Test for mean matrix in GMANOVA model under heteroscedasticity and non-normality for high-dimensional data

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Abstract

This paper is concerned with the testing bilateral linear hypothesis on the mean matrix in the context of the generalized multivariate analysis of variance (GMANOVA) model when the dimensions of the observed vector may exceed the sample size, the design may become unbalanced, the population may not be normal, or the true covariance matrices may be unequal. The suggested testing methodology can treat many problems such as the one- and two-way MANOVA tests, the test for parallelism in profile analysis, etc., as specific ones. We propose a bias-corrected estimator of the Frobenius norm for the mean matrix, which is a key component of the test statistic. The null and non-null distributions are derived under a general high-dimensional asymptotic framework that allows the dimensionality to arbitrarily exceed the sample size of a group, thereby establishing consistency for the testing criterion. The accuracy of the proposed test in a finite sample is investigated through simulations conducted for several high-dimensional scenarios and various underlying population distributions in combination with different within-group covariance structures. Finally, the proposed test is applied to a high-dimensional two-way MANOVA problem for DNA microarray data.

Keywords: Asymptotic distribution, GMANOVA model, Bilateral linear hypothesis on mean matrix,
1. Introduction

In this study, we examine high-dimensional tests for a bilateral linear hypothesis on the mean matrix in the generalized multivariate analysis of variance (GMANOVA) model. After establishing some prefatory notations, we focus on a more precise problem statement. Let $N_i$ independent $p$-dimensional observation vectors, $x_1^{(i)}, \ldots, x_{N_i}^{(i)}$, be drawn from the $i$-th population, where $i \in \{1, \ldots, g\}$ and $g$ denotes the number of underlying populations. For the complete observation matrix, defined as $X = (x_1^{(1)}, \ldots, x_{N_1}^{(1)}, x_2^{(1)}, \ldots, x_{N_2}^{(2)}, \ldots, x_{N_g}^{(g)}, \ldots, x_g^{(g)})'$, we assume the following generalized multivariate linear model:

$$X = A\Theta B' + E,$$  \hspace{1cm} (1)

where $A$ is a given $N \times k$ matrix with rank $k$, known as a between-group design matrix; $\Theta$ is a $k \times q$ unknown mean parameter matrix; $B$ is a given $p \times q$ matrix with rank $q$, known as a within-design matrix; and $E = (\varepsilon_1, \ldots, \varepsilon_N)'$ is an $N \times p$ error matrix with mean $O$. For details of the model (1), please refer to Rosen [17]. Our primary objective is to develop a test procedure for the matrix parameter $\Theta$ by relaxing the commonly adopted linear model assumptions, such as normality and homoscedasticity. We assume that $\text{Var} (\varepsilon_{N_{i+j}}) = \Sigma_i$, $j \in \{N_1\}$, $i \in \{g\}$,

where $\Sigma_i$ denotes a positive-definite $p \times p$ symmetric matrix and $N_i = N_0 + N_1 + \cdots + N_{i-1}$ with $N_0 = 0$, and the distribution of $X$ may be non-normal; in addition, $\Sigma_1, \ldots, \Sigma_g$ may not be equal and $N_1, \ldots, N_g$ are allowed not to be equal, thereby implying a heteroscedastic and unbalanced data design. Furthermore, in our setup, $p$ is much larger than $N_i$.

This paper proposes testing statistics for a bilateral linear hypothesis on the mean matrix, which is given as follows:

$$H_0 : L\Theta R' = O \text{ vs. } H_1 : L\Theta R' \neq O,$$  \hspace{1cm} (2)

where $L : \ell \times k$ and $R : r \times q$ are known matrices with ranks $\ell$ and $r$, respectively. The proposed tests are constructed to be well-defined for a high-dimensional GMANOVA model. We assume that $\ell$ and $r$ are fixed, even as the observation dimension $p$ and sample size $N_i$ tend toward infinity.

Owing to the general formulation and with various options for $L$ and $R$, the testing problem (2) incorporates numerous hypotheses of interest. For example, the test for a linear hypothesis on the mean matrix, particularly for testing the homogeneity of the means, is a special case.

The theory of multivariate inference proposes several solutions to this problem for the classical case of $p < N_i$, particularly assuming normality and homoscedasticity. The hypotheses of the mean in a multivariate linear model are usually tested based on the likelihood ratio (LR) criterion. An extensive overview of the results for a large $N_i$ and fixed $p$, along with the related results, is provided in classical
multivariate analysis literature; see, e.g., Muirhead [13], Srivastava [14], Anderson [1], and Fujikoshi et al. [8].

The classical methods for a one- and higher-way MANOVA collapse when \( p > N_i \), mainly from the singularity of the empirical covariance matrix involved, and thus, need to be modified. Several suggestions have been recently provided in the literature regarding the modification of the classical MANOVA tests for high-dimensional data. While most of these modifications relax only the normality assumption, such as those developed by Srivastava and Kubokawa [15] and Yamada and Himeno [20], there are more flexible approaches offering a completely nonparametric method to the problem, such as those developed by Ghosh and Biswas [9], Wang et al. [18], and Wang et al. [19].

Similarly, the results have been reported under homoscedasticity, i.e., \( \Sigma_i = \Sigma, \forall i \in \mathbb{G} \). For example, Yamada and Sakurai [21] compared the powers of three classical tests under an asymptotic framework in which \( p \) and \( N \) both increase such that \( p/N \) converges to a positive constant. Srivastava and Singull [16] provided a test procedure whose testing statistic was based on a modified maximum likelihood estimator of the mean matrix; Jana et al. [12] proposed the test statistic by replacing \( \Sigma^{-1} \) in the LR statistic derived under the condition that \( \Sigma \) is known with the Moore-Penrose inverse of the unbiased estimator of \( \Sigma \). All these results were derived for a homoscedastic GMANOVA under normality.

As another prominent issue, despite demonstrating several important results, most of the modifications of the classical theory have focused on some specific, high-dimensional testing problems, such as those proposed by Cai et al. [4], Cai and Xia [5], and Chen et al. [6], who restricted their attention to two-sample tests; Zhou et al. [22], who developed a test procedure for a hypothesis regarding linear combinations of the mean vectors; and Zhou et al. [23], who proposed a test based on the \( L^2 \)-norm for a high-dimensional two-way MANOVA.

The present study, however, aligns with a different approach. Instead of exploring a specific testing problem, we aim to develop a unified methodology that encompasses a wide spectrum of high-dimensional testing problems ranging from a one-way MANOVA to tests for parallelism in a profile analysis. The construction adopted for the unified theory is based on the bias-corrected estimator of the Frobenius norm of \( \mathfrak{A}^{1/2} L \Theta R' \mathfrak{B}^{1/2} \), where \( \mathfrak{A} \) and \( \mathfrak{B} \) are positive-definite symmetric matrices. The main distinguishing feature of the proposed methodology is that we simultaneously relax the classical linear model assumptions, such as normality, homoscedasticity, and low dimensionality, for all cases of a high-dimensional GMANOVA. In fact, our proposed tests can be applied for a distribution family that contains normal. The asymptotic framework of interest is one where \( p \) and \( N \) both increase, while \( p/N \) can converge to any arbitrarily large but bounded positive value. The asymptotic theory is developed under general assumptions that include an elliptical distribution and a family of distribution-specified conditions, such as those applied by Bai and Saranadasa [3] and Chen and Qin [7].

