The Infrared Boundary of Perturbative QCD

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Abstract

Evidence is reviewed suggesting that QCD remains a perturbative theory with a (relatively) small coupling constant down to a distinct infrared boundary on perturbative physics, a boundary corresponding to the momentum scale associated with a $\beta$-function pole.

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1 Supersymmetric Gluodynamics

Our information about the coupling constants characterising known interactions is necessarily perturbative. Many QCD calculations, for example, have been performed in the $\overline{\text{MS}}$ scheme in which the $\beta$-function for the couplant $x \equiv \alpha_s(\mu)/\pi$ is known to three subleading orders [1]:

$$\mu^2 \frac{dx}{d\mu^2} \equiv \beta(x) = -\beta_0 x^2 - \beta_1 x^3 - \beta_2 x^4 - \beta_3 x^5 - \beta_4 x^6 ...$$

(1)

$$\beta_0 = \frac{11}{4} - \frac{n_f}{6}$$

$$\beta_1 = \frac{51}{8} - \frac{19n_f}{24}$$

$$\beta_2 = \frac{2857}{128} - \frac{5033n_f}{1152} + \frac{325n_f^2}{3456}$$

$$\beta_3 = \frac{114.23033 - 27.133944n_f + 1.582379n_f^2 + 5.856696 \cdot 10^{-3}n_f^3}{4}$$

(2)

Thus, we can determine with high accuracy how the QCD couplant evolves with $\mu$ in perturbatively accessible processes. However, the behaviour of the couplant in the infrared region, the region where successive terms in the series (1) become comparable in magnitude, is not at all clear from the truncated perturbative series.

There is value in having theories in which the $\beta$-function is exactly known. One such theory is $N = 1$ supersymmetric $SU(N_c)$ Yang-Mills theory, a theory of gluons and gluinos often denoted as “supersymmetric gluodynamics.” The $\beta$-function for this theory has been computed to all orders by Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) using instanton calculus methods [2]; it is also obtainable algebraically by requiring that the supersymmetric anomaly multiplet characterising the theory satisfy the Adler-Bardeen theorem [3, 4]:

$$\mu^2 \frac{dx}{d\mu^2} \equiv \beta_{\text{NSVZ}}(x) = -\frac{3N_c x^2}{4} \left[ \frac{1}{1 - N_c x/2} \right]$$

(3)

The pole characterising the $\beta$-function (3) necessarily implies the existence of an infrared boundary for the asymptotically-free phase of the couplant [3]. Suppose we have some sufficiently small initial value $x_0 \left[ x_0 < 2/N_c \right]$ occurring at momentum scale $\mu_0$. We see that the $\beta$-function is negative, implying that $x(\mu)$ grows as $\mu$ decreases until $x = 2/N_c$, at which point $\beta(x)$ becomes singular. Alternatively $\mu$ can be regarded as a function of $x$, with initial value $\mu_0$ occurring at $x_0$ and with an extremum occurring when $x = 2/N_c$. This extremum is necessarily a minimum, a critical value $\mu_c$ that represents the lower bound on the domain of $x(\mu)$. In short, supersymmetric gluodynamics is a field theory characterised by a real perturbative interaction coupling constant only when $\mu > \mu_c$. Not only is $\mu_c$ the infrared boundary of this theory, but this boundary is characterised by a finite value of the couplant: $x(\mu_c) = 2/N_c$. 
The behaviour described above is highly suggestive of the qualitative picture we have of the strong interactions, whose perturbative character as a theory of quarks and gluons abruptly ends at near-GeV hadronic mass scales. At such mass scales, strong interactions are described by some effective-Lagrangian phenomenological theory distinct from perturbative QCD, a theory in which fundamental fields are no longer quarks and gluons, but hadrons.

2 Non-Supersymmetric Gluodynamics

For a function underlying an incompletely known perturbative series, such as the QCD β-function (1), one of the few available techniques for extracting information about the existence of poles or zeros of that function is to examine Pade approximants constructed entirely from the known terms of the series. Such approximants may exhibit a positive zero which precedes any positive poles, behaviour indicative of a nonzero infrared-stable fixed point. Alternatively, a leading pole, e.g., a positive pole which precedes any positive zeros in the approximant, is indicative of a β-function analogous to (2), a function characterised by an infrared boundary to perturbative dynamics in its asymptotically free phase. While one particular Pade approximant may exhibit a spurious pole (usually denoted as a “defect pole”), the occurrence of leading poles in many different Pade approximants all constructed from the same truncated series is strong evidence that such a pole genuinely occurs within the function underlying that series [6].

