THE OBSTACLE PROBLEM FOR A CLASS OF DEGENERATE FULLY NONLINEAR OPERATORS

JOÃO VITOR DA SILVA AND HERNÁN VIVAS

Abstract. We study the obstacle problem for fully nonlinear elliptic operators with an anisotropic degeneracy on the gradient:

\[
\begin{cases}
\min \left\{ f - |Du|^{\gamma} F(D^2 u), u - \phi \right\} = 0 & \text{in } \Omega \\
u = g & \text{on } \partial \Omega.
\end{cases}
\]

We obtain existence of solutions and prove sharp regularity estimates along the free boundary points, namely \(\partial \{u > \phi\} \cap \Omega\). In particular, for the homogeneous case \((f \equiv 0)\) we get that solutions are \(C^{1,1}\) at free boundary points, in the sense that they detach from the obstacle in a quadratic fashion, thus beating the optimal regularity allowed for such degenerate operators. We also present further features of the solutions and partial results regarding the free boundary.

These are the first results for obstacle problems driven by degenerate type operators in non-divergence form and they are a novelty even for the simpler scenario given by an operator of the form \(G[u] = |Du|^{\gamma} \Delta u\).

1. Introduction

1.1. Main proposal. This manuscript is concerned with existence and regularity issues for the obstacle problems governed by second order fully nonlinear elliptic equations of degenerate type as follows:

\[
\begin{cases}
\min \left\{ f - |Du|^{\gamma} F(D^2 u), u - \phi \right\} = 0 & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^n\) is a smooth, open and bounded domain, \(\phi\) and \(g\) are suitably smooth functions defined in \(\Omega\) and \(\partial \Omega\) respectively, \(f\) is a continuous and bounded function in \(\Omega\), \(\gamma > 0\) and \(F\) is a second order fully nonlinear uniformly elliptic operator. We recall that, for second order operators, uniform ellipticity means that for any pair of matrices \(X, Y \in \mathbb{R}^{n \times n}\)

\[
M_{\lambda,A}^{-}(X - Y) \leq F(X) - F(Y) \leq M_{\lambda,A}^{+}(X - Y)
\]

where \(M_{\lambda,A}^{-}\) and \(M_{\lambda,A}^{+}\) are the Pucci extremal operators given by

\[
M_{\lambda,A}^{-}(X) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i \quad \text{and} \quad M_{\lambda,A}^{+}(X) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i
\]

for some ellipticity constants \(0 < \lambda \leq \Lambda < \infty\) (here \(\{e_i\}_i\) are the eigenvalues of \(X\)). Throughout this work we will often refer to (1.1) as the \((F, \gamma)\)-obstacle problem.

Key words and phrases. Free boundary problems of obstacle type, degenerate elliptic equations, sharp regularity estimates.
Degenerate operators with an anisotropic degeneracy on the gradient such as the one appearing in (1.1) have interest both from the point of view of applied sciences and engineering and from a pure PDE perspective. We recommend the reader Birindelli and Demengel’s works [4], [5], [6] and [7] for a number of examples of degenerate fully nonlinear operators with similar structural properties. In fact, it is worth pointing out that the operator \( G(Du, D^2u) = |Du|^{\gamma} F(D^2u) \) is the simplest example of a more general class of operators dealt with in the aforementioned papers. Our results could be extended without (much) effort to that broader class (see [3] and [18]), but with decided to stick with \( G(Du, D^2u) = |Du|^{\gamma} F(D^2u) \) for the sake of simplicity and for ease of exposition.

It is worth highlighting that some of the major difficulties in dealing with our class of operators are: its non-divergence structure, in consequence, we are not allowed to make use of (nowadays) classical estimates from potential theory (cf. [2], [8], [10], [11], [15], [30] and [38] for some fundamental essays), and the degeneracy character that forces diffusion properties (e.g., uniformly ellipticity of operator) to collapse along an a priori unknown set of critical points of solution, namely the set

\[ \{ x \in \Omega : |Du(x)| = 0 \}. \]

These features produce significant constraints on the regularity that can be expected for solutions to such operators. Indeed, this is true even for non-degenerate and translation invariant operators. More precisely, it is known that viscosity solutions for fully nonlinear uniformly elliptic equations (with “frozen” coefficients)

\[ F(x_0, D^2u) = 0 \quad \text{in} \quad \Omega \]  

(1.2)

are locally \( C^{1,\alpha_F} \), for a constant \( \alpha_F \in (0,1) \) that depends only on dimension and ellipticity constants (cf. [9], [12] and [45] for some seminal surveys). Of course, the primary question is whether solutions of such equation are smooth enough to be classical, i.e. at least \( C^2 \).

Through the journey of finding these classical solutions, the result of Evans [21] and Krylov [27] was a pioneer paramount research on operators in non-divergence form; it states that under a concavity (or convexity) assumption on \( F \), solutions of (1.2) are locally \( C^{2,\alpha_0} \) for some \( 0 < \alpha_0 < 1 \) (see also Trudinger’s works [43] and [44] for similar researches). The question of whether any fully nonlinear elliptic operator would enjoy a \( C^2 \) a priori regularity theory eluded the mathematical community for the last three decades and in effect, the Nadirashvili-Vl˘aduţ’s counterexamples to \( C^{1,1} \) regularity in [33], [34], [35], [36] and [37] give a negative answer to such a challenging question.

In our (degenerate) case the situation becomes even more involved. We are dealing with equations of the form

\[ |Du|^{\gamma} F(D^2u) = f(x) \quad \text{in} \quad \Omega. \]  

(1.3)

Simple examples (cf. [3] Section 3 and [26] Example 1) show that, even for smooth right hand side, solutions are not better than \( C^{1,\alpha_F}_{\text{loc}} \) in general (even if \( F \) is concave/convex). For \( f \in L^\infty(\Omega) \), Imbert and Silvestre showed in [26] Theorem 1 that solutions to (1.3) are \( C^{1,\alpha} \) for some (small) \( \alpha \) in the interior of \( \Omega \). Afterwards in [3] Theorem 3.1], Araújo, Ricarte and Teixeira showed that in fact, given any \( \alpha \in (0,\alpha_F) \cap \left( 0, \frac{1}{1+\eta} \right] \) it is possible to show that \( u \in C^{1,\alpha}_{\text{loc}}(\Omega) \), even for a more general class of operators satisfying natural structural conditions. As a matter of fact, from their result the (optimal) \( C^{1,\frac{1}{1+\eta}} \) interior regularity follows when \( F \) is concave/convex (cf. [3] Corollary 3.2).
In this work, given $\gamma > 0$, $F$ a uniformly elliptic operator, $g \in C^{1,\beta}(\partial \Omega)$ and $\phi \in C^{1,\beta}(\Omega)$ for some $\beta \in (0, 1]$ we define the optimal exponent

$$\alpha := \min \left\{ \alpha_F - 1, \frac{1}{\gamma + 1}, \beta \right\}$$

(1.4)

where $\alpha_F$ stands for $\alpha - \varepsilon$ for any $\varepsilon > 0$. Then, we are interested in studying the obstacle problem

$$\begin{cases}
\min\{f - |Du|^\gamma F(D^2u), u - \phi\} = 0 \quad \text{in } \Omega \\
u = g \quad \text{on } \partial \Omega
\end{cases}$$

(1.5)

(see the next section for a brief historical remark on the obstacle problem).

The natural question in (1.5) (besides existence of solutions) is whether one can get $u \in C^{1,\beta}_{\text{loc}}(\Omega)$, that is if we can “transmit” the regularity across the “physical” free boundary $\partial\{u > \phi\}$. Crucial in the way to prove such a result is to obtain fine information about the behavior of the solution near free boundary points. Once this issue has been settled it is of interest to have some geometric information about free boundary itself; a natural first step is to prove that it has zero Lebesgue measure. We give a partial result in that direction as well.

