The local index formula for quantum $SU(2)$

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Abstract

The local index formula of Connes–Moscovici for the isospectral noncommutative geometry recently constructed on quantum $SU(2)$ [14, 20] is discussed. The cosphere bundle and the dimension spectrum as well as the local cyclic cocycles yielding the index formula, are presented.

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1 Introduction

In noncommutative geometry a’ la Connes [5], Riemannian spin geometry is described by spectral triple $(\mathcal{A}, \mathcal{H}, D)$, which consists of a (suitable) $\ast$-algebra $\mathcal{A}$, represented by bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator $D$ on $\mathcal{H}$ with compact resolvent and bounded commutators with $\mathcal{A}$. The canonical spectral triple associated to a spin manifold $M$ with a Riemannian metric $g$ is $(C^\infty(M, \mathbb{C}), L^2(\sigma, \text{vol}_g), \mathcal{D})$, where $\sigma$ is the Dirac spinor bundle over $M$ and $\mathcal{D}$ is the Dirac operator of the Levi-Civita connection of $g$. This classical spectral triple satisfies seven additional conditions which has been formulated and postulated also for noncommutative algebras $\mathcal{A}$ [7]. One of them, regularity (or smoothness), permits to introduce [11] the pseudodifferential calculus. Another one, dimension, allows to define generalized zeta functions and dimension spectrum, which are tools for the local index theorem of Connes-Moscovici [11], a powerful algorithm for performing complicated local computations by neglecting plethora of irrelevant details. This occurs because these formulae employ exotic traces given in terms of residues rather than the usual traces (which e.g. in position representation require multiple integrals over the whole $M$) employed in the index theorem formulated in terms of Fredholm modules.

An important area to implement and probe these ideas is quantum groups, to start with the best known $SU_q(2)$ (the quantum $SU(2)$). On $SU_q(2)$ a preliminary candidate [1] for the operator $D$ had unbounded commutators with $\mathcal{A}$ [16]. The first spectral triple, constructed in [3], was ‘singular’ (in the sense that it has no $\lim_{q \to 1}$). It has been extensively studied in [8] using the concept of a (quantum) cosphere bundle $S_q^*$ on $SU_q(2)$, that considerably simplifies the computations of local index formula by removing the irrelevant
Another postulated requirement is that of binomial expansion:

\[ \text{multiplicity} \]

\[ \mu_k = O(k^{-1}) \]

as \( k \to \infty \). Then for \( k > n \), \( D^{-k} \) is trace-class and the dimension spectrum \( \Sigma \) is the set of the singularities of zeta functions

\[ \zeta_{\beta}(z) = \text{Trace}_{\mathcal{H}}(\beta |D|^{-z}), \quad \forall \beta \in \Psi^0. \quad (1.1) \]

Assuming \( \Sigma \) to be discrete with simple poles only the Wodzicki-type residue functional

\[ \int T|D|^{-n} := \text{Res}_{z=n}\text{Trace}(T|D|^{-z}) \]

is tracial on \( T \in \Psi \).

I will present here an interesting isospectral bi-equivariant 3-summable spectral triple recently constructed in [14] and analyzed in [20] along the lines of [8]. The resulting cosphere bundle coincides with that in [8], as well as the dimension spectrum \( \Sigma \), given by the set \( \{1, 2, 3\} \). As computed in [18] the cyclic cohomology of the algebra \( \mathcal{A}(SU_q(2)) \) is given in terms of a single generator. This element can be expressed in terms of a single local cocycle, whose form in [20] is slightly different from the one in [8]. A nice exemplification of the theory is computation of the index of \( D \) coupled with the unitary representative of the generator of \( K_1(\mathcal{A}) \). The associated Fredholm module turns out to be 1-summable.

1.1 Preliminaries on the pseudodifferential calculus

The regularity (or smoothness) requirement for a spectral triple is that \( \mathcal{A} \cup \{D, \mathcal{A}\} \subset \cap_{n=1}^{\infty} \text{Dom } \delta^n \), where \( \delta(T) := |D|T - T|D| \). This condition permits to introduce [11] the pseudodifferential calculus as follows.

First, let \( \mathcal{H}^s := \text{Dom}(D^s) \) for \( s \in \mathbb{R} \) be the analogue of Sobolev spaces and let \( \mathcal{H}^\infty := \cap_{s \geq 0} \mathcal{H}^s \). (Assume for simplicity that that \( |D| \) is invertible; for the noninvertible see e.g. [2] and that \( \mathcal{H}^\infty \) is a core for \( |D| \)).

An operator \( T: \mathcal{H}^\infty \to \mathcal{H}^\infty \) such that \( |D|^{-\alpha}T \in \cap_{n=1}^{\infty} \text{Dom } \delta^n \), for \( \alpha \in \mathbb{R} \), is said to have order \( \alpha \). (Such \( T \) automatically extends to a bounded operator from \( \mathcal{H}^{\alpha+s} \) to \( \mathcal{H}^s \) for all \( s \geq 0 \)). Let \( \text{OP}^\alpha \) denote the set of operators of order \( \leq \alpha \). In particular, \( \text{OP}^0 = \cap_{n=1}^{\infty} \text{Dom } \delta^n \), the algebra of operators of order \( \leq 0 \) includes \( \mathcal{A} \cup \{D, \mathcal{A}\} \) and their iterated commutators with \( |D| \). Moreover, \( \{D^2, \text{OP}^\alpha\} \subset \text{OP}^{\alpha+1} \) and \( \text{OP}^{-\infty} := \cap_{\alpha \leq 0} \text{OP}^\alpha \) is a two-sided ideal in \( \text{OP}^0 \).

