Investigation on linear quadratic Gaussian control of semi-active suspension for three-axle vehicle

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Abstract
An 8-DOF three-axle vehicle model with semi-active suspension is built in this paper, of which the accuracy is verified through simulations and experiments. Based on the optimal control theory, the linear quadratic Gaussian controller for semi-active suspension is designed with 10 evaluation indicators. Considering the deficiency of linear quadratic Gaussian control weight coefficients based on experience, analytic hierarchy process is employed to determine the weight coefficients of each indicator. The control effect is analyzed through MATLAB/Simulink. The adaptability of proposed control strategy under 25 driving conditions is analyzed with different road grades and speeds. The driving condition of “70 km/h travel speed on the road of grade B” is selected, under which the comparison of vehicle responses between semi-active suspension and passive suspension is made. Results show that the vertical vibration is effectively diminished by using semi-active suspension with linear quadratic Gaussian controller. Compared with passive suspension, the riding comfort is improved and the adverse effect on handling stability is eliminated. The three-axle vehicle with semi-active suspension has good adaptability to various working conditions.

Keywords
Three-axle vehicle, semi-active suspension, linear quadratic Gaussian control, analytic hierarchy process, riding comfort

Introduction
With the evolution of highway haulage industry, people raise increasingly stringent requirements for the overall quality of commercial vehicle. Suspension system transfers all forces and moments between vehicle body and road, which significantly influences the riding comfort and handling stability of vehicle.¹⁻³ Since it is difficult to assure good performance with the fixed parameters of passive suspension, active and semi-active suspensions have attracted more attention in academic and industrial fields.⁴⁻⁵ Despite its excellent vibration isolation effect, active suspension cannot be extensively used because of its complexity and dependence on high power.⁶ Hence, semi-active suspension featuring high performance and low-power consumption has progressively turned into a focus of research.⁷⁻¹¹ Maciejewski proposed an active seat suspension with pneumatic muscles, results showed that a significant reduction in vibrations transmitted into the human body can be acquired by a slight increasing of the suspension travel[7]. In [9], a novel control strategy was applied to the horizontal seat suspension system, harmful vibrations transmitted to the driver in a wide range of excitation frequencies could be reduced. A variable damping control strategy was proposed by Gong, the vehicle ride comfort and handling stability could be substantially improved.¹⁰ The application of GPS system into MR/pneumatic suspension control is proposed by

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Morales, achieved the same roll angle levels as in a comparable passive vehicle while improving ride comfort by reducing accelerations up to 30%. 

The control strategies commonly used for semi-active suspension include optimal control, skyhook control, sliding mode control, fuzzy control, and PID control etc. Skyhook control and sliding mode control exhibit certain robustness, but they are not designed to realize the optimal system performance. Fuzzy control and Proportion-Integration-Differentiation (PID) control are adaptive to a certain extent, but the accuracy is not satisfactory. By contrast, optimal control takes into account various factors in vehicle system, the optimal overall performance can be acquired by multi-objective coordinated control. As a common sort of optimal control, LQG control is extraordinarily adaptive. The control indexes can be minimized by LQG control, by which comprehensive improvement effect could be obtained. However, the research findings of LQG control over semi-active suspension is principally focused on sedan cars and related studies are rare on multi-axle vehicles. Besides, the weight coefficients for each evaluation indexes are essential for LQG control, an analytic hierarchy process (AHP) method is used to gain the desirable weight coefficients for each indicator, which avoids subjective assume.

This work takes a three-axle vehicle as its object of study. Firstly, an 8-DOF three-axle vehicle model with semi-active suspension is built; the accuracy of the model is verified through comparison between simulation and experimental results. Secondly, the function of multi-objective LQG control for semi-active suspension is derived, 10 evaluation indexes are taken into account, and the evaluation indicator-specific weight coefficients are identified through AHP, based on which the LQG controller is designed for semi-active suspension. Thirdly, MATLAB/Simulink simulation is performed to analyze the adaptability and robustness of designed LQG controller on various driving conditions.

**Establishment and experimental verification of vehicle riding comfort model**

**Vehicle model**

An 8-DOF half vehicle model is built based on semi-active suspension, as shown in Figure 1. \( m_a, m_b, m_c, m_f, m_m, \) and \( m_r \) are the masses of cab, vehicle body, balance suspension, and three wheels. \( I_a, I_b, \) and \( I_c \) are the pitching-moment inertia of cab, vehicle body, and balance suspension. \( k_{af} \) and \( k_{ar} \) are the front and rear stiffness of cab, \( k_f \) and \( k_r \) are the leaf spring stiffness of steering suspension and balance suspension. \( k_{tf}, k_{tm}, \) and \( k_{tr} \) are the stiffness of three tires. \( c_{af} \) and \( c_{ar} \) are the front and rear damping of cab. \( F_f, F_m, \) and \( F_r \) are damping forces of shock absorbers at steering suspension and balance suspension. \( l_1 \) and \( l_2 \) are the distances from front and rear mountings.

