HOPF ALGEBRAS OF DIMENSION \(2p\)

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Abstract. Let \(H\) be a finite-dimensional Hopf algebra over an algebraically closed field of characteristic 0. If \(H\) is not semisimple and \(\dim(H) = 2n\) for some odd integer \(n\), then \(H\) or \(H^*\) is not unimodular. Using this result, we prove that if \(\dim(H) = 2p\) for some odd prime \(p\), then \(H\) is semisimple. This completes the classification of Hopf algebras of dimension \(2p\).

0. Introduction

In recent years, there has been some progress on the classification problems of finite-dimensional Hopf algebras over an algebraically closed field \(k\) of characteristic 0 (cf. [Mon98, And02]). It is shown in [Zhu94] that Hopf algebras of dimension \(p\), where \(p\) is a prime, are isomorphic to the group algebra \(k[\mathbb{Z}_p]\). In [Ng02b] and [Ng02a], the author completed the classification of Hopf algebras of dimension \(p^2\), which started in [AS98] and [Mas96]. They are group algebras and Taft algebras of dimension \(p^2\) (cf. [Taf71]). However, the classification of Hopf algebras of dimension \(pq\), where \(p, q\) are distinct prime numbers, remains open in general.

It is shown in [EG98, GW00] that semisimple Hopf algebras over \(k\) of dimension \(pq\) are trivial (i.e. isomorphic to either group algebras or the dual of group algebras). Most recently, Etingof and Gelaki proved that if \(p, q\) are odd prime such that \(p < q \leq 2p + 1\), then any Hopf algebra over \(k\) of dimension \(pq\) is semisimple [EG]. Meanwhile, the author proved the same result, using different method, for the case that \(p, q\) are twin primes [Ng]. In addition to that Williams settled the case of dimensions 6 and 10 in [Wil88], and Beattie and Dascaluescu did dimensions 14, 65 in [BD]. Hopf algebras of dimensions 6, 10, 14 and 65 are semisimple and so they are trivial.

In this paper, we prove that any Hopf algebra of dimension \(2p\), where \(p\) is an odd prime, over an algebraically closed field \(k\) of characteristic 0, is semisimple. By [Mas95], semisimple Hopf algebras of dimension \(2p\) are isomorphic to

\[ k[\mathbb{Z}_{2p}], \quad k[D_{2p}] \quad \text{or} \quad k[D_{2p}]^* \]

where \(D_{2p}\) is the dihedral group of order \(2p\). Hence, our main result Theorem 3.3 completes the classification of Hopf algebras of dimension \(2p\).

1. Notation and Preliminaries

Throughout this paper, \(p\) is an odd prime, \(k\) denotes an algebraically closed field of characteristic 0, and \(H\) denotes a finite-dimensional Hopf algebra over \(k\) with antipode \(S\). Its comultiplication and counit are respectively denoted by \(\Delta\) and \(\epsilon\). We will use Sweedler’s notation [Swe69]:

\[ \Delta(x) = \sum_i x_{(1)} \otimes x_{(2)}. \]
A non-zero element \( a \in H \) is called group-like if \( \Delta(a) = a \otimes a \). The set of all group-like elements \( G(H) \) of \( H \) is a linearly independent set, and it forms a group under the multiplication of \( H \). For the details of elementary aspects for finite-dimensional Hopf algebras, readers are referred to the references [Swe69] and [Mon93].

Let \( \lambda \in H^* \) be a non-zero right integral of \( H^* \) and \( \Lambda \in H \) a non-zero left integral of \( H \). There exists \( \alpha \in \text{Alg}(H, k) = G(H^*) \), independent of the choice of \( \Lambda \), such that \( \Lambda a = \alpha(a)\Lambda \) for \( a \in H \). Likewise, there is a group-like element \( g \in H \), independent of the choice of \( \lambda \), such that \( \beta \lambda = \beta(g)\lambda \) for \( \beta \in H^* \). We call \( g \) the distinguished group-like element of \( H \) and \( \alpha \) the distinguished group-like element of \( H^* \). Then we have Radford’s formula [Rad76] for \( S^4 \):

\[
S^4(a) = g(\alpha \rightarrow a \leftarrow \alpha^{-1})g^{-1} \quad \text{for} \quad a \in H,
\]

where \( \rightarrow \) and \( \leftarrow \) denote the natural actions of the Hopf algebra \( H^* \) on \( H \) described by

\[
\beta \rightarrow a = \sum a(1)\beta(a(2)) \quad \text{and} \quad a \leftarrow \beta = \sum \beta(a(1))a(2)
\]

for \( \beta \in H^* \) and \( a \in H \). In particular, we have the following proposition.

