ON EXTENDABILITY OF ADDITIVE CODE ISOMETRIES

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Abstract. For linear codes, the MacWilliams Extension Theorem states that each linear isometry of a linear code extends to a linear isometry of the whole space. But, in general, this is not the situation for nonlinear codes. In this paper codes over a vector space alphabet are considered. It is proved that if the length of such code is less than some threshold value, then an analogue of the MacWilliams Extension Theorem holds. One family of unextendable code isometries for the threshold value of code length is described.

1. Introduction

One of the main objectives of coding theory is to study the metric structure of a code, and the classification of code isometries forms an important part therein.

There is a full description of linear code isometries in a Hamming space. The famous MacWilliams Extension Theorem states that each linear isometry of a linear code extends to a linear isometry of the full space. The original proof of the theorem first appeared in [11] and was later refined in [2] and [14].

Unfortunately, in the case where the linearity of a code is not required, the situation is more complicated. There are nonlinear codes with isometries that do not extend to isometries of the whole space.

In general, it is a difficult task to describe codes that have only extendable isometries. Nevertheless, considering some classes of codes, this problem can be solved in particular cases. For example, in [1], [9] and [13] authors described several families of nonlinear codes with all isometries extendable. There they also observed various classes of codes that have unextendable isometries. Among the studied families there are some subclasses of codes that achieve the Singleton bound (MDS codes, see [12, p. 20]), some subclasses of codes with equal distance between codewords (equidistant codes) and some perfect codes (see [12, Ch. §11]).

There exist generalizations of the MacWilliams Extension Theorem for linear codes over ring and module alphabets. In [15] the author, using a character theoretic approach, proved that the extension theorem for codes over rings holds when the ring is Frobenius. It was also proved in [7] using a combinatorial approach. The results were later generalized for the case of codes over modules and different types of weights in [5], [6] and [16]. The authors also described different examples of unextendable isometries.

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In this paper, we focus our attention on the case when the alphabet is a vector space and a code is a subspace of a Hamming space. The importance of such codes is due to the fact that codes, which preserve addition, with the additional requirement of a special kind of self-orthogonality, naturally describe quantum stabilizer codes (see [8]).

The results presented in the paper are the following. Theorem 4.4 determines the threshold value of the code length for which an analogue of the MacWilliams Extension Theorem for codes over vector space alphabet holds. By providing Example 3, we prove that in general this result cannot be improved by increasing the bound on the code length.

2. ADDITIVE CODES AND ISOMETRIES

Let $L$ be a finite field and let $L^n$ be a Hamming space. There is a full description of linear isometries of linear codes in $L^n$. A map $f: L^n \rightarrow L^n$ is called monomial if it acts by permutation of coordinates and multiplications of coordinates by nonzero scalars.

**Theorem 2.1** (MacWilliams Extension Theorem, see [11]). Let $C \subseteq L^n$ be a linear code. Each linear isometry of $C$ extends to a monomial map.

A general analogue of the MacWilliams Extension Theorem does not exist for nonlinear codes. This means that there exists a nonlinear code and there exists an isometry of this code that does not extend to an isometry of the whole space. In [4] a full description of the isometries of the ambient space and in [1] an example of an unextendable code isometry can be found.

**Theorem 2.2** (see [4]). Let $X$ be a finite set with at least two elements and let $n$ be a positive integer. A map $f: X^n \rightarrow X^n$ preserves the Hamming distance if and only if there exist a permutation $\pi \in S_n$ and permutations $\sigma_1, \ldots, \sigma_n \in \text{Sym}(X)$ such that for any $x = (x_1, \ldots, x_n) \in X^n$,

$$f((x_1, \ldots, x_n)) = (\sigma_1(x_{\pi(1)}), \ldots, \sigma_n(x_{\pi(n)})).$$

**Example 1** (see [1]). Suppose $X = \{0, 1\}$. Two codes in $X^4$

$$C = \{(0, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (0, 1, 1, 0)\}$$

and

$$D = \{(0, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)\}$$

are isometric, i.e. there exists an isometry $f: C \rightarrow D$. Indeed, in both codes the distance between two different codewords is 2, thus any bijection $f: C \rightarrow D$ is an isometry. For any position, there exist two different codewords in $D$ that have different values in this position. But all the codewords in $C$ have equal values on the fourth position. According to Theorem 2.2, any isometry between these two codes cannot be extended to an isometry of the space $X^4$.

