GROUP-LIKE PROJECTIONS FOR LOCALLY COMPACT QUANTUM GROUPS

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Abstract. Let $G$ be a locally compact quantum group. We give a 1-1 correspondence between group-like projections in $L^\infty(G)$ preserved by the scaling group and idempotent states on the dual quantum group $\hat{G}$. As a byproduct we give a simple proof that normal integrable coideals in $L^\infty(G)$ which are preserved by the scaling group are in 1-1 correspondence with compact quantum subgroups of $G$.

1. Introduction

Let $G$ be a group, $X$ a non-empty subset of $G$ and $1_X : G \to \{0, 1\}$ its characteristic function. It is easy to check that $X$ is a subgroup of $G$ if and only if

$$1_X(st)1_X(t) = 1_X(s)1_X(t) \quad (1.1)$$

for all $s,t \in G$. Let $G$ be a locally compact group and $\Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$ the comultiplication on $L^\infty(G)$:

$$\Delta(f)(s,t) = f(st) \quad \text{for all } f \in L^\infty(G).$$

Suppose that $P \in L^\infty(G)$ is a non-zero group-like projection, i.e. $P$ satisfies

$$\Delta(P)(1 \otimes P) = P \otimes P. \quad (1.2)$$

Equation (1.2) implies that $P$ is a continuous function on $G$ (see Lemma 2.6). Denoting $X = \{s \in G : P(s) = 1\}$, we have $P = 1_X$ and $1_X$ satisfies (1.1). In particular $X$ is a subgroup of $G$ and the continuity of $1_X$ implies that $X$ is open. Thus we get a 1-1 correspondence between open subgroups of $G$ and group-like projections in $L^\infty(G)$.

Let $G$ be a locally compact group. The Banach dual $C_0(G)^*$ of $C_0(G)$ equipped with the convolution product is a Banach algebra. We say that a state $\omega \in C_0(G)^*$ is an idempotent state on $C_0(G)$ if $\omega \ast \omega = \omega$. In fact, as proved by Kelley [8, Theorem 3.4], there is a 1-1 correspondence between idempotent states on $C_0(G)$ and compact subgroups of $G$, where given a compact subgroup $H \subset G$ the corresponding state is of the form $\omega(f) = \int_H f(h)dh$ for all $f \in C_0(G)$.

Let $G \ni g \mapsto R_g \in B(L^2(G))$ be the right regular representation, $\vn(G) = \{R_g : g \in G\}''$ the group von Neumann algebra of $G$ and $\hat{\Delta} : \vn(G) \to \vn(G) \otimes \vn(G)$ the comultiplication, where $\hat{\Delta}(R_g) = R_g \otimes R_g$ for all $g \in G$. It is not difficult to see that $P = \int_H R_\xi dh \in \vn(G)$ is a group-like projection in $\vn(G)$, i.e. it satisfies

$$\hat{\Delta}(P)(1 \otimes P) = P \otimes P.$$

Theorem 4.3 and Kelley’s result show that all group-like projections in $\vn(G)$ are of this form. In other words we have a 1-1 correspondence between idempotent states on $C_0(G)$ and group-like projections in $\vn(G)$. Theorem 4.1 together with Theorem 4.3 yield a generalization of this correspondence to the context of locally compact quantum groups.

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A locally compact quantum group $G$ is a virtual object that is assigned with a von Neumann algebra $L^\infty(G)$ equipped with a comultiplication $\Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$. A projection $P \in L^\infty(G)$ is called a group-like projection if

$$\Delta(P)(1 \otimes P) = P \otimes P.$$  

A locally compact quantum group $G$ is also assigned with the $C^*$-algebra $C_0(G)$ and universal $C^*$-algebra $C^*_u(G)$. Both Banach duals $C^*_u(G)^*$ and $C_0(G)^*$ are in fact Banach algebras. We say that a state $\omega \in C^*_u(G)^*$ is an idempotent state (on $G$) if $\omega \ast \omega = \omega$. As already mentioned, our results establish a 1-1 correspondence between idempotent states on $G$ and group-like projections on the dual $\hat{G}$ which are preserved by the scaling group of $\hat{G}$. As a byproduct of our study we get a relatively simple proof that normal integrable codimensions in $L^\infty(G)$ which are preserved by the scaling group are in 1-1 correspondence with compact quantum subgroups of $G$. Our proof, unlike the previous proof [3, Theorem 5.15], uses only the von Neumann techniques and does not invoke the universal $C^*$-algebra $C^*_0(G)$.

2. Preliminaries

We will denote the minimal tensor product of $C^*$-algebras with the symbol $\otimes$. The ultraweak tensor product of von Neumann algebras will be denoted by $\hat{\otimes}$. For a $C^*$-subalgebra $B$ of a $C^*$-algebra the multipliers $M(A)$ of $A$, the norm closed linear span of the set $\{ba \mid b \in B, a \in A\}$ will be denoted by $\mathcal{M}(B)$. A morphism between two $C^*$-algebras $A$ and $B$ is a $\ast$-homomorphism $\pi$ from $A$ into the multiplier algebra $M(B)$, which is non-degenerate, i.e $\pi(A)B = B$. We will denote the set of all morphisms from $A$ to $B$ by $\text{Mor}(A,B)$. The non-degeneracy of a morphism $\pi$ yields its natural extension to the unital $\ast$-homomorphism $A \to M(B)$ also denoted by $\pi$. Let $B$ be a $C^*$-subalgebra of $M(A)$. We say that $B$ is non-degenerate if $BA = A$. In this case $M(B)$ can be identified with a $C^*$-subalgebra of $M(A)$. The symbol $\sigma$ will denote the flip morphism between tensor product of operator algebras. If $X \subset A$, where $A$ is a $C^*$-algebra then $X^{\text{norm-cls}}$ denotes the norm closure of the linear span of $X$; if $X \subset M$, where $M$ is a von Neumann algebra then $X^{\sigma\text{-weak cls}}$ denotes the $\sigma$-weak closure of the linear span of $X$. For a $C^*$-algebra $A$, the space of all functionals on $A$ and the state space of $A$ will be denoted by $A^*$ and $S(A)$ respectively. The predual of a von Neumann algebra $\mathcal{N}$ will be denoted by $\mathcal{N}^\ast$. For a Hilbert space $H$ the $C^*$-algebras of compact operators on $H$ will be denoted by $\mathcal{K}(H)$. The algebra of bounded operators acting on $H$ will be denoted by $B(H)$. For $\xi, \eta \in H$, the symbol $\omega_{\xi, \eta} \in B(H)_+$ is the functional $T \mapsto \langle \xi, T \eta \rangle$.

