On ID*-superderivations of Lie superalgebras

MENGMENG CAI and WENDE LIU*

School of Mathematical Sciences, Harbin Normal University
Harbin 150025, China

Abstract: Let $L$ be a Lie superalgebra over a field of characteristic different from 2, 3. Write $\text{ID}^*(L)$ for the Lie superalgebra consisting of superderivations mapping $L$ to $L^2$ and central elements to zero. In this paper we first give an upper bound for the superdimension of $\text{ID}^*(L)$. Then we characterize the $\text{ID}^*$-superderivation algebras for nilpotent Lie superalgebras of class 2 and model filiform Lie superalgebras.

Keywords: ID*-superderivation; nilpotent Lie superalgebras of class 2; model filiform Lie superalgebras

Mathematics Subject Classification 2010: 17B05, 17B30, 17B56

1. Introduction and preliminaries

In 2015, H. Arabynai and F. Saeedi studied derivation algebras of Lie algebras and proposed the notion $\text{ID}^*$-derivations of Lie algebras [AS15]. A $\text{ID}^*$-derivation of a Lie algebra $L$ is a derivation sending $L$ into $L^2$ and $Z(L)$ to zero, where $L^2$ and $Z(L)$ are the derived algebra and center of $L$, respectively. All the $\text{ID}^*$-derivations of a Lie algebra constitute a subalgebra of the full derivation algebra, which contains the inner derivation algebra.

The notion of $\text{ID}^*$-derivations may be naturally generalized to Lie superalgebra case. In this paper, we first give an upper bound for the superdimension of the $\text{ID}^*$-superderivation algebra for a Lie superalgebra $L$ in terms of superdimension of $L^2$ and the minimal generator number pair of $L/Z(L)$, where $L^2$ and $L/Z(L)$ are the derived algebra and central quotient of $L$, respectively. Then we show that the minimal generator number pairs are unique for finite-dimensional nilpotent Lie superalgebras. Finally, we characterize the $\text{ID}^*$-superderivation algebras of nilpotent Lie superalgebras of class 2 and model filiform Lie superalgebras, which prove that the upper bound we obtained for the superdimension of the $\text{ID}^*$-superderivation algebra for a Lie superalgebra $L$ is sharp.

Let $F$ be the ground field of characteristic different from 2, 3 and $\mathbb{Z}_2 := \{0, 1\}$ be the abelian group of order 2. For a homogeneous element $x$ in a vector superspace (superspace for short) $V = V_0 \oplus V_1$, write $\lvert x \rvert$ for the parity of $x$. In this paper the symbol $\lvert x \rvert$ implies that $x$ has been assumed to be a homogeneous element.

In $\mathbb{Z} \times \mathbb{Z}$, we define a partial order as follows:

$$(m, n) \leq (k, l) \iff m \leq k, n \leq l.$$ 

For $m, n \in \mathbb{Z}$, we write $\lvert (m, n) \rvert = m + n$. We also view $\mathbb{Z} \times \mathbb{Z}$ as an additive group in the usual way.

*Correspondence: wendeliu@ustc.edu.cn (W. Liu), supported by the NSF of China (11471090, 11701158)
Write $\text{sdim} V$ for the superdimension of a superspace $V$ and $\text{dim} V$ for the dimension of $V$ as an ordinary vector space. Note that
\[ \text{dim} V = |\text{sdim} V|. \]

A linear map of parity $\alpha \in \mathbb{Z}_2$, $D : L \to L$, is said to be a superderivation of $L$, if
\[ D[x, y] = [D(x), y] + (-1)^{|x|}[x, D(y)] \]
for all $x, y \in L$. Denote by $\text{Der}_\alpha(L)$ the set of all the superderivations of parity $\alpha$ of $L$, where $\alpha \in \mathbb{Z}_2$. Then the superspace $\text{Der}(L) := \text{Der}_0(L) \oplus \text{Der}_1(L)$ is a Lie superalgebra with respect to bracket
\[ [D, E] = DE - (-1)^{|D||E|}ED, \]
where $D, E \in \text{Der}(L)$. The elements of $\text{Der}(L)$ are called superderivations of $L$ and $\text{Der}(L)$ is called the superderivation superalgebra of $L$.