As we have already noted in the introduction, the studying of the extendability property for code isometries in general is difficult and only a few families of codes and their isometries have been properly described. In this paper we focus our attention on the extendability of code isometries in the case when the alphabet is a vector space. Everywhere in the paper we consider Hamming isometries, i.e., maps that preserves the Hamming distance.
Let $K$ be a finite field with $q$ elements and let $A$ be a finite $K$-linear vector space. Consider a Hamming space $A^n$. A code is called $K$-linear if it is a $K$-linear subspace of $A^n$. A $K$-linear isometry $f : C \to A^n$ is an isometry that is a $K$-linear homomorphism of vector spaces. Evidently, $f$ is a $K$-linear isometry if and only if $f$ preserves the Hamming weight.

Note that a $K$-linear code is closed under addition. Codes with this property are called additive. In the case $\dim_K A = 1$, a $K$-linear code is linear in the classical sense. If additionally $K = \mathbb{F}_p$, where $p$ is prime, the notions of additive and linear codes coincide.

It is sometimes convenient to represent the alphabet $A$ as a finite field. Denote $k = \dim_K A$. There exists an extension field $L$ of $K$ of degree $k$, which is isomorphic to $A$ as a $K$-linear vector space.

Example 2. Let $K = \mathbb{F}_2$ and let $A = \mathbb{F}_4$ be a field extension of $K$, where $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ and $\omega + 1 = \omega^2$. Note that $A$ is a two-dimensional $K$-linear vector space. Consider two codes $C_1 = \{(0, 0, 0), (1, 1, 0), (\omega, 0, 1), (\omega^2, 1, 1)\}$ and $C_2 = \{(0, 0, 0), (0, \omega^2, \omega), (1, 0, 1), (1, \omega^2, \omega^2)\}$ in $\mathbb{F}_4^3$. All the codes are $\mathbb{F}_2$-linear. Define a map $f : C_1 \to C_2$ in the following way: $f((0, 0, 0)) = (0, 0, 0)$, $f((1, 1, 0)) = (0, \omega^2, \omega)$, $f((\omega, 0, 1)) = (1, 0, 1)$ and $f((\omega^2, 1, 1)) = (1, \omega^2, \omega^2)$. Evidently, the map $f$ is $\mathbb{F}_2$-linear and it preserves the Hamming weight. Therefore $f$ is an $\mathbb{F}_2$-linear isometry of the $\mathbb{F}_2$-linear code $C_1$ in $\mathbb{F}_4^3$. Both codes $C_1$ and $C_2$ are not $\mathbb{F}_4$-linear.

We begin with the description of all $K$-linear isometries of the ambient space $A^n$. By $\text{GL}(A)$ we denote the group of all $K$-linear invertible maps from $A$ to itself.

Definition 2.3. A map $f : A^n \to A^n$ is called $K$-monomial if there exist a permutation $\pi \in S_n$ and maps $g_1, \ldots, g_n \in \text{GL}(A)$ such that for all $a \in A^n$,

\[ f(a) = f((a_1, a_2, \ldots, a_n)) = (g_1(a_{\pi(1)}), g_2(a_{\pi(2)}), \ldots, g_n(a_{\pi(n)})). \]

Proposition 1. A map $f : A^n \to A^n$ is $K$-monomial if and only if it is a $K$-linear isometry.

Proof. The only if part is obvious. In the other direction, use Theorem 2.2. Since $K$-linear permutations of $A$ are exactly elements of $\text{GL}(A)$, any $K$-linear isometry is a $K$-monomial map. \qed

We call a $K$-linear code isometry extendable if it is a restriction of a $K$-monomial map on the code. The following example shows an unextendable code isometry.

