On a rainbow extremal problem for color-critical graphs

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Abstract
Given $k$ graphs $G_1, \ldots, G_k$ over a common vertex set of size $n$, what is the maximum value of $\sum_{i \in [k]} e(G_i)$ having no "colorful" copy of $H$, that is, a copy of $H$ containing at most one edge from each $G_i$? Keevash, Saks, Sudakov, and Verstraëte denoted this number as $\text{ex}_k(n, H)$ and completely determined $\text{ex}_k(n, K_r)$ for large $n$. In fact, they showed that, depending on the value of $k$, one of the two natural constructions is always the extremal construction. Moreover, they conjectured that the same holds for every color-critical graphs, and proved it for 3-color-critical graphs. They also asked to classify the graphs $H$ that have only these two extremal constructions. We prove their conjecture for 4-color-critical graphs and for almost all $r$-color-critical graphs when $r > 4$. Moreover, we show that for every non-color-critical non-bipartite graphs, none of the two natural constructions is extremal for certain values of $k$.

KEYWORDS
color-critical graphs, rainbow extremal problem

1 | INTRODUCTION

For a given collection $F = \{F_1, \ldots, F_k\}$ of sets, a set $X \subseteq \bigcup F_i$ such that $|X \cap F_i| \leq 1$ for each $i \in [k]$ is often called a “colorful” set of $F$. If $|X \cap F_i|$ is exactly one for all $i$, then it is usually called a “transversal” of $F$. Colorful objects were considered over various types of mathematical objects. Bárány considered a transversal of a collection of convex subsets of Euclidean spaces and obtained a colorful version of Carathéodory’s theorem in [3]. Furthermore, more colorful variants of Carathéodory’s theorem [9] and Helly’s theorem [10] were obtained. Aharoni and Howard considered transversals of a set systems and obtained a colorful version of Erdős-Ko-Rado theorem in [2].
Perhaps the most famous transversals are the ones of Latin squares considered by Euler. In 1782, Euler [6] considered a Latin square, which is an \( n \times n \) array filled with numbers 1, \( \ldots \), \( n \), where every number appears exactly once in each of the rows and columns. We may consider a Latin square as a collection of rows, columns, and the sets of entries with the same numbers, then a transversal in a Latin square is a set of \( n \) entries such that no two are in the same row or in the same column or contain the same number.

A transversal in a Latin square is a special instance of a colorful subgraph in edge-colored graphs. By considering rows and columns as the bipartition of \( K_{n,n} \), a Latin square naturally corresponds to a proper edge-coloring of \( K_{n,n} \). We can consider this edge-colored graph as a collection \( G = \{ G_1, \ldots , G_n \} \) of graphs where each \( G_i \) is the graph consisting of edges of color \( i \). A transversal of \( G \) which is also a matching itself, is exactly the transversal of a Latin square that Euler considered. See [17] for a survey.

This motivates studies about colorful subgraphs of a collection of graphs. Indeed, various interesting results have been obtained. Aharoni et al. [1] considered a Turán type problem over graph collections, proving that there exists a colorful triangle of \( \{ G_1, G_2, G_3 \} \) if \( \min_{i \in [3]} e(G_i) \) is bigger than \( \frac{26-2\sqrt{7}}{81} n^2 \). Surprisingly, this irrational number is best possible. Besides triangles, other graphs like perfect matchings, Hamilton cycles, and \( F \)-factors were considered and transversal version of Dirac’s theorem and Hajnal-Szemerédi theorems were obtained [5,8,13].

In the above line of works, the results were obtained in terms of the restriction on \( \min_{i} e(G) \) or \( \min_{i} \delta(G) \). In other words, all graphs in the collection have to satisfy the given condition. However, enforcing conditions to all graphs in the graph collection seems quite restrictive. What if we reduce these conditions to the average behaviors of the graphs within the collection? In other words, in order to guarantee a colorful copy of \( H \), how large \( \sum_{i \in [k]} e(G_i) \) has to be? Indeed, such a problem was already considered by pioneering work of Keevash et al. [11] and also recently by Frankl [7]. They considered the collection \( G \) of graphs as one multi-graph with an edge-coloring where the set of edges with each color corresponds to a simple graph within the collection. We follow the notion below introduced by Keevash, Saks, Sudakov, and Verstraët.

In this article, a graph always means a simple graph. A simple \( k \)-coloring of a multigraph \( G \) is a decomposition of the edge multiset as a disjoint sum of \( k \) simple graphs which are referred as colors. A sub-(multi)graph \( H \) of a multigraph \( G \) is called multicolored if its edges receive distinct colors in a given simple \( k \)-coloring of \( G \). If \( G \) contains a multicolored copy of \( H \), we would also say in short that \( G \) contains a multicolored \( H \). The \( k \)-color Turán number, denoted by \( \mathrm{ex}_k(n,H) \), is the maximum number of edges in an \( n \)-vertex multigraph that has a simple \( k \)-coloring containing no multicolored copy of \( H \). The simply \( k \)-colored multigraphs that achieve this maximum are called the \( k \)-color extremal multigraphs of \( H \). We denote the set of the extremal multigraphs by \( \mathrm{Ex}_k(n,H) \), but if there is only one such multigraph up to graph isomorphism, then we abuse the notation and also refer to it by \( \mathrm{Ex}_k(n,H) \).

If \( k \leq e(H) - 1 \), then it is clear that \( \mathrm{Ex}_k(n,H) \) is the multigraph consisting of \( k \) copies of complete graphs. For \( k \geq e(H) \), there are two natural maximal\(^1\) simply \( k \)-colored multigraph \( G \) having no multicolored copy of \( H \). First, one can consider the multigraph consisting of \( e(H) - 1 \) copies of the complete graph. Second, one can consider \( k \) identical copies of a fixed extremal \( H \)-free graph. The first multigraph has \( (e(H) - 1) \binom{n}{2} \) edges, whereas the second construction has \( k \cdot \mathrm{ex}(n,H) \) edges. If \( k \) is close to \( e(H) \), then the first construction has more edges than the second one. However, as \( k \) grows (we allow \( k \) to depend on \( n \)), for certain value of \( k \) onward, the second construction has more edges than the first one. In [11, Theorem 1.1], it was shown for the multicolor Turán problem for \( H \) that whenever

\(^{1}\text{Maximal with respect to the subgraph relationship.}\)
\( k \geq \left( \frac{n}{2} \right) - \text{ex}(n, H) + e(H) \), the second construction always gives the unique extremal multigraph. Keevash, Saks, Sudakov, and Verstraëte proved that when \( H \) is a complete graph, the extremal multigraph is always one of these two natural constructions. It is well-known that the unique extremal \( K_r \)-free graph is the Turán graph \( T_{r-1}(n) \), which is the \( n \)-vertex balanced complete \((r - 1)\)-partite graph \([16]\).\

**Theorem 1.1** ([11, Theorem 1.2]). Suppose that \( r \geq 2 \), \( k \geq \left( \frac{r}{2} \right) \), and \( n > 10^4r^34 \). Let \( G \) be an \( n \)-vertex \( k \)-color extremal multigraph of \( K_r \). Then either all colors of \( G \) are identical Turán graphs \( T_{r-1}(n) \), or there are exactly \( \left( \frac{r}{2} \right) - 1 \) non-empty colors of \( G \), all of which are complete graphs \( K_n \). In particular,

\[
\text{ex}_k(n, K_r) = \begin{cases} 
  k \cdot t_{r-1}(n) & \text{for } k \geq \frac{1}{2} (r^2 - 1), \\
  \left( \binom{r}{2} - 1 \right) \binom{n}{2} & \text{for } \frac{1}{2} (r^2 - 1) < k < \frac{1}{2} (r^2 - 1).
\end{cases}
\]

It is natural to consider \( \text{ex}_k(n, H) \) for more general graphs \( H \). However, as the exact structures of the extremal graphs of \( H \) are not known for most of the graphs \( H \), it makes sense to first focus on the graphs \( H \) whose extremal graphs are well-understood. An important class of such graphs is the class of color-critical graphs. A graph or a multigraph \( H \) is called \( r \)-color-critical if it has chromatic number \( r \), and it has an edge (called a critical edge) whose removal reduces the chromatic number to \( r - 1 \). For an \( r \)-color-critical graph \( H \), Simonovits \([15]\, Theorem 1\) proved that \( T_{r-1}(n) \) is the unique extremal graph of \( H \) when \( n \) is sufficiently large. Indeed, Keevash, Saks, Sudakov, and Verstraëte conjectured that the above theorem can be extended to color-critical graphs.

**Conjecture 1.2** ([11, Conjecture 1.3]). Suppose \( r \geq 3 \) and \( k \geq h \). Let \( H \) be an \( r \)-color-critical graph with \( h \) edges. Then, there exists an \( n_0 = n_0(H) > 0 \) such that for all \( n \geq n_0 \), the \( k \)-color extremal multigraph of \( H \) either consists of exactly \( h - 1 \) nonempty colors where each of them is a copy of \( K_n \) or consists of \( k \) colors where all of them are identical copies of \( T_{r-1}(n) \). In particular,

\[
\text{ex}_k(n, H) = \begin{cases} 
  (h - 1) \binom{n}{2} & \text{for } h \leq \frac{r-1}{r-2}(h - 1), \\
  k \cdot t_{r-1}(n) & \text{for } \frac{r-1}{r-2}(h - 1) < k.
\end{cases}
\]

The \( r = 3 \) case of the conjecture was proved in [11, Theorem 1.4]. We further support the conjecture by proving the \( r = 4 \) case. When \( r \geq 5 \), we also prove this conjecture for “most” of the \( r \)-color-critical graphs.

**Theorem 1.3.** Conjecture 1.2 holds for \( r = 4 \).

**Theorem 1.4.** For \( r \geq 5 \) and a given \( \varepsilon > 0 \), there exists \( s_0 \) such that the following holds for all \( s \geq s_0 \). At least \((1 - \varepsilon)\)-fraction of all \( s \)-vertex \( r \)-color-critical graphs \( H \) on the vertex set \([s]\) satisfies Conjecture 1.2.

**Remark 1.5.** We also prove a stability result for the above two theorems. To see the precise meaning of “stability” in this context, refer to Lemma 7.1.

To prove Theorems 1.3 and 1.4, we develop the ideas in [11]. In order to overcome certain technical difficulties, we consider a certain \( r \)-vertex \( r \)-color-critical “multi” graph \( H_c \) instead of a simple graph \( H \).
By understanding the multicolor Turán problem for such a multigraph \( H_r \), we are able to deal with the reduced (multi)graphs obtained by applying regularity lemma to \( G_1, \ldots, G_k \) and deduce our desired results Theorems 1.3 and 1.4. See Section 3 for a rough sketch of the proofs of Theorems 1.3 and 1.4.

It was asked in [11] to identify the class of the graphs \( H \) that have only the two extremal constructions as above. Theorems 1.3 and 1.4 together with the result in [11] show that all \( r \)-color-critical graphs for \( r \in \{3, 4\} \) and almost all \( r \)-color critical graphs for \( r \geq 5 \) lie in the class. What about non-color-critical graphs? In [11], it was shown that for certain value of \( k \), the bowtie (the graph consisting of two edge-disjoint triangles intersecting in exactly one vertex) has an extremal construction different from the two natural ones. In fact, we are able to show the same for all non-color-critical graphs with chromatic number \( r \geq 3 \) by generalizing their construction for the bowtie. In contrast with [11, Theorem 1.1], the condition of \( n \) being sufficiently large compared to \( k \) in the following theorem is necessary.

**Proposition 1.6.** Let \( H \) be a non-color-critical graph with \( h \) edges and chromatic number at least three. Then, for any \( k \geq k^* = \frac{(r-1)(h-1)}{r-2} \) and sufficiently large \( n \), we have

\[
\text{ex}_k(n, H) > \max \left( k \cdot \text{ex}(n, H), (h - 1)\left(\frac{n}{2}\right) \right).
\]

**Organization.** The rest of the article proceeds as follows. We start by proving Proposition 1.6 in Section 2. We then give a few preliminaries along with a proof sketch of Theorems 1.3 and 1.4 in Section 3. Theorems 1.3 and 1.4 are proved through the next sections in the following way. In Sections 4 and 5, we prove the results for an \( r \)-vertex \( r \)-color-critical “multi”graph. In other words, we prove that all 4-vertex 4-color-critical multigraphs and most \( r \)-vertex \( r \)-color-critical multigraphs have two natural constructions as the multicolor extremal multigraphs. These results will be applied to regularity partition of \( G_1, \ldots, G_k \) to obtain approximate versions of Theorems 1.3 and 1.4. Appropriate stability versions of the results from Sections 4 and 5 will be proved in Sections 6 and 7. Finally, armed with everything we prove the exact statements of Theorems 1.3 and 1.4 in Section 8 along with their stability versions. We end with a few concluding remarks.

## 2 | CONSTRUCTION FOR NON-COLOR-CRITICAL GRAPHS

**Proof of Proposition 1.6.** Let \( H \) be a non-color-critical graph with chromatic number \( r \) for some \( r \geq 3 \). By Erdős–Stone–Simonovits theorem, we know that \( \text{ex}(n, H) = t_{r-1}(n) + o(n^2) \). By simple computations, it is clear that for \( k \geq k^* \) and sufficiently large \( n \), we have that \( k \cdot \text{ex}(n, H) > (h - 1)\left(\frac{n}{2}\right) \). Thus, to prove Proposition 1.6, it is enough to show that for \( k \geq k^* \), we have that \( \text{ex}_k(n, H) > k \cdot \text{ex}(n, H) \) for all sufficiently large \( n \). Fix \( k \geq k^* \).

