Monogamy of measurement-induced nonlocality

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Abstract
Measurement-induced nonlocality was introduced by Luo and Fu (2011 Phys. Rev. Lett. \textbf{106} 120401) as a measure of nonlocality in a bipartite state. In this paper, we will discuss the monogamy property of measurement-induced nonlocality for some three- and four-qubit classes of states. Unlike discord, we find quite surprising results in this situation. Both the GHZ- and W-states satisfy monogamy relations in the three-qubit case; however, in general there are violations of monogamy relations in both the GHZ-class and W-class states. In the case of the four-qubit system, monogamy holds for most of the states in the generic class. The four-qubit GHZ-state does not satisfy the monogamy relation, but the W-state does. We provide several numerical results including counterexamples regarding the monogamy nature of measurement-induced nonlocality. We will also extend our results of the generalized W-class to \textit{n}-qubit systems.

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(\textit{Some figures may appear in colour only in the online journal})

1. Introduction

Nonlocality is at the heart of the quantum world. From Bell’s theorem [1], it is understood that no local hidden variable theory could replace quantum theory as a theory of the physical world. The existence of entangled states in composite quantum systems assures the nonlocal behavior of quantum theory. Generally, violation of Bell’s inequality is taken as the signature of nonlocality. Entangled states play an important role in showing the violation of Bell’s inequality. However, nonlocality can be viewed from other perspectives. For example, there exist sets of locally indistinguishable orthogonal pure product states [2] used to show nonlocality without entanglement. Recently, introduced measurement-induced nonlocality (MIN) [3], is one of the ways to detect nonlocality in quantum states by some locally invariant
measurements. Locally invariant measurements cannot affect global states in classical theory but this is possible in quantum theory. So MIN is a type of nonlocal correlation which can only exist in the quantum domain. MIN is defined in such a way that it is non-negative for all states, invariant under local unitary and vanishes on the product state. In this sense, MIN could be observed as a type of nonlocal correlation which is induced by certain measurement. Although it is induced by some measurement, it is not a measure of quantum correlation in the true sense. But it can be treated as a measure of nonlocality, induced by some kind of measurement.

Now a natural question arises about the shareability of quantum correlations in multipartite states. It may be monogamous or polygamous. It is known that entanglement is monogamous [4]. In this work, we will investigate the monogamous behavior of MIN. We will check the monogamy property for some classes of states in both three and four-qubit systems. Unlike discord [5–7], here we will show that both the three-qubit generalized GHZ- and W-states satisfy monogamy relations; however, there are violations of the monogamy relation if we consider the generic whole class of pure three-qubit states. In the case of the pure four-qubit system, we consider the most important generic class of states. It contains the usual GHZ, maximally entangled states in the sense of Gour et al [8]. Most of the states satisfy monogamy relations. There are two important subclasses of the generic class, say, $M$ and $\tau_{\text{min}}$. In one subclass monogamy holds but in another it does not. In particular, the GHZ-state violates the monogamy relation and W-state satisfies it. Therefore, monogamous relations of MIN are quite different from that of some important measures of correlation [9, 10] and this acts as a distinguishing feature of some class of states. Our paper is organized as follows. In section 2, we will review some of the basic properties of MIN. In section 3, we will explain and discuss some four-qubit classes of states that we will require in our work. Section 4 is devoted to the notion of monogamy for MIN. Section 5 contains results on the pure three-qubit system and section 6 contains results on the four-qubit system. Several counterexamples and numerical figures are discussed in both of the above sections.

