Hyperheavenly spaces theory in application to Walker and para-Kähler spaces. I.

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Abstract.
Spaces equipped with congruences of null strings are considered. A special attention is paid to the spaces which belong to the two-sided Walker class and para-Kähler class. Properties of an intersection of self-dual and anti-self-dual congruences of null strings are used as an additional criterion for a classification of such spaces. Finally, a few examples of para-Kähler and para-Kähler-Einstein spaces are presented.

1 Introduction

1.1 Background
This article is thought of as a first part of more extensive work devoted to the para-Hermite and para-Kähler spaces (abbreviated by pH-spaces and pK-spaces, respectively; also pHE stands for para-Hermite-Einstein and pKE stands for para-Kähler-Einstein). PH structures were defined for the first time in [22]. Since that time pH structures and their modifications appeared in many geometrical problems. Recently, such spaces have been considered in the papers [1–3]. In [2] a relation between homogeneous pKE metric called the dancing metric and (2,3,5)-distributions with maximal algebra of infinitesimal symmetries has been found. In [1] it has been described how to construct a rank 2 distribution from 4-dimensional, conformal, neutral signature structures. Such a distribution is called twistor distribution. A fundamental invariant of twistor distributions, so-called Cartan quartic, depends on the components of the Weyl curvature of the conformal structure. The question arose: whether the root type of the Cartan quartic of twistor distribution agrees with the root type of the polynomial constructed from the anti-self-dual part of the Weyl tensor? In [3] it has been proven that the answer for this question is positive what is a highly nontrivial result.

In [3] the authors emphasize the importance of pKE metrics. To obtain examples of pKE metrics the authors directly integrated Cartan structure equations. They presented examples of pKE-spaces for which the anti-self-dual Weyl spinor is of the Petrov-Penrose (real) types [II], [III] and [N]. They also found a general form of the metric of pKE-space of the type [D] (to be more precise, it is the metric of the type [D]_{mn} \otimes [D]_{ce}, see Sections
1.3 and 2.2 for explanation of the symbols). This metric depends on 5 constants. Examples of pKE and pHE metrics of various Petrov-Penrose types have been presented also in our previous work [9]. These examples have been obtained as a real neutral slice of the 4-dimensional, complex space called the hyperheavenly space (HH-space). Note, that there are examples of the metrics of the type $[D]^{nn} \otimes [D]^{ee}$ in [9]. However, these examples depend on 3 complex constants only and they are not general. Thus, a progress made on this field by Bor, Makhmali and Nurowski is remarkable.

The results published in [1–3, 9] suggest that pKE-spaces play an important role in mathematical physics. Hence, the following research program seems to be reasonable:

(i) to classify pKE metrics for which the self-dual (SD) or the anti-self-dual (ASD) Weyl spinor is algebraically degenerated,

(ii) to solve vacuum Einstein equations with cosmological constant for different classes of pKE-spaces or - if these equations are too complicated to be solved completely - to reduce them as much as possible.

In this paper we have completed part (i). In fact, the classification we propose holds true also in pK, pH and pHE-spaces. We finalized also a tiny part of (ii). To explain which part of (ii) has been fulfilled we need to focus on the properties of the geometrical structures which are called congruences of null strings (also called foliations of null strings).

Congruences of null strings are families of totally null and geodesic 2-dimensional surfaces (the null strings). In real 4-dimensional spaces they appeared as the integral manifolds of the 2-dimensional, totally null distributions and they have been investigated since the fifties [35] (Walker spaces). Nevertheless, congruences of null strings are admitted only by neutral spaces. They cannot exist in Lorentzian spaces and in Riemannian spaces. However, congruences of null strings are also admitted by complex 4 dimensional spaces called the hyperheavenly spaces ($\mathcal{H}$H-spaces) [28, 29]. This fact is crucial for our further work.

Hyperheavenly spaces evolved from heavenly spaces ($\mathcal{H}$-spaces) in the seventies and among spinors [23, 25] and twistors they are a powerful tool in complex analysis of a spacetime. $\mathcal{H}$H-spaces are defined as a complex 4-dimensional manifolds equipped with a holomorphic metric which satisfies the vacuum Einstein equations with cosmological constant and for which SD (or ASD) part of the Weyl tensor is algebraically degenerated. Null strings in $\mathcal{H}$H-spaces are 2-dimensional holomorphic surfaces (in neutral spaces they are 2-dimensional real surfaces).

Properties of congruences of null strings have been investigated in [30]. For our purposes it is necessary to mention about their most important property which is called expansion of the congruence. In general, congruences of null strings are not parallelly

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1It is well-known that a general solution of the Einstein vacuum equations of the type $[D]$ in Lorentzian spaces depends on 7 constants (this solution is known as Plebański-Demiański solution [24]). Spaces equipped with a neutral signature metric which are of the (real) type $[D]$ splits into three classes. The first class is the type $[D_r]^{nn} \otimes [D_r]^{nn}$. It depends only on cosmological constant. The second class is the type $[D_r]^{mn} \otimes [D_r]^{ee}$ (it depends on 5 constants, [3]). The last class is the type $[D_r]^{ee} \otimes [D_r]^{ee}$. Such a space is not pKE anymore although it belongs to pHE class. The general metric of this last class remains unknown.

24-dimensional real manifolds equipped with a metric can be of three different types for which we use the following terminology. Spaces equipped with a metric of the signature $(+−−−)$ are called Lorentzian (or hyperbolic). Neutral spaces (also called split or ultrahyperbolic) are endowed with a metric of the signature $(+−−−)$. Finally, Riemannian spaces (also called proper-Riemannian or Euclidean) are equipped with a metric of the signature $(++++)$. 


propagated (for details see Section 2.3). Following Boyer, Finley and Plebański we call such congruences expanding \[4\]. The congruences which are parallely propagated we call nonexpanding. Additionally, congruences of null strings are SD or ASD\[3\] which depends on the orientation. After changing the orientation SD congruences become ASD congruences and vice-versa.

Real 4-dimensional Einstein spaces can be obtained as real slices \[34\] of $\mathcal{HH}$-spaces. It is quite hard to obtain Riemannian and Lorentzian spaces from $\mathcal{HH}$-spaces but real neutral slices of $\mathcal{HH}$-spaces can be obtained easily. Thus, $\mathcal{HH}$-spaces are useful tool in investigations of pHE and pKE-spaces \[9, 32\]. Also, pH and pK-spaces can be obtained as a real slice of weak $\mathcal{HH}$-spaces which are generalizations of $\mathcal{HH}$-spaces (see Section 2.6).

PH-spaces are usually defined as real neutral spaces equipped with an integrable almost para-complex structure\[4\] and a metric of split signature which satisfies certain compatibility condition (for a brief treatment of this topic see \[3\]). Equivalently, pH-spaces are equipped with two distinct expanding congruences of null strings of the same duality. PK-spaces are pH-spaces for which so called para-Kähler 2-form is closed. It is equivalent to the fact that both congruences of null strings are nonexpanding (see, e.g. \[32\]). Let the orientation be chosen in such a manner that both nonexpanding congruences of null strings are ASD. Hence, the ASD Weyl spinor of pK-spaces must be of the type $[D]$ or $[O]$. Consequently, pK-spaces can be of the types $[\text{any}] \otimes [D]^{nn}$ (if curvature scalar $R \neq 0$) or $[\text{any}] \otimes [O]^{n}$ (if $R = 0$, in this case a space is SD).

Consider now pK-spaces equipped with an additional congruence of SD null strings. This SD congruence can be expanding or nonexpanding. The present paper is devoted to the pK-spaces equipped with a nonexpanding congruence of SD null strings. It will be shown that these are spaces of the types $[\text{II}, D]^{n} \otimes [D]^{nn}$ or $[\text{III}, N]^{n} \otimes [O]^{n}$. Also, if more SD congruences of null strings exist in a space then this space is of one of the types $[\text{II}, D]^{nec} \otimes [D]^{nn}$, $[D]^{nn} \otimes [D]^{nn}$, $[\text{III}]^{nec} \otimes [O]^{n}$ and finally $[\text{II}]^{nec} \otimes [D]^{nn}$. However, this last type is not analyzed in this article\[5\]. Note, that among pKE-spaces only types $[\text{II}]^{n} \otimes [D]^{nn}$, $[D]^{nn} \otimes [D]^{nn}$ and $[\text{III}, N]^{n} \otimes [O]^{n}$ are possible and we found all metrics of these types.

Our considerations are local and in general complex. We consider complex manifolds of dimension four equipped with a holomorphic metric. The results can be easily carried over to the case of real manifolds with a neutral signature metric. We do not consider real Lorentzian slices and real Riemannian slices of the metrics presented in this paper.

1.2 Summary of main results

There are three main aims of this paper. The first aim is a detailed analysis of the spaces equipped with three distinct nonexpanding congruences of null strings (one SD and two ASD). Such spaces are two-sided Walker and para-Kähler. The results are presented in Sections 3.2 and 4. Especially interesting is the metric \[3.21\] which is a general metric of the space of the type $[\deg]^{n} \otimes [D]^{nn}$. This metric specialized to the Einstein case gives the metric \[4.2\].

3 In the sense that at each point $p$ of a null string a bivector tangent to the null string is SD or ASD.

4 An almost para-complex structure is an endomorphism $K : T\mathcal{M} \to T\mathcal{M}$ such that $K^{2} = id_{T\mathcal{M}}$ whose $\pm 1$-eigenspaces have rank 2.

5 Frankly, spaces equipped with more then 2 congruences of null strings of the same duality are so interesting that they deserve a separate paper.
Note, that a first step to obtain the metric (3.21) is to equip a weak $\mathcal{HH}$-space with an additional expanding congruence of ASD null strings (Section 3.1). Law and Matsushita called such spaces sesquiWalker spaces [21]. However, the Authors of [21] found only one class of metrics of such spaces (the metric (3.13) in our paper). The second (more generic) class remained undiscovered in [21]. We filled this gap (the metric (3.10)).

The second aim is an analysis of the SD spaces which can be obtained from the metric (3.21). The results are gathered is Sections 3.4 and 4.3. An advantage of our approach is the fact that we obtained SD metrics of the types $[III,N]^n \otimes [O]^n$ with maximally reduced number of arbitrary functions. A special attention is paid to the metric (3.50) which is the general metric of a space of the type $[N]^n \otimes [O]^n$. Such a space has a rare property of being two-sided conformally recurrent [27]. However, in [27] the authors listed three classes of such metrics with five arbitrary functions of two variables each. We proved that these three classes can be reduced to a single class which depends on two functions of two variables and one constant.

The third aim is a development of an approach to the problem of subclassification of the spaces equipped with congruences of null strings of a different duality. SD and ASD congruences of null strings intersect and this intersection constitutes the congruence of null (complex) geodesics. Properties of this intersection are used as a subcriterion in the classification. A foundation of our approach is presented in Section 2 (earlier in [10]) and full classification is listed in Appendix A.

The paper contains also examples of the spaces equipped with expanding and non-expanding congruences of null strings of the same duality. These are the metrics (3.19), (3.20), (3.42), (3.46) and (3.50) with constraints (3.53). According to our best knowledge these are the first explicit examples of such metrics.

Our paper is organized, as follows. In Section 2 the algebraic preliminaries of the subject are presented (formalism, Petrov-Penrose classification, congruences of null strings and their intersections, weak $\mathcal{HH}$-spaces). Section 3 is devoted to the sesquiWalker spaces, two-sided Walker spaces and pK-spaces. In Section 4 two-sided Walker-Einstein spaces and pKE-spaces are considered. Finally, in Appendix A the detailed classification of the spaces equipped with at most two congruences of SD and ASD spaces is presented.

1.3 Some basic abbreviations

A remark about abbreviations used in our paper is needed. We use abbreviations:

$\mathcal{C}$ – congruence of null strings

$\mathcal{C}s$ – congruences of null strings (if there is no need to establish
the number of congruences or the properties of congruences)

$\mathcal{C}_{m^A}$ – SD congruence of null strings generated by a spinor $m^A$

$\mathcal{C}_{m^\dot{A}}$ – ASD congruence of null strings generated by a spinor $m^\dot{A}$

$\mathcal{C}^n$ – nonexpanding congruence of null strings

$\mathcal{C}^e$ – expanding congruence of null strings

$\mathcal{C}^{nn}$ – two nonexpanding congruences of null strings of the same duality

$\mathcal{C}^{ne}$ – two congruences of null strings of the same duality,
one nonexpanding, one expanding

$\mathcal{C}^{ee}$ – two expanding congruences of null strings of the same duality
There are also possibilities $C^{nee}$, $C^{eee}$ and $C^{egee}$, but they do not appear in our paper. We also use "mixed" symbols with obvious meaning, like $C^{m,n}$ (nonexpanding SD congruence of null strings generated by a spinor $m^A$), etc.

Intersection of SD and ASD $C$s constitutes a congruence of null geodesics which properties are described by the optical scalars (shear, expansion and twist). For such structures we use the following abbreviations

$\mathcal{I}$ — congruence of null geodesics (I like Intersection)

$\mathcal{I}s$ — congruences of null geodesics (if there is no need to establish the number of intersections or the properties of intersections)

$\mathcal{I}(C_{m^A}, C_{m^A})$ — congruence of null geodesics which is an intersection of $C_{m^A}$ and $C_{m^A}$

$\mathcal{I}(C^n, C^n)$ — congruence of null geodesics which is an intersection of SD $C^n$ and ASD $C^n$

$\mathcal{I}(C^n, C^e)$ — congruence of null geodesics which is an intersection of SD $C^n$ and ASD $C^e$

$\mathcal{I}(C^e, C^e)$ — congruence of null geodesics which is an intersection of SD $C^e$ and ASD $C^e$

$\mathcal{I}^{--}$ — congruence of null geodesics which is nonexpanding and nontwisting

$\mathcal{I}^{+-}$ — congruence of null geodesics which is expanding but nontwisting

$\mathcal{I}^{++}$ — congruence of null geodesics which is nonexpanding but twisting

We also use equalities like $\mathcal{I}(C_{m^A}, C_{m^A}) = \mathcal{I}^{--}$ or $\mathcal{I}(C^n, C^e) = \mathcal{I}^{++}$ with obvious meaning.

A space which is equipped with SD (or ASD) $C^n$ is called Walker space $[35]$. If a space is equipped with $C^{mn}$ we deal with double Walker spaces $[19]$. If there are one SD $C^n$ and one ASD $C^n$, we call such a space two-sided Walker $[12]$. Consider a space equipped with SD $C^e$ and ASD $C^e$. Law and Matsushita called such spaces real AlphaBeta-geometries $[21]$. They also called spaces equipped with SD $C^n$ and ASD $C^e$ (or vice-versa) sesquiWalker spaces $[6]$.

At this point a set with reasonable names of the spaces has been exhausted. How a space equipped with SD $C^{mn}$ and ASD $C^{mn}$ should be called? The natural answer is two-sided double Walker space and it is somehow acceptable. But what about a space equipped with SD $C^n$ and ASD $C^{mn}$? The name one-sided Walker one-sided double Walker seems to be a little bit sloppy. If we admit the existence of $C$s of a different duality and different properties more problems with names of the spaces arise.

Realizing this, we propose uniform abbreviations. Note, that in Lorentzian spaces algebraic types of the SD and ASD parts of the Weyl spinor are the same (see Section $2.2$). Thus, to determine Petrov-Penrose type of the conformal curvature it is sufficient to use a single symbol (for example $[I]$ or $[D]$).

However, the SD and ASD Weyl spinors can be of a different Petrov-Penrose type in Riemannian, neutral and complex spaces. Thus, a single symbol of a type is not sufficient. Usually full data of a type of the conformal curvature is given in the following symbol

$$[\text{SD}_{\text{type}}] \otimes [\text{ASD}_{\text{type}}]$$

where

| SD Type | ASD Type |
|---------|----------|
| $I$, $II$, $D$, $III$, $N$, $O$ | $I$, $D$, $O$ | in complex spaces |
| $I_{r}$, $I_{rc}$, $I_{c}$, $II_{r}$, $II_{rc}$, $D_{r}$, $D_{c}$, $III_{r}$, $N_{r}$, $O_{r}$ | in neutral spaces |
For example, the symbol $[\text{D}] \otimes [\text{N}]$ means that the SD Weyl spinor is of the type $[\text{D}]$ and the ASD Weyl spinor is of the type $[\text{N}]$. The types $\text{I}$, $\text{I}_r$, $\text{I}_{rc}$ and $\text{I}_c$ are called algebraically general. All other types are called algebraically special or algebraically degenerated.

If, additionally, a space is equipped with one or more SD or ASD $\mathcal{C}$s, we add superscripts, i.e., we use a symbol

$$[\text{SD type}]^{i_1i_2\ldots} \otimes [\text{ASD type}]^{j_1j_2\ldots}$$

where the number of superscripts $i$ ($j$) carries information about the number of SD (ASD) $\mathcal{C}$s. $i_1, i_2, \ldots, j_1, j_2, \ldots = \{n, e\}$ where $n$ stands for $\mathcal{C}^n$, while $e$ stands for $\mathcal{C}^e$. Hence, the generic complex spaces are:

$$[\text{deg}]^n \otimes [\text{any}] - \text{weak nonexpanding } \mathcal{HH}\text{-spaces}$$
$$[\text{deg}]^e \otimes [\text{any}] - \text{weak expanding } \mathcal{HH}\text{-spaces}$$
$$[\text{deg}]^n \otimes [\text{any}], C_{ab} = 0 - \text{nonexpanding } \mathcal{HH}\text{-spaces}$$
$$[\text{deg}]^e \otimes [\text{any}], C_{ab} = 0 - \text{expanding } \mathcal{HH}\text{-spaces}$$

where $\text{deg}$ means that Petrov-Penrose type is algebraically special, $\text{any}$ means that Petrov-Penrose type is arbitrary, $C_{ab}$ is the traceless Ricci tensor. Similarly, for neutral spaces we get:

$$[\text{deg}]^n \otimes [\text{any}] - \text{Walker spaces, real weak nonexpanding } \mathcal{HH}\text{-spaces}$$
$$[\text{D}]^{nn} \otimes [\text{any}] - \text{double Walker spaces, para-Kähler spaces}$$
$$[\text{D}]^{ee} \otimes [\text{any}] - \text{para-Hermite spaces}$$
$$[\text{deg}]^n \otimes [\text{deg}]^n - \text{two-sided Walker spaces}$$
$$[\text{deg}]^e \otimes [\text{any}]^e - \text{sesquiWalker spaces}$$
$$[\text{any}]^e \otimes [\text{any}]^e - \text{AlphaBeta-geometries}$$
$$[\text{deg}]^e \otimes [\text{any}] - \text{real weak expanding } \mathcal{HH}\text{-spaces}$$
$$[\text{deg}]^n \otimes [\text{any}], C_{ab} = 0 - \text{Walker-Einstein spaces, real nonexpanding } \mathcal{HH}\text{-spaces}$$
$$[\text{deg}]^e \otimes [\text{any}], C_{ab} = 0 - \text{real expanding } \mathcal{HH}\text{-spaces}$$

Of course, our abbreviations hold true also if the orientation is changed. For example, $[\text{D}]^{nn} \otimes [\text{any}]$ and $[\text{any}] \otimes [\text{D}]^{nn}$ are essentially the same spaces.

