On the output of nonlinear systems excited by discrete prolate spheroidal sequences

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Abstract

Characterization of dynamical systems is an important aspect of many areas of science and engineering [1], [2]. Many inputs have been employed for system identification. Perhaps the two most prominent input categories are (i) uncorrelated broadband input, or (ii) narrowband input [3]. Broadband inputs distribute input signal energy evenly over the entire frequency domain [1], [4], while narrowband inputs focus signal energy into a small frequency interval [5]–[10]. Situations exist where the question is simply, “If a narrowband stimulus is applied as input, is there a narrowband response?” [6]. This situation arises in neuroscience in both the experimental and clinical settings. In the clinical setting there is a risk that stimulation will alter human cognitive ability [11]–[17]. In the experimental setting, stimulation may change the neural system under study, rendering subsequent inference invalid [18], or cause discomfort [19]. In all cases, it is desirable to minimize the input energy when inferring, or producing, narrowband response.

In the growing field of network science it is important to estimate the connectivity between the inputs and outputs of complicated systems [20]–[22]. Here knowledge of the network interconnectivity can be informative while detailed knowledge of the system dynamics is prohibitively difficult to obtain. In neuroscience this literature relates network properties to cognitive state, experimental condition, and pathology [23]–[27]. When attempting to detect a narrowband connection between system input and output, it is necessary to separate the relative influences of the inputs upon any one output. Generally, the strategies employed have resorted to the excitation of one input channel at a time [28]–[31].

In this paper, a theoretical characterization of the response of a Volterra MIMO system to DPSS input is provided. This characterization enables a strategy based upon multiple, simultaneous stimulation. The specific contributions are:

1) The discovery that each of the DPSSs, once passed through a nonlinear Volterra MIMO system, can be approximated by a quadratic generalized frequency response Volterra system representation [32] in the frequency domain, and by a linearly-transformed version of the input in the time domain.
2) The verification that the DPSSs, passed through a nonlinear Volterra MIMO system, remain approximately orthogonal.
3) The conditions under which Contributions 1 and 2 are valid, along with a quantification of approximation error.
4) An inner product based identification scheme exploiting Contributions 1–3 to estimate the linear narrowband connectivity of a nonlinear Volterra MIMO system.

Contribution 4 makes use of an inner-product detector to separate the relative influence of multiple simultaneous narrowband MIMO system inputs. This facilitates simultaneous network stimulation and connectivity inference.

The DPSSs and their continuous time analogues, the prolate spheroidal wave functions, are used in many areas. Example areas and publications include: time-series analysis [33], [34], signal processing [35]–[39], communication engineering [40], [41], theoretical physics [42], [43], and control [44].

Orthogonality plays a prominent role in system identification. In [4] the Wiener G-functionals result from a modified Gram-Schmidt orthogonalization procedure, facilitating cross-correlation based system identification. In [45], the kernels are orthogonalized with respect to the observed input. In [46]–[50] the Volterra kernels are expanded in terms of an orthogonal

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basis, allowing the conversion of a system of integrals to a linear-in-parameters algebraic equation. After conversion, parameters can be estimated by least squares. Standard bases for these expansions are provided by the Laguerre, Walsh, block-pulse, and Chebyshev functions (a generalized basis is considered in [50]).

The design of narrowband excitation has received considerable attention: [5]–[7], [8]–[10], [51]. In [5]–[7], [51] full system identification is addressed. In this work, motivated by the identification of linear narrowband response, emphasis is placed upon characterizing Volterra MIMO response to DPSS input. In [8]–[10] models are considered where the higher-order nonlinear system response is small and additive. This differs from the proposed work in that (i) sinusoidal input with random phases are used as opposed to the optimal in-band energy concentrated DPSSs and (ii) the nonlinear system response in the proposed work is suppressed by the use of DPSS as input.

By characterizing the Volterra MIMO response to DPSS input, progress is made towards developing improved methods of identifying linear narrowband MIMO Volterra response. To the best of our knowledge this is the first characterization of the response of a MIMO Volterra system to DPSS input, and the first use of the DPSSs for the purpose of linear, narrowband system identification of a nonlinear MIMO Volterra system.

The remainder of this paper is organized as follows. Following a review of relevant properties of the DPSS in II-A and nonlinear MIMO systems with memory in II-B the main results of the paper are presented in III & IV. In V-A DPSS excitation is used, in a comparison with sum of sinusoid and uncorrelated input to detect differences between the narrowband response of two 3rd order Volterra systems. The paper ends with a discussion in VI.

### Table I

| Eqn. | Symbol | Description |
|------|--------|-------------|
| (1)  | $W \in \mathbb{R}$ | The half-bandwidth of the DPSS energy-concentrated frequency interval. |
| (4)  | $\lambda_{\text{min}}$ | The minimum of the input DPSS in-band energy ratios. |
| (22) | $\Gamma(Q_m,*)$ | Suprema of the $Q$th order system response. |
| (23) | $V_{M,*}$ | Suprema of the Magnitudes of the input DP-SWFs. |
| (27) | $\Gamma(Q'_m,*)$ | Suprema of the $Q$th order system response’s in-band Taylor remainder. |
| (34) | $\Gamma(Q''_m,*,(0,f))$ | Suprema of the $Q$th order input-DC system responses at output frequency $f$. |
| (60) | $\Gamma(1')_{m,**}$ | Suprema of the derivative of the 1st order responses with respect to frequency. |

### II. Background & Preliminaries

For convenience Table II summarizes the key parameters used to establish the theory developed in Sections III & IV.

#### A. Discrete Prolate Spheroidal Sequences (DPSSs)

The zeroth-order discrete prolate spheroidal sequence (DPSS), or Slepian sequence [52], $v^{(0)}_t$, is the infinite, real-valued sequence that is index-limited to $[0, N - 1]$ and possesses the maximum fractional in-band energy concentration of all such sequences [52].

$$v^{(0)}_t = \arg \max_{V_N} \frac{\int_{-W}^{W} |V(f)|^2 df}{\int_{-\frac{1}{2}}^{\frac{1}{2}} |V(f)|^2 df}.$$  

(1)

where $V_N$ denotes the space of all infinite, real-valued sequences index-limited to $[0, N - 1]$, $W < \frac{1}{2}$ denotes the half-bandwidth (where, without loss of generality, a sampling period of 1 is assumed) and $V(f)$ is the discrete Fourier transform of an element of $V_N$. The higher-order DPSSs are the maximally in-band energy concentrated sequences that are mutually

1Index-limited to $[0, N - 1]$ means that any sequence element outside of the index-set $\{0, 1, \ldots, N - 1\}$ is zero.
The $k^{th}$ DPSS, $v_t^{(k)}$, satisfies eigenvalue/eigenvector equations in both the time and frequency domains. The discrete Fourier transform of the $k^{th}$ DPSS is $V_k(f)$, the $k^{th}$ discrete prolate spheroidal wave function (DPSWF),

$$
V_k(f) = \sum_{t=-\infty}^{\infty} v_t^{(k)} e^{-i2\pi ft},
$$

(2)

and

$$
V_k(f) = \sum_{t=0}^{N-1} v_t^{(k)} e^{-i2\pi ft}.
$$

(3)

The $k^{th}$ DPSWF satisfies the frequency domain eigenfunction equation

$$
\lambda_k V_k(f) = \int_{-W}^{W} D_N(f - f') V_k(f') df'.
$$

(4)

Here $D_N(f)$ is a Dirichlet ('sinc')-type kernel,

$$
D_N(f) = \frac{\sin(N\pi f)}{\sin(\pi f)} e^{-i\pi f(N-1)},
$$

(5)

and $\lambda_k$ is the eigenvalue associated with the $k^{th}$ DPSS. The $k^{th}$ eigenvalue is near one (i.e., $\lambda_k \approx 1$) for $k$ less than approximately $2NW$ (twice the dimensionless time-bandwidth product). The eigenvalues, $\lambda_k$, $k = 1, 2, \ldots$ monotonically decrease with increasing $k$ and are equal to the fraction of the DPSS energy within the $(-W, W)$ band of frequencies. That is

$$
\int_{-W}^{W} |V_k|^2 df' = \lambda_k.
$$

(6)

Thus the in-band signal energy of $v_t^{(k)}$ is $\lambda_k$. The DPSWFs are in-band orthogonal, i.e.,

$$
\int_{-W}^{W} V_k(f) V_k^*(f) df = \lambda_k \delta_{k,k'}.
$$

(7)

An additional inequality used to bound the in-band inner-product of two DPSWFs is

$$
\int_{-W}^{W} |V_k(f) V_k^*(f')| df' \leq \sqrt{\lambda_k \lambda_{k'}}.
$$

(8)

The DPSSs are functions of two parameters: the length, $N$, of the signal, and the user specified half-bandwidth parameter, $W$. For convenience, these dependencies are implied.