For $x \in L$, the map $\text{ad}_x : L \to L$ given by $y \mapsto [x, y]$ is a superderivation of $L$, called inner. The set of all inner superderivations of $L$ is denoted by $\text{ad}(L)$. It is a standard fact that $\text{ad}(L)$ is an ideal of $\text{Der}(L)$.

As in Lie algebra case, a superderivation of Lie superalgebra $L$ is called an ID-superderivation if it maps $L$ to the derived subsuperalgebra $[L, L]$. Denote by $\text{ID}(L)$ the set of all ID-superderivations of $L$,
\[ \text{ID}(L) = \{ \alpha \in \text{Der}(L) \mid \alpha(L) \subseteq [L, L] \}. \]
Hereafter, write $L^2$ for the derived subalgebra $[L, L]$. Denote by $\text{ID}^*(L)$ the set of all ID-superderivations mapping all central elements to 0,
\[ \text{ID}^*(L) = \{ \alpha \in \text{ID}(L) \mid \alpha(Z(L)) = 0 \}. \]
Obviously,
\[ \text{ad}(L) \leq \text{ID}^*(L) \leq \text{ID}(L) \leq \text{Der}(L). \]

2. An upper bound for the superdimension of $\text{ID}^*(L)$

To describe the upper bound for the superdimension of $\text{ID}^*$-superderivation algebra, we need the concept of minimal generator number pairs for a Lie superalgebra. As usual, we write $\{x_1, \ldots, x_p \mid y_1, \ldots, y_q\}$ implying that $x_i$ is even and $y_j$ is odd in a superspace.

**Definition 2.1.** A generator set of a Lie superalgebra $L$,
\[ \{x_1, \ldots, x_p \mid y_1, \ldots, y_q\}, \]
is said to be minimal if $L$ can not be generated by any subset
\[ \{a_1, \ldots, a_s \mid b_1, \ldots, b_t\} \]
with $(s, t) < (p, q)$. In this case, $(p, q)$ is called a minimal generator number pair of $L$.

For finite-dimensional nilpotent Lie superalgebras, the minimal generator number pairs are unique (see Proposition 3.3). However, this does not necessarily hold in general.

For a subalgebra $K$ of $L$ and a pair of nonnegative integers $(p, q)$, write $\lambda(K; p, q)$ for the number pair
\[ (p \cdot \text{dim}(K)_0 + q \cdot \text{dim}(K)_1, q \cdot \text{dim}(K)_0 + p \cdot \text{dim}(K)_1). \]
The following theorem gives an upper bound for the superdimension of $\text{ID}^*(L)$.
On ID*-superderivations of Lie superalgebras

**Theorem 2.2.** Suppose $L$ is a Lie superalgebra such that $\dim L^2 < \infty$ and $L/Z(L)$ is finitely generated. Then

$$\text{sdim} \lambda^* \leq \lambda(L^2; p, q),$$

where $(p, q)$ is a minimal generator number pair of $L/Z(L)$. In particular, $\lambda^*(L)$ is finite-dimensional.

**Proof.** Suppose $\{x_1, \ldots, x_p \mid y_1, \ldots, y_q\}$ is a subset of $L$ such that

$$\{x_1 + Z(L), \ldots, x_p + Z(L) \mid y_1 + Z(L), \ldots, y_q + Z(L)\}$$
is a minimal generator set of $L/Z(L)$. Define

$$\phi: \lambda^* \to (L^2)_{p} \oplus (L^2)_{q} \oplus (L^2)_{1},$$

$$\alpha \mapsto (\alpha(x_1), \ldots, \alpha(x_p), \alpha(y_1), \ldots, \alpha(y_q)).$$

Clearly, $\phi$ is an injective linear map. Therefore

$$\dim \lambda^*(L) \leq p \cdot \dim (L^2)_{p} + q \cdot \dim (L^2)_{q}.$$  \hfill (2.1)