Example 3. Let $n = |K| + 1$. Let $A = L$ be a field extension of $K$ of degree $[L : K] > 1$. Label all the elements in $K$, $x_1, x_2, \ldots, x_{|K|} \in K$. Let $\omega \in L \setminus K$. Consider two $K$-linear codes $C_1 = \langle v_1, v_2 \rangle_K$ and $C_2 = \langle u_1, u_2 \rangle_K$ in $L^n$ with

\[ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ x_1 & x_2 & \ldots & x_{|K|} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ \omega & \omega & \ldots & \omega \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \]

The $K$-linear map $f : C_1 \to C_2$, defined by $f(v_1) = u_1$ and $f(v_2) = u_2$, is an isometry. Indeed, let $\alpha v_1 + \beta v_2$ be an arbitrary element in $C_1 \setminus \{0\}$, where $\alpha, \beta \in K$. If $\beta = 0$, then $\omega(\alpha v_1 + \beta v_2) = n - 1$. If $\beta \neq 0$ then the equation $\alpha + \beta x_i = 0$, where $i \in \{1, \ldots, |K|\}$, has exactly one solution $x_i = -\alpha \beta^{-1} \in K$ and thus $\omega(\alpha v_1 + \beta v_2) = n - 1$. Therefore, all nonzero elements in $C_1$ have the weight equal to $n - 1$. It is easy to see that all nonzero codewords in $C_2$ also have the weight $n - 1$. The map $f$ maps nonzero elements of $C_1$ to nonzero elements of $C_2$ and hence is an isometry. At the same time, there is no $K$-monomial map that acts on...
C_1 in the same way as f. The last coordinates of all vectors in C_2 are always zero, but there is no such all-zero coordinate in C_1.

As it was noted by one of the referees, this example provides a counterpoint to the examples of unextendable isometries for linear codes over non-Frobenius rings in [7] and [16]. Examples similar to this, where one code has a zero column and another does not have, appeared earlier in [5] and particularly our example is a case of the construction observed in [16].

3. Tuples of spaces

Let C \subseteq A^n be a K-linear code. Let W be a K-linear vector space of the same dimension as C. Consider a parametrization of C. Let \lambda \in \text{Hom}_K(W, A^n) be a K-linear map \lambda : W \rightarrow A^n such that \text{Im} \lambda = C. We present \lambda in the form 

\lambda = (\lambda_1, \ldots, \lambda_n),

where \lambda_i is the projection of \lambda on ith coordinate, \ i \in \{1, \ldots, n\}. Obviously, \lambda_i \in \text{Hom}_K(W, A), for \ i \in \{1, \ldots, n\}.

For a map \lambda \in \text{Hom}_K(W, A^n) define its tuple of spaces \mathcal{V} = (V_1, \ldots, V_n), where 

V_i = \ker \lambda_i \subseteq W.

Example 4. Let K = F_2 and let A = F_4, as in Example 2. Let W = F_2^3 and let \lambda \in \text{Hom}_K(W, A^n) be defined as \lambda(w) = wG, for w \in W, where

\begin{equation}
G = \begin{pmatrix}
1 & 1 & 0 \\
\omega & \omega & 0 \\
1 & 0 & 1
\end{pmatrix}.
\end{equation}

Let (V_1, V_2, V_3) be the tuple of spaces of \lambda. The spaces are: V_1 = \langle (1, 0, 1) \rangle_{F_2}, V_2 = \langle (0, 0, 1) \rangle_{F_2} and V_3 = \langle (1, 0, 0), (0, 1, 0) \rangle_{F_2}.

Let f : C \rightarrow A^n be a K-linear map. Define a map \mu = f \lambda \in \text{Hom}_K(W, A^n). The following diagram is commutative,

\begin{equation}
\begin{array}{ccc}
W & \xrightarrow{\lambda} & A^n \\
\mu \downarrow & & \downarrow f \\
& & A^n
\end{array}
\end{equation}

Let \mathcal{U} = (U_1, \ldots, U_n) be the tuple of spaces of \mu.

Remark 1. Since the map \lambda : W \rightarrow A^n is injective, \ker \lambda = \bigcap_{i=1}^n \ker \lambda_i = \bigcap_{i=1}^n V_i = \{0\}. The map \mu = f \lambda is injective if and only if the map f is injective. For example, \bigcap_{i=1}^n U_i = \{0\} in the case f is an isometry.

