On a number of rational points on a convex curve

Let $\gamma$ be a bounded convex curve on a plane. Then $\sharp(\gamma \cap (\mathbb{Z}/n)^2) = o(n^{2/3})$. It strenghtens the classical result of Jarník [J] (an upper estimate $O(n^{2/3})$) and disproves a conjecture of Vershik on existence of the so-called universal Jarník curve.

In his famous paper [J] Jarník proved (among other results), that the maximal possible number of integer points, which may lie on a strictly convex plane curve of length $N$ grows like $cN^{2/3}$ (the exact constant $c$ was also computed in [J] and equals $3(2\pi)^{-1/3}$). In other words, the number of nodes of a lattice $L_N := (\mathbb{Z}/N)^2$, which lie on a strictly convex curve $\gamma$ of length 1, does not exceed $cN^{2/3}$; and for any $N$ there exists such strictly convex curve $\gamma^{(N)}$ of length 1, that

$$k(\gamma^{(N)}, N) := \sharp(\gamma^{(N)} \cap L_N) \geq cN^{2/3} \quad (*)$$

So, the natural question arises: does there exist a universal curve $\gamma$, for which (*) holds for infinite number of positive integers $N$? This question is formulated by A. M. Vershik; in the paper [P] it was attributed to J.-M. Deshouillers and G. Grekos. The conjecture about the existence of a universal curve was formulated in [P]. Indeed, the methods of papers [Ve], [Ba] (see below) show the nature of the typical convex lattice polygon, which, on the first glance, supports such conjecture. However, this conjecture fails and here we disprove it.

H. P. F. Swinnerton-Dyer [SD] has proved an estimate $k(\gamma, N) \leq cN^{3/5+\varepsilon}$ for $\gamma \in C^3$, and E. Bombieri and J. Pila [BP] proved an estimate $k(\gamma, N) \leq cN^{1/2+\varepsilon}$ ($\varepsilon > 0$ is arbitrary) for infinitely smooth $\gamma$. Here we do not mention further results in this direction in terms of smoothness, curvature conditions and other restrictions on a curve.

A. M. Vershik and I. Barany ([Ve], [Ba]) investigated limit shapes of large random polygons with vertices on a shallow lattice. The answers bring out the connection with affine geometry. Namely, let $l_\alpha(\gamma)$ denote an affine length of a curve $\gamma$ (an integral of cubic root of curvature by natural parameter). It appears that the number of polygons with vertices in nodes of $L_n$, lying in a small neighborhood of given curve $\gamma$, grows like $e^{-l_\alpha(\gamma)n^{2/3}}$, and the number of their vertices — like $c \cdot l_\alpha(\gamma)n^{2/3}$ (remarkably, it holds both for maximal and typical number of vertices, only constants differ). Polygons with vertices in nodes of $L_n$, which lie inside a given convex polygon, concentrate near the closed convex curve with maximal possible affine length ([Ba]). This curve is nothing else but the union of some parabola arcs, inscribed in polygon angles. Hence the number of $L_N$ nodes on such curve does not exceed $C \cdot N^{1/2}$. So, the typical curve is not universal. Here we prove that in fact the universal curve does not exist: for any bounded strictly convex curve an estimate $k(\gamma, n) = o(n^{2/3})$ holds.

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§1. Definitions and denotations.

Fix a cartesian coordinate system on a plane.

Let $S(F)$ denote the doubled area of a polygon $F$; $x \times y$ denote the pseudoscalar product of vectors $x$ and $y$ (i.e. an oriented area of a parallelogram, based on these vectors).

Fix a triangle $ABC$, oriented in such manner that $S = S(ABC) = +\overrightarrow{AC} \times \overrightarrow{CB}$.

Define following concepts:

1. $\text{An} = \text{An}(ABC)$ is an angle with vertex in the origin and sides collinear with rays $AC$ and $CB$ (value of this angle equals $\pi - \angle ACB$). We consider angle as a set of vectors $x \in \mathbb{R}^2$, which, being drawn from the origin, lie inside $\text{An}$. In other words, $\text{An}$ is a set of vectors $\{t_1 \cdot \overrightarrow{AC} + t_2 \cdot \overrightarrow{CB} | t_1, t_2 \geq 0\}$

2. Given a vector $x$, define its girth as

$$[x] = (x \times \overrightarrow{CB} + \overrightarrow{AC} \times x) / S.$$

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Girth is a linear function of a vector \( \mathbf{x} \in \mathbb{R}^2 \). Note that \( |\overrightarrow{AC}| = |\overrightarrow{CB}| = 1 \), \( |\overrightarrow{AB}| = 2 \). We also define a girth of a segment \( PQ \) as a value \( |\overrightarrow{PQ}| \).

3. Define \( ABC \)-radius of arbitrary triangle as a product of its sides girths, divided by quadruple area (it appears to be the usual circumradius if to take sides lengths instead girths). Note that \( ABC \)-radius of triangle \( ABC \) equals \( S^{-1} \).

4. Let \( \gamma : AC_1C_2 \ldots C_kB \) be a strictly convex broken line. We call it \( (AB, C) \)-broken line, if all its vertices lie inside triangle \( ABC \). If, additionally, all intermediate vertices \( C_i (i = 1, 2, \ldots, k) \) lie in nodes of a lattice \( L_n = (\frac{1}{n}\mathbb{Z})^2 \), we call \( \gamma \) an \( (AB, C; n) \)-broken line.

