ANALYTICITY OF FUNCTIONS ANALYTIC ON CIRCLES

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ABSTRACT Let $\Delta$ be the open unit disc in $\mathbb{C}$, let $p \in b\Delta$, and let $f$ be a continuous function on $\bar{\Delta}$ which extends holomorphically from each circle in $\bar{\Delta}$ centered at the origin and from each circle in $\bar{\Delta}$ which passes through $p$. Then $f$ is holomorphic on $\Delta$.

1. Introduction and the main result

Denote by $\Delta$ the open unit disc in $\mathbb{C}$. If $f$ is a continuous function on a circle $\Gamma$ then we say that $f$ extends holomorphically from $\Gamma$ if it extends holomorphically through the disc bounded by $\Gamma$.

Let $f$ be a continuous function on $\bar{\Delta}$ which extends holomorphically from every circle $|\zeta| = r$, $0 < r \leq 1$. A trivial example is a function constant on each circle $|\zeta| = r$. Obviously such a function is not necessarily holomorphic on $\Delta$. There are worse examples. For instance, the function

$$f(z) = \begin{cases} \frac{z^2}{\bar{z}} & (z \in \bar{\Delta} \setminus \{0\}) \\ 0 & (z = 0) \end{cases}$$

is continuous on $\bar{\Delta}$ and extends holomorphically from every circle $\Gamma$ in $\bar{\Delta}$ that either surrounds the origin or contains the origin, yet $f$ is not holomorphic on $\Delta$ [G2].

Let $p \in b\Delta$. In the present paper we show that if a continuous function on $\bar{\Delta}$ extends holomorphically from each circle centered at the origin and also from each circle contained in $\bar{\Delta}$ and passing through $p$ then it must be holomorphic on $\Delta$. In fact, we prove a somewhat better result:

**Theorem 1.1** Let $p \in b\Delta$ and let $\tau < 1/2$. Suppose that $f$ is a continuous function on $\bar{\Delta}$ such that

(i) $f$ extends holomorphically from each circle $|\zeta| = R$, $0 < R \leq 1$

(ii) $f$ extends holomorphically from each circle of radius $R \geq \tau$ which is contained in $\bar{\Delta}$ and passes through $p$.

Then $f$ is holomorphic on $\Delta$.

For each $z \in \mathbb{C}$, $r > 0$, denote $\Delta(z, r) = \{\zeta \in \mathbb{C}: |\zeta - z| < r\}$. Our family of circles can be written as $\{b\Delta(a(t), r(t)): 0 < t < 1\}$ where $t \mapsto a(t)$, $t \mapsto r(t)$ are piecewise
2. Semiquadrics and the related problem in $\mathbb{C}^2$

We begin the proof of our theorem. With no loss of generality assume that $p = -1$. As in [AG] and [G1] we introduce semiquadrics to pass to an associated problem in $\mathbb{C}^2$. Given $a \in \mathbb{C}$ and $r > 0$ let

$$\Lambda_{a,r} = \{(z, w) \in \mathbb{C}^2: (z - a)(w - \overline{a}) = r^2, \ 0 < |z - a| < r\}.$$ 

This is a closed complex submanifold of $\mathbb{C}^2 \setminus \Sigma$ where $\Sigma = \{(\zeta, \overline{\zeta}): \zeta \in \mathbb{C}\}$, which is attached to $\Sigma$ along $b\Lambda_{a,r} = \{(\zeta, \overline{\zeta}): \zeta \in b\Delta(a, r)\}$. A continuous function $g$ extends holomorphically from the circle $b\Delta(a, r)$ if and only if the function $G$, defined on $b\Lambda_{a,r}$ by $G(\zeta, \overline{\zeta}) = f(\zeta)$ ($\zeta \in b\Delta(a, r)$) has a bounded continuous extension to $\overline{\Lambda_{a,r}} = \Lambda_{a,r} \cup b\Lambda_{a,r}$ which is holomorphic on $\Lambda_{a,r}$. In fact, if we denote by the same letter $g$ the holomorphic extension of $g$ through $\Delta(a, r)$ we have

$$G\left(z, a + \frac{r^2}{z - a}\right) = g(z) \quad (z \in \overline{\Delta(a, r)} \setminus \{a\})$$

and, if we define $G(a, \infty) = g(a)$ we get a continuous function $G$ on $\Lambda_{a,r} = \overline{\Lambda_{a,r}} \cup \{(a, \infty)\}$, the closure of $\Lambda_{a,r}$ in $\mathbb{C}^2$.

