Constrained analysis
of topologically massive gravity

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Abstract

We quantize the Einstein gravity in the formalism of weak gravitational fields by using the constrained Hamiltonian method. Special emphasis is given to the 2+1 spacetime dimensional case where a (topological) Chern-Simons term is added to the Lagrangian.

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I. Introduction

One of the ultimate goals of the quantum field theory (QFT) is the quantization of gravity. It is well-known that usual rules of quantization when applied to the Einstein Lagrangian for the gravitational field do not lead to a finite theory in a sense that infinities cannot be eliminated by means of regularization and renormalization procedures [1]. With the advent of supersymmetry and strings [2] there was the expectation of solving all the problems that the usual QFT was not able to do. Unfortunately, this does not appear to be the case, unless till now. We already know that supersymmetry when applied to gravitation (supergravity) does not lead to a finite theory. On the other hand, the interesting advent of the strings is nowadays fighting with the complexibility of its mathematical structures and its second quantization is still far to be applied in order to obtain reasonable predictions.

It has been a common procedure in QFT to go to lower dimensions when we are trying to understand something in the usual four spacetime dimensions and we are not succeed. In the case of the Einstein gravity this procedure usually runs into problems. For example, going after the trivial one-dimensional case, we have that the Ricci tensor is identically zero in two-dimensional spacetime. This means that there is no way of including matter field in a Einstein bi-dimensional gravity. The three-dimensional case is also problematic. We find that the final theory does not have any physical degree of freedom. However, there is an important aspect in this last case. If one includes a Chern-Simons (CS) term, the final (topological) theory has one physical degree of freedom and the above mentioned inconsistency does not exist any more [3,4].

The interesting constraint structure of the Einstein plus CS theory in 2+1 dimensions is the main motivation of the present work. We shall consider the approximation of weak gravitational fields [4]. In order to better illustrate the general formalism, we consider this approximation in Sec. II for any space-time dimensions without the CS term. In Sec. III we add the CS term and restrict ourselves to the particular case of 2+1 dimensions. Since there are many first-class constraints in both cases, we have to introduce appropriate gauge-fixing conditions. We left Sec. IV for some concluding remarks.
II. Hilbert-Einstein theory in a weak gravitational field

The Hilbert-Einstein Lagrangian reads

\[ \mathcal{L} = \frac{1}{2\kappa^2} \sqrt{-g} \, R, \]  

(2.1)

where \( \kappa \) is the Einstein constant and we shall use the following flat metric convention \text{diag.}(+,−,\ldots,−). At it was said in the introduction, we are not making here any restriction to the spacetime dimensions \( D \).

Considering weak gravitational fields we take the expansion around the flat space geometry as

\[ g^{\mu\nu}(x) = \eta^{\mu\nu} - \kappa h^{\mu\nu}(x). \]  

(2.2)

Introducing (2.2) into (2.1) and just keeping free fields we get

\[ \mathcal{L} = \frac{1}{4} \partial_\lambda h^{\mu\nu} \partial^\lambda h^{\mu\nu} - \frac{1}{4} \partial_\lambda h^{\mu} \partial^\lambda h^{\nu} + \frac{1}{2} \partial_\lambda h^{\mu} \partial^\mu h^{\nu} - \frac{1}{2} \partial_\lambda h^{\nu} \partial^\mu h^{\mu\nu}, \]  

(2.3)

where it was considered that the metric is symmetric. Expression (2.3) is known as the Fierz-Pauli Lagrangian for massless particles of spin 2. It is a gauge theory and the corresponding gauge transformation reads

\[ \delta h^{\mu\nu} = \partial^\mu \zeta^\nu + \partial^\nu \zeta^\mu. \]  

(2.4)

We are going to study the resulting theory described by the Lagrangian density (2.3) making use of the constrained Hamiltonian procedure due to Dirac [5]. Since this is a noncovariant method, it might be convenient to separate time and space components in the expression above. We write down the result as

\[ \mathcal{L} = \frac{1}{4} (\dot{h}^{ij} \dot{h}_{ij} - \dot{h}^i_j \dot{h}^j_i) + (\partial^i \dot{h}^j_j - \partial_j \dot{h}^{ij}) \dot{h}^i_0 - V, \]  

