Precise Derivations of Radiative Properties of Porous Media Using Renewal Theory

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Abstract

This work uses the mathematical machinery of Renewal/Ruin (surplus risk) theory to derive preliminary explicit estimations for the radiative properties of dilute and disperse porous media otherwise only computable accurately with Monte Carlo Ray Tracing (MCRT) simulations. Although random walk and Lévy processes have been extensively used for modeling diffuse processes in various transport problems and porous media modeling, relevance to radiation heat transfer is scarce, as opposed to other problems such as probe diffusion and permeability modeling. Furthermore, closed-form derivations that lead to tangible variance reduction in MCRT are widely missing. The particular angle of surplus risk theory provides a richer apparatus to derive directly related quantities. To the best of the authors’ knowledge, the current work is the only work relating the surplus risk theory derivations to explicit computations of ray tracing results in porous media. The paper contains mathematical derivations of the radiation heat transfer estimates using the extracted machinery along with proofs and numerical validation using MCRT.

1. Introduction

At present, Monte Carlo Ray Tracing (MCRT) simulations are the most reliable non-experimental means for predicting radiative properties of heterogeneous structures[1]–[9]; such predictions are otherwise only computable directly via extremely limited and costly experiments[10]–[16] or approximate models rooted in extensive measurements[17]–[23]. Though only valid in geometric optics regimes where particle sizes and interspacing are large (so diffraction and dependent scattering are negligible), MCRT has been persistently invoked and improved over the past few decades [24]–[46]. However, due to stringent precision requirements, the high computational cost of setup and execution, as well as parameter-dependent uncertainty, the required time and computational cost of MC simulations are prohibitively high for obtaining accurate estimates[47]–[50].

The efforts to minimize MCRT cost fall under one of two categories: statistical characterization and variance reduction. In the face of dispersity and heterogeneity of geometry and material properties, the former attempts to treat the environment as an equivalent continuous and homogenous scattering domain[51]–[59] and insert the effective radiative statistics into the radiative transfer equations[60]–[62](RTE) solving for all modes of extinction, or as sampling seeds in MCRT. This way the need for full incorporation of the porous structure is obviated[2], [63], [64]. There is a more formal mathematical approach to the statistical treatment of heterogeneous materials called the “homogenization” technique. Rooted in vibration theory, it falls under the general umbrella of multiscale methods[65]–[69], though its applicability is limited to highly regular configurations.
Although not always directly intended for MCRT, the theories of mean-beam-length[70]–[72] and the study of configuration or view/shape/angle factors[60], [63], [73]–[75] are within the first category, attempting to make statistical sense of complex geometry by producing equivalent distribution functions. The work along these lines has also studied correction for the co-dependent scattering effects via the use of dependent factors[59], [70], [76] which, along with the incorporation of ray transport distance at dispersed media has led to uncommon RTE solutions such as the Dependent Included Discrete Ordinate Method (DI-DOM). For higher accuracy, statistical homogenizing can also be boosted with experimental measurements, e.g., through inverse analysis based on discrete ordinates radiative models[77]. The method of Radiation Distribution Function Identification (RDFI) has been proposed for facilitating MCRT in statistically homogeneous, isotropic porous media modeled as dispersed overlapping spheres[78] and has been extended to real[79]–[81], non-homogeneous and anisotropic beds[82], [83]. In RDFI, the porous medium is characterized by a cumulated extinction coefficient (CEC) distribution function, obtained by minimizing the difference between absorption and extinction distributions rendered by MCRT on a virtual medium and the corresponding experimental quantities. With the advent of computed tomography and imaging techniques, MCRT has been made more accurate and efficient via the so-called upscaling methodology[84], [85], where the characteristic functions of the radiative properties are estimated either through morphological tomography data or from a representative elementary volume (REV) of the medium treated as semi-transparent[20]–[22].

