Generalizations of Ekeland–Hofer and Hofer–Zehnder Symplectic Capacities and Applications

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Abstract
In this paper we construct analogues of the Ekeland–Hofer and the Hofer–Zehnder symplectic capacities based on a class of Hamiltonian boundary value problems motivated by Clarke’s and Ekeland’s work, and study generalizations of some important results about the original two capacities (for example, the famous Weinstein conjecture, representation formula for $c_{EH}$ and $c_{HZ}$, and a theorem by Evgeni Neduv).

Keywords Ekeland–Hofer symplectic capacity · Hofer–Zehnder symplectic capacity · Weinstein conjecture

Mathematics Subject Classification Primary 53D35 · 53C23; Secondary 70H05 · 37J05 · 57R17

1 Introduction and Main Results

Weinstein [48] and Rabinowitz [40] proved, respectively, the existence of periodic orbits on a convex energy surface and a strictly star-shaped hypersurface of a Hamiltonian system in $\mathbb{R}^{2n}$. Based on these, in 1978 Weinstein [49] proposed his famous conjecture: every hypersurface of contact type in symplectic manifolds carries a closed characteristic. Here a compact connected smooth hypersurface $S$ in a symplectic mani-
ifold \((M, \omega)\) is said to be of **contact type** if there exists a vector field \(X\) defined in an open neighborhood \(U\) of \(S\) in \(M\) which is transverse to \(S\) and satisfies \(L_X \omega = \omega\) in \(U\). Such a vector field \(X\) is called a **Liouville field**. (All compact manifolds or hypersurfaces in this paper are considered to be boundaryless without special statements.) A **closed characteristic** of \(S\) is an embedded circle \(P \subset S\) satisfying \(TP = L_S|_P\), where \(L_S \rightarrow S\) is the distinguished line bundle defined by:

\[
L_S = \left\{ (x, \xi) \in TS \mid \omega_x(\xi, \eta) = 0 \text{ for all } \eta \in T_x S \right\}.
\]

In 1986, Viterbo [47] first proved the Weinstein conjecture in \((\mathbb{R}^{2n}, \omega_0)\) with the global variational methods for periodic solutions of general Hamiltonian systems initiated by Rabinowitz [40, 41] and Weinstein [48]. Hereafter \(\omega_0\) denotes the standard symplectic structure given by \(\sum_i dq_i \wedge dp_i\) with the linear coordinates \((q_1, \ldots, q_n, p_1, \ldots, p_n)\).

Motivated by the above studies, Ekeland and Hofer [18] introduced a class of symplectic invariants (called symplectic capacities) for subsets in \((\mathbb{R}^{2n}, \omega_0)\) and reproved the famous Gromov’s nonsqueezing theorem in [23] and a \(C^0\) rigidity theorem due to Gromov and Eliashberg. Hofer and Zehnder [26] constructed a symplectic capacity for any symplectic manifold, called the Hofer–Zehnder capacity. The second named author of this article introduced the concept of pseudo symplectic capacities which is a mild generalization of symplectic capacities, constructed a pseudo symplectic capacity as a generalization of the Hofer–Zehnder capacity and established an estimate for it in terms of Gromov–Witten invariants ([35]).

For a symplectic matrix of order \(2n\), \(\psi \in \text{Sp}(2n, \mathbb{R})\), as a generalization of the existence of closed characteristics on the boundary \(S\) of a compact and convex set \(D\) in \((\mathbb{R}^{2n}, \omega_0)\) containing the origin in its interior, Clarke [10, 11] proved: there exists a nonconstant absolutely continuous curve \(z : [0, T] \rightarrow S\) for some \(T > 0\) such that \(J \dot{z}(t) \in \partial jD(z(t))\) a.e. and that \(z(T) = \psi z(0)\). Here \(\partial jD\) is subdifferential of the Minkowski functional \(j_D\) of \(D\) given by

\[
j_D(x) = \inf \left\{ \lambda > 0 \left| \frac{x}{\lambda} \in D \right\} \right.,
\]

and \(J\) is the standard complex structure on \(\mathbb{R}^{2n}\) given by the matrix

\[
J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}
\]

(1.1)

with the linear coordinates \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) on \(\mathbb{R}^{2n}\), where \(I_n\) denotes the identity matrix of order \(n\). (We also use \(J\) to denote the standard complex structure on \(\mathbb{R}^{2k}\) for different \(k \in \mathbb{N}_+\) without confusions.)

Clarke’s result means that any linear symplectic transformation is realized on some orbit of any convex energy surface, which was, in [10, page 356], viewed as a kind of converse to the Goldstein’s famous statement “the motion of a mechanical system corresponds to the continuous evolution or unfolding of a canonical (i.e., symplectic) transformation” [22, §8.6]. We generalize the Clarke’s results by constructing some
analyses of the Ekeland–Hofer and the Hofer–Zehnder symplectic capacities associated to symplectomorphisms. The finiteness of these “analogues” is closely related to (non-periodic) boundary value problems of Hamiltonian systems. The main difficulties in the constructions of these analogues are:

(i) how to adapt the classical definitions such as admissible functions, admissible deformations and nonresonant conditions in [18, 26] to fit them in the present non-periodic case;
(ii) how to solve new problems arisen in the related proofs by following the standard method in [18, 26].

We introduce the following definitions about characteristics.

**Definition 1.1** (i) For a smooth hypersurface $S$ in a symplectic manifold $(M, \omega)$ and $\Psi \in \text{Symp}(M, \omega)$, a $C^1$ embedding $z$ from $[0, T]$ (for some $T > 0$) into $S$ is called a $\Psi$-characteristic on $S$ if $z(T) = \Psi z(0)$ and $\dot{z}(t) \in (L_S)_{z(t)} \forall t \in [0, T]$. Clearly, $z(T - \cdot)$ is a $\Psi^{-1}$-characteristic, and for any $\tau > 0$ the embedding $[0, \tau T] \to S, t \mapsto z(t/\tau)$ is also a $\Psi$-characteristic.

(ii) If $S$ is the boundary of a compact convex set $D$ with nonempty interior in $(\mathbb{R}^{2n}, \omega_0)$, and $\Psi \in \text{Sp}(2n, \mathbb{R})$, corresponding to the definition of closed characteristics on $S$ in Definition 1 of [15, Chap.V, §1] we say a nonconstant absolutely continuous curve $z : [0, T] \to \mathbb{R}^{2n}$ (for some $T > 0$) to be a generalized characteristic on $S$ if $z([0, T]) \subset S$ and $\dot{z}(t) \in JN_S(z(t))$ a.e., where $N_S(x) = \{ y \in \mathbb{R}^{2n} | \langle u - x, y \rangle \leq 0 \ \forall u \in D \}$ is the normal cone to $D$ at $x \in S$. Moreover, if $z$ satisfies $z(T) = \Psi z(0)$ additionally, we call $z$ a generalized $\Psi$-characteristic on $S$.

Clearly, if $S$ in (ii) is also $C^{1,1}$ then generalized $\Psi$-characteristics on $S$ are $\Psi$-characteristics up to reparametrization. The notion of generalized characteristics might be defined on general symplectic manifolds via nonsmooth analysis on manifolds, but this is outside the scope of this paper and would appear elsewhere.

The action of an absolutely continuous curve $x : [0, T] \to \mathbb{R}^{2n}$ is defined by

$$A(x) = \frac{1}{2} \int_0^T \langle -J \dot{x}, x \rangle dt.$$  \hspace{1cm} (1.2)

Denote

$$\Sigma^\Psi_S = \{ A(x) > 0 | x \text{ is a generalized } \Psi \text{-characteristic on } S \}.$$  \hspace{1cm} (1.3)

The above Clarke’s result may be formulated as: the boundary of a compact convex set $D$ in $(\mathbb{R}^{2n}, \omega_0)$ containing the origin in its interior carries a generalized $\Psi$-characteristic. Motivated by this and the Weinstein conjecture ([49]) we state the following generalized version of the latter.

**Question** $\Psi$. Let $S$ be a hypersurface of contact type in a symplectic manifold $(M, \omega)$ and $\Psi \in \text{Symp}(M, \omega)$ such that $S \cap \text{Fix}(\Psi) \neq \emptyset$. Under what condition does there exist a $\Psi$-characteristic on $S$?
This question is closely related to the following.

**Leaf-Wise Intersection Problem** Given a compact hypersurface \( S \) and a symplectomorphism \( \Psi \in \text{Symp}(M, \omega) \), under what conditions on \( \Psi \) and \( S \) does there exist a point \( x \in S \) such that \( \Psi x \) lies on the leaf \( L_S(x) \) through \( x \)? Such \( x \) is called a leaf-wise intersection point for \( \Psi \) on \( S \).

Such a question was first addressed by Moser [37]. Since then various forms or generalizations of it were studied. See [14, 17, 21], [1, §1.1] and [29, §1.4] and references therein for a brief history of these problems.

Actually, the above leaf-wise intersection question for a hypersurface \( S \) of contact type is slightly weaker than Question \( \Psi \). Indeed, it is clear that a \( \Psi \)-characteristic \( \gamma : [0, T] \to S \) yields a leaf-wise intersection point \( \gamma(0) \) for \( \Psi \) on \( S \). Conversely, if \( x \) is a leaf-wise intersection point for \( \Psi \) on \( S \), we take a smooth function \( H : M \to \mathbb{R} \) having \( S \) as a regular energy surface, and obtain \( L_S(x) = \{ \varphi^t(x) \mid t \in \mathbb{R} \} \) and so \( \Psi(x) = \varphi^\tau(x) \) for some \( \tau \in \mathbb{R} \), where \( \varphi^t \) is the Hamiltonian flow of \( H \). If \( \tau > 0 \), then \( [0, \tau] \ni t \mapsto \varphi^{-t}(x) \) satisfies \( \dot{y}(t) = -X_H(y(t)) = X_{-H}(y(t)) \) and so it is a \( \Psi \)-characteristic on \( S \). However, it is possible that \( \tau = 0 \), i.e., \( \Psi(x) = x \), and we cannot get a \( \Psi \)-characteristic on \( S \) in this case.

As applications of our generalized capacities, some answers to Question \( \Psi \) and the leaf-wise intersection question above are given in Corollaries 1.25, 1.26 and Sect. 1.3. There exist several methods to study the Weinstein conjecture, which were developed based on pseudo-holomorphic curve theory, for example, Gromov-Witten invariants, symplectic (co)homology and contact homology. Our future work is to develop the corresponding theories matching to Question \( \Psi \).

**Notations and conventions.** A domain in \( \mathbb{R}^m \) is a connected open subset of \( \mathbb{R}^m \). For \( r > 0 \) and \( p = (p_1, \ldots, p_m) \in \mathbb{R}^m \) we write

\[
B^m(p, r) = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^m \mid \sum_{i=1}^m (x_i - p_i)^2 < r^2 \right\},
\]

\[
B^m(r) := B^m(0, r) \quad \text{and} \quad B^m := B^m(1).
\]

For \( R > 0 \) we write as usual

\[
Z^{2n}(R) = \{ (q, p) \in \mathbb{R}^n \times \mathbb{R}^n \mid q_1^2 + p_1^2 < R^2 \}
\]

with respect to the symplectic coordinates \( (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \) of \((\mathbb{R}^{2n}, \omega_0)\).

**1.1 An Extension of the Hofer–Zehnder Symplectic Capacity**

Let us recall the definition of the Hofer–Zehnder symplectic capacity. Given a symplectic manifold \((M, \omega)\) let \( \mathcal{H}(M, \omega) \) denote the set of smooth functions \( H : M \to \mathbb{R} \) for which there exists a nonempty open subset \( U = U(H) \) and a compact subset \( K = K(H) \subset M \setminus \partial M \) such that

\[\text{Springer}\]
(i) \( H|_{U} = 0 \),
(ii) \( H|_{M \setminus K} = m(H) := \max H \),
(iii) \( 0 \leq H \leq m(H) \).

Denote by \( X_{H} \) the Hamiltonian vector field defined by \( \omega(X_{H}, \cdot) = -dH \). A function \( H \in \mathcal{H}(M, \omega) \) is called admissible if \( \dot{x} = X_{H}(x) \) has no nonconstant periodic solutions of period less than or equal to 1. Let \( \mathcal{H}_{ad}(M, \omega) \) be the set of admissible Hamiltonians on \((M, \omega)\). The Hofer–Zehnder symplectic capacity \( c_{HZ}(M, \omega) \) of \((M, \omega)\) was defined in \([26]\) by

\[
c_{HZ}(M, \omega) = \sup \{ \max H \mid H \in \mathcal{H}_{ad}(M, \omega) \}.
\]

This symplectic invariant may be used to establish the existence of closed characteristics on an energy surface, and \( c_{HZ}(M, \omega) < \infty \) implies that the Weinstein conjecture holds in \((M, \omega)\).

Given a \( \Psi \in \text{Symp}(M, \omega) \) with \( \text{Fix}(\Psi) \neq \emptyset \), let

\[
\mathcal{H}_{\Psi}^{\Psi}(M, \omega) = \{ H \in \mathcal{H}(M, \omega) \mid U \cap \text{Fix}(\Psi) \neq \emptyset \},
\]

where \( U = U(H) \) is as in (i)–(iii). We call \( H \in \mathcal{H}_{\Psi}^{\Psi}(M, \omega) \) \( \Psi \)-admissible if all solutions \( x : [0, T] \to M \) of the Hamiltonian boundary value problem

\[
\begin{cases}
\dot{x} = X_{H}(x), \\
x(T) = \Psi x(0)
\end{cases}
\]  

with \( 0 < T \leq 1 \) are constant. The set of all such \( \Psi \)-admissible Hamiltonians is denoted by \( \mathcal{H}_{ad}^{\Psi}(M, \omega) \). As an analogue of the Hofer–Zehnder capacity of \((M, \omega)\) we call

\[
c_{HZ}^{\Psi}(M, \omega) := \sup \{ \max H \mid H \in \mathcal{H}_{ad}^{\Psi}(M, \omega) \} \tag{1.5}
\]

\( \Psi \)-Hofer–Zehnder capacity (Abb., \( \Psi \)-HZ capacity) or Hofer–Zehnder capacity relative to \( \Psi \) of \((M, \omega)\). Moreover, for an open subset \( O \subset M \) with \( O \cap \text{Fix}(\Psi) \neq \emptyset \), we also define \( \Psi \)-HZ capacity of \( O \) by

\[
c_{HZ}^{\Psi}(O, \omega) = \sup \{ \max H \mid H \in \mathcal{H}_{ad}^{\Psi}(O, \omega) \}, \tag{1.6}
\]

where \( \mathcal{H}_{ad}^{\Psi}(O, \omega) \) consists of \( H \in \mathcal{H}(O, \omega) \) such that \( U(H) \cap \text{Fix}(\Psi) \neq \emptyset \) and that the boundary value problem (1.4) has a nonconstant solution \( x : [0, T] \to O \) implies \( T > 1 \). It is not hard to check that \( c_{HZ}^{\Psi}(O, \omega) = c_{HZ}(O, \omega) \) if \( \Psi(O) = O \), where \( \Psi|_{O} \) is viewed as an element in \( \text{Symp}(O, \omega) \). Moreover, if \( \Psi = id_{M} \) we have clearly \( c_{HZ}^{\Psi}(M, \omega) = c_{HZ}(M, \omega) \) and \( c_{HZ}^{\Psi}(O, \omega) = c_{HZ}(O, \omega) \) for any open subset \( O \subset M \).

As \( c_{HZ} \), it follows immediately from the above definition that \( c_{HZ}^{\Psi} \) has inner regularity, i.e., for any precompact open subset \( O \subset M \) with \( O \cap \text{Fix}(\Psi) \neq \emptyset \), we have

\[
c_{HZ}^{\Psi}(O, \omega) = \sup \{ c_{HZ}^{\Psi}(K, \omega) \mid K \text{ open, } K \cap \text{Fix}(\Psi) \neq \emptyset, \overline{K} \subset O \}. \tag{1.7}
\]
It follows immediately from the definition that $c_{HZ}^\Psi$ has the following properties:

**Proposition 1.2**

(i) (Conformality). $c_{HZ}^\Psi(M, \alpha \omega) = \alpha c_{HZ}^\Psi(M, \omega)$ for any $\alpha \in \mathbb{R}_{>0}$, and $c_{HZ}^\Psi^{-1}(M, \alpha \omega) = -\alpha c_{HZ}^\Psi(M, \omega)$ for any $\alpha \in \mathbb{R}_{<0}$.

(ii) (Monotonicity). Suppose that $\Psi_i \in \text{Symp}(M_i, \omega_i)$ $(i = 1, 2)$. If there exists a symplectic embedding $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ of codimension zero such that $\phi \circ \Psi_1 = \Psi_2 \circ \phi$, then for open subsets $O_i \subset M_i$ with $O_1 \cap \text{Fix}(\Psi_i) \neq \emptyset$ $(i = 1, 2)$ and $\phi(O_1) \subset O_2$, it holds that $c_{HZ}^\Psi(O_1, \omega_1) \leq c_{HZ}^\Psi(O_2, \omega_2)$.

Clearly Proposition 1.2(ii) shows that $c_{HZ}^\Psi(M, \omega)$ is invariant for the centralizer of $\Psi$ in $\text{Symp}(M, \omega)$, denoted by $\text{Symp}_\Psi(M, \omega) := \{ \phi \in \text{Symp}(M, \omega) \mid \phi \circ \Psi = \Psi \circ \phi \}$ (i.e., the stabilizer at $\Psi$ for the adjoint action on $\text{Symp}(M, \omega)$). Moreover, for any $\Psi \in \text{Symp}(M, \omega)$ and any open subset $O \subset M$ with $O \cap \text{Fix}(\Psi) \neq \emptyset$, (ii) also implies

$$c_{HZ}^\Psi(O, \omega) = c_{HZ}^{\phi \circ \Psi \circ \phi^{-1}}(\Phi(O), \omega) \quad \forall \Phi \in \text{Symp}(M, \omega).$$

(1.8)

In this paper, we mainly consider the standard linear symplectic space $(\mathbb{R}^{2n}, \omega_0)$ and its linear symplectomorphisms. We make the following conventions: each symplectic matrix $\Psi \in \text{Sp}(2n, \mathbb{R})$ is identified with the linear symplectomorphism on $(\mathbb{R}^{2n}, \omega_0)$ which has the representing matrix $\Psi$ under the standard symplectic basis of $(\mathbb{R}^{2n}, \omega_0)$, $(e_1, \ldots, e_n, f_1, \ldots, f_n)$, where the $i$-th (resp. $(n+i)$-th) coordinate of $e_i$ (resp. $f_i$) is 1 and other coordinates are zero.

The following continuity holds for $c_{HZ}^\Psi$ where $\Psi \in \text{Sp}(2n, \mathbb{R})$.

**Proposition 1.3** For a bounded convex domain $A \subset \mathbb{R}^{2n}$, suppose that $\Psi \in \text{Sp}(2n, \mathbb{R})$ satisfies $A \cap \text{Fix}(\Psi) \neq \emptyset$. Then for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for any bounded convex domain $O \subset \mathbb{R}^{2n}$ intersecting with Fix($\Psi$), it holds that

$$|c_{HZ}^\Psi(O, \omega_0) - c_{HZ}^\Psi(A, \omega_0)| \leq \varepsilon$$

(1.9)

provided that $A$ and $O$ have Hausdorff distance $d_H(A, O) < \delta$.

**Proof** Let $p \in A \cap \text{Fix}(\Psi)$. Replacing $A$ and $O$ with $A - p$ and $O - p$ respectively, we may assume $0 \in A$. For any $0 < \varepsilon < 1$, by [43, Lemma 1.8.14] there exists $\delta > 0$ such that any bounded convex domain $O \subset \mathbb{R}^{2n}$ with $d_H(A, O) < \delta$ satisfies

$$(1 - \varepsilon)A \subset O \subset (1 + \varepsilon)A.$$  

Then the result easily follows from Proposition 1.2(i) and (ii).  

As in [5] we can also get more results on continuity of $c_{HZ}^\Psi$. For example, as in the proof of [5, Proposition 2.3] we have the following outer regularity of $c_{HZ}^\Psi$. Let $S$ be a smooth connected compact hypersurface of restricted contact type in $\mathbb{R}^{2n}$ with respect to a global Liouville vector field $X$ on $\mathbb{R}^{2n}$ and let $B_S$ be the bounded...
component of \( \mathbb{R}^{2n} \setminus \mathcal{S} \). Suppose that \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) satisfies \( B_{\mathcal{S}} \cap \text{Fix}(\Psi) \neq \emptyset \) and that \( X(\Psi(x)) = \Psi(X(x)) \) for all \( x \) near \( \mathcal{S} \). Then

\[
c_{\mathcal{H}}(B_{\mathcal{S}}, \omega_0) = \inf \{ c_{\mathcal{H}}(V, \omega_0) \mid V \subset \mathbb{R}^{2n} \text{ is open and } \overline{B_{\mathcal{S}}} \subset V \}.
\]

One of the main results of this paper is the following analogue of the representation formula for \( c_{\mathcal{H}} \) due to Hofer and Zehnder [26, Proposition 4].

**Theorem 1.4** For \( \Psi \in \text{Sp}(2n, \mathbb{R}) \), let \( \mathcal{D} \subset \mathbb{R}^{2n} \) be a convex bounded domain containing a fixed point \( p \) of \( \Psi \) and with boundary \( \mathcal{S} = \partial \mathcal{D} \). Then there is a generalized \( \Psi \)-characteristic \( x^* \) on \( \mathcal{S} \) such that

\[
A(x^*) = \min \{ A(x) > 0 \mid x \text{ is a generalized } \Psi \text{-characteristic on } \mathcal{S} \} \quad \text{(1.10)}
\]

\[
= c_{\mathcal{H}}(\mathcal{D}, \omega_0). \quad \text{(1.11)}
\]

If \( \mathcal{S} \) is of class \( C^{1,1} \), (1.10) and (1.11) become

\[
c_{\mathcal{H}}(\mathcal{D}, \omega_0) = A(x^*) = \inf \{ A(x) > 0 \mid x \text{ is a } \Psi \text{-characteristic on } \mathcal{S} \}. \quad \text{(1.12)}
\]

The proof of this theorem is in Sect. 3.

**Remark 1.5** A generalized \( \Psi \)-characteristic \( x^* \) on \( \mathcal{S} \) satisfying (1.10) and (1.11) is called a \( c_{\mathcal{H}} \)-carrier for \( \mathcal{D} \). The proof of Theorem 1.4 also shows that a generalized \( \Psi \)-characteristic on \( \mathcal{S} \) is a \( c_{\mathcal{H}} \)-carrier for \( \mathcal{D} \) if and only if it may be reparametrized as a solution \( x : [0, T] \to \mathcal{S} \) of \( -Jx^*(t) \in \partial H(x^*(t)) \) with \( T = c_{\mathcal{H}}(\mathcal{D}, \omega_0) \) and satisfying \( x(T) = \Psi(x(0)) \), where \( H = j_{\mathcal{D}}^2 \). Since \( \{ \partial H(x) \mid x \in \mathcal{S} \} \) is a bounded set in \( \mathbb{R}^{2n} \), it follows from the Arzela-Ascoli theorem that all \( c_{\mathcal{H}} \)-carriers for \( \mathcal{D} \) form a compact subset in \( C^0([0, T], \mathcal{S}) \) (and \( C^1([0, T], \mathcal{S}) \) if \( \mathcal{S} \) is \( C^1 \)), where \( T = c_{\mathcal{H}}(\mathcal{D}, \omega_0) \).

**Remark 1.6** Clearly, Theorem 1.4 implies the Clarke’s main result in [10]. When \( \Psi = I_{2n} \) and the boundary \( \mathcal{S} = \partial \mathcal{D} \) is smooth, Hofer and Zehnder [26, Proposition 4] proved Theorem 1.4, and then Künzle [31–33] removed the smoothness assumption of \( \mathcal{S} \) (also see Artstein-Avidan and Ostrover [4] for a different proof).

Fix a symplectic matrix \( \Psi \in \text{Sp}(2n, \mathbb{R}) \). Define

\[
g^\Psi : \mathbb{R} \to \mathbb{R}, s \mapsto \det(\Psi - e^{sJ}), \quad \text{(1.13)}
\]

where \( e^{sJ} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k J^k \). The set of zero points of \( g^\Psi \) in \( (0, 2\pi] \) is a nonempty finite set. Denote by

\[
t(\Psi) \quad \text{(1.14)}
\]

the smallest zero point of \( g^\Psi \) in \( (0, 2\pi] \). Then \( t(I_{2n}) = 2\pi \) and \( t(-I_{2n}) = \pi \). (See Lemma A.1.)

As a consequence of Theorem 1.4 we get

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Corollary 1.7 Let $E(q) := \{ z \in \mathbb{R}^{2n} \mid q(z) < 1 \}$ be the ellipsoid given by a positive definite quadratic form $q(z) = \frac{1}{2} (S_z z)$ on $\mathbb{R}^{2n}$, where $S \in \mathbb{R}^{2n \times 2n}$ is a positive definite symmetric matrix. Then for any $\Psi \in \text{Sp}(2n, \mathbb{R})$ there holds

$$\tag{1.15} c^\Psi_{HZ}(E(q), \omega_0) = \inf \{ T > 0 \mid \det(\exp(TJS) - \Psi) \neq 0 \}$$

$$\leq \frac{r_n^2}{2} \inf T \Phi \Psi \Phi^{-1},$$

where $\Phi \in \text{Sp}(2n, \mathbb{R})$ satisfies $\Phi(E(q)) = \{ z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j/r_j|^2 < 1 \}$ with $0 < r_1 \leq r_2 \leq \cdots \leq r_n$. In particular, (1.15) implies

$$\tag{1.16} c^\Psi_{HZ}(B^{2n}, \omega_0) = \frac{t(\Phi \Psi \Phi^{-1})}{2}.$$  

**Proof** Since the Hamiltonian vector field of the quadratic form $q(z)$ is $X_{q}(z) = JSz$, every $\Psi$-characteristic on $\partial E(q)$ may be parameterized as the form $[0, T] \ni t \mapsto \exp(tJS)z_0 \in \partial E(q)$, where $q(z_0) = 1$ and $\exp(TJS)z_0 = \Psi z_0$. Hence (1.15) follows from (1.10) and (1.11) immediately. Observe that $B^{2n}(1) = E(q)$ with $S = 2I_{2n}$. The definition of $t(\Psi)$ and (1.15) lead to (1.17) directly.

Note that for $\Psi \in \text{Sp}(2n, \mathbb{R})$ and an open set $O \subset (\mathbb{R}^{2n}, \omega_0)$ containing the origin, (i)–(ii) of Proposition 1.2 imply

$$\tag{1.18} c^\Psi_{HZ}(\alpha O, \omega_0) = \alpha^2 c^\Psi_{HZ}(O, \omega_0), \quad \forall \alpha \geq 0.$$  

Since

$$\Phi(E(q)) = \left\{ z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j/r_j|^2 < 1 \right\} \subset B^{2n}(0, r_n),$$

it follows from (1.18), Proposition 1.2(ii) and (1.17) that

$$c^\Psi_{HZ}(E(q), \omega_0) = c^\Phi_{HZ}(\Phi(E(q)), \omega_0)$$

$$\leq c^\Phi_{HZ}(B^{2n}(r_n), \omega_0) = \frac{r_n^2}{2} t(\Phi \Psi \Phi^{-1}).$$  

(1.16) follows immediately. \hfill \Box

Here is another important consequence of Theorem 1.4.

Corollary 1.8 Let $\Psi \in \text{Sp}(2n, \mathbb{R})$ and $D \subset \mathbb{R}^{2n}$ be a convex bounded domain with boundary $S = \partial D$. Suppose that $p \in D$ is a fixed point of $\Psi$.

(i) If $D$ contains a ball $B^{2n}(p, r)$, then for any generalized $\Psi$-characteristic $x$ on $S$ with positive action it holds that

$$A(x) \geq \frac{r^2}{2} t(\Psi).$$  

(1.19)
(ii) If \( D \subset B^{2n}(p, R) \), there exists a generalized \( \Psi \)-characteristic \( x^* \) on \( S \) such that
\[
0 < A(x^*) \leq \frac{R^2}{2} t(\Psi).
\]  

(1.20)

**Remark 1.9** When \( \Psi = I_{2n} \) and \( S \) is of class \( C^1 \), (i) and (ii) were obtained respectively by Croke and Weinstein in [12, Theorem C] and by Ekeland in Proposition 5 of [15, Chap.5, §1]. Then Künzle [32, 33] removed the \( C^1 \)-smoothness assumption of \( S \).

**Proof of Corollary 1.8** By a translation transformation (see the beginning of Sect. 3), we only need to consider the case \( p = 0 \).

For (i) of Corollary 1.8, \( B^{2n}(0, r) \subset D \) implies \( c_{HZ}^{\Psi}(B^{2n}(0, r), \omega_0) \leq c_{HZ}^{\Psi}(D, \omega_0) \). Moreover, \( c_{HZ}^{\Psi}(B^{2n}(0, r), \omega_0) = \frac{r^2}{2} t(\Psi) \) by (1.18) and (1.17), and \( c_{HZ}^{\Psi}(D, \omega_0) \) is equal to the minimum of actions of all \( \Psi \)-characteristics with positive actions on \( S \) by Theorem 1.4. Thus (1.19) follows immediately.

Similarly, for (ii) of Corollary 1.8 we have
\[
c_{HZ}^{\Psi}(D, \omega_0) \leq c_{HZ}^{\Psi}(B^{2n}(0, R), \omega_0) = \frac{R^2}{2} t(\Psi).
\]

By Theorem 1.4 there exists a generalized \( \Psi \)-characteristic \( x^* \) on \( S \) such that \( A(x^*) = c_{HZ}^{\Psi}(D, \omega_0) \).

\( \square \)

### 1.2 An Extension of the Ekeland–Hofer Capacity

For each symplectic matrix \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) and \( s \geq 0 \), there is a Hilbert space \( E_s^\Psi \) such that \( E_s^0 = L^2([0, 1], \mathbb{R}^{2n}) \) and for \( s \geq \frac{1}{2} \), \( x \in E_s^\Psi \) implies that \( x \) is continuous and satisfies \( x(1) = \Psi x(0) \). \( E_{\frac{1}{2}}^\Psi \) is the variational space we need in this article. When \( \Psi = I_{2n} \), \( E_{\frac{1}{2}}^I \) is exactly \( H^1(S^1, \mathbb{R}^{2n}) \). For details see Sect. 2.

In this subsection we fix a \( \Psi \in \text{Sp}(2n, \mathbb{R}) \). Then \( E := E_{\frac{1}{2}}^\Psi \) has an orthogonal splitting
\[
E = E^- \oplus E^0 \oplus E^+
\]
(see (2.11)). We closely follow Sikorav’s approach [44] to the Ekeland–Hofer capacity in [18] and construct an analogue of the classical Ekeland–Hofer capacity.

**Definition 1.10** A continuous map \( \gamma : E \to E \) is called an **admissible deformation** if there exists a homotopy \( (\gamma_\alpha)_{0 \leq \alpha \leq 1} \) such that \( \gamma_0 = \text{id}, \gamma_1 = \gamma \) and satisfies

(i) \( \forall u \in [0, 1], \gamma_\alpha(E\setminus(E^- \oplus E^0)) \subset E\setminus(E^- \oplus E^0) \), i.e. for any \( x \in E \) such that \( x^+ \neq 0 \), there holds \( \gamma_\alpha(x)^+ \neq 0 \).

(ii) \( \gamma_\alpha(x) = a(x, u)x^+ + b(x, u)x^0 + c(x, u)x^- + K(x, u) \), where \( (a, b, c, K) \) is a continuous map from \( E \times [0, 1] \) to \( (0, +\infty)^3 \times E \) and maps bounded sets to precompact sets.
Let $\Gamma$ be the set of all admissible deformations. For $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0})$ satisfying:

(H1) $\text{Int}(H^{-1}(0)) \neq \emptyset$ and contains a fixed point of $\Psi$,
(H2) there exists $z_0 \in \text{Fix}(\Psi)$, real numbers $a > \text{f}(\Psi)$ and $b$ such that $H(z) = a|z|^2 + \langle z, z_0 \rangle + b$ outside a compact subset of $\mathbb{R}^{2n}$.

we define $\Phi_H : \mathbb{E} \to \mathbb{R}$ by

$$
\Phi_H(x) = \frac{1}{2}(\|x^+\|_2^2 - \|x^-\|_2^2) - \int_0^1 H(x(t)) dt, \quad (1.21)
$$

and the $\Psi$-\textit{capacity} of $H$ by

$$
c_{\Psi EH}(H) = \sup_{\gamma \in \Gamma} \inf_{x \in \gamma(S^+)} \Phi_H(x), \quad \text{where} \quad S^+ = \{x \in \mathbb{R}^n_+ | \|x\| = 1\}. \quad (1.22)
$$

Then (H2) implies $c_{\Psi EH}(H) < +\infty$, and the conditions (H1)–(H2) imply $c_{\Psi EH}(H) > 0$ if $H$ is smooth. (See Propositions 4.2, 4.3.)