For ($N_c = 3$) gluodynamics, this truncated series is given by eqs. (1) and (2) with $n_f = 0$. The lowest approximants possessing both zeros and poles that are determined by the known and first unknown terms of the series (1) are

\[
\beta^{[2|1]}(x) = -\frac{11}{4} x^2 \left[ 1 - \frac{2.7996 x - 3.7475 x^2}{1 - 5.1178 x} \right]
\]

\[
\beta^{[1|2]}(x) = -\frac{11}{4} x^2 \left[ 1 - \frac{5.9672 x}{1 - 8.2854 x + 11.091 x^2} \right]
\]

\[
\beta^{[3|1]} = -\frac{11}{4} x^2 \left[ 1 + (2.31818 - 0.0087542 \beta_4) x + (8.11648 - 0.020294 \beta_4) x^2 \right]
+ \left( 41.5383 - 0.071053 \beta_4 \right) x^3 / [1 - 0.0087542 \beta_4 x]
\]

\[
\beta^{[2|2]}(x) = -\frac{11}{4} x^2 \left[ 1 + (13.4026 - 0.027715 \beta_4) x + (-22.9153 + 0.032788 \beta_4) x^2 \right]
/ \left[ 1 + (11.0844 - 0.027715 \beta_4) x + (-56.7275 + 0.097036 \beta_4) x^2 \right]
\]

\[
\beta^{[1|3]} = -\frac{11}{4} x^2 \left[ 1 + (9.56239 - 0.02222 \beta_4) x \right]
/ [1 + (7.24421 - 0.02222 \beta_4) x]
+ (-24.9099 + 0.005151 \beta_4) x^2 + (-42.5901 + 0.060939 \beta_4) x^3
\]

1 A demonstration of how Pade approximant poles for a series that differs only infinitesimally from a geometric series are able to reproduce the geometric-series pole is presented in ref. [4].
The Maclaurin expansions of the first two approximants reproduce the known terms \( \beta_0 - \beta_3 \) of the series (1) when \( n_f = 0 \). The Maclaurin expansions of the final three approximants also reproduce the unknown \( \beta_4 \) term of that series. In (4) and (5), the first positive pole \([x = 0.195 \text{ and } x = 0.151, \text{ respectively}]\) is seen to precede the first positive zero \([x = 0.264 \text{ and } x = 0.168]\). Remarkably, this behaviour persists in (6), (7) and (8), regardless of the value of the unknown coefficient \( \beta_4 \) in the \( \beta \)-function series (1), as is demonstrated graphically in Figs. 3-5 of ref. [7]. In (6), for example, a positive pole occurs at \( x = 114.2/\beta_4 \) provided \( \beta_4 \) is positive. This pole is always smaller than any positive zeros of the degree-3 numerator polynomial. Moreover, when \( \beta_4 \) is negative, eq.(6) exhibits no positive zeros or poles. Eqs. (7) and (8), which have higher-degree denominator polynomials, exhibit a positive pole for all values of \( \beta_4 \) which is seen to always precede any numerator zeros for that same value of \( \beta_4 \). Thus, for Pade approximants to the \( \beta \)-function capable of generating both zeros and poles, a zero never precedes a pole for \( N_c = 3 \) gluodynamics, as would be expected if gluodynamics were to have an infrared stable fixed point. Instead, all such approximants point strongly toward the existence of a \( \beta \)-function pole, as is known to characterise supersymmetric gluodynamics in the NSVZ renormalization scheme.

Such behaviour also characterises QCD in the 'tHooft \((N_c \to \infty)\) limit, a “gluodynamics” for any finite choice of \( n_f \). This is evident from the five approximants analogous to (4-8) constructed from the known and leading unknown coefficient in the \( \beta \)-function [1] describing the evolution of the (finite) couplant \( \lambda \equiv N_c \alpha_s(\mu)/4\pi \) as \( N_c \to \infty (\alpha_s \to 0) \):

\[
\mu^2 \frac{d\lambda}{d\mu^2} = -\frac{11}{3} \lambda^2 - \frac{34}{3} \lambda^3 - \frac{2857}{54} \lambda^4 - 315.49 \lambda^5 - \beta_4 \lambda^6 \ldots \tag{9}
\]

These five approximants are presented in the Appendix to ref. [7]. The pole/zero structure of these approximants is the same as that of \( N_c = 3 \) gluodynamics – a positive pole always precedes any positive zeros occurring within leading approximant versions of the \( \beta \)-function (9). These results strongly point to the existence of an infrared boundary to gluodynamics as a perturbative theory, a boundary which occurs at the value of \( \mu \) associated with such poles.