1.2. A brief historical overview. Geometric regularity for equations as the ones studied here have been subject to much interest in the PDE community in the last years, not only for its generality and several applications, but specially for its innate relation with a number of relevant free boundary problems in the literature (cf. [2], [14], [18], [19], [20], [23], [30] and [40] for some variational and non-variational examples). For this reason, understanding the “geometry” of the former model is an important step in comprising the behavior of solutions near their free boundary points.

Despite of the fact that there is a huge amount of literature on obstacle problems in divergence form and their qualitative features (cf. [2], [14], [18], [19], [20], [23], [30] and [38]), quantitative counterparts for non-variational (elliptic) models like (1.1) are far less studied due to the rigidity of the structure of such operators (cf. [28], [29] and [31] as some enlightening examples). Therefore, the treatment of such free boundary problems requires the development of new ideas and modern techniques. This lack of investigation has been our main impetus in studying fully nonlinear models with non-uniformly elliptic (anisotropic) structure like (1.1), which focus on a modern, systematic and non-variational approach for such a general class of operators.

In fact, it is worth pointing out that, to the best of the authors’ knowledge, the results presented here comprise the first known results of obstacle problems driven by degenerate equations in non-divergence form, and they are new even for simpler (linear second order operators) such as the Laplacian, e.g. for an operator of the form $G[u] = |Du|^\gamma \Delta u$.

We recall that obtaining quantitative/qualitative information of the solution close to the free boundary is a pivotal first step for addressing a number of analytic and geometric issues in free boundary problems such as blow-up analysis, free boundary regularity, geometric estimates, just to mention a few (cf. [18], [41] and [42] for some interesting examples).

Historically, the obstacle problem and its derivations have been a remarkable landmark concerning researches in the theory of free boundary problems and variational inequalities. As a matter of fact, their genesis dates back to the late 1960’s and early 1970’s to the works due to Stampacchia, Lewy, Kinderlehrer among others authors, see [38] Ch.1, §9 for a
historical overview. By way of motivation (cf. [38, Ch.1]), they have also arisen naturally as one of the simplest problems of unilateral classical (linear) elliptic theory of elasticity, where in a physical situation it is modeling the shape of an elastic membrane which is pushed by a certain obstacle (a physical body or constraint) from one side affecting its shape, with a fixed position on the boundary of a bounded and regular region $\Omega \subset \mathbb{R}^n$.

From a mathematical point of view, such a physical model can be summarized as follows: to find an equilibrium profile $v$ which fulfills in a weak sense the constrained free boundary problem:

$$
\begin{align*}
& v(x) \geq \varphi(x) \quad \text{in} \quad \Omega \\
& L_v(x) = f(x) \quad \text{in} \quad \Omega \cap \{v > \varphi\} \\
& L_v(x) \leq f(x) \quad \text{in} \quad \Omega \\
& v(x) = g(x) \quad \text{on} \quad \partial \Omega,
\end{align*}
$$

where $v$ is the function whose graph represents the shape of the membrane, $L$ is an elliptic operator, representing, to some extent, the heterogeneity or physical properties of the media, $\varphi$ is an obstacle (or physical constraint) and $g$ is a fixed boundary condition. From a variational viewpoint, solutions for such a free boundary problem can be obtained by minimizing the corresponding energy functional associated to $L$ (if this one enjoys of a certain structure, cf. [2]) within an appropriate space of functions whereas non-variational techniques include Perron’s method and comparison principles tools, see e.g. [28], [29], [31] and references therein.

Advancements of such researches concerning obstacle problems brought out not only important theoretical contributions, they also have proved to be pivotal in a wide range of models in applied mathematics like mechanics, engineering, mathematical programming, control and optimization theory, game theoretical methods in PDEs, etc (cf. [11], [22], [32], [38] and [39] for instrumental surveys and applications).

1.3. Statement of main results. In this section we present the main results in this manuscript. The sharp regularity exponent $\alpha$ is going to be fixed throughout and is defined by (1.4). Our first main result assures that we are able to obtain a viscosity solution to $(F, \gamma)$-obstacle problem in $\Omega$ under suitable assumptions on the data and, moreover, these solutions enjoy a “basic” (in the sense that is not optimal) regularity estimate.

**Theorem 1.1 (Existence of solutions with basic regularity).** Let $\phi \in C^{1,1}(\Omega)$, $f \in C^0(\Omega)$ and $g \in C^{1,\beta}(\Omega)$. Then, there exists $u$ a viscosity solution to the $(F, \gamma)$-obstacle problem in $\Omega$. Moreover, $u \in C^{1,\beta'}(\Omega)$ for any $\beta' < \beta$ and

$$
\|u\|_{C^{1,\beta'}(\Omega)} \leq C \left( n, \lambda, \Lambda, \gamma, \|\phi\|_{C^{1,1}(\Omega)}, \|f\|_{L^\infty(\Omega)}, \|g\|_{C^{1,\beta}(\Omega)} \right).
$$

**Remark 1.2.** Alternatively, one could prove existence of solutions via Perron’s method (see for instance [16] or [28]). This approach has the advantage of not requiring any smoothness on the obstacle other than continuity and the downside that it produces a solutions which is just continuous. Since from our perspective the main interest in this problem lies on the regularity issues, in particular the optimal regularity which is achieved when the obstacle is $C^{1,1}$, we opted for the more compact penalization scheme described in Section 2.

The next result establishes an optimal growth estimate along free boundary points. It is somewhat finer than the previous one. In effect, it states that if the obstacle has a $C^{1,\beta}$ modulus of continuity and the source term is bounded, then solutions to the $(F, \gamma)$-obstacle
problem in $\Omega$ are locally in $C^{1,\alpha}$ (with $\alpha$ given by (1.4)) along free boundary points. As mentioned before, it is of particular interest is the optimal case when $\phi$ is $C^{1,1}$ and $f \equiv 0$, where we obtain quadratic growth for solutions along the free boundary (see Corollary 1.5).

**Theorem 1.3 (Optimal regularity).** Let $\beta \in (0,1]$ and $u$ be a bounded viscosity solution to the $(F,\gamma)$-obstacle problem in $\Omega$ with obstacle $\phi \in C^{1,\beta}(\Omega)$ and $f \in L^\infty(\Omega)$ and let $\alpha$ be defined by (1.4).

Then, $u$ is $C^{1,\alpha}$ regular along free boundary points. More precisely, for any $\tilde{\Omega} \subset \subset \Omega$ and $x_0 \in \partial \{u > \phi\} \cap \tilde{\Omega}$ and for $r$ small enough there holds

$$
\sup_{B_r(x_0)} |u(x) - (u(x_0) + Du(x_0) \cdot (x - x_0))| \leq C \left( \|\phi\|_{C^{1,\beta}(\Omega)}^{\gamma+1} + \|f\|_{L^\infty(\Omega)} \right)^{\frac{1}{\gamma+1}} r^{1+\alpha}, \quad (1.6)
$$

where $C > 0$ is a universal constant.

In particular,

$$
\sup_{B_r(x_0)} |u(x) - \phi(x)| \leq C^* \left( \|\phi\|_{C^{1,\beta}(\Omega)}^{\gamma+1} + \|f\|_{L^\infty(\Omega)} \right)^{\frac{1}{\gamma+1}} r^{1+\alpha}, \quad (1.7)
$$

where $C^* > 0$ is a universal constant, i.e. $u$ detaches from the obstacle at the speed dictated by the obstacle’s modulus of continuity.

We recall that a constant is called universal if it depends only on the given data (and not on the solution).

As a consequence of the previous Theorem 1.3 we get, under suitable assumptions on the data, the same regularity for the obstacle problem as for the non-constrained problem:

**Corollary 1.4.** Let $u$ be a bounded viscosity solutions to $(F,\gamma)$-obstacle problem in $\Omega$ with obstacle $\phi \in C^{1,\frac{1}{\gamma+1}}(\Omega)$ and $f \in L^\infty(\Omega)$. Suppose further that $F$ is a convex (or concave) operator. Then, $u$ is $C^{1,\frac{1}{\gamma+1}}$ along free boundary points.

