The Fedosov deformation quantization for some induced symplectic connection

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Abstract

The Fedosov deformation quantization on a cotangent bundle with a symplectic connection induced by some linear symmetric connection on the base space is considered. A global construction of the symplectic homogeneous connection on the cotangent bundle modelled on the linear symmetric connection from the base space is proposed. Examples of the induced symplectic connection are given. A detailed analysis of the Abelian connection and flat sections representing special types of functions for this kind of symplectic connection is presented. Some properties of the \( \ast \)-product determined by the induced symplectic connection are shown.

Keywords: Fedosov deformation quantization, symplectic connection.

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1 Introduction

The first version of phase space formulation of quantum mechanics was proposed by Moyal [1], who adapted ideas of Weyl [2], Wigner [3] and Groenewold [4]. Unfortunately, this formalism can be applied only when the phase space of a system is \( \mathbb{R}^{2n} \). The successful generalisation of Moyal’s result was published by Bayen et al. [5, 6]. In these papers quantum mechanics was presented as a deformed version of classical mechanics with the Dirac constant \( \hbar \) being the deformation parameter. Since that moment this new branch of physics and mathematics called deformation quantization has been developed rapidly.

In his outstanding works [7, 8] Boris Fedosov presented a recurrent algorithm of construction a deformed \( \ast \)-product on any symplectic manifold. The \( \ast \)-product of functions obtained in this formalism is determined by the symplectic connection on the symplectic manifold. Unlike the Riemannian geometry the symplectic manifold can be equipped with many symplectic connections. The natural question arises which one of these symplectic connections is preferable. There are several criteria of the choice (see a review [11]). One of them is simplicity of recurrent formulas. Imagine that on a 6–D symplectic manifold in the 3rd step of the recurrence the number of elements in the Abelian connection exceeds 1500! Hence the most desirable seem to be symplectic connections generating finite Abelian connections. Unfortunately, as we proved [9], in many cases such a symplectic connection does not exist or has a very special form [9, 10]. Another criterion is compatibility of the symplectic connection with some extra structures present at the symplectic manifold. For example on pseudo-Kähler manifolds the preferred symplectic connection preserves the almost complex structure. It is also possible to select symplectic connections using a variational principle [12].

In the current paper we analyse symplectic manifolds which are cotangent bundles over paracompact manifolds. Symplectic manifolds of this type appear in physics as phase spaces of free systems or systems with constraints independent of time. Our aim is to equip them with symplectic connections which generate relatively simple Abelian connections and, moreover, are natural from the physical point of
view. Presented considerations are based on a conviction that Physics is determined by geometry of a configuration space.

The first part of the article is dedicated to some class of induced symplectic connections. Their construction has been done in a proper Darboux atlas which seems to be the set of physically preferable charts. Yet we emphasize that this construction is global and its result can be transformed easily into an arbitrary atlas. We show that in the proper Darboux coordinates we can choose the symplectic connection in such a way that its coefficients with two or three momenta indices vanish. Moreover, the symplectic connection components with one momentum index are determined by the linear symmetric connection on the configuration space of the system. This is the reason why we call this kind of symplectic connection the induced symplectic connection or the symplectic connection induced by the linear symmetric connection. Finally, the symplectic connection coefficients with three spatial indices are homogeneous functions of momenta. It is worth mentioning that the induced symplectic connection has clear geometrical interpretation. For every vector field $X$ transported parallely along some curve on a cotangent bundle $T^*M$ with respect to the induced symplectic connection its canonical projection on the base space $M$ is parallely propagated along the projected curve with respect to the linear connection inducing this symplectic connection.

A special case of our induced symplectic connection is the connection $\nabla^0$ introduced by Bordemann et al. and independently by J. F. Plebański et al. We analyse this example in details.

The second part of the paper is of a strictly technical character and is focused on the Fedosov $\ast$-product based on the induced symplectic connection. Thus we analyse several properties of the Abelian connection generated by the induced symplectic connection. We point out that in this case the recurrence relation leads to a relatively simple form of the Abelian connection series. There are only three types of elements appearing in the Abelian series $r$. The recurrence is completely determined by the curvature 2–form $R_{ij}^{k+r}$ components of the connection $\gamma + r$ standing at the elements $(y^i)^{1+} \cdots (y^n)^{r} dq^o \wedge dp^o$.

We propose several properties of series representing functions in the Fedosov deformation quantization. In the same section we consider the $\ast$-product generated by the induced symplectic connection of some special classes of functions. Among other things we compute the $\ast$-product of functions dependent only on spatial coordinates and conclude that it is the ‘usual’ pointwise product of them. We also consider the Moyal brackets of positions and momenta. The commutation rules for positions and momenta are like these ones following from the Dirac quantization scheme and they are invariant under proper Darboux transformations.

If it is possible, we use the Einstein summation convention. In situations in which the nature of coordinates (spatial or momenta) is not important we denote them by lower case Latin letters. The small Greek letters: $\alpha, \beta, \ldots$ denote spatial coordinates. The capital Latin letters $I, J, \ldots$ correspond to momenta coordinates.

## 2 A symplectic connection induced by some linear connection

Let $(\mathcal{W}, \omega)$ be a $2n$–D symplectic manifold and $\mathcal{A} = \{(U_z, \phi_z)\}_{z \in J}$ be an atlas on $\mathcal{W}$. By $\omega$ we mean the symplectic 2–form on $\mathcal{W}$.

**Definition 2.1.** The **symplectic connection** $\gamma$ on $\mathcal{W}$ is a torsionfree connection satisfying the conditions

$$\omega_{ij;k} = 0, \quad 1 \leq i, j, k \leq 2n,$$

where a semicolon ‘;’ stands for the covariant derivative.

For any point $p \in \mathcal{W}$ there exist local coordinates $(x^1, \ldots, x^{2n})$ on a neighbourhood of $p$ such that

$$\omega = \sum_{i=1}^{n} dx^i \wedge dx^{i+n}.$$  \hspace{1cm} (2.2)

The chart $(U_z, \phi_z)$ with coordinates $(x^1, \ldots, x^{2n})$ is called the **Darboux chart**.
In the Darboux coordinates the system of equations \((2.1)\) read
\[
\omega_{i;j:k} = -\gamma^l_{ik}\omega_{lj} - \gamma^l_{jk}\omega_{il} = \gamma_{ijk} - \gamma_{ij,k} = 0,
\] (2.3)
where \(\gamma_{ijk} \overset{\text{def}}{=} \gamma^l_{ijk}\omega_{il}\). Coefficients \(\gamma_{ijk}\) are symmetric with respect to the indices \(\{i, j, k\}\).

**Definition 2.2.** A symplectic manifold \((W, \omega)\) endowed with the symplectic connection \(\gamma\) is called the Fedosov manifold and it is denoted by \((W, \omega, \gamma)\).

A symplectic connection exists on any symplectic manifold. Moreover, every symplectic manifold may be equipped with many symplectic connections. In a chart \((U, \phi_z)\) the difference
\[
\Delta_{ijklk} \overset{\text{def}}{=} \gamma_{ijk} - \gamma_{ij,k}
\]
between coefficients \(\gamma_{ijk}\) and \(\gamma_{ij,k}\) of two symplectic connections \(\gamma\) and \(\gamma\) on \((W, \omega)\) is the tensor of the type \((0, 3)\) totally symmetric in its indices. Hence, starting from the same symplectic manifold \((W, \omega)\) we may construct many Fedosov manifolds.

Since now we will work in Darboux coordinates so locally every symplectic connection \(\gamma\) will be characterized by its coefficients \(\gamma_{ijk}\) totally symmetric in its indices. Then \(\gamma^i_{jk} = \omega^{il}\gamma_{ijkl}\), where
\[
\omega_{il}\omega^{lj} = \delta^l_k.
\]

The straightforward consequence of the symmetry of coefficients \(\gamma_{ijk}\) are the following relations:
\[
\gamma^i_{jk+n} = -\gamma^n_{j+i+k} = -\gamma^n_{k+l+j} = \gamma^n_{i+k+j} = -\gamma^l_{i+n+j},
\]
\[
\gamma^i_{k+n} = \gamma^n_{i+k} = \gamma^n_{i+k+n} = \gamma^n_{i+k+n},
\]
\[
1 \leq i, j, k \leq n.
\]

Locally the symplectic curvature tensor components are defined as
\[
K_{ijkl} \overset{\text{def}}{=} \omega_{il}K_{ijkl} = \frac{\partial \gamma_{ijl}}{\partial x^k} - \frac{\partial \gamma_{ijk}}{\partial x^l} + \omega^{st}\gamma_{ilt}\gamma_{jsk} - \omega^{st}\gamma_{itl}\gamma_{jsk},
\] (2.4)

The coefficients \(K_{ijkl}\) satisfy the following relations \([10]\)
\[
K_{ijkl} = -K_{ijlk}, \quad K_{ijkl} = K_{jikl},
\] (2.5)
\[
K_{ijkl} + K_{ijlk} + K_{iklj} = 0,
\] (2.6)
\[
K_{ijkl} + K_{ijlk} + K_{iklj} + K_{jikl} = 0.
\] (2.7)

Observe that the property \((2.7)\) follows from \((2.5)\) and \((2.6)\).

The Bianchi identity reads
\[
K_{ijkl;m} + K_{ijmn;kl} + K_{ijlm;nk} = 0,
\] (2.8)
where all covariant derivatives are calculated with respect to the symplectic connection.

Let the symplectic manifold \((W, \omega)\) be the cotangent bundle \(T^*M\) over some configuration space \(M\). Since now we assume that \(M\) is an \(n\)-D paracompact smooth differentiable manifold.

Moreover, let \((U, \phi_z)\) be a local chart on the base space \(M\). The local coordinates of a point \(p \in U\) in the chart \((U, \phi_z)\) are \(q^1, \ldots, q^n\), where \(n = \dim M\). Since there exists the bundle projection \(\pi : T^*M \to M\) we can introduce coordinates on \(\pi^{-1}(U)\) in a natural way. Indeed, let \(P = p dq^i\) be a cotangent vector at the point \(p\). The coordinates \((q^1, \ldots, q^{2n})\) of the point \((p, \tilde{P}) \in \pi^{-1}(U)\) are defined as:
\[
\tilde{q}^i = q^i, \quad i = 1, \ldots, n, \quad \tilde{q}^i = p_{i-n} \quad i = n+1, \ldots, 2n.
\]
The coordinates \((q^i, p_i)\) in the chart \((\pi^{-1}(U), (q^i, p_i))\) are known as the coordinates induced by the chart \((U, \phi_z)\). In these coordinates the symplectic form determined by the basic 1-form \(\theta = p_i dq^i\) reads
\[
\omega \overset{\text{def}}{=} -d\theta = dq^i \wedge dp_i
\] (2.9)
so the induced coordinates \((q^i, p_i)\) are also Darboux coordinates.
Definition 2.3. Let \( \{(U_z, \phi_z)\}_{z \in J} \) be an atlas on the symplectic manifold \( T^*M \) such that in every chart the coordinates \( q^i, 1 \leq i \leq n \) determine points on the base manifold \( M \) and \( q^{i+n} = p_i, 1 \leq i \leq n \), denote momenta in natural coordinates. Every atlas of this form is called the proper Darboux atlas and every chart of this atlas is known as the proper Darboux chart. The transition functions define the point transformations

\[
Q^k = Q^k(q^i), \quad P_i = \frac{\partial q^k}{\partial Q^i}p_k, \quad i, k = 1, \ldots, n. \tag{2.10}
\]

Assume that the base manifold \( M \) of the cotangent bundle \( T^*M \) is equipped with some linear connection \( \Gamma \). We propose some global construction of a symplectic connection on \( T^*M \) induced by the linear connection from the base space \( M \). We cover the symplectic manifold \( T^*M \) with a proper Darboux atlas. We require that in every proper Darboux atlas our symplectic connection fulfills two natural conditions:

1. it 'contains' the linear symmetric connection \( \Gamma \) from the base space \( M \) and
2. all of the symplectic connection coefficients take as simple form as possible.

In any proper Darboux chart the first requirement means that \( \forall_{\alpha, \beta} \gamma_{\alpha \beta}^a = \Gamma_{\alpha \beta}^a \).

At the beginning we analyse the transformation rule for a symplectic connection under proper Darboux transformations. As it is known, the general transformation rule for symplectic connection coefficients is of the form

\[
\gamma'_{ijk}(\tilde{Q}^1, \ldots, \tilde{Q}^{2n}) = \frac{\partial \tilde{q}^l}{\partial Q^i} \frac{\partial \tilde{q}^r}{\partial Q^j} \frac{\partial \tilde{q}^s}{\partial Q^k} \gamma_{lrs}(q^1, \ldots, q^{2n}) + \omega_{rd} \frac{\partial \tilde{q}^r}{\partial Q^i} \frac{\partial q^d}{\partial Q^j} \frac{\partial q^d}{\partial Q^k}. \tag{2.11}
\]

In Darboux coordinates the symplectic connection coefficients are symmetric with respect to all indices so we need to consider only four kinds of elements: \( \gamma_{IJK}, \gamma_{I\alpha \beta}, \gamma_{I\alpha}, \gamma_{\alpha \beta} \).