### 3 QCD

In ref. [7], the dynamics described above for gluodynamics are shown to apply to all approximant versions of the QCD \( \beta \)-function, even if \( n_f \) is as large as 5. When \( n_f \geq 6 \), \( \beta^{[2]} \) no longer exhibits a pole, and poles are seen to precede zeros in \( \beta^{[2]} \), \( \beta^{[2]} \) and \( \beta^{[3]} \) only if \( \beta_4 \) is larger than approximant-specific lower bounds. Such results corroborate a lattice study indicative of a similar \( n_f \) threshold \((n_f = 7)\) for infrared-stable fixed points to occur within QCD [8].

Pade approximant methods have been used successfully to predict unknown QCD \( \beta \)-function terms. For example, such methods, accompanied by explicit knowledge of
the $\mathcal{O}(n_f^3)$ term in $\beta_3$ led Ellis, Karliner and Samuel [9] to predict $\beta_3$ in (1):

$$\beta_{3 \text{pred}} = \frac{23,600 - 6400n_f + 350n_f^2 + 1.499n_f^3}{256}.$$ (10)

Comparison of this result with the exact result (2), as calculated explicitly in [4], demonstrates the power of Pade approximant methods. Similar methods have been used to obtain a corresponding prediction for the presently-unknown coefficient $\beta_4$ [10]:

$$\beta_{4 \text{pred}} = \frac{759,000 - 219,000n_f + 20,500n_f^2 - 49.8n_f^3 - 1.84n_f^4}{1024}.$$ (11)

In attempting to extract information about the infrared boundary of perturbative strong interaction physics, our interest is necessarily directed toward the $n_f = 3$ case of (11), for which $\beta_{4 \text{pred}} = 278$. The Pade approximant versions of the $n_f = 3$ QCD $\overline{\text{MS}}$ $\beta$-function whose Maclaurin expansions reproduce this estimate of $\beta_4$ as well as $\beta_1 - \beta_3$, as given in (2), are [11]

$$\beta_{[3|1]}(x) = -\frac{9}{4}x^2 \frac{1 - 4.116x - 6.006x^2 - 5.359x^3}{1 - 5.893x}$$ (12)
$$\beta_{[2|2]}(x) = -\frac{9}{4}x^2 \frac{1 - 5.498x - 1.972x^2}{1 - 7.276x + 6.492x^2}$$ (13)
$$\beta_{[1|3]}(x) = -\frac{9}{4}x^2 \frac{1 - 5.740x}{1 - 7.517x + 8.893x^2 - 3.190x^3}$$ (14)

All three approximants exhibit positive poles that precede any positive zeros. These poles $x_c$ respective occur at

$$x_{c[3|1]} = 0.170, \quad x_{c[2|2]} = 0.160, \quad x_{c[1|3]} = 0.162$$ (15)

and correspond to values of $\alpha_s$ between 0.50 and 0.53.

Both the agreement exhibited by these distinct approximants on the magnitude of their poles, as well as the perturbatively small value of the $\alpha_s$ corresponding to these poles, are quite striking. The mass scale $\mu_c$ associated with these approximant poles is easily obtained from integrating the differential equation

$$\mu^2 \frac{d}{d\mu^2} x = \beta_{[N|M]}(x)$$ (16)

from a physical initial value $x_0(\mu_0)$ to the pole value $x_{c[N|M]}$:

$$\mu_{c[N|M]} = \mu_0 \exp \left[ \frac{1}{2} \int_{x_0}^{x_{c[N|M]}} \frac{dx}{\beta_{[N|M]}(x)} \right].$$ (17)

If we identify $\alpha_s(m_\tau) = 0.314 \pm 0.010$ [12] and if we require the couplant to devolve with four active flavours from $\mu = m_\tau$ to a four-flavour threshold at 1.2 GeV (below which only three active flavours contribute), we find that

$$\mu_{c[3|1]} = 950 \pm 50 \text{ MeV}, \quad \mu_{c[2|2]} = 990 \pm 50 \text{ MeV}, \quad \mu_{c[1|3]} = 980 \pm 50 \text{ MeV}.$$ (18)
Alternatively, we can utilise the (three-active flavour) estimate $\alpha_s(m_\tau) = 0.33 \pm 0.02$ \cite{13} to find entirely via (12-14) and (17) that \cite{17}

$$\mu_c^{[3]} = 1.09 \pm 0.11 \text{ GeV}, \quad \mu_c^{[2]} = 1.14 \pm 0.11 \text{ GeV}, \quad \mu_c^{[1]} = 1.13 \pm 0.11 \text{ GeV}. \quad (19)$$

