GROUND STATES OF NONLINEAR SCHRÖDINGER EQUATIONS WITH FRACTIONAL LAPLACIANS

ZUPEI SHEN
Center for Applied Mathematics, Guangzhou University
Guangzhou 510405, China

ZHIQING HAN
School of Mathematical Sciences, Dalian University of Technology
Dalian 116024, China

QINQIN ZHANG*
Center for Applied Mathematics, Guangzhou University
Guangzhou 510405, China

Abstract. Inspired by Schaftingen [15], we develop a symmetric variational principle for the field equation involving a fractional Laplacians
\[
\begin{cases}
(−Δ)^α u + u = f(u), & x ∈ \mathbb{R}^N, \\
u(x) ≥ 0.
\end{cases}
\]
As an application, we prove the existence of symmetric ground states in the fractional Sobolev space \(H^α(\mathbb{R}^N)\). These results improve some known ones in the literature. An example is also given to illustrate our results.

1. Introduction. We are concerned with solutions to the following problem
\[
\begin{cases}
(−Δ)^α u + u = f(u), & x ∈ \mathbb{R}^N, \\
u(x) ≥ 0.
\end{cases}
\]
This equation is related to the fractional Schrödinger equations
\[
\begin{cases}
i\partial_t Ψ(x) + (−Δ)^α Ψ(x) = F(Ψ(x)), & x ∈ \mathbb{R}^N, \\
Ψ ≥ 0,
\end{cases}
\]
whose standing waves have been studied, see [22]. Problem (1) has been widely studied since it arises in several areas such as physics, biology, chemistry, and finance (see, for instance, [1, 4, 10, 20]). It was proved in [6] that \((−Δ)^α\) can be changed to the standard Laplacian \(-Δ\) as \(α → 1\), and hence Problem (1) can be reduced to the classical Schrödinger equations
\[
−Δu + u = f(u).
\]

2010 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Schrödinger equation, fractional Laplacian, ground state, rearrangement.

This work was supported by National Natural Science Foundation (11701114, 11471085) and the program for Changjiang scholars and Innovative Research Team in university (Grant No.IRT1226).

Corresponding author: Qinqin Zhang.
Over the past decades, the existence of solutions of Equation (2) has been extensively investigated under various conditions on the nonlinearity $f$ (see, for example, [3, 9, 19]). For many papers in this direction, the following classical condition of Ambrosetti and Rabinowitz is required:

(AR) There exists $\theta > 2$ such that $0 < \theta F(s) \leq sf(s)$ for all $s > 0$, where $F(s) = \int_0^s f(t) dt$ (see [3]). A crucial role of (AR) is to ensure the boundedness of Palais-Smale sequences. A natural question is whether the same results hold for the fractional Schrödinger equations. Dipierro et al. [7] obtained the existence of nontrivial solution of (1) with $f(s) = |s|^{p-1}s$ in $H^\alpha_r(\mathbb{R}^N)$, where $p \in (1, (N + 2\alpha)/(N - 2\alpha))$. Felmer et al. [8] obtained the existence of positive solutions for (1) under (AR) condition. Wei and Su [21] replaced (AR) with monotonicity condition and considered the problem in bounded domains. They proved the existence of nontrivial solutions of (1). Chang and Wang [5] proved the existence of ground state solutions under the general Berestycki-Lions type assumptions with $f \in C^1(\mathbb{R}, \mathbb{R})$.

A ground state solution is a nontrivial solution that minimizes the energy among all nontrivial solutions. We point out that in [17, 18], with monotonicity condition, an explicit characterization of the ground state, as a minimizer over the Nehari-Pankov manifold.

In this paper, we assume that the nonlinearity $f$ satisfies the following conditions:

(\text{f}_0) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(s) \geq 0$ if $s \geq 0$ and $f(s) \equiv 0$ if $s \leq 0$;
(\text{f}_1) There exists $1 < p < (N + 2\alpha)/(N - 2\alpha)$, such that $f(s) \leq C(1 + |s|^p)$;
(\text{f}_2) $\lim_{s \to 0} \frac{f(s)}{s} = 0$;
(\text{f}_3) $\frac{F(s)}{s^2} \to \infty$ as $s \to \infty$;
(\text{f}_4) $\frac{f(s)}{s}$ is increasing on $s \in (0, \infty)$.

We are now ready to state our results:

**Theorem 1.1.** Suppose that (\text{f}_0)-(\text{f}_4) are satisfied. Then (1.1) has at least one nonnegative symmetric mountain pass solution.

