PRESENTING QUEER SCHUR SUPERALGEBRAS

JIE DU AND JINKUI WAN

ABSTRACT. Associated to the two types of finite dimensional simple superalgebras, there are the general linear Lie superalgebra and the queer Lie superalgebra. The universal enveloping algebras of these Lie superalgebras act on the tensor spaces of the natural representations and, thus, define certain finite dimensional quotients, the Schur superalgebras and the queer Schur superalgebra. In this paper, we introduce the quantum analogue of the queer Schur superalgebra and investigate the presentation problem for both the queer Schur superalgebra and its quantum analogue.

1. Introduction

A superspace $V$ is a vector space over a field endowed with a $\mathbb{Z}_2$-grading (or a parity structure): $V = V_0 \oplus V_1$, where an element in $V_0$ is called even, while an element in $V_1$ is called odd. A superalgebra $A$ is a $\mathbb{Z}_2$-graded (associative) algebra with 1 over a field. Thus, the underlying space of $A$ is a superspace $A = A_0 \oplus A_1$ and the multiplication satisfies $A_i A_j \subseteq A_{i+j}$, for $i,j \in \mathbb{Z}_2$.

It is known (see, e.g., [2]) that a finite dimensional simple superalgebra over the complex field $\mathbb{C}$ is either isomorphic to the (full) matrix superalgebra $M = M_{m+n}(\mathbb{C})$ with even part $M_0 = \{ (A_0^0 | A \in M_m(\mathbb{C}), B \in M_n(\mathbb{C}) \}$ and odd part $M_1 = \{ (C^D | C \in M_{m,n}(\mathbb{C}), D \in M_{n,m}(\mathbb{C}) \}$, or isomorphic to the queer matrix superalgebra $Q = \{ (A^B | A,B \in M_n(\mathbb{C}) \}$ with even part $Q_0 = \{ (A^A | A \in M_n(\mathbb{C}) \}$ and odd part $Q_1 = \{ (B^B | B \in M_n(\mathbb{C}) \}$.

Associated to a superalgebra $A$, there is a Lie superalgebra $A^-$ equipped with the super bracket product (or super commutator) defined by

$$[x,y] := xy - (-1)^{\hat{x}\hat{y}}yx,$$

where $x, y \in A$ are homogeneous elements and $\hat{z} = i$ if $z \in A_i$. Thus, the two type simple superalgebras $M$ and $Q$ give rise to two Lie superalgebras $\mathfrak{gl}(m|n) := M^-$, the general linear Lie superalgebra, and $q(n) := Q^-$, the queer Lie superalgebra.

If $V$ denotes the natural representation of $\mathfrak{gl}(m|n)$ (resp., $q(n)$), then the tensor product $V^\otimes r$ is a representation of the universal enveloping algebra $U(\mathfrak{gl}(m|n))$ (resp., $U(q(n))$). The image of $U(\mathfrak{gl}(m|n))$ (resp., $U(q(n))$) in $\text{End}(V^\otimes r)$ is called the Schur superalgebra (resp. queer Schur superalgebra or Schur superalgebra of type $Q$, following [2]).

The Schur superalgebras and their representations were introduced and investigated by several authors including Donkin [6] and Brundan–Kujawa [4] almost over ten years ago. Recently, the study of quantum Schur superalgebras has made substantial progress; see [17, 11, 12, 8, 9]. In particular, in [12], El Turkey and Kujawa provided a presentation...
of the Schur superalgebras and their quantum analogues, which generalizes the work of Doty and Giaquinto [7] for (quantum) Schur algebras.

It is known that the queer Lie superalgebra \(q(n)\) differs drastically from the basic classical Lie superalgebras. For example, the Cartan subalgebra of \(q(n)\) is not purely even and there is no invariant bilinear form on \(q(n)\). On the other hand, \(q(n)\) behaves in many aspects as the Lie algebra \(gl(n)\). In particular, there exists a beautiful analogue of the Schur-Weyl duality discovered by Sergeev [20], often referred as Sergeev duality. In [18], Olshanski constructed a quantum deformation \(U_q(q(n))\) of the universal enveloping algebra \(U(q(n))\) and established a quantum analog of the Sergeev duality in the generic case.

The queer Schur superalgebra was introduced and studied by Brundan and Kleshchev [2], and then they determined the irreducible projective representations of the symmetric group \(S_r\), via Sergeev duality. In this paper, we will introduce the quantum analogue of queer Schur superalgebras and will follow the works [7] and [10] to determine a presentation for the queer Schur superalgebra (see [1] for a more general setting involving walled Brauer-Clifford superalgebras) and its quantum analogue, which was stated as an interesting problem in [12]. In particular, in the quantum case we also establish the existence of \(Z\)-form for the quantum superalgebra \(U_q(q(n))\). This is based on a lengthy but straightforward calculation of the commutation formulas for the divided powers of root vectors.

We organise the paper as follows. We first investigate a presentation of the queer Schur superalgebra in the first three sections. More precisely, we study in §2 the basics of the queer Lie superalgebra and its universal enveloping superalgebra, and establish the commutation formulas for divided powers of root vectors and a Kostant \(Z\)-form in §3. A presentation for the queer Schur superalgebra is given in §4. From §5 onwards, we investigate the quantum case. We start in §5 with the Olshanski presentation (via a certain matrix in \(\text{End}(V)^{\otimes 2}\) satisfying the quantum Yang-Baxter equation) and the Drinfeld–Jimbo type presentation for the quantum queer superalgebra and introduce all quantum root vectors. We compute all commutation formulas for these vectors in §6 and for those with higher order in §7. A Lusztig type form for the quantum queer superalgebra is introduced and certain quotients are investigated in §8. Finally, we solve the presentation problem for the quantum queer Schur superalgebra in the last section.

Throughout the paper, let \(\mathbb{Z}_2 = \{0, 1\}\). We will use a two-fold meaning for \(\mathbb{Z}_2\). We will regard \(\mathbb{Z}_2\) as an abelian group when we use it to describe a superspace. However, for a matrix or an \(n\)-tuple with entries in \(\mathbb{Z}_2\), we will regard it as a subset of \(\mathbb{Z}\).

2. THE QUEER LIE SUPERALGEBRA \(q(n)\) AND THE ASSOCIATED SCHUR SUPERALGEBRA \(Q(n, r)\)

The ground field in this section is the field \(\mathbb{Q}\) of rational numbers. It is known that the general linear Lie superalgebra \(gl(n|n)\) consists of matrices of the form

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

where \(A, B, C, D\) are arbitrary \(n \times n\) matrices, and the rows and columns of (2.0.1) are labelled by the set

\[I(n|n) = \{1, 2, \ldots, n, -1, -2, \ldots, -n\}.\]
For $i, j \in I(n|n)$, denote by $E_{i,j} \in \mathfrak{gl}(n|n)$ the matrix unit with 1 at the $(i, j)$ position and 0 elsewhere. The set $\{E_{i,j} \mid i, j \in I(n|n)\}$ is a basis of $\mathfrak{gl}(n|n)$ and the $\mathbb{Z}_2$-grading on $\mathfrak{gl}(n|n)$ is defined via $\hat{E}_{i,j} = 0$ if $ij > 0$ and $\hat{E}_{i,j} = 1$ if $ij < 0$. Then the Lie bracket in $\mathfrak{gl}(n|n)$ is given by

$$[E_{i,j}, E_{k,l}] = \delta_{jk}E_{i,l} - (-1)^{\hat{E}_{i,j}\hat{E}_{k,l}}E_{k,i}E_{k,j}. \quad (2.0.2)$$

The queer Lie superalgebra, denoted by $\mathfrak{g} = \mathfrak{q}(n)$, is the subalgebra of the general linear Lie superalgebra $\mathfrak{gl}(n|n)$ consisting of matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad (2.0.3)$$

where $A$ and $B$ are arbitrary $n \times n$ matrices. The even (resp., odd) part $\mathfrak{g}_0$ (resp., $\mathfrak{g}_1$) consists of those matrices of the form $(2.0.3)$ with $B = 0$ (resp., $A = 0$). We fix $\mathfrak{h}$ to be the standard Cartan subalgebra of $\mathfrak{g}$ consisting of matrices of the form $(2.0.3)$ with $A, B$ being arbitrary diagonal. Then the algebra $\mathfrak{h}_0$ has a basis $\{h_1, \ldots, h_n\}$ and $\mathfrak{h}_1$ has a basis $\{h_1, \ldots, h_n\}$, where

$$h_i = E_{i,i} + E_{-i,-i}, \quad h_i = E_{i,-i} + E_{-i,i}. \quad (2.0.4)$$

Fix the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) is the subalgebra of $\mathfrak{g}$ which consists of matrices of the form $(2.0.3)$ with $A, B$ being arbitrary upper triangular (resp., lower triangular) matrices. Observe that the even subalgebra $\mathfrak{h}_0$ of $\mathfrak{h}$ can be identified with the standard Cartan subalgebra of $\mathfrak{gl}(n)$ via the natural isomorphism

$$\mathfrak{q}(n)_0 \cong \mathfrak{gl}(n). \quad (2.0.5)$$

Let $\{\epsilon_i \mid i = 1, \ldots, n\}$ be the basis for $\mathfrak{h}_0^*$ dual to the standard basis $\{h_i \mid i = 1, \ldots, n\}$ for $\mathfrak{h}_0$ and we define a bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}_0^*$ via

$$(\epsilon_i, \epsilon_j) := \epsilon_j(h_i) = \delta_{ij}. \quad (2.0.6)$$

For $\alpha \in \mathfrak{h}_0^*$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}_0\}$. Then we have the root superspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ with the root system

$$\Phi = \{\alpha_{i,j} := \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}.$$

The set of positive roots corresponding to the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is

$$\Phi^+ = \{\alpha_{i,j} \mid 1 \leq i < j \leq n\}.$$  

Observe that each root superspace $\mathfrak{g}_\alpha$ has dimension vector $\mathbf{1}^4(1, 1)$ for $\alpha \in \Phi$. Let $\alpha_i = \alpha_{i,i+1}$ for $1 \leq i \leq n - 1$. Then the root space $\mathfrak{g}_{\alpha_i}$ is spanned by $\{e_i, e_j\}$ with

$$e_i = E_{i,i+1} + E_{-i,-i-1}, \quad e_j = E_{i,-i-1} + E_{-i,i+1}, \quad (2.0.7)$$

while the root space $\mathfrak{g}_{-\alpha_i}$ is spanned by $\{f_i, f_j\}$ with

$$f_i = E_{i+1,i+1} + E_{-i-1,-i-1}, \quad f_j = E_{i+1,-i} + E_{-i-1,i}. \quad (2.0.8)$$

Moreover, $\alpha_{i,j} = \alpha_i + \cdots + \alpha_{j-1}$ for all $1 \leq i < j \leq n$. Let

$$\mathcal{P} := \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i, \quad (\text{resp., } \mathcal{P}_{\geq 0} = \bigoplus_{i=1}^n \mathbb{N} \epsilon_i)$$

be the weight lattice (resp., positive weight lattice) of $\mathfrak{q}(n)$.

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$^1$The dimension vector of a superspace $V$ is the tuple $\mathbf{dim} V = (\dim V_0, \dim V_1)$. 

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The universal enveloping superalgebra $U = U(q(n))$ is obtained from the tensor algebra $T(q(n))$ by factoring out the ideal generated by the elements $[u, v] - u \otimes v + (-1)^{\bar{u}\bar{v}}v \otimes u$ for $u \in \mathfrak{g}, v \in \mathfrak{g}$ with $i, j \in \mathbb{Z}_2$, where $[u, v]$ denotes the Lie bracket of $u, v$ in $q(n)$. It inherits a $\mathbb{Z}_2$-grading from $q(n)$.

**Proposition 2.1** ([13] Proposition 1.1, cf.([15])). The universal enveloping superalgebra $U(q(n))$ is the associative superalgebra over $\mathbb{Q}$ generated by even generators $h_i, e_j, f_j$ and odd generators $h_i, e_j, f_j$, with $1 \leq i \leq n$ and $1 \leq j \leq n - 1$ subject to the following relations:

1. (QS1) $[h_i, h_j] = 0, \quad [h_i, f_j] = 0, \quad [h_i, [h_j, h_k]] = \delta_{ij}2h_k$;
2. (QS2) $[h_i, e_j] = (\epsilon_i, \alpha_j)e_j, \quad [h_i, f_j] = (\epsilon_i, \alpha_j)e_j, \quad [h_i, e_j] = -\epsilon_i, \epsilon_j)f_j, \quad [h_i, f_j] = -\epsilon_i, \epsilon_j)f_j$;
3. (QS3) $[h_i, e_j] = (\epsilon_i, \alpha_j)e_j, \quad [h_i, f_j] = -\epsilon_i, \alpha_j)f_j$,
   $$[h_i, e_j] = \begin{cases} 
   e_j, & \text{if } i = j \text{ or } j + 1, \\
   0, & \text{otherwise,}
   \end{cases} \quad [h_i, f_j] = \begin{cases} 
   f_j, & \text{if } i = j \text{ or } j + 1, \\
   0, & \text{otherwise;}
   \end{cases}$$
4. (QS4) $[e_i, f_j] = \delta_{ij}(h_i - h_{i+1}), \quad [e_i, f_j] = \delta_{ij}(h_i + h_{i+1}), \quad [e_i, f_j] = \delta_{ij}(h_i - h_{i+1})$;
5. (QS5) $[e_i, e_j] = [f_i, f_j] = [f_i, f_j] = 0$ for $|i - j| \neq 1$,
   $$[e_i, e_j] = [f_i, f_j] = 0 \quad \text{for } |i - j| > 1,$$
   $[e_i, e_{i+1}] = [e_i, f_{i+1}] = [e_i, f_{i+1}] = [f_{i+1}, f_i] = [f_{i+1}, f_i] = [f_{i+1}, f_i]$;
6. (QS6) $[e_i, e_j] = [e_i, e_j] = 0, \quad [f_i, f_j] = [f_i, f_j] = 0$ for $|i - j| = 1$.

Let $U^+$ (resp. $U^0$ and $U^-$) be the subalgebra of $U = U(q(n))$ generated by the elements $e_i, e_i$ (resp. $h_j, h_j$, and $f_i, f_i$), where $1 \leq i \leq n - 1, 1 \leq j \leq n$. Then, similar to Lie algebras, we have the Poincaré-Birkhoff-Witt (PBW) Theorem and triangular decomposition as follows; see [5] Theorem 1.32, [13] (1.3), [19] Theorem 2.1.

**Proposition 2.2.**

1. Suppose that $\{z_1, \ldots, z_p\}$ is a homogeneous basis for $q(n)$. Then the set
   $$\{z_1^{a_1}z_2^{a_2} \cdots z_p^{a_p} | a_1, \ldots, a_p \in \mathbb{N}, a_i \in \mathbb{Z}_2 \text{ if } z_i \text{ is odd, } 1 \leq i \leq p\}$$
   is a basis for $U(q(n))$.
2. The algebra $U(q(n))$ has the triangular decomposition $U(q(n)) \cong U^- \otimes U^0 \otimes U^+$.

Let $V$ be the vector superspace over $\mathbb{Q}$ with dimension vector $\text{dim}V = (n, n)$. Fix a basis $\{v_1, \ldots, v_n\}$ for $V_0$ and a basis $\{v_{-1}, \ldots, v_{-n}\}$ for $V_1$, respectively. Then there is a natural action of the algebra $gl(n|n)$ on $V$ given by left multiplication, that is,

$$E_{ij}v_k = \delta_{jk}v_i \quad (2.2.1)$$

for $i, j, k \in I(n|n)$. The restriction to the algebra $q(n)$ implies that $V$ naturally affords a representation of $U = U(q(n))$. As $U(q(n))$ admits a comultiplication $U(q(n)) \rightarrow U(q(n)) \otimes U(q(n))$ given on elements of $q(n)$ by $x \mapsto x \otimes 1 + 1 \otimes x$, we have that the $r$-fold tensor product $V^{\otimes r}$ of the natural module $V$ also affords a $U(q(n))$-module. Let $\phi_r$ denote the corresponding superalgebra homomorphism:

$$\phi_r : U(q(n)) \rightarrow \text{End}_Q(V^{\otimes r}).$$

Define the queer Schur superalgebra (also known as Schur superalgebra of type Q, cf. [2]) to be

$$Q(n, r) = \phi_r(U(q(n))), \quad (2.2.2)$$
that is, the image of $\phi_r$. Therefore, $Q(n, r)$ can be viewed as a quotient of $U(q(n))$.

Similar to Schur algebras associated to $gl(n)$, there is another way to define the queer Schur superalgebra $Q(n, r)$ via the known Schur-Sergeev duality as follows. Denote by $C_r$ the Clifford superalgebra generated by odd elements $c_1, \ldots, c_r$ subject to the relations

$$c_i^2 = -1, \quad c_ic_j = -c_jc_i, \quad 1 \leq i \neq j \leq r.$$  \hfill (2.2.3)

Denote by $\mathcal{S}_r^c = C_r \rtimes \mathfrak{S}_r$ the so-called Sergeev superalgebra, which is generated by the even elements $s_1, \ldots, s_{r-1}$ and the odd elements $c_1, \ldots, c_r$ subject to (2.2.3), and the additional relations:

$$s_i^2 = 1, \quad s_is_j = s_js_i, \quad 1 \leq i, j \leq r - 1, \quad |i - j| > 1,$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad 1 \leq i \leq r - 2,$$

$$s_ic_i = c_{i+1}s_i, \quad s_ic_j = c_js_i, \quad 1 \leq i \leq r - 1, \quad 1 \leq j \leq r, \quad j \neq i, i + 1.$$  

Then by [20, Lemma 2], we have a representation $(\psi_r, V^{\otimes r})$ of $\mathcal{S}_r^c$ on $V^{\otimes r}$ defined by

$$\psi_r(s_k)(v_{j_1} \otimes \cdots \otimes v_{j_k} \otimes v_{j_{k+1}} \otimes \cdots \otimes v_{j_r}) = (-1)^{\overline{j_k} + \overline{j_{k+1}}} v_{j_1} \otimes \cdots \otimes v_{j_{k+1}} \otimes v_{j_k} \otimes \cdots \otimes v_{j_r},$$

$$\psi_r(c_l)(v_{j_1} \otimes \cdots \otimes v_{j_l} \otimes \cdots \otimes v_{j_r}) = (-1)^{\overline{j_1} + \cdots + \overline{j_{l-1}}} v_{j_1} \otimes \cdots \otimes v_{j_{l-1}} \otimes J_V(v_{j_l}) \otimes \cdots \otimes v_{j_r},$$

for all $j_1, \ldots, j_r \in I(n|n), 1 \leq k \leq r - 1, 1 \leq l \leq r$, where $J_V \in \text{End}(V)$ satisfies $J_V(v_a) = v_{-a}$ and $J_V(v_{-a}) = -v_a$ for $1 \leq a \leq n$. Then a classical result [20, Theorem 3] of Sergeev says

$$Q(n, r) = \text{End}_{\mathcal{S}_r^c}(V^{\otimes r}),$$  \hfill (2.2.4)

which was introduced and studied in [2]. In particular, we note that $Q(n, r)$ is naturally a subsuperalgebra of the Schur superalgebra associated to the Lie superalgebra $gl(n|n)$.

3. Commutation formulas for root vectors and Kostant Z-form

For $\alpha_{i,j} = \epsilon_i - \epsilon_j \in \Phi$ with $1 \leq i \neq j \leq n$, we introduce the root vectors in $q(n)$ as follows:

$$x_{i,j} \equiv x_{\alpha_{i,j}} = E_{i,j} + E_{-i,-j}, \quad \bar{x}_{i,j} \equiv \bar{x}_{\alpha_{i,j}} = E_{h,-j} + E_{-i,j}.$$  \hfill (3.0.5)

Clearly, $x_\alpha$ is even and $\bar{x}_\alpha$ is odd for $\alpha \in \Phi$. Observe that the even element $x_\alpha$ for $\alpha \in \Phi$ correspond to the usual root vectors in $gl(n)$ under the identification (2.0.3). Moreover the set $\{x_\alpha, \bar{x}_\alpha \mid \alpha \in \Phi\} \cup \{h_i, h_{\bar{i}} \mid 1 \leq i \leq n\}$ is a homogeneous basis for $q(n)$.

By (3.0.5) and (2.0.2), a direct calculation gives rise to the following super commutator formulas.

**Lemma 3.1.** For $\alpha_{i,j}, \alpha_{k,l} \in \Phi$ satisfying $\alpha_{i,j} + \alpha_{k,l} \in \Phi$, let $\varepsilon = \varepsilon_{i,j;k,l} = \begin{cases} 1, & \text{if } j = k; \\ -1, & \text{if } i = l. \end{cases}$

Then we have in $q(n)$

\begin{align*}
(1) \quad [x_{i,j}, x_{k,l}] &= \begin{cases} h_i - h_j, & \text{if } \alpha_{i,j} + \alpha_{k,l} = 0; \\
\varepsilon x_\beta, & \text{if } \beta = \alpha_{i,j} + \alpha_{k,l} \in \Phi; \\
0, & \text{otherwise}, \end{cases} \\
(2) \quad [x_{i,j}, \bar{x}_{k,l}] &= \begin{cases} h_i - h_j, & \text{if } \alpha_{i,j} + \alpha_{k,l} = 0; \\
\varepsilon \bar{x}_\beta, & \text{if } \beta = \alpha_{i,j} + \alpha_{k,l} \in \Phi; \\
0, & \text{otherwise}, \end{cases} \\
(3) \quad [\bar{x}_{i,j}, \bar{x}_{k,l}] &= \begin{cases} h_i + h_j, & \text{if } \alpha_{i,j} + \alpha_{k,l} = 0; \\
x_\beta, & \text{if } \beta = \alpha_{i,j} + \alpha_{k,l} \in \Phi; \\
0, & \text{otherwise}, \end{cases}
\end{align*}
Proposition 3.2. We have the following.

\[ \alpha \]

\[ h \]

\[ \alpha \]

Now suppose, under the identification (2.0.5), part (1) follows from the classical case (cf. \( \Phi \)).

For \( \alpha \in \Phi \) and \( 1 \leq i \leq n \), we introduce the following elements:

\[ x^{(k)}_{\alpha} = \frac{x^k_{\alpha}}{k!} \quad \left( \begin{array}{c} h_i \\ -1 \\ \end{array} \right) = 1, \quad \text{and} \quad \left( \begin{array}{c} h_i \\ k \\ \end{array} \right) = \frac{h_i(h_i-1) \cdots (h_i-k+1)}{k!} \quad (k \geq 1). \]

It is known that \( \mathfrak{q}(n) \) can naturally be viewed as a subspace of the universal enveloping algebra \( U(\mathfrak{q}(n)) \) and moreover, for any homogeneous \( u, v \in \mathfrak{q}(n) \), we have \( uv - (-1)^{\bar{u}\cdot\bar{v}} vu = [u, v] \) in \( U(\mathfrak{q}(n)) \), where \([u, v]\) means the Lie bracket of \( u, v \) in \( \mathfrak{q}(n) \). Then by Lemma 3.1 we have the following.

