An Elementary Proof of the Quantum Adiabatic Theorem

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Abstract

We provide an elementary proof of the quantum adiabatic theorem.

1 Introduction

The model of adiabatic quantum computation is a new paradigm for designing quantum algorithms, proposed by Farhi et al. [5]. It was recently established that this model is polynomially equivalent to the standard model of quantum circuits [12, 1]. Nevertheless, this model provides a completely different way of constructing quantum algorithms and reasoning about them. Therefore, it is seen as a promising approach for the discovery of substantially new quantum algorithms.

Farhi et al. [4] have numerically studied adiabatic quantum algorithms on random instances of problems such as SAT and finding $k$-cliques, with promising results for small instances. Rigorous results are, however, quite scarce. Van Dam et al. [12] and Reichardt [10] have constructed examples of SAT formulas for which a certain natural adiabatic algorithm performs poorly. This suggests that, at least for certain cases, the quantum adiabatic algorithms are not much stronger than the classical method of simulated annealing. On the other hand, there are instances on which classical simulated annealing fails but the adiabatic quantum algorithm succeeds [3]. A rigorous analysis of adiabatic algorithms in the general case appears to be difficult.

The model of adiabatic quantum computation is based on a theorem known as the quantum adiabatic theorem [8]. Informally, this theorem says that if we take a quantum system whose Hamiltonian slowly changes from $H_1$ to $H_2$, then, under certain conditions on $H_1$ and $H_2$, the ground (lowest energy) state of $H_1$ gets transformed to the ground state of $H_2$. This is used to construct adiabatic algorithms for optimization problems, in the following way. We take a Hamiltonian $H_1$ whose ground state $|j_1\rangle$ we know and a Hamiltonian $H_2$ whose ground state corresponds to the solution of our optimization problem. Then, starting a quantum system in the state $|j_1\rangle$ and slowly changing the Hamiltonian from $H_1$ to $H_2$ will solve our optimization problem.

The adiabatic theorem has several proofs in the physics literature (see, e.g., [6, 2, 8]). However, these proofs are rather involved and seem to give very little intuition.
Even the correctness of the adiabatic theorem has recently been questioned. Marzlin and Sanders [7] and Tong et al. [11] have given counterexamples to some variants of the adiabatic theorem that were widely assumed to be true. Reichardt [10] recently claimed that none of the proofs examined by him contains a rigorous analysis of the convergence time. He includes in his paper another proof of the adiabatic theorem where he addresses this issue. His proof, however, follows the structure of previous proofs and does not seem to be more intuitive.

In this paper, we give a new proof of the adiabatic theorem. Unlike all previous proofs, our proof is elementary and should be much more accessible to computer scientists. Moreover, we believe that our proof gives a good insight into why the adiabatic theorem holds: essentially, the proof shows that the error in the adiabatic evolution can be written as a certain geometric sum and that if the evolution is performed slow enough, this geometric sum almost completely cancels out. We hope that our proof will lead to new adiabatic algorithms, a better understanding of existing algorithms and contribute to settling the controversy about the correctness of the adiabatic theorem.

2 Overview

2.1 Main result

Let \( H(s), 0 \leq s \leq 1 \), be a Hamiltonian dependent on a parameter \( s \). We refer to \( H \) as a time dependent Hamiltonian. We think of \( H(0) \) as the initial Hamiltonian and of \( H(1) \) as the final Hamiltonian. For a time dependent Hamiltonian \( H \), we use the notation \( kHk \) to denote \( \max_{s \in [0,1]} \| H(s)k \| \) where \( k \cdot k \) is the usual operator norm. We use a similar notation to denote the maximum norm (or absolute value) of other time dependent expressions.

