Nambu dynamics and its noncanonical Hamiltonian representation in many degrees of freedom systems

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1. Introduction

There are various ways to generalize the Hamiltonian dynamics. In the present paper, we focus on two generalized dynamics, the Nambu dynamics and the noncanonical Hamiltonian dynamics. The Nambu dynamics is a generalized Hamiltonian dynamics which is defined in the extended phase space spanned by $N(\geq 3)$ variables $(x_1, x_2, ..., x_N)$ [1]. Taking the Liouville theorem as a guiding principle, Nambu generalized the Hamilton equations of motion to the Nambu equations, which are defined by $N-1$ Hamiltonians and the Nambu bracket, an $N$-ary generalization of the canonical Poisson bracket. In order for the variable transformation including the time evolution to be consistent, the Nambu bracket must satisfy the fundamental identity, a generalization of the Jacobi identity [2–4]. On the other hand, the noncanonical Hamiltonian dynamics is also defined in the $N$-dimensional extended phase space, and the Hamilton equations of motion are generalized to the noncanonical ones, which are defined by one Hamiltonian and the noncanonical Poisson bracket [5]. Although the noncanonical Poisson bracket has the same structure as the canonical Poisson bracket, it is defined by means of the variable-dependent $N \times N$ Poisson matrix. The noncanonical Poisson bracket must satisfy the Jacobi identity for the consistent variable transformation including the time evolution. It has been shown that the Nambu dynamics can always be represented in the form of the noncanonical Hamiltonian dynamics with the noncanonical Poisson bracket defined by the Nambu bracket [4, 6].
Although the structure of the Nambu dynamics have impressed many authors, it has been revealed that the Nambu bracket exhibits serious difficulties in many degrees of freedom systems [1–4, 7]. This is because in such systems the Nambu bracket does not satisfy the fundamental identity. Since the fundamental identity is too strict, each degree of freedom must be decoupled to satisfy the identity. On the other hand, for the noncanonical Poisson bracket, whether or not the Jacobi identity holds is not a matter of the number of degrees of freedom, but rather a matter of the nature of the Poisson matrix.

In the present paper, we study the Nambu dynamics and the corresponding noncanonical Hamiltonian dynamics in many degrees of freedom systems, and show that even if the fundamental identity is violated, the Jacobi identity for corresponding dynamics could hold. That is, even if the consistent time evolution is broken in the Nambu dynamics, it could be restored in the corresponding noncanonical Hamiltonian dynamics. As an example we evaluate these two identities for a simplified Hénon–Heiles model [8], a system of two coupled oscillators whose semiclassical dynamics has been studied using the hidden Nambu formalism [9, 10].

The outline of this paper is as follows. In Sect. 2 we review the Nambu dynamics and its noncanonical Hamiltonian representation with proofs of the fundamental identity and the corresponding Jacobi identity. In Sect. 3 we show the violation of the fundamental identity for the Nambu bracket in many degrees of freedom systems, and give the condition under which the Jacobi identity for the corresponding noncanonical Poisson bracket holds. We also present an example of two degrees of freedom system. Our conclusions are given in the last section.

2. Nambu dynamics and noncanonical Hamiltonian dynamics

We begin with a brief review of the Nambu dynamics [1] and the relationship with the noncanonical Hamiltonian dynamics [4, 6] in one degree of freedom systems. Throughout this paper we treat the case of \( N = 3 \), and therefore we consider the dynamics of three Nambu variables \( (x_1, x_2, x_3) \) in this section. The generalization for arbitrary \( N \geq 3 \) is straightforward.

2.1. Nambu dynamics

In the Nambu dynamics, the canonical Poisson bracket is generalized to the Nambu bracket defined by means of the three-dimensional Jacobian,

\[
\{ A, B, C \} \equiv \frac{\partial (A, B, C)}{\partial (x_1, x_2, x_3)} = \epsilon_{ijk} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} \frac{\partial C}{\partial x_k},
\]

where \( A, B \), and \( C \) are any functions of the three variables \( (x_1, x_2, x_3) \) and \( \epsilon_{ijk} \) is the three-dimensional Levi–Civita symbol. We employ the summation convention over repeated indices throughout this paper. In terms of the Nambu bracket, the Nambu equation for any function \( f = f(x_1, x_2, x_3) \) can be written as

\[
\frac{df}{dt} = \{ f, H, G \} = \epsilon_{ijk} \frac{\partial f}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial G}{\partial x_k},
\]