**Proposition 3.2.** Maintain the notation defined in Lemma 3.1 and let \( m, s \in \mathbb{N} \). Then the following holds in \( U(\mathfrak{q}(n)) \):

1. \[ x^{(m)}_{i,j} \ x^{(s)}_{k,l} = \begin{cases} x^{(s-t)}_{k,t} x^{(m)}_{i,j} + \sum_{t=1}^{\min(m,s)} x^{(s-t)}_{k,t} \left( h_i - h_j - m - s + 2t \right) x^{(m-t)}_{i,j}, & \text{if } \alpha_{i,j} + \alpha_{k,l} = 0; \\ x^{(s)}_{k,t} x^{(m)}_{i,j}, & \text{otherwise.} \end{cases} \]

2. \[ x^{(m)}_{i,j} \ x^{(s)}_{k,l} = \begin{cases} x^{(s)}_{k,t} x^{(m)}_{i,j} + (h_i - h_j) x^{(m-1)}_{i,j} - \bar{\alpha}_{i,j} x^{(m-2)}_{i,j}, & \text{if } \alpha_{i,j} + \alpha_{k,l} = 0; \\ x^{(s)}_{k,t} x^{(m)}_{i,j} + \bar{\alpha}_{i,j} x^{(m-1)}_{i,j}, & \text{if } \beta = \alpha_{i,j} + \alpha_{k,l} \in \Phi; \\ \bar{x}_{k,t} x^{(m)}_{i,j}, & \text{otherwise.} \end{cases} \]

3. \[ x^{(m)}_{i,j} x^{(s)}_{k,l} = \begin{cases} -\bar{x}_{k,t} x^{(m)}_{i,j} + (h_i + h_j), & \text{if } \alpha_{i,j} + \alpha_{k,l} = 0; \\ -\bar{x}_{k,t} x^{(m)}_{i,j} + x^{(m)}_{i,j}, & \text{if } \beta = \alpha_{i,j} + \alpha_{k,l} \in \Phi; \\ \bar{x}_{k,t} x^{(m)}_{i,j}, & \text{otherwise.} \end{cases} \]

4. \[ x^{(m)}_{i,j} h^{(m)}_{k} = h^{(m)}_{k} x^{(m)}_{i,j} - (\epsilon_{k} \alpha_{i,j}) \bar{x}_{i,j} x^{(m-1)}_{i,j}, \quad \bar{x}_{i,j} h^{(m)}_{k} = -h^{(m)}_{k} \bar{x}_{i,j} - |(\epsilon_{k} \alpha_{i,j})| x^{(m)}_{i,j}. \]

5. \[ x^{(m)}_{i,j} h^{(m)}_{k} = (h^{(m)}_{k} - m(\epsilon_{k} \alpha_{i,j})) x^{(m)}_{i,j}, \quad \bar{x}_{i,j} h^{(m)}_{k} = (h^{(m)}_{k} - (\epsilon_{k} \alpha_{i,j})) \bar{x}_{i,j}. \]

**Proof.** Since the even root vectors \( x_{\alpha} \) can be identified with the usual root vectors in \( \mathfrak{gl}(n) \) under the identification (2.0.5), part (1) follows from the classical case (cf. [7, (5.11a-c)]). Now suppose \( \alpha_{i,j} + \alpha_{k,l} = 0 \). Then \( i = l, j = k \). By Lemma 3.1(2), the following holds:

\[ x^{m}_{i,j} x^{m}_{i,j} = x^{m}_{i,j} x^{m}_{i,j} + \sum_{d=0}^{m-1} x^{i,j} \left( h_i - h_j \right) x^{m-1-d}_{i,j}, \]

\[ = x^{m}_{i,j} x^{m}_{i,j} + \sum_{d=0}^{m-1} \left( (h_i - h_j) x^{d}_{i,j} - \sum_{t=0}^{d-1} x^{t}_{i,j} \cdot 2 \bar{x}_{i,j} x^{d-1-t}_{i,j} \right) x^{m-1-d}_{i,j}, \]

\[ = x^{m}_{i,j} x^{m}_{i,j} + \sum_{d=0}^{m-1} \left( (h_i - h_j) x^{d}_{i,j} - 2d x^{d-1}_{i,j} \right) x^{m-1-d}_{i,j}, \]

\[ = x^{m}_{i,j} x^{m}_{i,j} + m(h_i - h_j) x^{m-1}_{i,j} - m(m-1) \bar{x}_{i,j} x^{m-2}_{i,j}, \]
where the second and third equalities are due to the facts $x_{i,j}(h_i - h_j) = (h_i - h_j)x_{i,j} - 2\bar{x}_{i,j}$ and $x_{i,j}\bar{x}_{i,j} = \bar{x}_{i,j}x_{i,j}$ by Lemma \[3.1\]. Hence, the first case of part (2) holds. Similarly, the other cases can be verified.

Define the Kostant \(Z\)-form \(U_Z\) (cf. \[3\] Section 4]) to be the \(Z\)-subalgebra of \(U\) generated by

\[
\{ e_i^{(k)}, f_i^{(k)}, e_i, f_i \mid 1 \leq i \leq n - 1, k \in \mathbb{N} \} \cup \Big\{ \left( \frac{h_i}{k} \right), h_i \mid 1 \leq i \leq n, k \in \mathbb{N} \Big\}.
\]

Denote by \(U_Z^+\) (resp. \(U_Z^-\)) the \(Z\)-subalgebra of \(U\) generated by \(e_i^{(k)}, e_i\) (resp. \(f_i^{(k)}, f_i\)), where \(1 \leq i \leq n - 1\) and \(k \in \mathbb{N}\). Let \(U_Z^0\) be the \(Z\)-subalgebra of \(U\) generated by \(\left( \frac{h_i}{k} \right), h_i \mid 1 \leq i \leq n, k \in \mathbb{N} \).

For \(b \in \mathbb{N}^n\), set \(|b| = b_1 + \cdots + b_n\) and define

\[
\left( \frac{h}{b} \right) = \prod_{i=1}^n \left( \frac{h_i}{b_i} \right).
\]

For \(D = (D_1, \ldots, D_n) \in \mathbb{Z}_2^n\), set

\[
\pi_D = h_1^{D_1} \cdots h_n^{D_n}.
\]

For the subset \(\mathbb{Z}_2\) of \(\mathbb{N}\), let

\[
M_n(\mathbb{N}|\mathbb{Z}_2) := M_n(\mathbb{N}) \times M_n(\mathbb{Z}_2).
\]

For an \(n \times n\) matrix \(X\), let \(X = X^+ + X^0 + X^-\) be the decomposition of \(X\) into lower triangular, diagonal, and upper triangular parts of \(X\), and for each \(A = (A_0, A_1) \in M_n(\mathbb{N}|\mathbb{Z}_2)\), let \(A^\varepsilon = (A_0^\varepsilon, A_1^\varepsilon)\) for every \(\varepsilon \in \{ +, 0, - \}\) and define

\[
e_{A^+} = \prod_{1 \leq i < j \leq n} (x_{i,j}^{(a_j^{(0)} - a_i^{(1)} \bar{x}_{i,j}^1)}, \quad f_{A^-} = \prod_{1 \leq i < j \leq n} (x_{j,i}^{(a_i^{(0)} - a_j^{(1)} \bar{x}_{j,i}^1)},
\]

where \(A_0 = (a_j^{(0)}), A_1 = (a_j^{(1)})\), and the products are defined as follows (cf. \[10\]). For the \(j\)th row (reading to the right) \(a_{j+1}, \ldots, a_{jn}\) of \(A^+\), put

\[
\pi_j(A^+) = (x_{j,j+1}^{(a_j^{(0)} - a_{j+1}^{(1)} \bar{x}_{j,j+1}^1)} \cdots (x_{j,n}^{(a_j^{(0)} - a_{jn}^{(1)} \bar{x}_{j,n}^1})
\]

and let

\[
e_{A^+} = \pi_{n-1}(A^+) \cdots \pi_1(A^+).
\]

Similarly for the \(j\)th column (reading upwards) \(a_{nj}, \ldots, a_{j+1j}\) for \(A^-\), put

\[
\pi_j(A^-) = (x_{n,j}^{(a_j^{(0)} - a_{nj}^{(1)} \bar{x}_{n,j}}) \cdots (x_{j+1,j}^{(a_j^{(0)} - a_{j+1j}^{(1)} \bar{x}_{j+1,j}})
\]

and let

\[
f_{A^-} = \pi_{n-1}(A^-) \cdots \pi_1(A^-).
\]

Then we have the following.

**Proposition 3.3.**

1. As abelian groups, we have \(U_Z \cong U_Z^- \otimes U_Z^0 \otimes U_Z^+\).
2. The superalgebra \(U_Z\) is a free \(\mathbb{Z}\)-module with basis given by the set

\[
\{ m_A := f_{A^-} \left( \frac{h}{A_0} \right) \bar{h}_A e_{A^+} \mid A \in M_n(\mathbb{N}|\mathbb{Z}_2) \}.
\]
Proof. Applying the commutation formulas in Proposition 3.2 yields the inclusion $U_Z \subseteq U_Z^n U^n_Z$. Thus, the required isomorphism in part (1) follows from the restriction to $U_Z$ of the canonical isomorphism $U(q(n)) \to U^- \otimes U^0 \otimes U^+$ in Proposition 2.2(2).

Clearly by Propositions 2.2 and 3.2 (and noting (3.1.1)), the subalgebra $U^n_Z$ (resp. $U^-_Z$) is spanned by the set $\{e_A^+ \mid A \in M_n(N[Z_2]), A^- = A^0 = 0\}$ (resp. $\{f_A^- \mid A \in M_n(N[Z_2]), A^+ = A^0 = 0\}$). Furthermore, by (QS1) and (3.1.1) (together with a result for $U_Z(q(n)_u)$, we have that $U^0_Z$ is spanned by the set $\{(h^0_i)_{A^0_j} \mid A \in M_n(N[Z_2]), A^+ = A^- = 0\}$. Putting together we obtain that $U_Z$ is spanned by the set (3.3.1). Meanwhile by Proposition 2.2(1) it is easy to check that the set (3.3.1) is linearly independent. Hence the proposition is proved.

We define the degree of the generators $x_i^{(s)}, \bar{x}_{i,j}, h_i, (h^s_i)$ for $U_Z$ via

$$\deg(x_i^{(s)}) = s|j - i|, \quad \deg(\bar{x}_{i,j}) = |j - i|, \quad \deg(h_i) = 1, \quad \deg(h^s_i) = 0,$$

(3.3.2)

for $1 \leq i \neq j \leq n, s \in \mathbb{N}$. Then the degree of the element $m_A$ for $A = (A_0, A_1) \in M_n(N[Z_2])$ is given by

$$\deg(m_A) = \deg(A) := (a_{11}^1 + \cdots + a_{nn}^1) + \sum_{1 \leq i < j \leq n} (a_{ij} + a_{ji})|j - i|,$$

(3.3.3)

where $A_0 = (a_{ij}^0), A_1 = (a_{ij}^1)$ and $a_{ij} = a_{ij}^0 + a_{ij}^1$. By Proposition 3.3(2), each non-zero element $l$ in $U_Z$ can be written as a linear combination of $m_A$ with $A \in M_n(N[Z_2])$ and we define $\deg(l)$ to be the highest degree of the terms $m_A$ appearing in $l$. In particular, if $M \subseteq U_Z$ denotes the set of monomials $m$ in $x_i^{(s)}, \bar{x}_{i,j}, h_i, (h^s_i)$ ($1 \leq i \neq j \leq n, s \in \mathbb{N}$), then the degree of a non-zero monomial $m \in M$ is well-defined. Clearly by Proposition 3.2 $U_Z$ is a filtered algebra with respect to the degree defined in (3.3.2) and $\deg(m_A m_B) \leq \deg(m_A) + \deg(m_B) = \deg(A) + \deg(B)$ for $A, B \in M_n(N[Z_2])$. In particular, given a non-zero monomial $m \in M$ in which $x_i^{(s)}, \bar{x}_{i,j}, h_i$ appear $a_{1i}^1, b_{i,j}$ and $c_i$ times, respectively, for $1 \leq i \neq j \leq n, s \in \mathbb{N}$, then

$$\deg(m) \leq \sum_{s} \sum_{1 \leq i < j \leq n} s|j - i| a_{i,j}^s + \sum_{1 \leq i < j \leq n} |j - i| b_{i,j} + \sum_{i=1}^{n} c_i.$$  

(3.3.4)

Remark 3.4. Let

$$G = \{x_i^{(s)}, \bar{x}_{i,j}, h_i \mid s \in \mathbb{N}, 1 \leq i \neq j \leq n\}.$$

By (3.3.2) and (3.3.4), every commutator $ab - (-1)^{\hat{a} \cdot \hat{b}} ba$ in Proposition 3.2(1)-(4), where $a, b \in G$ belong to different triangular parts, has degree strictly less than $\deg(ab)$. The fact will be useful below.

4. Presenting the queer Schur superalgebra $Q(n, r)$

Denote by $\Lambda(n, r)$ the set of compositions of $r$ into $n$ parts, or equivalently we can view $\Lambda(n, r)$ as a subset of $P_{\geq 0}$ in the following way:

$$\Lambda(n, r) = \{\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_n \epsilon_n \in P_{\geq 0} \mid \lambda_1 + \cdots + \lambda_n = r\}.$$

For $\lambda \in \Lambda(n, r)$, denote by $\ell(\lambda)$ the number of nonzero parts in $\lambda$, that is, $\ell(\lambda) = \sharp\{i \mid \lambda_i \neq 0, 1 \leq i \leq n\}$. Recall that $\{v_1, \ldots, v_n\}$ and $\{v_{-1}, \ldots, v_{-n}\}$ are bases for $V_0$ and $V_1$. For a
Lemma 4.1. Suppose \( \underline{j} \in I(n|n)^r \) and \( \text{wt}(\underline{j}) = \mu \). We have, for \( 1 \leq i \leq n \),

1. \( \phi_r(h_i)(v_{\underline{j}}) = \mu_i v_{j_i} \);
2. \( \phi_r(h_i)(v_{\underline{j}}) = 0 \) if \( \mu_i = 0 \).

Proof. Firstly, by the definition of the action of \( q(n) \) on \( V \), we obtain

\[
\phi_1(h_i)(v_j) = (E_{i,i} + E_{-i,-i})(v_j) = \delta_{i,j} v_j, \quad \phi_1(h_i)(v_j) = (E_{i,-i} + E_{-i,i})(v_j) = \delta_{i,j} v_j.
\]

Then \( \text{wt}(\underline{j}) \in \Lambda(n, r) \) and the set \( \{ v_{\underline{j}} \mid \underline{j} \in I(n|n)^r \} \) is a basis for \( V^{\otimes r} \).

Let \( I \) be the ideal of the universal enveloping algebra \( U = U(q(n)) \) given by

\[
I = \langle h_1 + \cdots + h_n - r, \ h_1(h_1 - 1) \cdots (h_1 - r), \ldots, h_n(h_n - 1) \cdots (h_n - r) \rangle.
\]

Then by (4.1.1), we obtain for \( 1 \leq i \leq n \)

\[
h_i(h_i - 1) \cdots (h_i - r) = h_i^2(h_i - 1) \cdots (h_i - r) \in I.
\]

Define

\[
U(n, r) = U/I, \quad U_\mathbb{Z}(n, r) = U_\mathbb{Z}/I \cap U_\mathbb{Z} \quad U^0(n, r) = U^0/I \cap U^0, \quad U^0_\mathbb{Z}(n, r) = U^0_\mathbb{Z}/I \cap U^0_\mathbb{Z}.
\]

For notational simplicity, we will denote, by abuse of notation, the images of \( e_i, f_i, h_i, x_a, e_{A^+} \) etc. in \( U(n, r) \) by the same letters.

Proposition 4.2. The homomorphism \( \phi_r : U \to U(\mathbb{Q})(V^{\otimes r}) \) satisfies \( I \subseteq \ker \phi_r \). Hence there exists a natural surjective homomorphism

\[
\overline{\phi}_r : U(n, r) \twoheadrightarrow \mathbb{Q}(n, r).
\]
Observe that $0 \leq \mu_i \leq r$. If $\mu_i = 0$, then we have $\phi_r(h_i)(v_j) = 0$ by Lemma 4.1(2); otherwise, we have $1 \leq \mu_i \leq r$, which implies $(\mu_i - 1) \cdots (\mu_i - r) = 0$. Therefore, by (4.2.3), we deduce that $\phi_r(h_i(h_i-1)\cdots(h_i-r))(v_j) = 0$. This means

$$\phi_r(h_i(h_i-1)\cdots(h_i-r)) = 0 \quad (4.2.4)$$

for all $1 \leq i \leq n$. Therefore by (4.1.1), (4.2.2) and (4.2.3), the ideal $I$ is contained in $\ker \phi_r$ and hence the proposition is verified. \hfill \Box

For $\lambda \in \Lambda(n,r)$, write

$$1_\lambda = \left(\begin{array}{c} h \\ \lambda \end{array}\right) \in U^0_Z(n,r).$$

**Proposition 4.3.** The following hold in $U^0_Z(n,r)$:

1. The set $\{1_\lambda \mid \lambda \in \Lambda(n,r)\}$ is a set of pairwise orthogonal central idempotents in $U^0_Z(n,r)$ and $\sum_{\lambda \in \Lambda(n,r)} 1_\lambda = 1$.
2. $(h^b) = 0$ for any $b \in \mathbb{N}^n$ with $|b| > r$.
3. For $1 \leq i \leq n$, $\lambda \in \Lambda(n,r)$ and $b \in \mathbb{N}^n$, we have $h_i 1_\lambda = \lambda_i 1_\lambda$,

$$\left(\begin{array}{c} h \\ b \end{array}\right) 1_\lambda = \left(\begin{array}{c} \lambda \\ b \end{array}\right) 1_\lambda,$$

where $\left(\begin{array}{c} \lambda \\ b \end{array}\right) = \prod_{i=1}^n \left(\begin{array}{c} \lambda_i \\ b_i \end{array}\right)$.
4. Suppose $1 \leq i \leq n$ and $\lambda \in \Lambda(n,r)$. If $\lambda_i = 0$, then $h_i 1_\lambda = 0$.
5. The $\mathbb{Z}$-algebra $U^0_Z(n,r)$ is spanned by the set

$$\{1_\lambda h_D \mid \lambda \in \Lambda(n,r), D \in \mathbb{Z}_2^n, D_i \leq \lambda_i \text{ for } 1 \leq i \leq n\}.$$

**Proof.** By (4.1.1) and (4.1.2), we know the elements $h_1 + \cdots + h_n - r$ and $h_1(h_1-1)\cdots(h_1-r), \ldots, h_n(h_n-1)\cdots(h_n-r)$ are contained in the ideal $I$. Then under the identification (2.0.3), there is a natural algebra homomorphism from the algebra $T^0$ introduced in [7 (4.1)] for $\mathfrak{gl}(n)$ to $U^0(n,r)$, which sends the element $H_i$ in [7] to $h_i$ for $1 \leq i \leq n$. Therefore, parts (1)-(3) follow from [7 Proposition 4.2].

To prove part (4), we observe that the following holds for $\lambda \in \Lambda(n,r)$ and $1 \leq i \leq n$

$$(h_i - 1)(h_i - 2)\cdots(h_i - r) 1_\lambda = (\lambda_i - 1)(\lambda_i - 2)\cdots(\lambda_i - r) 1_\lambda \quad (4.3.1)$$

by part (3). Now suppose $\lambda_i = 0$. Then by (4.3.1) we obtain

$$(h_i - 1)(h_i - 2)\cdots(h_i - r) 1_\lambda = (-1)^r r! 1_\lambda.$$

This implies

$$(-1)^r r! h_i 1_\lambda = h_i(h_i - 1)(h_i - 2)\cdots(h_i - r) 1_\lambda = 0$$

since $h_i(h_i - 1)(h_i - 2)\cdots(h_i - r) \in I \cap U^0$. Therefore $h_i 1_\lambda = 0$.

Finally, the fact that the elements $(h^b)_D$ with $b \in \mathbb{N}^n, D \in \mathbb{Z}_2^n$ span $U^0_Z(n,r)$ implies that $U^0_Z(n,r)$ is spanned by the set $\{1_\lambda h_D \mid \lambda \in \Lambda(n,r), D \in \mathbb{Z}_2^n\}$. By part (4), we have $1_\lambda h_D = 0$ if there exists $1 \leq i \leq n$ such that $D_i > \lambda_i$. Therefore part (5) is proved. \hfill \Box
Proposition 4.4. Suppose \( 1 \leq i \leq n, \alpha \in \Phi, \lambda \in \Lambda(n, r) \). Then the following commutation formulas hold in \( U'_Z(n, r) \):

\[
\begin{align*}
\bar{x}_\alpha 1_\lambda &= \begin{cases} 1_{\lambda + \alpha} x_\alpha, & \text{if } \lambda + \alpha \in \Lambda(n, r), \\ 0, & \text{otherwise,} \end{cases} \\
1_\lambda x_\alpha &= \begin{cases} x_\alpha 1_{\lambda - \alpha}, & \text{if } \lambda - \alpha \in \Lambda(n, r), \\ 0, & \text{otherwise,} \end{cases} \\
h_i 1_\lambda &= 1_\lambda h_i.
\end{align*}
\]

Proof. The last equality follows from the relation (QS1). The proof of the remaining equalities is parallel to that of [7, Proposition 4.5] (see Lemma 3.1). Let us illustrate by checking in detail the second formula. Suppose \( \alpha = \alpha_{i, j} = \epsilon_i - \epsilon_j \) with \( i \neq j \). Then by Propositions 4.3(3) and 3.2(5) we obtain

\[
\bar{x}_\alpha 1_\lambda = h_i + 1_{\lambda + 1} h_i + 1_{\lambda} (h_1) (h_2) \ldots (h_n) \bar{x}_\alpha
\]

\[
= \frac{h_i}{\lambda_i + 1} \left( \begin{array}{c} \frac{h_i - 1}{\lambda_i} \\ \frac{h_i + 1}{\lambda_j} \end{array} \right) \prod_{l \neq i, j} \left( \begin{array}{c} h_i \\ \lambda_l \end{array} \right) \bar{x}_\alpha
\]

\[
= \left( \begin{array}{c} h_i \\ \lambda_i + 1 \\ \lambda_j \end{array} \right) \prod_{l \neq i, j} \left( \begin{array}{c} h_i \\ \lambda_l \end{array} \right) \bar{x}_\alpha.
\]

If \( \lambda + \alpha_{i, j} \notin \Lambda(n, r) \), then \( \lambda_j = 0 \) and (4.4.1) becomes

\[
\bar{x}_\alpha 1_\lambda = \left( \begin{array}{c} h_i \\ \lambda_i + 1 \end{array} \right) \prod_{l \neq i} \left( \begin{array}{c} h_i \\ \lambda_l \end{array} \right) \bar{x}_\alpha = \left( \begin{array}{c} h_i \\ \lambda + \epsilon_i \end{array} \right) \bar{x}_\alpha = 0,
\]

by Proposition 4.3(2). If \( \lambda + \alpha_{i, j} \in \Lambda(n, r) \), then \( \lambda_j \geq 1 \) and \( \left( \frac{h_i + 1}{\lambda_j} \right) = \left( \frac{h_i}{\lambda_j} \right) + \left( \frac{h_i}{\lambda_j - 1} \right) \).

