Uniqueness in potential scattering with reduced backscattering data

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Abstract

We consider inverse potential scattering problems where the source of the incident waves is located on a smooth closed surface outside of the inhomogeneity of the media. The scattered waves are measured on the same surface at a fixed value of the energy. We show that this data determines the bounded potential uniquely.

Key words: scattering data, inverse problem, backscattering, uniqueness.

1 Introduction

Consider a potential scattering problem

\[-\Delta u^\text{sc} - \lambda n(x)u^\text{sc} = \lambda [1 - n(x)]u^\text{inc}, \quad x \in \mathbb{R}^d, \quad \lambda = k^2 > 0,\]

where the support of \(n(x) - 1\) belongs to a bounded domain \(\mathcal{O}\), \(n(x)\) is positive and uniformly bounded in \(\mathcal{O}\). Here \(u^\text{inc}\) is an incident wave that satisfies the Helmholtz equation in \(\mathbb{R}^d\) or \(\mathbb{R}^d \setminus S\) where \(S\) is a set outside of the region \(\mathcal{O}\), where sources are distributed. Solution \(u^\text{sc}\) satisfies the radiation condition:

\[u^\text{sc} = u^\text{sc}_\infty(k, \theta) \frac{e^{ikr}}{r^{d-1}} + O \left( r^{-\frac{d+1}{2}} \right), \quad \theta = \frac{x}{r}, \quad r = |x| \to \infty.\]

There are many results on recovering information on the scatterer from the backscattering data. For example, results on the uniqueness of the solution of the inverse problem can be found in [3, 4, 5, 11, 12] and recovering of singularities was studied in

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In all the papers above, it was assumed that the echo data are available for incident waves coming from all the directions.

There are important applications when an observer has an access to the support of the potential only from one side. Additionally, the incident waves can be often emitted only from a bounded region, and not from infinitely remote points as in the classical backscattering problem. Recently, such a non-stationary potential scattering problem (with a potential that is smooth in $\mathbb{R}^3$) has been studied by Rakesh and Uhlmann [13]. They assumed that the incident waves were emitted from points $x$ varying in some sphere. They show the uniqueness for potentials with some restrictions on angular derivatives. In [8] we considered the scattering problem (1) when the incident waves were emitted from a smooth surface $S$ that is a boundary of a bounded domain $B$ located outside of $\mathcal{O}$. We assume that the receivers are also distributed over the same surface $S$, i.e., the following data are available:

$$\left\{u^\text{sc}|_S : u^\text{inc} \text{ emitted from } S\right\}. \quad (3)$$

We have shown that data (3) allows one to determine the interior eigenvalues of the scatterer. In this article, we prove a uniqueness result. Namely, let us fix $\lambda > 0$ that is not a Dirichlet Laplacian eigenvalue for the interior of $S$. We show that data (3) for a fixed value of $\lambda > 0$ determines the potential $n(\cdot)$ uniquely. We also will assume that $\lambda$ is not an eigenvalue of the Dirichlet problem for the equation $(-\Delta - \lambda n(x))u = 0$ in $\mathcal{O}$. Since the support of $n$ is bounded, the latter requirement can be enforced by a slight extension of $\mathcal{O}$. Without loss of the generality, we can assume that the boundary of $\mathcal{O}$ is infinitely smooth and the support of $n(x) - 1$ is located strictly inside of $\mathcal{O}$.

Note also that the problem we consider is different from the problem of recovering of the potential from partial Cauchy data (see e.g. [2]). In the latter problem, it is assumed that Cauchy data are available for all sufficiently regular solutions of the wave equation. On the other hand, in the backscattering problem one knows only the fields on $S$ that are produced by waves emitted from $S$.

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## 2 The main result

From now on, for the sake of simplicity of notations, we assume that $d = 3$. Define operator

$$\mathcal{L} : L_2(S) \to L_2(\partial\mathcal{O}), \quad \mathcal{L}^* : L_2(\partial\mathcal{O}) \to L_2(S),$$

$$(\mathcal{L}\varphi)(x) = \int_{S} e^{-ik|x-y|} \varphi(y) dS_y, \quad (\mathcal{L}^* \mu)(x) = \int_{\partial\mathcal{O}} e^{ik|x-y|} \mu(y) dS_y, \quad k = \sqrt{\lambda} > 0. \quad (4)$$

Function $u^\text{inc} = (\mathcal{L}\varphi)(x)$ represents the incident wave with the sources distributed over $S$ with density $\varphi$. 