If the SD (ASD) Weyl spinor vanishes, then the corresponding space is called ASD (SD) and it is equipped with infinitely many SD (ASD) $\mathcal{C}$s. If additionally $R \neq 0$ then all these $\mathcal{C}$s are expanding and in such a case we use the symbol $[\text{O}]^e$. If $R = 0$ then there are both expanding and nonexpanding $\mathcal{C}$s. In such a case we use the symbol $[\text{O}]^n$. Hence

$$[\text{any}] \otimes [\text{O}]^n - \text{SD spaces with } R = 0 \ (\text{if } C_{ab} = 0 : \text{heavenly spaces with } \Lambda = 0)$$
$$[\text{any}] \otimes [\text{O}]^e - \text{SD spaces with } R \neq 0 \ (\text{if } C_{ab} = 0 : \text{heavenly spaces with } \Lambda \neq 0)$$
$$[\text{O}]^n \otimes [\text{any}] - \text{ASD spaces with } R = 0 \ (\text{if } C_{ab} = 0 : \text{hellish spaces with } \Lambda = 0)$$
$$[\text{O}]^e \otimes [\text{any}] - \text{ASD spaces with } R \neq 0 \ (\text{if } C_{ab} = 0 : \text{hellish spaces with } \Lambda \neq 0)$$

We believe, that these abbreviations simplify considerably the rest of the article.
2 Preliminaries

2.1 The formalism

2.1.1 Generic complex case

In this section we present the foundations of the null tetrad and spinorial formalisms. The spinorial formalism used in this paper is Infeld - Van der Waerden - Plebański notation. For more details see [25,26] or [8,27] for a brief summary.

Let \((\mathcal{M}, ds^2)\) be a 4-dimensional complex analytic differential manifold equipped with a holomorphic metric. The metric \(ds^2\) can be written in the form

\[
ds^2 = 2e^1 e^2 + 2e^3 e^4 = -\frac{1}{2}g_{AB}g^{AB}\tag{2.1}
\]

where 1-forms \((e^1, e^2, e^3, e^4)\) are members of a complex null tetrad (a complex coframe) and they form a basis of 1-forms. The relation between \(e^a\) and \(g^{AB}\) reads

\[
(g^{AB}) := \sqrt{2} \begin{pmatrix}
e^4 & e^2 \\
e^1 & -e^3
\end{pmatrix}, \quad A = 1, 2, \quad B = 1, 2 \tag{2.2}
\]

A dual basis is denoted by \((\partial_1, \partial_2, \partial_3, \partial_4)\) and it is also called a complex null tetrad (a complex frame). A dual basis in spinorial formalism takes the form

\[
(\partial_{AB}) := -\sqrt{2} \begin{pmatrix}
\partial_4 & \partial_2 \\
\partial_1 & -\partial_3
\end{pmatrix}
\]

Spinorial indices are manipulated according to the following rules

\[
m_A = \varepsilon_{AB} m^B, \quad m^A = m_B \varepsilon^{BA}, \quad m_\dot{A} = \varepsilon_{\dot{A}B} \dot{m}^B, \quad \dot{m}^\dot{A} = \dot{m}_B \varepsilon^\dot{B}\dot{A} \tag{2.4}
\]

where \(\varepsilon_{AB}\) and \(\varepsilon_{\dot{A}B}\) are the spinor Levi-Civita symbols

\[
(\varepsilon_{AB}) := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =: (\varepsilon^{AB}), \quad (\varepsilon_{\dot{A}B}) := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =: (\varepsilon^{\dot{A}B}) \quad (\delta_{C}^{B}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Rules (2.4) imply the following rules for the objects from a tangent space

\[
\partial^A = \partial_B \varepsilon_{AB}, \quad \partial_A = \varepsilon_{BA} \partial^B, \quad \partial^\dot{A} = \partial_B \varepsilon_{\dot{A}B}, \quad \partial_A = \varepsilon_{\dot{B}A} \partial^{\dot{B}}, \quad \partial^A := \frac{\partial}{\partial x^A}, \quad \partial_A := \frac{\partial}{\partial x^A} \tag{2.6}
\]

Define 2-forms \(S^{AB}\) and \(S^{\dot{A}\dot{B}}\)

\[
(S^{AB}) := \begin{bmatrix}
2e^4 \wedge e^2 & e^1 \wedge e^2 + e^3 \wedge e^4 \\
e^1 \wedge e^2 + e^3 \wedge e^4 & 2e^3 \wedge e^1
\end{bmatrix} \tag{2.7a}
\]

\[
(S^{\dot{A}\dot{B}}) := \begin{bmatrix}
2e^4 \wedge e^1 & -e^1 \wedge e^2 + e^3 \wedge e^4 \\
-e^1 \wedge e^2 + e^3 \wedge e^4 & 2e^3 \wedge e^2
\end{bmatrix} \tag{2.7b}
\]
$S^{AB}$ are SD and $S^{AB}$ are ASD. They form a basis of 2-forms.

The first Cartan structure equations in the spinorial formalism read

$$dg^{AB} + \Gamma^A_C \wedge g^{CB} + \Gamma^B_C \wedge g^{AC} = 0$$  \hspace{1cm} (2.8)

where $\Gamma_{AB}$ and $\Gamma_{\dot{A}\dot{B}}$ are SD and ASD spinorial connection forms. Their relation with the spinorial forms $\Gamma_{ab}$ in the null tetrad formalism is well-known

$$(\Gamma_{AB}) = -\frac{1}{2}
\begin{bmatrix}
2 \Gamma_{42} & \Gamma_{12} + \Gamma_{34} \\
\Gamma_{12} + \Gamma_{34} & 2 \Gamma_{31}
\end{bmatrix}
$$  \hspace{1cm} (2.9)

If we use a decomposition

$$\Gamma_{AB} =: -\frac{1}{2}\Gamma_{ABC}g^{CD}, \quad \Gamma_{\dot{A}\dot{B}} =: -\frac{1}{2}\Gamma_{\dot{A}\dot{B}C}g^{\dot{D}D}, \quad \Gamma_{ab} =: \Gamma_{abc}e^c$$  \hspace{1cm} (2.10)

the explicit relations between $\Gamma_{ABM\dot{N}}, \Gamma_{\dot{A}\dot{B}M\dot{N}}$ and $\Gamma_{abc}$ read

$$\Gamma_{11\dot{A}\dot{B}} = \sqrt{2} \begin{bmatrix}
\Gamma_{424} & \Gamma_{422} \\
\Gamma_{421} & -\Gamma_{423}
\end{bmatrix}, \quad \Gamma_{22AB} = \sqrt{2} \begin{bmatrix}
\Gamma_{314} & \Gamma_{312} \\
\Gamma_{311} & -\Gamma_{313}
\end{bmatrix}$$  \hspace{1cm} (2.11)

The formula for the covariant derivative of an arbitrary spinor field in the spinorial formalism reads

$$\nabla_{MN} \Psi^{\dot{A}\dot{B}}_{CD} = \partial_{MN} \Psi^{\dot{A}\dot{B}}_{CD} + \Gamma^A_{SMN} \Psi^{\dot{A}\dot{B}}_{CD} - \Gamma^S_{CMN} \Psi^{\dot{A}\dot{B}}_{SD}$$

\hspace{1cm} \hspace{1cm} + \Gamma^\dot{B}_{SMN} \Psi^{A\dot{B}}_{CD} - \Gamma^\dot{S}_{DMN} \Psi^{A\dot{B}}_{CS}$$  \hspace{1cm} (2.12)

where

$$\nabla_{AB} := g_{aAB} \nabla^a, \quad \partial_{AB} := g_{aAB} \partial^a$$

and the matrices $g_{aAB}$ are defined by the relation $g^{AB} = g^a_{\dot{A}\dot{B}} e^a$.

The second Cartan structure equations read

$$R^A_B = d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B, \quad R^\dot{A}_\dot{B} = d\Gamma^\dot{A}_\dot{B} + \Gamma^\dot{A}_\dot{C} \wedge \Gamma^\dot{C}_\dot{B}$$  \hspace{1cm} (2.13)
\( R^A_B \) and \( \tilde{R}^\mathring{A}_\mathring{B} \) are the curvature 2-forms of the connection \( \Gamma^A_B \) or \( \Gamma^\mathring{A}_\mathring{B} \), respectively. Decomposition of \( R_{AB} = R_{(AB)} \) and \( R_{\mathring{A}\mathring{B}} = R_{(\mathring{A}\mathring{B})} \) reads

\[
R_{AB} = -\frac{1}{2} C^{ABCD} S^{CD} + \frac{R}{24} S_{AB} + \frac{1}{2} C_{ABCD} S^{CD} \tag{2.14}
\]

\[R_{\mathring{A}\mathring{B}} = -\frac{1}{2} C_{\mathring{A}\mathring{B}CD} S^{\mathring{C}\mathring{D}} + \frac{R}{24} S_{\mathring{A}\mathring{B}} + \frac{1}{2} C_{CD\mathring{A}\mathring{B}} S^{CD}\]

\( C_{ABCD} = C_{(ABCD)} \) (\( C_{\mathring{A}\mathring{B}CD} = C_{(\mathring{A}\mathring{B}CD)} \)) is the spinorial image of the SD (ASD) part of the Weyl tensor; \( C_{ABCD} = C_{(AB)\mathring{C}\mathring{D}} = C_{\mathring{A}B(\mathring{C}\mathring{D})} \) is the spinorial image of the traceless Ricci tensor and, finally, \( R \) is the curvature scalar.

Dotted and undotted spinors transform as follows

\[
m^rA_1...A_n = L^A_{R_1} ... L^A_{R_n} m^{R_1...R_n}, \quad m^{r\mathring{A}_1...\mathring{A}_n} = M^{\mathring{A}_1}_{\mathring{R}_1} ... M^{\mathring{A}_n}_{\mathring{R}_n} m^{\mathring{R}_1...\mathring{R}_n} \tag{2.15}
\]

where \( L^A_{R}, \quad M^{\mathring{A}}_{\mathring{R}} \in SL(2, \mathbb{C}) \).

### 2.1.2 Real neutral case

Any 4-dimensional real space can be obtained from a generic 4-dimensional complex space by the procedure of a real slice of a complex space \(^{[34]}\). To obtain such a slice one has to use reality conditions. We skip a discussion about the reality conditions in Lorentzian and Riemannian spaces, we focus only on neutral spaces. There are two different ways of obtaining neutral spaces from complex spaces. Following \(^{[33]}\) we call them \( S_r \) ("split real") and \( S_c \) ("split complex")

\[
\text{Reality conditions } S_r : \quad g^{AB} = g^{\mathring{A}\mathring{B}} \implies e^a = e^a, \quad a = 1, 2, 3, 4
\]

\[
\text{Reality conditions } S_c : \quad g^{AB} = \eta^{AC} \eta^{BD} g_{CD} \implies e^2 = e^1, \quad e^4 = -e^3
\]

where \( (\eta^{AB}) = (\eta^{\mathring{A}\mathring{B}}) := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \)

where "bar" means the complex conjugation. Application of \( S_r \) is easy: it is enough to replace all complex coordinates by real ones and all holomorphic functions by real smooth ones.

In the rest of the text we always use the reality conditions \( S_r \). The reason why we prefer \( S_r \) to \( S_c \) is as follows. We are interested in neutral slices of complex spaces which inherit \( \mathcal{C}s \) in the sense that complex \( \mathcal{C}s \) in a complex space become real \( \mathcal{C}s \) in neutral slice of the complex space. In other words, \( \mathcal{C}s \) must be characterized by the real index equal 2 \(^{[18,33]}\). We refer the Reader to Section 7.2 of \(^{[33]}\) where advantages of \( S_r \) over \( S_c \) in such a case have been clearly explained.

### 2.2 Petrov-Penrose classification

Algebraic classification of totally symmetric 4-index spinors has been presented in \(^{[23]}\). This classification applied to the SD (ASD) Weyl spinor leads to the Petrov-Penrose classification of the conformal curvature. A contraction of \( C_{ABCD} \) with an arbitrary 1-index spinor \( \xi^A \) such that \( \xi^2 \neq 0 \) yields

\[
C_{ABCD} \xi^A \xi^B \xi^C \xi^D = (\xi^2)^4 \mathcal{P}(z) \tag{2.16}
\]
where $\mathcal{P}(z)$ is a 4-th order polynomial in $z := \xi^1/\xi^2$. Due to the fundamental theorem of algebra $\mathcal{P}$ can be always brought to a factorized form. Hence

$$\left(\xi^2\right)^4 \mathcal{P}(z) = (a_A\xi^A)(b_B\xi^B)(c_C\xi^C)(d_D\xi^D) \quad (2.17)$$

Because of the arbitrariness of $\xi^A$ we find

$$C_{ABCD} = a_1 b_2 c_3 d_4 \quad (2.18)$$

Complex, 1-index, undotted spinors $a_A, b_B, c_C$ and $d_D$ are called Pennrse spinors. Penrose spinors are mutually linearly independent in general. In such a case the SD Weyl spinor is algebraically general. It corresponds to the case when the polynomial $\mathcal{P}(z)$ has four different roots. If at least two Penrose spinors are proportional to each other then the SD Weyl spinor is algebraically special. There are different patterns of the roots of polynomial $\mathcal{P}(z)$. Each of them corresponds to different Petrov-Penrose types of $C_{ABCD}$.

If $C_{ABCD}$ is complex then there are 6 different Petrov-Penrose types. In neutral spaces $C_{ABCD}$ is real. Hence, the scheme of the roots of $\mathcal{P}(z)$ is more complicated. There appear 10 different Petrov-Penrose types. The symbols which are usually used as abbreviations of the corresponding Petrov-Penrose types of spinor $C_{ABCD}$ and the scheme of the roots of the polynomial $\mathcal{P}(z)$ are gathered in the Table 1. Note, that in neutral spaces types [II], [III], [IIIr], [D] and [N] decompose into product of real 1-index spinors\(^7\). In [3] such types have been called the special real Penrose types.

Consider now a pair $\nu_A$ and $\mu_A$ of normalized spinors, $\nu^A\mu_A = 1$. Such spinors form a basis of 1-index undotted spinors. The SD Weyl spinor can be presented in the form

$$2C_{ABCD} =: C^{(1)}(\nu_A\nu_B\nu_C\nu_D) + 4C^{(2)}(\mu_A(\nu_B\nu_C\nu_D)) + 6C^{(3)}(\mu_A\mu_B\nu_C\nu_D)$$

$$+ 4C^{(4)}(\mu_A(\mu_B\nu_C\nu_D)) + C^{(5)}(\mu_A\mu_B\mu_C\mu_D) \quad (2.19)$$

where the scalars $C^{(i)}$, $i = 1, 2, 3, 4, 5$ are called the SD conformal curvature coefficients.

For our further purposes it is convenient to mention how a spinorial basis ($\nu_A, \mu_B$) can be adapted to the structure of the SD Weyl spinor. Let the SD Weyl spinor be algebraically degenerated, $C_{ABCD} = a_1(a_2b(c_3d))$ with $a_A$ being a multiple Penrose spinor. If we choose a spinorial basis of undotted spinors in such a manner that $\nu_A \sim a_A$, then $C^{(5)} = C^{(4)} = 0$. In such a case the conditions for the algebraic types of the SD Weyl spinor read

- type [II] : $C^{(3)} \neq 0, 2C^{(2)}C^{(2)} = 3C^{(1)}C^{(3)} \neq 0$ \quad (2.20)
- type [D] : $C^{(3)} \neq 0, 2C^{(2)}C^{(2)} = 3C^{(1)}C^{(3)} = 0$
- type [III] : $C^{(3)} = 0, C^{(2)} \neq 0$
- type [N] : $C^{(3)} = C^{(2)} = 0, C^{(1)} \neq 0$
- type [O] : $C^{(3)} = C^{(2)} = C^{(1)} = 0$

Consider now a neutral case. Let the SD Weyl spinor be algebraically degenerated with

\(^7\)Note, that the subscript $r$ in the symbols of the types [III], [N] and [O] is basically redundant. However, we keep this subscript to distinguish the types in complex spaces from the types in neutral spaces.
Table 1: Petrov-Penrose types of complex and real totally symmetric 4-index spinor. 

| Type | $C_{ABCD} =$ | Roots of $\mathcal{P}(z)$ | Type | $C_{ABCD} =$ | Roots of $\mathcal{P}(z)$ |
|------|--------------|------------------|------|--------------|------------------|
| [I]  | $a(ABCD)$   | $Z_1Z_2Z_3Z_4$  | [I]  | $m(abCD)$   | $R_1R_2R_3R_4$  |
|      |              |                  | [I]  | $m(aBCD)$   | $R_1R_2Z\bar{Z}$|
|      |              |                  | [I]  | $a(ABCD)$   | $Z_1\bar{Z}_1Z_2\bar{Z}_2$|
|      | $a(ACBD)$   | $Z_1^2Z_2Z_3$   | [II] | $m(ABCD)$   | $R_1^2R_2R_3$  |
|      |              |                  | [II] | $m(ABCD)$   | $R_1^2R_2^2$   |
|      |              |                  | [III] | $m(ABCD)$   | $R_1^3R_2^2$  |
|      | $a(ABCD)$   | $Z_1^3Z_2$      | [N]  | $m(ABCD)$   | $R^4$          |
| [O]  | 0            | $-$              | [O]  | 0            | $-$             |

|      |              |                  |      |              |                  |

Z means that a root is complex while $R$ stands for a real root; the power denotes the multiplicity of the corresponding root; spinors $a_A$, $b_A$, $c_A$ and $d_A$ are complex, spinors $m_A$, $n_A$, $r_A$ and $s_A$ are real; bar stands for the complex conjugation.

