Emergence of a fluctuation relation for heat in typical Landauer processes

Philip Taranto,† Felix A. Pollock,‡ and Kavan Modi‡

School of Physics & Astronomy, Monash University, Victoria 3800, Australia

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In a generalised framework for the Landauer erasure protocol, we study bounds on the heat dissipated in typical nonequilibrium quantum processes. In contrast to thermodynamic processes, quantum fluctuations are not suppressed in the nonequilibrium regime and cannot be ignored, making such processes difficult to understand and treat. Here, we derive an emergent fluctuation relation that virtually guarantees the average heat produced to be dissipated into the reservoir when either the system or reservoir is large (or both), or when the temperature is high. The implication of our result is that second law-like behaviour appears for nonequilibrium processes exponentially quickly in the dimension of the larger subsystem and linearly in the inverse temperature. We achieve these results by generalising a concentration of measure relation for subsystem states to the case where the global state is mixed.

In Mission Impossible, once Ethan Hunt hears the secret message, the tape self-destructs. This happens, of course, to ensure that the message doesn’t fall into the wrong hands. Burning the tape randomises the message, erasing its information and emitting heat in the process. Landauer’s Principle (LP) tells us that the latter is unavoidable, relating logically irreversible operations to a necessary energy expenditure. It lies at the interface between information theory and thermodynamics, with the profound consequence that “information is physical” [1].

LP underpins the technical challenge of managing the heat generated by computers and was initially postulated from classical thermodynamic considerations [1, 2]. Early research efforts aimed to either develop a microscopic, nonequilibrium version of LP [3–6] or extend it into the quantum domain [7–13]; however, the microscopic versions often relied on specific models, and many quantum extensions assumed the principle to hold a priori, before investigating its implications. Perhaps surprisingly, recent experiments demonstrate that LP applies to irreversible, nonequilibrium processes involving individual quantum systems [14–17]. This has sparked a revival of interest in developing a rigorous formulation of LP in the nonequilibrium, quantum setting, culminating in an equality form of LP derived by Reeb & Wolf (RW) within a minimal framework [18].

Despite the substantial work surrounding LP, little is known about the tightness of the Landauer bound at microscopic scales, nor about how Landauer heat can be tamed, i.e., how to minimise the heat required to process quantum information [19]. Minimising Landauer heat is crucial for our ability to manipulate quantum systems to outperform their classical counterparts, as the quantum advantage often relies on coherent control that suffers from heat fluctuations. An approach to resolving these outstanding issues uses the tools of nonequilibrium statistical physics [20–28]. In particular, RW [18] and Goold, Paternostro & Modi (GPM) [25] have derived tighter bounds on heat than that of Landauer. However, both of these bounds depend on details of the process and are, therefore, difficult to estimate in general.

In this Letter, we seek to make generic, process-independent statements about the heat exchanged during nonequilibrium, quantum processes. We prove the emergence of a fluctuation relation for the heat dissipated in a Landauer process, stating that the average heat dissipated into the environment through a generic open quantum process is almost always positive. We analytically prove that this fluctuation relation arises exponentially quickly as the dimensions of either subsystem grows, and linearly in the inverse temperature. Our result extends the minimal framework for describing Landauer processes [18] and is derived by examining fluctuations of the heat distribution [25]. We begin the Letter by introducing the former and constructing the latter.

Background.—Surprisingly, until recently there was no consensus on how LP should be quantitatively expressed. This changed when RW formally derived a bound for the dissipated heat under a minimal set of assumptions [18]: (i) the irreversible process involves a system s and a reservoir r; (ii) the initial joint state is uncorrelated: $\rho_{sr} = \rho_s \otimes \rho_r$; (iii) the reservoir is initially in a thermal (Gibbs) state $\rho_r := e^{-\beta H_r}/Z$, where $\beta$ is the inverse temperature, $H_r$ is the reservoir Hamiltonian, and $Z := \text{tr}(e^{-\beta H_r})$ is the partition function; and (iv) the joint state evolves unitarily: $\rho_{sr}' = U \rho_{sr} U^\dagger$. We call such processes Landauer processes.