It is known that if $(a, r) \neq (b, \rho)$ then $\Lambda_{a,r}$ meets $\Lambda_{b,\rho}$ if and only if $a \neq b$ and one of the circles $b\Delta(a, r)$, $b\Delta(b, \rho)$ surrounds the other [G1].

Let $\tau$ and $f$ be as in Theorem 1.1. By our assumption, $f$ extends holomorphically from two families of circles: $\{b\Delta(t, t + 1): -1 + \tau \leq t \leq 0\}$ and $\{b\Delta(0, R): 0 < R \leq 1\}$. Accordingly, there are two families of semiquadrics: $\{\Lambda_{t,t+1}: -1 + \tau \leq t \leq 0\}$ and $\{\Lambda_{0,R}: 0 < R \leq 1\}$ and the function $F(\zeta, \overline{\zeta}) = f(\zeta)$ ($\zeta \in \Delta$) has a bounded holomorphic extension through each of these semiquadrics. In each of these families the semiquadrics are pairwise disjoint. Let us look first at the first family and let $N$ be the closure of the union of $\Lambda_{t,t+1}$, $-1 + \tau \leq t \leq 0$ in $\mathbb{C} \times \overline{\mathbb{C}}$, that is,

$$N = \bigcup_{-1 + \tau \leq t \leq 0} [\Lambda_{t,t+1} \cup b\Lambda_{t,t+1} \cup \{(t, \infty)\}].$$

The continuity of $f$ together with the maximum principle implies that our function $(\zeta, \overline{\zeta}) \mapsto F(\zeta, \overline{\zeta}) = f(\zeta)$ defined on $\{((\zeta, \overline{\zeta})\zeta \in \Delta\} \setminus \Delta(-1 + \tau, \tau)$ continuously to $N$ so that the extension $F$ is holomorphic on each fiber $\Lambda_{t,t+1}$, $-1 + \tau \leq t \leq 0$. Note that the part $N_0$ of $N$ contained in $\mathbb{C} \times \mathbb{C}$ is a smooth CR manifold with piecewise smooth boundary consisting of three smooth pieces: $\Lambda_{-1 + \tau, \tau}, \Lambda_{0,1}$ and $N \cap \Sigma$ and the function $F$ is CR in the interior, that is,

$$\int_{N_0} f \bar{\partial} \omega = 0$$

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for each smooth \((2,0)\)-form \(\omega\) on \(\mathfrak{g}^2\) whose support intersects the interior of \(N_0\) in a compact set.

Now look at the second family and let \(L\) be the closure of the union of \(\Lambda_{0,R},\ 0 < R \leq 1,\) in \(\mathfrak{g} \times \mathfrak{g}\), that is
\[
L = \{0\} \times \mathfrak{g} \cup [\cup_{0 < R \leq 1} \Lambda_{0,R}].
\]
Again, our function \(F\) extends from \(L \cap \Sigma = \{(\zeta, \overline{\zeta}): \zeta \in \overline{\Delta}\}\) to a bounded continuous function on \(L\) which is holomorphic on each leaf \(\Lambda_{R,0},\ 0 < R \leq 1\). Again, the part \(L_0\) of \(L\) contained in \(\mathfrak{g}^2 \setminus (\{0\} \times \mathfrak{g})\) is a CR manifold with piecewise smooth boundary consisting of two pieces: \(\overline{\Lambda}_{0,1}\) and \(\{(\zeta, \overline{\zeta}): \zeta \in \overline{\Delta} \setminus \{0\}\}\) and the extension \(F\) is CR on the interior of \(L_0\).

