Total positivity and cluster algebras

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Abstract. This is a brief and informal introduction to cluster algebras. It roughly follows the historical path of their discovery, made jointly with A. Zelevinsky. Total positivity serves as the main motivation.

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Introduction

Cluster algebras are encountered in many algebraic and geometric contexts, with combinatorics providing a unifying framework. This short paper reviews the origins of cluster algebras, their deep connections with total positivity phenomena, and some of their recent manifestations in Teichmüller theory.

The introduction of cluster algebras, made in joint work with A. Zelevinsky [26], was rooted in the desire to understand, in a concrete and combinatorial way, G. Lusztig’s theory of total positivity and canonical bases in quantum groups (see, e.g., [44, 47]). Although this goal remains largely elusive (cf. [43]), the concept proved valuable due to its surprising ubiquity, and to the connections it helped uncover between diverse and seemingly unrelated areas of mathematics.

This paper gives a popular and quick introduction to the subjects in the title, aimed at an uninitiated reader, and roughly following the historical order of modern developments in the two related fields. Cumbersome technicalities involved in the usual definition of cluster algebras are largely omitted, giving way to prototypical examples from which the reader is invited to generalize, to discussions of underlying motivations, and to hints concerning further applications and extensions of the basic theory. Many important aspects are left out due to space limitations.

The style is rather informal, owing to the desire to see the forest through the trees, and to make the paper accessible to a general mathematical audience. There are no numbered formulas or theorems: results are stated as part of the general narrative. Some attributions are missing; they can be found in the sources quoted. The goal is to give the reader an intuitive feel for what cluster algebras are, and motivate her/him to read the more formal expositions elsewhere.

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Several survey/introductory papers dedicated to the subjects in the title, approached from various perspectives, have already appeared in the literature; see in particular \[1, 5, 19, 25, 29, 34, 39, 43, 55, 56, 57\]. An excellent introduction to applications of cluster algebras in representation theory is given in B. Leclerc’s contribution \[43\] to these proceedings. Besides consulting these sources and references therein, the reader is invited to visit the Cluster Algebras Portal \[18\], which provides numerous links to publications, conferences, seminars, thematic programs, software packages, etc.

Our presentation is loosely based on the papers \[2, 3, 20, 21, 23, 25, 26, 27, 29\], joint with A. Berenstein, M. Shapiro, D. Thurston, and A. Zelevinsky. Section 1 introduces total positivity and the idea of a positive/nonnegative part of an algebraic variety. Section 2 presents the basic notions of cluster algebra theory, emphasizing its roots in total positivity. Section 3 discusses the occurrence of cluster algebras in combinatorial topology of triangulated surfaces, and connections with Teichmüller spaces.

The format of this brief survey does not allow us to discuss several important directions of current research on cluster algebras and related fields. In particular, not covered here are the theory of cluster categories and the various facets of categorification \[39, 40, 41, 50\]; the connections between cluster algebras and Poisson geometry \[32, 33\]; closely related work on cluster varieties arising in higher Teichmüller theory \[10, 17\]; the polyhedral combinatorics of cluster fans and Cambrian lattices \[52\]; applications to discrete integrable systems \[13, 28, 37, 41\]; the machinery of quivers with potentials \[11, 12\]; connections with Donaldson-Thomas invariants \[42, 49\]; and other exciting topics.

Acknowledgments. The discovery of cluster algebras, the main work leading to it, and the development of fundamentals of the general theory were all done jointly with my longtime collaborator Andrei Zelevinsky. I am indebted to him, and to my co-authors Arkady Berenstein, Michael Shapiro, and Dylan Thurston for their invaluable contributions to our joint work discussed below. Catharina Stroppel persuaded me to give a talk in Bonn whose design this presentation follows. Bernhard Keller, George Lusztig, and Kelli Talaska made valuable editorial suggestions.

1. Total positivity

A matrix \(x\) with real entries is called totally positive (resp., totally nonnegative) if all its minors—that is, determinants of square submatrices—are positive (resp., nonnegative). The first systematic study of these classes of matrices was conducted in the 1930s by F. Gantmacher and M. Krein \[31\], following the pioneering work of I. Schoenberg \[53\]. In particular, they showed that the eigenvalues of an \(n \times n\) totally positive matrix are real, positive, and distinct.

Total positivity is a remarkably widespread phenomenon: matrices with positive/nonnegative minors play an important role in classical mechanics (theory of small oscillations), probability (one-dimensional diffusion processes), discrete potential theory (planar resistor networks), asymptotic representation theory (the
In pedestrian terms, each upper-triangular unipotent totally nonnegative \( G \) for totally positive matrices.

