Compare The Category of G-bornological group and G-
Topological Group

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Abstract

In this work, we compare the category of G-bornological
group G and G-topological group to consider the different
between them. Moreover, the category of G-bornological
set whose morphisms are called equivariant-bounded maps
are described. As well as, the main results that are a
bornological group acts on a bornological set by a
bornological isomorphism.

1. Introduction

In [1] considered bounded actions, when a bornological group acts on a
bornological set (G-bornological set) and that actions will be bounded.
This is indeed a generalization, since every group can be considered a
bornological group by using a discrete bornology, and morphisms
between G-bornological set to be bounded map. However the effect of
the bounded action is to partition a bornological set to class of orbital.
Moreover, the category of G-bornological sets is bounded map
compatible with the action. Also, it is proven before that the
boundedness of a bornological action can be deduced from its
boundedness at the identity. It is worth to mention that the
isomorphism map is in the category of bornological set. For more
information of bornological set and bornological group the reader can
see [2], [3], [4], [5], [6].
In the next, we give some basic facts that hold true only for bounded actions. Further the letters G and X stand for a bornological group \((G, \ast, \beta)\) and a bornological set \((X, \beta)\), respectively.

Definition 1-1: A bounded action (bornological group action) is a triple \((G, X, \emptyset)\) where G is a bornological group, X is a bornological set and \(\emptyset: G \times X \rightarrow X\) is a bounded function such that:

\[\emptyset(g_1, \delta(g_2)) = \emptyset(g_1 g_2, x)\] for all \(g_1, g_2 \in G\) and \(x \in X\).

b. \(\emptyset(e, x) = x\) for all \(x \in X\), where e is the identity element of G.

Then we say that the bornological group G acts on a bornological set X and X is called a left G-bornological set.

Moreover, we will note \(\emptyset(g, x)\) by the notion \(g \cdot x\) (or \(gx\)), so that (a) and (b) will be \(g_1 \cdot (g_2 \cdot x) = (g_1 g_2)\) and \(e \cdot x = x\).

If \(B\) is a bounded subset of a bornological group G and \(A\) is a bounded subset of a bornological set X we put \(B \cdot A = \{g \cdot x: g \in B, x \in A\} \subseteq X\).

2. Category of G-bornological Sets

Let G be a bornological group, X bornological set and G acts on X such that X is G-bornological sets. We deal with the category of G-bornological sets \((\text{G-Born})\) whose morphisms are called equivariant bounded maps and the objects are G-bornological sets.

Let \(\{X_i: i \in I\}\) be a collection of G-bornological sets. Then the product bornological sets \(X = \prod_{i \in I} X_i\) with the product bornology has a structure of G-bornological sets.

To define an action of G on X, take any \(g \in G\) and \(x = (x_i)_{i \in I}\), and put \(gx = gx_i\), then G acts on X, that the map

\[\delta: G \times \prod_{i \in I} X_i \rightarrow \prod_{i \in I} X_i\]

is bounded

**Proposition:** The coordinate-wise action of G on the product \(X = \prod_{i \in I} X_i\) of G-bounded, and X is a G-bornological set.
Proof: It is suffices to verify the bounded of the action of $G$ on $X$ at the identity element $e \in G$. Let $x = (x_i)_{i \in I} \in X$ be a point and $B_1 \times B_2 \subseteq G \times X$ be a bounded set consisting of pairs $(g, x)$ in $G \times X$. We assume that $B_1 = \prod_{i \in I} B_{i_1}$, where each $B_{i_1}$ is a bounded set containing $x_i$ in $X_i$ and the set $L = \{ i \in I : B_{i_1} \neq X_i \}$ is finite. Since all factors are $G$-bornological sets, we can choose, for every $i \in L$, bounded $C_i$ containing $ex_i$ in $X_i$, such that

$$B_{i_2}B_{i_1} \subseteq C_i$$

Let

$$B_2 = \cap_{i \in L} B_{i_2}$$

and

$$W = \prod_{i \in I} W_i$$

Where $W_i = C_i$ if $i \in L$ and $G_i = X_i$ otherwise. It follows immediately from the definition of the sets $B_2$ and $W$ that $B_2B_1 \subseteq W$. Therefore, the action of $G$ on $X$ is bounded.

