HÖLDER STABILITY OF DIFFEOMORPHISMS

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Abstract. We prove that a $C^2$ diffeomorphism $f$ of a compact manifold $M$ satisfies Axiom A and the strong transversality condition if and only if it is Hölder stable, that is, any $C^1$ diffeomorphism $g$ of $M$ sufficiently $C^1$ close to $f$ is conjugate to $f$ by a homeomorphism which is Hölder on the whole manifold.

1. Introduction

Let $M$ be a compact $C^\infty$ manifold, $\text{Diff}^1(M)$ be the group of $C^1$ diffeomorphisms of $M$. $f \in \text{Diff}^1(M)$ is structurally stable if for any $g \in \text{Diff}^1(M)$ sufficiently $C^1$ close to $f$, there is a homeomorphism $h$ of $M$ such that $g = hfh^{-1}$. Recall that $f$ satisfies Axiom A if the nonwandering set $\Omega$ of $f$ is hyperbolic and the set of periodic points of $f$ is dense in $\Omega$, $f$ satisfies the strong transversality condition if for any two points $x, y \in \Omega$ the stable manifold $W^s(x)$ intersects the unstable manifold $W^u(y)$ transversally. By the Structural Stability Theorem of Robbin, Robinson, Liao and Mañé [8, 9, 6, 7], $f \in \text{Diff}^1(M)$ is structurally stable if and only if $f$ satisfies Axiom A and the strong transversality condition. It is also known that in this case the conjugacy $h$ can be chosen to be Hölder on the nonwandering set $\Omega$ of $f$ (see [5, Theorem 19.1.2]).

In this paper, we prove that in the above case, the conjugacy $h$ can be chosen to be Hölder not only on $\Omega$ but also on the whole manifold $M$. We say that a diffeomorphism $f$ of $M$ is Hölder stable if for any $g \in \text{Diff}^1(M)$ sufficiently $C^1$ close to $f$, there is a Hölder homeomorphism $h$ of $M$ such that $g = hfh^{-1}$ (This notion should not be confused with the notion of $C^r$ structural stability of a $C^r$ diffeomorphism, for which $g$ is $C^r$ close to $f$ and the conjugacy $h$ is only required to be continuous). We prove that Axiom A plus the strong transversality condition is also equivalent to Hölder stability. For simplicity, we assume that $f$ is $C^2$.

Theorem 1.1. Let $f$ be a $C^2$ diffeomorphism of a compact $C^\infty$ manifold $M$. Then $f$ is Hölder stable if and only if $f$ satisfies Axiom A and the strong transversality condition.

Since Hölder stability implies structural stability, to prove Theorem 1.1 it is sufficient by the Structural Stability Theorem to prove that Axiom A plus the strong transversality condition implies Hölder stability.

To state the quantitative result, we recall the notion of hyperbolicity. The nonwandering set $\Omega$ of a diffeomorphism $f$ is hyperbolic if the restriction $TM|_\Omega$ of the tangent bundle $TM$ on $\Omega$ admits a $Tf$-invariant continuous splitting $TM|_\Omega = E^u \oplus E^s$ such that for some $\lambda \in (0, 1)$,

$$\|Tf^{-1}|_{E^s}\| \leq \lambda, \quad \|Tf|_{E^u}\| \leq \lambda.$$  

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Here the norm is evaluated with respect to some adapted smooth Riemannian metric on $M$.

**Theorem 1.2.** Let $f$ be a $C^2$ diffeomorphism of a compact $C^\infty$ manifold $M$ satisfying Axiom A and the strong transversality condition. Let $\lambda \in (0, 1)$ be as in \[\text{(1.1)}\], $\lambda = \max\{\text{Lip}(f), \text{Lip}(f^{-1})\}$. Suppose $\alpha \in (0, 1)$ satisfies $\lambda^\alpha < 1$. Then for any $C^\alpha$ neighborhood $V$ of the identity map in $C^\alpha(M, M)$, there exists a $C^1$ neighborhood $N$ of $f$ in $\text{Diff}^1(M)$ such that for every $g \in N$, there is a homeomorphism $h$ of $M$ in $V$ such that $g = hfh^{-1}$, and the assignment $g \mapsto h$ is $C^1$ as a map $N \to C^0(M, M)$ and sends $f$ to the identity.

Here $\text{Lip}(f)$ denotes the Lipschitz constant of $f$, $C^\alpha(M, M)$ and $C^0(M, M)$ are the Banach manifolds of $C^\alpha$ and $C^0$ maps on $M$, respectively.

Hölder stability over hyperbolic sets is well known (\[\text{Theorem 19.1.2}\]). It is also well known that the (un)stable distributions and (un)stable foliations over hyperbolic sets are Hölder continuous (\[\text{Section 19.1}\]). For more results on Hölder regularity for hyperbolic dynamical systems, see \[\text{Section 2.3}\].

One can not expect more regularity of the conjugacy $h$ than to be Hölder. For example, Lipschitz conjugacies almost never exist. But for dynamical systems of large group actions, $C^\alpha$ or $C^\infty$ conjugacies may exist (see \[\text{H}\] and the references therein).

Our proof of Theorem \[1.2\] follows the approach of Robbin-Robinson \[8, 9\], where the result that Axiom A plus the strong transversality condition implies structural stability is proved. As in Robbin \[8\], we divide the proof into three steps, which are the contents of the following three sections.

In Section 2, we prove that for each component $\Omega_i$ of $\Omega$, the splitting $TM|_{\Omega_i} = E^u|_{\Omega_i} \oplus E^s|_{\Omega_i}$ can be extended to a $Tf$-invariant splitting $TM|_{O(U_i)} = E^u_i \oplus E^s_i$ satisfying certain compatibility condition, where $U_i$ is a neighborhood of $\Omega_i$, $O(U_i) = \bigcup_{n=-\infty}^\infty f^n(U_i)$. The proof follows ideas in \[8, 9\]. But since we require that the extended splitting to be Hölder, and the metric $d$ on $M$, unlike Robbin’s metric $df$ \[8\], is not $f$-preserving, we need more careful topological arguments. Indeed, we can only prove that the extended bundles $E^u_i$ and $E^s_i$ are Hölder on $\bigcup_{n=-N}^N f^n(U_i)$ for every $N > 0$. But this is sufficient for us to derive further results. In Section 2 we only need the weaker restriction $\lambda^\alpha < 1$ on the Hölder exponent $\alpha$ comparing with Theorem \[1.2\] and the case of $\alpha = 1$ is allowed, which means as usual that the subbundles are Lipschitz.

Using the extended splitting in Section 2, we prove in Section 3 that the induced operator $f_1$ of $f$ on the Banach space of $C^0$ vector fields has a right inverse which restricts to a continuous linear operator on the Banach space of $C^\alpha$ and $df_1$-Lipschitz vector fields. The proof is also motivated by \[8\]. But as in Section 2, since $f$ does not preserve the metric $d$, some different topological arguments are needed. The condition of $\alpha \neq 1$ is not explicitly used in the proof. But since it is easy to see that $l \geq \lambda^{-1}$, the inequality $\lambda^\alpha < 1$ for $\alpha = 1$ never holds. So the case of $\alpha = 1$ is automatically excluded.

In Section 4 we finish the proof of Theorem \[1.2\]. We first prove a version of Implicit Function Theorem for Banach spaces involving non-closed subspaces. Then using the result in Section 3, we can apply the Implicit Function Theorem to the $C^1$ map $\Psi : \text{Diff}^1(M) \times C^0(M, M) \to C^0(M, M)$, $\Psi(g, h) = ghf^{-1}$ to obtain a fixed point $h$ of $\Psi(g, \cdot)$ for $g$ sufficiently $C^1$ close to $f$, and $h$ is sufficiently $C^\alpha$ and $df_1$-Lipschitz close to the identity. As in \[8, 9\], the fact that $h$ is $df_1$-Lipschitz close to the identity implies $h$ is a homeomorphism.
Most arguments concerning $C^0$ estimates in this paper are borrowed from [8] [9] except for a few changes of details. But to introduce notations in order to perform the $C^\alpha$ estimates, it seems necessary to repeat some of them.

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2. Extensions of the splitting

In this section we prove that the splitting $TM|_\Omega = E^u \oplus E^s$ can be extended to a neighborhood of each component of $\Omega$ and satisfies certain compatibility condition. This is motivated by [8, Theorem 8.4, C] and [9, Theorem 3.1, 5.1].