Similarly,

$$\varphi: \lambda^* \to (L^2)_{q} \oplus (L^2)_{p} \oplus (L^2)_{1},$$

$$\beta \mapsto (\beta(y_1), \ldots, \beta(y_q), \beta(x_1), \ldots, \beta(x_p))$$
is an injective linear map and then

$$\dim \lambda^*(L) \leq q \cdot \dim (L^2)_{q} + p \cdot \dim (L^2)_{p}.$$  \hfill (2.2)

It follows from (2.1) and (2.2) that $\text{sdim} \lambda^*(L) \leq \lambda(L^2; p, q)$. \qed

**Corollary 2.3.** Let $L$ be a Lie superalgebra. Then $\text{ad}(L)$ is finite-dimensional if and only if $\lambda^*(L)$ is finite-dimensional.

**Proof.** One direction is obvious. Suppose $\text{ad}(L)$ is finite-dimensional. Then $L/Z(L) \cong \text{ad}(L)$ is finite-dimensional and so is $L^2$. It follows from Theorem 2.2 that $\lambda^*(L)$ is finite-dimensional. \qed

3. Nilpotent Lie superalgebras of class 2 and model filiform Lie superalgebras

Let $L$ be a Lie superalgebra over $F$. Recall that the lower central series of $L$ is a sequence of ideals of $L$ defined inductively by $L^1 = L$ and $L^n = [L^{n-1}, L]$ for $n \geq 2$. If there exists $n \geq 2$ such that $L^n = 0$, then $L$ is called a nilpotent Lie superalgebra. The least integer $n$ for which $L^{n+1} = 0$ is called the (nilpotent) class of $L$. Clearly, a Lie superalgebra is of class 1 if and only if it is abelian. We should mention that nilpotent Lie (super)algebras of class 2 are of particular interest in both mathematics and physics.

We also use the notion of super-nilindex for a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$. Write

$$C^0(L_{\alpha}) = L_{\alpha}, C^{k+1}(L_{\alpha}) = [L_{\alpha}, C^k(L_{\alpha})],$$
where $\alpha \in \mathbb{Z}_2$ and $k \geq 0$. If $L$ is nilpotent, a pair $(p, q)$ of nonnegative integers is called the super-nilindex of $L$, if

\[ C^p(L_0) = 0, \ C^{p-1}(L_0) \neq 0; \ C^q(L_1) = 0, \ C^{q-1}(L_1) \neq 0. \]

Clearly, a Lie superalgebra is of super-nilindex $(1, 1)$ if and only if it is abelian with nontrivial even and odd parts.

A nilpotent Lie superalgebra of superdimension $(n + 1, m)$, where $n$ and $m$ are positive integers such that $n + m > 2$, is said to be filiform if its super-nilindex is $(n, m)$ (see [N16], for example).

Let $L^{n,m}$ be a Lie superalgebra with basis

\[ \{x_0, \ldots, x_n \mid y_1, \ldots, y_m\} \] (3.1)

and multiplication given by

\[ [x_0, x_i] = x_{i+1}, \ 1 \leq i \leq n - 1, \ [x_0, y_j] = y_{j+1}, \ 1 \leq j \leq m - 1. \]

It is easy to prove that $L^{n,m}$ is a filiform Lie superalgebra, which is called the model filiform Lie superalgebra of super-nilindex $(n, m)$ (see [N16], for example). We should mention that any filiform Lie superalgebra is a deformation of a model one in some sense (see [G04] for more details).

Among nilpotent Lie superalgebras of class 2, very interesting ingredients are the so-called generalized Heisenberg Lie superalgebras (see [NJ18] for non-superalgebra case).

**Definition 3.1.** A nonzero Lie superalgebra $H$ is called a generalized Heisenberg Lie superalgebra if $H^2 = Z(H)$.

We note that a Heisenberg Lie superalgebra is a generalized Heisenberg Lie superalgebra with center of dimension 1 (see [RSS11] for more details). Before considering $\text{ID}^*$-superderivation algebras of generalized Heisenberg Lie superalgebras, let us point out that the minimal generator number pairs of nilpotent Lie superalgebras are unique. To prove this fact, we need a fact as in the Lie algebra case (see [M67, Corollary 2], [T73, Lemma 2.1]).

