GAUGE THEORIES FROM WRAPPED AND FRACTIONAL BRANES 

P. Di Vecchia \textsuperscript{a}, H. Enger \textsuperscript{b}, E. Imeroni \textsuperscript{c, a} and E. Lozano–Tellechea \textsuperscript{d, a}

\textsuperscript{a} NORDITA, Blegdamsvej 17, DK-2100 Copenhagen \textit{\textTheta}, Denmark

\textsuperscript{b} Department of Physics, University of Oslo, P.O. Box 1048 Blindern, N-0316 Oslo, Norway

\textsuperscript{c} Dipartimento di Fisica Teorica, Università di Torino and I.N.F.N., Sezione di Torino, Via P. Giuria 1, I-10125 Torino, Italy

\textsuperscript{d} Instituto de Física Teórica, C-XVI, Universidad Autónoma de Madrid, E-28049 Madrid, Spain

Abstract

We compare two applications of the gauge/gravity correspondence to a non conformal gauge theory, based respectively on the study of D-branes wrapped on supersymmetric cycles and of fractional D-branes on orbifolds. We study two brane systems whose geometry is dual to $\mathcal{N} = 4$, $D = 2 + 1$ super Yang–Mills theory, the first one describing D4-branes wrapped on a two-sphere inside a Calabi–Yau two-fold and the second one corresponding to a system of fractional D2/D6-branes on the orbifold $\mathbb{R}^4/\mathbb{Z}_2$. By probing both geometries we recover the exact perturbative running coupling constant and metric on the moduli space of the gauge theory. We also find a general expression for the running coupling constant of the gauge theory in terms of the “stringy volume” of the two-cycle which is involved in both types of brane systems.

\textsuperscript{1}Work partially supported by the European Commission RTN programme HPRN-CT-2000-00131.
1 Introduction

The gauge/gravity correspondence has its origin on the fact that, on the one hand, D-branes are classical solutions of the low-energy string effective action and, on the other hand, they have a gauge theory living in their world-volume. This means that the low-energy dynamics of D-branes can be used to determine the properties of the gauge theory and vice-versa.

The most successful realization of this correspondence is the Maldacena conjecture \[1\], confirmed by all subsequent studies, according to which ten-dimensional type IIB string theory compactified on \( AdS_5 \times S^5 \) is dual to \( \mathcal{N} = 4 \) super Yang–Mills theory in four-dimensional Minkowski spacetime.

However, \( \mathcal{N} = 4 \) super Yang–Mills in four dimensions is a rather special theory due to its conformal properties and its high amount of supersymmetry. Therefore, a lot of effort has been recently devoted to find possible extensions of the Maldacena duality to non conformal and less supersymmetric gauge theories, or at least to use the low-energy brane dynamics to extract information on the properties of such more realistic theories.

Two approaches to this problem have been largely pursued, based respectively on the study of:
• D-branes wrapped on supersymmetric cycles \[2\];
• Fractional D-branes on orbifolds \[3, 4, 5, 6\].

After \(\mathcal{N} = 4\) super Yang–Mills, the next natural system to consider is \(\mathcal{N} = 2\) super Yang–Mills theory in four dimensions, also with matter. This system has been studied, on the one hand, by considering fractional D3-branes on orbifolds \[11, 12, 13\] and systems made of fractional D3/D7-branes \[14, 15\], and, on the other hand, by considering D5-branes wrapped on a two-cycle inside a Calabi–Yau two-fold \[16, 17\]. The low-energy dynamics of wrapped branes has also been recently used to study other gauge theories \[18, 19, 20, 21, 22, 23, 24\].

The use of fractional and wrapped branes presents some interesting similarities that are not surprising since fractional branes on orbifolds can be seen as D-branes wrapped on cycles that are vanishing in the orbifold limit of the ALE space which corresponds to the blow-up of the orbifold space \[3, 4, 5\]. In fact, by probing the supergravity solutions describing the two types of systems, one is able to recover all the relevant perturbative information on the Coulomb branch of the gauge theory living on the branes, namely the running coupling constant and the metric on the moduli space of the theory.

These two approaches, however, have not been able to provide information on the non-perturbative features of the gauge theories, as for instance on the instanton contribution to the moduli space of \(\mathcal{N} = 2\) super Yang–Mills in four dimensions. This is related to the existence at short distance of an enhançon \[25\] where the supergravity solution becomes inconsistent, because at this distance the probe brane becomes tensionless, signalling the appearance of new massless degrees of freedom. This means that the supergravity approximation is not valid anymore and that the region inside the enhançon is excised. This fact prevents to get information on the strong coupling regime of the gauge theory living in the world-volume of the branes, that is in fact determined by what happens inside the enhançon. To overcome this problem one must presumably also include the new massless degrees of freedom, as attempted for instance in Ref. \[26\].

In this paper we will apply both approaches discussed above to the study of \(\mathcal{N} = 4, D = 2 + 1\) super Yang–Mills theory, which is a theory with 8 Poincaré supercharges. The interest in this theory resides mainly in the fact that its properties, perturbative and nonperturbative, are well known \[30, 31\]. This is also the theory where the enhançon was first found \[25\] using a different approach based on the study of D6-branes wrapped on \(K3\) surfaces.

We will first consider a system made up of \(N\) D4-branes wrapped on a two-cycle inside a Calabi–Yau two-fold. The crucial property of this system, as of any other system of branes wrapped on supersymmetric cycles, is that the geometrical structure of the background forces the gauge theory living on the world-volume of the branes to be partially topologically

\[\text{Apart from the two approaches that we consider in this paper, other interesting ones are based on the study of fractional D-branes on conifolds (see Ref. \[6\] and Ref.s therein) and of M-branes wrapped on Riemann surfaces \[8, 9, 10\].}\n
\[\text{For recent developments concerning the physics of the enhançon see for instance Ref.s \[27, 28, 29\].}\]
twisted \cite{32} and this allows to preserve the desired amount of supersymmetry. To find the
supergravity solution describing the D4-branes, we will use the techniques introduced in
Ref. \cite{2}, which amount to find a solution of a lower dimensional gauged supergravity and
then uplift it to ten or eleven dimensions. We will then use the uplifted solution in a probe
computation in order to extract information on the Coulomb branch of the gauge theory
which lives on the flat three-dimensional part of the world-volume of the brane, which is
pure $\mathcal{N} = 4$, $D = 2 + 1$ super Yang–Mills with gauge group $SU(N)$.

Then, we will consider a system made of $N$ fractional D2-branes and $M$ D6-branes on
the orbifold $\mathbb{R}^4/\mathbb{Z}_2$ and, solving explicitly the equations of motion of type IIA supergravity,
we will find the corresponding classical solution. The probe computation will give us
information on the same three-dimensional theory, now also coupled to $M$ hypermultiplets
in the fundamental representation of the gauge group.

In both approaches we find that, as in other cases, the probe analysis correctly repro-
duces the perturbative part of the moduli space, giving the exact running coupling constant
of $\mathcal{N} = 4$ super Yang–Mills in three dimensions, but is unable to give the instanton con-
tribution. This analysis allows us to make some comments on the relation between the
two solutions and to see that in both cases the gauge coupling constant can be obtained
from a common expression representing the "stringy volume" of the two-cycle on which
the branes are wrapped. Moreover, in both cases the locus where the "stringy volume"
vanishes corresponds to the point where the Calabi–Yau two-fold in which the cycle is
embedded manifests an enhanced gauge symmetry, which is the origin of the enhançon
mechanism.

The structure of this paper is as follows. Section 2 and 3 are organized in an entirely
parallel way, and can be read independently from each other. They describe the two
different brane systems that we study in order to get information on $\mathcal{N} = 4$, $D = 2 + 1$
super Yang–Mills theory from supergravity, namely a system of $N$ D4-branes wrapped on
$S^2$ (in section 2) and a system of $N$ fractional D2-branes and $M$ D6-branes on the orbifold
$\mathbb{R}^4/\mathbb{Z}_2$ (in section 3). In section 4, we discuss and comment the results of the previous
two sections. Many details of the various computations are given in the appendices. In
appendix A we fix the conventions and discuss in detail how the two supergravity solutions
were found. In appendix B we discuss the world-volume actions for fractional branes.
Finally, in appendix C we give some details about the perturbative computation of the
running coupling constant of the gauge theory that we consider.

2 D4-branes wrapped on $S^2$

2.1 Setup

In this section we are going to consider a system made of $N$ D4-branes with two longitudinal
directions wrapped on a two-sphere.

As discussed in Ref. \cite{32}, the gauge theory living on the world-volume of wrapped branes
has to be topologically twisted. In this subsection we want to determine the topological
twist that is needed in order to obtain at low-energy on the flat part of the world-volume of the D4-branes $N = 4$ super Yang–Mills theory in three space-time dimensions, that is a theory with 8 supercharges. The twist which preserves 8 supercharges is exactly the one imposed by the geometrical structure of the background when the two-sphere is seen as a nontrivial two-cycle inside a Calabi–Yau two-fold.

The configuration that we are going to study is schematically shown in the following table, where the symbols – , ⌢ and · represent respectively unwrapped world-volume directions, wrapped world-volume directions and transverse directions:

| D4 | $\mathbb{R}^{1,2}$ | $S^2$ | $N_2$ | $\mathbb{R}^3$ |
|----|------------------|------|-------|-------------|
|     | – – – – ⌢ ⌢ · · |

In flat space, the presence of the D4-brane breaks spacetime Lorentz invariance in the following way: $SO(1,9) \rightarrow SO(1,4) \times SO(5)_R$. The fact that the D4-brane is wrapped on $S^2$ introduces an additional breaking of $SO(1,4)$ into $SO(1,2) \times SO(2)_{S^2}$. The twist is then introduced by breaking the $R$-symmetry group $SO(5)_R$ into $SO(2)_G \times SO(3)$ and by identifying $SO(2)_G$ with $SO(2)_{S^2}$. In conclusion our configuration breaks the original $SO(1,4) \times SO(5)_R$ into $SO(1,2) \times SO(2)_{S^2} \times SO(2)_G \times SO(3)$ with the two $SO(2)$ groups identified. The fields of the gauge theory living on the wrapped D4-branes transform according to the following representations of the above groups:

|             | $SO(1,4)$ $\rightarrow$ $SO(1,2) \times SO(2)_{S^2}$ | $SO(5)_R$ $\rightarrow$ $SO(2)_D \times SO(3)$ |
|-------------|-------------------------------------------------|----------------------------------|
| Vector      | $5 \rightarrow (3,1) \oplus (1,2)$              | $1 \rightarrow (1,1)$            |
| Scalars     | $1 \rightarrow (1,1)$                            | $5 \rightarrow (1,3) \oplus (2,1)$|
| Fermions    | $4 \rightarrow (2,+) \oplus (2,–)$              | $4 \rightarrow (+,2) \oplus (–,2)$|

Since we are interested in the three-dimensional theory living on the flat part of the world-volume at very low energies, we must keep only the massless states, which are the ones transforming as singlets under $SO(2)_D \equiv (SO(2)_{S^2} \times SO(2)_G)_{\text{diag}}$:

|             | $SO(1,2) \times SO(2)_D \times SO(3)$ |
|-------------|--------------------------------------|
| Vector      | $(3,1,1)$                             |
| Scalars     | $(1,1,3)$                             |
| Fermions    | $2 \times (2,1,2)$                    |

These states form exactly the vector multiplet of $N = 4$, $D = 2 + 1$ super Yang–Mills theory.

