In this paper we consider different classical effects in a model for a scalar field incorporating Lorentz symmetry breaking due to the presence of a single background vector $v^\mu$ coupled to its derivative. We perform an investigation of the interaction energy between stationary steady sources concentrated along parallel branes with an arbitrary number of dimensions, and derive from this study some physical consequences. For the case of the scalar dipole we show the emergence of a nontrivial torque, which is distinctive sign of the Lorentz violation. We also investigate a similar model in the presence of a semi-transparent mirror. For a general relative orientation between the mirror and the $v^\mu$, we are able to perform calculations perturbatively in $v^\mu$ up to second order. We also find results without recourse to approximations for two special cases, that of the mirror and $v^\mu$ being parallel or perpendicular to each other. For all these configurations, the propagator for the scalar field and the interaction force between the mirror and a point-like field source are computed. It is shown that the image method is valid in our model for the Dirichlet’s boundary condition, and we argue that this is a non-trivial result. We also show the emergence of a torque on the mirror depending on its orientation with respect to the Lorentz violating background. This is a new effect with no counterpart in theories with Lorentz symmetry in the presence of mirrors.

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and deserve investigations not only for their theoretical aspects, but also because of their possible relevance in the search for Lorentz symmetry breaking.

In this work, starting from the model studied in [33, 34], we initially consider stationary delta-like currents which are taken to be distributed along parallel $D$-branes, and calculate exactly their interaction energy, deriving from it some interesting particular cases. Afterwards, the same analysis is performed for a distribution of scalar dipoles. Finally, we investigate some consequences in our Lorentz violating model due to the presence of a two dimensional semi-transparent mirror in a $3+1$ dimensional spacetime. The calculations can be performed for a general orientation of the mirror and the background vector if they are treated perturbatively up to second order. Exact results are also obtained for two special cases: when the LV vector has only components parallel to the mirror, and when it has a single component perpendicular to the mirror. For all these configurations, we obtain the propagator for the scalar field and the interaction force between the plate and a point-like field source. We also compare the interaction forces with the ones obtained in the free theory (without the mirror) and we verify that the image method is valid in all the situations considered in this paper for Dirichlet’s boundary condition. This is a nontrivial result since, even if LV in this model clearly preserves the linearity of the equations of motion, the image method also is dependent of the symmetries of the problem, which are modified by the presence of the LV background. We show that a new effect arises when a point-like source is placed in the vicinity of the mirror, namely the existence of a small torque on the mirror, depending on its positioning relative to the background vector. This is an effect due to the Lorentz symmetry breaking, with no counterpart in standard scalar field theory.

The paper is organized as follows: in Section II, we develop a general setup considering effects of the presence of $N$ stationary field sources (scalar charges and dipoles distributions) concentrated at distinct regions of space, for arbitrary dimensions. In Section III, where we have the main results of the paper, we compute, in a $3+1$ spacetime, the propagator for the scalar field in the presence of a semi-transparent mirror considering different configurations for the background vector. We use the results we obtain to study the interaction energy between a point-like scalar charge and the mirror in Section IV. We obtain some new results with no counterpart in the standard Klein-Gordon theory, among them, we highlight a spontaneous torque acting on a setup where the distance between the charge and the mirror is kept as fixed. Section V is dedicated to the conclusions and final remarks.

II. INTERACTION BETWEEN EXTERNAL SOURCES

Along this section we shall deal with a model in $D + D_\perp + 1$ spacetime dimensions, where $D$ will denote the dimensionality of the sources considered, $D_\perp$ will be the number of orthogonal space directions, and the remaining coordinate $x^0$ represents time. It will be convenient to denote by $\mathbf{x}_\perp$ and $\mathbf{x}_\parallel$ the space directions perpendicular and parallel to the sources, so that the position four-vector is given by $x^\mu = (x^0, \mathbf{x}_\perp, \mathbf{x}_\parallel)$. We shall also use similar notations for the momenta $p^\mu$, as well as for any other four-vector whenever necessary. The spacetime metric have signature $(+1, -1, -1, \ldots, -1)$. We shall be dealing with sources represented by delta functions of different dimensionalities (or derivatives of those), thus representing charges evenly distributed on $D$ dimensional branes, in the most broad and general sense. Some particular cases will be considered after general results are obtained. To avoid the problematic case of coinciding sources, we shall always consider that $D_\perp = 1, 2, 3, \ldots$, while $D$ can be any integer, including zero, which would correspond to point-like sources.

Let us consider a massive real scalar field $\phi$ in a Lorentz-symmetry breaking scenario, defined by the following Lagrangian density [33, 34],

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} (v \cdot \partial \phi)^2 + J \phi , \quad (1)$$

where $m$ stands for the scalar field mass, $J$ is the external source and $v^\mu$ is the Lorentz violating background vector which is a dimensionless quantity, assumedly very small.

The specific LV model we consider is mainly motivated by simplicity, which allows to obtain general, and even some exact, results. The Lorentz violating background is parametrized by a single vector coefficient $v^\mu$, which justify the denomination of "aether-like" scalar model used for example in [33, 34]. A general parametrization for LV in a single scalar field theory have recently been proposed in [30], and the model studied by us can be seen as a particular case of the minimal (involving only operators of mass dimension not greater than four) and quadratic LV operator involving the Klein-Gordon field denoted as

$$\mathcal{L}_{LV} = \frac{1}{2} k^c_{\mu\nu} \partial_\mu \phi \partial_\nu \phi , \quad (2)$$

where $k^c_{\mu\nu}$ can be considered to be traceless, since its trace corresponds to a Lorentz invariant correction to the kinetic term which can be eliminated via a redefinition of the field an the parameters of the theory. Our model
then corresponds to the particular choice $k_{\mu\nu} = \nu^\mu\nu^\nu$, thus describing part of the effects of a minimal, CPT even Lorentz violating coefficient $k_{\mu\nu}$. Notice that the tracelessness condition of $k_{\mu\nu}$, in our particular case, is equivalent to $\nu^2 = \nu^\mu\nu_\mu = 0$, which is a condition we can impose without actually modifying any of the results we will present, except for the calculation presented in the Appendix. As a final note, it is known that in a single-field theory, the LV contained in Eq. (2) can actually be eliminated by means of a coordinate choice, absorbing $k_{\mu\nu}$ in the spacetime metric itself \[35, 36\]. However, in a general scenario, involving different fields and interactions among them, this can be done for only one field at a time. This is why it is still important to investigate the consequences of the LV described by Eq. (2), since we can always imagine the scalar field as belonging to a more complicated theory, where we are actually not allowed, or it is not preferred to use this coordinate freedom to eliminate $k_{\mu\nu}$ from the theory.

The free propagator $G_0(x, y)$ is the inverse of the kinetic operator $\mathcal{O}$,

$$\mathcal{O} = \Box + m^2 + (v \cdot \partial)^2,$$  

which can be calculated by standard field theory methods. In the Fourier representation, we can write

$$G_0(x, y) = \int \frac{d^{D+D_\perp+1} p}{(2\pi)^{D+D_\perp+1}} \frac{e^{ip(x-y)}}{[p^2 + (p \cdot v)^2 - m^2]}.$$  

This propagator is the basic ingredient we need to obtain several relevant physical quantities of the model. For example, since the theory is quadratic in the field variables $\phi$, it can be shown that the contribution of the source $J(x)$ to the vacuum energy of the system is given by \[37, 38\]

$$E = \frac{1}{2T} \int \int d^{D+D_\perp+1}x \ d^{D+D_\perp+1}y \ J(x) G_0(x, y) J(y),$$  

where $T$ is a time interval, which is to be taken to the limit $T \to \infty$.