It is easy to prove the following.

\textbf{Proposition 1.11} \textit{Let $H$, $K \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0})$ satisfy (H1) and (H2).}

(i) \textit{(Monotonicity)} If $H \leq K$ then $c_{\Psi EH}(H) \geq c_{\Psi EH}(K)$.

(ii) \textit{(Continuity)} $|c_{\Psi EH}(H) - c_{\Psi EH}(K)| \leq \sup\{|H(z) - K(z)| | z \in \mathbb{R}^{2n}\}$.

(iii) \textit{(Homogeneity)} $c_{\Psi EH}(\lambda^2 H(\cdot/\lambda)) = \lambda^2 c_{\Psi EH}(H)$ for $\lambda \neq 0$.

Let

$$
\mathcal{F}(\mathbb{R}^{2n}) = \{H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0}) | H \text{ satisfies (H2)}\}, \quad (1.23)
$$

$$
\mathcal{F}(\mathbb{R}^{2n}, B) = \{H \in \mathcal{F}(\mathbb{R}^{2n}) | H \text{ vanishes near } B\} \quad (1.24)
$$

for each $B \subset \mathbb{R}^{2n}$ such that $B \cap \text{Fix}(\Psi) \neq \emptyset$. We define

$$
c_{\Psi EH}(B) = \inf\{c_{\Psi EH}(H) | H \in \mathcal{F}(\mathbb{R}^{2n}, B)\} \in [0, +\infty) \quad (1.25)
$$

if $B$ is bounded and $B \cap \text{Fix}(\Psi) \neq \emptyset$, and

$$
c_{\Psi EH}(B) = \sup\{c_{\Psi EH}(B') | B' \subset B, \ B' \text{ is bounded and } B' \cap \text{Fix}(\Psi) \neq \emptyset\} \quad (1.26)
$$

if $B$ is unbounded and $B \cap \text{Fix}(\Psi) \neq \emptyset$. We call $c_{\Psi EH}(B)$ in (1.25) and (1.26) \textit{$\Psi$-Ekeland–Hofer capacity} (Abb., $\Psi$-EH capacity) or \textit{Ekeland–Hofer capacity relative to $\Psi$} of $B$.

We say $H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0})$ to be $\Psi$-\textit{nonresonant} if it satisfies (H2) with $\text{det}(e^{2\alpha J} - \Psi) \neq 0$. For each $B \subset \mathbb{R}^{2n}$ such that $B \cap \text{Fix}(\Psi) \neq \emptyset$ we write

$$
\mathcal{E}(\mathbb{R}^{2n}, B, \Psi) = \{H \in \mathcal{F}(\mathbb{R}^{2n}, B) | H \text{ is } \Psi\text{-nonresonant}\}.
$$
Note that each \( H \in \mathcal{E}(\mathbb{R}^{2n}, B, \Psi) \) satisfies (H1) and that \( \mathcal{E}(\mathbb{R}^{2n}, B, \Psi) \) is a **cofinal family** of \( \mathcal{F}(\mathbb{R}^{2n}, B) \), that is, for any \( H \in \mathcal{F}(\mathbb{R}^{2n}, B) \) there exists \( G \in \mathcal{E}(\mathbb{R}^{2n}, B, \Psi) \) such that \( G \geq H \).

**Remark 1.12** (i) \( c_{\psi}^{\Psi}(B) = c_{\psi}^{\Psi}(\bar{B}) \).

(ii) \( \mathcal{F}(\mathbb{R}^{2n}, B) \) in (1.25) and (1.26) can be replaced by its cofinal subset \( \mathcal{E}(\mathbb{R}^{2n}, B, \Psi) \), and can also be replaced by a smaller cofinal subset \( \mathcal{E}(\mathbb{R}^{2n}, B, \Psi) \cap C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0}) \).

(iii) \( c_{\psi}^{\Psi}(B + w) = c_{\psi}^{\Psi}(B) \) \( \forall w \in \text{Fix}(\Psi) \), where \( B + w = \{ z + w | z \in B \} \).

The following properties of the \( \Psi\text{-EH} \) capacity of subsets in \( \mathbb{R}^{2n} \) which contains fixed points of \( \Psi \) follow easily from its definition and Proposition 1.11.

**Proposition 1.13** Let \( B \subset B' \subset \mathbb{R}^{2n} \). Assume in addition that \( B \cap \text{Fix}(\Psi) \neq \emptyset \). Then

(i) (Monotonicity) \( c_{\psi}^{\Psi}(B) \leq c_{\psi}^{\Psi}(B') \).

(ii) (Conformality) \( c_{\psi}^{\Psi}(\lambda B) = \lambda^2 c_{\psi}^{\Psi}(B) \), \( \forall \lambda > 0 \).

(iii) (Exterior regularity) \( c_{\psi}^{\Psi}(B) = \inf \{ c_{\psi}^{\Psi}(U_\epsilon(B)) | \epsilon > 0 \} \), where \( U_\epsilon(B) \) is the \( \epsilon \)-neighborhood of \( B \).

Moreover, let \( \mathcal{S}, B_{\mathcal{S}} \subset \mathbb{R}^{2n} \) and \( X, \Psi \) be as below the proof of Proposition 1.3. Using a proof similar to that of [5, Proposition 2.3] we have the following inner regularity of \( c_{\psi}^{\Psi} \):

\[
c_{\psi}^{\Psi}(B_{\mathcal{S}}) = \sup \{ c_{\psi}^{\Psi}(V) | V \subset \mathbb{R}^{2n} \text{ is open and } \overline{V} \subset B_{\mathcal{S}} \}.
\]

The following theorem gives a variational explanation for \( c_{\psi}^{\Psi} \), which is important for proofs of several subsequent theorems.

**Theorem 1.14** If \( H \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}) \) satisfies (H1)–(H2), and is also \( \Psi \)-nonresonant, then \( c_{\psi}^{\Psi}(H) \) is a positive critical value of \( \Phi_H \) on \( \mathbb{E} \).

The proof of the above theorem is closely related to Sikorav’s approach in [44] and it will be completed by several propositions in Sect. 4.

For \( c_{\psi}^{\Psi} \) we have the following representation formula, which generalizes the one for \( c_{\psi}^{\Psi} \) in [18, 19, 44]. We give its proof in Sect. 5.

**Theorem 1.15** Let \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) and \( D \subset \mathbb{R}^{2n} \) be a convex bounded domain with \( C^{1,1} \) boundary \( S = \partial D \) and containing a fixed point \( p \) of \( \Psi \) in the closure of \( D \). Then there exists a \( \Psi \)-characteristic \( x^* \) on \( \partial D \) such that

\[
A(x^*) = \min \{ A(x) > 0 | x \text{ is a } \Psi \text{-characteristic on } S \} = c_{\psi}^{\Psi}(D). \tag{1.27}
\]

Moreover, if both \( \partial D \) and \( D \) contain fixed points of \( \Psi \), then

\[
c_{\psi}^{\Psi}(D) = c_{\psi}^{\Psi}(\partial D). \tag{1.28}
\]
Remark 1.16 An approximation argument shows that the condition “$c^{1,1}$” for $D$ in Theorem 1.15 is not needed if “$\Psi$-characteristic” is replaced by “generalized $\Psi$-characteristic” (for details see Sect. 3.4). Thus this and Theorem 1.4 imply $c_{EH}^{\Psi}(D) = c_{HZ}^{\Psi}(D, \omega_0)$ for any convex bounded domain $D \subset \mathbb{R}^{2n}$ containing a fixed point $p$ of $\Psi$. It follows from the definitions of both $c_{EH}^{\Psi}$ and $c_{HZ}^{\Psi}$ that $c_{EH}^{\Psi}(D) = c_{HZ}^{\Psi}(D, \omega_0)$ for any convex domain $D \subset \mathbb{R}^{2n}$ containing a fixed point $p$ of $\Psi$, not necessarily bounded. Hereafter we shall use $c_{ZH}^{\Psi}(D)$ to denote $c_{EH}^{\Psi}(D) = c_{HZ}^{\Psi}(D, \omega_0)$ without special statements. In this case a $c_{ZH}^{\Psi}$-carrier is also called a $c_{EH}^{\Psi}$-carrier.

Remark 1.17 Recently, Artstein-Avidan and Ostrover [3] established a Brunn-Minkowski type inequality for the Ekeland-Hofer-Zehnder symplectic capacity $c_{EHZ}$ of convex domains, and in [4] used it to derive several very interesting bounds and inequalities for the length of the shortest periodic billiard trajectory in a smooth convex body $\Omega$. In [28] we showed that a Brunn-Minkowski type inequality for the capacity $c_{EHZ}$ of convex domains is still true, and also proved some corresponding results for a larger class of (non-periodic) billiard trajectories in a smooth convex body in $\mathbb{R}^n$. These will be published elsewhere.

For integers $n_1 > 0$ and $n_2 > 0$, let $n = n_1 + n_2$ and

$$
\omega_0^{(1)} = \sum_{i=1}^{n_1} dq_i \land dp_i \quad \text{and} \quad \omega_0^{(2)} = \sum_{i=n_1+1}^{n} dq_i \land dp_i.
$$

For $S_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \text{Sp}(2n_i, \mathbb{R})$ with $A_i, B_i, C_i, D_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, 2$, denote

$$
S_1 \oplus S_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.
$$

Then $S_1 \oplus S_2 \in \text{Sp}(2n, \mathbb{R})$. Furthermore, $(S_1 \oplus S_2) \oplus S_3 = S_1 \oplus (S_2 \oplus S_3)$ for $S_i \in \text{Sp}(2n_i, \mathbb{R})$, $i = 1, 2, 3$. Thus we can define $S_1 \oplus \cdots \oplus S_k$ for $S_i \in \text{Sp}(2n_i, \mathbb{R})$, $i = 1, \ldots, k$.

As a generalization of [44, Th. 6.6.1] (also see [19, Prop.5] for special cases) we have the following theorem, which will be proved in Sect. 5.

**Theorem 1.18** Let $\Psi = \Psi_1 \oplus \cdots \oplus \Psi_k$, where $\Psi_i \in \text{Sp}(2n_i, \mathbb{R})$, $i = 1, \ldots, k$. Then for compact convex subsets $D_i \subset \mathbb{R}^{2n_i}$ containing fixed points of $\Psi_i$ $(1 \leq i \leq k)$ it holds that

$$
c_{EH}^{\Psi}(D_1 \times \cdots \times D_k) = \min_i c_{EH}^{\Psi_i}(D_i). \quad (1.29)
$$

Moreover, if both $\partial D_i$ and $\text{Int}(D_i)$ contain fixed points of $\Psi_i$ for each $i = 1, \ldots, k$, then

$$
c_{EH}^{\Psi}(\partial D_1 \times \cdots \times \partial D_k) = \min_i c_{EH}^{\Psi_i}(D_i). \quad (1.30)
$$
An immediate consequence is:

**Corollary 1.19** For \( \Psi := \Psi_1 \oplus \cdots \oplus \Psi_n \) where each \( \Psi_i \in \text{Sp}(2, \mathbb{R}) \) has the eigenvalue 1 and \( T^n = S^1(r_1) \times \cdots \times S^1(r_n) \subset \mathbb{R}^{2n} \), it holds that

\[
c_{\text{EH}}^{\Psi}(T^n) = \frac{1}{2} \inf_i \left\{ t(\Psi_i)r_i^2 \right\}.
\]

**Remark 1.20** Recall that a symplectic matrix \( M \in \text{Sp}(2, \mathbb{R}) \) has nonzero fixed points if and only if 1 is its unique eigenvalue. All such symplectic matrices have the form

\[
\begin{pmatrix}
  r & z \\
  z(1+z^2)/r & \cos \theta - \sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\]  

where \( r \in (0, \infty), z \in \mathbb{R} \) and \( \theta \in \mathbb{R}/(2\pi \mathbb{Z} - \pi) \) satisfy \((r^2 + z^2 + 1) \cos \theta = 2r\), see [34, page 48].

Note that (1.17) is a generalization for the normality \( c_{\text{EH}}(B^{2n}(1)) = c_{\text{HZ}}(B^{2n}(1), \omega_0) = \pi \). In order to get some kind of results similar to the normality \( c_{\text{EH}}(Z^{2n}(1)) = c_{\text{HZ}}(Z^{2n}(1), \omega_0) = \pi \) we need to make stronger restrictions for the symplectic matrix \( \Psi \in \text{Sp}(2n, \mathbb{R}) \).

**Theorem 1.21** For \( \Psi \in \text{Sp}(2n, \mathbb{R}) \), suppose that there exists \( P \in \text{Sp}(2n, \mathbb{R}) \) such that \( P^{-1}\Psi P = S_1 \oplus S_2 \) for some \( S_1 \in \text{Sp}(2, \mathbb{R}) \) and \( S_2 \in \text{Sp}(2n-2, \mathbb{R}) \). Then with \( W_{\Psi}^{2n}(1) := PZ^{2n}(1) \) it holds that

\[
c_{\text{HZ}}(W_{\Psi}^{2n}(1), \omega_0) = c_{\text{HZ}}^{S_1 \oplus S_2}(Z^{2n}(1), \omega_0) = c_{\text{EH}}^{S_1 \oplus S_2}(Z^{2n}(1)) = \frac{1}{2} t(S_1).
\]

**Proof** The first (resp. second) equality follows from (1.8) (resp. Remark 1.16) directly. Take \( n_1 = 1, n_2 = n - 1 \) and \( D_1 = B^2(1) \subset \mathbb{R}^2(q_1, p_1) \) and \( D_2 = \mathbb{R}^{2n-2}(q_2, \ldots, q_n; p_2, \ldots, p_n) \) in (1.29) and let \( \omega_0^{(1)} = dq_1 \wedge dp_1 \) and \( \omega_0^{(2)} = \sum_{i=2}^n dq_i \wedge dp_i \). We get

\[
c_{\text{EH}}^{S_1 \oplus S_2}(Z^{2n}(1)) = \min\{c_{\text{EH}}^S(B^2(1)), c_{\text{EH}}^S(\mathbb{R}^{2n-2})\} = c_{\text{EH}}^S(B^2(1)) = \frac{1}{2} t(S_1).
\]

\[\square\]

**Remark 1.22** For \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) satisfying the conditions in Theorem 1.21, there exists an unbounded domain \( W_{\Psi}^{2n}(1) := PZ^{2n}(1) \) with finite \( \Psi \)-HZ capacity.

**Remark 1.23** Every orthogonal symplectic matrix \( \Psi \) satisfies the conditions in Theorem 1.21. In fact, a symplectic matrix \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) is orthogonal if and only if

\[
\Psi = \begin{pmatrix} U & -V \\ V & U \end{pmatrix}
\]  

(1.32)
where $U, V \in \text{GL}(n, \mathbb{R})$ satisfies the $U + \sqrt{-1}V \in U(n, \mathbb{C})$. Moreover, the unitary matrix $U + \sqrt{-1}V$ can be diagonalized by another unitary matrix $S = T + \sqrt{-1}Q \in U(n, \mathbb{C})$, i.e.,

$$U + \sqrt{-1}V = S \text{diag}(e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_n}) S^{-1},$$  

where $0 < \theta_1 \leq \cdots \leq \theta_n \leq 2\pi$ are uniquely determined by $U + \sqrt{-1}V$. Then the orthogonal symplectic matrix

$$P := \begin{pmatrix} T & -Q \\ Q & T \end{pmatrix}$$  

satisfies

$$\Psi = P \tilde{\Psi} P^{-1},$$  

where

$$\tilde{\Psi} = \begin{pmatrix} \text{diag}(\cos \theta_1, \ldots, \cos \theta_n) & -\text{diag}(\sin \theta_1, \ldots, \sin \theta_n) \\ \text{diag}(\sin \theta_1, \ldots, \sin \theta_n) & \text{diag}(\cos \theta_1, \ldots, \cos \theta_n) \end{pmatrix}.$$  

Note that $\tilde{\Psi}_1 \oplus \cdots \oplus \tilde{\Psi}_n$, where

$$\tilde{\Psi}_i = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix}, \quad i = 1, \ldots, n.$$  

Then by Theorem 1.21 we obtain for $W_\Psi^{2n}(1) := P Z^{2n}(1)$

$$c_{HZ}(W_\Psi^{2n}(1), \omega_0) = c_{HZ}(Z^{2n}(1), \omega_0) = \frac{1}{2} \ell(\tilde{\Psi}_1) = \frac{\theta_1}{2}.$$  

Moreover, Lemma A.3 leads to $\ell(\tilde{\Psi}) = \ell(\Psi) = \theta_1$. Therefore (1.37) becomes

$$c_{HZ}(W_\Psi^{2n}(1), \omega_0) = \frac{\theta_1}{2} = \frac{\ell(\Psi)}{2}.$$  

A compact boundaryless smooth connected hypersurface $S$ in $(\mathbb{R}^{2n}, \omega_0)$ is said of **restricted contact type** if there exists a globally defined Liouville vector field $X$ (i.e. a smooth vector field $X$ on $\mathbb{R}^{2n}$ satisfying $L_X \omega_0 = \omega_0$) which is transversal to $S$. The following is a generalization of [18, Proposition 6]. Its proof is given in Sect. 6.

**Theorem 1.24** Let $\Psi \in \text{Sp}(2n, \mathbb{R})$ and $S$ be a hypersurface of restricted contact type in $(\mathbb{R}^{2n}, \omega_0)$ that admits a globally defined Liouville vector field $X$ transversal to it such that

$$X(\Psi z) = \Psi X(z), \quad \forall z \in \mathbb{R}^{2n}.$$  

\[ \square \] Springer
Suppose that $S$ contains a fixed point of $\Psi$. Then

$$c_{\Psi}^{E_{\text{EH}}}(B) = c_{\Psi}^{E_{\text{EH}}}(S) \in \Sigma^\Psi,$$

where $B$ is the bounded component of $\mathbb{R}^{2n} \setminus S$.

Bates [6] extended [18, Proposition 6] to certain domains whose boundaries are not of restricted contact type. The corresponding generalizations of Theorem 1.24 are also possible.

**Corollary 1.25** Under the assumptions of Theorem 1.24, $S$ carries a $\Psi$-characteristic $\gamma$ with action $c_{\Psi}^{E_{\text{EH}}}(S)$. In particular, there exists a leaf-wise intersection point $\gamma(0) \in S$ for $\Psi$.

Therefore we get a positive answer to Question $\Psi$ under the assumptions of Theorem 1.24. For a centrally symmetric hypersurface $S$ of restricted contact type in $(\mathbb{R}^{2n}, \omega_0)$, if $[0, 1] \ni t \mapsto \Psi_t$, is an isotopy of the identity in $\text{Symp}(\mathbb{R}^{2n}, \omega_0)$ which is odd, i.e., $\Psi_t(-x) = -\Psi_t(x)$ for all $(t, x) \in [0, 1] \times \mathbb{R}^{2n}$, Ekeland and Hofer proved in [17] that $S$ carries a leaf-wise intersection point for $\Psi_1$. Moreover, if this $S$ is also star-shaped, Albers and Frauenfelder [2] strengthened this result and showed that $S$ carries infinitely many leaf-wise intersection points or a leaf-wise intersection point which sits on a closed characteristic. Hence for any given $\Psi \in \text{Sp}(2n, \mathbb{R})$, every centrally symmetric hypersurface of restricted contact type in $(\mathbb{R}^{2n}, \omega_0)$ carries a leaf-wise intersection point for $\Psi$ by Ekeland–Hofer theorem, and every centrally symmetric star-shaped hypersurface in $(\mathbb{R}^{2n}, \omega_0)$ carries infinitely many leaf-wise intersection points or a leaf-wise intersection point which sits on a closed characteristic by Albers-Frauenfelder theorem. For a hypersurface $S$ of restricted contact type in $(\mathbb{R}^{2n}, \omega_0)$ and a $\Psi \in \text{Ham}_c(\mathbb{R}^{2n}, \omega_0)$, Hofer [24] showed that $S$ carries a leaf-wise intersection point if $\Psi$ has Hofer’s norm $\| \Psi \|_H \leq c_{E_{\text{EH}}}(S)$. The final restriction on norm $\| \Psi \|_H$ is not needed if the Rabinowitz Floer homology of $(\mathbb{R}^{2n}, S)$ does not vanish by [1, Theorem C]. Corollary 1.25 cannot be included in past results. Corollary 1.25 also implies the following generalization of the main result in [40] if the surfaces considered therein are smooth.

**Corollary 1.26** Let $\Psi \in \text{Sp}(2n, \mathbb{R}^{2n})$ and let $S \subset (\mathbb{R}^{2n}, \omega_0)$ be a smooth star-shaped hypersurface with respect to a fixed point $p$ of $\Psi$, that is, $p$ is an interior point of the bounded part surrounded by $S$ and has the property that every ray issuing from the point $p$ intersects $S$ in exactly one point and so transversally. Then $S$ carries a $\Psi$-characteristic with action $c_{\Psi}^{E_{\text{EH}}}(S)$, in particular, a leaf-wise intersection point for $\Psi$.

Indeed, let $\phi$ be the translation $x \mapsto x - p, \forall x \in \mathbb{R}^{2n}$. Then $\phi \in \text{Symp}(\mathbb{R}^{2n}, \omega_0)$ and commutes with $\Psi$. Then $\phi(S)$ is a star-shaped hypersurface with respect to the origin. As in the proof of Theorem 1.4 $\phi$ maps $\Psi$-characteristics on $S$ onto such characteristics on $\phi(S)$ in one-to-one and preserving-action way. Hence we can assume that $S$ is a star-shaped hypersurface with respect to the origin. Then $S$ is a hypersurface of restricted contact type in $(\mathbb{R}^{2n}, \omega_0)$ that admits a globally defined Liouville vector field $X$ given
by \(X(x) = \frac{\dot{x}}{x}\) for \(x \in \mathbb{R}^{2n}\). Clearly, \(X\) satisfies (1.38) and hence Corollary 1.25 yields Corollary 1.26.

Recently, Ekeland proposed a very closely related problem [16, Problem 4, §5], which may be formulated as follows in our notations.

**Problem E.** For a general Hamiltonian \(H\) on \(\mathbb{R}^{2n}\), assuming simply \(H(0) = 0\) and \(H(x) \to \infty\) when \(|x| \to \infty\), so that energy surfaces \(H(x) = h\) are bounded, what about the existence and multiplicity of solutions to the boundary-value problem (1.4) with \(\Psi \in \text{Sp}(2n, \mathbb{R})\)?

If \(H^{-1}(h)\) is regular and star-shaped, Corollary 1.26 implies that either the boundary-value problem (1.4) or
\[
\dot{x} = X_H(x) \quad \& \quad x(T) = \Psi^{-1}x(0)
\]
has a solution on \(H^{-1}(h)\).

### 1.3 Applications to Hamiltonian Dynamics

Recall that a thickening of a compact and regular energy surface \(S = \{ x \in M \mid H(x) = 0 \}\) in a symplectic manifold \((M, \omega)\) is an open and bounded neighborhood \(U\) of \(S\) which is filled with compact and regular energy surfaces having energy values near \(E = 0\), that is,
\[
U = \bigcup_{\lambda \in I} S_\lambda,
\]
where \(I = (-\varepsilon, \varepsilon)\) and \(S_\lambda = \{ x \in U \mid H(x) = \lambda \}\) is diffeomorphic to the given surface \(S = S_0\) for each \(\lambda \in I\). Suppose that \(\varepsilon \leq 1\). As a generalization of [27, p. 106, Theorem 1] we have:

**Theorem 1.27** Let \(S, S_\lambda\) and \(U\) be described as above, and \(\Psi \in \text{Symp}(M, \omega)\). Assume that \(S \cap \text{Fix}(\Psi) \neq \emptyset\) and that \(c_{HZ}^\Psi(U, \omega) < \infty\). Then there exists a sequence \(\lambda_j \to 0\) such that each energy surface \(S_{\lambda_j}\) carries a \(\Psi\)-characteristic.

The proof is standard. Pick an \(\varepsilon_0 \in (0, \varepsilon)\) and a \(C^\infty\) function \(f : \mathbb{R} \to \mathbb{R}_{\geq 0}\) such that
\[
f(s) = c_{HZ}^\Psi(U, \omega) + 1 \text{ for } \varepsilon_0 < |s| < \varepsilon, \quad f(s) = 0 \text{ for } |s| < \frac{\varepsilon_0}{2},
\]
\[
f'(s) < 0 \text{ for } -\varepsilon_0 < s < -\frac{\varepsilon_0}{2}, \quad f'(s) > 0 \text{ for } \frac{\varepsilon_0}{2} < s < \varepsilon_0.
\]
Let \(F(x) = f(H(x))\) for \(x \in U\). Note that \(S = S_0\) satisfies \(S \cap \text{Fix}(\Psi) \neq \emptyset\). Hence \(F \in \mathcal{H}^\Psi(U, \omega)\) with \(m(F) = c_{HZ}^\Psi(U, \omega) + 1\). By the definition of \(c_{HZ}^\Psi(U, \omega)\) there exists a nonconstant smooth curve \(x : [0, 1] \to \mathbb{R}^{2n}\) satisfying \(\dot{x} = X_F(x(t)) = f'(H(x(t)))X_H(x(t))\) and \(x(1) = \Psi x(0)\). Clearly \(H(x(t))\) is constant and \(x\) is a \(\Psi\)-characteristic sitting on \(S_{\varepsilon'}\), where \(\varepsilon_0/2 < |\varepsilon'| < \varepsilon_0\). By choosing \(\varepsilon_0\) sufficiently small, Theorem 1.27 follows.
Corollary 1.28 Let $\Psi \in \text{Symp}(M, \omega)$ and let $S \subset (M, \omega)$ be a hypersurface of restricted contact type that admits a globally defined Liouville vector field $X$ satisfying $X(\Psi(x)) = d\Psi(x)[X(x)]$ for all $x \in M$. If $S \cap \text{Fix}(\Psi) \neq \emptyset$ and $c_{\text{HZ}}^\Psi(U, \omega) < +\infty$ for some neighborhood $U$ of $S$ then there exists a $\Psi$-characteristic on $S$.

Indeed, let $\phi^t$ denote the local flow of the Liouville vector field $X$. Since $S$ is compact and $X$ is transversal to $S$, there exists a sufficiently small $\varepsilon > 0$ such that the map

$$\psi : S \times (-\varepsilon, \varepsilon) \to U \subset M, \quad (x, t) \mapsto \phi^t(x)$$

is a diffeomorphism (by shrinking $U$ if necessary), and that $\phi^t(\Psi(x)) = \Psi(\phi^t(x))$ for all $(t, x) \in (-\varepsilon, \varepsilon) \times S$ because of $X(\Psi(x)) = d\Psi(x)[X(x)]$ for all $x \in M$. Define $H : U \to \mathbb{R}$ by $H(x) = \lambda$ if $x = \psi(y, \lambda) \in U$. Let $S_{\lambda} = H^{-1}(\lambda) = \psi(S \times \{\lambda\})$ for $\lambda \in (-\varepsilon, \varepsilon)$. By Theorem 1.27 there exists $\lambda \in (-\varepsilon, \varepsilon)$ arbitrarily close to 0 such that $S_{\lambda}$ carries a $\Psi$-characteristic $y$. Note that $d\phi^\lambda : \mathcal{L}_S \to \mathcal{L}_{S_{\lambda}}$ is a bundle isomorphism. From these we derive that $x(t) = \phi^{-t}y(t)$ is a $\Psi$-characteristic on $S$. (See the arguments above Proposition 6.2 for details.)

Clearly, Corollary 1.28 may be applied to $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$ and $\Psi \in \text{Sp}(2n, \mathbb{R})$. But the result obtained is weaker than Theorem 1.24.

The same assumptions as under Theorem 1.27 may yield the following result for the leaf-wise intersection question, which can be viewed as a partial generalization of the result in [36] and will be proved in Sect. 7.

Theorem 1.29 Under the assumptions of Theorem 1.27 the following holds:

(i) There exists a subset $\Delta \subset (0, \varepsilon)$ of full Lebesgue measure $m(\Delta) = \varepsilon$ such that for every $\delta \in \Delta$ either $S_{\delta} \cup S_{-\delta}$ contains a fixed point of $\Psi$, or $S_{\delta}$ carries a $\Psi$-characteristic $y : [0, T] \to S_{\delta}$ satisfying $\dot{y} = X_H(y)$ or $S_{-\delta}$ carries a $\Psi$-characteristic $y : [0, T] \to S_{-\delta}$ satisfying $\dot{y} = -X_H(y)$. (If $\Psi$ has only finitely many fixed points in $U$, $\Delta$ may be chosen so that the first case does not occur.)

(ii) There exists a subset $\Lambda \subset I \setminus \{0\}$ of full Lebesgue measure $m(\Lambda) = m(I)$ such that for every nonzero parameter $\lambda \in \Lambda$ the associated energy surface $S_{\lambda}$ carries either a fixed point of $\Psi$ or a $\Psi$-characteristic $y : [0, T] \to S_{\lambda}$ satisfying $\dot{y} = X_H(y)$ or $\dot{y} = -X_H(y)$. (If $\Psi$ has only finite fixed points in $U$, $\Lambda$ may be chosen so that the first case does not occur.) Consequently, each $S_{\lambda}$ with $\lambda \in \Lambda$ carries a leaf-wise intersection point for $\Psi$.

Remark 1.30 Clearly, the statements (i) and (ii) in the above theorem cannot be contained in each other. When $\Psi = id_M$ the $\Psi$-characteristics become closed characteristics and Hofer and Zehnder showed in [27, p. 118, Theorem 4] that for some subset $\Lambda \subset I$ of full Lebesgue measure $m(\Lambda) = m(I)$ every energy surface $S_{\lambda}$ with $\lambda \in \Lambda$ carries a closed characteristic, provided $(M, \omega)$ has finite Hofer–Zehnder capacity and $S \subset (M, \omega)$ bounds a compact symplectic manifold. Macarini and Schlenk in [36] removed out the last additional assumption. (Actually, when $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$ and $\Psi = id_{\mathbb{R}^{2n}}$ Struwe [45] refined the arguments by Hofer and Zehnder [25] to prove such a result in 1990.)
Corollary 1.31 Let 0 be a regular value of \( H \in C^2(\mathbb{R}^{2n}) \) such that \( S := H^{-1}(0) \) is compact and connected. Let \( U = \cup_{\lambda \in I} S_\lambda \), where \( I = (-\varepsilon, \varepsilon) \) and \( S_\lambda = \{ x \in \mathbb{R}^{2n} \mid H(x) = \lambda \} \), be a thickening of \( S \) in \((\mathbb{R}^{2n}, \omega_0)\). Suppose that \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) and \( S \cap \text{Fix}(\Psi) \neq \emptyset \). Then the corresponding conclusions to those of Theorem 1.29 hold.

Since \( S \) is compact, there exists sufficiently large \( R > 0 \) such that \( U \subset B^{2n}(R) \). By the monotonicity and positive conformality property of the capacity \( c_{\text{HZ}} \) together with (1.17), we get that \( c_{\text{HZ}}(U, \omega_0) \leq c_{\text{HZ}}(B^{2n}(R), \omega_0) = R^2 t(\Psi)/2 \). Corollary 1.31 follows.

As a generalization of Struwe’s main result of [45], we have the following result which is stronger than Corollary 1.31. It is proved in Sect. 7.

Theorem 1.32 Suppose that 1 is a regular value of \( H \in C^2(\mathbb{R}^{2n}) \) and \( S := H^{-1}(1) \) is compact and connected. (Thus there exists \( \delta_0 > 0 \) such that each \( \beta \in [1 - \delta_0, 1 + \delta_0] \) is a regular value of \( H \) and \( S_\beta := H^{-1}(\beta) \) is diffeomorphic to \( S = S_1 \). Denote by \( \gamma \) the diameter of \( H^{-1}(1 - \delta_0, 1 + \delta_0) \).) Suppose for some \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) that the bounded component of \( \mathbb{R}^{2n} \setminus S \) contains a fixed point of \( \Psi \). Then for almost every \( \beta \in (1 - \delta_0, 1 + \delta_0) \) the associated energy surface \( S_\beta := H^{-1}(\beta) \) carries a \( \Psi \)-characteristic \( y \) satisfying \( \dot{y} = X_H(y) \) and with action \( 0 < A(y) < 16t(\Psi)\gamma^2 \).

Remark 1.33 Recently, Ginzburg and Gürel [21] showed that there exists a closed, smooth hypersurface \( S \subset \mathbb{R}^{2n} \) and a sequence of \( C^\infty \)-smooth autonomous Hamiltonians \( F_k \to 0 \) in \( \mathbb{C}^0 \), supported in the same compact set, such that \( S \) and \( \varphi_{F_k}(S) \) have no leafwise intersections. Here \( \varphi_{F_k} \) denotes the time-one map of the Hamiltonian flow of \( F_k \). This result suggests that Theorem 1.32 is best possible in some sense since there may exist \( \beta' \) near 1 such that \( H^{-1}(\beta') \) carries no \( \Psi \)-characteristics.