The remainder of this paper is organized as follows. In Section 2, we construct our class of statistics to test \( H_0 \) in (2). In Section 3, we show that the asymptotic distributions of the proposed test statistics become normal under the null and a specified alternative hypotheses. The proofs of the main results are provided in Appendix A, and additional theoretical results are given in the online supplementary...
material. In the online supplementary material, specific results contained in the proposed class of tests are reviewed, and the results of numerical evaluations conducted through various finite-sample simulation scenarios, along with an example using real data are also reported.

Hereafter, “⊥” denotes the independence, \( \Phi(\cdot) \) denotes the cumulative distribution function of the standard normal distribution, \( \Phi^{-1}(\cdot) \) denotes the inverse function of \( \Phi(\cdot) \), “\( \overset{P}{\to} \)” represents the convergence in probability, and “\( \overset{D}{\to} \)” denotes the convergence in distribution. In addition, we define “\( \asymp \)” as the asymptotic equivalence, as follows:

\[
A \asymp B \iff \frac{A}{B} = O(1) \quad \text{and} \quad \frac{B}{A} = O(1).
\]

2. Testing statistic and its variance

Before proposing the test statistics, we present a trivial, yet essential, aspect.

**Lemma 1.** Testing the hypothesis (2) is equivalent to testing

\[
H_0 : \text{tr}(\mathcal{B}^{1/2} R\Theta' L' A L \Theta R' \mathcal{B}^{1/2}) = 0 \quad \text{vs.} \quad H_1 : \text{tr}(\mathcal{B}^{1/2} R\Theta' L' A L \Theta R' \mathcal{B}^{1/2}) > 0,
\]

where \( \mathcal{A} \) and \( \mathcal{B} \) are positive-definite symmetric matrices.

Note that \( \mathcal{A} \) and \( \mathcal{B} \) in Lemma 1 are arbitrary; therefore, we set

\[
\mathcal{A} = \{L(A'A)^{-1} L'\}^{-1}, \quad \mathcal{B} = \{R(B'B)^{-1} R'\}^{-1}.
\]

Then, the expression in (3) is described as follows:

\[
Q = \text{tr}(\mathcal{B}^{1/2} R\Theta' L' A L \Theta R' \mathcal{B}^{1/2})
= \text{tr}(\{R(B'B)^{-1} R'\}^{-1/2} R\Theta' L'\{L(A'A)^{-1} L'\}^{-1} L \Theta R'(R(B'B)^{-1} R')^{-1/2}).
\]

We prepare an unbiased estimator of \( Q \) to define the testing statistic. A natural estimator of \( Q \) is given as follows:

\[
\hat{Q} = \text{tr}(\{R(B'B)^{-1} R'\}^{-1/2} R\hat{\Theta} L'\{L(A'A)^{-1} L'\}^{-1} L \hat{\Theta} R'(R(B'B)^{-1} R')^{-1/2}),
\]

where \( \hat{\Theta} = (A'A)^{-1} A' X B (B'B)^{-1} \), which is an unbiased estimator of \( \Theta \). Through

\[
\mathcal{P} = \{R(B'B)^{-1} R'\}^{-1/2} R(B'B)^{-1} B',
\]

\[
\Pi_H = A(A'A)^{-1} L'\{L(A'A)^{-1} L'\}^{-1} L(A'A)^{-1} A',
\]

we can simply denote \( \hat{Q} = \text{tr}(\mathcal{P} X' \Pi_H X \mathcal{P}') \). Note that \( \mathcal{P} \) is a \( r \times p \) matrix with rank \( r \), such that \( \mathcal{P}' \mathcal{P} \) is a projection matrix and \( \Pi_H \) is an \( N \times N \) projection matrix with rank \( \ell \). It can then be stated
that
\[ \hat{Q} = \text{tr}(\mathbf{PBD}'A'\Pi H \mathbf{A} \Theta B'\mathbf{P}') + 2 \text{tr}(\mathbf{PBD}'A'\Pi H \mathbf{E}'\mathbf{P}') \]
\[ + \sum_{i,j} h_{ij} \text{tr}(\mathbf{P}\varepsilon_i \varepsilon_j' \mathbf{P}') \]
\[ + \sum_{i=1}^{N} h_{1i} \varepsilon_i' \mathbf{P} \varepsilon_i, \quad (4) \]

where \((h_{ij}) = \Pi H\) and \(\sum_{i_1,\ldots,i_k}^N\) denote the sum over all different indices \(\{i_1,\ldots, i_k\}\); for example, \(\sum_{i,j,k}^N = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N\). It is clear that \(\sum_{i=1}^N h_{ii} \varepsilon_i' \mathbf{P} \varepsilon_i\) on the right side of equality (4) is an unnecessary term for an unbiased estimate of \(\hat{Q}\). To omit this term, we prepare an adjustment that is derived from the assumption that the linear system given below has the following solution:

\[ [(I_N - \Pi A) \circ (I_N - \Pi A)] \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} = \begin{pmatrix} h_{11} \\ \vdots \\ h_{NN} \end{pmatrix}, \quad (5) \]

where the notation “\(\circ\)” denotes the Hadamard product of the matrices. By letting \(D = \text{diag}(d_1, \ldots, d_N)\), the following formula can be obtained:

\[ \text{tr}(\mathbf{P}X'(I_N - \Pi A)D(I_N - \Pi A)\mathbf{X}'\mathbf{P}') = \text{tr}(\mathbf{P}X'(I_N - \Pi A)\mathbf{E}'\mathbf{P}') \]
\[ = \sum_{i=1}^{N} \left( \sum_{k=1}^{N} d_k(\delta_{ik} - p_{ik})^2 \right) \varepsilon_i' \mathbf{P} \varepsilon_i + \sum_{i,j} \left( \sum_{k=1}^{N} d_k(\delta_{ik} - p_{ik})(\delta_{jk} - p_{jk}) \right) \varepsilon_i' \mathbf{P} \varepsilon_j \]
\[ = \sum_{i=1}^{N} h_{ii} \varepsilon_i' \mathbf{P} \varepsilon_i + \sum_{i,j} \left( \sum_{k=1}^{N} d_k(\delta_{ik} - p_{ik})(\delta_{jk} - p_{jk}) \right) \text{tr}(\mathbf{P} \varepsilon_i \varepsilon_j' \mathbf{P}'), \]

where \(\delta_{ij}\) is the Kronecker delta and \((p_{ij}) = \Pi A\). Then, defining

\[ T = \hat{Q} - \text{tr}(\mathbf{P}X'(I_N - \Pi A)D(I_N - \Pi A)\mathbf{X}'\mathbf{P}'), \]

the following equalities hold.

\[ T = \text{tr}(\mathbf{PBD}'A'\Pi H \mathbf{A} \Theta B'\mathbf{P}') + 2 \text{tr}(\mathbf{PBD}'A'\Pi H \mathbf{E}'\mathbf{P}') + \sum_{i,j} \omega_{ij} \text{tr}(\mathbf{P} \varepsilon_i \varepsilon_j' \mathbf{P}') \]
\[ = \text{tr}(\mathbf{PBD}'A'\Omega \mathbf{A} \Theta B'\mathbf{P}') + 2 \text{tr}(\mathbf{PBD}'A'\Omega \mathbf{E}'\mathbf{P}') + \text{tr}(\mathbf{PE'ΩE}'\mathbf{P}') \]
\[ = \text{tr}(\mathbf{PX'ΩXP}'), \quad (6) \]

where \(\omega_{ij} = h_{ij} - \sum_{k=1}^{N} d_k(\delta_{ik} - p_{ik})(\delta_{jk} - p_{jk}),\) which yields \((\omega_{ij}) = \Omega = \Pi H - (I_N - \Pi A)D(I_N - \Pi A),\) and \(\Omega\) is a symmetric matrix such that each of its diagonal elements is zero. The second equality in
The expected value of $T$ is then obtained as follows:

$$E[T] = \text{tr}(PB\Theta'A'(I_N - \Pi_A)D(I_N - \Pi_A)) + 2E[\text{tr}(PB\Theta'A')\Omega]\text{PE}' = Q,$$

i.e., $T$ is an unbiased estimator of $Q$. From Lemma 1, the rejection of the null hypothesis $H_0$ results from the evidence indicating that the unbiased estimator of $Q$ is significantly larger than 0; hence, we propose the testing statistic as $T$.

