Equivalences for Linearizations of Matrix Polynomials

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ABSTRACT

One useful standard method to compute eigenvalues of matrix polynomials \( P(z) \in \mathbb{C}^{n \times n}[z] \) of degree at most \( t \) in \( z \) (denoted of grade \( t \), for short) is to first transform \( P(z) \) to an equivalent linear matrix polynomial \( L(z) = zB - A \), called a companion pencil, where \( A \) and \( B \) are usually of larger dimension than \( P(z) \) but \( L(z) \) is now only of grade 1 in \( z \). The eigenvalues and eigenvectors of \( L(z) \) can be computed numerically by, for instance, the QZ algorithm. The eigenvectors of \( P(z) \), including those for infinite eigenvalues, can also be recovered from eigenvectors of \( L(z) \) if \( L(z) \) is what is called a “strong linearization” of \( P(z) \). In this paper we show how to use algorithms for computing the Hermite Normal Form of a companion matrix for a scalar polynomial to direct the discovery of unimodular matrix polynomial cofactors \( E(z) \) and \( F(z) \) which, via the equation \( E(z)L(z)F(z) = diag(P(z), I_n, \ldots, I_n) \), explicitly show the equivalence of \( P(z) \) and \( L(z) \). By this method we give new explicit constructions for several linearizations using different polynomial bases. We contrast these new unimodular pairs with those constructed by strict equivalence, some of which are also new to this paper. We discuss the limitations of this experimental, computational discovery method of finding unimodular cofactors.

CCS CONCEPTS

- Computing methodologies → Hybrid symbolic-numeric methods: Linear algebra algorithms: Symbolic calculus algorithms.

KEYWORDS

linearization; matrix polynomials; polynomial bases; equivalence; Hermite normal form; Smith normal form

1 INTRODUCTION

Given a field \( \mathbb{F} \) and a set of polynomials \( \phi_k(z) \in \mathbb{F}[z] \) for \( 0 \leq k \leq t \) that define a basis for polynomials of grade \( t \) (“grade” is short for “degree at most”) then a square matrix polynomial \( P(z) \) is an element of \( \mathbb{F}^{n \times n}[z] \) which we can write as

\[
P(z) = \sum_{k=0}^{t} A_k \phi_k(z).
\]

The matrix coefficients \( A_k \) are elements of \( \mathbb{F}^{n \times n} \). For concrete exposition, take \( \mathbb{F} \) to be the field of complex numbers \( \mathbb{C} \). The case when the “leading coefficient” is singular—meaning the matrix coefficient of \( z^t \), once the polynomial is expressed in the monomial basis—can also be of special interest. Normally, only the case of regular square matrix polynomials, for which there exists some \( z^* \in \mathbb{F} \) with \( det(P(z^*)) \neq 0 \) is considered. Although our intermediate results require certain nonsingularity conditions, our results using strict equivalence are ultimately valid even in the case when the determinant of \( P(z) \) is identically zero.

Matrix polynomials are of significant classical and current interest: see the surveys [11] and [16] for more information about their theory and applications. In this present paper we use Maple to make small example computations in order to discover and prove certain facts about one method for finding eigenvalues of matrix polynomials, namely linearization, which means finding an equivalent grade 1 matrix polynomial \( L(z) \) when given a higher-grade matrix polynomial \( P(z) \) to start.

The paper [1] was the first to systematically study matrix polynomials in alternative polynomial bases. See [3] for a more up-to-date treatment.

2 DEFINITIONS AND NOTATION

A companion pencil \( L(z) = zB - A \) for a matrix polynomial \( P(z) \) has the property that \( det(L(z)) = \alpha det(P(z)) \) for some nonzero \( \alpha \in \mathbb{F} \). This means that the eigenvalues of the companion pencil are the eigenvalues of the matrix polynomial.

A linearization \( L(z) \) of a matrix polynomial \( P(z) \) has a stronger requirement: a linearization is a pencil \( L(z) = zB - A \) which is equivalent to \( P(z) \) in the following sense: there exist unimodular1 matrix polynomial cofactors \( E(z) \) and \( F(z) \) which satisfy \( E(z)L(z)F(z) = diag(P(z), I_n, \ldots, I_n) \). We write \( I_n \) for the \( n \times n \) identity matrix here.

Linearizations preserve information not only about eigenvalues, but also eigenvectors and invariant factors. Going further, a strong

1In this context, the matrix polynomial \( A(z) \) is unimodular if and only if \( det(A(z) \in \mathbb{F} \backslash \{0\} \) is a nonzero constant field element.
linearization is one for which the matrix pencil $-zL(1/z) = zA - B$ is a linearization for the reversal of $P(z)$, namely $z^{-1}P(1/z)$. Strong linearizations also preserve information about eigenstructure at infinity.

The Hermite normal form of a matrix polynomial $L$ is an upper triangular matrix polynomial $H$ unimodularly row equivalent to $L$, which is to say $L = EH$, where $E$ is a matrix polynomial with $\det(E)$ a nonzero constant. See [18] for properties and Hermite form algorithm description. We will chiefly use the computation of the Hermite normal form as a step in the process of discovering unimodular matrix polynomials $E(z)$ and $F(z)$ that demonstrate that $L(z)$ linearizes $P(z)$.

Another technique, which gives a stronger result (when you can do it) is to prove strict equivalence. We say matrix pencils $L_1(z)$ and $L_2(z)$ are strictly equivalent if there exist constant unimodular matrices $U$ and $W$ with $UL_1(z)W = L_2(z)$. We will cite and show some strict equivalence results in this paper, and prove a new strict equivalence for some Lagrange bases linearizations.

