Communication Efficient Decentralized Training with Multiple Local Updates

Xiang Li
School of Mathematical Sciences
Peking University
Beijing, 100871, China
smslixiang@pku.edu.cn

Wenhao Yang
Center for Data Science
Peking University
Beijing, 100871, China
yangwhsms@gmail.com

Shusen Wang
Department of Computer Science
Stevens Institute of Technology
Hoboken, NJ 07030, USA
shusen.wang@stevens.edu

Zhihua Zhang
School of Mathematical Sciences
Peking University
Beijing, 100871, China
zhzhang@math.pku.edu.cn

Abstract

Decentralized optimization has been demonstrated to be very useful in machine learning. This work studies the communication-efficiency issue in decentralized optimization. We analyze the Periodic Decentralized Stochastic Gradient Descent (PD-SGD) algorithm, a straightforward combination of federated averaging and decentralized SGD. For the setting of non-convex objective and non-identically distributed data, we prove that PD-SGD converges to a critical point. In particular, the number of local SGDs trades off communication and local computation. From an algorithmic perspective, we analyze a novel version of PD-SGD, which alternates between multiple local updates and multiple decentralized SGDs. We also show that when we periodically shrink the length of local updates, this generalized PD-SGD can better balance the communication-convergence trade-off both theoretically and empirically.

1 Introduction

The data (not necessarily identically distributed) are partitioned among $n$ work nodes. We seek to learn the model parameter (aka optimization variable) $x \in \mathbb{R}^d$ by solving the following distributed empirical risk minimization problem:

$$
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{k=1}^{n} f_k(x), \quad \text{where} \quad f_k(x) := \mathbb{E}_{\xi \sim \mathcal{D}_k} [F_k(x; \xi)].
$$

Here $\mathcal{D}_k$ is the distribution of data on the $k$-th node with $k \in [n] \triangleq \{1, \cdots, n\}$. Such a problem is traditionally solved under centralized optimization paradigms such as parameter servers [17]. Federated Learning (FL), which has a central parameter server, enables massive edge computing devices to jointly learn a centralized model while keeping all local data localized [13, 34, 26, 12, 18, 30, 57].

As opposed to the centralized optimization, decentralized optimization lets every worker node to collaborate only with their neighbors by exchanging information. In recent years, many decentralized algorithms have been proposed for solving the problem (1) [1, 50, 36, 3, 20, 15, 41]. A typical decentralized algorithm works in this way: a node collects its neighbors’ model parameters ($x$), take
the average, and then performs a (stochastic) gradient descent to update its local parameters [20].
Decentralized optimization can outperform the centralized under specific settings [20].

The communication costs can be the bottleneck of distributed optimization when the number of model parameters or the amount of worker nodes is large. It is well known that deep neural networks have a large number of parameters. For example, ResNet-50 [5] has 25 million parameters, so sending $x$ through a computer network can be expensive and time-consuming. Due to modern big data and big models, a large number of worker nodes can be involved in distributed optimization, which further increases the communication cost. The situation can be exacerbated if the worker nodes in distributed learning are remotely connected, which is the case in edge computing and other types of distributed learning.

In recent years, numerous communication-efficient algorithms have been developed for reducing the communication between the parameter server and worker nodes; we will discuss the prior work subsequently. Decentralized optimization, as well as the centralized, suffers from high communication costs. To directly save communication, many researchers let more local updates happen before each synchronization [26, 22, 39, 44, 43]. Periodic Decentralized SGD (PD-SGD) results from the application of local updates in decentralized optimization [44]. However, communication-efficiency for decentralized optimization has not been intensively studied, except for a few papers [20, 15, 41, 44, 45].

1.1 Contributions
In the paper, we analyze PD-SGD with a novel adaptive local update scheme in a non-convex stochastic optimization setting and non-identically distributed training data (i.e., $D_1, \cdots, D_n$ are not the same). This adaptive local update scheme alternates (precisely $I_1$ steps of) multiple local updates and multiple (precisely $I_2$ steps of) decentralized SGDs. When $I_2$ is set as 1, we recover the conventional PD-SGD. We argue that when the initial error (i.e., $f(x_1) - \min_x f(x)$) is sufficiently large, there exists an optimal $I_2$ that minimize the convergence error in terms of communication steps. From experiments, with a fixed $I_1$ larger $I_2$ often leads to higher convergence speed and lower error floor.

We empirically find the existence of a trade-off between convergence and communication. Specifically, more local computation (i.e., large $I_1/I_2$) will save communication by decreasing communication frequency but lower the convergence rate due to accumulated residual errors. We find that the decaying strategy that periodically halves the value of $I_1$ and keeps a moderate value of $I_2$ make PD-SGD better balance this trade-off. We also theoretically testify the convergence of this strategy in PD-SGD.

1.2 Related Work

Federated optimization. The optimization problem implicit in FL is referred to as federated optimization, drawing a connection (and contrast) to distributed optimization. Currently the state-of-the-art algorithm in federated optimization is Federated Averaging (FedAvg) [14, 26], which is a centralized optimization method. In a round of FedAvg, a small set of nodes is activated and alternates between running multiple local SGDs and sending updated parameters to the central server. With some ideal simplification (such as identical $D_k$’s or all activated nodes), the convergence of FedAvg has been analyzed by [56, 39, 44, 49]. Li et al. [19] is the first to analyze FedAvg in realistic federated setting (different $D_k$’s and partial activated nodes). PD-SGD is a natural extension of FedAvg (or Local SGD) towards decentralized optimization. The recent independent work, MATCHA [45], extends PD-SGD to a more federated setting by only activating a random subgraph of the network topology each round. Besides, many empirical studies are investigating the effect of system heterogeneity [31, 55], the power of scalability [22], strategies of preserving privacy [6, 27], and saving communication [9].

Decentralized stochastic gradient descent (D-SGD). Decentralized (stochastic) algorithms used to tackle the failure of being centralized as a compromise and used to be studied as consensus optimization in the control community [28, 50, 36]. Lian et al. [20] first justifies the potential advantage of Decentralized SGD (D-SGD) over its centralized counterpart. It not only reduces the communication cost but achieves the same linear speed-up as centralized counterparts when more
nodes are available [20]. This promising result pushes the research of distributed optimization from a sheer centralized mechanism to a more decentralized pattern [15, 41, 11, 45, 24].

**Communication efficient algorithms.** The current methodology towards communication-efficiency in distributed optimization could be divided into two categories. The more direct approach is to reduce the size of the messages through gradient compression or sparsification [32, 23, 51, 40, 42, 8]. An orthogonal one is to pay more local computation for less communication, e.g., one-shot aggregation [53, 54, 16, 21, 46], primal-dual algorithms [37, 38, 7] and distributed Newton methods [33, 52, 29, 35, 25]. Beyond them, a simple but powerful method is to reduce the communication frequency by allowing more local updates [58, 39, 22, 47, 44], which we focus on in this paper.

The most relevant works include [44, 45]. Wang and Joshi [44] proposes a unified framework termed as Cooperative SGD (C-SGD) that is able to combine decentralization and local updates. The algorithm we analyzed is an extension of C-SGD by introducing a new parameter (precisely $I_2$) controlling the length of decentralized SGDs. But they analyze C-SGD by assuming all work nodes have access to the underlying distribution (hence data is identically distributed). The recent independent work MATCHA [45] makes communication only among a random small portion of work nodes at each round\(^1\). When no nodes is activated, local updates come in. Consequently their result is formulated for random connecting matrix (i.e., $W_t$ in our case). The theories of this work and MATCHA [45] are independently developed based on C-SGD [44]. While MATCHA analyzes a random sequence of $W_t$, this work studies a deterministic sequence of $W_t$. The theories of MATCHA does not straightforwardly extend to deterministic sequence of $W_t$.

2 Notation and Preliminaries

**Decentralized system.** In Figure 1a, we illustrate a decentralized system that does not have a central parameter server. There are $n = 5$ nodes in the network where a node only communicates with its neighbors. Conventionally, the system can be described by a graph $G = ([n], W)$ where $W$ is a $n \times n$ doubly stochastic matrix describing the weights of the edges. A nonzero entry $w_{ij}$ indicates that the $i$-th and $j$-th nodes are connected.

**Definition 1.** We say a matrix $W = [w_{ij}] \in \mathbb{R}^{n \times n}$ to be symmetric and doubly stochastic, if $W$ is symmetric and each row of $W$ is a probability distribution over the vertex set $[n]$, i.e., $w_{ij} \geq 0$, $W = W^T$, and $W 1_n = 1_n$.

**Notation.** Let $x^{(k)} \in \mathbb{R}^d$ be the optimization variable (aka model parameters in machine learning language) held by the $k$-th node. The step is indicated by a subscript, e.g., $x^{(k)}_i$ is the parameter held by the $k$-th node in step $t$. Note that at any time moment, $x^{(1)}, \ldots, x^{(n)}$ may not be equal. Let $X := [x^{(1)}, \ldots, x^{(n)}] \in \mathbb{R}^{d \times n}$ be the concatenation of all the variables and $\bar{x} := \frac{1}{n} \sum_{k=1}^n x^{(k)}$ be the averaged variable. Let $\nabla F_k(x^{(k)}; \xi^{(k)})$ be the derivative of $F_k$ w.r.t. variable $x^{(k)}$.

\(^1\)Specifically, they first decompose the network topology into joint matchings (or subgraphs), then randomly activates a small portion of matchings.
and \( G(X; \xi) := [\nabla F_1(x^{(1)}; \xi^{(1)}), \ldots, \nabla F_n(x^{(n)}; \xi^{(n)})] \in \mathbb{R}^{d \times n} \) be the concatenated gradient evaluated at \( X \) with datum \( \xi \). We denote by \( [n] := \{1, 2, \ldots, n\} \). We term \( V_t = \mathbb{E}_n^{1/2} \sum_{k=1}^n \| x_t^{(k)} - \pi_t \|^2 \) as the residual error of \( X_t \).

**Decentralized Stochastic Gradient Descent.** D-SGD \([1, 15]\) works in the following way. At Step \( t \), the \( k \)-th node randomly chooses a local datum, \( \xi_t^{(k)} \), and uses its current local variable, \( x_t^{(k)} \), to evaluate the stochastic gradient \( \nabla F_k(x_t^{(k)}; \xi_t^{(k)}) \). Then each node performs stochastic gradient descent (SGD) to obtain an intermediate variable \( x_t^{(k)} \) and finally finishes the update by collecting and aggregating its neighbors’ intermediate variables:

\[
    x_t^{(k)} \leftarrow x_t^{(k)} - \eta \nabla F_k(x_t^{(k)}; \xi_t^{(k)}),
\]

\[
    x_{t+1}^{(k)} \leftarrow \sum_{l \in \mathcal{N}_k} w_{kl} x_t^{(l)},
\]

where \( \mathcal{N}_k = \{l \in [n] | w_{kl} > 0\} \) contains the indices of the \( k \)-th node’s neighbors. Noting that the communication is subsequent to local updates, we refer to this update rule as communication-after.