For the theory of locally compact quantum groups we refer to [9, 10, 11]. Let us recall that a von Neumann algebraic locally compact quantum group is a quadruple $G = (L^\infty(G), \Delta, \varphi, \psi)$, where $L^\infty(G)$ is a von Neumann algebra with a coassociative comultiplication $\Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$, $\varphi$ and $\psi$ are respectively, normal semifinite faithful left and right Haar weights on $L^\infty(G)$. The GNS Hilbert space of the right Haar weight $\psi_N$ will be denoted by $L^2(G)$ and the corresponding GNS map will be denoted by $\eta_{\psi}$. Let us recall that $\eta_{\varphi} : \mathcal{N}_\varphi \to L^2(G)$, where $\mathcal{N}_\varphi = \{x \in L^\infty(G) : \psi(x^\ast x) < \infty\}$. The antipode, the scaling group and the unitary antipode will be denoted by $S$, $(\tau_t)_{t \in \mathbb{R}}$ and $R$. We have $S = R \circ \tau_{-\frac{1}{2}}$. Moreover, for all $a, b \in \mathcal{N}_\varphi$ the following holds (see [10, Corollary 5.35])

$$S((\text{id} \otimes \varphi)(\Delta(a^\ast)(1 \otimes b))) = (\text{id} \otimes \varphi)((1 \otimes a^\ast)\Delta(b)). \tag{2.1}$$

We will denote $(\sigma^\varphi_t)_{t \in \mathbb{R}}$ and $(\sigma^\psi_t)_{t \in \mathbb{R}}$ the modular automorphism groups assigned to $\varphi$ and $\psi$ respectively.

The multiplicative unitary $W^G \in B(L^2(G) \otimes L^2(G))$ is a unique unitary operator such that

$$W^G(\eta_{\varphi}(x) \otimes \eta_{\psi}(y)) = (\eta_{\varphi} \otimes \eta_{\psi})(\Delta_G(x)(1 \otimes y))$$

for all $x, y \in D(\eta_{\varphi})$; $W^G$ satisfies the pentagonal equation $W^G_{12}W^G_{13}W^G_{23} = W^G_{23}W^G_{12}$ [11, 12]. Using $W^G$, $G$ can be recovered as follows:

$$L^\infty(G) = \{[\omega \otimes \text{id}]W^G \mid \omega \in B(L^2(G))\}^{\sigma\text{-weak cls}},$$

$$\Delta_G(x) = W^G(x \otimes 1)W^G^\ast.$$
A locally compact quantum group admits a dual object $\widehat{G}$. It can be described in terms of $W^G = \sigma(W^G)^*$

$$L^\infty(\widehat{G}) = \{(\omega \otimes \text{id})W^G \mid \omega \in B(L^2(\mathbb{G}))\}^{\sigma\text{-weak cls}},$$

$$\Delta_G(x) = W^G(x \otimes 1)W^G_\ast.$$

Note that $W^G \in L^\infty(\widehat{G}) \otimes L^\infty(G)$.

**Definition 2.1.** A von Neumann subalgebra $N$ of $L^\infty(G)$ is called

- **Left coideal** if $\Delta_G(N) \subset L^\infty(G) \otimes N$;
- **Invariant subalgebra** if $\Delta_G(N) \subset N \otimes N$;
- **Baaj-Vaes subalgebra** if $N$ is an invariant subalgebra of $L^\infty(G)$ which is preserved by the unitary antipode $R$ and the scaling group $(\tau_t)_{t \in \mathbb{R}}$ of $G$;
- **Normal** if $W^G(1 \otimes N)W^G_\ast \subset L^\infty(G) \otimes N$;
- **Integrable** if the set of integrable elements with respect to the right Haar weight $\psi_G$ is dense in $N^\ast$; in other words, the restriction of $\psi_G$ to $N$ is semifinite.

If $N$ is a coideal of $L^\infty(G)$, then $\tilde{N} = N \cap L^\infty(\widehat{G})$ is a coideal of $L^\infty(\widehat{G})$ called the **codual** of $N$; it turns out that $\tilde{N} = N$ (see [9, Theorem 3.9]).

The $C^\ast$-algebraic version $(C_0(\mathbb{G}), \Delta_G)$ of a given quantum group $G$ is recovered from $W^G$ as follows

$$C_0(\mathbb{G}) = \{(\omega \otimes \text{id})W^G \mid \omega \in B(L^2(\mathbb{G}))\}^{\text{norm-cl}\sigma},$$

$$\Delta_G(x) = W^G(x \otimes 1)W^G_\ast.$$

The comultiplication can be viewed as a morphism $\Delta_G \in \text{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ and we have $W^G \in M(C_0(\widehat{G}) \otimes C_0(\mathbb{G}))$. Since $M(C_0(\widehat{G}) \otimes C_0(\mathbb{G})) \subset M(K(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}))$ we conclude that for all $x \in L^\infty(G)$

$$\Delta_G(x) = W^G(x \otimes 1)W^G_\ast \in M(K(L^2(\mathbb{G})) \otimes C_0(\mathbb{G})). \quad (2.2)$$

Replacing $\Delta_G$ with $\Delta_{G \ast \text{id}}$ we also get that

$$\Delta_G(x) \in M(C_0(\mathbb{G}) \otimes K(L^2(\mathbb{G}))). \quad (2.3)$$

for all $x \in L^\infty(G)$.

Let $H$ be a Hilbert space and $U \in M(C_0(\mathbb{G}) \otimes K(H))$ a unitary. We say that $U$ is a representation of $G$ on $H$ if

$$(\Delta_G \otimes \text{id})(U) = U_{13}U_{23}.$$

Let us recall the definition of an action of a quantum group $G$ on a von Neumann algebra.

**Definition 2.2.** A (left) action of quantum group $G$ on a von Neumann algebra $N$ is a unital injective normal $\ast$-homomorphism $\alpha : N \to L^\infty(G) \otimes N$ s.t. $(\Delta_G \otimes \text{id}) \circ \alpha = (\text{id} \otimes \alpha) \circ \alpha$. If $M \subset N$ is a von Neumann subalgebra then we say that $M$ is preserved by $\alpha$ if $\alpha(M) \subset L^\infty(G) \otimes M$.