We first collect some standard facts that are used in the proof of Theorem 2.1 below. Most of them can be found in [11] [8] [10]. Let $f$ be a diffeomorphism of a compact manifold $M$ satisfying Axiom A and the strong transversality condition. Let $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ be the spectral decomposition of the nonwandering set $\Omega$ of $f$. Each $\Omega_i$ is a closed topological transitive hyperbolic $f$-invariant subset of $M$, and $E^u_i, E^s_i$ have constant ranks on $\Omega_i$. The components $\Omega_i$ can be ordered in such a way that $i < j$ implies $W^s(\Omega_i) \cap W^u(\Omega_j) = \emptyset$, where $W^\sigma(\Omega_i) = \bigcup_{x \in \Omega_i} W^\sigma(x)$, $\sigma = u, s$. For a subset $U$ of $M$, denote $O(U) = \bigcup_{n=-\infty}^{+\infty} f^n(U)$, $O^+(U) = \bigcup_{n=0}^{+\infty} f^n(U)$, $O^-(U) = \bigcup_{n=\infty}^{-\infty} f^{-n}(U)$. Then for $\Omega_i, \Omega_j$ such that $W^s(\Omega_i) \cap W^u(\Omega_j) = \emptyset$ and sufficiently small neighborhoods $U_i, U_j$ of $\Omega_i$ and $\Omega_j$, $O^-(U_i) \cap O^+(U_j) = \emptyset$. A subset $U$ of $M$ is called unrevisited if $x \in U$, $n > 0$, $f^n(x) \in U$ imply $f^m(x) \in U$ for $0 < m < n$. Then each $\Omega_i$ has arbitrarily small unrevisited open neighborhood.

We fix an $\Omega_i$. For $x \in \Omega_i$ and $\delta > 0$, let $W^\sigma_i(x)$ and $W^\sigma_i(x)$ be the local unstable and stable manifolds of size $\delta$ at $x$. Let $W^\sigma_i(\Omega_i) = \bigcup_{x \in \Omega_i} W^\sigma_i(x)$, $\sigma = u, s$. For $\delta$ sufficiently small, $W^\sigma_i(\Omega_i)$ has arbitrarily small unrevisited open neighborhood. Let $D = W^\sigma_i(\Omega_i) \setminus f(W^\sigma_i(\Omega_i))$. Then for $\delta$ sufficiently small, $D$ has arbitrarily small unrevisited open neighborhood, and for any open neighborhood $Q$ of $D$, the set $W^\sigma_i(\Omega_i) \cup O^+(Q)$ is an unrevisited open neighborhood of $\Omega_i$.

As in [8,9], we introduce the metric $d_f$ on $M$ by $d_f(x,y) = \sup_{n \in \mathbb{Z}} d(f^n(x), f^n(y))$, where $d$ is the metric induced from some Riemannian metric on $M$.

**Theorem 2.1.** Let $f$ be a $C^2$ diffeomorphism of $M$ satisfying Axiom A and the strong transversality condition, $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ be the spectral decomposition, and the components $\Omega_i$ are ordered as above. Let $\lambda \in (0,1)$ be as in (1.1), $l = \max\{\text{Lip}(f), \text{Lip}(f^{-1})\}$. Suppose $\alpha \in (0, 1]$ satisfies $\lambda^2 l^\alpha < 1$. Then for any $\lambda' \in (\lambda,1)$, there exist for each $1 \leq i \leq k$ an open neighborhood $U_i$ of $\Omega_i$ and two $f$-invariant continuous subbundles $E^\sigma_i$ of $TM|_{O(U_i)}$, $\sigma = u, s$, such that

(i) $TM|_{O(U_i)} = E^u_i \oplus E^s_i$;
(ii) $E^\sigma_i$ is $C^\alpha$ and $d_f$-Lipschitz on $\bigcup_{n=-N}^{N} f^n(U_i)$ for every $N > 0$;
(iii) $\|T f^{-1}|_{E^\sigma_i}\| \leq \lambda'$, $\|T f|_{(E^\sigma_i)^*}\| \leq \lambda'$ for $x \in U_i$;
(iv) for $i < j$, $O^-(U_i) \cap O^+(U_j) = \emptyset$, and $(E^\sigma_i)_x \subset (E^\sigma_j)_x$, $(E^\sigma_j)_x \subset (E^\sigma_i)_x$ for every $x \in O^+(U_i) \cap O^-(U_j)$.

**Proof.** We extend the definition of the bundles $E^u_i$ and $E^s_i$ on $\Omega$ as $E^u = \{v \in TM : \lim_{n \to +\infty} |T f^n(v)| = 0\}$, $E^s = \{v \in TM : \lim_{n \to -\infty} |T f^n(v)| = 0\}$, and denote $E^u_x = E^u \cap T_x M$, $\sigma = u, s$. By the strong transversality condition, $T_x M = E^u_x + E^s_x$ for every $x \in M$. For each $\Omega_i$, $E^\sigma_i|_{W^\sigma_i(\Omega_i)}$ is a continuous subbundle of $TM|_{W^\sigma_i(\Omega_i)}$ with constant rank.

As in [8 Section 10], to prove Theorem 2.1, it is sufficient to prove that under the conditions of Theorem 2.1, there exist for each $i$ an open neighborhood $U_i$ of $\Omega_i$ and a $f$-invariant continuous subbundle $E^\sigma_i$ of $TM|_{O(U_i)}$ such that
(i) $E^u_0|_{Q_1} = E^n|_{Q_1}$;
(ii) $E^u_0$ is $C^\alpha$ and $d_f$-Lipschitz on $\bigcup_{n=1}^{N} f^n(U_i)$ for every $N > 0$;
(iii') for $i < j$, $O^+(U_i) \cap O^+(U_j) = \emptyset$, and $(E^u_0)_x \subset (E^n)_x$ for every $x \in O^+(U_i) \cap O^+(U_j)$;
(iv') $T_x M = (E^u_0)_x + E^s_0$ for every $x \in O(U_i)$.

We prove this by induction on $i = 1, \cdots, k$. Let $1 \leq i \leq k$. Suppose that for $j < i$, $U_j$ and $E^u_j$ have been defined and satisfy (i')-(iv') (for $i = 1$ nothing is defined). We construct $U_i$ and $E^u_i$ satisfying (i')-(iv')

Let $\lambda < \lambda_1 < \lambda_2 < \lambda_3 < 1$ be such that $\lambda^2 s^{n^2} < 1$. Let $V_1$ be an open neighborhood of $\Omega_i$ such that $O^+(V_1) \cap O^-(U_j) = \emptyset$ for all $j < i$ (shrinking $U_j$, $j < i$ if necessary). Choose continuous subbundles $\tilde{E}^u, \tilde{E}^s$ of $TM|_{V_1}$ with $\tilde{E}^s|_{V_1 \cap W^s(\Omega_i)} = E^s|_{V_1 \cap W^s(\Omega_i)}$, $\sigma = u, s$. Since $TM|_{\Omega_i} = E^u|_{\Omega_i} \oplus E^s|_{\Omega_i}$, shrinking $V_1$ if necessary, we may assume that $TM|_{V_1} = \tilde{E}^u \oplus \tilde{E}^s$. Write $Tf|_{V_1 \cap f^{-1}(V_1)}$ as

$$
T_x f = \left( \begin{array}{cc} \tilde{F}_{ux} & \tilde{F}_{xs} \\ \tilde{F}_{us} & \tilde{F}_{ss} \end{array} \right)
$$

with respect to the splitting $TM|_{V_1} = \tilde{E}^u \oplus \tilde{E}^s$, $x \in V_1 \cap f^{-1}(V_1)$. Since $\|(\tilde{F}_{ux})\|_1 \leq \lambda$, $\|(\tilde{F}_{xs})\| \leq \lambda$, $\|(\tilde{F}_{us})\| = 0$ for $x \in \Omega_i$, by making $V_1$ smaller, we may assume that $\|(\tilde{F}_{ux})\|_1 \leq \lambda_1$, $\|(\tilde{F}_{xs})\| \leq \lambda_1$ for $x \in V_1 \cap f^{-1}(V_1)$. Note that since $\tilde{E}^s|_{V_1 \cap W^s(\Omega_i)} = E^s|_{V_1 \cap W^s(\Omega_i)}$ and $Tf(\tilde{E}^s) = \tilde{E}^s$, $\tilde{E}^s|_{V_1 \cap W^s(\Omega_i)} = 0$.

Choose $\delta > 0$ such that $W^s_\delta(\Omega_i) \subset V_1$, and such that $W^s_\delta(\Omega_i)$ has arbitrarily small unvisited open neighborhood. Let $D = \tilde{W}^s_\delta(\Omega_i) \setminus f(W^s_\delta(\Omega_i))$. Similar to the arguments in [3] page 488–491, we can prove (after possibly shrinking of $U_j$, $j < i$ in the induction hypothesis) that there exist an open neighborhood $Q_1 \subset V_1$ of $D$ and a $C^\alpha$ and $d_f$-Lipschitz subbundle $E^u_0$ of $TM|_{Q_1}$ such that

(1) $Tf|_{E^u_0(Q_1)} = (E^u_0)|_{f^{-1}(Q_1)}$;

(2) $T_x M = (E^u_0)_x + E^s_x$ and $T_x M = (E^u_0)_x + E^s_x$ for $x \in Q_1$;

(3) $(E^u_0)_x \subset (\tilde{E}^u)_x$ if $j < i$ and $x \in Q_1 \cap O^+(U_j)$.