1.4 An Extension of a Theorem by Evgeni Neduv

Evgeni Neduv [39, Theorem 4.4] showed that differentiability of the Hofer–Zehnder capacity can be used to derive some results on fixed period problem of Hamiltonian systems. Similar differentiability also holds for the \( \Psi \)-HZ capacity where \( \Psi \in \text{Sp}(2n, \mathbb{R}^{2n}) \) and can lead to a result on existence of solutions \( y : [0, T] \to \mathbb{R}^{2n} \) to the boundary value problem

\[
\dot{y}(t) = J \nabla H(y(t)), \quad y(T) = \Psi y(0).
\]

with fixed \( T \) and \( H \) a convex Hamiltonian satisfying certain asymptotic conditions. The main tool is the representation formula (Theorem 1.4).

For a proper and strictly convex Hamiltonian \( \mathcal{H} \in C^2(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0}) \) such that \( \mathcal{H}(0) = 0 \) and \( \mathcal{H}'' > 0 \) (which imply \( \mathcal{H} \geq 0 \) by the Taylor’s formula), if \( e_0 \geq 0 \) is a regular value of \( \mathcal{H} \) with \( \mathcal{H}^{-1}(e_0) \neq \emptyset \), the set \( D(e) := \{ \mathcal{H} < e \} \) is a strictly convex bounded domain in \( \mathbb{R}^{2n} \) with \( 0 \in D(e) \) and with \( C^2 \)-boundary \( \mathcal{S}(e) = \mathcal{H}^{-1}(e) \) for each number \( e \) sufficiently close to \( e_0 \). Given \( \Psi \in \text{Sp}(2n, \mathbb{R}) \), for any \( e \) near \( e_0 \) let \( \mathcal{C}(e) := \)
\[ c_{HZ}^\Psi(D(e), \omega_0) \]. By Remark 1.5 all \( c_{HZ}^\Psi \)-carriers for \( D(e) \) form a compact subset in \( C^1([0, C(e)], S(e)) \). Hence

\[ J(e) := \left\{ T_x = 2 \int_0^{C(e)} \frac{dt}{(\nabla \mathcal{H}(x(t)), x(t))} \mid x : [0, C(e)] \rightarrow S(e) \text{ is a } c_{HZ}^\Psi \text{-carrier for } D(e) \right\} \]

is a compact subset in \( \mathbb{R} \). Denote by \( T_{\text{max}}(e) \) and \( T_{\text{min}}(e) \) the largest and smallest numbers in \( J(e) \). Every \( c_{HZ}^\Psi \)-carrier \( x \) for \( D(e) \) can be reparameterized as a solution of

\[ -J \dot{x}(t) = \nabla \mathcal{H}(x(t)) \quad \forall t \in [0, T_x] \quad \text{and} \quad x(T_x) = \Psi x(0) \]

on \( S(e) = \mathcal{H}^{-1}(e) \), where \( T_x \in J(e) \), satisfying \( x(t) \neq \Psi x(0) \) for any \( t \in (0, T_x) \). The final property means that \( T_x \) is the minimal period of \( x \) if \( \Psi = I_{2n} \).

The following is our generalization for [39, Theorem 4.4]. Its proof is given in Sect. 8.

**Theorem 1.34** Let \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) and \( \mathcal{H} \in C^2(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0}) \) be as above. Then \( \mathcal{C}(e) \) has the left and right derivatives at \( e_0 \), \( \mathcal{C}_-(e_0) \) and \( \mathcal{C}_+(e_0) \), and they satisfy

\[ \mathcal{C}_-(e_0) = \lim_{\epsilon \to 0^-} T_{\text{max}}(e_0 + \epsilon) = T_{\text{max}}(e_0) \quad \text{and} \]

\[ \mathcal{C}_+(e_0) = \lim_{\epsilon \to 0^+} T_{\text{min}}(e_0 + \epsilon) = T_{\text{min}}(e_0). \]

Moreover, if \([a, b] \subset (0, \sup \mathcal{H}) \) is a regular interval of \( \mathcal{H} \) such that \( \mathcal{C}_+(a) < \mathcal{C}_-(b) \), then for any \( r \in (\mathcal{C}_+(a), \mathcal{C}_-(b)) \) there exists \( e' \in (a, b) \) such that \( \mathcal{C}(e) \) is differentiable at \( e' \) and \( \mathcal{C}_-(e') = \mathcal{C}_+(e') = r = T_{\text{max}}(e') = T_{\text{min}}(e') \).

As a monotone function on a regular interval \([a, b]\) of \( \mathcal{H} \) as above, \( \mathcal{C}(e) \) satisfies \( \mathcal{C}_-(e) = \mathcal{C}_+(e) \) for almost all values of \( e \in [a, b] \) and thus both \( T_{\text{max}} \) and \( T_{\text{min}} \) are almost everywhere continuous. Actually, both \( T_{\text{max}} \) and \( T_{\text{min}} \) have only at most countable discontinuous points and are also Riemann integrable on \([a, b]\) (see [7, Corollary 6.4]).

By Theorem 1.34, for any regular interval \([a, b] \subset (0, \sup \mathcal{H}) \) of \( \mathcal{H} \) with \( \mathcal{C}_+(a) \leq \mathcal{C}_-(b) \), if \( T \in [\mathcal{C}_+(a), \mathcal{C}_-(b)] \) then (1.41) has a solution \( y : [0, T] \rightarrow \mathcal{H}^{-1}([a, b]) \) such that \( y(T) = \Psi y(0) \) and \( y(t) \neq \Psi y(0) \) for any \( t \in (0, T) \). For example, we have

**Corollary 1.35** Suppose that a proper and strictly convex Hamiltonian \( \mathcal{H} \in C^2(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0}) \) satisfies the conditions:

(i) \( \mathcal{H}(0) = 0, \mathcal{H}'' > 0 \) and every \( e > 0 \) is a regular value of \( \mathcal{H} \),

(ii) there exist two positive definite symmetric matrices \( S_0, S_\infty \in \mathbb{R}^{2n \times 2n} \) such that

\[ \inf \{ t > 0 \mid \det(\exp(tJS_0) - \Psi) \neq 0 \} \leq \inf \{ t > 0 \mid \det(\exp(tJS_\infty) - \Psi) \neq 0 \} \]

(1.42)
and that $\mathcal{H}(x)$ is equal to $q(x) := \frac{1}{2} \langle S_0 x, x \rangle$ (resp. $Q(x) := \frac{1}{2} \langle S_{\infty} x, x \rangle$) for $|x|$ small (resp. large enough).

Then for every $T$ between the two numbers in (1.42) the corresponding system (1.41) has a solution $y : [0, T] \to \mathbb{R}^{2n}$ such that $y(T) = \Psi y(0)$ and $y(t) \neq \Psi y(0)$ for any $t \in (0, T)$.

In fact, if $e > 0$ is small (resp. large) enough then $D(e)$ is equal to $D_{q}(e) := \{ q < e \} = \sqrt{e} E(q)$ (resp. $D_{Q}(e) := \{ Q < e \} = \sqrt{e} E(Q)$) and so

$$c_{HZ}^{\Psi}(D(e), \omega_0) = c_{HZ}^{\Psi}(D_{q}(e), \omega_0) = e c_{HZ}^{\Psi}(E(q), \omega_0)$$

(resp. $c_{HZ}^{\Psi}(D(e), \omega_0) = c_{HZ}^{\Psi}(D_{Q}(e), \omega_0) = e c_{HZ}^{\Psi}(E(Q), \omega_0)$).

Then Corollary 1.7 implies

$$C'(a) = c_{HZ}^{\Psi}(E(q), \omega_0) = \inf \{ t > 0 \mid \det(\exp(tJS_0) - \Psi) \neq 0 \}$$

for $a > 0$ small enough, and

$$C'(b) = c_{HZ}^{\Psi}(E(Q), \omega_0) = \inf \{ t > 0 \mid \det(\exp(tJS_{\infty}) - \Psi) \neq 0 \}$$

for $b > 0$ large enough. The conclusions of Corollary 1.35 follow from Theorem 1.34 immediately.

**Organization of the paper.**

- Sect. 2: present our variational frame and related preparations.
- Sect. 3: prove Theorem 1.4 by improving the arguments in [25, 26].
- Sect. 4: provide the variational explanation for our extended Ekeland–Hofer capacity $c_{EH}^{\Psi}$ (the proof of Theorem 1.14).
- Sect. 5: prove Theorems 1.15, 1.18.
- Sect. 6: prove Theorem 1.24.
- Sect. 7: prove Theorems 1.29, 1.32.
- Sect. 8: prove Theorem 1.34.

### 2 Variational Frame and Related Preparations

In this section we shall give our variational frame by suitably modifying those in [18, 25–27]. For the sake of completeness some corresponding conclusions are also proved in details though part of them appeared in [13] in different forms.

For a given symplectic matrix $\Psi \in Sp(2n, \mathbb{R})$, consider the Hilbert subspace of $H^1([0, 1], \mathbb{R}^{2n})$,

$$X = \{ x \in H^1([0, 1], \mathbb{R}^{2n}) \mid x(1) = \Psi x(0) \}.$$

Since $C_c^{\infty}([0, 1]) \subset X$ is dense in $L^2([0, 1], \mathbb{R}^{2n})$ (cf. [8, Cor.4.23]), so is $X$ in $L^2([0, 1], \mathbb{R}^{2n})$. Consider the unbounded linear operator on $L^2([0, 1], \mathbb{R}^{2n})$ with
domain \( \text{dom}(\Lambda) = X \),

\[
\Lambda := -J \frac{d}{dt},
\]

(2.1)

which is also a bounded linear operator from \( X \) (with \( H^1 \) norm) to \( L^2([0, 1], \mathbb{R}^{2n}) \). Denote by \( E_1 \subset \mathbb{R}^{2n} \) the eigenvector space which belongs to eigenvalue \( 1 \) of \( \Psi \). By identifying \( a \in E_1 \) with the constant path in \( L^2([0, 1], \mathbb{R}^{2n}) \) given by \( \hat{a}(t) = a \) for all \( t \in [0, 1] \), we can identify \( \text{Ker}(\Lambda) \) with \( E_1 \). We write \( \text{Ker}(\Lambda) = E_1 \) without occurring of confusions. Denote by \( R(\Lambda) \) the range of \( \Lambda \). The following proposition is a standard exercise in functional analysis. But we still give its detailed proof for the sake of completeness.

**Proposition 2.1** (i) \( R(\Lambda) \) is a closed subspace in \( L^2([0, 1], \mathbb{R}^{2n}) \) and there exists the following orthogonal decomposition

\[
L^2([0, 1], \mathbb{R}^{2n}) = \text{Ker}(\Lambda) \oplus R(\Lambda).
\]

(2.2)

(ii) The restriction \( \Lambda_0 := \Lambda|_{R(\Lambda) \cap \text{dom}(\Lambda)} \) is a bijection onto \( R(\Lambda) \), and \( \Lambda_0^{-1} : R(\Lambda) \rightarrow R(\Lambda) \) is a compact and self-adjoint operator if \( R(\Lambda) \) is equipped with the \( L^2 \) norm.

**Proof** Step 1 (Prove that \( R(\Lambda) \) is closed in \( L^2([0, 1]) \)). Let \( E_1^\perp \) be the orthogonal complement of \( E_1 \) with respect to the standard Euclidean inner product in \( \mathbb{R}^{2n} \). Since \( \dim E_1 = 2n \) if and only if \( \Psi = I_{2n} \), the problem reduces to the periodic case studied in past if \( \dim E_1^\perp = 0 \). Hence we only consider the non-periodic case in which \( \dim E_1^\perp \geq 1 \). Then

\[
\Psi - I_{2n} : E_1^\perp \rightarrow (\Psi - I_{2n})\mathbb{R}^{2n}
\]

is continuously invertible. Denote by \( (\Psi - I)^{-1} \) its inverse and

\[
C := \sup\{|(\Psi - I_{2n})^{-1}x| \mid x \in (\Psi - I_{2n})(\mathbb{R}^{2n}) \text{ & } |x| = 1\},
\]

(2.3)

where \( |\cdot| \) denotes the standard norm in \( \mathbb{R}^{2n} \).

Let \( (x_k) \subset R(\Lambda) \) be a sequence converging to \( x \) in \( L^2([0, 1], \mathbb{R}^{2n}) \). For each \( x_k \), we may choose its preimage to be

\[
u_k(t) = J \int_0^t x_k(s)ds + u_k(0),
\]

where \( u_k(0) \in E_1^\perp \). Then \( (u_k) \) is a Cauchy sequence in \( X \) (with \( H^1 \) norm). In fact, since \( u_k(1) = \Psi u_k(0) \), we get

\[
J \int_0^1 x_k(s)ds = (\Psi - I)u_k(0),
\]
where $u_k(0) \in E_1^\perp$. Hence

$$|u_k(0) - u_m(0)| \leq C \left| \int_0^1 (x_k - x_m)(s) ds \right| \leq C \|x_k - x_m\|_{L^2},$$

and therefore

$$\|u_k - u_m\|_{L^2}^2 = \int_0^1 \left| \int_0^t (x_k - x_m)(s) ds + u_k(0) - u_m(0) \right|^2 dt \leq 2(C^2 + 1) \|x_k - x_m\|_{L^2}^2.$$

It is obvious that $\|\dot{u}_k - \dot{u}_m\|_{L^2} = \|x_k - x_m\|_{L^2}$ and thus

$$\|u_k - u_m\|_{H^1} \leq \sqrt{2C^2 + 3} \|x_k - x_m\|_{L^2} \to 0$$
as $k \to \infty$ and $m \to \infty$. Let $u_k \to u$ in $X$. Then $x_k = \Lambda u_k \to \Lambda u$ in $L^2([0, 1], \mathbb{R}^{2n})$, and so $\Lambda u = x$, i.e. $x \in R(\Lambda)$.

**Step 2** (Prove that $L^2([0, 1], \mathbb{R}^{2n})$ has the orthogonal decomposition as in (2.2)). Note that $\mathbb{R}^{2n}$ has the following orthogonal splitting:

$$\mathbb{R}^{2n} = J \ker(\Psi - I) \oplus R(\Psi - I),$$

(2.4)

where $R(\Psi - I) = (\Psi - I)(E_1^\perp) = (\Psi - I)(\mathbb{R}^{2n})$. In fact, for $a \in \ker(\Psi - I)$ and $b = (\Psi - I)c \in R(\Psi - I)$, we have

$$\langle Ja, b \rangle = \langle Ja, (\Psi - I)c \rangle = \langle J\Psi a, \Psi c \rangle - \langle Ja, c \rangle = \langle \Psi J\Psi a, c \rangle - \langle Ja, c \rangle = \langle Ja, c \rangle - \langle Ja, c \rangle = 0.$$

This and the dimension equality $\dim \ker(\Psi - I) + \dim R(\Psi - I) = \dim \mathbb{R}^{2n}$ lead to (2.4).

For any given $x \in L^2([0, 1])$, by (2.4) we can write

$$J \int_0^1 x(s) ds = Ja + b,$$

where $a \in \ker(\Psi - I)$ and $b = (\Psi - I)c \in R(\Psi - I)$. Let

$$u(t) = J \int_0^t (x(s) - a) ds + c \quad \forall t \in [0, 1].$$

Then $u \in X$ because

$$u(1) = J \int_0^1 (x(s) - a) ds + c = J \int_0^1 x(s) ds - Ja + c = \Psi c = \Psi u(0).$$

It follows from this and the definition of $u$ that $\Lambda u = x - a$. 

\[ \text{Springer} \]
Moreover, for \( a \in \text{Ker}(\Lambda) = \text{Ker}(\Psi - I) = E_1 \) and \( y = \Lambda w \in R(\Lambda) \), we compute
\[
\langle a, y \rangle_{L^2} = \int_0^1 \langle a, -J \dot{w} \rangle = \langle Ja, w(1) - w(0) \rangle = \langle J\Psi a, \Psi w(0) \rangle - \langle Ja, w(0) \rangle = 0
\]
because \( \Psi^t J \Psi = J \). Therefore the orthogonal decomposition in (2.2) follows immediately.

**Step 3 (Prove (ii)).** Firstly, we prove that \( R(\Lambda) \cap \text{dom}(\Lambda) \) is a closed subspace in \( X \) (with \( H^1 \) norm). Let \( (u_k) \subset R(\Lambda) \cap \text{dom}(\Lambda) \) be a Cauchy sequence in \( H^1 \) norm. Then it converges to some \( u \in X = \text{dom}(\Lambda) \) in the \( H^1 \) norm. Especially, \( (u_k) \) converges to \( u \) in the \( L^2 \) norm. Since \( R(\Lambda) \) is closed in \( L^2([0, 1], \mathbb{R}^{2n}) \) we get that \( u \in R(\Lambda) \).

The claim is proved.

Consider the operator \( \Lambda_0 := \Lambda|_{R(\Lambda) \cap \text{dom}(\Lambda)} \). Clearly, it is a bijective continuous linear map from a Hilbert subspace \( \text{dom}(\Lambda_0) \) of \( X \) to the Hilbert subspace \( R(\Lambda) \) of \( L^2([0, 1], \mathbb{R}^{2n}) \). Hence the Banach inverse operator theorem yields a continuous linear operator \( \Lambda_0^{-1} : R(\Lambda) \to \text{dom}(\Lambda_0) \). Note that \( i : \text{dom}(\Lambda_0) \hookrightarrow \hookrightarrow R(\Lambda) \) (as a restriction of the compact inclusion map \( H^1 \hookrightarrow L^2 \)) is compact. Hence \( i \circ \Lambda_0^{-1} : R(\Lambda) \to R(\Lambda) \) is compact.

We claim that \( i \circ \Lambda_0^{-1} \) is also self-adjoint. In fact, for any \( u, w \in X \) there holds
\[
\langle \Lambda u, w \rangle_{L^2} = \int_0^1 \langle -J \dot{u}, w \rangle dt
\]
\[
= \langle -Ju, w \rangle_{L^2}^1 - \int_0^1 \langle -Ju, \dot{w} \rangle dt
\]
\[
= -\langle J(u(1), w(1)) - J(u(0), w(0)) \rangle - \int_0^1 (J\dot{w}, u) dt = \langle u, \Lambda w \rangle_{L^2}.
\]

Note that \( \langle J(u(1), w(1)) - J(u(0), w(0)) \rangle = 0 \) since \( u, w \in X \) satisfy the boundary condition \( u(1) = \Psi u(0) \) and \( w(1) = \Psi w(0) \). For \( x, y \in R(\Lambda) \), let us choose \( u, w \in X \cap R(\Lambda) \) such that \( \Lambda u = x \) and \( \Lambda w = y \). Then \( \langle i \circ \Lambda_0^{-1} x, y \rangle = \langle u, \Lambda w \rangle_{L^2} = \langle \Lambda u, w \rangle_{L^2} = \langle x, i \circ \Lambda_0^{-1} y \rangle_{L^2} \). Hence we have proved that \( i \circ \Lambda_0^{-1} : R(\Lambda) \to R(\Lambda) \) is a compact self-adjoint operator. \( \square \)

**Remark 2.2** Since \( R(\Lambda) \) is a Hilbert subspace of \( L^2([0, 1], \mathbb{R}^{2n}) \) which is separable, by the standard linear functional analysis theory, there exists an orthogonal basis of \( R(\Lambda) \) which completely consists of eigenvectors of \( i \circ \Lambda_0^{-1} \). Note that \( \text{Ker}(i \circ \Lambda_0^{-1}) = 0 \) and that \( l \neq 0 \) is an eigenvalue of \( i \circ \Lambda_0^{-1} \) if and only if \( 1/l \) is an eigenvalue of \( \Lambda \) with the same multiplicity. Let
\[
\cdots \leq \lambda_{-k} \leq \cdots \leq \lambda_{-1} < 0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \tag{2.5}
\]
denote all eigenvalues of $\Lambda$, which satisfy $\lambda_k \to \pm \infty$ as $k \to \pm \infty$. By these and (2.2), we get a unit orthogonal basis of $L^2([0, 1], \mathbb{R}^{2n})$

$$\{e_j \mid \pm j \in \mathbb{N}\} \cup \{e_0^j\}_{j=1}^q$$

such that $\text{Ker}(\Lambda) = \text{Span}\{\{e_0^j\}_{j=1}^q\}$ and that each $e_j$ is an eigenvector corresponding to $\lambda_j$, $j = \pm 1, \pm 2, \ldots$.

**Remark 2.3** The nonzero eigenvalues of $\Lambda = -J \frac{d}{dt}$ are exactly the zero points of the function $g^\Psi$ defined in (1.13). In fact, let $\lambda$ be an eigenvalue of $\Lambda$ and $e \in X$ be an eigenvector associated with it. Then $-J \dot{e}(t) = \lambda e(t) \ \forall t \in [0, 1]$ and $e(1) = \Psi e(0)$. It follows that $e(t) = e^\lambda t J X$ and $e(1) = e^\lambda J X = \Psi e(0)$, where $e(0) \in \mathbb{R}^{2n} \setminus \{0\}$. Hence $\det(e^\lambda J - \Psi) = 0$. By Lemma A.1 function $g^\Psi$ has only finitely many zero points in $(0, 2\pi]$, denoted by $t_1 < \ldots < t_m$. Then all eigenvalues of $\Lambda$ are

$$\{t_l + 2k\pi \mid 1 \leq l \leq m, k \in \mathbb{Z}\}.$$  

Note that each eigenvector of $t_l + 2k\pi$ has the form

$$e(t) = e^{(t_l + 2k\pi)tJ} X,$$  

where $X \in \text{Ker}(e^{tJ} - \Psi)$. By requiring $|X| = 1$ we get that $|e(t)| \equiv 1$. Hence $e_j$ and $e_0^j$ in (2.6) can be chosen to satisfy

$$|e_j(t)| = |e_0^j(t)| \equiv 1 \ \forall t.$$

**Remark 2.4** If $\Psi \in \text{Sp}(2n, \mathbb{R}) \cap O(2n)$ is as in (1.32), by Lemma A.3 the eigenvalues of $\Lambda$ associated to $\Psi$ are

$$\{\theta_j + 2k\pi \mid 1 \leq j \leq n, k \in \mathbb{Z}\},$$

where $0 < \theta_1 \leq \ldots \leq \theta_n \leq 2\pi$ are as in (1.33), and the corresponding eigensubspace to $\theta_j + 2k\pi$ is generated by

$$e^{(\theta_j + 2k\pi)tJ} X_j \quad \text{and} \quad e^{(\theta_j + 2k\pi)tJ} J X_j$$

where $X_j$ is as in Lemma A.3.

Now we are in position to define the variational space needed in this article. Using the unit orthogonal basis given by (2.6) every $x \in L^2([0, 1], \mathbb{R}^{2n})$ can be uniquely written as

$$x = x^0 + \sum_{k < 0} x_k e_k + \sum_{k > 0} x_k e_k.$$
where \( x^0 \in \text{Ker}(\Lambda) \) and \( \{x_k | \pm k \in \mathbb{N}\} \in \mathbb{R} \). By Remark 2.3 we can always assume

\[ |e_k(t)| \equiv 1 \quad \text{for all} \quad t \in [0, 1] \quad \text{and} \quad k = \pm 1, \pm 2, \ldots. \]

For \( s \geq 0 \) define a linear subspace of \( L^2([0, 1], \mathbb{R}^{2n}) \) by

\[
E^s_\Psi = \left\{ x \in L^2([0, 1], \mathbb{R}^{2n}) \mid \sum_{k \neq 0} |\lambda_k|^{2s} x_k^2 < \infty \right\}. \tag{2.9}
\]

We omit the subscript \( \Psi \) in \( E^s_\Psi \) if \( \Psi \) is fixed and there is no confusion. It is easy to prove that

\[
\langle x, y \rangle_{E^s} = \langle x^0, y^0 \rangle_{\mathbb{R}^{2n}} + \sum_{k \neq 0} |\lambda_k|^{2s} x_k y_k, \quad x, y \in E^s \tag{2.10}
\]

defines a complete inner product on \( E^s \). Denote the associated norm by \( \| \cdot \|_{E^s} \). Note that \( E^0 = L^2 \) and \( \| \cdot \|_{E^0} = \| \cdot \|_{L^2} \).

Let \( E := E^{1/2} \) be defined by (2.9) with \( s = \frac{1}{2} \). It has the orthogonal splitting

\[
E = E^- \oplus E^0 \oplus E^+, \tag{2.11}
\]

where \( E^- = \text{span}\{e_k, k < 0\} \), \( E^0 = \text{Ker}(\Lambda) \) and \( E^+ = \text{span}\{e_k, k > 0\} \). Denote the associated projection on them by \( P^- \), \( P^0 \) and \( P^+ \). For \( x \in E \), write \( x = x^- + x^0 + x^+ \), where \( x^- \in E^- \), \( x^0 \in E^0 \) and \( x^+ \in E^+ \).

Similar to \( H^s(S^1, \mathbb{R}^{2n}) \) defined in [27], we have

**Proposition 2.5** Assume \( t > s \geq 0 \). Then the inclusion map \( I_{t,s} : E^t \rightarrow E^s \) is compact.

**Proof** Let \( P_N : E^t \rightarrow E^s \) be the finite rank operator defined by

\[
P_N(x) = x^0 + \sum_{0 < |k| \leq N} x_k e_k
\]

for \( x = x^0 + \sum_{k \neq 0} x_k e_k \). It is a compact linear operator. Moreover,

\[
\| (P_N - I_{t,s}) x \|^2_{E^s} = \sum_{|k| > N} x_k e_k \|^2_{E^t} = \sum_{|k| > N} |\lambda_k|^{2s} x_k^2 = \sum_{|k| > N} |\lambda_k|^{2(s-t)} |\lambda_k|^{2t} x_k^2
\]

and \( |\lambda_k|^{2(s-t)} | \leq \max(\lambda_N, |\lambda_N|)^{2(s-t)} \) for each \( |k| > N \), we deduce

\[
\sum_{|k| > N} |\lambda_k|^{2(s-t)} |\lambda_k|^{2t} x_k^2 \leq \max(\lambda_N, |\lambda_N|)^{2(s-t)} \|x\|^2_{E^t}.
\]

\[ \square \]
Since \( \lim_{k \to \pm \infty} \lambda_k = \pm \infty \), \( \lim_{N \to +\infty} \| P_N - I_{t,s} \|^{op} = 0 \). Hence \( I_{t,s} : E' \to E^s \) is compact. \( \square \)

**Proposition 2.6** Assume \( s > \frac{1}{2} \). If \( x \in E^s \), then \( x \) is continuous and satisfies \( x(1) = \Psi x(0) \). Moreover, there exists a constant \( c = c_s \) such that

\[
\sup_{t \in [0,1]} |x(t)| \leq c \| x \|_{E^s}. \tag{2.12}
\]

**Proof** For \( x = x^0 + \sum_{k \neq 0} x_k e_k \in E^s \), since

\[
|x^0| + \sum_{k \neq 0} |x_k e_k(t)| = |x^0| + \sum_{k \neq 0} |x_k|
= |x^0| + \sum_{k \neq 0} \frac{1}{|\lambda_k|^2 s} |\lambda_k|^s |x_k|
\leq |x^0| + \left( \sum_{k \neq 0} \frac{1}{|\lambda_k|^{2s}} \right)^{\frac{1}{2}} \left( \sum_{k \neq 0} |\lambda_k|^{2s} |x_k|^2 \right)^{-\frac{1}{2}}, \tag{2.13}
\]

the series of functions \( x^0 + \sum_{k \neq 0} x_k e_k(t) \) is absolutely uniformly convergent. In other words, \( x(t) \) is the uniform limit of the function sequence \( f_k(t) := x^0 + \sum_{0 < |j| < k} x_j e_j(t) \). It follows that \( x \) is continuous and \( x(1) = \Psi x(0) \) since \( f_k(1) = \Psi f_k(0) \) for all \( k \). Moreover, (2.12) is a direct consequence of (2.13). Note that we have used the fact that \( \sum_{k \neq 0} \frac{1}{|\lambda_k|^{2s}} \) is finite for \( s > \frac{1}{2} \) due to the form of the eigenvalues of \( \Lambda \) in (2.7). \( \square \)

Let \( a : E \to \mathbb{R} \) be the functional given by

\[
a(x) = \frac{1}{2} \left( \| x^+ \|^2_E - \| x^- \|^2_E \right). \tag{2.14}
\]

Then \( a \) is smooth and has gradient \( \nabla a(x) = x^+ - x^- \in E \).

**Remark 2.7** For \( x \in C^1([0,1], \mathbb{R}^{2n}) \) satisfying \( x(1) = \Psi x(0) \), there holds

\[
a(x) = \frac{1}{2} \int_0^1 \langle -J \dot{x}, x \rangle dt = A(x). \tag{2.15}
\]

In fact, write \( x = a^0 + \sum_{k \neq 0} a_k e_k \) and \( -J \dot{x} = b^0 + \sum_{k \neq 0} b_k e_k \) in \( L^2 \). For \( k \neq 0 \) we have

\[
b_k = \int_0^1 \langle -J \dot{x}, e_k \rangle dt
= \langle -J x, e_k \rangle_{L^2} - \int_0^1 \langle -J x, \dot{e}_k \rangle dt
\]
\[
\begin{align*}
&= - (\langle Jx(1), e_k(1) \rangle - \langle Jx(0), e_k(0) \rangle) - \int_0^1 \langle x, J\dot{e}_k \rangle dt \\
&= - (\langle J^* J\Psi x(0), e_k(0) \rangle - \langle Jx(0), e_k(0) \rangle) + \int_0^1 \langle x, \lambda_k e_k \rangle dt \\
&= \lambda_k a_k. \quad (2.16)
\end{align*}
\]

Moreover, for \( v \in \text{Ker}(\Lambda) \) we have \( \Psi v = v \) and thus
\[
\int_0^1 \langle -J\dot{x}, v \rangle dt = - (\langle Jx(1), v \rangle - \langle Jx(0), v \rangle) = 0. \quad (2.17)
\]
Hence \( b^0 = 0 \). It follows that
\[
\int_0^1 \frac{1}{2} \langle -J\dot{x}, x \rangle dt = \frac{1}{2} \sum_{k \neq 0} \lambda_k a_k^2 = \frac{1}{2} \left( \sum_{k > 0} |\lambda_k|a_k^2 - \sum_{k < 0} |\lambda_k|a_k^2 \right)
\]
\[
= \frac{1}{2} (\|x^+\|_E^2 - \|x^-\|_E^2) = a(x).
\]

Note that if \( x \) does not satisfy the boundary condition \( x(1) = \Psi x(0) \), the equality (2.15) does not hold in general since \( \|x\|_E \) is a norm associated to \( \Psi \).

From now on we assume that \( H : \mathbb{R}^{2n} \to \mathbb{R} \) is a smooth function satisfying the condition (H2) in Sect. 1.2. Then there exist positive numbers \( C_1 \) and \( C_2 \) such that
\[
|\nabla H(z)| \leq 2a|z| + C_1 \quad \text{and} \quad |H_{zz}| \leq C_2 \quad (2.18)
\]
for all \( z \in \mathbb{R}^{2n} \). Thus we have the well defined functional
\[
\hat{b} : L^2([0, 1]; \mathbb{R}^{2n}) \to \mathbb{R}, \quad x \mapsto \int_0^1 H(x(t)) dt.
\]
It is also differentiable and has \( L^2 \)-gradient \( \nabla \hat{b}(x) = \nabla H(x) \) for \( x \in L^2([0, 1]; \mathbb{R}^{2n}) \) (cf. [27]).

Let \( j : E \to L^2 \) be the inclusion map and \( j^* : L^2 \to E \) the adjoint operator of it, i.e. \( \langle j(x), y \rangle_{L^2} = \langle x, j^*(y) \rangle_E \) for all \( x \in E \) and \( y \in L^2 \). Define a functional
\[
b : E \to \mathbb{R}, \quad x \mapsto \hat{b}(j(x)).
\]
It is not hard to prove that \( b \) is differentiable and has \( E \)-gradient \( \nabla b(x) = j^* \nabla H(x) \) for \( x \in E \).

Arguing as in [27], we have the following propositions.

**Proposition 2.8** For \( y \in L^2 \), \( j^*(y) \in E^1 \) and \( j^* \) is a compact operator.
Proposition 2.9  The gradient $\nabla b : \mathbb{E} \to \mathbb{E}$ is compact and satisfies the global Lipschitz condition

$$\| \nabla b(x) - \nabla b(y) \|_{\mathbb{E}} \leq C_3 \| x - y \|_{\mathbb{E}} \quad \forall x, y \in \mathbb{E}$$

for some constant $C_3 > 0$. Moreover, there exist positive numbers $C_4$ and $C_5$ such that $|b(x)| \leq C_4 \| x \|_{L^2} + C_5$, $\forall x \in \mathbb{E}$.

Proposition 2.9 shows that the functional $\Phi_1 H := \alpha - b$ is differentiable and its gradient $\nabla \Phi_H$ satisfies a global Lipschitz condition. Hence the negative gradient flow of $\Phi_H$ defined by

$$\frac{d\phi^t(x)}{dt} = -\nabla \Phi_H(\phi^t(x)) \quad \text{and} \quad \phi^0(x) = x$$

exists for all $t \in \mathbb{R}$ and $x \in \mathbb{E}$ and it admits the representation

$$\phi^t(x) = e^t x^- + x^0 + e^{-t} x^+ + K(t, x),$$

where $K : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ is continuous and maps bounded sets into precompact sets (cf. [27]).