where \( H \) and \( G \) are Nambu Hamiltonians. The time evolution according to this equation preserves the three-dimensional phase space volume, and therefore the Liouville theorem holds in the Nambu dynamics.
The Nambu bracket of Eq. (1) must satisfy the following fundamental identity [2–4],
\[ \{A, B, C\}, D, E\} = \{\{A, D, E\}, B, C\} + \{A, \{B, D, E\}, C\} + \{A, B, \{C, D, E\}\}. \] (3)
Here \( D \) and \( E \) are any functions of the three variables, and play the roles of the generating functions of a variable transformation. In particular, if we choose them as the Nambu Hamiltonians, \((D, E) = (H, G)\), then the identity of Eq. (3) means that the distributive property of time derivatives holds,
\[ \frac{d}{dt}\{A, B, C\} = \{\frac{d}{dt}A, B, C\} + \{A, \frac{d}{dt}B, C\} + \{A, B, \frac{d}{dt}C\}. \] (4)
Therefore, if the fundamental identity is violated, the consistent time evolution is broken, at least in the sense that the distributive property does not hold.\(^1\)

The fundamental identity can be proved as follows [3]. The difference between the left-hand side and the right-hand side of Eq. (3) can be represented as
\[ \text{lhs} - \text{rhs} = -\frac{1}{2} \epsilon^{ijklmnp} \epsilon_{ilm} \epsilon_{njk} \partial_\mu A \partial_\nu B \partial_\rho C \partial_i (\partial_j D \partial_k E), \] (5)
which can be rewritten in terms of the generalized Kronecker delta,
\[ \text{lhs} - \text{rhs} = -\frac{1}{2} \epsilon^{ijklmnp} \epsilon_{ilm} \epsilon_{njk} \partial_\mu A \partial_\nu B \partial_\rho C \partial_i (\partial_j D \partial_k E) = -\frac{1}{2} \epsilon_{ijklmnp} \epsilon_{ilm} \epsilon_{njk} \partial_\mu A \partial_\nu B \partial_\rho C \partial_i (\partial_j D \partial_k E) = 0. \] (6)
Note that we do not distinguish upper and lower indices.

2.2. Noncanonical Hamiltonian representation

Start with the Nambu equation of Eq. (2). Using one of the Nambu Hamiltonians, \( G \), we define the Poisson matrix \( J_{ij}(x_1, x_2, x_3) \) as
\[ J_{ij} \equiv \epsilon_{ijk} \frac{\partial G}{\partial x_k}, \] (7)
which is anti-symmetric: \( J_{ji} = -J_{ij} \). In terms of this matrix, we define the noncanonical Poisson bracket as
\[ \{A, B\}_G \equiv J_{ij} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} = \{A, B, G\}, \] (8)
where \( A \) and \( B \) are any functions of \((x_1, x_2, x_3)\). Then we can rewrite the Nambu equation to the noncanonical Hamilton’s equation of motion,
\[ \frac{df}{dt} = \{f, H, G\} = \{f, H\}_G = J_{ij} \frac{\partial f}{\partial x_i} \frac{\partial H}{\partial x_j}. \] (9)
The Jacobi identity for the noncanonical Poisson bracket of Eq. (8) immediately follows from the fundamental identity. Let \( C = G, E = G, \) and \( D = C \) in the fundamental identity of Eq.

\(^1\) The violation of the Jacobi identity also implies the breaking of the consistent time evolution. It is an interesting subject to study how the violation of these identities affects the actual dynamics. For example, see [11].
The way to represent the Nambu dynamics in the form of the noncanonical Hamiltonian dynamics is not unique. For example, defining another Poisson matrix as
\[ \tilde{J}_{ij} \equiv -\epsilon_{ijk} \frac{\partial H}{\partial x_k}, \]
and another noncanonical Poisson bracket as
\[ \{ A, B \}_H \equiv \tilde{J}_{ij} \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial x_j} = \{ A, B, H \}, \]
then we obtain another expression of equation of motion,
\[ \frac{df}{dt} = \{ f, H, G \} = \tilde{J}_{ij} \frac{\partial f}{\partial x_i} \frac{\partial G}{\partial x_j}. \]

The bracket of Eq. (12) also satisfies the Jacobi identity. Note that the Liouville theorem holds in both dynamics of Eq. (9) and Eq. (13).

3. Many degrees of freedom systems

It is possible to extend the Nambu dynamics to many degrees of freedom systems. However, in general, the fundamental identity does not hold in such systems [1–4, 7]. Therefore it is nontrivial whether the Jacobi identity for the noncanonical Poisson bracket defined by the Nambu bracket holds or not. Here we give the conditions under which the identities hold. As an example we evaluate these identities for a semiclassical system of two coupled oscillators.

