Bose Operators, Coherent States, Truncation, Spin Coherent States, Lie Algebras and Spectrum

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Abstract. We study truncated Bose operators in finite dimensional Hilbert spaces. Spin coherent states for the truncated Bose operators and canonical coherent states for Bose operators are compared. The Lie algebra structure and the spectrum of the truncated Bose operators are discussed.

1 Introduction

Let \( \hat{b}^\dagger, \hat{b} \) be Bose creation and annihilation operators with the commutation relation \([b, b^\dagger] = I\), where \( I \) is the identity operator and \( \hat{b}|0\rangle = 0|0\rangle, \langle 0|0\rangle = 1 \). Then for the operators

\[
\hat{N} = \hat{b}^\dagger \hat{b}, \quad \hat{b}^\dagger, \quad \hat{b}, \quad I
\]

we find the commutators \([\hat{b}^\dagger \hat{b}, \hat{b}^\dagger] = \hat{b}^\dagger, [\hat{b}^\dagger \hat{b}, \hat{b}] = -\hat{b}, [\hat{b}^\dagger, \hat{b}] = -I\). All the other commutators are 0. It is well-known [2] that a non-hermitian faithful representation by \(3 \times 3\) matrices is given by

\[
\hat{b}^\dagger \hat{b} \rightarrow M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I \rightarrow M_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\hat{b}^\dagger \rightarrow M_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{b} \rightarrow M_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

since for the commutators we find

\[
[M_{22}, M_{23}] = M_{23}, \quad [M_{22}, M_{12}] = -M_{12}, \quad [M_{22}, M_{13}] = 0_3
\]
\[
[M_{23}, M_{12}] = -M_{13}, \quad [M_{23}, M_{13}] = 0, \quad [M_{12}, M_{13}] = 0.
\]

Note that the matrices \(M_{13}, M_{23}, M_{12}\) are nonnormal. With

\[
(\hat{b}^\dagger_1 \hat{b}^\dagger_2 \hat{b}^\dagger_3) M_{22} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{pmatrix} = \hat{b}^\dagger_1 \hat{b}_2, \quad (\hat{b}^\dagger_1 \hat{b}^\dagger_2 \hat{b}^\dagger_3) M_{13} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{pmatrix} = \hat{b}^\dagger_1 \hat{b}_3
\]

e tc we find the representation

\[
\hat{b}^\dagger_2 \hat{b}_2, \quad \hat{b}^\dagger_1 \hat{b}_3, \quad \hat{b}^\dagger_2 \hat{b}_3, \quad \hat{b}^\dagger_1 \hat{b}_2
\]
of the Lie algebra.

Here we consider the four operators

\[
\hat{N} = \hat{b}^\dagger \hat{b}, \quad \hat{b}^\dagger + \hat{b}, \quad \hat{b}^\dagger - \hat{b}, \quad I
\]

and truncations into finite dimensional Hilbert spaces \(\mathbb{C}^n\). The four operators \(\hat{N}, \hat{b}^\dagger + \hat{b}, \hat{b}^\dagger - \hat{b}, I\) form a basis of a Lie algebra. We obtain for the nonzero commutators

\[
[\hat{b}^\dagger \hat{b}, \hat{b}^\dagger + \hat{b}] = \hat{b}^\dagger - \hat{b}, \quad [\hat{b}^\dagger \hat{b}, \hat{b}^\dagger - \hat{b}] = \hat{b}^\dagger + \hat{b} \quad [\hat{b}^\dagger + \hat{b}, \hat{b}^\dagger - \hat{b}] = 2I.
\]

The Bargmann representation is

\[
\hat{b}^\dagger \leftrightarrow \zeta, \quad \hat{b} \leftrightarrow \frac{d}{d\zeta}, \quad (\hat{b}^\dagger)^n |0\rangle \leftrightarrow \zeta^n, \quad |0\rangle \leftrightarrow 1
\]

with \(n \in \mathbb{N}\) and the scalar product

\[
\langle f(\zeta)|g(\zeta) \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \overline{g(\zeta)} e^{-|\zeta|^2} dx dy
\]

where \(\zeta = x + iy\).