The Binet-Cauchy theorem implies that totally positive (resp., nonnegative) matrices in \( G = \text{SL}_n \), form a multiplicative semigroup, denoted by \( G_{\geq} \). In view of Cryer’s lemma, the study of \( G_{\geq} \) can be reduced to the investigation of its sub-semigroup \( N_{\geq} \subset G_{\geq} \) of upper-triangular unipotent totally nonnegative matrices.

The celebrated Loewner-Whitney Theorem \([45, 54]\) identifies the infinitesimal generators of \( N_{\geq} \) as the \textit{Chevalley generators} of the corresponding Lie algebra. In pedestrian terms, each upper-triangular unipotent totally nonnegative \( n \times n \) matrix can be written as a product of (totally nonnegative) matrices of the form

\[
x = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
* & 1 & 0 & \cdots & 0 \\
* & * & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
1 & * & \cdots & * \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

in which all three factors (lower-triangular unipotent, diagonal, and upper-triangular unipotent) are totally nonnegative. There is also a counterpart of this statement for totally positive matrices.

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\[
x_i(t) = \begin{bmatrix}
1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & t & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

here the matrix \( x_i(t) \) differs from the identity matrix by a single entry \( t \geq 0 \) in row \( i \) and column \( i+1 \). This led G. Lusztig \([46]\) to the idea of extending the notion of total positivity to other semisimple groups \( G \), by defining the set \( G_{\geq} \) of totally nonnegative elements in \( G \) as the semigroup generated by the Chevalley generators. Lusztig has shown that \( G_{\geq} \) can be described by inequalities of the form \( \Delta(x) \geq 0 \) where \( \Delta \) ranges over the appropriate dual canonical basis (at \( q = 1 \)). This set is infinite, and very hard to understand; fortunately, it can be replaced \([24]\) by a much simpler (and finite) set of \textit{generalized minors} \([22]\).

A yet more general (if informal) concept is one of a \textit{totally positive/nonnegative variety}. Vaguely, the idea is this: take a complex variety \( X \) together with a family \( \Delta \) of “important” regular functions on \( X \). The corresponding totally positive (resp., totally nonnegative) variety \( X_{>0} \) (resp., \( X_{\geq} \)) is the set of points at which all of these functions take positive (resp., nonnegative) values:

\[
X_{>0} = \{ x \in X : \Delta(x) > 0 \text{ for all } \Delta \in \Delta \}.
\]
If X is the affine space of matrices of a given size (or GL\(_n(\mathbb{C})\) or SL\(_n(\mathbb{C})\)), and \(\Delta\) is the set of all minors, then we recover the classical notion. One can restrict this construction to matrices lying in a given stratum of a Bruhat decomposition, or in a given double Bruhat cell \([22, 46]\). Another important example is the totally positive (resp., nonnegative) Grassmannian consisting of the points in a usual Grassmann manifold where all Plücker coordinates can be chosen to be positive (resp., nonnegative).

In each of these examples, the notion of positivity depends on a particular choice of a coordinate system: a basis in a vector space allows us to view linear transformations as matrices; a choice of reference flag determines a system of Plücker coordinates; and so on.

Why study totally nonnegative varieties? Besides the connections to Lie theory alluded to above, there are at least three more reasons.

First, some totally nonnegative varieties are interesting in their own right as they can be identified with important spaces, e.g. some of those arising in Teichmüller theory; cf. Section [3]. One can hope to gain additional insight into the structure of such spaces and their compactifications by “upgrading” them to complex varieties, studying associated quantizations, etc. The nascent “higher Teichmüller theory” \([7, 17]\) is one prominent expression of this paradigm.

Second, passing from a complex variety to its positive part can be viewed as a step towards its tropicalization. The deep connections between total positivity, tropical geometry, and cluster theory lie outside the scope of this short paper; see \([17, 21, 30]\) for some aspects of this emerging research area.

Yet another reason to study totally nonnegative varieties lies in the fact that their structure as semialgebraic sets reveals important features of related complex varieties. We illustrate this phenomenon using the example first studied in \([46]\) (cf. also \([2, 23]\)). Consider \(N \subset \text{SL}_n(\mathbb{C})\), the subgroup of \(n \times n\) unipotent upper-triangular matrices. The corresponding totally nonnegative variety is the semi-group \(N_{\geq 0}\) of totally nonnegative matrices in \(N\). Take \(n = 3\); then

\[
N_{\geq 0} = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x \geq 0, y \geq 0, \quad xz - y \geq 0 \right\}.
\]

The inequalities defining \(N_{\geq 0}\) are homogeneous in the following sense: replacing \((x, y, z)\) by \((ax, a^2y, az)\), with \(a > 0\), does not change them. Consequently, the space \(N_{\geq 0}\) is topologically a cone with the apex \(x = y = z = 0\) (the identity matrix) over the base \(M_{\geq 0} \subset N_{\geq 0}\) cut out by the plane \(x + z = 1\). Thus

\[
M_{\geq 0} \cong \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } y \leq x(1 - x)\}
\]

is the subset of the coordinate plane \(\mathbb{R}^2\) bounded by the \(x\) axis and the parabola \(y = x(1 - x)\), as shown in Figure 1(a).