With this set up, the algebra \( \mathcal{D} = \cup \mathcal{D}_k \) of differential operators is just the smallest algebra of operators on \( \mathcal{H}^\infty \) generated by \( \mathcal{A} \cup \{D, \mathcal{A}\} \) and filtered by the order \( k \in \mathbb{N} \) in such a way that \( \{D^2, \mathcal{D}_k\} \subset \mathcal{D}_{k+1} \). Then the algebra \( \Psi \) of pseudodifferential operators, is generated by operators, which modulo \( \text{OP}^\alpha \) for any \( \alpha \in \mathbb{R} \) are of the form \( T\ D^{-m} \) for some \( n \) and some \( T \in \mathcal{D} \). In particular \( \Psi^0 \) of order \( \leq 0 \) is the algebra generated by \( \cup_k \delta^k(\mathcal{A} \cup \{D, \mathcal{A}\}) \).

The algebra structure on \( \Psi \) can be read off in terms of an asymptotic expansion: \( T \sim \sum_{j=0}^{\infty} T_j \) whenever \( T \) and each \( T_j \) are operators from \( \mathcal{H}^\infty \) to \( \mathcal{H}^\infty \); and for each \( m \in \mathbb{Z} \), there exists \( N \) such that for all \( M > N \), the operator \( T - \sum_{j=1}^{M} T_j \) has order \( \leq m \).

For instance, for complex powers of \( |D| \) (e.g. defined by the Cauchy formula) there is a binomial expansion:

\[ [|D|^z, T] \sim \sum_{k=1}^{\infty} (k!)^{-1} z(z-1) \ldots (z-k+1) \delta^k(T) |D|^{z-k}. \]

Another postulated requirement is that of dimension: \( \exists \ n \in \mathbb{N} \) s.t. the eigenvalues (with multiplicity) of \( |D|^{-n} \), \( \mu_k = O(k^{-1}) \) as \( k \to \infty \). Then for \( k > n \), \( D^{-k} \) is trace-class and the dimension spectrum \( \Sigma \) is the set of the singularities of zeta functions

\[ \zeta_{\beta}(z) = \text{Trace}_{\mathcal{H}}(\beta |D|^{-z}), \quad \forall \beta \in \Psi^0. \quad (1.1) \]
2 The isospectral geometry of $SU_q(2)$

2.1 Spectral triple

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ of [14] can be written in the following form:

- $\mathcal{A}$ is the $*$-algebra [21] generated by $a$ and $b$ with
  \[ ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1. \]
  (Here $0 < q < 1$ and $a \leftrightarrow a^*$, $b \leftrightarrow -b$ are exchanged with respect to [3] and [8]).

- The Hilbert space of spinors $\mathcal{H}$ has an orthonormal basis $v_{x,y,s}^j$ where
  \[ j = 0, \frac{1}{2}, 1, \ldots; \quad x = 0, 1, \ldots, 2j; \quad y = 0, 1, \ldots, 2j + 1; \quad s = \uparrow, \downarrow; \quad (2.1) \]
  with the convention that the $\downarrow$ component is zero if $y = 2j$ or $2j + 1$.
  The spinor representation $\pi$ of $\mathcal{A}$ reads $\pi(a) := a_+ + a_-$, $\pi(b) := b_+ - b_-$, where
  \[
  a_+ v_{x,y,\uparrow}^j = q^{(x+y-2j-1)/2}[x+1]^{1/2} \left( \frac{q^{j+1/2}[y+1]}{[2j+1]} v_{x+1,y+1,\downarrow}^{j+1} \right),
  \]
  \[
  a_- v_{x,y,\uparrow}^j = q^{(x+y-2j-1)/2}[x+1]^{1/2} \left( -\frac{q^y}{2j} v_{x,y,\uparrow}^{j+1} \right),
  \]
  \[
  a_+ v_{x,y,\downarrow}^j = q^{(x+y-2j-1)/2}[2j-x]^{1/2} \left( \frac{q^{j+1/2}[y+1]}{[2j+1]} v_{x+1,y+1,\uparrow}^{j+1} \right),
  \]
  \[
  a_- v_{x,y,\downarrow}^j = q^{(x+y-2j-1)/2}[2j-x]^{1/2} \left( -\frac{q^y}{2j} v_{x,y,\downarrow}^{j+1} \right),
  \]
  \[
  b_+ v_{x,y,\uparrow}^j = q^{(x+y-2j-1)/2}[x+1]^{1/2} \left( \frac{q^{j+1/2}[2j-y+1]}{[2j+2]} v_{x+1,y-1,\uparrow}^{j+1} \right),
  \]
  \[
  b_- v_{x,y,\uparrow}^j = q^{(x+y-2j-1)/2}[x+1]^{1/2} \left( -\frac{q^{j+1/2}[2j-y+1]}{[2j+2]} v_{x+1,y-1,\uparrow}^{j+1} \right),
  \]
  \[
  b_+ v_{x,y,\downarrow}^j = q^{(x+y-2j-1)/2}[2j-x]^{1/2} \left( \frac{q^{j+1/2}[2j-y]}{[2j+1]} v_{x,y-1,\uparrow}^{j+1} \right),
  \]
  \[
  b_- v_{x,y,\downarrow}^j = q^{(x+y-2j-1)/2}[2j-x]^{1/2} \left( -\frac{q^{j+1/2}[2j-y]}{[2j+1]} v_{x,y-1,\uparrow}^{j+1} \right).
  \]
  (Here $[N] := (q^{-N} - q^N)/(q^{-1} - q)$.
  This representation is $U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$-equivariant and unique (up to phases) on $\mathcal{H}$.