Consider the simply \( k \)-colored multigraph \( G \) where all colors except one are equal to a fixed \( T_{r-1}(n) \) and the final color is equal to \( K_n \). We claim that this multigraph does not contain a multicolored copy of \( H \). Suppose not, then we can embed a multicolor copy of \( H \) in \( G \) and fix such an embedding. It is clear that in the embedding, \( H \) can have at most one edge outside of the fixed Turán graph \( T_{r-1}(n) \) (because there is only one color which contains edges outside of \( T_{r-1}(n) \)). Thus, there is an edge \( e \) in \( H \) such that \( H \setminus e \) can be embedded in \( T_{r-1}(n) \). Hence, \( H \setminus e \) is an \((r - 1)\)-partite graph, contradicting to the fact that \( H \) is a non-color-critical graph.
Finally, to finish the proof of Proposition 1.6, observe the following for sufficiently large \( n \):
\[
\text{ex}_k(n, H) \geq e(G) = (k - 1) \cdot t_{r-1}(n) + \binom{n}{2} > k \cdot \text{ex}(n, H).
\]

3 | PRELIMINARIES AND NECESSARY TOOLS

Let \( d_{r-1}(n) := \delta(T_{r-1}(n)) \) be the minimum degree of the Turán graph \( T_{r-1}(n) \). We usually use the following estimates:
\[
\frac{r}{r-1}(n - 1) \leq d_{r-1}(n) \leq \frac{r}{r-1}n \quad \text{and} \quad \frac{r}{r-1}(n) < t_{r-1}(n) \leq \frac{r}{r-1} \cdot n^2.
\]
We sometimes use the fact that \( e(H) \geq \binom{r}{2} \) for an \( r \)-color-critical multigraph \( H \), and in particular \( e(H) \geq 6 \) if \( r \geq 4 \).
Throughout this article, for brevity, we systematically avoid the floor and ceiling signs when they do not affect the underlying analysis.

3.1 | Notations

In the following, \( n, m, r, h, \) and \( k \) always denote positive integers. For a (multi)graph \( G \), let \( V(G) \) and \( E(G) \) denote the vertex set and the edge multiset of \( G \), respectively. We denote \( P(G) := \binom{V(G)}{2} \). The multiplicity of an edge \( e \) in a multigraph \( G \) is written as \( w_G(e) \) and the subscript will be omitted if the graph is clear from the context. For \( v \in V(G) \) and \( T \subseteq V(G) \), we write \( d_T(v) := \sum_{uv \in T} w_G(uv) \).
When we say a result holds if \( 0 < a \ll b, c \ll d < 1 \), it means that there exist non-decreasing functions \( f \) and \( g \) such that the result holds whenever \( b, c < f(d) \) and \( a < g(b, c) \). We will not compute those functions explicitly.

3.2 | Goodness and \( F_r \)

For the sake of convenience, we give a name to the property of having the two natural constructions as the multicolor extremal multigraphs.

**Definition 3.1.** We say a (multi)graph \( H \) with \( h \) edges and \( \chi(H) = r \) is good if there exist \( n_0(H) > 0 \) such that the following holds for all \( k \geq h, \ n \geq n_0(H) \), and \( k^* = \frac{r-1}{r-2}(h - 1) \).
If \( h \leq k < k^* \), then an \( n \)-vertex \( k \)-color extremal multigraph of \( H \) consists of exactly \( h - 1 \) non-empty colors, all of which are complete graphs \( K_r \), and if \( k \geq k^* \), then all colors of an \( n \)-vertex \( k \)-color extremal multigraph are the identical copies of an \( H \)-free graph.

Then Conjecture 1.2 says that for \( r \geq 3 \), all the \( r \)-color-critical graphs are good. Also, Proposition 1.6 says good graphs must be color-critical. We denote the former extremal multigraph by \((h - 1)K_n\). If \( H \) is a good \( r \)-color-critical graph with \( h \) edges, then \( T_{r-1}(n) \) is the unique extremal graph of \( H \), so we can denote the latter \( k \)-color extremal multigraph by \( kT_{r-1}(n) \). It is easy to confirm that the latter one has more edges if and only if \( k \geq k^* = \frac{r-1}{r-2}(h - 1) \). For any \( r \)-color-critical graph \( H \) with \( h \) edges, we set \( k^*(H) = k^*(r, h) := \frac{r-1}{r-2}(h - 1) \).

As mentioned in the introduction, we first study the problem for certain color-critical multigraphs. To obtain the desired result for simple graphs, we make a connection between an \( r \)-color-critical graph \( H \) and an \( r \)-color-critical multigraph \( H_e \) as follows. For an \( r \)-color-critical (multi-)graph \( H \), we call a proper coloring \( f \) with color classes \( V_1, \ldots, V_r \) a critical coloring if there exists two colors \( c', c'' \) such that \( e(V_{c'}, V_{c''}) = 1 \). For an \( r \)-color-critical graph \( H \) and its critical coloring \( f \), consider

an \( r \)-vertex multigraph \( H^f \) whose vertices are the color classes \( V_1, \ldots, V_r \) of \( f \) and the multiplicity of an edge \( V_iV_j \) is \( e(V_i, V_j) \) in \( H \) for each \( i \neq j \). By the choice of \( f \), it is clear that \( H^f \) is \( r \)-color-critical. We call \( H^f \) a color-reduced multigraph of \( H \). Note that this choice depends on the choice of the coloring \( f \). For an \( r \)-color-critical (multi)graph \( H \), we write \( H_r \) to denote an \( r \)-color-critical multigraph \( H^f \) where \( \max\{w_H(ij) : ij \in P(H^f)\} \) is minimized over all choices of critical coloring of \( f \). If there are several choices of \( f \) attaining the minimum, make an arbitrary choice. This choice will be convenient to define \( \mathcal{F}_r \) below.

We will first show that for the 4-color-critical graphs and most of the \( r \)-color-critical graphs \( H \), \( r \geq 5 \), their color-reduced multigraphs \( H_r \) are good. Using this fact together with the help of the multicolor version of the Szemerédi regularity lemma, we will prove that the original graphs \( H \) are good as well. For this, we first establish a stability version of the corresponding color-reduced graph and then use an appropriate embedding lemma for multicolored regularity lemma to embed the targeted graph into a simply \( k \)-colored multigraph \( G \) having more edges than the conjectured number.

In particular, for \( r \geq 5 \), we prove the goodness for those graphs in a specific class called \( \mathcal{F}_r \) which contains most of the \( r \)-color-critical graphs. An \( r \)-color-critical (multi)graph \( H \) with \( h \) edges is in \( \mathcal{F}_r \) if it has an \( r \)-color-critical color-reduced multigraph \( H_r \) whose edge multiplicities are at most

\[
\frac{2 + 2/r^2}{(r-1)(r-2)}(h-1) = \frac{1}{(r-1)2}(h-1) + O\left(\frac{h}{r^4}\right).
\]

Note that a color-reduced multigraph \( H_r \) of \( H \in \mathcal{F}_r \) is itself in \( \mathcal{F}_r \). Roughly speaking, for \( H \in \mathcal{F}_r \), the color-reduced multigraph \( H_r \) has balanced edge multiplicities, except for a few pairs including the pair having a critical edge. This allows to embed \( H_r \) in a certain \( r \)-vertex submultigraph \( G_0 \) of a simply \( k \)-colored multigraph \( G \) with some lower bounds on edge multiplicities of \( G_0 \). We will find such a \( G_0 \) when \( G \) is multicolored-\( H \)-free and \( e(G) \geq \text{ex}_k(\{G\}, H) \) but \( G \) does not have a desired structure, giving a contradiction. Moreover, it is not difficult to show that almost all \( r \)-color-critical graphs belong to \( \mathcal{F}_r \). We provide a proof sketch of the following proposition in Appendix A.2 of the online version of this article [4].

**Proposition 3.2.** For an integer \( r \) and a real \( \varepsilon > 0 \), there exists \( s_0 \) such that for all \( s \geq s_0 \) at least \((1 - \varepsilon)\)-fraction of all \( s \)-vertex \( r \)-color-critical graphs \( H \) on the vertex set \([s]\) are in \( \mathcal{F}_r \).

### 3.3 Minimum degree condition of the host graph

In many places, it is convenient to assume a minimum degree condition of the host graph \( G \). The following proposition allows us to assume such a minimum degree condition. We supply the proof in Appendix A.1 of the online version of this article [4].

**Proposition 3.3.** Let \( r \geq 3, k \geq 1, \) and \( H \) be an \( r \)-color-critical (multi)graph with \( h \) edges. For each \( n \in \mathbb{N} \), let

\[
A(n) := \begin{cases}
(h-1)K_n & \text{if } h \leq k < k^*(H), \\
kT_{r-1}(n) & \text{if } h \geq k^*(H).
\end{cases}
\]

Suppose there is an \( M_0 > 0 \) such that for all \( n > M_0 \), this \( A(n) \) is the unique \( n \)-vertex simply \( k \)-colored extremal multigraph with at least \( e(A(n)) \) edges and minimum degree \( \delta(A(n)) \).
Then there exists an \( n_0 = n_0(M_0, k) > 0 \) such that for all \( n > n_0 \), the multigraph \( A(n) \) is the unique \( n \)-vertex \( k \)-extremal multigraph of \( H \).

The minimum degree condition obtained from the previous proposition helps us to find a vertex set whose induced graph contains edges with high multiplicities. Such an induced subgraph is useful to build a multicolored copy of \( H \). The following proposition allows us to find a vertex having many edges towards a fixed vertex set. Repeating this proposition yields a desired vertex set containing many edges with high multiplicities.

**Proposition 3.4.** Suppose \( 0 < \frac{1}{n} \delta \ll \frac{1}{d}, \frac{1}{t}, \frac{1}{k} < 1 \). Suppose \( G \) is a simply \( k \)-colored multigraph of order \( n \) with \( \delta(G) \geq (1 - \delta)d(n - 1) \), and \( T \subseteq V(G) \) is a nonempty vertex set of order \( t \). Then there is a vertex \( v \in V(G) - T \) such that \( d_T(v) \geq dt \).

**Proof.** We have that \( e(T, V(G) - T) = \sum_{v \in T} d(v) - 2e(T) \geq t(1 - \delta)d(n - 1) - kt(t - 1) \). Thus there is a vertex \( v \in V(G) - T \) such that

\[
d_T(v) \geq \frac{1}{n - t}(t(1 - \delta)d(n - 1) - kt(t - 1)) \geq (1 - \delta)dt - \frac{kt^2}{n - t} > \lfloor dt \rfloor - 1.
\]

Note that \( \frac{1}{n} \delta \ll \frac{1}{d}, \frac{1}{t}, \frac{1}{k} \) implies the last inequality.\(^2\) Thus, it follows that \( d_T(v) \geq dt \). \( \blacksquare \)

### 3.4 Nested colorings

We say that a simple \( k \)-coloring is *nested* if its colors form a chain under inclusion. It is easy to see that the analogue of [11, Proposition 2.1] holds with the same proof even if we rather consider a multigraph \( H \) instead of a simple graph.

**Proposition 3.5.** Suppose \( G \) is a simply \( k \)-colored multigraph, and \( G \) does not contain a multicolored (multi)graph \( H \). Then there exists a simply \( k \)-nested-colored multigraph \( F \) on the same vertex set as \( G \) such that

1. \( F \) and \( G \) have the same edge set as multigraphs, and
2. \( F \) contains no multicolored \( H \).

The proof constructs \( F \) by applying the transformation to \( G \) finitely many times, which chooses two non-nested colors \( G_i \) and \( G_j \), and replace them by \( G_i \cap G_j \) and \( G_i \cup G_j \). From this proposition, it is easy to see the following. In the context of Theorems 1.3 and 1.4, to show that the two natural constructions for \( k \)-extremal multigraphs cover all possibilities, it is enough to consider nested colorings. This especially simplifies our analysis because there is a unique simple \( k \)-coloring of any given multigraph with nested coloring. In particular, for a given color set \([k]\), we can assume that any edge of multiplicity \( s \) has colors exactly \([s]\), and we will use such assumptions in Sections 7 and 8.

### 3.5 The condition for embedding a multicolored multigraph

For two multigraphs \( H \) and \( G \), we say that an injective map \( \phi : V(H) \to V(G) \) is an *embedding* if \( w_G(\phi(x)\phi(y)) \geq w_H(xy) \) for all \( xy \in P(H) \). For a multigraph \( G \), if any pair of multi-edges have disjoint sets of colors, then finding a multicolored copy of \( H \) in \( G \) boils down to finding an embedding of

\(^2\)To be more precise, \( \frac{1}{n} \delta < \frac{1}{10d^2} (dt + 1 - \lfloor dt \rfloor) \) ensures that the penultimate term is at least \( \frac{1}{2} (\lfloor dt \rfloor - 1 + 4dt) > \lfloor dt \rfloor - 1 \).
$H$ in $G$. However, it may no longer be true when the colors in the multi-edges in $G$ overlap and the most critical situation turns out to be the case when the coloring of $G$ is nested. In such a case, the embedding $\phi : H \hookrightarrow G$ yields a multicolored copy of $H$ if and only if we can greedily choose colors of edges in a certain ordering of $P(H)$. The following proposition captures this intuition and allows us to find a multicolored copy of a given multigraph. As it is easy to see that a greedy color choice with respect to the ordering $(e_1, \ldots, e_h)$ yields the proposition, we omit the proof.

**Proposition 3.6.** Let $G$ be a simply $k$-colored multigraph and $H$ be an $h$-edge multigraph. Suppose there is an embedding $\phi : H \hookrightarrow G$ which also gives $P(H) \hookrightarrow P(G)$. Then $\phi(H)$ yields a multicolored copy of $H$ if there is an enumeration $(e_1, \ldots, e_h)$ of $P(H)$ such that for each $1 \leq j \leq h$,

$$\sum_{i=1}^{j} w_H(e_i) \leq w_G(\phi(e_j)).$$

(1)

Note that the “only if” direction also holds if the simple $k$-coloring of $G$ is nested. When we want to argue that $G$ contains a multicolored $H$, we aim to find an embedding $\phi : H \hookrightarrow G$ and also an enumeration of $P(H)$ which satisfy (1). Note that among the embedding $\phi$, the enumeration of $P(H)$, and the corresponding enumeration of $P(\phi(H))$, two of them determine the other. We sometimes call the enumerations the edge embedding orders, and call them proper if they satisfy (1).

