LICHNEROWICZ-TYPE EQUATIONS ON COMPLETE MANIFOLDS

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Abstract. Under appropriate spectral assumptions we prove two existence results for positive solutions of Lichnerowicz-type equations on complete manifolds. We also give a priori bounds and a comparison result that immediately yields uniqueness for certain classes of solutions. No curvature assumptions are involved in our analysis.

1. Introduction

In the analysis of Einstein field equations in General Relativity the initial data set for the non-linear wave system plays an essential role. These initial data have to satisfy the Einstein constraint conditions that can be expressed in a geometric form as follows. Let \((M, \hat{g})\) be a Riemannian manifold and \(\hat{K}\) a symmetric 2-covariant tensor on \(M\). Then \((M, \hat{g})\) is said to satisfy the Einstein constraint equations with non-gravitational energy density \(\hat{\rho}\) and non-gravitational momentum density \(\hat{J}\) if

\[
\begin{aligned}
\left| \nabla_{\hat{g}} \hat{K} \right|^2 - \left( \text{tr}_{\hat{g}} \hat{K} \right)^2 &= S_{\hat{g}} - \hat{\rho} \\
\text{div}_{\hat{g}} \hat{K} - \nabla_{\hat{g}} \hat{\rho} &= \hat{J}.
\end{aligned}
\]

Here \(S_{\hat{g}}\) stands for the scalar curvature of the metric \(\hat{g}\). To look for solutions of (1.1) using the conformal method introduced by Lichnerowicz in [16] means that we generate an initial data set \((M, \hat{g}, \hat{K}, \hat{\rho}, \hat{J})\) satisfying (1.1) by first choosing the following conformal data: A Riemannian manifold \((M, \langle , \rangle)\); a symmetric 2-covariant tensor \(\sigma\) required to be traceless and transverse with respect to \(\langle , \rangle\), that is, for which \(\text{tr}_{\langle , \rangle} \sigma = 0\) and \(\text{div}_{\langle , \rangle} \sigma = 0\); a scalar function \(\tau\) (that will play the role of a non normalized mean curvature); a non-negative scalar function \(\rho\) and a vector field \(J\). Then one looks for a positive function \(u\) and a vector field \(W\) that solve the conformal constraint equations

\[
\begin{aligned}
\Delta u - c_m S u + c_m \left| \sigma + \mathcal{L} W \right|^2 u^{-N-1} - b_m \tau^2 u^{N-1} + c_m \rho u^{-N/2} &= 0 \\
\Delta_{\hat{g}} W + \frac{m-1}{m} u^N \nabla \tau + J &= 0.
\end{aligned}
\]

Where \(\Delta, S, \text{ and } |\cdot|\) denote respectively the Laplace-Beltrami operator, the scalar curvature, and the norm in the metric \(\langle , \rangle\). The operator \(\mathcal{L}\) is the traceless Lie derivative, that is, in a local orthonormal coframe

\[
\left( \mathcal{L} W \right)_{ij} = W_{ij} - W_{ji} - \frac{2}{m} \left( \text{div} W \right) \delta_{ij}
\]

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and $\Delta_L = \text{div} \circ \mathcal{L}$ is the vector laplacian. The constants appearing in (1.2) are respectively given by

$$N = \frac{2m}{m-2}, \quad c_m = \frac{m-2}{4(m-1)}, \quad b_m = \frac{m-2}{4m},$$

in particular we note that $N$ is the critical Sobolev exponent. If $(u, W)$ is a solution of (1.2) then the initial data set

$$\hat{g} = u^{-\frac{2}{m-2}} \langle , \rangle, \quad \hat{\rho} = u^{-\frac{3N+2}{2}} \rho, \quad \hat{J} = u^{-N} J,$$

$$\hat{K}^{ij} = u^{-\frac{2m+2}{m-2}} \left( \sigma + \mathcal{L}W \right)^{ij} + \frac{\tau}{m} u^{-\frac{4}{m-2}} \delta^{ij}$$

satisfies the Einstein constraints (1.1). For further informations on the physical content of the system (1.2) we refer to the recent surveys [4], [10], and the references therein.

The aim of this paper is to study the existence, $a$ priori bounds, and uniqueness of positive solutions of the Lichnerowicz-type equation

$$(1.3) \quad \Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0$$

with, at least, continuous coefficients, $a(x)$, $b(x)$, $c(x)$, $\sigma > 1$, $\tau < 1$, and with the sign restrictions

$$(1.4) \quad b(x) \geq 0, \quad c(x) \geq 0,$$

on a complete, non compact, connected manifold $(M, \langle , \rangle)$. Equation (1.3) is of the same form of the scalar equation of system (1.2) in case $\rho \equiv 0$, the coefficients $b(x)$ and $c(x)$ corresponding respectively to

$$b_m \tau^2 \quad \text{and} \quad c_m |\sigma + \mathcal{L}W|_g^2.$$ 

This latter fact justifies the sign condition (1.4). In recent years this type of equations has been studied by many authors, see for instance [17], [8], [18], and [11]. In the present work we significantly generalize many of the results obtained in the aforementioned papers.

We now introduce some notations and state our main existence results. Let $a(x) \in C^0(M)$ and $L = \Delta + a(x)$. If $\Omega$ is a non-empty open set, the first Dirichlet eigenvalue $\lambda^L_1(\Omega)$ is variationally characterized by means of the formula

$$(1.5) \quad \lambda^L_1(\Omega) = \inf \left\{ \int\nabla \varphi^2 - a(x) \varphi^2 : \varphi \in W^{1,2}_0(\Omega), \int\varphi^2 = 1 \right\}.$$

We recall that, if $\Omega$ is bounded and $\partial \Omega$ is sufficiently regular, the infimum is attained by the unique normalized eigenfunction $v$ on $\Omega$ satisfying

$$(1.6) \begin{cases} \Delta v + a(x)v + \lambda^L_1(\Omega)v = 0 & \text{on } \Omega \\ v = 0 & \text{on } \partial \Omega \\ v > 0 & \text{on } \Omega. \end{cases}$$

We then define the first eigenvalue of $L$ on $M$ as

$$\lambda^L_1(M) = \inf_{\Omega} \lambda^L_1(\Omega).$$
where $\Omega$ runs over all bounded domains of $M$. Observe that, due to the monotonicity of $\lambda^L_1$ with respect to the domain, that is

$$\Omega_1 \subseteq \Omega_2 \text{ implies } \lambda^L_1(\Omega_1) \geq \lambda^L_1(\Omega_2),$$

we have

$$\lambda^L_1(M) = \lim_{r \to +\infty} \lambda^L_1(B_r)$$

where, from now on, $B_r$ denotes the geodesic ball in the complete manifold $(M, \langle , \rangle)$ of radius $r$ centered at a fixed origin $o \in M$. Note that in case $\Omega_2 \setminus \Omega_1$ has non-empty interior the inequality in (1.7) becomes strict.

We need to extend definition (1.5) to an arbitrary bounded subset $B$ of $M$. We do this by setting

$$\lambda^L_1(B) = \sup_{\Omega} \lambda^L_1(\Omega)$$

where the supremum is taken over all open bounded sets $\Omega$ with smooth boundary such that $B \subseteq \Omega$. Observe that, by definition, if $B = \emptyset$ then $\lambda^L_1(B) = +\infty$.

We would like to remark that since the first Dirichlet eigenvalue for the Laplacian of a ball $B_r$ grows like $r^{-2}$ as $r \to 0^+$, $\lambda^L_1(B_r) \geq 0$ provided $r$ is sufficiently small and one may think of $\lambda^L_1(B) > 0$ as a condition expressing the fact that $B$ is small in a spectral sense. Of course, this condition also depends on the behaviour of $a(x)$ therefore small in a spectral sense does not necessarily mean, for instance, small in a Lebesgue measure sense. This is clear if $a(x) \leq 0$ because $\lambda^L_1(M) \geq 0$ in any complete manifold $(M, \langle , \rangle)$ so that in this case $\lambda^L_1(M) \geq 0$ and thus $\lambda^L_1(B) > 0$ on any bounded set $B \subset M$.

The main results of the paper are the following two existence theorems for positive solutions of equation (1.3).

**Theorem A.** Let $a(x), b(x), c(x) \in C^0_{\text{loc}}(M)$ for some $0 < \alpha \leq 1$. Assume (1.4) and suppose that $b(x)$ is strictly positive outside some compact set. Let

$$B_0 = \{x \in M : b(x) = 0\}$$

and suppose

$$\lambda^L_1(B_0) > 0$$

with $L = \Delta + a(x)$. Assume that

$$\lambda^L_1(M) < 0,$$

then (1.3) has a maximal positive solution $u \in C^2(M)$.

In this case maximal means that if $0 < w \in C^2(M)$ is a second solution of (1.3), then $w \leq u$. In the same vein we have the following result, where the spectral condition (1.12) is substituted by a spectral smallness requirement on the zero set of the coefficient $c(x)$ and a pointwise control on the coefficients. Recall that given the real function $\alpha(x)$, its positive and negative part are respectively defined by

$$\alpha_+(x) = \max \{\alpha(x), 0\},$$

$$\alpha_-(x) = -\min \{\alpha(x), 0\}.$$
Theorem B. Let \( a(x), b(x), c(x) \in C_{0,\alpha}^{0}(M) \) for some \( 0 < \alpha \leq 1 \). Assume the validity of (1.4) and let
\[
B_0 = \{x \in M : b(x) = 0\}, \quad C_0 = \{x \in M : c(x) = 0\},
\]
and
\[
\lambda_{1}^{\Delta+a}(B_0) > 0.
\]
Suppose that there exist two bounded open sets \( \Omega_1, \Omega_2 \) such that \( C_0 \subset \Omega_1 \subset \subset \Omega_2 \),
\[
\sup_{M \setminus \Omega_1} \frac{a_-(x) + b(x)}{c(x)} < +\infty,
\]
and
\[
\lambda_{1}^{\Delta-a}(\Omega_2) > 0.
\]
Then (1.3) has a maximal positive solution \( u \in C^{2}(M) \).

The proof of the theorems is the content of Section 2. These existence results should be compared with those obtained at the end of the very recent paper [18]. The remaining two sections of the paper are devoted to uniqueness of solutions. In particular, the results of Section 3 should be interpreted as Liouville-type theorems and compared with those obtained in [17], [8], and [11]. The main differences with previous work in the literature is that our geometric requirement on the manifold consist only in a mild volume growth assumption for geodesic balls and in the fact that we allow for non constant coefficients \( a(x), b(x), c(x) \) in equation (1.3). In this last setting in general there are no trivial solutions at hand. Thus, to provide a complete analysis of the problem in this case, we need to find an a priori estimate and use it to detect a trivial solution of (1.3). In particular, Corollary 3.12 is our main Liouville-type theorem.

In Section 4 we analyze another uniqueness result, this time under a spectral assumption on the manifold, in the spirit of the very recent [5] and [6].

2. PROOF OF THEOREMS A AND B

The main result of the paper is in fact the following proposition, whose proof will be the content of the section. At the end we will prove Theorems A and B as corollaries of the proposition.

Proposition 2.1. Let \( a(x), b(x), c(x) \in C_{0,\alpha}^{0}(M) \) for some \( 0 < \alpha \leq 1 \). Assume (1.4) and suppose that \( b(x) \) is strictly positive outside some compact set. Furthermore, suppose
\[
\lambda_{1}^{L}(B_0) > 0
\]
with \( L = \Delta + a(x) \). If \( 0 < u_- \in C^0(M) \cap W^{1,2}_{loc}(M) \) is a global subsolution of (1.3) on \( M \), then (1.3) has a maximal positive solution \( u \in C^{2}(M) \).

The proof of Proposition 2.1 is divided into several steps. In what follows we keep the notations of the proposition.

