$E_n$-cell attachments and a local-to-global principle for homological stability

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Abstract We define bounded generation for $E_n$-algebras in chain complexes and prove that this property is equivalent to homological stability for $n \geq 2$. Using this we prove a local-to-global principle for homological stability, which says that if an $E_n$-algebra $A$ has homological stability (or equivalently the topological chiral homology $\int_{\mathbb{R}^n} A$ has homology stability), then so has the topological chiral homology $\int_M A$ of any connected non-compact manifold $M$. Using scanning, we reformulate the local-to-global homological stability principle so that it applies to compact manifolds. We also give several applications of our results.

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1 Introduction

This paper establishes a local-to-global principle for homological stability. Since all sufficiently natural and local constructions on the category of $n$-manifolds are equivalent to topological chiral homology with coefficients in some $E_n$-algebra, our
local-to-global homological stability principle is formulated using topological chiral homology. To prove our result, we introduce a condition on \( E_n \)-algebras which we call \emph{bounded generation} and prove that this condition is equivalent to homological stability. This allows us to leverage homological stability for configuration spaces to prove that if an \( E_n \)-algebra has homological stability, then so does its topological chiral homology. We discuss several applications of this local-to-global homological stability principle as well as applications of the equivalence between bounded generation and homological stability.

1.1 Homological stability for framed \( E_n \)-algebras

When working in categories with some notion of homotopy theory, e.g. chain complexes or topological spaces, one can talk about algebras with commutativity conditions interpolating between associative and commutative; these are \( E_n \)-algebras. The typical example of an \( E_n \)-algebra is an \( n \)-fold based loop space \( \Omega^n X \). A framed \( E_n \)-algebra is an \( E_n \)-algebra with a compatible action of the special orthogonal group \( SO(n) \). See Sect. 2.2 for precise definitions.

Many framed \( E_n \)-algebras of interest in the category of spaces have \( \pi_0 \) isomorphic to \( \mathbb{N}_0 \), the non-negative integers, and one can compare the homology of different components. If the \( i \)th homology group of the \( k \)th component eventually becomes independent of \( k \), then the framed \( E_n \)-algebra is said to have homological stability, see Definition 32. This is the property of framed \( E_n \)-algebras we are interested in.

The basic example of a framed \( E_n \)-algebra that has homological stability is the configuration space of unordered particles in an \( n \)-dimensional Euclidean space, \( X = \bigsqcup_{k \geq 0} C_k(\mathbb{R}^n) \) [1] [2, Appendix A]. Here \( C_k(\mathbb{R}^n) \) is defined as \( \{(x_1, \ldots, x_k) \in (\mathbb{R}^n)^k \mid x_i \neq x_j \text{ if } i \neq j\}/\mathfrak{S}_k \) with \( \mathfrak{S}_k \) the symmetric group on \( k \) letters. Other examples of framed \( E_n \)-algebras with homological stability include symmetric powers of \( \mathbb{R}^n \) [3], bounded symmetric powers of \( \mathbb{R}^n \) [4, 5], various decorated configuration spaces [6], completions of certain partial \( E_n \)-algebras [7], some spaces of branched covers [8], classifying spaces of groups of diffeomorphisms fixing a disk [9–11], moduli spaces of manifolds embedded in \( \mathbb{R}^n \) [12], moduli spaces of instantons [13] and spaces of rational or holomorphic functions [2, 14–16]. In all of these examples, the map eventually inducing isomorphisms on homology is constructed using the framed \( E_n \)-algebra structure.

We believe the result of this paper can be put into an \( \infty \)-categorical context. However, we chose not to do this because it is not strictly necessary and would make the paper less accessible. As a convention, all our manifolds are smooth (but see Remark 2).

1.2 Homological stability for topological chiral homology

One can use a framed \( E_n \)-algebra as coefficients for a homology theory on oriented \( n \)-dimensional manifolds, which is called \emph{topological chiral homology}. The input is a framed \( E_n \)-algebra \( A \) in a symmetric monoidal \((\infty, 1)\)-category \( \mathsf{C} \) (for us spaces or chain complexes) and an oriented \( n \)-dimensional manifold \( M \), and the output is
an object \( \int_M A \) of \( C \). It is also known as factorization homology, higher Hochschild homology or configuration spaces with summable labels. References for topological chiral homology include [17–21].

Many of the examples of framed \( E_n \)-algebras with homological stability mentioned in the previous subsection are obtained by applying a geometric construction to \( \mathbb{R}^n \) and have natural analogues replacing \( \mathbb{R}^n \) by an arbitrary \( n \)-dimensional manifold \( M \). In many cases, the result of this geometric construction is weakly equivalent to the topological chiral homology of the manifold with coefficients in the framed \( E_n \)-algebra. For example, we have \( \bigsqcup_{k \geq 0} C_k(M) \simeq \int_M \bigsqcup_{k \geq 0} C_k(\mathbb{R}^n) \). See Example 4 and Sect. 6 for other examples.

If \( M \) is a connected manifold and \( X \) is a framed \( E_n \)-algebra in spaces, then there is an isomorphism \( \pi_0(X) \cong \pi_0(\int_M X) \). This leads us to the following question, originally posed by Ralph Cohen:

If the connected components of a framed \( E_n \)-algebra in spaces have homological stability, is the same true for its topological chiral homology on an oriented manifold?

The answer to the question turns out to be: Yes, as long as we restrict to non-compact connected manifolds (see Corollary 3). The assumption that the manifold is non-compact is used to construct maps \( t \) between components of the topological chiral homology by “bringing particles in from infinity.” However, see Sect. 1.6 for a reformulation that applies to compact manifolds.

1.3 Chain complexes and charge

The group of connected components of an \( E_n \)-algebra in topological spaces inherits a natural monoid structure. In this paper, we will often work in the category of non-negatively graded chain complexes because we are interested in statements about homology. However, in the category of chain complexes one cannot define connected components and this makes it harder to formulate homological stability. Our solution is to work with chain complexes with an extra grading which keeps track of the “connected components.” We will call this extra grading charge to differentiate it from the homological grading. In examples closely related to configuration spaces, it should be thought of as the number of particles counted with multiplicity. To make this precise, we define a charged space to be a space \( X \) together with a decomposition \( X = \bigsqcup_{c \geq 0} X(c) \), and a charged chain complex to be a chain complex \( A \) with a decomposition \( A = \bigoplus_{c \geq 0} A(c) \). If \( X \) is a charged space with each \( X(c) \) connected, then a charged algebra structure on \( X \) is a framed \( E_n \)-algebra structure on \( X \) which respects the decomposition. That is, we require the algebra structure maps to restrict to maps:

\[
E_n(m) \times X(c_1) \times \cdots X(c_m) \to X(c_1 + \cdots + c_m)
\]

One can similarly define charged algebras in the category of chain complexes. For precise definitions see Sect. 2.2.3. The requirement that the spaces \( X(c) \) be connected is needed for our formulation of homological stability.

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1.4 $E_n$-cell decompositions

Before we state our main results, we describe framed $E_n$-cell attachments which are the primary technical tool of this paper. The class of cellular framed $E_n$-algebras consists of those framed $E_n$-algebras that are built by successive framed $E_n$-cell attachments. Every charged algebra is weakly equivalent to a cellular algebra. This is similar to the fact that every topological space is weakly homotopy equivalent to a CW-complex. Studying framed $E_n$-cell decompositions allows us to characterize which charged algebras have homological stability and bound the homological stability range.

What are framed $E_n$-cell attachments? While we will primarily be interested in framed $E_n$-cell attachments in the category of chain complexes, the construction can also be defined for spaces. In that case the construction is more familiar: one can attach a cell (in the sense of CW-complexes) to a framed $E_n$-algebra in spaces, but the resulting space does not naturally carry the structure of framed $E_n$-algebra. For example, we do not know how to multiply elements in the interior of the cell. However, it is naturally a partial framed $E_n$-algebra and by freely adding those operations that are not defined yet, every partial framed $E_n$-algebra can be completed to a framed $E_n$-algebra. This is a framed $E_n$-cell attachment in spaces and a similar construction works in the category of chain complexes, see Sect. 3.

1.5 Main result

We can now state the main result of this paper. A charged algebra is called bounded generated if it is weakly equivalent to a cellular framed $E_n$-algebra obtained by $E_n$-cell attachments where all $E_n$-cells of a fixed dimension are attached in only finitely many charges.

**Theorem 1** Let $n \geq 2$ and $A$ be a charged algebra in chain complexes, then the following are equivalent:

(i) The charged algebra $A$ has homological stability as in Definition 32.

(ii) For all oriented connected non-compact $n$-dimensional manifolds $M$, $\int_M A$ has homological stability as in Definition 54.

(iii) The charged algebra $A$ is bounded generated as in Definition 71.

As $\int_{\mathbb{R}^n} A \simeq A$, the equivalence of (i) and (ii) can be captured by the slogan: “topological chiral homology has homological stability globally if and only if it has homological stability locally.” Also note that this equivalence between (i) and (ii) holds trivially for $n = 1$, since each non-compact connected 1-dimensional manifold is diffeomorphic to $\mathbb{R}$ and $\int_{\mathbb{R}} A \simeq A$.

Our proof of Theorem 1 is a simultaneous induction involving (i), (ii) and (iii). This shows that bounded generation is a useful concept even if one is only interested in homological stability and reinforces the idea that topological chiral homology is a useful construction even if one is only interested in the properties of $E_n$-algebras. We also remark that the use of condition (iii) to prove (ii) can be summarized by saying that we resolve topological chiral homology in the algebra variable, not in the manifold variable as is traditionally done in homological stability arguments. We note
that our argument does use homological stability for configuration spaces as input (which should be proven using traditional techniques). See Theorem 74 for a version of Theorem 1 involving homological stability with an explicit range.

**Remark 2** The methods of this paper can be used to prove various generalizations of Theorem 1. Two of these will be treated in this paper:

- We can replace the monoid \( \mathbb{N}_0 \) used to define charged algebras with any partial abelian monoid of charges \( C \) (see Definition 25). As we do not know of interesting applications of this level of generality, in this paper we only prove our theorems in the case where the monoid is \( \mathbb{N}_0^d \) with addition.
- One can replace orientations with other tangential structures \( \theta \), e.g. consider framed \( n \)-manifolds and \( E_n \)-algebras. We prove a version of Theorem 1 for any tangential structure with the property that the corresponding space of \( \theta \)-framed embeddings of \( \mathbb{R}^n \) into itself is connected (this includes framed manifolds, but not unoriented manifolds). It would be interesting to know if this connectivity assumption can be removed.

We will not give proofs of the following generalizations, but mention them as they may be of interest:

- One can generalize to other types of manifolds, e.g. topological manifolds and the corresponding version of framed \( E_n \)-algebras. This requires no change except replacing \( O(n) \) with \( \text{Top}(n) \) and the tangent bundle with the tangent microbundle, as well as using ideas from [22] to construct stabilization maps.
- One can replace the target category with chain complexes over any ring or more generally with the positive part \( C_{\geq 0} \) of a stable symmetric-monoidal \( (\infty, 1) \)-category \( C \) with a compatible \( t \)-structure. Examples include spectra and module spectra over an \( E_\infty \)-ring spectrum.

Theorem 1 has the local-to-global principle for homological stability as an easy corollary:

**Corollary 3** Suppose that \( X \) is a charged algebra in spaces, then \( X \simeq \int_{\mathbb{R}^n} X \) has homological stability if and only if \( \int_M X \) has homological stability for all oriented connected non-compact \( n \)-dimensional manifolds \( M \).

Note that to prove this theorem, our methods require that we work in a stable category, like chain complexes. Thus, even though we are primarily interested in \( E_n \)-algebras in spaces, it is helpful to also consider \( E_n \)-algebras in chain complexes.

The above results give us two new techniques for proving homological stability theorems. The first involves proving homological stability locally and then applying the local-to-global homological stability principle. In Sect. 6.1 we will give several examples of this. Here, we illustrate it by giving a new proof of Steenrod’s result that symmetric powers exhibit homological stability [3].

**Example 4** Let \( \text{Sym}_k(M) \) denote the quotient space \( M^k/\mathcal{S}_k \) with the symmetric group \( \mathcal{S}_k \) acting by permuting the terms. We have that

\[
\bigsqcup_{k \geq 0} \text{Sym}_k(M) \simeq \int_M \left( \bigsqcup_{k \geq 0} \text{Sym}_k(\mathbb{R}^n) \right)
\]
and since $\text{Sym}_k(\mathbb{R}^n)$ is contractible for all $k$ and $n$, trivially the framed $E_n$-algebra $\bigsqcup_{k \geq 0} \text{Sym}_k(\mathbb{R}^n)$ has homological stability. Corollary 3 implies that the spaces $\text{Sym}_k(M)$ have homological stability whenever $M$ is an oriented connected non-compact manifold of dimension $n \geq 2$. As symmetric powers are homotopy invariant, this actually proves homological stability for symmetric powers of any finite connected CW-complex by embedding it in some Euclidean space $\mathbb{R}^n$ and taking a regular open neighborhood. This argument extends to ENR’s by definition of an ENR and to arbitrary CW-complexes by exhaustion using finite subcomplexes.

The second new technique involves deducing homological stability from the existence of bounded $E_n$-cell decompositions. In Sect. 6.2, we give several examples of this, including an improvement over previously known homological stability ranges for bounded symmetric powers.

### 1.6 Stable homology and compact manifolds

After discussing the main theorem, two questions remain. What is the stable homology and what happens for compact manifolds? For $E_n$-algebras in spaces both questions are answered by the scanning map

$$s : \int_M^k X \to \Gamma^c_k(M, B^{TM}X)$$

Here $\int_M^k X$ denotes the charge $k$ component of $\int_M X$, $B^{TM}X$ is a bundle over $M$ with fiber given by the $n$-fold delooping $B^nX$ of $X$, and $\Gamma^c_k(-)$ denotes the space of compactly supported sections of degree $k$. In [23], the second author proved that if $X$ has homological stability and $M$ is connected and non-compact, then the scanning map is a homology equivalence in a range increasing with $k$. When $M$ is non-compact, all the components of $\Gamma^c(M, B^{TM}X)$ are homotopy equivalent and thus the stable homology of $\int_M^k X$ is equal to $H_*(\Gamma^c_0(M, B^{TM}X))$, see Theorem 85.

In Theorem 86, we show that when $X$ has homological stability, the scanning map is a homology equivalence in a range even for compact manifolds (for technical reasons, we only give the proof in the case that the manifold is framed). However, for compact $M$ it is no longer the case that the components of $\Gamma^c(M, B^{TM}X)$ are homotopy equivalent and thus there is no such thing as the stable homology. This reflects the fact that $H_*(\int_M^k X)$ need not stabilize when $M$ is compact, even if the components of $X$ have homological stability, see Page 467 of [24]. These results demonstrate the importance of $B^nX$. We show in Proposition 98 that attaching an $E_n$-cell of dimension $N$ to an $E_n$-algebra $X$ amounts to attaching an ordinary cell of dimension $n + N$ to $B^nX$.

### 2 Topological chiral homology and completions of partial algebras

In this section we define an operad $\mathcal{E}^\theta_n$ and topological chiral homology for partial $\mathcal{E}^\theta_n$-algebras. We also establish the existence of a useful spectral sequence and discuss
the notion of homological stability. In contrast to the introduction, we will work with general tangential structures and partial monoids of charges.

### 2.1 Operads, algebras and simplicial objects

We start with recalling some general definitions and results.

#### 2.1.1 Operads, monads and algebras

Operads are a general framework to encode algebraic structures and it hence is no surprise that we will use an operad to define the algebras of interest. An operad is a sequence of objects encoding for all integers \( k \geq 0 \) the \( k \)-ary operations in the algebraic structure of interest. Relevant references are [25,26].

Let \( (\mathcal{C}, \otimes, 1) \) be a symmetric monoidal category with all small colimits, small limits and such that \( \otimes \) preserves colimits. A (reduced) operad in \( \mathcal{C} \) is a sequence of objects \( \{\mathcal{O}(k)\}_{k \geq 0} \) such that \( \mathcal{O}(0) = 1 \) with the following additional data:

(i) an \( \mathfrak{S}_k \)-action on \( \mathcal{O}(k) \), where \( \mathfrak{S}_k \) denotes the symmetric group on \( k \) letters,
(ii) composition morphisms \( \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n) \), and
(iii) a unit map \( 1 \to \mathcal{O}(1) \).

These should satisfy appropriate unit, associativity and equivariance axioms.

The definition of an operad can be rephrased in terms of symmetric sequences. A symmetric sequence \( \mathcal{F} \) is a sequence \( \{\mathcal{F}(k)\}_{k \geq 0} \) of objects \( \mathcal{F}(k) \) with \( \mathfrak{S}_k \)-action. For a discrete group \( G \), \( X \) an object with a right \( G \)-action and \( Y \) an object with a left \( G \) action, let \( X \otimes_G Y \) denote the quotient of \( X \otimes Y \) by the diagonal action of \( G \). Then we can define a monoidal structure on the category of symmetric sequences by letting \( \otimes \) be

\[
(\mathcal{E} \otimes \mathcal{F})(k) := \bigsqcup_n E(n) \otimes_{\mathfrak{S}_n} \left( \bigsqcup_{k_1+\cdots+k_n=k} \mathcal{F}(k_1) \otimes \cdots \otimes \mathcal{F}(k_n) \right)
\]

An operad is the same as a unital monoid in symmetric sequences with respect to this monoidal structure which satisfies \( \mathcal{F}(0) = 1 \).

For any symmetric sequence \( \mathcal{F} \), one can construct a functor \( F : \mathcal{C} \to \mathcal{C} \) as follows.

**Definition 5** Let \( \mathcal{F} \) be an symmetric sequence in \( \mathcal{C} \). Then the functor \( F : \mathcal{C} \to \mathcal{C} \) is given by

\[
C \mapsto \bigsqcup_{n \geq 0} \mathcal{F}(k) \otimes_{\mathfrak{S}_k} C \otimes^k
\]

An object \( X \) of \( \mathcal{C} \) can be viewed as a symmetric sequence \( \mathcal{X} \) with \( \mathcal{X}(0) = X \) and \( \mathcal{X}(i) \) an initial object of \( \mathcal{C} \) for \( i > 0 \). Given a symmetric sequence \( \mathcal{F} \), we have that \( (\mathcal{F} \otimes \mathcal{X})(0) = FX \), and \( (\mathcal{F} \otimes \mathcal{X})(i) \) is an initial object for \( i > 0 \). When \( \mathcal{F} \) is a unital monoid in symmetric sequences, \( F \) will be a unital monoid in functors:
Definition 6 A monad in C is a unital monoid in the category of functors C → C. That is, it is a functor O with a unit natural transformation 1: id → O and a composition natural transformation c: O² → O. These are required to satisfy associativity and unit axioms in the sense that the following diagrams commute:

```
  OOO  O(c)  OO
   c    c
  OO    O
```

```
  O   O(1)   OO
    c
  O    O
```

Proposition 7 Let O be an operad and let O be the functor associated to O, viewed as a symmetric sequence. The operad structure on O endows O with the structure of a monad.

One can then define left modules and right modules over an operad O as left modules and right modules in the category of symmetric sequences over the symmetric sequence \{\mathcal{O}(k)\}.

An O-algebra structure on an object A is a left O-module structure on the associated symmetric sequence A. This agrees with the definition given in [25]. One can also define an algebra over a monad.

Definition 8 Let O be a monad in C. An O-algebra structure on an object A of C is a morphism a: O(A) → A satisfying associativity and unit axioms in the sense that the following two diagrams commute:

```
  OO(A)  O(a)  O(A)
    c    a
  O(A)    A
```

```
  A   1   O(A)
    a
  A    A
```

An O-algebra in C is an object with a fixed structure of an O-algebra.

Unwinding this definition in the case the monad comes from an operad, one finds that an O-algebra structure is the same as a O-algebra structure. Thus, we will not differentiate between algebras over an operad and algebras over the associated monad.

Example 9 Note that for any object X, OX is an O-algebra. Moreover, it is free, as O is the left adjoint to the forgetful functor from the category of O-algebras in C to the underlying category C.

A right O-module structure on a symmetric sequence F gives rise to a right functor structure on F over the monad O, a notion which is defined as follows:

Definition 10 A right functor F over a monad O is a functor F: C → C with a natural transformation : FO → F satisfying associativity and unit axioms in the sense that the following two diagrams commute:
One reason to spell out this definition is that we will occasionally be interested in some right functors which do not come from right modules, such as the \( n \)-fold suspension functor viewed as a right functor over the little \( n \)-disks operad.

### 2.1.2 Categories copowered over \( \text{Top} \)

We next describe the contexts in which we will work the remainder of this paper. Let \( \mathcal{C} \) be a symmetric monoidal category as before, i.e. with all small colimits and limits, and \( \otimes \) preserving colimits. Let \( \text{Top} \) denote the category of compactly generated weakly Hausdorff spaces. Then we further suppose \( \mathcal{C} \) has a lax-monoidal copowering (also known as tensoring) over \( \text{Top} \), i.e. there is a functor \( \otimes : \text{Top} \times \mathcal{C} \to \mathcal{C} \) and a natural transformation \( \alpha : X \otimes (Y \otimes C) \to (X \times Y) \otimes C \) satisfying associativity and unit axioms. We will only consider the following examples:

1. **Topological spaces**: \( \mathcal{C} = \text{Top} \), \( X \otimes Y = X \times Y \) and \( X \otimes (Y \otimes C) \to (X \times Y) \otimes C \) the identity.
2. **Simplicial sets**: \( \mathcal{C} = \text{sSet} \), \( X \otimes Y = \text{Sing}(X) \times Y \) and \( X \otimes (Y \otimes C) \to (X \times Y) \otimes C \) induced by the natural isomorphism \( \text{Sing}(X) \times \text{Sing}(Y) \cong \text{Sing}(X \times Y) \).
3. **Chain complexes (over \( \mathbb{Z} \))**: \( \mathcal{C} = \text{Ch} \), \( X \otimes A = C_*(X) \otimes A \) and \( X \otimes (Y \otimes C) \to (X \times Y) \otimes C \) induced by the Eilenberg–Zilber map \( \text{EZ} : C_*(X) \otimes C_*(Y) \to C_*(X \times Y) \).

These examples have well-known model structures: the Quillen model structure on \( \text{Top} \), the Quillen model structure on simplicial sets, and the projective model structure on \( \text{Ch} \). This allows us to talk about weak equivalences and cofibrations.

### 2.1.3 Bar constructions and homotopy-theoretic techniques

In this paper we will use several constructions (e.g. bar constructions or functors associated to symmetric sequences), and we will recall sufficient conditions for these constructions to preserve weak equivalences. These results are often well-known and we will apply them throughout the remainder of this paper.

We first discuss the homotopy invariance properties of functors associated to symmetric sequences. A symmetric sequence \( \mathcal{F} = \{ F(k) \}_{k \geq 0} \) is said to be \( \Sigma \)-cofibrant if each \( F(k) \) is cofibrant as an object with \( \Sigma_k \)-action (in the projective model structure). This is the case if \( F(k) \) is cofibrant as an object of \( \mathcal{C} \) and the action is free.

**Lemma 11** Suppose \( \mathcal{F} \) is \( \Sigma \)-cofibrant and \( X \) is cofibrant, then \( F(X) \) is cofibrant. If \( X \to Y \) is a weak equivalence between cofibrant objects of \( \mathcal{C} \), then \( F(X) \to F(Y) \) is a weak equivalence. Similarly, if \( X \) is cofibrant and \( \mathcal{F} \to \mathcal{G} \) is a weak equivalence between \( \Sigma \)-cofibrant symmetric sequence, then \( F(X) \to G(X) \) is a weak equivalence.
Furthermore, when $C = \text{Top}$ we can drop the cofibrancy assumptions on $X$, $Y$, and $F(k)$ and $G(k)$ for $k \geq 0$, if we additionally assume that (i) $F(k)$ and $G(k)$ are Hausdorff for $k \geq 0$, and (ii) the actions of $\mathcal{G}_k$ on $F(k)$ and $G(k)$ are free.