### 2.2 The supergravity solution

In this subsection we will construct a supergravity solution describing the system just introduced, made of $N$ D4-branes with two world-volume directions wrapped on $S^2$. One could in principle work in ten-dimensional type IIA supergravity, write a suitable Ansatz
for such a system, then solve the equations of motion and find the solution. This is, however, not a simple task because it is not easy to implement directly in ten dimensions the topological twist that we have discussed in the previous subsection from the point of view of the gauge theory living on the brane. One has to proceed in a longer, but more straightforward way that has been introduced in Ref. [2]. Instead of working directly in the ten dimensional theory one starts by considering, for the case of a p-brane, a \((p + 2)\)-dimensional gauged supergravity theory that is obtained by compactifying the original \(D\)-dimensional theory (where of course \(D = 10\) for the case of a D-brane or NS5-brane and \(D = 11\) for the case of an M-brane) on \(S^{D-p-2}\). The isometry group \(SO(D - p - 1)\) of \(S^{D-p-2}\) corresponds to the \(R\)-symmetry group that we discussed in the previous subsection. In gauged supergravity the \(R\)-symmetry group is gauged so that the theory contains \(SO(D - p - 1)\) gauge fields. In this theory one looks for a domain wall solution that preserves the desired amount of supersymmetry and breaks the original \(R\)-symmetry group in a way that implements the correct twist. In fact, in gauged supergravity the supersymmetry preserving condition contains also the gauge fields and can schematically be written as \((\partial_\mu + \omega_\mu + A_\mu) \epsilon = 0\). The discussed twist corresponds to the identification of some of the gauge fields with the spin connection of the manifold around which the brane is wrapped, \(A_\mu = -\omega_\mu\), so that the request of finding covariantly constant spinors is equivalent to that of just finding constant spinors. Once the solution with the correct properties has been found, the last step is to uplift it to \(D\) dimensions by using the formulas given in Refs. [33, 34].

In the following, in order to avoid many new calculations, we do not use directly a 6-dimensional gauged supergravity as it would be natural for a D4-brane. We will instead proceed in slightly different way by exploiting the fact that the solution of seven-dimensional gauged supergravity corresponding in eleven dimensions to an M5-brane wrapped on \(S^2\) and preserving 8 supercharges has already been constructed [2]. Therefore we proceed as follows. We start from the solution of 7-dimensional gauged supergravity given in Ref. [3], and uplifting it to eleven dimensions using the formulas found in Ref. [33] we obtain a solution of 11-dimensional supergravity describing \(N\) M5-branes wrapped on \(S^2\). Finally, upon compactification to ten dimensions we get the desired solution describing \(N\) D4-branes wrapped on \(S^2\). The details of this procedure are given explicitly in Appendix A.1. Here we write directly the ten-dimensional solution in the string frame, which reads:\(^4\)

\[\begin{align}
\mathrm{d} s^2_{st} &= \left( \frac{R_A}{R_0} \right)^3 \Delta^{1/2} e^{3\rho} \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta + \frac{R_A^3}{R_0^3} \Delta^{1/2} e^\rho (e^{2\rho} - \frac{1}{4}) (d\bar{\theta}^2 + \sin^2 \theta d\bar{\varphi}^2) \\
&\quad + \frac{R_A^3}{4R_0} \Delta^{-1/2} e^\rho \left( \frac{4\Delta}{e^{5A}} dp^2 + \Delta d\chi^2 + \cos^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) + e^{5\lambda} \sin^2 \chi \left( d\psi + \cos \theta d\bar{\varphi} \right)^2 \right), \\
\nonumber
\end{align}\]

\[\begin{align}
e^{2\phi} &= \left( \frac{R_A}{R_0} \right)^3 \Delta^{1/2} e^{3\rho},
\end{align}\]

\(^4\)This solution was partially given in appendix 7.4 of Ref. [3].
\[ C_3 = \frac{R_A^3}{8} \Delta e^{5\lambda} \cos^3 \chi \cos \theta \sin \theta \frac{d\tilde{\theta} \wedge d\tilde{\varphi} \wedge d\varphi}{\Delta} + \frac{R_A^3}{8} e^{5\lambda} (\Delta + 2) \cos^2 \chi \sin \chi \cos \theta \frac{d\chi \wedge d\varphi \wedge (d\psi + \cos \tilde{\theta} d\tilde{\varphi})}{\Delta^2} + \frac{R_A^3}{8} \partial_\rho (e^{5\lambda}) \cos^3 \chi \sin^2 \chi \cos \theta \frac{d\rho \wedge d\varphi \wedge (d\psi + \cos \tilde{\theta} d\tilde{\varphi})}{\Delta^2}, \] (2.1c)

where the functions \( e^{5\lambda} \) and \( \Delta \) entering the solution are given by:

\[
e^{5\lambda} = \frac{e^{2\rho} + ke^{-2\rho} - \frac{1}{2}}{e^{2\rho} - \frac{1}{4}},\quad (2.2a)
\]
\[
\Delta = e^{5\lambda} \cos^2 \chi + \sin^2 \chi.\quad (2.2b)
\]

Before proceeding let us give a short explanation of the various coordinates and constants appearing in eq.s (2.1)-(2.2):

- \( \xi^\alpha,\beta (\alpha, \beta = 0, 1, 2) \) are the coordinates along the unwrapped brane world-volume;
- \( \tilde{\theta}, \tilde{\varphi} \) are the coordinates along the wrapped world-volume;
- \( \rho \) is a radial coordinate transverse to the brane;
- \( \chi, \theta, \varphi, \psi \) parameterize the “twisted” four-sphere transverse to the brane;
- \( R_A \) is the radius of the \( \text{AdS}_7 \) space appearing in the near horizon geometry of the usual “flat” M5-brane solution (see appendix A.1), which is given in terms of ten dimensional quantities by \( R_A = 2\sqrt{\alpha'/\pi g_s N}^{1/3} \);
- \( R_0 \) is an arbitrary integration constant with dimension of a length that we will show to set the scale of the radius of the \( S^2 \) on which the D4-branes are wrapped;
- \( k \) is a dimensionless integration constant.

All coordinates are dimensionless except \( \xi^\alpha \) which have dimension of a length.

A D4-brane is coupled naturally to a 5-form potential while the solution given above contains a RR 3-form potential. However, the latter is related to \( C_5 \) by the duality relation \( dC_5 = * dC_3 \) (in the string frame). By using it we get:

\[
C_5 = \frac{R_0^6}{R_A^4} \Delta e^{4\rho} \left( e^{2\rho} - \frac{1}{4} \right) \sin \tilde{\theta} d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\tilde{\theta} \wedge d\tilde{\varphi} \]
\[
- \frac{R_0^6}{R_A^4} \frac{1}{2} e^{4\rho} \sin^2 \chi d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\rho \wedge \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right) \]
\[
- \frac{R_0^6}{R_A^4} \frac{1}{8} e^{4\rho} e^{5\lambda} \sin(2\chi) d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\chi \wedge \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right). \quad (2.3)
\]
A change of coordinates

The supergravity solution for the D4-branes wrapped on $S^2$ as given in eq. (2.1) is written in a way in which the role of the different coordinates and factors is not immediately clear. The first thing that we can do in order to clarify the role of the various terms appearing in the solution is to extract the warp factors for the longitudinal and transverse part of the metric in the string frame. They are given in terms of a function $H$ that for a D4-brane is related to the dilaton through the following relation:

$$ H = e^{-4\phi} = \left( \frac{R_0}{R_A} \right)^6 \Delta^{-1} e^{-6\rho}. \quad (2.4) $$

Using the previous definition of $H$, one can immediately see that the dependence on $H$ of the four longitudinal unwrapped directions of the metric is the one corresponding to four “flat” world-volume directions: $H^{-1/2} \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$, as expected. We also expect three transverse directions ($\theta$, $\varphi$ and a suitable combination of $\rho$ and $\chi$) to be flat, apart from the usual warp factor $H^{1/2}$. This can be seen to be correct by using instead of the coordinates $\rho$ and $\chi$ the following new coordinates:

$$ \begin{cases} 
  r = \frac{R_A^3}{2R_0^2} e^{2\rho} \cos \chi \\
  \sigma = \frac{R_A^3}{2R_0^2} \left[ e^{2\rho} \left( e^{2\rho} - \frac{1}{4} \right) e^{5\chi} \right]^{1/2} \sin \chi 
\end{cases} \quad (2.5) $$

which have dimensions of a length. In terms of the new coordinates in eq. (2.5), the solution for the metric, dilaton and R-R 5-form becomes:

$$ ds_{st}^2 = H^{-1/2} \left[ \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta + Z R_0^2 (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) \right] + H^{1/2} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) + \frac{1}{Z} \left( d\sigma^2 + \sigma^2 \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right)^2 \right) \right], \quad (2.6a) $$

$$ e^\phi = H^{-1/4}, \quad (2.6b) $$

$$ C_5 = d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge \left[ \frac{1}{H} Z R_0^2 \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi} - \frac{1}{Z} \sigma d\sigma \wedge \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right) \right], \quad (2.6c) $$

where the functions $H$ and $Z$ are (implicitly) defined as:

$$ H(r, \sigma) = \left( \frac{R_0}{R_A} \right)^6 \Delta^{-1}(r, \sigma) e^{-6\rho(r, \sigma)}, \quad Z(r, \sigma) = e^{-2\rho(r, \sigma)} \left( e^{2\rho(r, \sigma)} - \frac{1}{4} \right). \quad (2.7) $$

In the form given in eq. (2.4) the structure of the solution is much clearer. First of all one can clearly distinguish the trivial “flat” part of the solution from the nontrivial part

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5See also Refs. [35, 36] for related discussions.
coming from the internal directions of the four-dimensional Calabi-Yau space. In this sense, the coordinates $r$ and $\sigma$ that we have introduced represent two radial directions, respectively in the “flat” transverse space and in the space $N_2$ transverse to the brane but nontrivially fibered on the two-cycle on which the brane is wrapped. Moreover, the function $Z$ represents the “running volume” of the two-cycle, with the constant $R_0$ being the radius of the $S^2$ when $Z = 1$, while in the part of the metric containing $\sigma$ and $\psi$ we can easily see the twist which, as we have seen in section 2.1, is required for having a supersymmetric gauge theory living on the brane. Finally, also the R-R potential has a quite standard part ($H^{-1}$ times the volume form of the longitudinal space), plus an additional part due to the twist.

Another change of coordinates can be implemented to extract some additional piece of information about the solution. If we define a new coordinate $z$ and a function $\tilde{Z}$ as follows:

\[
\begin{align*}
    z &= R_0 \left(1 + \frac{\sigma^2}{R_0^2}\right)^{1/4} \\
    \tilde{Z} &= Z \left(1 + \frac{\sigma^2}{R_0^2}\right)^{-1/2}
\end{align*}
\]

the metric in eq. (2.6a) becomes:

\[
ds_{st}^2 = H^{-1/2} \left\{ \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta + \tilde{Z} \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right) \right\} + H^{1/2} \left\{ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + \frac{1}{\tilde{Z}} \left[ 4 \left( 1 - \frac{R_0^4}{z^4} \right)^{-1} dz^2 + z^2 \left( 1 - \frac{R_0^4}{z^4} \right) (d\psi + \cos \tilde{\theta} d\tilde{\phi})^2 \right] \right\}.
\]

The metric we have obtained on the four-dimensional space spanned by the coordinates $\{\tilde{\theta}, \tilde{\phi}, z, \psi\}$ is that of a “warped” Eguchi–Hanson space [37]. This fact provides additional evidence of the geometrical structure of the background: the D4-branes are wrapped on the two-sphere, of radius $R_0$, inside the simplest ALE space (which corresponds to the blow-up of an $\mathbb{R}^4/\mathbb{Z}_2$ orbifold).\footnote{As an aside, notice that, using eqs (2.4)-(2.5) and eq. (2.8), also the M5-brane solution (A.5) (from which we derived the D4-brane solution) can be brought into a form analogous to the one given in eq. (2.6) or in eq. (2.3). These forms for the classical solutions describing wrapped branes seem indeed to be quite general. For instance, changes of coordinates similar to the ones implemented here can be used to put in these forms also the solution found in Ref. [16] for D5-branes wrapped on $S^2$.}

### 2.3 Probing the wrapped brane solution

In order to extract information on the gauge theory living on the D4-branes, we will study the dynamics of a probe D4-brane wrapped on $S^2$ in the geometry generated by the solution found in the previous subsection (see Ref. [38] for a review on the probe technique). This
will allow us to study the Coulomb branch of pure \( N = 4 \) SYM theory in \( 2 + 1 \) dimensions with gauge group \( SU(N + 1) \rightarrow SU(N) \times U(1) \). The world-volume action for a single D4-brane in the string frame in the static gauge is given by:

\[
S_{\text{probe}} = -\frac{T_4}{\kappa} \int d^4 \xi d\bar{\theta} d\bar{\phi} e^{-\phi} \sqrt{-\text{det} [G_{ab} + 2\pi\alpha' F_{ab}]} + \frac{T_4}{\kappa} \int_{\mathcal{M}_5} (C_5 + 2\pi\alpha' C_3 \wedge F),
\]

(2.10)

where \( a, b = \{0, 1, 2, \bar{\theta}, \bar{\phi}\} \) and all the bulk fields are understood to be pullbacks onto the brane world-volume.

