Mathematical models of filtering in an extended loading layer

Y Chirkunov¹,* and Y Skolubovich¹

¹Novosibirsk State University of Architecture and Civil Engineering (Sibstrin), 113, Leningradskaya str., 630008, Novosibirsk, Russia

*E-mail: chr101@mail.ru

Abstract. The study of liquid or gas motion in a porous medium in the framework of classical models does not always adequately describe the real processes. This is due to the fact that in these models the presence of a non-stationary source is not taken into account. The same applies to the models of a filtering in the extended loading layer. For a more adequate description of the real processes of the filtering in the extended loading layer in the presence of a non-stationary source, we took the invariant submodels of a well-tested, adequately reflecting real processes without a source, Leibenzon model, to which we added a non-stationary source, which is singular at the initial time. Such sources are often found in a practice. We studied the invariant submodels of this model, which are described by exact solutions. We found their physical meaning. For particular values of the parameters that determine these submodels, a pressure distribution graphs are constructed.

1. Introduction

The classical model of a porous medium and some of its fairly simple complications describing the motion of liquid and gas in a porous medium have been studied in many works (see, for example, [1–7] and the large bibliography given in these works). A study of the three-dimensional models of porous medium in the presence of non-stationary singular source or absorption was performed in [8–10]. The present paper is devoted to the study of the mathematical models of filtering in the extended loading layer in the presence of a singular source, which are submodels of the following model

\[ \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \frac{\lambda}{t} p, \lambda \neq 0 \]  

(1)

where \( p = p(t, x) \) is the pressure at a point \( x \) at a time \( t \); \( \lambda \) – parameter characterizing the external impact.

In the present paper, as a model describing the motion of a liquid or gas in a porous medium in the presence of an unsteady source, we took a well-tested, adequately reflecting real processes without sources, one-dimensional Leibenzon model [11, 12], to which we added an unsteady singular at the initial time source.

The equation (1) describes the distribution of pressure in a liquid or gas along a ray drawn in a porous medium, in the presence of a source which is non-stationary singular at the initial time. Using the methods of group analysis [13, 14] of the differential equations, it was possible to establish all invariant submodels of the model defined by the equation (1) that are not connected by the point transformations. They are described by the invariant solutions of the nonlinear differential equation (1). In order not to clutter up the paper with the mathematical transformations for the reader’s convenience, the algorithm for obtaining these types of submodels is not given in the paper. In this article, we will explore some of these submodels. We will explore some of these submodels.
In all subsequent formulas the quantities \( c_m \ (m = 1, 2, 3, 4) \) are arbitrary real constants.

2. Invariant submodels

2.1. Self-similar invariant submodel

\[
p = x^{-\beta} q^{\frac{1}{2}} (\xi), \quad \xi = tx^{-\beta-2}
\]

is described by the equation

\[
(\beta + 2)^2 \xi^2 q'' + \left( (\beta + 2)(5\beta + 3)\xi - \frac{q}{2}\right) q' + 2\beta (2\beta + 1)q + \frac{\lambda}{\xi} q^2 = 0,
\]

obtained after substituting (2) in (1).

When \( \beta = -2 \) from the equation (3) it follows that the pressure is determined by the formula

\[
p = \begin{cases} 
-\left(\frac{1}{2} - 1\right) x^2 & \lambda = -1, \\
\frac{-x^2}{12t^{\lambda+1} + c_1 (\lambda+1)} & \lambda \neq -1.
\end{cases}
\]

Solution (2) has a physical meaning only in the following four cases:

For \( \lambda < -1, \ c_1 \leq 0 \), the solution (4) has a physical meaning for all \( t \in (0, \infty) \). The pressure at each point \( x > 0 \) decreases with time, and \( p \to 0 \) at \( t \to \infty \). Figure 1 shows the pressure distribution at \( \lambda = -2, \ c_1 = -12 \).

![Figure 1. Distribution of the pressure.](image)

It follows from this figure that, at the indicated parameters, the pressure at each point of the axis \( Ox \) drawn in a porous medium, quickly becomes close to zero. This leads to the fact that in a porous medium, a strong vacuum is rapidly created.
For \( \lambda < -1, \ c_1 > 0 \), the solution (4) has a physical meaning only for
\[
0 \leq t < \left( -\frac{1}{12}c_4 (\lambda + 1)^{1/(\lambda + 1)} \right). \]
At each point \( x > 0 \), the pressure \( p \to \infty \) at
\[
t \to \left( -\frac{1}{12}c_4 (\lambda + 1)^{1/(\lambda + 1)} \right) \quad \text{and at } t \to 0.
\]
Figure 2 shows the pressure distribution at \( \lambda = -2, \ c_1 = 1 \).

![Figure 2. Distribution of the pressure.](image)

It follows from this figure that, at the indicated parameters, the pressure at each point of the axis \( Ox \) drawn in a porous medium, at the initial moment of the time at \( t \to 0 \) is a very large, then rapidly decreases, reaches at the minimum value, and then rapidly increases and at \( t \to 12 \) again becomes a very large. This means that in a porous medium there is a very strong one-time pulsation of the pressure of a liquid or gas over a finite interval of the time.