Proof The first part follows from Lemma 11.5.2 and Proposition 11.5.3 of [26]. For the second part, we note that for a free action of a finite group $G$ on a Hausdorff space $Z$, the quotient map $Z \to Z/G$ is a covering map. For a proof, see e.g. Lemma 79.1 and Theorem 81.5 of [27] and note that the proof of 81.5 does not use the assumption of local path-connectedness in the proof that the quotient map is a covering map. Thus when the $\mathcal{G}_k$-action on $F(k)$ is free, there is a long exact sequence of homotopy groups for the fiber sequence $\mathcal{G}_k \to F(k) \times X^k \to F(k) \times \mathcal{G}_k X^k$. The five lemma now proves the second part. 

To avoid repeating arguments, one may use the following lemma.

Lemma 12 If for all $k \geq 0$ the action of $\mathcal{G}_k$ on $O(k)$ is free, then for $X \in \text{Top}$ or $sSet$ there is a natural weak equivalence $F(C_*(X)) \to C_*(F(X))$.

Proof The Eilenberg–Zilber map

$$C_* (\mathcal{F}(k)) \otimes C_* (X)^{\otimes k} \to C_* \left( \mathcal{F}(k) \times X^k \right)$$

is a weak equivalence by the Künneth theorem. This map is $\mathcal{G}_k$-equivariant and since the action of $\mathcal{G}_k$ on $F(k)$ is free, both are levelwise free chain complexes of $\mathbb{Z}[\mathcal{G}_k]$-modules. As a quasi-isomorphism between bounded below level free chain complexes is a homotopy equivalence, we get a weak equivalence upon applying $- \otimes_{\mathbb{Z}[\mathcal{G}_k]} \mathbb{Z}$. Now note that we have

$$\left( C_* (\mathcal{F}(k)) \otimes C_* (X)^{\otimes k} \right) \otimes_{\mathbb{Z}[\mathcal{G}_k]} \mathbb{Z} \cong C_* (\mathcal{F}(k)) \otimes_{\mathbb{Z}[\mathcal{G}_k]} C_*(X)^{\otimes k}$$

$$C_* \left( \mathcal{F}(k) \times X^k \right) \otimes_{\mathbb{Z}[\mathcal{G}_k]} \mathbb{Z} \cong C_* \left( \mathcal{F}(k) \times \mathcal{G}_k X^k \right)$$

We will next describe a simplicial object known as the double bar construction. For $C = \text{Top}$, $sSet$, or $\text{Ch}$, geometric realization of a simplicial object $C_\bullet$ in $C$ is defined to be the coend over $\Delta$ of $[k] \mapsto \Delta^k$ and $[k] \mapsto C_k$. For $C = sSet$, this is naturally weakly equivalent to the diagonal of the bisimplicial set. For $C = \text{Top}$, this is the ordinary (thin) geometric realization. For $C = \text{Ch}$ this is naturally weakly equivalent to taking the total chain complex of the associated normalized double chain complex.

Definition 13 Let $O$ be an operad, $A$ an $O$-algebra and $F$ a right $O$-functor. Let $B_\bullet (F, O, A)$ be the simplicial object in $C$ given by

$$[p] \mapsto F(O^p(A))$$

with face maps $B_{p}(F, O, A) \to B_{p-1}(F, O, A)$ induced by (i) for $d_0$ the natural transformation $FO \to F$, (ii) for $d_i$ with $0 < i < p$ the composition natural transformation $O^2 \to O$ applied to the $i$th and $(i+1)$st copy of $O$, and (iii) the action
map $OA \rightarrow A$ for $d_p$. Similarly, the degeneracies are induced by the unit natural transformation $id \rightarrow O$.

For $C = \text{Top}, \text{sSet}$, or $\text{Ch}$, we define

$$B(F, O, A) := |B_\bullet(F, O, A)|$$

with $|−|$ the (thin) geometric realization discussed above.

We will now discuss the homotopy invariance properties of this construction, and a spectral sequence computing its homology.

For a simplicial object $X_\bullet$ in $C$, the $p$th latching object is defined to be $L_p X := \text{colim}_{[r] \hookrightarrow [p]} X_r$ where the colimit is over all injective maps $[r] \hookrightarrow [p]$ in $\Delta^{op}$ not equal to $\text{id}_{[p]}$. A simplicial object $X_\bullet$ in $C$ is said to be Reedy cofibrant if each map $L_p X \rightarrow X_{p+1}$ is a cofibration. This implies each $X_p$ is cofibrant and is implied by all degeneracy maps being cofibrations. Also note all bisimplicial sets are Reedy cofibrant. If $C = \text{Top}$, a related definition is that of a proper simplicial space: this is Reedy cofibrancy with respect to the Strøm model structure, i.e. $L_p X \rightarrow X_{p+1}$ is required to be a Hurewicz cofibration. This is implied by each $s_i : X_p \rightarrow X_{p+1}$ being a Hurewicz cofibration. Note there is no condition on the individual $X_p$, as all spaces are cofibrant in the Strøm model structure.

**Lemma 14** If $X_\bullet \rightarrow Y_\bullet$ is a levelwise weak equivalence between Reedy cofibrant simplicial objects in $C$, then $|X_\bullet| \rightarrow |Y_\bullet|$ is a weak equivalence. When $C = \text{Top}$ and $X_\bullet \rightarrow Y_\bullet$ is a levelwise weak equivalence between proper simplicial objects in $C$, then $|X_\bullet| \rightarrow |Y_\bullet|$ is also a weak equivalence.

**Proof** This is discussed in Section A.2.9 of [28]. For proper simplicial objects in $C = \text{Top}$, this is Proposition A.1 of [29].

**Lemma 15** Associated to each Reedy cofibrant simplicial object or proper simplicial space $X_\bullet$ there is a geometric realization spectral sequence given by

$$E^1_{pq} = H_q(X_p) \Rightarrow H_\ast(|X_\bullet|)$$

with $d_1$-differential given by the alternating sum of the map induced on homology by the face maps, i.e. $d_1 = \sum_i (-1)^i (d_i)_\ast$. Given a simplicial map $f_\bullet : X_\bullet \rightarrow Y_\bullet$ as above, there is also a relative geometric realization spectral sequence given by

$$E^1_{pq} = H_q(Y_p, X_p) \Rightarrow H_\ast(|Y_\bullet|, |X_\bullet|)$$

with with $d_1$-differential given by the alternating sum of the map induced on homology by the face maps.

**Proof** This is the spectral sequence associated to the skeletal filtration, where the Reedy cofibrancy condition is used to identify the $E^1$-page. For proper simplicial spaces, Section 5 of [30] describes the spectral sequence for the so-called fat geometric realization, and the Reedy cofibrancy condition implies the fat realization is weakly equivalent to the geometric realization by Proposition A.1 of [29].

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Lemma 16  If $X_\bullet$ is a Reedy cofibrant simplicial object in $C = \text{Top}$ or $\text{sSet}$, or a proper simplicial object in $C = \text{Top}$, then $C_\ast(X_\bullet)$ is a Reedy cofibrant simplicial chain complex and there is a natural weak equivalence $|C_\ast[X_\bullet]| \to C_\ast(|X_\bullet|)$.

Proof  The map is induced by Eilenberg–Zilber maps, and that it is a weak equivalence is a consequence of spectral sequence comparison applied to the geometric realization spectral sequence. Also see Theorem 5.5.2.17 of [20].

Suppose that $F$ comes from a symmetric sequence $F$ and $O$ is an operad, then we can summarize the results as follows. Firstly, if $C = \text{sSet}$ we have that $|B_\ast(F, O, -)|$ preserves a weak equivalence $A \to B$ of $O$-algebras when $F(k)$ and $O(k)$ have free $S_k$-actions. If $C = \text{Ch}$, one additionally needs to demand that the underlying chain complexes of $A, B, O(k)$ and $F(k)$ are cofibrant. If $C = \text{Top}$ one additionally needs to demand that the inclusion $\{\text{id}\} \hookrightarrow O(1)$ is a Hurewicz cofibration and that the spaces $O(k)$ and $F(k)$ are Hausdorff.

2.2 Little disks operads, $\mathcal{E}_n^\theta$-algebras and charged algebras

In this subsection we define the algebras that are the subject of this paper. As discussed in the previous two subsections, we shall always assume that $C = \text{Top}, \text{sSet}$, or $\text{Ch}$.

2.2.1 Tangential structures

A tangential structure is a map $\theta : W \to BO(n)$. As a convention, we will demand that $W$ is 1-connected and $\theta : W \to BO(n)$ is a fibration. Recall that the classifying space of the $n$th orthogonal group $BO(n)$ has a universal vector bundle $\gamma$ over it. A $\theta$-framing of an $n$-dimensional manifold $M$ is a bundle map $\phi_M : TM \to \theta^*\gamma$, where by definition a bundle map is a fiberwise linear isomorphism. We fix once and for all a $\theta$-framing on $\mathbb{R}^n$. Because $W$ is non-empty, this exists. Because $W$ is 1-connected, Lemma 19 says this is essentially unique.

Let $\text{Bun}(TM, TN)$ denote the space of all bundle maps $TM \rightarrow TN$ with the compact-open topology, and $\text{Map}(-, -)$ the space of continuous maps with the compact-open topology.

Definition 17  If $M$ and $N$ are $\theta$-framed manifolds, then $\text{Bun}^\theta(TM, TN)$ is the space of triples $\Phi = (\phi, \alpha, \varphi) \subset \text{Bun}(TM, TN) \times \text{Map}(M, [0, \infty)) \times \text{Map}([0, \infty), \text{Bun}(TM, \theta^*\gamma))$ of a bundle map $\phi : TM \to TN$, a locally constant function $\alpha : M \to [0, \infty)$ and continuous map $\varphi : [0, \infty) \to \text{Bun}(TM, \theta^*\gamma)$ starting at $\phi_M$ and equal to $\phi_N \circ \phi$ on $[\alpha, \infty)$ (i.e. a Moore path).

This is homotopy equivalent to the homotopy fiber over $\phi_M$ of the map $\phi_N \circ - : \text{Bun}(TM, TN) \to \text{Bun}(TM, \theta^*\gamma)$.
**Definition 18** We define the space of $\theta$-framed embeddings as the pull back in the following diagram (where $\text{Emb}$ has the $C^\infty$-topology):

\[
\begin{array}{c}
\text{Emb}^\theta(M, N) \\
\downarrow \\
\text{Emb}(M, N)
\end{array} \rightarrow 
\begin{array}{c}
\text{Bun}^\theta(TM, TN) \\
\text{Bun}(TM, TN)
\end{array}
\]

**Lemma 19** For any two $\theta$-framings $\phi_1$ and $\phi_2$ on $\mathbb{R}^n$, the space $\text{Emb}^\theta((\mathbb{R}^n)_{\phi_1}, (\mathbb{R}^n)_{\phi_2})$ is path-connected.

**Proof** For any two $\theta$-framed manifolds the map $\text{Emb}^\theta(M, N) \rightarrow \text{Emb}(M, N)$ is a Serre fibration, because it is the pullback of the Serre fibration $\text{Bun}^\theta(TM, TN) \rightarrow \text{Bun}(TM, TN)$. In the case that $M = N = \mathbb{R}^n$, the map $\text{Emb}(M, N) \rightarrow \text{Bun}(TM, TN)$ is a weak equivalence. Thus it suffices to show that $\text{Bun}^\theta((\mathbb{T}\mathbb{R}^n)_{\phi_1}, (\mathbb{T}\mathbb{R}^n)_{\phi_2})$ is path-connected. It is weakly equivalent to the homotopy fiber of the map $\phi_2 \circ -: \text{O}(n) \rightarrow \text{Fr}(\theta^*\gamma)$ at the $\theta$-framing $\phi_1$, the latter being the frame bundle of $\theta^*\gamma$. This is in turn weakly equivalent to the homotopy fiber of the map $\ast \rightarrow W$ (the base point does not matter since $W$ is 0-connected), and hence to the based loop space $\Omega W$, which is path-connected since $W$ was assumed to be 1-connected. 

One can compose two $\theta$-framed embeddings $(f, \Phi): M \rightarrow N$ and $(g, \Psi): N \rightarrow P$ by composing the embeddings to $g \circ f$, and concatenating the triples $\Phi = (\phi, \alpha, \varphi)$ and $\Psi = (\psi, \beta, \upsilon)$ as follows. One composes the bundle maps to obtain a bundle $\psi \circ \phi: T M \rightarrow T P$, takes the locally constant function $\beta \circ f + \alpha: M \rightarrow [0, \infty)$ and takes the path of bundle maps $T M \rightarrow \theta^*\gamma$ to be

\[
t(t) = \begin{cases} 
\varphi(t) & \text{if } t \in [0, \alpha] \\
\upsilon(t - \alpha) \circ \phi & \text{if } t \in [\alpha, \alpha + \beta] \\
\phi_P \circ \psi \circ \phi & \text{if } t \in [\alpha + \beta, \infty)
\end{cases}
\]

This composition is associative. Composition with $\text{id}: M \rightarrow M$ with the length 0 Moore path at $\phi_M$ is the identity. We can use this to define a category of $\theta$-framed manifolds and embeddings.

**Definition 20** Let $\text{Emb}^\theta$ denote the topological category whose objects are $\theta$-framed manifolds and whose morphism spaces are spaces of $\theta$-framed embeddings. Disjoint union gives $\text{Emb}^\theta$ the structure of a symmetric monoidal category. On morphisms $(f, \Phi): M \rightarrow N$ and $(g, \Psi): P \rightarrow Q$, it is given by the map $(f \sqcup g, \Phi \sqcup \Psi)$, with $\Phi \sqcup \Psi$ defined by requiring its restriction to $M$ and $P$ to be $\Phi$ and $\Psi$ respectively.

The reason for requiring the length $\alpha$ of our Moore path to be locally constant on $M$ instead of constant, is to make $(- \sqcup -)$ a functor.

**Example 21** The examples of tangential structures relevant to this paper are:
the unoriented tangential structure $\pi_O : BO(n) \to BO(n)$ (note here $W = BO(n)$ is not 1-connected),

- the oriented tangential structure $\pi_{SO} : EO(n)/SO(n) \to BO(n)$, and

- the framed tangential structure $\pi_{pt} : EO(n) \to BO(n)$.

For $\pi_O$, we have $\Bun^O(TM, TN) \simeq \Bun(TM, TN)$ and $\Emb^O(M, N) \simeq \Emb(M, N)$. One can always forget a $\theta$-framing to a $\pi_O$-framing, using the canonical map $W \to BO(n)$ over $BO(n)$ to induce natural maps $\Bun^\theta(TM, TN) \to \Bun^O(TM, TN)$ and $\Emb^\theta(M, N) \to \Emb^O(M, N)$.

### 2.2.2 $\mathcal{E}_n^0$-algebras

The operad of $\theta$-framed little $n$-disks will be defined in terms of $\theta$-framed embeddings of copies of $\mathbb{R}^n$ into $\mathbb{R}^n$.

**Definition 22** The $\theta$-framed little $n$-disks operad $\mathcal{E}_n^\theta$ is the operad in the category $(\text{Top}, \times, \ast)$ given by $\mathcal{E}_n^\theta(k) = \Emb^\theta(\sqcup_k \mathbb{R}^n, \mathbb{R}^n)$. The symmetric group acts by permuting the Euclidean spaces in the domain, the operad composition map is given by composition of $\theta$-framed embeddings as described in the previous subsection, and the unit in $\mathcal{E}_n^\theta(1)$ is $\text{id} : \mathbb{R}^n \to \mathbb{R}^n$ with Moore path at $\phi_{\mathbb{R}^n}$ of length 0.

**Definition 23** Let $\mathbf{E}_n^\theta$ denote the monad in $\mathbf{C}$ associated to the operad $\mathcal{E}_n^\theta$.

The operad associated to the tangential structure $\pi_{SO} : EO(n)/SO(n) \to BO(n)$, $\mathcal{E}_n^{SO}$, is called the framed little $n$-disks operad, and the operad associated to the tangential structure $\pi_{pt} : EO(n) \to BO(n)$, $\mathcal{E}_n^{pt}$, is called the little $n$-disks operad.

**Remark 24** The nomenclature surrounding little disks operad can be confusing. We give some clarifications:

- It is standard but confusing that the framed little $n$-disks operad consists of orientation-preserving embeddings while the little $n$-disks operad consists of the framed embeddings.

- Some authors use a version of the framed little $n$-disks operad which we denote $(\mathcal{E}_n^{SO})^{\text{rect}}$, given by taking the subspace of $\Emb(\sqcup_k D^n, D^n)$ consisting of the orientation-preserving embeddings that are a composition of translation, dilation and rotations. Identifying $\text{int}(D^n)$ with $\mathbb{R}^n$ induce a weak equivalence of operads $(\mathcal{E}_n^{SO})^{\text{rect}} \to \mathcal{E}_n^{SO}$, which can used to obtain from every $(\mathcal{E}_n^{SO})^{\text{rect}}$-algebra a weakly equivalent $\mathcal{E}_n^{SO}$-algebra by double bar construction $B(\mathbf{E}_n^{SO}, (\mathbf{E}_n^{SO})^{\text{rect}}, \ast)$. To this algebra one can then apply the results of this paper.

- Some authors define the framed little $n$-disks operad using embeddings that do not necessarily preserve the orientation, i.e. they use $BO(n)$ instead of $BSO(n)$. Our proof does not work for this definition, cf. part (ii) of Remark 2.

- Finally, there is a distinction between the unital and non-unital $\mathcal{E}_n^\theta$-operads, which differ in whether $\mathcal{E}_n^\theta(0)$ is $\ast$ or $\emptyset$. Ours is the former.

### 2.2.3 Charged algebras

We discuss algebras with a generalization of charge as discussed in the introduction.
Definition 25 A partial monoid of charges is an abelian cancellative partial monoid \( C \) with unit.

We fix such a partial monoid \( C \) of charges. The reader may want to keep in mind the example \( C = \mathbb{N}_0 \), the non-negative integers under addition. Partial monoids that are not monoids will not be needed until Sect. 6.2. We next define \( C \)-charged algebras to remedy the fact connected components do not make sense for chain complexes.

Definition 26 (i) A \( C \)-charged space is a space \( X \) with a decomposition \( X = \bigsqcup_{c \in C} X(c) \). A morphism of \( C \)-charged spaces is a continuous map preserving the decomposition.

(ii) A \( C \)-charged simplicial set is a simplicial \( X \) with a decomposition \( X = \bigsqcup_{c \in C} X(c) \). A morphism of \( C \)-charged spaces is a simplicial map preserving the decomposition.

(iii) A \( C \)-charged chain complex is a non-negatively graded chain complex \( A \) with a decomposition \( A = \bigoplus_{c \in C} A(c) \) as chain complexes. A morphism of \( C \)-charged chain complexes is a chain map preserving the decomposition.

If \( x \in X \) is an element of \( X(c) \) or \( a \in A \) is an element of \( A(c) \), then we say it has charge \( c \). There is a \( C \)-charged space \( C \) with a single point in each charge, and a \( C \)-charged chain complex \( \mathbb{Z}[C] \) with a single \( \mathbb{Z} \)-summand in each charge. We will also view \( C \) as a \( C \)-charged simplicial set by taking 0-simplices to be \( C \) and all \( k \)-simplices for \( k \geq 1 \) to be degenerate.

Definition 27 (i) An augmented \( C \)-charged space \( X \) is a \( C \)-charged space \( X \) with a map \( \epsilon : X \to C \) of \( C \)-charged spaces, which we call the augmentation map. We denote the category of augmented \( C \)-charged spaces by \( \text{Top}_C \).

(ii) An augmented \( C \)-charged simplicial set \( X \) is a \( C \)-charged simplicial set \( X \) with a map \( \epsilon : X \to C \) of \( C \)-charged simplicial sets, which we call the augmentation map. We denote the category of augmented \( C \)-charged simplicial sets by \( \text{sSet}_C \).

(iii) An augmented \( C \)-charged chain complex \( A \) is a \( C \)-charged chain complex \( A \) with a map \( \epsilon : A \to \mathbb{Z}[C] \) of \( C \)-charged chain complexes, which we call the augmentation map. We denote the category of augmented \( C \)-charged spaces by \( \text{Ch}_C \).

We remark that a \( C \)-charged space or simplicial set has a unique augmentation, but this is not the case for \( C \)-charged chain complexes. Conversely, from a map \( \epsilon : X \to C \) one can recover the structure of a \( C \)-charged space or simplicial set on \( X \), but this is not the case for \( C \)-charged chain complexes.

Definition 28 (i) An augmented \( C \)-charged space \( X \) is said to be connected if \( \epsilon \) induces a \( \pi_0 \)-isomorphism.

(ii) An augmented \( C \)-charged simplicial set \( X \) is said to be connected if \( \epsilon \) induces a \( \pi_0 \)-isomorphism.

(iii) An augmented \( C \)-charged chain complex \( A \) is said to be connected if each chain complex \( A(c) \) is 0 in negative homological degrees and \( \epsilon \) induces a \( H_0 \)-isomorphism.
The chains on a (augmented) $C$-charged space have the structure of a (augmented) $C$-charged chain complex. Augmented $C$-charged spaces, simplicial sets or chain complexes are symmetric monoidal categories using Day convolution applied to the Cartesian and tensor products respectively: $(X \times Y)(e) = \bigsqcup_{e+e''=e} X(e') \times X(e'')$ and $(A \otimes B)(e) = \bigoplus e+e''=e A(e') \otimes B(e'')$. The new augmentations are obtained as compositions $X \times Y \to C \times C \to C$ and $A \otimes B \to \mathbb{Z}[C] \otimes \mathbb{Z}[C] \to \mathbb{Z}[C]$, with both right-hand maps coming from the partial monoid structure of $C$.

The operad $\mathcal{E}^\theta_n$ can be considered as an operad in $C$-charged spaces, by declaring all elements to have charge 0. Note that the forgetful functor from $C$-charged spaces, simplicial sets or chain complexes to spaces, simplicial sets or chain complexes is only symmetric monoidal if $C$ is a monoid. Thus, $\mathcal{E}_n^\theta$-algebras in $C$-charged spaces, simplicial sets or chain complexes are not necessarily $\mathcal{E}^\theta_n$-algebras in the underlying category.

**Definition 29**

(i) A $C$-charged $\mathcal{E}_n^\theta$-algebra in spaces is a connected augmented $C$-charged space $X$ with the structure of an algebra over $\mathcal{E}_n^\theta$ in $\text{Top}_C$.

(ii) A $C$-charged $\mathcal{E}_n^\theta$-algebra in simplicial sets is a connected augmented $C$-charged simplicial set $X$ with the structure of an algebra over $\mathcal{E}_n^\theta$ in $\text{sSet}_C$.

(iii) A $C$-charged $\mathcal{E}_n^\theta$-algebra in chain complexes is a connected augmented $C$-charged chain complex $A$ with the structure of an algebra over $C_*(\mathcal{E}_n^\theta)$ in $\text{Ch}_C$.

To guarantee that the constructions in this paper are invariant under weak equivalences, we will need to impose some cofibrancy conditions on our $C$-charged $\mathcal{E}_n^\theta$-algebra in chain complexes.

**Definition 30** A $C$-charged $\mathcal{E}_n^\theta$-algebra in chain complexes $A$ is cofibrant if the underlying chain complex $A$ is cofibrant.

Note that if $X$ is a $C$-charged $\mathcal{E}_n^\theta$-algebra in spaces or simplicial sets, then $C_*(X)$ is always cofibrant. Unless mentioned otherwise, charged algebras will be assumed to be cofibrant.

### 2.2.4 Homological stability for charged algebras

Let $A$ be a $C$-charged $\mathcal{E}_n^\theta$-algebra in chain complexes. The augmentation gives a preferred choice of a generator of $H_0(A(e_0))$. Assume $e + e_0$ is defined in $C$ and pick a representative $a \in A(e_0)$ of this generator. We can define a stabilization map

$$t_{e_0} : H_\ast(A(e)) \to H_\ast(A(e + e_0))$$

by multiplying with the homology class of $a$ using the ring structure on $H_\ast(A)$ induced by the $\mathcal{E}_n^\theta$-algebra structure. If $n \geq 2$, the ring structure is commutative, and the map $t_{e_0}$ is uniquely determined by the choice of augmentation. The singular chains on a $C$-charged $\mathcal{E}_n^\theta$-algebra $X$ in spaces are a $C$-charged $\mathcal{E}_n^\theta$-algebra in chain complexes, so this definition also gives us a stabilization map on the homology of $X$.