Variance reduction efforts have been more limited and sporadic, and often only applicable when there is prior knowledge of the structure of the radiation solution. The methods are generally either split-based or importance-sampling-based[86]–[88]. Some of the noteworthy methods include the binary spatial partitioning (BSP) grouping algorithm[89], probabilistic multiple-rays tracing technique[90], and the delta-scattering method[88], [91]. Analytical methods are also effective in reducing the variance of MCRT by providing rapid initial estimates. Albeit limited and nonuniversal, example work along those lines are those based on the Spatial Averaging Theorem (SAT)[92]–[94], as well as custom analytical and empirical models constructed from MCRT ground truth data[24], [25], [76], [95]–[97], though the scopes are limited to specific packing distributions and/or material compositions. Recent data-driven endeavors to use machine learning for MCRT estimations[98]–[104] can also be viewed as unorthodox branches of variance reduction efforts, though the work is in its very infancy.

The present work offers a novel approach to variance reduction based on the mathematics of Renewal/Ruin theory (Cramér-Lundberg), providing more generalizable and accurate results than the existing variance reduction schemes. The essence of the models is in approximating the travelling ray within a porous bed with a one- or two-sided renewal random walk process. Using the existing machinery of surplus risk theory and ruin analysis, we can then derive close upper bounds and explicit tight estimates for radiative metrics. Although random walk and Lévy processes have been extensively used for modeling diffuse processes[105]–[109] in various transport problems and porous media modeling, there are two shortcomings: (i) relevance to radiation heat transfer is scarce, as opposed to other problems such as probe diffusion and permeability modeling[110]–[113], (ii) closed-form derivations that lead to tangible variance reduction in MCRT are missing. The particular angle of surplus risk theory provides a richer apparatus to derive directly related quantities. To the best of the authors knowledge, the current work is the only work relating the surplus risk theory derivations to explicit computations of ray tracing results in porous media.
Summary of Results
We leverage key theorems of Ruin theory based on Markovian processes[114] to derive explicit characterizations of power extinction and reflectivity as would be derived by MCRT in flat one- and two-sided polydisperse porous media (Fig. 1). Considering a traveling beam that enters a flat (Beerian) porous medium and experiences independent mean free path and subsequent reflection events with known statistics $F_\ell(\cdot)$ (cumulative distribution function), the objective is to derive reflection, absorption, and transmission fractions. Assuming the media is an infinitely horizontally stretched slab, the sequence of vertical position (depth) of the ray at the time of scattering events can be modeled as a 1D Markov chain random walk. Existing results from Ruin theory (e.g., [114]) provide explicit calculations for the joint moment generating function of Ruin time random variables, including stopping time (see Lemmas 2 and 4 in Section 3 for a summary of relevant extractions). By invoking a series of techniques including Jensen[115] and Cauchy-Schwarz inequalities[116], Delta method[117], and Wald’s identity[118], we successfully relate the moment generating functions to approximate collective power dissipation and reflection in a statistical Beerian and non-Beerian porous media, both for one-sided and two-sided beds (See Theorems 2 and 3). We compare the derived analytical results with those obtained from exhaustive Monte Carlo ray tracing simulations and report astonishingly close matches, serving as validation for our theoretical results.

2. Mathematical Derivations
2.1. Part I. One-sided medium.

Definition 1. A one-sided $(x,p,F_\ell)$-renewal process (See Figure 2) is a random walk that starts at point $x \geq 0$, and takes consecutive steps of sizes $Y_i, i > 0$ with cumulative distribution $F_\ell$, where the step is to the right (increasing) with probability $p$ and to the left (decreasing) with probability $q = 1 - p$, and terminates as soon as it becomes less than 0. $T_x$ and $Z_x$ are defined as random variables describing the stopping (exit) time and overshoot at the stopping time respectively. In other words:
\[ T_x = \min\{T \geq 0 \mid \sum_{i=1}^{T} s_i Y_i < 0 \} \tag{1} \]

\[ Z_x = -\sum_{i=1}^{T_x} s_i Y_i - x, \tag{2} \]

where \( s_i \in \{1, -1\} \) is the side of the \( i^{th} \) step.

