A POINTED PRYM-PETRI THEOREM

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Abstract. We construct pointed Prym-Brill-Noether varieties parametrizing line bundles assigned to an irreducible étale double covering of a curve with a prescribed minimal vanishing at a fixed point. We realize them as degeneracy loci in type D and deduce their classes in case of expected dimension. Thus, we determine a pointed Prym-Petri map and prove a pointed version of the Prym-Petri theorem implying that the expected dimension holds in the general case. These results build on work of Welters and De Concini–Pragacz on the unpointed case. Finally, we show that Prym varieties are Prym-Tyurin varieties for Prym-Brill-Noether curves of exponent enumerating standard shifted tableaux times a factor of 2, extending to the Prym setting work of Ortega.

Introduction

Prym-Brill-Noether varieties parametrize line bundles assigned to an irreducible étale double covering of an algebraic curve with a prescribed minimal number of independent global sections. Introduced by Welters [Wel], they have been endowed with the scheme structure of a degeneracy locus by De Concini–Pragacz [DCP]. Specifically, following a result by Mumford [Mum1], Prym-Brill-Noether varieties are realized as the loci where two maximal isotropic subbundles of a vector bundle with a non-degenerate quadratic form intersect in a prescribed minimal dimension.

This study presents a refinement of Prym-Brill-Noether varieties obtained by imposing vanishing conditions at a fixed point. The goal is to extend some of the fundamental results on pointed Brill-Noether varieties from Eisenbud-Harris [EH2] and Ciliberto-Harris-Teixidor-i-Bigas [CHTiB] to the Prym setting. For this, we construct pointed Prym-Brill-Noether varieties as degeneracy loci with scheme structure induced by Schubert varieties in an orthogonal Grassmannian. Thus, results from Anderson-Fulton [AF] in type D apply to compute their expected dimension and class.

The varieties are described as follows. Let $C$ be a smooth algebraic curve of genus $g \geq 2$. A nontrivial 2-torsion point $\epsilon$ in $\text{Jac}(C)$ determines an irreducible étale double covering $\varphi: \tilde{C} \to C$, and this induces a norm map $\text{Nm}: \text{Pic}(\tilde{C}) \to \text{Pic}(C)$. The scheme-theoretic inverse image $\text{Nm}^{-1}(\omega_C)$ has two components:

$$\text{Pr}^+(C, \epsilon) := \left\{ L \in \text{Pic}^{2g-2}(\tilde{C}) : \text{Nm}(L) = \omega_C, h^0(\tilde{C}, L) \equiv 0 \mod 2 \right\},$$

$$\text{Pr}^-(C, \epsilon) := \left\{ L \in \text{Pic}^{2g-2}(\tilde{C}) : \text{Nm}(L) = \omega_C, h^0(\tilde{C}, L) \equiv 1 \mod 2 \right\}.$$
These are both translates of the Prym variety $\text{Pr}(C, \epsilon)$ given by the connected component of the origin in the kernel of $\text{Nm}: \text{Jac} (\tilde{C}) \to \text{Jac}(C)$. In particular, the Prym variety is an abelian subvariety of $\text{Jac}(\tilde{C})$ of dimension $g-1$, and the canonical polarization on $\text{Jac}(\tilde{C})$ restricts to twice a principal polarization $\Xi$ on $\text{Pr}(C, \epsilon)$ [Mum2].

For $a = (0 \leq a_0 < a_1 < \cdots < a_r)$ and a point $P$ in $\tilde{C}$, we consider the closed subset in $N_{m}^{-1}(\omega_C)$ assigned to $(C, \epsilon, P)$

$$V^a(C, \epsilon, P) := \left\{ L \in \text{Pic}^{2g-2}(\tilde{C}) \mid \begin{array}{c} \text{Nm}(L) = \omega_C, \\
\hbar^0(\tilde{C}, L) \equiv r + 1 \mod 2, \\
\hbar^0(\tilde{C}, L(-a_i P)) \geq r + 1 - i, \forall i \end{array} \right\}.$$ 

The Prym-Brill-Noether variety from [Wel] is recovered set-theoretically when no special condition is imposed at the point $P$, i.e., for $a = (0, 1, \ldots, r)$. In §1, we endow $V^a(C, \epsilon, P)$ with the scheme structure of a degeneracy locus in type D, generalizing the case $a = (0, 1, \ldots, r)$ from [DCP]. We call $V^a(C, \epsilon, P)$ the pointed Prym-Brill-Noether variety assigned to $(C, \epsilon, P)$.

We deduce that the expected dimension of $V^a(C, \epsilon, P)$ is

$$\beta(g, a) := g - 1 - |a|$$

where $|a| := \sum_{i=0}^{r} a_i$, and show that a Pfaffian formula from [AF] applies to compute the class of $V^a(C, \epsilon, P)$ when the expected dimension holds.

Specifically, let $\ell(a)$ be the number of positive elements of $a$. The theta divisor $\Xi$ on $\text{Pr}(C, \epsilon)$ translates to a theta divisor on $\text{Pr}^\pm(C, \epsilon)$. Let $\xi$ be the class of such theta divisor in the numerical equivalence ring $N^* (\text{Pr}^\pm(C, \epsilon), k)$ or in the singular cohomology ring $H^* (\text{Pr}^\pm(C, \epsilon), \mathbb{C})$. Define the class

$$B(g, a) := 2^{\ell(a)} \prod_{i=0}^{r} \frac{1}{a_i!} \prod_{0 \leq j < i \leq r} \frac{|a_i - a_j|}{a_i + a_j} \xi^{|a|}$$

in $N^* (\text{Pr}^\pm(C, \epsilon), k)$ for a ground field $k$ of characteristic different from 2, and in $H^* (\text{Pr}^\pm(C, \epsilon), \mathbb{C})$.

**Theorem 1.** One has $\dim(V^a(C, \epsilon, P)) \geq \beta(g, a)$. If equality holds, then $V^a(C, \epsilon, P)$ is a reduced Cohen-Macaulay and normal scheme, and

$$[V^a(C, \epsilon, P)] = B(g, a)$$

in $N^* (\text{Pr}^\pm(C, \epsilon), k)$ when $\text{char}(k) \neq 2$ and in $H^* (\text{Pr}^\pm(C, \epsilon), \mathbb{C})$.

For arbitrary $(C, \epsilon, P)$, the class $B(g, a)$ is supported on the pointed Prym-Brill-Noether variety also when $\dim(V^a(C, \epsilon, P)) > \beta(g, a)$. Since $\xi$ is ample, $B(g, a) \neq 0$ when $\beta(g, a) \geq 0$. Hence, we deduce:

**Corollary 1.** If $\beta(g, a) \geq 0$, then $V^a(C, \epsilon, P)$ is non-empty and of dimension at least $\beta(g, a)$ for all $(C, \epsilon, P)$ as above.
The above argument is modeled on the case $a = (0, 1, \ldots, r)$, first treated in [DCP]. Our main result here shows that the expected dimension is generically satisfied:

**Theorem 2.** For a general curve $C$ of genus $g$, an arbitrary nontrivial 2-torsion point $\epsilon$ in $\text{Jac}(C)$, and a general point $P \in \tilde{C}$, the variety $V^a(C, \epsilon, P)$ is either empty, or smooth of dimension $\beta(g, a)$ at any $L \in V^a(C, \epsilon, P)$ such that $h^0(\tilde{C}, L(−a_i P)) = r + 1 − i$ for $i = 0, \ldots, r$.

This statement extends the result on the Prym-Brill-Noether varieties from [Wel]. The proof involves an infinitesimal argument and the theory of limit linear series from Eisenbud-Harris [EH1]. In conclusion, we deduce:

**Corollary 2.** For a general curve $C$ of genus $g$, an arbitrary nontrivial 2-torsion point $\epsilon$ in $\text{Jac}(C)$, and a general point $P \in \tilde{C}$, one has $V^a(C, \epsilon, P) \neq \emptyset \iff \beta(g, a) \geq 0$.