$m_A$ being a real multiple Penrose spinor. Then $\nu_A \sim m_A$ yields $C^{(5)} = C^{(4)} = 0$ and

\[
\begin{align*}
type [\text{II}_r] : & \quad C^{(3)} \neq 0, 2C^{(2)}C^{(2)} - 3C^{(1)}C^{(3)} > 0 \\
type [\text{II}_r] : & \quad C^{(3)} \neq 0, 2C^{(2)}C^{(2)} - 3C^{(1)}C^{(3)} < 0 \\
type [\text{D}_r] : & \quad C^{(3)} \neq 0, 2C^{(2)}C^{(2)} - 3C^{(1)}C^{(3)} = 0 \\
type [\text{III}_r] : & \quad C^{(3)} = C^{(2)} = 0, C^{(1)} \neq 0 \\
type [\text{O}_r] : & \quad C^{(3)} = C^{(2)} = C^{(1)} = 0
\end{align*}
\]  

The classification presented in this Section can be applied, mutatis mutandis, to the ASD Weyl spinor $C_{ABCD}$.

### 2.3 Congruences of null strings

A structure which play a fundamental role in the further considerations is a congruence of null strings (abbreviated by $C$, see Section 1.3 for explanation of all the abbreviations). We recall only basic properties of $\mathcal{C}_s$, for deeper analysis of the topic, see, e.g., [8,30]).

Consider first a 2-dimensional SD holomorphic distribution $\mathcal{D}_{m^A} = \{m_Aa_A, m_AB\}$, $a_Ab^B \neq 0$. Such a distribution is defined by the Pfaff system

\[ m_Ag^{AB} = 0 \]  

$\mathcal{D}_{m^A}$ is integrable in the Frobenius sense if and only if the spinor field $m_A$ satisfies the equations

\[ m^Am^B\nabla_{A\dot{C}}m_B = 0 \]
Eqs. \((2.23)\) are called SD null string equations. If Eqs. \((2.23)\) hold true one says that the spinor \(m_A\) generates the congruence of SD null strings. The integral manifolds of the distribution \(D_{m^A}\) are totally null and geodesic, 2-dimensional SD holomorphic surfaces (SD null strings). The family of such surfaces constitute the congruence of SD null strings. Eqs. \((2.23)\) can be rewritten in the equivalent form

\[
\nabla_{AC}m_B = Z_{AC}m_B + \epsilon_{AB}M_C
\]

(2.24)

where \(Z_{AC}\) is the Sommers vector and \(M_C\) is the expansion of the congruence. To understand better the geometrical meaning of \(M_C\) let \(\mathcal{X} = X^{AB}\partial_{AB}\) be an arbitrary vector field and \(\mathcal{V} = V^{AB}\partial_{AB} \in D_{m^A} \iff V_{AB} = m_AV_B\). Then

\[
\nabla_{\mathcal{X}}\mathcal{V}_{MN} = m_MY_M + V_MX_{MB}M_B
\]

(2.25)

Hence, \(\nabla_{\mathcal{X}}\mathcal{V} \in D_{m^A}\) for any vector field \(\mathcal{V} \in D_{m^A}\) and for an arbitrary vector field \(\mathcal{X}\) (i.e., \(D_{m^A}\) is parallely propagated) if and only if \(M_A = 0\). According to Plebański - Robinson - Rozga terminology \(C\)s with vanishing expansion are called nonexpanding (or plane) while \(C\)s with \(M_A \neq 0\) are called expanding (or deviating).

ASD \(C\)s are similarly defined but they are generated by dotted spinors. If a spinor \(m_A\) generates an ASD \(C\) then it satisfies equations of ASD null strings \(m^Cm^B\nabla_{AC}m_B = 0\) or, equivalently

\[
\nabla_{AC}m_B = \dot{Z}_{AC}m_B + \epsilon_{CB}M_A
\]

(2.26)

where \(\dot{Z}_{AC}\) is the Sommers vector of \(C_{m^A}\) and \(M_A\) is the expansion of \(C_{m^A}\). Note, that expansion of SD \(C_{m^A}\) (ASD \(C_{m^A}\)) is given by the dotted \(M_A\) (undotted \(M_A\)) spinor field.

If \(C_{ABCD} \neq 0\) \((C_{ABCD} \neq 0)\) then the following facts hold true (see \([8,30]\) for proofs)

- if a spinor \(m_A\) \((m_A)\) generates a congruence of SD (ASD) null strings, then it is a undotted (dotted) Penrose spinor

- if a spinor \(m_A\) \((m_A)\) generates a nonexpanding congruence of SD (ASD) null strings, then it is a multiple undotted (dotted) Penrose spinor.

From these facts it follows that if \(C_{ABCD} \neq 0\) \((C_{ABCD} \neq 0)\) then the maximal number of distinct SD (ASD) \(C\)s is 4 and such a case is possible only if the SD (ASD) Weyl spinor is of the type \([I]\). All the possibilities are presented in the Scheme \([1]\).

A little more complicated scheme holds true for a neutral case. Note, that real \(C\) is generated by a real Penrose spinor. It implies that if the SD (ASD) Weyl spinor is of the types \([I_c]\) or \([D_c]\) then a space does not admit any SD (ASD) \(C\)s, see Scheme \([2]\).

### 2.4 Properties of intersections of SD and ASD congruences of null strings

Consider a space which is equipped with \(C_{m^A}\) (with the expansion \(M_A\)) and \(C_{m^A}\) (with the expansion \(M_A\)). These \(C\)s intersect and this intersection constitutes a congruence of (complex) null geodesics. Properties of such \(I\) have been investigated in \([30]\) and then

---

\(\text{Note: that expansion of the congruence of null strings is a different concept then the expansion of the congruence of null geodesics.}\)
Scheme 1: Types of the Weyl spinors in spaces equipped with different numbers of congruences of null strings in complex case.

\begin{align*}
0 \, Cs & : [I] \quad [II] \quad [III] \quad [D] \quad [N] \\
1 \, C & : [I]^e \quad [II]^e \quad [II]^n \quad [III]^e \quad [III]^n \quad [D]^e \quad [D]^n \quad [N]^e \quad [N]^n \\
2 \, Cs & : [I]^{ee} \quad [II]^{ee} \quad [II]^{en} \quad [III]^{ee} \quad [III]^{en} \quad [D]^{ee} \quad [D]^{en} \quad [D]^{nn} \\
3 \, Cs & : [I]^{eee} \quad [II]^{eee} \quad [II]^{een} \\
4 \, Cs & : [I]^{eeee}
\end{align*}

Scheme 2: Types of the Weyl spinors in spaces equipped with different numbers of congruences of null strings in neutral case.

\begin{align*}
0 \, Cs & : [I]_r \quad [I]_e \quad [I]_l \quad [I]_l \quad [II]_r \quad [II]_e \quad [II]_n \quad [III]_r \quad [III]_e \quad [III]_n \quad [D]_r \quad [D]_e \quad [D]_n \quad [N]_r \quad [N]_e \quad [N]_n \\
1 \, C & : [I]_r^e \quad [I]_e^e \quad [I]_l^e \quad [I]_l^e \quad [II]_r^e \quad [II]_e^e \quad [II]_n^e \quad [II]_l^e \quad [III]_r^e \quad [III]_e^e \quad [III]_n^e \quad [III]_l^e \quad [D]_r^e \quad [D]_e^e \quad [D]_n^e \quad [D]_l^e \quad [D]_r^n \quad [D]_e^n \quad [D]_n^n \quad [N]_r^e \quad [N]_e^e \quad [N]_l^n \\
2 \, Cs & : [I]_r^{ee} \quad [I]_e^{ee} \quad [I]_l^{ee} \quad [I]_l^{ee} \quad [II]_r^{ee} \quad [II]_e^{ee} \quad [II]_n^{ee} \quad [II]_l^{ee} \quad [III]_r^{ee} \quad [III]_e^{ee} \quad [III]_n^{ee} \quad [III]_l^{ee} \quad [D]_r^{ee} \quad [D]_e^{ee} \quad [D]_n^{ee} \quad [D]_l^{ee} \quad [D]_r^{en} \quad [D]_e^{en} \quad [D]_n^{en} \quad [D]_l^{en} \\
3 \, Cs & : [I]_r^{eee} \quad [I]_e^{eee} \quad [I]_l^{eee} \quad [I]_l^{eee} \\
4 \, Cs & : [I]_r^{eeee}
\end{align*}

in [10]. Let \( K_a \) be a null vector field along the \( \mathcal{I}(C_{m^A}, C_{m^A}) \). Hence, \( K_a \sim m_A m_{\bar{A}} \). Then one defines the expansion \( \theta \) and the twist \( \varrho \) of \( \mathcal{I}(C_{m^A}, C_{m^A}) \) as follows\(^9\)

\[\theta := \frac{1}{2} \nabla^a K_a \tag{2.27a}\]
\[\varrho^2 := \frac{1}{2} \nabla_c [a K_b] \nabla^a K^b \tag{2.27b}\]

It has been proven in [10] that if \( \mathcal{I}(C_{m^A}, C_{m^A}) \) is in an affine parametrization then \( \theta \) and \( \varrho \) are proportional to the following scalars

\[\theta \sim m_A M^A + m_{\bar{A}} M_{\bar{A}} \tag{2.28}\]
\[\varrho \sim m_A M^A - m_{\bar{A}} M_{\bar{A}}\]

\(^9\text{Note, that these definitions are the same like in the Lorentzian case. Nevertheless, the geometrical interpretation of } \theta \text{ and } \varrho \text{ in the complex case is not clear yet.}\)
Hence, \( \theta \) and \( \varrho \) of \( \mathcal{I}(C_{m^A}, C_{m^A}) \) depend on expansions \( M_A \) and \( M_A \). There are four possibilities for which we propose the following symbols

\[
\begin{align*}
[++] & : \theta \neq 0, \varrho \neq 0 \\
[++-] & : \theta \neq 0, \varrho = 0 \\
[-++] & : \theta = 0, \varrho \neq 0 \\
[-++] & : \theta = 0, \varrho = 0
\end{align*}
\]  

(2.29)

Consequently, one arrives at the Table 2. Note, that \( \mathcal{I}^- \) and \( \mathcal{I}^+ \) appear only as an intersection of expanding \( C_s \). There are two different types of \( \mathcal{I}^{++} \). Namely, \( \mathcal{I}^{++} = \mathcal{I}(C^n, C^n) \) or \( \mathcal{I}^{++} = \mathcal{I}(C^e, C^e) \). There are also three different types of \( \mathcal{I}^{--} \): \( \mathcal{I}^{--} = \mathcal{I}(C^n, C^n) \) or \( \mathcal{I}^{--} = \mathcal{I}(C^e, C^e) \) or \( \mathcal{I}^{--} = \mathcal{I}(C^e, C^e) \).

| Expansions | \( M_A = 0 \) | \( M_A \neq 0 \) |
|------------|---------------|---------------|
| \( M_A = 0 \) | [++] | [++] |
| \( M_A \neq 0 \) | [++] | [++] |

Table 2: Types of congruences of null geodesics via properties of congruences of null strings.

2.5 Classification of spaces equipped with at most 2 different congruences of null geodesics via SD and ASD null strings

Consider now four distinct \( C_s \) (two SD and two ASD): \( C_{m^A}, C_{n^A}, C_{m^A} \) and \( C_{n^A}; m^A n_A \neq 0, m^A n_A \neq 0 \). Their properties are gathered in the Table 3. Properties of the intersections of these \( C_s \) are listed in the Table 4.

| Congruence | Duality | Spinor | Expansion | Tangent vector |
|------------|---------|--------|-----------|---------------|
| \( C_{m^A} \) | SD | \( m_A \) | \( M_A \) | \( \{ m_A x_{\bar{a}} y_{\bar{b}} \} \), \( x_{\bar{b}} y_{\bar{b}} \neq 0 \) |
| \( C_{n^A} \) | SD | \( n_A \) | \( N_A \) | \( \{ n_A x_{\bar{a}} y_{\bar{b}} \} \), \( x_{\bar{b}} y_{\bar{b}} \neq 0 \) |
| \( C_{m^A} \) | ASD | \( m_A \) | \( M_A \) | \( \{ x_B m_A, y_B m_A \} \), \( x_{\bar{b}} y_{\bar{b}} \neq 0 \) |
| \( C_{n^A} \) | ASD | \( n_A \) | \( N_A \) | \( \{ x_B n_A, y_B n_A \} \), \( x_{\bar{b}} y_{\bar{b}} \neq 0 \) |

Table 3: Four distinct congruences of null geodesics.

Detailed classification of spaces equipped with such structures should contain information about the properties of \( C_s \) and \( I_s \). In the symbol (1.1) Petrov-Penrose types of both SD and ASD Weyl spinors as well as a number and properties of \( C_s \) are gathered. However, (1.1) does not cover the properties of \( I_s \). Thus, we extend the symbol (1.1) to the form

\[
\begin{align*}
\text{if } C_{m^A} \text{ and } C_{m^A} \text{ exist: } & \quad \{ [SD_{type}]^{i_1}_j \otimes [ASD_{type}]^{j_1}_i, [k_1] \} \quad \text{(2.30a)} \\
\text{if } C_{m^A}, C_{m^A} \text{ and } C_{n^A} \text{ exist: } & \quad \{ [SD_{type}]^{i_1}_j \otimes [ASD_{type}]^{j_1}_i, [k_1, k_2] \} \quad \text{(2.30b)} \\
\text{if } C_{m^A}, C_{n^A}, C_{m^A} \text{ and } C_{n^A} \text{ exist: } & \quad \{ [SD_{type}]^{i_1}_j \otimes [ASD_{type}]^{j_1}_i, [k_1, k_2, k_3, k_4] \} \quad \text{(2.30c)}
\end{align*}
\]
where the superscripts $i_1, i_2, j_1, j_2 = \{n,c\}$ carry information about the expansion of $C_{mA}, C_{nA}, C_{mA}$ and $C_{nA}$, respectively. Then, $k_1, k_2, k_3, k_4 = (-, -, +, +, +)$ are the properties of $\mathcal{I}(C_{mA}, C_{nA}), \mathcal{I}(C_{nA}, C_{mA}), \mathcal{I}(C_{nA}, C_{mA})$ and $\mathcal{I}(C_{nA}, C_{mA})$, respectively. Note, that the properties of $\mathcal{I}s$ are listed in order which is very important. Hence, $[-,-,+]$ and $[+,--,+]$ are not, in general, the same geometries. Subtleties hidden in the symbols \[2.30\] will be explained with help of the three examples.

**Example 1.** Consider the type
\[
\{[II]^e \otimes [D]^{nn}, [-, +, +]\}
\]
In this case the SD Weyl spinor is of the type $[II]$ and the SD Weyl spinor is of the type $[D]$. The space is equipped with $C_{mA}^{e}, C_{mA}^{n}$ and $C_{nA}^{m}$. $\mathcal{I}(C_{mA}, C_{nA}) = \mathcal{I}^{-}, \mathcal{I}(C_{mA}, C_{mA}) = \mathcal{I}^{+}$. Because both ASD Cs are nonexpanding, the types $\{[II]^e \otimes [D]^{nn}, [-, +, +]\}$ and $\{[II]^e \otimes [D]^{nn}, [++, --]\}$ are, in fact, the same geometries. Hence, in this case one can replace $[-, -, +] + [++, --]$ without changing the type. In general such a replacement leads to different types (see the next example).

**Example 2.** Consider now two types
\[
\{[\cdot]^e \otimes [\cdot]^{nn}, [-, +, +]\}, \{[\cdot]^{en} \otimes [\cdot]^{nn}, [++, --]\}
\]
Both these types are equipped with three Cs: $C_{mA}^{e}, C_{mA}^{n}$ and $C_{nA}^{m}$ and in both the cases $\mathcal{I}s$ are of the type $[+]$ and $[+]$. At the first glance, both types are the same geometries. However, closer investigation shows that in the first case $\mathcal{I}(C_{mA}^{e}, C_{mA}^{n}) = \mathcal{I}^{-}$ and $\mathcal{I}(C_{mA}^{e}, C_{nA}^{m}) = \mathcal{I}^{+}$ while in the second case $\mathcal{I}(C_{mA}^{e}, C_{mA}^{n}) = \mathcal{I}^{+}$ and $\mathcal{I}(C_{mA}^{e}, C_{nA}^{m}) = \mathcal{I}^{-}$. Hence, these types represent different geometries. This example shows how important is the order in which properties of $\mathcal{I}s$ are listed.

**Example 3.** The last example is the most subtle. Consider
\[
\{[\cdot]^{nn} \otimes [\cdot]^{en}, [-, -, +, +, +, +]\}, \{[\cdot]^{en} \otimes [\cdot]^{nn}, [-, -, +, +, +, +]\}
\]
In these cases all $\mathcal{I}s$ are $\mathcal{I}(C^{n}, C^{e})$. Also, both types are equipped with two $\mathcal{I}^{-}$ and two $\mathcal{I}^{+}$. However, the type $\{[\cdot]^{nn} \otimes [\cdot]^{en}, [-, -, +, +, +, +]\}$ is characterized by $\mathcal{I}(C_{mA}^{e}, C_{mA}^{n}) = \mathcal{I}^{-}$ while $\mathcal{I}(C_{mA}^{e}, C_{nA}^{m}) = \mathcal{I}^{+}$. Type $\{[\cdot]^{en} \otimes [\cdot]^{nn}, [-, -, +, +, +, +]\}$ is slightly different because $\mathcal{I}(C_{mA}^{e}, C_{mA}^{n}) = \mathcal{I}^{-}$ while $\mathcal{I}(C_{mA}^{e}, C_{nA}^{m}) = \mathcal{I}^{+}$.

In Appendix $A$ a detailed classification of spaces equipped with at most two SD and two ASD Cs is given. Also it is shown how a null tetrad can be adapted to $Cs$.

**Remark.** Note, that $\mathcal{I}(C^{n}, C^{e})$ is always $\mathcal{I}^{-}$. In such a case the symbols \[2.30\] can be simplified by omitting $[k_1], [k_1, k_2]$ and $[k_1, k_2, k_3, k_4]$ parts. Thus, in what follows the type $\{[\cdot]^{n} \otimes [\cdot]^{n}, [-, -]\}$ will be replaced by the simpler symbol $[\cdot]^{n} \otimes [\cdot]^{n}$; $\{[\cdot]^{n} \otimes [\cdot]^{nn}, [-, -]\}$ by $[\cdot]^{n} \otimes [\cdot]^{n}$ and finally $\{[\cdot]^{nn} \otimes [\cdot]^{nn}, [-, -]\}$ by $[\cdot]^{nn} \otimes [\cdot]^{nn}$. 

| Intersection | Tangent vector | Expansion | Twist |
|--------------|---------------|-----------|-------|
| $\mathcal{I}(C_{mA}, C_{nA})$ | $m_A m_A$ | $\theta \sim m_A M^A + m_A M^A$ | $\vartheta \sim m_A M^A - m_A M^A$ |
| $\mathcal{I}(C_{mA}, C_{nA})$ | $m_A n_A$ | $\theta \sim m_A N^A + n_A M^A$ | $\vartheta \sim m_A N^A - n_A M^A$ |
| $\mathcal{I}(C_{nA}, C_{mA})$ | $n_A m_A$ | $\theta \sim n_A M^A + m_A N^A$ | $\vartheta \sim n_A M^A - m_A N^A$ |
| $\mathcal{I}(C_{nA}, C_{nA})$ | $n_A n_A$ | $\theta \sim n_A N^A + n_A N^A$ | $\vartheta \sim n_A N^A - n_A N^A$ |

Table 4: Four intersections of congruences of null strings.
2.6 Weak hyperheavenly spaces

In this section we introduce basic definitions and notations of the weak hyperheavenly spaces. Such spaces have been defined in [12].