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Definition. We will call operator $F^{bs} = F^{bs}(\lambda) : L_2(S) \to L_2(S)$ the backscattering operator if

$$F^{bs} \varphi = u^{sc}|_S, \quad \varphi \in L_2(S),$$

where $u^{sc}$ is the solution of (1) with $u^{inc} = L \varphi$. Thus $F^{bs}$ maps the density of the incident wave, emitted from $S$, to the restriction of the scattered wave on $S$.

Theorem 2.1. Consider two real-valued bounded potentials $n_1$ and $n_2$ and their backscattering far-field operators $F^{bs}_i, i = 1, 2$. If $\lambda = \lambda_0$ is not a Dirichlet eigenvalue for the domain $S$, then the equality $F^{bs}_1 \varphi = F^{bs}_2 \varphi, \lambda = \lambda_0$, on a dense set $\{\varphi\}$ in $L_2(S)$ implies that $\|n_1 - n_2\|_{L^\infty} = 0$.

The following lemmas will be needed to prove the theorem above.

Lemma 2.2. Let $\lambda > 0$ be not an eigenvalue of the negative Dirichlet Laplacian in either of the domains $O$ or $B$ (with the boundary $S$). Then operators $L, L^*$ have dense ranges.

Proof. Let us prove that the range of $L$ is dense. Obviously, it is enough to show that the kernel of the operator $L^*$ is trivial. Assume that the opposite is true. Then there exists $\mu \in L^2(\partial O)$ such that $\mu \not\equiv 0$ and function

$$u := \int_{\partial O} \frac{e^{ik|x-y|}}{|x-y|} \mu(y) dS_y, \quad x \in \mathbb{R}^3, \quad k = \sqrt{\lambda} > 0,$$

which is defined on $\mathbb{R}^3$ and coincides with $L^* \mu$ on $S$, vanishes on $S$. Since

$$(-\Delta - \lambda)u = 0, \quad x \notin \partial O,$$

and $\lambda$ is not an eigenvalue of the Dirichlet problem in $B$, $u \equiv 0$ on $B$. Then from the equation above it follows that $u \equiv 0$ on $\mathbb{R}^3 \setminus O$.

If $\mu$ is continuous, the proof can be completed in a couple of lines using the potential theory. Indeed, $u$ is continuous in $\mathbb{R}^3$ in this case. Thus $u$ satisfies the Helmholtz equation and the homogeneous Dirichlet boundary condition in $O$. Since $\lambda$ is not an eigenvalue, it follows that $u \equiv 0$ in $O$, i.e., $u \equiv 0$ in $\mathbb{R}^3$. The latter contradicts the fact that the jump of the normal derivative of $u$ on $\partial O$ is equal to $-4\pi \mu \not\equiv 0$.

If $\mu \in L_2(\partial O)$, then we approximate $\mu$ in $L_2(\partial O)$ by smooth functions $\mu_n$. Consider

$$u_n = \int_{\partial O} \frac{e^{ik|x-y|}}{|x-y|} \mu_n(y) dS_y, \quad x \in \mathbb{R}^3. \quad (5)$$

If we restrict $u_n$ to $\partial O$, then operator (5) becomes a pseudo-differential operator on $\partial O$ of order $-1$, and therefore $u_n|_{\partial O}$ has a limit in $H^{1/2}(\partial O)$ as $n \to \infty$ (as well as in $H^1(\partial O)$). Functions $u_n$ satisfy the Helmholtz equation outside of $\partial O$, and they satisfy the radiation conditions. Thus the convergence of $u_n|_{\partial O}$ and standard a priori estimates in $H^1$ for the
solutions of the Helmholtz equation imply that functions $u_n$ converge in $H^1(\Omega)$ and in $H^1_{loc}(\mathbb{R}^3 \setminus \Omega)$. Obviously, they converge to $u \equiv 0$ in $H^1_{loc}(\mathbb{R}^3 \setminus \Omega)$. Thus

$$u_n|_{\partial \Omega} \to 0 \quad \text{in} \quad H^{1/2}(\partial \Omega) \quad \text{as} \quad n \to \infty.$$  

Hence $u_n$ converge in $H^1(\Omega)$ to a solution of the homogeneous Dirichlet problem. Since $\lambda$ is not an eigenvalue of the Dirichlet problem in $\Omega$, this implies that $u_n \to 0$ in $H^1(\Omega)$.