RW show that relaxing any one of the assumptions above can lead to violation of the bound

$$\beta \langle Q \rangle \geq \Delta S + R(\Delta S, d_r) := \omega, \quad (1)$$

where the inequality without $R(\Delta S, d_r)$ is Landauer’s bound [29]. Here, the average heat dissipated into the reservoir is $\langle Q \rangle := \text{tr}[H_r (\rho_r' - \rho_r)]$; the change in von Neumann entropy of the system is $\Delta S := S(\rho_s) - S(\rho'_s)$ with $S(\rho) := -\text{tr}[\rho \log(\rho)]$; and $R(\Delta S, d_r) \geq 0$ is a correction term that tightens the Landauer bound for finite-sized reservoirs.

Crucially, the above framework accounts for processes that will be employed by realistic quantum technologies – namely, nonequilibrium processes that lie outside the realm of traditional thermodynamics and are difficult to treat because of heat fluctuations that are not suppressed. In other words, the heat generated in a single run of the process can vary drastically from its average behaviour. The modern approach to
describing nonequilibrium processes employs fluctuation relations [30–45]. These relate thermodynamic quantities (e.g., free energy difference) to nonequilibrium quantities (e.g., work or heat), offering a promising route to understanding the thermodynamics of small systems whose relevant dynamics may occur on shorter timescales than equilibration. Recent work demonstrates that this formalism provides a tangible route for the experimental exploration of quantum thermodynamics [46–49], including measuring the heat distribution of a Landauer process [17].

Applying such tools to Landauer processes, GPM developed a novel bound for the average heat [25]. By taking projective measurements of the reservoir energy, the complete distribution of the heat exchanged can be constructed: 

\[ P(Q) := \frac{\sinh(\frac{\beta Q}{2})}{\cosh(\frac{\beta Q}{2})} \]

where \( \beta = \frac{1}{kT} \) and \( Q \) is the heat exchanged. The GPM bound is derived: 

\[ \beta(Q) \geq -\ln(\Gamma) \]

where \( \Gamma := \frac{\langle \mathcal{E} \rangle}{\langle \mathcal{S} \rangle} \). Invoking Jensen’s inequality [50], the GPM bound is derived: 

\[ \beta(Q) \geq -\ln(\Gamma) := \gamma \]

where \( \Gamma \) is the fluctuation relation for heat. Our main result shows that, for Haar-randomly sampled unitary operations, 

\[ \Box \]

Theorem 1. When either the system or the reservoir dimensions are much larger than the other, i.e., \( d_s \ll d_r \) or \( d_r \gg d_s \), the deviations of \( \Gamma \) from unity are at least exponentially suppressed in the dimension of the larger subsystem.

We can write Eq. (2) as 

\[ \Gamma := \frac{\langle \mathcal{E} \rangle}{\langle \mathcal{S} \rangle} = \frac{d_r \text{tr} [M_s \rho_s]}{d_r} \]

where 

\[ M_s := \text{tr} \left[ U^\dagger \begin{pmatrix} 1_s & \rho_s \end{pmatrix} d_s \right] \]

The following Lemma is a generalisation of standard concentration of measure results for quantum states (see, e.g., Refs. [53, 54]) to the case where the reduced density operators are generated from unitary orbits of mixed states:

**Lemma 2.** For any \( \sigma_{sr} = U \tau_{sr} U^\dagger \), where \( \tau_{sr} \) is a fixed system-reservoir density operator and \( U \) is a Haar-randomly sampled unitary operator

\[ \text{Prob} \left( \left\| \sigma_{sr} - \frac{1_s}{d_s} \right\|_1 \right) \geq \left( \frac{d_r}{d_s} \right)^2 \exp \left( -\frac{d_r \epsilon^2}{16} \right) \]

The same holds with system and reservoir labels swapped.