Lemma 2.2. Let \( \overline{a}(x), b(x), c(x) \in C_{0,\alpha}^{0}(M) \) for some \( 0 < \alpha \leq 1 \), and let (1.4) hold. Suppose that \( B_0 \) is bounded and
\[
\lambda_{1}^{\overline{a}}(B_0) > 0
\]
where $\mathcal{L} = \Delta + \varpi(x)$. If $\Omega$ is a bounded open set such that $B_0 \subset \Omega$, then there exists $v_+$ solution of
\begin{equation}
\begin{cases}
\Delta v_+ + \varpi(x)v_+ - b(x)v_+^\tau + c(x)v_+^\tau \leq 0 & \text{on } \Omega \\
v_+ > 0 & \text{on } \Omega.
\end{cases}
\end{equation}

Proof. Let $D$ and $D'$ be bounded open domains such that $B_0 \subset \subset D' \subset \subset D \subset \subset \Omega$, and $\lambda_1^T(D) > 0$. Let $u_1$ be a positive solution of 
\begin{equation}
\begin{cases}
\Delta u_1 + \varpi(x)u_1 + \lambda_1^T(D)u_1 = 0 & \text{on } D \\
u_1 = 0 & \text{on } \partial D.
\end{cases}
\end{equation}
Since $b(x) > 0$ on $M \setminus B_0$ and $\Omega \setminus D' \subset \subset M \setminus B_0$, 
$$
\beta = \inf_{\Omega \setminus D'} b(x) > 0.
$$
Define 
\begin{equation}
\alpha = \sup_{\Omega \setminus D'} \varpi(x), \quad \delta = \sup_{\Omega \setminus D'} c(x),
\end{equation}
and note that $\alpha, \delta < +\infty$ since $\Omega$ is bounded. Let $U$ be a positive constant. Then 
\begin{equation}
\Delta U + \varpi(x)U - b(x)U^\sigma + c(x)U^\tau = U \left( \varpi(x) - b(x)U^{\sigma-1} + c(x)U^{\tau-1} \right)
\leq U \left( \alpha - \beta U^{\sigma-1} + \delta U^{\tau-1} \right)
\end{equation}
on $\Omega \setminus D'$. We observe that the RHS of the above is non-positive for $U$ sufficiently large, say 
\begin{equation}
U \geq \Lambda_0 > 0.
\end{equation}
Next we choose a cut-off function $\psi \in C_0^\infty(M)$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $D'$, and $\text{supp } \psi \subset D$. Fix a positive constant $\gamma$ and define 
\begin{equation}
u = \gamma \left( \psi u_1 + (1 - \psi)\Lambda_0 \right).
\end{equation}
Since $b(x) \geq 0$ and $\lambda_1^T(D) > 0$, on $\overline{D'}$ we have 
\begin{equation}
\mathcal{L}u - b(x)u^\sigma + c(x)u^\tau = \mathcal{L}(\gamma u_1) - b(x)(\gamma u_1)^\sigma + c(x)(\gamma u_1)^\tau
= - \left[ \lambda_1^T(D)(\gamma u_1) + b(x)(\gamma u_1)^\sigma - c(x)(\gamma u_1)^\tau \right]
= - (\gamma u_1) \left[ \lambda_1^T(D) + b(x)(\gamma u_1)^{\sigma-1} - c(x)(\gamma u_1)^{\tau-1} \right].
\end{equation}
For the RHS of the above to be non-positive on $\overline{D'}$ it is sufficient to have 
\begin{equation}
c(x)(\gamma u_1)^{\tau-1} \leq \lambda_1^T(D) \quad \text{on } \overline{D'}.
\end{equation}
Towards this aim we note that, since $\overline{D'}$ is compact, $u_1 > 0$ on $\overline{D'}$, and $\tau < 1$, then (2.6) is satisfied for 
\begin{equation}
\gamma \geq \Gamma_0 = \Gamma_0(u_1) > 0,
\end{equation}
sufficiently large. We now consider $\Omega \setminus D$, since $\text{supp } \psi \subset D$, it follows that $u = \gamma \Lambda_0$ there. Thus, using $\Omega \setminus D \subset \Omega \setminus D'$, from (2.4) it follows that 
\begin{equation}
\Delta u + \varpi(x)u - b(x)u^\sigma + c(x)u^\tau \leq 0 \quad \text{on } \Omega \setminus D,
\end{equation}
and $u \equiv \Lambda_0$ on $\partial D \setminus \partial D'$.
is satisfied if we choose $\gamma \geq 1$, indeed in this case
\[
\gamma \Lambda_0 \geq \Lambda_0.
\]
It remains to analyze the situation on $D \setminus \overline{D'}$. First of all we note that, by standard elliptic regularity theory, $u_1 \in C^2(D)$. Thus, since $\text{supp} \psi \subset D$, it follows that $u \in C^2(\Omega)$, in particular this implies that there exists a positive constant $C_0$ such that
\[
(\Delta + \pi(x))u \leq \gamma C_0 \quad \text{on} \quad D \setminus \overline{D'}.
\]
Thus on $D \setminus \overline{D'}$ we have
\[
\Delta u + \pi(x) u - b(x)u^\sigma + c(x)u^\tau \leq \gamma C_0 - b(x)\gamma^\sigma (\psi u_1 + (1 - \psi) \Lambda_0)^\sigma + c(x)\gamma^\tau (\psi u_1 + (1 - \psi) \Lambda_0)^\tau.
\]
Now there exists constants $\varepsilon$ and $E$ such that
\[
\inf_{D \setminus \overline{D'}} b(x) (\psi u_1 + (1 - \psi) \Lambda_0)^\sigma = \varepsilon > 0,
\]
\[
\sup_{D \setminus \overline{D'}} c(x) (\psi u_1 + (1 - \psi) \Lambda_0)^\tau = E < +\infty.
\]
Therefore, on $D \setminus \overline{D'}$
\[
\Delta u + \pi(x) u - b(x)u^\sigma + c(x)u^\tau \leq \gamma (C_0 - \varepsilon \gamma^{\sigma - 1} + E \gamma^{\tau - 1}).
\]
Since $\sigma > 1$ and $\tau < 1$, it follows that there exists a positive constant $\Gamma_1$ depending only on $D$ and $D'$ such that
\[
C_0 - \varepsilon \gamma^{\sigma - 1} + E \gamma^{\tau - 1} \leq 0
\]
for $\gamma \geq \Gamma_1$.

Thus, by choosing
\[
\gamma \geq \max \{1, \Gamma_0, \Gamma_1\}
\]
u is the desired supersolution $v_+$ of (2.3) on $\Omega$.

**Definition 2.3.** We say that the property $(\Sigma)$ holds on $M$ (for equation (1.3)) if there exists $R_0 \in \mathbb{R}^+$ such that for all $R \geq R_0$ there exists a solution $\varphi \in C^0(\overline{B_R}) \cap W^{1,2}_{\text{loc}}(B_R)$ of
\[
(2.8) \quad \begin{cases}
\Delta \varphi + a(x)\varphi - b(x)\varphi^\sigma + c(x)\varphi^\tau \geq 0 & \text{on} \quad B_R \\
\varphi \geq 0 & \text{on} \quad B_R.
\end{cases}
\]

When $\tau < 0$, in order to avoid singularities, in the equation above it is assumed that $\text{supp} c(x) \subset \text{supp} \varphi$.

In Proposition 2.5 below we shall give some sufficient conditions for the validity of property $(\Sigma)$.

**Lemma 2.4.** Let $a(x), b(x), c(x) \in C^0_{\text{loc}}(M)$ for some $0 < \alpha \leq 1$, and let (1.4) hold. Suppose that $B_0$ is bounded and $\lambda_1^L(B_0) > 0$ with $L = \Delta + a(x)$. Furthermore assume that property $(\Sigma)$ holds on $M$. Let $\Omega$ be a bounded domain such that $B_0 \subset \Omega$.

Then, for each $n \in \mathbb{N}$, there exists a solution $u$ of the problem
\[
(2.9) \quad \begin{cases}
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 & \text{on} \quad \Omega \\
u > 0 & \text{on} \quad \Omega \\
u = n & \text{on} \quad \partial \Omega.
\end{cases}
\]
Proof. By the definition of $\lambda_1^+(B_0)$ and the assumption of positivity, there exists an open domain $D$ with smooth boundary such that $B_0 \subset D \subset \Omega$ and $\lambda_1^+(D) > 0$. Let $\rho \in C^\infty(M)$ be a cut-off function such that $0 \leq \rho \leq 1$, $\rho \equiv 1$ on $D$, $\rho \equiv 0$ on $M \setminus \Omega$. Fix $N \geq \max \{ \sup_{\Omega} |a(x)| + 1, \lambda_1^+(M \setminus \Omega) + 1 \}$.

Define
$$\overline{\pi}(x) = \rho(x)a(x) + N(1 - \rho(x))$$
and consider the operator $\overline{L} = \Delta + \overline{\pi}(x)$. Since $\overline{\pi}(x) = a(x)$ on $D$,
$$\lambda_1^+(B_0) = \lambda_1^+(B_0) > 0.$$  
(2.10)

Furthermore, from $N \geq \lambda_1^+(M \setminus \Omega) + 1$, we deduce $\lambda_1^+(M \setminus \Omega) \leq -1$ and it follows that there exists $R > 0$ sufficiently large such that
$$\Omega \subset B_R \quad \text{and} \quad \lambda_1^+(B_R) < 0.$$  
(2.11)

Fix $\varepsilon > 0$. Then $\lambda_1^+(B_{R+\varepsilon}) < 0$. Let $\varphi$ be the normalized eigenfunction of $\overline{L}$ on $B_{R+\varepsilon}$ relative to the eigenvalue $\lambda_1^+(B_{R+\varepsilon})$ (here, without loss of generality, that is, possibly substituting $B_{R+\varepsilon}$ with a slightly larger open set with smooth boundary, we are supposing $\partial B_{R+\varepsilon}$ smooth) so that

$$\begin{cases} \overline{L}\varphi + \lambda_1^+(B_{R+\varepsilon})\varphi = 0 & \text{on } B_{R+\varepsilon} \\ \varphi \equiv 0 & \text{on } \partial B_{R+\varepsilon} \\ \varphi > 0 & \text{on } B_{R+\varepsilon} \\ \|\varphi\|_{L^2(B_{R+\varepsilon})} = 1 \end{cases}$$

(2.12)

We fix $\gamma > 0$ sufficiently small that
$$\int_{B_{R+\varepsilon}} |\nabla \varphi|^2 \overline{\pi}(x) + \gamma |b(x) - c(x)| \varphi^2 = \lambda_1^+(B_{R+\varepsilon}) + \int_{B_{R+\varepsilon}} \gamma |b(x) - c(x)| \varphi^2 < 0.$$  

This shows that the operator $\overline{L} = \Delta + \overline{\pi}(x) - \gamma |b(x) - c(x)|$ satisfies $\lambda_1^+(B_{R+\varepsilon}) < 0$. Let $\psi$ be a positive eigenfunction corresponding to $\lambda_1^+(B_{R+\varepsilon})$. The $\psi$ satisfies

$$\begin{cases} \Delta \psi + \overline{\pi}(x)\psi - \gamma b(x)\psi + \gamma c(x)\psi \geq 0 & \text{on } B_{R+\varepsilon} \\ \psi \equiv 0 & \text{on } \partial B_{R+\varepsilon} \\ \psi > 0 & \text{on } B_{R+\varepsilon} \end{cases}.$$  

(2.13)

Thus
$$\begin{cases} \Delta \psi + \overline{\pi}(x)\psi - \gamma b(x)\psi + \gamma c(x)\psi \geq 0 & \text{on } B_R \\ \psi > 0 & \text{on } \overline{B_R} \end{cases}.$$  

Let $\mu > 0$ and define $v_- = \mu \psi$ on $B_R$. Choosing
$$\mu \leq \min \left\{ \gamma^{1-\tau} \left( \sup_{B_R} \psi \right)^{-1}, \gamma^{1-\sigma} \left( \sup_{B_R} \psi \right)^{-1} \right\}$$
we have
$$\begin{cases} \gamma \mu^{1-\sigma} \psi^{1-\sigma} \geq 1 \\ \gamma \mu^{1-\tau} \psi^{1-\tau} \leq 1 \end{cases}$$  
(2.14)

on $B_R$. Hence, using (2.13) and (2.14) we deduce
$$0 \leq \Delta v_- + \overline{\pi}(x)v_- - \gamma b(x)v_- \mu^{1-\sigma} \psi^{1-\sigma} + \gamma c(x)v_- \mu^{1-\tau} \psi^{1-\tau} \leq \Delta v_- + \overline{\pi}(x)v_- - b(x)v_-^\sigma + c(x)v_-^\tau,$$
that is,
\[
\begin{align*}
\Delta v_+ + \pi(x)v_+ - b(x)v_+^\sigma + c(x)v_+^\tau &\geq 0 \quad \text{on } B_R \\
v_+ &> 0 \quad \text{on } \overline{B_R}.
\end{align*}
\]
Because of the validity of (2.10), Lemma 2.2 yields the existence of \( v_+ \) satisfying
\[
\begin{align*}
\Delta v_+ + \pi(x)v_+ - b(x)v_+^\sigma + c(x)v_+^\tau &\leq 0 \quad \text{on } B_R \\
v_+ &> 0 \quad \text{on } \overline{B_R}.
\end{align*}
\]
Note that if \( 0 < \gamma \leq 1, \gamma v_- \) still satisfies (2.15); hence up to choosing a suitable \( \gamma \) we can suppose that
\[
\sup_{\overline{B_R}} v_- \leq \inf_{\overline{B_R}} v_+ \quad \text{on } \overline{B_R}.
\]
Let
\[\alpha_+ = \inf_{\partial B_R} v_+, \quad \alpha_- = \sup_{\partial B_R} v_- \]
and fix \( \alpha \in [\alpha_-, \alpha_+] \). Then, by the monotone iteration scheme, there exists a solution \( w \) of
\[
\begin{align*}
\Delta w + \pi(x)w - b(x)w^\sigma + c(x)w^\tau &= 0 \quad \text{on } B_R \\
w &\equiv \alpha > 0 \quad \text{on } \partial B_R,
\end{align*}
\]
and with the further property that
\[
0 < v_- \leq w \leq v_+ \quad \text{on } \overline{B_R}.
\]
Therefore, since \( \pi(x) \geq a(x) \) on \( \overline{B_R} \) and \( w > 0 \) we have
\[
\begin{align*}
\Delta w + a(x)w - b(x)w^\sigma + c(x)w^\tau &\leq 0 \quad \text{on } B_R \\
w &\equiv \alpha > 0 \quad \text{on } \partial B_R \\
w &> 0 \quad \text{on } \overline{B_R}.
\end{align*}
\]
Fix any \( n \in \mathbb{N} \). Let \( \zeta \in \mathbb{R} \) be such that
\[
\zeta \geq \max \left\{ 1, \frac{n}{\sup_{\partial \Omega} w} \right\},
\]
and define \( w_+ = \zeta w \). Then, because of (2.18), the fact that \( \Omega \subset \subset B_R \), and the signs of \( b(x) \) and \( c(x) \), \( w_+ \) satisfies
\[
\begin{align*}
\Delta w_+ + a(x)w_+ - b(x)w_+^\sigma + c(x)w_+^\tau &\leq 0 \quad \text{on } \Omega \\
w_+ &\geq n \quad \text{on } \partial \Omega \\
w_+ &> 0 \quad \text{on } \Omega.
\end{align*}
\]
We can suppose that the \( R \) chosen above is such that \( R \geq R_0 \), where \( R_0 \) is that of the property (\( \Sigma \)) in Definition 2.3. This choice implies that there exists a solution \( \psi \) of
\[
\begin{align*}
\Delta \psi + a(x)\psi - b(x)\psi^\sigma + c(x)\psi^\tau &\geq 0 \quad \text{on } B_R \\
\psi &\geq 0 \quad \text{on } \overline{B_R}.
\end{align*}
\]
If we define $w_- = \beta \psi$ where $0 < \beta \leq 1$, reasoning as above we can find $\beta$ so small that
\[
\begin{align*}
\Delta w_- + a(x) w_- - b(x) w_-^\sigma + c(x) w_-^\tau &\geq 0 \quad \text{on } \Omega \\
0 \leq w_- \leq w_+ &\quad \text{on } \Omega \\
w_- \leq n &\quad \text{on } \partial \Omega .
\end{align*}
\]
Using the monotone iteration scheme we easily arrange a solution $w$ of (2.9) between $w_-$ and $w_+$. We note that the positivity of $w$ is obvious in the case $\sup c(x) = M$, while in the general case it is a consequence of the strong maximum principle (see [12, 13]).