Tumanov’s condition that no circle \(b\Delta(a(s), r(s))\) is contained in the closed disc \(\overline{\Delta}(a(t), r(t))\) if \(s \neq t\) implies that the semiquadrics \(\Lambda_{a(t),r(t)}\) are pairwise disjoint so their union is a CR manifold through which the function \(F\) extends as a CR function. Tumanov then uses an argument of H.Lewy [L] and the Liouville theorem to show that the function \(F\) does not depend on the second variable, that is, that \(f\) is holomorphic. We want to follow the same idea but in our case the semiquadrics are no more pairwise disjoint and so their union is not a manifold. In particular, the manifolds \(L\) and \(N\) intersect. However, we show that our particular geometric setting allows to apply the reasoning of Tumanov on \((L \cup N) \setminus (L \cap N)\), a CR manifold to which \(F\) extends as a CR function, to be able to conclude that the function \(F\) does not depend on the second variable. We provide a detailed proof of Theorem 1.1.

3. The manifolds \(L\) and \(N\)

As we have already mentioned, the function \(F\) extends to \(L\) and to \(N\) so that the extensions are holomorphic on semiquadrics, the holomorphic fibers of \(L\) and \(N\). There is one piece of \(L \cap N\), namely \(\Lambda_{0,1}\) on which both extensions coincide. However, a semiquadric of \(L\) can intersect a semiquadric of \(N\). In fact, \(\Lambda_{a,R}\) intersects \(\Lambda_{s,t+1}\) if and only if \(R < 2t+1\). We know that in this case the intersection consists of one point \([G1]\). It is easy to see that it is of the form \((x, y)\) where \(x > 0\) and \(y > 0\). This implies that there is no problem in defining the extension of \(F\) in the part of \(N \cup L\) which is contained in \((\mathfrak{g} \times \overline{\mathfrak{g}}) \setminus ([0, 1] \times \overline{\mathfrak{g}})\). We denote this part of \(L \cup N\) by \(M\). Let \(S = \Delta \setminus \{\overline{\Delta}(\tau, 1+\tau) \cup [0,1]\}\). Given \(z \in S\) we shall study \(M_z = \{\zeta \in \mathfrak{g}: (z, \zeta) \in M\}\). We shall show that if \(S_z \neq 0\), \(M_z\) is a closed curve consisting of the segment joining \(\overline{z}\) and \(1/z\) and a circular arc joining \(1/z\) and \(\overline{z}\) and if \(z \in \mathbb{R}\) then \(M_z\) is the real axis in \(\mathfrak{g}\).

**Lemma 3.1** Let \(z \in S,\ z \notin \mathbb{R}\). The circle \(C_z\) passing through \(1/z,\ \overline{z}\) and \(-1\) is tangent to the real axis at \(-1\). Let \(\lambda_z\) be the arc of \(C_z\) with end points \(\overline{z}\) and \(1/z\) which does not contain \(-1\). Then \(M_z\) consists of \(\lambda_z\) and of the segment joining \(\overline{z}\) and \(1/z\).

**Proof.** We have \(L \cap \{\{z\} \times \mathfrak{g}\} = \{\{z, R^2/2\}: |z| \leq R \leq 1\}\) which is the segment joining \((z, \overline{z})\) and \((z, 1/z)\). To find what \(N \cap \{\{z\} \times \mathfrak{g}\}\) is we recall first that
\[
\Lambda_{t,t+1} = \{(z, w): w = t + \frac{(t+1)^2}{z-t}, |z-t| < 1+t\}.
\]
So we must determine \( \{ w(t) : t(z) \leq t \leq 0 \} \) where

\[
w(t) = t + \frac{(t+1)^2}{z-t} = \frac{(z+2)t+1}{z-t}
\]

and where \( t(z) \) is such that \( z \) lies on the circle \( |\zeta - t(z)| = t(z) + 1 \), that is, when \( w(t(z)) = \overline{z} \).

To find what circle \( \{ (z+2)t+1 : t \in \mathbb{R} \} \) is, write \( z = P + iQ \) with \( P, Q \) real, and assume that \( Q \neq 0 \). We have

\[
\frac{(P+iQ+2)t+1}{P+iQ-t} = \frac{[(P+iQ)t+2t+1][P-iQ-t]}{(P-t)^2 + Q^2} = \frac{(P^2+Q^2)t+2t(P-iQ)+P-iQ-(P+iQ)t^2-2t^2-t}{(P-t)^2 + Q^2}.
\]

This is real when

\[
0 = -2tQ - Qt^2 = Q(t+1)^2,
\]

that is, when \( t = -1 \) when

\[
w = \frac{(z+2)(-1)+1}{z-(-1)} = -1.
\]

It follows that (3.1) is a circle tangent to the real axis at \(-1\) and it also follows that the arc from \( \overline{z} \) to \( 1/z \) containing \(-1\) does not belong to \( \{ \zeta : (z, \zeta) \in N \} \). We already know that this circle must contain \( \overline{z} \) and \( 1/z \). This completes the proof.

