Abstract

This paper discusses a modified penetration theory where the time scale for the momentum wall layer is based on the onset of ejections from the wall but the time scale for the thermal layer is based on the unsteady state diffusion of heat from the wall into the streamwise flow, not the renewal of heat from eddies penetrating the wall layer.

A method for the determination of the thickness of the thermal wall layer has been developed. The theory correlates adequately experimental data for both the thickness of the thermal wall layer and the Nusselt number.

Key words: Heat transfer, penetration theory, turbulence, time scales.

1 Introduction

The turbulent transport of heat, mass and momentum near a wall has been investigated intensively for more than a century. Since Reynolds (1874) proposed that a mathematical analogy exists between the transport of these three quantities, much effort has focused on the extension of solutions of momentum to solutions for heat and mass transport. In view of the analogy any discussion on heat transfer will largely apply to mass transfer. There are a number of excellent reviews on heat transfer (Sideman and Pinczewski, 1975, Churchill, 1996, Mathpati and Joshi, 2007,
Pletcher, 1988). I will thus not be reviewing the existing work but will only mention the concepts and papers relevant to the topic at hand.

Essentially three approaches have been taken: a purely empirical approach typified by the Colburn analogy (1933), an analytical approach based on the solution time-averaged transport equations using turbulence models and an analogy approach where the Reynolds analogy is coupled with some model for the wall layer. In this latest category we need to differentiate between models based on steady time averaged profiles (Karman, 1930, Martinelli, 1947, Lyon, 1951, Metzner and Friend, 1958) and penetration models based on the unsteady state diffusion equations (Danckwerts, 1951, Harriott, 1962, Ruckensteine, 1968, Hughmark, 1968, McLeod and Ponton, 1977, Thomas and Fan, 1971, Loughlin et al., 1985, Fortuin et al., 1992, Hamersma and Fortuin, 2003, Kawase and Ulbrecht, 1983).

Most studies start with a definition of the transport flux (e.g. of heat) in terms of a diffusive and a turbulent (also called eddy) contribution (Boussinesq, 1877)

\[ q = -(k + \rho E_h) \frac{d\theta}{dy} \]  

where \( \theta \) is the temperature, \( y \) the normal distance from the wall, \( k \) the thermal conductivity, \( E_h \) the eddy thermal diffusivity, \( q \) the rate of heat transfer flux, \( \rho \) the fluid density.

Equation (1) may be rearranged as

\[ \theta^+ = \int_0^{y^+} \frac{q/q_w}{1 + E_h/k} dy^+ \]  

which is very similar to the equation for momentum transport

\[ U^+ = \int_0^{y^+} \frac{\tau/\tau_w}{1 + E_v/v} dy^+ \]  

Where $U^+ = U/u_*$, $y^+ = yu_*/\mu = yu_*/\nu$ and $\theta^+ = \rho C_p (\theta - \theta_w)u_*/q_w$ have been normalised with the friction velocity $u_* = \sqrt{\tau_w/\rho}$ and the fluid apparent viscosity $\mu$.

The suffixes w refer to the parameters at the wall, $\nu$ momentum, $h$ heat and $\tau$, $E_v$ are the shear stress and eddy diffusivity for momentum respectively.

In the analytical approach a model is proposed for the term $E_h$. Usually it involves a number of parameters which are assigned empirically.

Reynolds’ analogy states that

$$St = \frac{f}{2}$$

where

$$St = \frac{\dot{h}}{\rho C_p V}$$

is called the Stanton number,

$$f = \frac{2\tau_w}{\rho V^2}$$

the friction factor,

$$C_p$$ is the thermal capacity and

$V$ the average fluid velocity

This requires that the normalised velocity and temperature profiles be the same (Bird et al., 1960) p382.

$$\frac{d\theta^+}{dy^+} = \frac{dU^+}{dy^+}$$

It is normally assumed that Reynolds’ analogy implies two conditions

$$q/q_w = \tau/\tau_w$$

$$Pr_t = E_h/E_v = 1$$

where $Pr_t$ is called the turbulent Prandtl number. Equations (77) and (88) are at odds with experimental evidence. These are the paradoxes of the Reynolds analogy. The distribution of heat flux and shear stresses are not equal (Hinze, 1959, Churchill and Balzhiser, 1959, Seban and Shimazaki, 1951, Sleicher and Tribus, 1957) and the turbulent Prandtl number is not unity (Blom and deVries (Blom and deVries, 1968, McEligot et al., 1976, Malhotra and Kang, 1984, Kays, 1994, McEligot and Taylor, 1996, Weigand et al., 1997, Churchill, 2002). But if we move away from the wall into
the log-law and outer region then we can obtain equation (6) from (1) and (2) by applying the following simplifications (Trinh, 1969)

\[
\frac{E_h}{E_v} = \frac{q/q_w}{\tau/\tau_w}, \quad \frac{E_h}{k} >> 1 \quad \text{and} \quad \frac{E_v}{\nu} >> 1
\]  

and equation (6) in Reynolds’ analogy applies exactly outside a wall layer where the diffusive term predominates. The modelling of the diffusion layer in terms of pseudo-steady-state laminar flow (Karman, 1930, Martinelli, 1947, Lyon, 1951, Metzner and Friend, 1958, Reichardt, 1961, Levich, 1962) has received less attention by researchers in the last twenty years; many authors have switched to computer modelling investigate the velocity and temperature fields in greater details.