We note that the corresponding result for Yamabe-type equations, namely equations of the type (1.3) with $c(x) \equiv 0$, can be proved without the additional assumption of property ($\Sigma$). Indeed in this case this latter is automatically satisfied by the global subsolution $w_-=0$. The next proposition provides some sufficient conditions for the validity of property ($\Sigma$) on $M$.

**Proposition 2.5.** Assume the validity of one of the following

i) for some $\Lambda > 0$
\[
\lambda_1^{\Delta + a-b+\Lambda c}(M) < 0 ;
\]

ii) let $C_0 = \{ x \in M : c(x) = 0 \}$ be bounded and such that
\[
\lambda_1^{\Delta - a}(C_0) > 0 ;
\]

iii) there exists a positive subsolution $\varphi_- \in C^0(M) \cap W^{1,2}_{loc}(M)$ of (1.3).

Then property ($\Sigma$) holds on $M$.

**Proof.** If i) holds true, there exists $R_o > 0$ sufficiently large such that
\[
\lambda_1^{\Delta + a-b+\Lambda c}(B_{R_o}) < 0 ,
\]
for $R \geq R_o$. Accordingly there exists a corresponding positive eigenfunction $\psi$ on $B_{R+\varepsilon}$, $\varepsilon > 0$ small say, for which
\[
\begin{align*}
\Delta \psi + a(x) \psi - b(x) \psi + \Lambda c(x) \psi &\geq 0 \quad \text{on } B_{R} \\
\psi &> 0 \quad \text{on } \overline{B_{R}} .
\end{align*}
\]

We let
\[
0 < \mu \leq \min \left\{ \frac{\lambda_1}{1-\sigma} , \left( \sup_{B_{R}} \psi \right)^{-1} \right\}
\]
and we define
\[
\varphi = \mu \psi .
\]

Note that from (2.19)
\[
0 \leq \Delta \varphi + a(x) \varphi - b(x) \varphi^\sigma (\mu \psi)^{1-\sigma} + \Lambda c(x) \varphi^\tau (\mu \psi)^{1-\tau} .
\]

Now, because of (2.20)
\[
(\mu \psi)^{1-\sigma} \geq 1 \quad \text{and} \quad (\mu \psi)^{1-\tau} \leq 1
\]
on $\overline{B_{R}}$. Hence the above inequality implies the validity of (2.8) with $\varphi$ strictly positive on $\overline{B_{R}}$. 
If ii) holds true, then by Lemma 2.2, there exists $R_o > 0$ sufficiently large such that $C_0 \subset B_{R_o}$ and for $R \geq R_o$ there exists a solution $\psi$ of

$$
\begin{cases}
\Delta \psi - a(x)\psi - c(x)\psi^{2-\tau} + b(x)\psi^{2-\sigma} \leq 0 & \text{on } B_R \\
\psi > 0 & \text{on } \overline{B_R}.
\end{cases}
$$

Thus, defining $\varphi = \frac{1}{\tilde{T}}$, we have

$$
\Delta \varphi = -\varphi^2 \Delta \psi + 2\varphi^3 |\nabla \psi|^2 \geq -\varphi^2 \Delta \psi,
$$

which implies (2.8) always with $\varphi > 0$ on $\overline{B_R}$. Case iii) is obvious. \hfill \Box

In the sequel we shall need the following a priori estimate. Here $B_T(q)$ denotes the geodesic ball of radius $T$ centered at $q$.

**Lemma 2.6.** Let $a(x), b(x), c(x) \in C^0(M)$, $\sigma > 1$, $\tau < 1$, $0 < \tilde{T} < T$, and $\Omega \subset \subset B_{\tilde{T}}(q) \subset M$. Assume $b(x) > 0$ on $\overline{B_T(q)}$. Then there exists an absolute constant $C > 0$ such that any positive solution $u \in C^2(\overline{B_T(q)})$ of

$$
(2.21) \quad \Delta u + a(x)u - b(x)u^\tau + c(x)u^\sigma \geq 0
$$

satisfies

$$
(2.22) \quad \sup_\Omega u \leq C.
$$

**Proof.** We let $\rho(x) = \text{dist}(x, q)$ and, on the compact ball $\overline{B_T(q)}$, we consider the continuous function

$$
F(x) = \left[ T^2 - \rho(x)^2 \right]^{\frac{\sigma}{2\sigma - 1}} u(x)
$$

where $u(x)$ is any nonnegative $C^2$ solution of (2.21). Note that $F(x)|_{\partial B_T(q)} = 0$, thus, unless $u \equiv 0$ and in this case there is nothing to prove, $F$ has a positive absolute maximum at some point $\overline{u} \in \overline{B_T(q)}$. In particular $u(\overline{u}) > 0$. Now, proceeding as in the proof of Lemma 2.6 in [22], we conclude that, at $\overline{u}$,

$$
bu^{\sigma - 1} \leq \frac{8(\sigma + 1)\rho^2}{(\sigma - 1)^2(T^2 - \rho^2)^2} + \frac{4m + (m - 1)A\rho}{\sigma - 1}\frac{T^2 - \rho^2}{T^2 - \rho^2} + a_+ + cu^{\tau - 1},
$$

for some constant $A \geq 0$, independent of $u$. We now state an elementary lemma postponing its proof.

**Lemma 2.7.** Let $\alpha, \beta \in [0, +\infty)$, and $\mu, \nu \in (0, +\infty)$. If $t \in \mathbb{R}^+$ satisfies

$$
\nu \leq \frac{\alpha}{\mu} + \frac{\beta}{\nu},
$$

then

$$
(2.23) \quad t \leq \left( \alpha + \beta \frac{\nu}{\mu} \right)^{\frac{1}{\mu}}.
$$

Since $\sigma > 1$ and $\tau < 1$, from the lemma we conclude that, at $\overline{u}$,

$$
u \leq b^{-\frac{\sigma - 1}{\nu}} \left( \frac{8(\sigma + 1)\rho^2}{(\sigma - 1)^2(T^2 - \rho^2)^2} + \frac{4m + (m - 1)A\rho}{\sigma - 1}(T^2 - \rho^2) + a_+ + c\frac{T^2 - \rho^2}{\rho^2} \right)^{\frac{1}{\nu - \frac{\sigma - 1}{\nu}}}.
$$

Now the proof proceeds exactly as in Lemma 2.6 of [22] by substituting the $a_+$ there with $a_+ + c\frac{T^2 - \rho^2}{\rho^2}$. \hfill \Box
Proof of Lemma 2.7. If $t^\mu \leq \alpha$ we are done, since $\mu > 0$ and $\beta \geq 0$. In the other case set $s = t^\mu$; then $s > \alpha$ and thus

$$s \leq \alpha + \frac{\beta}{s^\mu} < \alpha + \frac{\beta}{(s - \alpha)^{\frac{1}{\mu}}}.$$  

Setting $r = s - \alpha$ we conclude that

$$r^{\frac{\mu + \nu}{\mu}} < \beta$$

and (2.23) follows. \hfill \qed

The next simple comparison result reveals quite useful.

**Lemma 2.8.** Let $\Omega \subseteq M$ be a bounded open set. Assume that $f_i : M \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$ are measurable functions such that for all $x \in M$

(2.24) \quad \frac{f_2(x,s)}{s} \geq \frac{f_1(x,t)}{t},

for $s \leq t$. Let $u, v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be solutions on $\Omega$ respectively of

(2.25) \quad \begin{cases} 
\Delta u + f_1(x,u) \geq 0; \\
\Delta v + f_2(x,v) \leq 0,
\end{cases}

with $u \geq 0, v > 0$. If $u \leq v$ on $\partial \Omega$, then $u \leq v$ on $\Omega$.

Proof. Set $\psi(x) = \frac{a(x)}{v(x)} \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, from (2.25) a standard computation yields

(2.26) \quad \Delta \psi \geq \frac{u}{v^2} f_2(x,v) - \frac{1}{v} f_1(x,u) - 2 \langle \nabla \psi, \nabla \log v \rangle.

Now, if we assume by contradiction that $u > v$ somewhere in $\Omega$. Then there exists $\varepsilon > 0$ such that

$$\Omega_\varepsilon = \{ x \in \Omega : \psi(x) > 1 + \varepsilon \} \neq \emptyset$$

and $\partial \Omega_\varepsilon \subset \Omega$. Thus it follows from (2.26) that the following inequality holds true on $\Omega_\varepsilon$

$$\Delta \psi + 2 \langle \nabla \psi, \nabla \log v \rangle \geq \psi \left[ \frac{f_2(x,v)}{v} - \frac{f_1(x,u)}{u} \right] \geq 0,$$

moreover $\psi \equiv 1 + \varepsilon$ on $\partial \Omega_\varepsilon$, and thus by the maximum principle $\psi \leq 1 + \varepsilon$ on $\Omega_\varepsilon$ contradicting the definition of $\Omega_\varepsilon$. \qed

**Remark 2.9.** We note that the hypotheses on $f_i$ of Lemma 2.8 are satisfied, for instance, if $f_1 = f_2 : M \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for all $x \in M$

$$s \mapsto \frac{f_i(x,s)}{s}$$

is a non increasing function and that the lemma can also be stated for $f_1 = f_2 : M \times \mathbb{R}^+ \to \mathbb{R}$ with $u, v > 0$. In particular this is the case for the Lichnerowicz-type nonlinearities considered in this paper, namely

$$f(x,s) = a(x)s - b(x)s^\sigma + c(x)s^\tau,$$

with $b(x), c(x)$ non negative and $\sigma > 1, \tau < 1$. Indeed, for any fixed $x \in M$ the function

$$g_\mu(s) = \frac{f(x,s)}{s} = a(x) - b(x)s^{\sigma - 1} + c(x)s^{\tau - 1}$$

is a non increasing function.
is smooth on $\mathbb{R}^+$ and its derivative is given by

$$g'_x(s) = -(\sigma - 1)b(x)s^{\sigma - 2} + c(x)(\tau - 1)s^{\tau - 2},$$

which is non positive by our assumptions on $b(x), c(x), \sigma,$ and $\tau$.

A reasoning similar to that in the proof of Lemma 2.8 will be used at the end of the argument in the proof of the next

**Lemma 2.10.** In the assumptions of Lemma 2.4 there exists a positive solution $u$ of the problem

$$\begin{cases}
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 & \text{on } \Omega \\
u = +\infty & \text{on } \partial \Omega.
\end{cases}$$

**Proof.** For $n \in \mathbb{N}$, let $u_n > 0$ on $\Omega$ be the solution of (2.28) obtained in Lemma 2.4 so that

$$\begin{cases}
\Delta u_n + a(x)u_n - b(x)u_n^\sigma + c(x)u_n^\tau = 0 & \text{on } \Omega \\
u_n > 0 & \text{on } \Omega \\
u_n = n & \text{on } \partial \Omega.
\end{cases}$$

First of all we claim that

$$u_n \leq u_{n+1}.$$ (2.29)

Indeed, $u_n = n < n + 1 = u_{n+1}$ on $\partial \Omega$. We then apply Lemma 2.8 with the choice $f_1 = f_2 = f$ and recalling Remark 2.9 we obtain the validity of (2.29).

If we show the convergence of the monotone sequence $u_n$ to a function $u$ solution of (2.27) we are done, indeed $u$ will certainly be positive. Towards this aim, by standard regularity theory, it is enough to show that the sequence $\{u_n\}$ is uniformly bounded on any compact subset $K$ of $\Omega$. If $K \subset \Omega \setminus B_0$, then we can find a finite covering of balls $\{B_i\}$ for $K$ such that $b(x) > 0$ on each $B_i$. Applying Lemma 2.6 we deduce the existence of a constant $C_1 > 0$ such that

$$u_n(x) \leq C_1 \quad \forall x \in K, \forall n \in \mathbb{N}.$$ (2.30)

It remains to find an upper bound on a neighborhood of $B_0$. Towards this end, for $\eta > 0$ we let

$$N_\eta = \{x \in M : d(x, B_0) < \eta\}$$

where $\eta$ is small enough that $\overline{N_\eta} \subset \Omega$. Furthermore, by the definition of $\lambda_1^L(B_0)$ and the fact that $\lambda_1^L(B_0) > 0$, we can also suppose to have chosen $\eta$ so small that

$$\lambda_1^L(N_\eta) > 0.$$ (2.30)

Now $\partial N_{\eta/2}$ is closed and bounded (because $B_0$ is so), hence compact by the completeness of $M$, this implies the existence of a constant $C_2$ such that

$$u_n(x) \leq C_2 \quad \forall x \in \partial N_{\eta/2}, \forall n \in \mathbb{N};$$

this follows from Lemma 2.6 by considering a finite covering of $\partial N_{\eta/2}$ with balls of radii less than $\eta/2$.

