Stabilization of arbitrary structures in a doubly degenerate reaction-diffusion system modeling bacterial motion on a nutrient-poor agar

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Abstract

A no-flux initial-boundary value problem for the doubly degenerate parabolic system

\[
\begin{align*}
    u_t &= \nabla \cdot (uv \nabla u) + \ell uv, \\
    v_t &= \Delta v - uv,
\end{align*}
\]

is considered in a smoothly bounded convex domain \(\Omega \subset \mathbb{R}^n\), with \(n \geq 1\) and \(\ell \geq 0\). The first of the main results asserts that for nonnegative initial data \((u_0, v_0) \in (L^\infty(\Omega))^2\) with \(u_0 \neq 0\), \(v_0 \neq 0\) and \(\sqrt{v_0} \in W^{1,2}(\Omega)\), there exists a global weak solution \((u, v)\) which, inter alia, belongs to \(C^0(\Omega \times (0, \infty)) \times C^{2,1}(\Omega \times (0, \infty))\) and satisfies \(\sup_{t>0} \|u(\cdot, t)\|_{L^p(\Omega)} < \infty\) for all \(p \in [1, p_0)\) with \(p_0 := \frac{n}{(n-2)+}\). It is next seen that for each of these solutions one can find \(u_\infty \in \bigcap_{p \in [1, p_0)} L^p(\Omega)\) such that, within an appropriate topological setting, \((u(\cdot, t), v(\cdot, t))\) approaches the equilibrium \((u_\infty, 0)\) in the large time limit. Finally, in the case \(n \leq 5\) a result ensuring a certain stability property of any member in the uncountably large family of steady states \((u_0, 0)\), with arbitrary and suitably regular \(u_0 : \Omega \to [0, \infty)\), is derived. This provides some rigorous evidence for the appropriateness of (⋆) to model the emergence of a strikingly large variety of stable structures observed in experiments on bacterial motion in nutrient-poor environments. Essential parts of the analysis rely on the use of an apparently novel class of functional inequalities to suitably cope with the doubly degenerate diffusion mechanism in (⋆).

Mathematics subject classification

35B36 (primary) · 35B40, 35K65, 35K59, 92C17 (secondary)
1 Introduction

The question to which extent simple reaction-diffusion models can describe essential aspects of structure evolution in nature, and especially in living systems, has been stimulating considerable parts of the literature on parabolic problems over the past decades. Within the realm of bounded solutions, a dissipation-induced trend toward asymptotic equilibration has rigorously been confirmed to predominate in large classes of such systems ([21], [22], [12], [14], [2] [9]). In line with this, substantial efforts have been focused on the identification of circumstances ensuring existence and favorable stability features of suitably structured steady states. Descriptions of structure-supporting properties on the basis of parabolic systems that exclusively involve reasonably regular ingredients seem accordingly constrained by natural limitations on richness of respectively relevant equilibria, despite a considerable collection of exceptions documented in the literature which assert the occurrence of impressively subtle stationary profiles especially near critical parameter settings (cf. [7], [8], [6], [4], [28], [24], [5] and also [25] for a very incomplete selection of findings in this regard).

The present study now addresses a modeling context which, despite a remarkable simpleness, seems characterized by an exceptionally large variety of evolutionary target states. Specifically, experimental findings reported in [11], [10] and [23] illustrate noticeably intricate facets in the collective behavior of populations of *Bacillus subtilis* when exposed to nutrient-poor environments; in particular, evolution into complex patterns, and even snowflake-like population distributions, appears as a generic feature rather than an exceptional event in such frameworks. Following experimentally gained indications for certain limitations of bacterial motility near regions of small nutrient concentrations, as a mathematical description for such processes the authors in [15] (cf. also [19] and [26]) propose the parabolic system

\[
\begin{align*}
 u_t &= D \nabla \cdot (uv \nabla u) + \ell uv, \\
 v_t &= d \Delta v - \partial uv,
\end{align*}
\]  

for the population density \( u = u(x, t) \) and the food resource distribution \( v = v(x, t) \). Forming the apparently most prominent ingredient herein, the diffusion operator in the first equation does not only degenerate in a standard porous medium-like manner at small values of \( u \), but moreover accounts for said motility restrictions by containing a second degeneracy that emerges as soon as \( v \) approaches the level zero, in view of the second equation clearly expected to dominate at least on large time scales.

This latter peculiarity evidently brings about significant challenges beyond those mastered by classical approaches to degenerate diffusion problems of porous medium type ([31], [2]), and also beyond those going along with the combination of two diffusion degeneracies due to the inclusion of diffusion rates that vanish both at small densities and at small gradient sizes of the diffusing quantity. In contrast to the situation in the latter class of doubly degenerate diffusion problems, meanwhile also fairly well-explored ([13], [16], [29], [30]), especially the presence of a *cross-degeneracy* marks a considerable distinctiveness in (1.1). Accordingly, already at the stage of questions from basic existence theory the available knowledge so far seems limited to a statement on global solvability in a one-dimensional version ([33]); built on a certain rather fragile energy structure, however, the corresponding arguments in [33] quite strongly rely not only on the presence of an additional cross-diffusive summand in the respective first equation and on strict positivity of \( \ell \), but beyond this also on the integrability assumption \( \int_\Omega \ln u(\cdot, 0) > -\infty \) which especially rules out an analysis of solutions emanating from compactly supported initial data.
Main results. The present manuscript now attempts to develop an approach which is not only capable of launching a fundamental theory of general nonnegative solutions for the simple system (1.1) in multi-dimensional settings, but which moreover also allows for conclusions addressing qualitative questions in such cases, in particular with regard to aspects related to stabilization of structures. For convenience setting $D = d = \vartheta = 1$ throughout the sequel, but keeping the parameter $\ell$ as an arbitrary nonnegative parameter in order to include the proliferation-free case $\ell = 0$, we shall subsequently examine this in the framework of the initial-boundary value problem

$$\begin{cases}
  u_t = \nabla \cdot (uv \nabla u) + \ell uv, & x \in \Omega, \ t > 0, \\
  v_t = \Delta v - uv, & x \in \Omega, \ t > 0, \\
  uv \frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega,
\end{cases}$$

(1.2)

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, with $n \geq 1$, and with given initial data such that

$$\begin{cases}
  u_0 \in L^\infty(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0, \\
  v_0 \in L^\infty(\Omega) \text{ is nonnegative and such that } v_0 \not\equiv 0 \text{ and } \sqrt{v_0} \in W^{1,2}(\Omega).
\end{cases}$$

(1.3)

It is fairly evident that in this context, any expedient analysis of (1.2) needs to appropriately quantify the potentially weakened diffusive action therein. In our first step toward this, by merely relying on basic $L^1$ bounds for both components we shall use evolution properties of the sublinear functionals $\int_{\Omega} u^q$ for $q \in (0, 1)$ to gain a priori estimates for expressions of the form $\int_0^T \int_{\Omega} u^{q-1} v |\nabla u|^2$ within this range of $q$ (Lemma 2.3), which will in turn imply bounds, inter alia, for

$$\int_0^T \int_{\Omega} \frac{u}{v} |\nabla v|^2$$

(Lemma 2.4). The core part of our analysis will then be concerned with the derivation of higher $L^p$ bounds for $u$, where in the course of standard testing procedures a key requirement will consist in suitably exploiting the respective diffusion-induced contributions, as quantified through weighted expressions of type

$$\int_{\Omega} u^{p-1} v |\nabla u|^2.$$ 

(1.5)

This will be achieved in Lemma 4.3, on the basis of the bounds known for the integral in (1.4), by means of two functional inequalities of the form

$$\left\| \nabla \left( \varphi^\alpha \sqrt{\psi} \right) \right\|_{L^{\frac{2p^*}{n+2m-t}}(\Omega)} \leq C(K) \cdot \left\{ \int_{\Omega} \varphi^{q-1} \psi |\nabla \varphi|^2 + \int_{\Omega} \frac{\varphi}{\psi} |\nabla \psi|^2 \right\}$$

(1.6)

and

$$\int_{\Omega} \varphi^p \psi \leq C(K) \cdot \left\{ \int_{\Omega} \varphi^{q-1} \psi |\nabla \varphi|^2 + \int_{\Omega} \frac{\varphi}{\psi} |\nabla \psi|^2 + \int_{\Omega} \varphi \psi \right\},$$

(1.7)

valid with some $C(K) > 0$ for arbitrary positive $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$ fulfilling

$$\int_{\Omega} \varphi^{p^*} \leq K,$$

throughout certain ranges of the parameters $\alpha, p, q$ and $p^*$ (Lemma 4.1 and Lemma 4.2).

Within the framework of an appropriately regularized variant of (1.2), the a priori information
thereby generated will firstly enable us to make sure that for any fixed initial data fulfilling (1.3), the problem (1.2) is solvable by functions inter alia belonging to $C^0(\Omega \times (0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$ and enjoying some time-independent bounds in $L^p(\Omega) \times L^\infty(\Omega)$:

**Theorem 1.1** Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, and suppose that $\ell \geq 0$ and that (1.3) holds. Then there exist functions

$$
\begin{cases}
    u \in L^\infty_{loc}(\overline{\Omega} \times [0, \infty)) \cap C^0(\overline{\Omega} \times (0, \infty)) \\
    v \in L^\infty(\Omega \times (0, \infty)) \cap L^4_{loc}((0, \infty); W^{1,4}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))
\end{cases}
$$

such that $u \geq 0$ and $v > 0$ in $\overline{\Omega} \times (0, \infty)$, and that $(u, v)$ forms a global weak solution of (1.2) in the sense of Definition 2.1 below. Moreover, this solution has the additional boundedness property that

$$
\sup_{t > 0} \|u(\cdot, t)\|_{L^p(\Omega)} < \infty \quad \text{for all } p \in \left[1, \frac{n}{(n - 2)_+}\right).
$$

By suitably taking into account respective dependences on time, our estimates gained in the context of (1.4), (1.5), (1.6) and (1.7) will, besides implying the above, already pave the way for our subsequent asymptotic analysis. By means of a duality-based argument, namely, the latter can be seen to ensure a decay feature of the form

$$
\int_0^\infty \|u_t(\cdot, t)\|_{(W^{1,\infty}(\Omega))^*}dt \leq C \cdot \left\{ \int_\Omega v_0 \right\}^\lambda,
$$

with $C = C(u_0, v_0)$ and some $\lambda > 0$, and with $(W^{1,\infty}(\Omega))^*$ denoting the dual space of $W^{1,\infty}(\Omega)$ (Lemma 5.1), and to thereby constitute the essential step toward our verification of the following statement on large time convergence of each individual among the trajectories obtained in Theorem 1.1.

