Surjective separating maps on noncommutative $L^p$-spaces

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Abstract
Let $1 \leq p < \infty$ and let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a bounded map between noncommutative $L^p$-spaces. If $T$ is bijective and separating, we prove the existence of decompositions $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ and maps

$$T_1 : L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1), \quad T_2 : L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2),$$

such that $T = T_1 + T_2$, $T_1$ has a direct Yeadon type factorisation and $T_2$ has an anti-direct Yeadon type factorisation. We further show that $T^{-1}$ is separating in this case. Next we prove that for any $1 \leq p < \infty$ (resp. any $1 \leq p \neq 2 < \infty$), a surjective separating map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is $S^1$-bounded (resp. completely bounded) if and only if there exists a decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorisation and $\mathcal{M}_2$ is subhomogeneous.

KEYWORDS
noncommutative $L^p$-spaces, operator spaces, separating maps, von Neumann algebras

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1 | INTRODUCTION

This paper deals with separating maps between noncommutative $L^p$-spaces, $1 \leq p < \infty$. These operators were investigated recently in [1, 4, 5], to which we refer for background, motivation and historical facts. Recall that a bounded map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ between two noncommutative $L^p$-spaces is called separating if for any $x, y \in L^p(\mathcal{M})$, the condition $x^*y = xy^* = 0$ implies that $T(x)^*T(y) = T(x)T(y)^* = 0$. It was shown in [4, Proposition 3.11] and [1, Theorem 3.3 & Remark 3.4] that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is separating if and only if there exists a $w^*$-continuous Jordan homomorphism $J : \mathcal{M} \to \mathcal{N}$, a positive operator $B$ affiliated with $\mathcal{N}$ and commuting with the range of $J$, as well as a partial isometry $w \in \mathcal{N}$ such that $w^*w = s(B) = J(1)$ and

$$T(x) = wBJ(x), \quad (x \in \mathcal{M} \cap L^p(\mathcal{M})).$$

Such a factorisation (which is necessarily unique) is called a Yeadon type factorisation in [4, 5]. We further say that $T$ admits a direct Yeadon type factorisation if the Jordan homomorphism $J$ in this factorisation is a $\ast$-homomorphism. It is proved in [5, Proposition 4.4] and [1, Theorem 3.6] that any separating map $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ with a direct Yeadon type factorisation is necessarily completely bounded. It is also proved in [5, Proposition 4.5] that any such map is $S^1$-bounded (see Section 2 below for the definition). The main purpose of the present paper is to establish a form of converse of these results for surjective maps. More precisely, we prove the following characterizations.
Theorem 1.1. Let $1 \leq p < \infty$, let $\mathcal{M}, \mathcal{N}$ be semifinite von Neumann algebras and let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a surjective separating map. The following are equivalent:

(i) $T$ is $S^1$-bounded;

(ii) There exists a direct sum decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that the restriction of $T$ to $L^p(\mathcal{M}_1)$ has a direct Yeadon type factorisation and $\mathcal{M}_2$ is subhomogeneous.

Moreover if $p \neq 2$, then (ii) is also equivalent to:

(iii) $T$ is completely bounded.

These results will be proved in Section 4. We also provide an example showing that the surjectivity assumption cannot be dropped. In Section 3, we establish a general decomposition result for bijective separating maps which plays a key role in the above characterization results. We prove in passing that the inverse of any bijective separating map is separating as well. Section 2 is preparatory.

2 BACKGROUND

In this section we recall some necessary background on semifinite noncommutative $L^p$-spaces and subhomogeneous von Neumann algebras.

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a normal semifinite faithful trace $\tau_\mathcal{M}$. Assume that $\mathcal{M} \subset B(H)$ acts on some Hilbert space $H$. Let $L^0(\mathcal{M})$ denote the $*$-algebra of all closed densely defined (possibly unbounded) operators on $H$, which are $\tau_\mathcal{M}$-measurable. Then for any $1 \leq p < \infty$, the noncommutative $L^p$-space associated with $\mathcal{M}$ can be defined as

$$L^p(\mathcal{M}) := \{ x \in L^0(\mathcal{M}) : \tau_{\mathcal{M}}(\vert x \vert^p) < \infty \}.$$ We set $\| x \|_p := \tau_\mathcal{M}(\vert x \vert^p)^{\frac{1}{p}}$ for any $x \in L^p(\mathcal{M})$. Then $L^p(\mathcal{M})$ equipped with $\| \cdot \|_p$ is a Banach space. The reader may consult [3, 8, 12] and the references therein for details and further properties.

We let $S^p$, $1 \leq p < \infty$, denote the noncommutative $L^p$-space built upon $B(\ell^2)$ with its usual trace; this is in fact the Schatten $p$-class of operators on $\ell^2$. For any $m \geq 1$, we let $S^p_m$ denote the Schatten $p$-class of $m \times m$ matrices. Whenever $E$ is an operator space, we let $S^p_m[E]$ denote the $E$-valued Schatten space introduced in [6, Chapter 1].

Recall that we may identify $L^p(\mathcal{M} \otimes M_m)$ with $L^p(\mathcal{M}) \otimes S^p_m$ in a natural way. Let $\mathcal{N}$ be, possibly, another semifinite von Neumann algebra. We say that an operator $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is completely bounded if there exists a constant $K \geq 0$ such that

$$\| T \otimes I_{S^p_m} : L^p(\mathcal{M} \otimes M_m) \to L^p(\mathcal{N} \otimes M_m) \| \leq K,$$

for any $m \geq 1$. In this case, the completely bounded norm of $T$ is the smallest such uniform bound and is denoted by $\| T \|_{cb}$. We further say that $T$ is a complete isometry if $T \otimes I_{S^p_m}$ is an isometry for any $m \geq 1$.

In [5, Section 3], we introduced $S^1$-valued noncommutative $L^p$-spaces, which naturally extend previous constructions from [2, 6]. We recall this definition here.

For $1 \leq p < \infty$, the $S^1$-valued noncommutative $L^p$-space, $L^p(\mathcal{M}; S^1)$, is the space of all infinite matrices $[x_{ij}]_{i,j \geq 1}$ in $L^p(\mathcal{M})$ for which there exist families $(a_{ik})_{i,k \geq 1}$ and $(b_{kj})_{k,j \geq 1}$ in $L^2(\mathcal{M})$ such that $\sum_{i,k} a_{ik}a^*_{ik}$ and $\sum_{k,j} b^*_{kj} b_{kj}$ converge in $L^p(\mathcal{M})$ and for all $i, j \geq 1$,

$$x_{ij} = \sum_{k=1}^\infty a_{ik} b_{kj}.$$
We equip $L^p(\mathcal{M} ; S^1)$ with the following norm

$$
\| [x_{ij}] \|_{L^p(\mathcal{M} ; S^1)} = \inf \left\{ \left\| \sum_{k,l=1}^\infty a_{lk} a_k^* \right\|_p \left\| \sum_{k,j=1}^\infty b_{kj}^* b_{kj} \right\|_p^{\frac{1}{p}} \right\},
$$

(2.1)

where the infimum is taken over all families $(a_{lk})_{i,k \geq 1}$ and $(b_{kj})_{k,j \geq 1}$ as above. The space $L^p(\mathcal{M} ; S^1)$ endowed with this norm is a Banach space.

