Propagation of perturbations along strings

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Abstract

A covariant formalism for physical perturbations propagating along a string in an arbitrary curved spacetime is developed. In the case of a stationary string in a static background the propagation of the perturbations is described by a wave-equation with a potential consisting of 2 terms: The first term describing the time-dilation and the second is connected with the curvature of space. As applications of the developed approach the propagation of perturbations along a stationary string in Rindler, de Sitter, Schwarzschild and Reissner-Nordström spacetimes are investigated.
1 Introduction

Recently there has been a lot of interest in string theory formulated in curved spacetime. The main complication, as compared to the case of flat Minkowski space, is related to the non-linearity of the equations of motion. It makes it possible to obtain a complete analytic solution only in a very few cases. These special cases include maximally symmetric spacetimes [1], conical spacetime [2] and gravitational shock-wave background [3]. Besides these only some special explicit analytic solutions are known. The consideration of small perturbations on their background is of interest, e.g. it allows one to investigate the stability-properties of these solutions [4]. More generally it is interesting to see how small waves propagate on a string or a membrane [5], and especially to see how they are affected by the curvature of space and time.

Small perturbations may play an important role for cosmic strings [6]. Cosmic strings may gain a significant part of their energy from their small scale structure. The effect of such small scale ”wiggles” on the equation of state of a cosmic string has been investigated by various authors [7], and also more mathematical aspects concerning separability of the equations of motion of ”wiggly” strings and ”noisy” strings in curved spacetimes [8] has been studied.

Finally string perturbations can be invoked for the purpose of a ”semi-classical” quantization procedure for strings in curved spacetime, by taking a classical solution to the equations of motion and then considering small perturbations as quantum fluctuations. This approach was developed by Vega and Sánchez in Ref. 9, with application to various gravitational backgrounds including black holes, de Sitter space, Rindler space and several others.

The main aim of the present paper is to develop the general theory of small perturbations propagating along a string in curved spacetime.

In section 2 we consider the first and second variations of the Polyakov action [10]. The first variation of the action gives the equations of motion for the string in a curved background, while the second variation can be used to obtain the equations describing the perturbations propagating in the background of the exact solution. From a geometrical point of view the problem we are interested in is the study of the variations of a minimal 2-surface embedded in a curved 4-space. In section 3 we consider perturbations propagating along a stationary string in a static spacetime. We exploit the fact that such a stationary configuration of the string can be described by
a geodesic equation in a properly chosen 3-dimensional ("unphysical") space \[11\]. This approach allows one to simplify general results of section 2, and to obtain a relatively simple "wave-equation" determining the propagation of perturbations. In section 4 we give a few examples to illustrate our general results. We consider perturbations propagating along stationary strings in Rindler, de Sitter, Schwarzschild and Reissner-Nordström spacetimes, as well as in the quasi-Newtonian gravitational field. As a curiosity we find that the perturbations propagating along a curved string in (flat) Rindler space and the perturbations propagating along a straight string in (curved) de Sitter space is determined by one and the same well-known equation from quantum mechanics: The Pöschl-Teller equation.

In the paper we use sign-conventions of Misner,Thorne,Wheeler \[12\].

2 General physical perturbations

In this section we derive an effective action and equations of motion for small perturbations on an arbitrary string-configuration in an arbitrary 4-dimensional gravitational background.

Our starting point is the Polyakov action for the relativistic string \[10\]:

\[
S = \int d\tau d\sigma \sqrt{-h} h^{AB} G_{AB}.
\] (2.1)

We use units in which beside \( G = 1, c = 1 \) the string tension \((2\pi\alpha')^{-1} = 1\). In Eq. (2.1) \( h_{AB} \) is the internal metric with determinant \( h \) whereas \( G_{AB} \) is the induced metric on the world-sheet:

\[
G_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^A} \frac{\partial x^\nu}{\partial \xi^B} = g_{\mu\nu} x^\mu A x^\nu B.
\] (2.2)

\( x^\mu (\mu = 0,1,2,3) \) are the spacetime coordinates and \( \xi^A (A = 0,1) \) are the world-sheet coordinates \((\xi^0, \xi^1) = (\tau, \sigma)\). Finally \( g_{\mu\nu} \) is the spacetime metric which in this section is completely arbitrary.

We now make variations \( \delta x^\mu \) and \( \delta h_{AB} \). The corresponding variation of the action (2.1) is conveniently written in the form:

\[
\delta S = \int d\tau d\sigma \sqrt{-h} \left( \frac{1}{2} h^{AB} G^C C - G^{AB} \right) \delta h_{AB} - 2g_{\mu\nu} \left[ \Box x^\nu + h^{AB} \Gamma_{\rho\sigma} x^\rho A x^\sigma B \right] \delta x^\mu,
\] (2.3)
where $G^C_C = h^{BC}G_{BC}$ is the trace of the induced metric on the world-sheet and $\Box$ is the d’Alambertian:

$$\Box = \frac{1}{\sqrt{-h}} \partial_A (\sqrt{-h} h^{AB} \partial_B).$$

From $\delta \mathcal{S}$ we get the usual equations of motion:

$$G_{AB} - \frac{1}{2} h_{AB} G^C_C = 0,$$  \hspace{1cm} (2.5)

$$\Box x^\mu + h^{AB} \Gamma^\mu_{\rho\sigma} x^\rho_A x^\sigma_B = 0.$$  \hspace{1cm} (2.6)

Equation (2.5) shows that the 2-dimensional world-sheet energy-momentum tensor vanishes. This equation is to be considered as a constraint on the solution of the equation of motion for $x^\mu$ (2.6).