The semialgebraic set \(M_{\geq 0}\) naturally decomposes into 5 algebraic strata: two of dimension 0, two of dimension 1, and one of dimension 2. Accordingly, the cone \(N_{\geq 0}\) decomposes into 6 algebraic strata of dimension 1 higher; the apex of \(N_{\geq 0}\) corresponds to the “empty face” of \(M_{\geq 0}\). See Figure 1(b).
Figure 1. (a) The base $M_{\geq 0}$ of the cone $N_{\geq 0}$. (b) The attachment of algebraic strata.

The adjacency of these strata is described by a partial order isomorphic to the Bruhat order on the symmetric group $S_3$. This happens in general, for any $n$: the decomposition of $N_{\geq 0}$ into algebraic strata produces a CW-complex with cell attachments described by the Bruhat order on $S_n$. Recall that the same partial order describes the attachment of Schubert cells in the manifold of complete flags in $\mathbb{C}^n$. The latter has rich topology, and is a central object of study in modern Schubert Calculus. By contrast, $N_{\geq 0}$ and $M_{\geq 0}$ have no topology to speak of (in fact, $M_{\geq 0}$ is expected to be homeomorphic to a ball) but has a cell decomposition with exactly the same cell attachments. The big difference of course is that the complex Schubert cells have twice the dimensions of their real (more precisely, positive real) counterparts. Still, the stratification of $M_{\geq 0}$ resulting from its semi-algebraic structure somehow “remembers” the Bruhat order—which is all one needs to know in order to reconstruct the topology of the flag manifold and its Schubert cells/varieties—including Schubert and Kazhdan-Lusztig polynomials, etc.

2. Cluster algebras

The discussion in Section 1 prompts one to ask: Which algebraic varieties $X$ have a natural notion of positivity? Which families $\Delta$ of regular functions should one consider in defining this notion? The concept of a cluster algebra can be viewed as an attempt to provide a general answer to these questions. Since the definition is fairly technical, we start with an example and then generalize.

Our prototypical example of a cluster algebra $\mathcal{A}$ is the coordinate ring of the base affine space for the special linear group $G = \text{SL}_n(\mathbb{C})$, defined as follows. The subgroup $N \subset G$ of unipotent upper-triangular matrices acts on $G$ by right multiplication. The algebra $\mathcal{A} = \mathbb{C}[G/N]$ consists of regular functions on $G$ which are invariant under this action of $N$. Thus elements of $\mathcal{A}$ can be viewed as polynomials in the entries $x_{ij}$ of a matrix $x = (x_{ij}) \in \text{SL}_n(\mathbb{C})$ which are invariant under column operations that add to a column of $x$ a linear combination of preceding columns. Classical invariant theory tells us that $\mathcal{A}$ is generated by the flag minors

$$\Delta_I : x \mapsto \det(x_{ij} | i \in I, j \leq |I|)$$
where $I$ ranges over nonempty proper subsets of $\{1, \ldots, n\}$. That is, $\Delta_I$ is a minor occupying the rows in $I$ and the first several columns. The generators $\Delta_I$ satisfy certain well known homogeneous quadratic identities sometimes called generalized Plücker relations.

A point in $G/N$ represented by a matrix $x$ is, by definition, totally positive/nonnegative if all flag minors $\Delta_I$ take positive/nonnegative values at $x$. Total positivity in $G/N$ is closely related to the classical notion of total positivity in $G$: it is not hard to deduce from Cryer’s lemma that a matrix $x$ is totally positive if and only if both $x$ and its transpose represent totally positive elements in $G/N$.

There are $2^n - 2$ flag minors; do we really have to test all of them to verify that a point $x \in G/N$ is totally positive? The answer is no: it suffices to test positivity of $\dim(G/N) = \frac{(n-1)(n+2)}{2}$ minors; one could hardly hope for a more efficient test.

To design such tests, we will need the notion of a pseudoline arrangement. The latter is a collection of $n$ “pseudolines” each of which is a graph of a continuous function on $[0,1]$; each pair of pseudolines must have exactly one crossing point in common. (See Figure 2.) The resulting arrangement is considered up to isotopy.

We label the pseudolines 1 through $n$ by numbering their left endpoints from the bottom up. To each region $R$ of a pseudoline arrangement, with the exception of the top and the bottom regions, we associate the chamber minor $\Delta_{I(R)}$ (cf. [2]) defined as the flag minor indexed by the set $I(R)$ of labels of the pseudolines passing below $R$. The $\frac{(n-1)(n+2)}{2}$ chamber minors associated with a given pseudoline arrangement form an extended cluster; we shall see that the positivity of these minors implies that all flag minors of a given matrix are positive.

