Optical theorem and indefinite metric in $\lambda \phi^4$ delta-theory

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A class of effective field theory called delta-theory, which improves ultraviolet divergences in quantum field theory, is considered. We focus on a scalar model with a quartic self-interaction term and construct the delta theory by applying the so-called delta prescription. We quantize the theory using field variables that diagonalize the Lagrangian, which include a standard scalar field and a ghost or negative norm state. As well known, the indefinite metric may lead to the loss of unitary of the $S$-matrix. We study the optical theorem and check the validity of the cutting equations for three processes at one-loop order, and found suppressed violations of unitarity in the delta coupling parameter of the order of $\xi^4$.

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I. INTRODUCTION

Initially, the formulation of delta theories was proposed to include new local gauge symmetries in non-abelian gauge theories [1]. The Batalin-Vilkovisky quantization method for gauge theories provided a basis to derive delta theories through a classical constraint of the equations of motion [2, 3]. Some years later, the formalism was applied to the gravitational field to explain the accelerated expansion of the universe [4] and to study some formal aspects of quantum field theories [5]. An appealing property of quantum delta-theories is the possibility to suppress the radiative corrections beyond one-loop order, thereby improving the convergence of the perturbative series. However, the theory produces a ghost or a negative norm state in the Hilbert space that might lead to the loss of unitarity. Despite this, the model has found many applications in gravity, including dark energy [4, 6], dark matter [7], and recently for cosmological fluctuations [8].

The concept of an indefinite metric plays an essential role in quantum field theories [9]. In gauge theories, it reflects the redundant degrees of freedom that later become necessary for covariant quantization and prove the Ward identities. In electrodynamics, one can get rid of the spurious degrees of freedom by following the Gupta-Bleuler formalism. Moreover, and as is well known, higher time derivative theories can lead to an indefinite metric [10].

Lee and Wick studied indefinite metric theories and proved that the $S$-matrix could be defined unitary by restricting the asymptotic space [11]. They postulated that the negative metric fields decay so fast that they never appear as an in or out states in the asymptotic Hilbert space. Cutkosky proposed a covariant formulation based on non-standard analyticity properties of amplitudes with extended cutting rules [12]. Lee-Wick theories have attracted a lot of attention in higher derivative extensions to the standard model since they allow to soften ultraviolet divergences and to solve the hierarchy problem [13]. Several quantum field theory models preserve unitarity under the application of the Lee-Wick formulation [14], even in the presence of Lorentz violation [15, 16]. Recently it has been proposed a modern formulation of Lee-Wick theories based on Wick rotated Euclidean theories [17].

In this work, we focus on a delta theory constructed from a $\lambda \phi^4$ self-interacting scalar model. We quantize the theory using field variables that diagonalize the Lagrangian, identify the ghost and define the physical asymptotic Hilbert space. As a central part of the work, we study the unitarity of the $S$-matrix by employing the techniques of cutting diagrams within the optical theorem.

The organization of the paper is as follows. In Sec. II we construct the scalar delta model. We quantize the theory and test the property that radiative corrections live at one-loop order. In Sec. III we diagonalize the Lagrangian and find the propagators for the standard particle and negative norm state. We also prove that the Hamiltonian is stable. Finally, in Sec IV we explore the one-loop unitarity of the model. We show that violations of unitarity are present, however suppressed by the delta coupling parameter to the order of $\xi^4$.

II. BASICS

In this section, we construct the delta theory and analyze its quantum corrections.

A. The scalar $\lambda \phi^4$ delta-theory

Consider the scalar Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda \phi^4}{4!},$$

with mass $m$ and coupling constant $\lambda$. 



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Now, we follow the delta prescription which consist to add to this Lagrangian the effective term
\[
\mathcal{L}'(\phi, \eta) = \mathcal{L}_0 + \xi \left[ \frac{\delta \mathcal{L}_0}{\delta \phi(x)} \right] \eta(x),
\]  
where the field \( \eta(x) \) is a new degree of freedom which has been coined the delta field \([11]\), and \( \xi \) is a small parameter that may be seen to arise from a more fundamental theory.