**Theorem 1.2** Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, let $\ell \geq 0$, and suppose that (1.3) holds. Then there exists a nonnegative $u_\infty \in \bigcap_{p \in [1, \frac{n}{(n - 2)_+})} L^p(\Omega)$ such that as $t \to \infty$, the solution $(u, v)$ of (1.2) from Theorem 1.1 satisfies

$$
v(\cdot, t) \to 0 \begin{cases}
    \text{in } L^\infty(\Omega) \\
    \text{in } L^p(\Omega)
\end{cases}
\quad \text{for all } p \in [1, \infty) \text{ if } n \geq 4,
$$

and with some $\gamma \in (0, 1)$, as $t \to \infty$ we have

$$
\begin{cases}
    u(\cdot, t) \to u_\infty \text{ in } L^p(\Omega) \quad \text{for all } p \in \left[1, \frac{n}{(n - 2)_+}\right) \text{ if } n \leq 5, \\
    u^\gamma(\cdot, t) \to u_\infty^\gamma \text{ in } L^\frac{p}{\gamma}(\Omega) \quad \text{for all } p \in \left[1, \frac{n}{(n - 2)_+}\right) \text{ if } n \geq 6.
\end{cases}
$$

Finally, the third of our main results provides some rigorous analytical evidence for the suitability of (1.1) in the considered modeling context: Namely, by adequately tracing the dependence of the constant $C = C(u_0, v_0)$ in (1.10) on the initial data we can identify the following stability property enjoyed by actually any member of the uncountable family of function pairs $(u_0, 0)$, which for all suitably regular $u_0$ indeed form steady states of (1.2):

**Theorem 1.3** Suppose that $n \leq 5$, and that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain with smooth boundary, and that $\ell \geq 0$ and $K > 0$. Then for each $\eta > 0$, there exists $\delta = \delta(\eta, K) > 0$ with the property that whenever (1.3) holds with

$$
\|u_0\|_{L^\infty(\Omega)} \leq K, \quad \|v_0\|_{L^\infty(\Omega)} \leq K \quad \text{and} \quad \int_\Omega \frac{|\nabla v_0|^2}{v_0} \leq K
$$

(1.13)
as well as
\[ \int_{\Omega} v_0 \leq \delta, \quad (1.14) \]
the solution \((u, v)\) of \((1.2)\) obtained in Theorem 1.1 satisfies
\[ \|v(\cdot, t) - v_0\|_{L^1(\Omega)} + \|u(\cdot, t) - u_0\|_{(W^{1,\infty}(\Omega))^*} \leq \eta \quad \text{for all } t > 0. \quad (1.15) \]
In particular, for the corresponding limit function \(u_\infty\) from Theorem 1.2 we then have
\[ \|u_\infty - u_0\|_{(W^{1,\infty}(\Omega))^*} \leq \eta. \quad (1.16) \]

### 2 Approximation by positive classical solutions

The following solution concept to be pursued below seems to generalize notions of classical solvability in a manner fairly natural in contexts in which due to the considered diffusion degeneracy, no substantial first order information on the solution component \(u\) is expected.

**Definition 2.1** Let \(n \geq 1\) and \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary, let \(\ell \geq 0\), and assume that \(u_0 \in L^1(\Omega)\) and \(v_0 \in L^1(\Omega)\) are nonnegative. Then a pair of nonnegative functions
\[
\begin{cases}
  u \in L^2_{loc}(\bar{\Omega} \times [0, \infty)) \quad \text{and} \\
  v \in L^\infty(\Omega \times (0, \infty)) \cap L^1_{loc}([0, \infty); W^{1,1}(\Omega))
\end{cases} \quad (2.1)
\]
such that
\[ u^2 \nabla v \in L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^n) \quad (2.2) \]
will be called a global weak solution of \((1.2)\) if
\[ -\int_0^\infty \int_{\Omega} u \varphi_t - \int_{\Omega} u_0 \varphi(\cdot, 0) = \frac{1}{2} \int_0^\infty \int_{\Omega} u^2 \nabla v \cdot \nabla \varphi + \frac{1}{2} \int_0^\infty \int_{\Omega} u^2 v \nabla \varphi + \int_0^\infty \int_{\Omega} \ell u v \varphi \quad (2.3) \]
for all \(\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))\) fulfilling \(\frac{\partial \varphi}{\partial v} = 0\) on \(\partial \Omega \times (0, \infty)\), and if
\[ \int_0^\infty \int_{\Omega} v \varphi_t + \int_{\Omega} v_0 \varphi(\cdot, 0) = \int_0^\infty \int_{\Omega} \nabla v \cdot \nabla \varphi + \int_0^\infty \int_{\Omega} u v \varphi \quad (2.4) \]
for any \(\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))\).

In order to construct such solutions through some convenient approximation by classical solutions also in the presence of possibly discontinuous initial data, given \(u_0\) and \(v_0\) merely fulfilling \((1.3)\) we first fix families \((u_{0\varepsilon})_{\varepsilon \in (0, 1)} \subset W^{1,\infty}(\Omega)\) and \((v_{0\varepsilon})_{\varepsilon \in (0, 1)} \subset W^{1,\infty}(\Omega)\) in such a way that
\[
\begin{cases}
  0 < u_{0\varepsilon} \leq \|u_0\|_{L^\infty(\Omega)} + 1 \quad \text{in } \bar{\Omega} \quad \text{for all } \varepsilon \in (0, 1), \\
  \frac{\varepsilon}{2} \leq v_{0\varepsilon} \leq \|v_0\|_{L^\infty(\Omega)} + \varepsilon \quad \text{in } \bar{\Omega} \quad \text{for all } \varepsilon \in (0, 1), \\
  \int_{\Omega} \frac{|\nabla u_{0\varepsilon}|^2}{v_{0\varepsilon}} \leq \int_{\Omega} \frac{|\nabla v_0|^2}{v_0} + 1 \quad \text{for all } \varepsilon \in (0, 1), \\
  u_{0\varepsilon} \to u_0 \quad \text{and} \quad v_{0\varepsilon} \to v_0 \quad \text{a.e. in } \Omega \quad \text{as } \varepsilon \searrow 0.
\end{cases} \quad (2.5)
\]
According to standard parabolic theory ([1]), the regularity and positivity assumptions on $u$ and $v$ and assume (1.3) and (2.5). Then for each $\varepsilon > 0$, allow for the following statement on global classical solvability by functions enjoying some rather expected basic features formally associated with (1.2):

**Lemma 2.2** Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $\ell \geq 0$, and assume (1.3) and (2.5). Then for each $\varepsilon \in (0, 1)$, there exist functions

\[
\begin{align*}
  u_\varepsilon & \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{and} \\
  v_\varepsilon & \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))
\end{align*}
\]

such that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times [0, \infty)$, and that $(u_\varepsilon, v_\varepsilon)$ solves (2.6) in the classical sense. This solution satisfies

\[
\int_\Omega u_0 \leq \int_\Omega u_\varepsilon(\cdot, t) \leq \int_\Omega u_0 + \ell \int_\Omega v_\varepsilon \quad \text{for all } t > 0
\]

and

\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \cdot e^{\ell \int_0^t v_\varepsilon} \quad \text{for all } t > 0
\]

and

\[
\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|v_\varepsilon(\cdot, t_0)\|_{L^\infty(\Omega)} \quad \text{for all } t_0 \geq 0 \text{ and any } t > t_0
\]

as well as

\[
\int_0^\infty \int_\Omega u_\varepsilon v_\varepsilon \leq \int_\Omega v_0.
\]

**Proof** According to standard parabolic theory ([1]), the regularity and positivity assumptions on $u_\varepsilon$ and $v_\varepsilon$ in (2.5) guarantee the existence of $T_{\max, \varepsilon} \in (0, \infty]$ and positive functions $u_\varepsilon$ and $v_\varepsilon$ from $C^0(\overline{\Omega} \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max, \varepsilon}))$ such that (2.6) is satisfied in $\Omega \times (0, T_{\max, \varepsilon})$, and that

\[
\text{either } T_{\max, \varepsilon} = \infty, \quad \text{or}
\]

\[
\limsup_{t \to T_{\max, \varepsilon}} \left\{ \|u_\varepsilon(\cdot, t)\|_{C^2(\overline{\Omega})} + \|v_\varepsilon(\cdot, t)\|_{C^2(\overline{\Omega})} + \frac{1}{\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}} + \frac{1}{\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}} \right\} = \infty.
\]

(2.12)

By nonnegativity of $u_\varepsilon v_\varepsilon$, the comparison principle ensures that this solution satisfies

\[
u_\varepsilon(x, t) \geq c_1(\varepsilon) := \min_{y \in \overline{\Omega}} u_0(y) \quad \text{for all } (x, t) \in \Omega \times (0, T_{\max, \varepsilon})
\]

and

\[
v_\varepsilon(x, t) \leq \|v_\varepsilon(\cdot, t_0)\|_{L^\infty(\Omega)} \leq c_2(\varepsilon) := \|v_0\|_{L^\infty(\Omega)} \quad \text{for all } t_0 \in [0, T_{\max, \varepsilon}) \text{ and any } (x, t) \in \Omega \times (t_0, T_{\max, \varepsilon}).
\]

(2.14)
where the latter especially asserts that
\[ u_{\varepsilon t} \leq \nabla \cdot (u_{\varepsilon} v_{\varepsilon} \nabla u_{\varepsilon}) + c_2(\varepsilon) \ell u_{\varepsilon} \quad \text{in } \Omega \times (0, T_{\text{max,}\varepsilon}). \]

Another comparison argument therefore shows that writing \( c_3(\varepsilon) := ||u_{0\varepsilon}||_{L^\infty(\Omega)} \), we have
\[ u_{\varepsilon}(x, t) \leq c_3(\varepsilon) e^{c_2(\varepsilon) t} \quad \text{for all } (x, t) \in \Omega \times (0, T_{\text{max,}\varepsilon}), \quad (2.15) \]
and that hence, in particular,
\[ v_{\varepsilon t} \geq \Delta v_{\varepsilon} - c_3(\varepsilon) e^{c_2(\varepsilon) t} v_{\varepsilon} \quad \text{in } \Omega \times (0, T_{\text{max,}\varepsilon}). \]

Again by comparison, we hence obtain that if we let \( y_{\varepsilon} \in C^1([0, \infty)) \) denote the solution of \( y_{\varepsilon}'(t) = -c_3(\varepsilon) e^{c_2(\varepsilon) t} y_{\varepsilon}(t), t > 0 \), with \( y_{\varepsilon}(0) = \min_{x \in \Omega} v_{0\varepsilon}(x) > 0 \), then
\[ v_{\varepsilon}(x, t) \geq y_{\varepsilon}(t) \quad \text{for all } (x, t) \in \Omega \times (0, T_{\text{max,}\varepsilon}). \quad (2.16) \]

Now assuming \( T_{\text{max,}\varepsilon} \) to be finite for some \( \varepsilon \in (0, 1) \), we could combine (2.14) with (2.15) to infer that \( u_{\varepsilon} v_{\varepsilon} \) would then be bounded in \( \Omega \times (0, T_{\text{max,}\varepsilon}) \), in view of standard parabolic regularity theory ([17]) applied to the second equation in (2.6) implying that \( v_{\varepsilon} \) would belong to \( C^{1+\theta_1,1+\theta_1}(\Omega \times [T_{\text{max,}\varepsilon}, T_{\text{max,}\varepsilon}])) \) for some \( \theta_1 = \theta_1(\varepsilon) \in (0, 1) \). Since \( u_{\varepsilon} v_{\varepsilon} \) would then also be uniformly bounded from below by a positive constant in \( \Omega \times (0, T_{\text{max,}\varepsilon}) \) by (2.13) and (2.16), and since thus the first equation in (2.6) would actually be uniformly parabolic in \( \Omega \times (0, T_{\text{max,}\varepsilon}) \), we could employ the same token once again to infer from the first equation in (2.6) that \( u_{\varepsilon} \in C^{1+\theta_2,1+\theta_2}(\Omega \times [T_{\text{max,}\varepsilon}, T_{\text{max,}\varepsilon}]) \) for some \( \theta_2 = \theta_2(\varepsilon) \in (0, 1) \). As thus \( u_{\varepsilon} v_{\varepsilon} \) would in fact lie in \( C^{\theta_3,\theta_3}(\Omega \times [T_{\text{max,}\varepsilon}, T_{\text{max,}\varepsilon}]) \) of both equations in (2.6) we could then invoke parabolic Schauder theory ([17]) to obtain \( \theta_4 = \theta_4(\varepsilon) \in (0, 1) \) such that \( \{u_{\varepsilon}, v_{\varepsilon}\} \subset C^{2+\theta_4,1+\theta_4}(\Omega \times [T_{\text{max,}\varepsilon}, T_{\text{max,}\varepsilon}]) \), and that hence, in particular,
\[ \sup_{t \in (T_{\text{max,}\varepsilon}^{\frac{1}{2}}, T_{\text{max,}\varepsilon})} \left\{ ||u_{\varepsilon}(\cdot, t)||_{C^2(\Omega)} + ||v_{\varepsilon}(\cdot, t)||_{C^2(\Omega)} \right\} < \infty. \]

Together with (2.13) and (2.16), in view of (2.12) this would contradict the hypothesis that \( T_{\text{max,}\varepsilon} \) be finite. In consequence, (2.9) and (2.10) immediately result from (2.15) and (2.14), while (2.8) and (2.11) directly follow upon integrating in (2.6).