Here, \( \sigma_{sr} := \text{tr}_s[\sigma_{sr}] \) and we use the trace norm [55]. Importantly, Lemma 2 bounds the trace distance of a reduced state from the maximally mixed state for Haar-randomly sampled joint interactions. While the bound in Eq. (4) is the same as in the usual case of pure joint states, which follows from Levy’s Lemma [56], the extension to mixed joint states is nontrivial, as the geometry of the corresponding space differs considerably. In fact, the following results do not hold for a naive application of the pure state result, because the trace distance from the identity of \( \sigma_{sr} \), generated from a convex mixture \( \sigma_{sr} \), cannot be directly upper bounded by an arbitrary component of the mixture. For a proof of this physically motivated application of Levy’s Lemma, see Appendix A.

**Proof.** (Theorem 1) Consider the case where \( d_s \gg d_r \). For a Haar-randomly chosen unitary \( U \), the state \( M_s \), defined in Eq. (3), is distributed exactly as \( \sigma_s \) in Lemma 2 with \( \tau_{sr} = (1_s/d_s) \otimes \rho_s \). Writing \( \mu_s := \| M_s - 1_s/d_s \|_1 \), it follows immediately that 

\[ \text{Prob} \left( \mu_s \geq \frac{d_r}{d_s} + \epsilon \right) \leq 2 \exp \left( -d_r \epsilon^2 / 16 \right) \]

Choosing \( \epsilon = 4 (\sqrt{x d_r} + \ln(2d_r)) / (d_r d_s) \) for some small \( x > 1/d_r \), gives

\[ \text{Prob} \left( \mu_s \geq \frac{d_r}{d_s} + 4 \sqrt{\frac{x d_r + \ln(2d_r)}{d_r d_s}} \right) \leq \exp \left( -d_r x \right) / d_s \]

As the reservoir dimension increases, independently of the inverse temperature \( \beta \) and for a fixed \( d_s \geq 2 \), the probability that \( \mu_s \) is greater than some vanishingly small quantity is exponentially diminishing in \( d_r \). Now, consider that \( \mu_s = \max_{\tau_{sr}} \| P(M_s - 1_s/d_s) \|_1 \), where the maximisation is taken over all projection operators \( P \). Using the fact that \( \rho_s \) is a convex mixture of projectors, and multiplying \( \mu_s \) by \( d_s \), we have

\[ d_s \mu_s \geq d_r \text{tr} \left[ \rho_s \left( M_s - \frac{1_s}{d_s} \right) \right] = |\Gamma - 1| =: \mu \]

Hence, \( \mu \) is upper bound by a number that has high probability of being extremely small, as shown in Eq. (5). It follows that \( |\Gamma - 1| \leq \mu \) at least exponentially in the limit \( d_s \gg d_r \). The Theorem can be proved for \( d_s \gg d_r \) using a similar argument with the state \( M_s \), defined in Eq. (3).
As the joint sr biased; that is, the matrix elements 

\[
\sum_{\lambda} \langle \lambda_{k}| \rho_{s} \| s_{k} \rangle \langle \lambda_{k}| \rho_{r} \| s_{k} \rangle \]

\( \Gamma \) over, from the above concentration of measure argument, it should be positive for a complex interaction like this. Moreover, in the converse case where \( \Gamma \) is not clear how the reservoir almost always heats up, even when the reservoir is hot and the initial energy of the system is low.

Proof. Consider the completely-positive trace-preserving (CPTP) map \( \mathcal{E}_{\sigma} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}) \): \( \mathcal{E}_{\sigma}(\sigma) := \text{tr}_{r} \left[ U^\dagger \mathbb{1}_{s} / d_{s} \otimes \sigma \right] U \). By the contractivity of the trace distance under CPTP operations, we have \( \tilde{\mu} := \| \rho_{r} - \mathbb{1}_{r} / d_{r} \| \geq \| \mathcal{E}_{\sigma}(\rho_{r} - \mathbb{1}_{r} / d_{r}) \| \). Expanding the action of \( \mathcal{E}_{\sigma} \) gives

\[
\tilde{\mu} \geq \left| \text{tr}_{r} \left[ U^\dagger \mathbb{1}_{s} / d_{s} \otimes \rho_{r} U \right] - \text{tr}_{r} \left[ U^\dagger \mathbb{1}_{s} / d_{s} \otimes \mathbb{1}_{r} / d_{r} U \right] \right| = \mu_{r},
\]

where \( \mu_{r} = \| M_{s} - \mathbb{1}_{s} / d_{s} \| \). Combining this with Eq. (6), we have \( d_{r} \mu_{r} \geq d_{s} \mu_{s} \).