2. Overview of MIN

Let $\rho$ be any bipartite state shared between two parties $A$ and $B$. Then MIN (denoted by $N(\rho)$) is defined as [3]

$$N(\rho) := \max_{\Pi^A} \| \rho - \Pi^A(\rho) \|^2,$$

where the maximum is over all von Neumann measurements $\Pi^A$ which do not disturb $\rho_A$, the local density matrix of $A$, i.e. $\sum_k \Pi_k^A \rho_A \Pi_k^A = \rho_A$, and $\| \cdot \|$ is taken as the Hilbert–Schmidt norm (i.e. $\|X\| = \|\text{Tr}(X^\dagger X)\|^{1/2}$). It is, in some sense, dual to that of the geometric measure of discord. Physically, MIN quantifies the global effect caused by locally invariant measurements. MIN has applications in general dense coding, quantum state steering, etc. MIN vanishes for the product state and remains positive for entangled states. For pure states, MIN reduces to linear entropy like geometric discord [9]. The explicit formula of MIN for the $2 \otimes n$ system, $m \otimes n$ (if $\rho_A$ is non-degenerate) system and an explicit upper bound for the $m \otimes n$ system were obtained by Luo and Fu in [3]. Later Mirafzali et al [11] formulated a way to reduce the problem of degeneracy in the $m \otimes n$ system and evaluate it for $(3 \otimes n)$-dimensional systems. MIN is invariant under local unitary, i.e. in the true sense, it is a nonlocal correlation measure. The set of all zero MIN states is non-convex. Guo and Hou [12] derived the conditions for the nullity of MIN. They have found that the set of states with zero MIN is a proper subset of the set of all classical-quantum states, i.e. zero discord states. MIN for classical-quantum states vanishes if each eigen-subspace of $\rho_A$ is one dimensional. It therefore reveals that non-commutativity
is the cause of this kind of nonlocality in quantum states. Recently, in [13], MIN has been quantified in terms of relative entropy to give it another physical interpretation. However, in our work, we have used the original definition of MIN.

Suppose $H_A, H_B$ are the Hilbert spaces associated with parties $A$ and $B$, respectively, and $L(H_A), L(H_B)$ denote the Hilbert space of linear operators acting on $H_A, H_B$ with the inner product defined by $(X|Y) := \text{tr}X^\dagger Y$. We state two important results which we will use in our work.

**Theorem 1** (Luo and Fu [3]). Let $|\psi\rangle_{AB}$ be any bipartite pure state with the Schmidt decomposition $|\psi\rangle_{AB} = \sum_i \sqrt{s_i} |\alpha_i\rangle_A |\beta_i\rangle_B$, then $N(|\psi\rangle_{AB}) = 1 - \sum_i s_i$.

**Theorem 2** (Luo and Fu [3]). Let $\rho_{AB}$ be any state of the $(2 \otimes n)$-dimensional system written in the form

$$\rho_{AB} = \frac{1}{\sqrt{2n}} \frac{p^A}{\sqrt{r}} \otimes \frac{p^B}{\sqrt{r}} + \sum_{i=1}^3 x_iX^A_i \otimes \frac{p^B}{\sqrt{r}} + \frac{p^A}{\sqrt{r}} \sum_{i=1}^{n-1} y_iY^B_i + \sum_{i=1}^3 \sum_{j=1}^{n-1} t_{ij}X^A_i \otimes Y^B_j,$$

where $\{|X^A_i\rangle : i = 0, 1, 2, 3\}$ and $\{|Y^B_j\rangle : j = 0, 1, 2, \ldots, n^2 - 1\}$ are the orthonormal Hermitian operator bases for $L(H_A)$ and $L(H_B)$, respectively, with $X^A_0 = I^A/\sqrt{2}, Y^B_0 = I^B/\sqrt{n}$. Then

$$N(\rho_{AB}) = \frac{\text{tr}TT^t}{\|x\|^2} x^t T T^t x \quad \text{if } x \neq 0$$

$$= \text{tr}TT^t - \lambda_3 \quad \text{if } x = 0 \quad (3)$$

where the matrix $T = (t_{ij})$ with $\lambda_3$ being minimum eigenvalue of $TT^t$ and $\|x\|^2 := \sum_i x_i^2$ with $x = (x_1, x_2, x_3)^t$.