Throughout the remaining part of this manuscript, unless otherwise stated we shall tacitly assume that \( \Omega \subset \mathbb{R}^n \) is a bounded convex domain with smooth boundary, and that \( \ell \geq 0 \), and whenever \((u_0, v_0)\) and families \((u_{\varepsilon 0}, v_{\varepsilon 0})_{\varepsilon \in (0, 1)}\) fulfilling (1.3) and (2.5) have been fixed, by \( T_{\text{max,}\varepsilon} \) and \((u_{\varepsilon}, v_{\varepsilon})\) we shall exclusively mean the objects introduced in Lemma 2.2.

### 2.1 Fundamental first-order estimates. Basic decay properties of \( v \)

The following simple observation forms a fundament for the main part of our first step toward deducing further regularity properties of the solutions gained above, to be accomplished in Lemma 2.4 below on the basis of the estimate in (2.17). Apart from that, (2.17) will later on be utilized as a basic piece of information in our derivation of \( L^p \) bounds for \( u_{\varepsilon} \) (Lemma 4.3), and (2.18) as well as (2.19) will prove useful in the course of our large time analysis in Lemma 5.5 and Lemma 5.7.

\[ \square \]
**Lemma 2.3** Let \( n \geq 1 \), assume (1.3) and (2.5), and let \( q \in (0,1) \). Then

\[
\int_0^\infty \int_\Omega u^q \varepsilon v_\varepsilon |\nabla u_\varepsilon|^2 \leq \frac{|\Omega|^{1-q}}{q(1-q)} \cdot \left\{ \int_\Omega u_0 \varepsilon + \ell \int_\Omega v_0 \varepsilon \right\}^q \quad \text{for all } \varepsilon \in (0,1) \tag{2.17}
\]

and

\[
\ell \int_0^\infty \int_\Omega u^q \varepsilon v_\varepsilon \leq \frac{|\Omega|^{1-q}}{q} \cdot \left\{ \int_\Omega u_0 \varepsilon + \ell \int_\Omega v_0 \varepsilon \right\}^q \quad \text{for all } \varepsilon \in (0,1), \tag{2.18}
\]

and furthermore we have

\[
\int_\Omega u^q (\cdot, t) \geq \int_\Omega u^q_0 \varepsilon \quad \text{for all } t > 0 \text{ and any } \varepsilon \in (0,1). \tag{2.19}
\]

**Proof** Let \( \varepsilon \in (0,1) \). Then using that \( u_\varepsilon \) is positive in \( \Omega \times (0,\infty) \), we may test the first equation in (2.6) against \( u^q \varepsilon \) to see that

\[
\frac{1}{q} \frac{d}{dt} \int_\Omega u^q \varepsilon = (1-q) \int_\Omega u^{q-1} \varepsilon |\nabla u_\varepsilon|^2 + \ell \int_\Omega u^q v_\varepsilon \quad \text{for all } t > 0, \tag{2.20}
\]

which firstly implies (2.19) as an immediate consequence of the restriction \( q \in (0,1) \). After an integration in time, we moreover infer from (2.20) that thanks to the Hölder inequality,

\[
(1-q) \int_0^t \int_\Omega u^{q-1} \varepsilon |\nabla u_\varepsilon|^2 + \ell \int_\Omega u^q v_\varepsilon = \frac{1}{q} \int_\Omega u^q (\cdot, t) - \frac{1}{q} \int_\Omega u^q_0 \varepsilon \\
\leq \frac{1}{q} \int_\Omega u^q (\cdot, t) \\
\leq \frac{|\Omega|^{1-q}}{q} \cdot \left\{ \int_\Omega u^q (\cdot, t) \right\}^q \quad \text{for all } t > 0.
\]

As \( \int_\Omega u^q \varepsilon \leq \int_\Omega u_0 \varepsilon + \ell \int_\Omega v_\varepsilon \) for all \( t > 0 \) by (2.8), this establishes both (2.17) and (2.18). \( \Box \)

Now in the course of a standard testing procedure performed on the second equation in (2.6), an application of (2.17) to \( q := \frac{1}{2} \) enables us to control the respective interaction-driven contribution by means of conveniently decaying quantities. In particular, the following conclusion can therefore be drawn in such a way that besides providing first order regularity features, due to its temporally global nature it furthermore already includes some information on temporal decay in the second solution component; this latter aspect will be of essential importance to our argument asserting \( L^p \) bounds for \( u_\varepsilon \) (see Lemma 4.3). The following lemma is the only place in this manuscript in which convexity of \( \Omega \) is explicitly made use of.

**Lemma 2.4** Let \( n \geq 1 \). Then for all \( K > 0 \) there exists \( C(K) > 0 \) such that whenever (1.3), (2.5) and (1.13) hold, we have

\[
\int_0^\infty \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v^2_\varepsilon} \leq C(K) \quad \text{for all } \varepsilon \in (0,1) \tag{2.21}
\]

and

\[
\int_0^\infty \int_\Omega \frac{u_\varepsilon}{v^2_\varepsilon} |\nabla v_\varepsilon|^2 \leq C(K) \quad \text{for all } \varepsilon \in (0,1). \tag{2.22}
\]
Therefore, (2.23) implies that convexity of the standard parabolic regularity theory, on the basis of the second equation in (2.6) we compute
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} = \int_{\Omega} \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla v_{\varepsilon t} - \frac{1}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} v_{\varepsilon t} = \int_{\Omega} \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon - \int_{\Omega} \frac{1}{v_\varepsilon} \nabla v_\varepsilon \cdot \nabla (u_\varepsilon v_\varepsilon) - \frac{1}{2} \int_{\Omega} \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 \Delta v_\varepsilon + \frac{1}{2} \int_{\Omega} \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 \Delta v_\varepsilon - \frac{1}{2} \int_{\Omega} \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 - \int_{\Omega} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1).
\]

(2.23)

Here we may combine the outcome of a straightforward rearrangement ([32, Lemma 3.2]) with a known functional inequality ([32, Lemma 3.3]) to see for all \( \varphi \in C^3(\Omega) \) such that \( \varphi > 0 \) in \( \Omega \) and \( \frac{\partial \varphi}{\partial n} = 0 \) on \( \partial \Omega \), writing \( c_1 := \frac{1}{(2+\sqrt{n})} \) and using that \( \frac{\partial |\nabla \varphi|^2}{\partial n} \leq 0 \) on \( \partial \Omega \) by convexity of \( \Omega \) ([20]) we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{1}{\varphi^2} |\nabla \varphi|^2 \Delta \varphi + \frac{1}{2} \int_{\Omega} \frac{1}{\varphi^2} |\nabla \varphi|^2 \Delta \varphi = \int_{\Omega} \varphi |D^2 \ln \varphi|^2 - \frac{1}{2} \int_{\partial \Omega} \varphi \frac{\partial |\nabla \varphi|^2}{\partial n}
\geq \int_{\Omega} \varphi |D^2 \ln \varphi|^2 
\geq c_1 \int_{\Omega} \frac{|\nabla \varphi|^4}{\varphi^3}.
\]

Therefore, (2.23) implies that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{1}{\varphi^2} |\nabla \varphi|^2 + c_1 \int_{\Omega} \frac{|\nabla \varphi|^4}{\varphi^2} + \frac{1}{2} \int_{\Omega} \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 \leq - \int_{\Omega} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1).
\]

(2.24)

and to derive the claimed conclusion from this, we now assume that (1.13) holds with some \( K > 0 \), observing that then
\[
\int_{\Omega} u_{0\varepsilon} \leq (K + 1)|\Omega| \quad \text{and} \quad \int_{\Omega} v_{0\varepsilon} \leq (K + 1)|\Omega|
\]
as well as
\[
\int_{\Omega} \frac{|\nabla v_{0\varepsilon}|^2}{v_{0\varepsilon}} \leq K + 1
\]
(2.25)
due to (2.5). Therefore, an application of Lemma 2.3 to \( q := \frac{1}{2} \) shows that
\[
\int_{0}^{\infty} \int_{\Omega} \frac{v_\varepsilon}{\sqrt{u_\varepsilon}} |\nabla u_\varepsilon|^2 \leq c_2(K) := 4\sqrt{(1 + \ell)(K + 1)|\Omega|} \quad \text{for all } \varepsilon \in (0, 1),
\]
(2.26)
while (2.11) entails that
\[
\int_{0}^{\infty} \int_{\Omega} u_\varepsilon v_\varepsilon \leq c_3(K) := (K + 1)|\Omega| \quad \text{for all } \varepsilon \in (0, 1).
\]
(2.27)
Since by twice employing Young’s inequality we can estimate
\[-\int_{\Omega} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \leq \frac{1}{4} \int_{\Omega} \frac{v_\varepsilon}{\sqrt{u_\varepsilon}} |\nabla u_\varepsilon|^2 + \int_{\Omega} \frac{\sqrt{u_\varepsilon}}{v_\varepsilon} |\nabla v_\varepsilon|^2 \]
\[\leq \frac{1}{4} \int_{\Omega} \frac{v_\varepsilon}{\sqrt{u_\varepsilon}} |\nabla u_\varepsilon|^2 + \frac{c_1}{2} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^2} + \frac{1}{2c_1} \int_{\Omega} u_\varepsilon v_\varepsilon \]
for all $t > 0$ and $\varepsilon \in (0, 1)$,

upon an integration in (2.24) we thus infer that due to (2.25), (2.26) and (2.27),

\[\int_{\Omega} \frac{|\nabla v_\varepsilon(\cdot, t)|^2}{v_\varepsilon(\cdot, t)} + c_1 \int_{0}^{t} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^2} + \int_{0}^{t} \int_{\Omega} \frac{u_\varepsilon}{v_\varepsilon} |\nabla v_\varepsilon|^2 \]
\[\leq \int_{\Omega} \frac{|\nabla v_{0\varepsilon}|^2}{v_{0\varepsilon}} + \frac{1}{2} \int_{0}^{t} \int_{\Omega} \frac{v_\varepsilon}{\sqrt{u_\varepsilon}} |\nabla u_\varepsilon|^2 + \frac{1}{c_1} \int_{0}^{t} \int_{\Omega} u_\varepsilon v_\varepsilon \]
\[\leq K + 1 + \frac{c_2(K)}{2} + \frac{(K + 1)|\Omega|}{c_1} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),\]

and conclude as intended. \(\square\)

### 3 Global existence

Besides on the boundedness properties documented in (2.9), (2.10) and (2.21), our construction of a solution to (1.2) through a limit procedure in Lemma 3.2 will rely on some further information on significantly enhanced regularity within finite time intervals bounded away from the initial instant. Due to its temporally local nature, the following statement in this regard, based on a combination of comparison arguments and standard parabolic theories, can apparently not be used in our large time analysis below, and hence does not trace dependencies of the obtained constants on the size of the initial data.