In particular, as mentioned before, we get the optimal regularity for the homogeneous obstacle problem:

**Corollary 1.5.** Let $u$ be a bounded viscosity solutions to the homogeneous $(F,\gamma)$-obstacle problem in $\Omega$ (that is $f \equiv 0$) with obstacle $\phi \in C^{1,1}(\Omega)$. Suppose further that $F$ is a convex (or concave) operator. Then, $u$ is $C^{1,1}$ along free boundary points.

A geometric interpretation of Theorem 1.3 is the following: if $u$ solves the $(F,\gamma)$-obstacle problem (1.1), and $x_0$ is a free boundary point, then near $x_0$ we obtain

$$
\sup_{B_r(x_0)} |u(x)| \leq |u(x_0)| + Cr^{1+\alpha}.
$$

On the other hand, from a (geometric) regularity viewpoint, it is a pivotal quantitative information to obtain the counterpart sharp lower estimate. Such a property is denominated non-degeneracy of solutions and we begin our discussion of it in the following Theorem:

**Theorem 1.6 (Non-degeneracy estimates).** Let $u$ be a bounded viscosity solution to the $(F,\gamma)$-obstacle problem in $\Omega$ with obstacle $\phi \in C^{1,\beta}(\Omega)$ and assume $f \in L^\infty(\Omega)$ is
bounded away from zero, i.e. \( f \geq m > 0 \) a.e. in \( \Omega \). Given \( \tilde{\Omega} \subset \subset \Omega \) there exists a universal constant \( c > 0 \), such that for and \( x_0 \in \partial \{ u > \phi \} \cap \tilde{\Omega} \) and \( r \) small enough

\[
\sup_{B_r(x_0)} (u(x) - \phi(x_0)) \geq cr^{1 + \frac{1}{\gamma + 1}}
\]

This result implies a \( \frac{1}{\gamma + 1} \)-growth estimate (on the gradient) away from free boundary points, with an extra correction term given by influence of the gradient of the corresponding obstacle. This is summarized in the following corollary:

**Corollary 1.7 (Non-degeneracy of the gradient).** Suppose that the assumptions of Theorem 1.6 are in force. If \( x_0 \in \{ u > \phi \} \cap \tilde{\Omega} \) and \( r := \text{dist}(x_0, \partial \{ u > \phi \}) \), then

\[
\sup_{B_r(x_0)} |Du| \geq c r^{\gamma + 1} - \frac{1}{2} \| D\phi \|_{L^\infty(B_r(x_0))}.
\]

From a “free boundary regularity” perspective, a natural non-degeneracy property states that the solutions of the homogeneous obstacle problem do not decay faster than quadratically close to the free boundary:

**Theorem 1.8 (Non-degeneracy for the homogeneous problem).** Let \( u \) be a bounded viscosity solution to the \((F, \gamma)\)-obstacle problem in \( \Omega \) with obstacle \( \phi \in C^2(\Omega) \) and \( f \equiv 0 \). Suppose further that

\[
|D\phi|^\gamma F(D^2\phi) \leq c < 0. \tag{1.8}
\]

Given \( x_0 \in \{ u > \phi \} \cap \tilde{\Omega} \) for \( \tilde{\Omega} \subset \subset \Omega \), there exists a universal constant \( c \) such that for \( r \) small enough

\[
\sup_{B_r(x_0)} (u(x) - \phi(x)) \geq cr^2
\]

As consequence of the non-degeneracy of Theorem 1.8 we can show that the free boundary has zero Lebesgue measure in the homogeneous case (and for an obstacle satisfying (1.8)). This requires us to recall the definition of *porosity*: a bounded measurable set \( E \) is porous if for any \( x \in E \) there exists a \( \delta \in (0, 1) \) such that for any ball \( B_r(x) \) there exists \( y \in B_r(x) \) such that

\[
B_{\delta r}(y) \subset B_r(x) \setminus E.
\]

Notice that if \( E \) is porous and \( x \in E \) then

\[
\frac{|B_r(x) \cap E|}{|B_r(x)|} = \frac{|B_r(x)| - |B_r(x) \setminus E|}{|B_r(x)|} \leq 1 - \delta^n,
\]

so that \( E \) has no points of density one and hence its Lebesgue measure is zero. Now we can state the following corollary:

**Corollary 1.9.** Suppose that the assumptions of Theorem 1.8. Then, the free boundary is porous and in particular it has zero Lebesgue measure.

The rest of the paper is organized as follows: in Section 2 we give the appropriate definition of viscosity solutions and proof Theorem 1.1, thus providing existence of such solutions for (1.1), together with a basic regularity estimate. In Section 3 we prove Theorem 1.3 together with Corollaries 1.4 and 1.5. In Section 4 we discuss non-degeneracy, and prove Theorems 1.6 and 1.8 and their respective corollaries. Finally, for the reader’s convenience, we gather in the Appendix some useful results that were used throughout the paper.
2. Existence and basic regularity

In this section we prove our existence result, namely Theorem 1.1. Let us first review the definition of viscosity solution for our operators (see for instance [16]).

**Definition 2.1 (Viscosity solutions).** A continuous function $u: \Omega \to \mathbb{R}$ is said to be a viscosity super-solution (resp. viscosity sub-solution) of $G(x, Du, D^2u) = f(x)$ if for every $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a strict minimum (resp. strict maximum) at the point $x_0$, then

$$G(x_0, \varphi(x_0), D^2\varphi(x_0)) \leq f(x_0) \quad (resp. \geq f(x_0))$$

Finally, we say that $u$ is a viscosity solution if it is simultaneously a viscosity sub-solution and a viscosity super-solution.

It is noteworthy to observe that on the above definition $G$ is asked to be non-decreasing on $X \in Sym(n)$ with respect to the usual partial ordering on symmetric matrices. For this very reason, the classical notions of solution, sub-solution and super-solution are equivalent to the viscosity ones, provided the function is regular enough, e.g. $u \in C^2(\Omega)$.

Now we give a proof of Theorem 1.1. While it is achieved by a rather standard penalization argument (cf. [14] and references therein), we sketch it here for the sake of completeness:

**Proof of Theorem 1.1.** Recall that we want to find (viscosity) solutions of (1.1). This equation can be conveniently be rewritten as the following system:

$$\begin{cases}
  u \geq \phi & \text{in } \Omega \\
  |Du|^\gamma F(D^2u) \leq f & \text{in } \Omega \\
  |Du|^\gamma F(D^2u) = f & \text{in } \Omega \cap \{u > \phi\} \\
  u = g & \text{on } \partial\Omega.
\end{cases} \quad (2.1)$$

The first step is to consider the following penalized problem:

$$\begin{cases}
  |Du_\varepsilon|^\gamma F(D^2u_\varepsilon) = f + \beta_\varepsilon(u_\varepsilon - \phi) & \text{in } \Omega \\
  u = g & \text{on } \partial\Omega.
\end{cases} \quad (2.2)$$

where for each $\varepsilon > 0$ we define $\beta_\varepsilon \in C^\infty(\mathbb{R})$ to be any smooth function satisfying

$$\beta_\varepsilon' \geq 0, \quad \beta_\varepsilon \leq 0, \quad \beta_\varepsilon(t) = 0 \text{ when } t \geq 0, \quad \beta_\varepsilon(t) = \frac{t}{\varepsilon} \text{ when } t \leq -\varepsilon.$$

In fact, as a technical intermediate step, let us consider the truncated version of the above: for $N > 0$ we define

$$\beta_{\varepsilon,N}(t) := \max\{-N, \beta_\varepsilon\}.$$

To get existence of solutions of (2.2) we consider the following problem first:

$$\begin{cases}
  |Du_\varepsilon|^\gamma F(D^2u_\varepsilon) = f + \beta_{\varepsilon,N}(v - \phi) & \text{in } \Omega \\
  u = g & \text{on } \partial\Omega.
\end{cases} \quad (2.3)$$

where $v$ is some given function (whose smoothness is going to be specified right below). This problem has a unique viscosity solution by Perron’s method whenever the right hand side is continuous (see for instance [16] Theorem 4.1).

