Interaction of point sources and vortices for incompressible planar fluids

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1 Introduction

We consider a new system of differential equations which is at the same time gradient and locally Hamiltonian. It is obtained by just replacing the factor $i$ by $i^2 = -1$ in the equations of interaction for $N$ point vortices, and it is interpreted as an interaction of $N$ point sources. Because of the local Hamiltonian structure and the symmetries it obeys, it does possess some of the first integrals that appear in the $N$ vortex problem. We will show that binary collisions are easily blown up in this case since the equations of motion are of first order. We just pass to relative coordinates and then a convenient change of time scale. This method may be easily generalized to the blow up of higher order collisions.

The author is indebted to Jesus Muciño for conjecturing that a sort of harmonic conjugate of the Hamiltonian function for point vortices could be a first integral of the vortex system. It turns out that this is not true, but nevertheless motivated the introduction of the model for a system of point sources. Furthermore, (M. Hampton, 2008) proposed to take complex vorticities instead of the standard real ones, which allows us to generalize the model to cover simultaneous interactions of sources and vortices.
2 Statement of the problem and main properties

The Hamiltonian for a system of planar point vortices is given by

\[ H = - \sum_{k=1, l>k}^{N} \Gamma_k \Gamma_l \ln |z_k - z_l|. \] (1)

If we construct the gradient vector field corresponding to \( H \), we get instead of the differential equations for \( N \) point vortices as in (Aref, 2007 and Newton, 2001).

\[ \dot{z}_k = i \sum_{l \neq k}^{N} \frac{\Gamma_l}{|z_k - z_l|^2} \frac{z_k - z_l}{z_k - z_l} = i \sum_{l \neq k}^{N} \frac{\Gamma_l}{\bar{z}_k - \bar{z}_l}, \] (2)

those corresponding to the interaction of a system of planar point sources, i.e.

\[ \Gamma_k \dot{z}_k = - \sum_{l \neq k}^{N} \Gamma_k \Gamma_l \frac{z_k - z_l}{|z_k - z_l|^2} = - \sum_{l \neq k}^{N} \frac{\Gamma_k \Gamma_l}{\bar{z}_k - \bar{z}_l}. \] (2)

This velocity vector field \( X_s \) can also be represented as a locally Hamiltonian vector field for the one–form

\[ \alpha = -d \left( \sum_{l>k}^{N} \Gamma_k \Gamma_l \arg(z_k - z_l) \right), \] (3)

i.e., \( i_{X_s} \omega = -\alpha \).

This \( \alpha \) is globally defined in \( \mathbb{C}^N \setminus \Delta \) where \( \Delta = \{(z_1, \ldots, z_N) \mid z_j \neq z_k \forall j, k \} \), and in fact it is a closed but not exact one–form, since the function in between the parenthesis is only locally defined. This is because we can rewrite

\[ \alpha = - \sum_{l>k}^{N} \Gamma_k \Gamma_l d(\arg(z_k - z_l)), \] (4)

and each polar angle \( \theta_{kl} = \arg(z_k - z_l) \) is defined only modulo \( 2\pi \) in \( \mathbb{C} \setminus \{0\} \), although its exterior differential \( d\theta_{kl} \) is globally defined there.

One can easily verify that the linear momentum

\[ Z = \sum_{k} \Gamma_k z_k, \] (5)
is a first integral of system \( (2) \), so if \( \sum_k \Gamma_k \neq 0 \) we can define an equivalent center for the sources by \( z_0 = Z / \sum_k \Gamma_k \). On the other hand, the angular momentum \( A = \sum_k \Gamma_k z_k \times \dot{z}_k \) is an invariant of the system. In fact, we have

**Proposition. 1.** The angular momentum \( A \) for the point sources system is identically zero.

**Proof.** If \( \cdot \) denotes the standard scalar product of \( \mathbb{R}^2 \) identified with \( \mathbb{C} \), we can write

\[
\sum_k \Gamma_k z_k \times \dot{z}_k = \sum_k \Gamma_k (iz_k) \cdot \dot{z}_k = - \sum_k \Gamma_k (iz_k) \cdot \left( \sum_{l \neq k} \Gamma_l \frac{z_k - z_l}{|z_k - z_l|^2} \right) \equiv 0,
\]

since \( \Gamma_k \Gamma_l (iz_k) \cdot (z_k - z_l) + \Gamma_k \Gamma_l (iz_l) \cdot (z_l - z_k) = 0 \), for any \( k \neq l \).