The paper [6] introduces a new notion in their Definition 2, that of a local linearization of rational matrix functions: this definition allows $E(z)$ and $F(z)$ to be rational unimodular transformations, and defines a local linearization only on a subset $\Sigma$ of $\mathbb{F}$. See also [14] which extended the matrix polynomial theory to use local linearization. Specifically, by this definition, two rational matrices $G_1(z)$ and $G_2(z)$ are locally equivalent if there exist rational matrices $E(z)$ and $F(z)$, invertible for all $z \in \Sigma$, such that $G_1(z) = E(z)G_2(z)F(z)$ for all $z \in \Sigma$.

This allows exclusion of poles of $E(z)$ or $F(z)$, for instance. It turns out that several authors, including [1], had in effect been using rational matrices and local linearizations for matrix polynomials. For most purposes, in skilled hands this notion provides all the analytical tools that one needs. However, as we will see, unimodular polynomial equivalence is superior in some ways. This paper therefore is motivated to see how far such local linearizations, in particular for the Bernstein basis and the Lagrange interpolational bases, can be strengthened to unimodular matrix polynomials. The point of this paper, as a symbolic computation paper, is to see how experiments with built-in code for Hermite normal form can help us to discover such cofactors.

We write

$$B_k^i(z) = \binom{\ell}{k} z^k (1-z)^{\ell-k} \quad 0 \leq k \leq \ell \quad (2.1)$$

for the Bernstein polynomials of degree $\ell$. This set of $\ell + 1$ polynomials forms a basis for polynomials of grade $\ell$.

We will use the convention of writing $A_k$ for the coefficients when the matrix polynomial is expressed in the monomial basis or in another basis with a three-term recurrence relation among its elements, $Y_k$ for the coefficients when the matrix polynomial is expressed in the Bernstein basis, and $P_k$ for the coefficients when the matrix polynomial is expressed in a Lagrange basis. We will use lower-case letters for scalar polynomials. We may write the (matrix) coefficient of $\phi_k(z)$ for an arbitrary basis element as $[\phi_k(z)]P(z)$; for instance, the "leading coefficient" of $P(z)$, considered as a grade $\ell$ polynomial, is $[z^\ell]P(z)$.

## 3 Explicit Cofactors

We find cofactors for orthogonal bases, the Bernstein basis, and Lagrange interpolational bases, all modulo certain exceptions. We believe all of these are new. These results are restricted to certain cases; say for the Bernstein basis when the coefficient $Y_k = [B_k^i(z)]P(z)$ (which is not the same as $[z^i]P(z)$) is nonsingular. The method of this paper does not succeed in that case to find cofactors that demonstrate universal linearization. In fact, however, it is known that this Bernstein companion pencil is indeed a true linearization; we will give two references with two different proofs, and give our own proof in section 5.2.

Similarly, this method produces new cofactors for Lagrange interpolational bases, but in this case restricted to when the coefficients $P_k$ (which are actually the values of the matrix polynomial at the interpolational nodes) are all nonsingular. We also have an algebraic proof of equivalence, which we have cut for space reasons; this gives another proof that the Lagrange interpolational basis pencils used here are, in fact, linearizations. In section 5.2 we prove strict equivalence.

### 3.1 Monomial Basis

Given a potential linearization $L(z) = zC_1 - C_0$, it is possible to discover the matrices $E(z)$ and $F(z)$ by computing the generic Hermite Form, with respect to the variable $z$, for instance by using a modestly sized example of $L(z)$ and a symbolic computation system such as Maple. The generic form that we obtain is of course not correct on specialization of the symbolic coefficients, and in particular is incorrect if the leading coefficient is zero; but we may adjust this by hand to find the following construction, which we will not detail in this section but will in the next.

By using the grade 5 scalar case, with symbolic coefficients which we write as $a_k$ instead of $A_k$ because they are scalar, we find that if $E(z)$ and $F(z)$ are

$$L(z) = \begin{bmatrix} 1 & h_4 & h_3 & h_2 & h_1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -z \\ 0 & 0 & -1 & -z & -z^2 \\ 0 & -1 & -z & -z^2 & -z^3 \end{bmatrix}$$

$$E(z) = \begin{bmatrix} z^4 & 0 & 0 & 0 & 1 \\ z^3 & 0 & 0 & 1 & 0 \\ z^2 & 0 & 1 & 0 & 0 \\ z & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3.2)

where $h_k = a_5$ and $h_k = a_k + zh_{k+1}$ for $k = 4, 3, 2, 1$ are the partial Horner evaluations of $p(z) = a_5z^5 + \cdots + a_0$, and if

$$L(z) = \begin{bmatrix} z a_5 + a_4 & a_3 & a_2 & a_1 & a_0 \\ -1 & z & 0 & 0 & 0 \\ 0 & -1 & z & 0 & 0 \\ 0 & 0 & -1 & z & 0 \\ 0 & 0 & 0 & -1 & z \end{bmatrix}$$

then we have $E(z)L(z)F(z) = \text{diag}(p(z), 1, 1, 1, 1)$. Moreover, $\det E(z) = \pm 1$ and $\det F(z) = \pm 1$, depending only on dimension.

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*The Smith form, which is related, is also useful here, but we found that the Maple implementation of the Hermite Form [18] gave a simpler answer, although we had to compute the matrix $F(z)$ separately ourselves because the Hermite form is upper triangular, not diagonal.*
Once this form is known, it is easily verified for general grades and quickly generalizes to matrix polynomials, establishing (as is well-known) that this form (known as the second companion form) is a linearization.

Note that the polynomial coefficients \( a_k \) appear linearly in \( E(z) \) and the unimodular matrix polynomials \( E \) and \( F \) are thus universally valid.