### A. Charges Distributions

As discussed in \[37, 38\], a stationary and uniform scalar charge distribution lying along $D$-dimensional parallel branes can be described by the external source

$$J_I(x) = \sum_{k=1}^{N} \sigma_k \delta^{D_\perp}(x_\perp - a_k),$$  

where we have $N$ fixed $D_\perp$-dimensional spatial vectors $a_k$ labelled by $k = 1, \ldots, N$, and the parameters $\sigma_k$ are the coupling constants between the field and the delta functions, playing the physical role of generalized charge densities on the branes. Substituting (6) into (5), discarding the self-interacting energies, we have

$$E_I = \frac{1}{2T} \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma_k \sigma_l (1 - \delta_{kl}) \int \int d^{D_\perp+1}x \ d^{D_\perp+1}y \ \delta^{D_\perp}(x_\perp - a_k) G_0(x, y) \delta^{D_\perp}(y_\perp - a_l),$$  

where $\delta_{kl}$ is the Kronecker delta. This expression can be simplified by using Eq. (4) and computing the integrals in the following order, $d^{D_\perp}x_\perp, d^{D_\perp}y_\perp, dx^0, d^Dx^\parallel$, then introducing the Fourier representation for the Dirac delta function and integrating in the momenta $dp^0, d^Dp^\parallel$, identifying the time interval as $T = \int dy^0$ and $L^D = \int d^Dx^\parallel$ as being the volume of a given brane, thus, we obtain

$$E_I = \frac{E_I}{L^D} = -\frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma_k \sigma_l (1 - \delta_{kl}) \int \frac{d^{D_\perp}p_\parallel}{(2\pi)^D_\perp} \frac{e^{-ip_\perp \cdot a_k}}{[p_\perp^2 - (v_\perp \cdot p_\perp)^2 + m^2]},$$  

where $a_{kl} = a_k - a_l$ and we have defined $E_I$ as the energy per unit of D-brane volume.

In order to calculate the remaining integral in (8), we need to take into account the relative orientation of the vector $p_\perp = (p^1, \ldots, p^{D_\perp})$ and the spatial components perpendicular to the sources of the Lorentz violating vector, i.e., $v_\perp = (v^1, \ldots, v^{D_\perp})$, hence we split the vector $p_\perp$ into two parts, one parallel and the other normal to $v_\perp$, namely $p_\perp = p_\perp^\parallel + p_\perp^\perp$, where

$$p_\perp^\parallel = v_\perp \left(\frac{v_\perp \cdot p_\perp}{v_\perp^2}\right), \quad p_\perp^\perp = p_\perp - v_\perp \left(\frac{v_\perp \cdot p_\perp}{v_\perp^2}\right),$$  

\[9\]
so that $\mathbf{p}_\perp \cdot \mathbf{v}_\perp = 0$ by construction. Now we define the vector $\mathbf{q}_\perp = (q^1, \ldots, q^{D_\perp})$ as follows,

$$
\mathbf{q}_\perp = \mathbf{p}_\perp + \mathbf{p}_{\perp p} \sqrt{1 - \mathbf{v}_\perp^2}.
$$

With these definitions one may write

$$
\mathbf{p}_\perp = \mathbf{q}_\perp + \frac{(\mathbf{v}_\perp \cdot \mathbf{q}_\perp)}{\mathbf{v}_\perp^2} \left( \frac{1}{\sqrt{1 - \mathbf{v}_\perp^2}} - 1 \right),
$$

leading to

$$
\mathbf{p}_\perp = \mathbf{q}_\perp + \frac{(\mathbf{v}_\perp \cdot \mathbf{q}_\perp)}{\mathbf{v}_\perp^2},
$$

and

$$
\mathbf{q}_\perp^2 = \mathbf{p}_\perp^2 - (\mathbf{v}_\perp \cdot \mathbf{p}_\perp)^2.
$$

Another definition which will be shown to be useful in what follows is

$$
\mathbf{b}_{kl} = \mathbf{a}_{kl} + \left( \frac{1 - \sqrt{1 - \mathbf{v}_\perp^2}}{\sqrt{1 - \mathbf{v}_\perp^2}} \right) \left( \frac{\mathbf{v}_\perp \cdot \mathbf{a}_{kl}}{\mathbf{v}_\perp^2} \right) \mathbf{v}_\perp,
$$

such that

$$
\mathbf{p}_\perp \cdot \mathbf{a}_{kl} = \mathbf{b}_{kl} \cdot \mathbf{q}_\perp.
$$

Finally, the Jacobian of the transformation from $\mathbf{p}$ to $\mathbf{q}$ can be calculated from (11), resulting in

$$
\det \left[ \frac{\partial \mathbf{p}_\perp}{\partial \mathbf{q}_\perp} \right] = \frac{1}{\sqrt{1 - \mathbf{v}_\perp^2}}.
$$

Putting all the previous expressions together, we end up with

$$
\mathcal{E}_I = -\frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma_k \sigma_l (1 - \delta_{kl}) \sqrt{1 - \mathbf{v}_\perp^2} \int \frac{d^{D_\perp} \mathbf{q}_\perp}{(2\pi)^{D_\perp}} \frac{e^{-i\mathbf{q}_\perp \cdot \mathbf{b}_{kl}}}{\mathbf{q}_\perp^2 + m^2},
$$

and now the integral can be solved exactly [37], leading to

$$
\mathcal{E}_I = -\frac{1}{2} \frac{m^{D_\perp - 2}}{(2\pi)^{D_\perp / 2}} \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma_k \sigma_l (1 - \delta_{kl}) (m b_{kl})^{1-(D_\perp/2)} K_{(D_\perp/2)-1} (m b_{kl}),
$$

where $K_n(x)$ stands for the K-Bessel function [39], and

$$
b_{kl} = |\mathbf{b}_{kl}| = \sqrt{\frac{a_{kl}^2 + (\mathbf{v}_\perp \cdot \mathbf{a}_{kl})^2}{1 - \mathbf{v}_\perp^2}}.
$$

Expression (18) is an exact result, which gives the interaction energy per unit of D-brane volume between $N D$-dimensional steady and uniform field sources for the model. As expected, for $v^H = 0$ or $\mathbf{v}_\perp = 0$ expression (18) reduces to the standard Lorentz invariant result obtained in [37]. In the final result, the presence of the LV amounts to the dependence of the energy not on the perpendicular distance between the sources, $a_{kl}$, but on the quantity $b_{kl}$ which depends not only on $a_{kl}$ but also on the orientation of the sources relative to the LV vector $\mathbf{v}_\perp$.

For the massless case, we have to consider separately the cases where $D_\perp = 2$ and $D_\perp \neq 2$. Taking $m = 0$ in (17), the relevant integral is written as

$$
\mathcal{E}_I (m = 0) = -\frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma_k \sigma_l (1 - \delta_{kl}) \sqrt{1 - \mathbf{v}_\perp^2} \int \frac{d^{D_\perp} \mathbf{q}_\perp}{(2\pi)^{D_\perp}} \frac{e^{-i\mathbf{q}_\perp \cdot \mathbf{b}_{kl}}}{\mathbf{q}_\perp^2},
$$
and for $D_\perp \neq 2$ we may directly integrate this expression, by analytic continuation \([37]\), obtaining

$$
\mathcal{E}_I (m = 0, D_\perp \neq 2) = -\frac{2(D_\perp/2)^{3/2}}{(2\pi)^{D_\perp}} \frac{1}{\sqrt{1 - v_\perp^2}} \Gamma \left( \frac{D_\perp}{2} - 1 \right) \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma_k \sigma_l (1 - \delta_{kl})
$$

\begin{equation}
\times \left[ \frac{a_k^2 + (\mathbf{v}_\perp \cdot \mathbf{a}_l)^2}{1 - v_\perp^2} \right]^{1-(D_\perp/2)},
\end{equation}