Obviously the identity operator \(I\) commutes with all other operators. Thus the Lie algebra generated by these operators is not semi-simple. The adjoint representation of this Lie algebra is given by

\[
\hat{b}^\dagger \hat{b} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{b}^\dagger + \hat{b} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad \hat{b}^\dagger - \hat{b} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}
\]
with the identity operator mapping to the $4 \times 4$ zero matrix. Let $|n\rangle$, $|\beta\rangle$, $|\zeta\rangle$ be the number states ($n = 0, 1, \ldots$), canonical coherent states ($\beta \in \mathbb{C}$) and squeezed states ($\zeta \in \mathbb{C}$), respectively. Then we find for the operators given by (1)  

$$
\langle n|\hat{b}^\dagger \hat{b}|n\rangle = n, \quad \langle \beta|\hat{b}^\dagger \hat{b}|\beta\rangle = \beta \beta^*, \quad \langle \zeta|\hat{b}^\dagger \hat{b}|\zeta\rangle = \sinh^2(|\zeta|)
$$

$$
\langle n|\hat{b}^\dagger + \hat{b}|n\rangle = 0, \quad \langle \beta|\hat{b}^\dagger + \hat{b}|\beta\rangle = 2\Re(\beta), \quad \langle \zeta|\hat{b}^\dagger + \hat{b}|\zeta\rangle = 0
$$

$$
\langle n|\hat{b}^\dagger - \hat{b}|n\rangle = 0, \quad \langle \beta|\hat{b}^\dagger - \hat{b}|\beta\rangle = -2\Im(\beta), \quad \langle \zeta|\hat{b}^\dagger - \hat{b}|\zeta\rangle = 0
$$

where $|\beta\rangle = D(\beta)|0\rangle$ and $|\zeta\rangle = S(\zeta)|0\rangle$ with the displacement operator $D(\beta)$ and squeezing operator $S(\zeta)$ given by

$$
D(\beta) = \exp(\beta \hat{b}^\dagger - \bar{\beta}\hat{b}), \quad S(\zeta) = \exp\left(-\frac{\zeta}{2}(\hat{b}^\dagger)^2 + \frac{\bar{\zeta}}{2} \hat{b}^2\right).
$$

Setting $\beta = e^{i\phi} \tan(\theta/2)$ with $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$ we obtain

$$
\langle \beta|\hat{b}^\dagger \hat{b}|\beta\rangle = \tan^2(\theta/2), \quad \langle \beta|\hat{b}^\dagger + \hat{b}|\beta\rangle = 2\tan(\theta/2) \cos(\phi).
$$

If $\theta = 0$, then $\langle \beta|\hat{b}^\dagger \hat{b}|\beta\rangle = 0$ and $\langle \beta|\hat{b}^\dagger + \hat{b}|\beta\rangle = 0$. If $\theta = \pi/2$, then $\langle \beta|\hat{b}^\dagger \hat{b}|\beta\rangle = 1$ and $\langle \beta|\hat{b}^\dagger + \hat{b}|\beta\rangle = 2\cos(\phi)$. These results can then be compared with the result from the truncated Bose operators and spin coherent states.

We study the $n \times n$ matrices which arise in the truncation of the four operators given by (1). Since the four operators given by (1) form a basis of a Lie algebra we ask the question whether the $n \times n$ matrices from the truncation form a basis of a Lie algebra. Furthermore we study the spectrum of the truncated operators. Coherent states in a finite-dimensional Hilbert space have been studied by Mira-nowicz et al [4][5]. Utilizing the spin coherent states we find the expectation values for the truncated operators of $\hat{b}^\dagger \hat{b}$ and $\hat{b}^\dagger + \hat{b}$.

We mention that this set of operators given in (1) can also be considered for Fermi systems. Let $\hat{c}^\dagger$, $\hat{c}$ be Fermi creation and annihilation operators with $[\hat{c}, \hat{c}^\dagger]_+ = I$, $[\hat{c}, \hat{c}]_+ = 0$, $[\hat{c}^\dagger, \hat{c}^\dagger]_+ = 0$, where $I$ is the identity operator and $\hat{c}|0\rangle = 0|0\rangle$ ($\langle 0|0\rangle = 1$). Then the operators

$$
\hat{N} = \hat{c}^\dagger \hat{c}, \quad \hat{c}^\dagger + \hat{c}, \quad \hat{c}^\dagger - \hat{c}, \quad I
$$

form a basis of a Lie algebra. We obtain the nonzero commutators

$$
[\hat{c}^\dagger \hat{c}^\dagger + \hat{c}] = \hat{c}^\dagger - \hat{c}, \quad [\hat{c}^\dagger \hat{c}^\dagger - \hat{c}] = \hat{c}^\dagger + \hat{c}, \quad [\hat{c}^\dagger + \hat{c}, \hat{c}^\dagger - \hat{c}] = 2I - 4\hat{c}^\dagger \hat{c}.
$$
Obviously the identity operator $I$ commutes with all other operators. So the Lie algebra is not semi-simple. A representation of these operators would be with $2 \times 2$ matrices

\[\hat{c}^\dagger \hat{c} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{c}^\dagger + \hat{c} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{c}^\dagger - \hat{c} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\]