Next we study the regularity of the critical points of $\Phi_H$.

Proposition 2.10  If $x \in \mathbb{E}$ is a critical point of $\Phi_H$ on $\mathbb{E}$, then $x$ is smooth and satisfies

$$\dot{x} = J \nabla H(x) \quad \text{and} \quad x(1) = \Psi x(0).$$

Proof  Let $x \in \mathbb{E}$ be a critical point of $\Phi_H$. Then

$$x^+ - x^- = j^*(\nabla H(x)). \quad (2.19)$$

Write in the space $L^2([0, 1], \mathbb{R}^{2n})$

$$x = a^0 + \sum_{k \neq 0} a_k e_k, \quad \nabla H(x) = b^0 + \sum_{k \neq 0} b_k e_k \quad \text{and} \quad y = y^0 + \sum_{k \neq 0} y_k e_k$$

for any $y \in \mathbb{E}$. Using $\langle j^*(\nabla H(x)), y \rangle_{\mathbb{E}} = \langle \nabla H(x), j(y) \rangle_{L^2}$, a direct computation yields $j^*(\nabla H(x)) = b^0 + \sum_{k \neq 0} \frac{1}{|\lambda_k|} b_k$ (cf. [28]). It follows that (2.19) becomes

$$0 = b^0 \quad \text{and} \quad \lambda_k a_k = b_k \quad \forall k \neq 0, \quad (2.20)$$

and therefore $x \in E^1$, where $E^1$ is as in (2.9). By Proposition 2.6 $x$ is continuous and satisfies $x(1) = \Psi x(0)$. Hence $\nabla H(x)$ is also continuous. Using (2.4), we may write

$$\int_0^1 J\nabla H(x) dt = Jd + (\Psi - I)c,$$
where \(d \in \text{Ker}(\Psi - I)\) and \(c \in \mathbb{R}^{2n}\). Define \(\xi(t) = \int_0^t (J \nabla H(x(s)) - Jd)ds + c\). Then \(\xi \in C^1([0, 1], \mathbb{R}^{2n})\) and \(\xi(1) = \Psi c = \Psi \xi(0)\). Writing \(\xi = \xi^0 + \sum_{k \neq 0} \xi_k e_k\) and computing as in (2.16) and (2.17), we get that

\[- J^t \dot{\xi} = \sum_{k \neq 0} \lambda_k \xi_k e_k. \quad (2.21)\]

On the other hand,

\[- J^t \dot{\xi} = \nabla H(x) - d = -d + \sum_{k \neq 0} b_k e_k. \quad (2.22)\]

Note here that \(b^0 = 0\) and \(d \in \text{Ker}(\Psi - I) = \text{Ker} \Lambda\). Comparing (2.21) and (2.22) we get that

\[d = 0 \quad \text{and} \quad \lambda_k \xi_k = b_k \, \forall k \neq 0. \quad (2.23)\]

Since \(\xi\) and \(x\) are continuous, the second equalities in (2.20) and (2.23) lead to \(\xi(t) - x(t) = \text{const}, \text{i.e.,} \), \(\xi(t) - x(t) = \xi(0) - x(0) = c - x(0)\). Hence

\[x(t) = \int_0^t (J \nabla H(x(s)) - Jd)ds + c - x(0) = \int_0^t J \nabla H(x(s))ds + x(0).\]

Therefore \(x \in C^1[0, 1]\) and satisfies \(\dot{x} = J \nabla H(x)\), which implies that \(x\) is smooth. \(\square\)

**Proposition 2.11** If \(H \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})\) is \(\Psi\)-nonresonant, i.e., for \(|z|\) sufficiently large, \(H(z) = a|z|^2 + \langle z, z_0 \rangle + b\) with \(z_0 \in \text{Fix}(\Psi)\) and \(a, b \in \mathbb{R}\) such that \(\det(e^{2nJ} - \Psi) \neq 0\), then each sequence \((x_k) \subset \mathbb{E}\) such that \(\nabla \Phi_H(x_k) \to 0\) has a convergent subsequence. In particular, \(\Phi_H\) satisfies the (PS) condition.

**Proof** Since \(\nabla \Phi_H(x) = x^+ - x^- - \nabla b(x)\) for any \(x \in \mathbb{E}\), we have

\[x_k^+ - x_k^- - \nabla b(x_k) \to 0. \quad (2.24)\]

**Case 1.** \((x_k)\) is bounded in \(\mathbb{E}\). Then \((x_k^0)\) is a bounded sequence in \(\text{Ker}(\Lambda)\) which has finite dimension. Hence \((x_k^0)\) has a convergent subsequence. Moreover, since \(\nabla b\) is compact, \((\nabla b(x_k))\) has a convergent subsequence and so both \((x_k^+)\) and \((x_k^-)\) have convergent subsequences in \(\mathbb{E}\). Hence \((x_k)\) has a convergent subsequence.

**Case 2.** \((x_k)\) is unbounded in \(\mathbb{E}\). Without loss of generality, we may assume

\[\lim_{k \to +\infty} \|x_k\|_E = +\infty.\]

Let \(y_k = \frac{x_k}{\|x_k\|_E} - \frac{1}{2a} z_0\) where \(z_0 \in \text{Fix}(\Psi)\). Then \(|y_k^0| \leq \|y_k\|_E \leq 1 + \frac{|z_0|}{2a}\) and (2.24) implies

\[y_k^+ - y_k^- - j^* \left( \frac{\nabla H(x_k)}{\|x_k\|_E} \right) \to 0. \quad (2.25)\]
Note that by (2.18) we have
\[
\left\| \frac{\nabla H(x_k)}{\|x_k\|_E} \right\|_{L^2}^2 \leq \frac{8a^2 \|x_k\|_{L^2}^2 + 2C_1^2}{\|x_k\|_E^2} \leq C_6
\]
for some constant \(C_6 > 0\), that is, \( (\nabla H(x_k)/\|x_k\|_E) \) is bounded in \(L^2\). Hence the sequence \( j^* \left( \frac{\nabla H(x_k)}{\|x_k\|_E} \right) \) is compact. (2.25) shows that \((y_k)\) has a convergent subsequence in \(E\). Without loss of generality, we may assume that \(y_k \to y\) in \(E\). Since (H2) implies
\[
H(z) = Q(z) := a|z|^2 + \langle z, z_0 \rangle + b
\]
for \(|z|\) sufficiently large, there exists a constant \(C_7 > 0\) such that
\[
|\nabla H(z) - \nabla Q(z)| \leq C_7, \quad \forall z \in \mathbb{R}^{2n}.
\]
It follows that
\[
\left\| \frac{\nabla H(x_k)}{\|x_k\|_E} - \nabla Q(y) \right\|_{L^2} \leq \left\| \frac{\nabla H(x_k)}{\|x_k\|_E} - \nabla Q(y_k) \right\|_{L^2} + \|\nabla Q(y_k) - \nabla Q(y)\|_{L^2} \\
\leq \left\| \frac{\nabla H(x_k) - \nabla Q(x_k)}{\|x_k\|_E} \right\|_{L^2} + \frac{|z_0|}{\|x_k\|_E} + 2a \|y_k - y\|_{L^2} \\
\leq \frac{C_7}{\|x_k\|_E} + \frac{|z_0|}{\|x_k\|_E} + 2a \|y_k - y\|_{L^2} \to 0
\]
as \(k \to \infty\). This implies that \( j^* \left( \frac{\nabla H(x_k)}{\|x_k\|_E} \right) \) tends to \( j^* (\nabla Q(y)) \) in \(E\), and thus we arrive at
\[
y^+ - y^- - j^*(\nabla Q(y)) = 0 \quad \text{and} \quad \left\| y + \frac{z_0}{2a} \right\|_E = 1.
\]
Arguments similar to Proposition 2.10 show that \(y\) is smooth and satisfies
\[
\dot{y} = J \nabla Q(y) \quad \text{and} \quad y(1) = \Psi y(0). \tag{2.26}
\]
Then we have
\[
y(t) + \frac{1}{2a} z_0 = e^{2a J t} (y(0) + \frac{1}{2a} z_0).
\]
Noting that \(z_0 \in \text{Fix}(\Psi)\), by the second condition in (2.26) we deduce that
\[
y(1) + \frac{1}{2a} z_0 = e^{2a J} (y(0) + \frac{1}{2a} z_0) = \Psi (y(0) + \frac{1}{2a} z_0).
\]
Since $H$ is $\Psi$-nonresonant, i.e. $\det(e^{2\alpha J} - \Psi) \neq 0$, we get that \(y(0) + \frac{1}{2\alpha} z_0 = 0\), which implies that \(y(t) + \frac{1}{2\alpha} z_0 \equiv 0\). However, we have already got that \(\|y + \frac{1}{2\alpha} z_0\| = 1\). This contradiction shows that the second case does not occur. \(\square\)

3 Proof of Theorem 1.4

By the assumptions in Theorem 1.4 $D$ contains a fixed point $p$ of $\Psi$. The symplectomorphism

\[
\phi : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0), \ x \mapsto x - p
\]

satisfies $\phi \circ \Psi = \Psi \circ \phi$. Hence $c_{\text{HZ}}(D, \omega_0) = c_{\text{HZ}}(\phi(D), \omega_0)$. Moreover, for a (generalized) $\Psi$-characteristic $z : [0, T] \to \partial D$, it is easily checked that $y = \phi \circ z$ is a (generalized) $\Psi$-characteristic on $\partial (\phi(D)) = \phi(\partial D)$ and satisfies $y(T) = \Psi(y(0))$ and $A(y) = A(z)$. Hence from now on we may assume $p = 0$, i.e. $0 \in \text{int}(D)$ in this section.

3.1 Proof of (1.10)

The goal of this subsection is to establish the existence of a generalized $\Psi$-characteristic with minimal action via the Clarke dual variational principle in [9] (see also [27, 38] in smooth case and [4, 15] in nonsmooth case for detailed arguments). The steps are classic besides that we need to take the boundary condition into consideration. For the sake of clearness, we present the detailed proof.

Let $j_D : \mathbb{R}^{2n} \to \mathbb{R}$ be the Minkowski (or gauge) functional associated to $D$. Then the Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $H(z) = (j_D(z))^2$ is convex (and so continuous by [42, Cor.10.1.1] or [30, Prop.2.31]). There exists some constant $R_1 \geq 1$ such that

\[
\frac{|z|^2}{R_1} \leq H(z) \leq R_1 |z|^2 \quad \forall z \in \mathbb{R}^{2n}.
\]

This implies that the Legendre transformation of $H$ defined by

\[
H^*(w) = \max_{\xi \in \mathbb{R}^{2n}} (\langle w, \xi \rangle - H(\xi)),
\]

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product, is a convex function from $\mathbb{R}^{2n}$ to $\mathbb{R}$ (and thus continuous). Moreover, there exists a constant $R_2 \geq 1$ such that

\[
\frac{|z|^2}{R_2} \leq H^*(z) \leq R_2 |z|^2 \quad \forall z \in \mathbb{R}^{2n}.
\]

Note that $H^*$ is also $C^{1,1}$ in $\mathbb{R}^{2n}$ with uniformly Lipschitz constant if $S$ is $C^{1,1}$ and strictly convex. (See [42, Cor.10.1.1].)
Recall that in Sect. 2 we denote by $E_1 \subset \mathbb{R}^{2n}$ the eigenvector space which belongs to the eigenvalue 1 of $\Psi$ and $E_1 \perp$ the orthogonal complement of $E_1$ with respect to the standard Euclidean inner product in $\mathbb{R}^{2n}$. When $\dim E_1 \perp = 0$, the problem reduces to the periodic case. Hence we only consider the non-periodic case in which $\dim E_1 \perp \geq 1$.

Consider the following subspace of $H^1([0, 1], \mathbb{R}^{2n})$

$$\mathcal{F} = \{ x \in H^1([0, 1], \mathbb{R}^{2n}) \mid x(1) = \Psi x(0) \& x(0) \in E_1 \perp \}$$  \hspace{1cm} (3.4)

and its subset

$$\mathcal{A} = \{ x \in \mathcal{F} \mid A(x) = 1 \}, \text{ where } A(x) = \frac{1}{2} \int_0^1 \langle -J\dot{x}, x \rangle dt.$$  \hspace{1cm} (3.5)

Then $\mathcal{A}$ is a regular submanifold of $\mathcal{F}$. In fact, for any $x \in \mathcal{F}$ and $\zeta \in T_x \mathcal{F} = \mathcal{F}$,

$$dA(x)[\zeta] = \int_0^1 \langle -J\dot{x}, x \rangle dt + \frac{1}{2} \langle -Jx, \zeta \rangle|_0^1 = \int_0^1 \langle -J\dot{x}, x \rangle dt$$

since

$$\langle Jx(1), \zeta(1) \rangle - \langle Jx(0), \zeta(0) \rangle = \langle J\Psi x(0), \Psi \zeta(0) \rangle - \langle Jx(0), \zeta(0) \rangle = \langle J\Psi Jx(0), \zeta(0) \rangle - \langle Jx(0), \zeta(0) \rangle = 0.$$  \hspace{1cm}

Thus $dA \neq 0$ on $\mathcal{A}$ because

$$dA(x)[x] = \int_0^1 \langle -J\dot{x}, x \rangle dt = 2, \quad \forall x \in \mathcal{A} = A^{-1}(1).$$

**Step 1** The functional $I : \mathcal{F} \rightarrow \mathbb{R}$ defined by

$$I(x) = \int_0^1 H^*(-J\dot{x}) dt$$

has a positive infimum on $\mathcal{A}$ denoted by $\mu := \inf_{x \in \mathcal{A}} I(x)$. In fact for $x \in \mathcal{F}$

$$\int_0^1 \dot{x} dt = x(1) - x(0) = (\Psi - I)x(0), \quad \text{where } x(0) \in E_1 \perp.$$  \hspace{1cm}

Hence $|x(0)| \leq C \|\dot{x}\|_{L^2}$ where $C$ is given by (2.3). Then it is easily estimated that

$$\|x\|^2_{L^2} = \int_0^1 \left| \int_0^t \dot{x}(s) ds + x(0) \right|^2 dt$$
\[
\leq 2 \int_0^1 \left( \left| \int_0^t \dot{x}(s) \, ds \right|^2 + |x(0)|^2 \right) \, dt \\
= 2(\|\dot{x}\|_{L^2}^2 + |x(0)|^2) \\
\leq 2(1 + C^2)\|\dot{x}\|_{L^2}^2.
\]

Moreover, if \( x \in A \) then there holds
\[
2 = 2A(x) \leq \|x\|_{L^2} \|\dot{x}\|_{L^2} \leq \sqrt{2(1 + C^2)}\|\dot{x}\|_{L^2}^2.
\]

It follows from these and (3.3) that for any \( x \in A \)
\[
I(x) = \int_0^1 H^*(\dot{x}) \, dt \geq \frac{1}{R^2} \|\dot{x}\|_{L^2}^2 \geq C_1 := \frac{2}{R_2\sqrt{2(1 + C^2)}}.
\]

**Step 2** There exists \( u \in A \) such that \( I(u) = \mu \). Let \( (x_n) \subset A \) be a sequence satisfying \( \lim_{n \to +\infty} I(x_n) = \mu \). By (3.8) for some \( C_2 > 0 \) we have
\[
R_2C_1 \leq \|x_n\|_{L^2}^2 \leq R_2I(x_n) \leq C_2, \quad n = 1, 2, \ldots.
\]

It follows from (3.6) and (3.7) that for all \( n \in \mathbb{N} \),
\[
\frac{2}{C_2} \leq \frac{2}{\|\dot{x}_n\|_{L^2}^2} \leq \|x_n\|_{L^2}^2 \leq 2(1 + C^2)\|\dot{x}_n\|_{L^2}^2 \leq 2(1 + C^2)C_2.
\]

Hence \( (x_n) \) is a bounded sequence in \( H^1([0, 1], \mathbb{R}^{2n}) \). After passing to a subsequence if necessary, we may assume that \( (x_n) \) converges weakly to some \( u \) in \( H^1([0, 1], \mathbb{R}^{2n}) \). By the Arzelá-Ascoli theorem, there also exists \( \hat{u} \in C_0^0([0, 1], \mathbb{R}^{2n}) \) such that
\[
\lim_{n \to +\infty} \sup_{t \in [0, 1]} |x_n(t) - \hat{u}(t)| = 0.
\]

Then a standard argument gives that \( \hat{u}(t) = u(t) \) almost everywhere. Since \( x_n \to u \) in \( C_0^0([0, 1], \mathbb{R}^{2n}) \), we get that \( u(1) = \Psi u(0) \) and \( u(0) \in E_1^\perp \). Moreover, \( u \in A \) because
\[
A(u) = \frac{1}{2} \int_0^1 \langle Ju, \dot{u} \rangle \, dt = \lim_{n \to +\infty} \frac{1}{2} \int_0^1 \langle Ju, \dot{x}_n \rangle \, dt \\
= \lim_{n \to +\infty} \frac{1}{2} \int_0^1 (\langle Jx_n, \dot{x}_n \rangle + \langle J(u - x_n), \dot{x}_n \rangle) \, dt = 1.
\]

Consider the functional
\[
\hat{I} : L^2([0, 1], \mathbb{R}^{2n}) \to \mathbb{R}, \quad u \mapsto \int_0^1 H^*(u(t)) \, dt.
\]
Then $I(x) = \hat{I}(-J\dot{x})$ for any $x \in \mathcal{F}$. Since $H^*$ is convex, so is $\hat{I}$. (3.3) also implies that $\hat{I}$ is continuous and thus has nonempty subdifferential $\partial \hat{I}(v)$ at each point $v \in L^2([0, 1], \mathbb{R}^{2n})$. Moreover, by Corollary 3 in [15, Chap. II, §3] we know

$$\partial \hat{I}(v) = \{w \in L^2([0, 1], \mathbb{R}^{2n}) \mid w(t) \in \partial H^*(v(t)) \text{ a.e. on } [0, 1]\}.$$ 

By definition of subdifferential it follows that

$$I(u) - I(x_n) = \hat{I}(-J\dot{u}) - \hat{I}(-J\dot{x}_n) \leq \int_0^1 \langle w(t), -J(\dot{u}(t) - \dot{x}_n(t)) \rangle \, dt \quad (3.9)$$

for any $w \in \partial \hat{I}(-J\dot{u}) = \{w \in L^2([0, 1], \mathbb{R}^{2n}) \mid w(t) \in \partial H^*(-J\dot{u}(t)) \text{ a.e. on } [0, 1]\}$. Since that $(x_n)$ converges weakly to some $u$ in $H^1([0, 1], \mathbb{R}^{2n})$ implies that $(\dot{x}_n)$ converges weakly to some $\dot{u}$ in $L^2([0, 1], \mathbb{R}^{2n})$, we deduce that the left hand of (3.9) converges to 0. Therefore

$$\mu \leq I(u) \leq \lim_{n \to +\infty} I(x_n) = \mu.$$ 

The desired claim is proved.

**Step 3** There exists a generalized $\Psi$-characteristic on $\mathcal{S}$, $x^* : [0, \mu] \to \mathcal{S}$, such that $A(x^*) = \mu$. Since $u$ is the minimum point of $I|_A$, applying the Lagrange multiplier theorem (cf. [11, Theorem 6.1.1]) we get some $\lambda \in \mathbb{R}$ such that $0 \in \partial(I + \lambda A)(u) = \partial I(u) + \lambda A'(u)$. Let us write $\Lambda_\mathcal{F}$ as the operator $\Lambda$ in (2.1) viewed as an operator from the Hilbert space $\mathcal{F}$ equipped with the $H^1$ norm to the Hilbert space $L^2$. It is a closed linear operator. Then $I = \hat{I} \circ \Lambda_\mathcal{F}$ and by Corollary 6 in [15, Chap.II, §2] we arrive at

$$\partial I(u) = (\Lambda_\mathcal{F})^* \partial \hat{I}(\Lambda_\mathcal{F}(u))$$

$$= \{(\Lambda_\mathcal{F})^* w \mid w \in L^2([0, 1], \mathbb{R}^{2n}) \& w(t) \in \partial H^*(-J\dot{u}(t)) \text{ a.e. on } [0, 1]\}.$$ 

Hence there exists a function $w \in L^2([0, 1], \mathbb{R}^{2n})$ with $w(t) \in \partial H^*(-J\dot{u}(t))$ a.e. on $[0, 1]$ such that $(\Lambda_\mathcal{F})^* w + \lambda A'(u) = 0$, i.e.,

$$0 = \int_0^1 \langle w(t), -J\dot{\zeta}(t) \rangle \, dt + \lambda \int_0^1 \langle u(t), -J\dot{\zeta}(t) \rangle \, dt \quad \forall \zeta \in \mathcal{F}.$$ 

This implies

$$w(t) + \lambda u(t) = a_0 \quad \text{a.e. on } [0, 1] \quad (3.10)$$

for some $a_0 \in \text{Ker}(\Psi - I)$. Then

$$\langle w, -J\dot{u} \rangle = \int_0^1 \langle w(t), -J\dot{u}(t) \rangle \, dt.$$ 

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\[
= \int_0^1 \langle a_0 - \lambda u(t), -J \dot{u}(t) \rangle dt = -2\lambda. \tag{3.11}
\]

Since the convex functional \( \hat{I} \) is 2-positively homogeneous, we use the Euler formula \([50, \text{Theorem 3.1}] \) to obtain

\[
\langle w, -J \dot{u} \rangle = 2 \hat{I}(\dot{J}u) = 2I(u) = 2\mu
\]
and thus \( -\lambda = \mu \) by (3.11). By (3.10), \( a_0 + \mu u(t) = w(t) \in \partial H^*(\dot{J}u(t)) \) and so \( -J \dot{u} \in \partial H(a_0 + \mu u(t)) \) a.e. on \([0,1]\). Define \( v : [0,\mu] \to \mathbb{R}^{2n} \) by

\[
v(t) := \mu u(t/\mu) + a_0. \tag{3.12}
\]

Then \( v \) satisfies

\[-J \dot{v}(t) \in \partial H(v(t)) \text{ a.e. on } [0,\mu] \tag{3.13}\]
and

\[
\langle v(\mu), -J \dot{v}(\mu) \rangle = \mu u(1) + a_0 = \Psi(\mu u(0) + a_0) = \Psi v(0).
\]

The Legendre reciprocity formula (cf. [15, Proposition II.1.15]) in convex analysis yields

\[
\int_0^\mu H(v(t))dt = -\int_0^\mu H^*(-J \dot{v}(t))dt + \int_0^1 \langle v(t), -J \dot{v}(t) \rangle dt
\]

\[= -\mu \int_0^1 H^*(-J \dot{u}(s))ds + \mu^2 \int_0^1 \langle -J \dot{u}(s), u(s) \rangle ds
\]

\[= -\mu^2 + 2\mu^2 = \mu^2. \tag{3.14}\]

By [32, Theorem 2] (3.13) implies \( H(v(t)) \) is constant and hence by (3.14) \( H(v(t)) \equiv \mu \) for all \( t \in [0,\mu] \). It follows that \( v \) is nonconstant and that

\[
x^* : [0,\mu] \to \mathcal{S}, \ t \mapsto \frac{v(t)}{\sqrt{\mu}} = \sqrt{\mu}u(t/\mu) + a_0/\sqrt{\mu} \tag{3.15}
\]
satisfies

\[-J \dot{x}^*(t) \in \partial H(x^*(t)) \text{ a.e. on } [0,\mu]\]

and

\[
x^*(\mu) = \Psi x^*(0), \ H(x^*(t)) \equiv 1, \ A(x^*) = \mu.
\]

That is to say, \( x^* \) is a \( \Psi \)-characteristic on \( \mathcal{S} \) with action \( A(x^*) = \mu \).
For any generalized $\Psi$-characteristic $y$ on $S$ with positive action, there holds $A(y) \geq \mu$. By Lemma 2 in [15, Chap.V,§1] (or its proof), after reparameterization we may assume that $y : [0, T) \mapsto S$ is an absolutely continuous map satisfying

$$- J\dot{y}(t) = \partial H(y(t)), \text{ a.e., } y(T) = \Psi y(0).$$

(3.16)

(See [28, Lemma 4.2] for details). Then

$$A(y) = T \text{ and } H(y(t)) = 1.$$  

(3.17)

Since $\{w \in \partial H(x) \mid x \in S\}$ is a bounded set in $\mathbb{R}^{2n}$ (by the proof of [28, Lemma 4.2]), $y$ is in $W^{1,\infty}([0, T], \mathbb{R}^{2n})$ and in particular $y \in W^{1,2}([0, T], \mathbb{R}^{2n})$. Choose $a \in \mathbb{R}$ and $b \in E_1$ so that

$$y^* : [0, 1] \to \mathbb{R}^{2n}, t \mapsto y^*(t) = ay(tT) + b$$

belongs to $A$. Then $1 = A(y^*) = a^2 A(y) = a^2 T$. Since

$$-J\dot{y}^*(t) = -aT J\dot{y}(tT) \in aT \partial H(y(Tt)) = \partial H(aT y(Tt)) \text{ a.e. on } [0, 1],$$

there holds $aT y(Tt) \in \partial H^*(-J\dot{y}^*(t))$ a.e. on $[0, 1]$ and the Legendre reciprocity formula (cf. [15, Proposition II.1.15]) in convex analysis yields

$$H^*(-J\dot{y}^*(t)) = -H(aT y(Tt)) + \langle -J\dot{y}^*(t), aT y(Tt) \rangle$$

$$= -(aT)^2 H(y(t)) + \langle -aT J\dot{y}(Tt), aT y(Tt) \rangle$$

$$= -(aT)^2 + (aT)^2 \langle -J\dot{y}(Tt), y(Tt) \rangle$$

$$= -(aT)^2 + 2(aT)^2 H(y(Tt)) = (aT)^2 = T, \text{ a.e. on } [0, 1],$$

where the fourth equality comes from the Euler formula [50, Theorem 3.1]. Hence

$$H^*(-J\dot{y}^*(t)) = T \text{ a.e. on } [0, 1]$$

so that

$$\int_0^1 H^*(-J\dot{y}^*(t))dt = T.$$  

The definition of $\mu$ implies $T \geq \mu$ and it follows that $A(y) \geq \mu$ by (3.17).

Remark 3.1 From the above proof we see that $u$ given by Step 2 satisfies

$$\min\{A(x) > 0 \mid x \text{ is a generalized } \Psi\text{-characteristic on } S\}$$

$$= A(x^*)$$

$$= I(u)$$

$$= \min\{I(x) \mid x \in F \& A(x) = 1\}.$$  

(3.18)

Moreover, as in the periodic case we have

$$\min\{I(x) \mid x \in F \& A(x) = 1\} = (\max\{A(x) \mid x \in F \& I(x) = 1\})^{-1}.$$  

(3.19)
3.2 A Key Lemma

Lemma 3.2. Let \( D \subset \mathbb{R}^{2n} \) be a compact convex domain with boundary \( S = \partial D \) and \( 0 \in \text{int}(D) \). If \( S \) is of class \( C^{2n+2} \), then the set

\[
\Sigma^\Psi_S := \{ A(x) \mid A(x) > 0 \text{ and } x \text{ is a } \Psi\text{-characteristic on } S \}
\]

is a nowhere dense set in \( \mathbb{R} \).

This lemma shows that the set of actions of \( \Psi\)-characteristics on a suitable smooth convex hypersurface has no interior point, which is crucial for the proof of (1.11) in Theorem 1.4. In the proof of [26, Proposition 4] (the case \( \Psi = I_{2n} \) of Theorem 1.4) the authors chose a smooth strictly convex surface \( \tilde{S} \) near \( S \) so that the set of actions of periodic orbits on \( \tilde{S} \) is discrete ([26, page 422]). We do not know whether there exists a similar result in our case. For the action spectrum of a Hamiltonian map with compact support there exists a similar result, Proposition 8 on the page 152 of [27], which was proved by Hofer and Zehnder in [27, pages 153–154] with an idea due to Sikorav [44]. In order to avoid the arguments of smoothness for the action functional on \( H^1(S^1, \mathbb{R}^{2n}) \) as given in [27, Appendix 3] we shall use new techniques to discuss the smoothness of the action functional on a suitable subspace in \( E^1 \).

Fix \( 1 < \alpha < 2 \). Since \( S \) is of class \( C^{2n+2} \), the Minkowski functional \( j_D : \mathbb{R}^{2n} \to \mathbb{R} \) is \( C^{2n+2} \) in \( \mathbb{R}^{2n} \setminus \{0\} \) so that the \( C^1 \) Hamiltonian function

\[
F: \mathbb{R}^{2n} \to \mathbb{R}, \quad z \mapsto (j_D(z))^\alpha
\]

is also \( C^{2n+2} \) in \( \mathbb{R}^{2n} \setminus \{0\} \). To clarify the elements in the set \( \Sigma^\Psi_S \), we only need to consider the action of \( \Psi\)-characteristic \( x : [0, T^*] \to S \) that satisfies

\[
\dot{x} = J \nabla F(x) \quad \text{and} \quad x(T^*) = \Psi(x(0)). \tag{3.20}
\]

Clearly such a \( \Psi\)-characteristic \( x \) has the action

\[
A(x) = \alpha T^*/2. \tag{3.21}
\]

We only need to prove for an arbitrarily fixed \( \sigma = \alpha T^*/2 \in \Sigma^\Psi_S \) that \( \Sigma^\Psi_S \cap (\sigma - \epsilon, \sigma + \epsilon) \) is a nowhere dense set for some sufficiently small positive number \( \epsilon \). To this end, let us choose \( 0 < \epsilon_1 < \epsilon_2 \) such that \( B^{2n}(\epsilon_1) \subset \subset B^{2n}(\epsilon_2) \subset \subset D \) and

\[
\max_{z \in B^{2n}(\epsilon_2)} F(z) < \left( \frac{2(\sigma + \epsilon)}{\alpha} \right)^{\sigma/2}.
\]

Take a smooth function \( f : \mathbb{R}^{2n} \to [0, 1] \) such that

\[
0 \leq f \leq 1, \quad f|_{B_{\epsilon_1}} = 0 \quad \text{and} \quad f|_{B_{\epsilon_2}} = 1
\]
and define a Hamiltonian $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$
\mathcal{F}(z) = f(z) F(z) = f(z)(j_D(z))^\alpha, \ \forall z \in \mathbb{R}^{2n}
$$

so that $\mathcal{F} \in C^{2n+2}(\mathbb{R}^{2n}, \mathbb{R})$. If $x \in C^1([0, T], \mathbb{R}^{2n})$ satisfies

$$
\begin{align*}
\dot{x} &= J \nabla F(x), \\
x(T) &= \Psi x(0), \\
\frac{aT}{2} &\in (\sigma - \epsilon, \sigma + \epsilon)
\end{align*}
$$

it is easily computed that

$$
y : [0, 1] \to \mathbb{R}^{2n}, \ t \mapsto y(t) = T \frac{1}{\sigma^2} x(tT)
$$

fulfils

$$
\dot{y}(t) = J \nabla F(y(t)), \ y(1) = \Psi y(0) \ \text{and} \ F(y(t)) = T \frac{\sigma}{\sigma^2} \geq \left( \frac{2(\sigma + \epsilon)}{\alpha} \right)^{\frac{\alpha}{\alpha-2}}.
$$

Hence $y(t) \subset (B_{\varepsilon_2})^c, \ \forall t \in [0, 1]$. Since $\mathcal{F} = F$ on $(B_{\varepsilon_2})^c$, we have

$$
\dot{y} = J \nabla \mathcal{F}(y), \ y(1) = \Psi y(0), \ \text{and} \ \mathcal{F}(y(t)) = F(y(t)) \ \forall t.
$$

Let $E = E^{\frac{1}{2}}$ be defined by (2.9). Then $y$ is a critical point of the functional

$$
\Phi_{\mathcal{F}} : E \to \mathbb{R}, \ x \mapsto \frac{1}{2} \|x^+\|_{\mathbb{E}}^2 - \frac{1}{2} \|x^-\|_{\mathbb{E}}^2 - \int_0^1 \mathcal{F}(x(t)) dt
$$

(which is well-defined since $E$ embeds continuously into $L^2$ and so into $L^\alpha$ for $1 < \alpha < 2$), and a direct computation yields

$$
\Phi_{\mathcal{F}}(y) = \frac{1}{2} \int_0^1 \langle -J \dot{y}, y \rangle - \int_0^1 F(y(t)) \\
= \left( \frac{\alpha}{2} - 1 \right) F(y(t)) \\
= \left( \frac{\alpha}{2} - 1 \right) T \frac{\sigma}{\sigma^2}.
$$

Note that all critical points of $\Phi_{\mathcal{F}}$ sit in the Banach space

$$
C_{\Psi}^{2n+2} := \{ z \in C^{2n+2}([0, 1], \mathbb{R}^{2n}) | z(1) = \Psi z(0) \}
$$

which is a subspace of $C^{2n+2}([0, 1], \mathbb{R}^{2n})$. In particular, $\Phi_{\mathcal{F}}|_E$ and $\Phi_{\mathcal{F}}|_{C_{\Psi}^1}$ have the same critical value sets.

