Dispersionless limit of Hirota equations in some problems of complex analysis

A.Zabrodin*

April 2001

Abstract

The integrable structure, recently revealed in some classical problems of the theory of functions in one complex variable, is discussed. Given a simply connected domain in the complex plane, bounded by a simple analytic curve, we consider the conformal mapping problem, the Dirichlet boundary problem, and to the 2D inverse potential problem associated with the domain. A remarkable family of real-valued functionals on the space of such domains is constructed. Regarded as a function of infinitely many variables, which are properly defined moments of the domain, any functional from the family gives a formal solution to the problems listed above. These functions are shown to obey an infinite set of dispersionless Hirota equations. This means that they are \( \tau \)-functions of an integrable hierarchy. The hierarchy is identified with the dispersionless limit of the 2D Toda chain. In addition to our previous studies, we show that, with a more general definition of the moments, this connection is not specific for any particular solution to the Hirota equations but reflects the structure of the hierarchy itself.

1 Introduction

Recently, it was found [1, 2, 3] that some problems of complex analysis, such as conformal mapping problem, Dirichlet boundary problem, and 2D inverse potential problem, have a hidden integrable structure. It turns out that there is an infinite hierarchy of non-linear integrable equations such that variations of sought-for solutions to the problems listed above under deformation of given data can be found by solving the hierarchy. The latter is a certain scaling limit, usually referred to as dispersionless limit, of the 2D Toda chain hierarchy.

The conformal mapping problem is to find a conformal map from a given simply connected domain in the complex plane to the unit disk. By the Riemann mapping theorem, such a map does exist and, after imposing some simple normalization conditions,
is unique (see, e.g., [4]). To be specific, let $\gamma$ be a closed analytic curve in the complex plane and $D_+$, $D_-$ be respectively the interior and exterior domains with respect to the curve. By $w(z)$ we denote the conformal map of $D_-$ to the exterior of the unit circle normalized in such a way that $\infty$ is taken to $\infty$ and the derivative at $\infty$ is real positive.

The Dirichlet boundary problem in $D_-$ sounds as follows: given a function $\psi(z)$ on the curve $\gamma$, to find a harmonic function $f(z, \bar{z})$ in $D_-$ such that $f(z, \bar{z}) = \psi(z)$ for $z \in \gamma$. The solution is given by the formula

$$f(z, \bar{z}) = -\frac{1}{\pi i} \oint_\gamma \psi(\zeta) \partial_\zeta G(z, \zeta) d\zeta$$

where $G(z, \zeta)$ is the Green function of the Dirichlet problem in $D_-$. The Green function is uniquely characterized by the properties [5]: (a) It is regular at infinity and harmonic everywhere in $D_-$ but at the point $z = \zeta$ where it has the logarithmic singularity: $G(z, \zeta) = \log |z - \zeta| + O(1)$ as $z \to \zeta$; (b) It is equal to zero on the boundary $\partial D_- = \gamma$, i.e., $G(z, \zeta) = 0$ for all $\zeta$ if $z \in \gamma$.

To specify what we mean by the inverse potential problem, we suppose, in the same setting as above, that the domain $D_+$ is filled with a continuously distributed background electric charge with uniform density. As is proved in the potential theory, the potential, $\Phi(z, \bar{z})$, generated by the charge, and its first derivatives (components of the electric field) are continuous functions at the boundary, i.e., $\Phi^+ = \Phi^-$, $\partial_z \Phi^+ = \partial_z \Phi^-$, $z \in \gamma$. Here $\Phi^\pm$ is the function $\Phi$ restricted to $D^\pm$. $\Phi^-$ is harmonic everywhere in $D_-$ but at infinity, where it has a logarithmic singularity, while $\Phi^+$ is harmonic up to a term proportional to $|z|^2$. Therefore, one can represent $\Phi^\pm$ by Taylor series (multipole expansions) in a neighbourhood of 0 and $\infty$ respectively. We suppose, without loss of generality, that 0 is in $D_+$. The coefficients of the multipole expansions are harmonic moments of $D_-$ and $D_+$, respectively. The inverse potential problem is to restore the shape of the domain given one of the series, either $\Phi^+$ or $\Phi^-$. For definiteness, we assume that the coefficients of $\Phi^+$ are given and address the problem how to find the coefficients of $\Phi^-$. (If one knows both sets of coefficients, the curve can be easily reconstructed.) For a review of inverse problems of potential theory see, e.g., [6] and references therein.

The three problems are closely related to each other. Given, say, the conformal map $w(z)$, the Green function can be found explicitly via the formula

$$G(z, \zeta) = \log \left| \frac{w(z) - w(\zeta)}{w(z)\overline{w(\zeta)} - 1} \right|$$

where the bar means complex conjugation. In turn, the holomorphic in $z$ part of the Green function is the conformal map, though normalized in a different way. To obtain the map $w(z)$, one should take the holomorphic part of $G(z, \infty)$. Concerning the inverse potential problem, we note that it is the Green function that provides a relation between $\Phi^+$ and $\Phi^-$. Indeed, let us modify the potential function and introduce $\tilde{\Phi}$ which is $\Phi$ with subtracted logarithmic singularity at infinity, so that $\tilde{\Phi}^-$ is harmonic in $D_- \cup \infty$. Then, according to (1), we can write

$$\tilde{\Phi}^-(z, \bar{z}) = -\frac{1}{\pi i} \oint_\gamma \tilde{\Phi}^+(\zeta, \bar{\zeta}) \partial_\zeta G(z, \zeta) d\zeta$$
Let us summarize the main results of \[1, 2, 3\]. Let \( t_k, k \geq 1 \), be the harmonic moments of \( D_+ \), and \( t_0 \) be the area of \( D_+ \) (divided by \( \pi \)): 

\[ t_k = \frac{1}{\pi k} \int_{D_+} z^{-k} d^2z, \quad t_0 = \frac{1}{\pi} \int_{D_+} d^2z. \]

Note that \( t_k \) are in general complex numbers while \( t_0 \) is real. In the subsequent formulas, the complex conjugate moments, \( \bar{t}_k \), are considered as independent variables. Introduce the operators 

\[
D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_k}, \quad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \partial_{\bar{t}_k}
\]

and 

\[
D(z, \bar{z}) = \partial_{t_0} + D(z) + \bar{D}(\bar{z})
\]

There exists a real-valued function \( F \) of \( t_0, \{t_k\}, \{\bar{t}_k\} \) such that the conformal map \( w(z) \) is given by 

\[
w(z) = z \exp \left( -\frac{1}{2} \partial_{t_0}^2 F - \partial_{t_0} D(z) F \right)
\]

This function admits an explicit integral representation. Moreover, the Green function of the Dirichlet boundary problem is represented as 

\[
G(z_1, z_2) = \log |z_1^{-1} - z_2^{-1}| + \frac{1}{2} D(z_1, \bar{z}_1) \bar{D}(\bar{z}_2, z_2) F
\]

Let us note that a formula of this type was first conjectured by L. Takhtajan. See \[4\] for a rigorous proof and discussion.