Next we let $\varphi$ be a positive eigenfunction corresponding to $\lambda_1^L(N_\eta)$. Then, since $\inf_{N_{\eta/2}} \varphi > 0$, it follows that there exists a constant $\mu_o > 0$ such that

$$\mu \varphi(x) > C_2 \quad \forall x \in \partial N_{\eta/2}, \forall \mu \geq \mu_o.$$
On $N_{\eta/2}$ we have
\begin{equation}
\Delta (\mu \varphi) + a(x) (\mu \varphi) = -\lambda_1^L(N_{\eta}) (\mu \varphi) < 0.
\end{equation}
We now choose $\mu \geq \mu_0$ sufficiently large that
\[ \mu^{-1} \left( \inf_{N_{\eta/2}} \varphi \right)^{\tau-1} \left( \sup_{N_{\eta/2}} c(x) \right) < \lambda_1^L(N_{\eta}), \]
this is possible since $\tau < 1$ and $\inf_{N_{\eta/2}} \varphi > 0$. Then, for each $\epsilon > 0$,
\begin{equation}
\mu^{-1} \left( \inf_{N_{\eta/2}} \varphi \right)^{\tau-1} \left( \sup_{N_{\eta/2}} c(x) \right) (1 + \epsilon)^{\tau-1} < \lambda_1^L(N_{\eta}).
\end{equation}

We let $\psi = \frac{u}{\mu \varphi}$ on $N_{\eta/2}$, where $u$ is any of the functions of the sequence $\{u_n\}$. The same computations as in the proof of Lemma 2.8 using (2.28) and (2.31) yields
\begin{equation}
\Delta \psi + 2 \langle \nabla \psi, \nabla \log (\mu \varphi) \rangle \geq \left( \lambda_1^L(N_{\eta}) - \left( \sup_{N_{\eta/2}} c(x) \right) (1 + \epsilon)^{\tau-1} \mu^{\tau-1} \varphi^{\tau-1} \right) \psi 
\end{equation}
Note that, accordingly to our choice of $\mu$, $\mu \varphi > C_2 > u$ on $\partial N_{\eta/2}$.

We claim that $\psi \leq 1$ on $\partial N_{\eta/2}$. By contradiction suppose the contrary. Then, for some $\epsilon_1 > 0$, the open set
\[ \Omega_{\epsilon_1} = \{ x \in N_{\eta/2} : \psi(x) > 1 + \epsilon_1 \} \neq \emptyset \]
and $\Omega_{\epsilon_1} \subset \subset N_{\eta/2}$. On $\Omega_{\epsilon_1}$
\[ u > (1 + \epsilon_1) \mu \varphi. \]
Therefore, since $\tau < 1$
\[ u^{\tau-1} \leq (1 + \epsilon_1)^{\tau-1} (\mu \varphi)^{\tau-1}. \]
Then, inserting this into (2.33), together with (2.32), we deduce
\[ \Delta \psi + 2 \langle \nabla \psi, \nabla \log (\mu \varphi) \rangle \geq \left( \lambda_1^L(N_{\eta}) - \left( \sup_{N_{\eta/2}} c(x) \right) (1 + \epsilon_1)^{\tau-1} \mu^{\tau-1} \varphi^{\tau-1} \right) \psi \geq 0. \]
By the maximum principle it follows that $\psi$ attains its maximum on $\partial \Omega_{\epsilon_1}$ but there $\psi(x) = 1 + \epsilon_1$ contradicting the assumption $\Omega_{\epsilon_1} \neq \emptyset$.
Thus $\psi \leq 1$ on $N_{\eta/2}$, that is, $u \leq \mu \psi$ on $N_{\eta/2}$. Hence, for all $n \in \mathbb{N}$
\[ u_n \leq \max \left\{ C_2, \sup_{N_{\eta/2}} \mu \varphi \right\}. \]
This completes the proof of the lemma. $\square$

We are now ready to prove Proposition 2.1. The proof, the same of Theorem 6.5 of [19], follows a standard argument and it is reported here for the sake of completeness.

Proof of Proposition 2.1. First of all we note that, by part iii) of Proposition 2.5, the existence of the global positive subsolution $u_-$ implies that the $\Sigma$-property holds on $M$. We fix an exhausting sequence $\{D_k\}$ of open, precompact sets with smooth boundaries such that
\[ B_0 \subset D_k \subset \overline{D_k} \subset D_{k+1}, \]
and for each $k$ we denote by $u_k^\infty$ the solution of the problem
\[
\begin{cases}
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 & \text{on } D_k; \\
u = +\infty & \text{on } \partial D_k,
\end{cases}
\]
which exists by Lemma 2.10. It follows from Lemma 2.8 that
\[
(2.34) \quad u_- \leq u_k^\infty \leq u_{k+1}^\infty \quad \text{on } D_k.
\]
Thus $\{u_k^\infty\}$ converges monotonically to a function $u$ solving (1.3), and satisfying, because of (2.34), $u \geq u_- > 0$. Let now $u_1 > 0$ be a second solution of (1.3) on $M$. By Lemma 2.8, $u_1 \leq u_k^\infty$ on $D_k$ for all $k$, and therefore $u_1 \leq u$, thus $u$ is a maximal positive solution.

We can now prove Theorem A using an existence result for solutions of Yamabe-type equations contained in [19].

Proof of Theorem A. By Proposition 2.1 it follows that to prove the theorem is sufficient to show that assumption (1.12) implies the existence of a global subsolution $u_-$ of (1.3). Toward this aim we consider the following Yamabe-type equation
\[
(2.35) \quad \Delta v + a(x)v - b(x)v^\sigma = 0 \quad \text{on } M,
\]
with $\sigma$, $a(x)$, and $b(x)$ as in Theorem A. Then, by the sign assumptions (1.4) it follows that a global subsolution $v$ of (2.35) is also a global subsolution of (1.3). Now we recall Theorem 6.7 of [19] which provides a positive solution $v$ of (2.35) under assumptions (1.11) and (1.12) to conclude the proof.

We conclude the section with the proof of Theorem B. The technique is the same of Theorem A: provide a global subsolution and then apply Proposition 2.1. In this case the subsolution is obtained by pasting a subsolution defined inside a compact set and another one defined in the complement of a compact set.

Proof of Theorem B. Reasoning as in Lemma 2.2, assumption (1.13) implies the existence of a solution $\psi \in C^2(\Omega_2)$ of the following problem
\[
\begin{cases}
\Delta \psi + a(x)\psi - b(x)\psi^\sigma + c(x)\psi^\tau \geq 0 & \text{on } \Omega_2 \\
\psi > 0 & \text{on } \Omega_2 \\
\psi = 0 & \text{on } \partial \Omega_2,
\end{cases}
\]
thus $u_1 = \psi$ is a subsolution of (1.3) in $\Omega_2$. In particular, since $\partial \Omega_1 \subset \Omega_2$, if we set $\nu = \min_{\partial \Omega_1} \psi$, we have that $\nu > 0$. Now we note that (1.14) implies that there exists $\mu \in \mathbb{R}^+$ such that
\[
\sup_{M \setminus \Omega_1} \frac{a_-(x) + b(x)}{c(x)} = \mu.
\]
Let us define \( \mu_* = \min \{ 1, \mu_\tau, \nu/2 \} \). Then on \( M \setminus \overline{\Omega}_1 \) we have that
\[
\Delta \mu_* + a(x)\mu_* - b(x)\mu_*^\sigma + c(x)\mu_*^\tau = a(x)\mu_* - b(x)\mu_*^\sigma + c(x)\mu_*^\tau \\
\geq -a_-(x)\mu_* - b(x)\mu_* + c(x)\mu_*^\tau \\
= c(x)\mu_* \left[ \mu_*^{\tau - 1} - \frac{a_-(x) + b(x)}{c(x)} \right] \\
\geq c(x)\mu_* \left[ \mu_*^{\tau - 1} - \mu \right] \\
\geq 0.
\]
Thus \( u_2 = \mu_* \) is a subsolution of (1.3) in \( M \setminus \overline{\Omega}_1 \). Set
\[
u_1 = \begin{cases} 
\max \{ u_1, u_2 \} & \text{on } \overline{\Omega}_1 \\
\max \{ 0, u_2 \} & \text{on } \Omega_2 \setminus \overline{\Omega}_1 \\
u_2 & \text{on } M \setminus \Omega_2 .
\end{cases}
\]
We claim that \( u_- \) is the required global subsolution. To prove the claim, we start by noting that the fact that \( 0 < \mu_* < \nu/2 \) implies \( 0 < u_- \in C^0(M) \cap W^{1,2}_{loc}(M) \). For the same reason it is clear that there exists \( \epsilon > 0 \) such that \( u_- \) is a subsolution of (1.3) on \( \overline{\Omega}_1 \setminus \epsilon \cup (M \setminus \Omega_2) \), where
\[
(U)_\epsilon = \bigcup_{x \in U} B_\epsilon(x)
\]
for any set \( U \subset M \) (\( B_\epsilon(x) \) denotes the geodesic ball of radius \( \epsilon \) centered in \( x \)). Thus we are left to show that \( u_- \) is a subsolution of (1.3) on \( \Omega_2 \setminus \overline{\Omega}_1 \), this is a rather standard fact but we sketch the proof here for the sake of completeness. First of all we set
\[
f(x, v) = a(x)v - b(x)v^\sigma + c(x)v^\tau,
\]
then we note that for any test function \( \varphi \in W^{1,2}_0(\Omega_2 \setminus \overline{\Omega}_1), \varphi \geq 0 \) we have that
\[
\int_{\Omega_2 \setminus \overline{\Omega}_1} \langle \nabla u_1, \nabla \varphi \rangle - \varphi f(x, u_1) \leq 0,
\]
and
\[
f(x, u_2) \geq 0 \quad \text{on } \Omega_2 \setminus \overline{\Omega}_1.
\]
Now, for any \( \varphi \in W^{1,2}_0(\Omega_2 \setminus \overline{\Omega}_1) \) and \( w \in W^{1,2}(\Omega_2 \setminus \overline{\Omega}_1) \) consider
\[
H(w, \varphi) = \int_{\Omega_2 \setminus \overline{\Omega}_1} \langle \nabla w, \nabla \varphi \rangle - \varphi f(x, u_-),
\]
It is clear that \( H(\cdot, \varphi) : W^{1,2}(\Omega_2 \setminus \overline{\Omega}_1) \to \mathbb{R} \) is a continuous functional, for any \( \varphi \).
We want to show that \( H(u_-, \varphi) \leq 0 \), for any test function \( \varphi \geq 0 \) on \( \Omega_2 \setminus \overline{\Omega}_1 \).
For \( \epsilon > 0 \), let
\[
u_\epsilon = \frac{1}{2} \left( u_1 + u_2 + \sqrt{(u_1 - u_2)^2 + \epsilon^2} \right),
\]
then \( \nu_\epsilon \) is smooth with
\[
\nabla \nu_\epsilon = \frac{1}{2} \left( 1 + \frac{u_1 - u_2}{\sqrt{(u_1 - u_2)^2 + \epsilon^2}} \right) \nabla u_1.
\]
moreover, by the definition of \( u_\varepsilon \), \( u_\varepsilon \in W^{1,2}_0(\Omega_2 \setminus \overline{\Omega}_1) \) and \( \varepsilon > 0 \), then
\[
\varphi_\varepsilon = \frac{1}{2} \left( 1 + \frac{u_1 - u_2}{\sqrt{(u_1 - u_2)^2 + \varepsilon^2}} \right) \varphi
\]
belongs to \( W^{1,2}_0(\Omega_2 \setminus \overline{\Omega}_1) \) and its gradient is given by
\[
\nabla \varphi_\varepsilon = \frac{1}{2} \left( 1 + \frac{u_1 - u_2}{\sqrt{(u_1 - u_2)^2 + \varepsilon^2}} \right) \nabla \varphi + \frac{\varepsilon^2}{2 \left( (u_1 - u_2)^2 + \varepsilon^2 \right)} \varphi \nabla u_1.
\]

The following computation uses the properties of \( u_\varepsilon, \varphi_\varepsilon \), and the fact that \( u_1 \) and \( u_2 \) are subsolutions
\[
H(u_\varepsilon, \varphi) = \int_{\Omega_2 \setminus \overline{\Omega}_1} \frac{1}{2} \left( 1 + \frac{u_1 - u_2}{\sqrt{(u_1 - u_2)^2 + \varepsilon^2}} \right) (\nabla u_1, \nabla \varphi) - \varphi f(x, u_-)
\]
\[
= \int_{\Omega_2 \setminus \overline{\Omega}_1} (\nabla u_1, \nabla \varphi_\varepsilon) - \frac{\varepsilon^2 \varphi |\nabla u_1|^2}{2 \left( (u_1 - u_2)^2 + \varepsilon^2 \right)} - \varphi f(x, u_-)
\]
\[
\leq \int_{\Omega_2 \setminus \overline{\Omega}_1} (\nabla u_1, \nabla \varphi_\varepsilon) - \varphi f(x, u_-)
\]
\[
\leq \int_{\Omega_2 \setminus \overline{\Omega}_1} \varphi \varepsilon f(x, u_1) - \varphi f(x, u_-).
\]

Now, since
\[
\varphi_\varepsilon \xrightarrow{L^2} \begin{cases} 
\varphi & \text{if } u_1 > u_2 \\
0 & \text{if } u_1 \leq u_2
\end{cases}
\]
from the continuity of \( H(\cdot, \varphi) \) we conclude that
\[
H(u_-, \varphi) = \lim_{\varepsilon \to 0} H(u_\varepsilon, \varphi) \leq - \int_{\{u_1 \leq u_2\} \cap \Omega_2 \setminus \overline{\Omega}_1} \varphi f(x, u_2) \leq 0,
\]
for any test function \( \varphi \).