**Claim 3.3** $\Phi_{\mathcal{F}}|_{C_{\Psi}^1}$ is of class $C^{2n+1}$.
In order to prove this we need the following result from the page 780 of [20].

**Proposition 3.4** Let $M$, $N$ and $P$ be finite-dimensional $C^\infty$-manifolds. If both $M$ and $N$ are compact then the map $\text{comp}: C^{r+s}(N, P) \times C^r(M, N) \to C^r(M, P)$ given by $\text{comp}(f, g) = f \circ g$ is $C^s$. In particular, the evaluation map $C^r(N, P) \times N \to P$ is $C^s$.

**Proof of Claim 3.3** It suffices to prove that $\Phi_\mathcal{F}$ is $C^{2n+1}$ in any ball $B(\hat{x}, R) \subset C^1_\Psi$ which is centered at $\hat{x} \in C^1_\Psi$ and has radius $R$. Observe that there exists $\hat{R} > 0$ such that

$$
x([0, 1]) \subset B^{2n}(0, \hat{R}) \quad \forall x \in B(\hat{x}, R).
$$

Hence $B(\hat{x}, R)$ is contained in

$$
C^1_\Psi([0, 1], B^{2n}(0, \hat{R})) := \{ z \in C^1([0, 1], B^{2n}(0, \hat{R})) \mid z(1) = \Psi z(0) \}
$$

which is an open subset of $C^1_\Psi$. Let $\bar{B}^{2n}(0, \hat{R})$ denote the closure of $B^{2n}(0, \hat{R})$. By Proposition 3.4 and the $C^{2n+2}$-smoothness of $\mathcal{F}$, the map

$$
C^1_\Psi([0, 1], \bar{B}^{2n}(0, \hat{R})) \ni x \mapsto \int_0^1 \mathcal{F}(x(t))dt \in \mathbb{R}.
$$

is $C^{2n+1}$ and so is

$$
C^1_\Psi([0, 1], \bar{B}^{2n}(0, \hat{R})) \ni x \mapsto \Phi_\mathcal{F}(x) = \frac{1}{2} \| x^+ \|_\mathcal{F}^2 - \frac{1}{2} \| x^- \|_\mathcal{F}^2 - \int_0^1 \mathcal{F}(x(t))dt
$$

is $C^{2n+1}$ since $C^1_\Psi([0, 1], \bar{B}^{2n}(0, \hat{R})) \hookrightarrow E^{1/2}$ is smooth. This implies the expected claim. $\square$

Take a smooth function $g : [0, 1] \to [0, 1]$ such that $g$ equals 1 (resp. 0) near 0 (resp. 1). Denote by $\phi^t$ the flow of $X_\mathcal{F}$. Then by the $C^{2n+1}$-smoothness of $X_\mathcal{F}$ we have a $C^{2n+1}$ map

$$
\psi : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \ (t, z) \mapsto g(t)\phi^t(z) + (1 - g(t))\phi^{t-1}(\Psi z).
$$

Clearly $\psi(0, z) = \phi^0(z) = z$, $\psi(1, z) = \phi^0(\Psi z) = \Psi z$ and thus $\psi(1, z) = \Psi \psi(0, z)$. Moreover, for each $z \in \mathbb{R}^{2n}$ satisfying $\phi^1(z) = \Psi z$, it holds that

$$
\psi(t, z) = g(t)\phi^t(z) + (1 - g(t))\phi^{t-1}(\Psi z)
$$

$$
= g(t)\phi^t(z) + (1 - g(t))\phi^{t-1}(\phi^1(z))
$$
\[
= g(t)\phi^t(z) + (1 - g(t))\phi^t(z) \\
= \phi^t(z), \quad \forall \ 0 \leq t \leq 1.
\]

Clearly, Proposition 3.4 implies that the composition

\[
\text{comp} : C^{r+s}(N \times M, P) \times C^r(M, N \times M) \to C^r(M, P), \ (f, g) \mapsto f \circ g
\]

is \(C^s\). Note that the map \(N \ni p \mapsto g_p \in C^r(M, N \times M)\) defined by \(g_p(m) = (p, m)\) for \(m \in M\) is smooth. Each map \(\varphi \in C^{r+s}(N \times M, P)\) gives rise to a \(C^s\) map \(N \to C^r(M, P), \ p \mapsto \text{comp}(\varphi, g_p) = \varphi(p, \cdot)\).

Applying this claim to \(M = [0, 1], N = \bar{B}^{2n}(0, R)\) for any given \(R > 0, P = \mathbb{R}^{2n}\) and the \(C^{2n+1}\) map

\[
\varphi = \psi|_{[0, 1] \times \bar{B}^{2n}(0, R)} : [0, 1] \times \bar{B}^{2n}(0, R), \ (t, z) \mapsto \psi(t, z) \in \mathbb{R}^{2n}
\]

we deduce that \(\bar{B}^{2n}(0, R) \ni z \mapsto \psi(\cdot, z) \in C^1_\psi\) is \(C^{2n}\) since \(\psi(\cdot, z) : [0, 1] \to \mathbb{R}^{2n}\) sits in \(C^1_\psi\). So

\[
\Omega : \mathbb{R}^{2n} \to C^1_\psi, \ z \mapsto \Omega(z) = \psi(\cdot, z) \quad (3.24)
\]

is \(C^{2n}\). (It is not hard to prove this directly!) This and Claim 3.3 show that the composition

\[
\Phi_{\bar{F}}|_{C^1_\psi} \circ \Omega : \mathbb{R}^{2n} \to \mathbb{R} \quad (3.25)
\]

is of class \(C^{2n}\). Note that every critical point \(y\) of the functional \(\Phi_{\bar{F}}|_{C^1_\psi}\) has the form \(y(t) = \phi^t(z_y)\) for some \(z_y \in \mathbb{R}^{2n}\) satisfying \(\phi^1(z_y) = \Psi z_y\) and thus \(y = \Omega(z_y)\). Hence \(z_y\) is a critical point of \(\Phi_{\bar{F}}|_{C^1_\psi} \circ \Omega\). In particular, the critical values of \(\Phi_{\bar{F}}|_{C^1_\psi}\) (and hence \(\Phi_{\bar{F}}\)) are contained in the set of critical values of \(\Phi_{\bar{F}}|_{C^1_\psi} \circ \Omega\) which is a nowhere dense set by Sard theorem.

Now since \(\{\Phi_{\bar{F}}(y) \mid y \in \text{Crit}(\Phi_{\bar{F}})\}\) is a nowhere dense set, so is

\[
\{\Phi_{\bar{F}}(y) \mid y(t) = T^{\frac{1}{n-2}} x(tT) \text{ and } x \text{ satisfies } (3.22)\}.
\]

By (3.21) and (3.23), this set is equal to

\[
\left\{ \left(\frac{\alpha}{2} - 1\right) \left(\frac{2}{\alpha} A(x)\right)^{\frac{\alpha}{n-2}} \mid x \text{ is a } \Psi\text{-characteristic on } S \text{ and } A(x) \in (\sigma - \epsilon, \sigma + \epsilon) \right\}.
\]

Hence

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\[
\sum_{\mathcal{S}}^{\Psi} \cap (\sigma - \epsilon, \sigma + \epsilon) = \left\{ x \mid x \text{ is a } \Psi\text{-characteristic on } \mathcal{S} \text{ and } A(x) \in (\sigma - \epsilon, \sigma + \epsilon) \right\}
\]

is a nowhere dense set. Then Lemma 3.2 follows.

3.3 Proof of (1.11) for Smooth and Strictly Convex \(D\)

The proof is similar to that of [26, Proposition 4] and [27] with slight change. We sketch the proof with some details omitted which can be found in [28].

**Step 1.** Prove

\[
c_{H/Z}^{\Psi}(D, \omega_0) \geq A(x^*). \tag{3.26}
\]

For small \(0 < \epsilon, \delta < 1/2\), pick a smooth function \(f : [0, 1] \to \mathbb{R}\) such that

\[
\begin{align*}
    f(t) &= 0, \quad t \leq \delta, \\
    f(t) &= A(x^*) - \epsilon, \quad 1 - \delta \leq t, \\
    0 &\leq f'(t) < A(x^*), \quad \delta < t < 1 - \delta.
\end{align*}
\]

Define \(H(x) = f(j_D^2(x))\) for \(x \in D\). Then \(H \in \mathcal{H}^{\Psi}(D, \omega_0)\). Let us prove that every solution \(x : [0, T] \to D\) of the boundary value problem

\[
\dot{x} = J\nabla H(x) = f'(j_D^2(x))J\nabla j_D^2(x) \quad \text{and} \quad x(T) = \Psi x(0) \tag{3.27}
\]

with \(0 < T \leq 1\) is constant. By contradiction we assume that \(x = x(t)\) is a nonconstant solution of (3.27). Then \(j_D(x(t))\) is equal to a nonzero constant and thus \(x(t) \neq 0\) for each \(t \in [0, T]\). Moreover, \(f'(j_D^2(x(t))) = a \in (0, A(x^*))\). Since \(\nabla j_D^2(\lambda z) = \lambda \nabla j_D^2(z)\) for all \((\lambda, z) \in \mathbb{R}_+ \times \mathbb{R}^{2n}\), multiplying \(x(t)\) by a suitable positive number we may assume that \(x([0, T]) \subset \mathcal{S} = \partial D\) and

\[
\dot{x} = a J\nabla j_D^2(x) \quad \text{and} \quad x(T) = \Psi x(0). \tag{3.28}
\]

Note that \(\langle \nabla j_D^2(z), z \rangle = j_D^2(z) = 1\) for any \(z \in \mathcal{S}\). We deduce from (3.28) that

\[
0 < A(x) = aT \leq a < A(x^*),
\]

which contradicts (1.10). This shows that \(H \in \mathcal{H}^{\Psi}(D, \omega_0)\) is \(\Psi\)-admissible and hence

\[
c_{H/Z}^{\Psi}(D, \omega_0) \geq m(H) = A(x^*) - \epsilon. \quad \text{Let } \epsilon \to 0 \text{ and we get (3.26).}
\]

**Step 2.** Prove

\[
c_{H/Z}^{\Psi}(D, \omega_0) \leq A(x^*). \tag{3.29}
\]
Let $H \in \mathcal{H}^\Psi(D, \omega_0)$ satisfy $m(H) > A(x^*)$. We wish to prove that the boundary value problem

$$\dot{x} = J \nabla H(x) \quad \text{and} \quad x(1) = \Psi x(0) \quad (3.30)$$

has a nonconstant solution $x : [0, 1] \to D$. By Lemma 3.2 we have a small number $\epsilon > 0$ such that $A(x^*) + \epsilon \notin \Sigma_S^\Psi$ and $m(H) > A(x^*) + \epsilon$. This means that the boundary value problem

$$\dot{x} = (A(x^*) + \epsilon)J \nabla \tilde{J}_D(x) \quad \text{and} \quad x(1) = \Psi x(0) \quad (3.31)$$

admits only the trivial solution $x \equiv 0$. (Otherwise, we have $x(t) \neq 0 \forall t \in [0, 1]$ as above. Thus after multiplying $x(t)$ by a suitable positive number we may assume that $x([0, 1]) \subset S = \partial D$, which leads to $A(x) = A(x^*) + \epsilon$.) For a fixed number $\delta > 0$ we take a smooth function $f : [1, \infty) \to \mathbb{R}$ such that

$$f(t) \geq (A(x^*) + \epsilon)t, \quad t \geq 1,$$

$$f(t) = (A(x^*) + \epsilon)t, \quad t \text{ large},$$

$$f(t) = m(H), \quad 1 \leq t \leq 1 + \delta,$$

$$0 \leq f'(t) \leq A(x^*) + \epsilon, \quad t > 1 + \delta.$$

With this $f$ we get an extension of $H$ as the following

$$\overline{H}(z) = \begin{cases} H(z), & \text{for } z \in D, \\ f(j_D^*(z)), & \text{for } z \notin D. \end{cases}$$

Let $\mathcal{E} = E^{\frac{1}{2}} = \mathcal{E}^- \oplus \mathcal{E}^0 \oplus \mathcal{E}^+$ be as in (2.11) and $\Phi_{\overline{H}}$ be as in (1.21), that is,

$$\Phi_{\overline{H}}(x) = \frac{1}{2}\|x^+\|^2_E - \frac{1}{2}\|x^-\|^2_E - \int_0^1 \overline{H}(x(t))dt. \quad (3.32)$$

If $x$ is a solution of $\dot{x}(t) = X_{\overline{H}}(x(t))$ satisfying $x(1) = \Psi x(0)$ and $\Phi_{\overline{H}}(x) > 0$, then it is nonconstant, sits in $D$ completely, and thus is a solution of $\dot{x} = X_H(x)$ on $D$ ( [28, Lemma 4.6]). Hence we only need to find a critical point of $\Phi_{\overline{H}}(x)$ with positive critical value (see [26, Lemma 4]).

The fact that $A(x^*) + \epsilon \notin \Sigma_S^\Psi$ implies the following lemma.

**Lemma 3.5** If a sequence $(x_k) \subset \mathcal{E}$ is such that $\nabla \Phi_{\overline{H}}(x_k) \to 0$ in $\mathcal{E}$, then it has a convergent subsequence in $\mathcal{E}$. In particular, $\Phi_{\overline{H}}$ satisfies the Palais-Smale condition.

**Proof** If $(x_k)$ is bounded in $\mathcal{E}$, as in the proof of Proposition 2.11 we deduce that $(x_k)$ has a convergent subsequence. Without loss of generality, we assume $\lim_{k \to +\infty} \|x_k\|_E = +\infty$. Let $y_k = \frac{x_k}{\|x_k\|_E}$. Then $\|y_k\|_E = 1$ and satisfies

$$y_k^+ - y_k^- - \frac{1}{\|x_k\|_E} \nabla b(x_k) = y_k^+ - y_k^- - j^* \left( \frac{\nabla \overline{H}(x_k)}{\|x_k\|_E} \right) \to 0 \text{ in } \mathcal{E}. \quad (3.33)$$
By the construction of $\overline{H}$ and the proof of Proposition 2.11, passing to a subsequence (if necessary) we may assume that $y_k \to y$ in $E$ so that $\|y\|_E = 1$. Since $H(z) = Q(z) := (A(x^*) + \epsilon)j^2_D(z)$ for $|z|$ large enough, arguing as in the proof of Proposition 2.11 we get

$$
\left\| \frac{\nabla H(x_k)}{\|x_k\|_E} - \nabla Q(y) \right\|_{L^2} \to 0
$$

so that (3.33) becomes

$$
y^+ - y^- - j^* \nabla Q(y) = 0.
$$

Hence $y$ satisfies the boundary value problem (3.31) and thus $y = 0$ because $A(x^*) + \epsilon \notin \Sigma^\Psi_S$. This contradicts the fact $\|y\|_E = 1$. That is, $(x_k)$ must be bounded in $E$. \qed

The $\Psi$-characteristic $x^*$ with minimal action in (1.10) can be reparametrization as $x_0 : [0, 1] \to \mathbb{R}^{2n}$ such that (cf. [28, §4.3])

$$
\begin{align*}
\dot{x}_0 &= A(x^*) J \nabla j^2_D(x_0), \\
x_0(1) &= \Psi x_0(0), \quad A(x_0) = A(x^*), \\
j_D(x_0(t)) &= 1, \quad \text{i.e., } x_0([0, 1]) \subset S.
\end{align*}
$$

(3.34)

Denote by $x_0^+$ the projections of $x_0$ onto $E^+$. Then $x_0^+ \neq 0$. (Otherwise, a contradiction occurs because $0 < A(x^*) = A(x_0) = -\frac{1}{2}\|x_\ominus^0\|_E^2$.) Following [26] we define for $s > 0$ and $\tau > 0$

$$
W_s := E^- \oplus E^0 \oplus sx_0^+, \\
\Sigma_\tau := \{x^- + x^0 + sx_0^+ \mid 0 \leq s \leq \tau, \quad \|x^- + x^0\|_E \leq \tau\}.
$$

Let $\partial \Sigma(\tau)$ denote the boundary of $\Sigma_\tau$ in $E^- \oplus E^0 \oplus \mathbb{R}x_0^+$. Then

$$
\partial \Sigma_\tau = \{x = x^- + x^0 + sx_0^+ \in \Sigma_\tau \mid \|x^- + x^0\|_E = \tau \text{ or } s = 0 \text{ or } s = \tau\}.
$$

(3.35)

Repeating the proofs of Lemmas 5, 6 in [26] leads to

**Lemma 3.6** There exists a constant $C > 0$ such that for any $s \geq 0$,

$$
\Phi_{\overline{H}}(x) \leq -\epsilon \int_0^1 j^2_D(x(t))dt + C, \quad \forall x \in W_s.
$$

**Lemma 3.7** $\Phi_{\overline{H}}|\partial \Sigma_\tau \leq 0$ if $\tau > 0$ is sufficiently large.

Arguing as in the proof of Lemma 9 in Chapter 3 of [27] we get
Lemma 3.8 For $z_0 \in \text{Fix}(\Psi) \cap H^{-1}(0)$, there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\Phi_H|\Gamma \geq \beta > 0,$$

where $\Gamma = \{z_0 + x \mid x \in \mathbb{E}^+ \& \|x\|_E = \alpha\}$.

Let $\phi^t$ be the negative gradient flow of $\Phi_H$. As in [27, pages 95–97] the following lemma can be proved by the standard topological degree method (cf. [28, Lemma 4.10]).

Lemma 3.9 $\phi^t(\Sigma_\tau) \cap \Gamma \neq \emptyset$, $\forall t \geq 0$.

Let $F = \{\phi^t(\Sigma_\tau) \mid t \geq 0\}$ and define

$$c(\Phi_H, F) := \inf_{t \geq 0} \sup_{x \in \phi^t(\Sigma_\tau)} \Phi_H(x).$$

Lemmas 3.8, 3.9 imply

$$0 < \beta \leq \inf_{x \in \Gamma} \Phi_H(x) \leq \sup_{x \in \phi^t(\Sigma_\tau)} \Phi_H(x) \forall t \geq 0,$$

and hence $c(\Phi_H, F) \geq \beta > 0$. On the other hand, since $\Sigma_\tau$ is bounded and Proposition 2.9 implies that $\Phi_H$ maps bounded sets into bounded sets we arrive at

$$c(\Phi_H, F) \leq \sup_{x \in \Sigma_\tau} \Phi_H(x) < \infty.$$

Using the Minimax Lemma on [27, page 79], we get a critical point $x$ of $\Phi_H$ with $\Phi_H(x) > 0$ and (1.11) is proved.

3.4 Completing the Proof of Theorem 1.4 for General Case

By Proposition 1.12 and Corollary 2.41 in [30] we may choose two sequences of $C^\infty$ strictly convex domains with boundaries, $(D_k^+)$ and $(D_k^-)$, such that

(i) $D_1^- \subset D_2^- \subset \cdots \subset D$ and $\bigcup_{k=1}^\infty D_k^- = D$,

(ii) $D_1^+ \supset D_2^+ \supset \cdots \supset D$ and $\bigcap_{k=1}^\infty D_k^+ = D$,

(iii) for any small neighborhood $O$ of $\partial D$ there exists an integer $N > 0$ such that

$$\partial D_k^+ \cup \partial D_k^- \subset O \forall k \geq N.$$

Denote by $j_D$, $j_{D_k^+}$ and $j_{D_k^-}$ the Minkowski functionals of $D$, $D_k^+$ and $D_k^-$ respectively. Let $H = (j_D)^2$, $H_k^+ = (j_{D_k^+})^2$ and $H_k^- = (j_{D_k^-})^2$ for each $k \in \mathbb{N}$. Their Legendre transformations are $H^*$, $H_k^{**}$ and $H_k^{-*}$, $k = 1, 2, \ldots$. Denote by

$$I(u) = \int_0^1 H^*(-J\dot{u}), \quad I_k^+(u) = \int_0^1 H_k^{**}(-J\dot{u}), \quad I_k^-(u) = \int_0^1 H_k^{-*}(-J\dot{u})$$

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for \( u \in \mathcal{A} \). Note that (i) and (ii) imply

(iv) \( j_{D_1^-} \geq j_{D_2^-} \geq \cdots \geq j_D \) and so \( H_1^{**} \leq H_2^{**} \leq \cdots \leq H^* \),

(v) \( j_{D_1^+} \leq j_{D_2^+} \leq \cdots \leq j_D \) and so \( H_1^{**} \geq H_2^{**} \geq \cdots \geq H^* \).

These lead to

\[
I_1^+(u) \geq I_2^+(u) \geq \cdots \geq I(u) \geq \cdots \geq I_2^-(u) \geq I_1^-(u), \quad \forall u \in \mathcal{A}. \tag{3.36}
\]

By the first three steps in Sect. 3.1 these functionals attain their minimums on \( \mathcal{A} \). It easily follows from (3.36) that

\[
\min_{\mathcal{A}} I^+_1 \geq \min_{\mathcal{A}} I^+_2 \geq \cdots \geq \min_{\mathcal{A}} I \geq \cdots \geq \min_{\mathcal{A}} I^-_2 \geq \min_{\mathcal{A}} I^-_1. \tag{3.37}
\]

Now (1.10) gives rise to

\[
\min_{\mathcal{A}} I = \min \{ A(x) > 0 \mid x \text{ is a generalized } \Psi \text{-characteristic on } S \} \tag{3.38}
\]

and (1.10) and (1.11) yield

\[
c_{\Psi HZ}^0(D^+_k, \omega_0) = \min_{\mathcal{A}} I^+_k \text{ and } c_{\Psi HZ}^0(D^-_k, \omega_0) = \min_{\mathcal{A}} I^-_k \tag{3.39}
\]

for each \( k \in \mathbb{N} \). By this, (3.37) and the monotonicity of \( c_{\Psi HZ}^0 \) we get

\[
\begin{align*}
\min_{\mathcal{A}} I^+_k \geq \min_{\mathcal{A}} I \geq \min_{\mathcal{A}} I^-_k.
\end{align*}
\]

Observe that \( \lim_{k \to \infty} c_{\Psi HZ}^0(D^+_k, \omega_0) = c_{\Psi HZ}^0(D, \omega_0) \) and \( \lim_{k \to \infty} c_{\Psi HZ}^0(D^-_k, \omega_0) = c_{\Psi HZ}^0(D, \omega_0) \) by Proposition 1.2(iii). Hence \( c_{\Psi HZ}^0(D, \omega_0) = \min_{\mathcal{A}} I \) by the squeezing theorem in calculus. The desired result follows from this and (3.38).

**Remark 3.10** For \( \Psi \in \text{Sp}(2n, \mathbb{R}) \) and an open set \( O \) in \( \mathbb{R}^{2n} \), let \( \mathcal{H}_0^\Psi \) consist of all \( H \in \mathcal{H}^\Psi \) which vanish near 0, and define

\[
c_{\Psi HZ}^0(O, \omega_0) = \sup \{ \max H \mid H \in \mathcal{H}_0^\Psi(O, \omega_0) \text{ and } H \text{ is } \Psi \text{-admissible} \}. \tag{3.40}
\]

Then \( c^0_{\Psi HZ}(O, \omega_0) \leq c_{\Psi HZ}^0(O, \omega_0) \) and \( c^0_{\Psi HZ}(O, \omega_0) \leq c_{\Psi HZ}^0(O^*, \omega_0) \) for an open subset \( O^* \supseteq O \). If \( B^{2n}(0, r) \subset O \subset B^{2n}(0, R) \) then

\[
c_{\Psi HZ}^0(O, \omega_0) \leq \left( \frac{r}{R} \right)^2 c_{\Psi HZ}^0(O, \omega_0) \tag{[28, Proposition 1.3]}
\]

In particular, if this \( O \) is convex, by the definition of \( c_{\Psi HZ}^0 \) it is easily proved that

\[
c_{\Psi HZ}^0(O, \omega_0) = \sup \{ c_{\Psi HZ}^0(K, \omega_0) \mid K \text{ convex bounded domain, } K \ni 0, \overline{K} \subset O \}. \tag{3.41}
\]
Note that (3.26) and (3.29) imply \( c_{\Psi HZ}(D, \omega_0) = c_{\Psi HZ}(D, \omega_0) \) for a \( C^\infty \) strictly convex bounded domain \( D \subset \mathbb{R}^{2n} \) containing 0. (3.41) and the inner regularity of \( c_{\Psi HZ} \) in (1.7) lead to \( c_{\Psi HZ}(O, \omega_0) = c_{\Psi HZ}(O, \omega_0) \).

### 4 Proof of Theorem 1.14

Our arguments are closely related to Sikorav’s approach in [44]. The proof will be completed by several propositions.

By definition of the admissible deformation \( \gamma \) (Definition 1.10) and the theory of topological degree, the following composition and intersection properties hold (see [44, Section 3.1] and [28, §3]).

**Proposition 4.1**

(i) For any \( \gamma \in \Gamma_1 \) and \( \tilde{\gamma} \in \Gamma_1 \) there holds
\[
\gamma \circ \tilde{\gamma} \in \Gamma_1
\]

(ii) Denote by \( S^+ \) the unit sphere in \( E^+ \). For any \( e \in E^+ \setminus \{0\} \) and \( \gamma \in \Gamma_1 \) there holds
\[
\gamma(S^+) \cap (E^- \oplus E^0_0 \oplus \mathbb{R}^+ e) \neq \emptyset.
\] (4.1)

The estimation for \( c_{\Psi EH}^\Psi(H) \) follows immediately, which is a slight change of Proposition 1 in Section 3.2 of [44].

**Proposition 4.2** If \( H \in C^0(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0}) \) then
\[
c_{\Psi EH}^\Psi(H) \leq \sup_{z \in \mathbb{C}^n} \left( \frac{t(\Psi)}{2} |z|^2 - H(z) \right)
\] (4.2)

where \( t(\Psi) \) is defined in (1.14). Moreover, if
\[
\Psi = \Psi_1 \oplus \Psi_2, \text{ where } \Psi_1 \in \text{Sp}(2, \mathbb{R}) \text{ and } \Psi_2 \in \text{Sp}(2n - 2, \mathbb{R}),
\] (4.3)

then we have
\[
c_{\Psi EH}^\Psi(H) \leq \sup_{z \in \mathbb{C}^n} \left( \frac{t(\Psi_1)}{2} |z_1|^2 - H(z) \right).
\] (4.4)

**Proof** Let \( e(t) = e^{t(\Psi)Jt} X \) where \( J \) is as in (1.1) and \( X \in \mathbb{R}^{2n} \) satisfies \( e^{t(\Psi)J} X = \Psi X \) and \( |X| = 1 \). For any \( x = y + \lambda e \), where \( y \in E^- \oplus E^0_0 \) and \( \lambda > 0 \), there holds
\[
\alpha(x) \leq \frac{1}{2} \|\lambda e\|_E^2 = \frac{t(\Psi)}{2} \lambda^2
\] (4.5)

and
\[
\int_0^1 \langle x(t), e^{t(\Psi)Jt} X \rangle dt = \int_0^1 \langle \lambda e^{t(\Psi)Jt} X, e^{t(\Psi)Jt} X \rangle dt = \lambda.
\] (4.6)
It follows that
\[ a(x) \leq \frac{t(\Psi)}{2} \left( \int_0^1 \langle x(t), e^{t(\Psi)J} X t \rangle dt \right)^2 \leq \frac{t(\Psi)}{2} \int_0^1 |x(t)|^2 dt. \]

By Proposition 4.1(ii), we get
\[
\inf_{x \in \gamma(S^+)} \Phi_H(x) \leq \sup_{x \in \mathbb{E}^- \oplus \mathbb{E}^0 \oplus \mathbb{R}^+} \Phi_H(x) \leq \sup_{z \in \mathbb{R}^{2n}} \frac{t(\Psi)}{2} |z|^2 - H(z), \quad \forall \gamma \in \Gamma
\]
and hence (4.2) is proved.

Now suppose that \( \Psi \) has the form in (4.3). Let \( \hat{e}(t) = e^{t(\Psi)J} \hat{X} \), where \( \hat{X} = (X_1, 0) \in \mathbb{R}^2 \times \mathbb{R}^{2n-2} \) satisfies \( e^{t(\Psi)J} \hat{X} = \Psi \hat{X} \) and \( |\hat{X}| = 1 \). For any \( x = y + \lambda \hat{e} \) where \( y \in \mathbb{E}^- \oplus \mathbb{E}^0 \) and \( \lambda > 0 \), write \( x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2 \times \mathbb{R}^{2n-2} \). Let \( J \) also denote the complex structure on \( \mathbb{R}^2 \). Then as the reasoning of (4.6) we get
\[
\int_0^1 \langle x_1(t), e^{t(\Psi)J} X_1 \rangle dt = \int_0^1 \langle x(t), e^{t(\Psi)J} \hat{X} \rangle dt
\]
\[
= \int_0^1 \langle \lambda e^{t(\Psi)J} \hat{X}, e^{t(\Psi)J} \hat{X} \rangle dt = \lambda.
\]
The same arguments as the proof of (4.2) lead to (4.4). □

As a generalization of [44, Proposition 2, Section 3.2] and [27, Lemma 9, Chapter 3] we have:

**Proposition 4.3** If \( H \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0}) \) satisfies (H1) and (H2), then \( c_{EH}(H) > 0 \).

**Proof** By the assumption (H1), we can take \( z_0 \in \text{int } H^{-1}(0) \cap \text{Fix}(\Psi) \). Define \( \gamma \in \Gamma \) by
\[
\gamma : \mathbb{E} \to \mathbb{E}, \quad x \mapsto \gamma(x) = z_0 + \varepsilon x
\]
where \( \varepsilon > 0 \) is a constant. Let us prove
\[
\inf_{y \in \gamma(S^+)} \Phi_H(y) > 0
\]
for sufficiently small \( \varepsilon \) as in the proof of [26, page 93, Lemma 9]. Since
\[
\Phi_H(z_0 + x) = \frac{1}{2} \|x\|^2_E - \int_0^1 H(z_0 + x) dt \quad \forall x \in \mathbb{E}^+,
\]
it suffices to prove that
\[
\lim_{\|x\|_E \to 0} \frac{\int_0^1 H(z_0 + x) dt}{\|x\|^2_E} = 0.
\]
Otherwise, suppose there exists a sequence \((x_j) \subset \mathbb{E}\) and \(d > 0\) satisfying
\[
\|x_j\|_E \to 0 \quad \text{and} \quad \frac{\int_0^1 H(z_0 + x_j(t))dt}{\|x_j\|_E^2} \geq d > 0 \quad \forall j.
\] (4.9)

Let \(y_j = \frac{x_j}{\|x_j\|_E}\) and hence \(\|y_j\|_E = 1\). By Proposition 2.5, \((y_j)\) has a convergent subsequence in \(L^2\). By a standard result in \(L^p\) theory (see [8, Th.4.9]), we have \(w \in L^2\) and a subsequence of \((y_j)\), still denoted by \((y_j)\), such that \(y_j(t) \to y(t)\) a.e. on \((0, 1)\) and that \(|y_j(t)| \leq w(t)\) a.e. on \((0, 1)\) for each \(j\). Recall that we have assumed that \(H\) vanishes near \(z_0\). By (H2) and the Taylor expansion of \(H\) at \(z_0 \in \mathbb{R}^n\), we have constants \(C_1 > 0\) and \(C_2 > 0\) such that \(H(z_0 + z) \leq C_1|z|^2\) and \(H(z_0 + z) \leq C_2|z|^3\) for all \(z \in \mathbb{R}^n\). It follows that
\[
\frac{H(z_0 + x_j(t))}{\|x_j\|_E^2} \leq C_1 \frac{|x_j(t)|^2}{\|x_j\|_E^2} = C_1 |y_j(t)|^2 \leq C_1 w(t)^2, \quad \text{a.e. on (0, 1), } \forall j,
\]
\[
\frac{H(z_0 + x_j(t))}{\|x_j\|_E^2} \leq C_2 \frac{|x_j(t)|^3}{\|x_j\|_E^2} = C_2 |x_j(t)| \cdot |y_j(t)|^2 \leq C_2 |x_j(t)| w(t)^2, \quad \text{a.e. on (0, 1), } \forall j.
\]

The first claim in (4.9) implies that \((x_j)\) has a subsequence such that
\[
x_{j_k}(t) \to 0, \quad \text{a.e. on (0, 1)}.
\]

Hence the Lebesgue dominated convergence theorem leads to
\[
\frac{\int_0^1 H(z_0 + x_{j_k}(t))dt}{\|x_{j_k}\|_E^2} = \int_0^1 \frac{H(z_0 + x_{j_k}(t))dt}{\|x_{j_k}\|_E^2} \to 0.
\]

This contradicts the second claim in (4.9). \(\square\)

Propositions 4.2, 4.3 show that \(c_{EH}^\psi(H)\) is a finite positive number for each \(H \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}_{\geq 0})\) satisfying (H1) and (H2). Based on this fact, the proof of Theorem 1.14 is given by a minmax argument as in [44, Section 3.4]. For the sake of completeness we give its details here.