The delta-Lagrangian is
\[
\mathcal{L}'(\phi, \eta) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda \phi^4}{4!} + \xi \partial_\mu \phi \partial^\mu \eta
- \xi m^2 \phi \eta - \xi \frac{\lambda}{3!} \phi^3 \eta.
\]  
where we have employed
\[
\frac{\delta \mathcal{L}_0}{\delta \phi(x)} = -\partial^\mu \partial_\mu \phi - m^2 \phi - \frac{\lambda \phi^3}{3!}.
\]  
The free equations of motion for the fields are
\[
\Box \phi + m^2 \phi = 0, \quad \Box \eta + m^2 \eta = 0.
\]  
The solutions correspond to the standard relation \( p^2 = m^2 \) for both fields. This is a notable difference with respect to the gravitational sector. For gravity, due to non-linearities and higher derivatives terms the delta-metric field satisfies a different equation of motion providing new solutions \([11]\).

### B. Fields and propagators

The fields can be expanded as follows
\[
\phi(\vec{x}, t) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( a_{\vec{p}} e^{-i\vec{p} \cdot \vec{x}} + a_{\vec{p}}^\dagger e^{i\vec{p} \cdot \vec{x}} \right) p_0 = E_p,
\]
\[
\eta(\vec{x}, t) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( c_{\vec{p}} e^{-i\vec{p} \cdot \vec{x}} + c_{\vec{p}}^\dagger e^{i\vec{p} \cdot \vec{x}} \right) p_0 = E_p,
\]
with \( E_p = \sqrt{p^2 + m^2} \).

To find the Hamiltonian we follow the canonical formulation. The canonical conjugate momenta associated to \( \phi \) and \( \eta \) are given by
\[
\pi_\phi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} + \xi \dot{\eta},
\]
\[
\pi_\eta(x) = \frac{\partial \mathcal{L}}{\partial \dot{\eta}} = \dot{\xi} \phi.
\]  
The Legendre transformation leads to the Hamiltonian
\[
H' = \int d^3 x \left( \frac{1}{4} \pi_\phi \pi_\eta - \frac{1}{2} \xi \pi_\phi \pi_\eta + \frac{1}{2} \left( \nabla \phi \right)^2 + \frac{1}{2} \left( \nabla \eta \right)^2 + \xi \nabla \phi \cdot \nabla \eta + \frac{1}{2} m^2 \phi^2 + \xi m^2 \phi \eta + \frac{\lambda}{4!} \phi^4 + \frac{\lambda \xi}{3!} \phi^3 \eta \right).
\]

Now, we impose the equal time commutations relations on the field operators as follows
\[
[\phi(\vec{x}, t), \pi_\phi(\vec{x}', t)] = i \delta(\vec{x} - \vec{x}'),
[\eta(\vec{x}, t), \pi_\eta(\vec{x}', t)] = i \delta(\vec{x} - \vec{x}').
\]  
Substituting the fields \([9]\) and \([11]\) in the above relations we find the nontrivial elements of the algebra of commutators
\[
[a_{\vec{p}}, c_{\vec{p}}^\dagger] = (2\pi)^3 \delta(\vec{p} - \vec{p}'),
\]
\[
[c_{\vec{p}}, a_{\vec{p}}^\dagger] = (2\pi)^3 \delta(\vec{p} - \vec{p}'),
\]
\[
[c_{\vec{p}}, c_{\vec{p}}^\dagger] = -(2\pi)^3 \delta(\vec{p} - \vec{p}').
\]  
The minus sign in \([13]\) is the first indication of a negative-norm state, which eventually leads to an indefinite metric in Hilbert space, as we show in the next section.

In terms of creation and annihilation operators the Hamiltonian is
\[
H' = \int \frac{d^3 \vec{p}}{(2\pi)^3} E_p \left( a_{\vec{p}}^\dagger a_{\vec{p}} + c_{\vec{p}}^\dagger c_{\vec{p}} + a_{\vec{p}}^\dagger c_{\vec{p}}^\dagger \right).
\]
We define the vacuum state \( |0\rangle \) to be annihilated by the operators
\[
a_{\vec{p}}|0\rangle = c_{\vec{p}}|0\rangle = 0.
\]
Let us write the Lagrangian \([10]\) with \( \lambda = 0 \) as
\[
\mathcal{L}_{\text{free}} = \frac{1}{2} \Psi^T S \Psi,
\]
defining the column field \( \Psi \)
\[
\Psi = \begin{pmatrix} \phi \\ \eta \end{pmatrix},
\]
and the non-diagonal matrix
\[
S = \begin{pmatrix} Q & \xi Q \\ \xi Q & 0 \end{pmatrix},
\]
with \( Q = -\Box - m^2 \).