**Theorem 1.2.** Suppose that (\text{f}_0)-(\text{f}_4) are satisfied. Then (1.1) has at least one nonnegative symmetric ground state solution.

The main difficulties in proving Theorem 1.1 and Theorem 1.2 are two-fold. Firstly, under Conditions (\text{f}_0)-(\text{f}_4), we cannot prove boundedness of PS sequences. To avoid this difficulty, we prove the boundedness of Cerami sequences rather than PS sequences. Secondly, the fractional Sobolev space embedding is non-compact, one would usually choose $H^\alpha_0(\mathbb{R}^N)$ as the suitable function space and combine with Strauss’s compactness lemma to get the compactness (see, for instance,[5, 7]). Since symmetrization (Schwarz symmetrization) is continuous in $H^\alpha(\mathbb{R}^N)$ (see [2]), a “rearrangement” of a mountain path then gives a symmetric Cerami sequence, we can recover the compactness.

The organization of this paper is as follows: In section 2, we introduce some notation and recall some known facts about fractional Sobolev space and symmetrization. Moreover, we develop a symmetric variational principle by symmetrization. In section 3, we give the proof of the main theorems. In section 4, we give an example to illustrate our results.

2. Preliminaries. Before proving Theorem 1.1 and Theorem 1.2, we first provide some preliminaries.
2.1. Fractional Sobolev space. For any $\alpha \in (0, 1)$, the fractional Sobolev space is defined by

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2} + \alpha}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$
endowed with norm

$$\|u\| = \left( \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2\alpha}} \, dx \, dy \right)^{\frac{1}{2}},$$

where the term $[u]_{H^\alpha(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2\alpha}} \, dx \, dy \right)^{\frac{1}{2}}$ is the so-called Gagliardo semi-norm of $u$. The space $H^\alpha(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} u \, v \, dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2\alpha}} \, dx \, dy.$$

Associated with Problem (1), the energy functional $\varphi : H^\alpha(\mathbb{R}^N) \to \mathbb{R}$ is

$$\varphi(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(u) \, dx.$$

Under Conditions $(f_0)$-$(f_4)$, it is easy to prove that $\varphi \in C^1(H^\alpha(\mathbb{R}^N))$ and the critical points of $\varphi$ are the corresponding weak solutions of Problem (1), moreover,

$$\varphi'(u)v = \langle u, v \rangle - \int_{\mathbb{R}^N} f(u)v \, dx.$$

For readers’ convenience, we recall an embedding result for fractional Sobolev spaces.

**Proposition 1.** Let $2 \leq q \leq 2^* = \frac{2N}{N - 2\alpha}$. Then we have

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C \|u\|_{H^\alpha(\mathbb{R}^N)},$$
for all $u \in H^\alpha(\mathbb{R}^N)$.

If, furthermore, $2 \leq q < \frac{2N}{N - 2\alpha}$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain, then every bounded sequence $\{u_k\} \subset H^\alpha(\mathbb{R}^N)$ has a convergent subsequence in $L^q(\Omega)$.

The following proposition is a version of the concentration compactness lemma proved in [8].

**Proposition 2.** Let $N \geq 2$. Assume that $\{u_k\}$ is bounded in $H^\alpha(\mathbb{R}^N)$, and satisfies

$$\lim_{k \to \infty} \sup_{\xi \in \mathbb{R}^N} \int_{B_R(\xi)} |u_k|^2 \, dx = 0,$$

for some $R > 0$. Then $u_k \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$.
2.2. Symmetrization. Firstly, we introduce some concepts.

**Definition 2.1.** If \( f : \mathbb{R}^N \to \mathbb{R} \) is a Lebesgue measurable function, then \( f \) is said to vanish at infinity if \( m \{ \{ x : |f(x)| > t \} \} \) is finite for all \( t > 0 \).

**Definition 2.2.** If \( f : \mathbb{R}^N \to \mathbb{R}^+ \) is a Lebesgue measurable function vanishing at infinity, we define
\[
 f^*(x) = \int_0^\infty \chi_{\{|f| > t\}}(x)dt.
\]

The rearrangement \( f^*(x) \) has a number of properties (see [13]).