### 3.2 Three-term Recurrence Bases

The monomial basis, the shifted monomial basis, the Taylor basis, the Newton interpolational bases, and many common orthogonal polynomial bases all have three-term recurrence relations that, for \( k \geq 1 \), can be written

\[
z \phi_k(z) = a_k \phi_{k+1}(z) + \beta_k \phi_k(z) + \gamma_k \phi_{k-1}(z). \tag{3.4}
\]

All such polynomial bases require \( a_k \neq 0 \). For instance, the Chebyshev polynomial recurrence is usually written \( T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z) \) but is easily rewritten in the above form by isolating \( z T_n(z) \), and all Chebyshev \( a_k = 1/2 \) for \( k > 1 \). We refer the reader to section 18.9 of the Digital Library of Mathematical Functions (dlmf.nist.gov) for more bases. See also [10].

For all such bases, we have the well-known linearization \(^3 \) \( L(z) = z C_1 - C_0 \) where

\[
C_1 = \begin{bmatrix}
a_2 \\
a_3 \\
\vdots \\
\alpha_{n-1}
\end{bmatrix}, \quad C_0 = \begin{bmatrix}
-\alpha_4 + \beta_4 \alpha_5 \\
-\alpha_3 + \gamma_3 \alpha_5 \\
\vdots \\
-\alpha_1 + \gamma_1 \alpha_0
\end{bmatrix} \tag{3.5}
\]

and

\[
C_0 = \begin{bmatrix}
a_4 \beta_1 \alpha_5 \\
\beta_2 \alpha_2 \\
\gamma_2 \\
\alpha_1 \beta_1 \gamma_1 \\
\gamma_1
\end{bmatrix} \tag{3.6}
\]

To illustrate our approach for discovering such linearizations, we construct a grade 5 example with symbolic coefficients for Hermite Normal Form computation by the following Maple commands:

```maple
with(LinearAlgebra):
m := 5:
poly := add(a[k]*ChebyshevT(k, z), k = 0 .. m):
(C0, C1) := CompanionMatrix(poly, z):
```

The CompanionMatrix procedure, written about 2005, does not use the same convention we use here and so we apply the standard involutory permutation (SIP) matrix to it and transpose it to place the polynomial coefficients in the top row.

```maple
J := Matrix( m, m, (i,j) -> if(i+j=m+1, 1, 0) ):
R := C1*z - C0:
JRJT := Transpose([J . R] . J):
```

Now compute the generic Hermite normal form:

```maple
(HH, UU) := HermiteForm( JRJT, z, output=['H', 'U']):
```

That returns matrices such that \( U L = H \), or \( L = U^{-1} H \). We now look at their structure.

\(^3\)For exposition, we follow Peter Lancaster’s dictum, namely that the \( 5 \times 5 \) case almost always gives the idea.

This produces

\[
\begin{bmatrix}
x & 0 & 0 & 0 & x \\
0 & x & 0 & 0 & x \\
0 & 0 & x & 0 & x \\
0 & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}\quad \begin{bmatrix}
x & x & x & x & x \\
x & x & x & 0 & 0 \\
x & 0 & x & 0 & 0 \\
0 & 0 & x & 0 & 0 \\
0 & 0 & 0 & x & 0
\end{bmatrix}. \tag{3.7}
\]

and a separate investigation shows the diagonal entries of the first row are (as is correct in the generic case) all just 1, until the lower corner entry which is a monic version of the original polynomial. Because the first \( m = 1 \) columns of \( H \) are the same as those of the identity matrix, the first \( m = 1 \) columns of \( U^{-1} \) are the same as the first \( m - 1 \) columns of \( L \). Since the final column of \( U^{-1} \) has only one nonzero entry, at the top, call that \( a_0 \). Call the entries in the final column of \( H \) in descending order \( h_{m-1}, h_{m-2}, \ldots, h_1 \); let us suppose that the final entry is a multiple \( c p(z) \) of the scalar polynomial.

Multiplying out \( U^{-1} H \) and equating its final column to the final column of \( L(z) \) gives us a set of (triangular, as it happens) equations in the remaining unknowns \( u_0, h_1, \ldots, h_{m-1} \). These allow us to deduce a general form for \( U^{-1}(z) \), and to prove it is unimodular.

Once we have \( U \) and \( H \) with \( U L(z) = H \), construction of unimodular \( E(z) \) and \( F(z) \) so that \( L(z) = E(z) \text{diag}(P(z), I_n, \ldots, I_n) F(z) \) is straightforward.

We present the matrices \( E(z) \) and \( F(z) \) found by our experiments; the proof is by multiplying out. \( E(z) \) is

\[
\begin{bmatrix}
I & A_1 & \cdots & A_{r-2} & \frac{(z-\beta_{r-2})A_{r-1}}{A_{r-1}} & A_{r-1} - A_{r-1} \\
0 & -\gamma_{r-2}I & \cdots & (z-\beta_{r-2})I & -\alpha_{r-1}I & (z-\beta_{r-2})I - A_{r-1} \\
0 & 0 & \cdots & -\gamma_{r-2}I & -\alpha_{r-1}I & -\alpha_{r-1}I
\end{bmatrix}
\]

while \( F(z) \) is

\[
\begin{bmatrix}
I & -\phi_1(z)I & \cdots & -\phi_{r-2}(z)I & -\phi_{r-1}(z)I
\end{bmatrix}
\]

\( E(z) \) is unimodular because no \( a_k \) is zero. \( F(z) \) is more obviously unimodular. Note that here we have chosen to write \( L(z) = E(z) \text{diag}(P(z), I_n, \ldots, I_n) F(z) \), whereas for equation (3.2) \( E \) and \( F \) were chosen to make \( E(z)L(z)F(z) = \text{diag}(P(z), I_n, \ldots, I_n) \). Since the matrices are unimodular the choice of what to call \( E \) or \( E^{-1} \) can be made for convenience. Notice also that the monomial basis is a special case of the three-term recurrence bases: the \( E(z) \) and \( F(z) \) in this section must contain those of equation (3.2) as a special case (when inverted).