The variance of $T$ is described as follows:

$$\sigma^2 = \text{Var}(T) = \sigma_0^2 + \text{Var}(\text{tr}(A\Theta'B'\text{PE} + \text{PE}'PB\Theta'A')),$$

where

$$\sigma_0^2 = \text{Var}(\text{tr}(\Omega\text{PE}'\text{PE}')) = 2 \sum_{i,j} \omega_{ij} E[(\varepsilon_i'\text{PE})^2],$$

$$\text{Var}(\text{tr}(\Omega(A\Theta'B'\text{PE} + \text{PE}'PB\Theta'A')) = 4 \sum_{i=1}^N m_i' E[\varepsilon_i\varepsilon_i'] m_i, \quad m_i = \text{PE}'PB\Theta'a_i, \quad (a_1, \ldots, a_N) = A'.$$

Note that $\sigma^2 = \sigma_0^2$ when $H_0$ is true. Defining

$$V = \begin{pmatrix}
(a_{1,1}\mathbf{1}_{N_1}'\mathbf{1}_{N_1} & b_{1,1}\mathbf{1}_{N_1}'\mathbf{1}_{N_1} & \cdots & b_{1,g}\mathbf{1}_{N_1}'\mathbf{1}_{N_g} \\
b_{2,1}\mathbf{1}_{N_1}'\mathbf{1}_{N_1} & (a_{2,1}\mathbf{1}_{N_1}'\mathbf{1}_{N_2} & \cdots & b_{2,g}\mathbf{1}_{N_1}'\mathbf{1}_{N_g} \\
\vdots & \vdots & \ddots & \vdots \\
b_{g,1}\mathbf{1}_{N_1}'\mathbf{1}_{N_1} & b_{g,2}\mathbf{1}_{N_1}'\mathbf{1}_{N_2} & \cdots & a_{g,2}\mathbf{1}_{N_g}'\mathbf{1}_{N_g}'
\end{pmatrix},$$

where

$$a_{i,j} = \text{tr}(\Psi_{i,j}), \quad b_{i,j} = b_{j,i} = \text{tr}(\Psi_i\Psi_j)$$

and $\Psi_i = \mathcal{P}\Sigma_i\mathcal{P}'$, the variance can be described as $\sigma_0^2 = 2 \text{tr}((\Omega \otimes \Omega)V)$.

3. Asymptotic distribution

This section presents an asymptotic distribution for testing statistic $T$ under the following assumptions:
A1: \( \limsup_{\min(N_1, \ldots, N_q) \to \infty} \rho N < \infty \), where \( \rho N = \max\{\omega_{ij}^2 \mid i, j \in [N], i < j \} / \min\{\omega_{ij}^2 \mid i, j \in [N], i < j, \omega_{ij} \neq 0 \} \).

A2: \( \lim_{p \to \infty} \max_{(i, j, k, \ell) \in J} \psi_{i, j, k, \ell} \left( \sum_{(i, j) \in J} b_{ij} \right) = 0 \), where \( \psi_{i, j, k, \ell} = \text{tr}(\Psi_i \Psi_j \Psi_k \Psi_{\ell}) \); \( \sum_{(i, j) \in J} b_{ij} \) denotes the sum of \( b_{ij} \) for indices \( i, j \in [g] \), such that \( (i, j) \in J = \{(i, j) \mid i, j \in [g], \Omega_{ij} \cap \Omega_{ij}^c \} \); and \( \Omega_{ij} \) is the \((i, j)\)-th block in \( \Omega \), which corresponds to that of \( V \) in [7].

A3: \( \lim_{\min(N_1, \ldots, N_q, p) \to \infty} \frac{\sum_{i=1}^g \sum_{j=1}^{N_i} (m'_{(s, r)} \Sigma m_{(s, r)})^2}{\mathcal{M} + (1 - \mathcal{M})} = 0 \), where

\[
\mathcal{M} = \begin{cases} 1, & m'_{(i, j)} m_{(i, j)} = 0 \text{ for } i \in [N], \\ 0, & \text{otherwise.}
\end{cases}
\]

For the two sample problems presented in online supplementary material, \( \mathcal{P} \) becomes the identity matrix and \( \mathcal{J} = \{1, 2\} \times \{1, 2\} \); thus, A2 is identical to the condition described by Chen and Qin [7]. From this perspective, A2 can be viewed as a modification of their condition. Note also that the assumption A3 is always satisfied under \( H_0 \).

We consider the distribution of \( T \) under the following model:

\[
\varepsilon_{N_i +j} = \Sigma_i^{1/2} z_{N_i +j}, \quad j \in [N], \quad i \in [g],
\]

where \( z_1, \ldots, z_N \) are independently and identically distributed (i.i.d.) as a \( p \)-dimensional distribution \( F \), with mean \( 0 \) and covariance matrix \( I_p \). In addition, we assume the following. For \( z = (z_1, \ldots, z_p)' \sim F \),

D1: \( \limsup_{p \to \infty} \max\{E[z_i^4] \mid i \in [p] \} < \infty \).

D2: \( E[\prod_{j=1}^p z_j^{\nu_j}] = 0 \) when there is at least one \( \nu_i = 1 \) whenever \( \nu_1 + \cdots + \nu_p = 4 \).

Notably, D2 is milder than the condition used by Bai and Saranadasa [3], without assuming that \( E[z_i^2 z_j^2] = 1 \) for \( i, j \in [g] \) with \( i \neq j \); moreover, the set of D1 and D2 is milder than the condition used by Chen and Qin [7], in which \( E[z_i^4] = 3 + \gamma \) and \( E[\prod_{j=1}^p z_j^{\nu_j}] = \prod_{i=1}^p E[z_i^{\nu_i}] \), where \( q \) is a positive integer such that \( \sum_{i=1}^q \nu_i \leq 8 \) and \( \ell_1 \neq \cdots \neq \ell_q \). In addition, conditions D1 and D2 are valid when \( F \) is an elliptical distribution with mean \( 0 \), covariance matrix \( I_p \), and a finite fourth moment. Thus, our assumption covers a wide range of multivariate distributions, including \( N_p(0, I_p) \).

**Remark 1.** Bai et al. [29] derived the asymptotic joint distributions of the eigenvalues in the MANOVA model as \( N, p \to \infty \) under the condition that \( p/N \to c \in (0, 1) \), based on the random matrix theory. For the error vector \( z = (z_1, \ldots, z_p) \), the authors assumed that \( z_1, \ldots, z_p \) are i.i.d.; \( E[z_i] = 0, E[z_i^2] = 1, \) and \( E[z_i^4] < \infty \). Because their assumption implies D1 and D2, our distribution family also includes their assumptions.