Next, notice that thanks to the global results of Birindelli and Demengel [6] Theorem 1.1], and the fact that $\beta_{\varepsilon,N}$ is bounded, for any $\beta' < \beta$ the operator

$$T : C^{1,\beta}(\Omega) \to C^{1,\beta}(\Omega)$$
given by \( T(v) = u \), where \( u \) is the solution of (2.3) has a fixed point in \( C^{1,\beta'}(\Omega) \). Indeed, the ball

\[
B := \{ v \in C^{1,\beta'}(\Omega) : v = g \text{ on } \partial\Omega, \|v\|_{C^{1,\beta}(\Omega)} \leq C \}
\]
is a convex and compact subset of \( C^{1,\beta}(\Omega) \). Moreover, thanks to the \textit{a priori} estimates \( T \) is continuous and, for an appropriate choice of \( C \), \( T(B) \subset B \). The existence of a fixed point is therefore guaranteed by Schauder’s Fixed Point Theorem (cf. [24, Theorem 11.1]). Let us label \( u_\varepsilon \) such a fixed point.

Next, we want to get a uniform bound for \( \beta_{\varepsilon,N} \) so that we are able to get a bound for \( u_\varepsilon \) which is uniform in \( \varepsilon \). Recall that \( \beta_{\varepsilon,N} \) is a non-positive, non-decreasing function, which means that its minimum is attained wherever \( u_\varepsilon - \phi \) attains its minimum. Let us thus consider \( x_0 \) a minimum point for \( u_\varepsilon - \phi \) and notice that

\[-\infty < (u_\varepsilon - \phi)(x_0) < 0 \quad \text{and} \quad x_0 \in \Omega.\]

Since \( u_\varepsilon - \phi \) is differentiable, there exists a horizontal plane \( P_{x_0} \) that satisfies

\[
P_{x_0}(x) \leq (u_\varepsilon - \phi)(x) \quad \text{in } \Omega \quad \text{and} \quad P_{x_0}(x_0) = (u_\varepsilon - \phi)(x_0).
\]

In the sequel, we can use \( P_{x_0} + \phi \) as a test function and get

\[
f(x_0) + \beta_{\varepsilon,N}((u_\varepsilon - \phi)(x_0)) \geq |D(P_{x_0} + \phi)(x_0)|^\gamma F(D^2(P_{x_0} + \phi)(x_0))
\]

which translates into

\[
\beta_{\varepsilon,N}((u_\varepsilon - \phi)(x_0)) \geq |D\phi(x_0)|^\gamma F(D^2\phi(x_0)) - f(x_0).
\]

Therefore,

\[
\beta_{\varepsilon,N} \geq C
\]

for some constant \( C \) depending only on the \( \|\phi\|_{C^{1,\gamma}(\Omega)} \) and \( \|f\|_{L^\infty(\Omega)} \). Hence we can drop the \( N \).

Now we take limits. Once again by the results in [6, Theorem 1.1] and the previous observation, the family \( \{u_\varepsilon\}_{\varepsilon > 0} \) of solutions to (2.2) are uniformly bounded in \( C^{1,\beta}(\Omega) \) and therefore Arzelà-Ascoli’s theorem ensures the existence of a function \( u \in C^{1,\beta'}(\Omega) \) such that

\[
u_\varepsilon \rightharpoonup u \quad \text{in } C^{1,\beta'}(\Omega) \quad \text{for any } \beta' < \beta.
\]

The bound \( |\beta_{\varepsilon}(u_\varepsilon - \phi)| \leq C \) ensures that \( u \geq \phi \), which is the first equation in (2.1). The other two equations are immediate by the stability of viscosity solutions under uniform limits (see e.g. [13, Theorem 3.8]).

3. Theorem 1.3 and its consequences

In this section we prove Theorem 1.3. We start with the following remarks that will simplify the notation hereafter:

\textbf{Remark 3.1 (Normalization assumptions).} Firstly, we may assume without loss of generality that \( u \) solves the \((F, \gamma)\)-obstacle problem in \( \Omega \) with obstacle \( \phi \) and source term \( f \) fulfilling

\[
\|\phi\|_{C^{1,\gamma}(\Omega)} \leq \frac{1}{2} \quad \text{and} \quad \|f\|_{L^\infty(\Omega)} \leq 1.
\]
As a matter of fact, let us consider the normalized function:

\[ v(x) := \frac{u(x)}{\left(2^{1+\gamma}\|\phi\|_{C^{1,\beta}(\Omega)}^{1+\gamma} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{1+\gamma}}} \]

\( v \) thus defined will satisfy an equation like (1.1) with \( F, f \) and \( \phi \) replaced by \( \hat{F}, \hat{f} \) and \( \hat{\phi} \) respectively where

\[
\begin{aligned}
\hat{F}(X) &:= \frac{1}{\left(2^{1+\gamma}\|\phi\|_{C^{1,\beta}(\Omega)}^{1+\gamma} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{1+\gamma}}} F \left(\left(2^{1+\gamma}\|\phi\|_{C^{1,\beta}(\Omega)}^{1+\gamma} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{1+\gamma}} X\right) \\
\hat{f}(x) &:= \frac{1}{\left(2^{1+\gamma}\|\phi\|_{C^{1,\beta}(\Omega)}^{1+\gamma} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{1+\gamma}}} f(x) \\
\hat{\phi}(x) &:= \frac{1}{\left(2^{1+\gamma}\|\phi\|_{C^{1,\beta}(\Omega)}^{1+\gamma} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{1+\gamma}}} \phi(x).
\end{aligned}
\]

Furthermore, \( \hat{F} \) is still elliptic with the same ellipticity constants as \( F \), and \( \hat{f} \) and \( \hat{\phi} \) fall into the desired statements.

This reduction implies, in particular, that we may assume that for any free boundary point \( z \in \partial\{u > \phi\} \) we have

\[ |Du(z)| = |D\phi(z)| \leq \frac{1}{2}. \]

For the sake of simplicity of notation we will also perform all our estimates in \( B_{\frac{1}{2}} \) without loss of generality. Also, throughout the proofs \( C \) will denote a universal that may change from line to line.

In the sequel, the proof will be divided into two technical lemmas, following the strategy set forth in [1], [2] and [17]. The basic idea is to deal with two regimes separately: either a free boundary point belongs to the critical zone (see below) where the gradient is small and the operator degenerates, or otherwise the operator is uniformly elliptic and classical results apply.

The critical zone of solutions (cf. [1, Section 3] and [17, Section 1.2] for similar ideas) \( C^\alpha_r \) is defined by

\[ C^\alpha_r(B) := \{ x \in B; |Du(x)| \leq r^\alpha \}, \]

where \( 0 < r \ll 1 \) is small, \( \alpha \) is defined by (1.4) and \( B \) is a ball.