Notice again that the result is linear in the unknown polynomial coefficients, and that we therefore have a universal equivalence.

### 4 Bernstein Basis

The set of Bernstein polynomials \( B_n^\ell(z) \) in equation (2.1) is a set of \( \ell + 1 \) polynomials each of exact degree \( \ell \) that together forms
a basis for polynomials of grade $\ell$ over fields $F$ of characteristic zero. Bernstein polynomials have many applications, for example in Computer Aided Geometric Design (CAGD), and many important properties including that of optimal condition number over all bases positive on $[0,1]$. They do not satisfy a simple three term recurrence relation of the form discussed in Section 3.2, although they satisfy an interesting and useful "degree-elevation" recurrence, namely

$$(j+1)B_{j+1}^F(z) + (\ell-j)B_j^F(z) = tB_{j-1}^F(z),$$

which specifically demonstrates that a sum of Bernstein polynomials of grade $\ell$ might actually have degree strictly less than $\ell$. See [7], [8], and [9] for more details of Bernstein bases.

A Bernstein linearization for $p_5(z) = \sum_{k=0}^5 y_k B_k^5(z)$ is $L(z) = [\frac{1}{2} y_2 z^2 + y_4 (1-z) y_3 (1-z) y_2 (1-z) y_1 (1-z) y_0 (1-z)]_{z=1}$. The resulting suggested form for $U^{-1}$ is

$${ }^{[x x x x x x \ x x 0 0 x \ 0 x 0 x 0 \ 0 x x x 0 0 x x x x]}$$

which is more complicated than before, because the final column is full (as before, the first $\ell - 1$ columns merely copy the companion pencil $L(z)$). Moreover, this time the entries in $U^{-1}$ are polynomial functions of $y_k$, $0 \leq k < \ell$, not just linear; but involve negative powers of $y_k$ (this equivalence appears to be new; the treatment in [17] is implicit, while the treatment in [1] only gives a local linearization). This means that in the case $Y_\ell$ is singular, $z = 1$ is an eigenvalue and something else must be done to linearize the matrix polynomial.

We find a recurrence relation for the unknown blocks in both the purported Hermite normal form and in the inverse of the cofactor. Put $U^{-1} = \begin{bmatrix} z Y_{\ell}/(1-z) Y_{\ell-1} & (1-z) Y_{\ell-2} & \cdots & (1-z) Y_1 & W_{\ell-1} \\ (z-1) I_n & 2 z I_n -(1-z) Y_1 I_n & \cdots & (\ell-1) z I_n /2 & W_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (z-1) I_n & \vdots & \cdots & (z-1) I_n & W_1 \\ (z-1) I_n & \vdots & \cdots & (z-1) I_n & W_0 \end{bmatrix}$

which, apart from the final column, is the same as the linearization $L(z)$, and the Hermite normal form analogue as

$$H = \begin{bmatrix} I_n & H_{\ell-1} & \cdots & H_1 \\ \vdots & \vdots & \ddots & \vdots \\ I_n & \vdots & \cdots & H_1 \\ P(z) \end{bmatrix}.$$  

Multiplying out $U^{-1} H$ we find that for the lower right corner block

$$(z-1) H_1 + W_0 P(z) = t z I_n.$$  

Evaluating that equation at $z = 1$ shows that $W_0(1) = t Y_\ell^{-1}$, so $P(1) = Y_\ell^{-1}$ must be nonsingular for this form to hold. Now it turns out that $W_0$ can be taken to be constant. Once $W_0 = t Y_\ell^{-1}$ is identified, equation (4.15) can be solved uniquely for the grade $\ell - 1$ matrix polynomial $H_1(z)$:

$$(1-z) H_1(z) = t \left( Y_\ell^{-1} P(z) - z I_n \right).$$

Division by $(1-z)$ is exact and $H_1(z)$ is a matrix polynomial. The other block entries in the multiplied-out equation give the recurrences

$$(z-1) H_{k+1} + z(\ell - k) H_k + W_k P(z) = 0$$

for $1 \leq k \leq \ell - 2$. These are solvable in the same way and under the same condition, namely that $Y_\ell$ must be nonsingular. Verifying that the first block row equation is also satisfied would complete the proof. We have only a partial proof of this, and computational proofs for arbitrary dimension $n$ with grades $2 \leq \ell \leq 10$, using the noncommuting variables feature of the Physics package in Maple, implemented by Edgardo Cheb-Terrab. For instance, when $\ell = 2$, computation with this package finds $W_1 = I - 2 Y_1 Y_2^{-1}$. When $\ell = 3$, computation with this package finds

$$W_2 = I - 3 Y_1 Y_3^{-1} - 6 Y_2 Y_3^{-1} + 9 Y_2 Y_3^{-1} - 2 Y_2 Y_3^{-1}$$

When $\ell = 4$, writing $Y_4 = Y_4 Y_4^{-1}$, we have

$$W_5 = I - 4 Y_1 - 18 Y_2 - 12 Y_3 + 24 Y_2 Y_3 + Y_3 Y_2 + 48 Y_3^2 - 64 Y_3^3.$$  

The package can simplify the first block row equation to zero, once $W_{\ell-1}$ is computed, providing a computational proof for arbitrary $n$ and the chosen grade $\ell$. We have done this for $\ell \leq 10$; it is the simplification to zero that requires the most computational effort. At $\ell = 10$ on a Microsoft Surface Pro running 64 bit Maple 2021, the simplification took over 103 minutes, including nearly 6 minutes of garbage collection time.
For analytical work to complete the proof for arbitrary grades, the change of variables $H_k(z) = B_k(z)U_k(z)$ transforms equation (4.17) to

$$U_{k+1} - U_k = -\frac{W_k P(z)}{(1-z)B_{k+1}^T(z)}$$

(4.20)

which gives explicit sums for the $U_m$. Substituting those sums into the first block row and simplifying the resulting double sum would complete the proof.