5. Let \( (AB, C) \)-broken line \( \gamma = AC_1C_2 \ldots C_kB \) be inscribed in an \( (AB, C) \)-broken line \( \gamma_1 = AD_1D_2 \ldots D_{k+1}B \) (i.e. points \( C_i \) lie on respective segments \( D_iD_{i+1} \) \( (i = 1, 2, \ldots, k) \)). Define a generalized affine length of a broken line \( \gamma \) with respect to \( \gamma_1 \) as a value

\[
I_A(\gamma : \gamma_1) := \sum_{i=0}^{k} S(C_iD_{i+1}C_{i+1})^{1/3} (C_0 = A, C_{k+1} = B),
\]

and a generalized affine length of a broken line \( \gamma \) as a value

\[
I_A(\gamma) = \sup_{\gamma_1} I_A(\gamma : \gamma_1),
\]

where the supremum is taken by all \( (AB, C) \)-broken lines \( \gamma_1 \), circumscribed around \( \gamma \).

The generalized affine length of \( (AB, C) \)-broken line \( \gamma \) may be also defined (up to some factor) as a supremum of affine lengths of smooth curves, circumscribed around \( \gamma \) and contained inside triangle \( ABC \). In classical affine geometry, the affine length of a broken line equals 0, so we have to add justifying "generalized" in our definition.

§ 2. Preliminary statements.

At first, we need an upper estimate for a number \( k \) of intermediate vertices of an \( (AB, C; n) \)-broken line for large \( n \). We use the known fact that an area of convex \( k \)-gon with integer vertices is not less than \( (8\pi^2)^{-1}k^3 \geq (k/5)^3 (k \geq 3) \). From this,

\[
k \leq \max(3, 5(Sn^2)^{1/3}) \tag{1}
\]

So the maximal quantity of nodes of lattice \( L_n \), which lie in a triangle of area \( S \), grows not faster than \( c(Sn^2)^{1/3} \).

At fact, it grows exactly as \( c(Sn^2)^{1/3} \). An example of \( (AB, C; n) \)-broken line, which has about \( c(Sn^2)^{1/3} \) vertices may be constructed as follows: arrange vectors of a set \( An \cap L_n \) in girth increasing order. Take \( c(Sn^2)^{1/3} \) vectors with minimal girths and construct a convex broken line, for which these vectors are vectors of edges. For small enough \( c \) (say, \( c = 1/100 \)) this broken line may be shifted inside triangle \( ABC \) in such way that its vertices will still lie in \( L_n \) nodes and will form, together with vertices \( A \) and \( B \), an \( (AB, C) \)-broken line. This construction is quite analogous to Jarník’s construction of broken line with given length and maximal number of integer points, and may be consider as its affine generalization. This example is needed not before paragraph 4.

The idea of our approach to the statement mentioned in abstract is as follows. Draw tangents to our curve in points, which lie in nodes of \( L_n \). For large \( n \), we get many small triangles, and their union contains our curve. It appears that if our curve contains quite many points on these lattices, then the sum of cubic roots of areas of these triangles decreases, with any new \( n \), some factor, which is impossible. For proving lemma 4 we need some technical statements form this paragraph.

Now consider all vectors of a set \( \mathbb{Z}^2 \cap An \) and arrange them by girths (in ascending order). Consider \( k \) such vectors \( z_1, z_2, \ldots, z_k \) with the least girths.

An array of vectors \( z_1, z_2, \ldots, z_k \) has the following structure: it contains all vectors, which lie in triangle \( OPQ = An \cap \{ z : |z| < r = |z_k| \} \), and some vectors, which girth equals \( r \). We have

\[
k = \frac{1}{2} S(OPQ) + o(r^2) = \frac{S}{2} r^2 + o(r^2)
\]
Further,

\[
\sum [z_i] = \int_{OPQ} [x] dx + o(r^3) = r^{-1} S^{-1} \int_{OPQ} (S(OPX) + S(OQX)) dx + o(r^3) = \frac{S}{3} r^3 + o(r^3).
\]

Hence

\[
\sum [z_i] \geq cS^{-1/2} k^{3/2} + o(k^{3/2}), \quad c = 2\sqrt{2}/3
\] (2)

The following elementary lemma is a basepoint of further considerations.

**Lemma 1.** Let points \( P, R \) be chosen on the sides \( AC \) and \( BC \) of a triangle \( ABC \) respectively, and a point \( Q \) — on a segment \( PR \). Then:

1°. \( S(AQP)^{1/3} + S(BQR)^{1/3} \leq S^{1/3} \). Moreover, there exists a function \( \varepsilon_1(\varepsilon) \), which tends to zero with \( \varepsilon \to 0 \), such that if

\[
\text{Err} := 1 - (S(APQ)/S)^{1/3} - (S(BQR)/S)^{1/3} < \varepsilon[AQ],
\]

then

2°. \( [AP] : [PQ] \in (1 - \varepsilon_1, 1 + \varepsilon_1) \)

3°. \( r(APQ) \in S^{-1}((1 - \varepsilon_1, 1 + \varepsilon_1) \), where \( r(APQ) \) denotes \( ABC \)-radius of a triangle \( APQ \).