Penetration theories originated with Higbie (1935) who used the equation for unsteady conduction to model the transport process in jets and packed columns.

\[
\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial y^2}
\]  

where \( \alpha = \rho C_p / k \) is the thermal diffusivity. The well-known solution is

\[
q_w = \frac{1}{\sqrt{\pi \alpha t_h}} \Delta \theta_m
\]  

where \( \Delta \theta_m = \theta_w - \theta_h \) is the maximum temperature drop between the wall and the edge of the laminar boundary layer and \( t_h \) is a time scale for the diffusion process. Higbie closed the derivation by assuming that the typical time scale over which equation (11) applies is the contact time is

\[
t_c = \frac{x}{U_{wo}}
\]  

where \( x \) is the swept length. In an effort to apply Higbie's approach to turbulent transport, Danckwerts (1951) assumed that the surface near the wall is periodically swept clean by eddies penetrating from the bulk stream. The rate of renewal of the surface fluid near the wall is a function of the probability of occurrence of eddies of various frequencies. Danckwerts assumed this probability distribution to be uniform. Subsequent postulates of the surface renewal distributions have been reviewed by (Mathpati & Joshi, 2007; Pletcher, 1988; Ruckenstein, 1987; Sideman & Pinczewski, 1975). Many of these postulates do not link the assumed distribution of eddies to the
improved understanding of the coherent structures or the wall structure but more recent work does e.g. Fortuin et al. (1992).

Ruckenstein (1968) first attempted to derive a physical model for the distribution function by modelling the eddy as a roll cell which circulates the fluid from the wall to the outer region. The motion close to the wall surface is assumed to obey the laminar transport equation

$$U \frac{\partial \theta}{\partial x} + V \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2}$$

(13)

Ruckenstein calls this state "pseudo-laminar flow" but does not elaborate about the relation between this state and the bursting phenomenon at the wall. Thomas and Fan (1971) used an eddy cell model proposed by Lamont and Scott (1970) in conjunction with a wall model by Black (1969) and the time scale measured by Meek and Baer (1970) to model the whole process. In both these approaches, the differentiation between the instantaneous fluxes and their time-averaged values is unclear and rough approximations are necessary to effect closure of the solution. Experimental measurements to vindicate these visualisations are difficult to obtain because the wall layer in mass transfer processes is extremely thin. Perhaps the most extensive studies have been attempted by Hanratty and his associates. Their ideas have evolved, along with improved experimental evidence, from a belief that the eddy diffusivity near the wall is proportional to \( y^4 \) at very high Schmidt numbers (Son and Hanratty, 1967), as predicted by Deissler (1955) to a belief that a more accurate power index is 3.38 (Shaw and Hanratty, 1964, 1977) to an argument that the analogy between heat and mass transfer breaks down completely very close to the wall (Na and Hanratty, 2000). The research of Hanratty showed that the characteristic length scale of mass transfer in the longitudinal direction is equal to that for momentum transfer (Shaw and Hanratty 1964, 1977) but the time scale for mass transfer is much shorter that for momentum transfer.

To explain this perplexing effect, Campbell and Hanratty (1983) have solved the unsteady mass transfer equations without neglecting the normal component of the convection velocity, which they model as a function of both time and distance. They found that only the low frequency components of the velocity fluctuations affect the
mass transfer rates and that the energetic frequencies associated with the bursting process have no effect. In their explanation, the concentration sub-boundary layer acts as a low pass filter for the effect of velocity fluctuations on the mass transport close to the wall.

The existence of two time scales in the wall region of heat or mass transfer has been noted by all modern investigators. Their explanation is varied. McLeod and Ponton (1977) differentiate between the renewal period and the transit time which is defined as the average time that an eddy takes to pass over a fixed observer at the wall. Loughlin et al. (1985) and more recently Fortuin et al. (1992) differentiate between the renewal time and the age of an eddy. However, the exact identification of the renewal time with particular physical phenomena observed in turbulent flows remains vague. This is why penetration theories are still regarded with scepticism. For example Mathpathi and Joshi (op. cit.) stated that “these models serve a very limited purpose, because of the limited understanding of the relationships among the model parameters (contact time, renewal rate, size and shape of fluid packet, penetration depth of surface renewing eddies, etc.), and the flow parameters”.

This paper discusses the relationship between the modern observations of the wall process in turbulent flows and the parameters of a physically realistic penetration theory.

2 Theory

2.1 Equivalence of the penetration and boundary layer approaches to heat transfer

The first paper that I published when I was allowed to leave Viet Nam was meant to lay the foundation for a penetration theory of turbulent transport. It dealt with a transformation of the unsteady state conduction equation into the classical laminar boundary layer solutions (Trinh and Keey, 1992b). For simplicity we deal with transfer of heat from a flat plate to a Newtonian fluid stream but the arguments hold equally well for other geometries, fluid rheological characteristics and diffused quantities.
The governing equation for unsteady flow past a flat plate with a zero pressure gradient is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

(14)

Stokes neglected the convection terms to obtain

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

(15)

which was to apply to a flat plate suddenly set in motion. Similarly, the governing equation for heat transfer is

$$\rho C_p \left( \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} \right) = k \nabla^2 \theta$$

(16)

For a laminar boundary layer past a flat plate, which can again be simplified to equation (10) by dropping the convection terms.

Figure 1  Diffusion path and convection velocities of an entity of heat after Trinh and Keey (1992).

We argued (Trinh and Keey, 1992b) that even in a steady state laminar boundary layer, heat and momentum continuously penetrate the fluid stream from the wall and furthermore each elemental packet of heat, called henceforth an entity of heat for...
convenience, travels across the boundary layer only once. Hence the appearance of steady state profiles can be seen as the result of an endless repetition of unsteady state movements of elements as shown in Figure 1.

Consider next the nature of forces acting on an entity of heat. At time t, the entity of heat enters an element of fluid at position (x, y), drawn in full line and coloured red in Figure 2, which has velocities, u and v. At time (t + δt), the element of fluid has moved by convection to a new position, not shown in Figure 2, and the thermal entity has diffused to an adjacent element at (x + δx, y + δy), drawn in dotted lines and coloured orange with a brick pattern. Since the thermal entity is a scalar property, it has no mass, feels the effect of diffusion forces only and convects with the velocity of the fluid element where it temporarily resides. The convective forces act on the host element of fluid, not on the entity of heat.

![Figure 2. Convection and diffusion of an entity of heat](image)

This physical analysis shows that the entities of fluid and heat move in different directions owing to different driving forces and it should be possible to separate the effects of diffusion and convection in the mathematical analysis. Consider now the mathematical description of this physical visualisation.

Equation (16) may be rewritten as
\[
\frac{D\theta}{Dt} = \alpha \frac{D^2\theta}{Dy^2}
\]  

where

\[
\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} = u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z}
\]

is called the substantial derivative.