Before discussing the monogamy properties of MIN, specifically for three- and four-qubit systems, we first mention some important classes of states in four-qubit systems with some discussions on their entanglement behavior.

### 3. Some special four-qubit classes

Four-qubit pure states can be classified into nine groups [14]. Among them the generic class is given by

$$\mathcal{A} = \left\{ \sum_{j=0}^3 z_j u_j : \sum_{j=0}^3 |z_j|^2 = 1, z_j \in \mathbb{C}, i = 0, 1, 2, 3 \right\}, \quad (4)$$

where $u_0 = |\phi^+\rangle = |\phi^+\rangle$, $u_1 = |\phi^-\rangle = |\phi^-\rangle$, $u_2 = |\psi^+\rangle = |\psi^+\rangle$, $u_3 = |\psi^-\rangle = |\psi^-\rangle$ and $|\phi^+\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, $|\psi^+\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. Consider two important subclasses $\mathcal{M}$ and $\tau_{\text{min}}$ of the generic class $\mathcal{A}$ which are defined as

$$\mathcal{M} = \left\{ \sum_{j=0}^3 z_j u_j : \sum_{j=0}^3 |z_j|^2 = 1, \sum_{j=0}^3 z_j^2 = 0 \right\}, \quad (5)$$

and

$$\tau_{\text{min}} = \left\{ \sum_{j=0}^3 x_j u_j : \sum_{j=0}^3 x_j^2 = 1, x_j \in \mathbb{R}, j = 0, 1, 2, 3 \right\}. \quad (6)$$

These two subclasses are important in the sense that $\mathcal{M}$ is the maximally entangled class and $\tau_{\text{min}}$ has the least amount of bipartite entanglement according to the definition of maximally
entangled states given by Gour et al [8]. Consider a pure bipartite state $|\psi_{AB}\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$. Then the tangle is defined as
\[
\tau_{AB} = 2(1 - \text{Tr}\rho_A^2)
\]
where $\rho_A = \text{Tr}_B|\psi_{AB}\rangle\langle\psi_{AB}|$. Now for a pure state $|\psi_{ABCD}\rangle$, shared between four parties, three quantities $\tau_1$, $\tau_2$, $\tau_{ABCD}$ are defined as
\[
\tau_1 = \frac{1}{4}(\tau_{A|BCD} + \tau_{B|ACD} + \tau_{C|ABD} + \tau_{D|ABC})
\]
\[
\tau_2 = \frac{1}{4}(\tau_{AB|CD} + \tau_{BC|AD} + \tau_{CA|BD} + \tau_{DA|BC})
\]
\[
\tau_{ABCD} = 4\tau_1 - 3\tau_2.
\]
For the above two subclasses $\mathcal{M}$ and $\tau_{\text{min}}$, we have $\tau_1(\mathcal{M}) = 1$, $\tau_2(\mathcal{M}) = \frac{3}{4}$, $\tau_{ABCD}(\mathcal{M}) = 0$ and $\tau_1(\tau_{\text{min}}) = 1$, $\tau_2(\tau_{\text{min}}) = 1$, $\tau_{ABCD}(\tau_{\text{min}}) = 1$. The four-qubit GHZ-state belongs to the class $\tau_{\text{min}}$.