When nonempty, $V^a(C, \epsilon, P)$ has dimension $\beta(g, a)$ and class equal to $B(g, a)$ in $\text{N}^+(\text{Pr}^\pm(C, \epsilon), k)$ when $\text{char}(k) \neq 2$ and in $\text{H}^*(\text{Pr}^\pm(C, \epsilon), \mathbb{C})$.

For example, when $\beta(g, a) = 0$, the variety $V^a(C, \epsilon, P)$ for a general $(C, \epsilon, P)$ consists of finitely many distinct points, counted by the degree of the class $B(g, a)$:

$$\deg B(g, a) := |a|! \cdot 2^{|a|−\ell(a)} \prod_{i=0}^{r} \frac{1}{a_i!} \prod_{0 \leq j < i \leq r} \frac{a_i - a_j}{a_i + a_j}.$$  

Here, the Poincaré formula yields $\deg \xi^{g−1} = (g − 1)! = |a|!$. Remarkably, the above count has an interpretation as enumeration of certain tableaux: comparing with [HH, Prop. 10.4] (see also Remarks 1.1 and 1.2), we have $\deg B(g, a) = n_a$ where

$$n_a := 2^{|a|−\ell(a)} \# \{\text{standard shifted tableaux of shape } (a_r, \ldots, a_0)\}.$$  

Standard shifted tableaux are defined as follows. For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 > \cdots > \lambda_\ell > 0$, the shifted diagram $S(\lambda)$ of shape $\lambda$ is a configuration of $\ell$ rows of boxes, with $\lambda_i$ boxes in the $i$-th row, such that, for each $i > 1$, the first box in row $i$ is placed underneath the second box in row $i − 1$. A standard shifted tableau of shape $\lambda$ is a filling of the boxes of $S(\lambda)$ by the numbers $1, 2, \ldots, |\lambda|$ such that the entries are strictly increasing down each column and from left to right along each row (as in Figure 1).

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1 2 4 6
3 5
7
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**Figure 1.** A standard shifted tableau of shape $(4, 2, 1)$.
In the final part of the paper, we show that $n_a$ has an incarnation also in the geometry of pointed Prym-Brill-Noether curves:

**Theorem 3.** When $\dim V^a(C, \epsilon, P) = \beta(g, \alpha) = 1$, the principally polarized abelian variety $(\text{Pr}(C, \epsilon), \Xi)$ is isomorphic to a Prym-Tyurin variety for the curve $V^a(C, \epsilon, P)$ of exponent $n_a$.

A polarized abelian variety $(A, \Xi)$ is said to be a *Prym-Tyurin variety* for a curve $X$ if there exists an embedding $i_A : A \hookrightarrow \text{Jac}(X)$ such that the pull-back of the canonical polarization $\Theta$ on $\text{Jac}(X)$ is $i_A^*[\Theta] = e[\Xi]$ for some integer $e$ called the *exponent* [BL, §12.2].

Theorem 3 translates to the Prym-Brill-Noether setting a result from Ortega [Ort] showing that for a general curve $C$ of odd genus, $\text{Jac}(C)$ is isomorphic as a principally polarized abelian variety to a Prym-Tyurin variety for the Brill-Noether curve. Here, $(\text{Jac}(C), \Theta)$ is replaced by $(\text{Pr}(C, \epsilon), \Xi)$, and the Brill-Noether curve by the Prym-Brill-Noether curve.

It would be interesting to extend this study to include vanishing conditions at more fixed points. In the case of Brill-Noether varieties of line bundles with prescribed vanishing at two points, a new determinantal formula in type $A$ was introduced in [ACT1] to compute their classes. For this, the study of two-pointed Prym-Brill-Noether varieties could lead to the computation of a new degeneracy locus formula in type D. Also, it would be interesting to compute the motivic invariants of Prym-Brill-Noether varieties, as done for the Brill-Noether varieties in [PP, ACT1, CP, ACT2].

Theorem 1 is proved in §1. We define a pointed Prym-Petri map in §2, and prove a pointed analogue of the Prym-Petri theorem in §3. Theorem 2 follows by combining Proposition 2.3 and Theorem 3.1. Theorem 3 is proved in §4. Throughout, we work over a field of characteristic different from 2.

1. The degeneracy locus structure

We define here the scheme structure for the pointed Prym-Brill-Noether varieties as a degeneracy locus in type D, generalizing the unpointed case from De Concini–Pragacz [DCP], also reviewed in Fulton-Pragacz [FP, Ch. 8]. We focus on the changes required in the pointed case, and refer to [DCP, FP] for the common aspects. We prove Theorem 1 at the end of the section by applying a result from Anderson-Fulton [AF, §4, Corollary].

Let $C$ be a genus $g$ curve, and $\epsilon$ a nontrivial 2-torsion point in $\text{Jac}(C)$. Let $\varphi : \tilde{C} \rightarrow C$ be the irreducible étale double covering induced by $\epsilon$. The maps used below are

$$
\begin{array}{c}
\text{Pic}^{2g-2}(\tilde{C}) \times \tilde{C} \xrightarrow{1 \times \varphi} \text{Pic}^{2g-2}(\tilde{C}) \times C \xrightarrow{\gamma} \text{Pic}^{2g-2}(\tilde{C}) \\
\downarrow \delta \quad \quad \quad \quad \downarrow \delta \\
\tilde{C} \xrightarrow{\varphi} \tilde{C} \quad \\
\end{array}
$$

where $\gamma, \delta$, and $\tilde{\delta}$ are the natural projections.
Let $L$ be a Poincaré line bundle on $\text{Pic}^{2g-2}(\bar{C}) \times \bar{C}$ appropriately normalized (specifically, as in [FP, pg. 93]). Consider the rank two bundle on $\text{Pic}^{2g-2}(\bar{C}) \times \bar{C}$:

$$\mathcal{E} := (1 \times \varphi)_* \mathcal{L}.$$ 

A major insight used below is provided by a result of Mumford [Mum2]: there exists a non-degenerate quadratic form

$$E_{|\text{Pr}^\pm(C, \epsilon)} \to \delta^* \omega_{\bar{C}}|_{\text{Pr}^\pm(C, \epsilon)}.$$ 

Select a divisor $D := \sum P_i$ on $C$ with $\deg(D) > 0$ and $P_i \neq P_j$ for $i \neq j$. Define

$$V := \gamma_*(\mathcal{E}(D)/\mathcal{E}(-D))|_{\text{Pr}^\pm(C, \epsilon)}, \quad U := \gamma_*(\mathcal{E}/\mathcal{E}(-D))|_{\text{Pr}^\pm(C, \epsilon)},$$

where $\mathcal{E}(\pm D) := \mathcal{E} \otimes \delta^* \mathcal{O}_{\bar{C}}(\pm D)$. One has $U \subset V$ with $\text{rank}(V) = 4 \deg(D)$ and $\text{rank}(U) = 2 \deg(D)$.

For $a = (0 \leq a_0 < a_1 < \cdots < a_r)$ and a point $P$ in $\bar{C}$, define

$$E_i := (1 \times \varphi)_* \mathcal{L}(-a_i P)$$

and

$$W_i := \gamma_*(\mathcal{E}_i(D))|_{\text{Pr}^\pm(C, \epsilon)}$$

for $i = 0, \ldots, r$. This gives a flag of bundles

$$W_r \subset \cdots \subset W_0 \subset V$$

with $\text{rank}(W_i) = \chi(E_i(D)) = 2 \deg(D) - a_i$ for each $i$.