For any integer $m \geq 1$, let $L^p(\mathcal{M} ; S^1_m)$ be the subspace of $L^p(\mathcal{M} ; S^1)$ of matrices $[x_{ij}]_{i,j \geq 1}$ with support in $\{1, ..., m\}^2$.

Following [5, Definition 3.8], we say that a bounded operator $\gamma : L^p(M) \to L^p(\mathcal{N})$ is $S^1$-bounded if there exists a constant $K > 0$ such that

$$
\| T \otimes I_{S^1_m} : L^p(\mathcal{M} ; S^1_m) \to L^p(\mathcal{N} ; S^1_m) \| \leq K,
$$

for any $m \geq 1$. In this case, the $S^1$-bounded norm of $T$ is the smallest such uniform bound and is denoted by $\| T \|_{S^1}$. We further say that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is an $S^1$-isometry if for each $m \geq 1$, $T \otimes I_{S^1_m}$ is an isometry.

We proved in [5] that for any $n \geq 1$, $L^p(M_n ; S^1) = S^p_n \otimes S^p_n$ isometrically. Further, if $\mathcal{M}, \mathcal{N}$ are hyperfinite, then $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is $S^1$-bounded if and only if it is regular in the sense of [7].

We note that any direct sum $\mathcal{M} = M_1 \oplus M_2$ induces isometric identifications $L^p(\mathcal{M}) = L^p(M_1) \oplus L^p(M_2)$ and $L^p(\mathcal{M} ; S^1) = L^p(M_1 ; S^1) \oplus L^p(M_2 ; S^1)$ (see [5, Lemma 5.2] for the last identification).

Recall that a $C^*$-algebra $\mathcal{A}$ is called subhomogeneous of degree $\leq N$ if all irreducible representations of $\mathcal{A}$ are of maximum dimension $N$. If $\mathcal{A}$ is subhomogeneous of degree $\leq N$, for some $N$, we simply say that $\mathcal{A}$ is subhomogeneous. It is well-known (see for example [9, Theorem 7.1.1]) that $\mathcal{M}$ is a subhomogeneous von Neumann algebra of degree $\leq N$ if and only if there exist $r \geq 1$, integers $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r \leq N$ and abelian von Neumann algebras $L^\infty(\Omega_1), ..., L^\infty(\Omega_r)$ such that

$$
\mathcal{M} \simeq \bigoplus_{1 \leq j \leq r} L^\infty(\Omega_j ; M_{n_j}).
$$

(2.2)

If a von Neumann algebra $\mathcal{M}$ is not subhomogeneous of degree $\leq N$, it is well-known that there is a non zero $*$-homomorphism $\gamma : M_{N+1} \to \mathcal{M}$. Lemma 2.1 below makes this more explicit in the semifinite case.

**Lemma 2.1.** Let $\mathcal{M}$ be a semifinite von Neumann algebra and let $N \geq 1$. If $\mathcal{M}$ is not subhomogeneous of degree $\leq N$, then there is a complete isometry from $S^p_{N+1}$ into $L^p(\mathcal{M})$ that is also an $S^1$-isometry.

**Proof.** Let $\mathcal{M} = M_1 \oplus M_2$ be the direct sum decomposition of $\mathcal{M}$ into a type I summand $M_1$ and a type II summand $M_2$ (see e.g. [11, Section 5]).

Assume that $M_2 \neq \{0\}$. Following the same lines as in [5, Lemma 2.3], there is a projection $e$ in $M_2$, a trace preserving von Neumann algebra identification

$$
\mathcal{M}_2 \simeq M_{N+1} \overline{\otimes} (e M_2 e)
$$

(2.3)

and a finite trace projection $\varepsilon$ in $e M_2 e$ such that the mapping

$$
\gamma : M_{N+1} \to \mathcal{M}_2 \subset \mathcal{M}; \quad \gamma(a) = a \otimes \varepsilon
$$

is a non zero $*$-homomorphism taking values in $L^1(\mathcal{M})$, and therefore $L^p(\mathcal{M})$.

For every $[a_{ij}]_{1 \leq i,j \leq m}$ in $S^p_{N+1} \otimes S^p_m$ we have

$$
\| [a_{ij} \otimes \varepsilon] \|_{L^p(\mathcal{M}_2 \otimes \mathcal{M}_m)} = \| \varepsilon \|_p \| [a_{ij}] \|_{L^p(M_{N+1} \otimes M_m)},
$$

where the infimum is taken over all families $(a_{ik})_{i,k \geq 1}$ and $(b_{kj})_{k,j \geq 1}$ as above.
and therefore $\|\varepsilon\|^{-1}_p \gamma$ is a complete isometry from $S_{N+1}^p$ into $L^p(M)$. By [5, Lemma 5.1],

$$\| [a_{ij} \otimes \varepsilon]_{L^p(M)} \|_{S_{N+1}^1} = \|\varepsilon\|_p \| [a_{ij}]_{S_{N+1}^p} \|_{S_{N+1}^1},$$

and therefore $\|\varepsilon\|^{-1}_p \gamma$ is also an $S^1$-isometry from $S_{N+1}^p$ into $L^p(M)$.

If $M_2 = \{0\}$, then $M$ is of type I. Since $M$ is not subhomogeneous of degree $\leq N$, it follows from [11, Theorem V.1.27] that there exist a Hilbert space $H$ with $\dim(H) \geq N+1$ and an abelian von Neumann algebra $W$ such that $M$ contains $B(H) \otimes W$ as a summand. Using this summand instead of (2.3) and arguing as above we obtain the result in this case as well.

$$\square$$

3 BIJECTIVE SEPARATING MAPS AND THEIR INVERSES

The goal of this section is to provide a decomposition for bijective separating maps that facilitates their study. We apply this decomposition to show that the inverse of a bijective separating map is separating as well.

First we recall some terminologies and results that we will use. A Jordan homomorphism between von Neumann algebras $M$ and $N$ is a linear map $J : M \to N$ such that $J(xy + yx) = J(x)J(y) + J(y)J(x)$ for all $x, y \in M$. It is plain that $*$-homomorphisms and anti-$*$-homomorphisms are Jordan homomorphisms. In fact, every Jordan homomorphism is a sum of a $*$-homomorphism and an anti-$*$-homomorphism, as we recall here.