4. Monogamy

Monogamy is an important aspect in our physical world which restricts the shareability of bipartite correlation. Entanglement is an example of quantum correlation which is monogamous w.r.t. the tangle. Mathematically, a correlation measure $Q$ is said to be monogamous iff for any $n$-party state $\rho_{A_1A_2\ldots A_n}$, the relation
\[
\sum_{k=1,k\neq i}^n Q(\rho_{A_kA_i}) \leq Q(\rho_{A_k|A_1A_2\ldots A_{i-1}A_{i+1}\ldots A_n})
\]
holds for all $i = 1, 2, \ldots, n$. Now consider an $n$-party state $\rho_{12\ldots n}$. Let the locally invariant measurement be done on party 1. Then MIN is defined as $N(\rho_{12\ldots n}) = \|\rho_{12\ldots n} - \Pi^*_i(\rho_{12\ldots n})\|^2$, where $\Pi^*_i = [\pi^*_i]$ is the optimal measurement done by party 1 which does not change its local density matrix, i.e. $\rho_i = \Sigma_\lambda \pi^*_i(\rho_1)\pi^*_i(\rho_2)$. On the other hand, since $\rho_i = \text{Tr}_{2,3,\ldots,n}(\rho_{12\ldots n}) = \text{Tr}_i(\text{Tr}_{12\ldots n})$, $j = 2, 3, \ldots, n$, the optimal measurement also does not change the local density matrices for all two-party reduced states of $\rho_{12\ldots n}$ of the kind $\rho_{ij} = \text{Tr}_{13,\ldots,j-1,j+1\ldots n}(\rho_{12\ldots n})$. Then $\sum_i N(\rho_{ij}) \geq \sum_i \|\rho_{ij} - \Pi^*_i(\rho_{ij})\|^2$. So in the case of polygamy $N(\rho_{12\ldots n}) < \sum_i \|\rho_{ij} - \Pi^*_i(\rho_{ij})\|^2$ then the state is monogamous w.r.t. MIN.

5. Monogamy in the three-qubit system

Any three-qubit pure state $|\psi_{ABC}\rangle$ has a generic form $\lambda_0|000\rangle + \lambda_1e^{i\theta}|010\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$, where $\lambda_i \in \mathbb{R}$, $\theta \in [0, \pi]$, $\sum_i \lambda_i^2 = 1$ [15]. This class includes the GHZ-class. The W-class can also be availed by putting $\lambda_4 = 0$. For the general state $|\psi_{ABC}\rangle$, the reduced density matrix $\rho_{AB} = \text{Tr}_C|\psi_{ABC}\rangle\langle\psi_{ABC}|$ has the form
\[
\rho_{AB} = \begin{bmatrix}
\lambda_0^2 & 0 & \lambda_0\lambda_1 e^{-i\theta} & \lambda_0\lambda_3 \\
0 & 0 & 0 & 0 \\
\lambda_0\lambda_1 e^{i\theta} & 0 & \lambda_1^2 + \lambda_2^2 & \lambda_1\lambda_3 e^{i\theta} + \lambda_2\lambda_4 \\
\lambda_0\lambda_3 & 0 & \lambda_1\lambda_3 e^{-i\theta} + \lambda_2\lambda_4 & \lambda_3^2 + \lambda_4^2
\end{bmatrix}
\]
where the correlation matrix $T = (t_{ij})$ is obtained from the relation $t_{ij} = \text{Tr}(\rho_{ij}^2)$, $i, j = 1, 2, 3$, $\sigma_i$ being the Pauli matrices:
\[
T = \begin{bmatrix}
\lambda_0\lambda_3 & 0 & \lambda_0\lambda_1 \cos \theta \\
0 & -\lambda_0\lambda_3 & -\lambda_0\lambda_1 \sin \theta \\
-\lambda_1\lambda_3 \cos \theta - \lambda_2\lambda_4 & -\lambda_1\lambda_3 \sin \theta & 0.5 - \lambda_4^2 - \lambda_3^2
\end{bmatrix}.
\]
The other reduced density matrix $\rho_{AB}$ and its corresponding correlation matrix could be written from the expressions of $\rho_{AB}$ and $T$ by only interchanging $\lambda_2$ and $\lambda_3$. The coherent vector for both reduced density matrices is $x = (\lambda_0\lambda_1\cos \theta, -\lambda_0\lambda_1 \sin \theta, \lambda_0^2 - 0.5)$. Clearly, $\|x\| = 0$ iff $\lambda_0^2 = 0.5, \lambda_1^2 = 0$. In case of $\|x\| = 0$, we have

$$N(\rho_{AB}) = 2a + c - \min \left\{ a, \frac{1}{2} (a + c - \sqrt{(a - c)^2 + 4b^2}) \right\}$$