**Definition 2.1.** Weak hyperheavenly space (weak HH-space) is a pair \((\mathcal{M}, ds^2)\) where \(\mathcal{M}\) is a 4-dimensional complex analytic differential manifold and \(ds^2\) is a holomorphic metric, satisfying the following conditions:

(i) there exists a 2-dimensional holomorphic totally null self-dual integrable distribution given by the Pfaff system

\[ m_A g^{AB} = 0, \ m_A \neq 0 \]

(ii) the SD Weyl spinor \(C_{ABCD}\) is algebraically degenerate and \(m_A\) is a multiple Penrose spinor i.e.

\[ C_{ABCD} m^A m^B m^C = 0 \]

Hence, weak HH-space is a space of the type \([\text{deg}]^\ast \otimes [\text{any}]\). In what follows we specialize the SD \(C\) to be nonexpanding. Note, that if \(C\) is nonexpanding then it is generated by a spinor which is a multiple Penrose spinor. Thus - for weak nonexpanding HH-spaces (i.e., types \([\text{deg}]^n \otimes [\text{any}]\)) - the condition (ii) in the definition 2.1 is implied by the condition (i).

The metric of weak nonexpanding HH-space can be brought to the form [12]

\[
\frac{1}{2} ds^2 = -dp^A dq_A + Q^{AB} dq_A dq_B = e^1 e^2 + e^3 e^4
\]  

(2.31)

where \((q_A, p_B)\) are local complex coordinates. The coordinate system \((q_A, p_B)\) is chosen in such a manner that \(p_A\) are coordinates on null strings while coordinates \(q_A\) label the null strings. \(Q^{AB} = Q^{(AB)}\) are holomorphic functions of variables \((q^A, p^B)\). After imposing the reality conditions \(S\), the metric (2.31) becomes the metric of the Walker space.

An appropriate choice of a null tetrad is essential for our further purposes. It is convenient to work with the null tetrad \((e^1, e^2, e^3, e^4)\) defined as follows

\[ [e^3, e^1] = -\frac{1}{\sqrt{2}} g^{2A} = dq_A \]
\[ [e^4, e^2] = \frac{1}{\sqrt{2}} g^{1A} = -dp^A + Q^{AB} dq_B \]  

(2.32)

The operators

\[ \partial_A := \frac{\partial}{\partial p^A}, \ \bar{\partial}_A := \frac{\partial}{\partial q^A} - Q_A{}^b \partial_B \]
\[ \partial^A := \frac{\partial}{\partial p_A}, \ \bar{\partial}^A := \frac{\partial}{\partial q_A} + Q^{AB} \partial_B \]  

(2.33)

form the dual basis

\[- \partial_A = [\partial_A, \partial_2], \ \bar{\partial}^A = [\partial_A, \partial_1], \ \Rightarrow \ \partial_{AB} = \sqrt{2} [\partial_B, \bar{\partial}_B] \]  

(2.34)

The null tetrad defined by (2.32) is called Plebański tetrad. It is easy to see that null strings (leaves of \(C\)) are spanned by the vectors \((\partial_2, \partial_4)\), i.e., \(C\) is generated by the spinor \(m_A = [0, m]\), \(m \neq 0\). In this sense Plebański tetrad is adapted to the SD \(C\).
From the first structure equations one finds that the only nonzero spinorial connection coefficients read
\[ \Gamma_{12\bar{D}} = -\frac{1}{\sqrt{2}} \partial^\lambda Q_{\bar{A}\bar{D}}, \quad \Gamma_{22\bar{D}} = -\sqrt{2} \partial^\lambda Q_{\bar{A}\bar{D}}, \quad \Gamma_{\bar{A}\bar{B}2\bar{D}} = \sqrt{2} \partial_{(\bar{A}}Q_{\bar{B})\bar{D}} \] (2.35)

Nonzero SD curvature coefficients \( C^{(i)} \), the curvature scalar \( R \) and the ASD Weyl spinor take the form
\[ C^{(3)} = \frac{R}{6} = -\frac{1}{3} \partial_A \partial_B Q^{AB}, \quad C^{(2)} = -\partial^A \partial^B Q_{AB}, \] \[ \frac{1}{2} C^{(1)} = -\partial^A \partial^B Q_{AB} + (\partial^A Q_{AB})(\partial_C Q^{BC}), \quad C_{\bar{A}B\bar{C}} = -\partial_{(\bar{A}}\partial_{\bar{B}}Q_{\bar{C}D)} \] (2.36)

The traceless Ricci tensor is given by the formulas
\[ C_{11\bar{A}\bar{B}} = 0, \quad C_{12\bar{A}\bar{B}} = -\frac{1}{2} \partial_{(\bar{A}}\partial_{\bar{C}}Q_{\bar{B})\bar{C}}, \quad C_{22\bar{A}B} = -\partial_{(\bar{A}}\partial_{\bar{B}}Q_{\bar{C}D)} \] (2.37)

The metric (2.31) remains invariant under the following transformations of the coordinates
\[ q^A = q^A(q^M), \quad p^A = D^{-1}_B A p^B + \sigma^A \] (2.38)

where \( \sigma^\lambda = \sigma^\lambda(q^\mu) \) are arbitrary functions and
\[ D_A^B := \frac{\partial q^A}{\partial q_B}, \quad D^{-1}_A{}^B := \Delta \frac{\partial q^A}{\partial q_B} = \Delta \frac{\partial p^A}{\partial p_B}, \quad \Delta := \det \left( \frac{\partial q^A}{\partial q_B} \right) = \frac{1}{2} D_{AB} D^{AB} \] (2.39)

Hence
\[ D_{\bar{A}}{}^\bar{B} := D_M{}^\bar{N} \in M^\bar{A} \epsilon_{\bar{B}N} = -\Delta^{-1} D^{-1}_{\bar{B}}{}^\bar{A} \] (2.40)
\[ D^{-1}_{\bar{A}}{}^\bar{B} := D^{-1}_M{}^N \in M^\bar{A} \epsilon_{\bar{B}N} = \frac{1}{\Delta} D_{\bar{B}}{}^\bar{A} \]

Functions \( Q^{AB} \) transform under (2.38) as follows
\[ Q^{\bar{A}\bar{B}} = D^{-1}_R A \bar{D} D^{-1}_S B Q^{RS} + D^{-1}_R A \frac{\partial p^B}{\partial q_R} \] (2.41)

Transformations (2.38) are equivalent to the spinorial transformations
\[ L_A^B = \begin{bmatrix} \Delta^{-\frac{1}{2}} & h \Delta^\frac{1}{2} \\ 0 & \Delta^\frac{1}{2} \end{bmatrix}, \quad 2h := \frac{\partial \sigma^R}{\partial q_R} \] (2.42)

\[ M_{\bar{A}}{}_{\bar{B}} = \Delta^\frac{1}{2} D^{-1}_{\bar{B}}{}^\bar{A} \]

Summarizing, (2.31) is the general metric of the spaces equipped with a single SD \( C^n \).

In the next Sections we consider spaces equipped with a richer structure. First we find the metric of a space equipped with SD \( C^n \) and ASD \( C^e \) or \( C^n \) (Section 3.1). Then we add one more ASD \( C^n \) (Section 3.2). Finally, spaces equipped with SD \( C^m \) or \( C^{ne} \) and ASD \( C^{mn} \) are considered (Section 3.3.3).
3 Two-sided (complex) Walker spaces

3.1 Spaces of the types \([\deg^n \otimes \text{any}^e]\) and \([\deg^n \otimes [\deg]^n]\)

3.1.1 ASD null strings equation

In this Section we equip a weak nonexpanding \(\mathcal{H}H\)-space with an additional structure, namely: ASD \(\mathcal{C}^e\). Let it be generated by a spinor \(m_A\) with an expansion \(M_A\). The ASD null string equations (2.26) contracted with \(m^B\) yield

\[
\frac{1}{\sqrt{2}} m_A M_1 = m^B \frac{\partial m_B}{\partial p^A} \quad (3.1a)
\]

\[
\frac{1}{\sqrt{2}} m_A M_2 = m^B \frac{\partial m_B}{\partial q^A} + m^S \frac{\partial}{\partial p^S} (m^B Q_{AB}) - m^B Q_{AB} \frac{\partial m^S}{\partial p^S} \quad (3.1b)
\]

It is easy to see that both expansion \(\theta\) and twist \(\varrho\) of \(I(\mathcal{C}^e_{m_A}, \mathcal{C}^e_{m_A})\) are proportional to \(M_1\) (compare Table 4). Hence, if \(M_1 \neq 0\) then we deal with a space of the type \([\deg^n \otimes \text{any}^e, \{\cdot\}]\) and if \(M_1 = 0\) then a space is of the type \([\deg^n \otimes [\deg]^n, \{-\}]\).

Before we solve Eqs. (3.1) we rewrite the spinor \(m_A\) in the form \(m_A = [z, 1]\) (a spinor which generates \(\mathcal{C}\) must be nonzero; one can assume \(m_A^2 \neq 0\) and re-scale \(m_A^2 = 1\)).

Hence, (3.1) contracted with \(m_A\) give

\[
zz_y - z_x = 0 \quad (3.2a)
\]

\[
z_q - zz_p - z_y Y + z \frac{\partial Y}{\partial y} - \frac{\partial Y}{\partial x} = 0, \quad Y := B + 2z Q + z^2 A \quad (3.2b)
\]

where \(z_y \equiv \frac{\partial z}{\partial y}, z_x \equiv \frac{\partial z}{\partial x},\) etc., with

\[
\frac{1}{\sqrt{2}} M_1 = -z_y \quad (3.3a)
\]

\[
\frac{1}{\sqrt{2}} M_2 = -z_p - \frac{\partial}{\partial x} (Q + zA) + z \frac{\partial}{\partial y} (Q + zA) - z_y (Q + zA) \quad (3.3b)
\]

where we denoted

\[
q^1 =: q, \quad q^2 =: p, \quad p^1 =: x, \quad p^2 =: y, \quad Q^{AB} =:
\begin{bmatrix}
A & Q \\
Q & B
\end{bmatrix}
\quad (3.4)
\]

According to (3.3a), \(\theta, \varrho \sim z_y\). Hence, further steps depend on \(z_y\).

3.1.2 Spaces of the types \([\deg^n \otimes \text{any}^e, \{\cdot\}]\)

Assume \(z_y \neq 0\). A general solution of (3.2a) can be obtained by multiplying (3.2a) by \(dx \wedge dy \wedge dp \wedge dq\), treating \(z\) as an independent variable and \(y\) as a function of \((q, p, x, z)\). Finally

\[
y = -xz + \Sigma(q, p, z) \quad (3.5)
\]

where \(\Sigma\) is an arbitrary function of its variables. Hence

\[
z_x = \frac{z}{\Sigma_z - x}, \quad z_y = \frac{1}{\Sigma_z - x}, \quad z_q = -\frac{\Sigma_q}{\Sigma_z - x}, \quad z_p = -\frac{\Sigma_p}{\Sigma_z - x} \quad (3.6)
\]
Denote
\[ \tilde{Y} = \tilde{Y}(q, p, x, z) = Y(q, p, x, y(x, z, q, p)) = Y(q, p, x, y) \] (3.7)

Eq. (3.2b) transformed to the coordinate system \((q, p, x, z)\) yields
\[ \tilde{Y} - (x - \Sigma_z)\tilde{Y}_x - z\Sigma_p + \Sigma_q = 0 \] (3.8)

with a general solution
\[ \tilde{Y} = (x - \Sigma_z)\Omega + z\Sigma_p - \Sigma_q \] (3.9)

where \(\Omega = \Omega(q, p, z)\) is an arbitrary function. The definition of \(Y\) implies a general solution for \(B\) in terms of \(A, Q, \Sigma\) and \(\Omega\). This is all we need to write an explicit form of the metric. Hence, one gets

**Theorem 3.1.** Let \((\mathcal{M}, ds^2)\) be a complex (neutral) space of the type \([\deg^n \otimes \text{any}^e, [+ +] \)\. Then there exists a local coordinate system \((q, p, x, z)\) such that the metric takes the form
\[
\frac{1}{2} ds^2 = -dpdx - z dqdx - (x - \Sigma_z) dqdz + A dp^2 + (\Sigma_p - 2Q) dpdq + ((x - \Sigma_z) \Omega + z\Sigma_p - 2zQ - z^2A) dq^2
\] (3.10)

where \(A = A(q, p, x, z), Q = Q(q, p, x, z), \Sigma = \Sigma(q, p, z)\) and \(\Omega = \Omega(q, p, z)\) are arbitrary holomorphic (real smooth) functions. ■

**Remark.** In [21] the Authors found a general metric of a space equipped with SD \(C^e_{m, \Lambda}\) and ASD \(C^e_{m, A}\) with an additional assumption of an integrability of a 3-dimensional distribution given by the Pfaff system \(m_A m_A g^{A_A} = 0\). They called such a space an integrable (sesquiWalker) \(\alpha\beta\)-geometry. An adjective "integrable" in the definition given by Law and Matsushita is equivalent to the vanishing of the twist of \(\mathcal{I}(C^e_{m, A}, C^e_{m, A})\), i.e., \(M_1 = 0\). Hence, the metric found in [21] is, in fact, the metric of a space of the type \([\deg^n \otimes \text{any}^e, [- -] \)\. We consider such spaces in the next Section. The metric (3.10) can be written in a simpler form but we do not enter this problem here.

3.1.3 Spaces of the types \([\deg^n \otimes \deg^e, [- -] \)\

Now let us consider a case with \(z_y = 0\). From (3.2a) one finds \(z_x = 0\) and consequently, \(z = z(q, p)\). A transformation law for \(z\) reads
\[ z' = \Delta^{-\frac{1}{2}}(zp'_p - p'_q) \] (3.11)

Hence, \(z\) can be gauged away without any loss of generality. From (3.2b) and (3.3b) one finds
\[ B_x = 0, \quad \frac{1}{\sqrt{2}} M_2 = -Q_x \] (3.12)

Thus we arrive at

**Theorem 3.2.** Let \((\mathcal{M}, ds^2)\) be a complex (neutral) space of the type \([\deg^n \otimes \text{any}^e, [- -] \)\. Then there exists a local coordinate system \((q, p, x, y)\) such that the metric takes the form
\[
\frac{1}{2} ds^2 = dqdy - dpdx + A(q, p, x, y) dp^2 - 2Q(q, p, x, y) dqdp + B(q, p, y) dq^2
\] (3.13)

where \(A = A(q, p, x, y), Q = Q(q, p, x, y)\) and \(B = B(q, p, y)\) are arbitrary holomorphic (real smooth) functions such that \(Q_x \neq 0\). ■

**Remark.** The metric (3.13) is exactly the metric found in [21].
3.1.4 Spaces of the types $[\text{deg}]^n \otimes [\text{deg}]^n$

From the metric \((3.13)\) one easily finds a metric of the two-sided (complex) Walker spaces. \(M_2 = 0\) implies \(Q_x = 0\). Hence

**Theorem 3.3.** Let \((\mathcal{M}, ds^2)\) be a complex (neutral) space of the type $[\text{deg}]^n \otimes [\text{deg}]^n$. Then there exists a local coordinate system \((q, p, x, y)\) such that the metric takes the form

\[
\frac{1}{2} ds^2 = dq dy - dp dx + A(q, p, x, y) \, dp^2 - 2Q(q, p, y) \, dq dp + B(q, p, y) \, dq^2 \quad (3.14)
\]

where \(A = A(q, p, x, y), Q = Q(q, p, y)\) and \(B = B(q, p, y)\) are arbitrary holomorphic (real smooth) functions.

**Remark.** Theorem 3.3 has been proven earlier in [12] but the proof presented here is much more concise.

Note, that from now on the gauge \((2.38)\) is restricted to the transformations such that

\[
q' = q'(q, p), \quad p' = p'(p) \quad (3.15)
\]

3.2 Spaces of the types $[\text{deg}]^n \otimes [\text{deg}]^{ne}$ and $[\text{deg}]^n \otimes [\text{D}]^{nn}$

In this Section we finally arrive at the metric of the para-Kähler space of the type $[\text{deg}]^n \otimes [\text{D}]^{nn}$. We complete this task in three steps. As a starting point we take the metric \((3.14)\). Recall, that this metric is equipped with \(C_{mA}^n\) and \(C_{mA}^n\). Now we assume the existence of the second ASD \(C_{mA}^n\) generated by the spinor \(n^A\) such that \(n^A m_A \neq 0 \implies n_1 \neq 0\). Hence, the spinor \(n^A\) can be brought to the form \(n^A = [1, w]\). ASD null string equations for the \(C_{mA}^n\) have the same form as \((3.1)\) but with \(m^A \rightarrow n^A\) and \(M^A \rightarrow N^A\). Explicitly

\[
w_y - w w_x = 0 \quad (3.16a)
\]
\[
w_p - w w_y + \frac{\partial Z}{\partial y} - w \frac{\partial Z}{\partial x} + w_x Z = 0, \quad Z := A + 2w Q + w^2 B \quad (3.16b)
\]

and

\[
\frac{1}{\sqrt{2}} \, N_1 = w_x \quad (3.17a)
\]
\[
\frac{1}{\sqrt{2}} \, N_2 = w_y + w \frac{\partial}{\partial x} (Q + w B) - w \frac{\partial}{\partial y} (Q + w B) - w_x (Q + w B) \quad (3.17b)
\]

and the properties of \(I(C_{mA}^n, C_{mA}^n)\) read \(\theta, \varrho \sim w_x\). Transformation for \(w\) reads

\[
w' = \Delta^{-\frac{1}{2}} (q'_w w - q'_p) \quad (3.18)
\]

The next steps are analogous like in Section 3.1. First we consider a case with \(w_x \neq 0\). This case corresponds to spaces of the types \{[$[\text{deg}]^n \otimes [\text{deg}]^{ne}, [--+, ++]\}. Then a case with \(w_x = 0\) (what implies \(w = 0\)) but with \(N_2 \neq 0\) leads to spaces of the types \{[$[\text{deg}]^n \otimes [\text{deg}]^{ne}, [---, -++]\}. Finally, a case with \(w = 0\) and \(N^A = 0\) leads to spaces of the types \{[$[\text{deg}]^n \otimes [\text{D}]^{nn}\}. We skip all the details and formulate three Theorems.
Theorem 3.4. Let \((\mathcal{M}, ds^2)\) be a complex (neutral) space of the type \([\text{deg}^n \otimes \text{deg}^{ne}, [-+, ++]]\). Then there exists a local coordinate system \((q, p, w, y)\) such that the metric takes the form

\[
\frac{1}{2} ds^2 = dqdy + w dpdy + (y - \Sigma_w) dpdw + \mathcal{B} dq^2 \\
- (2Q + \Sigma_y) dqdp + ((y - \Sigma_w) \Omega - w \Sigma_y - 2wQ - w^2 \mathcal{B}) dp^2
\]

where \(Q = Q(q, p, y)\), \(B = B(q, p, y)\), \(\Omega = \Omega(q, p, w)\) and \(\Sigma = \Sigma(q, p, w)\) are arbitrary holomorphic (real smooth) functions.