Bird, Stewart, & Lightfoot (1960) p.73 illustrate the difference between the Eulerian partial derivative \( \partial \theta/\partial t \) and the Lagrangian substantial derivative \( D\theta/Dt \) with the following example. Suppose you want to count the fish population in a river. The Eulerian partial derivative gives the rate of change in fish concentration at a fixed point (x,y) in the river as seen by an observer standing on the shore. An observer in a boat drifting with the current will see the change in fish concentration on the side of the boat as given by the substantial derivative.

However, a fish swimming in the river will have a different perception of the fish population, which not described by either of these derivatives. Clearly a Lagrangian derivative is required but the convection velocities \( u \) and \( v \) are no longer relevant to this case; the velocity and path of the fish are. An observer attached to the entity of heat moving across the boundary layer will perceive the changes in temperature according to an equation similar to equation (10) but the frame of reference in the penetration theory is not attached to the wall as implied by Higbie (1935).

A second concern lays in the neglect of the convection terms in equations (14) and (15). In a laminar boundary layer the terms \( \partial \theta/\partial t, \ u\partial \theta/\partial x \) and \( v\partial \theta/\partial y \) are of the same magnitude and the simplification seems unjustified. However the physical analysis presented above makes the decoupling of convective and diffusive forces in line with physical reality if we interpret equation (10) in terms of a Lagrangian derivative along the path of diffusion (Trinh, 2010a).

\[
\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial y^2}
\]

Then applying Taylor’s hypothesis

\[
\partial x = u \partial t
\]
transforms the solution of equation (10) into the classic solutions for steady state boundary layer heat transfer from a flat plate. Figure 3 shows that the unsteady state Lagrangian solution (in red dots) matches exactly the Eulerian solution of Polhausen (grid points on coloured background).

Figure 3 Diffusion path for a heat entity starting at the wall at $x = 0.1$ m and comparison with the Polhausen (1921) approximate solution for heat transfer in a laminar boundary layer.

We note that the entity of heat starting at the wall at location $x = 0.1m$ in this example reaches the edge of the thermal boundary layer at a position much further downstream
(x = 0.24m) as indicated in Figure 1. Thus clearly the sets of matching pairs (x, y) are not the same in the Eulerian Polhausen solution and in the Lagrangian solution.

Trinh and Keey (1992a) argued that the time scale of diffusion to be used in equation (19) must be linked with this distance Δx

$$t_h = \frac{\Delta x}{\bar{u}_B}$$

(21)

Where $\bar{u}_B$ is the average streamwise velocity in region B in Figure 1 and is related to the average velocity $\bar{u}_A$ in region A by

$$\frac{\bar{u}_A}{U_\infty} = 1 - \frac{\bar{u}_B}{U_\infty} = f(u)$$

(22)

Thus the time scale related to the entity of heat starting at the position (a) can be written as

$$t_h = \frac{x}{U_\infty f(u)}$$

(23)

Trinh and Keey showed that the function f(u) can be estimated by using the integral energy equation for boundary layer transport (Polhausen, 1921)

$$\frac{\partial}{\partial x} \left[ \frac{u}{U_\infty} \left( \frac{\theta - \theta_m}{\theta_w - \theta_m} \right) \right] dy = \frac{h}{\rho C_p U_\infty} = St$$

(24)

with a third order polynomial temperature profile as

$$f(u) = \frac{8}{3} M \frac{U_\infty}{U_h}$$

(25)

where

$$M = \frac{\delta_h^*}{\delta_h} = \frac{1}{\int_0 U_\infty \left( \frac{\theta - \theta_m}{\theta_w - \theta_m} \right) dy} \left( \frac{y}{\delta_h} \right)$$

(26)

is called usually a shape factor (Schlichting, 1979), p. 208.

$$\delta_h^* = \frac{1}{\int_0 U_\infty \left( \frac{\theta - \theta_m}{\theta_w - \theta_m} \right) dy}$$

(27)

is the integral energy thickness, $\delta_h$ the thermal boundary layer thickness and $U_\infty$ and $U_h$ are the velocities at the edge of the viscous and thermal boundary layers respectively.
To obtain the parameter $M$ we need the velocity distribution. Trinh and Keey again used Polhausen third order polynomial model

$$\frac{u}{U_\infty} = 1.5 \left( \frac{y}{\delta_v} \right) - \frac{1}{2} \left( \frac{y}{\delta_v} \right)^3$$  \hspace{1cm} (28)

Two situations must be distinguished depending on whether $U_h < U_\infty$ or $U_h > U_\infty$.

The first situation corresponds to $\Pr \gg 1$. Trinh and Keey showed that

$$M = \left( \frac{2}{5} \sigma - \frac{3}{280} \sigma^3 \right)$$  \hspace{1cm} (29)

where $\sigma = \frac{\delta_h}{\delta_v} = \Pr^{-b}$.

For $\Pr << 1$

$$M = \frac{3}{8} + \frac{1}{\sigma} \left( \frac{1}{4} - \frac{3}{5\sigma} + \frac{4}{35\sigma^3} \right)$$  \hspace{1cm} (30)

$$f(u) = 1 + \frac{1}{\sigma} \left( \frac{2}{3} - \frac{8}{5\sigma} + \frac{32}{105\sigma^3} \right)$$  \hspace{1cm} (31)

They obtain the parameter $b$ as a function of $\Pr$ as shown in Figure 4.