Let \( (s) \) be an eigenstate of \( H(s) \) with eigenvalue \( \langle s \rangle \) (in most applications, \( (s) \) is chosen to be the ground state of \( H(s) \)). When we say that we apply the adiabatic evolution given by \( H \) and \( \) applied for time \( \) we mean that we initialize a system in the state \( (0) \) and then apply the continuously varying Hamiltonian \( H(t) \) for times \( t \in [0,T] \). We expect the final state of the system to be close to \( (1) \). Our main result is

**Theorem 2.1** Let \( H(s), 0 \leq s \leq 1 \), be a time dependent Hamiltonian, let \( (s) \) be one of its eigenstates, and let \( \langle s \rangle \) be the corresponding eigenvalue. Assume that for any \( s \in [0,1] \) all other eigenvalues of \( H(s) \) are either smaller than \( \langle s \rangle \) or larger than \( \langle s \rangle + \) (i.e., there is a spectral gap of around \( \langle s \rangle \)). Consider the adiabatic evolution given by \( H \) and applied for time \( T \). Then, the following condition is enough to guarantee that the final state is at distance at most from \( (1) \):

\[
T > \frac{10^5}{2} \max_{s \in [0,1]} \left( \frac{kH\frac{k^3}{4}}{\langle s \rangle} ; \frac{\langle s \rangle}{3} \frac{kH\frac{k^3}{4}}{3} \right).
\]

In particular, this implies that as long as \( H \) has a \( 1=poly \) spectral gap around \( \), we can reach a state that is at most \( 1=poly \) away from \( (1) \) in polynomial time. We remark that it might be possible to improve the dependence on \( \) to \( 3 \) or even \( 2 \).

2.2 Overview of our proof

The main part of our paper is concerned with the special case of Theorem 2.1 in which \( (s) = 0 \) for all \( 0 \leq s \leq 1 \) (see Figure 1). The proof of this special case contains most of the important ideas and allows us
to avoid a few technical issues. Later, in Section 4, we complete the proof of Theorem 2.1 by showing how the general case reduces to this special case. In this overview, we concentrate on the proof of the special case. In order to emphasize the high level structure of the proof, some details are omitted.

Figure 1: Special case of constant eigenvalue 0 with a gap of \( \lambda \) on both sides

We start by discretizing the adiabatic evolution. Namely, we replace \( H(s) \) by a sequence of (time-independent) Hamiltonians \( H_0, \ldots, H_L \) each of which is applied for a small interval of time \( \tau = \frac{\lambda}{L} \).

Equivalently, we are applying the sequence of unitary transformations \( U_0 = e^{i\tau H_0}, U_1 = e^{i\tau H_1}, \ldots, U_L = e^{i\tau H_L} \). Let \( g_j = (j=L) \) be the corresponding discretization of \( (s) \). Our goal has now become the following: show that the unitary transformation \( U_L U_0 \) transforms \( g_0 \) into a state close to \( g_L \).

Figure 2: \( g_j \) and \( w_j \)

To show that, we consider a sequence \( w_1, \ldots, w_L \) where \( w_{j+1} \) is defined as the projection of \( g_j \) to the subspace orthogonal to \( g_{j+1} \) (see Figure 2). We will show that

\[
U_j g_j = g_j = g_{j+1} + w_{j+1} + O(1=L^2)
\]

where \( O(1=L^2) \) denotes a vector of norm \( O(1=L^2) \). Then, the final state is

\[
U_L U_0 g_0 = g_L + \sum_{j=1}^{L} U_L U_1 \ldots U_{j} w_j + O(1=L) =: X_L
\]

Showing that this expression is close to \( g_L \) is equivalent to showing that the norm of \( \sum_{j=1}^{L} U_L U_1 \ldots U_{j} w_j \) is small. We show this by proving that all but at most a small fraction of this sum cancels out. To show cancellations, we split the sum into smaller groups of \( \frac{L}{2} \) terms each. We then show that the norm of each group is at most \( \frac{\lambda}{L^2} \).

For simplicity, consider the first group

\[
X U_{L-1} \ldots U_{j} w_j: \quad (1)
\]

Since all terms start with the unitary \( U_{L-1} \), the norm of (1) is the same as the norm of

\[
X U_{L-1} \ldots \ldots U_{j} w_j:
\]
Next, we show that the $w_j$’s and the $U_j$’s change relatively slowly. More precisely, we show that if $\epsilon$ is sufficiently small compared to $L$, then we can make the following approximations:

- replace all $w_j$ by $w_1$;
- replace all $U_j$ by $U_1$.