(2.5)
where

\[
V = -\frac{1}{2} \partial_i h_{0j} \partial^j h^{0j} - \frac{1}{4} \partial_i h_{jk} \partial^j h^{jk} + \frac{1}{2} \partial_i h^{00} \partial^j h^j + \frac{1}{4} \partial_i h^j \partial^i h^k
- \frac{1}{2} \partial_i h^j \partial^i h^{00} - \frac{1}{2} \partial_i h^j \partial^i h^k + \frac{1}{2} \partial_i h^i \partial_j h^j + \frac{1}{2} \partial_i h_{ij} \partial^k h^{kj}.
\]

(2.6)

The canonical momenta are

\[
\begin{align*}
\pi^{00} &= \frac{\partial L}{\partial \dot{h}^{00}} = 0, \\
\pi^{0i} &= \frac{\partial L}{\partial \dot{h}^{0i}} = \frac{1}{2} (\partial^j h^j_j - \partial_j h^{ij}) , \\
\pi^{ij} &= \frac{\partial L}{\partial \dot{h}_{ij}} = \frac{1}{2} (\dot{h}^{ij} - \eta^{ij} \dot{h}^k_k).
\end{align*}
\]

(2.7)

(2.7a)\hspace{1cm} (2.7b)\hspace{1cm} (2.7c)

There are \(D\) primary (first-class) constraints [5]

\[
\begin{align*}
\Omega &= \pi^{00} \approx 0 , \\
\Omega^i &= \pi^{0i} + \frac{1}{2} (\partial_j h^{ij} - \partial^i h^j_j) \approx 0 .
\end{align*}
\]

(2.8)

(2.8a)\hspace{1cm} (2.8b)

This is in agreement with Castellani assumption that establishes that the number of symmetries of the theory (characterized by each one of the components of \(\zeta_\mu\)) is the same as the number of primary first-class constraints [6].

The next step is to look for secondary constraints. We first construct the primary Hamiltonian density

\[
H = \pi^{00} \dot{h}^{00} + 2 \pi^{0i} \dot{h}^{0i} + \pi^{ij} \dot{h}_{ij} - L + \lambda \Omega + \lambda_i \Omega^i ,
\]

\[
= \pi_{ij} \dot{h}^{ij} - \frac{1}{4} \dot{h}_{ij} \dot{h}^{ij} + \frac{1}{4} h^i_i \dot{h}^j_j + (\lambda + \dot{h}^{00}) \Omega + (\lambda_i + 2 \dot{h}^{0i}) \Omega^i + V .
\]

(2.9)
We notice that is possible to redefine the Lagrange multipliers $\lambda$ and $\lambda_i$ in order to absorb the velocities $\dot{h}_{00}$ and $\dot{h}_{0i}$. The remaining velocities can be eliminated by using the momentum expressions (2.7c). The final result reads

$$\mathcal{H} = \pi^{ij} \pi_{ij} + \frac{1}{2-D} \pi^i_i \pi^j_j + \lambda \Omega + \lambda_i \Omega^i + V. \quad (2.10)$$

Considering the fundamental Poisson brackets (*)

$$\begin{align*}
\{ h^{00}(x), \pi_{00}(y) \} &= \delta^{(D-1)}(\vec{x} - \vec{y}), \\
\{ h^{0i}(x), \pi_{0j}(y) \} &= \delta^i_j \delta^{(D-1)}(\vec{x} - \vec{y}), \\
\{ h^{ij}(x), \pi_{kl}(y) \} &= \frac{1}{2} \left( \delta^i_k \delta^j_l + \delta^i_l \delta^j_k \right) \delta^{(D-1)}(\vec{x} - \vec{y}).
\end{align*} \quad (2.11)$$

and using the Hamiltonian (2.10) we obtain from the consistency condition the following secondary-constraints

$$\begin{align*}
\Phi &= \nabla^2 h^i_i + \partial_i \partial_j h^{ij} \approx 0, \\
\Phi^i &= 2 \partial_j \pi^{ij} + \partial^i \partial_j h^{0i} + \nabla^2 h^{0i} \approx 0.
\end{align*} \quad (2.12)$$

Introducing these constraints into the Lagrangian by means of new Lagrange multipliers and using the consistency condition again we verify that there are no tertiary constraints. We notice that constraints $\Phi$ and $\Phi^i$ are also first-class.