![Figure 1. The 8-DOF semi-active suspension dynamic model of half vehicle.](image-url)
to the centroid of cab. \( l_3, l_4, l_5, l_6 \) are the distances from mountings of cab, centers of steering suspension, and balance suspension to the centroid of vehicle body. \( l_7 \) and \( l_8 \) are the distances from front and rear axles of balance suspension to its center. \( z_m, z_m, z_{tf}, z_{tm}, \) and \( z_r \) represent the vertical vibration of cab, vehicle body, and three wheels. \( \phi_m, \phi_m, \) and \( \phi_r \) denote the pitch vibration of cab, vehicle body and balance suspension. \( q_f, q_m, \) and \( q_r \) are the road excitation for three wheels.

Assuming that the basic damping of steering suspension and balance suspension are \( c_f \) and \( c_r \), the controllable damping forces of steering suspension, front-axle balance suspension, and rear-axle balance suspension are \( u_f, u_m, \) and \( u_r \), respectively.

As a rigid body structure, the pitch angles of the vertical body can be assumed as a change in a small angle range. Furthermore, the following equations are set as: \( \phi \approx \sin \phi \). Besides, some responses of vehicle are seen the same due to the symmetrical model.

The total forces \( F_f, F_m, \) and \( F_r \) of shock absorbers can be expressed as

\[
\begin{align*}
F_f &= c_f(\dot{z}_{bf} - \dot{z}_{gf}) + u_f \\
F_m &= c_r/2(\dot{z}_{br} - l_7\dot{\phi}_c - \dot{z}_{tm}) + u_m \\
F_r &= c_r/2(\dot{z}_{br} + l_8\dot{\phi}_c - \dot{z}_{tr}) + u_r
\end{align*}
\]  
(1)

When the cab pitch angle \( \phi_a \) changes within a relatively narrow range, the vertical displacements at both ends of cab are described by

\[
\begin{align*}
z_{af} &= z_a - l_3\phi_a + l_5\phi_b - z_b \\
z_{ar} &= z_a + l_3\phi_a + l_5\phi_b - z_b
\end{align*}
\]  
(2)

The displacement at the points of connection between vehicle body and suspension can be expressed as

\[
\begin{align*}
z_{bf} &= z_b - l_5\phi_b \\
z_{br} &= z_b + l_6\phi_b
\end{align*}
\]  
(3)

The displacements at both ends of balance suspension are

\[
\begin{align*}
z_{cm} &= z_{br} - l_7\phi_c = z_b + l_6\phi_b - l_7\phi_c \\
z_{cr} &= z_{br} + l_8\phi_c = z_b + l_6\phi_b + l_8\phi_c
\end{align*}
\]  
(4)

According to the D'Alembert principle, the differential equations of cab vertical and pitching motion can be derived

\[
m_a\ddot{z}_a + k_f \dot{z}_{af} + c_f \ddot{z}_{af} + k_r\dot{z}_{ar} + c_r\dot{z}_{ar} = 0
\]  
(5)

\[
I_a\dot{\phi}_a - (k_f \dot{z}_{af} + c_f \ddot{z}_{af})l_1 + (k_r \dot{z}_{ar} + c_r \ddot{z}_{ar})l_2 = 0
\]  
(6)

The equations of vertical, pitching motion at vehicle body centroid are

\[
(m_b + m_c)\ddot{z}_b - (k_f \dot{z}_{bf} - \dot{z}_{gf}) - (k_r \dot{z}_{bf} - \dot{z}_{bf}) + [k_f(\dot{z}_{bf} - \dot{z}_{gf}) + u_f + 1/2 k_r(\dot{z}_{bf} - \dot{z}_{gf}) + u_m] \\
+ \frac{1}{2} k_r(z_{cm} - z_{tm}) + c_r/2(z_{cm} - z_{tm}) + u_r = 0
\]  
(7)
\[
I_h \ddot{\varphi}_h + (k_{bf} \varphi_{bf} + c_{bf} \dot{\varphi}_{bf}) I_3 + (k_{ar} \varphi_{ar} + c_{ar} \dot{\varphi}_{ar}) I_4 - [k_f(z_{bf} - z_{gf}) + c_f(\dot{z}_{bf} - \dot{z}_{gf}) + u_f] I_5 \\
+ \left[ \frac{1}{2} k_r (\dot{z}_{cm} - \dot{z}_{im}) + \frac{1}{2} c_r (\ddot{z}_{cm} - \ddot{z}_{im}) + u_m \right] I_7 + \left[ \frac{1}{2} k_r (\dot{z}_{cr} - \dot{z}_{ir}) + \frac{1}{2} c_r (\ddot{z}_{cr} - \ddot{z}_{ir}) + u_r \right] I_8 = 0
\]