**Lemma 3.2.** Suppose that the assumptions of Theorem 1.3 (and Remark 3.1) are in force. Let \( x_0 \in \partial\{u > \phi\} \cap C^\alpha_r \left( B_{\frac{1}{2}} \right) \) with \( 0 < r < \frac{1}{4} \). Then,

\[ \sup_{B_r(x_0)} |u(x) - u(x_0)| \leq Cr^{1+\alpha}, \]  

where \( C > 0 \) is a universal constant. In particular

\[ \sup_{B_r(x_0)} |u(x) - (u(x_0) + Du(x_0) \cdot (x - x_0))| \leq Cr^{1+\alpha}. \]  

**Proof.** First, for a fixed \( 0 < r < \frac{1}{4} \) we define the re-scaled auxiliary functions:

\[ u_{r,x_0}(x) := \frac{u(rx + x_0) - u(x_0)}{r^{1+\alpha}} \quad \text{and} \quad \phi_{r,x_0}(x) := \frac{\phi(rx + x_0) - \phi(x_0)}{r^{1+\alpha}}. \]
Now, notice that $u_{r,x_0}$ is a viscosity solution to the $(F_{r,x_0}, \gamma)$–obstacle, i.e. it satisfies
\[ |Du_{r,x_0}|^{\gamma} F_{r,x_0}(D^2u_{r,x_0}) \leq f_{r,x_0} \quad \text{and} \quad u_{r,x_0} \geq \phi_{r,x_0} \quad (3.3) \]
in $B_1$ where
\[
\begin{cases}
F_{r,x_0}(X) = r^{1-\alpha} F\left(\frac{1}{r^{1-\alpha}} X\right) \\
f_{r,x_0}(x) = r^{1-\alpha(1+\gamma)} f(rx + x_0).
\end{cases}
\]
It is straightforward to check that $F_{r,x_0}$ is still uniformly elliptic (with the same ellipticity constants as $F$). Moreover, it follows by (1.4) that
\[ \|f_{r,x_0}\|_{L^\infty(B_1)} \leq 1. \]
Now, we may estimate the $L^\infty$–norm of $\phi_{r,x_0}$ as follows
\[
\|\phi_{r,x_0}\|_{L^\infty(B_1)} = \left\| \frac{\phi(rx+x_0)-\phi(x_0)}{r^{1+\alpha}} \right\|_{L^\infty(B_1)} 
\leq \left\| \frac{\phi(rx+x_0)-\phi(x_0)-rD\phi(x_0)(x-x_0)}{r^{1+\alpha}} \right\|_{L^\infty(B_1)} + \left\| \frac{Du(x_0)(x-x_0)}{r^{\alpha}} \right\|_{L^\infty(B_1)} 
\leq \frac{3}{2},
\]
so the function $u_{r,x_0}(x) + \frac{3}{2}$ is non negative and it is still a supersolution.
Hence, by applying the weak Harnack inequality (see [25, Theorem 2] or Theorem 5.1 in the Appendix) and recalling that $u_{r,x_0}(0) = \phi_{r,x_0}(0)$ we obtain
\[
\|u_{r,x_0} + \frac{3}{2}\|_{L^{p_0}(B_\frac{1}{4})} \leq C \left( \inf_{B_1} \left( u_{r,x_0}(x) + \frac{3}{2} \right) + \|f_{r,x_0}\|_{L^\infty(B_1)} \right) 
\leq C \left( \phi_{r,x_0}(0) + \frac{3}{2} + \|f_{r,x_0}\|_{L^\infty(B_1)} \right) 
\leq C, \quad (3.4)
\]
for some universal $p_0 > 0$ universal.
Now let us consider
\[ w(x) := \max \left\{ u_{r,x_0}(x), \sup_{B_1} \phi_{r,x_0}(x) \right\} + \frac{3}{2}. \]
Notice that $w$ is non-negative sub-solution (in the viscosity sense) to
\[ |Dw|^{\gamma} F_{r,x_0}(D^2w) = f_{r,x_0} \chi_{\{u_{r,x_0} > \sup \phi_{r,x_0}\}} \]
and hence by invoking the local maximum principle (see [25, Theorem 3] or Theorem 5.2 in the Appendix) we get that
\[ \sup_{B_\frac{1}{4}} w \leq C \left( \|w\|_{L^{p_0}(B_\frac{1}{4})} + \|f_{r,x_0}\|_{L^\infty(B_1)} \right). \]
Combining this estimate with the definition of $w$ and with equation (3.4) we get (upon relabeling the constants and some elementary manipulations)
\[ \sup_{B_\frac{1}{4}} u_{r,x_0}(x) \leq C. \]
But on the other hand, since
\[ u_{r,x_0}(x) \geq -\|\phi_{r,x_0}\|_{L^\infty(B_1)} \geq -\frac{3}{2} \]
we conclude that \( u_{r,x_0} \) is uniformly bounded in \( B_{\frac{1}{4}} \).

From here we get (3.1) for \( r \in (0, \frac{1}{4}) \) just by recalling the definition of \( u_{r,x_0} \). Since (3.2) follows by the triangle inequality this completes the proof. \( \square \)

In the following second Lemma, we will analyse the points outside the critical zone, where classical estimates apply.

**Lemma 3.3.** Suppose that the assumptions of Theorem 1.3 (and Remark 3.1) are in force. Let \( z_0 \in \partial \{ u > \phi \} \cap \left( B_{\frac{1}{2}} \setminus C^a \left( B_{\frac{1}{2}} \right) \right) \) with \( 0 < r < \frac{1}{4} \). Then,
\[ \sup_{B_{r,z_0}(z_0)} |u(x) - u(z_0)| \leq C r^{1+\alpha} , \] (3.5)
for \( 0 < r_s(r) < \frac{1}{4} \), where \( C > 0 \) is a universal constant.

**Proof.** Let \( 0 < r < \frac{1}{4} \) fixed and \( z_0 \in B_{\frac{1}{2}} \) such that \( |Du(z_0)| \geq r^\alpha \). By taking the particular case \( r_{z_0} = \sqrt[4]{|Du(z_0)|} \) we are allowed to apply the Lemma 3.2 and conclude that
\[ \sup_{B_{r_s(z_0)}(z_0)} |u(x) - u(z_0)| \leq C r_{z_0}^{1+\alpha} . \] (3.6)
As before, we define the re-scaled auxiliary functions:
\[
\begin{align*}
  u_{r,z_0}(x) &= \frac{u(rz_0,x+2z_0) - u(z_0)}{r^{6\alpha}} \\
  \phi_{r,z_0}(x) &= \frac{\phi(rz_0,x+2z_0) - \phi(z_0)}{r^{6\alpha}} \\
  f_{r,z_0}(x) &= r^{1-\alpha(\gamma+1)} f(rz_0 x + z_0) .
\end{align*}
\]
Notice that
\[ |Du_{r,z_0}(0)| = |D\phi_{r,z_0}(0)| = 1 \quad \text{and} \quad \|f_{r,z_0}\|_{L^\infty(B)} \leq 1 . \] (3.7)
Moreover, \( u_{r,z_0} \) is a viscosity solution to a \((F_{r,z_0}, \gamma)\)-obstacle problem as in (3.3). Particularly, \( u_{r,z_0} \) fulfils (in the viscosity sense) the following:
\[ \min \{ f_{r,z_0} - |Du_{r,z_0}| \gamma F(D^2 u_{r,z_0}), \ u_{r,z_0} - \phi_{r,z_0} \} = 0 . \]
Now, from the assumption \( \|\phi\|_{C^{1,\beta}(B_1)} \leq \frac{1}{2} \) (see Remark 3.1) we get
\[ \|\phi_{r,z_0}\|_{C^{1,\beta}(B_{\frac{1}{4}})} \leq C \] (universal constant).

Moreover, (3.6) assures us that \( u_{r,z_0} \) is uniformly bounded in the \( L^\infty(B_{\frac{1}{4}}) \) -topology. From estimates given in Theorem 1.3 it follows that
\[ \|u_{r,z_0}\|_{C^{1,\beta}(B_{\frac{1}{4}})} \leq C_{1} \] (universal constant).