The equations describing the propagation of perturbations in the background of an exact solution to Eqs. (2.5)-(2.6) can be obtained by variation of these equations. Technically it appears to be easier to make the second variation of these equations. Let $x^\mu = x^\mu (\xi^A)$ be a solution to Eqs. (2.5) and (2.6) describing a background strings configuration. In what follows it is convenient to introduce 2 vectors $n^\mu_R (R = 2, 3)$ normal to the surface of the string world-sheet:

$$g_{\mu\nu} n^\mu_R n^\nu_S = \delta_{RS}, \quad g_{\mu\nu} x^\mu_A n^\nu_R = 0.$$  \hspace{1cm} (2.7)

The general perturbation $\delta x^\mu$ can be decomposed as:

$$\delta x^\mu = \delta x^R n^\mu_R + \delta x^A x^\mu_A.$$  \hspace{1cm} (2.8)

It can be verified that the variations $\delta x^A x^\mu_A$ leave $\mathcal{S}$ unchanged. This is the result of the invariance of the string action with respect to reparametrizations of the world-sheet. That is why we restrict ourselves by preserving in the variation of the action only physical perturbations $\delta x^\mu$, which can be written as:

$$\delta x^\mu = \delta x^R n^\mu_R.$$  \hspace{1cm} (2.9)

Note that $R = (2, 3)$ is just a label for the normalvectors and summation over repeated indices is implied. Before carrying out the corresponding variation of Eq. (2.3) it is convenient to introduce the second fundamental form $\Omega_{R,AB}$
and the normal fundamental form $\mu_{RS,A}$ [13] defined for a given configuration of the strings world-sheet:

$$\Omega_{R,AB} = g_{\mu \nu} n_\mu^R x_\nu^A \nabla_\rho x_\nu^B,$$

(2.10)

$$\mu_{RS,A} = g_{\mu \nu} n_\mu^R x_\nu^A \nabla_\rho n_\nu^S,$$

(2.11)

where $\nabla_\rho$ is the spacetime covariant derivative. Note that $\Omega_{R,AB} = \Omega_{R,BA}$ whereas $\mu_{RS,A} = -\mu_{SR,A}$. Now the equations of motion (2.6) can be rewritten compactly as:

$$h_{AB} \Omega_{R,AB} = 0,$$

(2.12)

which is the well-known result [13] that a minimal surface has vanishing normal curvature in all directions normal to the strings world-sheet. One can also verify the following useful relation:

$$\delta G_{AB} = -2 \Omega_{R,AB} \delta x^R.$$

(2.13)

The variation of Eq. (2.3) turns out to be somewhat complicated leading to quadratic terms in $\delta x^R$ and $\delta h_{AB}$ as well as to mixed terms. After some algebra one finds:

$$\delta^2 S = \int d\tau d\sigma \sqrt{-h} \left( \delta h_{AB} [2 G^{BC} h^{AD} - \frac{1}{2} h^{AB} h^{CD} G^E_E - \frac{1}{2} h^{AB} G^{CD}] \delta h_{CD} + 4 \delta h_{AB} h^{AC} h^{BD} \Omega_{R,CD} \delta x^R - 2 \delta x^R [\delta h_{RS} - h^{AB} g_{\mu \nu} (x_\rho^A \nabla_\nu n_\rho^R)(x_\sigma^B \nabla_\sigma n_\nu^S) - 2 h^{AB} \mu_{RS,A} \partial_B - h^{AB} x_\rho^A x_\nu^B R_{\mu \rho \sigma \nu} n_\rho^R n_\sigma^S \delta x^S \right),$$

(2.14)

where $R_{\mu \rho \sigma \nu}$ is the Riemann curvature tensor in the spacetime in which the string is embedded. The variation of the internal metric $\delta h_{AB}$ obeys the equation:

$$\Omega_{R,AB} \delta x^R = \frac{1}{4} (G^C_C \delta h_{AB} - h_{AB} G^{CD} \delta h_{CD}).$$

(2.15)

This relation gives:

$$\delta h_{AB} \Omega^R_{AB} = -\frac{4}{G^C_C} \Omega_{R,AB} \Omega^S_{AB} \delta x^S.$$

(2.16)

Eq. (2.16) show that $\delta h_{AB}$ is not a dynamical field and it can be expressed in the algebraic way in terms of the perturbations $\delta x^S$. By using Eq. (2.16)
we can exclude the perturbations $\delta h_{AB}$ from the variation of the action. By using Eq. (2.16) and the identity:

$$h^{AB}g_{\mu\nu}(x^\rho_A, n_{R}^\mu) (x^\sigma_B, n^\sigma_S) = \mu_{RT}^A \mu_S^T A + \frac{2}{G_C C} \Omega_{R}^{AB} \Omega_{S,AB},$$  
(2.17)

we finally obtain the effective action for the physical perturbations in the form:

$$S_{\text{eff}} = \int d\tau d\sigma \sqrt{-h} \delta x^R (\delta x^S \Box + 2 \mu_{RS}^A \partial_A - \mu_{RT}^A \mu_S^T A + \frac{2}{G_C C} \Omega_{R}^{AB} \Omega_{S,AB} - h^{AB} x^\mu_A x^\nu_B R_{\mu\sigma\nu} n_{R}^\mu n_{S}^\sigma \delta x^S).$$  
(2.18)

