Aspects of a new class of braid matrices: roots of unity and hyperelliptic $q$ for triangularity, L-algebra, link-invariants, noncommutative spaces

A. Chakrabarti

Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France.

Abstract

Various properties of a class of braid matrices, presented before, are studied considering $N^2 \times N^2 (N = 3, 4, \ldots)$ vector representations for two subclasses. For $q = 1$ the matrices are nontrivial. Triangularity ($\hat{R}^2 = I$) corresponds to polynomial equations for $q$, the solutions ranging from roots of unity to hyperelliptic functions. The algebras of $L-$ operators are studied. As a crucial feature one obtains $2N$ central, group-like, homogenous quadratic functions of $L_{ij}$ constrained to equality among themselves by the $RLL$ equations. They are studied in detail for $N = 3$ and are proportional to $I$ for the fundamental $3 \times 3$ representation and hence for all iterated coproducts. The implications are analysed through a detailed study of the $9 \times 9$ representation for $N = 3$. The Turaev construction for link invariants is adapted to our class. A skein relation is obtained. Noncommutative spaces associated to our class of $\hat{R}$ are constructed. The transfer matrix map is implemented, with the $N = 3$ case as example, for an iterated construction of noncommutative coordinates starting from an $(N - 1)$ dimensional commutative base space. Further possibilities, such as multistate statistical models, are indicated.
1 Introduction:

A new class of braid matrices was presented in previous papers. The most convenient formulation can be found in Sec.3 of Ref.1. This is based on two previous works [2, 3]. For ready reference we summarize below the essential features. In succeeding sections different properties of such braid matrices will be studied. Remarkable new aspects will be encountered. We will always be concerned with $N^2 \times N^2$ vector representations of braid matrices ($N = 3, 4, ...$).

For proper appreciation one should start by noting explicitly the links and the crucial differences with the standard $SO_q(N)$ and $Sp_q(N)$ braid matrices. Our approach is consistently via spectral resolutions i.e. in terms of projectors.

The Baxterized braid matrices (depending on a spectral parameter $\theta$) satisfy

$$\hat{R}_{12}(\theta)\hat{R}_{23}(\theta + \theta')\hat{R}_{12}(\theta') = \hat{R}_{23}(\theta')\hat{R}_{12}(\theta + \theta')\hat{R}_{23}(\theta)$$

where

$$\hat{R}_{12} = \hat{R} \otimes I_N, \quad \hat{R}_{23} = I_N \otimes \hat{R}$$

and $I_N$ is the $N \times N$ identity matrix.

For the standard $SO_q(N)$ and $Sp_q(N)$ cases (see sources cited in Sec.2 of Ref.1) one has

$$\hat{R}(\theta) = P_+ v(\theta)P_- + w(\theta)P_0$$

where the projectors satisfy

$$P_i P_j = \delta_{ij} P_i, \quad P_+ + P_- + P_0 = I_{N^2}$$

All $\theta$-dependence is in $v(\theta)$ and $w(\theta)$. The projectors depend only on $q$.

The elegant canonical formulation (Sec.2, Ref.1) gives the following $v(\theta)$ and $w(\theta)$ where

$$q = \exp h$$

One has for each case (independently of $N$)

$$v(\theta) = \frac{\sinh(h - \theta)}{\sinh(h + \theta)}$$

and two solutions for $w(\theta)$.

$SO_q(2n + 1)$:

$$w(\theta) = \frac{\cosh((n + \frac{1}{2})h - \theta)}{\cosh((n + \frac{1}{2})h + \theta)}; \quad \frac{\sinh((n - \frac{1}{2})h - \theta)}{\sinh((n - \frac{1}{2})h + \theta)} v(\theta)$$

$SO_q(2n)$:

$$w(\theta) = \frac{\cosh(nh - \theta)}{\cosh(nh + \theta)}; \quad \frac{\sinh((n - 1)h - \theta)}{\sinh((n - 1)h + \theta)} v(\theta)$$
\[ Sp_q(2n): \]
\[
\begin{align*}
  w(\theta) &= \frac{\sinh((n+1)h-\theta)}{\sinh((n+1)h+\theta)}; \quad \frac{\cosh(nh-\theta)}{\cosh(nh+\theta)} v(\theta) \\
  & \quad \text{(1.7)}
\end{align*}
\]

In contrast, for our new class (Sec.3, Ref.1), conserving the projectors but changing the coefficients, we have
\[
\begin{align*}
  v(\theta) &= 1, \quad w(\theta) = \frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)} \\
  & \quad \text{(1.8)}
\end{align*}
\]

where
\[
\begin{align*}
  e^\eta + e^{-\eta} &= ([N - \epsilon] + \epsilon) = \frac{q^{N-\epsilon} - q^{-N+\epsilon}}{q - q^{-1}} + \epsilon \\
  & \quad \text{(1.9)}
\end{align*}
\]

and \(\epsilon = \pm 1\) when the \(P_i\) are those for \(SO_q(N)\) and \(Sp_q(N)\) respectively. (An overall ambiguity of sign for the right side of (1.9) has been fixed to assure real \(\eta\) for real \(q\). This will be maintained throughout, though complex \(q\) will be considered later.)

We adopt the following notations for our \(\hat{R}(\theta)\) when (1.8) is implemented:

(a): When \(P_i\) are those for \(SO_q\) our \(\hat{R}(\theta)\) is of type \(\hat{o}(N)\)

(b): When \(P_i\) are those for \(Sp_q\) our \(\hat{R}(\theta)\) is of type \(\hat{p}(N)\).

This is to signal the provenance of the projectors and also at the same time the fact that the coefficients (1.8) often lead to startlingly different properties as compared to the standard cases ( from (1.4) to (1.7) ).

For (1.8) we obtain
\[
\hat{R}(\theta) = P_+ + P_- + \frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)} P_0 = I + \left( \frac{\sinh(\eta - \theta)}{\sinh(\eta + \theta)} - 1 \right) P_0 \\
\text{(1.10)}
\]

For completeness we give \(P_0\) explicitly [4,5]. Let the n-tuple \((\rho_1, \rho_2, \ldots, \rho_N)\) be defined as follows for the respective cases indicated:

\[
\begin{align*}
  SO_q(2n+1) & : \quad (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{1}{2}) \\
  SO_q(2n) & : \quad (n - 1, n - 2, \ldots, 1, 0, 0, -1, \ldots, -n + 1) \\
  Sp_q(2n) & : \quad (n, n - 1, \ldots, 1, -1, \ldots, -n) \\
  & \quad \text{(1.11)} \quad \text{(1.12)} \quad \text{(1.13)}
\end{align*}
\]

Define correspondingly for

\[
\begin{align*}
  SO_q(N) & : \quad \epsilon = 1, \quad (i = 1, \ldots, N; \quad N = 2n, 2n + 1) \\
  Sp_q(2n) & : \quad \epsilon = 1 \quad (i \leq n), \quad \epsilon = -1 \quad (i > n) \quad (i = 1, \ldots, 2n) \\
  & \quad \text{(1.14)} \quad \text{(1.15)}
\end{align*}
\]

Set
\[
i' = N + 1 - i \\
\text{(1.16)}
\]
Then corresponding to (1.11), (1.12), in notations most suitable for us,

\[ ([N - 1] + 1) P_0 = \sum_{i,j=1}^{N} q^{(\rho_i - \rho_j)} (ij) \otimes (i'j') \equiv P'_0 \]  

(1.17)

and corresponding to (1.13) and (1.15)

\[ ([N + 1] - 1) P_0 = \sum_{i,j=1}^{N} q^{(\rho_i - \rho_j)} \epsilon_i \epsilon_j (ij) \otimes (i'j') \equiv P'_0 \]  

(1.18)

Here \((ij)\) denotes the \(N \times N\) matrix with 1 at (row-\(i\), col.-\(j\)) and zero elsewhere.

These standard \(P_0\) will be carried over to our \(\hat{o}(2n + 1), \hat{p}(2n)\). We do not indicate the \(q\)-dependence explicitly (by denoting \(\hat{o}_q(2n + 1)\) for example) since for \(q = 1\) our constructions remain nontrivial. Our \(\hat{R}\) matrices are not \(q\)-deformations of a "classical" limit for a particular value of \(q\) such as 1. This is just one of the remarkable features to be studied below.

For \(N = 3\), the \(9 \times 9\) projector \(P_0\) is given by

\[
(q + 1 + q^{-1}) P_0 \equiv P'_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-1} & 0 & q^{-\frac{1}{2}} & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & q^{\frac{1}{2}} & 0 & q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

(1.19)

This is the case that will be studied here extensively as the simplest example. For all \(N\) a basic feature is the proportionality of the rows with nonzero elements. This has important consequences. It remains here to display briefly the "pre-Baxterized" situation. Our canonical forms ensure, among various aspects studied in Ref.1,

\[
\hat{R}(-\theta) = (\hat{R}(\theta))^{-1}, \quad \hat{R}(0) = I_{N^2}
\]  

(1.20)

The limits

\[
\theta \to \pm \infty, \quad \hat{R}(\theta) \to \hat{R}^{\pm 1}
\]

satisfy the (non-Baxterized, \(\theta\)-independent) braid equation

\[
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}
\]  

(1.21)

where one can substitute \(\hat{R}^{-1}\) for \(\hat{R}\).
For the standard cases

\[ (SO_q(N)) : \hat{R}^{\pm 1} = P_+ - q^{\mp 2}P_+ + q^{\mp N}P_0 \]  \hspace{1cm} (1.22)

\[ (Sp_q(N)) : \hat{R}^{\pm 1} = P_+ - q^{\mp 2}P_+ - q^{\mp(N+2)}P_0 \]  \hspace{1cm} (1.23)

satisfy cubic equations. For our cases, with \( \eta \) given by (1.9), one has for all \( N \) the quadratic equation

\[ (e^{\eta}\hat{R}) - (e^{\eta}\hat{R})^{-1} = (e^{\eta} - e^{-\eta})I \]  \hspace{1cm} (1.24)

or

\[ (\hat{R} - I)(\hat{R} + e^{-2\eta}I) = 0 \]  \hspace{1cm} (1.25)

From (1.10), as \( \theta \to \pm \infty \),

\[ \hat{R}^{\pm 1} = I - (1 + e^{\mp 2\eta})P_0 = I - e^{\mp\eta}(e^{\eta} + e^{-\eta})P_0 = I - e^{\mp\eta}P'_0 \]  \hspace{1cm} (1.26)

The last equation follows from (1.9), (1.17) and (1.18). Note that though \( P'_0 \) is not a projector in (1.26), \( \hat{R} \) is inverted by inverting the coefficient of \( P'_0 \) due to the relation

\[ P'^2 = (e^{\eta} + e^{-\eta})P'_0 \]

Other properties of \( \hat{R} \) will be introduced later as they become directly relevant.

2 What \( q \) for triangularity ? :

A braid matrix for vector representation is called "triangular" if

\[ \hat{R}^2 = I \]  \hspace{1cm} (2.1)

For the standard cases \( (A, B, C, D)_q \) this is obtained trivially for \( q = 1 \). This is well-known. But for comparison with our case let us briefly indicate how this happens for \( SO_q(N) \) and \( Sp_q(N) \).

