On decay of entropy solutions to degenerate nonlinear parabolic equations with perturbed periodic initial data

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Abstract

Under a precise nonlinearity-diffusivity assumption we establish the decay of entropy solutions of a degenerate nonlinear parabolic equation with initial data being a sum of periodic function and a function vanishing at infinity (in the sense of measure).

1 Introduction

In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0, +\infty)$, we consider the nonlinear parabolic equation

$$ u_t + \text{div}_x (\varphi(u) - a(u)\nabla u) = 0, \quad (1.1) $$

where the flux vector $\varphi(u) = (\varphi_1(u), \ldots, \varphi_n(u))$ is merely continuous: $\varphi_i(u) \in C(\mathbb{R})$, $i = 1, \ldots, n$, and the diffusion matrix $a(u) = (a_{ij}(u))_{i,j=1}^n$ is Lebesgue measurable and bounded: $a_{ij}(u) \in L^\infty(\mathbb{R})$, $i, j = 1, \ldots, n$. We also assume that the matrix $a(u) \geq 0$ (nonnegative definite). This matrix may have nontrivial kernel. Hence (1.1) is a degenerate (hyperbolic-parabolic) equation. In particular case $a \equiv 0$ it reduces to a first order conservation law

$$ u_t + \text{div}_x \varphi(u) = 0. \quad (1.2) $$

Equation (1.1) can be written (at least formally) in the conservative form

$$ u_t + \text{div}_x \varphi(u) - D^2_x \cdot A(u) = 0, \quad (1.3) $$

where the matrix $A(u)$ is a primitive of the matrix $a(u)$, $A'(u) = a(u)$, and the operator $D^2_x$ is the second order “divergence”

$$ D^2_x \cdot A(u) \equiv \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} A_{ij}(u), \quad u = u(t,x). $$
Equation (1.1) is endowed with the initial condition
\[ u(0, x) = u_0(x) \in L^\infty(\mathbb{R}^n). \] (1.4)

Let \( g(u) \in BV_{\text{loc}}(\mathbb{R}) \) be a function of bounded variation on any segment in \( \mathbb{R} \). We will need the bounded linear operator \( T_g : C(\mathbb{R})/C \to C(\mathbb{R})/C \), where \( C \) is the space of constants. This operator is defined up to an additive constant by the relation
\[ T_g(f)(u) = g(u-)f(u) - \int_0^u f(s)dg(s), \] (1.5)

where \( g(u-) = \lim_{v \to u-} g(v) \) is the left limit of \( g \) at the point \( u \), and the integral in (1.5) is understood in accordance with the formula
\[ \int_0^u f(s)dg(s) = \text{sign } u \int_{J(u)} f(s)ds, \]

where \( \text{sign } u = 1 \), \( J(u) \) is the interval \([0, u]\) if \( u > 0 \), and \( \text{sign } u = -1 \), \( J(u) = [u, 0] \) if \( u \leq 0 \). Observe that \( T_g(f)(u) \) is continuous even in the case of discontinuous \( g(u) \). For instance, if \( g(u) = \text{sign}(u - k) \) then \( T_g(f)(u) = \text{sign}(u - k)(f(u) - f(k)) \). Notice also that for \( f \in C^1(\mathbb{R}) \) the operator \( T_g \) is uniquely determined by the identity \( T_g(f)'(u) = g(u)\frac{df}{du}(u) \) (in \( \mathcal{D}'(\mathbb{R}) \)).

We fix some representation of the diffusion matrix \( a(u) \) in the form \( a(u) = b^T(u)b(u) \), where \( b(u) = (b_{ij}(u)), i = 1, \ldots, r; j = 1, \ldots, n \) is a \( r \times n \) matrix-valued function (a square root of \( a(u) \)) with measurable and bounded entries, \( b_{ij}(u) \in L^\infty(\mathbb{R}) \). We recall the notion of entropy solution of the Cauchy problem (1.1), (1.4) introduced in [2].

**Definition 1.1.** A function \( u = u(t, x) \in L^\infty(\Pi) \) is called an entropy solution (e.s. for short) of (1.1), (1.4) if the following conditions hold:

(i) for each \( i = 1, \ldots, r \) the distributions
\[ \text{div}_x B_i(u(t, x)) \in L^2_{\text{loc}}(\Pi), \] (1.6)

where vectors \( B_i(u) = (B_{i1}(u), \ldots, B_{in}(u)) \in C(\mathbb{R}, \mathbb{R}^n) \), and \( B_{ij}'(u) = b_{ij}(u), i = 1, \ldots, r, j = 1, \ldots, n \);