Next, taking the limit \( \beta \rightarrow 0 \), we have \( \lim_{\beta \rightarrow 0} \tilde{\mu} = \lim_{\beta \rightarrow 0} \sum_{k} (e^{-\beta E_{k}} / Z - 1 / d_{k}) \). As \( \beta \rightarrow 0 \), \( Z \rightarrow d_{r} \) and we can expand the exponential. The zeroth order term cancels with the second term in the last equation, giving: \( \lim_{\beta \rightarrow 0} \tilde{\mu} = \left(1 / d_{r}\right)\sum_{k} (e^{-\beta E_{k}} / (d_{k}!)) \). Since \( \tilde{\mu} \) is behaves linearly with \( \beta \) in this limit, it follows that \( \Gamma \rightarrow 1 \) at least linearly.

Before discussing the implications of these results, we next demonstrate how quickly the fluctuation relation emerges in different regimes.

### Speed of convergence.
In order to test how fast \( \Gamma \) converges on the results of Theorems 1, 3, & 4, we now explore the statistics of simulated dynamics within the parameter space \( (d_{s}, d_{r}, \beta) \). We construct processes by Haar-randomly sampling unitary transformations from the joint space and use these unitary transformations to define the system and reservoir Hamiltonians

\[
H_{s} = i\text{tr}_{s} [\log(U)] / t \quad \text{and} \quad H_{r} = i\text{tr}_{r} [\log(U)] / t,
\]

where we choose \( t = 1 \) to fix the units of energy. Since we examine the effects of temperature and dimensionality on heat

The case where \( d_{s} \gg d_{r} \) is a familiar one: here, a small quantum system interacts with a large reservoir and we expect the reservoir to behave like a thermodynamic bath. Interestingly, in the converse case where \( d_{s} \nlessapprox d_{r} \), we also find that the reservoir almost always heats up, even when the reservoir is hot and the initial energy of the system is low.

Next, we consider the case where we have a large nonequilibrium system interacting with a large equilibrium reservoir, i.e., \( d_{s} \approx d_{r} \gg 2 \). One would not necessarily expect that heat should be positive for a complex interaction like this. Moreover, from the above concentration of measure argument, it is not clear how \( \Gamma \) behaves when \( d_{s} \) and \( d_{r} \) are comparable.

With the following Theorem, we show that a fluctuation relation also emerges when the overall dimension becomes large:

**Theorem 3.** When the system and reservoir dimensions are similar, we expect \( \Gamma \rightarrow 1 \) for large \( d_{s} = d_{r} \).

Proof. We can rewrite \( \Gamma \) in terms of the eigenbases of \( \rho_{s} = \sum_{k} \lambda_{k}^{(s)} |s_{k}\rangle \langle s_{k}| \) and \( \rho_{r} = \sum_{k} \lambda_{k}^{(r)} |r_{k}\rangle \langle r_{k}| \): \( \Gamma = \sum_{nmpq} \lambda_{n}^{(s)} \lambda_{p}^{(r)} |s_{n} r_{m} U| s_{p} r_{q}\rangle \langle s_{p} r_{q}| \langle s_{n} r_{m} U| s_{p} r_{q}\rangle / \sqrt{d_{s} d_{r}} \).

In this limit, we have \( \Gamma \rightarrow \sum_{nmpq} \lambda_{n}^{(s)} \lambda_{p}^{(r)} / (d_{s} d_{r}) = 1 \).

Finally, we analyse another scenario where we might expect classical thermodynamic intuition to hold: that of an interaction with a hot reservoir. In the following Theorem, we show that a fluctuation relation emerges in the high temperature limit:

**Theorem 4.** As temperature increases, \( \Gamma \rightarrow 1 \) at least linearly with inverse temperature \( \beta \).
dissipation in random processes, our conclusions should hold generically and describe typical behaviour.