- The Dirac operator
  \[ Dv_{x,y,\uparrow}^j = (2j + \frac{3}{2}) v_{x,y,\uparrow}^j, \quad Dv_{x,y,\downarrow}^j = -(2j + \frac{3}{2}) v_{x,y,\downarrow}^j, \quad (2.2) \]
  whose spectrum (with multiplicity) coincides with that of the classical Dirac operator on the round sphere $S^3$, is one of a family of operators in [14] ($U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$)-invariant, asymptotically diagonal with linear spectrum satisfying a modification of reality and first order condition).
2.2 Analytic properties

We shall use the polar decomposition $D = F|D|$, where $|D| := (D^2)^{1/2}$ and $F = D/|D|$, and the orthogonal projectors $P^\uparrow := \frac{1}{2}(1 + F)$, $P^\downarrow := \frac{1}{2}(1 - F) = 1 - P^\uparrow$, whose range spaces are respectively $\mathcal{H}^\uparrow$, spanned by $v^j_{x,y,\uparrow}$ and $\mathcal{H}^\downarrow$ spanned by $v^j_{x,y,\downarrow}$. Let $\delta^n(T) = \left[|D|, \ldots [|D|, T] \ldots \right]_n$.

The explicit computation in [20] shows that

$$\delta(a_+) = P^\uparrow a_+ P^\uparrow + P^\downarrow a_+ P^\downarrow, \quad \delta(a_-) = -P^\uparrow a_- P^\uparrow - P^\downarrow a_- P^\downarrow.$$  

Hence $\delta(\pi(a)) = \delta(a_+) + \delta(a_-)$ is bounded. Next, using $\delta([D, \cdot]) = [D, \delta(\cdot)]$,

$$\delta([D, a_+]) = P^\uparrow a_+ P^\uparrow - P^\downarrow a_+ P^\downarrow, \quad \delta([D, a_-]) = P^\uparrow a_- P^\uparrow - P^\downarrow a_- P^\downarrow.$$  

Calculations for $b$ give similar results and thus $\mathcal{A} \cup [D, \mathcal{A}]$ is contained in $\text{Dom} \delta$, and by iteration, also in $\text{Dom} \delta^k$ for $k \in \mathbb{N}$. Therefore, the triple $(\mathcal{A}, \mathcal{H}, D)$ is a $U_q(su(2)) \otimes U_q(su(2))$-equivariant regular $3^+$-summable spectral triple. (This holds also for a suitable completion of $\mathcal{A}(SU_q(2))$).

The details of the calculations above show that $\Psi^0$, the algebra spanned by $\delta^k(\mathcal{A})$ and $\delta^k([D, \mathcal{A}])$, for all $k \geq 0$, is in fact generated by the diagonal-corner operators $P^\uparrow a_+ P^\downarrow$, $P^\downarrow a_+ P^\uparrow$, $P^\uparrow b_+ P^\downarrow$, $P^\downarrow b_+ P^\uparrow$ together with the other-corner operators $P^\uparrow a_- P^\downarrow$, $P^\downarrow a_- P^\uparrow$, $P^\uparrow b_- P^\downarrow$, $P^\downarrow b_- P^\uparrow$.

The algebra $\mathcal{B}$ spanned by all $\delta^k(\mathcal{A})$ for $k \geq 0$ is generated by the diagonal operators

$$\tilde{a}_\pm := \pm \delta(a_+) = P^\uparrow a_+ P^\uparrow + P^\downarrow a_+ P^\downarrow, \quad (2.3)$$

$$\tilde{b}_\pm := \pm \delta(b_+) = P^\uparrow b_+ P^\uparrow + P^\downarrow b_+ P^\downarrow,$$

and by (off-diagonal) operators

$$\tilde{a}_j = P^\uparrow a_+ P^\uparrow + P^\downarrow a_- P^\downarrow, \quad \tilde{b}_j = P^\uparrow b_+ P^\uparrow + P^\downarrow b_- P^\downarrow. \quad (2.4)$$

Note that $\Psi^0$ is generated by (its subalgebra) $\mathcal{B}$ and by $P^\uparrow$.

In the sequel we shall need operators $\tilde{a}(a) := a_+ + a_-$ and $\tilde{b}(b) := b_+ + b_-$, where

$$a_+, v^j_{x,y,s} = \sqrt{1 - q^{2x+2}} \sqrt{1 - q^{2y+2}} v^j_{x+1,y+1,s},$$

$$a_-, v^j_{x,y,s} = q^{x+y+1} v^j_{x,y,s},$$

$$b_+, v^j_{x,y,s} = q^y \sqrt{1 - q^{2x+2}} v^j_{x+1,y,s},$$

$$b_-, v^j_{x,y,s} = -q^x \sqrt{1 - q^{2y+2}} v^j_{x,y-1,s}. \quad (2.5)$$

These formulae coincide with those in [3] Sec. 6] up to the exchange $a \leftrightarrow a^*$, $b \leftrightarrow -b$ and a doubling of the Hilbert space ($s = \uparrow, \downarrow$). Using a truncation coming from

$$(q^{-1} - q)[n]^{-1} - q^n = q^{3n} + O(q^{5n}),$$

$$1 - \sqrt{1 - q^n} \leq q^\alpha, \quad \text{for any } \alpha \geq 0;$$

it can be seen that, for $x = a$ or $x = b$, $\tilde{a}(x)$ approximate $\pi(x)$ up to operators (given by matrices) of rapid decay and so belonging to $\text{OP}^{-\infty}$. Hence, $\tilde{a}(x)$ can be used instead of $\pi(x)$ when dealing with the local cocycle in the local index theorem in the sequel.

