Adaptive Output Feedback based on Closed-loop Reference Models

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Abstract—This note presents the design and analysis of an adaptive controller for a class of linear plants in the presence of output feedback. This controller makes use of a closed-loop reference model as an observer, and guarantees global stability and asymptotic output tracking.

I. INTRODUCTION

While adaptive control has been studied since the 60’s, the evolution of its use in real systems and the extent to which we fully understand its behavior has only been elucidated within the last decade. Stability of adaptive control systems came only in the 70’s, with robustness and extensions to nonlinear systems coming in the 80’s and 90’s, respectively [1–3]. Recent directions in adaptive control pertain to guaranteed transient properties by using a closed-loop architecture for reference models [4–11]. In this paper, we focus on linear Multi Input Multi Output (MIMO) adaptive systems with partial state-feedback where we show that such closed-loop reference models can lead to a separation principle based adaptive controller which is simpler to implement compared to the classical ones in [1–3]. The simplification comes via the use of reference model states in the construction of the regressor, and not the classic approach where the regressor is constructed from filtered plant inputs and outputs.

In general, the separation principle does not exist for nonlinear systems and few authors have analyzed it. Relevant work on the separation principle in adaptive control can be found in [12], [13]. The structures presented in [12], [13] are very generic, and as such, no global stability results are reported in this literature. Also, due to the generic nature of the results it is a priori assumed (or enforced through a saturation function) that the control input and adaptive update law are globally bounded functions with respect to the plant state [12, Assumption 1.2]. No such assumptions are needed in this work and the stability results are global.

The class of MIMO linear plants that we address in this paper satisfy two main assumptions. The first is that the plant state is globally bounded functions with respect to the plant state [13], [14]. No such assumptions are needed in this work and the stability results are global.

The class of plants to be addressed in this paper is

\[ x = Ax + B\Delta u, \quad y = C^T x \]  

(1)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), and \( y \in \mathbb{R}^m \). \( A \) and \( \Delta \) are unknown, but \( B \) and \( C \) are assumed to be known, and only \( y \) is assumed to be available for measurement. The goal is to design a control input \( u \) so that \( x \) tracks the closed-loop reference model state

\[ x_m = A_m x_m + Br - L(y - y_m), \quad y_m = C^T x_m \]  

(2)
where \( r \in \mathbb{R}^m \) is the reference input and \( L \) is a feedback law that will be designed suitably. The reader is referred to references [4]–[7] for its motivation.

The following assumptions are made throughout.

**Assumption 1.** The product \( C^TB \) is full rank.

**Assumption 2.** The pair \( \{A_m, C^T\} \) is observable.

**Assumption 3.** The system in \( \mathbf{(1)} \) is minimum phase\(^1\)

**Assumption 4.** There exists a \( \Theta^* \in \mathbb{R}^{n \times m} \) such that \( A + B\Lambda\Theta^*T = A_m \) and \( K^* \in \mathbb{R}^{m \times m} \) such that \( \Lambda K^* = I \).

**Assumption 5.** \( \Lambda \) is diagonal with positive elements.

**Assumption 6.** The uncertain matching parameter \( \Theta^* \), and the input uncertainty matrix \( \Lambda \) have a priori known upper bounds \( \bar{\theta}^* \triangleq \sup \|\Theta^*\| \) and \( \bar{\Lambda} \triangleq \sup \|\Lambda\| \).

**Assumption 1** corresponds to one of the main assumptions mentioned in the introduction, and that is that the first Markov Parameter is nonsingular. The system in \( \mathbf{(1)} \) is square and therefore the other main assumption mentioned in the introduction is implicitly satisfied. The extension to non-square systems is presented later in the text. Assumption 2 is necessary as our result requires the use of an observer like gain in the reference model, notice the \( L \) in \( \mathbf{(2)} \). Assumption 3 is common in adaptive systems as the KYP Lemma does not hold for plants with a right half plane transmission zero.

Assumptions \( \mathbf{4} \) and \( \mathbf{5} \) imply that the pair \( \{A, B\} \) is controllable, and are such that a matching condition is satisfied. Such an assumption is commonly made in plants where states are accessible \( \mathbf{(1)} \), but is introduced in this problem when only certain outputs are accessible. One application area where such an assumption is routinely satisfied is in the area of aircraft control \( \mathbf{(10)} \). Extensions of Assumption \( \mathbf{3} \) to the case when the underlying regressor vector is globally Lipschitz are possible as well \( \mathbf{(10)} \). Assumption \( \mathbf{5} \) can be relaxed to \( \Lambda \) symmetric and full rank. Assumption \( \mathbf{6} \) facilitates an appropriate choice of \( L \). The specifics of the control design are now addressed.