**Figure 2.** One-sided renewal process.

**Lemma 1.** In a one-sided flat polydisperse porous medium with dissipation factor \( \beta \) and opaque particles, the following upper bound always holds for the reflected (non-absorbed) fraction \( \rho \) of the incident power:

\[ \rho \leq \rho^u = \mathbb{E}_x \left( e^{-\beta x} \sqrt{\mathbb{E}_{T_x} \left( \mathbb{E}_Y(e^{-2\beta Y}) \right)^{T_x} \mathbb{E}_{Z_x} e^{2\beta |Z_x|}} \right), \tag{3} \]

where \( x \) is the depth of the first scattering location inside the medium, and \( T_x, Z_x \) are the stopping time and overshoot random variables associated with the one-sided \((x, 0.5, F_Y)\)-renewal process constructed from the depth series of the consecutive scattering (reflecting) locations (see Figure 3).

Proof. Assuming that scattering is isotropic, there is an equal probability of decreases versus increases in the beam depth, \( i.e.\), \( p = q = 0.5 \). Therefore, considering that the medium has infinite height, the sequence of ray depth values at consecutive scattering locations is a one-sided \((x, 0.5, F_Y)\) renewal process. Denoting by \( \rho_x \) the reflected power of a beam of unity power that undergoes an isotropic scattering at a depth \( x \) inside the bed, the average reflect power is the expected value of \( e^{-\beta x} \rho_x \) for all possible initial scattering depth values \( x \). On the other hand, \( \rho_x \) can be expressed in terms of renewal process variables. Firstly, it can be calculated from the Beer law expressed as \( \rho_x = e^{-\beta L_x + \beta |Z_x|} \), where \( L_x \) and \( Z_x \) are the total random walk travel and exit overshoot by the stopping time, respectively. Furthermore, \( L_x = \sum_{i=1}^{T_x} y_i \) where \( T \) is the stopping time and \( y_i \)s are absolute changes in the depth of
the ray between every two consecutive scattering events. We can now invoke Cauchy-Schwarz inequality to conclude that:

\[ \mathbb{E}_x \rho_x = \mathbb{E}_x \left( e^{-\beta x} \mathbb{E} \left( e^{-\beta L_x + \beta |Z_x|} \right) \right) \leq \mathbb{E}_x \left( e^{-\beta x} \sqrt{\mathbb{E} e^{-2\beta L_x} \mathbb{E} e^{2\beta |Z_x|}} \right). \]

Furthermore, from Wald’s identity we have \( \mathbb{E} e^{-2\beta L_x} = \mathbb{E} e^{-2\beta \sum_{i=1}^{T_x} Y_i} = (\mathbb{E} Y (e^{-2\beta Y}))^{T_x} \) finalizing the proof.\( \square \)

**Figure 3.** Schematics of ray travel in a one-sided porous media, with consecutive scattering events. The ray travel can be modeled with a one-sided renewal process representing the depth (vertical) location of the ray at locations where it hits opaque particles.

**Theorem 1.** In a one-sided flat polydisperse porous medium with dissipation factor \( \beta \) and opaque particles, the following approximation holds for the reflected (non-absorbed) fraction \( \rho \) of the incident power:

\[ \rho \approx \hat{\rho} = \left( e^{-\beta x} \mathbb{E}_{T_x} \mathbb{E}_Y \left( e^{-\beta Y} \right)^{T_x} \right) \left( 1 - \Delta_{\epsilon} \mathbb{E} Z_x e^{-\epsilon Z_x/\bar{x}} \beta \right). \] (4)