### 3.6 $H$-friendly submultigraphs

When $k \geq k^*(H)$, we want to prove that our host multigraph $G$ is $(r-1)$-partite. For this, we seek to find some “skeleton,” which is an $(r-1)$-partite submultigraph $K$ of $G$ containing a copy of $H \setminus \{x\}$ for some vertex $x \in V(H)$. By analyzing how a vertex $v \in V(G)$ and $K$ interact, we can either obtain a multicolored copy of $H$ within $K \cup \{v\}$ or determine which part of the $(r-1)$-partition does $v$ belong to. The following concept of an $H$-friendly submultigraph provides such a skeleton structure that we need.

**Definition 3.7.** Consider an $r$-color-critical $h$-edge (multi)graph $H$. We say an $a(r-1)$-vertex simply $k$-colored $(r-1)$-partite multigraph $K$ with a vertex partition $W_1, \ldots, W_{r-1}$ of equal sizes is $H$-friendly if $K'$ obtained as follows always contains a multicolored $H$: add a new vertex $v$ to $K$ and add edges of multiplicity at most $k$ incident to $v$ so that $d(v) \geq \max\{(r-2)ak, (r-1)a(h-1)\} = (r-2)a \cdot \max\{k, k^*(H)\}$ and $d_{W_i}(v) \geq 1$ for all $i \in [r-1]$.

For example, for $|H| = m$, a simply colored complete $(r-1)$-partite multigraph $hK_{m,\ldots,m}$ is an $H$-friendly multigraph. Indeed, consider a critical edge $xy$ of $H$, and assume $d_H(x) \leq d_H(y)$. Then it is easy to check that there exists a multicolored copy of $H$ within $K' = K \cup \{v\}$ as above where $v$ plays the role of $x$.

One thing to note is that the definition of $H$-friendliness depends on the choice of $(r-1)$-partition $(W_1, \ldots, W_{r-1})$ of $K$. However, all $H$-friendly graphs we deal with in this article will be complete $(r-1)$-partite graphs with certain edge-multiplicities, which has the unique $(r-1)$-partition. Hence, we will not specify this vertex partition and just mention that such a multigraph $K$ is $H$-friendly.

The following lemma states that the existence of an $H$-friendly submultigraph of $G$ implies the desired global structure of $G$.

**Lemma 3.8.** Suppose $0 \ll \frac{1}{n} \ll \delta \ll \frac{1}{a} \frac{1}{m} \frac{1}{k} \leq 1$ and $r \geq 4$. Let $H$ be an $m$-vertex $r$-color-critical multigraph with $h$ edges and $k \geq k^*(H)$. Let $G$ be a simply $k$-colored multicolored-$H$-free multigraph of order $n$ with $\delta(G) \geq (1-\delta)kd_{r-1}(n)$. 
If $G$ contains an $a(r-1)$-vertex $H$-friendly multigraph $K$ as an induced subgraph, then $G$ is $(r-1)$-partite.

**Proof.** Write $V := V(G)$ and let $W_1, \ldots, W_{r-1}$ be a vertex partition of $K$ making it $H$-friendly, with $|W_i| = a$ for each $i$.

Suppose $v \in V - V(K)$ satisfies $\sum_{i=1}^{r-1} d_{W_i}(v) \geq (r-2)ak$. Then, as $K$ is $H$-friendly and $G$ contains no multicolored copy of $H$, we have $\min\{d_{W_i}(v) : 1 \leq i \leq r-1\} = 0$. In particular, since every edge has multiplicity at most $k$, this implies that $d_K(v) \leq (r-2)ak$ for all $v \in V - V(K)$. Let $A := \{v \in V - V(K) : d_K(v) < (r-2)ak\}$. As $d_K(v) = (r-2)ak$ for all $v \in V - A - K$, we have

$$\delta(G)|K| - k \cdot 2^{\left(\frac{r-1}{2}\right)} a^2 \leq e(K, V - V(K)) \leq ((r-2)ak - 1)|A| + (r-2)ak(n - |K| - |A|),$$

which gives $|A| \leq (r-2)ak(\delta(n-1) + 1) < \delta^{1/2}n$ by $|K| = a(r-1)$ and the minimum degree condition on $G$. The set $V - A$ is partitioned into $(V_1, \ldots, V_{r-1})$ where

$$V_i := \{v \in V - V(K) - A : d_{W_i}(v) = 0, \quad d_{W_i}(v) = ak \ \forall j \neq i\}.$$

We can check that for any $v \in V_i$ and $w \in W_i$, the submultigraph $G[(V(K) \setminus \{w\}) \cup \{v\}]$ is $H$-friendly. Indeed, as $W_i$ is an independent set of $K$, the degree sequence of $G[(V(K) \setminus \{w\}) \cup \{v\}]$ dominates that of $K$, so it is $H$-friendly. As $K$ is an induced subgraph of $G$, each $W_i$ is an independent set of $G$. Thus, if there is an edge $ab$ in $V_i \cup W_i$, then one of $a, b$ is in $V_i$ and $G[V(K) \cup \{a, b\}]$ contains a multicolored $H$, a contradiction. Thus each $V_i \cup W_i$ is an independent set.

Now we show that we can place the vertices in $A$ into one of the independent sets one by one while keeping them independent to conclude that $G$ is $(r-1)$-partite. Suppose to the contrary that at some step we have independent sets $\{U_i \supseteq V_i \cup W_i : 1 \leq i \leq r-1\}$ and there is $v \in V - \bigcup_{i \in [r-1]} U_i$ such that $d_{U_i}(v) \geq 1$ for each $i$, so that we cannot place $v$ in any of the $U_i$’s. Fix such a vertex $v$ and let $U := \bigcup_{i \in [r-1]} U_i$. Note that we have $|V - U| \leq |A| \leq \delta^{1/2}n$.

Note that for the function $f(x_1, \ldots, x_{r-1}) = \sum_{i \in [r-1]} x_i x_j$ with $\frac{x_i + x_j}{2}$, increase the value of $f$ by $\frac{(x_i - x_j)^2}{4}$. Hence, for any $i' \neq j' \in [r-1]$, we have

$$f(|U_1|, \ldots, |U_{r-1}|) \leq f\left(\frac{|U|}{r-1}, \ldots, \frac{|U|}{r-1}\right) - \frac{1}{4}(|U_i| - |U_j|)^2 \leq t_{r-1}(|U|) - \frac{1}{4}(|U_i| - |U_j|)^2.$$

Hence, if $|U_i| - |U_j| \geq 4\delta^{1/4}n$ for some $i' \neq j'$, then we have

$$e(G) \leq e(U) + \sum_{v \in V - U} d(v) \leq k\sum_{i \in J} |U_i||U_j| + \sum_{v \in V - U} d(v) \leq k\left(t_{r-1}(|U|) - (2\delta^{1/4}n)^2\right) + kn|V - U| < (1 - \delta)kt_{r-1}(n),$$

a contradiction as $\delta(G) \geq (1 - \delta)kd_{r-1}(n)$. Thus $\left|\frac{|U_i| - \frac{|U|}{r-1}}{r-1}\right| < 4\delta^{1/4}n$ for all $i$, giving

$$\left|\frac{|U_i| - \frac{n}{r-1}}{r-1}\right| < 4\delta^{1/4}n + |V - U| < \delta^{1/5}n.$$
Without loss of generality, let \( d_{U_1}(v) = \min \{ d_{U_i}(v) : 1 \leq i \leq r - 1 \} \). For each \( 2 \leq i \leq r - 1 \), let

\[ M_i := \{ u \in U_i : w(uv) \geq k/2 \} \subseteq U_i. \]

**Claim 1.** For every \( 2 \leq i \leq r - 1 \) we have \( |M_i| > n/(4k^3) \).

**Proof.** Suppose that Claim 1 does not hold for some \( 2 \leq i \leq r - 1 \). Then, since \( r \leq h \leq k \), we have

\[
d_{U_i}(v) \leq d_{U_i}(v) \leq k|M_i| + \frac{k - 1}{2}|U_i - M_i| = \frac{k - 1}{2}|U_i| + \frac{k + 1}{2}|M_i|
\]

\[
< \frac{k - 1}{2} \left( \frac{n}{r - 1} + \delta^{1/5} n \right) + \frac{k + 1}{2} \cdot \frac{n}{4k^3}
\]

\[
< \frac{k - 1/8}{2(r - 1)} n,
\]

which yields a contradiction, as the assumed minimality of \( d_{U_i}(v) \) yields

\[
d(v) \leq k(|V - U_1 - U_i|) + d_{U_i}(v) + d_{U_i}(v)
\]

\[
< k \left( n - 2 \left( \frac{n}{r - 1} - \delta^{1/5} n \right) \right) + 2 \cdot \frac{k - 1/8}{2(r - 1)} n
\]

\[
= \frac{r - 2}{r - 1} km - \left( \frac{1}{8(r - 1)} - 2\delta^{1/5}k \right) n < (1 - \delta)kd_{r-1}(n).
\]

As \( d_{U_1}(v) \geq 1 \), we can choose \( u_1 \in U_1 \) with \( w(\nu u_1) \geq 1 \). Let \( W'_1 \subseteq U_1 \) be a set of \( m \) vertices containing \( u_1 \). Consider a sequence \( W'_1, \ldots, W'_s \) of sets of \( m \) vertices with \( s \leq r - 1 \) such that \( W'_i \subseteq M_i \) for each \( 2 \leq i \leq s \) and that \( w(w_i w_j) = k \) for all \( 1 \leq j < i \leq s, w_i \in W'_i \), and \( w_j \in W'_j \). Take such a sequence with the maximum possible \( s \). Indeed, such a maximum choice exists as a sequence of one set \( W'_1 \) trivially satisfies this condition with \( s = 1 \).

We claim that \( s = r - 1 \). Suppose \( s < r - 1 \). For each \( i \in [s] \), take

\[
L_i := \{ u \in M_{i+1} : \exists w \in W'_i \text{ s.t. } w(uw) < k \}.
\]

Then we have

\[
m \cdot (1 - \delta)k \frac{r - 2}{r - 1} (n - 1) \leq m \cdot (1 - \delta)kd_{r-1}(n)
\]

\[
\leq \sum_{i \in W'_i} d(u) < km(|V - U_i| - |L_i|) + (km - 1)|L_i|
\]

\[
< km \left( n - \left( \frac{n}{r - 1} - \delta^{1/5} n \right) \right) \leq |L_i|,
\]

giving \( |L_i| \leq \delta^{1/6} n \). Thus

\[
|M_{s+1} \setminus \bigcup_{i \in [s]} L_i| \geq \frac{n}{4k^3} - r\delta^{1/6} n > m.
\]
Hence, we may choose a set $W_{s+1}$ of $m$ vertices from $M_{s+1} \cup \bigcup_{i \in [s]} L_i$ satisfying $w(uv) = k$ for all $u \in W_{s+1}$, $w \in W_i$, and $i \in [s]$, a contradiction to the maximality of $s$. Hence we have $s = r - 1$.

Let $xy$ be a critical edge of $H$. Then $d(x) + d(y) + \binom{r-2}{2} \leq h + 1$, so without loss of generality let $d(y) \leq h/2 < k/2$. This allows an embedding of a multicolored $H$ into $G((\bigcup_{i \in [r-1]} W'_i) \cup \{v\})$ with $x \mapsto u_1$ and $y \mapsto v$, using Proposition 3.6 with an embedding order of $P(H)$ starting with the edges incident to $y$, a contradiction. \hfill \Box

### 3.7 Ignoring large $k$

The final proposition of this section is an analogue of [11, Proposition 2.5], which helps us to reduce the multicolor Turán problems for sufficiently large $k$ to a single value of $k$. The same proof works almost line by line.

**Proposition 3.9.** Let $H$ be a graph. Suppose that there is an $n_0 = n_0(k) > 0$ such that for $n > n_0$, there exists a fixed $H$-free simple graph $F_n$ of order $n$ such that every $n$-vertex $k$-extremal multigraph $G$ of $H$ consists of $k$ identical copies of $F_n$. Then for all $\ell \geq k$ and $n > n_0$, every $n$-vertex $\ell$-extremal multigraph $G$ of $H$ consists of $\ell$ identical copies of $F_n$.

This allows us that in the case of $k \geq k^*(H)$, we can only consider the case of $k = \lceil k^*(H) \rceil$, so that the lower bound of $n$ only depends on $H$ rather than both $H$ and $k$.

### 4 4-VERTEX 4-COLOR-CRITICAL MULTIGRAPHS ARE GOOD

In this section, we prove the following theorem.

**Theorem 4.1.** Every 4-vertex 4-color-critical multigraph is good.

We start with a proof sketch of this result. Let $H$ be a 4-vertex 4-color-critical multigraph with $h$ edges and the fixed critical edge, and let $G$ be a rainbow-$H$-free multigraph of order $n$ such that

$$e(G) \geq \begin{cases} \binom{h-1}{2} & \text{if } k < k^* = k^*(H) = \frac{3}{2}(h-1), \\ kt_3(n) & \text{if } k \geq k^*. \end{cases}$$

Now we show that the above assumption implies either $G = (h - 1)K_n$ or $G = kT_3(n)$. By Proposition 3.3, we may assume that $\delta(G)$ is as large as both of $\delta((h - 1)K_n)$ and $\delta(kT_3(n))$. If $G \neq (h - 1)K_n$, then the above assumption on $e(G)$ yields an edge $v_1v_2$ of multiplicity at least $h$.

We wish to find a vertex $v_3$ such that 3-vertex graph $G[[v_1, v_2, v_3]]$ is $H$-friendly. If $G[[v_1, v_2, v_3]]$ is $H$-friendly, then we can prove that one of the following to obtain a desired conclusion.

1. If $k < k^*$, then Proposition 3.4 yields a vertex $v_4$ with $w(v_1v_4) + w(v_2v_4) + w(v_3v_4) \geq 3(h - 1)$. As $w(v_i v_4) \leq k$ holds for each $i$ and $k < k^*$, we obtain $\min_{i \in [3]} \{w(v_i v_4)\} \geq 1$. The $H$-friendliness of the existence of a multicolored copy of $H$, a contradiction. This contradiction yields $G = (h - 1)K_n$.

