Rotating Toroidal Branes in Supermembrane and Matrix Theory

M. Axenides\textsuperscript{a}, E. G. Floratos\textsuperscript{a}, L. Perivolaropoulos\textsuperscript{a},
\textsuperscript{b} C. Leandros\textsuperscript{c}

\textsuperscript{a} Institute of Nuclear Physics, N.C.S.R. Demokritos, 153 10, Athens, Greece,
e-mail:minos@inp.demokritos.gr,

\textsuperscript{b} National University of Athens, Athens, Greece, e-mail:manolis@inp.demokritos.gr

\textsuperscript{c} Physics Department, University of Patras, Patras, Greece, e-mail:leandros@physics.upatras.gr

(March 27, 2002)

In the lightcone frame, where the supermembrane theory and the Matrix model are strikingly similar, the equations of motion admit an elegant complexification in even dimensional spaces. Although the explicit rotational symmetry of the target space is lost, the remaining unitary symmetries apart from providing a simple and unifying description of all known solutions suggest new ones for rotating spherical and toroidal membranes. In this framework the angular momentum is represented by $\mathbb{U}(1)$ charges which balance the nonlinear attractive forces of the membrane. We examine in detail a six dimensional rotating toroidal membrane solution which lives in a 3-torus, $T^3$ and admits stable radial modes. In Matrix Theory it corresponds to a toroidal N-D\textsubscript{5} brane bound state. We demonstrate its existence and discuss its radial stability.

PACS:11.27 +d

1. INTRODUCTION

String Theory and the more speculative nonperturbative version of it, M-theory, still is the only surviving candidate for the unification of gravity with quantum mechanics and the other fundamental gauge interactions. The D-branes of string theory are the long searched for gravitational instantons and can be described together with their dualities by simple geometrical symmetries of compactified $M_2$ and $M_5$ branes. The problem for their interpretation as true gravitational solitons is that they are usually static and possess infinite mass and charge if not wrapped in compact submanifolds.

The fundamental degrees seem to be the $D_0$ branes and many attempts have been made to construct all the others as $D_0$ bound states. On the other hand in the Matrix model, bound states of $D_0$ branes with spherical topologies having finite mass, charge and angular momentum have been constructed either via background field improved Matrix model or by introducing rotational degrees of freedom. No supergravity solutions dual to these bound states have been constructed, although their existence is considered to be a valid hypothesis.

We adopt, as a conjecture to be proved, that the quantum supermembrane theory is identical with the M-theory. The true nonsingular gravitational solitons of 11-dimensional supergravity we expect them to be the duals of finite mass, flux, angular momentum solutions of supermembrane theory in flat eleven dimensional spacetime. The above picture is true for the infinite mass and charge $M_2$ and $M_5$ branes after compactification. Another possible interpretation for the compact solutions of the supermembrane theory in the lightcone frame and flat background is that of PP waves, i.e. infinite momentum boosts of analogous supergravity solutions with the same symmetries.

From this point of view the search for stable classical supermembrane solutions which can be used as true saddle points of the supermembrane path integral quantization, while particles and strings are singular saddle points appears to be worthwhile.

In this work we continue our search for supermembrane solutions introducing the complexification of the light cone frame equations of motion in flat space. We show that this framework encompasses all previously found spherical and toroidal rotating membranes and more importantly it suggests new ones. As an example we present a rotating toroidal membrane in six or eight dimensions ($C^2$ and $C^4$) which lives in three or four dimensional tori and the angular momentum is represented by three or correspondingly four charges which balance the nonlinear attractive forces of the membrane. We also demonstrate its radial stability.

In chapter II we describe the complexification of the lightcone membrane equations of motion in even dimensions and we present our generalized ansatz for the factorization of time. As a consequence we show that all previously known solutions in the literature can be deduced as particular examples.

In chapter III we exhibit a new interesting class of solutions which describe toroidal rotating membranes embedded in a $T^3$ complex torus. These solutions may be relevant for new supersymmetric toroidal compactifications of the M-theory. We demonstrate their stability against radial perturbations.

In chapter IV we utilize the isomorphism between the
nonperturbative sectors of Matrix and Supermembrane Theories and demonstrate the existence and radial stability of the corresponding Toroidal N-D_0 brane solutions.

We close by presenting some concluding remarks and speculations about the supergravity duals of our new configurations.

II. COMPLEX DIMENSIONS AND ROTATING MEMBRANES

The nonlinear attractive force as well as the absence of any coupling constant for the membrane is known to be the main difficulty for the analysis of its dynamics at the classical and quantum levels [13]. In order to understand the excitation spectrum of the membrane it was obvious that one has to find stable rotating solutions or to change the topology of space and wrap static membranes along topologically similar cycles.