**Lemma 3.1** Let $n \geq 1$, and assume (1.3) and (2.5). Then for all $\tau > 0$ and $T > \tau$, there exist $\theta = \theta(\tau, T; u_0, v_0) \in (0, 1)$ and $C(\tau, T) = C(\tau, T; u_0, v_0) > 0$ such that

\[\|v_\varepsilon\|_{C_t^{1+\theta}L_\Omega^\infty(\Omega \times (\tau, T))} \leq C(\tau, T) \quad \text{for all } \varepsilon \in (0, 1)\]  
(3.1)

and

\[\|u_\varepsilon\|_{C_t^{2+\theta,1+\theta}L_\Omega^\infty(\Omega \times (\tau, T))} \leq C(\tau, T) \quad \text{for all } \varepsilon \in (0, 1)\]  
(3.2)

as well as

\[v_\varepsilon(x, t) \geq C(\tau, T) \quad \text{for all } (x, t) \in \Omega \times (\tau, T) \text{ and any } \varepsilon \in (0, 1).\]  
(3.3)

**Proof** We fix $\tau > 0$ and $T > \tau$ and recall (2.9) to find $c_1(T) > 0$ such that

\[u_\varepsilon \leq c_1(T) \quad \text{in } \Omega \times (0, T) \quad \text{for all } \varepsilon \in (0, 1),\]

and that thus

\[v_{\varepsilon t} \geq \Delta v_\varepsilon - c_1(T)v_\varepsilon \quad \text{in } \Omega \times (0, T) \quad \text{for all } \varepsilon \in (0, 1).\]  
(3.4)

According to a comparison argument, this firstly implies that if we let $(e^{t\Delta})_{t \geq 0}$ denote the Neumann heat semigroup on $\Omega$, then

\[v_\varepsilon \left(\cdot, \frac{T}{4}\right) \geq e^{-c_1(T)\frac{T}{4}}e^{\frac{T}{4}}x_\varepsilon \Delta v_{0\varepsilon} \quad \text{in } \Omega \quad \text{for all } \varepsilon \in (0, 1),\]  
(3.5)
where observing that \( v_{0e} \rightarrow v_0 \) in \( L^1(\Omega) \) as \( \varepsilon \downarrow 0 \) due to (2.5) and the dominated convergence theorem, thanks to well-known regularization features of \( e^{t \Delta} \) we see that
\[
e^{t \Delta} v_{0e} \rightarrow e^{t \Delta} v_0 \quad \text{in } L^\infty(\Omega) \quad \text{as } \varepsilon \downarrow 0.
\]
Since our overall assumption \( v_0 \not\equiv 0 \) ensures that \( c_2(\tau) \equiv c_2(\tau; u_0, v_0) := e^{t \Delta} v_0 \) is positive throughout \( \Omega \) by the strong maximum principle, we can thus pick \( \varepsilon_1 = \varepsilon_1(\tau; u_0, v_0) \in (0, 1) \) in such a way that
\[
e^{t \Delta} v_{0e} \geq \frac{c_2(\tau)}{2} \quad \text{in } \Omega \quad \text{for all } \varepsilon \in (0, \varepsilon_1),
\]
so that, in particular,
\[
e^{t \Delta} v_{0e} \geq c_3(\tau) \equiv c_3(\tau; u_0, v_0) := \min \left\{ \frac{c_2(\tau)}{2}, \frac{\varepsilon_1}{2} \right\} \quad \text{in } \Omega \quad \text{for all } \varepsilon \in (0, 1), \quad (3.6)
\]
because by the comparison principle and (2.5), \( e^{t \Delta} v_{0e} \geq e^{t \Delta} (\frac{\varepsilon}{2}) = \frac{\varepsilon}{2} \) in \( \Omega \) for all \( \varepsilon \in (0, 1) \).

Now a second application of (3.4), again accompanied by a comparison argument, shows that for arbitrary \( t \in \left( \frac{\tau}{4}, T \right) \),
\[
v_e(\cdot, t) = e^{-c_1(T) - t} e^{t - \frac{\tau}{4} \Delta} v_e(\cdot, \frac{\tau}{4}) \quad \text{in } \Omega \quad \text{for all } \varepsilon \in (0, 1),
\]
so that combining (3.5) with (3.6) we find that for any such \( t \),
\[
v_e(\cdot, t) \geq e^{-c_1(T) - t} e^{t - \frac{\tau}{4} \Delta} \left\{ e^{-c_1(T) - t} c_3(\tau) \right\} \\
= c_3(\tau) e^{-c_1(T) - t} \\
\geq c_4(\tau, T) \equiv c_4(\tau, T; u_0, v_0) := c_3(\tau) e^{-c_1(T) - t} \quad \text{in } \Omega \quad \text{for all } \varepsilon \in (0, 1).
\]

Besides trivially implying (3.3), in conjunction with (2.9), (2.10) and (2.5) this shows that in the identity
\[
u_{el} = \nabla \cdot (a_e(x, t)u_e \nabla u_e) + \ell b_e(x, t), \quad (x, t) \in \Omega \times (0, \infty), \quad \varepsilon \in (0, 1),
\]
the functions \( a_e := v_e \) and \( b_e := u_e v_e \) satisfy
\[
c_4(\tau, T) \leq a_e \leq \|v_0\|_{L^\infty(\Omega)} + 1 \quad \text{in } \Omega \times \left( \frac{\tau}{4}, T \right) \quad \text{for all } \varepsilon \in (0, 1)
\]
and
\[
|b_e| \leq \left( \|v_0\|_{L^\infty(\Omega)} + 1 \right) \cdot e^{\varepsilon(\|v_0\|_{L^\infty(\Omega)} + 1) - T} \cdot \left( \|v_0\|_{L^\infty(\Omega)} + 1 \right) \quad \text{in } \Omega \times \left( \frac{\tau}{4}, T \right) \\
\quad \text{for all } \varepsilon \in (0, 1),
\]
whence a standard result on Hölder regularity in porous medium type scalar parabolic equations (27) becomes applicable so as to provide \( \theta_1 = \theta_1(\tau, T; u_0, v_0) \in (0, 1) \) and \( c_5(\tau, T) = c_5(\tau, T; u_0, v_0) > 0 \) such that
\[
\|u_e\|_{C^{\theta_1, \frac{\varepsilon_1}{2}}(\Omega \times \left[ \frac{\tau}{4}, T \right])} \leq c_5(\tau, T) \quad \text{for all } \varepsilon \in (0, 1). \quad (3.7)
\]
As quite a similar reasoning yields an analogous Hölder bound for the solutions \( v_e \) of \( v_{el} = \Delta v_e - b_e(x, t) \), this entails that with some \( \theta_2 = \theta_2(\tau, T; u_0, v_0) \in (0, 1) \) and \( c_6(\tau, T) = c_6(\tau, T; u_0, v_0) > 0 \) we moreover have
\[
\|b_e\|_{C^{\theta_2, \frac{\varepsilon_2}{2}}(\Omega \times \left[ \frac{\tau}{4}, T \right])} \leq c_6(\tau, T) \quad \text{for all } \varepsilon \in (0, 1).
\]
We may therefore draw on classical parabolic Schauder theory ([17]) to finally find \( \theta_3 = \theta_3(\tau, T; u_0, v_0) \in (0, 1) \) and \( c_7(\tau, T) = c_7(\tau, T; u_0, v_0) > 0 \) such that
\[
\| v_{\varepsilon} \|_{C^{2+\theta_3,1+\theta_1}(\Omega \times [\tau, T])} \leq c_7(\tau, T) \text{ for all } \varepsilon \in (0, 1),
\]
which precisely establishes (3.2), while (3.1) has already been contained in (3.7).

\[\square\]

A weak solution with the additional properties announced in (1.8) can now be obtained by means of a straightforward extraction using the Arzelà-Ascoli theorem.

**Lemma 3.2** Let \( n \geq 1 \), and assume (1.3) and (2.5). Then there exist \( (\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1) \) and functions \( u \) and \( v \) fulfilling (1.8) such that \( \varepsilon_j \downarrow 0 \) as \( j \to \infty \), that \( u \geq 0 \) and \( v > 0 \) in \( \Omega \times (0, \infty) \), that
\[
\begin{align*}
 u_{\varepsilon} &\to u \quad \text{in } L^p_{\text{loc}}(\Omega \times [0, \infty)) \text{ for all } p \geq 1 \text{ and in } C^0_{\text{loc}}(\Omega \times (0, \infty)), \\
v_{\varepsilon} &\to v \quad \text{in } L^p_{\text{loc}}(\Omega \times [0, \infty)) \text{ for all } p \geq 1 \text{ and in } C^{1,1}_{\text{loc}}(\Omega \times (0, \infty)),
\end{align*}
\]
and
\[
\begin{align*}
 \nabla v_{\varepsilon} &\to \nabla v \quad \text{in } L^4(\Omega \times (0, \infty))
\end{align*}
\]
as \( \varepsilon = \varepsilon_j \downarrow 0 \), and that \( (u, v) \) forms a global weak solution of (1.2) in the sense of Definition 2.1.

**Proof** According to the equicontinuity properties asserted by Lemma 3.1, the Arzelà-Ascoli theorem provides \( (\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1) \) as well as functions \( u \in C^0(\Omega \times (0, \infty)) \) and \( v \in C^{1,1}(\Omega \times (0, \infty)) \) such that \( \varepsilon_j \downarrow 0 \) as \( j \to \infty \), and that \( u_{\varepsilon} \to u \) in \( C^0_{\text{loc}}(\Omega \times (0, \infty)) \) and \( v_{\varepsilon} \to v \) in \( C^{1,1}_{\text{loc}}(\Omega \times (0, \infty)) \) as \( \varepsilon = \varepsilon_j \downarrow 0 \). Clearly, \( u \) is nonnegative, while strict positivity of \( v \) in \( \Omega \times (0, \infty) \) follows from (3.3). Apart from that, the estimates in (2.9) and (2.10) together with the Vitali convergence theorem imply that \( u_{\varepsilon} \to u \) and \( v_{\varepsilon} \to v \) in \( L^p_{\text{loc}}(\Omega \times [0, \infty)) \) for all \( p \geq 1 \) as \( \varepsilon = \varepsilon_j \downarrow 0 \), and that \( u \) belongs to \( L^\infty(\Omega \times [0, \infty)) \) and \( v \) lies in \( L^\infty(\Omega \times (0, \infty)) \). As (2.21) along with (2.10) ensures boundedness of \( (\nabla v_{\varepsilon})_{\varepsilon \in (0, 1)} \) in \( L^4(\Omega \times (0, \infty)) \), we readily obtain that also \( v \in L^4_{\text{loc}}((0, \infty); W^{1,4}(\Omega)) \), and that (3.10) holds as \( \varepsilon = \varepsilon_j \downarrow 0 \).