In some sense, the reason for this result is that we are dealing with a central velocity vector field. So, there is no simple way to introduce rotations into the system.

**Conjecture.** Along any fixed solution of \( (2) \), the polar angles \( \theta_{kl} \) undergo variations smaller than \( 2\pi \), so that \( g = - \sum_{l>l} \Gamma_k \Gamma_l \theta_{kl} \) is a first integral of the system along the given solution.

However, the moment of inertia \( I = \sum_k \Gamma_k |z_k|^2 = \sum_k \Gamma_k z_k \bar{z}_k \), which is a first integral in the case of \( N \)-vortices, is not always a first integral for \( N \)-point sources. Indeed,

\[
\dot{I} = \sum_k \Gamma_k z_k \dot{\bar{z}}_k + \sum_k \Gamma_k \dot{z}_k \bar{z}_k

= - \sum_k z_k \sum_{l<k} \Gamma_k \Gamma_l \frac{\bar{z}_k - \bar{z}_l}{|z_k - z_l|^2} - \sum_k \bar{z}_k \sum_{l<k} \Gamma_k \Gamma_l \frac{z_k - z_l}{|z_k - z_l|^2}

= - \sum_{l<k} \Gamma_k \Gamma_l,
\]

since \( z_k (\bar{z}_k - \bar{z}_l) + z_l (\bar{z}_l - \bar{z}_k) = |z_k - z_l|^2 \). So, \( \dot{I} \) is in general a nonzero constant. Hence, only the two point source problem is always integrable since it has enough first integrals. In order for the integrability to make sense, we have to definitely know that we are dealing with a globally Hamiltonian vector field. But this is the case for \( N = 2 \), since the angle \( \theta_{12} \) is not only bounded but constant, as it will become also clear from the analysis in the following section.
Proposition. 2. If \( \sum_{l<k} \Gamma_k \Gamma_l = 0 \), then \( I \) is a first integral. If \( \sum_{l<k} \Gamma_k \Gamma_l \neq 0 \), then all solutions are gradient–like with respect to \( I \).

This implies that only when \( \sum_{l<k} \Gamma_k \Gamma_l = 0 \) is there any chance of having periodic solutions.

Comparing with the \( N \)–vortex problem, we recall that in that case if the \( \Gamma_k \) denote vorticities, the angular momentum takes exactly the value \( \sum_{l<k} \Gamma_k \Gamma_l \), which is called the virial, while \( I \) is a first integral of the system.

Also, collisions are rather scarce in the \( N \)–vortex problem and depend on the values of the vorticities. In contrast, collisions are very likely to occur in positive or in negative time for a system of \( N \) sources. There is no simple way to continue solutions beyond collisions, unless we agree that any time a collision occurs the sources involved merge into one source whose intensity is the sum of the colliding source intensities. As a result, the solutions will only be continuous but not smooth when collision occurs and the dimension of the phase space jumps correspondingly.

3 Example: A single point source and two point sources.

The differential equations for a single source are

\[
\dot{z} = -\frac{\Gamma}{|z|^2} z = -\frac{\Gamma}{\bar{z}},
\]

where \( z \in \mathbb{C} \setminus \{0\} \), or in terms of real and imaginary parts

\[
\dot{x} = -\frac{\Gamma x}{x^2 + y^2}, \quad \dot{y} = -\frac{\Gamma y}{x^2 + y^2},
\]

where \( -\Gamma \) is the intensity of the point source. When \( -\Gamma < 0 \) the vector field points radially towards the origin, so that it is actually a sink; when \( -\Gamma > 0 \) it is really a source, since the fluid runs away from the origin.