3.1. Nambu dynamics

Consider a system of 3n Nambu variables \((x_1^1, x_2^1, x_3^1, ..., x_n^1, x_1^n, x_2^n, x_3^n)\). The time evolution of them can be given in the same form as Eq. (2) by extending the definition of the Nambu bracket,
\[ \{ A, B, C \} \equiv \sum_{\alpha=1}^{3n} \frac{\partial (A, B, C)}{\partial (x_1^\alpha, x_2^\alpha, x_3^\alpha)} = \sum_{\alpha=1}^{3n} \epsilon_{ijk} \frac{\partial A}{\partial x_i^\alpha} \frac{\partial B}{\partial x_j^\alpha} \frac{\partial C}{\partial x_k^\alpha}, \]
where \( A, B, \) and \( C \) are any functions of the \( 3n \) variables. In terms of this bracket, the Nambu equation for any function \( f = f(x_1^1, x_2^1, x_3^1, ..., x_n^1, x_1^n, x_2^n, x_3^n) \) can be written as
\[ \frac{df}{dt} = \{ f, H, G \} = \sum_{\alpha=1}^{3n} \epsilon_{ijk} \frac{\partial f}{\partial x_i^\alpha} \frac{\partial H}{\partial x_j^\alpha} \frac{\partial G}{\partial x_k^\alpha}, \]
where \( H \) and \( G \) are Nambu Hamiltonians. The Liouville theorem holds as well in this dynamics. Let us try to prove the fundamental identity for the Nambu bracket of Eq. (14). To simplify the equations, we employ the notation \( \partial A/\partial x_i^\alpha = \partial_i^\alpha A \). Using the definition of Eq. (14), the difference between the left and right hand sides of Eq. (3) can be represented as
\[ \text{lhs} - \text{rhs} = -\sum_{\alpha=1}^{3n} \sum_{\beta=1}^{3n} \left( \epsilon_{i\mu\nu} \epsilon_{\rho jk} \partial_i^\alpha A \partial_\mu^\beta B \partial_\rho^\beta C + \epsilon_{i\nu\rho} \epsilon_{\mu jk} \partial_i^\alpha A \partial_\nu^\beta B \partial_\rho^\beta C + \epsilon_{i\rho\mu} \epsilon_{\nu jk} \partial_i^\alpha A \partial_\rho^\beta B \partial_\mu^\beta C \right) \times \partial_i^\alpha \left( \partial_j^\beta D \partial_k^\beta E \right). \]
Unlike the case of one degree of freedom, the difference does not vanish in general, but vanishes under some conditions. For example, consider the case that \( 3n \) variables are decoupled
in the functions $D$ and $E$,

$$
D = \sum_{\alpha=1}^{n} D_{\alpha}(x_1^\alpha, x_2^\alpha, x_3^\alpha), \quad E = \sum_{\alpha=1}^{n} E_{\alpha}(x_1^\alpha, x_2^\alpha, x_3^\alpha),
$$

(17)

where $D_{\alpha}$ and $E_{\alpha}$ are only functions of $(x_1^\alpha, x_2^\alpha, x_3^\alpha)$. Then equation (16) reads

$$
\text{lhs} - \text{rhs} = - \sum_{\alpha=1}^{n} \left( \epsilon_{i\mu\nu} \epsilon_{\rho\mu\nu} + \epsilon_{i\mu\rho} \epsilon_{\rho\nu\nu} + \epsilon_{i\mu\rho} \epsilon_{\rho\nu\nu} \right) \partial_{\alpha}^{\mu} A \partial_{\alpha}^{\nu} B \partial_{\alpha}^{\rho} C \partial_{\alpha}^{\nu} \left( \partial_{\alpha}^{\mu} D \partial_{\alpha}^{\nu} E \right).
$$

(18)

We can show that this difference becomes zero in the same way as the Eq. (6). Although the fundamental identity holds in this case, it is almost meaningless as an identity for many degrees of freedom, because the decomposed $D$ and $E$ as in Eq. (17) mean that there is no interaction between the degrees of freedom. The functions $D$ and $E$ in Eq. (3) play the roles of the generating functions of a variable transformation, and in particular, they are the Hamiltonians in the time evolution. Therefore at least one of them must not be decomposed.

