POISSON STABLE MOTIONS OF MONOTONE NONAUTONOMOUS DYNAMICAL SYSTEMS

DAVID CHEBAN AND ZHENXIN LIU

Abstract. In this paper, we study the Poisson stability (in particular, stationarity, periodicity, quasi-periodicity, Bohr almost periodicity, almost automorphy, recurrence in the sense of Birkhoff, Levitan almost periodicity, pseudo periodicity, almost recurrence in the sense of Bebutov, pseudo recurrence, Poisson stability) of motions for monotone nonautonomous dynamical systems and of solutions for some classes of monotone nonautonomous evolution equations (ODEs, FDEs and parabolic PDEs). As a byproduct, some of our results indicate that all the trajectories of monotone systems converge to the above mentioned Poisson stable trajectories under some suitable conditions, which is interesting in its own right for monotone dynamics.

1. Introduction

The existence of Bohr almost periodic solutions of the equation

\[ x' = f(t, x) \]  

with Bohr almost periodic right-hand side \( f \) in \( t \), uniformly with respect to (shortly w.r.t.) \( x \) on every compact subset of \( \mathbb{R}^n \) was studied by many authors \[23, 26, 27, 29, 34, 37, 41\] (see also \[3, ChIV\], \[18, ChXII\], \[25, ChVII\] and the bibliography therein).

Z. Opial \[27\] consider the scalar (i.e. \( n = 1 \)) case of differential equation (1.1). If \( f \) is monotone w.r.t. spacial variable \( x \), he established that every bounded (on the whole axis \( \mathbb{R} \)) solution is Bohr almost periodic.

Recall that a function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) is called regular if for any \( g \in H(f) \) and \( v \in \mathbb{R}^n \) the limit equation

\[ x' = g(t, x) \]  

admits a unique solution \( \varphi(t, v, g) \) defined on \( \mathbb{R} \) with initial value \( x(0) = v \), where \( H(f) := \{ f^\tau : \tau \in \mathbb{R} \} \), \( f^\tau(t, x) := f(t+\tau, x) \) for any \( (t, x) \in \mathbb{R}^n \) and by bar we mean the closure in \( C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) which is equipped with the compact-open topology.

V. V. Zhikov \[41\] (see also \[25, ChVII\] and \[3, ChIV\]) studied the scalar equation (1.1) with regular \( f \) but without monotone assumption for \( f \). He obtained existence of at least one almost periodic solution of equation (1.1) if it admits one bounded and uniformly stable solution.

R. Sacker and G. Sell \[28\] (see also \[29, §3.6\] and \[18, ChXII\]) generalized V. V. Zhikov’s result, still for scalar equations, by replacing the regularity of \( f \) by
positive regularity. Namely: a function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is called positively regular if for any $g \in H(f)$ and $v \in \mathbb{R}^n$ the limit equation (1.2) admits a unique solution $\varphi(t, v, g)$ defined on $\mathbb{R}_+$ with initial value $x(0) = v$. In addition, R. Sacker and G. Sell [29] studied almost periodic solutions for general case (not necessarily scalar) of equation (1.1) in the framework of skew-product flows.

B. A. Shcherbakov [34] studied the Poisson stability (in particular, periodicity, Bohr almost periodicity, recurrence in the sense of Birkhoff, almost recurrence in the sense of Bebutov, Levitan almost periodicity) of solutions for scalar equation (1.1) with $f$ being monotone w.r.t. $x$ and Poisson stable in $t \in \mathbb{R}$ (uniformly w.r.t. $x$ on every compact subset of $\mathbb{R}$). That is, he generalized Z. Opial’s result to Poisson stable differential equations (1.1).

D. Cheban [13] considered the Poisson stable solutions for the scalar equation (1.1) with arbitrary Poisson stable (w.r.t. time $t$) $f$ and without monotonicity assumption for $f$, thus generalized the results of Z. Opial, V. V. Zhikov, R. Sacker and G. Sell, and B. A. Shcherbakov.

In the framework of monotone cocycles or nonautonomous dynamical systems the problem of almost periodicity and almost automorphy of solutions for equation (1.1) in general case (both finite and infinite dimensional cases) was studied in the works of W. Shen and Y. Yi [37], J. Jiang and X.-Q. Zhao [23], S. Novo et al [26], and the bibliography therein.

The aim of this paper is to study the existence of Poisson stable (e.g. stationary, periodic, quasi-periodic, Bohr/Levitan almost periodic, almost automorphic, almost recurrent, etc) solutions of equation (1.1) in both finite and infinite dimensional cases when (1.1) generates a monotone cocycle, which can be achieved, say, if $f$ is quasi-monotone. The existence of at least one such Poisson stable solution is obtained provided each solution of (1.1) is compact on $\mathbb{R}_+$ and uniformly stable. Meantime, as a byproduct our results (see Theorems 5.13, 5.16 and 5.19) show that all the solutions will converge to the Poisson stable ones, which is interesting on its own rights in monotone dynamics.

The paper is organized as follows.

In Section 2 we collect some notions and facts from the theory of nonautonomous dynamical systems: cocycles, nonautonomous dynamical systems, conditional compactness, Ellis semigroup etc.

Section 3 is dedicated to studying the structure of the $\omega$-limit set of noncompact semi-trajectories for autonomous and nonautonomous dynamical systems. The main result of this section is contained in Theorem 3.10 which states that the one-sided dynamical system $(X, \mathbb{R}_+, \pi)$ on $\omega_{x_0}$, the $\omega$-limit set of $x_0$, can be extended to a two-sided dynamical system provided the positive semi-trajectory $\Sigma_{x_0}^+$ of $x_0$ is conditionally precompact and $\omega_{x_0}$ is uniformly stable.

In Section 4 we give a survey of different classes of Poisson stable motions, B. A. Shcherbakov’s principle of comparability of motions by their character of recurrence and some generalization of this principle.

Section 5 is dedicated to the study of abstract monotone nonautonomous dynamical systems. The main results of the paper are contained in Theorems 5.8 and 5.10 which give sufficient conditions for existence of comparable and strongly comparable motions. Using these Theorems and Shcherbakov’s comparability principle of motions by character of recurrence we obtain a series of results of existence of stationary (respectively, periodic, quasi-periodic, Bohr almost periodic, almost
automorphic, Birkhoff recurrent, Levitan almost periodic, almost recurrent, pseudo recurrent, uniformly Poisson stable, Poisson stable) motions. As mentioned above, we also obtain the convergence of all the trajectories to Poisson stable ones under some suitable conditions.

In Section 6 we apply our abstract results obtained in Sections 3 and 5 to study different classes of Poisson stability (as listed above) of solutions for monotone differential equations (ODEs, FDEs and parabolic PDEs). In this way we obtain a series of new results (some of them coincides with the well-known results).

2. NDS: SOME GENERAL PROPERTIES

In this section we collect some notions and facts for nonautonomous dynamical systems which we will use below; the reader may refer to [8], [12] Ch. IX], [30] for details.

Throughout the paper, we assume that $X$ and $Y$ are metric spaces and for simplicity we use the same notation $\rho$ to denote the metrics on them, which we think would not lead to confusion. Let $R = (-\infty, +\infty)$, $R_+ = \{t \in R : t \geq 0\}$ and $R_- = \{t \in R : t \leq 0\}$. For given dynamical system $(X, R, \pi)$ and given point $x \in X$, we denote by $\Sigma_x$ (respectively, $\Sigma^+_x$) its trajectory (respectively, semi-trajectory), i.e. $\Sigma_x := \{\pi(t, x) : t \in R\}$ (respectively, $\Sigma^+_x := \{\pi(t, x) : t \in R_+\}$), and call the mapping $\pi(\cdot, x) : R \rightarrow X$ the motion through $x$ at the moment $t = 0$. For given set $A \subseteq X$, we denote $\Sigma_A := \{\pi(t, x) : t \in R, x \in A\}$; $\Sigma^+_A$ is defined similarly. We denote the hull (respectively, semi-hull) of a point $x$ by $H(x) := \overline{\Sigma_x}$ (respectively, $H^+(x) := \overline{\Sigma^+_x}$), where by bar we mean closure. A point $x \in X$ is called Lagrange stable, “st. $L^+$ in short, (respectively, positively Lagrange stable, “st. $L^+$+ in short) if $H(x)$ (respectively, $H^+(x)$) is compact.

2.1. Cocycles and NDS. Let $(Y, R, \sigma)$ be a two-sided dynamical system on $Y$ and $E$ a metric space.

Definition 2.1. A triplet $(E, \phi, (Y, R, \sigma))$ (or briefly $\phi$ if no confusion) is said to be a cocycle on state space (or fibre) $E$ with base $(Y, R, \sigma)$ if the mapping $\phi : R_+ \times Y \times E \rightarrow E$ satisfies the following conditions:

(i) $\phi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
(ii) $\phi(t + \tau, u, y) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in R_+, u \in E$ and $y \in Y$;
(iii) the mapping $\phi$ is continuous.

Definition 2.2. Let $(E, \phi, (Y, R, \sigma))$ be a cocycle on $E$, $X := E \times Y$ and $\pi$ be a mapping from $R_+ \times X$ to $X$ defined by $\pi := (\phi, \sigma)$, i.e. $\pi(t, (u, y)) = (\phi(t, u, y), \sigma(t, y))$ for all $t \in R_+$ and $(u, y) \in E \times Y$. The triplet $(X, R_+, \pi)$ is an autonomous dynamical system and called skew-product dynamical system.

Definition 2.3. Let $T_1 \subseteq T_2$ be two subsemigroups of the group $R$, $(X, T_1, \pi)$ and $(Y, T_2, \sigma)$ be two autonomous dynamical systems and $h : X \rightarrow Y$ be a homomorphism from $(X, T_1, \pi)$ to $(Y, T_2, \sigma)$ (i.e. $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in T_1$ and $x \in X$, and $h$ is continuous and surjective), then the triplet $(\pi_1, (X, T_1, \pi), (Y, T_2, \sigma), h)$ is called nonautonomous dynamical system (NDS).

Example 2.4. An important class of NDS are generated from cocycles. Indeed, let $(E, \phi, (Y, R, \sigma))$ be a cocycle, $(X, R_+, \pi)$ be the associated skew-product dynamical system $(X = E \times Y, \pi = (\phi, \sigma))$ and $h = pr_2 : X \rightarrow Y$ (the natural projection mapping), then the triplet $(\pi_1, (X, R_+, \pi), (Y, R, \sigma), h)$ is an NDS.
2.2. Conditional compactness. Lagrange stable (or called “compact”) motions have been studied comprehensively, but it is not the case for non-Lagrange stable motions. The following concept of conditional compactness introduced in \cite{5} is important for our study of noncompact motions.

**Definition 2.5.** Let \((X, h, Y)\) be a fiber space, i.e. \(X\) and \(Y\) be two metric spaces and \(h : X \to Y\) be a homomorphism from \(X\) onto \(Y\). A set \(M \subseteq X\) is said to be **conditionally precompact** if its intersection with the preimage of any precompact subset \(Y' \subseteq Y\), i.e. the set \(h^{-1}(Y') \cap M\), is a precompact subset of \(X\). A set \(M\) is called **conditionally compact** if it is closed and conditionally precompact.

**Remark 2.6.**

1. Let \((X, h, Y)\) be a compact space, \(Y \subseteq \mathbb{R}\) and \(h = \text{pr} \colon X \to Y\). Then the triplet \((X, h, Y)\) is a fiber space. The space \(X\) is conditionally compact, but it is not compact.

2. If \(Y\) is a compact set and \(M \subseteq X\) is conditionally precompact, then \(M\) is a precompact set.

The following result provides a useful criterion for conditional compactness in applications.

**Lemma 2.7.** Let \(\langle \mathcal{E}, \phi, (Y, \mathbb{R}, \sigma) \rangle\) be a cocycle and \(\langle (X, \mathbb{R}^+, \pi), (Y, \mathbb{R}, \sigma), h \rangle\) be the NDS generated by the cocycle \(\phi\) (cf. Example 2.4). Assume that \(x_0 := (u_0, y_0) \in X = E \times Y\) and the set \(Q^+_{\phi(u_0,y_0)} := \{\phi(t, u_0, y_0) : t \in \mathbb{R}^+\}\) is compact. Then the semi-hull \(H^+_{(x_0)}\) is conditionally compact.

Now we give a concrete example to illustrate the notion of conditional compactness for noncompact motions. To this end, we need to review some basic notions.

Denote by \(C(\mathbb{R})\) the family of all continuous functions \(f : \mathbb{R} \to \mathbb{R}\) equipped with the compact-open topology. This topology can be generated by Bebutov distance (see, e.g. \cite{1, 5, ChIV})

\[
d(f, g) := \sup_{l > 0} \min \{\max_{|t| \leq l} |f(t) - g(t)|, 1/l\}.
\]

Denote by \((C(\mathbb{R}), \mathbb{R}, \sigma)\) the shift dynamical system (or called Bebutov dynamical system), i.e. \(\sigma(\tau, f) := f^\tau\), where \(f^\tau(t) := f(t+\tau)\) for \(t \in \mathbb{R}\). Note that the function \(f \in C(\mathbb{R})\) is st. \(L^+\) (respectively, st. \(L\)) if and only if the function \(f\) is bounded and uniformly continuous on \(\mathbb{R}^+\) (respectively, on \(\mathbb{R}\)) (see, e.g. \cite{5} ChIV).

**Example 2.8.** Define \(h(t) := 2 + \cos t + \cos \sqrt{2}t\) for \(t \in \mathbb{R}\), then \(h\) is a Bohr almost periodic function. The function \(\varphi(t) := 1/h(t)\) (respectively, \(\psi(t) := \sin \varphi(t)\)) for \(t \in \mathbb{R}\) is Levitan almost periodic \cite{25, ChIV}, it is not Bohr almost periodic because it is not bounded (respectively, not uniformly continuous; see \cite{24} ChV, pp.212-213) or \cite{2} on \(\mathbb{R}\). Thus the function \(\varphi\) (respectively, \(\psi\)) is not st. \(L\). Denote by \(Y = H(\varphi) = \{\varphi^\tau : \tau \in \mathbb{R}\}\) (respectively, \(X = H(\psi, \varphi) = \{(\psi^\tau, \varphi^\tau) : \tau \in \mathbb{R}\}\)), where by bar we mean the closure in \(C(\mathbb{R})\) (respectively, in \(C(\mathbb{R}) \times C(\mathbb{R})\)).

**Lemma 2.9.** Let \(\varphi, \psi\) and \(X, Y\) be as in Example 2.8. Consider the NDS \((\{X, \mathbb{R}, \pi\}, (Y, \mathbb{R}, \sigma), h)\), where \(h = \text{pr} \colon X \to Y\), and \((Y, \mathbb{R}, \sigma)\) and \((X, \mathbb{R}, \pi)\) are the shift dynamical systems on \(Y\) and \(X\) respectively. Then the following statements hold:

(i) the set \(Y\) (respectively, \(X\)) is not compact in \(C(\mathbb{R})\) (respectively, \(C(\mathbb{R}) \times C(\mathbb{R})\))

1For the definitions of Bohr and Levitan almost periodic functions, see Section 4 for details.
(ii) the set $X$ is a conditionally compact.

Proof. The first statement follows from the construction of $Y$ (respectively, $X$) because the function $\varphi$ (respectively, $(\psi, \varphi)$) is not st. $L$.

To finish the proof of the lemma it is sufficient to establish that $X$ is conditionally precompact because it is closed. Let $K'$ be an arbitrary precompact subset of $Y$ and $K = h^{-1}(K') \subseteq X$. We need to show that $K$ is precompact. Consider an arbitrary sequence $\{(y_n, \varphi_n)\} \subset K$, then, by the definition of hull, for each $n \in \mathbb{N}$ there exists a number $\tau_n \in \mathbb{R}$ such that

$$d(\psi, \varphi_n) < 1/n \quad \text{and} \quad d(\varphi, \varphi_n) < 1/n.$$  

Since $\{\varphi_n\} = h(\{(y_n, \varphi_n)\}) \subseteq K'$ is precompact, we can extract a subsequence $\{\varphi_{n_k}\}$ such that $\varphi_{n_k} \rightarrow \varphi \in H(\varphi)$ as $k \rightarrow \infty$; note that we have $\varphi_{n_k} \rightarrow \varphi$ by (2.1). Taking into account that $\psi(t) = \sin(\varphi(t))$ for $t \in \mathbb{R},$ we obtain $\psi_{n_k} \rightarrow \psi := \sin(\varphi) \in H(\varphi)$. That is, $\{\psi_{n_k}, \varphi_{n_k}\} \rightarrow (\psi, \varphi) \in H(\psi, \varphi) = X$ as $k \rightarrow \infty$. Thus it follows from (2.1) that $\{\psi_{n_k}, \varphi_{n_k}\} \rightarrow (\psi, \varphi) \in X$ as $k \rightarrow \infty$. The proof is complete. \qed

2.3. Some general facts about NDS. In this subsection we recall some general facts about NDS, see [8] or [12, Ch. IX] for details.