Thus, we obtain that the norm of (1) is closely approximated by the norm of

$$
\sum_{j=0}^{\infty} U_j^2 w_1;
$$

(2)

We now arrive at the heart of the proof. Express $w$ as a sum of eigenvectors of $U_1$, $w = \sum_{k=1}^{d} a_k k$. Let $k$ be the eigenvalue of $H_1$ corresponding to the eigenvector $k$. Then, the above sum can be written as

$$
\sum_{k=1}^{d} a_k \sum_{j=0}^{\infty} e^{ij} k^n k:
$$

Recall that $w_1$ is orthogonal to $g_1$, and that $g_1$ has eigenvalue $0$ in $H_1$. Since we assumed $H_1$ has a spectral gap of $\epsilon$, all the $k$’s in the above sum (ignoring terms for which $a_k = 0$) are at least $1$ in absolute value. Hence, if we pick $\epsilon$ large enough compared to $1/k$, then most of the sum $\sum_{j=0}^{\infty} e^{ij} k^n$ cancels out, giving the desired result. These cancellations in the geometric sum are the essential reason why adiabatic evolution works.

In the next sections, we make these arguments precise.

## 3 Proof of a Special Case

In this section we prove Theorem 2.1 for the special case in which the eigenvalue of the eigenvector that we follow is always $0$ (see Figure 1). This case already captures the essential ideas in our proof. In Section 4 we will show how to reduce the general case to this special case.

Before we begin, we need to address a minor technical issue. Given some Hamiltonian with an eigenvector $(s)$, we would like to say that the adiabatic evolution closely follows $(s)$ in the $\ell_2$ norm. However, notice that the phase of $(s)$ is arbitrary. So, for example, $(s) = e^{is}$ is an equally good eigenvector and clearly, the adiabatic evolution cannot be close to both $(s)$ and $(s)$ in the $\ell_2$ norm as the distance between them is large. A possible solution is to use a distance measure that is insensitive to global phase.

We choose to take a different approach: we find a way to set the phase of $(s)$ so that the adiabatic evolution closely follows $(s)$ in the $\ell_2$ norm. As it turns out, the correct way to choose the phase is such that for all $s$, $\langle s, h \rangle = 0$; $(s)i = 0$. In the next claim, we show that this is possible. It can be seen that for any unit vector $(s)$, this inner product is a complex number that has zero real part. Intuitively, it indicates the speed by which the phase of the vector rotates.

**Claim 3.1** Let $(s)$ be a time-dependent unit vector in some Hilbert space, such that $(s)$ is a differentiable function of $s$. Then, there is another time-dependent unit vector $(s)$ that is identical to $(s)$ up to phase such that $\langle s, h \rangle = 0$; $(s)i = 0$ and $(0) = (0)$. 

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**Proof:** Write \( (s) = e^{i \cdot s} \) for some function \( s : [0,1] \to \mathbb{R} \). Taking derivative, we have
\[
0 \cdot (s) = e^{i \cdot s} \cdot 0 \cdot (s) + i e^{i \cdot s} \cdot (s) \cdot (s):
\]
Taking the inner product with \( (s) \), we obtain
\[
\langle h, 0 \rangle (s); (s) i = \langle h, 0 \rangle (s); (s) i + i \cdot 0 (s)
\]
since \( \langle h, 0 \rangle (s); (s) i = 1 \). In order to make this expression zero, we choose
\[
(s) = \frac{i}{\langle h, 0 \rangle (t); (t) i t}.
\]

We also need the following technical lemma. Essentially, it says that if \( H \) changes slowly, then so does \( H \).

**Lemma 3.2** Let \( H \) be a time dependent Hamiltonian and let \( (s) \) be an eigenvector whose eigenvalue is 0. Assume that the phase of the eigenvector is chosen such that \( \langle h, 0 \rangle (s); (s) i = 0 \) for all \( 0 \leq s \leq 1 \). Moreover, assume that all other eigenvalues are at least \( |m| \) in absolute value. Then,
\[
\frac{H_{0}^{k}}{k} + \frac{3 H_{0}^{k^{2}}}{2}:
\]
and
\[
\frac{H_{0}^{k}}{k} + \frac{3 H_{0}^{k^{2}}}{2}:
\]

The following is the main result of this section.