**Lemma 3.2.** Suppose $L$ is a finite-dimensional nilpotent Lie superalgebra. If $K$ is a sub-superalgebra of $L$ such that $K + L^2 = L$. Then $K = L$.

**Proposition 3.3.** Suppose $L$ is a finite-dimensional nilpotent Lie superalgebra. Then $\text{sdim}(L/L^2)$ is the unique minimal generator number pair of $L$.

**Proof.** Suppose $(p, q)$ is a minimal generator number pair of $L$. Then $(p, q) \geq \text{sdim}(L/L^2)$, since $L/L^2$ is abelian. Then by Lemma 3.2, we have $(p, q) = \text{sdim}(L/L^2)$. The proof is complete.

Now we are position to determine the $\text{ID}^*$-superderivation algebras of generalized Heisenberg Lie superalgebras.

**Proposition 3.4.** Suppose $H$ is a generalized Heisenberg Lie superalgebra of superdimension $\text{sdim}H = (m, n)$ and $\text{sdim}Z(H) = (m_1, n_1)$. Then

1. $\text{ID}^*(H)$ is isomorphic to the Lie superalgebra consisting of matrices

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
A & 0 & C & 0 \\
0 & 0 & 0 & 0 \\
D & 0 & B & 0
\end{pmatrix} \in \mathfrak{gl}(m|n),
\]

where $A, C, D \in \mathbb{Z}_2$ and $k \geq 0$. If $L$ is nilpotent, a pair $(p, q)$ of nonnegative integers is called the super-nilindex of $L$, if

\[ C^p(L_0) = 0, \ C^{p-1}(L_0) \neq 0; \ C^q(L_1) = 0, \ C^{q-1}(L_1) \neq 0. \]
where $A, B, C$ and $D$ are arbitrary matrices of formats $m_1 \times (m - m_1)$, $n_1 \times (n - n_1)$, $n_1 \times (m - n_1)$ and $m_1 \times (n - n_1)$, respectively.

(2) $\text{sdim}^{ID^*}(H)$ attains the upper bound $\lambda(H^2; p, q)$, where $(p, q)$ is the minimal generator number pair of $H/\mathbb{Z}(H)$, which coincides with the superdimension of $H/\mathbb{Z}(H)$.

Proof. Suppose 
\[
\{x_1, \ldots, x_{m-m_1+1}, \ldots, x_m \mid y_1, \ldots, y_{n-n_1+1}, \ldots, y_n\}
\]
is a basis of $H$ such that $\mathbb{Z}(H)$ is spanned by 
\[
\{x_{m-m_1+1}, \ldots, x_m \mid y_{n-n_1+1}, \ldots, y_n\}.
\]
Clearly, an even linear transformation of $H$ is an $ID^*$-superderivation if and only if its matrix with respect to basis (3.2) is of form
\[
\begin{pmatrix}
0 & 0 & A & 0 \\
A & 0 & 0 & 0 \\
0 & 0 & B & 0 \\
\end{pmatrix},
\]
where $A$ is an arbitrary $m_1 \times (m - m_1)$ matrix and $B$ is an arbitrary $n_1 \times (n - n_1)$ matrix. In particular, 
\[
\dim ID^*_0(H) = mn_1 + mn_1 - m_1^2 - n_1^2.
\]
Similarly, an odd linear transformation of $H$ is an $ID^*$-superderivation if and only if its matrix with respect to basis (3.2) is of form
\[
\begin{pmatrix}
0 & 0 & C & 0 \\
0 & 0 & 0 & 0 \\
D & 0 & 0 & 0 \\
\end{pmatrix},
\]
where $C$ is an arbitrary $n_1 \times (m - m_1)$ matrix and $D$ is an arbitrary $m_1 \times (n - n_1)$ matrix. Therefore, 
\[
\dim ID^*_1(H) = mn_1 + mn_1 - 2n_1 m_1.
\]
Since $H/\mathbb{Z}(H)$ is abelian and $\text{sdim}(H/\mathbb{Z}(H)) = (m - m_1, n - n_1)$, we have $(p, q) = (m - m_1, n - n_1)$ is the minimal generator number pair of $H/\mathbb{Z}(H)$. Hence 
\[
\lambda(H^2; m - m_1, n - n_1) = (mn_1 + mn_1 - m_1^2 - n_1^2, nm_1 + mn_1 - 2n_1 m_1).
\]
Then by (3.3), we have 
\[
\text{sdim}^{ID^*}(H) = \lambda(H^2; m - m_1, n - n_1).
\]
The proof is complete.