(ii) for every \( g(u) \in C^1(\mathbb{R}), i = 1, \ldots, r \)
\[ \text{div}_x T_g(B_i(u(t, x))) = g(u(t, x))\text{div}_x B_i(u(t, x)) \text{ in } \mathcal{D}'(\Pi); \] (1.7)

(iii) for any convex function \( \eta(u) \in C^2(\mathbb{R}) \)
\[ \eta(u)_t + \text{div}_x T_{\eta'}(\varphi)(u) - D_x^2 \cdot T_{\eta''}(A)(u) + \eta''(u)\sum_{i=1}^r (\text{div}_x B_i(u))^2 \leq 0 \text{ in } \mathcal{D}'(\Pi); \] (1.8)

(iv) \( \text{ess lim}_{t \to 0} u(t, \cdot) = u_0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^n) \).
In the isotropic case when the diffusion matrix is scalar the definition can be considerably simplified and was introduced earlier by J. Carrillo in [1].

Relation (1.8) means that for any non-negative test function \( f = f(t,x) \in C_0^\infty(\Pi) \)

\[
\int_\Pi \left[ \eta(u) f_t + T_\varphi'(u) \cdot \nabla_x f + T_\varphi'(A)(u) \cdot D_x^2 f - f \eta''(u) \sum_{i=1}^n (\text{div}_x B_i(u))^2 \right] dt dx \geq 0, \tag{1.9}
\]

where \( D_x^2 f \) is the symmetric matrix of second order derivatives of \( f \), and "." denotes the standard scalar multiplications of vectors or matrices (in particular, \( A \cdot B = \text{Tr} A^\top B \) for matrices \( A, B \)).

In the case of conservation laws (1.2) Definition 1.1 reduces to the known definition of entropy solutions in the sense of S. N. Kruzhkov [4]. Taking in (1.8) \( \eta(u) = \pm u \), we deduce that

\[ u_t + \text{div}_x \varphi(u) - D_x^2 \cdot A(u) = 0 \quad \text{in} \quad \mathcal{D}'(\Pi), \]

that is, an e.s. \( u \) is a weak solution of (1.3).

It is known that e.s. of (1.1), (1.4) always exists but in the case of only continuous flux may be nonunique, for conservations laws (1.2) the corresponding examples can be found in [5, 6]. Nevertheless, if initial function is periodic, the uniqueness holds: an e.s. of (1.1), (1.4) is unique and space-periodic, see [10, Theorem 1.3]. In general case there always exists the unique largest and smallest e.s., see [10, Theorem 1.1].

2 Formulation of the results

In the present paper we study the long time decay property of e.s. in the case when initial data is a perturbed periodic function. More precisely, we assume that the initial function \( u_0(x) = p(x) + v(x) \), where \( p(x) \) is periodic while \( v(x) \in L_0^\infty(\mathbb{R}^n) \), where the space \( L_0^\infty(\mathbb{R}^n) \) consists of functions \( v(x) \in L^\infty(\mathbb{R}^n) \) such that

\[
\forall \lambda > 0 \quad \text{meas}\{x \in \mathbb{R}^n : |v(x)| > \lambda\} < +\infty \tag{2.1}
\]

(here we denote by meas the Lebesgue measure on \( \mathbb{R}^n \)). Evidently, the space \( L_0^\infty(\mathbb{R}^n) \) contains functions vanishing at infinity as well as functions from the spaces \( L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), q > 0 \). Observe that the functions \( p, v \) are uniquely defined (up to equality on a set of full measure) by the function \( u_0 \). Let

\[
G = \{ e \in \mathbb{R}^n \mid p(x + e) = p(x) \quad \text{almost everywhere in} \quad \mathbb{R}^n \} \tag{2.2}
\]

be the group of periods of \( p \), it is not necessarily a lattice because \( p(x) \) may be constant in some directions. For example, if \( p \equiv \text{const} \) then \( G = \mathbb{R}^n \). The periodicity of \( p \) means
that the linear hull of $G$ coincides with $\mathbb{R}^n$, that is, there is a basis of periods of $p$. Denote by $H$ the maximal linear subspace contained in $G$. The dual lattice

$$G' = \{ \xi \in \mathbb{R}^n \mid \xi \cdot e \in \mathbb{Z} \forall e \in G \}$$

is indeed a lattice in the orthogonal complement $H^\perp$ of the space $H$ (we will prove this simple statement in Lemma 3.1 below). Observe also that $G' = L_0'$ in $H^\perp$, where $L_0 = G \cap H^\perp$, so that $G = H \oplus L_0$. It is rather well-known (at least for continuous periodic functions) that $L_0$ is a lattice in $H^\perp$. The case of measurable periodic functions requires some little modifications and, for the sake of completeness, we put the proof of this fact in Lemma 3.1. Notice that we use more general notion of period contained in (2.2). For the standard notion $p(x + e) \equiv p(x)$ (where the words “almost everywhere” are omitted) the group $G$ may have more complex structure. For example, the group of periods of the Dirichlet function on $\mathbb{R}$ is a set of rationals $\mathbb{Q}$, which is not a lattice in $\mathbb{R}$. We introduce the torus $T^d = \mathbb{R}^n / G = H^\perp / L_0$ of dimension $d = \dim H^\perp = n - \dim H$ equipped with the normalized Lebesgue measure $dy$. The periodic function $p$ can be considered as a function on this torus $T^d$: $p = p(y)$. Let