Moreover, the operators $\tilde{a}_\pm$, for $x = a, b$, satisfy simple commutation rules

$$[|D|, \tilde{a}_\pm] = \pm \tilde{a}_\pm, \quad [D, \tilde{a}_\pm] = \pm F \tilde{a}_\pm, \quad [F, \tilde{a}_\pm] = 0.$$
2.3 The cosphere bundle

The ‘cosphere bundle’ of $SU_q(2)$ constructed in [8] using the regular representation of $\mathcal{A}$ and the one obtained in [20] from the spinor representation are isomorphic. To see this let $\pi_{\pm}$ be two known (bounded) representations of $\mathcal{A}(SU_q(2))$ on the Hilbert space $\ell^2(N)$ with the standard orthonormal basis $\varepsilon_k, k \in \mathbb{N}$, determined by

$$\pi_{\pm}(a) \varepsilon_k := \sqrt{1-q^{2k+2}} \varepsilon_{k+1}, \quad \pi_{\pm}(b) \varepsilon_k := \pm q^k \varepsilon_k. \quad (2.6)$$

By sending

$$v^j_{x,y,s} \mapsto \varepsilon_{j,x,y,s} := \varepsilon_x \otimes \varepsilon_y \otimes \varepsilon_j \otimes \varepsilon_s \quad (2.7)$$

we identify the Hilbert space $\mathcal{H}$ with the subspace $\mathcal{H}' \subset \ell^2(N) \otimes \ell^2(N) \otimes \ell^2(\mathbb{Z})_2 \otimes \mathbb{C}^2$ given by the restrictions of indices (2.1).

This yields the correspondence

$$a_+ \leftrightarrow \pi_+(a) \otimes \pi_-(a) \otimes V \otimes 1_2, \quad \pi_- \leftrightarrow -q \pi_+(b) \otimes \pi_-(b^*) \otimes V^* \otimes 1_2, \quad b_+ \leftrightarrow -\pi_+(a) \otimes \pi_-(b) \otimes V \otimes 1_2, \quad b_- \leftrightarrow -\pi_+(b) \otimes \pi_-(a^*) \otimes V^* \otimes 1_2, \quad (2.8)$$

where $V$ is the unilateral shift operator $\varepsilon_{2j} \mapsto \varepsilon_{2j+1}$ in $\ell^2(\mathbb{Z})$.

Few remarks on (2.8) are in order:
- Note that the two first factors (taking into consideration the overall minus signs in the latter two equations) reproduce the Hopf tensor product of the representations $\pi_-$ and $\pi_+$, with respect to the standard coproduct of $SU_q(2)$.
- The shift $V$ is best encoded using the $\mathbb{Z}$-grading due to the one-parameter group $\gamma$ of automorphisms (playing the role of ‘geodesic flow’, see [6]), where $\gamma(t) : T \mapsto e^{it[D]}Te^{-it[D]}$ for any operator $T$ on $\mathcal{H}$. On the subalgebra of “diagonal” operators $T = P^i T P^i + P^i T P^i$, $\gamma$ detects the correct shift of $j$, for example, $\gamma(t) : x_{\pm} \mapsto e^{\pm it} x_{\pm}$, when $x = a, b$.
- (2.8) coincides with [8] (204) apart from the last factor $1_2$ (and different conventions).

In the rest of the section the meaning of the correspondence (2.8) will be clarified. Since $b - b^* \in \ker \pi_\pm$, the reps $\pi_\pm$ are not faithful. Let $\mathcal{A}(D^2_{q\pm})$ be the two quotient algebras defined by

$$0 \to \ker \pi_\pm \to \mathcal{A}(SU_q(2)) \xrightarrow{r_\pm} \mathcal{A}(D^2_{q\pm}) \to 0. \quad (2.9)$$

In $\mathcal{A}(D^2_{q\pm})$ one has (omitting the quotient maps $r_\pm$)

$$b = b^*, \quad ba = q ab, \quad a^* b = q ba^*, \quad a^* a + q^2 b^2 = 1, \quad aa^* + b^2 = 1. \quad (2.10)$$

These are just relations of the equatorial Podleś sphere $S^2_q$ [19] (modulo $b \mapsto q^{-1} b$ plus $q \mapsto q^2$). But the spectrum of $\pi_{\pm}(b)$ being ±tive, $\mathcal{A}(D^2_{q\pm})$ actually describe the two hemispheres of $S^2_q$ (thought of as quantum disks).

In [20] it was proven that there exists a $*$-homomorphism

$$\rho : \mathcal{B} \to \mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-}) \otimes \mathcal{A}(S^4) \quad (2.11)$$
defined on the generators (2.8,2.4) by
\[
\rho(\tilde{a}_+) := r_+(a) \otimes r_-(a) \otimes u, \quad \rho(\tilde{a}_-) := -q r_+(b) \otimes r_-(b^*) \otimes u^*, \\
\rho(\tilde{b}_+) := -r_+(a) \otimes r_-(b) \otimes u, \quad \rho(\tilde{b}_-) := -r_+(b) \otimes r_-(a^*) \otimes u^*,
\]

\[\rho(\tilde{a}_j) = 0 = \rho(\tilde{b}_j). \tag{2.12}\]
To see this, since the factors \(u, u^*\) take care of the \(j\)-dependence, it suffices to show that
\[
\rho_*(\tilde{a}_+) := \pi_+(a) \otimes \pi_-(a), \quad \rho_*(\tilde{a}_-) := -q \pi_+(b) \otimes \pi_-(b^*), \\
\rho_*(\tilde{b}_+) := -\pi_+(a) \otimes \pi_-(b), \quad \rho_*(\tilde{b}_-) := -\pi_+(b) \otimes \pi_-(a^*),
\]