**Definition 2.10.** A point $y \in Y$ is called positively (respectively, negatively) Poisson stable if there exists a sequence $t_n \rightarrow +\infty$ (respectively, $t_n \rightarrow -\infty$) such that $\sigma(t_n, y) \rightarrow y$ as $n \rightarrow \infty$. If $y$ is Poisson stable in both directions, it is called Poisson stable.

Denote $\mathcal{N}_y := \{(t_n) \subset \mathbb{R} : \sigma(t_n, y) \rightarrow y\}$, $\mathcal{N}_y^+ := \{(t_n) \in \mathcal{N}_y : t_n \rightarrow +\infty\}$, $\mathcal{N}_y^- := \{(t_n) \in \mathcal{N}_y : t_n \rightarrow -\infty\}$, and $\mathcal{N}_y^{\infty} := \{(t_n) \in \mathcal{N}_y : t_n \rightarrow \infty\}$.

Let $\langle (X, \mathbb{R}, \varphi), (Y, \mathbb{R}, \sigma), h \rangle$ be an NDS and $y \in Y$ be positively Poisson stable. Denote

$$\mathcal{E}_y^+ := \{\xi : \exists\{t_n\} \in \mathcal{N}_y^{\infty} \text{ such that } \pi(t_n, \cdot)|_{X_y} \rightarrow \xi\},$$

where $X_y := h^{-1}(y) = \{x \in X : h(x) = y\}$ and $\rightarrow$ means the pointwise convergence. If the NDS is two-sided, $y$ is negatively Poisson stable or Poisson stable and we replace $\mathcal{N}_y^+$ by $\mathcal{N}_y^-$ or $\mathcal{N}_y$ in (2.2), then we get the definition of $\mathcal{E}_y^-$ or $\mathcal{E}_y$.

Let $X^X_y$ denote the Cartesian product of $X$ copies of the space $X$, equipped with product topology. The set $X^X_y$ can be endowed with a semigroup structure with respect to composition of mappings from $X^X_y$ (for more details, see e.g. [9] Chl) and [17].

**Lemma 2.11.** Let $y \in Y$ be positively Poisson stable, $\langle (X, \mathbb{R}^+, \varphi), (Y, \mathbb{R}, \sigma), h \rangle$ be an NDS and $X$ be conditionally compact. Then $\mathcal{E}_y^+$ is a nonempty compact subsemigroup of the semigroup $X^X_y$.

**Corollary 2.12.** Let $y \in Y$ be negatively Poisson stable, $\langle (X, \mathbb{R}, \varphi), (Y, \mathbb{R}, \sigma), h \rangle$ be a two-sided NDS and $X$ be conditionally compact, then $\mathcal{E}_y^-$ is a nonempty compact subsemigroup of the semigroup $X^X_y$.

**Lemma 2.13.** Let $y \in Y$ be Poisson stable, $\langle (X, \mathbb{R}, \varphi), (Y, \mathbb{R}, \sigma), h \rangle$ be a two-sided NDS and $X$ be conditionally compact, then $\mathcal{E}_y$ is a nonempty compact subsemigroup of the semigroup $X^X_y$.
Lemma 2.16. Let \( \{ E_n \} \) be conditionally compact and \( \inf_{n \in \mathbb{N}} \rho(t_n, x_2) > 0 \) for all \( \{ t_n \} \in \mathbb{N} \) and \( x_1, x_2 \in X \) (\( x_1 \neq x_2 \)), then \( E_y^- \) is a subgroup of the semigroup \( E_y \).

**Corollary 2.14.** Under the conditions of Lemma 2.13, \( E^+_y \) and \( E^-_y \) are two nonempty subsemigroups of the semigroup \( E_y \).

**Lemma 2.15.** Under the conditions of Lemma 2.13 the following statements hold:

(i) if \( \xi_1 \in E^-_y \) and \( \xi_2 \in E^+_y \), then \( \xi_1 \cdot \xi_2 \in E^-_y \cap E^+_y \), where \( \xi_1 \cdot \xi_2 \) is the composition of \( \xi_1 \) and \( \xi_2 \);

(ii) \( E^-_y \cap E^+_y \) is a subsemigroup of the semigroups \( E^-_y \) and \( E^+_y \);

(iii) \( E^-_y \cdot E^-_y \subseteq E^-_y \) and \( E^+_y \cdot E^-_y \subseteq E^+_y \), where \( A_1 \cdot A_2 := \{ \xi_1 \cdot \xi_2 : \xi_i \in A_i, i = 1, 2 \} \) and \( A_i \subseteq E_y \);

(iv) if at least one of the subsemigroups \( E^-_y \) or \( E^+_y \) is a group, then \( E^-_y = E^+_y = E_y \).

**Lemma 2.16.** Let \( y \in Y \) be Poisson stable, \( (X, \mathbb{R}, \pi), (Y, \mathbb{R}, \sigma) \) be a two-sided NDS, \( X \) be conditionally compact and

\[
(2.3) \quad \inf_{n \in \mathbb{N}} \rho(t_n, x_1), \pi(t_n, x_2) > 0
\]

for all \( \{ t_n \} \in \mathbb{N} \) and \( x_1, x_2 \in X_y \) (\( x_1 \neq x_2 \)), then \( E_y^- \) is a subgroup of the semigroup \( E_y \).

**Corollary 2.17.** Under the conditions of Lemma 2.16 the following statements hold:

(i) \( E^-_y = E^+_y = E_y \);

(ii) \( E_y^+ \) (respectively, \( E^-_y \) and \( E_y \)) is a group.

**Lemma 2.18.** Assume that the conditions of Lemma 2.16 hold, then inequality \( (2.3) \) is also fulfilled for any \( \{ t_n \} \in \mathbb{N} \) and \( x_1, x_2 \in X_y \) (\( x_1 \neq x_2 \)).

**Definition 2.19.** Let \( (E, \phi, (Y, \mathbb{R}, \sigma)) \) (respectively, \( (X, \mathbb{R}^+, \pi) \)) be a cocycle (respectively, one-sided dynamical system). A continuous mapping \( \nu : \mathbb{R} \to E \) (respectively, \( \gamma : \mathbb{R} \to X \)) is called an entire trajectory of cocycle \( \phi \) (respectively, \( \gamma \) of dynamical system \( (X, \mathbb{R}^+, \pi) \)) passing through the point \( (u, y) \in E \times Y \) (respectively, \( x \in X \)) at \( t = 0 \) if \( \phi(t, \nu(s), \sigma(s, y)) = \nu(t + s) \) and \( \nu(0) = u \) (respectively, \( \pi(t, \gamma(s)) = \gamma(t + s) \) and \( \gamma(0) = x \)) for all \( t \in \mathbb{R}^+ \) and \( s \in \mathbb{R} \).

Denote by

- \( C(\mathbb{R}, X) \) the space of all continuous functions \( f : \mathbb{R} \to X \) equipped with the compact-open topology;
- \( \Phi_x \) the family of all entire trajectories of \( (X, \mathbb{R}^+, \pi) \) passing through the point \( x \in X \) at the initial moment \( t = 0 \) and \( \Phi := \bigcup \{ \Phi_x : x \in X \} \).

**Remark 2.20.** Note that:

(i) the compact-open topology on the space \( C(\mathbb{R}, X) \) is metrizable, for example by Bebutov distance

\[
d(\varphi, \psi) := \sup_{t > 0} d_t(\varphi, \psi),
\]

where \( d_t(\varphi, \psi) := \min \{ \max_{|t| \leq l} \rho(\varphi(t), \psi(t)), 1/l \} \);

(ii) if \( \gamma \in \Phi_x \) then \( \gamma^+ \in \Phi_{\gamma(\tau)} \), where \( \gamma^+ := \gamma(t + r) \) for \( t \in \mathbb{R} \), and consequently \( \Phi \) is a translation invariant subset of \( C(\mathbb{R}, X) \);

(iii) if \( \gamma_n \in \Phi_{x_n} \) and \( \gamma_n \to \gamma \) in \( C(\mathbb{R}, X) \) as \( n \to \infty \), then \( \gamma \in \Phi_x \) with \( x := \lim_{n \to \infty} x_n \) and consequently \( \Phi \) is a closed subset of \( C(\mathbb{R}, X) \).
Similar to the shift dynamical system \((C(\mathbb{R}), \mathbb{R}, \sigma)\) in Section 2.2, let \((C(\mathbb{R}, X), \mathbb{R}, \lambda)\) be the shift dynamical system (or Bebutov dynamical system, see e.g. \([3, 12, 30, 33]\)) on the space \(C(\mathbb{R}, X)\). By Remark 2.22 \(\Phi\) is a closed and invariant (with respect to shifts) subset of \(C(\mathbb{R}, X)\), and consequently on \(\Phi\) is defined a shift dynamical system \((\Phi, \mathbb{R}, \lambda)\) induced from \((C(\mathbb{R}, X), \mathbb{R}, \lambda)\).

3. Structure of the \(\omega\)-limit set

Let \(M\) be a subset of \(X\). We denote the \(\omega\)-limit set of \(M\) by

\[
\omega(M) := \bigcap_{t \geq 0} \bigcup \{\pi(t, M) : \pi \geq t\};
\]

for a singleton set, for simplicity we also write \(\omega(x)\) or \(\omega_z\) for \(\omega(\{x\})\) and denote \(\omega_q(M) := \omega(M) \cap h^{-1}(q)\). Note that \(x \in \omega(M)\) if and only if there exists sequences \(\{x_n\} \subset M\) and \(\{t_n\} \subset \mathbb{R}\) such that \(t_n \to +\infty\) as \(n \to \infty\) and \(\lim_{n \to \infty} \pi(t_n, x_n) = x\).

**Lemma 3.1.** Let \(((X, \mathbb{R}, \pi), (Y, \mathbb{R}, \sigma), h)\) be an NDS, \(M \subseteq X\) be a nonempty subset, and \(\Sigma^+ = \{\gamma(t, M) : t \geq 0\}\) be conditionally precompact. Then for any \(x \in \omega(M)\) there exists at least one entire trajectory \(\gamma\) of dynamical system \((X, \mathbb{R}, \pi)\) passing through the point \(x\) at \(t = 0\) and \(\gamma(\mathbb{R}) \subseteq \omega(M)\) \((\gamma(\mathbb{R}) := \{\gamma(t) : t \in \mathbb{R}\}\).

**Proof.** Let \(x \in \omega(M)\), then there are \(\{t_n\} \subset \mathbb{R}\) and \(\{x_n\} \subset M\) such that \(x = \lim_{n \to \infty} \pi(t_n, x_n)\) and \(t_n \to +\infty\) as \(n \to \infty\). We consider the sequence \(\{\gamma_n\} \subset C(\mathbb{R}, X)\) defined by

\[
\gamma_n(t) := \pi(t + t_n, x_n) \quad \text{for} \quad t \geq -t_n \quad \text{and} \quad \gamma_n(t) := x_n \quad \text{for} \quad t \leq -t_n.
\]

We now show that the above sequence \(\{\gamma_n\}\) is equicontinuous on any compact interval. If this is not true, then there exist \(\varepsilon_0, l_0 > 0, t^*_n \in [-l_0, l_0]\) and \(\delta_n \to 0\) such that

\[
|t^*_n - t_n^2| \leq \delta_n \quad \text{and} \quad \rho(\gamma_n(t_n^1), \gamma_n(t_n^2)) \geq \varepsilon_0.
\]

We may suppose that \(t^*_n \to 0\) \((i = 1, 2)\). Since \(t_n \to +\infty\) as \(n \to \infty\), there exists a number \(n_0 \in \mathbb{N}\) such that \(t_n \geq l_0\) for any \(n \geq n_0\). From (3.1) we obtain

\[
|t^*_n - t_n| \leq \delta_n \quad \text{and} \quad \rho(\gamma_n(t_n^1), \gamma_n(t_n^2)) \geq \varepsilon_0.
\]

Note that \(y_n := h(\pi(t_n, x_n)) \to h(x) := y\) and \(h(\pi(t_n - l_0, x_n)) = \sigma(-l_0, y_n) \to \sigma(-l_0, y) = \sigma(-l_0, y) \to \sigma(-l_0, y) = \) as \(n \to \infty\). Since \(\Sigma^+_M\) is conditionally precompact, the sequence \(\{\pi(t_n - l_0, x_n)\}\) is relatively compact. Without loss of generality we suppose that \(\{\pi(t_n - l_0, x_n)\}\) converges and denote by \(\bar{x}\) its limit. Then passing to limit in (3.2) we obtain

\[
\varepsilon_0 \leq \rho(\pi(t_0 + l_0, x), \pi(t_0 + l_0, \bar{x})) = 0,
\]
a contradiction.

Next we prove that the set \(\{\gamma_n(t) : t \in [-l, l], n \in \mathbb{N}\}\) is precompact for any \(l > 0\). To this end, note that for any \(n \geq n_0\) we have \(h(\gamma_n(t)) = h(\pi(t + t_n, x_n)) = \sigma(t, y_n)\), where \(y_n = h(\pi(t_n, x_n)) \to h(x) = y\) as \(n \to \infty\). So the set \(K := \{\sigma(t, y_n) : t \in [-l, l], n \in \mathbb{N}\} \subset Y\) is precompact. Since the set \(\Sigma^+_M\) is conditionally precompact and \(\{\gamma_n(t) : t \in [-l, l], n \in \mathbb{N}\} \subset h^{-1}(K) \cap \Sigma^+_M\), the set \(\{\gamma_n(t) : t \in [-l, l], n \in \mathbb{N}\}\) is precompact.

It follows from the Arzelà–Ascoli theorem that \(\{\gamma_n\}\) is a relatively compact sequence of \(C(\mathbb{R}, X)\). Let \(\gamma\) be a limit point of the sequence \(\{\gamma_n\}\), then there exists
a subsequence \( \{ \gamma_n \} \) such that \( \gamma(t) = \lim_{n \to \infty} \gamma_n(t) \) uniformly on every compact interval. In particular, \( \gamma(t) \in \omega(M) \) for any \( t \in \mathbb{R} \) because \( \gamma(t) = \lim_{n \to \infty} \pi(t + t_n, x_n) \). We note that

\[
\pi^t \gamma(s) = \lim_{n \to \infty} \pi^t \gamma_n(s) = \lim_{n \to \infty} \gamma_n(s + t) = \gamma(s + t)
\]

for all \( t \in \mathbb{R}_+ \) and \( s \in \mathbb{R} \). Finally, we see that \( \gamma(0) = \lim_{n \to \infty} \gamma_n(0) = \lim_{n \to \infty} \pi(t_n, x_n) = x \), i.e. \( \gamma \) is an entire trajectory of dynamical system \((X, \mathbb{R}_+, \pi)\) passing through point \( x \). The proof is complete. \( \square \)

**Remark 3.2.** 1. If \( \Sigma^+_h(M) \) is precompact, then so is \( \Sigma^+_M \).

2. If \( \Sigma^+_h(M) \) is not precompact, then \( \Sigma^+_M \), generally speaking, is not precompact.

**Theorem 3.3.** Let \((X, \mathbb{R}_+, \pi)\), \((Y, \mathbb{R}, \sigma), h)\) be an NDS, \( x_0 \in X, \Sigma^+_{x_0} \) be conditionally precompact and \( \omega_{y_0} \neq \emptyset \), where \( y_0 := h(x_0) \). Then the following statements hold:

(i) \( \omega_{x_0} \cap X_q \neq \emptyset \) for any \( q \in \omega_{y_0} \) (recall that \( X_q = h^{-1}(q) \)), and consequently \( \omega_{x_0} \neq \emptyset \);

(ii) \( h(\omega_{x_0}) = \omega_{y_0} \);

(iii) the set \( \omega_{x_0} \) is conditionally compact;

(iv) \( \pi(t, \omega_q(x_0)) = \omega_{\sigma(t, q)}(x_0) \) for any \( t \in \mathbb{R}_+ \) and \( q \in \omega_{y_0} \), recalling that \( \omega_q(x_0) = \omega(x_0) \setminus X_q \);

(v) \( \omega_{x_0} \) is invariant, i.e. \( \pi(t, \omega_{x_0}) = \omega_{x_0} \) for any \( t \geq 0 \).

**Proof.** (i) Let \( q \in \omega_{y_0} \), then there exists a sequence \( \{ \tau_n \} \subset \mathbb{R}_+ \) such that \( \tau_n \to +\infty \) and \( \sigma(\tau_n, y_0) \to q \) as \( n \to \infty \). Denote \( K := \{ \sigma(\tau_n, y_0) : n \in \mathbb{N} \} \), then the set \( \{ \pi(\tau_n, x_0) : n \in \mathbb{N} \} = \Sigma^+_{x_0} \cap h^{-1}(K) \) is precompact. Without loss of generality we suppose that the sequence \( \{ \pi(\tau_n, x_0) \} \) is convergent, and denote \( p := \lim_{n \to \infty} \pi(\tau_n, x_0) \).

Thus \( p \in \omega_{x_0} \cap X_q \neq \emptyset \).