In order to calculate the Dirac brackets we have to fix the gauge. Let us start from the corresponding of the Coulomb gauge of the electromagnetic Maxwell theory, i.e.

$$\partial_i h^{ij} \approx 0. \quad (2.13)$$

(*) It will always be understood that brackets are taken at the same time $x_0 = y_0$.  

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We also choose that

\begin{align}
    h^{00} & \approx 0, \quad (2.14a) \\
    h^{0i} & \approx 0. \quad (2.14b)
\end{align}

With the gauge condition \( h^{0i} \approx 0 \), the secondary constraints \( \Phi^i \) just turns to be \( \partial_j \pi^{ij} \approx 0 \). Compatibility with the equation of motion permit us to infer that

\[ \pi^i_{\ i} \approx 0. \quad (2.15) \]

Summarizing, the full set of (second-class) constraints is

\begin{align}
    \Omega &= \pi^{00} \approx 0, \quad (2.16a) \\
    \Psi &= h^{00} \approx 0, \quad (2.16b) \\
    \Omega^i &= \pi^{0i} \approx 0, \quad (2.16c) \\
    \Psi^i &= h^{0i} \approx 0, \quad (2.16d) \\
    \Phi &= h^i_i \approx 0, \quad (2.16e) \\
    \Sigma &= \pi^i_i \approx 0, \quad (2.16f) \\
    \Sigma^i &= \partial_j h^{ji} \approx 0, \quad (2.16g) \\
    \Phi^i &= \partial_j \pi^{ji} \approx 0. \quad (2.16h)
\end{align}

This set contains \( 4D \) constraints. There is no sense cases with \( D \leq 2 \) because the number of degrees of freedom will be negative. \( D=3 \) also corresponds to a nonphysical case where the number of degrees of freedom is zero. In general, the number of degrees of freedom is \( D(D-3)/2 \).

The calculation of the Dirac brackets is directly done by means of a hard algebraic work. The nonvanishing ones are
\[ \{h^{ij}(x), \pi_{kl}(y)\} = \left[ \frac{1}{2} \left( \delta^i_j \partial^j \partial_k + \delta^i_k \partial^j \partial_l + \delta^j_i \partial^i \partial_k + \delta^j_k \partial^i \partial_l \right) \frac{1}{\sqrt{2}} \right. \\
+ \frac{1}{2} \left( \delta^j_i \delta^i_j + \delta^j_k \delta^k_j \right) - \frac{1}{D - 2} \eta^{ij} \eta_{kl} \\\n- \frac{1}{D - 2} \left( \eta^{ij} \partial_k \partial_l + \eta_{kl} \partial^i \partial^j \right) \frac{1}{\sqrt{2}} \\\n+ \frac{D - 3}{D - 2} \frac{\partial^i \partial^j \partial_k \partial_l}{\nabla^4} \bigg] \delta^{(D-1)}(\vec{x} - \vec{y}). \quad (2.17) \]

We clearly notice that, in fact, the case with \(D=2\) does not make sense by virtue of terms with \((D-2)^{-1}\). Further, one can show that for \(D=3\) the above bracket is zero.

Looking at (2.17), we observe that there are no problem with ordering operators, so it can be directly transformed into commutator by means of the usual rule of quantization

\[ \{\text{Dirac brackets}\} \rightarrow \frac{1}{\imath \hbar} [\text{commutators}] \quad (2.18) \]

Considering the transformations above, one can calculate the propagators among the \(h^{ij}\) fields. The result is

\[ <0|T(h^{ij}(-k) h_{kl}(k))|0> = \frac{i}{k^2} \left[ \delta^i_k \delta^j_l + \delta^i_l \delta^j_k + \frac{1}{k^2} \left( \delta^i_k k^j k_l + \delta^i_l k^j k_k + \delta^j_i k^i k_l + \delta^j_k k^i k_k \right) \right. \\\n- \frac{2}{D - 2} \left( \eta^{ij} \eta_{kl} + \eta^{ij} \frac{k^k k_l}{k^2} - \eta_{kl} \frac{k^i k^j}{k^2} \right) \\\n+ \frac{2(D - 3)}{(D - 2)} \frac{k^i k^j k_k k_l}{k^4} \bigg]. \quad (2.19) \]

We notice that there is no dynamical massive pole.