The propagator follows by considering the inverse of \( S \). In momentum space, this is
\[
P = \begin{pmatrix} 0 & \Delta \\ \Delta & -\Delta \end{pmatrix},
\]
with
\[
\Delta = i \xi^2 \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-i p \cdot (x-y)}}{p^2 - m^2 + i\epsilon},
\]
and where we have included the \( i\epsilon \) prescription.

By considering the definition of propagator
\[
\Delta_{ij}(x-y) = \frac{1}{\xi^2} \langle 0 | T \Psi_i(x) \Psi_j(y) | 0 \rangle,
\]  
where \( T \) is the time ordering operator.
with \( \Psi_1 = \phi \) and \( \Psi_2 = \eta \), from Eqs. \((11)-(13)\) and \((15)\) we arrive at the same result \((20)\). Hence, from \((21)\) we can write

\[
\Delta_{11}(z) = 0
\]
\[
\Delta_{12}(z) = \frac{i}{\xi^2} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot z}}{p^2 - m^2 + i\epsilon},
\]
\[
\Delta_{22}(z) = -\frac{i}{\xi^2} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip \cdot z}}{p^2 - m^2 + i\epsilon},
\]
with \( z = x - y \).

The corrections to each matrix elements can be find with the perturbative series. We begin with the lowest order correction to \(\Delta_{12}(z)\)

\[
\delta^{(1)}\Delta_{12}(z) = \langle 0 | T \{ \phi(x)\eta(y) \} (-i\lambda) \int d^4w \left( \frac{\phi^4(w)}{4!} + \frac{\xi \phi^3(w)\eta(w)}{3!} \right) \{ 0 \}.
\]

According to Wick theorem and the definition of vacuum \((10)\) a contraction \(\phi\phi\) gives a possible contribution vanishes, due to \((22)\). Hence, one has that \(\delta^{(1)}\Delta_{12}(z) = 0\), and the same applies for higher order corrections to this element.

For the element \(\Delta_{22}(z)\), the first order correction \(\delta^{(1)}\Delta_{22}(z)\), however, is different from zero. It can be noted that the contraction of the external legs \(\eta(x)\eta(y)\) with \(\phi^3(w)\eta(w)\) gives a nonzero contribution, see Fig. 1.

![FIG. 1: The 1IP diagrams for each matrix element between fields \(\phi\) and \(\eta\) of the two-point function. The propagator \(\Delta_{12}(z) = \Delta_{21}(z)\) is represented by a segmented line and the propagator \(\Delta_{22}(z)\) by a broken line.](image)

However, at the order \(\lambda^2\) one has

\[
\delta^{(2)}\Delta_{22}(z) = \langle 0 | T \{ \eta(x)\eta(y) \} \frac{(-i\lambda)^2}{2!} \int d^4w d^4q \times \left( \xi^2 \frac{\phi^3(w)\eta(w)\phi^4(q)\eta(q)}{3!} + \frac{\phi^4(w)\phi^4(q)}{(4!)^2} + \frac{\xi \phi^3(w)\eta(w)\phi^4(q)}{3!} + \frac{\xi \phi^4(q)\phi^3(q)\eta(q)}{4!} \right) \{ 0 \},
\]

which vanishes due to the same previous arguments.

The only non-vanishing quantum corrections for the matrix propagator comes at one-loop order and so, we have tested one of the basic properties of delta theories in the scalar sector.

C. Effective action

Consider the vacuum-vacuum amplitude for the Lagrangian \(\mathcal{L}\) in the presence of the currents \(J\) and \(\tilde{J}\)

\[
Z[J, \tilde{J}] = \int D\phi D\eta e^{i\int d^4x (\mathcal{L}(\phi, \eta) + J\phi + \tilde{J}\eta)}.
\]

The equations of motion for \(\phi\) are

\[
\xi \Box \phi + \xi m^2 \phi + \xi \frac{\lambda}{3!} \phi^3 = \tilde{J}.
\]

and for \(\eta\)

\[
\Box \phi + m^2 \phi + \frac{\lambda}{3!} \phi^3 + \xi \Box \eta + \xi m^2 \eta + \xi \frac{\lambda}{2!} \phi^2 \eta = J.
\]