Let us first compute the static potential between the probe and the stack of \( N \) D4-branes, simply by substituting the solution (2.6) into eq. (2.10). The contribution of the Dirac–Born–Infeld part is given by:

\[
e^{-\phi} \sqrt{-\text{det} G_{ab}} = \sin \bar{\theta} Z R_0^2 H \left( 1 + \frac{\sigma^2 H}{Z^2 R_0^2} \right)^{1/2}.
\]

(2.11)

Adding to it the Wess–Zumino part, whose contribution is computed using the expression (2.6c) of the R-R 5-form, we get the following expression for the static potential:

\[
S_{\text{pot}} = -\frac{T_4}{\kappa} \int d^3 \xi d\bar{\theta} d\bar{\phi} \sin \bar{\theta} Z R_0^2 \left[ \left( 1 + \frac{\sigma^2 H}{Z^2 R_0^2} \right)^{1/2} - 1 \right].
\]

(2.12)

We see that in general there is a force between the branes, and this means that the configuration is not supersymmetric. This had to be expected in some way because we are allowing the probe brane to move in all its transverse directions, including also the ones which are inside the Calabi–Yau space. If instead we allow the probe brane to move only in the “flat” part of the transverse space spanned by \( \{r, \theta, \phi\} \), keeping it fixed at the locus \( \sigma = 0 \) in the “internal” transverse space, we see that the potential (2.12) vanishes, yielding a supersymmetric configuration. Therefore in the following we will always work at the “supersymmetric locus” \( \sigma = 0 \).

In order to study the dynamics of the probe brane, we will allow the transverse coordinates \( X^i = \{r, \theta, \phi\} \) to depend on the “flat” world-volume coordinates \( \xi^a \) but not on the “wrapped” ones \( x \) and \( y \). Moreover, the gauge field \( F_{\alpha\beta} \) is defined to be nonvanishing only on the “flat” part of the world-volume. Let us start from the DBI part of the action in eq. (2.10). By expanding the determinant, we find:

\[
S_{\text{DBI}} \simeq -\frac{T_4}{\kappa} \int d^3 \xi d\bar{\theta} d\bar{\phi} e^{-\phi} \left( -\text{det} G_{ab} \right)^{1/2}
\times \left\{ 1 + \frac{1}{2} G_{\alpha\beta} G_{ij} \partial_\alpha X^i \partial_\beta X^j + \frac{(2\pi\alpha')^2}{4} G^{\alpha\gamma} G^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \right\}.
\]

(2.13)

Inserting the expressions (2.6a) and (2.6b) for the metric and dilaton we get:

\[
S_{\text{DBI}} = -\frac{T_4}{2\kappa} \int d^3 \xi d\bar{\theta} d\bar{\phi} \sin \bar{\theta} Z R_0^2 \left( \frac{1}{H} \left[ (\partial r)^2 + r^2 (\partial \theta)^2 + \sin^2 \theta (\partial \phi)^2 \right] + \frac{(2\pi\alpha')^2}{4} H F^2 \right),
\]

(2.14)
where we have included an additional factor of $1/2$ due to the normalization of the generators of the gauge group.

Notice that when $Z = 0$, the effective tension of the brane vanishes, meaning that an enhançon mechanism is taking place [23]. Since, in order to preserve supersymmetry, we have fixed $\sigma = 0$, the enhançon radius is given by:

$$r_e = \frac{R_A^3}{8 R_0^2} = \frac{\pi g_s(\alpha')^{3/2} N}{R_0^2}. \quad (2.15)$$

In fact, this seems to be a general feature (albeit somewhat unnoticed in the literature) of the supergravity solutions corresponding to D-branes wrapped on cycles[7]. The fact that the solution is no longer valid inside the enhançon radius seems to prevent us from getting nonperturbative information on the gauge theory under study.

The transverse scalars have to be interpreted as Higgs fields for the gauge theory living on the brane: $X^i = 2\pi\alpha'\Phi^i$. Then, defining $\mu$ such that $r = 2\pi\alpha'\mu$ and integrating over the volume of the two-sphere on which the brane is wrapped, we obtain the final expression for the DBI part:

$$S_{\text{DBI}} = -\frac{4\pi T_4}{2\kappa} \int d^3 \xi \frac{Z R_0^2}{H} \left\{ 1 + \frac{(2\pi\alpha')^2}{2} H \left[ (\partial\mu)^2 + \mu^2 \left( (\partial\theta)^2 + \sin^2 \theta (\partial\varphi)^2 \right) \right] + \frac{(2\pi\alpha')^2}{4} H F^2 \right\}. \quad (2.16)$$

Turning now to the WZ part, the pullback of $C_3$ is given by:

$$C_3 = \frac{1}{8} R_A^3 \cos \theta \sin \tilde{\theta} \partial_\alpha \varphi d\xi^\alpha \wedge d\tilde{\theta} \wedge d\tilde{\varphi}. \quad (2.17)$$

Then from eq. (2.10) we get:

$$S_{\text{WZ}} = \frac{T_4}{\kappa} \int d^3 \xi d\tilde{\theta} d\tilde{\varphi} \sin \tilde{\theta} \left\{ \frac{Z R_0^2}{H} + \frac{2\pi\alpha' R_A^3}{16} \cos \theta \varepsilon^{\alpha\beta\gamma} \partial_\alpha \varphi F_{\beta\gamma} \right\}$$

$$= \frac{4\pi T_4}{\kappa} \int d^3 \xi \left\{ \frac{Z R_0^2}{H} + \frac{2\pi\alpha' R_A^3}{16} \cos \theta \varepsilon^{\alpha\beta\gamma} \partial_\alpha \varphi F_{\beta\gamma} \right\} \quad (2.18)$$

Putting eq.s (2.16) and (2.18) together and substituting the expressions for $T_4$, $\kappa$, $R_A$ and for the function $Z$, the probe action finally becomes:

$$S_{\text{probe}} = -\frac{R_0^2}{2\pi g_s \sqrt{\alpha'}} \int d^3 \xi \left( 1 - \frac{g_s \sqrt{\alpha'} N}{2 R_0^2} \right) \left\{ \frac{1}{2} [(\partial \mu)^2 + \mu^2 ((\partial \theta)^2 + \sin^2 \theta (\partial \varphi)^2)] + \frac{1}{4} F^2 \right\} + \frac{N}{8\pi} \int d^3 \xi \cos \theta \varepsilon^{\alpha\beta\gamma} \partial_\alpha \varphi F_{\beta\gamma}. \quad (2.19)$$

\(^7\text{Notice however that the nature and location of the singularities of the metric depend on the value of the constant } k \text{ appearing in eq. (2.2), as discussed in very similar cases in Refs [3, 14, 23]. Nonetheless, gauge theory physics as seen by the brane probe at the supersymmetric locus is independent of } k, \text{ and only feels the existence of the enhançon.}\)
From the coefficient of $F^2$ in eq. (2.19) we can read the running gauge coupling constant of the three-dimensional gauge theory as a function of the scale $\mu$. Defining the bare coupling as:

$$g^2_{YM} = \frac{2\pi g_s \sqrt{\alpha'}}{R_0^2},$$  \hspace{1cm} (2.20)

the running coupling constant is given by:

$$\frac{1}{g^2_{YM}(\mu)} = \frac{1}{g^2_{YM}} \left(1 - \frac{g^2_{YM} N}{4\pi \mu}\right),$$  \hspace{1cm} (2.21)

in perfect agreement with gauge theory expectations, as shown in appendix C.

Eq. (2.19) does not give explicitly the full metric on the moduli space of $\mathcal{N} = 4$, $D = 2 + 1$ SYM theory. In fact such a metric must be hyperKähler [30] and in eq. (2.19) we have only three moduli and not four as it should be in a hyperKähler metric. We need an extra modulus that can be obtained by dualising the vector field. In order to do that, we regard the original action in eq. (2.19) as a function of $F_{\alpha\beta}$ and we add to it a term:

$$- \int \Sigma \, dF,$$  \hspace{1cm} (2.22)

so that the equation of motion for the auxiliary field $\Sigma$ enforces the Bianchi identity for $F$ on shell. By partially integrating the additional term in eq. (2.22), we are left with the following action:

$$S_{\text{probe}} = - \int d^3\xi \frac{1}{g^2_{YM}(\mu)} \left\{ \frac{1}{2} \left[ (\partial \mu)^2 + \mu^2 \left( (\partial \theta)^2 + \sin^2 \theta (\partial \varphi)^2 \right) \right] + \frac{1}{4} F^2 \right\}$$

$$+ \frac{N}{8\pi} \int d^3\xi \cos \theta \varepsilon^{\alpha\beta\gamma} \partial_\alpha \varphi F_{\beta\gamma} + \frac{1}{2} \int d^3\xi \varepsilon^{\alpha\beta\gamma} \partial_\alpha \Sigma F_{\beta\gamma}.$$  \hspace{1cm} (2.23)

We can then eliminate $F$ by means of its equation of motion that follows from eq. (2.23):

$$F_{\beta\gamma} = g^2_{YM}(\mu) \varepsilon^{\alpha\beta\gamma} \left[ \frac{N}{4\pi} \cos \theta \partial_\alpha \varphi + \partial_\alpha \Sigma \right],$$  \hspace{1cm} (2.24)

and we arrive at an action that contains four moduli, given by:

$$S_{\text{probe}} = - \frac{1}{2} \int d^3\xi \left\{ \frac{1}{g^2_{YM}(\mu)} \left[ (\partial \mu)^2 + \mu^2 \left( (\partial \theta)^2 + \sin^2 \theta (\partial \varphi)^2 \right) \right] + g^2_{YM}(\mu) \left( \frac{N \cos \theta}{4\pi} \partial \varphi + \partial \Sigma \right)^2 \right\}.$$  \hspace{1cm} (2.25)
The complete metric on the moduli space $\mathcal{M}$ of the gauge theory, in terms of the 4 scalars $\mu$, $\theta$, $\phi$ and $\Sigma$ is finally given by:

$$ds^2_{\mathcal{M}} = \frac{1}{g_{YM}^2(\mu)} \left( d\mu^2 + \mu^2 d\Omega^2 \right) + g_{YM}^2(\mu) \left( d\Sigma + \frac{N \cos \theta}{4\pi} d\varphi \right)^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. The metric in eq. (2.26) is indeed hyperKähler since it has precisely the form of the Taub-NUT metric \[39\]. However, because of the form given in eq. (2.21) of the function $g_{YM}(\mu)$ our metric has a “negative mass” and thus is singular. This is due to the fact that in our probe analysis we are only able to reproduce the perturbative behaviour of the gauge theory. As discussed in Refs. \[25, 38\], the complete metric should also include the instanton contribution, becoming a completely nonsingular generalization of the Atiyah-Hitchin metric.

### 3 Fractional D2/D6-brane system

#### 3.1 Setup

In this section we consider a system of fractional branes on the orbifold:

$$\mathbb{R}^{1,5} \times \mathbb{R}^4/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts by changing sign to the last four coordinates:

$$\{x^6, x^7, x^8, x^9\} \rightarrow \{-x^6, -x^7, -x^8, -x^9\}.$$

To be precise, we are going to study a configuration of type IIA string theory\[8\], made of $N$ fractional D2-branes extended along $x^0, x^1, x^2$ and $M$ D6-branes extended along $x^0, x^1, x^2, x^6, \ldots, x^9$, as shown schematically in the following table, where the symbols – and · denote respectively coordinates which are longitudinal and transverse to the branes:

|        | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|--------|---|---|---|---|---|---|---|---|---|---|
| D2     | – | – | – | – | – | – | – | – | – | – |
| D6     | – | – | – | – | – | – | – | – | – | – |

A peculiar feature of the fractional branes transverse to the orbifold space (as the D2-branes that we are considering) is that they are stuck at the orbifold fixed point $x^6 = x^7 = x^8 = x^9 = 0$.

The orbifold projection breaks half of the supersymmetry of type IIA theory, and so does the considered D2/D6-brane system. We are then left with 8 supercharges. Thus, at low energy the theory living on $N$ fractional D2-branes is $\mathcal{N} = 4$, $D = 2 + 1$ SYM theory.

\[8\] Classical solutions describing fractional D-branes in type IIA orbifolds were constructed in Ref. \[40\].
with gauge group $SU(N)$. Moreover, from the point of view of this gauge theory the strings stretching from the D2 to the D6-branes and vice-versa make up $M$ hypermultiplets which transform in the fundamental representation of the gauge group.