### III. Topologically massive gravitation

When the spacetime dimension is odd, it is possible to introduce a CS term in the Lagrangian. In this section we concentrate on the 2+1 case. The corresponding CS Lagrangian reads
\[ \mathcal{L}_{CS} = \frac{1}{\mu} \epsilon^{\lambda\mu\nu} \Gamma^\sigma_{\lambda\sigma} \left( \partial_{\mu} \Gamma^\sigma_{\rho\nu} + \frac{2}{3} \Gamma^\sigma_{\mu \xi} \Gamma^\xi_{\nu \rho} \right). \]  

(3.1) 

Considering the expansion of weak gravitational fields given by (2.2) we get

\[ \mathcal{L}_{CS} = \frac{\kappa^2}{2\mu} \epsilon^{\lambda\mu\nu} \left( \partial_{\sigma} h^\rho_{\lambda} \partial_{\rho} \partial_{\mu} h^\sigma_{\nu} - \partial_{\sigma} h^\rho_{\kappa} \partial^\sigma \partial_{\mu} h^\kappa_{\rho \nu} \right). \]  

(3.2)

Making the separation of space and time components and adding the result to (2.5) the result is

\[ \mathcal{L} = \frac{1}{4} \left( \dot{h}_{ij} \dot{h}^{ij} - \dot{h}^i_i \dot{h}^j_j \right) + \left( \partial^i h^j_j - \partial_j h^i_j \right) \dot{h}^0_i + \frac{\kappa^2}{\mu} \epsilon^{ij} \left( \dot{h}^k_i \partial_k \partial_i h^0_j \right. \right. \]
\[- \partial_i \partial_k h^0_0 \dot{h}^k_j - \partial_i h^k_i \dot{h}_{k j} + \dot{h}^k_i \partial_k h^0_j + \frac{1}{2} \partial_k \partial_i \dot{h}_i^l \dot{h}_j^k \]
\[ + \frac{1}{2} \dot{h}^k_i \ddot{h}_{k j} + \frac{1}{2} \nabla^2 h^0_i \dot{h}^0_j + \frac{1}{2} \nabla^2 h^k_i \dot{h}^k_j \left) - V, \]  

(3.3)

where

\[ V = \frac{1}{2} \partial_i h_{0j} \partial^i h^{0j} + \frac{1}{4} \partial_i h_{jk} \partial^i h^{jk} - \frac{1}{2} \partial_i h^{00} \partial^i h^j_j - \frac{1}{4} \partial_i h^j_j \partial^i h^k_k \]
\[ + \frac{1}{2} \partial_i h^i_j \partial^0 h^0_0 + \frac{1}{2} \partial_i h^i_j \partial^i h^k_k - \frac{1}{2} \partial_i h^i_0 \partial_j h^0_j - \frac{1}{2} \partial_i h^i_j \partial_k h^{kj} \]
\[- \frac{\kappa^2}{\mu} \epsilon^{ij} \left( \partial_k h^0_i \partial_i \partial_j h^0_j - \nabla^2 h^0_i \partial_i h_{k j} - \nabla^2 h^0_0 \partial_i h_{0 j} \right). \]  

(3.4)