Proof. Directly follows from the use of Delta method:

\[ Z_m = \Delta_{\epsilon} \mathbb{E} e^{-\epsilon/\bar{x} Z} \]

\[ \mathbb{E} e^{\beta Z} \approx 1 - \Delta_{\epsilon} \mathbb{E} e^{-\epsilon Z/\bar{x}} \beta \]

\[ \rho = \mathbb{E}_x \left( e^{-\beta x} \mathbb{E} \left( e^{-\beta L_x + \beta |Z_x|} \right) \right) \approx \mathbb{E} \left( e^{-\beta x} \mathbb{E} e^{-\beta L_x} \right) \left( 1 - \Delta_{\epsilon} \mathbb{E} e^{-\epsilon Z/\bar{x}} \beta \right) \approx \hat{\rho} \]

\[ = \mathbb{E} \left( e^{-\beta x} \mathbb{E} \left( e^{-2\beta Y} \right)^{T_x} \right) \left( 1 - \Delta_{\epsilon} \mathbb{E} e^{-\epsilon Z/\bar{x}} \beta \right). \]
Lemma 2 (Combination of results from [114] for one-sided renewal process). The following explicit formula holds for the moment generating function of the overshoot $Z_x$ and stopping time $T_x$ of a one-sided $(x, p, F_y)$-renewal process:

$$
\mathbb{E}_{T_xZ_x}(\alpha^{T_x}e^{-\zeta Z_x}) = \sum_{i=1}^{m} c_i e^{\gamma_i x}, \forall 0 \leq \alpha \leq 1, \zeta \geq 0,
$$

where $\gamma_i$'s are the non-positive roots of the Cramer-Lundberg equation:

$$
pl_+(y) + qL_-(y) = \alpha^{-1},
$$

where $L_+$ and $L_-$ are one-sided Laplace transforms of the CDF $F_y$:

$$
L_+(v \in \mathbb{C}^+) = \mathcal{L}_{F_y}(v) = \frac{P_+(v)}{R_+(v)},
$$

$$
L_-(v \in \mathbb{C}^+) = \mathcal{L}_{F_y}(v) = \frac{P_-(v)}{R_-(v)},
$$

where $P_+, P_-, R_+, R_-$ are polynomials of degree $m$. The coefficients $c_i$'s are given by:

$$
c_i = \frac{R_+(-y_i)}{R_+(\zeta)} \frac{\Pi_{j \neq i}(\zeta - y_j)}{\Pi_{j \neq i}(\gamma_i - y_j)}
$$

Figure 4 summarized these relations in a compact way for convenience.

Theorem 2. If the homogenized beam length distribution in a Beerian porous media with opaque particles and dissipation factor $\beta$ can be expressed as a single exponential distribution with the rate parameter $\mu$, i.e., $F_\ell(y) = \mu e^{-\mu y}$, then the upper bound ($\rho^u$) and approximation $\hat{\rho}$ for the normal reflection become:

$$
\hat{\rho} = \frac{1 - \frac{2\eta}{2\eta + 1}}{\eta + 1} (1 + \eta) \approx \rho \leq \rho^u = \frac{1 - \frac{2\eta}{2\eta + 1}}{1 + \frac{1}{2\pi}} \left( \frac{1}{1 - 2\eta} \right)^{\frac{1}{2}},
$$

where $\eta = \beta/\mu$. The generalization to non-normal incident angle $\theta$ is as follows:
\[ \hat{\rho} = \left(1 - \sqrt{\frac{2\eta}{\gamma}}\right) \left(1 + \left(\eta + \sqrt{\frac{2\eta}{\gamma+1}}\cos \theta\right)\right)^{-1} \left(1 + \eta\right). \]  

(7)

Proof: Given the single exponential distribution assumption, (5) simplifies to \( \mathbb{E}_{\tau_x, Z_x}(\alpha^T_x e^{-\zeta Z_x}) = ce^{yx} \) where for the case of single mode distribution:

\[ c = \frac{R_+(\gamma)}{R_+(\zeta)} = \frac{\gamma + \mu}{\zeta + \mu} \]

\[ \gamma = -\mu \sqrt{1 - \alpha}, \quad \alpha = \frac{\mu}{\mu + 2\beta} \]