**Arbitrary coproduct exist:**

Let $(X_i)_{i \in I}$ be a collection of $G$-bornological sets. Then $\coprod X_i$ is a disjoint union of sets with the disjoint union bornology is the categorical product in $G$-Born (a set $B \subseteq \coprod X_i$ is bounded if and only if $B \cap X_i$ is bounded for all $X_i$).

To prove that the map $\delta: G \times \coprod X_i \to \coprod X_i$ is bounded. Take a bounded set $B = B_1 \times B_2 \subseteq G \times \coprod X_i$ and let $m_i: G \times X_i \to X_i$. Then $\delta(B) \cap X_i = m_i(B)$ is bounded. Thus $\delta(B)$ is bounded.

A $G$-subset $Y$ of a $G$-bornological sets $X$ (which is a subset that closed under action $G$, with the subspace bornology). Has bornological group action:

Indeed, let

$$\delta: G \times X \to X$$

And

$$\mu: G \times Y \to Y$$
be two actions and $W = B_1 \times B_2$ be a bounded subsets of $G \times Y$. Then
\[
\mu(B_1 \times B_2) = \{ g.y: (g,y) \in B_1 \times B_2 \} = \delta(B_1 \times B_2),
\]
because if $x \in X$ and $(g,x) = w \in W$, then $x = (g^{-1},w) \in Y$. Since $Y$ is
a G-subset, so $\mu(B_1 \times B_2)$ is bounded in $Y$.

**Fibered products exist:** let $B_1, B_2$ and $B_3$ be G-boronological sets
and let
\[
f: B_1 \rightarrow B_3
\]
And
\[
g: B_2 \rightarrow B_3
\]
be equivariant maps. The fibered product $B_1 \times_{B_3} B_2$ is the bornological
subset of $B_1 \times B_2$ consisting of pairs $(a,b)$ such that $f(a) = g(b)$. The
diagonal action $(g_1, (a,b)) \rightarrow (g_1 a, g_1 b)$, clearly preserved $B_1 \times_{B_3} B_2$
and thus makes it into a G-bornological set. The projections
\[
f: B_1 \times_{B_3} B_2 \rightarrow B_2
\]
And
\[
g: B_1 \times_{B_3} B_2 \rightarrow B_1
\]
are clearly equivariant. The fibered product $B_1 \times_{B_3} B_2$ satisfies the
universal property of pull backs i.e, if $D$ is a G-boronological and
\[
\psi: D \rightarrow B_1
\]
And
\[
\phi: D \rightarrow B_2
\]
are equivariant and such that
\[
f\psi = g\phi
\]
Then there is a unique map
\[
\theta: D \rightarrow B_1 \times_{B_3} B_2
\]
Actually, $\theta$ is given by $\theta(d) = (\psi(d), \psi(d))$. 
In the pull-back diagram,

\[ \begin{array}{ccc}
B_1 \times B_3^{B_2} & \xrightarrow{\hat{g}} & B_1 \\
\hat{f} & \downarrow & f \\
B_2 & \xrightarrow{g} & B_3 
\end{array} \]

We note that \( \hat{f} \) is bounded if \( f \) is bounded (and similarly as \( \hat{g} \) and \( g \)). To see this, let \((a,b)\) be in \( B_1 \times B_2 \) (that is \( f(a) = g(b) \)) and let \( L \) be a bounded set containing \( a \). Then \( f(L) \) is bounded and \( g \in g^{-1}(f(L)) \). Let \( K \subset g^{-1}(f(L)) \) be any bounded set containing \( b \). Now

\[(B_1 \times B_2^{B_2}) \cap (L \times K)\]

Clearly, projects onto \( K \) by \( \hat{f} \), which implies that \( \hat{f} \) is bounded. An important special case where \( C = A \setminus G \), we have

\[ f = \pi_{B_1}: B_1 \to B_1 \setminus G \]

Is the orbit function, and \( B_2 \) has trivial bornological group action. In this, the fibered will be denoted by \( g \ast B_1 \) and called the pullback of \( B_1 \) thorough \( g \) and we will have the diagram

\[ \begin{array}{ccc}
g \ast B_1 & \xrightarrow{\hat{g}} & B_1 \\
\hat{\pi} & \downarrow & \pi \\
B_2 & \xrightarrow{g} & B_1/G 
\end{array} \]