**Remark 2.** Yamada and Himeno [24] also used an assumption to specify the distribution family of \( F \) including multidimensional normal distribution. Note that their assumption is satisfied when A1-A3,
D1, and D2 hold, and thus, is milder than our own. However, we do not use Yamada and Himeno\textsuperscript{[20]}’s assumption because it is in a complex form.

**Theorem 1.** Suppose that the linear system in (5) has a solution, and that A1-A3, D1, and D2 hold. Then,

$$
\frac{T - Q}{\sqrt{\sigma^2}} \overset{D}{\rightarrow} N(0,1)
$$

under the asymptotic framework that \( N_i \approx N_j \) for \( i, j \in \{g\} \) as \( \min\{N_1, \ldots, N_g, p\} \) tends toward infinity.

The proof is given in the Appendix.

3.1. Asymptotic null distribution and proposed testing criterion

In this subsection, we present an asymptotic null distribution of \( T \). From Theorem 1 under the condition that the null hypothesis \( H_0 \) is true,

$$
\frac{T}{\sqrt{\sigma^2}} \overset{D}{\rightarrow} N(0,1).
$$

To analyze the real data, \( \sigma^2 \) must be estimated. Partition

\[
X = (X'_1, \ldots, X'_g)', \quad A = (A'_1, \ldots, A'_g)',
\]

where \( X_i = (x^{(i)}_1, \ldots, x^{(i)}_{N_i})' : N_i \times p \) and \( A_i : N_i \times k \). Define

\[
S_i = \frac{1}{N_i - k_i} \sum_{j=1}^{N_i} \mathcal{P}(x_j^{(i)} - \hat{x}_j^{(i)})(x_j^{(i)} - \hat{x}_j^{(i)})' \mathcal{P}' = \frac{1}{N_i - k_i} \mathcal{P}X_i'(I_{N_i} - \Pi_{A_i})X_i\mathcal{P}',
\]

\[
Q_i = \frac{1}{N_i - k_i} \sum_{j=1}^{N_i} \left\{ (x_j^{(i)} - \hat{x}_j^{(i)})' \mathcal{P}' \mathcal{P}(x_j^{(i)} - \hat{x}_j^{(i)}) \right\}^2,
\]

where \( \Pi_{A_i} = A_i(A'_iA_i)^+A'_i, k_i = \text{rank}(A_i), \)

\[
(\hat{x}_1^{(i)}, \ldots, \hat{x}_{N_i}^{(i)})' = \Pi_{A_i}(x_1^{(i)}, \ldots, x_{N_i}^{(i)})' = \Pi_{A_i}X_i.
\]

Here, \( A^+ \) is the Moore-Penrose inverse of the matrix \( A \). In the following lemma, we provide an expression for the unbiased estimator of \( a_{i,2} \) and that of \( b_{ij} \).

**Lemma 2.** The unbiased estimators of \( a_{i,2} \) and \( b_{ij} \) are expressed as follows:

\[
\hat{a}_{i,2} = \frac{1}{(N_i - k_i)\tau_{i,3}} \left[ \{(N_i - k_i)^2\tau_{i,2} - \tau_{i,3}^2\} \text{tr}(S_i^2) - \{(N_i - k_i)\tau_{i,2} - \tau_{i,3}^2\}(\text{tr}(S_i))^2 - (N_i - k_i - 1)\tau_{i,2}Q_i \right],
\]

\[
\hat{b}_{ij} = \text{tr}(S_iS_j),
\]

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respectively, where

\[ \tau_{i,j} = \text{tr}\left( (I_{N_i} - \Pi A_i) \odot (I_{N_i} - \Pi A_i)^j \right), \quad j = 1, 2, \]
\[ \tau_{i,3} = \frac{N_i - k_i - 1}{(N_i - k_i)^2} \{(N_i - k_i)(N_i - k_i + 2)\tau_{i,2} - 3\tau_{i,1}^2\}. \]

The unbiasedness of \( \hat{b}_{ij} \) is immediately followed from the fact that \( S_i \) and \( S_j \) are independent and that \( E[S_i] = \Psi_i \). The unbiasedness of \( \hat{a}_{i,2} \) is proved in the Supplementary Materials.

As an example of Lemma 2, set

\[
A = \begin{pmatrix}
1_{N_1} & 0 & \cdots & 0 \\
0 & 1_{N_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1_{N_g}
\end{pmatrix} = \text{diag}(1_{N_1}, \ldots, 1_{N_g}).
\]

(9)

It holds that

\[
\Pi A_i = \frac{1}{N_i} 1_{N_1} 1_{N_1}' = \Pi_{N_i}, \quad i \in [g],
\]

and thus,

\[
\Pi A = A \text{diag}(N_1^{-1}, \ldots, N_g^{-1}) A' = \text{diag}(\Pi_{N_1}, \Pi_{N_2}, \ldots, \Pi_{N_g}).
\]

From this result, the following equalities hold.

\[
\tau_{1,i} = \frac{(N_i - 1)^2}{N_i}, \quad \tau_{2,i} = \frac{(N_i - 1)(N_i^2 - 3N_i + 3)}{N_i^2}, \quad \tau_{3,i} = \frac{(N_i - 2)^2(N_i - 3)}{N_i},
\]
\[
\bar{x}^{(i)} = \frac{1}{N_i} \sum_{j=1}^{N_i} x^{(i)}_j, \quad j \in [N_i], \quad i \in [g].
\]

Therefore, \( \hat{a}_{i,2} \) in Lemma 2 can be described as

\[
\hat{a}_{i,2} = \frac{N_i - 1}{N_i(N_i - 2)(N_i - 3)} \left\{ (N_i - 1)(N_i - 2) \text{tr}(S_i^2) + (\text{tr}(S_i))^2 - N_i Q_i \right\}.
\]

(10)

This result coincides with the expression of Himeno and Yamada for the unbiased estimator of \( \text{tr}(\Sigma_i^2) \). Note that the expression (10) is identical to the following expression:

\[
\hat{a}_{i,2} = \frac{1}{P_{N_i,4}} \sum_{k,\ell,\alpha,\beta}^{N_i} \frac{((x^{(i)}_k - x^{(i)}_{\ell})(x^{(i)}_\alpha - x^{(i)}_\beta))^2}{4},
\]

where the symbol \( P_{n,k} \) denotes the number of such \( k \)-permutations of \( n \), i.e.,

\[
P_{n,k} = n(n-1) \cdots (n-k+1).
\]
Thus, we find that $\hat{a}_{i,2}$ always takes a positive value with probability 1.

**Remark 3.** For the general structure of $A$, $\hat{a}_{i,2}$ does not always take a non-negative value.

We mention that $\hat{a}_{2,i}$ and $\hat{b}_{ij}$ have rate consistencies, which are represented by the following theorem:

**Theorem 2.** Suppose that $\hat{a}_{2,i}$ and $\hat{b}_{ij}$ are defined as Lemma 2. Then, under the assumptions A1-A2 and D1-D2, $\hat{a}_{i,2}/a_{i,2}$ converges in probability to 1 under the asymptotic framework that $p/N_i$ converges to a non-negative constant as $\min\{p,N_i\}$ tends toward infinity for $i \in [g]$; $\hat{b}_{ij}/b_{ij}$ converges in probability to 1 as $\min\{p,N_i,N_j\}$ tends toward infinity for $i,j \in [g]$ with $i \neq j$.