Select a point $L$ in $\text{Pr}^\pm(C, \epsilon)$, and define $E := \varphi_* L$ and $E_i := \varphi_* L(-a_i P)$. A construction of Mumford [Mum1] (see also [FP, App. H]) shows

$$H^0(\bar{C}, L(-a_i P)) = H^0(C, E_i) = U \cap W_i \subset V,$$

where

$$W_i := H^0(C, E_i(D)), \quad U := H^0(C, E/E(D)), \quad V := H^0(C, E(D)/E(-D)).$$

Moreover, the non-degenerate quadratic form in (1.1) induces a non-degenerate quadratic form on the vector space $V$, and its subspaces $U$ and $W_r \subset \cdots \subset W_0$ are all isotropic. As in [DCP, FP] the construction globalizes over $\text{Pr}^\pm(C, \epsilon)$, hence

$$\gamma_* \mathcal{E}_i|_{\text{Pr}^\pm(C, \epsilon)} = U \cap W_i \subset V$$

and the vector bundle $V$ admits a non-degenerate quadratic form with values in $\mathcal{E}_{\text{Pr}^\pm(C, \epsilon)}$ such that $U$ and $W_r \subset \cdots \subset W_0$ are all isotropic subbundles.

We define $V^a(C, \epsilon, P)$ as the locus in $\text{Pr}^\pm(C, \epsilon)$ defined by the conditions

$$\text{rank}(W_i \cap U) \geq r + 1 - i, \quad \text{for } i = 0, \ldots, r.$$ 

This gives $V^a(C, \epsilon, P)$ the scheme structure induced by the Schubert variety corresponding to the partition $(a_r, a_{r-1}, \ldots, a_0)$ inside the orthogonal Grassmannian of $2 \deg(D)$-dimensional isotropic subspaces in a $4 \deg(D)$-dimensional vector space. We are now ready for:
**Proof of Theorem 1.** The first part of the statement follows from the degeneracy locus structure of $V^a(C, \epsilon, P)$ given above by means of [FP, Lemma, pg. 108]. Consequently, a Pfaffian formula from [AF, §4, Corollary] in terms of the Chern classes of $U$ and the $W_i$’s, and depending on the partition $(a_r, a_r-1, \ldots, a_0)$, yields a cohomology class supported on $V^a(C, \epsilon, P)$. When $\dim(V^a(C, \epsilon, P)) = \beta(g, a)$, the Pfaffian formula computes the class of $V^a(C, \epsilon, P)$.

As in [DCP, Lemma 5], $U$ has trivial Chern classes, and $c(W_i^\vee) = e(2\xi)$ (since $\deg(D) >> 0$, this is independent of $i$). The resulting Pfaffian formula for arbitrary $(a_r, a_r-1, \ldots, a_0)$ coincides with a Pfaffian computed in [DCP, Prop. 6] times $2^{-\ell(a)}(2\xi)^{|a|}$. This gives the class $B(g, a)$ in (0.2).

**Remark 1.1.** Comparing with [HH, pg. 216], $n_a$ can also be interpreted as the enumeration of marked shifted tableaux of shape $(a_r, a_r-1, \ldots, a_0)$ with unmarked diagonal entries. Such tableaux are defined as follows. For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 > \cdots > \lambda_\ell > 0$, the marked shifted tableaux of shape $\lambda$ are the standard shifted tableaux of shape $\lambda$ (as defined in the introduction) modified by marking some of the entries. Thus, the number of marked shifted tableaux of shape $\lambda$ is $2^{|\lambda|} \# \{\text{standard shifted tableaux of shape } \lambda\}$. The subset of those tableaux with unmarked diagonal entries has size $2^{|\lambda|-\ell} \# \{\text{standard shifted tableaux of shape } \lambda\}$. This equals $n_a$ for the shape $\lambda$ given by $(a_r, a_r-1, \ldots, a_0)$.

**Remark 1.2.** When $a = (0, 1, \ldots, r)$, the number $n_a$ equals the enumeration of standard Young tableaux of shape $(r, r-1, \ldots, 1)$, that is, the number of filling of the configuration of $r$ left-aligned rows of boxes, with $r - i + 1$ boxes in the $i$-th row, by the numbers $1, 2, \ldots, \frac{r(r+1)}{2}$, such that the entries are strictly increasing down each column and from left to right along each row.

### 2. The Pointed Prym-Petri Map

Here we construct a pointed Prym-Petri map, and apply it to find an upper bound for the dimension of pointed Prym-Brill-Noether varieties. The construction integrates the treatments of the unpointed case from [Wel] and the pointed Brill-Noether case from [CHTiB].

Let $C$ be a curve of genus $g$, and $\epsilon$ a non-trivial 2-torsion point in Jac($C$). Let $P \in \tilde{C}$, where $\varphi: \tilde{C} \to C$ is the irreducible étale double covering determined by $\epsilon$. One has a splitting of the space of differential forms

$$H^0(\tilde{C}, \omega_{\tilde{C}}) = H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \epsilon)$$

into invariants and anti-invariant sections under the action of the covering involution $\iota$.

The covering $\varphi$ induces a norm map $Nm: \text{Pic}(\tilde{C}) \to \text{Pic}(C)$. This satisfies $Nm \circ \varphi^* = 2 \text{id}_{\text{Pic}(C)}$ and $\varphi^* \circ Nm = \text{id}_{\text{Pic}(\tilde{C})} \otimes \iota^*$ [Mum2].
For \( \mathbf{a} = (0 \leq a_0 < a_1 < \cdots < a_r) \) and \( L \in \mathcal{V}^\mathbf{a}(C, \epsilon, P) \) such that 
\[ h^0(\tilde{C}, L(-a_i P)) = r + 1 - i \quad \text{for} \quad i = 0, \ldots, r, \]
select sections 
\[ \sigma_i \in H^0(\tilde{C}, L(-a_i P)) \setminus H^0(\tilde{C}, L(-a_{i+1} P)) \quad \text{for} \quad i = 0, \ldots, r - 1, \]
\[ \sigma_r \in H^0(\tilde{C}, L(-a_r P)). \]

Let \( M := \iota^* L \). Since \( \operatorname{Nm}(L) = \omega_C \), one has \( \varphi^* \omega_C = L \otimes M \), and since \( \varphi^* \omega_C = L \tilde{C} \), one deduces \( M = \omega_{\tilde{C}} \otimes L^\vee \). Consider the composition
\[
\bigoplus_{i=0}^r \langle \sigma_i \rangle \otimes H^0(\tilde{C}, M(a_i P)) \xrightarrow{\mu} H^0(\tilde{C}, \omega_{\tilde{C}}) \xrightarrow{p} H^0(C, \omega_C \otimes \epsilon)
\]
where \( \mu \) is the Petri map obtained by product of sections, and \( p \) is the projection onto the anti-invariant subspace under the action of \( \iota \). Explicitly, the composition \( p \circ \mu \) is given by
\[
\sigma \otimes \tau \mapsto \frac{1}{2} (\sigma \tau - \iota(\sigma)\iota(\tau)).
\]

For each \( i \), one has the following inclusions
\[
(2.1) \quad \iota^* H^0(\tilde{C}, L(-a_i P)) \hookrightarrow \iota^* H^0(\tilde{C}, L) \xrightarrow{\sim} H^0(\tilde{C}, M) \hookrightarrow H^0(\tilde{C}, M(a_i P)).
\]

In particular, the map \( p \circ \mu \) restricts to a map
\[
\overline{\pi} : \bigoplus_{i=0}^r \langle \sigma_i \rangle \otimes H^0(\tilde{C}, M(a_i P)) / \iota^* H^0(\tilde{C}, L(-a_i P)) \rightarrow H^0(C, \omega_C \otimes \epsilon).
\]

Here the source of \( \overline{\pi} \) is intended as a subspace of the source of \( p \circ \mu \). We call \( \overline{\pi} \) the pointed Prym-Petri map for \( L \).