In an interesting work by Nicolai and Hoppe [3] it became clear that rotating solutions for closed spherical and toroidal membranes exist in higher dimensional spherical spaces. With the discovery of the supermembrane Lagrangian, the search for supersymmetric membrane solutions became clear that rotating solutions for closed spherical and toroidal configurations [7,8]. The last few years rotating Matrix spherical and toroidal configurations were found [11]. For the spherical topology more detailed investigations uncovered a rich structure of moduli and stability in certain cases was shown [10].

In this section we review the existing solutions in a unified framework in the light cone frame and flat Euclidean spaces. The Hamiltonian for the bosonic part of the supermembrane using Darboux parametrization of the membrane surface (Area element dA = dσ_1 dσ_2) is:

\[ H = \frac{1}{8\pi T} \int d^2 \sigma \left[ \frac{1}{2} \dot{X}_i^2 + \frac{1}{4} \{X_i, X_j\}^2 \right] \quad i, j = 1, \ldots, d \]  

where \( T \) is the membrane tension. The corresponding equations of motion are given by:

\[ \ddot{X}_i = \{X_j, \{X_j, X_i\}\} \quad i, j = 1, \ldots, d \]  

where summation is implied in the \( j \) indices and \( \{ \} \) stands for the Poisson bracket with respect to the angular coordinates \( \sigma_1, \sigma_2 \). The Gauss constraint that also needs to be satisfied is

\[ \{\dot{X}_i, X_i\} = 0 \quad (2.3) \]

This constraint is preserved by the equations of motion and therefore if it is initially obeyed, as is the case with what follows, it will be obeyed at all times. If \( d = 2k \) we define \( Y_i = X_{i+k} \) with \( i = 1, \ldots, k \). The equations of motion are

\[ \ddot{X}_i = \{X_j, \{X_j, X_i\}\} + \{Y_j, \{Y_j, X_i\}\} \]

\[ \ddot{Y}_i = \{X_j, \{X_j, Y_i\}\} + \{Y_j, \{Y_j, Y_i\}\} \quad (2.4) \]

It is now convenient to introduce a convenient notation of complex coordinates \( Z_1, Z_2, \ldots, Z_k \) defined as

\[ Z_i = X_i + iY_i, \quad i = 1, \ldots, k \quad (2.5) \]

These coordinates transform the target space \( R^{2k} \) of membrane solutions into \( C^k \) simplifying in this way the field equations considerably. In the context of these complex coordinates of \( 2k \) the Hamiltonian takes the following form:

\[ H = \frac{1}{8\pi T} \int d^2 \sigma \left[ \frac{1}{2} |\dot{Z}|^2 + \frac{1}{2} |\partial_1 \dot{Z} \times \partial_2 \dot{Z}|^2 + \frac{1}{2} |\partial_1 \ddot{Z} \times \partial_2 \ddot{Z}|^2 \right] \quad (2.6) \]

where \( (\partial_1 \ddot{Z} \times \partial_2 \ddot{Z})_i = \epsilon_{ijk} \partial_1 \dot{Z}_j \partial_2 \dot{Z}_k = \frac{1}{2} \epsilon_{ijk} \{Z_j, Z_k\} \) and \( (\partial_\alpha \ddot{Z})_i = \frac{\partial}{\partial \sigma^\alpha} Z_i, \quad \alpha = 1, 2 \) and \( i = 1, 2, 3 \). Similarly the equations of motion transform to:

\[ \ddot{Z}_i = \ddot{X}_i + i\ddot{Y}_i = \frac{1}{2} \left[ \{Z_j^*, \{Z_j, Z_i\}\} + \{Z_j, \{Z_j^*, Z_i\}\} \right] \quad (2.7) \]

while the constraint becomes

\[ \{\dot{Z}_i^*, Z_i\} + \{\dot{Z}_j, Z_j^*\} = 0 \quad i, j = 1, \ldots, k \quad (2.8) \]

Although the target space rotational symmetry \( O(2k) \) becomes less obvious in the complexified equations, their explicit \( U(k) \) symmetry will simplify the description of the known solutions and it will facilitate the search for new ones. Below we exhibit the most general solution with factorization of time. To begin with our most general ansatz is:

\[ Z_i = \zeta_{ij}(t) f_j(\sigma_1, \sigma_2) \quad i, j = 1, 2, \ldots, k \quad (2.9) \]
where \( \zeta_{ij} \) is a \( k \times k \) non-singular complex matrix and \( f_j \) are \( k \)-complex linearly independent functions of membrane parameters. This ansatz provides a factorization of time in the equations of motion and the reparametrization constraint if the following conditions for the matrix \( \zeta \) and the functions \( f_i \) are satisfied,

\[
\zeta^\dagger \zeta = \eta_D \\
\zeta^\dagger \dot{\zeta} - \dot{\zeta}^\dagger \zeta = i \theta_D
\]  