If you do not put any conditions on $D$ and $E$, you have to impose restriction on $A$, $B$, and $C$. Consider the case that they are functions of a single degree of freedom, $A = A_{\alpha}$, $B = B_{\alpha}$, and $C = C_{\alpha}$. Then the left-hand side of the fundamental identity of Eq. (3) is $\{\{A_{\alpha}, B_{\alpha}, C_{\alpha}\}, D, E\}$, and equation (16) reads

$$
\text{lhs} - \text{rhs} = - \sum_{\alpha=1}^{n} \left( \epsilon_{i\mu\nu} \epsilon_{\rho\mu\nu} + \epsilon_{i\mu\rho} \epsilon_{\rho\nu\nu} + \epsilon_{i\mu\rho} \epsilon_{\rho\nu\nu} \right) \partial_{\alpha}^{\mu} A_{\alpha} \partial_{\alpha}^{\nu} B_{\alpha} \partial_{\alpha}^{\rho} C_{\alpha} \partial_{\alpha}^{\nu} \left( \partial_{\alpha}^{\mu} D \partial_{\alpha}^{\nu} E \right).
$$

(19)

The same calculation as in the Eq. (6) shows that this difference becomes zero.

3.2. Noncanonical Hamiltonian representation

Similar to the case of one degree of freedom, the Nambu dynamics of Eq. (15) can be represented in the form of the noncanonical Hamiltonian dynamics. Using the Hamiltonian $G$, we define the Poisson matrices $J_{ij}^{\alpha}(x_1^1, x_2^1, x_3^1, ..., x_1^n, x_2^n, x_3^n)$ as

$$
J_{ij}^{\alpha} \equiv \epsilon_{ijk} \frac{\partial G}{\partial x_k^{\alpha}}.
$$

(20)

In terms of these anti-symmetric matrices, we define the noncanonical Poisson bracket as

$$
\{A, B\}_G \equiv \sum_{\alpha=1}^{n} \{A, B\}_G^{\alpha} \equiv \sum_{\alpha=1}^{n} J_{ij}^{\alpha} \frac{\partial A}{\partial x_i^{\alpha}} \frac{\partial B}{\partial x_j^{\alpha}} = \{A, B, G\},
$$

(21)

and then we rewrite the Nambu equation to the noncanonical Hamilton’s equation of motion,

$$
\frac{df}{dt} = \{f, H, G\} = \{f, H\}_G = \sum_{\alpha=1}^{n} J_{ij}^{\alpha} \frac{\partial f}{\partial x_i^{\alpha}} \frac{\partial H}{\partial x_j^{\alpha}}.
$$

(22)

Since the Nambu bracket of Eq. (14) no longer satisfies the fundamental identity, it is nontrivial whether the noncanonical Poisson bracket of Eq. (21) satisfies the Jacobi identity. Let us find the conditions for the Jacobi identity to hold. The difference between the left
and right hand sides of Eq. (10) can be represented as

$$\text{lhs} - \text{rhs} = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \left( \left\{ \alpha, \{ A, B \} G^\beta \right\}_G - \left\{ \alpha, \{ A, C \} G^\beta \right\}_G - \left\{ \alpha, \{ B, C \} G^\beta \right\}_G \right), \quad (23)$$

where all the terms with $\alpha = \beta$ vanish, because the Jacobi identity holds for each degree of freedom. For the terms with $\alpha \neq \beta$, after a straightforward calculation we obtain

$$\left\{ \alpha, \{ A, B \} G^\beta \right\}_G - \left\{ \alpha, \{ A, C \} G^\beta \right\}_G - \left\{ \alpha, \{ B, C \} G^\beta \right\}_G + (\alpha \leftrightarrow \beta) = \partial_k \partial_{\beta} J_{ij}^\alpha \partial_{\alpha} A \partial_{\beta} B \partial_{\alpha} C \partial_{\beta} \partial_{\alpha} B \partial_{\beta} C \right\}_G + (\alpha \leftrightarrow \beta). \quad (24)$$

Therefore if the Poisson matrices satisfy

$$\frac{\partial}{\partial x_k} J_{ij}^\alpha = 0 \quad (\alpha \neq \beta), \quad (25)$$

then equation (24) becomes zero, and the Jacobi identity holds.