Figure 4  Exponent $b$ predicted by Trinh and Keey compared with Polhausen’s (1921)
For $\text{Pr} \gg 1$, $b \approx 1/3$. Trinh and Keey showed that the Higbie time scale can be obtained by assuming plug flow $u = U_\infty$. The time scale for $\text{Pr} \gg 1$ is

$$t_h = \frac{x}{U_\infty \left( \frac{2}{5} \text{Pr}^{-1/3} - \frac{1}{35} \text{Pr}^{-1/27} \right) \approx \frac{5x}{2U_\infty \text{Pr}^{-1/3}}}$$  \hspace{1cm} (32)

For $\text{Pr} = 1$, the thermal and momentum layer thicknesses are equal and

$$t_v = \frac{5x}{2U_\infty}$$  \hspace{1cm} (33)

Thus the time scales for the thermal and momentum boundary layers are not equal

$$\frac{t_h}{t_v} = \frac{13 \text{Pr}}{14 \text{Pr}^{-b} - 1} \approx \text{Pr}^{1/3}$$  \hspace{1cm} (34)

For $\text{Pr} \ll 1$

$$\frac{t_h}{t_v} = \frac{39}{105 + 70 \text{Pr}^b - 168 \text{Pr}^{-b} + 32 \text{Pr}^{4b}}$$  \hspace{1cm} (35)

### 2.2 Physical visualisation

![Schematic representation of the wall process in turbulent flow (Trinh, 2009).](image)

Figure 5. Schematic representation of the wall process in turbulent flow (Trinh, 2009).
The physical visualisation for turbulent transport underpinning this paper and all others in this theory of turbulence is based on the wall layer process first illustrated dramatically through hydrogen bubble tracers by Kline et al. (1967) and confirmed by many others. In plan view, Kline et al. observed a typical pattern of alternate low- and high-speed streaks. The low-speed streaks tended to lift, oscillate and eventually eject away from the wall in a violent burst.

In side view, they recorded periodic inrushes of fast fluid from the outer region towards the wall. This fluid was then deflected into a vortical sweep along the wall. The low-speed streaks appeared to be made up of fluid underneath the travelling vortex. The bursts can be compared to jets of fluid that penetrate into the main flow, and get slowly deflected until they become eventually aligned with the direction of the main flow.

The sweep phase, which lasts longest and dominates the statistics of the flow near the wall, can be modelled with the method of successive approximations borrowed from the analysis of oscillating laminar boundary layers (Schlichting, 1960, Tetlionis, 1981). The first approximation, called the solution of order $\varepsilon^0$, describes the diffusion of viscous momentum into the main stream. The solution of order $\varepsilon$ and higher only become important when the fast periodic velocity fluctuations have become strong enough to induce jets of fluid to be ejected from the wall i.e. during the bursting phase (Trinh, 2009).

In other words, the wall layer defined by the solution of order $\varepsilon^0$ is visualised as an unsteady state laminar sub-boundary layer which is interrupted by the emergence of the ejections. Mass, heat and momentum are contained in the same body of fluid ejected from the wall which explains, in my view, why there is an analogy between the laws of heat, mass and momentum in the outer region.

Since the low speed streaks occur randomly in time and space, a fixed probe will return a statistical average temperature and velocities of low-speed streaks of all ages. There are of course many other coherent structures in turbulent flows e.g. (Robinson, 1991) but they do not seem contribute the long term statistical averages and both the profiles of
velocity (and temperature) and the transport fluxes can be adequately be modelled with the two structures discussed here: the low speed streaks and the ejections (Trinh, 2010c).

2.3 The paradox of time scales

Most authors have long recognised that the Higbie time scale is unable to correlate steady state convective heat transfer. The differentiation between the residence time of a packet of fluid at the wall and the frequency at which it is renewed by fluid from the bulk flow proposed in existing penetration theories does not explain, in my view, the difference between the two time scales in equations Error! Reference source not found.31) and Error! Reference source not found.32) because surely the momentum, heat and mass in the wall layer must be renewed simultaneously by the incoming eddies. For example, if one explains the second time scale in the thermal wall layer in terms of the "age" of eddies sweeping the wall (Fortuin et al, 1991) one is faced with the question as to why the age of ejected lumps of fluid from the wall does not affect Reynolds' analogy. Indeed any argument based on convective forces which are specific to heat and mass transfer (i.e. not present in momentum transfer) raises a paradox when one applies Reynolds' analogy.

In the present theory, there is only one mechanism for agitation in a turbulent boundary layer. This agitation process relies on the intermittent ejection of wall fluid into the outer region and its time scale is \( t_v \), the time scale of the momentum wall layer. The second time scale \( t_u \) reflects a diffusion process within the wall layer and therefore does not relate to the agitation mechanism; it is related to the diffusion of heat and mass across the wall layer and best understood if one analyses equations Error! Reference source not found.15) and Error! Reference source not found.19) in a Lagrangian context. Because of the difference in momentum and thermal diffusivities, the depth of penetration of heat and viscous momentum from the wall differ as shown by the conceptual illustration of a mapping of the velocity and temperature contours in Figure 6. Contours of velocity and temperature in the wall layer. Figure 6.
In my view there is only one defining time scale for the wall layer $\tau_w$ because it defines the moment when the ejection occurs and therefore sets the life time of the low speed streak. Both the momentum and thermal content of the wall fluid are regenerated at the same time because of the subsequent inrush from the main stream. The surface renewal theory would require two different agitation events: the ejections and other separate rushes of fluid into the wall during the sweep phase. I have never seen any publication giving evidence of eddies or streams penetrating through the hair pin vortices into the low-speed streaks.

Most previous penetration theories postulate that turbulent eddies from the main flow penetrate into the wall layer to renew selectively its heat content without apparently affecting the momentum. The present visualisation argues that both heat and momentum penetrate from the wall into the main flow but at different rates.