### B. Volterra Expansion of Nonlinear MIMO System

Let $H$ be a nonlinear, time-invariant system formulated in discrete time with $M$ inputs and $M'$ outputs. Assume that $H$ admits a Volterra expansion such that the $m^{th}$ output, $y_{m,t}$, evaluated at time-index $t$, can be expressed as,

$$
y_{m,t} = y_{m}^{(0)} + \sum_{m'=1}^{M} \sum_{t'=1}^{\infty} \gamma_{m,m',t,t'} u_{m',t-t'} + \sum_{m_1,m_2=1}^{M} \sum_{t_1,t_2=1}^{\infty} \gamma_{m_1,m_2,t_1,t_2} u_{m_1,t-t_1} u_{m_2,t-t_2} + \ldots + \sum_{m_1,...,m_Q=1}^{M} \sum_{t_1,...,t_Q=1}^{\infty} \gamma_{m_1,...,m_Q,t_1,...,t_Q} \prod_{j=1}^{Q} u_{m_j,t-t_j}.
$$

(9)

Here, $\gamma_{m_1,...,m_a,t_1,...,t_a} \in \mathbb{R}$, $|\gamma_{m_1,...,m_a,t_1,...,t_a}| < \infty$, is a finite, order $a$ volterra kernel. It relates the product of $a$ inputs from channels $m_1$ to $m_a$ to the system output, $y_{m,t}$, on channel $m$ at time-index $t$. When forming this product input channel

Note: $v^{(0)} \in \ell^2(\infty)$. By truncation, $v^{(0)}$ defines $\tilde{v}^{(0)} \in \ell^2(N)$, and $V \in C^2((-\frac{1}{2}, \frac{1}{2}))$ and $V \in C^2(-W, W)$ is defined by the restriction of $V$ to $(-W, W)$. Each of these elements satisfies orthogonality relations with respect to the canonical inner-product associated with the Hilbert space to which they belong: $\ell^2(\infty)$, $\ell^2(N)$, $C^2(-W, W)$ and $C^2((-\frac{1}{2}, \frac{1}{2}))$. 
$m_j$ is evaluated at time-index $t_j$, for $j = 1, \ldots, a$. The number of input channels $M$ need not equal the number of output channels $M'$. The system, $\mathcal{H}$, is completely characterized by the collection of Volterra kernels $\gamma_{m,m_1, \ldots, m_j}^{(j)}$.

To facilitate development, (9) is re-written in terms of vectors. Specifically,

$$y_{m,t} = y_{m}^{(0)} + \sum_{m' = 1}^{M} \sum_{t' = 1}^{\infty} \gamma_{m,m',t'}^{(1)} u_{m',t-t'} + \sum_{m_2 = 1}^{M} \sum_{t_2 = 1}^{\infty} \gamma_{m,m_2,t_2}^{(2)} u_{m_1,t-t_1} u_{m_2,t-t_2} + \ldots + \sum_{m_Q = 1}^{M} \sum_{t_Q = 1}^{\infty} \gamma_{m,m_Q,t_Q}^{(Q)} \prod_{j=1}^{Q} u_{m_j,t-t_j},$$

(10)

where $m_Q = [m_1, m_2, \ldots, m_Q]^T$, $t_Q = [t_1, t_2, \ldots, t_Q]^T$, and $1_Q$ is a $Q$ dimension vector of ones. For convenience, in the following $y_{m}^{(0)}$ is set equal to zero (see Remark 3). In Section III investigation focuses upon the nature of $y_{m,t}$ when the input to the system is specified to be the DPSSs. This investigation is facilitated by the frequency domain representation of (10). Following the development in [58], [59], in Appendix A it is shown that:

$$Y_m(f) = \sum_{q=1}^{Q} T_{m,q}(f),$$

(11)

where

$$T_{m,1}(f) = \sum_{m' = 1}^{M} \Gamma_{m,m'}^{(1)}(f) U_{m'}(f),$$

(12)

$$T_{m,2}(f) = \sum_{m_1, m_2 = 1}^{M} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Gamma_{m_1,m_2}^{(2)}(f_1, f-f_1) \times U_{m_1}(f_1) U_{m_2}(f-f_1) \, df_1,$$

(13)

and

$$T_{m,Q}(f) = \sum_{m_Q = 1}^{M} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Gamma_{m,m_Q}^{(Q)}(f_Q-1, f-f_Q^T 1_Q-1) U_{m_Q}(f-f_Q^T 1_Q-1) \prod_{j=1}^{Q-1} U_{m_j}(f_j) \, df_Q-1.$$

(14)

Here, $f_Q = [f_1, f_2, \ldots, f_Q]^T$, and the Volterra kernels are specified to be causal. The discrete Fourier transform is taken over all time. Specifically:

$$U(f) = \sum_{t=-\infty}^{\infty} u_t e^{-j2\pi ft}.$$

(15)

As in the time domain, the collection of generalized frequency response functions, $\Gamma_{m,m}(f_{j-1}, f-f_{j-1}^T 1_{j-1})$, completely specify the nonlinear, time-invariant MIMO system $\mathcal{H}$.

### III. Suppression of Higher-Order MIMO Response to DPSS Input

The main result centers on investigating the orthogonality of the outputs of the nonlinear system $\mathcal{H}$ when its inputs are set to be the DPSSs. Under conditions to be described, the higher-order system responses to DPSS inputs are effectively suppressed. Recall the Volterra expansion of $\mathcal{H}$ (10) and set the $k^{th}$ input to $v_l^{(k)}$, the $k^{th}$ order DPSS for $k \leq K$. Here, $K$ is chosen such that $v_l^{(k)}$, $k \leq K$ possesses energy concentration within $(-W, W)$ near one. From (8), this is equivalent to specifying that the
first $K$ DPSS eigenvalues, $\lambda_k$, are as close to 1 as possible. Thus, of all DPSSs index-limited to the interval $[0, N - 1]$ with a given time-bandwidth parameter $NW$, these sequences are the most energy concentrated DPSSs within the frequency interval $(-W, W)$. When the number of input channels $M$ is greater than $K$, the remaining $M - K$ channel inputs are set to zero. In this way, each input is specified to be a strongly in-band energy-concentrated DPSS, or is otherwise set to zero and does not contribute to the output. Consider the $Q^{th}$ order Volterra kernel frequency response, $T_{m,Q}(f)$, due to this DPSS input:

$$T_{m,Q}(f) = \sum_{m_Q=1}^{M} \int_{-\frac{W}{2}}^{\frac{W}{2}} \Gamma_{m,m_Q}^{(Q)} \left( f_{Q-1}, f - f_{Q-1}^{T}1_{Q-1} \right) V_{m_Q} \left( f - f_{Q-1}^{T}1_{Q-1} \right) \prod_{j=1}^{Q-1} V_{m_j} \left( f_j \right) \, df_{Q-1} .$$

(16)

Decompose (16) into in-band and out-of-band components,

$$T_{m,Q}(f) = T_{m,Q}^{(i)}(f) + T_{m,Q}^{(o)}(f) ,$$

(17)

where

$$T_{m,Q}^{(i)}(f) = \sum_{m_Q=1}^{M} \int_{-W}^{W} \Gamma_{m,m_Q}^{(Q)} \left( f_{Q-1}, f - f_{Q-1}^{T}1_{Q-1} \right) V_{m_Q} \left( f - f_{Q-1}^{T}1_{Q-1} \right) \prod_{j=1}^{Q-1} V_{m_j} \left( f_j \right) \, df_{Q-1} ,$$

(18)

and

$$T_{m,Q}^{(o)}(f) = \sum_{m_Q=1}^{M} \int_{-\frac{W}{2}}^{\frac{W}{2}} \Gamma_{m,m_Q}^{(Q)} \left( f_{Q-1}, f - f_{Q-1}^{T}1_{Q-1} \right) V_{m_Q} \left( f - f_{Q-1}^{T}1_{Q-1} \right) \prod_{j=1}^{Q-1} V_{m_j} \left( f_j \right) \, df_{Q-1} .$$

(19)

The symbol $\frac{1}{2} \int_{-\frac{W}{2}}^{\frac{W}{2}} dx$ is used to specify the sum of two integrals. It is equal to $\int_{-\frac{W}{2}}^{\frac{W}{2}} dx + \int_{\frac{W}{2}}^{\frac{W}{2}} dx$. In Appendix (B) it is shown that,

$$\left| T_{m,Q}(f) - T_{m,Q}^{(i)}(f) \right| \leq A_{m,M,Q}(\lambda_{min}, V_{M,*}, \Gamma_{m,*}^{(Q)}),$$

