CONJECTURES ON SPECTRAL NUMBERS FOR
UPPER TRIANGULAR MATRICES AND FOR
SINGULARITIES

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Abstract. Cecotti and Vafa proposed in 1993 a beautiful idea how to associate spectral numbers \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) to real upper triangular \( n \times n \) matrices \( S \) with 1’s on the diagonal and eigenvalues of \( S^{-1}S^t \) in the unit sphere. Especially, \( \exp(-2\pi i \alpha_j) \) shall be the eigenvalues of \( S^{-1}S^t \).

We tried to make their idea rigorous, but we succeeded only partially. This paper fixes our results and our conjectures. For certain subfamilies of matrices their idea works marvellously, and there the spectral numbers fit well to natural (split) polarized mixed Hodge structures. We formulate precise conjectures saying how this should extend to all matrices \( S \) as above.

The idea might become relevant in the context of semiorthogonal decompositions in derived algebraic geometry. Our main interest are the cases of Stokes like matrices which are associated to holomorphic functions with isolated singularities (Landau-Ginzburg models). Also there we formulate precise conjectures (which overlap with expectations of Cecotti and Vafa). In the case of the chain type singularities, we have positive results.

We hope that this paper will be useful for further studies of the idea of Cecotti and Vafa.

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1. Introduction, conjectures and results

Cecotti and Vafa proposed in [CV93] a beautiful idea how to associate to upper triangular matrices in

\[ T(n, \mathbb{R}) := \{ S = (s_{ij}) \in M(n \times n, \mathbb{R}) \mid s_{ij} = 0 \text{ for } i > j, \]
\[ s_{ii} = 1, S^{-1}S^t \text{ has eigenvalues in } S^1 \} \quad (1.1) \]

(with \( n \in \mathbb{Z}_{\geq 1} \)) \( n \) spectral numbers \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) such that \( e^{-2\pi i\alpha_1}, \ldots, e^{-2\pi i\alpha_n} \) are the eigenvalues of \( S^{-1}S^t \). Furthermore they claim to have an almost rigorous proof that the recipe works and that in the case of Landau-Ginzburg models the spectral numbers of its Stokes matrices coincide with the true spectral numbers.

We consider the recipe as incomplete and see serious gaps in it and in the arguments that in the case of Landau Ginzburg models the spectral numbers coincide. We discuss this below. Still we find the idea fascinating.

This paper is the result of our efforts to make the recipe work. We succeeded only partially. We have certain subspaces of \( T(n, \mathbb{R}) \) where the recipe works and which are hopefully big enough to be useful for an extension of the recipe to all of \( T(n, \mathbb{R}) \). Below we formulate precise conjectures and results. The recipe is as follows.

Recipe 1.1. Start with some matrix \( S_1 \in T(n, \mathbb{R}) \). Choose a path from the unit matrix \( E_n \) to \( S_1 \) within \( T(n, \mathbb{R}) \), i.e. a continuous map \( S : [0, 1] \to T(n, \mathbb{R}) \) with \( S(0) = E_n \) and \( S(1) = S_1 \). Now choose in a natural way \( n \) continuous functions \( \alpha_j : [0, 1] \to \mathbb{R}, j \in \{1, \ldots, n\} \), such that \( \alpha_j(0) = 0 \) and \( e^{-2\pi i\alpha_1(r)}, \ldots, e^{-2\pi i\alpha_n(r)} \) are the eigenvalues of \( S(r)^{-1}S(r)^t \). Then \( \alpha_1(1), \ldots, \alpha_n(1) \) are defined to be the spectral numbers of \( S_1 \).

Remarks 1.2. (i) The recipe assumes that \( T(n, \mathbb{R}) \) is connected. Cecotti and Vafa conjecture this [CV93] first half of page 590, but have no proof for it. Our conjecture (a) below will imply this, but we also have no proof for it. But even if \( T(n, \mathbb{R}) \) is connected, the spectral numbers might depend on the chosen path.

(ii) Even if a path is given, it might happen that for some \( r \in (0, 1) \) several eigenvalues of \( S(r)^{-1}S(r)^t \) coincide. Then at this parameter \( r \) one can exchange the continuations at \( r \) of the functions \( \alpha_j \) for these eigenvalues. Then in general it is unclear whether and how to make a most natural choice and how to make the phrase in a natural way in the recipe precise. This holds especially if \( \alpha_i(r) - \alpha_j(r) \in 2\mathbb{Z} - \{0\} \).

(iii) Cecotti and Vafa proposed in [CV93] footnote 6 on page 583 to choose the path such that for \( r \in (0, 1) \) all eigenvalues of \( S(r)^{-1}S(r)^t \)
are different. This is within \( T(n, \mathbb{R}) \) for most matrices not possible because the eigenvalue \(-1\) has for all matrices in \( T(n, \mathbb{R}) \) even multiplicity because \( \det(S(r)^{-1}S(r)^t) = 1 \).

(iv) Only on the pages 589+590 in [CV93], it is demanded that the path is within \( T(n, \mathbb{R}) \), not yet on page 583. But if one chooses a path which leaves \( T(n, \mathbb{R}) \) there are two problems. The resulting spectral numbers might depend on the path. And the arguments with \( tt^* \)-geometry for the coincidence of the Stokes matrix spectral numbers with the true spectral numbers of a Landau-Ginzburg model will not work [CV93, first half of page 590]. Because of both problems we restrict to the recipe with paths within \( T(n, \mathbb{R}) \).

We have two subfamilies \( \text{THOR}1(n, \mathbb{R}) \) and \( \text{THOR}2(n, \mathbb{R}) \subset T(n, \mathbb{R}) \) for which the recipe 1.1 works. The families will be presented in section 4, but here we give their crucial properties and show how and why the recipe works for them.

**Theorem 1.3.** (a) The subspaces \( \text{THOR}1(n, \mathbb{R}) \) and \( \text{THOR}2(n, \mathbb{R}) \subset T(n, \mathbb{R}) \) which are defined in definition 4.4 (a) satisfy the following properties.

\[
\begin{align*}
(\gamma) & \text{ for } k \in \{1, 2\} \text{ can be represented by a closed simplex (the convex hull of } \dim \text{THOR}(n, \mathbb{R}) + 1 \text{ many points) in } \mathbb{R}^{\dim \text{THOR}(n, \mathbb{R})}. \text{ And } \\
& \text{dim THOR}1(n, \mathbb{R}) \quad \text{dim THOR}2(n, \mathbb{R}) \\
& n \text{ odd } \frac{n-1}{2} \quad \frac{n-1}{2} \\
& n \text{ even } \frac{n-2}{2} \quad \frac{n-2}{2} \\
(\beta) & \text{ For each } S \in \text{THOR}(n, \mathbb{R}), \text{ there is a regular matrix } \text{R}^{\text{mat}}_{(k)}(S) \in \text{GL}(n, \mathbb{R}) \text{ with eigenvalues in } S^1 \text{ and with } \\
& (-1)^k \cdot S^{-1}S^t = \text{R}^{\text{mat}}_{(k)}(S)^n. \\
(\delta) & \text{Regular means that } \text{R}^{\text{mat}}_{(k)}(S) \text{ has for each eigenvalue only one Jordan block. The map } \text{R}^{\text{mat}}_{(k)} : \text{THOR}(n, \mathbb{R}) \to \text{GL}(n, \mathbb{R}) \text{ is as a map to } M(n \times n, \mathbb{R}) \text{ affine linear. } \\
(\gamma) & \text{R}^{\text{mat}}_{(k)}(S) \text{ is semisimple (and thus has pairwise different eigenvalues) if and only if } S \in \text{int}(\text{THOR}(n, \mathbb{R})). \\
(\beta) & \text{E}_n \in \text{int}(\text{THOR}(n, \mathbb{R})) \text{ and } \text{R}^{\text{mat}}_{(k)}(E_n) \text{ has the eigenvalues } e^{-2\pi i(j-\frac{1}{2})/n}, j \in \{1, ..., n\}. \text{ Furthermore, } \bigcap_{k=1,2} \text{THOR}(n, \mathbb{R}) = \{E_n\}. \\
(\beta) & \text{The recipe 1.1 works well within } \text{THOR}(n, \mathbb{R}). \text{ For } S_1 \in \text{THOR}(n, \mathbb{R}) \text{ choose any continuous path } S : [0, 1] \to \text{THOR}(n, \mathbb{R}) \text{ with }
S(0) = E_n, S(1) = S_1 and S([0,1)) ⊂ \text{int}(T_{\text{HORk}}(n, \mathbb{R})). Then for r ∈ [0,1) the eigenvalues of R_{(k)}^{\text{mat}}(S(r)) are pairwise different and the paths α_1, ..., α_n : [0,1] → T_{\text{HORk}}(n, \mathbb{R}) can be chosen uniquely such that α_j(0) = 0 and e^{-2πi(α_j(r)+j−\frac{k}{2})/n} for j ∈ {1, ..., n} are the eigenvalues of R_{(k)}^{\text{mat}}(S(r)). The values α_1(1), ..., α_n(1) are independent of the chosen path S and give the spectrum Sp(S) = ∑_{j=1}^{n}(α_j(1)) ∈ Z_{≥0}(\mathbb{R}).

Proof: Part (a) will be proved in section 4. Part (b) follows immediately from part (a). In fact, part (a) implies existence and uniqueness of continuous functions α_j^{(k)} : T_{\text{HORk}}(n, \mathbb{R}) → \mathbb{R} such that α_j^{(k)}(E_n) = 0 and e^{-2πi(α_j^{(k)}(S)+j−\frac{k}{2})/n} for j ∈ {1, ..., n} are the eigenvalues of R_{(k)}^{\text{mat}}(S) for any S ∈ T_{\text{HORk}}(n, \mathbb{R}). For any S ∈ T_{\text{HORk}}(n, \mathbb{R}) the values α_j^{(k)}(S) at S are the spectral numbers of S. The only matrix in \bigcap_{k=1,2} T_{\text{HORk}}(n, \mathbb{R}) is E_n. Both cases k = 1 and k = 2 associate to E_n the spectrum Sp(E_n) = ∑_{j=1}^{n}(α_j(1)).

Remarks 1.4. (i) The crucial points are, that the matrices R_{(k)}^{\text{mat}}(S) for S ∈ \text{int}(T_{\text{HORk}}(n, \mathbb{R})) have pairwise different eigenvalues and the α_j^{(k)}(S) are determined by these eigenvalues and that the values e^{-2πiα_j^{(k)}(S)} are the eigenvalues of S^{-1}S^t because of (1.3).

(ii) HOR are the initials of the authors Horocholyn, Orlik and Randell of [Ho17] and [OR77]. In [Ho17, ch. 2] half of the matrices in \bigcup_{k=1,2} T_{\text{HORk}}(n, \mathbb{R}) were studied and the crucial equation (1.3) was proved for them. In [OR77, (4.1) Conjecture] it was conjectured that special matrices S in \bigcup_{k=1,2} T_{\text{HORk}}(n, \mathbb{Z}) turn up as Stokes matrices of the chain type singularities (sections 6 and 7). The main result Theorem (2.11) in [OR77] is that (−1)^kS^{-1}S^t is a monodromy matrix for such a singularity.

That the recipe 1.1 works for the matrices in \bigcup_{k=1,2} T_{\text{HORk}}(n, \mathbb{R}) is good news. It lead us to a number of conjectures and results which form the contents of this paper. We hope that they will be useful for a complete positive solution of recipe 1.1

The rest of this introduction has two purposes. It fixes some notions and proposes the conjectures 1.6, 1.7 and 1.9 which guide us through all of the paper. And it explains the structure of the paper and sketches some main results.

Section 4 introduces the subfamilies T_{\text{HORk}}(n, \mathbb{R}) ⊂ T(n, \mathbb{R}) for k ∈ {1,2} of HOR-matrices and proves theorem 1.3 (a). And it adds more precise information, especially, that the spectral pairs and the
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eigenspace decompositions of such a matrix give rise to a natural split polarized mixed Hodge structure.

Section 3 prepares this. It introduces isomorphic subspaces $T_{\text{Hor}}^\text{mat}(n, \mathbb{R})$ and it formalizes and studies the recipe

$$(\text{eigenvalues of } R_{(b)}^\text{mat}(S) \mapsto (\text{spectral numbers } \alpha_1, \ldots, \alpha_n \text{ of } S), \quad (1.4)$$

which is implicit in the proof of theorem 1.3 (b). This is elementary, but worth to be studied for itself. Properties of these spectral numbers give, combined with conjecture 1.9 on the spectral numbers of holomorphic functions, new features of these spectral numbers. The recipe (1.4) will also be extended to a recipe for spectral pairs $S_{pp}(S) = \sum_{j=1}^n (\alpha_j, k_j) \in \mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z})$.

Section 2 discusses spectral pairs from an abstract point of view. This is elementary, but must be provided. It also offers a review on the classification in [BH17] of the Seifert form pairs in definition 1.5. The notions in this definition are needed for the conjectures 1.6 and 1.7.

Definition 1.5. (a) A Seifert form pair $(H_\mathbb{R}, L)$ consists of a finite dimensional real vector space $H_\mathbb{R}$ and a nondegenerate bilinear form $L : H_\mathbb{R} \times H_\mathbb{R} \to \mathbb{R}$ (which is in general neither symmetric or antisymmetric). Its monodromy is the (unique) automorphism $M : H_\mathbb{R} \to H_\mathbb{R}$ with $L(Ma, b) = L(b, a)$ for $a, b \in H_\mathbb{R}$.

(b) Hermitian Seifert form pairs are classified in [Ne95]. The classification of real Seifert form pairs in [BH17] is reviewed in section 2.

(c) Trivial lemma: Any matrix $S \in GL(n, \mathbb{R})$ gives rise to the Seifert form pair $\text{Seif}(S) := (M(n \times 1, \mathbb{R}), L)$ with $L(a, b) := a^t \cdot S^t \cdot b$. Its monodromy $M$ is given by $M(a) = S^{-1}S^t \cdot a$.

(d) We define the sets $\text{Seif}(n), \text{Eig}(n)$, the projection $pr_{SE}$, and the maps $\Psi_{\text{Seif}}$ and $\Psi_{\text{Eig}}$ as follows.

$\text{Seif}(n) := \{\text{isomorphism classes of Seifert form pairs } (H_\mathbb{R}, L) \text{ with } \dim H_\mathbb{R} = n \text{ and with eigenvalues of the monodromy } M \text{ in } S^1 \}$, \hspace{1cm} (1.5)

$\text{Eig}(n) := \{\text{unordered tuples of numbers } \lambda_1, \ldots, \lambda_n \in S^1 \}$

$\quad := (S^1)^n / S_n$, \hspace{1cm} (1.6)

$pr_{SE} : \text{Seif}(n) \to \text{Eig}(n)$, \hspace{1cm} $[(H_\mathbb{R}, L)] \mapsto (\text{eigenvalues of } M)$, \hspace{1cm} (1.7)

$\Psi_{\text{Seif}} : T(n, \mathbb{R}) \to \text{Seif}(n)$,

$\quad S \mapsto [\text{Seif}(S)]$, \hspace{1cm} (1.8)

$\Psi_{\text{Eig}} := pr_{SE} \circ \Psi_{\text{Seif}} : T(n, \mathbb{R}) \to \text{Eig}(n)$. \hspace{1cm} (1.9)
(e) The group $G_{\text{sign}, n} := \{\pm 1\}^n$ acts on $T(n, \mathbb{R})$ by conjugation,

$$\varepsilon_1, \ldots, \varepsilon_n : S \mapsto \text{diag}(\varepsilon_1, \ldots, \varepsilon_n) \cdot S \cdot \text{diag}(\varepsilon_1, \ldots, \varepsilon_n)$$

(1.10)

for $(\varepsilon_1, \ldots, \varepsilon_n) \in G_{\text{sign}, n}$. The group $G_{\text{sign}, n}$ is called sign group. Of course, the maps $\Psi_{\text{Seif}}$ and $\Psi_{\text{Eig}}$ are $G_{\text{sign}, n}$-invariant.

(f) A Seifert form stratum in $T(n, \mathbb{R})$ is a union of components of one fiber of $\Psi_{\text{Seif}}$ which are permuted transitively by $G_{\text{sign}, n}$. An eigenvalue stratum in $T(n, \mathbb{R})$ is a union of components of one fiber of $\Psi_{\text{Eig}}$ which are permuted transitively by $G_{\text{sign}, n}$.

**Conjecture 1.6.** (a) $T_{\text{HOR}1}(n, \mathbb{R})$ intersects each eigenvalue stratum in $T(n, \mathbb{R})$.

(b) If $S_1, S_2 \in \bigcup_{k=1}^{2} T_{\text{HOR}k}(n, \mathbb{R})$ are in the same eigenvalue stratum of $T(n, \mathbb{R})$ then $\text{Sp}(S_1) = \text{Sp}(S_2)$.

(c) If $S_1, S_2 \in \bigcup_{k=1}^{2} T_{\text{HOR}k}(n, \mathbb{R})$ are in the same Seifert form stratum of $T(n, \mathbb{R})$ then $\text{Spp}(S_1) = \text{Spp}(S_2)$.

If it is true, conjecture 1.6 (a) implies that $T(n, \mathbb{R})$ is connected, conjecture 1.6 (a)+(b) gives spectral numbers $\text{Sp}(S)$ for any matrix $S \in T(n, \mathbb{R})$, and conjecture 1.6 (a)+(c) gives spectral pairs for any matrix $S$ in a Seifert form stratum which is met by $\bigcup_{k=1}^{2} T_{\text{HOR}k}(n, \mathbb{R})$. But these are not all Seifert form strata, as remark 2.11 (vii) and remark 5.3 (ii) will show. Unfortunately, for the other Seifert form strata, we have no precise idea how to lift $\text{Sp}(S)$ to $\text{Spp}(S)$.

**Conjecture 1.7.** Also for the matrices $S$ in the Seifert form strata which are not met by $\bigcup_{k=1}^{2} T_{\text{HOR}k}(n, \mathbb{R})$, $\text{Sp}(S)$ lifts in a natural way to $\text{Spp}(S)$.

**Remarks 1.8.** (i) For odd $n$, $T_{\text{HOR}1}(n, \mathbb{R})$ and $T_{\text{HOR}2}(n, \mathbb{R})$ are mapped by suitable elements of the sign group $G_{\text{sign}, n}$ to one another. For odd $n$ conjecture 1.6 (a) is equivalent to the analogous conjecture for $T_{\text{HOR}2}(n, \mathbb{R})$. But for even $n$ $\dim T_{\text{HOR}2}(n, \mathbb{R}) = \dim T_{\text{HOR}1}(n, \mathbb{R}) - 1$, and we expect that $T_{\text{HOR}2}(n, \mathbb{R})$ meets for large enough $n$ some other Seifert form strata than $T_{\text{HOR}1}(n, \mathbb{R})$.

In section 6 we will review some facts on holomorphic map germs $f : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 and on $M$-tame functions $f : X \to \mathbb{C}$ with $\dim X = m + 1$. Especially, we will discuss the following. In both cases, there is a Milnor number $\mu = \mu(f) \in \mathbb{Z}_{\geq 1}$. In both cases, there is a $\text{Br}_{\mu} \times G_{\text{sign}, \mu}$ orbit of matrices $S \in T(\mu, \mathbb{Z}) := T(\mu, \mathbb{R}) \cap GL(\mu, \mathbb{Z})$. Here $\text{Br}_{\mu}$ is the braid group with $\mu$ strings. We call these matrices Stokes matrices. Then $(-1)^{m+1} S^{-1} S'$ is
a matrix of the (classical global) monodromy. In both cases, there are $\mu$ spectral pairs $S_{pp}(f) = \sum_{j=1}^{\mu} (\alpha_j(f), l(f)) \in \mathbb{Z}_{\geq 0}(\mathbb{Q} \times \mathbb{Z})$ which come from natural mixed Hodge structures. The first entries are the spectral numbers $Sp(f) = \sum_{j=1}^{\mu} (\alpha_j(f)) \in \mathbb{Z}_{\geq 0}(\mathbb{Q})$. In a suitable numbering, the spectral numbers satisfy the symmetry $\alpha_j(f) + \alpha_{\mu+1-j}(f) = m - \frac{1}{2}$.

Building on the conjectures 1.6 and 1.7, we have a conjecture which embraces the claim of Cecotti and Vafa for Landau-Ginzburg models.

Conjecture 1.9. Suppose that the conjectures 1.6 and 1.7 are true. Let $f$ be a holomorphic map germ $f : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 or an $M$-tame function $f : X \to \mathbb{C}$ with $\dim X = m+1$. Then any Stokes matrix $S$ of $f$ satisfies

$$S_{pp}(S) = S_{pp}(f) - \left( m - \frac{1}{2}, m \right). \tag{1.11}$$

The resulting equality $Sp(S) = Sp(S) - \frac{m-1}{2}$ is in the case of $M$-tame functions equivalent to the claim in [CV93] that in the case of the Landau-Ginzburg models recipe 1.1 gives the central charges. Though the equality of spectral pairs is slightly stronger. And the case of an isolated hypersurface singularity is not covered by Landau-Ginzburg models, except for the quasihomogeneous singularities, they are $M$-tame on $\mathbb{C}^{m+1}$.

The results proved in the sections 5 and 7 can be summarized as follows.

Theorem 1.10. (a) (Section 5) In the cases $n = 2$ and $n = 3$, the conjectures 1.6 and 1.7 and the conjecture 1.9 for function germs are true.

(b) (Section 7) In the case of any chain type singularity $f(x_0, \ldots, x_m)$, the matrix $S = S_{\text{THOR},k}(\mu, \mathbb{Z})$ with $k \equiv m(2)$ which is considered in [OR77] (4.1) Conjecture, satisfies $Sp(S) = Sp(f) - \frac{m-1}{2}$.