(2.10)

where \( \eta_D \) is a positive real diagonal matrix and \( \theta_D \) is a real diagonal matrix. Here we assume also that \( \{ f_i^*, f_i \} = 0 \) for every \( i = 1, \ldots, k \). It is not difficult to show that the above conditions for the matrix \( \zeta \), which are sufficient but not necessary, imply that it must be complex diagonal up to multiplication on the left by a constant unitary matrix. We therefore adopt as an ansatz the diagonal form:

\[
Z_i = \zeta_i(t) f_i
\]  

(2.11)

It automatically satisfies the constraint and the NASC for the factorization of time is given by

\[
\frac{1}{2} \{ f_j, \{ f_j^*, f_i \} \} + \frac{1}{2} \{ f_j^*, \{ f_j, f_i \} \} = -\nu_{ji} f_i
\]  

(2.12)

for every \( i \) and \( j \). The time dependent complex scale factors \( \zeta_i \) satisfy

\[
\ddot{\zeta}_i = -\sum_j |\zeta_j|^2 \nu_{ji} \zeta_i
\]  

(2.13)

Firstly we treat the case of spherical membranes and we choose \( f_i = e_i, \; i = 1, 2, 3 \), where

\[
e_1 = \sin \theta \cos \phi, \quad e_2 = \sin \theta \sin \phi, \quad e_3 = \cos \theta
\]  

(2.14)

As a consequence \( \nu_{ij} = \delta_{jj} - \delta_{ij} \) and the equations of motion (2.13) become

\[
\ddot{\zeta}_i = -\sum_j |\zeta_j|^2 (\delta_{jj} - \delta_{ij}) \zeta_i = -|\zeta|^2 \zeta_i
\]  

(2.15)

In order to make explicit the conserved quantities, we separate the moduli and the phases

\[
\zeta_i = \lambda_i(t) e^{i \chi_i(t)} 
\]  

(2.16)

It is obvious that there are three \( U(1) \) conserved charges

\[
Q_i = -\frac{i}{2} (\dot{\zeta} \zeta_i^* - \zeta_i^* \dot{\zeta})
\]  

(2.17)

which in turn implies

\[
\dot{\chi}_i = \frac{Q_i}{\lambda_i^2}
\]  

(2.18)

By using (2.18) we find

\[
\ddot{\lambda}_i - \frac{Q_i^2}{\lambda_i^4} + (\lambda_i^2 - \lambda_i^2) \lambda_i = 0
\]  

(2.19)

and get the conserved energy of the system to be

\[
E = \sum_i \frac{1}{2} \dot{\lambda}_i^2 + \frac{1}{2} \sum_i Q_i^2 + \sum_i \frac{1}{2} (\lambda_i^2 \lambda_i^2 + \lambda_i^2 \lambda_i^2) + \lambda_i^2 \lambda_i^2
\]  

(2.20)

For a similar system see [10]. We now proceed to show that the Nicolai-Hoppe spherical membrane solution arises as a special case of the framework we have introduced. Indeed the Nicolai-Hoppe ansatz in the lightcone frame can be written as

\[
Z_i = \lambda(t) (U f)_i
\]  

(2.21)

where the unitary \( 3 \times 3 \) matrix \( U \) is an overall time dependent phase

\[
U = e^{i \lambda(t) I} \in U(3)
\]  

(2.22)

with \( I \) being identified as the identity \( 3 \times 3 \) matrix. In this case all scale factors \( \lambda_i \) as well as all phases are equal. The factorization of time for the three functions \( f_i \) imposes the condition

\[
\{ f_k^*, \{ f_k, f_i \} \} + \{ f_k, \{ f_k^*, f_i \} \} = -2 \nu f_i
\]  

(2.23)

where \( \nu \) is obviously positive and summation over the index \( k \) is implied. Nicolai and Hoppe choose \( f_i = e_i \) (2.14) which gives \( \nu = 2 \). With these assumptions we find,

\[
\ddot{\lambda} - \lambda \dot{\lambda}^2 + \nu \lambda^3 = 0
\]  

(2.24)

\[
\lambda \dot{x} + 2 \dot{\lambda} \dot{x} = 0
\]  

(2.25)

They imply

\[
\dot{x} = \frac{L}{\dot{\lambda}^2}
\]  

(2.26)

\[
\dot{\lambda} - \frac{L^2}{\dot{\lambda}^2} + \nu \lambda^3 = 0
\]  

(2.27)
where L is the conserved angular momentum of the configuration and therefore the conserved energy is

$$E = \frac{\dot{\lambda}^2}{2} + \frac{L^2}{2\lambda^2} + \frac{\nu}{4}\lambda^4$$

(2.28)

When $L = 0$ the collapsing membrane solution is obtained \[\] . When $L \neq 0$ the membrane can be stabilized at the minimum of the effective potential.