We should mention that there are many related researches on Heisenberg Lie superalgebras. For example, one may find a study on the cohomology of Heisenberg Lie superalgebras with coefficients in the trivial module [BL17].

Let us consider the $ID^*$-superderivation algebras of nilpotent Lie superalgebras of class 2. As in Lie algebra case, a nilpotent Lie superalgebra of class 2 is a direct sum of a generalized Heisenberg Lie superalgebra and an abelian Lie superalgebra [NJ18]:

---

On $ID^*$-superderivations of Lie superalgebras
Proposition 3.5. Let \( L \) be a finite-dimensional Lie superalgebra. Then \( L \) is nilpotent and of class 2 if and only if
\[
L = H \oplus S,
\]
where \( H \) is a generalized Heisenberg Lie subsuperalgebra and \( S \) is a central ideal of \( L \).

Proof. One direction is obvious. Suppose \( L \) is of nilpotent class 2. Then \( L^2 \subset Z(L) \) and there exists a central ideal \( S \) such that \( Z(L) = L^2 \oplus S \). Since \( L/L^2 \) is abelian, there exists an ideal \( H \) of \( L \) containing \( L^2 \) such that \( L/L^2 = H/L^2 \oplus Z(L)/L^2 \). Then
\[
L = H + Z(L) = H + L^2 + S = H + S.
\]
Note that \( H \cap Z(L) = L^2 \) and \( L^2 = H^2 \). Since \( S \cap L^2 = 0 \), we have \( S \cap H = S \cap H \cap Z(L) = S \cap L^2 = 0 \).

Hence \( L = H \oplus S \). We claim that \( Z(H) = L^2 \). In fact, it is clear that \( L^2 \subset Z(H) \). On the other hand, since \( [Z(H), L] = [Z(H), H + Z(H)] = 0 \), we have \( Z(H) \subset Z(L) \cap H = L^2 \). So \( Z(H) = L^2 = H^2 \) and then \( H \) is a generalized Heisenberg Lie superalgebra. The proof is complete.

We call (3.4) a standard composition for a nilpotent Lie superalgebra of class 2. Note that the superdimension of \( L/Z(L) \) is just the unique minimal generator number pair of \( L/Z(L) \), since \( L/Z(L) \) is abelian.

Theorem 3.6. Let \( L \) be a finite-dimensional Lie superalgebra of nilpotent class 2 and \( (p, q) \) the minimal generator number pair of \( L/Z(L) \). Suppose \( L = H \oplus S \) is a standard decomposition (Proposition 3.5) and
\[
\text{sdim}H = (m, n), \quad \text{sdim}Z(H) = (m_1, n_1), \quad \text{sdim}S = (s, t).
\]

Then

(1) \( \text{ID}^*(L) \) is isomorphic to the Lie superalgebra consisting of matrices
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
A & 0 & C & 0 \\
0 & 0 & 0 & 0 \\
D & 0 & B & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in \mathfrak{gl}(m|n),
\]
where \( A, B, C \) and \( D \) are arbitrary matrices of formats \( m_1 \times (m - m_1) \), \( n_1 \times (n - n_1) \), \( n_1 \times (m - m_1) \) and \( m_1 \times (n - n_1) \), respectively.