Theorem 1.10 (b) and conjecture (4.1) in [OR77], which says that the matrix $S$ there is a Stokes matrix of $f$, imply conjecture 1.9 for the chain type singularities. For them $S_{pp}(S) = S_{pp}(f) - \left( m - \frac{1}{2}, m \right)$ and $Sp(S) = Sp(f) - \frac{m-1}{2}$ are equivalent, as the monodromy is semisimple.

Section 8 formulates some critic with an explicit example on some arguments in [CV93] around recipe 1.1 which use $tt^*$ geometry. And it offers some speculations about approaches towards a positive solution of recipe 1.1.

We thank Duco van Straten and Martin Guest for discussions.
2. Enhanced real Seifert form pairs and spectral pairs

Matrices in $T(n, \mathbb{Z})$ turn up when one considers isolated hypersurface singularities. There such a matrix encodes the Seifert form on the Milnor lattice with respect to a distinguished basis. See section 6. Also spectral pairs turn up there. But their origin is different, it is a natural polarized mixed Hodge structure with semisimple automorphism.

Nemethi [Ne95] studied in the singularity case the relation between Seifert form and spectral pairs and proved that the isomorphism class of the real Seifert form is (within the singularity cases) equivalent to the spectral pairs modulo $2\mathbb{Z} \times \{0\}$. We recover this below, see lemma 2.9.

Here we will give a general abstract discussion of the relationship between (abstract) Seifert forms and spectral pairs. This builds especially on [BH17]. We will start with the classification of real Seifert form pairs. We will define an enhancement of a real Seifert form pair which includes spectral pairs. We will also discuss the triangular shape of the matrices in $T(n, \mathbb{R})$, it is related to a semiorthogonal decomposition. This leads to a reformulation of the question how to associate spectral pairs to matrices in $T(n, \mathbb{R})$.

Notations 2.1. Throughout the whole paper, $H_K$ is a finite dimensional vector space over a field $K$. If $H_{\mathbb{R}}$ is given, then $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = H_{\mathbb{R}} \oplus iH_{\mathbb{R}}$ is the complexification of $H_{\mathbb{R}}$.

If $L : H_K \times H_K \to K$ is a bilinear form then two subspaces $V_1, V_2 \subset H_K$ are $L$-orthogonal if $L(V_1, V_2) = L(V_2, V_1) = 0$. And then the left and right orthogonal subspaces to a subspace $U \subset H_K$ are $U^{L\bot} := \{a \in H_K \mid L(a, U) = 0\}$ and $U^{\bot L} := \{b \in H_K \mid L(U, b) = 0\}$.

If $M : H_K \to H_K$ is an automorphism, then $M_s, M_u, N : H_K \to H_K$ denote its semisimple, its unipotent and its nilpotent part with $M = M_sM_u = M_uM_s$ and $N = \log M_u = M_u$. If $K = \mathbb{C}$, denote $H_{\lambda} := \ker(M_s - \lambda \cdot \text{id}) : H_{\mathbb{C}} \to H_{\mathbb{C}}, H_{\neq 1} := \bigoplus_{\lambda \neq 1} H_{\lambda}, H_{\neq -1} := \bigoplus_{\lambda \neq -1} H_{\lambda}$.

Definition 2.2. (a) A Seifert form pair is a pair $(H_{\mathbb{R}}, L)$ where $L : H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathbb{R}$ is a nondegenerate bilinear form. It is called irreducible if $H_{\mathbb{R}}$ does not split into two nontrivial (i.e. both $\neq \{0\}$) $L$-orthogonal subspaces.

(b) The monodromy $M : H_{\mathbb{R}} \to H_{\mathbb{R}}$ of a Seifert form pair $(H_{\mathbb{R}}, L)$ is the unique automorphism with

\[ L(Ma, b) = L(b, a) \quad \text{for all } a, b \in H_{\mathbb{R}}. \quad (2.1) \]
The eigenvalues of a Seifert form pair are the eigenvalues of its monodromy. Two bilinear forms $I_s$ and $I_a$ on $H_\mathbb{R}$ are defined by

\[
\begin{align*}
I_s(a, b) &:= L(b, a) + L(a, b) = L((M + \text{id})a, b), \\
I_a(a, b) &:= L(b, a) - L(a, b) = L((M - \text{id})a, b),
\end{align*}
\]

(c) An $S^1$-Seifert form pair is a Seifert form pair with eigenvalues in $S^1$.

In [BH17], also four other bilinear forms $I_s^{(2)}, I_a^{(2)}, I_s^{(3)}$ and $I_a^{(3)}$ (on subspaces of $H_\mathbb{R}$) are associated to a Seifert form pair. Here we will use only $I_s$, and that only in the discussion of the case $n = 3$ in section 5.

Part (a) of the following theorem is an immediate consequence of the calculation $L(Ma, Mb) = L(Mb, a) = L(a, b)$ and the definitions of $I_s$ and $I_a$. The parts (b) and (c) give the classification of Seifert form pairs and are proved in [BH17] Theorem 2.5 and Theorem 2.9.

**Theorem 2.3.** (a) Let $(H_\mathbb{R}, L)$ be a Seifert form pair. The three bilinear forms $L$, $I_s$ and $I_a$ are monodromy invariant. The radical of $I_s$ is $\ker(M + \text{id})$, the radical of $I_a$ is $\ker(M - \text{id})$.

(b) Any Seifert form pairs splits into a direct and $L$-orthogonal sum of irreducible Seifert form pairs. The splitting is unique up to isomorphism.

(c) The irreducible Seifert form pairs are given by the types with the following names.

\[
\begin{align*}
\text{Seif}(\lambda, 1, n, \varepsilon) &\quad \text{with} \quad (\lambda = 1 & \& n \equiv 1(2)) \quad (2.3) \\
&\quad \text{or} \quad (\lambda = -1 & \& n \equiv 0(2)), \\
\text{Seif}(\lambda, 2, n) &\quad \text{with} \quad (\lambda = 1 & \& n \equiv 0(2)) \quad (2.4) \\
&\quad \text{or} \quad (\lambda = -1 & \& n \equiv 1(2)), \\
\text{Seif}(\lambda, 2, n, \zeta) &\quad \cong \quad \text{Seif}(\overline{\lambda}, 2, n, \overline{\zeta}) \quad (2.5) \\
&\quad \text{with} \quad \lambda, \zeta \in S^1 - \{\pm 1\}, \zeta^2 = \overline{\lambda} \cdot (-1)^{n+1}, \\
\text{Seif}(\lambda, 2, n) &\quad \text{with} \quad \lambda \in \mathbb{R}_{>1} \cup \mathbb{R}_{<1}, \quad (2.6) \\
\text{Seif}(\lambda, 4, n) &\quad \text{with} \quad \lambda \in \{\zeta \in \mathbb{C} \mid |\zeta| > 1, \text{Im} \zeta > 0\}. \quad (2.7)
\end{align*}
\]

Here $n \in \mathbb{Z}_{>1}, \varepsilon \in \{\pm 1\}$. The types are uniquely determined by the properties above of $\lambda$ and $n$ and the following properties. $H_\lambda$ and Jordan blocks are meant with respect to $M$.

\[2.3\] \text{Seif}(\lambda, 1, n, \varepsilon) : \text{dim} H_\mathbb{R} = n, H_\mathbb{C} = H_\lambda, \text{one Jordan block, for each } a \in H_\mathbb{R} - \text{Im} N

\[L(a, N^{n-1}a) \in \varepsilon \cdot \mathbb{R}_{>0}.\]
Seif(λ, 2, n) : dim \( H_\mathbb{R} = 2n \), \( H_\mathbb{C} = H_\lambda \), two Jordan blocks of size \( n \).

Seif(λ, 2, n, ζ) : dim \( H_\mathbb{R} = 2n \), \( H_\mathbb{C} = H_\lambda \oplus H_{\lambda^{-1}} \), two Jordan blocks of size \( n \), for each \( a \in H_\lambda - \text{Im} N \).

\( L(a, N^{n-1}a) \in \zeta \cdot \mathbb{R}_{>0} \).

Seif(λ, 2, n) : dim \( H_\mathbb{R} = 2n \), \( H_\mathbb{C} = H_\lambda \oplus H_{\lambda^{-1}} \), two Jordan blocks of size \( n \).

Seif(λ, 4, n) : dim \( H_\mathbb{R} = 4n \), \( H_\mathbb{C} = H_\lambda \oplus H_{\lambda^{-1}} \oplus H_{\lambda^2} \oplus H_{\lambda^{-2}} \), four Jordan blocks of size \( n \).

In this paper, only \( S^1 \)-Seifert form pairs will be relevant. So we will not need the types in (2.6) and (2.7). Only for completeness sake, we have given the full classification.

The signature of \( I_s \) will be useful in the case \( n = 3 \) in section 5 for determining the irreducible Seifert form pairs.

**Lemma 2.4.** [BH17, Lemma 2.10] The following table lists for the irreducible Seifert form pairs in theorem 2.3 (c) the signature of \( I_s \).

| type of a Seifert form pair | signature of \( I_s \) |
|-----------------------------|-------------------------|
| Seif(1, 1, n, ε) with \( n \equiv \varepsilon(4) \) | \( (\frac{n-1}{2}, 0, \frac{n-1}{2}) \) |
| Seif(1, 1, n, ε) with \( n \equiv -\varepsilon(4) \) | \( (\frac{n-1}{2}, 0, \frac{n+1}{2}) \) |
| Seif(-1, 1, n, ε) with \( n - 1 \equiv \varepsilon(4) \) | \( (\frac{n-1}{2}, 1, \frac{n-2}{2}) \) |
| Seif(-1, 1, n, ε) with \( n - 1 \equiv -\varepsilon(4) \) | \( (\frac{n-2}{2}, 1, \frac{n-2}{2}) \) |
| Seif(1, 2, n) with \( n \equiv 0(2) \) | \( (n, 0, n) \) |
| Seif(-1, 2, n) with \( n \equiv 1(2) \) | \( (n-1, 2, n-1) \) |
| Seif(λ, 2, n, ε) with \( n \equiv 0(2) \) | \( (n, 0, n) \) |
| (and \( λ \in S^1 - \{\pm1\} \)) | |
| Seif(λ, 2, n, ζ) with \( n \equiv 1(2) \) | \( (n-1, 0, n+1) \) |
| (and \( λ \in S^1 - \{\pm1\} \)) | |
| Seif(λ, 2, n, -ζ) with \( n \equiv 1(2) \) | \( (n+1, 0, n-1) \) |
| (and \( λ \in S^1 - \{\pm1\} \)) | |
| Seif(λ, 2, n) with \( λ \in \mathbb{R}_{>1} \cup \mathbb{R}_{<-1} \) | \( (n, 0, n) \) |
| Seif(λ, 4, n) with \( λ \in \{ζ \in \mathbb{C} \mid |ζ| > 1, \text{Im} ζ > 0\} \) | \( (2n, 0, 2n) \) |

Here \( n \in \mathbb{Z}_{\geq 1} \), \( ε \in \{\pm1\} \), and in the lines 7–9 \( ζ := \frac{\sqrt{n+1}}{|λ+1|} \cdot i^{n+1} \).

Now we turn to spectral pairs, first in an elementary abstract setting.

**Definition 2.5.** (a) A spectral pair is a pair \( (\alpha, k) \in \mathbb{R} \times \mathbb{Z} \). An unordered tuple of \( n \) spectral pairs is denoted by

\[ Spp = \sum_{(\alpha, k) \in \mathbb{R} \times \mathbb{Z}} d(\alpha, k)(\alpha, k) \in \mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z}) \subset \mathbb{Z}(\mathbb{R} \times \mathbb{Z}) \]
with \(|\text{Spp}| := \sum_{(\alpha, k)} d(\alpha, k) = n\). Here \(\mathbb{Z}(\mathbb{R} \times \mathbb{Z})\) is the group ring over \(\mathbb{R} \times \mathbb{Z}\). The number \(d(\alpha, k)\) is the multiplicity of \((\alpha, k)\) as a spectral pair. Any numbering of the \(n\) spectral pairs gives \(\text{Spp} = \sum_{j=1}^{n} (\alpha_j, k_j)\).

(b) (i) A **spectral pair ladder** (short: spp-ladder) consists of \(l + 1\) spectral pairs

\[(\alpha + k, m + l - 2k) \quad \text{with} \quad k \in \{0, 1, \ldots, l\}. \tag{2.8}\]

Here \(m \in \mathbb{Z}\) and \(l \in \mathbb{Z}_{\geq 0}\). The numbers \(m\) and \(l\) are uniquely determined by the spectral pair ladder. \(l + 1\) is its **length**, and \(m\) is its **center**. Its **first spectral pair** is the pair \((\alpha, m + l)\). Its **first spectral number** is \(\alpha\). The spp-ladder is determined by \(m\) and \(l\) and its first spectral number.

(ii) The **partner spp-ladder** is the spp-ladder \((m - l - 1 - \alpha + k, m + l - 2k)\) with \(k \in \{0, 1, \ldots, l\}\). \tag{2.9}

It has the same length and center. The **distance** of an spp-ladder to its partner is \(2\alpha + l + 1 - m\).

(iii) A spp-ladder is **single** if it is its own partner, i.e. if the distance to its partner is 0, i.e. if \(\alpha = \frac{m-l-1}{2}\).

(c) An unordered pair of spp-ladders (short: sppl-pair) consists of two spp-ladders which are partners of one another and which have distance \(\neq 0\).

**Lemma 2.6.** (a) Each sppl-pair and each single spp-ladder with center \(m\) are invariant under the Kleinian group \(\text{id}, \pi_1, \pi_2, \pi_3 : \mathbb{R} \times \mathbb{Z} \to \mathbb{R} \times \mathbb{Z}\) with

\[
\pi_1 : \left(\frac{m-1}{2} + \alpha, m + k\right) \mapsto \left(\frac{m-1}{2} - \alpha, m - k\right), \tag{2.10}\]

\[
\pi_2 : \left(\frac{m-1-k}{2} + \alpha, m + k\right) \mapsto \left(\frac{m-1-k}{2} - \alpha, m + k\right),
\]

\[
\pi_3 = \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 : (\alpha, m + k) \mapsto (\alpha + k, m - k).
\]

In the case of a sppl-pair, \(\pi_3\) maps each spp-ladder to itself, \(\pi_1\) and \(\pi_2\) map the two spp-ladders to one another.

(b) Suppose that a tuple \(\text{Spp} \in \mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z})\) of \(n\) spectral pairs is built from sppl-pairs and single spp-ladders with center \(m\). Then the sppl-pairs and the single spp-ladders are uniquely determined by \(\text{Spp}\).

(c) Suppose that a tuple \(\text{Spp} \in \mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z})\) of \(n\) spectral pairs is built from sppl-pairs and single spp-ladders with center \(m\). Then \(\text{Spp} \mod 2\mathbb{Z} \times \{0\}\) determines each spp-ladder uniquely up to simultaneous shift of its members by elements of \(2\mathbb{Z} \times \{0\}\), so it determines the lengths, the centers and the first spectral numbers modulo \(2\mathbb{Z}\) of all spp-ladders.
Proof: Trivial.

Definition 2.7. Let \((H_\mathbb{R}, L)\) be an \(S^1\)-Seifert form pair.

(a) An enhancement of it is a decomposition of \((H_\mathbb{R}, L)\) into a direct and \(L\)-orthogonal sum of Seifert form pairs \((H^{(j)}_\mathbb{R}, L^{(j)})\) with \(j \in \{1, \ldots, r\}\) for some \(r \in \mathbb{Z}_{\geq 1}\) together with spectral pairs \(\text{Spp}^{(j)} \in \mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z})\) with the following properties.

(i) \(\text{Spp}^{(j)}\) consists of finitely many copies of the same sppl-pair or the same single spp-ladder. Its length is called \(l_j\). All sppl-pairs and spp-ladders in \(\text{Spp} := \sum_{j=1}^{r} \text{Spp}^{(j)}\) have the same center \(m \in \mathbb{Z}\). This is also called the center of the enhancement. The first spectral number of the one of the two spp-ladders is called \(\alpha_j\) (if there are two, it does not matter which one).

\[|\text{Spp}^{(j)}| = \dim H^{(j)}_\mathbb{R}.\]

(ii) \((H^{(j)}_\mathbb{R}, L^{(j)})\) decomposes into copies of one irreducible Seifert form pair

\[
\text{Seif}((-1)^{m+1}e^{-2\pi i \alpha_j}, 2, l_j + 1, \zeta_j) \text{ in } (2.5) \quad \text{if } 2\alpha_j + l_j + 1 - m \in \mathbb{R} - \mathbb{Z},
\]

\[
\text{Seif}((-1)^{m+1}e^{-2\pi i \alpha_j}, 2, l_j + 1) \text{ in } (2.4) \quad \text{if } 2\alpha_j + l_j + 1 - m \in \mathbb{Z} - 2\mathbb{Z},
\]

\[
\text{Seif}((-1)^{m+1}e^{-2\pi i \alpha_j}, 1, l_j + 1, \varepsilon_j) \text{ in } (2.3) \quad \text{if } 2\alpha_j + l_j + 1 - m \in 2\mathbb{Z}.
\]

(b) An enhancement with center \(m\) is polarized if in (a)(ii)

\[
(\varepsilon_j \text{ resp. } \zeta_j) = e^{\frac{1}{2} \pi i (2\alpha_j + l_j + 1 - m)}.
\]

An enhancement with center \(m\) is signed polarized if in (a)(ii)

\[
(\varepsilon_j \text{ resp. } \zeta_j) = (-1)^{l_j} e^{\frac{1}{2} \pi i (2\alpha_j + l_j + 1 - m)}.
\]

Remarks 2.8. (i) Claim: An \(S^1\)-Seifert form pair \((H_\mathbb{R}, L)\) with (signed) polarized enhancement gives rise to and is equivalent to a split (signed) Steenbrink polarized mixed Hodge structure on \(H_C\).

The notions mixed Hodge structure and split mixed Hodge structure are standard, see e.g. [BH17, Def. 3.3 (a) and Remark 3.7]. The notion Steenbrink polarized mixed Hodge structure is defined in [BH17, Def. 3.3 (d)]. The signed version is defined in [BH17, Def. 6.1]. The signed version turns up in the case of isolated hypersurface singularities. The unsigned version turns up in \(M\)-tame functions. For both cases see section 6.

The claim follows easily from the results in [BH17], especially theorem 4.4. It builds on Deligne’s \(P^a\) of a mixed Hodge structure, on the polarizing form of a polarized mixed Hodge structure, and on the relation between Seifert form pairs and isometric triples, which is developed in [BH17].
(ii) Nemethi [Ne95] considered the case of an isolated hypersurface singularity \( f \) and studied there the relationship between the spectral pairs \( \text{Spp}(f) \) of Steenbrink’s mixed Hodge structure and the real Seifert form. He found that \( \text{Spp}(f) \mod 2\mathbb{Z} \times \{0\} \) is equivalent to the isomorphism class of the real Seifert form. The following lemma recovers this result modulo the claim above in (i).

(iii) But for this result one has to know a priori that \( \text{Spp}(f) \) comes from a signed Steenbrink polarized mixed Hodge structure, or that \( \text{Spp}(f) \) is part of a \emph{signed polarized} enhancement of the real Seifert form.

**Lemma 2.9.** Two \( S^1 \)-Seifert form pairs \( (H_i^R, L_i) \) for \( i \in \{1, 2\} \) with polarized enhancements (or with signed polarized enhancements) with centers \( m \) and spectral pairs \( \text{Spp}^i \) satisfy

\[
(H_1^R, L_1) \cong (H_2^R, L_2) \iff \text{Spp}^1 \equiv \text{Spp}^2 \mod 2\mathbb{Z} \times \{0\}. \tag{2.13}
\]

**Proof:** One can refine the decompositions of \( (H_1^R, L_1) \) and \( (H_2^R, L_2) \) in their enhancements to decompositions into sums of irreducible Seifert form pairs such that each comes equipped with a single spp-ladder or a sppl-pair. Then the irreducible Seifert form pair determines the length \( l \) of the single spp-ladder or of each spp-ladder in the sppl-pair. The center of the spp-ladder(s) is \( m \).

The first spectral number \( \alpha \) of the single spp-ladder or the first spectral numbers \( \tilde{\alpha} \) of the two spp-ladders in the sppl-pair are determined modulo \( \mathbb{Z} \) by \( e^{-2\pi i \alpha} = (-1)^{m+1} \lambda \) and \( e^{-2\pi i \tilde{\alpha}} = (-1)^{m+1} \bar{\lambda} \), where \( \lambda \) and \( \bar{\lambda} \) are the eigenvalue(s) of the irreducible Seifert form pair.

\( \alpha \) and \( \tilde{\alpha} \) are determined modulo \( 2\mathbb{Z} \) by the condition (2.11) respectively (2.12) in the cases (2.3) and (2.5). In the case (2.4), they satisfy \( \alpha \in \frac{1}{2} \mathbb{Z} \) and \( \tilde{\alpha} \equiv \alpha + 1(2) \).

Therefore the isomorphism class of \( (H_i^R, L_i) \) determines the union \( \text{Spp}^i \) of all spp-ladders in the enhancement modulo \( 2\mathbb{Z} \times \{0\} \). This proves \( \Rightarrow \).