This is the main result of this section. Note that each of the terms of equation (2.18) is spacetime and world-sheet invariant independently. It should be stressed that the various terms are not completely independent. It is well-known in differential geometry that there exists relations between the second fundamental form, the normal fundamental form and the Riemann tensor: The so-called Gauss-equation, the Codazzi-Mainardi-equation and the Ricci-equation [13]. These relations, however, do not seem to simplify the action (2.18). The equations of motion corresponding to the action (2.18) are:

$$\Box \delta x_R + 2 \mu_{RS}^A (\delta x^S)_{,A} + (\nabla_A \mu_{RS}^A) \delta x^S - \mu_{RT}^A \mu_S^T A \delta x^S + \frac{2}{G_C C} \Omega_{R}^{AB} \Omega_{S,AB} \delta x^S - h^{AB} x^\mu_A x^\nu_B R_{\mu\sigma\nu} n_{R}^\mu n_{S}^\sigma \delta x^S = 0,$$  
(2.19)

where $\nabla_A$ is the strings world-sheet covariant derivative. In general this is a complicated set of coupled partial (linear) second order differential equations. In the following sections we will however show that the solution can be found analytically in various interesting cases.

We conclude this section with the following remark. Obviously there is an ambiguity in the choice of normalvectors $n_{R}^\mu$ introduced in Eq. (2.7) connected with their local "rotations":

$$n_{R}^\mu \rightarrow n_{R}^\mu + \lambda(\tau, \sigma) \epsilon_{RS}^S n_{S}^\mu,$$  
(2.20)

$$\delta x^R \rightarrow \delta x^R + \lambda(\tau, \sigma) \epsilon_{RS}^S \delta x^S,$$  
(2.21)

where $\lambda(\tau, \sigma)$ is an arbitrary (small) function and $\epsilon_{RS}$ is the antisymmetric Levi-Civita symbol. The effective action (2.18) remains invariant under these
local rotations. Note that the last 2 terms in the bracket of Eq. (2.18) are
counts separately whereas the first 3 terms must be taken together. The
invariance with respect to transformations (2.20) and (2.21) can be used to
simplify the action by a suitable choice of the normal vectors. In the next
section we will show that for a stationary string in a static background one
can get rid of the terms in Eq. (2.18) involving the normal fundamental form
$\mu_{RS,A}$ by using this gauge freedom.

3 Stationary string in static background

We will now specialize to stationary strings in static spacetimes, i.e. we
take a stationary solution to Eqs. (2.5)-(2.6) and consider time-dependent
perturbations around it.

The metric of a static spacetime can in general be written:

$$g_{\mu\nu} = \begin{pmatrix} -F & 0 \\ 0 & F_{ij}/F \end{pmatrix},$$

where $\partial_t F = 0$, $\partial_t H_{ij} = 0$ and $i, j = 1, 2, 3$. The nonvanishing components
of the Christoffel symbol for the metric $g_{\mu\nu}$ are $\Gamma^i_{jk}$ and:

$$\Gamma^0_{i0} = \frac{F_i}{2F}, \quad \Gamma^i_{00} = \frac{F}{2} H^{ij} F_{ij}.$$

The solution describing a stationary string can be parametrized in the fol-
lowing way:

$$t = x^0 = \tau, \quad x^i = x^i(\sigma),$$

so that:

$$x^\mu_{,0} = (1, 0, 0, 0), \quad x^\mu_{,1} = (0, x^n).$$

(The prime denotes the differentiation with respect to $\sigma$.) For this parametriza-
tion the induced metric on the world-sheet is:

$$G_{00} = -F, \quad G_{01} = G_{10} = 0, \quad G_{11} = \frac{H_{ij}}{F} x^n x^j.$$

Working with the Nambu-Goto action [14] we identify the internal metric
and the induced metric:

$$G_{AB} = h_{AB}.$$
The equation of motion (2.6) in this case reduces to:

\[ x^{\prime\prime} + \tilde{\Gamma}^{i}_{jk} x^{\prime} x^{j} x^{k} = 0, \tag{3.7} \]

where \( \tilde{\Gamma}^{i}_{jk} \) is the Christoffel symbol for the metric \( H_{ij} \). It means that a configuration of a stationary string in a static spacetime coincides with a geodesic for the 3-dimensional "unphysical" space with the line-element:

\[ d\tilde{l}^2 = H_{ij} dx^{i} dx^{j} = F dl^2, \tag{3.8} \]

where \( dl \) is the physical distance in the static spacetime defined by Eq. (3.1) and \( \sigma \) is an affine parameter (proper length):

\[ H_{ij} x^{\prime \prime} x^{ij} = 1. \tag{3.9} \]

Here and later on a tilde indicates that an object is defined with respect to the 3-dimensional "unphysical" space (3.8). We use now this reduction to simplify the equations for the strings perturbations. To be more concrete we express the effective action (2.18) for the perturbations in terms of quantities defined in the "unphysical" space.