(2) \( \text{sdim} \text{ID}^*(L) \) attains the upper bound \( \lambda(L^2; p, q) \).

Proof. Since \( L = H \oplus S \) is a standard decomposition of \( L \), we have
\[
L^2 = H^2, \quad Z(L) = Z(H) \oplus S.
\]
Therefore, one sees that \( \text{ID}^*(L) \cong \text{ID}^*(H) \). Then our theorem follows from Proposition 3.5.
Finally, let us consider the ID*-superderivation algebras of model filiform Lie superalgebras. By Proposition 3.3, we have the following corollary.

**Corollary 3.7.** The minimal generator number pair of model filiform Lie superalgebra $L_{n,m}$ is $(2, 1)$.

**Theorem 3.8.** Let $L = L_{n,m}$ be a model filiform Lie superalgebra with a basis of $\mathfrak{sl}(n|m)$. 

1. If $m > n = 1$, then $\text{ID}^*(L)$ is isomorphic to the Lie superalgebra consisting of matrices

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
D & 0 & B & 0
\end{pmatrix} \in \mathfrak{gl}(m|n),
\]

where $B$ is of form

\[
\begin{pmatrix}
b_{1,1} \\
b_{2,1} & b_{1,1} \\
\vdots & \vdots & \ddots \\
b_{m-1,1} & b_{m-2,1} & \cdots & b_{1,1}
\end{pmatrix},
\] (3.5)

$D$ is of form

\[
\begin{pmatrix}
d_{1,1} \\
\vdots \\
d_{m-1,1}
\end{pmatrix}
\] (3.6)

with $b_{ij}, d_{kl}$ arbitrary elements in $\mathbb{F}$.

If $n > m = 1$, then $\text{ID}^*(L)$ is isomorphic to the Lie superalgebra consisting of matrices

\[
\begin{pmatrix}
0 & 0 \\
A & 0
\end{pmatrix} \in \mathfrak{gl}(m|n),
\]

where $A$ is of form

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} \\
& a_{2,2} & a_{1,2} \\
\vdots & \vdots & \ddots \\
a_{n-1,1} & a_{n-2,2} & \cdots & a_{1,2}
\end{pmatrix}
\] (3.7)

with $a_{ij}$ arbitrary elements in $\mathbb{F}$.

If $m \geq n \geq 2$, then $\text{ID}^*(L)$ is isomorphic to the Lie superalgebra consisting of matrices

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
A & 0 & C & 0 \\
0 & 0 & 0 & 0 \\
D & 0 & B & 0
\end{pmatrix} \in \mathfrak{gl}(m|n),
\]

where $A$ is of form (3.7), $B$ is of form (3.5), $D$ is of form

\[
\begin{pmatrix}
d_{1,1} & d_{1,2} \\
\vdots & \ddots \\
d_{n-1,1} & d_{n-2,2} & \cdots & d_{1,2} \\
\vdots & \ddots & \ddots \\
d_{m-1,1} & d_{m-2,2} & \cdots & d_{m-n+1,2}
\end{pmatrix}
\] (3.8)
and $C$ is of form
\[
\begin{pmatrix}
c_{1,1} & & \\
c_{2,1} & c_{1,1} & \\
 & & \\
& & \\
c_{n-1,1} & c_{n-2,1} & \cdots & c_{1,1}
\end{pmatrix}
\]
with $c_{st}$ and $d_{mn}$ being arbitrary elements in $\mathbb{F}$.

If $n > m \geq 2$, then $\text{ID}^*(L)$ is isomorphic to the Lie superalgebra consisting of matrices
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
A & 0 & C & 0 \\
0 & 0 & 0 & 0 \\
D & 0 & B & 0
\end{pmatrix}
\in \text{gl}(m|n),
\]
where $A$ is of form (3.7), $B$ is of form (3.5), $C$ is of form
\[
\begin{pmatrix}
c_{1,1} & & \\
c_{2,1} & c_{1,1} & \\
 & & \\
& & \\
c_{m-1,1} & c_{m-2,1} & \cdots & c_{1,1}
\end{pmatrix}
\]
and $D$ is of form
\[
\begin{pmatrix}
d_{1,1} & d_{1,2} & & \\
d_{2,1} & d_{2,2} & d_{1,2} & \\
 & & & \\
& & & \\
d_{m-1,1} & d_{m-2,1} & \cdots & d_{1,2}
\end{pmatrix}
\]
with $c_{st}$ and $d_{mn}$ being arbitrary elements in $\mathbb{F}$.

