Finiteness of integrable $n$-dimensional homogeneous polynomial potentials

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Abstract

We consider natural Hamiltonian systems of $n > 1$ degrees of freedom with polynomial homogeneous potentials of degree $k$. We show that under a genericity assumption, for a fixed $k$, at most only a finite number of such systems is integrable. We also explain how to find explicit forms of these integrable potentials for small $k$.

Key words: Hamiltonian systems; integrability; Kovalevskaya exponents; hypergeometric equation

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1 Introduction

At least half of the models which appear in physics, astronomy and other applied sciences have a form of a system of ordinary differential equations depending usually on several parameters. The question if the considered system possesses one or more first integrals is fundamental. First integrals give conservation laws for the model. Moreover, from an operational point of view, they simplify investigations of the system. In fact, we can always lower the dimension of the system by the number of its independent first integrals. If we know a sufficient number of first integrals, we can solve explicitly the considered system. As a rule, except possible obvious first integrals, as Hamiltonians for

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Hamilton’s equations, additional first integrals exist only for specific values of parameters of the considered systems. Thus, the problem is how to find these values of parameters, or, how to show that the system does not admit any additional first integral for specific values of the parameters. The problems mentioned above are generally very hard, and in spite of their basic physical importance there are no universal methods to solve them even for very special classes of differential equations.

In past the search for first integrals was based on the direct method due to Darboux, see e.g. [1]. Applying this method, we postulate a general form of the first integral. Usually, this first integral depends on some unknown functions. The condition that it is constant along solutions of the analysed system gives rise to a set of partial differential equations determining the unknown functions. Complexity of the obtained partial differential equations is the reason why it is usually assumed that the first integral is a polynomial with respect to momenta of low degree. For more information about the direct method see [2].

In the sixties of the previous century, Ablowitz, Ramani and Segur [3; 4] proposed a completely different method of searching for integrable systems. The kernel of this method, originating from the works of Kovalevskaya [5; 6] and Painlevé [7], is a conjecture that solutions of integrable systems after the extension to the complex time plane should be still simple, more precisely, single-valued. If all solutions of a given system are single-valued, then we say that it possesses the Painlevé property. But, to check if a given system possesses the Painlevé property, we must have at our disposal a single-valued particular solution of the system (or its appropriate truncation). Then the necessary condition for the Painlevé property is following: all solutions of the variational equations along this single-valued particular solution are single-valued. If for some specific values of parameters the considered system has the Painlevé property, then, assuming those values of parameters, we can look for first integrals applying the direct method. This means that the Painlevé property has played the role of necessary integrability conditions and, for this reason, it is sometimes called the Painlevé test. The results of Kovalevskaya and Lapunov [5; 6; 8] showed that checking the Painlevé property is in fact reduced to checking if a certain matrix, the so-called Kovalevskaya matrix (see the next section), is semisimple, and its eigenvalues, the so-called Kovalevskaya exponents, are integers. Yoshida [9] showed that Kovalevskaya exponents are related to the degrees of first integrals, and this fact simplifies the second step of the analysis, namely, finding the explicit form of the first integral. The Painlevé test appeared to be very effective and many new integrable systems were found thanks to its application. The main advantage of this method is its simplicity. Its weak point is the fact that there is no rigorous proof that the Painlevé property is directly related to the integrability. In fact, there are known examples of integrable systems that do not pass the Painlevé test, by
this reason the weak Painlevé test was introduced, see [10; 11; 12].

Let us remark that in Hamiltonian mechanics there exist a few other tools for testing the integrability, see [13], however, they usually work for very restricted classes of Hamiltonian systems.

Quite recently two mathematically rigorous approaches to the integrability problem formulated by Ziglin [14; 15] and Morales-Ruiz and Ramis [16; 17] have appeared. They explain relations between the existence of first integrals and branching of solutions as functions of the complex time and give necessary integrability conditions for Hamiltonian systems. It appears that the integrability is related to properties of the monodromy group or the differential Galois group of variational equations along a particular solution.

In this paper we apply the Morales-Ramis approach to the Hamiltonian systems defined in a linear symplectic space, e.g., \( \mathbb{R}^{2n} \) or \( \mathbb{C}^{2n} \) equipped with canonical variables \( \mathbf{q} = (q_1, \ldots, q_n), \mathbf{p} = (p_1, \ldots, p_n) \), and given by a natural Hamiltonian function

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(\mathbf{q}).
\]  

(1)

We assume that \( V(\mathbf{q}) \) is a homogeneous polynomial of degree \( k > 2 \). The integrability of Hamiltonian systems with Hamiltonian (1) was analysed by the direct method, the Painlevé analysis and some other techniques, see [18; 2; 11; 10]. Nevertheless, a quick overview of the literature shows that except for some “easy” cases only sporadic examples of integrable systems with two or three degrees of freedom governed by the Hamiltonian of the form (1) were found. In all integrable cases first integrals are polynomials and their degrees with respect to the momenta are not greater than four. Hence, it is natural to ask: do we know all integrable systems with Hamiltonian (1)? It is hard to believe that the answer to this question is positive. In fact, as far as we know, in all works only very limited families of such systems were investigated. Thus, what can we expect? Are there infinitely many integrable Hamiltonian systems which wait to be discovered?

The aim of this note is to give a necessarily limited answer to the above question. The main result of this paper shows that assuming that potential \( V \) is generic, the number of meromorphically integrable systems with Hamiltonian (1) is finite.

