LOCALIZATION NEAR THE EDGE FOR THE ANDERSON BERNOULLI MODEL ON THE TWO DIMENSIONAL LATTICE

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Abstract. We consider a Hamiltonian given by the Laplacian plus a Bernoulli potential on the two dimensional lattice. We prove that, for energies sufficiently close to the edge of the spectrum, the resolvent on a large square is likely to decay exponentially. This implies almost sure Anderson localization for energies sufficiently close to the edge of the spectrum. Our proof follows the program of Bourgain–Kenig, using a new unique continuation result inspired by a Liouville theorem of Buhovsky–Logunov–Malinnikova–Sodin.

1. Introduction

1.1. Anderson localization. We consider the Anderson–Bernoulli model on the lattice, which is the random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ given by

$$H = -\Delta + \delta V,$$

where $(\Delta u)(x) = \sum_{|y-x|=1}(u(y)-u(x))$ is the discrete Laplacian, $(Vu)(x) = V_x u(x)$ is a random potential whose values $V_x \in \{0,1\}$ for $x \in \mathbb{Z}^d$ are independent and satisfy $\mathbb{P}[V_x = 0] = \mathbb{P}[V_x = 1] = 1/2$, and $\delta > 0$ is the strength of the noise.

We are interested in the effect of the perturbation $\delta V$ on the spectral theory of the discrete Laplacian. We recall that the spectrum of the discrete Laplacian is the closed interval $\sigma(-\Delta) = [0,4d]$, as can be seen by taking the Fourier transform, $\tilde{-\Delta}(\xi) = 2 \sum_{1 \leq k \leq d}(1 - \cos(\xi_k))$. We recall that the spectrum of the random Hamiltonian is almost surely the closed interval $\sigma(H) = [0,4d + \delta]$, as can be seen by observing that, almost surely, every finite configuration appears in the random Bernoulli potential. While the spectrum of the discrete Laplacian is absolutely continuous, the random Hamiltonian may have eigenvalues. The perturbation $\delta V$ can create “traps” on which eigenfunctions are exponentially localized. This phenomenon is called Anderson localization.
To be precise, we say that $H$ has “Anderson localization” in the spectral interval $I \subseteq \sigma(H)$ if

$$\psi : \mathbb{Z}^d \to \mathbb{R}, \quad \lambda \in I, \quad H\psi = \lambda \psi, \quad \text{and} \quad \inf_{n>0} \sup_{x \in \mathbb{Z}^d} (1 + |x|)^{-n} |\psi(x)| < \infty$$

implies

$$\inf_{t>0} \sup_{x \in \mathbb{Z}^d} e^{t|x|} |\psi(x)| < \infty.$$  

That is, every polynomially bounded solution of the eigenfunction equation $H\psi = \lambda \psi$ with $\lambda \in I$ is an exponentially decaying eigenfunction. Recall that this implies the absence of continuous spectrum in $I$. (Rather, this implies that the continuous spectrum has spectral measure zero; see Kirsch [14, Section 7].)

We prove the following result.

**Theorem 1.1.** In dimension $d = 2$ there is an $\varepsilon > 0$, depending on $\delta > 0$, such that, almost surely, $H$ has Anderson localization in $[0, \varepsilon]$.

To put Theorem 1.1 in context, let us very briefly discuss some of the known results and open problems for the Anderson–Bernoulli model. Let us mention four related rigorous mathematical results:

- If $d = 1$, then $H$ almost surely has Anderson localization in all of $\sigma(H)$; see Kunz–Souillard [15] and Carmona–Klein–Martinelli [8].
- If the noise is continuous (that is, the random variables $V_x \in [0,1]$ have the same bounded density), then $H$ almost surely has Anderson localization in $[0, \varepsilon]$; see Fröhlich–Spencer [10].
- If the noise is continuous and $\delta \geq C$ is large, then $H$ almost surely has Anderson localization in all of $\sigma(H)$; see Aizenman–Molchanov [1] and Fröhlich–Martinelli–Scoppola–Spencer [9].
- If the lattice is replaced by the continuum $\mathbb{R}^d$, then $H$ almost surely has Anderson localization in $[0, \varepsilon]$; see Bourgain–Kenig [5].

These are only four results in a large literature. For a detailed discussion and references, see Aizenman–Warzel [2], Hundertmark [13], Kirsch [14], and Stolz [20].

According to Simon [17] (see also Bellissard–Hislop–Klein–Stolz [3]), the two most important open problems for the Anderson model (with Bernoulli or continuous noise) are the following.

**Problem 1.2.** In dimension $d = 2$, prove that $H$ almost surely has Anderson localization in all of $\sigma(H)$.

**Problem 1.3.** In dimensions $d \geq 3$, prove that, for every small $\varepsilon > 0$, there is a $\delta > 0$ such that $H$ almost surely has no eigenvalues in $[\varepsilon, 4d - \varepsilon]$.

In particular, our theorem is not on the list of most important open problems. However, understanding localization with Bernoulli noise does not appear to be a merely technical problem. The work of Bourgain–Kenig [5] handling the Bernoulli case in the continuum setting required a complete re-working of the multiscale analysis. Moreover, the Bernoulli case helps us understand localization as a universal phenomenon.

1.2. **Resolvent estimate.** We do not prove Theorem 1.1 directly. Instead, we rely on previous work to reduce the problem to proving bounds on the exponential decay of the resolvent. Our main theorem is the following.
Theorem 1.4. Suppose $d = 2$ and $\delta = 1$. For any $1/2 > \gamma > 0$, there are $\alpha > 1 > \varepsilon > 0$ such that, for every energy $\tilde{\lambda} \in [0, \varepsilon]$ and square $Q \subseteq \mathbb{Z}^2$ of side length $L \geq \alpha$,

$$
\mathbb{P}[(H_Q - \tilde{\lambda})^{-1}(x, y)] \leq e^{L^{1-\varepsilon} - |x-y|} \text{ for } x, y \in Q \geq 1 - L^{-\gamma}.
$$

Here $H_Q : \ell^2(Q) \to \ell^2(Q)$ denotes the restriction of the Hamiltonian $H$ to the square $Q$ with zero boundary conditions.

Our proof works, essentially verbatim, for any random potential $V : \mathbb{Z}^2 \to \mathbb{R}$ whose values $V(x)$ are i.i.d., bounded, and non-trivial. However, for simplicity, we argue only in the case of strength $\delta = 1$ and Bernoulli noise.

Proof of Theorem 1.4. Almost sure Anderson localization for $H$ in the interval $[0, \varepsilon]$ follows from Theorem 1.4 using the Peierls argument of Bourgain–Koenig [5] Section 7. See Germinet–Klein [11, Sections 6 and 7] for an axiomatic version of this. □

1.3. Outline. Our proof follows Bourgain–Koenig [5] fairly closely. We perform a multiscale analysis, keeping track of a list of “frozen” sites $F \subseteq \mathbb{Z}^2$ where the potential has already been sampled. The complementary “free” sites $\mathbb{Z}^2 \setminus F$ are sampled only to perform an eigenfunction variation on rare, “bad” squares. This strategy of frozen and free sites is used to obtain a version of the Wegner estimate [22] that is otherwise unavailable in the Bernoulli setting.

The eigenvalue variation of [5] relies crucially on an a priori quantitative unique continuation result. Namely, that if $u \in C^2_{loc}(\mathbb{R}^d)$ and $|\Delta u| \leq |u|$, then

$$
\int_{B_1(x)} |u| \geq ce^{-|x-y|^{1/3+\varepsilon}} \int_{B_1(y)} |u| \text{ for } x, y \in \mathbb{R}^d.
$$

The corresponding fact (even in qualitative form) is false on the lattice $\mathbb{Z}^d$.

To carry out the program on the lattice, we need a substitute for the above quantitative unique continuation result. For the two-dimensional lattice $\mathbb{Z}^2$, a hint of the missing ingredient appears in the paper of Buhovsky–Logunov–Malinnikova–Sodin [7]. In this paper, it is proved that any function $u : \mathbb{Z}^2 \to \mathbb{R}$ that is harmonic and bounded on a $1 - \varepsilon$ fraction of sites must be constant. One of the key components of this Liouville theorem is the following quantitative unique continuation result for harmonic functions on the two dimensional lattice.

Theorem 1.5 ([7]). There are constants $\alpha > 1 > \varepsilon > 0$ such that, if $u : \mathbb{Z}^2 \to \mathbb{R}$ is lattice harmonic in a square $Q \subseteq \mathbb{Z}^2$ of side length $L \geq \alpha$, then

$$
|\{x \in Q : |u(x)| \geq e^{-\alpha L} \|u\|_{L^\infty(\frac{1}{4}Q)}\}| \geq \varepsilon L^2.
$$

This implies that any two lattice harmonic functions that agree on a $1 - \varepsilon$ fraction of sites in a large square must be equal in the half square. Note that this result is false in dimensions three and higher.

Inspired by this theorem and its proof, we prove the following random quantitative unique continuation result for eigenvalues of the Hamiltonian $H$.

Theorem 1.6. There are constants $\alpha > 1 > \varepsilon > 0$ such that, if $\lambda \in [0, 1]$ is an energy and $Q \subseteq \mathbb{Z}^2$ is a square of side length $L \geq \alpha$, then $\mathbb{P}[\mathcal{E}] \geq 1 - e^{-\varepsilon L^{1/4}}$, where $\mathcal{E}$ denotes the event that

$$
|\{x \in Q : |\psi(x)| \geq e^{-\alpha L \log L} \|\psi\|_{L^\infty(\frac{1}{4}Q)}\}| \geq \varepsilon L^{3/2}(\log L)^{-1/2}
$$

holds whenever $\lambda \in \mathbb{R}$, $\psi : \mathbb{Z}^2 \to \mathbb{R}$, $|\lambda - \tilde{\lambda}| \leq e^{-\alpha (L \log L)^{1/2}}$, and $H\psi = \lambda\psi$ in $Q$. 



This is the main contribution of our work. Roughly speaking, this result says that, with high probability, every eigenfunction on a square $Q$ with side length $L$ is supported on at least $L^{3/2-\varepsilon}$ many points in $Q$. We in fact prove something slightly stronger, as our unique continuation result needs to be adapted to the “frozen” and “free” sites formalism. See Theorem 3.5 below.

In analogy with the Wegner estimate for continuous noise, we expect that, with probability $1-e^{-\varepsilon L}$, there are no $\psi : \mathbb{Z}^d \to \mathbb{R}$ satisfying $H \psi = \lambda \psi$ in $Q$, $|\lambda - \bar{\lambda}| < e^{-L}$, and $\psi = 0$ on $\mathbb{Z}^d \setminus Q$. That is, we expect the above unique continuation theorem to be vacuous in the case of Dirichlet data. Of course, there is (as of our writing) no such Wegner estimate available in the Bernoulli case. Moreover, we apply this result below for $\psi$ with non-zero boundary data. Still, it is worth keeping in mind that our unique continuation theorem is quite weak, and barely suffices for our application to Anderson localization.

Another contribution of our work is a generalization of Sperner’s theorem, see Theorem 4.2, which we need to handle the fact that our unique continuation result only gives support on a sparse set.

All of the essentially new ideas in this article are presented in the third, fourth, and fifth sections. The remaining sections consist of straightforward modifications of the ideas in [5].

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2. Preliminaries

2.1. Spectrum. As described above, it is a standard fact that the spectrum of $H$ is almost surely the interval $[0, 9]$. Henceforth, we only concern ourselves with energies in this interval. In particular, $\lambda$ always denotes a real number in the interval $[0, 9]$. Moreover, we fix a target energy $\bar{\lambda} \in [0, 9]$ throughout the article.

2.2. Squares. Unless otherwise specified, the letter $Q$ denotes a dyadic square in $\mathbb{Z}^2$. That is, a set $Q = x + [0, 2^n)^2 \subseteq \mathbb{Z}^2$ with $x \in \mathbb{Z}^2$.

The side length and area of $Q$ are $\ell(Q) = 2^n$ and $|Q| = \ell(Q)^2 = 2^{2n}$. The notations $\frac{1}{2}Q$ and $2Q$ denote the concentric halving and doubling of $Q$, respectively.

2.3. Restrictions to finite sets. We frequently consider the restriction $H_Q = 1_Q H 1_Q$ of the Hamiltonian $H$ to squares $Q \subseteq \mathbb{Z}^2$. We use the notation $R_Q = (H_Q - \bar{\lambda})^{-1}$ to indicate (when it exists) the unique operator on $\ell(\mathbb{Z}^2)$ such that $R_Q = 1_Q R_Q 1_Q$ and $R_Q (H_Q - \bar{\lambda}) = (H_Q - \lambda) R_Q = 1_Q$. Abusing notation, we sometimes think of $H_Q$ and $R_Q$ as elements of the space $S^2(\mathbb{R}^Q)$ of symmetric bilinear forms on $\mathbb{R}^Q$. Similarly, we sometimes think of the restriction $V_Q$ as an element of the vector space $\mathbb{R}^Q$. 
2.4. **Notation.** We use Hardy notation for constants, letting $C > 1 > c > 0$ denote universal constants that may differ in each instance. We use subscripts to denote additional dependencies, so that $C_{\epsilon}$ is allowed to depend on $\epsilon$.

We use $\|H_Q\|$ and $\|H_Q\|_{2}$ to denote the operator and Hilbert-Schmidt norms of $H_Q \in S^2(\mathbb{R}^d)$. For functions $\psi \in \mathbb{R}^Q$, we make frequent use of the bounds $\|\psi\|_{L^\infty(Q)} \leq \|\psi\|_{L^2(Q)} \leq |Q|^{1/2}\|\psi\|_{L^\infty(Q)}$ to absorb differences of norms into exponential prefactors.

When $\psi : Q \to \mathbb{R}$ and $t \in \mathbb{R}$, we use $\{\|\psi\| \geq t\}$ as shorthand for the set $\{x \in Q : \|\psi(x)\| \geq t\}$.

We occasionally use notation from functional calculus. In particular, if $A \in S^2(\mathbb{R}^n)$ and $I \subseteq \mathbb{R}$, then trace $1_I(A)$ is the number of eigenvalues of $A$ in the interval $I$.