Now the regularity requirements in (2.1) and (2.2) clearly result from the properties in (1.8) just asserted, and a verification of (2.3) and (2.4) can be achieved on the basis of (2.6) in a straightforward manner, using that due to (3.8) and (3.10) we have
\[
\begin{align*}
 u_{\varepsilon}^2 \nabla v_{\varepsilon} &\to u^2 \nabla v \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)) \quad \text{as } \varepsilon = \varepsilon_j \downarrow 0,
\end{align*}
\]
and that by (3.8) and (3.9),
\[
\begin{align*}
 u_{\varepsilon}^2 v_{\varepsilon} &\to u^2 v \quad \text{and } u_{\varepsilon} v_{\varepsilon} \to u v \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty))
\end{align*}
\]
as \( \varepsilon = \varepsilon_j \downarrow 0 \). \( \square \)

**4 Degeneracy control via functional inequalities. \( L^p \) bounds for \( u \)**

**4.1 Two functional inequalities**

Up to this point, temporally global bounds which exclusively refer to \( u_{\varepsilon} \), without any presence of weight functions involving \( v_{\varepsilon} \) as an expectedly decaying quantity such as in (2.18),
seem limited to the $L^1$ boundedness feature in (2.8). A key step toward a more substantial description of the large time behavior in (1.2), and especially toward an exclusion of asymptotic Dirac-type mass accumulation, will now consist in an adequate control of the diffusion degeneracy in the first equation from (2.6), and particularly its part stemming from the presence of the factor $v_\epsilon$ therein. In the framework of standard $L^p$ testing procedures, to be performed in Lemma 4.3, this specifically amounts to appropriately estimating integrals of the form

$$\int_\Omega u_\epsilon^p v_\epsilon |\nabla u_\epsilon|^2$$  \hspace{1cm} (4.1)

from below, and to thereby control the temporal growth of $\int_\Omega u_\epsilon^p$, as potentially driven by the forcing term $\ell u_\epsilon v_\epsilon$ in (2.6).

Of crucial importance to our approach in this direction will be the following observation on how far expressions of the form in (4.1), when added to integrals of the type appearing in (2.22), dominate gradients of certain products involving $u_\epsilon$ and $v_\epsilon$.

**Lemma 4.1** Let $n \geq 1$, $\alpha > \frac{1}{2}$, $p_* \geq 1$ and $K > 0$. Then there exists $C(\alpha, p_*, K) > 0$ with the property that whenever $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$ are positive in $\Omega$ and such that

$$\int_\Omega \varphi^{p_*} \leq K,$$$\hspace{1cm} (4.2)$$

for any choice of $q \in [0, 2\alpha - 1]$ the inequality

$$\left\| \nabla \left( \varphi^\alpha \sqrt{\psi} \right) \right\|^2_{L^{\frac{2p_*}{p_* + 2\alpha - 1}}(\Omega)} \leq C(\alpha, p_*, K) \cdot \left\{ \int_\Omega \varphi^{q-1} \psi |\nabla \varphi|^2 + \int_\Omega \frac{\varphi^\alpha}{\psi} |\nabla \psi|^2 \right\}$$  \hspace{1cm} (4.3)

holds.

**Proof** We first observe that for any such $\varphi$ and $\psi$, the pointwise estimate

$$\left| \nabla \left( \varphi^\alpha \sqrt{\psi} \right) \right|^2_{L^{\frac{2p_*}{p_* + 2\alpha - 1}}} = \left| \varphi^{\alpha - 1} \sqrt{\psi} \nabla \varphi + \frac{1}{2} \frac{\varphi^\alpha}{\sqrt{\psi}} \nabla \psi \right|^2_{L^{\frac{2p_*}{p_* + 2\alpha - 1}}} \leq c_1 \varphi \frac{p_*}{p_* + 2\alpha - 1} \psi \frac{p_*}{p_* + 2\alpha - 1} |\nabla \varphi|_{L^{\frac{2p_*}{p_* + 2\alpha - 1}}} + \varphi \frac{p_*}{p_* + 2\alpha - 1} \psi \frac{p_*}{p_* + 2\alpha - 1} |\nabla \psi|_{L^{\frac{2p_*}{p_* + 2\alpha - 1}}}$$  \hspace{1cm} (4.4)

holds throughout $\overline{\Omega}$ with $c_1 \equiv c_1(\alpha, p_*) := (2\alpha) \frac{p_*}{p_* + 2\alpha - 1}$. Since $\frac{p_* + 2\alpha - 1}{p_*} > 1$ thanks to our assumption that $\alpha > \frac{1}{2}$, we may employ the H"older inequality and rely on (4.2) to control the integral of the second summand on the right of (4.4) according to

$$\int_\Omega \varphi \frac{p_*}{p_* + 2\alpha - 1} \psi \frac{p_*}{p_* + 2\alpha - 1} |\nabla \psi|_{L^{\frac{2p_*}{p_* + 2\alpha - 1}}} \frac{2\alpha - 1}{p_*} \frac{p_*}{p_* + 2\alpha - 1} > \int_\Omega \varphi \frac{p_*}{p_* + 2\alpha - 1} \frac{2\alpha - 1}{p_*} \frac{p_*}{p_* + 2\alpha - 1} \int_\Omega \frac{\varphi^\alpha}{\psi} |\nabla \psi|^2_{L^{\frac{2p_*}{p_* + 2\alpha - 1}}} \leq \left\{ \int_\Omega \varphi \frac{p_*}{p_* + 2\alpha - 1} |\nabla \psi|^2_{L^{\frac{2p_*}{p_* + 2\alpha - 1}}} \right\} \left\{ \int_\Omega \frac{\varphi^\alpha}{\psi} \frac{p_*}{p_* + 2\alpha - 1} \right\} \leq K \frac{2\alpha - 1}{p_*} \frac{p_*}{p_* + 2\alpha - 1} \cdot \left\{ \int_\Omega \varphi^\alpha \frac{p_*}{p_* + 2\alpha - 1} \right\} \frac{2\alpha - 1}{p_*} \frac{p_*}{p_* + 2\alpha - 1} \cdot \left\{ \int_\Omega \frac{\varphi^\alpha}{\psi} \frac{p_*}{p_* + 2\alpha - 1} \right\} \frac{2\alpha - 1}{p_*} \frac{p_*}{p_* + 2\alpha - 1}.$$  \hspace{1cm} (4.5)
Similarly,
\[
\int_\Omega \frac{2p_\alpha q - 1}{q + \alpha - 1} |\nabla \varphi|^\frac{q}{q + \alpha - 1} \leq \int_\Omega \left\{ \varphi^{-1} |\nabla \varphi|^2 \right\}^{\frac{p_\alpha}{q + \alpha - 1}} \cdot \varphi^{-\frac{(2\alpha - q - 1)}{2\alpha - 1}} \cdot \left\{ \int_\Omega \varphi^{-\frac{(2\alpha - q - 1)}{2\alpha - 1}} \right\}^{\frac{2\alpha - 1}{q + \alpha - 1}}
\]
(4.6)
where using that the restrictions \( q \geq 0 \) and \( q \leq 2\alpha - 1 \) warrant that \( 0 \leq \frac{p_\alpha(2\alpha - q - 1)}{2\alpha - 1} \leq p_*, \)
we may invoke Young’s inequality along with (4.2) to see that
\[
\int_\Omega \frac{p_\alpha(2\alpha - q - 1)}{2\alpha - 1} \leq \int_\Omega (p_* + 1) \leq K + |\Omega|.
\]
In conjunction with (4.6) and (4.5), this shows that (4.4) implies the inequality
\[
\left\| \nabla \left( \varphi^{\alpha} \psi \right) \right\|_{L^{\frac{p_\alpha p_* + 2\alpha - 1}{2\alpha - 1}}(\Omega)} \leq c_1 \cdot (K + |\Omega|)^{\frac{2\alpha - 1}{2\alpha - 1}} \cdot \left\{ \int_\Omega \varphi^{-\frac{(2\alpha - q - 1)}{2\alpha - 1}} \right\}^{\frac{p_\alpha}{q + \alpha - 1}}
\]
from which (4.3) immediately follows if we let
\[
C(\alpha, p_*, K) := 2 \cdot \frac{p_\alpha + 2\alpha - 1}{p_*} \cdot \max \left\{ c_1 \cdot \frac{p_\alpha + 2\alpha - 1}{p_*} \cdot (K + |\Omega|)^{\frac{2\alpha - 1}{2\alpha - 1}} \right\},
\]
for instance.

Combined with suitable Sobolev embedding properties, the latter entails the following class of interpolation inequalities appropriate for our purposes.

**Lemma 4.2** Let \( n \geq 1 \), \( p_* \geq 1 \) and
\[
p := \begin{cases} 
p_* + 1 & \text{if } n = 1, \\
\frac{2p_* + n}{n} & \text{if } n \geq 2.
\end{cases}
\]
(4.7)
Then for all \( K > 0 \) one can find \( C(p_*, K) > 0 \) such that for any \( q \in [0, \min\{p_*, \frac{2p_*}{n}\}] \) and for each \( \varphi \in C^1(\Omega) \) and \( \psi \in C^1(\bar{\Omega}) \) fulfilling \( \varphi > 0 \) and \( \psi > 0 \) in \( \bar{\Omega} \) as well as
\[
\int_\Omega \varphi^{p_*} \leq K,
\]
we have
\[
\int_\Omega \varphi^{p} \psi \leq C(p_*, K) \cdot \left\{ \int_\Omega \varphi^{-\frac{(2\alpha - q - 1)}{2\alpha - 1}} \right\}^{\frac{p_*}{q + \alpha - 1}} \cdot \left\{ \int_\Omega \varphi^{-\frac{(2\alpha - q - 1)}{2\alpha - 1}} \right\}^{\frac{p_*}{q + \alpha - 1}}.
\]
(4.9)

**Proof** We first consider the case \( n \geq 2 \), in which \( W^{1, \frac{2n}{n+2}}(\Omega) \) is continuously embedded into \( L^2(\Omega) \), so that we can pick \( c_1 > 0 \) such that
\[
\int_\Omega \rho^2 \leq c_1 \|\nabla \rho\|^2_{L^{\frac{2n}{n+2}}(\Omega)} + c_1 \|\rho\|^2_{L^{\frac{2n}{n+2}}(\Omega)} \quad \text{for all } \rho \in W^{1, \frac{2n}{n+2}}(\Omega).
\]
(4.10)
We may then apply Lemma 4.1 to \( \alpha := \frac{2p_* + n}{2n} > \frac{1}{2} \) to infer the existence of \( c_2(p_*, K) > 0 \) such that whenever \( \varphi \) and \( \psi \) are positive functions from \( C^1(\bar{\Omega}) \) which satisfy (4.8), for any...
choice of \( q \in [0, 2\alpha - 1] \) we have

\[
\left\| \nabla \left( \varphi \frac{2^{p_\star n + n}}{2^{n}} \sqrt{\psi} \right) \right\|_{L^2(\Omega)}^2 \leq c_2(p_\star, K) \cdot \left\{ \int_{\Omega} \varphi^{q-1} \left| \nabla \varphi \right|^2 + \int_{\Omega} \frac{\varphi}{\sqrt{\psi}} \left| \nabla \psi \right|^2 \right\}.
\] (4.11)

Apart from that, we note that as a consequence of the Hölder inequality, any such pair \((\varphi, \psi)\) satisfies

\[
\left\| \varphi \frac{2^{p_\star n + n}}{2^{n}} \sqrt{\psi} \right\|_{L^2(\Omega)}^2 = \left\{ \int_{\Omega} \varphi^{\frac{n}{n+2}} \psi \frac{n}{n+2} \int_{\Omega} \varphi^{\frac{n}{n+2}} \right\}^{\frac{n+2}{n}} \leq \left\{ \int_{\Omega} \varphi \psi \right\} \cdot \left\{ \int_{\Omega} \varphi^{p_\star} \right\} \leq K^{\frac{2}{n}} \int_{\Omega} \varphi \psi
\] (4.12)

because of (4.8). Together with (4.11) and (4.10), this already establishes (4.9) with \( C(p_\star, K) := c_1 \cdot \max \left\{ c_2(p_\star, K), K^{\frac{2}{n}} \right\} \) in this case, for the restriction \( q \in [0, \frac{2p_\star}{n}] \) clearly warrants that \( q \leq 2\alpha - 1 \).