4. The main theorem

Recall the indicator function of a subset Y of a set X is a map \mathbb{1}_Y : X \rightarrow \{0, 1\}, such that \mathbb{1}_Y(x) = 1 if x \in Y and \mathbb{1}_Y(x) = 0 otherwise.

Lemma 4.1. Let \lambda \in \text{Hom}_K(W, A^n). For any w \in W the following equality holds,

\begin{equation}
\text{wt}(\lambda(w)) = n - \sum_{i=1}^n \mathbb{1}_{V_i}(w).
\end{equation}

Proof. For any w \in W, \text{wt}(\lambda(w)) = n - |\{i \mid \lambda_i(w) = 0\}| = n - \sum_{i=1}^n \mathbb{1}_{\ker \lambda_i}(w). \quad \square
Two tuples of spaces $\mathcal{U} = (U_1, \ldots, U_n)$ and $\mathcal{V} = (V_1, \ldots, V_n)$ are equal, denoted $\mathcal{U} = \mathcal{V}$, if they represent the same multiset of subspaces in $W$. Evidently, $\mathcal{U} = \mathcal{V}$ if and only if there exists a permutation $\pi \in S_n$ such that $V_i = U_{\pi(i)}$ for all $i \in \{1, \ldots, n\}$.

**Proposition 2.** Let $C \subseteq \mathbb{A}^n$ be a $K$-linear code and let $f : C \to \mathbb{A}^n$ be a $K$-linear map. The map $f$ is an isometry if and only if

$$
\sum_{i=1}^n I_{V_i} = \sum_{i=1}^n I_{U_i}.
$$

The map $f$ is extendable if and only if $\mathcal{V} = \mathcal{U}$.

*Proof.* By definition, a map $f$ is an isometry if for all $x \in C$, $\text{wt}(x) = \text{wt}(f(x))$, or the same for a $K$-linear map $f$, for all $w \in W$, $\text{wt}(\lambda(w)) = \text{wt}(\mu(w))$. Consequently, using Lemma 4.1, $f$ is an isometry if and only if eq. (1) holds.

The map $f$ is extendable if and only if there exist a permutation $\pi \in S_n$ and maps $g_1, \ldots, g_n \in \text{GL}(A)$ such that $\mu_i = g_i \lambda_{\pi(i)}$, for all $i \in \{1, \ldots, n\}$. To prove that this is equivalent to $U_i = V_{\pi(i)}$, for all $i \in \{1, \ldots, n\}$, it is enough to show that for any two maps $\sigma, \tau \in \text{Hom}_K(W, A)$, $\text{Ker} \sigma = \text{Ker} \tau$ if and only if there exist $g \in \text{GL}(A)$ such that $\tau = g \sigma$. In one direction, $\tau = g \sigma$ implies $\text{Ker} \tau = \text{Ker}(g \sigma) = \text{Ker} \sigma$. Conversely, define $h : \text{Im} \sigma \to \text{Im} \tau$ as $h(\sigma(w)) = \tau(w)$, for all $w \in W$. The condition $\text{Ker} \sigma = \text{Ker} \tau$ implies $h$ is a well-defined isomorphism of vector spaces. Let $g \in \text{GL}(A)$ be any map such that $g = h$ on the subspace $\text{Im} \sigma$. Obviously, $\sigma = g \tau$.

Proposition 2 shows that the task of description of $K$-linear isometries can be reformulated in terms of solutions of eq. (1). We call a pair of tuples of spaces $(\mathcal{U}, \mathcal{V})$ a solution, if $\mathcal{U}$ and $\mathcal{V}$ satisfy eq. (1). If $\mathcal{U} = \mathcal{V}$, then $(\mathcal{U}, \mathcal{V})$ is a solution. Call a solution $(\mathcal{U}, \mathcal{V})$ trivial if $\mathcal{U} = \mathcal{V}$. To illustrate Proposition 2 and give an example of a nontrivial solution, we consider the following example.