(20)

where

$$A_{m,M,Q}(\lambda_{min}, V_{M,*}, \Gamma_{m,*}^{(Q)}) = (1 - \lambda_{min})^{(Q-1)/2} V_{M,*} M^{Q} \Gamma_{m,*}^{(Q)} .$$

(21)

Here $V_{M,*}$ and $\Gamma_{m,*}^{(Q)}$ are respectively, bounds on the magnitude of the DPSWF (3) and on the magnitude of the $Q^{th}$ order Volterra kernel frequency response (16). These bounds are suprema over the appropriate domains:

$$\Gamma_{m,*}^{(Q)} = \sup_{m_Q \in \{1, 2, ..., M\}^{Q}} \left| \Gamma_{m,m_Q}^{(Q)} \left( f_{Q-1}, f - f_{Q-1}^{T}1_{Q-1} \right) \right| ,$$

(22)
and
\[
V_{M,*} = \sup_{m_Q \in \{1, 2, ..., M\}^Q, \ f_{Q-1} \in (-\frac{1}{2}, \frac{1}{2})^{Q-1}} \left| V_{m_Q} \left( f - \mathbf{1}_{Q-1}^T f_{Q-1} \right) \right|
\]  
(23)

From (21), when the product \(V_{M,*} \Gamma_{m,*}^{(Q)}\) is small relative to \(M^Q (1 - \lambda_{\text{min}})^{(Q-1)/2}\) the Qth order Volterra frequency response is approximated by \(T_{m,Q}^{(i)}(f)\). To offer some perspective, for DPSS with \(NW = 5, N = 200, K = 6\), yields \(1 - \lambda_{\text{min}} \approx 7 \times 10^{-5}\). Here \(W = NW/N\) is equal to 0.025.

Seeking to approximate the system’s Qth order response in terms of its response at zero input-frequency, consider the in-band portion of the system’s Qth order frequency response, \(T_{0,m,Q}^{(i)}(f)\). To this end, let
\[
T_{0,m,Q}^{(i)}(f) = \sum_{m_Q = 1}^{M} \Gamma_{m,m_Q}^{(Q)}(0_{Q-1}, f) J(W, Q, f, m_Q),
\]  
(24)

where
\[
J(W, Q, f, m_Q) = \int_{-W}^{W} V_{m_Q} \left( f - \mathbf{1}_{Q-1}^T f_{Q-1} \right) \prod_{j=1}^{Q-1} V_{m_j}(f_j) df_j.
\]  
(25)

It is useful to define the upper-bound, \(J_B\):
\[
|J(W, Q, f, m_Q)| \leq J_B(W, Q, M),
\]
\[
= \sup_{m_Q \in \{1, ..., M\}^Q, \ f \in (-W,W)} \int_{-W}^{W} V_{m_Q} \left( f - \mathbf{1}_{Q-1}^T f_{Q-1} \right) \prod_{j=1}^{Q-1} V_{m_j}(f_j) df_j.
\]  
(26)

Let
\[
\Gamma_{m,*}^{(Q)'} = \sup_{m_Q \in \{1, ..., M\}^Q, \ f_{Q-1} \in (-W,W)^{Q-1}} \int_{0}^{1} \left| \nabla f_{Q-1} \left\{ \Gamma_{m,m_Q}^{(Q)} (tf_{Q-1}, f - tf_{Q-1}^T f_{Q-1}) \right\} \right| dt.
\]  
(27)

By Taylor expanding (Appendix C), it can be seen that,
\[
\left| T_{m,Q}^{(i)}(f) - T_{0,m,Q}^{(i)}(f) \right| \leq B_{m,M,Q}(W, \Gamma_{m,*}^{(Q)'})
\]  
(28)

where
\[
B_{m,M,Q}(W, \Gamma_{m,*}^{(Q)'}) = W^{Q-1} \Gamma_{m,*}^{(Q)'} M^Q J_B(W, Q, M).
\]  
(29)
The integral specified in (25), $J(W, Q, f, m_Q)$, is approximately upper-bounded by (32). In this simulation $J(W, Q, f, m_Q)$ is numerically computed for $N = 256$ (black) and $N = 1000$ (red) as a function of Volterra order $Q$. The multi-index $m_Q$ is picked randomly 25 times for each order and $N$. The frequency $f$ is varied over the interval $(-W, W)$. Each mark corresponds to the maximum of the absolute value of $J$ (circles) over these frequencies, or the value of the bound $J_B$ (pluses). The solid lines are computed using the approximation (32) and usefully approximate an upper-bound for the actual maximum of the absolute integrals.

Combining (20) and (28) obtain,

\[
\begin{align*}
|T_{m,Q}(f) - T^{(i)}_{0,m,Q}(f)| & \leq A_{m,M,Q}(\lambda_{\min}, V_{M,*}, \Gamma_{m,*}) + B_{m,M,Q}(W, \Gamma^{(Q)}_{m,*}) . \\
\end{align*}
\]

(30)

Thus, (24) approximates the $Q^{th}$ order Volterra response to DPSS input. Contributing to (24) is the $Q-1$ dimensional integral over the DPSWFs, $J(W, Q, f, m_Q)$ specified in (25). Because

\[
\int_W^{-W} |V_j(f')|^2 df' = \lambda_j \approx 1 ,
\]

(31)

\[
|J(W, Q, f, m_Q)| \leq J_B(W, Q, M) ,
\]

\[
\approx (2W)^{\frac{Q-2}{2}} , \quad Q > 1 .
\]

(32)

Figure 1 shows (32), $J_B$ and $\max_{f \in (-W, W)} |J(W, Q, f, m_Q)|$ for various $Q$ and randomly chosen $m_Q$. 

Eqn. (32) approximately (exactly in the plot) upper-bounds $J(W, Q, f, m_Q^{(i)})$ for simulated conditions. Using (32), the in-band $Q^{th}$ order Volterra system response can be bounded:

\[
\begin{align*}
|T^{(i)}_{0,m,Q}(f)| & \leq \\
\sum_{m_Q=1}^{M} & \left| \Gamma^{(Q)}_{m,m_Q}(0, f) \right| |J(W, Q, f, m_Q)| , \\
\leq (2W)^{\frac{Q-2}{2}} & \sum_{m_Q=1}^{M} \left| \Gamma^{(Q)}_{m,m_Q}(0, f) \right| + \delta' , \\
\leq (2W)^{\frac{Q-2}{2}} & M^Q \Gamma_{m,*}(0, f) + \delta' .
\end{align*}
\]

(33)

\[\text{Here, } NW = 4, N = 256 \text{ (black curve) or } N = 1000 \text{ (red curve), for } Q = \{3, 4, 5, 6\} \text{ and } M = 6. \text{ Twenty five random draws (with replacement) from the sequence with elements one through six are made. The } i^{th} \text{ draw, } m_Q^{(i)} , \text{ is used to compute } J(W, Q, f, m_Q^{(i)}) \text{ for } f \text{ varied over the interval } (-W, W). \text{ For each } \max_{f \in (-W, W)} |J(W, Q, f, m_Q^{(i)})| \text{ a mark in Fig. 1 is made.}\]
Here

\[
\Gamma^{(Q)}_{m,*,*}(0,f) = \sup_{m,Q \in \{1, 2, \ldots, M\}^Q} |\Gamma^{(Q)}_{m,m_Q}(0,f)| .
\]

(34)

In the following the case \( \delta' = 0 \) is explicitly investigated. Combining (30) and (33),

\[
|T_{m,Q}(f)| \leq C_{m,M,Q}(f, \lambda_{\min}, W, V_{M,*}, \Gamma^{(Q)}_{m,*}, \Gamma^{(Q')}_{m,*}, \Gamma^{(Q)}_{m,*} (0,f)) ,
\]

(35)

\[
|T_{m,Q}(f)| \leq A_{m,M,Q}(\lambda_{\min}, V_{M,*}, \Gamma^{(Q)}_{m,*}) + B_{m,M,Q}(W, \Gamma^{(Q')}_{m,*}) + (2W)^{Q/2} M^Q \Gamma^{(Q)}_{m,*} (0,f) ,
\]

\[
= (1 - \lambda_{\min})^{Q/2} V_{M,*} M^Q \Gamma^{(Q)}_{m,*} + 2^{Q/2} W^{Q/2} M^Q \Gamma^{(Q')}_{m,*} + (2W)^{Q/2} M^Q \Gamma^{(Q)}_{m,*} (0,f) , \quad Q \geq 2 .
\]

(36)

Thus the \( Q^{th} \) order Volterra kernel system response due to the DPSS inputs is bounded by a sum of three positive terms. The first of these terms is due to the restriction of the integrals to \((-W, W)\). This term is largest when the DPSWFs are poorly concentrated within \((-W, W)\). In this case \( \lambda_{\min} \) is small. The second term in (36) results from the Taylor expansion of the truncated integrals. The third term bounds the in-band contribution to the \( Q^{th} \) order Volterra kernel response. Both the second and third term feature a further \( W \) dependence owing to the shifted multi-dimensional integral of the product DPSWFs (25), independent of system properties. For a fixed number of system inputs \( M \), and a fixed \( \lambda_{\min} \),

\[
|T_{m,Q}(f)| = O \left( (W)^{Q/2} \right) . \quad (37)
\]

Since \( W < \frac{1}{2} \), the higher-order responses are suppressed relative to the linear and quadratic responses when the Volterra system is driven with DPSS input.

