Wireless communications with diffuse waves

S.E. Skipetrov

CNRS/Laboratoire de Physique et Modélisation des Milieux Condensés,
38042 Grenoble, France

(Dated: November 21, 2018)

Wireless communications in a disordered environment have recently attracted considerable attention \[1, 2, 3\]. At first glance, scattering from randomly distributed heterogeneities disturbs the signal carried by the scattered wave (either acoustic or electromagnetic) and is expected to reduce the efficiency of communication. Indeed, the rate of error-free information transfer per Hertz of frequency bandwidth for a scalar, narrow-band communication channel between a transmitter and a receiver is bounded by the channel information capacity

\[
C = \max_Q \left\{ \log_2 \det \left[ I_n + G Q G^+ / N \right] \right\},
\]

where \( I_n \) is the \( n \times n \) unit matrix, \( G \) is a Green matrix (\( G_{io} \) gives the signal received by the receiver \( \alpha \) due to the transmitter \( i \)), and \( Q \) is a non-negative definite covariance matrix describing correlations between the transmitted signals (with the constraint on the maximum transmitted power \( Tr Q \leq n \)). Angular brackets in Eq. (1) denote averaging over realizations of disorder. A rigorous analysis of the information capacity \( C \) in a disordered medium cannot be based uniquely on the arguments of the information theory \[4, 5\] and requires the physical understanding of scattering undergone by the waves carrying the information from transmitters to receivers. In the present paper we analyze \( C \) in the framework of the model of diffuse multiple scattering, resulting in a complicated, seemingly random spatial distribution of scattered wave fields (so-called “speckles”). For the extreme cases of (a) a single speckle spot covering the whole antenna array (\( G \) has perfectly correlated entries) and (b) different antennas situated inside different speckle spots (entries of \( G \) are uncorrelated), we derive complete analytical expressions for \( C \). We show that \( C \) increases during the continuous transition from the case (a) to the case (b) upon increasing antenna spacing (or, equivalently, upon decreasing correlations between the entries of \( G \)) and changes its asymptotic scaling with \( n \) from \( C \propto \ln n \) to \( C \propto n \).

To be specific, we consider two identical linear arrays of equally-spaced transmitting/receiving antennas placed in a disordered medium. The distance \( L \) between the arrays is assumed to be much greater than the mean free path \( \ell \) for waves in the medium, while the array size \( d = (n - 1) a \ll \ell \) and \( d \ll (\lambda L)^{1/2} \) (Fresnel limit), where \( a \) is the spacing between adjacent antennas and \( \lambda \) is the wavelength. In the majority of practically important cases \( \lambda \ll \ell \), and hence the propagation of waves from transmitters to receivers is diffusive \[4\]. Entries \( G_{io} \) of the matrix \( G \) can be then treated as complex Gaussian random variables with zero mean, equal variances, and possibly nontrivial correlations \[8\]

\[
\langle G_{io} G_{i'\beta} \rangle = \sigma^2 \frac{\sin(k \Delta r_{ij})}{k \Delta r_{ij}} \frac{\sin(k \Delta r_{a \beta})}{k \Delta r_{a \beta}},
\]

where \( k = 2\pi / \lambda \), \( \Delta r_{ij} = |i - j| a \), and \( \Delta r_{a \beta} = |\alpha - \beta| a \).
FIG. 1: Average information capacity per antenna of a communication channel between two arrays of n antennas in a disordered medium assuming statistical independence of the entries of the Green matrix G: results obtained using the random matrix theory (solid line) and Monte Carlo simulation (symbols) are shown for the signal to noise ratio S/N = 100. Horizontal dashed line shows the asymptotic value of C/n for n → ∞. Inset: Capacity per antenna as a function of S/N for single transmitting and single receiving antennas (dashed line) and an infinitely large number of antennas (solid line).