### 2.4 Mathematical formulations

The equation for diffusion of viscous momentum
\[
\frac{\partial \tilde{u}}{\partial t} + \nu \frac{\partial^2 \tilde{u}}{\partial y^2} = 0
\]  \hspace{1cm} (36)

Can be solved by the method of Stokes (1851) for the conditions:

- **IC** \( t = 0 \) \( \text{all} \ y \) \( \tilde{u} = U_v \)
- **BC1** \( t > 0 \) \( y = 0 \) \( u = 0 \)
- **BC2** \( t > 0 \) \( y = \infty \) \( \tilde{u} = U_v \)

Where \( \tilde{u} \) is the smoothed phase velocity of the low speed streak. The well-known solution is

\[
\frac{\tilde{u}}{U_v} = \text{erf}(\eta_s)
\]  \hspace{1cm} (37)

where \( \eta_s = \frac{y}{\sqrt{4t}} \) \hspace{1cm} (38)

The average wall-shear stress is

\[
\tau_w = \frac{\mu U_v}{t_v} \left[ \frac{\partial \tilde{u}}{\partial y} \right]_{y=0} \text{dt} = \frac{\mu U_v}{t_v \sqrt{\pi}} \int_0^{\eta_s} \frac{1}{\sqrt{t}} \text{dt}
\]  \hspace{1cm} (39)

Equation Error! Reference source not found. may be rearranged as

\[
t_v^+ = \frac{2}{\sqrt{\pi}} U_v^+
\]  \hspace{1cm} (40)

The time-averaged velocity profile near the wall may be obtained by rearranging equation (37) as

\[
\frac{U^+}{U_v^+} = \int_0^t \text{erf} \left( \frac{y^+ \sqrt{\pi}}{4U_v^+ t_v^+} \right) d\left( \frac{t}{t_v^+} \right)
\]  \hspace{1cm} (41)

Equation (41) applies up to the edge of the wall layer where \( u/U_v = 0.99 \), which corresponds to \( y = \delta_v \) and \( \eta_s = 1.87 \). Substituting these values into equation Error! Reference source not found. gives

\[
\delta_v^+ = 3.74 t_v^+
\]  \hspace{1cm} (42)

where \( t_v^+ = u_* \sqrt{t_v^+} / \mu \). Back-substitution of equation Error! Reference source not found. into Error! Reference source not found. gives

\[
\delta_v^+ = 4.22 U_v^+
\]  \hspace{1cm} (43)
\[
\frac{\theta - \theta_h}{\theta_w - \theta_h} = \text{erf} \left( \eta_h \right) \tag{44}
\]

where \( \eta_h = \frac{y}{\sqrt{4\alpha t}} \tag{45} \)

which can be arranged as

\[
\eta_h = \frac{y^+ \text{Pr}^{1/2}}{2t^+} \tag{46}
\]

The time average wall heat flux is

\[
q_w = \frac{k(\theta_w - \theta_m)}{t_h} \int_0^{t_h} \int_0^h \frac{\partial (\theta_w - \theta)}{\partial y} \left( y = 0 \right) dt \tag{47}
\]

\[
qu = \frac{k(\theta_w - \theta_m)}{t_h \sqrt{\pi}} \int_0^{t_h} \frac{1}{\sqrt{\alpha t}} dt
\]

\[
qu = \frac{2}{\sqrt{\pi} \sqrt{\alpha t_h}} \tag{48}
\]

Equation (47) may be rearranged as

\[
t_h^+ = \frac{2}{\sqrt{\pi} \text{Pr} \theta_w^+} \tag{49}
\]

The time-averaged temperature profile near the wall may be obtained by rearranging equation Error! Reference source not found. as

\[
\frac{\theta^+}{\theta_h^+} = \int_0^t \text{erf} \left( \frac{y^+ \text{Pr} \sqrt{\pi}}{4\theta_w^+ \sqrt{t/t_h}} \right) dt \left( \frac{t}{t_h} \right)
\]

Equation (49) applies up to the edge of the wall layer where \( u/U_v = 0.99 \), which corresponds to \( y = \delta_h \) and \( \eta_h = 1.87 \). Substituting these values into equation (46) gives

\[
1.87 = \frac{\delta_h^+ \text{Pr}^{1/2}}{2t_h^+} \tag{50}
\]

Substituting equation (48) into (50) gives

\[
\theta_h^+ = \frac{\delta_h^+ \text{Pr}}{4.22} \tag{51}
\]

There are two alternative estimates for the time scale \( t_h \) that we will both investigate.

**2.4.1 Different time scales for diffusion of momentum and heat**

In this case, we follow the arguments of Trinh and Keey that the time scales for diffusion of heat and momentum in a boundary layer are different because the rates of
diffusion are governed by coefficients $\alpha$ and $\nu$ that do not have the same value. We must again differentiate two cases

$Pr > 1$

Combining equations (34), (42) and (50) gives

$$\frac{\delta^+_h}{Pr^{1/2}} \frac{t^+_h}{t^+_\nu} = \frac{\delta^+_\nu}{Pr^{1/2}} \sqrt{\frac{13 Pr}{14 Pr^{1/2} - 1}}$$

(52)

Taking $b \approx 1/3$ (Trinh and Keey 1992a), then

$$\frac{\delta^+_h}{Pr^{1/2}} \frac{t^+_h}{t^+_\nu} = \frac{\delta^+_\nu}{Pr^{1/2}} \sqrt{\frac{13 Pr}{14 Pr^{2/3} - 1}}$$

(53)

Now if we write for convenience

$$\delta^+_h = \delta^+_\nu Pr^a$$

(54)

equation (53) can be used to show that for $Pr > 1$, $a = -1/3$.

$Pr < 1$

Combining equations (35), (42) and (50) gives

$$\frac{\delta^+_h}{Pr^{1/2}} \frac{t^+_h}{t^+_\nu} = \frac{\delta^+_\nu}{Pr^{1/2}} \sqrt{\frac{34}{105 + 70 Pr^b - 168 Pr^{2b} + 32 Pr^{3b}}}$$

(55)

where $b \to 1/2$ as $Pr \to 0$ (Trinh and Keey, 1992a).

2.4.2 Same time scale for diffusion of heat and momentum

The argument here is that when the pocket of fluid at the wall is ejected it interrupts both before the diffusion of heat and momentum at the same time. Thus

$$t^+_h = t^+_\nu$$

(56)

Combining equations (42), (50) and (56)

$$\frac{\delta^+_h}{Pr^{1/2}}$$

(57)
3 Comparison with experimental measurements

Figure 7. Temperature data of Smith et al. (1967) for Pr = 5.7 at different Reynolds number and determination of thermal wall layer thickness. Smith et al. indicate that the transition region for heat transfer extends to higher Reynolds numbers than for momentum transfer as observed by many authors e.g. (Gnielinski, 1976, Trinh et al., 2010).

Figure 8. Zonal similarity profile of temperature. Same data as in Figure 7.
The temperature profiles become almost independent of Reynolds number above $\text{Re} > 20000$.