Let us explain here what does it mean a generic potential. Hamilton’s equations generated by (1) admit particular solutions of the form

\[
\mathbf{q}(t) = \varphi(t) \mathbf{d}, \quad \mathbf{p}(t) = \dot{\varphi}(t) \mathbf{d},
\]  

(2)
provided \( \mathbf{d} \) is a nonzero solution of
\[
V'(\mathbf{d}) = \mathbf{d},
\]
and \( \varphi(t) \) satisfies \( \ddot{\varphi} = -\varphi^{k-1} \). A direction \( \mathbf{d} \in \mathbb{C}^n \) defined by a solution of (3) is called a Darboux point of potential \( V \). We say that potential \( V \) is generic iff it admits exactly \( k \) different Darboux points. For details see Section 2.

To prove our finiteness result we combine the Morales-Ramis theory and a kind of global Kovalevskaya analysis of the auxiliary system
\[
\frac{d}{dt} \mathbf{q} = V'(\mathbf{q}).
\]

It appears that the Kovalevskaya exponents of the above system are closely related to the integrability of Hamiltonian system given by (1). The Morales-Ramis theory gives strong restrictions on their values. On the other hand, we can calculate the Kovalevskaya exponents for different particular solutions of (3). The key point is the fact that the Kovalevskaya exponents calculated for different solutions are not arbitrary, i.e., there exist certain relations among them.

Just to avoid misunderstanding let us fix terminology here. We consider complex Hamiltonian systems with phase space \( \mathbb{C}^{2n} \) equipped with the standard canonical structure. First integrals are always assumed to be meromorphic in appropriate domains. By saying that a potential \( V \) is integrable, we understand that the Hamilton equations generated by Hamiltonian (1) are integrable in the Liouville sense. It is easy to check that if potential \( V(\mathbf{q}) \) is integrable, then also \( V_A(\mathbf{q}) := V(A\mathbf{q}) \) is integrable for an arbitrary \( A \in \text{GL}(n, \mathbb{C}) \) satisfying \( AA^T = \alpha E \), \( \alpha \in \mathbb{C}^* \) and \( E \) is the identity matrix, see e.g. [2]. Potentials \( V \) and \( V_A \) are called equivalent, and the set of all potentials is divided into disjoint classes of equivalent potentials. Later a potential means a class of equivalent potentials in the above sense.

The plan of this paper is following. In the next section we briefly recall basic facts from the Kovalevskaya analysis and the Morales-Ramis theory. In Section 3 we formulate and prove our main results. In the last section we explain how our approach can be used for a systematic analysis of the integrability of homogeneous potentials.

2 Darboux points, Kovalevskaya exponents and Morales-Ramis theory

At the beginning we remind basic notions of the Kovalevskaya-Painlevé analysis, for more details and references see [13].
Let us consider a polynomial system
\[
\frac{d}{dt} x = F(x), \quad x \in \mathbb{C}^n, \tag{5}
\]
with homogeneous right hand sides $F = (F_1, \ldots, F_n)$, $\text{deg } F_i = k$, $k > 1$, for $i = 1, \ldots, n$. A direction $d \in \mathbb{C}^n$ (i.e., a non-zero vector) is called a Darboux point of system (5) if $d$ is parallel to $F(d)$ and $F(d) \neq 0$. Note that a Darboux point can be considered as a point $[d_1 : \cdots : d_n]$ in projective space $\mathbb{C}P^{n-1}$. The set of all Darboux points for system (5) is denoted by $\mathcal{D}_F$. It can be empty, finite or infinite. If all Darboux points are isolated, then $\mathcal{D}_F$ is finite, and in a generic case it has $D(n, k) := (k^n - 1)/(k - 1)$ elements, see Proposition 4 on page 348 in [20]. We always normalise Darboux points of system (5) in such a way that they satisfy the following nonlinear equations
\[
F(d) = d, \tag{6}
\]
but we must remember that different solutions of the above equations can define the same Darboux point.

**Remark 1.** Let us notice that if $d \neq 0$ is a solution of equation (6), then by homogeneity of $F$, also $\tilde{d} := \epsilon d$ is a solution of this equation provided $\epsilon$ is a $(k - 1)$-th root of the unity. Thus if equation (6) has $m$ different solutions, then they define only $m/(k - 1)$ different Darboux points.

The Kovalevskaya matrix $K(d)$ at a Darboux point $d \in \mathcal{D}_F$ is defined as
\[
K(d) := F'(d) - E, \tag{7}
\]
where $F'(d)$ is the Jacobian matrix of $F$ calculated at $d$. Eigenvalues $\Lambda_i = \Lambda_i(d), i = 1, \ldots, n$, of the Kovalevskaya matrix $K(d)$ are called the Kovalevskaya exponents. Using the homogeneity of $F$ it is easy to prove that one of the Kovalevskaya exponents, let us say $\Lambda_n$, is $k - 1$. We call this eigenvalue trivial.

If the general solution of system (5) is single-valued, then the Kovalevskaya exponents should be integer. However, as we have already mentioned the Painlevé test is not a correct integrability condition. The first strict relation between the existence of a first integral and the Kovalevskaya exponents was found by Yoshida [9], who proved that if system (5) possesses a polynomial first integral whose gradient does not vanish at the Darboux point, then the degree of the homogeneity of this first integral belongs to the spectrum of the Kovalevskaya matrix. This result was later generalised in [21; 22; 23] and the final relation is the following. If system (5) possesses a polynomial or rational first integral, then the Kovalevskaya exponents calculated at a certain Darboux point satisfy a resonance relation. However, in this paper we do not use this connection between the Kovalevskaya exponents and the integrability
in the class of polynomial or rational functions but we use a stronger result concerning the integrability in the wider class of meromorphic functions.