Components \( \gamma_{IJK} \) transform like tensors under proper Darboux transformations and are functions only of the connection coefficients of the same type. Hence if we assume that all coefficients \( \gamma_{IJK} \) vanish in some proper Darboux atlas, this property remains true in any proper Darboux atlas and it is compatible with the transformation rule.

Elements \( \gamma_{I\alpha} \) also transform like tensors under proper Darboux transformations. They are functions of coefficients of the same kind and \( \gamma_{IJK} \). Thus if all terms \( \gamma_{IJK} = 0 \) as we assumed before and all \( \gamma_{I\alpha} = 0 \) in some proper Darboux atlas, the choice of the connection coefficients is consistent.

Let us consider the coefficients \( \gamma_{I\alpha} \). In general they depend on coefficients of the same kind, \( \gamma_{IJK} \) and \( \gamma_{I\alpha} \). Remembering previous results we see that this relation reduces to the dependency on elements of the same type. The nontensorial part \( \omega_{rd} \frac{\partial \tilde{q}^r}{\partial Q^i} \frac{\partial q^d}{\partial Q^j} \frac{\partial q^d}{\partial Q^k} \) of the transformation formula does not vanish. It is a function only of spatial coordinates. These two facts suggest the following definition of \( \gamma_{I\alpha} \). Let \( \Gamma_{\alpha \beta}^d \) be linear symmetric connection coefficients on the base manifold \( M \). Then in proper Darboux coordinates we define

\[
\gamma_{I\alpha}(q^1, \ldots, q^n, p_1, \ldots, p_n) = -\Gamma_{\alpha\beta}^d(q^1, \ldots, q^n). \tag{2.12}
\]

This definition establishes the relation between the linear symmetric connection from the configuration space \( M \) and the symplectic connection on the phase space \( T^*M \). The symplectic connection coefficients \( \gamma_{I\alpha} \) are determined by the linear symmetric connection on \( M \) and depend only on the spatial coordinates \( q^1, \ldots, q^n \).

Finally we consider the symplectic connection coefficients \( \gamma_{\alpha \beta} \). From \((2.11)\) we see that they depend on the terms \( \gamma_{I\alpha} \) and \( \gamma_{\alpha \beta} \). Moreover, the transformation rule applied to coefficients \( \gamma_{I\alpha} \) lead to elements of the form \( P_{\delta} \gamma_{\alpha \beta}(Q^1, \ldots, Q^n) \). The consequence of the transformation rule \((2.11)\) is also presence of the expression \( P_{\delta} \gamma_{\alpha \beta}(Q^1, \ldots, Q^n) \) following from the nontensorial part of this rule. Thus it is expected that in every proper Darboux chart the symplectic connection coefficient \( \gamma_{\alpha \beta} \) be of the form

\[
\gamma_{\alpha \beta}(q^1, \ldots, q^n, p_1, \ldots, p_n) = p_{\varepsilon} f_{\alpha \beta}(q^1, \ldots, q^n). \tag{2.13}
\]
Quantities $f^c_{\alpha\beta\delta}(q^1, \ldots, q^n)$ are purely differentiable geometrical objects totally symmetric with respect to the indices $\{\alpha, \beta, \delta\}$. Under proper Darboux transformations they obey the following transformation rule

\[ f^c_{\alpha\beta\delta}(Q^1, \ldots, Q^n) = \frac{\partial q^1}{\partial Q^\alpha} \frac{\partial q^\tau}{\partial Q^\beta} \frac{\partial q^\nu}{\partial Q^\delta} \partial Q^c f^{\mu\nu\rho}_{\eta\tau\upsilon}(q^1, \ldots, q^n) + \sum_{\kappa=1}^n \frac{\partial^2 Q^\kappa}{\partial q^\mu\partial q^\nu} \left( \frac{\partial q^\mu}{\partial Q^\alpha} \frac{\partial q^\tau}{\partial Q^\beta} \frac{\partial q^\upsilon}{\partial Q^\delta} + \frac{\partial q^\tau}{\partial Q^\alpha} \frac{\partial q^\mu}{\partial Q^\beta} \frac{\partial q^\upsilon}{\partial Q^\delta} + \frac{\partial q^\upsilon}{\partial Q^\alpha} \frac{\partial q^\mu}{\partial Q^\beta} \frac{\partial q^\tau}{\partial Q^\delta} \right) \gamma_{\alpha\beta\gamma} + 2Q^c + \frac{\partial q^\mu}{\partial Q^\alpha} \frac{\partial q^\tau}{\partial Q^\beta} \frac{\partial q^\upsilon}{\partial Q^\delta} \partial q^\mu (q^1, \ldots, q^n) \tag{2.14} \]

The last step in our construction is to put together the coefficients $\gamma_{\alpha\beta\delta}$ on intersections of charts. Since the base manifold is paracompact we can use the partition of unity. Let $\{g_z\}_{z \in J}$ be a partition of unity corresponding to an atlas $\{(U_\xi, \phi_\xi)\}_{\xi \in J}$ on $\mathcal{M}$. Then we define

\[ \gamma_{\alpha\beta\delta} \overset{\text{def}}{=} \sum_{z \in J} g_z (\gamma_{\alpha\beta\delta})_z. \]

Locally these coefficients are of the form (2.13). Notice that they disappear on the base space $\mathcal{M}$.

The construction of the natural symplectic connection $\gamma$ on the cotangent bundle $T^*\mathcal{M}$ has been completed. As we demanded, the projection of this symplectic connection on the base space $\mathcal{M}$ is the linear symmetric connection $\Gamma$ and the symplectic connection seems to be as simple as possible. This natural symplectic connection will be called the symplectic connection induced by the linear symmetric connection or simply the induced symplectic connection.

It may be frustrating that our method of introducing the induced symplectic connection requires a proper Darboux atlas, but this fact results from the physical origin of our construction. It expresses the obvious difference between the configuration space and the fibres of the momenta. Once we obtain the induced symplectic connection we can transform it to an arbitrary atlas on the cotangent bundle. Moreover, we emphasise the fact that although our considerations have been made locally, the proposed construction of the induced symplectic connection is global (compare the paragraph about existence and extension of connections in [17]).

There are many induced symplectic connections on the cotangent bundle $T^*\mathcal{M}$. As one could see the choice of connection coefficients $\gamma_{\alpha\beta\delta}$ depends on the choice of the atlas on $\mathcal{M}$ and the partition of unity corresponding to this atlas. As we will show in an example it is possible to impose some geometrical condition to make the induced symplectic connection unique. Namely we can equip the cotangent bundle $T^*\mathcal{M}$ with a metric structure and then having a Levi–Civita connection on $T^*\mathcal{M}$ we define the induced symplectic connection. But the metric structure on the phase space does not have a clear physical interpretation so we prefer to consider the wider class of the induced symplectic connections.

A symplectic connection on a 2–D symplectic manifold in a Darboux chart is determined by $\binom{2n+2}{2n-1}$ coefficients. In the case of an induced symplectic connection only at the most $\frac{1}{2}n(n+1)(2n+1)$ of them are nonzero.

The induced symplectic conection coefficients $\gamma^i_{jk}$ on the cotangent bundle $T^*\mathcal{M}$ in a proper Darboux chart are:

\[
\begin{align*}
\gamma^0_{JK} &= \gamma^1_{JK} = \gamma^0_{J\beta} = 0, \\
\gamma^a_{\alpha+\beta} &= -\gamma^a_{\alpha-\beta} = \gamma_{a\alpha\beta} = -\gamma_{a\alpha\beta}, \\
\gamma^\alpha_{\alpha\beta} &= \gamma_{\alpha\beta}. \tag{2.15}
\end{align*}
\]

The induced symplectic connection has nice interpretation. Let $c : \tilde{q}^i = \tilde{q}^i(t)$, $i = 1, \ldots, 2n$, $t \in \mathbb{R}$ be a smooth curve on the cotangent bundle $T^*\mathcal{M}$ and $X$ a vector field parallel along this curve. Locally it means that

\[
\frac{dX^i}{dt} + \gamma^i_{jk} X^j \frac{dq^k}{dt} = 0. \tag{2.16}
\]
Then using the relations (2.15) and (2.12) we see that the set of equations (2.16) can be divided in two parts: the conditions
\[
\frac{dX^\alpha}{dt} + \gamma^\alpha_{\beta\gamma} X^\gamma \frac{dq^\beta}{dt} = \frac{dX^I}{dt} + \Gamma^\alpha_{\beta\delta} X^\beta \frac{dq^\delta}{dt} = 0
\]  
(2.17)
and some other relations of the kind
\[
\frac{dX^I}{dt} + \gamma^I_{jk} X^j \frac{dq^k}{dt} = 0.
\]

Now we project the curve $c$ in the canonical way on the base space $M$. This projected curve $\pi(c)$ is described by the system of equations $\pi(c) : q^\alpha = q^\alpha(t), \alpha = 1, \ldots, n$, $t \in \mathbb{R}$. The vector field $X$ transforms under this canonical projection into a vector field $\pi_*(X)$. As we can see from (2.17) the vector field $\pi_*(X)$ is pararelly propagated on the curve $\pi(c)$. Thus we conclude that the induced symplectic connection guarantees that for every vector field $X$ pararelly transported along the curve $c$ on the cotangent bundle $T^*M$ the canonical projection $\pi_*(X)$ of this field on the base space $M$ is pararelly propagated along the projected curve $\pi(c)$.

Moreover, if we consider the vector field $X^i = \frac{dq^i(t)}{dt}$ then we see that the canonical projection of any geodesic in $T^*M$ with respect to the linear symplectic connection $\gamma$ onto the base space $M$ is the geodesic with respect to the linear symmetric connection $\Gamma$ with the same affine parameter.

Let $\theta$ denote, as before, the basic 1–form on the cotangent bundle $T^*M$. The canonical vector field $V$ on $T^*M$ is defined by the relation
\[
i_V \omega = -\theta.
\]
Locally in Darboux coordinates $V = p_i \frac{\partial}{\partial p_i}$.

**Definition 2.4.** A connection $\nabla$ on the cotangent bundle $T^*M$ is called homogeneous iff for all vector fields $X, Y \in \Gamma(T(T^*M))$
\[\mathcal{L}_V \nabla X Y - \nabla \mathcal{L}_V X Y - \nabla X \mathcal{L}_V Y = 0\]
where $\mathcal{L}_V$ stands for the Lie derivative with respect to the canonical vector field $V$.

In a proper Darboux chart it means that for all indices $\alpha$ and $I$
\[
p_{\alpha} \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial p_\epsilon} X^\beta Y^\gamma + p_{\epsilon} \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial p_\epsilon} X^J Y^\beta + p_{\epsilon} \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial p_\epsilon} X^\beta Y^J + p_{\epsilon} \frac{\partial \Gamma^\alpha_{J\epsilon}}{\partial p_\epsilon} X^J Y^L +
\]
\[+ 2 \Gamma^\alpha_{J\epsilon} X^J Y^L + \Gamma^\alpha_{J\beta} X^J Y^\beta + \Gamma^\alpha_{J\beta} X^\beta Y^J = 0\]
(2.18)
and
\[
p_{\epsilon} \frac{\partial \Gamma^I_{\beta\gamma}}{\partial p_\epsilon} X^\beta Y^\delta + p_{\epsilon} \frac{\partial \Gamma^I_{\beta\gamma}}{\partial p_\epsilon} X^J Y^\beta + p_{\epsilon} \frac{\partial \Gamma^I_{\beta\gamma}}{\partial p_\epsilon} X^\beta Y^J + p_{\epsilon} \frac{\partial \Gamma^I_{J\epsilon}}{\partial p_\epsilon} X^J Y^L + \Gamma^I_{J\beta} X^\beta Y^\delta + \Gamma^I_{JL} X^J Y^L = 0\]
(2.19)
where symbols $\Gamma^I_{Jk}$ denote coefficients of the connection $\nabla$. Substituting (2.15) into (2.18) and (2.19) we obtain that every induced symplectic connection is homogeneous.

In proper Darboux coordinates the nonvanishing components of the symplectic curvature tensor $K$ of the symplectic connection (2.15) are $K^I_{\beta\gamma\delta}$, $K^\alpha_{\beta\gamma J}$ and $K^\alpha_{\beta\gamma\delta}$. The coefficients of the first group are determined by the curvature tensor of the linear symmetric connection from the base manifold $M$. As it can be checked
\[K^\alpha_{\alpha+n, \beta\gamma\delta}(q^1, \ldots, q^n, p_1, \ldots, p_n) = -R^\alpha_{\beta\gamma\delta}(q^1, \ldots, q^n)\]
so these terms do not depend on momenta. By $R^\alpha_{\beta\gamma\delta}$ we mean components of the curvature tensor of the linear symmetric connection $\Gamma$.