\( \Leftarrow \): Let \( (\alpha_j, m_j, l_j) \) for \( j \in \{1, ..., \rho^1\} \) be the first spectral numbers, the centers and the lengths minus one of the spectral pair ladders in \( \text{Spp}^1 \). By lemma 2.6(c), the triples \( (\alpha_j \mod 2\mathbb{Z}, m_j, l_j) \) are determined by \( \text{Spp}^1 \mod 2\mathbb{Z} \times \{0\} \). Definition 2.7 and (2.3) and (2.4) show that each such triple determines a unique irreducible Seifert form pair in \( (H_1^R, L_1) \). In the case of a sppl-pair, the triples of the two spp-ladders determine the same irreducible Seifert form pair. This shows \( \Leftarrow \). \( \square \)

Finally, we put the matrices in \( T(n, \mathbb{R}) \) into the frame of Seifert form pairs.
Lemma 2.10. Let \((H_\mathbb{R}, L)\) be an \(S^1\)-Seifert form pair with \(\dim H_\mathbb{R} = n \in \mathbb{Z}_{\geq 1}\). The following data are equivalent.

(A) A basis \(v = (v_1, \ldots, v_n)\) with \(L(v^t, v) \in T(n, \mathbb{R})\) up to the signs of the basis vectors \(v_j\).

(B) A splitting \(H_\mathbb{R} = \bigoplus_{j=1}^n H_\mathbb{R}^{(j)}\) with \(\dim H_\mathbb{R}^{(j)} = 1\), \(L(H_\mathbb{R}^{(i)}, H_\mathbb{R}^{(j)}) = 0\) for \(i < j\) and \(L(H_\mathbb{R}^{(j)}, H_\mathbb{R}^{(j)}) = \mathbb{R}_{\geq 0}\).

(C) A complete flag \(\{0\} \subset U_0 \subset U_1 \subset U_2 \subset \ldots \subset U_n = H_\mathbb{R}\) (complete flag means \(\dim U_j = j\)) with

\[
H_\mathbb{R} = \bigoplus_{j=1}^n H_\mathbb{R}^{(j)} \quad \text{where} \quad H_\mathbb{R}^{(j)} := U_j \cap U_{j-1}^{LR}, \quad (2.14)
\]

\[
L(H_\mathbb{R}^{(j)}, H_\mathbb{R}^{(j)}) = \mathbb{R}_{\geq 0}. \quad (2.15)
\]

Proof: (A)\(\Rightarrow\)(B): Put \(H_\mathbb{R}^{(j)} := \mathbb{R} \cdot v_j\).

(B)\(\Rightarrow\)(A): For each \(j\) choose a basis vector \(v_j\) of \(H_\mathbb{R}^{(j)}\) with \(L(v_j, v_j) = 1\). It exists and is unique up to the sign.

(B)\(\Rightarrow\)(C): Put \(U_j := \bigoplus_{i \leq j} H_\mathbb{R}^{(i)}\). Then \(U_{j-1}^{LR} = \bigoplus_{i \geq j} H_\mathbb{R}^{(i)}\).

(C)\(\Rightarrow\)(B): \(H_\mathbb{R}^{(j)}\) has because of \(\dim U_j + \dim U_{j-1}^{LR} = n + 1\) at least dimension 1. By (2.14) it has dimension 1. \(\square\)

Remarks 2.11. (i) A splitting as in (B) can be called a semiorthogonal decomposition. Such splittings are considered in a much richer context in derived algebraic geometry.

(ii) The complete flag in (C) and the positivity condition (2.15) might remind one of Hodge structures. But there is no close relationship.

(iii) In the case of isolated hypersurfaces the data in lemma 2.10 come from a distinguished basis, a refinement of the \(\mathbb{Z}\)-lattice structure. Steenbrink’s mixed Hodge structure is of a transcendent origin and has a clear relationship with the real structure, but no known relationship with distinguished bases.

(iv) Nevertheless, the wish to associate to matrices \(S \in T(n, \mathbb{R})\) spectral pairs, can now be interpreted as the wish to see in the data in lemma 2.10 a shadow of mixed Hodge structures.

(v) Let \((H_\mathbb{R}, L)\) be a real Seifert form pair. The set of all complete flags in \(H_\mathbb{R}\) is a real projective algebraic manifold \(M^{\text{flags}}\). For any complete flag \(U_*\), the condition (2.14) is equivalent to the condition

\[
U_j \oplus U_j^{LR} = H_\mathbb{R} \quad \text{for any} \quad j \in \{1, \ldots, n\}. \quad (2.16)
\]

Let us call complete flags which do not satisfy (2.14) degenerate. They form a Zariski closed subvariety \(M^{\text{degen}}\) in \(M^{\text{flags}}\), which separates the complement into components. For each component a tuple \((\varepsilon_1, \ldots, \varepsilon_n) \in \)
\(\{\pm 1\}^n\) with
\[
L(H_{\mathbb{R}}^{(j)}, H_{\mathbb{R}}^{(j)}) = \varepsilon_j \cdot \mathbb{R}_{\geq 0}
\]
exists, where \(U_\ast\) is in the component and \(H_{\mathbb{R}}^{(j)}\) is defined as in (2.14). This follows from the nondegeneracy of \(L\). The components with \((\varepsilon_1, ..., \varepsilon_n) = (1, ..., 1)\) give by (B) \(\Rightarrow\) (A) sets of matrices \(L(\mu^t, \nu)\) in \(T(n, \mathbb{R})\). The wish to associate to matrices \(S \in T(n, \mathbb{R})\) spectral pairs, is the wish to associate to each such component spectral pairs.

(vi) A refinement of it is the wish to associate to each complete flag in \(M^{\text{flags}} - M^{\text{deg}}\) in a component with \((\varepsilon_1, ..., \varepsilon_n) = (1, ..., 1)\) an enhancement of \((H_{\mathbb{R}}, L)\). In the case of \(S \in \bigcup_{k=1,2} T_{\text{HOR}k}(n, \mathbb{R})\), we will obtain such an enhancement.

(vii) There are Seifert form pairs \((H_{\mathbb{R}}, L)\) for which \(M^{\text{flags}} - M^{\text{deg}}\) has no components with \((\varepsilon_1, ..., \varepsilon_n) = (1, ..., 1)\), i.e. which are not isomorphic to \((M(n \times 1, \mathbb{R}), \tilde{L})\) with \(\tilde{L}(a, b) = a^t \cdot S \cdot b\) for any \(S \in T(n, \mathbb{R})\). Any sum of irreducible Seifert form pairs \(\text{Seif}(1, 1, 1, -1), \text{Seif}(-1, 1, 2, -1), \text{Seif}(-1, 2, 1), \text{Seif}(\lambda, 2, 1, \zeta)\)
(with \(\lambda \in S^1 - \{\pm 1\}\) and \(\zeta = \frac{i+1}{\sqrt{n+1}} \cdot i^{n+1}\)) has this property because then \(I_s\) is negative (semi)definite by lemma [2.3]. In the cases \(n \in \{2, 3\}\) the only other Seifert form pairs with this property are those which contain \(\text{Seif}(1, 1, 1, -1)\) or \(\text{Seif}(1, 1, 3, -1)\), see remark 5.3 (ii).

3. A Recipe for Spectral Pairs
Section 4 will present the subspaces \(T_{\text{HOR}k}(n, \mathbb{R})\) of \(T(n, \mathbb{R})\) for \(k \in \{1, 2\}\) and study the properties of the matrices in these subspaces. Here we prepare this. We will introduce isomorphic subspaces \(T^{\text{scal}}_{\text{HOR}k}(n, \mathbb{R}) \subset [0, 1]^n \subset \mathbb{R}^n\) and \(T^{\text{deg}}_{\text{HOR}k}(n, \mathbb{R}) \subset \mathbb{R}[x]_{\text{deg}=n}\) and propose for each of them a recipe for spectral pairs.

Definition 3.1. For \(n \in \mathbb{Z}_{\geq 1}\) define the spaces
\[
T^{\text{scal}}_{\text{HOR}1}(n, \mathbb{R}) := \{(\beta_1, ..., \beta_n) \in [0, 1]^n | \beta_1 \leq ... \leq \beta_n, \beta_j + \beta_{n+1-j} = 1\},
\]
\[
T^{\text{scal}}_{\text{HOR}2}(n, \mathbb{R}) := \{(\beta_1, ..., \beta_n) \in [0, 1]^n | 0 = \beta_1 \leq ... \leq \beta_n, \beta_j + \beta_{n+2-j} = 1 \text{ for } j \geq 2\},
\]
\[
T^{\text{simp}}(n) := \{(\beta_1, ..., \beta_n) \in [0, \frac{1}{2}]^n | \beta_1 \leq ... \leq \beta_n\}.
\]
Define the map
\[ \Pi : \bigcup_{k=1,2} T_{\text{HOR}k}^\text{scal}(n, \mathbb{R}) \to \mathbb{R}[x]_{\deg=n} \] (3.4)
\[ \beta = (\beta_1, ..., \beta_n) \mapsto \prod_{j=1}^n (x - e^{-2\pi i \beta_j}) \]
and the spaces
\[ T_{\text{HOR}k}^\text{pol}(n, \mathbb{R}) := \Pi(T_{\text{HOR}k}^\text{scal}(n, \mathbb{R})) \subset \mathbb{R}[x]_{\deg=n} \text{ for } k \in \{1, 2\}. \] (3.5)

Lemma 3.2. (a) \( T_{\text{simp}}^\text{scal}(n) \) is the \( n \)-simplex in \( \mathbb{R}^n \) with the \( n+1 \) corners \( (x_{1j}, ..., x_{nj}) \) for \( j \in \{0, 1, ..., n\} \) with \( x_{ij} = 0 \) for \( i \leq j \) and \( x_{ij} = \frac{1}{2} \) for \( i > j \).

(b) The following maps are affine linear isomorphisms. For odd \( n \)
\[ T_{\text{HOR}1}^\text{scal}(n, \mathbb{R}) \to T_{\text{simp}}^\text{scal}(\frac{n-1}{2}), \quad (\beta_1, ..., \beta_n) \mapsto (\beta_1, ..., \beta_{n-1}), \]
\[ T_{\text{HOR}2}^\text{scal}(n, \mathbb{R}) \to T_{\text{simp}}^\text{scal}(\frac{n-1}{2}), \quad (\beta_1, ..., \beta_n) \mapsto (\beta_2, ..., \beta_{n+1}). \]
For even \( n \)
\[ T_{\text{HOR}1}^\text{scal}(n, \mathbb{R}) \to T_{\text{simp}}^\text{scal}(\frac{n}{2}), \quad (\beta_1, ..., \beta_n) \mapsto (\beta_1, ..., \beta_n), \]
\[ T_{\text{HOR}2}^\text{scal}(n, \mathbb{R}) \to T_{\text{simp}}^\text{scal}(\frac{n-2}{2}), \quad (\beta_1, ..., \beta_n) \mapsto (\beta_2, ..., \beta_{n+1}). \]
(c) The map \( \Pi \) in (3.4) is injective, and
\[ T_{\text{HOR}1}^\text{pol}(n, \mathbb{R}) = \{ p \in \mathbb{R}[x] | \deg p = n, p_n = 1, p_j = p_{n-j}, \]
\[ \text{all zeros of } p \text{ are in } S^1 \}, \] (3.6)
\[ T_{\text{HOR}2}^\text{pol}(n, \mathbb{R}) = \{ p \in \mathbb{R}[x] | \deg p = n, p_n = 1, p_j = -p_{n-j}, \]
\[ \text{all zeros of } p \text{ are in } S^1 \}. \] (3.7)
If \( p \in T_{\text{HOR}k}^\text{pol}(n, \mathbb{R}) \) then \( p_0 = (-1)^{k-1} \), \( p_j = p_0 p_{n-j} \), \( x^np(x^{-1}) = p_0 \cdot p(x) \), \( \lambda \in S^1 \) and \( \overline{\lambda} \) have the same multiplicity as zeros of \( p \), and the multiplicity of 1 as a zero of \( p \) is even for \( k = 1 \) and odd for \( k = 2 \).

Proof: (a) Trivial.
(b) For \( \beta \in T_{\text{HOR}1}^\text{scal}(n, \mathbb{R}) \) the symmetry \( \beta_j + \beta_{n+1-j} \) is used. For odd \( n \) it implies \( \beta_{n+1} = \frac{1}{2} \). For \( \beta \in T_{\text{HOR}2}^\text{scal}(n, \mathbb{R}) \) \( \beta_1 = 0 \) and the symmetry \( \beta_j + \beta_{n+2-j} = 1 \) for \( j \geq 2 \) are used. For even \( n \) the symmetry implies \( \beta_{n+2} = \frac{1}{2} \).
(c) Trivial. \( \square \)
The following recipe formalizes the recipe
\[ \text{(eigenvalues of } R_{(k)}^\text{mat}(S)) \mapsto \text{(spectral numbers } \text{Sp}(S)) \]
which is implicit in the proof of theorem 1.3 (b). In definition 4.4 (c) and theorem 4.5 (d) it is connected with theorem 1.3 (b). It is completely elementary, but interesting in its own right.

**Recipe 3.3.** (a) The following recipe associates to any tuple \( \beta \in T_{\text{scal}}^\text{HOR} k(n, R) \) for \( k \in \{1, 2\} \) a spectrum \( \text{Sp}(\beta) = \sum_{j=1}^{n}(\alpha_j) \in \mathbb{Z}_{\geq 0}(\mathbb{R}) \).
Define for \( j \in \{1, \ldots, n\} \)
\[
\gamma_j := \frac{1}{n}(j - \frac{k}{2}) = \begin{cases} \frac{1}{n}(j - \frac{1}{2}) & \text{if } k = 1, \\ \frac{1}{n}(j - 1) & \text{if } k = 2, \end{cases} \tag{3.8}
\]
\[
\alpha_j := n(\beta_j - \gamma_j) = \begin{cases} n\beta_j - j + \frac{1}{2} & \text{if } k = 1, \\ n\beta_j - j + 1 & \text{if } k = 2. \end{cases} \tag{3.9}
\]

(b) The following extends the recipe in (a) to a recipe for spectral pairs \( \text{Spp}(\beta) := \sum_{j=1}^{n}(\alpha_j, k_j) \in \mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z}) \). See lemma 3.4 for the properties of \( \text{Spp}(\beta) \). Consider \( \kappa \in S^1 \) with \( \{\beta_j | e^{-2\pi i \beta_j} = \kappa\} \neq \emptyset \).
Then the recipe in (a) gives in fact
\[
\sum_{j: \exp(-2\pi i \beta_j) = \kappa} (\alpha_j) = \sum_{j=0}^{l}(\alpha + j) \text{ for some } \alpha \in \mathbb{R}, l \in \mathbb{Z}_{\geq 0} \tag{3.10}
\]
(if \( k = 1 \) and \( \beta_1 = 0 \) then \( (\alpha_1, \alpha_n) = \left(\frac{1}{2}, \frac{1}{2}\right) \), and if \( k = 2 \) and \( \beta_2 = 0 \) then \( (\alpha_1, \alpha_2, \alpha_n) = (0, -1, 1) \)). Extend this to the spp-ladder \( \sum_{j=0}^{l}(\alpha + j, 1 + l - 2j) \) of length \( l + 1 \) and center \( m = 1 \) as in definition 2.5 (b), and define \( \text{Spp}(\beta) \) as the sum of these spp-ladders.

(c) For a polynomial \( p \in T_{\text{HOR} k}^\text{pol}(n, \mathbb{R}) \) define the spectrum and the spectral pairs as follows,
\[
\text{Sp}(p) := \text{Sp}(\Pi^{-1}(p)), \quad \text{Spp}(p) := \text{Spp}(\Pi^{-1}(p)). \tag{3.11}
\]

The spectral numbers \( \alpha_1, \ldots, \alpha_n \) in this recipe are usually not ordered by size. But they satisfy the symmetry in part (b) of the following lemma. The lemma states also properties of the spectral pairs.

**Lemma 3.4.** (a) Denote \( \underline{\gamma} := (\gamma_1, \ldots, \gamma_n) \) in both cases \( k = 1 \) and \( k = 2 \). Then
\[
\underline{\gamma} \in T_{\text{HOR} k}^\text{scal}(n, \mathbb{R}), \quad \Pi(\underline{\gamma}) = x^n - (-1)^k, \tag{3.12}
\]
\[
\text{Spp}(\underline{\gamma}) = n \cdot (0, 1), \quad \text{Sp}(\underline{\gamma}) = n \cdot (0). \tag{3.13}
\]
(b) The spectral numbers $\alpha_1, ..., \alpha_n$ in the recipe satisfy the symmetry
\begin{equation}
\alpha_j + \alpha_{n+1-j} = 0 \quad \text{for } k = 1, \tag{3.14}
\end{equation}
\begin{equation}
\alpha_1 = 0, \alpha_j + \alpha_{n+2-j} = 0 \quad \text{for } k = 2 \text{ and } j \geq 2. \tag{3.15}
\end{equation}

$S_{\rm pl}(\beta)$ consists of $\text{sppl-pairs}$ and single $\text{spp-ladders}$ with center $m = 1$, for each value $\kappa \in S^1$ with $\{\beta_j \mid e^{-2\pi i \beta_j} = \kappa\} \neq \emptyset$ one $\text{spp-ladder}$. The partner of the $\text{spp-ladder}$ from $\kappa$ is the one from $\kappa$. The single $\text{spp-ladders}$ are those which come from $\kappa \in \{\pm 1\}$, so there are at most two of them.

(c) If $p \in T_{\text{HOR}k}^\text{pol}(n, \mathbb{R})$ then $(-1)^n p(-x) \in T_{\text{HOR}k}^\text{pol}(n, \mathbb{R})$ with $\tilde{k} \equiv k + n(2)$, and then
\begin{equation}
S_{\text{pp}}(p) = S_{\text{pp}}((-1)^n p(-x)). \tag{3.16}
\end{equation}

**Proof:** (a) Trivial.
(b) $\beta$ and $\gamma$ are both in $T_{\text{scal}k}^\text{pol}(n, \mathbb{R})$ and satisfy the same symmetry in (3.1) or (3.2). Thus the tuple $\frac{1}{n} \alpha = \beta - \gamma$ and the tuple $\gamma$ satisfy the symmetry in (3.14) or (3.15).
Consider as in part (b) of the recipe 3.3 $\kappa \in S^1$ with $\{\beta_j \mid e^{-2\pi i \beta_j} = \kappa\} \neq \emptyset$ and its $\text{spp-ladder}$. One sees easily with the symmetries (3.14) and (3.15) that the $\text{spp-ladders}$ for $\kappa$ and $\kappa$ are partners. Especially, those for $\kappa \in \{\pm 1\}$ are single $\text{spp-ladders}$.

(c) Write
\[
\tilde{p}(x) := (-1)^n p(-x), \quad \beta := \Pi^{-1}(p), \quad \tilde{\beta} = \Pi^{-1}(\tilde{p}), \quad \tilde{p} \in T_{\text{HOR}k}^\text{pol}(n, \mathbb{R}).
\]
Then $p_0 = (-1)^{k-1}$ and $\tilde{p}_0 = (-1)^{n+k-1}$ show the first line of (c).
For even $n$ and $k = 1$
\[
\tilde{\beta}_{\frac{n+1}{2}+j} = \frac{1}{2} + \beta_j \quad \text{and} \quad \tilde{\beta}_j = -\frac{1}{2} + \beta_{\frac{n+1}{2}+j} \quad \text{for } j = 1, ..., \frac{n}{2}.
\]
For odd $n$ and $k = 1$
\[
\tilde{\beta}_{\frac{n+1}{2}+j} = \frac{1}{2} + \beta_j \quad \text{for } j = 1, ..., \frac{n-1}{2},
\]
\[
\tilde{\beta}_j = -\frac{1}{2} + \beta_{\frac{n+1}{2}+j} \quad \text{for } j = 1, ..., \frac{n+1}{2}.
\]
For odd $n$ and $k = 2$
\[
\tilde{\beta}_{\frac{n+1}{2}+j} = \frac{1}{2} + \beta_j \quad \text{for } j = 1, ..., \frac{n+1}{2},
\]
\[
\tilde{\beta}_j = -\frac{1}{2} + \beta_{\frac{n+1}{2}+j} \quad \text{for } j = 1, ..., \frac{n-1}{2}.
\]
Observe that
\[
\tilde{\gamma} := \Pi^{-1} (x^n - (-1)^k) = \Pi^{-1} (x^n - (-1)^{\tilde{k}})
\]
is the $\gamma$-vector for $\tilde{k}$. As $\tilde{\gamma}$ is obtained from $\gamma$ as any $\tilde{\beta}$ from $\beta$, the tuples of differences $\tilde{\beta} - \tilde{\gamma}$ and $\beta - \gamma$ coincide up to reordering. Therefore $\text{Sp}(\tilde{\rho}) = \text{Sp}(\rho)$. Its extension to $\text{Spp}(\tilde{\rho}) = \text{Spp}(\rho)$ is rather obvious. □

It is interesting to ask about the images in $\mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z})$ and in $\mathbb{Z}_{\geq 0}(\mathbb{R})$ of the maps $\text{Spp}$ and $\text{Sp}$ from $T_{\text{HOR}}(n, \mathbb{R})$. The answer is not difficult, it is given in the following corollary. We omit the rather trivial proof.

**Corollary 3.5.** An unordered tuple $\sum_{\alpha \in \mathbb{R}} d(\alpha) (\alpha) \in \mathbb{Z}_{\geq 0}(\mathbb{R})$ of $n$ numbers (so $\sum_{\alpha \in \mathbb{R}} d(\alpha) = n$) is in $\text{Sp}(T_{\text{HOR}}(n, \mathbb{R}))$ if and only if the numbers can be ordered as $\alpha_1, ..., \alpha_n$ such that the symmetry in (3.14) respectively (3.15) holds and $\alpha_{j+1} \geq \alpha_j - 1$, and in the case $k = 1$ also $\alpha_1 \geq -\frac{1}{2}$.