There are two types of regions: the bounded regions entirely surrounded by pseudolines, and the unbounded ones, adjacent to the left and right borders. The $2(n - 1)$ chamber minors associated with unbounded regions are called frozen: these minors are present in every arrangement. For $n = 4$, the frozen minors are $\Delta_1, \Delta_{12}, \Delta_{123}, \Delta_4, \Delta_{34}$, and $\Delta_{234}$ (cf. Figure 2).
The chamber minors corresponding to the bounded regions form the *cluster* associated with the given pseudoline arrangement. (Thus an extended cluster is a cluster plus the frozen minors.) Each cluster contains \( \binom{n-1}{2} \) chamber minors. The two pseudoline arrangements shown in Figure 2 have clusters \( \{\Delta_2, \Delta_3, \Delta_{23}\} \) and \( \{\Delta_{13}, \Delta_3, \Delta_{23}\} \), respectively.

These two clusters differ in one element only. This is because the corresponding two arrangements are related to each other by a *local move* consisting in dragging one of the pseudolines through an intersection of two others; see Figure 3. As a result of such a move, one chamber minor (namely \( e \) in Figure 3, and \( \Delta_2 \) in Figure 2) disappears (we say that this minor is *flipped*), and a new one (namely \( f \) in Figure 3 and \( \Delta_{13} \) in Figure 2) is introduced.

\[
\begin{align*}
\text{Figure 3. A local move in a pseudoline arrangement}
\end{align*}
\]

It can be shown that for a local move as in Figure 3, the chamber minors associated with the regions where the action takes place satisfy the identity

\[
e f = ac + bd.
\]

This identity is one of the generalized Plücker relations alluded to above. We call it an *exchange relation*, as the chamber minors \( e \) and \( f \) are exchanged by the local move. For the local move shown in Figure 2, the exchange relation is

\[
\Delta_2 \Delta_{13} = \Delta_{12} \Delta_3 + \Delta_1 \Delta_{23}.
\]

The new chamber minor \( f \) produced by a local move is given by a simple rational expression \( f = \frac{ac + bd}{e} \) in the chamber minors of the original arrangement. Note that this expression is *subtraction-free* (no minus signs). One can now start with a particular pseudoline arrangement, label its regions by indeterminates, then use iterated local moves (combined with the corresponding birational transformations) to generate all possible arrangements, and in doing so write all flag minors as rational expressions in the initial extended cluster. All these expressions are clearly subtraction-free, and the claim follows: if the elements of the initial extended cluster evaluate positively at a given point in \( G/N \), then so do all flag minors.

Let \( \mathcal{F} \) denote the field of rational functions in the formal variables making up the initial extended cluster. Inside \( \mathcal{F} \), the rational expressions discussed in the previous paragraph generate the subalgebra \( \mathcal{A} \) canonically isomorphic to \( \mathbb{C}[G/N] \). Notice that our construction does not explicitly involve the group \( G \): we can pretend to be unaware that we are dealing with matrices, their minors, etc. Yet
the construction produces, by design, an algebra $\mathcal{A}$ equipped with a distinguished set of generators $\Delta$ (the rational expressions corresponding to the flag minors), and thus endowed with a notion of (total) positivity.

The example of a base affine space treated above displays, in a rudimentary form, the main features of a general cluster algebra set-up. We next proceed to describing the latter on an informal level, with details to be filled in later on.

Fix a field $F$ of rational functions in several variables, some of which are designated as “frozen.” Imagine a (potentially infinite) family of equinumerous finite collections (“clusters”) of elements in $F$. (These elements, called cluster variables, can be thought of as regular functions on some “cluster variety” $X$.) Each cluster can be “extended” by adjoining the frozen variables. The (extended) clusters are the vertices of a connected regular graph in which adjacent clusters are related by birational transformations of the most simple kind, replacing an arbitrary element of a cluster by a sum of two monomials divided by the element being removed. (By a monomial we mean a product of elements of a given extended cluster.) These transformations are subtraction-free, so positivity of the elements of a cluster at a point $x \in X$ does not depend on the choice of a cluster. The birational maps between adjacent clusters are encoded by appropriate combinatorial data, and the construction is made rigid by mandating that these data are transformed (as one moves to an adjacent cluster) according to certain canonical rules. These combinatorial rules define a discrete dynamics that drives the algebraic dynamics of cluster transformations. Consequently, the choice of initial combinatorial data (the pseudoline arrangement in the example of $G/N$) determines, in a recursive fashion, the entire structure of clusters and exchanges. The corresponding cluster algebra is then defined as the subring of the ambient field $F$ generated by the elements of all extended clusters.