4°. For any vector \( x \in \text{An}(APQ) \) we have \([x]_{APQ} : [x] \in (AP : AC) \cdot (1 - \varepsilon_1, 1 + \varepsilon_1) \) (here \([x]_{APQ} \) denotes an \( APQ \)-girth of a vector \( x \)).

**Proof.** We have

\[
\text{Err} = 1 - (S(APQ)/S)^{1/3} - (S(BRP)/S)^{1/3} = 1 - (\frac{AP}{AC} \cdot \frac{PQ}{RP} \cdot \frac{RC}{CB})^{1/3} - (\frac{PC}{AC} \cdot \frac{QR}{RP} \cdot \frac{BR}{BC})^{1/3} =
\]

\[
\left(\frac{AP}{AC} + \frac{PQ}{RP} + \frac{RC}{CB}\right)^{1/3} - \left(\frac{AP}{AC} \cdot \frac{PQ}{RP} \cdot \frac{RC}{CB}\right)^{1/3}
\]

\[
\left(\frac{PC}{AC} + \frac{QR}{RP} + \frac{BR}{BC}\right)^{1/3} - \left(\frac{PC}{AC} \cdot \frac{QR}{RP} \cdot \frac{BR}{BC}\right)^{1/3} \)
\]

(6)

Two last brackets have a form \((x + y + z)/3 - (xyz)^{1/3} \) and by AM-GM inequality, it implies p.1 1 of lemma. We have

\[
2(x + y + z) - 6(xy) = (x^{1/3} + y^{1/3} + z^{1/3}) \left((x^{1/3} - y^{1/3})^2 + (y^{1/3} - z^{1/3})^2 + (z^{1/3} - x^{1/3})^2 \right).
\]

So, if \( \max(x, y, z) \geq (1 + \delta) \min(x, y, z) \), then \( (x + y + z)/3 - (xyz)^{1/3} \geq c(\delta)(x + y + z) \), where \( c(\delta) \) is a positive function, which tends to 0 with \( \delta \to 0 \).

Applying this observation to the first summand in RHS of (6) \( x = AP/AC, y = PQ/RP, z = RC/CB \),

we get the following implication: if \( \text{Err} \leq \varepsilon[AQ] \), then (since

\[
[AQ] = [AP] + [PQ] = AP/AC + (PQ/PR) \cdot [PR] \leq (AP/AC) + 2(PQ/PR),
\]

the estimate \( \text{Err} \leq 2\varepsilon(x + y + z) \) holds, and therefore \( \max(x, y, z) \min(x, y, z) < 1 + \varepsilon_1 \), where \( \varepsilon_1 \) is chosen such that \( c(\varepsilon_1) > 2\varepsilon \). Such \( \varepsilon_1(\varepsilon) \) may be chosen, and it tends to zero with \( \varepsilon \).

Further,

\[
[PA] = x, [PQ] = (PQ/PR) \cdot [PR] = y([PC] + [CR]) = y(1 - x + z) = [PA] \cdot (y/x) \cdot (1 - x + z)
\]

We have

\[
y/x \in (1 - \varepsilon_1, 1 + \varepsilon_1), \quad 1 + x + z \in (1 - \varepsilon_1, 1 + ze \varepsilon_1) \subset (1 - \varepsilon_1, 1 + \varepsilon_1),
\]

hence

\[
[PA] : [PQ] \in ((1 - \varepsilon_1)^2, (1 + \varepsilon_1)^2) \subset (1 - \varepsilon_2, 1 + \varepsilon_2), \quad \varepsilon_2 := 2\varepsilon_1 + \varepsilon_1^2,
\]

which proves p.2 (with \( \varepsilon_2 \) instead \( \varepsilon_1 \)).

For proving p.3 we note that

\[
S \cdot r(ABC) = [AP] \cdot [PQ] \cdot \frac{[AP] + [PQ]}{2} \cdot (S/S(APQ)) = \frac{[AP] \cdot [PQ] \cdot ([AP] + [PQ])}{2xyz} \in (1 - \varepsilon_3, 1 + \varepsilon_3)
\]

where \( \varepsilon_3(\varepsilon) \to 0 \) if \( \varepsilon \to 0 \) (since mutual quotients of numbers \([AP] = x, [PQ], y, z \) lie in a segment \((1 - \varepsilon_2, 1 + \varepsilon_2)\)).

3
We pass to the p.4. Let \( \mathbf{x} = a \overrightarrow{AP} + b \overrightarrow{PQ}, \mathbf{x} \in \text{An}(APQ) \) \((a, b \geq 0)\) Then \( [\mathbf{x}]_{APQ} = a + b, [\mathbf{x}] = a[AP] + b[PQ] = (a + b)[AP] + b([PQ] - [AP]) = (AP/AC)(a + b)(1 + \frac{1}{a + b} \cdot ([PQ]/[AP] - 1)). \) The statement of p.4 follows.

**Note.** Point 1 of lemma may be founded, for instance, in book [F] \((p. 391)\).

**Corollary.** Applying the statement of p.1, many times, we get the following important fact: generalized affine length of any \((AB, C)-\)broken line \( \gamma \) satisfies an inequality \( l_A(\gamma) \leq S^{1/3}. \) Moreover, generalized affine length does not increase with adding new vertices to a broken line.