For instance, in the case \( \mathbf{a} = (0, 1, \ldots, r) \) where no special vanishing is imposed at \( P \), one has \( h^0(\tilde{C}, M(iP)) = h^0(\tilde{C}, M) \) for each \( i \) by the Riemann-Roch formula. Hence the source of \( \overline{\pi} \) is isomorphic to the image of \( \wedge^2 H^0(\tilde{C}, L) \) via the composition of
\[
\wedge^2 H^0(\tilde{C}, L) \hookrightarrow H^0(\tilde{C}, L)^\otimes 2 \xrightarrow{1 \otimes \iota^*} H^0(\tilde{C}, L) \otimes H^0(\tilde{C}, M)
\]
where the first map is given by \( \sigma \otimes \tau \mapsto \frac{1}{2} (\sigma \otimes \tau - \tau \otimes \sigma) \). It follows that the map \( \overline{\pi} \) recovers the Prym-Petri map in the unpointed case studied in [Wel].

Next, we show how the pointed Prym-Petri map encodes information on the tangent space of pointed Prym-Brill-Noethen varieties. The variety \( \mathcal{V}^\mathbf{a}(C, \epsilon, P) \) has a Zariski open subset
\[
\mathcal{V}^\mathbf{a}(C, \epsilon, P)^\circ := W_{2g-2}(\tilde{C}, P)^\circ \cap \text{Pr}^\bullet(C, \epsilon)
\]
where \( \bullet = + \) if \( r \) is odd, \( \bullet = - \) if \( r \) is even, and
\[
W_{2g-2}(\tilde{C}, P)^\circ := \left\{ L \in \text{Pic}^{2g-2}(\tilde{C}) : h^0(\tilde{C}, L(-a_i P)) = r + 1 - i, \forall i \right\}.
\]
The scheme structure of \( \mathcal{V}^\mathbf{a}(C, \epsilon, P) \) along \( \mathcal{V}^\mathbf{a}(C, \epsilon, P)^\circ \) coincides with the natural scheme structure on the intersection \( W_{2g-2}(\tilde{C}, P)^\circ \cap \text{Pr}^\bullet(C, \epsilon) \), as in [DCP, Prop. 4(1)].
For \( L \) in \( V^a(C, \epsilon, P) \), it follows that the tangent space of \( V^a(C, \epsilon, P) \) at \( L \) is the intersection of the tangent spaces of \( W^a_{2g-2}(\tilde{C}, P) \) and \( \text{Pr}^\bullet(C, \epsilon) \) at \( L \). We proceed to identify these tangent spaces.

As in [Wel, pg. 673], the tangent space of \( \text{Pr}^\bullet(C, \epsilon) \) at \( L \) is
\[
(2.2) \quad T_{\text{Pr}^\bullet(C, \epsilon)}(L) = H^0 \left( \tilde{C}, \omega_{\tilde{C}} \otimes \epsilon \right) ^\vee \hookrightarrow H^0 \left( \tilde{C}, \omega_{\tilde{C}} \right) ^\vee
\]
where the inclusion is the transpose of the projection \( p \). This follows from the fact that the tangent map of \( N \) at \( L \) is canonically given by twice the transpose of the pull-back \( \varphi^* \):
\[
2(\varphi^*)^t : H^0 \left( \tilde{C}, \omega_{\tilde{C}} \right) ^\vee \to H^0 \left( C, \omega_C \right) ^\vee.
\]
Moreover, from [CHTiB, Proof of (3.2)], the tangent space of \( W^a_{2g-2}(\tilde{C}, P) \) at \( L \) is
\[
(2.3) \quad T_{W^a_{2g-2}(\tilde{C}, P)}(L) = \text{Im}(\mu)^\perp \subset H^0 \left( \tilde{C}, \omega_{\tilde{C}} \right) ^\vee.
\]
Combining (2.2) and (2.3), we have
\[
T_{V^a(C, \epsilon, P)}(L) = \text{Im}(\mu)^\perp \cap H^0 \left( C, \omega_C \otimes \epsilon \right) ^\vee = \text{Im}(\overline{\mu})^\perp.
\]
To compute its dimension, note that since \( h^0(\tilde{C}, L(-a_i P)) = r + 1 - i \) for \( i = 0, \ldots, r \) and \( M = \omega_{\tilde{C}} \otimes L^\vee \), one has by the Riemann-Roch formula:
\[
h^0(\tilde{C}, M(a_i P)) = r + 1 - i + a_i, \quad \text{for } i = 0, \ldots, r.
\]
It follows that the source of \( \overline{\mu} \) has dimension equal to \( |a| \), hence
\[
\dim \left( \text{Im}(\overline{\mu})^\perp \right) = g - 1 - |a| + \dim \text{Ker}(\overline{\mu}).
\]
This implies:

**Proposition 2.1.** For \( L \in V^a(C, \epsilon, P) \), one has
\[
\dim_L(V^a(C, \epsilon, P)) \leq \beta(g, a) + \dim \text{Ker}(\overline{\mu}).
\]

Comparing with the inequality in Theorem 1, the following condition emerges:

**Definition 2.2.** A triple \((C, \epsilon, P)\) as above is said to satisfy the pointed Prym-Petri condition if the map \( \overline{\mu} \) is injective for all \( L \in V^a(C, \epsilon, P) \) such that \( h^0(\tilde{C}, L(-a_i P)) = r + 1 - i \) for \( i = 0, \ldots, r \).

Combining the inequality in Theorem 1 with Proposition 2.1, we deduce:

**Proposition 2.3.** If \((C, \epsilon, P)\) satisfies the pointed Prym-Petri condition, then \( V^a(C, \epsilon, P) \) is either empty, or smooth of dimension \( \beta(g, a) \) at any \( L \in V^a(C, \epsilon, P) \) such that \( h^0(\tilde{C}, L(-a_i P)) = r + 1 - i \) for \( i = 0, \ldots, r \).
3. The pointed Prym-Petri theorem in the general case

The main result of this section is the following pointed Prym-Petri Theorem:

**Theorem 3.1.** For a general curve $C$ of genus $g$, an arbitrary nontrivial 2-torsion point $\epsilon$ in $\text{Jac}(C)$, and a general point $P \in \tilde{C}$, the triple $(C, \epsilon, P)$ satisfies the pointed Prym-Petri condition.

Proposition 2.3 and Theorem 3.1 directly imply Theorem 2. For the proof of Theorem 3.1, we adapt and refine ideas from the unpointed case treated in [Wel] and the adjustment of the classical Brill-Noether case [EH1] to the pointed Brill-Noether case treated in [CHTiB]. First, we argue that:

**Lemma 3.2.** Exhibiting a single triple $(C, \epsilon, P)$ for each genus with $\epsilon \neq 0$ which satisfies the pointed Prym-Petri condition proves Theorem 3.1.

**Proof.** Satisfying the pointed Prym-Petri condition is an open condition on families of triples $(C, \epsilon, P)$, as in [Wel, (2.1)]. The statement follows from the irreducibility of the moduli space of triples $(C, \epsilon, P)$ of fixed genus with $\epsilon \neq 0$. □

For this, we consider a specific triple obtained as the geometric generic fiber of a family for each genus defined as follows.

3.1. The family $(\mathcal{C}, \epsilon, P)$. Let $T$ be the spectrum of a discrete valuation ring with parameter $t$, closed point 0, and generic point $\eta$. We will later use the fact that $T$ has trivial Picard group. Let $\pi: \mathcal{C} \rightarrow T$ be a flat projective family, with $\mathcal{C}$ a smooth surface, such that: 1) the generic fiber $\mathcal{C}_\eta$ is smooth and geometrically irreducible; and 2) the special fiber $\mathcal{C}_0$ is a reduced curve of arithmetic genus $g$ consisting of a chain of smooth components which are either rational or elliptic, and ordinary double points as only singularities (see Figure 2). Let $E_1, \ldots, E_g$ be the elliptic components of $\mathcal{C}_0$. For each $i = 1, \ldots, g$, we assume furthermore that $E_i$ meets the rest of the curve $\mathcal{C}_0$ in two points $P_i$ and $Q_i$ such that $P_i - Q_i$ is not a torsion point in $\text{Pic}(E_i)$.