**Proof of Theorem 1.14** Let us define
\[
\mathcal{F} := \{\gamma(S^+) \mid \gamma \in \Gamma \text{ and } \inf(\Phi_H | \gamma(S^+)) > 0\}.
\]
Then \(c_{EH}^\psi(H) = \sup_{F \in \mathcal{F}} \inf_{x \in F} \Phi_H(x)\) since \(c_{EH}^\psi(H) > 0\). Note that the flow \(\phi^u\) of \(\nabla \Phi_H\) has the form
\[
\phi^u(x) = e^{-u}x^- + x^0 + e^u x^+ + \tilde{K}(u, x),
\]
where $\tilde{K} : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ is compact. For a set $F = \gamma(S^+) \in \mathcal{F}$ where $\gamma \in \Gamma$, $\alpha := \inf(\Phi_H|\gamma(S^+)) > 0$ by definition of $\mathcal{F}$. Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth function such that $\rho(s) = 0$ for $s \leq 0$ and $\rho(s) = 1$ for $s \geq \alpha$. Define a vector field $V$ on $\mathbb{E}$ by

$$V(x) = x^+ - x^- - \rho(\Phi_H(x))\nabla b(x).$$

Clearly $V$ is locally Lipschitz and has linear growth so that $V$ has a unique global flow which we will denote by $\gamma^u$. Moreover, it is obvious that $\gamma^u$ has the same property as $\phi^u$ described above. For $x \in \mathbb{E}^- \oplus \mathbb{E}^0$, we have $\Phi_H(x) \leq 0$ and hence $V(x) = -x^-$ so that $\gamma^u(\mathbb{E}^- \oplus \mathbb{E}^0) = \mathbb{E}^- \oplus \mathbb{E}^0$ and $\gamma^u(\mathbb{E}^- \backslash (\mathbb{E}^- \oplus \mathbb{E}^0)) = \mathbb{E}^- \backslash (\mathbb{E}^- \oplus \mathbb{E}^0)$, since $\gamma^u$ is a homeomorphism for each $u \in \mathbb{R}$. Therefore, $\gamma^u \in \Gamma_1$ for all $u \in \mathbb{R}$.

Note that $V|\Phi_H^{-1}([\alpha, \infty]) = \nabla \Phi_H(x)$. We have $\gamma^u(F) = \phi^u(F)$ for $u \geq 0$. Since $\Gamma$ is closed for composition operation, $\mathcal{F}$ is positively invariant under the flow $\phi^u$ of $\nabla \Phi_H$. Using Proposition 2.11 we can prove Theorem 1.14 by a standard minimax argument. \hfill \Box

## 5 Proofs of Theorem 1.15, 1.18

Our proofs closely follow those of Theorems 6.5, 6.6 in [44].

### 5.1 Proof of Theorem 1.15

By assumption $Cl(D) \cap Fix(\Psi) \neq \emptyset$. We only consider the case $D \cap Fix(\Psi) \neq \emptyset$ because this may lead to the case $Cl(D) \cap Fix(\Psi) \subset \partial D$ by exterior regularity of the $\Psi$-EH capacity and a standard approximation argument as in Sect. 3.4. Moreover, we can also assume that $D$ contains 0 in its interior by a translation argument (see the beginning of Sect. 3).

In this section, we denote

$$a := \min \Sigma^\Psi_S = \min \{A(x) > 0 \mid x \text{ is a } \Psi\text{-characteristic on } S = \partial D\}.$$ 

Let $j_D : \mathbb{R}^{2n} \to \mathbb{R}$ be the Minkowski functional of $D$, $H(z) = j_D^2(z)$ and $H^*$ the Legendre transformation of $H$. Define $I$, $\mathcal{F}$ and $\mathcal{A}$ as in Sect. 3.1.

**Step 1** (Prove $c^\Psi_{EH}(D) \geq a$). For $\epsilon > 0$, let

$$\mathcal{E}_\epsilon(\mathbb{R}^{2n}, D)$$

consist of $H = f \circ H$, where $f \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies

$$f(s) = 0 \ \forall s \leq 1, \quad f'(s) \geq 0, \ \forall s \geq 1, \quad f'(s) = a \in \mathbb{R} \backslash \Sigma^\Psi_S \text{ if } f(s) \geq \epsilon$$

(5.2)
and \( \alpha \) is required to satisfy

\[
\alpha H(z) \geq \frac{t(\psi)}{2} |z|^2 - C \quad \text{for } |z| \text{ sufficiently large} \tag{5.3}
\]

where \( C > 0 \) is a constant. Then \( \mathcal{E}_\varepsilon(\mathbb{R}^{2n}, D) \) is a cofinal family of \( \mathcal{F}(\mathbb{R}^{2n}, D) \), i.e. for any \( G \in \mathcal{F}(\mathbb{R}^{2n}, D) \) there exists \( \overline{H} \in \mathcal{E}_\varepsilon(\mathbb{R}^{2n}, D) \) such that \( \overline{H} \geq G \). It follows that \( c_{\psi\text{EH}}^\psi(G) \geq c_{\psi\text{EH}}^\psi(\overline{H}) \).

For each \( \overline{H} \in \mathcal{E}_\varepsilon(\mathbb{R}^{2n}, D) \), by the last condition in (5.2) and (5.3) \( \Phi_{\overline{H}} \) satisfies the (PS) condition and \( c_{\psi\text{EH}}^\psi(\overline{H}) \) is a positive critical value of \( \Phi_{\overline{H}} \) (see Sect. 4 and [28, §5.1]). Corresponding to [44, Lemma 3, Section 6.5] we have

**Lemma 5.1** Any positive critical value \( c \) of \( \Phi_{\overline{H}} \) satisfies \( c > \min \Sigma_{\overline{S}}^\psi - \epsilon \). In particular, \( c_{\psi\text{EH}}^\psi(\overline{H}) > \min \Sigma_{\overline{S}}^\psi - \epsilon \).

**Proof** Let \( x \in E \) be a critical point of \( \Phi_{\overline{H}} \) with \( \Phi_{\overline{H}}(x) > 0 \). Then

\[
-J \dot{x}(t) = \nabla \overline{H}(x(t)) = f'(H(x(t))) \nabla H(x(t)), \quad x(1) = \Psi x(0),
\]

and \( H(x(t)) \equiv s_0 \) (a nonzero constant). It follows that

\[
\Phi_{\overline{H}}(x) = \frac{1}{2} \int_0^1 \langle Jx(t), \dot{x}(t) \rangle dt - \int_0^1 H(x(t)) dt
= \frac{1}{2} \int_0^1 \langle x(t), f'(s_0) \nabla H(x(t)) \rangle dt - \int_0^1 f(s) dt
= f'(s_0)s_0 - f(s_0).
\]

Since \( \Phi_{\overline{H}}(x) > 0 \), we get \( \beta := f'(s_0) > 0 \) and so \( s_0 > 1 \). Put

\[
y(t) = \frac{1}{\sqrt{s_0}} x(t/\beta).
\]

Then \( H(y(t)) = 1, -J \dot{y} = \nabla H(y(t)) \) and \( y(\beta) = \Psi y(0) \). These show that \( f'(s_0) = \beta = A(y) \in \Sigma_{\overline{S}}^\psi \). Therefore \( f(s_0) < \epsilon \) by definition of \( \overline{H} \). It follows from these that

\[
\Phi_{\overline{H}}(x) = f'(s_0)s_0 - f(s_0) > f'(s_0) - \epsilon \geq \min \Sigma_{\overline{S}}^\psi - \epsilon.
\]

By definition of the \( \Psi \)-EH capacity,

\[
c_{\psi\text{EH}}^\psi(D) = \inf \{ c_{\psi\text{EH}}^\psi(G) \mid G \in \mathcal{F}(\mathbb{R}^{2n}, D) \}
\geq \inf \{ c_{\psi\text{EH}}^\psi(\overline{H}) \mid \overline{H} \in \mathcal{E}_\varepsilon(\mathbb{R}^{2n}, D) \}
\geq \min \Sigma_{\overline{S}}^\psi - \epsilon = a - \epsilon, \forall \epsilon
\]

and hence \( c_{\psi\text{EH}}^\psi(D) \geq a \).
Step 2 (Prove \( c^\Psi_{EH}(D) \leq a \)). The proof of [44, Theorem 6.5] can be carried here verbatim. However, we give the details for the convenience of readers. We only need to prove that for each \( \varepsilon > 0 \), there exists \( H \in \mathcal{F}(\mathbb{R}^{2n}, D) \) such that

\[
 c^\Psi_{EH}(\tilde{H}) < a + \varepsilon \tag{5.4}
\]

which is reduced to prove: for any \( h \in \Gamma \), there exists \( x \in h(S^+) \) such that

\[
 \Phi_{\tilde{H}}(x) < a + \varepsilon. \tag{5.5}
\]

For \( \tau \) sufficiently large, choose \( H_\tau \in \mathcal{F}(\mathbb{R}^{2n}, D) \) satisfying

\[
 H_\tau \geq \tau \left( H - \left( 1 + \frac{\varepsilon}{2a} \right) \right). \tag{5.6}
\]

For \( h \in \Gamma \), in order to choose \( x \in h(S^+) \) satisfying (5.5) for \( \tilde{H} = H_\tau \), we make some preparations as in [44, Lemma 1, Section 6.5]. By arguments in Section 3.1, there exists \( w \in \mathcal{A} \) such that

\[
 a := \min \{ I(u) \mid u \in \mathcal{A} \} = I(w) = A(x^\ast) \quad \text{and} \quad A(w) = 1.
\]

Denote by \( w^\ast \) the projections of \( w \) onto \( E^\ast \) (according to the decomposition \( E = E^{1/2} = E^+ \oplus E^- \oplus E^0 \), \( \ast = 0, -, + \). Then \( w^+ \neq 0 \). (Otherwise, a contradiction occurs because \( 1 = A(w) = A(w^0 \oplus w^-) = -\frac{1}{2}\|w^\ast\|_E^2 \)). Put \( y = w/\sqrt{a} \) so that

\[
 I(y) = 1 \quad \text{and} \quad A(y) = \frac{1}{a}. \quad \text{Now for any} \ \lambda \in \mathbb{R} \ \text{and} \ x \in E \ \text{it holds that}
\]

\[
 \lambda^2 = I(\lambda y) = \int_0^1 H^*(\lambda J \dot{y}(t)) dt
\]

\[
 = \int_0^1 \sup_{\xi \in \mathbb{R}^{2n}} \{ \langle \xi, -\lambda J \dot{y}(t) \rangle - H(\xi) \} dt
\]

\[
 \geq \int_0^1 \{ \langle x(t), -\lambda J \dot{y}(t) \rangle - H(x(t)) \} dt.
\]

This leads to

\[
 \int_0^1 H(x(t)) dt \geq \int_0^1 \langle x(t), -\lambda J \dot{y}(t) \rangle dt - \lambda^2 = \lambda \int_0^1 \langle x(t), -J \dot{y}(t) \rangle dt - \lambda^2.
\]

Taking

\[
 \lambda = \frac{1}{2} \int_0^1 \langle x(t), -J \dot{y}(t) \rangle dt
\]
we arrive at
\[
\int_0^1 H(x(t))dt \geq \left( \frac{1}{2} \int_0^1 \langle x(t), -J \dot{y}(t) \rangle dt \right)^2, \quad \forall x \in E.
\] (5.7)

Note that \( y^+ \neq 0 \) and \( E^- \oplus E^0 + \mathbb{R}_+ y = E^- \oplus E^0 \oplus (\mathbb{R}_+ y^+) \). By the intersection property (ii) in Proposition 4.1 we derive
\[
h(S^+ \cap (E^- \oplus E^0 + \mathbb{R}_+ y)) \neq \emptyset, \quad \forall h \in \Gamma.
\]

For an \( h \in \Gamma \) and \( x \in h(S^+) \cap (E^- \oplus E^0 + \mathbb{R}_+ y) \), consider the polynomial
\[
P(t) = a(x + ty) = a(x) + t \left( \int_0^1 \langle x, -J \dot{y} \rangle dt \right) + a(y)t^2.
\]

Writing \( x = x^- + sy = x^- + sy^- + sy^+ \) where \( x^- \in E^- \oplus E^0 \), then \( P(t) = a(x^- + (t + s)y) \). Since \( a|_{E^- \oplus E^0} \leq 0 \) we deduce that \( P(-s) \leq 0 \). Moreover, \( a(y) = 1/a > 0 \) and we get
\[
P(t) \rightarrow +\infty \quad \text{as} \quad |t| \rightarrow +\infty.
\]

These imply that there exists \( t_0 \in \mathbb{R} \) such that \( P(t_0) = 0 \). It follows that
\[
\left( \int_0^1 \langle x, -J \dot{y} \rangle dt \right)^2 - 4a(y)a(x) \geq 0
\]
and so by (5.7) there holds
\[
a(x) \leq (a(y))^{-1} \left( \frac{1}{2} \int_0^1 \langle x, -J \dot{y} \rangle dt \right)^2
\]
\[
= a \left( \frac{1}{2} \int_0^1 \langle x, -J \dot{y} \rangle dt \right)^2
\]
\[
\leq a \int_0^1 H(x(t))dt.
\] (5.8)

**Proof** (Proof of the fact \( c_{EH1}^\Psi(D) \leq a \)). For any \( \varepsilon > 0 \), let \( H_\tau \) be a function in \( \mathcal{F}(\mathbb{R}^{2n}, D) \) satisfying (5.6). For any \( h \in \Gamma \), let \( x \in h(S^+) \cap (E^- \oplus E^0 + \mathbb{R}_+ y) \neq \emptyset \).

- If \( \int_0^1 H(x(t))dt \leq \left( 1 + \frac{\varepsilon}{a} \right) \), then by \( H_\tau \geq 0 \) and (5.8), we have
\[
\Phi_{H_\tau}(x) \leq a(x) \leq a \int_0^1 H(x(t))dt \leq a \left( 1 + \frac{\varepsilon}{a} \right) < a + \varepsilon.
\]
• If \( \int_0^1 H(x(t))dt > (1 + \frac{\varepsilon}{a}) \) then (5.6) implies
\[
\int_0^1 H_\tau(x(t))dt \geq \tau \left( \int_0^1 H(x(t))dt - \left(1 + \frac{\varepsilon}{2a}\right)\right) \\
\geq \tau \frac{\varepsilon}{2(a + \varepsilon)} \int_0^1 H(x(t))dt.
\] (5.9)

Choose \( \tau > 0 \) so large that
\[
\frac{\tau \varepsilon}{2(a + \varepsilon)} > a.
\]

Then (5.9) leads to
\[
\int_0^1 H_\tau(x(t))dt \geq a \int_0^1 H(x(t))dt
\]
and hence by (5.8) there holds
\[
\Phi_{H_\tau}(x) = a(x) - \int_0^1 H_\tau(x(t))dt \leq a(x) - a \int_0^1 H(x(t))dt \leq 0.
\]

In summary, in two cases we have \( \Phi_{H_\tau}(x) < a + \varepsilon \) and hence \( c_{EH}^\Psi(D) \leq a. \)

\( \square \)

**Step 3.** Prove \( c_{EH}^\Psi(\partial D)=a. \) Let
\[
\mathcal{E}_\varepsilon(\mathbb{R}^{2n}, \partial D)
\]
consist of \( \overline{H} = f \circ H \) where \( f \in C^\infty(\mathbb{R}, \mathbb{R}) \) satisfies
\[
f(s) = 0 \text{ for } s \text{ near } 1, \quad f'(s) \leq 0 \forall s \leq 1, \quad f'(s) \geq 0 \forall s \geq 1, \quad (5.11)
\]
\[
f'(s) = \alpha \in \mathbb{R} \setminus \Sigma^\Psi_S \text{ if } s \geq 1 \text{ and } f(s) \geq \epsilon, \quad (5.12)
\]
where \( \alpha \) is also required to be so large that
\[
\alpha H(z) \geq \frac{t(\Psi)}{2} |z|^2 - C
\] (5.13)
for some constant \( C > 0. \) Similar to Step 1, there holds that
\[
c_{EH}^\Psi(\overline{H}) > \min \Sigma^\Psi_{\partial D} - \epsilon, \quad \forall \overline{H} \in \mathcal{E}_\varepsilon(\mathbb{R}^{2n}, \partial D).
\]
It follows that \( c_{EH}^\Psi(\partial D) \geq a. \) On the other hand, by the monotonicity of \( c_{EH}^\Psi \) we get that \( c_{EH}^\Psi(\partial D) \leq c_{EH}^\Psi(D) = a. \) Therefore \( c_{EH}^\Psi(\partial D) = a. \)

\( \square \)
5.2 Proof of Theorem 1.18

The following lemma is a slight change of [44, Lemma 1, Section 6.6].

**Lemma 5.2** For a convex domain $D \subset \mathbb{R}^{2n}$ containing 0 and symplectic matrices $\Psi_1 \in \text{Sp}(2n, \mathbb{R})$ and $\Psi_2 \in \text{Sp}(2k, \mathbb{R})$, it holds that $c_{\Psi_1 \oplus \Psi_2}^E(D \times \mathbb{R}^{2k}) = c_{\Psi_1}^E(D)$.

**Proof** It suffices to prove this lemma for a convex bounded domain $D \subset \mathbb{R}^{2n}$ with $C^2$-smooth boundary $S$. Let $H = j_D^2$. By the definition and monotonicity of $\Psi$-EH capacity we have

$$c_{\Psi_1 \oplus \Psi_2}^E(D \times \mathbb{R}^{2k}) = \sup_R c_{\Psi_1 \oplus \Psi_2}^E(E_R),$$

where $E_R = \{(z, z') \in \mathbb{R}^{2n} \times \mathbb{R}^{2k} | H(z) + (|z'|/R)^2 < 1\}$.

Since $E_R$ is convex and $S_R := \partial E_R$ is of class $C^{1,1}$ because $H$ is of class $C^{1,1}$ on $\mathbb{R}^{2n}$, Theorem 1.15 gives rise to

$$c_{\Psi_1 \oplus \Psi_2}^E(E_R) = \min_{S_R} \Sigma_{\Psi_1 \oplus \Psi_2}^1.$$

Let $(x, \hat{x}) : [0, \lambda] \to S_R$ with $\lambda > 0$ satisfy

$$\dot{x} = X_H(x) \quad \text{and} \quad x(\lambda) = \Psi_1 x(0), \quad (5.14)$$

$$\dot{\hat{x}} = 2J \hat{x}/R^2 \quad \text{and} \quad \hat{x}(\lambda) = \Psi_2 \hat{x}(0). \quad (5.15)$$

Then $(x, \hat{x})$ is a $\Psi_1 \oplus \Psi_2$-characteristic on $S_R$ with action $\lambda$. By (5.15), $|\hat{x}|$ is constant. If $\hat{x} \neq 0$, then by the boundary condition in (5.15) we get $\lambda \geq R^2 t(\Psi_2)/2$, where $t(\Psi_2)$ is the smallest positive number satisfying $\det(\Psi_2 - e^{sJ}) = 0$ (see (1.14)). If $|\hat{x}| \equiv 0$, then $x$ lies on $S$ and $\lambda \in \Sigma_{S_R}^1$. Hence for $R > 0$ large enough we arrive at

$$c_{\Psi_1 \oplus \Psi_2}^E(E_R) = \min_{S_R} \Sigma_{\Psi_1 \oplus \Psi_2}^1 = \min_{S} \Sigma_{\Psi_1}^1 = c_{\Psi_1}^E(D)$$

and so the desired conclusion. \qed

Based on the proof of Proposition 4.3 and Lemma 5.1, [44, Lemma 2, Section 6.6] can be generalized to the following (cf. [28, Lemma 5.7]).

**Lemma 5.3** Let $D \subset \mathbb{R}^{2n}$ be a convex bounded domain with $C^2$-smooth boundary $S$ and containing 0. Let $\tilde{H} \in \mathcal{F}(\mathbb{R}^{2n}, D)$. Then for any $\epsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$\Phi_{\tilde{H}}|\gamma(B^+ \setminus B^+) \geq c_{\Psi_1}^E(D) - \epsilon \quad \text{and} \quad \Phi_{\tilde{H}}|\gamma(B^+) \geq 0, \quad (5.16)$$

where $B^+$ is the closed unit ball in $E^+$ and $S^+ = \partial B^+$.

Then Theorem 1.18 follows from Lemmas 5.2 and 5.3.
Proof of Theorem 1.18 Step 1 (Prove (1.29)). By the approximation arguments we may assume that each $D_i \subset \mathbb{R}^{2n_i}$ is a $C^2$ convex bounded domain containing a fixed point $p_i$ of $\Psi_i$, $1 \leq i \leq k$. Since $c_{\Psi_i}^{\Psi}(D_i - p_i) = c_{\Psi_i}^{\Psi}(D_i)$, $1 \leq i \leq k$, and

$$c_{\Psi_i}^{\Psi}((D_1 - p_1) \times \cdots \times (D_k - p_k)) = c_{\Psi_i}^{\Psi}(D_1 \times \cdots \times D_k),$$

we may also assume that each $D_i$ contains the origin of $\mathbb{R}^{2n_i}$, $1 \leq i \leq k$. Thus it follows from the monotonicity of $\Psi$-EH capacity and Lemma 5.2 that

$$c_{\Psi_i}^{\Psi}(D_1 \times \cdots \times D_k) \leq \min_i c_{\Psi_i}^{\Psi}(D_i).$$

(5.17)

In order to prove the converse inequality, note that for each $H \in \mathcal{F}(\mathbb{R}^{2n}, D_1 \times \cdots \times D_k)$ we may choose $\hat{H}_i \in \mathcal{F}(\mathbb{R}^{2n_i}, D_i)$, $i = 1, \ldots, k$, such that

$$\hat{H}(z_i) := \sum \hat{H}_i(z_i) \geq H(z) \quad \forall z.$$

For each $i = 1, \ldots, k$, by Lemma 5.3 there exists $\gamma_i \in \Gamma(\mathbb{R}^{2n_i})$ such that

$$\Phi_{\hat{H}_i} |_{\gamma_i}(B_1^+ \setminus (2k)^{-1}B_1^+) \geq c_{\Psi_i}^{\Psi}(D_i) - \epsilon,$$

and

$$\Phi_{\hat{H}_i} |_{\gamma_i}(B_1^+) \geq 0.$$

For any $x = (x_1, \ldots, x_k) \in S^+ \subset B_1^+ \times \cdots \times B_k^+$ there exists some $i_0$ such that

$$x_{i_0} \in B_{i_0}^+ \setminus (2k)^{-1}B_{i_0}^+.$$

Put $\gamma = \gamma_1 \times \cdots \times \gamma_k$ and we arrive at

$$\Phi_{\hat{H}}(\gamma(x)) = \sum \Phi_{\hat{H}_i}(\gamma_i(x)) \geq c_{\Psi_0}^{\Psi}(D_{i_0}) - \epsilon \geq \min_i c_{\Psi_i}^{\Psi}(D_i) - \epsilon$$

and hence

$$c_{\Psi}^{\Psi}(H) \geq c_{\Psi}^{\Psi}(\hat{H}) = \sup_{h \in \Gamma} \inf_{x \in h(S^+)} \Phi_{\hat{H}}(x) \geq \min_i c_{\Psi_i}^{\Psi}(D_i) - \epsilon.$$

This leads to

$$c_{\Psi_i}^{\Psi}(D_1 \times \cdots \times D_k) \geq \min_i c_{\Psi_i}^{\Psi}(D_i)$$

(5.18)

and by combining this with (5.17) we get (1.29).

Step 2 (Prove (1.30)). As in Step 1, for each $i = 1, \ldots, k$ we may assume: (i) $D_i \subset \mathbb{R}^{2n_i}$ is compact, convex, and has $C^2$-boundary; (ii) $\partial D_i$ contains a fixed point...
$p_i$ of $\Psi_i$ and $\text{Int}(D_i)$ contains the origin of $\mathbb{R}^{2n_i}$. Then Lemma 5.3 holds for every $\tilde{H} \in \mathcal{F}(\mathbb{R}^{2n}, S)$ with $S = \partial D_1 \times \cdots \times \partial D_k$. Arguing as in Step 1 we get that
\[ c_{E,H}^{\Psi_1 \otimes \cdots \otimes \Psi_k}(\partial D_1 \times \cdots \times \partial D_k) \geq \min_i c_{E,H}^{\Psi_i}(D_i). \]
Since $c_{E,H}^{\Psi_1 \otimes \cdots \otimes \Psi_k}(\partial D_1 \times \cdots \times \partial D_k) \leq c_{E,H}^{\Psi}(D_1 \times \cdots \times D_k) = \min_i c_{E,H}^{\Psi_i}(D_i)$ by the monotonicity property of $c_{E,H}^{\Psi}$ and (1.29), we obtain (1.30).

\section{6 Proof of Theorem 1.24}

The proof of [44, Th.7.5.1] is different from that of [18, Prop.6]. The former can be adapted to complete the proof for Theorem 1.24 conveniently.

Let $\lambda_0 = \frac{1}{2}(qd p - pd q)$, where $(q, p)$ are the standard coordinates on $\mathbb{R}^{2n}$. Then $d\lambda_0 = dq \wedge dp = \omega_0$ and for any $C^1$ path $x : [0, T] \rightarrow \mathbb{R}^{2n}$
\[ A(x) = \frac{1}{2} \int_0^T \langle -J \dot{x}, x \rangle dt = \int_x x^* \lambda_0. \] (6.1)
If $x$ is not closed, $\lambda_0$ can not be replaced by other primitives of $\omega_0$ in general.

\textbf{Lemma 6.1} For $\Psi \in Sp(2n, \mathbb{R})$, let $X$ be a vector field defined on $\mathbb{R}^{2n}$ such that
\[ X(\Psi(z)) = \Psi(X(z)), \quad \forall z \in \mathbb{R}^{2n} \] (6.2)
and suppose that $\lambda := i_X \omega_0$ is a primitive of $\omega_0$. Let $x : [0, T] \rightarrow \mathbb{R}^{2n}$ satisfy $x(T) = \Psi x(0)$. Then
\[ \int_x x^* \lambda_0 = \int_x x^* \lambda. \] (6.3)
\textbf{Proof} Let $X_0$ be the vector field on $\mathbb{R}^{2n}$ defined by $X_0(z) = \frac{1}{2}z$, $\forall z \in \mathbb{R}^{2n}$. Then we have
\[ \lambda_0 = i_{X_0} \omega_0, \quad \text{and} \quad X_0(\Psi(z)) = \Psi X_0(z), \quad \forall z \in \mathbb{R}^{2n}. \]
For a vector $Y \in T_z \mathbb{R}^{2n} = \mathbb{R}^{2n}$, we compute
\[ \Psi^* \lambda_0(z)[Y] = \lambda_0(\Psi z)[\Psi Y] = \omega_0(X_0(\Psi z), \Psi Y) = \omega_0(\Psi X_0(z), \Psi Y) \]
\[ = \Psi^* \omega_0(X_0(z), Y) = \omega_0(\Psi X_0(z), Y) = i_{X_0} \omega_0(z)[Y] = \lambda_0(z)[Y]. \]
Hence
\[ \Psi^* \lambda_0 = \lambda_0. \] (6.4)
The same arguments lead to

$$\Psi^*\lambda = \lambda.$$  \hfill (6.5)

Since $\lambda_0$ and $\lambda$ are primitives of $\omega_0$, $d(\lambda_0 - \lambda) = 0$ and there exists $F \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ such that $\lambda_0 - \lambda = dF$. (6.4) and (6.5) lead to $\Psi^*d F = \Psi^*(\lambda_0 - \lambda) = \lambda_0 - \lambda = dF$, which implies that there exists a constant $C$ such that $F(\Psi(z)) = F(z) = C$ for all $z \in \mathbb{R}^{2n}$. Since $\Psi(0) = 0$, $F(\Psi(0)) = F(0) = 0$ and we get that $C = 0$. Therefore

$$\int_x x^*\lambda_0 - \int_x x^*\lambda = \int_x x^*(dF) = F(x(1)) - F(x(0)) = F(\Psi(0)) - F(0) = 0.$$  \hfill \square

Note that the Liouville vector field $X$ in Theorem 1.24 satisfies the condition in Lemma 6.1. In the following part of this section, $X$ denotes the Liouville vector field in Theorem 1.24 and $\lambda = i_X\omega_0$. Let $\phi^t$ be the local flow of $X$. For $\epsilon > 0$ sufficiently small, the map

$$\psi : (-\epsilon, \epsilon) \times S \rightarrow \mathbb{R}^{2n}, (s, z) \mapsto \phi^s(z),$$  \hfill (6.6)

is well defined and $\mathbb{R}^{2n} \setminus \cup_{t \in (-\epsilon, \epsilon)} \phi^t(S)$ has two components. Moreover, since $X$ satisfies (6.2), there holds

$$\Psi(\phi^t(z)) = \phi^t(\Psi(z)), \forall (t, z) \in (-\epsilon, \epsilon) \times S.$$  \hfill (6.7)

Define $U := \cup_{t \in (-\epsilon, \epsilon)} \phi^t(S)$ and

$$K_\psi : U \rightarrow \mathbb{R}, w \mapsto \tau$$  \hfill (6.8)

if $w = \phi^\tau(z) \in U$ where $z \in S$. Let $X_{K_\psi}$ be the Hamiltonian vector field associated to $K_\psi$ defined by $\omega_0(\cdot, X_{K_\psi}) = dK_\psi$. Then for $w = \phi^\tau(z) \in U$ it holds that

$$\lambda_w(X_{K_\psi}) = (\omega_0)_w(X(w), X_{K_\psi}(w)) = 1$$  \hfill (6.9)

and

$$X_{K_\psi}(\phi^\tau(z)) = e^{-\tau}d\phi^\tau(z)[X_{K_\psi}(z)], \forall (\tau, z) \in (-\epsilon, \epsilon) \times S.$$  \hfill (6.10)

Clearly (6.7) and (6.10) show that $y : [0, T] \rightarrow S_\tau = \phi^\tau(S)$ satisfies

$$\dot{y}(t) = X_{K_\psi}(y(t)) \quad \text{and} \quad y(T) = \Psi y(0)$$

if and only if $y(t) = \psi(\tau, x(e^{-\tau}t))$, where $x : [0, e^{-\tau}T] \rightarrow S$ satisfies

$$\dot{x}(t) = X_{K_\psi}(x(t)) \quad \text{and} \quad x(e^{-\tau}T) = \Psi x(0).$$
In addition, there holds
\[ \int y^*\lambda = e^\tau \int x^*\lambda. \]

The following proposition is a generalization of Lemma 3.2. Its proof is a slight change of the proof of [44, Proposition 1, Section 7.4].

**Proposition 6.2** Let \( S \subset (\mathbb{R}^{2n}, \omega_0) \) be as in Theorem 1.24. Then the interior of
\[ \Sigma^\Psi_S = \{ A(x) > 0 \mid x \text{ is a } \Psi\text{-characteristic on } S \} \]
is empty.

**Proof** Assume that the neighborhood \( U \) of \( S \) defined above (6.8) is contained in a ball \( B^{2n}(0, R) \). Fix \( 0 < \delta < \epsilon \). Let \( A_\delta \) and \( B_\delta \) denote the unbounded and bounded components of \( \mathbb{R}^{2n} \setminus \bigcup_{t \in (-\delta, \delta)} \phi^t(S) \) respectively. Then
\[ \phi^\tau(S) \subset B_\delta \quad \text{for } -\epsilon < \tau < -\delta. \]

Define \( H \in \mathcal{F}(\mathbb{R}^{2n}) \) by
\[
H(x) = \begin{cases} 
C_0 \geq 0 & \text{if } x \in B_\delta, \\
f(\tau) & \text{if } x = \phi^\tau(y), \ y \in S, \ \tau \in [-\delta, \delta], \\
C_1 & \text{if } x \in A_\delta \cap B^{2n}(0, R), \\
h(|x|^2) & \text{if } x \in A_\delta \setminus B^{2n}(0, R)
\end{cases}
\]
(6.11)

where \( f : (-\epsilon, \epsilon) \to \mathbb{R} \) and \( h : [0, \infty) \to \mathbb{R} \) are smooth functions satisfying
\[
f|(-\epsilon, -\delta) = C_0, \quad f|([\delta, \epsilon)] = C_1, \quad sh'(s) - h(s) \leq 0 \quad \forall s.
\]
(6.12)
(6.13)

For a fixed \( T \in \Sigma^\Psi_S \), we choose the above function \( f \) such that for some constants \( 0 < \epsilon_1 < \delta \) and \( \overline{C} \),
\[
f(u) = Tu + \overline{C} \geq 0, \quad \forall u \in [-\epsilon_1, \epsilon_1].
\]
(6.14)

Clearly \( O := \{ e^{-\tau}T \mid \tau \in (-\epsilon_1, \epsilon_1) \} \) is an open neighborhood of \( T \). If \( e^{-\tau}T \in \Sigma^\Psi_S \) then there exists a \( \Psi\)-characteristic on \( S \) which can be parameterized as \( x : [0, 1] \to \mathbb{R}^{2n} \) satisfying
\[
\dot{x} = e^{-\tau}X_H(x), \quad x(1) = \Psi x(0).
\]

Let \( y = \phi^\tau(x) \). Then
\[
\dot{y} = X_H(y), \quad y(1) = \Psi y(0),
\]
and it follows that \( \Phi_H(y) = T - f(\tau) = T - T \tau - C \) is a critical value of \( \Phi_H \).