Such an estimate and the sentence (3.7), allow us to find a radius \( r_0 \ll 1 \) (universal) so that
\[ \forall x \in B_{r_0}(z_0) \quad \text{and} \quad \epsilon \in (0,1) \text{ fixed.} \]
Particularly, we obtain the following (in the viscosity sense)
\[
\min \{ K - F(D^2 u_{r_0}, z_0), \ u_{r_0} - \phi_{r_0} \} = 0,
\]
where \( K = K(\zeta, \alpha, \gamma, \| f_{r_0}, z_0 \|_{L^{\infty}(B_1)}) > 0. \)

Consequently, \( u_{r_0} \) is a viscosity solution (uniformly bounded) to an obstacle-type problem for a uniformly elliptic operator in \( B_{r_0}(z_0) \), with a \( C^{1, \beta} \) obstacle \( (\phi_{r_0}, z_0) \) and constant (positive) source term \( K \). From classical theory for obstacle-type problems (see \( \text{[28]} \) and \( \text{[29]} \))
\[
\| u_{r_0}, z_0 \|_{C^{1, \alpha}(B_{r_0})} \leq C(\beta, \gamma, \Lambda, \lambda, N)
\]
By scaling back we conclude that
\[
\| u_{r_0}, z_0 \|_{C^{1, \alpha}(B_{r_0})} \leq C(\beta, \gamma, \Lambda, \lambda, N),
\]
which particularly implies that
\[
\sup_{B_r(z_0)} | u(x) - (u(z_0) + Du(z_0) \cdot (x - z_0)) | \leq Cr^{1 + \alpha},
\]
for all \( r < r_0 r_{r_0} \).

Finally, we are in position to supply the proof of Theorem 1.3.

**Proof of Theorem 1.3** Remember that from Lemma 3.2 or 3.3 we have the following:
\[
\sup_{B_r(y_0)} | u(x) - (u(y_0) + Du(y_0) \cdot (x - y_0)) | \leq Cr^{1 + \alpha},
\]
for every \( r \in (0, \frac{1}{2}) \) such that \( r^\alpha > \| Du(y_0) \| \) (Lemma \( \text{[3.2]} \)) and \( r^\alpha \leq \sqrt{r_0} \| Du(y_0) \| = \sqrt{r_0} r_0 \) (Lemma \( \text{[3.3]} \), where \( y_0 \in \partial \{ u > \phi \} \cap B_{\frac{1}{2}} \)). Next prove the desired estimate when
\[
r \in \left( r_0 \sqrt{|r_0 u(y_0)|}, \sqrt{|Du(y_0)|} \right). \tag{3.8}
\]
For that purpose, suppose that \( r \) falls into interval specified in \( \text{[3.3]} \). Hence,
\[
\sup_{B_r(y_0)} | u(x) - (u(y_0) + Du(y_0) \cdot (x - y_0)) | \leq \sup_{B_{r_0}(y_0)} | u(x) - (u(y_0) + Du(y_0) \cdot (x - y_0)) |
\]
\[
\leq Cr_{r_0}^{1 + \alpha}
\]
\[
\leq \frac{C}{r_0} r^{1 + \alpha}.
\]
Thus, we obtain the estimate for all \( r \in (0, \frac{1}{2}) \).

This gives the result in \( B_{\frac{1}{2}} \). Getting it for \( \hat{\Omega} \) is just a standard covering procedure. Moreover, taking into account Remark 3.1 in order to obtain the desired \( C^{1, \alpha} \) regularity estimate for the original solution \( u \) (equation \( \text{[1.6]} \)), one just needs to multiply the constant \( C > 0 \) by the normalization factor \( \left( 2^{\gamma + 1} \| \phi \|^\gamma_{C^{1, \beta}(\hat{\Omega})} + \| f \|_{L^{\infty}(\hat{\Omega})} \right)^{\frac{1}{\gamma + 1}}. \)

Finally, to obtain \( \text{[1.7]} \) we just compute
\[
\sup_{B_r(y_0)} | u(x) - \phi(x) | \leq \sup_{B_r(y_0)} | u(x) - (u(y_0) + Du(y_0) \cdot (x - y_0)) |
\]
\[
+ \sup_{B_r(y_0)} | \phi(x) - (\phi(y_0) + D\phi(y_0) \cdot (x - y_0)) |
\]
\[
\leq (C + 1) r^{1 + \alpha}.
\]
Proof of Corollary 1.4. We simply note that if $F$ is convex or concave and $\phi \in C^{1+\frac{1}{\gamma}}(\Omega)$ then $\alpha = \frac{1}{\gamma+1}$. The result then follows by combining Theorem 1.3 with the estimates for the unconstrained problem obtained in [3, Corollary 3.1].

Proof of Corollary 1.5. It follows by noticing that if $f \equiv 0$ we can take $\alpha = 1$ when making the re-scaling.

4. Non-degeneracy results and Lebesgue measure of the free boundary

This section is devoted to prove Theorems 1.6 and 1.8 and their corollaries related to geometric non-degeneracy. They play an essential role in the description of solutions to free boundary problems of obstacle-type.

Proof of Theorem 1.6. Notice that, due to the continuity of solutions, it is sufficient to prove that such an estimate is satisfied just at point within $\{u > \phi\} \cap \Omega$. First of all, for $x_0 \in \{u > \phi\} \cap \Omega$ and $r$ small enough so that $B_r(x_0) \subset \subset \Omega$ let us define the scaled function

$$u_r(x) := \frac{u(x_0 + rx)}{r^{1+\frac{1}{\gamma+1}}} \quad \text{for} \quad x \in B_1.$$

Now, let us introduce the comparison function:

$$\varphi(x) := \left[ \frac{m(\gamma+1)\gamma+2}{2n\Lambda(\gamma+2)\gamma+1} \right]^{\frac{1}{\gamma+1}} |x|^{1+\frac{1}{\gamma+1}} + \frac{1}{r^{1+\frac{1}{\gamma+1}}} \phi(x_0).$$

Straightforward calculus shows that

$$|D\varphi|^\gamma \mathcal{G}(D^2\varphi) - \hat{f}(x) \leq 0 \quad \text{in} \quad B_1$$

and

$$|Du_r|^\gamma \mathcal{G}(D^2u_r) - \hat{f}(x) = 0 \quad \text{in} \quad B_1 \cap \{u_r > \phi_r\}$$

in the viscosity sense, where

$$\begin{align*}
\mathcal{G}(X) &:= r^{\frac{\gamma}{\gamma+1}} F \left( r^{-\frac{\gamma}{\gamma+1}} X \right) \\
\hat{f}(x) &:= f(x_0 + rx) \\
\phi_r(x) &:= \frac{\phi(x_0 + rx)}{r^{1+\frac{1}{\gamma+1}}}.
\end{align*}$$

Moreover, $\mathcal{G}$ is uniformly elliptic.

Finally, if $u_r \leq \varphi$ on $\partial(B_1 \cap \{u_r > \phi_r\})$ then the Comparison Principle (see Theorem 5.3 in the Appendix), would imply that

$$u_r \leq \varphi \quad \text{in} \quad B_1 \cap \{u_r > \phi_r\},$$

which clearly contradicts the fact that $u_r(0) > \phi_r(0)$. Therefore, there exists a point $y \in \partial(B_1 \cap \{u_r > \phi_r\})$ such that

$$u_r(y) > \varphi(y).$$

To conclude, we just notice that (by the choice of $r$) such a point must belong to $\partial B_1 \cap \{u_r > \phi_r\}$ so that then

$$\sup_{B_r(x_0)} u \geq u_r(y) \geq \epsilon + \frac{1}{r^{1+\frac{1}{\gamma+1}}} \phi(x_0).$$
for 
\[ c := \left[ \frac{m^2 - (\gamma + 1)^2 + 2\mu (\gamma + 2)^{\gamma + 1}}{2n\Lambda (\gamma + 2)^{\gamma + 1}} \right]^{\frac{1}{\gamma + 1}} \]
and we get the result by just scaling back.

Now, we will prove the non-degeneracy of the gradient.