We integrate \((27)\) with respect to the field \(\eta\), and obtain

\[
Z[J, \tilde{J}] = \int D\phi e^{i\int d^4x \left( \frac{1}{4!} \partial_{\mu} \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi_0^2 + \frac{\lambda \phi_0^4}{4!} + J\phi \right) \times \delta \left( -\xi \Box \phi - \xi m^2 \phi - \xi \frac{\lambda}{3!} \phi^3 + \tilde{J} \right)}.
\]

In terms of the classical solutions \(\phi_0\) of the equation of motion \((28)\), we can expand the delta as

\[
\delta \left( -\xi \Box \phi - \xi m^2 \phi - \xi \frac{\lambda}{3!} \phi^3 + \tilde{J} \right) = \operatorname{det}^{-1} \left( -\xi \Box - \xi m^2 - \xi \frac{\lambda}{2!} \phi^2 \right) |_{\phi = \phi_0} \delta (\phi - \phi_0).
\]

We substitute in \((30)\) and integrate, which yields

\[
Z[J, \tilde{J}] = \int D\phi e^{i\int d^4x \left( \frac{1}{4!} \partial_{\mu} \phi_0 \partial^\mu \phi_0 - \frac{1}{2} m^2 \phi_0^2 + \frac{\lambda \phi_0^4}{4!} + J\phi_0 \right) + i\operatorname{Tr} \log \left( -\xi \Box - \xi m^2 - \xi \frac{\lambda}{2!} \phi_0^2 \right)}.
\]

Consider the generating function of connected Green’s functions

\[
W[J, \tilde{J}] = -i \ln Z[J, \tilde{J}],
\]

which in \((32)\) produces

\[
W[J, \tilde{J}] = \int d^4x \left( \frac{1}{2!} \partial_{\mu} \phi_0 \partial^\mu \phi_0 - \frac{1}{2} m^2 \phi_0^2 + \frac{\lambda \phi_0^4}{4!} + J\phi_0 \right) + i\operatorname{Tr} \log \left( -\xi \Box - \xi m^2 - \xi \frac{\lambda}{2!} \phi_0^2 \right).
\]

We define as usual the effective action by

\[
\Gamma[\Phi, \tilde{\Phi}] = W[J, \tilde{J}] - \int d^4x \left( J\Phi + \tilde{J}\tilde{\Phi} \right),
\]

with the classical fields

\[
\Phi = \frac{\delta W}{\delta J} = \phi_0,
\]
\[
\tilde{\Phi} = \frac{\delta W}{\delta J} = 0.
\]
Finally, we have
\[ \Gamma[\Phi, \bar{\Phi}] = \int d^4x \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 + \frac{\lambda \Phi^4}{4!} \right) + i \text{Tr} \log(-\Box - m^2 - \frac{\lambda}{2!} \Phi^2) . \] (38)
Comparing with the quantum correction for an arbitrary standard theory \(\Gamma_{1\text{loop}}^{\text{Q}}\) involves just the one-loop correction, see diagrammatic arguments \([1]\). Let us introduce the notation, two, which is characteristic of delta theories.

For example, we have found for our model it has been amplified by a factor of two, which is characteristic of delta theories.

An alternative demonstration can be given using diagrammatic arguments \([3, 4]\). Let us introduce the notation, \(V_I\) for the number of vertices associated to the \(\phi^4\) interaction and \(V_2\) to number of vertices associated to the interaction \(\phi^n \eta\). We denote the number of internal lines \(I_1\) associated to the propagator \(\Delta_{12} = \Delta_{21}\), and \(I_2\) associated the propagator \(\Delta_{22}\).

Now, consider the general relation for a loop diagram
\[ L = I - V + 1 \] (41)
where \(L\) denotes the number of loops, \(I\) the number of internal lines and \(V\) the number of vertices. Since it is not possible to have external legs \(\phi\), the only vertex involved in the internal part of a diagram is \(V_2\). The contraction of \(\eta \phi\) in \(V_2\) produces a \(I_1\). Hence, one has
\[ L = I_1 - V_2 + 1 . \] (42)
The correspondence \(I_1 = V_2\) is also consequence of the contraction. We have that the maximum allowed number of loops is \(L = 1\).