We may also assume that $Q_1 \cap f^2(Q_1) = \emptyset$.

Since $TM|_{Q_1} = E^u_0 \oplus E^s|_{Q_1}$, there exists a continuous vector bundle morphism $\tilde{\tau}_0 : \tilde{E}^u|_{Q_1} \to \tilde{E}^s|_{Q_1}$, such that $E^u_0 = \text{Im}(id, \tilde{\tau}_0)$. By making $Q_1$ smaller, we may assume that $\|\tilde{\tau}_0\|$ is bounded, say $\|\tilde{\tau}_0\| \leq \tilde{\tau}$ for some $r \geq 1$.

Choose $\varepsilon > 0$ such that $r \varepsilon \leq \lambda_2^{1} - \lambda_3^{1}$. Since $\tilde{F}_{x}^{su}|_{W^s_\delta(\Omega_i)} = 0$, we may choose an unvisited open neighborhood $V_2 \subset V_1$ of $W^s_\delta(\Omega_i)$ such that $\|\tilde{F}_{x}^{su}\| \leq \tilde{\tau}$ for $x \in V_2$. Let $Q_2 \subset Q_1 \cap V_2$ be an unvisited open neighborhood of $D$. Choose $C^1$ approximations $\tilde{E}^u, \tilde{E}^s$ of $E^u|_{V_2}, E^s|_{V_2}$ such that

(1) $TM|_{V_2} = E^u \oplus E^s$, and if

$$
T_x f = \left( \begin{array}{cc} F_{ux} & F_{xs} \\ F_{us} & F_{ss} \end{array} \right)
$$

with respect to this splitting, then $\|F_{x}^{su}\| \leq \varepsilon$, $\|(F_{x}^{uu})^{-1}\| \leq \lambda_2$, $\|F_{x}^{ss}\| \leq \lambda_2$ for $x \in V_2 \cap f^{-1}(V_2)$;

(2) $TM|_{Q_2} = E^u_0|_{Q_2} \oplus E^s|_{Q_2}$, and if $\tau_0 : \tilde{E}^u|_{Q_2} \to \tilde{E}^s|_{Q_2}$ is the vector bundle morphism such that $E^u_0|_{Q_2} = \text{Im}(id, \tau_0)$, then $\|\tau_0\| \leq \varepsilon$;

(3) there exists a continuous vector bundle morphism $\tau_0 : \tilde{E}^u|_{V_2 \cap W^s(\Omega_i)} \to \tilde{E}^s|_{V_2 \cap W^s(\Omega_i)}$ such that $E^u_0|_{V_2 \cap W^s(\Omega_i)} = \text{Im}(id, \tau_0)$, and $\|\tau_0\| \leq \varepsilon$.

Note that since $f$ is $C^2$ and the splitting $TM|_{V_2} = E^u \oplus E^s$ is $C^1$, $F^{s\sigma'}$ is $C^1$. 

4 JINPENG AN
where $\sigma, \sigma' = u, s$. Note also that since $E^u_\alpha$ is $C^\alpha$ and $d_f$-Lipschitz, $\tau_0$ is $C^\alpha$ and $d_f$-Lipschitz.

Consider the smooth vector bundle $\mathcal{L}$ over $V_2$ whose fiber $\mathcal{L}_x$ at $x \in V_2$ is the space $\mathcal{L}(\tilde{E}^u_x, \tilde{E}^s_x)$ of linear maps from $\tilde{E}^u_x$ to $\tilde{E}^s_x$. A section $\tau$ of $\mathcal{L}$ is a vector bundle morphism from $\tilde{E}^u$ to $\tilde{E}^s$ covering the identity. Let $\mathcal{L}(r)_x$ be the disc $\{g \in \mathcal{L}_x : ||g|| \leq r\}$ in $\mathcal{L}_x$, and $\mathcal{L}(r) = \bigcup_{x \in V_2} \mathcal{L}(r)_x$ be the disc bundle of $\mathcal{L}$. Let $x \in V_2 \cap f^{-1}(V_2)$. For $g \in \mathcal{L}(r)_x$, define

$$\varphi^\sigma_g = F^\sigma_x + F^s_x \sigma g \in \mathcal{L}(\tilde{E}^u_x, \tilde{E}^s_{f(x)}),$$

$\sigma = u, s$. Then for $v \in \tilde{E}^u_x$, we have

$$|\varphi^u_g(v)| = |F^u_x(v) + F^s_x\sigma g(v)| \geq |F^u_x(v)| - |F^s_x\sigma g(v)| \geq (\lambda_2^{-1} - r\varepsilon)|v|.$$

So $\varphi^u_g$ is invertible, and

$$|| (\varphi^u_g)^{-1} || \leq (\lambda_2^{-1} - r\varepsilon)^{-1} \leq \lambda_3.$$ 

we also have

$$|| \varphi^u_g || = || F^u_x || + || F^s_x \sigma || \leq || F^u_x || + r || F^s_x || \leq \lambda_2 r.$$ 

Thus if we define the graph transform of $g \in \mathcal{L}(r)_x$ by

$$\Gamma(g) = \varphi^s_g (\varphi^u_g)^{-1} \in \mathcal{L}(f(x),$$

then

$$(2.1) \quad || \Gamma(g) || \leq \lambda_3 \lambda_2 r \leq r.$$

Note that since $F^\sigma\sigma'$ is $C^1$, the map $\Gamma : \mathcal{L}(r)|_{V_2 \cap f^{-1}(V_2)} \rightarrow \mathcal{L}(r)|_{f(V_2) \cap V_2}$ is $C^1$.

Let $x \in V_2 \cap f^{-1}(V_2)$, $g_1, g_2 \in \mathcal{L}(r)_x$. Then

$$|| \varphi^u_{g_1} - \varphi^u_{g_2} || = || F^u_x (g_1 - g_2) || \leq \varepsilon || g_1 - g_2 ||,$$

$$|| \varphi^s_{g_1} - \varphi^s_{g_2} || = || F^s_x (g_1 - g_2) || \leq \lambda_2 || g_1 - g_2 ||.$$ 

Hence

$$\begin{align*}
|| \Gamma(g_1) - \Gamma(g_2) || \\
\leq || \varphi^s_{g_1} (\varphi^u_{g_1})^{-1} - \varphi^s_{g_2} (\varphi^u_{g_2})^{-1} || + || \varphi^s_{g_1} (\varphi^u_{g_2})^{-1} - \varphi^s_{g_2} (\varphi^u_{g_2})^{-1} || \\
\leq || \varphi^s_{g_1} || || (\varphi^u_{g_1})^{-1} || || \varphi^u_{g_1} || - || \varphi^u_{g_2} || || (\varphi^u_{g_2})^{-1} || + || (\varphi^u_{g_2})^{-1} || || \varphi^s_{g_2} - \varphi^s_{g_1} || \\
\leq \lambda_3 r || (\varphi^u_{g_1})^{-1} || || \varphi^u_{g_1} - \varphi^u_{g_2} || + \frac{1}{\lambda_2^{-1} - r\varepsilon} || \varphi^s_{g_1} - \varphi^s_{g_2} || \\
\leq \frac{\lambda_3 r}{(\lambda_2^{-1} - r\varepsilon)^2 || \varphi^u_{g_1} - \varphi^u_{g_2} || + \frac{1}{\lambda_2^{-1} - r\varepsilon} || \varphi^s_{g_1} - \varphi^s_{g_2} ||} \\
\leq \lambda_3 || g_1 - g_2 ||.
\end{align*}$$

(2.2)

For the convenience of the following discussion, we embed $M$ isometrically into some Euclidian space $\mathbb{R}^N$. Then for $x \in V_2, \tilde{E}^u_x$ and $\tilde{E}^s_x$ can be viewed as subspaces of $\mathbb{R}^N$, and we have the identification

$$\mathcal{L}_x = \mathcal{L}(\tilde{E}^u_x, \tilde{E}^s_x) \cong \{ g \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) | \tilde{E}^u_x \oplus T_2 M \subset \ker(g), \text{Im}(g) \subset \tilde{E}^s_x \}.$$

Then for $g_1, g_2 \in \mathcal{L}$ with different base points, the summation $g_1 + g_2$ and its norm $||g_1 + g_2||$ make sense, as they are viewed as elements in $\mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$. Let $\Gamma_x = \Gamma|_{\mathcal{L}_x}$ be the restriction of $\Gamma$ on the fiber $\mathcal{L}_x$. Since the map $\Gamma : \mathcal{L}(r)|_{V_2 \cap f^{-1}(V_2)} \rightarrow$
$L(r)|_{f(V_2) \cap V_2}$ is $C^1$, it is Lipschitz and $C^\alpha$, which means that there exists $C > 0$ such that
\begin{equation}
||\Gamma_x(g_1) - \Gamma_y(g_2)|| \leq C \min\{||g_1 - g_2|| + d(x,y), (||g_1 - g_2|| + d(x,y))^{\alpha}\}
\end{equation}
for any $x, y \in V_2 \cap f^{-1}(V_2)$ and $g_1 \in L(r)_x, g_2 \in L(r)_y$. Note that since $\Gamma$ covers $f$ and $f$ is Lipschitz, we have indeed omitted a term $d(f(x), f(y))$ in the left hand side of (2.3).