**Remark 1.** In one step of D-SGD, we can exchange Step (2) and Step (3) so that we first average the local variable with neighbors and then update the local stochastic gradient into the local variable. The update rule becomes \( x_{t+1}^{(k)} \leftarrow \sum_{l \in \mathcal{N}_k} w_{kl} x_t^{(l)} - \eta \nabla F_k(x_t^{(k)}; \xi_t^{(k)}) \). The benefit is that the computation of stochastic gradients (i.e., \( \nabla F_k(x_t^{(k)}; \xi_t^{(k)}) \)) and communication (i.e., Step (3)) can be run in parallel. Our theory in Section 4 can be parallel to these cases. We provide the result in Appendix C.

**Periodic Decentralized SGD.** PD-SGD involves more local updates than D-SGD so as to lower the communication frequency. In particular, PD-SGD allows each node to perform multiple local updates before it communicates with its neighbors. The communication interval is the number of local updates between two consecutive synchronization. Larger the communication interval, less the communication frequency. PD-SGD is a special case of the framework Cooperative SGD \([44]\).

### 3 PD-SGD with adaptive update schemes

**The introduction of multiple decentralized SGDs.** To lower the communication frequency, we propose a new version of PD-SGD (Algorithm 1). In particular, each node repeatedly alternates between a local computation period and a communication period. In the local computation period, each node simply runs SGD, i.e., (2), for \( I_1(I_1 \geq 0) \) times in parallel. In the communication period, each node runs \( I_2(I_2 \geq 1) \) times D-SGD which is a combination of (2) and (3). Note that communication only happens in this period. In this way, a node performs \( \frac{I_2}{I_1+I_2}T \) communication per \( T \) total steps. Figure 1b illustrates one round of PD-SGD with \( I_1 = 3 \) and \( I_2 = 2 \).

The introduction of \( I_2 \) extends the scope of previous framework C-SGD \([44]\). Many existing algorithms become special cases when the period length \( I_1, I_2 \), and the connected matrix \( W \) are carefully determined. As an evident example, PD-SGD is reduced to D-SGD when \( I_1 = 0 \). Another important example is centralized Local SGD which is a case of PD-SGD with \( I_1 > 1, I_2 = 1 \), and \( W = Q \triangleq \frac{1}{n}1_n1_n^T \). See Table 1 for more examples. From this point of view, the theoretical analysis of PD-SGD provided in Section 4 naturally applies to these existing algorithms. We compare our results with their initial results in Appendix D.

| Algorithms                        | \( I_1 \) | \( I_2 \) | \( W \)       |
|----------------------------------|----------|----------|---------------|
| Fully synchronous SGD [2]        | 0        | 1        | Q             |
| PR-SGD [56, 49, 48] or Local SGD [22, 39, 44] | \( \geq 1 \) | 1        | Q             |
| Decentralized SGD (D-SGD) [10, 20] | 0        | 1        | Assumption 4  |
| Periodic Decentralized SGD (PD-SGD) [44] | \( \geq 0 \) | 1        | Assumption 4  |

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Algorithm 1 Periodic Decentralized SGD

1: **Input:** total steps $T$, step size $\eta$ and communication parameters $I_1 \geq 0, I_2 \geq 1$
2: **for** $t = 1$ to $T$ **do**
3: $\mathbf{X}_{t+\frac{1}{2}} \leftarrow \mathbf{X}_t - \eta \mathbf{G} (\mathbf{X}_t; \xi_t)$
4: **if** $t \ mod \ (I_1 + I_2) \in [I_1] \ **then**$
5: $\mathbf{X}_{t+1} \leftarrow \mathbf{X}_{t+\frac{1}{2}} \ **local** \ updates$
6: **else**
7: $\mathbf{X}_{t+1} \leftarrow \mathbf{X}_{t+\frac{1}{2}} \mathbf{W} \ **communication**$
8: **end if**
9: **end for**

Algorithm 2 PD-SGD with the decaying strategy

1: **Input:** total steps $T$, step size $\eta$, $I_1 \geq 0, I_2 \geq 1$ and decay interval $M$
2: $\mathcal{N}(t) = \arg\max \{j \leq t : j \in \mathcal{I}\}$, where $\mathcal{I} = \{M \cdot \sum_{i=0}^{[\frac{t}{M}]} [I_1 + I_2] : j \leq 1 + \lfloor \log_2 I_1 \rfloor \}$
3: **for** $t = 1$ to $T$ **do**
4: $\mathbf{X}_{t+\frac{1}{2}} \leftarrow \mathbf{X}_t - \eta \mathbf{G} (\mathbf{X}_t; \xi_t)$
5: **if** $(t - \mathcal{N}(t)) \ mod \ (I_1 + I_2) \in [I_1] \ **then**$
6: $\mathbf{X}_{t+1} \leftarrow \mathbf{X}_{t+\frac{1}{2}} \ **local** \ updates$
7: **else**
8: $\mathbf{X}_{t+1} \leftarrow \mathbf{X}_{t+\frac{1}{2}} \mathbf{W} \ **communication**$
9: **end if**
10: **if** $t \in \mathcal{I}$ **then**
11: $I_1 \leftarrow \lfloor \frac{I_1}{2} \rfloor \ **decay** \ I_1 \ **by** \ half
12: **end if**
13: **end for**

Decaying the length of local updates. Large local computation ratio (i.e., $I_1/I_2$) accumulates residual errors, which in turn slows down the convergence due to non-identically distributed data. A similar phenomena is also observed by an independent work [43], which finds that faster initial drop of global loss will result higher final error floor. From the experiment results (Figure 3d and 3c), we are inspired to decay $I_1$. Specifically, every $M$ rounds, we shrink $I_1$ by half empirically but fix $I_2$ (Algorithm 2). Note that $\mathcal{I}$ is the set of steps where we decay $I_1$ and $\mathcal{N}(t)$ returns the nearest step before $t$ after which the length of local updates is going to decline. In this way, we gradually half $I_1/I_2$ until it reaches zero (in the end, no local updates are performed). This simple strategy empirically performs better than vanilla PD-SGD.

4 Convergence analysis

4.1 Assumptions

In Eq. (1), we define $f_k (x) := \mathbb{E}_{\xi \sim \mathcal{D}_k} [F_k (x; \xi)]$ as the objective function of the $k$-th node. Here, $x$ is the optimization variable and $\xi$ is a data sample. Note that $f_k (x)$ captures the data distribution in the $k$-th node. We make a standard assumption: $f_1, \ldots, f_n$ are smooth.

**Assumption 1** (Smoothness). For all $k \in [n]$, $f_k (\cdot)$ is smooth with modulus $L$, i.e.,

$$\| \nabla f_k (x) - \nabla f_k (y) \| \leq L \| x - y \|, \ \forall \ x, y \in \mathbb{R}^d.$$  

We assume the stochastic gradients have bounded variance. The assumption has been made by the prior work [20, 44, 41, 40].

**Assumption 2** (Bounded variance). There exists some $\sigma > 0$ such that $\forall \ k \in [n]$,

$$\mathbb{E}_{\xi \sim \mathcal{D}_k} \| \nabla F_k (x; \xi) - \nabla f_k (x) \|^2 \leq \sigma^2, \ \forall \ x \in \mathbb{R}^d.$$  

Recall from Eq. (1) that $f (x) = \frac{1}{n} \sum_{k=1}^{n} f_k (x)$ is the global objective function. If the data distributions are not identical, that is, $\mathcal{D}_k \neq \mathcal{D}_l$ for $k \neq l$, then the global objective is not the same to the local objectives. In this case, we define $\kappa$ to quantify the degree of non-iid. If the data across nodes are iid, then $\kappa = 0$.

**Assumption 3** (Degree of non-iid). There exists some $\kappa \geq 0$ such that

$$\frac{1}{n} \sum_{k=1}^{n} \| \nabla f_k (x) - \nabla f (x) \|^2 \leq \kappa^2, \ \forall \ x \in \mathbb{R}^d.$$  

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Finally, we need to assume the nodes are well connected. Otherwise, the update in one node cannot be propagated to another node within a few iterations. In the worst case, if the system is not fully connected, the algorithm will not converge. We use $\rho = |\lambda_2|$ to quantify the connectivity where $\lambda_2$ is the second largest absolute eigenvalue. A small $\rho$ indicates nice connectivity. If the connections form a complete graph, then $W = \frac{1}{2} I_n I_n^T$ and thus $\rho = 0$.

**Assumption 4 (Nice connectivity).** The $n \times n$ connectivity matrix $W$ is symmetric doubly stochastic. Denote its eigenvalues by $1 = |\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n| \geq 0$. We assume the spectral gap $1 - |\lambda_2| \in (0, 1]$ and denote by $\rho = |\lambda_2| \in [0, 1)$.

### 4.2 Main Results

Recall that $\mathbf{x}_t = \frac{1}{N} \sum_{k=1}^{N} x^{(k)}_t$ is defined as the averaged variable in the $t$-th iteration. In the PD-SGD algorithm, $I_1$ and $I_2$ are respectively the numbers of local updates (i.e., simply (2)) and D-SGDS (i.e., a combination of (2) and (3)) in every round. Note that the objective function $f(x)$ is often non-convex when neural networks are applied. We thereby prove the convergence to a stationary point, e.g., a local minimum or saddle point. Theorem 1 shows the the gradient $\|\nabla f(\mathbf{x}_t)\|^2$ converges to zero.