We now turn to the case of the toroidal membrane. The four functions $f_i$ are chosen by Nicolai-Hoppe \[\] and Hoppe \[\] solutions, discussed previously by Taylor \[\] and more recently in \[\] .

In the framework of spherical matrix branes \[\] by working in a rotating basis of the SU(2) generators in the N-dimensional representation instead of external magnetic fluxes discussed by Myers \[\] , these configurations stabilize the attractive force of N D0 branes. Below we present the continuous membrane analog of the matrix rotating ellipsoid \[\] in the complex target space $C^3 \{1\}$.

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} e^{i\omega_1 t} & 0 & 0 \\ 0 & e^{i\omega_2 t} & 0 \\ 0 & 0 & e^{i\omega_3 t} \end{pmatrix} \begin{pmatrix} R_1 e_1(\theta, \phi) \\ R_2 e_2(\theta, \phi) \\ R_3 e_3(\theta, \phi) \end{pmatrix}$$

(3.3)

where the functions $e_i(\theta, \phi)$ are given by \[\] and form an SU(2) algebra.

$$\{e_i, e_j\} = -\epsilon_{ijk}e_k$$

(3.4)

The solution represents a rotating ellipsoid of fixed shape of three different axes $\lambda_i = R_i$, $i = 1, 2, 3$ with their respective angular frequencies given by:

$$\omega_1^2 = R_1^2 + R_2^2 , \quad \omega_2^2 = R_1^2 + R_3^2 , \quad \omega_3^2 = R_2^2 + R_3^2$$

(3.5)

The corresponding three U(1) charges are given by

$$Q_1 = \omega_1 R_2^2 , \quad Q_2 = \omega_2 R_3^2 , \quad Q_3 = \omega_3 R_1^2$$

(3.6)

and the energy (2.16) can be cast in terms of the $Q_i$ only given above. Its form determines the equilibrium parameters of our membrane configuration.

### III. NEW TOROIDAL MEMBRANE SOLUTIONS AND STABILITY

We shall show now that our generalized set up in complex $C^n$ flat spaces can easily accomodate toroidal solutions. They generalize anticipated toroidal solutions by Nicolai and Hoppe \[\] . We may also add that there is a keen interest in the mathematical literature regarding the various embeddings of Tori in $C^3 \{\}$.

A natural basis of functions on $T^2$ is

$$f_{\vec{n}} = e^{i\vec{n} \cdot \vec{\sigma}}$$

(3.1)

In this basis we have that

$$\{f_{\vec{n}}, f_{\vec{m}}\} = -(\vec{n} \times \vec{m}) f_{\vec{n} + \vec{m}}$$

(3.2)

where $\vec{n} \times \vec{m} = n_1 m_2 - n_2 m_1$.

This is the area preserving infinite dimensional symmetry of the torus $T^2$ namely $\text{SDiff}(T^2)$ \[\] . From eq.(2.1) the factorized form $Z_i(t) = \zeta_i(t)e^{i\vec{n}_i \cdot \vec{\sigma}}$ gives the following equations of motion for $\zeta_i$:

$$\dot{\zeta}_1 = -\zeta_1(R^2|\zeta_3|^2 + k^2|\zeta_2|^2)$$

$$\dot{\zeta}_2 = -\zeta_2(R^2|\zeta_4|^2 + k^2|\zeta_1|^2)$$

$$\dot{\zeta}_3 = -\zeta_3(R^2|\zeta_1|^2 + l^2|\zeta_2|^2)$$

(3.3)

The factorization condition is automatically satisfied for any triplet $\vec{n}_i \in Z^2$, $i, j = 1, 2, 3$

$$\frac{1}{2} \left\{ f_{\vec{n}_i}, \left\{ f_{\vec{m}_j}, f_{\vec{n}_k} \right\} \right\} + \frac{1}{2} \left\{ f_{\vec{m}_j}, \left\{ f_{\vec{n}_i}, f_{\vec{m}_k} \right\} \right\} = -\nu_{ij} f_{\vec{n}_i} \quad \nu_{ij} = (\vec{n}_i \times \vec{n}_j)^2$$