### III. PHYSICAL FIELDS AND THEIR PROPAGATORS

The diagonalization of the Lagrangian \((3)\) can be achieved by introducing the new fields
\[ \phi_1 = \phi + \eta , \]
\[ \phi_2 = \eta . \] (43)
Substituting in \((3)\) yields the Lagrangian
\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{m^2}{2} \phi_1^2 + \frac{m^2}{2} \phi_2^2 - \frac{\lambda}{4!} \left( \phi_1^4 - 3 \phi_2^4 - 6 \phi_1^2 \phi_2^2 + 8 \xi \phi_1 \phi_2^3 \right) , \] (44)
where we have absorbed \(\xi\) into the field \(\phi_2\) and \(\varepsilon = \pm 1\) depends on the sign of \(\xi\).

We write the new fields as
\[ \phi_1(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( b_{1\vec{p}} e^{-i p \cdot x} + b_{1\vec{p}}^\dagger e^{i p \cdot x} \right) \] (45)
\[ \phi_2(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( b_{2\vec{p}} e^{-i p \cdot x} + b_{2\vec{p}}^\dagger e^{i p \cdot x} \right) \] (46)
and define the new creation and annihilation
\[ b_{1\vec{p}} = a_{\vec{p}} + c_{\vec{p}} , \]
\[ b_{1\vec{p}}^\dagger = a_{\vec{p}}^\dagger + c_{\vec{p}}^\dagger , \]
\[ b_{2\vec{p}} = c_{\vec{p}} , \]
\[ b_{2\vec{p}}^\dagger = c_{\vec{p}}^\dagger . \] (47)
in terms of the ones in \([6, 7]\).

It can be checked by employing \((11)\), that they satisfy the commutations relations
\[ [b_{1\vec{p}}, b_{1\vec{q}}^\dagger] = (2\pi)^3 \delta(\vec{p} - \vec{q}) , \]
\[ [b_{2\vec{p}}, b_{2\vec{q}}^\dagger] = -(2\pi)^3 \delta(\vec{p} - \vec{q}) , \] (49)
with all others commutators being exactly zero.

The free Hamiltonian is found to be
\[ H = \int \frac{d^3p}{(2\pi)^3} E_p \left( b_{1\vec{p}}^\dagger b_{1\vec{p}} - b_{2\vec{p}}^\dagger b_{2\vec{p}} \right) . \] (51)
which can be expected due to the negative-norm state. Moreover, as a result of our previous definition of vacuum \([15]\), we have
\[ b_{1\vec{p}}|0\rangle = 0 , \]
\[ b_{2\vec{p}}|0\rangle = 0 . \] (52)
The number operators associated to the two types of particles are
\[ N_{1\vec{p}} = b_{1\vec{p}}^\dagger b_{1\vec{p}} , \]
\[ N_{2\vec{p}} = - b_{2\vec{p}}^\dagger b_{2\vec{p}} . \] (53)
In terms of the number operators, the Hamiltonian is given by
\[ H = \int \frac{d^3p}{(2\pi)^3} (E_p(N_{1\vec{p}} + N_{2\vec{p}})) , \] (54)
which is bounded from below and stable when the interactions are turned on. From the commutators \((49)\) and \((50)\), we identify the creation operator \(b_{1\vec{p}}^\dagger\) associated to a positive metric and \(b_{2\vec{p}}^\dagger\) associated to a negative metric \([11]\).

The propagators follows by the usual definition
\[ \Delta_1(x - y) = (0\langle T \phi_1(x) \phi_1(y) \rangle 0) = i \int_{c_1} d^4p \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} . \] (55)
and

\[ \Delta_2(x - y) = \langle 0 \vert T \phi_2(x) \phi_2(y) \vert 0 \rangle = -i \int_{C_1} d^4p \frac{e^{-ip \cdot (x - y)}}{(2\pi)^4 p^2 - m^2 + i\epsilon}, \]  

(56)

where the contour \( C_1 \) in the complex \( p_0 \)-plane lies below the negative pole and above the positive one. Both propagators as depicted in Fig. (2).

\[ \Delta_1(x - y) : \quad x \quad \quad \quad \quad y \]

\[ \Delta_2(x - y) : \quad x \quad \quad \quad \quad \quad y \]

FIG. 2: For the propagator \( \Delta_1(x - y) \) we consider a normal line and for the propagator \( \Delta_2(x - y) \) a segmented line.

We can also draw the vertices according to the interacting term in the Lagrangian (44), see Fig. (3). It is convenient to mention that the property of having radiative correction up to one loop order is lost in this new basis. However one may expect that writing the original quantum corrections at a given order in terms of the new fields these are cancelled out.

\[ = -i\lambda \]

\[ = 3i\lambda \]

\[ = i\lambda \]

\[ = 2i\epsilon \lambda \]

FIG. 3: The four vertices in the diagonal base \( \phi_1, \phi_2 \).