An unordered tuple $\sum_{(\alpha, k) \in \mathbb{R} \times \mathbb{Z}} d(\alpha, k) (\alpha, k) \in \mathbb{Z}_{\geq 0}(\mathbb{R} \times \mathbb{Z})$ of $n$ pairs is in $\text{Spp}(T_{\text{HOR}}(n, \mathbb{R}))$ if and only if the pairs can be ordered as $(\alpha_1, k_1), ..., (\alpha_n, k_n)$ such that the conditions above hold and the tuple $\sum_{j=1}^n (\alpha_j, k_j)$ is obtained from the tuple $\sum_{j=1}^n (\alpha_j)$ by part (b) of recipe 3.3.

**Remarks 3.6.** (i) Corollary 3.5 implies that there is no gap of size $> 1$ in the spectral numbers if one orders them by size. This follows (with the order in corollary 3.5) from $\alpha_{j+1} \geq \alpha_j - 1$, from $\alpha_1 \geq -\frac{1}{2}$, $\alpha_n \leq \frac{1}{2}$ for $k = 1$, and from $\alpha_1 = 0$, $\alpha_2 \geq -1$, $\alpha_n \leq 1$ for $k = 2$.

(ii) Now conjecture 1.9 implies that the spectral numbers of isolated hypersurface singularities and $M$-tame functions have no gap of size $> 1$. This is not a very strong claim in the case of isolated hypersurface singularities (there usually the gaps between spectral numbers are much smaller), but it is new in any case.

**Examples 3.7.** (i) It is also interesting to ask about the preimages in $T_{\text{HOR}}(n, \mathbb{R})$ of spectral numbers or spectral pairs, especially for $\text{Sp}(f)$ with $f$ an isolated hypersurface singularity or an $M$-tame function. In these cases $\text{Spp}(f) \in \mathbb{Z}_{\geq 0}(\mathbb{Q} \times \mathbb{Z})$, and, even stronger, the characteristic polynomial $p_{\text{ch}, M}(x) := \prod_{j=1}^n (x - e^{-2\pi i \alpha_j})$ is in $\mathbb{Z}[x]$, i.e. it is a product of cyclotomic polynomials.

(ii) In most cases, the preimages, the polynomials $p \in T_{\text{HOR}}^\text{pol}(/mu, \mathbb{R})$ with the correct spectrum $\text{Sp}(p) = \text{Sp}(f) - \frac{m-1}{2}$, are not in $\mathbb{Z}[x]$. If one looks only at the correct eigenvalues, and not at the correct spectral
numbers, one obtains the possibly bigger set
\[
\{ p \in T_{\text{HOR}}^\text{pol}(\mu, \mathbb{R}) \mid p(x) = \prod_{j=1}^\mu (x - \kappa_j), \}
\]
(3.17)

\[
p_{\text{ch}, M}(x) = \prod_{j=1}^\mu (x - (-1)^{k+m-1} \kappa_j^\mu) \}
\]

Even this set does often not contain polynomials in \(\mathbb{Z}[x]\), for example for the singularity \(E_6\), see below (v).

(iii) Remarkable exceptions are the chain type singularities, which are treated in section 7. For them distinguished polynomials \(p \in \mathbb{Z}[x]\) with the correct spectrum \(\text{Sp}(p) = \text{Sp}(f) - \frac{m-1}{2}\) exist. This will be proved in theorem 7.6. The polynomials \(p\) are given in (7.6). In fact, in the moment, the chain type singularities are the only candidates within isolated hypersurface singularities for which we know polynomials \(p\) in \(\mathbb{Z}[x] \cap T_{\text{HOR}}^\text{pol}(\mu, \mathbb{R})\) with the correct spectrum.

(iv) If \(f(x_0, x_1)\) (so \(m = 1\)) is one of the ADE-singularities, then the spectral numbers satisfy \(-\frac{1}{2} < \alpha_1 \leq \ldots \leq \alpha_\mu < \frac{1}{2}\). Then the number of \(\beta \in T_{\text{HOR}}^\text{scal}(\mu, \mathbb{R})\) with \(\text{Sp}(\beta) = \text{Sp}(f)\) is (here \(2N!! := 2N \cdot (N-2)!!\))

\[
\mu!! \quad \text{if } \mu \text{ is even and the singularity is not } D_\mu,
\]
\[
(\mu - 1)!! \quad \text{if } \mu \text{ is odd},
\]
\[
\mu!! \cdot \frac{1}{2} \quad \text{if } \text{the singularity is } D_\mu \text{ and } \mu \text{ is even}.
\]

The numbers \(\beta_j\) must satisfy
\[
\beta_j = \gamma_j + \frac{1}{\mu} \alpha_{\sigma(j)}, \quad (3.18)
\]
the symmetry in (3.1) or (3.2), including \(\beta_1 = 0\) in (3.1),

\[
0 \leq \beta_1 \leq \ldots \leq \beta_\mu \leq 1,
\]
here \(\sigma \in S_\mu\) is a permutation. Because of

\[
\max_j |\alpha_j| < \frac{1}{2} = 1 - \mu \gamma_\mu = \frac{\mu}{2} (\gamma_j - \gamma_{j-1}) = \mu \gamma_1 - 0,
\]
one can choose \(\sigma \in S_\mu\) almost arbitrarily. Only the symmetry in (3.1) or (3.2) has to be observed. For all ADE-singularities except \(D_\mu\) with \(\mu\) even, the spectral numbers are pairwise different. For \(D_\mu\) with \(\mu\) even, \(\alpha_{\mu} = \alpha_{\mu+2} = 0\).

Though most of the polynomials \(p = \Pi(\beta)\) are not in \(\mathbb{Z}[x]\). The singularities \(A_{\mu}, D_\mu\) and \(E_7\) can be written as chain type singularities. Therefore by theorem 7.6 at least the polynomial in (7.6) is in \(\mathbb{Z}[x]\).
(v) But the singularity $E_6$ and many other singularities have characteristic polynomials $p_{ch,M}$ such that not even the set in (3.17) contains any polynomial in $\mathbb{Z}[x]$. For $E_6$ as a curve singularity (so $m = 1$) $p_{ch,M} = \Phi_{12}\Phi_6$. For $E_8$ as a curve singularity $p_{ch,M} = \Phi_{15}$, and the set in (3.17) with $k = 2$ contains the polynomial $p = \Phi_{15} \in \mathbb{Z}[x]$. But $\text{Sp}(p) \neq \text{Sp}(f)$.

4. HOR-matrices

In this section we will introduce the two subspaces $T_{\text{HOR}}(n,\mathbb{R})$ for $k \in \{1, 2\}$ of $T(n,\mathbb{R})$ and study the properties of matrices in these spaces. We call the matrices HOR-matrices because of the initials of the authors Horocholyn, Orlik and Randell of [Ho17] and [OR77]. Horocholyn studied half of the matrices and proved the crucial formula (4.20) [Ho17, ch. 2]. Orlik and Randell considered a subfamily which is related to the chain type singularities [OR77, (4.1) Conjecture] and which we will treat in section 7. Before coming to the HOR-matrices, we recall a well known fact from Picard-Lefschetz theory. For the convenience of the reader, we present also a proof.

**Theorem 4.1.** Let $n \in \mathbb{Z}_{\geq 1}$, let $H_\mathbb{R}$ be an $\mathbb{R}$-vector space with a basis $\underline{e} = (e_1, ..., e_n)$, and let $S \in \text{GL}(n,\mathbb{R})$.

(a) The matrix $S$ defines on $H_\mathbb{R}$ a bilinear form $L$, which is called Seifert form, a symmetric bilinear form $I_s$, an antisymmetric bilinear form $I_a$ and an automorphism $M$, which is called monodromy, by the formulas

$$L(\underline{e}^t, \underline{e}) = S^t,$$

$$I_s(\underline{e}^t, \underline{e}) = S + S^t, \quad \text{so} \quad I_s(a, b) = L(b, a) + L(a, b),$$

$$I_a(\underline{e}^t, \underline{e}) = S - S^t, \quad \text{so} \quad I_a(a, b) = L(b, a) - L(a, b),$$

$$M \underline{e} = \underline{e} \cdot S^{-1}S^t, \quad \text{so} \quad L(Ma, b) = L(b, a).$$

$L$ determines $I_s, I_a$ and $M$. The monodromy $M$ respects all three bilinear forms $L, I_s$ and $I_a$.

(b) Define endomorphisms $s_a^{(1)}$ and $s_b^{(2)}$ on $H_\mathbb{R}$ for $a \in H_\mathbb{R}$ with $I_s(a, a) = 2$ and for arbitrary $b \in H_\mathbb{R}$ by

$$s_a^{(1)}(c) := c - I_s(a, c) \cdot a, \quad s_b^{(2)}(c) := c - I_a(b, c) \cdot b.$$  

Then $s_a^{(1)}$ respects $I_s$ and is a reflection (semisimple, eigenvalues $1, ..., 1, -1$). And $s_b^{(2)}$ respects $I_a$ and is a pseudo-reflection ($s_b^{(2)} = \text{id}$ or $s_b^{(2)} - \text{id}$ nilpotent with one single $2 \times 2$ Jordan block).
(c) Now let $S = (s_{ij}) \in T(n, \mathbb{R})$ (so $s_{ij} = 0$ for $i > j$, $s_{jj} = 1$, and the eigenvalues of $S^{-1}S^t$ are in $S^1$). Then

$$(-1)^k \cdot M = s_{e_1}^{(k)} \circ \ldots \circ s_{e_n}^{(k)}$$

for $k \in \{1, 2\}$.

**Proof:**
(a) $L(b, a) = L(Ma, b)$ is equivalent to $L(Me^t, e) = L(e^t, e)^t$ which holds:

$$L(Me^t, e) = L((e \cdot S^{-1}S^t)^t, e) = SS^{-t} \cdot S^t = S = L(e^t, e)^t.$$

$M$ respects $L$ because of

$$L(Ma, Mb) = L(Mb, a) = L(a, b).$$

$M$ respects $I_s$ and $I_a$ because of their relation to $L$ in (4.2) and (4.3).

(b) $s_a^{(1)}$ respects $I_s$ because of

$$I_s(s_a^{(1)}(b), s_a^{(1)}(c)) = I_s(b - I_s(a, b)a, c - I_s(a, c)a)$$
$$= I_s(b, c) - I_s(a, b)I_s(a, c) - I_s(a, c)I_s(b, a) + I_s(a, b)I_s(a, c)I_s(a, a)$$
$$= I_s(b, c).$$

$s_a^{(1)}$ is a reflection because its restriction to $\{c \in H_\mathbb{R} \mid I_s(a, c) = 0\}$ is id and because of $s_a^{(1)}(a) = -a$.

$s_b^{(2)}$ respects $I_a$ because of

$$I_a(s_b^{(2)}(c), s_b^{(2)}(d)) = I_a(c - I_a(b, c)b, d - I_a(b, d)b)$$
$$= I_a(c, d) - I_a(b, c)I_a(b, d) - I_a(b, d)I_a(c, b) + I_a(b, c)I_a(b, d)I_a(b, b)$$
$$= I_a(c, d).$$

$s_b^{(2)}$ is a pseudo-reflection because its restriction to $\{c \in H_\mathbb{R} \mid I_a(b, c) = 0\}$ is id and this space has dimension $n - 1$ or $n$ and contains $b$.

(c) Denote $D_{kl} := (\delta_{ik} \cdot \delta_{jl})_{i,j=1,\ldots,n} \in M(n \times n, \mathbb{Z})$. Denote by $E_n := (\delta_{ij}) = \sum_{j=1}^n D_{jj}$ the $n \times n$ unit matrix. Observe

$$D_{ij}D_{kl} = 0 \text{ if } j \neq k,$$

which implies

$$(E_n + D_{ij})(E_n + D_{kl}) = E_n + D_{ij} + D_{kl} \text{ if } j \neq k,$$

$$(E_n + D_{ij})^{-1} = E_n - D_{ij} \text{ if } i \neq j.$$
\[ S = \begin{pmatrix} 1 & \cdots & s_{n-1,n} \\ & \ddots & \vdots \\ & & 1 \\ \end{pmatrix} \begin{pmatrix} 1 & \cdots & s_{n-2,n-1} \\ & \ddots & \vdots \\ & & 1 \\ \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \end{pmatrix} \begin{pmatrix} 1 & \cdots & \cdots & s_{1} \\ & 1 & \cdots & \cdots \\ & & \ddots & \cdots \\ \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \end{pmatrix}, \]

\[ S^{-1}S^t = \begin{pmatrix} 1 & -s_{12} & \cdots & -s_{1n} \\ & 1 & \cdots & \cdots \\ & & \ddots & \cdots \\ & & & 1 \\ \end{pmatrix} \begin{pmatrix} 1 & \cdots & -s_{n-2,n-1} \\ & \ddots & \cdots \\ & & 1 \\ \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \end{pmatrix} \begin{pmatrix} 1 & \cdots & \cdots & -s_{1} \\ & 1 & \cdots & \cdots \\ & & \ddots & \cdots \\ \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \end{pmatrix}, \]

where the \( n \times n \)-matrix

\[ \left( s^{(2)}_{e_j} \right)^{mat} := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \\ \end{pmatrix} \begin{pmatrix} s_{1j} & \cdots & s_{j-1,j} & \cdots & -s_{j+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \ddots \\ \end{pmatrix} \begin{pmatrix} \vdots \\ \cdots \\ \cdots \\ \cdots \\ \end{pmatrix} \]
satisfies
\[ s_{e_j}^{(2)} e = e \cdot (s_{e_j}^{(2)})^{\text{mat}}. \]

This shows \( M = s_{e_1}^{(2)} \circ \ldots \circ s_{e_n}^{(2)}. \) Define the matrix \((s_{e_j}^{(1)})^{\text{mat}}\) by
\[ s_{e_j}^{(1)} e = e \cdot (s_{e_j}^{(1)})^{\text{mat}}. \]

Observe
\[
(s_{e_j}^{(1)})^{\text{mat}} = \begin{pmatrix}
1 & & \\
& \ddots & \\
-s_{1j} & \ldots & 1 & -s_{jj+1} & \ldots & -s_{jn} \\
-s_{j-1,j} & \ldots & -s_{jj} & 1 & \\
& \ddots & \\
-s_{1j} & \ldots & -s_{jj} & 1 & \\
\end{pmatrix}
\]
\[ = \left( -\sum_{i=1}^{j-1} D_{ii} + \sum_{i=j}^{n} D_{ii} \right) \cdot (s_{e_j}^{(2)})^{\text{mat}} \cdot \left( -\sum_{i=1}^{j} D_{ii} + \sum_{i=j+1}^{n} D_{ii} \right) \]

This shows
\[ -S^{-1} S^t = (s_{e_1}^{(1)})^{\text{mat}} \circ \ldots \circ (s_{e_n}^{(1)})^{\text{mat}} \quad \text{and} \quad -M = s_{e_1}^{(1)} \circ \ldots \circ s_{e_n}^{(1)}. \]

**Corollary 4.2.** Consider the same situation as in theorem 4.1. Define the cyclic automorphism \( C \) by
\[ C e_j = e_{j+1} \quad \text{for} \ 1 \leq j \leq n - 1, \quad \text{and} \quad C e_n = e_1, \quad \text{and} \quad C^n = \text{id}. \]

Define the automorphisms \( R_{(kj)} \) for \( k \in \{1, 2\}, j \in \{1, \ldots, n\} \) of \( H_R \) by
\[ R_{(kj)} := C^{-(j-1)} \circ s_{e_j}^{(k)} \circ C^j. \]

Then
\[ R_{(kj)} e = e \cdot R_{(kj)}^{\text{mat}}. \]
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with

\[
R_{(1j)}^{\text{mat}} = \begin{pmatrix}
-s_{j,j+1} & \cdots & -s_{jn} & -s_{1j} & \cdots & -s_{j-1,j} & -1 \\
\vdots & & & & & & \\
0 & & & & & & E_{n-1}
\end{pmatrix},
\]

(4.10)

and

\[
R_{(2j)}^{\text{mat}} = \begin{pmatrix}
-s_{j,j+1} & \cdots & -s_{jn} & s_{1j} & \cdots & s_{j-1,j} & 1 \\
\vdots & & & & & & E_{n-1}
\end{pmatrix},
\]

(4.11)

and

\[
(-1)^k \cdot S^{-1} S^t = R_{(k1)}^{\text{mat}} \cdot \cdots \cdot R_{(kn)}^{\text{mat}}
\]

and

\[
(-1)^k \cdot M = R_{(k1)} \circ \cdots \circ R_{(kn)}.
\]

(4.12)

**Proof:** \((-1)^k M = R_{(k1)} \circ \cdots \circ R_{(kn)}\) is an immediate consequence of \((-1)^k M = s_{1(e)}^{(k)} \circ \cdots \circ s_{n(e)}^{(k)}\) and the definition of \(R_{(kj)}\) and \(C^n = \text{id}\). The formulas for \(R_{(kj)}^{\text{mat}}\) follow from the formulas for \((s_{e}^{(k)})^{\text{mat}}\). \(\square\)

**Remarks 4.3.** The matrices \(R_{(kj)}^{\text{mat}}\) are companion matrices. A companion matrix is here a matrix (empty places mean zeros)

\[
R^{\text{mat}} = \begin{pmatrix}
-p_{n-1} & -p_{n-2} & \cdots & -p_1 & -p_0 \\
\vdots & & & & \\
0 & & & & & E_{n-1}
\end{pmatrix}
\]

(4.13)

with \(p_{n-1}, \ldots, p_0 \in \mathbb{C}\). Its characteristic polynomial is \(p(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1x + p_0\). For each eigenvalue \(\kappa \in \mathbb{C}\), it has only one Jordan block. A basis of a Jordan block of size \(l + 1\) with eigenvalue \(\kappa\) is

\[
v_j = \begin{pmatrix}
(n-1)_j \cdot \kappa^{n-1} \\
(n-2)_j \cdot \kappa^{n-2} \\
\vdots \\
(j)_j \cdot \kappa^j \\
0 \\
\vdots \\
0
\end{pmatrix}\quad \text{for } j = 0, 1, \ldots, l,
\]

(4.14)

with

\[
(a)_b := a(a-1) \cdot \cdots \cdot (a-b+1) \quad \text{for } a \in \mathbb{C}, b \in \mathbb{Z}_{\geq 0},
\]

(4.15)
(and \((a)_0 = 1\)) and
\[
\left( \kappa^{-1}R^{\text{mat}} - E_n \right)v_j = j \cdot v_{j-1} \quad \text{(with } v_{-1} = 0). \tag{4.16}
\]

Here we used that \(\kappa\) is a zero of \(p^{(j)}(x) = (n)_j x^{n-j} + p_{n-1} (n-1)_j x^{n-1-j} + \ldots + p_1 (j)_j x^0\) for \(0 \leq j \leq l\), and we used
\[
(a)_b - (a - 1)_b = b \cdot (a - 1)_{b-1} \quad \text{for } b \in \mathbb{Z}_{\geq 1}. \tag{4.17}
\]

**Definition 4.4.** Fix \(n \in \mathbb{Z}_{\geq 1}\) and \(k \in \{1, 2\} \).

(a) The space of polynomials \(T^{\text{pol}}_{\text{HOR}}(n, \mathbb{R}) \subset \mathbb{R}[x]_{\deg=n}\) was defined in definition 3.1. Define the map
\[
S^{(k)} : T^{\text{pol}}_{\text{HOR}}(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}) \tag{4.18}
\]
\[
p(x) = x^n + p_{n-1} x^{n-1} + \ldots + p_0 \mapsto \begin{pmatrix}
1 & p_{n-1} & \ldots & p_2 & p_1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & 1
\end{pmatrix}
\]
Define its image as \(T_{\text{HOR}}(n, \mathbb{R}) := S^{(k)}(T^{\text{pol}}_{\text{HOR}}(n, \mathbb{R}))\) (theorem 4.5 (a) will show that it is a subspace of \(T(n, \mathbb{R})\)). Define the map
\[
R^{\text{mat}}_{(k)} : T_{\text{HOR}}(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}) \tag{4.19}
\]
\[
S = S^{(k)}(p) \mapsto \begin{pmatrix}
-p_{n-1} & -p_{n-2} & \ldots & -p_1 & -p_0 \\
& & & \ddots & \ddots \\
& & & & E_{n-1}
\end{pmatrix}
\]
(recall \(p_0 = (-1)^{k-1}\)). \(R^{\text{mat}}_{(k)}(S)\) is a companion matrix, and its characteristic polynomial is \(p(x)\) by remark 4.3.

(b) For \(S \in T_{\text{HOR}}(n, \mathbb{R})\) take up the data in theorem 4.1. Define an automorphism \(R_{(k)}(S) : H_{\mathbb{R}} \rightarrow H_{\mathbb{R}}\) by \(R_{(k)}(S) \mathcal{E} := \mathcal{E} \cdot R^{\text{mat}}_{(k)}(S)\).

(c) For \(S \in T_{\text{HOR}}(n, \mathbb{R})\) define \(\text{Spp}(S) := \text{Spp}(p)\) and \(\text{Sp}(S) := \text{Sp}(p)\) where \(p \in T^{\text{pol}}_{\text{HOR}}(n, \mathbb{R})\) is the characteristic polynomial of \(R^{\text{mat}}_{(k)}(S)\) (or, equivalently, of \(R_{(k)}(S)\)), and where \(\text{Spp}(p)\) and \(\text{Sp}(p)\) are defined in recipe 3.3 (c).