Figure 9. Temperature profiles for air (Johnk and Hanratty, 1962), $\text{Pr} = 0.7$. Normalising the temperature and distance with their values at the edge of the wall layer removes the effect of the Reynolds number just as found for velocity profiles (Trinh, 2010d). Another example is shown for air with $\text{Pr} = 0.7$ in Figures 9 and 10.

Figure 10. Zonal similarity temperature profiles for air. Same data as in Figure 9.
Figure 11. Zonal similarity plot of velocity profile $W$ (Wei and Willmarth, 1989). This inability to collapse the temperature profiles highlights the fact that the mechanism in the log-law layer is based on a convection principle, which is independent of the Prandtl number whereas the mechanism in the wall layer is based on a diffusion principle which is dependent of the Prandtl number as shown in section 2.4.

Figure 12. Velocity and temperature defect plot showing Reynolds analogy. Data $G$ (Smith et al., 1967) $Pr=5.7$, $W$ (Wei and Willmarth, 1989), $L$ (Laufer, 1954) velocity,
However, Reynolds’ analogy applies well in the outer region as shown in the velocity and temperature defect plot in Figure 14.

A plot of \( \frac{\delta_h^+}{Pr^a} \) vs. \( Re \) where \( a \) is given by equations (54) and (55) collapses the heat and momentum transfer data as shown in Figure 13. In particular, the data support \( a \approx 1/3 \) for \( Pr \geq 5.7 \). By contrast, normalisation of \( \delta_h^+ \) with \( Pr^{1/2} \) as suggested by equation (57) does not collapse the thermal and momentum thicknesses as shown in Figure 14.

![Figure 13. Variation of thermal wall layer thickness with Reynolds and Prandtl numbers. Index \( a \) given by equations (54) and (55). Data JH (Johnk and Hanratty, 1962), G (Smith et al., 1967), J (Janberg, 1970), W (Wei and Willmarth, 1989), N (Nikuradse, 1932), B (Bogue, 1962), E (Eckelmann, 1974), La (Laufer, 1954), Lw (Lawn, 1971), T (Trinh, 2010b). Thermal data filled points. Velocity data hollow points.](image-url)
Applying Reynolds’ analogy in the outer region in the log-law region

$$\Theta^+ = 2.5 \ln \left( \frac{y^+}{\delta^+_h} \right) + \Theta^+_h$$  \hspace{1cm} (58)

For $Pr > 1$ we substitute $\delta^+_h = \delta^+_v Pr^{-1/3}$ and $\Theta^+_h = \delta^+_v Pr^{2/3} / 4.22$ into equation (58)

$$\Theta^+ = 2.5 \ln \left( \frac{y^+ Pr^{1/3}}{\delta^+_v} \right) + \frac{\delta^+_v Pr^{2/3}}{4.22}$$  \hspace{1cm} (59)

For fully turbulent flow $\delta^+_v = 64.8$ (Trinh, 2009) then

$$\Theta^+ = 2.5 \ln \left( y^+ Pr^{1/3} \right) + 15.35 Pr^{2/3} - 10.43$$  \hspace{1cm} (60)

At the pipe axis, $y^+ = R^+$, $\Theta^+ = \Theta^+_m$ and $R^+ = \frac{Re}{2 \sqrt{f}}$ and the Nusselt number can be derived from the temperature profile by standard techniques (e.g. Schlichting, 1960). Introducing

$$D(Pr, Re) = \Theta^+_m - \Theta^+_h$$  \hspace{1cm} (61)

where $\Theta^+_h$ is the mixing cup temperature given by

$$\Theta^+_h = \frac{1}{\pi R^+} \int \Theta^+ U^+ 2\pi (R^+ - y^+) dy^+$$  \hspace{1cm} (62)
The value of $D(Pr, Re)$ has been calculated by performing the integration indicated by equation (62) numerically and tabulated (Trinh, 1969).

$$\Theta_* = 2.5 \ln \left( \frac{Re \sqrt{f}}{2\sqrt{2}} Pr^{1/3} \right) + 15.35 Pr^{2/3} - 10.43 - D(Pr, Re) = \frac{\sqrt{f/2}}{St} \tag{63}$$

$$St = \frac{\sqrt{f/2}}{2.5 \ln Re \sqrt{f + 15.5 Pr^{2/3} - 13.03 + (2.5/3) \ln Pr - D(Pr, Re)}} \tag{64}$$

Equation (64) is shown against the data of Friend (1958) who measured both the Nusselt number ($Nu = St \cdot Re \cdot Pr$) and friction factor in Figure 15. It has the same level of accuracies as other analogy formulae derived in this theory of turbulence (Trinh, 2009). These will be discussed in a summarising review of heat transfer correlations.

![Figure 15. Comparison of equation (64) against data of Friend (1958)](image)

$$1.88 \leq Pr \leq 264.$$  

4 Conclusion

A method for determination the thickness of the thermal wall layer has been developed. A penetration theory is discussed where the time scale for the momentum wall layer is based on the onset of ejections from the wall but the time scale for the thermal layer is based on the unsteady state diffusion of heat from the wall into the
streamwise flow, not the renewal of heat from eddies penetrating the wall layer. The thickness of the thermal wall layer is well correlated by this technique. Reynolds’ analogy is confirmed in the region beyond the wall layer. Predictions for the Nusselt number also correlate well with experimental data. In contrast the predictions based on a unique time scale for both momentum and thermal transfer are found not accurate.