We now define homological stability when $C = \mathbb{N}_0^d$, whose elements we denote by $k$. We let $e_i$ denote the basis vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 is in the $i$th position) and call $t_i := t_{e_i}$ a basic stabilization map.
Definition 31 A map \( f : X \to Y \) of chain complexes, simplicial sets, or spaces is called an \( N \)-equivalence if the relative homology group \( H_*(Y, X) \) vanishes for \( * \leq N \). Equivalently \( f \) induces an isomorphism on homology groups \( H_* \) for \( * \leq N - 1 \) and a surjection for \( * \leq N \).

Definition 32 Let \( C = \mathbb{N}_0^d \).

(i) Let \( A \) be a \( C \)-charged \( \mathcal{E}_n^\theta \)-algebra in chain complexes, then we say that \( A \) has homological stability if there is a function \( \rho : \mathbb{N}_0 \to \mathbb{N}_0 \) with \( \lim_{j \to \infty} \rho(j) = \infty \) such that all basic stabilization maps \( t_i : H_*(A(k)) \to H_*(A(k + e_i)) \) are \( \rho(k_i) \)-equivalences.

(ii) Let \( X \) be a \( C \)-charged \( \mathcal{E}_n^\theta \)-algebra in spaces or simplicial sets, then we say that \( X \) has homological stability if \( C_*(X) \) has.

Remark 33 – In the case \( C = \mathbb{N}_0 \), we may remove the reference to a function \( \rho \) and instead say that all basic stabilization maps induce isomorphisms on \( H_* \) for \( k \gg * \).

– Definition 32 is not the most general definition of homological stability that one could give. However, for us it is not enough to demand that iterated stabilization maps are eventually isomorphisms, because we will need uniformity in the \( d \) different directions in \( \mathbb{N}_0^d \).

– To give a quantitative version, Theorem 74, of Theorem 1 we will impose extra conditions on the function \( \rho \). These extra conditions are easier to state if one extends the domain and codomain of \( \rho \) from \( \mathbb{N}_0 \) to \( \mathbb{R} \).

2.3 Topological chiral homology

In this subsection we will define topological chiral homology, first for \( \mathcal{E}_n^\theta \)-algebras on \( \theta \)-framed manifolds but eventually for partial \( \mathcal{E}_n^\theta \)-algebras on \( \theta \)-framed manifold bundles. We will also discuss homological stability and a useful spectral sequence. As before \( C = \text{Top}, \text{sSet} \) or \( \text{Ch} \), and \( C \)-charged \( \mathcal{E}_n^\theta \)-algebras in \( \text{Ch} \) are assumed to be cofibrant.

2.3.1 Topological chiral homology

To define topological chiral homology we define the following right \( \mathbf{E}_n^\theta \)-functor.

Definition 34 Suppose \( M \) is a \( \theta \)-framed manifold. Let \( \mathbf{M}^\theta \) be the right functor over \( \mathbf{E}_n^\theta \) given by

\[
C \mapsto \bigsqcup_{k \geq 0} \text{Emb}^\theta(\sqcup_k \mathbb{R}^n, M) \otimes \mathcal{E}_k C^{\otimes k}.
\]

The right functor structure is induced by composition of \( \theta \)-framed embeddings.

We will now define topological chiral homology of an \( \mathcal{E}_n^\theta \)-algebra \( A \) in \( C \) over a \( \theta \)-framed manifold \( M \) using the two-sided bar construction of Definition 13.
Definition 35 We define the topological chiral homology \( \int_M A \) of \( A \) over \( M \) to be the geometric realization

\[
\int_M A := B(M^\theta, E_n^\theta, A).
\]

We remark that \( M^\theta \) is an enriched functor from the category \( \text{Emb}^\theta \) of \( \theta \)-framed manifolds with spaces of \( \theta \)-framed embeddings as spaces of morphisms, to the category of right \( E_n^\theta \)-functors in \( \mathcal{C} \). Hence so is \( M \mapsto \int_M A \). This makes sense as \( \mathcal{C} \) has a copowering over \( \text{Top} \).

Definition 35 is one of many equivalent models for topological chiral homology and is a concrete instance of the homotopy coend in Definition 3.2 of [18]. One could also check that this model satisfies the axioms in Theorem 1.1 of [18], for example the following.

Lemma 36 We have that

\[
\int_{\mathbb{R}^n} A \simeq A.
\]

Proof For \( M = \mathbb{R}^n \), we have that \( M^\theta = E_n^\theta \). The unit of the monad endows the simplicial object \( B_\ast(E_n^\theta, E_n^\theta, A) \) with an extra degeneracy. □

Another property is that topological chiral homology preserves \( N \)-equivalences.

Lemma 37 Let \( A \to B \) is a map of \( E_n^\theta \)-algebras that induces an isomorphism on homology. Then \( \int_M A \to \int_M B \) also induces an isomorphism on homology. More generally, if \( f : A \to B \) is an \( N \)-equivalence then so is \( \int_M A \to \int_M B \).

Proof By Lemma 16 we may assume \( \mathcal{C} = \text{Ch} \). Since we constructed topological chiral homology by geometrically realizing a Reedy cofibrant simplicial object, Lemma 15 says there is a geometric realization spectral sequence converging to the relative homology \( H_\ast(\int_M B, \int_M A) \) with \( E^1 \)-page given by the relative homology

\[
E^1_{p,q} = H_q \left( \left( M^\theta\left( E_n^\theta \right)^P(B), M^\theta\left( E_n^\theta \right)^P(A) \right) \right)
\]

The lemma follows if the \( E^1 \) page vanishes for \( q \leq N \), and to prove this we prove a version of Lemma 11 to see that if \( f : X \to Y \) is an \( N \)-equivalence, then so are the maps \( M^\theta(X) \to M^\theta(Y) \) and \( E_n^\theta(X) \to E_n^\theta(Y) \). The functor \( E_n^\theta \) is a special case \( M = \mathbb{R}^n \) of the functor \( M^\theta \). Recall the definition

\[
M^\theta(X) = \bigoplus_{k \geq 0} C_\ast(\text{Emb}^\theta(\sqcup_k \mathbb{R}^n, M)) \otimes_{\mathbb{Z}[\mathcal{S}_k]} X^\otimes k
\]

Since the action of \( \mathcal{S}_k \) on \( \text{Emb}^\theta(\sqcup_k \mathbb{R}^n, M) \) is free, \( C_\ast(\text{Emb}^\theta(\sqcup_k \mathbb{R}^n, M)) \) is a levelwise free chain complex of \( \mathbb{Z}[\mathcal{S}_k] \)-modules. If \( C_\ast \) is a bounded below levelwise free chain complex of \( \mathbb{Z}[\mathcal{S}_k] \)-modules, then we have that \( C_\ast \otimes_{\mathbb{Z}[\mathcal{S}_k]} (-) : \text{Ch} \mathbb{Z}[\mathcal{S}_k] \to \text{Ch} \) preserves \( N \)-equivalences. By the Künneth theorem \( X^\otimes k \to Y^\otimes k \) is an \( N \)-equivalence if \( X \to Y \) is, which proves the desired result. □
2.3.2 Topological chiral homology of partial $\mathcal{E}_n^\theta$-algebras and completions

In later sections, we will construct $\mathcal{E}_n^\theta$-algebras by adding generators and relations to given $\mathcal{E}_n^\theta$-algebras. This is done by creating an intermediate object called a partial $\mathcal{E}_n^\theta$-algebra, and then freely completing the result to obtain an $\mathcal{E}_n^\theta$-algebra, a procedure described in this subsection.

**Definition 38** A structure of a partial $\mathcal{E}_n^\theta$-algebra on an object $A$ of $\mathbf{C}$ is a simplicial object $A_\bullet$ with the following properties:

(i) There exists a subobject $\text{Comp}_p \subset (E_n^\theta)^p(A)$ so that $A_p = E_n^\theta(\text{Comp}_p)$.

(ii) We have that $\text{Comp}_0 = A$.

(iii) If $p \geq 1$ and $0 \leq i < p$, then the face map $d_i : E_n^\theta(\text{Comp}_p) \to E_n^\theta(\text{Comp}_{p-1})$ is the restriction of the map

$$\left(E_n^\theta\right)^i \left(c \left(\mathbb{E}_2^\theta\right)^{p-i-1}(A)\right) : \left(E_n^\theta\right)^{p+1}(A) \to \left(E_n^\theta\right)^p(A)$$

with $c$ the composition natural transformation of the monad $E_n^\theta$.

(iv) If $p \geq 0$ and $0 \leq j \leq p$, the degeneracy map $s_j : E_n^\theta(\text{Comp}_p) \to E_n^\theta(\text{Comp}_{p+1})$ is the restriction of the map

$$\left(E_n^\theta\right)^{p+1} \left(1 \left(E_n^\theta\right)^{p-j}(A)\right) : \left(E_n^\theta\right)^{p+1}(A) \to \left(E_n^\theta\right)^{p+2}(A)$$

with $1$ the unit natural transformation of the monad $E_n^\theta$.

We denote the geometric realization $|A_\bullet|$ in $\mathbf{C}$ by $\tilde{A}$. When $\mathbf{C} = \text{Ch}$, we will again assume that $A_\bullet$ is levelwise cofibrant. In all relevant examples appearing in this paper, this is the case if $A$ is. Since $A_\bullet$ is an $\mathcal{E}_n^\theta$-algebra in simplicial objects in $\mathbf{C}$, $\tilde{A}$ is also an $\mathcal{E}_n^\theta$-algebra. Furthermore, there is a canonical map $A \to \tilde{A}$ given by including $A$ as $\text{id} \odot A$ viewed as 0-simplices of $\tilde{A}$.

**Definition 39** We call the $\mathcal{E}_n^\theta$-algebra $\tilde{A}$ the completion of $A_\bullet$ (or the completion of $A$, when the partial $\mathcal{E}_n^\theta$-algebra structure on $A$ is implicit).

**Example 40** If $A$ is $\mathcal{E}_n^\theta$-algebra, then it in particular can be considered as a partial $\mathcal{E}_n^\theta$-algebra $A_\bullet$ by taking $A_\bullet = B_\bullet(E_n^\theta, E_n^\theta, A)$ with face maps induced by the operad composition and the $\mathcal{E}_n^\theta$-algebra structure on $A$. In this case we have $\text{Comp}_p = (E_n^\theta)^p(A)$ and we claim its completion is canonically weakly equivalent to $A$ as a $\mathcal{E}_n^\theta$-algebra. The algebra structure map $E_n^\theta(A) \to A$ induces an augmentation map $A_\bullet \to A$. This induces a weak equivalence upon geometric realization, because the outermost unit maps $(E_n^\theta)^p(A) \to (E_n^\theta)^{p+1}(A)$ give $A_\bullet \to A$ an extra degeneracy, and hence $\tilde{A} = |A_\bullet| \to A$ is a weak equivalence.

**Example 41** In all cases that we consider, the partial $\mathcal{E}_n^\theta$-algebra on $A$ is obtained by constructing $\text{Comp}_p$ out of two pieces of data:

(a) a subobject $\text{Comp}_1 \subset E_n^\theta(A)$, and
(b) a map $c_1 : \text{Comp}_1 \to \text{Comp}_0 = A$ such that $c_1$ is associative and unital in the following sense:

(i) if $\text{Comp}_2, \text{Comp}'_2 \subset (E_n^\theta)^2(A)$ are defined as the pull backs

\[
\begin{array}{ccc}
\text{Comp}_2 & \xrightarrow{c_2} & E_n^\theta(\text{Comp}_1) \\
\searrow & & \searrow \\
\text{Comp}_1 & \xrightarrow{i} & E_n^\theta(A)
\end{array}
\quad \begin{array}{ccc}
\text{Comp}'_2 & \xrightarrow{\tilde{c}_2} & E_n^\theta(\text{Comp}_1) \\
\searrow & & \\
\text{Comp}_1 & \xrightarrow{i} & E_n^\theta(A)
\end{array}
\]

where $i$ is the inclusion and $c : (E_n^\theta)^2 \to E_n^\theta$ the monad composition natural transformation, then we have $\text{Comp}_2 \subset \text{Comp}'_2$ and $c_1 \circ c_2 = c_1 \circ \tilde{c}_2$ on $\text{Comp}_2$.

(ii) $\text{id} \odot A \subset \text{Comp}_1$ and following diagram commutes

\[
\begin{array}{c}
A \\
\searrow^{c_1} \\
A \\
\end{array} \quad \begin{array}{c}
A \\
\searrow^{\text{id} \odot A} \\
A \\
\end{array}
\]

We explain how to inductively obtain $\text{Comp}_p$ and $c_p$ from this. Suppose we have defined

$\text{Comp}_{p-1} \subset E_n^\theta(\text{Comp}_{p-2})$ and $c_{p-1} : \text{Comp}_{p-1} \to \text{Comp}_{p-2}$

then $\text{Comp}_p \subset E_n^\theta(\text{Comp}_{p-1})$ and $c_p : \text{Comp}_p \to \text{Comp}_{p-1}$ are obtained via the pull back

\[
\begin{array}{ccc}
\text{Comp}_p & \xrightarrow{c_p} & E_n^\theta(\text{Comp}_{p-1}) \\
\searrow & & \searrow \\
\text{Comp}_{p-1} & \xrightarrow{i} & E_n^\theta(\text{Comp}_{p-2})
\end{array}
\]

with $i$ the inclusion. Since $E_n^\theta$ preserves pullbacks, as it is constructed out of coproducts, quotients by group actions and limits, we see that we can also describe $\text{Comp}_p$ as the pullback

\[
\begin{array}{ccc}
\text{Comp}_p & \xrightarrow{(E_n^\theta)^{p-1}(\text{Comp}_1)} \\
\searrow & & \searrow \\
\text{Comp}_{p-1} & \xrightarrow{i} & (E_n^\theta)^{p-2}(\text{Comp}_1)
\end{array}
\]
Using this we can describe Comp$_p$ as those elements of $(E^\theta_n)^{p-1}$ (Comp$_1$) such that applying $c_1$ in the innermost most position $(p - 2)$ times we obtain an element of Comp$_1$, and $c_p$ as the restriction of $(E^\theta_n)^{p-1}$ ($c_1$).

Example 40 is a special case of Example 41; one declares all elements to be composable by taking Comp$_1 = E^\theta_n(A)$ and $c: \text{Comp}_1 = E^\theta_n(A) \to A$ to be given by the $E^\theta_n$-algebra structure on $A$.

The next lemma describes how to obtain a partial $E^\theta_n$-algebra structure from this example.

**Lemma 42** For any right $E^\theta_n$-functor $F$, we have that $F(\text{Comp}_p) \subset F((E^\theta_n)^p(A))$ is a simplicial object, if all face maps and degeneracy maps are the restrictions of those of the bar construction, with the exception of $d_p : F(\text{Comp}_p) \to F(\text{Comp}_{p-1})$ for $p \geq 1$, which is defined to be $F(c_p)$.

**Proof** It suffices to check that: (a) the face and degeneracy maps have images in the required codomain, (b) the last face map $d_p = F(c_p)$ satisfies the simplicial identities.

We start with (a) in the case of degeneracy maps. Recall that $F(\text{Comp}_p)$ consists of all elements of $F((E^\theta_n)^{p-1})$ (Comp$_1$) satisfying the following condition

(*) applying the map $d_p$ in the innermost position $(p - 2)$ times we obtain an element of $F(\text{Comp}_1)$.

Adding an additional identity element preserves this property, with the possible exception for the right-most degeneracy map $s_p : F(\text{Comp}_p) \to F(\text{Comp}_{p+1})$ where it is a consequence of property (ii).

For checking (a) in the case of face maps $d_i : F(\text{Comp}_p) \to F(\text{Comp}_{p-1})$, we distinguish $i = 0, 0 < i < p$ and $i = p$. For $i = 0$, one uses that $\text{Comp}_p$ is contained in $E^\theta_n(\text{Comp}_{p-1})$. For $0 < i < p$, one uses that associativity of the monad composition implies that (*) is preserved by $d_i$. The case $i = p$ follows directly from the definition of (*).

For checking (b), i.e. that $d_p$ satisfies the simplicial identities, one uses that $d_p$ is obtained from $F((E^\theta_n)^{p-2}(c_1))$ by restriction, so that applying $F((E^\theta_n)^{p-2}c_1)$ to properties (i) and (ii) gives $d_p \circ d_p = d_p \circ d_{p-1}$ and $d_p \circ s_p = \text{id}$ respectively. \hfill \Box

Using the completion of partial $E^\theta_n$-algebras, we define topological chiral homology of partial $E^\theta_n$-algebras as follows.

**Definition 43** For $A$ a partial $E^\theta_n$-algebra, with partial $E^\theta_n$-algebra structure given by $A_\bullet$ and completion $\bar{A}$, we define $\int_M \bar{A}$ to be $\int_M A$.

We now give a smaller equivalent construction in the setting of Example 41. To do so we consider $M^\theta(\text{Comp}_\bullet)$ with $M^\theta$ as in Definition 34, which was shown to be a simplicial object in Lemma 42.

**Lemma 44** Suppose $A$ has a partial $E^\theta_n$-algebra structure $A_\bullet$ as in Example 41. Then we have that $\int_M \bar{A} \simeq |M^\theta(\text{Comp}_\bullet)|$.

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Proof Recall that $\int_{M} \tilde{A}$ is the realization of the bisimplicial object

$$[p, q] \mapsto M^\theta \left(E_n^\theta\right)^{p+1} \left(\text{Comp}_q\right)$$

which admits an augmentation in the $p$-direction to $M^\theta \text{Comp}_\bullet$ by using the right $E_n^\theta$-functor structure of $M^\theta$. An extra degeneracy argument in the $p$-direction implies that for each $q$ the augmentation induces a weak equivalence

$$[p] \mapsto M^\theta \left(E_n^\theta\right)^{p+1} \left(\text{Comp}_q\right) \sim M^\theta \left(\text{Comp}_q\right)$$

Realizing this weak equivalence in the $q$-direction and using Lemma 14 gives the desired weak equivalence. $\square$

2.3.3 Topological chiral homology for manifold bundles

One may generalize topological chiral homology to manifold bundles, using embeddings of a collection of disks with image in a fiber.

Let $B$ be a paracompact space and $\pi : E \to B$ be a manifold bundle over $B$ with fibers diffeomorphic to $M$. Let $\tilde{\pi} : \tilde{E} \to B$ be the associated principal $\text{Diff}(M)$-bundle and let $T_v E$ be the vertical tangent bundle given by

$$T_v E = \tilde{E} \times_{\text{Diff}(M)} TM$$

A $\theta$-framing of $\pi : E \to M$ is a $\theta$-framing of its vertical tangent bundle, i.e. a bundle map $\phi_E : T_v E \to \theta^* \gamma$.

We shall define $\int_{E \downarrow \theta B} A$ for an $E_n^\theta$-algebra $A$ using a bar construction as in Sect. 2.3.1 but modifying the right $E_n^\theta$-functor $M^\theta$. To do so, we define a space $\text{Emb}_B(\sqcup_k \mathbb{R}^n, E)$ of embeddings $\sqcup_k \mathbb{R}^n \hookrightarrow E$ with image in some fiber $\pi^{-1}(b)$ of $\pi$:

$$\text{Emb}_B(\sqcup_k \mathbb{R}^n, E) = \tilde{E} \times_{\text{Diff}(M)} \text{Emb}(\sqcup_k \mathbb{R}^n, M)$$

Similarly define $\text{Bun}_B(\sqcup_k T \mathbb{R}^n, T_v E)$ as the space of vector bundle maps $\phi : \sqcup_k T \mathbb{R}^n \to T_v E$ with image in a single fiber $(T_v E)_b$, and $\text{Bun}_B^\theta(\sqcup_k T \mathbb{R}^n, T_v E)$ as the space of triples $(\phi, \alpha, \varphi)$ of a bundle map $\phi : \sqcup_k T \mathbb{R}^n \to T_v E$, a locally constant function $\alpha : \sqcup_k \mathbb{R}^n \to [0, \infty)$ and a path $\varphi_1 : [0, \infty) \to \text{Bun}_B(\sqcup_k T \mathbb{R}^n, \theta^* \gamma)$ starting at $\prod_k \phi \mathbb{R}^n$ and equal to $\phi_E \circ \phi$ on $[\alpha, \infty)$. Then $\text{Emb}_B^\theta(\sqcup_k \mathbb{R}^n, E)$ is defined to be the pull back

$$\text{Emb}_B^\theta(\sqcup_k \mathbb{R}^n, E) \longrightarrow \text{Bun}_B^\theta(\sqcup_k T \mathbb{R}^n, T_v E)$$

$$\Downarrow$$

$$\text{Emb}_B(\sqcup_k \mathbb{R}^n, E) \longrightarrow \text{Bun}_B(\sqcup_k T \mathbb{R}^n, T_v E)$$
Definition 45 Let $E \downarrow B^\theta$ be the following right functor over $E^\theta_n$

$$C \mapsto \bigsqcup_{k \geq 0} \text{Emb}^\theta_B\left(\sqcup_k \mathbb{R}^n, E\right) \otimes \mathbb{S}_k C \otimes^k$$

where the right functor structure comes from composition of $\theta$-framed embeddings.

Definition 46 The fiberwise topological chiral homology $\int_{E \downarrow B} A$ of the manifold bundle $E \to B$ is the geometric realization

$$\int_{E \downarrow B} A := |B_\bullet\left(E \downarrow B^\theta, E^\theta_n, A\right)|$$

There is a map $\pi_k : \text{Emb}^\theta_B\left(\sqcup_k \mathbb{R}^n, E\right) \to B$, mapping a $\theta$-framed embedding to the fiber containing its image. This commutes with the $\mathbb{S}_k$-action and is preserved under precomposition by $\theta$-framed embeddings, so if $C = \text{Top}$ or sSet it induces a well-defined map $\int_{E \downarrow B} A \to B$. This allows us to take the homology of fiberwise topological chiral homology with certain local coefficient system coming from the base $B$ when $C = \text{Top}$ or sSet.

For $C = \text{Ch}$, we will describe an analogous construction. We can pull back a local coefficient $L$ on $B$ along $\pi_k$ to $\text{Emb}^\theta_B\left(\sqcup_k \mathbb{R}^n, E\right)$, and form the twisted singular chains $C_\bullet\left(\text{Emb}^\theta_B\left(\sqcup_k \mathbb{R}^n, E\right); \pi_k^* L\right)$. Since the map $\pi_k$ commutes with the $\mathbb{S}_k$-action and the right $E^\theta_n$-functor structure, we have a right $E^\theta_n$-functor $E \downarrow B^\theta, L$ given by

$$C \mapsto \bigsqcup_{k \geq 0} C_\bullet\left(\text{Emb}^\theta_B\left(\sqcup_k \mathbb{R}^n, E\right); \pi_k^* L\right) \otimes \mathbb{S}_k C \otimes^k$$

If $L$ is the trivial local system $\mathbb{Z}$, then this coincides with $E \downarrow B^\theta$. The only other case we will need, is when there is a homomorphism $\pi_1(B) \to \mathbb{S}_r$ and $L$ is obtained from the sign representation $\mathbb{Z}_{\pm 1}$.

Definition 47 Given a local system $L$ on $B$, the fiberwise topological chiral homology with coefficients in $L$, denoted $\int_{E \downarrow B, L} A$, of the manifold bundle $E \to B$ is the geometric realization

$$\int_{E \downarrow B, L} A := |B_\bullet\left(E \downarrow B^\theta, L, E^\theta_n, A\right)|$$

Remark 48 For an $E^\theta_n$-algebra $A$ in $C = \text{Top}$ or sSet, we have that

$$H_\bullet\left(\int_{E \downarrow B, L} C_\bullet(A)\right) \cong H_\bullet\left(\int_{E \downarrow B} A; \pi^* L\right)$$

with $\pi : \int_{E \downarrow B} A \to B$ as above.
2.3.4 A spectral sequence for topological chiral homology of manifold bundles

In this section we will construct a spectral sequence for topological chiral homology of manifold bundles, analogous to the Serre spectral sequence. We start with a lemma concerning the map $\pi_k : \text{Emb}_B^\theta(\bigcup_k \mathbb{R}^n, E) \to B$ induced by the map $\pi : E \to B$.