**Proposition 3.** \( f^*(x) \) is nonnegative.

**Proposition 4.** For any Borel measurable function \( f : \mathbb{R}^N \to \mathbb{R}^+ \) such that \( f(0) = 0 \), we have
\[
 \int_{\mathbb{R}^N} f(u^*(x))dx = \int_{\mathbb{R}^N} f(u(x))dx.
\]

**Proposition 5.** Let \( \alpha \in (0, 1) \). For any \( u \in H^\alpha(\mathbb{R}^N) \), the following inequality holds
\[
 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u^*(x) - u^*(y)|^2}{|x - y|^{N + 2\alpha}}dxdy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2\alpha}}dxdy. \tag{5}
\]

2.3. Symmetry and variational principles. Symmetrization allows to restrict the search of a minimizer to the subset of symmetric functions. Similarly we show that on certain critical levels, there is a critical point which is symmetric. Let us first recall a general minimax principle.

**Theorem 2.3.** (Willem [23]). Define \( \Gamma = \{ \gamma \in C([0, 1], H^\alpha(\mathbb{R}^N)) | \gamma(0) = 0, \gamma(1) = e \} \). If \( \varphi \in C^1 (H^\alpha(\mathbb{R}^N), \mathbb{R}) \) satisfies
\[
 a = \max \{ \varphi(0), \varphi(e) \} < c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \varphi(\gamma(t)) < \infty,
\]
then, for every \( \epsilon \in (0, \frac{c - a}{2}) \), \( \delta > 0 \), and \( \gamma \in \Gamma \) with
\[
 \sup_{t \in [0, 1]} \varphi \circ \gamma(t) \leq c + \epsilon,
\]
there exists \( u \in H^\alpha(\mathbb{R}^N) \) such that
\[
 (a) \quad c - 2\epsilon \leq \varphi(u) \leq c + 2\epsilon,
(\overline{b}) \quad \text{dist}(u, \gamma([0, 1])) \leq \delta,
(\overline{c}) \quad (1 + \|u\|) \|\varphi'(u)\| \leq \frac{\delta}{\alpha}.
\]

If \( u \) is obtained by mountain pass theorem, we call \( u \) is a mountain pass solution. According to (4) and (5), we have \( \varphi(u^*) \leq \varphi(u) \). Combining with the fact that the symmetry operator \( * \) is continuous in \( H^\alpha(\mathbb{R}^N) \), we know that \( \gamma^* \) is still a path. If we replace path \( \gamma \) with \( \gamma^* \), then \( u \) given by Theorem 2.1 would be nearer to the set \( \gamma^*[0, 1] \), where \( \gamma^*[0, 1] = \{ u^* | u \in \gamma[0, 1] \} \). So we have

**Theorem 2.4.** Define \( \Gamma = \{ \gamma \in C([0, 1], H^\alpha(\mathbb{R}^N)) | \gamma(0) = 0, \gamma(1) = e \} \). If \( \varphi \in C^1 (H^\alpha(\mathbb{R}^N), \mathbb{R}) \) satisfies
\[
 a = \max \{ \varphi(0), \varphi(e) \} < c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \varphi(\gamma(t)) < \infty,
\]
then, for every \( \epsilon \in (0, \frac{c - a}{2}) \), \( \delta > 0 \), and \( \gamma \in \Gamma \) with
\[
 \sup_{t \in [0, 1]} \varphi \circ \gamma(t) \leq c + \epsilon,
\]
there exists \( u \in H^\alpha(\mathbb{R}^N) \) such that
(a) \( c - 2\epsilon \leq \varphi(u) \leq c + 2\epsilon \),
(b) \( |u - u^*|_{L^q} \leq C\delta \),
(c) \( (1 + \|u\|) \|\varphi'(u)\| \leq \frac{8\delta}{\alpha} \),
where \( q \in (2, \frac{2\star}{\alpha}) \).

3. Proofs of Theorem 1.1 and Theorem 1.2. In order to use the mountain pass theorem, we need to prove the next mountain pass geometry Lemma.

Lemma 3.1. Suppose that \((f_1)-(f_3)\) are satisfied. Then \( \varphi(u) \) has the mountain pass geometry.