Now we formulate a statement on asymptotic distribution of integer points on a surface \( ab - cd = \text{const} \).

**Lemma 2.** Consider pairs of vectors \((\mathbf{x}, \mathbf{y}): \mathbf{x}, \mathbf{y} \in \text{An} \cap \mathbb{Z}^2, \) for which \( \mathbf{x} \times \mathbf{y} = m \neq 0 (m \in \mathbb{Z} \) is a constant). For each such pair, consider a special point \((\lfloor x \rfloor, \lfloor y \rfloor) \in [0, \infty)^2. \) Then special points are equidistributed in a first quadrant in a following sense: for any bounded domain \( \Omega \subset (0, \infty)^2 \) with piecewise-smooth boundary, the number of special points in a domain \( N(\Omega) \) has an asymptotics (by \( N \to \infty)\)

\[
\frac{c(m)S \cdot N^2 S(\Omega) + o(N^2)}{m},
\]

where \( c(m) \) is a constant, which depends on \( m \) (namely, \( c(m) = (2\zeta(2))^{-1} \sigma(m)/m, \sigma(m) \) is a sum of positive integer divisors of \( m \)).

The proof of lemma 2 is given in appendix. Alex Gorodnik pointed out that these statements follow from known general results (for instance, [EM]).

**Corollary.** Consider triangles \( PQR \) such that

1. \( \overrightarrow{PQ}, \overrightarrow{QR} \in \text{An} \cap \mathbb{Z}^2 \)
2. \( |\overrightarrow{PR}| \leq N(n/S)^{1/3} \)
3. \( r(\overrightarrow{PQR}) \in 2nS^{-1}(t_1, t_2) (0 < t_1 < t_2) \)
4. \( S(\overrightarrow{PQR}) \leq m (m \in \mathbb{N}). \)

The number \( N(m, M, t_1, t_2) \) of such triangles may be bounded by above as follows \((n \to \infty):\)

\[
N(m, M, t_1, t_2) \leq c(m)(t_1 - t_2)^3(nS^2)^{1/3} + o(n^{2/3}).
\]

**Proof.** Just apply lemma 2 for domains \( \Omega(m', M, t_1, t_2) = \{(p, q): 0 < p, q < M, pq(p + q) \in 4m'(t_1, t_2)\} \) and \( N = (n/S)^{1/3} (m' = 1, 2, \ldots, m) \) and use a fact that domain \( \Omega(m', \infty, 0, 1) \) has a finite area.

**Lemma 3.** Consider convex quadrilateral \( PSTR \) such that \( \overrightarrow{PS}, \overrightarrow{ST}, \overrightarrow{TR} \in \text{An}. \) Let \( Q \) be a point on its side \( ST. \) Then the value of \( r(\overrightarrow{PQR}) \) lies between \( \frac{|\overrightarrow{PQ}|}{|\overrightarrow{PQ}|} r(\overrightarrow{PQS}) \) and \( \frac{|\overrightarrow{QR}|}{|\overrightarrow{QR}|} r(\overrightarrow{RQT}). \)

**Proof.** We have \( S(\overrightarrow{PQR}) = \frac{|\overrightarrow{QR}|}{|\overrightarrow{SQ}|} S(\overrightarrow{PQS}) + \frac{|\overrightarrow{PQ}|}{|\overrightarrow{QT}|} S(\overrightarrow{RQT}). \) This equality may seem unexpected: RHS a priori depends on a choice of linear function \( [\mathbf{x}] \). It may be proved, for instance, as follows: when points \( S \) and \( T \) move on fixed rays \( QS \) and \( QT, \) nor left, neither right sides of equality do not change. So, we may assume that \( |SQ| = |QR| \) and \( |QT| = |PQ|. \) First equality means that the line, which joins \( Q \) and a midpoint of \( RS, \) is parallel to a vector \( AC + BC. \) Analogously, second equality means that the line, which joins \( Q \) and the midpoint of \( PT \) is parallel to the same vector. So, the point \( Q \) lies on a Gauss line of a quadrilateral \( PRTS, \) and therefore satisfies an equation of this line \( S(\overrightarrow{PXR}) + S(\overrightarrow{TXS}) = S(\overrightarrow{SXP}) + S(\overrightarrow{RXT}) \) (of course, to get the Gauss line we must consider oriented areas in this equation), q.e.d. From here

\[
2r(\overrightarrow{PQR}) = \frac{|\overrightarrow{PQ}| \cdot |\overrightarrow{QR}| \cdot (|\overrightarrow{PQ}| + |\overrightarrow{QR}|)}{|\overrightarrow{SQ}| S(\overrightarrow{PQS}) + |\overrightarrow{QT}| S(\overrightarrow{RQT})} = \frac{|\overrightarrow{PQ}| + |\overrightarrow{QR}|}{2r(\overrightarrow{PQS}) + 2r(\overrightarrow{RQT})}.
\]

As it is known, the quotient \( \frac{x+y}{x+y'} \) lies between \( \frac{x}{y} \) and \( \frac{y}{y'} \). Applying this for \( x = |\overrightarrow{PQ}|, y = |\overrightarrow{QR}| \) and \( x' = \frac{|\overrightarrow{PQ}|}{|\overrightarrow{PQS}|}, y' = \frac{|\overrightarrow{QR}|}{|\overrightarrow{RQT}|}, \) we get desired statement.