**Theorem 1 (Convergence of PD-SGD).** Let Assumption 1, 2, 3, 4 hold and the constants $L$, $\kappa$, $\sigma$, and $\rho$ be defined therein. Let $\Delta = f(\mathbf{x}_0) - \min_{x} f(x)$ be the initial error and $K = \frac{L_1}{1 - \rho^2} + \frac{\rho}{1 - \rho}$. If the learning rate $\eta$ is small enough such that

$$\eta < \min \left\{ \frac{1}{2L}, \frac{1}{4\sqrt{2}LK} \right\},$$

then

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2\Delta}{\eta T} + \frac{\eta^2 \sigma^2}{n} + 4\eta^2 L^2 C_1 \sigma^2 + 4\eta^2 L^2 C_2 K^2.$$

where

$$C_1 = \frac{1}{2L} \left( \frac{1 + \rho^2 I_2}{1 - \rho^2 I_1} + \frac{1 + \rho^2}{1 - \rho^2} I_1 \right) + \frac{\rho^2}{1 - \rho^2}$$

and

$$C_2 = \min \left\{ 4K \left[ \frac{1}{2L} \left( \frac{1 + \rho^2 I_2}{1 - \rho^2 I_1} + \frac{1 + \rho^2}{1 - \rho I_1} \right) + \frac{\rho}{1 - \rho} \right], 4K^2 \right\}.$$  

**Theorem 2 (Convergence of PD-SGD with the decaying strategy).** Under the same condition and hyperparameters of Theorem 1, if we equip PD-SGD with the decaying strategy, then the bound (5) still holds by replacing $C_1$, $C_2$ with

$$\tilde{C}_1 = \frac{1}{T} \left( \frac{I_1}{1 - \rho^2 I_2^2} \rho^{2(T + I_2 - \max I - 1)} + (1 - \max I) \frac{\rho^2}{1 - \rho^2} \right)$$

and

$$\tilde{C}_2 = 4K \left[ \frac{1}{T} \left( \frac{I_1}{1 - \rho^2} \rho^{T I_2 - \max I - 1} + (1 - \max I) \frac{\rho}{1 - \rho} \right) \right].$$

where $I$ is the set of decay steps and $\max I = \max_{j \in I} j$.

If $T$ is fixed before running the algorithm, then we can set learning rate to $\eta = \mathcal{O}(\sqrt{T})$ and obtain the following corollary. The corollary shows the convergence against the number of total steps ($T$), the number of nodes ($n$), and the total period ($I = I_1 + I_1$).

**Corollary 1.** In the setting of Theorem 1, if we choose the learning rate to $\eta = \sqrt{\frac{T}{n}}$, then for PD-SGD we have

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2\Delta \sqrt{T}}{\sqrt{n} \sqrt{T}} + \frac{L_1 \sigma^2}{\sqrt{n} \sqrt{T}} + \frac{4C_1 L^2 \sigma^2 n + 16K^2 L^2 \kappa^2 n}{TI} \leq \mathcal{O} \left( \frac{\sqrt{T}}{\sqrt{n}} + \frac{nT}{T} \right).$$

**Remark 2.** Corollary 1 shows the convergence against computation. For fixed computation budge, bigger $I_1$ makes the convergence slower. Later on, we will show in (11) that for fixed communication budge $C = \frac{T I_2}{I_1 + I_2}$, reasonable $I_1$ makes the convergence faster.
4.3 Discussion

**Error decomposition.** From Theorem 1, the upper bound (5) is decomposed into two parts. The first part is exactly the same as the optimization error bound in fully synchronous SGD [2]. The second part is termed as residual errors as it results from performing periodic local updates and reducing inter-node communication. In the study of centralized parallel SGD, the application of local updates inevitably results the residual error [15, 39, 44, 4, 19, 48]. The residual error often grows with the number of local updates $I$. When data are independently and identically distributed $^2$ (i.e., $\kappa = 0$), [44] shows that the residual error grows only linearly in $I$. Haddapour et al. [4] also achieves the linear dependence on $I$ but only requires each node draws samples from its local partitions. When data are not identically distributed (i.e., $\kappa$ is strictly positive), both Yu et al. [49] and Zhou and Cong [56] show that the residual error grows quadratically in $I$. Theorem 1 shows that the residual error of PD-SGD is $O(I\sigma^2 + I^2\kappa^2)$ where the linear dependence comes from the stochastic gradients and the quadratic dependence results from the heterogeneity. $^3$ The similar dependence also established for centralized momentum SGD in [48].

**Effect of $I_1$ and $I_2$ on communication efficiency.** In every round, PD-SGD performs $I = I_1 + I_2$ steps of SGDs and $I_2$ steps of communications. From (10), large $I$ lowers the convergence rate since it increases the residual error. Given fixed $I$, traditional methods (see Table 1) simply set $I_2 = 1$ so that the communication frequency is reduced by a factor $I$. However, we argue that the optimal value of $I_2$ exists. To see that, let’s fix the communication budget as $C$. Replacing $T$ by $C\frac{I_2}{I_1}$, we obtain from (10) that

**Corollary 2.** In the setting of Theorem 1, if we choose the learning rate to $\eta = \sqrt{\frac{\kappa}{I_2}}$ and fix the total communication steps as $C$, then

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[\left\|\nabla f(\bar{x}_t)\right\|^2\right] \leq \frac{2\Delta \sqrt{T_2}}{\sqrt{\kappa} C} + \frac{L\sigma^2 \sqrt{T_2}}{\sqrt{\kappa} n C} + \frac{2\omega_1 n L^2 \sigma^2 I_2}{C} + \frac{16\omega_2 n L^2 \kappa^2 I_2}{C},$$

where $\omega_1 = \frac{1 + \rho^2 + \frac{1}{\rho \sqrt{2}}}{{1 - \rho \sqrt{2}} I_1} + \frac{\rho^2}{1 - \rho} I_2$ and $\omega_2 = \frac{1 + \rho^2 + \frac{1}{\rho \sqrt{2}}}{{1 - \rho \sqrt{2}} I_1} + \frac{\rho^2}{1 - \rho} \frac{1}{\kappa} I_2$.

When $\Delta$ is large enough, the right hand side of (11), as a function of $I_2$, first decreases and then increases (see Figure 2a), indicating the existence of optimal communication step $I_2^*$. While fixing the communication budget $C$, reasonably large $I_1$ is good for communication but too large $I_1$ may degrade the performance since $I_1$ will also affect $\omega_1$ and $\omega_2$.

**Effect of connectivity $\rho$.** The network connectivity also has impact on convergence rate (see Figure 2b). The connectivity is measured by $\rho$, the second largest absolute eigenvalue of $W$. If the graph is nicely connected, in which case $\rho$ is close to zero, then the update in one node will be propagated

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$^3$This is also possible if all nodes have access to the entire data, e.g., the distributed system may shuffle data regularly so that each node actually optimizes the same loss function.

$^2$As mentioned in Section 3, this conclusion also holds for all algorithms listed in Table 1.
to all the other nodes very soon, and the convergence is thereby fast. In this case, $\omega_1 \approx (\frac{1}{\rho})^2 + \frac{1}{T^2}$ and $\omega_2 \approx \frac{1}{T}$, (11) becomes $O\left(\sqrt{\frac{I_2}{n\rho}} + n\frac{I_1 I_2}{\rho^2}(\sigma^2 + \kappa^2) + n^2 I_1^2 \kappa^2\right)$. If the connection is very sparse, i.e., $\rho \approx 1$, then $\omega_1 \approx \omega_2 \approx \frac{1}{1-\rho}\frac{1}{T^2}$, and (11) becomes $O\left(\sqrt{\frac{I_2}{n\rho}} + \frac{n\sigma^2}{C(1-\rho)^2} + \frac{n^2 \kappa^2}{C(1-\rho)^2}\right)$. It shows that for a sparsely connected network only $I_2$ determines the bound (11).

**Effect of the variance $\sigma^2$ and $\kappa^2$.** The gradient variance is bounded by $\sigma^2$ which is defined in Assumption 2. Two terms in (11) are proportional to $\sigma^2$. Interestingly, locally running $I_1 > 1$ SGD's and setting the learning rate proportional to $1/\sqrt{T}$ alleviates the effect of variance.

The inter-node variance or the degree of non-iid is measured by $\kappa^2$ which is defined in Assumption 3. If the data across the nodes have identical distribution, then $\kappa$ will be zero. The nature of non-identical distribution negatively affects convergence.

When $\kappa = \sigma = 0$, we recover the convergence rate of fully synchronous GD. When $\kappa = 0$, we recover the result in Wang and Joshi [44]. We detail the discussion with their work in Appendix D.

### The decay strategy.
Comparing Theorem 2 and Theorem 1, we can find that the conclusion is very similar except the value of $C_1$, $C_2$ (and its counterparts). Obviously, with the decay strategy, both $C_1$ and $C_2$ will decrease when $T$ increase.

#### 5 Experiments

**Experiment setup.** We evaluate PD-SGD using the CIFAR-10 dataset which has ten classes of natural images. We set the number of worker nodes to $n = 100$ and connect every node with 10 nodes. The connection graph is sparse, and the second biggest eigenvalue is big: $\rho \approx 0.98$. To make the objective functions $f_1, \cdots, f_n$ heterogeneity, we let each node contain samples random selected from two classes. We build a small convolutional neural network (CNN) by adding the following layers one by one: $\text{Conv}(32, 5 \times 5) \Rightarrow \text{MaxPool}(2 \times 2) \Rightarrow \text{Conv}(64, 5 \times 5) \Rightarrow \text{MaxPool}(2 \times 2) \Rightarrow \text{Dense}(512) \Rightarrow \text{ReLU} \Rightarrow \text{Dense}(128) \Rightarrow \text{ReLU} \Rightarrow \text{Dense}(10) \Rightarrow \text{Softmax}$. There are totally 940,000 trainable parameters. We choose the best learning rate from $\{10^{-3}, 10^{-2}, 10^{-1}\}$. We set $T = 10,000$ and evaluate the averaged model every 10 global steps on the global loss (1).

**Convergence against computation.** Figure 3a shows that when $I_1$ is fixed as 10, larger $I_2$ leads to faster convergence in terms of computation. The setting of $I_1 = 0$ and $I_2 = 1$ uses the least amount of computation to converge.

**Convergence against communication.** For a fixed $I_2$, a big $I_1$ leads to fast convergence in the early stage in terms of communication. Figure 3b shows in the first 2000 rounds of communications, curves with a larger $I_1/I_2$ have a faster decrease of the global loss. However, in the late stage, large $I_1/I_2$, unfortunately, harms convergence. We speculate the reason is that at the beginning, the optimization parameters are far away from any stationary point, and more local updates will accelerate the move towards it. When it is close enough to a good parameter region (e.g., the neighborhood of stationary points), more local updates inevitably increases the residual errors and thus deteriorates the ultimate loss level. The empirical observation is different from the theory in Corollary 2. No optimal value of $I_2$ exists. We argue that this is because the initial error $\Delta$ is not large enough.

**Results of fixed round length $I$.** From our theory, the learning rate should be set as $\eta = O\left(\frac{1}{\sqrt{T^2}}\right)$.

As a supplementary, we fix $I_1 + I_2 = 15$ (which means the learning rate is same for all experiments) and find the similar phenomenon in Figure 3c and 3d. Larger the value of $\frac{I_1}{I_2}$, less total communication steps needed, faster the global loss decrease in terms of communication steps at the beginning but slower convergence rate in terms of total steps and higher loss level later. We may conclude that local updates are more favorable at the beginning, while communication should be more frequent near the

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4 This bound is almost an increasing function of $I_2$. This result is reasonable since in the extreme case where $\rho = 0$, all nodes are connected and any full average will not accumulate the residual error.
In this paper, we analyze a novel variant of PD-SGD under the setting of stochastic non-convex optimization and non-identically distributed training data. The variant extends PD-SGD by introducing a decay strategy that gradually decreases $\frac{I_1}{I_2}$.