Let $J : M \to N$ be a Jordan homomorphism and let $D \subset N$ be the $w^*$-closed $C^*$-algebra generated by $J(M)$. Then $J(1)$ is the unit of $D$. By e.g. [10, Theorem 3.3], there exist projections $e$ and $f$ in the center of $D$ such that $e + f = J(1)$, $x \mapsto J(x)e$ is a $*$-homomorphism, and $x \mapsto J(x)f$ is an anti-$*$-homomorphism. Let $N_1 = eN_2$ and $N_2 = fN_1$. Define $\pi : M \to N_1$ and $\sigma : M \to N_2$ by $\pi(x) = J(x)e$ and $\sigma(x) = J(x)f$, for all $x \in M$. Then $J$ is valued in $N_1 \oplus N_2$ and $J(x) = \pi(x) + \sigma(x)$, for all $x \in M$.

Assume that $M$ and $N$ are semifinite von Neumann algebras and let $1 \leq p < \infty$. In [4], inspired by Yeon’s fundamental description of isometries between noncommutative $L^p$-spaces, we say that a bounded operator $T : L^p(M) \to L^p(N)$ has a Yeon type factorisation if there exist a $w^*$-continuous Jordan homomorphism $J : M \to N$, a partial isometry $w \in N$, and a positive operator $B$ affiliated with $N$, which satisfy the following conditions:

(a) $w^*w = J(1) = s(B)$, the support projection of $B$;
(b) every spectral projection of $B$ commutes with $J(x)$, for all $x \in M$;
(c) $T(x) = wBJ(x)w$ for all $x \in M \cap L^p(M)$.

We call $(w, B, J)$ the Yeon triple associated with $T$. This triple is unique. Following [5], if $J$ is a $*$-homomorphism (respectively, anti-$*$-homomorphism), we say that $T$ has a direct (respectively, anti-direct) Yeon type factorisation.

Following [4], we say that a bounded operator $T : L^p(M) \to L^p(N)$ is separating if for every $x, y \in L^p(M)$ such that $x^*y = xy^* = 0$, we have $T(x^*)T(y) = T(x)T(y)^* = 0$. The following characterization has a fundamental role in the study of separating maps.

**Theorem 3.1** ([1, Theorem 3.3], [4, Theorem 3.5]). A bounded operator $T : L^p(M) \to L^p(N)$ admits a Yeon type factorisation if and only if it is separating.

It is easy to see that for a separating map $T : L^p(M) \to L^p(N)$ with Yeon triple $(w, B, J)$, we have that

$$T(z^*) = wT(z)^*w \quad (z \in L^p(M)).$$

(3.1)

Also, if $T$ has a direct (respectively, anti-direct) Yeon type factorisation, we get that

$$T(zm) = T(z)J(m) \quad (respectively, T(mz) = T(z)J(m)),$$

(3.2)

for every $z \in L^p(M)$ and $m \in M$. 

Remark 3.2. Let $T : L^p(M) \to L^p(N)$ be a separating map with Yeadon triple $(w, B, J)$. We observe that if $T$ is surjective, then $w$ is a unitary. Indeed on the one hand, we see that $T$ is valued in $wL^p(N)$. Since $ww^*w = w$, this implies that $T$ is valued in $ww^*L^p(N)$. Hence, if $T$ is surjective, we have $ww^*L^p(N) = L^p(N)$, which implies that $ww^* = 1$. On the other hand, $T(x) = T(x)J(1)$, for any $x \in L^p(M)$. Hence, $T$ is valued in $L^p(N)J(1)$, which implies $ww^* = J(1) = 1$.

Proposition 3.3. Let $T : L^p(M) \to L^p(N)$ be a separating map that is bijective. Then there exist direct sum decompositions

$$
\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2, \quad \text{and} \quad \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2,
$$

and bounded bijective separating maps $T_1 : L^p(M_1) \to L^p(N_1)$ with a direct Yeadon type factorisation and $T_2 : L^p(M_2) \to L^p(N_2)$ with an anti-direct Yeadon type factorisation such that $T = T_1 + T_2$.

Proof. Assume that $w = 1$. Consider a decomposition for $J$, induced by central projections $e$ and $f$, as recalled above. As detailed in [5, Remark 4.3], this induces a decomposition $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ and separating maps

$$
T_1 : L^p(M) \to L^p(N_1), \quad T_1(x) = T(x)e,
$$

with Yeadon triple $(e, Be, \pi)$, and hence a direct Yeadon type factorisation, and

$$
T_2 : L^p(M) \to L^p(N_2), \quad T_2(x) = T(x)f,
$$

with Yeadon triple $(f, Bf, \sigma)$, and hence an anti-direct Yeadon type factorisation, such that $T = T_1 + T_2$.

Let $\mathcal{M}_1 := \ker(\sigma)$ and $\mathcal{M}_2 := \ker(\pi)$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are $w^*$-closed ideals of $\mathcal{M}$, there exist central projections $\alpha, \beta \in \mathcal{M}$ such that $\mathcal{M}_1 = \alpha \mathcal{M}$, and $\mathcal{M}_2 = \beta \mathcal{M}$. Set $\mathcal{M}_3 := (1 - \alpha)(1 - \beta) \mathcal{M}$. Note that $\alpha \beta \in \ker(\sigma) \cap \ker(\pi)$, and therefore $J(\alpha \beta) = 0$. Since $T$ is one-to-one, by [4, Remark 3.14(a)], $J$ is one-to-one and therefore we must have that $\alpha \beta = 0$. Hence,

$$
1 = \alpha + \beta + (1 - \alpha)(1 - \beta).
$$

Consequently, $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$, and so we have the decomposition

$$
L^p(M) = L^p(M_1) \oplus L^p(M_2) \oplus L^p(M_3).
$$

The result will follow if we can show that

$$
L^p(M_1) = \ker(T_2), \quad L^p(M_2) = \ker(T_1) \quad \text{and} \quad \mathcal{M}_3 = \{0\}.
$$

To see that $L^p(M_1) \subseteq \ker(T_2)$, let $x \in \mathcal{M}_1 \cap L^p(M_1)$, then

$$
T_2(x) = B\sigma(x) = 0.
$$

Hence, $\mathcal{M}_1 \cap L^p(M_1) \subseteq \ker(T_2)$ and therefore $L^p(M_1) \subseteq \ker(T_2)$. Now suppose that $x$ belongs to $\ker(T_2)$. For any $n \geq 1$, let $p_n = \chi_{[-n, n]}(|x^*|)$, the projection associated with the indicator function of $[-n, n]$ in the Borel functional calculus of $|x^*|$, and $x_n := p_n x$. Then, using (3.2), we have

$$
T_2(x_n) = T_2(x)\sigma(p_n) = 0.
$$
Hence, $B \sigma(x_n) = 0$. Since $s(B) = 1$, this implies that $\sigma(x_n) = 0$, that is $x_n \in M_1$. Now because $x_n \to x$ in $L^p(M)$, we obtain that $x$ belongs to $L^p(M_1)$. Hence,

$$L^p(M_1) = \ker(T_2).$$

Similarly, we can show that $L^p(M_2) = \ker(T_2)$.