Given an action $\alpha : N \to L^\infty(G) \otimes N$ we have (see [9, Corollary 2.6])

$$N = \{(\mu \otimes \text{id})\alpha(x) : x \in N, \mu \in L^\infty(G)\}^{\sigma\text{-weak cls}},$$

which will be referred to as the Podlés condition. We can always find a unitary representation $U \in M(C_0(\mathbb{G}) \otimes K(H))$ on a Hilbert space $H$ and a normal faithful $\ast$-homomorphism $\pi : N \to B(H)$ such that

$$(\text{id} \otimes \pi)(\alpha(x)) = U^\ast(\text{id} \otimes \pi(x))U.$$

In this case we shall say that $U$ implements the action $\alpha$. For the construction of the canonical implementation see [13].

A locally compact quantum group $G$ is assigned with a universal version [9]. The universal version $C_0^u(G)$ of $C_0(G)$ is equipped with a comultiplication $\Delta^u_G \in \text{Mor}(C_0^u(G), C_0^u(G) \otimes C_0^u(G))$. The **coaction** is a $\ast$-homomorphism $\varepsilon : C_0^u(G) \to \mathbb{C}$ satisfying $(\text{id} \otimes \varepsilon) \circ \Delta^u_G = \text{id} = (\varepsilon \otimes \text{id}) \circ \Delta^u_G$.

Multiplicative unitary $W^G \in M(C_0(\widehat{G}) \otimes C_0(G))$ admits the universal lift $W^G \in M(C_0^u(\widehat{G}) \otimes C_0(G))$. 

For the construction of the canonical implementation see [13].
C^u_0(G)). The reducing morphisms for G and \(\hat{G}\) will be denoted by \(\Lambda_G \in \text{Mor}(C^0_0(G), C_0(G))\) and \(\Lambda_{\hat{G}} \in \text{Mor}(C^u_0(\hat{G}), C_0(\hat{G}))\) respectively. We have \((\Lambda_G \otimes \Lambda_{\hat{G}})(W^G) = W^{\hat{G}}\). We shall also use the half-lifted versions of \(W^G\), \(W^G = (\text{id} \otimes \Lambda_G)(\mathcal{W}^G) \in M(C^u_0(\hat{G}) \otimes C_0(G))\) and \(\mathcal{W}^G = (\Lambda_G \otimes \text{id})(W^G) \in M(C_0(\hat{G}) \otimes C^u_0(G))\). They satisfy the appropriate versions of pentagonal equation

\[
W^G_{12}W^G_{13}W^G_{23} = W^G_{23}W^G_{12},
\]

\[
W^G_{12}W^G_{13}W^G_{23} = \mathcal{W}^G_{23}W^G_{12}.
\]

The half-lifted versions of comultiplication is denoted by \(\Delta^u_{r,u} \in \text{Mor}(C_0(\hat{G}), C_0(\hat{G}) \otimes C^u_0(\hat{G}))\) and \(\hat{\Delta}_{r,u} \in \text{Mor}(C_0(\hat{G}), C_0(\hat{G}) \otimes C^u_0(\hat{G}))\), e.g.

\[\Delta^u_{r,u}(x) = \mathcal{W}^G(x \otimes 1)\mathcal{W}^G, \quad x \in C_0(G).\]

We have

\[
(\Lambda_G \otimes \text{id}) \circ \Delta^u_G = \Delta^u_{r,u} \circ \Lambda_G,
\]

\[
(\Lambda_{\hat{G}} \otimes \text{id}) \circ \hat{\Delta}^u_G = \hat{\Delta}^u_{r,u} \circ \Lambda_{\hat{G}}.
\]

If \(U \in M(C_0(G) \otimes \mathcal{K}(H))\) is a unitary representation of G on a Hilbert space then there exists a unique unitary \(U \in M(C^u_0(G) \otimes \mathcal{K}(H))\) such that \(U = (\Lambda_G \otimes \text{id})(U)\) and

\[
(\Delta^u \otimes \text{id})(U) = U_{23}U_{12}.
\]

Actually \(U_{23} = U^*_{13}(\Delta^u_{r,u} \otimes \text{id})(U)\).

Given a locally compact quantum group G, the comultiplications \(\Delta_G\) and \(\Delta^u_G\) induce Banach algebra structures on \(L^\infty(G)_*\) and \(C^u_0(G)_*\) respectively. The corresponding multiplications will be denoted by \(*\) and \(\mathfrak{T}\). We shall identify \(L^\infty(G)_*\) with a subspace of \(C^u_0(G)_*\) when convenient. Under this identification \(L^\infty(G)_*\) forms a two sided ideal in \(C^u_0(G)_*\). Following [3], for any \(\mu \in C^u_0(G)_*\) we define a normal map \(L^\infty(G) \to L^\infty(G)_*\) such that \(x \mapsto (\text{id} \otimes \mu)(\mathcal{W}^G(x \otimes 1)\mathcal{W}^G\) for all \(x \in L^\infty(G)\). We shall use a notation \(\mu \mathfrak{T} \sigma = (\text{id} \otimes \mu)(\mathcal{W}^G(x \otimes 1)\mathcal{W}^G\).

**Theorem 2.3.** Let \(N\) be a von Neumann algebra and \(\alpha : N \to L^\infty(G) \otimes N\) an action of \(G\) on \(N\). Let \(x \in N\), \(x^* = x\) and

\[N_x = \{(\mu \otimes \text{id})(\alpha(x)) : \mu \in L^\infty(G)_*\}.\]

Then \(N_x\) is the smallest unital von Neumann subalgebra of \(N\) preserved by \(G\) and containing \(x\).

**Proof.** Let us consider

\[S = \{(\mu \otimes \text{id})(\alpha(x)) : \mu \in L^\infty(G)_*\}.
\]

Then \(S\) forms a selfadjoint subset of \(S\). In particular \(N_x\) is (unital) von Neumann algebra generated by \(S\). Noting that

\[(\omega_1 \otimes \text{id})(\alpha((\omega_2 \otimes \text{id})(\alpha(x)))) = (\omega_2 \mathfrak{T} \omega_1 \otimes \text{id})(\alpha(x)) \in N_x\]

we conclude that \(N_x\) is preserved by \(G\).