(ii) For given \( p \in \omega_{x_0} \), there is a sequence \( t_n \to +\infty \) as \( n \to \infty \) such that \( \lim_{n \to \infty} \pi(t_n, x_0) = p \), and consequently \( q := h(p) = \lim_{n \to \infty} \sigma(t_n, y_0) \in \omega_{y_0} \). Thus we have \( h(\omega_{x_0}) \subseteq \omega_{y_0} \). Let now \( q \in \omega_{y_0} \) and \( \tau_n \to +\infty \) as \( n \to \infty \) such that \( q = \lim_{n \to \infty} \sigma(\tau_n, y_0) \). Denote by \( K := \{ \sigma(\tau_n, y_0) : n \in \mathbb{N} \} \). Since \( \Sigma^+_{x_0} \) is conditionally precompact and \( \{ \pi(\tau_n, x_0) \} \subseteq \Sigma^+_{x_0} \cap h^{-1}(K) \), the set \( \{ \pi(\tau_n, x_0) \} \) is precompact. Without loss of generality we suppose that the sequence \( \{ \pi(\tau_n, x_0) \} \) converges and denote by \( p := \lim_{n \to \infty} \pi(\tau_n, x_0) \). It is clear that \( q = h(p) \in h(\omega_{x_0}) \).

(iii) Let \( K \subseteq Y \) be a precompact subset, \( M := \omega_{x_0} \cap h^{-1}(K) \), \( \{ x_n \} \subseteq M \) be a sequence and \( y_n := h(x_n) \in K \cap \omega_{y_0} \) for \( n \in \mathbb{N} \). Then there exists a sequence \( \{ \tau_n \} \subset \mathbb{R}_+ \) such that \( \tau_n > n \) and

\[
\rho(y_n, \sigma(\tau_n, y_0)) < 1/n
\]

for \( n \in \mathbb{N} \). Since \( y_n = h(x_n) \in K \), we may suppose that the sequence \( \{ y_n \} \) converges and denote its limit by \( q \). Then we have \( q = \lim_{n \to \infty} \sigma(\tau_n, y_0) \) by (3.3). On the other hand \( \{ \pi(\tau_n, x_0) \} \subseteq \Sigma^+_{x_0} \cap h^{-1}(K) \) \( K' := \{ \sigma(\tau_n, y_0) \} \), and consequently the set \( \{ \pi(\tau_n, x_0) : n \in \mathbb{N} \} \) is precompact. Taking into consideration (3.3) we obtain that the sequence \( \{ x_n \} \) is also precompact. So the set \( \omega_{x_0} \) is conditionally precompact; furthermore, it is conditionally compact because it is closed.
We divide the proof into three steps.

(iv) Note first that \( \pi(t, \omega_{x_0} \cap X_q) \subseteq \pi(t, \omega_{x_0}) \cap \pi(t, X_q) \subseteq \omega_{x_0} \cap X_{\sigma(t,q)} = \omega_{\sigma(t,q)}(x_0) \) for any \( t \geq 0 \) and \( q \in \omega_{y_0} \). Let now \( t \geq 0 \), \( q \in \omega_{y_0} \) and \( x \in \omega_{\sigma(t,q)}(x_0) \), then by Lemma 3.7 there exists at least one entire trajectory \( \gamma \) passing through the point \( x \) at the moment \( t = 0 \) such that \( h(\gamma(\tau)) = \sigma(\tau, h(x)) = \sigma(\tau, \sigma(t,q)) \) for any \( \tau \in \mathbb{R} \). In particular, we have \( h(\gamma(-t)) = \sigma(-t, \sigma(t,q)) = q \), and consequently \( \gamma(-t) \in \omega_q(x_0) \). Therefore, we have \( x = \pi(t, \gamma(-t)) \in \pi(t, \omega_q(x_0)) \).

(v) It is sufficient to show \( \omega_{x_0} \subseteq \pi(t, \omega_{x_0}) \) because the inverse inclusion is evident. Let \( x \in \omega_{x_0} \), then by Lemma 3.7 there exists at least one entire trajectory \( \gamma \) lying on \( \omega_{x_0} \) and passing through the point \( x \) at the moment \( t = 0 \). Since \( \gamma(-t) \in \omega_{x_0} \) and \( x = \gamma(t - t) = \pi(t, \gamma(t)) \in \pi(t, \omega_{x_0}) \), we have \( \omega_{x_0} \subseteq \pi(t, \omega_{x_0}) \). The proof is now complete.

**Remark 3.4.** If \( y_0 \) is positively Lagrange stable, then the results of Theorem 3.3 are known. The novelty here is that \( y_0 \) is not necessarily Lagrange stable in the positive direction, and hence our results apply to noncompact Poisson stable motions.

**Definition 3.5.** Let \( \langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle \) be an NDS. A subset \( A \subseteq X \) is said to be (positively) uniformly stable if for arbitrary \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( \rho(x, a) < \delta \left( a \in A, \ x \in X \text{ and } h(a) = h(x) \right) \) implies \( \rho(\pi(t, x), \pi(t, a)) < \varepsilon \) for any \( t \geq 0 \). In particular, a point \( x_0 \in X \) is called uniformly stable if the singleton set \( \{x_0\} \) is so.

**Remark 3.6.** Let \( A \subseteq X \) be uniformly stable and \( B \subseteq A \), then \( B \) is also uniformly stable.

**Lemma 3.7.** ([3] ChIV, [4]) If the set \( A \subseteq X \) is uniformly stable and the mapping \( h : X \to Y \) is open, then the closure \( \overline{A} \) of \( A \) is uniformly stable.

**Corollary 3.8.** If \( \Sigma_{x_0}^+ \) is uniformly stable and \( h \) is open, then:

(i) \( H^+(x_0) \) is uniformly stable;

(ii) \( \omega_{x_0} \) is uniformly stable, because \( \omega_{x_0} \subseteq H^+(x_0) \).

**Remark 3.9.** Note that if an NDS is generated by a skew-product dynamical system (or equivalently by a cocycle) in which case the homomorphism \( h \) is given by the natural projection mapping, then clearly \( h \) is open.

**Theorem 3.10.** Let \( \langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle \) be an NDS with the following properties:

(i) there exists a point \( x_0 \in X \) such that the positive semi-trajectory \( \Sigma_{x_0}^+ \) is conditionally precompact;

(ii) the set \( \omega_{x_0} \) is positively uniformly stable.

Then all motions on \( \omega_{x_0} \) can be extended uniquely to the left and on \( \omega_{x_0} \) is defined a two-sided dynamical system \( (\omega_{x_0}, \mathbb{R}, \pi) \), i.e. the one-sided dynamical system \( (X, \mathbb{R}_+, \pi) \) generates on \( \omega_{x_0} \) a two-sided dynamical system \( (\omega_{x_0}, \mathbb{R}, \pi) \).

**Proof.** We divide the proof into three steps.

Step 1: we prove that the set \( \omega_{x_0} \subseteq X \) is distal in the negative direction with respect to the NDS \( \langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle \), i.e.

\[
\inf_{t \leq 0} \rho(\gamma_1(t), \gamma_2(t)) > 0
\]
for all $\gamma_i \in \tilde{\Phi}_x$, $(i = 1, 2$ and $x_1 \neq x_2)$, where $\tilde{\Phi}_x$ denotes the family of all entire trajectories of $(X, \mathbb{R}_+, \pi)$ passing through point $x$ and belonging to $\omega_{x_0}$. If this is not true, then there exist $y_0 \in Y, x_i^0 \in \omega_{x_0} \cap X_{y_0}$ ($h(x_1^0) = h(x_2^0), \ x_1 \neq x_2$), $\gamma_i^0 \in \tilde{\Phi}_x^0$ $(i = 1, 2)$ and $t_n \to +\infty$ such that

$$
\rho(\gamma_1^0(-t_n), \gamma_2^0(-t_n)) \to 0
$$

as $n \to \infty$. Let $\varepsilon_0 := \rho(x_1^0, x_2^0) > 0$ and $\delta_0 = \delta(\varepsilon_0) > 0$ be chosen from the positively uniform stability of $\omega_{x_0}$. Then we have

$$
\rho(\gamma_1^0(-t_n), \gamma_2^0(-t_n)) < \delta_0
$$

for sufficiently large $n$ by (3.5), so $\varepsilon_0 = \rho(x_1^0, x_2^0) = \rho(\pi(t_n, \gamma_1^0(-t_n)), \pi(t_n, \gamma_2^0(-t_n))) < \varepsilon_0$, a contradiction.

Step 2: we will show that for any $x \in \omega_{x_0}$ the set $\tilde{\Phi}_x$ is a singleton set. Let $\Phi := \{\tilde{\Phi}_x \mid x \in \omega_{x_0}\} \subset C(\mathbb{R}, X)$. It is immediate to check that $\Phi$ is a closed invariant subset of dynamical system $(C(\mathbb{R}, X), \mathbb{R}, \lambda)$, so on the set $\Phi$ is induced a dynamical system $(\Phi, \mathbb{R}, \lambda)$. Let $H$ be a mapping from $\Phi$ onto $\omega_{y_0}$, defined by $H(\gamma) := h(\gamma(0))$. Then it can be shown (see, e.g. [11, ChII]) that the triplet $(\Phi, \mathbb{R}, \lambda), (\omega_{y_0}, \mathbb{R}, \sigma), H)$ is an NDS. This NDS is distal in the negative direction, i.e.

$$
\inf_{t \leq 0} d(\gamma_1^t, \gamma_2^t) > 0
$$

for all $\gamma_1, \gamma_2 \in H^{-1}(y)$ $(\gamma_1 \neq \gamma_2)$ and $y \in \omega_{y_0}$, recalling that $\gamma^\tau := \sigma(\tau, \gamma)$, i.e. $\gamma^\tau(s) = \gamma(\tau + s)$ for $s \in \mathbb{R}$. Indeed, otherwise there exist $\bar{y}, \bar{\gamma}_1, \bar{\gamma}_2 \in H^{-1}(\bar{y})$ $(\bar{\gamma}_1 \neq \bar{\gamma}_2)$ and $t_n \to +\infty$ such that $d(\bar{\gamma}_1^{-t_n}, \bar{\gamma}_2^{-t_n}) \to 0$ as $n \to \infty$ and consequently

$$
\rho(\bar{\gamma}_1(-t_n), \bar{\gamma}_2(-t_n)) \leq d(\bar{\gamma}_1^{-t_n}, \bar{\gamma}_2^{-t_n}) \to 0.
$$

Since $\bar{\gamma}_1 \neq \bar{\gamma}_2$, there exists $t_0 \in \mathbb{R}$ such that $\bar{\gamma}_1(t_0) \neq \bar{\gamma}_2(t_0)$. Let $\bar{\gamma}_i(t) := \bar{\gamma}_i(t + t_0)$ for $t \in \mathbb{R}$, then $\bar{\gamma}_i \in \tilde{\Phi}_{\sigma(t_0, \bar{y})}$ and by (3.6) we have

$$
\rho(\bar{\gamma}_1(-t_n), \bar{\gamma}_2(-t_n)) \to 0 \quad \text{as} \quad n \to \infty.
$$

Thus we have found $q := h(\bar{\gamma}_1(t_0)), x_1 := \bar{\gamma}_i(t_0)$ $(i = 1, 2, \ h(x_1) = h(x_2) \ (x_1 \neq x_2)$ and the entire trajectories $\bar{\gamma}_i \in \tilde{\Phi}_x$, such that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are proximal (see (3.7)). But (3.7) and (3.3) are contradictory, so the negative distality of the NDS $(\Phi, \mathbb{R}, \sigma), (\omega_{y_0}, \mathbb{R}, \sigma), H)$ is proved.

If there exist $p \in \omega_{x_0}$ and two different trajectories $\gamma_1, \gamma_2 \in \tilde{\Phi}_p$, then in virtue of the distality of $\gamma_1$ and $\gamma_2$ we have

$$
\alpha(\gamma_1, \gamma_2) := \inf_{t \leq 0} d(\gamma_1^t, \gamma_2^t) > 0.
$$

So $\rho(\gamma_1(t), \gamma_2(t)) \geq \alpha(\gamma_1, \gamma_2) > 0$ for all $t \leq 0$. In particular $\gamma_1(0) \neq \gamma_2(0)$, a contradiction.

Step 3: let now $\bar{x} : \mathbb{R} \times \omega_{x_0} \to \omega_{x_0}$ be a mapping defined by

$$
\bar{x}(t, x) := \pi(t, x) \quad \text{if} \quad t \geq 0 \quad \text{and} \quad \bar{x}(t, x) := \gamma_x(t) \quad \text{if} \quad t < 0
$$

for $x \in \omega_{x_0}$, then $(\omega_{x_0}, \mathbb{R}, \bar{x})$ is a two-sided dynamical system. Here $\gamma_x$ is the unique entire trajectory of the dynamical system $(X, \mathbb{R}_+, \pi)$ passing through point $x$ and belonging to $\omega_{x_0}$. To prove that $(\omega_{x_0}, \mathbb{R}, \bar{x})$ is a two-sided dynamical system it suffices to check the continuity of the mapping $\bar{x}$. Let $x \in \omega_{x_0}, t \in \mathbb{R}_-, x_n \to x$
and $t_n \to t$. Then there is an $l_0 > 0$ such that $t_n \in [-l_0, l_0]$ for $n \in \mathbb{N}$ and consequently

$$\rho(\bar{T}(t_n, x_n), \bar{T}(t, x)) = \rho(\pi(t_n + l_0, \gamma_{x_n}(-l_0)), \pi(t + l_0, \gamma_{x}(-l_0)))$$

$$\leq \rho(\pi(t_n + l_0, \gamma_{x_n}(-l_0)), \pi(t_n + l_0, \gamma_{x}(-l_0)))$$

$$+ \rho(\pi(t_n + l_0, \gamma_{x}(-l_0)), \pi(t + l_0, \gamma_{x}(-l_0))).$$

We now show that the sequence $\{\gamma_{x_n}\}$ is relatively compact in $C(\mathbb{R}, \omega_{x_0})$, which amounts to checking that for arbitrary positive number $l$ the set $M := \{\gamma_{x_n}(t) : t \in [-l, l], \ n \in \mathbb{N}\}$ is precompact and $\{\gamma_{x_n}\}$ is equi-continuous on $[-l, l]$. Let $y_n := h(x_n)$. Since $y_n \to y := h(x)$ as $n \to \infty$, the set $K := \{\sigma(t, y_n) : t \in [-l, l], \ n \in \mathbb{N}\}$ is relatively compact. Since the set $\omega_{x_0}$ is conditionally precompact and $h(\gamma_{x_n}(t)) = \sigma(t, h(x_n)) = \sigma(t, y_n) \in K$ for $t \in [-l, l]$ and $n \in \mathbb{N}$, we have $\gamma_{x_n}(t) \in \omega_{x_0} \cap h^{-1}(K)$. Thus the set $M$ is relatively compact. If $\{\gamma_{x_n}\}$ is not equi-continuous on some compact interval, then there are $\varepsilon_0 \geq 0$, $l_0 > 0$ (without loss of generality, this $l_0$ can be taken the same as in (3.8)), $\delta_n \to 0$ ($\delta_n > 0$) and $t_n^i \in [-l_0, l_0]$ ($i = 1, 2$) such that

$$|t_n^1 - t_n^2| < \delta_n$$

and $\rho(\gamma_{x_n}(t_n^1), \gamma_{x_n}(t_n^2)) \geq \varepsilon_0$ for $n \in \mathbb{N}$. Note that

$$\varepsilon_0 \geq \rho(\gamma_{x_n}(t_n^1, \gamma_{x_n}(t_n^2)) = \rho(\pi(t_n^1 + l_0, \gamma_{x_n}(-l_0)), \pi(t_n^2 + l_0, \gamma_{x_n}(-l_0))).$$

Since the sequences $\{\gamma_{x_n}(-l_0)\}$ and $\{t_n^i\} \subset [-l_0, l_0]$ ($i = 1, 2$) are precompact, we may suppose that they are convergent. Denote $\bar{x} := \lim_{n \to \infty} \gamma_{x_n}(-l_0)$ and $t_0 := \lim_{n \to \infty} t_n^i$ ($i = 1, 2$). Passing to limit in (3.8) as $n \to \infty$ we obtain

$$\varepsilon_0 \leq \rho(\pi(t_0 + l_0, \bar{x}), \pi(t_0 + l_0, \bar{x})) = 0.$$ 

The obtained contradiction proves the equi-continuity of $\{\gamma_{x_n}\}$, and hence the relative compactness of $\{\gamma_{x_n}\}$ in $C(\mathbb{R}, X)$.

Note that every limit point $\gamma$ of the sequence $\{\gamma_{x_n}\}$ belongs to $\Phi$ and satisfies $\gamma(0) = x$. On the other hand, the set $\Phi_x$ consists of the single point $\gamma$ by Step 2, so we have

$$\lim_{n \to \infty} \gamma_{x_n} = \gamma \quad \text{in} \ C(\mathbb{R}, X).$$

In particular, $\gamma_{x_n}(-l_0) \to \gamma(-l_0)$ as $n \to \infty$. Taking limit in (3.8) as $n \to \infty$ we obtain the continuity of mapping $\bar{T}$ in $(t, x)$. The theorem is completely proved. $\square$

**Corollary 3.11.** Let $(\langle X, \mathbb{R}_+, \pi \rangle, \langle Y, \mathbb{R}, \sigma \rangle, h)$ be an NDS with the following properties:

(i) there exist two points $x_0^i \in X$ ($i = 1, 2$) such that the positive semitrajectories $\Sigma^+_x$ ($i = 1, 2$) are conditionally precompact;

(ii) the sets $\omega_{x_0^i}$ ($i = 1, 2$) are positively uniformly stable.