For the projectors in (1.2) and (1.3) denote

\[ (P_i)_{q=1} = P_i \hspace{1cm} (i = +, -, 0) \]  \hspace{1cm} (2.2)

and let

\[ P = \sum_{i,j} (ij) \otimes (ji) \hspace{1cm} P^2 = I \]  \hspace{1cm} (2.3)

(Acting on the left \( P \) permutes specific rows and acting on the right the corresponding columns. This evident feature is mentioned since it plays a crucial role below (2.40).)
From (1.22), (1.23) substituting the known explicit forms [4, 5] of the projectors for \( q = 1 \), with upper and lower signs for the two cases respectively,

\[
\hat{R} = P_+ - P_- \pm P_0 = \pm (I - 2P_\mp) = P
\]

Hence

\[
\hat{R}^2 = P^2 = I, \quad R = P\hat{R} = I
\]

for both cases.

For \( GL_q(N) \) one obtains the same result even more simply.

For our class \( q = 1 \) gives a quite nontrivial situation, as emphasized already in Ref.3. Denoting for all \( N = (3, 4, ... \) \)

\[
(\eta)_{q=1} = \hat{\eta}
\]

from (1.9)

\[
e^\hat{\eta} + (-)^\hat{\eta} = N, \quad (1 + e^{\mp 2\hat{\eta}}) = \frac{2N}{N \pm (N^2 - 4)^{\frac{1}{2}}} \neq 2
\]

and from (1.26)

\[
(\hat{R}^\pm)_{q=1} = I - (1 + e^{\mp 2\hat{\eta}})P_0, \quad \hat{R}^2 \neq 1
\]

The generalized Hecke condition is now

\[
(e^\hat{\eta}\hat{R}) - (e^\hat{\eta}\hat{R})^{-1} = (e^\hat{\eta} - (-)^\hat{\eta})I
\]

This cannot be conjugated to

\[
(\hat{R} - I)(\hat{R} + I) = 0
\]

For (2.1) we need for our case

\[
\eta = 0
\]

when

\[
\hat{R} = I - 2P_0, \quad \hat{R}^2 = I + 4(P_0^2 - P_0) = I
\]

Hence from (1.9) for \( \hat{o}(N) \) and \( \hat{p}(N) \) respectively

\[
[N \mp 1] \pm 1 = e^\eta + (-)^\eta = 2
\]

or respectively,

\[
(A) : \quad q^{N-2} + q^{N-4} + ... + q^{-N+4} + q^{-N+2} = 1, \quad (N = 3, 4, ... \)
\]

\[
(B) : \quad q^{N} + q^{N-2} + ... + q^{-N+2} + q^{-N} = 3, \quad (N = 4, 6, ... \)
\]

The degrees of the polynomials can be lowered by changing variables as follows for the different cases. To start with (A) is divided into two subclasses.

\[
(A)_1 : N = 2n + 2 \quad (n = 1, 2, ... \)
Set
\[ p = q^2, \quad Y = p + p^{-1} \quad \text{(2.14)} \]
when
\[ q = \pm p^{\frac{1}{2}} = \pm \frac{1}{\sqrt{2}} \left( Y + \sqrt{Y^2 - 4} \right)^{\frac{1}{2}} \quad \text{(2.15)} \]
From (2.12),
\[ S_n = (p^n + p^{-n}) + (p^{n-1} + p^{-n+1}) + \ldots + (p^2 + p^{-2}) + (p + p^{-1}) = 0 \quad \text{(2.16)} \]
Note that for \((A)\_1\) the right side cancels with \(p^0 = 1\). Now to express \(S_n\) in terms of \(Y\) implementing\[ S_{n+1} = YS_n - S_{n-1} + Y - 2 \quad \text{(2.17)} \]
one obtains finally
\[ S_n = Y^n + Y^{n-1} - \binom{n-1}{1} Y^{n-2} - \binom{n-2}{1} Y^{n-3} + \binom{n-2}{2} Y^{n-4} + \binom{n-3}{2} Y^{n-5} + \ldots \]
\[ + (-1)^r \left( \binom{n-r}{r} Y^{n-2r} + \binom{n-r-1}{r} Y^{n-2r-1} \right) + \ldots + c_1 Y + c_0 \quad \text{(2.18)} \]
where
\[ c_1 = (-1)^{s-1} s, \quad (n = 2s, r = s - 1); \quad c_1 = (-1)^s (s + 1), \quad (n = 2s + 1, r = s) \]
\[ c_0 = -2 \quad (n = 2 + 4m, 3 + 4m; \quad m = 0, 1, 2, \ldots); \quad c_0 = 0 \quad (n \neq 2 + 4m, 3 + 4m) \quad \text{(2.19)} \]
Explicitly
\[ S_1 = Y \]
\[ S_2 = Y^2 + Y - 2 = (Y - 1)(Y + 2) \]
\[ S_3 = Y^3 + Y^2 - 2Y - 2 = (Y + 1)(Y^2 - 2) \]
\[ S_4 = Y^4 + Y^3 - 3Y^2 - 2Y \]
\[ S_5 = Y^5 + Y^4 - 4Y^3 - 3Y^2 + 3Y \]
\[ S_6 = Y^6 + Y^5 - 5Y^4 - 4Y^3 + 6Y^2 + 3Y - 2 \]
\[ S_7 = Y^7 + Y^6 - 6Y^5 - 5Y^4 + 10Y^3 + 6Y^2 - 4Y - 2 \]
\[ S_8 = Y^8 + Y^7 - 7Y^6 - 6Y^5 + 15Y^4 + 10Y^3 - 10Y^2 - 4Y \]
\[ S_9 = Y^9 + Y^8 - 8Y^7 - 7Y^6 + 21Y^5 + 15Y^4 - 20Y^3 - 10Y^2 + 5Y \quad \text{(2.20)} \]
and so on.
\[ (A)_2: N = 2m + 1 \quad (m = 1, 2, \ldots) \]
Set
\[ z = q + q^{-1}, \quad q^{\pm 1} = \frac{1}{2}(z \pm \sqrt{z^2 - 4}) \] (2.21)

Retaining only the odd powers in (2.18), (2.20) and adapting notations one obtains
\[ \Sigma_{2m-1} = (q^{2m-1} + q^{-2m+1}) + (q^{2m-3} + q^{-2m+3}) + \ldots + (q + q^{-1}) \]
\[ = z^{2m-1} - \binom{2m-2}{1} z^{2m-3} + \binom{2m-3}{2} z^{2m-5} + \ldots \]
\[ + (-1)^r \binom{2m-r-1}{r} z^{2m-2r-1} + \ldots + (-1)^{m-1} m z \] (2.22)

Explicitly (noting that in contrast with (2.16) there is now 1 on the right side below)
\[ \Sigma_{N-2} = 1 \quad (N = 3, 5, \ldots) \] (2.23)

where
\[ \begin{align*}
\Sigma_1 &= z \\
\Sigma_3 &= z^3 - 2z \\
\Sigma_5 &= z^5 - 4z^3 + 3z \\
\Sigma_7 &= z^7 - 6z^5 + 10z^3 - 4z \\
\Sigma_9 &= z^9 - 8z^7 + 21z^5 - 20z^3 + 5z
\end{align*} \] (2.24)

and so on.

Comparing (2.12) and (2.13) one sees that (B) is obtained from (A)_1, with now a nonzero right side, as
\[ (B) : \quad S_n = 2 \quad (n = \frac{N}{2} = 2, 3, \ldots) \] (2.25)

Without trying to be exhaustive let us first point out some simple possibilities, particularly when (2.1) is obtained for \( q \) a root of unity. (See the relevant remarks in the concluding section.)

For \( \hat{\sigma}(3) \) one has from (2.23)
\[ \Sigma_1 = q + q^{-1} = 1, \quad q = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}, \quad q^6 = 1 \] (2.26)

For \( \hat{\sigma}(4) \) one has from (2.16) and (2.20)
\[ Y = q^2 + q^{-2} = 0, \quad q = e^{i\frac{\pi}{4}}, \quad q^8 = 1 \] (2.27)

Moreover, whenever \( c_0 = 0 \) from (2.19) i.e. for
\[ \hat{\sigma}(4), \hat{\sigma}(10), \hat{\sigma}(12), \ldots \] (2.28)
$Y$ can be factorized and (2.27) is a solution.

For $\hat{o}(6)$ the two roots of

$$S_2 = (Y - 1)(Y + 2) = 0$$

correspond to

$$q^{12} = 1, \quad q^4 = 1$$  \hspace{1cm} (2.29)

For $\hat{o}(8)$ the roots of $S_3$ give

$$q^6 = 1, \quad q^{16} = 1$$  \hspace{1cm} (2.30)

For $\hat{o}(10)$ and $\hat{o}(12)$ (apart from (2.27)) one has to solve cubic and quartic equations respectively. We do not present this standard algebra here. For odd $N$ one has again a cubic in $z$ for $\hat{o}(5)$.

For $\hat{o}(14)$ onwards for $\hat{o}(2n)$ one has polynomials of sixth and higher degrees (already for $Y$ before obtaining $q$ from (2.15)). Hence one needs hyperelliptic functions for $Y$.

For $\hat{o}(7)$ and $\hat{p}(10)$ one has quintics and elliptic solutions respectively for $z = (q + q^{-1})$ and $Y = (q^2 + q^{-2})$. For higher dimensions one again encounters hyperelliptic functions here.

It is known [6, 7] that the general case on a complex field

$$f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n = 0$$  \hspace{1cm} (2.31)

can be solved in terms of theta functions of zero arguments and the period matrix of the hyperelliptic curves

$$F^2 = x(x-1)f(x), \quad F^2 = x(x-1)(x-2)f(x)$$  \hspace{1cm} (2.32)

for odd $n$ and even $n$ respectively.

For quintics [7] one can, alternatively, implement further successive changes of variables (Tschirnhausen transformations) to obtain standard forms (the Bring-Jerrard quintic or the Brioschi quintic leading to the Jacobi sextic) which can be solved directly using elliptic functions. All this is however very complicated.

The coefficients $a_i$ of (2.31) are very special ones (binomial integers) for our case. What special (hopefully simplifying) features might they induce in the corresponding elliptic and hyperelliptic functions? An answer to this question is beyond the scope of this paper.

Let us contemplate the simplest case, that of $\hat{o}(3)$ with (2.26) giving

$$\hat{R} = I - \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-1} & 0 & q^{-\frac{1}{2}} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-\frac{1}{2}} & 0 & 1 & 0 & q^{\frac{1}{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & q^{\frac{1}{2}} & 0 & q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$  \hspace{1cm} (2.33)
where \( q = e^{i\pi/3} \).
Remarkably this satisfies

\[ \hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \]
along with

\[ \hat{R}^2 = I \]

2.1 A non-existence theorem:

For the standard cases (see (2.3), ((2.4), (2.5))

\[ (\hat{R})_{q=1} = P, \quad \hat{R}^2 = P^2 = I \]

For the nonstandard Jordanian case (see Refs. 8, 9 citing basic sources) considering again vector representations

\[ \hat{R} = F^{-1}PF, \quad \hat{R}^2 = I \]

(2.34)

where \( F \) is obtained through a ”contraction” [8, 9]. Thus the Yang-Baxter matrix

\[ R = P\hat{R} = (PF^{-1}P)F = F^{-1}_{21}F \]

(2.35)

is a ”Drinfeld twist” of unity. This leads to various interesting features [8, 9] making triangularity inherent without rendering \( \hat{R} \) trivial. Can our constructions above (for \( \eta = 0 \)) be expressed as a conjugation of \( P \) as in (2.34)? A priori such a possibility cannot be discarded.