Every \(M \subset N\) preserved by \(G\) and containing \(x\) must contain \(N_x\), so it remains to prove that \(x \in N_x\). For this we may assume that \(N \subset B(H)\) and \(\alpha\) is implemented by a unitary representation \(U \in M(C_0(G) \otimes \mathcal{K}(H))\)

\[\alpha(x) = U^*(1 \otimes x)U.
\]

Unitary implementation enables us to define a morphism \(\alpha_0 \in \text{Mor}(\mathcal{K}(H), C^u_0(\hat{G}) \otimes \mathcal{K}(H))\), where \(\alpha_0(x) = U^*(1 \otimes x)U\). Thus, using natural extension of the morphism \(\alpha_0\) to \(B(H) = M(\mathcal{K}(H))\) we can further extend \(\alpha\) to an action on \(B(H)\) and we shall assume in what follows that \(N = B(H)\). As the conclusion of the above observation we see that, given a C*-algebra \(B\), an element \(X \in M(B \otimes \mathcal{K}(H))\) and a functional \(\mu \in B^*\) we have

\[\alpha((\mu \otimes \text{id})(X)) = (\mu \otimes \text{id} \otimes \text{id})(\text{id} \otimes \alpha)(X))\]

Let \(U \in M(C^u_0(G) \otimes \mathcal{K}(H))\) be the universal lift of \(U\). Let us note that

\[M := \{(\mu \otimes \text{id})(U^*(1 \otimes x)U) : \mu \in C^u_0(G)_*\}\]
is a von Neumann subalgebra of $B(H)$ containing $x$ (for the latter take $\mu = \varepsilon$) and $N_x \subset M$. Furthermore, for every $\omega \in L^\infty(G)_*$ we have
\[
(\omega \otimes \text{id})(\alpha((\mu \otimes \text{id})(U^*(1 \otimes x)U))) = (\mu \tau \omega \otimes \text{id})(\alpha(x)) \in N_x \subset M
\]
for all $t \in \pi$ (see Eq. (2.6) we use $\mu \tau \omega \otimes \text{id} \in L^\infty(G)_*$). Since the action of $G$ on $M$ satisfies the Podleś condition, $M$ is generated by elements of the form $(\mu \tau \omega \otimes \text{id})(\alpha(x)), \mu \in C_0^u(G)^*, \omega \in L^\infty(G)_*$. Since $\mu \tau \omega \in L^\infty(G)_*$, we conclude that $M \subset N_x$ and in particular $x \in N_x$. \hfill \Box

**Remark 2.4.** If in the context of Theorem 2.3 we start with a not necessary self-adjoint $x \in N$, then the smallest von Neumann subalgebra of $N$ containing $x$ is given by
\[
N_x = \{(\mu \otimes \text{id})(\alpha(x)), (\mu \otimes \text{id})(\alpha(x^*)) : \mu \in L^\infty(G)_*\}''.
\]

**Definition 2.5.** Let $N$ be a von Neumann algebra with an action $\alpha : N \to L^\infty(G) \otimes N$ of a locally compact quantum group $G$ and let $x \in N$. We say that $N$ is $G$-generated by $x$ if $N_x = N$.

A state $\omega \in S(C_0^u(G))$ is said to be an idempotent state if $\omega \tau = \omega$. For a nice survey describing the history and motivation behind the study of idempotent states see [12]. For the theory of idempotent state we refer to [13]. We shall use [13] Proposition 4] which in particular states that an idempotent state $\omega \in S(C_0^u(G))$ is preserved by the universal scaling group $\tau_t^u$ and the universal unitary antipode $R^u : C_0^u(G) \to C_0^u(G)$, i.e.
\[
\omega \circ \tau_t^u = \omega = \omega \circ R^u
\]
for all $t \in \mathbb{R}$. An idempotent state $\omega \in S(C_0^u(G))$ yields a conditional expectation $E_\omega : C_0(G) \to C_0(G)$, (see [13])
\[
E_\omega(x) = \omega \tau x
\]
for all $x \in C_0(G)$. Using (2.7), we easily get
\[
\tau_t(E_\omega(x)) = E_\omega(\tau_t(x)).
\]
Conditional expectation extends to $E_\omega : L^\infty(G) \to L^\infty(G)$ and clearly (2.8) holds for all $x \in L^\infty(G)$. The image $N = E_\omega(L^\infty(G))$ of $E_\omega$ forms a coideal in $L^\infty(G)$.

Let $H$ and $G$ be locally compact quantum groups. A homomorphism $\pi \in \text{Mor}(C_0^u(G), C_0^u(H))$ such that
\[
(\pi \otimes \pi) \circ \Delta_H^u = \Delta_H^u \circ \pi
\]
is said to define a homomorphism from $H$ to $G$. If $\pi(C_0^u(G)) = C_0^u(H)$, then $H$ is called Woronowicz - closed quantum subgroup of $G$ [2]. A homomorphism from $H$ to $G$ admits the dual homomorphism $\hat{\pi} \in \text{Mor}(C_0^u(H), C_0^u(G))$ such that
\[
(id \otimes \pi)(\mathcal{W}^G) = (\hat{\pi} \otimes id)(\mathcal{W}^H).
\]
A homomorphism from $H$ to $G$ identifies $H$ as a closed quantum subgroup of $G$ if there exists an injective normal unital $*$-homomorphism $\gamma : L^\infty(H) \to L^\infty(G)$ such that
\[
\Lambda_G \circ \hat{\pi}(x) = \gamma \circ \Lambda_H(x)
\]
for all $x \in C_0^u(H)$. Let $H$ be a closed quantum subgroup of $G$, then $H$ acts on $L^\infty(G)$ (in the von Neumann algebraic sense) by the following formula
\[
\alpha : L^\infty(G) \to L^\infty(G) \otimes L^\infty(H), \ x \mapsto V(x \otimes 1)V^*,
\]
where
\[
V = (\gamma \otimes \text{id})(\mathcal{W}^H).
\]
(2.9)
The fixed point space of $\alpha$ is denoted by
\[
L^\infty(G/H) = \{x \in L^\infty(G) \mid \alpha(x) = x \otimes 1\}
\]
and referred to as the algebra of bounded functions on the quantum homogeneous space $G/H$. If $H$ is a compact quantum subgroup of $G$, then there is a conditional expectation $E : L^\infty(G) \to L^\infty(G)$ onto $L^\infty(G/H)$ which is defined by
\[
E = (id \otimes \psi_H) \circ \alpha,
\]
(2.10)
where $\psi_{\mathbb{H}}$ is the Haar state of $\mathbb{H}$.

According to [9, Definition 2.2] we say that $\mathbb{H}$ is an open quantum subgroup of $G$ if there is a surjective normal $*$-homomorphism $\rho : L^\infty(G) \to L^\infty(\mathbb{H})$ such that

$$\Delta_{\mathbb{H}} \circ \rho = (\rho \otimes \rho) \circ \Delta_G.$$ 

Every open quantum subgroup is closed [3, Theorem 3.6]. We recall that a projection $P \in L^\infty(G)$ is a group-like projection if $\Delta_G(P)(1 \otimes P) = P \otimes P$. Note that [2] implies that $\Delta_G(P)(1 \otimes P) \in M(C_0(\mathbb{H}) \otimes K(L^2(G)))$. In particular we have

**Lemma 2.6.** Let $P \in L^\infty(G)$ be a group-like projection. Then $P \in M(C_0(\mathbb{H}))$.

There is a 1-1 correspondence between (isomorphism classes of) open quantum subgroups of $G$ and central group-like projections in $\mathcal{G}$ [3, Theorem 4.3]. The group-like projection assigned to $\mathbb{H}$, i.e. the central support of $\rho$, will be denoted by $1_{\mathbb{H}}$.