IV. PERTURBATIVE UNITARITY

In this section, we study perturbative unitarity by verifying the optical theorem.

In terms of the physical fields \( \phi_1 \) and \( \phi_2 \), we focus on the process of two particles going to two particles with initial and final momenta \( (p_1, p_2) \) and \( (p_1', p_2') \), respectively

\[ p_1 + p_2 \rightarrow p_1' + p_2', \]  

(57)

as seen in Fig. (4).

We can reinstate the small coupling constant \( \xi \) for the propagator and the vertex, so that the diagrams in the first line, second and third line of Fig. 4 are of order \( \lambda^2, \lambda^2\xi^2 \) and \( \lambda^2\xi^2 \), respectively.

The amplitudes are given by

\[ i\mathcal{M}_1(p) = \frac{1}{2} i\mathcal{M}_1^{(1,1)}(p) + \frac{1}{2} i\mathcal{M}_1^{(2,2)}(p), \]

\[ i\mathcal{M}_2(p) = \frac{9}{2} i\mathcal{M}_2^{(2,2)}(p) + 4 i\mathcal{M}_2^{(1,2)}(p), \]

\[ i\mathcal{M}_3(p) = 4 i\mathcal{M}_3^{(2,2)}(p), \]  

(58)

where \( p = p_1 + p_2 \) and we have included the symmetry factors for each diagram. Each element is defined by

\[ \mathcal{M}_a^{(a,b)}(q) = (-i\lambda_i)^2 \mathcal{M}_a^{(a,b)} q^2 - m^2 + i\epsilon, \]

(59)

where \( a, b = 1, 2, i = 1, 2, 3 \), such that \( \lambda_1 = \lambda, \lambda_2 = \lambda\xi^2, \lambda_3 = \lambda\xi \), and the propagators for \( \phi_1 \) and \( \phi_2 \)

\[ \Delta_1(q) = \frac{i}{q^2 - m^2 + i\epsilon}, \]

\[ \Delta_2(q) = \frac{-i}{q^2 - m^2 + i\epsilon}. \]  

(60)

(61)

We compute the imaginary part of the amplitudes by computing its discontinuity. To begin with, let us write the loop integral

\[ \mathcal{M}_1^{(1,1)}(p) = (-i\lambda_1)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon}, \]

(62)
with two propagators $\Delta_1(q)$ and $\Delta_1(p-q)$ represented by the first diagram in Fig. [4].

We employ the residue theorem to compute the integral in the complex $q_0$-plane. For this, we integrate along the contour $C_1$ that encloses the poles of the lower half plane. Hence, we consider

$$\mathcal{M}_1^{(1,1)}(p) = \lambda^2 \int \frac{d^3\vec{q}}{(2\pi)^3} \int_{C_1} \frac{dq_0}{2\pi} \delta(q_0 - E_q - i\epsilon) \times \frac{1}{(q_0 - E_q + i\epsilon)(q_0 + E_q - i\epsilon)} \times \frac{1}{(q_0 - p_0 - E_{q-p} + i\epsilon)(q_0 - p_0 + E_{q-p} - i\epsilon)},$$

with $E_q = \sqrt{q^2 + m^2}$ and the poles at

$$q_0 = E_q - i\epsilon, \quad q_0 = p_0 + E_{q-p} - i\epsilon.$$

The integration gives

$$\mathcal{M}_1^{(1,1)}(p) = \lambda^2 \int \frac{d^3\vec{q}}{(2\pi)^3} (-2\pi i) \left(2E_q(-p_0 + E_q - E_{q-p})(-p_0 + E_q + E_{q-p} - i\epsilon) + \frac{1}{2E_{q-p}(p_0 + E_{q-p} - E_q)(p_0 + E_{q-p} + E_q - i\epsilon)} \right),$$

where we have rescaled $\epsilon$ and evaluated $\epsilon \to 0$ where it is not relevant for the discontinuity. It is convenient to compute its discontinuity and so, we employ the relation

$$\lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = \mathcal{P} \left( \frac{1}{x} \right) \mp i\pi\delta(x),$$

with $\mathcal{P}$ the principal value. This results in the discontinuity

$$\text{Disc}\mathcal{M}_1^{(1,1)}(p) = i\lambda^2 \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{2E_qE_{q-p}} \left(\mp i\pi\delta(p_0 - E_p - E_{q-p}) + i\pi\delta(p_0 + E_q - E_{q-p})\right).$$