3. Uniqueness results and "a priori" estimates

The aim of this section is to prove uniqueness of positive solutions of equation (1.3) on \( M \) or outside a relatively compact open set \( \Omega \). To avoid technicalities we suppose \( u \in C^2(M) \) or \( u \in C^2(M \setminus \overline{\Omega}) \) but this assumption can be relaxed as it will become clear from the arguments we are going to present. Also positivity of \( u \) can be relaxed as it will be remarked below. We begin by proving a further comparison result with the aid of the open form of the \( q \)-Weak Maximum Principle (q-WMP) introduced in [3], see also [2]. For the present purposes we let \( L \) be a linear operator of the form
\[
Lu = \Delta u - \langle X, \nabla u \rangle
\]
for some vector field \( X \) on \( M \). Let \( q(x) \in C^0(M) \) be such that \( q(x) > 0 \) on \( M \). We recall the following

\[
\text{L u} = \Delta u - \langle X, \nabla u \rangle
\]
Definition 3.1. We say that the $q$-Weak Maximum Principle holds on $M$ for the operator $L$ in (3.1) if, for each $u \in C^2(M)$ with $u^* = \sup_M u$ and for each $\gamma \in \mathbb{R}$, with $\gamma < u^*$, we have  

$$\inf_{\Omega_{\gamma}} (q(x)Lu) \leq 0,$$

where

$$\Omega_{\gamma} = \{ x \in M : u(x) > \gamma \}.$$

Next result is contained in Theorem 3.5 in [3].

Theorem 3.2. The $q$-WMP holds on $M$ for the operator $L$ if and only if the open $q$-WMP holds on $M$ for $L$, that is, for each $f \in C^0(\mathbb{R})$, for each open set $\Omega \in M$, with $\partial \Omega \neq \emptyset$ and for each $v \in C^0(\Omega) \cap C^2(\Omega)$ satisfying

$$\begin{aligned}
&i) \quad q(x)Lv \geq f(v) \quad \text{on } \Omega; \\
&ii) \quad \sup_{\Omega} v < +\infty,
\end{aligned}$$

we have that either

$$\begin{aligned}
&sup_{\Omega} v = sup_{\partial \Omega} v \\
\text{or}
\end{aligned}$$

$$f\left(\sup_{\Omega} v\right) \leq 0.$$

The following is a sufficient condition for the validity of the $q$-WMP (see Theorem 4.1 in [22]).

Theorem 3.3. Let $(M, \langle \cdot, \cdot \rangle)$ be complete and $q(x) \leq Cr(x)^{\mu}$ for some constants $C > 0$, $2 > \mu \geq 0$, and $r(x) >> 1$. Assume that

$$\liminf_{r \to +\infty} \frac{\log \text{vol}(B_r)}{r^{2-\mu}} < +\infty.$$  

Then the $q$-WMP holds on $M$ for the operator $\Delta$.

We observe that when $q$ is constant or more generally bounded between two positive constants then the $q$-WMP is equivalent to stochastic completeness of the manifold $(M, \langle \cdot, \cdot \rangle)$ this underlines the fact that the $q$-WMP does not require completeness of the metric and that Theorem 3.3 indeed gives a sufficient condition.

Theorem 3.4. Let $a(x), b(x), c(x) \in C^0(M)$ and $\sigma, \tau \in \mathbb{R}$ be such that $\sigma > 1$ and $\tau < 1$. Let $\Omega$ be a relatively compact open set in $M$. Assume

$$\begin{aligned}
&i) \quad b(x) > 0 \quad \text{on } M \setminus \Omega \\
&ii) \quad c(x) \geq 0 \quad \text{on } M \setminus \Omega \\
&iii) \sup_M \frac{a_-(x)}{b(x)} < +\infty \\
&iv) \sup_M \frac{c(x)}{b(x)} < +\infty
\end{aligned}$$

where, $a_-$ denotes the negative part of $a$. Let $u, v \in C^0(M \setminus \Omega) \cap C^2(M \setminus \Omega)$ be positive solutions of

$$\begin{aligned}
&Lu + a(x)u - b(x)u^\sigma + c(x)u^\tau \geq 0 \\
&Lv + a(x)v - b(x)v^\sigma + c(x)v^\tau \leq 0
\end{aligned}$$

Theorem 3.5 in [3].
on $M \setminus \overline{\Omega}$ satisfying

\begin{align}
\liminf_{x \to +\infty} v(x) &> 0, \\
\limsup_{x \to +\infty} u(x) &< +\infty,
\end{align}

and

\begin{equation}
0 < u(x) \leq v(x) \quad \text{on } \partial \Omega.
\end{equation}

Then

\begin{equation}
u(x) \leq v(x)\end{equation}
on $M \setminus \Omega$ provided that the $1/b$-WMP holds on $M \setminus \overline{\Omega}$ for $L$.

**Remark 3.5.** As it will be observed in the proof of the theorem, in case $0 \leq \tau < 1$ assumption (3.7) iv) can be dropped.

**Proof.** Without loss of generality we can suppose that $M \setminus \overline{\Omega}$ is connected. From positivity of $v$, (3.9), (3.10), and (3.11) there exist positive constants $C_1, C_2$ such that

\begin{equation}
v(x) \geq C_1 \quad u(x) \leq C_2 \quad \text{on } M \setminus \overline{\Omega}.
\end{equation}

We set $\zeta = \sup_{M \setminus \overline{\Omega}} \left( \frac{u}{v} \right)$, from the assumptions on $v$, $u$, and (3.13) it follows that $\zeta$ satisfies

\begin{equation}
0 < \zeta < +\infty.
\end{equation}

Note that if $\zeta \leq 1$ then $u \leq v$ on $M \setminus \overline{\Omega}$. Thus, assume by contradiction that $\zeta > 1$ and define

$\varphi = u - \zeta v$,

then $\varphi \leq 0$ on $M \setminus \overline{\Omega}$. It is not hard to realize, using (3.14) and the definition of $\zeta$, that

\begin{equation}
\sup_{M \setminus \overline{\Omega}} \varphi = 0.
\end{equation}

We now use (3.8) to compute

\begin{equation}
L \varphi \geq -a(x)\varphi + b(x) [u^\sigma - (\zeta v)^\sigma] - c(x) [u^\tau - (\zeta v)^\tau]
\end{equation}

\begin{equation}
+ b(x) \zeta v \left[ (\zeta v)^{\sigma-1} - \sigma v^{\sigma-1} \right] + c(x) \zeta v \left[ (\zeta v)^{\tau-1} - \tau v^{\tau-1} \right].
\end{equation}

We let

\begin{equation}
h(x) = \begin{cases}
\sigma u^{\sigma-1}(x) & \text{if } u(x) = \zeta v(x) \\
\frac{\sigma}{u(x) - \zeta v(x)} \int_{\zeta v(x)}^{u(x)} t^{\sigma-1} dt & \text{if } u(x) < \zeta v(x).
\end{cases}
\end{equation}

and, similarly, for $\tau \neq 0$,

\begin{equation}
j(x) = \begin{cases}
-\tau u^{\tau-1}(x) & \text{if } u(x) = \zeta v(x) \\
\frac{\tau}{\zeta v(x) - u(x)} \int_{\zeta v(x)}^{u(x)} t^{\tau-1} dt & \text{if } u(x) < \zeta v(x).
\end{cases}
\end{equation}
In case \( \tau = 0 \) choose \( j(x) \equiv 0 \). Observe that \( h \) and \( j \) are continuous on \( M \setminus \overline{\Omega} \) and \( h \) is non-negative. Using \( h \) and \( j \), and observing that \( -a(x) \varphi \geq a_-(x) \varphi \), from (3.16) we obtain

\[
L \varphi \geq \left[ a_-(x) + b(x) h(x) + c(x) j(x) \right] \varphi \\
+ b(x) \zeta \left[ (\zeta v)^{\sigma - 1} - v^{\sigma - 1} \right] + c(x) \zeta v [v^{\tau - 1} - (\zeta v)^{\tau - 1}] .
\]

Let

\[
\Omega_{-1} = \{ x \in M \setminus \overline{\Omega} : \varphi(x) > -1 \}.
\]

Since \( u \) is bounded above on \( M \setminus \overline{\Omega} \), there exists a constant \( C > 0 \) such that

\[
v(x) = \frac{1}{\zeta}(u(x) - \varphi(x)) \leq \frac{1}{\zeta}(C + 1)
\]
on \( \Omega_{-1} \). Using the definitions of \( h \) and \( j \), from the mean value theorem for integrals, we deduce

\[
h(x) = \sigma y_h^{\sigma - 1}, \quad j(x) = -\tau y_j^{\tau - 1}
\]
for some \( y_h = y_h(x) \) and \( y_j = y_j(x) \) in the range \([u(x), \zeta v(x)]\). Since \( u(x) \) and \( v(x) \)
are bounded above on \( \Omega_{-1} \)

\[
\max \{h(x), j(x)\} \leq C
\]
on \( \Omega_{-1} \) for some constant \( C > 0 \). Next we recall that \( b(x) > 0 \) on \( M \setminus \overline{\Omega} \) to rewrite

(3.17) in the form

\[
\frac{1}{b(x)} L \varphi \geq \left[ \frac{a_-(x)}{b(x)} + h(x) + \frac{c(x)}{b(x)} j_+(x) \right] \varphi \\
+ \zeta \left[ (\zeta v)^{\sigma - 1} - v^{\sigma - 1} \right] + \frac{c(x)}{b(x)} \zeta v [v^{\tau - 1} - (\zeta v)^{\tau - 1}] .
\]

Since \( \varphi \leq 0 \), (3.7) and (3.19) imply

\[
\left[ \frac{a_-(x)}{b(x)} + h(x) + \frac{c(x)}{b(x)} j_+(x) \right] \varphi \geq C \varphi
\]
for some constant \( C > 0 \) on \( \Omega_{-1} \). For further use we observe here that when \( 0 \leq \tau < 1 \), \( j_+(x) \equiv 0 \) so that in this case assumption (3.7) iv) is not needed to obtain this last inequality. Thus

\[
\frac{1}{b(x)} L \varphi \geq C \varphi + \zeta v \left[ (\zeta v)^{\sigma - 1} - v^{\sigma - 1} \right] + \frac{c(x)}{b(x)} \zeta v [v^{\tau - 1} - (\zeta v)^{\tau - 1}] 
\]
on \( \Omega_{-1} \). Recalling the elementary inequalities

\[
\begin{cases}
\frac{a^s - b^s}{s} \geq sb^{s-1}(a - b) & \text{for } s < 0 \text{ and } s > 1; \\
\frac{a^s - b^s}{s} \geq sa^{s-1}(a - b) & \text{for } 0 \leq s \leq 1,
\end{cases}
\]
a, b \in \mathbb{R}^+, coming from the mean value theorem for integrals (see Theorem 41 in [14]) we conclude

\[
\frac{1}{b(x)} L \varphi \geq C \varphi + \zeta^{\min(1, \sigma - 1)}(\zeta - 1)v^{\sigma} + (1 - \tau) \frac{c(x)}{b(x)} \zeta \frac{1}{\zeta^{1-\tau}} v^{\tau},
\]
on \( \Omega_{-1} \). Now we use the fact that \( \tau < 1, \ v \) is bounded from below by a positive constant, (3.7) i), ii), iv) to get (again if \( 0 \leq \tau < 1 \) we do not need (3.7) iv))

\[
\frac{1}{b(x)} L \varphi \geq C \varphi + B \quad \text{on } \Omega_{-1},
\]
for some positive constants $B$, $C$. Finally, we choose $0 < \varepsilon < 1$ sufficiently small that
\[ C\varphi > -\frac{1}{2}B \]
on\[ \Omega_{-\varepsilon} = \{ x \in M \setminus \overline{\Omega} : \varphi(x) > -\varepsilon \} \subset \Omega_{-1}. \]
Therefore
\[ (3.21) \quad \frac{1}{b(x)}L\varphi \geq \frac{1}{2}B > 0 \quad \text{on} \quad \Omega_{-\varepsilon}. \]
Furthermore, note that
\[ \varphi(x) \leq \min \left\{ -\varepsilon, \left(1 - \zeta\right) \min_{\partial \Omega} v \right\} < 0 \quad \text{on} \quad \Omega_{-\varepsilon}. \]
As a consequence $\sup_{\partial \Omega_{-\varepsilon}} \varphi < 0$ while $\sup_{\Omega_{-\varepsilon}} \varphi = 0$. By Theorem 3.2, (3.21) and the above fact, we obtain the required contradiction, proving that $\zeta \leq 1$. \hfill \Box

As an immediate consequence of Theorem 3.4 we obtain the following uniqueness result.