Theorem 3.5. Let \((\mathcal{M}, ds^2)\) be a complex (neutral) space of the type \([\text{deg}^n \otimes \text{deg}^{ne}, [--, --]]\). Then there exists a local coordinate system \((q, p, x, y)\) such that the metric takes the form

\[
\frac{1}{2} ds^2 = dqdy - dpdx + \mathcal{A}(q, p, x) dp^2 - 2Q(q, p, y) dqdp + \mathcal{B}(q, p, y) dq^2
\]

where \(A = A(q, p, x)\), \(Q = Q(q, p, y)\) and \(B = B(q, p, y)\) are arbitrary holomorphic (real smooth) functions such that \(Q_y \neq 0\).

\[
\text{If } Q_y = 0 \text{ then } Q = Q(q, p) \text{ and it can be gauged away without any loss of generality.}
\]

Thus we arrive at the Theorem

Theorem 3.6. Let \((\mathcal{M}, ds^2)\) be a complex (neutral) space of the type \([\text{deg}^n \otimes [D]^{nn}\) \([\text{deg}^n \otimes [D_r]^{nn}\]). Then there exists a local coordinate system \((q, p, x, y)\) such that the metric takes the form

\[
\frac{1}{2} ds^2 = dqdy - dpdx + \mathcal{A}(q, p, x) dp^2 + B(q, p, y) dq^2
\]

where \(A = A(q, p, x)\) and \(B = B(q, p, y)\) are arbitrary holomorphic (real smooth) functions.

**Remark.** The existence of ASD \(C^{nn}\) implies the existence of two distinct multiple dotted Penrose spinors. Thus the Petrov-Penrose type of the ASD Weyl spinor of the metric \((3.21)\) is \([D]^{nn}\).

Nonzero conformal curvature coefficients, the traceless Ricci tensor and the curvature scalar in Plebański tetrad read

\[
C^{(3)} = \frac{R}{6} = 2C_{1122} = -\frac{1}{3}(A_{xx} + B_{yy}), \quad C^{(2)} = -A_{qx} - B_{py}
\]

\[
\frac{1}{2} C^{(1)} = -B_{pp} - A_{qq} + B_{p}A_{x} - B_{y}A_{q}
\]

\[
C_{1212} = \frac{1}{4}(A_{xx} - B_{yy}), \quad C_{2212} = \frac{1}{2}(A_{xx} - B_{yy})
\]

It is worth to note that \(C_{1212}\) has a transparent geometrical meaning. If \(C_{1212} \neq 0\) the traceless Ricci tensor has four eigenvectors and two double eigenvalues. If \(C_{1212} = 0\) the traceless Ricci tensor has two eigenvectors and one quadruple eigenvalue.

The gauge freedom admitted by the metric \((3.21)\) is more restricted then \((3.15)\). Namely

\[
q' = q(q), \quad p' = p'(p), \quad \frac{dq'}{dq} = \mu(q), \quad \frac{dp'}{dp} = \nu(p)
\]

\[
x' = \frac{1}{\nu} x + \frac{1}{\nu} \frac{\partial \sigma}{\partial q}, \quad y' = \frac{1}{\mu} y + \frac{1}{\mu} \frac{\partial \sigma}{\partial q}
\]
where \( \sigma = \sigma(q,p) \), \( \mu = \mu(q) \) and \( \nu = \nu(p) \) are arbitrary functions. Under (3.23) the functions \( A \) and \( B \) transform as follows

\[
A' = \frac{1}{\nu^2} A - \frac{\nu^2}{\nu^3} x + \frac{1}{\nu} \frac{\partial q}{\partial p} \left( \frac{1}{\nu} \frac{\partial \sigma}{\partial p} \right) \quad \text{and} \quad B' = \frac{1}{\mu^2} B + \frac{\mu^2}{\mu^3} y - \frac{1}{\mu} \frac{\partial q}{\partial p} \left( \frac{1}{\mu} \frac{\partial \sigma}{\partial p} \right)
\] (3.24)

**Case** \( A_{xx} + B_{yy} \neq 0 \). If \( A_{xx} + B_{yy} \neq 0 \) the metric (3.21) is of the type \([II]^n \otimes [D]^{nn}\) or \([D]^{n} \otimes [D]^{nn}\). The type \([D]^{n} \otimes [D]^{nn}\) is given by the condition

\[
2C^{(2)}C^{(2)} - 3C^{(3)}C^{(1)} = 0 \quad (3.25)
\]

Eq. (3.25) written explicitly yields

\[
(A_{qx} + B_{py})^2 - (A_{xx} + B_{yy})(B_{pp} + A_{qq} - B_p A_x + B_y A_q) = 0 \quad (3.26)
\]

Interesting task is to find a solution of the condition (3.26) such that a space is still of the type \([D]^{n} \otimes [D]^{nn}\), i.e., there is no additional SD (such an additional structure is considered in Section 3.3). Namely, to find a solution of the condition (3.26) such that Eqs. (3.29) are not satisfied. This problem is quite advanced and we do not consider it here.

We mention only a special solution \( A = 0, B_p = g(B_y, q) \) where \( g \) is an arbitrary function. In this case Eq. (3.29b) is identically satisfied, but (3.29a) is not. Consequently, some \( g \)s lead to the type \([D]^{n} \otimes [D]^{nn}\) solutions but we were not able to find any explicit example.

**Case** \( A_{xx} + B_{yy} = 0 \). If \( A_{xx} + B_{yy} = 0 \) then \( C^{(3)} = 0 \) but at the same time \( C_{1122} = 0 \). Then a space becomes SD space of the type \([III]^n \otimes [O]^n\) or \([N]^n \otimes [O]^n\). Such spaces are considered in Section 3.4.

It is worth to point out that spaces with the metrics (3.19) and (3.20) are equipped with \( C^{ne} \). It has been proven in [32] that if an Einstein space is equipped with SD (ASD) \( C^{ne} \) then its SD (ASD) Weyl spinor must vanish. Thus, the traceless Ricci tensor of not conformally flat spaces equipped with SD (or ASD) \( C^{ne} \) must be nonzero. We believe, that metrics (3.19) and (3.20) could be the first explicit examples of not conformally flat spaces equipped with \( C \)s with “mixed” properties and of the same duality.

### 3.3 Spaces of the types \([II,D]^{ne} \otimes [D]^{nn}\) and \([D]^{nn} \otimes [D]^{nn}\)

#### 3.3.1 The second congruence of SD null strings

In this Section the structure of a space will be specialized deeper. The metric (3.21) is equipped with three \( C \)s:

\[
\text{SD} \quad C_{mA} \text{ generated by the spinor } m_A, \quad m_2 \neq 0 \quad (3.27)
\]
\[
\text{ASD} \quad C_{m\dot{A}} \text{ generated by the spinor } m_{\dot{A}}, \quad m_2 \neq 0
\]
\[
\text{ASD} \quad C_{n\dot{A}} \text{ generated by the spinor } n_{\dot{A}}, \quad n_1 \neq 0
\]

Now we assume the existence of the second SD \( C \). Let \( C \) be generated by the spinor \( n_A \) such that \( m_A n_A \neq 0 \implies n_1 \neq 0 \). Hence, the spinor \( n_A \) can be re-scaled to the form \( n_A = [1, n] \). The transformation law for \( n \) reads (compare (2.42))

\[
n' = \Delta^{-\frac{1}{2}}(n - \sigma_{pq}) \quad (3.28)
\]

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The SD null string equations \( n^{B} \nabla_{A \bar{M}} n_{B} = n_{A} N_{\bar{M}} \) written explicitly yield

\[
\begin{align*}
    n_{q} - n_{y} B - B_{p} + n B_{y} &= n n_{x} \quad (3.29a) \\
    n_{p} + n_{x} A + A_{q} - n A_{x} &= n n_{y} \quad (3.29b)
\end{align*}
\]

In general, \( C_{nA} \) is expanding and the expansion is given by the formula

\[
N_{\bar{M}} = \sqrt{2} \frac{\partial n}{\partial p^{\bar{M}}} \quad (3.30)
\]

It is easy to check, that the properties of the intersection of \( C_{nA} \) with ASD \( C_{s} \) read

\[
I(C_{nA}, C_{mA}) : \theta, \varrho \sim m^{4} A_{\bar{A}} \sim \frac{\partial n}{\partial x} \quad (3.31)
\]

\[
I(C_{nA}, C_{nA}) : \theta, \varrho \sim n^{4} A_{\bar{A}} \sim \frac{\partial n}{\partial y}
\]

Hence, we deal now with four \( C_{s} \) and four \( I_{s} \) with the properties (listed in order given by (2.30c)):

\[
\begin{align*}
[-+, -+, +, +] & \text{ if } n_{x} \neq 0, n_{y} \neq 0 \\
[-+, -+, -, +] & \text{ if } n_{x} = 0, n_{y} \neq 0 \\
[-+, -+, +, -] & \text{ if } n_{x} \neq 0, n_{y} = 0 \\
[-+, -+, -, -] & \text{ if } n_{x} = n_{y} = 0
\end{align*}
\]

The cases \([-+, -+, -, +] \) and \([-+, -+, +, -] \) are equivalent. In the case \([-+, -+, -, -] \), \( C_{nA} \) is nonexpanding.

The integrability condition of the system (3.29) (see, e.g., [8]) yields

\[
C^{(1)} - 4 C^{(2)} n + 6 C^{(3)} n^{2} = 0 \quad (3.32)
\]

Eq. (3.32) is a quadratic equation for \( n \). If \( C^{(3)} \neq 0 \) one finds a general solution in the form

\[
n_{\pm} = \frac{C^{(2)}}{3 C^{(3)}} \pm \frac{\sqrt{2 \delta}}{6 C^{(3)}}, \quad \delta := 2 C^{(2)} C^{(2)} - 3 C^{(3)} C^{(1)} \quad (3.33)
\]

Note, that vanishing of a discriminant of the quadratic equation (3.32) is equivalent to the type-D condition (3.25). Hence, if \( \delta \neq 0 \) and both \( n_{-} \) and \( n_{+} \) solve the system (3.29) one arrives at a space of the type \([II]^{ne} \otimes [D]^{mn} \). This is an extremely interesting case with three SD \( C_{s} \) and two ASD \( C_{s} \), but - as we mentioned earlier - we do not consider it in this paper. If \( n_{-} \) or \( n_{+} \) solve the system (3.29) then a space is of the type \([II]^{ne} \otimes [D]^{mn} \). If \( \delta = 0 \) then there is only one double solution of Eq. (3.32) and a space is of the type \([D]^{ne} \otimes [D]^{mn} \). Unfortunately, after putting \( n_{\pm} \) into the system (3.29) one gets very complicated equations for \( A \) and \( B \). These equations will be solved in a few special cases in the next Sections.

If \( C^{(3)} = 0 \), we obtain

\[
n = \frac{C^{(1)}}{4 C^{(2)}} \quad (3.34)
\]

Solutions of the system (3.29) with \( n \) given by (3.34) together with \( C^{(3)} = 0 \) imply that a space is of the type \([III]^{ne} \otimes [O]^{n} \). This case is considered in Section 3.4.2.
3.3.2 Spaces of the types $[D]^{nn} \otimes [D]^{nn}$

First we consider the case with $C_{\alpha \beta}^n$, i.e., $n = n(q, p)$. From (3.28) it follows that in this case the function $n$ can be gauged away, $n = 0$. It leaves us with the conditions $B_p = A_q = 0$ (compare (3.29)). Hence, $B = B(q, y)$ and $A = A(p, x)$. In this case the condition (3.25) is satisfied so a space is of the type $[D]^{nn} \otimes [D]^{nn}$. After suitable transformation of the variables the metric of such a space can be brought to the form

$$\frac{1}{2} ds^2 = dqdy - dpdx + A(x, p) dp^2 + B(y, q) dq^2$$

(3.35)

The metric (3.35) is a well known metric with an interesting property: it is two-sided conformally recurrent (see, e.g., [27]).

3.3.3 Spaces of the types $\{[II]^{ne} \otimes [D]^{nn}, [\ldots, -\ldots, -\ldots, +\ldots]\}$

Now we consider the case $\ldots, -\ldots, -\ldots, +\ldots$. Let $n_x = 0$. Differentiating (3.29b) with respect to $x$ one gets $A_{qx} = nA_{xx}$. Because $n_y \neq 0$ then $A_{xx} = A_{qx} = 0$. Hence, $A$ can be gauged away without any loss of generality (compare (3.24)). With $A = 0$, Eq. (3.29b) reduces to $n_p = nn_y$ with an implicit solution

$$y = -pn + S(q, n)$$

(3.36)

where $S = S(q, n)$ is an arbitrary function. Then we proceed analogously like in Section 3.1.2 i.e., we treat $n$ as an independent variable and $y$ as a function, $y = y(q, p, n)$. Note that

$$n_y = \frac{1}{S_n - p}, \; n_p = \frac{n}{S_n - p}, \; n_q = -\frac{S_q}{S_n - p}$$

(3.37)

Let us denote

$$\tilde{B} = \tilde{B}(q, p, n) = \tilde{B}(q, p, y(q, p, n)) = B(q, p, y)$$

(3.38)

Hence

$$B_y = \tilde{B}_n n_y, \; B_p = \tilde{B}_p + \tilde{B}_n n_p$$

(3.39)

Eq. (3.29a) reduces to the form

$$S_q + \tilde{B} + (S_n - p) \tilde{B}_p = 0$$

(3.40)

with a general solution

$$\tilde{B} = A(q, n) (p - S_n) - S_q$$

(3.41)

where $A = A(q, n)$ is an arbitrary function. Now we are ready to write an explicit form of the metric in coordinates $(q, p, x, n)$

$$\frac{1}{2} ds^2 = -dpdx - n dqdp - p dqdn + B dqdn + A(p - B) dq^2$$

(3.42)

where we denoted $B(q, n) := S_n$. The metric (3.42) is a general metric of a space of the type $\{[II]^{ne} \otimes [D]^{nn}, [\ldots, -\ldots, -\ldots, +\ldots]\}$. Because of the implicit relation between $n$ and $y$ (3.36) it is not straightforward to transform (3.42) into original "hyperheavenly" coordinate system $(q, p, x, y)$.
The SD conformal curvature coefficients and the traceless Ricci tensor read
\[
C^{(3)} = \frac{1}{3(B-p)^3} \left( A_{nn} (B-p)^2 + (B-p) \frac{\partial}{\partial n} (AB_n + B_q) - B_n (AB_n + B_q) \right)
\]
\[
C^{(2)} = 3nC^{(3)} + \frac{AB_n + B_q}{(B-p)^2}, \quad C^{(1)} = 4nC^{(2)} - 6n^2C^{(3)}
\]
\[
C_{121\dot{2}} = \frac{3}{4} C^{(3)}, \quad C_{221\dot{2}} = \frac{1}{2} C^{(2)}
\]
and
\[
\delta = 2 \left( \frac{AB_n + B_q}{(B-p)^2} \right)^2
\]
(3.43)

The SD Weyl spinor is of the type [II] if \( C^{(3)} \neq 0 \) and \( \delta \neq 0 \).

3.3.4 Spaces of the types \{[D]^{ne} \otimes [D]^{nn}, [-- , -- , -- , ++]\}

Reduction of the metric (3.42) to the type \{[D]^{ne} \otimes [D]^{nn}, [-- , -- , -- , ++]\} is realized by the condition \( \delta = 0 \). It yields
\[
AB_n + B_q = 0
\]
(3.45)

Multiplying (3.45) by \( dn \wedge dq \), treating \( B \) as a new variable and \( n \) as a function, \( n = n(q, B) \), one arrives at the solution \( A = n_q \). If we denote \( B \to z \) and \( n \to F \), the metric takes the form
\[
\frac{1}{2} ds^2 = -dpdx - F dqdp + (z-p)F_z dqdz
\]
(3.46)

where \( F = F(q, z) \) is an arbitrary function. The SD conformal curvature coefficients read
\[
C^{(3)} = \frac{\partial z \partial q \ln F_z}{3(z-p)F_z}, \quad C^{(2)} = 3FC^{(3)}, \quad C^{(1)} = 6F^2C^{(3)}
\]
(3.47)

3.3.5 Spaces of the types \{[II,D]^{ne} \otimes [D]^{nn}, [-- , -- , ++ , ++]\}

In this case \( n_x \neq 0 \) and \( n_y \neq 0 \). Such a case will be considered elsewhere.

3.4 Spaces of the types \{[III,N]^{n} \otimes [O]^{n}\}

3.4.1 Types \{[III,N]^{n} \otimes [O]^{n}\}

The standard approach to SD solutions via weak nonexpanding \( \mathcal{HH} \)-spaces uses the formula (2.36) [11, 15, 27]. By demanding that \( C_{ABCD} = 0 \) a solution for \( Q^{AB} \) can be obtained, but there are 15 arbitrary functions of two variables in this solution. Obviously, such an approach generates plenty of arbitrary functions. A bunch of them is gauge-dependent but it is not so straightforward to prove it.

There is an interesting geometrical reason why the "\( C_{ABCD} = 0 \)" - approach is not the optimal one for SD spaces. Weak nonexpanding \( \mathcal{HH} \)-spaces are equipped with a single SD \( C^n \) and there are no ASD \( C^n \) in general. However, self-duality condition \( C_{ABCD} = 0 \) is equivalent to the existence of infinitely many distinct ASD \( C^n \). The problem is that with \( C_{ABCD} = 0 \), Plebański tetrad is still adapted to the SD \( C^n \) but it is not adapted to the ASD \( C^n \) at all.
Our construction presented in Section 3.2 introduces a serious advantage. We assumed the existence of ASD $C^n$ and we adapted Plebaniśki tetrad to these two ASD $C$s. Thus, the metric (3.21) is a better starting point for finding SD metrics then the metric (2.31) because it has been already adapted to the pair of ASD $C$s. If we additionally demand $C_{ABCD} = 0$ a solution for $Q^{AB}$ is no longer full of functions which are arbitrary but gauge-dependent.