As we said, the idea of Kovalevskaya gave a strong impulse for searching a relation between the integrability and branching of solutions as functions of the complex time. This, somewhat mysterious, relation was fully explained for Hamiltonian systems by an elegant and powerful theory of S. L. Ziglin [14, 15]. The basic idea of this theory is following. To have a chance to describe possible branching, we need a particular single-valued solution \( \varphi(t) \) of the considered system. Knowledge about solutions close to \( \varphi(t) \) comes from the variational equations along \( \varphi(t) \). The monodromy group of these equations describes branching of solutions close to \( \varphi(t) \). The existence of integrals of the system puts a restriction on the monodromy group—it cannot be too “big”. At the end of the previous century, the strength of the Ziglin theory was considerably improved thanks to the application of the differential Galois theory [24]. The Morales-Ramis theory, see [17], is a kind of an algebraic version of the Ziglin theory—instead of the monodromy group the differential Galois group of the variational equations is used to find obstructions for the integrability. The main theorem of the Morales-Ramis theory states that if the investigated system is integrable in the Liouville sense, then the identity component of the differential Galois group of the variational equations along a particular solution is Abelian, see [16, 17].

Morales-Ruiz and Ramis used their theory to give the strongest known necessary conditions for the integrability of Hamiltonian systems with the homogeneous potential (1). Let us describe them shortly. To apply the Morales-Ramis theory we need a particular solution. As we explained in Introduction assuming that the considered potential has a Darboux point, such a solution is given by (3). It gives a family of phase curves \( \Gamma_\varepsilon \) of the form

\[
\dot{\varphi}^2 = \frac{2}{k}(\varepsilon - \varphi^k), \quad \varepsilon \neq 0
\]

These curves are elliptic for \( k = 3, 4 \) and hyperelliptic for \( k > 4 \).

It is easy to show that the variational equations along solution (3) have the form \( \ddot{x} = -\varphi(t)^{k-2}V''(d)x \). Thus, assuming that the Hessian matrix \( V''(d) \) is diagonalisable, we can find coordinates \((y_1, \ldots, y_n)\) such that in these variables equations read \( \ddot{y}_i = -\lambda_i \varphi(t)^{k-2}y_i \), for \( i = 1, \ldots, n \), where \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( V''(d) \). As was observed by Yoshida [25], the following change of the independent variable \( t \to z := \varphi(t)^k/\varepsilon \) transforms \( i \)-th variational equation into the Gauss hypergeometric equation with parameters dependent on \( k \) and \( \lambda_i \). But, for the hypergeometric equation the monodromy, as well as the differential Galois groups, are well known, see e.g., [26]. Basing on these facts, J. J. Morales-Ruiz and J. P. Ramis formulated in [16] a general theorem concerning the integrability of Hamiltonian systems with a homogeneous
potential. Here, we formulate this theorem for a polynomial homogeneous potential.

**Theorem 1.** If the Hamiltonian system given by (1) with the polynomial homogeneous potential \( V(q) \) of degree \( k > 2 \) is meromorphically integrable in the Liouville sense, then values of \((k, \lambda_i)\) for \( i = 1, \ldots, n \) belong to the following list

1. \( \left( k, \frac{k}{2}p(p - 1) + p \right) \),
2. \( \left( k, \frac{k - 1}{2k} + p(p + 1)\frac{k}{2} \right) \),
3. \( \left( 3, \frac{1}{6} (1 + 3p)^2 - \frac{1}{24} \right) \),
4. \( \left( 3, \frac{3}{32} (1 + 4p)^2 - \frac{1}{24} \right) \),
5. \( \left( 3, \frac{3}{50} (1 + 5p)^2 - \frac{1}{24} \right) \),
6. \( \left( 3, \frac{3}{50} (2 + 5p)^2 - \frac{1}{24} \right) \),
7. \( \left( 4, \frac{2}{9} (1 + 3p)^2 - \frac{1}{8} \right) \),
8. \( \left( 5, \frac{5}{18} (1 + 3p)^2 - \frac{9}{40} \right) \),
9. \( \left( 5, \frac{1}{10} (2 + 5p)^2 - \frac{9}{40} \right) \),

where \( p \) is an integer.

Let us notice that one eigenvalue of \( V''(d) \), let us say \( \lambda_n \), is \( k - 1 \), so it does not give any restriction to the integrability. For a typical situation when the investigated potential depends on some parameters, using the above theorem, we are able to distinguish infinite families (depending on parameters) of potentials which are suspected to be integrable.

**Remark 2.** If \( V'(d) = d \), then \( \tilde{d} = \gamma d \) satisfies \( V' (\tilde{d}) = \gamma^{k-2}d \), and using \( \tilde{d} \) we can find a particular solution as we did with \( d \). Although eigenvalues of \( V''(d) \) and \( V''(\tilde{d}) \) are different, we do not obtain a new restriction for the integrability. The reason of this is the fact that \( d \) and \( \tilde{d} \) define the same phase curves.