\[\rho_*(\tilde{a}_j) = 0 = \rho_*(\tilde{b}_j), \tag{2.13}\]
determine a \(*\)-homomorphism \(\rho_* : \mathcal{B} \to \mathcal{A}(D_{q+}^2) \otimes \mathcal{A}(D_{q-}^2)\).
(Here \(r_\pm(x)\) for \(x \in \mathcal{A}(SU_q(2))\), is identified with its faithful representant \(\pi_\pm(x)\).)
Let \(\Pi : \mathcal{H} \to \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})\) be (unbounded) operator \(\rho^j_{x,y,s} \mapsto \rho_{x,y,s} := \rho_{x} \otimes \rho_y\) (it ‘forgets’ the index \(j\) and \(s\)).
Define the map \(\rho_*\) by associating to \(T \in \mathcal{B}\) the operator \(\rho_*(T)\) on \(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})\), given by
\[\rho_*(T) \rho_{x,y} = \lim_{j \to \infty} \Pi(T \rho^j_{x,y,s}). \tag{2.14}\]
It is well-defined, since \(T\) is polynomial in generators of \(\mathcal{B}\) that are weighted shifts (uniformly bounded) in \(x, y, j\).
Now, using estimates of (14), it can be directly verified that \(\rho_*\) vanishes on the (rapidly decreasing) operators \(\tilde{a}_j, \tilde{b}_j, \tilde{a}_\pm - \tilde{a}_\pm\) and \(\tilde{b}_\pm - \tilde{b}_\pm\).
Hence the last eq. of (2.13) holds. Moreover \(\tilde{a}_\pm\) and \(\tilde{b}_\pm\) can be replaced by respectively \(a_\pm\) and \(b_\pm\) in the first four equations, which are then satisfied because the coefficients in \(a_\pm\) and \(b_\pm\) (c.f. (2.5)) are \(j\)-independent, e.g.
\[
\rho_*(\tilde{a}_+) \rho_{x,y} = \rho_{\tilde{a}_+} \rho_{x,y} = \lim_{j \to \infty} \sqrt{1 - q^{2x+2}} \sqrt{1 - q^{2y+2}} \Pi(v^j_{x+1,y+1, s}) \\
= \sqrt{1 - q^{2x+2}} / \sqrt{1 - q^{2y+2}} \rho_{x+1,y+1} \rho_{x,y} = (\pi_+(a) \otimes \pi_-(a)) \rho_{x,y}.
\]
Moreover, since the product of \(a_\pm\) and \(b_\pm\) still does not contain \(j\)-dependent coefficients, \(\rho_*\) respects the multiplication in \(\mathcal{B}\). Finally, \(\rho_*\) is an algebra homomorphism by linearity of \(\lim\).

The range of the map \(\rho\) in \(\mathcal{A}(D_{q+}^2) \otimes \mathcal{A}(D_{q-}^2) \otimes \mathcal{A}(S^1)\), denoted \(\mathcal{A}(S^*_q)\), is called (algebra of) \textit{cosphere bundle} on \(SU_q(2)\) (and \(\rho\) is ‘symbol map’).
It deserves its name since it corresponds to complete symbols, i.e. scalar pseudodifferential operators modulo the smoothing operators, which in our case coincide with diagonal pseudodifferential operators modulo the smoothing operators.
It should be mentioned that \(S^*_q\) for our geometry coincides with the cosphere bundle constructed in [8] and the symbol map \(\rho\) rectifies the correspondence (2.8). It can be extended to \(\Psi^0\) by setting \(\rho(P) = P\) and to appropriate smooth algebras.

Denote by \(Q\) the orthogonal projector on \(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2\) with range \(\mathcal{H}'\) (the Hilbert subspace identified with \(\mathcal{H}\)). Then (2.8) in combination with (2.11) implies that for all \(T \in \mathcal{B}\)
\[
T - Q(\rho(T) \otimes 1_2)Q \in \text{OP}^{-\infty}, \tag{2.15}\]
where the first \(T\) is viewed as an operator on \(\mathcal{H}'\) via (2.7) and \(\rho(T)\) acts on \(\ell^2(\mathbb{Z})\) via Fourier transform from \(S^1\) to \(\mathbb{Z}\).
2.4 The dimension spectrum

Let $\tau_1$ and $\tau_0^\uparrow$, $\tau_0^\downarrow$ be three functionals on $\mathcal{A}(D^2_{q\pm})$ defined by

\[
\tau_1(x) := \frac{1}{2\pi} \int_{S_1} \sigma(x),
\]
\[
\tau_0^\uparrow(x) := \lim_{N \to \infty} \left( \text{Tr}_N \pi_\pm(x) - (N + \frac{3}{2})\tau_1(x) \right),
\]
\[
\tau_0^\downarrow(x) := \lim_{N \to \infty} \left( \text{Tr}_N \pi_\pm(x) - (N + \frac{1}{2})\tau_1(x) \right),
\]

where

- $\sigma : \mathcal{A}(D^2_{q\pm}) \to \mathcal{A}(S^1)$ is the $\ast$-homomorphism (known as ‘symbol map’) that includes $S^1 = \partial D^2_{q\pm}$ (the equator of $S^2_q$) into $D^2_{q\pm}$

\[
\sigma(r_\pm(a)) := u; \quad \sigma(r_\pm(b)) := 0,
\]

with $u$ being the unitary generator of $\mathcal{A}(S^1)$;
- $\text{Tr}_N$ is the truncated trace

\[
\text{Tr}_N(T) := \sum_{k=0}^{N} \langle \varepsilon_k | T \varepsilon_k \rangle
\]

- the constants $\frac{3}{2}$ and $\frac{1}{2}$ are chosen to simplify the residues in the sequel.