**Lemma 49** For a manifold bundle $\pi : E \to B$ with $\theta$-framing $\phi_E$, the map

$$\pi_k : \text{Emb}_B^\theta(\bigcup_k \mathbb{R}^n, E) \to B$$

is a Serre fibration.

**Proof** The local trivializations of $\pi : E \to B$ induce local trivializations of the map $\pi_k : \text{Emb}_B^\theta(\bigcup_k \mathbb{R}^n, E) \to B$, so it is a Serre fibration. Since $\text{Bun}_B^\theta(\bigcup_k \mathbb{T}\mathbb{R}^n, T_v E)$ is a Serre fibration, so is its pull back $\text{Emb}_B^\theta(\bigcup_k \mathbb{R}^n, E) \to \text{Emb}_B^\theta(\bigcup_k \mathbb{R}^n, E)$. Hence the composition $\text{Emb}_B^\theta(\bigcup_k \mathbb{R}^n, E) \to B$ is also a Serre fibration. $\Box$

Before constructing the promised spectral sequences, we recall a construction of the Serre spectral sequence due to Dress and described in Section 6.4 of [31]. If $f : E \to B$ is a Serre fibration, consider the bisimplicial set $K_{\bullet, \bullet}(f)$ with $(p, q)$-simplices given by pairs $(u, v)$ of $u : \Delta^p \times \Delta^q \to E$ and $v : \Delta^p \to B$ such that the following diagram commutes

$$
\begin{array}{ccc}
\Delta^p \times \Delta^q & \xrightarrow{u} & E \\
\downarrow & & \downarrow^f \\
\Delta^p & \xrightarrow{v} & B
\end{array}
$$

This has the following properties:

(i) It follows from the definition that $K_{0, \bullet}(f) \cong \text{Sing}(E)$, the singular simplicial set.

(ii) By Lemma 6.48 of [31], precomposition with the unique map $\Delta^p \to \Delta^0$ induces a weak equivalence $K_{0, \bullet}(f) \to K_{p, \bullet}(f)$.

(iii) Given $v : \Delta^p \to B$, let $K_{p, \bullet}(f, v)$ denote the simplicial set of consisting of pairs $(u, v)$. If $e_v$ denotes the initial vertex of $v$, then by the argument on page 228 of [31], restricting to $e_v \times \Delta^q$ induces a map $K_{p, \bullet}(f, v) \to K_{0, \bullet}(f, e_v)$, which is a weak equivalence if $f$ is a Serre fibration. Under these identifications, if a face map does not preserve the initial vertex, one uses the two weak equivalences $K_{0, \bullet}(f, e_v) \leftarrow K_{1, \bullet}(f, s_v) \to K_{0, \bullet}(f, e'_{v})$ where $s_v$ is the 1-simplex connecting the old initial vertex $e_v$ and the new initial vertex $e'_{v}$. This zigzag will induce well-defined map on homology.

(iv) It follows from the definition that for each $e : \Delta^0 \to B$, we have that $K_{p, \bullet}(f, e) \cong \text{Sing}(f^{-1}(e))$.

**Proposition 50** Let $\pi : E \to B$ be a $\theta$-framed manifold bundle with $B$ path-connected and $n$-dimensional $\theta$-framed manifold $F$ as fiber, and $\mathcal{L}$ a local system on $B$ whose fibers are free as abelian groups. Then there is a spectral sequence
\[ E^2_{p,q} = H_p \left( B; \mathcal{H}_q \left( \int_F A \right) \otimes \mathcal{L} \right) \Rightarrow H_{p+q} \left( \int_{E \downarrow B, \mathcal{L}} A \right) \]

where \( \mathcal{H}_q \left( \int_F A \right) \) is the local system of \( B \) given by \( b \mapsto \mathcal{H}_q \left( \int_{\pi^{-1}(b)} A \right) \).

This is natural in fiberwise embeddings of manifold bundles over \( B \). One can restrict to fixed charge \( c \) to get a spectral sequence

\[ E^2_{p,q} = H_p \left( B; \mathcal{H}_q \left( \int_c F A \right) \otimes \mathcal{L} \right) \Rightarrow H_{p+q} \left( \int_c E \downarrow B, \mathcal{L} A \right) \]

**Proof** By Lemma 16 we may assume that \( \mathcal{C} = \text{Ch} \). Taking twisted singular chains of \([p] \mapsto S_p,*(\pi_k, \mathcal{L}) := C_*(K_p,*(\pi_k); \pi_k^* \mathcal{L}) \)

The condition that \( \mathcal{L} \) is free as an abelian group implies this chain complex is a levelwise free complex of \( \mathbb{Z}[\tilde{G}_k] \)-modules. Thus we can define a functor \( S_*(\pi, \mathcal{L}) : \text{Ch} \to \text{Fun}(\Delta^{op}, \text{Ch}) \) by

\[ C \mapsto \left( [p] \mapsto \bigcup_{k \geq 0} S_{p,\ast}(\pi_k, \mathcal{L}) \otimes_{\mathbb{Z}[\tilde{G}_k]} C^\otimes k \right) \]

whose values are cofibrant if \( C \) is. We remark for later use that \( S_p(\pi, \mathcal{L}) \) is a direct sum of functors \( S_p(\pi, \mathcal{L}, v) \) for \( v : \Delta^p \to B \), as \( K_{p,\ast}(\pi_k) \) is a disjoint union over \( v : \Delta^p \to B \) of the simplicial sets \( K_{p,\ast}(\pi_k, v) \).

Composition of \( \theta \)-framed embeddings endows the functor \( S_*(\pi, \mathcal{L}) \) and functors \( S_p(\pi, \mathcal{L}, v) \) with the structure of right \( \mathcal{E}^\theta_n \)-functors. Note that \( S_0(\pi, \mathcal{L}) \) is isomorphic to \( \mathcal{E} \downarrow \mathcal{B}^{\theta, L} \) as a right \( \mathcal{E}^\theta_n \)-functor. We next consider the bisimplicial object in \( \text{Ch} \) given by

\[ [p, q] \mapsto S_p(\pi, \mathcal{L}) \left( \mathcal{E}^\theta_n \right)^q (A) \]

Properties (i) and (ii) of \( K_{\ast,\ast} \) imply that \([p] \mapsto S_p(\pi, \mathcal{L}) \left( \mathcal{E}^\theta_n \right)^q (A) \) is weakly equivalent to the constant simplicial object \([p] \mapsto S_0(\pi, \mathcal{L}) \left( \mathcal{E}^\theta_n \right)^q (A) \cong \left( \mathcal{E} \downarrow \mathcal{B}^{\theta, L} \right) \left( \mathcal{E}^\theta_n \right)^q (A) \). Hence first realizing the \([p]\)-direction and using Lemma 14, and then the \([q]\)-direction we obtain

\[ \left| [p, q] \mapsto S_p(\pi, \mathcal{L}) \left( \mathcal{E}^\theta_n \right)^q (A) \right| \cong \left| [q] \mapsto \left( \mathcal{E} \downarrow \mathcal{B}^{\theta, L} \right) \left( \mathcal{E}^\theta_n \right)^q (A) \right| = \int_{E \downarrow B, \mathcal{L}} A \]
On the other hand, if we first realize the \([q]\)-direction and consider the geometric realization spectral sequence of Lemma 15 for the remaining \([p]\)-direction, we get

\[
E^1_{s,t} = \bigoplus_{v: \Delta^t \to B} H_t \left( \left[ [q] \mapsto S_p(\pi, \mathcal{L}, v) (E^\theta_n)^q (A) \right] \right)
\]

and using properties (iii) and (iv) of \(K_{\bullet, \bullet}\) and Lemma 14, we have identifications

\[
\left| S_p(\pi, v, \mathcal{L}) (E^\theta_n)^q (A) \right| \simeq \left| S_0(\pi, e_v, \mathcal{L}) (E^\theta_n)^q (A) \right| \simeq \left( \int_{\pi^{-1}(e_v)} A \right) \otimes \mathcal{L}
\]

Thus the entries on the \(E^1\)-page are given by

\[
E^1_{s,t} = \bigoplus_{v: \Delta^t \to B} H_t \left( \int_{\pi^{-1}(e_v)} A \right) \otimes \mathcal{L}
\]

and by Lemma 15 the \(d^1\)-differential is given by taking the alternating sum of maps induced by restricting to faces of the simplex. If a face does not have the same initial vertex as the larger simplex, then one also parallel transports along the 1-simplex connecting these two initial vertices. This is the definition of a chain complex computing the homology of \(B\) with local coefficients in the local system as described in the statement of this proposition.

For naturality, we note that the construction of \(S_{\bullet}(\pi, \mathcal{L})\) and the identifications in the proof are natural in fiberwise \(\theta\)-framed embeddings of \(\theta\)-framed manifold bundles over \(B\). By working in \(\text{Ch}_C\) to keep track of the charge, one proves that the spectral sequence can be restricted to components of fixed charge \(c\). \(\square\)

### 2.3.5 Stabilization maps

If \(M\) is a connected non-compact \(n\)-dimensional manifold, McDuff introduced a map \(C_k(M) \to C_{k+1}(M)\) of configuration spaces given by “bringing in a particle from infinity” \([32]\). This is usually called the \textit{stabilization map}. We generalize it to a map \(f^c_M A \to f^{c+c_0}_M A\) between the components of topological chiral homology of a charged algebra, as long as \(c + c_0\) is defined in the partial monoid \(C\). We shall give the construction only for \(C = \text{Ch}\), as the constructions for \(C = \text{Top}\) or \(\text{sSet}\) are similar.

We will use the fact that topological chiral homology is functorial with respect to \(\theta\)-framed embeddings to make space near the boundary to add in a new labeled embedded disk. The resulting map will depend on two choices; (i) an element \(a \in A(c_0)\), (ii) a \(\theta\)-framed embedding \(\psi: \mathbb{R}^n \sqcup M \to M\).

Topological chiral homology with coefficients in a fixed \(E^\theta_n\)-algebra \(A\) gives a symmetric monoidal functor from \(\text{Emb}^\theta\) to \(\text{C}\), the former defined in Sect. 2.2.1. This
functionality is induced by composition of embeddings. Thus the map \(\psi : \mathbb{R}^n \sqcup M \to M\) induces a map

\[
\psi_* : \int_{\mathbb{R}^n} A \otimes \int_M A \to \int_M A
\]

When \(A\) is a \(C\)-charged algebra, the monoidal structure is compatible with charge and so the maps \(\psi_*\) are additive with respect to charge. Using the natural equivalence \(A \to \int_{\mathbb{R}^n} A\) and restricting to specific charges we get maps

\[
\psi^0_* : A(c_0) \otimes \int_M A \to \int_M A
\]

as long as \(c + c_0\) is defined in \(C\). We can now define the stabilization maps.

**Definition 51** Suppose that \(c + c_0\) is defined in \(C\). Let \(a \in A(c_0)\) and \(\psi : \mathbb{R}^n \sqcup M \to M\), then \(t_{a,\psi} : \int_M c A \to \int_M c + c_0 A\) is the map given by \(t_{a,\psi}(b) = \psi^0_*(a, b)\).

This definition also makes sense for \(\theta\)-framed manifold bundles \(\pi : E \to B\) with fiberwise embedding \(\psi : (\mathbb{R}^n \times B) \sqcup E \to E\) over \(B\), as then one can repeat the construction above. The resulting map will similarly be denoted \(t_{a,\psi} : \int_{E \sqcup B} A \to \int_{E \sqcup B} A\).

### 2.3.6 Homological stability for topological chiral homology

We can now give a precise definition of homological stability for topological chiral homology in the case that \(C = \mathbb{N}_0^d\). To get well-defined stabilization maps on the manifold bundles involved, we will only use stabilization maps for embeddings \(\mathbb{R}^n \sqcup M \to M\) which can be thought of as making space for \(\mathbb{R}^n\) by moving part of \(M\) away from infinity:

**Definition 52** An end-like embedding is a path of embeddings \(\psi : [0, \infty) \to \text{Emb}^\theta(\mathbb{R}^n \sqcup M, M)\) such that there exists an exhausting smooth function \(h : M \to [0, \infty)\) such that \(\psi_t\) is the identity on \(h^{-1}([0, t])\).

**Remark 53** If \(M\) is the interior of a compact connected \(\theta\)-framed manifold with boundary, one can construct embeddings \(\psi_t\) as above that up to isotopy only depends on the choice of a component of the boundary. For general non-compact connected \(\theta\)-framed manifold the situation is more complicated: one can always construct an embedding \(\psi_t\) as above, and if \(\dim M \geq 5\) one can construct embeddings \(\psi_t\) as above that up to isotopy only depend on the choice of a component of the space of ends.

Suppose that \(A\) is a \(C\)-charged \(\mathcal{E}_n^\theta\)-algebra, and hence by definition connected. For each representative \(a_i \in A(e_i)\) of \(1 \in H_0(A(e_i)) \cong \mathbb{Z}\) (the isomorphism coming from the augmentation) and end-like embedding \(\psi_t\), we get a basic stabilization map

\[
t_{a_i,\psi_0} : \int_M A \to \int_M A.
\]
The fact that topological chiral homology is an enriched functor implies this map only depends on the isotopy class of $\psi_t$. From now on we will drop $a_i$ and $\psi_0$ from the notation.

**Definition 54** Let $C = \mathbb{N}_0^d$, $k = (k_1, \ldots, k_d)$, and let $M$ be a $\theta$-framed connected non-compact manifold.

(i) Let $A$ be a $C$-charged $\mathcal{E}_n^\theta$-algebra in chain complexes, then we say that $A$ has homological stability on $M$ if there is a function $\rho : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ with $\lim_{j \to \infty} \rho(j) = \infty$ such that all basic stabilization maps

$$t_i : H_*(\int_M^k A) \to H_*(\int_M^{k+e_i} A)$$

are $\rho(k_i)$-equivalences.

(ii) Let $X$ be a $C$-charged $\mathcal{E}_n^\theta$-algebra in spaces or simplicial sets, then we say that $X$ has homological stability on $M$ if $C_*(X)$ has.

### 3 Cell attachments

In this section we define $\mathcal{E}_n^\theta$-cell attachments and study their effect on homology. For the remainder of this section, we fix the tangential structure $\theta : W \to BO(n)$ and the monoid of charges $C$, and hence suppress them from our notation whenever convenient. For example, if we mention charged algebras, we mean $C$-charged $\mathcal{E}_n^\theta$-algebras.

#### 3.1 $\mathcal{E}_n^\theta$-cell attachments

Using the completion procedure of Sect. 2.3.2, we can now define $\mathcal{E}_n^\theta$-cell attachments in $\text{Ch}$ or $\text{Ch}_C$. This case is the one used in main theorem, and $\mathcal{E}_n^\theta$-cell attachments in $\text{Top}$ and $\text{sSet}$ are defined analogously. See Remark 58 for a discussion of the relationship between our construction and similar constructions in the literature.

Let $A$ be a charged algebra. Fixing a cycle $b_{N-1} \in A(c)$ of degree $N - 1$, we want to define the partial algebra $A \oplus e_N$ by declaring that $e_N$ is of charge $c$ and $d(e_N) = b_{N-1}$ and declaring no operation on $e_N$ is defined except the identity. We now make this precise:

**Definition 55** Let $D = A \oplus e_N$ be the chain complex with underlying graded abelian group $A \oplus e_N$ with differential extending that of $A$ by setting $d(e_N) = b_{N-1}$. This can be given the structure of a partial charged algebra by setting $D_p = \mathcal{E}_n^\theta \left( (\mathcal{E}_n^\theta)^p(A) \oplus (\text{id}^p \otimes e_N) \right) \subset B_p(\mathcal{E}_n^\theta, \mathcal{E}_n^\theta, D)$

This is a particular case of Example 41, where $\text{Comp}_1 = \mathcal{E}_n^\theta(A) \oplus (\text{id} \otimes e_N) \subset \mathcal{E}_n^\theta(A \oplus e_N)$ and $c_1 : \text{Comp}_1 \to A$ is given by the $\mathcal{E}_n^\theta$-action on $\mathcal{E}_n^\theta(A)$ and by $\text{id} \otimes e_N \mapsto e_N$. It is clear that condition (i) holds, and (ii) follows from the fact that $\text{Comp}_2 = \text{Comp}_2' = \left( \mathcal{E}_n^\theta \right)^2(A) \oplus (\text{id}^2 \otimes e_N)$. We then have that $\text{Comp}_\rho = \left( \mathcal{E}_n^\theta \right)^p(A) \oplus (\text{id}^p \otimes e_N)$. 

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We remark that $D_\bullet$ is Reedy-cofibrant if the underlying chain complex of $A$ is cofibrant. We can now define charged algebras obtained by $E^\theta_n$-cell attachments.

**Definition 56** Let $b_{N-1}$ and $A \oplus e_N$ be as before. We let the charged algebra $A \uplus e_N$ be the completion of the partial charged algebra $A \oplus e_N$, i.e. $A \uplus e_N = |D_\bullet|$, and say that $A \uplus e_N$ is obtained from $A \oplus e_N$ by attaching an $E^\theta_n$-cell $e_N$ to $A$.

Our next goal is to show that one may construct a map from $A \uplus e_N$ to $B$ out of a map $f : A \to B$ of charged algebras and an element $e \in B(k)_N$ such that $d(e) = f(b_{N-1})$.

**Lemma 57** Let $A \uplus e_N$ be the charged algebra obtained by attaching an $E^\theta_n$-cell $e_N$ to $A$ along $b_{N-1}$ in charge $k$, $f$ a map $A \to B$ and $e$ an element $e \in B(k)_N$ such that $d(e) = f(b_{N-1})$. There exists a map $\hat{f} : A \uplus e_N \to B$ of charged algebras such that $\hat{f}(e_N) = e$ and such that the following diagram commutes:

\[
\begin{array}{ccc}
A \uplus e_N & \xrightarrow{\hat{f}} & B \\
\uparrow & & \uparrow \\
\tilde{A} & \xrightarrow{\sim} & A \\
\end{array}
\]

where $\tilde{A} \to A \uplus e_N$ is induced by the inclusion $A_\bullet \to D_\bullet$ of in the notation of Example 40 and Definition 55 (which will be repeated in the proof).

**Proof** Recall that $A \uplus e_N$ is by definition given by

\[
A \uplus e_N := [(p) \mapsto D_p = E^\theta_n \left( (E^\theta_n)^p (A) \oplus \text{id}^p \otimes e_N \right)]
\]

By Example 40, $A$ and $B$ has partial charged algebra structures $A_\bullet = B_\bullet(E^\theta_n, E^\theta_n, A)$ and $B_\bullet = B_\bullet(E^\theta_n, E^\theta_n, B)$, and we have weak equivalences $\tilde{A} = |A_\bullet| \to A$ and $\tilde{B} = |B_\bullet| \to B$.

We will construct a map $D_\bullet \to B_\bullet$ out of the map $f : A \to B$ and the element $e \in B(k)_N$ such that $d(e) = f(b_{N-1})$. Then geometric realization and the augmentation gives us the map $A \uplus e_N \to B$ defined as $\tilde{D} \to \tilde{B} \to B$. To give a map $D_\bullet \to B_\bullet$ we only need to define compatible maps

\[
D_p = E^\theta_n \left( (E^\theta_n)^p (A) \oplus \text{id}^p \otimes e_N \right) \to (E^\theta_n)^{p+1} (B) = B_p
\]

This is done by applying $E^\theta_n$ to the direct sum of the maps $(E^\theta_n)^p (f)$ and $\text{id}^p \otimes e_N \mapsto \text{id}^p \otimes e$, which has the desired properties. \hfill \Box

**Remark 58** This construction hints at the fact that an alternative definition of an $E^\theta_n$-cell attachment is given by the push out

\[
A \cup_{E^\theta_n(S^{N-1})} E^\theta_n \left( D^N \right)
\]
in an \((\infty, 1)\)-category or a suitable model category of \(C\)-charged \(\mathcal{E}_n^\theta\)-algebras, see [21,26,33], or [34]. A definition of \(\mathcal{E}_n^\theta\)-cell structures in terms of push outs is used to define cofibrant replacement for algebras over operads, e.g. Section 12.3 of [26]. We also believe that there is an \((\infty, 1)\)-category (or model category) of partial algebras over an operad such that the forgetful functor from algebras to partial algebras and the completion functor from partial algebras to algebras form an \((\infty, 1)\)-adjunction (a Quillen adjunction).

Remark 59 The construction of this subsection can be generalized to the attachment of any collection of \(\mathcal{E}_n^\theta\)-cells \(\{e_N^j\}_{j \in J}\) attached along \(\{b_{N-1}^i\}_{j \in J}\): one defines the structure of a partial charged algebra on \(A \oplus \bigoplus_{j \in J} e_N^j\) in the same way and defines \(A \uplus \biguplus_{j \in J} e_N^j\) to be its completion. Lemma 57 generalizes in the expected way.

3.2 The homology after an \(\mathcal{E}_n^\theta\)-cell attachment

In this subsection we show how an \(\mathcal{E}_n^\theta\)-cell attachment affects homology. This will done by applying two spectral sequences, and we start by describing the first one. To do so we need to make a few definitions.

Let \(F_r(M)\) denote the configuration space of \(r\) ordered particles in \(M\) and \(C_r(M)\) the configuration space of \(r\) unordered particles in \(M\). Note that there is a manifold bundle \(E_r(M)\) over \(C_r(M)\) given by the subspace of \(M \times C_r(M)\) of \((m, \{m_1, \ldots, m_r\})\) such that \(m \neq m_i\) for all \(1 \leq i \leq r\). The fiber of this manifold bundle over a point \(\{m_1, \ldots, m_r\} \in C_r(M)\) is the punctured manifold \(M \setminus \{m_1, \ldots, m_r\}\). We will need to consider a variation of this manifold bundle, with \(C_r(M)\) replaced by a configuration space with labels \(C^\theta_r(M)\). The labels will come from the space \(\theta(\mathbb{R}^n) := \text{Emb}^\theta(\mathbb{R}^n, \mathbb{R}^n)\) over \(M\), with map \(\theta_{TM} : \theta(TM) \to M\) sending a \(\theta\)-framed embedding \(\psi\) to \(\psi(0)\), the image of the origin.

Lemma 60 The map \(\theta_{TM} : \theta(TM) \to M\) is a Serre fibration with path-connected fibers.

Proof The map \(\theta_{TM}\) equals the composition

\[
\text{Emb}^\theta(\mathbb{R}^n, M) \to \text{Emb}(\mathbb{R}^n, M) \to M
\]

The left-hand map is a Serre fibration as it is the pull back of the Serre fibration \(\text{Bun}^\theta(T\mathbb{R}^n, TM) \to \text{Bun}(T\mathbb{R}^n, TM)\) and the right-hand map is a Serre fibration by the parametrized isotopy extension theorem.

The fiber of \(m \in M\) is the space of \(\theta\)-framed embeddings \(\mathbb{R}^n \to M\) sending the origin to \(m\). Picking a chart around \(M\), this is weakly equivalent to a space of \(\theta\)-framed embeddings \(\mathbb{R}^n \to \mathbb{R}^n\), the latter with a possibly non-trivial \(\theta\)-framing. This is path-connected by Lemma 19.

We now define labeled configuration spaces.
**Definition 61** Fix $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ and $k = k_1 + \cdots + k_d$. Let $\pi : E \to M$ be a Serre fibration. The configuration space of $k$ ordered particles with labels in the fibration $\pi$, denoted $F_\pi^k(M)$, is given by the following subspace of $E^k$:

$$F_\pi^k(M) := \{(e_1, \ldots, e_k) | \pi(e_i) \neq \pi(e_j) \text{ for } i \neq j\}$$

The configuration space of $k$-colored particles with labels in the fibration $\pi$, denoted $C_\pi^k(M)$, is the quotient $F_\pi^k(M)/\mathcal{S}_k \subset E^k/\mathcal{S}_k$ where $\mathcal{S}_k = \prod_i \mathcal{S}_{k_i} \subset \mathcal{S}_k$ and $\mathcal{S}_k$ acts diagonally. For $d = 1$, we denote this by $C_\pi^k(M)$.