We observe that the inclusion of the CS Lagrangian leads to the appearance of higher derivative terms [7]. The correct Hamiltonian treatment requires a phase space as \( h^{\mu\nu} \oplus \dot{h}^{\mu\nu} \oplus \pi^{\mu\nu} \oplus s^{\mu\nu} \), where \( \pi^{\mu\nu} \) and \( s^{\mu\nu} \) are the canonical momenta conjugate to \( h_{\mu\nu} \) and \( \dot{h}_{\mu\nu} \), respectively. Now, velocities have to be considered as independent coordinates. Using the general expressions
\[ \pi_{\mu\nu} = \frac{\partial L}{\partial \dot{h}_{\mu\nu}} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \ddot{h}_{\mu\nu}} - 2\dot{\pi}_i \frac{\partial L}{\partial (\ddot{h}_{\mu\nu})}, \quad (3.5) \]

\[ s_{\mu\nu} = \frac{\partial L}{\partial h_{\mu\nu}}, \quad (3.6) \]

we get

\[ \pi^{00} = 0, \quad (3.7a) \]

\[ \pi^{0i} = \frac{1}{2} \left( \partial^i h^j_j - \partial_j h^{ji} \right) + \frac{\kappa^2}{2\mu} \left( 2\epsilon^{jk} \partial_j \partial^i h^0_k - \epsilon^{ij} \nabla^2 h^{0j} \right), \quad (3.7b) \]

\[ \pi^{ij} = \frac{1}{2} \left( \dot{h}^{ij} - \eta^{ij} h^k_k \right) + \frac{\kappa^2}{4\mu} \epsilon^{ik} \left( 2\partial_k \partial^i h^{00} - \partial^i \partial_j h^l_k + \nabla^2 h^{0j} \right) \]

\[ - \nabla^2 h^j_k - 2\partial_k \dot{h}^j_0 - 2\partial^j \dot{h}^0_k + (i \leftrightarrow j), \quad (3.7c) \]

\[ s^{00} = 0, \quad (3.7d) \]

\[ s^{0i} = 0, \quad (3.7e) \]

\[ s^{ij} = \frac{\kappa^2}{2\mu} \left[ \epsilon^{ik} \left( \partial_k h^0_j + \partial^j h_{0k} - \frac{1}{2} h^j_k \right) + (i \leftrightarrow j) \right], \quad (3.7f) \]

These lead to the following set of primary constraints

\[ \Omega = \pi^{00} \approx 0, \quad (3.8a) \]

\[ \Omega^i = \pi^{0i} + \frac{1}{2} \left( \partial_j h^{ji} - \partial^i h^j_j \right) - \frac{\kappa^2}{2\mu} \left( 2\epsilon^{jk} \partial_j \partial^i h^0_k - \epsilon^{ij} \nabla^2 h^{0j} \right) \approx 0, \quad (3.8b) \]

\[ \Lambda = \pi^i_i + \frac{1}{2} \dot{h}^i_i - \frac{\kappa^2}{2\mu} \epsilon^{ij} \partial_j \partial_k h^k_i \approx 0, \quad (3.8c) \]

\[ \Theta = s^{00} \approx 0, \quad (3.8d) \]

\[ \Theta^i = s^{0i} \approx 0, \quad (3.8e) \]

\[ \Theta^{ij} = s^{ij} - \left[ \frac{\kappa^2}{2\mu} \epsilon^{ik} \left( \partial_k h^j_0 + \partial^j h_{0k} - \frac{1}{2} h^j_k \right) + (i \leftrightarrow j) \right]. \quad (3.8f) \]

The total primary Hamiltonian density in this case reads
Developing the expression above, one can show that some terms can be absorbed by re-defining the Lagrange multipliers. One interesting point, that might be opportune to be mentioned, is that $\dddot{h}_{ij}$ are also eliminated in this way and not by using the expression (3.7c) that gives $\pi^{ij}$ in terms of $\dddot{h}_{ij}$ (this expression is not a constraint). This is providential because it is not possible to use (3.7c) in order to express $\dddot{h}_{ij}$ in terms of $\pi^{ij}$ and other components of the phase space. This is so because the coefficient of $\dddot{h}_{ij}$ does not have inverse.