Consider the 2 normalvectors \( n_{R}^{\mu} \) introduced in (2.7). Eqs. (3.4) and (2.7) imply that \( n_{R}^{\mu} \) are of the form:

\[ n_{R}^{\mu} = (0, n_{R}^{i}), \tag{3.10} \]

so that we can exclude from our consideration the zero-components and consider \( (x^{\prime i}, n_{R}^{i}) \) as an orthogonal system in the 3-dimensional "unphysical" space. The normalvectors are however not normalized with respect to \( H_{ij} \) and it is therefore convenient to rewrite the decompositions of the perturbations \( \delta x^{\mu} \) in the following way:

\[ \delta x^{i} = n_{R}^{i} \delta x^{R} = \left( \frac{n_{R}^{i}}{\sqrt{F}} \right)(\sqrt{F} \delta x^{R}) \equiv \tilde{n}_{R}^{i} \tilde{x}^{R}, \quad \delta x^{0} = 0. \tag{3.11} \]

In this case \( (x^{\prime i}, \tilde{n}_{2}^{i}, \tilde{n}_{3}^{i}) \) is an orthonormal system in the 3-dimensional space with metric \( H_{ij} \):

\[ H_{ij} \tilde{n}_{R}^{i} \tilde{n}_{S}^{j} = \delta_{RS}, \quad H_{ij} x^{\prime i} \tilde{n}_{R}^{j} = 0, \tag{3.12} \]

remembering that \( x^{\prime i} \) is already normalized because of Eq. (3.9).
The Christoffel symbols $\Gamma^i_{jk}$ and $\Gamma^i_{00}$ are given by Eq. (3.2). The conformal relation between the spacelike components of $g_{\mu \nu}$ and $H_{ij}$ leads to:

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} - \frac{1}{2F} (F_{,j} \delta^i_k + F_{,k} \delta^i_j - F_{,l} H^i_l H_{jk}).$$ \hspace{1cm} (3.13)

From this equation and Eqs. (2.10)-(2.11) we can now easily obtain the second fundamental form $\Omega_{R,AB}$ and the normal fundamental form $\mu_{RS,A}$ in terms of quantities defined in the unphysical space:

$$\Omega_{R,01} = \Omega_{R,10} = 0, \hspace{1cm} (3.14)$$

$$\Omega_{R,00} = F^2 \Omega_{R,11} = \frac{\sqrt{F}}{2} \tilde{n}^i_R F_{,i} \hspace{1cm} (3.15)$$

as well as:

$$\mu_{RS,0} = 0, \hspace{1cm} (3.16)$$

$$\mu_{RS,1} = H_{ij} \tilde{n}^i_R x^j \widehat{\nabla}_k \tilde{n}^j_S. \hspace{1cm} (3.17)$$

Finally we need a relation between the Riemann tensors. Generally it turns out to be very complicated but fortunately we only need the special projection appearing in equations (2.14) and (2.18)-(2.19), which after some tedious but straightforward algebra leads to:

$$\delta x^R h^{AB}_{,A,x^\nu} x^\nu A_{,B} R_{\mu \rho \sigma \nu} n^\rho_R n^\sigma_S \delta x^S = \delta x^R (x^x x^y \tilde{R}_{ijkl} \tilde{n}^k_R \tilde{n}^l_S + \frac{F_{,k} F_{,l}}{2F^2} \tilde{n}^k_R \tilde{n}^l_S) \delta x^S$$

$$+ \delta x_R x^x x^y (\tilde{\Gamma}^k_{ij} F_{,k} - \frac{F_{,ij}}{2F} + \frac{F_{,ij} F_{,y}}{4F^2}) \delta x^R. \hspace{1cm} (3.18)$$

Collecting everything we get the effective action Eq. (2.18) in the form:

$$S_{eff} = \int d\tau d\sigma \tilde{x}^R (\delta_{RS} (\partial^2_{\sigma} - \frac{\partial^2}{F^2}) - x^x x^y \tilde{R}_{ijkl} \tilde{n}^k_R \tilde{n}^l_S$$

$$+ 2\mu_{RS,1} \partial_{\sigma} - \mu_{RT,1} \mu_{S,T}) \delta x^S, \hspace{1cm} (3.19)$$

where we also used (from Eq. (2.4)):

$$\Box \equiv - \frac{\partial^2}{F} + F \partial^2_{\sigma} + F' \partial_{\sigma}. \hspace{1cm} (3.20)$$
This action allows further simplification by using the gauge-freedom related to the choice of the normalvectors discussed at the end of section 2. To fix this freedom we choose the basis \((x^\mu, \tilde{n}_2^i, \tilde{n}_3^i)\) obeying the normalization conditions (3.12) at a given point, and define the basis along the geodesic by means of parallel transport. For such a choice \(\mu_{RS,1} = 0\) and Eq. (3.19) reduces to:

\[
S_{eff} = \int d\tau d\sigma \tilde{x}^R \left( \delta_{RS} \left( \partial^2 - \frac{\partial^2}{F^2} \right) - x^\mu x^j \tilde{R}_{ijklj} \tilde{n}_R^k \tilde{n}_S^l \right) \delta \tilde{x}^S, \tag{3.21}
\]

with corresponding equations of motion for \(\tilde{x}_R^i\):

\[
(\partial^2 - \frac{\partial^2}{F^2}) \delta \tilde{x}_R^i = V_{RS} \delta \tilde{x}^S, \tag{3.22}
\]

where the matrix-potential \(V_{RS}\) is given by:

\[
V_{RS} = x^\mu x^j \tilde{R}_{ijklj} \tilde{n}_R^k \tilde{n}_S^l. \tag{3.23}
\]

There is a simple relation between Eqs. (3.22)-(3.23) and the so-called geodesic deviation equation. The geodesic deviation equation describing the rate of spread of 2 neighbouring geodesics is in our notation given by [15]:

\[
x'^j \tilde{\nabla}_j (x'^k \tilde{\nabla}_k \delta x^i) = \tilde{R}_{ijklj} x'^i \tilde{n}_R^k \tilde{n}_S^l. \tag{3.24}
\]