Hence, (4.4.1) becomes

\[
\bar{x}_\alpha 1_\lambda = \left( \begin{array}{c} h_i \\ \lambda_i + 1 \end{array} \right) \prod_{l \neq i, j} \left( \begin{array}{c} h_i \\ \lambda_l \end{array} \right) \bar{x}_\alpha + \left( \begin{array}{c} h_i \\ \lambda_i + 1 \end{array} \right) \prod_{l \neq i, j} \left( \begin{array}{c} h_i \\ \lambda_l \end{array} \right) \bar{x}_\alpha
\]

\[
= \left( \begin{array}{c} h_i \\ \lambda + \epsilon_i \end{array} \right) \bar{x}_\alpha + \left( \begin{array}{c} h_i \\ \lambda + \alpha \end{array} \right) \bar{x}_\alpha = 1_{\lambda + \alpha} \bar{x}_\alpha,
\]

as desired. \( \square \)

For \( A = (A_0, A_1) \in M_n(\mathbb{N}|\mathbb{Z}_2) \) with \( A_0 = (a_{i, j}^0), A_1 = (a_{i, j}^1) \) and \( a_{i, j} = a_{i, j}^0 + a_{i, j}^1 \), define

\[
\begin{align*}
\text{ro}(A) &:= \sum_{j}^{n} a_{1j} + \sum_{j}^{n} a_{nj}, \\
\text{co}(A) &:= \sum_{j}^{n} a_{j1} + \sum_{j}^{n} a_{jn}, \\
\chi(A) &:= a_{11} + \sum_{j=2}^{n} (a_{1j} + a_{j1}), a_{22} + \sum_{j=3}^{n} (a_{2j} + a_{j2}), \ldots, a_{nn} = \text{co}(A_0^0) + \text{co}(A_1^0) + \text{ro}(A^+) + \text{co}(A^-).
\end{align*}
\]
Let 

$$M_n(\mathbb{N}|\mathbb{Z}_2)' = \{ C \in M_n(\mathbb{N}|\mathbb{Z}_2) \mid C_0 = 0 \}.$$ 

Similar to (3.3.1), we define 

$$u_C = f_C - h C_0 e_0 \in U_Z$$

and 

$$u_{(C,\lambda)} = f_C - \lambda h C_0 e_0 \in U_Z(n, r)$$

for \( C \in M_n(\mathbb{N}|\mathbb{Z}_2)' \) and \( \lambda \in \Lambda(n, r) \). Clearly, by definition, the degree function defined in (3.3.3) satisfies \( \text{deg}(u_C) = \text{deg}(C) \).

By Proposition 4.3, we have in \( U_Z(n, r) \)

$$u_{(C,\lambda)} = 1 \lambda' u_C = u_C 1 \lambda'' \text{ if } u_{(C,\lambda)} \neq 0,$$

where \( \lambda' = \lambda + \text{ro}(C^-) - \text{co}(C^-) \) and \( \lambda'' = \lambda - \text{ro}(C^+) + \text{co}(C^+) \). Recall that \( \mathcal{M} \) consists of the monomials in \( x_{i,j}, x_{i,j}^h, h_{i,j}, (h_{s}) (1 \leq i \neq j \leq n, s \in \mathbb{N}) \).

**Lemma 4.5.** (1) Let \( 0 \neq m \in \mathcal{M} \). Then, in \( U_Z(n, r) \), \( m \) can be written as a linear combination of \( u_{(C,\lambda)} \) for \( C \in M_n(\mathbb{N}|\mathbb{Z}_2)' \), \( \lambda \in \Lambda(n, r) \) such that \( \text{deg}(C) \leq \text{deg}(m) \).

(2) Suppose \( C \in M_n(\mathbb{N}|\mathbb{Z}_2)' \). Then we have in \( U_Z(n, r) \)

$$f_C - h C_0 e_0 = \pm e_C + h C_0 f_C + \sum \zeta_{(G,\lambda)} u_{(G,\lambda)},$$

for some \( \zeta_{(G,\lambda)} \in \mathbb{Z} \), where the summation is over \( G \in M_n(\mathbb{N}|\mathbb{Z}_2)' \) and \( \lambda \in \Lambda(n, r) \) such that \( \text{deg}(G) < \text{deg}(C) \).

**Proof.** By Proposition 3.3 we see that in \( U_Z \) the monomial \( m \) can be written as

$$m = \sum_{A \in M_n(\mathbb{N}|\mathbb{Z}_2)} \xi_A^m m_A = \sum_{A \in M_n(\mathbb{N}|\mathbb{Z}_2)} \xi_A^m f_A - \left( h_{A_0} \right) h_{A_0} e_{A+}$$

(4.5.1)

where \( \xi_A^m \in \mathbb{Z} \) and \( \xi_A^m = 0 \) unless \( \text{deg}(m_A) \leq \text{deg}(m) \). Since \( \left( h_{A_0} \right) = \sum_{\lambda \in \Lambda(n, r)} \left( \lambda \right)_{A_0} 1 \lambda \) in \( U_Z(n, r) \) by Proposition 4.3, we have

$$m_A = \sum_{\lambda \in \Lambda(n, r)} \left( \lambda \right)_{A_0} 1 \lambda h_{A_0} e_{A+} = \sum_{\lambda \in \Lambda(n, r)} \left( \lambda \right)_{A_0} u_{(A',\lambda)},$$

where \( A' \in M_n(\mathbb{N}|\mathbb{Z}_2)' \) with \( (A')^+ = A^+, (A')^- = A^- \), \( (A')_0 = A_0 \) for each \( A \in M_n(\mathbb{N}|\mathbb{Z}_2) \).

Clearly \( \text{deg}(A') = \text{deg}(A) = \text{deg}(m_A) \). This together with (4.5.1) proves (1).

By applying a sequence of commutation formulas for generators from different triangular parts, we have in \( U_Z \)

$$f_C - h C_0 e_0 = \pm e_C + h C_0 f_C + g,$$

where \( g \) is a linear combination of monomials \( m \) in the generators for \( U_Z \) and, by Remark 3.3, \( \text{deg}(m) < \text{deg}(e_C + h C_0 f_C -) \). Now part (2) follows from part (1). \( \square \)

Given two elements \( \lambda = \sum_i \lambda_i \epsilon_i, \mu = \sum_i \mu_i \epsilon_i \in \mathcal{P}_+ \), we define

$$\lambda \preceq \mu \iff \lambda_i \leq \mu_i, \text{ for } 1 \leq i \leq n.$$ 

(4.5.2)

Set

$$\mathcal{B} = \{ u_{(C,\lambda)} \mid C \in M_n(\mathbb{N}|\mathbb{Z}_2)', \lambda \in \Lambda(n, r), \chi(C) \preceq \lambda \}.$$ 

Let \( M_n(\mathbb{N}|\mathbb{Z}_2)_r := \{ A = (A_0, A_1) = ((a_{i,j}^0), (a_{i,j}^1)) \in M_n(\mathbb{N}|\mathbb{Z}_2) \mid \sum_i (a_{i,j}^0 + a_{i,j}^1) = r \} \). Then, one can also check the following holds

$$\mathcal{B} = \{ u_A := f_A - 1 \chi(A) h A_0 e_{A+} \mid A \in M_n(\mathbb{N}|\mathbb{Z}_2)_r \}.$$
Proposition 4.6. The algebra $U_Z(n, r)$ is spanned by the set $\mathcal{B}$.

Proof. Let $U'_Z(n, r)$ be the $\mathbb{Z}$-submodule of $U_Z(n, r)$ spanned by $\mathcal{B}$. Clearly by Proposition 3.3 and (4.4.3), the algebra $U_Z(n, r)$ is spanned by monomials $m \in \mathfrak{M}$. Then by Lemma 4.5 and Proposition 4.4, it suffices to show that $u_{(C, \lambda)} \in U'_Z(n, r)$ for all $C \in M_n(\mathbb{N}[\mathbb{Z}_2])$ and $\lambda \in \Lambda(n, r)$. Now fix a $C \in M_n(\mathbb{N}[\mathbb{Z}_2])$ and $\lambda \in \Lambda(n, r)$. Write $C_0 = (c^0_{ij})$, $C_1 = (c^1_{ij})$ and let $c_{ij} = c^0_{ij} + c^1_{ij}$. If $\chi(C) \leq \lambda$, then $u_{(C, \lambda)} \in \mathcal{B}$ and we are done. Now assume $\chi(C) = (\chi_1(C), \ldots, \chi_n(C)) \notin \lambda$. Then by (4.5.2), there exists $1 \leq i \leq n$ such that $\lambda_i < \chi_i(C)$. We proceed on $\deg(C)$. If $\deg(C) = 0$, then the result holds as $u_{(C, \lambda)} = 1 \in \mathcal{B}$. Now assume that $\deg(C) \geq 1$ and let $i$ be the biggest $i$ such that $\lambda_i < \chi_i(C)$. Let $G$ be the submatrix $((c^0_{k,l})(c^1_{k,l}))_{1 \leq k, l \leq n}$ of $C$ at the bottom right corner. Then by (3.2.3) and (3.2.4) we can write $e_{G^+} = e_{G^+} \cdot \mathbf{m}_1$, $e_{G^-} = e_{G^-} \cdot \mathbf{m}_1 \cdot e_{G^-}$ and $\overline{h}_{C^0_1} = \overline{h} = \prod_{j=1}^{n-1} c_{i,j}$, where $\overline{h} = \sum_{i} \mathbf{m}_1 \overline{h} \cdot e_{(G^+)} \cdot e_{(G^-)} \cdot \mathbf{m}_1$.

We can assume $f_{G^-} - 1 \neq 0$. Otherwise, we are done. Then by (4.4.3) we have $f_{G^-} - 1 = 1 \cdot f_{G^-}$ and hence

$$u_{(C, \lambda)} = \pm \mathbf{m}_1 \overline{h} \cdot 1 \cdot (f_{G^-} - 1) \cdot e_{C^0_1} \cdot e_{G^+} \cdot e_{G^-} \cdot \mathbf{m}_1,$$

where $\lambda' = \lambda + \text{ro}(G^-) - \text{co}(G^-)$. Then applying Lemma 4.5(2) to $f_{G^-} \overline{h} e_{G^+}$, we have

$$u_{(C, \lambda)} = \pm \mathbf{m}_1 \overline{h} \cdot 1 \cdot (\pm e_{G^+} \overline{h} e_{G^+} - \sum \zeta_{(J, \gamma)} u_{(J, \gamma)} ) \cdot \mathbf{m}_1$$

$$= \pm \mathbf{m}_1 \overline{h} \cdot 1 \cdot e_{G^+} \overline{h} e_{G^+} - \sum \zeta_{(J, \gamma)} \mathbf{m}_1 \overline{h} u_{(J, \gamma)} \cdot \mathbf{m}_1,$$

where $\zeta_{(J, \gamma)} = 0$ unless $\deg(J) < \deg(G)$. Observe that $\deg(G) = \deg(G^+) = \deg(G^-) = \deg(G)$. Note that by (4.4.3), we can apply Lemma 4.5 to each term $\mathbf{m}_1 \overline{h} u_{(J, \gamma)} \cdot \mathbf{m}_1$ to obtain the following:

$$u_{(C, \lambda)} = \pm \mathbf{m}_1 \overline{h} \cdot 1 \cdot e_{G^+} \overline{h} e_{G^-} \cdot \mathbf{m}_1 + 1,$$

where 1 is a linear combination of $u_{(C', \mu)}$ with $\deg(C') < \deg(C)$. By induction, $1 \in U'_Z(n, r)$. Meanwhile, we claim that $\mathbf{m}_1 \overline{h} e_{G^+} \overline{h} e_{G^-} \cdot \mathbf{m}_1 = 0$. Indeed, we can assume $1 \cdot e_{G^+} \neq \overline{1}$. Otherwise, we are done. Then $1 \cdot e_{G^+} = e_{G^+} + 1 \cdot \mathbf{m}_1$ by (4.4.3), where $\lambda'' = \lambda' - \text{ro}(G^+) + \text{co}(G^+) = \lambda + \text{ro}(G^-) - \text{co}(G^-) - \text{ro}(G^+) + \text{co}(G^+)$ for some $\lambda' \in \Lambda(n, r)$. Since $G$ is the submatrix of $C$ consisting of the last $n - i + 1$ rows and columns, we have

$$\lambda'' = \lambda_i - (\sum_{j=1}^{n} c_{ji}) - (\sum_{j=1}^{n} c_{ij}) = \lambda_i - \chi_i(C) - c_{i1} = \lambda_i - \chi_i(C) + c_{i1}$$

by (4.4.2). This means $\lambda'' < \chi_i(C)$ since $\chi_i(C) > \lambda_i$, forcing $c_{i1} = 1$ and $\lambda'' = 0$ since $\lambda'' \in \Lambda(n, r)$. Hence, $1 \cdot e_{G^+} \overline{h} e_{G^-} = 0$ by Proposition 4.3(4), proving the claim. In conclusion, $u_{(C, \lambda)} \in U'_Z(n, r)$.

We remark that the proof above follows [10]. However, it is possible to modify the proof of [7] to give an alternative proof.  

---

2In the non super case, we have $1 \cdot e_{G^+} = 0$ as $\lambda'' < \chi_i(G^+)$. However, this inequality may not be true in the super case as the $i$-th entry of $C^0_1$ may not be zero.
**Theorem 4.7.** The homomorphism \( \overline{\phi}_r : U(n,r) \to Q(n,r) \) in (4.2.1) is an isomorphism. In other words, the Schur superalgebra \( Q(n,r) \) is the associative superalgebra generated by even generators \( h_i, e_j, f_j \), and odd generators \( h_i, e_j, f_j \), with \( 1 \leq i \leq n \) and \( 1 \leq j \leq n-1 \) subject to the relations (QS1)-(QS6) together with the following extra relations:

\[
\begin{align*}
&\text{(QS7)} \ h_1 + h_2 + \cdots + h_n = r; \\
&\text{(QS8)} \ h_i(h_i - 1) \cdots (h_i - r) = 0 \text{ for } 1 \leq i \leq n.
\end{align*}
\]

**Proof.** By [2, §4], we know that the dimension of the algebra \( Q(n,r) \) is equal to the number of monomials of total degree \( r \) in the free supercommutative algebra in \( n^2 \) even variables and \( n^2 \) odd variables. Thus,

\[
\dim Q(n,r) = |M_n(N|Z_2)_r|.
\]  

(4.7.1)

Hence, by (4.7.1), Proposition 4.2 and Proposition 4.6 we have

\[
\dim U(n,r) \leq |B| = \dim Q(n,r) \leq \dim U(n,r),
\]

which implies \( \dim U(n,r) = |B| = \dim Q(n,r) \). This forces that the surjective homomorphism \( \overline{\phi}_r \) is an isomorphism. \( \square \)

**Corollary 4.8.** The set

\[
B = \{ u_A \mid A \in M_n(N|Z_2)_r \}
\]

is a \( \mathbb{Z} \)-basis for \( U_Z(n,r) \). In particular, the set \( \{ 1_\lambda h_d \mid \lambda \in \Lambda(n,r), D \in \mathbb{Z}_+^n, D_i \leq \lambda_i, 1 \leq i \leq n \} \) is \( \mathbb{Z} \)-basis for \( U_Z^0(n,r) \).

**Corollary 4.9.** We have the following dimension formulas:

\[
\begin{align*}
(1) \ &\dim Q(n,r) = \sum_{k=0}^r \binom{n^2 + k - 1}{k} \binom{n^2}{r-k}; \\
(2) \ &\dim Q^0(n,r) = \sum_{\lambda \in \Lambda(n,r)} 2^{\ell(\lambda)}. \\
\end{align*}
\]

Using a similar argument in the proof of [7, Theorem 2.4], we obtain the following.

**Theorem 4.10.** The queer Schur superalgebra \( Q(n,r) \) is the unitary associative superalgebra generated by the even elements \( 1_\lambda, e_j, f_j \) and odd elements \( h_i, e_j, f_j \) for \( \lambda \in \Lambda(n,r), 1 \leq i \leq n, 1 \leq j \leq n-1 \) subject to (QS3) and (QS5)-(QS6) as well as the following relations:

\[
\begin{align*}
\text{(QS1')} \ &1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda; \\
&\sum_{\lambda \in \Lambda(n,r)} 1_\lambda = 1, h_i 1_\lambda = 1; \\
&h_i h_j h_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,r)} 2\lambda_i 1_\lambda, \\
&h_i 1_\lambda = 0 \text{ if } \lambda_i = 0; \\
\text{(QS2')} \ &e_j 1_\lambda = \begin{cases} \\
1_{\lambda+\alpha_j} e_j, & \text{if } \lambda + \alpha_j \in \Lambda(n,r), \\
0, & \text{otherwise},
\end{cases} \\
&f_j 1_\lambda = \begin{cases} \\
1_{\lambda-\alpha_j} f_j, & \text{if } \lambda - \alpha_j \in \Lambda(n,r), \\
0, & \text{otherwise},
\end{cases} \\
&1_\lambda e_j = \begin{cases} \\
e_j 1_{\lambda-\alpha_j}, & \text{if } \lambda - \alpha_j \in \Lambda(n,r), \\
0, & \text{otherwise},
\end{cases} \\
&1_\lambda f_j = \begin{cases} \\
f_j 1_{\lambda+\alpha_j}, & \text{if } \lambda + \alpha_j \in \Lambda(n,r), \\
0, & \text{otherwise};
\end{cases}
\]

\[
\text{(QS4')} \ e_if_j - f_j e_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,r)} (\lambda_i - \lambda_{i+1}) 1_\lambda; \\
e_i f_j + f_j e_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,r)} (\lambda_i + \lambda_{i+1}) 1_\lambda; \\
e_i f_j - f_j e_i = \delta_{ij} (h_i - h_{i+1});
\]

\[
e_i f_j - f_j e_i = \delta_{ij} (h_i - h_{i+1}').
\]
5. The quantum queer superalgebra $U_q(q(n))$ and its root vectors

The quantum queer superalgebra is more subtle than the enveloping algebra. In the next four sections, we will investigate the quantum root vectors and their commutation formulas. We then generalize Theorems 4.7 and 4.10 to quantum queer Schur algebras in the last section.

In [18], Olshanski introduced the quantum deformation $U_q = U_q(q(n))$ of the universal enveloping algebra $U(q(n))$ of $q(n)$ as follows. Let the symbol $\{\cdots\}$, where the dots standard for some inequalities, equal 1 if all these inequalities are satisfied and 0 otherwise. For $i, j, k \in I(n|n)$, put $\varphi(i, j) = \delta_{|i|,|j|}\text{sgn}(j)$,

$$p(i,j) = \begin{cases} 0, & \text{if } ij > 0, \\ 1, & \text{if } ij < 0, \end{cases} \quad \text{and} \quad \theta(i,j,k) = \text{sgn}(\text{sgn}(i) + \text{sgn}(j) + \text{sgn}(k)),$$

where $\text{sgn}(a) = 1$ if $a > 0$ and $\text{sgn}(a) = -1$ if $a < 0$ for an arbitrary nonzero integer $a$.

**Definition 5.1.** The quantum queer superalgebra $U_q(q(n))$ is the associative superalgebra over $\mathbb{Q}(q)$ generated by $L_{i,j}$ for $i,j \in I(n|n)$ with $i \leq j$ subject to the following relations

\begin{align*}
L_{i,i}L_{i,-i} = L_{-i,-i}L_{i,i} = 1, \\
(-1)^{p(i,j)p(k,l)} q^{\varphi(i,l)} L_{i,j} L_{k,l} + \{k \leq j < l\} \theta(i, j, k)(q - q^{-1}) L_{i,l} L_{k,j} \\
+ \{i \leq -l < j \leq -k\} \theta(-i, -j, k)(q - q^{-1}) L_{i,-l} L_{k,-j} \\
= q^{\varphi(i,k)} L_{k,l} L_{i,j} + \{k < l \leq j\} \theta(i, j, k)(q - q^{-1}) L_{i,l} L_{k,j} \\
+ \{-l \leq i < -k \leq j\} \theta(-i, -j, k)(q - q^{-1}) L_{i,-l} L_{k,-j}.
\end{align*}

(5.1.1)

The $\mathbb{Z}_2$-grading on $U_q(q(n))$ is defined via $\hat{L}_{i,j} = p(i,j)$ for $i, j \in I(n|n)$ with $i \leq j$.

Following [18] Remark 7.3, we introduce the following set of generators of $U_q(q(n))$:

\begin{align*}
K_i := L_{i,i}, & \quad K_i^{-1} := L_{-i,-i}, \quad K_i := -\frac{1}{q - q^{-1}} L_{-i,i}, \\
E_j := -\frac{1}{q - q^{-1}} K_{j+1} L_{j,-j-1,j}, & \quad E_j := -\frac{1}{q - q^{-1}} K_{j+1} L_{j-1,j}, \\
F_j := \frac{1}{q - q^{-1}} L_{j,j+1} K_{j+1}^{-1}, & \quad F_j := -\frac{1}{q - q^{-1}} L_{j,j+1} K_{j+1}^{-1}.
\end{align*}

(5.1.2)

for $1 \leq i \leq n$ and $1 \leq j \leq n-1$. Then the algebra $U_q(q(n))$ can be defined via a quantum analogue of the relations (QS1)-(QS6) using (5.1.1) as follows.

**Proposition 5.2** ([13] Theorem 2.1 cf. [14]). The quantum superalgebra $U_q(q(n))$ is isomorphic to the unital associative superalgebra over $\mathbb{Q}(q)$ generated by even generators $K_i^{\pm 1}, E_j, F_j$ and odd generators $K_i, E_j, F_j$, for $1 \leq i \leq n, 1 \leq j \leq n-1$, satisfying the following relations:

\begin{align*}
\text{(QQ1)} & \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i, \quad K_i K_j = K_j K_i, \\
& \quad K_i K_j + K_j K_i = 2\delta_{ij} \frac{K_i^2 - K_i^{-2}}{q^2 - q^{-2}}; \\
\text{(QQ2)} & \quad K_i E_j = q^{(\epsilon_i, \alpha_j)} E_j K_i, \quad K_i E_j = q^{(\epsilon_i, \alpha_j)} E_j K_i, \\
& \quad K_i F_j = q^{-(\epsilon_i, \alpha_j)} F_j K_i, \quad K_i F_j = q^{-(\epsilon_i, \alpha_j)} F_j K_i;
\end{align*}

for $1 \leq i \leq n, 1 \leq j \leq n-1$. Then the algebra $U_q(q(n))$ can be defined via a quantum analogue of the relations (QS1)-(QS6) using (5.1.1) as follows.
\document{\begin{align}
\text{(Q3)} &\quad K_i E_i - q E_i K_i = E_i K_i^{-1}, \quad q K_i E_{i-1} - E_{i-1} K_i = -K_i^{-1} E_{i-1}, \\
&\quad K_i F_i - q F_i K_i = -F_i K_i, \quad q K_i F_{i-1} - F_{i-1} K_i = K_i F_{i-1}, \\
&\quad K_i E_i + q E_i K_i = E_i K_i^{-1}, \quad q K_i E_{i+1} + E_{i+1} K_i = K_i^{-1} E_{i+1}, \\
&\quad K_i F_i + q F_i K_i = F_i K_i, \quad q K_i F_{i+1} + F_{i+1} K_i = K_i F_{i+1}, \\
&\quad K_i E_j - E_j K_i = K_i F_j - F_j K_i = K_i E_j + E_j K_i = K_i F_j + F_j K_i = 0 \text{ for } j \neq i, i-1; \\
\text{(Q4)} &\quad E_i F_j - F_j E_i = \delta_{ij} \left( \frac{K_i K_{i+1} - K_{i+1} K_i}{q - q^{-1}} \right), \\
&\quad E_i F_j + F_j E_i = \delta_{ij} \left( \frac{K_i K_{i+1} - K_{i+1} K_i}{q - q^{-1}} + (q - q^{-1}) K_i K_{i+1} \right), \\
&\quad E_i E_j - E_j E_i = F_i F_j - F_j F_i = 0 \text{ for } |i - j| \neq 1, \\
&\quad E_i E_j - E_j E_i = F_i F_j - F_j F_i = F_i E_j + E_j F_i = F_i E_j + E_j F_i = 0 \text{ for } |i - j| > 1, \\
&\quad q F_{i+1} F_i - F_i F_{i+1} = q F_{i+1} F_i + F_i F_{i+1} = q F_{i+1} F_i - F_i F_{i+1} = q F_{i+1} F_i - F_i F_{i+1}; \\
\text{(Q5)} &\quad E_i^2 = \frac{q - q^{-1}}{q + q^{-1}} E_i, \quad F_i^2 = \frac{q - q^{-1}}{q + q^{-1}} F_i^2, \\
&\quad E_i E_j - E_j E_i = F_i F_j - F_j F_i = 0 \text{ for } |i - j| \neq 1, \\
&\quad E_i E_j - E_j E_i = F_i F_j - F_j F_i = F_i E_j + E_j F_i = F_i E_j + E_j F_i = 0 \text{ for } |i - j| > 1, \\
&\quad q F_{i+1} F_i - F_i F_{i+1} = q F_{i+1} F_i + F_i F_{i+1} = q F_{i+1} F_i - F_i F_{i+1} = q F_{i+1} F_i - F_i F_{i+1}; \\
\text{(Q6)} &\quad E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \quad F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0, \\
&\quad E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \quad F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0, \\
&\quad \text{where } |i - j| = 1.
\end{align}

\begin{remark}
The generators in (5.1.2) are different from those in [13, Theorem 2.1]. Actually, setting
\begin{equation}
E'_j = K_{j+1}^{-1} E_j, \quad E'_j = K_{j+1}^{-1} E_j, \quad F'_j = F_j K_{j+1}, \quad F'_j = F_j K_{j+1}
\end{equation}
for $1 \leq j \leq n-1$, then by [14, Remark 1.2] the elements $K_{i}^{\pm 1}, K_{i}, E'_j, F'_j, E'_j, F'_j$ can be identified with $q^{k_i}, k_i, c_j, f_j, e_j, f_j$ in [13, Theorem 2.1], respectively. The relations in Proposition (5.2) are obtained by rewriting the whole defining relations among $q^{k_i}, k_i, c_j, f_j, e_j, f_j$ given by [13, Theorem 2.1] in terms of the elements $K_{i}^{\pm 1}, K_{i}, E'_j, F'_j, E'_j, F'_j$ by using (5.3.1).