Hence if \( O \cap \Sigma_S^{\psi} \) has an interior point, then
\[
\{ \tau \in (-\epsilon_1, \epsilon_1) \mid e^{-\tau} T \in \Sigma_S^{\psi} \} \subset \{ \tau \in (-\epsilon_1, \epsilon_1) \mid T - f(\tau) \text{ is a critical value of } A_H \}
\]
has nonempty interior and therefore the critical value set of \( A_H \) has nonempty interior. However, arguing as in Sect. 3.2, the critical value set of \( A_H \) has empty interior since \( H \) is a smooth function on \( \mathbb{R}^{2n} \). This is a contradiction. Hence \( \Sigma_S^{\psi} \) has empty interior.

\( \square \)

**Proof of Theorem 1.24** The proof of [44, Theorem 7.5] can be carried here almost verbatim. We give a sketch here.

For \( C > 0 \) large enough and \( \delta > 2\eta > 0 \) small enough, define \( H = H_{C, \eta} \in \mathcal{F}(\mathbb{R}^{2n}) \) adapted to \( \psi \) as the following:

\[
H_{C, \eta}(x) = \begin{cases} 
C & \text{if } x \in B_{\delta}, \\
 f_{C, \eta}(\tau) & \text{if } x = \psi(\tau, y), \ y \in S, \ \tau \in [-\delta, \delta], \\
 C & \text{if } x \in A_\delta \cap \mathcal{B}^{2n}(0, R), \\
h(|x|^2) & \text{if } x \in A_\delta \setminus \mathcal{B}^{2n}(0, R),
\end{cases}
\]

where \( \mathcal{B}^{2n}(0, R) \supseteq \overline{\psi((0, \varepsilon) \times S)} \) (the closure of \( \psi((0, \varepsilon) \times S) \)), \( f_{C, \eta} : (-\varepsilon, \varepsilon) \to \mathbb{R} \) and \( h : [0, \infty) \to \mathbb{R} \) are smooth functions satisfying

\[
\begin{align*}
 f_{C, \eta}(-\eta, \eta) &= 0, \quad f_{C, \eta}(s) = C \text{ if } |s| \geq 2\eta, \\
 f'_{C, \eta}(s) &> 0 \text{ if } \eta < |s| < 2\eta, \\
 f'_{C, \eta}(s) - f_{C, \eta}(s) &> c_{\psi}^{\psi}(S) + 1 \text{ if } s > 0 \text{ and } \eta < f_{C, \eta}(s) < C - \eta, \\
 h_{C, \eta}(s) &= a_H s + b \text{ for } s > 0 \text{ large enough, } a_H = C / R^2 > t(\Psi), \\
 sh'_{C, \eta}(s) - h_{C, \eta}(s) &\leq 0 \quad \forall s \geq 0.
\end{align*}
\]

Moreover, we assume
\[
\det \left( \exp \left( \frac{2C}{R^2} J \right) - \Psi \right) \neq 0.
\]

Such a family \( H_{C, \eta} \) \( (C \to +\infty, \eta \to 0) \) can be chosen to be cofinal in the set \( \mathcal{F}(\mathbb{R}^{2n}, S) \) and also have the property that
\[
C \leq C' \Rightarrow H_{C, \eta} \leq H_{C', \eta}, \quad \eta \leq \eta' \Rightarrow H_{C, \eta} \geq H_{C, \eta'}.
\]

It follows that
\[
c_{\psi}^{\psi}(S) = \lim_{\eta \to 0 \& C \to +\infty} c_{\psi}^{\psi}(H_{C, \eta}).
\]

By Proposition 1.11(i) and (6.17), \( \eta \leq \eta' \Rightarrow c_{\psi}^{\psi}(H_{C, \eta}) \leq c_{\psi}^{\psi}(H_{C, \eta'}), \) and hence
\[
\Upsilon(C) := \lim_{\eta \to 0} c_{\psi}^{\psi}(H_{C, \eta})
\]
exists and
\[ \Upsilon(C) = \lim_{\eta \to 0} c_{\Psi}^\psi(H_{C,\eta}) \geq \lim_{\eta \to 0} c_{\Psi}^\psi(H_{C',\eta}) = \Upsilon(C'), \]
i.e., \( C \mapsto \Upsilon(C) \) is non-increasing. Then it is obvious that
\[ c_{\Psi}^\psi(S) = \lim_{C \to +\infty} \Upsilon(C). \quad (6.19) \]

By the construction of \( H_{C,\eta}, c_{\Psi}^\psi(H_{C,\eta}) \) is a positive critical value of \( \Phi_{H_{C,\eta}} \) and the associated critical point \( x \in \mathbb{E} \) gives rise to a nonconstant \( \Psi \)-characteristic sitting in the interior of \( U_\delta := \bigcup_{t \in (-\delta, \delta)} \phi_t(S) \). It follows that
\[ c_{\Psi}^\psi(H_{C,\eta}) = \Phi_{H_{C,\eta}}(x) = f'_{C,\eta}(\tau) - f_{C,\eta}(\tau) \]
where \( f'_{C,\eta}(\tau) \in e^\tau \Sigma^\psi_S \) and \( \eta < \tau < 2\eta \). Choose \( C > 0 \) sufficiently large and \( \eta > 0 \) sufficiently small such that
\[ c_{\Psi}^\psi(H_{C,\eta}) < c_{\Psi}^\psi(S) + 1. \]
Then the choice of \( f \) below (6.15) implies

either \( f_{C,\eta}(\tau) < \eta \) or \( f_{C,\eta}(\tau) > C - \eta \).

Choose a sequence of positive numbers \( \eta_n \to 0 \). Passing to a subsequence we may assume two cases.

**Case 1.** Suppose that \( c_{\Psi}^\psi(H_{C,\eta_n}) = f'_{C,\eta_n}(\tau_n) - f_{C,\eta_n}(\tau_n) = e^{\tau_n} a_n - f_{C,\eta_n}(\tau_n) \), where \( a_n \in \Sigma^\psi_S, \eta_n < \tau_n < 2\eta_n \) and \( 0 \leq f_{C,\eta_n}(\tau_n) < \eta_n \). Since \( c_{\Psi}^\psi(H_{C,\eta_n}) \to \Upsilon(C) \), the sequence \( e^{-\tau_n}(c_{\Psi}^\psi(H_{C,\eta_n}) + f_{C,\eta_n}(\tau_n)) = a_n \) is a bounded sequence. Passing to a subsequence we may assume that \( (a_n) \) is convergent. Let \( a_n \to a_C \in \Sigma^\psi_S \) (the closure of \( \Sigma^\psi_S \)). Note that
\[ \lim_{n \to \infty} \left( e^{-\tau_n} (c_{\Psi}^\psi(H_{C,\eta_n}) + f_{C,\eta_n}(\tau_n)) \right) = \lim_{n \to \infty} e^{-\tau_n} \left( \lim_{n \to \infty} c_{\Psi}^\psi(H_{C,\eta_n}) + \lim_{n \to \infty} f_{C,\eta_n}(\tau_n) \right) = \Upsilon(C). \]
Hence
\[ \Upsilon(C) = a_C \in \Sigma^\psi_S. \quad (6.20) \]

Note that by standard arguments one can show that \( \Sigma^\psi_S = \Sigma^\psi_S \cup \{0\} \). Therefore \( \Sigma^\psi_S \) also has empty interior.

**Case 2.** Suppose that \( c_{\Psi}^\psi(H_{C,\eta_n}) = f'_{C,\eta_n}(\tau_n) - f_{C,\eta_n}(\tau_n) = e^{\tau_n} a_n - f_{C,\eta_n}(\tau_n) = e^{\tau_n} a_n - C - (f_{C,\eta_n}(\tau_n) - C) \), where \( a_n \in \Sigma^\psi_S, \eta_n < \tau_n < 2\eta_n \) and \( C - \eta_n <
\( f_{\eta_n}(\tau_n) \leq C \). As in Case 1 there holds
\[
\Upsilon(C) + C = a_C \in \Sigma_S^\psi. \tag{6.21}
\]

**Step 1.** Prove \( c_{\mathcal{EH}}(S) \in \Sigma_S^\psi \). Suppose that there exists a sequence \( C_n \uparrow +\infty \) such that \( \Upsilon(C_n) = a_{C_n} \in \Sigma_S^\psi \) for each \( n \). Then
\[
c_{\mathcal{EH}}(S) = \lim_{n \to \infty} \Upsilon(C_n) \in \Sigma_S^\psi.
\]

Otherwise, we have
\[
\text{there exists } \bar{C} > 0 \text{ such that (6.21) holds for each } C \in (\bar{C}, +\infty). \tag{6.22}
\]

Let us prove that this case does not occur. Note that (6.22) implies
\[
\text{Claim 6.3 If } C < C' \text{ belong to } (\bar{C}, +\infty) \text{ then}
\]
\[
\Upsilon(C) + C \geq \Upsilon(C') + C'. \tag{6.23}
\]

**Proof.** Assume that for some \( C' > C > \bar{C} \),
\[
\Upsilon(C) + C < \Upsilon(C') + C'. \tag{6.24}
\]

We shall prove:
\[
\text{for any given } d \in (\Upsilon(C) + C, \Upsilon(C') + C') \text{ there exists } C_0 \in (C, C') \text{ such that } \Upsilon(C_0) + C_0 = d. \tag{6.25}
\]

This contradicts the facts that \( \text{Int}(\Sigma_S^\psi) = \emptyset \) and (6.21) holds for all large \( C \).

Put \( \Delta_d = \{ C'' \in (C, C') | \Upsilon(C'') > d \} \). Since \( \Upsilon(C') + C' > d \) and \( \Upsilon(C') \leq \Upsilon(C''') \leq \Upsilon(C) \) for any \( C'' \in (C, C') \), we obtain \( \Upsilon(C'') + C'' > d \) if \( C'' \in (C, C') \) is sufficiently close to \( C' \). Hence \( \Delta_d \neq \emptyset \). Then \( C_0 := \inf \Delta_d \in [C, C') \). Let \( C'' \in \Delta_d \) satisfy \( C'' \downarrow C_0 \). By \( \Upsilon(C_n'') \leq \Upsilon(C_0) \) we have \( d < C_n'' + \Upsilon(C_n'') \leq \Upsilon(C_0) + C_n'' \) for each \( n \in \mathbb{N} \), and thus \( d \leq \Upsilon(C_0) + C_0 \) by letting \( n \to \infty \). Suppose that
\[
d < \Upsilon(C_0) + C_0. \tag{6.26}
\]
Since \( d > C + \Upsilon(C) \), this implies \( C \neq C_0 \) and so \( C_0 > C \). For \( \hat{C} \in (C, C_0) \), from \( \Upsilon(\hat{C}) \geq \Upsilon(C_0) \) and (6.26) we derive that \( \Upsilon(\hat{C}) + \hat{C} > d \) if \( \hat{C} \) is close to \( C_0 \). Therefore such \( \hat{C} \) belongs to \( \Delta_d \), which contradicts \( C_0 = \inf \Delta_d \) and it follows that (6.26) does not hold. That is, \( d = \Upsilon(C_0) + C_0 \). (6.25) is proved. Since (6.25) contradicts the fact that \( \Sigma_S^\psi \) has empty interior, (6.23) does hold for all \( \bar{C} < C < C' \). \qed
Since \( \Xi := \{ C > \tilde{C} \mid C \text{ satisfying (6.16)} \} \) is dense in \((\tilde{C}, +\infty)\), it follows from Claim 6.3 that \( \Upsilon(C') + C' \leq \Upsilon(C) + C \) if \( C' > C \) is in \( \Xi \). Fix a \( C^* \in \Xi \). Then

\[
\Upsilon(C') + C' \leq \Upsilon(C^*) + C^*, \quad \forall C' \in \{ C \in \Xi \mid C > C^* \}.
\]

Taking a sequence \( (C'_n) \subset \{ C \in \Xi \mid C > C^* \} \) such that \( C'_n \to +\infty \), we deduce that \( \Upsilon(C'_n) \to -\infty \). This contradicts the fact that \( \Upsilon(C'_n) \to c_{EH}(S) > 0 \). Hence (6.22) does not hold!

**Step 2. Prove**

\[
c_{EH}^\Psi(B) = c_{EH}^\Psi(S).
\] (6.27)

Construct

\[
\hat{h}_{C, \eta}(x) = \begin{cases} 
0 & \text{if } x \in B_\delta, \\
\hat{f}_{C, \eta}(\tau) & \text{if } x = \psi(\tau, y), \ y \in S, \ \tau \in [-\delta, \delta], \\
C & \text{if } x \in A_\delta \cap B^{2n}(0, R), \\
\hat{h}(\|x\|^2) & \text{if } x \in A_\delta \setminus B^{2n}(0, R)
\end{cases}
\] (6.28)

where \( B^{2n}(0, R) \supseteq \psi((-\varepsilon, \varepsilon) \times S) \), \( \hat{f}_{C, \eta} : (-\varepsilon, \varepsilon) \to \mathbb{R} \) and \( \hat{h} : [0, \infty) \to \mathbb{R} \) are smooth functions satisfying

\[
\begin{align*}
\hat{f}_{C, \eta}|(-\infty, \eta) & = 0, \quad \hat{f}_{C, \eta}(s) = C \text{ if } s \geq 2\eta, \\
\hat{f}_{C, \eta}'(s) & > 0 \text{ if } \eta < s < 2\eta, \\
\hat{f}_{C, \eta}(s) - \hat{f}_{C, \eta}(s) & > c_{EH}^\Psi(S) + 1 \text{ if } s > 0 \text{ and } \eta < \hat{f}_{C, \eta}(s) < C - \eta, \\
\hat{h}_{C, \eta}(s) & = a_H s + b \quad \text{for } s > 0 \text{ large enough, } a_H = C/R^2 > t(\Psi), \\
\hat{h}_{C, \eta}'(s) - \hat{h}_{C, \eta}(s) & \leq 0 \quad \forall s \geq 0, \\
det \left( \exp \left( \frac{2C}{R^2} J \right) - \Psi \right) & \neq 0.
\end{align*}
\]

Then

\[
c_{EH}^\Psi(B) = \inf_{\eta > 0, C > 0} c_{EH}^\Psi(\hat{H}_{C, \eta}).
\] (6.29)

For \( H_{C, \eta} \) in (6.15), we choose an associated \( \hat{H}_{C, \eta} \), where \( \hat{f}_{C, \eta}|[0, \infty) = f_{C, \eta}|[0, \infty) \) and \( \hat{h}_{C, \eta} = h_{C, \eta} \). Consider \( H_s = sH_{C, \eta} + (1-s)\hat{H}_{C, \eta}, 0 \leq s \leq 1 \) and put

\[
\Phi_s(x) := \Phi_H(s, x) \quad \forall x \in \Xi.
\]

It suffices to prove \( c_{EH}^\Psi(H_0) = c_{EH}^\Psi(H_1) \). If \( x \) is a critical point of \( \Phi_s \) with \( \Phi_s(x) > 0 \), then there holds \( x([0, 1]) \in S_\tau = \psi((\tau) \times S) \) for some \( \tau \in (\eta, 2\eta) \). The choice of \( H_{C, \eta} \) shows \( H_s(x(t)) = H_{C, \eta}(x(t)) \) for \( t \in [0, 1] \). This implies that each \( \Phi_s \) has the same positive critical values as \( \Phi_H(0, \eta) \). By the continuity in Proposition 1.11(ii),
s \mapsto c_{EH}^\Psi(H_s) is continuous and takes values in the set of positive critical values of \( \Phi_{H_{C,\eta}} \) which has measure zero (see Sect. 3.2). Hence \( s \mapsto c_{EH}^\Psi(H_s) \) is constant. In particular,

\[
c_{EH}^\Psi(\hat{H}_{C,\eta}) = c_{EH}^\Psi(H_0) = c_{EH}^\Psi(H_1) = c_{EH}^\Psi(H_{C,\eta}).
\]

Summarizing the above arguments we have proved that \( c_{EH}^\Psi(S) = c_{EH}^\Psi(B) \in \Sigma_\Psi^\Psi. \) Note that \( c_{EH}^\Psi(B) > 0. \) Hence \( c_{EH}^\Psi(S) = c_{EH}^\Psi(B) \in \Sigma_\Psi^\Psi. \)

### 7 Proofs of Theorems 1.29, 1.32

#### 7.1 Proof of Theorem 1.29

**Step 1** Let \( U = \bigcup_{\lambda \in I} S_\lambda \) be a thickening of a compact and regular energy surface \( S = \{ x \in M \mid H(x) = 0 \} \) in a symplectic manifold \((M, \omega)\) as in (1.39). Corresponding to [27, p. 109, Proposition 1] we have:

**Claim 7.1** For a converging sequence \( \lambda_j \to \lambda^* \) in the interval \( I \), suppose that for every \( \lambda_j \) the Hamiltonian boundary value problem

\[
\dot{x} = X_H(x), \quad x(T_j) = \Psi x(0), \quad 0 < T_j < \infty
\]

has a solution \( x_j : [0, T_j] \to S_{\lambda_j} \). If \( T_j \leq C \) for some constant \( C > 0 \) and for all \( j \), then \( S_{\lambda_j} \) carries either a fixed point of \( \Psi \) or a \( \Psi \)-characteristic \( y : [0, T] \to S_{\lambda^*} \) satisfying \( \dot{y} = X_H(y) \) and \( 0 < T \leq C \).

Indeed, for each \( j \), the map \( z_j : [0, 1] \to S_{\lambda_j} \) defined by \( z_j(t) = x_j(T_jt) \) satisfies \( \dot{z}_j(t) = T_j X_H(z_j(t)) \) and \( \Psi z_j(0) = z_j(1) \). By the Arzela-Ascoli theorem, passing to a subsequence we may assume that \( T_j \to T \) and \( z_j \to z \) in \( C^\infty([0, 1], M) \). Hence \( z(1) = \Psi z(0) \) and \( \dot{z}(t) = T X_H(z(t)) \) for \( 0 \leq t \leq 1 \). Clearly, \( 0 \leq T \leq C \). If \( T = 0 \) then \( z \) is constant and therefore \( z(0) \in S_{\lambda^*} \) is a fixed point of \( \Psi \). If \( T > 0 \) then \( y : [0, T] \to S_{\lambda^*} \) defined by \( y(t) := z(t/T) \) is the desired \( \Psi \)-characteristic.

**Step 2** Along the ideas in [36], for each \( n \in \mathbb{N} \) let \( G_n \) be the set of nonzero parameters \( \lambda \in I = (-\varepsilon, \varepsilon) \) for which \( S_\lambda \) contains either a fixed point of \( \Psi \) or a \( \Psi \)-characteristic \( y : [0, T] \to S_\lambda \) satisfying

\[
0 < T \leq n, \quad \dot{y} = X_H(y) \text{ if } \lambda > 0, \quad \dot{y} = -X_H(y) \text{ if } \lambda < 0.
\]

The above claim implies that \( G_n \cup \{0\} \) is closed and therefore that \( G = \bigcup_{n=1}^{\infty} G_n \) is a Lebesgue-measurable set. For \( 0 < \delta < \varepsilon \) we define

\[
U_\delta := \bigcup_{|\lambda| < \delta} S_\lambda = \{ x \in U \mid -\delta < H(x) < \delta \}
\]
which is an open subset in $U$. Since $S_0 = S$ has nonempty intersection with $\text{Fix}(\Psi)$, $U_\delta \cap \text{Fix}(\Psi) \neq \emptyset$. It follows that $c^\Psi_{HZ}(U_\delta, \omega)$ is well-defined and Proposition 1.2(ii) implies that $c^\Psi_{HZ}(U_{\delta_1}, \omega) \leq c^\Psi_{HZ}(U_{\delta_2}, \omega) \leq c^\Psi_{HZ}(U, \omega)$ for any $0 < \delta_1 < \delta_2 < \varepsilon$.

Claim 7.2 For $\delta^* \in (0, \varepsilon)$, if there exist positive numbers $L > 0$ and $\mu \in (\delta^*, \varepsilon)$ such that

$$c^\Psi_{HZ}(U_\delta, \omega) \leq c^\Psi_{HZ}(U_{\delta^*}, \omega) + L(\delta - \delta^*) \quad \forall \delta \in [\delta^*, \mu],$$

then $\delta^* \in G$ or $-\delta^* \in G$.

In fact, for a fixed $\delta \in (\delta^*, \mu)$, by definition of $c^\Psi_{HZ}$ we have $\tilde{H} \in \mathcal{H}^\Psi_{ad}(U_{\delta^*}, \omega)$ such that max $\tilde{H} > c^\Psi_{HZ}(U_{\delta^*}, \omega) - (\delta - \delta^*)$. As in [36] (or [51, p. 315]) we take a smooth function $f : [0, \varepsilon) \rightarrow \mathbb{R}$ such that

(a) $f(t) = \max \tilde{H}$ for $0 \leq t \leq \delta^*$,
(b) $f(t) = c^\Psi_{HZ}(U_\delta, \omega) + (\delta - \delta^*)$ for $\delta \leq t < \varepsilon$,
(c) $f'(t) > 0$ for $t \in (\delta^*, \delta)$ and $f'(t) \in [0, L + 3)$ for all $t \in [0, \varepsilon)$.

Define $F : U_\delta \rightarrow \mathbb{R}$ by setting $F = \tilde{H}$ on $U_{\delta^*}$ and $F(x) = f((H(x)))$ for $x \in U_\delta \setminus U_{\delta^*}$. Then $F \in \mathcal{H}^\Psi(\mathcal{U}_\delta, \omega)$ and max $F = c^\Psi_{HZ}(U_\delta, \omega) + (\delta - \delta^*) > c^\Psi_{HZ}(U_\delta, \omega)$. Hence for some $0 < T < 1$ we have a nonconstant differentiable path $\gamma : [0, T] \rightarrow U_\delta$ satisfying $\dot{\gamma} = X_F(\gamma)$ and $\Psi(\gamma(0)) = \gamma(T)$. Note that $\tilde{H} \in \mathcal{H}^\Psi_{ad}(U_{\delta^*}, \omega)$ may be naturally extended into an element in $\mathcal{H}^\Psi_{ad}(Cl(U_{\delta^*}), \omega)$ and that $c^\Psi_{HZ}(U_{\delta^*}, \omega) = c^\Psi_{HZ}(Cl(U_{\delta^*}), \omega)$, where $Cl(A)$ is the closure of $A$. Using the fact that $F$ is equal to a positive constant along $\gamma$ we deduce that $\gamma([0, T])$ is contained in $U_\delta \setminus Cl(U_{\delta^*})$. This implies that $H \circ \gamma$ is equal to a constant $c$ in $(\delta^*, \delta)$ or $(-\delta, -\delta^*)$ and

- $\dot{\gamma}(t) = f'(H(\gamma(t)))X_H(\gamma(t))$ on $[0, T]$ if $c \in (\delta^*, \delta)$,
- $\dot{\gamma}(t) = -f'(-H(\gamma(t)))X_H(\gamma(t))$ on $[0, T]$ if $c \in (-\delta, -\delta^*)$.

Let $\tau = f'(|c|)$ which belongs to $(0, L + 3)$. Note that the path $[0, \tau T] \ni t \rightarrow \gamma(t) = \gamma(t/\tau)$ sits in $S_c$ and satisfies $\Psi(\gamma(0)) = \gamma(\tau T)$ and

$$\begin{cases} \dot{y} = X_H(y) & \text{if } c > 0, \\ \dot{y} = -X_H(y) & \text{if } c < 0. \end{cases}$$

Moreover, $0 < \tau T \leq \tau \leq L + 3$.

Take a sequence $(\delta_j)$ in the interval $(\delta^*, \mu)$ such that $\delta_j \downarrow \delta^*$. By the arguments above we have sequences $\lambda_j \in (\delta^*, \delta_j)$ and $T_j \in (0, L + 3], j = 1, 2, \ldots$, such that for each $j$ there exists

- either a $\Psi$-characteristic $y_j : [0, T_j] \rightarrow S_{\lambda_j}$ satisfying $\dot{y} = X_H(y)$,
- or a $\Psi$-characteristic $y_j : [0, T_j] \rightarrow S_{-\lambda_j}$ satisfying $\dot{y} = -X_H(y)$.

Passing to a subsequence of $(\lambda_j)$ when necessary and using Claim 7.1 we obtain: either $S_{\delta^*}$ carries a $\Psi$-characteristic $y : [0, T] \rightarrow S_{\delta^*}$ satisfying $\dot{y} = X_H(y)$ and $0 < T \leq L + 3$, or $S_{-\delta^*}$ carries a $\Psi$-characteristic $y : [0, T] \rightarrow S_{-\delta^*}$ satisfying $\dot{y} = -X_H(y)$ and $0 < T \leq L + 3$, or $S_{\delta^*} \cup S_{-\delta^*}$ carries a fixed point of $\Psi$. Claim 7.2 is proved.
Step 3 Prove statement (i). For a nonzero $\lambda \in I$ let $\mathcal{P}(S_\lambda, \Psi)$ consist of fixed points of $\Psi$ and $\Psi$-characteristics on $S_\lambda$ satisfying $\dot{y} = \text{sign}(\lambda)X_H(y)$, where $\text{sign}(\lambda) = 1$ if $\lambda > 0$, and $\text{sign}(\lambda) = -1$ if $\lambda < 0$. Since the monotone nondecreasing function $(0, \varepsilon) \ni \delta \mapsto c_{\Psi}(U_\delta, \omega) \in \mathbb{R}$ is differentiable almost everywhere and thus Lipschitz continuous almost everywhere, we derive from Step 2 that

$$
\hat{G} := \{\delta \in (0, \varepsilon) \mid \mathcal{P}(S_\delta, \Psi) \neq \emptyset \text{ or } \mathcal{P}(S_{-\delta}, \Psi) \neq \emptyset\}
= \{\delta \in (0, \varepsilon) \mid \delta \in G \text{ or } -\delta \in G\}
$$

has Lebesgue measure $\varepsilon$ and thus satisfies the requirement in (i).

Step 4 Prove statement (ii). For each $n \in \mathbb{N}$ let $\Lambda_n$ be the set of nonzero parameters $\lambda \in (-\varepsilon, \varepsilon)$ for which $S_\lambda$ contains either a fixed point of $\Psi$ or a $\Psi$-characteristic $y : [0, T] \to S_\lambda$ satisfying

$$
0 < T \leq n, \quad \dot{y} = X_H(y) \quad \text{or} \quad \dot{y} = -X_H(y).
$$

Then $\Lambda_n$ contains $G_n$, and $\Lambda_n \cup \{0\}$ is closed in $I$ and so $\Lambda := \bigcup_{n=1}^{\infty} \Lambda_n$ is a Lebesgue-measurable set. By Step 3 the set $\{\delta \in (0, \varepsilon) \mid \delta \in \Lambda \text{ or } -\delta \in \Lambda\}$ has Lebesgue measure $\varepsilon$. As in the proof of [36] it follows from this that $\Lambda$ has Lebesgue measure $m(\Lambda) = 2\varepsilon$.

7.2 Proof of Theorem 1.32

We closely follow [45, 46]. By the assumptions there exists $\delta_0 > 0$ such that $\mathbb{R}^{2n} \setminus H^{-1}([1 - \delta_0, 1 + \delta_0])$ consists of a bounded component $B$ containing the fixed point $z_0$ of $\Psi$ and an unbounded one $A$. As the arguments at the beginning of Sect. 3 we may reduce to the case $z_0 = 0$. Then $r_0 := \sup\{\tau > 0 \mid B^{2n}(0, \tau) \subset B\} > 0$. Clearly, by shrinking $\delta_0$ we can also assume that each $\beta \in [1 - \delta_0, 1 + \delta_0]$ is a regular value of $H$ and $S_\beta := H^{-1}(\beta)$ is diffeomorphic to $S = S_1$. Let $\gamma$ be the diameter of $H^{-1}([1 - \delta_0, 1 + \delta_0])$. Then $\gamma > r_0$. Fix $r \in (\gamma, 2\gamma)$. Then $H^{-1}([1 - \delta_0, 1 + \delta_0]) \cup B$ is contained in the ball $B^{2n}(0, r)$.

Fix a number $\beta_0 \in (1 - \delta_0, 1 + \delta_0)$ and choose $0 < \delta < (\delta_0 - |1 - \beta_0|)/3$. Then the closure of $I_0 := (\beta_0 - \delta, \beta_0 + \delta)$ is contained in $(1 - \delta_0, 1 + \delta_0)$. Define

$$
U_\delta = H^{-1}([1 - \delta_0 + \delta, 1 + \delta_0 - \delta]).
$$

Let $A_\delta$ and $B_\delta$ be the unbounded and bounded components of $\mathbb{R}^{2n} \setminus U_\delta$ respectively. Then $U_\delta \cup B_\delta \subset B^{2n}(0, r)$. We modify the constant $b$ and the smooth functions $f, g$ in [25] such that

$$
\left(\frac{l(\Psi)}{2} + \varepsilon\right)r^2 < b < \frac{2l(\Psi) + 4\varepsilon}{3}r^2,
$$

$$
f(s) = 0 \text{ for } s \leq -\delta, \quad f(s) = b \text{ for } s \geq \delta, \quad f'(s) > 0 \text{ for } |s| < \delta,
$$

$$
g(s) = b \text{ for } s \leq r, \quad g(s) \geq \left(\frac{l(\Psi)}{2} + \varepsilon\right)s^2 \text{ for } s > r, \quad g(s) = \left(\frac{l(\Psi)}{2} + \varepsilon\right)s^2 \text{ for large } s.
$$
where $0 < \epsilon < 1$ satisfies $\det \left( \Psi - e^{(t\Psi + 2\epsilon)}J \right) \neq 0$.

$0 < g'(s) \leq (t(\Psi) + 2\epsilon)s$ for $s > r$. (So $g(s) \leq b + \left( \frac{t(\Psi)}{2} + \epsilon \right) (s^2 - r^2) \forall r$).

Following [46, Chap. II, §9], for $m \in \mathbb{N}$ and $\beta \in I_0$ we define

$$H_{\beta,m}(x) = \begin{cases} 
0 & \text{if } x \in B_\delta, \\
\tilde{f}(m(H(x) - \beta)) & \text{if } x \in U_\delta, \\
b & \text{if } x \in A_\delta \cap B^{2n}(0, r), \\
g(|x|) & \text{if } x \in A_\delta \setminus B^{2n}(0, r)
\end{cases}$$

and a functional $\Phi_{m,\beta}$ on the space $E$ in (2.11) by

$$\Phi_{m,\beta}(x) = \Phi_{H_{m,\beta}}(x) = \frac{1}{2} \left( \|x^+\|^2_E - \|x^-\|^2_E \right) - \int_0^1 H_{m,\beta}(x(t))dt.$$ 

Note that for any $m \in \mathbb{N}$ and $\beta \in I_0$ the function $H_{\beta,m}$ satisfies inequalities

$$-b + \left( \frac{t(\Psi)}{2} + \epsilon \right) |x|^2 \leq H_{\beta,m}(x) \leq \left( \frac{t(\Psi)}{2} + \epsilon \right) |x|^2 + b \quad \forall x \in \mathbb{R}^{2n}, \quad (7.1)$$

$$H_{\beta_{i-1},m} \leq H_{\beta_i,m} \quad \text{for } \beta_{i-1} \leq \beta_2, \beta_i \in I_0, i = 1, 2. \quad (7.2)$$

Since $B^{2n}(0, r_0) \subset B \subset B_\delta$, $H_{\beta,m} \equiv 0$ in $B^{2n}(0, r_0)$ and there exist constants $C_1 > 0$, $C_2 > 0$ independent of $m \in \mathbb{N}$ and $\beta \in I_0$ such that

$$|H_{\beta,m}(z_0 + x)| \leq C_1 |x|^2 \quad \text{and} \quad |H_{\beta,m}(z_0 + x)| \leq C_2 |x|^3 \quad \forall x \in \mathbb{R}^{2n}. \quad (7.3)$$

Moreover, for $|x|$ large enough we have uniformly (in $m \in \mathbb{N}$ and $\beta \in I_0$)

$$\nabla H_{\beta,m}(x) = (t(\Psi) + 2\epsilon)x \quad \text{and} \quad (H_{\beta,m})_{xx} = (t(\Psi) + 2\epsilon)I_{2n}.$$ 

It follows that for some positive constants $C_3$, $C_4$ independent of $m \in \mathbb{N}$ and $\beta \in I_0$ there holds

$$|\nabla H_{\beta,m}(x)| \leq (t(\Psi) + 2\epsilon)|x| + C_3 \quad \text{and} \quad |(H_{\beta,m})_{xx}| \leq C_4 \quad (7.4)$$

on $\mathbb{R}^{2n}$. Then $\Phi_{m,\beta}$ is still a $C^1$-functional on $E$ satisfying the (PS) condition, and each critical point $x$ of it is smooth and satisfies the Hamiltonian boundary value problem

$$\dot{x} = X_{H_{\beta,m}}(x) \quad \text{and} \quad x(1) = \Psi x(0). \quad (7.5)$$

(See Propositions 2.10 and 2.11.) By Proposition 2.9 the conditions in (7.4) also insures that $\nabla \Phi_{\beta,m}$ satisfies

$$\|\nabla \Phi_{\beta,m}(x) - \nabla \Phi_{\beta,m}(y)\|_E \leq \ell \|x - y\|_E \quad \forall x, y \in E \quad (7.6)$$
for some constant $\ell > 0$ independent of $m \in \mathbb{N}$ and $\beta \in I_0$.