Recall that $D \cap W^u(\Omega_1) = \emptyset$. So there exist $d_0 > 0$ and an unvisited open neighborhood $Q_3 \subset Q_2$ of $D$ such that $d(Q_3, W^u(\Omega_1)) \geq d_0$, and such that $x \in Q_3, y \in M, d(x,y) < d_0$ imply $y \in Q_2$. Let $V_3 = V_2 \cap (W^u(\Omega_1) \cup O^+(Q_3))$. Since $V_2$ and $W^u(\Omega_1) \cup O^+(Q_3)$ are unvisited open neighborhoods of $\Omega$, so is $V_3$.

To simplify notations, we denote
\[ \rho_f(x,y) = \min\{d(x,y), d_f(x,y)\} \]
for $x, y \in M$. Then a section $\tau$ of $L$ is $C^\alpha$ and $d_f$-Lipschitz if and only if
\[ \sup_{x, y \in V_2, x \neq y} \frac{||\tau(x) - \tau(y)||}{\rho_f(x, y)} < +\infty. \]
Now we choose
\[ K \geq \max \left\{ \frac{2rdiam(M)^{1-\alpha}}{d_0}, \frac{2r}{d_0}, \frac{C\tau}{1-\lambda_2^\alpha}, \frac{C}{1-\lambda_3^\alpha} \right\} \]
such that
\[ ||\tau_0(x) - \tau_0(y)|| \leq K\rho_f(x,y) \]
for $x, y \in Q_2$, where $\text{diam}(M)$ is the diameter of $M$. Let
\[ \Sigma = \{\text{continuous sections } \tau \in L(r)|_{V_3} : ||\tau(x) - \tau(y)|| \leq K\rho_f(x,y), \tau|_{Q_3} = \tau_0|_{Q_3}\}. \]
$\Sigma$ is a closed subset of the Banach space of continuous bounded sections of $L|_{V_3}$. By taking a bump function on $M$ which is 1 in $Q_3$ and 0 outside $Q_2$ and enlarging $K$ if necessary, it is easy to see that $\Sigma$ is nonempty. Define the graph transform $F_2(\tau)$ of $\tau \in \Sigma$ as the section
\[ F_2(\tau)(x) = \begin{cases} \Gamma(\tau(f^{-1}(x))), & x \in f(V_3) \cap V_3; \\ \tau(x), & x \in V_3 \setminus f(V_3) \end{cases} \]
of $L|_{V_3}$. We prove that $F_2$ maps $\Sigma$ into $\Sigma$ and is a contraction on $\Sigma$.

First we show that $V_3 = (f(V_3) \cap V_3) \cup Q_3$. Let $x \in V_3$. Recall that $V_3 = V_2 \cap (W^u(\Omega_1) \cup O^+(Q_3))$. If $x \in V_2 \cap W^u(\Omega_1)$, then there exists $n \geq 1$ such that $f^{-n}(x) \in V_2$. Since $V_2$ is unvisited, $f^{-1}(x) \in V_2$. We also have $f^{-1}(x) \in W^u(\Omega_1)$.

Hence $x \in f(V_3) \cup V_3$. If $x \in V_2 \cap O^+(Q_3)$, there exists $y \in Q_3$ such that $x = f^n(y)$ for some $n \geq 0$. If $n = 0$ then $x \in Q_3$. If $n \geq 1$, since $V_2$ is unvisited, $f^{-n}(y) \in V_2$. Hence $x \in f(V_3) \cup V_3$. This proves $V_3 = (f(V_3) \cap V_3) \cup Q_3$.

Let $\tau \in \Sigma$. We show that $F_2(\tau)|_{Q_3} = \tau_0|_{Q_3}$. Let $x \in Q_3$. If $x \in V_3 \setminus f(V_3)$, then $F_2(\tau)(x) = \tau(x) = \tau_0(x)$. If $x \in Q_3 \setminus (V_3 \setminus f(V_3)) = Q_3 \cap f(V_3)$, then $F_2^n(x) \in Q_3$ for some $n \geq 1$. Since $Q_3$ is unvisited, $f^{-1}(x) \in Q_3$. So $F_2(\tau)(x) = \Gamma(\tau(f^{-1}(x))) = \Gamma(\tau_0(f^{-1}(x))) = \tau_0(x)$. So $F_2(\tau)|_{Q_3} = \tau_0|_{Q_3}$.

Now $F_2(\tau)$ is continuous on $Q_3$ and $f(V_3) \cap V_3$. Since $f(V_3) \cap V_3$ and $Q_3$ are open in $V_3$ and $V_3 = (f(V_3) \cap V_3) \cup Q_3$, $F_2(\tau)$ is continuous on $V_3$.

By (2.1), $||F_2(\tau)(x)|| \leq r$ for $x \in f(V_3) \cap V_3$. So $||F_2(\tau)|| \leq r$.

Now we show that $||F_2(\tau)(x) - F_2(\tau)(y)|| \leq K\rho_f(x,y)$ for $\tau \in \Sigma$ and $x, y \in V_3$.

There are three cases.

1. $x, y \in V_3 \setminus f(V_3)$. This is obvious since $F_2(\tau)|_{Q_3} = \tau_0|_{Q_3}$ and $V_3 \setminus f(V_3) \subset Q_3$. 


(2) $x \in V_3 \setminus f(V_3)$, $y \in f(V_3) \cap V_3$. If $d(x, y) \geq d_0$, then
\[
\|F_z(\gamma)(x) - F_z(\gamma)(y)\| \leq 2r \leq \frac{2r}{d_0} d(x, y)
\]
\[
\leq \frac{2r \max \{\text{diam}(M)^{1-\alpha}, 1\}}{d_0} \rho_f(x, y) \leq K \rho_f(x, y).
\]
Suppose $d(x, y) < d_0$. Since $x \in Q_3$, we have $y \in Q_2$ and $y \notin W^u(\Omega_2)$. So there exists $n \geq 1$ such that $f^{-n}(y) \in Q_3$. But $Q_2$ is unvisited and $Q_2 \cap f^2(Q_2) = \emptyset$. So we must have $n = 1$ and then $F_z(\gamma)(y) = \Gamma(\gamma(f^{-1}(y))) = \Gamma(\gamma_0(f^{-1}(y))) = \gamma_0(y)$. So
\[
\|F_z(\gamma)(x) - F_z(\gamma)(y)\| = \|\gamma_0(x) - \gamma_0(y)\| \leq K \rho_f(x, y).
\]
(3) $x, y \in f(V_3) \cap V_3$. By (2.2) and (2.3),
\[
\|F_z(\gamma)(x) - F_z(\gamma)(y)\|=\|\Gamma(f^{-1}(x)) - \Gamma(f^{-1}(y))\|
\leq \|\Gamma(f^{-1}(x)) - \Gamma(f^{-1}(y))\|
\leq \|\Gamma(f^{-1}(x)) - \Gamma(f^{-1}(y))\| + C d(f^{-1}(x), f^{-1}(y))
\leq \lambda_d^3 K d(f^{-1}(x), f^{-1}(y)) + C d(f^{-1}(x), f^{-1}(y))
\leq (\lambda_d^3 K + C) d_f(x, y)
\leq K d_f(x, y)
\]
Similarly,
\[
\|F_z(\gamma)(x) - F_z(\gamma)(y)\|
\leq \lambda_d^3 K d_f(f^{-1}(x), f^{-1}(y)) + C d(f^{-1}(x), f^{-1}(y))
\leq (\lambda_d^3 K + C) d_f(x, y)
\leq K d_f(x, y)
\]
So $\|F_z(\gamma)(x) - F_z(\gamma)(y)\| \leq K \rho_f(x, y)$. This proves that $F_z$ maps $\Sigma$ into $\Sigma$. By (2.2), $F_z$ is a contraction on $\Sigma$. So there is a fixed point $\bar{\gamma}$ of $F_z$ in $\Sigma$.