In order to describe the above system by means of a supergravity solution, we have to study how the low-energy fields which appear in the effective action behave in the background (3.1). Our background is characterized by the presence of a 2-form $\omega_2$, Poincaré dual to the exceptional 2-cycle $\Sigma_2$ of the ALE space which is obtained by the resolution of the orbifold. In the orbifold limit, the volume of $\Sigma_2$ vanishes, but the background value of the integral of $B_2$ on it has to remain finite in order to define a sensible CFT [41, 42]:

$$
\int_{\Sigma_2} B_2 = \frac{(2\pi \sqrt{\alpha'})^2}{2}.
$$

(3.2)

The 2-form $\omega_2$ satisfies the following properties:

$$
\omega_2 = -\star_4 \omega_2, \quad \int_{\Sigma_2} \omega_2 = 1, \quad \int_{\text{ALE}} \omega_2 \wedge \omega_2 = -\frac{1}{2}.
$$

(3.3)

The supergravity fields can have components along the vanishing cycle, so the following decomposition holds for the NS-NS two-form and the R-R three-form:

$$
B_2 = \tilde{B}_2 + b \omega_2, \quad C_3 = \tilde{C}_3 + A_1 \wedge \omega_2.
$$

(3.4)

Since we will be looking for supergravity solutions which represent branes without a $B_2$ field in their world-volume, in the following we will put $\tilde{B}_2 = 0$, so we simply have:

$$
B_2 = b \omega_2, \quad C_3 = \tilde{C}_3 + A_1 \wedge \omega_2,
$$

(3.5)

where, because of eq. (3.2):

$$
b = \frac{(2\pi \sqrt{\alpha'})^2}{2} + \tilde{b},
$$

(3.6)

and $\tilde{b}$ represents the fluctuation around the background value of $b$. We will sometimes refer to the fields $b$ and $A_1$ in eq. (3.5) as “twisted” fields because they correspond to the massless states of the twisted sector of type IIA string theory on the orbifold.

Having given the main features of the system that we are going to study, we now turn to the supergravity solution.

### 3.2 The supergravity solution

In this subsection we will discuss the supergravity solution describing the fractional D2/D6 system that we have introduced in the previous subsection. The solution is derived in detail in appendix A.2. Here we will only summarize the procedure followed to find it.
The first step is to substitute the decompositions for $B_2$ and $C_3$ given in eq. (3.5) into the type IIA supergravity action, and to derive the equations of motion for the “untwisted” fields $G_{\mu\nu}$, $\phi$, $\tilde{C}_3$ and $C_1$ and for the “twisted” ones $b$ and $A_1$.

Then, we impose the standard Ansatz for the “untwisted” fields corresponding to a D2/D6 system:

\[
\begin{align*}
&ds^2 = H_2^{-5/8} H_6^{-1/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + H_2^{3/8} H_6^{7/8} \delta_{ij} dx^i dx^j + H_2^{3/8} H_6^{-1/8} \delta_{pq} dx^p dx^q, \\
&e^\phi = H_2^{1/4} H_6^{-3/4}, \\
&\tilde{C}_3 = (H_2^{-1} - 1) dx^0 \wedge dx^1 \wedge dx^2,
\end{align*}
\]  

(3.7a)

(3.7b)

(3.7c)

where the function $H_2$ depends on the radial coordinate $\rho = \sqrt{(x^3)^2 + \ldots + (x^9)^2}$ of the space transverse to the D2-brane, while the function $H_6$ depends only on the radial coordinate of the common transverse space $r = \sqrt{\delta_{ij} x^i x^j}$.

In order to write down a sensible Ansatz for the fields $A_1$ and $C_1$, we need to take into account the contribution coming from the boundary action describing the world-volume theory of the branes. After some calculation it is easy to get convinced that the following$^{[10]}$ is a sensible Ansatz for the fields $A_1$ and $C_1$:

\[
\begin{align*}
&dA_1 = C_1 \wedge db + \frac{1}{2} \varepsilon_{ijk} H_6 \partial_i b dx^j \wedge dx^k, \\
&dC_1 = \frac{1}{2} \varepsilon_{ijk} \partial_i H_6 dx^j \wedge dx^k,
\end{align*}
\]  

(3.8a)

(3.8b)

where $\varepsilon_{345} = \varepsilon^{345} = +1$.

Substituting our Ansätze in the equations of motion and computing all the relevant contributions coming from the boundary action $S_b$, the final solution for the fractional D2/D6 system can be expressed in the following form$^{[11]}$:

\[
\begin{align*}
&ds^2 = H_2^{-5/8} H_6^{-1/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + H_2^{3/8} H_6^{7/8} \delta_{ij} dx^i dx^j + H_2^{3/8} H_6^{-1/8} \delta_{pq} dx^p dx^q, \\
&e^\phi = H_2^{1/4} H_6^{-3/4}, \\
&\tilde{C}_3 = (H_2^{-1} - 1) dx^0 \wedge dx^1 \wedge dx^2, \\
&C_1 = \frac{g_s \sqrt{\alpha'} M}{2} \cos \theta d\phi, \\
&A_1 = -\pi^2 \alpha' g_s \sqrt{\alpha'} (4N - M) H_6 \cos \theta d\phi, \\
&b = \frac{Z}{H_6},
\end{align*}
\]  

(3.9a)

(3.9b)

(3.9c)

(3.9d)

(3.9e)

(3.9f)

$^{[10]}$The coordinates are labeled as: $\alpha, \beta = \{0, 1, 2\}$, $i, j = \{3, 4, 5\}$ and $p, q = \{6, 7, 8, 9\}$.

$^{[11]}$When $M = 0$, the Ansätze for $A_1$ and $C_1$ coincide with the ones given in Ref. [13].

Notice that, in order to easily express the fields $A_1$ and $C_1$ in eqs (3.9), we have changed coordinates in the common transverse space into polar coordinates: $(x^3, x^4, x^5) \rightarrow (r, \theta, \phi)$.
where:

\[
H_6(r) = 1 + \frac{g_s \sqrt{\alpha'}}{2r}, \quad Z(r) = \frac{(2\pi \sqrt{\alpha'})^2}{2} \left( 1 - \frac{g_s \sqrt{\alpha'(2N - M)}}{r} \right),
\]

(3.10)

and where \( H_2 \) is the solution of the following equation (see eq. (A.23a) in appendix A.2):

\[
(\delta^{ij} \partial_i \partial_j + H_6 \delta^{pq} \partial_p \partial_q) H_2 + \frac{1}{2} H_6 \delta^{ij} \partial_i b \partial_j b \delta(x^6) \cdots \delta(x^9) + \kappa T_2 N \delta(x^3) \cdots \delta(x^9) = 0.
\]

(3.11)

From the solution given in eq. (3.9) we can also compute the expressions for the fields \( C_7 \) and \( A_3 \) which appear naturally in the string theory. The duality relations are\(^{12}\):

\[
dC_7 = -e^{3\phi/2} dC_1, \quad (3.12a)
\]

\[
dA_3 = e^{\phi/2} \ast_6 G_2 - db \wedge \tilde{C}_3, \quad (3.12b)
\]

and the explicit computation gives:

\[
C_7 = (H_6^{-1} - 1) \, dx^0 \wedge \cdots \wedge dx^6, \quad (3.13a)
\]

\[
A_3 = \tilde{b} \, dx^0 \wedge dx^1 \wedge dx^2. \quad (3.13b)
\]

Notice that the field \( C_7 \) has a quite standard expression, due to the specific form of the Ansatz in eq. (3.8).

### 3.3 Probing the fractional brane solution

In this section we will study the world-volume theory of a probe fractional D2-brane, which is placed in the background given in eqs (3.9) at some finite distance \( r \) in the transverse space \( \{x^3, x^4, x^5\} \). This will give us information about the Coulomb branch of \( \mathcal{N} = 4 \) three-dimensional super Yang–Mills theory with gauge group \( SU(N + 1) \) broken into \( SU(N) \times U(1) \), coupled to \( M \) hypermultiplets in the fundamental representation of the gauge group.

Let us start from the world-volume action for a single fractional D2-brane, which is given by eq. (B.9) in the case of \( p = 2 \):

\[
S_{\text{probe}} = -\frac{T_2}{2\kappa} \int d^3\xi e^{-\phi/4} \sqrt{-\det \left[ G_{\alpha\beta} + e^{-\phi/2} 2\pi \alpha' F_{\alpha\beta} \right]} \left( 1 + \frac{\tilde{b}}{2\pi^2 \alpha'} \right) \left( C_3 + 2\pi \alpha' C_1 \wedge F \right),
\]

(3.14)

\(^{12}\)The duality relations can be derived from the equations of motion (A.13a) and (A.13c), for which the terms coming from the boundary action vanish (see appendix A.2).
where all bulk fields are understood to be pullbacks onto the brane world-volume and the fields $\mathcal{C}_3$ and $\mathcal{C}_1$ are given by:

$$
\mathcal{C}_3 = \bar{\mathcal{C}}_3 \left(1 + \frac{b}{2\pi^2 \alpha'}\right) + \frac{A_3}{2\pi^2 \alpha'} \frac{Z}{H_2 H_6} - 1, \quad (3.15a)
$$

$$
\mathcal{C}_1 = C_1 \left(1 + \frac{b}{2\pi^2 \alpha'}\right) + \frac{A_1}{2\pi^2 \alpha'} = -g_s \sqrt{\alpha'} (2N - M) \cos\theta d\phi. \quad (3.15b)
$$

The computation is analogous to the one performed in section 2.3. We fix the static gauge, and regard the coordinates $\{x^3, x^4, x^5\}$ transverse to the probe brane as Higgs fields of the dual gauge theory: $x^i = 2\pi \alpha' \Phi^i$. We also define polar coordinates $(\mu, \theta, \varphi)$ in the moduli space of the $\Phi^i$, so that $r = 2\pi \alpha' \mu$.

Expanding the DBI action for slowly varying world-volume fields and keeping only up to quadratic terms in their derivatives we get:

$$
S_{\text{DBI}} \simeq -\frac{\sqrt{\alpha'}}{2g_s} \int d^3 x \frac{Z}{2\pi^2 \alpha'} \left(\frac{1}{2} \eta^{\alpha\beta} \delta_{ij} \partial_\alpha \Phi^i \partial_\beta \Phi^j + \frac{1}{4} \eta^{\alpha\beta} \eta^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \right) - \frac{T_2}{2\kappa} \int d^3 x \frac{1}{2\pi^2 \alpha'} \frac{Z}{H_2 H_6}. \quad (3.16)
$$

Turning to the WZ part and substituting the expressions (3.15) into eq. (3.14) we get:

$$
S_{\text{WZ}} = \frac{T_2}{2\kappa} \left[ \int d^3 x \left( \frac{1}{2\pi^2 \alpha'} \frac{Z}{H_2 H_6} - 1 \right) + \int_{\mathcal{M}_3} 2\pi \alpha' \mathcal{C}_\varphi d\varphi \wedge F \right]. \quad (3.17)
$$

We easily see that the position-dependent terms cancel as expected because fractional D2-branes are BPS states and do not interact with the D2/D6 system. Ignoring the constant potential, the final result is:

$$
S_{\text{probe}} = -\frac{\sqrt{\alpha'}}{4g_s} \int d^3 x \frac{Z}{2\pi^2 \alpha'} \left\{ \frac{1}{2} \left[ (\partial \mu)^2 + \mu^2 \left( (\partial \theta)^2 + \sin^2 \theta (\partial \varphi)^2 \right) \right] + \frac{1}{4} F^2 \right\} - \frac{1}{16\pi} \int d^3 x (2N - M) \cos \theta \epsilon^{\alpha\beta\gamma} \partial_\alpha \varphi F_{\beta\gamma}. \quad (3.18)
$$

When $Z = 0$ the effective tension of the probe vanishes and this means that also in this case, as expected, an enhançon mechanism is taking place at the radius:

$$
r_e = \sqrt{\alpha'} g_s (2N - M). \quad (3.19)
$$

Substituting in (3.18) the expression of $Z$ in terms of $\mu$, we obtain:

$$
S_{\text{probe}} = -\frac{\sqrt{\alpha'}}{4g_s} \int d^3 x \left[ 1 - \frac{g_s (2N - M)}{2\pi \sqrt{\alpha'} \mu} \right] \left\{ \frac{1}{2} \left[ (\partial \mu)^2 + \mu^2 \left( (\partial \theta)^2 + \sin^2 \theta (\partial \varphi)^2 \right) \right] + \frac{1}{4} F^2 \right\} - \frac{1}{16\pi} \int d^3 x (2N - M) \cos \theta \epsilon^{\alpha\beta\gamma} \partial_\alpha \varphi F_{\beta\gamma}. \quad (3.20)
$$
From the coefficient of the gauge field kinetic term in the previous action we can read the running coupling constant:

\[
\frac{1}{g^2_{YM}(\mu)} = \frac{1}{g^2_{YM}} \left(1 - \frac{g^2_{YM}}{8\pi\mu} \right),
\]  
(3.21)

where we have defined the bare coupling as:

\[
g^2_{YM} = \frac{4g_s}{\sqrt{\alpha'}}.
\]  
(3.22)

Eq. (3.21) is exactly what expected for the gauge theory under consideration, as shown in appendix C.