3. **Unique continuation with a random potential**

3.1. **Statement.** We prove a quantitative unique continuation result for eigenfunctions of $H$. Our argument generalizes the unique continuation result of [7] for harmonic functions on $\mathbb{Z}^2$. The basic idea is that, with high probability, every eigenfunction in the square $Q$ is supported on $\ell(Q)^{3/2 - \epsilon}$ many sites. The precise statement is made more complicated by the presence of frozen sites, which we need in our application to Anderson localization.

In order to state our result precisely, we need to define the $45^\circ$ rotations of rectangles and lines.

**Definition 3.1.** A tilted rectangle is a set

$$R_{I,J} = \{(x,y) \in \mathbb{Z}^2 : x + y \in I \text{ and } x - y \in J\},$$

where $I, J \subseteq \mathbb{Z}$ are intervals. A tilted square $Q$ is a tilted rectangle $R_{I,J}$ with side length $\ell(Q) = |I| = |J|$.

**Definition 3.2.** Given $k \in \mathbb{Z}$, define the diagonals

$$D_k^\pm = \{(x, y) \in \mathbb{Z}^2 : x \pm y = k\}.$$

We need a notion of sparsity along diagonals.

**Definition 3.3.** Suppose $F \subseteq \mathbb{Z}^2$ a set, $\delta > 0$ a density, and $Q$ a tilted square. Say that $F$ is $\delta$-sparse in $Q$ if

$$|D_k^\pm \cap F \cap Q| \leq \delta|D_k^\pm \cap Q|$$

for all diagonals $D_k^\pm$.

We need a notion of sparsity at all scales.

**Definition 3.4.** Say that $F$ is $\delta$-regular in the set $E \subseteq \mathbb{Z}^2$ if $\sum_k |Q_k| \leq \delta|E|$ holds whenever $F$ is not $\delta$-sparse in each of the disjoint tilted squares $Q_1, \ldots, Q_n \subseteq E$.

We now state our unique continuation theorem. This is the same as Theorem 1.6 except that it has been adapted to allow for a regular set of “frozen” sites.

**Theorem 3.5.** For every small $\epsilon > 0$, there is a large $\alpha > 1$ such that, if

1. $Q \subseteq \mathbb{Z}^2$ a square with $\ell(Q) \geq \alpha$
2. $F \subseteq Q$ is $\epsilon$-regular in $Q$
3. $v : F \to \{0, 1\}$
(4) \( \mathcal{E}_{uc}(Q, F) \) denotes the event that
\[
\begin{cases}
|\lambda - \bar{\lambda}| \leq e^{-\alpha(\ell(Q) \log \ell(Q))^{1/2}} \\
H\psi = \lambda\psi \text{ in } Q \\
|\psi| \leq 1 \text{ in } a - \varepsilon(\ell(Q) \log(\ell(Q)))^{-1/2} \text{ fraction of } Q \setminus F,
\end{cases}
\]
implies \( |\psi| \leq e^{\alpha(\ell(Q) \log \ell(Q))} \) in \( \frac{1}{2}Q \).

then \( \mathbb{P}[\mathcal{E}_{uc}(Q, F) | V_F = \nu] \geq 1 - e^{-\varepsilon(\ell(Q))^{1/4}} \).

The rest of this section is devoted to the proof of Theorem \ref{thm:main}

3.2. Tilted coordinates. We work in the tilted coordinates
\[ (s, t) = (x + y, x - y) \]
The lattice is \( \{(s, t) \in \mathbb{Z}^2 : s - t \text{ even}\} \). The tilted rectangles are \( R_{I,J} = \{(s, t) \in I \times J : s - t \text{ even}\} \). The equation \( H\psi = \lambda\psi \) at the point \( (s, t) \) is
\[ (4 + V_{s,t} - \lambda)\psi_{s,t} - \psi_{s-1,t-1} - \psi_{s+1,t+1} - \psi_{s-1,t+1} - \psi_{s+1,t-1} = 0 \]

3.3. Basic lemmas. We recall and modify some elementary results from \cite{7}. These give a priori bounds on how information propagates from the boundary to the interior of a tilted rectangle.

Definition 3.7. The west boundary of a tilted rectangle is
\[ \partial^w R_{[a,b],[c,d]} = R_{[a,a+1],[c,d]} \cup R_{[a,b],[c,c+1]} \]
The main idea is that, if the equation \( H\psi = \lambda\psi \) holds in a tilted rectangle \( R \), then the values of \( \psi \) on \( R \) are determined by the values of \( \psi \) on the west boundary \( \partial^w R \). A qualitative version of this is the following.

Lemma 3.8. Every function \( \psi : \partial^w R_{[1,a],[1,b]} \to \mathbb{R} \) has a unique extension \( \psi : R_{[1,a],[1,b]} \to \mathbb{R} \) that satisfies \( H\psi = \lambda\psi \) in \( R_{[2,a-1],[2,b-1]} \).

Proof. First, observe that the equation \( H\psi = \lambda\psi \) at \( (s - 1, t - 1) \) rearranges to
\[ \psi_{s,t} = (4 + V_{s-1,t-1} - \lambda)\psi_{s-1,t-1} - \psi_{s+1,t+1} - \psi_{s-1,t+1} - \psi_{s+1,t-1} - \psi_{s-1,t+1} - \psi_{s+1,t-1} = 0. \]
Second, observe that, if \( (s, t) \in R_{[1,a],[1,b]} \setminus \partial^w R_{[1,a],[1,b]} \), then \( (s - 1, t - 1) \in R_{[2,a-1],[2,b-1]} \) and \( (s - 2, t), (s - t - 2), (s - 2, t - 2) \in R_{[1,a],[1,b]} \). In particular, we can recursively iterate \( (3.9) \) for \( (s - 1, t - 1) \in R_{[2,a-1],[2,b-1]} \) to determine the values of \( \psi \) on \( R \) from the values of \( \psi \) in \( \partial^w R \).

Figure 3.1 displays several examples of tilted rectangles for which one can easily verify the algorithm from the proof of Lemma 3.8. By quantifying the rate of growth in this algorithm, we obtain the following.

Lemma 3.10. If \( H\psi = \lambda\psi \) in \( R = R_{[2,a-1],[2,b-1]} \), then
\[ \|\psi\|_{\ell^\infty(R_{[1,a],[1,b]})} \leq e^{Ch \log a} \|\psi\|_{\ell^\infty(\partial^w R_{[1,a],[1,b]})}. \]

Proof. We show that, if \( H\psi = \lambda\psi \) in \( R_{[2,a-1],[2,b-1]} \) and \( |\psi| \leq 1 \in \partial^w R_{[1,a],[1,b]} \), then \( |\psi_{s,t}| \leq (\alpha a)^t \) for \( (s, t) \in R_{[1,a],[1,b]} \). Here \( \alpha \geq 1 \) is a universal constant to be determined. We prove this by induction on \( (s, t) \). First, we note that, if \( (s, t) \in \partial^w R_{[1,a],[1,b]} \), then \( |\psi_{s,t}| \leq 1 \leq (\alpha a)^t \) holds by assumption. Second, we note
that, if \((s, t) \in R_{[a,b],[c,d]}\), then we can use the equation (3.9) and the induction hypothesis to estimate
\[
|\psi_{s,t}| \leq C|\psi_{s-1,t-1}| + |\psi_{s-2,t}| + |\psi_{s,t-1}| + |\psi_{s,t-2}|
\]
\[
\leq C(\alpha(s-1))^t + (\alpha(s-2))^t + (\alpha(s-2))^t
\]
\[
\leq (\alpha s)^t((\frac{2-1}{2})^t + C(\alpha s)^{-1})
\]
\[
\leq (\alpha s)^t(1 - 2s^{-1} + C(\alpha s)^{-1})
\]
\[
\leq (\alpha s)^t.
\]
Here we used \(|4 + V_{s-1,t-1} - \lambda| \leq C\) and \(\alpha \geq 2C\).

Differentiating the expression for \(\psi\) on \(R\) in terms of \(\psi\) on \(\partial^w R\) with respect to \(\lambda\) and using Lemma 3.10 we obtain the following quantitative estimate of the dependence on \(\lambda\).

**Lemma 3.11.** If \(H\psi_0 = \lambda_0\psi_0\) and \(H\psi_1 = \lambda_1\psi_1\) in \(R_{[2,a-1],[2,b-1]}\) and \(\psi_0 = \psi_1\) in \(\partial^w R_{[1,a],[1,b]}\), then
\[
\|\psi_0 - \psi_1\|_{\ell^\infty(R_{[1,a],[1,b]})} \leq C^\alpha \log b \|\psi_0\|_{\ell^\infty(\partial^n R_{[1,a],[1,b]})})|\lambda_0 - \lambda_1|.
\]

**Proof.** Using Lemma 3.8 let \(\psi_\lambda : R_{[1,a],[1,b]} \rightarrow \mathbb{R}\) be the unique function such that \(H\psi_\lambda = \lambda\psi_\lambda\) in \(R_{[2,a-1],[2,b-1]}\) and \(\psi_\lambda = \psi_0\) in \(\partial^n R_{[1,a],[1,b]}\). Using (3.9), we see that, if \((s, t) \in R_{[3,a],[3,b]}\), then \(\frac{\partial}{\partial \lambda}(\psi_\lambda)_{s,t} = -(\psi_\lambda)_{s-1,t-1}\). In particular, by Lemma 3.10 we have
\[
|\frac{\partial}{\partial \lambda}(\psi_\lambda)_{s,t}| \leq C^\alpha \log a \|\psi_0\|_{\ell^\infty(\partial^n R_{[1,a],[1,b]})}.
\]
Integrating over \(\lambda \in [\lambda_0, \lambda_1]\) gives the desired estimate.

**3.4. Key lemma.** We recall a key ingredient [7] Lemma 3.4] used in the upper bound in the Liouville theorem for harmonic functions on the lattice.

**Lemma 3.12.** If \(a \geq Cb\), \(\Delta u = 0\) in \(R_{[2,a-1],[2,b-1]}\), \(|u| \leq 1\) in \(R_{[1,a],[1,2]}\), and \(|u| \leq 1\) in a 1/2 fraction of \(R_{[1,a],[b,b]}\), then \(|u| \leq C^\alpha \log a\) in \(R_{[1,a],[1,b]}\).

\[\square\]
Note that the bounds on \( u \) in the above lemma are on opposite sides of the rectangle. The lemma says that, if \( u \) is harmonic in a rectangle, bounded on the northwest boundary, and bounded on most of the southeast boundary, then \( u \) is bounded on the entire rectangle.

The main idea of the proof of the above lemma is the observation that, if \( u = 0 \) on \( R_{1,2} \) and \( u \) is harmonic in \( R_{1,2} \), then function \( u_{s,t} = (-1)^{(s+t)2}u_{s,t} \) is a polynomial of degree at most \( t - 2 \) in the variable \( s \). Using the polynomial structure, the Remez inequality on the southeast boundary \( R_{1,2} \) takes us from bounded on half of the points to bounded on all of the points. This argument is delicate, and appears to break down in the presence of a potential. Indeed, there is no reason to expect that a solution of \( H\psi = \lambda\psi \) with \( \psi = 0 \) on \( R_{1,2} \) will have polynomial structure.

Instead, we view the solution map \( \mathbb{R}^{2^{2^{\alpha^2}}} \to \mathbb{R}^2 \) given by Lemma 3.8 as a random linear operator. We bound the right inverse of the solution map using an \( \varepsilon \)-net in combination with a Martingale argument. Here we need that the rectangle \( R_{1,2} \) is extremely thin, requiring \( a \geq Cb^2 \log a \).

**Lemma 3.13.** For every small \( \varepsilon > 0 \) there is a large \( \alpha > 1 \) such that, if

1. \( a \geq ab^2 \log a \geq \alpha \)
2. \( F \subseteq \mathbb{Z}^2 \) is \( \varepsilon \)-sparse in \( R_{1,2} \)
3. \( v : F \to \{0, 1\} \)
4. \( \mathcal{E}_{ni}(R_{1,2}) \) denotes the event that

\[
\begin{align*}
|\lambda - \bar{\lambda}| &\leq e^{-ab \log a} \\
H\psi &= \lambda\psi \text{ in } R_{2\alpha - 1,2b - 1} \\
|\psi| &\leq 1 \text{ in } R_{1,2} \\
|\psi| &\leq 1 \text{ in a } 1 - \varepsilon \text{ fraction of } R_{1,2}
\end{align*}
\]

implies \( |\psi| \leq e^{ab \log a} \) in \( R_{1,2} \),

then \( \mathbb{P}[\mathcal{E}_{ni}(R_{1,2}) | V_F = v] \geq 1 - e^{-\alpha a} \).

**Proof.** Let \( \mathcal{E}_{ni}'(R_{1,2}) \) denote the event that

\[
\begin{align*}
H\psi &= \bar{\lambda}\psi \text{ in } R_{2\alpha - 1,2b - 1} \\
\psi &= 0 \text{ in } R_{1,2} \\
\max_{R_{1,2}} |\psi| &\geq 1
\end{align*}
\]

implies \( |\psi| \geq e^{-\frac{1}{2} \log a} \) in a \( 2\varepsilon \) fraction of \( R_{1,2} \).