(2) $\text{sdim}\text{ID}^*(L)$ attains the upper bound $\lambda(L^2; p, q)$, where $(p, q)$ is the minimal generator number pair of $L/\mathbb{Z}(L)$. Moreover
\[
(p, q) = \begin{cases} 
(1,1), & m > n = 1 \\
(2,0), & n > m = 1 \\
(2,1), & m \geq 2, n \geq 2
\end{cases}
\]

Proof. Case 1: $m > n = 1$. Then
\[
\text{sdim}L^2 = (0, m-1), \quad \text{sdim}\mathbb{Z}(L) = (1,1).
\]
It is easy to see that $\text{sdim}(L/\mathbb{Z}(L))/(L/\mathbb{Z}(L))^2 = (1,1)$. By Proposition 3.3, we have $(1,1)$ is the minimal generator number pair of $L/\mathbb{Z}(L)$ and
\[
\lambda(L^2; 1, 1) = (m-1, m-1).
\]

Clearly, an even linear transformation of $L$ is an $\text{ID}^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form
\[
\begin{pmatrix}
0 & 0 \\
B & 0
\end{pmatrix}
\]
where $B$ is of form (3.5). Hence, $\dim_{\mathcal{D}_0}(L) = m - 1$. Similarly, an odd linear transformation of $L$ is an $\mathcal{D}_0^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form
\[
\begin{pmatrix}
0 & 0 \\
D & 0
\end{pmatrix},
\]
where $D$ is of form (3.6). Therefore $\dim_{\mathcal{D}_0}(L) = m - 1$. Then by (3.12), we have
\[
\text{sdim}_{\mathcal{D}_0}(L) = \lambda(L^2; 1, 1).
\]

**Case 2:** $n > m = 1$. Then
\[
\text{sdim} L^2 = (n - 1, 0), \quad \text{sdim} Z(L) = (1, 1).
\]
It is easy to see that $\text{sdim}(L/Z(L))/(L/Z(L))^2 = (2, 0)$. By Proposition 3.3, we have (2, 0) is the minimal generator number pair of $L/Z(L)$ and
\[
\lambda(L^2; 2, 0) = (2n - 2, 0).
\]

Clearly, an even linear transformation of $L$ is an $\mathcal{D}_0^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form
\[
\begin{pmatrix}
0 & 0 \\
A & 0
\end{pmatrix},
\]
where $A$ is of form (3.7). Hence, $\dim_{\mathcal{D}_0}(L) = 2n - 2$. Similarly, an odd linear transformation of $L$ is an $\mathcal{D}_0^*$-superderivation if and only if its matrix with respect to basis (3.1) is 0. Therefore $\dim_{\mathcal{D}_0}(L) = 0$. Then by (3.13), we have
\[
\text{sdim}_{\mathcal{D}_0}(L) = \lambda(L^2; 2, 0).
\]

**Case 3:** $m \geq n \geq 2$. Then
\[
\text{sdim} L^2 = (n - 1, m - 1), \quad L/Z(L) \cong L^{n-1,m-1}.
\]
It follows from Corollary 3.4 that (2, 1) is the minimal generator number pair of $L/Z(L)$ and
\[
\lambda(L^2; 2, 1) = (2n + m - 3, 2m + n - 3).
\]