Then all motions on $K := \omega_{x_0^1} \cup \omega_{x_0^2}$ can be extended uniquely to the left and on $K$ is defined a two-sided dynamical system $(K, \mathbb{R}, \pi)$, i.e. the one-sided dynamical system $(X, \mathbb{R}_+, \pi)$ generates on $K$ a two-sided dynamical system $(K, \mathbb{R}, \pi)$.

**Remark 3.12.** Note that Theorem 3.10 is known if $Y$ is a compact minimal set (see, e.g., [26] and references therein) or if each point of $Y$ is Poisson stable (see [9]). In our case $Y$, generally speaking, can be non-compact and non-minimal, and there is no restriction on the element of $Y$. 

4. Poisson stable motions and their comparability

4.1. Classes of Poisson stable motions. Let \((X, \mathbb{R}, \pi)\) be a dynamical system. Let us recall the classes of Poisson stable motions we study in this paper, see \([30, 33, 36, 38]\) for details.

**Definition 4.1.** A point \(x \in X\) is called *stationary* (respectively, *\(\tau\)-periodic*) if \(\pi(t, x) = x\) (respectively, \(\pi(t + \tau, x) = \pi(t, x)\)) for all \(t \in \mathbb{R}\).

**Definition 4.2.** A point \(x \in X\) is called *quasi-periodic* if the associated function \(f(\cdot) := \pi(\cdot, x) : \mathbb{R} \to X\) satisfy the following conditions:

(i) the numbers \(\nu_1, \nu_2, \ldots, \nu_k\) are rationally independent;

(ii) there exists a continuous function \(\Phi : \mathbb{R}^k \to X\) such that \(\Phi(t_1 + 2\pi, t_2 + 2\pi, \ldots, t_k + 2\pi) = \Phi(t_1, t_2, \ldots, t_k)\) for all \((t_1, t_2, \ldots, t_k) \in \mathbb{R}^k\);

(iii) \(f(t) = \Phi(\nu_1 t, \nu_2 t, \ldots, \nu_k t)\) for \(t \in \mathbb{R}\).

**Definition 4.3.** For given \(\varepsilon > 0\), a number \(\tau \in \mathbb{R}\) is called a \(\varepsilon\)-shift of \(x\) (respectively, \(\varepsilon\)-almost period of \(x\)), if \(\rho(\pi(\tau, x), x) < \varepsilon\) (respectively, \(\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon\) for all \(t \in \mathbb{R}\)).

**Definition 4.4.** A point \(x \in X\) is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any \(\varepsilon > 0\) there exists a positive number \(l\) such that any segment of length \(l\) contains a \(\varepsilon\)-shift (respectively, \(\varepsilon\)-almost period) of \(x\).

**Definition 4.5.** If a point \(x \in X\) is almost recurrent and its trajectory \(\Sigma_x\) is precompact, then \(x\) is called *(Birkhoff)* recurrent.

**Definition 4.6.** A point \(x \in X\) is called *Levitan almost periodic* \([\underline{25}]\) (see also \([3, 9, 24]\)), if there exists a dynamical system \((Y, \mathbb{R}, \sigma)\) and a Bohr almost periodic point \(y \in Y\) such that \(\mathcal{N}_y \subseteq \mathcal{N}_x\).

**Definition 4.7.** A point \(x \in X\) is called *almost automorphic* if it is st. \(L\) and Levitan almost periodic.

**Definition 4.8.** A point \(x \in X\) is said to be *uniformly Poisson stable* or *pseudo periodic* in the positive (respectively, negative) direction if for arbitrary \(\varepsilon > 0\) and \(l > 0\) there exists a \(\varepsilon\)-almost period \(\tau > l\) (respectively, \(\tau < -l\)) of \(x\). The point \(x\) is said to be uniformly Poisson stable or pseudo periodic if it is so in both directions.

**Definition 4.9** \([\underline{31}, \underline{32}]\). A point \(x \in X\) is said to be *pseudo recurrent* if for any \(\varepsilon > 0\), \(p \in \Sigma_x\) and \(t_0 \in \mathbb{R}\) there exists \(L = L(\varepsilon, t_0) > 0\) such that

\[ B(p, \varepsilon) \bigcap \pi([t_0, t_0 + L], p) \neq \emptyset, \]

where \(B(p, \varepsilon) := \{x \in X : \rho(p, x) < \varepsilon\}\) and \(\pi([t_0, t_0 + L], p) := \{\pi(t, p) : t \in [t_0, t_0 + L]\}\).

**Definition 4.10.** A point \(x \in X\) is said to be *strongly Poisson stable* if \(p \in \omega_p\) for any \(p \in H(x)\).

**Remark 4.11.** It is known that:

(i) a strongly Poisson stable point is Poisson stable, but the converse is not true in general;

(ii) all the motions introduced above (Definitions 4.1–4.9) are strongly Poisson stable.
Definition 4.12 ([15][14]). A point \( x \in X \) is said to be asymptotically \( P \) if there exists a \( P \) point \( p \in X \) such that

\[
\lim_{t \to +\infty} \rho(\pi(t,x),\pi(t,p)) = 0.
\]

Here the property \( P \) can be stationary, \( \tau \)-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Levitan almost periodic, almost recurrent, pseudo periodic, pseudo recurrent, Poisson stable.

4.2. Comparability of motions by their character of recurrence.

4.2.1. Shcherbakov’s comparability principle of motions by their character of recurrence. In this subsection we present some notions and results stated and proved by B. A. Shcherbakov [33]–[36].

Let \((X,\mathbb{R},\pi)\) and \((Y,\mathbb{R},\sigma)\) be two dynamical systems.

Definition 4.13. A point \( x \in X \) is said to be comparable with \( y \in Y \) by character of recurrence if for any \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that every \( \delta \)-shift of \( y \) is a \( \varepsilon \)-shift for \( x \), i.e. \( \rho(\sigma(t,y),y) < \delta \) implies \( \rho(\pi(t,x),x) < \varepsilon \).

Theorem 4.14. The following conditions are equivalent:

(i) the point \( x \) is comparable with \( y \) by character of recurrence;
(ii) \( \mathcal{N}_y \subseteq \mathcal{N}_x \);
(iii) \( \mathcal{N}_y^\infty \subseteq \mathcal{N}_x^\infty \);
(iv) from any sequence \( \{t_n\} \subseteq \mathcal{N}_y \) we can extract a subsequence \( \{t_{n_k}\} \subseteq \mathcal{N}_x \);
(v) from any sequence \( \{t_n\} \subseteq \mathcal{N}_y^\infty \) we can extract a subsequence \( \{t_{n_k}\} \subseteq \mathcal{N}_x^\infty \).

Theorem 4.15. Let \( x \in X \) be comparable with \( y \in Y \). If the point \( y \) is stationary (respectively, \( \tau \)-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is the point \( x \).

Definition 4.16. A point \( x \in X \) is called uniformly comparable with \( y \in Y \) by character of recurrence if for any \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that every \( \delta \)-shift of \( \sigma(t,y) \) is a \( \varepsilon \)-shift of \( \pi(t,x) \) for all \( t \in \mathbb{R} \), i.e. \( \rho(\sigma(t+\tau,y),\sigma(t,y)) < \delta \) implies \( \rho(\pi(t+\tau,x),x) < \varepsilon \) for all \( t \in \mathbb{R} \) (or equivalently: \( \rho(\sigma(t_1,y),\sigma(t_2,y)) < \delta \) implies \( \rho(\pi(t_1,x),\pi(t_2,x)) < \varepsilon \) for all \( t_1, t_2 \in \mathbb{R} \)).

Denote \( \mathcal{M}_x := \{\{t_n\} \subseteq \mathbb{R} : \{\pi(t_n,x)\} \text{ converges}\} \), \( \mathcal{M}_x^\infty := \{\{t_n\} \in \mathcal{M}_x : t_n \to +\infty \text{ as } n \to \infty\} \) and \( \mathcal{M}_x^\infty := \{\{t_n\} \in \mathcal{M}_x : t_n \to \infty \text{ as } n \to \infty\} \).

Definition 4.17 ([7][10]). A point \( x \in X \) is said to be strongly comparable with \( y \in Y \) by character of recurrence if \( \mathcal{M}_y \subseteq \mathcal{M}_x \).

Theorem 4.18. (i) If \( \mathcal{M}_y \subseteq \mathcal{M}_x \), then \( \mathcal{N}_y \subseteq \mathcal{N}_x \), i.e. strong comparability implies comparability.

(ii) Let \( X \) be a complete metric space. If the point \( x \) is uniformly comparable with \( y \) by character of recurrence, then \( \mathcal{M}_y \subseteq \mathcal{M}_x \), i.e. uniform comparability implies strong comparability.

Theorem 4.19. Let \( y \) be Lagrange stable. Then \( \mathcal{M}_y \subseteq \mathcal{M}_x \) holds if and only if the point \( x \) is Lagrange stable and uniformly comparable with \( y \) by character of recurrence.

Theorem 4.20. Let \( X \) and \( Y \) be two complete metric spaces. Let the point \( x \in X \) be uniformly comparable with \( y \in Y \) by character of recurrence. If \( y \) is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, pseudo periodic, pseudo recurrent), then so is \( x \).
4.2.2. Some generalization of Shcherbakov’s results. In this subsection we present some generalization of Shcherbakov’s results concerning the (uniform) comparability of points by character of recurrence (see [6] or [14, Ch1] for more details).

Let $T_1 \subseteq T_2$ be two sub-semigroups of group $\mathbb{R}$ ($T_i = \mathbb{R}$ or $\mathbb{R}_+$ for $i = 1, 2$). Consider two dynamical systems $(X, T_1, \pi)$ and $(Y, T_2, \sigma)$.

**Theorem 4.21.** Let $y \in Y$ be positively Poisson stable. Then the following conditions are equivalent:

a. $\mathcal{M}_y \subseteq \mathcal{M}_x$;

b. $\mathcal{M}_y^\infty \subseteq \mathcal{M}_x^\infty$ and $\mathcal{N}_y^\infty \subseteq \mathcal{N}_x^\infty$;

c. there exists a continuous mapping $h : \omega_y \to \omega_x$ with the properties:

(i) $h(y) = x$;

(ii) $h(\sigma(t, q)) = \pi(t, h(q))$ for all $t \in T_1$ and $q \in \omega_y$.

**Theorem 4.22.** Let $y \in \omega_y$, then the following conditions are equivalent:

a. $\mathcal{N}_y^\infty \subseteq \mathcal{N}_x^\infty$;

b. $\mathcal{N}_y^{+\infty} \subseteq \mathcal{N}_x^{+\infty}$.

**Theorem 4.23.** Let $y \in \omega_y$, then the following conditions are equivalent:

a. $\mathcal{M}_y^\infty \subseteq \mathcal{M}_x^\infty$ and $\mathcal{N}_y^\infty \subseteq \mathcal{N}_x^\infty$;

b. $\mathcal{M}_y^{+\infty} \subseteq \mathcal{M}_x^{+\infty}$ and $\mathcal{N}_y^{+\infty} \subseteq \mathcal{N}_x^{+\infty}$.

5. Monotone NDS: existence of and convergence to Poisson stable motions

Assume that $E$ is an ordered space. A subset $U$ of $E$ is called lower-bounded (respectively, upper-bounded) if there exists an element $a \in E$ such that $a \leq U$ (respectively, $a \geq U$). Such an $a$ is said to be a lower bound (respectively, upper bound) for $U$. A lower bound $\alpha$ is said to be the greatest lower bound (g.l.b.) or infimum, if any other lower bound $a$ satisfies $a \leq \alpha$. Similarly, we can define the least upper bound (l.u.b.) or supremum.

**Definition 5.1.** Recall [21] that a bundle is a triplet $(X, h, Y)$, where $X, Y$ are topological spaces and $h : X \to Y$ is a continuous surjective mapping. The space $Y$ is called the base space, the space $X$ is called the total space, and the map $h$ is called the projection of bundle. For each $y \in Y$, the space $X_y := h^{-1}(y)$ is called the fiber of bundle over $y \in Y$.

**Example 5.2.** Let $X := W \times Y$. A triplet $(X, h, Y)$, where $h := pr_2$ is the projection on the second factor, is a bundle which is called the product bundle over $Y$ with fiber $W$.

A bundle $(X, h, Y)$ is said to be ordered if each fiber $X_y$ is ordered. Note that only points on the same fiber may be order related: if $x_1 \leq x_2$ or $x_1 < x_2$, then it implies $h(x_1) = h(x_2)$. We assume that the order relation and the topology on $X$ are compatible in the sense that $x \leq \tilde{x}$ if $x_n \leq \tilde{x}_n$ for all $n$ and $x_n \to x, \tilde{x}_n \to \tilde{x}$ as $n \to \infty$. 
Definition 5.3. For given bundle \((X, h, Y)\), an NDS \(\langle (X, \mathbb{R}^+, \pi), (Y, \mathbb{R}, \sigma), h \rangle\) defined on it is said to be monotone (respectively, strictly monotone) if \(x_1 \leq x_2\) (respectively, \(x_1 < x_2\)) implies \(\pi(t, x_1) \leq \pi(t, x_2)\) (respectively, \(\pi(t, x_1) < \pi(t, x_2)\)) for any \(t > 0\).

For given NDS \(\langle (X, \mathbb{R}^+, \pi), (Y, \mathbb{R}, \sigma), h \rangle\), let \(S \subseteq X\) be a nonempty ordered subset possessing the following properties:

(i) \(h(S) = Y\);
(ii) \(S\) is positively invariant with respect to \(\pi\), i.e. \(\langle (S, \mathbb{R}^+, \pi), (Y, \mathbb{R}, \sigma), h \rangle\) is an NDS.

Below we will use the following assumptions:

(C1) For every conditionally compact subset \(K\) of \(S\) and \(y \in Y\) the set \(K_y := h^{-1}(y) \cap K\) has both infimum \(\alpha_y(K)\) and supremum \(\beta_y(K)\).

(C2) For every \(x \in S\), the semi-trajectory \(\Sigma^+_x\) is conditionally precompact and its \(\omega\)-limit set \(\omega_x\) is positively uniformly stable.

(C3) The NDS
\[
\langle (S, \mathbb{R}^+, \pi), (Y, \mathbb{R}, \sigma), h \rangle
\]
is monotone.

(C4) Under condition (C1), both \(\alpha_y(K)\) and \(\beta_y(K)\) belong to \(K_y\) for any \(y \in Y\).

Remark 5.4. Note that condition (C4) holds if fibers of the bundle \((S, h, Y)\) are one-dimensional, i.e. \(S_y = h^{-1}(y) \cap S \subseteq \mathbb{R} \times \{y\}\) or \(S_y\) is homeomorphic to a subset of \(\mathbb{R} \times \{y\}\) for any \(y \in Y\).

5.1. (Uniform) comparability and existence of Poisson stable motions. Firstly, we state a simple result for two points to be asymptotic which will be frequently used below.

Lemma 5.5. Suppose that the following conditions are fulfilled:

(i) the points \(x, x_0 \in S\) with \(h(x) = h(x_0)\) are proximal, i.e. there is a sequence \(t_n \to +\infty\) as \(n \to \infty\) such that
\[
\lim_{n \to \infty} \rho(\pi(t_n, x), \pi(t_n, x_0)) = 0;
\]
(ii) the set \(\Sigma^+_{x_0} \in S\) is positively uniformly stable.

Then the points \(x, x_0\) are asymptotic, i.e. \(\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, x_0)) = 0\).

Proof. Let \(\Sigma^+_{x_0}\) be positively uniformly stable, \(\varepsilon > 0\) and \(\delta = \delta(x_0, \varepsilon) > 0\) the positive number figuring in the definition of uniform stability. By (B.2) there exists a number \(n_0 \in \mathbb{N}\) such that \(\rho(\pi(t_n, x), \pi(t_n, x_0)) < \delta\) for any \(n \geq n_0\). According to the choice of the number \(\delta\) we obtain \(\rho(\pi(t, x), \pi(t, x_0)) < \varepsilon\) for any \(t \geq t_{n_0}\). The lemma is proved.

Lemma 5.6. Assume that (C1)–(C3) hold. For given \(x_0 \in S\), let \(K := \omega_{x_0}\) and \(y_0 := h(x_0)\). Then:

(i) if \(q \in \omega_q \subseteq \omega_{y_0}, \alpha_q := \alpha_q(K), K^1_q := \omega_{\alpha_q}\) (respectively, \(\omega_{\beta_q} \cap h^{-1}(q)\)) consists of a single point \(\gamma_q\) (respectively, \(\delta_q\)), i.e. \(K^1_q = \{\gamma_q\}\) (respectively, \(\omega_{\beta_q} \cap h^{-1}(q) = \{\delta_q\}\));
(ii) let \(\gamma_q\) and \(\delta_q\) be as in (i), then we have
\[
\gamma_q \leq \alpha_q \leq \beta_q \leq \delta_q.
\]
Proof. Let $q$ be a point from $\omega_{\beta_q}$ with $q \in \omega_q$. We only consider the case of $\alpha_q$ because the proof for $\beta_q$ is similar.