The notions of high and low temperature depend on the energy level structure of $H_r$, so we must be careful in comparing different processes. At high temperature, we expect significant occupation of all reservoir states, which implies $\beta^{-1} \gg |E_N - E_0|$, where $E_N$ and $E_0$ are the highest and lowest reservoir eigenenergies respectively. On the other hand, in the low temperature regime, even the first excited state (with energy $E_1$) has little population, requiring $\beta^{-1} \ll |E_1 - E_0|$. Between these two regimes, the temperature is of the same order as the energy splittings in $H_r$. These considerations motivate the definition of the scaled temperature parameters

$$
\tilde{\beta}_{\text{low}} := (\beta |E_1 - E_0|)^{-1}, \quad \tilde{\beta}_{\text{high}} := (\beta |E_N - E_0|)^{-1},
$$

and

$$
\tilde{\beta}_{\text{mid}} := \frac{\beta}{N} \sum_{n=1}^{N-1} |E_n - E_{n-1}|,
$$

which are used in the low, high and intermediate temperature regimes respectively.

Theorems 1, 3 and 4 manifest themselves in Fig. 1. Plotted in each panel (temperature regime) is a fit of $\mu$ to data from a large number of Haar-randomly sampled interactions for a variety of system and reservoir dimensions. The validity of Theorems 1 and 3 can be seen in any temperature regime: for a variety of system and reservoir dimensions. The validity of Theorems 1, 3 and 4 manifests itself in Fig. 1. Plotting across the panels of Fig. 1 shows that as temperature increases, $\mu \to 0$ independently of $d_s, d_r$ (note the scale), demonstrating Theorem 4. Furthermore, in Appendix B we show that in the cases where the fluctuation relation arises, $\gamma$ almost always provides a tighter bound for the heat dissipated than previously known bounds.

Our result is counter-intuitive in two cases: (a) when the reservoir is at a high temperature and is of the same dimension as the system, a random interaction could cool it, producing negative heat; and (b) when the system is at a lower temperature than a smaller reservoir, we might also expect the reservoir to cool. However, in both cases the average heat is positive due to Theorems 1, 3 and 4. Intuition fails because the interactions between the system and the reservoir are not weak, as is generally assumed for thermodynamic systems. Randomly sampled unitary processes will generally correspond to Hamiltonians with significant interaction terms that are highly entangling [53]. For both of the above cases, the local state of the reservoir is therefore (highly likely to be) more mixed after the interaction than before, and so the average heat is positive. We now discuss broader implications of our findings.

**Discussion.**—Our ability to coherently control nonequilibrium quantum systems is pivotal if we are to leap into a world run on quantum technologies. Functional quantum technologies must implement irreversible operations, which necessarily generate heat, leading to decoherence that negatively impacts overall performance. In this Letter, we have demonstrated the emergence of a fluctuation relation for the heat generated in nonequilibrium Landauer processes. The implication being that the heat dissipated to the reservoir in a typical open process is almost always positive. This significantly enhances our understanding of Landauer heat, as previous studies have been unable to make process independent statements on the average heat exchanged during nonequilibrium interactions.

Admittedly, a real experiment does not have access to random unitary operations. However, much like burning Ethan Hunt’s tape, securely erasing quantum information requires generating highly random operations that mimic sampling from the full space of operations, e.g., by using a t-design [58]. In fact, many statistical properties of highly entangling quantum circuits are closely related to Haar-randomly sampling from the unitary group, meaning that our fluctuation relation will apply to realistic quantum technologies. Moreover, it will be increasingly important as these devices become larger and hotter, since the regimes in which the fluctuation relation arises are exactly those for which the GPM bound almost always provides the tightest bound on the heat generated (see Appendix B).

In addition to its practical significance, our result is of foundational importance, as it can be interpreted as a version of the second law of thermodynamics for heat dissipation. The fact that the relation emerges so quickly as the system and reservoir dimensions grow suggests that similar results may hold for systems with constrained interactions. For such small collections of particles, the concentration of physically relevant processes with respect to the Haar measure is unlikely to be particularly small – indeed, an extension of our work to consider restricted sets of operations is warranted.