Finally, we show that $M_3 = \{0\}$. Let $x \in L^p(M)$. By surjectivity of $T$, there is $y$ in $L^p(M)$ such that $T(y) = T_1(x)$. Writing $T(y) = T_1(y) + T_2(y)$, we obtain that $T_1(x - y) = 0$ and $T_2(y) = 0$, that is $x - y$ belongs to $\ker(T_1) = L^p(M_2)$ and $y$ belongs to $\ker(T_2) = L^p(M_1)$, thus $x$ belongs to $L^p(M_1) \oplus L^p(M_2)$. Hence, $M_3 = \{0\}$. This completes the proof in the case $w = 1$.

In the general case, consider the map $\tilde{T} := w^*T(\cdot)$, which takes any $x \in M \cap L^p(M)$ to $Bj(x)$. By Remark 3.2, $\tilde{T}$ is also a bijective separating map. Its Yeaden triple is $(1, B, J)$. We may apply the above decomposition to the map $\tilde{T}$ to obtain decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$ and bounded bijective separating maps $\tilde{T}_1 : L^p(M_1) \to L^p(N_1)$ with a direct Yeaden type factorisation and $\tilde{T}_2 : L^p(M_2) \to L^p(N_2)$ with an anti-direct Yeaden type factorisation such that $\tilde{T} = \tilde{T}_1 + \tilde{T}_2$. Since $\tilde{w}\tilde{T} = T$, we obtain the result. \qed

**Proposition 3.4.** Suppose that $T : L^p(M) \to L^p(N)$ is a bijective separating map, then

(i) $T^{-1} : L^p(N) \to L^p(M)$ is separating.

(ii) If $J : M \to N$ is the Jordan homomorphism associated with $T$, then $J$ is invertible and $J^{-1} : N \to M$ is the Jordan homomorphism associated with $T^{-1}$.

**Proof.** Using the decomposition given in Proposition 3.3, it is enough to show parts (i) and (ii) for a bijective separating map with a direct Yeaden type factorisation. So, throughout the proof we assume that this is the case. Note that by Remark 3.2, $J(1) = 1$.

(i) Suppose that $a, b \in L^p(N)$ such that $a^*b = ab^* = 0$. We show that $T^{-1}(a)^*T^{-1}(b) = T^{-1}(a)T^{-1}(b)^* = 0$. Let $x = T^{-1}(a)$ and $y = T^{-1}(b)$. Set $p_n := \chi_{[-n,n]}(\{y\})$, for any $n \geq 1$. We have

$$T(x^*y p_n)B = T(x^*)J(y p_n)B$$

by (3.2)

$$= T(x^*)w^*T(y p_n)$$

by (3.1)

$$= wT(x)^*T(y p_n)$$

by (3.2)

$$= wa^*bJ(p_n)$$

$$= 0.$$

Since $s(B) = J(1) = 1$, we obtain $T(x^*y p_n) = 0$. Because $T$ is one-to-one, we have that $x^*y p_n = 0$. Now, since $y p_n \to y$, we get that $x^*y = 0$. A similar argument using $ab^* = 0$ implies that $xy^* = 0$. Hence $T^{-1}$ must be separating.

(ii) By part (i), $T^{-1}$ is separating. We let $J'$ denote the Jordan homomorphism of its Yeaden triple. Let $e \in N$ be a projection with finite trace. For any $y \in eN e$, we have $T^{-1}(y) = T^{-1}(e)J'(y)$. Applying (3.2), we deduce that

$$y = TT^{-1}(y) = T(T^{-1}(e)J'(y)) = TT^{-1}(e)JJ'(y) = e JJ'(y).$$

Using the $w^*$-continuity of $J$ and $J'$, and the $w^*$-density of the union of the $eN e$, for $\tau_N(e) < \infty$, we deduce that $y = JJ'(y)$ for any $y \in N$. By [4, Remark 3.14(a)], since $T$ is one-to-one, $J$ must be one-to-one. Hence, $J$ is invertible with $J^{-1} = J'$.

**Remark 3.5.** Part (ii) of Proposition 3.4 shows that a separating invertible map $T : L^p(M) \to L^p(N)$ admits a direct Yeaden type factorisation if and only if $T^{-1}$ does.
4 | A CHARACTERIZATION OF COMPLETELY/$S^1$-BOUNDED SURJECTIVE SEPARATING MAPS

In this section we show that a separating map can always be reduced to a one-to-one separating map and therefore we may confine ourselves to the study of separating maps that are surjective rather than bijective. The goal of the section is to provide a characterization for surjective separating maps that are completely bounded (when $p \neq 2$) or $S^1$-bounded. We show that the surjectivity assumption is essential.

We require [5, Propositions 4.4 & 4.5] later on in our arguments in this section. We recall the statements for convenience.

**Proposition 4.1.** Let $T : L^p(M) \to L^p(N)$ be a bounded operator with a direct Yeadon type factorisation. Then $T$ is completely bounded and $\|T\|_{cb} = \|T\|$.

**Proposition 4.2.** Let $T : L^p(M) \to L^p(N)$ be a bounded operator with a direct Yeadon type factorisation. Then $T$ is $S^1$-bounded and $\|T\|_{S^1} = \|T\|$.

**Lemma 4.3.** Let $T : L^p(M) \to L^p(N)$ be a separating map. Then there exists a direct sum decomposition $M = M_0 \oplus \tilde{M}$ such that $\ker(T) = L^p(M_0)$.

**Proof.** Let $T : L^p(M) \to L^p(N)$ be a separating map and $J : M \to N$ be the Jordan homomorphism associated with $T$ via its Yeadon type factorisation. Let $M_0 := \ker(J)$. Then $M_0$ is an ideal. Since $J$ is $w^*$-continuous, $M_0$ is $w^*$-closed. Hence we have a direct sum decomposition

$$M = M_0 \oplus \tilde{M}.$$

It is clear that $L^p(M_0) \subset \ker T$. Further $J|_{\tilde{M}}$ is one-to-one. By [4, Remark 3.14(a)] this implies that $T|_{L^p(\tilde{M})}$ is one-to-one. This yields the result. \qed

For any von Neumann algebra $M$, we let $M^{op}$ denote its opposite von Neumann algebra. Recall that the underlying dual Banach space structure and involution on $M^{op}$ are the same as on $M$ but the product of $x$ and $y$ is defined by $yx$ rather than $xy$. Note that the Banach spaces $L^p(M)$ and $L^p(M^{op})$ are the same. It is evident that, for von Neumann algebras $M$ and $N$, $J : M \to N$ is a $*$-homomorphism if and only if

$$J^{op} : M^{op} \to N; \quad x \mapsto J(x),$$

is an anti-$*$-homomorphism. Hence, a separating map $T : L^p(M) \to L^p(N)$ has a direct Yeadon type factorisation if and only if

$$T^{op} : L^p(M^{op}) \to L^p(N); \quad x \mapsto T(x),$$

has an anti-direct Yeadon type factorisation.