with $\Gamma (x)$ standing for the Gamma Euler function. For the specific case of $D_\perp = 2$, this last expression is divergent, so a different regularization of the integral \([20]\) is needed. We proceed as in \([15, 37, 38]\), introducing a mass regulator $M$, as follows

$$
\mathcal{E}_I (m = 0, D_\perp = 2) = \lim_{M \to 0} \int \frac{d^2 \mathbf{q}_\perp}{(2\pi)^2} \frac{e^{-i\mathbf{q}_\perp \cdot \mathbf{b}_{kl}}}{\mathbf{q}_\perp^2 + M^2},
$$

so that we can use the integral \([37]\)

\begin{equation}
\int \frac{d^2 \mathbf{q}_\perp}{(2\pi)^2} \frac{e^{-i\mathbf{q}_\perp \cdot \mathbf{b}_{kl}}}{\mathbf{q}_\perp^2 + M^2} = \frac{1}{2\pi} K_0 (Mb_{kl}) ,
\end{equation}

as well as the asymptotic expression of the Bessel function for small arguments,

$$
-K_0 (Mb_{kl}) = \ln \left( \frac{Mb_{kl}}{2} \right) + \gamma ,
$$

\begin{equation}
= \ln \left( \frac{b_{kl}}{a_0} \right) - \ln 2 + \gamma + \ln (Ma_0) ,
\end{equation}

where $\gamma$ is the Euler constant and $a_0$ is an arbitrary constant length scale. Terms that not depend on the distances $a_{kl}$ (via $b_{kl}$, see Eq. \([19]\)) do not contribute to the force between the point-like currents, so they can be discarded. We therefore arrive at

$$
\mathcal{E}_I (m = 0, D_\perp = 2) = \frac{1}{4\pi \sqrt{1 - v_\perp^2}} \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma_k \sigma_l (1 - \delta_{kl}) \ln \left( \frac{b_{kl}}{a_0} \right) .
$$

Notice that in these manipulations, we exchanged the dependence on the arbitrary regulating mass $M$ for a regulating length $a_0$. Despite explicitly appearing in Eq. \([26]\) to keep the argument of the logarithm dimensionless, $a_0$ does not appear in derivatives of the energy, so it will not have any physical impacts.

In order to gain further insights and clarify the effects of the anisotropies generated by the Lorentz-symmetry breaking, we will now consider some examples derived from our general calculations. So, from now on we fix the dimensionality of spacetime to be $3 + 1$, and the number of sources to be $N = 2$. When $D_\perp = 3, D = 0$ we have two point-like sources in $3 + 1$ dimensions, and the energy \([18]\) becomes

$$
\mathcal{E}_I (D_\perp = 3, D = 0, N = 2) = -\frac{\sigma_1 \sigma_2}{4\pi \sqrt{1 - v^2}} e^{-mb} \frac{e^{-mb}}{b} ,
$$

where we discarded the sub-index $\perp$ for simplicity, and

$$
b = b_{12} = b_{21} = \sqrt{a^2 + (\mathbf{v} \cdot \mathbf{a})^2} = \sqrt{a^2 + (\mathbf{v} \cdot \mathbf{a})^2}.
$$

If $\mathbf{v} = 0$, the expression \([27]\) reduces to the well-known Yukawa potential, otherwise the factor proportional to $(\mathbf{v} \cdot \mathbf{a})^2$ in the definition of $b$ in \([28]\) implies in a dependence of the energy on the relative orientation of the two charges and the LV background. As a noteworthy particular case, if the distance vector $\mathbf{a}$ is perpendicular to the background vector $\mathbf{v}$, Eq. \([27]\) becomes

$$
\mathcal{E}_I (D_\perp = 3, D = 0, N = 2, \mathbf{v} \cdot \mathbf{a} = 0) = -\frac{\sigma_1 \sigma_2}{4\pi \sqrt{1 - v^2}} e^{-m|\mathbf{a}|} \frac{e^{-m|\mathbf{a}|}}{|\mathbf{a}|} .
$$
In this case the coefficient $1/\sqrt{1-\mathbf{v}^2}$ can be absorbed into the definition of the coupling constants $\sigma_1$ and $\sigma_2$, and Eq. (29) becomes the Yukawa potential with an effective coupling constant $\sigma \rightarrow \sigma (1-\mathbf{v}^2)^{-1/4}$.

Another interesting limit is the massless one, when we obtain

$$\mathcal{E}_I (D_\perp = 3, D = 0, N = 2, m = 0) = -\frac{\sigma_1 \sigma_2}{4\pi \sqrt{1-\mathbf{v}^2}} \left[ \mathbf{a}^2 + \frac{(\mathbf{v}_\perp \cdot \mathbf{a})^2}{1-\mathbf{v}^2} \right]^{-1/2}. \quad (30)$$

Now, if $\mathbf{v} = 0$, this reduces to the well-known Coulombian interaction with an overall minus signal, which is expected for the scalar field, in comparison with the electromagnetic one [37].

The force between two point-like scalar charges can be calculated from Eqs. (27) and (28), resulting in

$$\mathbf{F}_I (D_\perp = 3, D = 0, N = 2) = -\nabla \mathcal{E}_I (D_\perp = 3, D = 0, N = 2)$$

$$= -\frac{\sigma_1 \sigma_2}{4\pi \sqrt{1-\mathbf{v}^2}} \frac{e^{-mb}}{b^2} \left( m + \frac{1}{b} \right) \left[ \mathbf{a} + (\mathbf{v} \cdot \mathbf{a}) \mathbf{v} \right], \quad (31)$$

which depends on the direction of the background vector. When $m = 0$, the interaction force can be written in the following way

$$\mathbf{F}_I (D_\perp = 3, D = 0, N = 2, m = 0) = -\frac{\sigma_1 \sigma_2}{4\pi \mathbf{a}^2} \frac{1}{2} \left( \frac{1}{\sqrt{1-\mathbf{v}^2}} + \frac{3}{2} (\mathbf{v} \cdot \mathbf{a})^2 \right) \hat{a} + (\mathbf{v} \cdot \hat{a}) \mathbf{v}, \quad (32)$$

where $\hat{a}$ is an unit vector which points on the direction of the distance vector $\mathbf{a}$.

Notice that (32) is an anisotropic force that decays with the inverse square of the distance. In the special situation where $\mathbf{v}$ and $\hat{a}$ are perpendicular to each other, the force (32) becomes a Coulombian-like interaction with effective coupling constants $\sigma \rightarrow \sigma (1-\mathbf{v}^2)^{-1/4}$. Since $v$ is a small quantity, it is relevant to expand expression (32) in the lowest order in $v\mu$,

$$\mathbf{F}_I (D_\perp = 3, D = 0, N = 2, m = 0) \approx -\frac{\sigma_1 \sigma_2}{4\pi} \frac{1}{\mathbf{a}^2} \left[ \left( 1 + \frac{1}{2} \mathbf{v}^2 - \frac{3}{2} (\mathbf{v} \cdot \hat{a})^2 \right) \hat{a} + (\mathbf{v} \cdot \hat{a}) \mathbf{v} \right]. \quad (33)$$

The first term inside the brackets is proportional to $\hat{a}$, is a force in the same direction of the Lorentz invariant case, but modulated by a function of the angle between $\mathbf{a}$ and $\mathbf{v}$, the second term, however, is a new contribution proportional to the LV vector $\mathbf{v}$ itself.