Moreover, from (2.5) and (2.6) we see that
\[K^I_{\beta\gamma\delta} = K^I_{\beta\delta\gamma} J - K^I_{\beta\gamma\delta} J,\]
(2.20)
The components $K_{\beta\delta\gamma\epsilon}$ are functions of spatial coordinates only.

Finally the elements $K_{\alpha\beta\gamma\delta}$ are of the form

$$K_{\alpha\beta\gamma\delta} = \sum_{\epsilon=1}^{n} p_{\epsilon}(K_{\alpha\beta\gamma\delta})_{\epsilon},$$

where $\forall 1 \leq \epsilon \leq n (K_{\alpha\beta\gamma\delta})_{\epsilon}$ are some functions of $q^{1},\ldots,q^{n}$.

From the Bianchi identity (2.8) we get

$$(K_{\alpha\beta\gamma\delta})_{\epsilon} = \partial K_{\alpha\beta\gamma\delta\epsilon+n} - \partial K_{\alpha\beta\delta\epsilon+n} + \sum_{\epsilon=1}^{n} \gamma_{\epsilon+n \alpha \beta} K_{\beta\epsilon\gamma\delta\epsilon+n} - \sum_{\epsilon=1}^{n} \gamma_{\epsilon+n \alpha \gamma} K_{\beta\epsilon\delta\gamma\epsilon+n} +$$

$$+ \sum_{\epsilon=1}^{n} \gamma_{\epsilon+n \alpha \delta} K_{\beta\epsilon\gamma\delta\epsilon+n} - \sum_{\epsilon=1}^{n} \gamma_{\epsilon+n \beta \delta} K_{\epsilon\alpha\gamma\delta\epsilon+n} - \sum_{\epsilon=1}^{n} \gamma_{\epsilon+n \beta \gamma} K_{\epsilon\alpha\delta\gamma\epsilon+n} +$$

$$- \sum_{\epsilon=1}^{n} \gamma_{\epsilon+n \alpha \beta} K_{\epsilon\beta\gamma\delta\epsilon+n} - \sum_{\epsilon=1}^{n} \gamma_{\epsilon+n \beta \gamma} K_{\epsilon\alpha\delta\gamma\epsilon+n}.$$  \hspace{1cm} (2.21)

Hence we conclude that to obtain the complete symplectic curvature tensor we need to know all of the curvature tensor components $K_{\alpha\beta\gamma\epsilon}$ and the linear connection on the base manifold $M$. The fact that the base space $M$ is flat does not guarantee that the cotangent bundle $T^{*}M$ is symplectic flat.

The symplectic Ricci tensor $K_{ij} \overset{\text{def}}{=} \omega_{ls} K_{lisj}$.

In a cotangent bundle equipped with an induced symplectic connection only components

$$K_{\alpha\beta} = - \sum_{\epsilon=1}^{n} K_{\alpha\beta\epsilon+n}$$ \hspace{1cm} (2.22)

can be different from 0. Thus the Fedosov manifold $(T^{*}M,\omega,\gamma)$ equipped with an induced symplectic connection $\gamma$ is Ricci flat iff in a proper Darboux atlas

$$\forall \alpha,\beta \sum_{\epsilon=1}^{n} K_{\alpha\beta\epsilon+n} = 0.$$ \hspace{1cm} (2.23)

3 Examples of the induced symplectic connection

The first example of an induced symplectic connection is based on ideas presented in [15]. The similar problem was considered in [14]. It also appeared in another context in [13].

Assume that the $n$–D base manifold $M$ is endowed with a linear symmetric connection $\Gamma$. The tensor field $\tilde{g} \in \Gamma(T^0_2(T^{*}M))$ in proper Darboux coordinates is defined as

$$\tilde{g}_{jk} = \left( \begin{array}{cc} -2p_1 & 1 \\ 1 & 0 \end{array} \right).$$ \hspace{1cm} (3.24)

By 1 and 0 we denote the identity and zero matrices of the dimension $n \times n$ respectively. The coefficients $\Gamma_{\alpha\beta}$ are components of the linear symmetric connection on $M$.

The tensor field $\tilde{g}$ of the type $(0,2)$ is symmetric and nondegenerate. Its signature is $(+\cdots,+,-\cdots,-)$. Then $(T^{*}M,\tilde{g})$ is a $2n$–D Riemannian manifold. The Levi–Civita connection $\tilde{\Gamma}$ on it is determined by the tensor $\tilde{g}$ according to the well known relation

$$\tilde{\Gamma}^{i}_{jk} = \frac{1}{2} \tilde{g}^{il} \left( \frac{\partial \tilde{g}_{jk}}{\partial q^{l}} - \frac{\partial \tilde{g}_{jl}}{\partial q^{k}} - \frac{\partial \tilde{g}_{jk}}{\partial q^{l}} \right).$$ \hspace{1cm} (3.25)
By \( \hat{g}^{\beta \gamma} \) we mean components of the tensor inverse to the metric tensor \( \hat{g} \).

It can be easily checked that the coefficients of the Levi–Civita connection on the manifold \((T^*\mathcal{M}, \hat{g})\) are

\[
\Gamma^\alpha_{\beta \delta} = \Gamma^\alpha_{\beta \delta}, \quad \hat{\Gamma}^\alpha_{\beta \delta} = 0, \quad \hat{\Gamma}^\alpha_{I J} = 0,
\]

\[
\Gamma^\alpha_{\beta \delta} + n = \rho \left( \frac{\partial \Gamma^e_{\beta \delta}}{\partial q^a} - \frac{\partial \Gamma^e_{\alpha \beta}}{\partial q^a} - \frac{\partial \Gamma^e_{\alpha \delta}}{\partial q^a} + 2 \Gamma^e_{\alpha e} \Gamma^\nu_{\beta \delta} \right),
\]

\[
\hat{\Gamma}^\alpha_{\beta \delta} + n = -\delta^\alpha_{\beta \delta}, \quad \hat{\Gamma}^\alpha_{I J} = 0.
\]

(3.26)

Let us lower the upper index of Christoffel symbols \( \hat{\Gamma}^i_{jk} \) by their contraction with the symplectic form so \( \hat{\Gamma}^i_{jk} \equiv \omega_{ij} \hat{\Gamma}^i_{jk} \).

\[
\hat{\Gamma}^\alpha_{\beta + n} = -\delta^\alpha_{\beta \delta}, \quad \hat{\Gamma}^\alpha_{I I} = 0, \quad \hat{\Gamma}^\alpha_{I J K} = 0,
\]

\[
\hat{\Gamma}^\alpha_{\beta \delta} = \rho \left( \frac{\partial \Gamma^e_{\beta \delta}}{\partial q^a} - \frac{\partial \Gamma^e_{\alpha \beta}}{\partial q^a} - \frac{\partial \Gamma^e_{\alpha \delta}}{\partial q^a} + 2 \Gamma^e_{\alpha e} \Gamma^\nu_{\beta \delta} \right),
\]

\[
\hat{\Gamma}^\alpha_{\beta \delta \epsilon + n} = -\delta^\alpha_{\beta \delta}, \quad \hat{\Gamma}^\alpha_{\alpha I L} = 0.
\]

It is known (see \cite{[10]}), that having a symplectic manifold \((\mathcal{M}, \omega)\) endowed with some symmetric affine connection \( \Gamma \) we may define in a natural manner a symplectic connection on \((\mathcal{M}, \omega)\). In any Darboux chart the coefficients of that symplectic connection are equal

\[
\gamma_{ijk} \equiv \frac{1}{3} (\Gamma_{ijk} + \Gamma_{jik} + \Gamma_{kij}),
\]

(3.27)

Applying the formula (3.27) to the Levi–Civita connection \( \hat{\Gamma} \) on the manifold \((T^*\mathcal{M}, \hat{g})\) we obtain the symplectic connection on \(T^*\mathcal{M}\) induced by the Levi–Civita connection on it. Moreover, since the metric tensor \( \hat{g} \) (3.24) is a function of the linear symmetric connection \( \Gamma \) on the configuration space \( \mathcal{M} \) then in fact the symplectic connection \( \gamma \) is determined by the connection on the base space \( \mathcal{M} \).

In proper Darboux coordinates the coefficients of the induced symplectic connection on \(T^*\mathcal{M}\) read

\[
\gamma_{\alpha + n \beta \delta} = -\Gamma^\alpha_{\beta \delta}, \quad \gamma_{I I} = 0, \quad \gamma_{I J K} = 0,
\]

\[
\gamma_{\alpha \beta \delta} = -\frac{1}{3} \rho \left( \frac{\partial \Gamma^e_{\beta \delta}}{\partial q^a} + \frac{\partial \Gamma^e_{\alpha \beta}}{\partial q^a} + \frac{\partial \Gamma^e_{\alpha \delta}}{\partial q^a} - 2 \Gamma^e_{\alpha e} \Gamma^\nu_{\beta \delta} - 2 \Gamma^e_{\nu e} \Gamma^\nu_{\alpha \beta} \right).
\]

(3.28)

The induced symplectic connection \( \gamma \) and the Levi–Civita connection \( \hat{\Gamma} \) on \(T^*\mathcal{M}\) are different.

The nonvanishing components of the symplectic curvature tensor \( K \) of the symplectic connection (3.28) in proper Darboux coordinates are

\[
K_{\alpha + n \beta \gamma \delta} = -R^\alpha_{\beta \gamma \delta}, \quad K_{\alpha \beta \gamma \delta + n} = \frac{1}{3} \left( R^\delta_{\alpha \beta \gamma} + R^\delta_{\beta \alpha \gamma} \right),
\]

\[
K_{\alpha \beta \gamma \delta} = -\frac{1}{3} \rho \left( R^\epsilon_{\beta \gamma \delta ; \alpha} + R^\epsilon_{\alpha \gamma \delta ; \beta} + 3 \Gamma^\epsilon_{\nu \alpha} R^\nu_{\beta \gamma \delta} + 3 \Gamma^\epsilon_{\nu \beta} R^\nu_{\gamma \delta ; \alpha} + (R^\nu_{\alpha \gamma \delta} + R^\nu_{\beta \delta \alpha}) \Gamma^\nu_{\gamma \delta} - (R^\nu_{\alpha \beta \delta} + R^\nu_{\beta \alpha \delta}) \Gamma^\nu_{\gamma \delta} \right).
\]

(3.29)

By \( R^\alpha_{\beta \gamma \delta} \) we understand components of the Riemannian curvature tensor of the connection \( \hat{\Gamma} \).

From the formulas (3.28) and (3.29) we see that, unlike the general case of an induced symplectic connection, in the considered example the symplectic connection and its curvature are determined by components \( \gamma_{\alpha + n \beta \delta} \), \( K_{\alpha + n \beta \delta} \) and their derivatives only. Indeed,

\[
\gamma_{\alpha \beta \delta} = \frac{1}{3} \sum_{\epsilon, \nu = 1}^n \rho \left( \frac{\partial \gamma_{\epsilon + n \beta \delta}}{\partial q^a} + \frac{\partial \gamma_{\epsilon + n \alpha \beta}}{\partial q^a} + \frac{\partial \gamma_{\epsilon + n \alpha \delta}}{\partial q^a} + 2 \gamma_{\epsilon + n \nu \alpha} \gamma_{\nu + n \beta \delta} + 2 \gamma_{\epsilon + n \nu \beta} \gamma_{\nu + n \alpha \delta} \right)
\]

and

\[
K_{\alpha \beta \gamma \delta + n} = -\frac{1}{3} \left( K_{\delta + n \alpha \beta \gamma} + K_{\delta + n \beta \alpha \gamma} \right).
\]

(3.30)

(3.31)
It follows from (2.5), (2.6) and (3.31) that in any proper Darboux chart

\[ K_{\alpha\beta\gamma\delta} = \frac{1}{3} \sum_{r,s=1}^n p_r \left( \frac{\partial K_{r+s+n \beta\gamma\delta}}{\partial q^n} - \frac{\partial K_{r+s+n \alpha\gamma\delta}}{\partial q^n} \right) + 4\gamma_{r+s+n \alpha\delta}K_{r+s+n \beta\gamma\delta} + 4\gamma_{r+s+n \beta\gamma}K_{r+s+n \alpha\delta} + \\
\gamma_{r+s+n \gamma\delta}K_{r+s+n \alpha\beta} - \gamma_{r+s+n \gamma\delta}K_{r+s+n \alpha\beta} + \gamma_{r+s+n \delta\gamma}K_{r+s+n \alpha\beta} + \gamma_{r+s+n \alpha\gamma}K_{r+s+n \beta\delta} + \gamma_{r+s+n \alpha\delta}K_{r+s+n \beta\gamma} + \\
\gamma_{r+s+n \beta\delta}K_{r+s+n \alpha\gamma} - \gamma_{r+s+n \alpha\gamma}K_{r+s+n \beta\delta} - \gamma_{r+s+n \beta\gamma}K_{r+s+n \alpha\delta} - 2\gamma_{r+s+n \alpha\beta}K_{r+s+n \gamma\delta}. \] (3.32)

Using (3.30), we conclude that the symplectic connection induced by a Riemannian connection is determined by \( \frac{1}{2} n^2 (n+1) \) functions \( \gamma_{\alpha+n \beta\delta} \) and their partial derivatives. All of these functions depend only on spatial coordinates \( q^i, i = 1, \ldots, n \).