Lemma 4.4 below is the principal ingredient of our characterization theorems. Its proof relies on the relation between the completely bounded norm or $S^1$-norm of the identity map

$$I^{op} : L^p(M) \to L^p(M^{op})$$

and the norms of the transformations

$$[x_{ij}]_{1 \leq i,j \leq n} \mapsto [x_{ij}]_{1 \leq i,j \leq n}$$

either on $L^p(M \otimes M_n)$ or on $L^p(M; S^1_n)$, in particular in the specific case when $M = M_n$. We will use the fact that for any $n \geq 1$, we have $L^p(M_n \otimes M_m) \simeq S^p_m[S^p_n]$, isometrically, provided that $S^p_n$ is equipped with the operator space structure given in [6].
Let \( t_m \) denote the transposition map on scalar \( m \times m \) matrices. Assume that \( \mathcal{M} \) is semifinite. The map

\[
I_{\mathcal{M}^0} \otimes t_m : \mathcal{M}^0 \otimes M_m \to \mathcal{M}^0 \otimes M_m^0
\]
is a trace preserving \(*\)-homomorphism, and so

\[
I_{L^p(\mathcal{M}^0)} \otimes t_m : L^p(\mathcal{M}^0 \otimes M_m) \to L^p(\mathcal{M}^0 \otimes M_m^0)
\]
is an isometry. Moreover \( \mathcal{M}^0 \otimes M_m^0 = (\mathcal{M} \otimes M_m)^0 \), hence \( L^p(\mathcal{M}^0 \otimes M_m^0) = L^p(\mathcal{M} \otimes M_m) \) isometrically. For any \([x_{ij}]_{1 \leq i, j \leq m}\) in \( L^p(\mathcal{M}) \otimes S^p_m \), since \( I_{L^p(\mathcal{M}^0)} \otimes t_m \) maps \([x_{ij}]\) to \([x_{ji}]\), we get

\[
\| [x_{ij}] \|_{L^p(\mathcal{M}^0 \otimes M_m)} = \| [x_{ji}] \|_{L^p(\mathcal{M} \otimes M_m)}.
\]

(4.1)

We now show that similarly, for any \([x_{ij}]_{1 \leq i, j \leq m}\) in \( L^p(\mathcal{M}) \otimes S^1_m \),

\[
\| [x_{ij}] \|_{L^p(\mathcal{M}^0 \otimes S^1_m)} = \| [x_{ji}] \|_{L^p(\mathcal{M} \otimes S^1_m)}.
\]

(4.2)

To verify the identity (4.2), assume that \( \| [x_{ij}] \|_{L^p(\mathcal{M}^0 \otimes S^1_m)} < 1 \). Taking into account the opposite product and (2.1), we can write

\[
x_{ij} = \sum_k b_{kj} a_{ik}
\]

for some \( a_{ik}, b_{kj} \) in \( L^{2p}(\mathcal{M}) \) such that \( \sum_{i,k} a_{ik}^* a_{ik} \) and \( \sum_{k,j} b_{kj}^* b_{kj} \) have norm \( < 1 \) in \( L^p(\mathcal{M}) \). This exactly means that \( \| [x_{ij}] \|_{L^p(\mathcal{M} \otimes S^1_m)} < 1 \). This shows the inequality \( \geq \) in (4.2). Reversing the argument we find the other inequality.

Identities (4.1) and (4.2), respectively, imply

\[
\| I^0 : L^p(\mathcal{M}) \to L^p(\mathcal{M}^0) \|_{cb} = \sup_{m \geq 1} \| I_{L^p(\mathcal{M}^0)} \otimes t_m : L^p(\mathcal{M} \otimes M_m) \to L^p(\mathcal{M} \otimes M_m) \|.
\]

(4.3)

and

\[
\| I^0 : L^p(\mathcal{M}) \to L^p(\mathcal{M}^0) \|_{S^1} = \sup_{m \geq 1} \| I_{L^p(\mathcal{M}^0)} \otimes t_m : L^p(\mathcal{M} ; S^1_m) \to L^p(\mathcal{M} ; S^1_m) \|.
\]

(4.4)

When \( \mathcal{M} = M_n \), the above identities can be more specific. In fact, as we show below, we have that for any \( n \geq 1 \),

\[
\| I^0 : S^n_p \to \{ S^n_p \}^0 \|_{cb} = \| t_n : S^n_p \to S^n_p \|_{cb}
\]

(4.5)

and

\[
\| I^0 : S^n_p \to \{ S^n_p \}^0 \|_{S^1} = \| t_n : S^n_p \to S^n_p \|_{S^1}.
\]

(4.6)

Using (4.1) applied to \( \mathcal{M} = M_n \), to prove (4.5), it is enough to show that for any \([x_{ij}]_{1 \leq i, j \leq n}\) in \( S^n_p \otimes S^p_m \),

\[
\| [t_n(x_{ij})] \|_{S^n_p[S^p_m]} = \| [x_{ij}] \|_{S^n_p[S^p_m]}.
\]

(4.7)

This follows from the fact that \( t_m \otimes t_n = t_{nm} \) is an isometry on \( S^p_m[S^n_p] \cong S^n_{nm} \), and hence

\[
\| (t_m \otimes t_n)[t_n(x_{ij})] \|_{S^p_m[S^n_p]} = \| [t_n(x_{ij})] \|_{S^p_m[S^n_p]}.
\]

Since \( (t_m \otimes t_n)[t_n(x_{ij})] = [x_{ji}] \), this yields (4.7).
Likewise, using (4.2) applied to $\mathcal{M} = M_n$, to prove (4.6), it is enough to show that for any $[x_{ij}]_{1 \leq i,j \leq m}$ in $S^p_n \otimes S^1_m$,

$$\|\left[ t_n(x_{ij}) \right]\|_{S^p_n[S^1_m]} = \|\left[ x_{ji} \right]\|_{S^p_n[S^1_m]}.$$  \hfill (4.8)

Assume that $\|\left[ t_n(x_{ij}) \right]\|_{S^p_n[S^1_m]} < 1$. According to (2.1), we can write

$$t_n(x_{ij}) = \sum_k a_{ik} b_{kj}$$

for some $a_{ik}, b_{kj}$ in $S^2_p$ such that $\sum_{i,k} a_{ik}a^*_{ik}$ and $\sum_{k,j} b^*_{kj} b_{kj}$ have norm $< 1$ in $S^p_n$. Then we have

$$x_{ij} = \sum_k t_n(a_{ik} b_{kj}) = \sum_k t_n(b_{kj}) t_n(a_{ik})$$

hence

$$x_{ji} = \sum_k t_n(b_{kj}^*) t_n(a_{jk})$$

Further

$$\sum_{k,j} t_n(a_{jk}) t_n(b_{kj}) = t_n\left( \sum_{j,k} a_{jk} a^*_{jk} \right),$$

and $t_n$ is an isometry on $S^p_n$. Consequently, $\sum_{k,j} t_n(a_{jk}) t_n(b_{kj})$ has norm $< 1$ in $S^p_n$. Similarly, $\sum_{i,k} t_n(a_{ik}) t_n(b_{ki})$ has norm $< 1$ in $S^p_n$. This shows that $\|\left[ x_{ji} \right]\|_{S^p_n[S^1_m]} < 1$. We have thus proved the inequality $\geq$ in (4.8). Reversing the argument we find the other inequality.