Choose $\delta' > 0$ such that $W^u_{\delta'}(\Omega_2) \subset f(V_3) \cap V_3$. We prove that $\bar{\gamma}|_{W^u_{\delta'}(\Omega_2)} = \tau_0|_{W^u_{\delta'}(\Omega_2)}$. Let $x_0 \in W^u_{\delta'}(\Omega_2)$ be such that $\bar{\gamma}|_{W^u_{\delta'}(\Omega_2)}(x_0) - \tau_0|_{W^u_{\delta'}(\Omega_2)}(x_0)$ assumes maximal value at $x_0$. Then
\[
\|\bar{\gamma}(x_0) - \tau_0(x_0)\| = \|\Gamma(f^{-1}(x_0)) - \Gamma(\tau_0(f^{-1}(x_0)))\|
\leq \lambda_d^3 \|f^{-1}(x_0) - \tau_0(f^{-1}(x_0))\|
\leq \lambda_d^3 \|\bar{\gamma}(x_0) - \tau_0(x_0)\|.
\]
Hence $1 - \lambda_d^3 \|\bar{\gamma}(x_0) - \tau_0(x_0)\| \leq 0$, which implies that $\|\bar{\gamma}(x_0) - \tau_0(x_0)\| = 0$.

Define the $C^\alpha$ and $d_f$-Lipschitz subbundle $E^\alpha_{f_1}$ of $TM|_{f_1}(V_3)$ by $E^\alpha_{f_1} = \text{Im}(id, \bar{\gamma})$. Then $E^\alpha_{f_1}|_{V_3} = E^\alpha_{f_1}|_{V_3}$, $E^\alpha_{f_1}|_{W^u_{\delta'}(\Omega_2)} = E^u|_{W^u_{\delta'}(\Omega_2)}$, and $T f((E^\alpha_{f_1})_x) = (E^n_{f_1})_x$ if $x \in V_3 \cap f^{-1}(V_3)$.

Now consider the $C^\alpha$ and $d_f$-Lipschitz subbundle $T f^m((E^\alpha_{f_1})_x)$ of $TM|_{f^m(V_3)}$, $n \in \mathbb{Z}$. If for $n, m \in \mathbb{Z}$, $n < m$, $f^m(V_3) \cap f^m(V_3) \neq \emptyset$, then for $x \in f^m(V_3) \cap f^m(V_3)$, $f^{-n}(x) \in V_3$, $f^{-m}(x) \in V_3$. Since $V_3$ is unvisited, $f^{-n}(x) \in V_3$ for $n \leq p \leq m$. So for $n+1 \leq p \leq m$, $f^{-p}(x) \in V_3 \cap f^{-1}(V_3)$ and then $T f^p((E^\alpha_{f_1})_x) = T f^{p-1}(T f((E^\alpha_{f_1})_x)) = T f^{p-1}((E^\alpha_{f_1})_{f^{-p+1}(x)}) = T f^{p-1}(E^\alpha_{f_1})_x$. So $T f^n((E^\alpha_{f_1})_x) = T f^m((E^\alpha_{f_1})_x)$, and then the
bundles $Tf^n(E^n_{i2})$ ($n \in \mathbb{Z}$) patch together to a $Tf$-invariant subbundle $E^n_{i2}$ of $TM|_{\Omega_i}$. It is obviously continuous since $f^n(V_3)$ is open.

We have $Tf_i M = (E^n_{i2})_x + E^n_1$ for $x \in \mathcal{O}(V_3)$, as this holds for $x \in W^c_\alpha(\Omega_i) \cup Q_3$, $E^n_{i2}$ and $E^n_1$ are $Tf$-invariant, and $\mathcal{O}(V_3) = \mathcal{O}(W^n_\alpha(\Omega_i)) \cup \mathcal{O}(Q_3)$.

We prove that $(E^n_{i2})_x \subset (E^n_1)_x$ for every $j < i$ and $x \in \mathcal{O}^{-}(V_3) \cap \mathcal{O}^{+}(U_j)$. Let $x \in \mathcal{O}^{-}(V_3) \cap \mathcal{O}^{+}(U_j)$. Then $x \in \mathcal{O}(\Omega_i)$ and then $x \in \mathcal{O}(Q_3)$. Since $(E^n_{i2})_x \subset (E^n_1)_x$ for $x \in \mathcal{O}(Q_3) \cap \mathcal{O}^{+}(U_j)$, it also holds for $x \in \mathcal{O}(Q_3) \cap \mathcal{O}^{+}(U_j)$ by the $Tf$-invariance of $E^n_{i2}$ and $E^n_1$.

Finally, let $U_i$ be an open neighborhood of $\Omega_i$ such that $U_i \subset V_3$. Let $N > 0$. We prove that $E^n_{i2}$ is $C^\alpha$ and $d_f$-Lipschitz on $\bigcup_{n=-N}^N f^n(U_i)$. Consider the Grassmanian bundle $\mathcal{G}$ over $M$ consisting of all rank$(E^n_{i2})$-dimensional subspaces of the tangent spaces of $M$. Then $E^n_{i2}$ can be viewed as a $Tf$-invariant continuous section $s$ of $\mathcal{G}|_{\mathcal{O}(U_j)}$ which is $C^\alpha$ and $d_f$-Lipschitz on each $f^n(U_i)$. Embed the compact manifold $\mathcal{G}$ into some $\mathbb{R}^N$. Then $s$ can be viewed as a bounded map $s : \mathcal{O}(V_3) \to \mathbb{R}^N$. Since $\overline{f^n(U_i)} \subset f^n(V_3)$ for all $n$, there exists $d_1 > 0$ such that for $-N \leq n \leq N$, $x \in f^n(U_i), y \in M, d(x, y) < d_1$ imply that $y \in f^n(V_3)$. Let $K' > 0$ be such that $|s(x) - s(y)| \leq K'|f|_\infty(x, y)$ for $x, y \in f^n(V_3), -N \leq n \leq N$. We prove that $s$ is $C^\alpha$ and $d_f$-Lipschitz on $\bigcup_{n=-N}^N f^n(U_i)$. Let $x, y \in \bigcup_{n=-N}^N f^n(U_i)$. Since $s$ is bounded, we may assume that $d(x, y) < d_1$. Suppose $x \in f^n(U_i)$. Then $y \in f^n(V_3)$. Hence $|s(x) - s(y)| \leq K'|f|_\infty(x, y)$. So the neighborhood $U_i$ of $\Omega_i$ and the bundle $E^n_1 = E^n_{i2}|_{\mathcal{O}(U_i)}$ satisfy the conditions (i')–(iv'). The proof of Theorem 2.1 is finished.

3. Existence of right inverses

Let $f$ be a $C^2$ diffeomorphism of a compact manifold $M$, $\alpha \in (0, 1)$. Let $\mathfrak{X}^0(M)$ denote the Banach space of continuous vector fields on $M$ with the $C^0$ norm $\| \cdot \|$, and let $\mathfrak{X}^\alpha_f(M)$ be the subspace of $\mathfrak{X}^0(M)$ consisting of $C^\alpha$ and $d_f$-Lipschitz vector fields. As in the previous section, suppose $M$ is isometrically embedded into some Euclidian space $\mathbb{R}^N$. For $\eta \in \mathfrak{X}^\alpha_f(M)$, denote

$$L_\alpha(\eta) = \sup_{x, y \in M, x \neq y} \frac{|\eta(x) - \eta(y)|}{d(x, y)'^\alpha},$$

$$L_f(\eta) = \sup_{x, y \in M, x \neq y} \frac{|\eta(x) - \eta(y)|}{d_f(x, y)}.$$

Then $\mathfrak{X}^\alpha_f(M)$, being endowed with the norm

$$\|\eta\|_{\alpha, f} = \max\{\|\eta\|, L_\alpha(\eta), L_f(\eta)\},$$

is a Banach space. For $\eta \in \mathfrak{X}^0(M)$, define the vector field $f_\eta(\eta)$ on $M$ by

$$f_\eta(\eta)(x) = Tf(\eta(f^{-1}(x))).$$

Then $f_\eta(\eta)$ is in $\mathfrak{X}^0(M)$, and in $\mathfrak{X}^\alpha_f(M)$ if $\eta \in \mathfrak{X}^\alpha_f(M)$.

The following theorem is motivated by [8, Theorem B] and [9, Section 8].

**Theorem 3.1.** Let $f$ be a $C^2$ diffeomorphism of $M$ satisfying Axiom A and the strong transversality condition, $\lambda, l$ be as in Theorem 2.1. Suppose $\alpha \in (0, 1)$ satisfies $\lambda'^\alpha < 1$. Then there exists a continuous linear operator $J$ on $\mathfrak{X}^0(M)$ such that

(i) $J$ is a right inverse of $1 - f_\eta$;
(ii) $J$ maps $\mathfrak{X}^\alpha_f(M)$ into $\mathfrak{X}^\alpha_f(M)$ and restricts to a continuous linear operator on $\mathfrak{X}^\alpha_f(M)$ with respect to the norm $\| \cdot \|_{\alpha, f}$. 