We may compute the center of the circle in Lemma 3.1 by intersecting the real line \( \{(1+\overline{z})/2 + i\lambda(-1-\overline{z})/2 : \lambda \in \mathbb{R} \} \) with the vertical line \( \Re \zeta = -1 \). Again, write \( z = P + iQ \) with \( P, Q \) real. We compute \( \lambda \) at the point of intersection from the condition \((1/2)(-1+P-\lambda Q) = -1\) which gives \( \lambda = (P+1)/Q \) and a short computation shows that the center of the circle is \(-1-i|z+1|^2/(2\overline{3}z)\).

We now look at what \( M_T \) is when \( T \in \mathcal{S} \) is real, that is, when \(-1+2\tau < T < 0\). Observe first that \( \Lambda_{0,R} \) intersects \( \{ T \} \times \mathfrak{C} \) if \(|T| < R < 1\) and \( \{ (T) \times \mathfrak{C} \} \cap \Lambda_{0,R} = \{(T, R^2/T)\} \). When \( R \) moves from \(|T|\) to 1 the point \( R^2/T \) moves on the real axis from \( T \) to \( 1/T \). This takes care of the intersection of \( \{ T \} \times \mathfrak{C} \) with \( L \). To find the intersection with \( N \) we have to see what

\[
w = t + \frac{(1+t)^2}{T-t} = \frac{t(T+2)+1}{T-t} = w(t)
\]

does when \( t \) decreases from 0 to \((T-1)/2\), that is, when \( \Lambda_{t,t+1} \) meets \( \{ T \} \times \mathfrak{C} \). At \( t = 0 \) we have \( w(0) = 1/T \) and as \( t \) decreases from 0 to \( T \), \( w(t) \) moves from \( 1/T \) to \(-\infty \) along
the real axis. When \( t \) decreases from \( T \) to \((T-1)/2\), \( w(t) \) decreases from \(+\infty\) to \( T \). Thus, \( M_T = \mathbb{R} \cup \{\infty\} \).

4. Completion of the proof of Theorem 1.1

Denote by \( \pi(z, w) = z \) the projection onto the first coordinate axis. Let \( U \subset S \) be a small open disc and consider \( \pi^{-1}(U) \cap M \). This set consists of two smooth manifolds \( \pi^{-1}(U) \cap N \) and \( \pi^{-1}(U) \cap L \) with common boundary \( \{(\zeta, 1/\zeta): \zeta \in U \} \cup \{(\zeta, \overline{\zeta}): \zeta \in U \} \) along which they meet transversely. The set \( \pi^{-1}(U) \cap M \) is a topological manifold which can be oriented as part of the boundary of \( \cup_{z \in U} \{z\} \times D_z \) where, for each \( z \in S \setminus \mathbb{R} \) and as \( z \) approaches \( a \in b\Delta \setminus \mathbb{R} \) they shrink to the point \( \overline{a} \).

Since we want to provide a detailed proof of Theorem 1.1, we shall need

**Lemma 4.1** Let \( B \) be an open ball in \( \mathbb{C}^2 \) and let \( E \subset B \) be a closed two-dimensional smooth submanifold of \( B \) which is the common boundary of two closed three-dimensional smooth submanifolds \( \Sigma_1 \) and \( \Sigma_2 \) of \( B \setminus E \) such that \( M = \Sigma_1 \cup \Sigma_2 \cup E \) is a topological submanifold of \( B \). Let \( f \) be a continuous function on \( M \) which is CR on \( \Sigma_1 \) and \( \Sigma_2 \), that is, \( \int_{\Sigma_i} f \overline{\partial} \alpha = 0 \) for each smooth, \((2,0)\) form on \( B \) whose support intersects \( \Sigma_i \) in a compact set, \( i = 1, 2 \). Then \( f \) is CR on \( M \), that is, \( \int_M f \overline{\partial} \alpha = 0 \) for every smooth, \((2,0)\) form on \( B \) whose support intersects \( M \) in a compact set.