**Corollary 3.6.** In the assumptions of Theorem 3.4 the equation
\[ Lu + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 \quad \text{on} \quad M \setminus \overline{\Omega}, \]
admits at most a unique solution $u \in C^0(M \setminus \Omega) \cap C^2(M \setminus \overline{\Omega})$ with assigned boundary data on $\partial \Omega$ and satisfying
\[ (3.22) \quad C_1 \leq u(x) \leq C_2 \quad \text{on} \quad M \setminus \overline{\Omega} \]
for some positive constants $C_1$, $C_2$, provided that the $1/b$-WMP holds on $M$ for the operator $L$.

We observe that the two main assumptions in Corollary 3.6 are the validity of the $1/b$-WMP on $M$ for $L$ and the validity of the bounds (3.22). In case $L = \Delta$ in Theorem 3.3 we have given a sufficient condition for the validity of the $1/b$-WMP, for $b > 0$ everywhere it remains to analyze (3.22). Towards this aim we recall the following result companion of Theorem 3.3 and whose proof can be easily adapted from [22] and [21].

**Theorem 3.7.** Let $(M, \langle , \rangle)$ be a complete Riemannian manifold and $a(x), b(x) \in C^0(M)$. Assume that $\|a_+\|_\infty < +\infty$, $b(x) > 0$ on $M$, and
\[ (3.23) \quad b(x) \geq \frac{C}{r(x)^\mu} \]
outside a compact set $K$ for some constants $C > 0$ and $\mu < 2$. Assume (3.6) and
\[ \sup_M \frac{a_+(x)}{b(x)} < +\infty. \]
Let $u \in C^2(M)$ be a non-negative solution of
\[ \Delta u + a(x)u - b(x)u^\sigma \geq 0 \quad \text{on} \quad \Omega_\gamma, \]
where $\sigma > 1$ and
\[ \Omega_\gamma = \{ x \in M : u(x) > \gamma \} \]
for some $\gamma < u^* \leq +\infty$. Then $u^* < +\infty$. Furthermore, having set

$$H_\gamma = \sup_{\Omega_\gamma} \frac{a_+(x)}{b(x)},$$

we have

(3.24) $$u^* \leq H_\gamma^{1/(\sigma-1)}.$$  

We are now ready to prove

**Proposition 3.8.** Let $(M, \langle \ , \rangle)$ be a complete Riemannian manifold. Let $a(x), b(x), c(x) \in C^0(M)$, and assume $\|a_+ + c_+\|_\infty < +\infty$, that $b(x) > 0$ on $M$ and that it satisfies (3.23) for some $\mu < 2$ outside a compact set. Suppose the validity of (3.6) and of

(3.25) $$\sup_M \frac{a_+(x) + c_+(x)}{b(x)} < +\infty.$$  

Let $\sigma > 1$, $\tau < 1$, and $u \in C^2(M)$ be a positive solution of

(3.26) $$\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau \geq 0$$

on

(3.27) $$\Omega_\gamma = \{ x \in M : u(x) > \gamma \},$$

for some $\gamma < u^* \leq +\infty$. Then $u^* < +\infty$ and indeed

(3.28) $$u^* \leq \max \{ \gamma^*, H_\gamma^{1/(\sigma-1)} \}$$

where $\gamma^* = \max \{ 1, \gamma \}$ and

$$H_\gamma = \sup_{\Omega_\gamma} \frac{a_+(x) + c_+(x)}{b(x)}.$$  

**Proof.** First we show that $u^* < +\infty$. We can suppose $u^* > 1$. If $\gamma < 1$ we let $\tilde{\gamma}$ be such that $1 \leq \tilde{\gamma} < u^*$ and note that $\Omega_{\tilde{\gamma}} \subset \Omega_\gamma$. It follows that (3.26) holds on $\Omega_{\tilde{\gamma}}$. Thus, without loss of generality, we can suppose $\gamma \geq 1$. Since $u^\tau \leq u$ on $\Omega_\gamma$, from (3.26) we have

$$\Delta u + a_+(x)u - b(x)u^\sigma + c_+(x)u^\tau \geq \Delta u + a_+(x)u - b(x)u^\sigma + c_+(x)u^\tau$$

$$\geq \Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau$$

$$\geq 0,$$

on $\Omega_\gamma$; in other words

$$\Delta u + [a_+(x) + c_+(x)] u - b(x)u^\sigma \geq 0 \text{ on } \Omega_\gamma.$$  

Applying Theorem 3.7 we deduce $u^* < +\infty$. To prove (3.28) first let $\gamma \geq 1$ so that $\gamma^* = \gamma$. $\Omega_{\gamma^*} = \Omega_\gamma$, $H_{\gamma^*} = H_\gamma$ and (3.28) follows directly from (3.24) of Theorem 3.7. Suppose now $\gamma < 1$. Then $\gamma^* = 1$ and $\Omega_{\gamma^*} = \Omega_1$. If $\Omega_1 = \emptyset$ then $u^* \leq \gamma^*$. If $\Omega_1 \neq \emptyset$ then (3.26) holds on $\Omega_1$ and applying again Theorem 3.7 we deduce the validity of (3.28). \qed

Now, as in case ii) of Proposition 2.5, we are going to exploit the simmetry of equation (1.3) to obtain a bilateral \textit{a priori} estimate. This is the content of the next crucial
Let \((M,\langle\cdot,\cdot\rangle)\) be a complete Riemannian manifold. Let \(a(x),b(x),c(x)\in C^0(M),\|a_+ + c\|_\infty < +\infty,\|a_- + b\|_\infty < +\infty.\) Moreover assume that \(b(x) > 0\) and \(c(x) > 0\) on \(M,\) and that both satisfy (3.33) for some \(\mu < 2.\) Suppose the validity of (3.6) and of

\[
\sup_M \frac{a_+(x) + c(x)}{b(x)} = H < +\infty,
\]

and

\[
\sup_M \frac{a_-(x) + b(x)}{c(x)} = K < +\infty.
\]

Let \(\sigma > 1,\tau < 1.\) Then any positive, \(C^2\) solution of

\[
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 \quad \text{on } M,
\]

satisfies

\[
K \leq u(x) \leq \mathcal{H} \quad \text{on } M,
\]

where

\[
\mathcal{K} = \min \left\{ 1, K^{1/(\tau - 1)} \right\}, \quad \mathcal{H} = \max \left\{ 1, H^{1/(\sigma - 1)} \right\}.
\]

Proof. Suppose \(\Omega_1 = \{x \in M : u(x) > 1\} \neq \emptyset,\) then the validity of (3.31) implies that of

\[
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau \geq 0 \quad \text{on } \Omega_1,
\]

thus the estimate from above in (3.32) follows from Proposition 3.8. In case \(\Omega_1 = \emptyset\) the same estimate is trivially true because of the definition (3.33) of \(\mathcal{H}.\) For the estimate from below we consider the function \(v = \frac{1}{u} \in C^2(M),\) since \(u > 0\) on \(M.\) Since \(\Delta v = -v^2 \Delta u + 2v^3|\nabla u|^2,\) using (3.31) we have

\[
\Delta v + \tilde{a}(x)v - \tilde{b}(x)v^\tilde{\sigma} + \tilde{c}(x)v^\tilde{\tau} \geq 0 \quad \text{on } M,
\]

where we have set \(\tilde{a}(x) = -a(x),\tilde{b}(x) = c(x),\tilde{c}(x) = b(x),\tilde{\sigma} = 2 - \tau > 1,\) and \(\tilde{\tau} = 2 - \sigma < 1.\) Now, since

\[
\frac{a_-(x) + b_+(x)}{c(x)} = \frac{\tilde{a}_+(x) + \tilde{c}_+(x)}{\tilde{b}(x)},
\]

we can reason as above and deduce

\[
v \leq \max \left\{ 1, K^{1/(\tilde{\sigma} - 1)} \right\} = \max \left\{ 1, K^{1/(1-\tau)} \right\}
\]

and the lower bound in (3.32) follows immediately from the definition of \(v.\) \qed

We note that the existence of solutions for equation (1.3) can be easily obtained under the hypotheses of Theorem 3.9 by direct application of the monotone iteration scheme of H. Amann (see for instance [27] or [19]). Indeed in this case it is relatively easy to find an ordered pair of global sub and supersolutions. The key point is that their existence is a consequence of the \textit{a priori} estimate. This is the content of the following

Lemma 3.10. Let \((M,\langle\cdot,\cdot\rangle)\) be a complete Riemannian manifold. Let \(a(x),b(x),c(x),\sigma,\tau,H,K,\mathcal{H},\) and \(\mathcal{K}\) be as in Theorem 3.9. Then \(u^+ \equiv \mathcal{H}\) and \(u^- \equiv \mathcal{K}\) are respectively a global supersolution and a global subsolution of (3.31). Moreover \(u^- \leq u^+.\)
Proof. First of all we note that since \( \mathcal{H} \geq 1 \) and \( \tau < 1 \), then it follows that \( \mathcal{H}^{\tau-1} \leq 1 \). This implies that
\[
\Delta u^+ + a(x)u^+ - b(x)(u^+)^\sigma + c(x)(u^+)^\tau = \mathcal{H} \left[ a(x) - b(x)\mathcal{H}^{\sigma-1} + c(x)\mathcal{H}^{\tau-1} \right] \\
\leq \frac{\mathcal{H}}{b(x)} \left[ \frac{a_+(x) + c(x)}{b(x)} - \mathcal{H}^{\sigma-1} \right] \\
\leq 0
\]
where in the last passage we have used (3.29) and the fact that \( \mathcal{H} \geq H^\frac{1}{\tau} \), thus \( u^+ \) is a global supersolution. The proof of the fact that \( u^- \) is a subsolution is analogous and \( u^- \leq u^+ \) follows from the definitions of \( \mathcal{H} \) and \( \mathcal{K} \). \( \square \)

From this we immediately deduce the next existence result (see also [24] for a similar result).

**Theorem 3.11.** Let \((M, \langle \cdot, \cdot \rangle)\) be a complete Riemannian manifold. Let \( a(x), b(x), c(x), \sigma, \tau \) be as in Theorem 3.9 and assume that \( a(x), b(x), c(x) \in C^{0,\alpha}(M) \) for some \( \alpha > 0 \). Then (1.3) has a positive solution \( u \in C^2(M) \).

**Proof.** Let \( \{\Omega_k\}_{k \in \mathbb{N}} \) be a family of bounded open sets with smooth boundaries such that
\[
\Omega_k \subset \subset \Omega_{k+1} ; \\
\bigcup_{k \in \mathbb{N}} \Omega_k = M .
\]

For each \( k \in \mathbb{N} \) consider the Dirichlet problem
\[
\begin{aligned}
\Delta v + a(x)v - b(x)v^\sigma + c(x)v^\tau &= 0 \quad \text{on } \Omega_k ; \\
v &= u^+ \quad \text{on } \partial \Omega_k ,
\end{aligned}
\]
where \( u^+ = \mathcal{H} \) is the global supersolution of Lemma 3.10. Since \( u^+ \) and \( u^- \) of Lemma 3.10 are respectively a supersolution and a subsolution of (3.34) for any \( k \in \mathbb{N} \), it follows from the monotone iteration scheme (see for instance Theorem 2.1 of [27]) that for any \( k \) there exists a solution \( v_k \in C^{2,\alpha}(\Omega_k) \) of (3.34) such that \( u^- \leq v_k \leq u^+ \). From Lemma 2.8 it follows that
\[
u_i \leq v_i \leq v_j \leq u^+ \quad \text{on } \Omega_k
\]
for all \( i, j \in \mathbb{N} \) such that \( i \geq j \geq k \). Thus, from the Schauder interior estimates and the compactness of the embedding \( C^{2,\alpha}(\Omega_k) \subset C^2(\Omega_k) \) it follows that the \( v_k \) converge uniformly on compact sets to a solution \( u \in C^2(M) \) of (1.3). Moreover \( u(x) \geq u^- > 0 \). \( \square \)

Putting together Theorem 3.3, Corollary 3.6 with \( \Omega = \emptyset \), Theorem 3.9, and Theorem 3.11 we have the following

**Corollary 3.12.** In the assumptions of Theorem 3.9 with \( 0 \leq \mu < 2 \) the equation
\[
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 \quad \text{on } M .
\]
admits a unique positive solution \( u \in C^2(M) \).

The next corollary deals with the easier case where \( a(x), b(x), \) and \( c(x) \) are of the form \( \zeta f(x) \) where \( 0 < f(x) \in C^0(M) \cap L^\infty(M) \) and \( \zeta \in \mathbb{R} \). It generalizes Theorem 2 of [17] and Theorem 7 of [18]. Furthermore it should be compared with Theorem 3.15 and Example 3.18 of [11].
Corollary 3.13. Let \((M, \langle \cdot, \cdot \rangle)\) be a complete Riemannian manifold. Let \(\alpha, \beta, \gamma \in \mathbb{R}\) such that \(\beta, \gamma > 0\). Let \(\sigma > 1, \tau < 0, 0 < f(x) \in C^0(M) \cap L^\infty(M)\) satisfying (3.23) outside a compact set for some \(\mu < 2\), and assume the validity of (3.6). Then the unique positive solution of
\[
\Delta u + f(x) (\alpha u - \beta u^\sigma + \gamma u^\tau) = 0 \quad \text{on } M
\]
is given by \(u \equiv \lambda\), where \(\lambda \in \mathbb{R}^+\) satisfies
\[
p(t) = \alpha + \beta t^{\sigma - 1} - \gamma t^{\tau - 1}.
\]

We conclude the section with a second uniqueness result whose proof is based on that of Theorem 4.1 of [7], see also Theorem 5.1 in [19].