In the example of the base affine space, one key feature of the set-up described above is lacking: we do not always know how to exchange an element of a cluster. If a region in a pseudoline arrangement is bounded by more than three pseudolines, then the corresponding chamber minor cannot be readily flipped by a local move. For instance, how do we exchange the chamber minor $\Delta_{23}$ in Figure 2 on the left? There is in fact a “hidden” exchange relation of the form $\Delta_{23} \odot = \odot + \odot$ —but how do we guess what those $\odot$’s are?

The answer to this question will fall into our lap once we replace the language of pseudoline arrangements, too specialized for a general theory, by a more universal combinatorial language of quivers. (Using quivers somewhat restricts the generality of the cluster theory, but is general enough for the purposes of this paper.) Developing this language will take a little time—but will pay off quickly.

A quiver is a finite oriented graph. We allow multiple edges, but not loops (i.e., edges connecting a vertex to itself) or oriented 2-cycles (i.e., edges of opposite orientation connecting the same pair of vertices). We will need a slightly richer notion, with some vertices in a quiver designated as frozen. The remaining vertices are called mutable. We assume that no edges connect frozen vertices to each other. (Such edges would make no difference in what follows.)

Quivers play the role of the aforementioned combinatorial data accompanying
the clusters. We think of the vertices of a quiver as labeled by the elements of an extended cluster, so that the frozen vertices are labeled by the frozen variables, and the mutable vertices by the cluster variables.

We next describe the quiver analogue of a local move. Let \( z \) be a mutable vertex in a quiver \( Q \). The quiver mutation \( \mu_z \) transforms \( Q \) into a new quiver \( Q' = \mu_z(Q) \) via a sequence of three steps. At the first step, for each pair of directed edges \( x \to z \to y \) passing through \( z \), we introduce a new edge \( x \to y \) (unless both \( x \) and \( y \) are frozen, in which case do nothing). At the second step, we reverse the direction of all edges incident to \( z \). At the third step, we repeatedly remove oriented 2-cycles until unable to do so. See Figure 4. It is easy to check that mutating \( Q' \) at \( z' \) recovers \( Q \).

![Figure 4. A quiver mutation. Vertices \( u \) and \( v \) are frozen.](image)

Quiver mutation can be viewed as a generalization of the notion of a local move: there is a combinatorial rule associating a quiver with an arbitrary pseudoline arrangement so that local moves translate into quiver mutations. Rather than stating this rule precisely, we refer to Figure 5 and let the reader guess.

![Figure 5. The quivers corresponding to the pseudoline arrangements shown in Figure 2.](image)

The chambers of an arrangement correspond to the vertices of the associated quiver.

Let us now define cluster exchanges using the language of quivers. This turns out to be very simple. Consider a quiver \( Q \) accompanied by an extended cluster \( z \), a finite collection of algebraically independent elements in our ambient field of rational functions \( \mathcal{F} \). (Such a pair \( (Q, z) \) is called a seed.) Pick a mutable vertex \( z \). A seed mutation at \( z \) replaces \( (Q, z) \) by the seed \( (Q', z') \) whose quiver is \( Q' = \mu_z(Q) \) and whose extended cluster is \( z' = z \cup \{z'\} \setminus \{z\} \); here the new cluster variable \( z' \) is determined by the exchange relation

\[
z z' = \prod_{z \leftarrow y} y + \prod_{z \to y} y.
\]

(The products are over the edges directed at/from \( z \), respectively.) For example,
the exchange relation associated with the quiver mutation shown in Figure 4 is
\[ zz' = vx + uy; \]
applying mutation \( \mu_x \) to the quiver on the right would invoke the exchange relation \( xx' = z' + u^2 \).

Following the blueprint outlined earlier, we now define a \textit{cluster algebra} \( \mathcal{A}(Q) \) associated to an arbitrary quiver \( Q \). Assign a formal variable to each vertex of \( Q \); these variables form the initial extended cluster \( z \), and generate the ambient field \( F \). Starting with the initial seed \( (Q, z) \), repeatedly apply seed mutations in all possible directions. The cluster algebra \( \mathcal{A}(Q) \) is defined as the subring of \( F \) generated by all the elements of all extended clusters obtained by this recursive process.