**Proposition 1.1** ([Rad76]). Let \( H \) be a finite-dimensional Hopf algebra with antipode \( S \) over the field \( k \). Suppose that \( g \) and \( \alpha \) are distinguished group-like elements of \( H \) and \( H^* \) respectively. Then the order of \( S^4 \) divides the least common multiple of the order of \( g \) and the order of \( \alpha \).

For any \( a \in H \), the linear operator \( r(a) \in \text{End}_k(H) \) is defined by \( r(a)(b) = ba \) for \( b \in H \). The semisimplicity of a finite-dimensional Hopf algebra can be characterized by the antipode.

**Theorem 1.2** ([LR87], [LR88], [Rad94]). Let \( H \) be a finite-dimensional Hopf algebra with antipode \( S \) over the field \( k \). Then the following statements are equivalent:

1. \( H \) is not semisimple;
2. \( H^* \) is not semisimple;
3. \( S^2 \neq \text{id}_H \);
4. \( \text{Tr}(S^2) = 0 \);
5. \( \text{Tr}(S^2 \circ r(a)) = 0 \) for all \( a \in H \).

**Proposition 1.3** ([Ng Corollary 2.2]). Let \( H \) be a finite-dimensional Hopf algebra over \( k \) with antipode \( S \) and \( g \) the distinguished group-like element of \( H \). If \( \text{lcm}(o(S^4),o(g)) = n \) is an odd integer greater than 1, then the subspace

\[
H_- = \{ u \in H \mid S^{2n}(u) = -u \}
\]

has even dimension.

The following lemma is useful in our remaining discussion.

**Lemma 1.4.** Let \( V \) be a finite-dimensional vector space over the field \( k \). If \( T \) is a linear automorphism on \( V \) such that \( \text{Tr}(T) = 0 \) and \( o(T) = q^n \) for some prime \( q \) and positive integer \( n \), then

\[
q \mid \dim(V).
\]
Proof. Let \( \omega \in k \) be a primitive \( q^n \)th root of unity and
\[
V_i = \{ u \in V \mid T(u) = \omega^i u \} \quad \text{for} \quad i = 0, \ldots, q^n - 1.
\]
Consider the integral polynomial
\[
f(x) = \sum_{i=0}^{q^n - 1} \dim(V_i) x^i.
\]
Since
\[
0 = \text{Tr}(T) = f(\omega),
\]
there exists \( g(x) \in \mathbb{Z}[x] \) such that
\[
f(x) = \Phi_{q^n}(x) g(x)
\]
where \( \Phi_{q^n}(x) \) is the \( q^n \)th cyclotomic polynomial. Hence,
\[
\dim(V) = f(1) = \Phi_{q^n}(1) g(1).
\]
Since \( \Phi_{q^n}(x) = \Phi_{q}(x^{q^n-1}) \), \( \Phi_{q^n}(1) = \Phi_{q}(1) = q \). Thus we have
\[
q \mid \dim(V).
\]
\( \square \)

2. Unimodularity of Hopf algebras of dimension \( 2n \)

In this section, we prove that if \( H \) is a non-semisimple Hopf algebra over \( k \) of dimension \( 2n \), where \( n \) is an odd integer, then \( H \) or \( H^* \) is not unimodular. This result is essential to the proof of our main result in the next section.