**Claim 3.14.** If \( \alpha \geq C \), then \( \mathcal{E}_{ni} \supseteq \mathcal{E}_{ni}' \).

Suppose \( |\lambda - \bar{\lambda}| \leq e^{-ab \log a} \). If \( H\psi = \lambda\psi \) in \( R_{2\alpha - 1,2b - 1} \), \( |\psi| \leq 1 \) in \( R_{1,2} \), and \( |\psi| \leq 1 \) in a \( 1 - \varepsilon \) fraction of \( R_{1,2} \). By Lemma 3.8 there are unique \( \psi, \psi' : R_{1,2} \to \mathbb{R} \) such that

\[
\begin{align*}
H\psi' &= \lambda\psi' \text{ in } R_{2\alpha - 1,2b - 1} \\
\psi' &= 0 \text{ in } R_{1,2} \\
\psi' &= \psi \text{ in } R_{1,2}
\end{align*}
\]
and
\[
\begin{cases}
H\psi'' = \lambda\psi'' & \text{in } R_{[2,a-1],[2,b-1]} \\
\psi'' = 0 & \text{in } R_{[1,a],[1,2]} \\
\psi'' = \psi & \text{in } R_{[1,2],[3,6]}.
\end{cases}
\]

Assuming $E'_{ni}$ holds, we see that $|\psi''| \geq e^{-\frac{1}{2}\alpha b \log a} \max_{R_{[1,2],[3,6]}} |\psi|$ in a $2\epsilon$ fraction of $R_{[1,a],[b-1,b]}$. Since Lemma 3.11 gives
\[
\max_{R_{[1,a],[1,b]}} |\psi' - \psi''| \leq e^{(C-\alpha)b \log a} \max_{R_{[1,2],[3,6]}} |\psi|,
\]
we see that
\[
|\psi'| \geq |\psi''| - |\psi' - \psi''| \\
\geq (e^{-\frac{1}{2}\alpha b \log a} - e^{(C-\alpha)b \log a}) \max_{R_{[1,2],[3,6]}} |\psi| \\
\geq \frac{1}{2} e^{-\frac{1}{2}\alpha b \log a} \max_{R_{[1,2],[3,6]}} |\psi|
\]
in a $2\epsilon$ fraction of $R_{[1,a],[b-1,b]}$. Since Lemma 3.10 gives
\[
\max_{R_{[1,a],[1,b]}} |\psi' - \psi| \leq e^{Cb \log a},
\]
we obtain that $|\psi| \geq \frac{1}{2} e^{-\frac{1}{2}\alpha b \log a} \max_{R_{[1,2],[3,6]}} |\psi| - e^{Cb \log a}$ in a $2\epsilon$ fraction of $R_{[1,a],[b-1,b]}$. Since $|\psi| \leq 1$ on a $1 - \epsilon$ fraction of $R_{[1,a],[b-1,b]}$, it follows that $\max_{R_{[1,2],[3,6]}} |\psi| \leq e^{(C+\frac{1}{2}\alpha)\log a}$. Another application of Lemma 3.10 gives $|\psi| \leq e^{(C+\frac{1}{2}\alpha)\log a}$ in $R_{[1,a],[1,6]}$. In particular, we see that $E'_{ni}$ implies $E_{ni}$.

We now estimate the probability that $E'_{ni}$ holds. Recall that, if $H\psi = \lambda\psi$ in $R_{[2,a-1],[2,b-1]}$ and $\psi = 0$ in $R_{[1,a],[1,2]}$, then the values of $\psi$ on the whole tilted rectangle $R_{[1,a],[1,6]}$ are determined by the potential $V$ and the values of $\psi$ on the southwest boundary $R_{[1,2],[3,6]}$. Let us write $\psi^0$ and $\psi^1$ for the restriction of $\psi$ to $R_{[1,2],[3,6]}$ and $R_{[1,a],[b-1,b]}$, respectively. We prove the lemma by studying the random linear mapping $\psi^0 \mapsto \psi^1$. Our goal is to show that, with high probability, if $|\psi^0| \geq 1$ on at least one site, then $|\psi^1| \geq e^{-\alpha b \log a}$ on an $\epsilon$ fraction of its domain.

In particular, the lemma follows from Claim 3.17 below.

**Claim 3.15.** For any fixed $\psi^0 : R_{[1,2],[3,6]} \to \mathbb{R}$, there is a $(s_0, t_0) \in R_{[1,2],[3,6]}$ such that
\[
|\psi| \geq e^{-Cb \log a} \|\psi^0\|_\infty \quad \text{in } R_{[1,a],[t_0,t_0]}
\]
holds for all choices of potential $V$.

For $\beta \geq 1$ to be determined, let $(s_0, t_0) \in R_{[1,2],[3,6]}$ maximize $e^{-\beta t_0} |\psi_{s_0,t_0}|$. It is enough to prove $|\psi| \geq 1/2$ in $R_{[1,2],[t_0,t_0]}$ under the assumption $|\psi_{s_0,t_0}| = 1$. For $t \in [1, t_0]$, let $m_t = \|\psi\|_{L^\infty(R_{[1,a],[1,t]})}$. Using (3.9), we observe that, if $(s, t) \in R_{[3,a],[3,t_0]}$, then
\[
|\psi_{s,t}| \leq |\psi_{s-2,t}| + Cm_{t-1} + Cm_{t-2}.
\]
Since $|\psi_{s,t}| \leq e^{\beta(t-t_0)}$ for $(s, t) \in R_{[1,2],[3,t_0]}$, induction gives
\[
m_t \leq e^{\beta(t-t_0)} + Cam_{t-1} \quad \text{for } t \geq 3.
\]
Since $m_1 = m_2 = 0$, assuming $\beta \geq C \log a$ gives $m_t \leq 2e^{\beta(t-t_0)}$. Using (3.13) again, we observe that, if $(s, t_0) \in R_{[3,a],[t_0,t_0]}$, then
\[
|\psi_{s,t_0}| \geq |\psi_{s-2,t_0}| - Ce^{-\beta}.
\]
Since \(|\psi_{s_0,t_0}| = 1\), we obtain, by induction, \(|\psi| \geq 1 - Cae^{-\beta}\) on \(R_{[1,2],\bar{t}_0}\). Assuming again that \(\beta \geq C \log a\), we obtain \(|\psi| \geq 1 - e^{-\beta/2}\) on \(R_{[1,2],\bar{t}_0}\). Since \(a \geq C\), we obtain the claim.

**Claim 3.16.** For any fixed \(\psi^0 : R_{[1,2],\bar{t}_0} \to \mathbb{R}\),
\[
P\{\|\psi^0\| \geq e^{-C\log a}\|\psi^0\|_{\infty}\} \geq \varepsilon a|V_F = v| \geq 1 - e^{-ca}.
\]

Select \((s_0, t_0) \in R_{[1,2],\bar{t}_0}\) using the previous claim. Suppose for the moment that \((s_1, t_1) \in R_{[3,\bar{a}],\bar{t}_1}\), and \((s_1 - 1, t_0) \in R_{[2,\bar{a}],\bar{t}_0} \setminus F\). Forming the alternating sum of the equation (3.6) at the points \((s, t_1 - 1) \in R_{[3,\bar{a}],\bar{t}_1}\) and using \(\psi = 0\) on \(R_{[1,2],\bar{t}_1}\), we obtain
\[
\psi_{s_1,t_1} = -\psi_{s_1-2,t_1} + \sum_{0 \leq k \leq \frac{t_1}{2}} (-1)^k (4 - \bar{\lambda} + V_{s_1-1,t_1-1-2k})\psi_{s_1-1,t_1-1-2k}.
\]

See Figure 3.2 for a schematic of this computation.

Since the values \(\psi_{s_1-1,t_1-1-2k}\) depend only on \(\psi^0\) and the potential \(V\) on the tilted rectangle \(R_{[1,\bar{a}],\bar{t}_1}\), we see that \(\psi_{s_1,t_1}\) depends on \(V_{s_1-1,\bar{t}_0}\) only through the term
\[
(-1)^{t_1-\bar{t}_0-1}(4 - \bar{\lambda} + V_{s_1-1,\bar{t}_0})\psi_{s_1-1,\bar{t}_0}.
\]

Since \(|\psi_{s_1-1,\bar{t}_0}| \geq e^{-C\log a}\|\psi^0\|_{\infty}\) holds almost surely, we obtain
\[
P\{|\psi_{s_1,t_1}| \geq e^{-C\log a}\|\psi^0\|_{\infty}\|V_F \cup R_{[1,\bar{a}],\bar{t}_1}| \geq 1/2.
\]
That is, \(\psi_{s_1,t_1}\) is sensitive to the variation of \(V_{s_1-1,\bar{t}_0}\).
Now, let \( s_1, ..., s_K \in [1, a] \) be an increasing list of all the integers \( s_k \in [1, a] \) such that \((s_k, t_k) \in E[3, a], [b-1, b]\) such that \((s_k - 1, t_k) \in E[2, a], [t_0, t_0] \setminus F\). Since \( F \) is \( \varepsilon \)-sparse, there are at least \( K \geq (\frac{2}{3} - \varepsilon) a - 3 \geq ca \) such integers. Let \( F_k \) denote the \( \sigma \)-algebra generated by \( V_F \cup E[1, s_k - 1, 1, \beta] \). By the above, we see that \( \psi_{s_k, t_k} \) is \( F_k \)-measurable and that
\[
P[|\psi_{s_k, t_k}| \geq e^{-Ch^2} \| \psi^0 \|_{\infty} | F_{k-1}] \geq 1/2.
\]
Therefore, assuming \( \varepsilon > 0 \) is a small universal constant, we may conclude the claim by Azuma’s inequality.

**Claim 3.17.** If \( X \) denotes the space of \( \psi^0 : E[1,3,3,1,1] \to \mathbb{R} \) with \( \| \psi^0 \|_{\infty} = 1 \), then
\[
P[ \inf_{\psi^0 \in X} \{ |\psi^1| \geq e^{-\alpha \log a} \} \geq \varepsilon a | V_F = v] \geq 1 - e^{-ca}.
\]

For any \( \beta \geq 1 \), we can choose a finite subset \( \tilde{X} \subseteq X \) such that \( |\tilde{X}| \leq e^{C \beta^2 \log a} \) and, for any \( \psi^0 \in X \), there is a \( \tilde{\psi}^0 \in \tilde{X} \) with \( \| \psi^0 - \tilde{\psi}^0 \|_{\infty} \leq e^{-\beta \log a} \). By Lemma 3.10, \( \| \psi^0 - \tilde{\psi}^0 \|_{\infty} \leq e^{-\beta \log a} \) implies \( \| \psi^1 - \tilde{\psi}^1 \|_{\infty} \leq e^{(C - \beta) \log a} \). In particular, we have
\[
\inf_{\psi^0 \in X} \{ |\psi^1| \geq e^{-\alpha \log a} - e^{(C - \beta) \log a} \} \leq \min_{\psi^0 \in X} \{ |\tilde{\psi}^1| \geq e^{-Ch^2 \log a} \}.
\]

By the previous claim and a union bound,
\[
P[ \inf_{\psi^0 \in X} \{ |\psi^1| \geq e^{-Ch^2 \log a} \} \geq \varepsilon a | V_F = v] \geq 1 - e^{C \beta^2 \log a - ca}.
\]

Assuming \( \beta \geq C, \alpha \geq \beta + C, \) and \( a \geq \alpha b^2 \log a \geq \alpha \), we obtain the claim. \( \square \)

3.5. **Growth lemma.** Using our key lemma, we prove a “growth lemma” suitable for use in a Calderon–Zygmund stopping time argument. Our proof is similar to that of [\textit{31}] Lemma 3.6 except that we are forced to use extremely thin rectangles. This leads to large support on only \( \ell (Q)^{3/2 - \varepsilon} \) many points.

**Lemma 3.18.** For every small \( \varepsilon > 0 \), there is a large \( \alpha > 1 \) such that, if
\begin{enumerate}
  \item \( Q \) tilted square with \( \ell (Q) \geq \alpha \)
  \item \( F \subseteq \mathbb{Z}^2 \) is \( \varepsilon \)-sparse in \( 2Q \)
  \item \( v : F \rightarrow \{0, 1\} \)
  \item \( E_{\text{ex}}(Q, F) \) denotes the event that
    \[
    \begin{aligned}
    |\lambda - \tilde{\lambda}| &\leq e^{-\alpha \ell (Q) \log \ell (Q)} e^{1/2} \\
    H \psi = \lambda \psi &\text{ in } 2Q \\
    |\psi| &\leq 1 \text{ in } \frac{1}{2} Q \\
    \psi &\leq 1 \text{ in a } 1 - \varepsilon (\ell (Q) \log \ell (Q))^{-1/2} \text{ fraction of } 2Q \setminus F
    \end{aligned}
    \]
\end{enumerate}
then \( \mathbb{P}[E_{\text{ex}}(Q, F) | V_F = v] \geq 1 - e^{-\varepsilon \ell (Q)} \).

**Proof.** Let \( E_{\text{ex}}' \) denote the event that
\[
\begin{aligned}
|\lambda - \tilde{\lambda}| &\leq e^{-\alpha (a \log a)} e^{1/2} \\
H \psi = \lambda \psi &\text{ in } 4Q[1, a], [1, a] \\
|\psi| &\leq 1 \text{ in } R[1, a], [1, a] \\
|\psi| &\leq 1 \text{ in a } (1 - \varepsilon (a \log a)^{-1/2}) \text{ fraction of } 4R[1, a], [1, a] \setminus F
\end{aligned}
\]
implies $|\psi| \leq e^{\alpha \ell(Q) \log \ell(Q)}$ in $R_{[1,a],[1,2a]}$. By the 90° symmetry of our problem and a covering argument, it is enough to prove that, for every small $\varepsilon > 0$, there is a large $\alpha > 1$ such that $\mathbb{P}[E_{ex}|V_F = v] \geq 1 - e^{-\varepsilon a}$.

Let $\alpha' > 1 > \varepsilon' > 0$ denote a valid pair of constants from Lemma 3.13. By a union bound, the event

$$E_{ni} = \bigcap_{|c,d| \leq \frac{1}{2}, a} E_{ni}(R_{[1,a],[c,d]}),$$

satisfies $\mathbb{P}[E_{ni}|V_F = v] \geq 1 - e^{-\varepsilon' a + C \log a}$. It suffices to prove that, for all $\varepsilon \in (0, c\varepsilon'(\alpha')^{-1/2})$, there is a large $\alpha > \alpha'$ such that $E_{ex} \supseteq E_{ni}$. Henceforth we assume that $E_{ni}$ and (3.19) hold. Our goal is to show that $|\psi| \leq e^{\alpha \ell(Q) \log \ell(Q)}$ in $R_{[1,a],[1,2a]}$.