IV. ORTHOGONALITY OF MIMO RESPONSE TO DPSS INPUT

The system output \( Y_m(f) \) is approximated by the contributions from \( T_{m,1}(f) \) and \( T_{m,2}(f) \),

\[
|Y_m(f) - T_{m,1}(f) - T_{m,2}(f)| \leq y_m^{(0)} + \sum_{j=3}^{Q} |T_{m,j}(f)| .
\]

(38)

As in (31) consider the situation where the DC response \( y_m^{(0)} \) is set to zero and the bound (32) is exact (i.e. \( \delta' = 0 \)). As a consequence of the higher order Volterra kernel suppression demonstrated in (31) the system output \( Y_m(f) \) can be further bounded:

\[
|Y_m(f) - T_{m,1}(f) - T_{m,2}(f)| \leq \sum_{j=3}^{Q} |T_{m,j}(f)| ,
\]

\[
\leq \sum_{j=3}^{Q} C_{m,M,j}(f, \lambda_{\min}, W, V_{M,*}, \Gamma^{(j)}_{m,*}, \Gamma^{(j')}_{m,*}, \Gamma^{(j)}_{m,*} (0,f)) .
\]

(39)

In general the Volterra expansion of \( \mathcal{H} \) may have a finite or infinite number of terms. Any Volterra expansion, however, may be represented as an infinite series by adding kernels of value zero if needed. This representation is referred to as the infinite Volterra expansion.

**Definition 1.** Exponential DPSS System

Let \( \mathcal{H} \) be a nonlinear, time-invariant MIMO system equal to the Volterra expansion in the limit as \( Q \to \infty \). Then the
for all \( j \in \mathbb{Z}^+ \) and all \( m \in \{1, \ldots, M'\} \).

**Definition 2. \( \epsilon \)-Quadratic System**

Fix an \( \epsilon > 0 \) and a Volterra system \( H \). Let \( T_{m,1}(f) \) and \( T_{m,2}(f) \) be the first two Volterra kernel responses contributing to the \( m \)-th output channel of \( H \), as defined in Equation (11). Then \( H \) is an \( \epsilon \)-Quadratic System if for every output channel \( m \in \{1,\ldots,M'\} \) the channel output \( Y_m(f) \) satisfies

\[
|Y_m(f) - T_{m,1}(f) - T_{m,2}(f)| \leq \epsilon.
\]

**Theorem 1. Exponential DPSS System, \( \epsilon \)-Quadratic Equivalence**

Let \( H \) be an exponential DPSS System. Then there exists an \( \epsilon > 0 \) such that \( H \) is an \( \epsilon \)-quadratic system.

**Proof:** The proof follows from direct calculation. Consider,

\[
|Y_m(f) - T_{m,1}(f) - T_{m,2}(f)| \leq \\
V_{M,\star} \sum_{j=3}^{\infty} \Gamma_{m,\star}^{(j)} M^j \left(1 - \lambda_{\min}\right)^{j/2} + \\
W^{-2} \sum_{j=3}^{\infty} \Gamma_{m,\star}^{(j)} M^j \left(1 - \lambda_{\min}\right)^{j/2} + \\
\sum_{j=3}^{\infty} \Gamma_{m,\star}^{(j)} (0,f) M^j (2W)^{-j/2},
\]

\[
\leq V_{M,\star} \sum_{j=3}^{\infty} \frac{\alpha^j}{j!} M^j \left(1 - \lambda_{\min}\right)^{j/2} + \\
2^{-\frac{j}{2}} W^{-2} \sum_{j=3}^{\infty} \frac{\gamma^j}{j!} M^j (2W)^{-j/2} + \\
(2W)^{-1} \sum_{j=3}^{\infty} \frac{\beta^j}{j!} M^j (2W)^{-j/2}.
\]

Continuing,

\[
|Y_m(f) - T_{m,1}(f) - T_{m,2}(f)| \leq \\
V_{M,\star} \left[ e^{\alpha M \sqrt{1 - \lambda_{\min}}} - (1 + \alpha M \sqrt{1 - \lambda_{\min}} + \alpha^2 M^2 (1 - \lambda_{\min})) + \\
W^{-2} 2^{-\frac{j}{2}} e^{\sqrt{2}\gamma M W^2} - \\
\left(1 + \sqrt{2}\gamma M W^2 + 2\gamma^2 M^2 W^2\right) + \\
(2W)^{-1} e^{\sqrt{2}W \beta M} - \\
\left(1 + \sqrt{2}W \beta M + 2W \beta^2 M^2\right)\right],
\]

\[
\equiv \epsilon.
\]
Remark 1. Linear System Response

An \( \epsilon \)-Quadratic System driven by DPSSs approximates a linear system for \( W \ll 1 \). The system response in the time-domain due to the first-order Volterra kernel is,

\[
\mathcal{F}^{-1}\{T_{m,1}(f)\} = \left| \sum_{m_1=1}^{M} \int_{-\frac{W}{2}}^{\frac{W}{2}} \Gamma_{m,m_1}^{(1)}(f)V_{m_1}(f)e^{i2\pi ft} df \right|, \\
\approx \left| \sum_{m_1=1}^{M} \Gamma_{m,m_1}^{(1)}(0) \int_{-W}^{W} V_{m_1}(f)e^{i2\pi ft} df \right|, \\
\leq \sum_{m_1=1}^{M} \left| \Gamma_{m,m_1}^{(1)}(0) \right| \int_{-W}^{W} \left| V_{m_1}(f) \right| df, \\
= O\left(\sqrt{W}\right),
\]

since

\[
\int_{-W}^{W} \left| V_{m_1}(f) \right|^2 df \approx 1, \\
\left| V_{m_1}(f) \right|^2 \approx \frac{1}{2W}, \\
\left| V_{m_1}(f) \right| \approx (2W)^{-\frac{1}{2}},
\]

and

\[
\int_{-W}^{W} \left| V_{m_1}(f) \right| df \approx 2W(2W)^{-\frac{1}{2}}, \\
= \sqrt{2W}.
\]

Similarly,

\[
\mathcal{F}^{-1}\{T_{m,2}(f)\} = \left| \sum_{m_1,m_2=1}^{M} \int_{-\frac{W}{2}}^{\frac{W}{2}} \int_{-\frac{W}{2}}^{\frac{W}{2}} \Gamma_{m,m_1,m_2}^{(2)}(f_1,f_1-f_2) \times \right. \\
V_{m_1}(f-f_1)V_{m_2}(f_1)e^{i2\pi ft} df_1df \left|,
\approx \left| \sum_{m_1,m_2=1}^{M} \Gamma_{m,m_1,m_2}^{(2)}(0,0) \times \\
\int_{-W}^{W} \int_{-W}^{W} V_{m_1}(f-f_1)V_{m_2}(f_1)e^{i2\pi ft} df_1df \right|, \\
\leq \sum_{m_1,m_2=1}^{M} \left| \Gamma_{m,m_1,m_2}^{(2)}(0,0) \right| \times \\
\int_{-W}^{W} \int_{-W}^{W} \left| V_{m_1}(f-f_1)V_{m_2}(f_1) \right| df_1df, \\
= O\left(W\right).
\]

As motivated in the introduction, in parameter identification applications aimed at determining the existence of connections between MIMO system inputs and outputs, the focus is upon the inner-product, \( I_{m,m'} \), between the \( m \) channel output, \( Y_m(f) \), of the \( \epsilon \)-Exponential DPSS system and the \( m' \)-order test DPSS. Let,

\[
I_{m,m'} = \int_{-\frac{W}{2}}^{\frac{W}{2}} Y_m(f)V_{m'}^*(f) df.
\]
Once again, split the integral into in-band and out-of-band components:

\[ I_{m,m'} = \int_{-\frac{W}{2}}^{\frac{W}{2}} Y_m(f) V_{m'}^*(f) \, df , \]

\[ = \int_{-W}^{W} Y_m(f) V_{m'}^*(f) \, df + \int_{-\frac{W}{2}}^{\frac{W}{2}} Y_m(f) V_{m'}^*(f) \, df , \]

\[ = I_{m,m'}^{(i)} + I_{m,m'}^{(o)}. \] (48)

Because \( T_{m,1}(f) + T_{m,2}(f) \) approximates the output of an \( \epsilon \)-exponential DPSS system, we can now state the following Lemma.