**§3. Main part**

**Theorem.** Let \( \gamma \) be a bounded convex curve. Denote \( k(\gamma, n) = \sharp(\gamma \cap L_n). \) Then \( k(\gamma, n) = o(n^{2/3}). \)
Lemma 4. For any $c > 0$ there exists such a number $a(c) > 0$, that for any triangle $\triangle ABC$ for large enough $n > N(c, \triangle ABC)$ the following statement holds: any $(AB, C; n)$-broken line $\gamma$, which have at least $\geq c(n^2S(ABC))^{1/3}$ vertices, have not too much generalzied affine length: $l_A(\gamma) \leq (1 - a(c))S(ABC)^{1/3}$.

At first, we show how the lemma 4 implies a theorem.

Assume that the theorem is not valid and for some bounded convex curve $\gamma$ we have $k(\gamma, q_n) \geq cq_n^{2/3}$ for some increasing sequence of positive integers $q_1 < q_2 < q_3 \ldots$. Without loss of generality, curve $\gamma$ joins points $A$ and $B$, and lies inside triangle $\triangle ABC$ (every bounded convex curve may be partitioned onto finite number of such curves). Let $S(ABC) = 1$. Define $\gamma_n$ as an $(AB, C)$-broken line, inscribed in $\gamma$, for which the set $\gamma \cap (\cup_{i=1}^n L_i)$ is a set of intermediate vertices.

We fix a support line in each point of curve $\gamma$. Draw these lines in vertices of $\gamma_n$, we get an $(AB, C)$-broken line $\gamma_n'$, circumscribed around $\gamma_n$. Denote by $\triangle_1, \triangle_2, \ldots, \triangle_k$, triangles formed by intersecting support lines in neighbour vertices of $\gamma_n$. The line $\gamma$ lies inside the union $\cup_{i=1}^k \triangle_i$. Denote $S_i = S(\triangle_i)$. Let $q = q_m$ be such large integer, that for any triangle $\triangle_i (1 = 1, 2, \ldots, k)$ the alternative of lemma 4 holds: either

1. $\sum_{i=1}^k (\gamma \cap \triangle_i \cap L_q) \leq (c/2) \cdot S_i^{1/3} \cdot q^{2/3}$, or
2. $l_A(\gamma_m \cap \triangle_i) \leq (1 - a)S_i^{1/3}$.

Note once more, that number $a$ depends only on $c$. Denote by $M_1$ the set of indecies $i$, for which the case (1) holds, and by $M_2$ — the set of the ther indecies $i$ (for them, case (2) holds). Note that

$$c\left(\sum_{i=1}^k S_i^{1/3}\right)q^{2/3} \leq cS(ABC)^{1/3}q^{2/3} = cq^{2/3} \leq \sum_{i=1}^k \sum_{i \in M_1} (\gamma \cap L_q \cap \triangle_i) +$$

$$+ \sum_{i \in M_2} \sum_{i \in M_1} (\gamma \cap L_q \cap \triangle_i) \leq (c/2)\left(\sum_{i \in M_1} S_i^{1/3}\right)q^{2/3} + 5\left(\sum_{i \in M_2} S_i^{1/3}\right)q^{2/3} \leq (c/2)\left(\sum_{i \in M_1} S_i^{1/3}\right)q^{2/3} + 5\left(\sum_{i \in M_2} S_i^{1/3}\right)q^{2/3},$$

hence

$$\sum_{i \in M_2} S_i^{1/3} \geq \frac{c}{10}\left(\sum_{i \in M_1} S_i^{1/3}\right).$$

Furthermore,

$$l_A(\gamma_m : \gamma_n') \leq \sum_{i=1}^k l_A(\gamma_m \cap \triangle_i) = \sum_{i \in M_1} + \sum_{i \in M_2} \leq$$

$$\leq \sum_{i \in M_1} S_i^{1/3} + (1 - a)\sum_{i \in M_2} S_i^{1/3} \leq (1 - \frac{ac}{10})\left(\sum_{i=1}^k S_i^{1/3}\right) = (1 - \frac{ac}{10})l_A(\gamma_n : \gamma_n').$$

From here we may deduce that $\lim_{n \to \infty} l_A(\gamma_n : \gamma_n') = 0$, which contradicts to our assumptions (from which $l_A(\gamma_n : \gamma_n') \geq c/5$).

So, it suffices to prove lemma 4.

Let $\gamma = C_0C_1C_2 \ldots C_kC_{k+1}$ ($C_0 = A, C_{k+1} = B, C_i \in L_n (i = 1, 2, \ldots, k)$) be an $(AB, C; n)$—broken line, and $k \geq cS^{1/3}n^{2/3}$. Let’s fix support lines $l_i$ in points $C_i (i = 1, 2, \ldots, k)$, define also lines $l_0 = AC$ and $l_{k+1} = BC$. Put $l_i \cap AC = B_i (i = 1, 2, \ldots, k+1), l_i \cap l_{i+1} = D_i (i = 0, 1, \ldots, k)$.