Proof. Choose \( \lambda < X < \rho = \kappa X < 1 \) such that \( \rho^\alpha < 1 \). Let \( \Omega = \Omega_1 \cup \cdots \cup \Omega_k \) be the spectral decomposition ordered as in Theorem 2.1. Let \( U_i \) be an open neighborhood of \( \Omega_i \), \( E_i^\sigma \) be two \( T^f \)-invariant subbundles of \( TM|_{O(U_i)} \) satisfying (i)–(iv) in Theorem 2.1 for the above \( \lambda, \sigma = u, s \). It is well known that \( \bigcup_{i=1}^k O(U_i) = M \). So there exists \( N > 0 \) such that \( \{ \bigcup_{n=-N}^N f^n(U_1), \cdots, \bigcup_{n=-N}^N f^n(U_k) \} \) is a cover of \( M \). Shrinking \( U_i \) if necessary, we may assume that they are unrevisited. Then it is easy to see that for every \( x \in M \), the set \( \{ n \in \mathbb{Z} : f^n(x) \notin \bigcup_{i=1}^k U_i \} \) contains at most \( n_0 = 2kN \) elements. Let \( \theta_1, \cdots, \theta_k \) be a smooth partition of unity subordinate to the above cover. For \( \eta \in \mathcal{X}^0(M) \), let \( \eta_{\sigma} = P_{E_\sigma}^{\theta}(\theta \eta) \), where \( P_{E_\sigma}^{\theta} \) (resp. \( E_\sigma^\theta \)) is the projection of \( TM|_{O(U_i)} \) onto \( E_i^\sigma \) (resp. \( E_i^\sigma \)) along \( E_i^\sigma \) (resp. \( E_i^\sigma \)), and define \( J_\sigma(\eta) = \sum_{n=0}^{+\infty} f_n^\sigma(\eta_{\sigma}) \), \( J_{iu}(\eta) = -\sum_{n=1}^{+\infty} f_n^\alpha(\eta_{iu}) \), \( J_i(\eta) = \sum_{\sigma=us} N_{x=i} J_{\sigma}(\eta) \). Robbin [8] proved that these series converge uniformly, and then \( J \) is a continuous right inverse of \( 1-f_i \). We prove in the following that \( J \) maps \( \mathcal{X}^0(M) \) into \( \mathcal{X}^0(M) \) and restricts to a continuous linear operator on \( \mathcal{X}^0(M) \). As in [8], it is sufficient to prove this property for each \( J_{iu} \).

Let \( \eta \in \mathcal{X}^0(M) \). Fix \( i = 1, \cdots, k \), and denote \( \zeta = \eta_{iu} = P_{E_i^u}(\theta \eta) \). Then \( \text{supp}(\zeta) \subset \bigcup_{n=-N}^N f^n(U_1) \) and \( \zeta(x) \in (E_i^u)_x \) for \( x \in \bigcup_{n=-N}^N f^n(U_1) \). Since \( E_i^u \) are \( C^\alpha \) and \( d_{\sigma}-\text{Lipschitz on} \bigcup_{n=-N}^N f^n(U_1), \zeta \in \mathcal{X}^0(M) \). Let \( K = (\|Tf\|_\rho)^{-2\alpha} \). It is proved in [8] Section 6] that

\[
|f_n^\sigma(\zeta)(x)| \leq \left( \frac{\|Tf\|}{\rho} \right)^{n_0+\kappa} \rho^\alpha |\zeta(f^{-n}(x))| \leq K \rho^\alpha |\zeta(f^{-n}(x))|
\]

for all \( x \in M \) and \( n \geq 0 \) (note that we always have \( \|Tf\| > \rho \)). Hence

\[
\|f_n^\sigma(\zeta)\| \leq K \rho^\alpha \|\zeta\|
\]

for all \( n \geq 0 \). Let

\[
C = \|Tf\| \max\{L_\alpha(P_{E_i^u}|_{U_i}) : 1 \leq j \leq k\} + L_\alpha(Tf),
\]

where \( L_\alpha(Tf) \) is the Hölder constant of \( Tf \) as a map \( x \mapsto T_xf \) for \( x \in M \), \( L_\alpha(P_{E_i^u}|_{U_i}) \) is the Hölder constant of \( P_{E_i^u}|_{U_j} \) as a map \( x \mapsto P_{E_i^u}|_{U_j} \) for \( x \in U_j \). We prove that

\[
L_\alpha(f_n^\sigma(\zeta)) \leq K (\rho^\alpha)^n L_\alpha(\zeta) + C'((\rho^\alpha)^n - \rho^n) \|\zeta\|
\]

for all \( n \geq 0 \), where \( C' = \frac{C K^2 \rho^\alpha}{\rho^\alpha - \frac{1}{2}} \).

We first prove some inequalities on individual tangent vectors. Let \( p, q \in M \), \( v_p \in T_pM, v_q \in T_qM \). Then

\[
\|T_p f(v_p) - T_q f(v_q)\| \leq |T_p f(v_p - v_q)| + |T_p f - T_q f|(v_q) \leq \|Tf\| |v_p - v_q| + L_\alpha(Tf) |v_q| d(p, q)^\alpha \leq \|Tf\| |v_p - v_q| + C |v_q| d(p, q)^\alpha.
\]

Recall that a smooth adapted Riemannian metric on \( M \) can be obtained by approximating a \( C^0 \) adapted metric for which the bundles \( E^u \) and \( E^s \) are mutually orthogonal on \( \Omega \). So after choosing a better approximation of the \( C^0 \) metric and shrinking the \( U_j \)'s, we may assume that for each \( j \), \( \|P(E_j^s)\| \leq \kappa \) for every \( p \in U_j \), where \( \kappa > 1 \) is as in the beginning of the proof. So for \( p, q, v_p, v_q \) as above, if
moreover we have $p, q \in U_j$ for some $j$, and $v_p \in (E^+_q)_p, v_q \in (E^+_q)_q$, then

$$|T_p f(v_p) - T_q f(v_q)|$$

(3.5) \[ \leq |T_p f(P^1_{E^+_q}(v_p - v_q)) + |T_p f(P^1_{E^+_q} - P^1_{E^+_q}(v_p)) + |(T_p f - T_q f)(v_q)| \]

\[ \leq \lambda|v_p - v_q| + (|T f| L_\alpha (P^1_{E^+_q}(U_j) \cup L_\alpha (T f))|v_q|d(p, q)^\alpha \]

\[ \leq \rho|v_p - v_q| + C|v_q|d(p, q)^\alpha. \]

Now we prove (3.3). Let $x, y \in M, n \geq 0$. If one of $f^{-n}(x)$ and $f^{-n}(y)$ does not belong to $\bigcup_{n=-N}^N f^n(U_i)$, say $f^{-n}(x) \notin \bigcup_{n=-N}^N f^n(U_i)$, then by (3.1), we have

$$|f^n(x) - f^n(y)|$$

(3.5) \[ = |f^n(x) - f^n(y)| \leq K\rho^n|\zeta(f^n(x)) - \zeta(f^n(y))| \leq K(\rho^n)^n L_\alpha (\zeta) d(x, y)^\alpha. \]

So (3.3) holds in this case. Suppose $f^{-n}(x), f^{-n}(y) \in \bigcup_{n=-N}^N f^n(U_i)$. Let $1 \leq m \leq n$. Then by letting $p = f^{-m}(x), q = f^{-m}(y), v_p = f^{-m}(\zeta(f^{-m}(x))), v_q = f^{-m}(\zeta(f^{-m}(y)))$ in (3.3) and using (3.2), we get

$$|f^n f^{n+1}(\zeta(f^{-m+1}(x)) - f^n f^{n+1}(\zeta(f^{-m+1}(y)))|$$

(3.6) \[ \leq |T f||f^n f^{n-1}(\zeta(f^{-m}(x)) - f^n f^{n+1}(\zeta(f^{-m}(y)))| + C||f^n f^{n-1}(\zeta)||d(f^{-m}(x), f^{-m}(y))\]

\[ \leq |T f||f^n f^{n-1}(\zeta(f^{-m}(x)) - f^n f^{n-1}(\zeta(f^{-m}(y)))| + C K\rho^{n-m}\alpha ||\zeta||d(x, y)^\alpha. \]

If moreover $f^{-m}(x), f^{-m}(y) \in U_j$ for some $j \geq i$, then $f^n f^{n-1}(\zeta(f^{-m}(x)) \in (E^+_q) f^{-m}(x), f^n f^{n-1}(\zeta(f^{-m}(y)) \in (E^+_q) f^{-m}(y)$. By (3.6) and (3.2), we have

$$|f^n f^{n+1}(\zeta(f^{-m+1}(x)) - f^n f^{n+1}(\zeta(f^{-m+1}(y)))|$$

(3.7) \[ \leq \rho|f^n f^{n-1}(\zeta(f^{-m}(x)) - f^n f^{n-1}(\zeta(f^{-m}(y)))| + C||f^n f^{n-1}(\zeta)||d(f^{-m}(x), f^{-m}(y))\]

\[ \leq \rho|f^n f^{n-1}(\zeta(f^{-m}(x)) - f^n f^{n-1}(\zeta(f^{-m}(y)))| + C K\rho^{n-m}\alpha ||\zeta||d(x, y)^\alpha. \]