For the plant in \( \mathbf{(1)} \) and \( \mathbf{(2)} \) satisfying the six assumptions above, we propose the following adaptive controller:

\[
\begin{align*}
    u &= \Theta^T(t)x_m + K^T(t)r \\
    \hat{\Theta} &= -\Gamma_\theta x_m e_y^T \Theta M \\
    \hat{K} &= -\Gamma_k r e_y^T M
\end{align*}
\]

where \( M \triangleq C^T B \), \( e_y = y - y_m \) and \( \Gamma_\theta, \Gamma_k \) are both positive diagonal free design matrices. The matrix \( M \) is referred to as the mixing matrix throughout.

The reason for the choice of the control input in \( \mathbf{(3)} \) is simply because \( x \) is not available for measurement, and the reference model state \( x_m \) serves as an observer-state. Historically, the use of such an observer has always proved to be quite difficult, as the non-availability of the state proves to be a significant obstacle in determining a stable adaptive law. In the following, it is shown that these obstacles can be overcome for the specific class of multivariable plants that satisfy Assumptions 1 through \( \mathbf{6} \).

From \( \mathbf{(1)}, \mathbf{(2)}, \) and \( \mathbf{(4)} \), it is easy to show that the state error \( e = x - x_m \) satisfies the dynamics

\[
\dot{e} = (A_m + LC^T) e + BA(\hat{\Theta}^T x_m + \hat{K}^T r - \Theta^* T e)
\]

\[
e_y = C^T e
\]

The structure of \( \mathbf{(1)} \) and the adaptive laws suggest the use of the following Lyapunov function:

\[
V = e^T P e + \text{Tr}(\Lambda \hat{\Theta}^T \Gamma_{\theta}^{-1} \hat{\Theta}) + \text{Tr}(\Lambda \hat{K}^T \Gamma_{k}^{-1} \hat{K})
\]

\[
\text{where for now it is assumed that } P = P^T > 0 \text{ satisfies the following equation}
\]

\[
(A_m + LC^T)^T P + P(A_m + LC^T) = -Q
\]

\[
P B = C M
\]

where \( Q = Q^T > 0 \). Taking the derivative of \( \mathbf{(5)} \) and using \( \mathbf{(5)}, \mathbf{(7)} \) and \( \mathbf{(9)} \) it can be shown that

\[
\dot{V} = -e^T Qe + 2e^T P B A \Theta^* T e.
\]

Establishing sign-definiteness of \( \dot{V} \) is therefore non-trivial as the size of the sign-indefinite term in \( \mathbf{(10)} \) is directly proportional to the parametric uncertainty \( \Theta^* \), and \( P \) and \( Q \) are necessarily correlated by \( \mathbf{(9)} \). In what follows, we will show how \( L \) and \( M \) can be chosen such that a \( P \) and \( Q \) satisfying \( \mathbf{(9)} \) exist and furthermore, \( \lim_{t \to \infty} e_y(t) = 0 \). It will be shown that stability for the above adaptive system can only be insured if \( Q > 0 \) is sufficiently weighted along the \( C^T \) direction.

**III. Stability Analysis**

**A. Stability in the SISO Case**

The choice of \( L \) is determined in two steps. First, an observer gain \( L_s \) and mixing matrix \( M \) are selected so that the transfer function \( M^T C^T(sI - A - L_s C^T)B \) is Strict Positive Real (SPR)\(^2\). Then the full observer gain \( L \) is defined.

**Lemma 1.** For a SISO \((m = 1)\) system in \( \mathbf{(1)} \) satisfying Assumptions \( \mathbf{1}-\mathbf{4} \) there exists an \( L_s \) such that

\[
C^T(sI - A_m - L_s C^T)^{-1} B = \frac{a}{s + \rho}
\]

where \( \rho > 0 \) is arbitrary and \( a = C^T B \).

**Proof.** Given that \( C^T B \) is non-zero \( C^T(sI - A_m - L_s C^T)^{-1} B \) is a relative degree one transfer function. In order to see this fact, consider a system in control canonical form, and compute the coefficient for \( s^{n-1} \) in the numerator. By Assumption 2, all zeros of the transfer function \( C^T(sI - A)^{-1} B \) are stable, and since zeros are invariant under feedback, \( C^T(sI - A_m)^{-1} B \) is minimum phase as well. Assumption 2 implies that the eigenvalues of \( A_m + L_s C^T \) can be chosen arbitrarily. Therefore, one can place \( n - 1 \) of the eigenvalues of \( A_m + L_s C^T \) at the \( n - 1 \) zeros of \( C^T(sI - A_m)^{-1} B \) and its \( n \)-th eigenvalue clearly at \(-\rho\).