In the sequel, $E(x)$ denotes the integer part of $x$.

**Lemma 4.4.** Suppose that $\mathcal{M}$ is a semifinite von Neumann algebra.

(i) If $\mathcal{M}$ is subhomogeneous of degree $\leq N$ for some $N \geq 1$, then for all $[x_{ij}] \in M_m \otimes L^p(\mathcal{M}), m \geq 1$, we have

$$\|\left[ x_{ji} \right]\|_{L^p(\mathcal{M} \otimes M_m)} \leq N^{1/2 - 1/p} \|\left[ x_{ij} \right]\|_{L^p(\mathcal{M} \otimes M_m)},$$

and

$$\|\left[ x_{ji} \right]\|_{L^p(\mathcal{M} \otimes S^1_m)} \leq N \|\left[ x_{ij} \right]\|_{L^p(\mathcal{M} \otimes S^1_m)}.$$  \hfill (4.9)

(ii) Suppose that there exists $K \geq 1$ such that for all $[x_{ij}] \in L^p(\mathcal{M}) \otimes S^p_m, m \geq 1$,

$$\|\left[ x_{ji} \right]\|_{L^p(\mathcal{M} \otimes M_m)} \leq K \|\left[ x_{ij} \right]\|_{L^p(\mathcal{M} \otimes M_m)}.$$  \hfill (4.10)

Then if $p \neq 2$, $\mathcal{M}$ is subhomogeneous of degree $\leq N$ with $N = E\left( \frac{1}{K^{2/3}} \right)$.

(iii) Suppose that there exists $K \geq 1$ such that for all $[x_{ij}] \in L^p(\mathcal{M}) \otimes S^p_m, m \geq 1$,

$$\|\left[ x_{ji} \right]\|_{L^p(\mathcal{M} \otimes S^1_m)} \leq K \|\left[ x_{ij} \right]\|_{L^p(\mathcal{M} \otimes S^1_m)}.$$  \hfill (4.10)

Then $\mathcal{M}$ is subhomogeneous of degree $\leq N$ with $N = E(K)$. 
Proof.

(i) Assume that $\mathcal{M} = L^\infty(\Omega; M_n)$. Let $m \geq 1$ be given. We have

$$L^p(\mathcal{M} \otimes M_m) \simeq L^p(\Omega; S^p_n[S^p_n]).$$

By Pisier–Fubini theorem [6, (3.6)],

$$L^p(\mathcal{M}; S^1_m) \simeq L^p(\Omega; S^p_n[S^1_m]).$$

Consequently,

$$\| I_{L^p(\mathcal{M})} \otimes t_m : L^p(\mathcal{M} \otimes M_m) \to L^p(\mathcal{M} \otimes M_m) \| = \| t_m \otimes I_{S^p_n : S^p_n[S^1_m] \to S^p_n[S^1_n]} \|.$$  (4.11)

and

$$\| I_{L^p(\mathcal{M})} \otimes t_m : L^p(\mathcal{M}; S^1_m) \to L^p(\mathcal{M}; S^1_m) \| = \| I_{S^p_n} \otimes t_m : S^p_n[S^1_m] \to S^p_n[S^1_n] \|.  \quad (4.12)$$

Applying (4.3) to both sides of (4.11), we deduce

$$\| I_{L^p(\mathcal{M})} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op}) \|_{cb} = \| I_{L^p(\mathcal{M})} : S^p_n \to S^p_n[S^1_n]^{op} \|_{cb},$$

and applying (4.4) to both sides of (4.12), we deduce that

$$\| I_{L^p(\mathcal{M})} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op}) \|_{S^1} = \| I_{L^p(\mathcal{M})} : S^p_n \to S^p_n[S^1_n]^{op} \|_{S^1}.$$

By [5, Lemma 5.3],

$$\| t_n : S^p_n \to S^p_n \|_{cb} = n^{2(1/p-1/2)} \quad \text{and} \quad \| t_n : S^p_n \to S^p_n \|_{S^1} = n,$$

hence we obtain by (4.5) and (4.6) that

$$\| I_{L^p(\mathcal{M})} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op}) \|_{cb} = n^{2(1/p-1/2)} \quad \text{and} \quad \| I_{L^p(\mathcal{M})} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op}) \|_{S^1} = n.$$

When $\mathcal{M}$ is subhomogeneous of degree $\leq N$, there exist $r \geq 1$, integers $1 \leq n_1 \leq n_2 \leq \cdots \leq n_r \leq N$ and abelian von Neumann algebras $L^\infty(\Omega_1), \ldots, L^\infty(\Omega_r)$ such that (2.2) holds. Then for any $m \geq 1$, we have

$$L^p(\mathcal{M} \otimes M_m) \simeq \bigoplus_{1 \leq j \leq r} L^p(\Omega_j; S^p_m[S^p_{n_j}]) \quad \text{and} \quad L^p(\mathcal{M}; S^1_m) \simeq \bigoplus_{1 \leq j \leq r} L^p(\Omega_j; S^p_m[S^1_{n_j}]).$$

Using our previous argument and direct sums we deduce that

$$\| I_{L^p(\mathcal{M})} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op}) \|_{cb} \leq N^{2(1/p-1/2)} \quad \text{and} \quad \| I_{L^p(\mathcal{M})} : L^p(\mathcal{M}) \to L^p(\mathcal{M}^{op}) \|_{S^1} \leq N.$$

The result follows from (4.1) and (4.2).