If \( n = 1 \), we proceed quite similarly, replacing (4.10) with the inequality

\[
\int_{\Omega} \rho^2 \leq c_3 \| \rho \|_{L^1(\Omega)}^2 + c_3 \| \rho \|_{L^1(\Omega)},
\]
valid for some \( c_3 > 0 \) and all \( \rho \in W^{1,1}(\Omega) \), substituting (4.12) by the observation that for all positive \( \varphi \in C^1(\Omega) \) and \( \psi \in C^1(\Omega) \) we have

\[
\left\| \varphi \frac{n+1}{2} \sqrt{\psi} \right\|_{L^1(\Omega)}^2 = \left\{ \int_{\Omega} \sqrt{\varphi \psi} \cdot \varphi \frac{n+1}{2} \right\}^2 \leq \left\{ \int_{\Omega} \varphi \psi \right\} \cdot \left\{ \int_{\Omega} \varphi^{p_\star} \right\}
\]

and applying Lemma 4.1 to \( \alpha := \frac{p_\star + 1}{2} \).

\( \square \)

### 4.2 Global \( L^p \) estimates for \( u \). Proof of theorem 1.1

Having Lemma 4.2 at hand, we are now prepared for our derivation of the following statement on \( L^p \) boundedness of \( u_\varepsilon \), yet conditional in presupposing the existence of corresponding bounds in \( L^{p_\star} \) with some \( p_\star \geq 1 \), by means of the announced testing-based argument.

**Lemma 4.3** Let \( n \geq 1 \), \( p_\star \geq 1 \) and \( K > 0 \), and let \( p \) be as defined in Lemma 4.2. Then there exists \( C(p_\star, K) > 0 \) such that if (1.3), (2.5) and (1.13) hold, and if

\[
\int_{\Omega} u_\varepsilon^{p_\star}(\cdot, t) \leq K \quad \text{for all} \ t > 0 \ \text{and} \ \varepsilon \in (0, 1),
\] (4.13)

then

\[
\int_{\Omega} u_\varepsilon^p(\cdot, t) \leq C(p_\star, K) \quad \text{for all} \ t > 0 \ \text{and} \ \varepsilon \in (0, 1)
\] (4.14)

and

\[
\int_0^\infty \int_{\Omega} u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 \leq C(p_\star, K) \quad \text{for all} \ \varepsilon \in (0, 1)
\] (4.15)
as well as
\[
\int_0^\infty \int_\Omega u^p_\varepsilon v_\varepsilon \leq C(p_*, K) \quad \text{for all } \varepsilon \in (0, 1).
\] (4.16)

**Proof** We fix any \( q \in (0, 1) \) such that \( q \leq \min\{p_*, \frac{2p_*}{n}\} \), and combine Lemma 2.3 with Lemma 2.4 and (2.11) to see that thanks to the hypothesis (1.13) we can find \( c_1(K) > 0 \) fulfilling
\[
\int_0^\infty \int_\Omega u^{p-1}_\varepsilon \varepsilon^2 \leq C(p_*, K) \quad \text{for all } \varepsilon \in (0, 1).
\] (4.17)

On the basis of (4.13) and our selection of \( p \) and \( q \), from Lemma 4.2 we thus infer the existence of \( c_2(p_*, K) > 0 \) such that
\[
\int_0^\infty \int_\Omega u^p_\varepsilon v_\varepsilon \leq c_2(p_*, K) \quad \text{for all } \varepsilon \in (0, 1).
\] (4.18)

As a multiplication of the first equation in (2.6) by \( u^{p-1}_\varepsilon \), followed by an integration by parts, shows that
\[
\frac{1}{p} \int_\Omega \left( u^p_\varepsilon + (p - 1) \int_0^t \int_\Omega u^{p-1}_\varepsilon \right) \leq \ell \int_\Omega u^p_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),
\]
and hence
\[
\frac{1}{p} \int_\Omega u^p_\varepsilon + (p - 1) \int_0^t \int_\Omega u^{p-1}_\varepsilon \right) \leq \ell \int_\Omega u^p_\varepsilon \leq \frac{1}{p} \left( \|u_0\|_{L^\infty(\Omega)} + 1 \right) \ell \cdot |\Omega| + c_2(p_*, K) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)
\]
according to (2.5) and (4.18), the claim directly follows. \( \square \)

Upon repeated application of the latter, we obtain bounds in all the \( L^p \) spaces appearing in the claim from Theorem 1.1. Apart from that, the additional decay features expressed in (4.20) and (4.21) will play essential roles in our derivation of stabilization in the first solution component in the next section (see Lemma 5.1).

**Corollary 4.4** Let \( n \geq 1, p \in (1, \frac{n}{(n-2)+}) \) and \( K > 0 \). Then there exists \( C(p, K) > 0 \) such that if (1.3), (2.5) and (1.13) are satisfied, then
\[
\int_\Omega u^p_\varepsilon \leq C(p, K) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)
\] (4.19)
and
\[
\int_0^\infty \int_\Omega u^{p-1}_\varepsilon v_\varepsilon \leq C(p, K) \quad \text{for all } \varepsilon \in (0, 1)
\] (4.20)
as well as
\[
\int_0^\infty \int_\Omega u^p_\varepsilon v_\varepsilon \leq C(p, K) \quad \text{for all } \varepsilon \in (0, 1).
\] (4.21)

**Proof** We recursively define \((p_k)_{k \geq 0}\) by letting
\[
p_0 := 1 \quad \text{and} \quad p_{k+1} := \begin{cases} p_k + 1, & k \geq 0, \text{ if } n = 1, \\ \frac{2p_{k+n}+n}{n}, & k \geq 0, \text{ if } n \geq 2. \end{cases}
\]
Assuming (1.3), (2.5) and (1.13) to hold, from Lemma 4.3 we then immediately obtain $(c_k(K))_{k \geq 1} \subset (0, \infty)$ such that for each $k \geq 1$,

$$\int_{\Omega} u^p_k (\cdot, t) \leq c_k(K) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1)$$

(4.22)
as well as

$$\int_{0}^{\infty} \int_{\Omega} u^{p_k-1}_k \nu_\varepsilon |\nabla u_\varepsilon|^2 + \int_{0}^{\infty} \int_{\Omega} u^p_k \nu_\varepsilon \leq c_k(K) \quad \text{for all } \varepsilon \in (0, 1).$$

(4.23)

Since it can readily be verified that $p_k \nearrow \frac{n}{(n-2)_+}$ as $k \to \infty$, for arbitrary $p \in (1, \frac{n}{(n-2)_+})$ the estimates in (4.19), (4.20) and (4.21) easily result from (4.22) and (4.23) upon an interpolation using the basic bounds provided by (2.8), (2.17) and (2.11).

In particular, the latter completes our reasoning with regard to solvability and global $L^p$ regularity in (1.2):

Proof of Theorem 1.1 The part concerning existence and regularity has been completely covered by Lemma 3.2. The additional boundedness feature in (1.9) immediately follows from (4.19) when combined with (3.8).

Remark For initial data enjoying regularity and positivity features beyond those in (1.3), the existence result from Theorem 1.1 can be supplemented by a corresponding uniqueness statement by a straightforward combination of the reasoning from Lemma 2.2 with the standard theory developed in [1]: Indeed, if beyond (1.3) it was required that both $u_0$ and $v_0$ belong to $\bigcup_{q>n} W^{1,q}(\Omega)$ and satisfy $u_0 > 0$ and $v_0 > 0$ in $\overline{\Omega}$, then (1.2) could actually be seen to admit a global classical solution which is unique in the class of functions fulfilling $\{u, v\} \subset \bigcup_{q>n} C^0([0, \infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$. Under the present mild hypotheses in (1.3) on the initial data, however, we do not expect solutions to be uniquely determined by the requirements in Definition 2.1, nor by the additional regularity features obtained in Lemma 3.2 which are yet fairly poor near the initial instant; we cannot even rule out the possibility that different choices of approximate initial data in (2.5) may lead to different limits.

5 Large time behavior

5.1 Large time convergence of $u$ in dual Sobolev spaces when $n \leq 5$

Fortunately, in all physically relevant space dimensions the respective restrictions on $p$ in Corollary 4.4 are mild enough so as to allow for the following conclusion on large time decay of $u_\varepsilon t$ on the basis of (4.20), (4.21) and, again, the basic integrability property from (2.11).

Lemma 5.1 Let $n \leq 5$. Then there exists $\lambda = \lambda(n) > 0$ such that for all $K > 0$ one can find $C(K) > 0$ with the property that if (1.3), (2.5) and (1.13) hold, we have

$$\int_{0}^{\infty} \|u_\varepsilon t (\cdot, t)\|_{(W^{1,\infty}(\Omega))^{\cdot}} dt \leq C(K) \cdot \left\{ \int_{\Omega} v_0 \right\}^{\lambda} \quad \text{for all } \varepsilon \in (0, 1).$$