\( ^1 \)A MIMO system is minimum phase if all of its transmission zeros are in the strict left half of the complex plane.

\( ^2 \)M is denoted the mixing matrix, as it mixes the outputs of \( C^T(sI - A - L_s C^T)B \) so as to achieve strict positive realness.
The choice of $L_s$ in Lemma 1 results in a relative degree one transfer function with a single pole not canceling the zeros. This system however need not be SPR as $a$ may be negative; however $a + \rho > 0$ is SPR and thus the following Corollary holds.

**Corollary 1.** If $L_s$ is chosen as in (11) and $M$ selected as in (6), the SISO transfer function $M^T C^T (sI - A_m - Ls C^T)^{-1} B$ is SPR. Therefore, there exists $P = F^T > 0$ and $Q_s = Q^T_s > 0$ such that

$$
(A_m + Ls C^T) T P + P(A_m + Ls C^T) = -Q_s \quad PB = CM.
$$

(12)

**Lemma 2.** Choosing $L = L_s - \rho BM^T$ where $L_s$ is defined in Lemma 1 and $\rho > 0$ is arbitrary, the transfer function $M^T C^T (sI - A_m - LC^T)^{-1} B$ is SPR and satisfies:

$$
(A_m + LC^T) T P + P(A_m + LC^T) = -Q \quad Q \overset{\Delta}{=} Q_s + 2 \rho C M M^T C^T
$$

(13)

where $P$ and $Q_s$ are defined in (12) and $M$ is defined in (6).

**Proof.** Starting with the first equation in (12) and adding the term $-\rho \{PB M T C^T + CMB^T P\}$ on both sides of the inequality results in the following equality

$$(A_m + LC^T) T P + P (A_m + LC^T) = -Q_s - \rho \{PB M T C^T + CMB^T P\}.$$  

Using the second equality in (12) the above equality simplifies to

$$
\rho^* = \frac{\lambda^2 \bar{\theta}^2}{2 \lambda_{\min} (Q_s)}.
$$

(14)

where $\lambda$ and $\bar{\theta}^*$ are a priori known bounds defined in (5).

**Theorem 1.** The closed-loop adaptive system specified by (1), (2), (4) and (5), satisfying assumptions 1 to 6, with $\rho > 0$ has globally bounded solutions with $\lim_{t \to \infty} e_y(t) = 0$ with

$$
Q(\rho) = \begin{bmatrix} 2 \rho M M^T M A \Theta^T T \\ \Theta^* \Lambda M^T \\ Q_s \end{bmatrix} \quad \mathcal{E} = \begin{bmatrix} e_y \\ e \end{bmatrix}. 
$$

Given that $\rho > \rho^* > 0$, $2 \rho M M^T - M A \Theta^T T Q_s^{-1} \Theta^* \Lambda M^T > 0$ by (14) and $Q_s$ is positive definite by design. By Schur complement, $Q(\rho)$ is positive definite. Therefore $V \leq 0$ and thus $e_y, e, \Theta, \bar{K} \in L_\infty$. Furthermore, given that $M$ is positive definite $e_y \in L_2$. Using Barbalat Lemma it follows that $\lim_{t \to \infty} e_y(t) = 0$.

**Remark 1.** Theorem 1 implies that a controller as in (4) with the state replaced by the observer state $x_m$ will guarantee stability, thereby illustrating that the separation principle based adaptive control design can be satisfactorily deployed. It should be noted however that two key parameters $L$ and $M$ had to be suitably chosen. If $L = L_s$ then stability is not guaranteed. That is, simply satisfying an SPR condition is not sufficient for stability to hold. It is imperative that $Q$ be chosen as in (13), i.e. be sufficiently positive along the output direction $C^T \bar{C}$ so as to contend with the sign indefinite term $2e^T PB \Theta \Theta^T e$ in $V$. The result does not require that $L_s$ be chosen so that perfect pole zero cancellation occurs in Lemma 1 all that is necessary is that the phase lag of $C^T (sI - A_m - Ls C^T)^{-1} B$ never exceeds 90 degrees. Finally, it should be noted that any finite $\rho > \rho^*$ ensures stability.

**B. Stability in the MIMO Case**

Stability in the MIMO case follows the same set of steps as in the SISO case. First, an $L_s$ and $M$ are defined such that the transfer function $M^T C^T (sI - A_m - Ls C^T)^{-1} B$ is SPR. Then $L$ is defined such that the underlying adaptive system is stable. The following Lemmas mirror the results from Corollary 1 and Lemma 2.