A straightforward calculation shows that on the basis $a^l b^m$ of $D^2_{q\pm}$, where $m \in \mathbb{N}$, $l \in \mathbb{Z}$ and $a^{-l} := (a^*)^l$ for $l > 0$,

\[
\tau_1(a^l b^m) = \delta_l \delta_m,
\]
\[
\tau_0^\uparrow(a^l b^m) = \frac{1}{1 - \lambda} \delta_l (1 - \delta_m) - \frac{1}{2} \delta_l \delta_m,
\]
\[
\tau_0^\downarrow(a^l b^m) = \frac{1}{1 - \lambda} \delta_l (1 - \delta_m) + \frac{1}{2} \delta_l \delta_m
\]

where $\lambda = (\pm q)^m$ and $\delta_k = 1$ when $k = 0$ and 0 otherwise. Moreover (by checking on the basis)

\[
\text{Tr}_N(\pi_\pm(x)) = (N + \frac{3}{2})\tau_1(x) + \tau_0^\uparrow(x) + O(N^{-k})
= (N + \frac{1}{2})\tau_1(x) + \tau_0^\downarrow(x) + O(N^{-k}) \quad \text{for all} \quad x \in D^2_{q\pm}; k > 0.
\]

For the following result, we shall use the following notation:
- denote the Wodzicki-type residue functional as in [11]:

\[
\int T := \text{Res}_{z=0} \text{Tr}(T|D|^{-z}).
\]
- denote $r$ the projection onto the first two factors in $\mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-}) \otimes \mathcal{A}(S^1)$; in particular, it yields a map

\[
r : \mathcal{A}(S^*_q) \to \mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-}).
\]
- denote $T^0$ the grade-zero part (with respect to the geodesic flow $\gamma(t)$ transported via (2.7) to $\mathcal{H}'$) of a diagonal operator $T$. 

7
A theorem in [20] states that the dimension spectrum of the spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is simple and given by \(\{1, 2, 3\}\) with residues

\[
\begin{align*}
\int T|D|^{-3} &= 2(\tau_1 \otimes \tau_1)(r\rho(T)^0), \\
\int T|D|^{-2} &= (\tau_1 \otimes (\tau_0^+ + \tau_0^1) + (\tau_0^+ + \tau_0^1) \otimes \tau_1)(r\rho(T)^0), \\
\int T|D|^{-1} &= (\tau_0^+ \otimes \tau_0^1 + \tau_0^1 \otimes \tau_0^1)(r\rho(T)^0).
\end{align*}
\]

To see this recall that the dimension spectrum consists of the poles of the zeta function \(\zeta_T(z) := \text{Tr}(T|D|^{-z})\) for all \(T \in \Psi^0\), but in our case, under the trace, it suffices to consider only \(T \in P^\dagger \mathcal{B}\) or \(T \in P^\dagger \mathcal{B}\).

Moreover, in \(\zeta_T(z)\) we can replace \(T\) by \(Q(r\rho(T) \otimes 1_2)\) since their difference is a smoothing operator by (2.15). Calculating first for \(P^\dagger T\) (fixing \(s = \uparrow\), splitting the overall trace into traces over \(j, x, y\) and using the tracial property) yields, up to holomorphic term,

\[
\text{Tr} P^\dagger T|D|^{-z} = \text{Tr}(P^\dagger Q(r\rho(T) \otimes 1_2)QP^+ |D|^{-z})
\]

\[
= \sum_{2j=0}^{\infty} (2j + \frac{3}{2})^{-z}(\text{Tr}_{2j} \otimes \text{Tr}_{2j+1})(r\rho(T)^0)
\]

\[
= (\tau_1 \otimes \tau_1)(r\rho(T)^0) \zeta(z) - 2 + (\tau_1 \otimes \tau_0^+ + \tau_0^1 \otimes \tau_1)(r\rho(T)^0) \zeta(z - 1) + (\tau_0^+ \otimes \tau_0^1)(r\rho(T)^0) \zeta(z). \tag{2.18}
\]

Thus

\[
\begin{align*}
\int P^\dagger T|D|^{-3} &= (\tau_1 \otimes \tau_1)(r\rho(T)^0), \\
\int P^\dagger T|D|^{-2} &= (\tau_1 \otimes \tau_0^+ + \tau_0^1 \otimes \tau_1)(r\rho(T)^0), \tag{2.19} \\
\int P^\dagger T|D|^{-1} &= (\tau_0^+ \otimes \tau_0^1)(r\rho(T)^0).
\end{align*}
\]

Similar calculation for \(P^\dagger\), by shifting the summation index \(j \mapsto j + \frac{1}{2}\), yields just (2.19) with permuted tensor product. Hence \(\zeta_T\) has simple poles at 1, 2 and 3 with the residues as stated.

### 2.5 Local index formula

Recall that with a general (odd) spectral triple \((\mathcal{A}, \mathcal{H}, D)\) the index of \(D\) defines an additive map

\[
K_1(\mathcal{A}) \to \mathbb{Z}, \quad [U] \mapsto \text{Index}(PUP), \tag{2.20}
\]

where \(U \in \text{Mat}_r(\mathcal{A})\) is a unitary representative of the \(K_1\) class, and \(P = \frac{1}{2}(1 + F)\) with \(F = D/|D|\) \((PUP\) is automatically Fredholm).