Definition 4.4 (a) and the next formula (4.20) are essentially due to Horocholyn [Ho17, ch. 2] (he considered half of the cases). He also studied the signature of \(S + S'\). Theorem 4.5 and corollary 4.6 encompass his results. In cases relevant for chain type singularities (see section 7), the matrices \(S\) and \(R^{\text{mat}}_{(k)}(S)\) are also given in [OR77]. But
there (4.20) is not even mentioned, although the authors are certainly aware of it.

**Theorem 4.5.** Choose \( S \in T_{\text{HOR}}(n, \mathbb{R}) \) and take up the data in theorem 4.4.

\((a)\) \( (-1)^k : S^{-1} S^t = R_{(k)}^{\text{mat}}(S)^n \) and \( (-1)^k : M = R_{(k)}(S)^n \). \hfill (4.20)

The generalized eigenspaces of \( R_{(k)}(S) \) are the spaces \( H_\kappa^{(R)} := \text{ker}((R_{(k)}(S) - \kappa \cdot \text{id})^n) \subset H_\mathbb{C} \) with \( p(\kappa) = 0 \). The generalized eigenspaces of \( M \) are the spaces \( H_\lambda = \bigoplus_{\kappa: (-1)^k \kappa^n = \lambda} H_\kappa^{(R)} \). Especially, \( T_{\text{HOR}}(n, \mathbb{R}) \subset T(n, \mathbb{R}) \). The monodromy \( M \) and the automorphism \( R_{(k)}(S) \) have a single Jordan block on \( H_\kappa^{(R)} \) (because of remark 4.3).

\((b)\) \( R_{(k)}(S) \) respects \( L \). Therefore \( H_\mathbb{R} \) decomposes \( L \)-orthogonally into the Seifert form pairs \( (H_\kappa^{(R)} \cap H_\mathbb{R}, L), (H_\bar{\kappa}^{(R)} \cap H_\mathbb{R}, L) \), and \( ((H_\kappa^{(R)} \oplus H_{\bar{\kappa}}^{(R)}) \cap H_\mathbb{R}) \) for each \( \kappa \in S^1 \) with \( \text{Im} \kappa > 0 \) and \( H_\kappa^{(R)} \neq \{0\} \).

\((c)\) \( \text{Spp}(S) \) and the decomposition of \( (H_\mathbb{R}, L) \) in \((b)\) give a polarized enhancement of \( (H_\mathbb{R}, L) \) (definition 2.7): \( \text{Spp}(S) \) consists of spp-ladders, one for each eigenvalue \( \kappa \) of \( R_{(k)}(S) \). The spp-ladder for \( \kappa \) has length \( l + 1 = \dim H_\kappa^{(R)} \), center \( m = 1 \), and first spectral number \( \alpha \) with \( e^{-2\pi i \alpha} = \kappa \). Furthermore

\[
L(a, N^{(\pi)}) \in e^{\frac{1}{2} \pi i (2a + l)} \cdot \mathbb{R}_{>0} \quad \text{for} \quad a \in H_\kappa^{(R)} - N(H_\kappa^{(R)}). \hfill (4.21)
\]

If \( \kappa = \pm 1 \), it is a single spp-ladder. If \( \kappa \neq \pm 1 \), the partner spp-ladder is the one for \( \bar{\kappa} \).

\((d)\) The underlying spectrum \( \text{Sp}(S) \) is the one which recipe 4.1 gives for \( S \) if it is applied to \( T_{\text{HOR}}(n, \mathbb{R}) \) (see part \((c)\) of theorem 4.3).

**Proof:** \((a)\) The coefficients \( p_{n-1}, \ldots, p_1 \) in the matrix

\[
S = \begin{pmatrix}
1 & p_{n-1} & \cdots & p_1 \\
& \ddots & \ddots & \vdots \\
& & \ddots & p_{n-1} \\
& & & 1
\end{pmatrix} \in T_{\text{HOR}}(n, \mathbb{R})
\]

satisfy \( p_{n-j} = (-1)^{k-1} p_j \). Therefore the matrices \( R_{(k)}^{\text{mat}}(S) \) for \( j \in \{1, \ldots, n\} \) in corollary 4.2 are all equal to one another and to \( R_{(k)}(S)^n \). Thus \( (-1)^k : M = R_{(k)}(S)^n \) and (4.20). The other statements are immediate consequences of (4.20).
(b) We have to prove $R_{(k)}^{\text{mat}}(S)^t \cdot S^t \cdot R_{(k)}^{\text{mat}}(S) = S^t$. Equivalent is $S \cdot R_{(k)}^{\text{mat}}(S) = R_{(k)}^{\text{mat}}(S)^{-t} \cdot S$. Recall $p_{n-j} = p_0 \cdot p_j$ and observe

$$R_{(k)}^{\text{mat}}(S)^{-1} = \begin{pmatrix} E_{n-1} \\ -p_0 & -p_1 & \cdots & -p_{n-1} \end{pmatrix},$$

$$R_{(k)}^{\text{mat}}(S)^{-t} = \begin{pmatrix} E_{n-1} \\ -p_0 & -p_1 & \vdots & \vdots \\ -p_{n-1} \end{pmatrix}.$$

One calculates $S \cdot R_{(k)}^{\text{mat}}(S)$ and $R_{(k)}^{\text{mat}}(S)^{-t} \cdot S$ and finds in both cases

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & p_{n-1} & \cdots & p_2 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & p_{n-1} \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$  

(c) All statements in part (c) except that the enhancement is polarized follow immediately from part (b) and from lemma 3.4(b).

It rests to show that the enhancement is polarized, i.e. $(4.21)$. $(-1)^k \cdot M = R_{(k)}(S)^n$ gives $N = n \cdot (\text{nilpotent part of } R_{(k)}(S))$. On $H_{\kappa}^{(R)}$

$$N^l = n^l \cdot (\text{nilpotent part of } R_{(k)}(S))^l = n^l \cdot (\kappa^{-1} R_{(k)}(S) - \text{id})^l.$$  

The vector $v_l$ in remark 4.3 corresponds to an element $a \in H_{\kappa}^{(R)} - N(H_{\kappa}^{(R)})$. We have to calculate the phase of

$$L(a, N^l) = v_l^t \cdot S^t \cdot (\kappa^{-1} R_{(k)}^{\text{mat}}(S) - E_n)^l \cdot \overline{\theta} = v_l^t \cdot S^t \cdot l! \cdot \overline{\theta}$$

and want to find $e^{\frac{1}{2} \pi i (2\alpha + l)}$. We denote $p_n := 1$.  

$$v_l^t \cdot S^t \cdot \overline{\theta} = \begin{pmatrix} (n-1)_{l \kappa^{n-1}} \\ (n-2)_{l \kappa^{n-2}} \\ \vdots \\ (l)_{l \kappa^l} \\ 0 \end{pmatrix}^t \begin{pmatrix} 1 & \cdots & \cdots & \cdots \\ p_{n-1} & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ p_1 & \cdots & p_{n-1} & 1 \end{pmatrix} \begin{pmatrix} \kappa^{n-1} \\ \kappa^{n-2} \\ \vdots \\ \kappa^l \\ 0 \end{pmatrix}.$$
Indeed, if

\[(\kappa)_{l+1} = \Pi_{n-1}^{n-2} + p_{n-1} \cdot \overline{\kappa}^{n-2} \]

\[+ \ldots + (l)_l \cdot \kappa^l \cdot (p_{l+1} \cdot \overline{\kappa}^{n-1} + p_{l+2} \cdot \overline{\kappa}^{n-2} + \ldots + p_n \cdot \overline{\kappa}^0) \]

\[= \left( (n-1)_l + (n-2)_l + \ldots + (l)_l \right) p_n \cdot \overline{\kappa}^0 + \ldots + (l+1)_l \cdot p_{l+1} \cdot \overline{\kappa}^{n-l-1} \]

\[= \frac{1}{l+1} \left[ (n)_{l+1} \cdot p_n \cdot \overline{\kappa}^0 + (n-1)_{l+1} \cdot p_{n-1} \cdot \overline{\kappa}^1 \right. \]

\[+ \ldots + \left. (l+1)_{l+1} \cdot p_{l+1} \cdot \overline{\kappa}^{n-l-1} \right] \]

\[= \frac{1}{l+1} \cdot \overline{\kappa}^{n-l-1} \cdot p(l+1)(\kappa). \]

The last equality uses

\[(l+1)(n)_{l+1} = (n-1)_l + (n-2)_l + \ldots + (l)_l, \quad (4.22)\]

which is an immediate consequence of (4.17).

Now write \(\beta = (\beta_1, \ldots, \beta_n) := \Pi^{-1}(p(x)) \) and \(\kappa_j := e^{-2\pi i \beta_j} \). Then \(p(x) = \prod_{j=1}^{n} (x - \kappa_j) \) and \(\kappa \) is a zero of it of order \(l + 1\). Thus

\[p(l+1)(\kappa) = (l+1)! \cdot \prod_{j: \kappa_j \neq \kappa} (\kappa - \kappa_j). \]

If \(\kappa = \pm 1\) then a single spp-ladder is associated to \(H^{(R)}_\kappa\). It satisfies

\[2a + l = 0, \text{ so then (4.21) predicts } L(a, \overline{\kappa}^0) > 0, \text{ so } v^t \cdot S^t \cdot \overline{\kappa}^0 > 0. \]

Indeed, if \(\kappa = 1\) then the \(\kappa_j \neq \kappa\) come in complex conjugate pairs or are equal to \(-1\), so \(p(l+1)(1) > 0\) and \(v^t \cdot S^t \cdot \overline{\kappa}^0 > 0\). If \(\kappa = -1\) then the \(\kappa_j \neq \kappa\) come in complex conjugate pairs or are equal to \(1\). Thus the multiplicity of \(1\) is congruent to \(n - l - 1 \mod 2\). Therefore \(p(l+1)(-1) \in (-1)^{n-l-1} \cdot \mathbb{R}_{>0}^0 \) and \(v^t \cdot S^t \cdot \overline{\kappa}^0 > 0\).

It rests to consider the case \(\kappa \neq \pm 1\). We can suppose \(\text{Im } \kappa < 0\). Then an index \(a\) exists with \(\beta_{a-1} < \beta_a = \ldots = \beta_{a+l} < \beta_{a+l+1}\) and \(a + l \leq \frac{n}{2}\) and \(\kappa = \kappa_a = \ldots = \kappa_{a+l}\).

We have the four cases \((k = 1 \& n \equiv 0(2))\), \((k = 1 \& n \equiv 1(2))\), \((k = 2 \& n \equiv 0(2))\) and \((k = 2 \& n \equiv 1(2))\). We treat only the case \((k = 2 \& n \equiv 0(2))\). The other cases are analogous. Then \(\kappa_1 = 1\),
\[ \kappa_{n+2} = -1 \text{ and} \]
\[ \prod_{j : \kappa_j \neq \kappa} (\kappa - \kappa_j) \]
\[ \prod_{2 \leq j \leq n, \kappa_j \neq \kappa} (\kappa^2 - \kappa (\kappa_j + \kappa_j) + 1) \]
\[ \prod_{2 \leq j \leq n, \kappa_j \neq \kappa} (\kappa + \kappa - (\kappa_j + \kappa_j)) \]
\[ \in \mathbb{R}_{>0}. \]

Here \( \alpha = \alpha_{a+l} \) by the recipe \( \text{[3.3]} \) and
\[ \prod_{j : \kappa_j \neq \kappa} (e^{2\pi i \beta_0})^{n/2} = e^{\pi i (\kappa_{a+l} + \kappa_{a+l})} = e^{\pi i (\kappa_{a+l} + 1)}, \]
\[ \prod_{j : \kappa_j \neq \kappa} (e^{2\pi i \beta_0})^{l/2} = (-i)^{l+2} \cdot (-1)^{a+2} = e^{\pi i (2a+l)}. \]

(d) This was essentially proved in the proof of theorem \( \text{[1.3]} \) (b). Define
\[ \beta_j = (\beta_1, ..., \beta_n) := (S_{\kappa} \circ \Pi)^{-1} : T_{\text{HOR}}(n, \mathbb{R}) \rightarrow T_{\text{scal}}(n, \mathbb{R}). \]

Then the functions \( \beta_j : T_{\text{HOR}}(n, \mathbb{R}) \rightarrow [0, 1] \) and the function \( \alpha_k \) in the proof of theorem \( \text{[1.3]} \) (b) are related by the recipe \( \text{[3.3]} \) (a), i.e. by \( \alpha_j = n \beta_j(S) - j + \frac{1}{2} \).

The following corollary of theorem \( \text{[4.5]} \) gives an example, what is in the polarized enhancement in theorem \( \text{[4.5]} \) (c). It was proved in a more elementary way in \( \text{[Ho17]} \) (for the cases considered there).

**Corollary 4.6.** Choose a matrix \( S \in T_{\text{HOR}}(n, \mathbb{R}) \) and take up the data in theorem \( \text{[4.4]} \). The symmetric form \( I_s \) is nondegenerate on \( H_{\neq -1} \). Its signature on \( H_{\mathbb{R}} \cap H_{\neq -1} \) is \( (s_+, s_0, s_-) \) with
\[ s_+ = \mid \{ \alpha_j \mid \alpha_j \in \left( -\frac{1}{2}, \frac{1}{2} \right) \mod 2\mathbb{Z} \} \mid, \]
\[ s_- = \dim H_{\neq -1} - s_+, \quad s_0 = 0. \]

**Proof:** The polarized enhancement of \( (H_{\mathbb{R}}, L) \) in theorem \( \text{[4.5]} \) (c) is (by remark \( \text{[2.8]} \)) a split Steenbrink polarized mixed Hodge structure on \( H_{\mathbb{R}} \cong M(n \times 1, \mathbb{R}) \) of weight \( m = 1 \). Such structures are studied in \( \text{[BH17]} \). Theorem 4.6 in \( \text{[BH17]} \) gives a square root of a Tate twist, which allows to go from weight \( m = 1 \) to an arbitrary weight \( \tilde{m} \in \mathbb{Z} \). In \( \text{[CKS86]} \) Corollary 3.13] (see also \[ \text{[He03]} \) Theorem 7.5]) an equivalence between a polarized mixed Hodge structure and a nilpotent orbit of polarized pure Hodge structures is given. Especially, they have the same spectral numbers and the same polarizing form. Therefore we can work with a polarized pure Hodge structure of even weight \( \tilde{m} \). In
that case, the polarizing form on $H_{\neq 1}$ is $I_s$, and \((4.23)\) is an immediate consequence of the polarization. 

\begin{remark}
In corollary \((4.6)\), when is $I_s$ positive definite on $H_{\mathbb{R}}$? Only if all spectral numbers are in $(-\frac{1}{2}, \frac{1}{2})$ mod $2\mathbb{Z}$. But by corollary \((3.5)\) and remark \((3.6)\), the gaps between subsequent spectral numbers (if they are ordered by size) are $\leq 1$. This enforces that all spectral numbers are in $(-\frac{1}{2}, \frac{1}{2})$. And this implies that the numbers $\beta_j$ in $\overline{\beta} = (\Pi \circ S^{(k)})^{-1}(S) \in T_{\text{HOR}}^{\text{scal}}(n, \mathbb{R})$ are interlacing with the numbers $\gamma_1, ..., \gamma_n$. Their pairwise distances are $|\beta_j - \gamma_j| < \frac{1}{2n}$. Such an interlacing is also discussed in \cite{Ho17}.
\end{remark}

\begin{remark}
For $S \in T_{\text{HOR}}(n, \mathbb{R})$ take up the data in theorem \((4.1)\) and define $H_{\mathbb{Z}} := M(n \times 1, \mathbb{Z})$. Then $L : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ is unimodular, and $R_{(k)}(S)$ and $M = (-1)^k R_{(k)}(S)^n$ are $L$-orthogonal automorphisms of $H_{\mathbb{Z}}$. For $R_{(k)}(S)$ this follows from theorem \((4.5)\) (b).

Furthermore, let $\varepsilon^*$ be the $\mathbb{Z}$-basis of $H_{\mathbb{Z}}$ which is left $L$-dual to the standard basis $\varepsilon$, i.e. with $L((\varepsilon^*)^t, \varepsilon) = E_n$. Then the matrix $R_{(k)}^{\text{mat}*}(S)$ of $R_{(k)}(S)$ with respect to $\varepsilon^*$, so with $R_{(k)}(S)(\varepsilon^*) = \varepsilon^* \cdot R_{(k)}^{\text{mat}*}(S)$, is

$$R_{(k)}^{\text{mat}*}(S) = R_{(k)}^{\text{mat}}(S)^{-t} = \begin{pmatrix}
-p_0 \\
-p_1 \\
\vdots \\
-p_{n-1}
\end{pmatrix}
$$

by the proof of theorem \((4.5)\) (b).

This implies $R_{(k)}(S)(\varepsilon_j^*) = \varepsilon_{j+1}^*$ for $j \in \{1, ..., n-1\}$. So $R_{(k)}(S)$ is a cyclic automorphism of $H_{\mathbb{Z}}$. This applies to the chain type singularities and is a remarkable fact there (remark \((7.4)\) (iv)).

\section{The cases $n = 2$ and $n = 3$}

\subsection{The case $n = 2$}
Consider an upper triangular matrix $S = 
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}
$ with $a \in \mathbb{R}$ and consider the matrix $R_{(1)}^{\text{mat}}(S) = \begin{pmatrix}
-a & -1 \\
1 & 0
\end{pmatrix}$.

By the proof of theorem \((4.5)\) (a) (or a direct calculation)

$$-S^{-1}S^t = R_{(1)}^{\text{mat}}(S)^2. \quad (5.1)$$

The characteristic polynomial of $R_{(1)}^{\text{mat}}(S)$ is $p(x) = x^2 + ax + 1$. Thus $R_{(1)}^{\text{mat}}$ and $S^{-1}S^t$ have eigenvalues in $S^1$ if and only if $|a| \leq 2$. Therefore

$$T(2, \mathbb{R}) = T_{\text{HOR}}(2, \mathbb{R}) = \left\{ \begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix} \mid a \in [-2, 2] \right\} \cong [-2, 2] \quad (5.2)$$

$$\cong T_{\text{HOR}}^{\text{scal}}(2, \mathbb{R}) = \{ (\beta_1, \beta_2) \mid \beta_1 \in [0, \frac{1}{2}], \beta_2 = 1 - \beta_1 \} \cong [0, \frac{1}{2}].$$
The recipe 3.3 gives for \( p(x) = x^2 + ax + 1 \) with \( |a| \leq 2 \)

\[
\beta_1 \in [0, \frac{1}{2}], \quad \beta_2 = 1 - \beta_1 \in \left[ \frac{1}{2}, 1 \right] \quad \text{with} \quad 2 \cos(2\pi\beta_1) = -a,
\]

(5.3)

\[
\gamma_1 = \frac{1}{4}, \quad \gamma_2 = \frac{3}{4}.
\]

(5.4)

\[
\alpha_1 = 2\beta_1 - \frac{1}{2} \in \left[ -\frac{1}{2}, \frac{1}{2} \right], \quad \alpha_2 = 2\beta_2 - \frac{3}{2} = -\alpha_1.
\]

(5.5)

\( \alpha_1 \) is determined by \( 2 \sin(\pi\alpha_1) = a \). \( R_{\text{mat}}^{(1)}(S) \) and \( S^{-1}S^t \) are not semisimple precisely at the boundary of \( T_{\text{HOR1}}(2, \mathbb{R}) \). There they have a \( 2 \times 2 \) Jordan block and the following eigenvalues, and the spectral pairs are:

| \( a = -2 \) | \( a = 2 \) |
|-----------------|-----------------|
| eigenvalue of \( R_{\text{mat}}^{(1)}(S) \) | eigenvalue of \( S^{-1}S^t \) |
| \( e^{2\pi i \beta_1} = 1 \) | \( e^{-2\pi i \alpha_1} = -1 \) |
| \( e^{-2\pi i \beta_1} = 1 \) | \( e^{-2\pi i \alpha_1} = -1 \) |
| \( e^{2\pi i \alpha_1} = 1 \) | \( e^{-2\pi i \alpha_1} = -1 \) |

(5.6)

The following table lists the types of the Seifert form pairs which one obtains by theorem 4.5 (c) for each \( a \in [-2, 2] \).}

| \( a = 0 \) | \( a \in ]-2, 2[ \backslash \{ 0 \} \) | \( a = \pm 2 \) |
|----------------|----------------------|----------------|
| \( 2 \cdot \text{Seif}(1, 1, 1, 1) \) | \( \text{Seif}(e^{-2\pi i \alpha_1}, 2, 1, e^{\pi i \alpha_1}) \) | \( \text{Seif}(-1, 1, 2, 1) \) |

(5.7)

The eigenvalue strata and the Seifert form strata (definition 1.5 (f)) in \( T(2, \mathbb{R}) \) coincide. One is \( \{ E_2 \} \), the others are \( \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \right\} \) for \( a \in [-2, 2] \backslash \{ 0 \} \).