Claim 3.20. There is a sequence $b_0 \leq \cdots \leq b_K \in [a, \frac{5}{2}a]$ such that

1. $b_0 = a$
2. $b_K \geq 2a$
3. $\frac{1}{2}a \leq \alpha'(b_{k+1} - b_k + 2)^2 \log a \leq a$ for $0 \leq k < K$
4. $|\psi| \leq 1$ on a $1 - \varepsilon'$ fraction of $R_{[1,a],[b_{k+1}-1,b_{k+1}]}$ for $0 \leq k < K$

Let $B$ denote the largest even integer such that $\alpha'B^2 \log a \leq a$. Assume that $b_k$ is already defined. Decompose the rectangle $R_{[1,a],[b_k+B/2,b_k+B]}$ as a disjoint union of diagonals,

$$R_{[1,a],[b_k+B/2,b_k+B]} = \bigcup_{b \in [b_k+B/2,b_k+B]} R_{[1,a],[b,b]}.$$

By hypothesis, $|\psi| > 1$ on at most $\varepsilon a^{3/2}(\log a)^{-1/2}$ many points in $4R_{[1,a],[1,a]} \setminus F$. On the other hand, there are at least $cBa \geq c a^{3/2}(\alpha' \log a)^{-1/2}$ many points in $R_{[1,a],[b_k+B/2,b_k+B]}$ and at most $\varepsilon cBa \leq C \varepsilon a^{3/2}(\alpha' \log a)^{-1/2}$ many points in $R_{[1,a],[b_k+B/2,b_k+B]} \cap F$. It follows that, since $\varepsilon \leq c\varepsilon'(\alpha')^{-1/2}$, there is a $b_{k+1} \in [b_k+B/2,b_k+B]$ such that $|\psi| \leq 1$ on a $1 - \varepsilon'$ fraction of $R_{[1,a],[b_{k+1}-1,b_{k+1}]}$. 

![Figure 3.3. A schematic for the proof of Lemma 3.18](image)
With the claim in hand, we apply $E_{ni}(R_{[1,a],[b_{k-1},b_{k+1}]})$ to conclude

$$
\|\psi\|_{L^∞(R_{[1,a],[b_{k-1},b_{k+1}]})} \leq e^{Cα(b_{k+1}−b_k)}\log a (1 + \|\psi\|_{L^∞(R_{[1,a],[b_{k-1},b_{k+1}]}))}.
$$

By induction, we obtain

$$
\|\psi\|_{L^∞(R_{[1,a],[1,2a]})} \leq (e^{Cα B \log a}) Cα/B \leq e^{αa \log a}.
$$

3.6. Covering argument. Theorem 3.5 is proved using a Calderon–Zygmund stopping time argument. This is a random version of [7, Proposition 3.9].

Proof of Theorem 3.5. For $Q \subseteq \mathbb{Z}^2$ a tilted square and $β > 1 > δ > 0$, let $E_{uc}'(Q, F)$ denote the event that

$$
(3.21)
\begin{cases}
|λ - \bar{λ}| \leq e^{-β(ℓ(Q)\log ℓ(Q))^{1/2}} \\
Hψ = λψ in Q \\
|ψ| \leq 1 in a 1 − δ(ℓ(Q)\log Q)^{-1/2} fraction of Q \setminus F,
\end{cases}
$$

implies $|ψ| \leq e^{β(ℓ(Q)\log ℓ(Q))}$ in $\frac{1}{16} Q$. By a covering argument, it is enough to prove that, if $F$ is $δ$-regular in $Q$ and $ℓ(Q) ≥ β$, then $P[E_{uc}'(Q, F) = v] ≥ 1 - e^{-δ(ℓ(Q))^{1/4}}$.

Indeed, for any square $Q \subseteq \mathbb{Z}^2$ we can find a list of tilted squares $Q_1, ..., Q_N \subseteq Q$ such that $\frac{1}{16} Q ≤ \bigcup Q_k, N ≤ C$, and $\min_k ℓ(Q_k) ≥ cℓ(Q)$. Now, if the conclusion of the theorem holds for each $Q_k'$ then it is also true for $Q$.

Let $α > 1 > ε > 0$ denote a valid pair of constants for Lemma 3.18. We may assume $ε^2 > δ$ and $β ≥ 2α$. Let $Q$ denote the set of tilted squares $Q' \subseteq Q$ such that

(1) $ℓ(Q') ≥ ℓ(Q)^{1/4}$
(2) $2Q' \subseteq \frac{1}{4} Q$
(3) $F$ is $ε$-sparse in $2Q'$
(4) $\frac{1}{4} Q' \cap \frac{1}{16} Q \neq \emptyset$.

By a union bound, the event

$$
E_{ex} = \bigcap_{Q' \in Q} E_{ex}(Q', F) \cap E_{ex}(2Q', F)
$$

satisfies $P[E_{ex}'(Q, F) = v] ≥ 1 - e^{-ε(ℓ(Q))^{1/4} + C\log ℓ(Q)} ≥ 1 - e^{-δ(ℓ(Q))^{1/4}}$. Thus, it suffices to prove $E_{uc}'(Q, F) \supseteq E_{ex}$.

Henceforth we assume that $E_{ex}$ and (3.21) hold. Our goal is to prove $|ψ| ≤ e^{β(ℓ(Q)\log ℓ(Q))}$ in $\frac{1}{16} Q$. Let $Q_g ⊆ Q$ denote the set of tilted squares $Q' \in Q$ such that

$$
\|\psi\|_{L^∞(Q')} \leq e^{β(ℓ(Q')\log ℓ(Q'))}.
$$

Let $Q_{mg} \subseteq Q_g$ denote the $Q' \in Q_g$ that are maximal with respect to inclusion.

Claim 3.22. If $Q' \in Q_{mg}$, then one of the following holds.

(1) $4Q' \not\subseteq \frac{1}{2} Q$
(2) $F$ is not $δ$-sparse in $4Q'$
(3) $|\{ |ψ| ≥ 1 \} \cap 4Q' | ≥ ε(ℓ(Q)\log ℓ(Q))^{1/2}|4Q'|$.

Suppose $Q' \in Q_g$ and all three conditions are false. Observe that $4Q' \subseteq Q$ and, since $F$ is $δ$-sparse in $4Q'$ and we have assumed that $δ ≤ ε^2 < ε$, we see that $F$ is
Claim 3.23. If $Q$, an application of Lubell’s proof \[16\] of the Lubell–Yamamoto–Meshalkin inequality. Recall where the three alternatives from the previous claim hold, then $\varepsilon$-sparse in $4Q'$. Thus, the event $E_{\text{ex}}(2Q', F)$ holds and we see that

$$
\|\psi\|_{\ell^\infty(2Q')} \leq e^{\alpha(t(2Q') \log t(2Q')} \max \{1, \|\psi\|_{\ell^\infty(Q')} \}
\leq e^{\alpha(t(2Q') \log t(2Q')} e^{\beta(t(Q') \log t(Q')}
\leq e^{\beta(t(2Q') \log t(2Q')}.
$$

Here we used $\beta \geq 2\alpha$. Since we also have $\frac{1}{4}(2Q') \cap \frac{1}{64}Q \supseteq \frac{1}{4}Q' \cap \frac{1}{64}Q \neq \emptyset$, we see that $2Q' \in \mathcal{Q}_\varepsilon$. In particular, $Q'$ is not maximal with respect to inclusion.

Claim 3.25. If $Q^k_{mg} \subseteq Q$, for $k = 1, 2, 3$, denote the sets of maximal good squares where the three alternatives from the previous claim hold, then $|\cup Q^2_{mg}| + |\cup Q^3_{mg}| \leq C(\delta + \delta/\varepsilon)|Q|$. Here $\cup Q_{mg}$ is shorthand for $\cup_{Q \in \mathcal{Q}_{mg}} Q$.

By the Vitali covering lemma (see for example Stein \[19\] Section 3.2), we can find $\tilde{Q}^k_{mg} \subseteq Q^k_{mg}$ such that $|\cup \tilde{Q}^k_{mg}| \geq c|\cup Q^k_{mg}|$ and $Q', Q'' \in \tilde{Q}^k_{mg}$ implies $4Q' \cap 4Q'' = \emptyset$. Since $F$ is $\delta$-regular, we must have $|\cup \tilde{Q}^k_{mg}| \leq C\delta|Q|$. Since $|\psi| \leq 1$ on a $1 - \delta(t(Q) \log t(Q))^{-1/2}$ fraction of $Q$, we must have $|\cup \tilde{Q}^3_{mg}| \leq C(\delta/\varepsilon)|Q|$.\hfill\square

Claim 3.24. If $\delta > 0$ are sufficiently small, then $|\cup Q_{sl}| \geq c|Q|$.

Observe that we can find a list of tilted squares $Q'_1, \ldots, Q'_K \subseteq \frac{1}{64}Q$ such that $2Q'_k$ are disjoint, $t(Q'_k) \geq t(Q)^{1/4}$, and $K \geq c\ell(Q)^{3/2}$. Since $F$ is $\delta$-regular, we have $F$ is $\varepsilon$-sparse in $2Q'_k$ for a $1 - C\delta$ fraction of the $Q'_k$. Since $|\psi| > 1$ on at most $\delta(t(Q)^{3/2}(\log t(Q))^{-1/2}$ many points in $Q$ and there are at least $c\ell(Q)^{3/2}$ squares $Q'_k$, we must have $|\psi| \leq 1$ on at least half of the $Q'_k$. Assuming $\delta > 0$ is sufficiently small, we see that $Q_{sl}$ contains at least $c|Q|^{3/2}$ disjoint squares with volume $|Q|^{1/2}$.

Claim 3.25. $|\psi| \leq e^{\beta(t(Q) \log t(Q)}$ in $\frac{1}{64}Q$.

Since we assumed $\varepsilon^2 > \delta > 0$ and $\varepsilon > 0$ small, Claim 3.23 and Claim 3.24 together imply there is a $Q' \in Q^2_{mg} \cap (Q^2_{mg} \cup Q^3_{mg})$. Together $Q' \in Q_s$, $Q' \notin Q^2_{mg}$, $Q' \notin Q^3_{mg}$, and $E_{\text{ex}}(2Q', F)$ imply that $|\psi|_{\ell^\infty(2Q')} \leq e^{\beta(t(2Q') \log t(2Q')}$. Since $4Q' \subseteq \frac{1}{2}Q$ and $Q' \cap \frac{1}{64}Q \neq \emptyset$, we have $\frac{1}{64}Q \subseteq 2Q'$ and the claim.\hfill\square

4. SPERNER’S THEOREM

We prove a generalization of Sperner’s theorem \[18\]. Our argument is a modification of Lubell’s proof \[19\] of the Lubell–Yamamoto–Meshalkin inequality. Recall that a Sperner family is a set $\mathcal{A}$ of subsets of $\{1, \ldots, n\}$ that form an antichain with respect to inclusion. We consider a relaxation of this condition.

Definition 4.1. Suppose $\rho \in (0, 1]$. A set $\mathcal{A}$ of subsets of $\{1, \ldots, n\}$ is $\rho$-Sperner if, for every $A \in \mathcal{A}$, there is a set $B(A) \subseteq \{1, \ldots, n\} \setminus A$ such that $|B(A)| \geq \rho(n - |A|)$ and $A \subseteq A' \in \mathcal{A}$ implies $A' \cap B(A) = \emptyset$.

Note that a Sperner family is 1-Sperner with $B(A) = \{1, \ldots, n\} \setminus A$. In particular, the following result is a generalization of Sperner’s theorem.

Theorem 4.2. If $\rho \in (0, 1]$ and $\mathcal{A}$ is a $\rho$-Sperner set of subsets of $\{1, \ldots, n\}$, then

$$
|\mathcal{A}| \leq 2^n n^{-1/2} \rho^{-1}.
$$

Proof. Let $\Pi_n$ denote the set of permutations of $\{1, \ldots, n\}$. For $\sigma \in \Pi_n$, let $\mathcal{A}_\sigma = \{\sigma_1, \ldots, \sigma_k \in \mathcal{A} : k = 0, \ldots, n\}$. For $k \geq 0$, let $\mathcal{A}_k = \{A \in \mathcal{A} : |A| = k\}$.\hfill\square
Claim 4.3. \(_{0}^{n} \leq 2^{n}n^{-1/2}\) for \(k = 0, \ldots, n\).

This is standard.

Claim 4.4. If \(A \in A_{k}\), then \(|\{\sigma \in \Pi_{n} : A \in A_{\sigma}\}| = k!(n - k)!\).

This follows because \(A \in A_{\sigma}\) if and only if \(\sigma\) is a permutation of \(A\) concatenated with a permutation of \(\{1, \ldots, n\} \setminus A\).

Claim 4.5. If \(|j| \geq 0\), then \(|\{\sigma \in \Pi_{n} : |A_{\sigma}| \geq j + 1\}| \leq n!(1 - \rho)^{j}\).

Sample a uniform random \(\sigma \in \Pi_{n}\) one element at a time. In order to have \(|A_{\sigma}| \geq j + 1\), there must be a least \(k \geq 0\) such that \(\{\sigma_{1}, \ldots, \sigma_{k}\} \in A\). Moreover, by the \(\rho\)-Sperner property, it must also be the case that \(\sigma_{k+i} \notin B(\{\sigma_{1}, \ldots, \sigma_{k}\})\) for \(i = 1, \ldots, j\). Each time we sample the next \(\sigma_{k+i}\), the probability that \(\sigma_{k+i} \notin B(\{\sigma_{1}, \ldots, \sigma_{k}\})\) is at most \(1 - \rho\). In particular, the probability a uniform random \(\sigma \in \Pi_{n}\) has \(|A_{\sigma}| \geq j + 1\) is at most \((1 - \rho)^{j}\).