(i) It follows from the definition of $\alpha_q$ that

$$\alpha_q \leq x \quad \text{for any } x \in K_q = K \bigcap h^{-1}(q).$$

Since $\pi(t, K_q) = K_{\sigma(t,q)}$ by Theorem 3.3 we have

$$\alpha_{\sigma(t,q)} \leq \pi(t, x) \quad \text{for any } x \in K_q \text{ and } t \geq 0.$$ (5.3)

We now prove that

$$\pi(t, \alpha_q) \leq \alpha_{\sigma(t,q)} \leq \pi(t, x) \quad \text{for any } x \in K_q \text{ and } t \geq 0.$$ (5.4)

Since $K$ is invariant we have

$$\gamma(-t) \in K \quad \text{for any } \gamma \in \Phi_x := \{ \gamma \in \Phi : \gamma(\mathbb{R}) \subseteq K \}, \quad x \in K_{\sigma(t,q)} \text{ and } t \geq 0.$$ Note that $h(\gamma(-t)) = q$, consequently $\alpha_q \leq \gamma(-t)$. Since the NDS (5.1) is monotone, we obtain

$$\pi(t, \alpha_q) \leq \pi(t, \gamma(-t)) = \gamma(0) = x \in K_{\sigma(t,q)}.$$ (5.5)

This implies that $\pi(t, \alpha_q) \leq \alpha_{\sigma(t,q)}$ for any $t \geq 0$ because $x \in K_{\sigma(t,q)}$ is arbitrary.

Let $x_1 \in K_q^1$, then there is a sequence $t_n \to +\infty$ such that

$$\pi(t_n, \alpha_q) \to x_1 \quad \text{and} \quad \sigma(t_n, q) \to q$$
as $n \to +\infty$. By (5.4), we have

$$\pi(t_n, \alpha_q) \leq \alpha_{\sigma(t_n,q)}.$$ (5.6)

Denote by $\tilde{K} := K \bigcup K^1$. By Theorem 3.3 both $K$ and $K^1$ are conditionally compact, and hence $\tilde{K}$ is conditionally compact. So without loss of generality we suppose that the sequence $\pi(t_n, \cdot)_{\mid_{\tilde{K}}}$ is convergent and denote by $\xi$ its limit; note that $\xi \in E^+_q$ (with $\pi$ being restricted on $\tilde{K}$ in the definition of $E^+_q$). By Theorem 3.10 $\pi$ can be extended to a two-sided dynamical system on $\tilde{K}$, and by the proof of Theorem 3.10 the required negative separation property (2.3) in Corollary 2.17 also holds. Then it follows from Corollary 2.17 that $E^+_q$ is a group, so we have $\xi(\tilde{K}_q) = \tilde{K}_q$, $\xi(K_q) = K_q$ and $\xi(K^1_q) = K^1_q$. Thus, for any point $x_2 \in K_q$ and $\xi \in E^+_q$ there exists a (unique) point $\tilde{x}_2 \in K_q$ such that $\xi(\tilde{x}_2) = x_2$. We have

$$\sigma(t_n, q) \to q \quad \text{and} \quad x_2 = \lim_{n \to +\infty} \pi(t_n, \tilde{x}_2).$$ (5.7)

Combining (5.6) and (5.7), we conclude that

$$\pi(t_n, \alpha_q) \leq \alpha_{\sigma(t_n,q)} \leq \pi(t_n, \tilde{x}_2).$$ (5.8)

Letting $n \to +\infty$ in (5.7), we get by (5.6)

$$x_1 \leq x_2$$

for any $x_2 \in K_q$, and hence

$$x \leq \alpha_q$$

for all $x \in K^1_q$.

Finally we will show under condition (2.3) that the set $K^1_q$ consists of a single point. In fact, if $x', x'' \in K^1_q$ and (5.8) holds, then reasoning as above we can choose a sequence $t_n \to +\infty$ as $n \to +\infty$ and $\tilde{x}_2 \in K_q$ such that $\sigma(t_n, q) \to q$, $\pi(t_n, \alpha_q) \to x'$ and
and \( \pi(t_n, x'') \to x'' \). Since \( x'' \leq \alpha_q \), we have \( \pi(t_n, x'') \leq \pi(t_n, \alpha_q) \). Consequently, \( x'' \leq x' \). Since \( x', x'' \in K^1_\omega \) are arbitrary, we have \( x' = x'' \), i.e. \( K^1_\omega \) consists of a single point \( \gamma_q \).

(ii) The fact \( \gamma_q \leq \alpha_q \) follows from (5.8) and the fact \( K^1_\omega = \{ \gamma_q \} \).

The proof is complete. \( \square \)

**Corollary 5.7.** Assume that the conditions of Lemma 5.6 hold. Then \( \gamma_q \) satisfies 
\( \pi(t, \gamma_q) = \pi_{\sigma(t, q)} \) for \( t \in \mathbb{R} \), i.e. the mapping \( t \mapsto \gamma_{\sigma(t, q)} \) is an entire trajectory of the dynamical system \( (X, \mathbb{R}, \pi) \) passing through the point \( \gamma_q \) at \( t = 0 \). The same result holds for \( \delta_q \).

**Proof.** It follows from Theorem 5.5 that \( \omega_{\alpha_q} \) is an invariant set, and by Theorem 5.10 the one-sided dynamical system \( (X, \mathbb{R}, \pi) \) generates on \( \omega_{\alpha_q} \) a two-sided dynamical system \( (\omega_{\alpha_q}, \mathbb{R}, \pi) \). On the other hand, Lemma 5.6 yields that \( \omega_{\alpha_q}(\alpha_q) = \{ \gamma_q \} \), which enforces that the required result holds. \( \square \)

**Theorem 5.8 (Comparability).** Assume that (C1)–(C3) hold. For given \( x_0 \in \mathcal{S} \), let \( y_0 := h(x_0) \). If \( y_0 \in \omega_{y_0} \), then the point \( \gamma_{y_0} \) (respectively, \( \delta_{y_0} \)) is comparable with \( y_0 \) by character of recurrence and

\[
\lim_{t \to +\infty} \rho(\pi(t, \alpha_{y_0}), \pi(t, \gamma_{y_0})) = 0
\]

(respectively, \( \lim_{t \to +\infty} \rho(\pi(t, \beta_{y_0}), \pi(t, \delta_{y_0})) = 0 \)).

**Proof.** We will only prove the result for \( \gamma_{y_0} \) because the proof for \( \delta_{y_0} \) is similar.

Let \( \{t_n\} \in \mathcal{S}^+_{\gamma_{y_0}} \), then \( \sigma(t_n, y_0) \to y_0 \) and \( t_n \to +\infty \) as \( n \to \infty \). By condition (C2) the set \( \Sigma_{\gamma_{y_0}}^+ \) is conditionally precompact, then the sequence \( \{\pi(t_n, \gamma_{y_0})\} \) is precompact. Let \( z \) be a limit point of the sequence \( \{\pi(t_n, \gamma_{y_0})\} \), then there is a subsequence \( \{t'_n\} \subseteq \{t_n\} \) such that \( \pi(t'_n, \gamma_{y_0}) \to z \) as \( n \to \infty \). On the other hand, \( \sigma(t_n, y_0) \to y_0 \) as \( n \to \infty \), so \( z \in \omega_{\gamma_{y_0}} \cap h^{-1}(y_0) \subseteq \omega_{\alpha_{y_0}} \cap h^{-1}(y_0) = \{ \gamma_{y_0} \} \) by Lemma 5.6 i.e. \( z = \gamma_{y_0} \). Since \( \{\pi(t_n, \gamma_{y_0})\} \) is precompact and \( \gamma_{y_0} \) is its unique limit point, we have \( \pi(t_n, \gamma_{y_0}) \to \gamma_{y_0} \) as \( n \to \infty \). That is, \( \{t_n\} \in \mathcal{S}^+_{\gamma_{y_0}} \) and hence \( \mathcal{S}^+_{\gamma_{y_0}} \subseteq \mathcal{S}^+_{\gamma_{y_0}} \).

The first statement then follows from Theorems 4.22 and 4.31.

Since \( h(\alpha_{y_0}) = y_0 \in \omega_{y_0} \), there exists a sequence \( t_n \to +\infty \) as \( n \to \infty \) such that

\[
\sigma(t_n, y_0) \to y_0 \quad \text{as} \quad n \to \infty .
\]

Taking into consideration that \( \Sigma_{\alpha_{y_0}}^+ \) is conditionally precompact, without loss of generality we can suppose that the sequence \( \{\pi(t_n, \alpha_{y_0})\} \) converges. Denote by \( \bar{x} := \lim_{n \to \infty} \pi(t_n, \alpha_{y_0}) \), then \( h(\bar{x}) = y_0 \) and \( \bar{x} \in \omega_{\alpha_{y_0}} \), i.e. \( \bar{x} \in \omega_{\alpha_{y_0}} \cap h^{-1}(y_0) \). By Lemma 5.6 we have \( \omega_{\alpha_{y_0}} \cap h^{-1}(y_0) = \{ \gamma_{y_0} \} \), so \( \bar{x} = \gamma_{y_0} \) and consequently

\[
\pi(t_n, \alpha_{y_0}) \to \gamma_{y_0} \quad \text{as} \quad n \to \infty .
\]

On the other hand, by the first statement of the theorem and (5.10) we obtain

\[
\pi(t_n, \gamma_{y_0}) \to \gamma_{y_0} \quad \text{as} \quad n \to \infty .
\]

From (5.11) and (5.12) we get

\[
\lim_{n \to \infty} \rho(\pi(t_n, \alpha_{y_0}), \pi(t_n, \gamma_{y_0})) = 0 .
\]

Now to finish the proof of equality (5.9) it is sufficient to apply Lemma 5.5 The proof is complete. \( \square \)
By Theorems 5.8 and 4.13 we have the following result:

**Corollary 5.9.** Under the conditions (C1)–(C3) if the point \( y_0 \) is stationary (respectively, \( \tau \)-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then:

(i) the point \( \gamma_{y_0} \) has the same recurrent property as \( y_0 \);
(ii) the point \( \alpha_{y_0} \) is asymptotically stationary (respectively, asymptotically \( \tau \)-periodic, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically Poisson stable).

To get the existence of more classes of Poisson stable motions, we need to establish uniform comparability (cf. Theorem 4.20) and this reduces to verifying strong comparability when the base space is compact (cf. Theorem 4.19). This is what we are doing in the following

**Theorem 5.10 (Strong comparability).** Assume that (C1)–(C3) hold, \( x_0 \in S \) and \( y_0 := h(x_0) \in Y \) is strongly Poisson stable. Then the point \( \gamma_{y_0} \) (respectively, \( \delta_{y_0} \)) is strongly comparable with \( y_0 \) by character of recurrence and

\[
\lim_{t \to +\infty} \rho(\pi(t, \alpha_{y_0}), \pi(t, \gamma_{y_0})) = 0
\]

(respectively, \( \lim_{t \to +\infty} \rho(\pi(t, \beta_{y_0}), \pi(t, \delta_{y_0})) = 0 \)).

**Proof.** We only consider the case of \( \gamma_{y_0} \) because the proof for \( \delta_{y_0} \) is similar.

Let \( q \in H(y_0) \) be an arbitrary point. Then \( q \in \omega_q \) and by Lemma 5.6 we have

\[
\omega_q(\alpha_{y_0}) = \omega(\alpha_{y_0}) \cap h^{-1}(q) = \{\gamma_q\}.
\]

Now we will show that \( \mathcal{M}_t^{+\infty} \subseteq \mathcal{M}_{\gamma_{y_0}}^{+\infty} \). Let \( \{t_n\} \in \mathcal{M}_{\gamma_{y_0}}^{+\infty} \), then there exists \( q \in \omega_{y_0} \) such that \( \sigma(t_n, y_0) \to q \) and \( t_n \to +\infty \) as \( n \to \infty \). Since the set \( \omega_{\gamma_{y_0}} \) is conditionally compact, the sequence \( \{\pi(t_n, \gamma_{y_0})\} \) is compact. Let \( z \) be a limit point of the sequence \( \{\pi(t_n, \gamma_{y_0})\} \), then there is a subsequence \( \{t_n'\} \subseteq \{t_n\} \) such that \( \pi(t_n', \gamma_{y_0}) \to z \) as \( n \to \infty \). On the other hand \( \sigma(t_n', y_0) \to q \) as \( n \to \infty \), so \( z \in \omega_{\gamma_{y_0}} \cap h^{-1}(q) \subseteq \omega_{\alpha_{y_0}} \cap h^{-1}(q) = \{\gamma_q\} \), i.e. \( z = \gamma_q \). Since \( \{\pi(t_n, \gamma_{y_0})\} \) is relatively compact and \( \gamma_q \) is its unique limit point, we have \( \pi(t_n, \gamma_{y_0}) \to \gamma_q \) as \( n \to \infty \), i.e. \( \{t_n\} \in \mathcal{M}_{\gamma_{y_0}}^{+\infty} \). The first statement then follows from Theorems 4.23 and 4.21.

To prove (5.13) it is sufficient to apply Theorem 5.8 (the second statement). The proof is complete. \( \square \)

By Theorems 5.10, 4.19 and 4.20 we have

**Corollary 5.11.** Under the conditions (C1)–(C3) if the point \( y_0 \) is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent), then:

(i) the point \( \gamma_{y_0} \) (respectively, \( \delta_{y_0} \)) has the same recurrent property as \( y_0 \);
(ii) the point \( \alpha_{y_0} \) (respectively, \( \beta_{y_0} \)) is asymptotically quasi-periodic (respectively, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically Birkhoff recurrent).

If in addition \( y_0 \) is Lagrange stable, then the above items (i) and (ii) also hold for pseudo periodic and pseudo recurrent case.

The following result has its independent interest, so we formulate it here in spite that it will not be used in what follows.
Proposition 5.12. Assume that the hypotheses of Lemma 5.7 hold. Then for any $x$ satisfying $\gamma_q \leq x < \omega_q(x_0)$, we have
\[
\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, \gamma_q)) = 0.
\]
Similarly, for any $x$ satisfying $\omega_q(x_0) < x \leq \delta_q$, we have
\[
\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, \delta_q)) = 0.
\]

Proof. We only need to prove the result for $\gamma_q \leq x < \omega_q(x_0)$. Take $\{t_n\} \in \mathcal{N}^+_{\gamma_q}$. Since the set $\{\pi(t_n, \alpha_q) : n \in \mathbb{N}\}$ is conditionally compact by the condition (C2) and $\lim_{n \to \infty} \sigma(t_n, q) = q$, the set $\{\pi(t_n, \alpha_q) : n \in \mathbb{N}\}$ is precompact. But it follows from Lemma 5.6 that $\omega_q(\alpha_q) = \{\gamma_q\}$, so
\[
\lim_{n \to \infty} \rho(\pi(t_n, \alpha_q), \gamma_q) = 0.
\]
On the other hand, since $\mathcal{N}^+_{\gamma_q} \subseteq \mathcal{M}_{\gamma_q}$ by Theorem 5.8 we get
\[
\lim_{n \to \infty} \rho(\pi(t_n, \gamma_q), \gamma_q) = 0.
\]
Since $\gamma_q \leq x < \omega_q(x_0)$, by the monotonicity it follows that
\[
\pi(t_n, \gamma_q) \leq \pi(t_n, x) \leq \pi(t_n, \alpha_q), \quad \text{for } n \in \mathbb{N}.
\]
Letting $n \to \infty$, we obtain from (5.14) and (5.15) that
\[
\lim_{n \to \infty} \rho(\pi(t_n, x), \gamma_q) = 0.
\]
This together with (5.13) yields that
\[
\lim_{n \to \infty} \rho(\pi(t_n, x), \pi(t_n, \gamma_q)) = 0.
\]
Since $\omega(\alpha_q)$ is uniformly stable by (C2) and $\Sigma^+_{\gamma_q} \subseteq \omega(\alpha_q)$, the result now follows from Lemma 5.5. The proof is complete. \qed

5.2. Convergence to Poisson stable motions. In this subsection, we give some sufficient conditions which imply the convergence of all motions to Poisson stable ones. This kind of convergence is fundamental in classical monotone dynamics (see, e.g. [22, 40]).

Theorem 5.13. Assume that (C1)–(C4) hold. For given $x_0 \in \mathcal{S}$, let $y_0 := h(x_0)$. If $y_0 \in \omega_{y_0}$, then the following statements hold:
(i) \( \gamma_{y_0} \in \omega_{x_0} \);
(ii) the point $\gamma_{y_0}$ is comparable with $y_0$ by character of recurrence and
(iii)
\[
\lim_{t \to +\infty} \rho(\pi(t, x_0), \pi(t, \gamma_{y_0})) = 0.
\]
The same result holds for $\delta_{y_0}$, i.e. items (i)–(iii) hold with $\gamma_{y_0}$ replaced by $\delta_{y_0}$.