The equation of pitching motion of balance suspension is
\[
I_c \ddot{\varphi}_c - \left[ \frac{1}{2} k_r (\dot{z}_{cm} - \dot{z}_{im}) + \frac{1}{2} c_r (\ddot{z}_{cm} - \ddot{z}_{im}) + u_m \right] I_7 + \left[ \frac{1}{2} k_r (\dot{z}_{cr} - \dot{z}_{ir}) + \frac{1}{2} c_r (\ddot{z}_{cr} - \ddot{z}_{ir}) + u_r \right] I_8 = 0
\]

The equations of vertical motion for the front, middle, and rear wheels are
\[
m_j \ddot{z}_j - \left[ k_f(z_{bf} - z_{gf}) + c_f(\dot{z}_{bf} - \dot{z}_{gf}) + u_f \right] + k_{gf} (z_{gf} - q_{gf}) = 0
\]
\[
m_m \ddot{z}_m - \left[ \frac{1}{2} k_r (\dot{z}_{cm} - \dot{z}_{im}) + \frac{1}{2} c_r (\ddot{z}_{cm} - \ddot{z}_{im}) + u_m \right] + k_{im} (z_{im} - q_m) = 0
\]
\[
m_r \ddot{z}_r - \left[ \frac{1}{2} k_r (\dot{z}_{cr} - \dot{z}_{ir}) + \frac{1}{2} c_r (\ddot{z}_{cr} - \ddot{z}_{ir}) + u_r \right] + k_{ir} (z_{ir} - q_r) = 0
\]

**Road model**

The statistical characteristics of pavement roughness are frequently expressed by power spectral density function \( G_q(n) \)

\[
G_q(n) = G_q(n_0) \left( \frac{n}{n_0} \right)^{-\omega}
\]

where \( n \) is the spatial frequency, \( n_0 \) indicates the standard spatial frequency, \( G_q(n_0) \) is the road roughness coefficient, and \( \omega \) is the frequency index, which determines the frequency structure of road power spectral density.

When \( \omega = 2 \)

\[
G_q(f) = G_q(n_0) \left( \frac{n_0}{f} \right)^2
\]

The velocity power spectral density function is

\[
\dot{G}_q(f) = 4\pi^2 G_q(n_0)(n_0)^2 \nu
\]

Low-pass filter white noise signal of Gaussian distribution is used to simulate the road excitation\(^{24}\)

\[
\begin{align*}
\dot{q}_f(t) &= -2\pi f_0 q_f(t) + 2\pi \sqrt{G_q(n_0)\nu} w_f(t) \\
\dot{q}_m(t) &= -2\pi f_0 q_m(t) + 2\pi \sqrt{G_q(n_0)\nu} w_m(t) \\
\dot{q}_r(t) &= -2\pi f_0 q_r(t) + 2\pi \sqrt{G_q(n_0)\nu} w_r(t)
\end{align*}
\]

where, \( f_0 \) is lower cutoff frequency, which is set equal to 0.1; \( G_q(n_0) \) represents road irregularity coefficient; \( n_0 \) stands for reference spatial frequency, \( n_0 = 0.1 \text{ m}^{-1} \); \( \nu \) means the travel speed of vehicle; \( w_f(t), w_m(t), \) and \( w_r(t) \) are road input white noise signals of front wheel, middle wheel, and rear wheel, respectively. For the three-axle vehicles, the time lag of road excitation for middle wheel and rear wheel is \((l_5 + l_6 - l_7)/v\) and \((l_5 + l_6 + l_8)/v\), respectively.
Experimental verification of model

In order to reduce difficulties and workload in analysis and calculation, scholars built various models respectively as to vertical dynamics, lateral dynamics, and longitudinal dynamics neglecting certain trivial movements. However, the model verification has been rarely mentioned in current studies, which makes the proposed control effects less convincing, especially for heavy-duty vehicles.

A Dongfeng commercial vehicle is researched in this work, the vehicle parameters are listed in Table 1. In order to add the actual value of the proposed model and verify the rationality of vehicle parameter selection, a field test on vehicle dynamic responses was performed at the Daqing–Guangzhou Expressway in Hebei province, China. Figure 2 shows the details of experimental testing.

A riding comfort model is built based on MATLAB/Simulink, which is shown in Figure 3. The comparison of simulations and experiments under 50 km/h and B-class road surface is shown in Figure 4 and Table 2.