A superscript $\theta$ means we take $\pi = \theta_{TM} : \theta(TM) \to M$. The quotient map $q : F_\theta^k(M) \to C_\theta^k(M)$ is a principal $\mathcal{S}_k$-bundle. This can be seen by noting that there is a map $f : C_\theta^k(M) \to C_r(M)$ which forgets the labels, and that $q$ is obtained by pulling back the principal $\mathcal{S}_k$-bundle $F_r(M) \to C_r(M)$ along $f$. We let $E_\theta^k(M)$ denote the pullback of $E_r(M)$ along $f$. The fiber of $E_\theta^k(M)$ over $\{\psi_1, \ldots, \psi_r\} \in C_r^k(M)$ is given by $M \setminus \{\psi_1(0), \ldots, \psi_r(0)\}$.

**Lemma 62** Every end-like embedding $\mathbb{R}^n \sqcup M \hookrightarrow M$ as in Definition 52 induces a unique isotopy class of fiberwise embeddings $\mathbb{R}^n \times C_\theta^k(M) \sqcup E_\theta^k(M) \to E_\theta^k(M)$.

**Proof** Recall that an end-like embedding is a map $\psi : [0, \infty) \to \text{Emb}^\theta(\mathbb{R}^n \sqcup M, M)$ such that there exists an exhausting proper smooth function $h : M \to [0, \infty)$ with $\psi_t$ equal to the identity on $h^{-1}([0, t])$. Fix such an $h$ and pick a smooth function $\eta : C_\theta^k(M) \to [0, \infty)$ such that the configuration $x \in C_r(M)$ is contained in $h^{-1}([0, \eta(x)])$. Consider the map

$$\mathbb{R}^n \times C_\theta^k(M) \sqcup E_\theta^k(M) \to E_\theta^k(M)$$

defined over $x \in C_\theta^k(M)$ as the map induced by $\psi_{\eta(x)}$. This is a fiberwise embedding over $C_\theta^k(M)$ and it is independent of the choice of $h$ and $\eta$ up to isotopy. To see this, note that for any pair $h_i, \eta_i, i = 0, 1$ we can find $h'$ and $\eta'$ such that $h' \leq \min(h_0, h_1)$ and $\eta' \geq \max(\eta_0, \eta_1)$. Then linearly interpolating the functions $h_i$ and $h'$, and the functions $\eta_i$ and $\eta$ gives isotopies from the fiberwise embedding constructed out of $h_i$ and $\eta_i$ to that constructed out of $h'$ and $\eta'$.

As topological chiral homology is an enriched functor on $\text{Emb}^\theta$, the upshot of the lemma is that given a basic stabilization map for $M$ there is a map

$$t_i : \int_{E_\theta^k(M) \sqcup C_\theta^k(M)} A \to \int_{E_\theta^k(M) \sqcup C_\theta^k(M)} A$$

that is well-defined on homology. We will apply the following spectral sequence to it, which is constructed by applying Proposition 50 to the $\theta$-framed manifold bundle $E_\theta^k(M) \to C_\theta^k(M)$. The local coefficients on $C_\theta^k(M)$ in a $\mathbb{Z}[\mathcal{S}_r]$-module $\mathcal{L}$ comes from the homomorphism $\pi_1(C_\theta^k(M)) \to \mathcal{S}_r$ induced by the $\mathcal{S}_r$-principal bundle map $q : F_\theta^k(M) \to C_\theta^k(M)$.
Proposition 63  Let $L$ be a $\mathbb{Z}[\mathcal{S}_r]$-module that is free as an abelian group. Then there is a spectral sequence

$$E^2_{p,q} = H_p \left( C^0_r(M); \mathcal{H}_q \left( \int_{M \setminus \{r\ \text{points}\}}^c A \right) \otimes L \right) \Rightarrow H_{p+q} \left( \int_{E^0_r(M) \downarrow C^0_r(M), L}^c \mathcal{L} \right)$$

where $\mathcal{H}_q(\int_{M \setminus \{r\ \text{points}\}}^c A)$ is the local system over $C^0_r(M)$ obtained by pulling back the local system over $C_r(M)$ with fiber over the point $\{m_1, \ldots, m_r\} \in C_r(M)$ given by $H_q(\int_{M \setminus \{m_1, \ldots, m_r\}}^c A)$. A stabilization map induces a map of spectral sequences.

Now we continue to describe the second spectral sequence involved in the study of the homology of a $E^\theta_n$-cell attachment. To identify its $E^1$-page we need to introduce a new right $E^\theta_n$-functor. We let

$$\text{Emb}_{F^\theta_r(M)}^{\theta, \text{disj}} \left( \sqcup_k \mathbb{R}^n, q^* E^\theta_r(M) \right) \subset \text{Emb}_{F^\theta_r(M)}^{\theta, \text{disj}} \left( \sqcup_k \mathbb{R}^n, q^* E^\theta_r(M) \right)$$

be the subspace of those $\theta$-framed embedding $(\phi_1, \ldots, \phi_k)$ over those points $(\psi_1, \ldots, \psi_r) \in F^\theta_r(M)$ such that the images of the $\phi_i$ for $1 \leq i \leq k$ and $\psi_j$ for $1 \leq j \leq r$ are all pairwise disjoint. Since this property is preserved by pre-composition of the $\phi_i$ with $\theta$-framed embeddings, we can define a right $E^\theta_n$-functor

$$E^\theta_r(M) \downarrow C^\theta_r(M)^{\text{disj}, L}: \text{Ch} \to \text{Ch}$$

by

$$C \mapsto \bigoplus_{k \geq 0} \left( C_*(\text{Emb}_{F^\theta_r(M)}^{\theta, \text{disj}} (\sqcup_k \mathbb{R}^n, q^* E^\theta_r(M))) \otimes \mathbb{Z}[\mathcal{S}_r] \mathcal{L} \right) \otimes \mathbb{S}_k C^{\otimes k}$$

Lemma 64  The inclusion induces a weak equivalence of right $E^\theta_n$-functors

$$E^\theta_r(M) \downarrow C^\theta_r(M)^{\text{disj}, L} \to E^\theta_r(M) \downarrow C^\theta_r(M)^L$$

which is natural in $M$.

Proof  By Lemma 11 it suffices to prove that for all $r \geq 0$ the inclusion

$$\text{Emb}_{F^\theta_r(M)}^{\theta, \text{disj}} (\sqcup_k \mathbb{R}^n, q^* E^\theta_r(M)) \hookrightarrow \text{Emb}_{F^\theta_r(M)}^{\theta, \text{disj}} (\sqcup_k \mathbb{R}^n, q^* E^\theta_r(M))$$

is a $\mathcal{S}_r$-equivariant weak equivalence. To see this is case, consider a commutative diagram

$$\begin{array}{ccc}
\partial D^i & \xrightarrow{f} & \text{Emb}_{F^\theta_r(M)}^{\theta, \text{disj}} (\sqcup_k \mathbb{R}^n, q^* E^\theta_r(M)) \\
\downarrow & & \downarrow \\
D^i & \xrightarrow{f} & \text{Emb}_{F^\theta_r(M)}^{\theta, \text{disj}} (\sqcup_k \mathbb{R}^n, q^* E^\theta_r(M))
\end{array}$$

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We may assume that there is an open neighborhood $U$ of $\partial D^i$ in $D^i$ such that $F|_U$ lands in $\Emb^{\theta, \text{disj}}_{\eta^p}(\bigcup_k \mathbb{R}^n, q^*E^p_\eta(M))$. Pick a family of origin-preserving self-embeddings $\lambda_t : \mathbb{R}^n \to \mathbb{R}^n$ for $t \in [0, \infty)$ such that $\lambda_0 = \text{id}$ and $\lambda_t$ has image in the open ball of radius $1/t$. Since $\Emb^{\theta}(\mathbb{R}^n, \mathbb{R}^n) \to \Emb(\mathbb{R}^n, \mathbb{R}^n)$ is a fibration, we can lift this to a family of $\theta$-framed embeddings starting at the identity.

Then there exists a real number $T \geq 0$ such that after precomposing each of the $(k+r)$ embeddings in $F(d)$ with $\lambda_{T'}$ to $T' \geq T$ their images are pairwise disjoint. Now pick a continuous function $\eta : D^i \to [0, 1]$ with support in $U$ and $1$ on $\partial D^i$, and let $(\eta_t)_*$ denote precomposition of all $k$ embeddings of $\mathbb{R}^n$ with $\lambda_t$. Then $(\lambda_{(1-\eta)}T)_* \circ F$ is $\mathcal{G}_r$-equivariantly homotopic to $F$ rel $\partial D^i$ and has image in $\Emb^{\theta, \text{disj}}_{E^\theta_\eta(M)}(\bigcup_k \mathbb{R}^n, q^*F^\theta_\eta(M))$. \hfill $\Box$

The statement of the next proposition will involve the homology of components of charge $c-pk$, which by definition is the unique element of $C$ such that $(c-pk)+p \cdot k = c$, where $p \cdot k$ means adding $k$ to itself $p$ times. This may not exist for large $p$ and if it does not, the homology will be defined to be 0. Let $\mathbb{Z}_{\pm 1}$ be the $\mathbb{Z}[\mathcal{G}_r]$-module induced by the sign homomorphism $\mathcal{G}_r \to \{\pm 1\}$.

**Proposition 65** Suppose $e_N$ is attached to $A$ in charge $k$ along $b_{N-1}$, then there is a spectral sequence

$$E^1_{pq} = H_{q-p(N-1)}\left(\int_c^{c-pk} E^p_\eta(M) \downarrow C^0_\eta(M) \otimes \mathbb{Z}_{\pm 1}^\otimes \right) \Rightarrow H_{p+q}(\int_c^c A \uplus e_N)$$

and if $b_{N-1} = 0$ the spectral sequence collapses at the $E^1$-page. If $M$ is path-connected, the map $d^1 : E^1_{1q} \to E^1_{0q}$ is induced by the map $H_{N-1}^s\left(\int_c^{c-k} M \setminus m \right) \to H^s_{\mathcal{F}_{\eta} M} A)$ by adding a particle with label $b_{N-1}$ at $m \in M$. Stabilization maps induce a map of spectral sequences.

**Proof** By Lemma 44 we have that $\int_c^c A \uplus e_N \simeq |\mathcal{M}^\theta(\text{Comp}_\cdot)|$. Recall that $\text{Comp}_p = (\mathcal{E}_n^p)^\eta(\mathcal{A}) \uplus (\text{id}^p \otimes e_N)$, so that we can define a filtration of simplicial chain complexes by setting $\mathcal{F}_j M^\theta(\text{Comp}_p)$ to be

$$\bigoplus_{0 \leq r \leq j} \bigoplus_{k \geq 0} C_*\left(\Emb^{\theta}(\bigcup_{k+r} \mathbb{R}^n, M)\right) \otimes_{\mathcal{G}_k \times \mathcal{G}_r} \left((\mathcal{E}_n^\eta)^{\theta}(\mathcal{A})^{\otimes k} \otimes (\text{id}^p \otimes e_N)^{\otimes r}\right)$$

That is, we allow at most $j$ copies of the new cell.

This induces a filtration of chain complexes upon realization. The spectral sequence associated to this filtration is one in the statement of this proposition, and is constructed as in Section 2.2 of [31]. Its convergence follows from the finiteness of the filtrations in each homological degree. The associated graded is independent of $b_{N-1}$, so it suffices to identify it in the case $b_{N-1} = 0$. In that case the filtration exactly comes from a direct sum decomposition with summands the subcomplexes of $|\mathcal{M}^\theta(\text{Comp}_\cdot)|$ with exactly $r$ copies of $e_N$. More precisely, we have that
and $|\mathcal{M}^\theta(\text{Comp}_p)|$ is isomorphic to the direct sum over $r$ of the realization of simplicial chain complexes on the last line. If $b_{N-1} = 0$, this direct sum decomposition causes the spectral sequence to collapse at the $E^1$-page. Furthermore, it allows us to identify the $E^1$-page, as Lemmas 11 and 64 show that the inclusion

$$B_* \left( E^\theta_r(M) \downarrow C^\theta_r(M \text{ dist.}_{\mathbb{Z}_\pm 1}^N), E^\theta_n, A \right) \rightarrow B_* \left( E^\theta_r(M) \downarrow C^\theta_r(M \text{ dist.}_{\mathbb{Z}_\pm 1}^N), E^\theta_n, A \right)$$

is a weak equivalence. The $r$th column on the $E^1$-page is given by taking the homology of the associated chain complex, so that by comparing to the definition of topological chiral homology of manifold bundles with local coefficients we obtain the identification

$$E^1_{pq} = H_{* - pN + p} \left( \int^{c - p\mathbf{k}}_{E^\theta_p(M) \downarrow C^\theta_p(M), \mathbb{Z}_\pm 1^N} A \right)$$

We recall that the $d^1$-differential is defined on a class $[x] \in E^1_{pq}$ on the $E^1$-page as follows: one lifts it to a cycle $x$ in the associated graded $\mathcal{F}_p / \mathcal{F}_{p-1},$ i.e. $d(x) \in \mathcal{F}_{p-1}.$ Then $d_1([x]) = [d(x)].$ In our case, an element of $E^1_{0q}$ can be represented by taking a cycle in $\int^{c - k}_{M \setminus m} A$ and adding a copy of $e_N$ at $m.$ The differential then vanishes on the cycle and sends $e_N$ to $b_{N-1}.$

An important consequence is that we can easily compute the effect of $\mathcal{E}^\theta_{N}$-cell attachments in homology in degrees less than or equal to the degree of the cell.

**Corollary 66** Suppose that $e_N$ is attached in charge $k$ along $b_{N-1}$.

(i) If $c < k$ (that is, there exists no $c'$ such that $c = c' + k$), we have that

$$H_*(A(c)) \rightarrow H_*(((A \cup e_N)(c))$$

is an isomorphism.
(ii) In general we have that

\[ H_*(A) \to H_*(A \cup e_N) \]

is an isomorphism for \( * \leq N - 2 \).

(iii) If \( c \geq k \), (that is, there exists a \( c' \) such that \( c = c' + k \)), there is an exact sequence

\[ H_N(A(c)) \to H_N((A \cup e_N)(c)) \to \mathbb{Z} \to H_{N-1}(A(c)) \to H_{N-1}((A \cup e_N)(c)) \to 0 \]

where the map \( \mathbb{Z} \to H_{N-1}(A(c)) \) is induced by \( 1 \mapsto t_c(b_{N-1}) \). In particular, attaching an \( E_0^n \)-cell of degree \( N \) with trivial attaching map does not affect \( H_{N-1} \).

(iv) If \( b_{N-1} = 0 \) the map \( H_N(A(c)) \to H_N((A \cup e_N)(c)) \) is split injective and thus

\[ H_N((A \cup e_N)(c)) \cong H_N(A(c)) \oplus \mathbb{Z}. \]

**Proof** Parts (i), (ii) and (iii) are a consequence of studying the initial parts of spectral sequences in Propositions 63 and 65 as follows. For general \( M \), on the \( E^1 \)-page of the spectral sequence of Proposition 65 we have that (a) \( E_{0,q}^1 = H_q(\int_M^k A) \), (b) \( E_{p,q}^1 = 0 \) if \( p \geq 1 \) and \( q \leq N - 2 \), (c) \( E_{1,q}^1 = 0 \) if \( p \geq 1 \) if \( c < k \), and (d) \( E_{1,n-1}^1 = \mathbb{Z} \) if there exists a \( c' \) such that \( c = c' + k \). Now specialize to \( M = \mathbb{R}^n \) and use that \( A(c) = \int_{\mathbb{R}^n}^c A \).

For part (iv), one just needs to note that if \( b_{N-1} = 0 \) there is a map \( A \cup e_N \to A \) sending \( e_N \) to 0, which on homology splits the map \( A \cong \tilde{A} \to A \cup e_N \).

**Remark 67** The results in this subsection have analogues for the attachment of any collection of \( E_0^n \)-cells of the same dimension and charge, as in Remark 59. In that case \( C_r^g(M) \) and its relatives get replaced by configuration spaces with labels in the indexing set of the \( e_0^n \)-cells, and further modifications are as one expects.

### 3.3 \( E_0^n \)-cell attachments preserve homological stability

Computing the homology in higher degrees is harder, but we can use the spectral sequences to deduce that \( E_0^n \)-cell attachments preserve homological stability. To do so, we restrict ourselves to monoids of charges given by \( C = \mathbb{N}_0^d \). We will prove a slightly stronger statement than we will eventually need.

**Corollary 68** Let \( C = \mathbb{N}_0^d \), \( n \geq 2 \) and let \( M \) be a \( \theta \)-framed non-compact connected manifold. Suppose that \( A \) is a charged algebra such that for each \( r \geq 0 \) the space \( \int_M^{A \cup e_N} A \) has homological stability in a range that does not depend on \( r \), then \( \int_M^A(A \cup e_N) \) has homological stability.

Moreover, fix \( 1 \leq i \leq d \) and let \( \rho : C \to \mathbb{R}_{\geq 0} \) be a function satisfying (i) \( \rho(a) < \rho(b) \) if \( a_i < b_i \) and \( a_j \geq b_j \) for \( j \neq i \), (ii) \( \rho(a + b) \leq \rho(a) + \rho(b) \) for all \( a, b \in C \).

Let \( e_N \) be attached in charge \( k \) with \( \rho(k) \leq N \). Assume that for all \( r \geq 0 \) we have

\[ t_i : \int_{M \setminus \{r \text{ points}\}}^c A \to \int_{M \setminus \{r \text{ points}\}}^{c + e_i} A \]

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is a $\rho(\mathbf{c})$-equivalence. Then

$$t_i: \int_M^c A \cup e_N \to \int_M^{c+e_i} A \cup e_N$$

is also a $\rho(\mathbf{c})$-equivalence.

**Proof** First note that the second part of this corollary implies the first. The idea of the proof is to apply the two spectral sequences of Propositions 63 and 65 above.

Firstly, we prove that $H_\ast \left( \int_j E_\theta^r(M) \downarrow C_\theta^r(M), L \right)$ has homological stability in the same range as $H_\ast \left( \int_j M \{r \text{ points} \} A \right)$. To do so we apply Proposition 63 to the stabilization map

$$H_\ast \left( \int_j E_\theta^r(M) \downarrow C_\theta^r(M), L \right) \to H_\ast \left( \int_j^{j+e_i} E_\theta^r(M) \downarrow C_\theta^r(M), L \right).$$

The stabilization map induces a map of spectral sequences converging to the domain and target of the stabilization map, which on the $E^2$-pages

$$E^2_{p,q} = H_p \left( C_r^0(M); \mathcal{H}_q \left( \int_M \{r \text{ points} \} A \right) \otimes L \right)$$

$$\Rightarrow$$

$$(E')^2_{p,q} = H_p \left( C_r^0(M); \mathcal{H}_q \left( \int_M \{r \text{ points} \} A \right) \otimes L \right)$$

is induced by the map $\mathcal{H}_q \left( \int_M \{r \text{ points} \} A \right) \otimes L \to \mathcal{H}_q \left( \int_M^{j+e_i} \{r \text{ points} \} A \right) \otimes L$ of local systems. By hypothesis, if we fix a $Q$ for $j \gg 0$ this is an isomorphism for all $q \leq Q$, and hence the map on $E^2$-pages is an isomorphism for $q \leq Q$ as well. A spectral sequence comparison argument then implies the stabilization map is an isomorphism for $q \leq Q$.

If we make this argument quantitative, we get that for all $r$ and all $j$, the map

$$t: \int_j^{j+e_i} E_\theta^r(M) \downarrow C_\theta^r(M), L \to \int_j^{j+e_i} E_\theta^r(M) \downarrow C_\theta^r(M), L$$

is a $\rho(j)$-equivalence.

Next, we prove the corollary by applying the spectral sequence of Proposition 65. The stabilization map induces a map of spectral sequences converging to the induced map $H_\ast \left( \int_M A \cup e_N \right) \to H_\ast \left( \int_M^{e+e_i} A \cup e_N \right)$. On the $E^1$-pages it is given by the following.
\[
E^1_{p,q} = H_{q-(N-1)p} \left( \int \frac{c-pk}{E^p(M) \downarrow C^p(M), \mathbb{Z}^{\pm_1}_N} A \right) \\
(E')^1_{p,q} = H_{q-(N-1)p} \left( \int \frac{c+e-pk}{E^p(M) \downarrow C^p(M), \mathbb{Z}^{\pm_1}_N} A \right)
\]

We will first prove it is a surjection for \( p+q \leq \rho(c) \). There are now three cases to consider: (i) \( c_i - pk_i < -1 \) or \( c_j - pk_j \leq -1 \) for all \( j \neq i \), (ii) \( c_i - pk_i = -1 \) and \( c_j - pk_j \geq 0 \) for all \( j \neq i \), and (iii) \( c_j - pk_j \geq 0 \) for all \( j \).

(i) In the first case we are considering a map between zero chain complexes, which is an isomorphism.

(ii) In the second case we are considering a map from a zero chain complex to a non-zero one. This is obviously an isomorphism in negative homological degrees, i.e., if \( q - (N-1)p < 0 \), but we need it to be a surjection for all \( p+q \leq \rho(c) \). Note that \( p\rho(k) \geq \rho(pk) \) by part (ii) of our assumptions on \( \rho \) and \( \rho(pk) > \rho(c) \) by part (i) of our assumptions on \( \rho \) since \( c_i = pk_i - 1 < pk_i \) but \( c_j = pk_j \). So it suffices to prove that it is a surjection for \( p+q \leq p\rho(k) \) if \( q - (N-1)p \geq 0 \). This is true because the inequality \( q - (N-1)p \geq 0 \) is equivalent to \( p+q \leq pN \), and we assumed that \( \rho(k) \leq N \).

(iii) Finally, in the third case, the map on the \((p, q)\)-entry of \( E^1 \)-page is an surjection if \( q - (N-1)p \leq \rho(c-pk) \), or in other words if \( p+q \leq \rho(c-pk) + Np \). We want a surjection for all \( p+q \leq \rho(c) \), which happens if \( \rho(c) \leq \rho(c-pk) + Np \) for all \( p \). By our assumption regarding where the \( E^\theta_n \)-cell is attached, \( N \geq \rho(k) \) and this inequality is implied by \( \rho(c) \leq \rho(c-pk) + \rho(k)p \) for all \( p \). This follows from \( \rho(a+b) \leq \rho(a) + \rho(b) \) for all \( a, b \in C \).

Thus the map on the \( E^1_{pq} \)-pages is a surjection for \( p+q \leq \rho(c) \). A similar argument shows that it is an isomorphism for \( p+q \leq \rho(c) - 1 \). The corollary now follows from spectral sequence comparison. \( \square \)

**Remark 69** Using Remark 67, the results in this subsection have analogues for the attachment of any collection of \( \varepsilon^\theta_n \)-cells of the same dimension and charge, as in Remark 59.

### 3.4 Bounded generation

The definitions of the previous subsection lead us to the definition of bounded generation, which we will only give for \( C = \mathbb{N}_0^d \). We treat the degree zero case separately and put restriction on the \( E^\theta_n \)-cells of degree one to make sure we stay in the category of charged algebras.

**Definition 70** Let the monoid of charges be given by \( C = \mathbb{N}_0^d \).

(i) We say that a charged algebra \( A \) is obtained from \( A' \) by \( \varepsilon^\theta_n \)-cell attachments of degree \( N \) in charge \( \leq c \) if there exists a possibly infinite collection of \( \varepsilon^\theta_n \)-cells...
{e_i^N}_{i \in I} attached along \{b_i\}_{i \in I} of homological degree $N - 1$ and charge $\leq c$, such that there is an isomorphism

$$A' \cup \bigoplus_{i \in I} e_i^N \to A$$

If $N = 1$ we require the attaching maps to be trivial, so that the underlying $C$-charged chain complexes remain connected.

(ii) We say that a $C$-charged algebra $A$ is bounded cellular of degree $\leq 0$ if $A = 0 \cup \bigcup_{i=1}^d e_i^0$ with each $e_i^0$ of degree 0.