5 References

BIRD, R. B., STEWART, W. E. & LIGHTFOOT, E. N. 1960. Transport Phenomena, New York, Wiley and Sons.,
BLACK, T. J. 1969. Viscous drag reduction examined in the light of a new model of wall turbulence. In: C.S., W. (ed.) Viscous Drag Reduction. New York: Plenum Press.
BLOM, J. & DEVRIES, D. A. 1968. On the values of the turbulent Prandtl number. Third All-Union Heat and Mass Transfer Conference.
BOGUE, D. C. 1962. Velocity profiles in turbulent non-Newtonian pipe flow, Ph.D. Thesis, University of Delaware.
BOUSSINESQ, J. 1877. Théorie de l’écoulement tourbillant. Mémoires Présentés à l’Académie des Sciences, 23, 46.
CAMPBELL, J. A. & HANRATTY, T. J. 1983. Turbulent velocity fluctuations that control mass transfer at a solid boundary layer. AIChE J., 29, 215.
CHURCHILL, S. W. 1996. A Critique of Predictive and Correlative Models for Turbulent Flow and Convection. Ind. Eng. Chem. Res. , 35, 3122-3140.
CHURCHILL, S. W. 2002. A Reinterpretation of the Turbulent Prandtl Number. Ind. Eng. Chem. Res. , 41, 6393-6401.
CHURCHILL, S. W. & BALZHISER, R. E. 1959. The Radial Heat Flux. Chem. Eng. Prog. Symp. Ser., 55, 127.
COLBURN, A. P. 1933. A method of correlating forced convection heat transfer data and a comparison with fluid friction. Trans. AIChE, 29, 174.
DANCKWERTS, P. V. 1951. Significance of liquid film coefficients in gas absorption. Ind. Eng. Chem., 43, 1460.
DEISSLER, R. G. 1955. Analysis of turbulent heat transfer, mass transfer, and friction in smooth tubes at high Prandtl and Schmidt numbers. NACA Rep. 1210.
ECKELMANN, H. 1974. The structure of the viscous sublayer and the adjacent wall region in a turbulent channel flow. Journal of Fluid Mechanics, 65, 439.
FORTUIN, J. M. H., MUSSCHENGA, E. E. & HAMERSMA, P. J. 1992. Transfer processes in turbulent pipe flow described by the ERSR model. AIChE J., 38, 343.
FRIEND, P. S. 1958. M.Ch.E. Thesis. Univ. of Delaware, Newark.,
GNIELINSKI, V. 1976. NEW EQUATIONS FOR HEAT AND MASS-TRANSFER IN TURBULENT PIPE AND CHANNEL FLOW. International Chemical Engineering, 16, 359-368.
GOWEN, R. A. & SMITH, J. W. 1967. The effect of the Prandtl number on temperature profiles for heat transfer in turbulent pipe flow. Chem. Eng. Sci., 22, 1701-1711.
HAMERSMA, P. J. & FORTUIN, J. M. H. 2003. Effect of Drag Reducers on Transport Phenomena in Turbulent Pipe Flow. *AIChE Journal*, 49, 335-349.

HARRIOTT, P. 1962. A Random Eddy Modification of the Penetration Theory. *Chemical Engineering Science*, 17, 149-154.

HIGBIE, R. 1935. Rate of absorption of a gas into a still liquid during short periods of exposure. *Transactions AIChE*, 31, 365.

HINZE, J. O. 1959. Turbulence, New York, McGraw-Hill.

HUGHMARK, G. A. 1968. Eddy diffusivity close to a wall. *AIChE J.*, 14, 325.

JANBERG, K. 1970. Etude expérimentale de la distribution des températures dans les films visqueux, aux grands nombres de Prandtl. *Int. J. Heat Mass transfer*, 13, 1234.

JOHNK, R. E. & HANRATTY, T. J. 1962. Temperature profiles for turbulent flow of air in a pipe. *Chemical Engineering Science*, 17, 867.

KARMAN, V. T. 1930. Mechanische Ähnlichkeit und Turbulenz. *Proc. 3rd. Int. Congress Appl. Mech.*, Stockholm also NACA TM 611 (1931). Stockholm: also NACA TM 611 (1931).

KAWASE, Y. & ULBRECHT, J. J. 1983. Turbulent Heat And Mass-Transfer In Non-Newtonian Pipe-Flow - A Model Based On The Surface Renewal Concept. *Physicochemical Hydrodynamics*, 4, 351-368.

KAYS, W. M. 1994. Turbulent Prandtl number. Where are we? *Journal of Heat Transfer, Transactions ASME* 116, 284-295.

KLINE, S. J., REYNOLDS, W. C., SCHRAUB, F. A. & RUNSTADLER, P. W. 1967. The structure of turbulent boundary layers. *Journal of Fluid Mechanics*, 30, 741.

LAMONT, J. C. & SCOTT, D. S. 1970. An eddy cell model of mass transfer into the surface of a turbulent fluid. *AIChE Journal*, 16.

LAUFER, J. 1954. The structure of turbulence in a fully developed pipe. NACA TN No 2945.

LAWN, C. J. 1971. The Determination of the Rate of Dissipation in Turbulent Pipe Flow. *Journal of Fluid Mechanics*, 48, 477-505.

LEVICH, V. G. 1962. *Physico-Chemical Hydrodynamics*, New York, Prentice-Hall.

LOUGHLIN, K. F., ABUL-HAMAYEL, M. A. & THOMAS, L. C. 1985. The surface rejuvenation theory of wall turbulence for momentum, heat and mass transfer: application to moderate and high schmidt (Prandtl) numbers. *AIChE J.*, 31, 1614-1620.

LYON, R. N. 1951. Heat transfer at high fluxes in confined spaces. *Chem. Eng. Prog.*, 47, 75.

MALHOTRA, A. & KANG, S. S. 1984. Turbulent Prandtl number in circular pipes. *Int. J. Heat and Mass Transfer*, 27, 2158-2161.

MARTINELLI, R. C. 1947. Heat transfer to molten metals. *Trans. ASME*, 69, 947.

MATHPATI, C. S. & JOSHI, J. B. 2007. Insight into Theories of Heat and Mass Transfer at the Solid-Fluid Interface Using Direct Numerical Simulation and Large Eddy Simulation. *Ind. Eng. Chem. Res.*, 46, 8525-8557.

MCELIGOT, D. M., PICKETT, P. E. & TAYLOR, M. F. 1976. Measurement of wall region turbulent Prandtl numbers in small tubes. *Int. J. Heat Mass Transfer*, 19, 799-803.