(12)

$$N(\rho_{MC}) = 2g + k - \min \left\{ g, \frac{1}{2} (g + k - \sqrt{(g - k)^2 + 4f^2}) \right\}$$

(13)

$$N(\rho_{ABC}) = 0.5$$

(14)

where

$$a = \lambda_0^2 \lambda_3^2$$

(15)

$$b = -\lambda_0 \lambda_2 \lambda_3 \lambda_4$$

(16)

$$c = \lambda_2^2 \lambda_4^2 + (0.5 - \lambda_2^2)^2$$

(17)

$$g = \lambda_0^2 \lambda_2^2$$

(18)

$$k = \lambda_2^2 \lambda_4^2 + (0.5 - \lambda_2^2)^2$$

(19)

Now, the minimum value of $N(\rho_{AB}) + N(\rho_{MC})$ is 0.5. So in case of $\|x\| = 0$ monogamy is violated for most of the states. For example, we can consider a state with $\lambda_2 = \lambda_3 = \lambda_4 = \sqrt{\frac{1}{2}}$. In this case $N(\rho_{AB}) + N(\rho_{MC}) = 0.516046 > 0.5$. Numerical simulation of $10^6$ random states (the states are generated by choosing random $\lambda_i, i = 0, 1, \ldots, 5$ with uniform distribution) shows that around 0.02% of the GHZ-class states and around 20% of the W-class states satisfy equality of the monogamy relation.

When $\|x\| \neq 0$, we have

$$N(\rho_{ABC}) = 2 \lambda_0^2 \lambda_3^2 (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)$$

(20)

$$N(\rho_{AB}) = a + b + c - \frac{1}{\|x\|^2} x^T T^T x$$

(21)

$$N(\rho_{MC}) = g + f + k - \frac{1}{\|x\|^2} x^T T^T x$$

(22)

where

$$a = \lambda_0^2 \lambda_3^2 + \lambda_0^2 \lambda_1^2 \cos^2 \theta$$

(23)

$$b = \lambda_0^2 \lambda_3^2 + \lambda_0^2 \lambda_1^2 \sin^2 \theta$$

(24)

$$c = (\lambda_2 \lambda_4 + \lambda_1 \lambda_3 \cos \theta)^2 + \lambda_2^2 \lambda_3^2 \sin^2 \theta + (0.5 - \lambda_2^2 - \lambda_3^2)^2$$

(25)

$$g = \lambda_0^2 \lambda_2^2 + \lambda_0^2 \lambda_3^2 \cos^2 \theta$$

(26)

$$f = \lambda_0^2 \lambda_2^2 + \lambda_0^2 \lambda_1^2 \sin^2 \theta$$

(27)

$$k = (\lambda_3 \lambda_4 + \lambda_1 \lambda_2 \cos \theta)^2 + \lambda_3^2 \lambda_4^2 \sin^2 \theta + (0.5 - \lambda_3^2 - \lambda_4^2)^2.$$  

(28)

Now the maximum value of $N(\rho_{AB}) + N(\rho_{MC})$ is 0.5. Hence, three-qubit pure states with $\|x\| \neq 0$ satisfy the monogamy relation. Specifically, the three-qubit generalized GHZ-class of pure states $(\alpha|000\rangle + \beta|111\rangle)$ is monogamous in the region $\alpha \neq \beta$ and the monogamy relation holds good with equality if $\alpha = \beta = \frac{1}{\sqrt{2}}$. In the three-qubit generalized W-class states $(\alpha|001\rangle + \beta|010\rangle + \gamma|001\rangle)$, we have $N(\rho_{AB}) + N(\rho_{MC}) = N(\rho_{ABC}) = 2|\alpha|^2 (1 - |\alpha|^2)$. That is, the monogamy relation holds with equality.
6. Monogamy in the four-qubit system