Clearly, an even linear transformation of $L$ is an $\mathcal{D}_0^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form
\[
\begin{pmatrix}
0 & 0 \\
A & 0
\end{pmatrix},
\]
\[
\begin{pmatrix}
0 & 0 \\
D & 0
\end{pmatrix},
\]
\[
\begin{pmatrix}
0 & 0 \\
B & 0
\end{pmatrix}.
\]
where $A$ is of form (3.7) and $B$ is of form (3.5). Hence, $\dim \text{ID}_0^*(L) = 2n + m - 3$. Similarly, an odd linear transformation of $L$ is an ID$^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form

$$\begin{pmatrix}
0 & 0 & C & 0 \\
0 & 0 & 0 & 0 \\
D & 0 & 0 & 0
\end{pmatrix},$$

where $D$ is of form (3.8) and $C$ is of form (3.9). Therefore $\dim \text{ID}_1^*(L) = 2m + n - 3$. Similarly, an even linear transformation of $L$ is an ID$^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
A & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

where $A$ is of form (3.7) and $B$ is of form (3.5). Hence, $\dim \text{ID}_0^*(L) = 2n + m - 3$. Similarly, an odd linear transformation of $L$ is an ID$^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & C & 0 \\
D & 0 & 0 & 0
\end{pmatrix},$$

where $C$ is of form (3.10) and $D$ is of form (3.11). Therefore, $\dim \text{ID}_1^*(L) = 2m + n - 3$. Then by (3.14), we have

$$\text{sdim} \text{ID}^*(L) = \lambda(L^2; 2, 1).$$

Case 4: $n > m \geq 2$. Then

$$\text{sdim}L^2 = (n - 1, m - 1), \quad L/Z(L) \cong L^{n-1,m-1}.$$ 

It follows from Corollary (3.7) that (2, 1) is the minimal generator number pair of $L/Z(L)$ and

$$\lambda(L^2; 2, 1) = (2n + m - 3, 2m + n - 3).$$

Clearly, an even linear transformation of $L$ is an ID$^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
A & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

where $A$ is of form (3.7) and $B$ is of form (3.5). Hence, $\dim \text{ID}_0^*(L) = 2n + m - 3$. Similarly, an odd linear transformation of $L$ is an ID$^*$-superderivation if and only if its matrix with respect to basis (3.1) is of form

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & C & 0 \\
D & 0 & 0 & 0
\end{pmatrix},$$

where $C$ is of form (3.10) and $D$ is of form (3.11). Therefore, $\dim \text{ID}_1^*(L) = 2m + n - 3$. Then by (3.14), we have

$$\text{sdim} \text{ID}^*(L) = \lambda(L^2; 2, 1).$$

The proof is complete.

For a more detailed description of the first cohomology group of model filiform Lie superalgebra with coefficients in the adjoint module, the reader is referred to [LY18].

References

[AS15] H. Arabyani, F. Saeedi, On dimensions of derived algebra and central factor of a Lie algebra. Bull. Iranian Math. Soc. 41 (2015): 1093–1102.
[BL17] W. Bai, W. Liu, Cohomology of Heisenberg Lie superalgebra. J. Math. Phys. 58 (2017): 021701-1–021701-14.

[G04] M. Gilg, On deformations of the filiform Lie superalgebra $L_{n,m}$. Comm. Algebra 32 (2004): 2099–2115.

[LY18] W. Liu, Y. Yang, Cohomology of model filiform Lie superalgebra. J. Algebra Appl. 17 (2018): 1850074-1–1850074-13.

[M67] E. I. Marshall, The Frattini subalgebra of a Lie algebra. J. London Math. Soc. 42 (1967): 416–422.

[N16] R. M. Navarro, Low-dimensional filiform Lie superalgebras. J. Geom. Phys. 108 (2016): 71–82.

[NJ18] P. Niroomand and F. Johari, The structure, capability and the Schur multiplier of generalized Heisenberg Lie algebras J. Algebra 505 (2018): 482–489.

[RSS11] M.C. Rodriguez-Vallarte, G. Salgado, O.A. Sanchez-Valenzuela, Heisenberg Lie superalgebras and their invariant superorthogonal and supersymplectic forms. J. Algebra 332 (2011): 71–86.

[T73] D. A. Towers, A frattini theory for algebras. Proc. London Math. Soc. 27 (3) (1973): 440–462.