For $1 \leq m \leq n$, denote

$$\nu_m = \left\{ \begin{array}{ll} \rho, & \text{if } f^{-m}(x), f^{-m}(y) \in U_j \text{ for some } j \geq i; \\ \|T f\|, & \text{otherwise.} \end{array} \right. $$

Then by (3.6) and (3.7), we have

$$|f^n f^{n+1}(\zeta(f^{-m+1}(x)) - f^n f^{n+1}(\zeta(f^{-m+1}(y)))|$$

(3.8) \[ \leq \nu_m|f^n f^{n-1}(\zeta(f^{-m}(x)) - f^n f^{n-1}(\zeta(f^{-m}(y)))| + C K\rho^{n-m}\alpha ||\zeta||d(x, y)^\alpha. \]

Since we have supposed that $f^{-n}(x), f^{-n}(y) \in \bigcup_{n=-N}^N f^n(U_i)$, we have $f^{-(n-N)}(x), f^{-(n-N)}(y) \in O^+(U_i)$. But $O^+(U_i) \cap O^-(U_j) = \emptyset$ for $j < i$. So each of the sets
\[ 1 \leq m \leq n - N: f^{-m}(x) \not\in \bigcup_{j=1}^{k} U_j \text{ and } 1 \leq m \leq n - N: f^{-m}(y) \not\in \bigcup_{j=1}^{n} U_j \]

consists of at most \( n_0 \) elements. Then for all but at most \( 2n_0 + N \) integers \( m \) in \( \{1, \cdots, n\} \), \( f^{-m}(x), f^{-m}(y) \in U_j \) for some \( j \geq i \), that is, at most \( 2n_0 + N \) numbers \( \nu_m(1 \leq m \leq n) \) equal to \( \|Tf\| \). So we have \( \nu_1 \nu_2 \cdots \nu_m \leq \left( \frac{\|Tf\|}{\rho} \right)^{2n_0+N} \rho^m = K \rho^m \). Then by (3.8), we get

\[ \|f^n_\ast(\zeta)(x) - f^n_\ast(\zeta)(y)\|
\leq \nu_1 \nu_2 \cdots \nu_n \|f^{-n}(x) - \zeta(f^{-n}(y))\|
+ CK(\rho^{-1}l^\alpha + \nu_1 \rho^{-2}l^{2\alpha} + \nu_1 \nu_2 \rho^{-3}l^{3\alpha}
+ \cdots + \nu_1 \nu_2 \cdots \nu_n \nu_1^{n\alpha})\|d(x,y)\|^\alpha
\leq K\rho^n |\zeta(f^{-n}(x)) - \zeta(f^{-n}(y))|
+ CK^2 \rho^{-1}(l^\alpha + l^{2\alpha} + \cdots + l^{n\alpha})\|d(x,y)\|^\alpha
\leq K(\rho l^n) \alpha L_\alpha(\zeta)d(x,y)^n + \frac{CK^2 l^n}{\rho(l^n - 1)}((\rho l^n)^n - \rho^n)|\zeta|d(x,y)^\alpha. \]

This proves (3.3).

Recall that \( \zeta = \eta_{is} \). By (3.3), we have

\[ (3.9) \quad L_\alpha(J_{is}(\eta)) \leq \sum_{n=0}^{\infty} L_\alpha(f^n_\ast(\eta_{is})) \leq \frac{K}{1 - \rho l^n} L_\alpha(\eta_{is}) + \left( \frac{C'}{1 - \rho l^n} - \frac{C'}{1 - \rho} \right)\|\eta_{is}\|. \]

Similarly, we can prove that

\[ (3.10) \quad L_f(J_{is}(\eta)) \leq AL_f(\eta_{is}) + B\|\eta_{is}\| \]

for some constant \( A, B > 0 \) (see [8, Section 6]). Since the bundles \( E_i^s \) and \( E_i^u \) are \( C^\alpha \) and \( d_f \)-Lipschitz on \( \bigcup_{n=N}^{\infty} J_{is}(U_i) \), the operator on \( \mathcal{X}_f^s(M) \) which maps \( \eta \) to \( \eta_{is} \) is continuous. So by (3.2), (3.9) and (3.10), the operator \( J_{is} \) maps \( \mathcal{X}_f^s(M) \) into \( \mathcal{X}_f^s(M) \) and is continuous on \( \mathcal{X}_f^s(M) \). This proves the theorem. \( \square \)

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. As indicated in the introduction, Theorem 1.1 follows from Theorem 1.2.

We first extract some analytical arguments in [8, 9] to the following lemma, which can be viewed as a generalization of the usual Implicit Function Theorem for Banach spaces.

**Lemma 4.1.** Let \( (X, \|\cdot\|) \) be a Banach space, \( X' \) be a linear subspace of \( X \) with a complete norm \( \|\cdot\|' \) such that the inclusion \( (X', \|\cdot\|') \hookrightarrow (X, \|\cdot\|) \) is continuous, and such that the closed unit ball \( \{x \in X' : \|x\|' \leq 1\} \) in \( X' \) is a closed subset of \( (X, \|\cdot\|) \). Let \( \mathcal{M} \) be a Banach manifold, \( f \in \mathcal{M} \), \( \mathcal{U} \) be an open set in \( X \) containing \( 0 \in X \). Let \( \Psi : \mathcal{M} \times \mathcal{U} \to X \) be a \( C^1 \) map satisfying \( \Psi(f,0) = 0 \) and \( \Psi(\mathcal{M} \times (\mathcal{U} \cap X')) \subset X' \). Denote by \( A = D_2\Psi(f,0) : X \to X \) the partial derivative of \( \Psi \) at the point \( (f,0) \) along the second variable. Suppose

(1) \( A(X') \subset X' \);

(2) \( 1 - A \) has a continuous linear right inverse \( J \) which maps \( X' \) into \( X' \) and restricts to a continuous linear operator on \( X' \);

(3) for any \( \varepsilon > 0 \), there exist a neighborhood \( \mathcal{M}_\varepsilon \) of \( f \) in \( \mathcal{M} \) and a neighborhood \( \mathcal{U}_\varepsilon \) of \( 0 \) in \( \mathcal{U} \) such that

\[ \|\Psi(g, x) - A(x)\|' \leq \varepsilon(1 + \|x\|') \]
for all \( g \in \mathcal{M} \), \( x \in \mathcal{U} \cap X' \).

Then for any neighborhood \( \mathcal{V} \subset X' \) of 0 in \((X', \| \cdot' \|)\), there exist a neighborhood \( \mathcal{N} \) of \( f \) in \( \mathcal{M} \) and a map \( c : \mathcal{N} \to \mathcal{V} \) such that

(i) \( c(f) = 0 \);
(ii) \( \Psi(g, c(g)) = c(g) \) for all \( g \in \mathcal{N} \);
(iii) as a map \( \mathcal{N} \to X \), \( c \) is \( C^1 \).

**Proof.** Denote the norm of \( J \) as a operator on \( X \) by \( \| J \| \), and the norm of \( J \mid_{X'} \) as a operator on \( X' \) by \( \| J \|' \). Choose \( 0 < \epsilon \leq 1 \) such that the closed ball \( B'(\epsilon) = \{ x \in X' : \| x \|' \leq \epsilon \} \) lies in \( \mathcal{V} \). By the condition (3) and the continuous differentiability of \( \Psi \), we may choose an open neighborhood \( \mathcal{N} \) of \( f \) in \( \mathcal{M} \) and \( r > 0 \) such that the closed ball \( B(r) = \{ x \in X : \| x \| \leq r \} \) lies in \( \mathcal{U} \), and such that

\[
(4.1) \quad \| D_2 \Psi(g, x) - A \| \leq \frac{1}{2 \| J \|}
\]

for all \( g \in \mathcal{N}, x \in B(r) \), and

\[
(4.2) \quad \| \Psi(g, x) - A(x) \|' \leq \frac{\epsilon}{2 \| J \|'} (1 + \| x \|')
\]

for all \( g \in \mathcal{N}, x \in B(r) \cap X' \). By making \( \mathcal{N} \) smaller, we may also assume that

\[
(4.3) \quad \| \Psi(g, 0) \| \leq \frac{r}{2 \| J \|}
\]

for all \( g \in \mathcal{N} \).