(ii) Suppose that $\mathcal{M}$ is not subhomogeneous of degree $\leq N = E(K^{2/2-1/p})$. By Lemma 2.1, there exists a complete isometry

$$S^p_{N+1} \hookrightarrow \mathcal{M}.$$

This embedding implies that for any $m \geq 1$,

$$\| t_m \otimes I_{S^p_{N+1}} : S^p_m[S^p_{N+1}] \to S^p_m[S^p_{N+1}] \| \leq \| I_{L^p(\mathcal{M})} \otimes t_m : L^p(\mathcal{M} \otimes M_m) \to L^p(\mathcal{M} \otimes M_m) \|.$$
According to (4.3) and (4.5), this implies that

$$\|t_{N+1} : S^p_{N+1} \rightarrow S^p_{N+1}\|_{cb} \leq \|I^{op} : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M}^{op})\|_{cb}.$$ 

Hence

$$\|I^{op} : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M}^{op})\|_{cb} \geq (N + 1)^{\frac{2}{p} - \frac{1}{2}}.$$ 

Comparing this with inequality (4.9) above and applying (4.1), we get a contradiction.

(iii) The proof is similar to the proof of part (ii).

\[ \Box \]

**Proposition 4.5.** Let $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ be separating. If $\mathcal{M}$ is subhomogeneous then $T$ is completely bounded and $S^1$-bounded.

**Proof.** Changing $T$ to $w^*T$, we can assume that $w = J(1)$. By [5, Remark 4.3], we can write $T$ as a sum $T = T_1 + T_2$ such that $T_1$ has a direct Yeadon type factorisation and $T_2$ has an anti-direct Yeadon type factorisation. By Propositions 4.1 and 4.2, $T_1$ is completely bounded and $S^1$-bounded. Hence it suffices to show that $T_2$ is completely bounded and $S^1$-bounded. Let $I^{op} : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M}^{op})$ be the identity map and set $T_2^{op} := T_2 \circ I^{op \! - \! 1}$. Since $T_2$ has an anti-direct Yeadon type factorisation, $T_2^{op}$ has a direct Yeadon type factorisation. So, by Propositions 4.1 and 4.2, $T_2^{op}$ is completely bounded and $S^1$-bounded. Since $\mathcal{M}$ is subhomogeneous, part (i) of Lemma 4.4 and its proof show that $I^{op}$ is completely bounded and $S^1$-bounded. By composition, we obtain that $T_2 = T_2^{op} \circ I^{op}$ is completely bounded and $S^1$-bounded. \[ \Box \]

**Proposition 4.6.** Suppose that $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$ is a bijective separating map with an anti-direct Yeadon type factorisation.

(i) If $p \neq 2$ and $T$ is completely bounded then $\mathcal{M}$ is subhomogeneous.

(ii) If $T$ is $S^1$-bounded then $\mathcal{M}$ is subhomogeneous.

**Proof.**

(i) Suppose that $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, is a bijective separating map with an anti-direct Yeadon type factorisation. Assume that $T$ is completely bounded. Let $I^{op} : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M}^{op})$ be the identity map and set $T^{op} := T \circ I^{op \! - \! 1}$. Since $T$ is bijective with an anti-direct Yeadon type factorisation, $T^{op}$ is bijective with a direct Yeadon type factorisation. By part (i) of Proposition 3.4 and Remark 3.5, $T^{op \! - \! 1}$ is also separating with a direct Yeadon type factorisation. Therefore, by Proposition 4.1, $T^{op \! - \! 1}$ is completely bounded. Hence, $I^{op} := T^{op \! - \! 1} \circ T$ is completely bounded. It now follows from part (ii) of Lemma 4.4 and (4.1) that $\mathcal{M}$ is subhomogeneous.

(ii) The same argument as in part (i) with $S^1$-bounded (norm) replacing completely bounded (norm), Proposition 4.2 replacing Proposition 4.1, part (iii) of Lemma 4.4 replacing its part (ii) and (4.2) replacing (4.1) yields the result. \[ \Box \]

**Remark 4.7.** Suppose that $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, $1 \leq p < \infty$, is a surjective separating isometry with an anti-direct Yeadon type factorisation. The proof of Proposition 4.6 shows that when $T$ is completely bounded and $p \neq 2$, $\mathcal{M}$ is subhomogeneous of degree $\leq E(\frac{1}{\|T\|_{cb}^{\frac{1}{2} - \frac{1}{p}}})$. When $T$ is $S^1$-bounded, $\mathcal{M}$ is subhomogeneous of degree $\leq E(\|T\|_{S^1})$.

**Theorem 4.8.** Let $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, be a bounded separating map that is surjective. Then the following are equivalent.

(i) $T$ is completely bounded.

(ii) There exists a decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorisation and $\mathcal{M}_2$ is subhomogeneous.
Proof. (i) ⇒ (ii) Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p \neq 2 < \infty$, is a surjective completely bounded separating map. In view of Lemma 4.3, we may assume $T$ is bijective.

By Proposition 3.3, there exist decompositions $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ and surjective separating maps $T_1 : L^p(\mathcal{M}_1) \to L^p(\mathcal{N}_1)$ and $T_2 : L^p(\mathcal{M}_2) \to L^p(\mathcal{N}_2)$ such that $T_1$ has a direct Yeadon type factorisation, $T_2$ has an anti-direct Yeadon type factorisation and $T = T_1 + T_2$. Since $T$ is completely bounded, $T_2$ is also completely bounded. By part (i) of Proposition 4.6, $\mathcal{M}_1$ must be subhomogeneous.

(ii) ⇒ (i) This is a consequence of Propositions 4.1 and 4.5. □

Theorem 4.9. Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$, $1 \leq p < \infty$, be a separating map that is surjective. Then the following are equivalent.

(i) $T$ is $S^1$-bounded.

(ii) There exists a decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that $T|_{L^p(\mathcal{M}_1)}$ has a direct Yeadon type factorisation and $\mathcal{M}_2$ is subhomogeneous.

Proof. The proof is similar to Theorem 4.8, replacing completely bounded with $S^1$-bounded, part (i) of Proposition 4.6 by its part (ii) and Proposition 4.1 by Proposition 4.2. □

The following example shows the surjectivity assumption in Theorems 4.8 and 4.9 is essential. In fact in this example, on a non-subhomogeneous semifinite von Neumann algebra $\mathcal{M}$ and for a given $\varepsilon > 0$, we construct a separating isometry $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ such that $T$ is not surjective, $\|T\|_{cb} \leq 1 + \varepsilon$, $\|T\|_{S^1} \leq 1 + \varepsilon$ and part (ii) of Theorems 4.8 and 4.9 is not satisfied.