Proof. We only need to prove the result for $\gamma_{y_0}$. Under conditions (C1)–(C4) we have $\alpha_{y_0} \in \omega_{x_0} \cap h^{-1}(y_0)$, and by Lemma 5.6 we have $\{\gamma_{y_0}\} = \omega_{\alpha_{y_0}} \cap h^{-1}(y_0) \subseteq \omega_{x_0} \cap h^{-1}(y_0)$. This means that there exists a sequence $t_n \to +\infty$ as $n \to \infty$ such that $\sigma(t_n, y_0) \to y_0$ and $\pi(t_n, x_0) \to \gamma_{y_0}$. Since $\mathcal{N}_{y_0} \subseteq \mathcal{M}_{y_0}$ by Theorem 5.8 we have
\[
\rho(\pi(t_n, x_0), \pi(t_n, \gamma_{y_0})) \leq \rho(\pi(t_n, x_0), y_0) + \rho(y_0, \pi(t_n, \gamma_{y_0})) \to 0
\]
as $n \to \infty$. Now to finish the proof it is sufficient to apply Lemma 5.5. \qed
Corollary 5.14. Assume that the conditions of Theorem 5.13 hold, then \( \omega_{y_0}(x_0) \) is a singleton set and hence we have

\[
\omega_{y_0}(x_0) = \{\alpha_{y_0}\} = \{\beta_{y_0}\} = \{\gamma_{y_0}\} = \{\delta_{y_0}\}.
\]

Proof. Let \( \{t_n\} \in \mathcal{N}_y^{+\infty} \). Then \( \{t_n\} \in \mathcal{N}_y^{+\infty} \cap \mathcal{N}_{y_0}^{+\infty} \) since \( \gamma_{y_0} \) and \( \delta_{y_0} \) are comparable with \( y_0 \) by character of recurrence. So we have \( (\pi(t_n, \gamma_{y_0}), \pi(t_n, \delta_{y_0})) \rightarrow (\gamma_{y_0}, \delta_{y_0}) \) as \( n \rightarrow \infty \). On the other hand, it follows from (5.16) for \( \gamma_{y_0} \) and \( \delta_{y_0} \) that

\[
\lim_{t \rightarrow +\infty} \rho(\pi(t, \gamma_{y_0}), \pi(t, \delta_{y_0})) = 0.
\]

This enforces that \( \gamma_{y_0} = \delta_{y_0} \).

Recall that

\[
\alpha_{y_0} \leq x \leq \beta_{y_0} \quad \text{for any } x \in \omega_{y_0}(x_0)
\]

and \( \{\gamma_{y_0}\} = \{\omega_{y_0}(\alpha_{y_0}), \omega_{y_0}(\beta_{y_0})\} \) by Lemma 5.6. On the other hand \( \gamma_{y_0}, \delta_{y_0} \in \omega_{y_0}(x_0) \) by Theorem 5.13 so we have

\[
\gamma_{y_0} \leq x \leq \delta_{y_0} \quad \text{for any } x \in \omega_{y_0}(x_0)
\]

by the monotonicity of the NDS and the invariance of \( \omega(x_0) \). Thus it follows that \( \omega_{y_0}(x_0) = \{\gamma_{y_0}\} = \{\delta_{y_0}\} \). The proof is complete. \( \square \)

Corollary 5.15. Under the conditions (C1)--(C4) if the point \( y_0 \) is stationary (respectively, \( \tau \)-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then:

(i) the point \( \gamma_{y_0} \) (respectively, \( \delta_{y_0} \)) has the same recurrent property as \( y_0 \);

(ii) the point \( x_0 \) is asymptotically stationary (respectively, asymptotically \( \tau \)-periodic, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically Poisson stable).

By Theorems 5.13, 5.10 and Corollary 5.11 we have

Theorem 5.16. Assume that (C1)--(C4) hold, \( x_0 \in \mathcal{S} \) and \( y_0 := h(x_0) \in Y \) is strongly Poisson stable. Then the following statements hold:

(i) \( \gamma_{y_0} \in \omega(x_0) \);

(ii) the point \( \gamma_{y_0} \) is strongly comparable with \( y_0 \) by character of recurrence;

(iii) we have

\[
\lim_{t \rightarrow +\infty} \rho(\pi(t, x_0), \pi(t, \gamma_{y_0})) = 0;
\]

(iv) the point \( \gamma_{y_0} \) has the same recurrent property mentioned in Corollary 5.14 as \( y_0 \) and the point \( x_0 \) has the same asymptotically recurrent property as \( \alpha_{y_0} \).

The same result holds for \( \delta_{y_0} \), i.e. items (i)--(iv) hold with \( \gamma_{y_0} \) replaced by \( \delta_{y_0} \).

Remark 5.17. When the point \( y_0 \) is Bohr almost periodic, then the results of Corollary 5.14 and Theorem 5.16 coincide with that of J. Jiang and X.-Q. Zhao [23, Theorem 4.1], i.e. \( \omega(x_0) \) is isomorphic to \( \omega(y_0) \) and \( x_0 \) is asymptotically almost periodic.

In Theorems 5.13 and 5.16 we get that the solutions will converge to the Poisson stable ones by mainly monotone and uniformly stable conditions. Now we give
another criterion, adapted from W. Shen and Y. Yi [37], for the convergence by
Lyapunov functions. To this end, let us denote
\[ \hat{S} := \{(x_1, x_2) : x_1, x_2 \in S \text{ and } h(x_1) = h(x_2)\} \]
and introduce

**Definition 5.18.** A continuous function \( L : \hat{S} \to \mathbb{R}_+ \) is called a Lyapunov function if it satisfies the following two conditions:

(i) \( L(x_1, x_2) = 0 \) if and only if \( x_1 = x_2 \);

(ii) \( L(\pi(t, x_1), \pi(t, x_2)) < L(x_1, x_2) \) for \( x_1 \neq x_2 \) and \( t > 0 \).

The NDS (5.1) is said to be contracting if it admits a Lyapunov function.

**Theorem 5.19** (Global attracting property). Assume (C2) and that the NDS (5.1) is contracting. For given \( x_0 \in S \), if \( y_0 := h(x_0) \) is Poisson stable, then the following statements hold for any \( q \) satisfying \( q \in \omega(q) \subset \omega(y_0) \):

(i) \( \omega_q(x_0) = \{\gamma_q^0\} \), a singleton set;

(ii) for any \( x \in S \) with \( h(x) = q \), we have

\[ \lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, \gamma_q^0)) = 0. \]

**Proof.** (i) Firstly it follows from Theorem 5.16 that \( \omega(x_0) \) is a nonempty conditionally compact invariant set, and by Theorem 3.10 the NDS generates on \( \omega(x_0) \) a two-sided dynamical system. Assume that \( x_1, x_2 \in \omega_q(x_0) \), then by the uniform stability of \( \omega(x_0) \) and the proof of Theorem 3.10 we know that the trajectories on \( \omega(x_0) \) is negatively distal, i.e. \( \inf_{t \in \mathbb{R}} \rho(\pi(t, x_1), \pi(t, x_2)) > 0 \). By Lemma 2.16 and Corollary 2.17 \( E^+_{y_0} \) is a group, so there exists \( \{t_n\} \in \mathbb{N}^+\) such that \( \pi(t_n, x) \to e \) with \( e \) being the identity element of \( E^+_{y_0} \). In particular,

\[ \pi(t_n, x_i) \to x_i \quad \text{as } n \to \infty, \ i = 1, 2. \]

Then it follows that

\[ L(x_1, x_2) = \lim_{n \to \infty} L(\pi(t_n, x_1), \pi(t_n, x_2)) < L(\pi(t_n_0, x_1), \pi(t_n_0, x_2)) < L(x_1, x_2) \]

for some \( n_0 \in \mathbb{N} \), a contradiction. Therefore, \( \omega_q(x_0) \) is a singleton set.

(ii) Note that, like \( \omega(x_0) , \omega(x) \neq \emptyset \) for any \( x \in S \) with \( h(x) = q \in \omega(q) \subset \omega(y_0) \). We claim that \( \omega_q(x) = \omega_q(x_0) \) for all \( q \). Indeed, if not, then similar to (i) \( \omega_q(x) = \{\gamma_q\} \) is a singleton set with \( \gamma_q \neq \gamma_q^0 \). Let \( E = \omega(x_0) \bigcup \omega(x) \), then \( E \) is conditionally compact and uniformly stable. By the same proof in (i) we can conclude that \( E_q \) is a singleton set, i.e. \( \gamma_q = \gamma_q^0 \), a contradiction.

If

\[ \lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, \gamma_q^0)) \neq 0, \]

then this enforces that \( \gamma_q = \gamma_q^0 \) for all \( q \in \omega(q) \subset \omega(y_0) \) does not hold. This contradiction proves our result. \( \square \)

6. Applications

6.1. Ordinary differential equations. Let \( \mathbb{R}^n \) be an \( n \)-dimensional real Euclidean space with the norm \(|\cdot|\). Let us consider a differential equation

\[ u' = f(t, u), \]

\[ \text{Ordinary differential equations. Let } \mathbb{R}^n \text{ be an } n \text{-dimensional real Euclidean space with the norm } |\cdot|. \text{ Let us consider a differential equation} \]
where \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \). Along with equation (6.1), we consider its \( H \)-class, i.e. the family of equations

\[
(6.2) \quad u' = g(t, v),
\]

where \( g \in H(f) := \{ f\tau : \tau \in \mathbb{R}\} \), \( f^\tau(t, u) := f(t + \tau, u) \) for all \((t, u) \in \mathbb{R} \times \mathbb{R}^n\) and by bar we mean the closure in \( C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \), which is equipped with the compact-open topology and this topology can be generated by the following metric

\[
d(f_1, f_2) := \sum_{k \geq 1} \frac{1}{2^k} \frac{d_k(f_1, f_2)}{1 + d_k(f_1, f_2)},
\]

where \( d_k(f_1, f_2) := \sup_{|t| \leq k, |x| \leq k} |f_1(t, x) - f_2(t, x)| \).

**Condition (A1).** The function \( f \) is regular, that is, for every equation (6.2) the conditions of existence, uniqueness and extendability on \( \mathbb{R}_+ \) are fulfilled.

Denote by \( \varphi(\cdot, v, g) \) the solution of equation (6.2), passing through the point \( v \in \mathbb{R}^n \) at the initial moment \( t = 0 \). Then the mapping \( \varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n \) is well defined and satisfies the following conditions (see, e.g. [3, 30]):

(i) \( \varphi(0, v, g) = v \) for all \( v \in \mathbb{R}^n \) and \( g \in H(f) \);

(ii) \( \varphi(t, \varphi(t, v, g), g^\tau) = \varphi(t + \tau, v, g) \) for all \( v \in \mathbb{R}^n \), \( g \in H(f) \) and \( t, \tau \in \mathbb{R}_+ \);

(iii) the mapping \( \varphi : \mathbb{R}_+ \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n \) is continuous.

Denote by \( Y := H(f) \) and \( (Y, \mathbb{R}, \sigma) \) the shift dynamical system on \( Y \) induced from \( (C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \mathcal{S}) \), i.e. \( \sigma(\tau, g) = g^\tau \) for \( \tau \in \mathbb{R} \) and \( g \in Y \). Then the equation (6.1) generates a cocycle \( \langle \mathbb{R}^n, \varphi, (Y, \mathbb{R}, \sigma) \rangle \) and an NDS \( \langle (X, \mathbb{R}_+), (Y, \mathbb{R}, \sigma), h \rangle \), where \( X := \mathbb{R}^n \times Y, \pi := (\varphi, \sigma) \) and \( h = pr_2 : X \to Y \).

Let \( \mathbb{R}^n_+ := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, 2, \ldots, n \} \). Then it defines a partial order on \( \mathbb{R}^n_+ \): \( u \leq v \) if and only if \( v - u \in \mathbb{R}^n_+ \).

**Condition (A2).** Equation (6.1) is monotone. This means that the cocycle \( \langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma) \rangle \) (or shortly \( \varphi \)) generated by (6.1) is monotone: if \( u, v \in \mathbb{R}^n_+ \) and \( u \leq v \) then \( \varphi(t, u, g) \leq \varphi(t, v, g) \) for all \( t \geq 0 \) and \( g \in H(f) \).

Let \( K \) be a closed cone in \( \mathbb{R}^n \). The dual cone to \( K \) is the closed cone \( K^* \) in the dual space \( (\mathbb{R}^n)^* \) of linear functions on \( \mathbb{R}^n \), defined by

\[
K^* := \{ \lambda(x) : \lambda(x) \geq 0 \text{ for any } x \in K \},
\]

where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathbb{R}^n \).

Recall (33, 34 ChV) that a function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) is said to be quasi-monotone if for any \( (t, u), (t, v) \in \mathbb{R} \times \mathbb{R}^n \) and \( \phi \in (\mathbb{R}^n)^* \) we have: \( u \leq v \) and \( \phi(u) = \phi(v) \) imply \( \phi(f(t, u)) \leq \phi(f(t, v)) \).

**Lemma 6.1.** Let \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) be a regular and quasi-monotone function, then the following statements hold:

(i) if \( u \leq v \), then \( \varphi(t, u, f) \leq \varphi(t, v, f) \) for any \( t \geq 0 \);

(ii) any function \( g \in H(f) \) is quasi-monotone;

(iii) equation (6.1) is monotone.

**Proof.** The first statement is proved in [22 ChIII].

Let \( g \in H(f) \), then there exists a sequence \( \{ h_k \} \subset \mathbb{R} \) such that \( g(t, u) = \lim_{k \to \infty} f^{h_k}(t, u) \) for any \( (t, u) \in \mathbb{R} \times \mathbb{R}^n \). Let \( u \leq v \) \((u, v \in \mathbb{R}^n)\) and \( \phi \in (\mathbb{R}^n)^* \) such that \( \phi(u) = \phi(v) \). Since \( f \) is quasi-monotone, we have

\[
\phi(f(t + h_k, u)) \leq \phi(f(t + h_k, v)),
\]
Passing to the limit in (6.3) as \( k \to \infty \) we obtain that \( g \) is quasi-monotone.

Finally, the third statement follows from the first and second statements. The proof is complete. \( \square \)

**Definition 6.2.** A solution \( \varphi(t, u_0, f) \) of equation (6.1) is said to be:

- (positively) uniformly stable, if for arbitrary \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( |\varphi(t_0, u, f) - \varphi(t_0, u_0, f)| < \delta \) \( t_0 \in \mathbb{R}, u \in \mathbb{R}^n \) implies \( |\varphi(t, u, f) - \varphi(t, u_0, f)| < \varepsilon \) for any \( t \geq t_0 \);

- compact on \( \mathbb{R}_+ \) if the set \( Q := \varphi(\mathbb{R}_+, u_0, f) \) is a compact subset of \( \mathbb{R}^n \), where \( \varphi(\mathbb{R}_+, u_0, f) := \{ \varphi(t, u_0, f) : t \in \mathbb{R}_+ \} \).

**Definition 6.3** (10, 33, 36). A solution \( \varphi(t, u_0, f) \) of equation (6.1) is called comparable (respectively, strongly comparable, uniformly comparable) if the motion \( \pi(t, x_0) \) (here \( x_0 = (u_0, f) \)) is comparable (respectively, strongly comparable, uniformly comparable) with \( \sigma(t, f) \) by character of recurrence.

Recall that a function \( \varphi \in C(\mathbb{R}, \mathbb{R}^n) \) is said to possess the property (A), if the motion \( \sigma(\cdot, \varphi) \) generated by \( \varphi \) possesses this property in the shift dynamical system \((C(\mathbb{R}, \mathbb{R}^n), \mathbb{R}, \sigma)\). A function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) is said to possess the property (A) in \( t \in \mathbb{R} \) uniformly w.r.t. \( u \) on every compact subset of \( \mathbb{R}^n \) if the motion \( \sigma(\cdot, f) \) generated by \( f \) possesses this property in the shift dynamical system \((C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)\). In the quality of the property (A) there can stand Lagrange stability, periodicity, asymptotic periodicity, almost periodicity, asymptotic almost periodicity and so on.

If \( x_0 = (u_0, y_0) \in X = \mathbb{R}^n \times Y \) and \( \alpha_{y_0} \) (respectively, \( \gamma_{y_0} \)) is the point from \( X \) defined in Lemma 5.6, then we denote by \( \alpha_{u_0} \) (respectively, \( \gamma_{u_0} \)) the point from \( \mathbb{R}^n \) such that \( \alpha_{y_0} = (\alpha_{u_0}, y_0) \) (respectively, \( \gamma_{y_0} = (\gamma_{u_0}, y_0) \)). The we have the following

**Theorem 6.4.** Suppose that the following assumptions are fulfilled:

- the function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) is positively Poisson stable in \( t \in \mathbb{R} \) uniformly w.r.t. \( u \) on every compact subset from \( \mathbb{R}^n \), i.e. there exists a sequence \( t_n \to +\infty \) as \( n \to \infty \) such that \( f^n \) converges to \( f \) in \( C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \);

- each solution \( \varphi(t, u_0, f) \) of equation (6.1) is bounded on \( \mathbb{R}_+ \) and uniformly stable.