These estimates are all suggestive of the $\mu_c \simeq 4\pi f_\pi \, (\approx 1.17 \text{ GeV})$ Georgi-Manohar boundary on effective-Lagrangian strong interaction physics \cite{14}. These results also show remarkable consistency across the approximants (12-14) in the coordinates of the infrared terminus ($\mu_c, x_c \equiv \alpha_s(\mu_c)/\pi$) of QCD as a perturbative theory of quarks and gluons.

Of course, these estimates rely on the estimate (11) for $\beta_4$. Because of the alternation of sign in this expression, the estimate itself is considerably less likely to be accurate than the individual estimates for polynomial coefficients of powers of $n_f$. An alternative approach toward acquiring insight into the infrared boundary of QCD is to assume that $\alpha_s(\mu_c) = \pi/4$, the threshold value associated with dynamical chiral symmetry breaking \cite{13}. Individual $[31], [22],$ and $[13]$ approximants to the $n_f = 3$ $\beta$-function when $\beta_4$ is taken to be arbitrary are given by

$$\beta^{[3]}(x) = -\frac{9}{4} x^2 \left[ 1 + (1.7778 - 0.021174\beta_4)x + (4.4471 - 0.037642\beta_4)x^2 \right]$$

$$+ \left( 20.990 - 0.094670\beta_4 \right)x \right] / \left[ 1 - 0.021174\beta_4 x \right] \quad (20)$$

$$\beta^{[2]}(x) = -\frac{9}{4} x^2 \left[ 1 + (7.19456 - 0.045604\beta_4)x + (-11.3292 + 0.033619\beta_4)x^2 \right]$$

$$\left[ 1 + (5.41678 - 0.045604\beta_4)x + (-25.4301 + 0.11469\beta_4)x^2 \right] \quad (21)$$

$$\beta^{[1]}(x) = -\frac{9}{4} x^2 \left[ 1 + (5.80845 - 0.041491\beta_4)x + (4.03067 - 0.041491\beta_4)x \right]$$

$$+ \left( -11.6367 + 0.073762\beta_4 \right)x^2 + (-18.3242 + 0.054377\beta_4)x \right] \quad (22)$$

These approximants are respectively seen to develop a leading positive pole at $x = 1/4$ (i.e. $\alpha_s = \pi/4$) for the following values of $\beta_4$:

$$\beta_4^{[3]} = 189, \quad \beta_4^{[2]} = 181, \quad \beta_4^{[1]} = 202, \quad (23)$$

values surprisingly uniform across the three distinct approximants. Upon substituting such values for $\beta_4$ into their corresponding approximants (20-22), we can estimate via (17) concomitant values for the infrared-boundary momentum scale at the $x = 1/4$ pole. We set $x_c^{[3]} = 1/4$ in (17) and utilize the extreme values of the 3-active-flavour range $0.33 \pm 0.02$ already quoted for $\alpha_s(\pm \beta_4)$ in order to find that \cite{17}

$$\alpha_s(m_\tau) = 0.31 : \mu_c^{[3]} = 775 \text{ MeV}, \quad \mu_c^{[2]} = 785 \text{ MeV}, \quad \mu_c^{[1]} = 778 \text{ MeV}$$

$$\alpha_s(m_\tau) = 0.33 : \mu_c^{[3]} = 863 \text{ MeV}, \quad \mu_c^{[2]} = 874 \text{ MeV}, \quad \mu_c^{[1]} = 867 \text{ MeV}$$

$$\alpha_s(m_\tau) = 0.35 : \mu_c^{[3]} = 948 \text{ MeV}, \quad \mu_c^{[2]} = 960 \text{ MeV}, \quad \mu_c^{[1]} = 952 \text{ MeV}.$$
There is remarkable uniformity in these estimates obtained from distinct approximants, suggesting that the infrared boundary is genuine, rather than a single-approximant artefact. A more precise estimate of the boundary, however, requires more precise knowledge of $\alpha_s$ at $\mu = m_T$. We can conclude, however, that there is clear evidence from Pade approximant methods for QCD to be a computationally-perturbative gauge theory of quarks and gluons right up to an $O(1 \text{ GeV})$ boundary mass scale, an explicit lower bound on the domain of $\alpha_s(\mu)$ below which the description of the strong interactions must necessarily be quite different.

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