But in fact this proof has been superseded by the proof of strict equivalence in the next section.

### 4.2 Strict Equivalence

For the pencil in equation (4.11) we exhibit the strict equivalence by the unimodular matrices $U$ and $W$ giving $L_m(z) = UL_B(z)W$, below; here are $W$ and $U^{-1}$:

\[
\begin{bmatrix}
5 & 0 & 0 & 0 & 0 \\
-10 & 10 & 0 & 0 & 0 \\
10 & -20 & 10 & 0 & 0 \\
-5 & 15 & -15 & 5 & 0 \\
1 & -4 & 6 & -4 & 1
\end{bmatrix}
&
\begin{bmatrix}
1 & h_3 & h_2 & h_1 & -y_0 \\
0 & 5 & 0 & 0 & 0 \\
0 & -10 & 10 & 0 & 0 \\
0 & 10 & -20 & 10 & 0 \\
0 & -5 & 15 & -15 & 5
\end{bmatrix},
\]

where $h_3 = -10y_3 + 20y_2 - 15y_1 + 4y_0$, $h_2 = -10y_2 + 15y_1 - 6y_0$, and $h_1 = -5y_1 + 4y_0$. Generalizing these to matrix polynomials is straightforward, as is generalizing these to arbitrary grade.

The paper [17] contains, in its equation (3.6), a five by five example including tensor products that shows how to find a strict equivalence of the pencil here to the strong linearizations they construct in their paper. Both the results of that paper and the construction given by example above demonstrate that the Bernstein linearization here is a strong linearization, independently of the regularity of the matrix polynomial.

### 4.3 A new reversal

We here present a slightly different reversal, namely rev $p(z) = (z + 1)^{\ell}p(1/(z + 1))$ of a polynomial of grade $\ell$ expressed in a Bernstein basis, instead of the standard reversal $z^{\ell}p(1/z)$. This new reversal has a slight numerical advantage if all the coefficients of $p(z)$ are the same sign. We also give a proof that the linearization of this reversal is the corresponding reversal of the linearization, thus giving a new independent proof that the linearization is a strong one. This provides the details of the entries in the unimodular matrix polynomials $E(z)$ and $F(z)$ with $E(z)L(z)F(z) = \text{diag}(P(z), I_n, \ldots, I_n)$ constructed above.

A short computation shows that if

$$p(z) = \sum_{k=0}^{\ell} y_k B_k^T(z)$$

(4.22)

then

$$\text{rev } p(z) = (z + 1)^{\ell}p\left(\frac{1}{z + 1}\right) = \sum_{k=0}^{\ell} d_k B_k^T(z)$$

(4.23)

where

$$d_k = \sum_{j=0}^{k} \binom{k}{j} y_{\ell-j}.$$  

(4.24)

whereas the coefficients of the standard reversal are, in contrast,

$$e_k = \sum_{m=0}^{\ell-k} (-1)^m \binom{\ell-k}{m} y_{\ell-m-k}$$

(4.25)

which has introduced sign changes, which may fail to preserve numerical stability if all the $y_k$ are of one sign. A further observation is that the coefficient $d_0$ only involves $y_\ell$, while $e_0$ involves all $y_k$: $d_1$ involves $y_\ell$ and $y_{\ell-1}$ while $e_1$ involves all but $y_\ell$, and so on; in that sense, this new reversal has a more analogous behaviour to the monomial basis reversal, which simply reverses the list of coefficients.

For interest, we note that if $(A, B)$ is a linearization for $p(z)$ so that $p(z) = \det(zB - A)$, then reversing the linearization by this transformation is not a matter of simply interchanging $B$ and $A$:

$$\begin{align*}
(z + 1)^{\ell}p\left(\frac{1}{z + 1}\right) &= (z + 1)^{\ell} \det\left(\frac{1}{z + 1} B - A\right) \\
&= \det(B - (z + 1)A) \\
&= \det(B - A - zA)
\end{align*}$$

(4.26)

and so the corresponding reversed linearization is $(A, B - A)$. The sign change is of no importance.

Suppose that the Bernstein linearization of $p(z)$ is $(A, B)$ and that the Bernstein linearization of rev $p(z)$ is $(A_R, B_R)$. That is, the matrices $A_R$ and $B_R$ have the same form as that of $A$ and $B$, but where $(A, B)$ contain $y_k$s the matrices $(A_R, B_R)$ contain $d_k$s. To give a new proof that the Bernstein linearization is actually a strong linearization, then, we must find a pair of unimodular matrices $(U, W)$ which have $U A_R W = B - A$ and $U B_R W = A$, valid for all choices of coefficients $y_k$ (which determine the corresponding reversed coefficients $d_k$ by the formula above).

First, it simplifies matters to deal not with $W$ but rather with $W^{-1}$. Then, our defining conditions become

$$U A_R = (B - A) W^{-1}$$

(4.27)

$$U B_R = A W^{-1},$$

(4.28)

which are linear in the unknowns (the entries of $U$ and of $W^{-1}$). By inspection of the first few dimensions, we find that $U$ and $W^{-1}$ have the following form (using the six-by-six case, for variation, to demonstrate). The anti-diagonal of the general $U$ has entries $-(\ell - i + 1)/i$ for $i = 1, 2, \ldots, \ell$.

$$U = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -6 \\
0 & 0 & 0 & 0 & -\frac{5}{2} & u_{2,6} \\
0 & 0 & -\frac{4}{3} & u_{5,5} & u_{3,6} \\
0 & -\frac{3}{4} & u_{5,4} & u_{4,5} & u_{4,6} \\
-\frac{1}{6} & u_{6,3} & u_{5,4} & u_{5,5} & u_{5,6} \\
\frac{1}{5} & u_{6,2} & u_{6,3} & u_{6,4} & u_{6,5} & u_{6,6}
\end{bmatrix}$$