**Lemma 3.** For the MIMO system in (1) with $M$ chosen as in (6) there always exists an $L_s$ such that $M^T C^T (sI - A_m - Ls C^T)^{-1} B$ is SPR.

**Proof.** An algorithm for the existence and selection of such an $L_s$ is given in (13).

**Remark 2.** In order to apply the results from (13), the MIMO system of interest must be 1) minimum phase and 2) $M^T C^T B$ must be symmetric positive definite. By Assumption 5, $C^T (sI - A)^{-1} B$ is minimum phase, and therefore $C^T (sI - A_m)^{-1} B$ is minimum phase as well. Also, given that $M$ is full rank, the transmission zeros of $C^T (sI - A_m)^{-1} B$ are equivalent to the transmission zeros of $M^T C^T (sI - A)^{-1} B$, see Lemma 10 in Appendix A. Therefore, condition 1 of this remark is satisfied. We now move on to condition 2.

By Assumption 1 $C^T B$ is full rank, and by the definition of $M$ in (6) it follows that $M^T C^T B = B^T C M > 0$, which is a necessary condition for $M^T C^T (sI - A_m)^{-1} B$ to be SPR. See Corollary 5 in Appendix A. A similar explicit construction of an $L_s$ such that $M^T C^T (sI - A_m - Ls C^T)^{-1} B$ is SPR can be found in (19).

**Lemma 4.** Choosing $L = L_s - \rho BM^T$ where $L_s$ is defined in Lemma 1 and $\rho > 0$ is arbitrary, the transfer function $M^T C^T (sI - A_m - LC^T)^{-1} B$ is SPR and satisfies:

$$
(A_m + LC^T) T P + P(A_m + LC^T) = -Q \quad PB = CM
$$

(17)
where $P = P^T > 0$ and $Q_s = Q_s^T > 0$ are independent of $\rho$ and $M$ is defined in $[4]$. 

**Proof.** This follows the same steps as in the proof of Lemma $[2]$. □

**Theorem 2.** The closed-loop adaptive system specified by $(1)$, $(2)$, $(4)$ and $(5)$, satisfying assumptions $1$ to $4$ with $L$ as in Lemma $[2]$, $M$ chosen as in $(6)$, and $\rho > \rho^*$ has globally bounded solutions with $\lim_{t \to \infty} e_y(t) = 0$ where $\rho^*$ is defined in $(14)$.

**Proof.** This follows the same steps as in the proof of Theorem $[1]$. □

**IV. Extensions**

In the previous section a method was presented for choosing $L$ in $(2)$ and $M$ in $(5)$ so that the overall adaptive system is stable and $\lim_{t \to \infty} e(t) = 0$. For the SISO and MIMO cases the proposed method, thus far, is a two step process. First a feedback gain and mixing matrix are chosen such that a specific transfer function is SPR. Then, the feedback gain in the first step is augmented with an additional feedback term of sufficient magnitude along the direction $BM^T$ so that stability of the underlying adaptive system can be guaranteed.

In this section, the method is extended to two different cases. In the first case, we apply this method to an LQG/LTR system, thus far, is a two step process. First a sufficient magnitude along the direction $\nu$ is chosen as in $(10)$, with $\nu > 0$, with $Q_0 = Q_0^T > 0$ in $\mathbb{R}^m$ and $\nu > 0$, with $Q_0 = Q_0 + (1 + \frac{1}{\nu}) B^T C B$ and $R_0 = \frac{\nu}{\nu + 1} R_0$.

Note that $(19)$ can also be represented as

$$A^T \dot{P}_\nu + \dot{P}_\nu A + A^T M P_\nu - P_\nu C R^{-1} C^T P_\nu + Q_\nu = 0$$

where $P_\nu$ is the solution to the Riccati Equation

$$P_\nu A^T_m + A_m P_\nu - P_\nu C R^{-1} C^T P_\nu + Q_\nu = 0$$

(19)

where $Q_0 = Q_0^T > 0$ in $\mathbb{R}^a$ and $R_0 = R_0^T > 0$ in $\mathbb{R}^m$ and $\nu > 0$, with $Q_\nu = Q_0 + (1 + \frac{1}{\nu}) B^T C B$ and $R_\nu = \frac{\nu}{\nu + 1} R_0$.

Given that our system is observable and $Q$ is symmetric and positive definite, the Riccati equation has a solution $P_\nu$ for all fixed $\nu$. We are particularly interested in the limiting solution when $\nu$ tends to zero. The Riccati equation in $(19)$ is very similar to those studied in the LTR literature, with one very significant difference. In LTR methods the state weighting matrix is independent of $\nu$ where as in our application $Q_\nu$ tends to infinity for small $\nu$.