(3.4)

where

$$k^2 = (\vec{n}_1 \times \vec{n}_2)^2 , \quad l^2 = (\vec{n}_2 \times \vec{n}_3)^2 , \quad m^2 = (\vec{n}_3 \times \vec{n}_1)^2$$

(3.5)

We get the toroidal solutions if we choose $\zeta_i = R_i e^{i\omega_i t}$, $i = 1, 2, 3$ with the angular frequencies being related to the amplitudes as follows:

$$\omega_1^2 = k^2 R_3^2 + m^2 R_3^2$$

(3.6)

$$\omega_2^2 = k^2 R_1^2 + l^2 R_3^2$$

(3.7)

$$\omega_3^2 = m^2 R_1^2 + l^2 R_2^2$$

(3.8)

We observe that contrary to the case of the spherical membrane solutions it is possible to choose $\vec{n}_i$ and for unequal $R_i$, $i = (1, 2, 3)$ such that all of the $\omega_i$ are equal.
In this case we observe that \(|Z_i|^2 = R_i^2\), \(i = 1, 2, 3\) the toroidal membrane moves in a \(T^3 = S^1 \times S^1 \times S^1\) torus with coordinates at any time \(t\) determined by the phases

\[
\begin{align*}
\phi_1 &= \omega_1 t + n_1^1 \sigma_1 + n_1^2 \sigma_2 \\
\phi_2 &= \omega_2 t + n_2^1 \sigma_1 + n_2^2 \sigma_2 \\
\phi_3 &= \omega_3 t + n_3^1 \sigma_1 + n_3^2 \sigma_2
\end{align*}
\]

(3.10)

We have implicitly used \(U(1)^3\) as the remaining global symmetry of eq.(3.3) to fix the initial conditions for \(\phi_i\). With these phases, at any moment of time, the embedding of \(T^2\) inside \(T^3\) can be expressed as \(Z_i = R_i e^{i \phi_i(t)}\). In order to visualize the motion of \(T^2\) inside \(T^3\) it is helpful to write the equation of the embedding in the periodic space of the phases \(\phi_i, \ i = 1, 2, 3\) which is a \(T^3\) of length \(2\pi\) as:

\[
I\phi_1 + m\phi_2 + k\phi_3 = (\omega_1 l + \omega_2 m + \omega_3 k) t
\]

(3.11)

This equation is derived by eliminating \(\sigma_1, \sigma_2\) from eq.(3.10). We see that it is possible to have a time periodicity if and only if \(\omega_i = (q_i/p_i)\omega\) where the \(q_i\) and \(p_i\) are relative prime integers and \(i = 1, 2, 3\). This is possible when the \(R_i\)'s, which are the radii of \(T^3\) take appropriate values. In this case the period is \(T = 2\pi \frac{q}{p}\) and \(p = \text{lcm}(p_1, p_2, p_3)\). We herein denote "\(\text{lcm}\)" the least common multiple. We also note that for the special case of \(\vec{n}_1 + \vec{n}_2 + \vec{n}_3 = 0\) then \(k^2 = l^2 = m^2\). In this case the dynamical equations of motion (3.3) are similar to the spherical case and thus eq.(3.11) simplifies considerably.

The solution is invariant under the modular group \(SL(2, Z)\) which acts on \(\vec{n}_i\). In other words the energy of the system possesses an \(SL(2, Z)\) degeneracy. We may also note that in this case we have a "twisted" \(SU(2)\) Poisson algebra

\[
\{e^{i\vec{n}_1 \vec{\sigma}}, e^{i\vec{n}_3 \vec{\sigma}}\} = -(\vec{n}_1 \times \vec{n}_2) e^{-i\vec{n}_3 \vec{\sigma}}
\]

(3.12)

and cyclic permutations.

In the following we shall demonstrate the radial stability of these toroidal solutions. The linearized equations of motion around \(Z^\circ_i = R_i e^{i\vec{n}_i \vec{\sigma} + i\omega_i t}\) are easily found. The variations for radial excitations \(\delta \xi_i = \xi_i - \xi_i^\circ\) satisfy the corresponding linear equation which can be obtained directly from eq.(3.3). For example for the case \((i = 1)\):

\[
\begin{align*}
\delta \xi_1 &= -\delta \xi_1 \omega_1^2 - \xi_1^\circ [2\omega_1^2 \xi_1^\circ + \delta \xi_1^\circ + \delta \xi_2^\circ + \delta \xi_2^\circ] \\
&+ m^2(\xi_1^\circ \cdot \delta \xi_1^\circ + \delta \xi_3^\circ \cdot \xi_3^\circ)]
\end{align*}
\]

(3.13)

In order to organize better our eigenvalue equation we go to the body frame by introducing the transformation

\[
n_i = e^{-i \omega_i t} \delta \xi_i, \ i = 1, 2, 3
\]