Using the claims, compute

\[
|A| = \sum_{k \geq 0} |A_{k}|
\]

\[
\leq 2^{n}n^{-1/2} \sum_{k \geq 0} \frac{k!(n - k)!}{n!} |A_{k}|
\]

\[
= 2^{n}n^{-1/2} \sum_{\sigma \in \Pi_{n}} \frac{1}{n!} |A_{\sigma}|
\]

\[
= 2^{n}n^{-1/2} \sum_{j \geq 0} \frac{1}{n!} |\{\sigma \in \Pi_{n} : |A_{\sigma}| \geq j + 1\}|
\]

\[
\leq 2^{n}n^{-1/2} \sum_{j \geq 0} (1 - \rho)^{j}
\]

\[
= 2^{n}n^{-1/2} \rho^{-1}.
\]

Here the second, third, and fifth steps follow from claims, in order. \(\square\)

5. Wegner Estimate

We recall the Courant–Fischer–Weyl min-max principle, which says that, for \(A \in S^{2}(\mathbb{R}^{n})\), the eigenvalues \(\lambda_{1}(A) \geq \cdots \geq \lambda_{n}(A)\) can be computed by

\[
\lambda_{k}(A) = \max_{V \subseteq \mathbb{R}^{n}} \min_{v \in V} \|Av\| = \min_{V \subseteq \mathbb{R}^{n}} \max_{v \in V} \|Av\|.
\]

We use this to prove the following eigenvalue variation result.

Lemma 5.1. Suppose the real symmetric matrix \(A \in S^{2}(\mathbb{R}^{n})\) has eigenvalues \(\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{n} \in \mathbb{R}\) with orthonormal eigenbasis \(v_{1}, v_{2}, \ldots, v_{n} \in \mathbb{R}^{n}\). If

1. \(0 < r_{1} < r_{2} < r_{3} < r_{4} < r_{5} < 1\)
2. \(r_{1} \leq \min\{r_{3}r_{5}, r_{2}r_{3}/r_{1}\}\)
3. \(0 < \lambda_{j} \leq \lambda_{i} \leq r_{1} < r_{2} < \lambda_{i-1}\)
4. \(r_{i,k}^{2} \geq r_{3}\)
5. \(r_{i,k}^{2} \leq r_{4}\) whenever \(r_{2} < \lambda_{i} < r_{3}\),

then \(\text{trace} 1_{[r_{1}, \infty)}(A) < \text{trace} 1_{[r_{1}, \infty)}(A + e_{k} \otimes e_{k})\), where \(e_{k} \in \mathbb{R}^{n}\) is the \(k\)th standard basis element.
Lemma 5.2

Hilbert-Schmidt norm $\| \cdot \|_{HS}$ implies that $\| A \|^2 \geq \text{tr}(A^2)$ for any Hermitian matrix $A$. This is a direct consequence of the Cauchy-Schwarz inequality and the fact that the trace of a matrix is equal to the sum of its eigenvalues.

**Proof.** It is enough to show $\lambda_1(A') \geq r_1$, where $A' = A + \varepsilon_k \otimes e_k$. Let $W_{i,j}$ denote the span of $\{v_1, \ldots, v_{i-1}, v_j\}$. The min-max principle gives

$$\lambda_i(A + \varepsilon_k \otimes e_k) = \max_{W \subseteq \mathbb{R}^n} \min_{w \in W} \langle w, A' w \rangle \geq \min_{w \in W_{i,j}} \langle w, A' w \rangle.$$ 

Every unit vector $w \in W_{i,j}$ can be written $\alpha_1 v_j + \alpha_2 w_2 + \alpha_3 w_3$, where $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, $w_2 \in \text{Span}\{v_j : r_2 < \lambda_j < r_3\}$ a unit vector, and $w_3 \in \text{Span}\{v_j : r_5 \leq \lambda_j\}$ a unit vector. We break into three cases. If $\alpha_2^2 \geq cr_3/r_4$, then $\langle w, A' w \rangle \geq \alpha_2^2 (w_2, Aw_2) \geq \alpha_2^3 r_2 \geq cr_2 r_3/r_4 \geq r_1$. If $\alpha_3^2 \geq cr_3$, then $\langle w, A' w \rangle \geq \alpha_3^2 (w_3, Aw_3) \geq \alpha_3^2 r_5 \geq cr_3 r_5 \geq r_1$. If $\alpha_2^2 < cr_3/r_4$ and $\alpha_3^2 < cr_3$, then $\langle w, A' w \rangle \geq (\alpha_1 v_j, k + \alpha_2 w_2, k + \alpha_3 w_3, k)^2 \geq \frac{1}{2} \alpha_1^2 r_3 - 2 \alpha_2^2 r_4 - 2 \alpha_3^2 \geq cr_3 r_1$. Here we used the fact that $(x + y + z)^2 \geq \frac{1}{2} x^2 - 2 y^2 - 2 z^2$ holds for all $x, y, z \in \mathbb{R}$ and that our hypotheses give $v_{2,3}^2 \geq r_3$ and $w_{2,3}^2 \leq r_4$. \hfill $\square$

We need the following bound on the size of a family of almost orthonormal vectors. An almost identical version of the following lemma and proof appears in Tao [21] as a step in a proof of a version of the Kabatjanskii–Levenstein [12] bound.

**Lemma 5.2 (Tao [21]).** If $v_1, \ldots, v_m \in \mathbb{R}^n$ satisfy $|v_i \cdot v_j - \delta_{ij}| \leq (5n)^{-1/2}$, then $m \leq (5 - \sqrt{5})n/2$.

**Proof.** Let $A \in S^2(\mathbb{R}^m)$ be given by $A_{ij} = v_i \cdot v_j$. Since the matrix $A$ has rank at most $n$, the matrix $A - I$ has eigenvalue $-1$ with multiplicity at least $m - n$ and Hilbert-Schmidt norm $\|A - I\|_2^2 \geq m - n$. On the other hand, the hypothesis gives $\|A - I\|_2^2 = \sum_{i,j} (v_i \cdot v_j - \delta_{ij})^2 \leq m^2/(5n)$. In particular, $m - n \leq m^2/(5n)$ which implies that $m$ cannot be in the interval $(5 - \sqrt{5})n/2, (5 + \sqrt{5})n/2$. Since this interval contains a positive integer whenever $n \geq 1$, we must have that $m$ is less than or equal to its left endpoint. \hfill $\square$

We need the following weak unique continuation bound. The advantage over Theorem 5.3 is that it holds a priori.

**Lemma 5.3.** For every integer $K \geq 1$, if

1. $L \geq C_K L'$ and $L' \geq C_K$
2. $Q \subseteq \mathbb{Z}^2$ with $\ell(Q) = L$
3. $Q_k \subseteq Q$ with $\ell(Q_k') = L'$ for $k = 1, \ldots, K$
4. $H_Q \psi = \lambda \psi$,

then

$$\|\psi\|_{L^\infty(Q')} \geq e^{-C_K L'} \|\psi\|_{L^\infty(Q)}$$

holds for some $Q' \subseteq Q \cup \bigcup_k Q_k'$ with $\ell(Q') = L'$.

**Proof.** We recall two elementary observations, both of which can be found in Bourgain–Klein [6]. The first is that, if $H \psi = \lambda \psi$ in a rectangle $[0,a] \times [0,b]$, then

$$\max_{[0,a] \times [0,b]} |\psi| \leq e^{C(a+b)} \max_{[2,a-2] \times [2,b-2]} |\psi|$$

The second is that, if $H \psi = \lambda \psi$ in the rectangle $[0,a] \times [-a,a]$, then

$$\max_{0 \leq x \leq a-|y|} |\psi(x,y)| \leq e^{Ca} \max_{0 \leq x \leq 1} |\psi(x,y)|.$$

Both of these observations follow by iterating the equation $H \psi = \lambda \psi$. We use them to prove the following two claims, from which the lemma easily follows.
Figure 5.1. A schematic for the proof of Lemma 5.3. The dotted line is the boundary of the set $E$. The dashed lines and arrows indicate where the second observation is being used to bound $|\psi|$ on the annulus $Q \cap 4Q' \setminus 2Q'$.

Claim 5.4. $\|\psi\|_{\ell^\infty(Q \cup_k 8Q'_k)} \leq e^{CL} \|\psi\|_{\ell^\infty(E)}$, where $E = \bigcup_{Q' \subseteq Q \cup_k Q'_k, \ell(Q') = L'} \frac{1}{2}Q'$.

Observe that, if $x \in Q \cup_k 8Q'_k$, then either $x \in E$ or there is a $Q' \subseteq \mathbb{Z}^2$ with $x \in \frac{1}{2}Q' \subseteq Q$, $\ell(Q') = L'$, and $8Q' \cap \cup_k Q'_k = \emptyset$. The second observation gives $|\psi| \leq e^{CL'} \|\psi\|_{\ell^\infty(E)}$ in $(4Q' \setminus 2Q') \cap Q$. The first observation then gives $|\psi| \leq e^{CL} \|\psi\|_{\ell^\infty(E)}$ in $2Q' \cap Q$. See Figure 5.1 for a schematic of this computation.

Claim 5.5. $\|\psi\|_{\ell^\infty(Q)} \leq e^{CL'} \|\psi\|_{\ell^\infty(Q \cup_k 8Q'_k)}$.

To estimate $\psi$ on $8Q'_k \cap Q$, we observe that there is a square $Q''_k \subseteq \mathbb{Z}^2$ such that $8Q'_k \subseteq Q''_k$, $2Q''_k \cap \cup_j Q'_j = Q''_k \cap \cup_j Q'_j$, and $\ell(Q''_k) \leq C_{K} L'$. (In fact, we may take $\ell(Q''_k) \leq CS_{K} L'$.) In particular, $|\psi| \leq e^{CL'} \|\psi\|_{\ell^\infty(Q \cup_k 8Q'_k)}$ on $Q \cap 2Q''_k \setminus Q''_k$. The first observation then gives $|\psi| \leq e^{CL'} \|\psi\|_{\ell^\infty(Q \cup_k 8Q'_k)}$ in $8Q'_k \cap Q$. \hfill \Box

We now prove our analogue of the Wegner estimate [5, Lemma 5.1]. This brings together all of our new ingredients.

Lemma 5.6. If

1. $\eta > \varepsilon > \delta > 0$ small
2. $K \geq 1$ integer
3. $L_0 \geq \cdots \geq L_5 \geq C_{\eta, \varepsilon, \delta, K}$
4. $Q \subseteq \mathbb{Z}^2$ with $\ell(Q) = L_0$
5. $Q'_1, \ldots, Q'_K \subseteq Q$ with $\ell(Q'_k) = L_3$
6. $G \subseteq \cup_k Q'_k$ with $0 < |G| \leq L_0^3$
7. $F \subseteq \mathbb{Z}^2$ is $\eta$-regular in every $Q' \subseteq Q \setminus \cup_k Q'_k$ with $\ell(Q') = L_3$
(8) \( V_F = v \), \( |\lambda - \bar{\lambda}| \leq e^{-L_5} \), and \( H_Q \psi = \lambda \psi \) implies
\[
e^{-L_4} \|\psi\|_{L^2(Q \cup Q')} \leq \|\psi\|_{L^2(Q)} \leq (1 + L_0^{-\delta}) \|\psi\|_{L^2(Q)},
\]
then
\[
\mathbb{P} [\|R_Q\| \leq e^{L_1} |V_F = v| \geq 1 - L_0^{-C_\varepsilon^{-1/2}}.
\]

**Remark 5.7.** The scales in the above lemma have the following interpretations:
- \( L_0 \): large scale
- \( e^{-L_1} \): large scale resolvent bound
- \( e^{-L_2} \): unique continuation lower bound
- \( L_3 \): small scale
- \( e^{-L_4} \): localization smallness
- \( e^{-L_5} \): localization interval
- \( L_6 \): localization support

These are set up to be compatible with the multiscale analysis below.

**Proof.** Throughout the proof, we allow the constants \( C > 1 > c > 0 \) to depend on \( \eta, \varepsilon, \delta, K \). Let \( \lambda_1(H_Q) \geq \cdots \geq \lambda_{L_0^2}(H_Q) \) denote the eigenvalues of \( H_Q \). Choose eigenvectors \( \psi_k(H_Q) \in \mathbb{R}^Q \) such that \( \|\psi_k\|_{L^\infty(Q)} = 1 \) and \( H_Q \psi_k = \lambda_k \psi_k \). We may assume that \( \lambda_k \) and \( \psi_k \) are deterministic functions of the potential \( V_Q \in [0, 1]^Q \).

**Claim 5.8.** We may assume \( \cup_k Q_k' \subseteq F \).

Let \( F' = \cup_k Q_k' \setminus F \) and observe that
\[
\mathbb{P} [\mathcal{E} | V_F = v] = 2^{-|F'|} \sum_{v': F' \to \{0, 1\}} \mathbb{P} [\mathcal{E} | V_{F \cup F'} = v \cup v']
\]
holds for all events \( \mathcal{E} \). Thus, it suffices to estimate each term in the sum.

**Claim 5.9.** \( \mathbb{P} [\mathcal{E}_{uc} | V_F = v] \geq 1 - e^{-L_3^2} \), where \( \mathcal{E}_{uc} \) denotes the event that
\[
\{\{\|\psi\| \geq e^{-L_2} \|\psi\|_{L^\infty(Q)}\} \setminus F \} \geq L_4^{3/2}
\]
holds whenever \( |\lambda - \bar{\lambda}| \leq e^{-L_5} \) and \( H_Q \psi = \lambda \psi \).

Let \( \alpha > 1 > \varepsilon' > 0 \) be constants that work in Theorem 5.3. We may assume \( \varepsilon' > \eta \). By Theorem 5.3, the event
\[
\mathcal{E}_{uc}' = \bigcap_{Q' \subseteq Q \setminus \cup_k Q_k'} \mathcal{E}_{uc}(Q', F)
\]
satisfies
\[
\mathbb{P} [\mathcal{E}_{uc}' | V_F = v] \geq 1 - e^{-\varepsilon' L_4^{1/4} - C \log L_0} \geq 1 - e^{-L_6}.
\]
Thus, it suffices to show \( \mathcal{E}_{uc} \supseteq \mathcal{E}_{uc}' \).

Let us suppose that \( \mathcal{E}_{uc}' \) holds, \( |\lambda - \bar{\lambda}| \leq e^{-L_5} \), and \( H_Q \psi = \lambda \psi \).