The induced symplectic curvature tensor is completely described, via (3.31) and (3.32), by its \( \frac{1}{2} n^2 (n^2 - 1) \) components being only functions of \( q^i, i = 1, \ldots, n \):

1. \( K_{\alpha+n \beta\gamma}, \beta < \gamma \),
2. \( K_{\alpha+n \gamma\beta}, \gamma < \beta \),
3. \( K_{\alpha+n \beta\gamma} \) and \( K_{\alpha+n \delta\gamma} \) for \( \beta < \gamma < \delta \),

their derivatives and the symplectic connection coefficients \( \gamma_{\alpha+n \beta\delta} \). Remember that from (2.6)

\[ K_{\alpha+n \gamma\beta\delta} = K_{\alpha+n \beta\gamma\delta} + K_{\alpha+n \delta\beta\gamma}, \beta < \gamma < \delta. \]

It follows from (2.5), (2.6) and (3.31) that in any proper Darboux chart

\[ K_{\alpha\beta\gamma\delta+n} + K_{\gamma\alpha\beta\delta+n} + K_{\beta\gamma\alpha\delta+n} = 0. \] (3.33)

Applying (2.5), (2.6) and (3.33) we get back (3.31). Thus properties (3.31) and (3.33) are equivalent.

The immediate consequence of the identity (3.33) is the relation

\[ K_{\alpha\alpha\alpha+n} = 0. \]

Among all of symplectic connections fulfilling in a proper Darboux chart the conditions:

1. \( \gamma_{\alpha+n \beta+n \delta} = 0, \gamma_{\alpha+n \beta+n \delta+n} = 0 \),
2. \( \gamma_{\alpha+n \beta\delta} \) are some functions of spatial coordinates \( q^1, \ldots, q^n \),
3. on the base space \( M \) all of coefficients \( \gamma_{\alpha\beta\delta} \) disappear

only the curvature tensor of the induced symplectic connection (3.28) satisfies (3.33). Indeed, for the fixed indices \( \alpha, \beta, \gamma, \delta \) we obtain from (2.4) and (3.33) that

\[ 0 = \frac{\partial \gamma_{\alpha\beta\delta}}{\partial p_\gamma} - \frac{\partial \gamma_{\gamma+n+n \alpha\beta}}{\partial q^n} + \omega^{\nu\gamma} \gamma_{\nu+n+n \alpha\beta} - \omega^{\nu\gamma} \gamma_{\nu\alpha\beta} \gamma_{\nu+n+n \epsilon\delta} + \frac{\partial \gamma_{\gamma+n+n \alpha\delta}}{\partial p_\gamma} - \frac{\partial \gamma_{\gamma+n+n \delta}}{\partial q^n} + \omega^{\nu\gamma} \gamma_{\nu+n+n \alpha\beta} \gamma_{\nu+n+n \epsilon\delta}. \] (3.34)

Solving the system of differential equations (3.31) numerated by \( \gamma \) for the fixed \( \alpha, \beta, \delta \) and applying the 3rd condition we see that in Darboux coordinates every coefficient \( \gamma_{\alpha\beta\delta} \) must be of the form (3.30).

From (2.22) and (2.24) we conclude that potentially nonzero components of the symplectic Ricci tensor are

\[ K_{\alpha\beta} = \frac{1}{3} \left( R_{\alpha\beta} + R_{\beta\alpha} \right), \]

where \( R_{\alpha\beta} \) are elements of the Ricci tensor on the base manifold. In the case when \( R_{\alpha\beta} \) is a Ricci tensor of some Levi–Civita connection, it is symmetric. Then the symplectic manifold \( T^* M \) is Ricci flat iff the Riemannian manifold \( M \) is Ricci flat.
Let us consider the case that the base manifold $M$ is a Riemannian manifold with a metric structure determined locally by the metric tensor $g_{\alpha\beta}$. This metric structure introduces the unique Levi–Civita connection on $M$ (see formula (3.25)). A smooth curve $c : \mathbb{R} \to M$ is locally characterized by the system of equations $q^\alpha = q^\alpha(t), t \in \mathbb{R}$. The lift of $c$ from the Riemannian manifold $M$ to the cotangent bundle $T^*M$ is the smooth curve $\tilde{c} : \mathbb{R} \to T^*M$ locally expressed by the set of equations

$$\begin{cases}
q^\alpha = q^\alpha(t), \\
p_\alpha = g_{\alpha\beta}(q^\delta(t))\frac{dq^\delta(t)}{dt},
\end{cases} t \in \mathbb{R}.$$  

It can be checked (compare (14)) that among all symplectic connections induced by the Levi–Civita connection from $M$ only for the symplectic connection locally characterised by the coefficients (3.28) the lift of any geodesic on $M$ with respect to the Levi–Civita connection on $M$ is a geodesic with respect to the induced symplectic connection (3.28) on $T^*M$.

The second proposed choice of an induced symplectic connection can be applied only in some very special situation. Let us assume that an $n$–D base manifold $M$ can be covered with an atlas in which at all chart intersections the transition functions are linear i.e.

$$\forall \alpha \quad Q^\alpha = a^\alpha_\beta q^\beta.$$  

By $a^\alpha_\beta$ we denote elements of the matrix of transformation between ‘old’ coordinates $q^\beta$ and ‘new’ ones $Q^\alpha$. The matrix $a$ does not depend on the point and it is nonsingular. The choice on an atlas described above can be made on a sphere when we use spherical coordinates.

Then in the proper Darboux atlas on the cotangent bundle $T^*M$ we propose the following induced symplectic connection:

$$\forall_{IJK} \forall_{\alpha\beta\delta} \quad \gamma_{IJK} = \gamma_{IJa} = \gamma_{\alpha\beta\delta} = 0 \quad \gamma_{Ia\beta} = -\Gamma^{I-n}_{a\beta}(q^1, \ldots, q^n).$$  

(3.35)

Hence the induced symplectic connection is completely characterized by its $\frac{1}{2}n^2(n + 1)$ coefficients depending only on spatial coordinates. The straightforward consequence of this fact is the observation that among symplectic curvature tensor components different from 0 can be only $K_{I\alpha\beta\delta}$ and $K_{\alpha\beta\delta}I$ being functions of $q^1, \ldots, q^n$ only. These two classes of components satisfy the condition (2.20). All of them are functions of spatial coordinates exclusively.

4 The Fedosov deformation quantization with an induced symplectic connection

Considerations presented in this part of our paper have been divided in two subsections. In the first one we analyse the construction and properties of an Abelian connection determined by the symplectic connection introduced in the second chapter. The next subsection is devoted to flat sections of the Weyl bundle and some examples of the $*$-product with the induced symplectic connection.

We assume that the Reader is familiar with the Fedosov quantization algorithm. For details see [7, 8].

4.1 The Abelian connection

Let $(\mathcal{W}, \omega, \gamma)$ be a Fedosov manifold covered by an atlas $\mathcal{A} = \{U_z, \phi_z\} \in \mathcal{J}$, By $\hbar$ we denote a deformation parameter. We assume that it is positive. In physics the deformation parameter $\hbar$ is identified with the Dirac constant. The symbols $y^1, \ldots, y^{2n}$ represent the components of an arbitrary vector $y$ belonging to the tangent space $T_p\mathcal{W}$ at the point $p \in \mathcal{W}$ with respect to the natural basis $(\frac{\partial}{\partial x^i})_p$ determined by the chart $(U_z, \phi_z)$ such that $p \in U_z$. 

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We introduce the formal series
\[ a \overset{\text{def}}{=} \sum_{l=0}^{\infty} \hbar^k a_{k,j_1...j_l} y^{j_1} ... y^{j_l}, \quad k \geq 0 \] (4.1)

at the point \( p \). For \( l = 0 \) we put \( a = \hbar^k a_k \). By \( a_{k,j_1...j_l} \) we mean the components of a covariant tensor totally symmetric with respect to the indices \{\( j_1, ..., j_l \)\} in the natural basis \( dx^{j_1} \otimes ... \otimes dx^{j_l} \).

The part of the series \( a \) standing at \( \hbar^k \) and containing \( l \) components of the vector \( y \) will be denoted by \( a[k,l] \). Thus
\[ a = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a[k,l]. \] (4.2)
The degree \( \text{deg}(a[k,l]) \) of the component \( a[k,l] \) equals \( 2k + l \).

Notice that since \( a_{k,j_1...j_l} \) are totally symmetric in the indices \{\( j_1, ..., j_l \)\}, the element \( a \) defined by the formula (2.39) can be understood as the polynomial
\[ a = \sum_{z=0}^{\infty} \sum_{k=0}^{z} \hbar^k \tilde{a}_{k,i_1...i_{2n}} (y^1)^{i_1} ... (y^{2n})^{i_{2n}}, \] (4.3)
where
\[ 0 \leq i_1, ..., i_{2n} \leq z - 2k, \quad i_1 + ... + i_{2n} = z - 2k. \]
The symbol \( \lfloor z \rfloor \) denotes the floor of \( z \). The relation between the tensor components \( a_{k,j_1...j_l} \) and the polynomial coefficients \( \tilde{a}_{k,i_1...i_{2n}} \) reads
\[ \tilde{a}_{k,i_1...i_{2n}} = \frac{(z - 2k)!}{i_1!...i_{2n}!} a_k \omega^{i_1j_1} ... \omega^{i_{2n}j_{2n}} \] (4.4)

Let \( P^*_{\mathfrak{p}} \mathcal{W}[\hbar] \) denote a set of all elements \( a \) of the form (4.1) at the point \( p \). The product \( \circ: P^*_{\mathfrak{p}} \mathcal{W}[\hbar] \times P^*_{\mathfrak{p}} \mathcal{W}[\hbar] \to P^*_{\mathfrak{p}} \mathcal{W}[\hbar] \) of two elements \( a, b \in P^*_{\mathfrak{p}} \mathcal{W}[\hbar] \) is the mapping
\[ a \circ b \overset{\text{def}}{=} \sum_{t=0}^{\infty} \frac{1}{t!} \left( -\frac{i\hbar}{2} \right)^t \omega^{i_1j_1} ... \omega^{i_{2n}j_{2n}} \frac{\partial^t a}{\partial y^{i_1}...\partial y^{i_t}} \frac{\partial^t b}{\partial y^{j_1}...\partial y^{j_t}}. \] (4.5)
As it has been proved in [3], in Darboux coordinates we have
\[
\begin{align*}
(y^1)^r (y^{i+n})^j \circ (y^1)^s (y^{i+n})^k &= r! j! s! k! \sum_{t=0}^{\min[r,k]+\min[j,s]} \left( \frac{i\hbar}{2} \right)^t (y^1)^{r-s-t} (y^{i+n})^{k+j-t} \times \\
& \times \sum_{a=\max[t-r,t-k,0]}^{-\min[j,s,t]} (-1)^a \frac{1}{a!(t-a)!(r-t+a)!(j-a)!(s-a)!(k-t+a)!}.
\end{align*}
\] (4.6)

The pair \( (P^*_{\mathfrak{p}} \mathcal{W}[\hbar], \circ) \) is a noncommutative associative algebra called the Weyl algebra. Taking a set of the Weyl algebras \( (P^*_{\mathfrak{p}} \mathcal{W}[\hbar], \circ) \) at all points of the manifold \( \mathcal{W} \) we obtain the Weyl bundle
\[ P^* \mathcal{W}[\hbar] \overset{\text{def}}{=} \bigcup_{p \in \mathcal{W}} (P^*_{\mathfrak{p}} \mathcal{W}[\hbar], \circ). \]

Geometrical structure of the Fedosov deformation quantization is based on the \( m \)-differential form calculus with values in the Weyl bundle. Locally such a form can be written as follows
\[ a = \sum_{l=0}^{\infty} \hbar^k a_{k,j_1...j_{l+s_1...s_m}} (x^1, ..., x^{2n}) y^{j_1} ... y^{j_l} dx^{s_1} \wedge \cdots \wedge dx^{s_m}, \] (4.7)
where \(0 \leq m \leq 2n\). Now \(a_{k,i_1\ldots i_t} (x^1, \ldots, x^{2n})\) are components of smooth tensor fields on \(\mathcal{W}\) and \(C^\infty (T\mathcal{W}) \ni y \mapsto y^i \frac{\partial}{\partial x^i}\) is a smooth vector field on \(\mathcal{W}\). We use the same symbol for the vector field \(y \in C^\infty (T\mathcal{W})\) and the vector \(y \in T_y \mathcal{W}\).