Returning to our running example, we illustrate this definition by describing the cluster algebra structure in \( \mathbb{C}[\text{SL}_4/N] \). Let us start with the quiver shown on the left in Figure 5. We view the 9 variables \( \Delta_I \) labeling the vertices of this quiver as formal indeterminates (secretly, they are chamber minors). We declare the variables \( \Delta_2, \Delta_3, \) and \( \Delta_{23} \) mutable; the remaining six variables are frozen. There are three possible mutations out of this seed; we use the quiver to write the corresponding exchange relations:
\[
\begin{align*}
\Delta_2 \Delta_{13} &= \Delta_{12} \Delta_3 + \Delta_1 \Delta_{23}, \\
\Delta_3 \Delta_{24} &= \Delta_4 \Delta_{23} + \Delta_{34} \Delta_2, \\
\Delta_{23} \Omega &= \Delta_{123} \Delta_{34} \Delta_2 + \Delta_{12} \Delta_{234} \Delta_3.
\end{align*}
\]

At this point, these relations merely define \( \Delta_{13}, \Delta_{24}, \) and \( \Omega \) as rational functions in the original extended cluster. The first two relations look familiar: they correspond to the two local moves that can be applied to the given pseudoline arrangement. The third relation is new: it enables us to flip the chamber minor \( \Delta_{23} \), something we could not do before. Although the resulting cluster does not correspond to a pseudoline arrangement, we can still determine its associated quiver using the definition of quiver mutation. Continuing this process recursively \textit{ad infinitum} yields more and more extended clusters; taken together, they generate a cluster algebra.

If one interprets the elements of the initial cluster as actual flag minors, then the generators produced by this process become rational functions on the base affine space. Remarkably, all these generators are regular functions, and generate the ring of all such functions. This holds for any \( n \), resulting in a cluster algebra structure in \( \mathbb{C}[\text{SL}_n/N] \); see, e.g., \cite{Fomin02, Fomin04, Fomin05}.

In the special case \( n = 4 \), this recursive process produces a \textit{finite} number of distinct extended clusters, 14 of them to be exact. Altogether they contain 15 generators: in addition to the \( 2^4 - 2 = 14 \) flag minors \( \Delta_I \), there is a single new cluster variable
\[
\Omega = -\Delta_1 \Delta_{234} + \Delta_2 \Delta_{134}
\]
that already appeared in the third exchange relation above.

Figure 6 shows the 14 clusters for \( \mathbb{C}[\text{SL}_4/N] \) as vertices of a planar graph; note that there is one additional vertex at infinity, so that the graph should be viewed as drawn on a sphere rather than a plane. The regions are labeled by cluster variables. Each cluster consists of the three elements labeling the regions adjacent to the corresponding vertex. The edges of the graph correspond to seed mutations. The 6 frozen variables are not shown.
What do we gain by introducing a cluster algebra structure into a commutative ring that already appears well understood? One reason has been given earlier: such a structure gives rise to a well-defined notion of the (totally) positive part of the associated algebraic variety. Another reason has to do with defining a “canonical basis” in the algebra at hand; the next paragraph hints at a possible approach.

Let us call two generators of a cluster algebra compatible if they appear together in some extended cluster. A cluster monomial is a product of pairwise compatible (not necessarily distinct) generators. It is not too hard to show that in the cluster algebra \( \mathcal{A} = \mathbb{C}[\text{SL}_4 / N] \), the cluster monomials form a linear basis. This is a particular instance of the dual canonical basis of G. Lusztig (called the “upper global basis” by M. Kashiwara).

Unfortunately, the general picture (for arbitrary \( \text{SL}_n \)) is much more complicated: the cluster monomials seem to form just a part of the dual canonical (or dual semicanonical) basis; see [43]. The challenge of describing the rest of the dual canonical basis in concrete terms remains unmet.

Many other algebraic varieties of representation-theoretic importance turn out to possess a natural structure of a cluster algebra (hence the notions of positivity, cluster monomials, perhaps canonical bases, etc.). The list includes Grassmannians, flag manifolds, Schubert varieties, and double Bruhat cells in arbitrary semisimple Lie groups. See [22, 29, 34, 39, 43, 55, 56].

We conclude this section by mentioning some of the most basic structural results in the general theory of cluster algebras. The first such result is the Laurent phenomenon: the cluster variables are not merely rational functions in the elements of the initial extended cluster—all of them are in fact Laurent polynomials! We conjectured [26] that these Laurent polynomials always have positive coefficients; many instances of this conjecture have been proved (see in particular [3, 14, 48, 50]) but the general case seems out of reach at the moment.
Another basic structural result is the classification [27] of the cluster algebras of finite type, i.e., those with finitely many seeds (equivalently, finitely many generators). In the generality presented here, the classification theorem states that a cluster algebra has finite type if and only if one of its seeds has a quiver whose subquiver formed by the mutable vertices is an orientation of a disjoint union of simply-laced Dynkin diagrams. (The full-blown version of the cluster theory leads to a complete analogue of the Cartan-Killing classification.)

The combinatorial scaffolding for a cluster algebra is provided by its cluster complex, a simplicial complex whose vertices are the cluster variables, and whose maximal simplices are the clusters. In the finite type case, this simplicial complex can be identified as the dual complex of a generalized associahedron, a remarkable convex polytope [6, 28] associated with the corresponding root system. In particular, the cluster complex of finite type is homeomorphic to a sphere. This can be observed in our running example of $\mathbb{C}[\text{SL}_4/N]$: the cluster complex is the dual simplicial complex of the spherical cell complex shown in Figure 6.