2. If $k \geq k^*$, Lemma 3.8 implies that $G$ is 3-partite, concluding that $G = kT_3(n)$.

For this purpose, we want to find a vertex $v_3$ such that the existence of $v_4$ with $\sum_{i \in [3]} w(v_i v_4) \geq \max\{3(h - 1), 2k\}$ and $\min_{i \in [3]} \{w(v_i v_4)\} \geq 1$ implies a multicolored copy of $H$. As this depends on $H$,
we consider the two cases based on the multiplicity of the edge in \( H \) not adjacent to the critical edge (which is denoted by \( h_5 \) in Lemma 4.2).

If \( h_5 \) is small\(^3\) (smaller than the number \( b_1 \) in Lemma 4.2), then we utilize Proposition 3.4 to obtain a vertex \( v_3 \) satisfying the following:

\[
w(v_1v_3) + w(v_2v_3) \geq \max \left\{ 2(h-1), \frac{4}{3}k \right\}.
\]

Then Lemma 4.2 applies to show that \( G[\{v_1, v_2, v_3\}] \) is \( H \)-friendly.

If \( h_5 \) is large, then we can exploit the fact that other edges of \( H \) has relatively small multiplicities as their sum is fixed to be \( h \). By planning to embed the edge \( h_5 \) into \( v_1v_2 \) and the critical edge to \( v_3v_4 \), we consider the two cases based on the multiplicity of the other pairs \( v_i \) with \( (i, j) \in \{1, 2\} \times \{3, 4\} \) become more relaxed, so finding a multicolored copy of \( H \) becomes less difficult. This relaxed condition allows us to find a vertex \( v_3 \) where \( G[\{v_1, v_2, v_3\}] \) is \( H \)-friendly. In Lemma 4.4, we formalize this intuition by considering a set \( S \) of the vertices \( v \in G \) with “large” value of \( w(v_1v) + w(v_2v) \). This completes the proof sketch.

The following lemma encapsulates a lot of case-studies in our argument. To describe this lemma more conveniently, we introduce one notation. Let \( p_1, \ldots, p_\ell \) be real numbers. For each \( j \in \{\ell\} \), we denote the \( j \)th smallest number among \( \{p_1, \ldots, p_\ell\} \) by \( \min_j \{p_i : 1 \leq i \leq \ell\} \). In particular, if \( p_1 \leq \ldots \leq p_\ell \), then \( \min_j \{p_i\} = p_j \).

**Lemma 4.2.** Let \( H \) be a 4-vertex 4-color-critical multigraph with \( h \) edges, and \( h \leq k \). Let \( G \) be a simply \( k \)-colored 3-vertex multigraph with the multiplicities \( w(v_1v_2) = a, w(v_1v_3) = b_1, \) and \( w(v_2v_3) = b_2 \) such that \( a \geq h \) and \( b_1 + b_2 \geq \max \{2(h-1), \frac{4}{3}k\} \). Suppose the edge multiplicities of \( H \) are given as in Figure 1. If \( h_5 < \min\{b_1, b_2\} \), then \( G \) is \( H \)-friendly.

**Proof.** To show the \( H \)-frinedliness of \( G \), we assume that \( v_4 \) is a vertex with \( w(v_1v_4) = c_i \) for each \( i \in \{3\} \) and \( c_1 + c_2 + c_3 \geq \max \{3(h-1), 2k\} \) and \( \min_{i \in \{3\}} \{c_i\} \geq 1 \) and argue that the graph \( G_0 \) obtained by adding such a vertex \( v_4 \) to \( G \) contains a multicolored copy of \( H \). See Figure 1. Without loss of generality, assume that \( b_1 \leq b_2 \). We first claim that the following inequalities hold.

\[
a \geq h; \quad b_1 \geq \frac{h-1}{2}; \quad b_2 \geq h-1; \\quad c_1 + c_2 + c_3 \geq \max \{3h-3, 2k\}; \quad \min \{c_i\} \geq 1; \quad \min_2 \{c_i\} \geq \frac{3h-3}{4}; \quad \max \{c_i\} \geq h-1.
\]

---

\(^3\)For each edge, we often use its label to refer its multiplicity if there is no risk of confusion.
Indeed, \(b_1 + b_2 \geq \max\{2(h-1), \frac{4}{3}k\}\) and \(b_1 \leq b_2 \leq k\) imply the first line above, and \(c_1 + c_2 + c_3 \geq \max\{3(h-1), 2k\}\) and \(\max\{c_1, c_2, c_3\} \leq k\) imply the second line above.

We denote the critical edge \(x_1x_2\) by \(I\). We consider several cases, and for each case we determine the proper edge embedding orders using Proposition 3.6. We will decide two enumerations \((e_1, \ldots, e_6)\) and \((f_1, \ldots, f_6)\) of \(P(H)\) and \(P(G_0)\), respectively, and embed each \(e_i\) into \(f_i\). Except the last one case, we will set

\[
e_1 = I \quad \text{and} \quad (f_i)_{i=1}^6 = (\min\{c_i\}, b_1, \min_2\{c_i\}, \max\{c_i\}, b_2, a) .
\]

The embedding orders will be determined so that they are consistent with respect to the vertex-edge incidence relations and they satisfy the condition in Proposition 3.6. In particular, since \((f_i)_{i=1}^6\) is fixed, if we determine the positions of two incident edges within \((e_1, \ldots, e_6)\), this determines how three vertices (thus, indeed all four vertices) must be mapped into \(G_0\) and determine the entire sequence \((e_1, \ldots, e_6)\). This concludes the following useful fact.

For two incident edges \(f_i, f_j\) of \(G_0\) and two incident edges \(e_i, e_j\) of \(H\), there exists exactly one embedding \(\phi : H \leftrightarrow G_0\) such that \(\phi(e_i) = f_i\) and \(\phi(e_j) = f_j\).

(2)

Note that the above choice of \((f_1, \ldots, f_6)\) implies

\[
f_1 \geq 1 = I, \quad f_4, f_5 \geq h - 1 \geq h - e_6 = e_1 + \cdots + e_5, \quad \text{and} \quad f_6 \geq h .
\]

(3)

Thus, to verify that the orders we decide is proper, we only need to check

\[
f_2 \geq 1 + e_2 \quad \text{and} \quad f_3 \geq 1 + e_2 + e_3 .
\]

(4)

**Case 1**: \(h_1, \ldots, h_5 < (h-1)/2\) In this case, as every \(h_i\) is at least 1 and they sum up to \(h - 1\), we see

\[
\min\{h_i\} \leq \frac{h - 1}{5} \quad \text{and} \quad \min_2\{h_i\} \leq \frac{h - 2}{4} .
\]

We consider two subcases as follows.

**Case 1-(1):** \(f_1 = \min\{c_i\} \neq c_2\). In this case, we let \(e_2 := \min\{h_1, \ldots, h_4\}\). As \(e_1\) and \(e_2\) are fixed and they are incident edges of \(H\) and \(f_1\) and \(f_2\) are also incident in \(G_0\), (2) yields a unique embedding.\(^4\) Moreover, this yields proper embedding orders because the orderings satisfy (4) as

\[
f_2 \geq \frac{h - 1}{2} \geq 1 + \min_2\{h_i\} \quad \text{and} \quad f_3 \geq \left\lfloor \frac{3h - 3}{4} \right\rfloor \geq 1 + \left\lfloor \frac{h - 1}{2} \right\rfloor + \left\lfloor \frac{h - 2}{4} \right\rfloor \geq 1 + \max\{h_i\} + \min_2\{h_i\} \geq 1 + e_2 + e_3.
\]

\(^4\)For example, if \(f_1 = c_1\) and \(e_2 = h_1\), then we should embed \(x_1, x_2, x_3, x_4\) to \(v_1, v_4, v_3, v_2\) in this order. So, \((e_i)_{i=1} = (I, h_1, h_4, h_2, h_3, h_3)\) if \(\min_2\{c_i\} = c_2\) and \((e_i)_{i=1} = (I, h_1, h_2, h_4, h_3, h_3)\) otherwise.
Here, the second inequality for \( f_3 \) can be checked by considering the value of \( h \mod 4 \). In other words, for \( \ell \in \mathbb{N} \),

\[
\left\lfloor \frac{3(4\ell - 3) - 3}{4} \right\rfloor = 3\ell \geq 1 + (2\ell - 1) + (\ell - 1) = 1 + \left\lfloor \frac{(4\ell - 1) - 1}{2} \right\rfloor + \left\lfloor \frac{(4\ell - 2) - 2}{4} \right\rfloor.
\]

\[
\left\lfloor \frac{3(4\ell + 1) - 3}{4} \right\rfloor = 3\ell \geq 1 + (2\ell) + (\ell - 1) = 1 + \left\lfloor \frac{(4\ell + 1) - 1}{2} \right\rfloor + \left\lfloor \frac{(4\ell + 1) - 2}{4} \right\rfloor.
\]

\[
\left\lfloor \frac{3(4\ell + 2) - 3}{4} \right\rfloor = 3\ell + 1 \geq 1 + (2\ell) + (\ell) = 1 + \left\lfloor \frac{(4\ell + 2) - 1}{2} \right\rfloor + \left\lfloor \frac{(4\ell + 2) - 2}{4} \right\rfloor.
\]

\[
\left\lfloor \frac{3(4\ell + 3) - 3}{4} \right\rfloor = 3\ell + 2 \geq 1 + (2\ell + 1) + (\ell) = 1 + \left\lfloor \frac{(4\ell + 3) - 1}{2} \right\rfloor + \left\lfloor \frac{(4\ell + 3) - 2}{4} \right\rfloor.
\]

Case 1-(2): \( f_1 = \min\{c_i\} = c_2 \). In this case, we let \( e_2 := h_5 \) and \( e_3 := \min\{h_1, \ldots, h_4\} \). As \( e_1 \) and \( e_3 \) are fixed and they are incident edges of \( H \) and \( f_1 \) and \( f_3 \) are also incident in \( G_0 \), (2) yields a unique embedding. Moreover, this yields proper embedding orders because the orderings satisfy (4) as

\[
f_2 \geq \left\lfloor \frac{h - 1}{2} \right\rfloor \geq 1 + h_5 \quad \text{and}
\]

\[
f_3 \geq \left\lfloor \frac{3h - 3}{4} \right\rfloor \geq 1 + \left\lfloor \frac{h - 1}{2} \right\rfloor + \left\lfloor \frac{h - 1}{5} \right\rfloor \geq 1 + \max\{h_1\} + \min\{h_i\} \geq 1 + e_2 + e_3.
\]

Here, by noting that \( \left\lfloor \frac{h - 2}{4} \right\rfloor \geq \left\lfloor \frac{h - 1}{5} \right\rfloor \), the second inequality for \( f_3 \) is implied by \( \left\lfloor \frac{3h - 3}{4} \right\rfloor \geq 1 + \left\lfloor \frac{h - 1}{2} \right\rfloor + \left\lfloor \frac{h - 2}{4} \right\rfloor \) shown in the previous case.

Case 2: \( \max\{h_1, \ldots, h_4\} \geq (h - 1)/2 \) In this case, without loss of generality suppose that

\[
h_1 = \max\{h_1, \ldots, h_4\} \geq (h - 1)/2 \quad \text{and thus} \quad h_2 + h_3 + h_4 + 5 \leq (h - 1)/2.
\]

Hence, we have

\[
\min\{h_2, \ldots, h_5\} \leq \frac{h - 5}{4} \quad \text{and} \quad \max\{h_2, \ldots, h_5\} \leq \frac{h - 7}{2}.
\]

If \( f_1 = \min\{c_i\} \neq c_3 \), then we set \( e_6 := h_1 \) so that we will embed \( h_1 \) to a which has the largest multiplicity lower bound. As \( e_6 = h_1 \) is incident with \( e_1 = I \) and \( f_1 \) and \( f_6 \) are incident in \( G_0 \), (2) yields a unique embedding.

If \( f_1 = \min\{c_i\} = c_3 \), then we want to embed \( I \) to \( c_3 \). As \( h_1 \) is incident with \( I \), we cannot embed \( h_1 \) to \( a \) which is not incident with \( c_3 \). Hence we let \( e_5 := h_1 \) so that we embed \( h_1 \) to the edge of \( G_0 \) whose multiplicity lower bound is the second largest. As \( e_5 \) is incident with \( e_1 = I \) and \( f_1 \) and \( f_5 \) are incident in \( G_0 \), (2) yields a unique embedding.