**Lemma 1.** The inner-product, \( I_{m,m'} \), can be approximated by the inner product of \( V_{m'} \) with the first and second order Volterra responses of \( H \).

**Proof:** Let \( X_{m,m'} \) be the inner-product of the first two Volterra system responses with the input \( V_{m'}(f) \). The out-of-band component \( X_{m,m'}^{(o)} \) of the inner product, \( X_{m,m'} \), is approximated by

\[ X_{m,m'}^{(o)} = X_{m,m'}^{(o,1)} + X_{m,m'}^{(o,2)} , \] (49)

where

\[ X_{m,m'}^{(o,1)} = \left| \sum_{m''=1}^{M} \int_{-\frac{W}{2}}^{\frac{W}{2}} \Gamma_{m,m''}^{(1)}(f) V_{m''}(f) V_{m'}^*(f) \, df \right| , \]

\[ \leq \Gamma_{m,m''}^{(1)} \sum_{m''=1}^{M} \sqrt{1 - \lambda_{m''}} \sqrt{1 - \lambda_{m'}}, \]

\[ \leq M \Gamma_{m,m''}^{(1)} (1 - \lambda_{\text{min}}). \] (50)

The second-order term is bounded in a similar fashion,

\[ X_{m,m'}^{(o,2)} = \left| \sum_{m_1, m_2}^{M} \int_{-\frac{W}{2}}^{\frac{W}{2}} \left[ \int_{-W}^{W} \Gamma_{m_1, m_2}^{(2)}(f_1, f - f_1) \right] V_{m_2}(f - f_1) V_{m_1}(f) V_{m'}^*(f) \, df \right| , \]

\[ \leq M^2 \Gamma_{m,m''}^{(2)} \sqrt{1 - \lambda_{m'}}. \] (51)

Then, the in-band contribution \( X_{m,m'}^{(i)} \) to \( X_{m,m'} \) is approximated as:

\[ X_{m,m'}^{(i)} = X_{m,m'}^{(i,1)} + X_{m,m'}^{(i,2)} . \] (52)

By Taylor expanding the linear transfer function \( \Gamma_{m,m'}^{(1)}(f) \), the in-band contribution \( X_{m,m'}^{(i,1)} \) can be expressed as

\[ X_{m,m'}^{(i,1)} = \int_{-W}^{W} \sum_{m''=1}^{M} \Gamma_{m,m''}^{(1)}(f) V_{m''}(f) V_{m'}^*(f) \, df , \]

\[ = \left( \sum_{m''=1}^{M} \Gamma_{m,m''}^{(1)}(0) \lambda_{m''} \delta_{m',m''} \right) + R , \]

\[ = \Gamma_{m,m''}^{(1)}(0) \lambda_{m'} + R. \] (53)

The remainder \( R \), due to Taylor’s theorem is,

\[ R = \sum_{m''=1}^{M} \frac{d\Gamma_{m,m''}^{(1)}(\zeta)}{d\zeta} \int_{-W}^{W} f V_{m''}(f) V_{m'}^*(f) \, df , \] (54)
for some $\zeta \in (0, f)$. Next, the second order contribution can be upper bounded as follows:

$$
\left| X^{(i, 2)} \right| = \left| \int_{-W}^{W} \sum_{m_1, m_2 = 1}^{M} \Gamma^{(2)}_{m_1, m_2} (f_1, f - f_1) \times V_{m_2} (f - f_1) V_{m_1} (f) \, df \, df \right| ,
$$

$$
\leq \sqrt{2W M^2} \Gamma^{(2)}_{m, *}. \tag{55}
$$

Then

$$
\left| X^{(i)}_{m, m'} - \Gamma^{(1)}_{m, m'} (0) \lambda_{m'} \right| - \sqrt{2W M^2} \Gamma^{(2)}_{m, *} \leq |R|, 
$$

$$
\leq \sum_{m'' = 1}^{M} \left| d \Gamma^{(1)}_{m, m''} (\zeta) \right| \left| \int_{-W}^{W} |f V_{m''} (f) V_{m'} (f)| \, df \right| ,
$$

$$
\leq W \sum_{m'' = 1}^{M} \left| \Gamma^{(1)}_{m, m''} (\zeta) \right| \sqrt{\lambda_{m''} \lambda_{m'}},
$$

$$
\leq W \sum_{m'' = 1}^{M} \left| \frac{d \Gamma^{(1)}_{m, m''} (\zeta)}{d \zeta} \right|. \tag{56}
$$

Moving $\sqrt{2W M^2} \Gamma^{(2)}_{m, *}$ to the right-hand side of (56) yields

$$
\left| X_{m, m'} - \Gamma^{(1)}_{m, m'} (0) \lambda_{m'} \right| \leq |R| + \sqrt{2W M^2} \Gamma^{(2)}_{m, *} + M \Gamma^{(1)}_{m, *} (1 - \lambda_{min}) + M^2 \Gamma^{(2)}_{m, *} \sqrt{1 - \lambda_{m'}},
$$

$$
\leq W \sum_{m'' = 1}^{M} \left| \Gamma^{(1)}_{m, m''} (\zeta) \right| + \left( \sqrt{2W} + \sqrt{1 - \lambda_{m'}} \right) M^2 \Gamma^{(2)}_{m, *} + M \Gamma^{(1)}_{m, *} (1 - \lambda_{min}). \tag{57}
$$

Finally,

$$
\left| I_{m, m'} - X_{m, m'} \right| \leq \int_{-\frac{W}{2}}^{\frac{W}{2}} \left| Y_{m} (f') - T_{m, 1} (f') - T_{m, 2} (f') \right| \times |V_{m'} (f')| \, df',
$$

$$
\leq \epsilon \sqrt{\lambda_{m'}}, \tag{58}
$$

so that

$$
\left| I_{m, m'} - \Gamma^{(1)}_{m, m'} (0) \lambda_{m'} \right| \leq \epsilon \sqrt{\lambda_{m'}} + W \sum_{m'' = 1}^{M} \left| \frac{d \Gamma^{(1)}_{m, m''} (\zeta)}{d \zeta} \right| + M \Gamma^{(1)}_{m, m''} (1 - \lambda_{min}),
$$

$$
\leq \epsilon \sqrt{\lambda_{min}} + W M \Gamma^{(1)}_{m, *} + M \Gamma^{(1)}_{m, m''} (1 - \lambda_{min}) + \left( \sqrt{2W} + \sqrt{1 - \lambda_{m'}} \right) M^2 \Gamma^{(2)}_{m, *}. \tag{59}
$$
Here

\[ \Gamma_{m',m'}^{(1)/}\delta = \sup_{\zeta \in (0, f)} \left| \frac{d\Gamma_{m',m'}^{(1)/}\delta(\zeta)}{d\zeta} \right| . \quad (60) \]

Eqn. (59) establishes an upper bound on the difference between the inner-product of the system output on channel \( m \) with the DPSS input on channel \( m' \) for the case \( \delta' = 0 \) (see (53)). This situation is only approximately valid (see (52), and Eqns. (55)-(59) are more generally (\( \delta' \neq 0 \)) approximate. The bound (59) provides conditions under which the DPSS remain orthogonal, even after passing through \( \mathcal{H} \). As described in the Introduction, and as will be demonstrated in V below, this result provides the basis for a detector capable of separating the relative influences of inputs on the system outputs.

V. LINEAR NARROWBAND IDENTIFICATION

The theory in Sections III-IV can be applied for the purposes of identifying the existence of linear narrowband connections between inputs and outputs. Specifically, for an \( \epsilon \)-quadratic MIMO, the DPSS can be applied to the input to determine the existence of a linear narrowband connection from input channel \( m' \) to output channel \( m \) through the use of the inner-product (59). Because the inner product of the \( \epsilon \)-quadratic MIMO system with the DPSS supplied to input \( m' \) is approximately equal to the average, linear impulse response connecting \( m' \) with \( m \), the inner product determines the existence of in-band linear responses that do not average to zero (but see Remark 2). Eqn. (59) determines the accuracy of this approximation in terms of the quantities listed in Table II and the number of input channels \( M \).