The above remark justifies introducing the notion of a Darboux point of a homogeneous polynomial potential \( V \). We say that \( d \in C^n \) is a Darboux point of a \( V \in C[q] \), if it is a Darboux point of the auxiliary system

\[
\frac{d}{dt}q = V'(q).
\]  

We always normalise coordinates of a Darboux point \( d \) in such way that they satisfy [3]. We denote by \( D_V \) the set of Darboux points of system [3]. Notice that for \( d \in D_V \), the Kovalevskaya exponents \( \Lambda_i(d) \) are given by \( \Lambda_i(d) = \lambda_i(d) - 1 \), where \( \lambda_i(d) \) are eigenvalues of \( V''(d) \).
3 Main result

It is obvious that the more Darboux points we have for a given potential, the more obstructions for its integrability follow from Theorem 1. Investigating the integrability of two dimensional potentials, see [27, 28], we noticed the following fact. For a generic case the number of Darboux points is \( k = \deg V \). We have two eigenvalues of \( V''(d_i) \) for every \( d_i \in D_V \). One of them is \( k - 1 \), the remaining one we denote by \( \lambda_i \). It appears that non-trivial eigenvalues \( \lambda_1, \ldots, \lambda_k \) calculated at all Darboux points are not independent, i.e., they satisfy a certain relation. This relation has the simplest form if we express it in terms of \( \Lambda_i = \lambda_i - 1 \), i.e., in terms of the Kovalevskaya exponents of the auxiliary system (8), and it reads

\[
\sum_{i=1}^{k} \frac{1}{\Lambda_i} = -1. \tag{9}
\]

Now, if the system is integrable, then \( \Lambda_i = \lambda_i - 1 \) take values determined by Theorem 1. In [28] we showed that for an arbitrary \( k > 2 \) there exist at most a finite number of \( \Lambda_1, \ldots, \Lambda_k \), satisfying this requirement. Moreover, from our considerations in [28] it follows that the number of potentials of degree \( k \) which have specified values of \( (\Lambda_1, \ldots, \Lambda_k) \), is finite. All the above facts imply that for a fixed \( k > 2 \), the number of integrable homogeneous potentials of degree \( k \) with the maximal number of Darboux points is finite. Thus, it is natural to ask if we can prove a similar fact for a case when \( n > 2 \). Unfortunately, the methods we used in [28] are applicable only when \( n = 2 \). Now our aim is to show how to overcome this difficulty.

A theorem proved in [20] plays the central role in our considerations. Here we formulate it in a form adapted to our needs. Let us return to a general first order homogeneous system (5) and assume that it has the maximal number of Darboux points. As it was mentioned, one of the Kovalevskaya exponents at an arbitrary Darboux point \( d \) is \( k - 1 \); we denote the remaining ones by \( \Lambda(d) = (\Lambda_1(d), \ldots, \Lambda_{n-1}(d)) \). Let \( \tau_i \) for \( 0 \leq i \leq n - 1 \), denote the elementary symmetric polynomial in \( (n - 1) \) variables of degree \( i \) i.e.

\[
\tau_r(x) := \tau_r(x_1, \ldots, x_{n-1}) = \sum_{1 \leq i_1 < \cdots < i_r \leq n-1} \prod_{s=1}^{r} x_{i_s}, \quad 1 \leq r \leq n - 1.
\]

and \( \tau_0(x) := 1 \). The theorem below is in fact a reformulation of Corollary 12 on page 359 in [20].

**Theorem 2.** Assume that system (5) with homogeneous polynomial right hand sides of degree \( k \) has the maximal number of Darboux points and let \( S \) be a symmetric homogeneous polynomial in \( n - 1 \) variables of degree less than
Then, the number

\[ R := \sum_{d \in \mathcal{D}} \frac{S(\Lambda(d))}{\tau_{n-1}(\Lambda(d))}. \tag{10} \]

depends only on the choice of \( S \), dimension \( n \) and homogeneity degree \( k \).

In other words, \( R = R(S, n, k) \) does not depend on a specific choice of \( F \), provided that \( F \) has the maximal number of Darboux points. The above theorem shows that there exist \( n - 1 \) different universal relations among “non-trivial” Kovalevskaya exponents calculated at all Darboux points. In order to use the above theorem effectively, we have to know the values of \( R(S, n, k) \) for arbitrary \( n, k \) and a chosen set of \( n \) independent symmetric homogeneous polynomials \( S_i \) of degree \( i \) for \( i = 0, \ldots, n - 1 \). In [20] one can find these values for \( n = 3 \) and \( k = 2 \). Fortunately, the method used in [20] works also in the general case. To calculate \( R(S, n, k) \) it is enough to choose a system for which one can easily determine the Kovalevskaya exponents, but the system must be defined for arbitrary \( n > 2 \) and \( k > 2 \), and, of course, it must have the maximal number of Darboux points. These requirements are satisfied by the \( n \)-dimensional generalisation of the Jouanolou system

\[ \dot{x}_i = x_{i+1}^k, \quad 1 \leq i \leq n, \quad x_{n+1} \equiv x_i, \tag{11} \]

see [29]. For this system the Kovalevskaya exponents do not depend on a Darboux point and can be written explicitly. We show this in the following lemma.

**Lemma 1.** Let \( \Lambda = (\Lambda_1, \ldots, \Lambda_{n-1}) \) denotes the non-trivial Kovalevskaya exponents calculated at a Darboux point of system (11). Then the elementary symmetric polynomials of \( \Lambda \) take the following values

\[ \tau_r(\Lambda) = (-1)^r \sum_{i=0}^{r} \binom{n-i-1}{r-i} k^i, \tag{12} \]

for \( 0 \leq r \leq n - 1 \).