**Lemma 5.** If Assumptions $1$ through $5$ are satisfied then $\lim_{\nu \to 0} \nu P_\nu = 0$, $\lim_{\nu \to 0} P_\nu = P_0$ where $0 < P_0^T = P_0 < \infty$, and the following asymptotic relation holds

$$P_\nu = P_0 + P_1 \nu + O(\nu^2).$$

Furthermore, there exists a unitary matrix $W$ in $\mathbb{R}^{m \times m}$ such that

$$P_0 C = BW^T \sqrt{R_0}, \quad \text{and} \quad \tilde{P}_0 B = CR_0^{-1/2} W$$

where $\tilde{P}_0 = P_0^{-1}$ and $W = (UV)^T$ with $B^T C R_0^{-1/2} = U \Sigma V$. Finally, the inverse $P_\nu = P_0^{-1}$ is well defined in limit of small $\nu$ and

$$\tilde{P}_\nu = \tilde{P}_0 + \tilde{P}_1 \nu + O(\nu^2).$$

A full proof of this result is omitted to save space. The following two facts, 1) $\lim_{\nu \to 0} \nu P_\nu = 0$, and 2) $\lim_{\nu \to 0} P_\nu = P_0$ where $0 < P_0^T = P_0 < \infty$ follow by analyzing the integral cost

$$x^T(0) P_\nu x(0) = \min \int_0^\infty x^T(\tau) Q_\nu x(\tau) + u^T(\tau) R_\nu u(\tau) \, d\tau$$

in the same spirit as was done in $(20)$. In order to apply the results from $(20)$ the system must be observable (Assumption 2), controllable (Assumptions 4 and 5), minimum phase (Assumption 3), and $C^T B$ must be full rank (Assumption 1). For a detailed analysis of the asymptotic expansions $P_\nu = P_0 + P_1 \nu + O(\nu^2)$ and $\tilde{P}_\nu = \tilde{P}_0 + \tilde{P}_1 \nu + O(\nu^2)$ see $(10)$, §13.3, Theorem 13.2, Corollary 13.1.

The update law for the adaptive parameters is then given as

$$\dot{\Theta} = -\Gamma_\Theta x_m^T e^T R_0^{-1/2} W$$

$$\dot{K} = -\Gamma_k r e_y^T R_0^{-1/2} W$$

(24)

where $W$ is defined just below $(23)$.

**Theorem 3.** The closed-loop adaptive system specified by $(1)$, $(2)$, $(4)$ and $(24)$, satisfying assumptions $1$ to $6$, with $L$ as in $(18)$, and $\nu$ sufficiently small has globally bounded solutions with $\lim_{t \to \infty} e_y(t) = 0$.

**Proof.** Consider the Lyapunov candidate

$$V = e^T \tilde{P}_0 e + \text{Tr}(\tilde{\Theta} \Theta^T) + \text{Tr}(\tilde{K} K^T)$$

Taking the derivative along the system trajectories and substitution of the update laws in $(24)$ results in

$$\dot{V} = e^T A^T \tilde{P}_0 e + e^T \tilde{P}_0 A e + e^T \tilde{P}_0 B A \Theta^T e + e^T \tilde{P}_0 B A \tilde{\Theta}^T x_m + 2 \text{Tr}(\tilde{\Theta} \Theta^T x_m e_y^T R_0^{-1/2} W)$$

+ $e^T \tilde{P}_0 B A \tilde{K}^T r + 2 \text{Tr}(\tilde{K} K^T e_y^T R_0^{-1/2} W)$

(25)

The first step in the analysis of the above expression is to replace the elements $A^T \tilde{P}_0$ and $\tilde{P}_0 A$ with bounds in terms
Remark 3. The same discussion for the SISO and MIMO cases is valid for the LQG/LTR based selection of $L$. Stability follows do to the fact that the Lyapunov candidate suitably includes the “fast dynamics” along the $e_y$ error dynamics. This fact is illustrated in [20] with the term $CR_v^{-1}CT$ appearing on the right hand, which when expanded in terms of $v$ takes the form $\frac{1}{\nu}CR_0^{-1}CT$. By directly comparing $\frac{1}{\nu}CR_0^{-1}CT$ to the term $2pCMMT \frac{C}{\nu}$ on the right hand side of (13), increasing $\rho$ and decreasing $\nu$ have the same affect on the underlying Lyapunov equations. Thus, stability is guaranteed so long as $\rho$ is sufficiently large or equivalently, $\nu$ sufficiently small.