If $C_{1222}$ is 0 in (3.22) one arrives at spaces of the types $[III,N] \otimes [O]^n$. These are the only possible SD spaces equipped with SD $C^n$ (SD spaces equipped with SD $C^e$ will be considered elsewhere). Because in spaces of the types $[III,N]^n \otimes [O]^n$ there are infinitely many $I$s it does not make any sense to list their properties.

With $C^{(3)} = 0 = C_{1122}$ one finds that\footnote{In general $B = -M y^2 + N y + S$, where $S = S(p,q)$ but $S$ can be gauged away without any loss of generality what we treat as done.}

$$
A = M x^2 + P x + \Omega, \quad B = -M y^2 + N y
$$

(3.48)

where $M$, $P$, $\Omega$ and $N$ are arbitrary functions of variables $(q,p)$. Then

$$
C^{(2)} = 2M_p y - 2M_q x - N_p - P_q
$$

(3.49)

$$
\frac{1}{2} C^{(1)} = xy(2MM_q x - 2M_M p y + 2MN_p + 2M P_q) + y^2(M_{pp} - MP_p)
$$

$$
- x^2(M_{qq} + NM_q) + y(- N_{pp} + PN_p + 2M \Omega_q)
$$

$$
- x(P_{qq} + NP_q) - N \Omega_q - \Omega_{qq}
$$

$C_{1212} = M$

$C_{2212} = M_p y + M_q x - \frac{1}{2} N_p + \frac{1}{2} P_q$

Finally, one arrives at

**Theorem 3.7.** Let $(\mathcal{M}, ds^2)$ be a complex (neutral) space of the type $[III, N]^n \otimes [O]^n$ ($[III, N]^n \otimes [O]^n$). Then there exists a local coordinate system $(q,p,x,y)$ such that the metric takes the form

$$
\frac{1}{2} ds^2 = dq dy - dp dx + (M x^2 + P x + \Omega) dp^2 + (- M y^2 + N y) dq^2
$$

(3.50)

where $M = M(q,p)$, $P = P(q,p)$, $\Omega = \Omega(q,p)$, $N = N(q,p)$ and $\Sigma = \Sigma(q,p)$ are functions such that

- for the type $[III]^n \otimes [O]^n$ ($[III]^n \otimes [O]^n$) : $|M_p| + |M_q| + |N_p| + |P_q| \neq 0$;
- for the type $[N]^n \otimes [O]^n$ ($[N]^n \otimes [O]^n$) : $M = M_0 = \text{const}$, $P = \Sigma_p$, $N = -\Sigma_q$.

Hence, the general metric of spaces of the type $[III]^n \otimes [O]^n$ has been reduced to a form with four arbitrary functions of two variables. For the type $[N]^n \otimes [O]^n$ we have two arbitrary functions of two variables and one constant.

**Remark.** Spaces of the type $[N]^n \otimes [O]^n$ are two-sided conformally recurrent. Such spaces have been considered by Plebański and Przanowski in [27]. Plebański and Przanowski listed three classes of such solutions which depend on five arbitrary functions of two variables each. Our result prove that all these metrics can be reduced to the metric (3.50) with two functions of two variables and one constant. Consequently, our approach is a serious improvement of results published in [27].
3.4.2 Type $[\text{III}]^{ne} \otimes [\text{O}]^{n}$

The only additional structure in which a space of the type $[\text{III}]^{n} \otimes [\text{O}]^{n}$ can be equipped is one more SD $C$. This second congruence must be $C^\epsilon$. With such a congruence an algebraic reduction to the type $[\text{N}]^{n} \otimes [\text{O}]^{n}$ is not possible anymore. Thus, we assume the existence of the function $n$ such that it has the form (3.34) with $C^{(1)}$ and $C^{(2)}$ given by (3.49). Inserting $n$ into Eqs. (3.29) one finds that $M = M_0 = \text{const}$ and

$$
\begin{align*}
2 & (ba_q - ab_q) - 4a^2 N_p - 2M_0af + cb = 0 \\
2 & (ac_q - ca_q) + 2Na c - c^2 = 0 \\
2 & (af_q - fa_q) + 2Na f - fc = 0 \\
2 & (ba_p - ab_p) - 4M_0a^2 \Omega + 2Pba - b^2 = 0 \\
2 & (ac_p - ca_p) + 4a^2 P_q - 2M_0af + cb = 0 \\
2 & (af_p - fa_p) + 4a^2 \Omega_q + 2\Omega ac - 2Pfa + fb = 0
\end{align*}
$$

(3.51)

where we denoted

$$
a := N_p + P_q, \quad b := PN_p - N_{pp} + 2M_0 \Omega_q, \quad c := NP_q + P_{qq}, \quad f := \Omega_{qq} + N \Omega_q
$$

(3.52)

(3.51) is a system of six equations for three functions $N$, $P$ and $\Omega$ of two variables $(p, q)$. This system is surprisingly complicated. We were able to find only three different special solutions

(i) $N = 0$, $M_0$ is arbitrary, $P = \frac{4}{4p - q}$, $\Omega = \xi P - \xi_p - M_0 \xi^2$, $\xi = \xi(p)$

(ii) $N = M_0 = \Omega = 0$, $P = \frac{4}{4p - q} + \xi$, $\xi = \xi(p)$

(iii) $M_0 = P = \Omega = 0$, $N = \frac{\xi q}{p - \xi}$, $\xi = \xi(q)$, $\xi_q \neq 0$

where $\xi$ is an arbitrary function of its variable. Hence, the solutions (3.53) are the only known explicit solutions of the type $[\text{III}]^{ne} \otimes [\text{O}]^{n}$.

4 Para-Kähler Einstein spaces

4.1 General case

From Theorem 3.6 and from the formulas (3.22) one easily obtains Einstein spaces of the types $[\text{deg}]^{n} \otimes [\text{D}]^{n}$. With $C_{1212} = C_{2212} = 0$ and $R = -4\Lambda$ one arrives at the solution

$$
A = \frac{\Lambda}{2} x^2 + \Phi_p x + \Omega, \quad B = \frac{\Lambda}{2} y^2 + \Phi_q y + \Sigma
$$

(4.1)

where $\Phi(q, p)$, $\Sigma(q, p)$ and $\Omega(q, p)$ are arbitrary functions of their variables. Hence, the general metric of the type $[\text{deg}]^{n} \otimes [\text{D}]^{n}$ Einstein spaces can be brought to the form

$$
\frac{1}{2} ds^2 = dqdy - dpdx + \left( \frac{\Lambda}{2} x^2 + \Phi_p x + \Omega \right) dp^2 + \left( \frac{\Lambda}{2} y^2 + \Phi_q y + \Sigma \right) dq^2
$$

(4.2)

The SD conformal curvature coefficients read

$$
C^{(3)} = 2C_{1122} = -\frac{2}{3} \Lambda, \quad C^{(2)} = -2\Phi_{pq}
$$

(4.3)

$$
\frac{1}{2} C^{(1)} = (\Phi_p \Phi_{pq} - \Phi_{cpp} - \Lambda \Omega_p)y - (\Phi_q \Phi_{pq} + \Phi_{pqq} - \Lambda \Sigma_p)x - \Phi_q \Omega_q + \Phi_p \Sigma_p - \Omega_{qq} - \Sigma_{pp}
$$
Remark. The solution \((4.1)\) corresponds to the \(\mathcal{HH}\)-space generated by the key function
\[
\Theta = -\frac{\Lambda}{12} x^2 y^2 - \frac{1}{6} \Phi_p x y^2 - \frac{1}{6} \Phi_q y x^2 - \frac{1}{2} \Sigma x^2 - \frac{1}{2} \Omega y^2 + \alpha_A p^A + \beta \quad (4.4)
\]
where \(\alpha_A\) and \(\beta\) are arbitrary functions of \((q, p)\). Also,
\[
F^i = \Phi_p, \quad F^2 = \Phi_q, \quad N_1 = \Sigma_p, \quad N_2 = -\Omega_q, \quad \gamma = -\Lambda \beta + \Sigma \Omega - \frac{\partial \Omega^M}{\partial q^M} + \alpha_M F^M \quad (4.5)
\]
The structural functions \(F^A, N^A\) and \(\gamma\) are well-known in \(\mathcal{HH}\)-spaces theory [see, e.g., (7)]. From \((3.24)\) one finds
\[
\Phi' = \Phi - \Lambda \sigma + \ln \frac{\mu}{\nu} + \Phi_0 \quad (4.6)
\]
\[
\nu^2 \Omega' = \Omega - \Phi'_p \sigma_p - \frac{\Lambda}{2} \sigma_p^2 + \nu \frac{\partial}{\partial p} \left( \frac{\sigma_p}{\nu} \right)
\]
where \(\sigma = \sigma(q, p), \mu = \mu(q)\) and \(\nu = \nu(p)\) are arbitrary gauge functions and \(\Phi_0\) is an arbitrary gauge constant. Hence, \(\Phi, \Sigma\) or \(\Omega\) can be gauged away but for different Petrov-Penrose types different choices are optimal.

4.2 Types \([\Pi]^n \otimes [D]^{nn}\) and \([D]^{nn} \otimes [D]^{nn}\)

4.2.1 General case

For the types \([\Pi]^n \otimes [D]^{nn}\) and \([D]^{nn} \otimes [D]^{nn}\) the cosmological constant \(\Lambda \neq 0\) so \(\Phi\) can be always gauged away. Additionally, type-[D] condition \((3.25)\) implies \(C^{(1)} = 0 \iff \Sigma_p = \Omega_q = 0\). Hence, \(\Sigma = \Sigma(q)\) and \(\Omega = \Omega(p)\). Thus, using gauge functions \(\mu(q)\) and \(\nu(p)\) one can always set \(\Sigma = \Omega = 0\). Finally, one arrives at

**Theorem 4.1.** Let \((\mathcal{M}, ds^2)\) be complex (neutral) Einstein space of the type \([\Pi]^n \otimes [D]^{nn}\) or \([D]^{nn} \otimes [D]^{nn}\) \(([\Pi_r]_n \otimes [D_r]^{nn}, [\Pi_{rc}]_n \otimes [D_r]^{nn}\) or \([D_r]^{nn} \otimes [D_r]^{nn}\)). Then there exists a local coordinate system \((q, p, x, y)\) such that the metric takes the form
\[
\frac{1}{2} ds^2 = dq dy - dp dx + \left( \frac{\Lambda}{2} x^2 + \Omega \right) dp^2 + \left( \frac{\Lambda}{2} y^2 + \Sigma \right) dq^2 \quad (4.7)
\]
where \(\Omega = \Omega(q, p)\) and \(\Sigma = \Sigma(q, p)\) are functions such that

for the type \([\Pi]^n \otimes [D]^{nn}\) \(([\Pi_r]_n \otimes [D_r]^{nn}, [\Pi_{rc}]_n \otimes [D_r]^{nn})\) : \(|\Sigma_p| + |\Omega_q| \neq 0\);

for the type \([D]^{nn} \otimes [D]^{nn}\) \(([D_r]^{nn} \otimes [D_r]^{nn})\) : \(\Sigma = \Omega = 0\).

\[\blacksquare\]

**Remark.** Consider the metric of the type \([D]^{nn} \otimes [D]^{nn}\). Performing the coordinate transformation
\[
\frac{1}{y} \rightarrow \frac{1}{y} + \frac{\Lambda}{2} q, \quad \frac{1}{x} \rightarrow -\frac{1}{x} - \frac{\Lambda}{2} p \quad (4.8)
\]
one arrives at the well-known form

\[ ds^2 = \frac{2dx dp}{\left(1 + \frac{\Lambda}{2} x p\right)^2} + \frac{2dy dq}{\left(1 + \frac{\Lambda}{2} y q\right)^2} \] (4.9)

There are two different classes of homogeneous pKE spaces. A neutral slice of the metric (4.9) corresponds to one of them [3]. The second class is so-called dancing metric but the dancing metric is of the type [O]n ⊗ [D]mn and it cannot be obtained from the metric (4.7).

4.2.2 Types [II]n ⊗ [D]mn and [D]mn ⊗ [D]mn with symmetries

Symmetries in nonexpanding \( \mathcal{H} \)-spaces have been analyzed in [7, 16]. It has been proven that Eqs. \( \nabla_a K_b = \chi_0 g_{ab} \) in nonexpanding \( \mathcal{H} \)-spaces can be reduced to a single, first order, partial, linear differential equation called the master equation for the key function \( \Theta \). Feeding the master equation with (4.4) and (4.5) with \( \Phi = 0 \) we find that any Killing vector admitted by the metric (4.7) has the form

\[
K = \delta^1 \frac{\partial}{\partial q} + \delta^2 \frac{\partial}{\partial p} - \left( \frac{d\delta^2}{dp} x + \frac{1}{\Lambda} \frac{d^2\delta^2}{dp^2} \right) \frac{\partial}{\partial x} - \left( \frac{d\delta^1}{dq} y - \frac{1}{\Lambda} \frac{d^2\delta^1}{dq^2} \right) \frac{\partial}{\partial y} \tag{4.10}
\]

where \( \delta^1 = \delta^1(q) \), \( \delta^2 = \delta^2(p) \). The vector \( K \) cannot be null and it cannot be proper homothetic. The master equation reduces to the system of two differential equations

\[
\delta^1 \Sigma_q + \delta^2 \Sigma_p + 2\Sigma \frac{d\delta^1}{dq} + \frac{1}{\Lambda} \frac{d^2\delta^1}{dq^2} = 0 \tag{4.11}
\]

\[
\delta^1 \Omega_q + \delta^2 \Omega_p + 2\Omega \frac{d\delta^2}{dp} + \frac{1}{\Lambda} \frac{d^2\delta^2}{dp^2} = 0
\]

Detailed analysis of (4.11) is long and tedious. We skip all the details and present only final results. The metric (4.7) does not admit any Killing vector in general. If an existence of a single Killing vector is assumed it can be brought to the form \( K_1 = \partial_q \) or \( K_1 = \partial_x + \partial_p \) without any loss of generality. The maximal number of Killing vectors admitted by the metric (4.7) of the type [II]n ⊗ [D]mn is 2. Type [D]mn ⊗ [D]mn is equipped with 6 Killing vectors. The results are presented in the Table 5 (all quantities in the Table 5 with a subscript 0 are constants).

4.3 Types [III, N]n ⊗ [O]n

4.3.1 General case

For the type [III]n ⊗ [O]n the cosmological constant \( \Lambda = 0 \) and \( \Phi \neq 0 \). However, \( \Sigma \) or \( \Omega \) can be gauged away (compare (4.6)). Let \( \Sigma = 0 \) what remains valid for both types [III]n ⊗ [O]n and [N]n ⊗ [O]n.

\[11\] We use the following terminology: if a vector \( K \) satisfies Eqs. \( \nabla_a K_b = \chi_0 g_{ab} \) then \( K \) is called a homothetic vector; if \( \chi_0 \neq 0 \) then \( K \) is called a proper homothetic vector; if \( \chi_0 = 0 \) the \( K \) is called a Killing vector.
| Killing vectors | Functions in the metric |
|-----------------|------------------------|
| $K_1 = \partial_q$ | $\Sigma = \Sigma(p)$, $\Sigma_p \neq 0$, $\Omega = 0$ |
| $K_1 = \partial_q$, $K_2 = q \partial_q - y \partial_y + \zeta_0 \partial_p + \xi_0(p \partial_p - x \partial_x)$ | $\Omega = 0$, $\Sigma(p) = \exp \left( \int \frac{-2 dp}{\gamma_0 p^2 + \xi_0 p + \zeta_0} \right)$ |
| $\pm \gamma_0(p^2 \partial_p - 2(p x + \Lambda^{-1}) \partial x)$ | $|\gamma_0| + |\xi_0| + |\zeta_0| \neq 0$ |
| $\pm \gamma_0 = 0$ then $\gamma_0 = 1$, $\xi_0 = 0$, $\zeta_0$ is arbitrary |
| $\pm \gamma_0 = 0$ and $\xi_0 \neq 0$ then $\zeta_0 = 0$ |
| $\pm \gamma_0 = \xi_0 = 0$ then $\zeta_0 = 1$ |
| $K_1 = \partial_q + \partial_p$, $K_2 = q \partial_q + p \partial_p - x \partial_x - y \partial_y$ | $\Sigma = \Sigma(z)$, $\Omega = \Omega(z)$, $z := q - p$, $|\Sigma_z| + |\Omega_z| \neq 0$ |
| $\Sigma(z) = \Sigma_0 z^{-2}$, $\Omega(z) = \Omega_0 z^{-2}$, $z := q - p$, $|\Sigma_0| + |\Omega_0| \neq 0$ |
| $K_1 = \partial_q + \partial_p$, $K_2 = e^{a_0 q} (\partial_q + (-a_0 y + a_0^2 \Lambda^{-1}) \partial_y)$ | $\Sigma(z) = \frac{\Sigma_0}{(1 - e^{a_0 z})^2} - \frac{a_0^2}{2 \Lambda}$ |
| $+ e^{a_0 p} (\partial_p - (a_0 x + a_0^2 \Lambda^{-1}) \partial_x)$ | $\Omega(z) = \frac{\Omega_0}{(1 - e^{a_0 z})^2} - \frac{a_0^2}{2 \Lambda}$ |
| $z := q - p$, $a_0 \neq 0$, $|\Sigma_0| + |\Omega_0| \neq 0$ |
| $K_1 = \partial_q + \partial_p$, $K_2 = e^{a_0 q} (\partial_q + (-a_0 y + a_0^2 \Lambda^{-1}) \partial_y)$ | $\Sigma(z) = \Sigma_0 e^{-2 a_0 z} - \frac{a_0^2}{2 \Lambda}$, $z := q - p$, $\Omega = \Omega_0$, $a_0 \neq 0$, $\Sigma_0 \neq 0$ |

Table 5: Killing vectors in spaces of the types $[\Pi]^n \otimes [O]^{nn}$ or $[D]^{nn} \otimes [D]^{nn}$.