Theorem 3.14. Let \((M, \langle \cdot, \cdot \rangle)\) be a complete manifold, \(a(x), b(x), c(x) \in C^0(M)\), \(\sigma > 1, \tau < 1\), and assume (1.4), that is,
\[
b(x) \geq 0, \quad c(x) \geq 0,
\]
and
\[
b(x) + c(x) \not\equiv 0 \quad \text{on } M.
\]
Let \(u, v \in C^2(M)\) be positive solutions of
\[
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0 \quad \text{on } M,
\]
such that
\[
\left\{ \int_{\partial B_r} (u - v)^2 \right\}^{-1} \notin L^1(+\infty).
\]
Then \(u \equiv v\) on \(M\).

Remark 3.15. Note that condition (3.36) is implied by \(u - v \in L^2(M)\) or even by the weaker request
\[
\int_{B_r} (u - v)^2 = o(r^2) \quad \text{as } r \rightarrow +\infty.
\]
See for instance Proposition 1.3 in [25].

Proof. The proof follows, mutatis mutandis, that reported in Theorem 5.1 of [19] up to equation (5.7) that now becomes
\[
\int_M b(x) (v^2 - u^2) (v^{\sigma - 1} - u^{\sigma - 1}) + \int_M c(x) (v^2 - u^2) (v^{\tau - 1} - u^{\tau - 1}) + 
\]
\[
+ \int_M \left\{ \left| \nabla v - \frac{v}{u} \nabla u \right|^2 + \left| \nabla u - \frac{u}{v} \nabla v \right|^2 \right\} = 0.
\]
Because of (1.4) we deduce \(|\nabla u - \frac{v}{u} \nabla v| \equiv 0\) on \(M\) so that \(u = Av\) for some constant \(A > 0\). Substituting into (3.37) yields
\[
(1 - A^2) (1 - A^{\sigma - 1}) \int_M b(x)v^{\sigma + 1} = 0
\]
and
\[
(1 - A^2) (A^{\tau - 1} - 1) \int_M c(x)v^{\tau + 1} = 0.
\]
Since \(v > 0\), (3.35) implies \(A = 1\), that is, \(u = v\) on \(M\).

Remark 3.16. The exponent 2 in (3.36) is sharp, see the discussion after Theorem 5.1 in [19].
4. A FURTHER COMPARISON AND UNIQUENESS RESULT

In this section we prove a comparison result and a corresponding uniqueness result based on a spectral property of the operator $L = \Delta + a(x)$. As we have seen, the request $\lambda_1^L (M) < 0$ facilitates the search of solutions of equation (1.3). Somehow the opposite request seems to limitate the existence of solutions.

We recall that $L$ has finite index if and only if there exists a positive solution $u$ of the differential inequality

$$L u \leq 0,$$

outside a compact set $K$. In what follows we shall denote with $(M, \langle , \rangle, G)$ a triple with the following properties: $(M, \langle , \rangle)$ is a complete manifold with a preferred origin $o$ and $G \in C^2(M \setminus \{o\})$, $G : M \setminus \{o\} \to \mathbb{R}_+$ is such that

$$\begin{align*}
 & i) \Delta G \leq 0 \quad \text{on } M \setminus \{o\}; \\
 & ii) G(x) \to +\infty \quad \text{as } x \to o; \\
 & iii) G(x) \to 0 \quad \text{as } x \to +\infty,
\end{align*}$$

Clearly a good candidate for $G$ is the (positive) Green kernel at $o$ on a non-parabolic complete manifold, which, however, might not satisfy (4.2) iii). Observe that, for instance by the work of Li and Yau, \cite{15}, iii) is satisfied by the Green kernel if $\text{Ric} \geq 0$. Other examples always concerning the Green kernel, are given by non-parabolic complete manifolds supporting a Sobolev inequality of the type

$$S(\alpha)^{-1} \left( \int_M v^{1-\alpha} \right)^{1-\alpha} \leq \int_M |\nabla v|^2 \quad \text{for all } v \in C_\infty(M)$$

for some $\alpha \in (0, 1)$, $S(\alpha) > 0$, and for all $v \in C_\infty(M)$. For further examples see \cite{20} and the references therein. Note that, in these results, the authors also describe the behavior of $G(x)$ at infinity from above and below. That of $|\nabla G(x)|$ from above can often be obtained by classical gradient estimates. This is helpful for instance in Theorem 4.5 below.

However, since we only require superharmonicity of $G$, under a curvature assumption we can use transplantation from a non-parabolic model. The argument is as follow. Assume $(M, \langle , \rangle)$ is a $m$-dimensional manifold with a pole $o$ and with radial sectional curvature (with respect to $o$) $K_{\text{rad}}$ satisfying

$$K_{\text{rad}} \leq -F(r(x)) \quad \text{on } M,$$

with $r(x) = \text{dist}_M(x, o)$, $F \in C^0(\mathbb{R}_0^+)$. Let $g$ be a $C^2$-solution of the problem

$$\begin{align*}
 & i) g'' - F(r) g \leq 0 \\
 & ii) g(0) = 0, \quad g'(0) = 1,
\end{align*}$$

and suppose that $g > 0$ on $\mathbb{R}_+$. Note that this request is easily achieved by bounding appropriately $F$ from above. See for instance \cite{5}. Then by the Laplacian comparison theorem

$$\Delta r \geq (m - 1) \frac{g'(r)}{g(r)} \quad \text{on } M \setminus \{o\},$$

and weakly on $M$. Consider the $C^2$-model $M_g$ defined by $g$ with the metric

$$\langle , \rangle_g = dr^2 + g^2(r) d\theta^2$$
on $M \setminus \{o\} = \mathbb{R}^+ \times S^{m-1}$, $S^{m-1}$ the unit sphere, $d\theta^2$ its canonical metric. Then $M \setminus \{o\}$ is non-parabolic if and only if $\frac{1}{g(r)} \in L^1(+)$. Now we transplant the positive Green function on $M \setminus \{o\}$ evaluated at $(y,o)$ to $M$, that is, we let

$$G(x) = \int_{r(x)}^{+\infty} \frac{ds}{g(s)^{m-1}} > 0 \quad \text{on} \quad M \setminus \{o\}.$$ 

An immediate computation yields

$$\Delta G(x) = -\frac{1}{g(r(x))^{m-1}} \left\{ \Delta r(x) - (m-1) \frac{g'(r(x))}{g(r(x))} \right\} \quad \text{on} \quad M \setminus \{o\}.$$ 

Hence, (4.5) implies (4.2) i). The remaining of (4.2) be trivially satisfied. 

Thus we solve the problem by looking for a solution of (4.4) satisfying

$$\frac{1}{g(r)} \in L^1(+) \quad \text{on} \quad M \setminus \{o\}.$$ 

We are now ready to prove the following

**Theorem 4.1.** Let $(M, \langle , \rangle, G)$ be as above and suppose that $a(x) \in C^0(M)$ satisfy

$$a(x) \leq \left\{ 1 + \frac{1}{\log^2 G(x)} \left[ 1 + \frac{1}{\log^2 (\log G(x))} \right] \right\} \frac{\|\nabla \log G(x)\|^2}{4},$$

outside a compact set $K$. Then the operator $L = \Delta + a(x)$ has finite index.

**Remark 4.2.** Observe that condition (4.9) is meaningful outside a sufficiently large compact set $K$ because of (4.2) iii).

**Proof.** On $\mathbb{R}^+$ we define the function

$$\kappa(s) = 1 + \frac{1}{4s^2} \left[ 1 + \frac{1}{\log^2 s} \right],$$

so that inequality (4.9) can be rewritten as

$$a(x) \leq \kappa(t(x)) \frac{\|\nabla \log G(x)\|^2}{4} \quad \text{on} \quad M \setminus K.$$ 

To prove the theorem we need to provide a positive solution $u$ of (4.1) on $M \setminus \hat{K}$ for some compact $\hat{K}$. Towards this aim we look for $u$ of the form

$$u(x) = \sqrt{G(x)} \beta(t(x)) = e^{-t(x)} \beta(t(x)).$$
on $M \setminus \Lambda_T$ for some $T > 0$ sufficiently large and with $\beta : [T, +\infty) \to \mathbb{R}^+$. Now a simple computation shows that $u$ satisfies

$$(4.13) \quad \Delta u + \left[ 1 - \frac{\ddot{\beta}}{\beta}(t(x)) \right] \frac{|\nabla \log G(x)|^2}{4} u = \frac{1}{2} \frac{\Delta G}{\sqrt{G}(x)} \left[ \beta(t(x)) - \dot{\beta}(t(x)) \right]$$

on $M \setminus \Lambda_T$, where $\dot{\beta}$ means the derivative with respect to $t$. Thus, using (4.11) and (4.13) we obtain

$$\Delta u + a(x)u \leq \left[ \kappa(t(x)) - 1 + \frac{\ddot{\beta}}{\beta}(t(x)) \right] \frac{|\nabla \log G(x)|^2}{4} u + \frac{\Delta G}{2\sqrt{G}}(x) \left[ \beta(t(x)) - \dot{\beta}(t(x)) \right].$$

Hence using (4.2) i) we have that (4.1) is satisfied on $M \setminus \Lambda_T$ for $u$ as in (4.12) if we show the existence of a positive solution $\beta$ of

$$(4.14) \quad \ddot{\beta} + \left[ \kappa(t) - 1 \right] \beta = 0$$

satisfying the further requirement

$$(4.15) \quad \beta - \dot{\beta} \geq 0$$

on $[T, +\infty)$ for some $T > 0$; in other words we have to show that (4.14) is non-oscillatory and that (4.15) holds at least in a neighborhood of $+\infty$. As for non-oscillation, applying Theorem 6.44 of [5], we see that this is the case if

$$\kappa(t) - 1 \leq \frac{1}{4t^2} \left[ 1 + \frac{1}{\log^2 t} \right]$$

on $[T, +\infty)$ for some $T > 0$ sufficiently large. This is guaranteed by the definition (4.10) of $\kappa$. To show the validity of (4.15) we use the following trick. Fix $n \geq 3$ and define $\rho \in \mathbb{R}^+$ via the prescription

$$(4.16) \quad t = t(\rho) = \log \left( \sqrt{n - 2\rho} \frac{n-2}{2} \right).$$

Note that

$$(4.17) \quad t(0^+) = -\infty, \quad t(+\infty) = +\infty, \quad t'(\rho) = \frac{n-2}{2} \frac{1}{\rho} \quad \text{on } \mathbb{R}^+.$$

We then define

$$(4.18) \quad z(\rho) = e^{-t(\rho)} \beta(t(\rho)).$$

If $\beta$ is a solution of (4.14) on $[T, +\infty)$, having set $R = \rho(T) > 0$ with $\rho(t)$ the inverse function of $t(\rho)$, $z$ satisfies

$$(4.19) \quad (\rho^{n-1}z')' + \kappa(t(\rho)) \frac{(n-2)^2}{4\rho^2} \rho^{n-1}z = 0 \quad \text{on } [R, +\infty).$$

we can also fix the initial conditions

$$(4.20) \quad z(R) = 1, \quad z'(R) = 0.$$
But
\[
z'(\rho) = \frac{\sqrt{n} - 2}{2} \frac{1}{\rho^2} \left\{ \dot{\beta}(t(\rho)) - \beta(t(\rho)) \right\}
\]
and therefore (4.15) is satisfied.
This completes the proof of the Theorem. \(\square\)

**Remark 4.3.** We have just proved that the equation

\[\dot{\beta} + \frac{1}{4t^2} \left[ 1 + \frac{1}{\log^2 t} \right] \beta = 0 \quad \text{on } [T, +\infty)\]  

(say \(T \geq e\)) is non-oscillatory. This is not a consequence of the usual Hille-Nehari criterion (see [28]). Indeed, setting \(h(t)\) to denote the coefficient of the linear term in (4.21), the condition of the classical criterion to guarantee the non-oscillatory character of the equation is that \(h(t) \geq 0\) for \(t >> 1\) and

\[
\limsup_{t \to +\infty} t \int_{t}^{+\infty} h(s)ds < \frac{1}{4}.
\]

However, in this case we have

\[
\frac{1}{4} < t \int_{t}^{+\infty} \frac{ds}{4s^2} < t \int_{t}^{+\infty} h(s)ds < \frac{1}{4} + \frac{1}{4} \int_{t}^{+\infty} \frac{ds}{s \log^2 s} = \frac{1}{4} + \frac{1}{4 \log t}
\]

so that (4.22) is not satisfied.

We shall now see how to get non-oscillation of (4.21) following the idea in the proof of the mentioned Theorem 6.44 of [5]. This will enable us to determine the asymptotic behavior of a solution \(\beta\) of (4.21) at \(+\infty\) and therefore of \(u\) defined in (4.12) and solution of (4.1). This will be later used in Theorem 4.5.