From now on we will omit the variables \(x^1, \ldots, x^{2n}\) in \(a_{k,i_1\ldots i_t} (x^1, \ldots, x^{2n})\).

Differential forms of the type \(4.1\) are smooth sections of the direct sum \(\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda \overset{\text{def}}{=} \oplus_{m=0}^{2n} (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^m)\). By \(\Lambda^m\) we mean the space of smooth \(m\)-forms on the manifold \(\mathcal{W}\).

The commutator of forms \(a, b \in C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^m)\) is defined as
\[
[a, b] \overset{\text{def}}{=} a \circ b - (-1)^{m_1 m_2} b \circ a.
\]

**Definition 4.1.** The antiderivation operator \(\delta : C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^m) \to C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^{m+1})\) is defined by
\[
\delta a \overset{\text{def}}{=} dx^k \wedge \frac{\partial a}{\partial y^k}.
\]

The operator \(\delta\) lowers the degree \(\deg(a)\) of the elements of \(\mathcal{P}^* \mathcal{W}[\hbar] \Lambda\) by 1.

Every two forms \(a, b \in C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^m)\) and \(b \in C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda)\) satisfy
\[
\delta (a \circ b) = (\delta a) \circ b + (-1)^{m_1} a \circ (\delta b).
\]

The operator \(\delta^{-1} : C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^m) \to C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^{m-1})\) is defined by
\[
\delta^{-1} a = \left\{
\begin{array}{ll}
\frac{1}{l+m} y^k \frac{\partial}{\partial x^k} a & \text{for } l + m > 0, \\
0 & \text{for } l + m = 0,
\end{array}
\right.
\]

where \(l\) is the degree of \(a\) in \(y^i\)'s i.e. the number of \(y^i\)’s. The operator \(\delta^{-1}\) raises the degree of the forms of \(\mathcal{P}^* \mathcal{W}[\hbar] \Lambda\) in the Weyl algebra by 1.

The exterior covariant derivative \(\partial_{\gamma}\) of a form \(a \in C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^m)\) determined by a symplectic connection \(\gamma\) is the linear operator \(\partial_{\gamma} : C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^m) \to C^\infty (\mathcal{P}^* \mathcal{W}[\hbar] \otimes \Lambda^{m+1})\) defined in a Darboux chart by the formula
\[
\partial_{\gamma} a \overset{\text{def}}{=} da + \frac{i}{\hbar} [\gamma, a].
\]

The 1–form \(\gamma\) standing at the commutator
\[
\gamma \overset{\text{def}}{=} \frac{1}{2} \gamma_{ijk} y^j y^k dx^i.
\]

In the case when the connection \(\gamma\) is an induced symplectic connection, the 1–form \(\gamma\) contains three kinds of elements (indices \(\alpha, \beta, \epsilon, I\) are fixed!):

\begin{enumerate}
\item \(\frac{2 - \delta^m \alpha}{2} \gamma_{\alpha \beta \epsilon} y^\alpha y^\beta dq^\epsilon, \ \alpha \leq \beta,\)
\item \(\gamma_{I \beta \epsilon} y^\beta y^I dq^\epsilon,\)
\item \(\frac{2 - \delta^m \beta}{2} \gamma_{I \beta \epsilon} y^\beta y^I dp_{1-n}, \ \beta \leq \epsilon.\)
\end{enumerate}

For every symplectic connection 1–form \(\gamma\) its antiderivation \(\delta \gamma = 0\). Remember that the coefficients \(\gamma_{\alpha \beta \epsilon}\) are of the form \(2.13\).

The curvature 2–form \(R_{\gamma}\) of \(\gamma\) in a Darboux chart can be expressed by the formula
\[
R_{\gamma} = d\gamma + \frac{i}{2\hbar} [\gamma, \gamma] = d\gamma + \frac{i}{\hbar} \gamma \circ \gamma.
\]

In the case when \(\gamma\) is determined by the induced symplectic connection we obtain that \(R_{\gamma}\) consists of three types of terms (all indices are fixed!):
1. \( \frac{2-\delta_{\alpha\beta}}{2}K_{\alpha\beta}ey^\alpha y^\beta dq^e \wedge dq^\nu, \quad \alpha \leq \beta, \quad \epsilon < \nu, \)
2. \( K_I_{\alpha\beta}ey^\alpha y^\beta dq^\alpha \wedge dq^\nu, \quad \beta < \epsilon, \)
3. \( \frac{2-\delta_{\alpha\beta}}{2}K_{\alpha\beta}ey^{n+1}y^\alpha dq^\alpha \wedge dp_\nu, \quad \alpha \leq \beta. \)

The terms \( K_{\alpha\beta}e \) are homogeneous functions of momenta \( p_\alpha. \)

The property \( \delta R_\gamma = 0 \) follows from (4.14) and (4.10). It is equivalent to (2.6).

Let us introduce a new symbol. The \( a[v|i_1, \ldots, i_n|\tau[j, k]] \) is the coefficient standing at
\( a[v|i_1, \ldots, i_n|\tau[j, k]] p_\nu (y^1)^{i_1} \cdots (y^n)^{i_n} y^{j+n} dx^j \wedge dx^k, \quad 1 \leq j < k \leq 2n. \)

The crucial role in Fedosov’s deformation quantization is played by an Abelian connection \( \tilde{\gamma}. \) From the definition the Abelian connection \( \tilde{\gamma} \) is the connection in the Weyl algebra bundle which curvature is a central form so for any \( a \in C^\infty(P^*\mathcal{W}[h] \otimes \Lambda) \) we obtain \( \partial_\gamma (\partial_\gamma a) = 0. \)

The Abelian connection proposed by Fedosov is of the form
\( \tilde{\gamma} = \omega_{ij} y^i dx^j + \gamma + r. \)

Its curvature
\( R_{\tilde{\gamma}} = -\frac{1}{2} \omega_{ij} dx^i \wedge dx^j + R_\gamma - \delta r + \partial_\gamma r. \)

The requirement \( R_{\tilde{\gamma}} = -\frac{1}{2} \omega_{ij} dx^i \wedge dx^j \) imposes the following condition on the series \( r \)
\( \delta r = R_\gamma + \partial_\gamma r + \frac{i}{\hbar} \gamma r. \)

Fedosov has proved [7] that the equation (4.16) has a unique solution
\( r = \delta^{-1} R_\gamma + \delta^{-1} \left( \partial_\gamma r + \frac{i}{\hbar} \gamma r \right) \)
fulfilling the conditions: \( \delta^{-1} r = 0 \) and \( \deg(r) \geq 3. \)

Let \( r[z] \) denote the component \( r[z] \) \( \frac{\text{def}}{} \sum_{k=0}^{\lfloor \frac{z-1}{2} \rfloor} h^k r_m[2k, z-4k] dx^m, \quad z \geq 3 \) of \( r \) of the degree \( z. \) As it was shown [13] [16] [19]
\( r[z] = \delta^{-1} R_\gamma, \)
\( r[z] = \delta^{-1} \left( \partial_\gamma r[z-1] + \frac{i}{\hbar} \sum_{j=3}^{z-2} r[j] o r[z+1-j] \right), \quad z \geq 4. \)

Of course the foregoing equations may be written in a compact form
\( r[z] = \delta^{-1} R_{\gamma+r}[z-1], \quad z \geq 3. \)

The expression \( R_{\gamma+r}[z-1] \) is the part of the curvature of the connection \( \gamma + \sum_{i=3}^{z-1} r[i] \) of the degree \( (z-1). \) From the relation (4.16) we deduce that \( \delta R_{\gamma+r}[z] = 0 \) for \( z \geq 2. \) Moreover, the 2–form \( R_{\gamma+r} \) fulfills the Bianchi identity
\( dR_{\gamma+r} + \frac{i}{\hbar} [\gamma + r, R_{\gamma+r}] = 0. \)

In the case when \( \gamma \) is the induced symplectic connection not containing \( h, \forall z \geq 2 \frac{\partial R_{\gamma+r}[z]}{\partial h} = 0 \) (compare [13]). Moreover, we see that \( R_{\gamma+r}[z] \) is a sum of three kinds of elements:
1. \( R[0|i_1, \ldots, i_n|\alpha, \beta + n] (y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dp_\beta, \)
\( 0 \leq i_1, \ldots, i_n \leq z, \quad i_1 + \cdots + i_n = z. \) There are \( n^2 \left( \frac{z+n-1}{z} \right) \) elements of this type.
2. \( R[0|i_1, \ldots, i_n|\tau|\alpha, \beta] (y^1)^{i_1} \cdots (y^n)^{i_n} y^r dq^\alpha \wedge dq^\beta, \)
\[
0 \leq i_1, \ldots, i_n \leq z - 1, i_1 + \cdots + i_n = z - 1, \alpha < \beta. \] We get \( n \binom{(z - 1) + n - 1}{z - 1} \binom{n}{2} \) terms of this form.

3. \( R[v|i_1, \ldots, i_n|0|\alpha, \beta] p_v (y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dq^\beta, \)
\[
0 \leq i_1, \ldots, i_n \leq z, i_1 + \cdots + i_n = z, \alpha < \beta. \] There are \( n \binom{z + n - 1}{z} \binom{n}{2} \) elements of this kind.

Every function \( R[v|i_1, \ldots, i_n|\tau|j, k] \) depends only on spatial coordinates \( q^1, \ldots, q^n. \)

There are some constraints imposed on these functions. All of elements from the first class are chosen to be independent. Every coefficient standing at a term from the second group is determined by two coefficients belonging to the first class (see formula (4.22)). Among the third set we can choose a special group of \( n(z + 1) \binom{n + z}{z + 2} \) elements. The selection method will be presented below. Any other coefficient from the third group is a linear function of these selected ones.

Let us consider consequences of the restriction \( \delta R_{\tau + \tau}[z] = 0, z \geq 2. \) We start from
\[
n \binom{(z - 1) + n - 1}{z - 1} \binom{n}{2} \] nontrivial equations containing terms of the first and the second kind.

\[
(i_\alpha + 1) R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta, \ldots, i_n|0|\beta, \tau + n] - (i_\beta + 1) R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta + 1, \ldots, i_n|0|\alpha, \tau + n] + \]
\[
+ R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta, \ldots, i_n|\tau|\alpha, \beta] (y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dq^\beta \wedge dp_\tau = 0, \ (4.21)\]
where \( i_1 + \cdots + i_n = z - 1, \alpha < \beta. \) The number of equations (4.21) equals the number of coefficients of the second type. Moreover, in each equation only one coefficient of the type \( R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta, \ldots, i_n|\tau|\alpha, \beta] \) appears and each term \( R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta, \ldots, i_n|\tau|\alpha, \beta] \) is present in exactly one of these equations. Hence we conclude that every coefficient from the second set can be uniquely expressed by elements from the first collection. Indeed, from (4.21)
\[
R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta, \ldots, i_n|\tau|\alpha, \beta] =
\]
\[
= (i_\beta + 1) R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta + 1, \ldots, i_n|0|\alpha, \tau + n] - (i_\alpha + 1) R[0|i_1, \ldots, i_\alpha + 1, \ldots, i_n|0|\beta, \tau + n],
\]
\[
0 \leq i_1, \ldots, i_n \leq z - 1, \ i_1 + \cdots + i_n = z - 1, \alpha < \beta. \ (4.22)\]
The conditions
\[
(i_\alpha + 1) R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta + 1, \ldots, i_n|\tau|\beta, \kappa] - (i_\beta + 1) R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta + 1, \ldots, i_n|\tau|\alpha, \kappa] + \]
\[
+ (i_\kappa + 1) R[0|i_1, \ldots, i_\alpha, \ldots, i_\beta, \ldots, i_n|\tau|\alpha, \beta] \times \]
\[
x (y^1)^{i_1} \cdots (y^n)^{i_n} \{y^\gamma y^r dq^\alpha \wedge dq^\beta \wedge dq^\kappa = 0,\]
\[
0 \leq i_1, \ldots, i_n \leq z - 2, \ i_1 + \cdots + i_n = z - 2, \alpha < \beta < \kappa \ (4.23)\]
are nontrivial for \( 3 \leq n. \) They are imposed only on terms of the second kind. However, applying the relations (4.22) we turn them into identities.