For optimal (in sense of generalized affine length definition) choice of support lines we have

$$l_A(\gamma) = \sup_{i=0}^k S(C_iD_iC_{i+1})^{1/3},$$

We have

$$S(ABC)^{1/3} - l_A(\gamma) = \sup_{i=1}^k x_i, x_i = S(AC_{i+1}B_{i+1})^{1/3} - S(C_iC_{i+1}D_i)^{1/3} - S(ACB_i)^{1/3}.$$
Numbers \( x_i \) are non-negative by p.1 of lemma 1. Our goal is to find \( N \geq c_1 n^{2/3} S(ABC)^{1/3} \) indecies \( i \), for which \( x_i \geq c_2 [C_i C_{i+1}] S(ABC)^{1/3} \) (with some constants \( c_1 \) and \( c_2 \), which depend only on \( c \)). If we succeed, the lemma 4 will be proved because of estimate (2) for the sum of \( N \) least girths.

Note also that if we find \( N_i \geq \epsilon_0 n^{2/3} S(ABC)^{1/3} \) indecies \( i \), for which \( x_i \geq c_1 [C_i C_{i+1}] S(ABC)^{1/3} \), it is also enough for our goal. Indeed, using inequality (5), we have an estimate like

\[
100n^2 S(AC_{i+1} B_{i+1}) \geq \varepsilon_i^3,
\]

so for \( i \geq N_i/2 \) we have \( S(AC_{i+1} B_{i+1})^{1/3} \geq \epsilon_0 S(ABC)^{1/3} \), and hence for at least \( N_i/2 \) indecies \( i \) desired estimate \( x_i \geq c_1 [C_i C_{i+1}] S(ABC)^{1/3} \) holds.

Applying lemma 1 at first stage for triangle \( ABC \) and points \( P = B_{i+1}, Q = C_{i+1} \), and then for triangle \( AC_{i+1} B_{i+1}, P = B_i, Q = C_i \), we see, that it suffices for some \( \epsilon_0(c) \) to find \( N_i \geq \epsilon_0 n^{2/3} S(ABC)^{1/3} \) indecies \( i \), for which either \( \max([C_i D_i], [D_i C_{i+1}]) \geq 1 + \epsilon_0 \) or \( r(C_i D_i C_{i+1}) \notin S^{-1}(1 - \epsilon_0, 1 + \epsilon_0) \).

We call the index \( i \), satisfying at least one of these two conditions, \( \varepsilon_0 \)-nice. Lemma 3 (for \( PSTR = C_i D_i D_{i+1} C_{i+2}, Q = C_{i+1} \) implies the following

**Statement.** Assume that indecies \( i \) and \( i + 1 \) are not \( \varepsilon_0 \)-nice. Then for some \( \epsilon_1(\epsilon_0) \) we have

\[
r(C_i C_{i+1} C_{i+2}) \geq 2S^{-1}(1 - \epsilon_1, 1 + \epsilon_1),
\]

and \( \epsilon_1 \) tends to 0 together with \( \epsilon_0 \).

So, it suffices to prove that for \( \varepsilon > 0 \) the number \( N_\varepsilon \) of indecies \( i \), for which \( r(C_i C_{i+1} C_{i+2}) \in 2S^{-1}(1 - \varepsilon, 1 + \varepsilon) \), admits an upper bound \( N_\varepsilon \leq \epsilon_1(Sn^2)^{1/3} \), where \( \epsilon_1 \) tends to 0 together with \( \varepsilon \).

Note that \( \sum S(C_i C_{i+1} C_{i+2})^{1/3} \leq 2S^{1/3} \) (sums by odd and even \( i \) do not exceed \( S^{1/3} \) by corollary of lemma 1), so the number of indecies \( i \), for which \( S(C_i C_{i+1} C_{i+2}) \geq mn^{-2} \), does not exceed (by Chebyshev inequality) \( 2m^{-1/3}(SN^2)^{1/3} \). Moreover, since for the sum of girths we have \( \sum_{i=0}^n [C_i C_{i+1}] \leq 2 \), we may also assume that \( [C_i C_{i+2}] \leq M(n^2 S)^{-1/3} \) (Chebyshev inequality again) for some large \( M \).

Now it suffices to use the corollary of lemma 2.

**Note.** The statement of a theorem may be slightly strengthened by letting \( L_n = (\frac{1}{n} Z)^2 + x_n \), where \( x_n \) is some vector of a shift, and chosing \( n \) not necessary integral.

§4. About possible number of integer points on a convex curve

The following example was constructed in [P]: there exists a strictly convex bounded curve \( \gamma \), for which

\[
k(\gamma, q_n) \geq c_n q_n^{2/3},
\]

where \( q_1 < q_2 < \ldots \) is arbitrary fast increasing integer sequence, and coefficients \( c_n \) decrease as \( K^{-n} (K > 1 \) is some explicit constant). Here we strengthen this result: it suffices to suppose that \( c_n < \infty \). But our method is quite rigorous, and it seems probable that this condition may be reduced to some weaker one (may be even to the necessary condition \( \lim c_n = 0 \)).

**Theorem 2.** Let \( \sum c_k < \infty \) be a convergent positive series; \( M \subset \mathbb{N} \) be an infinite subset of positive integers. Then there exists a sequence \( q_1 > q_2 < \ldots, q_i \in M \) of positive integers and a strictly convex bounded curve \( \gamma \) such that \( k(\gamma, q_n) \geq c_n q_n^{2/3} \).