In either case, the embedding orders are proper embedding orders because (4) holds as

\[
f_2 \geq \frac{h - 1}{2} \geq 1 + \frac{h - 7}{2} \geq 1 + \max\{h_2, \ldots, h_5\} \geq 1 + e_2 \quad \text{and}
\]

\[
f_3 \geq \frac{3h - 3}{4} \geq 1 + \frac{h - 7}{2} + \frac{h - 5}{4} \geq 1 + \max\{h_2, \ldots, h_5\}
\]

\[
+ \min\{h_2, \ldots, h_5\} \geq 1 + e_2 + e_3.
\]
Case 3: \( h_5 \geq (h - 1)/2 \) and \( b_1 \geq h_5 + 1 \). In this case, we have \( h_1 + h_2 + h_3 + h_4 \leq (h - 1)/2 \). Without loss of generality, we assume \( h_1 = \min\{h_1, \ldots, h_4\} \). Note that \( e_1 = I \) is incident with \( h_1 \). We have

\[
h_1 \leq \frac{h - 1}{8} \quad \text{and} \quad \max\{h_1, \ldots, h_4\} \leq \frac{h - 7}{2}.
\]

Case 3-(1): \( f_1 = \min\{c_i\} \neq c_2 \). Let \( e_2 := h_1 \). Then \( e_1 = I \) and \( e_2 = h_1 \) are incident in \( H \) while \( f_1 \) and \( f_2 \) are incident in \( G_0 \). Hence (2) yields a unique embedding. These embedding orders are proper because they satisfy (4) as

\[
f_2 \geq \frac{h - 1}{2} \geq 1 + \frac{h - 1}{8} \geq 1 + h_1 = 1 + e_2 \quad \text{and}
\]
\[
f_3 \geq \frac{3h - 3}{4} \geq 1 + \frac{h - 1}{8} + \frac{h - 7}{2} \geq 1 + h_1 + \max\{h_1, \ldots, h_4\} \geq 1 + e_2 + e_3.
\]

Case 3-(2): \( f_1 = \min\{c_i\} = c_2 \) and \( \min_2\{c_i\} \geq 1 + h_1 + h_5 \). As \( f_1 = c_2 \) and \( f_2 = b_1 \) are not incident, we choose \( e_2 \) first instead of \( e_2 \). Also, let \( e_3 := h_1 \). As \( e_1 = I \) and \( e_3 = h_1 \) are incident in \( H \) while \( f_1 \) and \( f_3 \) are incident in \( G_0 \), (2) yields a unique embedding, and we have \( e_2 = h_5 \) in this unique ordering. This uniquely determines \( (e_i)_i \), which is proper as

\[
f_2 \geq 1 + h_5 = 1 + e_2 \quad \text{and}
\]
\[
f_3 \geq 1 + h_5 + h_1 = 1 + e_2 + e_3.
\]

Case 3-(3): \( \min\{c_i\} = c_2 \) and \( \min_2\{c_i\} \leq h_1 + h_5 \). This is the only case in which the prescribed \( (e_i)_i \) and \( (f_i)_i \) are not used. Suppose first that \( c_2 \leq c_1 \leq c_3 \). From

\[
\frac{3h - 3}{4} \leq c_1 \leq h_1 + h_5 \leq \frac{h - 1}{8} + h_5,
\]

we see \( h_5 \geq (5h - 5)/8 \). As \( h_1 + \cdots + h_5 = h - 1 \), this yields new bounds

\[
h_1 = \min\{h_1, \ldots, h_4\} \leq \frac{3h - 3}{32} \quad \text{and} \quad \max\{h_1, \ldots, h_4\} \leq \frac{3h - 27}{8}.
\]

In addition, from \( c_1 + c_2 + c_3 \geq \max\{3h - 3, 2k\} \) we have

\[
c_1 + c_2 \geq \begin{cases} 
(3h - 3) - \frac{3h - 4}{2} & \text{if } h \leq \frac{3h - 4}{2}, \\
2k - k & \text{if } k \geq \frac{3h - 3}{2}.
\end{cases}
\]

With this and \( c_1 \leq h_1 + h_5 \leq h - (1 + h_2 + h_3 + h_4) \leq h - 4 \), it follows \( c_2 \geq (h + 5)/2 \). Set \( (e_i)_i := (I, h_1, h_3, h_5, h_2) \) and \( (f_i)_i := (c_1, b_1, c_2, c_3, b_2, a) \). This is proper as the inequalities in (3) holds along with the following ones:

\[
f_2 \geq \frac{h - 1}{2} \geq 1 + \frac{3h - 3}{32} \geq 1 + h_1 = 1 + e_2,
\]
\[
f_3 \geq \frac{h + 5}{2} \geq 1 + \frac{3h - 3}{32} + \frac{3h - 27}{8} \geq 1 + h_1 + \max\{h_1, \ldots, h_4\} \geq 1 + e_2 + e_3.
\]
On the other hand, suppose $c_2 \leq c_3 \leq c_1$. Then we essentially follow the above argument as follows, except for the way we set $(f_i)_i$. From

$$\frac{3h-3}{4} \leq c_3 \leq h_1 + h_5 \leq \frac{h-1}{8} + h_5,$$

we see $h_5 \geq (5h-5)/8$. As $h_1 + \ldots + h_5 = h - 1$, this yields new bounds

$$h_1 = \min\{h_1, \ldots, h_4\} \leq \frac{3h-3}{32} \quad \text{and} \quad \max\{h_1, \ldots, h_4\} \leq \frac{3h-27}{8}.$$

In addition, from $c_1 + c_2 + c_3 \geq \max\{3h-3, 2k\}$ we have

$$c_3 + c_2 \geq \begin{cases} \frac{(3h-3) - \frac{h-1}{2} - \frac{3h-3}{2}}{2k} & \text{if } h \leq k \leq \frac{3h-4}{2}, \\ \frac{h-1}{2} & \text{if } k \geq \frac{3h-3}{2}. \end{cases}$$

With this and $c_3 \leq h_1 + h_5 \leq h - (1 + h_2 + h_3 + h_4) \leq h - 4$, it follows $c_2 \geq (h + 5)/2$. Set $(e_i)_i := (I, h_1, h_4, h_2, h_3, h_5)$ and $(f_i)_i := (c_3, b_1, c_2, c_1, b_2, a)$. This is proper as the inequalities in (3) holds along with the following ones:

$$f_2 \geq \frac{h-1}{2} \geq \frac{2h-3}{32} \geq 1 + h_1 = 1 + e_2,$$
$$f_3 \geq \frac{h+5}{2} \geq \frac{3h-3}{32} + \frac{3h-27}{8} \geq 1 + h_1 \geq \max\{h_1, \ldots, h_4\} \geq 1 + e_2 + e_3.$$

The only remaining case is when $h_5 \geq (h - 1)/2$ and $b_1 \leq h_5$, but we assume $b_1 > h_5$ in the statement of the lemma. This completes the proof of Lemma 4.2.

In order to prove Theorem 4.1, we first prove the following stronger lemma which implies the theorem straightforwardly. We will use this lemma later in the proof of Theorem 6.1.

**Lemma 4.3.** Suppose $k \geq h$ and $0 < \frac{1}{n} \ll \delta \ll \frac{1}{k} < 1$. Let $H$ be a 4-vertex 4-color-critical multigraph with $h$ edges, and $k^* := k^*(H) = \frac{3}{2}(h - 1)$. Let $G$ be a simply $k$-colored multicolored-$H$-free multigraph of order $n$ such that

$$\delta(G) \geq \begin{cases} (1 - \delta)(h-1)(n-1) & \text{if } h \leq k < k^*, \\ (1 - \delta)kd_5(n) & \text{if } k \geq k^*. \end{cases}$$

1. If $h \leq k < k^*$, then $w(e) \leq h - 1$ for all $e \in P(G)$.
2. If $k \geq k^*$ and $G$ has an edge with multiplicity at least $h$, then $G$ is 3-partite.

**Proof of Theorem 4.1 using Lemma 4.3.** Let $H$ be a 4-vertex 4-color-critical multigraph with $h$ edges, and $k^* := k^*(H) = \frac{3}{2}(h - 1)$.

First consider when $h \leq k < k^*$. Let $G$ be a simply $k$-colored multigraph of order $n$ not containing a multicolored copy of $H$, such that $e(G) \geq (h - 1)\left(\binom{n}{2}\right)$. By Proposition 3.3, we may assume $\delta(G) \geq (h - 1)(n - 1)$. Then Lemma 4.3 applies so that every edge in $G$ has multiplicity at most $h - 1$, which gives the desired result.

Next, consider when $k \geq k^*$. By Proposition 3.9, it is enough to prove Theorem 4.1 for $k = [k^*]$. Indeed, then, it would also follow for all $k \geq k^*$. Let $G$ be a simply $k$-colored
multigraph of order $n$ not containing a multicolored copy of $H$, such that $e(G) \geq k\delta_3(n)$. By Proposition 3.3 assume $\delta(G) \geq k\delta_3(n)$. There is an edge with multiplicity at least $h$, since otherwise $e(G) \leq (h-1) \binom{n}{2} < k\delta_3(n)$, a contradiction. Then Lemma 4.3 applies so that $G$ is 3-partite, which gives the desired result.

To prove Lemma 4.3, we use the following lemma. It implies that, if there is one edge in $G$ of large multiplicity, then we can find an $H$-friendly subgraph for a 4-vertex 4-color-critical multigraph $H$ within a multigraph $G$ with large minimum degree.

**Lemma 4.4.** Suppose $k \geq h$ and $0 < \frac{1}{n} \ll \delta \ll \frac{1}{k} < 1$, and $k^* := k^*(4,h) = \frac{3}{2}(h-1)$. Let $H$ be a 4-vertex 4-color-critical multigraph with $h$ edges and $G$ be a simply $k$-colored multicolored-$H$-free multigraph of order $n$ with

$$\delta(G) \geq \begin{cases} 
(1 - \delta)(h-1)(n-1) & \text{if } h \leq k < k^*, \\
(1 - \delta)k\delta_3(n) & \text{if } k \geq k^*.
\end{cases}$$

Suppose there is an edge $v_1v_2$ of multiplicity at least $h$. Then there is $v_3 \in V(G) - \{v_1, v_2\}$ such that $G[\{v_1, v_2, v_3\}]$ is $H$-friendly.

**Proof.** Label the edge multiplicities of $H$ as in Figure 2. Let $k' := \max\{k^*, k\}$.

Suppose first that $h_5 < (h-1)/2$. As $(1-\delta)k^*d_3(n) \geq (1-\delta)(h-1)(n-1)$, using Proposition 3.4, we can choose $v_3 \in V - \{v_1, v_2\}$ such that $w(v_1v_3) + w(v_2v_3) \geq \max\{2(h-1), \frac{4}{3}k\}$. Then Lemma 4.2 implies that $G[\{v_1, v_2, v_3\}]$ is $H$-friendly.

Next, suppose $h_5 \geq (h-1)/2$. Define a set

$$S := \left\{ v \in V - \{v_1, v_2\} : w(vv_1) + w(vv_2) \geq k' + \frac{h-h_5+1}{2} \right\}.$$  

**Claim 1.** $S$ is independent.

**Proof.** Suppose to the contrary that there is an edge $v_3v_4$ in $S$. Let $G_0 := G[\{v_1, \ldots, v_4\}]$, and label the edge multiplicities as in Figure 2. Without loss of generality assume $m_1 \leq m_2$.

As in the proof of Lemma 4.2, for each edge, we use the label referring its multiplicity to refer the edge itself, and we denote $I := x_1x_2$. We claim $G_0$ contains a multicolored $H$, so we determine the proper edge embedding orders $(e_i)_{i=1}^6$ and $(f_i)_{i=1}^6$ of $P(H)$ and $P(G_0)$, respectively. We assign: $e_1 = I, f_1 = v_3v_4$ and $e_6 = h_5, f_6 = v_1v_2$. 

![Figure 2](image-url) The labelings of $G_0$ and $H.$
Suppose $m_4 \leq m_3$. Then as $m_1 \leq m_2$, we have $m_1 + m_4 \leq m_2 + m_3$,

$$m_1, m_4 \geq \left( k' + \frac{h - h_5 + 1}{2} \right) - k \geq \frac{h - h_5 + 1}{2},$$

$$m_2, m_3 \geq \frac{1}{2} \left( \frac{3}{2} (h - 1) + \frac{h - h_5 + 1}{2} \right) = h - \frac{h_5}{4} - \frac{1}{2} \geq h - h_5 = h_1 + h_2 + h_3 + h_4 + 1,$$

and $\min\{h_1 + h_4, h_2 + h_3\} \leq (h - h_5 - 1)/2$. Without loss of generality, suppose $h_1 + h_4 \leq h_2 + h_3$. Then set $(e_2, \ldots, e_5) = (h_1, h_4, h_2, h_3)$ and $(f_2, \ldots, f_3) = (m_1, m_4, m_2, m_3)$. It is straightforward to check that these embedding orders are consistent with respect to the vertex-edge incidence relations and they satisfy (1).

On the other hand, suppose $m_3 < m_4$. We essentially follow the above argument while switching $m_3$ and $m_4$ as follows. As $m_1 \leq m_2$, we have $m_1 + m_3 \leq m_2 + m_4$,

$$m_1, m_3 \geq \left( k' + \frac{h - h_5 + 1}{2} \right) - k \geq \frac{h - h_5 + 1}{2},$$

$$m_2, m_4 \geq \frac{1}{2} \left( \frac{3}{2} (h - 1) + \frac{h - h_5 + 1}{2} \right) = h - \frac{h_5}{4} - \frac{1}{2} \geq h - h_5 = h_1 + h_2 + h_3 + h_4 + 1,$$

and $\min\{h_1 + h_3, h_2 + h_4\} \leq (h - h_5 - 1)/2$. Without loss of generality, suppose $h_1 + h_3 \leq h_2 + h_4$. Then set $(e_2, \ldots, e_5) := (h_1, h_3, h_2, h_4)$ and $(f_2, \ldots, f_3) := (m_1, m_3, m_2, m_4)$. Again it is straightforward to check that these embedding orders are consistent with respect to the vertex-edge incidence relations and they satisfy (1).

Therefore $G_0$ contains a multicolored $H$ in both cases, a contradiction.

We claim that there exists a vertex $v_3 \in S$ such that $w(v_1 v_3) + w(v_2 v_3) \geq \frac{4k'}{3} + \frac{h_5 - 1}{2}$. Suppose not. Then we have $w(v_1 v) + w(v_2 v) \leq \frac{4k'}{3} + \frac{h_5}{2} - 1$ for all $v \in S$. Then, by the minimum degree condition of $G$ and the fact that $d_S(n) \geq \frac{2}{3}(n - 1)$, we have that $\delta(G) \geq (1 - \delta)k' \cdot \frac{2}{3}(n - 1)$. Thus,

$$2(1 - \delta)k' \cdot \frac{2}{3}(n - 1) \leq d(v_1) + d(v_2)$$

$$= e(\{v_1, v_2\}, S) + e(\{v_1, v_2\}, V - (S \cup \{v_1, v_2\})) + 2w(v_1 v_2)$$

$$\leq \left( \frac{4k'}{3} + \frac{h_5}{2} - 1 \right) |S| + \left( k' + \frac{h_5}{2} \right) (n - |S| - 2) + 2k.$$

Here, the final inequality holds by the definition of $S$. Using $h - h_5 = h_1 + h_2 + h_3 + h_4 + 1 \geq 5$ and $0 < \frac{1}{n} \ll \delta \ll \frac{1}{k'} < 1$ and $k' \geq 3(h - 1)/2$ and $h_5 \geq (h - 1)/2 = (h + 1)/2 - 1 \geq h - h_5 - 1 \geq 4$, we have the following.

$$|S| \geq \frac{(2k' - 3h + 3h_5 - 8\delta k')n - 12k + 4(1 + 2\delta)k' + 6(h - h_5)}{2k' - 3h + 6h_5 - 6}$$

$$> \frac{(2k' - 3h + 3h_5 - 1)n}{2k' - 3h + 6h_5 - 6} = \left( 1 - \frac{3h_5 - 5}{2k' - 3h + 3h_5 - 6} \right) n$$

$$\geq \left( 1 - \frac{3h_5 - 5}{3h_5 - 9} \right) n > \frac{n}{2}.$$
This fact together with Claim 1 implies that for any \( v \in S \),
\[
d(v) \leq k|V - S| < \frac{kn}{2} < (1 - \delta)k^2 \frac{2}{3} n \leq \delta(G),
\]
a contradiction. Thus, there exists \( v_3 \in S \) such that \( w(v_1v_3) + w(v_2v_3) \geq \frac{4k^2}{3} + \frac{h_i - 1}{2} \geq \max\{2(h - 1), \frac{4}{3} k\} \). Fix such a vertex \( v_3 \). Then we have \( w(v_1v_2) \geq h \) and \( \min\{w(v_2v_3), w(v_1v_3)\} = \frac{4k^2}{3} + \frac{h_i - 1}{2} - k' \geq \frac{h + h_i - 2}{2} > h_5 \), hence Lemma 4.2 implies that \( G[\{v_1, v_2, v_3\}] \) is \( H \)-friendly.