(iii) We say that a charged algebra $A$ is bounded cellular of degree $\leq N$ if $A$ is obtained from an $E_\theta^n$-algebra which is bounded cellular of degree $\leq 0$, by $E_\theta^n$-cell attachments of $E_\theta^n$-cells such that those of homological degrees $n \leq N$ are attached in finitely many charges $k$.

We can now define bounded generation, the notion appearing in the main theorem.

**Definition 71** (i) A charged algebra $A$ is said to be bounded generated in degrees $\leq N$ if there is an $(N+1)$-equivalence $f : B \to A$ with $B$ bounded cellular of degree $\leq N$.

(ii) It is said to be bounded generated if it is bounded generated in degrees $\leq N$ for all $N \geq 0$.

**Remark 72** The previous definitions and constructions can be modified for $E_\theta^n$-algebras in Top and sSet. For Top the input thus consists of an $E_\theta^n$-algebra $X$ and a map $S^{N-1} \to X$. One simply replaces the chain complex with a single generator $e_{N-1}$ in degree $N - 1$ by $S^{N-1}$ and the chain complex with two generators $e_N$ and $e_{N-1}$ in degrees $N$ and $N - 1$ respectively, with $d(e_N) = e_{N-1}$ by $D^N$. Then one can define a partial $E_\theta^n$-algebra $X \cup D^N$ and its completion $X \cup D^N$, so that the analogues of Corollaries 66 and 68 hold. For sSet, a subtlety is that $X$ may not be fibrant as a simplicial set; we need to attach an iterated subdivision of $\Delta^N$ along an iterated subdivision of $\partial \Delta^N$.

4 The proof of the main theorem

In this section, we prove a strengthening of Theorem 1 by proving a result with explicit ranges. In this section we fix the tangential structure $\theta$ and set the monoid of charges $C$ to be $\mathbb{N}_0^d$, and hence suppress them from our notation whenever convenient.

**Definition 73** For $k \in \mathbb{N}_0^d$, we let $|k|$ denote $\max \{k_i\}$. We call this the maximal charge.

**Theorem 74** Let $n \geq 2$, $C = \mathbb{N}_0^d$, $A$ be a charged algebra and $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a strictly increasing function with (i) $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{R}_{\geq 0}$, (ii) $\rho(x) \leq x/2$ and (iii) $\rho^{-1}([0, \infty)) \subset \mathbb{N}_0$. The following are equivalent:

(i) For all $k \in C$ and each $i$, the basic stabilization maps $t_i : A(k) \to A(k + e_i)$ are $\rho(k_i)$-equivalences.
(ii) For any \( \theta \)-framed connected non-compact \( n \)-dimensional manifold, all \( k \in C \) and each \( i \), the basic stabilization maps \( t_i : \int_M^k A \to \int_M^{k+e_i} A \) are \( \rho(k_i) \)-equivalences.

(iii) There is a weak equivalence \( B \to A \) where \( B \) is bounded cellular such that if \( B \) has an \( E^n_\theta \)-cell in positive degree \( N \) and with charge \( k \), then \( N \geq \rho(|k|) \).

Here are some remarks about the consequences and assumptions of this theorem:

- Theorem 74 implies Theorem 1. To see this, note that in Definition 71, if all \( E^n_\theta \)-cells of degree \( \leq N \) are attached in finitely many charges, then there exists a function \( \rho : \mathbb{N}_0 \to \mathbb{R}_{\geq 0} \) with \( \lim_{j \to \infty} \rho(j) = \infty \) such that all \( E^n_\theta \)-cells of degree \( N \) are attached in charges \( k \) with \( \rho(|k|) \leq N \). Next remark that for any function \( \rho : \mathbb{N}_0 \to \mathbb{R}_{\geq 0} \) with \( \lim_{j \to \infty} \rho(j) = \infty \), there exists another function \( \rho' : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) with \( \lim_{j \to \infty} \rho'(j) = \infty \) which moreover satisfies the conditions in Theorem 74 and has the property that \( \lfloor \rho(j) \rfloor \geq \lfloor \rho'(j) \rfloor \) for all \( j \in \mathbb{N}_0 \).

- The homological stability range of \( \int_M A \) in general is not as good as that of \( A \). In the case of Example 4, note that even though \( \text{Sym}(\mathbb{R}^n) \) has a stability range with infinite slope, its topological chiral homology \( \text{Sym}(M) \) has a stability range with slope 1.

- Note that (i) only depends on the algebra structure on the homology of \( A \). Thus if \( A \) has \( E^n_\theta \)-algebra and \( E^n_\theta' \)-algebra structures inducing the same algebra structures on homology, this theorem can be used to transfer information about \( A \) as an \( E^n_\theta \)-algebra to \( A \) as an \( E^n_\theta' \)-algebra.

- The assumption \( n \geq 2 \) is required for homological stability with labels in a fibration, the base case of an induction in the proof. No generality is lost, as the case \( n = 1 \) is trivial per the remarks following Theorem 1.

- Closely related to bounded generated \( E^n_\theta \)-algebras are degreewise finitely generated framed \( E^n_\theta \)-algebras. These are cellular algebras with finitely many cells of a given degree. The same proof shows that having homological stability in conjunction with finite type homology, i.e. in each homological degree and charge the homology is finitely generated as an abelian group, is equivalent to degreewise finite generation.

We continue with remarks about \( \rho \):

- The definition of \( \rho \) as a function \( \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) instead of a function \( \mathbb{N}_0 \to \mathbb{N}_0 \) merely makes it easier to state its required properties.

- One can give a version of this theorem with different \( \rho_i \) for each stabilization map, which is allowed to depend on more than just the \( i \)th coordinate of the charge.

- The condition that \( \rho(x) \leq x/2 \) stems from the fact that this is the stability range for free \( E^n_\theta \)-algebras. If one were to invert 2 in the coefficients of homology, this can be relaxed to \( \rho(x) \leq x \) when \( n \geq 3 \) [22,35]. Rationally this is Theorem B of [6].

Assuming Theorem 1, we will give the proof of the local-to-global homological stability principle, i.e. Corollary 3. Recall that \( X \) was an \( E^n_{SO} \)-algebra in \( \text{Top} \) with homological stability and \( \pi_0(X) = \mathbb{N}_0 \). We will instead prove it for any tangential structure \( \theta \) and \( C = \mathbb{N}^d_0 \).
Proof (Proof of Corollary 3) Since $X$ has homological stability, so does $C_\ast(X)$. Thus by Theorem 1, $\int_M^k C_\ast(X)$ has homological stability with $M$ a $\theta$-framed connected non-compact manifold. It now suffices to note that we have that:

$$C_\ast \left( \int_M X \right) = C_\ast \left( |B_\ast (M^n, E_n^0, X)| \right) \sim \left| B_\ast (M^n, E_n^0, C_\ast(X)) \right| = \int_M C_\ast(X)$$

The only content here is in the middle weak equivalence, which is Lemma 16. □

The proof of Theorem 74 involves an induction over the degrees in which we have homological stability. This requires us to define the following four classes of charged algebras. The proof will then follow from determining their interplay.

**Definition 75** Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a strictly increasing function with (i) $\rho(x+y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{R}_{\geq 0}$, (ii) $\rho(x) \leq x/2$ and (iii) $\rho^{-1}(\mathbb{N}_0) \subset \mathbb{N}_0$. We define the following classes of charged algebras:

(i) $C^\rho_N$ is the class of bounded cellular algebras (see Definition 70) with $E_n^0$-cells of degree $j \leq N$ and the $E_n^0$-cells of degree $j$ attached in charge $k$ only if $j \geq \rho(|k|)$.

(ii) $B^\rho_N$ is the class of charged algebras $A$ such that there is an $N$-equivalence $B \to A$ with $B \in C^\rho_N$.

(iii) $S^\rho_N$ is the class of charged algebras $A$ that has homological stability range $\rho$ up to degree $N$. That is, the map $t_i : A(k) \to A(k+e_i)$ is a $\min(N, \rho(k_i))$-equivalence.

(iv) $T^\rho_N$ is the class of charged algebras $A$ which on $\theta$-framed connected non-compact $n$-dimensional manifolds $M$ have homological stability range $\rho$ up to degree $N$.

That is, for each $i$ the map $t_i : \int_M^k A \to \int_M^{k+e_i} A$ is a $\min(N, \rho(k_i))$-equivalence.

Also let $B^\rho_\infty$, $S^\rho_\infty$ or $T^\rho_\infty$ denote the intersection of respectively $B^\rho_N$, $S^\rho_N$ or $T^\rho_N$ over all $N \in \mathbb{N}_0$.

We deduce Theorem 74 from the propositions which follow later in this section.

Proof (Proof of Theorem 74) Combining Propositions 77, 78, 79 and 81, we see that $C^\rho_N = S^\rho_N = T^\rho_N$ for all $N \geq 0$. Thus $C^\rho_\infty = S^\rho_\infty = T^\rho_\infty$ which is equivalent to Theorem 74. □

### 4.1 Closure properties

We start with an easy proposition.

**Lemma 76** Let $B \to A$ be an $N$-equivalence. If $B \in B^\rho_N$, $S^\rho_N$ or $T^\rho_N$, then respectively so is $A$.

Proof The statement about $B^\rho_N$ is true because a composition of maps which are $N$-equivalences is an $N$-equivalence.
Let $B \rightarrow A$ be an $N$-equivalence and assume $B \in S^\rho_N$. Consider the diagram

$$
\begin{array}{ccc}
B(k) & \longrightarrow & B(k + e_i) \\
\downarrow & & \downarrow \\
A(k) & \longrightarrow & A(k + e_i)
\end{array}
$$

Since the top horizontal map is a $\min(N, \rho(k))$-equivalence and the vertical maps are $N$-equivalences, the bottom horizontal map is a $\min(N, \rho(k))$-equivalence and so we may conclude that $A \in S^\rho_N$.

Now let $B \rightarrow A$ be an $N$-equivalence but instead assume $B \in T^\rho_N$. Consider the diagram

$$
\begin{array}{ccc}
\int_M^k B & \longrightarrow & \int_M^{k+e_i} B \\
\downarrow & & \downarrow \\
\int_M^k A & \longrightarrow & \int_M^{k+e_i} A
\end{array}
$$

The top horizontal map is a $\min(N, \rho(k))$-equivalence by assumption. Lemma 37 implies that the vertical maps are $N$-equivalences. Thus the bottom horizontal map is a $\min(N, \rho(k))$-equivalence and so we may conclude that $A \in T^\rho_N$. \hfill \Box

4.2 The case $N = 0$

Since our proof of Theorem 74 is an induction over $N$, we start with $N = 0$.

**Proposition 77** Any charged algebra is in $B^\rho_0 = T^\rho_0 = S^\rho_0$.

**Proof** By definition, charged algebras are augmented and connected. Thus we have a chosen isomorphism $H_0(A) \cong \mathbb{Z}[\mathbb{N}^d_0]$, which is in $S^\rho_0$. We claim it is in $B^\rho_0$ as well. To see this, take any representative $a_i \in A(e_i)$ of the chosen generator of $H_0(A(e_i)) \cong \mathbb{Z}$. Let $B$ be the free $\mathcal{E}^\rho_0$-algebra on $d$ generators in degree 0. This is in $C^\rho_0$ by definition. The choice of $a_i \in A(e_i)$ gives a map $B \rightarrow A$ which induces an isomorphism on $H_0$ (and hence is a 0-equivalence). Thus, $A \in B^\rho_0$.

By the proof of Proposition 79, $H_0(\int_M^k B) \cong \mathbb{Z}$ for $M$ connected. This shows that $B \in T^\rho_0$. Since $B \rightarrow A$ is a 0-equivalence, Lemma 76 shows that $A$ is in $T^\rho_0$ as well. By Proposition 78, we have $T^\rho_0 \subset S^\rho_0$.

4.3 Stability for topological chiral homology implies stability

Another easy proposition says that homological stability of topological chiral homology on any open manifold implies it for $\mathbb{R}^n$.

**Proposition 78** For any $N \geq 0$ we have that $T^\rho_N \subset S^\rho_N$.
**Proof** We have $\int_{\mathbb{R}^n} A \simeq A$ by Lemma 36, and $\mathbb{R}^n$ is a $\theta$-framed connected non-compact manifold. \hfill $\Box$

### 4.4 Level wise bounded generation implies stability for topological chiral homology

Next we prove that bounded generation implies homological stability for topological chiral homology of $A$ on any $\theta$-framed connected non-compact $n$-dimensional manifold $M$. See Definition 61 for the definition of configuration spaces labeled in a fibration.

**Proposition 79** For any $N$ we have that $\mathcal{C}_N^0 \subset \mathcal{T}_\infty^\rho$ and thus $\mathcal{C}_N^0 \subset \mathcal{S}_\infty^\rho$.

**Proof** We prove this by induction on $N$. For $N = 0$, we can assume that the $\mathcal{E}_n^\theta$-algebra $A$ is of the form

$$
\bigoplus_{k=(k_1, \ldots, k_d)} C_* \left( \mathcal{E}_n^\theta(k_1 + \cdots + k_d) / \mathcal{S}_k \right)
$$

We need to establish homological stability for $\int_M A$.

Recall from Sect. 3.2 that $\mathcal{C}_k^0(M)$ is quotient by $\mathcal{S}_k$ of the space of $\theta$-framed embeddings $\bigsqcup_k \mathbb{R}^n \to M$ whose centers are disjoint, which is equals the configuration space of $k$-colored particles with labels in the fibration $\theta(TM) \to M$.

By an extra degeneracy argument $\int_M A$ is weakly equivalent to chains on the space

$$
\bigcup_{k=(k_1, \ldots, k_d)} \text{Emb}^\theta(\bigsqcup_{k_1 + \cdots + k_d} \mathbb{R}^n, M) / \mathcal{S}_k
$$

The inclusion of this into $\mathcal{C}_k^0(M)$ is a weak equivalence, by a similar shrinking argument as in Lemma 64. It thus suffices to establish homological stability for the spaces $\bigoplus_k C_*(\mathcal{C}_k^0(M))$. This holds with function $\rho(x) = x/2$ by a generalization of Proposition B.4 of [35]: this generalization says that if $E \to M$ is a fiber bundle over $M$ with path-connected fibers, then the configuration space of unordered of points in $M$ with labels in $E$ exhibits homological stability. The necessary generalizations involve replacing a single color by multiple colors and replacing fiber bundle with fibration. They prove a stability range of $k^2 - 1$ but this can be improved to a range of $k^2$ using the arguments appearing in the last paragraph of page 9 of [22].

1 We conclude that $A \in \mathcal{T}_\infty^\rho$ and so $\mathcal{C}_0^0 \subset \mathcal{T}_\infty^\rho$.

Assume the statement is true for $N - 1$ and choose $A \in \mathcal{C}_N^0$. Then there are cellular $\mathcal{E}_n^\theta$-algebras $A_0, A_1, \ldots$ such that $A_0 \in \mathcal{C}_{N-1}^0$, $A_{i+1}$ is constructed from $A_i$ by attaching cells in degree $N$ and charge $i$, and $A_i = A$ for $i > \rho(N)$. By our induction hypothesis, $A_0 \in \mathcal{T}_\infty^\rho$. Corollary 68 and Remark 69 imply that if $A_i \in \mathcal{T}_\infty^\rho$ and $i \leq \rho(N)$, then $A_{i+1} \in \mathcal{T}_\infty^\rho$. Thus, $A \in \mathcal{T}_\infty^\rho$. \hfill $\Box$

---

1 See also the appendix of the arXiv-version of this paper.
Proposition 80  For any \( N \) we have that \( \mathcal{B}^0_N \subset T^\rho_N \) and thus \( \mathcal{B}^0_N \subset S^0_N \).

Proof  By definition, \( A \in \mathcal{B}^0_N \) implies the existence of a \( N \)-equivalence \( B \to A \) such that \( B \in \mathcal{C}^0_N \). But we just saw that \( \mathcal{C}^0_N \subset T^\rho_N \) and hence \( B \in T^\rho_N \). The conclusion \( A \in T^\rho_N \) now follows from Lemma 76.

\[ \Box \]

4.5 Stability implies bounded generation

Finally we prove the hardest step in the induction, that homological stability in the range \( * \leq N + 1 \) implies bounded generation in the range \( * \leq N + 1 \) given that result for the range \( * \leq N \). The strategy is that of relative CW approximation of maps.

Proposition 81  If \( S^0_N = \mathcal{B}^0_N \), then \( S^0_{N+1} \subset \mathcal{B}^0_{N+1} \).

Proof  Let \( A \in S^0_{N+1} \), then our hypothesis implies \( A \in \mathcal{B}^0_N \) and there exists an \( N \)-equivalence \( f : B \to A \) with \( B \in \mathcal{C}^0_N \). Let \( R_{N+1} = \rho^{-1}(N + 1) \), which is a single non-negative integer since \( \rho : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) is strictly increasing and satisfies \( \rho^{-1}(\mathbb{N}_0) \subset \mathbb{N}_0 \). We will extend the map \( f : B \to A \) by attaching to \( B \) a collection of \( \mathcal{E}_n \)-cells of degree \( N + 1 \) and charge \( k \) satisfying \( |k| \leq R_{N+1} \). The end result will be an \( \mathcal{E}_n \)-algebra \( B' \) and an \((N + 1)\)-equivalence \( f' : B' \to A \) of \( \mathcal{E}_n \)-algebras extending \( f : B \to A \).

We will build \( f' : B' \to A \) by induction on maximal charge \( c = |k| \). Let \( B^\mathrm{surj}_{-1} : B \) and \( f_{-1} = f \). For \( 0 \leq c \leq R_{N+1} \), we will define \( \mathcal{E}_n \)-algebras with maps of \( \mathcal{E}_n \)-algebras extending \( f : B \to A \):

\[
\begin{align*}
f^\mathrm{surj}_c : B^\mathrm{surj}_c &\to A \quad \text{and} \quad f^\mathrm{iso}_c : B^\mathrm{iso}_c &\to A
\end{align*}
\]

with the following properties

- Both \( B^\mathrm{surj}_c \) and \( B^\mathrm{iso}_c \) will be in \( \mathcal{C}^0_{N+1} \).
- The algebra and map \( f^\mathrm{surj}_c : B^\mathrm{surj}_c \to A \) will extend \( f^\mathrm{iso}_c : B^\mathrm{iso}_c \to A \) and have the property that \( B^\mathrm{surj}_c(k) \to A(k) \) is an \((N + 1)\)-equivalence for \( |k| \leq c \) and an \( N \)-equivalence for \( |k| > c \).
- The algebra and map \( f^\mathrm{iso}_c : B^\mathrm{iso}_c \to A \) will extend \( f^\mathrm{surj}_{c-1} : B^\mathrm{surj}_{c-1} \to A \) and have the property that \( B^\mathrm{iso}_c(k) \to A(k) \) is an \( N \)-equivalence for all \( k \) and \( H_N(B^\mathrm{iso}_c(k)) \to H_N(A(k)) \) is an isomorphism for \( |k| \leq c \).

The idea is that the algebra and map \( f^\mathrm{surj}_c : B^\mathrm{surj}_c \to A \) correct the problem that \( f^\mathrm{iso}_c : B^\mathrm{iso}_c \to A \) is not a surjection on \( H_{N+1} \) and maximal charge \( |k| = c \), while the algebra and map \( f^\mathrm{iso}_c : B^\mathrm{iso}_c \to A \) correct the problem that the map \( f^\mathrm{surj}_{c-1} : B^\mathrm{surj}_{c-1} \to A \) is not an isomorphism on \( H_N \) and maximal charge \( |k| = c \). Eventually \( f' : B' \to A \) will be \( f^\mathrm{surj}_{R_{N+1}} : B^\mathrm{surj}_{R_{N+1}} \to A \).

It suffices to assume we have constructed \( f^\mathrm{surj}_c : B^\mathrm{surj}_c \to A \) for \( c \leq R_{N+1} \), and construct first \( f^\mathrm{iso}_c : B^\mathrm{iso}_c \to A \) and then \( f^\mathrm{surj}_c : B^\mathrm{surj}_c \to A \).

Step 1—isomorphism

\[ \circledast \text{ Springer} \]
Assume we have defined $f_{c-1}^{\text{surj}} : B_{c-1}^{\text{surj}} \to A$ with the desired properties listed above. Consider the kernels of $H_N(B_{c-1}^{\text{surj}}(k)) \to H_N(A(k))$ for each $k$ satisfying $|k| = c$. Take cycles $\{b_p\}_{p \in P_k}$ with $b_p \in B_{c-1}^{\text{surj}}(k)$ representing a generating set of these kernels. Because the homology class of $b_p$ is in the kernel of the map to $H_*(A(k))$, $f_{c-1}^{\text{surj}}(b_p)$ is a boundary $d(x_p)$.

For each $p \in P_k$ we attach an $E_n^0$-cell $((e^{N+1})_p)$ to $B_{c-1}^{\text{surj}}$ in charge $k$ along $b_p$. We call the resulting algebra $B_c^{\text{iso}}$ and using Lemma 57 extend the map $B_{c-1}^{\text{surj}} \to A$ to a map

$$f_c^{\text{iso}} : B_c^{\text{iso}} := B_{c-1}^{\text{surj}} \cup \bigcup_{p \in P} ((e^{N+1})_p) \to A$$

by sending $((e^{N+1})_p)$ to $x_p$.

Note that $B_c^{\text{iso}} \in C_{N+1}^p$ and that $B_c^{\text{iso}} \to A$ induces an isomorphism on $H_*$ for $* \leq N - 1$ by Property (ii) of Corollary 66 and is an $(N+1)$-equivalence for charges $k$ with $|k| = c - 1$ by Property (i) of Corollary 66.

We also claim it induces an isomorphism on $H_N$ for charges $k$ with $|k| = c$. This will be a consequence of Property (iii) of Corollary 66, which says there is an exact sequence

$$H_{N+1}(B_{c-1}^{\text{surj}}(k)) \to H_{N+1}(B_c^{\text{iso}}(k)) \to \bigoplus_{p \in P_k'} \mathbb{Z} \to H_N(B_{c-1}^{\text{surj}}(k))$$

$$\quad \to H_N(B_c^{\text{iso}}(k)) \to 0$$

where $k'$ ranges over all $k'$ such that $k' \leq k$ and the generator of the $\mathbb{Z}$-summand corresponding to $p \in P_k$ goes to $i_{k-k'}(b_p)$. We conclude that $H_N(B_c^{\text{iso}}(k)) = H_N(B_{c-1}^{\text{surj}}(k)) / \text{im} \left( \bigoplus_{p \in P_k'} \mathbb{Z} \right)$ and note that this image contains the kernel $\text{ker} (H_N(B_{c-1}^{\text{surj}}(k)) \to H_N(A(k)))$. Furthermore, by construction the surjective map $H_N(B_{c-1}^{\text{surj}}(k)) \to H_N(A(k))$ factors over $H_N(B_c^{\text{iso}}(k))$. Hence the map $H_N(B_c^{\text{iso}}(k)) \to H_N(A(k))$ is a surjection with trivial kernel and hence an isomorphism.

**Step 2—surjection**

Assume we have defined $f_c^{\text{iso}} : B_c^{\text{iso}} \to A$ with the desired properties. Take cycles $\{a_q\}_{q \in Q_k}$ with $a_q \in A(k)$ representing a generating set of $H_{N+1}(A(k))$ for each $k$ satisfying $|k| = c$.

For each $q \in Q_k$ we attach an $E_n^0$-cell $((e^{N+1})_q)$ to $B_c^{\text{iso}}$ in charge $k$ with trivial attaching map (that is, setting $d((e^{N+1})_q) = 0$). We call the resulting algebra $B_c^{\text{surj}}$ and using Lemma 57 extend the map $B_c^{\text{iso}} \to A$ to a map

$$f_c^{\text{surj}} : B_c^{\text{surj}} := B_c^{\text{iso}} \cup \bigcup_{q \in Q} ((e^{N+1})_q) \to A$$
by sending \((e^{N+1})_q\) to \(a_q\). This is possible since \(d(a_q) = 0\) because \(a_q\) is a cycle.

Note that \(B_c^{\text{surj}} \in C_{N+1}^\rho\) and that \(B_c^{\text{surj}} \to A\) induces an isomorphism on \(H_*\) for 
\(* \leq N\) by properties (ii) and (iii) of Corollary 66 and a surjection on \(H_{N+1}\) for charges \(k\) with \(|k| \leq c - 1\) by Property (i) of Corollary 66.