It can be seen from Figure 4 that the simulation results of the three-axle vehicle are basically consistent with the experimental test results at various points, especially the vehicle body centroid. The root mean square (RMS) values of acceleration between simulation and test results are close to each other, the maximum values of

| Symbol | Value | Symbol | Value | Symbol | Value |
|--------|-------|--------|-------|--------|-------|
| m_a/kg | 557.5 | c_w/(N·s·m⁻¹) | 3620 | l_1/m | 1.2 |
| l_a/(kg·m²) | 810 | c_f/(N·s·m⁻¹) | 50,636 | l_2/m | 1.0 |
| m_f/kg | 11523 | c_r/(N·s·m⁻¹) | 25,320 | l_3/m | 4.0 |
| l_f/(kg·m²) | 55502 | k_a/(N·m⁻¹) | 36,230 | l_4/m | 1.8 |
| m_b/kg | 117 | k_c/(N·m⁻¹) | 36,230 | l_5/m | 3.64 |
| l_b/(kg·m²) | 351 | k_d/(N·m⁻¹) | 251,380 | l_6/m | 2.71 |
| m_t/kg | 412 | k_e/(N·m⁻¹) | 20,64,000 | l_7/m | 0.65 |
| m_tr/kg | 676 | k_f/(N·m⁻¹) | 874,590 | l_8/m | 0.65 |
| m_tr/kg | 676 | k_v/(N·m⁻¹) | 17,49,180 | c_a/(N·m⁻¹) | 3620 |

Figure 2. Site conditions of experimental test. (a) test vehicle, (b) acquisition instrument, (c) measuring point in cab, and (d) measuring point of front wheel.
acceleration shown in Figure 4 are within a reasonable range, which illustrates that the combination of vehicle and road model can generate an ideal working condition. The percentage difference of four measuring points, cab, vehicle body centroid, front wheel, and middle wheel is 15.85%, 4.99%, 16.58%, and 13.16%, respectively. Although there exist some error, the dynamic model is able to substantially reflect the overall vibration properties of the vehicle. Therefore, the model can be used for simulation analysis and control study for riding comfort.

**Design of LQG controller for semi-active suspension**

**System state equation**

Cab acceleration, cab pitching angular acceleration, vehicle body acceleration, vehicle body pitching angular acceleration, dynamic displacements of suspensions, and dynamic deflections (dynamic loads) of tires are essential for evaluating the riding comfort performance of three-axle vehicles. In addition, dynamic loads are also the evaluation index of tire-road contact safety and road friendliness. Hence, the state variable \( X \) selected for system is

\[
X = [x_1, x_2, x_3, \cdots, x_{16}]^T
\]

where \( x_1 = \ddot{z}_a, x_2 = \dot{\phi}_a, x_3 = \ddot{\phi}_b, x_4 = \dot{\phi}_c, x_5 = \ddot{\phi}_c, x_6 = \ddot{z}_f, x_7 = \dot{z}_m, x_8 = \ddot{z}_m, x_9 = z_{tf}, x_{10} = z_{tr}, x_{11} = z_{bf} - z_{tf}, x_{12} = z_{cm} - z_{tm}, x_{13} = z_{ct} - z_{tr}, x_{14} = \ddot{q}_f, x_{15} = z_{mf} - q_{tf}, x_{16} = z_{it} - q_r. \) In this work, all of the evaluation indicators above can be obtained by the 8-DOF model.

Perform full-state observation of the system, its output variable \( Y = X \). Considering the motion differential equation of vehicle system and the road input model, the state equation of the system can be expressed as

\[
\begin{aligned}
\dot{X} &= AX + BU + FW \\
Y &= CX + DU
\end{aligned}
\] (17)
Figure 4. Comparison of simulation and test results. (a) Simulated cab acceleration. (b) Tested cab acceleration. (c) Simulated vehicle body acceleration. (d) Tested vehicle body acceleration. (e) Simulated front wheel acceleration. (f) Tested front wheel acceleration. (g) Simulated middle wheel acceleration. (h) Tested middle wheel acceleration.

Table 2. Comparison of simulation and test in RMS of acceleration.

| Measuring points          | RMS of acceleration (m/s²) | Simulation | Test  | Error rate (%) |
|---------------------------|----------------------------|------------|-------|----------------|
| Cab (driver’s seat)       | 0.1149                     | 0.0978     |       | 15.85          |
| Vehicle body centroid     | 0.2131                     | 0.2243     |       | 4.99           |
| Front wheel (left)        | 0.9449                     | 1.1328     |       | 16.58          |
| Middle wheel (left)       | 1.1099                     | 0.9808     |       | 13.16          |
where $X$ is state variable. $U = [u_x, u_m, u_e]^T$, which means control force matrix. $W = [\dot{q}_1(t), \dot{q}_2(t), \dot{q}_3(t)]^T$ represents road input matrix. $A$ stands for system matrix. $B$ stands for control matrix. $F = [O_{13} \times E_{3 \times 3}]^T$ means perturbation matrix. $C = E_{16 \times 16}$ is output matrix. $D = O_{16 \times 3}$ is transfer matrix. In this paper, $E$ represents identity matrix, and $O$ represents all-zero matrix.