Consider the case that $3n$ variables are coupled in the Hamiltonian $H$, but decoupled in the Hamiltonian $G$,

$$G = \sum_{\alpha=1}^{n} G_\alpha (x_1^\alpha, x_2^\alpha, x_3^\alpha). \quad (26)$$

Then the corresponding Poisson matrices of Eq. (20) are functions of the single degree of freedom, $J_{ij}^\alpha = J_{ij}^\alpha (x_1^\alpha, x_2^\alpha, x_3^\alpha)$, which satisfy the condition of Eq. (25), and the Jacobi identity holds. In this case the consistent time evolution is broken in the original Nambu dynamics, but restored in the corresponding noncanonical Hamiltonian dynamics. On the other hand, if we define the Poisson matrices by mean of the Hamiltonian $H$,

$$\tilde{J}_{ij}^\alpha = -\epsilon_{ijk} \frac{\partial H}{\partial x_k^\alpha}, \quad (27)$$

and rewrite the Nambu equation as

$$\frac{df}{dt} = \{ f, H, G \} = \{ f, G \}_H = \sum_{\alpha=1}^{n} J_{ij}^\alpha \frac{\partial f}{\partial x_i^\alpha} \frac{\partial G}{\partial x_j^\alpha}, \quad (28)$$

then the Jacobi identity does not hold, and the consistent time evolution cannot be restored. This is because $3n$ variables are not decoupled in the Hamiltonian $H$, and therefore $H$ cannot be written in the decomposed form, $H = \sum_{\alpha=1}^{n} H_\alpha (x_1^\alpha, x_2^\alpha, x_3^\alpha)$. It should be noted that the Liouville theorem holds in both dynamics of Eq. (22) and Eq. (28).

3.3. Example: semiclassical coupled oscillators

As an example of many degrees of freedom systems, consider a one-dimensional system of two quantum oscillators whose Hamiltonian is given by

$$\hat{H} = \frac{1}{2m_1} \hat{p}_1^2 + \frac{1}{2m_2} \hat{p}_2^2 + \frac{m_1 \omega_1^2}{2} \hat{q}_1^2 + \frac{m_2 \omega_2^2}{2} \hat{q}_2^2 + \lambda \hat{q}_1 \hat{q}_2. \quad (29)$$

This is a simplified version of the quantum Hénon–Heiles model [8]. The semiclassical equations of motion for the quantum expectation values $\langle \hat{q}_1 \rangle, \langle \hat{p}_1 \rangle, \langle \hat{q}_2 \rangle, \langle \hat{p}_2 \rangle, \langle \hat{q}_2 \rangle$ are given
by approximating the higher order expectation values by means of the lower ones [12],
\[
\frac{d}{dt} \langle \hat{q}_1 \rangle = \frac{1}{m_1} \langle \hat{p}_1 \rangle, \quad \frac{d}{dt} \langle \hat{q}_2 \rangle = \frac{1}{m_2} \langle \hat{p}_2 \rangle,
\]
\[
\frac{d}{dt} \langle \hat{p}_1 \rangle = -m_1 \omega_1^2 \langle \hat{q}_1 \rangle - \lambda \langle \hat{q}_2^2 \rangle, \quad \frac{d}{dt} \langle \hat{p}_2 \rangle \simeq -m_2 \omega_2^2 \langle \hat{q}_2 \rangle - 2 \lambda \langle \hat{q}_1 \rangle \langle \hat{q}_2 \rangle,
\]
\[
\frac{d}{dt} \langle \hat{q}_1 \rangle \simeq 2 \frac{1}{m_1} \langle \hat{q}_1 \rangle \langle \hat{p}_1 \rangle, \quad \frac{d}{dt} \langle \hat{q}_2 \rangle \simeq 2 \frac{1}{m_2} \langle \hat{q}_2 \rangle \langle \hat{p}_2 \rangle.
\]
(30)

This semiclassical dynamics can be formulated as the Nambu dynamics using the hidden Nambu formalism [9, 10]. We choose \( n = 2, \) \( N = 3 \) Nambu variables as follows:
\[
\begin{pmatrix}
    x_1^1 \\
    x_2^1 \\
    \langle \hat{q}_1^1 \rangle
\end{pmatrix} = \begin{pmatrix}
    \langle \hat{q}_1 \rangle \\
    \langle \hat{p}_1 \rangle \\
    \langle \hat{q}_1^2 \rangle
\end{pmatrix}, \quad \begin{pmatrix}
    x_1^2 \\
    x_2^2 \\
    \langle \hat{q}_2^2 \rangle
\end{pmatrix} = \begin{pmatrix}
    \langle \hat{q}_2 \rangle \\
    \langle \hat{p}_2 \rangle \\
    \langle \hat{q}_2^2 \rangle
\end{pmatrix}, \quad \tag{31}
\]