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## Appendix A: Proof of Lemma 2

The typical application of Levy’s Lemma to Reduced quantum states is as follows [53]

**Lemma 5.** For any pure state $\phi_r = U\psi_r U^\dagger$, where $\psi_r$ is a fixed system-reservoir pure state and $U$ is a Haar-randomly sampled unitary, and for arbitrary $\epsilon > 0$, the distribution of distances between the reduced density matrix of the system $\phi_s = \text{tr}_r[\phi_r]$ and the maximally mixed state $1/d_s$, satisfies

$$\text{Prob} \left[ \left\| \phi_s - \frac{1}{d_s} \right\|_1 \geq \sqrt{\frac{d_s}{d_s} + \epsilon} \right] \leq 2 \exp \left( -\frac{d_s d_s \epsilon^2}{16} \right).$$

(A1)

If our states $M_s$ and $M_s$ were being generated from pure joint states, we could directly apply Lemma 5 to achieve the desired result in proving Theorem 1. However, they are instead being generated from mixed joint states: i.e., $M_s$ is distributed as $\text{tr}_r[1/d_s \otimes \rho_s]$ (and similarly for $M_s$). The following argument shows why the standard version of Levy’s Lemma does not necessarily hold for the mixed case and motivates our proof of Theorem 2.

The initial $sr$ state can always be decomposed as a mixture
of pure states: \( \tau_{\psi} = \sum_{kl} \lambda_{kl} |kl\rangle \langle kl| \). We can therefore write
\[
\mu_s = \left\| \operatorname{tr} \left[ \sum_{kl} \lambda_{kl} U |kl\rangle \langle kl| U^\dagger \right] - 1/d_s \right\|_1. \tag{A2}
\]
Defining
\[
\mu_{kl} := \left\| \operatorname{tr} \left[ U |kl\rangle \langle kl| U^\dagger \right] - 1/d_s \right\|_1, \tag{A3}
\]
we have that \( \mu_s = \sum_{kl} \lambda_{kl} \mu_{kl} (\leq \max_{kl} \mu_{kl}) \), since the partial trace and the trace norm are both convex functions. While Eq. (A1) would apply to each of the \( \mu_{kl} \) with \( U \) sampled independently, the upper bound on \( \mu_{kl} \) depends on the full set \( \{\mu_{kl}\} \) for each given \( U \). Lemma 5 makes no statistical statements about the latter.

Furthermore, since the space of density matrices with fixed spectrum is a geodesically different space from that of pure states (it is a flag manifold rather than a complex projective space), the usual proof of Lemma 5 cannot be trivially modified. We now proceed to prove our Theorem 2.

**Proof.** The proof hinges on a version of the well known Lemma by Levy [56]:

**Lemma 6 (Levy).** Consider a manifold \( M \) endowed with a metric \( g \) and measure \( \mu \), and a Lipschitz continuous function \( f : M \to \mathbb{R} \) with Lipschitz constant \( \eta \), i.e., \( f \) satisfies \( |f(x) - f(y)| \leq \eta |x - y| \), \( \forall x, y \in M \). The value of the function is concentrated around its expectation value \( \mathbb{E}_x f \) according to the distribution
\[
\operatorname{Prob} \left( f(x) \geq \varepsilon + \mathbb{E}_x f \right) \leq 2 \alpha_M (\varepsilon/\eta), \tag{A4}
\]
where \( \alpha_M(x) \) is a concentration function for \( M \), defined as (an upper bound on) the measure of the set of points in the space more than a distance \( x \) from the minimal-boundary volume enclosing half the space.