Therefore, there exists \( v_3 \in V - \{v_1, v_2\} \) making \( G[\{v_1, v_2, v_3\}] \) \( H \)-friendly.

We finish this section by proving Lemma 4.3 as follows.

**Proof of Lemma 4.3.** **Case 1:** \( h \leq k < k^* \). Suppose to the contrary that there is an edge \( v_1v_2 \) with multiplicity at least \( h \). By Lemma 4.4, there exists vertices \( v_1, v_2, v_3 \) such that \( G[\{v_1, v_2, v_3\}] \) is \( H \)-friendly.

Using Proposition 3.4, we can find a vertex \( v \in V - \{v_1, v_2, v_3\} \) such that \( w(v_1v) + w(v_2v) + w(v_3v) \geq 3(h - 1) \). As \( k < k^* = \frac{3}{2}(h - 1) \), we have \( 3(h - 1) = \max\{2k, 3(h - 1)\} \). Moreover, as each of \( w(v_1v), w(v_2v) \leq k \), we have that for each \( i \in [3] \), \( w(v_i v) \geq 3(h - 1) - 2k \geq 1 \). As \( G[\{v_1, v_2, v_3\}] \) is \( H \)-friendly, this implies that \( G[\{v_1, v_2, v_3, v\}] \) contains a multicolored copy of \( H \), a contradiction. Hence every edge of \( G \) has multiplicity at most \( h - 1 \), completing the proof.

**Case 2:** \( k \geq k^* \) and \( G \) has an edge with multiplicity at least \( h \). By applying Lemma 4.4, we obtain a vertex set \( \{v_1, v_2, v_3\} \) such that \( K := G[\{v_1, v_2, v_3\}] \) is an \( H \)-friendly multigraph. By applying Lemma 3.8 with \( r = 4 \) and \( K \), we conclude that \( G \) is 3-partite. This finishes the proof.

\[ \]

5 | \( r \)-VERTEX MULTIGRAPHS IN \( P_r \) ARE GOOD

Here we prove that for \( r \geq 5 \), the \( r \)-vertex multigraphs in \( P_r \) are good.

**Theorem 5.1.** For \( r \geq 5 \), the \( r \)-vertex multigraphs in \( P_r \) are good.

We again prove a stronger lemma which implies Theorem 5.1 straightforwardly as in the proof of Theorem 4.1. Again, this stronger lemma will be useful later when we prove Theorem 6.1.

**Lemma 5.2.** Suppose \( k \geq h \) and \( 0 \ll \frac{1}{n} \ll \delta \ll \frac{1}{k} < 1 \). Let \( H \) be an \( r \)-vertex multigraph in \( P_r \), with \( h \) edges and \( r \geq 5 \), and \( k^* := k^*(H) \). Let \( G \) be a simply \( k \)-colored multicolored-\( H \)-free multigraph of order \( n \) such that
\[
\delta(G) \geq \begin{cases} (1 - \delta)(h - 1)(n - 1) & \text{if } h \leq k < k^*, \\ (1 - \delta)kd_{r-1}(n) & \text{if } k \geq k^*. \end{cases}
\]

1. If \( h \leq k < k^* \), then \( w(e) \leq h - 1 \) for all \( e \in P(G) \).
2. If \( k \geq k^* \) and \( G \) has an edge with multiplicity at least \( h \), then \( G \) is \( (r - 1) \)-partite.

**Proof.** We use the following lemma which allows to find an \( H \)-friendly submultigraph of \( G \). Note that the edge multiplicities of \( H \) are almost uniform except the critical edge. We
will take an ordering of the vertices $x_1, \ldots, x_r$ of $H$ and will embed the vertex $x_i$ to a vertex $v_i$ in $G$. Once we embed the vertices, we will choose a color for each edge $v_iv_j$ in the lexicographic order on $(j, i)$. This sequential procedure suggests natural sufficient conditions, which are lower bounds on the certain sums of the multiplicities of edges in $G$. If the edge multiplicities of $H$ are almost uniform, then such lower bounds form a sequence which is very close to an arithmetic progression. This intuition is captured by the following lemma.

**Lemma 5.3.** Let $H$ be an $r$-vertex multigraph in $F_r$ with $h$ edges, and let $k \geq h$ and $k^* := k^*(H)$. Let $K$ be a simply $k$-colored multigraph on $\{v_1, \ldots, v_r\}$ such that $w(v_1v_2) \geq h$ and for each $3 \leq j \leq r - 1$,

$$\sum_{i=1}^{j-1} w(v_iv_j) \geq (j - 1) \max \left\{ (h - 1), \frac{r - 2}{r - 1}k \right\}.$$

Then $K$ is $H$-friendly.

First let $h \leq k < k^*$. Suppose to the contrary that there is an edge $v_1v_2$ with multiplicity at least $h$. By Proposition 3.4, we can find $v_3, \ldots, v_r$ such that $\sum_{i=1}^{r-1} e(v_iv_j) \geq (j - 1)(h - 1)$ for each $3 \leq j \leq r$. By Lemma 5.3, we see $G[\{v_1, \ldots, v_{r-1}\}]$ is $H$-friendly, so $G[\{v_1, \ldots, v_{r-1}, v_r\}]$ contains a multicolored $H$, a contradiction.

Next let $k \geq k^*$ and $v_1v_2$ be an edge with multiplicity at least $h$. By Proposition 3.4, we can find $v_3, \ldots, v_{r-1}$ such that $\sum_{i=1}^{r-1} e(v_iv_j) \geq (j - 1)\frac{r-2}{r-1}k$ for each $3 \leq j \leq r - 1$. Lemma 5.3 says that $G[\{v_1, \ldots, v_{r-1}\}]$ is $H$-friendly, thus Lemma 3.8 applies so that $G$ is $(r - 1)$-partite.

We finish this section with the following proof of Lemma 5.3.

**Proof of Lemma 5.3.** Let $v_j$ be a vertex not in $K$ such that $\sum_{i=1}^{r-1} w(v_iv_j) \geq (r-2)\max\{k, k^*\}$ and $\min\{w(v_iv_1)\} \geq 1$, and let $G_0 := G[V(K) \cup \{v_j\}]$. We need to show $G_0$ contains a multicolored $H$. For each $2 \leq j \leq r$, enumerate $\{v_jv_j : 1 \leq i \leq j - 1\}$ into an ascending order $e_{ij_1}, \ldots, e_{ij_{j-1}}$ of multiplicities. Since $w(e) \leq k$ for each $e \in P(G)$, we see that for all $2 \leq i \leq r$ and $1 \leq i \leq j - 1$,

$$w(e_{ij}) \geq \frac{1}{i} \left( \sum_{\ell=1}^{j-1} w(v_{\ell}v_j) - \sum_{\ell=i+1}^{j-1} w(e_{\ell j}) \right) \geq \frac{1}{i} \left( (j - 1) \max \left\{ h - 1, \frac{r - 2}{r - 1}k \right\} - (j - 1 - i)k \right) \geq \frac{1}{i} \left( (j - 1) \cdot \max \left\{ h - 1, \frac{r - 2}{r - 1}k \right\} - (j - 1 - i) \cdot \frac{r - 1}{r - 2} \max \left\{ h - 1, \frac{r - 2}{r - 1}k \right\} \right) \geq \frac{i(r-1) - (j-1)}{i(r-2)} \max \left\{ h - 1, \frac{r - 2}{r - 1}k \right\} \geq \frac{i(r-1) - (j-1)}{i(r-2)} (h - 1) =: b_{ij}.$$

We have $w(e_{2,1}) = w(v_1v_2) \geq h$ by the assumption. If $h \leq k < k^*$, then the third inequality is strict, so the above gives $w(e_{r,1}) > b_{r,1} = 0$, that is, $w(e_{r,1}) \geq 1$; if $k \geq k^*$ we have $w(e_{r,1}) \geq 1$ by the assumption. We embed $H$ in $G_0$ so that a critical edge of $H$ is embedded
in $e_{r,1}$ and the vertex-edge incidence relation is preserved, with the edge embedding order of $P(H)$ given as

\[
e_{r,1}, e_{r-1,1}, \ldots, e_{4,1}, e_{3,1}, e_{r,2}, e_{r-1,2}, \ldots, e_{4,2}, \\
\vdots \\
e_{r,r-3}, e_{r-1,r-3}, \\
e_{r,r-2}, \\
e_{r,r-1}, e_{r-1,r-2}, \ldots, e_{2,1}.
\]

Since $w(e_{j,j-1}) \geq b_{j,j-1} = h - 1$ for $3 \leq j \leq r$ and $w(e_{2,1}) \geq h$, this is a proper embedding order if the $m$th edge for $2 \leq m \leq \binom{r}{2}-(r-1)$ has multiplicity at least $(m-1) \cdot \alpha_r(h-1) + 1$

where $\alpha_r := \frac{2+2/r^2}{(r-1)(r-2)}$ (recall that $w(e) \leq \alpha_r(h-1)$ for all edges $e$ of $H$ by the definition of $F_r$). Since each $(b_{j,i})_{i=2}^{r+2}$ for a fixed $i$ is an arithmetic progression, it suffices to check this for the edges

$e_{r,i}$ and $e_{i+2,j}$ for each $1 \leq i \leq r - 2$, as well as $e_{r-1,1}$.

We start with the edges $e_{r,i}$. As we embed a critical edge of $H$ in $e_{r,1}$, we only need to consider $2 \leq i \leq r - 2$. For $2 \leq i \leq r - 2$, the edge $e_{r,i}$ is the $1 + \sum_{i=1}^{i-1}(r-(\ell+1)) = (i-1)(r-1) - \frac{i(i-1)}{2} + 1 =: m_{1,i}$th element in the list. Then we need to show

\[F_{1,i} := b_{r,i} - (m_{1,i} - 1) \alpha_r(h-1) > 0.\]

Note that for $2 \leq i \leq r - 2$, we have

\[\frac{b_{r,i}}{h-1} = \frac{r-1}{r-2} - \frac{r-1}{i(r-2)} = \frac{i-1}{r-2} \cdot \frac{r-1}{i},\]

and

\[
(m_{1,i} - 1) \alpha_r = (i-1) \left( (r-1) - \frac{i}{2} \right) \frac{1}{r-2} \left( \frac{2}{r-1} + \frac{2}{r^2(r-1)} \right) \\
= \frac{i-1}{r-2} \left( 2 - \frac{i}{r-1} + \frac{2}{r^2} - \frac{i}{r^2(r-1)} \right).
\]

Hence, we have

\[
\frac{r-2}{i-1} \cdot \frac{F_{1,i}}{h-1} = \frac{r-2}{i-1} \left( \frac{b_{r,i}}{h-1} - (m_{1,i} - 1) \alpha_r \right) \\
= \left( \frac{r-1}{i} - 2 + \frac{i}{r-1} \right) - \left( \frac{2}{r^2} - \frac{i}{r^2(r-1)} \right) \\
= \frac{(r-1-i)^2}{(r-1)i} + \frac{i^2 - 2(r-1)i}{r^2(r-1)i} \\
= \frac{r^2(r-1-i)^2}{r^2(r-1)i} + \frac{(r-1-i)^2 - (r-1)^2}{r^2(r-1)i} > 0.
\]
The final inequality holds as \( r^2(r - 1 - i)^2 \geq (r - 1)^2. \)

Next we consider the edges of the form \( e_{i+2,i}. \) As the edge \( e_{r,r-2} \) is already considered in the above case, we only need to deal with the edges \( e_{i+2,i} \) for \( 1 \leq i \leq r - 3. \) For \( 1 \leq i \leq r - 3, \) the edge \( e_{i+2,i} \) is the \( \sum_{\ell=1}^{i} (r - (\ell + 1)) = i(r - 1) - \frac{i(i+1)}{2} =: m_{2,i} \)th element in the list. Then we need to show

\[
F_{2,i} := b_{i+2,i} - (m_{2,i} - 1)\alpha_r(h - 1) > 0.
\]

We have

\[
\frac{b_{i+2,i}}{h - 1} = \frac{i(r - 1) - i - 1}{i(r - 2)} = \frac{i}{r - 2} \left( \frac{r - 2 - i - 1}{i} \right),
\]

and

\[
(m_{2,i} - 1)\alpha_r = \left( \frac{i(r - 1) - \frac{i(i + 1)}{2}}{i} + 1 \right)\alpha_r
= \frac{i}{r - 2} \left( \frac{(r - 1) - \frac{i(i + 1) + 2}{2i}}{r - 1} + \frac{2}{r^2} \right)
\leq \frac{i}{r - 2} \left( \frac{2 - \frac{i(i + 1) + 2}{i} + \frac{2}{r^2}}{r - 1} \right)
= \frac{i}{r - 2} \left( 2 - \frac{i + 1}{r - 1} - \frac{2}{i(r - 1)} + \frac{2}{r^2} \right).
\]

Hence, we have

\[
\frac{r - 2}{i} \cdot \frac{F_{2,i}}{h - 1} = \frac{r - 2}{i} \left( \frac{b_{i+2,i}}{h - 1} - (m_{2,i} - 1)\alpha_r \right)
\geq \left( \frac{r - 2}{i} - \frac{1}{i^2} \right) - \left( \frac{2 - \frac{i + 1}{r - 1} - \frac{2}{i(r - 1)} + \frac{2}{r^2}}{r - 1} \right)
\geq \left( \frac{r - 2}{i} - \frac{i + 1}{r - 1} - 2 \right) + \left( \frac{2}{i(r - 1)} - \frac{2}{(r - 1)^2} \right)
\geq \frac{r(i - 1)(r - i - 2) - 1}{i(r - 1)}
= \frac{i(r - i - 1)(r - i - 2) - (r - 1)}{i^2(r - 1)} \geq 0.
\]

The last inequality holds as we have \( i(r - i - 1)(r - i - 2) - (r - 1) \geq 0 \) because \( 1 \leq i \leq r - 3. \)

Finally, observe that \( w(e_{r-1,1}) \geq \frac{r - 1}{r - 2} > \alpha_r(h - 1). \) This finishes the verification that the edge embedding is indeed a proper one, which ends the proof of Lemma 5.3.  