However, using our diagonalizers (see App. B of Ref. 1 for explicit constructions) one can prove quite simply and generally that no invertible \( F \) exists that can realize (2.34).

It is sufficient to to consider \( \hat{o}(3) \). Higher dimensions can be treated in a strictly parallel fashion. The essential result for us is that the diagonalizer \( M \) gives for (1.26)

\[ M\hat{R}^{\pm 1}M^{-1} = (-e^{2\eta}, 1, 1, 1, 1, 1, 1, 1, 1)_{(\text{diag})} \]

(2.36)

For \( \eta = 0 \), when \( \hat{R}^2 = I \), one thus obtains

\[ M\hat{R}^{\pm 1}M^{-1} = (-1, 1, 1, 1, 1, 1, 1, 1, 1)_{(\text{diag})} \equiv D \]

(2.37)

Now assume that an \( F \) exists for our \( \hat{R} \) satisfying (2.34). Then

\[ MF^{-1}PFM^{-1} = D \]

(2.38)

Defining

\[ G = FM^{-1} \]

(2.39)
since $D^2 = I$

\[ G = PGD \quad (2.40) \]

The action of $P$ here for the $9 \times 9$ case (see (2.3)) leaves the rows $(1, 5, 9)$ untouched and interchanges the pairs of rows

$$(2, 4), (3, 7), (6, 8)$$

Now consider the action of $D$ after parametrizing $G$ in terms of parameters arbitrary to start with. In rows $(1, 5, 9)$ the first element is constrained to be zero the others being unrestricted. If row-2 is parametrized as

\[ (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) \quad (2.41) \]

then row-4 must be

\[ (-a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) \quad (2.42) \]

Hence with arbitrary $a_1$, $r_i$ denoting the row-$i$,$$

(r_2 - r_4) = (2a_1, 0, 0, 0, 0, 0, 0, 0, 0) \quad (2.43)$$

Similarly, in evident notations,

\[ (r_3 - r_5) = (2b_1, 0, 0, 0, 0, 0, 0, 0, 0) \quad (2.44) \]

\[ (r_6 - r_8) = (2c_1, 0, 0, 0, 0, 0, 0, 0, 0) \quad (2.45) \]

This evidently implies that the determinant

\[ \Delta G = 0 \quad (2.46) \]

Hence $G$ is not invertible. Hence neither is $F = GM$.
This contradicts the assumption that an invertible $F$ exists giving (2.34) for our $\hat{R}$.

### 3 L-algebra (group-like central elements):

Before writing down the $RLL$-equations and the implied constraints we signal the most remarkable features to emerge in Secs.3 and 4. We will study them mostly in the context of the simplest case $\hat{o}(3)$ i.e. $N = 3$.

1: In the $L^+$ subalgebra one obtains $2N$ central, group-like elements constrained to equality by the $RLL$-equations. There are $2N$ corresponding ones for the $L^-$ subalgebra.

2: In standard cases group-like elements are usually associated to ”quantum determinants”. But our above-mentioned sets have no determinant-like structure at all. Each one is the sum of $N$ quadratic terms (no negative signs).
(3): In the $3 \times 3$ fundamental representation of the $L$-operators for $\hat{o}(3)$ these elements are proportional to $I_3$. Consistently with their group-like property and centrality they are proportional to $I_9$ for the $9 \times 9$ coproduct representations. The explicit verification of this involves remarkable cancellations. Iterated coproducts of course lead to $I_{3^p}$ at the $p$-th stage.

(4): In the standard cases we are used to the coproducts being reducible. Thus the $9 \times 9$ coproducts $\Delta L^\pm_{ij}$ for $SO_q(3)$ can be conjugated to block-diagonal forms corresponding to the familiar irreducible components ($9 \times 9 \rightarrow 5 \times 5 \oplus 3 \times 3 \oplus 1 \times 1$) or in terms of angular momenta ($1 \times 1 \rightarrow 2 \oplus 1 \oplus 0$). But here one encounters obstructions in a systematic search for block-diagonalizations. This is of course consistent with the central elements announced above being proportional to $I$. But since such a search reveals special features of the generators this aspect of the $9 \times 9$ coproduct mentioned above will be treated explicitly in the next section.

Let us now formulate the $RLL$-constraints. The $FRT$-equations [4] for the $L^\pm$ operators are, in our notations,

$$\hat{R}L^\pm L^\pm_1 = L^\pm_2 L^\pm_1 \hat{R}$$

(3.1)

$$\hat{R}L^\pm_2 L^\pm_1 = L^\pm_2 L^\pm_1 \hat{R}, \quad (\hat{R}^{-1} L^\pm_2 L^\pm_1 = L^\pm_2 L^\pm_1 \hat{R}^{-1})$$

(3.2)

Here $\hat{R}$ is a $N^2 \times N^2$ matrix satisfying (1.21) and $L^\pm$ have each $N^2$ components $L^\pm_{ij}$, $(i, j = 1, 2, ..., N)$ arranged in a $N \times N$ matrix form with

$$L^\pm_2 = I_N \otimes L^\pm, \quad L^\pm_1 = L^\pm \otimes I_N$$

(3.3)

From (1.26), with $P'_0$ given by (1.17),(1.18),

$$\hat{R}^1 = I + \lambda_\pm P'_0, \quad \lambda_\pm = -e^{\mp \eta}$$

(3.4)

Here, from (1.9), $\lambda_\pm$ are the roots of

$$\lambda + \lambda^{-1} + ([N \mp 1] \pm 1) = 0$$

(3.5)

and, in particular, for $\hat{o}(3)$ of

$$\lambda + \lambda^{-1} + (q + 1 + q^{-1}) = 0$$

(3.6)

This simple change of notation ($\eta \rightarrow \lambda$) will permit below a compact, unified treatment of $L^+$ and $L^-$ due to the symmetry

$$(\lambda_+, L^+, L^-) \Rightarrow (\lambda_-, L^-, L^+)$$

(3.7)

This is a special feature of our class. We will often suppress below the superscripts and subscripts of $L$ and $\lambda$ respectively when dealing with (3.1) and write $L_{ij}$ and $\lambda$. 

12
From (3.1) and (3.4) one has

\[ P'_0 L'_2 L'_1 = L'_2 L'_1 P'_0, \quad (\epsilon = \pm) \]  

(3.8)

In these equations \( \lambda_{\pm} \) do not appear explicitly. They do however appear for (3.2).

### 3.1 \( L^+ \) and \( L^- \) subalgebras:

The group-like elements belong to these subalgebras since the coproducts are defined separately for each. The RLL constraints lead to \((2N^2 - 1)\) equations for each subalgebra which separate into three subsets of \(N(N - 1), N(N - 1), (2N - 1)\) respectively. The total number is easily understood as follows.

The diagonalizer \( M \) (see App.B, Ref.1) gives

\[ MP'_0 M^{-1} \approx (1, 0, 0, \ldots, 0)_{(\text{diag})} \]  

(3.9)

Conjugating the factors on both sides of (3.8) by \( M \) only the row-1 survives on the left and only the col.-1 on the right. These have each \(N^2\) elements and one element in common. Hence the result.

It is convenient to start by deriving some results in a form valid, more generally, for the whole algebra as follows. We consider \( \hat{o}(N) \) for definiteness, the modifications for \( \hat{p}(N) \) are evident.

Along with (1.17) and (3.3), for \( N \geq 3 \), we write

\[ L_2 L'_1 = \sum_{i,j,k,l} L_{ik} L'_{ij} (ij) \otimes (kl) \]  

(3.10)

where

\[ L_2 L'_1 = L'_2 L'_1, \quad (\epsilon, \epsilon') = (+, +), (-, -), (+, -), (-, +) \]  

One can show that (with \( i' = N - i + 1 \))

\[ P'_0 L'_2 L'_1 = \sum_{i',k,l} q^{-\rho_j} S^{(1)}_{ik} (ik) \otimes (i'l) \]  

(3.11)

\[ L'_2 L'_1 P'_0 = \sum_{i',k,l} q^{-\rho_j} S^{(2)}_{ik} (ki) \otimes (li') \]  

(3.12)

where

\[ S^{(1)}_{ik} = \sum_j q^{-\rho_j} L'_{jk} L'_j \]  

(3.13)

\[ S^{(2)}_{ik} = \sum_j q^{-\rho_j} L'_{ij} L'_{kj} \]  

(3.14)
We now go back to our subalgebras by setting \( \epsilon = \epsilon' \) i.e. \( L' = L \).

For \( l \neq k' \) \( (l + k \neq N + 1) \) one obtains from (3.8), due to the structure of \( P_0 \),

\[
S_{lk}^{(1)} = \sum_j q^{-\rho_j} L_{j'l} L_{jk} = 0 \tag{3.15}
\]

and

\[
S_{lk}^{(2)} = \sum_i q^{-\rho_i} L_{i'k} L_{ki} = 0 \tag{3.16}
\]

Each set \((S^{(1)}, S^{(2)})\) corresponds to \(N(N-1)\) equations and

\[
S^{(1)} \rightarrow S^{(2)} \Rightarrow L_{ij} \rightarrow L_{ji} \tag{3.17}
\]

For \( l + k = N + 1 \)

one obtains \((i \text{ and } j \text{ assuming each value independently with } i + i' = j + j' = N + 1)\)

\[
q^{-\rho_i} S_{ii'}^{(1)} = q^{-\rho_j} S_{jj'}^{(2)} \quad (i = 1, \ldots N; j = 1, \ldots N) \tag{3.18}
\]

The equality of these \(2N\) quadratic expressions give \((2N-1)\) equations. We will denote this set as \(\hat{S}_3\). This provides the group-like central elements. We will study this set in detail for \(\hat{o}(3)\). For \(N = 3\) one obtains from (3.18)

\[
q^{-\frac{1}{2}} S_{13}^{(1)} = S_{22}^{(1)} = q^{\frac{1}{2}} S_{31}^{(1)} = q^{-\frac{1}{2}} S_{13}^{(2)} = S_{22}^{(2)} = q^{\frac{1}{2}} S_{31}^{(2)} \tag{3.19}
\]

Or explicitly \((\text{with } L \text{ all } L^+ \text{ or all } L^- \text{ below})\)

\[
(\hat{S}_3) : \quad L_{11} L_{33} + q^{-\frac{1}{2}} L_{21} L_{23} + q^{-1} L_{31} L_{13} = q^{\frac{1}{2}} L_{12} L_{32} + L_{22} L_{22} + q^{-\frac{1}{2}} L_{32} L_{12} = q L_{13} L_{31} + q^{\frac{1}{2}} L_{23} L_{21} + L_{33} L_{11} = L_{11} L_{33} + q^{-\frac{1}{2}} L_{12} L_{32} + q^{-1} L_{13} L_{31} = q^{\frac{1}{2}} L_{21} L_{23} + L_{22} L_{22} + q^{-\frac{1}{2}} L_{23} L_{21} = q L_{31} L_{13} + q^{\frac{1}{2}} L_{32} L_{12} + L_{33} L_{11} \tag{3.20}
\]

We denote the sets (3.15) and (3.16) by \(\hat{S}_1\) and \(\hat{S}_2\) respectively. For \(N = 3\) the first set is

\[
\hat{S}_1 : \quad q^{-\frac{1}{2}} L_{31} L_{11} + L_{21} L_{21} + q^{\frac{1}{2}} L_{11} L_{31} = 0
\]
\[ q^{-\frac{1}{2}}L_{32}L_{11} + L_{22}L_{21} + q^{\frac{1}{2}}L_{12}L_{31} = 0 \]

\[ q^{-\frac{1}{2}}L_{31}L_{12} + L_{21}L_{22} + q^{\frac{1}{2}}L_{11}L_{32} = 0 \]

\[ q^{-\frac{1}{2}}L_{33}L_{12} + L_{23}L_{22} + q^{\frac{1}{2}}L_{13}L_{32} = 0 \]

\[ q^{-\frac{1}{2}}L_{32}L_{13} + L_{22}L_{23} + q^{\frac{1}{2}}L_{12}L_{33} = 0 \]

\[ q^{-\frac{1}{2}}L_{33}L_{13} + L_{23}L_{23} + q^{\frac{1}{2}}L_{13}L_{33} = 0 \] (3.21)

The set \( \hat{S}_2 \) is obtained immediately from \( \hat{S}_1 \) via (3.17).