Consider the function \( f(U) = \| \sigma_s(U) - 1/d_s \|_1 \), the trace distance of the reduced state \( \sigma_s(U) = \operatorname{tr} [U \tau_s(U^\dagger) \] from the maximally mixed state. Using a reverse triangle inequality and the contractivity of the trace norm under partial trace, we have, for any \( U, V \in \operatorname{SU}(d_s) \),
\[
|f(U) - f(V)| = \left\| \sigma_s(U) - \frac{1}{d_s} \right\|_1 - \left\| \sigma_s(V) - \frac{1}{d_s} \right\|_1 \leq \left\| \sigma_s(U) - \sigma_s(V) \right\|_1 \leq \left\| U \sigma_s U^\dagger - V \sigma_s V^\dagger \right\|_1 \leq 2 \| U - V \|_2, \tag{A5}
\]
where, for the final inequality, we have used Lemma 1 from Ref. [59], which relates the penultimate quantity to the Hilbert-Schmidt distance \( \|X\|_2 = \sqrt{\operatorname{tr} [XX^\dagger]} \) between the two unitaries. Importantly, this distance induces the Haar measure on the group manifold. Eq. (A5) demonstrates that \( f(U) \) is a Lipschitz continuous function on the unitaries with Lipschitz constant \( \eta = 2 \).

Calculating the expectation value of \( f(U) \) follows a standard argument [60, 61] and is the same as for the case of pure system-reservoir states. The expected trace distance is related to the expected Hilbert-Schmidt distance squared using Jensen’s inequality
\[
\mathbb{E}_U \left\| \sigma_s(U) - \frac{1}{d_s} \right\|_1 \leq \sqrt{d_s \mathbb{E}_U \left\| \sigma_s(U) - \frac{1}{d_s} \right\|_2} \leq \sqrt{d_s \mathbb{E}_U \left\| \sigma_s(U) - \frac{1}{d_s} \right\|_2}. \tag{A6}
\]
The Hilbert-Schmidt distance can then be expanded in terms of the purity of \( \sigma \), as
\[
\mathbb{E}_U \left\| \sigma_s(U) - \frac{1}{d_s} \right\|_2 = \mathbb{E}_U \operatorname{tr} \left[ \sigma_s^2 \right] - \frac{1}{d_s}, \tag{A7}
\]
Lastly, the expectation value of the purity can be calculated for the Haar measure by utilising properties of the swap operator (though the calculation usually involves an average over pure states, it is the same for the unitary orbit of any state); it is
\[
\mathbb{E}_U \operatorname{tr} \left[ \sigma_s^2 \right] = \frac{d_s + d_r}{d_s d_r + 1}, \tag{A8}
\]
which, using Eqs. (A6) & (A7), leads to
\[
\mathbb{E}_U f \leq \sqrt{d_s \mathbb{E}_U \left\| \sigma_s(U) - \frac{1}{d_s} \right\|_2} \leq \sqrt{d_s \mathbb{E}_U \left\| \sigma_s(U) - \frac{1}{d_s} \right\|_2} \leq \sqrt{d_s \mathbb{E}_U \left\| \sigma_s(U) - \frac{1}{d_s} \right\|_2}. \tag{A9}
\]
Using Lemma 6, we have that
\[
\operatorname{Prob} \left( f(U) \geq \varepsilon + \sqrt{\frac{d_s}{d_r}} \right) \leq 2 \alpha_{d_s} (\varepsilon/2), \tag{A10}
\]
where \( \alpha_{d_s}(x) \) is the concentration function on the group manifold of \( \operatorname{SU}(d_s) \) equipped with the Haar measure. We can relate this to the concentration function for the sphere, using a Theorem by Gromov [62, 63]:

**Theorem 7 (Gromov).** Let \( S^{n+1}(R) \) be the \((n + 1)\)-sphere of radius \( R \), and let \( M \) be a closed \((n + 1)\)-dimensional Riemannian manifold with \( \text{Ric}(M) \geq n/R^2 = \text{Ric}(S^{n+1}(R)) \), where \( \text{Ric}(X) \) is the infimum of diagonal elements of the Ricci curvature tensor on \( X \). Choose \( M_0 \subset M \) to be a domain with smooth boundary and let \( B \) be a round ball in \( S^{n+1}(R) \) such that
\[
\frac{\operatorname{Vol}(M_0)}{\operatorname{Vol}(M)} = \frac{\operatorname{Vol}(B)}{\operatorname{Vol}(S^{n+1}(R))}. \tag{A11}
\]
It then follows that
\[
\frac{\operatorname{Vol}(\partial M_0)}{\operatorname{Vol}(M)} \geq \frac{\operatorname{Vol}(\partial B)}{\operatorname{Vol}(S^{n+1}(R))}. \tag{A12}
\]
That is, if the Ricci curvature is everywhere greater than that of some sphere, then the rate at which volume is enclosed as one moves away from the boundary of a region is at least as great as the corresponding rate for a similar region on the sphere. Thus, if the inequality in Eq. (A12) holds, the corresponding concentration functions are related as $\alpha_M(x) \leq \alpha_{S^{d_{sr} - 1}(R)}(x)$.