**Decaying $I_1$.** The above empirical observation suggests using a big $I_1$ in the beginning and a small $I_1$ in the end. We decay $I_1$ by half every 50 rounds, i.e., about 1000 steps initially. Figure 3f shows that $I_1 = 10, I_2 = 1$ with the decay strategy is the most efficient method.

### 6 Conclusion

In this paper, we analyze a novel variant of PD-SGD under the setting of stochastic non-convex optimization and non-identically distributed training data. The variant extends PD-SGD by introducing...
$I_2$ consecutive steps of decentralized SGDs after local updates. Our theory suggests that reasonably bigger $I_1$ leads to more computation but less communication. It also suggests that there is a nontrivial optimal $I_2$ when the initial error is large enough. Empirical study shows that larger $I_2$ often fasten convergence and Our theory and experiments show that an good communication-efficient strategy is to set $I_1$ big in the beginning and gradually decay $I_1$ with a fixed moderate $I_2$.

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A Proof of Theorem 1

A.1 Additional notation

In the proofs we will use the following notation. Let \( G(X; \xi) \) be defined in Section 2 previously. Let

\[
  g(X; \xi) := \frac{1}{n} G(X; \xi) 1_n = \frac{1}{n} \sum_{k=1}^{n} F_k(x^{(k)}; \xi^{(k)}) \in \mathbb{R}^d
\]

be the averaged gradient. Recall from (1) the definition \( f_k(x) := \mathbb{E}_{\xi \sim \mathcal{D}_k} [F_k(x; \xi)] \). We analogously define

\[
  \nabla f(X) := \mathbb{E}[G(X; \xi)] = \left[ \nabla f_1(x^{(1)}), \cdots , \nabla f_n(x^{(n)}) \right] \in \mathbb{R}^{d \times n},
\]

\[
  \nabla f(X) := \mathbb{E}[g(X; \xi)] = \frac{1}{n} \nabla f(X) 1_n = \frac{1}{n} \sum_{k=1}^{n} \nabla f_k(x^{(k)}) \in \mathbb{R}^d,
\]

\[
  \nabla f(\mathbf{x}) := \nabla f(X) = \frac{1}{n} \sum_{k=1}^{n} \nabla f_k(\mathbf{x}) \in \mathbb{R}^d.
\]

Let \( Q = \frac{1}{n} 1_n 1_n^T \) and \( \mathbf{w}_t = \frac{1}{n} \sum_{k=1}^{n} x_t^{(k)} \). Define the residual error as

\[
  V_t = \frac{1}{n^2} \| X_t (I - Q) \|_F^2 = \mathbb{E}_{\xi} \frac{1}{n} \sum_{k=1}^{n} \| x_t^{(k)} - \mathbf{w}_t \|_2^2.
\]

where the expectation is taken with respect to all randomness of stochastic gradients or equivalently \( \xi = (\xi_1, \cdots , \xi_t, \cdots ) \) where \( \xi_s = (\xi_s^{(1)}, \cdots , \xi_s^{(n)})^T \in \mathbb{R}^n \). Except where noted, we will use notation \( \mathbb{E} (\cdot) \) in stead of \( \mathbb{E}_{\xi} (\cdot) \) for simplicity. Hence \( V_t = \frac{1}{n^2} \sum_{k=1}^{n} \| x_t^{(k)} - \mathbf{w}_t \|_2^2 \).

PD-SGD can be equivalently written in matrix form which will be used in the convergence analysis. Specifically,

\[
  X_{t+1} = (X_t - G(X_t; \xi_t)) W_t
\]

where \( X_t \in \mathbb{R}^{d \times n} \) is the concatenation of \( \{x_t^{(k)}\}_{k=1}^{n} \), \( G(X_t; \xi_t) \in \mathbb{R}^{d \times n} \) is the concatenated gradient evaluated at \( X_t \) with the sampled datum \( \xi_t \), and \( W_t \in \mathbb{R}^{n \times n} \) is the connected matrix defined by

\[
  W_t = \left\{ \begin{array}{ll}
  I_n & \text{if } t \mod I \in [I], \\
  \mathbf{W} & \text{if } t \mod I \notin [I].
\end{array} \right.
\]

A.2 Useful lemmas

The main idea of proof is to express the iterated sequence in terms of gradients and then develop upper bound on residual errors. This proof technique can be found in [43, 44, 45, 49, 48]. Our contribution is the new way to bound the residual error (Lemma 6).

Lemma 1 (One step recursion). Let Assumptions 1 and 2 hold and \( L \) and \( \sigma \) be defined therein. Let \( \eta \) be the learning rate. Then the iterate obtained from the update rule (13) satisfies

\[
  \mathbb{E}[f(\mathbf{x}_{t+1})] \leq \mathbb{E}[f(\mathbf{x}_t)] - \frac{\eta}{2} (1 - \eta L) \mathbb{E}[\| \nabla f(X_t) \|^2] - \frac{\eta}{2} \mathbb{E}[\| \nabla f(X_t) \|^2] + \frac{L \sigma^2 \eta^2}{4n} + \frac{\eta L^2}{2} V_t,
\]

where the expectations are taken with respect to all randomness in stochastic gradients.

Proof. Recall that from the update rule (13) we have

\[
  \mathbf{x}_{t+1} = \mathbf{x}_t - \eta G(X_t; \xi_t).
\]

When Assumptions 1 and 2 hold, it follows directly from Lemma 8 in Tang et al. [41] that

\[
  \mathbb{E}[f(\mathbf{x}_{t+1})] \leq \mathbb{E}[f(\mathbf{x}_t)] - \frac{\eta}{2} \mathbb{E}[\| \nabla f(X_t) \|^2] - \frac{\eta}{2} (1 - \eta L) \mathbb{E}[\| \nabla f(X_t) \|^2] + \frac{L \sigma^2 \eta^2}{4n}.
\]
where (a) follows from Jensen’s inequality, (b) follows from Assumption 1, and $V_t$ is defined in (12).

**Lemma 2 (Residual error decomposition).** Let $X_1 = x_1 1_n^\top \in \mathbb{R}^{d \times n}$ be the initialization. If we apply the update rule (13), then for any $t \geq 2$,

$$X_t(I_n - Q) = -\sum_{s=1}^{t-1} G(X_s; \xi_s)(\Phi_{s,t-1} - Q)$$

(16)

where $\Phi_{s,t-1}$ is defined in (17) and $W_t$ is given in (14).

**Proof.** For convenience, we denote by $G_t = G(X_t; \xi_t) \in \mathbb{R}^{d \times n}$ the concatenation of stochastic gradients at iteration $t$. According to the update rule, we have

$$X_t(I_n - Q) = (X_{t-1} - \eta G_{t-1})W_{t-1}(I_n - Q)$$

$$= X_{t-1}(I_n - Q)W_{t-1} - \eta G_{t-1}(W_{t-1} - Q)$$

$$= X_{t-1}(I_n - Q)\prod_{s=t}^{t-1} W_s - \eta \sum_{s=t}^{t-1} G_s(\Phi_{s,t-1} - Q)$$

(17)

where (a) follows from $W_{t-1}Q = QW_{t-1}$; (b) results by iteratively expanding the expression of $X_s$ from $s = t - 1$ to $s = t - l + 1$ and plugging in the definition of $\Phi_{s,t-1}$ in (14); (c) follows simply by setting $l = t - 1$. Finally, the conclusion follows from the initialization $X_1 = x_1 1_n^\top$ which implies $X_1(I - Q) = 0$.

**Lemma 3 (Gradient variance decomposition).** Given any sequence of deterministic matrices $\{A_s\}_{s=1}^t$, then for any $t \geq 1$,

$$E_\xi \left\| \sum_{s=1}^{t} [G(X_s; \xi_s) - \nabla f(X_s)] A_s \right\|_F^2 = \sum_{s=1}^{t} E_{\xi_s} \left\| [G(X_s; \xi_s) - \nabla f(X_s)] A_s \right\|_F^2$$

(18)

where the expectation $E_\xi(\cdot)$ is taken with respect to the randomness of $\xi = (\xi_1, \cdots, \xi_t, \cdots)$ and $E_{\xi_s}(\cdot)$ is with respect to $\xi_s = (\xi_s^{(1)}, \cdots, \xi_s^{(n)})^\top \in \mathbb{R}^n$.

**Proof.**

$$E_\xi \left\| \sum_{s=1}^{t} [G(X_s; \xi_s) - \nabla f(X_s)] A_s \right\|_F^2 = \sum_{s=1}^{t} E_{\xi_s} \left\| [G(X_s; \xi_s) - \nabla f(X_s)] A_s \right\|_F^2$$
\[
+ 2 \sum_{1 \leq s < t \leq l} \mathbb{E}_{\xi_t, \xi_t} \left[ \left( [G(X_s; \xi_s) - \nabla f(X_s)] A_s, [G(X_t; \xi_t) - \nabla f(X_t)] A_t \right) \right]
\]

Since different nodes work independently without interference, for \( s \neq t \in [l] \), \( \xi_s \) is independent with \( \xi_t \). Let \( \mathcal{F}_s = \sigma(\{\xi_t\}_{t=1}^s) \) be the \( \sigma \)-field generated by all the random variables until iteration \( t \). Then for any \( 1 \leq s < t \leq l \), we obtain

\[
\mathbb{E}_{\xi_t, \xi_t} \left[ \left( [G(X_s; \xi_s) - \nabla f(X_s)] A_s, [G(X_t; \xi_t) - \nabla f(X_t)] A_t \right) \mid \mathcal{F}_{t-1} \right] = \mathbb{E}_{\xi_t} \left[ \left( [G(X_s; \xi_s) - \nabla f(X_s)] A_s, [G(X_t; \xi_t) - \nabla f(X_t)] A_t \right) \mid \mathcal{F}_{t-1} \right] = 0
\]

where (a) follows from the tower rule by noting that \( X_s \) and \( \xi_s \) are both \( \mathcal{F}_{t-1} \)-measurable and (b) uses the fact that \( \xi_t \) is independent with \( \mathcal{F}_s(s < t) \) and \( G(X_t; \xi_t) \) is an unbiased estimator of \( \nabla f(X_t) \). \( \Box \)