Remark 4. The stability analysis of this method was first presented in [10]. This remark illustrates why the stability analysis presented in [10] resulted in $e(t)$ converging to a compact set for finite $\nu$. Consider the Lyapunov candidate from [10] (14.43) repeated here in

$$V = e^T P_0 e + Tr(\Lambda \Theta \Gamma^{-1} \Theta) + Tr(\Lambda \tilde{K} T \Gamma^{-1} \tilde{K}).$$

Taking the time derivative along the system trajectories

$$\dot{V} = -e^T \dot{Q}_e e - e^T CR_v^{-1}CT \dot{e} + 2e^T \dot{P}_0 B A \Theta^* T \dot{e}$$
$$+ 2e^T \dot{P}_0 B A \Theta^* T x_m + 2Tr(\Lambda \Theta^* T x_m e^T R_0^{-1/2}W)$$
$$+ 2e^T \dot{P}_0 B A \tilde{K} T \dot{r} + 2Tr(\Lambda \tilde{K} T e^T R_0^{-1/2}W),$$

which can be simplified to

$$\dot{V} \leq -e^T \dot{Q}_e e - e^T CR_v^{-1}CT \dot{e} + 2e^T \dot{P}_0 B A \Theta^* T \dot{e}$$
$$+ O(\nu)\|e\|\|x_m\| + O(\nu)\|e\|\|r\|$$
as $\nu \to 0$. Note that $x_m$ is a function of $e$. Therefore, it is difficult to bound $x_m$ before the boundedness of $e$ is obtained. Furthermore, the presence of $r(t)$ on the righthand side will always perturb $V$ away from 0 for all finite $\nu$. In Theorem 3 we overcame this issue by selecting a slightly different Lyapunov function, $\tilde{P}_0$, replaced by the limiting solution of $P_0$. It would appear to be a rather benign change to the Lyapunov candidate. This change however allows us to go from stability to the model following error converging to zero.

B. Extension to Non-square Systems

Consider dynamics of the following form

$$\dot{x} = Ax + B_1 u, \quad y = C^T x$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $p > m$. $B_1 \in \mathbb{R}^{n \times m} \text{ and } C \in \mathbb{R}^{n \times p}$ are known. $A \in \mathbb{R}^{n \times n}$ and $A \in \mathbb{R}^{m \times m}$ are unknown. To address the non-square aspect Assumption 1 is replaced with the following:

Assumption 7. Rank($C$) = $p$ and Rank($C^T B_1$) = $m$.

Again, the goal is to design a controller such that $x(t)$ follows the reference model

$$\dot{x}_m = A_m x_m + B_1 r - L e_y, \quad y_m = C^T x_m$$

where $C^T(sI - A_m)^{-1} B_1$ represents the ideal behavior corresponding to a command $r$.

Lemma 6. For a non-square system in the form of (30) and (37) that satisfies Assumptions 2, 3, and 7, there exists a $B_2 \in \mathbb{R}^{n \times (p-m)}$ such that the “squared-up” system
\( C^T(sI - A_m)^{-1}B \) is minimum phase, and \( C^T B \) is full rank, where
\[
B = [B_1 \ B_2].
\] (32)

**Proof.** The reader is referred to [21] for further details. \( \square \)

We now consider the squared-up plant \( \{A_m, B, C^T\} \) and state the lemmas corresponding to Lemma 3 and Lemma 4.

**Lemma 7.** For the MIMO system in (30) satisfying Assumptions 3, 4, and 7 with \( M \) chosen as in (6) with \( B \) as defined in (32), there exists an \( L_s \) such that \( M^T C^T(sI - A_m - L_s C^T)^{-1}B \) is SPR.

**Lemma 8.** Choosing \( L = L_s - \rho BM^T \) where \( L_s \) is defined in Lemma 8 and \( \rho > 0 \) is arbitrary, the transfer function \( M^T C^T(sI - A_m - LC)^{-1}B \) is SPR and satisfies:
\[
(A_m + LC)^TP + P(A_m + LC)^T = -Q
\]
\[
Q \triangleq Q_s + 2\rho CMM^T C^T
\]
\[
PB = CM
\] (33)

where \( P = \bar{P}^T > 0 \) and \( Q_s = \bar{Q}^T_s > 0 \) are independent of \( \rho \) and \( M \) is defined in (6).

We should note that the \( B \) matrix above corresponds to additional \( p-m \) inputs which are fictitious. The following corollary helps in determining controllers that are implementable.