We now show how the sets \((\hat{S}_1, \hat{S}_2, \hat{S}_3)\) imply the two basic properties of the members of \( \hat{S}_3 \), that they are (1) central and (2) group-like.

(1): Exploiting systematically the sets \( \hat{S}_i \) one can pass through different chains of intermediate steps. One possible sequence is as follows.

\[ q^{\frac{1}{2}}L_{11}L_{23}L_{21} = q^{-\frac{1}{2}}L_{12}L_{32}L_{11} - q^{-\frac{1}{2}}L_{13}L_{21}L_{21} + q^{\frac{1}{2}}L_{12}L_{12}L_{31} \] (3.22)

\[ qL_{11}L_{13}L_{31} = q^{-1}L_{13}L_{31}L_{11} + q^{-\frac{1}{2}}L_{13}L_{21}L_{21} - q^{\frac{1}{2}}L_{12}L_{12}L_{31} \] (3.23)

Summing these with \( L_{11}L_{33}L_{11} \) one obtains (using again (3.20) in the last step)

\[ L_{11}(qL_{13}L_{31} + q^{\frac{1}{2}}L_{23}L_{21} + L_{33}L_{11}) = (L_{11}L_{33} + q^{-\frac{1}{2}}L_{12}L_{32} + q^{-1}L_{13}L_{31})L_{11} \]

\[ = (qL_{13}L_{31} + q^{\frac{1}{2}}L_{23}L_{21} + L_{33}L_{11})L_{11} \] (3.24)

Thus \( L_{11} \), and similarly each \( L_{ij} \) can be shown to commute with the members of \( \hat{S}_3 \). Hence the latter are central.

(2): The rule for coproducts [4] (with \( L \) either \( L^+ \) or \( L^- \) throughout) is

\[ \Delta L_{ij} = \sum_k L_{ik} \otimes L_{kj} \] (3.25)

Hence

\[ \Delta L_{ab} \Delta L_{cd} = \sum_{i,j} L_{ai}L_{cj} \otimes L_{ib}L_{jd} \] (3.26)

Now let us start with the first member of \( \hat{S}_3 \) and compute the sum

\[ \Sigma = \Delta L_{11} \Delta L_{33} + q^{-\frac{1}{2}}\Delta L_{21} \Delta L_{23} + q^{-1} \Delta L_{31} \Delta L_{13} \] (3.27)
Collecting terms and systematically implementing $\hat{S}_1, \hat{S}_2$ many terms cancel leaving

$$\Sigma = (L_{11}L_{33} + q^{-\frac{1}{2}}L_{21}L_{23} + q^{-1}L_{31}L_{13}) \otimes (L_{11}L_{33} + q^{-\frac{1}{2}}L_{21}L_{23} + q^{-1}L_{31}L_{13})$$  (3.28)

Thus this and similarly the other members of $\hat{S}_3$ are group-like.

For $\hat{o}(4)$ the 8 members of $\hat{S}_3$ satisfy

$$q^{-1}S_{14}^{(1)} = S_{23}^{(1)} = S_{32}^{(1)} = qS_{41}^{(1)}$$
$$q^{-1}S_{14}^{(2)} = S_{23}^{(2)} = S_{32}^{(2)} = qS_{41}^{(2)}$$  (3.29)

Their centrality and group-like property can be established in strict analogy to the $\hat{o}(3)$ case. Moreover they indicate how the generalization for higher $N$ can be carried out. Our presentation here will be limited to $\hat{o}(3)$.

### 3.2 Beyond the $L^\pm$ subalgebras:

We now consider the ”mixed” case (3.2). The major new feature now is the explicit involvement of $\lambda_\pm$ (see (3.4),(3.5)) in the constraints

$$(I + \lambda_+ P_0^r) L_2^±L_1^- = L_2^-L_1^+(I + \lambda_+ P_0^r)$$  (3.30)

$$(I + \lambda_- P_0^r) L_2^-L_1^+ = L_2^+L_1^-(I + \lambda_- P_0^r)$$  (3.31)

But even when $P_0^r$ does not contribute, namely at

$$(ij) \otimes (kl), \quad (kl) \neq (i'j')$$

one obtains simple but probing constraints. Retaining only such rows and columns one obtains a ”reduced” matrix of $N(N-1) \times N(N-1)$ dimensions. For $N = 3$ this corresponds to the suppression of rows and columns (3, 5, 7) leaving a $6 \times 6$ matrix. Using for this reduced case, for all $N$, the subscript $r$ one extracts from (3.30),(3.31)

$$(L_2^\epsilon L_1^\epsilon)_{(r)} = (L_2^\epsilon L_1^\epsilon)_{(r)}$$  (3.32)

For $\epsilon = \epsilon'$ this is trivial. But not now and one can go further as follows.

Since $\lambda$ satisfies a quadratic equation one can linearize all polynomials in $\lambda$ using $\lambda_+$ and $\lambda_- = (\lambda_+)^{-1}$. The symmetry of (3.30),(3.31),

$$(\lambda_+ \rightarrow \lambda_-) \Rightarrow (L^+ \rightarrow L^-)$$  (3.33)
indicates the parametrization where the \( \lambda \)-dependence is explicitly (and only) in the coefficient as

\[
L_{ij} = A_{ij} + \lambda B_{ij}
\]

Now injecting (3.34) in (3.32) one obtains for each element of the reduced matrix \( (L_{ab}^+, L_{cd}^-, \text{and so on}) \)

\[
(\lambda_+ - \lambda_-(A_{ab}B_{cd} - B_{ab}A_{cd}) = 0, \quad (\lambda_+ \neq \lambda_-) \tag{3.35}
\]

Thus the \( \lambda \)-dependence is simply factored out for this reduced matrix. Even when \( L^+ \) and \( L^- \) are considered in the context of the respective subalgebras they must satisfy (3.35). This will indeed be found to be the case in the explicit realizations of the following section.

We now come to parts where both \( I \) and \( P'_0 \) contribute and hence \( \lambda \) is directly involved. Instead of \( \hat{S}_3 \) of (3.20) one now has for \( \hat{o}(3) \) 9 relations of the type

\[
\lambda_+ \left( (qL_{31}^I L_{13} + q^{L_{32}^J} L_{12} + L_{33}^I L_{11}) - (qL_{31}^J L_{13} + q^{L_{32}^I} L_{12} + L_{33}^J L_{11}) \right) = q(L_{33} L_{11}' - L_{33}' L_{11})
\]

Instead of \( \hat{S}_1, \hat{S}_2 \) one now has equations of the type

\[
\lambda_+(q^{L_{31}^I L_{11}'} + L_{21}' L_{21} + q^{L_{11}^I L_{31}'})
\]

\[
= q^{L_{31}^J L_{11}' - L_{33} L_{11}'} = (L_{21}' L_{21} - L_{21} L_{21}') = q^{L_{11}' L_{31} - L_{11} L_{31}'}
\] (3.37)

For \( \epsilon = \epsilon' \) one recovers the results for the subalgebras. We have obtained a systematic formulation of the full set of 81 constraints for \( \hat{o}(3) \) exploiting certain symmetries. This will not be reproduced here. The generalizations of the results of this subsection for \( N > 3 \) can be obtained fairly systematically.

### 3.3 From \( L^\pm \) to \( L(\theta) \):

Since our \( \hat{R} \) satisfies a quadratic equation (1.24) all the three FRT equations ((3.1),(3.2)) can be condensed into a single one by defining in analogy to

\[
\hat{R}(\theta) = \frac{e^{\eta+\theta} \hat{R} - e^{-\eta-\theta} \hat{R}^{-1}}{e^{\eta+\theta} - e^{-\eta-\theta}} \tag{3.38}
\]

\[
L(\theta) = \frac{e^{\eta+\theta} L^+ - e^{-\eta-\theta} L^-}{e^{\eta+\theta} - e^{-\eta-\theta}} \tag{3.39}
\]

It can be shown that [10,11]

\[
\hat{R}(\theta - \theta') L_2(\theta)L_1(\theta') = L_2(\theta')L_1(\theta)\hat{R}(\theta - \theta') \tag{3.40}
\]
contains effectively all the three FRT equations. One can write (3.38) i.e. (1.10) as

\[ \hat{R}(\theta) = I - \frac{\sinh \theta}{\sinh(\eta + \theta)} P'_0 \] (3.41)

4 Fundamental and coproduct representations:

Here we will study the \( \hat{o}(3) \) fundamental \((3 \times 3)\) and the coproduct \((9 \times 9)\) representations. They illustrate the significance of the remarks (3, 4) at the beginning of Sec.3. This will be commented upon at the end. Specific symmetries of the matrices obtained will be displayed. They might be helpful in a more systematic study of representations.

The general prescription for the fundamental representations \((N \times N\) blocks for all \(N\)) is as follows. (See App.B of Ref.11 for a systematic presentation citing basic sources.)

\[ L(\theta) = \hat{R}(\theta) P, \quad (L(\theta))_{q \to \pm \infty} = L^\pm = \hat{R}^\pm P \] (4.1)

A Hopf algebra can be defined \([4, 10]\) for \(L\) using

\[ (L^\pm)^{-1} = PL^\mp P = (L^\mp)_{21} \] (4.2)

From (3.4, 3.6) and (4.1) one obtains \((\text{for the fund. repr.})\)

\[
L = \begin{pmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{pmatrix}
\]

\[
L_{11} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \lambda
\end{pmatrix}, \quad L_{12} = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & q^{1/2} & 0
\end{pmatrix}, \quad L_{13} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
(1 + q^{-1} \lambda) & 0 & 0
\end{pmatrix}
\]

\[
L_{21} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & q^{1/2} \lambda \\
0 & 0 & 0
\end{pmatrix}, \quad L_{22} = \begin{pmatrix}
0 & 0 & 0 \\
0 & (1 + \lambda) & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad L_{23} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
q^{-1/2} \lambda & 0 & 0
\end{pmatrix}
\]

\[
L_{31} = \begin{pmatrix}
0 & 0 & (1 + q \lambda) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad L_{32} = \begin{pmatrix}
0 & q^{1/2} & \lambda \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad L_{33} = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (4.3)

One sets \(\lambda = (\lambda_+, \lambda_-, \lambda(\theta))\) for \(L = (L^+, L^-, L(\theta))\) respectively. One obtains \(\lambda(\theta)\) directly from (3.39).