### 3. FROM IDEMPOTENT STATES TO GROUP-LIKE PROJECTIONS

Let $G$ be a locally compact quantum group and $\omega \in C_G^\sigma(G)^*$ an idempotent state on $G$ and let $E_\omega : L^\infty(G) \to L^\infty(G)$ be the conditional expectation assigned to $\omega$:

$$E_\omega(x) = \omega \tau x.$$ 

We note that

$$\eta_G(E_\omega(x)) = \eta_G(\omega \tau x) = (\text{id} \otimes \omega)(W) \eta_G(x),$$

where in the last equality we use [3, Proposition 7.4]. The element $(\text{id} \otimes \omega)(W) \in L^\infty(\hat{G})$ is a hermitian projection which we denote by $P_\omega$. In particular

$$\eta_G(E_\omega(x)) = P_\omega \eta_G(x). \quad (3.1)$$

Let $N = E_\omega(L^\infty(G))$ be the coideal assigned to $\omega$. The set

$$E_\omega(\{ (\mu \otimes \text{id})(W) : \mu \in L^\infty(\hat{G})_* \}) = \{ (P_\omega \cdot \mu \otimes \text{id})(W) : \mu \in L^\infty(\hat{G})_* \} \quad (3.2)$$

is weakly dense in $N$.

Let us recall that $\tilde{N} \subset L^\infty(\hat{G})$ denotes the codual coideal of $N$. Since $N$ is preserved by $\tau^G$, $\tilde{N}$ is preserved by $\tau^\delta_G$.

**Theorem 3.1.** Adopting the above notation we have

$$N = \{ x \in L^\infty(G) : P_\omega x = x P_\omega \}$$

and

$$\tilde{N} = \{ y \in L^\infty(\hat{G}) : \Delta_G(y)(1 \otimes P_\omega) = y \otimes P_\omega \}.$$ 

Moreover, $P_\omega \in \tilde{N}$ is a minimal central projection of $\tilde{N}$ and it satisfies

- $\tau^\delta_t(P_\omega) = P_\omega$ for all $t \in \mathbb{R}$;
- $R^\delta_t(P_\omega) = P_\omega$;
- $\sigma^{\hat{G}}_t(P_\omega) = P_\omega$ for all $t \in \mathbb{R}$;
- $\sigma^\delta_t(P_\omega) = P_\omega$ for all $t \in \mathbb{R}$;
- $\Delta^\delta_G(P_\omega)(1 \otimes P_\omega) = P_\omega \otimes P_\omega = \Delta^\delta_G(P_\omega)(P_\omega \otimes 1)$.

**Proof.** The equalities $\tau^\delta_t(P_\omega) = P_\omega$ and $R^\delta_t(P_\omega) = P_\omega$ follow easily from [2,4].

Let $x \in L^\infty(G)$. Using (3.1) we see that the condition

$$P_\omega x = x P_\omega \quad (3.3)$$

holds if and only if

$$\eta_G(E_\omega(xz)) = \eta_G(x E_\omega(z))$$

for all $z \in \mathcal{N}_\psi$. The latter is equivalent to the identity $E_\omega(xz) = x E_\omega(z)$ holding for all $z \in \mathcal{N}_\psi$. Since $\mathcal{N}_\psi \subset L^\infty(G)$ forms a dense subset of $L^\infty(G)$, we see that (3.3) is equivalent with $E_\omega(x) = x$. 


Using (3.2), we can see that \( y \in \hat{N} \) if and only if
\[
(\mu \otimes \text{id})((1 \otimes y)W(P_\omega \otimes 1)) = (\mu \otimes \text{id})(W(P_\omega \otimes y))
\]
for all \( \mu \in L^\infty(\hat{G})_\ast \). Equivalently \( y \in \hat{N} \) if and only if
\[
W^*(1 \otimes y)W(P_\omega \otimes 1) = P_\omega \otimes y
\]
which is in turn equivalent with
\[
\Delta^\hat{G}(y)(1 \otimes P_\omega) = y \otimes P_\omega.
\]
Since \( P_\omega \in \hat{N} \) we get \( \Delta^\hat{G}(P_\omega)(1 \otimes P_\omega) = P_\omega \otimes P_\omega \).

Using Podleś condition \( \hat{N} = \{(\mu \otimes \text{id})(\Delta^\hat{G}(y)) : y \in \hat{N}, \mu \in L^\infty(\hat{G}), \}^{\ast\text{-weak cls}} \) we conclude that
\( P_\omega \) is a minimal central projection in \( \hat{N} \). Indeed, for all \( y \in \hat{N} \) and \( \mu \in L^\infty(\hat{G})_\ast \) we have
\[
(\mu \otimes \text{id})(\Delta^\hat{G}(y))P_\omega = \mu(y)P_\omega = P_\omega(\mu \otimes \text{id})(\Delta^\hat{G}(y)).
\]
Thus \( \hat{N}P_\omega = \mathbb{C}P_\omega \) (i.e. \( P_\omega \) is minimal in \( \hat{N} \)) and \( P_\omega \in Z(\hat{N}) \). Minimality and centrality of \( P_\omega \in \hat{N} \) yields a unique normal character \( \varepsilon_\omega : \hat{N} \to \mathbb{C} \) such that \( yP_\omega = \varepsilon_\omega(y)P_\omega \) for all \( y \in \hat{N} \).