The final expression for the primary hamiltonian density reads

$$
\mathcal{H} = \pi^{ij} \dot{h}_{ij} - \frac{1}{4} \left( \dot{h}_{ij} \dot{h}_{ij} + 2 \partial_i h_{0j} \partial^i h^{0j} + \partial_i h_{jk} \partial^i h^{jk} - \dot{h}_{ij} \dot{h}_{ij} - 2 \partial_i h^{00} \partial^i h^0_j - \partial_i h^0_j \partial^i h^k_k + 2 \partial_i \dot{h}_{ij} \partial^i h^{00} + 2 \partial_i \dot{h}_{ij} \partial^i h^{k_k} \right) \\
+ \frac{\kappa^2}{2 \mu} \epsilon^{ij} \left( 2 \partial_i \partial_k h^{00} \dot{h}_{ij} - 2 \partial_k h^0_0 \partial_i \partial_i h^{k_k} + 2 \nabla^2 h^0_k \partial_i h_{k j} + 2 \nabla^2 h^0_0 \partial_i h_{0j} - \partial_k \partial_i h^0_j \dot{h}_{ij} - \nabla^2 h^0_i \dot{h}_{k j} \right) \\
+ \lambda \Omega + \lambda_i \Omega^i + \rho \Theta + \rho_i \Theta^i + \rho_{ij} \Theta^{ij} + \xi \Lambda .
$$

Using the consistency condition and making properly combinations of the final constraints in order to obtain the biggest number of first-class ones we get the final set

(i) **1st class contraints**

$$
\Omega = \pi^{00} \approx 0 , \quad \Theta = s^{00} \approx 0 , \quad \Theta^i = s^{0i} \approx 0 .
$$
In fact we notice that there is just one degree of freedom for $D=3$.

We use the Dirac method iteratively. The elimination of the second-class constraints above results in the following preliminary brackets

\begin{align}
\{h^{ij}(x), \pi_{kl}(y)\}^* &= \frac{1}{2} (\delta^i_k \delta^j_l + \delta^i_l \delta^j_k) \delta^{(2)}(\vec{x} - \vec{y}) , \\
\{\dot{h}^{ij}(x), s_{kl}(y)\}^* &= \frac{1}{4} (\delta^i_k \delta^j_l + \delta^i_l \delta^j_k - \eta^{ij} \eta_{kl}) \delta^{(2)}(\vec{x} - \vec{y}) , \\
\{\dot{h}^{ij}(x), \dot{h}_{kl}(y)\}^* &= -\frac{\kappa^2}{4\kappa^2} (\epsilon^i_k \delta^j_l + \epsilon^i_l \delta^j_k + \epsilon^j_k \delta^i_l + \epsilon^j_l \delta^i_k) \delta^{(2)}(\vec{x} - \vec{y}) , \\
\{s^{ij}(x), s_{kl}(y)\}^* &= -\frac{\kappa^2}{16\mu} (\epsilon^i_k \delta^j_l + \epsilon^i_l \delta^j_k + \epsilon^j_k \delta^i_l + \epsilon^j_l \delta^i_k) \delta^{(2)}(\vec{x} - \vec{y}) , \\
\{\dot{h}^{ij}(x), \dot{h}_{kl}(y)\}^* &= -\eta^{ij} \eta_{kl} \delta^{(2)}(\vec{x} - \vec{y}) , \\
\{\dot{h}^{ij}(x), \pi_{kl}(y)\}^* &= -\frac{\kappa^2}{4\kappa^2} \eta^{ij} (\epsilon^m_k \partial_m \partial_{k} + \epsilon^m_k \partial_m \partial_{l}) \delta^{(2)}(\vec{x} - \vec{y}) , \\
\{h^{00}(x), \pi_{00}(y)\}^* &= \delta^{(2)}(\vec{x} - \vec{y}) , \\
\{\dot{h}^{00}(x), s_{00}(y)\}^* &= \delta^{(2)}(\vec{x} - \vec{y}) , \\
\{h^{0i}(x), \pi_{0j}(y)\}^* &= \frac{1}{2} \delta^i_j \delta^{(2)}(\vec{x} - \vec{y}) .
\end{align}

\text{(ii) 2nd class constraints}

\begin{align}
\Theta^{ij} &= s^{ij} - \left[ \frac{\kappa^2}{2\mu} \epsilon^{ik} \left( \partial_k h^j_0 + \partial^j h_{0k} - \frac{1}{2} \dot{h}^j_k \right) + (i \leftrightarrow j) \right] \approx 0 , \quad (3.12a) \\
\Lambda &= \pi^i_i + \frac{1}{2} \dot{h}^i_i - \frac{\kappa^2}{2\mu} \epsilon^{ij} \partial_j \partial_k h^k_i \approx 0 . \quad (3.12b)
\end{align}

In fact we notice that there is just one degree of freedom for $D=3$.