6 STABILITY FOR \( r \)-VERTEX \( r \)-COLOR-CRITICAL MULTIGRAPHS

In previous two sections, we have determined \( \text{Ex}_k(n,H) \) for all 4-vertex 4-color-critical multigraphs and \( r \)-vertex multigraphs in \( P_r \) with \( r \geq 5. \)

We now prove that for such a graph \( H, \) if a simply \( k \)-colored multigraph \( G \) with no multicolored copy of \( H \) has close-to-maximum number of edges, then \( G \) is close to one of two natural extremal
graphs. We use the ideas in [14, Lemma 2.3] to prove this. For the convenience of writing, we use the following notation. For multigraphs \( G_1 \) and \( G_2 \) of the same order, define their symmetric difference

\[
|G_1 \triangle G_2| := \min_{\substack{e \in E(G_2) \setminus E(G_1) \setminus E(G_2) \setminus E(G_1)}} \sum_{e \in E(G_1)} |w_{G_1}(e) - w_{G_2}(e)|.
\]

as the minimum number of edges needed to be changed from \( G_1 \) to \( G_2 \).

**Theorem 6.1.** Suppose \( h \leq k \) and \( 0 < \frac{1}{n} < \eta \ll \varepsilon, \frac{1}{k} < 1 \). Let \( H \) be an \( r \)-vertex \( r \)-color-critical multigraph with \( h \) edges with \( r \geq 4 \). Furthermore, if \( r \geq 5 \), then assume \( H \in T_r \). Let \( G \) is a simply \( k \)-colored multicolored-\( H \)-free multigraph of order \( n \) such that

\[
e(G) \geq \begin{cases} 
\left(h - 1\right)\left(\frac{n}{2}\right) - \eta n^2 & \text{if } k < k^* := k^*(H), \\
KT_{r-1}(n) - \eta n^2 & \text{if } k \geq k^*.
\end{cases}
\]

Then we have:

1. if \( k = k^* \), then either \( |G \triangle (h - 1)Kn| \leq \varepsilon n^2 \) or \( |G \triangle KT_{r-1}(n)| \leq \varepsilon n^2 \);  
2. if \( k \neq k^* \), then \( |G \triangle \text{Ex}_k(n, H)| \leq \varepsilon n^2 \).

We remark that when \( k = k^* \), the difference between the numbers of edges in \((h - 1)Kn\) and \( KT_{r-1}(n)\) is \( o(n^2) \) (in fact, it is \( O(n) \)), thus in this case, we consider two possibilities in Theorem 6.1.

In order to prove this theorem, we will show that for those choices of \( H \), the graph \( G \) in Theorem 6.1 have a large minimum degree after deleting a small number of vertices. Then we may apply Lemma 4.3 or Lemma 5.2 to the remaining graph with high minimum degree.

We first collect the following useful proposition.

**Proposition 6.2.** Suppose \( 0 < \frac{1}{n} \ll \delta < 1 \). Let \( G \) be an \( n \)-vertex multigraph with \( d \left( \frac{n}{2} \right) \) edges. If \( B \subseteq V(G) \) is a vertex set with \( |B| = \frac{n}{2} \delta n \) and every \( v \in B \) satisfies \( d_G(v) < (1 - \delta)dn \), then \( G - B \) has at least \( \left(1 + \frac{1}{2} \delta^2\right)d \left( \frac{|G - B|}{2} \right) \) edges.

**Proof.** Consider \( J := G - B \), then we have

\[
e(J) \geq e(G) - \sum_{v \in B} d(v) > d \left( \frac{|G - B| + |B|}{2} \right) - (dn - \delta dn)|B|
\]

\[
= d \left( \frac{|G - B|}{2} \right) + d|G - B||B| + d \left( \frac{|B|}{2} \right) - d(|G - B| + |B|)|B| + \delta dn|B|
\]

\[
> d \left( \frac{|G - B|}{2} \right) + d|B|\delta n - |B|
\]

\[
= d \left( \frac{|G - B|}{2} \right) + \frac{1}{4}d \delta^2 n^2 > (1 + \frac{1}{2} \delta^2)d \left( \frac{|G - B|}{2} \right).
\]

**Proof of Theorem 6.1.** Theorems 4.1 and 5.1 show that \( H \) is good in either of the cases. Note that if \( k \neq k^* \), then \( |k - k^*| \geq \frac{1}{r-2} \) from the integrality of \( k \) and the definition of \( k^* \).

Note that the minimum degree \( \delta(\text{Ex}_k(n, H)) \) is \( \max \left\{ \frac{r-2}{r-1}k, h - 1 \right\}(n - 1) + O(1) = \max \left\{ k, \frac{r-2}{r-1}(h - 1) \right\}d_v(n) + O(1) \).
Let $\delta$ be such that $0 < \eta \ll \delta \ll \varepsilon, \frac{1}{k} < 1$. We first show that there is a submultigraph of $G$ of order at least $(1 - \delta^{1/2})n$ with minimum degree at least $(1 - \delta^{1/2}) \max \left\{ k, \frac{r-1}{r-2} (h-1) \right\} d_r(n)$. 

Let $d' > 0$ be the number such that $G$ contains $d'(\binom{n}{2})$ edges, then $d' \geq \max \left\{ \frac{r-2}{r-1} k, (h-1) \right\} - 2\eta$. Define

$$L := \{ v \in G : d(v) < (1-\delta)d' \}.$$ 

If $|L| \geq \delta n/2$, then choose any $B \subseteq L$ with $|B| = \delta n/2$. By Proposition 6.2, we conclude that the multigraph $G - B$ contains at least the following number of edges:

$$\left( 1 + \frac{1}{2} \delta^2 \right) d'(\binom{|G-B|}{2}) > \left( \max \left\{ \frac{r-2}{r-1} k, (h-1) \right\} - 2\eta + \frac{1}{2} \delta^2 \right) (\binom{|G-B|}{2}) > \max \left\{ k, \frac{r-1}{r-2} (h-1) \right\} t_{r-1}(|G-B|).$$

On the other hand, $G - B$ does not contain a multicolored copy of $H$ and $|G - B| > n/2 > \eta^{-1}/2$ with $\eta \ll 1/h$. Thus since $H$ is good, we have that $e(G - B) \leq \max \left\{ (h-1) \binom{|G-B|}{2}, k t_{r-1}(|G-B|) \right\} \leq \max \left\{ k, \frac{r-1}{r-2} (h-1) \right\} t_{r-1}(|G-B|)$, a contradiction. Hence, we have $|L| \leq \delta n/2$.

Now consider the graph $J := G - L$. Then $\sum_{v \in L} d(v) \leq k n |L|$, and

$$\delta(J) \geq (1-\delta) \max \left\{ k, \frac{r-1}{r-2} (h-1) \right\} d_r(n) - k |L| > (1 - \delta^{1/2}) \max \left\{ k, \frac{r-1}{r-2} (h-1) \right\} d_r(n).$$

By Lemma 4.3 for $r = 4$ or Lemma 5.2 for $r \geq 5$, we obtain that $J$ is either a subgraph of $(h-1)K_{|J|}$ or an $(r-1)$-partite graph.

First, suppose that $J$ is a subgraph of $(h-1)K_{|J|}$. If $k > k^*$, then

$$e(G) \geq e_k(n, H) - \eta n^2 \geq \left( k^* + \frac{1}{r-2} \right) t_{r-1}(n) - \eta n^2 > (h-1) \binom{n}{2},$$

a contradiction. Hence we have $k \leq k^*$ and as we have $e(J) \geq \delta(J)|J|/2 \geq (h-1) \binom{|J|}{2} - \delta^{1/3} n^2$ from (5), we conclude that

$$|G \Delta (h-1) K_n| \leq \sum_{e \in P(G) - P(J)} |(h-1) - w(e)| + ((h-1) \binom{|J|}{2} - e(J)) \leq kn |L| + ((h-1) \binom{|J|}{2} - ((h-1) \binom{|J|}{2} - \delta^{1/3} n)) \leq 2\delta^{1/3} n^2 < \varepsilon n^2.$$ 

If $J$ is not a subgraph of $(h-1)K_{|J|}$, then there is an edge $v_1 v_2$ in $J$ with multiplicity at least $h$. By Lemma 4.3 for $r = 4$ or Lemma 5.2 for $r \geq 5$, we obtain that $k \geq k^*$ and $J$ is an $(r-1)$-partite graph with parts $V_1, \ldots, V_{r-1}$. If $|V_i| - |V_j| \geq 2a$ for some $i \neq j$, then $e(J) \leq t_{r-1}(|J|) - a^2$, thus this together with (5) implies that two classes can differ in size by at most $2\delta^{1/3} n$. It follows that $|V_i| - |V_j|_{r=1} < 2\delta^{1/3} n$ for each $i$ and so by deleting at most
For $\epsilon > 0$ and a color $\rho$, the pair $(X, Y)$ is \((\epsilon; \rho)\)-regular if for every $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \epsilon |X|$ and $|Y'| \geq \epsilon |Y|$, we have $|d_\rho(X, Y) - d_\rho(X', Y')| \leq \epsilon$. Then $(X, Y)$ is $\epsilon$-regular if it is $(\epsilon; \rho)$-regular for all colors $\rho$. For $\epsilon, \gamma > 0$ and a color $\rho$, the pair $(X, Y)$ is \((\epsilon, \gamma; \rho)\)-lower-regular if for every $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \epsilon |X|$ and $|Y'| \geq \epsilon |Y|$, we have $d_\rho(X', Y') \geq \gamma$.

A partition $P = (V_1, \ldots, V_m)$ of $V(G)$ is an $\epsilon$-regular partition of a simply $k$-colored multigraph $G$ if

1. $|V_i| - |V_j| \leq 1$ for all $\{i, j\} \in \left(\begin{array}{c} m \\ 2 \end{array}\right)$;
2. all but at most $\epsilon m^2$ of the pairs $(V_i, V_j)$, $\{i, j\} \in \left(\begin{array}{c} m \\ 2 \end{array}\right)$ are $\epsilon$-regular.
For \( \varepsilon, \gamma > 0 \) and a given such an \( \varepsilon \)-regular partition \( \mathcal{P} \) of \( G \), consider a simply \( k \)-colored multigraph \( R \) with colors \( \{R_1, \ldots, R_k\} \) where the vertex set is \( \{v_1, \ldots, v_m\} \) and for each \( i \neq j \in \{m\} \) and \( \rho \in \{k\} \), we have \( v_i v_j \in E(R_\rho) \) if and only if \( (V_i, V_j) \) is \( (\varepsilon, \gamma; \rho) \)-lower-regular. We call this multigraph \( R \) the \( (\varepsilon, \gamma, \mathcal{P}) \)-reduced multigraph of \( G \). Consider a simply \( k \)-colored multigraph \( G^\mathcal{P} = G^\mathcal{P}(\varepsilon, \gamma) \) with vertex set \( V(G) \) where for all \( \varepsilon \neq j \in \{m\} \) and \( \rho \in \{k\} \) the bipartite graph \( G^\mathcal{P}_\rho[V_i, V_j] \) is a complete bipartite graph if \( v_i v_j \in E(R_\rho) \), and an empty bipartite graph if \( v_i v_j \notin E(R_\rho) \).

If the simple \( k \)-coloring of \( G \) is nested, then for any \( \varepsilon, \gamma > 0 \) and an \( \varepsilon \)-regular partition \( \mathcal{P} \), the definition of lower-regularity ensures that \( R \) and \( G^\mathcal{P} \) is also nested.

The following is easily derived from the proof outline of \cite[Theorem 1.18 (Many-color regularity lemma)]{12}.

**Theorem 7.2** (Multicolor regularity lemma). For any \( \varepsilon > 0 \) and integers \( k, M_0 \geq 1 \), there exists \( M \) such that every simply \( k \)-colored multigraph \( G \) with \( n \geq M \) vertices admits an \( \varepsilon \)-regular partition \( \mathcal{P} = \{V_1, \ldots, V_m\} \) with \( M_0 \leq m \leq M \). Moreover, for \( \gamma > 0 \), the \( (\varepsilon, \gamma, \mathcal{P}) \)-reduced multigraph \( R \) of \( G \) satisfies \( d(R) \geq d(G) - 2(\varepsilon + \gamma) \).

In fact the setting in \cite{12} is about a \( k \)-edge-colored simple graph, but the proof in \cite{12} exactly yields the above result in our settings. The moreover part of the above statement can be also derived by a routine computation.

On the other hand, as all but at most \( \varepsilon m^2 \) pairs \( (V_i, V_j) \) are \( \varepsilon \)-regular, one can easily see the following holds.