**Proposition 2.1.** Let \( H \) be a finite-dimensional Hopf algebra over the field \( k \) with antipode \( S \). If \( H \) is unimodular and \( o(S^2) = 2 \), then
\[
4 \mid \dim(H).
\]

*Proof.* Let \( \lambda \) be a non-zero right integral of \( H^* \). Since \( H \) is unimodular, by [Rad94, Proposition 2],
\[
\lambda(ab) = \lambda(S^2(b)a)
\]
for all \( a, b \in H \). Let
\[
H_i = \{ u \in H \mid S^2(u) = (-1)^i u \} \quad \text{for} \quad i = 0, 1.
\]
We claim that \( (a, b) = \lambda(ab) \) defines a non-degenerate alternating form on \( H_1 \). For any \( a, b \in H_1 \),
\[
\lambda(ab) = \lambda(S^2(b)a) = -\lambda(ba).
\]
Since \( \lambda(u) = \lambda(S^2(u)) = -\lambda(u) \) for all \( u \in H_1, \lambda(H_1) = \{0\} \). Let \( a \in H_1 \) such that \( \lambda(ab) = 0 \) for all \( b \in H_1 \). Then for all \( b \in H_0, ab \in H_1 \) and so \( \lambda(ab) = 0 \). By the non-degeneracy of \( \lambda \) on \( H, a = 0 \). Therefore, \( (a, b) = \lambda(ab) \) defines an non-degenerate alternating bilinear form on \( H_1 \) and hence \( \dim(H_1) \) is even. Since \( o(S^2) = 2 \), by Theorem 1.2 \( \text{Tr}(S^2) = 0 \) and so \( \dim(H_0) = \dim(H_1) \). Therefore
\[
\dim(H) = \dim(H_0) + \dim(H_1) = 2 \dim(H_1)
\]
is a multiple of 4. \( \square \)

**Corollary 2.2.** Let \( H \) be a Hopf algebra over \( k \) of dimension \( 2n \) where \( n \) is an odd integer. If \( H \) is not semisimple, then \( H \) or \( H^* \) is not unimodular.
Proof. If both $H$ and $H^*$ are unimodular, by Proposition 1.1, $S^4 = id_H$. Since $H$ is not semisimple, by Theorem 1.2, $o(S^2) = 2$. It follows from Proposition 2.1 that $\dim(H)$ is then a multiple of 4 which contradicts $\dim(H) = 2n$.

3. HOPF ALGEBRAS OF DIMENSION 2$p$

In this section, we prove, by contradiction, that non-semisimple Hopf algebras over $k$ of dimension $2p$, $p$ an odd prime, do not exist. By [Mas95], semisimple Hopf algebras of dimension $2p$ are $k[\mathbb{Z}_{2p}]$, $k[D_{2p}]$ and $k[D_{2p}]^*$ where $D_{2p}$ is the dihedral group of order $2p$. Our main result completes the classification of Hopf algebras of dimension $2p$. We begin to prove our main result with the following lemma.

Lemma 3.1. Let $H$ be a non-semisimple finite-dimensional Hopf algebra over $k$ of dimension $2p$ where $p$ is an odd prime. Suppose that $g$ and $\alpha$ are the distinguished group-like element of $H$ and $H^*$ respectively. Then

$$\text{lcm}(o(g), o(\alpha)) = 2 \text{ or } p.$$

Proof. Since $H$ is not semisimple, by Theorem 1.2, $H^*$ is also not semisimple. Therefore $|G(H)|$ and $|G(H^*)|$ are strictly less than $2p$. By Nichols-Zoeller theorem [NZ89],

$$|G(H)|, |G(H^*)| \in \{1, 2, p\}.$$ 

It follows from [Ng02b, Lemma 5.1] that

$$\text{lcm}(|G(H)|, |G(H^*)|) = 1, 2 \text{ or } p.$$ 

Since $\text{lcm}(o(g), o(\alpha))$ divides $\text{lcm}(|G(H)|, |G(H^*)|)$, we obtain

$$\text{lcm}(o(g), o(\alpha)) = 1, 2 \text{ or } p.$$ 

By Corollary 2.2, $\text{lcm}(o(g), o(\alpha)) > 1$ and so the result follows. \qed

Lemma 3.2. Let $H$ be a finite-dimensional Hopf algebra over $k$ and $a \in G(H)$ of order $d$. Let $\omega \in k$ be a primitive $d$th root of unity and

$$e_i = \frac{1}{d} \sum_{j=0}^{d-1} \omega^{-ij} a^j \quad (i = 0, \ldots, d - 1).$$

Then $\dim(H)/d$ is an integer, $\dim(\text{He}_i) = \dim(H)/d$ and $S^2(\text{He}_i) = \text{He}_i$ for $i = 0, \ldots, d - 1$. In addition, if $H$ is not semisimple, then

$$\text{Tr}(S^2|_{\text{He}_i}) = 0$$

for $i = 0, \ldots, d - 1$.