In this case we can take \( \omega = dx \wedge dy \) and \( \alpha = -\Gamma \frac{-ydx + xdy}{x^2 + y^2} \). The one form \( \alpha \) is closed but not exact on \( \mathbb{C} \setminus \{0\} \), since we can write \( \alpha = -\Gamma d\theta \), but only when the polar angle \( \theta \) is defined in an open interval of length smaller or equal than \( 2\pi \).
The equations of 2 point sources of nonzero real intensities $\Gamma_1$ and $\Gamma_2$ are given by

$$
\dot{z}_1 = -\Gamma_2 \frac{z_1 - z_2}{|z_1 - z_2|^2}, \quad \dot{z}_2 = -\Gamma_1 \frac{z_2 - z_1}{|z_2 - z_1|^2}.
$$

(7)

In terms of relative coordinates $z = z_1 - z_2$, they reduce to the single source equations of intensity $-(\Gamma_1 + \Gamma_2)$:

$$
\dot{z} = -(\Gamma_1 + \Gamma_2) \frac{z}{|z|^2}.
$$

(8)

If $-(\Gamma_1 + \Gamma_2) < 0$ this is an equivalent sink, so that the 2 point sources approach each other. On the contrary, if $-(\Gamma_1 + \Gamma_2) > 0$ it is an equivalent source and the 2 point sources move away from each other. Finally, if $\Gamma_1 + \Gamma_2 = 0$, then $z$ is constant, so that the two sources will move with the same velocity and keeping a constant distance, as we can see from (7).

In the cases where $\Gamma_1 + \Gamma_2 \neq 0$ we verify that indeed $I$ is not a first integral. All the solutions will then either end in collision or begin at collision in finite time. Indeed, equation (8) may be written as

$$
\frac{d}{dt} |z|^2 = \dot{z} \ddot{z} + \dot{\ddot{z}} z = -2(\Gamma_1 + \Gamma_2),
$$

which implies $|z|^2 = -2(\Gamma_1 + \Gamma_2)t$.

4 Regularization and blow up of binary collisions of point sources

In this section we show how to desingularize any binary collision of point sources.

We begin with the single point source (6), which as we saw in the last section is equivalent to a two point source. Upon multiplication of the equation by $\dot{z}$, we get $\ddot{z} \dot{z} = -\Gamma$, suggesting the change of time scale $\frac{dt}{d\tau} = z$, producing the new equation

$$
\frac{dz}{d\tau} = -\Gamma
$$

in terms of the time $\tau$. Moreover, we may assume $z \in \mathbb{R}$, since any motion is radial. This equation can be integrated as $z(\tau) = -\Gamma(\tau - \tau_0) + z_0$, and from $dt = z(\tau)d\tau$ we get the quadratic equation relating both times $t =$
\[- \frac{\Gamma^2}{2} + (\Gamma \tau_0 + z_0) = -\frac{1}{2} \Gamma (\tau - \tau_0 - z_0 / \Gamma)^2 + \frac{1}{2} \Gamma (\tau_0 + z_0 / \Gamma)^2. \] This may be considered as a regularization since the collision is not a singularity any more and it is reached in finite time. However, the physical time is bounded and it is not defined beyond the collision time value \( t = \frac{1}{2} \Gamma (\tau_0 + z_0 / \Gamma)^2 \), which is reached when \( \tau = \tau_0 + z_0 / \Gamma \). This is clear, since in contrast with the gravitational 2–body problem, we would not expect here the motion to be defined beyond a collapse of two sources since they merge into a single source.

If we multiply instead equation (9) by \(|z|^2| \) we get a new time rescaling \( \frac{dt}{ds} = |z|^2 \) which produces the differential equation

\[\frac{dz}{ds} = -\Gamma z,\]

whose integration gives an exponential solution \( z(s) = e^{-\Gamma s} \). This new desingularization corresponds to a blow up, since the singularity \( z = 0 \) is now an equilibrium point and so it is reached in infinite time.

Because for 3 or more sources collision does not occur in general along a straight line, it is the blow up which can be used for desingularizing binary collisions. We will illustrate the transformation for the case of 3 sources.

The differential equations for 3 sources located at \( z_1, z_2, z_3 \in \mathbb{C} \) with respective intensities \( \Gamma_1, \Gamma_2, \Gamma_3 \) are given by

\[
\begin{align*}
\dot{z}_1 &= -\Gamma_2 \frac{1}{\bar{z}_1 - \bar{z}_2} - \Gamma_3 \frac{1}{\bar{z}_1 - \bar{z}_3}, \\
\dot{z}_2 &= -\Gamma_1 \frac{1}{\bar{z}_2 - \bar{z}_1} - \Gamma_3 \frac{1}{\bar{z}_2 - \bar{z}_3}, \\
\dot{z}_3 &= -\Gamma_1 \frac{1}{\bar{z}_3 - \bar{z}_1} - \Gamma_2 \frac{1}{\bar{z}_3 - \bar{z}_2}.
\end{align*}
\]

