstein’s equations as a solutions of our model. However, in this case, the classical solution for \( E_\alpha^a \) bears no relation to the background dreibein.

This model does have several advantages over the Einstein-Hilbert theory. From a quantum theoretic point of view, because the metric is treated as a background field, the measure of the path integral of the gauge fields is easier to define. This is not the case with the integration over metrics in a path integral of the Einstein-Hilbert action where the metric is itself considered the fundamental quantum field.

In conclusion, we have shown that Einstein gravity in 2+1 dimensions may be formulated as a gauge theory coupled to a scalar in an external metric. Furthermore, this model illustrates how the Planck scale can arise from the spontaneous breaking of a discrete symmetry. This precludes the existence of additional massless particles in the low-energy spectrum. Matter may be coupled to gravity in this model in a straightforward manner.
REFERENCES

1. A. Achucarro and P. Townsend, *Phys. Lett.* **180B** (1986) 89.

2. E. Witten, *Nucl. Phys.* **B311** (1988) 46.

3. S. Deser and P. van Nieuwenhuizen, *Phys. Rev.* **D10** (1974) 401; P. Minkowski, *Phys. Lett.* **71B** (1978) 419; L. Smolin, *Nucl. Phys.* **B160** (1979) 253; A. Zee, *Phys. Rev. Lett.* **42** (1979) 417, *Phys. Rev. Lett.* **44** (1980) 703; additional references may be found in S. L. Adler, *Rev. Mod. Phys.* **54** (1982) 729.

4. C. W. Misner, *Phys. Rev.* **D18** (1978) 4510.

5. For reviews see A. Ashtekar, *Non-perturbative Canonical Quantum Gravity*, (World Scientific, Singapore, 1991); C. Rovelli, *Class. Quan. Grav.* **8** (1991) 1613.

6. I. Bengtsson, *Phys. Lett.* **220B** (1989) 51.

7. N. Manojlović and A. Miković, “Ashtekar Formulation of 2 + 1 Gravity on a Torus”, preprint #’s Imperial/TP/91-92/7 and QMW/PH/92/7 (April 1992).

8. For a review see D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rept.* **209** (1991) 129.