**Remark 4.** Because $\hat{a}_{i,2}$ and $\hat{b}_{ij}$ are location-free estimators, their unbiasedness and rate consistency hold under $H_1$.

Let $\hat{\sigma}_0^2$ be the estimator of $\sigma_0^2$ obtained by replacing the unknown parameters in $\sigma_0^2$ with their unbiased estimators given in Lemma 2, i.e., $\hat{\sigma}_0^2 = \text{tr}((\Omega \odot \Omega)\hat{V})$, where $\hat{V}$ is defined by $V$ through a replacement of the unknown parameters with their unbiased estimators. Then,

$$\frac{\hat{\sigma}_0^2}{\sigma_0^2} \xrightarrow{p} 1 \quad (11)$$

under the asymptotic framework that $\max\{p/N_i : i \in [g]\}$ converges to a non-negative constant as $\min\{p,N_1,\ldots,N_g\}$ tends toward infinity.

**Remark 5.** As mentioned in Remark 3, $\hat{a}_{i,2}$ does not always take a non-negative value, which indicates that $\hat{\sigma}_0^2$ also does not always take a non-negative value.

**Theorem 3.** Suppose that the assumptions in Theorem 2 are satisfied. Then, under the null hypothesis $H_0$ given in (2),

$$I(\hat{\sigma}_0^2 > 0) \frac{T}{\sqrt{\sigma_0^2}} \xrightarrow{d} N(0,1)$$

under the asymptotic framework that $\max\{p/N_i : i \in [g]\}$ converges to a non-negative constant as $\min\{p,N_1,\ldots,N_g\}$ tends toward infinity and $N_i \asymp N_j$ for $i,j \in [g]$, where $I(.)$ is the indicator function, and $\hat{\sigma}_0^2$ is defined as $\sigma_0^2$ by replacing $a_{i,2}$ and $b_{ij}$ with $\hat{a}_{i,2}$ and $\hat{b}_{ij}$, respectively, as defined in Lemma 2.

We propose the following testing criterion with the significance level $\varepsilon$ for the case in which the linear system in (5) has a solution. If $\text{tr}((\Omega \odot \Omega)\hat{V}) > 0$, then

$$\frac{T}{\sqrt{\sigma_0^2}} > \frac{T}{\sqrt{2\text{tr}((\Omega \odot \Omega)\hat{V})}} = \Phi^{-1}(1 - \varepsilon) \implies \text{Reject the null hypothesis } H_0. \quad (12)$$
3.2. Asymptotic power for the proposed test

In this section, the asymptotic power for the proposed test is considered. Note that the following equivalence holds.

\[ I(\hat{\sigma}_0^2 > 0) \frac{T}{\sqrt{\hat{\sigma}_0^2}} > \Phi^{-1}(1 - \varepsilon) \iff T - \frac{Q}{\sqrt{\sigma^2}} > I(\hat{\sigma}_0^2 > 0)\sqrt{\frac{\hat{\sigma}_0^2}{\sigma^2}} \Phi^{-1}(1 - \varepsilon) - \frac{Q}{\sqrt{\sigma^2}}. \]

Because the ratio consistency of \( \hat{\sigma}_0^2 \) given in (11) also holds under \( H_1 \), Theorem 1 yields the asymptotic power of the proposed test, which is described as follows.

**Theorem 4.** Suppose that the assumptions in Theorem 2 are satisfied. Under the asymptotic framework \( A \) that \( \max\{p/N_1, \ldots, p/N_g\} \) converges to a non-negative constant as \( \min\{p, N_1, \ldots, N_g\} \) tends toward infinity and \( N_i \approx N_j \) for \( i, j \in [g] \), the asymptotic power becomes 1 if \( Q/\sqrt{\sigma^2} \to \infty \). In addition,

\[ \lim_{A} P \left( I(\hat{\sigma}_0^2 > 0) \frac{T}{\sqrt{\hat{\sigma}_0^2}} > \Phi^{-1}(1 - \varepsilon) \right) = \Phi \left( \lim_{A} \left( -\sqrt{\frac{\hat{\sigma}_0^2}{\sigma^2}} \Phi^{-1}(1 - \varepsilon) + \frac{Q}{\sqrt{\sigma^2}} \right) \right) \]

if \( Q/\sqrt{\sigma^2} \) converges to a non-negative constant, where the notation \( \lim_{A} \) denotes the limit under the asymptotic framework \( A \), and \( \hat{\sigma}_0^2 \) is the same as those in Theorem 3.

4. Concluding remarks

In this paper, we proposed a test for the bilateral linear hypothesis of the mean matrix in the heteroscedastic GMANOVA model where the dimensions may exceed the sample sizes. The proposed test is valid even for the case in which the underlying population distribution is not normal. In Theorem 2 we showed the rate consistency of \( \hat{a}_{i,2} \), which is defined in Lemma 2. This consistency was proved under the asymptotic framework that \( p/N_i \) converges to a non-negative constant as \( \min\{N_1, \ldots, N_g, p\} \) tends toward infinity; thus, a future study will show the consistency without assuming the convergence of \( p/N_i \), to guarantee the performance for a case in which \( p \) is arbitrarily larger than \( N_i \).

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Appendix A. Derivation of asymptotic distribution

In this section, we show the asymptotic normality of the random variable

\[ S = \sum_{i,j} \omega_{ij} \left( z_i \Sigma_{ij}^{1/2} P \Sigma_j^{1/2} z_j + m_i \Sigma_j^{1/2} z_j + z_i \Sigma_i^{1/2} m_j \right), \]

where \( \Sigma_i \) is a positive-definite symmetric matrix for \( i \in [N] \). Note that

\[ T - Q = \text{tr}(\Omega E' \varPhi E) + 2 \text{tr}(\Omega E' \varPhi B \Theta A') = S, \]

for the case in which \( \bar{\Sigma}_j = \Sigma_i, \ j \in \{\bar{N}_i + 1, \ldots, \bar{N}_i + N_i\}, \ i \in [g]. \) (A.1)

To simplify the notation, hereafter, we provide a proof for the case in which \( \omega_{ij} \neq 0, \ i > j \). Using the same derivation approach, this can be proved for a general case under which there exist \( i \) and \( j \) such that \( \omega_{ij} = 0 \). Define

\[ \bar{\Psi}_i = P \bar{\Sigma}_i P', \ \bar{\Upsilon}_i = P \bar{\Sigma}_i^{1/2}, \ i \in [N]. \]

Instead of assuming A1, A2, and A3, we assume

\[ \bar{A}1: \limsup_{N \to \infty} \rho_N < \infty. \]

\[ \bar{A}2: \max \{M_i : i = 1, \ldots, 6\} \to 0 \ (\min\{N, p\} \to \infty), \]

where

\[ M_1 = \sum_{i,j} (\text{tr}(\bar{\Psi}_i \bar{\Psi}_j))^2, \ M_2 = \sum_{i,j,k} \text{tr}(\bar{\Psi}_i \bar{\Psi}_j) \text{tr}(\bar{\Psi}_i \bar{\Psi}_k), \]

\[ M_3 = \sum_{i,j,k} \sqrt{\text{tr}(\bar{\Psi}_i \bar{\Psi}_j \bar{\Psi}_k) \text{tr}(\bar{\Psi}_j \bar{\Psi}_k)}, \]

\[ M_4 = \sum_{i,j,k} \text{tr}(\bar{\Psi}_i \bar{\Psi}_j \bar{\Psi}_k \bar{\Psi}_\ell), \ M_5 = \sum_{i,j,k} \text{tr}(\bar{\Psi}_i \bar{\Psi}_j \bar{\Psi}_k), \]

\[ M_6 = \sum_{i,j,k} \sqrt{\text{tr}(\bar{\Psi}_i \bar{\Psi}_j \bar{\Psi}_k \bar{\Psi}_\ell) \text{tr}(\bar{\Psi}_i \bar{\Psi}_j \bar{\Psi}_k \bar{\Psi}_\ell)}, \ M_7 = \sum_{i,j} \text{tr}(\bar{\Psi}_i \bar{\Psi}_j). \]

\[ \bar{A}3: \frac{\sum_{i=1}^N (m_i' \Sigma_i m_i)}{M + (1 - M)(\sum_{i=1}^N m_i' \Sigma_i m_i)^2} \to 0 \ (\min\{N, p\} \to \infty). \]