Then under the conditions (A1)–(A2) the following statements hold:

1. for any solution \( \varphi(t, u_0, f) \) of equation (6.1) there exists a solution \( \varphi(t, \gamma_{u_0}, f) \) of (6.1) defined and bounded on \( \mathbb{R} \) such that:
   (a) \( \varphi(t, \gamma_{u_0}, f) \) is a comparable solution of (6.1);
   (b) \( \lim_{t \to +\infty} |\varphi(t, \alpha_{u_0}, f) - \varphi(t, \gamma_{u_0}, f)| = 0 \);

2. if the function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) is stationary (respectively, \( \tau \)-periodic, Levitan almost periodic, almost recurrent, Poisson stable) in \( t \in \mathbb{R} \) uniformly w.r.t. \( u \) on every compact subset of \( \mathbb{R}^n \), then \( \varphi(t, \gamma_{u_0}, f) \) has the same recurrent property in \( t \) and hence the solution \( \varphi(t, \alpha_{u_0}, f) \) has the same asymptotic recurrent property in \( t \).

**Proof.** Let \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) and \((C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)\) be the shift dynamical system on \( C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \). Denote by \( Y := H(f) \) and \((Y, \mathbb{R}, \sigma)\) the shift dynamical system on \( H(f) \) induced from \((C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)\). Consider the cocycle
where \( u (t, u_0, f) \) is bounded on \( \mathbb{R}_+ \), it follows from Lemma 6.7 that \( \Sigma^t_{u_0} \) is conditionally precompact; on the other hand, since \( \varphi (t, u_0, f) \) is uniformly stable, it follows from Corollary 6.8 and Remark 6.9 that \( \omega_{u_0} \) is uniformly stable. Now to finish the proof it is sufficient to apply Theorems 5.8 and 4.15. \( \square \)

**Condition (A3).** For any compact subset \( K \subset \mathbb{R}^n \) the function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) is bounded and uniformly continuous on \( \mathbb{R} \times K \).

**Remark 6.5.** Note that if a function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) satisfies condition (A3), then the set \( \{ f^h : h \in \mathbb{R} \} \) is precompact in \( C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) and vice versa.

**Theorem 6.6.** Suppose that the following assumptions are fulfilled:
- the function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) is strongly Poisson stable in \( t \in \mathbb{R} \) uniformly w.r.t. \( u \) on every compact subset of \( \mathbb{R}^n \);
- each solution \( \varphi (t, u_0, g) \) of every equation (6.2) is bounded on \( \mathbb{R}_+ \) and uniformly stable.

Then under the conditions (A1)-(A3) the following statements hold:

1. for any solution \( \varphi (t, v_0, g) \) of equation (6.2) there exists a solution \( \varphi (t, \gamma_{v_0}, g) \) of (6.2) defined and bounded on \( \mathbb{R} \) such that:
   (a) \( \varphi (t, \gamma_{v_0}, g) \) is a uniformly comparable solution of (6.2);
   (b) \( \lim_{t \to +\infty} |\varphi (t, \alpha_{v_0}, g) - \varphi (t, \gamma_{v_0}, g)| = 0 \).

2. if the function \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, pseudo recurrent, uniformly Poisson stable) in \( t \in \mathbb{R} \) uniformly w.r.t. \( u \) on every compact subset of \( \mathbb{R}^n \), then the solution \( \varphi (t, \gamma_{u_0}, f) \) has the same recurrent property in \( t \) and hence the solution \( \varphi (t, \alpha_{u_0}, f) \) has the same asymptotic recurrent property in \( t \).

**Proof.** Note that under the condition (A3) the hull \( H(f) \) is compact, so uniform comparability is equivalent to strong comparability by Theorem 4.10. Then this theorem can be proved similarly as Theorem 6.4 using Lemma 2.7, Corollary 3.8 and Theorems 6.10, 4.20. \( \square \)

### 6.2. Functional-differential equations with finite delay.
Let us first recall some notions and notations from [10]. Let \( r > 0 \), \( C([a, b], \mathbb{R}^n) \) be the Banach space of all continuous functions \( \varphi : [a, b] \to \mathbb{R}^n \) equipped with the sup–norm. If \( [a, b] = [-r, 0] \), then we set \( \mathcal{C} := C([-r, 0], \mathbb{R}^n) \). Let \( \sigma \in \mathbb{R} \), \( A \geq 0 \) and \( u \in C([\sigma - r, \sigma + A], \mathbb{R}^n) \). We will define \( u_t \in \mathcal{C} \) for any \( t \in [\sigma, \sigma + A] \) by the equality \( u_t(\theta) := u(t + \theta), \ -r \leq \theta \leq 0 \). Consider a functional differential equation

\[
\dot{u} = f(t, u_t),
\]

where \( f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n \) is continuous.

Denote by \( C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n) \) the space of all continuous mappings \( f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n \) equipped with the compact-open topology. On the space \( C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n) \) is defined (see, e.g. [12] ChI and [13] ChI) a shift dynamical system \( (C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n), \mathbb{R}, \sigma) \), where \( \sigma(\tau, f) := f^\tau \) for any \( f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n) \) and \( \tau \in \mathbb{R} \) and \( f^\tau \) is \( \tau \)-translation of \( f \), i.e. \( f^\tau (t, \phi) := f(t + \tau, \phi) \) for any \( (t, \phi) \in \mathbb{R} \times \mathcal{C} \). Let us set \( H(f) := \{ f^\tau : \tau \in \mathbb{R} \} \).
Along with equation (6.4) let us consider the family of equations
\begin{equation}
\dot{v} = g(t, v_t),
\end{equation}
where \( g \in H(f) \).

**Condition (F1).** In this subsection, we suppose that equation (6.4) is regular, i.e. the conditions of existence, uniqueness and extendability on \( \mathbb{R}_+ \) are fulfilled.

**Remark 6.7.** Denote by \( \tilde{\varphi}(t, u, f) \) the solution of equation (6.4) defined on \([-r, +\infty)\) (respectively, on \( \mathbb{R} \)) with the initial condition \( u \in C \). By \( \varphi(t, u, f) \) we will denote below the trajectory of equation (6.4), corresponding to the solution \( \tilde{\varphi}(t, u, f) \), i.e. a mapping from \( \mathbb{R}_+ \) (respectively, \( \mathbb{R} \)) into \( C \), defined by \( \varphi(t, u, f)(s) := \tilde{\varphi}(t + s, u, f) \) for any \( t \in \mathbb{R}_+ \) (respectively, \( t \in \mathbb{R} \)) and \( s \in [-r, 0] \). Below we will use the notions of “solution” and “trajectory” for equation (6.4) as synonymous concepts.

It is well-known [3, 20] that the mapping \( \varphi : \mathbb{R}_+ \times C \times H(f) \to C \) possesses the following properties:

(i) \( \varphi(0, v, g) = v \) for any \( v \in C \) and \( g \in H(f) \);

(ii) \( \varphi(t + \tau, v, g) = \varphi(t, \varphi(\tau, v, g), \sigma(\tau, g)) \) for any \( t, \tau \in \mathbb{R}_+ \), \( v \in C \) and \( g \in H(f) \);

(iii) the mapping \( \varphi \) is continuous.

Thus equation (6.4) generates a cocycle \( \langle C, \varphi, (Y, \mathbb{R}, \sigma) \rangle \) and an NDS \( (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle \), where \( Y := H(f) \), \( X := C \times Y \), \( \pi := (\varphi, \sigma) \) and \( h := pr_2 : X \to Y \).

**Remark 6.8.** Denote by \( B := \{ f \in C(\mathbb{R} \times C, \mathbb{R}^n) : f \) is continuous in \( t \) uniformly w.r.t \( u \) on any bounded subset of \( C \) and \( f(B) \) is a bounded subset of \( \mathbb{R}^n \) for any bounded subset \( B \subset \mathbb{R} \times C \}, \) equipped with the topology of uniform convergence on every bounded subset of \( \mathbb{R} \times C \). This topology can be defined by the following metric
\begin{equation}
d(f_1, f_2) := \sum_{k \geq 1} \frac{1}{2^k} \frac{d_k(f_1, f_2)}{1 + d_k(f_1, f_2)},
\end{equation}
where \( d_k(f_1, f_2) := \sup_{|\phi| \leq k} |f_1(t, \phi) - f_2(t, \phi)| \). Note that

(i) the metric space \( (B, d) \) is complete;

(ii) the subset \( B \subset C(\mathbb{R} \times C, \mathbb{R}^n) \) is translation invariant, i.e. \( f^h \in B \) for any \( f \in B \) and \( h \in \mathbb{R} \);

(iii) the mapping \( \sigma : \mathbb{R} \times B \to B \), defined by equality \( \sigma(h, f) := f^h \) for any \( (h, f) \in \mathbb{R} \times B \), is continuous.

Thus on the space \( (B, d) \) is defined a shift dynamical system \( (B, \mathbb{R}, \sigma) \).

Let \( C_+ := \{ \phi \in C : \phi \geq 0 \}, \) i.e. \( \phi(t) \geq 0 \) for any \( t \in [-r, 0] \} \) be the cone of nonnegative functions in \( C \). By \( C_+ \) on the space \( C \) is defined a partial order: \( u \leq v \) if and only if \( v - u \in C_+ \).

**Condition (F2).** Equation (6.4) is monotone, that is, the cocycle \( \langle C, \varphi, (H(f), \mathbb{R}, \sigma) \rangle \) generated by (6.4) possesses the following property: if \( u \leq v \), then \( \varphi(t, u, g) \leq \varphi(t, v, g) \) for any \( t \geq 0 \) and \( g \in H(f) \).

Recall (see, e.g. [22], [10]) that a function \( f \in C(\mathbb{R} \times C, \mathbb{R}^n) \) is said to be quasi-monotone if \( (t, u), (t, v) \in \mathbb{R} \times C, u \leq v \), and \( u_i(0) = v_i(0) \) for some \( i \), then \( f_i(t, u) \leq f_i(t, v) \).

**Lemma 6.9.** Let \( f \in C(\mathbb{R} \times C, \mathbb{R}^n) \) be a quasi-monotone function, then the following statements hold:
Theorem 6.12. Let \( f \in H(f) \) be quasi-monotone, then for any \( t \geq 0 \) and \( g \in H(f) \).

Proof. The first statement is proved in [39, ChV].

Let \( g \in H(f) \), then there exists a sequence \( \{h_k\} \subset \mathbb{R} \) such that \( g(t, u) = \lim_{k \to \infty} f^{h_k}(t, u) \) for any \( (t, u) \in \mathbb{R} \times C \). Let \( u \leq v \) \((u, v \in C)\) and \( u_i(0) = v_i\) for some \( i \). Since \( f \) is quasi-monotone, we have

\[
(6.7) \quad f_i(t + h_k, u) \leq f_i(t + h_k, v)
\]

and passing to limit in \((6.7)\) as \( k \to \infty \) we obtain that \( g \) is quasi-monotone too.

Finally, the third statement follows from the first and second statements, and this concludes the proof. \( \square \)

Condition (F3). For any bounded subset \( A \subset C \) the set \( f(\mathbb{R} \times A) \) is bounded in \( \mathbb{R}^n \).

Lemma 6.10. Let \( \varphi(t, u, f) \) be a bounded on \( \mathbb{R}_+ \) solution of equation \((6.4)\), then under the condition (F3) the set \( \varphi(\mathbb{R}_+, u, f) \subset C \) is precompact.

Proof. This statement follows from the Lemmas 2.2.3 and 3.6.1 in [19]. \( \square \)

Definition 6.11. A solution \( \varphi(t, u, f) \) of equation \((6.4)\) is said to be:

- \( (\text{positively}) \) uniformly stable, if for arbitrary \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( \|\varphi(t_0, u, f) - \varphi(t_0, u_0, f)\| \leq \delta \) \((t_0 \in \mathbb{R}_+, u, u_0 \in C)\) implies \( \|\varphi(t, x, f) - \varphi(t, x_0, f)\| \leq \varepsilon \) for any \( t \geq t_0 \);

- compact on \( \mathbb{R}_+ \) if the set \( Q := \varphi(\mathbb{R}_+, u_0, f) \) is a compact subset of \( C \), where \( \varphi(t, u_0, f) := \{\varphi(t, u_0, f) : t \in \mathbb{R}_+\} \).

Condition (F4). Every solution \( \varphi(t, u, g) \) of every equation \((6.5)\) is bounded on \( \mathbb{R}_+ \) and uniformly stable.

Let \( f \in B, \sigma(t, f) \) be the motion (in the shift dynamical system \((B, \mathbb{R}, \sigma)\)) generated by \( f \), \( u_0 \in C \), \( \varphi(t, u_0, f) \) be a solution of equation \((6.4)\), \( x_0 := (u_0, f) \in X := C \times H(f) \) and \( \pi(t, x_0) := (\varphi(t, u_0, f), \sigma(t, f)) \) be the motion of skew-product dynamical system \((X, \mathbb{R}_+, \pi)\).

Like in ODE case, a solution \( \varphi(t, u_0, f) \) of equation \((6.4)\) is called comparable (respectively, strongly comparable or uniformly comparable) if the motion \( \pi(t, x_0) \) is comparable (respectively, strongly comparable or uniformly comparable) with \( \sigma(t, f) \) by character of recurrence.

Applying the results from Sections 4.5 we can obtain a series of results for functional differential equation \((6.4)\). Below we formulate some of them.

Theorem 6.12. Suppose that the following assumptions are fulfilled:

- the function \( f \in B \) is positively Poisson stable in \( t \in \mathbb{R} \) uniformly w.r.t. \( u \) on every bounded subset from \( C \);

- each solution \( \varphi(t, u_0, f) \) of equation \((6.4)\) is bounded on \( \mathbb{R}_+ \) and uniformly stable.

Then under the conditions (F1)–(F3) the following statements hold:

1. for any solution \( \varphi(t, u_0, f) \) of equation \((6.4)\) there exists a solution \( \varphi(t, \gamma_{u_0}, f) \) of \((6.4)\) defined and bounded on \( \mathbb{R} \) such that:

   \( \varphi(t, \gamma_{u_0}, f) \) is a comparable solution of \((6.4)\).
be the Banach space of continuous functions $f$ in $t \in \mathbb{R}$ uniformly w.r.t. $u$ on every bounded subset from $C$, then $\varphi(t, \gamma_{u_0}, f)$ has the same recurrent property and hence $\varphi(t, \alpha_{u_0}, f)$ has the same asymptotic recurrent property in $t$.

Here the notations $\varphi(t, \alpha_{u_0}, f)$ and $\varphi(t, \gamma_{u_0}, f)$ have the similar meaning as in Section 6.1

Proof. Let $f \in \mathcal{B}$ and $(\mathcal{B}, \mathbb{R}, \sigma)$ be the shift dynamical system on $\mathcal{B}$. Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the shift dynamical system on $H(f)$ induced from $(\mathcal{B}, \mathbb{R}, \sigma)$. Consider the cocycle $\langle C, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ generated by equation (6.4) (see Condition (F1)). Now to finish the proof of Theorem it is sufficient to apply Lemma 2.7 and Theorems 5.8, 4.15.

Theorem 6.13. Suppose that the following assumptions are fulfilled:

- the function $f \in \mathcal{B}$ is strongly Poisson stable in $t \in \mathbb{R}$ uniformly w.r.t. $u$ on every bounded subset from $\mathcal{B}$;
- the set $H(f)$ is compact in $\mathcal{B}$.

Then under the conditions (F1)–(F4) the following statements hold:

1. for any solution $\varphi(t, v_0, g)$ of every equation (6.3) there exists a solution $\varphi(t, \gamma_{v_0}, g)$ of (6.3) defined and bounded on $\mathbb{R}$ such that:

   (a) $\varphi(t, \gamma_{v_0}, g)$ is a uniformly comparable solution of (6.4);
   (b) $\lim_{t \to +\infty} ||\varphi(t, \alpha_{v_0}, g) - \varphi(t, \gamma_{v_0}, g)||_c = 0$.

2. if the function $f \in \mathcal{B}$ is quasi-periodic (respectively, Bohr almost periodic, Birkhoff recurrent, pseudo recurrent, uniformly Poisson stable) in $t \in \mathbb{R}$ uniformly w.r.t. $u$ on every bounded subset from $C$, then $\varphi(t, \gamma_{u_0}, f)$ has the recurrent property and hence $\varphi(t, \alpha_{u_0}, f)$ has the same asymptotic recurrent property in $t$.

Proof. This theorem can be proved similarly as Theorem 6.12 using Lemma 2.7 and Theorems 6.10, 4.20.