Example 4.10. Let $1 < p < \infty$. Consider the von Neumann algebra

$$\mathcal{M} = \ell^\infty \{ M_n \} = \{ (x_n)_{n \geq 1} : \forall n \geq 1, x_n \in M_n \text{ and } \sup_{n \geq 1} \|x_n\|_\infty < \infty \},$$

the infinite direct sum of all $M_n$, $n \geq 1$. Let $\mathcal{N} := \mathcal{M} \oplus \mathcal{M}$, the direct sum of two copies of $\mathcal{M}$. The noncommutative $L^p$-space associated with $\mathcal{M}$ is

$$\ell^p \{ S^p_n \} = \{ (x_n)_{n \geq 1} : \forall n \geq 1, x_n \in S^p_n \text{ and } \sum_{n \geq 1} \|x_n\|_p^p < \infty \},$$

equipped with the norm

$$\|(x_n)_{n \geq 1}\|_p = \left( \sum_{n=1}^{\infty} \|x_n\|_p^p \right)^{\frac{1}{p}},$$

and so the noncommutative $L^p$-space associated with $\mathcal{N}$ is $\ell^p \{ S^p_n \} \oplus \ell^p \{ S^p_n \}$. Let $(\beta_n)_{n \geq 1}$ be a sequence in the interval (0,1). We may define two operators

$$T_1 : \ell^p \{ S^p_n \} \to \ell^p \{ S^p_n \} \quad \text{and} \quad T_2 : \ell^p \{ S^p_n \} \to \ell^p \{ S^p_n \}$$

by setting

$$T_1((x_n)_{n \geq 1}) = \left( (1 - \beta_n)^{\frac{1}{p}} x_n \right)_{n \geq 1} \quad \text{and} \quad T_2((x_n)_{n \geq 1}) = \left( \beta_n^\frac{1}{p} t_n(x_n) \right)_{n \geq 1}.$$
for any \( x = (x_n)_{n \geq 1} \in \ell^p \{S_n^p\} \). Consider

\[
T : \ell^p \{S_n^p\} \to \ell^p \{S_n^p\} \oplus \ell^p \{S_n^p\}, \quad T(x) = (T_1(x), T_2(x)).
\]

It is plain that \( T \) is an isometry. Indeed for any \( x = (x_n)_{n \geq 1} \in \ell^p \{S_n^p\} \), we have

\[
\|T(x)\|_p^p = \|T_1(x)\|_p^p + \|T_2(x)\|_p^p
= \sum_{n=1}^{\infty} (1 - \beta_n)\|x_n\|_p^p + \sum_{n=1}^{\infty} \beta_n\|x_n\|_p^p
= \sum_{n=1}^{\infty} \|x_n\|_p^p = \|x\|_p^p.
\]

Given \( \varepsilon > 0 \), consider the above construction with

\[
\beta_n = \frac{(1 + \varepsilon)^p - 1}{n^p - 1}.
\]

We show that \( T \) is \( S^1 \)-bounded with \( \|T\|_{S^1} \leq 1 + \varepsilon \). Indeed consider an integer \( m \geq 1 \). We have

\[
\ell^p \{S_n^p\} [S^1_m] = \ell^p \{S_n^p[S^1_m]\},
\]

and therefore, we also have that

\[
\left( \ell^p \{S_n^p\} \oplus \ell^p \{S_n^p\} \right)[S^1_m] = \ell^p \{S_n^p[S^1_m]\} \oplus \ell^p \{S_n^p[S^1_m]\}.
\]

Now let \( x = (x_n)_{n \geq 1} \in \ell^p \{S_n^p[S^1_m]\} \) (here each \( x_n \) is an element of \( S_n^p[S^1_m]\) ). Then

\[
(I_{S^1_m} \otimes T)(x) = \left( (1 - \beta_n)\frac{1}{p} x_n \right)_{n \geq 1}, \left( \beta_n \left( t_n \otimes I_{S^1_m} \right)(x_n) \right)_{n \geq 1}.
\]

Consequently,

\[
\| (I_{S^1_m} \otimes T)(x) \|_p^p = \sum_{n=1}^{\infty} (1 - \beta_n)\|x_n\|_{S_n^p[S^1_m]}^p + \sum_{n=1}^{\infty} \beta_n\|(t_n \otimes I_{S^1_m})(x_n)\|_{S_n^p[S^1_m]}^p
\leq \sum_{n=1}^{\infty} (1 - \beta_n)\|x_n\|_{S_n^p[S^1_m]}^p + n^p\beta_n\|x_n\|_{S_n^p[S^1_m]}^p \text{ by [5, Lemma 5.3 (ii)]}
\leq (1 + \varepsilon)^p \sum_{n=1}^{\infty} \|x_n\|_{S_n^p[S^1_m]}^p
= (1 + \varepsilon)^p \|x\|_p^p.
\]

It is clear that \( T \) is separating and that the Jordan homomorphism \( J : \mathcal{M} \to \mathcal{N} \) in its Yeadon triple is given by

\[
J( (x_n)_{n \geq 1} ) = ( (x_n)_{n \geq 1}, (t_n(x_n))_{n \geq 1} ) .
\]

It follows that whenever \( \mathcal{M}_1 \) is a non zero summand of \( \mathcal{M} \), the Yeadon factorisation of the restriction of \( T \) to \( L^p(\mathcal{M}_1) \) is neither direct nor indirect. A fortiori, \( T \) does not satisfy the assertion (ii) of Theorem 4.9.
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REFERENCES
[1] G. Hong, S. K. Ray, and S. Wang, Maximal ergodic inequalities for positive operators on noncommutative $L^p$-spaces, (preprint, 2020), arXiv:1907.12967v5.
[2] M. Junge, Doob’s inequality for non-commutative martingales, J. Reine Angew. Math. 549 (2002), 149–190.
[3] M. Junge and Q. Xu, Noncommutative maximal ergodic theorems, J. Amer. Math. Soc. 20 (2007), 385–439.
[4] C. Le Merdy and S. Zadeh, $\ell^1$-contractive maps on noncommutative $L^p$-spaces, J. Operator Theory 85 (2021), no. 2, 417–442.
[5] C. Le Merdy and S. Zadeh, On factorization of separating maps on noncommutative $L^p$-spaces, Indiana Univ. Math. J. (to appear), arXiv:2007.04577.
[6] G. Pisier, Non-commutative vector valued $L^p$-spaces and completely $p$-summing maps, Astérisque, vol. 247, 1998.
[7] G. Pisier, Regular operators between non-commutative $L_p$-spaces, Bull. Sci. Math. 119 (1995), 95–118.
[8] G. Pisier and Q. Xu, Non-commutative $L_p$-spaces, Handbook of the Geometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1459–1517.
[9] V. Runde, Amenable Banach algebras, Springer-Verlag, New York, 2020.
[10] E. Størmer, On the Jordan structure of $C^*$-algebras, Trans. Amer. Math. Soc. 120 (1965), 438–447.
[11] M. Takesaki, Theory of operator algebras. I, Encyclopaedia of Mathematical Sciences, vol. 124, Springer-Verlag, Berlin, 2002.
[12] M. Terp, $L_p$-spaces associated with von Neumann algebras, Notes, Math. Institute, Copenhagen Univ., 1981.
[13] F. Y. Yeadon, Isometries of noncommutative $L_p$-spaces, Math. Proc. Cambridge Philos. Soc. 90 (1981), 41–50.

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