Proof. Let $B = k[a]$. Then $B$ is a Hopf subalgebra of $H$ and $\dim(B) = d$. By Nichols-Zoeller theorem, $H$ is a free $B$-module. In particular, $\dim(H)$ is a multiple of $d$ and

$$H \cong B^{\dim(H)/d}$$

as right $B$-modules. Note that $e_0, \ldots, e_{d-1}$ are orthogonal idempotents of $B$ such that

$$1 = e_0 + \cdots + e_{d-1},$$

and $B e_i = k e_i$. Therefore,

$$\text{He}_i \cong B^{\dim(H)/d} e_i = (B e_i)^{\dim(H)/d} = (k e_i)^{\dim(H)/d}$$
and so \( \dim(He_i) = \frac{\dim(H)}{d} \) for \( i = 0, \ldots, d - 1 \). Since \( S^2(a) = a \), \( S^2(e_i) = e_i \) for \( i = 0, \ldots, d - 1 \). Therefore,
\[
S^2(He_i) = HS^2(e_i) = He_i.
\]
If, in addition, \( H \) is not semisimple, by Theorem 1.2,
\[
\Tr(S^2|_{He_i}) = \Tr(S^2 \circ r(e_i)) = 0.
\]
for \( i = 0, \ldots, d - 1 \). □

**Theorem 3.3.** If \( p \) is an odd prime, then any Hopf algebra of dimension \( 2p \) over the field \( k \) is semisimple.

**Proof.** Suppose there exists a non-semisimple Hopf algebra \( H \) of dimension \( 2p \). Let \( g \) and \( \alpha \) be the distinguished group-like elements of \( H \) and \( H^* \) respectively. By Corollary 2.2, \( g \) and \( \alpha \) can not be both trivial. By Theorem 1.2, we may simply assume that \( g \) is not trivial and \( o(g) = d \). Let \( \omega \in k \) be a primitive \( d \)th of unity and
\[
e_i = \frac{1}{d} \sum_{j=0}^{d-1} \omega^{-ij}g^j \quad (i = 0, \ldots, d - 1).
\]
By Lemma 3.1,
\[
\text{lcm}(o(g), o(\alpha)) = 2 \text{ or } p.
\]
If \( \text{lcm}(o(g), o(\alpha)) = 2 \), then \( d = 2 \) and \( S^8 = \text{id}_H \) by Proposition 1.1. It follows from Theorem 1.2 that
\[
o(S^2) = 2 \text{ or } 4.
\]
It follows from Lemma 3.2 that
\[
\dim(He_i) = p \quad \text{and} \quad \Tr(S^2|_{He_i}) = 0 \quad (i = 0, 1).
\]
Since \( o(S^2) = 2 \) or \( 4 \), there exists \( j \in \{0, 1\} \) such that
\[
o(S^2|_{He_j}) = 2 \text{ or } 4.
\]
By Lemma 1.4, \( \dim(He_j) \) is even which contradicts that \( \dim(He_j) = p \).

If \( \text{lcm}(o(g), o(\alpha)) = p \), then \( d = p \) and \( S^{4p} = \text{id}_H \). By Proposition 1.3, the subspace
\[
H = \{ u \in H | S^{2p}(u) = -u \}
\]
has even dimension. On the other hand, by Lemma 3.2, we have
\[
\dim(He_i) = 2 \quad \text{and} \quad \Tr(S^2|_{He_i}) = 0 \quad (i = 0, \ldots, p - 1).
\]
Thus, for \( i \in \{0, \ldots, p - 1\} \), there is a basis \( \{ u_i^+, u_i^- \} \) for \( He_i \) such that
\[
S^2(u_i^+) = \pm \zeta_i
\]
for some \( p \)th root of unity \( \zeta_i \). Thus, \( \{ u_0^-, u_1^-, \ldots, u_{p-1}^- \} \) forms a basis of \( H^- \) and so
\[
\dim(H^-) = p,
\]
a contradiction! □

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