Consider a four-qubit generic pure state \(|\psi_{ABCD}\rangle\) shared between four parties A, B, C, D i.e. from the class \(A\) where

\[
|\psi_{ABCD}\rangle = \sum_{j=0}^{3} z_j u_j, \quad \sum_{j=0}^{3} |z_j|^2 = 1. \tag{29}
\]

We consider the two-qubit reduced density matrices \(\rho_{AB}, \rho_{AC}\) and \(\rho_{AD}\) of \(\rho = |\psi_{ABCD}\rangle\langle\psi_{ABCD}|\). Each reduced density matrix is of the form

\[
\begin{bmatrix}
\alpha & 0 & 0 & \beta \\
1 & 0 & \gamma & \delta \\
\bar{\alpha} & \bar{\gamma} & \bar{\delta} & 0 \\
\beta & 0 & 0 & \alpha
\end{bmatrix}
\]

where \(\alpha, \beta, \gamma, \delta\) are some suitable functions of \(z_0, z_1, z_2, z_3\) such that \(\alpha, \beta, \gamma, \delta \in \mathbb{R}\) and \(\alpha + \gamma = 2\). We define four quantities

\[
a = z_0 + z_1, \quad b = z_0 - z_1, \quad c = z_2 + z_3, \quad d = z_2 - z_3.
\]

Then, for \(\rho_{AB}\)

\[
\begin{align*}
\alpha &= 2(|z_0|^2 + |z_1|^2) \tag{30} \\
\beta &= 2(|z_0|^2 - |z_1|^2) \tag{31} \\
\gamma &= 2(|z_2|^2 + |z_3|^2) \tag{32} \\
\delta &= 2(|z_2|^2 - |z_3|^2), \tag{33}
\end{align*}
\]

for \(\rho_{AC}\)

\[
\begin{align*}
\alpha &= (|a|^2 + |c|^2) \tag{34} \\
\beta &= 2 \text{Re}(\bar{a}c) \tag{35} \\
\gamma &= (|b|^2 + |d|^2) \tag{36} \\
\delta &= 2 \text{Re}(\bar{b}d) \tag{37}
\end{align*}
\]

and for \(\rho_{AD}\)

\[
\begin{align*}
\alpha &= (|a|^2 + |d|^2) \tag{38} \\
\beta &= 2 \text{Re}(\bar{a}d) \tag{39} \\
\gamma &= (|b|^2 + |c|^2) \tag{40} \\
\delta &= 2 \text{Re}(\bar{b}c). \tag{41}
\end{align*}
\]

The elements of the correlation matrix \(T\) can be obtained from \(t_{ij} = \text{tr}(\rho_{AX} X_i \otimes Y_j), X = B, C, D\). Eigenvalues of the matrix \(TT^t\) are of the form \(k(\beta + \delta)^2, k(\alpha - \gamma)^2\) with \(k = \frac{1}{2}\). The coherent vector \(x = (x_1, x_2, x_3)^t\) as obtained from the relation \(x_i = \text{tr}(\rho_{AX} X_i \otimes I), X = B, C, D\) is zero for all three cases. Hence, we have (by theorem 2)

\[
N(\rho_{AX}) = k[2(\beta^2 + \delta^2) + (\alpha - \gamma)^2 - \lambda_3] \quad \text{where} \quad X = B, C, D \tag{42}
\]

\[
\lambda_3 = \min\{(\beta + \delta)^2, (\alpha - \gamma)^2, (\beta - \delta)^2\}. \tag{43}
\]

On the other hand, we can write the state \(\rho_{ABCD}\) in the form \(\frac{1}{\sqrt{2}}(|0\phi_0\rangle + |1\phi_1\rangle)\) where \(|\phi_0\rangle, |\phi_1\rangle\) are mutually orthonormal. Since this is a pure state we have \(N(\rho_{ABCD}) = 0.5\) (using
Figure 1. $10^5$ random simulation for the states, generated by choosing random $\lambda_i$, $i = 0, 1, 2, 3$ with uniform distribution shows about 66% violation of monogamy for the generic class $A$. The red line marks the departure from monogamy.