We use the Dirac method iteratively. The elimination of the second-class constraints above results in the following preliminary brackets
The remaining brackets are zero. To calculate the final Dirac brackets we have to fix the gauge. Here we also choose the corresponding of the Coulomb gauge (2.13), plus the further conditions (2.14a) and (2.14b). Consistency with the equations of motion permit us also to choose

\[
\dot{h}^{00} \approx 0, \\
\dot{h}^{0i} \approx 0, \\
\dot{h}_i^i \approx 0, \\
\partial_x h^{ij} \approx 0. \quad (3.14)
\]

With this gauge choice, the set of constraints given by (3.11) becomes second-class. It is then possible to calculate the Dirac brackets. After a hard algebraic work we get

\[
\begin{align*}
\{ h^{ij}(x), \pi_{kl}(y) \}^D &= - \left[ \eta^{ij} \eta_{kl} + \eta^{ij} \frac{k}{\nabla^2} \partial_k \partial_l + \eta_{kl} \frac{k}{\nabla^2} \partial^i \partial^j \\
&\quad - \frac{1}{2} \left( \delta^i_k \delta^j_l + \delta^j_l \delta^i_k + \delta^i_l \partial_k \partial_l + \delta^j_l \partial_i \partial_l + (i \leftrightarrow j) \right) \delta^{(2)}(\vec{x} - \vec{y}) \right], \quad (3.17a) \\
\{ \pi^{ij}(x), \pi_{kl}(y) \}^D &= - \frac{k^2}{4\mu} \left[ (\epsilon_{ml} \partial^m \partial_k + \epsilon_{mk} \partial^m \partial_l) (\eta^{ij} + 2 \frac{k}{\nabla^2} \partial^i \partial^j) \\
&\quad - 2(\eta_{kl} + 2 \frac{k}{\nabla^2} \partial_k \partial_l) (\epsilon_{mi} \partial^m \partial^j + i \leftrightarrow j) \right] \delta^{(2)}(\vec{x} - \vec{y}), \quad (3.17b) \\
\{ \dot{h}^{ij}(x), \dot{h}_{kl}(y) \}^D &= - \frac{\mu}{4k^2} \left( \epsilon^i_l \partial^j_k + \epsilon^i_k \partial^j_l + i \leftrightarrow j \right) \delta^{(2)}(\vec{x} - \vec{y}), \quad (3.17c) \\
\{ \dot{h}^{ij}(x), s_{kl}(y) \}^D &= \frac{1}{4} \left( \delta^i_k \delta^j_l + \delta^j_l \delta^i_k - \eta^{ij} \eta_{kl} \right) \delta^{(2)}(\vec{x} - \vec{y}), \quad (3.17d) \\
\{ s^{ij}(x), s_{kl}(y) \}^D &= - \frac{k^2}{16\mu} \left( \epsilon^i_l \partial^j_k + \epsilon^i_k \partial^j_l + i \leftrightarrow j \right) \delta^{(2)}(\vec{x} - \vec{y}), \quad (3.17e) \\
\{ h^{ij}(x), \dot{h}_{kl}(y) \}^D &= \left( \eta^{ij} + \frac{k}{\nabla^2} \partial^i \partial^j \right) \left( \eta_{kl} + 2 \frac{k}{\nabla^2} \partial_k \partial_l \right) \delta^{(2)}(\vec{x} - \vec{y}), \quad (3.17f) \\
\{ \dot{h}^{ij}(x), \pi_{kl}(y) \}^D &= \frac{k^2}{2\mu} \left( \epsilon^m_i \partial_m \partial_k + l \leftrightarrow k \right) \left( \eta^{ij} + \frac{k}{\nabla^2} \partial^i \partial^j \right) \delta^{(2)}(\vec{x} - \vec{y}), \quad (3.17g) \\
\{ \dot{h}^{ij}(x), s_{kl}(y) \}^D &= - \frac{k^2}{4\mu \sqrt{2}} \eta^{ij} \left( \epsilon^m_i \partial_m \partial_k + l \leftrightarrow k \right) \left( \eta^{ij} + \frac{k}{\nabla^2} \partial^i \partial^j \right) \delta^{(2)}(\vec{x} - \vec{y}) \quad (3.17h) \\
\{ \pi^{ij}(x), s_{kl}(y) \}^D &= \frac{k^4}{8\mu^2} \left( \delta^i_k \partial^j_l + \frac{k}{\nabla^2} \partial^i \partial^j \partial_k \partial_l \right) + i \leftrightarrow j; l \leftrightarrow k \right) \delta^{(2)}(\vec{x} - \vec{y}), \quad (3.17i)
\end{align*}
\]
\[ \{ h^{ij}(x), \pi_{0k}(y) \}^D = \left( \frac{1}{2} \eta^{ij} + \frac{\partial^i \partial^j}{\nabla^2} \right) \delta^{(2)}(\vec{x} - \vec{y}), \]  