Using Eq. (3.11) and multiplying both sides by \(H_{ij} \tilde{n}_S^i\) gives:

\[
\frac{d^2}{d\sigma^2} \tilde{x}_R^i + H_{ij} \tilde{n}_S^i \left( x'^k \tilde{\nabla}_k (x'^i \tilde{\nabla}_i \tilde{n}_R^j) + 2(x'^k \tilde{\nabla}_k \tilde{n}_R^j) \frac{d}{d\sigma} \right) \tilde{x}^R = x'^i x'^j \tilde{R}_{ijklj} \tilde{n}_R^k \tilde{n}_S^l \delta \tilde{x}^S. \tag{3.25}
\]

With our special choice of the normalvectors the terms in the square bracket vanish and the result is equivalent to Eq. (3.22) except for the \(\partial_\tau\)-term. For time-independent perturbations our result therefore leads to the geodesic deviation equation, as it of course should, but due to the possibility of having time-dependent perturbations we get an extra term. This extra term can be written in a more convenient form if we re-introduce the variations \(\delta x^R = \tilde{\delta} x^R / \sqrt{F}\) (c.f. Eq. (3.11)) and define what we will call the conformal string-parameter \(\sigma_c\):

\[
d\sigma_c = \frac{d\sigma}{F}. \tag{3.26}
\]
In this case Eq. (3.22) becomes:

\[
(\partial_{\sigma_c}^2 - \partial_{\tau}^2)\delta x_R = U_{RS}\delta x^S, \tag{3.27}
\]

where the potential \( U_{RS} \) is given by:

\[
U_{RS} = V\delta_{RS} + F^2V_{RS}. \tag{3.28}
\]

Here \( V_{RS} \) is the potential (3.23) while \( V \) is given by:

\[
V = \frac{3}{4F^2}\left(\frac{dF}{d\sigma_c}\right)^2 - \frac{1}{2F}\frac{d^2F}{d\sigma_c^2} = \frac{1}{4}(F'^2 - 2FF''). \tag{3.29}
\]

Equations (3.27)-(3.29) finally represent the desired results of this section. Since we consider time-dependent perturbations propagating along a stationary string in a static background, we can split the \( \tau \) and \( \sigma_c \)-dependence of \( \delta x^R \) so that Eq. (3.27) leads to a Schrödinger-like matrix equation. The potential \( U_{RS} \) consists of 2 terms: The first (diagonal) term \( V\delta_{RS} \) is defined by the red-shift factor \( F \) and it is connected with the time-delay effect in a static gravitational field. The second (generally non-diagonal) term \( V_{RS} \) is connected with the curvature of the 3-dimensional "unphysical" space.

We conclude this section with the following remark. In the general case the presence of the spacetimes curvature may result in the mixture of polarizations of the perturbations during their propagation along the string. This mixture may be absent if the spacetime under consideration possesses symmetries. Let us suppose that the metric \( H_{ij} \) is invariant under the discrete symmetry transformation \( (x^1, x^2, x^3) \rightarrow (x^1, x^2, -x^3) \). It is evident that for a static string lying in the plane \( x^3 = 0 \) the perturbations in the \( x^3 \)-direction (perpendicular to the string) and in the strings plane cannot be mixed. For the corresponding choice of \( \tilde{n}_2^i = n_\perp^i \) and \( \tilde{n}_3^i = n_\parallel^i \) the potential \( U_{RS} \) in Eq. (3.27) becomes diagonal.

In the next section we will consider equation (3.27) in a few special cases.

## 4 Special cases

### 4.1 Quasi-Newtonian gravitational field

As a first example we consider the perturbations propagating along a string in a static gravitational field described in the quasi-Newtonian approximation.
In this approximation the line-element is [12]:

\[ ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2), \]

(4.1)

where \( \phi = \phi(x, y, z) \) is a gravitational potential.

In the notation of Eq. (3.1) we find:

\[ F = 1 + 2\phi, \quad H_{ij} = (1 - 2\phi)(1 + 2\phi)\delta_{ij} \approx \delta_{ij}, \]

(4.2)

i.e. the 3-dimensional ”unphysical” space is flat and the stationary string is represented simply by a straight line, according to Eq. (3.7). As the result the term \( V_{RS} \) in the effective potential \( U_{RS} \) (3.27) vanishes and the equation for the propagation of perturbations takes the form:

\[ (\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x_R = V \delta x_R, \]

(4.3)

where \( V = -d^2\phi/d\sigma_c^2 \) to first order in \( \phi(x^i) \). After Fourier-expanding \( \delta x_R \):

\[ \delta x_R(\tau, \omega_c) = \int e^{-i\omega \tau} D^R_\omega(\sigma_c)d\omega, \]

(4.4)

equation (4.3) reduces to:

\[ \left( \frac{d^2}{d\sigma_c^2} + \omega^2 - V \right) D^R_\omega = 0. \]

(4.5)

In the adiabatic approximation we formally obtain the solution to this equation in the form:

\[ D^R_\omega(\sigma_c) = e^{\pm i \int^{\sigma_c} \sqrt{\omega^2 - V} d\sigma'_c}. \]

(4.6)

### 4.2 Rindler space

In this subsection we consider a stationary string in Rindler space, corresponding to a static homogeneous gravitational background. This case, and the equivalent problem of a uniformly accelerating string in Minkowski space, has earlier been investigated from a different point of view in Ref.4. The line-element of Rindler space may be written [16]:

\[ ds^2 = -a^2x^2dt^2 + dx^2 + dy^2 + dz^2, \]

(4.7)
where $a$ is a positive constant.