Exactly as in the case of the wrapped branes described in section 2, eq. (3.20) does not give explicitly the full hyperKähler metric on the moduli space of the gauge theory. We can obtain the needed extra modulus by dualising the vector field into a scalar, using exactly the same procedure which brought us from eq. (2.19) to eq. (2.26) in section 2. The final result is:

\[
ds^2_M = \frac{1}{g^2_{YM}(\mu)} \left(d\mu^2 + \mu^2 d\Omega^2\right) + g^2_{YM}(\mu) \left(d\Sigma + \frac{(2N - M) \cos \theta}{8\pi} d\varphi\right)^2,
\]  
(3.23)

where \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2\). If we put \(M = 0\) this metric coincides with the one found in eq. (2.20) by probing the geometry of \(N\) D4-branes wrapped on \(S^2\). Again, we have found the hyperKähler Taub-NUT metric, but with a “negative mass” which makes it singular. Also in this case we have only recovered the perturbative behaviour of the gauge theory.

**Complete action for a D6-brane extended along the orbifold**

As in the case of the D7-brane analyzed in Ref. [15], the supergravity solution corresponding to our D2/D6 system can also be used to get the form of the complete world-volume action for a D6-brane extended along the whole orbifold space. In fact, in deriving the classical solution corresponding to the D2/D6 system it was enough to consider only the linear terms in the bulk fields. However, since the D2/D6 system is BPS, we expect that when we insert the corresponding classical solution into the world-volume action of either the D2-brane or the D6-brane, we obtain a constant result. In the previous subsection we have seen that this is the case for a fractional D2-brane. If we instead insert the classical solution into the action of a D6-brane given by (as it follows from eq. (A.18) for \(p = 2\)):

\[
S_6 = \frac{T_6}{\kappa} \left\{ - \int d^7 \xi \sqrt{-G_{\rho\sigma}} G_{\rho\sigma} + \int_{M_7} C_7 \right\}
+ \frac{T_2}{2\kappa} \frac{1}{2(2\pi\alpha')^2} \left\{ \int d^3 \xi \sqrt{-G_{\alpha\beta}} \tilde{b} - \int_{M_3} A_3 \right\} + \ldots,
\]  
(3.24)

\[\text{In this case, the dualisation procedure can also be done directly in the original three-dimensional world-volume action, as in Ref.s [44, 25]. The result that one obtains coincides with that in eq. (3.23).}\]
we get terms that are dependent on the distance $r$ between the D6-brane and the system D2/D6 described by the classical solution. The situation here is exactly the same as the one found in Ref. [15], and as in that case we must add to the previous action higher order terms that restore the no-force condition. Including them we arrive at the following boundary action for a D6-brane extended along the whole orbifold space:

$$S_6 = \frac{T_6}{\kappa} \left\{ - \int d^7x e^{4\phi} \sqrt{-\det G_{\mu\nu}} + \int_{\mathcal{M}_7} C_7 \right\} + \frac{T_2}{2\kappa} \frac{1}{2(2\pi \sqrt{\alpha'})^2} \left\{ \int d^3\xi e^{-\phi/4} \sqrt{-\det G_{\alpha\beta}} \tilde{b} \left( 1 + \frac{\tilde{b}}{4\pi^2\alpha'} \right) \right\} - \int_{\mathcal{M}_3} A_3 \left( 1 + \frac{\tilde{b}}{4\pi^2\alpha'} \right) - \int_{\mathcal{M}_3} \tilde{C}_3 \tilde{b} \left( 1 + \frac{\tilde{b}}{4\pi^2\alpha'} \right) \right\}. \tag{3.25}$$

4 Discussion and conclusions

In this final section we want to compare the two approaches that we have described in the previous section. Let us compare the two cases dual to the pure gauge theory, that is, in absence of D6-branes ($M = 0$) in the fractional brane case.

Let us start by summarizing some of the properties and differences of the two systems:

- Both systems are able to capture only the perturbative dynamics of $\mathcal{N} = 4$, $D = 2+1$ SYM theory.

- The role of the running coupling constant is played in the two supergravity solutions by two parameters: the “running volume” $Z$ of the 2-cycle for the wrapped D4-branes and the “twisted $B$-field” $b$ for the fractional D2-branes.

- The enhançon, where the gauge coupling $g_{YM}$ diverges, is located at the locus where respectively $Z = 0$ and $b = 0$.

Does it exist a closer relationship between the two setups? Both systems consist of wrapped branes. In fact, on the one hand, as we have seen in section 3, a fractional Dp-brane can be seen as a D($p+2$)-brane wrapped on the vanishing two-cycle of the ALE space which corresponds to the blow-up of the orbifold. On the other hand we have also seen that the D4-branes considered in section 2 are wrapped on a two-sphere inside the same ALE space, as explicitly shown in eq. (2.9). In fact, the two systems provide exactly the same information about the gauge theory living on their world-volume. In order to see the connection between the two systems it is useful to write down a general formula that provides the perturbative running coupling constant of a general $(p+1)$-dimensional gauge theory living on the flat part of the world-volume of a D($p+2$)-brane wrapped on a (vanishing or not) two-cycle $\Sigma_2$. It is given by the following expression:

$$\frac{1}{g_{YM}^2(\mu)} = \frac{V_{ST}(\Sigma_2)}{g_{Dp}^2}, \tag{4.1}$$
where \( g_{Dp}^2 = 2(2\pi)^{p-2} g_s \alpha'^{(p-3)/2} \) is the usual (bare) coupling constant of the gauge theory living on a Dp-brane in flat space and the dimensionless “stringy volume” \( V_{ST} \) is given by:

\[
V_{ST}(\Sigma_2) = \frac{1}{(2\pi \sqrt{\alpha'})^2} \int d^2 \zeta \sqrt{-\det (G_{AB} + B_{AB})},
\]

(4.2)

where \( \zeta^{1,2} \) parameterizes the cycle \( \Sigma_2 \), while \( G_{AB} \) and \( B_{AB} \) \((A,B = 1,2)\) are the bulk metric without any warp factors and the \( B \)-field on the cycle.

We can easily see that the formula in eq. (4.1) holds for the systems considered in sections 2 and 3. For the three-dimensional gauge theory at hand, we have \( g_{D2}^2 = 2g_s^2 \sqrt{\alpha'} \).

For the case of the D4-branes wrapped on \( S^2 \), we have:

\[
G^w = Z R_0^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \tilde{\theta} \end{pmatrix}, \quad B^w = 0,
\]

(4.3)

so that the “stringy volume” is given by:

\[
V_{ST}^w(\Sigma_2) = \frac{Z R_0^2}{\pi^2 \alpha'}.
\]

(4.4)

Substituting eq. (4.4) in eq. (4.1) we get the correct running for \( N = 4, D = 2 + 1 \) SYM theory with \( SU(N) \) gauge group:

\[
\frac{1}{(g^w_{YM})^2(\mu)} = \frac{1}{(g^w_{YM})^2} \left( 1 - \frac{(g^w_{YM})^2 N}{4\pi \mu} \right),
\]

(4.5)

where the bare coupling is defined as in eq. (2.20) as follows: \( (g^w_{YM})^2 = \frac{2\pi g_s \sqrt{\alpha'}}{R_0^2} \).

If instead we consider the fractional D2-branes as D4-branes wrapped on the vanishing cycle \( \Sigma_2 \), we find:

\[
G^f = 0, \quad B^f = Z \tilde{\omega}_2.
\]

(4.6)

Now the “stringy volume” is:

\[
V_{ST}^f(\Sigma_2) = \frac{Z}{4\pi^2 \alpha'}.
\]

(4.7)

and substituting eq. (4.7) in eq. (4.1) we get again the correct running coupling constant:

\[
\frac{1}{(g^f_{YM})^2(\mu)} = \frac{1}{(g^f_{YM})^2} \left( 1 - \frac{(g^f_{YM})^2 N}{4\pi \mu} \right).
\]

(4.8)

where now the bare coupling constant is defined as \( (g^f_{YM})^2 = \frac{4g_s^2}{\sqrt{\alpha'}} \) as in eq. (3.22).

Notice also that if we choose the “background value” \( V_0 \) of the geometrical volume on which the D4-branes are wrapped (that is, the volume of the two-sphere inside the Eguchi–Hanson space in eq. (2.9), once we remove the branes setting \( H = Z = 1 \)) in such
a way that it coincides with the background value of the $B$-field of the fractional brane case, $V_0 = 4\pi R_0^2 = (2\pi\sqrt{\alpha'})^2$, the bare coupling constants (and enhançon radii) computed in the two cases become exactly the same in terms of string parameters.

One can see that the formula (4.1) works perfectly also, for instance, for the case of $\mathcal{N} = 2$ SYM in four dimensions, applying it to the fractional D3-brane solution of Refs. [11, 12] and to the wrapped D5-brane solution of Ref. [16].

Although the two systems give the same perturbative gauge coupling constant, there seems not to be a “physical” limit in which one can obtain the fractional brane solution from the wrapped one or vice-versa, playing with the volume of the cycle. This is due to the fact that the two Ansätze are radically different in the warp factors, which are respectively the ones of a D4 and of a D2-brane, and is also due to the absence of a $B$-field on the world-volume of the D4-branes wrapped on $S^2$. On the other hand, if we look at the whole moduli space of the four-dimensional ALE space, we see that it is characterized by the volume of the exceptional cycle and by the flux of the $B$-field on it. These are the two moduli that are combined into the “stringy volume” in eq. (4.2) which, as we have seen, provides the running coupling constant. The situation in the two cases can then be summarized by the following diagram:

\[
\begin{array}{ccc}
B = 0, & V \neq 0 & \text{Wrapped branes} \\
& \rightarrow & \\
& B = 0, & V = 0 & \text{Enhanced gauge symmetry} \\
& \text{Fractional branes} & \leftarrow & B \neq 0, & V = 0
\end{array}
\]

where, in the case of the wrapped D4-branes, we are keeping the size of the cycle finite and the $B$-flux vanishing, while in the case of the fractional D2-branes the geometrical volume shrinks to zero size and in order to have a conformal orbifold we must give a definite fixed background value to the $B$-flux, which in the case of the $\mathbb{Z}_2$ orbifold is $(2\pi\sqrt{\alpha'})^2/2$.

The limiting case in which both the geometrical volume of the cycle and the $B$-flux are taken to vanish is the point where the theory on the Calabi–Yau space manifests an enhanced gauge symmetry, which is at the origin of the enchançon mechanism. This is the point where the “stringy volume” vanishes and the supergravity solutions break down.

Acknowledgements

We would like to thank M. Bertolini, M. Frau, A. Lerda, P. Merlatti and T. Ortín for enlightening discussions. H.E. wishes to thank NORDITA for kind hospitality. The work of E.I. and E.L.–T. is supported in part by a European Commission Marie Curie Training Site Fellowship under the Contract No. HPMT-CT-2000-00010.

A Finding the supergravity solutions

In the two sections of this appendix we will describe the way in which we have obtained the supergravity solutions describing, respectively, D4-branes wrapped on $S^2$ and fractional
D2/D6-branes on the orbifold $\mathbb{R}^4/\mathbb{Z}_2$.