The function \( F \) is the dispersionless limit of the logarithm of the \( \tau \)-function of the 2D Toda chain hierarchy. It obeys the dispersionless Hirota equations 

\[
(z_1 - z_2) e^{D(z_1)D(z_2)F} = z_1 e^{-\partial_{t_0} D(z_1)F} - z_2 e^{-\partial_{t_0} D(z_2)F}
\]

\[
z_1 \bar{z}_2 \left(1 - e^{-D(z_1)D(\bar{z}_2)F} \right) = e^{\partial_{t_0} (\partial_{t_0} + D(z_1) + \bar{D}(\bar{z}_2)) F}
\]

The dispersionless Toda hierarchy is an example of the universal Whitham hierarchy introduced in \[8\]. It is a multi-dimensional extension of the hierarchies of hydrodynamic type \[3, 9\]. Equations of the Toda hierarchy are known to have infinitely many solutions. The solutions can be parametrized \[3\] by canonical transformations in a two-dimensional phase space in such a way that any solution corresponds to a canonical pair of functions. (In \[3\], the canonical pair is called twistor data of the solution.) In fact, this is equivalent to the characterization of the solutions via string equations.

In the approach developed in \[3\], it was one particular solution to the dispersionless 2D Toda hierarchy that described conformal maps. The place, if any, of other solutions in the conformal map context was not clear. In this paper we address exactly this question. Extending the previous result, we show that the same class of conformal maps can be described by any generic solution to the hierarchy.

This description is achieved by another, more general definition of moments of the domain. The moments are again the independent flows of the hierarchy. However, their relation to the shape of the domain is different, and some additional data is required for their definition. The additional data is a real analytic function on the complex plane, \( \sigma(z, \bar{z}) \). In the context of the inverse potential problem, this function is density of the background charge, which was constant in the standard formulation of the problem. More precisely, we suppose that \( \sigma(z, \bar{z}) \) is given in the whole complex plane and fixed once and
for all while the shape of the domain may vary. The background charge has density $\sigma(z, \bar{z})$ inside the domain and 0 outside. The inverse potential problem is posed in the same way as before. The moments, however, are now defined with respect to the density $\sigma$. In the sequel we refer to this problem as generalized inverse potential problem.

The result shows that it is the hierarchy of Hirota equations (rather than particular string equations or other solution-specific ingredients) which reflects the integrable structure of conformal maps and the Dirichlet problem.

In Sec. 2 we introduce the function $F$ with the help of a variational principle and find its first derivatives. The first derivatives give a formal solution to the generalized inverse potential problem. Formal solutions to the conformal map problem and the Dirichlet problem are given by second derivatives of $F$. There is an infinite set of relations for the second derivatives, which can be combined into the (dispersionless limit of) Hirota equation. This stuff is discussed in Sec. 3. At last, Sec. 4 is devoted to an alternative approach to the dispersionless integrable hierarchy, which allows one to prove the Lax-Sato equations for conformal maps.

## 2 Generalized inverse potential problem

The inverse potential problem is perhaps the most convenient context to introduce the $\tau$-function for curves. Following the idea of [3], we define the $\tau$-function with the help of an electrostatic variational principle. The set-up and the notation are the same as in Sec. 1.

We recall that the domain $D_+$ is filled with a background electric charge. Let us consider non-uniform charge distributions such that density of the charge is characterized by a real analytic function $\sigma(z, \bar{z})$. We assume that this function is defined all over the complex plane, the charge density being equal to $-\sigma$ inside $D_+$ and 0 outside. With a fixed function $\sigma$, the 2D electrostatic potential $\Phi$ generated by the domain is a functional of the shape of the domain. The potential $\Phi$ obeys the Poisson equation

$$-\partial_z \partial_{\bar{z}} \Phi(z, \bar{z}) = \begin{cases} \sigma(z, \bar{z}) & \text{if } z = x + iy \in D_+ \\ 0 & \text{if } z = x + iy \in D_- \end{cases}$$  \(10\)

The potential $\Phi$ can be written as an integral over the domain $D^+$:

$$\Phi(z, \bar{z}) = -\frac{2}{\pi} \int_{D_+} d^2 z' \sigma(z', \bar{z}') \log |z - z'|$$  \(11\)

In $D_-$, the potential is a harmonic function represented by the series

$$\Phi^-(z, \bar{z}) = -2t_0 \log |z| + 2\Re \sum_{k > 0} \frac{v_k}{k} z^{-k},$$  \(12\)

as $z \to \infty$, where

$$v_k = \frac{1}{\pi} \int_{D_+} z^k \sigma(z, \bar{z}) d^2 z \quad (k > 0)$$  \(13\)

are moments of the interior domain $D_+$ with respect to the density $\sigma$, and

$$\pi t_0 = \int_{D_+} \sigma(z, \bar{z}) d^2 z$$  \(14\)
is the total charge of $D_+$. 

In $D_+$, the potential is represented by a series $\Phi^+$ which has both harmonic and non-harmonic terms. The latter is due to the background charge. We denote the non-harmonic part by $U(z, \bar{z})$. The function $U$ obeys the Poisson equation $\partial_z\partial_{\bar{z}}U(z, \bar{z}) = \sigma(z, \bar{z})$ and is fixed by the condition that its expansion around 0 (recall that $0 \in D_+$) does not contain harmonic terms. For example, if $\sigma(z, \bar{z}) = 1$ then $U(z, \bar{z}) = z\bar{z}$. We adopt the following general form of $U$:

$$U(z, \bar{z}) = -\sum_{m,n \geq 1} T_{mn} z^m \bar{z}^n$$

The expansion of $\Phi$ around $z = 0$ can be obtained from (11) in a direct way. The result is:

$$\Phi^+(z, \bar{z}) = -U(z, \bar{z}) + v_0 + 2\text{Re} \sum_{k>0} t_k z^k$$

Here the non-harmonic term, $U$, is determined by the background charge, as is explained above, and does not depend on the form of the domain. It is the harmonic contribution that takes the boundary into account. The coefficients are:

$$t_k = \frac{1}{2\pi i k} \oint_\gamma z^{-k} \partial_z U \, dz \quad (k > 0)$$

They provide a complimentary (to $v_k$) set of parameters characterizing the domain $D_+$. Here and below we use the same notation for the moments ($t_0$, $t_k$, etc) as in Sec. 1. In the case $\sigma = 1$ they are equal to harmonic moments of the exterior domain $D_–$. Finally,

$$v_0 = \frac{2}{\pi} \int_{D_+} \log |z| \sigma(z, \bar{z}) \, d^2z.$$ 

Note that the moments $v_k$ and $t_0$ also admit a similar contour integral representation:

$$v_k = \frac{1}{2\pi i} \oint_\gamma z^k \partial_z U \, dz \quad (k > 0), \quad t_0 = \frac{1}{2\pi i} \oint_\gamma \partial_z U \, dz$$

which is a simple consequence of the Green formula.

Since the potential and the electric field are continuous at the boundary of the domain, the two complementary sets of moments are related by the conditions

$$\Phi^+ = \Phi^-, \quad \partial_z \Phi^+ = \partial_z \Phi^-, \quad z \in \gamma$$

The generalized inverse potential problem is to find the curve $\gamma$ given the function $\sigma$ and one of the functions $\Phi^+$ or $\Phi^-$, i.e., one of the infinite sets of moments. We will choose as independent variables the total charge $\pi t_0$ and the coefficients $t_k$, $t_k$ ($k \geq 1$) of the multipole expansion of $\Phi$ around the origin. Under certain conditions, which we do not discuss here, they uniquely determine the form of the curve, at least if the deformed curve to be described by $t_k$ is close enough to a fixed given curve. (Some important results concerning uniqueness of solution to inverse potential problems can be found in [12].) One such condition, specific for the case of non-uniform density of the background charge, is worth mentioning. For the problem to be well-posed, one should require that $\sigma(z, \bar{z}) \neq 0$ at any point of the curve $\gamma$. Otherwise a small perturbation of the curve
around the zero of \( \sigma \) does not result, at least in the first order, in the variation of \( \Phi \). To avoid this difficulty, one may assume, for example, that \( \sigma(z, \bar{z}) > 0 \).