Note that (3.34) is evidently satisfied with

\[
L_{11}^\pm = \begin{pmatrix}
1 & 0 & 0 + \lambda_+ \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \equiv A_{11} + \lambda_+ B_{11}
\]
and so on.

Now we consider the $9 \times 9$ coproducts of (4.3) given by

$$\Delta L_{ij}^\pm = \sum_k L_{ik}^\pm \otimes L_{kj}^\pm$$

We will not present the easily obtained $9 \times 9$ matrices but the symmetries they exhibit for the reason mentioned before. Define

$$(r_1, r_2) : \text{reflections about the diagonal and the antidiagonal respectively}$$

$$f : (q \to q^{-1})(r_2 r_1), \quad f(A) \equiv fA$$

Then in terms of $3 \times 3$ blocks $A_{ij}, ..., E_3$ (not exhibited here) one obtains with $(ij = 11, 12, 13)$ and $(i'j' = 33, 32, 31)$ respectively (and noting that $\lambda$ is invariant for $q \to q^{-1}$)

$$\Delta L_{ij} = \begin{vmatrix} A_{ij} & 0 & 0 \\ B_{ij} & 0 & 0 \\ C_{ij} & q^{-\frac{1}{2}}\lambda B_{ij} & \lambda A_{ij} \end{vmatrix}, \quad f \Delta L_{ij} = \Delta L_{i'j'} = \begin{vmatrix} \lambda f A_{ij} & q^{\frac{1}{2}}\lambda f B_{ij} & f C_{ij} \\ 0 & 0 & f B_{ij} \\ 0 & 0 & f A_{ij} \end{vmatrix} \quad (4.4)$$

$$\Delta L_{21} = \begin{vmatrix} 0 & D_1 & 0 \\ q^{-\frac{1}{2}}\lambda D_2 & 0 & q^{\frac{1}{2}}\lambda D_1 \\ 0 & D_2 & 0 \end{vmatrix}, \quad f \Delta L_{21} = \Delta L_{23} = \begin{vmatrix} 0 & f D_2 & 0 \\ q^{-\frac{1}{2}}\lambda f D_1 & 0 & f D_3 \\ 0 & f D_1 & 0 \end{vmatrix} \quad (4.5)$$

$$\Delta L_{22} = f \Delta L_{22} = \begin{vmatrix} 0 & E_1 & 0 \\ q^{-\frac{1}{2}}\lambda f E_1 & 0 & q^{\frac{1}{2}}\lambda E_1 \\ 0 & f E_1 & 0 \end{vmatrix} \quad (4.6)$$

Having displayed the symmetries we now study in more detail the three generators $\Delta L_{ii}$. For the standard cases the $L_{ii}$ can be obtained directly in diagonal forms for irreducible representations [4] and through appropriate conjugations for reducible ones. For our $3 \times 3$ representation also they are diagonal with their sum proportional to $I$. But for the $9 \times 9$ coproducts above they are not diagonal. They do not commute mutually and hence cannot be diagonalized simultaneously. To better understand the structure encountered let us try to diagonalize the sum $(\sum_i \Delta L_{ii})$. Define

$$\mu = (q^\frac{1}{2} - q^{-\frac{1}{2}}), \quad z = (q^\frac{1}{2} + q^{-\frac{1}{2}}), \quad k = (q + 4 + q^{-1})^{-\frac{1}{2}} \quad (4.7)$$
Set

\[
\sqrt{2} N = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & zk & 0 & 2k & 0 & zk & 0 & 0 \\
0 & 0 & \sqrt{2} k & 0 & -\sqrt{2} zk & 0 & \sqrt{2} k & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (4.8)

This is orthogonal i.e.

\[
N^{-1} = N^T
\]

One obtains

\[
N(\Delta L_{11} + \Delta L_{22} + \Delta L_{33})N^{-1} = \lambda(-y, y, y, -y, 3, 3, -y, -y, -y)_{(\text{diag})}, \quad y = (q + 1 + q^{-1})
\] (4.9)

The eigenvalues can be permuted through evident supplementary conjugations. We have thus diagonalized the sum. Now let us look at the component terms. One has (denoting by \((bd)\) a block diagonal structure)

\[
N(\Delta L_{ii})N^{-1} = (\alpha_i, \beta_i, \gamma_i, \delta_i)_{(bd)} \quad (i = 1, 2, 3)
\] (4.10)

where

\[
\alpha_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad \alpha_2 = \frac{1}{2} \begin{bmatrix} \lambda^2 + 1 & \lambda^2 - 1 \\ -\lambda^2 + 1 & -\lambda^2 - 1 \end{bmatrix}
\] (4.11)

\[
\beta_1 = \frac{\lambda^2}{2} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \beta_3 = \frac{1}{2} \begin{bmatrix} \lambda^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \beta_2 = \frac{1}{2} \begin{bmatrix} -\lambda^2 - 1 & \lambda^2 - 1 \\ -\lambda^2 + 1 & \lambda^2 + 1 \end{bmatrix}
\] (4.12)

\[
\delta_1 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^2 \end{bmatrix}, \quad \delta_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \delta_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\] (4.13)

\[
\gamma_1 = \frac{\lambda}{2} \begin{bmatrix} 3 & \mu k & \sqrt{2}(2 + q^{-1})k \\ \mu k & 3z^2 k^2 & \sqrt{2}(2 + q^{-1})\mu k^2 \\ -\sqrt{2}(2 + q)k & -\sqrt{2}(2 + q)\mu k^2 & -2(z^2 - 1)k^2 \end{bmatrix}
\] (4.14)

\[
\gamma_3 = \frac{\lambda}{2} \begin{bmatrix} 3 & \mu k & -\sqrt{2}(2 + q)k \\ \mu k & 3(1 - 2k^2) & -\sqrt{2}(2 + q)\mu k^2 \\ \sqrt{2}(2 + q^{-1})k & \sqrt{2}(2 + q^{-1})\mu k^2 & -2(1 - 3k^2) \end{bmatrix}
\] (4.15)
\[ \gamma_2 = \frac{\lambda}{2} \begin{pmatrix} 0 & -2\mu k & \sqrt{2}\mu z k \\ -2\mu k & 12k^2 & \sqrt{2}\mu^2 z k^2 \\ \sqrt{2}\mu z k & \sqrt{2}\mu^2 z k^2 & -2(z^2 - 1)z^2 k^2 \end{pmatrix} \]  

(4.16)

Note that

\[ \alpha_1^2 = \alpha_3^2 = \beta_1^2 = \beta_3^2 = 0 \]  

(4.17)

Such nilpotent matrices are **nondiagonalizable**.

It has been explicitly verified that not only \( N(\Delta L_{ij})N^{-1} \) \((i \neq j)\) are not correspondingly block diagonalized but all their nonzero elements lie systematically outside the blocks arising for \( N(\Delta L_{ii})N^{-1} \). One can examine larger blocks, say, \((6 \otimes 6 \oplus 3 \otimes 3)\) after permuting the \( \gamma_i \) and \( \delta_i \) blocks. But one finds that the whole \( 9 \times 9 \) space is needed for \( N(\Delta L_{ii})N^{-1} \). These results will not be displayed here though particularly for \( q = 1 \) they acquire relatively simple forms. One can implement further conjugations and permutations but the essential features persist.

For the \( 3 \times 3 \) representations all the members of \( \hat{S}_3 \) in (3.20) say, for example,

\[ L_{11}L_{33} + q^{-\frac{1}{2}}L_{21}L_{23} + q^{-1}L_{31}L_{13} = \lambda I_3 \]  

(4.18)

Also

\[ L_{11} + L_{22} + L_{33} = (1 + \lambda)I_3 \]  

(4.19)

The members of \( \hat{S}_3 \) being group-like (4.18) gives for the \( 9 \times 9 \) coproducts \( \lambda^2 I_9 \). This has been verified explicitly. But (\( \sum \Delta L_{ii} \)) behaves quite differently as shown above.

The obstructions encountered in reduction of the \( 9 \times 9 \) coproducts to smaller dimensional irreducible components (via block diagonalization in a fashion analogous, say, to the case of \( SO_q(3) \)) is consistent with the central \( \hat{S}_3 \) operators being proportional to \( I \). But our study of \( \Delta L_{ii} \) reveals specific properties of these generators for higher dimensional representations (such as symmetries and nondiagonalizable blocks). This can be helpful in a more systematic study of representations. The symmetries displayed in (4.4, 4.5, 4.6) stem from those of \( P_0 \) and hence should be significant more generally.

## 5 Link invariants (Turaev construction):

### 5.1 Construction of ”enhanced” operators:

Given a matrix satisfying the braid equation the Turaev construction[12] of an enhanced Yang Baxter operator (\( EYB \)) leads to explicit construction of invariants (invariant under Markov moves of first and second types) for oriented links. Such an enhanced system [12, 13]
consists of a $N^2 \times N^2$ braid matrix $\hat{R}$, an $N \times N$ matrix $f$ and elements $(a, b)$, all invertible, satisfying the relations

$$\hat{R} \pm 1 f \otimes f = f \otimes f \hat{R} \pm 1$$  \hspace{1cm} (5.1)$$

$$tr_2(\hat{R} \pm 1 f \otimes f) = a \pm 1 bf$$  \hspace{1cm} (5.2)$$

where one defines

$$tr_2(\sum_{ijkl} c_{ij,kl}(ij) \otimes (kl)) = \sum_{ij} \left( \sum_k c_{ij,kk} \right)(ij)$$  \hspace{1cm} (5.3)$$

Let us first obtain $(f, a, b)$ for our class of $\hat{R}$. Our spectral resolution and the properties of the projector $P_0$ (and hence of $P'_0$ defined in (1.17, 1.18)) render the constructions particularly transparent.