For \( \lambda > -1, \ c_1 < 0 \), the solution (4) has a physical meaning only for
\[
0 \leq t < \left( -\frac{1}{12}c_4 (\lambda + 1)^{1/(\lambda + 1)} \right). \]
At each point \( x > 0 \), the pressure \( p \to \infty \) at
\[
t \to \left( -\frac{1}{12}c_4 (\lambda + 1)^{1/(\lambda + 1)} \right), \quad \text{Figure 3 shows the pressure distribution at } \lambda = 2, \ c_1 = -500.\]
It follows from this figure that, at the indicated parameters, the pressure at each point of the ray $Ox$ drawn in a porous medium, ubmodel at the initial moment of the time is zero, then rapidly increases, and at $t \to 5$ becomes very large. This means that a very strong single soliton arises in a porous medium over a finite period of time for pressure.

For $\lambda = -1$ this solution has physical meaning only at $t \in \left( 0, \exp\left( -\frac{c_2}{T^2} \right) \right)$. At each point $x > 0$, the pressure $p \to \infty$ at $t \to \exp\left( -\frac{c_2}{T^2} \right)$. Figure 4 shows the pressure distribution at $\lambda = -1, c_1 = -24$.

2.2. Invariant submodel

$$ p = \frac{1}{t} q(x) $$

is described by the equation

$$ 2 \left( q q^* + q^2 \right) + \left( \lambda + 1 \right) q = 0. $$

When $\lambda \neq -1$, a function $q(x)$ is implicitly determined by a quadrature
The integral in formula (5) is linearly expressed \([15]\) through the elliptic integrals of the first and second kind and a linear-fractional function of the variable \(\arccos q\).

For \(c_3 = c_4 = 0\), a solution exists only \(\lambda < -1\). The pressure in this case is determined by the formula

\[
p = -\frac{(\lambda + 1)x^2}{12t}.
\]

(6)

Figure 5 shows the pressure distribution for \(\lambda = -2\).

Figure 5. Distribution of the pressure.

It follows from this figure that at each point of the axis \(Ox\) drawn in a porous medium, the pressure decreases over time and becomes close to zero. Liquid or gas moves to a point \(O\).

When \(\lambda = -1\) the pressure is determined by the formula

\[
p = \frac{\sqrt{c_3x + c_4}}{t}.
\]

At each point \(x > 0\), pressure decreases over time, and \(p \to 0\) at \(t \to \infty\). Figure 5 shows the pressure distribution at \(\lambda = -1\), \(c_3 = 1\), \(c_4 = 0\). It follows from this figure that, at the indicated parameters, the pressure at each point of the ray \(Ox\) drawn in a porous medium, quickly becomes close to zero. This leads to the fact that in a porous medium creates a strong vacuum. The liquid or gas in this case moves to the point \(O\).
3. Conclusion
The mechanical significance of the investigated models is as follows: 1) these models describe nonlinear processes of filtering in the extended loading layer, in the presence of a non-stationary source, singular at the initial moment of a time, 2) these models can use as tests in the numerical calculations performed in the filtering in the extended loading layer.

Also, these models can be used to study the processes associated with an underground flow of liquid or gas, with the engineering surveys in the construction of the buildings, as well as in a shale oil and gas production.

Acknowledgments
The study was carried out with the financial support of RFBR and the Government of the Novosibirsk region in the framework of the Project № 19-41-540004.

References
[1] Vázquez J L 2007 The porous medium equation Mathematical theory, Oxford University Press, Oxford
[2] Vázquez J L 2007 The porous medium equation Mathematical theory, Oxford University Press, Oxford
[3] Caffarelli L, Vázquez J L 2011 Nonlinear porous medium flow with fractional potential pressure Arch Ration Mech Anal, 202, pp 537 – 565
[4] Allard J F, Atalla N 2009 Propagation of Sound in Porous Media: Modelling Sound Absorbing Materials, Wiley Textbook Series
[5] Barenblatt G I, Vazquez J L 1998 A new free boundary problem for unsteady flows in porous media, European J Appl Math, 9(1), pp 37 – 54
[6] Szimkiewicz A 2013 Modelling Water Flow in Unsaturated Porous Media: Accounting for Nonlinear Permeability and Material Heterogeneity Springer-Verlag, Berlin
[7] Otto F 2001 The geometry of dissipative evolution equations: The porous medium equation Commun Partial Differ Equations, 26 (1–2), pp 101–174
[8] Chirkunov Yu A, Skolubovich Yu L 2018 Nonlinear three-dimensional diffusion models of porous medium in the presence of non-stationary source or absorption and some exact solutions Int J Non-Linear Mech., 106, pp 29 – 37
[9] Chirkunov Yu A, Skolubovich Yu L 2019 Investigation of a nonlinear three dimensional diffusion model of a porous medium Conference Series, J Phys, 1268 012074
[10] Chirkunov Yu A 2018 Self-similar waves in a three-dimensional porous medium in the presence of non-stationary singular source or absorption, *Int J Non-Linear Mech.*, **117**, 103205

[11] Leibenzon L S 1953 *Underground Hydro and Gas Dynamics, Collection of works*, 2, Publishing house of the Acad of Scienc of the USSR, Moscow

[12] Leibenzon L S 1955 *Oil field mechanics, Collection of works*, 4, (M : Publishing house of the Acad of Scienc of the USSR, Moscow

[13] Chirkunov Yu A, Khabirov S V 2012 *Elements of symmetry analysis of differential equations of continuum mechanics*, NSTU, Novosibirsk

[14] Chirkunov Yu A 2007 *Group analysis of linear and quasi-linear differential equations*, NSUEM, Novosibirsk

[15] Gradstein I S, Ryzhik I M 1971 *Tables of integrals, sums, series and products*, Nauka, Moscow