We also claim it induces a surjection on \(H_{N+1}\) for charges \(k\) with \(|k| = c\). This is a consequence of Property (iv) of Corollary 66, since in this case we get an exact sequence

\[
H_{N+1}(B_c^{\text{iso}}(k)) \to H_{N+1}(B_c^{\text{surj}}(k)) \to \bigoplus_{q \in Q_k} \mathbb{Z} \to 0
\]

where \(k'\) ranges over all \(k'\) such that \(k' \leq k\). The first map is canonically split by sending the \((e^{N+1})_q\) to 0, which implies we have an isomorphism

\[
H_{N+1}(B_c^{\text{surj}}(k)) \cong H_{N+1}(B_c^{\text{iso}}(k)) \oplus \bigoplus_{q \in Q_k} \mathbb{Z}.
\]

By construction we have that the map \(H_{N+1}(B_c^{\text{surj}}(k)) \to H_{N+1}(A(k))\) extends the map \(H_{N+1}(B_c^{\text{iso}}(k)) \to H_{N+1}(A(k))\) by sending the generator of the \(\mathbb{Z}\)-summand corresponding to \(q \in Q_k\) to \(a_q \in H_{N+1}(A(k))\). This makes the map surjective since we hit all generators of the cokernel.

Let \(f': B' \to A\) be \(f_{R_{N+1}}^{\text{surj}} : B_{R_{N+1}}^{\text{surj}} \to A\), thus obtaining an \(\mathcal{E}_n^\theta\)-algebra \(B' \in C_{N+1}^\rho\) with a map \(B' \to A\) such that the map is an \((N+1)\)-equivalence for charges \(r\) satisfying \(|r| \leq R_{N+1}\). We will prove it is an \((N+1)\)-equivalence for all charges.

By definition, \(A\) has homological stability range \(\rho\) and by Proposition 79, so does \(B'\). Let \(r\) satisfy \(|r| = R_{N+1}, j \in \mathbb{N}_0^d\) and consider the following commutative diagram:

\[
\begin{array}{ccc}
B'(r) & \xrightarrow{\delta} & B'(r + j) \\
\downarrow & & \downarrow \\
A(r) & \xrightarrow{\delta} & A(r + j)
\end{array}
\]

The two stabilization maps (the horizontal maps) are \((N+1)\)-equivalences by homological stability. The leftmost vertical map is an \((N+1)\)-equivalence by the argument above. Therefore, the rightmost vertical map is an \((N+1)\)-equivalence as well. This shows that \(B' \to A\) is an \((N+1)\)-equivalence and so \(A \in B_{N+1}^\rho\).

\[\square\]

Remark 82 The technique used in the proof of Proposition 81 can be used to show that any connected augmented charged algebra in chain complexes can be obtained by iterated \(\mathcal{E}_n^\theta\)-cell attachments up to weak equivalence.

5 Extending the main theorem to compact manifolds

In this section we prove a restatement of the main theorem, which extends to connected compact manifolds. In this section we restrict to \(\mathcal{E}_n^\rho\)-algebras in \(\text{Top}\) or \(s\text{Set}\) so that
we can use the results of [23], but we believe that these results also hold for $\mathsf{Ch}$ and other tangential structures.

For us, the $n$-fold delooping $B^n X$ will be defined to be $B(\Sigma^n, E^n_{\mathcal{PL}}, X)$, with $\Sigma^n$ as defined below. There is a natural zig-zag of weak equivalences between this model of the $n$-fold delooping and that of May [25].

**Definition 83** Let $\Sigma^n_+$ be the right functor over $E^n_{\mathcal{PL}}$ in $\mathsf{Top}$ given by

$$X \mapsto \Sigma^n X_+$$

where we consider the sphere in $\Sigma^n X_+ = S^n \wedge X_+$ as the one-point compactification of $\mathbb{R}^n$. The right module structure $a : \Sigma^n_+ E^n_{\mathcal{PL}} X \to \Sigma^n_+ X$ is defined as follows. A point of $\Sigma^n_+ E^n_{\mathcal{PL}} X$ is either the base point or determined by a triple $(r, (e, \psi), x)$ of $r \in \mathbb{R}^n$, $(e, \psi) \in \text{Emb}^\theta(\sqcup_k \mathbb{R}^n, \mathbb{R}^n)$ and $x \in X$. Here $e$ is an embedding and $\phi$ is a path of bundle maps. If $r$ is not in the image of $e$, we define $a(r, (e, \psi), x) \in \Sigma^n_+ X$ to be the base point. Otherwise, suppose that $r$ is contained in the image under $e$ of the $i$th component of $\sqcup_k \mathbb{R}^n$, let $e_i : \mathbb{R}^n \to \mathbb{R}^n$ be the restriction of $e$ to the component whose image contains $r$ and set

$$a(r, (e, \psi), x) := \left( e_i^{-1}(r), x \right) \in \mathbb{R}^n \times X.$$

Since $\Sigma^n_+$ is a right $E^n_{\mathcal{PL}}$-functor, so is $F \circ \Sigma^n_+$ for any functor $F$. Let $\text{Map}^c(\Sigma^n_+, \mathcal{N})$ denote the right $E^n_{\mathcal{PL}}$-functor whose value on a space $Y$ is $\text{Map}^c(\Sigma^n_+, Y)$, the space of compactly supported maps. See [23] for a description of a natural transformation $\mathbb{M} \to \text{Map}^c(\Sigma^n_+, \mathcal{N})$. We now recall the definition of the scanning map in this context.

**Definition 84** For a framed manifold $M$ and an $E^n_{\mathcal{PL}}$-algebra $X$, the **scanning map**

$$s : \int_M X \to \text{Map}^c(\Sigma^n_+, \mathbb{M} B^n X)$$

is the composition of the map $B(\mathbb{M} B^n, E^n_{\mathcal{PL}}, X) \to B(\text{Map}^c(\Sigma^n_+, \mathbb{M}), E^n_{\mathcal{PL}}, X)$ induced by the map of right modules, with the natural map $B(\text{Map}^c(\Sigma^n_+, \mathbb{M}), E^n_{\mathcal{PL}}, X) \to \text{Map}^c(\Sigma^n_+, \mathbb{M} B^n X)$. 

When $X$ is grouplike, the scanning map is a weak equivalence. This is known as **non-abelian Poincaré duality** [20, 21, 36]. In particular, there is a weak-equivalence $\int_M \Omega^n B^n X \to \text{Map}^c(M, \mathbb{M} B^n X)$ for any $E^n_{\mathcal{PL}}$-algebra $X$. For connected $M$, we have that $\pi_0(\text{Map}^c(M, \mathbb{M} B^n X))$ is the Grothendieck group of $\pi_0(X)$. Given $k$ in the Grothendieck group of $\pi_0(X)$, we let $\text{Map}^c_k(M, \mathbb{M} B^n X)$ denote the corresponding connected component.

For $M = \mathbb{R}^n$, the scanning map gives a map $g : X \to \Omega^n B^n X$ which is the group-completion map. We now assume $C = \mathbb{N}^+_{\mathcal{PL}}$ for simplicity. When $X$ has homological stability, the map $g : X \to \Omega^n B^n X$ is a homology equivalence in the same range.
by the group-completion theorem [37]. The analogous result for non-compact path-connected framed manifolds was proven in [23].

**Theorem 85** (Miller) Let \( M \) be a non-compact path-connected framed manifold of dimension \( n \geq 2 \). Let \( X \) be a \( E_n^\text{pl} \)-algebra with homological stability range \( \rho \). The scanning map \( s: \int_M^k X \to \text{Map}^c_k(M, B^n X) \) is a \( \rho(\min\{k_i\}) \)-equivalence.

We will extend this result to compact manifolds. In the compact case, the spaces \( \text{Map}^c_k(M, B^n X) \) need not have the same homotopy type as \( k \) varies and thus the spaces \( \int_M^k X \) need not exhibit homological stability even if \( X \) has homological stability.

Naturality of topological chiral homology gives us a map \( \int_M^g \), which fits into the following diagram:

\[
\begin{array}{ccc}
\int_M^k X & \xrightarrow{\ s\ } & \text{Map}^c_k(M, B^n X) \\
\downarrow f \gamma & & \downarrow \simeq \\
\int_M^k \Omega^n B^n X & & \\
\end{array}
\]

For technical reasons, it is easier to study the map \( f \gamma \) than the scanning map \( s \).

**Theorem 86** Let \( n \geq 2 \). For any framed path-connected manifold \( M \) and an \( E_n^\text{pl} \)-algebra \( X \) that has homological stability with range \( \rho \), the map \( s: \int_M^k X \to \text{Map}^c_k(M, B^n X) \) is a \( \rho(\min\{k_i\}) \)-equivalence.

**Proof** The proof uses a resolution by punctures argument introduced in [6].

First of all we replace \( s \) by \( f \gamma \). Let \( \text{Emb}^\text{pl}_\infty(\bigsqcup_k \mathbb{R}^n, M) \) be the subspace of \( \text{Emb}^\text{pl}(\bigsqcup_k \mathbb{R}^n, M) \) consisting of embeddings and paths such that the closure of the image is an embedded closed disk and its complement has infinite cardinality. We use it to modify Definition 34 and Definition 35 by using the right \( E_n^\text{pl} \)-functor \( M^\text{pt}_\infty \) given by \( X \mapsto \bigsqcup_{k \geq 0} \text{Emb}^\text{pl}_\infty(\mathbb{R}^n, M) \circ \otimes_k X^k \) and defining

\[
\int_{M, \infty} X = B \left( M^\text{pt}_\infty, E_n^\text{pl}, X \right)
\]

The inclusion \( \text{Emb}^\text{pl}_\infty(\bigsqcup_k \mathbb{R}^n, M) \hookrightarrow \text{Emb}^\text{pl}(\bigsqcup_k \mathbb{R}^n, M) \) is a weak equivalence by a similar argument to 64, so the inclusion \( \int_{M, \infty} X \to \int_M X \) is a weak equivalence as well by Lemmas 11 and 14.

We define a semisimplicial space with \( q \)-simplices given by pairs \((\phi, (m_0, \ldots, m_q))\) of an ordered \((q + 1)\)-tuple of points \( m_i \) in \( M \) and a framed embedding \( \phi: \bigsqcup_k \mathbb{R}^n \to M \setminus \{m_0, \ldots, m_q\} \). We put the usual topology on the embeddings and paths of bundle maps, but put the discrete topology on the points in \( M \). The face map \( d_i \) forgets the point \( m_i \). This semisimplicial space has an augmentation to \( \text{Emb}^\text{pl}_\infty(\bigsqcup_k \mathbb{R}^n, M) \) by forgetting all the \( m_i \). It defines semisimplicial right \( E_n^\text{pl} \)-functor \( M^\text{pt}_\infty \) and thus an augmented semisimplicial simplicial space \( I_{*, *}(M, X) \) given by

\[
[p, q] \mapsto B_p \left( M^\text{pt}, E_n^\text{pl}, X \right) = M^\text{pt}_{q, \infty} \left( E_m^\text{pl} \right)^p X
\]
This satisfies

\[
\left\lfloor [p] \mapsto I_{p,q}(M, X) \right\rfloor \simeq \begin{cases} \int_{M,\infty} X & \text{if } q = -1 \\ \bigsqcup_{(m_0,\ldots,m_q)} \int_{M\setminus(m_0,\ldots,m_q),\infty} X & \text{if } q \geq 0 \end{cases}
\]

Realizing in the other direction, we claim that the augmentation

\[
\epsilon_p : \left\lfloor [q] \mapsto M_{q,\infty}^\text{pt} \left( E_n^\text{pt}\right)^p X \right\rfloor \to M_{\infty}^\text{pt} \left( E_n^\text{pt}\right)^p X
\]

is a weak equivalence. To prove this, we start by noting that because the complement of the closure of the images of the outermost embeddings in \( \int_{M,\infty} X \) has infinitely many points (this was the point of using \( \text{Emb}_{\infty}^\text{pt} \)), that the point inverses of \( \epsilon_p \) are contractible by the argument on Page 17 of [6] or Lemma 5.7 of [5]. Since every point has a neighborhood not intersecting the image of a disk (this was the point of using closures in our definition of \( \text{Emb}_{\infty}^\text{pt} \)), a standard argument like Proposition 5.8 of [5] proves that \( \epsilon_p \) is a microfibration as in Section 2 of [38]. By Lemma 2.2 of [38], the map \( \epsilon_p \) is a Serre fibration and hence a weak equivalence. Using Lemma 14 conclude that \( |I_{\bullet,\bullet}(M, X)| \to \int_{M,\infty} X \) is a weak equivalence.

Our construction gives a map

\[
(f g)_{\bullet,\bullet} : I_{\bullet,\bullet}(M, X) \to I_{\bullet,\bullet} (M, \Omega^n B^n X)
\]

After realizing in the \( p \)-direction, this is given by \( f g \) on the augmentation and each of the simplicial levels. Restricting to charge \( k \) and using that the augmentation maps are weak equivalences, we get a relative geometric realization spectral sequence as in Lemma 15 for the \( p \)-direction:

\[
E^1_{pq} = \bigoplus_{\{m_0,\ldots,m_p\}} H_q \left( \int_{M\setminus(m_0,\ldots,m_p)}^k \Omega^n B^n X, \int_{M\setminus(m_0,\ldots,m_p)}^k X \right) \\
\Rightarrow H_{p+q} \left( \int_{M}^k \Omega^n B^n X, \int_{M}^k X \right)
\]

The result of [23] says that for each of the summands, the map \( (f g)_p \) is a \( \rho(\min\{k_i\}) \)-equivalence on the \( k \)-component (this uses \( n \geq 2 \) to guarantee that \( M \) remains connected after removing points). This means that the \( E^1 \)-page vanishes for \( q \leq \rho(\min\{k_i\}) \) and hence so does the \( E^\infty \)-page. This proves the result.\[\square\]

Thus the local-to-global principle can be rephrased as saying that the group-completion map induces a homology equivalence in a range tending to infinity if and only if the scanning map for topological chiral homology does.
6 Applications of the main theorem

In this section we discuss examples of applications of Theorem 1. In the first subsection, we discuss applications of the implication (i) \( \Rightarrow \) (ii) and in the second subsection we discuss applications of the implication (iii) \( \Rightarrow \) (ii).

We will consider many examples of labeled configuration spaces in this section. All of these examples are homeomorphic to the configuration spaces \( C(M; A) \) considered in [21] for some choice of partial algebra \( A \). The techniques of Section 3.3 of [39] show that \( \int_M C(\mathbb{R}^n; A) \simeq C(M; A) \) and so we can use topological chiral homology to study these spaces.

6.1 Applications of the local-to-global homological stability principle

In this subsection, we apply the local-to-global homological stability principle to give new proofs of homological stability for bounded symmetric powers and the divisor spaces which appear in Segal’s work on spaces of rational functions [2]. Recall that we already discussed symmetric powers in Example 4.

6.1.1 Bounded symmetric powers and divisor spaces associated to projective spaces

We start by defining bounded symmetric powers.

**Definition 87** Let \( \text{Sym}^{\leq d}_k (M) \) denote the subspace of \( \text{Sym}_k (M) \) where at most \( d \) particles occupy the same point in \( M \) and define

\[
\text{Sym}^{\leq d} (M) := \bigsqcup_k \text{Sym}^{\leq d}_k (M).
\]

The spaces \( \text{Sym}^{\leq d} (\mathbb{R}^2) \) can be identified with the space of monic polynomials with roots of multiplicity of order at most \( d \). They have applications to the study of \( J \)-holomorphic curves [39] and were also studied in [40] in relation to a motivic analogue of homological stability. Since there is a canonical map of operads \( \mathcal{E}^\theta_n \rightarrow \mathcal{E}^\text{SO}_n \), the functoriality of \( \text{Sym}^{\leq d} \) with respect to embeddings makes \( \text{Sym}^{\leq d} (\mathbb{R}^n) \) into an \( \mathcal{E}^\text{SO}_n \)-algebra for all \( \theta \), but we will only use the \( \mathcal{E}^\text{SO}_n \)-algebra structure. Note that \( \pi_0 (\text{Sym}^{\leq d} (\mathbb{R}^n)) \cong \mathbb{N}_0 \) and let \( t \) denote the stabilization map that increases charge by one.

Before describing corollaries of the local-to-global homological stability principle for bounded symmetric powers, we recall the definition of certain divisor spaces. These divisor spaces appear in the study of holomorphic maps from a curve to complex projective space \( \mathbb{C} P^d \). Using the local-to-global homological stability principle and the results of [41], we will give a new proof of homological stability for divisor spaces. One can consider \( (\text{Sym}(M))^{d+1} \) as a configuration space of particles labeled by particles with \( d + 1 \) different colors.

**Definition 88** Let \( \text{Div}^d (M) \) denote the subspace of \( (\text{Sym}(M))^{d+1} \) where any given point in \( M \) is occupied by particles of at most \( d \) different colors.

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The functoriality of \( \text{Div}^d \) with respect to embeddings makes \( \text{Div}^d(\mathbb{R}^n) \) into an \( \mathbb{E}^d_n \)-algebra. Note that \( \tau_0(\text{Div}^d(\mathbb{R}^n)) = \mathbb{N}^{d+1}_0 \). For \( k = (k_1, \ldots, k_{d+1}) \), we define \( \text{Div}^d_k(M) \) to be the subspace of \( \text{Div}^d(M) \) where there are exactly \( k_i \) particles of color \( i \).

The space \( \text{Div}^d(\mathbb{R}^2) \) can be thought of as the space of \( d + 1 \) monic polynomials with no common root. Let \( \delta(k) \) denote \( (k, k, \ldots, k) \in \mathbb{N}^{d+1}_0 \). In [2], Segal noted that \( \text{Div}^d_{\delta(k)}(\mathbb{R}^2) \) is homeomorphic to \( \text{Hol}^*_k(\mathbb{C}P^1, \mathbb{C}P^d) \), the space of based holomorphic degree \( k \) maps from \( \mathbb{C}P^1 \) to \( \mathbb{C}P^d \). More generally, there is a map

\[
\pi : \text{Div}^d_{\delta(k)}(\Sigma_g) \to (J_g)^d
\]

with \( \Sigma_g \) a curve of genus \( g \) and \( J_g \) its Jacobian. The fiber of \( \pi \) at 0 is homeomorphic to \( \text{ Hol}^*_k(\Sigma_g, \mathbb{C}P^d) \) and the map \( \pi \) is a homology fibration in a range [2]. There are \( d \) different basic stabilization maps, with

\[
t_i : \text{Div}^d_k(\mathbb{R}^2) \to \text{Div}^d_{k + e_i}(\mathbb{R}^2)
\]

the basic stabilization map that adds a point of the \( i \)th color. Note that on Page 46 of [2], it is shown that if \( k_i \geq k_j \) for all \( j \) then \( t_i \) is a homotopy equivalence. Thus, the relevant map to consider is \( t = t_1 \circ t_2 \circ \ldots \circ t_d : \text{Div}^d_{\delta(k)}(\mathbb{R}^2) \to \text{Div}^d_{\delta(k+1)}(\mathbb{R}^2) \); that is, we can restrict to the diagonal.

These divisor spaces and bounded symmetric powers are closely related when \( M = \mathbb{R}^2 \). In [42], Vassiliev proved the following.

**Theorem 89** (Vassiliev) There is a homology equivalence compatible with the stabilization map between \( \text{Div}^d_{\delta(k)}(\mathbb{R}^2) \) and \( \text{Sym}^{\leq d}_{k(d+1)}(\mathbb{R}^2) \). Additionally, the stabilization map

\[
t : \text{Sym}^{\leq d}_k(\mathbb{R}^2) \to \text{Sym}^{\leq d}_{k+1}(\mathbb{R}^2)
\]

is a homotopy equivalence unless \( k + 1 \) is divisible by \( d + 1 \).

These results say that homological stability for bounded symmetric powers of \( \mathbb{R}^2 \) is equivalent to stability for divisors in \( \mathbb{R}^2 \). In [41], Cohen, Cohen, Mann and Milgram computed the cohomology of the spaces \( \text{Div}^d_{\delta(k)}(\mathbb{R}^2) \) with field coefficients and also described the effect of the stabilization map. See also [43] for this calculation. From these explicit calculations, one sees that \( t : \text{Div}^d_{\delta(k)}(\mathbb{R}^2) \to \text{Div}^d_{\delta(k+1)}(\mathbb{R}^2) \) induces a homology isomorphism in the range \( * \leq k(2d - 1) \). This implies that \( t : \text{Sym}^{\leq d}_k(\mathbb{R}^2) \to \text{Sym}^{\leq d}_{k+1}(\mathbb{R}^2) \) induces a homology isomorphism in the range \( * \leq \frac{k(2d-1)}{d+1} \). Applying the local-to-global homological stability principle we get the following corollaries.

**Corollary 90** For \( \Sigma \) an orientable connected non-compact surface,

\[
t : \text{Sym}^{\leq d}_k(\Sigma) \to \text{Sym}^{\leq d}_{k+1}(\Sigma)
\]

induces an isomorphism on \( H_* \) for \( * \leq k/2 \).
This greatly improves on the range of $* \leq \frac{k}{d+1} - d + 3$ established in [4] and agrees with the range recently proved in [5].

**Corollary 91** For $\Sigma$ an orientable connected non-compact surface,

$$t : \text{Div}^d_{\delta(k)}(\Sigma) \to \text{Div}^d_{\delta(k+1)}(\Sigma)$$

induces an isomorphism on $H_*$ for $* \leq k/2$.

Homological stability for the spaces $\text{Div}^d_{\delta(k)}(\Sigma)$ was originally proven in [2]. Note that although Segal’s range has a higher slope, our range has a higher constant term. Although $\text{Div}^d_{\delta(k)}(\mathbb{R}^2)$ and $\text{Sym}^{d\leq d}_{k(d+1)}(\mathbb{R}^2)$ are homology equivalent, $\text{Div}^d_{\delta(k)}(M)$ and $\text{Sym}^{d\leq d}_{k(d+1)}(M)$ are not homology equivalent for $M$ a surface not homeomorphic to $\mathbb{R}^2$. Thus the previous two corollaries are not equivalent.

### 6.1.2 Divisor spaces associated to toric varieties

Let $T$ be a toric variety with $H_2(T)$ torsion free. In [16], Guest introduced a configuration space $\text{Div}^T(M)$ such that $\text{Hol}^*(\mathbb{C}P^1, T)$ is a collection of components of $\text{Div}^T(\mathbb{R}^2)$. Guest proved homological stability for the components of the spaces $\text{Div}^T(\Sigma)$. Using the local-to-global homological stability theorem, we can conclude that the components of $\text{Div}^T(\Sigma)$ have homological stability for $\Sigma$ a non-compact connected orientable surface. As far as we know, this is a new result. This is a first step towards generalizing Guest’s theorem on the topology of holomorphic maps to a toric variety to the case of maps out of a higher genus surface.

### 6.1.3 A mysterious example

In the introduction we mentioned other examples of $\mathcal{E}_n^\theta$-algebras where homological stability is known. Often there is no geometric interpretation of their topological chiral homology, as is the case in the following example. If $\Sigma_{g,1}$ denotes a genus $g$ surface with one boundary component and $\text{Diff}^\theta(\Sigma_{g,1})$ the topological group of diffeomorphisms fixing the boundary topologized with the $C^\infty$-topology, then

$$\bigcup_{g \geq 0} B\text{Diff}^\theta(\Sigma_{g,1})$$

is an $\mathcal{E}_2^{SO}$-algebra. Since not all Browder operations vanish by Theorem 2.5 of [44], this action does not extend to an action of the $\mathcal{E}_3^{SO}$-operad. Homological stability for this $\mathcal{E}_2^{SO}$-algebra was proven in [9] (see also [10]). Thus, the spaces

$$\int_{\Sigma}^k \left( \bigcup_{g \geq 0} B\text{Diff}^\theta(\Sigma_{g,1}) \right)$$
have homological stability for \(\Sigma\) any orientable connected non-compact surface. We know of no geometric interpretation of this result. Using [11], one can construct similar higher-dimensional examples.