Define the diagonal $N \times N$ matrices $f$ for the cases

(1) : $\hat{o}(N)$, \hspace{1cm} $N = 2n + 1$, \hspace{1cm} $n = 1, 2, \ldots$

(2) : $\hat{o}(N)$, \hspace{1cm} $N = 2n$, \hspace{1cm} $n = 2, 3, \ldots$

(3) : $\hat{p}(N)$, \hspace{1cm} $N = 2n$, \hspace{1cm} $n = 2, 3, \ldots$

respectively as follows

(1) : $f = (q^{-(2n-1)}, q^{-(2n-3)}, \ldots, q^{-1}, 1, q, \ldots, q^{(2n-3)}, q^{(2n-1)})_{(\text{diag})}$  \hspace{1cm} (5.4)$$

(2) : $f = (q^{-(2n-2)}, q^{-(2n-4)}, \ldots, q^{-2}, 1, 1, q^2, \ldots, q^{(2n-4)}, q^{(2n-2)})_{(\text{diag})}$  \hspace{1cm} (5.5)$$

(3) : $f = (q^{-2n}, q^{-(2n-2)}, \ldots, q^{-2}, q^2, \ldots, q^{(2n-2)}, q^{2n})_{(\text{diag})}$  \hspace{1cm} (5.6)$$

Note the following facts:

(1): The $N$ diagonal elements of $f$, in each case, are the nonzero diagonal elements of the corresponding $P'_0$ (related to the projector as $P_0 = (trP'_0)^{-1}P'_0$), the remaining $N(N - 1)$ diagonal elements of $P'_0$ being zero. Hence (with upper and lower signs for $\hat{o}(N)$ and $\hat{p}(N)$ respectively),

$$T \equiv trf = trP'_0 = [N \mp 1] \pm 1 = -(\lambda_+ + \lambda_-) = e^{-\eta!} + e^{\eta!}$$  \hspace{1cm} (5.7)$$

where $e^{\pm \eta!}$ are the roots of

$$e^{2\eta!} - Te^{\eta!} + 1 = 0$$  \hspace{1cm} (5.8)$$

(2): The $N^2 \times N^2$ matrix $f \otimes f$, in each case, has 1 on rows and columns
These are precisely ones on which $P_0'$ has nonzero elements. Hence directly ( without further computations ) we obtain

\[ P_0' f \otimes f = f \otimes f P_0' \]  \hspace{1cm} (5.9)

(3): Using (1.17, 1.18, 5.3) one obtains

\[ \text{tr}_2 (P_0' f \otimes f) = f \]  \hspace{1cm} (5.10)

Now from (5.9, 5.10) it follows immediately

\[ \hat{R}^{\pm 1} f \otimes f = (I - e^{\mp \eta} P_0') f \otimes f = f \otimes f (I - e^{\mp \eta} P_0') = f \otimes f \hat{R}^{\pm 1} \]  \hspace{1cm} (5.11)

and

\[ \text{tr}_2 (\hat{R}^{\pm 1} f \otimes f) = \text{tr}_2 ((I - e^{\mp \eta} P_0') f \otimes f) = (T - e^{\mp \eta}) f = e^{\pm \eta} f \]  \hspace{1cm} (5.12)

Hence

\[ \text{tr}_2 (\hat{R}^{\pm 1} f \otimes f) = a^{\pm 1} bf, \quad (a = e^\eta, b = 1) \]  \hspace{1cm} (5.13)

Thus we have obtained for our $\hat{R}$, in terms of $f$ introduced above, the enhanced operator

\[ (\hat{R}, f, e^\eta, 1) \]

Our $f$ is strictly analogous to those of Turaev for $SO_q(2n + 1), SO_q(2n), Sp_q(2n)$ respectively. But whereas for the standard cases $a$ ( $\alpha$ in the notation of [12] ) is also a simple power of $q$, for us it involves the square root of a Laurent polynomial in $q$. One obtains with $\delta = (1, 2, 0)$ for $\hat{o}(2n + 1), \hat{o}(2n), \hat{p}(2n)$ respectively

\[ T = (q^{-2n+\delta} + q^{-2n-2+\delta} + ... + q^{2n+2-\delta} + q^{2n-\delta} + \delta) \]  \hspace{1cm} (5.14)

and

\[ a^{\pm 1} = e^{\pm \eta} = \frac{1}{2} (T \pm \sqrt{T^2 - 4}) \]  \hspace{1cm} (5.15)

This is the crucial new aspect for our class of $\hat{R}$. For the simplest case $\hat{o}(3)$ one obtains

\[ a^{\pm 1} = \frac{1}{2} (q + 1 + q^{-1}) \pm \frac{1}{2} \left( (q + 3 + q^{-1})(q - 1 + q^{-1}) \right)^{\frac{1}{2}} \]  \hspace{1cm} (5.16)

giving for $q = 1$,

\[ a^{\pm 1} = \frac{3}{2} \pm \frac{1}{2} \sqrt{5} \]  \hspace{1cm} (5.17)
In general, for \( q = 1 \), as for the standard case \( f \) reduces to the \( N \times N \) unit matrix but as emphasized before our \( \hat{R} \) remains nontrivial and

\[
a^{\pm 1} = \frac{1}{2} N \pm \frac{1}{2} \sqrt{N^2 - 4}, \quad (N = 3, 4, \ldots) \tag{5.18}
\]

The discussion of Sec.2 shows that for solutions of (2.10)

\[
\eta = 0, \quad a = 1 \tag{5.19}
\]

implying a complex root of unity \( q \) for \( N = 3 \) but finally elliptic and hyperelliptic ones as \( N \) increases. (Overcrossings and undercrossings degenerate for \( \hat{R} = \hat{R}^{-1} \).)

[Comparison of notations: The present author is often confused by different significances of the same symbol (and vice versa) encountered elsewhere. The following points might be helpful in our context.

Turaev’s \( R \) [12] satisfying the braid equation (his eqn. 1)

\[
R_1 R_2 R_1 = R_2 R_1 R_2
\]

is our \( \hat{R} \). Our \( R \) is \( P \hat{R} \) where

\[
P(ij) \otimes (kl) = (kj) \otimes (il), \quad (ij) \otimes (kl) P = (il) \otimes (kj), \quad P(ij) \otimes (kl) P = (kl) \otimes (ij)
\]

and \( R \) satisfies the Yang-Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}
\]

It must also be clearly be noted that if \( \sigma \) (or \( \tau \)) is defined as

\[
\sigma(ij) \otimes (kl) = (kl) \otimes (ij) = P(ij) \otimes (kl) P
\]

then

\[
\sigma R = P R P = R_{21} = \hat{R} P
\]

does not stisfy the braid equation above (satisfied by \( \hat{R} \)). Moreover, since

\[
P(f \otimes f) P = f \otimes f, \quad P^2 = I
\]

the condition (5.1) implies also (for the YB-matrix \( R \))

\[
R^{\pm 1} f \otimes f = f \otimes f R^{\pm 1}
\]

This form is presented in Sec.15.2.2 of Ref.13. But \( \hat{R} \) cannot be replaced by \( R \) in (5.2). The symbol \( I \) denotes \( \sigma R \) in Sec.15.2.2 of Ref.13 and \( -\sigma R \) at the end of Sec.15.2.5. ]
5.2 Link Invariants and skein relation:

We follow the presentation of Ref.12 and Sec.15 of Ref.13 with some changes of notations. Let \( \rho(\beta) \) be the representation of the braid \( \beta \) associated to \( \hat{R} \) and let \( \alpha(\beta) \) be the ” augmentation homomorphism” changing by \( \pm 1 \) corresponding to the actions of \( T^\pm \), the generators of the braid group.

Define for our case ( with \( b = 1 \) )

\[
P(\beta) = a^{-\alpha(\beta)} \text{tr} \left( \rho_m(\beta) \cdot f^\otimes m \right)
\]

(5.20)

\( \rho_m \) being the endomorphism of \( V^\otimes m \) associated to \( \hat{R} \).

Using appropriately the properties (5.1) and (5.2) of \( f \) such a \( P(\beta) \) can be shown to be Markov invariant and provide an invariant of oriented links. Markov moves are defined, for example, in Sec.15.1 of Ref.13 and the proof of invariance of \( P(\beta) \) is given in Sec.15.2 following Ref.12.

For an ”unknot” ( no crossing) one has

\[
P(\bigcirc) = \text{tr} f = T
\]

(5.21)

Using standard notations ( \( \bar{L}_+ \), \( \bar{L}_- \), \( \bar{L}_0 \) ) corresponding to one point of the projection of a braid differing by an overcrossing, undercrossing and nocrossing respectively one obtains in our case ( following the steps below eqn.(6), Sec.15.2 of Ref.13 )

\[
xP(\bar{L}_+) + yP(\bar{L}_-) + zP(\bar{L}_0)
\]

\[
= a^{-\alpha(\beta)} \text{tr} \left( (x\hat{R}^{-1} + y\hat{R} + zI) \otimes \text{id}^\otimes 2 \otimes \text{id}^\otimes (m-2) \cdot \rho_m(\beta) \cdot f^\otimes m \right)
\]

(5.22)

Now for our case

\[
e^\eta \hat{R} - e^{-\eta} \hat{R}^{-1} = (e^\eta - e^{-\eta}) I
\]

(5.23)

Hence setting

\[
x = -e^{-\eta}, \quad y = e^\eta, \quad z = -(e^\eta - e^{-\eta})
\]

(5.24)

one obtains the skein relation

\[
e^{-\eta}P(\bar{L}_+) - e^{\eta}P(\bar{L}_-) = (e^{-\eta} - e^{\eta})P(\bar{L}_0)
\]

(5.25)

One can now exploit this relation along with (5.21) in well-known fashions to construct invariant polynomials. ( See also Ref.14 where a large number of sources are cited. ) We will not present a full study of this aspect. Our aim has been to indicate the roles played by our coefficients \( e^{\pm \eta} \) ( an ingredient of our \( \hat{R} \) ) in this context. This has been achieved in our brief treatment.
6 From \( \hat{R} \) to noncommutative spaces:

6.1 Coordinates, differentials and mobile frames:

We implement well-known prescriptions \([15, 16, 17]\) in the context of our class of \( \hat{R} \). For the \( N^2 \times N^2 \) matrix \( \hat{R} \) satisfying

\[
(\hat{R} - I)(\hat{R} + e^{-2\eta}I) = 0 \tag{6.1}
\]

where for \( \hat{o}(N) \), \( \hat{p}(N) \) respectively

\[
e^{\eta} + e^{-\eta} = [N \mp 1] \pm 1
\]

let the coordinates \( (x_1, x_2, ..., x_N) \) and the differentials \( (\xi_1, \xi_2, ..., \xi_N) \) be ordered in \( N \)-columns \( x \) and \( \xi \) respectively.

The prescriptions for the associated covariant differential geometries satisfying the Leibnitz rule \([15, 16, 17]\) are

\[
(1) : \quad (\hat{R} - I)x \otimes x = 0, \quad x \otimes \xi = e^{2\eta}\hat{R}\xi \otimes x, \quad (\hat{R} + e^{-2\eta}I)\xi \otimes \xi = 0 \tag{6.2}
\]

\[
(2) : \quad (\hat{R} + e^{-2\eta}I)x \otimes x = 0, \quad x \otimes \xi = -\hat{R}\xi \otimes x, \quad (\hat{R} - I)\xi \otimes \xi = 0 \tag{6.3}
\]

We concentrate below on (6.2). The set (6.3) can be treated analogously, essentially interchanging the roles of \( x \) and \( \xi \) (except for the \( (x, \xi) \) commutators).