There exists a set \( E \subseteq E(G) \) of at most \( \left( \varepsilon + \frac{1}{m} + \gamma \right) n^2 \) edges of \( G \) such that \( G - E \subseteq G^\mathcal{P} \). \hfill (6)

One advantage of the regularity lemma is that it is useful to prove the existence of certain subgraph of \( G \). A slight modification of the proof of \cite[Theorem 2.1 (Key lemma)]{12} gives the following.

**Theorem 7.3** (Multicolor embedding lemma). Suppose \( 0 < \frac{1}{n} \ll \varepsilon \ll \gamma \ll \frac{1}{h} \leq 1 \). Let \( H \) be a multigraph on \( r \) vertices and \( h \) edges. Let \( G \) be a simply nestedly \( k \)-colored multigraph, and \( \mathcal{P} \) be its \( \varepsilon \)-regular partition. If \( G^\mathcal{P}(\varepsilon, \gamma) \) contains a multicolored copy of \( H \), then \( G \) contains a multicolored copy of \( H \).

Note that the above theorem is only true when the coloring of \( G \) is nested which can be assumed as mentioned in the preliminaries. Otherwise, all edges of \( G \) might have multiplicity 1 while \( H \) has an edge of larger multiplicity.

**Proof of Lemma 7.1.** Let \( 0 < \frac{1}{n} \ll \frac{1}{m_0} \ll \varepsilon \ll \gamma \ll \eta \ll \eta' \ll \mu \ll \frac{1}{k} \), and for \( s \geq 1 \), set

\[
G'(s) := \begin{cases} 
(h - 1)K_s & \text{if } h \leq k < k^*, \\
KT_{r-1}(s) & \text{if } k \geq k^*.
\end{cases}
\]

We show that if \( e(G) \geq e(G'(n)) - \eta m^2 \), then either of the following holds:

\[
|G \bigtriangleup G'(n)| < \mu n^2 \quad \text{if } k \neq k^*, \\
|G \bigtriangleup (h - 1)K_n| < \mu n^2 \quad \text{or} \quad |G \bigtriangleup KT_{r-1}(n)| < \mu n^2 \quad \text{if } k = k^*.
\] \hfill (7)

Apply Theorem 7.2 to \( G \) with the constants \( \varepsilon, k, 1/\varepsilon, m_0, M \) playing the roles of \( \varepsilon, k, M_0, M \) to obtain an \( \varepsilon \)-regular partition \( \mathcal{P} = \{V_1, \ldots, V_m\} \) with \( 1/\varepsilon \leq m \leq m_0 \).
Let \( R \) be an \((\varepsilon, \gamma, P)\)-reduced multigraph of \( G \). Then both \( R \) and \( G^P \) have a nested simple \( k \)-coloring. Then \( R \) is multicolored-\( H_c \)-free, since otherwise \( G^P \) contains a multicolored \( H \), and so does \( G \) by the embedding lemma.

By (6), there exists a set \( E \subseteq E(G) \) of edges of \( G \) with \(|E| \leq (\varepsilon + 1/m + \gamma/n^2) \leq \eta n^2 \) such that \( G - E \subseteq G^P \). Also, by the moreover part of Theorem 7.2, we have \( e(R) \leq e(G^P(m)) - 2n^2 \). By Theorem 6.1, we have the following: if \( k \neq k^* \), then \( |R \triangle G^P(m)| < \eta m^2 \). Otherwise, if \( k = k^* \), then \(|R \triangle (h - 1)K_m| < \eta m^2 \) or \( |R \triangle kT_{r-1}(m)| < \eta m^2 \).

If \(|R \triangle (h - 1)K_m| < \eta m^2 \), then

\[
|G^P \triangle (h - 1)K_m| < 2\eta n^2.
\]

This implies \( e(G^P) \leq (h - 1)\left(\frac{n}{2}\right) + 2\eta n^2 \leq e(G) + 3\eta n^2 \), thus \( |G \triangle G^P| \leq |E| + |E| + 3\eta n^2 \leq 5\eta n^2 \). Hence, we have

\[
|G \triangle (h - 1)K_m| \leq |G \triangle G^P| + |G^P \triangle (h - 1)K_m| \leq 8\eta n^2 \leq \mu n^2.
\]

If \(|R \triangle kT_{r-1}(m)| < \eta m^2 \), then

\[
|G^P \triangle kT_{r-1}(n)| < 2\eta n^2 + \frac{(r - 1)n^2}{m} \leq 3\eta n^2.
\]

This implies \( e(G^P) \leq kt_{r-1}(n) + 3\eta n^2 \leq e(G) + 4\eta n^2 \), thus \( |G \triangle G^P| \leq |E| + |E| + 4\eta n^2 \leq 6\eta n^2 \). Hence, we have

\[
|G \triangle kT_{r-1}(n)| \leq |G \triangle G^P| + |G^P \triangle kT_{r-1}(n)| \leq 10\eta n^2 \leq \mu n^2.
\]

From these, we can conclude that (7) holds.

We remark that the above proof implies that for any \( r \)-color-critical graph \( H \), not necessarily in \( T_r \), if its color-reduced multigraph \( H_c \) satisfies the statement of Theorem 6.1, then \( H \) also satisfies the conclusion of Lemma 7.1.

8 | PROOF OF THEOREMS 1.3 AND 1.4

In this section, using the results from the previous sections, we prove Theorems 1.3 and 1.4. Let \( H \) be an \( h \)-edge \( r \)-color-critical graph with \( r \geq 4 \). By Proposition 3.2, we may assume that \( H \) is in \( T_r \) if \( r \geq 5 \). We may further assume that \( H \) has no isolated vertices, so it has at most \( 2h \) vertices. Choose constants \( \varepsilon, \mu > 0 \) so that we have \( 0 < 1/n \ll \varepsilon \ll \mu \ll 1/h < 1 \). Let \( k^* = k^*(H) = \frac{r - 1}{r - 2}(h - 1) \). By Proposition 3.9, we may assume \( h \leq k \leq [k^*] \).

Assume that \( G \) is simply \( k \)-colored multigraph with

\[
e(G) \geq \begin{cases} 
(h - 1)\left(\frac{n}{2}\right) & \text{if } h \leq k < k^*, \\
k \cdot t_{r-1}(n) & \text{if } k \geq k^*.
\end{cases}
\]
By Proposition 3.5 and the discussion after it, assume the coloring of $G$ is nested. In addition, by Proposition 3.3, we may assume that the following holds.

$$\delta(G) \geq \begin{cases} (h-1)(n-1) & \text{if } k < k^*, \\ k \cdot d_{r-1}(n) & \text{if } k \geq k^*. \end{cases} \quad \text{(8)}$$

By Lemma 7.1, we have one of the following two cases.

**Case 1:** $|G \triangle (h-1)K_n| \leq \epsilon n^2$ and $k \leq k^*$

Note that $k^* \cdot d_{r-1}(n) \geq (h-1)(n-1)$, so (8) implies $\delta(G) \geq (h-1)(n-1)$ even if $k = k^*$. Let

$$E_1 := \{ e \in P(G) : w(e) \geq h \} \quad \text{and} \quad E_2 := \{ e \in P(G) : w(e) \leq h - 2 \}.$$

As $|G \triangle (h-1)K_n| \leq \epsilon n^2$, we have

$$|E_1 \cup E_2| \leq \epsilon n^2. \quad \text{(9)}$$

As $k \leq k^* < 2h - 1$, we have

$$0 \leq e(G) - (h-1)\binom{n}{2} \leq (k - (h-1))|E_1| - |E_2| < h|E_1| - |E_2|,$$

giving

$$|E_1| > \frac{1}{h}|E_2|. \quad \text{(10)}$$

We claim that either

(i) there exists an edge $v_1v_2$ and $A \subseteq V(G) - \{v_1, v_2\}$ of size at least $\frac{n}{2} - 2$ such that $w(v_1v_2) \geq h$ and $w(v_1u) \geq h - 1$ for all $i = 1, 2$ and $u \in A$, or

(ii) there exists a vertex $v$ and $B \subseteq V(G) - \{v\}$ of size at least $\frac{n}{4h}$ such that $w(vu) \geq h$ for all $u \in B$.

Suppose neither of the two cases hold. We then count the number of the 2-paths each of which consists of an edge in $E_1$ and an edge in $E_2$. As (i) does not hold, the number is at least $|E_1| \cdot \frac{n}{2}$. On the other hand, since (ii) does not hold, the number is at most $|E_2| \cdot 2^{\binom{n}{2}}$. However, (10) implies that $|E_2| \cdot 2^{\binom{n}{2}} < |E_1| \cdot \frac{n}{2}$, a contradiction.

In either case, by (9) and Turán’s theorem, there is a clique $C$ of size $|H| < \frac{1}{4hk}$ in $A$ or $B$ whose edges have multiplicity $h - 1$. Then we can find a multicolored copy of $H$ in either $G[\{v_1, v_2\} \cup V(C)]$ or $G[\{v\} \cup V(C)]$, a contradiction. Therefore $E_1 = E_2 = \emptyset$, implying $G = (h-1)K_n$.

**Case 2:** $|G \triangle kT_{r-1}(n)| \leq \epsilon n^2$ and $k \geq k^*$

Note in this case $k = \lceil k^* \rceil$. By (8), we have $\delta(G) \geq kd_{r-1}(n)$. Let $G'$ be a graph on the vertex set $V(G)$ with $uv \in E(G')$ if and only if $w_G(uv) = k$. Since $|G \triangle kT_{r-1}(n)| \leq \epsilon n^2$, we know that

$$e(G') \geq t_{r-1}(n) - \epsilon n^2 > t_{r-2}(n) + \epsilon n^2. \quad \text{By Erdős–Stone–Simonovits theorem, } G' \text{ contains a copy } K \text{ of an complete } (r-1)\text{-partite graph } K_{2h,2h,...,2h} \text{ with vertex partition } W_1, \ldots, W_{r-1}.$$

We claim that $K$ is $H$-friendly. Suppose that as in the definition of $H$-friendly multigraph, we added a new vertex $v$ to $K$ and added edges of multiplicity at most $k$ incident to $v$ so that $d(v) \geq (r-2)(2h) \cdot \max\{k, k^*\}$. and $d_{W_i}(v) \geq 1$ for all $i \in [r-1]$. Without loss of generality, assume $d_{W_1}(v) = \min\{d_{W_1}(v), \ldots, d_{W_{r-1}}(v)\}$. If $\min\{d_{W_2}(v), \ldots, d_{W_{r-1}}(v)\} < k/2$, then

$$d(v) < 2 \cdot (k/2) + (r-3) \cdot 2hk \leq ((r-3)(2h) + 1)k^* \leq (r-2)(2h)k^* \leq d(v),$$
a contradiction. Here the second inequality follows from $k = \lceil k^n \rceil$. Thus $\min\{d_{W_2}(v), \ldots, d_{W_{r-1}}(v)\} \geq k/2$.

Let $xy$ be a critical edge of $H$. Then $d(x) + d(y) + \binom{r-2}{2} \leq h + 1$, so without loss of generality assume $d(y) \leq h/2 < k/2$. This allows an embedding of a multicolored $H$ into $G[K \cup \{v\}]$ with $x$ being embedded in $W_1$ and $y \mapsto v$, using Proposition 3.6 with an embedding order of $P(H)$ starting with the edges incident to $y$.

Moreover, $K$ is an induced subgraph of $G$, as an additional edge within a color class of $K$ yields a copy of $H$. Hence, $G$ contains an $H$-friendly subgraph with $2h(r-1)$ vertices as an induced subgraph. Now Lemma 3.8 implies that $G$ is $(r-1)$-partite. Since $e(G) \geq kT_{r-1}(n)$, we conclude that $G = kT_{r-1}(n)$.

If $k \neq k^n$, we have identified the unique extremal graph as above. If $k = k^n$, we have concluded that $G$ is either $(h-1)K_n$ or $kT_{r-1}(n)$. However, the former graph contains less number of edges than $kT_{r-1}(n)$, thus the latter is the unique $k$-extremal graph for $H$ in this case.

9 | CONCLUDING REMARKS

One obvious remaining question is to determine the $k$-color extremal numbers for $r$-color-critical graphs not in $T_r$. Also, determining $k$-color extremal numbers for non-color-critical graphs is also a natural question. In fact, proof techniques in this article provides asymptotics of $k$-color extremal number of some non-color critical graphs. Consider an $r$-partite $h$-edge graph $H$ with a vertex partition $(X_1, \ldots, X_r)$ such that $\min_{i \in \{1, \ldots, r\}} e(X_i, X_j) = m$ and $e(X_i, X_j) \leq \binom{r}{2}^{-1}(h-m) + O((h-m)/r^3)$.

Additionally assume that for any $(r-1)$-partition $(X_1', \ldots, X_r')$ of the vertex set of $H$ the total number of edges in a same part is at least $m$ (i.e., $\sum_{i \in I} e(G[X_i']) \geq m$). Although these conditions might seem artificial at the first sight, some natural graphs like the balanced complete $r$-partite graphs or the so-called generalized book graphs satisfy them. For such a graph, $m-1$ copies of $K_n$ together with $k-m+1$ copies of identical $T_{r-1}(n)$ provides a lower bound for $k$-color Turán number of $H$. The proofs from Sections 4 and 5 can be extended to $r$-vertex multigraph with one edge of multiplicity at most $m$ instead of 1. Together with the techniques used in Section 7, such a result implies that if $1/n \ll 1/k$ and $k \geq \binom{r-1}{r/2}(h-m) + m - 1$, then we have $\text{ex}_k(n, H) = (m-1)\binom{n}{2} + (k-m+1)t_{r-1}(n) + o(n^2)$.

For example, this asymptotically determines $k$-color extremal number of balanced complete $r$-partite graphs. Note that the assumption $1/n \ll 1/k$ is necessary here because if $k$ is large compared to $n$, as proved in [11, Theorem 1.1], the $k$-color extremal graph consists of $k$ identical copies of $\text{Ex}(n, H)$ and has more edges than $(m-1)K_n + (k-m+1)t_{r-1}(n)$. Determining what relations between $k$ and $n$ ensure $\text{Ex}_k(n, H) = (m-1)K_n + (k-m+1)t_{r-1}(n)$ is also an interesting question.

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