In other words, under the conditions mentioned above, \( \{t_k\}_{k=0}^{\infty} \) is a good set of local coordinates in the space of analytic curves. The moments \( v_j \) are also determined by \( \{t_k\}_{k=0}^{\infty} \), so any moment of the interior can be regarded as a function of \( t_k \). For simplicity, we assume in this paper that only a finite number of \( t_k \) and \( T_{mn} \) are non-zero. In this case the series (16) is a polynomial in \( z, \bar{z} \) and, therefore, it defines the function \( \Phi^+ \) for all \( z \in D_+ \). As soon as \( \sigma \) is real, \( t_0, v_0 \) are real numbers while all other moments are in general complex quantities.

We now pass to the electrostatic variational principle mentioned in the beginning of this section. Consider the energy functional describing a charge with a density \( \rho(z, \bar{z}) \) in the background potential \( \Phi \) (11) generated by the charge distributed in \( D_+ \) with the density \( \sigma \):

\[
\mathcal{E}\{\rho\} = -\frac{1}{\pi^2} \int \int d^2z \, d^2z' \, \rho(z, \bar{z}) \log |z - z'| \rho(z', \bar{z}') - \frac{1}{\pi} \int d^2z \, \rho(z, \bar{z}) \, \Phi(z, \bar{z}).
\]  

(21)

The first term is the 2D “Coulomb” energy of the charge while the second one is the energy of interaction with the background charge. Clearly, the distribution of the charge neutralizing the background charge gives minimum to the functional. We denote density of this distribution by \( \rho_0 \): \( \rho_0 = \sigma \) inside the domain and \( \rho_0 = 0 \) outside, so that the total charge density be zero. At the minimum the functional is equal to minus electrostatic energy \( E \) of the background charge \( \min_{\rho} \mathcal{E}(\rho) = -E \), where

\[
E = -\frac{1}{\pi^2} \int \int_{D_+} d^2z \, d^2z' \, \sigma(z, \bar{z}) \log |z - z'| \sigma(z', \bar{z}')
\]  

(22)

Varying \( \rho \) in (21) and then setting \( \rho = \rho_0 \), we obtain eq. (16).

Let us find derivatives of \( E \) with respect to \( t_k \). This can be done in different ways. One of them would be to take the derivative of (22) directly. In doing so, one should somehow control the contribution coming from an infinitesimal change of the domain under the variation of \( t_k \). To avoid this problem, we use the same argument as in the case \( \sigma = 1 \) (see [3]). Namely, instead of taking the derivative of (22), we differentiate \( E \) represented as \( -\mathcal{E}(\rho) \) in the form (21) at the extremum. There are contributions of two kinds: one from the explicit \( t_k \)-dependence of \( \Phi^+ \) and another from the implicit \( t_k \)-dependence of \( \rho_0 \):

\[
\frac{\partial E}{\partial t_k} = -\left. \frac{\partial \mathcal{E}(\rho)}{\partial t_k} \right|_{\rho=\rho_0} = \frac{1}{\pi} \int_{D_+} d^2z \sigma(z, \bar{z}) \partial_k \Phi^+(z, \bar{z}) - \int d^2z \left. \frac{\delta \mathcal{E}(\rho)}{\delta \rho(z, \bar{z})} \right|_{\rho=\rho_0} \frac{\delta \rho_0(z, \bar{z})}{\delta t_k}
\]

Since we are at the extremum, the variational derivative in the second term is zero, and this term does not contribute. In other words, the derivative is equal to the partial derivative of \( -\mathcal{E} \) calculated at the extremum. Let us treat, for a while, \( v_0 \) and \( t_k \) as independent variables, then the partial derivatives of \( \Phi^+ \) are especially simple: \( \partial_k \Phi^+(z, \bar{z}) = z^k \), \( \partial_{v_0} \Phi^+(z, \bar{z}) = 1 \). Plugging them into the integral, we arrive at the relations

\[
\frac{\partial E}{\partial t_k} = v_k, \quad \frac{\partial E}{\partial t_k} = \bar{v}_k, \quad \frac{\partial E}{\partial v_0} = -t_0,
\]

(23)
where the partial derivative with respect to $t_k$ is taken at fixed $v_0$ and $t_j \ (j \neq 0, k)$. Therefore, the differential $dE$ is $dE = \sum_{k>0}(v_k dt_k + \bar{v}_k d\bar{t}_k) - t_0 dv_0$. Note that there are two contributions to the $t_k$-derivative of (22): one from $\partial_{t_k} \Phi$ and another from the change of the domain. The former is easily calculated to be $\frac{1}{2}v_k$, thus the latter is $\frac{1}{2}v_k$, too. However, it is not so easy to obtain this result directly (see the appendix for an example of such a calculation). A direct proof of the formulas for first derivatives in the case $\sigma = 1$ can be found in [7].

It is more natural, however, to treat the total charge rather than $v_0$ as an independent variable, i.e., to apply the variational principle at a fixed total charge. This is achieved via the Legendre transformation. Let us introduce the function $F = E + t_0v_0$, whose differential is $dF = \sum_{k>0}(v_k dt_k + \bar{v}_k d\bar{t}_k) + v_0 dt_0$. The integral representation of this function follows from (22):

$$F = -\frac{1}{\pi^2} \int \int_{D_+} d^2 z d^2 z' \sigma(z, \bar{z}) \log |z^{-1} - z'^{-1}| \sigma(z', \bar{z}')$$

(24)

The function $F$ plays a major role in what follows. It is a real-valued function of the moments $t_0, t_1, t_2, \ldots$. Rewriting (23) in the new variables, we get the main property of the function $F$:

$$\frac{\partial F}{\partial t_k} = v_k, \quad \frac{\partial F}{\partial \bar{t}_k} = \bar{v}_k, \quad \frac{\partial F}{\partial t_0} = v_0$$

(25)

where the derivative with respect to $t_k$ is taken at fixed $t_j \ (j \neq k)$. The very existence of the common potential function for moments implies the symmetry relations for their derivatives, $\partial_{t_k} v_k = \partial_{\bar{t}_k} v_n, \partial_{t_k} \bar{v}_k = \partial_{\bar{t}_k} \bar{v}_n$, which were first obtained in [1] (for the case $\sigma = 1$) within a different approach.