Remark 2. Generalization to Non-zero Frequency

The results presented in II-B and III generalize to frequency shifted, or modulated DPSS input. In II-B and III instead of Taylor expanding about zero frequency, the Taylor expansions are carried out about a carrier frequency \( f_0 \). That is, the DPSS are multipled by the phase factor \( e^{i2\pi f_0 t} \), and the truncated frequency interval changes from \((-W, W)\) to \((f_0 - W, f_0 + W)\) (ignoring the contribution from the negative frequencies). Thus, the results of this work allow for linear narrowband response at frequency intervals differing from baseband.

While it is outside the scope of this paper, note that these conditions are expected to approximate those required for the inner-product detector to achieve the performance of a matched-filter when the system response is added to measurement noise prior to observation (60). This scenario is studied in simulation in V-A, where the DPSS based inner-product detector is found effective.

Remark 3. In the theory discussed so far, the zeroth-order Volterra system response, is assumed equal to zero. Such a response contributes to the system DC offset. In situations where this is not guaranteed, restriction to the odd-ordered DPSS ensures that the inner-product will not respond to this offset.

A. Linear Narrowband Identification: Simulation

To explore the utility of the proposed method two 3rd order Volterra SISO systems are simulated. The first system is a null system containing a white, or constant response as a function of frequency, and the second, or alternate, system possesses an elevated response about 2 Hz. In this simulation four methods of detecting a narrowband system response are compared. Specifically, the detector responses resulting from null system excitation are compared with detector responses resulting from alternate system excitation.

Each simulation involves 240 measurements (\( n = 240 \)) and the time-index is an element in the set of time-indices: \( t \in \{0, \ldots, n - 1\} \). This set is used in all simulations, each of which involves only a single trial of simulated data. The sample period \( \Delta \) is equal to 1/30 s, Nyquist frequency \( f_N \) is equal to 15 Hz, the duration of observation is 8 s and the Rayleigh resolution \( f_R \), is equal to 3/8 Hz.

Let \( \gamma_{(j,n)} \) be the \( j \)-th order kernel for the null system evaluated at time-index \( t \), and let \( \gamma_{(j,a)} \) be the \( j \)-th order kernel for the alternate system. The null system is specified by inverse discrete Fourier transforming \( \Gamma_{(1,n)} \), set to a constant function of frequency equal to 3/4. The alternate system is specified by inverse Fourier transforming \( \Gamma_{(1,a)} \) specified as:

\[ \Gamma_{(1,a)}(f) = \begin{cases} \Gamma_{(1,n)}(f) + 10^{-3}/f^{-3}, & 3f_R \leq |f| \leq f_N \\ 10^{-3}/f_R^{-3}, & |f| < 3f_R \end{cases} . \quad (61) \]

To illustrate the merits of the proposed method narrowband response detection is compared against the least-squares kernel identification procedure presented in [61], [62]. This procedure makes use of a Laguerre polynomial basis. For comparison, and to facilitate the specification of the higher-order response functions, let \( c_k^{(n,1)} \) (\( c_k^{(a,1)} \)) be the \( k \)-th Laguerre expansion coefficient for the null (alternate) system, multiplying

\[ g_{k,t} = P_{100(k-1)+1,t} \quad (62) \]
in the representation:

\[ \gamma^{(1,n)}_t = \sum_{k=1}^{50} c^{(1,n)}_k g_{k,t}. \]  

(63)

Here \( P_{k,t} \) is the \( k \)-th order discretized Laguerre polynomial, \( P_{k,t} = L_k(\Delta t) \), where \( L_k \) is the \( k \)-th Laguerre polynomial:

\[ L_k(x) = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k}{j} x^j. \]  

(64)

The expansion coefficients, \( c^{(1,n)}_k \) (\( c^{(1,a)}_k \)), \( k = 1, \ldots, 50 \), are computed as a least-squares solution to (63) after specifying \( \gamma^{(1,n)} \) (\( \gamma^{(1,a)} \)). They are plotted in Fig. (3) (top row). The first order kernel is depicted in the time and frequency domains in Fig. (2) (top row).

The second-order kernels, \( \gamma^{(2,n)}_t \), \( \gamma^{(2,a)}_t \) are specified in a fashion akin to that for the first-order kernels, save that after computing the expansion coefficients \( c^{(2,n)}_k \), \( c^{(2,a)}_k \) as above,

\[ \gamma^{(n,2)}_{t_1,t_2} = \sum_{k=1}^{50} c^{(2,n)}_k g_{k,t_1} g_{k,t_2}. \]  

(65)

In this procedure \( c^{(2,a)}_k \) (Fig. (3)) is chosen such that the linear combination

\[ \sum_{k=1}^{50} c^{(2,a)}_k g_{k,t}, \]  

(66)

plotted in the time (Fig. (2), middle row, left) and in the frequency (Fig. (2), middle-row, right) domains, result in the second order kernel shown in Fig. (4). The third-order Volterra kernel is specified in a similar fashion:

\[ \gamma^{(n,3)}_{t_1,t_2,t_3} = \sum_{k=1}^{50} c^{(n,3)}_k g_{k,t_1} g_{k,t_2} g_{k,t_3}. \]  

(67)

The resulting coefficients \( c^{(n,3)}_k \), \( c^{(a,3)}_k \) are plotted in Fig. (3) and the time and Fourier representations of the associated linear combination of Laguerre polynomials is plotted in Fig. (4). The 3\( \text{rd} \) order kernel for the alternate system is chosen such that there is an elevated response at 2 Hz and at 6 Hz such that narrowband system excitation centered upon 2 Hz excites a strong 3\( \text{rd} \)-order Volterra response.

For both the null and alternate systems, narrowband response detection is performed using four methods. These are direct kernel identification (or estimation) using (i) white Gaussian input, and (ii) white M-sequence input. Narrowband response excitation results from (iii) an input sequence equal to a sum of sinusoids (SSR) with in-band frequencies, and (iv) discrete prolate spheroidal stimulation. In cases (i)-(iv) the output is added to Gaussian white noise with a variance of \( 1 \) Hz, and the frequencies \( 2 \times 10^{-6} \), \( 4 \times 10^{-6} \), or \( 6 \times 10^{-6} \). Ten repetitions of each input, signal energy and higher-order response scale are performed.

For input types (i) and (ii) kernel identification is performed assuming a third-order Volterra system using the least-squares identification procedure specified in [61]. Unlike in [61], instead of optimizing the \( \alpha \) parameter parameterizing the Laguerre basis (here it is set to zero), the Laguerre expansion is restricted to include Laguerre polynomials with an order equal to an integer multiple of 100.\(^4\) For input (i) and (ii), linear narrowband response is taken to be the collection of values \( |\Gamma^{(n,2)}(f)| \) \( (|\Gamma^{(a,2)}(f)|) \), \( f = 2 - W, 2 - W + df \ldots, 2 + W - df, 2 + W \). Here \( W \) is varied over the values .5 Hz, .75 Hz, and 1 Hz, corresponding respectively to \( NW \) equal to 4, 6, and 8. Due to sampling, input types (i), (ii) depend on \( W \); however, for simplicity this dependence is ignored and the results of simulations involving input (i), (ii) using different values of \( W \) are combined. In total, \( 3 \times 3 \times 10 \) simulations are performed for each of the system types (null & alternate) resulting in 270 simulations involving input types (i) and (ii). For input (iii), (iv) this number is 810, as every value of \( W \) is simulated. The SSR input (iii) is comprised of a sum of cosines. Each cosine is at a frequency, \( f \in (2 - W, 2 + W) \) and the frequencies are spaced by \( f_W \). Input type (iv) is defined in Section II-A. System response to input (iv) is studied and characterized in Sections III, IV. The narrowband responses to these input are taken to be the inner-product of the output with the scaled input (scaled to possess an energy equal to 1).

When testing for narrowband response, some performance is due to the choice of detector (choice of hypothesis test). To attribute performance to the method of system identification, for each input method, null system responses are used to normalize the responses of the alternate system. Let, respectively, \( a \), and \( \sigma \) be the sample average and the sample standard deviation of the null system responses. The alternate system responses are reduced by \( a \) and then divided by \( \sigma \) to obtain

\(^4\)More accurately, the Associated Laguerre polynomials are parameterized by \( \alpha \). The Laguerre polynomials result when \( \alpha \) equals zero.
normalized alternate system responses. Under the null hypothesis that there is no difference in narrowband response between
the null and the alternate systems, the resulting normalized alternate system responses are realizations of independent standard
normal random variables. Deviations of the normalized observations from that expected of a standard Gaussian random variable
provides evidence in favour of rejecting the hypothesis that the observations are the result of the null system.