Proof. By \((f_2)\) and \((f_3)\), for any \( \epsilon > 0 \), there exists a constant \( C_\epsilon > 0 \) such that
\[
f(s) \leq \epsilon s + C_\epsilon s^p,
\]
which yields
\[
\varphi(u) \geq \frac{1}{2}\|u\|^2 - \epsilon \int_{\mathbb{R}^N} u^2 dx - C_\epsilon \int_{\mathbb{R}^N} u^{p+1} dx
\geq \left( \frac{1}{2} - \epsilon C_1 \right)\|u\|^2 - C_2 \|u\|^{p+1},
\]
As \( \epsilon \) is small enough, then there exist \( \rho > 0 \) and \( \sigma > 0 \) such that
\[
\varphi(u) \geq \sigma > 0 \text{ with } \|u\| = \rho.
\]
Let \( G(s) = F(s) - \frac{s^2}{2} \). Combining \((f_3)\) with \( F(0) = 0 \), we know that there exist two positive constants \( \zeta \) and \( \kappa \) such that
\[
G(\zeta) = F(\zeta) - \frac{1}{2} \zeta^2 \geq \kappa.
\]
For \( r > 0 \), define
\[
u_r(x) = \begin{cases} 
\zeta, & \text{if } |x| < r, \\
\zeta(r + 1 - |x|), & \text{if } r \leq |x| \leq r + 1, \\
0, & \text{if } |x| > r + 1.
\end{cases}
\]
Clearly, \( \nu_r(x) \in H^\alpha(\mathbb{R}^N) \). Let \( \delta = \max_{[0, \zeta]} G(s) \), by a simple estimation, we obtain
\[
\int_{\mathbb{R}^N} G(\nu_r) dx \geq \frac{\pi \|u\|^2}{\Gamma(\frac{N}{2} + 1)} \left[ G(\zeta) - \delta \left( (1 + \frac{1}{r})^\frac{N}{2} - 1 \right) \right] r^N,
\]
which implies that there exists \( r_0 \) large enough such that \( \int_{\mathbb{R}^N} G(\nu_{r_0}) dx > 0 \). Let \( \overline{u}_\theta(x) = \nu_{r_0}(\frac{x}{\theta}) \). It is easy to see that
\[
\varphi(\overline{u}_\theta(x)) = \frac{\theta^{N-2\alpha}}{2} [u_{r_0}]^2 - \theta^N \int_{\mathbb{R}^N} G(\nu_{r_0}) dx.
\]
Then there exists \( \overline{\theta} > 0 \), such that for any \( \theta > \overline{\theta} \), we have
\[
\varphi(\overline{u}_\theta(x)) < 0.
\]
According to \((7)\) and \((8)\), we have proved that \( \varphi(u) \) has a mountain pass geometry. \(\square\)
Since the functional \( \varphi \) has a mountain pass geometry, by Theorem 2.4, we can deduce the existence of a Cerami sequence at the mountain pass level \( c \) (\( (C)_c \) sequence for short), namely, a \( \{u_n\} \in H^\alpha(\mathbb{R}^N) \) satisfying
\[
\varphi(u_n) \to c, \quad \text{and} \quad (1 + \|u_n\|) \|\varphi'(u_n)\| \to 0, \quad \text{as} \quad n \to \infty.
\]
In order to prove the boundedness of \( (C)_c \) sequences, we start with the following lemma.

**Lemma 3.2.** Suppose that (f4) is satisfied. Then for any \( t \in \mathbb{R} \), we have
\[
\mathcal{F}(t) = \frac{1}{2}tf(t) - F(t) \geq 0 \quad (9)
\]
and
\[
\frac{1}{2}(s^2 - 1)f(t)t + F(t) - F(st) \leq 0, \quad \text{for} \quad s \in [0,1]. \quad (10)
\]

**Proof.** It is clear to see that (9) and (10) hold for \( t \leq 0 \). For \( t \geq 0 \), the next inequality
\[
F(t) = \int_0^t f(x)dx = \int_0^t \frac{f(x)}{x} xdx \leq \frac{f(t)}{t} \frac{1}{2}t^2 = \frac{1}{2}tf(t)
\]
implies that (9) is true. Let
\[
g(s) = \frac{1}{2}(s^2 - 1)f(t)t + F(t) - F(st).
\]
Then for \( s \in [0,1] \),
\[
g'(s) = sf(t)t - f(st)t
\]
\[
= su^2 \left( \frac{f(t)}{t} - \frac{f(st)}{st} \right)
\]
\[
\geq 0,
\]
which yields
\[
g(s) \leq g(1) \leq 0
\]
and so (10) is also true. The proof is complete. \( \square \)

**Remark 1.** Some similar inequalities were proved for some indefinite variational problems in [16]. For example, it was proved that
\[
f(w) \left[ s \left( \frac{s}{2} + 1 \right)w + (1 + s)w \right] + F(w) - F((1 + s)w + v) < 0
\]
for some \( s \geq -1 \). This inequality played a crucial role in proving boundedness of (PS)_c or (C)_c sequences in [14, 16].