Here, $A = [A_1, A_2]$

\[
A_1 = \begin{bmatrix}
-\frac{c_{af} + c_{ar}}{m_a} & \frac{c_{af}l_1 - c_{ar}l_2}{m_a} & \frac{c_{af} + c_{ar}}{m_a} & \frac{c_{af}l_3 - c_{ar}l_4}{m_a} & 0 & 0 & 0 & 0 \\
\frac{c_{af}l_1 - c_{ar}l_2}{m_a} & -\frac{c_{af}l_1 - c_{ar}l_2}{m_a} & \frac{c_{af}l_1 - c_{ar}l_2}{m_a} & \frac{c_{af}l_1 - c_{ar}l_2}{m_a} & 0 & 0 & 0 & 0 \\
\frac{c_{af} + c_{ar}}{m_b} & -\frac{c_{af} + c_{ar}}{m_b} & \frac{c_{af} + c_{ar}}{m_b} & \frac{c_{af} + c_{ar}}{m_b} & 0 & 0 & 0 & 0 \\
\frac{c_{af} + c_{ar}}{m_c} & -\frac{c_{af} + c_{ar}}{m_c} & \frac{c_{af} + c_{ar}}{m_c} & \frac{c_{af} + c_{ar}}{m_c} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{c_r(l_1 - l_b)}{2l_c} & \frac{c_r(l_1 - l_b)}{2l_c} & 0 & 0 & \frac{c_r}{2l_c} & \frac{c_r}{2l_c} \\
0 & 0 & \frac{c_r}{2l_c} & \frac{c_r}{2l_c} & 0 & 0 & \frac{c_r}{2l_c} & \frac{c_r}{2l_c} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -l_i & -1 & l_i & 0 & 0 & 0 & 0 \\
1 & l_i & -1 & l_i & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & l_i & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & l_i & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
-\frac{k_{af}}{m_a} & \frac{k_{af}l_1 - k_{ar}l_2}{m_a} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{k_{af}l_1 - k_{ar}l_2}{m_a} & -\frac{k_{af}l_1 - k_{ar}l_2}{m_a} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{k_{af} + k_{ar}}{m_b} & -\frac{k_{af} + k_{ar}}{m_b} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{k_{af} + k_{ar}}{m_c} & -\frac{k_{af} + k_{ar}}{m_c} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{k_f}{m_f} & 0 & 0 & 0 & \frac{k_f}{m_f} & 0 \\
0 & 0 & 0 & \frac{k_r}{2m_m} & 0 & 0 & \frac{k_r}{2m_m} & 0 \\
0 & 0 & 0 & 0 & \frac{k_r}{2m_m} & 0 & \frac{k_r}{2m_m} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{k_r}{2m_m} & \frac{k_r}{2m_m} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-k_{af}}{m_a} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{k_r}{2m_m} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{k_r}{2m_m} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{k_r}{2m_m} & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & -\frac{1}{m_a + m_c} & \frac{1}{m_c} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{m_a + m_c} & \frac{1}{m_c} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{m_a + m_c} & \frac{1}{m_c} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{m_a + m_c} & \frac{1}{m_c} & 0 & 0 & 0 & 0
\end{bmatrix}
\]

where $x_1 = \frac{c_{af} + c_{ar} + c_f + c_r}{m_a}$, $x_2 = \frac{c_{af}l_3 + c_{ar}l_4 + c_fl_5 - c_fl_6}{m_a}$, $x_3 = \frac{c_{af}l_3^2 + c_{ar}l_4^2 + c_fl_5^2 + c_f^2}{m_a}$.
LQG controller

According to the large mass, high centroid and long-distance travel of three-axle vehicles, 10 evaluating indicators are selected in terms of riding comfort. The first four indicators are vertical acceleration, pitching acceleration of cab, vertical acceleration, pitching acceleration of vehicle body, which indicate the vibration of goods. The other indicators include displacements of steering and balance suspensions, dynamic loads of front wheel, middle wheel and rear wheel, which indicate the vehicle safety and tire-ground contact.