$$m = \int_{T^d} p(y) dy$$

be the mean value of this function. Clearly, this value coincides with the mean value of the initial data:

$$m = \lim_{R \to \infty} \frac{1}{|B_R|} \int_{B_R} u_0(x) dx,$$

where $B_R$ is the ball $|x| < R$, and $|B_R| = \text{meas } B_R$ is its Lebesgue measure. The latter follows from the fact that functions $v(x)$ from $L_0^\infty(\mathbb{R}^n)$ always have zero mean value. More precisely,

$$\lim_{|A| \to \infty} \frac{1}{|A|} \int_A |v(x)| dx = 0,$$  \hspace{1cm} (2.3)

where $A$ runs over Lebesgue measurable sets of finite measure $|A|$. In fact, let $\varepsilon > 0$, $E = \{ x \in \mathbb{R}^n : |v(x)| > \varepsilon \}$. Then $p = \text{meas } E < +\infty$. Obviously

$$\int_A |v(x)| dx = \int_E |v(x)| dx + \int_{A \setminus E} |v(x)| dx \leq p\|v\|_\infty + \varepsilon |A|.$$

This implies that

$$\frac{1}{|A|} \int_A |v(x)| dx \leq \frac{p\|v\|_\infty}{|A|} + \varepsilon \to \varepsilon \quad \text{as } |A| \to \infty.$$
Hence,
\[ \limsup_{|A| \to \infty} \frac{1}{|A|} \int_A |v(x)| \, dx \leq \varepsilon, \]
and since \( \varepsilon > 0 \) is arbitrary, we conclude that
\[ \lim_{|A| \to \infty} \frac{1}{|A|} \int_A |v(x)| \, dx = 0, \]
as was to be proved.

We will study the long time decay property of e.s. with respect to the following shift-invariant Stepanov norm on \( L^\infty(\mathbb{R}^n) \):
\[
\|u\|_X = \sup_{y \in \mathbb{R}^n} \int_{|x-y| < 1} |u(x)| \, dx \tag{2.4}
\]
(where we denote by \(|z|\) the Euclidean norm of a finite-dimensional vector \(z\)). As was demonstrated in [8], this norm is equivalent to each of more general norms
\[
\|u\|_V = \sup_{y \in \mathbb{R}^n} \int_{y+V} |u(x)| \, dx, \tag{2.5}
\]
where \(V\) is any bounded open set in \(\mathbb{R}^n\) (the original norm \(\| \cdot \|_X\) corresponds to the unit ball \(|x| < 1\)). For the sake of completeness we repeat the proof of this result in Lemma 3.2 below. Obviously, norm (2.4) generates the stronger topology than one of \(L^1_{loc}(\mathbb{R}^n)\).

We denote by \(F\) the closed set of points \(u \in \mathbb{R}\) such that the flux components \(\varphi(u) \cdot \xi\) are not affine on any vicinity of \(u\) where the diffusion coefficients \(a(u)\xi \cdot \xi = 0\) (almost everywhere in this vicinity) for all \(\xi \in G', \xi \neq 0\). In the case when \(G = H = \mathbb{R}^n\) (and nonzero vectors \(\xi \in G'\) do not exist), we define \(F\) as the set of \(u\) such that there is no such vicinity of \(u\) where the entire vector \(\varphi(u)\) is affine while the entire diffusion matrix \(a(u) = 0\) (a.e. in this vicinity). Our main result is the following decay property.

**Theorem 2.1.** Assume that the following nonlinearity-diffusivity condition is satisfied: for all \(a < m, b > m\) the intervals \((a,m), (m,b)\) intersect with \(F\): \((a,m) \cap F \neq \emptyset, (m,b) \cap F \neq \emptyset\). Suppose that \(u(t,x)\) is an e.s. of (1.1), (1.4). Then
\[
\text{ess lim}_{t \to +\infty} \|u(t,\cdot) - m\|_X = 0. \tag{2.6}
\]

The condition \(H = \mathbb{R}^n\) means that the function \(p(x)\) is constant, \(p \equiv m\). In this case, the requirement of Theorem 2.1 reduces to the condition that on any semivici


(a, m), (m, b) of the mean m either the flux vector \( \varphi(u) \) is not affine or the diffusion matrix \( a(u) \neq 0 \). When \( p \equiv m = 0 \), Theorem 2.1 was proved in [10, Theorem 1.4] (and in [8] for conservation laws (1.2)). The case of arbitrary \( m \) reduces to the case \( m = 0 \) by the change \( u \to u - m \), \( \varphi(u) \to \varphi(u + m) \), \( a(u) = a(u + m) \). Thus, we may suppose in the sequel that \( p \neq \text{const} \) and therefore the lattice \( G' \) is not trivial.