Additionally, for the type $[N]^n \otimes [O]^n$ the SD conformal curvature coefficient $C^{(2)} = 0$ what implies $\Phi_{pq} = 0$. Consequently, $\Phi = \Phi_1(q) + \Phi_2(p)$ and using $\mu$ and $\nu$ both $\Phi_i$ can be gauged away. Finally, we arrive at $^{12}$

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$^{12}$A short historical remark about exact solutions of algebraically degenerated heavenly spaces is needed. All metrics of SD Einstein spaces of the type $[N]^n \otimes [O]^n$ and $[N]^n \otimes [O]^n$ have been found for the first time in $^{17}$ (the metrics (5.20) and (5.24) in $^{17}$). A great progress in the subject have been done by Fete, Janis and Newman. They have found all algebraically degenerated heavenly metrics $^{13,14}$. Finally, an original approach towards algebraically degenerated heavenly spaces have been used by Finley and Plebański $^{15}$. Consequently, they have found more compact forms of algebraically degenerated heavenly metrics.
Theorem 4.2. Let \((\mathcal{M}, ds^2)\) be a complex (neutral) Einstein space of the type \([III, N]^n \otimes [O]^n ([III_r, N_r]^n \otimes [O_r]^n)\). Then there exists a local coordinate system \((q, p, x, y)\) such that the metric takes the form

\[
\frac{1}{2} ds^2 = dq dy - dp dx + (\Phi_p x + \Omega) dp^2 + \Phi_q y dq^2
\]  

(4.12)

where \(\Phi = \Phi(q, p)\) and \(\Omega = \Omega(q, p)\) are functions such that

for the type \([III]^n \otimes [O]^n ([III_r]^n \otimes [O_r]^n) : \Phi_{pq} \neq 0, \Omega\) is arbitrary;

for the type \([N]^n \otimes [O]^n ([N_r]^n \otimes [O_r]^n) : \Phi = 0, \Omega_{qq} \neq 0\).

\(\blacksquare\)

4.3.2 Types \([III, N]^n \otimes [O]^n\) with symmetries

If the metric (4.12) admits homothetic vector then it can be always brought to the form

\[
K = \delta^1 \frac{\partial}{\partial q} + \delta^2 \frac{\partial}{\partial p} + \left(2 \chi_0 x - \frac{\partial \delta^2}{\partial p} x + \frac{\partial \delta^1}{\partial p} y + \epsilon^1\right) \frac{\partial}{\partial x} + \left(2 \chi_0 y + \frac{\partial \delta^2}{\partial q} x - \frac{\partial \delta^1}{\partial q} y + \epsilon^2\right) \frac{\partial}{\partial y}
\]  

(4.13)

where \(\delta^A = \delta^A(p, q), \epsilon^A = \epsilon^A(p, q)\). The master equation splits into system of equations

\[
\frac{\partial \delta^1}{\partial p} = a_0 e^\Phi, \quad \frac{\partial \delta^2}{\partial q} = b_0 e^{-\Phi}; \quad a_0, b_0 \text{ are constants}
\]

(4.14a)

\[
\delta^1 \Phi_q + \delta^2 \Phi_p + \frac{\partial \delta^2}{\partial p} - \frac{\partial \delta^1}{\partial q} = \text{const},
\]

(4.14b)

\[
\epsilon^2 \Phi_q = -\frac{\partial \epsilon}{\partial q};
\]

(4.14c)

\[
\delta^1 \Omega_q + \delta^2 \Omega_p - 2\Omega \left(\chi_0 - \frac{\partial \delta^2}{\partial p}\right) + \epsilon^1 \Phi_p = \frac{\partial \epsilon^1}{\partial p},
\]

(4.14d)

\[
2\Omega \frac{\partial \delta^2}{\partial q} = \frac{\partial \epsilon^1}{\partial q} - \frac{\partial \epsilon^2}{\partial p}
\]

(4.14e)

with the transformation rules

\[
\delta'^1 = \mu(q) \delta^1, \quad \delta'^2 = \nu(p) \delta^2,
\]

(4.15)

\[
\nu \epsilon^1 = \epsilon^1 - \partial_p (\delta^1 \sigma_q - \delta^2 \sigma_p + 2 \chi_0 \sigma) + 2 \sigma_{qp} \delta^1
\]

\[
\mu \epsilon^2 = \epsilon^2 + \partial_q (\delta^1 \sigma_q - \delta^2 \sigma_p - 2 \chi_0 \sigma) + 2 \sigma_{qp} \delta^2
\]

Let the vector \(K\) (4.13) be non-null, i.e., \(\delta^A \neq 0\). If \(\Phi \neq 0\) then from (4.14a)-(4.14b) it follows that there is an algebraic condition for \(\Phi\). Hence, \(\Phi\) is not arbitrary anymore. Consider now a case with \(\Phi = 0\). Then \(\delta^A\) become linear in \(p\) and \(q\) and (4.14d)-(4.14e) constitute an algebraic condition for \(\Omega\) which implies that \(\Omega\) is not arbitrary anymore.
Consequently, if \( \delta^A \neq 0 \) then Eqs. (4.14) are not satisfied for arbitrary \( \Phi \) and \( \Omega \). Hence, the metric (4.12) does not admit any non-null homothetic vector in general.

Comprehensive analysis of a symmetry algebra of the metric (4.12) is outside a scope of this text. Instead, we focus on null homothetic vectors. The paper [6] has been devoted to the issue of an existence of null homothetic vectors in Einstein spaces. Here we would like to itemize a few additional remarks which have not been noticed in [6].

The homothetic vector (4.13) is null if and only if \( \delta^\dot{A} = 0 \). Thus, any null homothetic vector admitted by the metric (4.12) takes the form

\[
K = (2\chi_0 x + \epsilon_p) \frac{\partial}{\partial x} + (2\chi_0 y + \epsilon_q) \frac{\partial}{\partial y} \tag{4.16}
\]

where \( \epsilon = \epsilon(p, q) \) is an arbitrary function which transforms as follows

\[
\epsilon' = \epsilon - 2\chi_0 \sigma + \epsilon_0 \tag{4.17}
\]

The master equation reduces to the system of two equations

\begin{align*}
\epsilon_q \Phi_q &= -\epsilon_{qq} \tag{4.18a} \\
-2\chi_0 \Omega + \epsilon_p \Phi_p &= \epsilon_{pp} \tag{4.18b}
\end{align*}

4.3.3 Type \([\text{III}]^n \otimes [\text{O}]^n\) with null symmetries

For the type \([\text{III}]^n \otimes [\text{O}]^n\) the function \( \Phi \neq 0 \). Thus, from (4.18) it follows that the metric (4.12) does not admit any null homothetic vector in general. Existence of such a symmetry puts additional constraints on the functions \( \Phi \) and \( \Omega \).

Assume first the existence of null and proper homothetic vector\textsuperscript{13}. With \( \chi_0 \neq 0 \) one can always gauge away \( \epsilon \) what implies \( \Omega = 0 \). Hence, \( K = 2\chi_0(x\partial_x + y\partial_y) \). Finally, the metric reads

\[
\frac{1}{2} ds^2 = dq dy - dp dx + \Phi_p x dp^2 + \Phi_q y dq^2 \tag{4.19}
\]

Obviously, the metric (4.19) depends on one function of two variables \( \Phi = \Phi(p, q) \).

The metric of a space of the type \([\text{III}]^n \otimes [\text{N}]^e\) equipped with null and proper homothetic vector has been already found in [6] (it is the metric (4.14) of [6]). However, there is a dependence on two functions of two variables in (4.14) of [6]. Here we proved that one of these functions is, in fact, gauge-dependent what is a serious "upgrade" of the results of [6].

Consider now a case with \( \chi_0 = 0 \). After substitution \( \Phi = \ln Q \) a null Killing vector can be brought to the form \( K = Q\partial_x + Q^{-1}\partial_y \) and the master equation yields \( Q^2 Q_q + Q_p = 0 \). A general solution of this equation reads \( q = pQ^2 + \phi(Q) \). It suggests a change of the variables \( q \rightarrow Q \). Finally, the metric depends on one function of two variables \( \Omega = \Omega(p, Q) \) and one function of one variable \( \phi = \phi(Q) \). We do not write down this metric explicitly because its equivalent form has been already written in [6] (the metric (5.28) in [6]).

\textsuperscript{13} Such a symmetry is quite rare and among Einstein spaces it is admitted only by spaces of the types \([\text{III}]^n \otimes [\text{N}]^e\), \([\text{III}]^n \otimes [\text{O}]^n\) and \([\text{N}]^e \otimes [\text{O}]^n\), see [6].
4.3.4 Type $[N]^n \otimes [O]^n$ with null symmetries

In this case $\Phi = 0$. If $\chi_0 \neq 0$ then (4.18) implies $\Omega_{qq} = 0$ and a space becomes flat. Thus, $\chi_0 = 0$. Hence, any null Killing vector admitted by the metric

$$\frac{1}{2} ds^2 = dq dy - dp dx + \Omega dp^2$$

(4.20)

has the form $K = \epsilon_0 (q \partial_x + p \partial_y) + a_0 \partial_x + b_0 \partial_y$ and it is admitted for an arbitrary $\Omega$. Therefore, the metric (4.20) is automatically equipped with three different null Killing vectors, $K_1 = \partial_x$, $K_2 = \partial_y$ and $K_3 = q \partial_x + p \partial_y$.

It is worth to mention that there is an interesting geometrical difference between vectors $K_1$ ($K_2$) and $K_3$. Any null vector has the form $K_A k^A = k^A k_A$. If $k_A$ is a null Killing vector and a space is Einstein space then both $k_A$ and $k_A$ generate a SD (ASD, respectively) $C$. The metric (4.20) is equipped with a single SD $C$ and infinitely many ASD $C$s. The SD $C$ is generated by the spinor $m^A$ (compare (3.27)), hence $k^A \sim m^A$ for both $K_1$ ($K_2$) and $K_3$. Dotted spinor $k_A$ generates the ASD $C_{kA}$ and this particular $C$ is somehow distinguished between infinitely many ASD $C$s. However, for $K_1$($K_2$) we find $C_{kA} = C^n$ while for $K_3$, $C_{kA} = C^c$.

Metrics of the type $[N]^n \otimes [O]^n$ equipped with $K_1$ ($K_2$) or $K_3$ were listed as two different metrics in [6] ((5.28) and (5.33) of [6]). Here we proved that these are, in fact, the same metrics equipped with three different null Killing vectors (although it is quite problematic to show that (5.28) of [6] can be brought to the form (5.33) of [6]).

5 Concluding remarks.

This paper is a first part of more extensive work devoted to pK and pKE-spaces. In this part we have found all pK and pKE metrics equipped with a single nonexpanding congruence of SD null strings. In our formalism these are metrics of spaces of the types $[deg]^n \otimes [D]^m$. Also, a few other interesting metrics with nonzero traceless Ricci tensor equipped with congruences of SD and ASD null strings have been constructed. The results are gathered in the Table 6 (the metrics marked by * have been already known, the rest of the metrics are new results).

The second part of our work will be devoted to the pK and pKE-spaces equipped with expanding congruence of SD null strings and algebraically degenerated SD Weyl spinor. These are spaces of the types $[deg]^n \otimes [D]^m$.

There is a third family of pKE-spaces, i.e., spaces with algebraically general SD Weyl spinor which is equivalent to the lack of existence of congruences of SD null strings. These are spaces of the types $[I] \otimes [D]^m$ or $[I] \otimes [O]^n$. Special examples of the type $[I] \otimes [O]^n$ have been found in [5] but we are still far from the full solution of such a problem. Explicit examples of pKE-spaces of the type $[I] \otimes [D]^m$ are even greater challenge. According to our best knowledge such examples are not known nowadays. It is only known that a general solution depends on two holomorphic functions of three variables each. However, a thorough analysis of the problem gives hope that an explicit example of the type $[I] \otimes [D]^m$ will be constructed. This problem is now intensively studied but the results will be presented elsewhere.
### A Appendix. Classification of spaces equipped with congruences of SD and ASD null strings.

In this Appendix we present a detailed classification of spaces equipped with at most two SD and two ASD $C$s. Such a classification can be given in terms of the properties of $C$s and $I$s without specification of a spinorial basis. In the Tables 9, 10 and 11 we call such a classification "spinorial".

A special choice of a spinorial basis allows to adapt the null tetrad to the structure of $C$s. Let the spinorial basis be chosen in such a manner that $m_A = [0, m], n_A = [n, 0], m_A = [0, \hat{m}]$ and $n_A = [\hat{n}, 0]$. With such a choice of the spinorial basis the null tetrad is adapted to the $C$s in a sense that $C_{m_A}$ is spanned by $(\partial_2, \partial_1), C_{n_A}$ is spanned by $(\partial_1, \partial_3)$, etc. Also, $I(C_{m_A}, C_{m_A}) \sim \partial_4, I(C_{n_A}, C_{n_A}) \sim \partial_2$, etc. For details see the Tables 7, 8 and the Scheme 1. In the Tables 9, 10 and 11 we call such a classification "tetradial".

It is worth to note that not all the types listed in the Tables 9, 10 and 11 are admitted by an arbitrary space. We enumerate only two restrictions:

(i) $C^n$ are admitted only by the spaces with nonzero traceless Ricci tensor

(ii) if the curvature scalar $R \neq 0$ then $C^n$ are admitted only by the spaces of the types [II] and [D]

| Type | Metric | Functions in the metric |
|------|--------|-------------------------|
| $[\deg^n \otimes [\text{any}]]$ | (2.31) * | 3 functions of 4 variables |
| $\{[\deg^n] \otimes [\text{any}]^c, [++]\}$ | (3.10) | 2 functions of 4 variables, 2 functions of 3 variables |
| $\{[\deg^n] \otimes [\text{any}]^c, [- -]\}$ | (3.13) * | 2 functions of 4 variables, 1 function of 3 variables |
| $[\deg^n] \otimes [\deg^n]$ | (3.14) * | 1 function of 4 variables, 2 functions of 3 variables |
| $\{[\deg^n] \otimes [\deg^{nc}, [- -], + +]\}$ | (3.19) | 4 functions of 3 variables |
| $\{[\deg^n] \otimes [\deg^{nc}, [+ -], [- -]]\}$ | (3.20) | 3 functions of 3 variables |
| $[\deg^n] \otimes [D]^{nn}$ | (3.21) | 2 functions of 3 variables |
| $\{[\II]^{nc} \otimes [D]^{nn}, [- - - -], + +\}$ | (3.42) | 2 functions of 2 variables |
| $\{[D]^{nn} \otimes [D]^{nn}, [- - - -], + +\}$ | (3.46) | 1 function of 2 variables |
| $[D]^{nn} \otimes [O]^{nn}$ | (3.50) | 2 functions of 2 variables |
| $[N]^n \otimes [O]^{nn}$ | (3.50) | 2 functions of 2 variables, 1 constant |

| Einstein spaces |
|----------------|
| $[\II]^{n} \otimes [D]^{nn}$ | (4.7) | 2 functions of 2 variables, 1 constant |
| $[D]^{nn} \otimes [D]^{nn}$ | (4.9) * | 1 constant |
| $[\II]^{n} \otimes [O]^{nn}$ | (4.12) * | 2 functions of 2 variables |
| $[N]^{n} \otimes [O]^{nn}$ | (4.12) * | 1 function of 2 variables |

Table 6: Summary of main results.
An interesting question arises: are there any subtypes which cannot exist at all? We are going to dig this problem deeper.

Figure 1: Congruences of null strings and congruences of null geodesics in adapted null tetrad.

| Congruence | Spanned | Spinor | Expansion | Conditions |
|------------|---------|--------|-----------|------------|
| $C_{mA}$  | $(\partial_2, \partial_4)$ | $m_A = [0, m], m \neq 0$ | $M_A = \sqrt{2} m [\Gamma_{421}, -\Gamma_{423}]$ | $\Gamma_{422} = \Gamma_{424} = 0$ |
| $C_{nA}$  | $(\partial_1, \partial_3)$ | $n_A = [n, 0], n \neq 0$ | $N_{\dot{A}} = \sqrt{2} n [\Gamma_{314}, \Gamma_{312}]$ | $\Gamma_{311} = \Gamma_{313} = 0$ |
| $C_{m\dot{A}}$ | $(\partial_1, \partial_4)$ | $\dot{m}_A = [0, \dot{m}], \dot{m} \neq 0$ | $M_{\dot{A}} = \sqrt{2} \dot{m} [\Gamma_{412}, -\Gamma_{413}]$ | $\Gamma_{411} = \Gamma_{414} = 0$ |
| $C_{n\dot{A}}$ | $(\partial_2, \partial_3)$ | $\dot{n}_{\dot{A}} = [\dot{n}, 0], \dot{n} \neq 0$ | $N_{A} = \sqrt{2} \dot{n} [\Gamma_{324}, \Gamma_{321}]$ | $\Gamma_{322} = \Gamma_{323} = 0$ |

Table 7: Four distinct congruences of null strings in adapted null tetrad.

| Intersection | Tangent vector | Expansion | Twist |
|-------------|---------------|-----------|-------|
| $I(C_{mA}, C_{m\dot{A}})$ | $\sim \partial_4$ | $\theta \sim \Gamma_{412} + \Gamma_{421}$ | $\vartheta \sim \Gamma_{412} - \Gamma_{421}$ |
| $I(C_{mA}, C_{nA})$ | $\sim \partial_2$ | $\theta \sim \Gamma_{324} + \Gamma_{423}$ | $\vartheta \sim \Gamma_{324} - \Gamma_{423}$ |
| $I(C_{nA}, C_{mA})$ | $\sim \partial_1$ | $\theta \sim \Gamma_{413} + \Gamma_{314}$ | $\vartheta \sim \Gamma_{413} - \Gamma_{314}$ |
| $I(C_{nA}, C_{nA})$ | $\sim \partial_3$ | $\theta \sim \Gamma_{321} + \Gamma_{312}$ | $\vartheta \sim \Gamma_{321} - \Gamma_{312}$ |