An interesting consequence of the anisotropy in the interaction energy (27) is the emergence of an spontaneous torque on a scalar dipole, depending on its orientation relative to the LV background. To see this, we consider a typical scalar dipole composed by two opposite coupling constants $\sigma_1 = -\sigma_2 = \sigma$, placed at positions $\mathbf{a}_1 = \mathbf{R} + \frac{A}{2}$ and $\mathbf{a}_2 = \mathbf{R} - \frac{A}{2}$, $\mathbf{A}$ taken to be a fixed vector. From Eq. (27), we obtain

$$\mathcal{E}_I^{\text{dipole}} (D_\perp = 3, D = 0, N = 2) = \frac{\sigma^2}{4\pi \sqrt{1-\mathbf{v}^2}} e^{-m|\mathbf{A}|f(\theta)} |\mathbf{A}| f(\theta), \quad (34)$$

where

$$f(\theta) = \sqrt{1 + \frac{\mathbf{v}^2 \cos^2(\theta)}{1-\mathbf{v}^2}}, \quad (35)$$

and $0 \leq \theta \leq \pi$ stands for the angle between $\mathbf{A}$ and the background vector $\mathbf{v}$. This interaction energy leads to an spontaneous torque on the dipole as follows,

$$\tau_I^{\text{dipole}} (D_\perp = 3, D = 0, N = 2) = -\frac{\partial \mathcal{E}_I^{\text{dipole}} (D_\perp = 3, D = 0, N = 2)}{\partial \theta}$$

$$= -\frac{\sigma^2}{8\pi |\mathbf{A}| (1-\mathbf{v}^2)^{3/2}} \frac{1}{f^2(\theta)} \left( m |\mathbf{A}| + \frac{1}{f(\theta)} \right) \sin(2\theta) e^{-m|\mathbf{A}|f(\theta)}. \quad (36)$$

This spontaneous torque on the scalar dipole is an exclusive effect due to the Lorentz violating background. A similar effect was also described in electrodynamics modified with a CPT even LV term (author?) [15] and in a
non-minimal (higher derivative) LV modification in [40]. If \( v^\mu = 0 \) (or, more specifically, \( v_\perp = 0 \)), the torque vanishes, as it should, as well as for the specific configurations \( \theta = 0, \pi/2, \pi \). For the massless case the torque becomes

\[
\tau^\text{dipole}_{I} (D_\perp = 3, D = 0, N = 2, m = 0) = -\frac{\sigma^2}{8\pi |A|} \frac{v^2 \sin(2\theta)}{(1 - v^2 \sin^2(\theta))^{3/2}} \]

\[
\cong -\frac{\sigma^2 v^2}{8\pi |A|} \sin(2\theta),
\]

which exhibits a maximum value at \( \theta = \pi/4 \).

The final examples we consider involve one and two dimensional sources, i.e., strings and planes. For \( D_\perp = 2, D = 1 \) and \( N = 2 \) we have two delta-like scalar charges distributions concentrated along two different parallel strings placed at a distance \( a \) from each other. In this case, from Eq. (18) the energy per string length reads

\[
\mathcal{E}_I (D_\perp = 2, D = 1, N = 2) = -\frac{\sigma_1 \sigma_2}{2\pi \sqrt{1 - v_\perp^2}} K_0 (mb),
\]

which is reduced, in the case \( m = 0 \), to the expression

\[
\mathcal{E}_I (D_\perp = 2, D = 1, N = 2, m = 0) = -\frac{\sigma_1 \sigma_2}{2\pi \sqrt{1 - v_\perp^2}} \ln \left( \frac{b}{a_0} \right),
\]

where we used (26).

Finally, for \( D_\perp = 1, D = 2 \) and \( N = 2 \), corresponding to two delta currents concentrated on parallel planes, we have

\[
\mathcal{E}_I (D_\perp = 1, D = 2, N = 2) = -\frac{\sigma_1 \sigma_2}{2m \sqrt{1 - v_\perp^2}} e^{-mb},
\]

or, in the massless limit,

\[
\mathcal{E}_I (D_\perp = 1, D = 2, N = 2, m = 0) = \frac{\sigma_1 \sigma_2}{2\sqrt{1 - v_\perp^2}} \sqrt{\frac{a^2 + \left( \frac{v_\perp \cdot a}{1 - v_\perp^2} \right)^2}{1 - v_\perp^2}}.
\]

**B. Point-like Dipoles**

The technique developed in this section can be applied to other interesting systems, such as dipole distributions, when the relevant currents involve derivatives of delta functions. In this subsection we provide some results in the case of two steady point-like dipoles placed at fixed points in 3 + 1 dimensions. This setup can be described by external sources given by the directional derivatives of the Dirac delta function [37], as follows

\[
J_{II} (x) = \sum_{k=1}^{2} V_{(k)} \cdot \nabla \left[ \delta^4 (x - a_k) \right],
\]

where \( V_\mu (k) \) designates the dipole moments 1 and 2, taken to be fixed in the reference frame we are performing the calculations. Following the same steps presented in the previous section, we obtain for the interaction energy between the two scalar dipoles

\[
E_{II} = -\int \frac{d^3 p_\perp}{(2\pi)^3} e^{-ip_\perp \cdot a} \frac{(V_\perp (1) \cdot p_\perp) (V_\perp (2) \cdot p_\perp)}{p_\perp^2 - (v \cdot p_\perp)^2 + m^2},
\]

where we defined \( a = a_1 - a_2 = a_{12} \).

Performing the same change in the integration variables as used in the previous section, using the definition (14)
of the vector $b_{kl}$ and making some manipulations we end up with

\[ E_{II} = \frac{1}{4\pi \sqrt{1-v^2}} e^{-mb} \left[ \frac{(mb)^2 + 3(mb + 1)}{b^2} \right] \left[ (V_{(1)} \cdot a) (V_{(2)} \cdot a) + \frac{(v \cdot a)}{1-v^2} \left[ (V_{(1)} \cdot a) (V_{(2)} \cdot v) + (V_{(2)} \cdot a) (V_{(1)} \cdot v) \right] \right. 
\[ \left. + \left( \frac{v \cdot a}{1-v^2} \right)^2 (V_{(1)} \cdot v) (V_{(2)} \cdot v) \right] - (mb + 1) \left[ (V_{(1)} \cdot V_{(2)}) + \frac{(V_{(1)} \cdot v) (V_{(2)} \cdot v)}{1-v^2} \right] \right]. \tag{45} \]

In the massless case, we can use the definition \[14\] and write

\[ E_{II}(m = 0) = \frac{1}{4\pi \sqrt{1-v^2}} \left[ a^2 + \frac{(v \cdot a)^2}{1-v^2} \right]^{-3/2} \left\{ \frac{3}{a^2 + \frac{(v \cdot a)^2}{1-v^2}} \right\} \times \left[ (V_{(1)} \cdot a) (V_{(2)} \cdot a) + \frac{(v \cdot a)}{1-v^2} \left[ (V_{(1)} \cdot a) (V_{(2)} \cdot v) + (V_{(2)} \cdot a) (V_{(1)} \cdot v) \right] \right. 
\[ \left. + \left( \frac{v \cdot a}{1-v^2} \right)^2 (V_{(1)} \cdot v) (V_{(2)} \cdot v) \right] - (V_{(1)} \cdot V_{(2)}) - \frac{(V_{(1)} \cdot v) (V_{(2)} \cdot v)}{1-v^2} \right\}. \tag{46} \]

For the case where $v = 0$ or $v = 0$, we have the well-known result obtained in standard scalar field theory \[37\],

\[ E_{II}(m = v = 0) = \frac{\sigma_1 \sigma_2}{4\pi |a|^3} \left[ 3 \left( \frac{V_{(1)} \cdot a\cdot V_{(2)} \cdot a}{a^2} \right) - (V_{(1)} \cdot V_{(2)}) \right]. \tag{47} \]

Different particular cases can be analyzed, and torques depending on the orientation of the dipoles relative to the LV background can be deduced. Since these results follow directly from the approach outlined in the previous subsection, we will not quote the explicit expressions here.