Lemma 5.3 provides an \( L_3 \)-square \( Q' \subseteq Q \setminus \cup_k Q_k' \)
\[
\|\psi\|_{L^\infty(Q')} \geq e^{-C L_3} \|\psi\|_{L^\infty(Q)}.
\]
Since \( \mathcal{E}_{uc}(Q', F) \) holds and \( |\lambda - \bar{\lambda}| \leq e^{-L_5} \leq e^{-\alpha(L_3 \log L_3)^{1/2}} \), we see that
\[
\{\{\|\psi\| \geq e^{-\alpha L_3 \log L_3} \|\psi\|_{L^\infty(Q')}\} \cap Q' \setminus F \} \geq \varepsilon' L_3^{3/2} (\log L_3)^{-1/2}.
\]
Thus
\[
\{\{\|\psi\| \geq e^{-L_2} \|\psi\|_{L^\infty(Q)}\} \cap Q \setminus F \} \geq L_4^{3/2},
\]
which proves the inclusion and the claim.

**Claim 5.10.** For $1 \leq k_1 \leq k_2 \leq L_0^3$ and $0 \leq \ell \leq CL_0^3$, we have

$$\Pr[\mathcal{E}_{k_1,k_2,\ell} \cap \mathcal{E}_{uc} | V_F = v] \leq CL_0L_4^{-3/2}$$

where $\mathcal{E}_{k_1,k_2,\ell}$ denotes the event that

$$|\lambda_{k_1} - \bar{\lambda}|, |\lambda_{k_2} - \bar{\lambda}| < s_\ell \quad \text{and} \quad |\lambda_{k_1-1} - \bar{\lambda}|, |\lambda_{k_2+1} - \bar{\lambda}| \geq s_{\ell+1},$$

and $s_\ell = e^{-L_4 + (L_2 - L_4 + C)\ell}.$

Since we are conditioning on $V_F = v$, we can view the events $\mathcal{E}_{uc}$ and $\mathcal{E}_{k_1,k_2,\ell}$ as subsets of $\{0,1\}^{Q\setminus F}$. For $1 \leq k_1 \leq k_2 \leq L^2$ and $i = 0,1$, let $\mathcal{E}_{k_1,k_2,\ell,i}$ denote the event that

$$\mathcal{E}_{k_1,k_2,\ell} \quad \text{and} \quad \{||\psi_{k_1}| \geq e^{-L_2} \cap \{V_Q = i\} \setminus F| \geq \frac{1}{2}L_4^{3/2}\}.$$

Observe that

$$\mathcal{E}_{k_1,k_2,\ell} \cap \mathcal{E}_{uc} \subseteq \mathcal{E}_{k_1,k_2,\ell,0} \cup \mathcal{E}_{k_1,k_2,\ell,1}.$$

Next, observe that if $w \in \mathcal{E}_{k_1,k_2,\ell,i}$, $x \in Q \setminus F$, $w(x) = i$, and $|\psi_{k_1}(x)| \geq e^{-L_2}$, then $w^x \notin \mathcal{E}_{k_1,k_2,\ell,i}$, where

$$w^x(y) = \begin{cases} w(y) & \text{if } y \neq x \\ 1 - w(y) & \text{if } y = x. \end{cases}$$

Indeed, this follows by applying Lemma 5.1 centered at $\bar{\lambda} - s_\ell$ (and its version for $-H_Q$ centered at $s_\ell - \bar{\lambda}$) and with radii $r_1 = s_\ell$, $r_2 = s_{\ell+1}$, $r_3 = e^{-L_2}$, $r_4 = e^{-L_4}$, and $r_5 = e^{-L_5}$. Here we need $\ell \leq CL_0^3$ and $L_5 \geq C$ to guarantee that the hypotheses of Lemma 5.1 hold.

By definition of $\mathcal{E}_{k_1,k_2,\ell,i}$, if $w \in \mathcal{E}_{k_1,k_2,\ell,i}$, then the set of $x \in Q \setminus F$ for which $w(x) = i$ and $|\psi_{k_1}(x)| \geq e^{-L_2}$ has size at least $\frac{1}{2}L_4^{3/2}$. Since $Q \setminus F$ has size at most $L_0^3$, we see that $\mathcal{E}_{k_1,k_2,\ell,i}$ is $\frac{1}{2}L_0^{-2}L_4^{3/2}$-Sperner. Applying Theorem 4.2, we obtain

$$\Pr[\mathcal{E}_{k_1,k_2,\ell,i} | V_F = v] \leq CL_0L_4^{-3/2}.$$ 

This gives the claim.

**Claim 5.11.** There is a set $K \subseteq \{1,...,L_0^3\}$, depending only on $F$ and $v$, such that $|K| \leq CL_0^3$ and

$$\{|\|R_Q\| > e^{L_1}\} \cap \{V_F = v\} \subseteq \bigcup_{k_1,k_2 \in K} \mathcal{E}_{k_1,k_2,\ell}.$$

Since we are conditioning on $V_F = v$, we can view $\lambda_k$ and $\psi_k$ as functions on $\{0,1\}^{Q\setminus F}$. Let $1 \leq k_1 < \cdots < k_m \leq L_0^3$ list all indices $k_i$ for which there is at least one $w \in \{0,1\}^{Q\setminus F}$ such that $|\lambda_{k_i}(w) - \bar{\lambda}| \leq e^{-L_2}$. To prove the claim, it suffices to prove that $m \leq CL_0^3$. Indeed, we can always find an $0 \leq \ell \leq m$ such that the annulus $[\lambda - s_{\ell+1}, \lambda + s_{\ell+1}] \setminus [\lambda - s_\ell, \lambda + s_\ell]$ contains no eigenvalue of $H_Q$.

Since $\cup_k Q_k^c \subseteq F$, the left-hand side of hypothesis (8) says that $w \in \{0,1\}^{Q\setminus F}$ and $|\lambda_k(w) - \bar{\lambda}| \leq e^{-L_5}$ implies $|\psi_k(w)|_{\ell^\infty(Q\setminus F)} \leq e^{-L_4}$. In particular, the min-max principle implies that

$$|\lambda_k(w) - \bar{\lambda}| \leq e^{-L_1} + |Q|e^{-L_4} \leq e^{-L_5}$$

holds for all $w \in \{0,1\}^{Q\setminus F}$. 

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Lemma 6.1. geometric resolvent identity. ministic part of the multiscale analysis. We need the following conseque nce of the

Proof. Write

\[ P[\| R_Q \| > e^{L_1}|V_F = v] \]
\[ \leq P[\mathcal{E}_{uc}|V_F = v] + \sum_{k_1,k_2 \in K} P[\mathcal{E}_{k_1,k_2,\ell} \text{ and } \mathcal{E}_{uc}|V_F = v] \]
\[ \leq e^{-L_0^\delta} + CL_0^{1+3\delta} L_4^{-3/2} \]
\[ \leq L_0^{10\epsilon - 1/2} \]
using the above claims. \qed

6. The Geometric Resolvent Identity

We prove a discrete analogue of [5, Section 2], which encapsulates the dete-

Lemma 6.2. If \( x \in Q' \subseteq Q \) and \( y \in Q \), then there are \( u \in Q' \) and \( v \in Q \setminus Q' \) such that \( |u - v| = 1 \) and \( |R_Q(x,y)| \leq |R_Q'(x,y)| + |Q||R_Q'(x,u)||R_Q(v,y)| \).

Proof. Write

\[ H_Q = H_Q' + H_{Q \setminus Q'} - 1_{Q'} \Delta 1_{Q \setminus Q'} - 1_Q \Delta 1_{Q'} \]

Multiply on the left by \( R_{Q'} \) and on the right by \( R_Q \) to obtain

\[ R_{Q'} = 1_{Q'} R_Q - R_{Q'} \Delta 1_{Q \setminus Q'} R_Q. \]

Expanding the last term in coordinates gives

\[ R_Q(x,y) = R_{Q'}(x,y) + \sum_{u \in Q',\ell} \int_{v \in Q \setminus Q',|u-v| = 1} R_{Q'}(x,u) R_{Q}(v,y). \]

Since the sum has at most \(|Q|\) terms, the lemma follows. \qed

We show propagation of exponential decay from small scales to large scales, even in the presence of finitely many defects.

Lemma 6.2. If

1. \( \epsilon > \delta > 0 \) are small
2. \( K \geq 1 \) an integer
3. \( L_0 \geq \cdots \geq L_6 \geq C_{\epsilon,\delta,K} \) dyadic scales with \( L_k^{1-\delta} \geq L_{k+1} \)
4. \( 1 \geq m \geq 2L_5^{-\delta} \) represents the exponential decay rate
5. \( Q \subseteq \mathbb{Z}^2 \) a square with \( \ell(Q) = L_0 \)
6. \( Q_1',\ldots,Q_K' \subseteq Q \) disjoint \( L_2 \)-squares with \( |R_{Q_k'}| \leq e^{L_4} \)
7. for all \( x \in Q \) one of the following holds
   a. there is a \( Q_k' \) such that \( x \in Q_k' \) and \( \text{dist}(x,Q \setminus Q_k') \geq \frac{1}{8} \ell(Q_k') \)
   b. there is an \( L_5 \)-square \( Q'' \subseteq Q \) such that \( x \in Q'' \) and \( \text{dist}(x,Q \setminus Q'') \geq \frac{1}{8} \ell(Q'') \)
then \(|R_Q(x,y)| \leq e^{L_1 - m|y - y|} \) for \( x,y \in Q \) where \( m = m - L_5^{-\delta} \).
Remark 6.3. The scales in the above lemma have the following interpretations:

- \( L_0 \) large scale
- \( e^{L_1} \) large scale resolvent bound
- \( L_2 \) defect scale
- \( -L_3 \) defect edge weight
- \( e^{L_4} \) defect resolvent bound
- \( L_5 \) small scale
- \( e^{L_6} \) small scale resolvent bound
- \( 2L_7^{-\delta} \) exponential decay bound
- \( L_7^{-\delta} \) exponential decay loss

These are set up to be compatible with the multiscale analysis below.

Proof. Throughout the proof, we let \( C > 1 > c > 0 \) depend on \( \varepsilon, \delta, K \).

Put a weighted directed multigraph structure on \( Q \) as follows. If \( x, y \in Q \), then add the edge \( x \to y \) with weight \(|x - y|\). If \( x \in Q'_k, y \in Q \setminus Q'_k \), dist\( (x, Q \setminus Q'_k) \geq \ell(Q'_k)/8 \), and dist\( (y, Q'_k) = 1 \), then add the edge \( x \to y \) with negative weight \(-L_3\).

Since \( \ell(Q'_k) = L_2 \gg L_3 \) and the \( Q'_k \) disjoint, this directed graph has no cycles with negative total weight. In particular, there is a well defined directed weighted distance \( d(x, y) \) that satisfies

\[
d(x, y) \leq d(x, z) + d(z, y)
\]

and, using the finiteness of \( K \),

\[
|x - y| \geq d(x, y) \geq |x - y| - CL_2.
\]

We estimate the quantity

\[
\alpha = \max_{x,y \in Q} e^{\tilde{m}d(x,y)} |R_Q(x, y)|
\]

in terms of itself.

Suppose \( x, y \in Q \) and that \( x \) falls into case (a) in the hypotheses. Using the geometric resolvent identity, we can find points \( u \in Q'_k \) and \( v \in Q \setminus Q'_k \) such that \(|u - v| = 1\) and

\[
|R_Q(x, y)| \leq |R_{Q'_k}(x, y)| + L^2 |R_{Q'_k}(x, u)||R_Q(v, y)|.
\]

Using the definition of \( \alpha \), \( d(x, y) \leq -L_3 + d(v, y) \), and \( \tilde{m}L_3 \geq cL_3^{1-\delta} \gg L_4 \), estimate

\[
|R_Q(x, y)| \leq e^{L_4} |Q'_k(y) + L^2 \cdot e^{L_4} \cdot \alpha e^{-\tilde{m}d(v, y)}
\]

\[
\leq e^{L_4} |Q'_k(y) + L^2 \cdot e^{L_4} \cdot \alpha e^{-\tilde{m}d(x,y) + L_3)}
\]

\[
\leq e^{L_4} |Q'_k(y) + L^2 \cdot e^{L_4 - \tilde{m}L_3} \cdot \alpha e^{-\tilde{m}d(x,y)}
\]

\[
\leq e^{L_4} |Q'_k(y) + \frac{1}{2} \alpha e^{-\tilde{m}d(x,y)}.
\]

Since \( \tilde{m}d(x,y) \leq CL_2 \) when \( y \in Q'_k \), we obtain

\[
e^{\tilde{m}d(x,y)} |R_Q(x, y)| \leq e^{CL_2} + \frac{1}{2} \alpha.
\]

Suppose \( x, y \in Q \) and that \( x \) falls into case (b) in the hypotheses. Using the geometric resolvent identity, we can find points \( u \in Q'' \) and \( v \in Q \setminus Q'' \) such that \(|u - v| = 1\) and

\[
|R_Q(x, y)| \leq |R_{Q''}(x, y)| + L^2 |R_{Q''}(x, u)||R_Q(v, y)|.
\]
Using the definition of $\alpha$ and $(m - \tilde{m})|x - u| \geq c L_{k}^{1-\delta} \gg L_{6}$, we estimate

$$|R_{Q}(x, y)| \leq e^{L_{6}}1_{Q''}(y) + L^{2} \cdot e^{L_{6} - m|y - u|} \cdot \alpha e^{-\tilde{m}d(v, y)} \leq e^{L_{6}}1_{Q''}(y) + L^{2} \cdot e^{L_{6} - (m - \tilde{m})|x - u| + 1} \cdot \alpha e^{-\tilde{m}d(x, y)}$$

Since $\tilde{m}d(x, y) \leq CL_{5}$ if $y \in Q''$, we obtain

$$e^{\tilde{m}d(x, y)}|R_{Q}(x, y)| \leq e^{CL_{5} + \frac{1}{2} \alpha}.$$ 

Combining the above estimates, we see that

$$\alpha \leq e^{CL_{2} + \frac{1}{2} \alpha}.$$ 

In particular,

$$|R_{Q}(x, y)| \leq e^{CL_{2} - \tilde{m}d(x, y)} \leq e^{L_{1} - \tilde{m}|x - y|}$$

for all $x, y \in Q$. \hfill \Box

We also need the continuity of exponential resolvent bounds.