3. Triangulations and laminations

Cluster algebras owe much of their appeal to the ubiquity of the combinatorial and algebraic dynamics that underlies them. A priori, one might not expect the fairly rigid axioms governing quiver mutations and exchange relations to be satisfied in a large variety of contexts. Yet this is exactly what happens. Moreover, in each instance the framework of clusters and mutations seems to arise organically rather than artificially. A case in point is discussed in this section: the classical (by now) machinery of triangulations and laminations on bordered Riemann surfaces, which goes back to W. Thurston, can be naturally recast in the language of quiver mutations. The resulting connection between combinatorial topology and cluster theory is bound to benefit both.

This section is based on the papers [20, 21], which were in turn inspired by the work of V. Fock and A. Goncharov [16, 17], M. Gekhtman, M. Shapiro, and A. Vainshtein [32, 33], and R. Penner [51].

Let $S$ be a connected oriented surface with boundary. (A few simple cases must be ruled out.) Fix a finite nonempty set $M$ of marked points in the closure of $S$. An arc in $(S, M)$ is a non-selfintersecting curve in $S$, considered up to isotopy, which connects two points in $M$, does not pass through $M$, and does not cut out an unpunctured monogon or digon. Arcs are compatible if they have non-intersecting realizations. Collections of pairwise compatible arcs are the simplices of the arc complex of $S$. The facets of this simplicial complex correspond to (ideal) triangulations. Note that these triangulations may contain self-folded triangles. See Figure 7.

The vertices of the dual graph of the arc complex correspond to the triangulations; the edges in this graph correspond to flips. A flip replaces an arc in a triangulation by another (uniquely defined) arc. Note that an edge inside a self-folded triangle cannot be flipped. The situation is akin to pseudoline arrangements,
Figure 7. The arc complex of a once-punctured triangle. Its 10 two-dimensional simplices correspond to ideal triangulations. Among them, 6 contain self-folded triangles.

which are likewise related to each other by flips (of a different kind).

This analogy goes much deeper than it might appear at first. To see that, we translate the setting into the *lingua franca* of quivers. Let us define the quiver $Q(T)$ associated to a triangulation $T$. The vertices of $Q(T)$ are labeled by the arcs in $T$. If two arcs belong to the same triangle, we connect the corresponding vertices of the quiver $Q(T)$ by an edge whose orientation is determined by the clockwise orientation of the boundary of the triangle. See Figure 8. For triangulations containing self-folded triangles, the definition is more complicated but is nevertheless completely elementary and explicit.

Figure 8. A triangulation $T$ of a once-punctured hexagon and the associated quiver $Q(T)$.

As the reader may have guessed by now, flips in ideal triangulations translate into mutations of the associated quivers. Furthermore, the quiver language suggests what we should do about the “forbidden” flips (of interior edges in self-folded
triangles): forget about triangulations and just mutate the corresponding quivers.

It is easy to check that a quiver mutation corresponding to an edge inside a self-folded triangle transforms any quiver into an isomorphic one. Another simple observation is that the number of different (up to isomorphism) quivers $Q(T)$ associated to triangulations $T$ of a given surface is finite (because the action of the mapping class group on triangulations has finitely many orbits). Combining these two observations, one concludes that any quiver $Q(T)$ associated to a triangulated surface is of finite mutation type: its iterated mutations produce finitely many distinct (non-isomorphic) quivers. In fact, as shown in [15], all connected quivers of finite mutation type, with a few exceptions, are of the form $Q(T)$, for some triangulation $T$ of some marked bordered surface $(S, M)$. (We assume that there are no frozen vertices.) The complete list of exceptions consists of (a) quivers with two vertices and more than one edge, and (b) 11 quivers listed in [10].

The construction of quivers $Q(T)$ can be generalized by involving W. Thurston’s machinery of laminations on Riemann surfaces. An integral (unbounded measured) lamination on $(S, M)$ is a finite collection of non-selfintersecting and pairwise non-intersecting curves in $S$, considered modulo isotopy. The curves in a lamination must satisfy certain constraints. In particular, each of them is either closed, or runs from boundary to boundary, or spirals into an interior marked point (a puncture). See Figure 9.

![Figure 9](image-url)

Figure 9. (a) A lamination; (b) curves not allowed in a lamination.

Let $L$ be an integral lamination, and $T$ a triangulation without self-folded triangles. For an arc $\gamma$ in $T$, the shear coordinate $b_\gamma(T, L)$ is the signed number of curves in $L$ which intersect $\gamma$ and in doing so, connect the opposite sides of the quadrilateral surrounding $\gamma$. The sign depends on which pair of opposite sides the curves connect; see Figure 10.