For convenience, let us write down some of the relations in terms of the generators $E'_j, F'_j, E'_j, F'_j$ as follows, which will be useful later on.

\begin{lemma}[13, Theorem 2.1] The following holds in $U_q(q(n))$: for $1 \leq k \leq n-2, 1 \leq a \leq n$ and $1 \leq i, j \leq n-1$ with $|i - j| > 1$,
\begin{align*}
E'_{k+1} E_k - E_k E'_{k+1} &= E'_{k+1} E_k - E_k E'_{k+1}, \quad F'_{k+1} F_k - F_k F'_{k+1} = F'_{k+1} F_k - F_k F'_{k+1}, \\
K_a E'_i &= q^{(\epsilon_a, \alpha_i)} E'_i K_a, \quad K_a F'_i = q^{-(\epsilon_a, \alpha_i)} F'_i K_a, \quad K_a E'_i = q^{(\epsilon_a, \alpha_i)} E'_i K_a, \quad K_a F'_i = q^{-(\epsilon_a, \alpha_i)} F'_i K_a, \\
E'_j E'_i - E'_i E'_j &= 0.
\end{align*}
\end{lemma}

By [13, §4] (cf. [13, (2.9)]), the comultiplication $\Delta: U_q(q(n)) \to U_q(q(n)) \otimes U_q(q(n))$ is defined by
\begin{equation}
\Delta(L_{i,j}) = \sum_{k=i}^{j} L_{i,k} \otimes L_{k,j},
\end{equation}
for $i, j = 0, 1, \ldots, n$.}
for \( i, j \in I(n|n) \) with \( i \leq j \). Then by (5.1.2) we have
\[
\Delta(K_i) = K_i \otimes K_i, \\
\Delta(E_j) = 1 \otimes E_j + E_j \otimes K_j^{-1}K_{j+1}, \\
\Delta(F_j) = K_jK_{j+1}^{-1} \otimes F_j + F_j \otimes 1.
\]
(5.4.2)

for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n - 1 \).

By Proposition 5.2, it is routine to check that there is an anti-involution \( \Omega : U_q(\mathfrak{q}(n)) \rightarrow U_q(\mathfrak{q}(n)) \) given by
\[
\Omega(q) = q^{-1}, \quad \Omega(K_i) = K_i^{-1}, \quad \Omega(K_i) = K_i, \\
\Omega(E_j) = F_j, \quad \Omega(F_j) = E_j, \quad \Omega(E_j) = F_j, \quad \Omega(F_j) = E_j.
\]
(5.4.3)

for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n - 1 \).

As for Lie algebras, we have the following PBW Theorem for \( U_q(\mathfrak{q}(n)) \) due to Olshanski.

**Proposition 5.5** ([18] Theorem 6.2). Fix an order on \( L_{i,j} \) for \( i, j \in I(n|n) \) with \( i < j \). Then the set
\[
\left\{ L_{m_1}^{i_1} \cdots L_{m_n}^{i_n} \bigg| m_1, \ldots, m_n \in \mathbb{Z}, m_{ij} \in \mathbb{N}, m_{ij} \in \mathbb{Z}_2 \quad \text{if} \quad \hat{L}_{i,j} = 1 \right\},
\]
(5.5.1)
is a \( \mathbb{Q}(q) \)-basis of the algebra \( U_q(\mathfrak{q}(n)) \).

**Remark 5.6.** In [18], a particular order on the elements \( L_{i,j} \) for \( i, j \in I(n|n) \) with \( i < j \) was chosen to prove the PBW Theorem. Actually, it is easy to check that the arguments in the proof of [18] Theorem 6.2 do not depend on the choice of the order.

For \( \alpha_{i,j} = \epsilon_i - \epsilon_j \in \Phi \) with \( 1 \leq i \neq j \leq n \), we introduce inductively the root vectors as follows. For \( 1 \leq i \leq n - 1 \), we set
\[
X_{i,i+1} = E_i, \quad X_{i+1,i} = F_i, \quad X_{i,i+1} = E_i, \quad X_{i+1,i} = F_i.
\]
For \( |j - i| > 1 \), we define
\[
X_{\alpha_{i,j}} \equiv X_{i,j} := \begin{cases} 
X_{k,k}X_{k,j} - qX_{k,j}X_{k,k} & \text{if} \ i < j, \\
X_{k,k}X_{k,j} - q^{-1}X_{k,j}X_{k,k} & \text{if} \ i > j,
\end{cases}
\]
\[
\Xi_{\alpha_{i,j}} \equiv \Xi_{i,j} := \begin{cases} 
X_{k,k}\Xi_{k,j} - q\Xi_{k,j}X_{k,k} & \text{if} \ i < j, \\
X_{k,k}\Xi_{k,j} - q^{-1}X_{k,j}\Xi_{k,k} & \text{if} \ i > j,
\end{cases}
\]
(5.6.1)

where \( k \) is strictly between \( i \) and \( j \). It is straightforward to check that \( X_{i,j}, \Xi_{i,j} \) are independent of the choice of \( k \). Clearly \( X_{i,j} \) are even elements while \( \Xi_{i,j} \) are odd elements. By (5.1.2) and (QQ2) in Proposition 5.2 one can check that for \( \alpha \in \Phi, 1 \leq i \leq n \)
\[
K_iX_{\alpha}K_i^{-1} = q^{(\alpha, \alpha)}X_{\alpha}, \quad K_i\Xi_{\alpha}K_i^{-1} = q^{(\alpha, \alpha)}\Xi_{\alpha}.
\]
(5.6.2)

Therefore the elements \( X_{\alpha} \) and \( \Xi_{\alpha} \) can be viewed as the quantum analog of the root vectors \( x_{\alpha} \) and \( \bar{x}_{\alpha} \) introduced in (3.0.5). Meanwhile, by (5.4.3), we obtain
\[
\Omega(X_{i,j}) = X_{j,i}, \quad \Omega(\Xi_{i,j}) = \Xi_{j,i}.
\]
(5.6.3)

for \( 1 \leq i \neq j \leq n \).

By (5.1.2) and (5.1.1), a direct calculation shows that
\[
K_aL_{i,j} = \begin{cases} 
q^{-1}L_{i,j}K_a, & \text{if} \ |i| = a, \\
qL_{i,j}K_a, & \text{if} \ |j| = a, \\
K_aL_{i,j}, & \text{otherwise},
\end{cases}
\]
\[
K_a^{-1}L_{i,j} = \begin{cases} 
qL_{i,j}K_a^{-1}, & \text{if} \ |i| = a, \\
q^{-1}L_{i,j}K_a^{-1}, & \text{if} \ |j| = a, \\
L_{i,j}K_a^{-1}, & \text{otherwise}.
\end{cases}
\]
(5.6.4)

for \( 1 \leq a \leq n \) and \( i, j \in I(n|n) \) with \( i \leq j \) and \( |i| \neq |j| \).
Lemma 5.7. The following holds for $1 \leq i < j \leq n$:

\[
X_{i,j} = \frac{-1}{q - q^{-1}} K_j L_{j-i}, \quad \overline{X}_{i,j} = \frac{-1}{q - q^{-1}} K_j L_{j-i},
\]

\[
X_{j,i} = \frac{1}{q - q^{-1}} L_{i,j} K_j^{-1}, \quad \overline{X}_{j,i} = \frac{-1}{q - q^{-1}} L_{i,j} K_j^{-1}.
\]

Proof. Suppose $1 \leq i < j \leq n$. Recall the generators (5.3.1) used in [13, Theorem 2.1]. Then by (2.4) and Remark 5.3, we have

\[
L_{j-i} = (-1)^{j-i}(q - q^{-1}) \left( \prod_{a=i+1}^{j-1} K_a \right) \cdot \prod_{a=i+1}^{j-1} \text{ad} E_a'(E_i'),
\]

\[
L_{j-i} = (-1)^{j-i}(q - q^{-1}) \left( \prod_{a=i+1}^{j-1} K_a \right) \cdot \prod_{a=i+1}^{j-1} \text{ad} E_a'(E_i'),
\]

\[
L_{i,j} = (q - q^{-1}) \left( \prod_{a=i+1}^{j-1} K_a^{-1} \right) \cdot \prod_{a=i+1}^{j-1} \text{ad} F_a'(F_i'),
\]

\[
L_{i,j} = - (q - q^{-1}) \left( \prod_{a=i+1}^{j-1} K_a^{-1} \right) \cdot \prod_{a=i+1}^{j-1} \text{ad} F_a'(F_i'),
\]

(5.7.1)

where $\text{ad} G_a(G_i) := G_a G_i - G_i G_a$ and $\prod_{a=j+1}^{j-1} \text{ad} G_a(G_i) := \text{ad} G_{j-1} \cdots \text{ad} G_{i+1}(G_i)$ if $j \geq i + 2$ and $\prod_{a=i+1}^{j-1} \text{ad} G_a(G_i) = G_i$ if $j = i + 1$, for $G_i = E_i', E_i', F_i'$, $F_i'$ and $G_a = E_a', F_a'$ with $i + 1 \leq a \leq j - 1$. The first formula in (5.7.1) and Lemma 5.3 imply

\[
L_{j-i} = (-1)^{j-i}(q - q^{-1}) \left( \prod_{a=i+1}^{j-1} K_a \right) \cdot \prod_{a=i+1}^{j-1} \text{ad} E_a'(E_i') \left( \text{ad} E_{E_i'}(E_{i+1}') \right)
\]

\[
= (-1)^{j-i}(q - q^{-1}) \left( \prod_{a=i+1}^{j-1} K_a \right) \cdot \text{ad} E_i' \left( \prod_{a=i+1}^{j-1} \text{ad} E_a'(E_{i+1}') \right)
\]

\[
= K_{i+1} \cdot \text{ad} E_i'(L_{j-i-1}),
\]

then, by (5.6.3) and (5.3.1),

\[
L_{j-i} = K_{i+1} E_i'L_{j-i-1} - K_{i+1} L_{j-i-1} E_i'
\]

\[
= K_{i+1} E_i'L_{j-i-1} - q L_{j-i-1} K_{i+1} E_i'
\]

\[
= E_i L_{j-i-1} - q L_{j-i-1} E_i.
\]

(5.7.2)

Similarly, by (5.7.1) and the corresponding equalities in Lemma 5.3, one can prove

\[
L_{j-i} = E_i L_{j-i-1} - q L_{j-i-1} E_i,
\]

\[
L_{i,j} = L_{i+1,j} F_i - q^{-1} F_i L_{i+1,j}, \quad L_{i,j} = L_{i-1,j} F_i - q^{-1} F_i L_{i-1,j}.
\]

(5.7.3)

We now prove the first formula in the lemma by induction on $j - i$. Indeed, if $j = i + 1$, then

\[
X_{i,j} = E_{i+1} = \frac{-1}{q - q^{-1}} K_j L_{j-i+1}
\]
by (5.1.2). Now assume \( j \geq i + 2 \), then by induction and (5.6.1) we have

\[
X_{i,j} = X_{i,i+1}X_{i+1,j} - qX_{i+1,j}X_{i,i+1}
\]

\[
= E_i - 1 - q^{-1}K_jL_{j,-i-1} - q^{-1}K_jL_{j,-i-1}E_i
\]

\[
= \frac{-1}{q - q^{-1}}K_j(E_iL_{j,-i-1} - qL_{j,-i-1}E_i) \quad \text{(by (5.6.4) as } j \geq i + 2)\]

\[
= \frac{-1}{q - q^{-1}}K_jL_{j,-i} \quad \text{(by (5.7.2))}
\]

as desired. By a parallel argument, we can prove the remaining three formulas.

Fix an order in \( \Phi^+ \), or equivalently in the set \( \{(i,j) \mid 1 \leq i < j \leq n\} \). Recall that \( M_n(\mathbb{N}|\mathbb{Z}_2)' \) is the set consisting of \( C \in M_n(\mathbb{N}|\mathbb{Z}_2) \) such that \( C^0 = 0 \). Given \( C = (C_0, C_1) \in M_n(\mathbb{N}|\mathbb{Z}_2)' \), we can introduce the elements

\[
X_{C^+} = \prod_{1 \leq i < j \leq n} (X_{i,j}^0 X_{i,j}^1), \quad X_{C^-} = \prod_{1 \leq i < j \leq n} (X_{j,i}^0 X_{j,i}^1), \quad K_{C_1} = K_1^{c_{11}} \cdots K_n^{c_{nn}},
\]

where \( C_0 = (c_{0ij}), C_1 = (c_{1ij}) \). For \( \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{Z}^n \), let \( K_\sigma = K_1^{\sigma_1} \cdots K_n^{\sigma_n} \).

**Proposition 5.8.** The set

\[
\{ X_{C^-}K_\sigma \overline{K}_{C_1}X_{C^+} \mid C \in M_n(\mathbb{N}|\mathbb{Z}_2)', \sigma \in \mathbb{Z}^n \}
\]

is a \( \mathbb{Q}(q) \)-basis of \( U_q(\mathfrak{q}(n)) \). In particular, if \( U_q(\mathfrak{q}(n)_0) \) denotes the subalgebra generated by \( E_1, \ldots, E_{n-1}, F_1, \ldots, F_{n-1}, K_1^{\pm 1}, \ldots, K_n^{\pm 1} \), then

\[
U_q(\mathfrak{q}(n)_0) \cong U_q(\mathfrak{gl}(n)),
\]

which is a Hopf algebra isomorphism.

**Proof.** Suppose \( C \in M_n(\mathbb{N}|\mathbb{Z}_2)' \) and \( C_0 = (c_{0ij}), C_1 = (c_{1ij}) \). By Lemma 5.7 and (5.6.4), the following holds

\[
X_{C^-}K_\sigma \overline{K}_{C_1}X_{C^+} = g(C, \sigma)L_{1,1}^{m_1} \cdots L_{n,n}^{m_n} \prod_{1 \leq i < j \leq n} (L_{i,j}^{c_{0ij}} L_{j,i}^{c_{1ij}}).
\]

\[
L_{i,j}^{c_{0ij}} \cdots L_{n,n}^{c_{nn}} \prod_{1 \leq i < j \leq n} (L_{j,i}^{c_{0ij}} L_{i,j}^{c_{1ij}}),
\]

where \( g(C, \sigma) \in \mathbb{Q}(q) \) is nonzero, \( m_j = \sigma_j + \sum_{i=1}^j (c_{0ij} + c_{1ij} - c_{0ij} - c_{1ij}) \) for \( 1 \leq j \leq n \), and the product is taken with the fixed order on \( \{(i,j) \mid 1 \leq i < j \leq n\} \). This implies, up to nonzero scalars, the set (5.8.1) actually coincides with the set (5.5.1) where the order on \( L_{i,j} \) for \( i, j \in I(n|n) \) with \( i < j \) is taken to be compatible with the product on the right hand side of (5.8.3). Then by Proposition 5.5, the first assertion is verified. For the second assertion, we observe that the relations in Proposition 5.2 involving \( E_i, F_i, K_j \) for \( 1 \leq i \leq n - 1 \) and \( 1 \leq j \leq n \) are the same as the standard relations for \( U_q(\mathfrak{gl}(n)) \). This gives a homomorphism from \( U_q(\mathfrak{gl}(n)) \) to \( U_q(\mathfrak{q}(n)_0) \) which is an isomorphism by the first assertion. The last assertion is clear.

**Remark 5.9.** Under the identification (5.8.2), the elements \( X_\alpha \) and \( X_{-\alpha} \) up to scalar multiplication coincides with the root vectors defined in [7] (3.1)]. More precisely, let \( X_\alpha' = q^{j-i+1}X_\alpha \) and \( X_{-\alpha}' = q^{j-i-1}X_{-\alpha} \) for \( \alpha = \epsilon_i - \epsilon_j \in \Phi^+ \). Then \( X_\alpha' \) and \( X_{-\alpha}' \) correspond to the root vectors \( X_\alpha \) and \( X_{-\alpha} \) in [7] (3.1)] (cf. [21]).
Let $U_q^0$ be the subalgebra of $U_q(q(n))$ generated by $K_i^\pm 1, K_i$ for $1 \leq i \leq n$, and let $U_q^+$ (resp. $U_q^-$) be the subalgebra of $U_q(q(n))$ generated by the elements $E_j, E_j$ (resp. $F_j, F_j^+$) for $1 \leq j \leq n - 1$. We have reproduced the following.

**Proposition 5.10 ([13] Theorem 2.3).** There is a $\mathbb{Q}(q)$-linear isomorphism

$$U_q(q(n)) \cong U_q^- \otimes U_q^0 \otimes U_q^+.$$

6. **Commutation formulas for quantum root vectors**

We divide the commutation formulas into four groups which will be discussed in four cases below. Each of the first three cases consists of two lemmas, dealing with the case of two positive (or negative) roots and the case of one positive and one negative roots. The three cases are divided according to whether the pair of root vectors are even-even, even-odd, or odd-odd. Moreover, by the anti-automorphism $\Omega$ given in (5.6.3), it suffices to look at the commutation formulas of positive root vector $X_{i,j}, X_{i,j}$ ($i < j$) with others.

**Case 1**—Commutation formulas for two even-even root vectors $X_{i,j}X_{k,l}$ ($i < j, k \neq l$).

**Lemma 6.1.** The following holds for $1 \leq i, j, k, l \leq n$ satisfying $i < j, k < l$:

$$X_{i,j}X_{k,l} = \begin{cases} X_{k,l}X_{i,j}, & (i < j < k < l \text{ or } i < k < l < j), \\ q^{-1}X_{k,l}X_{i,j}, & (i < k < j = l \text{ or } i = k < j < l), \\ qX_{k,l}X_{i,j} + X_{i,l}, & (i < k = j < l), \\ X_{k,l}X_{i,j} - (q - q^{-1})X_{k,j}X_{i,l}, & (i < k < j < l). \end{cases}$$

**Proof.** Suppose $1 \leq i, j, k, l \leq n$ and $i < j, k < l$. Using the formula (5.11), a straightforward calculation shows that

$$L_{j,i}L_{i,j,k} = \begin{cases} L_{l,k}L_{j,i,j}, & (i < j < k < l \text{ or } i < k < l < j), \\ q^{-1}L_{l,k}L_{j,i,j}, & (i < k < j = l \text{ or } i = k < j < l), \\ L_{l,k}L_{j,i,j} - (q - q^{-1})L_{j,i,j} - L_{l,k}L_{j,i,j}, & (i < k = j < l), \\ L_{l,k}L_{j,i,j} - (q - q^{-1})L_{j,i,j} - L_{l,k}L_{j,i,j}, & (i < k < j < l). \end{cases}$$

Then the lemma is proved case-by-case using Lemma 5.7. Let us illustrate by checking in detail the case when $i < k = j < l$. In this case, by (5.12), Lemma 5.7 and the above formula, we have

$$X_{i,j}X_{k,l} = \frac{-1}{q - q^{-1}}K_j L_{j,i,j} \cdot \frac{-1}{q - q^{-1}}K_k L_{l,k} = \frac{1}{(q - q^{-1})^2}K_j K_k L_{j,i,j} - L_{l,k} L_{j,i,j}$$

$$= \frac{1}{(q - q^{-1})^2}K_j K_k L_{l,k} L_{j,i,j} - (q - q^{-1})L_{j,i,j} - L_{l,k} L_{j,i,j}$$

$$= \frac{1}{(q - q^{-1})^2}K_j L_{j,i,j} \cdot qK_j L_{j,i,j} - \frac{1}{q - q^{-1}}K_j L_{j,i,j} K_j L_{l,i,j}$$

$$= qX_{k,l}X_{i,j} + X_{i,l},$$

where the second equality and fourth equality are due to (5.6.4). The remaining cases can be verified similarly and we skip the detail. \square

Observe that we can derive another set of commutation formulas for $X_{i,j}$ and $X_{k,l}$ by solving for $X_{k,l}X_{i,j}$ in Lemma 6.1 and then interchanging $(i, j)$ and $(k, l)$. Namely, we
have

$$X_{i,j}X_{k,l} = \begin{cases} 
X_{k,l}X_{i,j}, & (k < l < i < j \text{ or } k < i < j < l), \\
qX_{k,l}X_{i,j}, & (k < i < l = j \text{ or } k = i < l < j), \\
q^{-1}X_{k,l}X_{i,j} - q^{-1}X_{k,j}, & (k < i = l < j), \\
X_{l,k}X_{i,j} + (q - q^{-1})X_{k,j}X_{i,l}, & (k < i < l < j). 
\end{cases} \quad (6.1.1)$$

This together with Lemma 6.1 gives a complete commutation formulas for even positive root vectors $X_{i,j}$ and $X_{k,l}$ with $i < j, k < l$.