(4.29)
and

\[
W^{-1} = \begin{bmatrix}
0 & 0 & 0 & 0 & z_{1,5} & z_{1,6} \\
0 & 0 & 0 & z_{2,4} & z_{2,5} & z_{2,6} \\
0 & 0 & z_{3,3} & z_{3,4} & z_{3,5} & z_{3,6} \\
0 & z_{4,2} & z_{4,3} & z_{4,4} & z_{4,5} & z_{4,6} \\
z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} & z_{5,5} & z_{5,6} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  

(4.30)

One can then guess the explicit general formulae

\[
u_{i,j} = \frac{(t - i + 1)}{i} \left( \frac{i}{t + 1 - j} \right) \quad 1 \leq i, j \leq t
\]

(4.31)

\[
z_{i,j} = -\frac{(t - i)}{j} \left( \frac{i}{t - j} \right) \quad 1 \leq i, j \leq t - 1
\]

(4.32)

\[
z_{i,t} = d_t - \frac{(t - i)}{t} y_t, \quad 1 \leq i \leq t - 1
\]

(4.33)

and then prove that these are not only necessary for the equations above, but also sufficient. The matrix \(B - A\) is diagonal and gives a direct relationship between the triangular block in \(W^{-1}\) and a corresponding portion of \(U\); the other equation gives a recurrence relation for the entries of \(U\). Comparison of the final columns of the products gives an explicit formula for the final column of \(W^{-1}\) and an explicit formula for the entries of \(U\) by comparison of the coefficients of the symbols \(y_t\); this formula can be seen to verify the recurrence relation found earlier, closing the circle and establishing sufficiency. Both matrices \(U\) and \(W\) have determinant \(\pm 1: [t/2] \) row-permutations brings \(U\) to upper triangular form and the determinant \((-1)^t\) times \((-1)^{[t/2]}\) can be read off as the product of the formerly anti-diagonal elements, and similarly for the \([1 : t - 1, 1 : t - 1]^{\text{block of } W^{-1}}\) which gives a sign \((-1)^{t-1+[t/2]}\).

5 LAGRANGE INTERPOLATIONAL BASES

A useful arrowhead companion matrix pencil for polynomials given in a Lagrange basis was given in [4, 5]. Later, Piers Lawrence recognized that the mathematically equivalent pencil re-ordered by similarity transformation by the standard involutory permutation (SIP) matrix so that the arrow was pointing up and to the left was numerically superior, in that one of the spurious infinite eigenvalues will be immediately deflated—without rounding errors—by the standard QZ algorithm [15]. We shall use that variant in this paper.

For expository purposes, consider interpolating a scalar polynomial \(p(z)\) on the four distinct nodes \(\tau_k\) for \(0 \leq k \leq 3\). In the first barycentric form [2] this is

\[
p(z) = w(z) \sum_{k=0}^{3} \frac{\beta_k p_k}{z - \tau_k} = \sum_{k=0}^{3} p_k w_k(z),
\]

(5.34)

where the node polynomial \(w(z) = (z - \tau_0)(z - \tau_1)(z - \tau_2)(z - \tau_3)\), and \(w_k(z) = \beta_k w(z)/(z - \tau_k)\), and the barycentric weights \(\beta_k\) are

\[
\beta_k = \prod_{j=0, j \neq k}^{3} \frac{1}{\tau_k - \tau_j}.
\]

(5.35)

Then a Schur complement with respect to the bottom right \(4 \times 4\) block shows that if

\[
L(z) = \begin{bmatrix}
0 & -p_3 & -p_2 & -p_1 & -p_0 \\
\beta_3 & z - \tau_3 & 0 & 0 & 0 \\
\beta_2 & 0 & z - \tau_2 & 0 & 0 \\
\beta_1 & 0 & 0 & z - \tau_1 & 0 \\
\beta_0 & 0 & 0 & 0 & z - \tau_0
\end{bmatrix}
\]

(5.36)

then \(\det L(z) = p(z)\). By exhibiting a rational unimodular equivalence, the paper [1] showed that the general form of this was at least a local linearization for matrix polynomials \(P(z)\), and also demonstrated that this was true for the reversal as well, showing that it was a strong (local) linearization. They also gave indirect arguments, equivalent to the notion of patched local linearizations introduced in [6], showing that the construction gave a genuine strong linearization. Here, we wish to see if we can explicitly construct unimodular matrix polynomials \(E(z)\) and \(F(z)\) which show equivalence, directly demonstrating that this is a linearization.

5.1 Hermite form of the linearization

As we did for the monomial basis, we start with a scalar version. We compute the Hermite normal form \(H\), and the transformation matrix \(U\) so that \(UL(z) = H\), with Maple to give clues to find the general form. When we do this for the grade 3 example above we find that the form of \(U\) is not helpful, but that the form of \(U^{-1}\) and \(H\) are:

\[
L(z) = \begin{bmatrix}
0 & x & x & x & 0 \\
x & 0 & 0 & 0 & x \\
x & 0 & x & 0 & x \\
x & 0 & x & x & x \\
x & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(5.37)

As is usual, the generic Hermite normal form contains \(p(z)\) in the lower right corner; on specialization of the polynomial coefficients this can change, of course. We return to this point later.

This gives us enough information to conjecture the general form in the theorem below, and a proof follows quickly.