First, we prove the left-hand side of (6). We have from (4) that:

\[ \hat{\rho} = \mathbb{E}\left(e^{-\beta x} \mathbb{E}x \alpha^T x\right) \left(1 - \Delta_\epsilon \mathbb{E}e^{-\epsilon \xbar}/\beta \right) \]

where \( \alpha = \mathbb{E}_\gamma(e^{-2\beta y}), \quad x = \mathbb{E}x \). For the case of exponential distribution with parameter \( \mu, \alpha = \frac{\gamma}{\beta + \gamma} \) and \( x = 1/\mu \), and therefore we can write:

\[ -\Delta_\epsilon \mathbb{E}e^{-t/(\epsilon \xbar)} = -\frac{c_1(2\epsilon/\xbar) - c_1(\epsilon/\xbar)}{\epsilon/\xbar} = -\frac{2\epsilon}{\xbar + \mu} - \frac{\mu}{\xbar + \mu} = \frac{1}{\mu} \]

Furthermore, from the moment generating results of Lemma 2 we know that:

\[ \mathbb{E}x \alpha^T x = c_1 e^{yx}, \]

where

\[ c_1(\zeta = 0) = \frac{R_+(\gamma)}{R_+(\zeta = 0)} = \frac{\gamma + \mu}{\mu}, \]

\[ \gamma = -\mu \sqrt{1 - \alpha} = -\mu \sqrt{\frac{2\beta}{2\beta + \mu}}. \]

Therefore:

\[ \hat{\rho} = \mathbb{E}(c_1(0)e^{-(\beta - \gamma)x}) \left(1 + \frac{\beta}{\mu}\right) = \frac{\gamma + \mu}{\beta + \mu - \gamma} \left(1 + \frac{\beta}{\mu}\right) = \frac{-\mu}{\beta + \mu} \frac{\sqrt{2\beta + \mu}}{2\beta + \mu} \left(1 + \frac{\beta}{\mu}\right) \]

\[ = \frac{1 - \sqrt{2\eta}}{\eta + 1 + \frac{2\eta}{\sqrt{2\eta + 1}}} (1 + \eta). \]
The case of non-zero incident angle is a very straightforward modification by considering:

$$\rho_\theta = \mathbb{E}_x \rho_{\bar{x}},$$

where $\bar{x} = x \cos \theta$. Therefore:

$$\rho_\theta = \int_0^\infty \mu e^{-(\mu + \beta \cos \theta)x} \mathbb{E}(e^{-\beta|x|Z_x}) dx.$$

The tight estimate $\hat{\rho}_\theta$ can therefore be written as:

$$\hat{\rho}_\theta = \mathbb{E}(e^{-\beta\bar{x}} \mathbb{E}\alpha_T x)(1 - \Delta e^{-\epsilon_Z/\epsilon_e} \beta).$$

where $\alpha = \mathbb{E}_x e^{-\beta\gamma}$. Using similar calculations as before, we can write:

$$\hat{\beta} = \mathbb{E}(c_1(0)e^{-(\beta - \gamma)x\cos \theta}) \left(1 + \frac{\beta}{\mu}\right) = \frac{\gamma + \mu}{\mu + (\beta - \gamma) \cos \theta} \left(1 + \frac{\beta}{\mu}\right)$$

$$= \frac{1 - \sqrt{\frac{2\eta}{\eta + 1 + 2\eta} \cos \theta}}{1 + \left(\frac{\eta + 1 + 2\eta}{\eta + \frac{2\eta}{1 + 2\eta}}\right)} (1 + \eta),$$

which proves (5). We now turn to the right-hand side of (4). We can write:

$$\rho^{(u)} = \mathbb{E}_x \left(e^{-\beta x} \mathbb{E}\alpha_T x \mathbb{E} e^{2\beta |Z_x|} \right).$$