**Lemma 6.2.** Assume that $1 \leq i, j, k, l \leq n$ satisfy $i < j, k > l$. Then

$$X_{i,j}X_{k,l} = \begin{cases} 
X_{k,l}X_{i,j} + \frac{K_iK_j^{-1} - K_j^{-1}K_i}{q - q^{-1}}, & (i = l, j = k), \\
X_{k,l}X_{i,j}, & (i < j \leq l < k \text{ or } i < l < k < j), \\
X_{k,l}X_{i,j} - K_i^{-1}K_jX_{k,j}, & (i = l < j < k), \\
X_{k,l}X_{i,j} + K_i^{-1}K_jX_{i,l}, & (i < l < j = k), \\
X_{k,l}X_{i,j} + (q - q^{-1})K_i^{-1}K_jX_{k,j}X_{i,l}, & (i < l < j < k). 
\end{cases}$$

**Proof.** Suppose $1 \leq i, j, k, l \leq n$ and $i < j, k > l$. By (5.1.1), it is easy to check that

$$L_{j,-i}L_{l,k} = L_{l,k}L_{j,-i}, \text{ if } i < j < l < k \text{ or } i < l < k < j.$$ 

Then in this situation, by Lemma 5.7 and (5.6.4), we have $X_{i,j}X_{k,l} = X_{k,l}X_{i,j}$. Similarly, in other cases, again by the formula (5.1.1), we obtain

$$L_{j,-i}L_{l,k} = \begin{cases} 
L_{l,k}L_{j,-i} + (q - q^{-1})(L_{j,j}L_{i,-i} - L_{j,-i}L_{i,j}), & (i = l, j = k), \\
L_{l,k}L_{j,-i} + (q - q^{-1})L_{j,k}L_{l,-i}, & (i = l < j < k), \\
qL_{l,k}L_{j,-i}, & (i < l = j < k), \\
L_{l,k}L_{j,-i} + (q - q^{-1})L_{j,j}L_{l,-i}, & (i < l < j = k), \\
L_{l,k}L_{j,-i} + (q - q^{-1})L_{j,k}L_{l,-i}, & (i < l < j < k). 
\end{cases}$$

As before, the lemma is proved case-by-case using Lemma 5.7. Let us illustrate by checking in detail the case when $i < l < j = k$. In this case, by (5.1.2), Lemma 5.7 and the above formulas we have

$$X_{i,j}X_{k,l} = \frac{-1}{q - q^{-1}}K_jL_{j,-i} \cdot \frac{1}{q - q^{-1}}L_{l,k}K_k^{-1}$$

$$= \frac{-1}{(q - q^{-1})^2}K_j(L_{l,k}L_{j,-i} + (q - q^{-1})L_{j,j}L_{l,-i})K_k^{-1}$$

$$= \frac{-1}{(q - q^{-1})^2}L_{l,k}K_jq \cdot q^{-1}K_k^{-1}L_{j,-i} - \frac{1}{q - q^{-1}}K_jL_{j,j}K_k^{-1}L_{l,-i}$$

$$= \frac{1}{q - q^{-1}}L_{l,k}K_k^{-1} \cdot \frac{-1}{q - q^{-1}}K_jL_{j,-i} + K_jK_k^{-1}L_{l,-i}$$

$$= X_{k,l}X_{i,j} + K_jK_k^{-1}X_{i,l},$$

where the third equality is due to (5.6.4) and the assumption $i < l < j = k$. The remaining cases can be verified similarly and we skip the detail. □

Observe that applying the anti-automorphism $\Omega$ to the formulas in Lemma 6.2 and interchanging $(i,j)$ and $(k,l)$, one can obtain another set of commutation formulas for
Proof. by checking in detail the case when $X_{i,j}$ and $X_{k,l}$ as follows:

$$X_{i,j}X_{k,l} = \begin{cases} X_{k,l}X_{i,j}, & (l < k \leq i < j \text{ or } l < i < j < k), \\ X_{k,l}X_{i,j} - X_{k,j}K_iK_k^{-1}, & (l = i < k < j), \\ X_{k,l}X_{i,j} + X_{i,j}K_kK_i^{-1}, & (l < i < k = j), \\ X_{k,l}X_{i,j} - (q - q^{-1})X_{k,j}X_{i,l}K_iK_k^{-1}, & (l < i < k < j). \end{cases}$$

This together with Lemma 6.2 gives a complete commutation formulas for even positive root vectors $X_{i,j}$ and the even negative root vectors $X_{k,l}$ with $i < j, k > l$.

**Case 2**—Commutation formulas for two even-odd root vectors $X_{i,j}X_{k,l}$ ($i < j, k \neq l$).

**Lemma 6.3.** Suppose that $1 \leq i, j, k, l \leq n$ with $i < j$ and $k < l$.

1. If $i = k, j = l$ or $i < j < k < l$ or $k < l < i < j$ or $k < i < j < l$, then

$$X_{i,j}X_{k,l} = \overline{X}_{k,l}X_{i,j}.$$  

2. In other cases, the following formulas hold:

$$X_{i,j}X_{k,l} = \begin{cases} \overline{X}_{k,l}X_{i,j} + (q - q^{-1})(\overline{X}_{k,j}X_{i,l} - X_{k,j}\overline{X}_{i,l}), & (i < k < l < j), \\ q^{-1}\overline{X}_{k,l}X_{i,j}, & (i = k < j < l), \\ q\overline{X}_{k,l}X_{i,j} + \overline{X}_{i,l}, & (i < k = j < l), \\ q\overline{X}_{k,l}X_{i,j} - (q - q^{-1})X_{k,j}\overline{X}_{i,l}, & (i < k < j = l), \\ q^{-1}\overline{X}_{k,l}X_{i,j} + q^{-1}(q - q^{-1})\overline{X}_{k,j}X_{i,l}, & (k = i < l < j), \\ q^{-1}\overline{X}_{k,l}X_{i,j} - q^{-1}\overline{X}_{k,j}, & (k < i = l < j), \\ q\overline{X}_{k,l}X_{i,j}, & (k < i < l = j), \\ \overline{X}_{k,l}X_{i,j} + (q - q^{-1})\overline{X}_{k,j}X_{i,l}, & (k < i < l < j). \end{cases}$$

Proof. Let $i, j, k, l \in \{1, 2, \ldots, n\}$ satisfy $i < j$ and $k < l$. As before by (5.1.1), one can check that if $i = k, j = l$ or $i < j < k < l$ or $k < l < i < j$ or $k < i < j < l$, then

$$L_{j,-i}L_{l,k} = L_{l,k}L_{j,-i}. \text{ Then part (1) is proved by Lemma 5.7 and (5.6.4). To prove part (2), again by (5.1.1) we obtain}$$

$$L_{j,-i}L_{l,k} = \begin{cases} L_{l,k}L_{j,-i} + (q - q^{-1})(L_{j,k}L_{l,-i} - L_{j,-k}L_{l,i}), & (i < k < l < j), \\ q^{-1}L_{l,k}L_{j,-i}, & (i = k < j < l), \\ L_{l,k}L_{j,-i} - (q - q^{-1})L_{j,-j}L_{j,i}, & (i < k < j = l), \\ qL_{l,k}L_{j,-i} - (q - q^{-1})L_{j,-k}L_{l,i}, & (i < k < j < l), \\ L_{l,k}L_{j,-i} - (q - q^{-1})L_{j,k}L_{l,-i}, & (k = i < l < j), \\ q^{-1}L_{l,k}L_{j,-i} + q^{-1}(q - q^{-1})L_{j,k}L_{l,-i}, & (k < i < l < j), \\ L_{l,k}L_{j,-i} + (q - q^{-1})L_{j,k}L_{l,-i}, & (k < i < l < j), \\ qL_{l,k}L_{j,-i}, & (k < i < l < j). \end{cases}$$

As before, part (2) of the lemma is proved case-by-case using Lemma 5.7. Let us illustrate by checking in detail the case when $k = i < l < j$. In this case, by the above formula and
Lemma 5.7 we have
\[
X_{i,j} \overline{X}_{k,l} = \frac{-1}{q - q^{-1}} K_j L_{j,-i} \cdot \frac{-1}{q - q^{-1}} K_l L_{l,-k} = \frac{1}{(q - q^{-1})^2} K_j K_l L_{j,-i} L_{l,-k}
\]
\[
= \frac{1}{(q - q^{-1})^2} K_j K_l (q^{-1} L_{-l,k} L_{j,-i} + q^{-1} (q - q^{-1}) L_{j,k} L_{l,-i})
\]
\[
= \frac{1}{(q - q^{-1})^2} (q^{-1} K_l L_{l,-k} K_j L_{j,-i} + q^{-1} (q - q^{-1}) K_j L_{j,-k} K_l L_{l,-i})
\]
\[
=q^{-1} X_{k,l} X_{i,j} + q^{-1} (q - q^{-1}) \overline{X}_{k,j} X_{i,l},
\]
where the second and fourth equalities are due to (5.6.4). The remaining cases can be verified similarly and we skip the detail. \(\square\)

**Lemma 6.4.** Suppose that \(1 \leq i, j, k, l \leq n\) with \(i < j\) and \(k > l\). Then

(1) If \(i < j \leq l < k\) or \(l < k \leq i < j\) and \(k > l\), then

\[
X_{i,j} \overline{X}_{k,l} = \overline{X}_{k,l} X_{i,j}.
\]

(2) In other cases, the following formula holds:

\[
X_{i,j} \overline{X}_{k,l} = \begin{cases} 
\overline{X}_{k,l} X_{i,j} - (K_j K_i^{-1} - K_j^{-1} K_i) , & (i = l, j = k), \\
\overline{X}_{k,l} X_{i,j} + q(q - q^{-1}) K_i^{-1} K_j^{-1} (X_{k,j} X_{i,l} - X_{k,j} \overline{X}_{i,l}) , & (i < j < k), \\
\overline{X}_{k,l} X_{i,j} - K_i^{-1} K_j \overline{X}_{i,l} , & (i = l < j < k), \\
\overline{X}_{k,l} X_{i,j} + (q - q^{-1}) K_i^{-1} K_j X_{i,l} + K_i^{-1} \overline{X}_{i,l} , & (i < j < k = l), \\
\overline{X}_{k,l} X_{i,j} - \overline{X}_{k,j} K_i^{-1} K_j^{-1} - (q - q^{-1}) X_{k,j} K_i^{-1} K_j^{-1} , & (l < i < k < j), \\
\overline{X}_{k,l} X_{i,j} + X_{i,l} K_i^{-1} K_j^{-1} , & (l < i < k = j), \\
\overline{X}_{k,l} X_{i,j} - (q - q^{-1}) K_i K_j^{-1} X_{k,j} \overline{X}_{i,l} , & (l < i < k < j).
\end{cases}
\]

**Proof.** Assume \(1 \leq i < j \leq n\), \(1 \leq l < k \leq n\). By (5.1.1) one can prove

\[
L_{j,-i} L_{l,-k} = \begin{cases} 
L_{l,k} L_{j,-i} , & \text{if } i < j < l < k \text{ or } l < k < i < j \text{ or } l < i < j < k, \\
q L_{l,k} L_{j,-i} , & \text{if } i < l = j < k, \\
q^{-1} L_{l,k} L_{j,-i} , & \text{if } l < i = k < j.
\end{cases}
\]

Then part (1) can be proved by (5.6.4) and Lemma 5.7. Otherwise, again by (5.1.1) it is straightforward to check that the following holds

\[
L_{j,-i} L_{l,-k} = \begin{cases} 
L_{l,k} L_{j,-i} + (q - q^{-1}) (L_{j,-i} - L_{j,-i} L_{i,-i}) , & (i = l, j = k), \\
L_{l,k} L_{j,-i} + (q - q^{-1}) (L_{j,-i} - L_{j,-i} L_{i,-i}) , & (i < l < k < j), \\
L_{l,k} L_{j,-i} + (q - q^{-1}) L_{j,-i} L_{i,-i} , & (i < l < k < j), \\
L_{l,k} L_{j,-i} - (q - q^{-1}) (L_{j,-i} - L_{j,-i} L_{i,-i}) , & (l = i < k < j), \\
L_{l,k} L_{j,-i} - (q - q^{-1}) L_{j,-i} L_{i,-i} , & (l < i < k < j).
\end{cases}
\]
This together with Lemma 5.7 and (5.6.4) gives rise to part (2). Let us explain in detail the case when \( i < l < k < j \). In this case, by the above formula and Lemma 5.7 we have

\[
X_{i,j}X_{k,l} = \frac{-1}{q - q^{-1}} K_j L_{j,-i} \cdot \frac{-1}{q - q^{-1}} L_{l,k} K_k^{-1}
\]

\[
= \frac{1}{(q - q^{-1})^2} K_j (L_{l,k} L_{j,-i} + (q - q^{-1})(L_{j,k} L_{l,-i} - L_{j,-k} L_{l,i})) K_k^{-1}
\]

\[
= \frac{1}{(q - q^{-1})^2} L_{l,k} K_k^{-1} K_j L_{j,-i}
\]

\[
+ \frac{1}{q - q^{-1}} q K_k^{-1} K_j^{-1}(K_j L_{j,-k} K_l L_{l,-i} - K_j L_{j,-k} K_l L_{l,i})
\]

\[
= X_{k,l} X_{i,j} + q(q - q^{-1}) K_j^{-1} K_k^{-1}(X_{k,j} X_{i,l} - X_{k,j} X_{i,l}),
\]

where the third and fourth equalities are due to (5.6.4). □

**Case 3**—Commutation formulas for two odd-odd root vectors \( X_{i,j} X_{k,l} \) (\( i < j, k \neq l \)).

**Lemma 6.5.** Let \( i, j, k, l \in \{1, 2, \ldots, n\} \) satisfy \( i < j \) and \( k < l \). Then we have

\[
X_{i,j} X_{k,l} = \begin{cases}
-q - q^{-1} X_{i,j}^2, & (i = k, j = l), \\
|q + q^{-1}| X_{i,j}, & (i < j < k < l), \\
X_{k,l} X_{i,j} - (q - q^{-1})(X_{k,j} X_{i,l} + X_{k,j} X_{i,l}), & (i < k < l < j), \\
-q X_{k,j} X_{i,l} - q(q - q^{-1}) X_{k,j} X_{i,l}, & (i = k < j < l), \\
-q X_{k,j} X_{i,j} + X_{i,l}, & (i < k = j < l), \\
-q X_{k,j} X_{i,j} - (q - q^{-1}) X_{k,j} X_{i,l}, & (i < k < j < l).
\end{cases}
\]

**Proof.** Suppose that \( i, j, k, l \in \{1, 2, \ldots, n\} \) satisfy \( i < j \) and \( k < l \). As before by (5.1.1), we get

\[
L_{j,-i} L_{l,-i} = \begin{cases}
-q - q^{-1} L_{j,-i}^2, & (i = k, j = l), \\
q + q^{-1} L_{j,-i}, & (i < j < k < l), \\
L_{l,-k} L_{j,-i}, & (i < k < l < j), \\
-q L_{l,-k} L_{j,-i} - q(q - q^{-1}) L_{j,-k} L_{l,-i}, & (i = k < j < l), \\
L_{l,-k} L_{j,-i} - (q - q^{-1}) L_{j,-k} L_{l,-i}, & (i < k = j < l), \\
-q L_{l,-k} L_{j,-i} - (q - q^{-1}) L_{j,-k} L_{l,-i}, & (i < k < j < l).
\end{cases}
\]

Then the lemma is proved case-by-case as before. We will illustrate by checking in detail the case when \( i < k = j < l \). In this case, by the above formulas, (5.1.2) and Lemma 5.7.
we have
\[ \mathcal{X}_{i,j} \mathcal{X}_{k,l} = \frac{-1}{q - q^{-1}} K_j L_{j,l} - \frac{-1}{q - q^{-1}} K_l L_{j,l} \]
\[ = \frac{1}{(q - q^{-1})^2} K_j K_l L_{j,l} \]
\[ = \frac{1}{(q - q^{-1})^2} K_j K_l (- L_{l,k} L_{j,i} - (q - q^{-1}) L_{j,j} L_{l,l}) \]
\[ = - \frac{1}{(q - q^{-1})^2} q K_l L_{l,k} K_j L_{j,i} - \frac{1}{q - q^{-1}} K_j L_{j,j} K_l L_{l,l} \]
\[ = - q \mathcal{X}_{k,l} \mathcal{X}_{i,j} + X_{i,l}, \]
where the second and fourth equalities are due to (5.6.4). The remaining cases can be verified similarly, and we omit the detail. \( \square \)

As before, by solving for \( \mathcal{X}_{k,l} \mathcal{X}_{i,j} \) in Lemma 6.5 and then interchanging \((i,j)\) and \((k,l)\), we obtain

\[ \mathcal{X}_{i,j} \mathcal{X}_{k,l} = \begin{cases} 
- \mathcal{X}_{k,l} \mathcal{X}_{i,j}, & (k < l < i < j), \\
- \mathcal{X}_{k,l} \mathcal{X}_{i,j} + (q - q^{-1})(\mathcal{X}_{k,j} \mathcal{X}_{i,l} - X_{k,j} X_{i,l}), & (k < i < j < l), \\
- q^{-1} \mathcal{X}_{k,l} \mathcal{X}_{i,j} - q^{-1}(q - q^{-1}) X_{k,j} X_{i,l}, & (k = i < l < j), \\
- q^{-1} \mathcal{X}_{k,j} \mathcal{X}_{i,l} + q^{-1} X_{k,j}, & (k < i < j), \\
- q^{-1} \mathcal{X}_{k,j} \mathcal{X}_{i,l} - (q - q^{-1}) X_{k,j} X_{i,l}, & (k < i < l < j), \\
- \mathcal{X}_{k,l} \mathcal{X}_{i,j} - (q - q^{-1}) X_{k,j} X_{i,l}, & (k < i < l < j). 
\end{cases} \]

This together with Lemma 6.5 gives a complete commutation formula for two odd positive root vectors.

**Lemma 6.6.** The following holds for \( i, j, k, l \in \{1, 2, \ldots, n\} \) satisfying \( i < j \) and \( k > l \):

1. If \( i = l, j = k \), then
\[ \mathcal{X}_{i,j} \mathcal{X}_{i,j} = - \mathcal{X}_{j,i} \mathcal{X}_{i,j} + \frac{K_i K_j - K_i^{-1} K_j^{-1}}{q - q^{-1}} + (q - q^{-1}) K_i K_j. \]

2. In other cases, the following formulas hold:
\[ \mathcal{X}_{i,j} \mathcal{X}_{k,l} = \begin{cases} 
- \mathcal{X}_{k,l} \mathcal{X}_{i,j}, & (i < j \leq l < k), \\
- \mathcal{X}_{k,l} \mathcal{X}_{i,j} - q(q - q^{-1}) K_i^{-1} K_j^{-1} (\mathcal{X}_{k,j} \mathcal{X}_{i,l} + X_{k,j} X_{i,l}), & (i < l < k < j), \\
- \mathcal{X}_{k,j} \mathcal{X}_{i,j} + q^{-1} X_{k,j} K_i K_j - q^{-1}(q - q^{-1}) \mathcal{X}_{k,j} K_i K_j, & (i = l < j < k), \\
- \mathcal{X}_{k,l} \mathcal{X}_{i,j} - (q - q^{-1}) K_i^{-1} K_j^{-1} \mathcal{X}_{i,l} + K_i^{-1} K_j^{-1} \mathcal{X}_{i,l}, & (i < l < j), \\
- \mathcal{X}_{k,l} \mathcal{X}_{i,j} - (q - q^{-1}) K_i^{-1} K_j^{-1} \mathcal{X}_{i,l} + \mathcal{X}_{k,l} \mathcal{X}_{i,j}, & (i < l < j < k). 
\end{cases} \]

*Proof.* Suppose that \( i, j, k, l \in \{1, 2, \ldots, n\} \) satisfy \( i < j \) and \( k > l \). If \( i = l, j = k \), then by (5.1.1) we have
\[ L_{j,l} L_{j,i} = - L_{i,j} L_{j,i} + (q - q^{-1}) (L_{j,j} L_{i,i} - L_{j,l} L_{l,i}) - (q - q^{-1}) L_{j,j} L_{i,j}. \] (6.6.1)
Otherwise still by (5.1.1), we get

\[
L_{-j,i}L_{-l,i} = \begin{cases} 
-L_{-l,k}L_{-j,i}, & (i < j < l < k), \\
-L_{-l,k}L_{-j,i} - (q - q^{-1})(L_{-j,k}L_{-l,i} + L_{-j,-k}L_{-l,-i}), & (i < l < k < j), \\
-L_{-l,k}L_{-j,i} + (q - q^{-1})(L_{-j,k}L_{-i,i} - L_{-j,-k}L_{-i,-i}), & (i = l < j < k), \\
-qL_{-l,k}L_{-j,i}, & (i = l = j < k), \\
-L_{-l,k}L_{-j,i} - (q - q^{-1})(L_{-j,-j}L_{-l,i} + L_{-j,-j}L_{-l,-i}), & (i < l < j = k), \\
-L_{-l,k}L_{-j,i} - (q - q^{-1})L_{-j,-j}L_{-l,-i}, & (i < l < j < k).
\end{cases}
\]

Then as before the lemma is proved using Lemma 5.7. We leave the detail to the reader. \(\square\)

As before, by applying the anti-automorphism \(\Omega\) given in (5.4.3) and (5.6.3) to the formulas in Lemma 6.6.2 and interchanging \((i, j)\) and \((k, l)\), the following holds:

\[
X_{i,j}X_{k,l} = \begin{cases} 
-\bar{X}_{k,l}X_{i,j}, & (l < k \leq i < j), \\
-\bar{X}_{k,l}X_{i,j} + q^{-1}(q - q^{-1})(X_{k,j}X_{i,l} - \bar{X}_{k,j}\bar{X}_{i,l})K_{i}K_{j}, & (l < i < j < k), \\
-\bar{X}_{k,l}X_{i,j} - (q - q^{-1})\bar{X}_{k,j}K_{i}^{-1}K_{k}^{-1} + X_{k,j}K_{i}^{-1}K_{k}^{-1}, & (l = i < k < j), \\
\bar{X}_{k,l}X_{i,j} + q^{-1}K_{i}K_{j}X_{i,l} - q^{-1}(q - q^{-1})K_{i}K_{j}\bar{X}_{i,l}, & (l < i < k = j), \\
\bar{X}_{k,l}X_{i,j} - (q - q^{-1})\bar{X}_{k,j}K_{i}K_{k}^{-1}, & (l < i < k < j).
\end{cases}
\]

This together with Lemma 6.6 gives a complete commutation formula between odd positive root vectors and odd negative root vectors.

**Case 4**—Commutation formulas between \(X_{i,j}\) and \(K_{a}\) and between \(\bar{X}_{i,j}\) and \(K_{a}\) where \(1 \leq i < j \leq n\) and \(1 \leq a \leq n\).

**Lemma 6.7.** Suppose \(i, j, a \in \{1, 2, \ldots, n\}\) satisfy \(i < j\). Then

\[
X_{i,j}K_{a} = \begin{cases} 
K_{a}X_{i,j}, & (a < i \text{ or } a > j), \\
q^{-1}K_{i}X_{i,j} - q^{-1}\bar{X}_{i,j}K_{i}^{-1}, & (a = i), \\
qK_{j}X_{i,j} + q\bar{X}_{i,j}K_{j}^{-1}, & (a = j), \\
K_{a}X_{i,j} + q(q - q^{-1})(\bar{X}_{a,j}X_{i,a} - X_{a,j}\bar{X}_{i,a})K_{a}^{-1}, & (i < a < j),
\end{cases}
\]

\[
\bar{X}_{i,j}K_{a} = \begin{cases} 
-\bar{K}_{a}\bar{X}_{i,j}, & (a < i \text{ or } a > j), \\
-q^{-1}\bar{K}_{i}\bar{X}_{i,j} + q\bar{X}_{i,j}\bar{K}_{i}^{-1}, & (a = i), \\
-q\bar{K}_{j}\bar{X}_{i,j} + qX_{i,j}\bar{K}_{j}^{-1}, & (a = j), \\
-\bar{K}_{a}\bar{X}_{i,j} - q(q - q^{-1})(\bar{X}_{a,j}\bar{X}_{i,a} + X_{a,j}X_{i,a})K_{a}^{-1}, & (i < a < j).
\end{cases}
\]

**Proof.** Let \(i, j, a \in \{1, 2, \ldots, n\}\) and suppose \(i < j\). Then by (5.1.1), one can deduce the following commutation formula:

\[
L_{-j,-i}L_{-a,a} = \begin{cases} 
L_{-a,a}L_{-j,-i}, & (a < i \text{ or } a > j), \\
q^{-1}L_{-a,a}L_{-j,-i} + q^{-1}(q - q^{-1})L_{-j,-i}L_{-a,a}, & (a = i), \\
qL_{-a,a}L_{-j,-i} - q(q - q^{-1})L_{-j,-i}L_{-a,a}, & (a = j), \\
L_{-a,a}L_{-j,-i} + (q - q^{-1})(L_{-j,a}L_{-a,a}L_{-j,-i} - L_{-j,-a}L_{-a,i}), & (i < a < j),
\end{cases}
\]

\[
L_{-j,i}L_{-a,a} = \begin{cases} 
-L_{-a,a}L_{-j,i}, & (a < i \text{ or } a > j), \\
-qL_{-a,a}L_{-j,i} - q^{-1}(q - q^{-1})L_{-j,i}L_{-a,a}, & (a = i), \\
-qL_{-a,a}L_{-j,i} - q(q - q^{-1})L_{-j,i}L_{-a,a}, & (a = j), \\
-L_{-a,a}L_{-j,i} - (q - q^{-1})(L_{-j,a}L_{-a,a}L_{-j,i} + L_{-j,-a}L_{-a,i}), & (i < a < j).
\end{cases}
\]

This together with Lemma 5.7 and (5.1.2) proves the lemma as before. \(\square\)
7. Quantum Commutation Formulas of Higher Order

We now derive the commutation formulas for higher order quantum root vectors. We only need to consider the cases corresponding to Cases 1, 2, and 4 in §6. This is because \( \overline{x}_{ij} \in U_q(q(n)) \) (see Lemma 6.5) for \( 1 \leq i \neq j \leq n \). We need some preparation.