Note that the assertion \( \text{“A1, A2, A3 } \Rightarrow \bar{A}1, \bar{A}2, \bar{A}3 \text{” is true for the case in which (A.1) holds; thus, the set of } \bar{A}1, \bar{A}2, \bar{A}3 \text{ is milder than that of A1, A2, and A3. The following lemma is essential for the proof; however, because it is trivial, its proof has been omitted here.} \]

**Lemma 3.** Let \( F \) be a \( p \)-dimensional distribution that satisfies D2. Assume that \( z_1 \) and \( z_2 \) are i.i.d. as \( F \). Then, for the non-negative definite symmetric matrices \( A \) and \( B \) and the square matrix \( C \),
there exists a numeric constant $C_1$ such that

\[
E \left[ (z_1'Az_1)^2 \right] \leq C_1\delta (\text{tr}(A))^2,
\]
\[
E [z_1'Az_1z_1'Bz_1] \leq C_1\delta \text{tr}(A) \text{tr}(B),
\]
\[
E [(z_1'Cz_2)^4] \leq C_2\delta^2 (\text{tr}(CC'))^2,
\]
where $\delta = \max\{E[z_i^4] : i \in [p]\}$.

It is observed that

\[
E[S] = 0, \quad \sigma^2 = \text{Var}(S) = 2\sum_{i,j}^{N} \text{tr}(\bar{\Psi}_i\bar{\Psi}_j) + 4\sum_{i=1}^{N} m'_i \bar{\Sigma}_i m(i).
\]

Define

\[
\eta_1 = \frac{2}{\sigma} z_1' \bar{\Sigma}_1^{1/2} m(1),
\]
\[
\eta_i = \frac{2}{\sigma} z_i' \bar{\Sigma}_i^{1/2} \left( \sum_{j=1}^{i-1} \omega_{ij} P' P \bar{\Sigma}_j^{1/2} z_j + m(i) \right), \quad i \in \{2, \ldots, N\}.
\]

It can thus be stated that

\[
\frac{S}{\sigma} = \sum_{i=1}^{N} \eta_i.
\]

Let $\mathcal{F}_i$ be a $\sigma$-algebra generated by $\{z_1, \ldots, z_i\}$, and let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, where $\emptyset$ is the empty set and $\Omega$ is the possibility space. It holds that

\[
E[\eta_i|\mathcal{F}_{i-1}] = 0, \quad i \in [N].
\]

This indicates that $\{\eta_i\}$ is a martingale difference sequence, and $\sum_{i=1}^{N} E[\eta_i^2] = 1$. From Heyde and Brown [10], the following inequality holds:

\[
\sup_{x \in \mathbb{R}} \left| P \left( \sum_{i=1}^{N} \eta_i \leq x \right) - \Phi(x) \right| \leq K \left[ \sum_{i=1}^{N} E[\eta_i^2] + \text{Var} \left( \sum_{i=1}^{N} \eta_i^2 \right) \right]^{1/5},
\]

where $K$ is a numeric constant. To establish the asymptotic normality of $S/\sigma$, it is sufficient to show the following conditions:

C1: $\sum_{i=1}^{N} E[\eta_i^2] \to 0$ as $\min\{N, p\} \to \infty$.

C2: $\text{Var} \left( \sum_{i=1}^{N} \eta_i^2 \right) \to 0$ as $\min\{N, p\} \to \infty$.

In the following subsections (Appendix A.1. and Appendix A.2.), we show C1 and C2.

Summarizing the aforementioned results, we obtain the following theorem:

**Theorem 5.** Under the assumptions $\bar{A}1-\bar{A}3$ and $D1-D2$, $S/\sqrt{\sigma^2}$ converges to a standard normal
distribution as \( \min\{N, p\} \) tends to infinity.

**Appendix A.1. Proof of C1**

In this section, we show C1. From the Cauchy-Schwarz inequality,

\[
E[\eta_i^4] \leq \frac{16}{\sigma^4} \cdot 8 \left( E \left[ \sum_{j=1}^{i-1} \omega_{ij} z_i' \bar{\Upsilon}_i \bar{\Upsilon}_j z_j \right]^4 \right) + E \left[ (z_i' \bar{\Sigma}_i^{1/2} m_{(i)})^4 \right].
\]

It can then be described that

\[
\sum_{i=1}^{N} E[\eta_i^4] \leq 128(A_1 + A_2),
\]

where

\[
A_1 = \frac{1}{\sigma^4} \sum_{i=2}^{N} E \left[ \left( \sum_{j=1}^{i-1} \omega_{ij} z_i' \bar{\Upsilon}_i \bar{\Upsilon}_j z_j \right)^4 \right] > 0, \quad A_2 = \frac{1}{\sigma^4} \sum_{i=1}^{N} E \left[ (z_i' \bar{\Sigma}_i^{1/2} m_{(i)})^4 \right] > 0.
\]

To prove that \( A_1 \to 0 \), we write

\[
A_1 = \frac{1}{\sigma^4} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \omega_{ij}^4 E[ (z_i' \bar{\Upsilon}_i \bar{\Upsilon}_j z_j)^4 ] + \frac{3}{\sigma^4} \sum_{i=2}^{N} \sum_{k,\ell} \omega_{ik}^2 \omega_{\ell i}^2 E[ z_i' \bar{\Upsilon}_i \bar{\Upsilon}_i z_1 z_1' \bar{\Upsilon}_i \bar{\Upsilon}_i z_1 ].
\]

There exists a numeric constant \( C_3 \) such that

\[
A_1 < C_3 \rho_N^2 (A_3 + A_4),
\]

where

\[
A_3 = \sum_{i,j} E \left[ \left( z_i' \bar{\Upsilon}_i \bar{\Upsilon}_j z_j \right)^4 \right] \frac{M_2}{M_2^2} > 0,
\]

\[
A_4 = \sum_{i,j,k} E \left[ z_i' \bar{\Upsilon}_i \bar{\Upsilon}_j z_1 z_1' \bar{\Upsilon}_k \bar{\Upsilon}_i z_1 \right] \frac{M_2}{M_2^2} > 0.
\]

Note that the condition C1 is established if both \( A_3 \) and \( A_4 \) converge to 0. From Lemma 3,

\[
A_3 < C_4^2 s^2 \frac{M_1}{M_2^2} \to 0 \quad (\min\{N, p\} \to \infty),
\]

where the convergence follows from the assumption \( \bar{A}_2 \). Using the same method of derivation, it holds
that

\[ A_4 < C_1 \delta \frac{M_2}{M^2_\delta} \to 0 \quad (\min\{N, p\} \to \infty). \]

Next, we prove that \( A_2 \to 0 \). It follows from Lemma 3 that

\[
A_2 \leq \frac{C_1 \delta}{\sigma^4} \sum_{i=1}^{N} (m'_i \Sigma_i m(i))^2
\]

\[
< C_1 \frac{\sum_{i=1}^{N} (m'_i \Sigma_i m(i))^2}{MM^2_\delta + (1 - M) \left( \sum_{i=1}^{N} m'_i \Sigma_i m(i) \right)^2} \to 0 \quad (\min\{N, p\} \to \infty)
\]

under the assumption that \( \tilde{A}_3 \).