Again, as shown above, $\mathbb{E}\alpha_T x = \frac{\gamma + \mu}{\mu} e^{\gamma x}$. Furthermore, we can use the moment-generating function of $Z_x$ from Lemma 2 to conclude that $Z_x$ has an exponential distribution with rate parameter $\mu$. Therefore, (note that it is only valid for $\mu > 2\beta$):

$$\mathbb{E} e^{2\beta Z_x} = \frac{\mu}{\mu - 2\beta}.$$

Putting all these together along with $\gamma = -\mu \sqrt{\frac{2\beta}{2\beta + \mu}}$ we get:

$$\rho^{(u)} = \mathbb{E}_x \left(e^{-\beta x} e^{\frac{\gamma + \mu}{\mu} \left(\frac{\mu}{\mu - 2\beta}\right)} \right) = \kappa \mathbb{E} e^{-x(\beta - \gamma/2)} = \frac{\kappa \mu}{\mu + \beta + \gamma/2},$$

where $\kappa = \sqrt{\left(\frac{\gamma + \mu}{\mu}\right) \left(\frac{\mu}{\mu - 2\beta}\right)}$, which can be simplified as:

$$\rho^{(u)} = \frac{\sqrt{\left(\frac{\gamma + \mu}{\mu}\right) \left(\frac{\mu}{\mu - 2\beta}\right)} \mu}{\mu + \beta - \gamma}.$$
\[
\left(1 - \sqrt{\frac{2\beta}{2\beta + \mu}}\right)^{0.5} \left(\frac{\mu}{\mu - 2\beta}\right)^{0.5} \frac{\mu}{\mu + \beta + 0.5\mu\sqrt{\frac{2\beta}{2\beta + \mu}}} = \left(1 - \sqrt{\frac{2\eta}{2\eta + 1}}\right)^{0.5} \left(\frac{1}{1 - 2\eta}\right)^{0.2} \frac{1}{1 + \eta + 0.5\sqrt{\frac{2\eta}{2\eta + 1}}}
\]

thus completing the proof \(\Box\)

**Remark.** Note that the dissipation factor in a Beerian environment is given by \(\beta = \frac{2\pi k}{\lambda}\), where \(k\) is the complex component of the refractive index. Therefore, the upper bound (right hand side) in Theorem 2 is only valid for

\[
\mu > 2\beta \rightarrow \mu > \frac{4\pi k}{\lambda} \rightarrow \lambda > 4\pi kL,
\]

where \(L\) is the average mean free beam length penetration length \(\mathbb{E}\ell, \ell \sim \text{Exp}(\mu)\). So this means the upper bound is only valid in microscale porous media in which the average distance to closest scattering particle within the participating medium is comparable to wavelength (near field).

2.2. Part II. Two-sided medium.

**Definition 2.** A two-sided \((x, h, p, F_y)\)-renewal process (See Fig. 5) is a random walk that starts at point \(x \geq 0\), and takes consecutive steps of sizes \(Y_i, i \geq 0\) with cumulative distribution \(F_y\), where steps are to the right (increasing) with probability \(p\) and to the left with probability \(q = 1 - p\), and terminates as soon as it becomes less than 0 or larger than \(h\). \(T_{x,h}, Z_{x,h}^+, Z_{x,h}^-\) are defined as random variables describing the stopping time and positive and negative overshoots at stopping time, respectively. In other words:

\[
T_x = \min\{T \geq 0 | \sum_{i=1}^{T} s_i Y_i < 0 \text{ or } \sum_{i=1}^{T} s_i Y_i > h\},
\]

\[
Z_{x,h}^+ = \max\left(\sum_{i=1}^{T_x} s_i Y_i - h, 0\right),
\]

\[
Z_{x,h}^- = \max\left(-\sum_{i=1}^{T_x} s_i Y_i - x, 0\right),
\]

where \(s_i \in \{1, -1\}\) is the side of the \(i^{th}\) step.
**Figure 5.** Two-sided renewal process.

**Lemma 3.** In a two-sided flat polydisperse non-Beerian porous bed of height $h$, the reflectivity fraction $\rho_\theta$ of the incident power at angle $\theta$ is given by:

$$
\mathbb{E}_x \rho_\bar{x} = \mathbb{P}(X_\infty \leq 0) = \frac{h - x + \mathbb{E}Z^+_{h, \bar{x}}}{\mathbb{E}Z^-_{h, \bar{x}} + \mathbb{E}Z^+_{h, \bar{x}} + h'},
$$

where $x$ is a random variable representing the first scattering depth inside the medium and $\bar{x} = x \cos \theta$, and $Z^+_{h, \bar{x}}, Z^-_{h, \bar{x}}$ are the overshoot variables defined for the equivalent two-sided renewal process of Definition 2.