**Proof.** The proof, suggested by E.L.Stout, uses the fact obtained by G. Lupacciolu [Lu] and C. Laurent-Thiebaut [LT], which in our simple case reduces to the fact that if a continuous function \( f \) is CR on \( \Sigma_i \), then for any smoothly bounded domain \( D \) in \( \Sigma_i \), compactly contained in \( \Sigma_i \), we have \( \int_D f \overline{\partial} \beta = \int_{bD} f \beta \) for every smooth, \((2,0)\) form on \( \mathbb{C}^2 \), \( i = 1, 2 \). The statement in our theorem is local so we may assume that \( E \) is a small perturbation of a piece of a two dimensional plane passing through the center \( T \) of \( B \) and that the smooth form \( \alpha \) has support contained in a small neighbourhood of \( T \). Let \( P \) be a small ball centered at \( T \) containing the support of \( \alpha \) in its interior and, for small \( \varepsilon > 0 \), let \( P_\varepsilon \) consist of those points of \( P \) whose distance from \( E \) exceeds \( \varepsilon \). For \( i = 1, 2 \) let \( P_{\varepsilon,i} = P_\varepsilon \cap \Sigma_i \) and let \( S_{\varepsilon,i} = \{z \in P \cap \Sigma_i : \text{dist}(z,E) = \varepsilon\} \). The sets \( S_{\varepsilon,i} \) are the only parts of the boundaries of \( P_{\varepsilon,i}, \ i = 1, 2 \) which intersect support of \( \alpha \) and as \( \varepsilon \) tends to zero, they, as oriented pieces of the boundaries of \( P_{\varepsilon,i}, \ i = 1, 2 \), converge to \( E \cap P \) with the opposite orientations.
Now,
\[ \int_M f \overline{\partial} \alpha = \lim_{\varepsilon \to 0} \int_{P_{\varepsilon}} f \overline{\partial} \alpha \]
\[ = \lim_{\varepsilon \to 0} \left[ \int_{P_{\varepsilon,1}} f \overline{\partial} \alpha + \int_{P_{\varepsilon,2}} f \overline{\partial} \alpha \right] \]
\[ = \lim_{\varepsilon \to 0} \left[ \int_{bP_{\varepsilon,1}} f \alpha + \int_{bP_{\varepsilon,2}} f \alpha \right] \]
\[ = \lim_{\varepsilon \to 0} \left[ \int_{S_{\varepsilon,1}} f \alpha + \int_{S_{\varepsilon,2}} f \alpha \right] \]
\[ = 0 \]

This completes the proof.

**LEMMA 4.2** Suppose that \( U \) is a small open disc whose closure is contained in \( S \setminus \mathbb{R} \) and that \( G \) is a continuous function on \( \pi^{-1}(U) \cap M \) which is holomorphic on each holomorphic leaf of \( \pi^{-1}(U) \cap N \) and on each holomorphic leaf of \( \pi^{-1}(U) \cap L \). Then \( G \) is CR on \( \pi^{-1}(U) \cap M \), that is
\[ \int_{\pi^{-1}(U) \cap M} G \overline{\partial} \beta = 0 \]
for each smooth two zero form \( \beta \) on \( \mathbb{C}^2 \) whose support meets \( \pi^{-1}(U) \cap M \) in a compact set.

The lemma says that if \( G \) is CR on \( [\pi^{-1}(U) \cap L] \setminus (L \cap N) \) and on \( [\pi^{-1}(U) \cap N] \setminus (L \cap N) \) then \( G \) is CR on \( \pi^{-1}(U) \cap M \). This obviously follows from Lemma 4.1 and the fact that if a continuous function is holomorphic on each holomorphic leaf then it is CR.

**LEMMA 4.3** Let \( U \subset S \setminus \mathbb{R} \) be an open disc and let \( M_U = \pi^{-1}(U) \cap M \). Suppose that \( G \) is a continuous CR function on \( M_U \), that is, given a smooth two-zero form \( \omega \) whose support intersects \( M_U \) in a compact set, we have
\[ \int_{M_U} G \overline{\partial} \omega = 0. \]

Then the function \( z \mapsto \int_{M_z} G(z, w) dw \) is holomorphic on \( U \).