We will repeatedly use the following property: Given a dominant morphism $T' \rightarrow T$ of spectra of discrete valuation rings, the family $\mathcal{C}' \rightarrow T'$ obtained by base extension and minimal resolution of singularities has special fiber $\mathcal{C}'_0$ satisfying the same assumptions as $\mathcal{C}_0$.

Up to extending the base, we can assume that there exists a line bundle $\epsilon$ on $\mathcal{C}$ such that $\epsilon^2 \cong \mathcal{O}_\mathcal{C}$, with nontrivial restriction $\epsilon_\eta$ on $\mathcal{C}_\eta$, and with restriction to $\mathcal{C}_0$ nontrivial only over $E_g$. The line bundle $\epsilon$ gives rise to an irreducible étale double covering $\varphi: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ over $T$, where $\tilde{\mathcal{C}} := \text{Spec}(\mathcal{O}_\mathcal{C} \oplus \epsilon)$, the ring structure induced by $\epsilon^2 \cong \mathcal{O}_\mathcal{C}$. Let $\iota$ be the covering involution on $\tilde{\mathcal{C}}$. The generic fiber $\tilde{\mathcal{C}}_\eta$ is smooth and geometrically irreducible, and the special fiber $\tilde{\mathcal{C}}_0$ is a reduced curve of arithmetic genus $2g - 1$ consisting of a chain of smooth components which are either rational or elliptic, and ordinary double points as only singularities. The elliptic component can be
denoted $E'_1, E''_1, \ldots, E'_{g-1}, E''_{g-1}, \tilde{E}_g$ such that for $i = 1, \ldots, g-1$, the restriction of the map $\varphi$ over $E_i$ is a reducible double covering $E'_i \sqcup E''_i \rightarrow E_i$, hence $E'_i \cong E''_i \cong E_i$, and the restriction of the map $\varphi$ over $E_g$ is an irreducible double covering $\tilde{E}_g \rightarrow E_g$ (see Fig. 2). In particular, for each of the two points where $E_g$ meets the rest of the curve $\mathcal{C}_0$, the difference of the two preimages is a 2-torsion point in $\text{Jac}(\tilde{E}_g)$.

Finally, we select a section $P : T \rightarrow \tilde{\mathcal{C}}$ with the property: the corresponding point $P_0$ in $\tilde{\mathcal{C}}_0$ lies in a rational component splitting the chain $\tilde{\mathcal{C}}_0$ so that there is a connected component of arithmetic genus $2g - 1$ (as in Fig. 2).

![Figure 2](image-url)

**Figure 2.** A graphical representation of the curves $\mathcal{C}_0$, $\tilde{\mathcal{C}}_0$, and the étale double covering $\tilde{\mathcal{C}}_0 \rightarrow \mathcal{C}_0$. Ellipses and line segments stand for elliptic and rational components, respectively.

The maps are summarized in the following diagram:

![Diagram](image-url)

### 3.2. On the triple $(\mathcal{C}_\eta, \epsilon_\eta, P_\eta)$.

To show Theorem 3.1, it is enough to prove that the geometric generic fiber $(\mathcal{C}_\eta, \epsilon_\eta, P_\eta)$ of $(\mathcal{C}, \epsilon, P)$ satisfies the pointed Prym-Petri condition, where $\mathcal{C}_\eta := \mathcal{C}_\eta \otimes k(\eta)$, $\epsilon_\eta := \epsilon_\eta \otimes k(\eta)$, and $P_\eta$ is the point in $\tilde{\mathcal{C}}_\eta := \tilde{\mathcal{C}}_\eta \otimes k(\eta)$ induced by $P$. As in [EH1, pg. 272], a line bundle on $\mathcal{C}_\eta$ comes from a line bundle defined over some finite extension of $k(\eta)$. After extending the base and changing the notation, it is enough to prove:

**Theorem 3.3.** The triple $(\mathcal{C}_\eta, \epsilon_\eta, P_\eta)$ satisfies the pointed Prym-Petri condition.

Theorem 3.1 thus follows from Lemma 3.2 and Theorem 3.3. The remainder of this section is dedicated to the proof of Theorem 3.3.
We proceed to describe the relevant pointed Prym-Petri map. Let $\mathcal{L}_\eta$ in $V^a(\mathcal{E}_\eta, \epsilon, P_\eta)$ and $\mathcal{M}_\eta := \iota^* \mathcal{L}_\eta$. Consider the sequence of sub-line bundles

$$\mathcal{L}_\eta^i \hookrightarrow \cdots \hookrightarrow \mathcal{L}_\eta^0 \hookrightarrow \mathcal{L}_\eta,$$

$$\mathcal{M}_\eta \hookrightarrow \mathcal{M}_\eta^0 \hookrightarrow \cdots \hookrightarrow \mathcal{M}_\eta^r,$$

where

$$L^i := \mathcal{L}_\eta(-a_i P_\eta), \quad M^i := \mathcal{M}_\eta(a_i P_\eta).$$

Assume $h^0(\mathcal{E}_\eta, L^i) = r + 1 - i$ for each $i$, and select sections

$$\sigma_i \in \pi_* L^i_\eta \setminus \pi_* L^{i+1}_\eta \quad \text{for} \quad i = 0, \ldots, r - 1, \quad \sigma_r \in \pi_* L^r_\eta.$$

In particular, $\{\sigma_i\}_i$ is a basis of $\pi_* L^0_\eta$. For each $i$, consider the composition of the inclusions

$$t^* \pi_* \mathcal{L}_\eta^i \hookrightarrow t^* \pi_* \mathcal{M}_\eta \cong \pi_* \mathcal{M}_\eta^i$$

as in $(2.1)$. The pointed Prym-Petri map for $\mathcal{L}_\eta$ is defined as in $\S 2$:

$$\pi_\eta : \bigoplus_{i=0}^r (\sigma_i) \otimes \pi_* \mathcal{M}_\eta^i / t^* \pi_* \mathcal{L}_\eta^i \longrightarrow \pi_* \left( \omega_{E\eta} \otimes \epsilon_\eta \right).$$

### 3.3. Extending $\pi_\eta$ over $T$

First, we extend the map $\pi_\eta$ to a map over $T$. The source of the map $\pi_\eta$ is a vector subspace of

$$\bigoplus_{i=0}^r (\sigma_i) \otimes \pi_* \mathcal{M}_\eta^i \subseteq \bigoplus_{i=0}^r \pi_* \mathcal{L}_\eta^i \otimes \pi_* \mathcal{M}_\eta^i.$$

Thus we proceed by extending the line bundles $\mathcal{L}_\eta, L^i_\eta$, and $\mathcal{M}_\eta^i$ over $\mathcal{E}_\eta$.

After base extension and minimally resolving the singularities, the line bundle $\mathcal{L}_\eta$ on $\mathcal{E}_\eta$ extends to a line bundle $\mathcal{L}$ on $\mathcal{E}$. Since $\text{Nm}(\mathcal{L}_\eta) \cong \omega_{E\eta}$, one has $\text{Nm}(\mathcal{L}) \cong \omega_{E/T} \otimes \mathcal{F}$, where $\mathcal{F}$ is a line bundle assigned to a linear combination of the components of $\mathcal{E}_0$. Since $T$ has trivial Picard group, $\mathcal{O}_{\mathcal{E}}(\mathcal{E}_0) \cong \mathcal{O}_{\mathcal{E}}$, hence we can assume that $E_0$ does not appear in the linear combination defining $\mathcal{F}$. Moreover, since $\text{Nm}(\mathcal{O}_{\mathcal{E}}(E_i')) \cong \text{Nm}(\mathcal{O}_{\mathcal{E}}(E_i')) \cong \mathcal{O}_{\mathcal{E}}(E_i)$ for $i = 1, \ldots, g - 1$, and similarly for all rational components, we can assume $\text{Nm}(\mathcal{L}) \cong \omega_{E/T}$ after an appropriate twist.