(3.14)

By taking the appropriate time derivatives we eliminate the time dependence of the coefficients of our eigenvalue equation through

\[
e^{-i \omega_i t} \delta \xi_1 = \dot{n}_1 + 2i \omega_1 \dot{n}_1 - \omega_1^2 m
\]

(3.15)

We formulate the perturbation eqs. of motion for \(\delta \xi_1\) in terms of real and imaginary parts

\[
\dot{n}_{1R} - 2\omega_1 \dot{n}_{1I} = -2R_1 \left(k^2 R_2 n_{2R} + m^2 R_3 n_{3R}\right)
\]

\[
\dot{n}_{1I} + 2\omega_1 \dot{n}_{1R} = 0
\]

(3.16)

(3.17)

As a consequence we have that

\[
\frac{d\dot{n}_{1R}}{dt} - 2\omega_1 \dot{n}_{1I} = -2R_1 \left(k^2 R_2 n_{2R} + m^2 R_3 n_{3R}\right)
\]

(3.18)

Similar equations can be obtained for \(i = 2, 3\). We eliminate the imaginary part and in order to avoid dealing with many indices we define

\[
\dot{n}_{iR} = u_i, \ i = 1, 2, 3
\]

(3.19)

The eigenvalue eqs of motion take the form

\[
\begin{align*}
\ddot{u}_1 + (2\omega_1)^2 u_1 &= -2R_1 \left(k^2 R_2 u_2 + m^2 R_3 u_3\right) \\
\ddot{u}_2 + (2\omega_2)^2 u_2 &= -2R_2 \left(k^2 R_1 u_1 + l^2 R_3 u_3\right) \\
\ddot{u}_3 + (2\omega_3)^2 u_3 &= -2R_3 \left(m^2 R_1 u_1 + l^2 R_2 u_2\right)
\end{align*}
\]

(3.20)

We study the stability of the radial mode of the above equation. We set \(u_i(t) = \xi_i \exp(i\lambda t)\) and obtain the following matrix whose eigenvalues should be positive definite:

\[
M = \begin{bmatrix}
4\omega_1^2 & 2R_1 R_2 k^2 & 2R_1 R_3 m^2 \\
2R_2 R_1 k^2 & 4\omega_2^2 & 2R_2 R_3 l^2 \\
2R_1 R_3 m^2 & 2R_2 R_3 l^2 & 4\omega_3^2
\end{bmatrix}
\]

(3.21)

In order to demonstrate positive definiteness for the eigenvalues of matrix \(M\) it is enough to show that \(\xi . M . \xi > 0\) for every real vector \(\xi \in R^3\).

Indeed in our case we have that the corresponding expression takes the form

\[
\xi . M . \xi = 4k^2 [(R_2 \xi_1 + R_1 \xi_2)^2 - R_1 R_2 \xi_1 \xi_2] + 4m^2 [(R_3 \xi_1 + R_1 \xi_3)^2 - R_1 R_3 \xi_1 \xi_3] + 4l^2 [(R_3 \xi_2 + R_2 \xi_3)^2 - R_2 R_3 \xi_2 \xi_3]
\]

(3.22)

which is manifestly positive. With regard to the remaining three dimensions \((i = 7, 8, 9)\) the analysis for a general perturbation, not necessarily radial, leads to bounded harmonic motion.
IV. TOROIDAL BOUND STATES OF N-D0 BRANES

We introduce $N \times N$ hermitian matrices $X_i$, $i = 1, \ldots, 6$. In the complex notation we introduced previously we define

$$Z_1 = X_1 + iX_4$$

$$Z_2 = X_2 + iX_5$$

$$Z_3 = X_3 + iX_6$$ (4.1)

The equations of motion take the following form

$$\ddot{Z}_i = -\frac{1}{2} \left[ Z^\dagger_i, [Z_j, Z_i] \right] - \frac{1}{2} \left[ Z_j, [Z^\dagger_i, Z_i] \right]$$ (4.3)

Gauss’s Law equation of the constraint is given by

$$\left[ Z_i, Z^\dagger_i \right] + \left[ Z^\dagger_i, Z_i \right] = 0$$ (4.4)

In analogy with (2.11) of our membrane ansatz we make the following matrix ansatz:

$$Z_i = \zeta_i(t) M_i \quad i = 1, 2, 3$$ (4.5)

where $\zeta_i$ are the diagonals of a general nonsingular complex matrix $\zeta_{ij}$ and $M_i$ is a general matrix that corresponds to the $f_i$ functions of the membrane parameters. The constraint equation (4.4) is automatically satisfied with the NASC for the factorization of time given by

$$\frac{1}{2} \left[ M^\dagger_i, [M_j, M_i] \right] + \frac{1}{2} \left[ M_j, [M^\dagger_i, M_i] \right] = \nu_{ji} M_i$$ (4.6)

for every $i$ and $j$, see J.Hoppe in [1]. The $\zeta_i$ factors satisfy equation (2.13).