(5.1)
Proof Since \( n \leq 5 \), we have \( 3 - \frac{n}{(n-2)_+} < \frac{n}{(n-2)_+} \), so that it is possible to pick \( p = p(n) \in (1, 2) \) fulfilling
\[
p < \frac{n}{(n-2)_+} \quad \text{and} \quad 3 - p < \frac{n}{(n-2)_+}, \tag{5.2}
\]
where using a continuity argument we may rely on the latter inequality in choosing \( r = r(n) \in (0, 1) \) suitably small such that
\[
\frac{3 - p - r}{1 - r} < \frac{n}{(n-2)_+} \tag{5.3}
\]
Noting that the restriction \( p < 2 \) warrants that then \( \frac{3 - p - r}{1 - r} > 1 \), given \( K > 0 \) we can draw on Corollary 4.4 to see that according to (5.3) and the first inequality in (5.2) we can fix \( c_1(K) > 0 \) and \( c_2(K) > 0 \) such that whenever (1.3), (2.5) and (1.13) holds,
\[
\int_0^\infty \int_\Omega u_{e1}^{p-1} v_e |\nabla u_e|^2 \leq c_1(K) \quad \text{for all } \varepsilon \in (0, 1) \tag{5.4}
\]
and
\[
\int_0^\infty \int_\Omega u_{e1}^{3-p-r} v_e \leq c_2(K) \quad \text{for all } \varepsilon \in (0, 1). \tag{5.5}
\]
Under these hypotheses, we now use the first equation in (2.6) to find that for all \( t > 0 \) and any \( \psi \in W^{1,\infty}(\Omega) \) such that \( \| \psi \|_{W^{1,\infty}(\Omega)} \equiv \max \{ \| \psi \|_{L^\infty(\Omega)}, \| \nabla \psi \|_{L^\infty(\Omega)} \} \leq 1 \),
\[
\left| \int_\Omega u_{e1} \psi \right| = - \int_\Omega u_{e1} v_e \nabla u_e \cdot \nabla \psi + \ell \int_\Omega u_{e1} v_e \psi \leq \int_\Omega u_{e1} v_e |\nabla u_e| + \ell \int_\Omega u_{e1} v_e \quad \text{for all } \varepsilon \in (0, 1),
\]
so that
\[
\| u_{e1} \|_{W^{1,\infty}(\Omega)} \leq \int_\Omega u_{e1} v_e |\nabla u_e| + \ell \int_\Omega u_{e1} v_e \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \tag{5.6}
\]
Since for all \( T > 0 \) and \( \varepsilon \in (0, 1) \) we have
\[
\int_0^T \int_\Omega u_{e1} v_e |\nabla u_e| \leq \left( \int_0^T \int_\Omega u_{e1}^{p-1} v_e |\nabla u_e|^2 \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_\Omega u_{e1}^{3-p} v_e \right)^{\frac{1}{2}} \leq \left( \int_0^T \int_\Omega u_{e1}^{p-1} v_e |\nabla u_e|^2 \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_\Omega u_{e1}^{3-p-r} v_e^{1-r} \right)^{\frac{1}{2}} \leq \left( \int_0^T \int_\Omega u_{e1}^{p-1} v_e |\nabla u_e|^2 \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_\Omega u_{e1} v_e \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_\Omega u_{e1}^{3-p-r} v_e^{1-r} \right)^{\frac{1}{2}}
\]
due to the Hölder inequality, from (5.4), (5.5) and (5.6) we thus infer that thanks to (2.11),
\[
\int_0^T \| u_{e1}(\cdot, t) \|_{W^{1,\infty}(\Omega)} dt \leq c_1'(K)c_2'(K) \cdot \left( \int_0^T \int_\Omega u_{e1} v_e \right)^{\frac{\nu}{2}} + \ell \int_0^T \int_\Omega u_{e1} v_e \leq c_1'(K)c_2'(K) \cdot \left( \int_\Omega v_{e1} \right)^{\frac{\nu}{2}} + \ell \int_\Omega v_{e1}
\]
for all \( T > 0 \) and \( \varepsilon \in (0, 1) \).
As
\[
\int_{\Omega} v_{0\varepsilon} \leq \left\{ \int_{\Omega} v_{0\varepsilon} \right\}^{\frac{\varepsilon}{2}} \cdot (\|v_{0\varepsilon}\|_{L^\infty(\Omega)} \cdot |\Omega|)^{\frac{1}{2}} \leq \left\{ \int_{\Omega} v_{0\varepsilon} \right\}^{\frac{\varepsilon}{2}} \cdot (K + 1)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}}
\]
for all \( \varepsilon \in (0, 1) \)

by (1.13) and (2.5), this implies the claim with \( \lambda(n) := \frac{\varepsilon}{2} \) and \( C(K) := c_1^{\frac{1}{2}}(K)c_2^{\frac{1}{2}}(K) + \ell \cdot (K + 1)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \).

\[\square\]

As (5.1) involves \( L^1 \) norms with respect to the time variable only, in order to avoid any discussion of possibly measure-valued parts appearing in the time derivatives \( u_t \) of the corresponding limits we prefer to formulate a conclusion of Lemma 5.1 for \( u \) which is sufficient for our subsequent reasoning in a version including temporal BV norms only, and hence exclusively involving the zero-order expression \( u \) itself:

**Corollary 5.2** Let \( n \leq 5 \) and \( K > 0, \) let \( \lambda = \lambda(n) \) and \( C(K) \) be as in Lemma 5.1, and assume (1.3), (2.5) and (1.13). Then for any \( (t_k)_{k \in \mathbb{N}} \subset [0, \infty) \) such that \( t_{k+1} \geq t_k \) for all \( k \in \mathbb{N}, \) we have

\[
\sum_{k=1}^{\infty} \left\| u(\cdot, t_{k+1}) - u(\cdot, t_k) \right\|_{(W^{1,\infty}(\Omega))^*} \leq C(K) \cdot \left\{ \int_{\Omega} v_{0\varepsilon} \right\}^{\lambda}, \tag{5.7}
\]

where we have set \( u(\cdot, 0) := u_0. \)

**Proof** Fixing any such nondecreasing sequence \( (t_k)_{k \in \mathbb{N}}, \) under the present hypotheses we infer from Lemma 5.1 that for each \( N \in \mathbb{N}, \)

\[
\sum_{k=1}^{N} \left\| u_k(\cdot, t_{k+1}) - u_k(\cdot, t_k) \right\|_{(W^{1,\infty}(\Omega))^*} = \sum_{k=1}^{N} \left\| \int_{t_k}^{t_{k+1}} u_{\varepsilon t}(\cdot, t) dt \right\|_{(W^{1,\infty}(\Omega))^*} \leq \sum_{k=1}^{N} \int_{t_k}^{t_{k+1}} \left\| u_{\varepsilon t}(\cdot, t) \right\|_{(W^{1,\infty}(\Omega))^*} dt \leq C(K) \cdot \left\{ \int_{\Omega} v_{0\varepsilon} \right\}^{\lambda} \text{ for all } \varepsilon \in (0, 1), \tag{5.8}
\]

because \( (t_k, t_{k+1}) \cap (t_l, t_{l+1}) = \emptyset \) for each \( k, l \in \mathbb{N} \) with \( k \neq l. \) Since from (2.5) and Lemma 3.2 we know that with \( (\varepsilon_j)_{j \in \mathbb{N}} \) as provided by the latter we have \( u_{\varepsilon}(\cdot, t_k) \rightarrow u(\cdot, t_k) \) in \( L^1(\Omega) \leftrightarrow (W^{1,\infty}(\Omega))^* \) as \( \varepsilon = \varepsilon_j \searrow 0 \) for each \( k \in \mathbb{N}, \) we obtain (5.7) from (5.8) upon taking \( \varepsilon = \varepsilon_j \searrow 0 \) and then \( N \rightarrow \infty \) there. \( \square \)

When considering individual trajectories, we may here yet ignore any of the information about dependencies on initial data enclosed in (5.1), and to thereby obtain the following as a particular consequence.

**Corollary 5.3** Let \( n \leq 5, \) and suppose that (1.3) and (2.5) hold. Then there exists a nonnegative \( u_\infty \in (W^{1,\infty}(\Omega))^* \) such that

\[
u(\cdot, t) \rightarrow u_\infty \text{ in } (W^{1,\infty}(\Omega))^* \quad \text{as } t \rightarrow \infty. \tag{5.9}\]
Proof We only need to observe that whenever $(t_k)_{k \in \mathbb{N}} \subset (0, \infty)$ is nondecreasing and unbounded,
\[
\|u(\cdot, t_{k+m}) - u(\cdot, t_k)\|_{(W^{1,\infty}(\Omega))^*} \leq \sum_{l=k}^{k+m-1} \|u(\cdot, t_{l+1}) - u(\cdot, t_l)\|_{(W^{1,\infty}(\Omega))^*}
\]
for all $k \geq 1$ and $m \geq 1$,

and that thus (5.7) particularly asserts that $(u(\cdot, t_k))_{k \in \mathbb{N}}$ forms a Cauchy sequence in $(W^{1,\infty}(\Omega))^*$.

Beyond this, however, the particular quantitative form of the right-hand side in (5.1) enables us to also make sure that initially small $v_0$ enforce solutions to remain near $u_0$ in their first component.

Corollary 5.4 Let $n \leq 5$ and $K > 0$. Then given any $\eta > 0$, one can find $\delta = \delta(K, \eta) > 0$ such that whenever (1.3), (2.5) and (1.13) hold as well as
\[
\int_{\Omega} v_0 \leq \delta, \tag{5.10}
\]
we have
\[
\|u(\cdot, t) - u_0\|_{(W^{1,\infty}(\Omega))^*} \leq \eta \quad \text{for all } t > 0. \tag{5.11}
\]

Proof As the number $\lambda(n)$ provided by Corollary 5.2 is positive, the claim immediately results upon applying the latter to $(t_k)_{k \in \mathbb{N}}$ defined by $t_1 := 0$ and $t_k := t$ for $k \geq 2$ and any fixed $t > 0$.

5.2 Stabilization of $u^\gamma$ in dual Sobolev spaces for all $n \geq 6$ and some $\gamma \in (0, 1)$

In higher-dimensional cases in which the estimates provided by Corollary 4.4 seem insufficient for the above argument, with regard to large time asymptotics we rather resort to an analysis of certain sublinear powers of $u$. Indeed, for suitably small $\gamma$ the expressions $u^\gamma$ can be seen to enjoy a stabilization feature similar to that in Lemma 5.1, albeit apparently without information about comparably favorable effects induced by small $v_0$ here:

Lemma 5.5 There exists $\gamma \in (0, 1)$ with the property that whenever $n \geq 6$ and (1.3) as well as (2.5) hold, there exists $C = C(u_0, v_0) > 0$ such that
\[
\int_0^\infty \|\partial_t u^\gamma \|_{(W^{1,\infty}(\Omega))^*} dt \leq C \quad \text{for all } \varepsilon \in (0, 1). \tag{5.12}
\]

Proof We fix any $\gamma \in (0, \frac{1}{2})$ and observe that then for all $t > 0$ and each $\psi \in W^{1,\infty}(\Omega)$, according to (2.6) we have
\[
\int_{\Omega} \partial_t u^\gamma \cdot \psi = \gamma(1 - \gamma) \int_{\Omega} u^{\gamma-1}_\varepsilon |\nabla u_\varepsilon|^2 \psi - \gamma \int_{\Omega} u^\gamma_\varepsilon \nabla u_\varepsilon \cdot \nabla \psi
\]
\[
+ \gamma \varepsilon \int_{\Omega} u^\gamma_\varepsilon \psi \quad \text{for all } \varepsilon \in (0, 1),
\]
from which it follows that with some $c_1 > 0$,
\[
\int_0^T \|\partial_t u^\gamma(\cdot, t)\|_{(W^{1,\infty}(\Omega))^*} dt \leq c_1 \cdot \left( \int_0^T \int_{\Omega} u^{\gamma-1}_\varepsilon |\nabla u_\varepsilon|^2 + \int_0^T \int_{\Omega} u^\gamma_\varepsilon |\nabla u_\varepsilon| + \varepsilon \int_0^T \int_{\Omega} u^\gamma_\varepsilon \right)
\]
for all $T > 0$ and $\varepsilon \in (0, 1)$. Since here

$$
\int_0^T \int_\Omega u_\varepsilon^\gamma v_\varepsilon |\nabla u_\varepsilon| \leq \int_0^T \int_\Omega u_\varepsilon^{2\gamma-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon v_\varepsilon \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0, 1)
$$

by Young’s inequality, the claim readily results upon combining (2.17) with (2.18) and (2.11), because both $\gamma$ and $2\gamma$ belong to $(0, 1)$. 