Using \( \Delta^\hat{G} \circ \sigma^\hat{G}_t = (\tau^\hat{G}_{\varepsilon_\omega t}) \circ \Delta^\hat{G} \) (see [10] Proposition 6.8]) we get
\[
\Delta^\hat{G}(\sigma^\hat{G}_t(P_\omega))(1 \otimes P_\omega) = (\sigma^\hat{G}_t \otimes \tau^\hat{G}_{\varepsilon_\omega t})(\Delta^\hat{G}(P_\omega))(1 \otimes P_\omega)
\]
\[
= (\sigma^\hat{G}_t \otimes \tau^\hat{G}_{\varepsilon_\omega t})(\Delta^\hat{G}(P_\omega)(1 \otimes P_\omega))
\]
\[
= \sigma^\hat{G}_{\varepsilon_\omega t}(P_\omega) \otimes P_\omega
\]
and \( \sigma^\hat{G}_{\varepsilon_\omega t}(P_\omega) \in \hat{N} \). In particular \( P_\omega \sigma^\hat{G}_{\varepsilon_\omega t}(P_\omega) = \varepsilon_\omega(\sigma^\hat{G}_t(P_\omega))P_\omega \), where \( \varepsilon_\omega(\sigma^\hat{G}_t(P_\omega)) \in \{0, 1\} \) for all \( t \in \mathbb{R} \). Since the map \( \mathbb{R} \ni t \mapsto \varepsilon_\omega(\sigma^\hat{G}_t(P_\omega)) \in \mathbb{R} \) is continuous and \( \varepsilon_\omega(\sigma^\hat{G}_t(P_\omega))|_{t=0} = 1 \), we conclude that \( P_\omega \sigma^\hat{G}_{\varepsilon_\omega t}(P_\omega) = P_\omega \), i.e. \( \sigma^\hat{G}_{\varepsilon_\omega t}(P_\omega) \geq P_\omega \) for all \( t \in \mathbb{R} \). Thus also \( \sigma^\hat{G}_{\varepsilon_\omega t}(P_\omega) \leq P_\omega \) for all \( t \in \mathbb{R} \) and \( \sigma^\hat{G}_{\varepsilon_\omega t}(P_\omega) = P_\omega \).

Since \( P_\omega \) is preserved by \( R^\hat{G} \), the identity \( \Delta^\hat{G}(P_\omega)(1 \otimes P_\omega) = P_\omega \otimes P_\omega \) implies that
\[
\Delta^\hat{G}(P_\omega)(P_\omega \otimes 1) = P_\omega \otimes P_\omega.
\]

Finally using \( \sigma^\hat{G}_t = R^\hat{G} \circ \sigma^\hat{G}_{\varepsilon_\omega t} \circ R^\hat{G} \) we get \( \sigma^\hat{G}_t(P_\omega) = P_\omega \) for all \( t \in \mathbb{R} \).

For the concept of \( \hat{G} \)-generation used in the next Lemma, see Definition [2.5].

Lemma 3.2. Let \( \omega \in C^\ast_m(G) \) be an idempotent state, \( N = E_\omega(L^\infty(G)) \) the corresponding coideal and \( \tilde{N} \subset L^\infty(\hat{G}) \) the coideal of \( N \). Then \( \tilde{N} \) is \( \hat{G} \)-generated by \( P_\omega \in \tilde{N} \).

Proof. Let us recall that \( x \in N \) if and only if \( x \in L^\infty(G) \) and \( xP_\omega = P_\omega x \). Let \( \hat{V} = (J \otimes J)W^*(J \otimes J) \in L^\infty(\hat{G}) \otimes L^\infty(\hat{G}) \) where \( J : L^2(G) \to L^2(G) \) is the Tomita-Takesaki antiunitary conjugation assigned to \( \psi \). Then for all \( y \in L^\infty(\hat{G}) \) we have
\[
\Delta^\hat{G}(y) = \hat{V}^*(1 \otimes y)\hat{V}.
\]
In particular if \( x \in L^\infty(G) \) and \( P_\omega x = xP_\omega \) then
\[
\Delta^\hat{G}(P_\omega)(1 \otimes x) = \hat{V}^*(1 \otimes P_\omega)\hat{V}(1 \otimes x) = (1 \otimes x)\Delta^\hat{G}(P_\omega).
\]
Conversely, if (3.3) holds then
\[
P_\omega \otimes P_\omega x = \Delta^\hat{G}(P_\omega)(1 \otimes x)(P_\omega \otimes 1) = (1 \otimes x)\Delta^\hat{G}(P_\omega)(P_\omega \otimes 1) = P_\omega \otimes xP_\omega
\]
and we get \( P_\omega x = xP_\omega \). In particular \( \mathcal{N} = S' \cap L^\infty(G) \), where
\[
\mathcal{S} = \{(\mu \otimes \text{id})(\Delta^\hat{G}(P_\omega)) : \mu \in L^\infty(\hat{G})_\ast \}.
\]

Let us note that \( \mathcal{S}' \) is the smallest coideal of \( L^\infty(\hat{G}) \) containing \( P_\omega \) (see Theorem [2.3]). Since \( \mathcal{N} \subset \mathcal{S}' \cap L^\infty(G) = (\mathcal{S}'')' \cap L^\infty(G) \) we get \( \mathcal{S}'' = \tilde{N} \).

Lemma 3.3. Adopting the above notation we have \( \tau^\hat{G}_t(x) = \sigma^\hat{G}_t(x) \) for all \( x \in \tilde{N} \) and \( t \in \mathbb{R} \).
Theorem 3.4. Adopting the assumptions and notation of Lemma 3.2 we have

$$P$$

From this, it follows that

$$(\text{note that for the latter we also use } \tau^\hat{G}\text{-invariance and } \sigma^\hat{G}\text{-invariance of } P_\omega).$$

Since $\hat{N}$ is $\hat{G}$-generated by $P_\omega$, we are done.

Next result is a strengthening of Lemma 3.2

**Theorem 3.4.** Adopting the assumptions and notation of Lemma 3.2 we have

$$\hat{N} = \{ (\mu \otimes \text{id})(\Delta^\hat{G}(P_\omega)) : \mu \in L^\omega(\hat{G}) \}.$$  (3.5)

**Proof.** From $\tau^\hat{G}$-invariance of $\hat{N}$ it follows that $\hat{N} \cap D(S^{-1}_G)$ is a dense subset of $\hat{N}$. Suppose that $x \in \hat{N} \cap D(S^{-1}_G)$. We shall prove that

$$\Delta^\hat{G}(P_\omega)(1 \otimes x) = \Delta^\hat{G}(P_\omega)(S^{-1}_G(x) \otimes 1).$$  (3.6)