**Lemma 3.3** Let \( H \) be a time dependent Hamiltonian and let \( (s) \) be an eigenvector whose eigenvalue is 0 such that \( \langle h, 0 \rangle (s); (s) i = 0 \) and assume that all other eigenvalues are at least \( |m| \) in absolute value. Let \( T \) denote the time along which we apply the Hamiltonian. Then, the following condition is enough to guarantee that an adiabatic evolution starting from \( (0) \) is within distance \( 1000 \) of \( (1) \):
\[
T \leq \frac{\max (k H_{0}^{k}, k H_{0}, k H_{0}^{k})}{k H_{0}^{k}}.
\]

**Proof:** For ease of presentation, we discretize time into infinitesimally small units of size \( 1=L \). One should think of \( L \) as a quantity going to infinity while all other quantities remain constant. We use the \( O() \) notation to describe the asymptotic behavior of an expression as a function of \( L \); all other quantities are regarded as constants, e.g., \( k H_{0}^{k=3}=L = O(L=1) \).

We start by discretizing the adiabatic evolution. Let \( U_{j} = e^{i \cdot \frac{T}{L} \cdot H (j=L)} \)
be the unitary obtained by applying \( H (j=L) \) for \( T=L \) time units. Then the adiabatic evolution is closely approximated by the unitary
\[
U_{L} = (0)^{U}
\]
The error in this approximation goes down to 0 with \( L \) and can therefore be ignored (see, e.g., [12]).

Next, let \( g_j = (j=L) \) be the discretized eigenvectors. In other words, \( g_j \) is the eigenstate with eigenvalue 0 of \( H \) (j=L). The adiabatic evolution starts with \( g_0 = (0) \). Notice that \( U_j g_j = g_j \). Define

\[
\omega_{j+1} = P g_{j+1}^\perp (g_j + \omega_j) + O (1=\sqrt{L})
\]

where \( P g_{j+1}^\perp \) is the projection on the subspace orthogonal to \( g_{j+1} \) (see Figure 8). By taking the Taylor series of \( \omega_j \) about \( (j+1)=L \), we obtain

\[
\omega_j = \frac{0((j+1)=L)}{L} + O (1=\sqrt{L})
\]

By applying \( P g_{j+1}^\perp \) to both sides of the equality, we get

\[
\omega_{j+1} = \frac{0((j+1)=L)}{L} + O (1=\sqrt{L})
\]

where the \( 0((j+1)=L) \) term remains unchanged because we chose \( h^0(s) ; \) \( (s)i = 0 \) for all \( s \) and, in particular, for \( s = (j+1)=L \). By combining the two equations above, we obtain

\[
g_j = g_{j+1} + \omega_{j+1} + O (1=\sqrt{L})
\]

Therefore, we can write

\[
U_L = \sum_{j=1}^{L} (g_j + \omega_j) = g_L + \sum_{j=1}^{L} U_L g_j + \omega_j + O (1=\sqrt{L})
\]

Our goal is to show that the above is very close to \( g_L = (1) \). The term \( O (1=\sqrt{L}) \) is negligible since \( L \) goes to infinity. Therefore, it is enough to show in the following that

\[
\omega_j = \frac{0((j+1)=L)}{L} + O (1=\sqrt{L})
\]

First, according to Lemma 3.2

\[
kw_j = kH L^\frac{k}{L}
\]

Hence, if we try to bound the left side of (4) by a straightforward application of the triangle inequality we obtain a bound of \( kH L^\frac{k}{L} \). In the remainder of the proof we will show how to improve this bound to .