(9)

Passing to the relative coordinates \( \xi = z_3 - z_1, \eta = z_3 - z_2 \), we can write \( z_2 - z_1 = \xi - \eta \), getting the following system of differential equations for \( \xi, \eta \)

\[
\begin{align*}
\dot{\xi} &= -\frac{\Gamma_1 + \Gamma_3}{\xi} - \frac{\Gamma_2}{\bar{\eta}} - \frac{\Gamma_2}{\xi - \bar{\eta}}, \\
\dot{\eta} &= -\frac{\Gamma_1}{\xi} - \frac{\Gamma_2 + \Gamma_3}{\bar{\eta}} - \frac{\Gamma_1}{\bar{\eta} - \xi}.
\end{align*}
\]
The change of time scale $\frac{dt}{ds} = |\xi|^2$ yields the blow up system
\[
\frac{d\xi}{ds} = -(\Gamma_1 + \Gamma_3)\xi - \frac{\Gamma_2|\xi|^2}{\eta} - \frac{\Gamma_2|\xi|^2}{\xi - \eta} \\
\frac{d\eta}{ds} = -\Gamma_1\xi - \frac{(\Gamma_2 + \Gamma_3)|\xi|^2}{\eta} - \frac{\Gamma_1|\xi|^2}{\eta - \xi}
\]
which is already regular at binary collision $\xi = 0$. If we want the solution in terms of the original coordinates when $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ is nonzero we need to solve the system
\[
\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 = \Gamma z_0 \\
-z_1 + z_3 = \xi \\
-z_2 + z_3 = \eta,
\]
where $z_0$ is the equivalent source center for the system, getting
\[
z_1 = z_0 + (\xi + \eta)/\Gamma - \xi \\
z_2 = z_0 + (\xi + \eta)/\Gamma - \eta \\
z_3 = z_0 + (\xi + \eta)/\Gamma
\]

If we want to blow up at the same time binary collisions $\xi = 0$ and $\eta = 0$, we change the rescaling by $\frac{dt}{ds} = |\xi|^2|\eta|^2$. This situation is generalized as follows.

**Proposition. 3.** For the general case of $N$ sources we change to $N - 1$ relative coordinates $\xi_1, \xi_2, \cdots, \xi_{N-1}$ and take a time scale factor $\frac{dt}{ds}$ which is the product of each of the $|\xi_i|^2$ corresponding to the collisions $\xi_i = 0$ we want to desingularize.

Passing to relative coordinates is equivalent to factoring out the translation symmetry corresponding to the first integral $Z$. One can show that for $N$ sources, only $N - 1$ suitable chosen relative coordinates generate all the remaining ones.

If we try to formally apply this regularization and blow up transformations to the differential equation $\dot{z} = i\frac{\Gamma}{\xi}$ for the velocity vector field of one vortex, we get in the first case $\ddot{z} = i\Gamma$, but rescaling does not have any sense here if we want to keep a real time; but by taking real parts this equation yields $\frac{d}{dt}|z|^2 = 0$. In the second case we get $\frac{d\xi}{ds} = |z|^2\dot{z} = i\Gamma z$, which upon integration gives $z(s) = Ke^{i ts}$ and finally $\frac{dt}{ds} = |z|^2 = K^2$ and so $t = K^2 s + t_0$. In both cases, we see that solutions are concentric circles as it is well known.
5 Interaction of point sources and point vortices

We can generalize the models for interaction of point vortices and of point sources to a model where we consider simultaneous interaction of point vortices and point sources.

The idea is to formally write the differential equations for point vortices where we assume now that the vorticities are nonzero complex numbers, that is

\[ \dot{z}_k = i \sum_{l \neq k} \Gamma_l \frac{z_k - z_l}{|z_k - z_l|^2}, \]

with \( \Gamma_k \in \mathbb{C} \setminus 0 \). This new velocity vector field \( X \) can be written as the sum of a Hamiltonian vector field \( X_1 \) with respect to the function \( H = -\sum_{k=1}^{N} \sum_{l \geq k} \text{Re}(\Gamma_k)\text{Re}(\Gamma_l) \ln |z_k - z_l| \), plus a gradient vector field \( X_2 \) with respect to the function \( G = -\sum_{k=1}^{N} \sum_{l \geq k} \text{Im}(\Gamma_k)\text{Im}(\Gamma_l) \ln |z_k - z_l| \) if we decompose the system as

\[ \dot{z}_k = i \sum_{l \neq k} \text{Re}(\Gamma_k) \frac{z_k - z_l}{|z_k - z_l|^2} - \sum_{l \neq k} \text{Im}(\Gamma_l) \frac{z_k - z_l}{|z_k - z_l|^2}. \]