**Example 5.** Let $K = \mathbb{F}_2$, let $A = \mathbb{F}_4$, let $W = \mathbb{F}_2^3$ and let $C = \text{Im} \lambda$ as in Example 4. Define an $\mathbb{F}_2$-linear map $f : C \to \mathbb{F}_2^3$ on the following way: $f((1,1,0)) = (1,1,0)$, $f((\omega,\omega,0)) = (1,0,1)$ and $f((1,0,1)) = (\omega,\omega,0)$. Consider the following matrices $G$ and $G'$,

$$
G = \begin{pmatrix} 1 & 1 & 0 \\ \omega & \omega & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad G' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \omega & \omega & 0 \end{pmatrix}.
$$

The map $\mu = f \lambda$ can be presented as $\mu(w) = wG'$, $w \in W$. Calculate the tuples of spaces $V_1, V_2, V_3 \subseteq \mathbb{F}_2^3$ and $U_1, U_2, U_3 \subseteq \mathbb{F}_2^3$. The spaces are: $V_1 = \langle (1,0,1) \rangle_{\mathbb{F}_2}$, $V_2 = \langle (0,0,1) \rangle_{\mathbb{F}_2}$ and $V_3 = \langle (1,0,0), (0,1,0) \rangle_{\mathbb{F}_2}$. In the same way, $U_1 = \langle (1,1,0) \rangle_{\mathbb{F}_2}$, $U_2 = \langle (0,1,0) \rangle_{\mathbb{F}_2}$ and $U_3 = \langle (1,0,0), (0,0,1) \rangle_{\mathbb{F}_2}$. The spaces $V_1, V_2, V_3$ and $U_1, U_2, U_3$ satisfy eq. (1). By Proposition 2, the map $f : C \to \mathbb{F}_2^3$ is an $\mathbb{F}_2$-linear isometry. Since $(V_1, V_2, V_3) \neq (U_1, U_2, U_3)$, the isometry $f$ is unextendable.

From Proposition 2, we see that a $K$-linear isometry is extendable if and only if the corresponding solution of eq. (1) is trivial. Nontrivial solutions of the equation must satisfy specific requirements on the subspace coverings.

**Theorem 4.2 (see [10]).** Let $V$ be a nonzero $K$-linear vector space of dimension $k \geq 2$ and let $U_i \subset V$ be proper subspaces, for $i \in \{1, \ldots, n\}$. If $V = \bigcup_{i=1}^n U_i$, then $n \geq |K| + 1$ and the equality is attained if and only if there exists a subspace $S \subset V$.
of dimension \(k - 2\) such that \(\{U_1, \ldots, U_{|K|+1}\}\) is the set of all hyperplanes in \(V\) that contain \(S\).

In [10] the author gives a proof which does not require any deep knowledge of finite geometry. As it was noted by one of the referees, Theorem 4.2 is also an easy consequence of a much more general result on blocking sets by Bose and Burton that one can find in [3].

**Lemma 4.3.** Let \(r, s > 0\) and let \(U_1, U_2, \ldots, V_1, \ldots, V_s\) be different \(K\)-linear spaces. Assume that \(a_1, \ldots, a_r, b_1, \ldots, b_s > 0\) and

\[
\sum_{i=1}^{r} a_i \mathbb{1}_{U_i} = \sum_{i=1}^{s} b_i \mathbb{1}_{V_i}.
\]

Then \(\max\{r, s\}\) is greater than the cardinality of \(K\).

**Proof.** Among the spaces \(V_1, \ldots, V_s, U_1, \ldots, U_r\) choose one that is maximal under inclusion. It is either \(V_i\) for some \(i \in \{1, \ldots, s\}\), or \(U_j\) for some \(j \in \{1, \ldots, r\}\). In the first case \(V_i = \bigcup_{j=1}^{r} (V_i \cap U_j)\). Since for all \(j \in \{1, \ldots, r\}\), \(U_j \neq V_i\) and \(V_i \not\subset U_j\), we have, for all \(j \in \{1, \ldots, r\}\), \(V_i \cap U_j \subset V_i\). Hence \(\dim_K V_i > 1\). From Theorem 4.2, \(r > |K|\). Similarly, in the second case \(s > |K|\).