![Figure 10](image-url)

Figure 10. A (signed) contribution of a curve in $L$ to the shear coordinate $b_\gamma(T, L)$. 

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By a theorem of W. Thurston, the shear coordinates coordinate integral laminations in the following sense: for a fixed triangulation $T$, the map

$$L \mapsto (b_\gamma(T, L))_{\gamma \in T}$$

is a bijection between integral laminations and $\mathbb{Z}^n$.

A multi-lamination $L$ on $(S, M)$ is an arbitrary finite family of laminations. Given such $L$ and a triangulation $T$ of the surface $(S, M)$, we construct the “extended” quiver $Q(T, L)$ by adding vertices and oriented edges to $Q(T)$ as follows. For each lamination $L$ in $L$, we introduce a new vertex labeled by $L$. We then connect this vertex to each vertex in $Q(T)$, say labeled by an arc $\gamma$, by $|b_\gamma(T, L)|$ edges whose direction is determined by the sign of $b_\gamma(T, L)$. See Figure 11.

Figure 11. (a) Shear coordinates of a lamination $L$; (b) the quiver $Q(T, \{L\})$.

Amazingly, the same property as before holds: for a fixed multi-lamination $L$, a flip in a triangulation $T$ translates into the corresponding mutation in the quiver $Q(T, L)$. (The definition of the latter can be generalized to allow for self-folded triangles.) This strongly suggests the existence of a cluster algebra structure associated with any given marked surface $(S, M)$ and any multi-lamination $L$ on it.

This class of cluster algebras can be understood on several levels. On the combinatorial level, the cluster complex of such an algebra can be explicitly described in terms of tagged arcs, which are ordinary arcs adorned with very simple combinatorial decorations. This description represents the cluster complex as a finite covering space for the arc complex. The cluster complex turns out to be either contractible or homotopy equivalent to a sphere. Unlike the generalized associahedra mentioned above, these cluster complexes are usually not compact; moreover, with a few exceptions, they exhibit exponential growth. See [20].

The coordinatization theorem implies that any quiver $Q$ whose mutable part can be interpreted as a quiver $Q(T)$ corresponding to a triangulation $T$ of some marked surface $(S, M)$, there exists a (unique) multi-lamination $L$ on $(S, M)$ such that $Q = Q(T, L)$. In view of the discussion above, the cluster algebra $\mathcal{A}(Q)$ associated with such a quiver $Q$ depends only on $(S, M)$ and $L$ but not on the triangulation $T$. Consequently, one should be able to understand this cluster algebra in terms of the topology of the surface $(S, M)$ and the multi-lamination $L$. 
We illustrate this construction by returning, once again, to the example of the cluster algebra $\mathcal{A} = \mathbb{C}[[\text{SL}_4/N]]$. The mutable part of any quiver $Q$ defining this algebra (see, e.g., Figure 6) has 3 vertices, and is isomorphic to a quiver $Q(T)$ associated to a triangulation of a hexagon, i.e., a disk with 6 marked points on the boundary. Thus, we can let $(S, M)$ be a hexagon. Due to the absence of marked points in the interior of $S$, the construction simplifies considerably: there are no self-folded triangles, and the cluster complex coincides with the arc complex. The underlying combinatorics of $\mathcal{A}$ is thus modeled as follows: cluster variables correspond to arcs (that is, the diagonals of the hexagon), clusters correspond to triangulations, and exchanges correspond to flips. It remains to determine the appropriate multi-lamination $L$. This is done by interpreting the multiplicities of edges connecting the frozen vertices in $Q$ to the mutable ones as shear coordinates of laminations, and then constructing the unique laminations having those shear coordinates. The result is shown in Figure 12.

![Figure 12](image)

Figure 12. (a) Labeling the cluster variables in $\mathbb{C}[[\text{SL}_4/N]]$ by the diagonals of a hexagon. (b) Labeling the frozen variables by laminations, each consisting of a single curve.

It is natural to ask whether cluster variables in the cluster algebra associated with a multi-lamination on a bordered surface can be given an intrinsic geometric interpretation. The answer is yes: each cluster variable can be viewed as a suitably renormalized lambda length [51] (a.k.a. Penner coordinate) of the corresponding (tagged) arc. For a given arc, such a lambda length is a real function on (an appropriate generalization of) the decorated Teichmüller space for $(S, M)$; see [21] for further details. Thus in this geometric realization, the decorated Teichmüller space plays the role of the corresponding totally positive variety. This brings us back full circle to the problems discussed at the end of Section 1, namely to the challenges of understanding the stratification of a totally nonnegative variety (in this case, a compactified decorated Teichmüller space) and the singularities of its boundary.
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