### III. THE PROPAGATOR IN THE PRESENCE OF A SEMI-TRANSPARENT MIRROR

In this section we compute the propagator for the model \[11\] in the presence of a two-dimensional semi-transparent mirror for the model. We keep spacetime $3 + 1$ dimensional hereafter, and take a coordinate system where the mirror is perpendicular to the $x^3$ axis, located on the plane $x^3 = 0$. Its partial transparency is described by the potential $\frac{\mu}{2} \delta(x^3)$, where $\mu > 0$ is a coupling constant with appropriate dimension, establishing the degree of transparency of the mirror: the limit $\mu \to \infty$ corresponds to a perfect mirror \[41\]. Therefore, the Lagrangian density is given by

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} (v \cdot \partial \phi)^2 - \frac{1}{2} \mu \delta(x^3) \phi^2 + J \phi. \tag{48} \]

The propagator for this theory, $G(x, y)$, satisfies the differential equation

\[ \left[ \square + m^2 + (v \cdot \partial)^2 + \mu \delta(x^3) \right] G(x, y) = -\delta^4(x - y), \tag{49} \]

and also a kind of Bethe-Salpeter equation

\[ G(x, y) = G_0(x, y) + \int d^4 z \ G(x, z) \mu \delta(z^3) G_0(z, y), \tag{50} \]

where $G_0(x, y)$ is the free propagator given by the Eq. \[4\], which solves \[49\] without the potential. From now on, we define $x^\mu_p = (x^0, x^1, x^2)$ and $p^\mu_p = (p^0, p^1, p^2)$ as the coordinates and momentum parallel to the mirror, respectively.
In order to solve Eq. (49), it is convenient to write \( G(x, y) \) and \( G_0(x, y) \) as Fourier transforms in the parallel coordinates, as follows,

\[
G(x, y) = \int \frac{d^3p}{(2\pi)^3} e^{ip_x(x - x_p) + ip_y(y - y_p)} G(p; x^3, y^3), \tag{51a}
\]

\[
G_0(x, y) = \int \frac{d^3p}{(2\pi)^3} e^{ip_x(x - x_p) + ip_y(y - y_p)} G_0(p; x^3, y^3), \tag{51b}
\]

where \( G(p; x^3, y^3) \) and \( G_0(p; x^3, y^3) \) stand for the reduced Green’s functions \(^{[41, 42]}\). Substituting (51) in (50) and performing some manipulations we arrive at

\[
G(p; x^3, y^3) = G_0(p; x^3, y^3) + \mu G(p; x^3, 0) G_0(p; 0, y^3). \tag{52}
\]

Setting \( y^3 = 0 \) in (52), we can obtain \( G(p; x^3, 0) \) strictly in terms of \( G_0(p; x^3, 0) \). Using the result back again in Eq. (52), we obtain

\[
G(p; x^3, y^3) = G_0(p; x^3, y^3) + \frac{\mu G_0(p; x^3, 0) G_0(p; 0, y^3)}{1 - \mu G_0(p; 0, 0)}. \tag{53}
\]

Substituting (53) in equation (51) leads to

\[
G(x, y) = G_0(x, y) + \tilde{G}(x, y), \tag{54}
\]

where

\[
\tilde{G}(x, y) = \int \frac{d^3p}{(2\pi)^3} e^{ip_x(x - x_p) + ip_y(y - y_p)} \frac{\mu G_0(p; x^3, 0) G_0(p; 0, y^3)}{1 - \mu G_0(p; 0, 0)}. \tag{55}
\]

The propagator (54) is composed of the sum of the free propagator (4) with the correction (55), which accounts for the presence of the semi-transparent mirror. From now on, we will calculate \( G(x, y) \) for different configurations of the mirror with respect to the background vector.

### A. The propagator in the lowest order in \( v \)

Since \( v^3 \) is assumedly a very small parameter, we will perform the calculations perturbatively up to the second order in \( v^3 \), which is the lowest order in which the background vector appears non-trivially. Expanding the propagator (4), we obtain

\[
G_0(x, y) = \int \frac{d^4p}{(2\pi)^4} e^{ip_x(x - x_p) + ip_y(y - y_p)} \left[ 1 - \frac{(p \cdot v)^2}{(p^2 - m^2)} \right]. \tag{56}
\]

Splitting \( G_0(x, y) \) into parallel and perpendicular coordinates with respect to the mirror, we have

\[
G_0(x, y) = \int \frac{d^3p}{(2\pi)^3} e^{ip_x(x - x_p) + ip_y(y - y_p)} \left[ \int \frac{dp^3}{2\pi} e^{-ip^3(x^3 - y^3)} \left( 1 - \frac{(p \cdot v)^2}{(p^2 - m^2)} \right) \right], \tag{57}
\]

where \( p^3 \) stands for the momentum perpendicular to the mirror. From Eq. (51), we identify the term between brackets on the right hand side of Eq. (57) as being \( G_0(p; x^3, y^3) \).

Using the fact that \(^{[41]}\)

\[
\int \frac{dp^3}{2\pi} e^{-ip^3(x^3 - y^3)} = -\frac{e^{-\lambda|x^3 - y^3|}}{2\lambda}, \tag{58}
\]

where \( \lambda = \sqrt{m^2 - p^2} \), we are taken to,

\[
G_0(p; x^3, y^3) = -\frac{e^{-\lambda|x^3 - y^3|}}{2\lambda} \left\{ 1 + \frac{1}{2} \left[ \frac{(p_p \cdot v_p)^2}{\lambda^2} (1 + \lambda |x^3 - y^3|) 
- 2iv^3 (x^3 - y^3)(p_p \cdot v_p) + (v^3)^2 (1 - \lambda |x^3 - y^3|) \right] \right\}, \tag{59}
\]
with \(v_\mu = (v^0, v^1, v^2)\) and \(v^3\) standing for the background vector parallel and perpendicular to the mirror respectively. Substitution of this last expression into Eq. [55], and taking into account only contributions up to second order in \(v^\mu\), leads to

\[
\tilde{G}(x, y) = \int \frac{d^3p_p}{(2\pi)^3} e^{ip_p \cdot (x_p - y_p)} \left\{ 1 + \frac{(p_p \cdot v_p)^2}{2\lambda^2} \left[ \left( \frac{4\lambda + \mu}{2\lambda + \mu} \right) + \lambda(|x^3| + |y^3|) \right] \right. \\
\left. - i\sqrt{2}v^3(x^3 - y^3)(p_p \cdot v_p) + \left( \frac{v^3|^2}{2} \left[ \left( \frac{4\lambda + \mu}{2\lambda + \mu} \right) - \lambda(|x^3| + |y^3|) \right] \right) \right\} \frac{\mu e^{-\lambda(|x^3| + |y^3|)}}{2\lambda(2\lambda + \mu)}. \tag{60}
\]

As expected in this perturbative result, the limit \(v^\mu \to 0\) correctly reproduces the standard result for the scalar field theory in the presence of a semi-transparent mirror [41].

### B. Exact propagators

There are two special cases for which we carry out the calculations without the necessity of treating the background vector perturbatively, corresponding to the spacial part of \(v^\mu\) being parallel and perpendicular to the mirror. In this subsection we obtain the exact propagator in the presence of a semi-transparent mirror in these cases.