And \( \hat{\pi} \) is equivariant from the \( G \)-bornological set \( g \ast B_1 \) to the trivial \( G \)-bornological sets \( B \). Then \( \hat{\pi} \) induces a map

\[ \sigma: (g \ast B_1)/G \to B_2 \]
Here \( \pi \) is bounded and onto, therefore \( \sigma \) is bounded and onto. If \((a,b)\) and \((\hat{a}, \hat{b})\) are both in \( g * B_1 \), then \( \pi(a) = g(b) = \pi(\hat{a}) \). So that \( a \) and \( \hat{a} \) are in the same orbit, where \((a,b)\) and \((\hat{a}, \hat{b})\) are in the same orbit. This shows \( \sigma \) that is one-to-one, and hence the \( \sigma: (g * B_1)/ G \to B_2 \) is a bounded isomorphism. Since \( \sigma \) is canonical, we may regard \( B_2 \) as the orbital bornological set \( (g * B_1)/ G \).

**Equalizers exist:** Consider \( f, g: X \to Y \), we have

\[
X \times_Y X = \{(x, \hat{x}): f(x) = g(\hat{x})\}.
\]

with the subspace bornology of \( X \times X \) and diagonal action, since products of \( G \)-bornological sets are \( G \)-bornological set and \( G \)-subsets of \( G \)-bornological sets are \( G \)-bornological sets, \( X \times_Y X \), is \( G \)-bornological set.

This means that equalizer is the limit of diagram consisting of two morphisms

\[
f: X \to Y, \\
g: X \to Y,
\]

Such that they satisfy universal properties.

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![Diagram](image-url)
Coequalizers do not always exist: but given bounded maps $f, \hat{f}: X \to Y$ in G-Born the coequalizer

$$h: Y \to Z$$

is given by $z = Y/\sim$ where $\sim$ is the equivalence relation generated by identifying $f(x) \sim \hat{f}$ for all $x \in X$, and $h$ is the quotient map. The set $Y/\sim$ has the group action $g \cdot [y] = [g \cdot y]$. This is well-defined: since $y = f(x)$ and $\hat{y} = \hat{f}(x)$ then for any $g_o \in G$ we have

$$g_o \cdot y = g_o \cdot f(x) = f(g_o \cdot x) \sim \hat{f}(g_o \cdot x) = g_o \hat{f} = g_o \cdot \hat{y}.$$ 

In general, if $y \sim \hat{y}$ then there is a finite string of equivalence correcting them, and by induction $g_o \cdot y \sim g_o \cdot \hat{y}$. As a space, $Z$ has the quotient bornology. It is well-known that this space has the universal property as a bornological set, and it is easy to check that it satisfies the universal property in the category of G-sets. It is clear that the quotient map $h: Y \to Z$ is a bounded. So for a bounded set $B \subseteq Y$,

$$(h(B)) = \{ y \in Y: \exists b \in B, y \sim b \}$$

Is the union of $f \left( \hat{f}^{-1}(B) \right), \hat{f}(f^{-1}(B))$, where each of them is bounded.

There is an inclusion functor from bornological sets to G-bornological sets taking a bornological set $X$ to the G-bornological X with trivial bornological group action (the map $G \times X \to X$ is bounded since the image of a bounded set $G \times B$ is bounded set $B$). Clearly, G-bornological set $G \times X$ with bornological group action $g(x, \hat{g}) = (x, g \hat{g})$ bornological group action.

**Proposition:** Let $B$ be a discreet bornology on a finite group $G$ and $X$ a G-bornological set. The action of $G$ is bounded if and only if $g.B$ is bounded for all bounded sets $B \subseteq X$ and $g \in G$.

In the end, in this paper we try to compare the category of G-bornological and G-topological group, such that in G-topological group it will be happen, when a topological group acts on a topological set (G-topological group) and the actions will be continuous. Same can indeed a generalization, when every group can be considered a topological group by using the discrete topology. Furthermore, the morphisms between
G-topological set to be continuous map. Moreover, the category of G-topological sets is continuous map compatible with the action of G.

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