**Lemma 6.4.** If $Q \subseteq \mathbb{Z}^{2}$ a square, $\lambda \in \mathbb{R}$, $\alpha > \beta > 0$, and

$$|(H_{Q} - \lambda)^{-1}(x, y)| \leq e^{\alpha - \beta|x - y|} \quad \text{for } x, y \in Q,$$

then, for all $|\lambda' - \lambda| < C|Q|^{-1}e^{-\alpha}$,

$$|(H_{Q} - \lambda')^{-1}(x, y)| \leq 2e^{\alpha - \beta|x - y|} \quad \text{for } x, y \in Q.$$

**Proof.** Recall the resolvent identity

$$(H_{Q} - \lambda')^{-1} = (H_{Q} - \lambda)^{-1} + (H_{Q} - \lambda)^{-1}(\lambda - \lambda')(H_{Q} - \lambda')^{-1}$$

formally obtained by multiplying the equation

$$(H_{Q} - \lambda') = (H_{Q} - \lambda) + (\lambda - \lambda')$$

on the left and right by $(H_{Q} - \lambda)^{-1}$ and $(H_{Q} - \lambda')^{-1}$. Using exponential decay, we can estimate the operator norm of $(H_{Q} - \lambda)^{-1}$ by

$$\|(H_{Q} - \lambda)^{-1}\| \leq \|(H_{Q} - \lambda)^{-1}\|_{2} \leq C\beta^{-1}|Q|^{1/2}e^{\alpha}.$$ 

Thus, if $|\lambda' - \lambda| \leq C\beta|Q|^{-1/2}e^{-\alpha}$, then $|(H_{Q} - \lambda)^{-1}(\lambda' - \lambda)| \leq 1/2$ and we can solve the resolvent identity for $(H_{Q} - \lambda')^{-1}$ by fixed point iteration. To obtain the decay estimate, we define

$$\gamma = \max_{x, y \in Q} e^{\beta|x - y| - \alpha}|(H_{Q} - \lambda')^{-1}(x, y)|.$$ 

We can use the resolvent identity, the exponential decay hypothesis, and $|\lambda' - \lambda| \leq \frac{1}{2}|Q|^{-1}e^{-\alpha}$ to estimate

$$|(H_{Q} - \lambda')^{-1}(x, y)| \leq e^{\alpha - \beta|x - y|} + |\lambda' - \lambda|\sum_{z \in Q} e^{\alpha - \beta|x - z|}e^{\alpha - \beta|z - y|}\gamma \leq e^{\alpha - \beta|x - y|} + |\lambda' - \lambda||Q|e^{\alpha - \beta|x - y|}\gamma \leq e^{\alpha - \beta|x - y|} + \frac{1}{2}e^{\alpha - \beta|x - y|}\gamma.$$ 

Dividing through by $e^{\alpha - \beta|x - y|}$ and computing the maximum over $x, y \in Q$, we obtain $\gamma \leq 1 + \frac{1}{2}\gamma$ and the lemma. \hfill \Box
7. Principal eigenvalue

We give a maximum principle version of the results of [5, Section 4]. The base case of the multiscale analysis relies on an estimate of the principal eigenvalue of $H$ on a large square. The basic idea is that, if the set $\{V = 1\}$ is an $R$-net in the square $Q$, then the principal eigenvalue is bounded below by $cR^{-2}(\log R)^{-1}$. The same argument also yields exponential decay of the Green’s function.

**Definition 7.1.** If $X \subseteq Y \subseteq \mathbb{Z}^2$ and $R > 0$, then $X$ is an $R$-net in $Y$ if $\max_{y \in Y} \min_{x \in X} |y - x| \leq R$.

In the next lemma, we use the $R$-net property to construct a barrier to bound the Green’s function.

**Lemma 7.2.** If $Q \subseteq \mathbb{Z}^2$ is a square, $R \geq 2$ a distance, $Q \cap \{V = 1\}$ is an $R$-net in $Q$, and $L = R^2 \log R$, then

$$|H^{-1}_Q(x, y)| \leq e^{CL - cL^{-1}|x - y|} \text{ for } x, y \in Q.$$

**Proof.** Recall that the principal eigenvalue can be computed via

$$\lambda_1 = \sup_{\psi : Q \to (0, \infty)} \inf_Q \frac{H\psi}{\psi}.$$

To estimate $\lambda_1$, we construct a test function $\psi$. Since we only care about the square $Q$, we may assume that $X = \{V = 1\}$ is an $R$-net in all of $\mathbb{Z}^2$. We may also assume that $R \geq C$ is large.

**Claim 7.3.** There is a $\psi : \mathbb{Z}^2 \to \mathbb{R}$ such that $H\psi \geq cR^{-2}$ and $1 \leq \psi \leq C \log R$.

Let $G : \mathbb{Z}^2 \to \mathbb{R}$ denote lattice Green’s function. That is, $G$ is the unique function that satisfies $G(0) = 0$, $G \leq 0$, and $-\Delta G = 1_{\{0\}}$. Observe that there is a small $\varepsilon > 0$ such that, if $R \geq C$, the function

$$\varphi(x) = 1 - G(x) - \varepsilon R^{-2}|x|^2$$

satisfies

$$(-\Delta + 1_{\{0\}})\varphi \geq R^{-2}\varepsilon \quad \text{and} \quad 1 \leq \varphi \leq C \log R \quad \text{in } B_{3R}$$

and

$$\min_{B_{3R} \setminus B_{2R}} \varphi \geq \max_{B_R} \varphi.$$

We define $\psi : \mathbb{Z}^2 \to \mathbb{R}$ by

$$\psi(y) = \min_{x \in B_{3R}(y) \cap X} \varphi(y - x).$$

Observe that, since $X$ is an $R$-net and $\min_{B_{3R} \setminus B_{2R}} \varphi \geq \max_{B_R} \varphi$, we have

$$\psi(y) = \min_{x \in B_{3R}(y) \cap X} \varphi(y - x).$$

Thus, for any $y \in \mathbb{Z}^2$, we can pick an $x \in B_{2R}(y) \cap X$ such that

$$\psi(y) = \varphi(y - x) \quad \text{and} \quad \psi(z) \geq \varphi(z - x) \quad \text{for } |z - y| = 1.$$

This implies $1 \leq \psi(y) \leq C \log R$ and $H\psi(y) \geq H\varphi(y) \geq \varepsilon R^{-2}$.

**Claim 7.4.** $0 \leq H^{-1}(x, y) \leq e^{CL - cL^{-1}|x - y|}$ for all $x, y \in \mathbb{Z}^2$. 

Note that, for any \( u, v : \mathbb{Z}^2 \to \mathbb{R} \),
\[
H(uv)(x) = u(x)Hv(x) + \sum_{|y-x|=1} (u(y) - u(x))v(y).
\]
It follows that there is a universal small \( \varepsilon > 0 \) such that, for all \( y \in \mathbb{Z}^2 \), the function \( \rho_y(x) = e^{-\varepsilon R^{-2}(\log R)^{-1}|x-y|}\psi(x) \) satisfies
\[
H\rho_y(x) \geq e^{-\varepsilon R^{-2}(\log R)^{-1}|x-y|}(H\psi(x) - C\varepsilon R^{-2})
\geq e^{-\varepsilon R^{-2}(\log R)^{-1}|x-y|}(cR^{-2} - C\varepsilon R^{-2})
\geq 0
\]
Next, observe that \( \rho_y(y) \geq 1 \) and there is a universal large \( \alpha > 1 \) such that, if \( |x-y| \geq R' = \alpha(R \log R)^2 \), then \( \rho_y(x) \leq 1/2 \). Define the functions
\[
\eta_y(x) = 1 - \frac{|x-y|^2}{4R'}
\]
and
\[
\xi_y(x) = \begin{cases} 
\min \{ \rho_y(x), \eta_y(x) \} & \text{if } |x-y| \leq 2R' \\
\rho_y(x) & \text{otherwise.}
\end{cases}
\]
Note that, since \( \eta_y \geq 3/4 \) in \( \mathbb{Z}^2 \cap B_{2R'}(y) \) and \( \rho_y \leq 1/2 \) in \( \mathbb{Z}^2 \cap B_{2R'}(y) \setminus B_{R'}(y) \), we have \( \xi_y = \rho_y \in \mathbb{Z}^2 \setminus B_{R'}(y) \). Thus, for \( x \in \mathbb{Z}^2 \) with \( |x-y| \geq R' \) or \( \xi_y(x) = \rho_y(x) \), we have \( H\xi_y(x) \geq H\rho_y(x) \geq 0 \). For \( x \in \mathbb{Z}^2 \) with \( |x-y| \leq R' \) and \( \xi_y(x) = \eta_y(x) \), we have \( H\xi_y(x) \geq H\eta_y(x) = c(R')^{-2} \). Since \( \xi_y(y) = \eta_y(y) \), we conclude that
\[
H\xi_y \geq c(R')^{-2} 1_{\{ y \}}.
\]
Since we also have \( \xi_y \geq 0 \) in \( \mathbb{Z}^2 \) and \( \xi_y(x) \to 0 \) as \( |x| \to \infty \), we see that \( \xi_y \) is a supersolution of the equation solved by the Green’s function \( x \mapsto H^{-1}(x,y) \). Since the potential \( V \) is non-negative, the Hamiltonian \( H \) has a maximum principle. We conclude that \( 0 \leq H^{-1}(x,y) \leq c(R')^2 \xi_y(x) \leq e^{cL^{-1}|x-y|} \).

8. Multiscale analysis

We now assemble our ingredients into a proof of Theorem 1.4 by following the outline of [5]. In this section, we assume that all squares have dyadic side length and are half-aligned. That is, all squares have the form
\[
Q = x + [0, 2^n)^2 \text{ for } x \in 2^{n-1}\mathbb{Z}^2.
\]
We need a simple covering lemma.

Lemma 8.1. If \( K \geq 1 \) an integer, \( \alpha \geq C^K \) a dyadic integer, \( L_0 \geq \alpha L_1 \geq L_1 \geq \alpha L_2 \geq L_2 \) dyadic scales, \( Q \subseteq \mathbb{Z}^2 \) and \( L_0 \)-square, and \( Q_1', ..., Q_K' \subseteq Q \) are \( L_2 \)-squares, then there is a dyadic scale \( L_3 \in [L_1, \alpha L_1] \) and disjoint \( L_3 \)-squares \( Q_1', ..., Q'_K \subseteq Q \) such that
\[
(8.2) \text{ for every } Q'_K, \text{ there is } Q'_j, \text{ such that } Q'_k \subseteq Q'_j \text{ and } \text{dist}(Q'_k, Q \setminus Q'_j) \leq \frac{1}{8}L_3.
\]

Proof. Start with \( L_3 = L_1 \) and select any list of \( L_3 \)-squares \( Q_1', ..., Q'_K \subseteq Q \) so that \( (8.2) \) holds. Initially, the \( Q'_k \) may not be disjoint. We modify this family, decreasing the size of the family while increasing the size of the squares. We iterate the following: If \( Q'_j \cap Q'_k \neq \emptyset \) for some \( j < k \), then we delete \( Q'_k \) from the list.
and increase the size of all the squares by a constant (universal) factor to maintain (8.2). This process must stop after at most $K - 1$ stages. Thus, having $\alpha \geq C^K$ is enough room to find a scale $L_0$ that works. Finally, let $Q'_{K+1}, ..., Q'_K \subseteq Q$ be any additional $L_3$-squares such that $Q'_1, ..., Q'_K$ are disjoint. □

Theorem 8.3. For every $\gamma \in (\frac{1}{4}, \frac{1}{2})$, there are

1. small $1 > \varepsilon > \nu > \delta > 0$
2. integer $M \geq 1$
3. dyadic scales $L_k$, for $k \geq 0$, with $\lfloor \log_2 L_k^{1-6\varepsilon} \rfloor = \log_2 L_{k-1}$
4. decay rates $1 \geq m_k \geq L_k^{-\delta}$
5. random sets $F_k \subseteq F_{k+1} \subseteq \mathbb{Z}^2$

such that

6. $F_k$ is $\eta_k$-regular in $Q$ for $\ell(Q) \geq L_k$, where $\eta_k = \varepsilon^2 + L_0^{-\varepsilon} + \cdots + L_k^{-\varepsilon} < \varepsilon$.
7. $F_k \cap Q$ is $V_{F_k \cap Q}$-measurable for $\ell(Q) \geq L_k$
8. if $\ell(Q) = L_k$, $0 \leq \lambda \leq e^{-L_k^\delta}$, and $\mathcal{E}_g(Q)$ denotes the event that $|(H_Q - \lambda)^{-1}(x,y)| \leq e^{L_k^{1-\varepsilon} - m_k|x-y|}$ for $x,y \in Q$

holds, then

$$\mathbb{P}[[\mathcal{E}_g(Q) | V_{F_k \cap 2Q} = 1]] = 1 - L_k^{-\gamma}.$$ 

9. $m_k \geq m_{k-1} - L_k^{-\nu}$ for $k > M$.

Proof. **Step 1.** We set up the base case. Assume $\varepsilon, \nu, \delta, M, L_k$ are as in (1-3). We impose constraints on these objects during the proof. For the base case, we define $F_k = [\varepsilon^{-1}] \mathbb{Z}^2$ and $m_k = L_k^{-\delta}$ for $k = 0, ..., M$.

**Claim 8.4.** If $L_0 \geq C_{\varepsilon, \delta}$, then (1-9) hold for $k = 0, ..., M$.

Fix $k = 0, ..., M$. Every ball $B_{L_k^{\delta/3}}(x)$ ball contains $c \varepsilon^2 L_k^{2\delta/3}$ elements of $F_k$. A union bound gives

$$\mathbb{P} \left[ \max_{x \in Q} \min_{y \in Q \cap V = 1} |x - y| \geq C L_k^{\delta/3} \right] \leq L_k^2 e^{-c \varepsilon^2 L_k^{2\delta/3}}.$$

Thus, by Lemma 7.2 and Lemma 6.4, we see that, for every $0 \leq \lambda \leq e^{-L_k^\delta} \leq e^{-L_k^\delta}$ and $L_k$-square $Q$,

$$\mathbb{P}[[\mathcal{E}_g(Q) | V_{F_k \cap 2Q} = 1]] = 1 \geq 1 - e^{-L_k^{-\delta/3}} \geq 1 - L_k^{-\gamma}.$$ 

In particular, (8) holds. Since $F_k$ is $\varepsilon^2$-regular in all large squares and making $L_0$ larges gives $\varepsilon^2 < \eta_k < \varepsilon$, we see that (6) holds. The other properties are immediate.