**Corollary 2.** Choosing \( L = L_s - \rho BM^T \) where \( L_s \) is defined in Lemma 8 and \( \rho > 0 \) is arbitrary, the transfer function \( M^T C^T(sI - A_m - LC)^{-1}B \) is SPR and \( M \) is defined by the partition \( M = [M_1 \ M_2] \) which satisfies \( P[B_1 \ B_2] = C[M_1 \ M_2] \).

Accordingly, we propose the following adaptive law:
\[
\dot{\Theta} = -\Gamma \epsilon \epsilon^T M_1,
\]
\[
\dot{K} = -\Gamma_0 r^T y M_1
\] (34)

The following theorem shows that the overall system is globally stable and \( \lim_{t \to \infty} \epsilon(t) = 0 \).

**Theorem 4.** The closed-loop adaptive system specified by (30), (31), (32) and (33), satisfying assumptions 2 to 7, with \( B \) chosen as in (32), \( L \) as in Lemma 8 and \( M \) chosen as in Equation (6), with \( M_1 \) defined in Corollary 2 and \( \rho > \rho^* \) has globally bounded solutions with \( \lim_{t \to \infty} \epsilon(t) = 0 \), where \( \rho^* \) is defined as
\[
\rho^* = \frac{\bar{\lambda}^2 \bar{\theta}^2}{2\lambda_{min}(Q_s)\lambda_{min}(MM^T)}.
\] (35)

**Proof:** The proof follows as that of Theorem 1.

V. SIMULATION STUDY

For the simulation study we compare the performance of a combined linear and adaptive LQG controller to an LQR controller, which is full states accessible by definition. The uncertain system to be controlled is defined as
\[
x_p = A_p x_p + B_p u \quad \text{and} \quad y_p = C_p^T x_p
\]
where \( x_p = [V \ a \ q \ \theta]^T \) is the state vector for the plant consisting of: velocity in ft/s, angle of attack in radians, pitch rate in radians per second, and roll angle in radians. The control input consists of \( u = [T \ \delta]^T \), the throttle position percentage and elevator position in degrees. The measured outputs are \( y_p = [V \ a \ q \ h]^T \) where \( h \) is height measured in feet. We note that two of the states for this example are not available for measurement, the angle of attack and the pitch angle. The pitch angle is never directly measurable and is always reconstructed from the pitch rate through some filtering process. The angle of attack however is usually available for direct measurement in most classes of aircraft. There are several classes of vehicles however where this information is hard to obtain directly: weapons, munitions, small aircraft, hypersonic vehicles, and very flexible aircraft, just to name a few.

In this example we intend to control the altitude of the aircraft, and for this reason an integral error is augmented to the plant. The extended state plant is thus defined as
\[
\dot{x} = Ax + B_1u + B_2r \quad \text{and} \quad y = CTx
\]
where \( y_s = h \) is the desired altitude,
\[
x = \begin{bmatrix} x_p \\ \int(y - r) \end{bmatrix}, \quad A = \begin{bmatrix} A_p & 0_{4 \times 1} \\ C_z & 0_{1 \times 1} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_p \\ 0_{1 \times 2} \end{bmatrix},
\]
\[
B_2 = \begin{bmatrix} 0_{4 \times 1} \\ -I_{1 \times 1} \end{bmatrix}, \quad C^T = \begin{bmatrix} CT_1 & 0_{3 \times 1} \\ 0_{1 \times 4} & I_{1 \times 1} \end{bmatrix}, \quad y = \begin{bmatrix} y_p \\ \int(y - r) \end{bmatrix}
\]
The reference system is defined as
\[
\dot{x}_m = A_m x_m + B_2 r - L_\nu(y_m - \bar{y}_m) \quad \text{and} \quad y_m = CT x_m
\]
where \( A_m = A_{nom} + B_1 K_{R}^T \) with \( K_{R}^T = -R_R^{-1}B_p P_R \) the solution to the algebraic Riccati equation
\[
A_{nom}^T P_R + P_R A_{nom} - P_R B_R R_R^{-1} B_R^T P_R + Q_R = 0
\]
and
\[
A_{nom} = \begin{bmatrix} A_{nom} & 0_{4 \times 1} \\ C_z & 0_{1 \times 1} \end{bmatrix}
\]
The closed-loop reference model gain \( L_\nu \) is defined as in (18) where we have squared up the input matrix through the artificial selection of a matrix \( B_2 \) and defined \( B = [B_1 B_2] \) so that \( CTB \) is square, full rank, and \( C^T(sI - A_m)^{-1}B \) is minimum phase. The control input for the linear and adaptive LQG controller is defined as
\[
u = K_{R}^T x_m + \Theta^T x_m
\]
where the update law for the adaptive parameters is defined as
\[
\dot{\Theta} = -\Gamma_0 x_m \epsilon^T M_1,
\]
with \( M_1 \) the first \( n \) columns of \( R_0^{-1/2} W \) where \( W \) is defined just below (22). The LQR controller is defined as
\[
u = K_{R}^T x.
\]