Moreover, \( n^2 \) coefficients of the type \( R[0|0, \ldots, 0, i_\alpha |z = 0, \ldots, 0|0|\alpha, \tau + n] \) disappear because
\[
\delta \left( R[0|0, \ldots, 0, i_\alpha = z, 0, \ldots, 0|0|\alpha, \tau + n](y^n)^z dq^\alpha \wedge dp_\tau \right) = 0. \ (4.24)\]

Finally, let us investigate consequences of the condition \( \delta R_{\tau + \tau}[z] = 0, \ z \geq 2 \) for the elements of the third type. As it can be easily checked, each of these elements appears in at most \( \min[z, n - 2] \) equations.
Assume that the element $R[v, i_1, \ldots, i_n, 0][\alpha, \beta] p_{\nu} \,(y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dq^\beta$, is present exactly in $f$ equations, $0 \leq f \leq \min[z, n-2]$ following from the general condition $\delta R_{\gamma\tau}[z] = 0$. It means that among indices $i_\eta$ there are exactly $f$ numbers $i_\eta$ such that $\eta \neq \alpha, \eta \neq \beta$ and $i_\eta \neq 0$. The total number of terms present in exactly $f$ equations equals

$$n \left( \begin{array}{c} n \\ 2 \end{array} \right) \left( \begin{array}{c} n - 2 \\ f \end{array} \right) \left( \begin{array}{c} z + 1 \\ z - f \end{array} \right).$$

Each of the constraints following from the condition $\delta R_{\gamma\tau}[z] = 0$ contains three different coefficients $R[v, i_1, \ldots, i_n, 0][\alpha, \beta]$ and each of these coefficients appears in the same number of equations following from the constraint $\delta R_{\gamma\tau}[z] = 0$. Moreover, any arbitrary pair of coefficients appears in at most one equation.

Hence the set of

$$n \left( \begin{array}{c} n - 1 + (z - 1) \\ z - 1 \end{array} \right) \left( \begin{array}{c} n \\ 3 \end{array} \right)$$

equations following from the requirement $\delta R_{\gamma\tau}[z] = 0$ can be divided in $\min[z, n-2]$ separate classes containing only coefficients $R[v, i_1, \ldots, i_n, 0][\alpha, \beta]$ appearing in the analysed formulas for exactly $f$ times each, where $1 \leq f \leq \min[z, n-2]$. Each class consists of

$$n \left( \begin{array}{c} n \\ f + 2 \end{array} \right) \left( \begin{array}{c} z + 1 \\ z - f \end{array} \right)$$

independent blocks. Every block is a system of \( f + 1 \) \((f + 2)\) linear equations containing \((f + 1)\) coefficients. Among these equations only \((f + 1)\) conditions are linearly independent. Therefore from

$$\left( \begin{array}{c} f + 1 \\ 2 \end{array} \right)$$

coefficients only \((f + 1)\) terms are independent. The choice of these \((f + 1)\) elements is not arbitrary. We propose it below.

Each expression $R[v, i_1, \ldots, i_n, 0][\alpha, \beta] p_{\nu} \,(y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dq^\beta$ belongs to exactly one block so each block may be characterized by the quantity $p_{\nu} \,(y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dq^\beta$. However, the block $p_{\nu} \,(y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dq^\beta$ can be equivalently determined by the expression

$$p_{\nu} \,(y^1)^{i_1} \cdots (y^n)^{i_n} + \cdots + (y^n)^{i_n} + \cdots + (y^n)^{i_n}.$$  

In the next paragraph we use this latter characterization of blocks.

Let us consider the block $p_{\nu} \,(y^1)^{i_1} \cdots (y^{(l+2)})^{i_{s+j}}$, where for $l = 1, \ldots, f + 2$ the indices $1 \leq i_{s_j}, 1 \leq s_j < s_1 < \cdots < s_{f+j+2} \leq n$. As the independent $f + 1$ coefficients $R[v, i_1, \ldots, i_n, 0][\alpha, \beta]$ we choose these ones standing at the exterior products $dq^\alpha \wedge dq^\beta, \ldots, dq^{i_j} \wedge dq^{i_{f+j+2}}$. After simple but tedious calculations we arrive at the following relation:

$$R[v, i_{s_j} - 1, \ldots, i_{s_j} - 1, \ldots, i_{s_k} - 1, \ldots, i_{s_{f+j+2}}, \ldots, 0][s_j, s_k] = \frac{i_{s_k} - i_{s_{f+j+2}}}{i_{s_{f+j+2}} - i_{s_k}} R[v, \ldots, i_{s_j} - 1, \ldots, i_{s_k} - 1, \ldots, i_{s_{f+j+2}} - 1, \ldots, 0][s_j, s_k, s_{f+j+2}].$$

The formula (4.25) can be applied in the cases when $j < k < f + 2$.

What is amazing, the elements $R[v, i_1, \ldots, i_n, 0][\alpha, \beta]$, $\sum_{j=1}^n i_j = z$ are determined by $R[0, i_1, \ldots, i_n, 0][\alpha, \beta + n]$ i.e. elements of the first type and the Abelian connection components of the degree less than $z$. The explicit form of this relation is contained in Appendix A.

From (4.19) we compute the element $r[z + 1], 2 \leq z$. We use the notation analogous to that applied in the previous considerations and by $r[v, i_1, \ldots, i_n, r[j]]$ we mean the coefficient standing at

$$p_{\nu} \,(y^1)^{i_1} \cdots (y^n)^{i_n} r[z + 1] dx^j, 1 \leq j \leq 2n.$$

There are three kinds of components of $r[z + 1]$:
1. \( r[0]|i_1, \ldots, i_n|0|\alpha + n| (y^1)^{i_1} \cdots (y^n)^{i_n} dp_\alpha, \)

\[ 0 \leq i_1, \ldots, i_n \leq z + 1, \ i_1 + \cdots + i_n = z + 1. \] We have \( n \left( \frac{z + n}{z + 1} \right) \) elements of this type. They appear as images of the elements \( R_{\gamma + r}[z] \) of the first kind in the mapping \( \delta^{-1} \).

2. \( r[0]|i_1, \ldots, i_n|0|\alpha| (y^1)^{i_1} \cdots (y^n)^{i_n} dy^n + dq^\alpha, \)

\[ 0 \leq i_1, \ldots, i_n \leq z, \ i_1 + \cdots + i_n = z. \] The number of expressions of this form is \( n^2 \left( \frac{z + n - 1}{z} \right) \). They come from applying the \( \delta^{-1} \) operator to the terms \( R_{\gamma + r}[z] \) of the first and the second type. And finally

3. \( r[y]|i_1, \ldots, i_n|0|\alpha| p^y(y^1)^{i_1} \cdots (y^n)^{i_n} dy^n, \)

\[ 0 \leq i_1, \ldots, i_n \leq z + 1, \ i_1 + \cdots + i_n = z + 1. \] There are \( n^2 \left( \frac{z + n}{z + 1} \right) \) elements generated by components of \( R_{\gamma + r}[z] \) of the third type.

The same classification can be applied for the 1-form of the symplectic connection \( \gamma \) (see page 12). The total symmetry of components \( \gamma_{ijk} \) in indices \( \{i, j, k\} \) implies

\[ \forall_{\alpha, \beta, \tau} \gamma[0]|0, \ldots, i_\alpha = 1, 0, \ldots |\tau|\beta = \gamma[0]|0, \ldots, i_\beta = 1, 0, \ldots |\tau|\alpha, \] (4.26)

\[ \forall_{\alpha, \tau} \gamma[0]|i_1, \ldots, i_\alpha, \ldots, i_n|\tau|\alpha = (i_\alpha + 1)\gamma[0]|i_1, \ldots, i_\alpha + 1, \ldots, i_n|\tau + n], \] (4.27)

\[ \forall_{\alpha < \beta} \forall_v 0 < i_\beta \gamma[v]|i_1, \ldots, i_\alpha, \ldots, i_\beta, \ldots, i_n|0|\alpha = \frac{i_\alpha + 1}{i_\beta} \gamma[v]|i_1, \ldots, i_\alpha + 1, \ldots, i_\beta - 1, \ldots, i_n|0|\beta. \] (4.28)

Let us consider some relations between three classes of components of \( r[z + 1], \ 2 \leq z. \)

After simple calculations we conclude that

\[ r[0], \ldots, i_k, \ldots, i_n|0|\alpha + n| = \frac{1}{z + 2} \sum_{l=1}^{u} R[0], \ldots, i_s, \ldots, i_n - 1, \ldots, i_s, \ldots, 0|s, \alpha + n]. \] (4.29)

We assume that \( 1 \leq u \leq n, \ \forall_l \ i_{s_l}, \ i_{s_1} + \cdots + i_{s_u} = z + 1. \)

In a special case \( r[0], \ldots, i_\beta = z + 1, 0, \ldots |0|\alpha + n| = \frac{1}{z^2} R[0], \ldots, (i_\beta - 1) = z, 0, \ldots |0|\beta, \alpha + n]. \)

But we know that \( \forall_\beta R[0], \ldots, (i_\beta - 1) = z, 0, \ldots |0|\beta, \alpha + n| = 0 \) so

\[ r[0], \ldots, i_\beta = z + 1, 0, \ldots |0|\alpha + n] = 0. \] (4.30)

The elements of the second kind are determined by the formula

\[ r[0], \ldots, i_\alpha, \ldots, i_n|\tau|\alpha = -\frac{1}{z + 2} R[0], \ldots, i_s, \ldots, i_n, \ldots |0|\alpha, \tau + n| + \]

\[ + \frac{1}{z + 2} \sum_{l \text{ that } s_l < \alpha} R[0], \ldots, i_{s_{l}}, \ldots, i_n - 1, \ldots, i_\alpha, \ldots, i_n, \ldots |\tau|s_l, \alpha| + \]

\[ - \frac{1}{z + 2} \sum_{l \text{ that } s_l > \alpha} R[0], \ldots, i_{s_1}, \ldots, i_\alpha, \ldots, i_{s_l}, \ldots, i_n - 1, \ldots, i_n, \ldots |\tau|s_l, \alpha| \]

\[ = \frac{i_\alpha - z - 1}{z + 2} R[0], \ldots, i_{s_1}, \ldots, i_n, \ldots |0|\alpha, \tau + n| + \]

\[ + \frac{i_\alpha + 1}{z + 2} \sum_{l \text{ that } s_l \neq \alpha} R[0], \ldots, i_{s_{l}}, \ldots, i_n - 1, \ldots, i_\alpha + 1, \ldots, i_n, \ldots |0|s_l, \tau + n|, \] (4.31)
Remember that for all \( l \) there is \( s_l \neq \alpha \) and \( 0 < s_l \) but it may be \( i_\alpha = 0 \).

The straightforward consequence of the relation (4.31) is the statement that for all \( \alpha, \tau \)
\[
r[0, \ldots, 0, i_\tau] = 0.
\] (4.32)

Applying (4.29) to the result (4.31) we obtain
\[
r[0, \ldots, i_{s_1}, \ldots, i_{s_n}, |\alpha] = (i_\alpha+1)r[0, \ldots, i_{s_1}, \ldots, i_\alpha+1, \ldots, i_{s_n}, |\alpha] - R[0, \ldots, i_{s_1}, \ldots, i_{s_n}, \ldots, |\alpha, \tau+n].
\] (4.33)

Finally we present formulas determining components of the correction \( r \) belonging to the 3rd category.
\[
r[v, \ldots, i_{s_1}, \ldots, i_{s_n}, |\alpha] = \frac{1}{z+2} \sum_{\text{all } l \text{ that } s_l < \alpha} R[v, \ldots, i_{s_1}, \ldots, i_{s_l}, i_l - 1, \ldots, i_{s_n}, \ldots, |\alpha, s_l] + \frac{1}{z+2} \sum_{\text{all } l \text{ that } s_l > \alpha} R[v, \ldots, i_{s_1}, \ldots, i_{s_l}, i_l - 1, \ldots, i_{s_n}, \ldots, |\alpha, s_l].
\] (4.34)

The straightforward consequence of (4.33) is the equality
\[
r[|\tau, 0, \ldots, 0, i_\alpha] = \frac{1}{z+2} \sum_{\text{all } l \text{ that } s_l < \alpha} R[v, \ldots, i_{s_1}, \ldots, i_{s_l}, i_l - 1, \ldots, i_{s_n}, \ldots, |\alpha, s_l].
\] (4.35)

which also results from the condition \( \delta^{-1}r = 0 \).

The relations (4.30), (4.32) and (4.35) yield (4.24).