From our previous definitions

\[
\hat{R} = I - e^{\mp\eta}P'_0
\]

Hence from (6.2)

\[
P'_0x \otimes x = 0 \tag{6.4}
\]

The set of constraints (6.4) reduces to a single one due to the proportionality of the nonzero rows of \( P'_0 \). This one is easy to write down for all \( N \) from (1.17, 1.18). One obtains for

\[
\hat{o}(3) : \quad q^{-\frac{1}{2}}x_1x_3 + x_2x_2 + q^2x_3x_1 = 0 \tag{6.5}
\]

\[
\hat{o}(4) : \quad q^{-1}x_1x_4 + x_2x_3 + x_3x_2 + qx_4x_1 = 0 \tag{6.6}
\]

\[
\hat{p}(4) : \quad q^{-2}x_1x_4 + q^{-1}x_2x_3 - qx_3x_2 - q^2x_4x_1 = 0 \tag{6.7}
\]
and so on. Consider now the constraints involving $\xi$ with the $\hat{o}(3)$ case as example. Define

$$\Pi = (q^{-\frac{1}{2}}\xi_1 x_3 + \xi_2 x_2 + q^{\frac{1}{2}}\xi_3 x_1)$$  \hspace{1cm} (6.8)$$

$$\Pi' = (q^{-\frac{1}{2}}\xi_1 x_3 + \xi_2 x_2 + q^{\frac{1}{2}}\xi_3 x_1)$$  \hspace{1cm} (6.9)$$

Now from (6.2) for $N = 3$ with

$$e^\eta + e^{-\eta} = (q + 1 + q^{-1})$$

one has

$$x_i \xi_j = e^{2\eta} \xi_i x_j, \quad (i + j \neq 4)$$  \hspace{1cm} (6.10)$$

$$x_1 \xi_3 = e^{2\eta} \xi_1 x_3 - e^\eta q^{-\frac{1}{2}} \Pi, \quad x_2 \xi_2 = e^{2\eta} \xi_2 x_2 - e^\eta \Pi, \quad x_3 \xi_1 = e^{2\eta} \xi_3 x_1 - e^\eta q^{\frac{1}{2}} \Pi$$  \hspace{1cm} (6.11)$$

$$\xi_i \xi_j = 0, \quad (i + j \neq 4)$$  \hspace{1cm} (6.12)$$

$$q^{-\frac{1}{2}}\xi_1 x_3 = y^{-1} q^{-1} \Pi', \quad \xi_2 x_2 = y^{-1} \Pi', \quad q^{\frac{1}{2}}\xi_3 x_1 = y^{-1} q \Pi', \quad y = (q + 1 + q^{-1})$$  \hspace{1cm} (6.13)$$

Note the consistency of the sum of the equations (6.13). For $N > 3$

$$\xi_i \xi_j = 0, \quad (i + j \neq (N + 1))$$  \hspace{1cm} (6.14)$$

The coefficients of $(\Pi, \Pi')$ are now obtained from (1.17, 1.18) they being proportional to the nonzero elements in a row of $P_0'$. Also $\eta$ will now be as below (6.1).

We now briefly consider the construction of mobile frames \cite{17, 18} ( or ”stehbeins” in the terminology of the authors cited ). Let

$$\theta = \sum_i \theta_i \xi_i$$  \hspace{1cm} (6.15)$$

such that

$$[\theta, x_i] = 0$$  \hspace{1cm} (6.16)$$

From (6.2, 6.16), remembering that in our conventions

$$\hat{R} = \hat{R}_{ij,kl}(ij) \otimes (kl)$$
and indicating all summations explicitly (and using the $L^\pm$ of Secs. (3, 4))

\[
(x_i \theta_j) = e^{-2\eta} \sum_{k,a,b} \theta_k \tilde{R}_{ka,ib} x_a \xi_b
\]

\[
= e^{-2\eta} \sum_{k,a,b} \theta_k (\tilde{R}^{-1} P)_{kb,ia} x_a \xi_b
\]

\[
= e^{-2\eta} \sum_{k,a,b} (\theta_k L^-_{kb,ia}) \xi_b
\]

\[
= x_i (\sum_b \theta_b \xi_b)
\]

Hence

\[
x_i \theta_j = e^{-2\eta} \sum_{k,l} \theta_k L^-_{kj,il} x_l
\]

Thus one obtains the commutators of $(x_i, \theta_j)$. For $N = 3$, setting

\[
\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1 = \tau
\]

\[
x_i \theta_i = e^{-2\eta} \tau - e^{-\eta} \theta_i x_i', \quad (i = 1, 2, 3)
\]

\[
x_1 \theta_1 = -e^{-\eta} q^{\frac{1}{2}} \theta_3 x_2, \quad x_1 \theta_2 = -e^{-\eta} \theta_3 x_1
\]

\[
x_2 \theta_1 = -e^{-\eta} q^{\frac{3}{2}} \theta_2 x_3, \quad x_2 \theta_3 = -e^{-\eta} q^{-\frac{3}{2}} \theta_2 x_1
\]

\[
x_3 \theta_1 = -e^{-\eta} q^{-\frac{1}{2}} \theta_1 x_3, \quad x_3 \theta_2 = -e^{-\eta} q^{-\frac{1}{2}} \theta_1 x_2
\]

Generalizations for $N > 3$ are obtained analogously.

In Ref.17 different solutions of $\theta$ (3 solutions for $SO_q(3)$) are presented. They involve a dilatation operator and inverses of coordinates, a radius $r$ being defined. We consider no extensions of our algebras or such solutions for $\theta$ in the present work.

### 6.2 Towers of noncommutative $(x_1, \ldots, x_N)$ on a commutative $(N - 1)$-base space:

In eqn. (1.3) of Ref.4 it is pointed out that the mapping, implementing the transfer matrix $t$,

\[
\delta(x_i) = \sum_k t_{ik} \otimes x_k \equiv (tx)_i
\]
provides an iterative sequence of solutions. Since

\[ \hat{R}(t \otimes I)(I \otimes t) = (t \otimes I)(I \otimes t) \hat{R} \]  \hspace{1cm} (6.22)

for any polynomial of \( f(\hat{R}) \), if

\[ f(\hat{R})(x \otimes x) = 0 \]

\[ (t \otimes I)(I \otimes t)f(\hat{R})(x \otimes x) = f(\hat{R})(t \otimes I)(I \otimes t)(x \otimes x), \quad = f(\hat{R})(tx) \otimes (tx) = 0 \]  \hspace{1cm} (6.23)

This is the mapping (6.21). It can evidently be iterated as

\[ \delta((t^n x) \otimes (t^n x)) = (t^{n+1} x) \otimes (t^{n+1} x) \]  \hspace{1cm} (6.24)

We denote this as

\[ x^{(n)} \rightarrow x^{(n+1)} \]

We note here a remarkable possibility, starting again with \( \hat{o}(3) \) as an example. Choose as the starting point a commutative solution of (6.5) as follows:

With parameters \((a, b, c) \geq 0\) the following surface satisfies (6.5),

\[ x_1^{(0)} = a, \quad x_3^{(0)} = -b, \quad x_2^{(0)} = \pm \left( (q^{1/2} + q^{-\frac{1}{2}})ab \right)^{\frac{1}{2}} \]

\[ x_1^{(0)} = -a, \quad x_3^{(0)} = b, \quad x_2^{(0)} = \pm \left( (q^{1/2} + q^{-\frac{1}{2}})ab \right)^{\frac{1}{2}} \]  \hspace{1cm} (6.25)

As \((a, b, c)\) varies through real, non-negative values one obtains a double cone whose projections on the \((1, 3)\) plane covers the second and the fourth quadrants. The vertices meet at the origin. The projections of the contours \( x_2 = \text{const.} \) on the \((1, 3)\) plane are parabolas. The origin is invariant under \( \delta \).

Now we implement \( \delta \) as in (6.21) to obtain

\[ x^{(1)}_i = (tx^{(0)})_i \]  \hspace{1cm} (6.26)

Consider for simplicity the \( 3 \times 3 \) fundamental \( t \)-matrices. These are given by (compare (4.1) and see the reference cited above it)

\[ t = t^+ = P \hat{R} = R = P(\hat{R}P)P = L^+_{21} \]  \hspace{1cm} (6.27)

and

\[ t = t^- = P \hat{R}^{-1} = P(R^{-1}P) = (\hat{R}^{-1})_{21} = L^-_{21} \]  \hspace{1cm} (6.28)
We treat below $t^\pm$ together by setting correspondingly (for $N = 3$)

\[
\lambda = \lambda_\pm = \lambda^{\pm 1}, \quad (\lambda + \lambda^{-1} + y = 0) \quad (6.29)
\]

For the fundamental rep. of $t$ our map gives

\[
x_1^{(1)} = \begin{pmatrix}
 x_1^{(0)} & 0 & 0 \\
 x_2^{(0)} & 0 & 0 \\
 (1 + q\lambda)x_3^{(0)} & q^{\frac{3}{2}}\lambda x_2^{(0)} & \lambda x_1^{(0)}
\end{pmatrix} \quad (6.30)
\]

\[
x_2^{(1)} = \begin{pmatrix}
 0 & x_1^{(0)} & 0 \\
 q^{\frac{3}{2}}\lambda x_3^{(0)} & (1 + \lambda)x_2^{(0)} & q^{-\frac{3}{2}}\lambda x_1^{(0)} \\
 0 & x_3^{(0)} & 0
\end{pmatrix} \quad (6.31)
\]

\[
x_3^{(1)} = \begin{pmatrix}
 \lambda x_3^{(0)} & q^{-\frac{3}{2}}\lambda x_2^{(0)} & (1 + q^{-1}\lambda)x_1^{(0)} \\
 0 & 0 & x_2^{(0)} \\
 0 & 0 & x_3^{(0)}
\end{pmatrix} \quad (6.32)
\]

The symmetries signalled above (4.4) reappear. Iteration now may proceed as

\[
(x_i^{(0)}, x_i^{(1)}) \rightarrow (x_i^{(n)}, x_i^{(n+1)})
\]

At each stage, given only (6.5) for $x_i^{(n)}$ (and (6.29) for $\lambda$), $x_i^{(n+1)}$ also satisfies (6.5) and hence

\[
(\hat{R} - I)x_i^{(n+1)} \otimes x_i^{(n+1)} = 0 \quad (6.33)
\]

This has been verified explicitly. Moreover at each stage one can implement any chosen representation of $t$ (say, the $9 \times 9$ rather than the $3 \times 3$). Thus one may obtain varied sequences in the iterations. The illustration above is sufficient for our purpose. Let us compare this construction with a parallel possibility for $SO_q(3)$. We refer to the results of Ex.4.1.22 of Ref.17. But for easier comparison with our results above we change the basis from the circular components to our type as

\[
(x_-, y, x_+) \rightarrow (x_1, x_2, x_3)
\]

Now (4.1.61) of Ref.17 is

\[
x_1x_2 = qx_2x_1, \quad x_3x_2 = q^{-1}x_2x_3, \quad (x_3x_1 - x_1x_3) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x_2^2 \quad (6.34)
\]

There are now three constraints as compared to a single one for (6.5). But one can choose the commutative $(1, 3)$ plane (as compared to the double cone before) as the starting point by setting

\[
x_2^{(0)} = 0, \quad x_1^{(0)}x_3^{(0)} = x_3^{(0)}x_1^{(0)} \quad (6.35)
\]
This satisfies all the three constraints (6.34). Using the $3 \times 3$ $t$-matrix blocks for $SO_q(3)$ (and setting $\kappa = (q - q^{-1})$) one obtains

$$
x^{(n+1)}_1 = \begin{vmatrix} qx^n_1 & 0 & 0 \\ 0 & x^n_1 & 0 \\ 0 & 0 & q^{-1}x^n_1 \end{vmatrix}
$$

(6.36)

$$
x^{(n+1)}_2 = \begin{vmatrix} x^n_2 & \kappa x^n_1 & 0 \\ 0 & x^n_2 & q^{-\frac{1}{2}}\kappa x^n_1 \\ 0 & 0 & x^n_2 \end{vmatrix}
$$

(6.37)

$$
x^{(n+1)}_3 = \begin{vmatrix} q^{-1}x^n_3 & q^{-\frac{1}{2}}\kappa x^n_2 & \kappa(1-q^{-1})x^n_1 \\ 0 & x^n_3 & \kappa x^n_2 \\ 0 & 0 & q x^n_3 \end{vmatrix}
$$

(6.38)

Starting the iteration from (6.35) at each step $x^n_i$ satisfies (6.34). As before one can use more general realizations of the $t$-matrix at any step. One may note that for this case

$$
(q = 1) \rightarrow x^{(n+1)}_i = I_3 \otimes x^{(n)}_i
$$

This is consistent with $\hat{R} = I$ for $q = 1$ in the standard cases. But, as emphasized before, there is no such triviality for our class for any value of $q$ (including 1). Note also that for our class the matrices $x^{(1)}_i$ and the iterated ones are non-invertible. This is a general feature for our class.