The normalized responses for inputs (i) and (ii) are plotted in Fig. 5. The responses due to inputs with an energy of \(4 \times 10^4\) (bottom row) are consistent with those expected from the null system, while the responses (top row) associated with an input energy of \(4 \times 10^6\) are not. The normalized responses for input types (iii) and (iv) are plotted in Fig. 6 (top row: input energy equal to \(4 \times 10^4\), bottom row: input energy equal to \(4 \times 10^5\)). The normalized inner-product responses due to the SSR and DPSS input are significantly different from that expected due to the null system. Narrowband system response is successfully detected with the inner-product detector using input types (iii) and (iv) that is not detected by the kernel identification approach associated with white input. The DPSS input yields detections comparable to that of the SSR input, while possessing a larger

\[\text{normalized alternate system responses. Under the null hypothesis that there is no difference in narrowband response between the null and the alternate systems, the resulting normalized alternate system responses are realizations of independent standard normal random variables. Deviations of the normalized observations from that expected of a standard Gaussian random variable provides evidence in favour of rejecting the hypothesis that the observations are the result of the null system.}

\[\text{The normalized responses for inputs (i) and (ii) are plotted in Fig. 5. The responses due to inputs with an energy of } 4 \times 10^4 \text{ (bottom row) are consistent with those expected from the null system, while the responses (top row) associated with an input energy of } 4 \times 10^6 \text{ are not. The normalized responses for input types (iii) and (iv) are plotted in Fig. 6 (top row: input energy equal to } 4 \times 10^4 \text{, bottom row: input energy equal to } 4 \times 10^5 \text{). The normalized inner-product responses due to the SSR and DPSS input are significantly different from that expected due to the null system. Narrowband system response is successfully detected with the inner-product detector using input types (iii) and (iv) that is not detected by the kernel identification approach associated with white input. The DPSS input yields detections comparable to that of the SSR input, while possessing a larger}\]

\[\text{\footnote{For all plots containing horizontal black lines, the probability of lying within the horizontal lines is .95 (under the hypothesis that the responses are due to the null system).}}\]
fraction of in-band energy for $W$ equal to .5 or .75 (Fig. 6). When $W$ is equal to 1 the DPSS input yields superior detection performance to that of the SSR input.

Fig. (7) depicts the average of the normalized cross-product responses. Here a cross-product response is defined to be the absolute value of the inner-product between the response to DPSS stimulation $v_j$ with DPSS $v_{j'}$, $j' \neq j$. The normalization is performed by dividing the cross-response by the square-root of the absolute value of the product of the self-responses. The cross-responses become smaller with increasing $W$, with decreasing input signal energy and with decreasing higher-order system response scale.

VI. DISCUSSION & CONCLUSIONS

In this work, conditions are provided under which nonlinear system output due to discrete prolate spheroidal sequence (DPSS) input remains approximately orthogonal to other DPSS input. These conditions are developed under two assumptions. Specifically, that the system admits the Volterra MIMO system representation Eqn. (9) with $y_m^{(0)}$ equal to zero (but see Remark
Fig. 4. **The magnitude of the second order general frequency response function**, $|\Gamma^{(2)}|$. Estimates corresponding to Gaussian white input (left), and M-sequence input (middle). The input signal-to-output-noise ratio (SNR) is equal to $4 \times 10^6$. The system response due to the second and third order kernels is scaled to have an output sample standard deviation equal to $1.4 \times 10^2$ (variance equal to $2 \times 10^4$). Right: The magnitude of the actual second-order frequency response function.

Fig. 5. **Null system normalized magnitude of the collection of in-band generalized frequency responses**. Alternate system is detected for large input energy, but not for low-input energy. Compare to Fig. (6). The dependence on higher-order system response is evident.

For the DPSS stimulation, the DPSSs are modulated to 2 Hz by multiplication with a cosine oscillating with a frequency of 2 Hz [5]. As discussed in Section (V), only the odd-ordered DPSS are input to the system, up to a maximum order of $2NW - 1$. The responses associated with the $j^{th}$ order modulated DPSS are associated, in Fig. (6), with $v_{(j-1)/2}$, $j \in \{0,2,\ldots,2[(2NW - 1)/2]\}$.

3), and (ii) that Eqn. (32) is an exact bound. The conditions relate the DPSS bandwidth parameter $W$, the DPSS eigenvalues (through $\lambda_{\min}$), and suprema over the Volterra kernels to higher-order system response.

These properties facilitate linear narrowband detection in network settings using multiple simultaneous DPSS input, at both baseband, and higher frequencies. The performance of these detectors, owing to higher-order nonlinear suppression, are expected to approximate matched-filter type detectors with optimal performance where system output is added to noise prior to observation. The further development of these notions is left for future study.

Important limitations to this work are (i) due to higher-order nonlinear system response suppression, the utility of the results of this work for understanding the full nonlinear system dynamics is expected to be limited and (ii) the results presented in this work require that the Volterra model accurately describes system dynamics. This precludes, for example, sub-harmonic generation.

In summary, this work provides a mathematical quantification of DPSS stimulation of MIMO Volterra nonlinear systems. It provides a mathematically principled foundation from which to further develop nonlinear system identification methodology and from which to build applications in engineering and network science.
Fig. 6. **Null system normalized magnitude of the collection of in-band inner-product detector responses.** Alternate system is detected in all cases. There is a clear dependence on input energy and on $W$.

Fig. 7. **DPSS orthogonality depends on $W$, input energy, and on higher-order response.** Vertical axis is the sample average of the magnitude of the cross-DPSS products, normalized by the square root of the product of the self-DPSS products. Here $1 - \lambda_j$, for a given DPSS order $j$, decreases with increasing $W$, decreasing out-of-band contribution.

### APPENDIX A

**GENERALIZED FREQUENCY RESPONSE FUNCTION**

On output channel $m$, the $Q^{th}$-order system response $T_{m,Q}$ as a function of frequency $f$ is the discrete Fourier transform of the $Q^{th}$ order Volterra response:

$$ T_{m,Q}(f) = \mathcal{F}\left\{ \sum_{m_1}^{M} \sum_{m_2}^{M} \ldots \sum_{m_Q}^{M} \gamma^{(Q)}_{m,m_2,...,m_Q} \times u_{m_1,t-\tau_1} u_{m_2,t-\tau_2} \ldots u_{m_Q,t-\tau_Q} \right\} , $$

$$ = \sum_{t=-\infty}^{\infty} \left( \sum_{m_1}^{M} \sum_{m_2}^{M} \ldots \sum_{m_Q}^{M} \gamma^{(Q)}_{m,m_2,...,m_Q} \times u_{m_1,t-\tau_1} u_{m_2,t-\tau_2} \ldots u_{m_Q,t-\tau_Q} \right) e^{-i2\pi ft} . $$

Here the Volterra kernels are indexed at negative lags. To avoid an acausal system response, a necessary condition is that these kernels are set equal to zero for negative lag index. With this further stipulation (68) is equivalent to (9).
Substituting the Fourier representation

\[ u_{m,j,t} = \int_{-\frac{1}{2}}^{\frac{1}{2}} U_{m,j}(f') e^{i2\pi f' t} df', \quad (69) \]

into (68) results in:

\[
T_{m,Q}(f) = \sum_{\tau_1, \ldots, \tau_Q = -\infty}^{\infty} \left( \sum_{m_1, \ldots, m_Q = 1}^{1} \sum_{\gamma_1(Q)}^{\gamma_1} \Gamma_{m,m_Q,\tau_1,\ldots,\tau_Q} \times \prod_{j=1}^{Q} U_{m,j}(f_{j}) e^{i2\pi f_{j}(t-\tau_{j})} df_{j} \right) e^{-i2\pi f t},
\]

\[
= \sum_{\tau_1, \ldots, \tau_Q = -\infty}^{\infty} \left( \sum_{m_1, \ldots, m_Q = 1}^{1} \sum_{\gamma_1(Q)}^{\gamma_1} \Gamma_{m,m_Q,\tau_1,\ldots,\tau_Q} \times \prod_{j=1}^{Q} U_{m,j}(f_{j}) e^{-i2\pi f_{j} \tau_{j}} \times \right.
\]

\[
\left. \sum_{t=\infty}^{t=-\infty} e^{-i2\pi f t} \right) df_{j}.
\]

(70)