**Proof.** Consider a semicircle of length \( \sum c_i \). Partion it into arcs of lengths \( c_i \), denote by \( A_i \) and \( A_{i+1} \) endpoints of arc of length \( c_i \). Draw tangents to a semicircle in endpoints of these arcs. Each arc lies in some triangle \( A_i B_i A_{i+1} \) (\( B_i \) is a point, in which tangents in \( A_i \) and \( A_{i+1} \) intersect). An area of triangle \( A_i B_i A_{i+1} \) is not less than \( Cc_i^3 \), where \( C \) does not depend on \( i \). Without loss of generality, \( C > 100 \). (else make a homothety, which increase areas in 100/(C times).) Now take a single triangle \( A_i B_i A_{i+1} \) and construct such an (\( A_i A_{i+1}, B; q_i \))-broken line, that number of intermediate vertices of this broken line is not less than \( c_i q_i^{2/3} \) (it is possible for large enough \( q_i \), as it was discussed in paragraph 1). The union of all these broken lines is desired curve \( \gamma \) (to be more precise, we must replace straigh edges of broken lines to some smoothly strictly convex curves for making \( \gamma \) strictly convex).

Appendix. Distribution of integer points on a surface \( ab - cd = const \).

Consider bounded domains \( \Omega_1, \Omega_2 \subset \mathbb{R}^2 \), with piecewise-smooth boundary. We study the asymptotics of a quantity \( M(\Omega_1, \Omega_2; n) \) of pairs of vectors \( x_1 \in n \Omega_1 \cap \mathbb{Z}^2, x_2 \in n \Omega_1 \cap \mathbb{Z}^2 \) such that \( x_1 \times x_2 = 1 \) (here \( x_1 \times x_2 \) denotes an oriented area of parallelogramm based on vectors \( x_1, x_2 \)). At first, we consider the case of triangles

\[
\Omega_i = \{(x, y) : 0 < y < x \leq a_i \} \quad (i = 1, 2, a_i > 0)
\]

In this case, the problem reduces to the following question: how many solutions does an equality

\[
x_1 y_2 - y_1 x_2 = 1
\]

(a1)
with conditions
\[ 0 < y_1 < x_1 < n a_1, \quad 0 < y_2 < x_2 < n a_2 \]

have? It is well-known, that for fixed coprime \( x_1, x_2 \) an equation (*) has a unique solution in \( y_1, y_2 \), for which \( 0 < y_1 \leq x_1, \quad 0 < y_2 \leq x_2 \). Cases of equality \( y_1 = x_1 \) or \( y_2 = x_2 \) may be realized only if \( x_1 = 1 \) or \( x_2 = 1 \), i.e. for at most \( C(a_1, a_2) \cdot n \) variants. The number of integer points \( (x_1, x_2) \) with coprime coordinates in the rectangle \( 0 < x_1 < n a_1, \quad 0 < x_2 < n a_2 \) is
\[ \zeta(2)^{-1} n^2 a_1 a_2 + o(n^2), \]
hence for the case (*) we have asymptotics
\[ M(\Omega_1, \Omega_2; n) = \zeta(2)^{-1} n^2 a_1 a_2 + o(n^2). \]

In next, we need the following reformulation of gotten result. Denote by \( l(\Omega, \varphi) \) 1-dimensional Lebesgue measure of the intersection of domain \( \Omega \) and a line \( y = \tan \varphi \cdot x \) (the line which form angle \( \varphi \) with X-axis). Then
\[ M(\Omega_1, \Omega_2; n) = \zeta(2)^{-1} \int_0^\pi l(\Omega_1, \varphi) \cdot l(\Omega_2, \varphi) \, d\varphi \cdot n^2 + o(n^2) \]  \hspace{1cm} (a3)

A triangle \( OAB \) with area \( 1/2 \) and integer vertices \( A \) and \( B \) is called a basic triangle.

Note that formula (a3) holds also for domains, which may be gotten from (*) by \( SL(2, \mathbb{Z}) \)-element affine action. In other words, formula (a3) holds for triangles, which may be gotten from some basic triangle by homotheties.

Our further plan is approximation of quite generic domains by unions of such “basic” domains, almost disjoint in central projection to unit circle.

Fix a number \( \varepsilon > 0 \). The basic triangle \( OAB \) is called \( \varepsilon \)-suitable, if
\begin{enumerate}
  \item \( |OA/OB - 1| < \varepsilon \)
  \item \( \angle AOB < \varepsilon \).
\end{enumerate}

We need the following

**Lemma.** Almost every (in the sense of Lebesgue measure on a unit circle) ray, arising from the origin, intersects interiors of infinitely many \( \varepsilon \)-suitable basic triangles.

**Proof.** Without loss of generality, the ray is defined as \( 0 < y = \alpha x, \quad 0 < \alpha < 1 \). Consider a continued fraction for \( \alpha \): \( \alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \). For almost all \( \alpha \) elements \( a_i \) are unbounded (it follows from Gauss-Kuzmin formula, but may be gotten easier, see [Kh]). In terms of convergents \( \frac{B_k}{q_k}(k = 1, 2, \ldots) \) it means that the ratio \( q_{k+1}/q_k \) of neighbour denominators is unbounded \( q_{k+1} = a_k q_k + q_{k-1} \). Each pair of neighbour convergents corresponds to some basic triangle \( OAB \) \( A = (q_k, p_k), \quad B = (q_{k+1}, p_{k+1}) \), and our ray intersects the segment \( AB \). If the ratio \( q_{k+1}/q_k \) is large, than \( OB >> OA \).