When the component of the background vector perpendicular to the mirror is equal to zero \((v^3 = 0)\), we have (see the Appendix)

\[
G_0(p_p; x^3, y^3) = -\frac{e^{-L|x^3 - y^3|}}{2L}, \tag{61}
\]

where \(L = \sqrt{m^2 - [p_p^2 + (p_p \cdot v_p)^2]}\). Substituting (61) in (55), we arrive at

\[
\tilde{G}(x, y) = \int \frac{d^3p_p}{(2\pi)^3} e^{ip_p \cdot (x_p - y_p)} \frac{\mu e^{-L(|x^3| + |y^3|)}}{2L(2L + \mu)}. \tag{62}
\]

On the other hand, when \(v_p^3 = 0\) and \(v^3 \neq 0\), we can write (see the Appendix)

\[
G_0(p_p; x^3, y^3) = -\frac{e^{-\lambda\sqrt{(1 - (v^3)^2)^{-1}}} |x^3 - y^3|}}{2\lambda\sqrt{1 - (v^3)^2}}, \tag{63}
\]

which leads to

\[
\tilde{G}(x, y) = \int \frac{d^3p_p}{(2\pi)^3} e^{ip_p \cdot (x_p - y_p)} \frac{\mu e^{-\lambda\sqrt{(1 - (v^3)^2)^{-1}}} (|x^3| + |y^3|)}}{2\lambda\sqrt{1 - (v^3)^2} \left( \frac{2\lambda\sqrt{1 - (v^3)^2} + \mu} \right)}. \tag{64}
\]

It is easy to see that these expressions reproduce the result previously obtained when expanded up to the second order in \(v^\mu\).

### IV. CHARGE-MIRROR INTERACTION

Having calculated the relevant propagator in the previous section, here we consider the interaction energy between a point-like current and the semi-transparent mirror, which is given by [41]

\[
E = \frac{1}{2T} \int \int d^4x \ d^4y \ J(x) \tilde{G}(x, y) J(y). \tag{65}
\]

Without loss of generality (due to translation invariance in the directions parallel to the mirror) and for simplicity, we choose a point-like scalar charge placed at \(a = (0, 0, a)\), corresponding to the source \(J(x) = \sigma \delta^4(x - a)\). Again, we will present a result perturbative in \(v\) for the general case, and also exact results for particular cases.
A. Perturbative results

Expanding the expressions up to second order of \( v \), following the same steps presented in the previous sections, we obtain

\[
E_{MC} = \frac{\mu \sigma^2}{8\pi^2} \int d^2 p \left\{ 1 + \left( \frac{p_p \cdot v_p}{2 (p_p^2 + m^2)} \right) \left( \frac{4 \sqrt{p_p^2 + m^2 + \mu}}{2 \sqrt{p_p^2 + m^2 + \mu}} + 2a \sqrt{p_p^2 + m^2} \right) \right. \\
+ \left( \frac{v^3}{2} \right)^2 \left[ \frac{4 \sqrt{p_p^2 + m^2 + \mu}}{2 \sqrt{p_p^2 + m^2 + \mu}} - 2a \sqrt{p_p^2 + m^2} \right] \left( \frac{4 \sqrt{p_p^2 + m^2 + \mu}}{2 \sqrt{p_p^2 + m^2 + \mu}} + \frac{2a \sqrt{p_p^2 + m^2}}{2 \sqrt{p_p^2 + m^2 + \mu}} \right) \right\},
\]

where \( a > 0 \) is the distance between the mirror and the charge. The sub-index \( MC \) means that we have the interaction energy between the mirror and the charge.

Equation (66) can be simplified by using polar coordinates, integrating out in the solid angle and performing the change of integration variable \( p \to \frac{y}{y + \mu} = \frac{2}{m} \left( \frac{y}{y + \mu} - ay \right) \).

The relevant integrals can be found in [43],

\[
\int_2^\infty dy \frac{e^{-ay}}{(y + \mu)} = e^{\mu a} Ei (1, 2ma + \mu) ,
\]

\[
\int_2^\infty dy \frac{e^{-ay}}{y^2 (y + \mu)} \left( \frac{y}{y + \mu} \right) = \frac{1}{2} e^{\mu a} Ei (1, 2ma + \mu) ,
\]

and

\[
\int_2^\infty dy \frac{e^{-ay}}{(y + \mu)} \left( \frac{(2y + \mu)}{(y + \mu)} - ay \right) = 2 \left[ (\mu a + 1) e^{\mu a} Ei (1, 2ma + \mu) - \frac{m + \mu}{(2m + \mu)} e^{-2ma} \right] ,
\]

where \( Ei (u, s) \) is the exponential integral function [39], and therefore the interaction energy reads

\[
E_{MC} = \frac{\mu \sigma^2}{16\pi} \left\{ e^{\mu a} Ei (1, 2ma + \mu) + \frac{v^2}{2} e^{\mu a} Ei (1, 2ma + \mu) \right. \\
+ \left( \frac{v^3}{2} \right)^2 \left[ (\mu a + 1) e^{\mu a} Ei (1, 2ma + \mu) - \frac{(m + \mu)}{(2m + \mu)} e^{-2ma} \right] \right\}.
\]

This is a perturbative result and gives the interaction energy between a point-like scalar charge and a semi-transparent mirror in the massive case. The first term on the right hand side reproduces the result of the standard (Lorentz invariant) Klein-Gordon field [11], the remaining terms are corrections due to the Lorentz symmetry breaking.

From the energy (71), we derive two kinds of physical phenomena. The first one is a force between the mirror and the charge,

\[
F_{MC} = -\frac{\partial E_{MC}}{\partial a} = -\frac{\mu \sigma^2}{16\pi a} \left[ 1 + \frac{v^2}{2} \right] \left( \mu a e^{\mu a} Ei (1, 2ma + \mu) - e^{-2ma} \right) \\
+ \left( \frac{v^3}{2} \right)^2 \left( (2 + \mu a) e^{\mu a} Ei (1, 2ma + \mu) - (\mu a + 1) e^{-2ma} + \frac{m + \mu}{2m + \mu} e^{-2ma} \right) ,
\]

and
which is always attractive, provided that \( v^2_p (v^3)^2 << 1 \).

Let us define the following dimensionless functions,

\[
F_p(x, y) = \frac{x}{2} \left[ e^{-2y} - xe^x Ei(1, 2y + x) \right],
\]

\[
F_3(x, y) = x \left[ (x + 1)e^{-2y} - (x + 2) xe^x Ei(1, y + x) - 2 \frac{y + x}{2y + x} ye^{-2y} \right],
\]

and rewrite the force (72) in the form

\[
F_{MC} = \sigma^2 \frac{v^2}{16\pi a^2} \left[ \frac{1}{a} \right] \left( \mu a \left( e^{-2\mu a} - \mu a e^{\mu a} Ei(1, 2\mu a + \mu a) \right) + \frac{v^2_p}{2} F_p(\mu a, ma) + (v^3)^2 F_3(\mu a, ma) \right],
\]

where we have a Coulombian behavior modulated by the expression inside brackets. The correction due to the Lorentz symmetry breaking is given by the functions \( F_p \) and \( F_3 \), the first one is associated with the components of the background vector parallel to the mirror and the second one, with the component perpendicular to the mirror. \( F_p \) is positive along its domain and \( F_3 \) assume positive and negative values, as shown in the Figures 1 and 2. Both functions vanishes in the limit \( \mu = 0 \), where we have no mirror present.

**Figure 1:** Function \( F_p \), appearing in the force described in Eq. (76), where the vertical axis is in arbitrary units.