Since \( \sum_{t=\infty}^{t=-\infty} e^{-i2\pi f t} \) is equal to the dirac-delta function \( \delta(f) \):

\[
T_{m,Q}(f) = \sum_{m_1, \ldots, m_Q = 1}^{1} \sum_{\tau_1, \ldots, \tau_Q = -\infty}^{\infty} \gamma_1(Q)_{m,m_Q,\tau_1,\ldots,\tau_Q} \times \prod_{j=1}^{Q} U_{m,j}(f_{j}) \delta \left( f - \sum_{j=1}^{Q} f_{j} \right) df_{j},
\]

\[
= \sum_{m_1, \ldots, m_Q = 1}^{1} \prod_{j=1}^{Q} \Gamma_{m,m_Q,\tau_1,\ldots,\tau_Q} \delta \left( f - \sum_{j=1}^{Q} f_{j} \right) \times \prod_{j=1}^{Q} U_{m,j}(f_{j}) \delta \left( f - \sum_{j=1}^{Q-1} f_{j} \right)
\]

\[
= \sum_{m_1, \ldots, m_Q = 1}^{1} \prod_{j=1}^{Q} \Gamma_{m,m_Q,\tau_1,\ldots,\tau_Q} \delta \left( f - \sum_{j=1}^{Q-1} f_{j} \right) \prod_{j=1}^{Q-1} U_{m,j}(f_{j}) \delta f_{Q-1}.
\]

(71)

After notational changes, (71) is identical to (14).
APPENDIX B
OUT-OF-BAND RESPONSE OF THE $Q^{th}$ ORDER VOLTERRA KERNEL

\[ |T_{m,Q}(f) - T_{m,Q}^{(i)}(f)| = |T_{m,Q}^{(o)}(f)| \]

\[
= \left| \sum_{m_Q=1}^{M} \frac{1}{2} \Gamma_{m,m_Q}^{(Q)} (f_{Q-1}, f - f_{Q-1}^T 1_{Q-1}) \right| \]

\[
V_{m_Q} (f - f_{Q-1}^T 1_{Q-1}) \prod_{j=1}^{Q-1} V_{m_j} (f_j) \; df_{Q-1} \]

\[
\leq \sum_{m_Q=1}^{M} \frac{1}{2} \left| \Gamma_{m,m_Q}^{(Q)} (f_{Q-1}, f - f_{Q-1}^T 1_{Q-1}) \right| \]

\[
V_{m_Q} (f - f_{Q-1}^T 1_{Q-1}) \prod_{j=1}^{Q-1} V_{m_j} (f_j) \; df_{Q-1} \]

\[
\leq \Gamma_{m,*,V_{M,*}}^{(Q)} \sum_{m_Q=1}^{M} \prod_{j=1}^{Q-1} \sqrt{\lambda_{m}} \]

\[
\leq \Gamma_{m,*,V_{M,*}}^{(Q)} (1 - \lambda_{min})^{(Q-1)/2} ,
\]

\[
= A_{m,M,Q} (\lambda_{min}, V_{M,*}, \Gamma_{m,*}^{(Q)}) .
\] (72)

Here

\[ \Gamma_{m,*}^{(Q)} = \]

\[
\sup_{m_Q \in \{1,2,...,M\}^Q} \left| \Gamma_{m,m_Q}^{(Q)} (f_{Q-1}, f - f_{Q-1}^T 1_{Q-1}) \right| , \]

(73)

and

\[ V_{M,*} = \]

\[
\sup_{m_Q \in \{1,2,...,M\}^Q} \left| V_{m_Q} (f - f_{Q-1}^T 1_{Q-1}) \right| .
\] (74)

Note that both $\lambda_{min}$ and $V_{M,*}$ depend upon $W$ implicitly.
APPENDIX C
IN-BAND RESPONSE OF THE $Q$TH ORDER VOLterra KERNEL

The bound in (28) is established using the multidimensional version of Taylor’s remainder theorem applied to the Taylor expansion of a complex valued function of several real valued variables. Specifically,

$$
\left| T_{m,Q}^{(i)}(f) - T_{0,m,Q}^{(i)}(f) \right| \leq \\
\sum_{m_Q=1}^{M} \int_{-W}^{W} f_{Q-1}^{T} \times \\
\int_{0}^{1} \nabla f_{Q-1} \Gamma_{m,m_Q}^{(Q)} \left( t f_{Q-1}, f - t f_{Q-1}^{T} 1_{Q-1} \right) dt \times \\
V_{m_Q} \left( f - f_{Q-1}^{T} 1_{Q-1} \right) \prod_{j=1}^{Q-1} V_{m_j} \left( f_j \right) df_{Q-1} .
$$

(75)

More manipulations yield,

$$
\left| T_{m,Q}^{(i)}(f) - T_{0,m,Q}^{(i)}(f) \right| \leq \\
\int_{-W}^{W} \left| f_{Q-1}^{T} \right| \sum_{m_Q=1}^{M} \times \\
\int_{0}^{1} \left| \nabla f_{Q-1} \Gamma_{m,m_Q}^{(Q)} \left( t f_{Q-1}, f - t f_{Q-1}^{T} 1_{Q-1} \right) \right| dt \times \\
\left| V_{m_Q} \left( f - f_{Q-1}^{T} 1_{Q-1} \right) \prod_{j=1}^{Q-1} V_{m_j} \left( f_j \right) \right| df_{Q-1} ,
$$

$$
\leq W^{Q-1} \sum_{m_Q=1}^{M} \int_{-W}^{W} \times \\
\int_{0}^{1} \left| \nabla f_{Q-1} \Gamma_{m,m_Q}^{(Q)} \left( t f_{Q-1}, f - t f_{Q-1}^{T} 1_{Q-1} \right) \right| dt \times \\
\left| V_{m_Q} \left( f - f_{Q-1}^{T} 1_{Q-1} \right) \prod_{j=1}^{Q-1} V_{m_j} \left( f_j \right) \right| df_{Q-1} .
$$

(76)

Let

$$
\Gamma_{m,*}^{(Q)} = \\
\sup_{m_Q \in \{1, \ldots, M\}^Q} \sup_{f \in (-W,W)^Q} \sup_{f_{Q-1} \in (-W,W)^{Q-1}} \int_{0}^{1} \left| \nabla f_{Q-1} \left\{ \Gamma_{m,m_Q}^{(Q)} \left( t f_{Q-1}, f - t f_{Q-1}^{T} 1_{Q-1} \right) \right\} \right| dt .
$$

(77)
Then, using (26),

\[
T_{m,Q}^{(i)}(f) - T_{0,m,Q}^{(i)}(f) \leq W^{-1} \sum_{m=1}^{M} W^{-1} \int_{-W}^{W} V_{mQ} \left( f - e_j^T Q^{(i)} \right) \prod_{j=1}^{Q-1} V_{mj} \left( f_j \right) d\xi_{Q-1},
\]

\[
= W^{-1} \sum_{m=1}^{M} W^{-1} \int_{-W}^{W} V_{mQ} \left( f - e_j^T Q^{(i)} \right) M^Q J_B \left( W, Q, M \right),
\]

\[
= B_{m,M,Q} \left( W, Q^{(i)} \right).
\]

(78)

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Higher-Order MIMO Relative Response Suppression w. DPSS Input

\[ C_{M,Q}(N,NW) \]

Volterra Kernel Order, \( Q \)

- \((1-\lambda_{\text{min}}, N, NW, M) = 9e-6, 250, 3, 2\)
- \((1-\lambda_{\text{min}}, N, NW, M) = 7e-5, 250, 5, 6\)
- \((1-\lambda_{\text{min}}, N, NW, M) = 1e-16, 250, 10, 6\)
- \((1-\lambda_{\text{min}}, N, NW, M) = 9e-6, 500, 3, 2\)
- \((1-\lambda_{\text{min}}, N, NW, M) = 7e-5, 500, 5, 6\)
- \((1-\lambda_{\text{min}}, N, NW, M) = 2e-16, 500, 10, 6\)
- \((1-\lambda_{\text{min}}, N, NW, M) = 9e-6, 1000, 3, 2\)
- \((1-\lambda_{\text{min}}, N, NW, M) = 7e-5, 1000, 5, 6\)
- \((1-\lambda_{\text{min}}, N, NW, M) = 0e+00, 1000, 10, 6\)
Higher-Order Kernel Suppression

\[ \log_{10}\left( \frac{s_{q+q'}}{s_2} \right) \]

\( N = 200 \)

No. Of Available Tapers

\( NW = 3.00, \lambda_{\text{min}} = 1.0000 \)
\( NW = 5.00, \lambda_{\text{min}} = 0.9999 \)
\( NW = 8.00, \lambda_{\text{min}} = 0.9998 \)
\( NW = 11.00, \lambda_{\text{min}} = 0.9996 \)
Higher-Order Kernel Suppression

\[ \log_{10}(s_{q+q'}/s_2) \]

No. of Available Tapers

\[ N = 2000 \]

\[ \text{NW} = 3.00, \lambda_{\text{min}} = 1.0000 \]
\[ \text{NW} = 5.00, \lambda_{\text{min}} = 0.9999 \]
\[ \text{NW} = 8.00, \lambda_{\text{min}} = 0.9998 \]
\[ \text{NW} = 11.00, \lambda_{\text{min}} = 0.9996 \]