Since the group manifold of $SU(d_{sr})$ is compact and simply connected, it has constant, positive Ricci curvature (with respect to the Hilbert-Schmidt distance) [64]. This can be calculated to be $\text{Ric}({\mathcal{U}}) = d_{sr}/2$ (see, e.g., Chapter 18 of Ref. [65]). The manifold dimension is $d_{sr}^2 - 1$, so we must compare it with $S^{d_{sr}^2 - 1}(R)$, finding that the radius must be at least $R_0 = \sqrt{2(d_{sr}^2 - 2)/d_{sr}}$ in order for Theorem 7 to apply. Choosing the minimal case, we use the Theorem to upper bound $\alpha_{\mathcal{U}}(x)$ by $\alpha_{S^{d_{sr}^2 - 1}(R_0)}(x) = \exp(-d_{sr}x^2/4)$) [56]. Combining this with Eq. (A10) leads to the desired result in Eq. (4).

The same argument follows for a function $g(U) = \|\sigma_{1}(U) - 1/d_{sr}\|_1$; therefore the inequality holds under exchange of system and reservoir labels.

Appendix B: Bound Comparison

Although no discernible hierarchy between $\omega$ and $\gamma$ exists, here we compare the tightness of these bounds to $\beta_{\langle Q \rangle}$. In Fig. 2, we sample 5,000 Haar-random interactions and initial states to analyse general behaviours in regions of the parameter space. The average heat dissipated in these cases is non-negative, so $\gamma - \omega > 0$ implies that $\gamma$ is a tighter bound.

The only distribution of $\gamma - \omega$ that has a significant proportion of negative data points occurs for small-scale interactions where the process occurs at low temperature (see (a) in Fig. 2). This shows that $\omega$ can provide a tighter bound to the average heat than $\gamma$ in this regime ($\omega$ outperforms $\gamma$ for $\sim 35\%$ of such interactions). However, regardless of the temperature, when either dimension is much larger than the other (or both are large), $\gamma$ almost always provides a tighter bound to the average heat (see (b) – (h) in Fig. 2). It is interesting to note that as the dimension of the system increases, $\gamma - \omega$ tends to peak around a specific value (see (c), (d), (g) & (h) in Fig. 2).

The tightness of $\gamma$ with respect to the average heat itself can be understood from the distribution of $\beta_{\Delta Q} - \gamma$. Independent of subsystem dimensions, $\gamma$ provides a tight bound when the process occurs at high temperatures (see (b), (d), (f) & (g) in Fig. 2), but a rather poor bound for interactions at low temperatures (where all other known bounds also perform poorly). It is also interesting to note that as the dimension of the reservoir increases, $\beta_{\Delta Q} - \gamma$ tends to 0 and so the GPM bound is tight.
FIG. 2: We sample 5,000 Haar-random interactions and compute $\beta \Delta Q$, $\omega$ and $\gamma$ for a variety of dimensions and temperature regimes. Approximate confidence polytopes are calculated by peeling off convex hull layers centred on the bivariate median to highlight general behaviour. The only region where $\omega$ significantly outperforms $\gamma$ in tightness to $\beta \Delta Q$ is the small dimension, low temperature regime, seen in panel (a), where $\sim 35\%$ of interactions generate heat that is closer to $\omega$ than to $\gamma$. In all remaining cases, where fluctuation relations quickly emerge, $\gamma$ almost always provides a tighter bound to the average heat. Furthermore, we see that for any interaction at high temperature, $\gamma$ is a tight bound on $\beta \Delta Q$. For interactions occurring at low temperatures, $\gamma$ is not a particularly tight bound regardless of the dimension (although neither is any previously known bound).