**Appendix A.2. Proof of C2**

It can be described that

\[
\sum_{i=1}^{N} (\eta_i^2 - E[\eta_i^2]) = 4(B_1 + B_2 + B_3 + 2B_4),
\]

where

\[
B_1 = \frac{1}{2\sigma^2} \sum_{i=1}^{N} \omega_{ij}^2 \left\{ (z'_i \bar{\Upsilon}'_i \bar{\Upsilon}_j z_j)^2 - \text{tr} \bar{\Psi}_i \bar{\Psi}_j \right\},
\]

\[
B_2 = \frac{1}{\sigma^2} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \omega_{ij} \omega_{ik} z'_i \bar{\Upsilon}_i \bar{\Upsilon}_j z_j z_k \bar{\Upsilon}'_i \bar{\Upsilon}_k \bar{\Upsilon}_k,
\]

\[
B_3 = \frac{1}{\sigma^2} \sum_{i=1}^{N} \left\{ (z'_i \bar{\Upsilon}_i \Sigma_i m(i))^2 - m'_i \Sigma_i m(i) \right\},
\]

\[
B_4 = \frac{1}{\sigma^2} \sum_{i=2}^{N} \sum_{j=1}^{i-1} z'_i \bar{\Upsilon}_i \Sigma_i \Sigma_i m(i) \omega_{ij} z'_i \bar{\Upsilon}_i \bar{\Upsilon}_j z_j.
\]

The following inequality then holds.

\[
\text{Var} \left( \sum_{i=1}^{N} \eta_i^2 \right) \leq 16E[4(B_1^2 + B_2^2 + B_3^2 + 4B_4^2)] = 64(E[B_1^2] + E[B_2^2] + E[B_3^2] + 4E[B_4^2]).
\]
Here, the expectations on the right-hand side of the equality can be described as follows:

\[
E[B_1^2] = \frac{1}{2\sigma^4} \sum_{i,j}^N \omega_{ij}^2 \{ E[(z_i' \tilde{Y}_i \tilde{Y}_j z_2)^4] - (\text{tr}(\tilde{\Psi}_i \tilde{\Psi}_j))^2 \}
+ \frac{1}{\sigma^4} \sum_{i,j,k}^N \omega_{ij}^2 \omega_{ik}^2 \{ E[z_i' \tilde{Y}_i \tilde{Y}_j \tilde{Y}_k z_2] - (\text{tr}(\tilde{\Psi}_i \tilde{\Psi}_j \tilde{\Psi}_k))^2 \},
\]

\[
E[B_2^2] = \frac{2}{\sigma^4} \sum_{i,j,k}^N \omega_{ij}^2 \omega_{jk}^2 \{ E[z_i' \tilde{Y}_i \tilde{Y}_j \tilde{Y}_k z_2] - (\text{tr}(\tilde{\Psi}_i \tilde{\Psi}_j \tilde{\Psi}_k))^2 \}
+ \frac{4}{\sigma^4} \sum_{i,j,k}^N \omega_{ij}^2 \omega_{jk}^2 \omega_{lk}^2 \text{tr}(\tilde{\Psi}_i \tilde{\Psi}_j \tilde{\Psi}_k \tilde{\Psi}_l),
\]

\[
E[B_3^2] = \frac{1}{\sigma^4} \sum_{i=1}^N \{ E[(m_i' \Sigma_i z_1)^4] - (m_i' \Sigma_i m_i)^2 \},
\]

\[
E[B_4^2] = \frac{1}{\sigma^4} \sum_{i=1}^N \{ E[(m_i' \Sigma_i z_1)^2 z_i' \tilde{Y}_i \tilde{Y}_j \tilde{Y}_k z_2] - (m_i' \Sigma_i m_i)^2 \}
+ \frac{2}{\sigma^4} \sum_{i=1}^N \omega_{ij}^2 \omega_{jk}^2 m_i' \Sigma_i \tilde{Y}_i \tilde{Y}_j \tilde{Y}_k \Sigma_j m_j.
\]

From Lemma 3 and assumptions A1–A3, we have

\[
E[B_1^2] < \rho_N^2 \frac{(C_1^2 \delta^2 + 1)M_1 + 2(C_1^2 + 1)M_2}{2M_1^2} \to 0 \quad (\min\{N,p\} \to \infty),
\]

\[
E[B_2^2] < \rho_N^2 \frac{2C_1 \delta M_1 + 4M_2}{M_1^2} \to 0 \quad (\min\{N,p\} \to \infty),
\]

\[
E[B_3^2] \leq \frac{(C_1 + 1) \sum_{i=1}^N (m_i' \Sigma_i m_i)^2}{M + (1 - M) \sum_{i=1}^N m_i' \Sigma_i m_i} \to 0 \quad (\min\{N,p\} \to \infty),
\]

\[
E[B_4^2] \leq \rho_N \frac{C_1 \delta \sqrt{(M_1 + M_2) \sum_{i=1}^N (m_i' \Sigma_i m_i)^2}}{M + (1 - M) M_1 \sum_{i=1}^N m_i' \Sigma_i m_i}
+ 2\rho_N \frac{\sqrt{M_5 + M_6 \sum_{i=1}^N m_i' \Sigma_i m_i}}{M + (1 - M) M_7 \sum_{i=1}^N m_i' \Sigma_i m_i} \to 0 \quad (\min\{N,p\} \to \infty),
\]

which imply that C2 is satisfied.
Appendix B. Rate consistency

In this section, we prove the rate consistency stated in Theorem 2. It is sufficient to show that

$$\frac{\hat{a}_{1,2}}{a_{1,2}} \xrightarrow{p} 1 \quad (\min\{N_1, p\} \to \infty \text{ and } p/N_1 \to c \in [0, \infty)), $$

$$\frac{\hat{b}_{12}}{b_{12}} \xrightarrow{p} 1 \quad (\min\{N_1, N_2, p\} \to \infty),$$

where

$$\hat{a}_{1,2} = \nu_1 \text{tr}(S^2_1) - \nu_2 (\text{tr}(S_1))^2 - \nu_3 Q_1, \quad \hat{b}_{12} = \text{tr}(S_1S_2).$$

Here,

$$\nu_1 = \frac{(N_1 - k_1)^2 \tau_{1,2} - \tau_{1,1}^2}{(N_1 - k_1) \tau_{1,3}}, \quad \nu_2 = \frac{(N_1 - k_1) \tau_{1,2} - \tau_{1,1}^2}{(N_1 - k_1) \tau_{1,3}}, \quad \nu_3 = \frac{(N_1 - k_1 - 1) \tau_{1,1}}{(N_1 - k_1) \tau_{1,3}}.$$ 

Let

$$Y_i = \mathcal{P}^{1/2} \Sigma_i, \quad i = 1, 2.$$ 

Appendix B.1. Proof of the rate consistency for $a_{1,2}$

To show the rate consistency, we use the following lemmas (Lemma 4-6):

**Lemma 4.** The following four results hold.

$$\lim_{N_1 \to \infty} \nu_1 = 1, \quad \nu_2 = O(N_1^{-2}), \quad \nu_3 = O(N_1^{-1}), \quad \nu_1 - (N_1 - k_1) \nu_3 = O(N_1^{-1}).$$

**Lemma 5.** Under the assumptions D1 and D2, the following probability convergences hold:

(i) \( \frac{\text{tr}(S_1)}{\text{tr}(\Sigma_1)} \xrightarrow{p} 1 \) as \( \min\{N_1, p\} \) tends toward infinity.