In the notation of section 3 we can then identify:

$$F = a^2 x^2, \quad H_{ij} = a^2 x^2 \delta_{ij},$$

(4.8)
i.e. the "unphysical" space is conformally flat. The corresponding Christoffel symbols and Riemann tensor components are found to be:

$$\tilde{\Gamma}_x^x = -\tilde{\Gamma}_y^y = -\tilde{\Gamma}_z^z = \tilde{\Gamma}_x^y = \tilde{\Gamma}_x^z = \frac{1}{x^2},$$

(4.9)
as well as:

$$\tilde{R}_{xyxy} = \tilde{R}_{xzxz} = -\tilde{R}_{yzyz} = a^2.$$  

(4.10)
Next we have to solve the geodesic equation (3.7) in order to determine the stationary string configuration. In order to exclude possible misunderstanding we emphasize that a free string cannot be at rest in the Rindler spacetime. In order to be able to study a stationary configuration one needs to assume that there are additional (non-gravitational) forces acting either on the string or on its endpoints. We do not consider the delicacies concerning this point here but refer to Ref. [4]. Due to the symmetry of the problem we may consistently take $z = 0$. After one integration the other 2 equations of (3.7) then lead to:

$$y' = \frac{b}{x^2}, \quad x'^2 + \frac{b^2}{x^4} = \frac{1}{a^2 x^2},$$

(4.11)
where $b$ is an integration constant. These equations can be easily solved by introducing the conformal string parameter (3.26). The solution reads:

$$x(\sigma_c) = ba \cosh(a\sigma_c), \quad y(\sigma_c) = ba^2 \sigma_c.$$  

(4.12)
We choose the 2 normalvectors $n^i_\perp$ and $n^i_\parallel$ in the following way:

$$n^i_\perp = \frac{1}{ax}(0, 0, 1), \quad n^i_\parallel = (-y', x', 0).$$  

(4.13)
(The prime as earlier denotes the derivative with respect to $\sigma$.) It can be shown that $n^i_\perp$ and $n^i_\parallel$ are covariantly constant in the 3-dimensional "unphysical" space. Note that $n^i_\perp$ is pointing in the $z$-direction perpendicular to the plane of the string while $n^i_\parallel$ is lying in the plane of the string, c.f. the discussion at the end of section 3.
To determine the time-dependent perturbations of the string in the 2 directions we have to calculate the components of the potential $U_{RS}$. Using Eqs. (4.8) and (4.10)-(4.13) one finds:

$$V_{\perp\perp} = \frac{2b^2a^2 - x^2}{a^2x^6}, \quad V_{\parallel\parallel} = \frac{-1}{a^2x^4}, \quad V_{\perp\parallel} = V_{\parallel\perp} = 0, \quad V = \frac{a^2}{x^2}(x^2 - 2a^2b^2).$$ \tag{4.14}

The equations of motion for the perturbations (3.27) become:

$$\left(\partial^2_{\tau} - \partial^2_{\sigma_{c}}\right)\delta x_{\perp} = 0, \tag{4.15}$$

$$\left(\partial^2_{\sigma_{c}} - \partial^2_{\tau}\right)\delta x_{\parallel} = -\frac{2a^4b^2}{x^2}\delta x_{\parallel} = -\frac{2a^2}{\cosh^2(a\sigma_{c})}\delta x_{\parallel}. \tag{4.16}$$

The perturbations connected with the oscillations of the string in the direction perpendicular to its plane are described by simple plane waves in the $(\tau, \sigma_{c})$-coordinates, while the perturbations in the plane of the string (4.16) are somewhat more complicated. After Fourier expanding $\delta x_{\parallel}$:

$$\delta x_{\parallel}(\tau, \sigma_{c}) = \int e^{-i\omega \tau} D_{\omega}(\sigma_{c}) d\omega,$$ \tag{4.17}

equation (4.16) leads to:

$$\frac{d^2}{d\sigma_{c}^2} D_{\omega} + \left(\omega^2 + \frac{2a^2}{\cosh^2(a\sigma_{c})}\right) D_{\omega} = 0, \tag{4.18}$$

which is a well-known equation in quantum mechanics; the so-called Pöschl-Teller equation [17]. The complete solution may be written in terms of hypergeometric functions but we shall not go into any details here. It suffices to say that if we consider a scattering-process, i.e. we consider solutions of the asymptotic form:

$$D_{\omega}(\sigma_{c}) = \begin{cases} 
e^{-i\omega\sigma_{c}} + R_{\omega} e^{-i\omega\sigma_{c}} & \text{for } \sigma_{c} \to -\infty, \\ T_{\omega} e^{i\omega\sigma_{c}} & \text{for } \sigma_{c} \to \infty, \end{cases} \tag{4.19}$$

then it can be shown that the reflection-coefficient $R_{\omega}$ vanishes [17]. The only effect of the potential is then the appearance of a phase factor in the transmitted wave $T_{\omega} = e^{2i\phi_{c}}$ [17]:

$$\phi_{c}(\omega) = \arg \left( \frac{\Gamma\left(\frac{i\omega}{a}\right)e^{-i\frac{\omega^2}{4a}}\log 2}{\Gamma(1 + \frac{i\omega}{2a})\Gamma\left(\frac{i\omega}{2a}\right)} \right). \tag{4.20}$$
4.3 de Sitter space