For simplicity we consider $p_0 > 0$ and we relabel the two momenta of each propagator as $q_1$ and $q_2$, such that

$$\vec{q} = \vec{q}_1, \quad \vec{p} - \vec{q} = \vec{q}_2.$$

We can write

$$\text{Disc}\mathcal{M}_1^{(1,1)}(p) = i\lambda^2 \int \frac{d^3\vec{q}_1}{(2\pi)^3} \frac{d^3\vec{q}_2}{(2\pi)^3} \delta^3(\vec{p} - \vec{q}_1 - \vec{q}_2) \frac{1}{2E_{q_1}E_{q_2}} (i\pi)\delta(p_0 - E_{q_1} - E_{q_2}),$$

or

$$\text{Disc}\mathcal{M}_1^{(1,1)}(p) = i\lambda^2 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \delta^4(p - q_1 - q_2) \delta(q_1^2 - m^2) \delta(q_2^2 - m^2)\theta(q_{01}\theta(q_{02}).$$

Using the relation $\text{Disc}\mathcal{M} = 2i\text{Im}\mathcal{M}$ we have

$$2\text{Im}\mathcal{M}_1^{(1,1)}(p) = \lambda^2 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \delta^4(p - q_1 - q_2) \delta(q_1^2 - m^2) \delta(q_2^2 - m^2) \theta(q_{01}\theta(q_{02}).$$

The second and third type of integral follow in a similar way, such to have

$$\frac{\text{Im}\mathcal{M}_1^{(1,1)}(p)}{\lambda^2} = -\frac{\text{Im}\mathcal{M}_2^{(1,2)}(p)}{\lambda^2} = \frac{\text{Im}\mathcal{M}_1^{(2,2),2,3}(p)}{\lambda^2}.$$

Let us define the corresponding cutted diagrams by $A_1^{(1,1)} = A_1^{(2,2)} = i\lambda, \quad A_2^{(1,2)} = A_3^{(2,2)} = 2i\lambda, \quad A_2^{(2,2)} = 3i\lambda.$

Now, the first and third processes give the correct cutting equation, being

$$2\text{Im}\mathcal{M}_1 = \lambda^2 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \delta^4(p - q_1 - q_2) \left( |A_1^{(1,1)}|^2 + |A_1^{(2,2)}|^2 \right),$$

and

$$2\text{Im}\mathcal{M}_3 = \xi^2 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \delta^4(p - q_1 - q_2) |A_3^{(2,2)}|^2.$$

The second process, due to the wrong sign of $\text{Im}\mathcal{M}_2^{(1,2)}(p)$, is responsible for a violation of unitarity, which however is very suppressed in the delta coupling parameter of the order of $\lambda^2\xi^4$. Hence, as long as the theory is understood to be effective one should have departures from unitarity only becoming relevant beyond the region of validity at which our scalar theory has been defined.

For completeness we can explore alternative approaches to study perturbative unitarity. Due to the presence of an indefinite metric and the ghost character of the field $\phi_2$, it is natural to explore whether the prescription that excludes negative metric states from the asymptotic Hilbert space allows restoring unitarity at one-loop level. By applying the Lee-Wick prescription, the only nontrivial process is the first process in [58]. However, since the cut diagram associated to the amplitude $\mathcal{M}_1^{(2,2)}(p)$ vanishes, in this case there is also violation of unitarity of the order $\lambda^2$. 
V. CONCLUSIONS AND OUTLOOK

In this work, we have verified the delta theory’s property of suppressing its radiative corrections beyond one-loop order. In particular, we have constructed the delta-theory associated to a scalar model with a quartic self-interaction term. We have quantized the model and find the corresponding propagators for the positive and negative metric fields. We have tested unitarity at one-loop order by employing the optical theorem and the cutting equations. We have found suppressed violations of unitarity of the order of $\lambda^2 \xi^4$ which provides a safe region to set up a meaningful effective theory based on the delta approach. The application of the Lee-Wick prescription, unfortunately increases the order of violations of unitarity to the order $\lambda^2$. Regarding the effective point of view and the richer structure provided by gauge models, we believe that relevant studies on perturbative unitarity for delta theories may come from analyzing the gravity sector, which we leave for future work.

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