We will be using the following conventions:

- A metric in $D$ dimensions has signature $(-, +^{D-1})$.
- $\varepsilon$-symbols in $D$ dimensions are defined in such a way that $\varepsilon^{012\ldots(D-1)} = -\varepsilon^{012\ldots(D-1)} = +1$.
- A $p$ form is defined as $\omega_p = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$.
- The Hodge dual $^D \star$ in $D$ dimensions is defined as
  
  $$^D \omega_p = \sqrt{-\det G_D} \frac{1}{p! (D-p)!} \varepsilon_{\nu_1 \ldots \nu_{D-p} \mu_1 \ldots \mu_p} \omega_{\mu_1 \ldots \mu_p} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{D-p}}.$$
  
  Moreover, $^*$ denotes $^*_{10}$ and $\hat{\star}$ denotes $^*_{11}$.

A.1 D4-branes wrapped on $S^2$

In this appendix we explain how we have obtained the type IIA supergravity solution describing $N$ D4-branes wrapped on $S^2$, using the techniques and the solutions given in Ref. [2].

Our procedure will be the following. We want to obtain the solution for the D4-branes by compactifying the solution for the M5-branes wrapped on $S^2$. The latter is found by uplifting to eleven dimensions a solution of 7-dimensional gauged supergravity with the correct identification between spin connection and gauge connection, by means of the formulas given in Ref. [33].

The seven dimensional solution

The starting point is the seven dimensional gauged supergravity considered in Ref. [2]. Following that paper, we consider a $U(1) \times U(1)$ consistent truncation of the $SO(5)$ gauged supergravity arising when one compactifies eleven dimensional supergravity on $S^4$. The bosonic field content of the truncated theory consists of two $U(1)$ gauge fields ($A^{(1,2)}$), two scalar fields ($\lambda_{1,2}$) and a metric.

The full solution of seven dimensional gauged supergravity is:

$$ds_{(7)}^2 = \left(\frac{R_A}{R_0}\right)^2 e^{2\rho} e^{\lambda} n_{ij} d\xi_i d\xi_j + R_A^2 \left(e^{\lambda}(e^{2\rho} - \frac{1}{4})(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) + e^{-4\lambda} d\rho^2\right),$$

$$A^{(1)} = \frac{R_A}{4} \cos \tilde{\theta} d\tilde{\varphi}, \quad A^{(2)} = 0,$$

$$\lambda \equiv \lambda_2, \quad 2\lambda_1 + 3\lambda_2 = 0,$$

$$e^{5\lambda} = \frac{e^{2\rho} + ke^{-2\rho} - \frac{1}{2}}{e^{2\rho} - \frac{1}{4}}.$$  

This solution is exactly the one given in Ref. [2], although we have kept track of units and we have used standard spherical coordinates $\tilde{\theta}$ and $\tilde{\varphi}$ for the two-sphere on which the
branes are wrapped. \( R_A = 2(\pi N)^{1/3} l_p \) is the radius of the \( AdS_7 \) space appearing in the near horizon limit of the usual flat M5-brane solution, and \( R_0 \) is an arbitrary integration constant with dimension of a length (which is \((C_2)^{-1/2}\) of eq. (24) in Ref. [2]). Finally, \( k \) is a (dimensionless) integration constant, which was called \( C_1 \) in Ref. [2]. All the coordinates entering in the above solution are dimensionless, except those spanning the unwrapped part of the world-volume of the brane, \( \xi^i, i = 0, \ldots, 3 \), which have dimensions of a length.

**Uplift formulas and eleven dimensional solution**

The seven-dimensional solution can be lifted to eleven dimensions with the help of eq.s (4.1) and (4.2) of Ref. [33], which we rewrite here:

\[
ds^2 = \tilde{\Delta}^{1/3} ds_{(7)}^2 + g^{-2} \tilde{\Delta}^{-2/3} \left( X_0^{-1} d\mu_0^2 + \sum_{i=1}^{2} X_i^{-1} \left( d\mu_i^2 + \mu_i^2 \left( d\phi_i + gA^{(i)} \right)^2 \right) \right), \tag{A.2a}
\]

\[
\hat{\star} d\hat{C}_3 = 2g \sum_{\alpha=0}^{2} \left( X_\alpha^2 \mu_\alpha^2 - \tilde{\Delta} X_\alpha \right) \varepsilon_{(7)} + g\tilde{\Delta} X_0 \varepsilon_{(7)} + \frac{1}{2g} \sum_{\alpha=0}^{2} X_\alpha^{-1} \star_7 dX_\alpha \wedge d(\mu_\alpha^2) + \frac{1}{2g^2} \sum_{i=1}^{2} X_i^{-2} d(\mu_i^2) \wedge \left( d\phi_i + gA^{(i)} \right) \wedge \star_7 F^{(i)}. \tag{A.2b}
\]

Here and below, hats will always refer to eleven-dimensional quantities. The above formulas are written in the notation of Ref. [33]: \( g \) is the seven dimensional gauged supergravity coupling constant, \( \varepsilon_{(7)} \) is the seven dimensional volume form, \( A^{(1,2)} \) are the two \( U(1) \) gauge fields, the \( X_\alpha \) are a suitable parameterization of the 2 scalars present in the theory and \( \tilde{\Delta} \) is given by:

\[
\tilde{\Delta} \equiv \sum_{\alpha=0}^{2} X_\alpha \mu_\alpha^2,
\]

where \( \mu_\alpha \) parameterize a two-sphere: \( \mu_0^2 + \mu_1^2 + \mu_2^2 = 1 \). The quantities appearing in the uplift formulas are given in terms of those appearing in eq. (A.1) by the following expressions:

\[
\frac{1}{g^2} = \left( \frac{R_A}{2} \right)^2, \\
X_0 = X_1 = e^{2\lambda}, \\
X_2 = e^{-3\lambda}, \\
A^{(1,2)} = 2A^{(2,1)}, \\
\Delta \equiv e^{3\lambda} \tilde{\Delta} = e^{5\lambda} \cos^2 \chi + \sin^2 \chi, \\
\varepsilon_{(7)} = -\sqrt{-\det G_{(7)}} \, d\xi_0 \wedge \cdots d\xi_3 \wedge d\tilde{\theta} \wedge d\tilde{\phi} \wedge d\rho,
\]

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where we have chosen the following parameterization for $\mu_i$:

\[
\begin{align*}
\mu_0 &= \cos \chi \cos \theta, \\
\mu_1 &= \cos \chi \sin \theta, \\
\mu_2 &= \sin \chi.
\end{align*}
\]  

By using the above expressions we are now ready to write the full solution in eleven dimensions:

\[
\begin{align*}
d\hat{s}^2 &= \Delta^{1/3} \left( \left( \frac{R_A}{R_0} \right)^2 e^{2\rho} \delta_{ij} d\xi^i d\xi^j + R_A^2 (e^{2\rho} - \frac{1}{4}) (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) \right) \\
&\quad + \Delta^{-2/3} \left( \frac{R_A}{2} \right)^2 \left( \frac{4\Delta}{e^{5\lambda}} d\rho^2 + \Delta d\chi^2 + \cos^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) + e^{5\lambda} \sin^2 \chi \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right)^2 \right), \\
&= \Delta^{1/3} \left( \left( \frac{R_A}{R_0} \right)^2 e^{2\rho} \delta_{ij} d\xi^i d\xi^j + R_A^2 (e^{2\rho} - \frac{1}{4}) (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) \right) \\
&\quad + \Delta^{-2/3} \left( \frac{R_A}{2} \right)^2 \left( \frac{4\Delta}{e^{5\lambda}} d\rho^2 + \Delta d\chi^2 + \cos^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) + e^{5\lambda} \sin^2 \chi \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right)^2 \right), \\
&= \Delta^{1/3} \left( \left( \frac{R_A}{R_0} \right)^2 e^{2\rho} \delta_{ij} d\xi^i d\xi^j + R_A^2 (e^{2\rho} - \frac{1}{4}) (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) \right) \\
&\quad + \Delta^{-2/3} \left( \frac{R_A}{2} \right)^2 \left( \frac{4\Delta}{e^{5\lambda}} d\rho^2 + \Delta d\chi^2 + \cos^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) + e^{5\lambda} \sin^2 \chi \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right)^2 \right), \\
&= \Delta^{1/3} \left( \left( \frac{R_A}{R_0} \right)^2 e^{2\rho} \delta_{ij} d\xi^i d\xi^j + R_A^2 (e^{2\rho} - \frac{1}{4}) (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) \right) \\
&\quad + \Delta^{-2/3} \left( \frac{R_A}{2} \right)^2 \left( \frac{4\Delta}{e^{5\lambda}} d\rho^2 + \Delta d\chi^2 + \cos^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) + e^{5\lambda} \sin^2 \chi \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right)^2 \right)
\end{align*}
\]

where we have also relabeled the angles appearing in (A.2): $\phi_1 = \varphi, \phi_2 = \psi$. This solution describes the near horizon geometry of an M5-brane wrapped on a two-sphere. The unwrapped world-volume coordinates are $\xi^0, \ldots, \xi^3$, the wrapped ones are $\tilde{\theta}$ and $\tilde{\varphi}$, and the remaining coordinates are transverse to the M5. It can be easily seen that the metric in eq. (A.5) reduces to the one given in eq. (26) of Ref. [2] if we restrict ourselves to work at the IR fixed point analysed there.

From eq. (A.5b) we can compute the three-form potential. It is equal to:

\[
\hat{C}_3 = \frac{R_A^3}{8} e^{5\alpha} \frac{\cos^3 \chi \cos \theta \sin \tilde{\theta}}{\Delta} d\tilde{\theta} \wedge d\tilde{\varphi} \wedge d\varphi \\
+ \frac{R_A^3}{8} \frac{e^{5\alpha} \Delta + 2}{\Delta^2} \frac{\cos^2 \chi \sin \chi \cos \theta}{\Delta^2} d\chi \wedge d\varphi \wedge \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right) \\
+ \frac{R_A^3}{8} \frac{\partial_\rho (e^{5\alpha})}{\Delta^2} \frac{\cos^3 \chi \sin^2 \chi \cos \theta}{\Delta^2} d\rho \wedge d\varphi \wedge \left( d\psi + \cos \tilde{\theta} d\tilde{\varphi} \right).
\]  

The last step consists in compactifying to ten dimensions the M5-brane solution just obtained along one of its non-wrapped world-volume coordinates (that we choose to be $\xi^3$) to get the solution describing the geometry of $N$ wrapped D4-branes. The compactification
is obtained by means of the standard expressions in the ten dimensional string frame:

\[
(G_{st})_{\mu\nu} = (\hat{G}_{33})^{1/2}\hat{G}_{\mu\nu},
\]

\[
e^{2\phi} = (\hat{G}_{33})^{3/2},
\]

\[
(C_{3})_{\mu\nu\rho} = (\hat{C}_{3})_{\mu\nu\rho},
\]

(A.7a) \quad \text{(A.7b)} \quad \text{(A.7c)}

where we have split eleven dimensional indices in \( \hat{\mu} = \{ \mu, \xi^3 \} \). This is all we need to get the final expression for the wrapped D4-brane solution presented in section 2.2.

### A.2 Fractional D2/D6-brane system

In this section we will describe in some detail how to find the supergravity solution describing a fractional D2/D6-brane system. We will always work in the Einstein frame. The Type IIA effective action in the orbifold background (3.1) is given, in our conventions, by:

\[
S_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} R - \frac{1}{2} \int \left( d\phi \wedge * d\phi + e^{-\phi} H_3 \wedge H_3 \\
- e^{3\phi/2} F_2 \wedge * F_2 - e^{\phi/2} F_4 \wedge * F_4 + B_2 \wedge F_4 \wedge F_4 \right),
\]

(A.8)

where the field strengths are given by:

\[
H_3 = dB_2, \quad F_2 = dC_1, \quad F_4 = dC_3, \quad F_4 = F_4 - C_1 \wedge H_3,
\]

(A.9)

and \( \kappa = 8\pi^{7/2}g_s\alpha'^2 \). In order to find a D-brane solution we must add to the previous bulk action a boundary action whose corresponding Lagrangian we call \( \mathcal{L}_b \). The equations of motion are then obtained by varying the total action \( S_{\text{IIA}} + S_b \).