**Lemma 4** (Bound on second moments of gradients). For any \( n \) points: \( \{ x^{(k)}_t \}_{k=1}^n \), define \( X_t = [x^{(1)}_t, \ldots, x^{(n)}_t] \) as their concatenation, then under Assumption 1 and 3,

\[
\frac{1}{n} \mathbb{E} \| \nabla f(X_t) \|^2_F \leq 8L^2 V_t + 4\kappa^2 + 4 \mathbb{E} \| \nabla f(X_t) \|^2_F.
\] (19)

**Proof.** By splitting \( \nabla f(X_t) \) into four terms, we obtain

\[
\mathbb{E} \| \nabla f(X_t) \|^2_F = \mathbb{E} \| \nabla f(X_t) - \nabla f(x_t 1_n^T) + \nabla f(x_t 1_n^T) - \nabla f(x_t 1_n^T) \|_F^2 F
\]

\[
+ \| \nabla f(x_t 1_n^T) - \nabla f(x_t 1_n^T) + \nabla f(x_t 1_n^T) \|_F^2 F
\]

\[
\leq 4 \mathbb{E} \| \nabla f(X_t) - \nabla f(x_t 1_n^T) \|^2_F + 4 \mathbb{E} \| \nabla f(x_t 1_n^T) - \nabla f(x_t 1_n^T) \|^2_F
\]

\[
+ 4 \mathbb{E} \| \nabla f(x_t 1_n^T) - \nabla f(x_t 1_n^T) \|^2_F + 4 \mathbb{E} \| \nabla f(x_t 1_n^T) \|^2_F
\]

\[
= 8L^2 V_t + 4 \mathbb{E} \| \nabla f(x_t 1_n^T) - \nabla f(x_t 1_n^T) \|^2_F + 4L^2 n V_t + 4n \mathbb{E} \| \nabla f(X_t) \|^2_F
\]

where (a) follows from the basic inequality \( \| \sum_{i=1}^n A_i \|^2_F \leq n \sum_{i=1}^n \| A_i \|^2_F \); (b) follows from the smoothness of \( \{ f_k \}_{k=1}^n \) and \( f = \frac{1}{n} \sum_{k=1}^n f_k \) (Assumption 1) and the definition of \( V_t \) in (12); (c) follows from Assumption 3 as a result of the fact \( \| \nabla f(x_t 1_n^T) - \nabla f(x_t 1_n^T) \|^2_F = \sum_{k=1}^n \| \nabla f_k(x_t) - \nabla f(x_t) \|^2_F \). \( \Box \)

**Lemma 5** (Bound on residual errors). Let \( \rho_{s,t-1} = \| \Phi_{s,t-1} - Q \| \) where \( \Phi_{s,t-1} \) is defined in (17). Then the residual error can be upper bounded, i.e.,

\[
V_t \leq 2\eta^2 U_t
\]

where

\[
U_t = \sigma^2 \sum_{s=1}^{t-1} \rho_{s,t-1}^2 + \left( \sum_{s=1}^{t-1} \rho_{s,t-1} \right) \left( \sum_{s=1}^{t-1} \rho_{s,t-1} \left( 8L^2 V_s + 4\kappa^2 + 4 \mathbb{E} \| \nabla f(X_s) \|^2 \right) \right).
\] (20)

**Proof.** Again we denote by \( G_t = G(X_t; \xi_t) \) for simplicity. From Lemma 2, we can obtain a closed form of \( V_t \). Then it follows that

\[
nV_t = \mathbb{E} \| X_t (I - Q) \|^2_F = \eta^2 \mathbb{E} \left\| \sum_{s=1}^{t-1} G_s (\Phi_{s,t-1} - Q) \right\|_F^2
\]
Assumption 2 and Lemma 4.

where (a) follows from the basic inequality \( \|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2) \) and \( \mathbb{E}G_s = \nabla f(X_s) \); (b) follows from Lemma 3; (c) follows from the triangle inequality \( \|\sum_{s=1}^{t-1} A_s\|_F \leq \sum_{s=1}^{t-1} \|A_s\|_F \); (d) follows from the basic inequality \( \|AB\|_F \leq \|A\|_F \|B\| \) for any matrix \( A \) and \( B \); (e) directly follows from the notation \( \rho_{s,t-1} = \|\Phi_{s,t-1} - Q\| \); (f) follows from the Cauchy inequality; (g) follows from Assumption 2 and Lemma 4.

Lemma 6 is the most important lemma in the paper, since it captures the accumulation rate of residual errors. What’s more, the task of proving convergence for different communication patterns can be reduced to how residual errors are accumulated.

**Lemma 6 (Manipulation on \( \rho_{s,t-1} \)).** Define \( \rho_{s,t-1} = 1 \) for any \( t \leq s \) and \( \rho_{s,t-1} = \|\Phi_{s,t-1} - Q\| \) when \( s < t \). The following properties hold for \( \rho_{s,t-1} \):

1. \( \rho_{s,t-1} = \prod_{l=0}^{t-1} \rho_l \) with \( \rho_l = 1 \) if \( l \) mod \( I \in [I_1] \), else \( \rho_l = \rho \) where \( I = I_1 + I_2 \) and \( \rho \) is defined in Assumption 4. As a direct consequence, \( \rho_{s,t-1} = \rho_{s-1,l-1} \rho_{l,t-1} \) for any \( s \leq l \leq t \).

2. Define

\[
\alpha_j = \sum_{t=1}^{j+1} \sum_{s=1}^{t-1} \rho_{s,t-1}. \tag{21}
\]

Then for all \( j \geq 0 \),

\[
\alpha_j \leq \frac{1}{2} \left( \frac{1 + \rho^2}{1 - \rho^2} I_1 + \frac{1 + \rho}{1 - \rho} I_1 \right) + I \frac{\rho}{1 - \rho}. \tag{22}
\]

3. Define

\[
\beta_j = \sum_{t=1}^{j+1} \sum_{s=1}^{t-1} \rho_{s,t-1}^2. \tag{23}
\]
Then for all \( j \geq 0 \),
\[
\beta_j \leq \frac{1}{2} \left( \frac{1 + \rho^{2l_2} I_2}{1 - \rho^{2l_2} I_2} + \frac{1 + \rho^2}{1 - \rho^2} I_1 \right) + I_1 \frac{\rho^2}{1 - \rho^2}.
\] (24)

4. For any \( t \geq 1 \), \( \sum_{s=1}^{t-1} \rho_{s,t-1} \leq K \) where
\[
K = \frac{I_1}{1 - \rho^2} + \frac{\rho}{1 - \rho}.
\] (25)

As a direct corollary, \( \alpha_j \leq IK \).

5. Define
\[
\gamma_j = \sum_{t=j+1}^{t-1} \left( \sum_{s=1}^{t-1} \rho_{s,t-1} \right)^2.
\] (26)

Then \( \gamma_j \leq K \alpha_j \), where \( K \) is given in (25).

6. Assume \( T = (R + 1)I \) for some non-negative integer \( R \). Define
\[
w_s = \sum_{t=s+1}^{T} \rho_{s,t-1}
\] (27)

Then for all \( s \in [T] \), \( w_s \leq K \) where \( K \) is given in (25).

**Proof.** We prove these properties one by one:

1. By definition, we have \( \rho_{s,t-1} = \| \Phi_{s,t-1} - Q \| = \| \prod_{l=s}^{t-1} W_l - Q \|. \) Since for any positive integer \( l \), \( W_l Q = Q W_l \), then \( W_l \) and \( Q \) can be simultaneously diagonalized. From this it is easy to see that \( \| \prod_{l=s}^{t-1} W_l - Q \| = \prod_{l=s}^{t-1} \rho_I \) where \( \rho_I \) is the second largest absolute eigenvalue of \( W_l \). Note that \( W_l \) is either \( W \) or \( I \) according to the value of \( l \) as a result of the definition (14). Hence \( \rho_I = 1 \) if \( l \) mod \( I \in [I_1] \), else \( \rho \).

2. We now directly compute \( \alpha_j = \sum_{t=j+1}^{j+1} \sum_{s=1}^{t-1} \rho_{s,t-1} \). Without loss of generality, assume \( t = jI + i \) with \( j \geq 0 \), \( 1 \leq i \leq I \). (i) When \( 1 \leq i \leq I_1 + 1 \), then
\[
\sum_{s=1}^{t-1} \rho_{s,t-1} = (i - 1) + I_1 \sum_{r=0}^{j-1} \rho_{r}^{l_2(j-r)} + \sum_{r=0}^{j-1} \sum_{l=1}^{l_2} \rho_{r}^{l_2(j-r)+1-l}
\]
\[
= (i - 1) + I_1 \frac{\rho_{l_2} - \rho_{l_2}^{j+1}}{1 - \rho_{l_2}} + \frac{\rho - \rho_{l_2}^{j+1}}{1 - \rho} \leq (i - 1) + I_1 \frac{\rho_{l_2} - \rho}{1 - \rho_{l_2}} + \frac{\rho}{1 - \rho}.
\] (28)

(ii) When \( I_1 + 1 \leq i \leq I \), then
\[
\sum_{s=1}^{t-1} \rho_{s,t-1} = \rho^{i-I_1-1} \left[ I_1 \sum_{r=0}^{j} \rho_{r}^{l_2(j-r)} + \sum_{r=0}^{j-1} \sum_{l=1}^{l_2} \rho_{r}^{l_2(j-r)+1-l} \right] + \sum_{l=1}^{i-I_1-1} \rho^{i-I_1-1} \left[ I_1 \frac{\rho - \rho_{l_2}^{j+1}}{1 - \rho_{l_2}} + \frac{\rho_{l_2}^{j+1}}{1 - \rho} \right] \leq I_1 \rho^{i-I_1-1} \frac{\rho_{l_2} - \rho}{1 - \rho_{l_2}} + \frac{\rho}{1 - \rho}.
\] (29)

Therefore, by combining (i) and (ii), we obtain
\[
\alpha_j = \sum_{t=jI+1}^{(j+1)I} \sum_{s=1}^{t-1} \rho_{s,t-1}
\]

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Lemma 7 (Bound on average residual error). Assume \( T = (R + 1)I \) and the learning rate is so small that \( 16\eta^2 I^2 K^2 < 1 \), then

\[
\frac{1}{T} \sum_{t=1}^{T} \rho_i V_i \leq \frac{2\eta^2}{1 - 16\eta^2 I^2 K^2} \left[ C_1 \sigma^2 + C_2 \kappa^2 + 4K^2 \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \nabla J(X_t) \|^2 \right]
\]
where $K$ is given in (25) and
\[
C_1 = \frac{1}{2I} \left( \frac{1 + \rho^2 I_2}{1 - \rho^2 I_2} I_1 + \frac{1 + \rho^2}{1 - \rho^2 I_1} \right) + \frac{\rho^2}{1 - \rho^2} \tag{31}
\]
\[
C_2 = \min \left\{ 4K \left[ \frac{1}{2I} \left( \frac{1 + \rho^2 I_2}{1 - \rho^2 I_2} I_1 + \frac{1 + \rho}{1 - \rho} I_1 \right) + \frac{\rho}{1 - \rho} \right], 4K^2 \right\} \tag{32}
\]