**Step 2.** We set up the induction step. We choose $M \geq 1$ so that

$$L_k^\delta \geq L_{k-M} \geq L_k^{\delta/2} \quad \text{for } k > M.$$

We call an $L_k$-square $Q$ “good” if

$$\mathbb{P}[[\mathcal{E}_g(Q) | V_{F_k \cap 2Q} = 1]] = 1.$$ 

That is, an $L_k$-square is good if, after observing the potential on the frozen sites $F_k \cap 2Q$, we see that $(H_Q - \lambda)^{-1}$ is well-behaved. Note that the event that $Q$ is good is $V_{F_k \cap 2Q}$-measurable. An $L_k$-square that is not good is “bad.” We must control the bad squares in order to apply Lemma 6.2.
Suppose that $Q$ is an $L_k$-square and we have chain 

$$Q \supseteq Q_1 \supseteq \cdots \supseteq Q_j$$

with $Q_i$ bad and $\ell(Q_i) = L_{k-i}$. We call each $Q_i$ a “hereditary bad subsquare” of $Q$. Note that the set of hereditary bad subsquares of $Q$ is a $V_{F_k \cup 2Q}$-measurable random variable. We control the number of hereditary bad subsquares using the following claim.

**Claim 8.5.** If $\varepsilon < c$ and $N \geq C_{M,\gamma,\delta}$, then, for all $k > M$,

$$\mathbb{P} \{Q \text{ has fewer than } N \text{ hereditary bad } L_{k-M} \text{-subsquares} \} \geq 1 - L_k^{-1}.$$

Writing $N = (N')^M$, we can use the induction hypothesis to estimate

$$\mathbb{P} \{Q \text{ has more than } N' \text{ bad } L_{j-1} \text{-subsquares} \} \leq \sum_{k-M\leq j \leq k} \mathbb{P} \{Q' \text{ has more than } N' \text{ bad } L_{j-1} \text{-subsquares} \} \leq \sum_{k-M\leq j \leq k} (L_j/L_{j-1})^{CN'}(L_{j-1}^{-\gamma})^{cN'} \leq CML_k^{C+(C\varepsilon-c\gamma)\delta N'}.$$

The claim follows making $\varepsilon < c$ and $N' \geq C_{M,\gamma,\delta}$.

We now fix an integer $N \geq 1$ as in the claim. We call an $L_k$-square $Q$ “ready” if $k > M$ and $Q$ has fewer than $N$ hereditary bad $L_{k-M}$-subsquares. Note the event that $Q$ is ready is $V_{F_k \cup 2Q}$-measurable.

Suppose the $L_k$-square $Q$ is ready. Let $Q''_1, \ldots, Q''_N \subseteq Q$ be a list of $L_{k-M}$-squares that includes every hereditary bad $L_{k-M}$-subsquare of $Q$. Let $Q''_1, \ldots, Q''_N \subseteq Q$ be a list of $L_{k-1}$-squares that includes every hereditary bad $L_{k-1}$-subsquare of $Q$. Applying Lemma 8.1, we can choose a dyadic scale $L' \in [cNL_k^{-2\varepsilon}, L_k^{1-2\varepsilon}]$ and disjoint $L'$-squares $Q'_1, \ldots, Q'_N \subseteq Q$ such that, for every $Q''_i$, there is a $Q'_j$ such that $Q''_i \subseteq Q'_j$ and $\text{dist}(Q''_i, Q \setminus Q''_i) \geq \frac{1}{8}L'$. Note that we can choose $Q''_1, Q''_2, Q''_3$ in a $V_{F_k \cup 2Q}$-measurable way.

For $k > M$, we define $F_k$ to be the union of $F_{k-1}$ and the subsquares $Q'_1, \ldots, Q'_K \subseteq Q$ of each ready $L_k$-square $Q$. For $k > M$, we define $m_k = m_{k-1} - L_k^{-\varepsilon}$.

**Step 3.** Having verified properties (1-9) for $k = 0, \ldots, M$, we now verify properties (1-9) for $k > M$ by induction. Note that (1-5) and (9) are automatic from the definitions. We must verify (6-8) for $k > M$, assuming (1-9) holds for all $j < k$.

**Claim 8.6.** Properties (6) and (7) hold.

For each $L_k$-square $Q$, the event that $Q$ is ready, the scale $L'$, and the $L'$-squares $Q'_i \subseteq Q$ are all $V_{F_k \cup 2Q}$ measurable. Thus, $F_k \cap Q$ is $V_{F_k \cup 2Q}$ measurable. Note that we have $4Q$ in place of $2Q$ because each $L_k$-square $Q$ intersects 8 other half-aligned $L_k$-squares. In particular, (7) holds.

To see (6), observe that, for each $L_k$-square $Q$, the set $Q \cap F_k \setminus F_{k-1}$ is covered by at most $9N$ squares $Q'_i$ of size less than $L_k^{1-2\varepsilon}$. In particular, if $Q''_i$ is tilted
square, $Q \cap F_{k-1}$ is $\varepsilon_{k-1}$-sparse in $Q''$, and $Q \cap F_k$ is not $\varepsilon_k$-sparse in $Q''$, then $Q''$ must intersect one of the $Q'_i$ and have size at most $L_k^{1-\varepsilon}$. This implies that $F_k \cap Q$ is $\varepsilon_k$-regular in $Q$.

Claim 8.7. If the $L_k$-square $Q$ is ready, $|\lambda - \bar{\lambda}| \leq e^{-L_k^{1-\varepsilon}}$, and $H_{Q_i} \psi = \lambda \psi$, $E = Q'_i \cup Q''_i$, and $G = Q'_i \cap \cup_j Q''_j$, then
\[
e^{eL_k^{1-\varepsilon}} \|\psi\|_{L^2(E)} \leq \|\psi\|_{L^2(Q'_i)} \leq (1 + e^{-cL_k^{1-\varepsilon}})\|\psi\|_{L^2(G)}.
\]

If $x \in Q'_i$ and there is a good $L_{k-1}$-square $Q'' \subseteq Q'_i$ with $x \in Q''$ and $\text{dist}(x, Q'_i \setminus Q'') \geq \frac{1}{5}L_{k-1}$, then by the definition of good and Lemma 6.4,
\[
|\psi(x)| \leq e^{eL_k^{1-\varepsilon}} \frac{1}{5}m_{k-1}L_{k-1} \|\psi\|_{L^2(Q'_i)} \leq e^{-cL_k^{1-\varepsilon}} \|\psi\|_{L^2(Q'_i)}.
\]

Similarly, if $x \in Q'_i$ and there is a good $L_{k-2}$-square $Q'''' \subseteq Q'_i$ with $x \in Q''''$ and $\text{dist}(x, Q'_i \setminus Q''') \geq \frac{1}{5}L_{k-2}$, then
\[
|\psi(x)| \leq e^{-cL_k^{1-\varepsilon}} \|\psi\|_{L^2(Q'_i)}.
\]

Next, observe that the first case holds for all $x \in E$ and either the first or second case holds for all $x \in Q'_i \setminus G$. The claim follows.

Claim 8.8. If $Q$ is an $L_k$-square and $\mathcal{E}_i(Q)$ denotes the event that
\[
Q \text{ is ready and } P(||(H_{Q'_i} - \bar{\lambda})^{-1}|| \leq e^{-L_k^{1-\varepsilon}} |V_{F_k \cap Q}^n)_1 = 1,
\]
then $P[\mathcal{E}_i(Q)] \geq 1 - L_k^{C\varepsilon - 1/2}$.

Recall the event $Q$ ready and squares $Q'_i \subseteq Q$ are $V_{F_{k-1} \cap 2Q}$-measurable. We may assume $i = 1$. We apply Lemma 5.4 to the square $Q'_i$ with scales $L' \geq L_k^{1-\varepsilon} \geq L_k^{1-2\varepsilon} \geq L_k^{1-3\varepsilon} \geq L_k^{1-4\varepsilon}$, frozen set $F_{k-1}$, defects $\{Q''_j : Q''_j \subseteq Q'_i\}$, and $G = \cup\{Q''''_j : Q''''_j \subseteq Q'_i\}$. Assuming $\varepsilon > 2\delta$, the previous claim provides the localization required to verify the hypotheses of Lemma 5.6. Since $Q'_i \subseteq F_k$ when $Q$ is ready, the claim follows.

Claim 8.9. If $Q$ is an $L_k$-square and $\mathcal{E}_1(Q), ..., \mathcal{E}_N(Q)$ hold, then $Q$ is good.

We apply Lemma 6.2 to the square $Q$ with small parameters $\varepsilon > \nu > 0$, scales $L_k \geq L_k^{1-\varepsilon} \geq L_k^{1-2\varepsilon} \geq L_k^{1-3\varepsilon} \geq L_k^{1-4\varepsilon} \geq L_k^{1-\varepsilon}$, and defects $Q'_1, ..., Q'_N$. We conclude that
\[
||H_{Q'_j} - \bar{\lambda}||^{-1}(x, y) \leq e^{L_k^{1-\varepsilon}} |x - y|.
\]

Since the events $\mathcal{E}_i(Q)$ are $V_{F_k \cap 4Q}$-measurable, we see that $Q$ is good.

Claim 8.10. Property (8) holds.

Combining the previous two claims, for any $L_k$-square $Q$, we have $P[\mathcal{E}_g(Q)] \geq 1 - N L_k^{C\varepsilon - 1/2} \geq 1 - L_k^{-\gamma}$, provided that $\gamma < 1/2 - C\varepsilon$. \hfill \Box

REFERENCES

[1] Michael Aizenman and Stanislav Molchanov, *Localization at large disorder and at extreme energies: an elementary derivation*, Comm. Math. Phys. 157 (1993), no. 2, 245–278. MR1244867

[2] Michael Aizenman and Simone Warzel, *Random operators*, Graduate Studies in Mathematics, vol. 168, American Mathematical Society, Providence, RI, 2015. Disorder effects on quantum spectra and dynamics. MR3364516
[3] J. Bellissard, P. D. Hislop, A. Klein, and G. Stolz, Random Schrödinger operators: universal localization, correlations and interactions (Banff International Research Station, April 2009), available at http://www.birs.ca/workshops/2009/09w5116/report09w5116.pdf

[4] Borislav Bojanov, Elementary proof of the Remez inequality, Amer. Math. Monthly 100 (1993), no. 5, 483–485, DOI 10.2307/2324304. MR1215537

[5] Jean Bourgain and Carlos E. Kenig, On localization in the continuous Anderson-Bernoulli model in higher dimension, Invent. Math. 161 (2005), no. 2, 389–426, DOI 10.1007/s00222-004-0435-7. MR2180453

[6] Jean Bourgain and Abel Klein, Bounds on the density of states for Schrödinger operators, Invent. Math. 194 (2013), no. 1, 41–72, DOI 10.1007/s00222-012-0440-1. MR3103255

[7] Lev Bulovskiy, Alexander Logunov, Eugenia Malinnikova, and Mikhail Sodin, A discrete harmonic function bounded on a large portion of $\mathbb{Z}^2$ is constant. arXiv:1712.07902.

[8] René Carmona, Abel Klein, and Fabio Martinelli, Anderson localization for Bernoulli and other singular potentials, Comm. Math. Phys. 108 (1987), no. 1, 41–66. MR872140

[9] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, Constructive proof of localization in the Anderson tight binding model, Comm. Math. Phys. 101 (1985), no. 1, 21–46. MR814541

[10] Jürg Fröhlich and Thomas Spencer, Absence of diffusion in the Anderson tight binding model for large disorder or low energy, Comm. Math. Phys. 88 (1983), no. 2, 151–184. MR696803

[11] François Germinet and Abel Klein, A comprehensive proof of localization for continuous Anderson models with singular random potentials, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 1, 53–143, DOI 10.4171/JEMS/356. MR2998830

[12] G. A. Kabatjanskii and V. I. Levenstein, Bounds for packings on the sphere and in space, Problemy Peredachi Informacii 14 (1978), no. 1, 3–25 (Russian). MR0547423

[13] Dirk Hundertmark, A short introduction to Anderson localization, Analysis and stochastics of growth processes and interface models, Oxford Univ. Press, Oxford, 2008, pp. 194–218, DOI 10.1093/acprof:oso/9780199229252.003.0009. MR2603225

[14] Werner Kirsch, An invitation to random Schrödinger operators, Random Schrödinger operators, Panor. Synthèses, vol. 25, Soc. Math. France, Paris, 2008, pp. 1–119 (English, with English and French summaries). With an appendix by Frédéric Klopp. MR2509110

[15] Hervé Kunz and Bernard Souillard, Sur le spectre des opérateurs aux différences finies aléatoires, Comm. Math. Phys. 78 (1980/81), no. 2, 201–246 (French, with English summary). MR597748

[16] D. Lubell, A short proof of Sperner’s lemma, J. Combinatorial Theory 1 (1966), 299. MR0194348

[17] Barry Simon, Schrödinger operators in the twenty-first century, Mathematical physics 2000, Imp. Coll. Press, London, 2000, pp. 283–288, DOI 10.1142/9781848160224_0014. MR1773049

[18] Emanuel Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928), no. 1, 544–548, DOI 10.1007/BF01711114 (German). MR1544925

[19] Elias M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR1232192

[20] Günter Stolz, An introduction to the mathematics of Anderson localization, Entropy and the quantum II, Contemp. Math., vol. 552, Amer. Math. Soc., Providence, RI, 2011, pp. 71–108, DOI 10.1090/conm/552/10911. MR2868042

[21] Terrance Tao, A cheap version of the Kabatjanskii-Levenstein bound for almost orthogonal vectors. https://terrytao.wordpress.com.

[22] Franz Wegner, Bounds on the density of states in disordered systems, Z. Phys. B 44 (1981), no. 1-2, 9–15, DOI 10.1007/BF01292646. MR639135