The vector field \( X_2 \) is also locally Hamiltonian with respect to the one–form \( \alpha = -\sum_{l \geq k} \Gamma_k \Gamma_l d(\text{arg}(z_k - z_l)), \) globally defined in \( \mathbb{C}^N \setminus \Delta \). Here, \( \text{Re}(\Gamma_k) \) represent the vorticities and \( -\text{Im}(\Gamma_k) \) represent the source intensities.

If not all the real numbers \( \text{Re}(\Gamma_k) \) are nonzero, we reduce correspondingly the dimension of phase space in order to define the corresponding symplectic structure to produce the Hamiltonian vector field \( X_1 \).

5.1 Example, 2 point source-vortices

The equations of 2 point source-vortices of nonzero complex intensities \( \Gamma_1 \) and \( \Gamma_2 \) are given by

\[ \dot{z}_1 = -\Gamma_2 \frac{z_1 - z_2}{|z_1 - z_2|^2}, \quad \dot{z}_2 = -\Gamma_1 \frac{z_2 - z_1}{|z_2 - z_1|^2}. \]  

(10)

In terms of relative coordinates \( \mathbf{z} = z_1 - z_2 \), they reduce to the single source-vortex equations of intensity \( -(\Gamma_1 + \Gamma_2) \):

\[ \dot{\mathbf{z}} = -(\Gamma_1 + \Gamma_2) \frac{\mathbf{z}}{|\mathbf{z}|^2}. \]  

(11)
The velocity vector field corresponding to these equations is considered in Chapter 11 of (Needham, 1997) in a different context. This time equation (11) implies \( |z|^2 = -2\text{Re}(\Gamma_1 + \Gamma_2)t + r_0^2 \). But in fact, if we write \( z = re^{i\theta} \) then \( \dot{z} = (\dot{r} + ir\dot{\theta})e^{i\theta} \) or \( \ddot{z} = (r\ddot{r} + 2r\dot{r}\dot{\theta}) = -(\Gamma_1 + \Gamma_2) \), i.e. \( r\ddot{r} = -\text{Re}(\Gamma_1 + \Gamma_2), r^2\dot{\theta} = -\text{Im}(\Gamma_1 + \Gamma_2) \). Integrating, we get

\[
\frac{r}{2} = \sqrt{-2t\text{Re}(\Gamma_1 + \Gamma_2) + r_0^2}
\]

and

\[
\theta = \int \frac{-\text{Im}(\Gamma_1 + \Gamma_2)dt}{-2t\text{Re}(\Gamma_1 + \Gamma_2) + r_0^2} + \theta_0
\]

This corresponds to a logarithmic spiral of the form \( r = Ke^{\alpha\theta} \). So that a collision occurs in finite time with infinite spiralling, that is, while the argument \( \theta \) remains unbounded. If \( \text{Re}(\Gamma_1 + \Gamma_2) = 0 \), then \( r = r_0 \) is constant and \( \theta \) is linear in \( t \), corresponding to an equivalent pure vortex. Likewise, if \( \text{Im}(\Gamma_1 + \Gamma_2) = 0 \), then \( \theta = \theta_0 \) is constant and \( r^2 \) is linear in \( t \), which is equivalent to a pure source.

If we apply now a blow up to equation (11), we get \( |z|^2\ddot{z} = -(\Gamma_1 + \Gamma_2)z \). The change of time scale \( \frac{ds}{dt} = |z|^2 \) gives \( \frac{dz}{ds} = -(\Gamma_1 + \Gamma_2)z \), which is integrated as \( z(s) = Ke^{-(\Gamma_1 + \Gamma_2)s} = Ke^{-\text{Re}(\Gamma_1 + \Gamma_2)s}e^{-i\text{Im}(\Gamma_1 + \Gamma_2)s} \), giving another description of the logarithmic spiral in terms of time \( s \). We remark that an infinite spiraling while going to collision in finite time occurs also in the Manev problem which is an approximate model for the Schwarschild equation in general relativity.

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