Define \( k(8= )LkH L^\frac{k}{L} (1=\sqrt{L}) \). Notice that \( = O (L) \). We start by partitioning the sum in (4) to sections containing \( k \) terms each. Namely, (4) will follow by showing that for any \( k \),

\[
\omega_j = \frac{0((j+1)=L)}{L} + O (1=\sqrt{L})
\]
For simplicity, let us consider the case $k = 1$; essentially the same proof works for any $k$. So, in the following we will show that

\[ \sum_{j=1}^{L} U_{1j} \sum_{j=1}^{L} w_j : \]

Since $U_{1j} : : U$ are unitary and are applied to every component of this sum, this is equivalent to

\[ \sum_{j=1}^{L} U_{1j} \sum_{j=1}^{L} w_j : \]

Later, we will show that

\[ \sum_{j=1}^{L} U_{1j} \sum_{j=1}^{L} w_j : \]

Assuming (7), we can now complete the proof of the theorem. Let $g$ be any eigenvector of $H$ (1=L) such that $g \notin g_1$ and let $\lambda$ denote its eigenvalue. The corresponding eigenvalue of $g$ in $U_1$ is $e^{i \tau_{1L}}$. Since $g \notin g_1$, we can write

\[ \sum_{j=0}^{L} U_{1j}^\dagger g = \sum_{j=0}^{L} e^{i \tau_{1L}} = \sum_{j=0}^{L} e^{i \tau_{1L}} \sum_{j=0}^{L} \frac{4L}{1j} \sum_{j=0}^{L} \frac{2L}{4L} \]

where we used that $\sum_{j=0}^{L} \frac{1}{j} = 2$ for any small enough $L$. These cancellations in the geometric sum are at the heart of the adiabatic theorem. Using (5), we obtain that

\[ \sum_{j=0}^{L} U_{1j}^\dagger w_1 = \sum_{j=0}^{L} \frac{4L}{1j} \sum_{j=0}^{L} \frac{2L}{4L} \]

where the first inequality follows by writing $w_1$ in the basis of eigenvectors of $H$ (1=L) and recalling that $w_1$ is orthogonal to $g_1$. Combined with (7), this proves (6) and completes the proof of the theorem.

It remains to prove (7). We will prove it in two steps. First, we will show that we can replace all $w_j$'s with $w_1$ and later we will show that we can replace all $U_j$'s with $U_1$.

**Lemma 3.4** For all $j,k$,

\[ kw_j w_k \leq \frac{k}{L^2} \frac{k \theta_k}{L^2} + 3 \frac{k \theta_k^2}{L^2} + O (1=L^2) \]

**Proof:** Using Eq. (5),

\[ w_{j+k} \cdot w_j = \frac{1}{L} \sum_{0}^{L} (j+k) + O (1=L^2) \]

By the mean value theorem and the second claim in Lemma 3.2, the norm of the above is at most

\[ \frac{k}{L^2} \frac{\theta_k}{L^2} + 3 \frac{k \theta_k^2}{L^2} + O (1=L^2) \]

\[ \blacksquare \]
By the above lemma and the triangle inequality, we obtain

\[
X \sum_{j=1}^{k} U_{j}^{k} X \sum_{j=1}^{k} U_{j}^{k} X \sum_{j=1}^{k} U_{j}^{k} U_{1} \sum_{j=1}^{k} U_{j}^{k} k
\]

\[
= X \sum_{j=1}^{k} U_{j}^{k} w_{j} k
\]

\[
\frac{2}{L^2} k L \frac{\delta k}{L^2} + 3 \frac{k H_{0}^2}{L^2} + o \left( L^{-3} \right)
\]

where the last inequality is by our choice of \( T \).

In order to complete the proof, it is enough to show (notice that \( P_{j=0}^{1} U_{j}^{j} = P_{j=0}^{j-1} U_{j}^{j} \)):

\[
X \sum_{j=1}^{k} U_{j}^{k} X \sum_{j=1}^{k} U_{j}^{k} U_{1} \sum_{j=1}^{k} U_{j}^{k} \sqrt{4L} = \sqrt{4L}
\]

(9)

**Lemma 3.5** *For all \( j \)*

\[
k U_{j+1} U_{j} k \frac{T k H_{0}^2}{L^2} + o \left( L^{-3} \right)
\]

**Proof:** Let \( J = H ((j+1)-L) H (j-L) \). Then, using the Trotter formula [9], we can write

\[
U_{j+1} = e^{i \frac{T}{L} H ((j+1)-L)} = e^{i \frac{T}{L} H (j-L)} e^{i \frac{T}{L} H} J + o \left( L^{-3} \right)
\]

where we used \( kJk = o \left( L^{-3} \right) \). Then,

\[
k U_{j+1} U_{j} k = k e^{i \frac{T}{L} H} J k + o \left( L^{-3} \right) = \frac{T}{L} k J k + o \left( L^{-3} \right) \frac{T k H_{0}^2}{L^2} + o \left( L^{-3} \right)
\]

By combining this lemma with the triangle inequality, we obtain that for all \( k \),

\[
k U_{k} U_{1} k \frac{k T k H_{0}^2}{L^2} + o \left( k^{-3} \right)
\]

We now prove Equation 9 using a sequence of 1 triangle inequalities, as illustrated in the following diagram:
That is, we use the triangle inequality to bound the left side of (9) by the sum of 1 terms where the k'th term is given by (notice that all terms not containing $U_k$ cancel and that the unitaries $U_1;\ldots;U_{k+1}$ appear in all remaining terms and can therefore be ignored):

$$
\sum_{j=1}^{\kappa} U_k U_1^{j-1} w_1 \sum_{j=1}^{\kappa} U_k^{j-1} w_1 = (U_k U_1) \sum_{j=1}^{\kappa} U_1^{j-1} w_1
$$

$$
= \sum_{j=1}^{\kappa} k U_k U_1^{j-1} w_1 + \frac{k T k H \theta_k}{L^2} + \mathcal{O}(k=L^3) + \frac{4L}{T} \frac{k H \theta_k}{L} + \frac{4k k H \theta^2}{L^2} + \mathcal{O}(k=L^3)
$$

where the last inequality follows from (5) and an argument similar to the one used after (8). Summing over $k = 1;\ldots;1$ we obtain that the left side of (9) can be upper bounded by:

$$
\frac{10}{2} \frac{k H \theta^2}{L^2} + \frac{4L}{T} \frac{k H \theta_k}{L} + \frac{4k k H \theta^2}{L^2} + \mathcal{O}(k=L^3)
$$

by our choice of $T$.

3.1 Proof of Lemma 3.2

The equality

$$
H(s) (s) = 0
$$

holds for all $0 \leq s \leq 1$. Take the derivative according to $s$,

$$
H^0 (s) (s) + H (s) ^0 (s) = 0:
$$

and by taking the norm we obtain,

$$
k H (s) ^0 (s) k = k H (s) (s) k
$$

On the other hand,

$$
k H (s) ^0 (s) k = k P (s) ^0 (s) k = k ^0 (s) k
$$

where we used the fact that all other eigenvalues of $H(s)$ are at least in absolute value and $P (s)$ denotes the projection on the space orthogonal to $(s)$. By combining the two inequalities, we obtain the first claim. For the second claim, let us consider the derivative of Equation (10)

$$
H^0 (s) (s) + 2 H^0 (s) (s) + H (s) ^0 (s) = 0
$$

and by taking the norm we obtain

$$
k H (s) ^0 (s) k = k P (s) ^0 (s) k + \frac{k H \theta^2}{L^2}
$$

On the other hand we have,

$$
k H (s) ^0 (s) k = k P (s) ^0 (s) k
$$
from which we obtain

$$ kP(s) \delta(s) k = \frac{kH}{2} + \frac{2}{2} kH^2 $$

(11)

Now, by taking the derivative of $h(s); \delta(s)i = 0$,

$$ h(s); \delta(s)i + h(s); \delta(s)i = 0 $$

and hence,

$$ j \ h(s); \delta(s)i j = k (s)k^2 $$

We complete the proof by combining the last equality with (11):

$$ k \ h(s); \delta(s)i k + j \ h(s); \delta(s)i j \ (\text{using the triangle inequality}) $$

$$ k \ h(s); \delta(s)i k + j \ h(s); \delta(s)i j \ (\text{using the first claim}) $$

4 Reducing to a special case

Lemma 4.1 Let $H(s)$ be a time dependent Hamiltonian and let $(s)$ be an eigenvector with eigenvalue $(s)$. Then, for any $0 \leq s \leq 1$,

$$ \delta(s) kH \delta(s) $$

and

$$ \delta(s) kH \delta(s) + 4kH \delta(s) $$

Before proving this lemma, let us see why it implies the main theorem.

Proof of Theorem 2.1 Define the Hamiltonian $H^*(s) = H(s)(I)_T$. Since $H$ and $H^*$ differ by a multiple of the identity, they both describe the same adiabatic evolution up to some global phase. Moreover, by Lemma 4.1

$$ kH \delta(s) $$

and

$$ kH \delta(s) + j \delta(s) + 2kH \delta(s) $$

Hence, according to Lemma 4.3 it is enough to choose $T$ to be at least

$$ \frac{1000}{2} \max \left( \frac{kH \delta(s)^3}{4}; \frac{kH \delta(s) J \delta(s)}{3} \right); \frac{105}{2} \max \left( \frac{kH \delta(s)^3}{4}; \frac{kH \delta(s) J \delta(s)}{3} \right); $$

Proof of Lemma 4.1 Using the Taylor expansion, we can write:

$$ (ds) = (0) + ds \delta(0) + \frac{1}{2} ds^2 \delta(0) + O(ds^3) = ds \delta(0) + \frac{1}{2} ds^2 \delta(0) + O(ds^3) $$

$$ H(ds) = H(0) + dsH \delta(0) + \frac{1}{2} ds^2 H \delta(0) + O(ds^3) $$
We know that
\[ H (ds)j (ds)i = (ds)j (ds)i \]
and by multiplying with \( h (0)j \) we obtain:
\[ h (0)J (ds)j (ds)i = (ds)h (0)j (ds)i \]

With the above equalities, this simplifies to
\[ h (0)J dsH 0 (0) + 1 = 2 ds^2 H 0 (0) + O (ds^3) j (ds)i = (ds 0 (0) + 1 = 2 ds^2 0 (0) + O (ds^3)) h (0)j (ds)i \]
and we can divide by \( ds \):
\[ h (0)J 0 (0) + 1 = 2 ds H 0 (0) + O (ds^2) j (ds)i = (0 (0) + 1 = 2 ds 0 (0) + O (ds^2)) h (0)j (ds)i \]
Since \( h (0)j (ds)i = 1 \ O (ds^2) \) we can hide all error terms inside the \( O (ds^2) \):
\[ h (0)J 0 (0) + 1 = 2 ds H 0 (0) j (0)i + h (0)J dsH 0 (0) j 0 (0)i = (0 (0) + 1 = 2 ds 0 (0) + O (ds^2)) \]

Now, \( (ds) = 0 (0) + ds 0 (0) + O (ds^2) \):
\[ h (0)J 0 (0) + 1 = 2 ds H 0 (0) j (0)i + h (0)J dsH 0 (0) j 0 (0)i = (0 (0) + 1 = 2 ds 0 (0) + O (ds^2)) \]

So by equating the coefficients of the polynomials, we obtain
\[ 0 (0) = h (0)J 0 (0) j (0)i \ kH 0 k \]
and
\[ 0 (0) = h (0)J 0 (0) j (0)i + 2 h (0)J 0 (0) j 0 (0)i \ kH 0 k + 2 kH 0 k 0 k : \]

In order to bound the last expression, define a Hamiltonian \( H (s) = H (s) \ (s)I \). Then, it is clear that \( (s) \) is an eigenvector of \( H (s) \) with eigenvalue \( 0 \) and all other eigenvalues are at least \( 0 \) in absolute value. Therefore, according to Lemma \ref{lemma3.2}
\[ k 0 (s)k \ kH 0 k \ kH 0 k + m ax s j 0 (s) j \ 2 kH 0 k \]
and we obtain
\[ 0 (0) \ kH 0 k + 4 kH 0 k^2 = \]

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