To represent the result (23) in a more compact form, let us modify the potential $\Phi$ and introduce $\tilde{\Phi}(z, \bar{z}) = \Phi(z, \bar{z}) + 2t_0 \log |z| + v_0$, so that

$$\tilde{\Phi}(z, \bar{z}) = -\frac{2}{\pi} \int_{D_+} d^2 \zeta \sigma(\zeta, \bar{\zeta}) \log |z^{-1} - \zeta^{-1}|$$

(26)

and

$$F = \frac{1}{2\pi} \int_{D_+} d^2 z \sigma(z, \bar{z}) \tilde{\Phi}^+(z, \bar{z})$$

(27)

The modification amounts to adding the neutralizing point-like charge at the origin. The expansion of the modified potential around the origin is

$$\tilde{\Phi}^+(z, \bar{z}) = -U(z, \bar{z}) + 2t_0 \log |z| + \sum_{k \geq 1}(t_k z^k + \bar{t}_k \bar{z}^k)$$

(28)

From now on, this formula will serve as a definition of the variables $t_k$. Likewise $\Phi$, the function $\tilde{\Phi}$ and its first derivatives are continuous on the boundary $\gamma = \partial D_+$. For $z$ in $D_-$ the function $\tilde{\Phi}(z, \bar{z})$ is harmonic. In terms of $\tilde{\Phi}$, formulas (27) can be compactly written in the form

$$\mathcal{D}(z, \bar{z})F = \tilde{\Phi}^- (z, \bar{z})$$

(29)

where we use the operator $\mathcal{D}$ introduced in (3), and $z$ is assumed to be in $D_-$. Both sides are to be understood as series in $z$ and $\bar{z}$. It is our assumption that they converge in $D_-$. 

7
To conclude this section, let us discuss homogeneity properties of the function $F$. Under the assumption that only a finite number of $t_k$'s are non-zero, we can substitute (16) into (24) and perform the term-wise integration. We get the following relation:

$$2F = -\frac{1}{4\pi} \int_{D_+} d^2 z U \Delta U + t_0 v_0 + \sum_{k>0} (t_k v_k + \bar{t}_k \bar{v}_k),$$

(30)

where $\Delta = 4\partial_z \partial_{\bar{z}}$ is the Laplace operator. Generally speaking, (30) does not have any definite homogeneity properties as a function of $t_k$. However, for homogeneous functions $\sigma$, $F$ is quasihomogeneous. For instance, if $\sigma(z, \bar{z}) = |z|^{2M-2}$ with an integer $M$, then (30) can be brought into the form

$$4MF = -t_0^2 + 2Mt_0 \partial_{t_0} F + \sum_{k\geq 1} (2M - k)(t_k \partial_{t_k} F + \bar{t}_k \partial_{\bar{t}_k} F)$$

where the relations (25) are taken into account. This relation reflects homogeneity properties of the moments under the rescaling $z \to \lambda z$ and means that $F$ is a quasihomogeneous function of degree $4M$.

A more fundamental homogeneity property, valid for any $\sigma$, can be revealed by extending the definition of the function $F$. Let us treat parameters $T_{mn}$ defining $U$ or $\sigma$ (see (15)) on equal footing with the variables $t_k$. To wit, let us identify $t_k = T_{k0}$, $\bar{t}_k = T_{0k}$, $t_0 = T_{00}$, and allow $F$ to be a function of all $T_{mn}$, $m, n \geq 0$, regarded as an extended set of independent variables. Using the same variational argument as before, one can deduce the relations

$$\frac{\partial F}{\partial T_{mn}} = \frac{1}{\pi} \int_{D_+} z^n \bar{z}^m \sigma(z, \bar{z}) d^2 z$$

which generalize (25). Then (30) acquires the form of the pure homogeneity condition:

$$2F = \sum_{n, m \geq 0} T_{mn} \frac{\partial F}{\partial T_{mn}}$$

(31)

The $\tau$-function can be defined as $\tau = e^F$. This notation, however, is introduced here only for the purpose to stress the relation to Hirota equations. It is $F$, i.e., the logarithm of the $\tau$-function, which we only deal with in the sequel. Recall that the $\tau$-function itself does not have a good dispersionless limit. Only its logarithm, multiplied by a small dispersion parameter, makes sense in this limit. The bilinear Hirota equations for $\tau$ are rewritten as highly non-linear equations for $\log \tau$. In the next section, we derive these equations starting from the definition of the function $F$.

## 3 Dirichlet problem and dispersionless Hirota equations for the function $F$

The dispersionless Hirota equations are certain relations between second derivatives of logarithm of the $\tau$-function with respect to $t_k$. The second derivatives of $F$ are coefficients of the expressions like $D(z_1)D(z_2)F$, where the operator $D(z)$ is introduced in (4). Recall that we already know first derivatives of $F$, see (29). Therefore, what we need to find is $D(z_1, \bar{z}_1)\tilde{\Phi}(z_2, \bar{z}_2)$ for $z_1, z_2 \in D_-$. For this we use the following general argument.
Consider a small deformation of the domain \( D_+ \) obtained from the original one by adding to it an infinitesimal bump (of arbitrary form) with area \( \epsilon \) located at a point \( \xi \in \gamma \). By \( \delta_{\epsilon(\xi)} \), denote variation of any quantity under such deformation in the first order in \( \epsilon \). The potential generated by the deformed domain is then \( \tilde{\Phi} + \delta_{\epsilon(\xi)} \tilde{\Phi} \), where

\[
\delta_{\epsilon(\xi)} \tilde{\Phi}(z, \bar{z}) = -\frac{\epsilon}{\pi} \sigma(\xi, \bar{\xi}) \log |z^{-1} - \xi^{-1}|^2
\]  

(32)
is the potential generated by the bump, as it is clear from the explicit formula (26). Comparing the expansion of \( \delta_{\epsilon(\xi)} \tilde{\Phi}(z, \bar{z}) \) in \( z \) as \( z \to 0 \) with (28), we find variations of \( t_k \) under the deformation:

\[
\delta_{\epsilon(\xi)} t_0 = \frac{\epsilon}{\pi} \sigma(\xi, \bar{\xi}), \quad \delta_{\epsilon(\xi)} t_k = \frac{\epsilon}{\pi k} \sigma(\xi, \bar{\xi}) \xi^{-k}, \quad k \geq 1
\]

Given any function \( A \) of the moments, its variation, \( \delta_{\epsilon(\xi)} A \), is

\[
\delta_{\epsilon(\xi)} A = \partial_{\epsilon A} \delta_{\epsilon(\xi)} t_0 + \sum_{k \geq 1} (\partial_{t_k} A \delta_{\epsilon(\xi)} t_k + \partial_{t_k} A \delta_{\epsilon(\xi)} t_k)
\]

which can be written as

\[
\delta_{\epsilon(\xi)} A = \frac{\epsilon}{\pi} \sigma(\xi, \bar{\xi}) D(\xi, \bar{\xi}) A, \quad \xi \in \gamma
\]

(33)

So we see that for \( \xi \) on the boundary curve \( \gamma = \partial D_+ \), the operator \( D(\xi, \bar{\xi}) \) has a clear geometrical meaning: the result of its action on any quantity is proportional to the variation of this quantity under attaching the bump at the point \( \xi \). To put it differently, we can say that the boundary value of the function \( D(z, \bar{z}) A \) is equal to \( \pi \delta_{\epsilon(\xi)} A / (\epsilon \sigma(z, \bar{z})) \), \( z \in \gamma \). For functions \( A \) such that the series \( D(z) A \) converges everywhere in \( D_- \) up to the boundary, this remark gives a usable method to find the function \( D(z, \bar{z}) A \). To wit, this function is harmonic in \( D_- \), and its boundary value is given by (33). It is then just the subject of the Dirichlet boundary problem to find the harmonic function given its boundary value. This method appears to be especially useful when one is able to find the left hand side of (33) independently.