This map is computed by pairing \(K_1(\mathcal{A})\) with “nonlocal” cyclic cocycles \(\chi_n\) given in terms of \(F\)

\[
\chi_n(a_0, \ldots, a_n) = \lambda_n \text{Tr}(a_0 [F, a_1] \ldots [F, a_n]), \quad \text{for all } a_j \in \mathcal{A}, \tag{2.21}
\]

where \(\lambda_n\) is a normalization constant and the (smallest) integer \(n \geq p\) is determined by the degree \(p\) of summability of the Fredholm module \((\mathcal{H}, F)\) over \(\mathcal{A}\). In our case it is...
1-summable, since the commutators \([F, \pi(x)]\), for \(x \in \mathcal{A}\), are trace-class (they are off-diagonal operators given by matrices of rapid decay). Thus we need only the first Chern character \(\chi_1(a_0, a_1) = \lambda_1 \text{Tr}(a_0 [F, a_1])\), with \(a_1, a_2 \in \mathcal{A}\). An explicit expression for this cyclic cocycle on the basis of \(SU_q(2)\) was obtained in [18].

On the other hand, the Connes–Moscovici local index theorem expresses the index map in terms of a local cocycle \(\phi_{\od}\) in the \((b, B)\) bicomplex of \(\mathcal{A}\), where

\[
\begin{align*}
    b\varphi(a_0, a_1, \ldots, a_{n+1}) := \\
    \sum_{j=0}^{n} (-1)^j \varphi(a_0, \ldots, a_ja_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1}a_0, a_1, \ldots, a_n)
\end{align*}
\]

and \(B = NB_0\), with

\[
\begin{align*}
    (N\psi)(a_0, \ldots, a_{n-1}) := \sum_{j=0}^{n-1} (-1)^{(n-1)}j \psi(a_j, \ldots, a_{n-1}, a_0, \ldots, a_{j-1}), \\
    (B_0\varphi)(a_0, \ldots, a_{n-1}) := \varphi(1, a_0, \ldots, a_{n-1}) - (-1)^n \varphi(a_0, \ldots, a_{n-1}, 1),
\end{align*}
\]

satisfy \(b^2 = 0, B^2 = 0\) and \(bb + Bb = 0\), so that \((b + B)^2 = 0\).

The cocycle \(\phi_{\od}\) is a local representative of the cyclic cohomology class of \(\chi_3\) (Chern character). The pairing of the cyclic cohomology class \([\phi_{\od}] \in HC_{odd}(\mathcal{A})\) with \(K_1(\mathcal{A})\) gives the index of \(D\) with coefficients in \(K_1(\mathcal{A})\). The (finite number of) components of \(\phi_{\od} = (\phi_1, \phi_3, \ldots)\) are explicitly given in [11] in terms of the operator \(D\).

In our case \((\mathcal{A}, \mathcal{H}, D)\) has metric dimension 3 so the local cocycle \(\phi_{\od}\) has two components

\[
\phi_1(a_0, a_1) = \int a_0 [D, a_1] |D|^{-1} - \frac{1}{4} \int a_0 \nabla([D, a_1]) |D|^{-3} + \frac{1}{8} \int a_0 \nabla^2([D, a_1]) |D|^{-5},
\]

\[
\phi_3(a_0, a_1, a_2, a_3) = \frac{1}{12} \int a_0 [D, a_1] [D, a_2] [D, a_3] |D|^{-3},
\]

where \(\nabla(T) := [D^2, T]\). (The cocycle condition \((b + B)\phi_{\od} = 0\) reads \(B\phi_1 = 0, b\phi_1 + B\phi_3 = 0, b\phi_3 = 0\).

Whenever \([F, a]\) is traceclass \(\forall a \in \mathcal{A}\), which is our case, \(\phi_1, \phi_3\) can be rewritten, using the binomial expansion, as

\[
\phi_1(a_0, a_1) = \int a_0 \delta(a_1) F|D|^{-1} - \frac{1}{2} \int a_0 \delta^2(a_1) F|D|^{-2} + \frac{1}{4} \int a_0 \delta^3(a_1) F|D|^{-3},
\]

\[
\phi_3(a_0, a_1, a_2, a_3) = \frac{1}{12} \int a_0 \delta(a_1) \delta(a_2) \delta(a_3) F|D|^{-3}. \tag{2.22}
\]

Moreover Prop. 2 proved in [8] applies to our case: the Chern character \(\chi_1\) is equal to \(\phi_{\od} - (b + B)\phi_{ev}\) where the cochain \(\phi_{ev} = (\phi_0, \phi_2)\) is given by

\[
\phi_0(a) := \text{Tr}(Fa |D|^{-2}) |_{z=0},
\]

\[
\phi_2(a_0, a_1, a_2) := \frac{1}{24} \int a_0 \delta(a_1) \delta^2(a_2) F|D|^{-3}.
\]

Remarks:
- For the definition of \(\phi_0\) it is necessary that \(0 \notin \Sigma\).
- The cochain \( \phi_{ev} = (\phi_0, \phi_2) \) is named \( \eta \)-cochain in [8].
- In components, the equality of the characters means
  \[
  \phi_1 = \chi_1 + b\phi_0 + B\phi_2, \quad \phi_3 = b\phi_2.
  \]