(ii) \( \frac{\text{tr}(S_1^2)}{\text{tr}(\Sigma_1)} - \frac{1}{(N_1 - k_1)^2} \frac{\text{tr}(Z Y_1' Y_1 Z')^2}{\text{tr}(\Sigma_1^2)} \xrightarrow{p} 0 \) under the asymptotic framework that \( p/N_1 \) converges to a non-negative constant as \( \min\{N_1, p\} \) tends toward infinity.

(iii) \( \frac{Q_1}{p \text{tr}(\Sigma_1)} - \frac{1}{N_1 - k_1} \frac{\text{tr}((Z Y_1' Y_1 Z') \odot (Z Y_1' Y_1 Z'))}{p \text{tr}(\Sigma_1)} \xrightarrow{p} 0 \) under the asymptotic framework that \( p/N_1 \) converges to a non-negative constant as \( \min\{N_1, p\} \) tends toward infinity.

Here, \( Z = (z_1, \ldots, z_{N_1})' = (z_1^{(1)}, \ldots, z_{N_1}^{(1)}).$$

**Lemma 6.** Under the assumptions D1 and D2, the following probability of convergences holds:

(i) \( \frac{1}{N_1} \sum_{i=1}^{N_1} \frac{(z_i' Y_1 Y_i z_i)^2}{\text{tr}(\Sigma_1^2)} \xrightarrow{p} 0 \) as \( \min\{N_1, p\} \) tends toward infinity.

(ii) \( \frac{1}{N_1(N_1 - 1)} \sum_{i,j} \frac{(z_i' Y_1 Y_i z_j)^2}{\text{tr}(\Sigma_1^2)} \xrightarrow{p} 1 \) as \( \min\{N_1, p\} \) tends toward infinity.
Lemma 7. Under the assumptions

Here, \((z_1, \ldots, z_{N_1}) = (z_1^{(1)}, \ldots, z_{N_1}^{(1)})\).

Proofs of these lemmas are simple but tedious; therefore, they have been omitted (they are provided in the supplementary material).

It can be described that

\[
\frac{\hat{a}_{1,2}}{\text{tr}(\Sigma_1^2)} = \nu_1 \frac{\text{tr}(S_1^2)}{\text{tr}(\Sigma_1^2)} - \frac{p}{N_1} \frac{(N_1^2 \nu_2) \left( \frac{\text{tr}(S_1)}{\text{tr}(\Sigma_1)} \right)^2}{p \text{tr}(\Sigma_1^2)} - N_1 \nu_3 \frac{p}{N_1} \frac{Q_1}{\text{tr}(\Sigma_1^2)}.
\]

Applying Lemma 3\(^{[3]}\),

\[
\frac{\hat{a}_{1,2}}{\text{tr}(\Sigma_1^2)} \rightarrow \left\{ \frac{\nu_1}{(N_1 - k_1)^2} \frac{\text{tr}((Z\Psi_1' Y_1 Z')^2)}{\text{tr}(\Sigma_1^2)} - \frac{\nu_3}{N_1 - k_1} \frac{\text{tr}((Z\Psi_1' Y_1 Z') \odot (Z\Psi_1' Y_1 Z'))}{\text{tr}(\Sigma_1^2)} \right\} = 0
\]

under the asymptotic framework that \(p/N_1\) converges to a non-negative constant as \(\min \{N_1, p\} \rightarrow \infty\). Here, the expression in the braces can be expressed as follows.

\[
\frac{\nu_1}{(N_1 - k_1)^2} \sum_{i,j} \left( \frac{\text{tr}(Y_i' Y_j z_i z_j)}{\text{tr}(\Sigma_1^2)} \right)^2 + \left\{ \frac{\nu_1}{(N_1 - k_1)^2} - \frac{\nu_3}{N_1 - k_1} \right\} \sum_{i=1}^{N_1} \frac{(z_i' Y_i Y_1 z_i)^2}{\text{tr}(\Sigma_1^2)},
\]

which converges to 1 in probability, where the convergence is followed from Lemma 4, Lemma 6, and \(0 \leq (\text{tr}(\Sigma_1^2))^2 / \{p \text{tr}(\Sigma_1^2)\} \leq 1\). Summarizing the aforementioned results, we can see that \(\hat{a}_{1,2}/a_{1,2}\) converges to 1 in probability.

Appendix B.2. Proof of rate consistency for \(b_{12}\)

Let us put \(Z_1 = (z_1, \ldots, z_{N_1})' = (z_1^{(1)}, \ldots, z_{N_1}^{(1)})\) and \(Z_2 = (z_{N_1+1}, \ldots, z_{N_1+N_2})' = (z_1^{(2)}, \ldots, z_{N_2}^{(2)})\).

To show the rate consistency, we use the following lemmas.

Lemma 7. Under the assumptions D1 and D2,

\[
\frac{\text{tr}(S_1 S_2)}{\text{tr}(\Sigma_1 \Sigma_2)} \rightarrow \left. \frac{1}{(N_1 - k_1)(N_2 - k_2)} \right. \frac{\text{tr}(Z_1 \Psi_1' Y_1 Z_1 \odot Z_2 \Psi_2' Y_1 Z_1)}{\text{tr}(\Sigma_1 \Sigma_2)} \rightarrow 0
\]

as \(\min \{N_1 N_2, p\} \rightarrow \infty\).

Lemma 8. Under the assumptions D1 and D2,

\[
\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (z_i' Y_i' Y_2 z_j)^2 \rightarrow 1
\]

as \(\min \{N_1 N_2, p\} \rightarrow \infty\).

Lemma 7 can be proved using the derivation method applied to prove Lemma 6(ii) and Lemma 8 can be proved using the derivation method used to prove Lemma 6(ii); these proofs have, therefore, been omitted from this paper.
From Lemma 7-8,

\[
\frac{\text{tr}(S_1S_2)}{\text{tr}(\Psi_1\Psi_2)} - 1 = \frac{\text{tr}(S_1S_2)}{\text{tr}(\Psi_1\Psi_2)} - \frac{1}{(N_1 - k_1)(N_2 - k_2)} \frac{\text{tr}(Z_1'Y_1'Z_2'Y_2')}{\text{tr}(\Psi_1\Psi_2)} \\
+ \frac{k_2N_1 + k_1N_2 - k_1k_2}{(N_1 - k_1)(N_2 - k_2)} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (y_i'y_jy_j'y_i')^2 \\
+ \frac{\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (y_i'y_jy_j'y_i')^2}{N_1N_2\text{tr}(\Psi_1\Psi_2)} - 1
\]

\( P \to 0. \)

**Supplementary materials**

In the supplementary material, we prove Lemma 2 and Lemma 4-6. In addition, specific results contained in the proposed class of tests are reviewed, and the results of numerical evaluations conducted through various finite-sample simulation scenarios, along with an example using real data are also reported.

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