The simplest cases to consider are (a) kd → 0 and (b) ka = mπ (m = 1, 2, ...). In the former case, all G_{ij} are perfectly correlated and one finds
\[ C = \frac{1}{2} \log \left( 1 + \frac{S}{N} \right), \]
where C is the average power received at each receiver assuming independent signals from transmitters, and C ∝ 1/n for nS/N ≫ 1. In the case (b), Q = I_n in Eq. (1) and the machinery of the random matrix theory [10] can be employed for further analysis of the problem. In particular, it is useful to rewrite Eq. (1) as
\[ C = \frac{1}{2} \log \left( 1 + \frac{S}{N} \right), \]
where \( \mu_i \) are the squares of the singular values of the matrix \( G^{+} / S^{1/2} \) with the joint probability density function
\[ p_n(\mu_1, \ldots, \mu_n) = A_n \exp \left( -n \sum_{i=1}^{n} \mu_i \right) \prod_{i<j} \left( \mu_i - \mu_j \right)^2, \]
where \( A_n \) is a normalization constant. The moment generating function \( F(x) \) of a random variable c, defined by Eq. (3) without averaging, is obtained by averaging \( \exp(xc) \) with help of Eq. (3) and \( C = \langle c \rangle \) is then calculated as
\[ C = \frac{d}{dx} F(x) \bigg|_{x=0} = \text{Tr} \left( P^{-1} R \right), \]
where \( P \) and \( R \) are \( n \times n \) matrices:
\[ P_{ij} = (i + j - 2)! n^{i+j-2}, \]
\[ R_{ij} = \int_{0}^{\infty} d\mu \ln \left[ 1 + (S/N) \mu \right] \mu^{i+j-2} \exp \left( -n \mu \right). \]

Equations (3)-(7) provide an efficient way for capacity calculation at arbitrary n and S/N (see the solid line in Fig. 1) and agree perfectly with the direct Monte Carlo simulation of capacity using Eq. (5) (symbols in Fig. 1). Note that the growth of C with n is linear at large n and hence is much faster than in the case (a).

An alternative way of calculating capacity consists in performing averaging directly in Eq. (1) using Eq. (4). This yields
\[ C/n = \int_{0}^{\infty} d\mu \log \left[ 1 + (S/N) \mu \right] f_n(\mu), \]
where \( f_n(\mu) = \int_{0}^{\infty} d\mu_2 \ldots d\mu_n p_n(\mu_1, \ldots, \mu_n) \) can be evaluated by a direct integration, at least at moderate n: \( f_1(\mu) = \exp(-\mu), f_2(\mu) = 2 \exp(-2\mu)[1 - 2\mu + 2\mu^2], f_3(\mu) = 3 \exp(-3\mu)[1 - 6\mu + 18\mu^2 - 18\mu^3 + (27/4)\mu^4], \) etc. The values of capacity obtained then from Eq. (8) coincide exactly with that following from Eq. (5). An asymptotic expression for \( f_n(\mu) \) at n ≫ 1 can be found in the framework of the random matrix theory [10]: \( f_n(\mu) = (2\pi)^{-1}(4/\mu-1)^{1/2} \) for \( 0 < \mu < 4 \) and \( f_n(\mu) = 0 \) otherwise. Eq. (8) then yields
\[ C/n = \langle S/N \rangle_3 F_2 \left( 1, 1, 3/2; 2, 3; -4 S/N \right) / \ln 2, \]
where \( 3F_2 \) is the generalized hypergeometric function. This result is shown in the main plot of Fig. 1 by a dashed horizontal line. We find C/n ≈ log_2(S/N) − log_2 e for S/N ≫ 1 (see also the inset of Fig. 1). It is worthwhile to note that C/n decreases monotonically with n, while the difference between the values of C/n at n = 1 and n > 1 never exceeds 7%.

We now allow ka to take arbitrary values, thus introducing correlations between the entries of the Green matrix G. Eq. (1) can again be reduced to Eq. (3) with \( \mu_1 \) denoting the squares of the singular values of the matrix \( Q^{1/2} G^{+} / S^{1/2} \), where Q is chosen to maximize the result. The joint probability density function of \( \mu_i \), analogous to Eq. (4), is not known in this case, but one can still calculate C in the large-n limit [11]. The idea is to represent the moment generating function F(x) of a random variable c, defined by Eq. (3) without averaging, as a multiple Gaussian integral (the so-called “replica trick”) and then to perform the integrations using saddle point methods in the limit n ≫ 1. The maximization of \( C = \langle d/dx F(x) \rangle_{x=0} \) over Q is then accomplished by requiring δC ≤ 0 for all allowed small variations δQ of the optimal matrix Q. This yields a system of nonlinear
FIG. 2: Average information capacity per antenna of a communication channel between two identical linear arrays of \( n \) antennas in a disordered medium assuming antenna spacing \( a \) and \( S/N = 100 \). Solid lines correspond to \( n = 1 \) (exact result), \( n = 2 \) (Monte Carlo simulation) and \( n = 100 \) (asymptotic large-\( n \) result). Dashed line shows the result obtained using the asymptotic large-\( n \) theory at \( n = 2 \). Inset: Capacity per antenna as a function of \( n \) for different antenna spacings \( a \). Results for \( ka = \pi/2 \) and \( \pi/4 \) are asymptotic large-\( n \) results.