The second phenomena is obtained when we fix the distance between the charge and the mirror and vary the orientation of the whole system with respect to the background vector. In this case, we can show that a torque emerges on the system. In order to calculate this torque, we define as \( 0 \leq \alpha \leq \pi \) the angle between the normal to the mirror (\( \hat{x}_3 \)) and the background vector, in such a way that

\[
(v^3)^2 = v^2 \cos^2(\alpha), \quad v^2_p = v^2 \sin^2(\alpha),
\]

then the torque can be computed from Eq. (71) computed as follows,

\[
\tau_{MC} = -\frac{\partial E_{MC}}{\partial \alpha} = -\frac{\mu a \sigma v^2}{16\pi} \sin(2\alpha) \left[ \left( \mu a + \frac{1}{2} \right) e^{\mu a} Ei(1, 2\mu a + \mu a) - \frac{(m + \mu)}{(2m + \mu)} e^{-2ma} \right].
\]

Equation (78) is a new effect with no counterpart in standard scalar field theory in the presence of a semi-transparent mirror. Defining the function

\[
T(x, y) = x \left[ \frac{y + x}{(2y + x)} e^{-2y} - \left( x + \frac{1}{2} \right) e^x Ei(1, 2y + x) \right],
\]

we can write Eq. (78) in the form

\[
\tau_{MC} = \frac{\sigma^2 v^2}{16\pi a} \sin(2\alpha) T(\mu a, ma).
\]
In Figure 3, we show the behavior of the function $T$ in terms of $\mu a$ and $ma$. The function is positive except in a very small region around $\mu a = ma = 0$, and goes to zero if $ma$ is large or $\mu a$ approaches zero. This behavior can also be seen from the graphic, where we have three plots, with three different values for the mass, in the vicinity of $\mu a = 0$. In the limit $\mu a \to 0$, the function (79) vanishes, as expected. This torque and the force modulation contained in Eq. (76) are phenomenological signatures of the Lorentz violation in our model, and might be relevant in experimental setups involving mirrors.

For the massless case the energy (71) becomes

$$E_{MC} (m = 0) = \frac{\mu \sigma^2}{16\pi} \left\{ e^{\mu a} Ei(1, \mu a) + \frac{v^2}{2} e^{\mu a} Ei(1, \mu a) + (v^3)^2 [(\mu a + 1) e^{\mu a} Ei(1, \mu a) - 1] \right\}. \quad (81)$$

A particular case of interest is the limit $\mu \to \infty$, corresponding physically to the field subjected to Dirichlet boundary
conditions in the plane. In this limit, we have a perfect two-dimensional mirror and, from Eq. (71), we obtain

\[ E_{MC}(\mu \to \infty) = \frac{\sigma^2 e^{-2ma}}{16\pi a} \left( 1 + \frac{v^2}{2} - ma(v^3)^2 \right). \] (82)

The first term on the right hand side is the three-dimensional Yukawa potential between two charges at a distance 2a apart. The second and third terms are corrections due to the Lorentz symmetry breaking up to second order in \( v^\mu \).

The corresponding interaction force between the point-like charge and the perfect mirror is given by

\[ F_{MC}(\mu \to \infty) = -\frac{\partial E_{MC}(\mu \to \infty)}{\partial a} = \frac{\sigma^2 e^{-2ma}}{8\pi a} \left[ \left( 1 + \frac{v^2}{2} \right) \left( m + \frac{1}{2a} \right) - m^2 a(v^3)^2 \right]. \] (83)

In Eq. (83) we have the interaction force between two point-like scalar charges for the model (1). Expanding this expression up to second order in \( v^\mu \), we can obtain the interaction force for the special case where we have two opposite point-like charges, \( \sigma_1 = \sigma \) and \( \sigma_2 = -\sigma \), placed at a distance 2a apart. In this specific situation, this force turns out to be equivalent to Eq. (83). The interesting conclusion is that the image method is valid for the Lorentz violation theory (1) up to second order in \( v^\mu \) for the Dirichlet boundary condition.

Taking the limit when \( \mu \to \infty \) in Eq. (81) or equivalently putting \( m = 0 \) in (82), we obtain the interaction energy between a point charge and a perfect mirror for the massless scalar field, and consequently the interaction force,

\[ F_{MC}(\mu \to \infty, m = 0) = \frac{\sigma^2}{16\pi a^2} \left( 1 + \frac{v^2}{2} \right). \] (84)

Equation (84) is the usual Coulombian force with an overall minus sign between the scalar charge and its image, placed at a distance 2a apart. With the same analysis, one can argue that Eq. (84) is in agreement with Eq. (33), and again the validity of the image method is verified. In the same limit, from Eq. (78), we have

\[ \tau_{MC}(\mu \to \infty, m = 0) = -\frac{\partial E_{MC}(\mu \to \infty, m = 0)}{\partial \alpha} = \frac{\sigma^2 v^2}{32\pi a} \sin(2\alpha). \] (85)

When \( \alpha = 0, \pi/2, \pi \), corresponding to the mirror being parallel, perpendicular or antiparallel to the background vector \( v \), the torque (85) vanishes. The configurations \( \alpha = 0, \pi \) are stable equilibrium situations, while for \( \alpha = \pi/2 \) we have an unstable equilibrium point. When \( \alpha = \pi/4, 3\pi/4 \), the torque (85) exhibits its maximum and minimum values, respectively. The equilibrium situation is attained when the mirror is parallel or antiparallel to the background vector.
B. Exact results

The first case in which we can provide exact results is when \( v^\mu = v_p^\mu \), what leads to

\[
E_{MC} = \frac{\mu \sigma^2}{8\pi^2} \int d^2 p_p \frac{e^{-2a\sqrt{p_p^2 - (p_p \cdot v_p)^2} + m^2}}{2 \sqrt{p_p^2 - (p_p \cdot v_p)^2} + m^2} \cdot \frac{2\sqrt{p_p^2 - (p_p \cdot v_p)^2} + m^2}{2\sqrt{2\sqrt{q^2 + m^2} + \mu}}. \tag{86}
\]

Performing a change in the integration variables similar to the one we have made in the Appendix, and then using polar coordinates, we have

\[
E_{MC} = \frac{\mu \sigma^2}{4\pi \sqrt{1 - v_p^2}} \int_0^\infty dq q e^{-2a\sqrt{q^2 + m^2}} \cdot \frac{2\sqrt{q^2 + m^2} + \mu}{2\sqrt{2\sqrt{q^2 + m^2} + \mu}}. \tag{87}
\]

Now, carrying out the change of integration variable \( y = \sqrt{q^2 + m^2} \), we obtain

\[
E_{MC} = \frac{\mu \sigma^2}{16\pi \sqrt{1 - v_p^2}} e^{\mu a} Ei(1,2ma + \mu a). \tag{88}
\]

Equation (88) gives the exact expression for the interaction energy between a point-like current and a semi-transparent mirror for the special case where the background vector has only the parallel components to the mirror. We notice that (88) is the usual result found in standard scalar field theory with an effective coupling constant \( \sigma \rightarrow \sigma (1 - v_p^3)^{-1/4} \).

Taking the limit \( \mu \rightarrow \infty \) in Eq. (88) and computing the interaction force, we arrive at

\[
F_{MC} (\mu \rightarrow \infty) = \frac{\sigma^2}{16\pi \sqrt{1 - v_p^2}} e^{2ma} \frac{e^{-2ma} a}{a} \left( 2m + \frac{1}{a} \right), \tag{89}
\]

which is the interaction force characterized by the Dirichlet’s boundary condition.

In Eq. (91), we have the exact interaction force between two point-like currents. For the special situation where \( v^3 = 0, \sigma_1 = \sigma, \sigma_2 = -\sigma \) and \( a \rightarrow 2a \), this result turns out to be equivalent to Eq. (89). Thus, we again verify that for this special case, \((v^3 = 0)\), the image method is also valid for the Dirichlet boundary condition.