After all, convergence of such powers $u^{\gamma}$ for each fixed trajectory can be asserted within this topological setting:

**Corollary 5.6** Let $n \geq 6$ and $\gamma > 0$ be as in Lemma 5.5, and assume that (1.3) and (2.5) hold. Then there exists a nonnegative $u_\infty \in (W^{1,\infty}(\Omega))^*$ such that

$$
u^{\gamma}(\cdot, t) \to u_\infty^{\gamma} \quad \text{in } (W^{1,\infty}(\Omega))^* \quad \text{as } t \to \infty.
$$

**Proof** This can be concluded from Lemma 5.5 in much the same manner as Corollary 5.3 was derived from Lemma 5.1. 

5.3 $L^1$ decay of $v$

By now making use of the lower bound for sublinear $L^q$ quasi-norms from Lemma 2.3, irrespective of the spatial dimension we can derive the following decay property of $v$ from (2.11) and (2.21).

**Lemma 5.7** Let $n \geq 1$ and assume (1.3) as well as (2.5). Then

$$
\int_\Omega v(\cdot, t) \to 0 \quad \text{as } t \to \infty.
$$

**Proof** We fix any $q \in (0, 1)$ such that $\frac{4q}{3} \leq 1$, and from (2.8) we then obtain $c_1 = c_1(u_0, v_0) > 0$ such that

$$
\|u_\varepsilon\|_{L^{\frac{4q}{3}}(\Omega)} \leq c_1 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1).
$$

We moreover employ a Poincaré inequality to pick $c_2 > 0$ fulfilling

$$
\left\|\varphi - \frac{1}{|\Omega|} \int_\Omega \varphi\right\|_{L^4(\Omega)} \leq c_2 \|\nabla \varphi\|_{L^4(\Omega)} \quad \text{for all } \varphi \in W^{1,4}(\Omega),
$$

and abbreviating $c_3 := \frac{1}{|\Omega|} \int_\Omega u_0^q > 0$, given $\eta > 0$ we choose $t_0 = t_0(\eta; u_0, v_0) > 1$ large enough such that

$$
\frac{1}{c_3} \int_{t-1}^t \int_\Omega u^q v \leq \frac{\eta}{2} \quad \text{for all } t > t_0
$$

and

$$
\frac{c_1 c_2}{c_3} \cdot \left\{\int_{t-1}^t \int_\Omega |\nabla v|^4\right\}^{\frac{1}{4}} \leq \frac{\eta}{2} \quad \text{for all } t > t_0;
$$

here we note that such a selection is possible due to the fact that (2.11), Lemma 2.4 and (2.10) together with Lemma 3.2 and Fatou’s lemma guarantee that both $\int_0^\infty \int_\Omega u v$ and $\int_0^\infty \int_\Omega |\nabla v|^4$
are finite, and that hence, by the Hölder inequality and again due to (2.10) and Lemma 3.2,
\[
\int_{t-1}^{t} \int_{\Omega} u^q v \leq \left( \|v_0\|_{L^\infty(\Omega)} + 1 \right)^{1-q} \int_{t-1}^{t} \int_{\Omega} u^q v^q \\
\leq \left( \|v_0\|_{L^\infty(\Omega)} + 1 \right)^{1-q} \left| \Omega \right|^{-1-q} \left\{ \int_{t-1}^{t} \int_{\Omega} u v \right\}^q \\
\to 0 \quad \text{as } t \to \infty.
\]

For \( t > t_0 \) and \( \varepsilon \in (0, 1) \), we may then decompose
\[
\int_{t-1}^{t} \int_{\Omega} u^q v_{\varepsilon} = \int_{t-1}^{t} \int_{\Omega} u^q(x, s) \cdot \left\{ \frac{1}{|\Omega|} \int_{\Omega} v_{\varepsilon}(y, s) dy \right\} dx ds \\
+ \int_{t-1}^{t} \int_{\Omega} u^q(x, s) \cdot \left\{ v_{\varepsilon}(x, s) - \frac{1}{|\Omega|} \int_{\Omega} v_{\varepsilon}(y, s) dy \right\} dx ds, \tag{5.18}
\]
where by the Hölder inequality, (5.15) and (5.14),
\[
\left| \int_{t-1}^{t} \int_{\Omega} u^q(x, s) \cdot \left\{ v_{\varepsilon}(x, s) - \frac{1}{|\Omega|} \int_{\Omega} v_{\varepsilon}(y, s) dy \right\} dx ds \right| \\
\leq \int_{t-1}^{t} \left| u_{\varepsilon}(\cdot, s) \right|^q \left| L \frac{4q}{q} (\Omega) \right| \left| v_{\varepsilon}(\cdot, s) \right| - \frac{1}{|\Omega|} \int_{\Omega} v_{\varepsilon}(\cdot, s) \left| L \frac{4q}{q} (\Omega) \right| ds \\
\leq c_1^q c_2 \int_{t-1}^{t} \left\| \nabla v_{\varepsilon}(\cdot, s) \right\|_{L^4(\Omega)} ds \\
\leq c_1^q c_2 \cdot \left\{ \int_{t-1}^{t} \int_{\Omega} \left| \nabla v_{\varepsilon} \right|^4 \right\}^{\frac{1}{4}}.
\]

Since furthermore
\[
\int_{t-1}^{t} \int_{\Omega} u^q(x, s) \cdot \left\{ \frac{1}{|\Omega|} \int_{\Omega} v_{\varepsilon}(y, s) dy \right\} dx ds \\
= \frac{1}{|\Omega|} \int_{t-1}^{t} \left\{ \int_{\Omega} u^q_{\varepsilon}(x, s) dx \right\} \cdot \left\{ \int_{\Omega} v_{\varepsilon}(y, s) dy \right\} ds \\
\geq \frac{1}{|\Omega|} \cdot \left\{ \int_{\Omega} u^q_{\varepsilon} \right\} \cdot \left\{ \int_{\Omega} v_{\varepsilon}(y, t) dy \right\}
\]
according to Lemma 2.3 and the evident nonpositivity of \( \frac{d}{dt} \int_{\Omega} v_{\varepsilon} \) throughout \((0, \infty)\), from (5.18) we altogether conclude that
\[
\frac{1}{|\Omega|} \cdot \left\{ \int_{\Omega} u^q_{\varepsilon} \right\} \cdot \left\{ \int_{\Omega} v_{\varepsilon}(x, t) dx \right\} \leq \int_{t-1}^{t} \int_{\Omega} u^q v_{\varepsilon} \\
+ c_1^q c_2 \cdot \left\{ \int_{t-1}^{t} \int_{\Omega} \left| \nabla v_{\varepsilon} \right|^4 \right\}^{\frac{1}{4}} \quad \text{for all } t > t_0 \text{ and } \varepsilon \in (0, 1).
\]

With \((\varepsilon_j)_{j \in \mathbb{N}}\) taken from Lemma 3.2, we let \( \varepsilon = \varepsilon_j \searrow 0 \) here to infer from (2.5) and the locally uniform convergence properties of \((u_{\varepsilon_j})_{j \in \mathbb{N}}, (v_{\varepsilon_j})_{j \in \mathbb{N}}\) and \((\nabla v_{\varepsilon_j})_{j \in \mathbb{N}}\) asserted by
Lemma 3.2 that in line with our definition of $c_3$,
\[
\int_{\Omega} v(x, t) \, dx \leq \frac{1}{c_3} \int_{t-1}^{t} \int_{\Omega} u^q v + \frac{c_4 c_2}{c_3} \left\{ \int_{t-1}^{t} |\nabla v|^4 \right\} ^{\frac{1}{4}}
\]
\[
\leq \frac{\eta}{2} + \frac{\eta}{2} = \eta \quad \text{for all } t > t_0.
\]
As $\eta > 0$ was arbitrary, this establishes the claim. \hfill \Box

5.4 Proofs of Theorems 1.2 and 1.3

Large time convergence in the claimed topological settings can now be obtained from Corollary 5.3, Corollary 5.6 and Lemma 5.7 by suitable interpolation using the boundedness features asserted by Theorem 1.1.

Proof of Theorem 1.2 As $\sup_{t > 0} \| v(\cdot, t) \|_{L^\infty(\Omega)}$ is finite according to Theorem 1.1, for $n \geq 4$ the convergence statement in (1.11) can be derived from Lemma 5.7 by a simple interpolation based on the Hölder inequality. In the case $n \leq 3$, we once again use that thanks to (2.21) and Lemma 3.2, the integral $\int_{0}^{\infty} \int_{\Omega} |\nabla v|^4$ is finite, so that with some $c_1 > 0$ and some $(t_k)_{k \in \mathbb{N}} \subset (0, \infty)$ fulfilling $t_k \to \infty$ as $k \to \infty$ we have $\| v(\cdot, t_k) \|_{W^{1,4}(\Omega)} \leq c_1$ for all $k \in \mathbb{N}$. Since for any such $n$ the Gagliardo-Nirenberg inequality provides $c_2 > 0$ fulfilling
\[
\| \varphi \|_{L^\infty(\Omega)} \leq c_2 \| \varphi \|_{W^{1,4}(\Omega)}^{\frac{4-n}{2n}} \| \varphi \|_{L^1(\Omega)}^{\frac{4-n}{2n}} \quad \text{for all } \varphi \in W^{1,4}(\Omega),
\]
in view of Lemma 5.7 this implies that
\[
\| v(\cdot, t_k) \|_{L^\infty(\Omega)} \leq c_1^{\frac{4n}{2n-4}} c_2^{\frac{4-n}{2n}} \to 0 \quad \text{as } k \to \infty,
\]
and that thus (1.11) also holds in this case, because being a classical solution of its respective sub-problem of (1.2) in $\bar{\Omega} \times [1, \infty)$ by Theorem 1.1, thanks to the maximum principle the function $v$ has the property that $0 < t \mapsto \| v(\cdot, t) \|_{L^\infty(\Omega)}$ is nonincreasing.

To deduce (1.12), we only need to combine the outcomes of Corollary 5.3 and Corollary 5.6 with the observation that for each $\kappa > 0$ and any $p \in \left( 1, \frac{n}{(n-2)\kappa} \right)$, the family $(u^\kappa(\cdot, t))_{t > 0}$ is bounded in $L^{\frac{2}{p}}(\Omega)$ by (1.9), and hence relatively compact with respect to the weak topology in this space. \hfill \Box

Our main result on stability of arbitrary equilibria in (1.2) has actually been covered by Corollary 5.4 in its essence:

Proof of Theorem 1.3 It is sufficient to apply Corollary 5.4, and to note that with $(\varepsilon_j)_{j \in \mathbb{N}}$ taken from Lemma 3.2 we have
\[
\| v(\cdot, t) - v_0 \|_{L^1(\Omega)} \leq \lim_{\varepsilon \to \varepsilon_j \to 0} \left\{ \int_{\Omega} v_\varepsilon(\cdot, t) + \int_{\Omega} v_0 \right\} \leq 2 \lim_{\varepsilon \to \varepsilon_j \to 0} \int_{\Omega} v_0 = 2 \int_{\Omega} v_0 \quad \text{for all } t > 0
\]
due to the evident nonincrease of $0 \leq t \mapsto \int_{\Omega} v_\varepsilon(\cdot, t)$ for all $\varepsilon \in (0, 1)$. \hfill \Box

Acknowledgements The author warmly thanks the anonymous reviewer and Frederic Heihoff for numerous insightful comments which significantly improved this manuscript.

Funding Open Access funding enabled and organized by Projekt DEAL. He furthermore acknowledges support of the Deutsche Forschungsgemeinschaft in the context of the project Emergence of structures and advantages in cross-diffusion systems (Project No. 411007140, GZ: WI 3707/5-1).
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