As an example we consider the case of a spherical N-D0 brane bound state. In this case the N-D0 branes constitute a fuzzy sphere in six dimensions described by $N \times N$ hermitian matrices $M_i$. More specifically we consider the case $M_i = J_i$ where $J_i$ are the three generators of the $N = 2j + 1$ dimensional irreducible representation of $SU(2)$. By plugging into both the equations of motion and that of the constraint we find that $\nu_{ij} = 2$, $i, j = 1, 2, 3$. The analysis is identical with the spherical membrane case (2.15-2.20) and (2.30-2.32).

We now proceed with the case of the torus $T^2$. We herein consider $N \times N$ matrices

$$J_{\tilde{n}} = \omega^{\frac{n_1 n_2}{2}} P^{n_1} Q^{n_2}$$ (4.7)

where $\omega = e^{2\pi i/N}$, $\tilde{n} = (n_1, n_2) \in Z_N \times Z_N$

From their definition $J_{\tilde{n}}$, $P$ and $Q$ are unitary and periodic matrices. They satisfy the conditions for a quantum discrete torus [17]:

$$Q P = \omega P Q$$

$$P^N = I, \quad Q^N = I$$ (4.9)

and

$$J_{\tilde{n}} J_{\tilde{m}} = \omega^{-\frac{n \cdot m}{N}} J_{\tilde{n} + \tilde{m}}$$ (4.10)

from which it also follows that

$$[J_{\tilde{n}}, J_{\tilde{m}}] = -2 i \sin \frac{\pi}{N} (m \times n ) \cdot J_{\tilde{n} + \tilde{m}}$$ (4.11)

In the present case of the Matrix model we take as ansatz $M_i = J_0^{\tilde{n}_i}$, $i = 1, 2, 3$ [11]. One can show that in the limit $N \to \infty$, $\omega \to 1$ we recover the membrane parametrization, namely $J_{\tilde{n}_i} = f_i = e^{i\tilde{n}_i \cdot \tilde{\sigma}} \quad i = 1, 2, 3$ [15].

It is straightforward to check that the above ansatz gives us

$$\nu_{ij} = 4 \left[ \sin \frac{\pi}{N} (\tilde{n}_j \times \tilde{n}_i) \right]^2$$

Our matrix ansatz (1.5) implies the following equations of motion for the $\zeta_i$:

$$\ddot{\zeta}_1 = -4\zeta_1 \sum_{j \neq i} \left[ \sin \frac{\pi}{N} (\tilde{n}_j \times \tilde{n}_i) \right]^2 |\zeta_j|^2$$ (4.12)

They translate into an identical set of equations for each $i = 1, 2, 3$ as with the toroidal membrane case (3.3) with the proper identification for $\nu_{ij}$ namely

$$2 \sin \left( \frac{\pi}{N} k \right) \leftrightarrow k = \tilde{n}_1 \times \tilde{n}_2$$

$$2 \sin \left( \frac{\pi}{N} m \right) \leftrightarrow m = \tilde{n}_2 \times \tilde{n}_3$$ (4.13)

$$2 \sin \left( \frac{\pi}{N} l \right) \leftrightarrow l = \tilde{n}_3 \times \tilde{n}_1$$

By a complete similarity with the toroidal membrane if $\tilde{n}_i + \tilde{n}_2 + \tilde{n}_3 = 0$ then $k = m = l$. The algebra gets simplified and we get a twisted $SU(2)$ trigonometric algebra $[J_{\tilde{n}_1}, J_{\tilde{n}_2}] = -4 i \sin \left( \frac{\pi}{N} k \right) J_{\tilde{n}_3}$ along
with their cyclic permutations. With this correspondence in mind it can be observed that for the case of $\zeta_i(t) = R_i e^{i\omega t}$, $i = 1, 2, 3$ the angular frequencies for the $N - D0$ Brane bound state ansatz admits a similar dependence on their amplitudes. For the case $i = 1$, for example, we have that

$$\omega_1^2 = 4 \left[ R_2^2 \sin^2 \left( \frac{\pi}{N} k \right) + R_3^2 \sin^2 \left( \frac{\pi}{N} m \right) \right] \tag{4.14}$$

In this case the matrices $Z_i$ satisfy $Z_i Z_i^\dagger = R_i^2$, $i = 1, 2, 3$ namely the $N - D0$ bound state has the topology of a quantum $T^2$ torus embedded in a quantum $T^3$ torus with coordinates determined at any time $t$ by appropriate phases which are formally given by (3.10).