Towards this aim we consider the function

\[w(t) = \sqrt{t} \log t\]

solution of Euler equation

\[\dot{w} + \frac{1}{4t^2} w = 0\]

on \([T, +\infty), T > 0,\) and positive on \([T, +\infty)\) for \(T > 1\). Then the function

\[z = \frac{\beta}{w}\]

satisfies

\[(w^z\dot{z}) + \left( \kappa(t) - 1 - \frac{1}{4t^2} \right) w^2 z = 0 \quad \text{on } [T, +\infty)\]

for \(T >> 1\). Since \(\frac{1}{w^2} \in L^1(+\infty)\) we can define the critical curve \(\chi_{w^2}\) relative to \(w^2\) as in (4.21) of [5]. A computation yields

\[\chi_{w^2}(t) = \frac{1}{4} \frac{1}{t^2 \log^2 t} \quad \text{for } t >> 1,\]

so that

\[\kappa(t) - 1 - \frac{1}{4t^2} = \chi_{w^2}(t).\]
Hence from Theorem 5.1 and Proposition 5.7 of [5] we deduce that the solution \( z(t) \) of (4.26) satisfies
\[
z(t) \sim \frac{C}{\sqrt{\log t}} \log \log t \quad \text{as} \quad t \to +\infty,
\]
for some constant \( C > 0 \) and therefore
\[
(4.27) \quad \beta(t) \sim C \sqrt{\log t} \log \log t \quad \text{as} \quad t \to +\infty.
\]
Using the above, we finally obtain the asymptotic behavior of \( u \) in (4.12), that is,
\[
(4.28) \quad u(x) \sim \varphi(x) \quad \text{as} \quad x \to \infty \quad \text{in} \quad M,
\]
with
\[
(4.29) \quad \varphi(x) = C \sqrt{G(x)} \sqrt{-\log \sqrt{G(x)}} \log \log \left( -\log \sqrt{G(x)} \right) \log \log \left( -\log \sqrt{G(x)} \right)
\]
as \( x \to \infty \) on \( M \).

In particular the behavior of \( u \) at infinity is known once that of \( G(x) \) is known.

Next we prove a version of Theorem 5.20 of [5] for equation (1.3).

**Theorem 4.4.** Let \((M, (\cdot, \cdot))\) be a complete manifold, \(a(x), b(x), c(x) \in C^0(M), \sigma > 1, \tau < 1\), and assume (1.4) and (3.35). Let \(\Omega\) be a relatively compact open set and assume the existence of \(w \in C^2(M \setminus \overline{\Omega})\) positive solution of
\[
L w = \Delta w + a(x)w \leq 0 \quad \text{on} \quad M \setminus \overline{\Omega}.
\]
Suppose that \(u \) and \(v \) are positive \(C^2\) solutions on \( M \) of
\[
\begin{align*}
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau & \leq 0 \\
\Delta v + a(x)v - b(x)v^\sigma + c(x)v^\tau & \geq 0.
\end{align*}
\]
If
\[
(4.32) \quad u - v = o(w) \quad \text{as} \quad x \to \infty,
\]
then \( v \leq u \) on \( M \).

**Proof.** The idea of the proof is the same as that of Theorem 5.20 of [5]. We report it here for the sake of completeness and for some minor differences. First we extend \( w \) to a positive function \( \tilde{w} \) on \( M \). Towards this end let \( \Omega' \) be a relatively compact open set such that \( \overline{\Omega'} \subset \Omega \). Fix a cut-off function \( \psi \), \( 0 \leq \psi \leq 1 \) such that \( \psi \equiv 1 \) on \( \Omega \) and \( \psi \equiv 0 \) on \( M \setminus \overline{\Omega'} \). Define \( \tilde{w} = \psi + (1 - \psi)w \). Note that \( \tilde{w} > 0 \) on \( M \) and \( \tilde{w} = w \) on \( M \setminus \overline{\Omega'} \) so that \( L \tilde{w} \leq 0 \) on \( M \setminus \overline{\Omega'} \). For notational convenience we write again \( w \) and \( \Omega \) in place of \( \tilde{w} \) and \( \tilde{\Omega} \), but this time \( \tilde{w} > 0 \) on \( M \).

Let \( \varepsilon > 0 \) and define \( u_\varepsilon = u + \varepsilon w \) on \( M \). Then \( u_\varepsilon \) is a solution on \( M \) of
\[
\Delta u_\varepsilon + a(x)u_\varepsilon \leq b(x)u^\sigma - c(x)u^\tau + \varepsilon L w.
\]
Therefore, interpreting the differential inequality in the weak sense, we have that for each \( \varphi \in \text{Lip}_{loc}(M) \), \( \varphi \geq 0 \)
\[
- \int_M \langle \nabla u_\varepsilon, \nabla \varphi \rangle +\int_M a(x)u_\varepsilon \varphi \leq \int_M b(x)u^\sigma \varphi - \int_M c(x)u^\tau \varphi + \varepsilon \int_M \varphi L w.
\]
Now, by the second Green formula
\[
\int_M \varphi L w = \int_M a(x)w \varphi + \int_M w \Delta \varphi = \int_M wL \varphi
\]
and therefore we can rewrite the above inequality as

\[ -\int_M \langle \nabla u_\varepsilon, \nabla \varphi \rangle + \int_M a(x)u_\varepsilon \varphi \leq \int_M b(x)u^\sigma \varphi - \int_M c(x)u^\tau \varphi + \varepsilon \int_M wL \varphi. \]

Similarly, interpreting the second differential inequality of (4.31) in the weak sense

\[ -\int_M \langle \nabla v, \nabla \varphi \rangle + \int_M a(x)v \varphi \geq \int_M b(x)v^\sigma \varphi - \int_M c(x)v^\tau \varphi, \]

with \( \varphi \) as above.

Next, by contradiction suppose that

\[ \Gamma = \{ x \in M : v(x) > u(x) \} \neq \emptyset. \]

Then, for \( \varepsilon > 0 \) sufficiently small

\[ \Gamma_\varepsilon = \{ x \in M : v_\varepsilon(x) > u_\varepsilon(x) \} \neq \emptyset. \]

We now consider the Lipschitz function \( \gamma_\varepsilon = (v^2 - u_\varepsilon^2)_+ \). Condition (4.32) implies that \( \gamma_\varepsilon \) has compact support in \( M \) and it is not identically zero because of (4.35). Thus the functions \( \varphi_1 = \frac{\gamma_\varepsilon}{u_\varepsilon} \) and \( \varphi_2 = \frac{\gamma_\varepsilon}{v} \) are admissible, respectively for (4.33) and (4.34). Substituting we have

\[ -\int_M \left( \frac{\nabla u_\varepsilon}{u_\varepsilon} , \nabla \gamma_\varepsilon \right) - \left| \frac{\nabla u_\varepsilon}{u_\varepsilon} \right|^2 \gamma_\varepsilon - a(x)\gamma_\varepsilon \leq \int_M b(x)\frac{u^\sigma}{u_\varepsilon} \gamma_\varepsilon - c(x)\frac{u^\tau}{u_\varepsilon} \gamma_\varepsilon + \varepsilon wL \left( \frac{\gamma_\varepsilon}{u_\varepsilon} \right), \]

and

\[ -\int_M \left( \frac{\nabla v}{v} , \nabla \gamma_\varepsilon \right) - \left| \frac{\nabla v}{v} \right|^2 \gamma_\varepsilon - a(x)\gamma_\varepsilon \geq \int_M b(x)v^{\sigma-1} \gamma_\varepsilon - c(x)v^{\tau-1} \gamma_\varepsilon. \]

Thus, subtracting the second from the first we deduce

\[ -\int_{\Gamma_\varepsilon} \left( \frac{\nabla u_\varepsilon}{u_\varepsilon} - \frac{\nabla v}{v} , \nabla \gamma_\varepsilon \right) + \int_{\Gamma_\varepsilon} \left( \frac{\nabla u_\varepsilon}{u_\varepsilon} - \frac{\nabla v}{v} \right)^2 \gamma_\varepsilon \leq \int_{\Gamma_\varepsilon} b(x) \left( \frac{u^\sigma}{u_\varepsilon} - v^{\sigma-1} \right) \gamma_\varepsilon - \int_{\Gamma_\varepsilon} c(x) \left( \frac{u^\tau}{u_\varepsilon} - v^{\tau-1} \right) \gamma_\varepsilon + \varepsilon \int_M wL \left( \frac{\gamma_\varepsilon}{u_\varepsilon} \right). \]

Inserting the expression for \( \gamma_\varepsilon \) and rearranging, we finally have

\[ \int_{\Gamma_\varepsilon} \left| \nabla u_\varepsilon - \frac{u_\varepsilon}{v} \nabla v \right|^2 - \left| \nabla v - \frac{v}{u_\varepsilon} \nabla u_\varepsilon \right|^2 \leq \int_{\Gamma_\varepsilon} b(x) \left( \frac{u^\sigma}{u_\varepsilon} - v^{\sigma-1} \right) \gamma_\varepsilon - \int_{\Gamma_\varepsilon} c(x) \left( \frac{u^\tau}{u_\varepsilon} - v^{\tau-1} \right) \gamma_\varepsilon + \varepsilon \int_M wL \left( \frac{\gamma_\varepsilon}{u_\varepsilon} \right). \]

Let \( V \) be a relatively compact open set with smooth boundary such that \( \overline{\Omega} \subset V \) and let \( \psi, 0 \leq \psi \leq 1 \) be a cut-off function such that \( \psi \equiv 1 \) on \( \Omega \) and \( \psi \equiv 0 \) on \( M \setminus \overline{\Omega} \). Then, using again the second Green formula and (4.30) we have

\[ \int_M wL \left( \frac{\gamma_\varepsilon}{u_\varepsilon} \right) = \int_M wL \left( \frac{\gamma_\varepsilon}{u_\varepsilon} \right) + \int_M wL \left( 1 - \psi \right) \frac{\gamma_\varepsilon}{u_\varepsilon} \]

\[ = \int_M wL \left( \frac{\gamma_\varepsilon}{u_\varepsilon} \right) + \int_M \left( 1 - \psi \right) \frac{\gamma_\varepsilon}{u_\varepsilon} wL \]

\[ \leq \int_M wL \left( \frac{\gamma_\varepsilon}{u_\varepsilon} \right). \]
Theorem 4.5. Let $(M, (\cdot, \cdot), G)$ as in (4.2), $a(x), b(x), c(x) \in C^0(M), \sigma > 1, \tau < 1$, and assume (1.4), (3.35), and
\[
a(x) \leq \left\{ 1 + \frac{1}{\log^2 G(x)} \left[ 1 + \frac{1}{\log^2 \left( -\log \sqrt{G(x)} \right) } \right] \right\} |\nabla \log G(x)|^2
\]
outside a compact set. If $u$ and $v$ are positive $C^2$ solutions of
\[
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0
\]
such that
\[
u(x) - v(x) = o(\varphi(x)) \quad \text{as } x \to \infty
\]
with $\varphi(x)$ as in (4.29), then $u \equiv v$ on $M$.

It is reasonable that if we strenghten the upper bound (4.11) on $a(x)$ the growth of $u$ defined in (4.12) should improve in (4.29).

For the sake of simplicity let us suppose
\[
a(x) \leq \lambda |\nabla \log G(x)|^2
\]
on $M \setminus K$ for some constant $\lambda \in (-\infty, 1]$. We proceed as in the proof of Theorem 4.1 to arrive to (4.14) that now reads
\[
\dot{\beta} + [\lambda - 1] \beta = 0
\]
on $[T, +\infty)$ for some $T > 0$. Positive solutions of the above are immediately obtained. Indeed, for $\lambda = 1$ we let $\beta(t) = Ct$ for some constant $C > 0$ while for
\( \lambda \in (-\infty, 1) \) we let \( \beta(t) = C e^{t/M}, C > 0. \) Thus the positive solution \( u(x) \) of \( \text{Lu} \leq 0 \) given in (4.12) satisfies

\[
(4.40) \quad u(x) \sim \begin{cases} 
C \sqrt{G(x)} \log \frac{1}{G(x)} & \text{if } \lambda = 1 \\
CG(x)^{\frac{1}{2-2\lambda}} & \text{if } \lambda \in (-\infty, 1) 
\end{cases}
\]

as \( x \to \infty \) for some constant \( C > 0. \)

Thus, going back to Theorem 4.5 we obtain the following version

**Theorem 4.6.** Let \((M, \langle , \rangle, G)\) as in (4.2), \(a(x), b(x), c(x) \in C^0(M), \sigma > 1,\) \( \tau < 1, \) and assume (1.4), (3.35), and

\[
a(x) \leq \lambda \frac{\lvert \nabla \log G(x) \rvert^2}{4}
\]

outside a compact set, for some constant \( \lambda \in (-\infty, 1]. \) If \( u \) and \( v \) are positive \( C^2 \) solutions of

\[
\Delta u + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0
\]

such that

\[
 u(x) - v(x) = \begin{cases} 
 o \left( \sqrt{G(x)} \log \frac{1}{G(x)} \right) & \text{if } \lambda = 1 \\
 o \left( G(x)^{\frac{1}{2-2\lambda}} \right) & \text{if } \lambda \in (-\infty, 1) 
\end{cases} \quad \text{as } x \to \infty,
\]

then \( u \equiv v \) on \( M. \)

As a final remark we observe that finiteness of the index of \( L = \Delta + a(x) \) can be also deduced by the validity of a Sobolev-type inequality on \( M. \) Indeed, according to Lemma 7.33 of [23], the validity of (4.3) and the assumption

\[
a_+ (x) \in L^{1/\alpha} (M)
\]

imply that \( L \) has finite index.

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