**Theorem 5.1.** If

\[
L(z) = \begin{bmatrix}
0 & -P_t & -P_{t-1} & \ldots & -P_0 \\
\beta_t I_n & (z - \tau_t) I_n & 0 & \ldots & 0 \\
\beta_{t-1} I_n & 0 & (z - \tau_{t-1}) I_n & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\beta_0 I_n & 0 & 0 & 0 & (z - \tau_0) I_n
\end{bmatrix}
\]

(5.38)

and no \(P_k\) is singular then an invertible \(W\) exists with \(U = W^{-1}\) and

\[
L(z) = WH \text{ where } W =
\]

\[
\begin{bmatrix}
0 & -P_t & -P_{t-1} & \ldots & -P_1 & 0 \\
\beta_t I_n & (z - \tau_t) I_n & 0 & \ldots & U_t \\
\beta_{t-1} I_n & 0 & (z - \tau_{t-1}) I_n & 0 & \ldots & U_{t-1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\beta_1 I_n & 0 & 0 & \ldots & (z - \tau_1) I_n & U_1 \\
\beta_0 I_n & 0 & 0 & \ldots & 0 & U_0
\end{bmatrix}
\]
and $H$ is the identity matrix with its final column replaced with a block matrix that can be partitioned as

$$H = \begin{bmatrix}
I_n & I_n & \cdots & I_n & G & H_l & \cdots & H_{l-1} & \cdots & \cdots & \cdots & \cdots & I_n & H_1 & P(z)
\end{bmatrix}.$$  

Moreover,

$$G = -\frac{1}{\beta_0}(z - \tau_0)I_n, \quad (5.39)$$

for $1 \leq k \leq \ell$, and

$$H_k = \frac{1}{z - \tau_k} (\beta_k G - U_k P(z)). \quad (5.41)$$

Note that the definition of $U_k$ in equation (5.40) ensures that the division in equation (5.41) is exact, and therefore $H_k$ is a matrix polynomial of grade $\ell - 1$. Furthermore, $W$ is unimodular so $U = W^{-1}$ is also unimodular.

**Proof.** Block multiplication of the forms of $W$ and $H$ gives $\ell + 1$ block equations:

$$- \sum_{k=1}^{\ell} P_k H_k = P_0 \quad (5.42)$$

$$\beta_k G + (z - \tau_k) H_k + U_k P(z) = 0, 1 \leq k \leq \ell \quad (5.43)$$

$$\beta_0 G = (z - \tau_0) I_n \quad (5.44)$$

The last block equation identifies $G$. Putting $z = \tau_k$ in each of the middle block equations identifies each $U_k$. Once that has been done, the middle block equations define each $H_k$. All that remains is to show that these purported solutions satisfy equation (5.42) and that the resulting matrix $W$ is unimodular.

We now substitute equations (5.41)–(5.40) into the left hand side of equation (5.42) to get

$$\sum_{k=1}^{\ell} \frac{1}{z - \tau_k} P_k \left( \beta_k \frac{G}{\beta_0} (z - \tau_0) + \beta_0 \frac{E_k - P_k}{\beta_0} P(z) \right). \quad (5.45)$$

Expanding this, we have

$$LHS = - \left( \frac{z - \tau_0}{\beta_0} \right) \sum_{k=1}^{\ell} \frac{\beta_k}{\beta_0} P_k + \sum_{k=1}^{\ell} \frac{\beta_0 (\tau_k - \tau_0) - \beta_0}{\beta_0} (z - \tau_k) P(z)$$

$$= - \left( \frac{z - \tau_0}{\beta_0} \right) \left[ \frac{1}{w(z)} P(z) - \frac{\beta_0}{\beta_0} P_0 \right]$$

$$+ \left[ \frac{z}{w(z)} - \frac{\beta_0 \tau_0}{\beta_0} \right] P(z) \left( \frac{\tau_0}{w(z)} - \frac{\tau_0}{\beta_0} \right) P(z)$$

$$= P_0. \quad (5.46)$$

This shows that we have found a successful factoring analogous to the scalar Hermite normal form.

All that remains is to show that $W(z)$ is unimodular. We use the Schur determinantal formula,\(^9\) and identify the $\beta_k (\tau_k - \tau_0)$ as the barycentric weights on the nodes with $\tau_0$ removed and we see that the determinant simplifies to 1. This completes the proof. \(\square\)

As a corollary, we can explicitly construct unimodular matrix polynomials $E(z)$ and $F(z)$ from these factors showing that, if each $P_k$ is nonsingular, $L(z)$ is equivalent to $\text{diag}(P(z), I_n, \ldots, I_n)$.

As in [1] we may use $LU$ factoring to show that $P$ is also equivalent to $L$ at the nodes, essentially using Proposition 2.1 of [6]. We also have a purely algebraic proof based on local equivalence of Smith forms, not shown here.

### 5.2 Strict Equivalence

We prove the following theorem, establishing the strict equivalence of the companion pencil of equation (5.38), for a matrix polynomial $P(z)$ determined to grade $\ell$ by interpolation at $\ell + 1$ points, to the monomial basis linearization for $P(z) = 0 z^{\ell + 1} + 0 z^{\ell + 1} + \cdots + A_k z^k$ considered as a grade $\ell + 2$ matrix polynomial. This establishes that the Lagrange basis pencil is in fact a linearization, indeed a strong linearization, independently of the singularity or not of any of the values of the matrix polynomial, and independently of the regularity of the matrix pencil.