**Proof of Corollary 1.7.** In effect, let \( x_0 \in \{u > \phi\} \cap B_{\frac{1}{2}} \) and \( y_0 \in \partial \{u > \phi\} \) such that 
\[ r := \text{dist}(x_0, \partial \{u > \phi\}) = |x_0 - y_0|. \]

Now, from non-degeneracy, Theorem 1.6 there exists \( z \in \partial B_r(x_0) \) such that 
\[ u(z) - \phi(x_0) = cr^{\gamma+1}. \]

Moreover, from local Lipschitz regularity of \( u \) and \( \phi \) we get 
\[
\begin{align*}
    u(z) - \phi(x_0) &= u(z) - u(y_0) + \phi(y_0) - \phi(x_0) \\
    &\leq \|Du\|_{L^\infty(B_r(x_0))}|z - y_0| + \|D\phi\|_{L^\infty(B_r(x_0))}|y_0 - x_0| \\
    &\leq 2r\|Du\|_{L^\infty(B_r(x_0))} + r\|D\phi\|_{L^\infty(B_r(x_0))}
\end{align*}
\]

Therefore, by using the previous estimates we obtain the desired result 
\[ \|Du\|_{L^\infty(B_r(x_0))} \geq cr^{\gamma+1} - \frac{1}{2}\|D\phi\|_{L^\infty(B_r(x_0))}. \]

**Remark 4.1.** An interesting piece of information about previous Corollary 1.7 is the following: when the gradient of the obstacle is “flat” enough, i.e. \( \|D\phi\|_{L^\infty(B_r(x_0))} \ll 1 \), then solutions to our obstacle problem present an almost \( \frac{1}{\gamma+1} \)-growth away from free boundary points (cf. [3, Theorem 3.3] for a similar non-degeneracy property). Furthermore, a legitimate \( \frac{1}{\gamma+1} \)-behavior is brought to light once we suppose that \( \|D\phi\|_{L^\infty(B_r(x_0))} \leq c_0 r^{\frac{1}{\gamma+1}} \) and \( c > 2c_0 \), which is not a restrictive assumption, due to explicit universal dependence of constant \( c \).

Next we will prove our second non-degeneracy result.

**Proof of Theorem 1.8.** Let \( y \in \{u > \phi\} \cap \Omega \) and \( v(x) = \phi(x) + \varepsilon|x - y|^2 \), where \( \varepsilon \ll 1 \) is chosen such that \( |Du|^\gamma F(D^2v) < 0 \), which is possible since the second order degenerate fully nonlinear operator in force is continuous with respect to parameter \( \varepsilon \).

Now, by putting \( r < \text{dist}(x_0, \partial\Omega) \), we obtain that 
\[ |Dv|^\gamma F(D^2v) < 0 = |Du|^\gamma F(x, D^2u) \quad \text{in} \quad \{u > \phi\} \cap B_r(x_0) \]
in the viscosity sense. Furthermore, \( u(y) \geq \phi(y) = v(y) \). By invoking the comparison principle it follows that there is \( z_y \in \partial \{u > \phi\} \cap B_r(x_0) \) such that \( u(z_y) \geq v(z_y) \). Since \( u < v \) on \( B_r(x_0) \cap \partial \{u > \phi\} \) it must hold that \( z_y \in \{u > \phi\} \cap \partial B_r(x_0) \). We conclude the proof by continuity, by letting \( y \to x_0 \).

As mentioned before, the porosity of the free boundary is a consequence of the non-degeneracy in the homogeneous case:
Proof of Corollary 1.9. Let $x_0 \in \partial \{u > \phi\} \cap \hat{\Omega}$ and pick $r$ small enough so that $B_{2r}(x_0) \subset \subset \hat{\Omega}$. By Theorem 1.8 we have that there exists some $y \in \partial B_r(x_0)$ such that

$$u(y) - \phi(y) \geq cr^2$$  \hspace{1cm} (4.1)

for some (universal) constant $c$.

On the other hand, the growth control proved in Theorem 1.3 gives

$$u(y) - \phi(y) \leq C \left( \text{dist}(y, \partial \{u > \phi\}) \right)^2$$  \hspace{1cm} (4.2)

(4.1) and (4.2) together imply

$$\text{dist}(y, \partial \{u > \phi\}) > \left( \frac{c}{C} \right)^{1/2} r =: \tilde{C}r$$  \hspace{1cm} (4.3)

and taking $\delta := \frac{\tilde{C}}{4}$ we obtain the that $B_{2\delta r}(y) \cap B_{2r}(x_0) \subset \{ u > \phi \} \cap \hat{\Omega}$ and the result is proved. \hfill $\square$

5. Appendix

In this Appendix we gather, for the reader’s convenience, the statements of the important results that have been cited throughout the paper. First we restate two results by Imbert, namely the weak Harnack inequality and the local maximum principle used in the proof of Lemma 3.2.

Theorem 5.1 (Weak Harnack inequality, [25 Theorem 2]). Let $u$ be a non-negative continuous function satisfying

$$|Du|^\gamma G(D^2 u) \leq f \quad \text{in} \quad B_1$$

in the viscosity sense (i.e. $u$ is a super-solution). Assume that $G$ is uniformly elliptic and $f \in C^0(B_1)$.

Then,

$$\|u\|_{L^{p_0}(B_{1/2})} \leq C \left( \inf_{B_1} u + \|f\|_{L^\infty(B_1)} \right)$$

for some (universal) $p_0$ and a universal constant $C$.

Theorem 5.2 (Local Maximum principle, [25 Theorem 3]). Let $u$ be a continuous function satisfying

$$|Du|^\gamma G(D^2 u) \geq f(x) \quad \text{in} \quad B_1$$

in the viscosity sense (i.e. $u$ is a sub-solution). Assume that $G$ is uniformly elliptic and $f \in C^0(B_1)$.

Then, for any $p > 0$

$$\sup_{B_{1/2}} u \leq C \left( \|u^+\|_{L^p(B_1)} + \|f\|_{L^\infty(B_1)} \right)$$

where $C$ is a universal constant depending also on $p$.

The next result is a comparison principle due to Birindelli and Demengel (see [4] for details).
Theorem 5.3 (Comparison Principle). Let $u_1$ and $u_2$ be continuous functions in $\overline{\Omega}$ and $f \in C^0(\overline{\Omega})$ fulfilling
\[
|Du_1|^\gamma F(D^2 u_1) - f(x) \leq 0 \leq |Du_1|^\gamma F(D^2 u_1) - f(x) \quad \text{in} \quad \Omega
\]
in the viscosity sense for some uniformly elliptic operator $F$. If $u_1 \geq u_2$ on $\partial \Omega$, then $u_1 \geq u_2$ in $\Omega$.

Acknowledgements. This work was partially supported by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET-Argentina) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES-Brazil). J.V. da Silva thanks FCEyN/CEMIM from Universidad Nacional de Mar del Plata for its warm hospitality and for fostering a pleasant scientific atmosphere during his visit where part of this work was written.