The function of aggregative performance indicator for LQG controller is determined based on modern control theory in consideration of above-noted evaluation indicators and control factors

\[
J = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( q_1 z_a^2 + q_2 \dot{\varphi}_a^2 + q_3 \dot{\varphi}_b^2 + q_4 (z_\text{bf} - z_g)^2 
+ q_5 (z_\text{cm} - z_m)^2 + q_7 (z_{cr} - z_r)^2 + q_8 (z_\text{tf} - q_\text{tf})^2 + q_9 (z_\text{tm} - q_m)^2 + q_{10} (z_\text{tr} - q_r)^2 \right) dt
\]  

where \( q_1 \) represents the weight coefficient of cab vertical acceleration, and is set to 1. \( q_2, q_3, \) and \( q_4 \) stand for the weight coefficient of pitching angular acceleration of cab, vertical acceleration, and pitching angular acceleration of vehicle body. \( q_5, q_6, \) and \( q_7 \) are weight coefficients of dynamic displacements of suspensions. \( q_8, q_9, \) and \( q_{10} \) are weight coefficients of tire dynamic loads of the three wheels, where the dynamic load is replaced with tire dynamic deflection.

Equation (15) can be changed into the standard quadratic form

\[
J = \lim_{T \to \infty} \frac{1}{T} \int_0^T (X^T Q X + U^T R U + 2X^T N U) dt
\]

where \( Q \) means state weighting matrix; \( R \) represents control weighting matrix; \( N \) stands for cross term weighting matrix. The elements of the matrix are shown below

According to the extremum principle, the optimal control force is

\[
U(t) = -R^{-1}(B^T P + N^T)X(t) = -KX(t)
\]

where \( K \) is the optimal feedback gain matrix; \( P \) means constant positive definite matrix. It can be determined with Riccati algebraic equation

\[
PA + A^T P - (PB + N)R^{-1}(B^T P + N^T) + Q = 0
\]

Determine indicator weight coefficient

As a method for multi-objective planning and decision-making, AHP is employed to determine the weight coefficients of each evaluation indicators so as to effectively avoid repeating pilot calculation and improve the adaptability of LQG control.\(^{25}\) The computational procedure is as follows:

1. **Equi-scale quantification.** Equi-scale quantification is needed since it is impossible to make direct comparison between performance indicators due to their significance difference in the order of magnitude. LQG control effect of semi-active suspension is normally evaluated based on the mean square \( \sigma_i^2 \) of performances under corresponding operating condition of passive suspension. The equi-scale quantification scale factors \( \beta_i \) for performance indicators can be determined by the equation (19)

\[
\sigma_i^2 \cdot \beta_1 = \sigma_1^2 \cdot \beta_i
\]

where the equi-scale quantification scale factor \( \beta_1 \) of cab acceleration mean square \( \sigma_1^2 \) is set to 1 as the reference for other indicators.
2. Determine weight coefficient

(a) Build judgment matrix $H$. The importance ratio $h_{ij}$ between indicators is shown in Table 3. If its relative importance is between the two ratios, it takes 2, 4, 6, and 8.

Build judgment matrix $H$ based on the importance of indicators

$$H = (h_{ij})_{n \times n} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ 1/h_{12} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/h_{1n} & 1/h_{2n} & \cdots & h_{nn} \end{bmatrix}$$

(b) Determine the weight order vector $W$ of each indicator. Calculate the element multiplication vector $M$ for line $H$ of judgment matrix and its $n$th root vector $\bar{W}$ using equations (20)–(21) based on literature, and calculate the weight order vector $W$ of $\bar{W}$ based on equation (22), it’s the weight coefficient of each indicator.

$$\begin{cases} M = [M_1, M_2, \cdots, M_n]^T \\ M_k = \prod_{j=1}^{n} h_{ij}, \ (i, j = 1, 2, \cdots, n) \end{cases}$$

$$\begin{cases} \bar{W} = [\bar{W}_1, \bar{W}_2, \cdots, \bar{W}_n]^T \\ \bar{W}_i = \sqrt[n]{M_i}, \ (i, j = 1, 2, \cdots, n) \end{cases}$$

$$W = \bar{W} / \sum_{i=1}^{n} W_i, \ (i, j = 1, 2, \cdots, n)$$

(c) Check the consistency of judgment matrix. The random consistency ratio $CR$ is calculated using equation (23)

$$\begin{cases} CR = \frac{\lambda_{\text{max}} - n}{RI(n - 1)} \\ \lambda_{\text{max}} = \frac{\sum_{i=1}^{n} (HW)^i}{nW_1}, \ (i, j = 1, 2, \cdots, n) \end{cases}$$

where $\lambda_{\text{max}}$ and $RI$ represent the maximum eigenvalue and random consistency index of judgment matrix $H$. When $n = 10$, $RI$ takes 1.56. When $CR < 0.1$, it means consistency check is passed; otherwise, consistency checking shall be performed by the method stated in Ref. 27