In eq. (1) eliminate equal terms from different sides. After that, group equal terms on each side and make a renumbering of the spaces on both sides of the equation. After this procedure we get the reduced equality,

\[
\sum_{i=1}^{r} a_i \mathbb{1}_{U_i'} = \sum_{i=1}^{s} b_i \mathbb{1}_{V_i'},
\]

where \(V_i', U_j' \subset W\) are \(K\)-linear spaces, \(a_i, b_j > 0\), and the spaces \(V_i', U_j'\) are all different, for \(i \in \{1, \ldots, r\}\), \(j \in \{1, \ldots, s\}\). Note that \(r, s \leq n\), and if the solution is nontrivial, then \(r, s > 0\).

**Theorem 4.4.** Let \(K\) be a finite field and let \(A\) be a \(K\)-linear vector space of dimension greater than one. Let \(n \leq |K|\) and let \(C \subseteq A^n\) be a \(K\)-linear code. Any \(K\)-linear Hamming isometry of \(C\) extends to a \(K\)-monomial map. Moreover, for any \(n > |K|\) there exists a code in \(A^n\) that has an unextendable \(K\)-linear isometry.

**Proof.** Assume that there exist a \(K\)-linear code \(C \subseteq A^n\) and an unextendable \(K\)-linear isometry \(f : C \to A^n\). Proposition 2 implies that there exists a nontrivial solution \((\mathcal{U}, \mathcal{V})\) of eq. (1). Consider eq. (2), for which all the conditions of Lemma 4.3 are satisfied and therefore \(n \geq \max\{r, s\} > |K|\).

For \(n = |K| + 1\) we have already introduced a \(K\)-linear code in \(A^n\) with unextendable \(K\)-linear isometry in Example 2. Evidently, for \(n > |K| + 1\) such a pair of codes and an isometry are constructed by adding a set of arbitrary columns to the generator matrices of the two codes from Example 2.

Of course, the techniques developed in the paper can be used to prove the classical MacWilliams Extension Theorem for linear codes.

**Proof of the MacWilliams Extension Theorem.** Classical linear codes are the case when \(\dim_K A = 1\). Due to Proposition 2 it is enough to show that all solutions \((\mathcal{U}, \mathcal{V})\) of eq. (1) are trivial. From the definition of a tuple of spaces, for all \(i \in \{1, \ldots, n\}\), \(\dim_K V_i = \dim_K \text{Ker} \lambda_i \geq \dim_K W - \dim_K A = \dim_K W - 1\). In the same way,
\( \dim_K U_i \geq \dim_K W - 1 \). Therefore the spaces in \( \mathcal{U} \) and \( \mathcal{V} \) are either hyperplanes of \( W \) or the space \( W \) itself.

Consider eq. (2) and assume that the solution \(( \mathcal{U}, \mathcal{V} )\) is nontrivial. Without loss of generality, assume that \( U'_i \) is maximal with respect to inclusion among all the spaces in eq. (2). Calculate the restriction of eq. (2) on the space \( U'_i \),

\[
a_1 \mathbb{I}_{U'_i} + \sum_{i=2}^r a_i \mathbb{I}_{U'_i \cap U'_i} = \sum_{i=1}^s b_i \mathbb{I}_{V'_i \cap U'_i}.
\]

Now calculate the sum over all the points in \( W \) of the left and the right side. Denote \( k = \dim_K U'_i \). Since \( U_1 \) is either \( W \) or a hyperplane of \( W \) we directly calculate the size of all the intersections,

\[
a_1 |K|^k + \sum_{i=2}^r a_i |K|^{k-1} = \sum_{i=1}^s b_i |K|^{k-1}.
\]

Note that in both cases the formula is the same. Equation (2) evaluated in \( \{0\} \) gives the equality \( \sum_{i=1}^r a_i = \sum_{i=1}^s b_i \), and thus we get \( a_1 (|K| - 1) = 0 \), which is impossible. Therefore the solution of eq. (1) must be trivial. \( \square \)

5. Unextendable additive isometries

Recall \( |K| = q \). In this section we give a description of one family of nontrivial solutions of eq. (1) in the case of \( n = q + 1 \). As we mentioned above, nontrivial solutions of eq. (1) correspond to unextendable \( K \)-linear code isometries (see Proposition 2).