From this, it follows that $\{ (\mu \otimes \text{id})(\Delta^\hat{G}(P_\omega)) : \mu \in L^\omega(\hat{G}) \}^{\text{weak}}$ is an ideal in $\hat{N}$ (in particular a von Neumann subalgebra of $\hat{N}$). It is also easy to check that the right hand side of Eq. (3.6) is $\hat{G}$-invariant. By ergodicity of the action of $\hat{G}$ on $\hat{N}$, we conclude that Eq. (3.5) holds (here we use the same argument as in the final part of the proof of [3, Theorem 3.3]). It remains to prove Eq. (3.6). To this end, we continue assuming that $x$ is $\tau^\hat{G}$-analytic. Note that by Corollary 3.3, it is also $\sigma^\hat{G}$-analytic. Let $a, b \in N^\hat{G}$. We compute

$$(\text{id} \otimes \tilde{\varphi})(1 \otimes a^*)(\Delta^\hat{G}(bP_\omega)(S_G^\omega(x) \otimes 1)) = S_G^\omega((\text{id} \otimes \tilde{\varphi})(1 \otimes b^*)(\Delta^\hat{G}(a^*))(1 \otimes bP_\omega))$$

$$= S_G^\omega((\text{id} \otimes \tilde{\varphi})(1 \otimes P_\omega)(\Delta^\hat{G}(x^a))(1 \otimes b))$$

$$= S_G^\omega((\text{id} \otimes \tilde{\varphi})(1 \otimes P_\omega)(\Delta^\hat{G}(x^a))(1 \otimes b))$$

$$= S_G^\omega((\text{id} \otimes \tilde{\varphi})(1 \otimes x^a)(\Delta^\hat{G}(bP_\omega))$$

$$= (\text{id} \otimes \tilde{\varphi})(1 \otimes x^a)(\Delta^\hat{G}(bP_\omega))$$

$$= (\text{id} \otimes \tilde{\varphi})(1 \otimes x^a)(\Delta^\hat{G}(bP_\omega)(1 \otimes \sigma^\hat{G}_i(x)))$$

where in the first and the fifth equality, we use Eq. (2.11) and in the second and the fourth equality, we use $\sigma^\hat{G}$-invariance of $P_\omega$. Thus we get

$$\Delta^\hat{G}(P_\omega)(S_G^\omega(x) \otimes 1) = \Delta^\hat{G}(P_\omega)(1 \otimes \sigma^\hat{G}_i(x)).$$

Replacing $x$ with $\sigma^\hat{G}_i(x)$ and using Corollary 3.3, we get (3.6) for $\tau^\hat{G}$-analytic $x$. Since the space of $\tau^\hat{G}$-analytic elements forms a core of $S^{-1}_G$, we get (3.5). \qed

Theorem 3.4 is a generalization of [3, Theorem 3.3]. Note that in the proof of [3, Theorem 3.3], which treats the case of central $P_\omega$, a small mistake was done where instead of Eq. (3.6) the following formula was derived:

$$\Delta^G(P_\omega)(1 \otimes x) = \Delta^G(P_\omega)(R_G^\omega(x) \otimes 1).$$

The next theorem was first proved in [3, Theorem 5.15]. The previous proof strongly uses the universal $C^*$-version of $G$. In what follows we give a simpler proof which is basically based on the von Neumann version of $G$.

**Theorem 3.5.** Let $N \subset L^\omega(G)$ be an integrable normal coideal preserved by $\tau^G$. Then there exists a unique compact quantum subgroup $H \subset G$ such that $N = E_\omega(L^\omega(G))$. Let $\hat{N}$ be the codual coideal. Then, since $N$ is preserved by $\tau^\hat{G}$, $\hat{N}$ is
preserved by $\tau_\hat{G}$ (see [7, Proposition 3.2]). Normality of $N$ is equivalent with $\Delta_\hat{G}(N) \subset \hat{N} \otimes \hat{N}$ (see [7, Proposition 4.3]). Moreover, using Theorem [3,1] we see that

$$S = \{(\mu \otimes \text{id})(\Delta_\hat{G}(P_\sigma)) : \mu \in L^\infty(\hat{G})\}$$

is weakly dense in $\hat{N}$. Let us note that $R_\hat{G}(\mu \otimes \text{id})(\Delta_\hat{G}(P_\sigma)) = (\text{id} \otimes \mu \circ R_\hat{G})(\Delta_\hat{G}(P_\sigma))$. Since $P_\sigma \in N$ we have $\Delta_\hat{G}(P_\sigma) \in \hat{N} \otimes N$ and we see that $R_\hat{G}(S) \subset N$. Thus we conclude that $R_\hat{G}(N) \subset N$.

Summarizing $N$ forms a Baaj-Vaes subalgebra of $L^\infty(\hat{G})$ and there exists $\mathbb{H} \subset G$ such that $N = L^\infty(\hat{G} \mid \mathbb{H})$. Since $N = L^\infty(G/H)$ is integrable, we use [5, Theorem A.3] for concluding that $\mathbb{H}$ is a compact quantum group.

\[ \square \]

4. From group-like projections to idempotent states

Let $\psi$ be an n.s.f. weight on a von Neumann algebra $N$ and $\sigma : \mathbb{R} \to \text{Aut}(N)$ the KMS-group of automorphisms assigned to $\psi$. We denote

$$T_\psi = \{x \in N_\psi \cap N_\psi^* : x \text{ is } \sigma-\text{analytic and } \sigma_z(x) \in N_\psi \cap N_\psi^* \text{ for all } z \in \mathbb{C}\}.$$

Note that if $x \in T_\psi$ then $\sigma_z(x) \in T_\psi$ for all $z \in \mathbb{C}$. Let us recall that the KMS-condition for $\sigma$ yields that if $x \in N_\psi \cap \text{Dom}(\sigma_1)$ then $\sigma_1(x)^* \in N_\psi$ and

$$\psi(x^*x) = \psi(\sigma_1(x)\sigma_1(x)^*). \quad (4.1)$$

**Lemma 4.1.** Let $x \in T_\psi$ and suppose that $y$ is $\sigma$-analytic. Then $yx \in T_\psi$.

**Proof.** Let $x \in T_\psi$. Clearly $yx$ is $\sigma$-analytic. Since $N_\psi$ forms a left ideal in $N$ we have $yx \in N_\psi$. Moreover $(yx)^*$ is also $\sigma$-analytic and

$$\psi((yx)^*(yx)^*) = \psi(\sigma_1((yx)^*)\sigma_1((yx)^*))$$

$$= \psi(\sigma_{-\frac{1}{2}}(yx)^*\sigma_{-\frac{1}{2}}(yx))$$

$$= \psi(\sigma_{-\frac{1}{2}}(yx)^*\sigma_{-\frac{1}{2}}(y)\sigma_{-\frac{1}{2}}(x))$$

$$\leq \|\sigma_{-\frac{1}{2}}(y)\|^2 \psi(\sigma_{-\frac{1}{2}}(x)^*\sigma_{-\frac{1}{2}}(x)) < \infty.$$

Thus we get $yx \in N_\psi \cap N_\psi^*$. Replacing $x$ with $\sigma_z(x)$ and $y$ with $\sigma_z(y)$ in the above reasoning, we conclude that $\sigma_z(yx) \in N_\psi \cap N_\psi^*$. Thus $yx \in T_\psi$ and we are done.

\[ \square \]

**Remark 4.2.** Let $G$ be a locally compact quantum group. In the course of the proof of the next theorem, the symbol $\hat{\eta}$ denotes the GNS map for the Haar weight $\hat{\psi}$ on $\hat{G}$. We will use the fact that if $a, b \in T_{\hat{\psi}}$, then the slice $(\mu \otimes \text{id})(W)$ for $\omega \in C_0^\infty(G)$ such that $P = (\text{id} \otimes \omega)(W)$.