For \( m \geq 1 \), let

\[
[m]! = [m][m-1] \cdots [1], \quad \text{where} \quad [m] = \frac{q^m - q^{-m}}{q - q^{-1}}.
\]

We also use the convention \([0] = [0]! = 1\). For \( c \in \mathbb{Z}, t \geq 1 \), set

\[
\left[\frac{c}{m}\right] = \frac{[c][c-1] \cdots [c-m+1]}{[m]!}, \quad \left[\frac{c}{0}\right] = 1.
\]

Generally, for an element \( Z \) in an associative \( \mathbb{Q}(q) \)-algebra \( \mathcal{R} \) and \( m \in \mathbb{N} \), let

\[
Z^{(m)} = \frac{Z^m}{[m]!}.
\]

If \( Z \) is invertible, define, for \( t \geq 1 \) and \( c \in \mathbb{Z} \),

\[
\left[\frac{Z}{t}; c\right] = \prod_{s=1}^{t} \frac{Zq^{c-s+1} - Z^{-1}q^{-c+s-1}}{q^s - q^{-s}}, \quad \text{and} \quad \left[\frac{Z}{0}; c\right] = 1. \tag{7.0.1}
\]

Lemma 7.1 (cf. [16, Lemma 1.6]). Let \( \mathcal{R} \) be an associative algebra over \( \mathbb{Q}(q) \) and let \( X,Y,Z \in \mathcal{R} \). Then the following holds for any positive integers \( m, s \).

1. If \( X,Y,Z \) satisfy \( XY = YX + Z \), \( XZ = q^{-2}ZX \), \( ZY = q^{-2}YZ \), then

\[
X^{(m)}Y^{(s)} = \sum_{t=0}^{\min(m,s)} q^{-(m+s)t+\frac{t(3s+1)}{2}}Y^{(s-t)}Z^{(t)}X^{(m-t)}.
\]

2. If \( X,Y,Z \) satisfy \( XY = qYX + Z \), \( XZ = q^{-1}ZX \), \( ZY = q^{-1}YZ \), then

\[
X^{(m)}Y^{(s)} = \sum_{t=0}^{\min(m,s)} q^{(m-t)(s-t)}Y^{(s-t)}Z^{(t)}X^{(m-t)}.
\]

Proof. Suppose that \( X,Y,Z \) satisfy the relations \( XY = YX + Z \), \( XZ = q^{-2}ZX \), \( ZY = q^{-2}YZ \), and \( m, s \) are positive integers. If \( m \geq 1 \), then we have

\[
X^{m}Y = YX^{m} + \sum_{t=0}^{m-1} X^tZX^{m-1-t} = YX^{m} + \sum_{t=0}^{m-1} q^{-2t}ZX^tX^{m-1-t} \tag{7.1.1}
\]

\[
= YX^{m} + \frac{1 - q^{-2m}}{1 - q^2}ZX^{m-1} = YX^{m} + q^{-m+1}[m]ZX^{m-1}
\]
and hence $X^{(m)}Y = YX^{(m)} + q^{-m+1}ZX^{(m-1)}$. This proves the lemma in the case $s = 1$. By induction on $s$,

$$X^{(m)}Y^{(s+1)} = X^{(m)}Y^{(s)} Y = \frac{1}{s+1} \min(m,s) \sum_{t=0}^{\min(m,s)} q^{(m+s)t + \frac{(s+1)\cdot 1}{2}} Y^{(s-t)} Z^{(t)} X^{(m-t)} Y $$

$$= \frac{1}{s+1} \sum_{t=0}^{\min(m,s)} q^{(m+s)t + \frac{(s+1)\cdot 1}{2}} Y^{(s-t)} Z^{(t)} \left( YX^{(m-t)} + q^{-m+t+1}ZX^{(m-t-1)} \right) $$

$$= \frac{1}{s+1} \sum_{t=0}^{\min(m,s)} q^{(m+s)t + \frac{(s+1)\cdot 1}{2}} Y^{(s-t)} Y^{(s-1-t)} q^{-2t} Z^{(t)} X^{(m-t)} $$

$$+ \frac{1}{s+1} \sum_{t=0}^{\min(m,s)+1} q^{(m+s)(t-1) + \frac{(t-1)(s+1)-2}{2}} Y^{(s+1-t)} q^{-m+t+1} \left[ t + 1 \right] X^{(m-t)} $$

$$= \sum_{t=0}^{\min(m,s)+1} q^{(m+s+1)t + \frac{(s+1)\cdot 1}{2}} Y^{(s+1-t)} Z^{(t)} X^{(m-t)},$$

where the last equality is due to the fact that $q^{-t}[s+1-t] + q^{s+1-t}[t] = [s+1]$. This proves part (1) of the lemma. Similarly, it is easy to prove part (2).

By a proof similar to (7.11), we have a special case of Lemma 7.1(2) with the condition $ZY = q^{-1}YZ$ dropped.

**Corollary 7.2.** Let $\mathcal{R}$ be an associative algebra over $\mathbb{Q}(q)$ and let $X,Y,Z \in \mathcal{R}$. Suppose $m$ is a positive integer. If $X,Y,Z$ satisfy $XY = qYX - Z$, $XZ = q^{-1}ZX$. Then

$$X^{(m)}Y = q^m YX^{(m)} - ZX^{(m-1)}.$$

**Lemma 7.3.** Let $\mathcal{R}$ be an associative algebra over $\mathbb{Q}(q)$. Suppose that the elements $X, Y, I, J, H \in \mathcal{R}$ satisfy

$$XY = YX + H - I, \quad XH = q^2HX - J, \quad XI = q^{-2}IX + q^{-2}J, \quad XJ = JX.$$

Then the following holds for any positive integer $m$:

$$X^{(m)}Y = YX^{(m)} + q^{m-1}HX^{(m-1)} - q^{-m+1}IX^{(m-1)} - q^{-1}JX^{(m-2)}.$$

**Proof.** Let $m$ be a positive integer. Observe that the following holds for $0 \leq t \leq m-1$:

$$X^t H = q^{2t}HX^t - \sum_{s=0}^{t-1} X^s \cdot J \cdot (q^2X)^{t-1-s} = q^{2t}HX^t - [t]q^{-1}JX^{t-1},$$

$$X^t I = q^{-2t}IX^t + \sum_{s=0}^{t-1} X^s \cdot q^{-2}J \cdot (q^{-2}X)^{t-1-s} = q^{-2t}IX^t + [t]q^{-1}JX^{t-1}.$$
Then one can deduce that

\[
X^m Y = Y X^m + \sum_{t=0}^{m-1} X^t (H - I) X^{m-1-t} = Y X^m + \sum_{t=0}^{m-1} X^t H X^{m-1-t} - \sum_{t=0}^{m-1} X^t I X^{m-1-t}
\]

\[
= Y X^m + \sum_{t=0}^{m-1} \left( q^{2t} H X^{m-1} - [t] q^{t-1} J X^{m-2} - q^{-2t} I X^{m-1} - [t] q^{-t-1} J X^{m-2} \right)
\]

\[
= Y X^m + q^{m-1} [m] H X^{m-1} - q^{-m+1} [m] I X^{m-1} - q^{-1} [m] [m-1] J X^{m-2}.
\]

Hence the lemma is verified. \(\square\)

We now apply the formulas in the previous lemmas to derive the commutation formulas of higher order. First, we deal with the even-even case for commuting \(X_{i,j}^{(m)} X_{k,l}^{(s)}\).

**Proposition 7.4.** Assume that \(i, j, k, l \in \{1, 2, \ldots, n\}\) satisfy \(i < j\) and \(k < l\). Then the following holds for positive integers \(m, s\).

1. If \(i < j < k < l\) or \(i < k < l < j\), then
   \[
   X_{i,j}^{(m)} X_{k,l}^{(s)} = X_{k,l}^{(s)} X_{i,j}^{(m)}.
   \]

2. If \(i < k < j = l\) or \(i = k < j < l\), then
   \[
   X_{i,j}^{(m)} X_{k,l}^{(s)} = q^{-ms} X_{k,l}^{(s)} X_{i,j}^{(m)}.
   \]

3. If \(i < k = j < l\), then
   \[
   X_{i,j}^{(m)} X_{k,l}^{(s)} = q^{ms} X_{k,l}^{(s)} X_{i,j}^{(m)} + \sum_{t=1}^{\min(m,s)} q^{(m-t)(s-t)} X_{k,l}^{(s-t)} X_{i,l}^{(t)} X_{i,j}^{(m-t)}.
   \]

4. If \(i < k < j < l\), then
   \[
   X_{i,j}^{(m)} X_{k,l}^{(s)} = X_{k,l}^{(s)} X_{i,j}^{(m)} + \sum_{t=1}^{\min(m,s)} (-1)^t [t]! (q - q^{-1})^t q^{-(m+s)t + \frac{t(t+1)}{2}} X_{k,l}^{(s-t)} X_{i,l}^{(t)} X_{i,j}^{(m-t)}.
   \]

**Proof.** Let \(i, j, k, l \in \{1, 2, \ldots, n\}\). Clearly, parts (1) and (2) follow from the first two formulas in Lemma 6.1 respectively.

Assume that \(i < k = j < l\) and let \(X = X_{i,j}, Y = X_{k,l}, Z = X_{i,l}\). Then by Lemma 6.1 we have \(XY = qYX + Z, XZ = q^{-1}ZX, ZY = q^{-1}YZ\). Thus by Lemma 7.1(2), one can deduce that part (3) is proved.

Finally, if \(i < k < j < l\), then by Lemma 6.1 the elements \(X = X_{i,j}, Y = X_{k,l}, Z = -(q-q^{-1})X_{k,j} X_{i,l}\) satisfy \(XY = YX + Z, XZ = q^{-2}ZX, ZY = q^{-2}YZ\) and hence part (4) of the proposition follows from Proposition 7.4(1). \(\square\)

**Remark 7.5.** As before, we can obtain another set of formulas for the commutation of two even divided powers \(X_{i,j}^{(m)}\) and \(X_{k,l}^{(s)}\) by solving for \(X_{k,l}^{(s)} X_{i,j}^{(m)}\) in Proposition 7.4 and then interchanging \((i, j)\) and \((k, l)\). These newly obtained formulas together with Proposition 7.4 exhaust the possibilities for a commutation of two even divided powers \(X_{i,j}^{(m)}\) and \(X_{k,l}^{(s)}\) associated to positive root vectors.

**Proposition 7.6.** Assume that \(i, j, k, l \in \{1, 2, \ldots, n\}\) satisfy \(i < j\) and \(k > l\). Let \(m, s\) be positive integers. Then
Proposition 7.8. Assume that Lemma 7.1(1) with 30 DU AND W AN between X. This together with Proposition 7.6 exhaust the possibilities for a commutation relation.

By applying the anti-automorphism to the formulas in Proposition 7.6, we get Remark 7.7.

Then by Lemma 6.2, part (1) of the proposition can be directly checked by induction on k, l.

It is easy to see that part (2) follows from Lemma 6.2.

If i = k < j < l, then by (5.6.2), Lemmas 6.2 and 6.1, the elements X = X_{i,j}, Y = X_{k,l}, Z = −K_i^{-1}K_jX_{k,j} satisfy XY = YX + Z, ZX = q^2ZX. Furthermore, by Lemma 6.1 we have X_{i,k}X_{j,k} = q^{-1}X_{j,k}X_{i,k} and hence by applying Ω in (5.4.3) we obtain X_{k,j}X_{k,l} = qX_{k,l}X_{k,j}. This leads to ZY = q^2YZ by (5.6.2). Meanwhile again by (5.6.2) we have X_{k,j}K_{i}^{-1}K_j = qK_{i}^{-1}K_jX_{k,j}, which implies Zt = (−1)^{t}q^{\frac{t(t+1)}{2}}K_i^{-1}K_j^{t}X_{k,j}^{t}. Therefore, by Lemma 7.1(1) with q^{-1} being replaced by q, part (3) holds. Similarly, the formulas in parts (4) and (5) can be checked by Lemma 6.2, (5.6.2) and (6.1.1) and Lemma 7.1(1).

Now it remains to prove part (1). Assume l = i, k = j. By (5.6.2) and (7.0.1), we have

\[ X_{i,j} \cdot \left[ K_i K_j^{-1}, c \right] = \left[ K_i K_j^{-1}, c - 2 \right] \cdot X_{i,j}, \quad X_{j,i} \cdot \left[ K_i K_j^{-1}, c \right] = \left[ K_i K_j^{-1}, c + 2 \right] \cdot X_{j,i}. \]

Then by Lemma 6.2 part (1) of the proposition can be directly checked by induction on s, which is similar to the classical case. 

\[ \square \]

Remark 7.7. By applying the anti-automorphism to the formulas in Proposition 7.6, we can obtain another set of commutation formulas for the divided powers \( X_{i,j}^{(m)} \) and \( X_{k,l}^{(s)} \). This together with Proposition 7.6 exhaust the possibilities for a commutation relation between \( X_{i,j}^{(m)} \) and \( X_{k,l}^{(s)} \).

Next, we derive the commutation formulas for \( X_{i,j}^{(m)} X_{k,l} \).

Proposition 7.8. Assume that \( i, j, k, l \in \{1, 2, \ldots, n\} \) satisfy \( i < j \) and \( k < l \). Let \( m \) be a positive integer. Then

1. If \( i = k, j = l, \) or \( i < j < k < l \) or \( k < l < i < j \) or \( k < i < j < l \), then

\[ X_{i,j}^{(m)} X_{k,l} = X_{k,l} X_{i,j}^{(m)}. \]
(2) If \( i < k < l < j \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = X_{k,l}X_{i,j}^{(m)} + q^{m-1}(q - q^{-1})X_{k,j}X_{i,l}X_{i,j}^{(m-1)}
- q^{-m+1}(q - q^{-1})X_{k,j}\overline{X}_{i,l}X_{i,l}X_{i,j}^{(m-1)}
- q^{-1}(q - q^{-1})^2X_{k,j}\overline{X}_{i,j}X_{i,l}X_{i,j}^{(m-2)}.
\]

(3) If \( i = k < j \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = q^{-m}\overline{X}_{k,l}X_{i,j}^{(m)}.
\]

(4) If \( i < k = j < l \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = q^m\overline{X}_{k,l}X_{i,j}^{(m)} + \overline{X}_{i,l}X_{i,j}^{(m-1)}.
\]

(5) If \( i < k < j = l \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = q^m\overline{X}_{k,l}X_{i,j}^{(m)} - (q - q^{-1})X_{k,j}\overline{X}_{i,l}X_{i,j}^{(m-1)}.
\]

(6) If \( i < k < j < l \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = \overline{X}_{k,l}X_{i,j}^{(m)} - q^{-m+1}(q - q^{-1})X_{k,j}\overline{X}_{i,l}X_{i,j}^{(m-1)}.
\]

(7) If \( k = i < l < j \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = q^{-m}\overline{X}_{k,l}X_{i,j}^{(m)} + q^{-1}(q - q^{-1})\overline{X}_{k,j}X_{i,l}X_{i,j}^{(m-1)}.
\]

(8) If \( k < i = l < j \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = q^{-m}\overline{X}_{k,l}X_{i,j}^{(m)} - q^{-1}\overline{X}_{k,j}X_{i,j}^{(m-1)}.
\]

(9) If \( k < i < l = j \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = q^m\overline{X}_{k,l}X_{i,j}^{(m)}.
\]

(10) If \( k < i < l < j \), then

\[
X_{i,j}^{(m)} \overline{X}_{k,l} = \overline{X}_{k,l}X_{i,j}^{(m)} + q^{-m+1}(q - q^{-1})\overline{X}_{k,j}X_{i,l}X_{i,j}^{(m-1)}.
\]

**Proof.** Clearly, the equalities in parts (1), (3) and (9) can be checked directly by taking advantage of the corresponding formulas in Lemma 6.3 respectively.

If \( i < k < l < j \), we set

\[
X = X_{i,j}, \quad Y = \overline{X}_{k,l}, \quad H = (q - q^{-1})\overline{X}_{k,j}X_{i,l}, \quad I = (q - q^{-1})X_{k,j}\overline{X}_{i,l},
\]

\[
J = (q - q^{-1})^2X_{k,j}\overline{X}_{i,j}X_{i,l}.
\]

Then firstly, by the formula in the first case in Lemma 6.3(2) we have \( XY = YX + H - I \). By the formula in the fourth case in Lemma 6.3(2) and Lemma 6.1, we see that \( XH = q^2HX - J \). By the formula in the sixth case in Lemma 6.3(2) and Lemma 6.1, one can check that \( XI = q^{-2}IX + q^{-2}J \). Furthermore, by Lemma 6.1 and Lemma 6.3(1) we obtain \( XJ = JX \). Putting together, part (2) is proved by using Lemma 7.3.

If \( i < k = j < l \), then by the formulas in the second and third cases in Lemma 6.3 the elements \( X = X_{i,j}, \ Y = \overline{X}_{k,l}, \ Z = -\overline{X}_{i,l} \) satisfy \( XY = qYX - Z, \ XZ = q^{-1}ZX \). Hence part (4) is verified by using Lemma 7.2.

If \( i < k < j = l \), then the elements \( X = X_{i,j}, \ Y = \overline{X}_{k,l}, \ Z = (q - q^{-1})X_{k,j}\overline{X}_{i,l} \) satisfy \( XY = qYX - Z, \ XZ = q^{-1}ZX \) by the formula in the fourth case in Lemmas 6.3(2), 6.3(1) and 6.1. Therefore part (5) follows from Corollary 7.2.
Similarly, the remaining parts can be proved case-by case using Lemmas 6.3, 6.1 and 7.1 and Corollary 7.2. To save some space, we omit the detail.

\[ \square \]

**Proposition 7.9.** Assume that \( i, j, k, l \in \{1, 2, \ldots, n\} \) satisfy \( i < j \) and \( k > l \). Let \( m \) be a positive integer. Then

1. If \( i < j \leq l < k \) or \( l < k \leq i < j < k \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j}. \]

2. If \( i = l \) and \( j = k \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j} - q^{m-1}K_i^{-1}K_j^{-1}X^{(m-1)}_{i,j} \]
   \[ + q^{-m+1}K_j^{-1}K_i^{-1}X^{(m-2)}_{i,j}. \]

3. If \( i < l < k < j \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j} + q^m(q - q^{-1})K_i^{-1}K_j^{-1}X_{i,l}X^{(m-1)}_{i,j} \]
   \[ - q^{-m+2}(q - q^{-1})K_j^{-1}K_i^{-1}X_{k,j}X_{i,l}X^{(m-1)}_{i,j} \]
   \[ - (q - q^{-1})^2K_i^{-1}K_j^{-1}X_{i,l}X_{i,j}X^{(m-2)}_{i,j}. \]

4. If \( i = l < j < k \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j} - q^{m-1}K_j^{-1}K_i^{-1}X^{(m-1)}_{i,j}. \]

5. If \( i < l < j < k \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j} + q^m(q - q^{-1})K_i^{-1}K_j^{-1}X_{i,l}X^{(m-1)}_{i,j} \]
   \[ + q^{-m+1}K_i^{-1}K_j^{-1}X_{i,l}X^{(m-1)}_{i,j} \]
   \[ + q^{-1}(q - q^{-1})K_i^{-1}K_j^{-1}X_{i,l}X^{(m-2)}_{i,j}. \]

6. If \( i < l < j < k \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j} + q^{m-1}(q - q^{-1})K_i^{-1}K_j^{-1}X_{i,l}X^{(m-1)}_{i,j}. \]

7. If \( l = i < k < j \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j} - q^{m-1}X_{k,j}X^{(m-1)}_{i,j} \]
   \[ - q^{-m+1}(q - q^{-1})X_{k,j}X^{(m-1)}_{i,j} \]
   \[ + q^{-1}(q - q^{-1})X_{k,j}X^{(m-2)}_{i,j}. \]

8. If \( l < i < k = j \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j} + q^{-m+1}X_{i,l}K_i^{-1}X^{(m-1)}_{i,j}. \]

9. If \( l < i < k < j \), then
   \[ X^{(m)}_{i,j} X_{k,l} = X_{k,l}X^{(m)}_{i,j} - q^{m-1}(q - q^{-1})K_i^{-1}K_j^{-1}X_{k,j}X^{(m-1)}_{i,j}. \]

**Proof.** Clearly, part (1) follows directly from Lemma 6.4(1).

If \( i = l, j = k \), we set
\[
X = X_{i,j}, \quad Y = X_{j,i}, \quad H = -K_jK_i^{-1}, \quad I = -K_j^{-1}K_i, \quad J = qX_{i,j}K_j^{-1}K_i^{-1} = K_j^{-1}X_{i,j}K_i^{-1}.
\]
Then we have $XY = YX + H - I$, $XH = q^2HX - J$, $XI = q^{-2}IX + q^{-2}J$, $XJ = JX$ by (5.6.2) and Lemmas 6.3(1), 6.3(1) and 6.7. Hence part (2) follows from Lemma 7.3.

If $i < l < k < j$, we set

$$X = X_{i,j}, \quad Y = X_{k,l}, \quad H = (q - q^{-1})K_i^{-1}K_k^{-1}X_{k,j}X_{i,l},$$

$$I = q(q - q^{-1})K_i^{-1}K_k^{-1}X_{k,j}X_{i,l}, \quad J = q(q - q^{-1})^2K_i^{-1}K_k^{-1}X_{k,j}X_{i,l}.$$

Then clearly by Lemma 6.4(2), we have $XY = YX + H - I$. By the formula in the fourth case in Lemma 6.3(2) and (6.1.1), one can check $XH = q^2HX - J$. Meanwhile, by the formulas in the sixth case in Lemma 6.4(2) and Lemma 6.1, we see $XI = q^{-2}IX + q^{-2}J$. Finally, by Lemma 6.3(1), 6.1(1) and Lemma 6.1, $XJ = JX$ holds. Therefore part (3) follows from Lemma 7.3.

If $i = l < j < k$, then by the formula in the third case in Lemma 6.4(2), Lemma 6.4(1) and (5.6.2), the following elements $X = X_{i,j}$, $Y = X_{k,l}$, $Z = -K_i^{-1}K_jX_{k,j}$ satisfy $XY = YX + Z$, $XZ = q^2ZX$ and hence part (4) can be verified by a proof similar to (7.1.1) with $q^{-1}$ being replaced by $q$.

If $i < l < j < k$, we set

$$X = X_{i,j}, \quad Y = X_{k,l}, \quad H = (q - q^{-1})K_i^{-1}K_jX_{k,l}, \quad I = -K_i^{-1}K_j^{-1}X_{i,l},$$

$$J = -(q - q^{-1})K_i^{-1}K_j^{-1}X_{i,j}X_{i,l}.$$

Then by the formula in fourth case in Lemma 6.4(2) we see $XY = YX + H - I$. By (6.1.1), (5.6.2) and Lemma 6.7, we get $XH = q^2HX - J$ and by the formula in the sixth case in Lemma 6.3(2) and (5.6.2), we obtain $XI = q^{-2}IX + q^{-2}J$. Finally, by Lemma 6.3(1), 6.6.2 and (6.1.1), one can check that $XJ = JX$ holds. Putting together, we obtain that part (5) follows from Lemma 7.3.