In the examples above one starts with a classical surface and iterating as above makes it more and more "fuzzy" in this specific sense. This should be compared with the "fuzzy sphere" of Ref.17 (Sec.7.2) where one starts fuzzy and a smooth surface is approached as a limit. One moves in opposite senses in the two formalisms.

So far we have studied the coordinate space $(x_i)$ only. It must be noted carefully that one cannot obtain a consistent nontrivial set $\xi^{(0)}_i$ corresponding to $x^{(0)}_i$. This is evident from (6.10) to (6.13) where $q$ is not restricted. So the whole covariant prescription can be introduced at a noncommutative stage only. This however does not alter the fact that one can build sequences of noncommutative $(x_i)$ starting from a smooth surface.

For $N > 3$, with (6.6) and (6.7) as simplest examples, one has evidently more flexibility in choosing $\xi^{(0)}_i$. Without going into details we indicate below for $\hat{o}(4)$ the structures (valid more generally) induced by our type of iterations.

$$
\hat{o}(4) : \quad \lambda + \lambda^{-1} + (q^2 + 2 + q^{-2}) = 0
$$

31
\begin{align*}
x_1^{(n+1)} &= \begin{bmatrix}
& x_1^{(n)} & 0 & 0 & 0 \\
& x_2^{(n)} & 0 & 0 & 0 \\
& x_3^{(n)} & 0 & 0 & 0 \\
(1 + \lambda)x_4^{(n)} & q^{-1}\lambda x_3^{(n)} & q^{-1}\lambda x_2^{(n)} & q^{-2}\lambda x_1^{(n)}
\end{bmatrix} \quad (6.39)
\end{align*}

\begin{align*}
x_2^{(n+1)} &= \begin{bmatrix}
0 & x_1^{(n)} & 0 & 0 \\
0 & x_2^{(n)} & 0 & 0 \\
q\lambda x_4^{(n)} & (1 + \lambda)x_3^{(n)} & \lambda x_2^{(n)} & q^{-1}\lambda x_1^{(n)} \\
0 & x_4^{(n)} & 0 & 0
\end{bmatrix} \quad (6.40)
\end{align*}

\begin{align*}
x_3^{(n+1)} &= \begin{bmatrix}
0 & 0 & x_1^{(n)} & 0 \\
q\lambda x_4^{(n)} & \lambda x_3^{(n)} & (1 + \lambda)x_2^{(n)} & q^{-1}\lambda x_1^{(n)} \\
0 & 0 & x_3^{(n)} & 0 \\
0 & 0 & x_4^{(n)} & 0
\end{bmatrix} \quad (6.41)
\end{align*}

\begin{align*}
x_4^{(n+1)} &= \begin{bmatrix}
q^2\lambda x_4^{(n)} & q\lambda x_3^{(n)} & q\lambda x_2^{(n)} & (1 + \lambda)x_1^{(n)} \\
0 & 0 & 0 & x_2^{(n)} \\
0 & 0 & 0 & x_3^{(n)} \\
0 & 0 & 0 & x_4^{(n)}
\end{bmatrix} \quad (6.42)
\end{align*}

The analogous structures for \( \hat{p}(4) \) have also been obtained. The structures of the matrices are quite similar to those for \( \hat{o}(4) \). The \( q \)-dependence of the coefficients show typical differences along with sign changes. They will not be reproduced here. In each case, for all \( N \), there is in each matrix one row and one column with nonzero elements. How they shift with \( i \) of \( x_i^{(n)} \) should already be fairly apparent from the two preceding examples.

### 7 Remarks:

We have started to explore the contents of a class of braid matrices presented previously. Unsurprisingly our study remains incomplete in all directions. We briefly indicate below perspectives of further developments.

For the standard cases ( \( q \)-deformed unitary and orthogonal algebras ) we have studied extensively elsewhere representations for \( q \) a root of unity [19,20]. ( These two references cite many other sources.) We introduced "the method of fractional parts" for this purpose. Here we have noted (Sec.2) how certain roots of unity give triangularity \( (\hat{R}^2 = I) \) for different dimensions. A study of our \( L \)-algebras for such cases along similar lines can be of interest.

Our other solutions for triangularity involve elliptic and hyperelliptic \( q \) with integer (binomial) coefficients in the defining equations. The roles of such functions deserve further explorations. Higher genus curves have appeared before [21, 22] in the construction of statistical models as solutions of star-triangle (or Yang-Baxter) relations. There such curves
are necessary ingredients of the solutions. In our case the situation is quite different. Our class of braid matrices have been obtained for any $q$. One can even set $q = 1$ and still have interesting solutions. Our special values of $q$ appear only when the additional constraint of triangularity is imposed and depend on the dimension $N$.

We have obtained some important general features of our $L$-algebras (see for example the eqns. from (3.11) to (3.18)). But the explicit study of realizations is limited to $3 \times 3$ and $9 \times 9$ ones for $\hat{o}(3)$. This has already shown the crucial role of the central, group-like elements we have constructed, thus achieving a principal goal. But a more general study of the $L$-algebras is desirable. Quadratic [23] and higher degree [24] homogeneous algebras have been studied from a functional point of view yielding, for example, the Poincare series. (More sources are cited in Ref.24.) The Poincare series for our algebra would show whether, and if so what, irreducible representations interpolate the $N^{2p} \times N^{2p}$ dimensional coproduct representations (corresponding, as pointed out in Sec.4, to the central $S_3$ elements proportional to $I_{N^p})$ obtained by iterating the coproduct prescription. We are unable to answer this question definitively at present, though attempts to realize intermediate dimensional ones (between $3 \times 3$ and $9 \times 9$) exploiting the symmetries pointed out in Sec.4 seem to encounter obstructions. Our detailed study of the $9 \times 9$ case gives an idea of the features to be expected more generally.

The Turaev construction for link invariants turns out to be elegantly adaptable to our case. Systematic construction of invariant polynomials and possibility of generalizations to invariants of $3$-manifolds will be studied elsewhere.

Noncommutative geometries associated with our $\hat{R}$ have been presented indicating possible constructions of “noncommutative towers” on classical base spaces of dimensions $< N$. Here again a deeper study of the differential geometries remains to be done. Possible roles of our special values of $q$ corresponding to triangularity should be interesting to explore in this context.

It follows from (1.10) or (3.41) along with (1.17) that for $\hat{o}(N)$, real positive $q$ and

$$-\eta < \theta < 0$$

the elements of $\hat{R}(\theta)$ are all non-negative (either zero or real positive). Hence such an $\hat{R}(\theta)$ along with the corresponding transfer matrix $t(\theta)$ (obtainable from $\hat{R}(\theta)$) can furnish the basis of a multistate statistical model. The elements of $\hat{R}(\theta)$ provide the Boltzmann weights. This class of models will be studied in a following paper.

References

[1] A. Chakrabarti, J. Math. Phys. 44, 5320 (2003)

[2] A. Chakrabarti and R. Chakrabarti, J. Math. Phys. 44, 785 (2003)

[3] A. Chakrabarti, J. Math. Phys. 43, 1589 (2002)
[4] L.D.Faddeev, N.Yu.Reshetikhin and L.A.Takhtadzhyan, Leningrad Math.J. 1,193 (1990)
[5] A.Klymik and K.Schmudgen, Quantum Groups and Their Representations, Springer (1997)
[6] Tata Lectures on Theta II, D.Mumford, Birkhauser (1984) (App.by H.Umemura)
[7] R.Bruce King, Beyond the quartic equation, Birkhauser (1996) (Sec.8.3.-1)
[8] B.Abdesselam, A.Chakrabarti and R.Chakrabarti, Mod.Phys.Lett. A13,779 (1998)
[9] A.Chakrabarti and R.Chakrabarti, J.Phys.A:Math.Gen. 33,1(2000)
[10] A.P.Isaev, Sov.J.Part.Nucl. 26, 501 (1995)
[11] A.Chakrabarti, A nested sequence of projectors and corresponding braid matrices $\hat{R}(\theta)$: (I) Odd dimensions, math.QA/0401207 (App.B)
[12] V.G.Turaev, Invent.Math. 92, 527 (1988)
[13] V.Chari and A.Pressley, Quantum Groups, C.U.P. (1994) (Secs.15.1, 15.2)
[14] F.Y.Wu, Rev.Mod.Phys. 64, 1099 (1992)
[15] J.Wess and B.Zumino, Nucl.Phys.B (Proc.Suppl.) 18B, 302 (1990)
[16] L.Hlavaty, J.Phys.A: Math.Gen. 25, 485 (1992)
[17] J.Madore, An Introduction to Noncommutative Differential Geometry, C.U.P. (1999)
[18] B.L.Cerchiai, G.Fiore and J.Madore, Geometrical tools for quantum Euclidean spaces, math.QA/0002007
[19] B.Abdesselam, D.Arnaudon and A.Chakrabarti, J.Phys.A: Math.Gen. 28, 5495 (1995)
[20] B.Abdesselam, D.Arnaudon and A.Chakrabarti, J.Phys.A: Math.Gen. 28, 3701 (1995)
[21] H.Au-Yang, B.M.McCoy, J.Perk, S.Tang and M.L.Yan, Phys.Lett. A 123, 219 (1987)
[22] R.J.Baxter, J.H.H.Perk and H.Au-Yang, Phys.Lett. A 128, 138 (1988)
[23] Yu.I.Manin, Quantum Groups and Non-Commutative Geometry, CRM Univ. de Montreal (1988)
[24] M.Dubois-Violette and T.Popov, Lett.Math.Phys. 61, 159 (2002)