Theorem 1. Numerical simulation for $10^6$ generic states shows that about 66% of them satisfy the monogamy relation. (Figure 1 represents a simulation of $10^5$ states.) Hence, in general four-qubit generic class is not monogamous w.r.t. MIN. So there exist quantum states whose locally shared nonlocality exceeds the amount of globally shared nonlocality.

However, for the subclasses $\mathcal{M}$ and $\tau_{\min}$ we can still check whether the monogamy relation holds or not, and if not then what is the amount of violation. These will be illustrated in the next few results.

**Theorem 3.** Consider a four-qubit system. Then for any pure state $\rho$, belonging to the generic class $A$ we have $N(\rho_{XY}) \leq \sum_{i=1}^{3} \lambda_i$ where $\rho_{XY}$ denotes any bipartite reduced density matrix of $\rho$ and $\lambda_i$ are the eigenvalues of $TT^t$, $T$ being the correlation matrix of $\rho_{XY}$.

**Proof.** The proof of the theorem is very easy as it directly follows from theorem 2 keeping in mind that the eigenvalues of $TT^t$ are all positive and the Bloch vector $\|x\| = 0$ for all bipartite reduced states of any generic state.

We will use this theorem in our proof of the next two results. \hfill $\Box$

**Theorem 4.** Let $\rho_{ABCD}$ be any four-qubit pure state belonging to the class $\mathcal{M}$. Then $N(\rho_{AB}) + N(\rho_{AC}) + N(\rho_{AD}) \leq 1$.

**Proof.** Let us consider any four-qubit pure state $\rho_{ABCD} = |\psi\rangle_{ABCD}\langle\psi|$ belonging to the class $\mathcal{M}$ i.e. $|\psi\rangle_{ABCD} = \sum_{j=0}^{3} z_j u_j$ with $\sum_{j=0}^{3} |z_j|^2 = 1, \sum_{j=0}^{3} z_j^2 = 0$. Then, taking $z_j = x_j + iy_j \forall j = 0, 1, 2, 3; x_j, y_j \in \mathbb{R}$ and utilizing the above restrictions, we obtain $\sum_{j=0}^{3} x_j^2 = \sum_{j=0}^{3} y_j^2 = \frac{1}{4}$ and $\sum_{j=0}^{3} x_j y_j = 0$. The fruitful implementation of the result of the previous theorem and simple algebraic manipulations using these results lead to the fact $N(\rho_{AB}) + N(\rho_{AC}) + N(\rho_{AD}) \leq \frac{1}{4}$. \hfill $\Box$

**Theorem 5.** Let $\rho_{ABCD}$ be any four-qubit pure state belonging to the class $\tau_{\min}$. Then $\frac{1}{2} \leq N(\rho_{AB}) + N(\rho_{AC}) + N(\rho_{AD}) \leq \frac{3}{4}$. 