As we know (see page 15), not all of elements \( R[v, \ldots, i_{s_1}, \ldots, i_{s_n}, \ldots, |\alpha, \beta] \) are independent. Assume that \( s_n = \alpha \) in (4.34). Then
\[
r[v, \ldots, i_{s_1}, \ldots, i_{s_n}, \ldots, |\alpha] = \frac{1}{z+2} \sum_{l=1}^{n-1} R[v, \ldots, i_{s_1}, \ldots, i_{s_l}, i_l - 1, \ldots, i_{s_n}, \ldots, |\alpha, s_l].
\] (4.36)

The relation determining \( r[v, \ldots, i_{s_1}, \ldots, i_{s_n}, \ldots, |\alpha] \) for \( s_u < \alpha \) is a slight modification of (4.36)
\[
r[v, \ldots, i_{s_1}, \ldots, i_{s_u}, \ldots, |\alpha] = \frac{1}{z+2} \sum_{l=1}^{u} R[v, \ldots, i_{s_1}, \ldots, i_{s_l}, i_l - 1, \ldots, i_{s_{u}}, \ldots, |\alpha, s_l].
\] (4.37)

For \( s_u > \alpha \) from (4.25) we obtain
\[
r[v, \ldots, i_{s_1}, \ldots, i_{s_u}, \ldots, |\alpha] = \frac{i_\alpha + 1}{i_{s_u}(z+2)} \sum_{l=1}^{u-1} R[v, \ldots, i_{s_1}, \ldots, i_{s_l}, i_l - 1, \ldots, i_{\alpha+1}, \ldots, i_{s_u} - 1, \ldots, |\alpha, s_l] + \frac{1}{i_{s_u}} R[v, \ldots, i_{s_1}, \ldots, i_{\alpha}, \ldots, i_{s_u} - 1, \ldots, |\alpha, s_u] + \frac{1}{i_{s_u}} R[v, \ldots, i_{s_1}, \ldots, i_{\alpha}, \ldots, i_{s_u} - 1, \ldots, |\alpha, s_u].
\] (4.38)

Remember, that although the formulas (4.36), (4.37) and (4.38) contain the curvature 2-form components of the third kind, due to the Bianchi identity they are in fact determined exclusively by the elements \( R[0, i_{\alpha}, \ldots, i_{n} |\alpha, \beta + n] \).

Then we are ready to construct the iterative formula determining \( R_{\gamma+r}[z] \) by all \( R_{\gamma+r}[v] \), \( 2 \leq v \leq z-1 \). We see that it is sufficient to find the relation defining components \( R_{\gamma+r}[z] \) of the first kind.

Starting from the definition of the curvature \( R_{\gamma+r} \) and applying the formula (4.30) we obtain that
The element \( a = \sigma^{-1}(a_0) \) can be found by the iteration

\[
a = a_0 + \delta^{-1} \left( \partial_y a + \frac{i}{\hbar} [r, a] \right). \tag{4.40}
\]

This relation means that

\[
a[0] = a_0,
\]

\[
a[z] = \delta^{-1} \left( \partial_y a[z - 1] + \frac{i}{\hbar} \sum_{l=1}^{z-2} \left[ r[z + 1 - l], a[l] \right] \right), \quad z \geq 1. \tag{4.41}
\]

Since now we restrict our considerations to the situation when the symplectic manifold \((W, \omega)\) is a cotangent bundle \(T^*M\) and \(\gamma\) is an induced symplectic connection on it. We focus on some properties of the series \(\sigma^{-1}(a_0)\) in this case.

Let us start from the following observation.

**Corollary 4.1.** An element \(h^k g(y^1)^{i_1} \cdots (y^{2n})^{i_{2n}} \in C^\infty(P^*W[[\hbar]]), \ g \in C^\infty(T^*M), \ i_1 + \cdots + i_{2n} \geq 1\) is given. The expression

\[
\delta^{-1} \left( \frac{i}{\hbar} \left[ \gamma + r, h^k g(y^1)^{i_1} \cdots (y^{2n})^{i_{2n}} \right] \right) = \sum_{d=0}^{\infty} \sum_{\text{all possible } j_1 + \cdots + j_{2n} = 1} h^{k+2d} b_{k+2d} j_1 \cdots j_{2n} (y^1)^{j_1} \cdots (y^{2n})^{j_{2n}}
\]

The coefficients \(\gamma_{\alpha\beta\delta}\) influence the Abelian connection \(\gamma + r\) only indirectly through the elements \(K_{\alpha\beta\delta l}\).
where $\gamma$ is the 1–form representing the induced symplectic connection, $r$ is the Abelian connection series generated by $\gamma$ and $b_{k+2d,j_1...j_{2n}}$ are some smooth functions on $T^*\mathcal{M}$. Moreover

$$\sum_{l=1}^{n} j_{l+n} = \sum_{l=1}^{n} i_{l+n} - 2d$$

for elements obtained from commutators with $\gamma$ or $r$ of the first and of the second kind (see the classification on page 17) and

$$\sum_{l=1}^{n} j_{l+n} = \sum_{l=1}^{n} i_{l+n} - 2d - 1$$

if components of $\gamma$ or $r$ were of the third kind.

This corollary is the straightforward consequence of two definitions: of the commutator and of the operator $\delta^{-1}$.

Thus we see that the commutators appearing in the recurrence (4.40) do not increase the number of $y$'s with momenta indices. Moreover $\forall_{K} \ j_{K} \leq i_{K} + 1$ if the commutators are calculated with elements of $\gamma + r$ of the 1st and 2nd kind and $\forall_{K} \ j_{K} \leq i_{K}$ if the commutators with $\gamma + r$ of the 3rd type are considered. We recall that the capital letters correspond to momenta coordinates.

Therefore we observe that the total number of momenta-like elements $(y^{K})^{i_{K}}$ may increase only in the operation $\delta^{-1}(\hbar^{k}g_{y}(y^{1}_{i_{1}} \cdots (y^{2n})_{i_{2n}}))$.

In contrary, let an element $g(y^{1}_{i_{1}} \cdots (y^{2n})_{i_{2n}}), \ g \in C^{\infty}(T^*\mathcal{M}), \ \sum_{s=1}^{n} i_{s} = l$ be given. Then every term generated from this element by the recurrence (4.40) contains at least $l$ position-like components $y^{\alpha}$.

**Corollary 4.2.** Let $a_{0} = (p_{1})^{i_{1}} \cdots (p_{n})^{i_{n}} f(q^{1}, \ldots, q^{n})$ be some smooth function defined on the cotangent bundle $T^*\mathcal{M}$. Then $\sigma^{-1}(a_{0})$ consists only of elements of the form

$$\hbar^{2d}d_{2d}l_{1} \ldots l_{n} \ j_{1} \ldots j_{2n}(q^{1}, \ldots, q^{n})p_{1}^{i_{1}} \cdots p_{n}^{i_{n}}(y^{1})^{j_{1}} \cdots (y^{2n})^{j_{2n}}$$

such that

$$2d + \sum_{s=1}^{n} l_{s} + \sum_{s=1}^{n} j_{n+s} = \sum_{s=1}^{n} i_{s}.$$

The proof of this corollary can be done with the use of the mathematical induction and the Corollary 4.1

Hence we conclude that if $a_{0}$ is a function of the spatial coordinates only then the series $\sigma^{-1}(a_{0})$ contain neither powers of the deformation parameter $\hbar$ nor $y^{K}$. Thus $\sigma^{-1}(a_{0})[z]$ is a polynomial in $y^{\alpha}$ of the degree $z$. Moreover, there are not any terms $(p_{i})^{l_{i}}, \ l_{i} > 0$ in this series. Degrees of partial derivatives of $a_{0}$ in $\sigma^{-1}(a_{0})[z]$ are from the ordered set $\{1, \ldots, z\}$.

In the process of generating $\sigma^{-1}(a_{0})$ in this case only two kinds of elements can be different from 0: the exterior derivatives and the commutators with components $\gamma + r$ of the 2nd category.

For an function $a_{0} \in C^{\infty}(T^*\mathcal{M})$ the element $a[z], z \geq 1$ consists of the terms of the form

$$\hbar^{2d}(p_{1})^{i_{1}} \ldots (p_{n})^{i_{n}} f(q^{1}, \ldots, q^{n}) \frac{\partial^{j_{1}+\cdots+i_{2n}}}{\partial^{q^{1}_{j_{1}}}\cdots\partial^{q^{n}_{j_{n}}}} a_{0}(y^{1})^{j_{1}} \cdots (y^{2n})^{j_{2n}}.$$
4. \[ \sum_{s=1}^{n} i_{s+n} = 2d + \sum_{s=1}^{n} j_{s+n} + \sum_{s=1}^{n} l_s. \] (4.42)

The Corollary is compatible with this statement.

5. The maximal value of the sum \( \sum_{s=1}^{n} l_s = \left\lfloor \frac{z}{2} \right\rfloor \). If \( \sum_{s=1}^{n} l_s = \frac{z}{2} \), where \( z \) is an even number, then \( 2d + \sum_{s=1}^{n} j_{s+n} = 0 \) and \( \sum_{s=1}^{n} i_{s+n} = \frac{z}{2} \).

6. Every function \( f(q^1, \ldots, q^n) \) is a polynomial in symplectic connection coefficients \( \gamma_{\alpha\beta\delta} \). Partial derivatives of components \( \gamma_{\alpha\beta\delta} \) with respect to momenta and partial derivatives of these both groups of quantities with respect to spatial coordinates.

All of these observations follow from simple but boring analysis of the Fedosov recurrence.

Using the one-to-one correspondence between the collection of the flat sections \( \mathcal{P}^* \mathcal{W}[[h]] \) and the set \( C^\infty(\mathcal{W}) \) we introduce the associative star product \( \ast \) of functions \( a_0, b_0 \in C^\infty(\mathcal{W}) \)

\[ a_0 \ast b_0 \overset{\text{def.}}{=} \sigma^{-1}(a_0) \circ \sigma^{-1}(b_0). \] (4.43)

This \( \ast \)-product is natural and of the Weyl type.

Moreover, applying the definition (4.43) to the \( \ast \)-product of functions depending only on spatial coordinates we see that their \( \ast \)-product is the usual pointwise multiplication of functions. But in the case we multiply two elements of the form

\[ [(p_1)^{i_1} \cdots (p_n)^{i_n} f(q^1, \ldots, q^n)] \ast [(p_1)^{j_1} \cdots (p_n)^{j_n} g(q^1, \ldots, q^n)] \]

we see from the Corollary that the maximal power of \( h \) appearing in this product do not exceed \( h^{i_1+\cdots+i_n+j_1+\cdots+j_n} \).

Then for example in the case of 3-D base space

\[ p_1 \ast p_2 = p_1 p_2 + \frac{\hbar^2}{4} \left( \gamma_{411} \gamma_{412} + \gamma_{422} \gamma_{511} + \gamma_{412} \gamma_{512} + \gamma_{512} \gamma_{522} + \gamma_{423} \gamma_{611} + \gamma_{413} \gamma_{612} + \gamma_{523} \gamma_{612} + \gamma_{513} \gamma_{622} + \gamma_{613} \gamma_{623} \right). \]

What is even more interesting also \( p_\alpha \ast p_\beta \neq p_\alpha \cdot p_\beta \) in general.

The part of the product \( p_\alpha \ast p_\beta \) standing at \( h^2 \) depends only on the coefficients of the linear connection from the base space (see (2.12)). Hence if the configuration space \( \mathcal{M} \) is flat then we can always choose the spatial coordinates so that for all momenta canonically conjugated with them \( p_\alpha \ast p_\beta = p_\alpha \cdot p_\beta \).