**Proof.** Note first that if \( K \subset U \) is a compact set then \( \pi^{-1}(K) \) intersects \( M_U \) in a compact set. Let \( \alpha \) be a smooth function of one complex variable \( z \) with compact support contained in \( U \). Then \( \beta(z, w) = \alpha(z) dz \wedge dw \) is a smooth form on \( \mathbb{C}^2 \) whose support intersects \( M_U \) in a compact set so
\[ \int_M G(z, w) \frac{\partial \alpha(z)}{\partial \overline{z}} d\overline{z} \wedge dz \wedge dw = 0 \]
which, by Fubini, implies that
\[ \int_U \left( \int_{M_z} G(z, w) dw \right) \frac{\partial \alpha(z)}{\partial \overline{z}} dz = 0. \]
Since this holds for every smooth function \( \alpha \) with compact support contained in \( U \) it follows that the function \( z \mapsto \int_{M_z} G(z,w)dw \) is holomorphic on \( U \). This completes the proof.

We now show that for each \( z \in S \setminus \mathbb{R} \) the function \( w \mapsto F(z,w) \), defined on \( M_z \), extends holomorphically through \( D_z \). Consider the function

\[
H(z,w,W) = \frac{F(z,w)}{w-W}.
\]

For each \( \eta > 0 \) there is an \( R(\eta) < \infty \) such that if \( |W| > R(\eta) \), the function \( (z,w) \mapsto H(z,w,W) \) is well defined and continuous on \( P_\eta = \{(z,w) \in M : z \in S_\eta \} \) where \( S_\eta = z \in S, |\Im z| > \eta \}, and is holomorphic on each holomorphic leaf in \( P_\eta \) so by Lemma 4.2 it is CR on \( P_\eta \). Lemma 4.3 now implies that for each fixed \( W \), \( |W| > R(\eta) \), \( z \mapsto \frac{1}{2\pi i} \int_{M_z} F(z,w) \frac{1}{w-W} dw = \Theta(z,W) \)

is holomorphic on \( P_\eta \). Since \( z \mapsto \Theta(z,W) \) is continuous on \( S_\eta \) and since the curves \( M_z \) shrink to \( \overline{a} \) when \( z \in S_\eta \) approaches \( a \in b\Delta \setminus \mathbb{R} \) it follows that \( \Theta(z,W) = 0 \) for each \( z \in S_\eta \cap b\Delta \setminus \mathbb{R} \) and thus \( \Theta(z,W) \equiv 0 \) (\( z \in S_\eta \)). Thus, for each \( z \in S_\eta \) we have \( \Theta(z,W) = 0 \) for all \( W, |W| > R(\eta) \) which implies that

\[
\frac{1}{2\pi i} \int_{M_z} F(z,w) \frac{1}{w-W} dw = 0
\]

for all \( W \in \mathbb{C} \setminus \overline{D_z} \) which implies that the function \( w \mapsto F(z,w) \) extends from \( M_z \) holomorphically through \( D_z \) for each \( z \in S_\eta \). Since \( \eta > 0 \) was arbitrary this holds for each \( z \in S \setminus \mathbb{R} \).

Recall that \( (-1 + 2\tau, 0) \times \{\infty\} \subset M \) and that \( F \) is continuous on \( M \). Given \( T, -1 + 2\tau < T < 0 \), we will show that \( F \) is constant on \( \{T\} \times M_T \). To do this, we use the reasoning of Tumanov. Fix \( T \in (-1 + 2\tau, 0) \) and observe that for small \( \eta > 0 \), \( M_{T+i\eta} \) are simple closed curves bounding \( D_{T+i\eta} \) which depend continuously on \( \eta \) and, as domains in \( \overline{T} \), continuously tend to the halfplane \( \Im \zeta < 0 \) as \( \eta \) tends to 0. Since for each small \( \eta > 0 \) the function \( \zeta \mapsto F(T+i\eta, \zeta) \) extends from \( M_{T+i\eta} \) holomorphically through \( D_{T+i\eta} \), the continuity of \( F \) implies that \( t \mapsto F(T,t) \) has a bounded holomorphic extension from \( \mathbb{R} \) through the upper halfplane. Thus, \( t \mapsto F(T,t) \) has a bounded holomorphic extension through \( \mathbb{C} \) which, by the Liouville theorem, must be constant. Thus, for each \( T, -1 + 2\tau < T < 0 \), the holomorphic extensions of \( f \) from all circles in our family which surround \( T \), coincide. This implies that \( f \) is holomorphic in a neighbourhood of the segment \( (-1 + 2\tau, 0) \) and it is easy to see that the analyticity propagates along the circles so it follows that \( f \) is holomorphic on \( \Delta \). This completes the proof of Theorem 1.1.