As a very simple example, one can easily find \( \delta_{\epsilon(\xi)} F = \frac{\epsilon}{\pi} \sigma(\xi, \bar{\xi}) \tilde{\Phi}(\xi, \bar{\xi}) \) from the integral representation of \( \tilde{\Phi} \). In this case, the harmonic continuation does not require any additional care since the function \( \tilde{\Phi} \) is already harmonic in \( D_- \). Whence we reproduce the result (29).

Let us turn to second derivatives. As it follows from (33), the boundary value of the function \( D(\xi, \bar{\xi}) D(z, \bar{z}) F = D(\xi, \bar{\xi}) \tilde{\Phi}(z, \bar{z}) \), regarded as a function of \( \xi \), is given by eq. (32) for \( z \) in \( D_- \):

\[
D(\xi, \bar{\xi}) D(z, \bar{z}) F = -2 \log |z^{-1} - \xi^{-1}|, \quad \xi \in \gamma
\]

(34)

Here we do not discuss convergence of the series \( D(\xi) \tilde{\Phi} \) in \( D_- \), which is the necessary assumption for the harmonic continuation rule explained above. To be sure that the class of domains having this property is not empty, one may think of domains and background charges that are close enough to the uniformly charged disk, in which case the proof of the convergence is the matter of some routine estimates. The right hand side of (34) is not harmonic in \( D_- \) as it stands because of the logarithmic singularity at \( \xi = z \). To find the
harmonic continuation of this function, we add to (34) a function harmonic everywhere in \( D \), but at the point \( z \), where it has the logarithmic singularity +2 log \(|z^{-1} - \xi^{-1}|\) (thus cancelling the singularity of (34)), and equal to zero on the boundary. By definition of the Green function of the Dirichlet problem (see Sec. 1), it is the Green function \( G(z, \xi) \) (2). Hence we obtain formula (7). Tending \( z_2 \) to \( \infty \) and then separating the holomorphic part in \( z_1 \), one arrives at the expression (8) for the conformal map. Separating holomorphic and antiholomorphic parts of (7) in both variables, and using (2), we obtain:

\[
\log \frac{w(z_1) - w(z_2)}{z_1 - z_2} = -\frac{1}{2} \partial_{t_0}^2 F + D(z_1)D(z_2) F
\]

(35)

\[
\log \left(1 - \frac{1}{w(z_1)w(z_2)}\right) = -D(z_1)D(z_2) F
\]

(36)

The limiting case of (35) as \( z_2 \to \infty \) gives another useful formula for the conformal map:

\[
w(z) = e^{-\frac{1}{2}\partial_{t_0}^2 F} (z - (\partial_{t_0} + D(z))\partial_1 F)
\]

(37)

All these formulas have the same form for any function \( \sigma \).

It is now straightforward to see how the Hirota equations come into play. They are obtained by plugging (3) into equalities of the type (35), (36), thus excluding \( w(z) \) from them. In this way one obtains eqs. (3), (4), in which one recognizes dispersionless limit of bilinear Hirota equations for the 2D Toda chain hierarchy. Another Hirota equation does not even require (3) for its derivation: take three copies of eq. (33) for each pair of the points \( z_1, z_2, z_3 \), take the exp-function of both sides of each equation and then sum them up. One arrives at the equation

\[
(z_1 - z_2)e^{D(z_1)D(z_2)F} + (z_2 - z_3)e^{D(z_2)D(z_3)F} + (z_3 - z_1)e^{D(z_3)D(z_1)F} = 0
\]

(38)

This is the dispersionless KP hierarchy (which is a holomorphic, with respect to the variables \( t_k \), sector of the Toda chain) written in the most symmetric form. Another version, known in the literature [14, 11], is obtained from here as \( z_3 \to \infty \).

The three equations, (8), (9), and (38) (along with their bar-versions) are to be understood as an infinite set of relations for second derivatives of \( F \). Altogether, they form the integrable hierarchy. These relations are obtained by expanding both sides in a power series in \( z_i \) and comparing the coefficients. For example, comparing leading terms in both sides of (3) as \( z_1, z_2 \to \infty \), one obtains the first equation of the hierarchy.

It is the dispersionless version of the Toda equation:

\[
\partial_{t_1 t_3}^2 F = e^{\partial_{t_0}^2 F} \phi = \partial_{t_0}^2 e^{\phi}.
\]

(46)

Next-to-leading terms give the equation

\[
\partial_{t_2 t_1}^2 F = 2\partial_{t_0}^2 F \partial_{t_1 t_1}^2 F.
\]

An important comment is in order. All the formulas for second derivatives of \( F \), obtained in this section, are of the same form for any function \( \sigma \), i.e., \( \sigma \) does not enter them explicitly. Indeed, consider, for instance, (33), (36). Their left hand sides, being expressed through the conformal map only, are determined nearly by the shape of the domain. For a given domain, they are the same for any density of the background charge. Clearly, this also holds true for the generating formula for second derivatives (7). Note that \( F \) itself does depend on the choice of \( \sigma \).
This remark can be reformulated as a covariance property of the second derivatives. Let \( \hat{F}, \hat{\ell}_k \) be the dependent and independent variables related to another density function, \( \hat{\sigma}(z, \bar{z}) \). As it follows from the above comment, second derivatives of \( F \) are covariant in the following sense:

\[
\frac{\partial^2 \hat{F}}{\partial t_j \partial t_k} = \frac{\partial^2 F}{\partial t_j \partial t_k}
\]

when the both sides are calculated for the same domain \( D_+ \). A similar formula for mixed \( t_i, \ell_k \)-derivatives also holds true. We stress that this covariance takes place only for second derivatives. For instance, as it is seen from (29), in general \( \partial_{\ell_j} \hat{F} \neq \partial_{\ell_j} F \), since the potential \( \hat{\Phi}^- \) does depend on \( \sigma \) for a given domain.

The Hirota equations are invariant with respect to varying the density \( \sigma \). At the same time, the particular solution describing the conformal maps and the Dirichlet problem is determined by the background charge. We conclude that the function \( \sigma \) parametrizes different solutions of the Hirota equations. We come back to this point again, within a different approach, in Sec. 5. Regarded as a function of all \( T_{mn} \) (see the end of Sec. 2), \( F \) solves the extended Toda chain hierarchy introduced in [13] in the context of the 2-matrix model.