All simulation and design parameters are given in Appendix B. Note that the free design parameter \( \Gamma \) has zero for the last entry, this is due to the fact that for an uncertainty in \( A_p \) feedback from the integral error state is not needed for a matching condition to exist. The simulation results are now presented.
Figure 1 contains the trajectories of the state space for the adaptive controller (black), linear controller (gray), reference model $x_m$ (black dotted), and reference command height (gray dashed). The reference command in height was chosen to be a filtered step, as can be seen by the gray dashed line. The plant when controlled only by the full state linear optimal controller is unable to maintain stability as can be seen by the diverging trajectories. The reference model trajectories are only visibly different from the plant state trajectories under adaptive control in the angle of attack subplot and the pitch angle subplot, the two states which are not measurable. Figure 2 contains the control input trajectories for the adaptive controller and Figure 3 contains the adaptive control parameters. There are two points to take away form the simulation example. First, the adaptive output feedback controller is able to stabilize the system while the full state accessible linear controller is not. Second, the state trajectories, control input, and adaptive parameters exhibit smooth trajectories. This smooth behavior is rigorously justified in [4] for a simpler class of closed-loop reference models.

VI. Conclusions

This note presents methods for designing output feedback adaptive controllers for plants that satisfy a states accessible matching condition, thus recovering a separation like principle for this class of adaptive systems, similar to linear plants.

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APPENDIX A

THE SPR CONDITION, KYP LEMMA AND TRANSMISSION ZEROS

This section contains relevant definitions for linear systems that were assumed to be familiar to the reader. They have been included for completeness. We begin with two definitions of positive realness. The KYP Lemma is then introduced. The section closes with a few rank conditions related to transfer matrices.

Definition 1 ([1], [22]). An $n \times n$ matrix $Z(s)$ of complex variable $s$ is Positive Real if

1) $Z(s)$ is analytic when $\text{Re}(s) > 0$ (Re $\triangleq$ real part)
2) $Z^T(s) = Z(s^*)$ when $\text{Re}(s) > 0$ ($^*$ denotes complex conjugation)
3) $Z^T(s^*) + Z(s)$ is positive semidefinite for $\text{Re}(s) > 0$.

Definition 2. An $n \times n$ matrix $Z(s)$ of complex variable $s$ is Strictly Positive Real (SPR) if $Z(s - \epsilon)$ is positive real for some $\epsilon > 0$.

Throughout the remainder of this section the following transfer matrix is referred to

$$Z(s) = C^T(sI - A)^{-1}B.$$  \hspace{1cm} (36)

Lemma 9 (Kalman Yakubovich Popov (KYP), [1] Lemma 2.5). A $Z(s)$ as defined in (36) that is minimal is SPR iff there exists $P = P^T > 0$ and $Q = Q^T > 0$ s.t. $A^TP + PA = -Q$ and $PB = C$.

Corollary 3. If $B \in \mathbb{R}^{n \times m}$, $m \leq n$ is rank $m$ and $Z(s)$ is SPR, then $C^TB = (C^TB)^T > 0$.

Proof. Given that $PB = C$, it also follows that $B^TP = C^T$ and thus $B^TPB = C^TB$ is symmetric, rank $m$ and positive.

Definition 3. For $Z(s)$ as defined in (36) that is minimal and square, the transmission zeros are the zeros of the polynomial $\psi(s) = \det(sI - A)\det[C^T(sI - A)^{-1}B]$ ([22] Theorem 1.19).

Lemma 10. For $G \in \mathbb{R}^{m \times m}$ and full rank, the location of the transmission zeros for a square $Z(s)$ in (36) are equivalent to the location of the transmission zeros of $GZ(s)$.

Proof. If $s_0 \in \mathbb{C}$ is a transmission zero, then $\det(s_0I - A)\det[GC^T(s_0I - A)^{-1}B] = 0$, and recalling the product rule for determinants $\det[GC^T(s_0I - A)^{-1}B] = \det(G)\det[C^T(s_0I - A)^{-1}B]$. $G$ is full rank and thus $\det(G) \neq 0$. Therefore, $s_0$ is a solution to $\det(s_0I - A)\det[C^T(s_0I - A)^{-1}B] = 0$ as well.