Let \( V \) be a \( K \)-linear space of dimension \( k \geq 2 \) and let \( S \subset V \) be a subspace of codimension 2. Define two tuples of spaces \( \mathcal{U}^* = (U^*_1, \ldots, U^*_{q+1}) \) and \( \mathcal{V}^* = (V^*_1, \ldots, V^*_{q+1}) \) in the following way. Let \( V^*_1 = \cdots = V^*_q = S \), \( V^*_{q+1} = V \) and let \( U^*_1, \ldots, U^*_{q+1} \) be all different hyperplanes in \( V \) that contain \( S \).

**Proposition 3.** Let \( \mathcal{U} \) and \( \mathcal{V} \) be two tuples of spaces such that

\[
\max_{i \in \{1, \ldots, q+1\}} \dim_K V_i > \max_{i \in \{1, \ldots, q+1\}} \dim_K U_i.
\]

The pair \(( \mathcal{U}, \mathcal{V} )\) is a nontrivial solution of eq. (1) if and only if there exist spaces \( V \) and \( S \) of dimension \( k \) and \( k - 2 \) correspondingly, such that \( \mathcal{U} = \mathcal{U}^* \) and \( \mathcal{V} = \mathcal{V}^* \).

**Proof.** Prove the only if part. Without loss of generality, assume that \( \dim_K V_{q+1} = k = \max_{i \in \{1, \ldots, q+1\}} \dim_K V_i \) and put \( V = V_{q+1} \). Obviously, \( k \geq 2 \) and from eq. (1),

\[
V = \bigcup_{i=1}^{q+1} (U_i \cap V),
\]

where \( U_i \cap V \subset V \) for all \( i \in \{1, \ldots, q+1\} \). From Theorem 4.2 there exists a subspace \( S \subset V \) such that \( \dim_K S = k - 2 \) and \( S \subset U_i \cap V \subset V \), \( \dim_K U_i \cap V = k - 1 \) and all the spaces \( U_i \cap V \) are different for \( i \in \{1, \ldots, q+1\} \).

From the conditions \( \dim_K U_i \cap V = k - 1 \), \( \dim_K \mathcal{U} \subset k \) and \( U_i \cap V \subset V \) we deduce \( U_i = U_i \cap V \subset V \), where \( i \in \{1, \ldots, q+1\} \). Hence \( \mathcal{U} = \mathcal{U}^* \). Since \( V = \bigcup_{i=1}^{q+1} U_i \) it is easy to see that \( \mathbb{I}_V + q \mathbb{I}_S = \sum_{i=1}^{q+1} \mathbb{I}_{U_i} \). Equation (1) becomes

\[
\mathbb{I}_V + \sum_{i=1}^q \mathbb{I}_{V_i} = \sum_{i=1}^q \mathbb{I}_{U_i} = \mathbb{I}_V + q \mathbb{I}_S.
\]

Subtracting \( \mathbb{I}_V \) from both sides we get

\[
\sum_{i=1}^q \mathbb{I}_{V_i} = q \mathbb{I}_S.
\]
From Lemma 4.3, considering the fact that the number of terms from both sides is less than \( q + 1 \), \( V_i = S \), for all \( i \in \{1, \ldots, q\} \). Therefore, we proved \( V = V^* \).

In the other direction, easy to see that the pair \((U^*, V^*)\) is really a nontrivial solution of eq. (1).

Having a nontrivial solution of eq. (1) for \( n = q + 1 \), which satisfies the requirements of Remark 1, we can build an unextendable \( K \)-linear code isometry of a code of length \( q + 1 \). The unextendable isometry presented in Example 3 is a particular case, which corresponds to the solution \((U^*, V^*)\) with \( V = K^2 \) and \( S = \{0\} \).

The full description of nontrivial solutions of eq. (1) for the case when \( n = q + 1 \), if \( \max_{i \in \{1, \ldots, n\}} \dim_K V_i = \max_{i \in \{1, \ldots, n\}} \dim_K U_i \), will appear in a future paper.

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