**Proof.** Denote $Z_s = 8L^2V_s + 4E\|\nabla f(X_s)\|^2$ for short. From Lemma 5, $V_t \leq 2\eta^2 U_t$, then
\[
\frac{1}{T} \sum_{t=1}^{T} V_t \leq 2\eta^2 \cdot \frac{1}{T} \sum_{t=1}^{T} U_t \tag{33}
\]
and
\[
\sum_{t=1}^{T} U_t = \sum_{t=1}^{T} \left[ \sigma^2 \sum_{s=1}^{t-1} \rho_{s,t-1}^2 + \left( \sum_{s=1}^{t-1} \rho_{s,t-1} \right) \left( \sum_{s=1}^{t-1} \rho_{s,t-1} \left( Z_s + 4K^2 \right) \right) \right] \tag{20}
\]
\[
\overset{(a)}{=} \sigma^2 \sum_{j=0}^{R} \beta_j + 4K^2 \sum_{j=0}^{R} \gamma_j \sum_{j=1}^{T} Z_s \sum_{s=1}^{t-1} \rho_{s,t-1} \tag{20}
\]
\[
\overset{(b)}{=} \sigma^2 \sum_{j=0}^{R} \beta_j + 4K^2 \sum_{j=0}^{R} R \alpha_j \sum_{j=0}^{R} Z_s \sum_{j=1}^{T} \rho_{s,t-1} \tag{20}
\]
\[
\overset{(c)}{=} \sigma^2 \sum_{j=0}^{R} \beta_j + 4K^2 \sum_{j=0}^{R} R \alpha_j \sum_{j=0}^{R} Z_s \sum_{j=1}^{T} \rho_{s,t-1} \tag{20}
\]
\[
\overset{(d)}{=} \sigma^2 \sum_{j=0}^{R} \beta_j + 4K^2 \sum_{j=0}^{R} R \alpha_j \sum_{j=0}^{R} Z_s \sum_{j=1}^{T} \rho_{s,t-1} \tag{20}
\]
\[
\overset{(e)}{=} T \left[ C_1 \sigma^2 + C_2 \kappa^2 + K^2 \frac{1}{T} \sum_{t=1}^{T} U_t + 4K \frac{1}{T} \sum_{t=1}^{T} E\|\nabla f(X_t)\|^2 \right] \tag{34}
\]
\[
\overset{(f)}{\leq} T \left[ C_1 \sigma^2 + C_2 \kappa^2 + 16\eta^2 L^2 K^2 \frac{1}{T} \sum_{t=1}^{T} U_t + 4K \frac{1}{T} \sum_{t=1}^{T} E\|\nabla f(X_t)\|^2 \right] \tag{34}
\]
where (a) follows from the definition of $\beta_j$ and $\gamma_j$ (see (23) and (26)); (b) follows from 4 and 5 in Lemma 6 ($K$ is given in (25)); (c) follows from the basic inequality $\sum_{t=1}^{T} \sum_{s=1}^{t} \rho_{s,t-1} = \sum_{s=1}^{t} \sum_{t=s+1}^{T} \rho_{s,t-1}$; (d) follows from 6 in Lemma 6; (e) follows because $C_1$ and $C_2$ are upper bounds of $\frac{1}{T} \sum_{j=0}^{R} \beta_j$ and $\frac{4K}{T} \sum_{j=0}^{R} \alpha_j$ respectively. Indeed, recall that $T = (R + 1)I$ and it follows from 2, 3 and 4 in Lemma 6 that
\[
\frac{1}{T} \sum_{j=0}^{R} \beta_j \leq \frac{1}{2I} \left( \frac{1 + \rho^2 I_2}{1 - \rho^2 I_2} I_1 + \frac{1 + \rho^2}{1 - \rho^2 I_1} \right) + \frac{\rho^2}{1 - \rho^2} = C_1,
\]
\[
\frac{4K}{T} \sum_{j=0}^{R} \alpha_j \leq 4K \left[ \frac{1}{2I} \left( \frac{1 + \rho^2 I_2}{1 - \rho^2 I_2} I_1 + \frac{1 + \rho}{1 - \rho} I_1 \right) + \frac{\rho}{1 - \rho} \right] \text{ and } \frac{4K}{T} \sum_{j=0}^{R} \alpha_j \leq 4K^2.
\]
Finally (f) follows by adding an additional non-negative $Z_T$ and plugging into the notation of $Z_s$.

By arranging (34) and assuming the learning rate is small enough such that $16\eta^2 L^2 K^2 < 1$, then we have
\[
\frac{1}{T} \sum_{t=1}^{T} U_t \leq \frac{1}{1 - 16\eta^2 L^2 K^2} \left[ C_1 \sigma^2 + C_2 \kappa^2 + 4K \frac{1}{T} \sum_{t=1}^{T} E\|\nabla f(X_t)\|^2 \right]. \tag{35}
\]

Our conclusion then follows by combining (33) and (35). \qed
A.3 Completing the Proof of Theorem 1

Proof. From Lemma 1, it follows that
\[
\mathbb{E}[f(\mathbf{x}_{t+1})] \leq \mathbb{E}[f(\mathbf{x}_t)] - \frac{\eta}{2} (1 - \eta L) \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] - \frac{\eta}{2} \nabla f(\mathbf{x}_t) + \frac{L \sigma^2 \eta^2}{2n} + \frac{\eta L^2}{2} V_t.
\]
Note that the expectation is taken with respect to all randomness of stochastic gradients, i.e., \( \xi = (\xi_1, \xi_2, \ldots) \). Arranging this inequality, we have
\[
\mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] \leq \frac{2}{\eta} \left\{ \mathbb{E}[f(\mathbf{x}_t)] - \mathbb{E}[f(\mathbf{x}_{t+1})] \right\} - (1 - \eta L) \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] + \frac{L \sigma^2 \eta^2}{n} + L^2 V_t. \tag{36}
\]
Then it follows that
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] \\
\leq \frac{2}{\eta T} \left\{ \mathbb{E}[f(\mathbf{x}_1)] - \mathbb{E}[f(\mathbf{x}_{T+1})] \right\} + \frac{L \sigma^2 \eta}{n} + \frac{L^2}{T} \sum_{t=1}^{T} V_t - \frac{1 - \eta L}{T} \sum_{t=1}^{T} \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] \tag{a}
\leq \frac{2}{\eta T} \left\{ \mathbb{E}[f(\mathbf{x}_1)] - \mathbb{E}[f(\mathbf{x}_{T+1})] \right\} + \frac{L \sigma^2 \eta}{n} - \frac{1 - \eta L}{T} \sum_{t=1}^{T-1} \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] \\
+ \frac{2 \eta^2 L^2}{1 - 16 \eta^2 L^2 K^2} \left( C_1 \sigma^2 + 4 K^2 \sigma^2 + 4 K^2 \frac{1}{T} \sum_{t=1}^{T-1} \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] \right) \tag{b}
\leq \frac{2}{\eta T} \left\{ \mathbb{E}[f(\mathbf{x}_1)] - \mathbb{E}[f(\mathbf{x}_{T+1})] \right\} + \frac{L \sigma^2 \eta}{n} + 4 \eta^2 L^2 C_1 \sigma^2 + 16 \eta^2 L^2 K^2 \sigma^2 \\
- (1 - \eta L - 16 \eta^2 L^2 K^2) \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] \tag{c}
\leq \frac{2}{\eta T} \left\{ \mathbb{E}[f(\mathbf{x}_1)] - \mathbb{E}[f(\mathbf{x}_{T+1})] \right\} + \frac{L \sigma^2 \eta}{n} + 4 \eta^2 L^2 C_1 \sigma^2 + 16 \eta^2 L^2 K^2 \sigma^2 \tag{d}
\]
where (a) follows by telescoping and averaging (36); (b) follows from the upper bound of \( \frac{1}{T} \sum_{t=1}^{T} V_t \) in Lemma 7 (here we don’t use \( C_2 \) but its upper bound \( 4 K^2 \sigma^2 \)); (c) follows from the choice of the learning rate \( \eta \) which satisfies \( \frac{1 - \eta L}{1 - 16 \eta^2 L^2 K^2} \leq 2 \) (since \( 16 \eta^2 L^2 K^2 \leq \frac{1}{2} \) from (4)) and rearrangement; (d) follows the requirement that the learning rate \( \eta \) is small enough such that \( \eta L + 16 \eta^2 L^2 K^2 < 1 \) (which is satisfied since \( \eta L \leq \frac{1}{2} \) and \( 16 \eta^2 L^2 K^2 \leq \frac{1}{2} \)). \( \square \)

B Proof of Theorem 2

In this section, we will give the convergence result of Theorem 2 which states that the convergence will be fastened if we use the decaying strategy. The framework used to prove Theorem 1 can still apply here. To that end, we need a modified version of Lemma 6 which reveals how the residual errors are accumulated.

B.1 Notation

But before that, we first explain in detail how we decay \( I_1 \), though the process has already been depicted in Algorithm 2. This will help readers better understand the proof of our new Lemma 8. In short, we halve \( I_1 \) every \( M \) rounds. That is we first run \( M \) rounds of PD-SGD with parameters \( I_1 \) and \( I_2 \), then run another \( M \) rounds of PD-SGD with parameters \( \left\lfloor \frac{I_1}{2} \right\rfloor \) and \( I_2 \), and keep this process going on until we reach the \( 1 + \left\lfloor \log_2 I_2 \right\rfloor \) th run, where \( I_1 \) shrinks to zero and we only run D-SGD.