As in the proofs of [46, page 138, Lemma 9.2] we have:

Claim 7.3 Let $x \in \mathbb{E}$ be a critical point of $\Phi_{\beta,m}$ with $\Phi_{\beta,m}(x) > 0$. Then $H(x(t)) \equiv h \in \mathbb{R}$ with $|h - \beta| < \delta/m$,

$$T_x = mf'(m(H(x) - \beta)) > 0,$$

and $y : [0, T_x] \to \mathbb{R}^{2n}$, $t \mapsto x(t/T_x)$ satisfies: $\dot{y} = X_H(y)$ and $y(T_x) = \Psi y(0)$.

Let $\mathbb{E}^+, \mathbb{E}^0, \mathbb{E}^-$ be as in (2.11). Corresponding to [46, page 138, Lemma 9.3] there holds:

Claim 7.4 There exist numbers $\alpha > 0$, $\rho > 0$ independent of $m \in \mathbb{N}$ and $\beta \in I_0$ such that $\Phi_{\beta,m}(x) \geq \alpha$ for $x \in S^+_\rho = \{x \in \mathbb{E}^+ | \|x\|_{\mathbb{E}} = \rho\}$.

Proof Since we do not know whether the space $\mathbb{E}$ in (2.11) can be embedded into some $L^p([0, 1], \mathbb{R}^{2n})$ with $p > 2$, the proof of [46, page 138, Lemma 9.3] does not work in the present case. However, because of estimates in (7.3), as in the proof of Proposition 4.3 we may still use the method in the proof of [26, page 93, Lemma 9]. Indeed, it suffices to prove that

$$\lim_{\|x\|_{\mathbb{E}} \to 0} \frac{\int_0^1 H_{\beta,m}(x(t)) dt}{\|x(t)\|^2_{\mathbb{E}}} = 0 \quad (7.7)$$

uniformly $m \in \mathbb{N}$ and $\beta \in I_0$. Otherwise, suppose there exist sequences $(x_j) \subset \mathbb{E}$, $(m_j) \subset \mathbb{N}$ and $(\beta_j) \subset I_0$, and $d > 0$ satisfying

$$\|x_j\|_{\mathbb{E}} \to 0 \quad \text{and} \quad \frac{\int_0^1 H_{\beta_j,m_j}(x_j(t)) dt}{\|x_j\|^2_{\mathbb{E}}} \geq d > 0 \quad \forall j. \quad (7.8)$$

Let $y_j = \frac{x_j}{\|x_j\|_{\mathbb{E}}}$. Then $\|y_j\|_{\mathbb{E}} = 1$ and hence $(y_j)$ has a convergent subsequence in $L^2$. By [8, Th.4.9] we have $w \in L^2$ and a subsequence of $(y_j)$, still denoted by $(y_j)$, such that $y_j(t) \to y(t)$ a.e. on $(0, 1)$ and that $|y_j(t)| \leq w(t)$ a.e. on $(0, 1)$ and for each $j$.

As in the proof of Proposition 4.3 it follows from (7.3) that

$$\frac{H_{\beta_j,m_j}(x_j(t))}{\|x_j\|^2_{\mathbb{E}}} \leq C_1 w(t)^2, \quad \text{a.e. on } (0, 1), \forall j,$$

$$\frac{H_{\beta_j,m_j}(x_j(t))}{\|x_j\|^2_{\mathbb{E}}} \leq C_2 |x_j(t)| w(t)^2, \quad \text{a.e. on } (0, 1), \forall j,$$

where $C_1, C_2$ are independent of $m_j$ and $\beta_j$. Moreover, by the first claim in (7.8) $(x_j)$ has a subsequence $x_{j_i}(t) \to 0$ a.e. on $(0, 1)$. Using the Lebesgue dominated convergence theorem we deduce

$$\frac{\int_0^1 H_{\beta_{j_i},m_{j_i}}(x_{j_i}(t)) dt}{\|x_{j_i}\|^2_{\mathbb{E}}} = \int_0^1 \frac{H_{\beta_{j_i},m_{j_i}}(x_{j_i}(t)) dt}{\|x_{j_i}\|^2_{\mathbb{E}}} \to 0.$$
which contradicts the second claim in (7.8).

Similar to Proposition 4.2, let \( \hat{e}(t) = \frac{1}{(\Psi(t))^J} e^{t(\Psi(t))} X \) where \( X \in \mathbb{R}^{2n} \) satisfies \( e^{t(\Psi(t))} X = \Psi X \) and \( |X| = 1 \). Then \( \hat{e} \in \mathbb{E}^+ \), \( \|\hat{e}\|_E = 1 \) and \( \|\hat{e}\|_{L_2} = \frac{1}{(\Psi(t))} \). Define

\[
Q_R := \{ x = s\hat{e} + x^0 + x^- \in \mathbb{E} | \|x^0 + x^-\|_E \leq R, 0 \leq s \leq R \}.
\]

Let \( \partial Q_R \) denote the relative boundary of \( Q_R \) in \( \mathbb{E}^- \oplus \mathbb{E}^0 \oplus \mathbb{R}\hat{e} \).

**Claim 7.5** There exists a number \( R > \rho \), independent of \( m \in \mathbb{N} \) and \( \beta \in I_0 \), such that \( \Phi_{\beta,m} \mid \partial Q_R \leq 0 \).

**Proof** For \( x = s\hat{e} + x^0 + x^- \) with \( s = R \) or \( \|x^0 + x^-\|_E = R \), by (7.1) we have

\[
\alpha(x) = \frac{1}{2} \left( \|s\hat{e}\|_E^2 - \|x^-\|_E^2 \right) = \frac{1}{2} \left( s^2 - \|x^-\|_E^2 \right),
\]

\[
\int_0^1 H_{\beta,m}(x(t))dt \geq -b + \left( \frac{t(\Psi)}{2} + \epsilon \right) \int_0^1 |x(t)|^2 dt.
\]

Note that \( \|s\hat{e} + x^0 + x^-\|_{L_2}^2 = \|x^0\|_{L_2}^2 + \|x^-\|_{L_2}^2 + \frac{\epsilon^2}{t(\Psi)} \). Hence we arrive at

\[
\Phi_{\beta,m}(x) \leq \frac{1}{2} \left( s^2 - \|x^-\|_E^2 \right) + b - \left( \frac{t(\Psi)}{2} + \epsilon \right) \left( \|x^0\|_{L_2}^2 + \|x^-\|_{L_2}^2 + \frac{s^2}{t(\Psi)} \right)
\]

\[
= -\epsilon s^2 \frac{1}{t(\Psi)} - \frac{1}{2} \|x^-\|_E^2 - \left( \frac{t(\Psi)}{2} + \epsilon \right) \left( \|x^0\|_{L_2}^2 + \|x^-\|_{L_2}^2 + b \right)
\]

\[
\leq 0
\]

if \( R > 0 \) is sufficiently large. Moreover, it is clear that \( \Phi_{\beta,m}(x) \leq 0 \) for \( x = s\hat{e} + x^0 + x^- \) with \( s = 0 \). \( \square \)

As in [46, page 134] let \( \hat{\Gamma} \) be the class of maps \( h \in C^0(\mathbb{E}, \mathbb{E}) \) such that \( h \) is homotopic to the identity through a family of maps \( h_t = L_t + K_t, 0 \leq t \leq T, \) where \( L_0 = \text{id}_E, K_0 = 0 \) and \( L_t : \mathbb{E} \to \mathbb{E} \) is a Banach space isomorphism satisfying \( L_t(\mathbb{E}^*) = \mathbb{E}^*, ++ = 0, +, - \), and where \( K_t \) is compact and \( h_t(\partial Q_R) \cap S^+_{\rho} = \emptyset \) for each \( t \). Repeating the proofs of [46, Lemmas 8.10,8.11] we can obtain that \( \partial Q_R \) and \( S^+_{\rho} \) link with respect to \( \hat{\Gamma} \) and that the gradient flow \( G : \mathbb{E} \times [0, \infty) \to \mathbb{E} \) given by

\[
\frac{\partial}{\partial t} G(x, t) = -\nabla \Phi_{\beta,m}(G(x, t)) \quad \text{and} \quad G(x, 0) = x
\]

exists globally and \( G(\cdot, T) \in \hat{\Gamma} \) for any \( T \geq 0 \). Hence \( G(\partial Q_R, t) \cap \hat{S}^+_{\rho} \neq \emptyset \) for all \( t \geq 0 \). By the standard arguments we deduce that

\[
c(H_{\beta,m}) := \inf_{t \geq 0} \sup_{x \in \partial Q_R} \Phi_{\beta,m}(G(x, t)) \geq \inf_{x \in \hat{S}^+_{\rho}} \Phi_{\beta,m}(x) \geq \alpha \quad (7.9)
\]
are positive critical values for all $\beta, m$. On the other hand, for any $t \geq 0$ it holds that
\[
\sup_{x \in Q_R} \Phi_{\beta, m}(G(x, t)) \leq \sup_{x \in Q_R} \Phi_{\beta, m}(G(x, 0)) = \sup_{x \in Q_R} \Phi_{\beta, m}(x) \leq \sup_{x \in E^\perp \oplus E_0 \oplus R \geq 0} \Phi_{H, \beta, m}(x) \leq \sup_{z \in C^n} \left( \frac{t(\Psi)}{2} |z|^2 - H_{\beta, m}(z) \right)
\]
where the final inequality is obtained as in the proof of (4.2) in Proposition 4.2. By this and (7.1) we also have
\[
c(H_{\beta, m}) \leq \sup_{z \in C^n} \left( \frac{t(\Psi)}{2} |z|^2 - H_{\beta, m}(z) \right) \leq b. \tag{7.10}
\]

**Claim 7.6** For fixed $x \in E$ and $m \in \mathbb{N}$, $I_0 \ni \beta \mapsto \Phi_{\beta, m}(x)$ is monotone non-increasing by (7.2) and there holds
\[
\frac{\partial}{\partial \beta} \Phi_{\beta, m}(x) = m \int_0^1 f'(m(H(x(t)) - \beta))dt.
\]
In particular, for a critical point $x$ of $\Phi_{\beta, m}$ with $\Phi_{\beta, m}(x) > 0$, $\frac{\partial}{\partial \beta} \Phi_{\beta, m}(x)$ is equal to $T_x$ given by Claim 7.3.

Corresponding to the critical value $c(H_{\beta, m})$ in (7.6) we have a critical point $x_{\beta, m} \in E$. Since for each $m \in \mathbb{N}$ the map $I_0 \ni \beta \mapsto c(H_{\beta, m}) = \Phi_{\beta, m}(x_{\beta, m})$ is non-decreasing, as in the arguments on [46, page 140] we have $C_{\beta} := \lim_{m \to \infty} \frac{\partial}{\partial \beta} c(H_{\beta, m}) < \infty$ for almost every $\beta \in I_0$. Fixing such a $\beta$ we get a subsequence $\Lambda \subset \mathbb{N}$ such that $\frac{\partial}{\partial \beta} c(H_{\beta, m}) \to C_{\beta}$ as $m \in \Lambda$ and $m \to \infty$. Repeating the proof of [46, Lemma 9.4] yields

**Claim 7.7** For any $m \in \Lambda$ there exists a critical point $x_{\beta, m}$ of $\Phi_{\beta, m}$ such that $\Phi_{\beta, m}(x_{\beta, m}) = c(H_{\beta, m}) \geq \alpha$ and $T_{\beta, m} := \frac{\partial}{\partial \beta} \Phi_{\beta, m}(x_{\beta, m}) \leq C_{\beta} + 4$.

By Claim 7.3, $x_{\beta, m}$ is smooth and satisfies $H(x_{\beta, m}(t)) \equiv h_{\beta, m} \in (\beta - \delta/m, \beta + \delta/m)$ and
\[
\begin{align*}
\dot{x}_{\beta, m} &= X_{H_{\beta, m}}(x_{\beta, m}) = mf'(m(H(x_{\beta, m}) - \beta)) X_H(x_{\beta, m}) = T_{\beta, m} X_H(x_{\beta, m}), \\
x_{\beta, m}(1) &= \Psi x_{\beta, m}(0). \tag{7.11}
\end{align*}
\]
It follows that the sequences $(x_{\beta, m})$ and $(\dot{x}_{\beta, m})$ are uniformly bounded and equi-continuous. Since $(T_{\beta, m})$ is bounded we may assume $T_{\beta, m} \to T \leq C_{\beta} + 1$. By the
Arzéla-Ascoli theorem we get a subsequence \( x_{\beta,m_j} \) converging in \( C^1([0, 1], \mathbb{R}^{2n}) \) to a solution of

\[
\dot{x} = TX_{H}(x) \quad \text{and} \quad x(1) = \Psi x(0)
\]

(7.12)

with \( H(x(t)) \equiv \beta \). Note that \( A(x_{\beta,m_j}) \geq \Phi_{\beta,m_j}(x_{\beta,m_j}) \geq \alpha \). Let \( j \to \infty \) and we get \( A(x) \geq \alpha \). This implies that \( x \) is non-constant and \( T > 0 \). Since \( H(x(t)) \equiv \beta \), we obtain that \( x([0, 1]) \subset U_\delta \) and so

\[
\int_0^1 H_{\beta,m_j}(x_{\beta,m_j}(t)) dt \leq b, \quad \forall j.
\]

This and (7.10) lead to

\[
\alpha \leq A(x_{\beta,m_j}) = \Phi_{\beta,m_j}(x_{\beta,m_j}) + \int_0^1 H_{\beta,m_j}(x_{\beta,m_j}(t)) dt \leq 2b < \frac{16t(\Psi) + 32\epsilon}{3} y^2.
\]

Clearly \( 0 < \epsilon \ll 1 \) can be chosen to satisfy \( 0 < \epsilon < t(\Psi) \). Hence \( \alpha \leq A(x) < 16t(\Psi)\gamma^2 \). Finally, \( y(t) := x(t/T) \) sits in \( S_\delta \) with action \( A(y) = A(x) < 16t(\Psi)\gamma^2 \) satisfying \( \dot{y} = X_{H}(y) \) and \( y(T) = \Psi y(0) \).

8 Proof of Theorem 1.34

Under the assumptions of Theorem 1.34, for each number \( \epsilon \) with \( |\epsilon| \) small enough the set \( D_\epsilon := D(e_0 + \epsilon) \) is a strictly convex bounded domain in \( \mathbb{R}^{2n} \) with \( 0 \in D_\epsilon \) and with \( C^2 \)-boundary \( S_\epsilon = S(e_0 + \epsilon) \). Following the notations in Theorem 1.34 and Sect. 3 let \( H_\epsilon = \langle j_{D_\epsilon} \rangle^2 \) and \( H_\epsilon^* \) denote the Legendre transform of \( H_\epsilon \). Both \( H_\epsilon \) and \( H_\epsilon^* \) are \( C^{1,1} \) on \( \mathbb{R}^{2n} \), \( C^2 \) on \( \mathbb{R}^{2n} \setminus \{0\} \) and have positive Hessian matrices at every point on \( \mathbb{R}^{2n} \setminus \{0\} \). Recall that \( H_\epsilon^*(x) = \langle \xi_\epsilon(x), x \rangle - H_\epsilon(\xi_\epsilon(x)) \), where \( \nabla H_\epsilon(\xi_\epsilon(x)) = x \) and \( \nabla H_\epsilon^*(x) = \xi_\epsilon(x) \). It was proved in [39] that \( \epsilon \mapsto \xi_\epsilon \) is \( C^1 \) and \( H_\epsilon(x) \) and \( H_\epsilon^*(x) \) are \( C^2 \) functions of \( \epsilon \) for every fixed \( x \in \mathbb{R}^{2n} \setminus \{0\} \).

Let \( x^* : [0, \mu] \to S_0 \) satisfying

\[
\dot{x} = J\nabla H_0(x), \quad x(\mu) = \Psi x(0)
\]

be a \( c_{HZ}^\Psi \)-carrier for \( D_0 \). Then \( \mu = A(x^*) = c_{HZ}^\Psi(D_0, \omega_0) \). By the proof in Step 3 of Sect. 3.1, for some \( a_0 \in \text{Ker}(\Psi - I_{2n}) \subset \mathbb{R}^{2n} \),

\[
u : [0, 1] \to \mathbb{R}^{2n}, \quad t \mapsto \frac{1}{\sqrt{\mu}} x^*(\mu t) - \frac{a_0}{\mu}
\]

belongs to \( \mathcal{F} \) in (3.4) and satisfies \( A(\nu) = 1 \) and

\[
-J\dot{\nu}(t) = \nabla H_0(\mu \nu(t) + a_0) \quad \forall t \in [0, 1].
\]

(8.1)
Moreover, there holds
\[ c^\Psi_{HZ}(D_0, \omega_0) = \int_0^1 H_0^*(-J \dot{u}) \, dt \]
and the arguments of Sect. 3.1 also imply
\[ \mathcal{C}(\epsilon) := \mathcal{C}(e_0 + \epsilon) = c^\Psi_{HZ}(D_\epsilon, \omega_0) \leq \int_0^1 H_\epsilon^*(-J \dot{u}) \, dt. \quad (8.2) \]

By the Taylor’s formula
\[ H_\epsilon^*(-J \dot{u}(t)) = H_0^*(-J \dot{u}(t)) + \frac{\partial H_\epsilon^*}{\partial \epsilon} \bigg|_{\epsilon=0} (-J \dot{u}(t)) \epsilon + \frac{1}{2} \frac{\partial^2 H_\epsilon^*}{\partial \epsilon^2} \bigg|_{\epsilon=\tau} (-J \dot{u}(t)) \epsilon^2 \quad (8.3) \]
where \( 0 < \tau < \epsilon \). Let
\[ T_{x^*} = 2 \int_0^{\mathcal{C}(\epsilon)} \frac{dt}{\langle \nabla \mathcal{H}(x^*(t)), x^*(t) \rangle}. \]

Computing as in [39] we deduce that
\[ \int_0^1 \frac{\partial H_\epsilon^*}{\partial \epsilon} \bigg|_{\epsilon=0} (-J \dot{u}(t)) \, dt = T_{x^*} \]
and there exists a constant \( K \) only depending on \( S_0 \) and \( H_\epsilon \) with \( \epsilon \) near 0 such that
\[ \left| \frac{\partial^2 H_\epsilon^*}{\partial \epsilon^2} \bigg|_{\epsilon=\tau} (-J \dot{u}(t)) \right| \leq 2K, \quad \forall t \in [0, 1]. \]

Then for \( \epsilon \) near 0 there holds
\[ \mathcal{C}(\epsilon) \leq c^\Psi_{HZ}(D_0, \omega_0) + T_{x^*}\epsilon + K \epsilon^2. \quad (8.4) \]

Recall that \( T^{\text{max}}(e_0 + \epsilon) \) and \( T^{\text{min}}(e_0 + \epsilon) \) are the largest and smallest numbers in the compact set \( J(e_0 + \epsilon) \) defined by (1.40). By [39, Lemma 4.1] and [39, Corollary 4.2], both are functions of bounded variation in \( \epsilon \) (and thus bounded near \( \epsilon = 0 \)), and \( \epsilon \mapsto \mathcal{C}(e_0 + \epsilon) \) is continuous. As in the proof of [39, Theorem 4.4], using these and (8.4) we can show that \( \mathcal{C}(\epsilon) \) has left and right derivatives at \( \epsilon = 0 \), i.e.,
\[ \mathcal{C}'_-(0) = \lim_{\epsilon \to 0^-} T^{\text{max}}(e_0 + \epsilon) = T^{\text{max}}(e_0) \quad \text{and} \]
\[ \mathcal{C}'_+(0) = \lim_{\epsilon \to 0^+} T^{\text{min}}(e_0 + \epsilon) = T^{\text{min}}(e_0). \]
which complete the proof of the first part of Theorem 1.34. The final part is a direct consequence of the first one and a modified version of the intermediate value theorem (cf. [39, Theorem 5.1]).

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Declarations

Conflict of interest The authors have no conflicts of interest.

A Appendix: Some Facts on Symplectic Matrixes

For a symplectic matrix $\Psi \in \text{Sp}(2n, \mathbb{R})$, recall that

$$g^\Psi : \mathbb{R} \to \mathbb{R}, \ s \mapsto \det(\Psi - e^{sJ}),$$

and $t(\Psi)$ is the smallest zero point of $g^\Psi$ in $(0, 2\pi]$. $\Psi$ is the smallest zero point of $g^\Psi$ in $(0, 2\pi]$. Therefore $e^{sJ}$ is unitarily similar to

$$\begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix} \in GL(2n, \mathbb{C}).$$

Hence $e^{sJ}$ is unitarily similar to

$$\begin{pmatrix} e^{s\sqrt{-1}}I_n & 0 \\ 0 & e^{-s\sqrt{-1}}I_n \end{pmatrix} \in GL(2n, \mathbb{C}).$$

Therefore

$$\det \left[ I - \begin{pmatrix} e^{s\sqrt{-1}}I_n & 0 \\ 0 & e^{-s\sqrt{-1}}I_n \end{pmatrix} \right] = (1 - e^{s\sqrt{-1}})^n(1 - e^{-s\sqrt{-1}})^n$$

and $\det(I - e^{sJ}) = 0$ if and only if $s \in 2\mathbb{Z}\pi$. Similarly,

$$\det \left[ -I - \begin{pmatrix} e^{s\sqrt{-1}}I_n & 0 \\ 0 & e^{-s\sqrt{-1}}I_n \end{pmatrix} \right] = (-1 - e^{s\sqrt{-1}})^n(-1 - e^{-s\sqrt{-1}})^n$$
and \( \det(-I - e^{sJ}) = 0 \) if and only if \( s \in \pi + 2\mathbb{Z}\pi \). Hence the second claim in the lemma follows.

**Remark A.2** In general, if \( \Psi \) is not symplectic, \( \det(\Psi - e^{sJ}) \) may not have finitely many zero points in \((0, 2\pi]\). For example, the matrix \( \Psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) is not symplectic and it is easy to compute that

\[
\det\left( \frac{1}{2} \Psi - e^{sJ} \right) = \frac{3}{4} \quad \forall s \in \mathbb{R} \quad \text{and} \quad \det(\Psi - e^{sJ}) \equiv 0 \quad \forall s \in \mathbb{R}.
\]

**Lemma A.3** Let

\[
\Psi = \begin{pmatrix} U & -V \\ V & U \end{pmatrix} \in \text{Sp}(2n, \mathbb{R}) \cap O(2n)
\]

and \( e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_n} \) \((0 < \theta_1 \leq \cdots \leq \theta_n \leq 2\pi)\) be eigenvalues of \( U + iV \). Then the set of zero points of the function \( g^\Psi \) in \((0, 2\pi]\) is \( \{\theta_1, \ldots, \theta_n\} \) and \( t(\Psi) = \theta_1 \).

**Proof** For \( x, y \in \mathbb{R}^n \),

\[
(e^{tJ} - \Psi) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff (U + \sqrt{-1}V)(x + \sqrt{-1}y) = e^{\sqrt{-1}t}(x + \sqrt{-1}y)
\]

and thus \( \det(\Psi - e^{tJ}) = 0 \iff t = \theta_j \) for some \( 1 \leq j \leq n \).

**Remark A.4** Let \( \{e_1, f_1 = Je_1, \ldots, e_n, f_n = Je_n\} \) be the standard basis of \( \mathbb{R}^{2n} \), i.e. \( e_j \in \mathbb{R}^{2n} \) is the unit vector whose the \( j \)-th component equals 1 and others are zero. For \( P \) and \( \widetilde{\Psi} \) as in (1.34), (1.35) and 1.36), define \( X_j = Pe_j \) and \( Y_j = Pf_j \). Then \( \widetilde{\Psi} e_j = e^{\theta_j} e_j \) and \( \widetilde{\Psi} f_j = e^{\theta_j} f_j \) for \( j = 1, \ldots, n \). So \( \Psi X_j = e^{\theta_j} X_j \), \( \Psi Y_j = e^{\theta_j} Y_j \), \( j = 1, \ldots, n \), and

\[
\{X_j, Y_j = JX_j\}_{1 \leq i \leq n} \quad \text{(A.1)}
\]

is a symplectic and orthogonal basis of \((\mathbb{R}^{2n}, \omega_0, J)\).

**References**

1. Albers, P., Frauenfelder, U.: Leaf-wise intersections and Rabinowitz Floer homology. J. Topol. Anal. 2(1), 77–98 (2010)
2. Albers, P., Frauenfelder, U.: On a theorem by Ekeland-Hofer. Isr. J. Math. 187, 485–491 (2012)
3. Artstein-Avidan, S., Ostrover, Y.: A Brunn-Minkowski inequality for symplectic capacities of convex domains. Int. Math. Res. Not. 13, 31 (2008)
4. Artstein-Avidan, S., Ostrover, Y.: Bounds for Minkowski billiard trajectories in convex bodies. Int. Math. Res. Not. 1, 165–193 (2014)
5. Bates, S. M.: Some simple continuity properties of symplectic capacities, The Floer memorial volume, 185–193, Progr. Math., 133, Birkhäuser, Basel (1995)
6. Bates, S.M.: A capacity representation theorem for some non-convex domains. Math. Z. 227(4), 571–581 (1998)
7. Blot, J.: On the almost everywhere continuity. arXiv:1411.3582v1
8. Brezis, H.: Functional Analysis. Sobolev Spaces and Partial Differential Equations. Universitext. Springer, New York (2011)
9. Clarke, F.H.: A classical variational principle for periodic Hamiltonian trajectories. Proc. Am. Math. Soc. 76(1), 186–188 (1979)
10. Clarke, F.H.: On Hamiltonian flows and symplectic transformations. SIAM J. Control Optim. 20(3), 355–359 (1982)
11. Clarke, F.H.: Optimization and Nonsmooth Analysis. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. Wiley, New York (1983)
12. Croke, C.B., Weinstein, A.: Closed curves on convex hypersurfaces and periods of nonlinear oscillations. Invent. Math. 64(2), 199–202 (1981)
13. Dong, Y.: P-index theory for linear Hamiltonian systems and multiple solutions for nonlinear Hamiltonian systems. Nonlinearity 19(6), 1275–1294 (2006)
14. Dragnev, D.: Symplectic rigidity, symplectic fixed points and global perturbations of Hamiltonian systems. Commun. Pure Appl. Math. 61, 346–370 (2008)
15. Ekeland, I.: Convexity Methods in Hamiltonian Mechanics, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 19. Springer-Verlag, Berlin (1990)
16. Ekeland, I.: Hamilton-Jacobi on the symplectic group. Rend. Istit. Mat. Univ. Trieste 49, 137–146 (2017)
17. Ekeland, I., Hofer, H.: Two symplectic fixed-points theorems with applications to Hamiltonian dynamics. J. Math. Pure Appl. 68(4), 467–489 (1989)
18. Ekeland, I., Hofer, H.: Symplectic topology and Hamiltonian dynamics. Math. Z. 200, 355–378 (1989)
19. Ekeland, I., Hofer, H.: Symplectic topology and Hamiltonian dynamics II. Math. Z. 203, 553–567 (1990)
20. Eells, J.: A setting for global analysis. Bull. Am. Math. Soc. 72, 791–807 (1966)
21. Ginzburg, V.L., Gürel, B.Z.: Fragility and persistence of leafwise intersections. Math. Z. 280(3–4), 989–1004 (2015)
22. Goldstein, H.: Classical Mechanics. Addison-Wesley, Reading, MA (1950)
23. Gromov, M.: Pseudoholomorphic curves in symplectic manifolds. Inv. Math. 82, 307–347 (1985)
24. Hofer, H.: On the topological properties of symplectic maps. Proc. R. Soc. Edinb. Sect. A 115(1–2), 25–38 (1990)
25. Hofer, H., Zehnder, E.: Periodic solutions on hypersurfaces and a result by C. Viterbo. Invent. Math. 90(1), 1–9 (1987)
26. Hofer, H., Zehnder, E.: A new capacity for symplectic manifolds. Anal. Cetera 1990, 405–429 (1990)
27. Hofer, H., Zehnder, E.: Symplectic Invariants and Hamiltonian dynamics. Birkhäuser Advanced Texts, Basler Lehrbücher, Birkhäuser Verlag, Basel (1994)
28. Jin, R., Lu, G.: Generalizations of Ekeland-Hofer and Hofer-Zehnder symplectic capacities and applications. arXiv:1903.01116v2 (2019)
29. Kang, J.: Generalized Rabinowitz Floer homology and coisotropic intersections. Int. Math. Res. Not. IMRN 10, 2271–2322 (2013)
30. Krantz, S.G.: Convex Analysis. Textbooks in Mathematics, CRC Press, Boca Raton, FL (2015)
31. Künzle, A. F.: Une capacité symplectique pour les ensembles convexes et quelques applications.” Ph. D. thesis, Université Paris IX Dauphine (1990)
32. Künzle, A.F.: Singular Hamiltonian Systems and Symplectic Capacities Singularities and Differential Equations (Warsaw, 1993). Banach Center Publications, pp. 171–187. Polish Academy of Sciences, Warsaw (1996)
33. Künzle, A. F.: Symplectic capacities in manifolds. (English summary) Symplectic singularities and geometry of gauge fields (Warsaw, 1995), Banach Center Publ., 39, Polish Acad. Sci. Inst. Math., Warsaw, pp. 77–87(1997)
34. Long, Y.: Index Theory for Symplectic Paths with Applications. Progress in Mathematics, vol. 207. Birkhäuser Verlag, Basel (2002)
35. Lu, G.: Gromov-Witten invariants and pseudo symplectic capacities. Isr. J. Math. 156, 1–63 (2006)
36. Macarini, L., Schlenk, F.: A refinement of the Hofer-Zehnder theorem on the existence of closed characteristics near a hypersurface. Bull. Lond. Math. Soc. 37(2), 297–300 (2005)
37. Moser, J.: A fixed point theorem in symplectic geometry. Acta Math. 141(1–2), 17–34 (1978)
38. Moser, J., Zehnder, E.J.: Notes on Dynamical Systems, Courant Lecture Notes in Mathematics, vol. 12. New York University, Courant Institute of Mathematical Sciences, American Mathematical Society, New York, Providence, RI (2005)
39. Nedev, E.: Prescribed minimal period problems for convex Hamiltonian systems via Hofer-Zehnder symplectic capacity. Math. Z. 236(1), 99–112 (2001)
40. Rabinowitz, P.: Periodic solutions of Hamiltonian systems. Commun. Appl. Math. 31, 157–184 (1978)
41. Rabinowitz, P.: Periodic solutions of Hamiltonian systems on a prescribed hypersurface. J. Differ. Equ. 33, 336–352 (1979)
42. Rockafellar, R.T.: Convex Analysis. Princeton Mathematical Series, vol. 28. Princeton University Press, Princeton, NJ (1970)
43. Schneider, R.: Convex Bodies: The Brunn-Minkowski Theory, Encyclopedia of Mathematics and its Applications, vol. 44. Cambridge University Press, Cambridge (1993)
44. Sikorav, J.-C.: Systèmes Hamiltoniens et topologie symplectique. Dipartimento di Matematica dell’Università di Pisa. ETS, EDITRICE PISA (1990)
45. Struwe, M.: Existence of periodic solutions of Hamiltonian systems on almost every energy surface. Bol. Soc. Brasil. Mat. (N.S.) 20(2), 49–58 (1990)
46. Struwe, M.: Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 4th edn. Springer-Verlag, Berlin (2008)
47. Viterbo, C.: A proof of Weinstein’s conjecture in $\mathbb{R}^{2n}$. Ann. Inst. H. Poincaré Anal. Non Linéaire 4(4), 337–356 (1987)
48. Weinstein, A.: Periodic orbits for convex Hamiltonian systems. Ann. Math. 108, 507–518 (1978)
49. Weinstein, A.: On the hypotheses of Rabinowitz’s periodic orbit theorems. J. Differ. Equ. 33, 353–358 (1979)
50. Yang, F., Wei, Z.: Generalized Euler identity for subdifferentials of homogeneous functions and applications. J. Math. Anal. Appl. 337(1), 516–523 (2008)
51. Zehnder, E.: Lectures on Dynamical Systems. Hamiltonian Vector Fields and Symplectic Capacities. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich (2010)

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