The set \( T_{\text{HOR2}}(2, \mathbb{R}) \) has dimension 0 by (1.2). It is \( T_{\text{HOR2}}(2, \mathbb{R}) = \{ E_2 \} \), and

\[
R_{(2)}^{\text{mat}}(E_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_{(2)}^{\text{mat}}(E_2)^2 = E_2.
\]

(5.8)

Recipe 3.3 gives in the case \( k = 2 \) for \( S = E_2 \)

\[
\beta_1 = 0, \quad \beta_2 = \frac{1}{2}, \quad \gamma_1 = 0, \quad \gamma_2 = \frac{1}{2}, \quad \alpha_1 = 0, \quad \alpha_2 = 0.
\]

(5.9)

In the case \( n = 2 \) conjecture 1.6 is satisfied (and conjecture 1.7 is empty). The only singularity up to suspension with \( \mu = 2 \) is \( A_2 \). It is a chain type singularity. Theorem 7.6 implies for \( n = 2 \) conjecture 1.9 for function germs.
5.2. The case \( n = 3 \). The following theorem describes the set \( T(3, \mathbb{R}) \), its Seifert form strata and its eigenvalue strata (definition \( 1.5 \) (f)). Define

\[
f^C : \mathbb{C}^3 \to \mathbb{C}, \quad f(a_1, a_2, a_3) := 4 + a_1a_2a_3 - (a_1^2 + a_2^2 + a_3^2), \quad (5.10)
\]

\[
f := f^C|_{\mathbb{R}} : \mathbb{R}^3 \to \mathbb{R},
\]

\[
S^{[3]} : \mathbb{R}^3 \to M(3 \times 3, \mathbb{R}),
\]

\[
a = (a_1, a_2, a_3) \mapsto S^{[3]}(a) = \begin{pmatrix} 1 & a_1 & a_3 \\ 1 & a_2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (5.11)
\]

\[
M(3, \mathbb{R})_{\text{tri}} := S^{[3]}(\mathbb{R}^3) \subset M(3 \times 3, \mathbb{R}),
\]

\[
\text{ray}(S) := S^{[3]}(\mathbb{R} \cdot a) \text{ for } S = S^{[3]}(a) \neq E_3 \text{ (i.e. for } a \neq 0) . \quad (5.12)
\]

**Theorem 5.1.** \( T(3, \mathbb{R}) \) is the closed semialgebraic subset of \( M(3, \mathbb{R})_{\text{tri}} \)

\[
T(3, \mathbb{R}) = \{ S^{[3]}(a) \in M(3, \mathbb{R})_{\text{tri}} \mid 0 \leq f(a_1, a_2, a_3) \leq 4 \}. \quad (5.13)
\]

Consider the subsets

\[
T(3, \mathbb{R})_{\text{pos}} := \{ S \in T(3, \mathbb{R}) \mid S + S^t \text{ pos. def. or pos. semidefinite} \},
\]

\[
T(3, \mathbb{R})_{\text{exc}} := \{ S^{[3]}(2, 2, 2), S^{[3]}(-2, -2, 2), S^{[3]}(-2, -2, -2), S^{[3]}(2, -2, -2) \},
\]

\[
T(3, \mathbb{R})_{\text{ind}} := T(3, \mathbb{R}) - T(3, \mathbb{R})_{\text{pos}}.
\]

\( T(3, \mathbb{R})_{\text{pos}} \) is homeomorphic to a 3-ball and \( G_{\text{sign},3} \)-invariant (\( G_{\text{sign},n} \): definition \( 1.5 \) (e)).

\[
\overline{T(3, \mathbb{R})}_{\text{ind}} = T(3, \mathbb{R})_{\text{ind}} \cup T(3, \mathbb{R})_{\text{exc}}, \quad (5.15)
\]

\[
\overline{T(3, \mathbb{R})}_{\text{ind}} \cap T(3, \mathbb{R})_{\text{pos}} = T(3, \mathbb{R})_{\text{exc}}.
\]

\( T(3, \mathbb{R})_{\text{ind}} \) is homeomorphic to four copies of \([0, 1] \times \mathbb{R}^2\). These components are permuted by the group \( G_{\text{sign},3} \). Each component is in the open quadrant in \( M(3, \mathbb{R})_{\text{tri}} \cong \mathbb{R}^3 \) which contains one of the points in \( T(3, \mathbb{R})_{\text{exc}} \). The boundary \( \partial T(3, \mathbb{R}) \) is smooth and transversal to the rays \( \text{ray}(S) \) for \( S \in M(3, \mathbb{R})_{\text{tri}} - \{ E_3 \} \) except at the 4 points in \( T(3, \mathbb{R})_{\text{exc}} \). At each of the 4 points in \( T(3, \mathbb{R})_{\text{exc}} \) it is isomorphic to a cone.

For each type of a Seifert form pair of rank 3, at most one Seifert form stratum exists. The following table lists those which exist.
| Type of a Seifert form pair | description of Seifert form stratum |
|-----------------------------|-----------------------------------|
| $3 \cdot \text{Seif}(1, 1, 1, 1)$ | $\{E_3\}$ |
| $\text{Seif}(1, 1, 1, 1)$ + $\text{Seif}(e^{-2\pi i \alpha_1}, 2, 1, e^{\pi i \alpha_1})$ | diffeomorphic to a 2-sphere in $\text{int}(T(3, \mathbb{R})_{\text{pos}})$ |
| $\text{Seif}(1, 1, 1, 1)$ + $\text{Seif}(-1, 1, 2, 1)$ | $\partial T(3, \mathbb{R})_{\text{pos}} - T(3, \mathbb{R})_{\text{exp}}$ $\approx 2$-sphere $- 4$ points |
| $\text{Seif}(1, 1, 1, 1)$ + $\text{Seif}(-1, 2, 1)$ | $T(3, \mathbb{R})_{\text{exc}}$ |
| $\text{Seif}(1, 1, 1, 1)$ + $\text{Seif}(-1, 1, 2, -1)$ | the 4 components of $\partial T(3, \mathbb{R})_{\text{ind}}$ whose closures contain points of $T(3, \mathbb{R})_{\text{exp}}$ $\approx 4$ copies of $\mathbb{R}^2 - \{0\}$ |
| $\text{Seif}(1, 1, 1, 1)$ + $\text{Seif}(e^{-2\pi i \alpha_1}, 2, 1, -e^{\pi i \alpha_1})$ | diffeomorphic to 4 copies of $\mathbb{R}^2$, one in each component of $\text{int}(T(3, \mathbb{R})_{\text{ind}})$ |
| $\text{Seif}(1, 1, 3, 1)$ | the 4 components of $\partial T(3, \mathbb{R})_{\text{ind}}$ which do not intersect $T(3, \mathbb{R})_{\text{exc}}$ $\approx 4$ copies of $\mathbb{R}^2$ |

The three Seifert form strata with eigenvalues $(1, -1, -1)$ form one eigenvalue stratum. It is one component of $\partial T(3, \mathbb{R})$. The other Seifert form strata are eigenvalue strata. The following is a rough picture of a part of $T(3, \mathbb{R})$. The thick line indicates $T_{\text{HOR1}}(3, \mathbb{R})$, which will be discussed below.
Proof: The characteristic polynomial of $S^{-1}S^t$ for $S = S^{[3]}(a)$ is

$$p_{ch,S^{-1}S^t}(x) = \det(xE_3 - S^{-1}S^t) = \det(xS - S^t) = x^3 - (f(a) - 1)x^2 + (f(a) - 1)x - 1 = (x - 1)(x^2 - (f(a) - 2)x + 1).$$

(5.16)

This shows (5.13). The boundary $\partial T(3, \mathbb{R})$ of $T(3, \mathbb{R})$ is $\{S \in M(3, \mathbb{R})_{tri} | f(a) = 0$ or $f(a) = 4 \} - \{E_3\}$. For any $S = S^{[3]}(a) \in M(3, \mathbb{R})_{tri} - \{E_3\}$, consider the function

$$g^{ray,S} : \mathbb{R}_+ \to \mathbb{R}$$

$$g^{ray,S}(r) := r \cdot (3r \cdot a_1 a_2 a_3 - 2(a_1^2 + a_2^2 + a_3^2)).$$

Claim 1:

(i) If $a_1 a_2 a_3 \leq 0$, then $g^{ray,S}$ is strictly decreasing with the limit $-\infty$, so then ray$(S)$ intersects $\partial T(3, \mathbb{R})$ only in one point.

(ii) If $a_1 a_2 a_3 > 0$ and $S \notin$ ray$(T(3, \mathbb{R})_{exc})$, then $g^{ray,S}$ is first strictly decreasing to a minimum < 0 and then strictly increasing with limit $+\infty$. Then ray$(S)$ intersects $\partial T(3, \mathbb{R})$ at three points.

(iii) If $S \in T(3, \mathbb{R})_{exc}$, then $g^{ray,S}$ is first strictly decreasing with minimum = 0 at $S$ and then strictly increasing with limit $+\infty$. Then ray$(S)$ intersects $\partial T(3, \mathbb{R})$ at $S$ and at one other point.

Proof of claim 1: (i) is clear. (ii) and (iii):

$$(g^{ray,S})'(r) = r \cdot (3r \cdot a_1 a_2 a_3 - 2(a_1^2 + a_2^2 + a_3^2)),$$

$$r_0 := 2(a_1^2 + a_2^2 + a_3^2)/(3a_1 a_2 a_3), \quad \text{so that} \quad (g^{ray,S})'(r_0) = 0,$$

$$g^{ray,S}(r_0) = \frac{4}{27(a_1 a_2 a_3)^2} (27a_1^2 a_2^2 a_3^2 - (a_1^2 + a_2^2 + a_3^2)^3)$$

$$\begin{cases}
= 0 & \text{for } S \in T(3, \mathbb{R})_{exc}, \\
< 0 & \text{for } a_1 a_2 a_3 > 0, S \notin \text{ray}(T(3, \mathbb{R})_{exc}).
\end{cases}$$

(*) is an easy exercise. This finishes the proof of claim 1. \hfill (\Box)

The eigenvalue map $\Psi_{Eig} : T(3, \mathbb{R}) \to \text{Eig}(3)$ has the same fibers as the map $T(3, \mathbb{R}) \to \mathbb{R}$, $S(a) \to f(a)$. Claim 1 shows that the fibers are smooth and transversal to the rays ray$(S)$, except at the point $E_3$ and the four points in $T(3, \mathbb{R})_{exc}$. At each of these four points the fiber is locally diffeomorphic to a cone, because $f$ has
at \((a_1, a_2, a_3) \in \{(2, 2, 2), (-2, -2, 2), (-2, 2, -2), (2, -2, -2)\}\) an \(A_1\)-singularity, and the signature of the Hessian

\[
\text{Hess}(f)(a) = \left( \frac{\partial^2 f}{\partial a_i \partial a_j} \right)(a) = \begin{pmatrix} -2 & a_3 & a_2 \\ a_3 & -2 & a_1 \\ a_2 & a_1 & -2 \end{pmatrix}
\]

is \((1, 0, 2)\), because \(\det \text{Hess}(f)(a) = 32 > 0\) and \(-2 < 0\).

Claim 1 shows that \(M(3, \mathbb{R})_{\text{tri}} - \{S(a) \mid f(a) = 0\}\) has six components: the component \(C_1\) which contains \(E_3\), the component \(C_2\) which contains all of the four quadrants with \(a_1a_2a_3 < 0\) except their intersection with \(C_1\), and the four components \(C_3, C_4, C_5, C_6\) which contain each one of the partial rays in \(S^{[3]}(\mathbb{R}_{>1}) \cdot (S^{[3]} \cdot (T(3, \mathbb{R})_{\text{exc}}))\).

\((5.16)\) implies

\[
\det(S + S^t) = 2 \cdot f(a) \begin{cases} > 0 & \text{on } C_1, C_3, C_4, C_5 \text{ and } C_6, \\
< 0 & \text{on } C_2. \end{cases}
\]

\(2 \cdot E_3\) has signature \((3, 0, 0)\), any matrix \(S + S^t\) with \(S \in S^{[3]}(\mathbb{R}_{>1}) \cdot (S^{[3]})^{-1}(T(3, \mathbb{R})_{\text{exc}})\) has signature \((1, 0, 2)\) because \(\det \begin{pmatrix} 2 & a_1 \\ a_1 & 2 \end{pmatrix} < 0\) for such matrices. Therefore

\[
\text{signature}(S + S^t) = \begin{cases} (3, 0, 0) & \text{on } C_1, \\
(2, 0, 1) & \text{on } C_2, \\
(1, 0, 2) & \text{on } C_3, C_4, C_5 \text{ and } C_6, \end{cases}
\]

\[
\text{signature}(S + S^t) = \begin{cases} (2, 1, 0) & \text{on the part of } \partial T(3, \mathbb{R}) \text{ between } C_1 \text{ and } C_2, \\
(1, 2, 0) & \text{on } T(3, \mathbb{R})_{\text{exc}}, \\
(1, 1, 1) & \text{on the part of } \partial T(3, \mathbb{R}) \text{ between } C_2 \text{ and } C_3, C_4, C_5, C_6. \end{cases}
\]

Thus \(\dim \text{Rad}(S + S^t) = 1\) for \(S\) in \(\{S(a) \mid f(a) = 0\} - T(3, \mathbb{R})_{\text{exc}}\).

Therefore \(S^{-1}S^t\) has for such an \(S\) a \(2 \times 2\) Jordan block with eigenvalues \(-1\). For \(S \in T(3, \mathbb{R})_{\text{exc}}\) it is semisimple with eigenvalues \(1, -1, -1\).

Finally, consider the set \(\{S(a) \mid f(a) = 4\} - \{E_3\}\). It is the union of the four boundary components of \(T(3, \mathbb{R})\) which do not contain \(T(3, \mathbb{R})_{\text{exc}}\). For \(S \in \{S(a) \mid f(a) = 4\} - \{E_3\}\), claim 1 gives \(a_1a_2a_3 > 0\).

This implies \(\text{rk}(S - S^t) = \text{rk} \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix} = 2\) and \(\dim \text{Rad}(S - S^t) = 1\) and that \(S^{-1}S^t\) has a single \(3 \times 3\) Jordan block with eigenvalue \(1\).

The proof up to now gives all statements in theorem 5.1 except the table with Seifert form pairs and Seifert form strata. The proof gives also the eigenvalue strata and the signature of \(I_s\) at each point of \(M(3, \mathbb{R})_{\text{tri}}\).
The table with Seifert form pairs and Seifert form strata can now be deduced from the eigenvalues of $S^{-1}S'$, its Jordan block structure, claim 1, the signature of $S + S'$, and from lemma 2.4.

Now we study the subvarieties $T_{\text{HOR}k}(3, \mathbb{R}) \subset T(3, \mathbb{R})$ for $k \in \{1, 2\}$.

$$p(x) = x^3 + p_2 x^2 + p_1 x + p_0$$
$$= x^3 + (-1)^{k-1} p_1 x^2 + p_1 x + (-1)^{k-1} \in T_{\text{HOR}k}^{\text{pol}}(3, \mathbb{R})$$
$$= \begin{cases} x^3 + p_1 x^2 + p_1 x + 1 & \text{for } k = 1, \\ x^3 - p_1 x^2 + p_1 x - 1 & \text{for } k = 2. \end{cases}$$

$$T_{\text{HOR}1}(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & p_1 \\ 1 & p_1 \\ 1 \end{pmatrix} \mid p_1 \in [-1, 3] \right\},$$

$$T_{\text{HOR}2}(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & -p_1 \\ 1 & -p_1 \\ 1 \end{pmatrix} \mid p_1 \in [-1, 3] \right\}. $$

The element $g = (1, -1, 1) \in G_{\text{sign}, 3}$ exchanges $T_{\text{HOR}1}(3, \mathbb{R})$ and $T_{\text{HOR}2}(3, \mathbb{R})$. In the picture after theorem 5.1, the thick line indicates $T_{\text{HOR}1}(3, \mathbb{R})$.

By lemma 3.3 (c) $\text{Spp}(g(S)) = \text{Spp}(S)$ for $S \in \bigcup_{k=1,2} T_{\text{HOR}k}(3, \mathbb{R})$. Therefore we restrict in the following to $T_{\text{HOR}1}(3, \mathbb{R})$.

**Theorem 5.2.** (a) $T_{\text{HOR}1}(3, \mathbb{R})$ intersects the Seifert form stratum of type $\text{Seif}(1, 1, 1, 1) + \text{Seif}(e^{-2\pi i \alpha_1}, 2, 1, e^{\pi i \alpha_1})$ twice, the Seifert form stratum of type $\text{Seif}(1, 1, 1, 1) + \text{Seif}(-1, 2, -1)$ not at all and each other Seifert form stratum once.

(b) Recipe 3.3 gives for $T_{\text{HOR}1}(3, \mathbb{R})$ numbers $\beta_j, \gamma_j, \alpha_j$ for $j = 1, 2, 3$ with

$$\beta_1 \in [0, \frac{1}{2}], \quad \beta_2 = \frac{1}{2}, \quad \beta_3 = 1 - \beta_1 \in [\frac{1}{2}, 1],$$

$$\gamma_1 = \frac{1}{6}, \quad \gamma_2 = \frac{1}{2}, \quad \gamma_3 = \frac{5}{6};$$

$$\alpha_1 = 3\beta_1 - \frac{1}{2} \in [-\frac{1}{2}, 1], \quad \alpha_2 = 3\beta_2 - \frac{3}{2} = 0,$$

$$\alpha_3 = 3\beta_3 - \frac{5}{2} = -\alpha_1 \in [-1, \frac{1}{2}].$$

$\beta_1$ is determined by $\beta_1 \in [0, \frac{1}{2}]$ and $\cos(2\pi \beta_1) = \frac{1 - p_1}{2}$. Thus $\beta_1$ and $\alpha_1$ are monotonically increasing with $p_1 \in [-1, 3]$. 


The conjectures 1.6 and 1.7 hold. The Seifert form strata in $T(3,\mathbb{R})_{pos}$ have spectral numbers in $[-\frac{1}{2}, \frac{1}{2}]$, the Seifert form strata in $T(3,\mathbb{R})_{ind}$ have spectral numbers in $[-1, -\frac{1}{2}] \cup \{0\} \cup [\frac{1}{2}, 1]$. The two Seifert form strata with eigenvalues $1, -1, -1$ and a $2 \times 2$ Jordan block for the eigenvalue $-1$ have the same spectral pairs $(0, 1), (-\frac{1}{2}, 2), (\frac{1}{2}, 0)$. The Seifert form stratum $\{S^3(a) \mid f(a) = 4\} - \{E_3\}$ with a $3 \times 3$ Jordan block has the spectral pairs $(-1, 3), (0, 1), (1, -1)$.

(d) Conjecture 1.9 for function germs holds in the case $n = 3$. 

Proof: (a) $T_{HOR1}(3,\mathbb{R})$ is the intersection of $T(3,\mathbb{R})$ with the line through $E_3 = S^{[3]}(0, 0, 0)$ and $S^{[3]}(2, 2, 2) \in T(3,\mathbb{R})_{exc}$. This and theorem 5.1 show part (a).

(b) $\beta_1$ in recipe 3.3 for $S \in T_{HOR1}(3,\mathbb{R})$ is determined by $\beta_1 \in [0, \frac{1}{2}]$ and $(x - e^{2\pi i \beta_1})(x - e^{-2\pi i \beta_1}) = x^2 + (p_1 - 1)x + 1$, which is $\cos(2\pi \beta_1) = \frac{1}{2}$. This shows all of (b).

(c) This follows from (a) and (b) and the following observation. At the boundary points of $T_{HOR1}(3,\mathbb{R})$, the monodromy $S^{-1}S'$ and $R_{(1)}^{\text{mat}}(S)$ have for each eigenvalue of $R_{(1)}^{\text{mat}}(S)$ one Jordan block.

(d) The only singularity up to suspension with $\mu = 3$ is $A_3$. It is a chain type singularity. Theorem 7.6 implies for $n = 3$ conjecture 1.9 for function germs.

Remarks 5.3. (i) By theorem 5.2 (a), $T_{HORk}(3,\mathbb{R})$ for $k \in \{1, 2\}$ does not intersect the Seifert form stratum of type $\text{Seif}(1, 1, 1, 1) + \text{Seif}(-1, 1, 2, -1)$. This is consistent with theorem 4.5 (c): On this
Seifert form stratum, the single spp-ladder \((\frac{-1}{2}, 2), (\frac{1}{2}, 0)\) has first spectral number \(\alpha = \frac{-1}{2}\) and \(l = 1\), and
\[
L(a, N^l a) \in (-1) \cdot \mathbb{R}_{>0} = (-1) \cdot e^{\frac{\pi i}{2}(2\alpha + l)} \cdot \mathbb{R}_{>0}.
\]
Theorem 4.5 (c) forbids the existence of a matrix in \(T_{\text{Hor}}(n, \mathbb{R})\) and in this Seifert form stratum.

(ii) The table (5.5) for \(n = 2\) and the table in theorem 5.2 for \(n = 3\) show that precisely the following \(S^1\)-Seifert form pairs have no realization as \((M(n \times 1, \mathbb{R}), L)\) with \(L(a, b) = a^t S^t b\) and \(S \in T(n, \mathbb{R})\): All those for which \(I_s\) is negative semidefinite (cf. lemma 2.4 and remark 2.11 (vii)), and all those which contain Seif(1, 1, 1, -1) or Seif(1, 1, 3, -1).

It is an interesting question what holds for \(n \geq 4\).

6. ISOLATED HYPERSURFACE SINGULARITIES AND \(M\)-TAME FUNCTIONS

The purpose of this section is merely to give references for facts mentioned in the introduction, namely that holomorphic functions germs \(f : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)\) with isolated singularity at 0 (short: ihs) and \(M\)-tame functions \(f : X \to \mathbb{C}\) come equipped with \(\text{Br}_\mu \ltimes G_{\text{sign}, \mu}\)-orbits of Stokes matrices in \(T(\mu, \mathbb{Z})\) and with spectral pairs \(\text{Spp}(f)\). First we recall the definition of an \(M\)-tame function.

Definition 6.1. [NZ90] [NS99] A function \(f : X \to \mathbb{C}\) is \(M\)-tame if \(X\) is an affine manifold (of some dimension \(m+1\)) and if for some closed embedding \(X \hookrightarrow \mathbb{C}^N\) the following holds. For any \(\eta > 0\) an \(R(\eta) > 0\) exists such that the fibers \(f^{-1}(\tau)\) with \(|\tau| < \eta\) are transversal to all spheres \(S_{R}^{2N+1} = \{z \in \mathbb{C}^N \mid |z| = R\}\) with \(R \geq R(\eta)\).