Since $\mu_n$ is smooth, the jump on $\partial \Omega$ of the normal derivative of the potential $u_n$ defined by (5) is equal to $-4\pi \mu_n \neq 0$. On the other hand, the normal derivatives of weak (in $H^1$) solutions of the Helmholtz equation are well defined, and from the weak (in $H^1$) convergence of $u_n$ to zero it follows that this jump (which is equal to $\mu_n$) tends to zero in $H^{-1/2}(\partial \Omega)$. Since $\mu_n$ approximates $\mu$ in $L^2(\partial \Omega)$, it follows that $\mu = 0$. This contradicts the assumption made in the first lines of the proof. Thus the density of the range of the operator $L$ is proved. Similar arguments are valid for $L^*$.

Denote by $F_0(\lambda), F^{\text{out}}(\lambda)$ Dirichlet-to-Neumann maps for the Helmholtz equation in the interior and exterior of $\Omega$, respectively. The solutions are assumed to satisfy the radiation condition when $F^{\text{out}}$ is defined. Let $F_n$ be the Dirichlet-to-Neumann map for the equation $(\Delta + \lambda_n)u = 0$ in $\Omega$. The normal vector in all the cases is assumed to be directed outside of $\Omega$.

**Lemma 2.3.** The backscattering far-field operator has the following representation:

$$F^{\text{bs}} = \frac{1}{4\pi} \mathcal{L}^*(F_0^{\text{in}} - F^{\text{out}})(F_n^{\text{in}} - F^{\text{out}})^{-1}(F_0^{\text{in}} - F_n^{\text{in}}) \mathcal{L}. \quad (6)$$

**Remark.** These formulas are direct analogues of the formulas for the scattering amplitude in the problem of scattering of the plane waves (see [7, Th.2.3] in the case of the transmission problem). The only difference is that a plane wave is defined by the direction $\omega$ of the incident wave, and $S$ is replaced by the unit sphere $S^2 = \{\omega : |\omega| = 1\}$ in this case. The operators $\mathcal{L}, \mathcal{L}^*$ are also slightly different in the case of the plane waves. In particular,

$$\mathcal{L} : L_2(S^2) \to L_2(\partial \Omega), \quad \mathcal{L} \varphi(x) = \int_{S^2} e^{ik\omega \cdot x} \varphi(\omega) dS_\omega. \quad (7)$$

**Proof.** Let us prove (6). Note that $u^{\text{inc}} = \mathcal{L} \varphi$. We will look for $u^{\text{sc}}$ outside of $\Omega$ in the form of the potential $u^{\text{sc}} = \mathcal{L}^* \mu$ with an unknown density $\mu$. Function $\mu$ must be chosen in such a way that $u^{\text{sc}}$ allows an extension in $\Omega$ that satisfies (1).

Every solutions of the Schrödinger equation with a bounded potential belongs to $H^2(\Omega')$ for any bounded domain $\Omega'$. Therefore functions $u^{\text{sc}}, u^{\text{inc}}$ and their normal derivatives are well defined on $\partial \Omega$. We reduce the scattering problem (1), (2) to the following equation on $\partial \Omega$ for unknown $\mu$:

$$F_n(u^{\text{sc}} + u^{\text{inc}}) = F^{\text{out}} u^{\text{sc}} + F_0 u^{\text{inc}}. \quad (8)$$
We note that operator $F_n$ is symmetric, and the imaginary part of the quadratic form of operator $F_{\text{out}}$ coincides with the total cross section, and therefore is positive (see [7, Lemma 2.1]). Thus, operator $F_n - F_{\text{out}}$ is invertible, and equation (8) implies that

$$u^{sc} = (F_n - F_{\text{out}})^{-1}(F_0 - F_n) u^{\text{inc}}. \quad (9)$$

Evidently $\mu = \frac{1}{4\pi}(F_0 - F_{\text{out}})u^{sc}$. It remains only to substitute (9) for $u^{sc}$ in the latter equation for $\mu$ and note that $F^{bs}\varphi = u^{sc}\big|_S = L^{*}\mu\big|_S$.