**Theorem 5.2.** Provided that the nodes $\tau_k$ are distinct, then the Lagrange basis linearization in equation (5.38), namely $L_1(z) = zC_{L,1} - C_{L,0}$, is strictly equivalent to the monomial basis linearization $L_M(z) = zC_{M,1} - C_{M,0};$ in other words, there exist nonsingular constant matrices $U$ and $W$ such that both $UC_{L,1}W = C_{M,1}$ and $UC_{L,0}W = C_{M,0}$. Explicitly, if $U$ is the Vandermonde matrix with $(i, j)$ entry $u_{ij} = \tau_{i+1}^{j} - 1$ for $1 \leq i, j \leq \ell + 1$, then we have

$$U = \begin{bmatrix}
I_n & 0 & V \otimes I_n
\end{bmatrix} \quad (5.47)$$

and, if the node polynomial $w(z) = \prod_{k=1}^{\ell} (z - \tau_k) = z^{\ell + 1} + q_1 z^\ell + \cdots + q_0$ has coefficients $q_k$, which we place in a row vector $q = [q_0, q_{-1}, \ldots, q_\ell]$, we have

$$W = \begin{bmatrix}
I_n & 0 & V^{-1} \otimes I_n
\end{bmatrix}. \quad (5.48)$$

**Proof.** Notice first that $\det U = \det V \otimes I_n = \prod_{k=1}^{n} (\tau_k - \tau_i)^n$ is not zero, and that $\det W$ is the reciprocal of that. Both these matrices are therefore nonsingular if the nodes are distinct.

Next, it is straightforward to verify that $UC_{L,1}W = C_{M,1}$ by multiplying out, and using the fact that the upper left corner block of each is the zero block, and that the rest is the identity because $(V \otimes I_n)(V^{-1} \otimes I_n) = I_n_{n(\ell + 1)}$.

The remainder of the proof consists of detailed examination of the consequences of multiplying out the more complicated $UC_{L,0}W$. From now on we drop the $\otimes I_n$ notation as clutter, and the proof is considered only for the scalar case, but the tensor products should be kept in mind as the operations are read. Writing

$$C_{L,0} = \begin{bmatrix}
0 & -P
\beta & D
\end{bmatrix} \quad (5.49)$$

nonisospectral, the relevant Schur complement is $0 - [\beta I, 0, 0, \ldots] R^{-1} [0; U_1; \ldots; U_\ell]$.

We use continuity afterwards to assure that the result is universally valid.
where \( D = \text{diag}(\tau, \tau-1, \ldots, \tau_0) \), we have (using \( PV^{-1} = A \))
\[
C_{L,0} W = \begin{bmatrix} 0 & -A \\ \beta & \beta q + DV^{-1} \end{bmatrix}.
\] (5.50)

We seek to establish that
\[
U^{-1} C_{M,0} = \begin{bmatrix} 1 \\ V^{-1} \end{bmatrix} \begin{bmatrix} 0 & -A_t \\ 1 & 0 \\ 1 & 0 \\ \vdots \\ 1 & 0 \end{bmatrix}
\] (5.51)
is the same matrix. This means establishing that
\[
[\beta \quad \beta q + DV^{-1}] = [V^{-1} \quad 0].
\] (5.52)

Begin by relating the monomial basis to the Lagrange basis:
\[
\begin{bmatrix} z^\ell \\ z^{\ell-1} \\ \vdots \\ z^1 \\ 1 \end{bmatrix} = \beta \begin{bmatrix} z^\ell \\ z^{\ell-1} \\ \vdots \\ z^1 \\ 1 \end{bmatrix} = \beta \begin{bmatrix} w_\ell(z) \\ wr_{\ell-1}(z) \\ \vdots \\ w_1(z) \\ w_0(z) \end{bmatrix},
\] (5.53)
where the \( w_k(z) \) are the Lagrange basis polynomials \( w_k(z) = \beta_k \prod_{j \neq k} (z - \tau_j) \). The matrix with the powers of \( \tau \) is, of course, just \( V \).

Name the columns of \( V^{-1} = [A_t, A_{t-1}, \ldots, A_0] \). We thus have to show that the final column of equation (5.52) is 0 and that
\[
A_{j-1} = \beta q_j + DA_j
\] (5.54)
for \( j = 1, 2, \ldots, \ell \), and that \( A_t = \beta \). But this is just a translation into matrix algebra of the \( \ell + 2 \) polynomial coefficients of
\[
\beta_i w_j(z) = (z - \tau_j) w_i(z) \quad 0 \leq i \leq \ell,
\] (5.55)
or \( zw_j(z) = \beta_i w(z) + \tau_i w_i(z) \). The coefficients of different powers of \( z \) in these identities provide the columns in the matrix equation (5.52). This completes the proof. \( \square \)

6 CONCLUDING REMARKS

This paper shows how to use scalar matrix tools and computation with low-dimensional examples to solve problems posed for general matrix polynomials.

The mathematical problems that we examined included the explicit construction of unimodular matrix polynomial cofactors \( E \) and \( F \) which show that \( L(z) \) is, indeed, a linearization of \( P(z) \).

We showed that the approach discovered cofactors that were valid generically. This is because the primary tool used here, namely the Hermite normal form, is discontinuous at special values of the parameters (and indeed discovery of those places of discontinuity is the main purpose of the Hermite and Smith normal forms).
Nonetheless, in the case of the monomial basis and others we were able afterwards to guess a universal form from the HNF. A separate investigation, based on the change-of-basis matrix but also inspired by experimental computation, found new explicit universal cofactors for all bases considered here. We also introduced a new reversal for polynomials expressed in the Bernstein basis which may have better numerical properties than the standard reversal does.

Our experimental method is analogous to a modular method: we have projected the general (arbitrary dimension, arbitrary block size) problem down to a much smaller particular one, where the solution is obtainable by direct computation. In contrast to modular methods where the lifting process is algorithmic, here the lifting process has human elements and indeed requires some guesswork followed by proof; success of this technique is not at all guaranteed.

In future work, we plan to expand the discussion to include more details, and to apply this method to Hermite interpolational bases, where (as previously for Lagrange interpolational bases) only local linearizations are currently known in the literature.

ACKNOWLEDGMENTS

We thank all the referees for their thoughtful comments, and their work in this time of pandemic. We thank Froilán Dópico for several very useful discussions on linearization, giving several references, and pointing out the difference between local linearization and linearization. The support of Western University’s New International Research Network grant is also gratefully acknowledged. This work was supported by NSERC.

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