Table 8: Four intersections of congruences of null strings in adapted null tetrad.
| Type / Subtype | Conditions          | Spinorial | Tetradial |
|---------------|---------------------|-----------|-----------|
| Type \([\cdot]^{n} \otimes [\cdot]^{n}\) | \(M^A = M^A = 0\) | \(\Gamma_{42} = \Gamma_{41} = 0\) |          |
| \([-,-]\)     | -                   |           | -         |
| Type \([\cdot]^{n} \otimes [\cdot]^{e}\) | \(\tilde{M}^A = 0, M^A \neq 0\) | \(\Gamma_{42} = 0, \Gamma_{41} \neq 0\) |          |
| \([-,-]\)     | \(m_A M^A = 0\)    | \(\Gamma_{412} = 0\) |          |
| \([++,+]\)    | \(m_A M^A \neq 0\) | \(\Gamma_{412} \neq 0\) |          |
| Type \([\cdot]^{e} \otimes [\cdot]^{e}\) | \(\tilde{M}^A \neq 0, M^A \neq 0\) | \(\Gamma_{42} \neq 0, \Gamma_{41} \neq 0\) |          |
| \([-,-]\)     | \(m_A M^A = m_A \tilde{M}^A = 0\) | \(\Gamma_{412} = 0, \Gamma_{421} = 0\) |          |
| \([+-]\)      | \(m_A M^A = m_A \tilde{M}^A \neq 0\) | \(\Gamma_{412} = \Gamma_{421} \neq 0\) |          |
| \([-+\]\)    | \(m_A M^A = -m_A \tilde{M}^A \neq 0\) | \(\Gamma_{412} = -\Gamma_{421} \neq 0\) |          |
| \([++,+]\)    | \(m_A M^A \pm m_A \tilde{M}^A \neq 0\) | \(\Gamma_{412} \pm \Gamma_{421} \neq 0\) |          |
| **Table 9:** Types of spaces equipped with one SD and one ASD congruence of null strings. | | |

| Type / Subtype | Conditions          | Spinorial | Tetradial |
|---------------|---------------------|-----------|-----------|
| Type \([\cdot]^{n} \otimes [\cdot]^{nn}\) | \(M^A = N^A = M^A = 0\) | \(\Gamma_{42} = \Gamma_{32} = \Gamma_{41} = 0\) |          |
| \([-,-,-,-]\) | -                   |           | -         |
| Type \([\cdot]^{n} \otimes [\cdot]^{ne}\) | \(\tilde{M}^A = M^A = 0, N^A \neq 0\) | \(\Gamma_{42} = \Gamma_{41} = 0, \Gamma_{32} \neq 0\) |          |
| \([-,-,-]\)  | \(m_A N^A = 0\)    | \(\Gamma_{324} = 0\) |          |
| \([-,-,++]\) | \(m_A N^A \neq 0\) | \(\Gamma_{324} \neq 0\) |          |
| Type \([\cdot]^{e} \otimes [\cdot]^{nn}\) | \(M^A = 0, M^A \neq 0, N^A \neq 0\) | \(\Gamma_{42} = 0, \Gamma_{41} \neq 0, \Gamma_{32} \neq 0\) |          |
| \([-,-,-]\)  | \(m_A M^A = m_A N^A = 0\) | \(\Gamma_{412} = \Gamma_{324} = 0\) |          |
| \([-,-,++]\) | \(m_A M^A = 0, m_A N^A \neq 0\) | \(\Gamma_{412} = 0, \Gamma_{324} \neq 0\) |          |
| \([++,++]\)  | \(m_A M^A \neq 0, m_A N^A \neq 0\) | \(\Gamma_{412} \neq 0, \Gamma_{324} \neq 0\) |          |
| \([++,++]\)  | \(m_A M^A \neq 0, n_A M^A \neq 0\) | \(\Gamma_{421} = 0, \Gamma_{423} \neq 0\) |          |
| Type \([\cdot]^{e} \otimes [\cdot]^{ne}\) | \(M^A \neq 0, M^A = N^A = 0\) | \(\Gamma_{42} \neq 0, \Gamma_{41} = \Gamma_{32} = 0\) |          |
| \([-,-,++]\) | \(m_A M^A = 0, n_A M^A \neq 0\) | \(\Gamma_{421} = 0, \Gamma_{423} \neq 0\) |          |
| \([++,++]\)  | \(m_A M^A \neq 0, n_A M^A \neq 0\) | \(\Gamma_{421} \neq 0, \Gamma_{423} \neq 0\) |          |
| Type \([\cdot]^{e} \otimes [\cdot]^{ne}\) | \(M^A \neq 0, M^A = 0, N^A \neq 0\) | \(\Gamma_{42} \neq 0, \Gamma_{41} = 0, \Gamma_{32} \neq 0\) |          |
| \([-,-,++]\) | \(m_A M^A = 0, m_A N^A = n_A M^A \neq 0\) | \(\Gamma_{421} = 0, \Gamma_{324} = \Gamma_{423} \neq 0\) |          |
| \([-,-,+]\)  | \(m_A M^A = 0, m_A N^A \neq 0, n_A M^A \neq 0\) | \(\Gamma_{421} = 0, \Gamma_{324} = -\Gamma_{423} \neq 0\) |          |
| \([-,-,++]\) | \(m_A M^A = 0, m_A N^A = n_A M^A \neq 0\) | \(\Gamma_{421} \neq 0, \Gamma_{324} \neq \Gamma_{423} \neq 0\) |          |
| \([++,+-]\)  | \(m_A M^A \neq 0, m_A N^A \neq 0, n_A M^A = 0\) | \(\Gamma_{421} \neq 0, \Gamma_{324} = \Gamma_{423} = 0\) |          |
| \([++,++]\)  | \(m_A M^A \neq 0, m_A N^A = n_A M^A \neq 0\) | \(\Gamma_{421} \neq 0, \Gamma_{324} = \Gamma_{423} \neq 0\) |          |

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| Type / Subtype | Conditions | Type $\cdot [\cdot]_{\text{nn}}$ $\otimes [\cdot]_{\text{nn}}$ | Spinorial | Tetradial |
|---------------|------------|----------------------------------|-----------|-----------|
| [−−, −−, −−, −−] | $M^A = N^A = M^A = 0$ | $m_A M^A = m_A M^A = 0, m_A N^A = n_A M^A \neq 0$ | $\Gamma_{42} = 0, \Gamma_{32} \neq 0$ | $\Gamma_{42} = 0, \Gamma_{41} = 0, \Gamma_{32} \neq 0$ |
| [−−, ++, −−, ++] | $m_A N^A = 0, n_A N^A \neq 0$ | $m_A M^A = 0, m_A N^A = n_A M^A \neq 0$ | $\Gamma_{412} = 0, \Gamma_{324} \neq 0, \Gamma_{423} \neq 0$ | $\Gamma_{42} \neq 0, \Gamma_{41} = 0, \Gamma_{32} = 0$ |
| [−−, ++, −−, +++] | $m_A N^A = 0, n_A N^A \neq 0$ | $m_A M^A = 0, m_A N^A = n_A M^A = 0$ | $\Gamma_{412} = 0, \Gamma_{324} = -\Gamma_{423} \neq 0$ | $\Gamma_{41} = 0, \Gamma_{42} \neq 0, \Gamma_{32} \neq 0$ |
| [−−, −−, +++, +++] | $m_A M^A = 0, m_A N^A = n_A M^A \neq 0$ | $m_A M^A = 0, m_A N^A = n_A M^A = 0$ | $\Gamma_{412} = 0, \Gamma_{324} = -\Gamma_{423} \neq 0$ | $\Gamma_{42} \neq 0, \Gamma_{41} = 0, \Gamma_{32} \neq 0$ |
| [−−, −−, −−, ++] | $m_A N^A = 0, n_A N^A \neq 0$ | $m_A M^A = 0, m_A N^A \neq 0$ | $\Gamma_{412} = 0, \Gamma_{324} = -\Gamma_{423} \neq 0$ | $\Gamma_{42} \neq 0, \Gamma_{41} = 0, \Gamma_{32} \neq 0$ |
| [−−, −−, +++, −−] | $m_A N^A = 0, n_A N^A \neq 0$ | $m_A M^A = 0, m_A N^A \neq 0$ | $\Gamma_{412} = 0, \Gamma_{324} = -\Gamma_{423} \neq 0$ | $\Gamma_{42} \neq 0, \Gamma_{41} = 0, \Gamma_{32} \neq 0$ |
| [−−, −−, +++, +++] | $m_A N^A = 0, n_A N^A \neq 0$ | $m_A M^A = 0, m_A N^A = n_A M^A = 0$ | $\Gamma_{412} = 0, \Gamma_{324} = -\Gamma_{423} \neq 0$ | $\Gamma_{42} \neq 0, \Gamma_{41} = 0, \Gamma_{32} \neq 0$ |

Table 10: Types of spaces equipped with one SD and two ASD congruences of null strings.
| Type \([\cdot]^\text{nc} \otimes [\cdot]^\text{cc}\) | \(M^A = 0, N^A \neq 0, M^A \neq 0, N^A \neq 0\) | \(\Gamma_{42} = 0, \Gamma_{41} \neq 0, \Gamma_{31} \neq 0, \Gamma_{32} \neq 0\) |
|---|---|---|
| for all types below | \(m_A M^A = m_A N^A = 0\), \(\Gamma_{412} = \Gamma_{324} = 0\), | | 
| \([-,-,-,+,+] \) | \(n_A M^A = -m_A N^A \neq 0, n_A N^A = -n_A N^A \neq 0\) | \(\Gamma_{413} = -\Gamma_{314} \neq 0, \Gamma_{321} = -\Gamma_{312} \neq 0\) |
| \([-,-,-,++,+] \) | \(n_A M^A = -m_A N^A \neq 0, n_A N^A = n_A N^A \neq 0\) | \(\Gamma_{413} = -\Gamma_{314} \neq 0, \Gamma_{321} = \Gamma_{312} \neq 0\) |
| \([-,-,-,++,+] \) | \(n_A M^A = -m_A N^A \neq 0, n_A N^A \pm n_A N^A \neq 0\) | \(\Gamma_{413} = -\Gamma_{314} \neq 0, \Gamma_{321} \pm \Gamma_{312} \neq 0\) |
| \([-,-,-,++,+] \) | \(n_A M^A = m_A N^A \neq 0, n_A N^A = n_A N^A \neq 0\) | \(\Gamma_{413} = \Gamma_{314} \neq 0, \Gamma_{321} = \Gamma_{312} \neq 0\) |
| \([-,-,-,++,+] \) | \(n_A M^A \pm m_A N^A \neq 0, n_A N^A \pm n_A N^A \neq 0\) | \(\Gamma_{413} = \Gamma_{314} \neq 0, \Gamma_{321} \pm \Gamma_{312} \neq 0\) |

| for all types below | \(m_A M^A = 0, m_A N^A \neq 0\), \(\Gamma_{412} = 0, \Gamma_{324} \neq 0\), | | 
| \([-,-,+,+,+] \) | \(n_A M^A = -m_A N^A \neq 0, n_A N^A = n_A N^A \neq 0\) | \(\Gamma_{413} = -\Gamma_{314} \neq 0, \Gamma_{321} = \Gamma_{312} = 0\) |
| \([-,-,+,+,+] \) | \(n_A M^A = -m_A N^A \neq 0, n_A N^A = -n_A N^A \neq 0\) | \(\Gamma_{413} = -\Gamma_{314} \neq 0, \Gamma_{321} = -\Gamma_{312} \neq 0\) |
| \([-,-,+,+,+] \) | \(n_A M^A = -m_A N^A \neq 0, n_A N^A \pm n_A N^A \neq 0\) | \(\Gamma_{413} = -\Gamma_{314} \neq 0, \Gamma_{321} \pm \Gamma_{312} \neq 0\) |
| \([-,-,+,+,+] \) | \(n_A M^A = m_A N^A \neq 0, n_A N^A = n_A N^A \neq 0\) | \(\Gamma_{413} = -\Gamma_{314} \neq 0, \Gamma_{321} = \Gamma_{312} = 0\) |
| \([-,-,+,+,+] \) | \(n_A M^A \pm m_A N^A \neq 0, n_A N^A \pm n_A N^A \neq 0\) | \(\Gamma_{413} = -\Gamma_{314} \neq 0, \Gamma_{321} \pm \Gamma_{312} \neq 0\) |

| for all types below | \(m_A M^A \neq 0, m_A N^A \neq 0\), \(\Gamma_{412} \neq 0, \Gamma_{324} \neq 0\), | | 
| \([-,-,+,-,+] \) | \(n_A M^A = m_A N^A \neq 0, n_A N^A \neq -n_A N^A \neq 0\) | \(\Gamma_{413} = \Gamma_{314} \neq 0, \Gamma_{321} = \Gamma_{312} = 0\) |
| \([-,-,+,-,+] \) | \(n_A M^A = m_A N^A = 0, n_A N^A \neq -n_A N^A \neq 0\) | \(\Gamma_{413} = \Gamma_{314} \neq 0, \Gamma_{321} = \Gamma_{312} = 0\) |
| \([-,-,+,-,+] \) | \(n_A M^A = m_A N^A = 0, n_A N^A \neq n_A N^A \neq 0\) | \(\Gamma_{413} = \Gamma_{314} \neq 0, \Gamma_{321} = \Gamma_{312} \neq 0\) |
| \([-,-,+,-,+] \) | \(n_A M^A = -m_A N^A \neq 0, n_A N^A \pm n_A N^A \neq 0\) | \(\Gamma_{413} = \Gamma_{314} \neq 0, \Gamma_{321} = \Gamma_{312} \neq 0\) |
| \([-,-,+,-,+] \) | \(n_A M^A \neq m_A N^A \neq 0, n_A N^A \neq n_A N^A \neq 0\) | \(\Gamma_{413} = \Gamma_{314} \neq 0, \Gamma_{321} \neq \Gamma_{312} \neq 0\) |

| Type \([\cdot]^\text{cc} \otimes [\cdot]^\text{cc}\) | \(M^A \neq 0, N^A \neq 0, M^A \neq 0, N^A \neq 0\) | \(\Gamma_{42} = 0, \Gamma_{41} \neq 0, \Gamma_{31} \neq 0, \Gamma_{32} \neq 0\) |
|---|---|---|
| for all types below | \(m_A M^A = m_A M^A = m_A N^A = n_A N^A = 0\), \(\Gamma_{412} = \Gamma_{324} = 0\), | | 
| \([-,-,-,+-,+] \) | \(m_A N^A = -n_A M^A \neq 0, n_A M^A = n_A M^A \neq 0\) | \(\Gamma_{322} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0\) |
| \([-,-,-,+-,+] \) | \(m_A N^A = -n_A M^A \neq 0, n_A M^A = m_A N^A \neq 0\) | \(\Gamma_{322} = -\Gamma_{423} \neq 0, \Gamma_{413} = \Gamma_{314} \neq 0\) |
| \([-,-,-,+-,+] \) | \(m_A N^A = -n_A M^A \neq 0, n_A M^A \pm m_A N^A \neq 0\) | \(\Gamma_{322} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq \Gamma_{314} \neq 0\) |
| \([-,-,-,+-,+] \) | \(m_A N^A = n_A M^A \neq 0, n_A M^A = m_A N^A \neq 0\) | \(\Gamma_{322} = \Gamma_{423} \neq 0, \Gamma_{413} = \Gamma_{314} \neq 0\) |
| Case | Equation | Condition | Solution for \( \Gamma \) |
|------|----------|-----------|------------------------|
| \([-+, ++, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A \pm m_A N^A \neq 0 \) | \( \Gamma_{324} = \Gamma_{423} \neq 0, \Gamma_{413} \pm \Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A \pm n_A M^A \neq 0, n_A M^A \neq m_A N^A \neq 0 \) | \( \Gamma_{324} \pm \Gamma_{423} \neq 0, \Gamma_{413} \pm \Gamma_{314} \neq 0 \) |
| for all types below | \( m_A M^A = m_A M^A = 0, n_A N^A = -n_A N^A \neq 0 \) | \( \Gamma_{412} = \Gamma_{421} = 0, \Gamma_{321} = -\Gamma_{312} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = -n_A M^A \neq 0, n_A M^A = -m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = -n_A M^A \neq 0, n_A M^A = m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = -n_A M^A \neq 0, n_A M^A \pm m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A = m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A \pm m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A = n_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A M^A = m_A M^A = 0, n_A N^A = n_A N^A \neq 0 \) | \( \Gamma_{412} = \Gamma_{421} = 0, \Gamma_{321} = -\Gamma_{312} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A = -m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = -n_A M^A \neq 0, n_A M^A = n_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A \pm n_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A \neq n_A M^A \neq 0, n_A M^A = m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A \neq m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A \neq n_A M^A \neq 0, n_A M^A \pm m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A \neq n_A M^A \neq 0, n_A M^A = n_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} \neq -\Gamma_{314} \neq 0 \) |
| for all types below | \( m_A M^A = m_A M^A = 0, n_A N^A = n_A N^A \neq 0 \) | \( \Gamma_{412} = \Gamma_{421} = 0, \Gamma_{321} = -\Gamma_{312} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A = -m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A = m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A \neq m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A \neq n_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A M^A = n_A M^A \neq 0, n_A N^A = -n_A N^A \neq 0 \) | \( \Gamma_{412} = -\Gamma_{421} \neq 0, \Gamma_{321} = -\Gamma_{312} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A = -m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A N^A = n_A M^A \neq 0, n_A M^A = m_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| \([-+, +, +, +]\) | \( m_A M^A \neq n_A M^A \neq 0, n_A N^A = n_A N^A \neq 0 \) | \( \Gamma_{324} = -\Gamma_{423} \neq 0, \Gamma_{413} = -\Gamma_{314} \neq 0 \) |
| [-+, +++, ++] | $m_A N^A = -n_A M^A \neq 0$, $n_A M^A \pm m_A N^A \neq 0$ | $\Gamma_{324} = -\Gamma_{423} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |
| [-++, +++, ++] | $m_A N^A = n_A M^A \neq 0$, $n_A M^A \pm m_A N^A \neq 0$ | $\Gamma_{324} = -\Gamma_{423} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |
| [--, +++, ++] | $m_A N^A = n_A M^A \neq 0$, $n_A M^A \pm m_A N^A \neq 0$ | $\Gamma_{324} = -\Gamma_{423} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |
| [++, +++, ++] | $m_A N^A \pm n_A M^A \neq 0$, $n_A M^A \pm m_A N^A \neq 0$ | $\Gamma_{324} \pm \Gamma_{423} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |

for all types below $m_A M^A = m_A M^A \neq 0$, $n_A N^A = n_A N^A \neq 0$, $\Gamma_{412} = \Gamma_{421} \neq 0$, $\Gamma_{432} \pm \Gamma_{312} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |

| [+--, +++, ++] | $m_A N^A = n_A M^A \neq 0$, $n_A M^A \pm m_A N^A \neq 0$ | $\Gamma_{324} = -\Gamma_{423} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |
| [+--, +++, ++] | $m_A N^A = n_A M^A \neq 0$, $n_A M^A \pm m_A N^A \neq 0$ | $\Gamma_{324} = -\Gamma_{423} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |
| [++, +++, ++] | $m_A N^A \pm n_A M^A \neq 0$, $n_A M^A \pm m_A N^A \neq 0$ | $\Gamma_{324} \pm \Gamma_{423} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |

for all types below $m_A M^A = m_A M^A \neq 0$, $n_A N^A = n_A N^A \neq 0$, $\Gamma_{412} = \Gamma_{421} \neq 0$, $\Gamma_{432} \pm \Gamma_{312} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |

| [++, +++, ++] | $m_A N^A \pm n_A M^A \neq 0$, $n_A M^A \pm m_A N^A \neq 0$ | $\Gamma_{324} \pm \Gamma_{423} \neq 0$, $\Gamma_{413} \pm \Gamma_{314} \neq 0$ |

Table 11: Types of spaces equipped with two SD and two ASD congruences of null strings.

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