MCELIGOT, D. M. & TAYLOR, M. F. 1996. The turbulent Prandtl number in the near-wall region for low-Prandtl-number gas mixtures. *Int. J. Heat Mass Transfer*, 39, 1287--1295.
MCLEOD, N. & PONTON, J. W. 1977. A model for turbulent transfer processes at a solid-fluid boundary. *Chem. Eng. Sci.*, 32.

MEEK, R. L. & BAER, A. D. 1970. The Periodic viscous sublayer in turbulent flow. *AIChe J.*, 16, 841.

METZNER, A. B. & FRIEND, W. L. 1958. Turbulent Heat Transfer Inside Tubes and the Analogy Among Heat, Mass and Momentum Transfer. *A. I. Ch. E. J.*, 4, 393-402.

NA, Y. & HANRATTY, T. J. 2000. Limiting behavior of turbulent scalar transport close to a wall. *International Journal of Heat and Mass Transfer* 43, 1749-1758.

NIKURADSE, J. 1932. Gesetzmäßigkeit der turbulenten Strömung in glatten Rohren. *Forsch. Arb. Ing.-Wes.* N0. 356.

PLETCHER, R. H. 1988. Progress in Turbulent Forced Convection. *J. Heat Transfer*, 110, 1129.

POLHAUSEN, K. 1921. Zur näherungsweisen Integration der Differentialgleichung der laminaren Reibungsschicht. *ZAMM*, 1, 252-268.

REICHARDT, H. 1961. In: HARTNETT, J. P. (ed.) *Recent Advances in Heat and Mass Transfer*. McGraw-Hill.

REYNOLDS, O. 1874. On the extent and action of the heating surface for steam boilers. *Proc. Manchester Lit. Phil. Soc.*, 14, 7-12.

ROBINSON, S. K. 1991. Coherent Motions in the Turbulent Boundary Layer. *Annual Review of Fluid Mechanics*, 23, 601.

RUCKENSTEIN, E. 1968. A generalised penetration theory for unsteady convective mass transfer *Chem. Eng. Sci.*, 23, 363.

SCHLICHITING, H. 1960. *Boundary Layer Theory*, New York., MacGrawHill.

SEBAN, R. A. & SHIMAZAKI, T. T. 1951. Heat transfer to a fluid in a smooth pipe with walls at a constant temperature. *Trans. ASME*, 73, 803.

SHAW, D. A. & HANRATTY, T. J. 1964. Fluctuations in the Local Rate of Turbulent Mass Transfer to a Pipe Wall. *AIChe J.*, 10, 475-482.

SHAW, D. A. & HANRATTY, T. J. 1977. Influence of Schmidt number on the fluctuations of turbulent mass transfer to a wall. *AIChe J.*, 23, 160.

SIDEMAN, S. & PINCZEWSKI, W. 1975. Turbulent heat and mass transfer at interfaces. In: GUTFINGER, C. (ed.) *Topics in Transport Phenomena*. Hemisphere Publishing House, Wiley

SLEICHER, C. A. & TRIBUS, M. 1957. Heat Transfer in a Pipe with Turbulent Flow and Arbitrary Wall-Temperature Distribution. *Trans. ASME* 79, 765.

SMITH, J. W., GOWEN, R. A. & WASMUND, B. O. 1967. Eddy diffusivities and temperature profiles for turbulent heat transfer to water in pipes. *Chem. Eng. Prog. Symp. Ser.*, 63, 92.

SON, S. M. & HANRATTY, T. J. 1967. Limiting relation for the eddy diffusivity close to a wall. *AIChe J.*, 13, 689-696.

STOKES, G. G. 1851. On the effect of the internal friction of fluids on the motion of pendulums. *Camb. Phil. Trans.*, IX, 8.

TETLIONIS, D. M. 1981. Unsteady Viscous Flow. *SpringerVerlag, New York.*

THOMAS, L. C. & FAN, L. T. 1971. A model for turbulent transfer in the wall region. *J. Eng. Mech. Div.*, 97, 169.

TRINH, K. T. 1969. A boundary layer theory for turbulent transport phenomena, *M.E. Thesis*, New Zealand, University of Canterbury.

TRINH, K. T. 2009. A Theory Of Turbulence Part I: Towards Solutions Of The Navier-Stokes Equations. arXiv.org 0910.2072v1 [physics.flu.dyn] [Online].
TRINH, K. T. 2010a. The Fourth Partial Derivative In Transport Dynamics. *arXiv.org 1001.1580 [math-ph.]* [Online].

TRINH, K. T. 2010b. A Theoretical Closure for Turbulent Flows Near Walls. *arXiv.org 1001.2353 [physics.flu-dyn]* [Online].

TRINH, K. T. 2010c. Turbulent Couette Flow: An analytical solution. *arXiv.org 1001.2353 [physics.flu-dyn]* [Online]. Available: http://arxiv.org/abs/1009.0969.

TRINH, K. T. 2010d. A Zonal Similarity Analysis of Velocity Profiles in Wall-Bounded Turbulent Shear Flows. *arXiv.org 1001.1594 [phys.fluid-dyn]* [Online].

TRINH, K. T. & KEEY, R. B. 1992a. A Modified Penetration Theory and its Relation to Boundary Layer Transport. *Trans. IChemE, ser. A, 70*, 596-603.

TRINH, K. T. & KEEY, R. B. 1992b. A Time-Space Transformation for Non-Newtonian Laminar Boundary Layers. *Trans. IChemE, ser. A, 70*, 604-609.

TRINH, K. T., WILKINSON, B. H. P. & KIKA, N. K. 2010. A Similarity Analysis for Heat Transfer in Newtonian and Power Law Fluids Using the Instantaneous Wall Shear Stress. *arXiv.org* [Online].

WEI, T. & WILLMARTH, W. W. 1989. Reynolds-number Effects on the Structure of a Turbulent Channel Flow. *Journal of Fluid Mechanics, 204*, 57-95.

WEIGAND, B., FERGUSON, J. R. & CRAWFORD, M. E. 1997. An extended Kays and Crawford turbulent Prandtl number model. *Inf. J. Heal Mass Transfer. Vol. 40, No. 17, pp. 4191-4196, 1997, 40, 4191-4196.*