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Proof. Let us consider any four-qubit pure state $\rho_{ABCD} = |\psi\rangle_{ABCD}\langle\psi|$, belonging to the class $\tau_{\min}$ i.e. $|\psi\rangle_{ABCD} = \sum_{j=0}^{N-1} x_j |u_j\rangle$, with $\sum_{j=0}^{N-1} x_j^2 = 1$ and $x_j \in \mathbb{R}$, $j = 0, 1, 2, 3$. Now let us denote the sets of eigenvalues of $\rho_{AB}$, $\rho_{AC}$ and $\rho_{AD}$ as $\{\lambda_{1}^{AB}, \lambda_{2}^{AB}, \lambda_{3}^{AB}\}$, $\{\lambda_{1}^{AC}, \lambda_{2}^{AC}, \lambda_{3}^{AC}\}$ and $\{\lambda_{1}^{AD}, \lambda_{2}^{AD}, \lambda_{3}^{AD}\}$ respectively. Without loss of generality, we can consider, $\lambda_{i}^{AB} \geq \lambda_{j}^{AB} \geq \lambda_{k}^{AB}$, $\forall i, j, k = 1, 2, 3$ with $i \neq j \neq k$. This consideration gives natural connections among the eigenvalues of $\rho_{AC}$ and also $\rho_{AD}$. These are $\lambda_{i}^{AC} \geq \lambda_{j}^{AC} \geq \lambda_{k}^{AC}$ and $\lambda_{i}^{AD} \geq \lambda_{j}^{AD} \geq \lambda_{k}^{AD}$, $\forall i, j, k = 1, 2, 3$ with $i \neq j \neq k$. Observing the behavior of the eigenvalues and straightforward derivation quite easily establish the desired result, i.e. $\frac{1}{2} \leq N(\rho_{AB}) + N(\rho_{AC}) + N(\rho_{AD}) \leq \frac{3}{2}$. \hfill $\Box$

These two results give us information about the monogamous behavior of MIN in two important classes of the four-qubit system. Since, $N(\rho_{ABCD}) = \frac{1}{2}$ in the whole class $\mathcal{A}$, therefore, MIN is monogamous in the subclass $\mathcal{M}$ but is polygamous in other class $\tau_{\min}$. Furthermore, all states that are connected to these classes by LU share the same fate. The four-qubit GHZ-state belongs to the class $\tau_{\min}$. Hence, GHZ and their LU equivalent states are not monogamous w.r.t MIN. Another two states (and obviously their LU equivalent states) $|L\rangle = \frac{1}{\sqrt{2}}(u_0 + u_1 + \omega u_2)$ where $\omega = e^{\pi i}$ and $|M\rangle = \frac{1}{\sqrt{2}}(u_0 + \frac{1}{\sqrt{6}}(u_1 + u_2 + u_3)$ which maximize the Tsallis $\alpha$-entropy for different regions of $\alpha$ [8] satisfy the monogamy relation of MIN. On the other hand, the four-qubit cluster states satisfy the monogamy relation as their two-party reduced density matrices are completely mixed (i.e. MIN is zero). For four-qubit generalized W-states $\rho^W = \alpha|1000\rangle + \beta|0100\rangle + \gamma|0010\rangle + \delta|0001\rangle$ where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$, the monogamy relation holds with equality, i.e. it can be easily shown (as in the three-qubit case) that $N(\rho^W_{AB}) + N(\rho^W_{AC}) + N(\rho^W_{AD}) = N(\rho^W_{ABCD}) = 2|\alpha|^2(1 - |\alpha|^2)$, Hence, for this type of states nonlocality shows an additive property with respect to each party. This result of the generalized W-class can be further extended to $n$-qubit systems with the same conclusion.

7. Conclusion

We have explored the monogamy nature of MIN and found certain classes of states on which MIN shows a monogamous nature. Unlike geometric discord [7], MIN can be polygamous for pure states as revealed in some subclasses of three- and four-qubit generic classes. On the other hand, the W-class seems to satisfy the monogamy relation with equality in the $n$-qubit system. So for the W-class, MIN becomes additive in terms of sharing between the parties. The monogamous nature of the W-class states w.r.t. MIN in any dimension indicates a distinguishing feature of this class of states. Thus, the existence of monogamy of this type of correlation puts a restriction on the amount of shared nonlocality. The monogamous nature of MIN for these classes of states can be exploited in providing some cryptographic protocol.

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