The second exact case we discuss is when only \( v^3 \) is nonzero, what leads to the result

\[
E_{MC} = \frac{\sigma^2}{16\pi [1 - (v^3)^2]} e^{\mu a[1-(v^3)^2]^{-1}} Ei\left(1,2ma \left[1-(v^3)^2\right]^{-1} + \mu a \left[1-(v^3)^2\right]^{-1}\right). \tag{90}
\]

Eq. (90) is equivalent to the result obtained in standard scalar field theory with an effective mass \( m \rightarrow m \left[1-(v^3)^2\right]^{-1} \) and an effective degree of transparency of the mirror \( \mu \rightarrow \mu \left[1-(v^3)^2\right]^{-1} \). From Eq. (90) we can compute the interaction force in the limit \( \mu \rightarrow \infty \), resulting in

\[
F_{MC} (\mu \rightarrow \infty) = \frac{\sigma^2}{16\pi} e^{-2m[1-(v^3)^2]^{-1}a} \frac{1}{a} \left( \frac{1}{a} + 2m \left[1-(v^3)^2\right]^{-1}\right). \tag{91}
\]

For the massless case, the interaction force (91) becomes the corresponding Coulombian interaction between two charges at a distance \( 2a \) apart with an overall minus sign. Thus, in this particular scenario, Lorentz violation effects disappear from the end result. As before, taking \( v_p^\mu = 0, \sigma_1 = \sigma, \sigma_2 = -\sigma \) and \( a \rightarrow 2a \) in Eq. (91), we reproduce the result in Eq. (91). Thus, the image method is also valid for the case where \( v^\mu = (0, 0, 0, v^3) \).

It is important to mention that the validity of the image method in a Lorentz-violating scenario is a non-trivial result, since the presence of the LV background reduces the symmetry of the problem, which is a key element in the application of the method. This suggests that the presence of mirrors in Lorentz-violating scenarios is a subject which deserves more investigation.

V. FINAL REMARKS

In this paper, we investigated the interactions between external sources for a massive real scalar field in the presence of an aether-like CPT-even Lorentz symmetry breaking term. First we performed an analysis in \( D_1 + D + 1 \) dimensions
where we considered steady field sources concentrated along parallel $D$-branes, without recourse to any approximation schemes. We discussed some particular instances of our general results and observed effects with no counterpart in the standard (Lorentz invariant) scalar field theory and that have not been explored in the literature up to now. For example, we have shown the emergence of an exclusive effect due to the Lorentz symmetry breaking, agreeing with results obtained in different, more complicated models such as [40].

Afterwards, some consequences of the Lorentz violation theory [1] due to the presence of a semi-transparent mirror were studied in $3 + 1$ dimensions. We considered different configurations of the background vector, starting by taking into account all the components of the background vector, and treating it perturbatively up to second order. Next, we provided exact results for two special cases, specifically when the background vector has only components parallel and perpendicular to the mirror. For all these configurations of the background vector, we obtained the propagator for the scalar field and the interaction force between the mirror and a point-like current. We showed that the image method is valid in the considered theory for Dirichlet boundary condition, which is a non-trivial result. We also showed that a new effect arises from the obtained results, a torque acting on the mirror according to its positioning with respect to the background vector.

These results suggest that the extension of these studies to more general LV models is a very interesting prospect. Despite not being directly applicable to the phenomenological search of Lorentz violation established within the formalism of the Standard Model extension [1–4], the Klein-Gordon field can still be explored as a prototype, establishing interesting effects of LV yet to be explored. A first natural extension of our results would be explored more general LV backgrounds as described by Eq. (2). The extension of these studies for non-minimal (higher-derivative) LV models would also be of interest.

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Appendix A: The Eqs. (61) and (63)

In this appendix we provide additional details on the computation of Eqs. (61) and (63). We note that in some of the intermediate expressions that follow, the condition $v^2 = 0$ cannot be imposed to ensure the tracelessness of the LV coefficient $k_{\mu \nu}$ defined in Eq. (2), however, this condition can be safely imposed in the final result, from which one can obtain, in the proper limiting cases, the perturbative results previously obtained, thus ensuring the consistency of the calculation.

Starting from Eq. (4), in order to put $G_0(x, y)$ in an appropriated form, we have to carry out a change of the integration variables. We split the four-vector momentum $p^\mu$ into two parts, one parallel, $p_{pa}^\mu$, and the other normal, $p_{no}^\mu$, to the Lorentz violation parameter $v^\mu$,

$$p^\mu = p_{no}^\mu + p_{pa}^\mu, \quad p = \left(\frac{v \cdot p}{v^2}\right)v^\mu, \quad p_{no}^\mu = p^\mu - \left(\frac{v \cdot p}{v^2}\right)v^\mu, \quad (A1)$$

where $p_{no} \cdot v = 0$ and $(p \cdot v)^2 = p_{pa}^2 v^2$. Now, we define the four-vector $q^\mu$

$$q^\mu = p_{no}^\mu + p_{pa}^\mu \sqrt{1 + v^2} = p^\mu + \left(\frac{v \cdot p}{v^2}\right)(\sqrt{1 + v^2} - 1)v^\mu. \quad (A2)$$

With definitions (A1) and (A2), we have

$$p_{pa}^\mu = \frac{(v \cdot q)}{v^2} \sqrt{1 + v^2}, \quad p_{no}^\mu = q^\mu - \frac{(v \cdot q)}{v^2}v^\mu, \quad (A3)$$

$$p^\mu = q^\mu + \frac{(v \cdot q)}{v^2} \left(\frac{1}{\sqrt{1 + v^2}} - 1\right)v^\mu, \quad (A4)$$
With the aid of the definition
\[ b^\mu = (x^\mu - y^\mu) + \left( \frac{1 - \sqrt{1 + v^2}}{\sqrt{1 + v^2}} \right) \left( \frac{v \cdot (x - y)}{v^2} \right) v^\mu, \quad (A5) \]
and Eq. \((A3)\), we obtain
\[ p \cdot (x - y) = b \cdot q \quad (A6) \]
The Jacobian of the transformation from \( p^\mu \) to \( q^\mu \) can be calculated from Eq. \((A3)\)
\[ \det \left[ \frac{\partial p^\mu}{\partial q^\nu} \right] = -\frac{1}{\sqrt{1 + v^2}}. \quad (A7) \]
Using these results, we obtain
\[ G_0 (x, y) = -\frac{1}{\sqrt{1 + v^2}} \int \frac{d^4 q}{(2\pi)^4} e^{ib \cdot q} \]
\[ = -\frac{1}{\sqrt{1 + v^2}} \int \frac{d^3 q_p}{(2\pi)^3} e^{ib_{p \cdot q}} \int \frac{dq^3}{2\pi} e^{-iq^3b^3}. \quad (A8) \]
The first integral in Eq. \((A8)\) is given by
\[ \int \frac{d^3 q_p}{(2\pi)^3} e^{ib_{p \cdot q}} = -\sqrt{1 + v^2} \int \frac{d^3 p_p}{(2\pi)^3} e^{ip_p \cdot (x_p - y_p)} \quad (A9) \]
where we used the Eqs. \((A6)\) and \((A7)\), while the last integral is given by
\[ \int \frac{dq^3}{2\pi} e^{-iq^3b^3} = -e^{-L|b^3|} / 2L. \quad (A10) \]
where \( L = \sqrt{m^2 - q^2_p} \) or, from Eq. \((A4)\), \( L = \sqrt{m^2 - \left[ p^2_p + (p \cdot v)^2 \right]} \), and \( b^3 \) is found by taking \( \mu = 3 \) in \((A5)\), as follows,
\[ b^3 = (x^3 - y^3) + \left( \frac{1 - \sqrt{1 + v^2}}{\sqrt{1 + v^2}} \right) \left( \frac{v \cdot (x - y)}{v^2} \right) v^3. \quad (A11) \]
Collecting terms, we write
\[ G_0 (x, y) = \int \frac{d^3 p_p}{(2\pi)^3} e^{ip_p \cdot (x_p - y_p)} \left[ -\frac{1}{2} \sqrt{\frac{1 + v^2}{1 + v^2}} e^{-L|b^3|} / L \right]. \quad (A12) \]
Finally, taking \( v^3 = 0 \) in the term between brackets on the right-hand side of the Eq. \((A12)\), we obtain the Eq. \((61)\).
In the same way, taking \( v^\mu_p = 0 \), we obtain Eq. \((63)\).

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