The general form of a \( \ast \)-product of smooth functions on a Poisson manifold is

\[ a_0 \ast b_0 = \sum_{i=0}^{\infty} \hbar^i B_i(a_0, b_0), \]

where \( B_i(\cdot, \cdot) \) are some bidirectional operators. Let us write \( \sigma^{-1}(a_0)[z] \) in the following form

\[ \sigma^{-1}(a_0)[z] = a[z] = \sum_{d_a=0}^{2d_a} \hbar^{2d_a} a_{2d_a} [z_a - 4d_a]. \]

Then for every \( \ast \)-product calculated according to the Fedosov method the term \( B_i(a_0, b_0) \) depends only on elements \( a[z_a], b[z_b] \), which satisfy the following relations

\[ 1 \leq z_a, z_b \leq 2i - 1, \]
\[ z_a + z_b = 2i, \]
\[ d_a + d_b \leq \frac{2i}{4}, \]
\[ z_b - z_a = 4(d_b - d_a), \]
\[ z_a - 4d_a \leq i, \quad z_b - 4d_b \leq i. \] (4.44)
In the case when the \(*\)-product is generated on a cotangent bundle according to Fedosov’s algorithm with some induced symplectic connection, in a proper Darboux chart all of expressions $B_i(a_0, b_0)$ for a fixed $i \geq 1$ are sums of elements

$$f(q^1, \ldots, q^n) \frac{\partial^{k_1+\cdots+k_{2n}} a_0}{\partial (q^{1})^{k_1} \cdots \partial (p_n)^{k_{2n}}} \frac{\partial^{j_1+\cdots+j_{2n}} b_0}{\partial (q^{1})^{j_1} \cdots \partial (p_n)^{j_{2n}}} \quad \text{and}$$

$$p_1^{s_1} \ldots p_n^{s_n} g(q^1, \ldots, q^n) \frac{\partial^{k_1+\cdots+k_{2n}} a_0}{\partial (q^{1})^{k_1} \cdots \partial (p_n)^{k_{2n}}} \frac{\partial^{j_1+\cdots+j_{2n}} b_0}{\partial (q^{1})^{j_1} \cdots \partial (p_n)^{j_{2n}}}.$$  \hfill (4.45)

Moreover, $\sum_{v=1}^n s_v \leq \left[ \frac{i}{2} \right]$. Let us assume that $a_0(q^1, \ldots, q^n)$ is some function only of spatial coordinates. Then the component $B_i(a_0, b_0)$ of the product of $a_0$ with an arbitrary function $b_0(q^1, \ldots, p_n)$ satisfies the following properties:

1. $z_b - z_a = 4d_b$ and therefore $z_b \geq z_a$, $i \geq z_a$, $z_b \geq i$.
2. $z_b = i + 2d_b$ so $z_a, z_b$ and $i$ are of the same parity,
3. only components of $\sigma^{-1}(b_0)$ containing exclusively momenta-like $y^K$ appear in the product $a_0 * b_0$. Thus the terms of the kind \textbf{(4.46)} from the series $\sigma^{-1}(b_0)$ do not appear in the deformed multiplication.
4. Hence

$$\forall_{i \geq 1} \quad B_i(a_0, b_0) = \sum_{\text{all possible}} g_{i_1 \ldots i_{2n}}(q^1, \ldots, q^n) \times$$

$$\times \frac{\partial^{|i_1+\cdots+i_{2n}|} a_0}{\partial (q^{1})^{i_1} \cdots \partial (p_n)^{i_{2n}}} \frac{\partial^{j_1+\cdots+j_{2n}} b_0}{\partial (p_1)^{j_1} \cdots \partial (p_n)^{j_{2n}}}.$$  \hfill (4.46)

Moreover, if $a_0(q^1, \ldots, q^n)$ is some function only of spatial coordinates and $b_0 = b_1(q^1, \ldots, q^n) \cdot b_2(p_1, \ldots, p_n)$ then

$$a_0 * b_0 = b_1 \cdot (a_0 * b_2).$$

The canonical variables $q^1, \ldots, p_n$ fulfill commutation relations consistent with the Dirac quantization rules i.e.

$$\{q^\alpha, q^\beta\}_M = 0, \quad \{q^\alpha, p_\beta\}_M = -ih\delta^\alpha_\beta, \quad \{p_\alpha, p_\beta\}_M = 0.$$  \hfill (4.47)

By the symbol $\{\cdot, \cdot\}_M$ we denote the Moyal bracket of functions

$$\{a_0, b_0\}_M \overset{\text{def}}{=} a_0 * b_0 - b_0 * a_0.$$

Observe that the sign in the second equation \textbf{(4.47)} is the consequence of the Fedosov sign convention. The commutation relations \textbf{(4.47)} are invariant under the point transformations \textbf{(2.10)}.

**Examples**

Let us consider the Fedosov construction in the case when the induced symplectic connection is given by the relations \textbf{(5.38)}. As we remember, in this situation the symplectic curvature tensor satisfies the property \textbf{(3.33)}. Moreover, from the relation \textbf{(4.39)} we can see that this symmetry is inherited by the coefficients $R[0][i_1, \ldots, i_n][\alpha, \beta + n](y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dp_\beta$ of higher degrees. Thus we conclude that
the Abelian connection series generated by the induced symplectic connection \((3.28)\) is determined by the relatively simple formulas:

\[
\forall i_1, \ldots, i_n \forall \alpha \ r[0|i_1, \ldots, i_n|0|\alpha + n] = 0,
\]

\[
\forall i_1, \ldots, i_n \forall \alpha, \tau \ r[0|i_1, \ldots, i_n|\tau|\alpha] = -R[0|i_1, \ldots, i_n|0|\alpha, \tau + n],
\]

\[
\forall i_1, \ldots, i_n \forall \alpha, \beta \ R[0|i_1, \ldots, i_n|0|\alpha, \beta + n] = -r[\beta|i_1, \ldots, i_n|0|\alpha] +
\]

\[
- \sum_{m=1}^{n} \sum_{g_1 + \ldots + g_n = 1 \atop \forall i_n \leq g_n \leq \text{Min}[i_n, 1] \}} (g_1 + 1)\gamma_0 g_1, \ldots, g_m + 1, \ldots, g_n|0|\beta + n\cdot R[0|i_1 - g_1, \ldots, i_m - g_m, \ldots, i_n - g_n|0|\alpha, m + n].
\]

Unfortunately the relations determining \(R[v|i_1, \ldots, i_n|0|\alpha, \beta]\) and hence \(r[\beta|i_1, \ldots, i_n|0|\alpha]\) are still complicated. Thus the formula representing \(\sigma^{-1}(a_0)\) does not have a compact form.

Now we are going to mention the case when the symplectic connection is of the form \((3.35)\). The Abelian connection consists only of elements of the first and of the second type. Then every component \(B_l(a_0, b_0)\) of the product \(a_0 \ast b_0\) is a sum of terms of the kind \((4.45)\) only. All functions \(k(q^1, \ldots, q^n)\) from the formula \((4.3)\) are polynomials in the linear connection \(\Gamma^\alpha_{\beta\delta}\) coefficients and their partial derivatives.

### 5 Conclusions

We introduced some class of symplectic connections on a cotangent bundle \(T^*M\). These connections are constructed in a proper Darboux atlas but their construction is global. Since they are modelled by a linear connection from the base space \(M\) we call them induced symplectic connections. The induced symplectic connections are homogeneous. Among all symplectic connections with which the cotangent bundle \(T^*M\) can be equipped, the induced symplectic connections seem to be the most natural. The reasons are that they contain the linear connection from the base manifold \(M\) and are of simple structure.

There is a deep geometrical relation between the induced symplectic connection and the linear connection inducing it. For every smooth curve on \(T^*M\) and every vector field \(X\) transport parallely along this curve with respect to the induced symplectic connection we obtain that the projected vector field \(\pi_*(X)\) is parallely propagated along the projected curve on \(M\) with respect to the linear connection inducing our symplectic connection.

For the induced symplectic connections the recurrent formulas from the Fedosov scheme give relatively simple results. Having completed the curvature \(R_{\gamma+r}\) and the Abelian connection \(\gamma + r\) up to the degree \(z\) to realize the \(z+1\) step of the recurrence it is sufficient to find all terms

\[
R[0|i_1, \ldots, i_n|0|\alpha, \beta + n](y^1)^{i_1} \cdots (y^n)^{i_n} dq^\alpha \wedge dp_\beta.
\]

The coefficients \(R[0|i_1, \ldots, i_n|0|\alpha, \beta + n]\) are functions of spatial coordinates only.

The Abelian correction \(r\) does not contain the deformation parameter \(\hbar\) and is a sum of three groups of elements.

The Fedosov scheme of calculating the Abelian connection reduces to the loop:

1. the Abelian connection 1–form \(\gamma + \sum_{l=3}^{v} r[l]\), the curvature 2–form \(\sum_{l=2}^{v} R_{\gamma+r}[l]\) components of the 1st kind and the curvature 2–form \(\sum_{l=2}^{v} R_{\gamma+r}[l]\) components \(R[v, \ldots, i_2, \ldots, i_{s_\alpha}, \ldots|0|\alpha, \beta]\), \(s_\alpha \leq \beta\) of the 3rd kind are known.

2. Using the formula \((4.29)\) one finds all of coefficients \(r[0|i_1, \ldots, i_n|0|\alpha + n]\), \(i_1 + \cdots + i_n = z + 1\).

3. Then from \((4.33)\) one obtains the elements of the form \(r[0|i_1, \ldots, i_n|\tau|\alpha]\), \(i_1 + \cdots + i_n = z\).

4. In the next step one calculates \(r[v|i_1, \ldots, i_n|0|\alpha]\), \(i_1 + \cdots + i_n = z + 1\) applying the relations \((4.36)\), \((4.37)\) and \((4.38)\).
5. From (4.39) one gets  
\[ R[0|i_1, \ldots, i_n|0|\alpha, \beta + n], \quad i_1 + \cdots + i_n = z + 1. \]

6. Finally from the formula presented in the Appendix A one has \( R[v|, i_1, \ldots, i_n|0|\alpha, \beta] \) for \( s_u \leq \beta, \quad i_{s_1} + \cdots + i_{s_n} = z + 1. \)

7. One comes back to the 1st step of the loop.

Simplicity of the Abelian connection influences on the form of a series \( \sigma^{-1}(a_0) \) although the general formula of this series is still complicated. But for example in the case \( a_0 \) is a function of spatial coordinates only, its series \( \sigma^{-1}(a_0) \) contains neither any powers of \( \hbar \) nor elements \( y^k \). The element \( \sigma^{-1}(a_0)[z] \) is a polynomial of the degree \( z \) with respect to \( y^\alpha \).

For functions of the form \( (p_1)^{i_1} \cdots (p_n)^{i_n} f(q^1, \ldots, q^n) \) series representing them consist only of elements \( \hbar^{2d} (p_1)^{i_1} \cdots (p_n)^{i_n} g(q^1, \ldots, q^n)(y^1)^{j_1} \cdots (y^{2n})^{j_2n} \), \( d \geq 0 \), where 2\( d + \sum_{l=1}^{n} l_i + \sum_{k=1}^{n} k_i + n = \sum_{l=1}^{n} i_l. \)

The \(*\)-product of functions depending only on spatial coordinates is the usual commutative product of them. In more general situation when \( a_0(q^1, \ldots, q^n) \) is a function of spatial coordinates and \( b_0(q^1, \ldots, p_n) \) is an arbitrary function, coefficients standing at nonzero differential operators in \( B_2(a_0, b_0) \) are functions only of spatial coordinates. Moreover, only partial derivative operators with respect to momenta acting on \( b_0 \) are present.

The maximal power of \( \hbar \) in the \(*\)-product of polynomials in momenta does not exceed the sum of degrees of these polynomials.

Finally, the commutation relations determined by the Dirac quantization for position and momenta are fulfilled and they are invariant under the proper Darboux transformations.

### A The formula determining \( R[v|i_1, \ldots, i_n|0|\alpha, \beta] \)

After simple but tedious calculations based on the Bianchi identity (4.20) we get

\[
R[v|i_1, \ldots, i_n|0|\alpha, \beta] = \\
\sum_{l=1}^{n} \sum_{s_1+\cdots+s_n\geq z+1} (s_l+1)R[0|s_1, \ldots, s_l+1, \ldots, s_n|0|\alpha, \beta + n, \gamma|0|i_1-s_1, \ldots, i_l-s_l, \ldots, i_n-s_n|l|\beta] + \\
\sum_{l=1}^{n} \sum_{s_1+\cdots+s_n\leq z-2} (s_l+1)R[0|s_1, \ldots, s_l+1, \ldots, s_n|0|\alpha, \beta, \gamma|0|i_1-s_1, \ldots, i_l-s_l, \ldots, i_n-s_n|l|\beta] + \\
\sum_{l=1}^{n} \sum_{s_1+\cdots+s_n\geq z+1} (s_l+1)R[0|s_1, \ldots, s_l+1, \ldots, s_n|0|\alpha, \beta, \gamma|0|i_1-s_1, \ldots, i_l-s_l, \ldots, i_n-s_n|l|\alpha] + \\
\sum_{l=1}^{n} \sum_{s_1+\cdots+s_n\leq z-2} (s_l+1)R[0|s_1, \ldots, s_l+1, \ldots, s_n|0|\beta, \gamma|0|i_1-s_1, \ldots, i_l-s_l, \ldots, i_n-s_n|l|\alpha] + \\
\sum_{l=1}^{n} \sum_{s_1+\cdots+s_n\geq z+1} (i_l-s_l+1)R[0|s_1, \ldots, s_l+1, \ldots, s_n|l|\alpha, \beta, \gamma|0|i_1-s_1, \ldots, i_l-s_l+1, \ldots, i_n-s_n|0|\nu+n] + \\
\sum_{l=1}^{n} \sum_{s_1+\cdots+s_n\leq z-2} (i_l-s_l+1)R[0|s_1, \ldots, s_l+1, \ldots, s_n|l|\alpha, \beta, \gamma|0|i_1-s_1, \ldots, i_l-s_l+1, \ldots, i_n-s_n|0|\nu+n].
\]

It is assumed that \( \sum_{l=1}^{n} i_l = z > 2 \). Remember that elements \( R[0|s_1, \ldots, s_n|l|\alpha, \beta] \) are linear functions of components of the 1st kind (4.22).
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