Let \( N_0 = \left\lfloor \log_2 I_1 \right\rfloor \) and recall that
\[
\mathcal{T} = \left\{ M \cdot \sum_{i=0}^{j} \left\lfloor \frac{I_1}{2^i} \right\rfloor + I_2 : 0 \leq j \leq 1 + N_0 \right\} \cup \{0\} \tag{37}
\]
and denote by $\max \mathcal{I}$ the maximum element in $\mathcal{I}$. For convenience, we denote by

$$R_k = \left\{ l : M \cdot \sum_{i=0}^{k-1} \left[ \frac{I_1}{2^i} + I_2 \right] < l \leq M \cdot \sum_{i=0}^{k} \left[ \frac{I_1}{2^i} + I_2 \right] \right\}$$

the set of all steps which locate in the $k$ th $M$ rounds of run where the length of local updates is $\left\lfloor \frac{t}{2^k} \right\rfloor$. $\mathcal{N}(t) = \text{argmax}\{ j \leq t : j \in \mathcal{I} \}$ returns the latest step before $t$ after which $I_1$ is going to decay. (We define argmax) Therefore, according to the strategy, $\mathcal{W}_t$ is renewed by

$$\mathcal{W}_t = \left\{ \begin{array}{ll}
\mathcal{I}_s & \text{if } \exists k \text{ s.t. } t \in R_k \text{ and } t - \mathcal{N}(t) \mod \left\lfloor \frac{t}{2^k} \right\rfloor \notin \left\lfloor \frac{t}{2^k} \right\rfloor \\
\mathcal{W} & \text{otherwise.}
\end{array} \right.$$

\subsection*{B.2 Important lemma and missing proof}

\textbf{Lemma 8.} Recall $M$ is the decay interval, $T$ the total steps and $\rho_{s, t-1} = \| \Phi_{s, t-1} - Q \|$ where $\Phi_{s, t-1}$ is defined in (17) with $\mathcal{W}_t$ given in (39). Let $I_1, I_2$ be the initialized communication parameters. Assume $T$ is a multiple of $M$ satisfying $T \geq \max \mathcal{I}$. Then for PD-SGD with the decaying strategy, we have that

1. \( \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{t-1} \rho_{s, t-1} \leq \frac{1}{T} \sum_{s=1}^{T-1} \rho_{s, t-1} \leq \frac{1}{T} \sum_{s=1}^{T} \rho_{s, t-1} \)
2. \( \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{t-1} \rho_{s, t-1}^2 \leq \frac{1}{T} \sum_{s=1}^{T} \rho_{s, t-1}^2 \)
3. For any $t \geq 1, \sum_{s=1}^{t} \rho_{s, t-1} \leq K$ where $K = \frac{I_1}{1 - \rho^2} + \frac{\rho}{1 - \rho}$
4. For any $T > s \geq 1, \sum_{t=s+1}^{T} \rho_{s, t-1} \leq K$.

\textbf{Proof.} We verify each inequality by directly computation:

1. By exchanging the order of sum, we have

$$\sum_{t=1}^{T} \sum_{s=1}^{t-1} \rho_{s, t-1} = \sum_{s=1}^{T-1} \sum_{t=s}^{T} \rho_{s, t} = \sum_{s=1}^{T-1} \sum_{t=s}^{T} \rho_{s, t} + \sum_{s=\max \mathcal{I} + 1}^{T-1} \sum_{t=s}^{T} \rho_{s, t}$$

$$\leq \sum_{s=1}^{T-1} \sum_{t=s}^{T} \rho_{s, t} + \sum_{s=\max \mathcal{I} + 1}^{T-1} \sum_{t=s}^{T} \rho_{s, t}$$

2. One can complete the proof by replacing $\rho$ with $\rho^2$ in the latest argument.

3. If $t \in R_k$, let $t_0 = \mathcal{N}(t-1) = \max \{ j \in \mathcal{I} : j \in [t-1] \} = \min R_k - 1$, then we have

$$\sum_{s=1}^{t-1} \rho_{s, t-1} = \sum_{s=1}^{t-1} \rho_{s, t_0} + \sum_{s=t_0+1}^{t-1} \rho_{s, t-1}$$

$$\leq \rho_{t_0+1, t-1} \sum_{s=0}^{t_0} I_2 \sum_{s=0}^{M-1} \rho^s I_2 \left( \frac{I_1}{2^s} + \frac{\rho - \rho I_2 + 1}{1 - \rho} \right) + \sum_{s=t_0+1}^{t-1} \rho_{s, t-1}$$

$$\leq \rho_{t_0+1, t-1} \left( \frac{1}{1 - \rho I_2} I_1 + \frac{\rho(1 - \rho I_2)}{1 - \rho} \right) + \sum_{s=t_0+1}^{t-1} \rho_{s, t-1}$$
\[
\frac{I_1}{1 - \rho I_2} + \frac{\rho}{1 - \rho} = K
\]
where (a) uses \(1 \) in Lemma 6; (b) follows from \( |\frac{I_1}{T}| \leq I_1 \); to obtain (c), one can conduct a similar discussion like what we have done in 2 in Lemma 6 by discussing whether \( t \) locates in the local update phase or the communication phase. The case is more complicated since we should also think about which round \( t \) locates. No matter which case here, (c) always holds. We leave the tedious check for the readers.

4. The idea here is very similar to the that for the latest statement. If \( s \geq \max I \), then $\sum_{t=s+1}^{T} \rho_{s,t-1} \leq \sum_{t=1}^{\infty} \rho^t = \frac{\rho^s}{1 - \rho} \leq K$. Otherwise, local updates are involved in. Similarly, one can imitate what we have done in 6 in Lemma 6 by discussing which round and which phase \( s \) locates in. We leave the tedious check for the readers.

\[
1 \sum_{t=1}^{T} V_t \leq \frac{2\eta^2}{1 - 16\eta^2L^2K^2} \left[ \tilde{C}_1 \sigma^2 + \tilde{C}_2 \kappa^2 + 4K^2 \sum_{t=1}^{T} E\|\nabla f(X_t)\|^2 \right]
\]

where
\[
\tilde{C}_1 = \frac{1}{T} \frac{I_1}{1 - \rho^2 I_2} \rho^{2(T+I_2 - \max I - 1)} + (1 - \frac{\max I}{T}) \frac{\rho^2}{1 - \rho},
\]
\[
\tilde{C}_2 = 4K \left[ \frac{1}{T} \frac{I_1}{1 - \rho I_2} \rho^{T+I_2 - \max I - 1} + (1 - \frac{\max I}{T}) \frac{\rho}{1 - \rho} \right].
\]

**Proof**. One can replace Lemma 6 with Corollary 8 in the proof of Lemma 7 to achieve the conclusion.

**Proof of Theorem 2**. To prove Theorem 2, one can simply replace Lemma 7 with Lemma 9 in the proof of Appendix A.3.

## C Convergence of another update rule for PD-SGD

### C.1 Main result

For completeness, in this section, we study another update rule in this section:

\[
X_{t+1} = X_t + W_t - \eta G(X_t; \xi_t)
\]

where \( W_t \) is given in (14). Since in this update rule, the stochastic gradient descent happens after each node communicates with its neighbors, we call this type of update as communication-before. By contrast, what we have analyzed in the body of this paper is termed as communication-after. A lot of previous efforts study the communication-before update rule, including [10, 20]. Fortunately, the technique of proving Theorem 1 is so powerful that the convergence result for this new update rule can be easily parallel.

**Theorem 3** (Convergence rate of PD-SGD with the update rule (43)). Let Assumption 1, 2, 3, 4 hold and the constants \( L, \kappa, \sigma, \) and \( \rho \) be defined therein. Let \( \Delta = f(x_0) - \min_x f(x) \) be the initial error; \( K = \frac{I_1}{1 - p^2 I_2} + \frac{1}{1 - \rho} = K + 1 \), and \( \tilde{C}_1 = \frac{1}{2(1 + I_2)} \left( I_2^2 + \frac{1 + \rho^2}{1 - \rho^2} I_1^2 + \frac{1 + \rho^2}{1 - \rho^2} I_1 \right) + \frac{1}{1 - \rho} = C_1 + 1 \) where \( K \) and \( \tilde{C}_1 \) have already given in Theorem 1. If the learning rate \( \eta \) is small enough such that

\[
\eta \leq \min \left\{ \frac{1}{2L}, \frac{1}{4\sqrt{2LK}} \right\},
\]

then

\[
\frac{1}{T} \sum_{t=1}^{T} E\|\nabla f(x_t)\|^2 \leq \frac{2\Delta}{\eta T} + \eta L \sigma^2 + 4\eta^2 L^2 \tilde{C}_1 \sigma^2 + 16\eta^2 L^2 K^2 \kappa^2.
\]
Remark 3. Comparing the difference of results between Theorem 1 and Theorem 3, one can find that only the value of \( \hat{K} \) and \( C_1 \) have been modified. In this way, one can parallel the conclusions derived for the update rule (13) to those with the update rule (43) by simply substituting \( K, C_1 \) with \( \hat{K}, \hat{C}_1 \).

Note that \( \hat{K}, \hat{C}_1 \) is always strictly larger than \( K, C_1 \). This may be an indicator that the communication-allow update rule (13) converges faster than the communication-before update rule (43).

C.2 Useful lemmas and missing proof

Lemma 10 (Residual error decomposition). Let \( X_1 = x_11_n^T \in \mathbb{R}^{d \times n} \) be the initialization, then for any \( t \geq 2 \),

\[
X_t(I - Q) = -\eta \sum_{s=1}^{t-1} G(X_s; \xi_s) (\Phi_{s+1,t-1} - Q) \tag{46}
\]

where \( \Phi_{s,t-1} \) is already given in (17).

Proof. We still denote the gradient \( G(X_s; \xi_s) \) as \( G_t \). According to the update rule, we have

\[
X_t(I_n - Q) = (X_{t-1} W_{t-1} - \eta G_{t-1})(I_n - Q)
\]

(a) \( = X_{t-1}(I_n - Q) W_{t-1} - \eta G_{t-1}(I_n - Q) \)

(b) \( = X_{t-1}(I_n - Q) \prod_{s=t-1}^{t-1} W_s - \eta \sum_{s=t-1}^{t-1} G_s(\Phi_{s+1,t-1} - Q) \)

(c) \( = X_1(I_n - Q) \Phi_{t,t-1} - \eta \sum_{s=1}^{t-1} G_s(\Phi_{s+1,t-1} - Q) \)

where (a) follows from \( W_{t-1} Q = Q W_{t-1} \); (b) results by iteratively expanding the expression of \( X_s \) from \( s = t - 1 \) to \( s = t - l + 1 \) and plugging in the definition of \( \Phi_{s,t-1} \) in (17); (c) follows simply by setting \( l = t - 1 \). Finally, the conclusion follows from the assumption \( X_1(I_n - Q) = 0 \).

Lemma 11 (Bound on residual errors). Let \( \rho_{s,t-1} = \| \Phi_{s,t-1} - Q \| \) where \( \Phi_{s,t-1} \) is defined in (17). Then the residual error can be upper bounded, i.e., \( V_t \leq 2\eta^2 U_t \) where

\[
U_t = \sigma^2 \sum_{s=1}^{t-1} \rho_{s+1,t-1}^2 + \left( \sum_{s=1}^{t-1} \rho_{s+1,t-1} \right) \left( \sum_{s=1}^{t-1} \rho_{s+1,t-1} \right) \left( 8L^2 V_s + 4\kappa^2 + 4E\| \nabla f(X_s) \|_2^2 \right).
\]

Proof. The proof can be simply parallel by replacing \( \rho_{s,t-1} \) with \( \rho_{s+1,t-1} \) in Lemma 5.