We apply the known procedure of “noses stretch”. Namely, we define a sequence of points \( B_0 = B, \quad \overline{OB}_i = \overline{OB}_{i-1} + \overline{OA} \). One of segments \( B_{i-1}B_i \) intersects our ray. The triangle \( \overline{OB}_{i-1}B_i \) is clearly basic, and, if \( a_k \) and \( k \) are quite large, it is \( \varepsilon \)-suitable.

Let domains \( \Omega_1, \Omega_2 \) be homothetic triangles \( \Omega_1 = OCD, \quad \Omega_2 = \lambda \Omega_1 (\lambda > 0) \), where the line \( CD \) is vertical and points \( C, \ D \) lie in the domain \( 0 < y < x \) (\( C \) lower than \( D \)).

Using Vitali theorem and our lemma, we may find \( \varepsilon \)-suitable triangles \( OA_iB_i \) \( i = 1, 2, \ldots, n \) such that rays \( OC, OA_i, \ OB_i, \ OA_{i+1}, \ OB_{i+1}, \ldots, OA_n, \ OB_n, \ OD \) go counterclockwise in given order and
\[ \sum_{i=1}^n \angle A_iOB_i > \angle COD - \varepsilon. \]

Let \( \Delta_i \) be the largest triangle, homothetic (with centre \( O \)) to \( OA_iB_i \), which is contained in \( \Omega_1 \) \( i = 1, 2, \ldots, n \).

Then we may sum up the estimates of type (a3) for triangles \( \Delta_i, \ \lambda \Delta_i, \) and get the following lower bound:
\[ M(\Omega_1, \Omega_2; n) \geq \left( \lambda \sum_{i=1}^n 2S(\Delta_i) \right) n^2 + o(n^2) \]

For small \( \varepsilon \), we have \( S(\Delta_i) \geq c(\varepsilon)S(\angle A_iOB_i \cap \triangle OCD) \), where \( c(\varepsilon) \to 1 \) if \( \varepsilon \to 0 \).
So, we let $\varepsilon$ tend to zero and for given domains $\Omega_1, \Omega_2$ we get a lower bound of type (a3):

$$M(\Omega_1, \Omega_2; n) \geq \zeta(2)^{-1} \int_0^\pi l(\Omega_1, \varphi) \cdot l(\Omega_2, \varphi) d\varphi \, n^2 + o(n^2)$$  \hspace{1cm} (a4)

Consider points $C_1$ and $D_1$, in which line $CD$ meets X-axis and line $x = y$, respectively. Consider triangles $\triangle_1 = OC_1C$, $\triangle_2 = OCD$, $\triangle_3 = ODD_1$, $\triangle_0 = OC_1D_1$ and triangles $\lambda \triangle_i$. We have

$$M(\triangle_0, \lambda \triangle_0; n) \leq \sum_{i=1}^3 \sum_{j=1}^3 M(\triangle_i, \lambda \triangle_j; n).$$

LHS has an asymptotic of type (a3), three summands in RHS (with $i = j$) satisfy the lower bound (a4). Combining these two observations, we see that the lower bound is an upper bound as well in all three cases, and crossing terms give a contribution $o(n^2)$.

So, the asymptotic (a3) is gotten for domains of described type.

Now all "quite good" (for example, with piecewise-smooth boundary) domains may be approximated from both sides by sums and differences of such domains.

Let’s now consider slightly more general problem. Namely, replace the condition on a pair of vectors $x_1 \times x_2 = 1$ to the condition $x_1 \times x_2 = m = \text{const} \neq 0$. Again we start from the same special case, when

$$\Omega_i = \{(x, y) : 0 < y < x \leq a_i \} (i = 1, 2, a_i > 0).$$

It’s easy problem to find answer in this case. Indeed, let’s fix the greatest common divisor $\gcd(x_1, x_2) = d|m$ ($x_1, x_2$ are abscissas of vectors $x_1, x_2$). We get

$$d^{-1} \zeta(2)^{-1} n^2 a_1 a_2 + o(n^2)$$

our pairs. Sum up by all divisors of $m$ and get asymptotics

$$M(\Omega_1, \Omega_2; m; n) = \sigma(m) \cdot |m|^{-1} \cdot \zeta(2)^{-1} \int_0^\pi l(\Omega_1, \varphi) \cdot l(\Omega_2, \varphi) d\varphi \, n^2 + o(n^2)$$ \hspace{1cm} (a3’)

The generalization for generic domains $\Omega_1, \Omega_2$ does not differ from the one for case $m = 1$.

Lemma 2 easily follows from the proven fact.

**Proof of lemma 2.** Without loss of generality, we assume that $\Omega$ is a rectangle $\{0 < x < A, 0 < y < B\}$. In this case, the number of blue points in a rectangle $\Omega$ is a number of pairs of vectors $(x, y) : x \times y = m$ such that $x \in N\Omega_x, y \in N\Omega_y$, where $\Omega_x$ and $\Omega_y$ are domains defined as

$$\Omega_x = \{x \in \mathbb{A} : |x| \leq A\}, \Omega_y = \{y \in \mathbb{A} : |x| \leq B\}.$$

Applying (3’), we get the desired result.

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