3. Determine subjective weighting scale factor. The subjective weighting scale factor $\gamma_1$ of cab acceleration is set to 1, and the subjective weighting scale factors of other indicators are determined by

$$W_i / \gamma_1 = W_i / \gamma_1, \ (i = 1, 2, \cdots n)$$

---

**Table 3. Comparison values of importance between indicators.**

| $h_{ij}$ | Equally | Moderately | Strongly | Very | Extremely |
|----------|---------|------------|----------|------|-----------|
| $h_{ij}$ | 1       | 3          | 5        | 7    | 9         |

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4. **Determine the final weight coefficient.** The final weight coefficients $q_i$ of LQG control are determined with equation (25) based on the equi-scale quantification scale factors $\beta_i$ and subjective weighting scale factors $\gamma_i$ of optimal control evaluation indicators.

$$q_i = \beta_i \cdot \gamma_i, \quad (i = 1, 2, \ldots, n)$$ (28)

**Calculate optimal feedback gain matrix**

The cab acceleration and its pitching angular acceleration are the most important indexes for evaluating riding comfort, the vehicle body acceleration and pitching angular acceleration are relatively important, the dynamic displacements of suspensions and the tire dynamic loads are relatively not important. On this basis, judgment matrix $H$ is built with AHP method

$$H = \begin{bmatrix}
1 & 1 & 2 & 2 & 3 & 3 & 3 & 5 & 5 & 5 \\
1 & 1 & 2 & 2 & 3 & 3 & 3 & 5 & 5 & 5 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
\frac{2}{1} & \frac{2}{2} & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
\frac{1}{2} & \frac{1}{2} & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 2 \\
\frac{1}{3} & \frac{1}{3} & \frac{2}{2} & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\frac{1}{3} & \frac{3}{3} & \frac{2}{2} & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{3} & \frac{3}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\
\frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{3} & \frac{3}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\
\frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1}
\end{bmatrix}$$

The weight coefficient of each performance indicator is calculated by the procedures given in Determine indicator weight coefficient section. Matrices $A$, $B$, $Q$, $R$, and $N$ are obtained by substituting the weight coefficients and vehicle model parameters into equation (16)–(18). The value of optimal feedback gain matrix $K$ can be obtained with equation (26) by calling the quadratic optimal control function $\text{lqr}$ from MATLAB Toolbox.

$$[K, P, E] = \text{lqr}(A, B, Q, R, N)$$ (29)

**Simulation analysis**

In this paper, the control effects and adaptabilities of semi-active suspension under 25 driving conditions are analyzed. The road grades A, B, C, D, and E and the vehicle speeds are 10, 30, 50, 70, and 90 km/h, respectively. Furthermore, the effect of the designed LQG controller is testified by comparing passive suspension and semi-active suspension responses at 70 km/h driving condition on the road of grade B.

**Analysis of LQG control adaptability**

Figure 5 shows the effects of proposed LQG control on vehicle responses under different road surface roughness and vehicle speeds. It can be seen from Figure 5(a) that the control effect on cab acceleration under each operating condition is approx 80%, which implies that the proposed method can improve the riding comfort with good adaptability. Figure 5(c) shows that the control effects on vehicle body acceleration range from 20.02% to 22.35%. Accordingly, the trend of pitching angular acceleration is consistent with acceleration, which can be
Figure 5. Control effects of each performance indicators under 25 driving conditions. (a) Control effect of cab acceleration (%). (b) Control effect of cab pitching angular acceleration (%). (c) Control effect of vehicle body acceleration. (d) Control effect of vehicle body pitching angular acceleration. (e) Control effect of displacement of steering suspension. (f) Control effect of front axle displacement of balance suspension. (g) Control effect of rear axle displacement of balance suspension. (h) Control effect of front tire dynamic load. (i) Control effect of middle tire dynamic load. (j) Control effect of rear tire dynamic load.

obviously observed in Figure 5(b) and (d). Figure 5(e), (f), and (g) indicates that the control effects on dynamic displacements of steering suspension, front-axle balance suspension and rear-axle balance suspension are 70.02–71.20%, 53.99–55.01%, and 56.62–57.80%, respectively. Figure 5(h), (i), and (j) shows that the control effects on tire dynamic loads of front wheel, middle wheel, and rear wheel are 65.70–66.27%, 56.78–57.74%, and 53.75–54.51%, respectively. Remarkable improvements are observed on the overall vehicle system. Besides, AHP-based semi-active suspension system with LQG control is favorably adaptable to different driving conditions.