Now we will treat the case 1: \(f\) an ihs, and the case 2: \(f\) \(M\)-tame, almost simultaneously. In both cases the definition domain shall have dimension \(m+1\).

In case 1, see e.g. [Lo84], [AGV88] or [El07] for the construction of a good representative \(f : Y \to \Delta_\eta\) where \(\Delta_\eta := \{\tau \in \mathbb{C} \mid |\tau| < \eta\}\) is a sufficiently small disk. In case 2, one can similarly construct a good representative \(f : Y \to \Delta_\eta\) for \(\eta > 0\) sufficiently large. The Milnor number \(\mu\) is in case 1 the Milnor number of the singularity at 0 \(\in Y\) which is then the only singularity of \(f : Y \to \Delta_\eta\). In case 2, \(\mu\) is the sum of the Milnor numbers of all singularities of \(f : Y \to \Delta_\eta\), which are all singularities of \(f : X \to \mathbb{C}\).

In both cases, the relative homology groups (reduced if \(m = 0\)) \(\text{Ml}(f, \zeta) := H_{n+1}(Y, f^{-1}(\zeta \eta), \mathbb{Z})\) with \(\zeta \in S^1\) are isomorphic to \(\mathbb{Z}^\mu\) [Lo84 (5.11)] [AGV88 ch. 2], and some generators of them can be
called (classes of) Lefschetz thimbles. They form a flat $\mathbb{Z}$-lattice bundle on $S^1$. An intersection form for Lefschetz thimbles is well defined on relative homology groups with different boundary parts. It is for any $\zeta \in S^1$ a $(-1)^{m+1}$ symmetric unimodular bilinear form

$$I_{\text{Lef}} : M_l(f, \zeta) \times M_l(f, -\zeta) \to \mathbb{Z} \quad (6.1)$$

Let $\gamma_\pi$ (respectively $\gamma_-\pi$) be the isomorphism $M_l(f, -\zeta) \to M_l(f, \zeta)$ by flat shift in mathematically positive (respectively negative) direction. Then the classical Seifert form is given by

$$L : M_l(f, \zeta) \times M_l(f, \zeta) \to \mathbb{Z}, \quad L(a, b) := (-1)^{m+1} I_{\text{Lef}}(a, \gamma_\pi b). \quad (6.2)$$

The classical monodromy $M$ and the intersection form $I$ on $M_l(f, \zeta)$ are given by

$$L(Ma, b) = (-1)^{m+1} L(b, a), \quad (6.3)$$

$$I(a, b) = -L(a, b) + (-1)^{m+1} L(b, a) = L((M - \text{id})a, b). \quad (6.4)$$

We define a normalized Seifert form $L^{\text{nor}}$ and a normalized monodromy $M^{\text{nor}}$ by

$$L^{\text{nor}} := (-1)^{(m+1)(m+2)/2} \cdot L, \quad (6.5)$$

$$M^{\text{nor}} := (-1)^{m+1} M. \quad (6.6)$$

Thus $M^{\text{nor}}$ is the monodromy of $L$ and of $L^{\text{nor}}$ in the sense of definition 2.2 (b).

Finally, we refer to [AGV88, ch. 2] or [Eb07, 5.5] for the definition of a distinguished basis $\delta = (\delta_1, \ldots, \delta_\mu)$ of $M_l(f, 1)$. The set of distinguished bases forms one orbit of the group $\text{Br}_{\mu} \rtimes G_{\text{sign}, \mu}$. Here $\text{Br}_{\mu}$ is the braid group with $\mu$ strings, and $G_{\text{sign}, \mu}$ was defined in definition 1.5 (e). See [AGV88] or [Eb07] for the action of $\text{Br}_{\mu}$. The group $G_{\text{sign}, \mu}$ acts componentwise by sign changes. Each distinguished basis $\delta$ gives rise to one matrix

$$S := L^{\text{nor}}(\delta^t, \delta^t) \in T(\mu, \mathbb{Z}). \quad (6.7)$$

We call these matrices Stokes matrices because some of them encode certain Stokes structures (which will not be discussed here). These matrices form also one $\text{Br}_{\mu} \rtimes G_{\text{sign}, \mu}$-orbit. In the case of the ihs, this orbit is finite only for the simple and the simple elliptic singularities, and the orbit of distinguished bases is finite only for the simple singularities [Eb16].

Now we come to the spectral pairs. In the case of an ihs $f$, spectral pairs $\text{Spp}(f)$ were first defined by Steenbrink [St77] as invariants of his natural mixed Hodge structure on the space dual to $M_l(f, 1)$ (see also [AGV88]). It is in the notation of [BH17] a signed Steenbrink polarized
mixed Hodge structure of weight $m$. For the polarization see [He02] or [BH17].

In the case of an M-tame function $f$, the spectral pairs are defined in the same way as invariants of Sabbah’s natural mixed Hodge structure [Sa98] on the space dual to $Ml(f, \zeta)$. A certain twist of Sabbah’s Hodge filtration is a part of a Steenbrink polarized mixed Hodge structure of weight $m$ [HS07, Corollary 11.4] in the notation of [BH17].

In both cases, $f$-ihs or $f$ M-tame function, $\text{Spp}(f)$ is a union of single spp-ladders and sppl-pairs with center $m$ (as the spectral pairs of any Steenbrink mixed Hodge structure in the sense of [BH17]).

7. Chain type singularities and their spectra

We used the initials of Horocholyn, Orlik and Randell in the name HOR-matrices because a good part of these matrices turns up in the papers [Ho17] and [OR77]. See the beginning of section 4 for [Ho17]. Orlik and Randell studied the chain type singularities (definition 7.1 below). They conjectured that each of them has a distinguished basis whose Stokes matrix $S$ is a certain HOR-matrix $S$ [OR77, Conjecture (4.1)] (=conjecture 7.3). This conjecture and theorem 4.5 (a) would imply that the matrix of the monodromy for this distinguished basis is $(R_{(k)}^\text{mat})^\mu$ with $k \equiv m(2)$ (remark 7.4 (iii)). The main result theorem (2.11) in [OR77] says that the matrix of the monodromy for some basis of the Milnor lattice is this matrix.

We will recall the definition of a chain type singularity and the HOR-matrix of Orlik and Randell. Then we will show that the spectrum $\text{Sp}(S)$ of this HOR-matrix from definition 4.4 (c) (or theorem 4.3 (b), see theorem 4.5 (c)) is up to the shift $\frac{m-1}{2}$ the correct spectrum of the singularity, $\text{Sp}(S) = \text{Sp}(f) - \frac{m-1}{2}$. This is positive evidence for conjecture 1.9. It is the main result of this section. Of course, the evidence would be stronger if somebody would prove conjecture (4.1) in [OR77].

Definition 7.1. (a) A chain type singularity is a function germ on $(\mathbb{C}^{m+1}, 0)$ which is defined by a polynomial

$$f(x_0, ..., x_m) = x_0^{a_0} + x_0 x_1^{a_1} + ... + x_{m-1} x_m^{a_m} = x_0^{a_0} + \sum_{j=1}^{m} x_{j-1} x_j^{a_j}$$

with $a_0 \in \mathbb{Z}_{\geq 2}, a_1, ..., a_m \in \mathbb{Z}_{\geq 1}$. 
(b) Define the function
\[ \rho : \bigcup_{k=0}^{\infty} \mathbb{Z}^k \rightarrow \mathbb{Z}, \]  
(7.1)
\[ \rho(a_0, a_1, \ldots, a_{k-1}) := a_0 a_{k-1} - a_1 a_{k-1} + \ldots + (-1)^{k-1} a_{k-1} + (-1)^k \]
(the case \( k = 0 \) is \( \rho(\emptyset) = 1 \)).

**Lemma 7.2.** Consider \( f \) in definition 7.1 (a). It has indeed an isolated singularity at 0. It is a quasihomogeneous polynomial of weighted degree 1 with respect to weights \((w_0, \ldots, w_m)\) which are determined as follows. Define
\[ r_{-1} := 1, \quad r_k := a_0 \ldots a_k = r_{k-1} a_k \quad \text{for} \quad 0 \leq k \leq m, \]  
(7.2)
\[ \mu_{-1} := 1, \quad \mu_k := \rho(a_0, \ldots, a_k) = r_k - \mu_{k-1} \quad \text{for} \quad 0 \leq k \leq m, \]  
(7.3)
\[ w_{-1} := 0, \quad w_k := \frac{\mu_{k-1}}{r_k} = 1 - \frac{w_{k-1}}{r_k} \quad \text{for} \quad 0 \leq k \leq m. \]  
(7.4)
Its Milnor number is \( \mu = \mu_m \).

**Proof:** The partial derivatives of \( f \) are
\[ \frac{\partial f}{\partial x_0} = a_0 x_0^{a_0-1} + x_1^{a_1}, \]  
(7.5)
\[ \frac{\partial f}{\partial x_1} = a_1 x_0^{a_1-1} + x_2^{a_2}, \ldots, \]
\[ \frac{\partial f}{\partial x_{m-1}} = a_{m-1} x_{m-2}^{a_{m-1}-1} + x_m^{a_m}, \]
\[ \frac{\partial f}{\partial x_m} = a_m x_{m-1}^{a_m-1}. \]

Suppose that \( x \in \mathbb{C}^{m+1} \) is a zero of all partial derivatives. Then
\[ x_0 \neq 0 \Rightarrow x_1 \neq 0 \Rightarrow x_2 \neq 0 \Rightarrow \ldots \Rightarrow x_m \neq 0 \]
\[ \Rightarrow \frac{\partial f}{\partial x_m}(x) \neq 0, \text{ a contradiction.} \]
\[ x_0 = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow \ldots \Rightarrow x_m = 0. \]
Therefore the singularity \( x = 0 \) of \( f \) is the only singularity in \( \mathbb{C}^{m+1} \).
The weights \((w_0, \ldots, w_m)\) are uniquely determined by
\[ w_0 = \frac{1}{a_0} = \frac{\mu_{-1}}{r_0}, \]
\[ w_k = \frac{1 - w_{k-1}}{a_k} = \frac{1 - \frac{\mu_{k-2}}{r_{k-1}}}{a_k} = \frac{r_{k-1} + \mu_{k-2}}{r_{k-1} a_k} = \frac{\mu_{k-1}}{r_k}. \]
In the following calculation of the Milnor number, \(\mu\) is a well known formula for all quasihomogeneous singularities.

\[
\mu = \prod_{k=0}^{m} \left( \frac{1}{w_k} - 1 \right) = \prod_{k=0}^{m} \frac{r_k - \mu_{k-1}}{\mu_{k-1}} = \prod_{k=0}^{m} \frac{\mu_k}{\mu_{k-1}} = \mu_m. \quad \square
\]

**Conjecture 7.3.** [OR77] Conjecture (4.1) The chain type singularity \( f = x_0^{a_0} + x_0 x_1^{a_1} + \ldots + x_{m-1} x_m^{a_m} \) has a distinguished basis whose Stokes matrix \( S \) is the HOR-matrix \( S \) (definition 4.4 (a)) with polynomial

\[
p(x) = x^\mu + p_{\mu-1} x^{\mu-1} + \ldots + p_0 = \prod_{k=1}^{m} (x^r_k - 1)^{(-1)^{m-k}}.
\]  

**Remarks 7.4.** (i) In conjecture 7.3 \( p(x) \) has only simple eigenvalues, namely all zeros of \( x^m - 1 \) minus certain gaps, which are most zeros of \( x^{m-1} - 1 \).

(ii) In conjecture 7.3 \( p_0 = (-1)^{m+1} \) and \( S \in T_{\text{HOR},k}(\mu, \mathbb{R}) \cap T(\mu, \mathbb{Z}) \) with \( k \equiv m(2) \).

(iii) Theorem (2.11) in [OR77] says that for a suitable basis of \( Ml(f) \), the monodromy matrix is \( T_{(k)}(S)^\mu \) with \( k \equiv m(2) \). This is compatible with conjecture 7.3 and theorem 4.5 (a), which give this for a distinguished basis with Stokes matrix \( S \). Here recall that in the singularity case the monodromy in theorem 4.1 is the normalized monodromy \( M^{n or} \) in 6.6 and that the true monodromy is \( (-1)^{m+1} M^{n or} \).

(iv) Conjecture 7.3 and definition 4.4 (b) give the automorphism \( R_k(S) : Ml(f) \to Ml(f) \) (with \( k \equiv m(2) \)) with characteristic polynomial \( p(x) \). It respects \( L \) by theorem 4.5 (b), it satisfies \( R_k(S)^\mu = M \) by theorem 4.5 (a), and it is cyclic by remark 4.8.

**Remarks 7.5.** Here we will argue that it is almost always (and especially in the proof of theorem 7.6) sufficient to consider chain type singularities \( f = x_0^{a_0} + x_0 x_1^{a_1} + \ldots + x_{m-1} x_m^{a_m} \) with \( a_0 \in \mathbb{Z}_{\geq 2}, a_1, \ldots, a_m \in \mathbb{Z}_{\geq 2}, \) and the \( A_1 \)-singularity \( x_0^2 \).

(i) \( f(x) \) is right equivalent to \( c_0 \cdot x_0^{a_0} + c_1 \cdot x_0 x_1^{a_1} + \ldots + c_m \cdot x_m x_m^{a_m} \) for arbitrary \( c_0, \ldots, c_m \in \mathbb{C}^* \).

(ii) Let \( f(x_0, \ldots, x_m) \) be a chain type singularity with \( a_0 = 2 \). Consider the new coordinates \( \tilde{x}_0 = x_0 + \frac{1}{2} x_1^{a_1}, \tilde{x}_k = x_k \) for \( 1 \leq k \leq m \). Then

\[
f(x_0, \ldots, x_m) = (x_0 + \frac{1}{2} x_1^{a_1})^2 - \frac{1}{4} x_1^{2a_1} + x_1 x_2^{a_2} + \ldots + x_{m-1} x_m^{a_m}
\]

\[
= \tilde{x}_0^2 - \frac{1}{4} \tilde{x}_1^{2a_1} + \tilde{x}_1 \tilde{x}_2^{a_2} + \ldots + \tilde{x}_m \tilde{x}_m^{a_m} \quad (7.7)
\]
This is (up to a rescaling in $\tilde{x}_1$) a 1-fold suspension of the chain type singularity $\tilde{f}(y_0, \ldots, y_{m-1}) = y_0^{2a_1} + y_0y_1^{a_2} + \ldots + y_{m-2}y_{m-1}^{a_m}$ with

$$\tilde{a}_0 = 2a_1, \quad \tilde{a}_k = a_{k+1} \text{ for } 1 \leq k \leq m-1,$$

$$\tilde{r}_k = r_{k+1} \text{ for } 0 \leq k \leq m-1, \quad \text{all } \tilde{r}_k \equiv 0(2),$$

$$p(x) = (x + 1)^{-m} \cdot \prod_{k=1}^{m} (x^{r_k} - 1)^{(-1)^{m-k}},$$

$$\tilde{p}(x) = (x - 1)^{-m} \cdot \prod_{k=1}^{m} (x^{r_k} - 1)^{(-1)^{m-k}} = (-1)^{m} \cdot p(-x),$$

$$\text{Sp}(\tilde{f}) = \text{Sp}(f) - \frac{1}{2}.$$

Lemma 3.4 (c) implies $\text{Sp}(\tilde{p}(x)) = \text{Sp}(p(x))$.

(iii) Let $f(x_0, \ldots, x_m)$ be a chain type singularity with $a_0 = 3$. Suppose that it has an exponent $a_j = 1$ and that $a_1, \ldots, a_{j-1} \geq 2$. Consider the new coordinates $x_{j-1} = x_{j-1} + x_{j+1}^{a_{j+1}}$ and $\tilde{x}_k = x_k$ for $k \neq j-1$.

Then

$$f(x_0, \ldots, x_m) = \tilde{x}_0^{a_0} + x_0\tilde{x}_1^{a_1} + \ldots + x_{j-2}\tilde{x}_{j-1}^{a_{j-1}} + (x_{j-1} + x_{j+1}^{a_{j+1}})x_j + x_{j+1}^{a_{j+2}} + \ldots + x_{m-1}^{a_m}$$

$$= \tilde{x}_0^{a_0} + \tilde{x}_0\tilde{x}_1^{a_1} + \ldots + (-1)^{a_j-1}\tilde{x}_{j-2}\tilde{x}_{j+1}^{a_{j+1}} + \tilde{x}_{j+1}\tilde{x}_{j+2}^{a_{j+2}} + \ldots + \tilde{x}_{m-1}\tilde{x}_m^{a_m}$$

$$+ \tilde{x}_{j-1}\tilde{x}_j + \tilde{x}_{j-2} \cdot \left( \sum_{k=0}^{a_{j-1} - 1} (-1)^k \binom{a_{j-1}}{k} \tilde{x}_{j-1}^{a_{j-1} - k} \tilde{x}_{j+1}^{k} \right). \quad (7.8)$$

The first line of (7.8) is a chain type singularity $\tilde{f}(y_0, \ldots, y_{m-2})$ with

$$\tilde{a}_k = a_k \text{ for } 0 \leq k \leq j-2, \tilde{a}_{j-1} = a_{j-1}a_{j+1},$$

$$\tilde{a}_k = a_{k+2} \text{ for } j \leq k \leq m-2,$$

$$\tilde{r}_k = r_k \text{ for } 0 \leq k \leq j-2, \tilde{r}_k = r_{k+2} \text{ for } j-1 \leq k \leq m-2,$$

$$\tilde{p}(x) = p(x).$$

The first monomial $\tilde{x}_{j-1}\tilde{x}_j$ in the second line of (7.8) gives a 2-fold suspension of $\tilde{f}$. The second part $\tilde{x}_{j-2} \cdot (\ldots)$ consists of monomials of weighted degree $> 1$ if one associates to $\tilde{x}_{j-1}$ and to $\tilde{x}_j$ the degree $\frac{1}{2}$,
because for $k = a_{j-1} - 1$

$$a_{j+1}a_{j-1}\tilde{w}_{j+1} = 1 - \tilde{w}_{j-2} \text{ and}$$

$$\tilde{w}_{j-2} + \tilde{w}_{j-1} + a_{j+1}(a_{j-1} - 1)\tilde{w}_{j+1} = \frac{1}{2} + 1 - a_{j+1}\tilde{w}_{j+1} = \frac{1}{2} + 1 - \frac{1 - \tilde{w}_{j-2}}{a_{j-1}} > 1 \text{ because } a_{j-1} \geq 2.$$ 

Therefore $f$ is right equivalent to a 2-fold suspension of $\tilde{f}$. This implies

$$\text{Sp}(\tilde{f}) = \text{Sp}(f) - 1.$$ 

(iv) One transforms a chain type singularity with $a_0 = 2$ with (ii) to a 1-fold suspension of a chain type singularity with one variable less. One repeats (ii) until one arrives either at the $A_1$-singularity $x_0^2$ or at a chain type singularity with $a_0 \geq 3$. Then one repeats (iii) until one arrives at a chain type singularity with $a_0 \geq 3, a_1, ..., a_m \geq 2$. Then

$$\text{Sp}(\tilde{p}(x)) = \text{Sp}(p(x)).$$

**Theorem 7.6.** Consider a chain type singularity $f(x) = x_0^{a_0} + x_0 x_1^{a_1} + ... + x_{m-1} x_m^{a_m}$. The spectrum of the HOR-matrix $S$ in conjecture 7.3 (see definition 4.4 (c) for $\text{Sp}(S)$) satisfies

$$\text{Sp}(S) = \text{Sp}(f) - \frac{m - 1}{2}. \quad (7.9)$$

**Proof:** For the $A_1$-singularity $x_0^2$ $S = (1)$ and $\text{Sp}(S) = (0)$ and $\text{Sp}(f) = (-\frac{1}{2})$ and $m = 0$, so (7.9) holds. Because of this and the remarks 7.5 (ii)–(iv), it is sufficient to prove theorem 7.6 for the cases $a_0 \in \mathbb{Z}_{\geq 3}, a_1, ..., a_m \in \mathbb{Z}_{\geq 2}$. The spectrum $\text{Sp}(f) = (\alpha_1(f), ..., \alpha_\mu(f))$ (with an arbitrary numbering) of a quasihomogeneous singularity with weights $w_0, ..., w_m \in \mathbb{Q} \cap (0, 1)$ such that $\deg_w f = 1$ can be given in several ways:

(A) By the generating function

$$\sum_{j=1}^\mu t^{\alpha_j(f)+1} = \prod_{k=0}^m t - t^{w_k}. \quad (7.10)$$

(B) If $m_1, ..., m_\mu \in \mathbb{C}[x]$ are weighted homogeneous polynomials which represent a basis of the Jacobi algebra then

$$\alpha_j(f) = -1 + \sum_{k=0}^m w_k + \deg_w m_j \quad \text{for } j = 1, ..., \mu. \quad (7.11)$$

Here (B) is more convenient than (A). Claim 1 is the first of four steps of the main part of the proof.
Step 1 = Claim 1: The following monomials represent a basis of the Jacobi algebra:

\[ x_0^{b_0} x_1^{b_1} \cdot \ldots \cdot x_m^{b_m} \]  
with \[ 0 \leq b_j \leq a_j - 1 \text{ for } j \in \{0, 1, \ldots, m - 1\} \]  
and \[ 0 \leq b_m \leq a_m - 2, \]

\[ x_0^{b_0} x_1^{b_1} \cdot \ldots \cdot x_{m-2}^{b_{m-2}} x_m^{a_m-1} \]  
with \[ 0 \leq b_j \leq a_j - 1 \text{ for } j \in \{0, 1, \ldots, m - 3\} \]  
and \[ 0 \leq b_{m-2} \leq a_{m-2} - 2, \]

\[ \vdots \]

for \( m \equiv 0(2) : \)

\[ x_0^{b_0} x_2^{a_2-1} x_4^{a_4-1} \cdot \ldots \cdot x_m^{a_m-1} \]  
with \[ 0 \leq b_0 \leq a_0 - 2, \]

for \( m \equiv 1(2) : \)

\[ x_0^{b_0} x_1^{b_1} x_3^{a_3-1} \cdot \ldots \cdot x_m^{a_m-1} \]  
with \[ 0 \leq b_0 \leq a_0 - 1 \]  
and \[ 0 \leq b_1 \leq a_1 - 2, \]

\[ x_1^{a_1-1} x_3^{a_3-1} \cdot \ldots \cdot x_m^{a_m-1}. \] (7.12)

Proof of claim 1: Their number is

\[ \mu = a_0 a_{m-1}(a_m - 1) + a_0 a_{m-3}(a_{m-2} - 1) + \ldots + \begin{cases} a_0 - 1 & \text{for } m \equiv 0(2) \\ a_0(a_1 - 1) + 1 & \text{for } m \equiv 1(2) \end{cases} \]

Therefore for claim 1 it is sufficient to prove that any monomial in \( \mathbb{C}\{x\} \) is a linear combination of the monomials above and of an element of the Jacobi ideal \( J_f = \left( \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m} \right) \). The generators \( \frac{\partial f}{\partial x_j} \) of \( J_f \) are given in (7.5). Obviously also

\[ x_{m-1}^{a_{m-1}} x_m^{a_m} \cdot x_{m-2}^{a_{m-2}} \cdot \ldots \cdot x_1^{a_2} \cdot x_0^{a_0} \]

are in \( J_f \). Start with any monomial in \( \mathbb{C}\{x\} \). Using \( \frac{\partial f}{\partial x_{m-1}}, \frac{\partial f}{\partial x_{m-2}}, \ldots, \frac{\partial f}{\partial x_0} \), and \( x_0^{a_0} \) (in this order), one can reduce it modulo \( J_f \) to 0 or to a monomial \( x_0^{b_0} \cdot \ldots \cdot x_m^{b_m} \) with \( 0 \leq b_j \leq a_j - 1 \) for all \( j \).