and define the Nambu Hamiltonians \( H \) and \( G \) as
\[
H = \frac{1}{2m_1} (x_2^1)^2 + \frac{1}{2m_2} (x_2^2)^2 + \frac{m_1 \omega_1^2}{2} x_3^1 + \frac{m_2 \omega_2^2}{2} x_3^2 + \lambda x_1^1 x_3^1, \quad \tag{32}
\]
\[
G = \sum_{\alpha=1}^{2} \left( x_3^\alpha - (x_1^\alpha)^2 \right), \quad \tag{33}
\]

then it can be shown that the Nambu equation of Eq. (15) reproduces the semiclassical equations of Eq. (30). This is a semiclassical dynamics with constraints that the quantum fluctuation of each mode, \( \langle \hat{q}_n^2 \rangle - \langle \hat{q}_n \rangle^2 \), is constant in time. Therefore the Hamiltonian \( G \) can be written in the decomposed form of Eq. (33). Since the Hamiltonian \( H \) has the interaction term between two degrees of freedom, the fundamental identity for these \( H \) and \( G \) does not hold [10]. For example, if we choose \( (A, B, C) = (x_1^1, x_2^1, x_2^2) \) and \( (D, E) = (H, G) \), then the left-hand side of Eq. (3) is zero, whereas the right-hand side is \(-\lambda\). This implies that the consistent time evolution is broken in this Nambu dynamics.

Let us see if the Jacobi identity holds in corresponding two types of noncanonical Hamiltonian dynamics. First, if we define the Jacobi matrices using the Hamiltonian \( G \) as in Eq. (20), they can be written as
\[
J^1 = \begin{pmatrix}
    0 & 1 & 0 \\
    -1 & 0 & -2x_1^1 \\
    0 & 2x_1^1 & 0
\end{pmatrix}, \quad J^2 = \begin{pmatrix}
    0 & 1 & 0 \\
    -1 & 0 & -2x_2^2 \\
    0 & 2x_2^2 & 0
\end{pmatrix}. \quad \tag{34}
\]

They satisfy Eq. (25), and therefore the Jacobi identity holds. For example, if we choose \( (A, B) = (x_1^1, x_2^2) \) and \( C = H \), then the both sides of Eq. (10) are zero. On the other hand, if we define the Jacobi matrices in another way using the Hamiltonian \( H \) as in Eq. (27), they read
\[
\tilde{J}^1 = \begin{pmatrix}
    0 & -\frac{m_1 \omega_1^2}{2} & \frac{1}{m_1} x_1^1 \\
    \frac{m_1 \omega_1^2}{2} & 0 & -\lambda x_3^1 \\
    -\frac{1}{m_1} x_1^1 & \lambda x_3^1 & 0
\end{pmatrix}, \quad \tilde{J}^2 = \begin{pmatrix}
    0 & -\frac{m_2 \omega_2^2}{2} - \lambda x_1^1 & \frac{1}{m_2} x_2^2 \\
    \frac{m_2 \omega_2^2}{2} + \lambda x_1^1 & 0 & 0 \\
    -\frac{1}{m_2} x_2^2 & 0 & 0
\end{pmatrix}. \quad \tag{35}
\]

They do not satisfy Eq. (25), and therefore the Jacobi identity is violated. If we choose \( (A, B) = (x_1^1, x_2^2) \) and \( C = H \) again, then the left-hand side of Eq. (10) is zero, whereas the right-hand side is \(-\lambda m_1 \omega_1^2 x_1^1\). The consistent time evolution is restored in the noncanonical Hamiltonian dynamics with the Poisson matrices of Eq. (34), but remains broken in the dynamics with Eq. (35).
4. Conclusions

It is well known that the Nambu bracket does not satisfy the fundamental identity in many degrees of freedom systems. In the present paper, we have shown that the noncanonical Poisson bracket defined by the Nambu bracket could satisfy the Jacobi identity, and derived the condition for it, Eq. (25). We have given an example of two degrees of freedom system to show the breaking and restoration of the consistent time evolution in the Nambu dynamics and the corresponding noncanonical Hamiltonian dynamics.

The violation of the Jacobi identity has been an important subject in the generalized Hamiltonian dynamics [13, 14]. As for the fundamental identity, its violation in many degrees of freedom systems implies the difficulty to formulate the statistical mechanics of Nambu variables. Therefore it would be interesting to see if we could construct the effective statistical mechanics of Nambu variables by means of the noncanonical Hamiltonian representation or its analogs.

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