6.3. Parabolic systems. Consider the following system of parabolic differential equations

\begin{equation}
\partial_t w_j = \nu_j \Delta w_j + f_j(t, x, w_1, \ldots, w_n), \quad j = 1, \ldots, n
\end{equation}

in a smooth bounded domain $D \subset \mathbb{R}^d$, $d \leq 3$, with the Neumann boundary conditions. Here $\Delta$ is the Laplace operator and $\nu_j$ are some positive constants and $f = (f_1, \ldots, f_n)$ is a function satisfying certain conditions (specified below). Let $\mathcal{B}$ be the Banach space of continuous functions $h : \overline{D} \times \mathbb{R}^n_+ \to \mathbb{R}^n$ such that all derivatives $\partial_{w_i} h_i$ are continuous on $\overline{D} \times \mathbb{R}^n_+$ and $||h||_\mathcal{B} < \infty$, where

\[ ||h||_\mathcal{B} := \sup \{ (1 + |w|)^{-1} \sum_{i \geq 1} |h_i(x, w)| + \sum_{i,j \geq 1} |\partial_{w_i} h_i(x, w)| : (x, w) \in \overline{D} \times \mathbb{R}^n_+ \}. \]

Denote by $C(\mathbb{R}, \mathcal{B})$ the space of all continuous functions $f : \mathbb{R} \to \mathcal{B}$ equipped with the compact-open topology and $(C(\mathbb{R}, \mathcal{B}), \mathbb{R}, \sigma)$ the shift dynamical system on $C(\mathbb{R}, \mathcal{B})$.\]
Remark 6.15. Hence a monotone NDS \( \langle \cdot \rangle \) then every function \( \langle \cdot \rangle \) uniformly w.r.t. \( (x, w) \in \overline{D} \times \mathbb{R}_n^+ \).

Condition (P2). The function \( f = (f_1, \ldots, f_n) \in C(\mathbb{R}, \mathfrak{B}) \), i.e. \( f : \mathbb{R} \times \overline{D} \times \mathbb{R}_n^+ \rightarrow \mathbb{R}^n \) is continuous in \( t \in \mathbb{R} \) uniformly w.r.t. \( (x, w) \in \overline{D} \times \mathbb{R}_n^+ \).

Condition (P3). The function \( f = (f_1, \ldots, f_n) \in C(\mathbb{R}, \mathfrak{B}) \) is positively Poisson stable in \( t \in \mathbb{R} \) uniformly w.r.t. \( (x, w) \in \overline{D} \times \mathbb{R}_n^+ \), i.e. there exists a sequence \( t_k \rightarrow +\infty \) as \( k \rightarrow \infty \) such that \( f(t + t_k, x, w) \rightarrow f(t, x, w) \) uniformly w.r.t. \( (x, w) \in [-l, l] \times \overline{D} \times \mathbb{R}_n^+ \) for any \( l \in \mathbb{N} \).

Remark 6.14. Note that if the function \( f \in C(\mathbb{R}, \mathfrak{B}) \) possesses the property (P3), then every function \( g \in H(f) \) possesses the same property.

Let \( C(\overline{D}, \mathbb{R}^n) \) (respectively, \( C(\overline{D}, \mathbb{R}_n^+) \)) be the space of all continuous functions \( f : \overline{D} \rightarrow \mathbb{R}^n \) (respectively, \( f : \overline{D} \rightarrow \mathbb{R}_n^+ \)) and \( V := C(\overline{D}, \mathbb{R}^n) \) (respectively, \( V := C(\overline{D}, \mathbb{R}_n^+) \)) equipped with the norm \( ||f||_V := \max\{|f(x)| : x \in \overline{D}\} \) (respectively, \( V_+ := C(\overline{D}, \mathbb{R}_n^+) \) with the same norm). By \( V_+ \) it induces a partial order on \( C(\overline{D}, \mathbb{R}^n) \): \( f \leq g \) if and only if \( g - f \in V_+ \).

Under the conditions (P1)–(P3) it can be proved (see, for example, [10], [20], ChIII and [40], ChVII) for any \( g \in H(f) \) and \( v \in V_+ \) that the problem (6.9) admits a unique solution \( \varphi(t, v, g) \) belonging to the space \( C(\mathbb{R}_+, V_+) \). Denote by \( Y := H(f) \) and \( (Y, \mathbb{R}, \sigma) \) the shift dynamical system on \( H(f) \). From general properties of solutions of (6.9) we have:

(i) \( \varphi(0, v, g) = v \) for all \( v \in V_+ \) and \( g \in H(f) \):
(ii) \( \varphi(t, \varphi(\tau, v, g), g) = \varphi(t + \tau, v, g) \) for \( v \in V_+ \), \( g \in H(f) \) and \( t, \tau \in \mathbb{R}_+ \):
(iii) the mapping \( \varphi : \mathbb{R}_+ \times V_+ \times H(f) \rightarrow V_+ \) is continuous;
(iv) if \( u \leq v \), then \( \varphi(t, u, g) \leq \varphi(t, v, g) \) for all \( t \geq 0 \) and \( g \in H(f) \).

Therefore, the problem (6.8) generates a monotone cocycle \( \langle V_+, \varphi, (Y, \mathbb{R}, \sigma) \rangle \) and hence a monotone NDS \( \langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle \), where \( X := V_+ \times Y \), \( \pi := (\varphi, \sigma) \) and \( h := pr_2 : X \rightarrow Y \).

Remark 6.15. Since for \( d \leq 3 \) the Sobolev space \( H^2(D) \) is compactly embedded into \( C(\overline{D}, \mathbb{R}^n) \), one can prove (see, e.g. [20]) that the set \( \varphi(\mathbb{R}_+, u, f) \) is precompact if it is bounded.

Like in ODE case, the compactness, uniform stability of a solution \( \varphi(t, u_0, f) \) to (6.8) can be defined similarly, and the (strong, uniform) comparability of \( \varphi(t, u_0, f) \) can also be defined similarly. Then we are in the position to state the following

Theorem 6.16. Suppose that each solution \( \varphi(t, u_0, f) \) of (6.8) is compact on \( \mathbb{R}_+ \) and uniformly stable. Then under the conditions (P1)–(P3) the following statements hold:

1. For any solution \( \varphi(t, u_0, f) \) of equation (6.8) there exists a solution \( \varphi(t, \gamma u_0, f) \) of (6.8) defined and compact on \( \mathbb{R} \) such that:
(a) $\varphi(t, \gamma_{u_0}, f)$ is a comparable solution of (6.8);
(b) $\lim_{t \to +\infty} ||\varphi(t, \alpha_{u_0}, f) - \varphi(t, \gamma_{u_0}, f)||_V = 0$.

2. If the function $f \in C(\mathbb{R}, \mathcal{B})$ is stationary (respectively, $\tau$-periodic, Levitan almost periodic, almost recurrent, Poisson stable) in $t \in \mathbb{R}$, then the solution $\varphi(t, \gamma_{u_0}, f)$ has the same recurrent property and hence the solution $\varphi(t, \alpha_{u_0}, f)$ has the same asymptotic recurrent property in $t$.

Here the notations $\varphi(t, \alpha_{u_0}, f)$ and $\varphi(t, \gamma_{u_0}, f)$ have the similar meaning as in Section 6.1.

Proof. Let $f \in C(\mathbb{R}, \mathcal{B})$ and $(C(\mathbb{R}, \mathcal{B}), \mathbb{R}, \sigma)$ be the shift dynamical system on $C(\mathbb{R}, \mathcal{B})$. Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the shift dynamical system on $H(f)$ induced from $(C(\mathbb{R}, \mathcal{B}), \mathbb{R}, \sigma)$. Consider the cocycle $\langle V_+, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ generated by (6.8). Now to finish the proof it is sufficient to apply Lemma 2.7 and Theorems 5.8, 4.15.

Theorem 6.17. Suppose that the following assumptions are fulfilled:
- the set $H(f)$ is a compact subset of $C(\mathbb{R}, \mathcal{B})$;
- the function $f \in C(\mathbb{R}, \mathcal{B})$ is strongly Poisson stable in $t \in \mathbb{R}$;
- each solution $\varphi(t, v_0, g)$ of every problem (6.9) is compact on $\mathbb{R}_+$ and uniformly stable.

Then under the conditions (P1)–(P3) the following statements hold:

1. For any solution $\varphi(t, v_0, g)$ of every problem (6.9) there exists a solution $\varphi(t, \gamma_{v_0}, g)$ of (6.8) defined and compact on $\mathbb{R}$ such that:
   (a) $\varphi(t, \gamma_{v_0}, g)$ is a uniformly comparable solution of (6.8);
   (b) $\lim_{t \to +\infty} ||\varphi(t, \alpha_{v_0}, g) - \varphi(t, \gamma_{v_0}, g)||_V = 0$.

2. If the function $f \in C(\mathbb{R}, \mathcal{B})$ is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, pseudorecurrent, uniformly Poisson stable) in $t \in \mathbb{R}$, then the solution $\varphi(t, \gamma_{u_0}, f)$ has the same recurrent property and hence the solution $\varphi(t, \alpha_{u_0}, f)$ has the same asymptotic recurrent property in $t$.

Proof. This theorem can be proved similarly to Theorem 6.16 using Lemma 2.7 and Theorems 5.10, 4.20.

Acknowledgements

This work is partially supported by NSFC Grants 11271151, 11522104, and the startup and Xinghai Jieqing funds from Dalian University of Technology.

References

[1] V. M. Bebutov, On the shift dynamical systems on the space of continuous functions, Bull. of Inst. of Math. of Moscow University 2;5 (1940), pp.1-65. (in Russian)
[2] B. Basit and H. Gnizler, Spectral criteria for solutions of evolution equations and comments on reduced spectra, arXiv preprint (2010), arXiv:1006.2169
[3] I. U. Bronshtein, Extensions of Minimal Transformation Group. Kishinev, Stiintsa, 1974 (in Russian). [English translation: Extensions of Minimal Transformation Group, Sijthoff & Noordhoff, Alphen aan den Rijn, 1979]
[4] I. U. Bronshtein and V. F. Cherny, About extensions of dynamical systems with the uniformly asymptotically stable points, Differential Equations 10 (1974) no. 7, 1225–1230.
[5] T. Caraballo and D. Cheban, Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard’s separation condition. I, *J. Differential Equations*, **246** (2009), 108–128.

[6] T. Caraballo and D. Cheban, Almost periodic motions in semi-group dynamical systems and Bohr/Levitan almost periodic solutions of linear difference equations without Favard’s separation condition, *J. Difference Equ. Appl.* **19** (2013), 872–897.

[7] D. Cheban, On the comparability of the points of dynamical systems with regard to the character of recurrence property in the limit, *Mathematical Sciences* Issue No. 1, Kishinev, “Shiintsa”, 1977, pp. 66–71.

[8] D. Cheban, Global pullback attractors of C-analytic nonautonomous dynamical systems, *Stoch. Dyn.* **1** (2001), no. 4, 511–535.

[9] D. Cheban, Levitan almost periodic and almost automorphic solutions of V-monotone differential equations, *J. Dynam. Differential Equations* **20** (2008), 669–697.

[10] D. Cheban, *Asymptotically Almost Periodic Solutions of Differential Equations*. Hindawi Publishing Corporation, New York, 2009, ix+186 pp.

[11] D. Cheban, *Global Attractors of Set-Valued Dynamical and Control Systems*. Nova Science Publishers Inc, New York, 2010, xvii+269 pp.

[12] D. Cheban, *Global Attractors of Nonautonomous Dynamical and Control Systems*. 2nd Edition. Interdisciplinary Mathematical Sciences, vol. 18, River Edge, NJ: World Scientific, 2015, xxv+589 pp.

[13] D. Cheban, Levitan/Bohr almost periodic and almost automorphic solutions of the scalar differential equations, *Mathematics & Information Technologies: Research and Education (MITRE-2016)* Chişinău, Republic of Moldova, June 23-26, 2016, pp.18–19. The corresponding full-length paper: submitted.

[14] D. Cheban, Bohr/Levitan Almost Periodic Motions and Global Attractors of Almost Periodic Dynamical Systems. In preparation, 2017.

[15] D. Cheban and B. A. Shcherbakov, Poisson asymptotic stability of motions of dynamical systems and their comparability with regard to the recurrence property in the limit, *Differential Equations* **13** (1977), no. 5, 898–906.

[16] I. Chueshov, Order-preserving skew-product flows and nonautonomous parabolic systems, *Acta Appl. Math.* **65** (2001), no. 1-3, 185–205.

[17] R. Ellis, *Lectures on Topological Dynamics*. W. A. Benjamin, Inc., New York 1969 xv+211 pp.

[18] A. M. Fink, *Almost Periodic Differential Equations*. Lecture Notes in Mathematics, Vol. 377, Springer-Verlag, Berlin-New York, 1974. viii+336 pp.

[19] J. K. Hale, *Theory of Functional Differential Equations*. Second edition. Applied Mathematical Sciences, Vol. 3. Springer-Verlag, New York-Heidelberg, 1977. x+365 pp.

[20] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981. iv+348 pp.

[21] D. Husemoller, *Fibre Bundles*. Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994. xii+353 pp.

[22] M. Hirsch and H. Smith, *Monotone Dynamical Systems*. Handbook of differential equations: ordinary differential equations. Vol. II, 239–357, Elsevier B. V., Amsterdam, 2005.

[23] J. Jiang and X. Zhao, Convergence in monotone and uniformly stable skew-product semiflows with applications, *J. Reine Angew. Math.* **589** (2005), 21–55.

[24] B. M. Levitan, *Almost Periodic Functions*. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 396 pp. (in Russian)

[25] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*. Moscow State University Press, Moscow, 1978, 204 pp. (in Russian). [English translation: Almost Periodic Functions and Differential Equations. Cambridge University Press, Cambridge, 1982, xi+211 pp.]

[26] S. Novo, R. Obaya, and Ana M. Sanz, Stability and extensiability results for abstract skew-product semiflows, *J. Differential Equations* **235** (2007), no. 2, 623–646.

[27] Z. Opial, Sur les solutions presque-périodiques des équations différentielles du premier et du second ordre (in French), *Ann. Polon. Math.* **7** (1959), 51–61.

[28] R. Sacker and G. Sell, Skew-product flows, finite extensions of minimal transformation groups and almost periodic differential equations, *Bull. Amer. Math. Soc.* **79** (1973), 802–805.
[29] R. Sacker and G. Sell, *Lifting properties in skew-product flows with applications to differential equations*, Mem. Amer. Math. Soc. 11 (1977), no. 190, iv+67 pp.

[30] G. Sell, *Lectures on Topological Dynamics and Differential Equations*. Van Nostrand Reinhold Mathematical Studies, No. 33. Van Nostrand Reinhold Co., London, 1971. ix+199 pp.

[31] B. A. Shcherbakov, Classification of Poisson-stable motions, Pseudo-recurrent motions (in Russian), *Dokl. Akad. Nauk SSSR* 146 (1962), 322–324.

[32] B. A. Shcherbakov, On classes of Poisson stability of motion, Pseudorecurrent motions (in Russian), *Bul. Akad. Stiince RSS Moldoven* 1963 (1963), no. 1, 58–72.

[33] B. A. Shcherbakov, *Topologic Dynamics and Poisson Stability of Solutions of Differential Equations*, Știința, Chișinău, 1972, 231 pp. (in Russian)

[34] B. A. Shcherbakov, The compatible recurrence of the bounded solutions of first order differential equations (in Russian), *Differencial’nye Uravnenija* 10 (1974), 270–275.

[35] B. A. Shcherbakov, The comparability of the motions of dynamical systems with regard to the nature of their recurrence, *Differencial’nye Uravnenija* 11 (1975), no. 7, 1246–1255 (in Russian). [English translation: *Differential Equations* 11 (1975), no. 7, 937–943].

[36] B. A. Shcherbakov, *Poisson Stability of Motions of Dynamical Systems and Solutions of Differential Equations*. Știința, Chișinău, 1985, 147 pp. (in Russian)

[37] W. Shen and Y. Yi, Almost automorphic and almost periodic dynamics in skew-product semiflows, *Mem. Amer. Math. Soc.* 136 (1998), no. 647, x+93pp.

[38] K. S. Sibirsky, *Introduction to Topological Dynamics*. Kishinev, RIA AN MSSR, 1970, 144 pp. (in Russian). [English translation: Introduction to Topological Dynamics. Noordhoff, Leiden, 1975. ix+163 pp.]

[39] H. L. Smith, Monotone semiflows generated by functional differential equations, *J. Differential Equations* 66 (1987), no. 3, 420–442.

[40] H. L. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems*. Mathematical surveys and monographs, Volume 41. American Mathematical Society, Providence, RI, 1995. x+174 pp.

[41] V. V. Zhikov, On problem of existence of almost periodic solutions of differential and operator equations (in Russian), *Nauchnye Trudy VVPI, Matematika* 8 (1969), 94–188.

D. CHEBAN: SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN 116024, P. R. CHINA

E-mail address: cheban@usm.md; davidcheban@yahoo.com

Z. LIU: SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN 116024, P. R. CHINA

E-mail address: zxliu@dlut.edu.cn

\[\text{2Permanent address: State University of Moldova, Faculty of Mathematics and Informatics, Department of Mathematics, A. Mateevich Street 60, MD-2009 Chișinău, Moldova}\]