If $i < l < j < k$, then by the formula in the fifth case in Lemma 6.4(2), Lemma 6.4(1), and the equalities (5.6.2) and (6.1.1), the elements $X = X_{i,j}$, $Y = X_{k,l}$, $Z = -(q - q^{-1})K_i^{-1}K_jX_{k,j}$ satisfy $XY = YX + Z$, $XZ = q^2ZX$ and hence part (6) follows from a proof similar to (7.1.1) with $q^{-1}$ being replaced by $q$.

If $l = i < k < j$, we let

$$X = X_{i,j}, \quad Y = X_{k,l}, \quad H = -X_{k,j}K_i^{-1}K_k^{-1}X_{i,l}, \quad I = (q - q^{-1})X_{k,j}K_iK_k^{-1}X_{i,l},$$

$$J = -(q - q^{-1})X_{k,j}X_{i,j}K_i^{-1}K_k^{-1}X_{i,l}.$$

By the formula in the sixth case in Lemma 6.4(2), we have $XY = YX + H - I$. Meanwhile, by the formula in the fourth case in Lemma 6.3(2) and (5.6.2), one can check $XH = q^2HX - J$ and by Lemma 6.1 and Lemma 6.7, we obtain $XI = q^{-2}IX + q^{-2}J$. Finally, by Lemma 6.3(1), Lemma 6.1 and (5.6.2), we see $XJ = JX$. Hence part (7) follows from Lemma 7.3.

If $l < i < k = j$, then by the formula in the seventh case in Lemma 6.4(2), (5.6.2) and Lemma 6.4(1) we find that the elements $X = X_{i,j}$, $Y = X_{k,l}$, $Z = X_{i,l}K_iK_j^{-1}$ satisfy $XY = YX + Z$, $XZ = q^{-2}ZX$. Then by a proof similar to (7.1.1), part (8) holds. Similarly, the last part can be verified by Lemma 6.4(2), Lemma 6.4(1), (5.6.2) and Lemma 6.1.

Finally, it remains to derive the commutation formulas between $X^{(m)}_{i,j}$ and $K_a$.

**Proposition 7.10.** The following holds for $i, j, a \in \{1, 2, \ldots, n\}$ satisfying $i < j$: □
If \( a < i \) or \( a > j \), then
\[
X_{i,j}^{(m)} K_a = K_a X_{i,j}^{(m)}.
\]

2. If \( a = i \), then
\[
X_{i,j}^{(m)} K_i = q^{-m} K_i X_{i,j}^{(m)} - K_i^{-1} X_{i,j} X_{i,j}^{(m-1)}.
\]

3. If \( a = j \), then
\[
X_{i,j}^{(m)} K_j = q^m K_j X_{i,j}^{(m)} + K_j^{-1} X_{i,j} X_{i,j}^{(m-1)}.
\]

4. If \( i < a < j \), then
\[
X_{i,j}^{(m)} K_a = K_a X_{i,j}^{(m)} + q^m (q - q^{-1}) X_{a,j} X_{i,a} K_a^{-1} X_{i,j}^{(m-1)} - q^{-m+2} (q - q^{-1}) X_{a,j} X_{i,a} K_a^{-1} X_{i,j}^{(m-1)} - (q - q^{-1})^2 X_{a,j} X_{i,a} K_a^{-1} X_{i,j}^{(m-2)}.
\]

Proof. As before, the proposition can be verified case-by-case using Lemma 6.1, Lemma 6.3, Corollary 7.2 and Lemma 7.3. We will check the case when \( i < a < j \). In this case, we set
\[
X = X_{i,j}, \quad Y = K_a, \quad H = q(q - q^{-1}) X_{a,j} X_{i,a} K_a^{-1}, \quad I = q(q - q^{-1}) X_{a,j} X_{i,a} K_a^{-1},
\]
\[
J = q(q - q^{-1}) X_{a,j} X_{i,a} K_a^{-1}.
\]

Then by Lemma 6.7, we have \( XY = YX + H - I \). Moreover, by the formula in the fourth case in Lemma 6.3 and 6.1, we have \( XH = q^2HX - J \). Meanwhile, by the formula in the sixth case in Lemma 6.3 and 6.1, one can deduce that \( XI = q^{-2}IX + q^{-2}J \). Finally, by Lemma 6.3 and 6.1, we obtain \( XJ = JX \). Putting together, part (4) is proved by Lemma 7.3. Similarly, the other parts can be verified. \( \square \)

Remark 7.11. As before, by applying the anti-automorphism \( \Omega \) to the formulas in Propositions 7.8, 7.9 and 7.10, we will obtain a complete set of commutation formulas involving an even divided power \( X_{i,j}^{(m)} \) and an odd root vector \( \bar{X}_{k,l} \) or the element \( K_a \) for \( i > j, k \neq l, 1 \leq a \leq n \).

8. Lusztig type form for \( U_q(q(n)) \)

We now present two applications of the commutation formulas. The first application is the existence of an integral form (or Lusztig form) for \( U_q(q(n)) \). Recall from (7.0.1) the notation
\[
\begin{bmatrix}
K_i \\
K_t
\end{bmatrix} = \begin{bmatrix}
K_i; 0 \\
0; t
\end{bmatrix}.
\]

Let \( Z = \mathbb{Z}[q, q^{-1}] \). Define \( U_{q,Z} \) to be the \( Z \)-subsuperalgebra of \( U_q(q(n)) \) generated by
\[
K_{i}^{\pm 1}, \quad \begin{bmatrix}
K_i \\
K_t
\end{bmatrix}, \quad E_{j}^{(m)}, \quad F_{j}^{(m)}, \quad K_i, \quad E_j, \quad F_j \quad (1 \leq i \leq n, 1 \leq j \leq n - 1, t, m \in \mathbb{N}).
\]

By Proposition 7.4 and 3, we obtain
\[
X_{i,j}^{(m)} = X_{i,k}^{(m)} X_{k,j}^{(m)} X_{i,k}^{(m)} - q^{2} X_{k,j}^{(m)} X_{i,k}^{(m)} - \sum_{t=1}^{m-1} q^{(m-t)^2} X_{k,j}^{(m-t)} X_{i,j}^{(t)} X_{i,k}^{(m-t)}
\]
for \( m \geq 1 \) and \( 1 \leq i < j \leq n \) such that \( |j - i| > 1 \), where \( k \) is strictly between \( i \) and \( j \). Hence by induction on \( |j - i| \) and \( m \), we obtain \( X_{i,j}^{(m)} \in U_{q,Z} \) for \( 1 \leq i < j \leq n \) as the classical case (cf. [16, Proposition 2.17]). Then using (5.6.3), one can deduce that
$X_{i,j}^{(m)} \in U_{q,Z}$ for $1 \leq j < i \leq n$. Then we will also consider the set of generators involving $K_{i}^{\pm 1}$, $\left[ K_{i} \right]_{t}$ $(1 \leq i \leq n, t \in \mathbb{N})$ and the set (cf. Remark 3.4):

$$G_q = \{ X_{i,j}^{(m)}, \ X_{i,j}, \ K_{i} \ | \ 1 \leq i \neq j \leq n, m \in \mathbb{N} \} \quad (8.0.1)$$

Let $U_{q,Z}$ be the $Z$-subsuperalgebra of $U_q(q(n))$ generated by $K_{i}^{\pm 1}$, $\left[ K_{i} \right]_{t}$, $K_{i}$ with $1 \leq i \leq n, t \in \mathbb{N}$. Denote by $U_{q,Z}^{+}$ (resp. $U_{q,Z}^{-}$) the $Z$-subsuperalgebra of $U_q(q(n))$ generated by $E_{j}^{(m)}, E_{j}$ (resp. by $F_{j}^{(m)}, F_{j}$) with $1 \leq j \leq n-1, m \in \mathbb{N}$.

**Lemma 8.1.** For $1 \leq i \leq n$, we have

$$K_{i}^{2} = q^{-1}K_{i}\left[ \frac{K_{i}}{1} \right] - q^{-1}(q^{-1})\left[ \frac{K_{i}}{2} \right].$$

**Proof.** By (8.0.1), we have

$$\left[ \frac{K_{i}}{1} \right] = \frac{K_{i} - K_{i}^{-1}}{q - q^{-1}}, \quad \left[ \frac{K_{i}}{2} \right] = \frac{K_{i} - K_{i}^{-1}}{q - q^{-1}}, \quad \frac{K_{i}q^{-1} - K_{i}^{-1}q}{q^{2} - q^{-2}}.$$

Then a direct calculation shows that

$$q^{-1}K_{i}\left[ \frac{K_{i}}{1} \right] - q^{-1}(q^{-1})\left[ \frac{K_{i}}{2} \right] = \frac{K_{i}^{2} - K_{i}^{-2}}{q^{2} - q^{-2}}.$$

Hence the lemma is proved using the relation (QQ1) in Proposition 5.2. \(\square\)

Observe that by Lemma 6.5 and applying the anti-automorphism (5.4.3), we have

$$X_{i,j}^{2} = \frac{q - q^{-1}}{q + q^{-1}}X_{i,j}^{2} = -(q - q^{-1})X_{i,j}^{(2)}; \quad (8.1.1)$$

for $1 \leq i < j \leq n$.

Similarly to (3.2.2), we introduce the following elements: for $A \in M_{n}(\mathbb{N}|Z_{2})$,

$$E_{A^{+}} = \prod_{1 \leq i < j \leq n} \left( X_{i,j}^{(a_{0_{ij}})}, X_{i,j}^{a_{1_{ij}}} \right) \quad \text{and} \quad F_{A^{-}} = \prod_{1 \leq i < j \leq n} \left( X_{j,i}^{(a_{0_{ji}})}, X_{j,i}^{a_{1_{ji}}} \right), \quad (8.1.2)$$

where $A_{0} = (a_{0_{ij}}), A_{1} = (a_{1_{ij}})$, and the order in products is the same as those in (3.2.3) and (3.2.4), respectively.

Recall that for $\sigma \in \mathbb{Z}^{n}$ and $A \in M_{n}(\mathbb{N}|Z_{2})$, we have $K_{\sigma} = K_{1}^{\sigma_{1}} \cdots K_{n}^{\sigma_{n}}$ and $K_{A_{i}}^{a_{1_{ij}}} = K_{1}^{a_{1_{ij}}} \cdots K_{n}^{a_{n_{ij}}}$, where $A_{1} = (a_{1_{ij}})$. For $b = (b_{1}, \ldots, b_{n}) \in \mathbb{N}^{n}$, define

$$\left[ \begin{array}{c} K \\ b \end{array} \right] = \prod_{i=1}^{n} \left[ \begin{array}{c} K_{i} \\ b_{i} \end{array} \right].$$

**Proposition 8.2.**

1. There is a $Z$-linear isomorphism $U_{q,Z} \cong U_{q,Z}^{+} \otimes U_{q,Z}^{-} \otimes U_{q,Z}^{0}$.

2. The following set

$$\left\{ m_{A,\tau}^{q} := F_{A^{-}}\left( \prod_{i=1}^{n} K_{i}^{\tau_{i}} \right) \prod_{i=1}^{n} K_{A_{i}}^{a_{1_{ij}}} E_{A^{+}} \ | \ A \in M_{n}(\mathbb{N}|Z_{2}), \tau = (\tau_{1}, \ldots, \tau_{n}) \in \mathbb{Z}^{n} \right\}$$

is a $Z$-basis for $U_{q,Z}$. 


Proof. By an argument parallel to the proof of Proposition 3.3 (1), one can check that part (1) of the proposition holds.

By Lemma 5.8, the set \( \{ E_A^+ \mid A \in M_n(\mathbb{N}|\mathbb{Z}_2), 0 = A^+ = A^0 \} \) is linearly independent. Meanwhile by Lemma 6.5 (8.1.1), and Propositions 7.4 and 7.8 we obtain that \( U_{q,z}^+ \) is spanned by the set \( \{ E_A^+ \mid A \in M_n(\mathbb{N}|\mathbb{Z}_2), 0 = A^+ = A^0 \} \) using a proof similar to [16, Proposition 1.13]. Hence \( \{ E_A^+ \mid A \in M_n(\mathbb{N}|\mathbb{Z}_2), 0 = A^+ = A^0 \} \) is a \( Z \)-basis for \( U_{q,z}^+ \).

Similarly, one can show that the set \( \{ F_A^- \mid A \in M_n(\mathbb{N}|\mathbb{Z}_2), 0 = A^- = A^0 \} \) is a \( Z \)-basis for \( U_{q,z}^- \). Finally by (5.8.2) the subalgebra \( U_{q,z}^0(\mathfrak{q}(n)_0) \) of \( U_{q,z}^0 \) generated by \( K_i^\pm, \left[ \begin{array}{c} K_i \\ t \end{array} \right] \) with \( 1 \leq i \leq n \) and \( t \in \mathbb{N} \) can be identified with the corresponding algebra \( U_{q,z}^0(\mathfrak{g}(n)) \) associated to \( \mathfrak{g}(n) \). Then it follows from the classical case that \( U_{q,z}^0(\mathfrak{q}(n)_0) \) has a \( Z \)-basis given by \( \{ (\prod_{i=1}^n K_i^{	au_i}) \left[ \begin{array}{c} K \\ b \end{array} \right] \mid b \in \mathbb{N}^n, \tau_i \in \mathbb{Z}_2, 1 \leq i \leq n \} \). This together with Lemma 8.1 and the relation (QWI) in Proposition 5.2 we obtain that \( U_{q,z}^0 \) is spanned by the set \( \{ (\prod_{i=1}^n K_i^{	au_i}) \left[ \begin{array}{c} K \\ b \end{array} \right] \left| D \right. \mid b \in \mathbb{N}^n, D \in \mathbb{Z}_2^n, \tau_i \in \mathbb{Z}_2, 1 \leq i \leq n \} \). Again by Lemma 5.8, this set is linearly independent and hence it is a \( Z \)-basis for \( U_{q,z}^0 \). Putting all together, part (2) follows from part (1). \( \square \)

Remark 8.3. Observe the commutation formula in Lemma 6.6 (1). If we use a degree function similarly defined as in (3.3.3), then \( [E_i, F_i] \) is a linear combination of monomials of degree not strictly less. In order to obtain the analog of Lemma 6.5 in quantum case, we introduce the following twisted degree function:

\[
\deg'(X_{i,j}^{(s)}) = 2s|j - i|, \quad \deg'(\overline{X}_{i,j}) = 2|j - i|, \quad \deg'(K_i) = 1, \quad \text{and} \quad \deg'(K_i) = 0 \quad (8.3.1)
\]

for \( 1 \leq i \neq j \leq n, s \in \mathbb{N} \). Then we have

\[
\deg'(m_{A,\tau}^s) = \deg'(A) := (a_{11}^s + \cdots + a_{nn}^s) + \sum_{i \neq j} 2(a_{ij} + a_{ji})|j - i|, \quad (8.3.2)
\]

where \( \tau \in \mathbb{Z}_2^n \) and \( A = (A_0, A_1) \in M_n(\mathbb{N}|\mathbb{Z}_2) \) with \( A_0 = (a_{0 ij}), A_1 = (a_{1 ij}) \) and \( a_{ij} = a_{ij}^0 + a_{ij}^1 \). Let \( \mathfrak{M}_q \subseteq U_q^0 \) be the set of monomial \( m_q \) in \( X_{i,j}^{(s)}, \overline{X}_{i,j}, K_i, K_i^{\pm 1}, \left[ \begin{array}{c} K_i \\ s \end{array} \right] \) for \( 1 \leq i \neq j \leq n, s \in \mathbb{N} \). Then similar to non-quantum case, the degree of a non-zero monomial \( m_q \in \mathfrak{M}_q \) is defined accordingly. With \( \deg' \), we see that commuting the product of two generators \( x, y \in \mathcal{G}_q \) from different triangular parts yields a linear combination of certain monomials with smaller twisted degree. See Lemma 6.6 Propositions 7.6 and 7.9 for \( x, y \) in different \( \pm \)-part, noting the following holds for \( i < j, k > l \) satisfying \( i \leq l < j \) or \( l \leq i < k \):

\[
2 < 2|j - i| + 2|k - l|, \quad \text{if} \quad l = i, k = j,
\]

\[
2|k - j| + 2|l - i| + 1 < 2|j - i| + 2|k - l|, \quad \text{if} \quad l \neq i \text{ or } k \neq j,
\]

and see Propositions 6.7 and 7.10 for \( x, y \) in different 0-part and \( \pm \)-part.

The next application is a spanning set of a certain quotient superalgebra of \( U_{q,z}^0 \). Similar to the non-quantum situation, let \( I_q \) be the ideal of the quantum superalgebra \( U_q(\mathfrak{q}(n)) \) given by

\[
I_q = \langle K_1 \cdots K_n - q^r, (K_i - 1)(K_i - q) \cdots (K_i - q^r), K_i(K_i - q) \cdots (K_i - q^r) \forall i \rangle. \quad (8.3.3)
\]
Define
\[ U_q(n, r) = U_q/I_q, \quad U_q(n, r)z = U_qz/I_q \cap U_qz, \]
\[ U_q^0(n, r) = U_q^0/I_q \cap U_q^0, \quad U_q^0(n, r)z = U_q^0z/I_q \cap U_q^0z. \]

Clearly, \( U_q(n, r) \) is a homomorphic image of \( U_q(q(n)) \).

As before, we also denote by abuse of notation the image of \( K_i, E_j, F_j \) etc. in \( U_q(n, r) \) by the same letters. We will write, for \( \lambda \in \Lambda(n, r) \),
\[ 1_\lambda = \begin{bmatrix} K \\ \lambda \end{bmatrix} \in U_q^0(n, r)z. \]

**Proposition 8.4.** The following holds in \( U_q^0(n, r)z \):

(1) The set \( \{1_\lambda \mid \lambda \in \Lambda(n, r)\} \) is a set of pairwise orthogonal central idempotents in \( U_q^0(n, r)z \) satisfying \( 1 = \sum_{\lambda \in \Lambda(n, r)} 1_\lambda \).

(2) \[ \begin{bmatrix} K \\ b \end{bmatrix} = 0 \text{ for } b \in \mathbb{N}^n \text{ with } |b| > r. \]

(3) For \( 1 \leq i \leq n, \lambda \in \Lambda(n, r) \) and \( b \in \mathbb{N}^n \), we have \( K_i^{\pm} 1_\lambda = q^{\pm \lambda_i} 1_\lambda \),
\[ \begin{bmatrix} K_i; c \\ t \end{bmatrix} 1_\lambda = \begin{bmatrix} \lambda_i + c \\ t \end{bmatrix} 1_\lambda, \quad \begin{bmatrix} K \\ b \end{bmatrix} 1_\lambda = \begin{bmatrix} \lambda \\ b \end{bmatrix} 1_\lambda, \quad \text{and} \quad \begin{bmatrix} K \\ b \end{bmatrix} = \sum_{\lambda \in \Lambda(n, r)} \begin{bmatrix} \lambda \\ b \end{bmatrix} 1_\lambda, \]

where \( \begin{bmatrix} \lambda \\ b \end{bmatrix} = \prod_{i=1}^{n} \begin{bmatrix} \lambda_i \\ b_i \end{bmatrix} \).

(4) Suppose \( 1 \leq i \leq n \) and \( \lambda \in \Lambda(n, r) \). If \( \lambda_i = 0 \), then \( K_i 1_\lambda = 0 \).

(5) The \( \mathbb{Z} \)-algebra \( U_q^0(n, r)z \) is spanned by
\[ \{1_\lambda K_D \mid \lambda \in \Lambda(n, r), D \in \mathbb{Z}_2^n, D_i \leq \lambda_i, 1 \leq i \leq n\}. \]

**Proof.** By the identification (8.3.2), since the elements \( K_1 K_2 \cdots K_n - q^r \) and \((K_i - 1)(K_i - q) \cdots (K_i - q^r)\), \( 1 \leq i \leq n \) are contained in the ideal \( I_q \), we observe that there is a natural algebra homomorphism from the algebra \( T^0 \) introduced in [7] (8.1) to \( U_q^0(n, r)z \), which sends the generator \( K_i \) in [7] to our generator \( K_i \) for \( 1 \leq i \leq n \). Hence parts (1)-(3) follow directly from [7] Propositions 8.2-8.3.

To prove part (4), firstly we observe that the following holds
\[ (K_i - q) \cdots (K_i - q^r)1_\lambda = (q^{\lambda_i} - q) \cdots (q^{\lambda_i} - q^r)1_\lambda \]
for \( 1 \leq i \leq n \) by part (3) of the proposition. Now assume \( \lambda_i = 0 \). Then we have
\[ (K_i - q) \cdots (K_i - q^r)1_\lambda = (1 - q) \cdots (1 - q^r)1_\lambda. \]

Hence
\[ (1 - q) \cdots (1 - q^r)K_i 1_\lambda = K_i (K_i - q) \cdots (K_i - q^r)1_\lambda = 0, \]
which is due to the fact \( K_i (K_i - q) \cdots (K_i - q^r) \in I_q \cap U_q^0z \). Therefore \( K_i 1_\lambda = 0 \).

Finally, by part (3), the relation (QQ1) and Lemma 8.1, we know the algebra \( U_q^0(n, r)z \) is spanned by the set \( \{K_D 1_\lambda = 1_\lambda K_D \mid \lambda \in \Lambda(n, r), D \in \mathbb{Z}_2^n\} \). By part (4), we have \( K_D 1_\lambda = 0 \) if there exists \( 1 \leq i \leq n \) such that \( D_i = 1 \) and \( \lambda_i = 0 \). Hence part (5) holds. \( \square \)
As observed in Remark 8.3, commuting two elements

\[ X_\alpha 1_\lambda = \begin{cases} 1_{\lambda + \alpha} X_\alpha, & \text{if } \lambda + \alpha \in \Lambda(n, r), \\ 0, & \text{otherwise}, \end{cases} \]

\[ 1_\lambda X_\alpha = \begin{cases} X_\alpha 1_{\lambda - \alpha}, & \text{if } \lambda - \alpha \in \Lambda(n, r), \\ 0, & \text{otherwise}, \end{cases} \]

Proof. Then similar to the non-quantum case, the following holds:

\[ \mathcal{B}_q = \{ u_{(C, \lambda)}^q \mid C \in M_n(\mathbb{N}|\mathbb{Z}_2)', \lambda \in \Lambda(n, r), \chi(C) \leq \lambda \}. \]

Then similar to the non-quantum case, the following holds:

\[ \mathcal{B}_q = \{ u_A^q := F_A^{-1} 1_{\chi(A)} K_{A'} E_{A'} \mid A \in M_n(\mathbb{N}|\mathbb{Z}_2)_r \}. \]

**Proposition 8.5.** Suppose \( 1 \leq i \leq n, \alpha \in \Phi \) and \( \lambda \in \Lambda(n, r) \). Then we have in \( U_q(n, r)_Z \):

\[ X_\alpha 1_\lambda = \begin{cases} 1_{\lambda + \alpha} X_\alpha, & \text{if } \lambda + \alpha \in \Lambda(n, r), \\ 0, & \text{otherwise}, \end{cases} \]

\[ 1_\lambda X_\alpha = \begin{cases} X_\alpha 1_{\lambda - \alpha}, & \text{if } \lambda - \alpha \in \Lambda(n, r), \\ 0, & \text{otherwise}, \end{cases} \]

\[ K_i 1_\lambda = 1_\lambda K_i. \]

Proof. The last equality follows from the relation (QQ1). The proof of the remaining formulas is parallel to that of Proposition 4.4. We skip the detail here.