As a third example we consider a slightly more complicated case, namely the stationary string in the de Sitter space. We use static coordinates in which the de Sitter metric takes the form:

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.21) \]

Here \( f(r) = 1 - H^2r^2 \), and \( H \) is the constant Hubble-parameter. In this case we find:

\[ F = f(r), \quad H_{ij} = \text{diag}(1, r^2f(r), r^2f(r) \sin^2 \theta), \quad (4.22) \]

with the following expressions for non-vanishing components of the Christoffel symbols \( \tilde{\Gamma}^i_{jk} \) and Riemann tensor \( \tilde{R}^{ijkl} \):

\[
\begin{align*}
\tilde{\Gamma}^r_{\phi\phi} &= \sin^2 \theta \tilde{\Gamma}^r_{\theta\theta} = -r(1 - 2H^2r^2) \sin^2 \theta, \quad \tilde{\Gamma}^\phi_{\theta\phi} = \cot \theta, \\
\tilde{\Gamma}^\phi_{r\phi} &= \tilde{\Gamma}^\theta_{r\theta} = \frac{1 - 2H^2r^2}{rf(r)}, \quad \tilde{\Gamma}^\phi_{\phi\phi} = -\cos \theta \sin \theta \quad (4.23)
\end{align*}
\]

and:

\[
\begin{align*}
\tilde{R}^{r\phi r\phi} &= \sin^2 \theta \tilde{R}^{r\theta r\theta} = \frac{H^2r^2(3 - 2H^2r^2)}{f(r)} \sin^2 \theta, \\
\tilde{R}^{\theta\phi\theta\phi} &= H^2r^4(3 - 4H^2r^2). \quad (4.24)
\end{align*}
\]

Consider now the geodesic equation (3.7). Without loss of generality we assume that the string lies in the equatorial plane \( \theta = \pi/2 \) and find after one integration:

\[
\begin{align*}
\phi' &= \frac{bH^2}{r^2f(r)}, \quad (4.25) \\
r'^2 &= 1 - \frac{b^2H^4}{r^2f(r)}, \quad (4.26)
\end{align*}
\]

with \( b \) an integration constant. The normalvectors \( n^i_\perp \) and \( n^i_\parallel \) are conveniently chosen in the form:

\[
\begin{align*}
n^i_\perp &= \frac{1}{r\sqrt{f(r)}}(0, 1, 0), \quad n^i_\parallel = \frac{1}{r\sqrt{f(r)}}(-bH^2, 0, r'). \quad (4.27)
\end{align*}
\]

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As in the case of Rindler space (Section 4.2) for this choice \( n^i \parallel \) lies in the plane of the string while \( n^i \perp \) is perpendicular to this plane. Note also that \( n^i \parallel \) and \( n^i \perp \) are covariantly constant in the "unphysical" space. The components of the potential \( U_{RS} \) read:

\[
V_{\parallel \parallel} = \frac{H^2}{f^3(r)}(-3 + 2H^6b^2 + 5H^2r^2 - 2H^4r^4), \quad V_{\parallel \perp} = V_{\perp \parallel} = 0, \\
V_{\perp \perp} = -\frac{H^2}{f^2(r)}(3 - 2H^2r^2), \quad V = H^2(1 - \frac{2b^2H^6}{f(r)}).
\] (4.28)

The equations of motion for the time-dependent perturbations are of the form (3.27):

\[
(\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x_\perp = -2H^2f(r)\delta x_\perp, \quad (4.29)
\]

\[
(\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x_\parallel = -2H^2(f(r) + \frac{b^2H^6}{f(r)})\delta x_\parallel, \quad (4.30)
\]

where \( f(r) = 1 - H^2r^2(\sigma_c) \), and (from Eqs. (3.26) and (4.26)):

\[
r^2(\sigma_c) = \frac{1}{H^4}\wp(\sigma_c + z_o) + \frac{2}{3H^2}. \quad (4.31)
\]

Here \( \wp(z) \) is the Weierstrass elliptic \( \wp \)-function [18] with invariants:

\[
g_2 = 4H^4(\frac{1}{3} - b^2H^6), \quad g_3 = \frac{4}{3}H^6(-\frac{2}{9} + b^2H^6). \quad (4.32)
\]

and \( z_o \) is an integration constant which specifies the solution.

Let us now consider 2 special configurations (4.31) where we can write down explicit analytic solutions to Eqs. (4.29)-(4.30).

For \( 2bH^3 = 1 \) (4.31) reduces to:

\[
r^2(\sigma_c) = \frac{1}{2H^2}, \quad (4.33)
\]

which is just a circular string with constant radius. In this case the perturbations are simply determined by:

\[
(\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x_\perp + H^2\delta x_\perp = 0, \quad (4.34)
\]

\[
(\partial^2_{\sigma_c} - \partial^2_{\tau})\delta x_\parallel + 2H^2\delta x_\parallel = 0. \quad (4.35)
\]
From Eqs. (3.26), (4.25) and (4.33) follows that:

\[ \phi = H \sigma_c, \quad (4.36) \]

so that \( \sigma_c \) is periodic with period \( 2\pi/H \). This periodicity is to be considered as boundary conditions for the solutions to Eqs. (4.34)-(4.35), which are then represented by plane waves in the form \((n \in \mathbb{Z})\):

\[ \delta x_\perp \sim e^{-iH(\tau \sqrt{n^2-1} \pm n \sigma_c)}, \quad (4.37) \]