**Lemma 3.3.** Under the conditions of Theorem 1.1, every \( (C)_c \) sequence \( \{u_n\} \) is bounded.

**Proof.** By contradiction, suppose \( \{u_n\} \) is unbounded. We may assume \( \|u_n\| \to \infty \). Let \( v_n = \frac{u_n}{\|u_n\|^2} \), \( s_n = \|u_n\| \), and \( t = \sqrt{2c + 2} + 1 \). Then by a simple computation
\[
\varphi(tv_n) - \varphi(u_n) = \frac{1}{2} \left( \frac{t^2}{\|u_n\|^2} - 1 \right)\|u_n\|^2 - \int_{\mathbb{R}^N} F(tv_n)d\rho + \int_{\mathbb{R}^N} F(u_n)d\rho,
\]
\[
\frac{1}{2} \left( \frac{t}{\|u_n\|} + 1 \right)\varphi'(u_n)(tv_n - u_n) = \frac{1}{2} \left( \frac{t^2}{\|u_n\|^2} - 1 \right)\|u_n\|^2 - \frac{1}{2} \left( \|u_n\|^2 - 1 \right) \int_{\mathbb{R}^N} f(u_n)u_n d\rho.
\]
By (12), as \( n \) is large enough, it is easy to see that
\[
\frac{1}{2} \left( \frac{t^2}{\|u_n\|^2} - 1 \right) \|u_n\|^2 \leq 1 + \frac{1}{2} \left( \frac{t^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} f(u_n) u_n \, dx.
\] (13)

Combining (10), (11), and (13) gives us
\[
\varphi(t v_n) - \varphi(u_n) \leq 1 + \frac{1}{2} \left( \frac{t^2}{\|u_n\|^2} - 1 \right) \int_{\mathbb{R}^N} F(t v_n) \, dx - \int_{\mathbb{R}^N} F(s_n u_n) \, dx + \int_{\mathbb{R}^N} F(u_n) \, dx
\]
\[
\leq 1 \text{ as } n \to \infty.
\] (14)

Let
\[
\delta = \lim_{n \to \infty} \sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |v_n|^2 \, dx.
\] (15)

We claim that \( \delta > 0 \). In fact, if \( \delta = 0 \), by Proposition 2, if follows that \( v_n \to 0 \) in \( L^q(\mathbb{R}^N) \), as \( n \to \infty \).

For large enough \( n \), according to (14), we obtain
\[
c + 1 \geq \varphi(u_n) + 1 \geq \varphi(t v_n) = \frac{t^2}{2} \|v_n\|^2 - \int_{\mathbb{R}^N} F(v_n) \, dx
\]
\[
\geq \frac{t^2}{2}.
\]

This is a contradiction to \( t = \sqrt{2c + 2} + 1 \), and hence the claim has been proved.

Going if necessary to a subsequence, we may assume that existence of \( y_n \in \mathbb{R}^N \) such that
\[
\int_{B(y_n, 1)} |v_n|^2 \, dx > \frac{\delta}{2}.
\]

Let us define \( w_n = v_n(\cdot - y_n) \). Then we get
\[
\int_{B(0, 1)} |w_n|^2 \, dx > \frac{\delta}{2}.
\]

Since \( \{ w_n \} \) is bounded in \( H^\alpha(\mathbb{R}^N) \), without loss of generality, we may assume that
\[
w_n \to w \neq 0 \text{ in } L^2_{loc}(\mathbb{R}^N), \text{ as } n \to \infty,
\]
\[
w_n \to w \neq 0 \text{ a.e in } \mathbb{R}^N, \text{ as } n \to \infty.
\]

It follows that
\[
|u_n(x - y_n)| \to \infty, \text{ as } n \to \infty.
\]

By \((f_1)\), we obtain
\[
\frac{F(u_n(x - y_n))}{u_n^2(x - y_n)} w_n^2(x) \to \infty, \text{ as } n \to \infty.
\]

Consequently, using Fatou lemma, we have
\[
\frac{1}{2} \frac{c + o(1)}{\|u_n\|^2} = \frac{\int_{\mathbb{R}^N} F(u_n) \, dx}{\|u_n\|^2}
\]
\[
\geq \int_{w \neq 0} \frac{F(u_n(x - y_n))}{u_n^2(x - y_n)} w_n^2 \, dx \to \infty
\]
as \( n \to \infty \), which is impossible. This completes the proof.