**Proof.** Solutions \( d = (d_1, \ldots, d_n) \) of equation (6) describing Darboux points can be written as

\[ d_n = s, \quad d_{n-1} = s^k, \quad d_{n-2} = s^{k^2}, \ldots, d_2 = s^{k^{n-2}}, \quad d_1 = s^{k^{n-1}}, \tag{13} \]

where \( s \) is a primitive root of unity of degree \( k^{n-1} \), i.e. \( s \) is a solution of the cyclotomic equation

\[ s^{k^{n-1}} - 1 = 0. \tag{14} \]

Equation (13) has \( k^n - 1 \) complex solutions, hence by Remark II system (11) has \( (k^n - 1)/(k - 1) \) Darboux points. The Kovalevskaya matrix at point \( d \) has
the form

\[
K(d) = \begin{pmatrix}
-1 & kd_2^{k-1} & 0 & \cdots & 0 \\
0 & -1 & kd_3^{k-1} & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & -1 & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & kd_n^{k-1} \\
kd_1^{k-1} & 0 & \cdots & 0 & -1 & -1
\end{pmatrix}.
\]

Thus its characteristic polynomial is

\[
P(\Lambda) = \det (K(d) - \Lambda E) = (-1)^n(\Lambda + 1)^n + (-1)^{n-1}k^n d_1^{k-1} \cdots d_n^{k-1}
= (-1)^n[(\Lambda + 1)^n - k^n].
\]  \hspace{1cm} (15)

In this way we showed that all Darboux points have the same Kovalevskaya exponents given by

\[
\Lambda_i = k\varepsilon^{n-i} - 1, \quad 1 \leq i \leq n,
\]  \hspace{1cm} (16)

where \(\varepsilon\) is a primitive \(n\)-th root of the unity. In order to find elementary symmetric functions of nontrivial eigenvalues \(\Lambda := (\Lambda_1, \ldots, \Lambda_{n-1})\) we factorise characteristic polynomial (15) in the following way

\[
P(\Lambda) = (-1)^n(\Lambda + 1 - k)Q(\Lambda, n, k),
\]

where

\[
Q(\Lambda, n, k) = \sum_{p=0}^{n-1} \sum_{i=p}^{n-1} \binom{i}{p} k^{n-1-i} \Lambda_p = \sum_{q=0}^{n-1} \sum_{i=n-1-q}^{n-1} \binom{i}{n-1-q} (n-1-q) k^{n-1-i} \Lambda^{n-1-q}.
\]

Thus symmetric functions \(\tau_r(\Lambda)\) are up to the sign coefficients of the above polynomial

\[
\tau_r(\Lambda) = (-1)^r \sum_{i=n-1-r}^{n-1} \binom{i}{n-1-r} k^{n-1-i}, \quad r = 0, \ldots, n-1.
\]

From the above formula we obtain (12) by a simple change of indices. \(\square\)

By the above lemma we have in particular \(\tau_1(\Lambda) = 1 - n - k\), and

\[
\tau_{n-1}(\Lambda) = (-1)^{n-1} \frac{k^n - 1}{k - 1}, \quad \tau_{n-2}(\Lambda) = (-1)^{n} \frac{k^n - n(k - 1) - 1}{(k - 1)^2}.
\]  \hspace{1cm} (17)

Knowing (12) we can calculate \(R(S, n, k)\) for an arbitrary choice of \(S\). In what follows we need explicit formulas for \(S = \tau_1^r\) and \(S = \tau_r\) for \(0 \leq r \leq n - 1\).
**Proposition 1.** For $0 \leq r \leq n - 1$ we have
\[
R(\tau^r, n, k) = (-1)^{n-1}(1-n-k)^r, \tag{18}
\]
and
\[
R(\tau, n, k) = (-1)^{r+1-n} \sum_{i=0}^{r} \binom{n-i-1}{r-i} k^i. \tag{19}
\]

**Proof.** It is enough to insert formulas (12) into (10) and perform elementary simplification. \[\square\]

Let us notice that in particular we have
\[
R(\tau_{n-2}, n, k) = -\frac{k^n - n(k-1) - 1}{(k-1)^2}. \tag{20}
\]

Now, the question is if we can use the above facts for our problem. As it was mentioned, the integrability conditions of a homogeneous potential are given in terms of the Kovalevskaya exponents of the auxiliary gradient system (8). The $\mathbb{C}$-linear space of polynomial homogeneous systems of a given degree has the dimension greater than the space of polynomial homogeneous gradient systems of the same degree. Thus it seems to be possible that the number of isolated Darboux points of system (8) with the potential of degree $k$ is always smaller than $D(n, k-1)$. But it is not like that—there exist potentials of degree $k$ which have $D(n, k-1)$ Darboux points. The simplest example is following
\[
V_0 = \sum_{i=1}^{n} q_i^k. \tag{21}
\]

We prove that potentials of degree $k$ with $D(n, k-1)$ Darboux points are generic. Let us precise the meaning of a generic potential. The $\mathbb{C}$-linear space $\mathcal{H}_k$ of all homogeneous polynomials of degree $k$ has dimension
\[
d = \binom{n+k-1}{k}.
\]

Let us fix an monomial ordering $\prec$ of variables $q_1, \ldots, q_n$. Then every homogeneous polynomial $V$ of degree $k$ can be uniquely written in the form
\[
V = \sum_{i=1}^{d} v_i q^{\alpha_i},
\]
where $q^{\alpha_1} \prec \cdots \prec q^{\alpha_d}$ are all monomials of degree $k$. Hence we identify $\mathcal{H}_k$ with $\mathbb{C}^d$ identifying $V$ with $(v_1, \ldots, v_d) \in \mathbb{C}^d$. We convert $\mathcal{H}_k$ into a complete normed space fixing in $\mathbb{C}^d$ an arbitrary norm. Now we show the following.
Lemma 2. Let $G_k \subset H_k$ be a set of all homogeneous potentials of degree $k > 2$ such that

1. if $V \in G_k$, then $V$ has maximal number of Darboux points $d_1, \ldots, d_s$, where
   $$s = D(n, k - 1) := \frac{(k - 1)^n - 1}{k - 2},$$
   and
2. for each Darboux point all the Kovalevskaya exponents are different from zero.