(3.17j)

\[ \{ s^{ij}(x), \pi_{0k}(y) \}^D = \frac{\kappa^2}{4\mu} \left( e^{mi} \partial_m \partial^j + e^{mj} \partial_m \partial^i \right) \delta^{(2)}(\vec{x} - \vec{y}). \]  

(3.17k)

One can check the validity of the brackets above by showing that they are strongly consistent with all the constraints.

We notice that there are no problem with ordering operators. So, the above brackets can be transform without problems to commutators by means of the usual rule of quantization given by (2.18). The Feynman propagators among the fields \( h^{ij} \) are (details of calculating Feynman propagators when there are higher derivatives can be found in references [8])

\[
< 0|T(h^{ij}(x)h_{kl}(y))|0> = i \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik(x-y)}}{k^2(\frac{\mu^2}{\kappa^4} - 4k_0^2)} \left[ 4\kappa^2 k_0^2 \frac{k_k k_l}{k^2} \eta^{ij} 
- \frac{\mu^2}{\kappa^4} \left( \eta^{ij} \eta_{kl} + \frac{k^i k^j}{k^2} \eta_{kl} + 2 \frac{k_k k_l}{k^2} \eta^{ij} + 2 \frac{k^i k^j k_k k_l}{k^4} \right) 
- \frac{\mu^2}{2\kappa^2} \left( \eta^l \eta^i_k + \eta^l_j \eta^i_k \eta^j_k - 2 \eta_{kl} \eta^{ij} \right) 
+ 8k_0^2 \left( \frac{1}{2} \eta_{ij} \eta_{kl} + \frac{k_k k_l}{k^2} \eta_{ij} + \frac{1}{2} \frac{k_i k_j}{k^2} \eta_{kl} + \frac{k_i k_j k_k k_l}{k^4} \right) 
- \frac{\mu^2}{\kappa^2 k^2} \left( k_k k_l \eta^{ij} + k^i k^j \eta_{kl} - (k^i k_l \eta^j_k + i \leftrightarrow j; k \leftrightarrow l) \right) 
+ \frac{i\mu k_0}{k^2} \left( \frac{1}{2} \epsilon^{pi} k_p k_j \eta_{kl} + \epsilon^i_k k^j k_l + i \leftrightarrow j; k \leftrightarrow l \right) 
- \frac{i\mu k_0}{2} \left( \epsilon^{k l} \eta^{ij}_l + i \leftrightarrow j; k \leftrightarrow l \right) \right].
\]  

(3.18)

We observe that there is a massive (topological) pole given by \( \mu/2\kappa^2 \). This is in agreement with previous results found in literature [3,4,9] and show the consistency of the quantization procedure we have used.
IV. Conclusion

We have studied the quantization of the Einstein gravitational theory in the formalism of weak gravitational fields. This is a high constrained system and we have used the Hamiltonian Dirac method to quantize it. We have given a particular emphasis to the case of the spacetime dimension D=2+1 where a CS term can be added. The final propagators in this case exhibit a massive (topological) pole. This result is in agreement with previous one found in literature that use other quantization procedure.

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