\[\square\]

**Proof of Theorem 2.1.** We will reduce the statement of the theorem to the Gelfand-Calderon problem, which is solved in [9, Th.1] when $d = 3$ and in [11, Th.2.1] when $d = 2$.

We preserve notations $F_0, F_{\text{out}}$ for the Dirichlet-to-Neumann maps for the Helmholtz equation in the interior and exterior of $\mathcal{O}$, respectively, and we denote by $F_{\text{in}}^{n_1}, F_{\text{in}}^{n_2}$ the Dirichlet-to-Neumann maps for the Schrödinger equations in $\mathcal{O}$ with potentials $\lambda n_1$ and $\lambda n_2$, respectively.

Operators

$$(F_0^{\text{in}} - F_{\text{out}})(F_{n_1}^{\text{in}} - F_{\text{out}})^{-1}(F_0^{\text{in}} - F_{n_1}^{\text{in}}) : L_2(\partial\mathcal{O}) \to L_2(\partial\mathcal{O}), \quad i = 1, 2, \quad (10)$$

are bounded (and also compact). Indeed, each of the Dirichlet-to-Neumann operators introduced above is a pseudo-differential operator of the first order (non-smoothness of the potential does not play any role here, since the support of the potential is strictly inside of the domain). Their full symbols were calculated in [6, Sect.3]. From this calculation it follows that operator $F_0^{\text{in}} - F_{\text{out}}$ has order one, operator $(F_{n_1}^{\text{in}} - F_{\text{out}})^{-1}$ has order $-1$, and a couple of the first terms of the full symbol of operator $F_0^{\text{in}} - F_{n_1}^{\text{in}}$ vanish, i.e., the latter operator is compact. Thus (10) is compact.

Assume that data (3) for $n_1$ and $n_2$ coincide on a dense set $\{\varphi\}$ in $L_2(\mathcal{S})$. Then from Lemma 2.2 it follows that operators (10) are equal. The first factor from the left in (10) is an invertible operator (see the justification of the transition from (8) to (9)). Hence, the equality of operators in (10) implies that

$$(F_{n_1}^{\text{in}} - F_{\text{out}})^{-1}(F_0^{\text{in}} - F_{n_1}^{\text{in}}) = (F_{n_2}^{\text{in}} - F_{\text{out}})^{-1}(F_0^{\text{in}} - F_{n_2}^{\text{in}}) \quad \text{as operators in } L_2(\partial\mathcal{O}).$$

Adding and subtracting $F_{\text{out}}$ in the right factors, we get

$$(F_{n_1}^{\text{in}} - F_{\text{out}})^{-1}(F_0^{\text{in}} - F_{\text{out}}) = (F_{n_2}^{\text{in}} - F_{\text{out}})^{-1}(F_0^{\text{in}} - F_{\text{out}}) \quad \text{as operators in } L_2(\partial\mathcal{O}).$$

Hence, operators

$$(F_{n_1}^{\text{in}} - F_{\text{out}})^{-1}, (F_{n_2}^{\text{in}} - F_{\text{out}})^{-1} : H^{-1}(\partial\mathcal{O}) \to L_2(\partial\mathcal{O}),$$

are equal, and therefore,

$$F_{n_1}^{\text{in}}, F_{n_2}^{\text{in}} : L_2(\partial\mathcal{O}) \to H^{-1}(\partial\mathcal{O})$$

are equal. Thus

$$F_{n_1}^{\text{in}}\varphi = F_{n_2}^{\text{in}}\varphi$$

for every $\varphi \in L_2(\partial\mathcal{O})$.

Now uniqueness follows from [11] if $d = 2$ and [9] if $d > 2$. 

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