**Theorem 4.3.** Let $G$ be a locally compact quantum group and let $P \in L^\infty(\hat{G})$ be a non-zero group-like projection such that $\tau_\hat{G}(P) = P$ for all $t \in \mathbb{R}$. Then there exists an idempotent state $\omega \in C_0^\infty(G)$ such that $P = (\text{id} \otimes \omega)(W)$.

**Proof.** Let us consider $\hat{N} \subset L^\infty(\hat{G})$, where

$$\hat{N} = \{y \in L^\infty(\hat{G}) : \Delta_\hat{G}(y)(1 \otimes P) = y \otimes P \text{ and } \Delta_\hat{G}(y^*)(1 \otimes P) = y^* \otimes P\}.$$

We will show that $\hat{N}$ forms a coideal in $L^\infty(\hat{G})$ and we will focus on its codual $N \subset L^\infty(G)$. Let us first note that $P \in N$ and $\hat{N}$ is $\tau_\hat{G}$-invariant. Moreover it is easy to see that $\hat{N}$ is a von Neumann subalgebra of $L^\infty(\hat{G})$. Let us check that $\hat{N}$ forms a coideal of $L^\infty(\hat{G})$. For $y \in \hat{N}$ we have

$$(\text{id} \otimes \Delta_\hat{G})(\Delta_\hat{G}(y)(1 \otimes 1 \otimes P)) = (\Delta_\hat{G} \otimes \text{id})(\Delta_\hat{G}(y))(1 \otimes 1 \otimes P)$$

$$= (\Delta_\hat{G} \otimes \text{id})(\Delta_\hat{G}(y)(1 \otimes P))$$

$$= (\Delta_\hat{G} \otimes \text{id})(y \otimes P) = \Delta_\hat{G}(y) \otimes P.$$

Similarly we show that $(\text{id} \otimes \Delta_\hat{G})(\Delta_\hat{G}(y)^*)(1 \otimes 1 \otimes P) = \Delta_\hat{G}(y)^* \otimes P$ and we get $\Delta_\hat{G}(y) \in L^\infty(\hat{G}) \otimes \hat{N}$. Repeating the reasoning presented in the fourth paragraph of the proof of Theorem 3.1 we conclude
that \( P \) is a minimal central projection of \( \hat{\mathcal{N}} \). Using \( \tau^\omega \) invariance of \( P \) and repeating the reasoning presented in the fifth paragraph of the proof of Theorem 3.1, we see that \( \sigma_t^\omega (P) = P \). In particular \( P \) is \( \sigma^\omega \) - analytic.

Let \( \mathcal{N} \subset L^\infty (\hat{\mathcal{G}}) \) denote the codual of \( \hat{\mathcal{N}} \). Since \( \hat{\mathcal{N}} \) is preserved by \( \tau^\hat{\omega} \), \( \mathcal{N} \) is preserved by \( \tau^\omega \). Moreover following backward the reasoning presented in the third paragraph of the proof of Theorem 3.1 we show that \( (P \cdot \mu \otimes \id) (W) \in \mathcal{N} \) for all \( \mu \in L^\infty (\hat{\mathcal{G}})_\ast \).

Let \( a,b \in \mathcal{T}_\psi \) and let us consider \( \mu = \mu^{\hat{\eta}(a),\hat{\eta}(b)} \in L^\infty (\hat{\mathcal{G}})_\ast \) and \( x = (P \cdot \mu \otimes \id) (W) \) (note that \( P \cdot \mu = \mu^{\hat{\eta}(a),\hat{\eta}(pb)} \)). Using Lemma 3.1 we see that \( Pb \in \mathcal{T}_{\hat{\psi}} \). In particular, as explained in Remark 4.1, \( x \in N^{\tau^\psi} \). Clearly there exists \( a,b \in \mathcal{T}_\psi \) such that the corresponding \( x \) is non-zero. Indeed, suppose the converse holds: \( (P \cdot \mu^{\hat{\eta}(a),\hat{\eta}(b)} \otimes \id) (W) = 0 \) for all \( a,b \in \mathcal{T}_\psi \). Then \( P \cdot \mu^{\hat{\eta}(a),\hat{\eta}(b)} (y) = 0 \) for all \( y \in L^\infty (\hat{\mathcal{G}}) \). Thus, taking \( y = 1 \) we get \( \langle \hat{\eta}(a) | P \hat{\eta}(b) \rangle = 0 \) for all \( a,b \in \mathcal{T}_\psi \). Since \( \hat{\eta}(\mathcal{T}_\psi) \) is dense in \( L^2 (\hat{\mathcal{G}}) \), we conclude that \( P = 0 \), contradiction. In particular \( \mathcal{N} \) contains a nonzero element \( x \in N \cap N^{\tau^\psi} \). Since \( (\psi \otimes \id) \Delta_G (x^*x) = \psi (x^*x) \) we see that \( \mathcal{N} \) contains a non-zero integrable element with respect to the action \( \Delta_G |_N \) and using [3 Proposition 3.2.] we conclude that \( \mathcal{N} \) is integrable.

Summarizing, \( \mathcal{N} \) is an integrable coideal of \( L^\infty (\hat{\mathcal{G}}) \) preserved by \( \tau^\omega \). Using [4 Theorem 4.2] we see that there exists an idempotent state \( \omega \in C^\omega_0 (\mathcal{G})^\ast \) such that \( \mathcal{N} = E_\omega(L^\infty (\mathcal{G})) \), where \( E_\omega \) is the conditional expectation assigned to \( \omega \).

Let \( P_\omega = (\id \otimes \omega)(\mathcal{W}) \). Then \( P_\omega \in \hat{\mathcal{N}} \) is a minimal central projection. Moreover,

\[
(P \cdot \mu \otimes \id) (W) = E_\omega ((P \cdot \mu \otimes \id) (W)) = (P_\omega P \cdot \mu \otimes \id) (W)
\]

for all \( \mu \in L^\infty (\hat{\mathcal{G}})_\ast \). Thus \( P = P_\omega P \) and we see that \( P_\omega \geq P \). Using the minimality of \( P_\omega \) we get \( P_\omega = P \).

\[\Box\]

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