The stability analysis of our configuration subjected to radially symmetric fluctuations $\delta \zeta_i = \zeta_i - \zeta_i^0$ around the solutions $\zeta_i^0 = R_i e^{i n x + \pi \omega t}$ proceeds identically with the membrane case and is given by eqs(3.11–3.20) always taking properly into account the correspondence given by [11,13]. Indeed we associate positive numbers to positive numbers the positive definiteness of the eigenvalues of the matrix $M$ of radial fluctuations does not get modified.

V. CONCLUSIONS

We have complexified the membrane and Matrix model dynamical equations in even dimensional flat real spaces simplifying the description of the existing known solutions as well as the search for new solutions of rotating closed membranes and $N - D0$ brane bound states correspondingly. Indeed we presented constructions of a new toroidal rotating membrane which is radially stable and the motion of which is restricted in a $T^3$ torus. Its physical interpretation is that of rotating black hole solutions of the corresponding eleven dimensional supergravity $\mathfrak{g}$. This interpretation extrapolates the gravitational duality of D-branes as solutions of open string theory in flat spacetime with boundaries to the case of supermembranes. This is easily understood if we use as an intermediate step the Matrix Model. Indeed the latter can be used to connect supergravity with the supermembrane theory. The stability consideration of classical solutions is relevant to the quantization of supermembrane and matrix model which has been recently in the focus of attention [14].

VI. ACKNOWLEDGEMENTS

We thank A.Kehagias for useful discussions. This work was supported by the EU under the TMR No CT2000-00122,00431. L.P. acknowledges support from NCSR Demokritos where most of this work was completed and by the C.Carathéodory Research Grant no. 2793 from the U.of Patras. This work has also benefited from the network “Cosmology in the Lab” which is supported by the European Science Foundation.

[1] P. A. Collins and R. W. Thucker, Nucl. Phys. B112, 150(1976).
[2] K. Kikkawa and M. Yamasaki, Progr.Theor.Phys. 76, 1379 (1986).
[3] J. Hoppe and H. Nicolai, Phys. Letts. B196, 451(1987).
[4] M. Duff, TASI Lectures 1996, hep-th/9611203.
[5] P. Townsend, Phys.Lett. B350, 184(1995); hep-th/961212; E. Witten, Nucl. Phys. B443, 85(1995); ibid B460, 335(1995); J. H. Schwarz, Phys. Lett. B360, 13(1995); M. Duff, Int.J.Mod.Phys. A11, 5623(1996).
[6] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys.Rev. D55, 5112(1997); N. Ishibasi, H. Kawai, Y. Kitazawa, A. Tsuchiya, Nucl. Phys. B498, 467(1997); W. Taylor, NATO Lectures, hep-th/0001016.
[7] W. Taylor, Rev. Mod. Phys. 73, 419(2001).
[8] D. Kabat and W. Taylor, Advances in Theoretical Mathematical Physics 2, 181(1998); W. Taylor and M. Van Raamsdonk, hep-th/9904005; R. C. Myers, JHEP 9912, 022 (1999).
[9] J. Hoppe, hep-th/9702169; W. Taylor and M. V. Raamsdonk, Nucl. Phys. B532, 227(1998); S. G. Rey, hep-th/9711081; R. G. Cai and K. S. Soh, Mod. Phys. Lett. A14, 1895(1999); T. Harmark and N. A. Obers, JHEP 0003:024(2000); D. Mateos, S. Ng and P. K. Townsend, hep-th/0112054.
[10] T. Harmark and K. G. Savvidy, Nucl. Phys. B585, 567 (2000); hep-th/0002150; G. K. Savvidy and K. G. Savvidy, Phys.Lett. B501, 283(2001).
[11] M. Axenides, E. G. Floratos and L. Perivolaropoulos, JHEP 0011, 020(2000); M. Axenides, E. G. Floratos and L. Perivolaropoulos, Phys. Rev. D64, 107901(2001).
[12] H. Ma and Yunjie Ma, math-DG/0106141.
[13] E. G. Floratos and J. Iliopoulos, Phys. Lett. B201, 237(1988).
[14] A. Dasgupta, H. Nicolai and J. Plefka, Grav.Cosmol. 8, 1(2002); M. Hayakawa and N. Ishibashi, Nucl. Phys. B614, 171(2001).
[15] D. B. Fairlie, P. Fletcher and C. K. Zachos, Phys.Lett. B218, 203 (1989).