The next thing is to bound the average residual error, i.e., \( \frac{1}{T} \sum_{t=1}^{T} V_t \). To that end, we should first figure out how the error is propagated in this case, as we have done in Lemma 6.

Corollary 3 (Manipulation on \( \rho_{s+1,t-1} \)). Noting that

\[
\sum_{s=1}^{t-1} \rho_{s+1,t-1} = \sum_{s=2}^{t} \rho_{s,t-1} \leq \sum_{s=1}^{t-1} \rho_{s,t-1} + 1, \tag{47}
\]

we can immediately deduce from Lemma 6 that

1. \( \hat{\alpha}_j = \sum_{t=1}^{(j+1)l} \sum_{s=1}^{t-1} \rho_{s+1,t-1} \leq \frac{1}{2} \left( \frac{1+\rho_j^2}{1-\rho_j^2} I_j^2 + \frac{1+\rho_j^2}{1-\rho_j^2} I_j \right) + I \frac{1}{1-\rho_j^2} \).

2. \( \hat{\beta}_j = \sum_{t=(j+1)l}^{(j+1)l} \sum_{s=1}^{t-1} \rho_{s+1,t-1} \leq \frac{1}{2} \left( \frac{1+\rho_j^2}{1-\rho_j^2} I_j^2 + \frac{1+\rho_j^2}{1-\rho_j^2} I_j \right) + I \frac{1}{1-\rho_j^2} \).

3. Let \( \hat{K} = \frac{I}{1-\rho_j^2} + \frac{1}{1-\rho_j^2} = K + 1 \), then \( \sum_{s=1}^{t-1} \rho_{s+1,t-1} \leq \hat{K} \) and \( \hat{\alpha}_j \leq \hat{K} \hat{\alpha}_j \).

4. \( \hat{\gamma}_j = \sum_{t=(j+1)l}^{(j+1)l} \sum_{s=1}^{t-1} \rho_{s+1,t-1} \leq \hat{K} \hat{\alpha}_j \).
5. If $T = (R + 1)I$, we have $w_s = \sum_{t=s+1}^{T} \rho_{s+1,t-1} = 1 + \sum_{t=s+1}^{T} \rho_{s+1,t-1} = 1 + w_{s+1} \leq 1 + K = \tilde{K}$.

**Lemma 12** (Bound on average residual error). Assume $T = (R + 1)I$ and the learning rate is so small that $16\eta^2 L^2 \tilde{K}^2 < 1$, then

$$
\frac{1}{T} \sum_{t=1}^{T} V_t \leq \frac{2\eta^2}{1 - 16\eta^2 L^2 \tilde{K}^2} \left[ \tilde{C}_1 \sigma^2 + \tilde{C}_2 \kappa^2 + 4\tilde{K}^2 \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \nabla f(x_t) \|^2 \right]
$$

(48)

where $\tilde{K} = K + 1$ with $K$ given in (25) and

$$
\tilde{C}_1 = \frac{1}{2T} \left( \frac{1 + \rho^2 I_2}{1 - \rho^2 I_2} I_1^2 + \frac{1 + \rho^2}{1 - \rho} I_1 \right) + \frac{1}{1 - \rho^2}
$$

(49)

$$
\tilde{C}_2 = \min \left\{ 4\tilde{K} \left[ \frac{1}{2T} \left( \frac{1 + \rho^2 I_2}{1 - \rho^2 I_2} I_1^2 + \frac{1 + \rho}{1 - \rho} I_1 \right) + \frac{1}{1 - \rho^2} \right], 4\tilde{K}^2 \right\}
$$

(50)

**Proof.** One can replace Lemma 6 with Corollary 3 in the proof of Lemma 7 to achieve the conclusion.

**Proof of Theorem 3.** To prove Theorem 3, one can simply replace Lemma 7 with Lemma 12 in the proof of Appendix A.3.

\**D** Discussion on others’ convergence results

PD-SGD with our adaptive update scheme incorporates many previous algorithms from Section 3. Based on Theorem 1 or Theorem 3, we could give convergence results for them (see Table 2). It is natural to compare our results with their original ones.

| Algorithms         | $I_1$ | $I_2$ | $\rho$ | $\sigma$ | $\kappa$ | Convergence Rate                                                                 |
|--------------------|-------|-------|--------|----------|---------|---------------------------------------------------------------------------------|
| SGD [2]            | 0     | 1     | 0      | 0        | 0       | $\frac{2\Delta}{\eta T} + \frac{\eta L \sigma^2}{n}$                           |
| PR-SGD [49]        | $\geq 1$ | 1     | 0      | $> 0$    | $> 0$   | $\frac{2\Delta}{\eta T} + \frac{\eta L \sigma^2}{n} + 2\eta^2 L^2 \sigma^2 I_1 + 16\eta^2 L^2 \kappa^2 I_1^2$ |
| D-SGD [20]         | 0     | 1     | 0, 1   | $> 0$    | $> 0$   | $\frac{2\Delta}{\eta T} + \frac{\eta L \sigma^2}{n} + 4\eta^2 L^2 \sigma^2 I_1 + \frac{16\eta^2 L^2 \kappa^2}{(1 - \rho^2) I_1}$ |
| DPA-SGD [44]       | $\geq 1$ | 1     | 0, 1   | $> 0$    | $> 0$   | $\frac{2\Delta}{\eta T} + \frac{\eta L \sigma^2}{n} + 2\eta^2 L^2 \sigma^2 (\frac{1 + \rho^2}{1 - \rho^2} I_1 - 1)$ |

**Convergence for PR-SGD** PR-SGD [56, 49, 48] is the special case of PD-SGD when $I_2 = 1$ and $\rho = 0$ (i.e., $W = Q = \frac{1}{n} 1_n 1_n^\top$). Yu et al. [49] derives its convergence (Theorem 4) by requiring Assumption 5 which is definitely stronger than our Assumption 3. Roughly speaking we always have bound $\kappa^2 \leq 4G^2$ since $\frac{1}{n} \sum_{k=1}^{n} \| \nabla f_k(x) - \nabla f(x) \|^2 \leq \frac{2}{n} \sum_{k=1}^{n} \| \nabla f_k(x) \|^2 + 2\| \nabla f(x) \|^2 \leq 4G^2$. Then our bound matches theirs up to constant factors. Another interesting thing is in this case our bound only depends on $I_1 = I - 1$ while Yu et al. [49]’s relies on $I$. Though they are the same asymptotically, our refined analysis shows that the step of model averaging doesn’t account for the accumulation of residual errors.

**Assumption 5.** (Bounded second moments) There are exist some $G > 0$ such that for all $k \in [n],$

$$
\mathbb{E}_{x \sim D_\Delta} \| \nabla F_k(x; \xi) \|^2 \leq G^2
$$

**Theorem 4** (Yu et al. [49]). Let Assumption 1, 2 and 5 hold and $I, \sigma, G$ defined therein. Let $\{x_t\}_{t=1}^{T}$ denote by the sequence obtained by PR-SGD and $\Delta = f(x_0) - \min_x f(x)$ be the initial error. If $0 \leq \frac{1}{\eta T} \leq \frac{1}{T}$, then for all $T$, we have

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \| \nabla f(x_t) \|^2 \leq \frac{2\Delta}{\eta T} + \frac{\eta L \sigma^2}{n} + 4\eta^2 I^2 L^2
$$
Convergence for D-SGD  

D-SGD [10, 20] is the special case of PD-SGD where $I_1 = 0$, $I_2 = 1$, $1 > \rho \geq 0$ and the communication-after update rule (introduced in Appendix C) is applied. The original paper [20] provides an analysis for D-SGD, which we simplify and translate into Theorem 5 in our notation. To guarantee convergence at a neighborhood of stationary points, [20] requires a smaller learning rate $O\left(\frac{1}{\sqrt{T}}\right)$ than our $O\left(\frac{1}{T}\right)$. By contrast their residual error is larger than ours up to a factor of $O(n)$. They could achieve as similar bounds on residual errors as ours by shrinking the learning rate, but the convergence would be slowed down.

Theorem 5 (Lian et al. [20]).  Let Assumption 1, 2, 3 and 4 hold and $L, \sigma, \kappa$ defined therein. Let $\{x_t\}_{t=1}^T$ denote by the sequence obtained by D-SGD and $\Delta = f(x_0) - \min\limits_x f(x)$ be the initial error. When the learning rate is small enough such that $\eta \leq \frac{1-\rho}{2\sqrt{6L}}$, then for all $T$, we have

$$
\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left\| \nabla f(x_t) \right\|^2 \leq \frac{4\Delta}{\eta T} + \frac{2\eta L \sigma^2}{n} + n \left[ \frac{6\eta^2 L^2 \kappa^2}{1-\rho^2} + \frac{54\eta^2 L^2 \kappa^2}{(1-\rho)(1-\rho^2)} \right]
$$

Convergence for PD-SGD  

PD-SGD is derived as a byproduct of the framework of Cooperative SGD (C-SGD) in [44]. In that paper, Wang and Joshi [44] term PD-SGD as Decentralized Periodic Averaging SGD (PDA-SGD). In our paper, PD-SGD (or DPA-SGD) is the case when $I_2 = 1$ and $1 > \rho \geq 0$. We translate their original analysis into Theorem 6 for ease of comparison.

First, our residual error is exactly the same with theirs up to constant factors. Second, they didn’t consider the case when the data is non-identically distributed. Third, we allow more flexible communication pattern design by introducing parameters $I_2$.

Theorem 6 (Wang and Joshi [44]).  Let Assumption 1, 2 and 4 hold and $L, \sigma$ defined therein. Let $\{x_t\}_{t=1}^T$ denote by the sequence obtained by PD-SGD and $\Delta = f(x_0) - \min\limits_x f(x)$ be the initial error. When the learning rate is small enough such that $\eta \leq \min\left\{ \frac{1}{2T}, \frac{1-\rho}{\sqrt{10L}} \right\}$, then for all $T$, we have

$$
\frac{1}{T} \sum_{t=1}^T \mathbb{E} \left\| \nabla f(x_t) \right\|^2 \leq \frac{2\Delta}{\eta T} + \frac{\eta L \sigma^2}{n} + \frac{\eta^2 L^2 \sigma^2}{1-\rho^2} + \frac{1+\rho^2}{1-\rho^2} - 1
$$

---

In this way, their $D_2 \geq \frac{2}{3}$ and $D_1 \geq \frac{1}{3}$, and this result follows from replacing $D_1, D_2$ with these constant lower bounds.