By the theory of limit linear series [EH1], for every component $Y$ of $\mathcal{E}_0$, there exists a line bundle $\mathcal{L}_Y$ extending $\mathcal{L}_\eta$ over $\mathcal{E}$ which has degree 0 on all components of $\mathcal{E}_0$ except $Y$. The line bundle $\mathcal{L}_Y$ is obtained from $\mathcal{L}$ by twisting with a linear combination of the components of $\mathcal{E}_0$. Similarly, one has extensions over $\mathcal{E}$ of line bundles on $\mathcal{E}_\eta$. As noted in [Wel, pg. 677], one has $t^*(\mathcal{L}_Y) \cong (t^* \mathcal{L}_{i(Y)})$ and $\text{Nm}(\mathcal{L}_Y) \cong \text{Nm}(\mathcal{L}_{i(Y)})$.

For each $i$, consider the extension $\mathcal{L}^i_{E_0}$ of $\mathcal{L}^i_\eta$ over $\mathcal{E}$ which has degree 0 on all components of $\mathcal{E}_0$ except $E_0$, and similarly: the extension $\mathcal{M}^i_{E_0}$ of $\mathcal{M}_\eta$; the extension $\mathcal{L}_{E_0}$ of $\mathcal{M}_\eta$; and the extension $\mathcal{M}_{E_0}$ of $\mathcal{L}_\eta$. We identify $\pi_* \mathcal{L}^i_{E_0}$
with the $\mathcal{O}_T$-submodule of $\bar{\pi}_* \mathcal{L}^i_{E_g}$ which spans this space, and similarly for $\bar{\pi}_* \mathcal{M}^i_{E_g}$. Thus one has inclusions of $\mathcal{O}_T$-submodules

$$\bar{\pi}_* \mathcal{L}^i_{E_g} \hookrightarrow \bar{\pi}_* \mathcal{L}^i_{\eta} \quad \text{and} \quad \bar{\pi}_* \mathcal{M}^i_{E_g} \hookrightarrow \bar{\pi}_* \mathcal{M}^i_{\eta}.$$

After possibly replacing the basis $\{\sigma_i\}_i$ of $\bar{\pi}_* \mathcal{L}^0_{\eta}$, we can assume in addition to (3.2) that there exist integers $\alpha_i$ such that

$$t^{\alpha_i} \sigma_i \in \bar{\pi}_* \mathcal{L}^i_{E_g} \setminus \bar{\pi}_* \mathcal{L}^{i+1}_{E_g} \quad \text{for} \quad i = 0, \ldots, r - 1, \quad t^{\alpha_r} \sigma_r \in \bar{\pi}_* \mathcal{L}^r_{E_g}$$

as in [EH1, 1.2]. In particular, $\{t^{\alpha_i} \sigma_i\}_i$ is a basis of $\bar{\pi}_* \mathcal{L}^0_{E_g}$. To simplify the notation, let

$$\hat{\sigma}_i := t^{\alpha_i} \sigma_i \quad \text{for} \quad i = 0, \ldots, r.$$

The map

$$\overline{\mu} : \bigoplus_{i=0}^r (\hat{\sigma}_i) \otimes \bar{\pi}_* \mathcal{M}^i_{E_g} \big/ (t^* \bar{\pi}_* \mathcal{L}^i_{E_g} \to \bar{\pi}_* (\omega_{\mathcal{E}} \otimes \epsilon))$$

induced by product of sections gives (3.4) over $\eta$. Here, the inclusion of $\mathcal{O}_T$-submodules

$$(3.5) \quad t^* \bar{\pi}_* \mathcal{L}^i_{E_g} \hookrightarrow \bar{\pi}_* \mathcal{M}^i_{E_g}$$

is induced from (3.3).

3.4. On the kernel of $\overline{\mu}_\eta$. Let $\rho \in \text{Ker}(\overline{\mu}_\eta)$. We will show $\rho = 0$. Assuming $\rho \neq 0$, there exists a unique integer $\gamma$ such that

$$t^\gamma \rho \in \bigoplus_{i=0}^r (\hat{\sigma}_i) \otimes \bar{\pi}_* \mathcal{M}^i_{E_g} \setminus (t^* \bar{\pi}_* \mathcal{L}^i_{E_g} \big/ (t^* \bar{\pi}_* \mathcal{L}^i_{E_g} \to \bar{\pi}_* (\omega_{\mathcal{E}} \otimes \epsilon))).$$

Since $\rho \in \text{Ker}(\overline{\mu}_\eta)$, one has $t^\gamma \rho \in \text{Ker}(\overline{\mu})$. Next, for each $i$, we restrict to

$$(3.6) \quad L^i := \text{Im} \left( \bar{\pi}_* \mathcal{L}^i_{E_g} \to H^0 \left( \mathcal{L}^i_{E_g} \otimes \mathcal{O}_{\mathcal{E}_g} \right) \right),$$

$$M^i := \text{Im} \left( \bar{\pi}_* \mathcal{M}^i_{E_g} \to H^0 \left( \mathcal{M}^i_{E_g} \otimes \mathcal{O}_{\mathcal{E}_g} \right) \right).$$

Let $\varpi_i \in L^i$ be the image of $\hat{\sigma}_i$ via the restriction map $\bar{\pi}_* \mathcal{L}^i_{E_g} \to L^i$. One has $L^i = \langle \varpi_i, \ldots, \varpi_r \rangle$. Similarly, let $\varpi$ be the image of $t^\gamma \rho$ via the map

$$\bigoplus_{i=0}^r (\hat{\sigma}_i) \otimes \bar{\pi}_* \mathcal{M}^i_{E_g} \to \bigoplus_{i=0}^r (\varpi_i) \otimes M^i.$$

By assumption, we have

$$(3.7) \quad \varpi \neq 0 \quad \text{in} \quad \bigoplus_{i=0}^r (\hat{\sigma}_i) \otimes M^i / t^* L^i.$$

Here, the inclusion $t^* L^i \hookrightarrow M^i$ is induced from (3.5).
Let $P'_g$ be the point where $\tilde{E}_g$ meets the connected components in the rest of $\tilde{\mathcal{C}}_0$ which contains the point $P_0$, and let $P''_g := \iota(P'_g)$ (as in Fig. 2).

**Lemma 3.4.** The assumption $\rho \in \text{Ker}(\overline{\eta}_g)$ implies

$$\text{ord}_{P'_g}(\overline{\rho}) \geq 2g - 2 \quad \text{and} \quad \text{ord}_{P''_g}(\overline{\rho}) \geq 2g - 2.$$  \hspace{1cm} (3.8)

This means that $\overline{\eta}$ is a linear combination of elements $\overline{\sigma}_i \otimes \overline{\tau}_j$ with $\overline{\tau}_j \in M^i$ such that $\text{ord}_{P'_g}(\overline{\sigma}_i) + \text{ord}_{P''_g}(\overline{\tau}_j) \geq 2g - 2$ for all $i, j$, and similarly at $P''_g$.

In the proof of the lemma, we combine results from [EH1, pp. 278-280], [CHTiB, Proof of 3.2], and [Wel, p. 679].

**Proof.** First we set the notation. Let $k \in \{1, \ldots, g - 1\}$. We can assume that there exist integers $\alpha_{i,k}$ such that

$$t^{\alpha_{i,k}} \sigma_i \in \overline{\pi}_* \mathcal{L}^i_{E'_k} \setminus \overline{\pi}_* \mathcal{L}^{i+1}_{E'_k} \quad \text{for} \quad i = 0, \ldots, r - 1, \quad t^{\alpha_{r,k}} \sigma_r \in \overline{\pi}_* \mathcal{L}^r_{E'_k}$$

as in [EH1, 1.2]. Let $\overline{\sigma}_{i,k} := t^{\alpha_{i,k}} \sigma_i$ for $i = 0, \ldots, r$. Then there exists a unique integer $\gamma_k$ such that

$$t^{\gamma_k} \rho \in \bigoplus_{i=0}^r \left( \langle \overline{\sigma}_{i,k} \rangle \otimes \overline{\pi}_* \mathcal{M}^i_{E'_k} \setminus t \left( \langle \overline{\sigma}_{i,k} \rangle \otimes \overline{\pi}_* \mathcal{M}^i_{E'_k} \right) \right).$$

When $k = g$, we set $E'_g := \tilde{E}_g$, $\overline{\sigma}_{i,g} := \overline{\sigma}_i$, and $\gamma_g := \gamma$.