equations for the eigenvalues of \( Q \) and some auxiliary variables that can be solved numerically. We refer the reader to Refs. \([11, 12]\) for an exhaustive account of the theoretical approach and the algorithm of the numerical calculation. The results that we obtained for identical linear arrays of equally-spaced antennas are presented in Fig. 2. As follows from the figure, at \( ka > \pi \) correlations between the entries of \( G \) are too weak to affect \( C \) significantly and the latter remains very close to its maximum value, given by Eqs. (5) or (8). In contrast, correlations become important when \( ka \) decreases below \( \pi \), leading to a significant drop of \( C \). The dashed line in Fig. 2 results from the asymptotic large-\( n \) theory \([1, 11, 12]\) with \( n = 2 \) and is shown for illustration purposes only. Its closeness to the Monte Carlo result (solid line for \( n = 2 \)) testifies a qualitative validity of the large-\( n \) theory even at small \( n \). As \( ka \) increases, the scaling of \( C \) with \( n \) changes from \( C \propto \ln n \) at \( ka = 0 \) to \( C \propto n \) for \( ka > \pi \) (see the inset of Fig. 2). We note that even at \( 0 < ka < \pi \), there is still a significant gain in capacity as compared to the case of \( ka = 0 \): e.g., at \( n = 100 \), \( C/n \) for \( ka = \pi/2 \) (\( \pi/4 \)) is almost 20 (13) times larger than for \( ka = 0 \).

In conclusion, we have studied the information capacity \( C \) of a wireless communication channel in a disordered medium, assuming multiple diffuse scattering of waves that carry information, and taking into account fluctuations of wave fields in space (speckles). Although multiple scattering reduces the received signal, it allows for a dramatic increase of capacity in the case of communication between two antenna arrays, provided that antenna spacing \( a > \lambda/2 \), where \( \lambda \) is the wavelength. Namely, for identical linear arrays of \( n \) equally spaced receiving/transmitting antennas, scaling of \( C \) with \( n \) changes from \( C \propto \ln n \) for \( kd \ll \pi \) to \( C \propto n \) for \( ka \gtrsim \pi \), where \( d = a(n - 1) \) is the array size and \( k = 2\pi/\lambda \). Even at \( 0 < ka < \pi \) an important gain in capacity is possible as compared to \( ka = 0 \).

The author is indebted to Prof. R. Maynard for helpful discussions and critical reading of the manuscript.

[1] A.L. Moustakas, H.U. Baranger, L. Balents, A.M. Sengupta, S.H. Simon, Science 287, 287 (2000).
[2] M.R. Andrews, P.P. Mitra, and R. deCarvalho, Nature 409, 316 (2001).
[3] S.H. Simon, A.L. Moustakas, M. Stoytchev, and H. Safar, Phys. Today 54(9), 38 (2001).
[4] C.E. Shannon, Bell Syst. Tech. J. 27, 379 & 623 (1948).
[5] T.M. Cover and J.A. Thomas, Elements of Information Theory (Wiley, N.Y., 1991).
[6] G.J. Foschini and M.J. Gans, Wireless Personal Communications 6, 311 (1998).
[7] A. Ishimaru, Wave Propagation and Scattering in Random Media (Academic, N.Y., 1978).
[8] B. Shapiro, Phys. Rev. Lett. 57, 2186 (1986).
[9] I.E. Telatar, Eur. Trans. Telecommun. 10, 585 (1999).
[10] M.L. Mehta, Random Matrices (Academic, N.Y., 1991).
[11] A.M. Sengupta and P.P. Mitra, Physics/0010081 (2000).
[12] A.M. Sengupta and P.P. Mitra, Phys. Rev. E 60, 3389 (1999).
[13] For simplicity, we consider the transmitting and receiving arrays to consist of the same number \( n \) of antennas.