References

[1] Amaral, M., da Silva, J.V., Ricarte, G.C. and Teymurazyan, R. Sharp regularity estimates for quasilinear evolution equations. To appear in Israel J. Math., DOI: https://doi.org/10.1007/s11856-019-1842-1
[2] Andersson, J., Lindgren, E. and Shahgholian, H. Optimal regularity for the obstacle problem for the $p$–Laplacian. J. Differential Equations 259 (2015), no. 6, 2167-2179.
[3] Araújo, D., Ricarte, G.C. and Teixeira, E. Geometric gradient estimates for solutions to degenerate elliptic equations. Calc. Var. Partial Differential Equations 53 (2015), no. 3-4, 605-625.
[4] Birindelli, I. and Demengel, F. Comparison principle and Liouville type results for singular fully nonlinear operators. Ann. Fac. Sci. Toulouse Math. (6) 13 (2004), no. 2, 261-287.
[5] Birindelli, I. and Demengel, F. The Dirichlet problem for singular fully nonlinear operators. Discrete Contin. Dyn. Syst. 2007, Dynamical systems and differential equations. Proceedings of the 6th AIMS International Conference, suppl., 110-121. ISBN: 978-1-60133-010-9; 1-60133-010-3.
[6] Birindelli, I. and Demengel, F. $C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations. ESAIM Control Optim. Calc. Var. 20 (2014), no. 4, 1009-1024.
[7] Birindelli, I. and Demengel, F. Hölder regularity of the gradient for solutions of fully nonlinear equations with sub linear first order term. Geometric methods in PDE’s, 257-268, Springer INdAM Ser., 13, Springer, Cham, 2015.
[8] Caffarelli, L.A. The regularity of free boundaries in higher dimension. Acta Math. 139 (1977) 155-184.
[9] Caffarelli, L.A. Interior a priori estimates for solutions of fully nonlinear equations. Ann. of Math. (2) 130 (1989), no. 1, 189-213.
[10] Caffarelli, L.A. The obstacle problem. Lezioni Fermiane. [Fermi Lectures] Accademia Nazionale dei Lincei, Rome; Scuola Normale Superiore, Pisa, 1998. ii+54 pp.
[11] Caffarelli, L.A. The obstacle problem revisited. J. Fourier Anal. Appl. 4 (1998), no. 4-5, 383-402.
[12] Caffarelli, L.A. and Cabré, X. Fully nonlinear elliptic equations. American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, RI, 1995. vi+104 pp. ISBN: 0-8218-0437-5.
[13] Caffarelli, L.A., Crandall, M.G., Kocan, M. and Świȩch, A. On viscosity solutions of fully nonlinear equations with measurable ingredients. Comm. Pure Appl. Math. 49 (1996), no. 4, 365-397.
[14] Caffarelli, L.A., Duque, L. and Vivas, H. The two membranes problem for fully nonlinear operators. Discrete & Continuous Dynamical Systems - A 2018, 38 (12): 6015-6027. doi:10.3934/dcds.2018152.
[15] Caffarelli, L.A. and Kinderlehrer, D. Potential methods in variational inequalities. J. Analyse Math. 37 (1980), 285-295.
[16] Crandall, M. G., Ishii H. and Lions P.-L. User’s guide to viscosity solutions of second order partial differential equations Bulletin of the American mathematical society 27.1 (1992): 1-67.
[17] da Silva, J.V. Geometric $C^{1+\alpha}$ regularity estimates for nonlinear evolution models. Nonlinear Anal. 184 (2019), 95-115.
[18] da Silva, J.V., Leitão, R.A. and Ricarte, G.C. Geometric regularity estimates for fully nonlinear elliptic equations with free boundaries. Preprint.
da Silva, J.V. Rossi, J.D. and Salort, A. *Regularity properties for p-dead core problems and their asymptotic limit as p → ∞*. J. London Math. Soc. (2) 99 (2019) 69-96.

da Silva, J.V. and Salort, A., *Sharp regularity estimates for quasi-linear elliptic dead core problems and applications*. Calc. Var. Partial Differential Equations 57 (2018).

Evans, L.C. *Classical solutions of fully nonlinear, convex, second-order elliptic equations*. Comm. Pure Appl. Math., 35(3): 333-363, 1982.

Figalli, A. *Regularity of interfaces in phase transitions via obstacle problems*. Proceedings of the International Congress of Mathematicians, 2018.

Figalli, A., Krummel, B. and Ros-Oton, X. *On the regularity of the free boundary in the p-Laplacian obstacle problem*. J. Differential Equations 263 (2017), no. 3, 1931-1945.

Gilbarg, D. and Trudinger, N.S. *Elliptic partial differential equations of second order*. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

Krylov, N.V. *Boundedly inhomogeneous elliptic and parabolic equations in a domain*. Izv. Akad. Nauk SSSR Ser. Mat., 47(1): 75-108, 1983.

Lee, K.-A. *Obstacle problems for the fully nonlinear elliptic operators*. Thesis (Ph.D.)-New York University. 1998. 53 pp. ISBN: 978-0-599-04972-7.

Lee, K.-A. and Shahgholian, H. *Regularity of a free boundary for viscosity solutions of nonlinear elliptic equations*. Comm. Pure Appl. Math. 54 (2001), no. 1, 43-56.

Lee, K.-A. and Shahgholian, H. *Hausdorff measure and stability for the p-obstacle problem (2 < p < ∞)*. J. Differential Equations 195 (2003), no. 1, 14-24.

Lee, K.-A. and Park, J. *Obstacle problem for a non-convex fully nonlinear operator*. J. Differential Equations 265 (2018), no. 11, 5809-5830.

Manfredi, J.J., Rossi, J.D. and Somersille, S. *An obstacle problem for tug-of-war games*. Commun. Pure Appl. Anal. 14 (2015), no. 1, 217-228.

Nadirashvili, N. and Vlăduț, S. *Nonclassical solutions of fully nonlinear elliptic equations*. Geometric and Functional Analysis 17.4 (2007): 1283-1296.

Nadirashvili, N. and Vlăduț, S. *Singular viscosity solutions to fully nonlinear elliptic equations*. J. Math. Pures Appl. (9), 89 (2): 107-113, 2008.

Nadirashvili, N. and Vlăduț, S. *Octonions and singular solutions of Hessian elliptic equations*. Geom. Funct. Anal. 21 (2011), no. 2, 483-498.

Nadirashvili, N. and Vlăduț, S. *Singular solutions of Hessian fully nonlinear elliptic equations*. Adv. Math. 228 (2011), no. 3, 1718-1741.

Nadirashvili, N. and Vlăduț, S. *Singular solutions of Hessian elliptic equations in five dimensions*. J. Math. Pures Appl. (9) 100 (2013), no. 6, 769-784.

Rodrigues, J.F. *Obstacle problems in mathematical physics*. North-Holland Mathematics Studies, 134. Notas de Matemática [Mathematical Notes], 114. North-Holland Publishing Co., Amsterdam, 1987.

Ros-Oton, X. *Obstacle problems and free boundaries: an overview*. SeMA J. 75 (2018), 399-419.

Teixeira, E. *Regularity for the fully nonlinear dead-core problem*. Math. Ann. 364 (2016), no. 3-4, 1121-1134.

Teixeira, E. *Geometric regularity estimates for elliptic equations*. Mathematical Congress of the Americas, 185-201, Contemp. Math., 656, Amer. Math. Soc., Providence, RI, 2016.

Trudinger, N.S. *Fully nonlinear, uniformly elliptic equations under natural structure conditions*. Trans. Amer. Math. Soc. 278 (1983), no. 2, 751-769.

Trudinger, N.S. *Regularity of solutions of fully nonlinear elliptic equations*. Boll. Un. Mat. Ital. A (6) 3 (1984), no. 3, 421-430.
[45] Trudinger, N. S. *Hölder gradient estimates for fully nonlinear elliptic equations*. Proc. Roy. Soc. Edinburgh Sect. A 108 (1988), no. 1-2, 57-65.

Departamento de Matemática - Instituto de Ciências Exatas - Universidade de Brasília.
Campus Universitário Darcy Ribeiro, 70910-900, Brasília - Distrito Federal - Brazil.
*E-mail address:* J.V.Silva@mat.unb.br

Instituto de Investigaciones Matemáticas Luis A. Santaló (IMAS) - CONICET (Argentine), Ciudad Universitaria, Pabellón I (1428) Av. Cantilo s/n - Buenos Aires
*E-mail address:* jdasilva@dm.uba.ar

Instituto de Investigaciones Matemáticas Luis A. Santaló (IMAS) - CONICET (Argentine), Ciudad Universitaria, Pabellón I (1428) Av. Cantilo s/n - Buenos Aires

Centro Marplatense de Investigaciones matemáticas/Conicet, Dean Funes 3350, 7600 Mar del Plata, Argentina
*E-mail address:* havivas@mdp.edu.ar