\[ \delta x_\parallel \sim e^{-iH(\tau \sqrt{n^2-2} \pm n \sigma_c)}, \quad (4.38) \]

so that the perturbations in the 2 normal directions propagate with different frequencies (for fixed wavenumber). Note that there are unstable modes also, corresponding to imaginary frequencies. This is easy to understand physically since the stationary circular string needs an exact balancing of the string tension and the expansion of the universe. Considering for instance the perturbation in the plane of the string (4.38), we find that the \((n = 1)\)-mode corresponds to a wavelength that is larger than the circumference of the string (using Eq. (4.33)):

\[ \lambda = \frac{2\pi}{Hn} \bigg|_{n=1} = \frac{2\pi}{H} = \sqrt{2}(2\pi r). \quad (4.39) \]

For such perturbations the string will either collapse to a point or expand towards the horizon.

As another simple configuration in de Sitter space we take a ”straight” string which is obtained by the choice \(b = 0\). In this case the Weierstrass function in Eq. (4.31) reduces to a hyperbolic function:

\[ r^2(\sigma_c) = \frac{1}{H^2} \tanh^2(H\sigma_c). \quad (4.40) \]

The equations for the 2 different polarizations of the perturbations are now identical and read:

\[ (\partial^2_{\sigma_c} - \partial^2_\tau) \delta x_R + \frac{2H^2}{\cosh^2(H\sigma_c)} \delta x_R = 0. \quad (4.41) \]

After Fourier expansion:

\[ \delta x_R(\tau, \sigma_c) = \int e^{-i\omega \tau} D^R(\omega)(\sigma_c) d\omega, \quad (4.42) \]
we re-discover the Pöschl-Teller equation (4.18) with $a$ replaced by $H$:

$$\frac{d^2}{d\sigma^2_c} D^R_\omega + \left( \omega^2 + \frac{2H^2}{\cosh^2(H\sigma_c)} \right) D^R_\omega = 0,$$

(4.43)

and with similar conclusions as after (4.18).

### 4.4 Black holes

As a final example we now consider a stationary string in the background of a black hole. The stationary configurations of the string in the black hole metrics were described in Ref. [11]. For a stationary string lying in the equatorial plane ($\theta = \pi/2$) of a Schwarzschild black hole with the metric:

$$ds^2 = -(1 - \frac{2m}{r}) dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(4.44)

where $m$ is the mass of the black hole, $r = r(\sigma_c)$ is defined by the equation:

$$\left( \frac{dr}{d\sigma_c} \right)^2 = \frac{\Delta(\Delta - b^2)}{r^4}.$$

(4.45)

Here $\Delta = r^2 - 2mr$ and $b$ is an integration constant.

The calculation of the various parts of the potential $U_{RS}$ proceeds in the same way as in subsections 4.1-4.3. The details are however a little more involved due to the complexity of the Christoffel symbols, Riemann tensors etc, so we just give here the result. The two polarizations of perturbations (parallel and perpendicular to the strings plane) propagate independently. The corresponding equations read:

$$(\partial^2_{\sigma_c} - \partial^2_{\tau}) \delta x_\| = \frac{m}{r^5}(2\Delta - 3b^2) \delta x_\|, \quad (4.46)$$

$$(\partial^2_{\sigma_c} - \partial^2_{\tau}) \delta x_\perp = \left[ \frac{m}{r^5}(2\Delta - 3b^2) - \frac{2m^2b^2}{\Delta r^4} \right] \delta x_\perp. \quad (4.47)$$

The solution of these equations is supposed to be analyzed somewhere else.

The above considerations can be easily generalized to the case of a charged black hole. We shall not present here the results but restrict ourselves by the
following interesting observation. The equations for the string perturbations as well as the equation for the equilibrium configuration of a string are greatly simplified for an extreme Reissner-Nordström black hole. The line-element for this case is:

\[ ds^2 = -\frac{(r-m)^2}{r^2}dt^2 + \frac{r^2}{(r-m)^2}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.48) \]

where \( m \) is the mass of the black hole (the charge \( Q = m \)). It follows that:

\[ F = \frac{(r-m)^2}{r^2}, \quad H_{ij} = \text{diag}(1, (r-m)^2, (r-m)^2 \sin^2 \theta). \quad (4.49) \]

\( H_{ij} \) is therefore the metric of ordinary spherical coordinates, with \( (r-m) \) being the radial coordinate, i.e. the 3-dimensional "unphysical" space is flat and the stationary string is represented by a straight line [11]. This is seen explicitly by solving equation (3.7), which in the equatorial plane \( (\theta = \pi/2) \) leads to:

\[ \phi' = \frac{b}{(r-m)^2}, \quad r'^2 = 1 - \frac{b^2}{(r-m)^2}, \quad (4.50) \]

i.e.:

\[ \phi(r-m) = \pm \arcsin \left( \frac{\sqrt{(r-m)^2 - b^2}}{r-m} \right), \quad (4.51) \]

where \( b \) is an integration constant. The potential \( U_{RS} \) which enters in the equations of motion for the perturbations (3.27) contains only the term from the time-dilation part:

\[ V = \frac{m}{r^6}(2(r-m)^3 - b^2(3r-2m)). \quad (4.52) \]

It follows that the perturbations are identical for both polarizations (perpendicular to the plane of the string and in the plane of the string). The solutions of Eq. (3.27) with the potential (4.52) are to be discussed elsewhere.

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