2 Truncation and Lie Algebras

To find the matrix representation of $\hat{N} = \hat{b}^\dagger \hat{b}$, $\hat{b}^\dagger + \hat{b}$, $\hat{b}^\dagger - \hat{b}$ we are applying number states $|n\rangle$ ($n = 0, 1, \ldots$) with the properties

\[\hat{b}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad \hat{b} |n\rangle = \sqrt{n} |n-1\rangle.\]

The number operator $\hat{N}$ is unbounded. Since $\hat{b}^\dagger \hat{b} |n\rangle = n |n\rangle$ we obtain the infinite dimensional unbounded diagonal matrix $\text{diag}(0, 1, 2, \ldots)$. Using the number states $|n\rangle$ we find the matrix representation of the unbounded operators $\hat{B} = \hat{b}^\dagger + \hat{b}$ as

\[\hat{B} = \hat{b}^\dagger + \hat{b} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.\]

Finally $\hat{b}^\dagger - \hat{b}$ is given by the matrix

\[\hat{C} = \hat{b}^\dagger - \hat{b} = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.\]

The identity operator $I$ is represented by the infinite dimensional unit matrix. Now we truncate these infinite dimensional matrices. The truncation could also be found as follows. Let $n \geq 1$ and $\{|0\rangle, |1\rangle, \ldots, |n\rangle\}$ be an orthonormal basis in $\mathbb{C}^{n+1}$. Note that

\[\sum_{\ell=0}^{n} |\ell\rangle \langle \ell| = I_{n+1}.\]

Consider the linear operators ($n \times n$ matrices)

\[b_n = \sum_{j=1}^{n} \sqrt{j} |j - 1\rangle \langle j|, \quad b_n^\dagger = \sum_{k=1}^{n} \sqrt{k} |k\rangle \langle k - 1|\]
with
\[ b_n^\dagger + b_n = \sum_{k=1}^{n} \sqrt{k} (|k - 1\rangle \langle k| + |k\rangle \langle k - 1|). \]

Then
\[
\begin{align*}
b_n b_n^\dagger &= \sum_{j=1}^{n} \sum_{k=1}^{n} \sqrt{j} \sqrt{k} (|j - 1\rangle \langle k| + |k\rangle \langle j - 1|), \\
b_n^\dagger b_n &= \sum_{k=1}^{n} \sum_{j=1}^{n} \sqrt{k} \sqrt{j} (|k - 1\rangle \langle j| + |j\rangle \langle k - 1|)
\end{align*}
\]

and we obtain the commutator
\[
[b_n, b_n^\dagger] = b_n b_n^\dagger - b_n^\dagger b_n = I_{n+1} - (n + 1)|n\rangle \langle n|.
\]

If we select the standard basis as the orthonormal basis we obtain the truncated matrices we consider in the following.

Now we truncate the infinite-dimensional matrices to \( n \times n \) matrices acting on the Hilbert space \( \mathbb{C}^n \), where \( n \geq 2 \). For \( n = 2 \) we obtain the matrices
\[
N_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad C_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2
\]
when we truncate the infinite dimensional unbounded matrices \( \hat{b}^\dagger \hat{b}, \hat{b}^\dagger + \hat{b} \) and \( \hat{b}^\dagger - \hat{b} \), where \( \sigma_1, \sigma_2, \sigma_3 \) denote the three Pauli spin matrices. For the commutator \([N_2, B_2], [N_2, C_2], [B_2, C_2]\) we find
\[
\begin{align*}
[N_2, B_2] &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = C_2 = -i\sigma_2 \\
[N_2, C_2] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B_2 = \sigma_1 \\
[B_2, C_2] &= 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2I_1 \oplus (-1) = 2\sigma_3
\end{align*}
\]
where \( I_1 \) is the \( 1 \times 1 \) identity matrix and \( \oplus \) denotes the direct sum. For \( n = 3 \) we obtain the \( 3 \times 3 \) matrices
\[
N_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}
\]
when we truncate the infinite dimensional unbounded matrices \( \hat{b}^\dagger \hat{b}, \hat{b}^\dagger + \hat{b}, \hat{b}^\dagger - \hat{b} \).
We find the commutator \([N_3, B_3], [N_3, C_3], [B_3, C_3]\) as