**Lemma 4.** (Combination of results from [114] for two-sided renewal processes). The following explicit formula holds for the moment generating function of the overshoots $Z^+_{x,h}, Z^-_{x,h}$ and stopping time $T_{x,h}$ of a two-sided $(x, h, p, F_y)$-renewal process:

$$
\mathbb{E}(A\alpha^{T_{x,h}}e^{-\zeta Z_{x,h}^-}) + \mathbb{E}(B\alpha^{T_{x,h}}e^{-\zeta Z_{x,h}^+}) = \sum_{i=1}^{m} c_i e^{\gamma_i x} + \sum_{i=1}^{m} d_i e^{\delta_i x},
$$

$$
\forall 0 \leq \alpha \leq 1, \zeta, \xi \geq 0, A, B \in \mathbb{R}
$$

where $\gamma_i$s and $\delta_i$s are the non-positive roots of the Cramer-Lundberg equation:

$$
pL_+(\gamma) + qL_-(\gamma) = \alpha^{-1},
$$

where $L_+$ and $L_-$ are one-sided Laplace transforms of the CDF $F_y$:

$$
L_+(\nu \in \mathbb{C}^+) = \mathcal{L}_{F_y}^+(\nu) = \sum_{i=1}^{m} a_i \frac{\mu_i}{\mu_i + \nu},
$$

$$
L_-(\nu \in \mathbb{C}^+) = \mathcal{L}_{F_y}^-(\nu) = \sum_{i=1}^{l} b_i \frac{\psi_i}{\psi_i + \nu}.
$$

The coefficients $c_i$s and $d_i$s are the solutions of the following linear system of equations:

$$
\sum_{i=1}^{m} \frac{c_i}{\mu_t + \gamma_i} + \sum_{i=1}^{l} \frac{d_i}{\mu_t + \delta_i} = \frac{A}{\mu_t + \zeta}, \quad 1 \leq t \leq m
$$
\[ \sum_{i=1}^{m} c_i e^{y_i h} + \sum_{i=1}^{l} d_i e^{\delta_i h} = B, \quad 1 \leq t \leq l \]

Fig. 6 summarized these relations in a compact way for convenience.

Theorem 3. If the homogenized beam length distribution in a non-Beerian two-sided porous bed of height \( h \) with opaque particles can be expressed as a single exponential distribution with rate parameter \( \mu \), i.e., \( F_x(y) = \mu e^{-\mu y} \), then the reflection fraction \( \rho_\theta \) for incident angle \( \theta \) is precisely equal to:

\[ \rho_\theta = \frac{(1 - \cos \theta)(1 - e^{-h \mu \cos \theta}) + h \mu}{h \mu + 2}. \]  

Proof. We can easily follow Lemma 4 to conclude that \( \mathbb{E}Z_{x,h}^+ = Z_{x,h}^- = 1/\mu \). Using that directly in Lemma 3 results in:

\[ \mathbb{E}\rho_{x \cos \theta} = \mathbb{E}_x \left( \frac{h - x \cos \theta + 1/\mu}{2/\mu + h} \middle| x \cos \theta \leq h \right). \]

The rest of the proof follows mechanically from calculating the conditional expected value given the distribution of \( x \).