Remark that the nonlinearity-diffusivity requirement in Theorem 2.1 implies that \( m \in F \) because of closeness of this set. Generally, under this weaker condition \( m \in F \) the decay property fails, cf. section 4 below. But, in periodic case \( v \equiv 0 \), the decay property (2.6) holds under the weaker condition \( m \in F \), that is,

\[ \forall \xi \in G', \xi \neq 0 \text{ either the flux components } \varphi(u) \cdot \xi \text{ are not affine or} \]
\[ \text{the diffusion coefficients } a(u)\xi \cdot \xi \neq 0 \text{ on any vicinity of } m. \]  

In the standard case when \( G \) is a lattice (that is, when \( \dim H = 0 \)) it was proved in [9, Theorem 1.1], see also earlier paper [3], where the decay property was established under a more restrictive nonlinearity-diffusivity assumption. The general case of arbitrary \( H \) easily reduces to the case \( \dim H = 0 \). We provide the details in the following theorem.

**Theorem 2.2.** Suppose that the initial function \( u_0 = p(x) \) is periodic with a group of periods \( G \), and condition (2.7) is satisfied. Then the e.s. \( u = u(t, x) \) of problem (1.1), (1.4) exhibits the decay property

\[ \text{ess lim}_{t \to +\infty} u(t, \cdot) = m = \int_{T^d} u_0(x) dx \quad \text{in } L^1(T^d). \]  

**Proof.** Observe that for all \( e \in G \) \( u_0(x + e) = u_0(x) \) a.e. in \( \mathbb{R}^n \). Obviously, \( u(t, x + e) \) is an e.s. of (1.1), (1.4) with the same initial data \( u_0(x) \). By the uniqueness of an e.s., known in the case of periodic initial function, we claim that \( u(t, x + e) = u(t, x) \) a.e. in \( \Pi \), that is, \( u(t, x) \) is \( G \)-periodic in the space variables. In particular, \( u(t, \cdot) \in L^1(T^d) \) for a.e. \( t > 0 \) and relation (2.8) is well-defined. We choose a non-degenerate linear operator \( Q \) in \( \mathbb{R}^n \), which transfers the space \( H \) into the standard subspace

\[ \mathbb{R}^{n-d} = \{ x = (x_1, \ldots, x_n) \mid x_i = 0 \forall i = 1, \ldots, d \}. \]

After the change \( y = Qx \) our problem reduces to the problem

\[ v_t + \text{div}_y(\tilde{\varphi}(v) - \tilde{a}(v) \nabla_y v) = 0, \quad v(0, y) = v_0(y) = u_0(Q^{-1}(y)), \]  

where \( \tilde{\varphi}(v) = Q \varphi(v), \tilde{a}(v) = Qa(v)Q^* \), where \( Q^* = Q^\top \) is a conjugate operator. As is easy to verify, \( u(t, x) = v(t, Qx) \), where \( v(t, y) \) is an e.s. of (2.9). Observe that \( v_0(y) \) is
periodic with the group of periods $\bar{G} = Q(G)$. Therefore, the e.s. $v(t, y)$ is space periodic with the group of periods containing $\bar{G}$. In particular, the functions $v_0(y)$, $v(t, y)$ are constant in directions $\mathbb{R}^{n-d} = Q(H)$: $v_0(y) = v_0(y_1, \ldots, y_d)$, $v(t, y) = v(t, y_1, \ldots, y_d)$ with $v_0(y') \in L^\infty(\mathbb{R}^d)$, $v(t, y') \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$. This readily implies that $v(t, y')$ is an e.s. of the low-dimensional problem

$$v_t + \text{div}_y' (\tilde{\varphi}(v) - \tilde{a}(v) \nabla_y v) = 0, \quad v(0, y') = v_0(y'). \quad (2.10)$$

Observe that $v_0(y')$ is periodic with the lattice of periods

$$\bar{L} = \bar{G} \cap \mathbb{R}^d = \{ y = (y_1, \ldots, y_d) \in \mathbb{R}^d \mid (y_1, \ldots, y_d, 0, \ldots, 0) \in \bar{G} \}$$

(by Lemma 3.1 it is indeed a lattice). