On K-K-W Type Theorems for Conformal Perturbations of Twisted Dirac Operators

Jian Wang\textsuperscript{a}, Yong Wang\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a}School of Science, Tianjin University of Technology and Education, Tianjin, 300222, P.R.China
\textsuperscript{b}School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R.China

Abstract

In this paper, we prove two Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and conformal perturbations of twisted signature operators on four-dimensional manifolds with (resp. without) boundary.

Keywords: Conformal perturbations of twisted Dirac operators; noncommutative residue; non-unitary connection.

1. Introduction

The noncommutative residue found in \cite{1, 2} plays a prominent role in noncommutative geometry. In \cite{3}, Connes used the noncommutative residue to derive a conformal four-dimensional Polyakov action analogy. In \cite{4}, Connes proved that the noncommutative residue on a compact manifold $M$ coincided with Dixmier’s trace on pseudodifferential operators of order $-\dim M$. Several years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which is called the Kastler-Kalau-Walze theorem now. Kastler\textsuperscript{5} gave a brute-force proof of this theorem. Kalau and Walze\textsuperscript{6} proved this theorem in the normal coordinates system simultaneously. Ackermann\textsuperscript{7} gave a note on a new proof of this theorem by means of the heat kernel expansion. Moreover, Fedosov etc.\textsuperscript{8} constructed a noncommutative residue on the algebra of classical elements in Boutet de Monvel’s calculus on a compact manifold with boundary of dimension $n > 2$. For Dirac operators and signature operators on manifolds with boundary, Wang\textsuperscript{9} gave an operator-theoretic explanation of the gravitational action for manifolds with boundary and proved a Kastler-Kalau-Walze type theorem.

In \cite{10}, Bismut and Zhang introduced the de-Rham Hodge operator twisted by a flat vector bundle with a non-metric connection, and extended the famous Cheeger-Müller theorem to the non-unitary case. In \cite{11}, Zhang considered the sub-signature operators twisted by a non-unitary flat vector bundle and proved the associated Riemann-Roch theorem. In \cite{12}, we proved the Lichnerowicz formula for Dirac operators and signature operators twisted by a vector bundle with a non-unitary connection and got two Kastler-Kalau-Walze type theorems for twisted Dirac operators and twisted signature operators on four-dimensional manifolds with boundary. It is important that Wang proved a Kastler-Kalau-Walze type theorem for perturbations of Dirac operators on compact manifolds with or without boundary in \cite{13}. The motivation of this paper is to establish two Kastler-Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and perturbations of twisted signature operators.

This paper is organized as follows. In Section 2, we prove a Kastler-Kalau-Walze type theorem for conformal perturbations of twisted Dirac operators on 4-dimensional compact manifolds with or without boundary. In Section 3, we prove a Kastler-Kalau-Walze type theorem for conformal perturbations of twisted signature operators on 4-dimensional compact manifolds with or without boundary.

\textsuperscript{*}Corresponding author.

Email addresses: wangj484@nenu.edu.cn (Jian Wang), wangy581@nenu.edu.cn (Yong Wang)
2. A Kastler-Kalau-Walze Type Theorem for Conformal Perturbations of twisted Dirac Operators

2.1. Boutet de Monvel’s calculus and noncommutative residue

In this section, we recall some basic facts and formulae about Boutet de Monvel’s calculus as follows. Let

\[ F : L^2(\mathbb{R}) \to L^2(\mathbb{R}); \quad F(u)(v) = \int e^{-itv}u(t)dt \]

denote the Fourier transformation and \( \varphi(\mathbb{R}^+) = r^+\varphi(\mathbb{R}) \) (similarly define \( \varphi(\mathbb{R}^-) \)), where \( \varphi(\mathbb{R}) \) denotes the Schwartz space and

\[ r^+ : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^+); \quad f \to f(\mathbb{R}^+); \quad \mathbb{R}^+ = \{ x \geq 0; x \in \mathbb{R} \}. \]  

(2.1)

We define \( H^+ = F(\varphi(\mathbb{R}^+)); \quad H^- = F(\varphi(\mathbb{R}^-)) \) which are orthogonal to each other. We have the following property: \( h \in H^+ (H^-) \) if \( h \in C^\infty(\mathbb{R}) \) which has an analytic extension to the lower (upper) complex half-plane \( \{ \text{Im} \xi < 0 \} (\{ \text{Im} \xi > 0 \}) \) such that for all nonnegative integer \( l \),

\[ \frac{d^l h}{d\xi^l}(\xi) = \sum_{k=1}^\infty \frac{d^l f_k}{d\xi^l}(\xi) \]  

(2.2)

as \( |\xi| \to +\infty, \text{Im} \xi \leq 0 \) (\( \text{Im} \xi \geq 0 \)).

Let \( H' \) be the space of all polynomials and \( H^- = H_0^- \bigoplus H^+; \quad H = H^+ \bigoplus H^- \). Denote by \( \pi^+ \) (\( \pi^- \)) respectively the projection on \( H^+ \) (\( H^- \)). For calculations, we take \( H = \tilde{H} = \{ \text{rational functions having no poles on the real axis} \} \ (\tilde{H} \text{ is a dense set in the topology of } H) \). Then on \( \tilde{H} \),

\[ \pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \to 0} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \]  

(2.3)

where \( \Gamma^+ \) is a Jordan close curve included \( \text{Im} \xi > 0 \) surrounding all the singularities of \( h(\xi) \) in the upper half-plane and \( \xi_0 \in \mathbb{R} \). Similarly, define \( \pi^- \) on \( \tilde{H} \),

\[ \pi^- h = \frac{1}{2\pi} \int_{\Gamma^-} h(\xi) d\xi, \]  

(2.4)

So, \( \pi^-(H^-) = 0 \). For \( h \in H \cap L^1(\mathbb{R}), \pi^+ h = \frac{1}{2\pi} \int R h(v) dv \) and for \( h \in H^+ \cap L^1(\mathbb{R}), \pi^- h = 0 \). Denote by \( B \) Boutet de Monvel’s algebra (for details, see Section 2 of [14]).

An operator of order \( m \in \mathbb{Z} \) and type \( d \) is a matrix

\[ A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} : C^\infty(\mathbb{R}, E_1) \bigoplus C^\infty(\partial X, F_1) \to C^\infty(\mathbb{R}, E_2) \bigoplus C^\infty(\partial X, F_2). \]

where \( X \) is a manifold with boundary \( \partial X \) and \( E_1, E_2 (F_1, F_2) \) are vector bundles over \( X \) (\( \partial X \)). Here, \( P : C^\infty_0(\Omega, \overline{E_1}) \to C^\infty(\Omega, \overline{E_2}) \) is a classical pseudodifferential operator of order \( m \) on \( \Omega \), where \( \Omega \) is an open neighborhood of \( X \) and \( \overline{E_i}[X = E_i \ (i = 1, 2)] \). \( P \) has an extension: \( E'(\Omega, \overline{E_1}) \to D'(\Omega, \overline{E_2}) \), where \( E'(\Omega, \overline{E_1}) \) (\( D'(\Omega, \overline{E_2}) \)) is the dual space of \( C^\infty(\Omega, \overline{E_1}) \) (\( C^\infty(\Omega, \overline{E_2}) \)). Let \( e^+ : C^\infty(\mathbb{R}, E_1) \to E'(\Omega, \overline{E_1}) \) denote extension by zero from \( X \) to \( \Omega \) and \( r^+ : D'(\Omega, \overline{E_2}) \to D'(\Omega, \overline{E_2}) \) denote the restriction from \( \Omega \) to \( X \), then define

\[ \pi^+ P = r^+ P e^+ : C^\infty(\mathbb{R}, E_1) \to D'(\Omega, E_2). \]

In addition, \( P \) is supposed to have the transmission property; this means that, for all \( j, k, \alpha \), the homogeneous component \( p_j \) of order \( j \) in the asymptotic expansion of the symbol \( p \) of \( P \) in local coordinates near the boundary satisfies:

\[ \partial_{x_n}^k \partial_{\xi^i}^k p_j(x', 0, 0, +1) = (-1)^{j-\alpha} \partial_{x_n}^k \partial_{\xi^i}^k p_j(x', 0, 0, -1), \]
then \( \pi^+ P : C^\infty(X, E_1) \to C^\infty(X, E_2) \) by Section 2.1 of [14].

In the following, write \( \pi^+ D^{-1} = \begin{pmatrix} \pi^+ D^{-1} & 0 \\ 0 & 0 \end{pmatrix} \), we will compute \( \widetilde{\mathrm{Wres}}[\pi^+ (\tilde{D}_F)^{-1} \circ \pi^+ \tilde{D}_F^{-1}] \). Let \( M \) be a compact manifold with boundary \( \partial M \). We assume that the metric \( g^M \) on \( M \) has the following form near the boundary

\[
g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,
\]

(2.5)

where \( g^{\partial M} \) is the metric on \( \partial M \). Let \( U \subseteq M \) be a collar neighborhood of \( \partial M \) which is diffeomorphic \( \partial M \times [0, 1) \). By the definition of \( h(x_n) \in C^\infty([0, 1)) \) and \( h(x_n) > 0 \), there exists \( \tilde{h} \in C^\infty((-\varepsilon, 1)) \) such that \( \tilde{h}|_{(0,1)} = h \) and \( \tilde{h} > 0 \) for some sufficiently small \( \varepsilon > 0 \). Then there exists a metric \( \tilde{g} \) on \( \tilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0] \) which has the form on \( U \cup_{\partial M} \partial M \times (-\varepsilon, 0] \)

\[
\tilde{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + dx_n^2,
\]

(2.6)

such that \( \tilde{g}|_M = g \). We fix a metric \( \tilde{g} \) on the \( \tilde{M} \) such that \( \tilde{g}|_M = g \). Note \( \tilde{D}_F \) is the twisted Dirac operator on the spinor bundle \( S(TM) \otimes F \) corresponding to the connection \( \tilde{\nabla} \).

Now we recall the main theorem in [8].

**Theorem 2.1. (Fedosov-Golse-Leichtnam-Schrohe)** Let \( X \) and \( \partial X \) be connected, \( \dim X = n \geq 3 \), \( A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B} \), and denote by \( p, b \) and \( s \) the local symbols of \( P, G \) and \( S \) respectively. Define:

\[
\widetilde{\mathrm{Wres}}(A) = \int_X \int_S \mathrm{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx \\
+ 2\pi \int_{\partial X} \int_S \{ \mathrm{tr}_E [(\mathrm{tr} b_{-\nu})(x', \xi')] + \mathrm{tr}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx',
\]

(2.7)

Then

a) \( \widetilde{\mathrm{Wres}}([A, B]) = 0 \), for any \( A, B \in \mathcal{B} \); b) It is a unique continuous trace on \( \mathcal{B}/\mathcal{B}^{-\infty} \).

2.2. A Kastler-Kalau-Walze Type Theorem for Conformal Perturbations of twisted Dirac Operators

In this section, we shall prove a Kastler-Kalau-Walze type formula for conformal perturbations of twisted Dirac Operators on four-dimensional compact manifolds with boundary. Let \( S(TM) \) be the spinors bundle and \( F \) be an additional smooth vector bundle equipped with a non-unitary connection \( \tilde{\nabla}^F \). Let \( \tilde{\nabla}^F,^* \) be the dual connection on \( F \), and define

\[
\nabla^F = \frac{\tilde{\nabla}^F + \tilde{\nabla}^F,^*}{2}, \quad \Phi = \frac{\tilde{\nabla}^F - \tilde{\nabla}^F,^*}{2},
\]

(2.8)

then \( \nabla^F \) is a metric connection and \( \Phi \) is an endomorphism of \( F \) with a 1-form coefficient. We consider the tensor product vector bundle \( S(TM) \otimes F \), which becomes a Clifford module via the definition:

\[
c(a) = c(a) \otimes \text{id}_F, \quad a \in TM,
\]

(2.9)

and which we equip with the compound connection:

\[
\nabla^{S(TM) \otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \tilde{\nabla}^F.
\]

(2.10)

The corresponding twisted Dirac operator \( \tilde{D}_F \) is locally specified as follows:

\[
\tilde{D}_F = \sum_{i=1}^n c(e_i) \tilde{\nabla}_{e_i}^{S(TM) \otimes F}.
\]

(2.11)
Let
\[ \nabla^{S(TM)\otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla^F, \]  
then the spinor connection \( \tilde{\nabla} \) induced by \( \nabla^{S(TM)\otimes F} \) is locally given by
\[ \tilde{\nabla}^{S(TM)\otimes F} = \nabla^{S(TM)} \otimes \text{id}_F + \text{id}_{S(TM)} \otimes \nabla^F + \text{id}_{S(TM)} \otimes \Phi. \]

**Definition 2.2.** Let \( \{e_i\}(1 \leq i, j \leq n) \) be the orthonormal frames (natural frames respectively) on \( TM \), then the form of Dirac operators as following
\[ D_F = \sum_{i,j} g^{ij} \partial_i \nabla^{S(TM)\otimes F} = \sum_{i,j} c(e_i) \nabla^{S(TM)\otimes F}, \]
where \( \nabla^{S(TM)\otimes F}_\partial_j = \partial_j + \sigma_j^a + \sigma_j^e \) and \( \sigma_j^a = \frac{1}{4} \sum_{k,l} (\nabla^{TM}_e c_{ij} c(e_j)c(e_k), \sigma_j^e \) is the connection matrix of \( \nabla^F \).

**Definition 2.3.** For sections \( \psi \otimes \chi \in S(TM) \otimes F \), then the twisted Dirac operators \( \tilde{D}_F, \tilde{D}_F^* \) associated to the connection \( \tilde{\nabla} \) as follows
\[ \tilde{D}_F(\psi \otimes \chi) = D_F(\psi \otimes \chi) + \sum_{i=1}^n c(e_i) \otimes \Phi(e_i)(\psi \otimes \chi), \]
\[ \tilde{D}_F^*(\psi \otimes \chi) = D_F(\psi \otimes \chi) - \sum_{i=1}^n c(e_i) \otimes \Phi^*(e_i)(\psi \otimes \chi). \]

Here \( \Phi^*(e_i) \) denotes the adjoint of \( \Phi(e_i) \).

In the following, we will compute the more general case \( \tilde{\text{Wres}}[\pi^+ (f \tilde{D}_F^{-1}) \circ \pi^+ (f^{-1} (\tilde{D}_F)^{-1})] \) for nonzero smooth functions \( f, f^{-1} \). Denote by \( \sigma_j(A) \) the \( l \)-order symbol of an operator \( A \). An application of (3.5) and (3.6) in [14] shows that
\[ \tilde{\text{Wres}}[\pi^+ (f \tilde{D}_F^{-1}) \circ \pi^+ (f^{-1} (\tilde{D}_F)^{-1})] = \text{Wres}[f \tilde{D}_F^{-1} \circ f^{-1} (\tilde{D}_F)^{-1}] + \int_{\partial M} \Psi, \]
where
\[ \Psi = \int_{[\xi']} \int_{-\infty}^{\infty} \sum_{j,k=0}^{\infty} \sum_{a=0}^{\infty} \frac{(i)^{a+j+k+\ell}}{a!} \text{trace}_{S(TM)\otimes F} \left[ \partial_x^j \partial_{\xi'}^k \partial_{\xi_n}^a \sigma_{\eta_r} (f \tilde{D}_F^{-1})(x', 0, \xi', \xi_n) \times \partial_x^j \partial_{\xi'}^k \partial_{\xi_n}^a \sigma_l (f^{-1} (\tilde{D}_F)^{-1})(x', 0, \xi', \xi_n) \right] \text{d}\xi_n \sigma(\xi') \text{d}x', \]
and the sum is taken over \( r - k + |a| + \ell + j - 1 = -n, r \leq -1, \ell \leq -1 \).

Note that
\[ \text{Wres}[f \tilde{D}_F^{-1} f^{-1} (\tilde{D}_F)^{-1}] = \text{Wres}[(\tilde{D}_F f \tilde{D}_F f^{-1})^{-1}] = \text{Wres}[(\tilde{D}_F^* \tilde{D}_F - \tilde{D}_F^* c(df)f^{-1})^{-1}]. \]

In order to calculate the symbol of operators \( \tilde{D}_F \tilde{D}_F - \tilde{D}_F^* c(df)f^{-1} \), we recall the basic notions of Laplace type operators in Section 1 of [15]. Let \( V \) be a bundle vector over \( M \). Any differential operator \( P \) of Laplace type has locally the form
\[ P = -(g^{ij} \partial_i \partial_j + A^i \partial_i + B), \]
where \( \partial_i \) is a natural local frame on \( TM \), and \( (g^{ij})_{1 \leq i, j \leq n} \) is the inverse matrix associated with the metric matrix \( (g_{ij})_{1 \leq i, j \leq n} \) on \( M \), and \( A^i \) and \( B \) are smooth sections of \( \text{End}(V) \) on \( M \) (Endomorphism). If \( P \)
is a Laplace type operator of the form (2.20), then there is a unique connection \( \nabla \) on \( V \) and a unique Endomorphism \( E \) such that

\[
P = -\left[ g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla_{\partial_i} \partial_j}) + E \right],
\]

where \( \nabla^L \) denotes the Levi-Civita connection on \( M \). Moreover (with local frames of \( T^*M \) and \( V \)), \( \nabla_{\partial_i} = \partial_i + \omega_i \) and \( E \) are related to \( g^{ij}, A^i \), and \( B \) through

\[
\omega_i = \frac{1}{2} g_{ij} \left( A^i + g^{kl} \Gamma_{kl}^i \text{Id} \right),
\]

\[
E = B - g^{ij} \left( \partial_i (\omega_i) + \omega_i \omega_j - \omega_k \Gamma_{ij}^k \right),
\]

where \( \Gamma_{ij}^k \) is the Christoffel coefficient of \( \nabla^L \).

The next task then is to prove \( \tilde{D}_F \tilde{D}_F \) has the Laplace type form. Let \( \partial^i = g^{ij} \partial_j, \sigma^i = g^{ij} \sigma_j, \Gamma^k = g^{ij} \Gamma_{ij}^k \). From (6a) in \( \mathbb{E} \), we have

\[
\tilde{D}_F^* \tilde{D}_F = D_F^2 - c(\Phi^*) D_F + D_F c(\Phi) - c(\Phi^*) c(\Phi),
\]

and

\[
-c(\Phi^*) D_F + D_F c(\Phi) = -\sum_j c(\Phi^*) c(e_j) \left[ e_j + \sigma_j^{S(TM) \otimes F} \right] + \sum_j c(e_j) \otimes c(\Phi) e_j
\]

\[
+ \sum_j c(e_j) \otimes e_j (c(\Phi)) + \sum_j \left[ c(e_j) \sigma_j^{S(TM) \otimes F} c(\Phi) + c(e_j) \otimes \sigma_j^{F} c(\Phi) \right].
\]

Combining (2.24)-(2.25), we obtain the specification of \( \tilde{D}_F^* \tilde{D}_F - \tilde{D}_F^* c(df)f^{-1} \).

In terms of local coordinates \( \{ \partial_i \} \) inducing the coordinate transformation \( e_j = \sum_{k=1}^n (e_j, dx^k) \partial_k \), let \( \Gamma^k = g^{ij} \Gamma_{ij}^k \), then

\[
\omega_j = \sigma_j^{S(TM) \otimes F} + \sigma_j^F + g^{ij} \partial_i c(df) f^{-1} + \frac{1}{2} \sum_{i,j} \left[ \langle e_k, dx^i \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^i \rangle c(e_k) c(\Phi) + \Gamma^i \right].
\]

For a smooth vector field \( X \in \Gamma(M, TM) \), let \( c(X) \) denote the Clifford action. By direct computation in normal coordinates, we obtain

\[
\tilde{\nabla}_X = \nabla_X^{S(TM) \otimes F} + \frac{1}{2} \left[ c(\Phi^*) c(X) - c(X) c(\Phi) \right] + c(X) c(df)f^{-1}.
\]
We now compute $E$. Regrouping the terms and inserting (2.26), (2.27) into (2.23), we obtain

$$E = g^{ij} \left[ \partial_i (\sigma^j_S(T) \otimes \Phi) + \sigma^j_S(T) \otimes \Phi \sigma^i_S(T) \Phi - \Gamma^k_{ij} \sigma^k_S(T) \Phi \right] + \sum_j \left[ c(\Phi^*) c(e_j) \right] \sigma^j_S(T) \Phi$$

$$- \sum_j \left[ c(e_j) \otimes e_j (\Phi) - \sum_j \left[ c(e_j) \sigma^j_S(TM) \otimes c(\Phi) + c(e_j) \otimes \sigma^j_S(TM) \right] \right]$$

$$+ c(\Phi^*) c(e_j) - \frac{1}{4} s - \frac{1}{2} \sum_{j \neq j} \Gamma_{ij} \sigma^k_S(TM) \Phi$$

$$- \frac{1}{2} \partial_j (\sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi))$$

$$- \frac{1}{2} g^{ij} \left( \sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi) \right)$$

$$- \frac{1}{4} g^{ij} \left( \sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi) \right)$$

$$\times \left( \sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi) \right)$$

$$+ \left[ \sigma^j_S(TM) \Phi + \frac{1}{2} \sum_{l=1}^n \langle e_l, dx^k \rangle c(\Phi^*) c(e_l) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_l) c(\Phi) \right] \Gamma^k_{ij}$$

$$+ \frac{1}{2} g^{ij} \left( \sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi) \right)$$

$$- \frac{1}{4} g^{ij} \left( \sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi) \right)$$

$$- \frac{1}{4} g^{ij} \left( \sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi) \right)$$

$$\times \left( \sum_{k=1}^n \langle e_k, dx^j \rangle c(\Phi^*) c(e_k) - \sum_{k=1}^n \langle e_k, dx^j \rangle c(e_k) c(\Phi) \right)$$

$$+ c(\partial_i) \sigma^j_S(TM) \Phi c(df) f^{-1} + c(\partial_i) \frac{\partial c(df) f^{-1}}{\partial x_j} - c(\Phi^*) c(df) f^{-1}$$

$$- g^{ij} \left( \frac{1}{2} c(\partial_i) c(df) f^{-1} - \frac{1}{4} g^{ij} c(\partial_i) c(df) f^{-1} c(\partial_i) c(df) f^{-1} + \frac{1}{2} g^{ij} c(\partial_k) c(df) f^{-1} \Gamma^k_{ij} \right)$$

$$- g^{ij} \sigma^j_S(TM) \Phi c(\partial_i) c(df) f^{-1} - \left[ c(\Phi^*) c(e_i) - c(e_i) c(\Phi) \right] c(\partial_i) c(df) f^{-1}. \quad (2.29)$$

Since $E$ is globally defined on $M$, so we can perform computations of $E$ in normal coordinates. In terms of normal coordinates about $x_0$ one has: $\sigma^j_S(TM) (x_0) = 0$ $c(e_i)) (x_0) = 0$, $\Gamma^k_{ij} (x_0) = 0$, we conclude that

$$E(x_0) = - \frac{1}{4} - \frac{1}{2} \sum_{j \neq j} \Gamma_{ij} \sigma^k_S(TM) \Phi$$

$$- \frac{1}{2} \sum_j \left[ \nabla^j c(\Phi^*) c(e_j) - \frac{1}{2} \sum_j \langle e_j, \nabla^j c(\Phi^*) \rangle \right]$$

$$+ c(\partial_i) \frac{\partial c(df) f^{-1}}{\partial x_i} - c(\Phi^*) c(df) f^{-1}$$

$$- \frac{1}{4} g^{ij} c(\partial_i) c(df) f^{-1} c(\partial_i) c(df) f^{-1} \Gamma^k_{ij}$$

$$- c(\Phi^*) c(e_i) - c(e_i) c(\Phi) \right] c(\partial_i) c(df) f^{-1}. \quad (2.30)$$

From Theorem 1 in [3] and Theorem 1 in [4], for $M$ a compact $n$ dimensional ($n \geq 4$, even) Riemannian manifold and $\tilde{D}_F \tilde{D}_F$ a generalized Laplacian acting on sections of vector bundle on $M$, the following relation holds:

$$Wres[(\tilde{D}_F \tilde{D}_F - \tilde{D}_F \tilde{D}_F c(df) f^{-1})^{-1}] \left( x_0 \right) = \left( \frac{2\pi}{2\pi - 2} \right)! \int_M \text{Tr} \left( \frac{S}{6} + E \right) d\text{vol}_M, \quad (2.31)$$
where Wres denotes the noncommutative residue.

**Lemma 2.4.** The following identity holds

\[
\text{Tr}[c(\Phi^*)c(df)] = -\text{Tr}_F[\Phi^*(\text{grad}_M f)]\text{Tr}[id];
\]

\[
\text{Tr}[c(\partial_i) \frac{\partial(c(df) f^{-1})}{\partial x_i}](x_0) = [-f^{-1}\Delta(f) - \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle](x_0)\text{Tr}[id];
\]

\[
\text{Tr}[\partial_i(c(\partial_i)c(df) f^{-1})](x_0) = [-f^{-1}\Delta(f)(x_0) - \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle](x_0)\text{Tr}[id];
\]

\[
\text{Tr}[\text{grad}_i\text{grad}_j(\Phi)] = f^{-2}[[\text{grad}_M(f)]^2 + 2\Delta(f)](x_0)\text{Tr}[id];
\]

\[
\text{Tr}[c(\Phi^*)c(e_i)c(\partial_i)c(df) f^{-1}](x_0) = -f^{-1}\text{Tr}_F[\Phi^*(\text{grad}_M f)](x_0)\text{Tr}[id].
\]

\[
\text{Tr}[c(e_i)c(\Phi)c(\partial_i)c(df) f^{-1}](x_0) = f^{-1}\text{Tr}_F[\Phi(\text{grad}_M f)](x_0)\text{Tr}[id].
\]

**Proof.** By the relation of the Clifford action and \( \text{tr}AB = \text{tr}BA \),

\[
\text{Tr}[c(\xi^*)c(dx_\alpha)] = 0; \quad \text{Tr}[c(dx_\alpha)^2] = -4; \quad \text{Tr}[c(\xi^*)^2](x_0)|_{\xi^* = 1} = -4;
\]

\[
\text{Tr}[\partial_{x_\alpha}c(\xi^*)c(dx_\alpha)] = 0; \quad \text{Tr}[\partial_{x_\alpha}c(\xi^*)c(dx_\alpha)](x_0)|_{\xi^* = 1} = -2\hbar'(0).
\]

Let \( c(\partial_i) = \sum_{j=1}^4 \partial_i \tilde{e}_k c(\tilde{e}_k) \) and \( c(\Phi^*) = \sum_{j=1}^4 e_j \otimes \Phi^*(e_j) \), then

\[
\text{Tr}[c(\Phi^*)c(df)](x_0) = \text{Tr}[\sum_{j=1}^4 (c(e_j) \otimes \Phi^*(e_j)) c(df)](x_0) = \sum_{j=1}^4 \text{Tr}[c(e_j)c(df)]\text{Tr}[\Phi^*(e_j)](x_0)
\]

\[
= -\sum_{j=1}^4 g(e_j, \text{grad}_M f)\text{Tr}[id]\text{Tr}[\Phi^*(e_j)](x_0) = -\sum_{j=1}^4 e_j(f)\text{Tr}[id]\text{Tr}[\Phi^*(e_j)](x_0)
\]

\[
= -\text{Tr}[id]\text{Tr}_F[\Phi^*(\sum_{j=1}^4 e_j(f)e_j)](x_0) = -\text{Tr}_F[\Phi^*(\text{grad}_M f)](x_0)\text{Tr}[id]
\]

\[
= -\text{Tr}_F[\Phi^*(\text{grad}_M f)](x_0)\text{Tr}[id],
\]

7
and
\[
\text{Tr} \left[ \partial_t (c(\partial_t c(df)f^{-1})) \right](x_0) = \\
\text{Tr} \left[ \partial_t (c(\partial_t c(df)f^{-1})) \right](x_0) + \text{Tr} \left[ c(\partial_t \partial_t c(df))f^{-1} \right](x_0) + \text{Tr} \left[ c(\partial_t c(df))\partial_t (f^{-1}) \right](x_0)
\]
\[
= \text{Tr} \left[ \partial_t (\sum_{k=1}^{4} (\partial_t \tilde{c}_k c(\tilde{c}_k))c(df)f^{-1}) \right](x_0) + f^{-1} \text{Tr} \left[ c(\partial_t \partial_t (c(df))) \right](x_0)
\]
\[
- g_{TM}(\partial_t, grad_M f) \partial_t (f^{-1})(x_0) \text{Tr}[id]
\]
\[
= \sum_{k=1}^{4} \text{Tr} \left[ \partial_t \left( g_{TM}(\partial_t, \tilde{c}_k) \right) \right] c(\tilde{c}_k)(c(df)f^{-1})(x_0) + f^{-1} \sum_{j=1}^{4} \frac{\partial (\tilde{c}_j(f))}{\partial x_i} c(\tilde{c}_j)(x_0)
\]
\[
= \sum_{k=1}^{4} \text{Tr} \left[ \partial_t \sum_{j=1}^{4} \frac{\partial (\tilde{c}_j(f))}{\partial x_i} c(\tilde{c}_j)(x_0) \right] + f^{-1} \sum_{j=1}^{4} \frac{\partial (\tilde{c}_j(f))}{\partial x_i} c(\tilde{c}_j)(x_0)
\]
\[
- (\text{grad}_M f, \text{grad}_M f^{-1})(x_0) \text{Tr}[id]
\]
\[
= - \sum_{k=1}^{4} \text{Tr} \left[ \partial_t \left( g_{TM}(\partial_t, \tilde{c}_k) \right) \right] c(\tilde{c}_k)(f)(x_0) \text{Tr}[id] + f^{-1} \sum_{j=1}^{4} \frac{\partial (\tilde{c}_j(f))}{\partial x_i} c(\tilde{c}_j)(x_0) \text{Tr}[id]
\]
\[
= - \sum_{k=1}^{4} \text{Tr} \left[ \partial_t \left( g_{TM}(\partial_t, \tilde{c}_k) \right) c(\tilde{c}_k)(f)(x_0) \right] + f^{-1} \Delta(f)(x_0) - (\text{grad}_M f, \text{grad}_M f^{-1})(x_0) \text{Tr}[id]
\]
\[
= - f^{-1} \Delta(f)(x_0) - (\text{grad}_M f, \text{grad}_M f^{-1})(x_0) \text{Tr}[id].
\]

And similarly we have proved this lemma. For more trace expansions, we can see [3, 4, 5].

From (2.30), Lemma 2.4 and Tr(\epsilon_i \epsilon_j) = 0 (i \neq j), we find for the trace
\[
\text{Tr}(E(x_0)) = \text{Tr} \left[ - \frac{1}{4} s + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_i [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)]^2 \right]
\]
\[
- \frac{1}{2} \sum_j \nabla^F_j (c(\Phi^*))c(e_j) - \frac{1}{2} \sum_j c(e_j) \nabla^F_j (c(\Phi))
\]
\[
- 2f^{-1} \Delta(f) + 4f^{-1} \text{Tr}_F [\Phi(\text{grad}_M f)] - f^{-2} [\text{grad}_M(f)]^2 + 2\Delta(f),
\]
where \(\Delta\) denotes the Laplacian operator.

Substituting (2.32) into (2.31), we obtain

**Theorem 2.5.** For even \(n\)-dimensional compact spin manifolds without boundary, the following equality holds:
\[
Wres \left[ f \tilde{D}^{-1} \text{Tr}(\tilde{D}^{-1}) \right]^\frac{n^2}{n-2} = \frac{(2\pi)^\frac{n}{2}}{\left( \frac{n}{2} - 1 \right)!} \int_M \left\{ \text{Tr} \left[ - \frac{s}{12} + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_i [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)]^2 \right] \right.
\]
\[
- \frac{1}{2} \sum_j \nabla^F_j (c(\Phi^*))c(e_j) - \frac{1}{2} \sum_j c(e_j) \nabla^F_j (c(\Phi))
\]
\[
- 2f^{-1} \Delta(f) + 4f^{-1} \text{Tr}_F [\Phi(\text{grad}_M f)] - f^{-2} [\text{grad}_M(f)]^2 + 2\Delta(f) \right\} \text{vol}_M.
\]

where \(s\) is the scaler curvature.
Locally we can use Theorem 2.5 to compute the interior term of (2.17), then for conformal perturbations of twisted Dirac Operators on four-dimensional compact manifolds with boundary,

\[
\int_M \int_{\xi = 1} \text{trace}_{S(TM) \otimes F}[\sigma^{-4}((\tilde{D}_F^* \tilde{D}_F - \tilde{D}_F^* e(df)f^{-1})^{-1})\sigma(\xi)dx = 4\pi^2 \int_M \left\{ \text{Tr} \left[ -\frac{s}{12} + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_i [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)]^2 \right. \right. \\
- \frac{1}{2} \sum_j \nabla^F_c(e_j)c(e_j) - \frac{1}{2} \sum_j c(e_j)\nabla^F_c(e(\Phi)) \left. \right] \right. \\
- 2f^{-1}\Delta(f) + 4f^{-1}\text{Tr}_F[\Phi(\text{grad}_M f)] - f^{-2}[\text{grad}_M (f)^2 + 2\Delta(f)] \right\} \text{dvol}_M. \tag{2.34}
\]

So we only need to compute \(\int_{\partial M} \Psi\). Let us now turn to compute the symbol expansion of \(\tilde{D}_F^{-1}\). Recall the definition of the twisted Dirac operator \(\tilde{D}_F\) in Definition 2.3. Let \(\nabla^{TM}\) denote the Levi-civita connection about \(g^M\). In the local coordinates \(\{x_i; 1 \leq i \leq n\}\) and the fixed orthonormal frame \(\{\tilde{e}_1, \ldots, \tilde{e}_n\}\), the connection matrix \((\omega_{s,i})\) is defined by

\[
\nabla^{TM}(\tilde{e}_1, \ldots, \tilde{e}_n) = (\tilde{e}_1, \ldots, \tilde{e}_n)(\omega_{s,i}). \tag{2.35}
\]

Let \(c(\tilde{e}_i)\) denote the Clifford action. Let \(g^{ij} = g(dx_i, dx_j)\) and

\[
\nabla_{\partial_i}^T \partial_j = \sum_k \Gamma^k_{ij} \partial_k; \quad \Gamma^k = g^{ij} \Gamma^k_{ij}. \tag{2.36}
\]

Let the cotangent vector \(\xi = \sum \xi_j dx_j\) and \(\xi^i = g^{ij} \xi_j\). By Lemma 1 in [14] and Lemma 2.1 in [13], for any fixed point \(x_0 \in \partial M\), we can choose the normal coordinates \(U\) of \(x_0\) in \(\partial M\) (not in \(M\)). By the composition formula and (2.2.11) in [13], we obtain

**Lemma 2.6.** Let \(\tilde{D}_F^*, \tilde{D}_F\) be the twisted Dirac operators on \(\Gamma(S(TM) \otimes F)\), then

\[
\sigma_{-1}(\tilde{D}_F^*)^{-1} = \sigma_{-1}(\tilde{D}_F)^{-1} = \frac{\sqrt{-1}c(\xi)}{||\xi||^2} \tag{2.37}
\]

\[
\sigma_{-2}(\tilde{D}_F^*)^{-1} = \frac{c(\xi)c_0(\tilde{D}_F^*)c(\xi)}{||\xi||^4} + \frac{c(\xi)}{||\xi||^6} \sum_j c(dx_j) \left[ \partial_{x_j}c(\xi)||\xi||^2 - c(\xi)\partial_{x_j}(||\xi||^2) \right] \tag{2.38}
\]

\[
\sigma_{-2}(\tilde{D}_F)^{-1} = \frac{c(\xi)c_0(\tilde{D}_F)c(\xi)}{||\xi||^4} + \frac{c(\xi)}{||\xi||^6} \sum_j c(dx_j) \left[ \partial_{x_j}c(\xi)||\xi||^2 - c(\xi)\partial_{x_j}(||\xi||^2) \right], \tag{2.39}
\]

where

\[
\sigma_0(\tilde{D}_F^*) = \sigma_0(\tilde{D}_F) + \sum_{j=1}^{n} c(e_j)(\sigma_j^F - \Phi^*(e_j)); \tag{2.40}
\]

\[
\sigma_0(\tilde{D}_F) = \sigma_0(\tilde{D}_F) + \sum_{j=1}^{n} c(e_j)(\sigma_j^F + \Phi(e_j)). \tag{2.41}
\]

Let us now turn to compute \(\Psi\) (see formula (2.18) for definition of \(\Psi\)). Since the sum is taken over \(-r - \ell + k + j + ||\alpha|| = 3, r, \ell \leq -1\), then we have the boundary term of (2.17) is the sum of the following five terms.

**Case a)** \(r = -1, \; l = -1 \; k = j = 0, \; ||\alpha|| = 1\)
From (2.18) we have

\[\text{case a) I)}\]
\[-\int_{|\xi'|=1}^{\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi'}^2 \pi \sigma^{-1}(D_{\xi'}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[= -f \sum_{j \leq n} \partial_j (f^{-1}) \int_{|\xi'|=1}^{\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi'}^2 \pi \sigma^{-1}(D_{\xi'}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[= 0. \quad (2.42)\]

And similarly we get

\[\text{case a) II)} \quad r = -1, \quad l = -1, \quad k = |\alpha| = 0, \quad j = 1 \]

\[\text{case a) II)} \]
\[-\frac{1}{2} \int_{|\xi'|=1}^{\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi_n} \pi \sigma^{-1}(D_{\xi_n}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[-f \sum_{j \leq n} \partial_j (f^{-1}) \int_{|\xi'|=1}^{\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi_n} \sigma^{-1}(D_{\xi_n}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[= -\frac{3}{8} \pi h'(0) \text{dim} F \Omega_3 dx' - \frac{\pi i}{2} \Omega_3 f^{-1} \partial_{x_n}(f^{-1}) dx'. \quad (2.43)\]

\[\text{case a) III)} \quad r = -1, \quad l = -1, \quad j = |\alpha| = 0, \quad k = 1 \]

\[\text{case a) III)} \]
\[-\frac{1}{2} \int_{|\xi'|=1}^{\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi_n} \pi \sigma^{-1}(D_{\xi_n}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[-f \sum_{j \leq n} \partial_j (f^{-1}) \int_{|\xi'|=1}^{\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi_n} \sigma^{-1}(D_{\xi_n}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[= -\frac{3}{8} \pi h'(0) \text{dim} F \Omega_3 dx' + \frac{\pi i}{2} \Omega_3 f \partial_{x_n}(f^{-1}) dx'. \quad (2.44)\]

\[\text{case b)} \quad r = -2, \quad l = -1, \quad k = j = |\alpha| = 0 \]

\[\text{case b)} \]
\[-i \int_{|\xi'|=1}^{\infty} \sum_{|\alpha|=1} \text{trace} \left[ \pi \sigma^{-2}(f D_{\xi_n}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[= \frac{9}{8} h'(0) \text{dim} F - \frac{1}{4} \text{Tr} \left( \text{id} \otimes (\sigma F - \Phi^*(e_j)) \right) \pi \Omega_3 dx'. \quad (2.45)\]

\[\text{case c)} \quad r = -1, \quad l = -2, \quad k = j = |\alpha| = 0 \]

\[\text{case c)} \]
\[-i \int_{|\xi'|=1}^{\infty} \sum_{|\alpha|=1} \text{trace} \left[ \pi \sigma^{-2}(f D_{\xi_n}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[= \left[ \frac{9}{8} h'(0) \text{dim} F + \frac{1}{4} \text{Tr} \left( \text{id} \otimes (\sigma F + \Phi(e_n)) \right) \right] \pi \Omega_3 dx'. \quad (2.46)\]

We note that \(\text{dim} S(TM) = 4\), now \(\Psi\) is the sum of the case \(a, b, c\), so

\[\sum \text{case a, b, c} = \frac{\pi i}{2} \Omega_3 \left[ f \partial_{x_n}(f^{-1}) - f^{-1} \partial_{x_n}(f) \right] dx' + \text{Tr}_F (\Phi^*(e_n) + \Phi(e_n)) \pi \Omega_3 dx'. \quad (2.47)\]

Hence we conclude that
Theorem 2.7. Let $M$ be a 4-dimensional compact manifolds with the boundary $\partial M$, for conformal perturbations of twisted Dirac operators $\tilde{D}_F$, then

$$\text{Wres}[\pi^+(f\tilde{D}_F^*)] \circ \pi^+(f^{-1}(\tilde{D}_F)^{-1}) = 4\pi^2 \int_M \{\text{Tr} \left[ -\frac{s}{12} + c(\Phi^*)c(\Phi) - \frac{1}{4} \sum_i [c(\Phi^*)c(e_i) - c(e_i)c(\Phi)]^2 \right]$$

$$- \frac{1}{2} \sum_j \nabla^F_{e_j} (c(\Phi^*))c(e_j) - \frac{1}{2} \sum_j c(e_j)\nabla^F_{e_j} (c(\Phi)) \right]$$

$$- 2f^{-1}\Delta(f) + 4f^{-1}\text{Tr}_F[\Phi(\text{grad}_M f)] - f^{-2} [\text{grad}_M(f)^2 + 2\Delta(f)] \} \text{vol}_M$$

$$+ \int_{\partial M} \frac{\pi_i}{2} \Omega_3 [f\partial_{e_i}(f^{-1}) - f^{-1}\partial_{e_i}(f)] \text{d}x' + \text{Tr}_F(\Phi^*(e_n) + \Phi(e_n))\pi\Omega_3 \text{d}x', \quad (2.48)$$

where $s$ is the scalar curvature.

3. A Kastler-Kalau-Walze Type Theorem for Conformal Perturbations of twisted signature Operators

Let us recall the definition of twisted signature operators. We consider a $n$-dimensional oriented Riemannian manifold $(M, g^M)$. Let $F$ be a real vector bundle over $M$. Let $g^F$ be an Euclidean metric on $F$. Let

$$\wedge^* (T^*M) = \bigoplus_{i=0}^n \wedge^i (T^*M) \quad (3.1)$$

be the real exterior algebra bundle of $T^*M$. Let

$$\Omega^*(M, F) = \bigoplus_{i=0}^n \Omega^i(M, F) = \bigoplus_{i=0}^n C^\infty(M, \wedge^* (T^*M) \otimes F) \quad (3.2)$$

be the set of smooth sections of $\wedge^*(T^*M) \otimes F$. Let $*$ be the Hodge star operator of $g^{TM}$. It extends on $\wedge^*(T^*M) \otimes F$ by acting on $F$ as identity. Then $\Omega^*(M, F)$ inherits the following standardly induced inner product

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha \wedge^* \beta \rangle_F, \quad \alpha, \beta \in \Omega^*(M, F). \quad (3.3)$$

Denote by $\hat{\nabla}^F$ the non-Euclidean connection on $F$. Let $d^F$ be the obvious extension of $\nabla^F$ on $\Omega^*(M, F)$. and $\delta^F = d^{F*}$ be the formal adjoint operator of $d^F$ with respect to the inner product. Then the differential operator $\hat{D}^F$ acting on $\Omega^*(M, F)$ can be defined by

$$\hat{D}^F = d^F + \delta^F. \quad (3.4)$$

Let

$$\omega(F, g^F) = \hat{\nabla}^{F,*} - \hat{\nabla}^F, \quad \nabla^{F,e} = \nabla^F + \frac{1}{2} \omega(F, g^F). \quad (3.5)$$

Then $\nabla^{F,e}$ is an Euclidean connection on $(F, g^F)$.

Let $\nabla^{\wedge* (T^*M)}$ be the Euclidean connection on $\wedge^*(T^*M)$ induced canonically by the Levi-Civita connection $\nabla^TM$ of $g^{TM}$. Let $\nabla^e$ be the Euclidean connection on $\wedge^*(T^*M) \otimes F$ obtained from the tensor product of $\nabla^{\wedge^* (T^*M)}$ and $\nabla^{F,e}$. Let $\{e_1, \cdots, e_n\}$ be an oriented (local) orthonormal basis of $TM$. The following result was proved by Proposition in [10].

Proposition 3.1. [10] The following identity holds

$$d^F + \delta^F = \sum_{i=1}^n c(e_i)\nabla^e_{e_i} = \frac{1}{2} \sum_{i=1}^n c(e_i)\omega(F, g^F)(e_i). \quad (3.6)$$
Let
\[ D_F^* = \sum_{j=1}^{n} c(e_j) \nabla_{e_j}, \] (3.7)
then the twisted signature operators \( \hat{D}_F, \hat{D}_F^* \) as follows.

**Definition 3.2.** For sections \( \psi \otimes \chi \in \wedge^*(T^*M) \otimes F, \)
\[ \hat{D}_F(\psi \otimes \chi) = D_F^*(\psi \otimes \chi) - \frac{1}{2} \sum_{i=1}^{n} \hat{c}(e_i)\omega(F,g^F)(e_i)(\psi \otimes \chi), \] (3.8)
and
\[ \hat{D}_F^*(\psi \otimes \chi) = D_F^*(\psi \otimes \chi) - \frac{1}{2} \sum_{i=1}^{n} \hat{c}(e_i)\omega(F,g^F)(e_i)(\psi \otimes \chi). \] (3.9)

Here \( \omega^*(F,g^F)(e_i) \) denotes the adjoint of \( \omega(F,g^F)(e_i) \).

In the following, we will compute the more general case \( \hat{W}_{\text{res}}[\pi^+(f D_F^*) \circ \pi^+(f^{-1}(\hat{D}_F^*)^{-1})] \) for nonzero smooth functions \( f, f^{-1}. \) Denote by \( \sigma_l(A) \) the \( l \)-order symbol of an operator \( A. \) An application of (3.5) and (3.6) in [14] shows that
\[ \hat{W}_{\text{res}}[\pi^+(f D_F^*) \circ \pi^+(f^{-1}(\hat{D}_F^*)^{-1})] = W_{\text{res}}[f \hat{D}_F^{-1} f^{-1}(\hat{D}_F^*)^{-1}] + \int_{\partial M} \tilde{\Psi}, \]
where
\[ \tilde{\Psi} = \int_{|s| = 1} \int_{-\infty}^{\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{|a|+j+k+\ell}}{\alpha !(j + k) !} \text{trace}_{S(T^*M) \otimes F} \left[ \partial_{x_n}^{\alpha} \partial_{\xi_n}^{\alpha} \sigma_l^+(f \hat{D}_F^*)^{-1}(x^l, 0, \xi^s, \xi_n) \right. \]
\[ \times \partial_{x_n}^{\alpha} \partial_{\xi_n}^{\alpha} \sigma_l(f^{-1}(\hat{D}_F^*)^{-1})(x^l, 0, \xi^s, \xi_n) d\xi_n d(\xi^s) dx^l, \] (3.11)
and the sum is taken over \( r - k + |a| + \ell - j - 1 = -n, r \leq -1, \ell \leq -1. \)

Let \( \hat{c}(\omega) = \sum_{i} c(e_i)\omega(F,g^F)(e_i) \) and \( \hat{c}(\omega^*) = \sum_{i} c(e_i)\omega^*(F,g^F)(e_i) \), then
\[ \hat{D}_F^* \hat{D}_F = \hat{D}_F^* c(df)^{-1} = -g^{ij} \partial_i \partial_j - 2\sigma_{\Lambda_1(T^*M) \otimes F}^{ij} \partial_j - c(e_i)c(df)^{-1} \partial_i - \frac{1}{2} \sum_{j} \left[ \hat{c}(\omega^*)c(e_j) + c(e_j) \otimes \hat{c}(\omega) \right] e_j \]
\[ -g^{ij} \left[ \partial_i(\sigma_{\Lambda_1(T^*M) \otimes F}^{k\ell} + \sigma_{\Lambda_1(T^*M) \otimes F}^{k\ell} \otimes F_{\sigma_{\Lambda_1(T^*M) \otimes F}}^{ij} - \Gamma^{k\ell}_{ij} \sigma_{\Lambda_1(T^*M) \otimes F}^{k\ell} \right] \]
\[ -\frac{1}{2} \sum_{j} g^{ij} \hat{c}(\omega^*)c(e_j)\sigma_j^{\Lambda_1(T^*M) \otimes F} - \frac{1}{2} \sum_{j} g^{ij} c(e_j) \otimes e_j(\hat{c}(\omega)) \]
\[ + \frac{1}{4} s + \frac{1}{2} \sum_{i \neq j} R_{ij \ell}^k c(e_i, e_j) c(e_j) \]
\[ -\frac{1}{4} \sum_{i} g^{ij} \ell \sum_{s,t} \omega_{s,t}(e_i) \left[ \hat{c}(\omega^*)c(e_j) + c(e_j)\hat{c}(\omega) \right] c(df)^{-1} \]
\[ - \sum_{i} g^{ij} c(e_j) \sigma_j^{\Lambda_1(T^*M) \otimes F} c(df)^{-1} = -\frac{1}{2} \hat{c}(e_j) \omega^*(F,g^F)c(df)^{-1}. \] (3.12)
In terms of local coordinates \( \{\partial_i\} \) inducing the coordinate transformation \( e_j = \sum_{k=1}^n (e_j, dx^k)\partial_k \), then

\[
\omega_j = \sigma^j\Lambda_{(T^*M)} + \sigma^j_F + c(e_j)c(df)f^{-1} + \frac{1}{2} \omega_j + \frac{1}{4} \left( \sum_{k=1}^n (e_k, dx^k)\partial_k \right) \omega_j + \left( \sum_{k=1}^n (e_k, dx^j)\partial_k \right) c(e_k)\partial_k (\omega(F,g^F)) + \right).
\]

(3.13)

For a smooth vector field \( X \in \Gamma(M, TM) \), then

\[
\nabla_X = \nabla_X^{(T^*M)\otimes F,e} + c(X)c(df)f^{-1} + \frac{1}{4} \left( \partial_k (\omega(F,g^F))c(X) - c(X)c(\omega(F,g^F)) \right).
\]

(3.14)

Since \( E \) is globally defined on \( M \), so we can perform computations of \( E \) in normal coordinates. In terms of normal coordinates about \( x_0 \) one has: \( \sigma^j\Lambda_{(T^*M)}(x_0) = 0, e_j(c(e_i))(x_0) = 0, \Gamma^k(x_0) = 0 \). From (3.12) and (3.13), we obtain

\[
E(x_0) = -\frac{1}{s} - \frac{1}{2} \sum_{i \neq j} R^{F,e}(e_i, e_j)c(e_j)c(e_j) - \frac{1}{16} \sum_{i} \left[ \partial_k (\omega^*)c(e_i) - c(e_i)\partial_k (\omega^*) \right] - \frac{1}{4} \partial_k (\omega^*)\partial_k (\omega)
\]

\[
- \frac{1}{4} \sum_j (\omega^{*})(\partial)_j (\omega^*)c(e_j) + \frac{1}{4} \sum_j c(e_j)(\partial)_j (\omega^*)
\]

\[
+ \frac{1}{4} \sum_i c(e_i)(\partial)_i (\omega^*) (c(e_i)\partial_i (\omega^*))c(e_i) + \frac{1}{4} \sum_k (e_k, dx^k)\partial_k (\partial_k (\omega^*)) + \left( \sum_{k=1}^n (e_k, dx^j)\partial_k (\omega^*))c(e_k)\right) - \frac{1}{4} \sum_k (e_k, dx^j)\partial_k (\omega^*))c(e_k)\partial_k (\omega) + \right).
\]

(3.15)

From (3.15) and Lemma 2.4 we obtain

\[
\text{Tr}(E(x_0)) = \text{Tr}\left[ -\frac{1}{s} - \frac{n}{16} \left[ \partial_k (\omega^*) - \partial_k (\omega) \right] - \frac{1}{4} \partial_k (\omega^*)\partial_k (\omega) - \frac{1}{4} \sum_j (\partial)_j (\omega^*)c(e_j) + \right.
\]

\[
+ \frac{1}{4} \sum_j c(e_j)(\partial)_j (\omega^*)
\]

\[
+ \left. \text{Tr}\left[ -\partial_j (c(\partial)_j c(df)f^{-1}) - \frac{5}{4} c(e_i)c(df)f^{-1}c(\partial)_i c(df)f^{-1} \right] \right)
\]

\[
= \text{Tr}\left[ -\frac{1}{s} - \frac{n}{16} \left[ \partial_k (\omega^*) - \partial_k (\omega) \right] - \frac{1}{4} \partial_k (\omega^*)\partial_k (\omega) - \frac{1}{4} \sum_j (\partial)_j (\omega^*)c(e_j) + \right.
\]

\[
+ \frac{1}{4} \sum_j c(e_j)(\partial)_j (\omega^*)
\]

\[
+ 4f^{-1}\Delta(f) + 8(\text{grad}_M(f), \text{grad}_M(f^{-1})) - 5f^{-2}[(\text{grad}_M(f))^2 + 2\Delta(f)].
\]

(3.16)

Hence we conclude
Theorem 3.3. For even $n$-dimensional oriented compact Riemannian manifolds without boundary, the following equality holds:

$$\text{Wres}(f \hat{D}_F^{-1} \circ f^{-1}(\hat{D}_F)^{-1}) = \frac{(2\pi)^\frac{n}{2}}{(4-2)!} \int_M \text{Tr} \left[ -\frac{s}{12} + \frac{n}{16} \hat{\epsilon}(\omega^*) - \hat{\epsilon}(\omega) \right] - \frac{1}{4} \hat{\epsilon}(\omega^*) \hat{\epsilon}(\omega)$$

$$- \frac{1}{4} \sum_j \nabla^F e_j (\hat{\epsilon}(\omega^*)) c(e_j) + \frac{1}{4} \sum_j c(e_j) \nabla^F e_j (\hat{\epsilon}(\omega))$$

$$+ 4f^{-1}\Delta(f) + 8\langle \text{grad}_M(f), \text{grad}_M(f^{-1}) \rangle - 5f^{-2}||\text{grad}_M(f)||^2 + 2\Delta(f) \right] d\omega_M.$$

(3.17)

Locally we can use Theorem 3.3 to compute the interior term of (3.10), then

$$\int_M \int_{|\xi|=1} \text{trace}_{\Lambda^* (T^* M) \otimes F} [\sigma_4((f \hat{D}_F^{-1} \circ f^{-1}(\hat{D}_F)^{-1})|\sigma(\xi)) d\sigma$$

$$= 4\pi^2 \int_M \text{Tr} \left[ -\frac{s}{12} + \frac{1}{4} \hat{\epsilon}(\omega^*) - \hat{\epsilon}(\omega) \right] - \frac{1}{4} \hat{\epsilon}(\omega^*) \hat{\epsilon}(\omega) - \frac{1}{4} \sum_j \nabla^F e_j (\hat{\epsilon}(\omega^*)) c(e_j)$$

$$+ 4f^{-1}\Delta(f) + 8\langle \text{grad}_M(f), \text{grad}_M(f^{-1}) \rangle - 5f^{-2}||\text{grad}_M(f)||^2 + 2\Delta(f) \right] d\omega_M.$$

(3.18)

So we only need to compute $\int_{\partial M} \hat{\Psi}$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\hat{e}_1, \ldots, \hat{e}_n\}$, the connection matrix $(\omega_{i,s})$ is defined by

$$\hat{\nabla}(\hat{e}_1, \ldots, \hat{e}_n) = (\hat{e}_1, \ldots, \hat{e}_n)(\omega_{s,t}).$$

(3.19)

Let $M$ be a 4-dimensional compact oriented Riemannian manifold with boundary $\partial M$ and the metric of (2.6). $\hat{D}_F = d^F + \delta^F : C^\infty(M, \Lambda^*(T^* M) \otimes F) \rightarrow C^\infty(M, \Lambda^*(T^* M) \otimes F)$ is the twisted signature operator. Take the coordinates and the orthonormal frame as in Section 2. Let $\epsilon(\hat{e}_s^*)$, $\iota(\hat{e}_s^*)$ be the exterior and interior multiplications respectively. Write

$$\epsilon(\hat{e}_s) = \epsilon(\hat{e}_s^*) - \iota(\hat{e}_s^*) ; \quad \hat{\epsilon}(\hat{e}_s) = \epsilon(\hat{e}_s^*) + \iota(\hat{e}_s^*).$$

(3.20)

We'll compute $\text{tr}_{\Lambda^* (T^* M) \otimes F}$ in the frame $\{e_i^* \wedge \ldots \wedge e_i^* | 1 \leq i_1 < \ldots < i_k \leq 4\}$. By (3.2) in [9], we have

$$\hat{D}_F = d^F + \delta^F = \sum_{i=1}^n \epsilon(e_i) \nabla^e_i - \frac{1}{2} \sum_{i=1}^n \epsilon(e_i) \omega(F, g^F)(e_i)$$

$$= \sum_{i=1}^n \epsilon(\hat{e}_s) \left( \nabla^e_{\hat{e}_s} (T^* M) \otimes id_F + id_{\Lambda^* (T^* M) \otimes \nabla^e_{\hat{e}_s}} - \frac{1}{2} \sum_{i=1}^n \epsilon(\hat{e}_s) \omega(F, g^F)(e_i) \right)$$

$$= \sum_{i=1}^n \epsilon(\hat{e}_s) \left[ \hat{e}_s + \frac{1}{4} \sum_{s,t} \omega_s,t(\hat{e}_s) \hat{\epsilon}(\hat{e}_s) \hat{\epsilon}(\hat{e}_s) - c(\hat{e}_s) c(\hat{e}_s) \right] \otimes id_F$$

$$+ id_{\Lambda^* (T^* M) \otimes \sigma_i^{F^*}} - \frac{1}{2} \sum_{i=1}^n \epsilon(\hat{e}_s) \omega(F, g^F)(e_i),$$

(3.21)

$$\hat{D}_F^* = \sum_{i=1}^n \epsilon(\hat{e}_s) \left[ \hat{e}_s + \frac{1}{4} \sum_{s,t} \omega_s,t(\hat{e}_s) \hat{\epsilon}(\hat{e}_s) \hat{\epsilon}(\hat{e}_s) - c(\hat{e}_s) c(\hat{e}_s) \right] \otimes id_F$$

$$+ id_{\Lambda^* (T^* M) \otimes \sigma_i^{F^*}} - \frac{1}{2} \sum_{i=1}^n \epsilon(\hat{e}_s) \omega^*(F, g^F)(e_i).$$

(3.22)
The dual metric of $g$ then we have
\[ \sigma_0(\tilde{D}_F) = \sum_{i=1}^{n} c(\tilde{e}_i) \left[ \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) [c(\tilde{e}_s)c(\tilde{e}_t)] - c(\tilde{e}_s)c(\tilde{e}_t) ] \otimes \id_F + id_{\Lambda^*T^*M} \otimes \sigma_{1,F} \right] - \frac{1}{2} \sum_{i=1}^{n} \tilde{e}_i \omega(F,g^F)(\tilde{e}_i); \] (3.24)
\[ \sigma_0(\tilde{D}_F^*) = \sum_{i=1}^{n} c(\tilde{e}_i) \left[ \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) [c(\tilde{e}_s)c(\tilde{e}_t)] - c(\tilde{e}_s)c(\tilde{e}_t) ] \otimes \id_F + id_{\Lambda^*T^*M} \otimes \sigma_{1,F} \right] - \frac{1}{2} \sum_{i=1}^{n} \tilde{e}_i \omega(F,g^F)(\tilde{e}_i). \] (3.25)

By the composition formula of pseudodifferential operators in Section 2.2.1 of [9], we have

**Lemma 3.4.** The symbol of the twisted signature operators $\tilde{D}_F, \tilde{D}_F^*$ as follows:

\[
\sigma_1((\tilde{D}_F)^{-1}) = \sigma_1((\tilde{D}_F)^{-1}) = \sqrt{-1}c(\xi); \\
\sigma_2((\tilde{D}_F)^{-1}) = \frac{c(\xi)\sigma_0(\tilde{D}_F)c(\xi)}{|\xi|^4} + \frac{c(\xi)|\xi|^6}{|\xi|^6} \sum_j c(\partial_j)c(\xi)\partial_j(\xi^2) - c(\xi)\partial_j(\xi^2); \\
\sigma_2((\tilde{D}_F^*)^{-1}) = \frac{c(\xi)\sigma_0(\tilde{D}_F^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)|\xi|^6}{|\xi|^6} \sum_j c(\partial_j)c(\xi)\partial_j(\xi^2) - c(\xi)\partial_j(\xi^2). \] (3.26) (3.27) (3.28)

Since $\tilde{\Psi}$ is a global form on $\partial M$, so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates of $x_0$ in $\partial M$ (not in $M$) and compute $\tilde{\Psi}(x_0)$ in the coordinates $\tilde{U} = U \times [0,1]$ and the metric $\frac{1}{h(x)}g^M + dx^2$. The dual metric of $g^M$ on $\tilde{U}$ is $\frac{1}{h(x)}g^M + dx^2$. Write $g^{ij}_M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $g^{ij}_M = g^M(dx_i, dx_j)$, then
\[ [g^{ij}_M] = \begin{bmatrix} \frac{1}{h(x)} & 0 \\ 0 & 1 \end{bmatrix}; \quad [g^{ij}_{M}(x_0)] = \begin{bmatrix} h(x_0) & 0 \\ 0 & 1 \end{bmatrix}. \] (3.29) (3.30)

Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame field in $U$ about $g^M$ which is parallel along geodesics and $e_i = \frac{\partial}{\partial x_i}(x_0)$, then $\tilde{e}_1 = \sqrt{h(x)}e_1, \ldots, \tilde{e}_{n-1} = \sqrt{h(x_0)}e_{n-1}, \tilde{e}_n = dx_n$ is the orthonormal frame field in $U$ about $g^M$. Locally $\Lambda^*(T^*M)|\tilde{U} \cong \tilde{U} \times \Lambda^n_C(\frac{\tilde{\omega}}{2})$. Let $\{f_1, \ldots, f_n\}$ be the orthonormal basis of $\Lambda^n_C(\frac{\tilde{\omega}}{2})$. Take a spin frame field $\sigma : \tilde{U} \to Spin(M)$ such that $\pi^\sigma = \{\tilde{e}_1, \ldots, \tilde{e}_n\}$ where $\pi : Spin(M) \to O(M)$ is a double covering, then $\{[\sigma, f_i], 1 \leq i \leq 4\}$ is an orthonormal frame of $\Lambda^*(T^*M)|\tilde{U}$. In the following, since the global form $\tilde{\Psi}$ is independent of the choice of the local frame, so we can compute $tr_{\Lambda^*(T^*M)}$ in the frame $\{[\sigma, f_i], 1 \leq i \leq 4\}$. Let $\{E_1, \ldots, E_6\}$ be the canonical basis of $R^6$ and $c(E_i) \in cl_C(n) \cong \Hom(\Lambda^*_C(\frac{\tilde{\omega}}{2}), \Lambda^n_C(\frac{\tilde{\omega}}{2}))$ be the Clifford action. By [9], then
\[ c(\tilde{e}_i) = [\sigma, c(E_i)]; \quad c(\tilde{e}_i)([\sigma, f_i]) = [\sigma, c(E_i)]f_i; \quad \frac{\partial}{\partial x_i} = c(\tilde{e}_i), \] (3.31)
then we have $\frac{\partial}{\partial x_i}c(\tilde{e}_i) = 0$ in the above frame. By Lemma 2.2 in [9], we have
Lemma 3.5. With the metric $g^M$ on $M$ near the boundary

$$
\partial_x j(|\xi|^2 g^x)(x_0) = \begin{cases} 
0, & \text{if } j < n; \\
h'(0)\|\xi\|^2 g^M, & \text{if } j = n.
\end{cases}
$$

(3.32)

$$
\partial_x [c(\xi)](x_0) = \begin{cases} 
0, & \text{if } j < n; \\
\partial x_n(c(\xi))(x_0), & \text{if } j = n.
\end{cases}
$$

(3.33)

where $\xi = \xi' + \xi_n dx_n$

Then an application of Lemma 2.3 in [9] shows

Lemma 3.6. The symbol of the twisted signature operators $\tilde{D}_F^r, \tilde{D}_F$

$$
\sigma_0(\tilde{D}_F^r) = \frac{-3}{4} h'(0)c(dx_n) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i)c(\tilde{e}_n)\tilde{c}(\tilde{e}_i)(x_0) \otimes id_F
$$

+ \sum_{i=1}^{n} c(\tilde{e}_i)\sigma_{F,e} - \frac{1}{2} \sum_{i=1}^{n} \tilde{c}(e_i)\omega(F, g^F)(e_i);

(3.34)

$$
\sigma_0(\tilde{D}_F) = \frac{-3}{4} h'(0)c(dx_n) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i)c(\tilde{e}_n)\tilde{c}(\tilde{e}_i)(x_0) \otimes id_F
$$

+ \sum_{i=1}^{n} c(\tilde{e}_i)\sigma_{F,e} - \frac{1}{2} \sum_{i=1}^{n} \tilde{c}(e_i)\omega(F, g^F)(e_i);

(3.35)

Now we can compute $\tilde{\Psi}$ (see formula (3.11) for definition of $\tilde{\Psi}$), since the sum is taken over $-r - \ell + k + j + |\alpha| = 3, r, \ell \leq -1$, then we have the following five cases:

Case a(I): $r = -1, \ell = -1, k = j = 0, |\alpha| = 1$

From (3.11) we have

$$
case a) I) \quad \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1}^{\infty} \text{trace} \left[ \partial^2 F \pi_{E, \alpha_1} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_x \pi_{E, \alpha_1} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'
$$

$$
- \int_{j<n} \partial_x (f^{-1}) \int_{\xi'=1}^{+\infty} \sum_{|\alpha|=1}^{\infty} \text{trace} \left[ \partial^2 F \pi_{E, \alpha_1} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_x \pi_{E, \alpha_1} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'
$$

$$= 0.
$$

(3.36)

And similarly we get

Case a) II) $r = -1, \ell = -1 k = |\alpha| = 0, j = 1$

$$
case a) II) \quad \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1}^{\infty} \text{trace} \left[ \partial^2 F \pi_{E, \alpha_1} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_x \pi_{E, \alpha_1} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'
$$

$$
- \int_{j<n} \partial_x (f^{-1}) \int_{\xi'=1}^{+\infty} \sum_{|\alpha|=1}^{\infty} \text{trace} \left[ \partial^2 F \pi_{E, \alpha_1} \sigma_{-1}(\tilde{D}_F^{-1}) \times \partial_x \pi_{E, \alpha_1} \sigma_{-1}(\tilde{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'
$$

$$= -\frac{3}{2} \pi h'(0) \Omega_3 dx' - \frac{\pi i}{2} \Omega_3 f^{-1} \partial_x (f) dx'.
$$

(3.37)
case a) III) $r = -1$, $l = -1$, $j = |\alpha| = 0$, $k = 1$

\[
\begin{align*}
\text{case a) III)} & = - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \hat{\partial}_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\hat{D}_F^{-1}) \times \hat{\partial}_{\xi_n} \sigma_{-1}(\hat{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& - f \hat{\partial}_{\xi_n}(f^{-1}) \left[ \hat{\partial}_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\hat{D}_F^{-1}) \times \hat{\partial}_{\xi_n} \sigma_{-1}(\hat{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& = \frac{3l}{2} \pi h'(0) \Omega_3 dx' + \frac{\pi^2}{2} \Omega_3 f \hat{\partial}_{\xi_n}(f^{-1}) dx'.
\end{align*}
\] (3.38)

case b) $r = -2$, $l = -1$, $k = j = |\alpha| = 0$

\[
\begin{align*}
\text{case b) } & = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-2}(f \hat{D}_F^{-1}) \times \hat{\partial}_{\xi_n} \sigma_{-1}(f \hat{D}_F^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& = \left[ \frac{9}{2} h'(0) - 4 \text{tr}_{\mathbb{R}}[\sigma^F_{\mathbb{R}}] \right] \pi_\Omega dx'.
\end{align*}
\] (3.39)

case c) $r = -1$, $l = -2$, $k = j = |\alpha| = 0$

\[
\begin{align*}
\text{case c) } & = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(f \hat{D}_F^{-1}) \times \hat{\partial}_{\xi_n} \sigma_{-2}(f^{-1}(\hat{D}_F^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& = \left[ - \frac{9}{2} h'(0) + 4 \text{tr}_{\mathbb{R}}[\sigma^F_{\mathbb{R}}] \right] \pi_\Omega dx'.
\end{align*}
\] (3.40)

We note that $\dim S(TM) = 4$, now $\bar{\Psi}$ is the sum of the case (a, b, c), so

\[
\sum \text{case a, b, c} = \frac{\pi^2}{2} \Omega_3 [f \hat{\partial}_{\xi_n}(f^{-1}) - f^{-1} \hat{\partial}_{\xi_n}(f)] dx'.
\] (3.41)

Hence we conclude that

**Theorem 3.7.** Let $M$ be a 4-dimensional compact manifolds with the boundary $\partial M$, for Perturbations of twisted signature Operators $\hat{D}_F$, then

\[
\begin{align*}
\bar{W} & \approx \pi^+ (\hat{D}_F)^{-1} \circ \pi^+ (\hat{D}_F^{-1}) \\
& = 4 \pi^2 \int_M \left\{ \text{Tr} \left[ - \frac{s}{12} + \frac{1}{4} \hat{c}(\omega^*) - \hat{c}(\omega) \right]^2 - \frac{1}{4} \hat{c}(\omega^*)\hat{c}(\omega) - \frac{1}{4} \sum_j \nabla_{\epsilon_j}^F (\hat{c}(\omega^*)) c(e_j) \\
& + \frac{1}{4} \sum_j c(e_j) \nabla_{\epsilon_j}^F (\hat{c}(\omega)) \right\} \\
& + 4f^{-1} \Delta(f) + 8 \text{grad}_M(f), \text{grad}_M(f^{-1}) \right\} d\text{vol}_M \\
& + \int_{\partial M} \frac{\pi^2}{2} \Omega_3 [f \hat{\partial}_{\xi_n}(f^{-1}) - f^{-1} \hat{\partial}_{\xi_n}(f)] dx'.
\end{align*}
\] (3.42)

where $s$ is the scalar curvature.

**Acknowledgements**

This work was supported by the National Natural Science Foundation of China No. 11501414 and No. 11771070. The authors also thank the referee for his (or her) careful reading and helpful comments.
References

[1] V. W. Guillemin.: A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues. Adv. Math. 55, no. 2, 131-160, (1985).
[2] M. Wodzicki.: local invariants of spectral asymmetry. Invent. Math. 75(1), 143-178, (1984).
[3] A. Connes.: Quantized calculus and applications. XIth International Congress of Mathematical Physics(Paris,1994), Internat Press, Cambridge, MA, 15-36, (1995).
[4] A. Connes.: The action functional in Noncommutative geometry. Comm. Math. Phys. 117, 673-683, (1998).
[5] D. Kastler.: The Dirac Operator and Gravitation. Comm. Math. Phys. 166, 633-643, (1995).
[6] W. Kalau and M. Walze.: Gravity, Noncommutative geometry and the Wodzicki residue. J. Geom. Phys. 16, 327-344, (1995).
[7] T. Ackermann.: A note on the Wodzicki residue. J. Geom. Phys. 20, 404-406, (1996).
[8] B. V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe.: The noncommutative residue for manifolds with boundary. J. Funct. Anal. 142, 1-31, (1996).
[9] Y. Wang.: Gravity and the Noncommutative Residue for Manifolds with Boundary. Lett. Math. Phy. 80, 37-56, (2007).
[10] J. M. Bismut and W. Zhang.: An Extension of a theorem by Cheeger and Müller, Astérisque, No. 205, Paris, (1992).
[11] W. Zhang.: Sub-signature operators, $\eta$ invariants and a Riemann-Roch theorem for flat vector bundles. Chin. Ann. Math. 25B: 1, 7-36, (2004).
[12] J. Wang and Y. Wang:Twisted Dirac operators and the noncommutative residue for manifolds with boundary. J.Pseudo-Differ.Appl.7:181-211, (2016).
[13] Y. Wang.: A Kastler-Kalau-Walze Type Theorem and the Spectral Action for Perturbations of Dirac Operators on Manifolds with Boundary. Abstract and Applied Analysis, vol. 2014, Article ID 619120, 13 pages, (2014).
[14] Y. Wang.: Diffential forms and the Wodzicki residue for Manifolds with Boundary. J. Geom. Phys. 56, 731-753, (2006).
[15] P. Gilkey, K. Kirsten, JH. Park.: Heat content asymptotics for operators of Laplace type with spectral boundary conditions. Lett. Math. Phy. 68(2), 67-76, (2004).