For \( g \in \mathcal{N} \), define a map \( R_g : B(r) \to X \) by

\[
R_g(x) = J(\Psi(g, x) - A(x)).
\]

Then for \( x \in B(r) \), by (4.1), (4.3) and the Mean Value Theorem, we have

\[
\left\| R_g(x) \right\| \\
\leq \| J \| \left( \| \Psi(g, 0) \| + \| (\Psi(g, x) - A(x)) - (\Psi(g, 0) - A(0)) \| \right) \\
\leq \| J \| \left( \frac{r}{2 \| J \|'} + \frac{1}{2 \| J \|'} \| x \| \right) \\
\leq r.
\]

So \( R_g \) maps \( B(r) \) into \( B(r) \). For \( x, y \in B(r) \), also by (4.1) and the Mean Value Theorem, we have

\[
\left\| R_g(x) - R_g(y) \right\| \\
\leq \| J \| \left( \| (\Psi(g, x) - A(x)) - (\Psi(g, y) - A(y)) \| \right) \\
\leq \| J \| \frac{1}{2 \| J \|'} \| x - y \| \\
= \frac{1}{2} \| x - y \|.
\]

So \( R_g \) is a contraction on \( B(r) \). By the Contraction Principle, there is a unique fixed point \( c(g) \) of \( R_g \) in \( B(r) \). This means that \( (1 - A)(c(g)) = (1 - A)(R_g(c(g))) = \Psi(g, c(g)) - A(c(g)) \). So \( \Psi(g, c(g)) = c(g) \). It is obvious that \( c(f) = 0 \).

We prove that \( c(g) \in \mathcal{V} \). Let \( x_0 = R_g^n(0) \in B(r), n \geq 0 \). Then \( \| x_n - c(g) \| \to 0 \), and it is obvious by induction that \( x_n \in X' \). We have \( x_{n+1} = R_g(x_n) = J(\Psi(g, x_n) - A(x_n)) \). By (4.2), we get \( \| x_{n+1} \|' \leq \frac{\epsilon}{2} (1 + \| x_n \|') \), which is equivalent to

\[
\| x_{n+1} \|' - \frac{\epsilon}{2 - \epsilon} \leq \frac{\epsilon}{2} (\| x_n \|' - \frac{\epsilon}{2 - \epsilon}).
\]
By induction we easily get $\|x_n\| - \frac{x_n}{2^n} \leq 0$ for all $n \geq 0$. Hence $\|x_n\| \leq \frac{x_n}{2^n} \leq \varepsilon$.

But the closed ball $B'(\varepsilon)$ in $X'$ is closed in $X$ and $x_n \to c(g)$ in $X$. So $c(g) \in B'(\varepsilon) \subset V$.

The proof of the fact that $c$ as a map $\mathcal{N} \to X$ is $C^1$ is the same as the proof of the corresponding result in the usual Implicit Function Theorem. We omit the details here. \hfill \Box

**Proof of Theorem 1.2.** The map $\Psi : \text{Diff}^1(M) \times C^0(M, M) \to C^0(M, M)$ between Banach manifolds defined by

$$\Psi(g, h) = ghf^{-1}$$

is $C^1$ (see, for example, [2]). Let $(U_0, \varphi)$ be a coordinate chart around the identity map $id$ in $C^0(M, M)$, where the coordinate $\varphi : U_0 \to \mathcal{X}^0(M)$ is provided by the exponential map associated with some Riemannian metric on $M$, that is, $\varphi(h)(x) = \exp_{x}^{-1}(h(x))$. $\varphi$ maps the set of $C^{\alpha}$ and $d_{l}$-Lipschitz maps in $U_0$ onto $\varphi(U_0) \cap \mathcal{X}^{\alpha}_{l}(M)$. Let $U \subset U_0$ be an open neighborhood of $id$ in $C^0(M, M)$, $M$ be an open neighborhood of $f$ in $\text{Diff}^1(M)$, such that $\Psi(M \times U) \subset U_0$. By abuse of language, we identify $U_0$ with $\varphi(U_0)$ via the coordinate $\varphi$. But we denote an element in $U_0$ by $h$ when we view it as a map, and by $\eta$ if it is regarded as a vector field.

The partial derivative $D_{2}\Psi(f, id) : \mathcal{X}^0(M) \to \mathcal{X}^0(M)$ of $\Psi$ at the point $(f, id)$ equals to $f_{\circ}$. Since $f$ is $C^{2}$, $D_{2}\Psi(f, id)$ maps $\mathcal{X}^1_{r}(M)$ into $\mathcal{X}^0_{r}(M)$. By Theorem 3.1, $1 - D_{2}\Psi(f, id)$ has a right inverse $J$ which restricts to a continuous linear operator on $\mathcal{X}^0_{r}(M)$.

To apply Lemma 4.1 we need to verify the following two conditions.

1. The closed unit ball in $\mathcal{X}^0_{r}(M)$ is a closed subset in $\mathcal{X}^{0}(M)$;

2. For every $\varepsilon > 0$, there exist a neighborhood $M_{\varepsilon} \subset M$ of $f$ and $\delta > 0$ such that $\|\Psi(g, \eta) - f_{\circ}(\eta)\|_{\alpha, f} \leq \varepsilon (1 + \|\eta\|_{\alpha, f})$ for all $g \in M_{\varepsilon}, \eta \in U \cap \mathcal{X}^0_{r}(M)$ with $\|\eta\| < \delta$.

To prove (1), let $(\eta_n)_{n=1}^{\infty}$ be a sequence in the closed unit ball in $\mathcal{X}^0_{r}(M)$, that is, $\|\eta_n\|_{\alpha, f} = \max\{\|\eta_n\|, L_{\alpha}(\eta_n), L_{f}(\eta_n)\} \leq 1$ for all $n$. Suppose $\eta \in \mathcal{X}^0(M)$ such that $\|\eta\| - \|\eta_n\| \to 0$. Then $\|\eta\| \leq 1$. By letting $n \to \infty$ in the inequality $\frac{L_{\alpha}(\eta_n) - L_{f}(\eta_n)}{d(x, y)} \leq 1$, we get $L_{\alpha}(\eta) \leq 1$. Similarly, $L_{f}(\eta) \leq 1$. So $\|\eta\|_{\alpha, f} \leq 1$. (1) is proved.

Denote $Q(g, \eta) = \Psi(g, \eta) - f_{\circ}(\eta)$. Let $\varepsilon' > 0$. Then

$$(4.4) \quad \|Q(g, \eta)\| \leq \varepsilon'$$

for $g$ sufficiently $C^1$ close to $f$ and $\|\eta\|$ sufficiently small. By considering the partial differentials of the $C^{1}$ map $\mathcal{M} \times TM \to TM, (g, x, v) \mapsto (f(x), \exp_{f(x)}^{-1}(g(\exp_{x}(v))))$ along the directions of $x$ and $v$ (see [8] Lemma 3.2, Lemma 3.4 or [9] Lemma 8.4), we have

$$|Q(g, \eta)(x) - Q(g, \eta)(y)| \leq \varepsilon'(d(f^{-1}(x), f^{-1}(y)) + |\eta(f^{-1}(x)) - \eta(f^{-1}(y))|)$$

whenever $g$ is sufficiently $C^1$ close to $f$ and $\|\eta\|$ is sufficiently small, from which we easily get

$$(4.5) \quad L_{\alpha}(Q(g, \eta)) \leq \varepsilon' l(\text{diam}(M)^{1-\alpha} + l^{\alpha} L_{\alpha}(\eta)),$$

$$(4.6) \quad L_{f}(Q(g, \eta)) \leq \varepsilon'(1 + L_{f}(\eta))$$

for such $g$ and $\eta$ if $\eta \in \mathcal{X}^0_{r}(M)$, where $\text{diam}(M)$ is the diameter of $M$. By (4.4), (4.5) and (4.6), we have

$$\|Q(g, \eta)\|_{\alpha, f} \leq \varepsilon' \max\{l(\text{diam}(M)^{1-\alpha} + l^{\alpha})(1 + \|\eta\|_{\alpha, f})$$

for $g$ sufficiently $C^1$ close to $f$ and $\eta \in \mathcal{X}^0_{r}(M)$ with $\|\eta\|$ sufficiently small. This proves (2).
Let $\mathcal{V}$ be a $C^\alpha$ neighborhood of $id$ in $C^\alpha(M, M)$ as in Theorem 1.2. Then we may choose a neighborhood $\mathcal{V}_f$ of $id$ in the Banach manifold $C^\alpha_f(M, M)$ of $C^\alpha$ and $d_f$-Lipschitz maps on $M$ such that $\mathcal{V}_f \subset \mathcal{V}$, and such that elements in $\mathcal{V}_f$ are sufficiently $d_f$-Lipschitz close to the identity. Applying Lemma 4.1 to the map $\Psi$, we get a $C^1$ neighborhood $\mathcal{N}$ of $f$ in $\mathcal{M} \subset \text{Diff}^1(M)$ and a function $c : \mathcal{N} \to \mathcal{V}_f$ with $c(f) = id$ such that $\Psi(g, c(g)) = gc(g)f^{-1} = c(g)$ for every $g \in \mathcal{N}$, and $c$ is $C^1$ as a map $\mathcal{N} \to C^0(M, M)$. It is easy to show that if $c(g)$ is sufficiently $d_f$-Lipschitz close to the identity, then $c(g)$ is a homeomorphism (see [8, 9]). So $g = c(g)f c(g)^{-1}$.

This proves Theorem 1.2.

□

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