The first step in order to find a supergravity solution in the orbifold background (see for instance Ref. [11]) is substituting in the action (A.8) the form (3.5) of the fields:

\[
B_2 = b\omega_2, \quad C_3 = \bar{C}_3 + A_1 \wedge \omega_2.
\]

(A.10)

Recalling that the 2-form \( \omega_2 \) is normalized as in eq. (3.3), one obtains:

\[
S'_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} R - \frac{1}{2} \int \left( d\phi \wedge * d\phi - e^{3\phi/2} dC_1 \wedge * dC_1 - e^{\phi/2} d\bar{C}_3 \wedge * d\bar{C}_3 \right) \\
- \frac{1}{4} \int_{\mathbb{R}^1,5} \left( e^{-\phi} db \wedge * db - e^{\phi/2} G_2 \wedge * G_2 - 2b \wedge d\bar{C}_3 \wedge dA_1 \right),
\]

(A.11)

where we have introduced the quantity:

\[
G_2 \equiv dA_1 - C_1 \wedge db.
\]

(A.12)
By varying the previous action one finds the equations of motion for the fields $C_1$, $\tilde{C}_3$, $A_1$, $b$ and $\phi$ respectively:

\begin{align}
 d \left( e^{\phi/2} \ast dC_1 \right) - \frac{1}{2} e^{\phi/2} db \wedge \ast^6 G_2 \wedge \Omega_4 + 2\kappa^2 \frac{\delta L_b}{\delta C_1} &= 0, \\
 d \left( e^{\phi/2} \ast d\tilde{C}_3 \right) + \frac{1}{2} db \wedge dA_1 \wedge \Omega_4 + 2\kappa^2 \frac{\delta L_b}{\delta \tilde{C}_3} &= 0, \\
 d \left( e^{\phi/2} \ast eG_2 \right) + db \wedge d\tilde{C}_3 + 4\kappa^2 \frac{\delta L_b}{\delta A_1} &= 0, \\
 d \left( e^{-\phi} \ast eC_1 \wedge \ast^6 G_2 \right) + d\tilde{C}_3 \wedge dA_1 + 4\kappa^2 \frac{\delta L_b}{\delta b} &= 0, \\
 d^* d\phi + \frac{3}{4} e^{3\phi/2} dC_1 \wedge \ast dC_1 + \frac{1}{4} e^{\phi/2} d\tilde{C}_3 \wedge \ast d\tilde{C}_3 + \frac{1}{4} \left[ e^{-\phi} db \wedge \ast \ast^6 G_2 \wedge \ast \ast eG_2 \right] \wedge \Omega_4 + 2\kappa^2 \frac{\delta L_b}{\delta \phi} &= 0,
\end{align}

where we have defined $\Omega_4 = \delta(x^6) \cdots \delta(x^9) \, dx^6 \wedge \cdots \wedge dx^9$. By varying the action with respect to the metric one gets also the Einstein equations that it is convenient to split into three separate equations (according to which components of the metric are involved in each case). By denoting with $x^{\rho,\sigma,\ldots} = \{x^0, \ldots, x^5\}$ the coordinates on $\mathbb{R}^{1,5}$ and by $x^{\mu, \nu, \ldots} = \{x^6, \ldots, x^9\}$ the orbifolded coordinates we find the following equations:

\begin{align}
 R_{\rho\sigma} - \frac{1}{2} RG_{\rho\sigma} + 2\kappa^2 \frac{\delta L_b}{\delta G^{\rho\sigma}} &= T^u_{\rho\sigma} + \Omega_4 T^t_{\rho\sigma}, \\
 R_{\rho\nu} - \frac{1}{2} RG_{\rho\nu} + 2\kappa^2 \frac{\delta L_b}{\delta G^{\rho\nu}} &= T^u_{\rho\nu}, \\
 R_{\nu\sigma} - \frac{1}{2} RG_{\nu\sigma} + 2\kappa^2 \frac{\delta L_b}{\delta G^{\nu\sigma}} &= T^u_{\nu\sigma}.
\end{align}

The energy-momentum tensors above refer separately to those of the “twisted” and “untwisted” fields, and are given by:

\begin{align}
 T^u_{\mu\nu} &= \frac{1}{2} (\partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} (\partial \phi)^2 G_{\mu\nu}) + \frac{1}{2} e^{3\phi/2} (F_{2_{\mu A} F_{2_{\nu} A}} - \frac{1}{4} (F_2)^2 G_{\mu\nu}) + \frac{1}{2} \cdot 3! (F_{4_{\mu A B C} F_{4_{\nu} A B C}} - \frac{1}{8} (F_4)^2 G_{\mu\nu}), \\
 T^t_{\rho\sigma} &= \frac{1}{2} \sqrt{-G} \left( \frac{1}{2} e^{\phi/2} (\partial_{\rho} b \partial_{\sigma} b - \frac{1}{2} (\partial b)^2 G_{\rho\sigma}) + \frac{1}{2} e^{\phi/2} (G_{2_{\mu A} G_{2_{\nu} A}} - \frac{1}{4} (G_2)^2 G_{\rho\sigma}) \right),
\end{align}

where, in the expression for $T^u_{\mu\nu}$, indices $\mu, \nu$ run over the appropriate coordinates (according to the equation in which they are used) and, in all cases, summed indices $(A, B, \ldots)$ run over all ten dimensional coordinates. In the expression for $T^t_{\rho\sigma}$, $G_{(6)}$ refers to the determinant of the restriction of the ten dimensional metric to the 6-dimensional subspace $\mathbb{R}^{1,5}$. 

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As explained in section 3.1, we are interested in a bound state of $N$ fractional D2-branes and $M$ D6-branes. The world-volume of the D2-branes extends in the directions $x^0, x^1, x^2$, while these branes are stuck at the orbifold fixed point $x^6 = x^7 = x^8 = x^9 = 0$. The D6-branes extend in the directions $x^0, x^1, x^2$ as well as along the orbifolded directions $x^6, x^7, x^8, x^9$.

For the “untwisted” fields we consider the following standard Ansatz for a D2/D6 system:

\begin{align}
\frac{ds^2}{H} & = H^{-5/8} H_6^{-1/8} \eta_{\alpha \beta} dx^\alpha dx^\beta + H_2^{3/8} H_6^{7/8} \delta_{ij} dx^i dx^j + H_2^{3/8} H_6^{-1/8} \delta_{pq} dx^p dx^q, \\
\phi & = H_2^{1/4} H_6^{-3/4}, \\
\bar{C}_3 & = (H_2^{-1} - 1) dx^0 \wedge dx^1 \wedge dx^2,
\end{align}

where we have divided the coordinates in three groups: $x^{\alpha, \beta, \ldots} = \{x^0, x^1, x^2\}$ denote the coordinates along the world-volume of both branes, $x^{i, j, \ldots} = \{x^3, x^4, x^5\}$ denote the ones transverse to both, while $x^{p, q, \ldots} = \{x^6, x^7, x^8, x^9\}$ denote the (orbifolded) coordinates along the world-volume of the D6-branes and transverse to the D2-branes. The function $H_2$ depends on the radial coordinate $\rho = \sqrt{(x^3)^2 + \ldots + (x^9)^2}$ of the space transverse to the D2-brane, while the function $H_6$ depends only on the radial coordinate of the common transverse space $r = \sqrt{\delta_{ij} x^i x^j}$.

In order to find a sensible Ansatz for the fields $A_1$ and $C_1$ we need to take a more careful look at the contributions coming from the boundary action describing the world-volume theory of the branes. For our system, such an action is the sum of a term describing the D2-branes and a term describing the D6-branes:

\begin{equation}
S_b = NS_2 + MS_6, \quad (A.17)
\end{equation}

The relevant parts of the action (as explained in Ref. \[45\], only the linear terms contribute) are given by (see appendix B):

\begin{align}
S_2 & = \frac{T_2}{2 \kappa} \left\{ - \int d^3 x \, e^{-\phi/4} \sqrt{-\det G_{\alpha \beta}} \left( 1 + \frac{\bar{b}}{2 \pi^2 \alpha'} \right) + \int_{\mathcal{M}_3} \bar{C}_3 \left( 1 + \frac{\bar{b}}{2 \pi^2 \alpha'} + \frac{A_3}{2 \pi^2 \alpha'} \right) \right\}, \quad (A.18a) \\
S_6 & = \frac{T_6}{\kappa} \left\{ - \int d^7 x \, e^{4 \phi} \sqrt{-\det G_{\rho \sigma}} + \int_{\mathcal{M}_7} C_7 \right\} + \frac{T_2}{2 \kappa 2(2 \pi \sqrt{\alpha'})^2} \left\{ \int d^4 \xi \sqrt{-\det G_{\alpha \beta}} \bar{b} - \int_{\mathcal{M}_3} A_3 \right\} + \ldots, \quad (A.18b)
\end{align}

where the indices $\alpha, \beta, \ldots$ run along the common world-volume $\mathcal{M}_3$ and $\rho, \sigma, \ldots$ along the whole D6-brane world-volume $\mathcal{M}_7$. 

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We notice that the previous boundary actions do not depend on the fields $C_1$ and $A_1$. This means that eq.s (A.13a) and (A.13c) will not contain the contribution coming from the boundary action:

\[ d\left(e^{3\phi/2} \ast dC_1\right) - \frac{1}{2} e^{3\phi/2} db \wedge \ast eG_2 \wedge \Omega_4 = 0, \]  
\[ d\left(e^{3\phi/2} \ast eG_2\right) + db \wedge d\bar{C}_3 = 0. \]  

Taking into account the expression in eq. (A.16c) for $\bar{C}_3$, we see that the second equation is easily satisfied by imposing:

\[ e^{\phi/2} \ast eG_2 = H^{-1} \frac{1}{2} db \wedge dx^0 \wedge dx^1 \wedge dx^2. \]  

Eq. (A.20) implies that the second term of eq. (A.19a) vanishes. Then, eq. (A.19a) can be satisfied by imposing:

\[ e^{3\phi/2} \ast dC_1 = -d \left(H_6^{-1}\right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^6 \wedge \cdots \wedge dx^9. \]  

Eq.s (A.20) and (A.21) imply after some manipulations the following expressions for $A_1$ and $C_1$:

\[ dA_1 = C_1 \wedge db + \frac{1}{2} \varepsilon_{ijk} H_6 \partial_i b dx^j \wedge dx^k, \]  
\[ dC_1 = \frac{1}{2} \varepsilon_{ijk} \partial_i H_6 dx^j \wedge dx^k, \]

where $\varepsilon_{ijk}$ is such that $\varepsilon_{345} = \varepsilon_{345}^* = +1$.

We are now ready to find the complete solution. Inserting the Ansätze (A.16)-(A.22) into the equations of motion (A.13) and computing all the relevant contributions coming from the boundary action $S_b$, after some algebra we get:

\[ (\delta^{ij} \partial_i \partial_j + H_6 \delta^{pq} \partial_p \partial_q) H_2 + \frac{1}{2} H_6 \delta^{ij} \partial_i b \partial_j b \delta(x^6) \cdots \delta(x^9) + \kappa T_2 N \delta(x^3) \cdots \delta(x^9) = 0, \]  

from the eq. (A.13b) for $\bar{C}_3$,

\[ H_2^{-1} \left(H_6 \delta^{ij} \partial_i \partial_j b + 2 \delta^{ij} \partial_i H_6 \partial_j b\right) - \frac{\kappa T_2}{4\pi^2 \alpha'} (4N - M) \delta(x^3) \cdots \delta(x^5) = 0, \]  

from the eq. (A.13d) for $b$ and

\[ \frac{1}{4} H_2^{-1} \left(\left(\delta^{ij} \partial_i \partial_j + H_6 \delta^{pq} \partial_p \partial_q\right) H_2 + \frac{1}{2} H_6 \delta^{ij} \partial_i b \partial_j b \delta(x^6) \cdots \delta(x^9)\right) - \frac{3}{4} H_6^{-1} \delta^{ij} \partial_i H_6 + \frac{\kappa T_2}{4} \left(N - \frac{6T_6}{T_2} M\right) \delta(x^3) \cdots \delta(x^9) = 0, \]  

\[ 28 \]
from the eq. (A.13e) for \( \phi \). Plugging eq. (A.23a) into eq. (A.23c) we get the following equation for the function \( H_6 \):

\[
\delta^{ij} \partial_i \partial_j H_6 = -2\kappa T_6 M \delta(x^3) \cdots \delta(x^5),
\]

(A.24)

whose solution is given by the following harmonic function:

\[
H_6(r) = 1 + \frac{g_s \sqrt{\alpha'} M}{2r}.
\]

(A.25)

Analogously, using eq. (A.24) into eq. (A.23b) we obtain the following equation for the function \( Z \equiv H_6 b \):

\[
\delta^{ij} \partial_i \partial_j Z = \frac{\kappa T_2}{2\pi^2 \alpha'} (2N - M) \delta(x^3) \cdots \delta(x^5),
\]

(A.26)

which is solved by:

\[
Z(r) = \frac{(2\pi \sqrt{\alpha'})^2}{2} \left( 1 - \frac{g_s \sqrt{\alpha'}(2N - M)}{r} \right),
\]

(A.27)

where we have chosen the constant term in order to satisfy the condition (3.6) for the background value of the field \( b \). We are now left with eq. (A.23a), which is in general difficult to solve. Finally, after some computation one can show that with our Ansatz the equations (A.14) for the metric are also satisfied provided that eq. (A.23a) holds.