A short comment on the bilinear Hirota equations and their dispersionless limit is in order. Let \( \hbar \) be a parameter. The equations are written for a \( \tau \)-function \( \tau_\hbar \) which depends on the times \( t_k \) and on the parameter \( \hbar \). The list of equations is parallel to the one for the dispersionless case. We use the same notation. Eqs. (38), (39) are replaced by

\[
z_1 \left( e^{\hbar (\partial_{\hbar_0} - D(z_1))} \tau_\hbar \right) \left( e^{-\hbar D(z_1)} \tau_\hbar \right) - z_2 \left( e^{\hbar (\partial_{\hbar_0} - D(z_2))} \tau_\hbar \right) \left( e^{-\hbar D(z_1)} \tau_\hbar \right) = 0
\]

\[
= (z_1 - z_2) \left( e^{-\hbar (D(z_1) + D(z_2))} \tau_\hbar \right) \left( e^{\hbar \partial_{\hbar_0} \tau_\hbar} \right)
\]

\[
\left( e^{-\hbar D(z_1)} \tau_\hbar \right) \left( e^{-\hbar D(z_2)} \tau_\hbar \right) - \tau_\hbar \left( e^{\hbar (D(z_2) - D(z_1))} \tau_\hbar \right) =
\]

\[
= (z_1 z_2)^{-1} \left( e^{-\hbar (\partial_{\hbar_0} + D(z_1))} \tau_\hbar \right) \left( e^{\hbar (\partial_{\hbar_0} + D(z_2))} \tau_\hbar \right)
\]

respectively. The dispersionful analog of eq. (38) is

\[
(z_1 - z_2) \left( e^{-\hbar (D(z_1) + D(z_2))} \tau_\hbar \right) \left( e^{-\hbar D(z_1)} \tau_\hbar \right) + \text{cyclic per-s of } z_1, z_2, z_3 = 0
\]

The operators \( e^{\hbar D(z)} \) are shift operators, so the above equations are difference ones. To make this explicit, introduce the vector fields \( \partial_{x_i} = -D(z_i) \). In the variables \( x_i \) the equations acquire a more familiar form. For example, eq. (12) reads

\[
(z_1 - z_2) \tau_\hbar (x_1 + \hbar, x_2 + \hbar, x_3) \tau_\hbar (x_1, x_2, x_3 + \hbar) + \text{cyclic per-s of } 1, 2, 3 = 0
\]

These bilinear difference equations for the \( \tau \)-function enjoy many remarkable properties. Eq. (12) first appeared in Hirota’s paper [15]. For a review of the difference Hirota equations see e.g. [16].

As is clear from these formulas, the parameter \( \hbar \) plays the role of lattice spacing. In the dispersionless limit, difference equations become differential ones. The limit is well-defined on the class of solutions such that there exists a finite limit

\[
F = \lim_{\hbar \to 0} \hbar^2 \log \tau_\hbar
\]

(44)
Rewriting the Hirota equations for $\log\tau_h$ and extracting the leading terms as $\hbar \to 0$, one arrives at the dispersionless Hirota equations for $F$. Different aspects of the mathematical theory of dispersionless integrable equations were discussed in [17, 18, 11, 14].

### 4 Integrable structure of conformal maps in the Takedasaki-Takebe formalism

The approach of the previous section is maybe the easiest way to see the link between Dirichlet problem and Hirota equations. However, such more customary attributes of integrability like Lax representation remain obscure, from this point of view. In this section we discuss an alternative approach which makes the integrable structure explicit via the Lax-type equations for the inverse conformal map, $z(w)$. In our normalization, the general form of the inverse map is

$$z(w) = rw + \sum_{j=0}^{\infty} u_j w^{-j}. \quad (45)$$

where $r$ is real positive. For $w$ on the unit circle, eq.(45) gives a parametrization of the curve.

We introduce one more new object. Let $S(z)$ be the analytic continuation of the function $\partial_z U$ away from the curve:

$$S(z) = \partial_z U(z, \bar{z}), \quad z \in \gamma \quad (46)$$

For analytic curves, $S(z)$ can be proved to be holomorphic in some strip-like neighbourhood of the curve $\gamma$. This function turns out to be a very useful object from technical point of view. We call it generalized Schwarz function, or simply Schwarz function for brevity. (The latter name is commonly used in the case $\sigma = 1$, see e.g. the book [19].) Comparing its definition with the definition of the moments through contour integrals (17), (19), one obtains the Laurent series representation of the Schwarz function:

$$S(z) = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1}. \quad (47)$$

We thus see that $S(z)$ is the generating function of all moments $t_k, v_k$. We regard it as a function of the moments $t_k$: $S(z) = S(t_k, z)$. Given the coefficients $T_{mn}$ (13) and the Schwarz function, the curve $\gamma$ is determined by eq. (46).

Following the lines of ref. [11], we introduce the dispersionless 2D Toda hierarchy starting with the following data. Suppose we are given with four Laurent series of the form

$$L = rw + \sum_{j=0}^{\infty} u_j w^{-j}, \quad M = \sum_{k=1}^{\infty} k t_k L^k + t_0 + \sum_{k=1}^{\infty} v_k L^{-k} \quad (48)$$

$$\bar{L} = r\bar{w}^{-1} + \sum_{j=0}^{\infty} \bar{u}_j w^j, \quad \bar{M} = \sum_{k=1}^{\infty} k \bar{t}_k \bar{L}^k + t_0 + \sum_{k=1}^{\infty} \bar{v}_k \bar{L}^{-k} \quad (49)$$
The coefficients \( t_0, t_k, \bar{t}_k \) are regarded as \textit{independent variables} while all other coefficients \( (r, u_j, \bar{u}_j, v_j, \bar{v}_j) \) are \textit{dependent variables}. Imposing some conditions on the Laurent series, we are going to study the latter as functions of the former.

Recall now the Takasaki-Takebe theorem \cite{14}. Let the Poisson bracket \{,\} be defined as

\[
\{f, g\} = w \frac{\partial f}{\partial w} \frac{\partial g}{\partial t_0} - w \frac{\partial g}{\partial w} \frac{\partial f}{\partial t_0} \quad (50)
\]

and let \( f, g \) be a canonical pair, \( \{f(w, t_0), g(w, t_0)\} = f(w, t_0) \). Suppose the four Laurent series \( L, \bar{L}, M, \bar{M} \) introduced above are connected by the functional relations

\[
\bar{L} = f^{-1}(L, M), \quad \bar{M} = g(L, M) \quad (51)
\]

Then the following is true:

(A) The pairs \( L, M \) and \( (\bar{L})^{-1}, \bar{M} \) are canonical: \( \{L, M\} = L, \{\bar{L}^{-1}, \bar{M}\} = \bar{L}^{-1} \);

(B) The four Laurent series satisfy the Lax-Sato equations

\[
\frac{\partial X}{\partial t_n} = \{H_n, X\}, \quad \frac{\partial X}{\partial \bar{t}_n} = -\{\bar{H}_n, X\} \quad (52)
\]

where \( X \) stands for any one of \( L, \bar{L}, M, \bar{M} \) and

\[
H_n = (L^n)_{\geq 1} + \frac{1}{2} (L^n)_0, \quad \bar{H}_n = (\bar{L}^n)_{\leq -1} + \frac{1}{2} (\bar{L}^n)_0, \quad n \geq 1 \quad (53)
\]

(C) There exists a function \( F \) of \( t_k, \bar{t}_k, \) and \( t_0 \) such that \( v_k = \partial_{t_k} F, \bar{v}_k = \partial_{\bar{t}_k} F \), and the function \( F \) obeys the dispersionless Hirota equations.