Using (4) and (5), since \( \{u_n\} \) is bounded in \( H^\alpha(\mathbb{R}^N) \), it is easy to see that \( \{u_n^*\} \) is bounded in \( H^\alpha(\mathbb{R}^N) \). Then we may assume, going if necessary to a subsequence,

\[
u_n^* \rightharpoonup u \quad \text{in} \quad H^\alpha(\mathbb{R}^N) \quad \text{and} \quad u_n^* \to u \quad \text{a.e in} \quad \mathbb{R}^N, \quad \text{as} \quad n \to \infty.
\]

Moreover, for \( q \in (2, 2^*_\alpha) \), we may assume

\[
u_n^* \to u \quad \text{in} \quad L^q(\mathbb{R}^N), \quad \text{as} \quad n \to \infty.
\]

According to (b) of Theorem 2.4, we have

\[
u_n \to u \quad \text{in} \quad L^q(\mathbb{R}^N), \quad \text{as} \quad n \to \infty.
\]

In order to prove Theorem 1.1, we make use of a variation of Strauss’s compactness lemma introduced by Chang and Wang [5].

**Lemma 3.4.** Let \( X \) be a Banach space such that \( X \) is embedded respectively continuously and compactly into \( L^q(\mathbb{R}^N) \) for \( q \in [q_1, q_2] \) and \( q \in (q_1, q_2) \), where \( q_1, q_2 \in (0, \infty) \). Assume that \( \{u_n\} \subset X \), \( v: \mathbb{R}^N \to \mathbb{R} \) is a measurable function and \( P \in C(\mathbb{R}, \mathbb{R}) \) such that

\[
\lim_{|s| \to \infty} \frac{P(s)}{|s|^{q_2}} = 0;
\]

\[
\lim_{|s| \to 0} \frac{P(s)}{|s|^{q_1}} = 0;
\]

\[
sup_n \|u_n\|_X < \infty;
\]

\[
\lim_{n \to \infty} P(u_n(x)) = v(x) \quad \text{for a.e} \quad x \in \mathbb{R}^N.
\]

Then, up to a subsequence, we have

\[
\lim_{n \to \infty} \|P(u_n(x)) - v\|_{L^1(\mathbb{R}^N)} = 0.
\]

**Remark 2.** Although the embedding \( H^\alpha(\mathbb{R}^N) \) to \( L^q(\mathbb{R}^N) \) is not compact. However, Theorem 2.4 provides an alternative to concentration-compactness for the Problems 1.1. We can prove that \( u_n \to u \in L^q(\mathbb{R}^N) \). By a same argument of the proof of Lemma 3.4 in [5], we know that the conclusion of Lemma 3.4 is still valid for \( \{u_n\} \).

**Proof of Theorem 1.1.** Let \( \{u_n\} \) be given by Theorem 2.2. Suppose \( f(s)s = P(s) \), \( q_1 = 2, \ q_2 = 2^*_\alpha \). Apparently, \( f(s)s \) satisfies the requirement of Lemma 3.4. According to Remark 3.2, we get

\[
\int_{\mathbb{R}^N} f(u_n)u_n dx \to \int_{\mathbb{R}^N} f(u)u dx, \quad \text{as} \quad n \to \infty.
\]

By (16), it is easy to see that \( \|u_n\|^2 \to \|u\|^2 \). Together with \( u_n \to u \), we obtain

\[
u_n \to u \quad \text{in} \quad H^\alpha(\mathbb{R}^N), \quad \text{as} \quad n \to \infty.
\]

Then we have \( c = \varphi(u) \). By a similar argument as the proof of part (i) of Theorem 1.3 and part (i) of Theorem 1.4 in [8], we obtain \( u \geq 0 \).

To get ground state solutions, we adapt the argument of [12] (see also [14]). We need to prove the following lemma.

**Lemma 3.5.** The zero function 0 is an isolated critical point.
Proof. Let $v$ be a critical point other than 0. Combining with Sobolev inequality and (6), we get

$$
\|v\|^2 = \int_{\mathbb{R}^N} f(v)v \, dx \\
\leq \epsilon |v|^2_{L^2} + C_c |v|^{p+1}_{L^{p+1}} \\
\leq C_1 \epsilon \|v\|^2 + C_2 C_c \|v\|^{p+1}.
$$

Since $p + 1 > 2$, then $\|v\| \geq \gamma > 0$. This implies that 0 is an isolated critical point.