Then set $G_k$ is open and non-empty.

Proof. First we show that $H_k$ is not empty. It is an easy exercise to check that $V_0$ satisfies condition (1) and (2) so it is an element of $G_k$.

To prove that that $G_k$ is open we have to show that for every $V \in G_k$ all potentials close enough to $V$ also belongs to $G_k$. To this end we notice that a Darboux point $d$ of $V \in H_k$ is a zero of

$$G(q) := V'(q) - q.$$  \hspace{1cm} (22)

We claim that if $V \in G_k$, then a Darboux point $d$ of $V$ is an isolated zero of $G$. In fact, the Jacobian of $G$ calculated at $d$ is not singular as $G'(d) = K(d)$ and by assumption $\det K(d) \neq 0$. Let $V \in G_k$ and

$$W = \sum_{i=1}^{d} \varepsilon_i q^{\alpha_i}.$$ Darboux points of $V + W$ are solutions $d(\varepsilon)$ of

$$G(d, \varepsilon) := V'(d) + W'(d) - d = 0, \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d).$$

By assumption that $V \in G_k$ we have that for $\varepsilon = 0$ the above equation has $s$ isolated solutions $d(0)$. Hence for $\|\varepsilon\|$ small enough there exist $s$ solutions $d(\varepsilon)$. This exactly means that for an arbitrary $V \in G_k$ there exists an open subset of $G_k$ containing $V$, i.e., $G_k$ is open. \hfill $\square$

Thanks to the above lemma we can apply directly Theorem 2 to the auxiliary system and this gives the following.

Theorem 3. Let us assume that a homogeneous polynomial potential $V \in \mathbb{C}[q]$ of degree $k$ has $D(n, k - 1)$ Darboux points $d \in D_V$. Then non-trivial Kovalevskaya exponents $\Lambda(d)$ satisfy the following relations:

$$\sum_{d \in D_V} \frac{\tau_1(\Lambda(d))^r}{\tau_{n-1}(\Lambda(d))} = (-1)^{n-1}(2 - n - k)^r,$$ \hspace{1cm} (23)
or, alternatively
\[
\sum_{d \in D_V} \frac{\tau_r(\Lambda(d))}{\tau_{n-1}(\Lambda(d))} = (-1)^p \sum_{i=0}^{r} \binom{n-i-1}{r-i}(k-1)^i, \tag{24}
\]
for \(0 \leq r \leq n - 1\).

Now, if a Hamiltonian system (1) with potential \(V\) satisfying assumptions of the above theorem is integrable, then \(\Lambda_i(d)\) for \(1 \leq i \leq n - 1\) and \(d \in D_V\) take rational values given in Theorem 1 and satisfy relations (23) and (24). These are really strong restrictions. This fact is shown by the following theorem.

**Theorem 4.** Among Hamiltonian systems given by (1) with homogeneous potentials of fixed degree \(k > 2\) admitting the maximal number of Darboux points only a finite number is integrable.

In order to prove this theorem we recall Lemma B.1 from [28].

**Lemma 3.** Let us consider the following equation
\[
X_1 + \cdots + X_m = c, \quad c > 0, \tag{25}
\]
and look for its solutions \(X = (X_1, \ldots, X_m) \in \mathcal{X}^m\) where \(\mathcal{X}\) is a set of all sequences \(\{x_i\}_{i \in \mathbb{N}}\) of non-negative real numbers such that \(\lim_{n \to \infty} x_n = 0\). Then for an arbitrary \(c > 0\) equation (25) has at most a finite number of solutions in \(\mathcal{X}^m\).

To prove Theorem 4 we only need one relation given in Theorem 3. Namely, relation (24) for \(r = n - 2\) reads
\[
\sum_{d \in D_V} \frac{n-1}{\Lambda_i(d)} = -\frac{(k-1)^n - n(k-2) - 1}{(k-2)^2}. \tag{26}
\]

Let us define \((n-1)D(n, k-1)\) quantities \(X_i = 1/\Lambda_i(d)\) where \(i = 1, \ldots, n - 1\) and \(d \in D_V\). Then from Theorem 1 it follows that for a fixed \(k\), \(X_i\) belong to an appropriate set \(\mathcal{X}_k\) possessing the properties

1. \(\mathcal{X}_k = \{x^{(k)}_n \in \mathbb{Q} \setminus \{0\} \mid x^{(k)}_n \in (-\infty, -1) \cup (0, \infty), \ n \in \mathbb{N}\}\),
2. For each \(k\) the sequence \(\{x^{(k)}_n\}\) has only one accumulation point at 0. In particular, for each \(k\) only a finite number of \(x^{(k)}_n\) take negative values not greater than -1.

From relation (26) it follows that at least one of \(X_i\) is negative. However, if \(X_i\) is negative, then it cannot be greater than -1. Hence, not all of \(X_i\) are negative. So assume that \(X_i\) for \(i = m + 1, \ldots, l\), where \(l = (n-1)D(n, k-1)\), are negative for some \(0 < m < l\). There is only finitely many choices for \(X_{m+1}, \ldots, X_l\). For each of them we can rewrite relation (26) in the form (25).
But, by the above lemma, this means that (26) has only a finite number of solutions.