As in the non-quantum case, set

\[ u_C^q = F_C^{-1} K_{C'} E_{C'} \in U_q.Z \] and \( u_{(C, \lambda)}^q = F_C^{-1} 1_{\lambda} K_{C'} E_{C'} \in U_q(n, r)_Z \)

for \( C \in \mathcal{M}_n(\mathbb{N}|\mathbb{Z}_2)' \) and \( \lambda \in \Lambda(n, r) \). Then, by (8.3.2) we have \( \deg'(u_C^q) = \deg(C) \). By Proposition 8.5, we have in \( U_q(n, r)_Z \)

\[ u_{(C, \lambda)}^q = 1_\lambda u_C^q = u_C^q 1_{\lambda'} \text{ if } u_{(C, \lambda)}^q \neq 0, \]

(8.5.1)

where \( \lambda' = \lambda + \text{ro}(C^-) - \text{co}(C^-) \) and \( \lambda'' = \lambda - \text{ro}(C^+) + \text{co}(C^+) \).

**Lemma 8.6.** (1) Let \( 0 \neq m_q \in \mathcal{M}_q \). Then, in \( U_q(n, r)_Z \), \( m_q \) can be written as a linear combination of \( u_{(C, \lambda)}^q \) for \( C \in \mathcal{M}_n(\mathbb{N}|\mathbb{Z}_2)', \lambda \in \Lambda(n, r) \) such that \( \deg'(C) \leq \deg'(m_q) \).

(2) Suppose \( C \in \mathcal{M}_n(\mathbb{N}|\mathbb{Z}_2)' \). Then we have in \( U_q(n, r)_Z \)

\[ F_C^{-1} K_{C'} E_{C'} = \pm E_{C'} K_{C'} F_C^{-1} + \sum \eta_{(G, \lambda)}^C u_{(G, \lambda)}^q, \]

for some \( \eta_{(G, \lambda)}^C \in \mathbb{Z} \), the summation is over \( G \in \mathcal{M}_n(\mathbb{N}|\mathbb{Z}_2)' \) and \( \lambda \in \Lambda(n, r) \) such that \( \deg'(G) < \deg'(C) \).

Proof. As observed in Remark 8.3, commuting two elements \( x, y \in \mathcal{G}_q \) from different triangular parts strictly decreases the twisted degree. Thus, with Proposition 8.2 at hand, the arguments in the proof of Lemma 4.5 are applicable here. We leave the detail to the reader.

Set

\[ \mathcal{B}_q = \{ u_{(C, \lambda)}^q \mid C \in \mathcal{M}_n(\mathbb{N}|\mathbb{Z}_2)', \lambda \in \Lambda(n, r), \chi(C) \leq \lambda \}. \]

Then similar to the non-quantum case, the following holds:

\[ \mathcal{B}_q = \{ u_A^q := F_A^{-1} 1_{\chi(A)} K_{A'} E_{A'} \mid A \in \mathcal{M}_n(\mathbb{N}|\mathbb{Z}_2)_r \}. \]

**Proposition 8.7.** The algebra \( U_q(n, r)_Z \) is spanned by the set \( \mathcal{B}_q \).

Proof. With Lemma 8.6 at hand, it is easy to see the arguments in the proof of Proposition 4.6 are also applicable in the quantum case. We leave the detail to the reader.
9. A presentation for the quantum queer Schur superalgebras $Q_q(n, r)$

By base change to the superspace $V$ of §2, we now assume that $V$ is a superspace over the field $\mathbb{Q}(q)$. Thus, $V$ has basis $\{v_1, \ldots, v_n, v_{-1}, \ldots, v_{-n}\}$. Following [18], we set

$$\Theta = \sum_{1 \leq a \leq n} (E_{-a,a} - E_{a,-a}),$$

$$T = \sum_{i, j \in I(n|n)} \text{sgn}(j) E_{i,j} \otimes E_{j,i},$$

$$S = \sum_{i \leq j \in I(n|n)} S_{i,j} \otimes E_{i,j} \in \text{End}_{\mathbb{Q}(q)}(V^{\otimes 2}),$$

where $S_{i,j}$ for $i \leq j$ are defined as follows:

$$S_{a,a} = \sum_{b=1}^{n} q^{\delta_{ab}}(E_{b,b} + E_{-b,-b}) = 1 + (q - 1)(E_{a,a} + E_{-a,-a}), \quad 1 \leq a \leq n,$$

$$S_{-a,-a} = \sum_{b=1}^{n} q^{-\delta_{ab}}(E_{b,b} + E_{-b,-b}) = 1 + (q^{-1} - 1)(E_{a,a} + E_{-a,-a}), \quad 1 \leq a \leq n,$$

$$S_{b,a} = (q - q^{-1})(E_{a,b} + E_{-a,-b}), \quad 1 \leq b < a \leq n,$$

$$S_{-b,-a} = -(q - q^{-1})(E_{a,b} + E_{-a,-b}), \quad 1 \leq a < b \leq n,$$

$$S_{-b,a} = -(q - q^{-1})(E_{-a,b} + E_{a,-b}), \quad 1 \leq a, b \leq n.$$

To endomorphisms $Y \in \text{End}_{\mathbb{Q}(q)}(V)$ and $Z = \sum_{t} H_t \otimes I_t \in \text{End}_{\mathbb{Q}(q)}(V)^{\otimes 2} = \text{End}_{\mathbb{Q}(q)}(V^{\otimes 2})$, we associate the following elements in $\text{End}_{\mathbb{Q}(q)}(V^{\otimes r})$:

$$Y_{(k)} = \text{id}^{\otimes k-1} \otimes Y \otimes \text{id}^{\otimes r-k}, \quad 1 \leq k \leq r,$$

$$Z_{(j,k)} = \sum_{t} (H_t)_{(j)}(I_t)_{(k)}, \quad 1 \leq j \neq k \leq r.$$

Define $\Phi_1 : U_q(\mathfrak{q}(n)) \to \text{End}_{\mathbb{Q}(q)}(V)$ by

$$\Phi_1(L_{i,j}) = S_{i,j}, \quad \text{for } i, j \in I(n|n), i \leq j. \quad (9.0.3)$$

It is known [18, §4] that $\Phi_1$ is an algebra homomorphism and hence defines a representation $(\Phi_1, V)$ of $U_q(\mathfrak{q}(n))$. Then using the comultiplication (5.4.1), we obtain a representation $(\Phi_r, V^{\otimes r})$, which can be viewed as a deformation of the representation $(\phi_r, V^{\otimes r})$ of $U(\mathfrak{q}(n))$. Define the quantum queer Schur superalgebra $Q_q(n, r)$ (or $q$-Schur superalgebra of queer type) by

$$Q_q(n, r) = \Phi_r(U_q(\mathfrak{q}(n))), \quad (9.0.4)$$

that is, the image of the homomorphism $\Phi_r$. As before, the superalgebra $Q_q(n, r)$ can be viewed as a quotient of $U_q(\mathfrak{q}(n))$.

Similar to the classical case, there exists another explanation of the superalgebra $Q_q(n, r)$ via an analog of Jimbo-Schur duality for $U_q(\mathfrak{q}(n))$ as follows. The Hecke-Clifford algebra $H^c_r$ is the associative superalgebra over $\mathbb{Q}(q)$ with even generators $T_1, \ldots, T_{r-1}$
and odd generators $c_1, \ldots, c_r$ subject to (2.2.3) and the following relations:

\[
(T_i - q)(T_i + q^{-1}) = 0, \quad T_i c_j = c_j T_i \quad (j \neq i, i + 1),
\]

\[
T_i T_{i'} = T_{i'} T_i \quad (|i - i'| > 1), \quad T_i c_i = c_{i+1} T_i,
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i c_{i+1} = c_i T_i - (q - q^{-1})(c_i - c_{i+1}),
\]

for all $1 \leq i, i' \leq r - 1$ and $1 \leq j \leq r$.

Let $\delta = TS \in \text{End}_{\mathbb{Q}(q)}(V_{\otimes 2})$. Then $\delta S_{j(j+1)} \in \text{End}_{\mathbb{Q}(q)}(V_{\otimes r})$. It follows from theorems 5.2-5.3 that there exists a representation $(\Psi_r, V_{\otimes r})$ of the Hecke-Clifford algebra $\mathcal{H}_r$ defined by

\[
\Psi_r(T_i) = \bar{S}_{j(j+1)} \quad \text{and} \quad \Psi_r(c_k) = \Theta(k),
\]

where $1 \leq j \leq r - 1$ and $1 \leq k \leq r$. Moreover, $\Phi_r(U_q(q(n))) = \text{End}_{\mathcal{H}_r}(V_{\otimes r})$ and hence, by (9.0.3), the following holds

\[
Q_q(n, r) = \text{End}_{\mathcal{H}_r}(V_{\otimes r}).
\]

Let $V_Z$ (resp. $V_{\bar{Z}}$) be the free $\mathbb{Z}$-supermodule (resp. $\bar{\mathbb{Z}}$-supermodule) with basis $v_1, \ldots, v_n, v_{-1}, \ldots, v_{-n}$. Let $\mathcal{H}_{r,Z}$ (resp. $\mathcal{H}_{r,\bar{Z}}$) be the corresponding Hecke-Clifford algebra (resp. Sergeev superalgebra) defined over $\mathbb{Z}$ (resp. $\bar{\mathbb{Z}}$), and let

\[
Q_q(n, r) = \text{End}_{\mathcal{H}_{r,Z}}(V_{\otimes r}).
\]

Then the specialization of $\mathcal{H}_{r,Z}$ at $q = 1$ coincides the Sergeev superalgebra $\mathcal{H}_{r,Z}$. Clearly, $Q(q) \otimes \mathbb{Z} \text{End}_{\mathcal{H}_{r,Z}}(V_{\otimes r}) \subseteq Q_q(n, r)$. Meanwhile by [21 Theorem 4.5] we have $Q(n, r) = Q \otimes \mathbb{Z} \text{End}_{\mathcal{H}_{r,Z}}(V_{\otimes r})$. As $\text{End}_{\mathcal{H}_{r,Z}}(V_{\otimes r})$ specializes to $\text{End}_{\mathcal{H}_{r,Z}}(V_{\otimes r})$ at $q = 1$, we have

\[
\dim Q(q) \otimes \mathbb{Z} \text{End}_{\mathcal{H}_{r,Z}}(V_{\otimes r}) \geq \dim Q \otimes \mathbb{Z} \text{End}_{\mathcal{H}_{r,Z}}(V_{\otimes r})
\]

and hence

\[
\dim Q_q(n, r) \geq \dim Q(n, r).
\]

**Lemma 9.1.**

1. Let $a, j \in I(n|n)$ and $1 \leq b \leq n$. If $-b < a$ and $|j| \neq b$, then

\[
(\Phi_1(L_{-b,a}))(v_j) = 0.
\]

2. Suppose $j \in I(n|n)^r$ and $\text{wt}(j) = \mu$. We have, for $1 \leq i \leq n$,

   (a) $\Phi_r(K_i)(v_j) = q^{\mu_i}v_j$;

   (b) $\Phi_r(K_i)(v_j) = 0$ if $\mu_i = 0$.

**Proof.** Part (1) follows from definitions (9.0.2), (9.0.3) and (2.2.1), since $|j| \neq b$ implies

\[
(\Phi_1(L_{-b,a}))(v_j) = -(q - q^{-1})(E_{a,-b} + E_{-a,b})(v_j) = 0.
\]

Similarly, by definitions (9.0.3), (9.0.2), (2.2.1), and (5.1.2), we obtain

\[
\Phi_1(K_i)(v_j) = \Phi_1(L_{i,i})(v_j) = (1 + (q - 1)(E_{i,i} + E_{-i,-i}))(v_j) = q^{\delta_{i,j}}v_j
\]

for $1 \leq i \leq n$ and $j \in I(n|n)$. Hence, the comultiplication (5.4.2) on $U_q(q(n))$ (and noting $K_i = L_{i,i}$) gives

\[
\Phi_r(K_i)(v_j^\lambda) = (\Phi_1(K_i) \otimes \Phi_1(K_i) \otimes \cdots \otimes \Phi_1(K_i))(v_j)
\]

\[
= (\prod_{k=1}^{r} q^{\delta_{i,j+k}}) v_j = q^{\delta_{i,j}}v_j
\]

(see (4.0.1)),

proving part (2a).
Next, suppose $1 \leq b \leq n$ and $\mu_b = 0$. Then, by (4.0.1), $|j_k| \neq b$ for each $1 \leq k \leq r$. Suppose $\Phi_r(K_i)(v_j) \neq 0$. Then $\Phi_r(L_{-b,b})(v_j) = -(q - q^{-1})\Phi_r(K_b)(v_j) \neq 0$ by (5.1.2). On the other hand, by the comultiplication (5.4.1) and the definition of $\Phi_r$,

$$\Phi_r(L_{-b,b})(v_j) = \left( \sum_{k_1 = -b}^{b} \sum_{k_2 = k_1}^{b} \cdots \sum_{k_{r-1} = k_{r-2}}^{b} \Phi_1(L_{-b,k_1}) \otimes \Phi_1(L_{k_1,k_2}) \otimes \cdots \otimes \Phi_1(L_{k_{r-2},k_{r-1}}) \otimes \Phi_1(L_{k_{r-1},b}) \right)(v_j).$$

Thus, there exist $-b \leq k_1 \leq k_2 \leq \cdots \leq k_{r-1} \leq b$ such that

$$\Phi_1(L_{-b,k_1})(v_j) \neq 0, \quad \Phi_1(L_{k_1,k_2})(v_j) \neq 0, \quad \cdots, \quad \Phi_1(L_{k_{r-1},b})(v_j) \neq 0. \quad (9.1.1)$$

Since $|j_k| \neq b$, repeatedly applying part (1) to the first, second, ... and second last inequality forces $-b = k_1$, $-b = k_2$, ... and $-b = k_{r-1}$. Then by part (1) again, $\Phi_1(L_{k_{r-1},b})(v_j) = \Phi_1(L_{-b,b})(v_j) = 0$ (since $|j_r| \neq b$), which is in contradiction to the last inequality in (9.1.1). Hence, part (2) is verified.

We are now ready to establish the quantum version of Theorem 4.7. Recall the ideal $I_q$ defined in (8.3.3).

**Theorem 9.2.** The homomorphism $\Phi_r : U_q(\mathfrak{q}(n)) \to \text{End}_{\mathbb{Q}(q)}(V^{\otimes r})$ satisfies $I_q \subseteq \ker \Phi_r$ and induces an algebra isomorphism

$$\overline{\Phi}_r : U_q(n,r) \xrightarrow{\sim} Q_q(n,r). \quad (9.2.1)$$

In particular, the Schur superalgebra $Q_q(n,r)$ is the associative superalgebra generated by even generators $K_i^{\pm 1}, E_j, F_j$, and odd generators $K_i, E_j, F_j$, with $1 \leq i \leq n$ and $1 \leq j \leq n - 1$ subject to the relations (QQ1)-(QQ6) together with the following extra relations:

(QQ7) $K_1 \cdots K_n = q^r$;

(QQ8) $(K_i - 1)(K_i - q) \cdots (K_i - q^r) = 0$, where $1 \leq i \leq n$;

(QQ9) $K_i(K_i - q) \cdots (K_i - q^r) = 0$, where $1 \leq i \leq n$.

**Proof.** Fix an arbitrary $j \in I(n|n)$. Suppose $\text{wt}(j) = \mu \in \Lambda(n,r)$. Then by Lemma 9.1(1) we have

$$\Phi_r(K_1 \cdots K_n)(v_j) = q^{\mu_1 + \cdots + \mu_n}v_j = q^rv_j,$$

$$\Phi_r((K_i - 1)(K_i - q) \cdots (K_i - q^r))(v_j) = (q^{\mu_i} - 1)(q^{\mu_i} - q) \cdots (q^{\mu_i} - q^r)v_j = 0$$

since $0 \leq \mu_i \leq r$ for $1 \leq i \leq n$ and $\mu_1 + \cdots + \mu_n = r$. This means

$$\Phi_r(K_1 \cdots K_n - q^r) = 0, \quad \Phi_r((K_i - 1)(K_i - q) \cdots (K_i - q^r)) = 0 \quad (9.2.2)$$

for $1 \leq i \leq n$. Meanwhile still by Lemma 9.1(1) we obtain

$$\Phi_r(K_i(K_i - q) \cdots (K_i - q^r))(v_j) = (q^{\mu_i} - q) \cdots (q^{\mu_i} - q^r)\Phi_r(K_i)(v_j). \quad (9.2.3)$$

Similar to non-quantum case, if $\mu_i = 0$, then $\Phi_r(K_i)(v_j) = 0$ due to Lemma 9.1(2). Otherwise we have $1 \leq \mu_i \leq r$. Then $(q^{\mu_i} - q) \cdots (q^{\mu_i} - q^r) = 0$. Putting together, by (9.2.3) we obtain $\Phi_r(K_i(K_i - q) \cdots (K_i - q^r))(v_j) = 0$. Therefore, we have proved

$$\Phi_r(K_i(K_i - q) \cdots (K_i - q^r)) = 0. \quad (9.2.4)$$

In Theorem 4.7, the counterpart of (QQ8) is the relation $h_i(h_i - 1) \cdots (h_i - r) = 0$. This was not displayed, as it can easily be derived from (QS8).
In summary, by (8.3.3), (9.2.2) and (9.2.4), the ideal \(I_q\) is contained in the kernel of the homomorphism \(\Phi_r\). Hence, \(\Phi_r\) induces a surjective homomorphism
\[
\overline{\Phi}_r : U_q(n, r) \twoheadrightarrow \mathcal{Q}_q(n, r).
\] (9.2.5)

Now a dimensional comparison gives the required isomorphism since, by (9.0.6) and Proposition 8.7,
\[
\dim U_q(n, r) \leq |\mathcal{B}_q| = |\mathcal{B}| = \dim \mathcal{Q}(n, r) \leq \dim \mathcal{Q}_q(n, r) \leq \dim U_q(n, r),
\]
The last assertion follows from the definition of \(U_q(n, r)\) in (8.3.4).

Propositions 8.7 and 8.4(5) can now be strengthened as follows.

**Corollary 9.3.** The set \(\mathcal{B}_q\) defined in (8.6.1) forms a \(\mathcal{Z}\)-basis for \(U_q(n, r)\)\(\mathcal{Z}\). In particular, the set \(\{1_{\lambda}D^i | \lambda \in \Lambda(n, r), D \in \mathbb{Z}_2^n, D_i \leq \lambda_i, 1 \leq i \leq n\}\) is a \(\mathcal{Z}\)-basis for \(U_q^0(n, r)\). Moreover, \(\dim_{\mathcal{Q}(n)} \mathcal{Q}_q(n, r) = \dim_{\mathcal{Q}} \mathcal{Q}(n, r)\) and \(\dim_{\mathcal{Q}(q)} \mathcal{Q}_q(n, r) = \dim_{\mathcal{Q}} \mathcal{Q}(n, r)\) are given as in Corollary 4.9.

**Remark 9.4.** Since \(\Phi_r(U_q) \subseteq \mathcal{Q}_q(n, r)\) by (9.0.4) and (9.0.5), the restriction of \(\overline{\Phi}_r\) in (9.2.5) induces a superalgebra monomorphism \(\overline{\Phi}_{r, \mathcal{Z}} : U_q(n, r)\)\(\mathcal{Z}\) \(\rightarrow \mathcal{Q}_q(n, r)\)\(\mathcal{Z}\). It is natural to conjecture that \(\overline{\Phi}_{r, \mathcal{Z}}\) is surjective.

Similar to the non-quantum case, by an argument parallel to the proof of [7 Theorem 3.4] we have the following.

**Theorem 9.5.** The quantum queer Schur superalgebra \(\mathcal{Q}_q(n, r)\) is the unitary associative superalgebra generated by the even elements \(1_{\lambda}, E_j, F_j\) and odd elements \(K_i, E_j, F_j\) for \(\lambda \in \Lambda(n, r), 1 \leq i \leq n, 1 \leq j \leq n - 1\) subject to the relations:

(QQ1') \(1_{\lambda}1_\mu = \delta_{\lambda, \mu}1_\lambda, \sum_{\lambda \in \Lambda(n, r)} 1_\lambda = 1, K_i1_\lambda = 1_\lambda K_i,\)

\[K_iK_j + K_jK_i = \delta_{ij}\sum_{\lambda \in \Lambda(n, r)} \frac{2(q^{2\lambda_i} - q^{-2\lambda_j})}{q^2 - q^{-2}}1_\lambda,\]

\[K_i1_\lambda = 0 \text{ if } \lambda_i = 0;\]

(QQ2') \(E_j1_\lambda = \begin{cases} 1_{\lambda+\alpha_j}E_j, & \text{if } \lambda + \alpha_j \in \Lambda(n, r), \\ 0, & \text{otherwise}, \end{cases}\)

\[F_j1_\lambda = \begin{cases} 1_{\lambda-\alpha_j}F_j, & \text{if } \lambda - \alpha_j \in \Lambda(n, r), \\ 0, & \text{otherwise}, \end{cases}\]

\[1_\lambda E_j = \begin{cases} E_j1_{\lambda-\alpha_j}, & \text{if } \lambda - \alpha_j \in \Lambda(n, r), \\ 0, & \text{otherwise}, \end{cases}\]

\[1_\lambda F_j = \begin{cases} F_j1_{\lambda+\alpha_j}, & \text{if } \lambda + \alpha_j \in \Lambda(n, r), \\ 0, & \text{otherwise}; \end{cases}\]

(QQ3') \(K_iE_i - qE_iK_i = \sum_{\lambda \in \Lambda(n, r)} E_iq^{-\lambda_1}1_\lambda, \quad qK_iE_{i-1}E_{i-1}K_i = -\sum_{\lambda \in \Lambda(n, r)} q^{-\lambda_1}1_\lambda E_{i-1},\)

\[K_iF_i - qF_iK_i = -\sum_{\lambda \in \Lambda(n, r)} q^{\lambda_1}F_i1_\lambda, \quad qK_iF_{i-1}E_{i-1}K_i = \sum_{\lambda \in \Lambda(n, r)} q^{\lambda_1}1_\lambda F_{i-1},\]

\[K_iE_i + qE_iK_i = \sum_{\lambda \in \Lambda(n, r)} q^{\lambda_1}E_i1_\lambda, \quad qK_iE_{i+1}E_{i-1}K_i = \sum_{\lambda \in \Lambda(n, r)} q^{-\lambda_1}1_\lambda E_{i+1},\]

\[K_iF_i + qF_iK_i = \sum_{\lambda \in \Lambda(n, r)} q^{\lambda_1}F_i1_\lambda, \quad qK_iF_{i+1}F_{i-1}K_i = \sum_{\lambda \in \Lambda(n, r)} q^{\lambda_1}1_\lambda F_{i+1},\]

\[K_iE_j - E_jK_i = K_iF_j - F_jK_i = K_iE_j + E_jK_i = K_iF_j + F_jK_i = 0 \text{ for } j \neq i, i - 1;\]

(QQ4') \(E_iF_j - F_jE_i = \delta_{ij}\sum_{\lambda \in \Lambda(n, r)} [\lambda_i - \lambda_{i+1}]1_\lambda,\)

\[E_iF_j + F_jE_i = \delta_{ij}\sum_{\lambda \in \Lambda(n, r)} [\lambda_i + \lambda_{i+1}]1_\lambda + \delta_{ij}(q - q^{-1})K_iK_{i+1},\]

\[E_iF_j - F_jE_i = \delta_{ij}\sum_{\lambda \in \Lambda(n, r)} (q^{-\lambda_i+1}K_i - q^{-\lambda_i}K_{i+1})1_\lambda,\]
\[ E_iF_j - F_jE_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,r)} (q^{\lambda+1}K_i - q^\lambda K_{i+1})1_{\lambda}, \]

and the relations (QQ5)-(QQ6).

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(DU) SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, UNSW SYDNEY 2052, AUSTRALIA.
E-mail address: j.du@unsw.edu.au

(WAN) DEPARTMENT OF MATHEMATICS, BEIJING INSTITUTE OF TECHNOLOGY, BEIJING, 100081, P.R. CHINA.
E-mail address: wjk302@gmail.com