\[
[N_3, B_3] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = C_3
\]

\[
[N_3, C_3] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = B_3
\]

\[
[B_3, C_3] = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 2I_2 \oplus 2(-2)
\]

where \(I_2\) is the \(2 \times 2\) identity matrix. For \(n = 4\) we obtain the \(4 \times 4\) matrices

\[
N_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}
\]

when we truncate the infinite dimensional unbounded matrices \( \hat{b}^\dagger \hat{b} \) and \( \hat{b}^\dagger + \hat{b} \) to \(4 \times 4\) matrices. We find the commutators \([N_4, B_4], [N_4, C_4], [B_4, C_4]\) as

\[
[N_4, B_4] = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = C_4
\]

\[
[N_4, C_4] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = B_4
\]

\[
[B_4, C_4] = 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = 2I_3 \oplus 2(-3)
\]

where \(I_3\) is the \(3 \times 3\) identity matrix. The commutators of a truncation for arbitrary \(n\) is now obvious. We find

\[
[N_n, B_n] = C_n, \quad [N_n, C_n] = B_n, \quad [B_n, C_n] = 2I_{n-1} \oplus 2(-n + 1)
\]

where \(I_{n-1}\) is the \((n-1) \times (n-1)\) identity matrix. Thus the commutation relations for \([\hat{b}^\dagger \hat{b}, \hat{b}^\dagger + \hat{b}], [\hat{b}^\dagger \hat{b}, \hat{b}^\dagger - \hat{b}]\) are preserved for the truncation to finite dimensional matrices, whereas the commutator \([\hat{b}^\dagger + \hat{b}, \hat{b}^\dagger - \hat{b}]\) is not preserved, i.e. we do not find 2 times the \(n \times n\) identity matrix \(I_n\), but the direct sum of \(2I_{n-1}\) and \(2(-n+1)\).
3 Truncation and Spin Coherent States

Spin coherent states have been introduced by Radcliffe [6] in 1971. Arecchi et al [7] also studied spin coherent states under the name atomic coherent states in 1972. The spin coherent states for spin-$\frac{1}{2}$, spin-1, spin-$\frac{3}{2}$ are given by

$$|\theta, \phi\rangle_{1/2} = \left( \frac{\cos(\theta/2)}{\sin(\theta/2)} e^{i\phi} \right), \quad |\theta, \phi\rangle_{1} = \left( \frac{\cos^2(\theta/2)}{\sqrt{2} \cos(\theta/2) \sin(\theta/2)} e^{i\phi} \right),$$

$$|\theta, \phi\rangle_{3/2} = \left( \frac{\cos^3(\theta/2)}{\sqrt{3} \cos^2(\theta/2) \sin(\theta/2)} e^{i\phi} \right).$$

For spin-$\frac{1}{2}$ we find

$$\langle \theta, \phi|N_2\rangle|\theta, \phi\rangle = \sin^2(\theta/2), \quad \langle \theta, \phi|B_2\rangle|\theta, \phi\rangle = \sin(\theta) \cos(\phi).$$

As in the case for $\hat{b}^\dagger \hat{b}$ the expectation value is independent of $\phi$ and as in the case for $\hat{b}^\dagger + \hat{b}$ the expectation value depends on $\phi$ in the form $\cos(\phi)$. For $\theta = \pi/2$ we have $\langle \theta, \phi|B_2\rangle|\theta, \phi\rangle = \cos(\phi)$.

For spin-1 we obtain

$$\langle \theta, \phi|N_3\rangle|\theta, \phi\rangle = 2 \sin^2(\theta/2), \quad \langle \theta, \phi|B_3\rangle|\theta, \phi\rangle = \sin(\theta)(\sqrt{2} \cos^2(\theta/2) + 2 \sin^2(\theta/2)) \cos(\phi).$$

As in the case for $\hat{b}^\dagger \hat{b}$ the expectation value is independent of $\phi$ and as in the case for $\hat{b}^\dagger + \hat{b}$ the expectation value depends on $\phi$ in the form $\cos(\phi)$. For $\theta = \pi/2$ we obtain $\langle \theta, \phi|B_3\rangle|\theta, \phi\rangle = \frac{1}{\sqrt{2}}(1 + \sqrt{2}) \cos(\phi)$.