### 6.2 Homological stability via \(E^n_{\theta}\)-cell decompositions

Sometimes it is possible to identify \(E^n_{\theta}\)-cell decompositions (or at least bound where \(E^n_{\theta}\)-cells are attached) and then use Theorem 1 to deduce homological stability. In this subsection we will apply this idea to \(E^n_{\theta}\)-algebras built out of completions of partial \(E^n_{\theta}\)-algebras under certain assumptions about the partial monoid of connected components.

#### 6.2.1 Completing partial monoids

Any partial abelian monoid \(C\) can be completed to an abelian monoid \(c(C)\). Abstractly this is the left adjoint to the inclusion of abelian monoids into partial abelian monoids, and concretely it is given by setting \(c(C)\) to be given by formal finite sums \(c_1 \boxplus \cdots \boxplus c_k\) of elements of \(C\) under the equivalence relation generated by the fact that \(\boxplus\) is commutative and \(c_1 \boxplus c_2 \sim c_1 + c_2\) if \(c_1 + c_2\) is defined. In all cases we consider, \(c(C) = \mathbb{N}^d\). To distinguish between the monad \(E^n_{\theta}\) in the category of \(C\)-charged chain complexes and \(c(C)\)-charged chain complexes, we denote the former by \(E^n_{\theta}, C\) and the latter by \(E^n_{\theta}\).

If \(C \to D\) is a map of partial monoids, we get an induced functor from the category of \(C\)-charged algebras to the category of partial \(D\)-charged algebras. Thus, any \(C\)-charged algebra \(A\) can be viewed as partial \(c(C)\)-charged algebra and then completed to form an actual \(c(C)\)-charged algebra. To describe this completion procedure concretely, we define a partial \(c(C)\)-charged algebra structure on a \(C\)-charged algebra \(A\) by setting

\[
A_p = B_p \left( E^n_{\theta} \circ \iota, E^n_{\theta}, C, A \right)
\]

where \(\iota\) is the inclusion of \(\text{Ch}_C\) into \(\text{Ch}_{c(C)}\).

**Definition 92** Let \(A\) be a \(C\)-charged algebra, then the \(c(C)\)-completion \(\tilde{A}\) is defined to be the realization \(|A_\bullet|\). This is a \(c(C)\)-charged algebra.

Suppose \(A\) is a \(C\)-charged algebra and we have an element \(b_{N-1} \in A(k)_{N-1}\), then there are two things we can do. First, we could define \(A \uplus e_N\) in \(C\)-charged algebras by attaching an \(E^n_{\theta}\)-cell to \(b_{N-1}\), view it as a partial \(c(C)\)-charged algebra and complete it to a \(c(C)\)-charged algebra \(A \uplus e_N\). Alternatively, we could view \(A\) as a partial \(c(C)\)-charged algebra and complete it to a \(c(C)\)-charged algebra \(\tilde{A}\) and then attach an \(E^n_{\theta}\)-cell to the image of \(\text{id} \otimes b_{N-1}\) in simplicial degree 0 to get a \(c(C)\)-charged algebra \(A \uplus e_N\). A formal argument should tell us that \(\tilde{A} \uplus e_N\) and \(\tilde{A} \uplus e_N\) are the equivalent (they are the values of two functors that are derived left adjoint to the same functor). Since we have not set up the categorical framework needed to make this precise, we instead give a concrete simplicial argument.
Proposition 93 Let $A$ be a $C$-charged algebra. The $c(C)$-charged algebras $\tilde{A} \sqcup e_N$ and $\bar{A} \sqcup e_N$ are weakly equivalent.

Proof As both $\tilde{A} \sqcup e_N$ and $\bar{A} \sqcup e_N$ are completions of completions, they are naturally realizations of bisimplicial chain complexes. We will not compare them directly but instead compare both of them to a simplicial chain complex $\bar{A} \oplus e_N$ obtained as the completion of the partial $c(C)$-charged algebra $A \oplus e_N$. To define a partial $c(C)$-charged algebra structure on $A \oplus e_N$, we take $\text{Comp}_1$ to be the subspace of $E_n^\theta (A \oplus e_N)$ spanned by $E_n^\theta, C (A)$ and $\text{id} \otimes e_N$, i.e. the only compositions that are defined involve elements of $A$ where the total charge is in $C$ or the identity acting on $e_N$. More precisely, $\bar{A} \oplus e_N$ is the geometric realization of the simplicial object $P_\bullet$ defined by

$$[p] \mapsto P_p := E_n^\theta \left[ \left( E_n^\theta, C \right)^p (A) \oplus (\text{id}^p \otimes e_N) \right] \subset B_p \left( E_n^\theta \circ \iota, E_n^\theta, C, A \oplus e_N \right)$$

We will now prove using an extra degeneracy argument that there is a weak equivalence

$$\tilde{A} \sqcup e_n \to \bar{A} \oplus e_N$$

As discussed above, the $c(C)$-charged algebra $\tilde{A}$ is the realization of the simplicial object $B_\bullet \left( E_n^\theta (A), E_n^\theta, C, A \right)$ and by Definition 55, $c(C)$-charged algebra $\bar{A} \sqcup e_N$ is the realization of the simplicial object $[[p] \mapsto E_n^\theta \left[ \left( E_n^\theta, C \right)^p (\tilde{A}) \oplus (\text{id}^p \otimes e_N) \right]]$. Thus, $\tilde{A} \sqcup e_N$ is the realization of the bisimplicial object $P_1^\bullet$, given by

$$[p, q] \mapsto P_{pq} := E_n^\theta \left[ \left( E_n^\theta, C \right)^q B_q \left( E_n^\theta \circ \iota, E_n^\theta, C, A \right) \right] \oplus (\text{id}^p \otimes e_N)$$

We will now define an augmentation map for its $p$-direction

$$\epsilon : P_{0q}^1 = E_n^\theta \left[ \left( E_n^\theta, \left( E_n^\theta, C \right)^q (A) \right) \oplus e_N \right] \to P_q = E_n^\theta \left[ \left( E_n^\theta, C \right)^q (A) \oplus (\text{id}^q \otimes e_N) \right]$$

There are maps

$$E_n^\theta \left[ \left( E_n^\theta, \left( E_n^\theta, C \right)^q (A) \right) \oplus e_N \right] \to E_n^\theta \left[ E_n^\theta \left( E_n^\theta, C \right)^q (A \oplus e_N) \right] \to E_n^\theta \left( E_n^\theta, C \right)^q (A \oplus e_N)$$

the first of which is induced by the monad unit maps $1 \to E_n^\theta$ and $1 \to E_n^\theta, C$, and the second of which is induced by the monad multiplication map $E_n^\theta \otimes E_n^\theta \to E_n^\theta$. The image of the composition of these two maps is contained in $P_q$ and so we get a map $P_{0q}^1 \to P_q$. The two maps $\epsilon \circ d_0, \epsilon \circ d_1 : P_{1q}^1 \to P_q$ agree as they only involve monad composition in the copies of $E_n^\theta$, so this is an augmentation map.

We thus get an augmented simplicial object $P_{0q}^1$ by setting $P_{-1, q}^1 = P_q$ and this has an extra degeneracy coming from the unit map $1 \to E_n^\theta$. More specifically, to define
the extra degeneracy $P^1_{p,q} = P^1_{p+1,q}$ for $p \geq 0$, we insert this identity to the left of the $E_n^\theta$ factor coming from $B_q(E_n^\theta \circ \iota, E_n^{\theta,C}, A)$ in

$$P^1_{p,q} = E_n^\theta \left[ (E_n^\theta)^p \left( B_q \left( E_n^\theta \circ \iota, E_n^{\theta,C}, A \right) \right) \oplus (\text{id}^p \otimes e_N) \right]$$

Similarly, the unit gives us a map $P^{-1,q} = P_q \to P^0_{0,q}$. Thus the augmentation induces a weak equivalence $|P^1_{\bullet q}| \to P_q$. Since level wise weak equivalences realize to weak equivalence, we conclude that $\bar{A} \cup e_N \simeq |P^1_{\bullet \bullet}| \to |P_{\bullet \bullet}| \simeq \bar{A} \oplus e_N$ is a weak equivalence.

We next prove that $\bar{A} \cup e_N \simeq \bar{A} \oplus e_N$. One can use an argument as above, but we shall quote a result from the literature. The algebra $\bar{A} \cup e_N$ is the realization of the bisimplicial object

$$[p,q] \mapsto B_p \left[ E_n^\theta \circ \iota, E_n^{\theta,C}, E_n^{\theta,C} \left( \left( E_n^{\theta,C} \right)^q (A) \oplus (\text{id}^q \otimes e_N) \right) \right]$$

Theorem 9.10.3 of [25] implies that this is equivalent to the realization of the simplicial object with $q$ simplices given by $[q] \mapsto E_n^\theta \left( \left( E_n^{\theta,C} \right)^q (A) \oplus (\text{id}^q \otimes e_N) \right)$, which is exactly $P_{\bullet \bullet}$.

Thus $\bar{A} \cup e_N$ and $\bar{A} \cup e_N$ are both weakly equivalent to $\bar{A} \oplus e_N = |P_{\bullet \bullet}|$ and hence to each other. \qed

Using this proposition, we prove homological stability for $E_n^\theta$-algebras formed by completing certain partial algebras. Let $|d| = \{0, 1, \ldots, d\} \subset \mathbb{N}_0$ viewed as a partial monoid with addition. Completions of partial algebras with monoid of components $|d|$ were studied before in [7].

**Proposition 94** Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a strictly increasing function satisfying $\rho(a + b) \leq \rho(a) + \rho(b)$ for $a, b \in \mathbb{R}_{\geq 0}$, $\rho(s) \leq s/2$ for all $s \in \mathbb{R}_{\geq 0}$ and $\rho^{-1}(\mathbb{N}_0) \subset \mathbb{N}_0$. Let $\bar{A}$ be a $|d|$-charged algebra with $t : A_k \to A_{k+1}$ a $\rho(k)$-equivalence for $k < d$ and let $\bar{A}$ denote its completion as a $\mathbb{N}_0$-charged algebra. For any oriented connected non-compact $n$-dimensional manifold $A$, we have that

$$t : \int_M^k \bar{A} \to \int_M^{k+1} \bar{A}$$

is a $\rho(k)$-equivalence.

**Proof** Mimicking the proof of Proposition 81 and stopping after attaching $E_n^\theta$-cells in charge $d$, we can construct a cellular $E_n^\theta$-algebra $B$ and a map $B \to \bar{A}$ such that $B_k \to \bar{A}_k$ is an equivalence for all $k \leq d$ and all $E_n^\theta$-cells of $B$ are attached above $\rho$ and in charge $\leq d$. By the implication (iii) $\Rightarrow$ (ii) of Theorem 74, $t : \int_M^k B \to \int_M^{k+1} B$ is a $\rho(k)$-equivalence. Let $B^{\leq d}$ denote the partial $E_n^\theta$-algebra formed by only considering components of charge $\leq d$. By an iterated application of Proposition 93, we have...
that $B^{\leq d} \simeq B$. The map $B^{\leq d} \to A$ is an equivalence. Thus, $B^{\leq d} \to B \to \tilde{A}$ is an equivalence. By Theorem 74 we have that $t : \int_M^k \tilde{A} \to \int_M^{k+1} \tilde{A}$ is a $\rho(k)$-equivalence.

This proposition has implications for completions (in the category of $\mathbb{N}_0$-charged algebras) of $[d]$-charged algebras without making any assumptions on their homology. Such a result was proven in Theorem 1.1 of [7] but with a worse range. The techniques of [7] are completely different than those used here and follow the traditional approach to proving homological stability introduced by Quillen.

**Corollary 95** Let $A$ be a $[d]$-charged algebra, $\tilde{A}$ the completion in the category of $\mathbb{N}_0$-charged algebras and $M$ a $\theta$-framed connected non-compact $n$-dimensional manifold, then we have that

$$
t : \int_M^k \tilde{A} \to \int_M^{k+1} \tilde{A}
$$

is a $\rho(k)$-equivalence for $\rho(k) = \min(k/2, k/d)$.

**Proof** We note that for any $[d]$-charged algebra $A$, $t : A_k \to A_{k+1}$ is a $\rho(k)$-equivalence for $k < d$.

In [7], it was remarked that the range established in Corollary 95 is optimal.

### 6.2.2 Bounded symmetric powers revisited

We can apply Proposition 94 to the case of bounded symmetric powers.

**Corollary 96** Let $M$ be a oriented path-connected non-compact manifold of dimension at least 2. The stabilization map $t : \text{Sym}^{\leq d}_k(M) \to \text{Sym}^{\leq d}_{k+1}(M)$ induces an isomorphism on $H^*$ for $* \leq k/2$.

**Proof** For $k \leq d$, $\text{Sym}^{\leq d}_k(\mathbb{R}^n)$ is contractible, so Proposition 94 applies with $\rho(k) = k/2$.

This result greatly improves on the range of $* \leq \frac{k}{2d}$ established in [7] which was the only integral homological stability range known for bounded symmetric powers in high dimensions. We note that the arguments of [4] and Sect. 6.1.1 all use that $M$ is two-dimensional in a crucial way—either using that $\text{Sym}(\mathbb{C})$ is a manifold or that $\Omega^2 S^3 \times \mathbb{Z} \simeq \Omega^2 S^2$—and thus those techniques cannot be used to achieve this range in higher dimensions. A rational range with slope 1 was proven in [5] using the fact that $\text{Sym}(M)$ is an orbifold. Using the techniques of this paper, we can reprove this rational range and show that it also holds with $\mathbb{Z}[1/2]$-coefficients.

Using Theorem 86, we also conclude that the scanning map

$$s : \text{Sym}^{\leq d}_k(M) \to \text{Map}_k^c(\tilde{M}^n, B^n \text{Sym}^{\leq d}(\mathbb{R}^n))$$
induces an isomorphism on homology in the range \(* \leq k/2\) for \(M\) not necessarily non-compact, but framed. This improves on the best previously known range, a range of \(* \leq \frac{k-2d}{2d}\) established in [7].

Recall that in Remark 72, we described cell attachments in \(\text{Top}\). In the case \(n = 2\) and trivial tangential structure, it is actually possible to completely describe an \(E_n\)-cell structure on \(\text{Sym}^{\leq d} (\mathbb{R}^n)\). This result was known to Søren Galatius and Jacob Lurie, and we learned it from them through personal communication. It is related to the results in Section 5 of [45], which also discusses \(BU\) as an \(E^\text{pt}_2\)-algebra.

**Proposition 97** There exists an \(E^\text{pt}_2\)-cell decomposition of \(\text{Sym}^{\leq d} (\mathbb{R}^2)\) with exactly one \(E^\text{pt}_2\)-cell in dimensions \(0, 2, \ldots, 2(d-1)\). The cell of dimension \(2i\) is attached in charge \(i + 1\).

**Proof** We proceed by induction on \(d\). The fact that we are working with framings implies that the statement is true for \(d = 1\). By Lemma 2.7 of [46], \(\text{Sym}^{\leq d} (\mathbb{R}^2) \simeq S^{2d-1}\). Attaching an \(E^\text{pt}_2\)-cell to this sphere makes \(\text{Sym}^{\leq d+1} (\mathbb{R}^2)\) contractible and hence equivalent to \(\text{Sym}^{\leq d+1} (\mathbb{R}^2)\) for \(k \leq d\). \(\text{Sym}_k^{\leq d} (\mathbb{R}^2)\) and \(\text{Sym}_k^{\leq d} (\mathbb{R}^2)\) are both contractible and hence homotopy equivalent. Using Proposition 93, we see that \(\text{Sym}^{\leq d+1} (\mathbb{R}^2)\) is equivalent to \(\text{Sym}^{\leq d} (\mathbb{R}^2)\) with a \(2d-2\)-dimensional \(E^\text{pt}_2\)-cell attached to the sphere representing a generator of \(\pi_{2d-3} (\text{Sym}^{\leq d+1} (\mathbb{R}^2))\). This proves the inductive step. \(\square\)

In the next proposition we will show that an \(E^\text{pt}_n\)-cell decomposition of an \(E^\text{pt}_n\)-algebra \(A\) induces an ordinary cell decomposition of \(B^n A\). A similar result should hold for \(E^\text{st}_n\)-algebras and cells in spaces with an action of \(\text{Bun}^0 (T \mathbb{R}^n, T \mathbb{R}^n) \simeq \Omega W\).

**Proposition 98** Let \(X\) be an \(E^\text{pt}_n\)-algebra in \(\text{Top}\), then we have that

\[ B^n (X \cup D^N) \simeq (B^n X) \cup_{S^{n-1}} D^{n+N} \]

**Proof** The \(E^\text{pt}_n\)-algebra \(X \cup D^N\) is defined as the realization \(|E^\text{pt}_n (\text{Comp}_\bullet)|\) with

\[ \text{Comp}_p = \left( E^\text{pt}_n \right)^p (X) \cup_{S^{n-1}} D^N \]

where the attaching map \(S^{n-1} \to \left( E^\text{pt}_n \right)^p (X)\) is induced from the map \(S^{n-1} \to X\) by applying the unit natural transformation of the monad \(E^\text{pt}_n\) \(p\) times. One model for \(B^n\) is given by \(|B \_ (\Sigma^n, E^\text{pt}_n, -)|\), we get that \(B^n (X \cup D^N)\) is the realization of the bisimplicial object

\[ [p, q] \mapsto B_p (\Sigma^n, E^\text{pt}_n, E^\text{pt}_n (\text{Comp}_q)) \]

Let us first realize in the \(B_\bullet\) direction. Theorem 9.10.3 of [25] implies that \(B (\Sigma^n, E^\text{pt}_n, E^\text{pt}_n Y) \simeq \Sigma^n Y_+\) for any \(E^\text{pt}_n\)-algebra \(Y\). This implies that \(B^n (X \cup D^N)\) is weakly equivalent to the realization of \(\Sigma^n (\text{Comp}_\bullet)_+\).
As in Definition 55, remark that $\text{Comp}_p$ can be obtained as the push out

\[
\begin{array}{ccc}
S^{N-1} & \longrightarrow & D^n \\
\downarrow & & \downarrow \\
\left(\mathcal{E}_n^{\text{pt}}\right)^p(X) & \longrightarrow & \text{Comp}_p
\end{array}
\]

Since $\Sigma^n_+ -$ preserves push outs, $\Sigma^n_+(\text{Comp}_p)_+$ can be obtained as a push out as well:

\[
\begin{array}{ccc}
S^{n+N-1} & \longrightarrow & D^{n+N} \\
\downarrow & & \downarrow \\
\Sigma^n_+\left(\mathcal{E}_n^{\text{pt}}\right)^p(X) & \longrightarrow & \Sigma^n(\text{Comp}_p)_+
\end{array}
\]

Since geometric realization preserves colimits, we get that $|\Sigma^n(\text{Comp}_\bullet)_+|$ is obtained as a push out

\[
\begin{array}{ccc}
S^{n+N-1} & \longrightarrow & D^{n+N} \\
\downarrow & & \downarrow \\
B^n X & \longrightarrow & |\Sigma^n(\text{Comp}_\bullet)_+| \simeq B^n(X \cup D^N)
\end{array}
\]

This theorem also holds in $\mathbf{sSet}$ and $\mathbf{Ch}$, and in the latter case attaching an $\mathcal{E}_n^{\text{pt}}$-cell of dimension $N$ to $A$ amounts to attaching an ordinary $N$-cell to the cotangent space $L(A) \simeq B^n A[-n]$ (see [47] for more discussion of the cotangent space). Since the number of generators of homology serves as a lower bound for the number of cells, we get the following lemma.

**Lemma 99** Let $X$ be an $\mathcal{E}_n^{\text{pt}}$-algebra in $\mathbf{Top}$. Suppose $X$ is equivalent to a cellular $\mathcal{E}_n^{\text{pt}}$-algebra with $k \mathcal{E}_n^{\text{pt}}$-cells of dimension $i$. Then $k$ is at least as large as the number of generators of $\tilde{H}_{n+i}(B^n X)$ as an abelian group.

**Example 100** It is well known that $B^2\text{Sym}^{\leq d}(\mathbb{R}^2) \simeq \mathbb{C}P^d$, and the lemma thus implies that any $\mathcal{E}_2^{\text{pt}}$-cell decomposition of $\text{Sym}^{\leq d}(\mathbb{R}^2)$ must have at least one $\mathcal{E}_2^{\text{pt}}$-cell in dimension $0, 2, \ldots, 2(d - 1)$. Proposition 97 says that this lower bound can be achieved for $\text{Sym}^{\leq d}(\mathbb{R}^2)$.

### 6.2.3 Divisor spaces revisited

Using ideas similar to those in Sect. 6.2.2, one can prove homological stability for the divisor spaces of [2,16] and the spaces of coprime polynomials considered in [48].
as well as higher dimensional versions of these spaces. For simplicity of notation, we restrict our attention to the case relevant to holomorphic maps to \( \mathbb{C}P^1 \), the case of \( \text{Div}^1(M) \).

**Definition 101** Let \( A \) and \( B \) be \( E^0_n \)-algebras in \( \text{Top} \). Let \( A \lor B \) be the partial \( E^0_n \)-algebra with underlying space \( A \lor B \) and composition only defined if all elements are either in \( A \) or all in \( B \).

This construction is relevant as \( \text{Div}^1(\mathbb{R}^n) \cong N_0 \lor N_0 \). Proposition 93 in this case says the following.

**Proposition 102** Let \( A \) and \( B \) be \( N_0 \)-charged algebras with choice of \( E^0_n \)-cell structures. There is an \( E^0_n \)-cell structure on \( A \lor B \) with the following properties:

(i) There are no \( E^0_n \)-cells attached in charge \((k, j)\) unless either \( k \) or \( j \) is equal to zero.

(ii) There is a bijection between the \( E^0_n \)-cells of homological degree \( i \) of charge \( k \) of \( A \) with the \( E^0_n \)-cells of homological degree \( i \) of charge \((k, 0)\) of \( A \lor B \).

(iii) There is a bijection between the \( E^0_n \)-cells of homological degree \( i \) of charge \( k \) of \( B \) with the \( E^0_n \)-cells of homological degree \( i \) of charge \((0, k)\) of \( A \lor B \).

We now prove homological stability for \( \text{Div}^1(M) \).

**Corollary 103** Let \( M \) be a \( \theta \)-framed connected non-compact manifold of dimension at least 2. The stabilization map \( t: \text{Div}^1_{(k, j)}(M) \to \text{Div}^1_{(k+1, j+1)}(M) \) induces an isomorphism on \( H_* \) for \( * \leq \min(k/2, j/2) \).

**Proof** Since the components of \( N_0 \) have homological stability, by the implication (i) \( \Rightarrow \) (iii) of Theorem 74, we conclude that \( C_*(N_0) \) has an \( E^0_n \)-cell decomposition with cells of charge \( k \) having homological degree at least \( k/2 \). Using Proposition 102, we conclude that \( C_*(N_0 \lor N_0) \) has an \( E^0_n \)-cell decomposition with cells of charge \((k, j)\) having homological degree at least \( \max(k/2, j/2) \). Thus, by the implication (iii) \( \Rightarrow \) (ii) of Theorem 74, \( t: \int_M N_0 \lor N_0 \to \int_M N_0 \lor N_0 \) induces an isomorphism in homology in the range \( * \leq \min(k/2, j/2) \). The claim now follows since \( \text{Div}^1(M) \cong \int_M N_0 \lor N_0 \).

This result is new in the case \( M \) has dimension greater than 2. We note that this proposition combined with non-abelian Poincaré duality [23] gives a proof of Segal’s result that \( \text{Hol}_k^*(\mathbb{C}P^1, \mathbb{C}P^1) \to \text{Map}_k^*(\mathbb{C}P^1, \mathbb{C}P^1) \) is a homology equivalence in a range tending to infinity with \( k \) whose only input consists of the basic properties of holomorphic functions and the fact that \( N_0 \) has homological stability.

Gravesen [14] and Boyer, Hurtubise, Mann and Milgram [15] gave configuration space models for spaces of based holomorphic maps from \( \mathbb{C}P^1 \) into generalized flag varieties. We believe that our techniques can also be applied to those cases.

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