This theorem is proved in \cite{14}. The proof of (A) consists in differentiating \((51)\) with respect to \( w, t_0 \) and combining the results. Taking derivatives of \((51)\) with respect to \( t_n, \bar{t}_n \) and using (A), we obtain (B), where \((...)_{\geq 1} \ ( ...)_{\leq -1} \) means the part of the Laurent series with strictly positive (strictly negative) powers of \( w \), and the constant term is denoted by \((...)_0 \). To prove (C), one should take \( \partial M / \partial t_k \) at fixed \( w \). Using the Lax-Sato equations \((52)\), one gets \( \partial_k v_k = (wL^k dH_j)_0 \). Further, this representation, together with \((53)\), implies the symmetry \( \partial_j v_k = \partial_k v_j \), which, in its turn, implies the existence of a common potential function for \( v_k \). The proof of the Hirota equations for \( F \) from these data is implicit in \cite{14}. More explicit arguments are given in \cite{14} (see also \cite{3}).

It then follows that \( L \) and \( \bar{L} \) are the two Lax functions of the dispersionless Toda hierarchy. Equations of the hierarchy are written for the “potentials” \( r, \{u_n\}, \{\bar{u}_n\} \) as functions of the times \( t_0, \{t_n\}, \{\bar{t}_n\} \). These times appear explicitly in the Laurent series \( M, \bar{M} \) which are the Orlov-Shulman functions \cite{20} of the hierarchy. The Takasaki-Takebe theorem says that solutions to the hierarchy are in one-to-one correspondence with canonical pairs of functions \( f, g \).

Remarkably, this formalism turned out to be very convenient for describing the integrable structure of conformal maps. The identification of the objects related to conformal maps with the above Laurent series goes as follows:

\[
L(w) = z(w), \quad \bar{L}(w) = \bar{z}(w^{-1}), \quad M(L) = zS(z), \quad \bar{M}(\bar{L}) = \bar{z}\bar{S}(\bar{z})
\]

13
(for a series \( f(z) = \sum f_j z^j \) we write \( \bar{f}(z) = \sum \bar{f}_j z^j \)). Together with \( S(z) = \partial_z U(z, \bar{z}) \) this means that we have the following functional relations between \( L, \bar{L}, M, \bar{M} \):

\[
M = L \frac{\partial U(L, \bar{L})}{\partial L}, \quad \bar{M} = \bar{L} \frac{\partial U(L, \bar{L})}{\partial \bar{L}} \tag{54}
\]

Notice that (54) gives an equivalent description of the canonical pair from the Takasaki-Takebe theorem. Namely, (54) tells us that \( U(L, \bar{L}) \) is generating function of the corresponding canonical \( f, g \)-pair. Indeed, let in general \( M = F(L, \bar{L}), \bar{M} = G(L, \bar{L}) \), and find the Poisson bracket between \( \bar{L}^{-1} \) and \( \bar{M} \). For brevity we write \( \bar{L} = \varphi(L, M) \), where \( \varphi \) is a function implicitly determined by the equality \( M = F(L, \bar{L}) \). We have:

\[
\frac{\partial \bar{L}}{\partial M} \frac{\partial \bar{M}}{\partial L} - \frac{\partial \bar{L}}{\partial \bar{M}} \frac{\partial \bar{M}}{\partial \bar{L}} = \frac{\partial \varphi}{\partial L} \frac{\partial G}{\partial L} - \frac{\partial \varphi}{\partial \bar{M}} \frac{\partial G}{\partial \bar{M}} = - \frac{\partial \varphi}{\partial M} \frac{\partial G}{\partial \bar{L}}
\]

Therefore,

\[
\{ \bar{L}^{-1}, \bar{M} \} = L \bar{L}^{-2} \frac{\partial \varphi}{\partial \bar{M}} \frac{\partial G}{\partial \bar{L}} = \frac{L}{\bar{L}^2} \frac{\partial G}{\partial \bar{L}},
\]

where the last equality follows from \( \partial \varphi/\partial M = (\partial \bar{L}/\partial M)_{L=\text{const}} = (\partial F/\partial \bar{L})^{-1} \). Now, on taking \( F, G \) from (54), we get \( \{ \bar{L}^{-1}, \bar{M} \} = \bar{L}^{-1} \), i.e. the transformation (54) is indeed canonical.

The Poisson bracket \( \{ L, \partial_L U(L, \bar{L}) \} = 1 \) follows from \( \{ L, M \} = L \). This can be rewritten as

\[
\{ L, \bar{L} \} = \frac{1}{\sigma(L, \bar{L})} \tag{55}
\]

which is one of the forms of the so-called string equation.

## 5 Conclusion

We have demonstrated that some classical problems of complex analysis have an integrable structure which is the same as the one arising in the dispersionless limit of soliton equations. Given a simply connected domain in the complex plane and an auxiliary real-analytic function \( \sigma(z, \bar{z}) \) defined all over the complex plane, we have constructed a solution to the dispersionless 2D Toda chain hierarchy. The independent flows of the hierarchy are moments of the domain, or multipole coefficients, defined with respect to the function \( \sigma \) which in the context of potential theory plays the role of density of background charge. The set of dependent variables is then the complimentary set of multipole coefficients. The energy of the background charge in the domain, regarded as a function of the moments, is (logarithm of) the \( \tau \)-function of the hierarchy. This function obeys the dispersionless Hirota equations. Taking derivatives of this function with respect to the moments, one obtains formal solutions to the conformal mapping problem, the Dirichlet boundary problem, and to the 2D inverse potential problem. An interesting question, to be discussed elsewhere, is an inverse problem connected with the scheme outlined above: given a solution to the dispersionless Hirota equations, associate with it a boundary or conformal mapping problem, and to restore the function \( \sigma \). Such an interpretation may be helpful for better understanding dispersionless integrable hierarchies.
Acknowledgements

The author is grateful to the organizers of the Protvino conference “Classical and quantum integrable systems” for the opportunity to present these results, and to I.Krichever, A.Marshakov, M.Mineev-Weinstein and P.Wiegmann for useful discussions. This work was supported in part by CRDF grant RP1-2102, by grant INTAS-99-0590 and by RFBR grant 00-02-16477.

Appendix

In the appendix we give a direct proof of the formulas (25) for first derivatives of the function \( F \) with respect to \( t_k \). We use the notation of Sec. 2.

Consider the variation \( \delta_k F \) \((k > 0)\) of the function \( \delta_k F = \frac{1}{2\pi} \int_{D_+} \tilde{\Phi}(z, \bar{z}) \sigma(z, \bar{z}) d^2z \) under a small change \( t_k \to t_k + \delta t_k \): \( F \to F + \delta_k F \). All other moments \( t_j \) with \( j \neq k \), \( j \geq 0 \), are kept constant. We have:

\[
\delta_k F = \frac{1}{2\pi} \int_{D_+} \delta_k \tilde{\Phi}^+(z, \bar{z}) \sigma(z, \bar{z}) d^2z + \frac{1}{2\pi} \int_{\delta_k D_+} \tilde{\Phi}(z, \bar{z}) \sigma(z, \bar{z}) d^2z
\]

where \( \delta_k D_+ \) is the variation of the domain due to the change of \( t_k \)'s. (To be more precise, \( \int_{\delta_k D_+} \) stands for the difference \( \int_{D_+, (t_k + \delta t_k)} - \int_{D_+, (t_k)} \).) By \( I_1 \) and \( I_2 \) we denote the first and the second terms of this formula. \( I_1 \) is easy to calculate. Since \( \delta_k \tilde{\Phi}^+(z, \bar{z}) = z^k \delta t_k + \bar{z}^k \delta \bar{t}_k \), the first term is

\[
I_1 = \frac{1}{2\pi} \int_{D_+} \delta_k \tilde{\Phi}^+(z, \bar{z}) \sigma(z, \bar{z}) d^2z = \frac{1}{2} (v_k \delta t_k + \bar{v}_k \delta \bar{t}_k)
\]