5. Remarks
A careful examination of the proof of Theorem 1.1 shows that to prove that \( f \) is holomorphic there is no need to assume that \( f \) extends holomorphically from each circle centered at the origin. In fact, the same proof gives

**THEOREM 5.1** Let \( f \) be a continuous function on \( \overline{\Delta} \) and let \( p \in b\Delta, \ 0 < r < 1, 0 < \rho < 1 \). Assume that \( f \) extends holomorphically from each circle of radius \( R, \ r \leq R \leq 1 \), centered at the origin, and from each circle of radius \( R, \ \rho \leq R \leq 1 \), passing through \( p \) and contained in \( \overline{\Delta} \). If the smallest circles of these two families are disjoint then \( f \) is holomorphic on \( \Delta \).

Let \( p_1, p_2 \in b\Delta, \ p_1 \neq p_2 \). In a way similar to the way above we prove that a continuous function on \( \overline{\Delta} \) which extends holomorphically from each circle contained in \( \overline{\Delta} \) and passing through \( p_1 \) and which extends holomorphically from each circle contained in \( \overline{\Delta} \) and passing through \( p_2 \) then \( f \) is holomorphic on \( \Delta \). In fact, again, fewer circles suffice:

**THEOREM 5.2** Let \( p_1, p_2 \in b\Delta, \ p_1 \neq p_2 \). Let \( 0 < r_j < 1, \ j = 1, 2 \), and assume that \( f \) is a continuous function on \( \overline{\Delta} \) which, for each \( j = 1, 2 \), extends holomorphically from each circle of radius \( \rho, \ r_j \leq \rho \leq 1 \), contained in \( \overline{\Delta} \) and passing through \( p_j \). If the smallest circles of these two families are disjoint then \( f \) is holomorphic on \( \Delta \).

Note that, after applying an automorphism of \( \Delta \) one can, with no loss of generality assume that \( p_1 = -1, \ p_2 = 1 \). The domains \( D_z \) now are bounded by two circular arcs. Note that the example in Section 1 shows that in both theorems the condition that the smallest circles of the families be disjoint cannot be omitted.

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**References**

[A] M. L. Agranovsky: Propagation of boundary CR foliations and Morera type theorems for manifolds with attached analytic discs.
Adv. Math. 211 (2007) 284–326.

[AG] M. L. Agranovsky and J. Globevnik: Analyticity on circles for rational and real-analytic functions of two real variables.
J. d’Analyse Math. 91 (2003) 31-65

[G1] J. Globevnik: Holomorphic extensions from open families of circles.
Trans. Amer. Math. Soc. 355 (2003) 1921-1931

[G2] J. Globevnik: Analyticity on families of circles.
Israel J. Math. 142 (2004) 29-45

[G3] J. Globevnik: Analyticity on translates of a Jordan curve.
[L] H. Lewy: On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables. Ann. of Math. 64 (1956) 514-522

[Lu] G. Lupacciolu: A theorem on holomorphic extensions of CR-functions. Pacif. J. Math. 124 (1986) 177-191

[LT] C. Laurent-Thiebaut: Sur l’extension des fonctions CR dans une variete de Stein. Ann. Mat. Pura Appl. 150 (1988) 141–151

[R] H. Rossi: A generalization of a theorem of Hans Lewy. Proc. Amer. Math. Soc. 19 (1968) 436-440

[T1] A. Tumanov: A Morera type theorem in the strip. Math. Res. Lett. 11 (2004) 23-29

[T2] A. Tumanov: Testing analyticity on circles. Amer. J. Math. 129 (2007) 785-790

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