Instance analysis

The effectiveness of designed LQG controller is probed based on simulation results obtained at 70 km/h and the road of grade B. The comparison of performance indicators between semi-active suspension and passive
obviously observed in Figure 5(b) and (d). Figure 5(e), (f), and (g) indicates that the control effects on dynamic displacements of steering suspension, front-axle balance suspension and rear-axle balance suspension are 70.02–71.20%, 53.99–55.01%, and 56.62–57.80%, respectively. Figure 5(h), (i), and (j) shows that the control effects on tire dynamic loads of front wheel, middle wheel, and rear wheel are 65.70–66.27%, 56.78–57.74%, and 53.75–54.51%, respectively. Remarkable improvements are observed on the overall vehicle system. Besides, AHP-based semi-active suspension system with LQG control is favorably adaptable to different driving conditions.

**Instance analysis**

The effectiveness of designed LQG controller is probed based on simulation results obtained at 70 km/h and the road of grade B. The comparison of performance indicators between semi-active suspension and passive
suspension is made, which is shown in Figure 6 and Table 4. Figure 7 shows the control forces of semi-active suspension.

As shown in Figure 6(a) and (b), semi-active suspension can effectively diminish the vibration of cab and vehicle body, which means better riding comfort is assured, the maximum absolute value of the cab acceleration is within [0, 0.1]. On the contrast, the maximum absolute value of cab acceleration with passive suspension is beyond 0.2, which can be observed in Figure 6(a). As can be clearly seen in Figure 6(c) and (d), the obvious reduction in dynamic displacements of each suspension indicates that controller can effectively reduce shock from the collision between suspension and stop block during the travel of vehicle. The maximal displacement of steering suspension with the proposed control is \(-0.56\) mm; the maximal displacement with passive suspension is 3.88 mm. And from the well-accepted formula of \(F = k \Delta x\), it can be concluded that better riding comfort is obtained by a smaller control force. From Figure 6(e) and (f), dynamic load on each tire is reduced to a certain extent, which means the tire-road contact safety and road friendliness are also improved. As shown in Figure 7(a) and (b), the control forces for steering suspension and front-axle balance suspension are within a reasonable range, the maximal control forces of both are less than 3.2 kN.

As shown in Table 4, all performance indicators of vehicle with semi-active LQG control decrease significantly compared with those with passive control. The suppressions of cab acceleration and cab pitching angular acceleration are the most obvious, while that of vehicle body acceleration and its pitching angular acceleration are not so apparent due to the complex structure, bigger inertia and longer wheelbase of the three-axle vehicle. In addition to the improvement in riding comfort, the handling stability and tire-ground contact are well improved. Summarily, the semi-active suspension with LQG controller shows excellent control effects on riding comfort and handling stability of the vehicle.

### Table 4. Comparison of RMS for semi-active and passive suspensions.

| Indicators            | Passive       | Semi-active  | Effect (%) |
|-----------------------|---------------|--------------|------------|
| \(\ddot{z}_g/(m \cdot s^{-2})\) | 0.1733        | 0.0346       | -80.09     |
| \(\ddot{z}_b/(m \cdot s^{-2})\) | 0.0789        | 0.0145       | -81.36     |
| \(\ddot{z}_b/(m \cdot s^{-2})\) | 0.2339        | 0.1843       | -21.07     |
| \(\ddot{z}_b/(m \cdot s^{-2})\) | 0.1385        | 0.0931       | -32.77     |
| \(z_M - z_{tr}/mm\)  | 0.9703        | 0.2821       | -70.95     |
| \(z_M - z_{tr}/mm\)  | 1.3043        | 0.5919       | -54.61     |
| \(z_M - z_{tr}/mm\)  | 1.3702        | 0.5871       | -57.21     |
| TDLg/kN               | 0.7455        | 0.2535       | -66.02     |
| TDLg/kN               | 1.6416        | 0.7037       | -57.08     |
| TDLg/kN               | 1.9787        | 0.8996       | -54.18     |

**Figure 7.** Semi-active suspension control forces. (a) Control force of steering suspension. (b) Control force of front axle of balance suspension.
Conclusion
The main conclusions are as follows: (1) The presented 8-DOF three-axle vehicle model is accurate enough to study vehicle riding comfort. (2) The semi-active suspension with LQG controller provides vehicle with favorable robustness and adaptability under different driving conditions. (3) The riding comfort, tire-road contact safety, and road friendliness of three-axle vehicle can be improved by LQG controller using AHP method, and the control forces are within a reasonable range. As an extension of this work, the proposed control algorithm will be used to research the multi-objective integrated control of chassis for balance of riding comfort, tire-road contact safety, road friendliness, and handling stability.

Declaration of Conflicting Interests
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