Furthermore there is a useful variant, shown in [20] for \( P \) of metric dimension 3, following Proposition 3 in [8] valid for \( P := \frac{1}{2}(1 + F) \) of metric dimension 2: the local Chern character \( \phi_{odd} \) is equal to \( \psi_1 - (b + B)\phi_{ev}' \), where

\[
\psi_1(a_0, a_1) := 2 \int a_0 \delta(1) P |D|^{-1} - \int a_0 \delta^2(a_1) P |D|^{-2} + \frac{2}{3} \int a_0 \delta^3(a_1) P |D|^{-3},
\]
and \( \phi_{ev}' = (\phi_0, \phi_2') \) is given by

\[
\phi_0'(a) := \text{Tr}(a |D|^{-5})|_{z=0},
\]
\[
\phi_2'(a_0, a_1, a_2) := -\frac{1}{24} \int a_0 \delta(a_1) \delta^2(a_2) F |D|^{-3}.
\]

Remarks:
- The term in \( \psi_1 \) involving \( P |D|^{-3} \) is not present in [8] (as there the metric dimension of the projector \( P \) is 2).
- Since \( \phi_2 = -\phi_2' \) the last two propositions jointly show that the cyclic 1-cocycle \( \chi_1 \) can be given (up to coboundaries) in terms of one single \((b, B)\)-cocycle \( \psi_1 \), more precisely

\[
\chi_1 = \psi_1 - b\beta,
\]
where

\[
\beta(a) = 2 \text{Tr}(Pa |D|^{-5})|_{z=0}.
\] (2.23)

To exemplify the theory above compute \( \text{Index}(PUP) \) when \( P = P^\dagger \) and \( U \) is the unitary operator

\[
U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix},
\] (2.24)
acting on the doubled Hilbert space \( \mathcal{H} \otimes \mathbb{C}^2 \) via the representation \( \pi \otimes 1_2 \). Calculating \( \psi_1(U^{-1}, U) \), with the local cyclic cocycle \( \psi_1 \) extended by \( \text{Tr}_\mathbb{C}^2 \), yields

\[
2 \sum_{kl} \int U_{kl}^* \delta(U_{lk}) P |D|^{-1} - \sum_{kl} \int U_{kl}^* \delta^2(U_{lk}) P |D|^{-2} + \frac{2}{3} \sum_{kl} \int U_{kl}^* \delta^3(U_{lk}) P |D|^{-3}.
\]

Using \( (2.19) \), simplifying due to \( \rho(\delta^2(U_{kl})) = \rho(U_{kl}) \), computing (using the relations in \( \mathcal{A}(D_{q_{d\pm}}^2) \))

\[
\rho(U_{kl}^* \delta(U_{lk}))^0 = 2(1 - q^2) 1 \otimes r_+(b)^2
\]
and substituting \( (2.17) \) produces

\[
\psi_1(U^{-1}, U) = 2(1 - q^2)(2\tau_0^+ \otimes \tau_0^+ + 2/3 \tau_1 \otimes \tau_1)(1 \otimes r_+(b)^2) - (\tau_1 \otimes \tau_0^+ + \tau_0^+ \otimes \tau_1)(1 \otimes 1) = -2.
\]

Thus, including the correct normalization constant \( -\frac{1}{2} \),

\[
\text{Index}(PUP) = -\frac{1}{2} \psi_1(U^{-1}, U) = 1.
\]
3 Final comments

Since $HC^1(\mathcal{A}(SU_q(2)))$ is one-dimensional, the characters of the 1-summable Fredholm modules found in [8] and in [20], are all cohomologous to the cyclic cocycle found in [18].

As signaled in the text, the polynomial algebras $\mathcal{A}$ employed above admit a smooth completion (consisting of series in the (ordered) generators with coefficients of rapid decay. The fact that these pre $C^*$-algebras are e.g. closed under holomorphic functional calculus, follows from Lemma 2 of [9] and the homomorphism properties of the maps (2.9), (2.16) and (2.11).

It may be interesting to mention another class of infinitesimal operators with several interesting properties, given by the two-sided ideal $\mathcal{G}$ of $\mathcal{B}(\mathcal{H})$, that was denoted by $\mathcal{K}_q$ in [14], [20]. It is generated by one operator $L_q$, where $L_q v_{x,y,s} := q^j v_{x,y,s}$, but using the methods of [15] it can be seen that it is in fact independent on the choice of $0 < q < 1$. The ideal $\mathcal{G}$ is strictly contained in the ideal of infinitesimals of arbitrary order, that is, compact operators whose $n$-th singular value $\mu_n$ satisfies $\mu_n = O(n^{-\alpha})$, for all $\alpha > 0$. It seems however that it is too big for $q$-deformed groups, for instance it is not so useful for the purpose of (2.11) since the homomorphism $\rho_\bullet$ does not extend to $\mathcal{G}$, even though $\rho_\bullet(L_q) = 0$.

An interesting question is if the cochain $\beta$ (2.23) (whose coboundary is the difference between the original Chern character and the local one), is given by the remainders in the rational approximation of the logarithmic derivative of the Dedekind eta function as in [8]. The quite involved preliminary computations indicate that the higher derivatives are needed in the case at hand.

As far as the homogeneous spaces of $SU_q(2)$ are regarded, it should be added that local index formula on the equatorial Podleś quantum sphere $S^2_q$ has been worked out in [12] and in [13] on the general family $S^2_{qc}$ of Podleś spheres.

A further study of these geometries and of $SU_q(2)$ presented in this note, which are characterized by $D$ with linearly growing spectrum and the real structure and first order condition satisfied up to rapidly decaying operators, will be highly appreciated. This in particular regards the investigation, along the lines of [10], of the internal perturbations of the spectral action, which present some difficulties (non-vanishing tadpole graph) for these examples.

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