Let us note that for \( n = 2 \) relation (26) transforms into (9).

4 Discussion and Comments

Fact that there exist some relations between the Kovalevskaya exponents was observed earlier in a study of Painlevé property of multi-parameter systems, see e.g. relation (3.12) in [30]. For two degrees of freedom and a homogeneous potential of degree 3, one can find all Darboux points and the respective Kovalevskaya exponents explicitly and then check directly that relation (9) holds. Exactly in this way relation (9) was found for \( k = 3 \) in [27]. However this method fails for \( k > 4 \) as there is no way to find explicit solutions of non-linear equations (3). For an arbitrary \( k \) relation (9) was found in [28] but the method used there cannot be directly generalised to higher dimensional systems. Nevertheless it allows to find certain relations between the Kovalevskaya exponents for non-generic potentials.

Let us underline that Theorem 4 is only one, and not the most important, consequence of relations (23) and (24). These relations give also a possibility to investigate completely the integrability problem of potentials with a given degree \( k \) in an algorithmic way. At first, for a given \( k \) we have to find all solutions \( \Lambda_{i,j} = \Lambda_i(d_j) \), \( i = 1,\ldots,n-1, \ j = 1,\ldots,D(n,k-1) \) of relations (23) or (24) such that the corresponding quantities \( \lambda_{i,j} = \Lambda_{i,j} + 1 \) belong to an item in the table given in Theorem 1. Such solutions are called admissible. For this a computer algebra program is needed. For \( n = 2 \) and relatively small values of \( k \) such solutions can be find quickly. Tables 1 and 2, taken from [27,28] give all such solutions for \( k = 3 \) and \( k = 4 \), respectively.

\[
\begin{align*}
\{\Lambda_1,\Lambda_2,\Lambda_3\} \\
\{-1,-1,1\} \\
\{-2/3,4,4\} \\
\{-7/8,14,14\} \\
\{-2/3,7/3,14\}
\end{align*}
\]

Table 1
Admissible solutions of (9) for \( k = 3 \).
\{A_1, A_2, A_3, A_4\}
\{-1, -1, 2, 2\}
\{-5/8, 5, 5, 5\}
\{-5/8, 2, 20, 20\}
\{-5/8, 27/8, 27/8, 135\}
\{-5/8, 2, 14, 35\}

Table 2
Admissible for solutions of (9) \(k = 4\).

It must be said however that for bigger values of \(k\) finding all admissible solutions starts to be a computer time demanding problem. For \(k > 5\) it is known that relation (9) has always at least two solutions.

\[ A_k^{(1)} = \{-1, -1, (k - 2), \ldots, k - 2\}, \quad A_k^{(2)} = \left\{-\frac{k + 1}{2k}, k + 1, \ldots, k + 1\right\}. \]

However for \(k = 14, 17, 19, 26, \ldots\) additional solutions appear, see [31]. For \(n > 2\) finding all admissible solutions of (23) or (24) is a very difficult problem even for \(k = 3\).

In the next step we have to find the potentials which give admissible Ko- valevskaya exponents. As we have already mentioned, for a given admissible solution \(\{\Lambda_{ij}\}\) the number of the corresponding non-equivalent potentials is finite. A procedure of potentials reconstruction reduces to finding all solutions of a system on polynomial equations. For \(n = 2\) and small \(k\) this problem can be solved explicitly. For bigger values of \(k\) we do not know if it is possible to find explicit solutions. For \(n > 2\) finding all admissible solutions of (23) or (24) is a very difficult problem even for \(k = 3\). The algorithm applied for \(n = 2\) is useless and new more efficient one is needed. For \(A_k^{(1)}\) and \(A_k^{(2)}\) one can find the corresponding potentials solving certain linear differential equations, for details see [31].

Among selected potentials we can find integrable, as well as not integrable ones. At this point we need a tool stronger than Theorem 1 to prove that those non-integrable are really non-integrable. Fortunately, there exists such a tool. It is yet another theorem due to Morales and Ramis which says that if the system is integrable, then the identity component of the differential Galois group of the \(i\)-th order variational equations is Abelian for any \(i \in \mathbb{N}\). For more details see [17, 32]. However this theorem can be used effectively only when \(k = 3\) or \(k = 4\). For example of applications and details see [27].

We have already mentioned that the integrability is a highly non-generic phe-
nomenon. Our Theorem 4 concerns only potentials with the maximal number of Darboux points. But of course there exist potentials with not isolated Darboux points, potentials which have no the maximal number of isolated Darboux points, and potentials without Darboux points. Among them one can find integrable ones. An example of an integrable potential without Darboux points is given in [33]. In this example the additional first integral is of the fourth degree with respect to the momenta. Nevertheless, we conjecture that Theorem 4 is true without any assumptions.

In the end we mention that relations (23) and (24), and their generalisations for nongeneric cases, can be derived in a way different from that used in [20]. It is exposed, together with an integrability analysis of three dimensional potentials, in our forthcoming paper [34].