As a consequence of Proposition 1, we have the following lemma.

Lemma 3.6. Let $\{u_k\} \subset H^\alpha(\mathbb{R}^N)$ be a $C^c$ sequence. If $u_k \rightharpoonup u \in H^\alpha(\mathbb{R}^N)$, then $u$ is a solution of (1.1).

We denote by $\mathfrak{K}$ the set of critical points of $\varphi$. By Theorem 1.1, we know that $\mathfrak{K}$ is non-empty. Set

$$
m = \inf \{\varphi(u) | u \in \mathfrak{K}\}.
$$

Then, for any $u \in \mathfrak{K}$, we have

$$
\varphi(u) = \varphi(u) - \frac{1}{2} \varphi'(u)u = \mathfrak{F}(u) \geq 0.
$$

That is

$$
0 \leq m \leq \varphi(v),
$$

where $v$ is a mountain pass solution in Theorem 1.1. Suppose $\{u_n\} \in \mathfrak{K}$ such that $\varphi(u_n) \to m$. Then $\{u_n\}$ is a $(C)_m$ sequence. By Lemma 3.3, $\{u_n\}$ is bounded. For this sequence $\{u_n\}$, we denote $\delta$ as in (15).

We now claim that $\delta > 0$. If $\delta = 0$, then $u_n \to 0$ in $L^p$ for any $p \in (2, 2^*_\alpha)$. Using (6), we can deduce

$$
\left| \int_{\mathbb{R}^N} f(u_n)u_n \, dx \right| \leq \epsilon |u_n|^2_{L^2} + C_c |u_n|^{p+1}_{L^{p+1}}.
$$

It follows that

$$
\lim_{n \to \infty} \left| \int_{\mathbb{R}^N} f(u_n)u_n \, dx \right| \leq \epsilon \sup_n |u_n|^2_{L^2} \to 0 \text{ as } \epsilon \to 0.
$$

Then

$$
\int_{\mathbb{R}^N} f(u_n)u_n \, dx \to 0, \text{ as } n \to \infty.
$$

Consequently,

$$
\|u_n\|^2 = \int_{\mathbb{R}^N} f(u_n)u_n \, dx \to 0, \text{ as } n \to \infty.
$$

This contradicts with the fact that 0 is an isolated critical point. Therefore, $\delta > 0$, and the claim is proved.

After a suitable translation, we may assume

$$
u_n \rightharpoonup u \neq 0, \text{ as } n \to \infty.$$

By Lemma 3.6, we know that \( u \) is a nontrivial critical point of \( \varphi \). Using Fatou lemma, we get

\[
\varphi(u) = \varphi(u) - \frac{1}{2} \varphi'(u)u = \int_{\mathbb{R}^N} \mathfrak{F}(u)dx
\]

\[
\leq \lim_{n \to \infty} \int_{\mathbb{R}^N} \mathfrak{F}(u_n)dx
\]

\[
= \lim_{n \to \infty} \left\{ \varphi(u_n) - \frac{1}{2} \varphi'(u_n)u_n \right\} = m.
\]

Hence \( u \) is a ground state solution. To get symmetric ground state solutions, we show that the mountain pass value gives the least energy level by the argument from [11, 17, 18]

**Proof of Theorem 1.2.** Note that \( f(s)/s \) is increasing on \((0, \infty)\). This property enables us to make use of the Nehari manifold:

\[
\mathfrak{H} = \{ u \in H^\alpha(\mathbb{R}) \setminus 0 | \varphi'(u)u = 0 \}.
\]

The least energy level \( m \) is characterized as

\[
m = \inf_{u \in \mathfrak{H}} \varphi(u).
\]

We claim that \( c = m \), where \( c \) is the mountain pass value defined in Theorem 2.4.

From [8], we observe that

\[
m = \inf_{u \in H^\alpha(\mathbb{R})} \sup_{\theta \geq 0} \varphi(\theta u).
\]

Given any \( u \in \mathfrak{H} \), we may define a path \( \gamma_u \) as \( \gamma_u(t) = tTu \), where \( \varphi(Tu) < 0 \). It is easy to see that \( \gamma_u \in \Gamma \). Thus, \( m \geq c \). On other hand, since all the solutions of Problem (1) belong to \( \mathfrak{H} \), we have \( m \leq c \).