Finally, the fields \( C_1 \) and \( A_1 \) are obtained by integrating eqs (A.22). In order to do so, we change the coordinate system into polar coordinates in the common transverse space: \((x^3, x^4, x^5) \rightarrow (r, \theta, \varphi)\). Then we obtain:

\[
C_1 = \frac{g_s \sqrt{\alpha'} M}{2} \cos \theta d\varphi,
\]

(A.28a)

\[
A_1 = -\pi^2 \alpha' \frac{g_s \sqrt{\alpha'}(4N - M)}{1 + \frac{g_s \sqrt{\alpha'} M}{2r}} \cos \theta d\varphi.
\]

(A.28b)

The supergravity solution that we have found is summarized in eq. (3.9).

### B World-volume actions for fractional branes

The world-volume action for a fractional Dp-brane \((p \leq 5)\) transverse to the orbifold space \( \mathbb{R}^4/\mathbb{Z}_2 \) can be obtained in several ways. Recalling that a fractional Dp-brane is a D\((p + 2)\)-brane wrapped\(^{14}\) on the vanishing cycle \( \Sigma_2 \) defined in section B.1, one can get the action

\(^{14}\)To be precise, we are considering fractional branes of “type 1”, which have a B-flux on the shrinking cycle but not an F-flux.
for a fractional Dp-brane starting from the one of a D(p + 2)-brane, which in the Einstein frame is:

\[ S_{p+2} = S_{\text{DBI}} + S_{\text{WZ}}, \] (B.1)

with:

\[ S_{\text{DBI}} = -\frac{T_{p+2}}{\kappa} \int d^{p+3} \xi e^{\frac{p+1}{4} \phi} \sqrt{-\det [G_{\alpha\beta} + e^{-\frac{\phi}{2}} 2\pi \alpha' F_{\alpha\beta}]} \], (B.2a)

\[ S_{\text{WZ}} = \frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} \sum_q C_q \wedge e^{B+2\pi\alpha' F}, \] (B.2b)

where \( \xi^{a,b,...} = \{\xi^0, \ldots, \xi^{p+2}\} \) are the coordinates of the brane world-volume and \( T_p = \sqrt{\pi}(2\pi \sqrt{\alpha'})^{3-p} \). All bulk fields in eq.s (B.2) are pullbacks onto the world-volume \( \mathcal{M}_{p+3} \) of the brane.

Let us start by considering the DBI part of the action. In order to wrap the brane on the cycle \( \Sigma_2 \) we have to impose the decomposition in eq. (3.4) for the field \( B_2 \) (we suppose that it has no components outside the cycle). The metric has no support on \( \Sigma_2 \), so eq. (B.2a) becomes:

\[ S_{\text{DBI}} = -\frac{T_{p+2}}{\kappa} \int d^{p+1} \xi e^{\frac{p+1}{4} \phi} \sqrt{-\det [G_{\alpha\beta} + e^{-\frac{\phi}{2}} 2\pi \alpha' F_{\alpha\beta}]} e^{-\frac{\phi}{2}} \int_{\Sigma_2} b \omega_2, \] (B.3)

where we have used eq.s (3.3). Considering now \( C_p \), one gets:

\[ \frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} C_p \wedge B \rightarrow \frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} \tilde{C}_{p+1} b \wedge \omega_2 = \frac{T_p}{2\kappa} \int_{\mathcal{M}_{p+3}} \tilde{C}_{p+1} \left(1 + \frac{\tilde{b}}{2\pi^2 \alpha'} \right), \] (B.6)

(notice that \( \tilde{C}_{p+3} \) vanishes). To discuss what is the result of wrapping, let us first consider the highest rank field \( C_{p+3} \), whose contribution to the WZ action is given by:

\[ \frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} C_{p+3} = \frac{T_{p+2}}{\kappa} \int_{\mathcal{M}_{p+3}} A_{p+1} \wedge \omega_2 = \frac{T_p}{2\kappa} \int_{\mathcal{M}_{p+3}} A_{p+1} \frac{2\pi \alpha'}{2\pi^2 \alpha'}, \] (B.5)
Therefore the first term of the WZ action is:

$$\frac{T_p}{2\kappa} \int_{\mathcal{M}_{p+1}} \left[ \bar{C}_{p+1} \left( 1 + \frac{\tilde{b}}{2\pi^2\alpha'} \right) + \frac{A_{p+1}}{2\pi^2\alpha'} \right].$$

(B.7)

If we consider any other lower rank potential, one can see that the relevant contributions to the WZ action always involve the following combinations of fields:

$$C_q = \bar{C}_q \left( 1 + \frac{\tilde{b}}{2\pi^2\alpha'} \right) + \frac{A_q}{2\pi^2\alpha'},$$

(B.8)

This means that the world-volume action for a fractional Dp-brane can always be put in the form:

$$S_p = S_{\text{DBI}} + S_{\text{WZ}},$$

(B.9)

where:

$$S_{\text{DBI}} = -\frac{T_p}{2\kappa} \int d^{p+1} \xi e^{\frac{\nu_p}{4} - \phi} \sqrt{-\det \left[ G_{\rho\sigma} + e^{-\frac{\nu_p}{2} 2\pi \alpha' F_{\rho\sigma}} \right]} \left( 1 + \frac{\tilde{b}}{2\pi^2\alpha'} \right),$$

(B.10a)

$$S_{\text{WZ}} = \frac{T_p}{2\kappa} \int_{\mathcal{M}_{p+1}} \sum_q C_q \wedge e^{2\pi \alpha' F}.$$  

(B.10b)

The precise expression of the action for a fractional Dp-brane is confirmed by the couplings of the brane to the bulk fields, computed with the boundary state formalism \[11\] and with explicit computation of string scattering amplitudes on a disk \[16\].

In this paper we also consider D-branes whose world-volume directions extend along the whole orbifold space, namely D(p + 4)-branes with four longitudinal directions along \( x^6, \ldots, x^9 \) and \( p + 1 \) along \( x^0, \ldots, x^p \). In this case the terms linear in the bulk fields of the boundary action can be inferred from the couplings computed with the boundary state \[15\], and one gets:

$$S_{p+4} = \frac{T_{p+4}}{2\kappa} \left\{ -\int d^{p+5} \xi e^{\frac{\nu_{p+1}}{4} - \phi} \sqrt{-\det G_{\rho\sigma}} + \int_{\mathcal{M}_{p+5}} C_{p+5} \right\}$$

$$+ \frac{T_p}{2\kappa} \left( \frac{1}{2(2\pi \sqrt{\alpha'})^2} \right) \int d^{p+1} \xi \sqrt{-\det G_{\alpha\beta}} \tilde{b} - \int_{\mathcal{M}_{p+1}} A_{p+1} \right\} + \ldots,$$

(B.11)

where \( x^{\rho,\ldots} \) are the coordinates of the brane world-volume \( \mathcal{M}_{p+5} \), while \( x^{\alpha,\beta,\ldots} \) are the coordinates along the part \( \mathcal{M}_{p+1} \) of the world-volume which lie outside the orbifold directions. The ellipses in the action \((B.11)\) stand for terms of higher order in the fields, not accounted by the boundary state approach.

\[\text{One has also to take into account the fact that the boundary state sees the fields correctly normalized on the covering space, while we are using fields that are correctly normalized on the orbifold \[13\].}\]
C The running coupling constant of $\mathcal{N} = 4$, $D = 2 + 1$ SYM theory

In this appendix, we briefly compute the expression of the running gauge coupling constant of $\mathcal{N} = 4$, $D = 2 + 1$ super Yang–Mills theory, in order to compare the perturbative gauge theory result [47] (see Ref. [48] for a review) with what we obtain from the supergravity solutions for the wrapped and the fractional brane systems.

The one-loop effective action for a $D$-dimensional field theory expanded around a background which is a solution of the classical field equations can be expressed as:

$$S_{\text{eff}} = \frac{1}{4 g_{\text{YM}}^2} \int d^D x \left\{ \tilde{F}_{\mu\nu}^a \tilde{F}^a_{\mu\nu} + \frac{1}{2} \text{Tr} \log \Delta_1 + \left( \frac{N_s}{2} - 1 \right) \text{Tr} \log \Delta_0 - N_f \text{Tr} \log \Delta_{1/2} \right\}, \quad (C.1)$$

where $N_s$ and $N_f$ are respectively the number of scalars and Dirac fermions and where:

$$(\Delta_1)^{ab}_{\mu\nu} = - (\tilde{D}^2)^{ab}_{\mu\nu} + 2 f^{abc} \tilde{F}^c_{\mu\nu}, \quad (\Delta_0)^{ab} = - (\tilde{D}^2)^{ab}, \quad \Delta_{1/2} = i \tilde{D}, \quad (C.2)$$

$D_\mu$ being the covariant derivative and $f^{abc}$ the gauge group structure constants. The part of the determinants in eq. (C.1) quadratic in the gauge fields can be extracted obtaining:

$$S_{\text{eff}} = \frac{1}{4} \int d^D x F^2 \left\{ \frac{1}{g_{\text{YM}}^2} + I \right\}, \quad (C.3)$$

where:

$$I = \frac{1}{(4\pi)^{D/2}} \int_0^\infty ds \frac{d_s}{s^{D/2-1}} e^{-\mu^2 s} R, \quad (C.4)$$

where $\mu$ is the mass of the fields, and

$$R = 2 \left[ \frac{N_s}{12} c_s + \frac{D - 26}{12} c_v + \frac{2[D/2] N_f}{6} c_f \right], \quad (C.5)$$

where $[D/2] = D/2$ if $D$ is even and $[D/2] = \frac{D-1}{2}$ if $D$ is odd, and where the constants $c$ set the normalization of the generators of the gauge group ($Tr(\lambda^a \lambda^b) = c \delta^{ab}$) in the representations under which the scalars, the vector and the fermions respectively transform. Concentrating on the case $D = 3$, we get:

$$I = \frac{1}{(4\pi)^{3/2}} \int_0^\infty ds \frac{d_s}{s^{1/2}} e^{-\mu^2 s} R = \frac{1}{8\pi\mu} R, \quad (C.6)$$

$^16$A bar on an operator in eqs. (C.1) and (C.2) indicates that the operator is evaluated at the background value $\tilde{A}_\mu^a$ of the gauge field.
with

\[ R = 2 \left[ \frac{N_s}{12} c_s - \frac{23}{12} c_v + \frac{N_f}{3} c_f \right]. \tag{C.7} \]

In our case we have a theory with gauge group \( SU(N) \) and 8 supercharges, coupled to \( M \) hypermultiplets in its fundamental representation. The vector multiplet contains 2 Dirac fermions and 3 scalars, while each hypermultiplet is made up of 2 Dirac fermions and 4 scalars. Recalling that \( c = \frac{1}{2} \) for the fundamental representation and \( c = N \) for the adjoint representation, eq. (C.7) gives:

\[ R = -2N + M, \tag{C.8} \]

and the running effective coupling given by eq.s (C.3)-(C.6) is equal to:

\[ \frac{1}{g_{\text{YM}}^2(\mu)} = \frac{1}{g_{\text{YM}}^2} \left( 1 - g_{\text{YM}}^2 \frac{2N - M}{8\pi\mu} \right), \tag{C.9} \]

This expression of the one-loop running coupling constant is in complete agreement with both our results in eqs (2.21) and (3.21).

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