If \( b_m \leq a_m - 2 \) stop here. Suppose \( b_m = a_m - 1 \). If \( b_{m-1} \geq 1 \) the monomial is in \( J_f \). Suppose \( b_{m-1} = 0 \). If \( b_{m-2} \leq a_{m-2} - 2 \) stop here. Suppose \( b_{m-2} = a_{m-2} - 1 \). If \( b_{m-3} \geq 1 \), the monomial is modulo \( \mathbb{C} \cdot \frac{\partial f}{\partial x_{m-2}} \) congruent to a monomial \( x_0^{b_0} \cdot \ldots \cdot x_m^{b_m} \) with \( b_{m-1} \geq a_{m-1} \), \( b_m = a_m - 1 \), so it is in \( J_f \). Suppose \( b_{m-3} = 0 \). The claim is proved by repeating these arguments. \( \square \)
Step 2: The second step is the definition of a directed graph $G$ whose directed edges are labelled by the monomials in (7.12). Before defining the directed edges, consider the following $m + 1$ Laurent monomials in $\mathbb{C}[x_0^{\pm 1}, \ldots, x_m^{\pm 1}]$: 

\[ x^{(m)}_2 = x_{m-1}^{a_{m-2}}, \quad x^{(m-1)}_2 = x_{m-2}^{a_{m-2}}, \quad \ldots, \quad x^{(0)}_2 = x_0^{a_{m-1} - (a_{m-1}) x_0^{a_{m-1}}}, \]

Now an edge labelled by $g$ goes from $x^b = x_0^{b_0} \cdot \ldots \cdot x_m^{b_m}$ to $x^c = x_0^{c_0} \cdot \ldots \cdot x_m^{c_m}$ if $x^b \cdot x^g(j) = x^c$. This defines a directed graph $G$ with vertices labelled by the monomials in (7.12) and edges labelled by $g(0), \ldots, g(m)$.

Claim 2: (a) The graph is a chain. If $m \equiv 0(2)$ it starts at $x_0^{a_0 - 1}x_1^{a_2 - 1} \cdot \ldots \cdot x_{m-2}^{a_m - 2}$ and ends at $x_1^{a_1 - 1}x_2^{a_3 - 1} \cdot \ldots \cdot x_{m-1}^{a_m - 1}$. If $m \equiv 1(2)$ it starts at $x_1^{a_1 - 1}x_2^{a_3 - 1} \cdot \ldots \cdot x_{m-1}^{a_m - 1}$ and ends at $x_0^{a_0 - 1}x_2^{a_2 - 1} \cdot \ldots \cdot x_{m-1}^{a_m - 1}$.

(b) The weight of the Laurent monomial $x^g(j)$ is

\[ \deg_w x^g(j) = \left\{ \begin{array}{ll}
-w_m & \text{if } j \equiv m(2), \\
1 - 2w_m & \text{if } j \equiv m + 1(2).
\end{array} \right. \]  

(7.13)

Proof of claim 2: (a) Careful inspection of the set of monomials in (7.12).

(b) In both cases use $w_{k-1} + a_kw_k = 1$. \[ \square \]

Step 3: The third step consists in making precise the recipe 3.3 in the case of the HOR-matrix $S$ respectively its polynomial $p(x)$ in conjecture 7.3. Because $p_0 = (-1)^{m+1}$, the case $m \equiv 0(2)$ is the case $k = 1$ in recipe 3.3, and the case $m \equiv 1(2)$ is the case $k = 2$ in recipe
Then
\[ \alpha_j = \mu(\beta_j - \gamma_j) \quad \text{for } j = 1, \ldots, \mu \]
\[ \mu \cdot \gamma_j = \begin{cases} j - \frac{1}{2} & \text{in the case } m \equiv 0(2) \\ j - 1 & \text{in the case } m \equiv 1(2) \end{cases} \]
\[ \mu \cdot \beta = \frac{\mu}{r_m} (\delta_1, \ldots, \delta_\mu) \quad \text{with } \delta_{j+1} = \delta_j + 1 \text{ or } \delta_{j+1} = \delta_j + 2, \]
\[ \{\delta_1, \ldots, \delta_\mu\} \subset \{0, 1, 2, \ldots, r_m - 1\}, \text{ namely} \]
\[ \prod_{j=1}^{\mu} (x - e^{-2\pi i \delta_j / r_m}) = p(x) = \prod_{l=0}^{m}(x^{r_l} - 1)^{(-1)^{m-1}}. \]

\[ \alpha_1, \ldots, \alpha_\mu \text{ denote now the spectral numbers in } \text{Sp}(S) \text{ with the order from recipe 3.3. We find:} \]

If \( \delta_{j+1} = \delta_j + 1 \) then \( \alpha_{j+1} - \alpha_j = \frac{\mu}{r_m} - 1 = \frac{\mu - r_m}{r_m} = -\frac{\mu - w_m}{r_m} = -w_m \)

If \( \delta_{j+1} = \delta_j + 2 \) then \( \alpha_{j+1} - \alpha_j = 2 \frac{\mu}{r_m} - 1 = 1 - 2w_m. \)

We have to show that \( \alpha_1, \ldots, \alpha_\mu \) coincide up to the shift by \( \frac{m - 1}{2} \) with the spectral numbers of \( f \) which are given by (7.11) and (7.12).

**Step 4 = Claim 3:** Denote the monomials in (7.12) by \( m_1, \ldots, m_\mu \) with the numbering as the chain \( G \) prescribes it. Denote \( \alpha_1(f), \ldots, \alpha_\mu(f) \) according to (7.11). Then
\[ \alpha_j = \alpha_j(f) - \frac{m - 1}{2}, \text{ so } \text{Sp}(S) = \text{Sp}(f) - \frac{m - 1}{2}. \]

**Proof of claim 3:** If the vertices \( m_j \) and \( m_{j+1} \) in the chain \( G \) are connected by an edge of type \( g(l) \) then
\[ \alpha_{j+1}(f) - \alpha_j(f) = \deg_w m_{j+1} - \deg_w m_j = \deg_w x^{g(l)} = \begin{cases} -w_m & \text{if } l \equiv m(2) \\ 1 - 2w_m & \text{if } l \equiv m + 1(2) \end{cases} \]

Therefore it rests to see two points:
\[ \alpha_1(f) = \frac{m - 1}{2} + \alpha_1, \]
\[ \delta_{j+1} = \delta_j + 2 \iff \text{the edge from } m_j \text{ to } m_{j+1} \text{ is of type } \gamma_l \text{ with } l \equiv m + 1(2). \]

We carry out the first point in both cases \( m \equiv 0(2) \) and \( m \equiv 1(2) \) and leave the second point to the reader.
The case $m \equiv 0(2)$: Then
\[
\alpha_1 = \mu (\beta_1 - \gamma_1) = \frac{\mu}{r_m} - \frac{1}{2} = \frac{1}{2} - w_m,
\]
\[
\alpha_1(f) = -1 + \sum_{k=0}^{m} w_k + \text{deg}_w x_0^{a_0-1} x_2^2 \ldots x_m^{a_m-2}
\]
\[
= -1 + \text{deg}_w x_0^{a_0} x_1^2 x_3 \ldots x_m^{a_m-1}
\]
\[
= \frac{m}{2} - w_m = \frac{m - 1}{2} + \alpha_1.
\]

The case $m \equiv 1(2)$: Then
\[
\alpha_1 = \mu (\beta_1 - \gamma_1) = 0
\]
\[
\alpha_1(f) = -1 + \sum_{k=0}^{m} w_k + \text{deg}_w x_1^{a_1-1} x_3 a_3 - 1 \ldots x_m^{a_m-1}
\]
\[
= -1 + \text{deg}_w x_0^{a_1} x_2 x_3 \ldots x_m^{a_m-2} x_{m-1}^{a_m}
\]
\[
= \frac{m - 1}{2} = \frac{m - 1}{2} + \alpha_1. \quad (\square)
\]

This finishes the proof of theorem 7.6 \(\square\)

8. SOME REMARKS AND SPECULATIONS

In the following three subsections, we offer a critical discussion of some arguments in [CV93] with a counterexample, we make a few comments on flat vector bundles and a few comments on Thom-Sebastiani formulas.

8.1. Arguments in [CV93] for conjecture 1.9. The arguments concern the case of M-tame functions respectively Landau-Ginzburg models. They are given precisely in [CV93] pages 589 and 590]. They use \(tt^*\)-geometry.

Indeed, any matrix $S \in T(n, \mathbb{R})$ gives together with arbitrary values $(u_1, \ldots, u_n)$ with $u_i \neq u_j$ for $i \neq j$ and a sufficiently generic value $\xi \in S^1$ rise to a TERP-structure in the sense of [He03], more precisely, it gives a semisimple mixed TERP-structure of weight 1 [HS07, Lemma 10.1], which we call now $TERP(S, (u_1, \ldots, u_n), \xi)$.

But for conjecture 1.9 Cecotti and Vafa want to consider a limit TERP-structure for $(u_1, \ldots, u_n) \to (0, \ldots, 0)$. This should be the UV limit (ultraviolet limit). They assume that it exists and that it has good properties, especially it should be pure and polarized and have the correct spectrum. In [CV93, ch. 5, page 601], they conclude that
the UV limit is well defined and nondegenerate (in a certain sense), if $S^{-1}S'$ is semisimple.

We agree neither with the assumption nor with the conclusion. The following example serves for both as a counterexample.

Therefore we do not consider conjecture 1.9 (for the M-tame case respectively the Landau-Ginzburg models) as proved in [CV93]. Though we do believe that $tt^*$-geometry is a promising road. But a much more precise analysis of the limit behaviour seems to be needed.

**Example 8.1.** Consider a family of exceptional unimodal singularities, e.g. the family $E_{12}$:

$$f_{t\mu}(x, y) = x^3 + y^7 + t\mu \cdot xy^5 \quad \text{with} \quad \mu = 12. \quad (8.1)$$

$f_0$ is quasihomogeneous of weighted degree 1 with respect to the weights $(w_x, w_y) = \left(\frac{1}{3}, \frac{1}{7}\right)$, and $f_{t\mu}$ for $t\mu \neq 0$ is semiquasihomogeneous.

The TERP-structures $TERP(f_{t\mu})$ were studied in [He03, 8.3 (C)]: There is a bound $r_2 \in \mathbb{R} > 0$ such that $TERP(f_{t\mu})$ is not pure for $|t\mu| = r_2$, it is pure and polarized for $|t\mu| < r_2$, and it is pure, but not polarized for $|t\mu| > r_2$. The spectral numbers (from Steenbrink’s MHS) are called $\alpha_1, ..., \alpha_{\mu}$ and satisfy here

$$\alpha_j + \alpha_{\mu+1-j} = 0,$$

$$\alpha_1 = -\frac{11}{21} < -\frac{1}{2} < \alpha_2 = -\frac{8}{21} < ... < \alpha_{\mu-1} = \frac{8}{21} < \frac{1}{2} < \frac{11}{21} = \alpha_{\mu}. \quad (8.2)$$

The eigenvalues of the supersymmetric index $Q$ are for $|t\mu| \neq r_2$

$$\alpha_2, ..., \alpha_{\mu-1} \text{ and } \pm \left(1 - \frac{|t\mu|^2}{r_2^2}\right)^{-1} \left(\alpha_1 - \frac{|t\mu|^2}{r_2^2}(-1 - \alpha_1)\right). \quad (8.3)$$

The last two eigenvalues of $Q$ tend for $|t\mu| \to 0$ to $\pm \alpha_1 = \mp \frac{11}{21}$ and for $|t\mu| \to \infty$ to $\pm (-1 - \alpha_1) = \mp \frac{10}{21}$.

Now consider a universal unfolding

$$F_t(x, y) = f_{t\mu}(x, y) + \sum_{j=1}^{\mu-1} t_j m_j, \quad t \in M \subset \mathbb{C}^\mu, \quad (8.4)$$

for suitable monomials $m_j$ with weighted degree $\deg_w(m_j) < 1$. Here $M \subset \mathbb{C}^\mu$ is an open set which contains $\mathbb{C}^{\mu-1} \times \{0\} \cup \{(0, ..., 0)\} \times \mathbb{C}$ and which is invariant under the flow of the Euler field $E = \sum_{j=1}^{\mu} \deg_w(t_j) \cdot t_j \frac{\partial}{\partial m_j}$.

Choose $\xi \in S^1$ and choose for any $(u_1, ..., u_\mu) \in \mathbb{C}^\mu$ with $\text{Re}\left(\frac{u_i - u_j}{\xi}\right) \neq 0$ for $i \neq j$ a special distinguished system of paths: They shall go
straight in the direction $\xi$ to $\partial \Delta_\eta$ and then turn on $\partial \Delta_\eta$ to $\xi \cdot \eta$. The set

$$\{ t \in M \mid \text{the critical values } u_1, ..., u_\mu \text{ of } F_t \text{ satisfy} \quad \Re\left( \frac{u_i - u_j}{\xi} \right) \neq 0 \text{ for } i \neq j \}$$

consists of finitely many regions, the Stokes regions. Each Stokes region gives one $G_{\text{sign}, \mu}$-orbit of Stokes matrices $S$. For $t$ in one region

$$\text{TERP}(F_t) = \text{TERP}(S, (u_1, ..., u_\mu), \xi),$$

and rescaling $(u_1, ..., u_\mu)$ to $(r \cdot u_1, ..., r \cdot u_\mu)$ with $r > 0, r \to 0$, corresponds to moving $t$ along $-\Re E$. There are now two severe problems.

(I) For $t \in M$ in one region $\text{TERP}(F_t)$ tends to $\text{TERP}(f_0)$ only if $t_\mu = 0$. If $t_\mu \neq 0$ then for $r \to 0 \text{TEPP}(F_t)$ approximates $\text{TERP}(f_\mu)$ for larger and larger $t_\mu$, so it will become pure, but not polarized, and the eigenvalues of its supersymmetric index $Q$ will tend to $\alpha_2, ..., \alpha_{\mu-1}, \pm(-1 - \alpha_1)$.

(II) The $\text{Br}_\mu \ltimes G_{\text{sign}, \mu}$-orbit of all Stokes matrices is infinite. The $G_{\text{sign}, \mu}$-orbits of the Stokes matrices from the finitely many Stokes regions in $M$ form only a finite subset. For $S$ not in this subset it is not at all clear how $\text{TERP}(S, (u_1, ..., u_\mu), \xi)$ will behave for $r \to 0$.

Both problems show that the assumption and the conclusion about existence and good properties of the UV limit are not justified in the generality in which they are claimed in [CV93].

8.2. (Harmonic) vector bundles. We hope that the conjectures 1.6, 1.7 and 1.9 are true and will be proved in the future. The special cases of the HOR-matrices made crucial use of the formulas (4.20) $(-1)^k \cdot S^{-1}S^t = R^{\text{mat}}(k)(S)^n$ for $k \in \{1, 2\}$. They are special cases of the formulas (4.12) $(-1)^k \cdot S^{-1}S^t = R^{\text{mat}}(k_1) \circ ... \circ R^{\text{mat}}(k_n)$. Here the matrices $R^{\text{mat}}(k_{ij})$ are obtained by a certain twist from matrices for Picard-Lefschetz transformations, and they are companion matrices (remark 4.3). We hope that the formulas (4.12) will be useful for an approach to the conjectures 1.6, 1.7 and 1.9.

Certainly, it will also be useful to consider the flat vector bundle on $\mathbb{C} - \{u_1, ..., u_n\}$ of rank $n$ whose monodromy is given by these matrices $R^{\text{mat}}(k_{ij})$ at $u_j$ for $j \in \{1, ..., n\}$ and by $(-1)^kS^{-t}S$ at $\infty$. The vector bundle whose monodromy is given by the Picard-Lefschetz transformations and $(-1)^kS^{-t}S$ is very familiar, it arises as homology bundle of a suitable function with $A_1$-singularities only. We hope that the
local monodromies given by the companion matrices $R^{\text{mat}}_{(kj)}$ will become useful beyond the special case of HOR-matrices.

In the special case of HOR-matrices, the flat bundle decomposes because of (4.20) into flat subbundles, for each eigenvalue $\kappa$ of $R^{\text{mat}}_{(k)}$ one. In the semisimple case, these are flat line bundles. Then one can understand the $\beta$ and $Sp(S)$ in terms of natural holomorphic extensions of these line bundles on $\mathbb{C} - \{u_1, \ldots, u_n\}$ to $\mathbb{P}^1\mathbb{C}$.

But how this observation might extend to the general case of arbitrary matrices $S \in T(n, \mathbb{R})$ is not clear to us. Possibly work on harmonic bundles, tame or wild at $\{u_1, \ldots, u_n, \infty\}$, by Biquard, Boalch, Mochizuki and Sabbah might be useful. And this might have connections to the TERP structures.

8.3. Thom-Sebastiani formulas. In the case of isolated hypersurface singularities (short: ihs), an important technique for obtaining new ihs is, to consider the sum $f(x_0, \ldots, x_m) + g(x_{m+1}, \ldots, x_{m+n+1})$ of two ihs $f$ and $g$ in different variables. This is discussed in [AGV88, I.2.7] and reviewed (in notations closer to this paper) in [GH17]. There is a canonical isomorphism

$$\Phi : Ml(f + g, 1) \xrightarrow{\cong} Ml(f, 1) \otimes Ml(g, 1), \quad (8.7)$$

with $M(f + g) \cong M(f) \otimes M(g)$ (8.8)

and $L^{\text{nor}}(f + g) \cong L^{\text{nor}}(f) \otimes L^{\text{nor}}(g).$ (8.9)

If $\delta = (\delta_1, \ldots, \delta_{\mu(f)})$ and $\gamma = (\gamma_1, \ldots, \gamma_{\mu(g)})$ are distinguished bases of $f$ and $g$ with Stokes matrices $S(f)$ and $S(g)$, then

$$\Phi^{-1}(\delta_1 \otimes \gamma_1, \ldots, \delta_{\mu(f)} \otimes \gamma_{\mu(g)}, \delta_2 \otimes \gamma_1, \ldots, \delta_{\mu(f)} \otimes \gamma_1, \ldots, \delta_{\mu(f)} \otimes \gamma_{\mu(g)})$$

is a distinguished basis of $Ml(f + g, 1)$, that means, one takes the vanishing cycles $\Phi^{-1}(\delta_i \otimes \gamma_j)$ in the lexicographic order. Then by (6.7) and (8.9), the matrix

$$S(f + g) = S(f) \otimes S(g) \quad (8.10)$$

(where the tensor product is defined so that it fits to the lexicographic order) is the Stokes matrix of this distinguished basis.

In [SS85, ch. 8] a Thom-Sebastiani for Steenbrink’s mixed Hodge structure is stated. It is fine if the monodromy is semisimple, but it needs a correction in the general case. That correction is an interesting and nontrivial twist [BH17, Corollary 6.5], which comes from a Fourier-Laplace transformation. Anyway, the resulting Thom-Sebastiani formula in [SS85] for the spectral pairs of $f$, $g$ and $f + g$ is correct.

The set of HOR-matrices is not invariant under the tensor product of matrices. It might be a good idea to check whether there are natural
modifications for the recipe how the HOR-matrices give rise to spectral numbers, which are compatible with the Thom-Sebastiani formulas.

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