3. Numerical Results

The analytical upper bound and estimate values of Eqn. (4) are compared with those derived from exhaustive Monte Carlo simulations in Fig. 7, assuming a statistical Beerian (\( \beta = 1 \)) infinite medium with an exponential mean free beam length distribution with parameter \( \mu \). Two regimes of \( \mu \) are considered, namely near- and far-field (Fig.s 7.A and 7.B). The reflectivity estimate \( \hat{\rho} \) is accurate down to an astonishing 0.01\%, as it is directly stemming from analytical derivations.
To demonstrate the efficacy of using statistical homogenization in deriving approximate radiative properties, we studied a one-sided Beerian packed bed (infinite slab, $\beta > 0$) filled with random overlapping circular particles (Figure 8), assuming 100% opacity, and computed the empirical distribution of mean free beam length using an in-house pixel-based MCRT code. Fig. 9 shows the average radiation flux for the normal incident angle inside the medium obtained using MCRT using 10000 iterations. The empirical distribution of the mean free beam length, $f_\ell(x) = \partial F_\ell / \partial x$, namely, the random variable describing the distance between every two consecutive scattering events was then fit using a single-rate exponential distribution with Maximum-Likelihood and least-square PDF fitting. The empirical and analytical distributions are shown in Fig. 10. An analytical estimate of the reflection of upward incident radiation was obtained by introducing the fitted distribution $\hat{F}_\ell(\cdot)$ to the results of Theorem 2 and is compared to computational values from MCRT and pack-free MC using 10000 iterations, as depicted in Fig. 11. The curves indicate a high estimation accuracy for the purely analytical model.
Figure 8. Sample random one-sided porous medium filled with overlapping circular opaque particles.

Figure 9. Average radiation flux calculated via ray tracing simulations with 10000 iterations for the normal incident (from the bottom side) for the one-sided medium of Figure 8.
Figure 10. Empirical homogenized probability distribution function (PDF) of the mean free beam length inside the porous medium of Figure 8, along with exponential approximations obtained via curve fitting and maximum likelihood approaches.

Figure 11. Reflectivity estimates from MCRT, pack-free MC, and purely analytical model using Theorem 2 with the numerical fitted distributions for the porous medium of Figure 8.

Finally, Fig. 12 compares the analytical reflectivity formula of Theorem 3 with exhaustive MC simulations for a non-Beerian two-sided porous medium with an exponential ($\mu$) mean-free beam length distribution for zero incident angle. The curve indicates a precise match, therefore validating the derivation.
Figure 12. Comparison of the analytical reflectivity formula of Theorem 3 with exhaustive MC simulations for a non-Beerian two-sided porous medium with an exponential ($\mu$) mean-free beam length distribution.

4. Conclusion & Glimpse of Future Work

We extracted the existing machinery of Renewal, Ruin (Cramér-Lundberg), and surplus risk theories and used them to obtain preliminary precise geometric optics radiation estimations in porous media. Future work shall seek to expand Theorems 2&3 for probability decompositions that are more complex than a single rate exponential. Erlang mode decomposition can capture the complexity of arbitrary heterogeneous geometric factors. Future work shall strive to derive analytically simplified moment-generating function expressions and use techniques similar to those of this work to approximate spectral power reflection as a function of mean free beam length mode decomposition. As suggested by more recent renewal model literature[119], the overshoots and stopping times can be modeled by mixtures of Erlangs $\sum_{i=1}^{\infty} \pi'_i n_{k'_i,\mu'_i} x^{k'_i-1} e^{-x\mu'_i}$ when the step size distribution is also an Erlang mixture $\sum_{i=1}^{\infty} \pi_i n_{k_i,\mu} x^{k_i-1} e^{-x\mu}$. The literature machineries[114], [119] provide tools for the mapping of model mixture parameters ($\pi_i, k_i, \mu_i$) to the solution mixture parameters ($\pi'_i, k'_i, \mu'_i$) which, in the most complex case, involves finding roots of a polynomial equation of the order of the mixture size. Our future work shall strive to compile a thorough portfolio of derivations and possible closed forms, particularly in asymptotic limit regimes of parameter ratios using renewal processes[120]–[123]. Several other key results of Renewal theory can precisely explain asymptotic power behavior, uncertainty estimates of the derived calculations, and exact distribution functions. The work was limited to porous media with opaque particles. Future work shall consider refraction and other scattering modes. More advanced future work shall also consider two-dimensional renewal processes allowing for explicit modeling of other media shapes and more advanced parameters.
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