For $k = 1, \ldots, g - 1$, let $E'_k$ be the point where $E'_k$ meets the connected components in the rest of $\tilde{\mathcal{C}}_0$ which contains the point $P_0$ (as in Fig. 2). From [CHTiB, Proof of 3.2], which adapts to the pointed case the argument first treated in the unpointed case in [EH1, pp. 278-280], the assumption $\rho \in \text{Ker}(\overline{\eta}_g)$ implies

$$\text{ord}_{P'_{k+1}} \left( t^{\gamma_{k+1}} \rho \right) \geq \text{ord}_{P'_k} \left( t^{\gamma_k} \rho \right) + 2 \quad \text{for} \quad k = 1, \ldots, g - 1.$$  \hspace{1cm} (3.9)

Here one uses that the difference of the two nodal points in $E'_k$ is not a torsion point in $\text{Jac}(E'_k)$ for $k = 1, \ldots, g - 1$. Actually, the proof in [EH1, CHTiB] uses families of curves with special fiber obtained from $\tilde{\mathcal{C}}_0$ by replacing each elliptic component with a rational component meeting the rest of the curve in the same two points and meeting in a third point a new chain of rational curves ending with an elliptic component. However, as it was first shown in [Wel], the argument extends to the case of families of curves with special fiber given by a chain of rational and elliptic curves, and gives (3.9) for a given $k$, as long as the difference of the two nodal points in $E'_k$ is not a torsion point in $\text{Jac}(E'_k)$. Moreover, such families allow one to work more generally over an arbitrary field of characteristic different from 2 [Wel].

The inequality (3.9) for $k = 1, \ldots, g - 1$ implies $\text{ord}_{P'_g} (\overline{\rho}) \geq 2g - 2$. A similar argument holds after replacing $P'_a$ with $P''_a := \iota(P'_a)$, and $E'_a$ with $E''_a := \iota(E'_a)$, for each $a = 1, \ldots, g$, hence the statement. \hfill \Box
We emphasize that the difference $P'_g - P''_g$ is a 2-torsion point in $\text{Jac}(\tilde{E}_g)$, hence the above application of the argument from [EH1, CHTiB] does not yield anything more than (3.8).

While the conclusion of Lemma 3.4 is similar to [Wel, (2.20)], in the next step we need to account for the additional complexity given by the more complicated source of $\mu$.

3.5. A vanishing statement. For $i = 0, \ldots, r$, one has from (3.1):

\begin{align}
\mathcal{L}_{\tilde{E}_g}^i \otimes \mathcal{O}_{\tilde{E}_g} &\cong \mathcal{L}_{E_g} \otimes \mathcal{O}_{E_g} (-a_i P'_g), \\
\mathcal{M}_{\tilde{E}_g}^i \otimes \mathcal{O}_{\tilde{E}_g} &\cong \mathcal{M}_{E_g} \otimes \mathcal{O}_{E_g} (a_i P'_g).
\end{align}

(3.10)

From [Wel, (2.21)], the line bundles $\mathcal{L}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g}$ and $\mathcal{M}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g}$ are both isomorphic to

\begin{align}
\mathcal{O}_{\tilde{E}_g}((2g - 2)P'_g) \quad \text{or} \quad \mathcal{O}_{\tilde{E}_g}((2g - 3)P'_g + P''_g).
\end{align}

(3.11)

This follows from the isomorphisms

\begin{align}
\text{Nm} \left( \mathcal{L}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g} \right) &\cong \mathcal{O}_{E_g}((2g - 2)P'_g), \\
\text{Nm} \left( \mathcal{M}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g} \right) &\cong \mathcal{O}_{E_g}((2g - 2)P'_g),
\end{align}

and the fact that the line bundles in (3.11) are the two distinct line bundles in the inverse image of $\mathcal{O}_{E_g}((2g - 2)P'_g)$ via the $(2 : 1)$ norm map

\[ \text{Nm} : \text{Pic}^{2g-2}(\tilde{E}_g) \to \text{Pic}^{2g-2}(E_g) \]

and are both invariant by $\iota$. For this, one uses that $P'_g - P''_g$ is a 2-torsion point in $\text{Jac}(\tilde{E}_g)$. Combining with (3.10), we deduce the following:

**Lemma 3.5.** Let $\mathcal{L}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g}$ and $\mathcal{M}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g}$ be both isomorphic to either one of the two line bundles in (3.11) and assume

\[ \mathfrak{p} \in \bigoplus_{i=0}^r \langle \mathfrak{p}_i \rangle \otimes M^i / \iota^* L^i \]

satisfies (3.8). Then $\mathfrak{p} = 0$.

**Proof.** From (3.6), (3.10), and the assumption $\mathcal{L}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g} \cong \mathcal{M}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g}$, all spaces $L^0, \ldots, L^r$ and $M^0, \ldots, M^r$ inject in $V := H^0(\mathcal{M}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g}(a_r P'_g))$.

We select a basis $\{e_n\}_n$ of $V$ such that the vectors $e_n$ have distinct orders of vanishing at $P'_g$ and distinct orders of vanishing at $P''_g$.

For this, consider first the case $\mathcal{L}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g} \cong \mathcal{M}_{\tilde{E}_g} \otimes \mathcal{O}_{\tilde{E}_g} \cong \mathcal{O}_{\tilde{E}_g}((2g - 2)P'_g)$.

Recall that $2P'_g = 2P''_g$ on $\tilde{E}_g$. Let $Q$ and $R$ be points in $\tilde{E}_g \setminus \{P'_g, P''_g\}$ such
that $P_g + P_g'' = Q + R$. Define the following divisors on $\tilde{E}_g$

$$D_{2k} := (2g - 2 + a_r - 2k)P_g' + 2kP_g''$$
for $k = 0, \ldots, g - 1 + \left\lfloor \frac{a_r}{2} \right\rfloor$.

(3.12) $$D_{2k+1} := (2g - 5 + a_r - 2k)P_g' + (2k + 1)P_g'' + Q + R$$
for $k = 0, \ldots, g + \left\lceil \frac{a_r - 5}{2} \right\rceil$.

For $n \in \{0, \ldots, 2g - 4 + a_r, 2g - 2 + a_r\}$, let $e_n$ be a section in $V$ with divisor $D_n$. The sequence of orders of vanishing of such $2g - 2 + a_r$ sections $e_n$ at $P_g'$ is $(0, \ldots, 2g - 4 + a_r, 2g - 2 + a_r)$, and similarly at $P_g''$. It follows that $\{e_n\}_n$ is a basis of $V$ with the desired property.

Write $\overline{e} = \bigoplus_{i=0}^r \overline{e}_i$ with $\overline{e}_i \in \langle \overline{e}_i \rangle \otimes M^i$ for each $i$. Select $i$ such that $\overline{e}_i \neq 0$. The assumption (3.8) implies that

(3.13) $\ord_{P_g'}(\overline{e}_i) \geq 2g - 2$ and $\ord_{P_g''}(\overline{e}_i) \geq 2g - 2$.

The basis $\{e_n\}_n$ of $V$ induces

- a basis $\{\lambda_\ell\}_\ell$ of the subspace $H^0\left(\mathcal{L}_{E_g} \otimes \mathcal{O}_{E_g}(-a_i P_g')\right)$, and
- a basis $\{\mu_m\}_m$ of the subspace $H^0\left(\mathcal{M}_{E_g} \otimes \mathcal{O}_{E_g}(a_i P_g')\right)$.