For spin-$\frac{3}{2}$ we obtain

$$\langle \theta, \phi|N_4\rangle|\theta, \phi\rangle = 3 \sin^2(\theta/2),$$

$$\langle \theta, \phi|B_4\rangle|\theta, \phi\rangle = \sin(\theta)(\sqrt{3} \cos^4(\theta/2) + 3\sqrt{2} \sin^2(\theta/2) \cos^2(\theta/2) + 3 \sin^4(\theta/2)) \cos(\phi).$$

As in the case for $\hat{b}^\dagger \hat{b}$ the expectation value is independent of $\phi$ and as in the case for $\hat{b}^\dagger + \hat{b}$ the expectation value depends on $\phi$ in the form $\cos(\phi)$. For $\theta = \pi/2$ we obtain

$$\langle \theta, \phi|B_4\rangle|\theta, \phi\rangle = \frac{\sqrt{3}}{4}(1 + \sqrt{2}\sqrt{3} + \sqrt{3}).$$
4 Truncation and Spectrum

It is well known that the spectrum of the unbounded operator $\hat{b}^\dagger + \hat{b}$ is the whole real axis $\mathbb{R}$ [8]. Truncating the matrix representation of the unbounded operator $\hat{b}^\dagger + \hat{b}$ up to the $6 \times 6$ matrices we obtain the symmetric matrices over $\mathbb{R}$

$$B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$B_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix}.$$

We find the eigenvalues and eigenvectors of these matrices. Since the matrices are symmetric over the real number the eigenvalues must be real. Furthermore the sum of the eigenvalues must be 0 since the trace of the matrices is 0 and the eigenvalues are symmetric around 0 [9]. For $B_n$ with $n$ odd one of the eigenvalues is always 0. We order the eigenvalues from largest to smallest. For $B_2$ we obtain the eigenvalues $1, -1$ with the eigenvectors

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvalues of the matrix $B_3$ are $\sqrt{3}, 0, -\sqrt{3}$ with the corresponding unnormalized eigenvectors

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1/\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} -\sqrt{3} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The eigenvalues of the matrix $B_4$ are

$$\sqrt{3 + \sqrt{6}}, \quad \sqrt{3 - \sqrt{6}}, \quad -\sqrt{3 - \sqrt{6}}, \quad -\sqrt{3 + \sqrt{6}}$$

with the corresponding unnormalized eigenvectors

$$\begin{pmatrix} 1 \\ \sqrt{3 + \sqrt{2}\sqrt{3 + 3}} \\ \frac{\sqrt{2} + \sqrt{3}}{\sqrt{3 + \sqrt{2}\sqrt{3}}} \end{pmatrix}, \quad \begin{pmatrix} \frac{\sqrt{3 - \sqrt{2}\sqrt{3}}}{\sqrt{2} - \sqrt{3}} \\ -\sqrt{3 - \sqrt{2}\sqrt{3}} \end{pmatrix}.$$
The eigenvalues of the matrix $B_5$ are

$$\sqrt{5 + \sqrt{10}}, \quad \sqrt{5 - \sqrt{10}}, \quad 0, \quad -\sqrt{5 - \sqrt{10}}, \quad -\sqrt{5 + \sqrt{10}}$$

with the corresponding unnormalized eigenvectors: The eigenvectors of $B_5$ are (for $\lambda = -\sqrt{5 + \sqrt{10}}, -\sqrt{5 - \sqrt{10}}, 0, \sqrt{5 - \sqrt{10}}, \sqrt{5 + \sqrt{10}}$)

$$\begin{pmatrix} 1 \\ -\sqrt{5 + \sqrt{10}}(4 + \sqrt{10})/\sqrt{2} \\ \sqrt{5 + \sqrt{10}(2 + \sqrt{10})}/\sqrt{6} \\ 1/\sqrt{3}(2 + \sqrt{10}) \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\sqrt{5 - \sqrt{10}}(4 - \sqrt{10})/\sqrt{2} \\ \sqrt{5 - \sqrt{10}(2 - \sqrt{10})}/\sqrt{6} \\ 1/\sqrt{3}(2 - \sqrt{10}) \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1/\sqrt{2} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ \sqrt{5 - \sqrt{10}}(4 - \sqrt{10})/\sqrt{2} \\ \sqrt{5 - \sqrt{10}(\sqrt{10} - 2))/\sqrt{6} \\ 1/\sqrt{3}(2 - \sqrt{10}) \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \sqrt{5 + \sqrt{10}}(4 + \sqrt{10})/\sqrt{2} \\ \sqrt{5 + \sqrt{10}(2 + \sqrt{10})}/\sqrt{6} \\ 1/\sqrt{3}(2 + \sqrt{10}) \end{pmatrix}.$$
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