References

[1] E. T. Whittaker, G. N. Watson, A Treatise on the Analytical Dynamics of Particle and Rigid Bodies with an Introduction to the Problem of Three Bodies, 4th Edition, Cambridge University Press, London, 1965.
[2] J. Hietarinta, Direct methods for the search of the second invariant, Phys. Rep. 147 (2) (1987) 87–154.
[3] M. J. Ablowitz, A. Ramani, H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type. I, J. Math. Phys. 21 (4) (1980) 715–721.
[4] M. J. Ablowitz, A. Ramani, H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type. II, J. Math. Phys. 21 (5) (1980) 1006–1015.
[5] S. Kowalevski, Sur le problème de la rotation d’un corps solide autour d’un poit fixe, Acta Math. 12 (1889) 177–232.
[6] S. Kowalevski, Sur une propriétée du système d’équations différentielles qui définit la rotation d’un corps solide autour d’un poit fixe, Acta Math. 14 (1890) 81–93.
[7] P. Painlevé, Sur les équations différentielles du second ordre et d’ordre supérieur dont l’intégrale générale est uniforme, Acta Math. 25 (1902) 1–85.
[8] A. M. Lyapunov, On a certain property of the differential equations of the problem of motion of a heavy rigid body, having a fixed point., Soobshch. Har’kovsk. Mat. Obschh. 4 (3) (1894) 123–140, in Russian.
[9] H. Yoshida, Necessary condition for the existence of algebraic first integrals. I. Kowalevski’s exponents, Celestial Mech. 31 (4) (1983) 363–379.
[10] A. Ramani, B. Dorizzi, B. Grammaticos, Painlevé conjecture revisited, Phys. Rev. Lett. 49 (21) (1982) 1539–1541.
[11] A. Ramani, B. Grammaticos, T. Bountis, The Painlevé property and
singularity analysis of integrable and nonintegrable systems, Phys. Rep. 180 (3) (1989) 159–245.

[12] B. Grammaticos, A. Ramani, Integrability—and how to detect it, in: Integrability of nonlinear systems (Pondicherry, 1996), Vol. 495 of Lecture Notes in Phys., Springer, Berlin, 1997, pp. 30–94.

[13] V. V. Kozlov, Symmetries, Topology and Resonances in Hamiltonian Mechanics, Springer-Verlag, Berlin, 1996.

[14] S. L. Ziglin, Branching of solutions and non-existence of first integrals in Hamiltonian mechanics. I, Functional Anal. Appl. 16 (1982) 181–189.

[15] S. L. Ziglin, Branching of solutions and non-existence of first integrals in Hamiltonian mechanics. II, Functional Anal. Appl. 17 (1983) 6–17.

[16] J. J. Morales-Ruiz, J. P. Ramis, A note on the non-integrability of some Hamiltonian systems with a homogeneous potential, Methods Appl. Anal. 8 (1) (2001) 113–120.

[17] J. J. Morales Ruiz, Differential Galois theory and non-integrability of Hamiltonian systems, Vol. 179 of Progress in Mathematics, Birkhäuser Verlag, Basel, 1999.

[18] J. Hietarinta, A search for integrable two-dimensional Hamiltonian systems with polynomial potential, Phys. Lett. A 96 (6) (1983) 273–278.

[19] M. Lakshmanan, R. Sahadevan, Painlevé analysis, Lie symmetries, and integrability of coupled nonlinear oscillators of polynomial type, Phys. Rep. 224 (1-2) (1993) 93.

[20] A. Guillot, Un théorème de point fixe pour les endomorphismes de l’espace projectif avec des applications aux feuilletages algébriques, Bull. Braz. Math. Soc. (N.S.) 35 (3) (2004) 345–362.

[21] S. D. Furta, On non-integrability of general systems of differential equations, Z. Angew. Math. Phys. 47 (1) (1996) 112–131.

[22] A. Nowicki, On the nonexistence of rational first integrals for systems of linear differential equations, Linear Algebra Appl. 235 (1996) 107–120.

[23] A. Goriely, Integrability, partial integrability, and nonintegrability for systems of ordinary differential equations, J. Math. Phys. 37 (4) (1996) 1871–1893.

[24] M. van der Put, M. F. Singer, Galois theory of linear differential equations, Vol. 328 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2003.

[25] H. Yoshida, A criterion for the nonexistence of an additional integral in Hamiltonian systems with a homogeneous potential, Phys. D 29 (1-2) (1987) 128–142.

[26] K. Iwasaki, H. Kimura, S. Shimomura, M. Yoshida, From Gauss to Painlevé, A modern theory of special functions, Aspects of Mathematics, E16, Friedr. Vieweg & Sohn, Braunschweig, 1991.

[27] A. J. Maciejewski, M. Przybyska, All meromorphically integrable 2D Hamiltonian systems with homogeneous potential of degree 3, Phys. Lett. A 327 (5-6) (2004) 461–473.
[28] A. J. Maciejewski, M. Przybylska, Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential, J. Math. Phys. 46 (6) (2005) 062901, 33 pp.

[29] A. J. Maciejewski, J. M. Ollangnier, A. Nowicki, J.-M. Strelcyn, Around Jouanolou non-integrability theorem, Indag. Math., N.S. 11 (2) (2000) 239–254.

[30] B. Grammaticos, B. Dorizzi, A. Ramani, Integrability of Hamiltonians with third- and fourth-degree polynomial potentials, J. Math. Phys. 24 (9) (1983) 2289–2295.

[31] M. Przybylska, Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential II In preparation.

[32] J. J. Morales-Ruiz, Kovalevskaya, Liapounov, Painlevé, Ziglin and the differential Galois theory, Regul. Chaotic Dyn. 5 (3) (2000) 251–272.

[33] K. Nakagawa, A. J. Maciejewski, M. Przybylska, New integrable Hamiltonian system with quartic in momenta first integral, Phys. Lett. A 343 (1-3) (2005) 171–173.

[34] M. Przybylska, Darboux points and integrability of homogenous Hamiltonian systems with three degrees of freedom In preparation.