Let $c_\ell := \ord_{P_g'}(\lambda_\ell)$. The construction of the basis via the divisors in (3.12) implies

(3.14) $\ord_{P_g'}(\lambda_\ell) = \begin{cases} 2g - 2 - a_i - c_\ell & \text{if } a_i + c_\ell \equiv 0 \mod 2, \\ 2g - 4 - a_i - c_\ell & \text{if } a_i + c_\ell \equiv 1 \mod 2. \end{cases}$

Similarly, let $b_m := \ord_{P_g''}(\mu_m)$. Then

(3.15) $\ord_{P_g''}(\mu_m) = \begin{cases} 2g - 2 + a_i - b_m & \text{if } a_i + b_m \equiv 0 \mod 2, \\ 2g - 4 + a_i - b_m & \text{if } a_i + b_m \equiv 1 \mod 2. \end{cases}$

Since $\langle \overline{e}_i \rangle \subseteq H^0(\mathcal{L}_{E_g} \otimes \mathcal{O}_{E_g}(-a_i P_g'))$ and $M^i \subseteq H^0(\mathcal{M}_{E_g} \otimes \mathcal{O}_{E_g}(a_i P_g'))$, we can then write $\overline{e}_i = \sum_{\ell,m} d_{\ell,m} \lambda_\ell \otimes \mu_m$, for some coefficients $d_{\ell,m}$. From (3.13), and since the basis $\{\lambda_\ell\}_\ell$ (respectively, $\{\mu_m\}_m$) has distinct orders of vanishing at $P_g'$, and similarly at $P_g''$, one has $d_{\ell,m} = 0$ for those $\ell, m$ which fail the conditions

$\ord_{P_g'}(\lambda_\ell) + \ord_{P_g''}(\mu_m) \geq 2g - 2$ and $\ord_{P_g'}(\lambda_\ell) + \ord_{P_g''}(\mu_m) \geq 2g - 2$.

Furthermore, these conditions imply first $c_\ell + b_m \geq 2g - 2$, and then

(3.16) $c_\ell + b_m = 2g - 2$ and $a_i + c_\ell \equiv a_i + b_m \equiv 0 \mod 2$.

It follows that

$\text{div}(\lambda_\ell) = c_\ell P_g' + (b_m - a_i)P_g''$ and $\text{div}(\mu_m) = b_m P_g' + (a_i + c_\ell)P_g''$. 
Hence, the image of $\iota^*\lambda_\ell$ via the composition of the inclusions
\[ \iota^*H^0\left(\mathcal{L}_{E_g} \otimes \mathcal{O}_{E_g}(-a_i P'_g)\right) \rightarrow \iota^*H^0\left(\mathcal{L}_{E_g} \otimes \mathcal{O}_{E_g}\right) \xrightarrow{\cong} H^0\left(\mathcal{M}_{E_g} \otimes \mathcal{O}_{E_g}\right) \rightarrow H^0\left(\mathcal{M}_{E_g} \otimes \mathcal{O}_{E_g}(a_i P'_g)\right) \]
lies in $\langle \mu_m \rangle$. We deduce $\overline{\mathcal{P}}_g \in \langle \mathcal{P}_i \rangle \otimes \text{Im} \left( \iota^*L^i \hookrightarrow M^i \right)$, hence the statement.

Finally, in the case $\mathcal{L}_{E_g} \otimes \mathcal{O}_{E_g} \cong \mathcal{M}_{E_g} \otimes \mathcal{O}_{E_g} \cong \mathcal{O}_{E_g}\left((2g - 3)P'_g + P''_g\right)$, replace (3.12) with
\[ D_{2k+1} := (2g - 3 + a_r - 2k)P'_g + (2k + 1)P''_g \]
for $k = 0, \ldots, g + b + \left\lfloor \frac{a_r - 3}{2} \right\rfloor$,
\[ D_{2k} := (2g - 4 + a_r - 2k)P'_g + 2kP''_g + Q + R \]
for $k = 0, \ldots, g - 2 + b + \left\lfloor \frac{a_r}{2} \right\rfloor$.

After making these changes, the conclusion (3.16) becomes
\[ c_\ell + b_m = 2g - 2 \quad \text{and} \quad a_i + c_\ell \equiv a_i + b_m \equiv 1 \mod 2. \]
Otherwise the argument proceeds as in the previous case. 

3.6. **Proof of Theorem 3.3.** For $\mathcal{L}_\eta \in V^a(\mathcal{C}_\eta, \epsilon_\eta, P_\eta)$ with $h^0(\mathcal{L}_\eta) = r + 1 - i$ for each $i$, consider the pointed Prym-Petri map $\overline{\mathcal{P}}_\eta$ for $\mathcal{L}_\eta$ as in (3.4). Let $\rho \in \text{Ker}(\overline{\mathcal{P}}_\eta)$. Assuming $\rho \neq 0$, one deduces an element $\rho \neq 0$ as in (3.7). By Lemmata 3.4 and 3.5, one deduces $\overline{\mathcal{P}} = 0$, a contradiction.

4. **Prym-Tyurin varieties for Prym-Brill-Noether curves**

Here we prove Theorem 3. The statement is inspired by a result from Ortega [Ort], showing that for a general curve $C$ of genus $2a + 1$, Jac($C$) is isomorphic as a principally polarized abelian variety to a Prym-Tyurin variety for the Brill-Noether curve consisting of line bundles of degree $a + 2$ with at least two independent global sections, of exponent computed by the Catalan number $\frac{(2a)!}{a!(a+1)!}$. The proof follows the argument from [Ort] after appropriately translating the geometric objects from the Brill-Noether to the Prym-Brill-Noether setting.
Proof of Theorem 3. Select $L_0 \in V^a(C, \epsilon, P)$, and consider the embedding 

$$V^a(C, \epsilon, P) \hookrightarrow \text{Pr}(C, \epsilon), \quad L \mapsto L \otimes L^{-1}_0.$$ 

From [FL, Remark 1.9], this induces a surjective map

$$(4.1) \quad H_1(V^a(C, \epsilon, P), \mathbb{Z}) \twoheadrightarrow H_1(\text{Pr}(C, \epsilon), \mathbb{Z}).$$

Since $H_1(\text{Pr}(C, \epsilon), \mathbb{Z}) \cong H_1(\tilde{C}, \mathbb{Z})^{-}$, where $H_1(\tilde{C}, \mathbb{Z})^{-}$ is the space of anti-invariant cycles, the rational extension of (4.1) induces a surjection

$$\text{Jac}(V^a(C, \epsilon, P)) \twoheadrightarrow \text{Pr}(C, \epsilon).$$

By duality, using the respective principal polarizations

$$\text{Pr}(C, \epsilon) \cong \text{Pr}(C, \epsilon)^{\vee} \quad \text{and} \quad \text{Jac}(V^a(C, \epsilon, P)) \cong \text{Jac}(V^a(C, \epsilon, P))^{\vee},$$

one has an embedding

$$\text{Pr}(C, \epsilon) \hookrightarrow \text{Jac}(V^a(C, \epsilon, P)).$$

Welters’s criterion [BL, 12.2.2] yields that $(\text{Pr}(C, \epsilon), \Xi)$ is isomorphic to a Prym-Tyurin variety for the curve $V^a(C, \epsilon, P)$, and its exponent $e$ is determined by $[V^a(C, \epsilon, P)] = \frac{e}{(g-2)!} [\Xi]^{g-2}$. From Theorem 1, we deduce

$$e = (g-2)!^2 |\alpha|-\ell(\alpha) \prod_{i=0}^{r} \frac{1}{a_i!} \prod_{0 \leq j < i \leq r} \frac{a_i - a_j}{a_i + a_j}.$$ 

The assumption $\beta(g, a) = 1$ implies $g - 2 = |\alpha|$, hence the statement. \qed

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