Let us turn to the second one, \( I_2 = \frac{1}{2\pi} \int_{\delta_k D_+} \tilde{\Phi}(z, \bar{z}) \sigma(z, \bar{z}) d^2z \). Since the integral goes over a small neighbourhood of the curve \( \gamma = \partial D_+ \), we can substitute \( \tilde{\Phi} \) by \( \tilde{\Phi}^- \), which, by virtue of (20), coincides in this neighbourhood with \( \tilde{\Phi}^+ \) up to second order terms. Recall that \( \tilde{\Phi}^-(z, \bar{z}) = v_0 + 2 \Re \omega(z) \), where the function

\[
\omega(z) = \sum_{k \geq 1} \frac{v_k}{k} z^{-k}
\]

is holomorphic in \( D_- \) and regular at infinity. Under our assumptions, all singularities of \( \omega \) are in a domain \( B \subset \subset D_+ \) such that \( R = D_+ \setminus B \) is a strip-like domain. In other words, \( \omega(z) \) can be analytically extended across \( \gamma \) to the strip-like domain \( R \). Moreover, recall that we are working in the space of analytic curves. It can be proved that for curves that belong to a small but finite neighbourhood of the point in this space that corresponds to a given curve, the domain \( B \) can be chosen to be independent of \( t_k \).

We can write, in the first order in \( \delta t_k \):

\[
I_2 = \frac{v_0}{2\pi} \int_{\delta_k D_+} \sigma(z, \bar{z}) d^2z + \frac{1}{\pi} \Re \int_{\delta_k R} \omega(z) \sigma(z, \bar{z}) d^2z
\]
The first integral in the right hand side is the variation of \( t_0 \), and, therefore, equals zero as soon as \( t_0 \) is kept constant. So, we are left with the second one,

\[
I_2 = \frac{1}{\pi} \text{Re} \int_{D_+(t_k+\delta t_k)\backslash B} \omega(z)\sigma(z,\bar{z})d^2z - \frac{1}{\pi} \text{Re} \int_{D_+(t_k)\backslash B} \omega(z)\sigma(z,\bar{z})d^2z =
\]

\[
= \text{Re} \oint_{\partial R(t_k+\delta t_k)} \frac{dz}{2\pi i} \omega(z)\partial_z U(z,\bar{z}) - \text{Re} \oint_{\partial R(t_k)} \frac{dz}{2\pi i} \omega(z)\partial_z U(z,\bar{z})
\]

The second line follows from the Stokes formula. The boundary of \( R \) consists of two curves: inner and outer ones. The inner boundary is the same in both integrals, so its contribution cancel. The outer boundary is \( \gamma(t_k+\delta t_k) \) in the first integral and \( \gamma(t_k) \) in the second one. Therefore, using the definition of the generalized Schwarz function (46), we can write:

\[
I_2 = \text{Re} \oint_{\gamma(t_k+\delta t_k)} \frac{dz}{2\pi i} \omega(z)S(t_k+\delta t_k,z) - \text{Re} \oint_{\gamma(t_k)} \frac{dz}{2\pi i} \omega(z)S(t_k,z)
\]

Since \( S(z) \) is holomorphic in \( R \), the contour can be taken to be \( \gamma = \gamma(t_k) \) in both integrals. Thus we obtain, by virtue of (47):

\[
I_2 = \text{Re} \left( \delta t_k \oint \frac{dz}{2\pi i} \omega(z) \frac{\partial S(z)}{\partial t_k} \right) = \frac{1}{2} (v_k \delta t_k + \bar{v}_k \delta \bar{t}_k)
\]

Summing up the contributions \( I_1 \) and \( I_2 \), we get \( \delta_k F = v_k \delta t_k + \bar{v}_k \delta \bar{t}_k \), which is the desired result for \( k \geq 1 \). The calculation of the \( t_0 \)-derivative can be done in a similar way.

References

[1] M.Mineev-Weinstein, P.B.Wiegmann and A.Zabrodin, Phys. Rev. Lett. 84 (2000) 5106-5109

[2] P.B.Wiegmann and A.Zabrodin, Commun. Math. Phys. 213 (2000) 523-538

[3] I.Kostov, I.Krichever, M.Mineev-Weinstein, P.B.Wiegmann and A.Zabrodin, \( \tau \)-function for analytic curves, [nep-th/0005259]

[4] A.Hurwitz and R.Courant, Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. Herausgegeben und ergänzt durch einen Abschnitt über geometrische Funktionentheorie, Springer-Verlag, 1964 (Russian translation, adapted by M.A.Evgrafov: Theory of functions, Nauka, Moscow, 1968)

[5] E.Hille, Analytic function theory, v.II, Ginn and Company, 1962

[6] L.Zalcman, Contemp. Math. 63 (1987) 337-349

[7] L.Takhtajan, Free bosons and tau-functions for compact Riemann surfaces and closed smooth Jordan curves. I. Current correlation functions, e-Print Archive: [math.QA/0102164]
[8] I.Krichever, Funk. Anal. i ego Pril. 22:3 (1988) 37-52 (English translation: Funct. Anal. Appl. 22 (1989) 200-213); 
I.Krichever, Comm. Pure Appl. Math. 47 (1992) 437-476

[9] B.A.Dubrovin and S.P.Novikov, Soviet Math. Dokl. 27 (1983) 665-669

[10] S.P.Tsarev, Soviet Math. Dokl. 31 (1985) 488-491

[11] K.Takasaki and T.Takebe, Rev. Math. Phys. 7 (1995) 743-808

[12] P.S.Novikov, C.R. (Dokl.) Acad. Sci. URSS (N.S.) 18 (1938) 165-168; 
M.Sakai, Proc. Amer. Math. Soc. 70 (1978) 35-38; 
V.Strakhov and M.Brodsky, SIAM J. Appl. Math. 46 (1986) 324-344

[13] L.Bonora and C.S.Xiong, Nucl. Phys. B 344 (1995) 408-444

[14] R.Carroll and Y.Kodama, J. Phys. A: Math. Gen. A28 (1995) 6373-6388

[15] R.Hirota, J. Phys. Soc. Japan 50 (1981) 3785-3791

[16] A.Zabrodin, Teor. Mat. Fiz. 113 (1997) 179-230 (English translation: Theor. Math. Phys. 113 (1997) 1347-1392, solv-int/9704001)

[17] J.Gibbons and Y.Kodama, Phys. Lett. 135 A (1989) 167-170; 
J.Gibbons and Y.Kodama, Proceedings of NATO ASI, 'Singular Limits of Dispersive Waves' ed. N.Ercolani, Plenum 1994.

[18] B.A.Dubrovin, Commun. Math. Phys. 145 (1992) 195-207; 
B.Dubrovin, In 'Montecatini Terme 1993, Integrable systems and quantum groups' 120-348, e-Print Archive: hep-th/9407018

[19] P.J.Davis, The Schwarz function and its applications, The Carus Mathematical Monographs, No. 17, The Math. Assotiation of America, Buffalo, N.Y., 1974

[20] A.Orlov and E.Shulman, Lett. Math. Phys. 12 (1986) 171-179