4. **An example.** The role of (AR) is crucial in applying the critical point theory. However, although (AR) is a quite natural condition, it is somewhat restrictive many nonlinearities. In fact, (AR) condition implies that \( F(x, t) \geq C|t|^\theta \) for some \( C > 0 \), and \( \theta > 2 \). Thus, 2-superlinear functions do not satisfy (AR) condition. For example, the function

\[
f(t) = \begin{cases} t\log(1 + t), & t > 0, \\ 0, & t \leq 0, \end{cases}
\]

does not satisfy (AR) condition. However, it satisfies our conditions.

**REFERENCES**

[1] S. Abe and S. Thurner, Anomalous diffusion in view of Einsteins 1905 theory of Brownian motion, *Physica A*, 356 (2005), 403–407.

[2] F. Almgren and E. Lieb, Symmetric decreasing rearrangement is sometimes continuous, *J. Amer. Math. Soc.*, 2 (1989), 683–773.

[3] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, 14 (1973), 349–381.

[4] D. Applebaum, Lévy processes–from probability to finance and quantum groups, *Not. Am. Math. Soc.*, 51 (2004), 1336–1347.

[5] X. Chang and Z.-Q. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity, *Nonlinearity*, 26 (2013), 479–494.

[6] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional sobolev spaces, *Bull. Sci. Math.*, 136 (2012), 521–573.
[7] S. Dipierro, G. Palatucci and E. Valdinoci, Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, *Le Matematiche*, LXVIII, 68 (2013), 201–216.

[8] P. Felmer, A. Quaas and J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A*, 142 (2012), 1237–1262.

[9] P. Felmer, A. Quaas, M. Tang and J. Yu, Monotonicity properties for ground states of the scalar field equation, *Ann. I. Poincare-AN*, 25 (2008), 105–119.

[10] M. Jara, Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps, *Comm. Pure Appl. Math.*, 62 (2009), 198–214.

[11] L. Jeanjean and K. Tanaka, A remark on least energy solutions in $\mathbb{R}^N$, *Proc. Amer. Math. Soc.*, 131 (2002), 2399–2408.

[12] L. Jeanjean and K. Tanaka, A positive solution for an asymptotically linear elliptic problem on $\mathbb{R}^N$ autonomous at infinity, *ESAIM Control Optim. Calc. Var.*, 7 (2002), 597–614.

[13] E. Lieb and M. Loss, *Analysis*, Grad. Stud. Math., vol. 14, Amer. Math. Soc, Providence, RI, 2001.

[14] S. Liu, On superlinear Schrödinger equations with periodic potential, *Calc. Var. Partial Differential Equations*, 45 (2012), 1–9.

[15] J. Van Schaftingen, Symmetrization and minimax principles, *Commun. Contemp. Math.*, 7 (2005), 463–481.

[16] A. Szulkin and T. Weth, Ground state solutions for some indefinite variational problems, *J. Funct. Anal.*, 257 (2009), 3802–3822.

[17] X. Tang, Non-Nehari manifold method for asymptoticallyperiodic Schrödinger equations, *Sci. China Math.*, 58 (2015), 715–728.

[18] X. Tang, Non-Nehari-manifold method for asymptoticallylinear Schrödinger equation, *J. Aust. Math. Soc.*, 98 (2015), 104–116.

[19] X. Tang, X. Lin and J. Yu, Nontrivial solutions for Schrödinger equation with local supper-quadratic conditions, *J. Dyn. Diff. Equat.*, accepted for publication.

[20] L. Vlahos, H. Isliker, Y. Kominis and K. Hizonidis, *Normal and Anomalous Diffusion: A Tutorial, Order and Chaos*, Patras University Press, 2008.

[21] Y. Wei and X. Su, Multiplicity of solutions for non-local elliptic equations driven by the fractional Laplacian, *Calc. Var. Partial Differential Equations*, 52 (2015), 95–124.

[22] H. Weitzner and G. M. Zaslavsky, Some applications of fractional equations, Chaotic transport and complexity in classical and quantum dynamics, *Commun. Nonlinear Sci. Numer. Simul.*, 8 (2003), 273–281.

[23] M. Willem, *Minimax Theorems*, Progr. Nonlinear Differential Equations Appl., vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.

Received December 2017; revised April 2018.

*E-mail address*: pershen@126.com  
*E-mail address*: hanzhiq@dlut.edu.cn  
*E-mail address*: qqzhang@gzhu.edu.cn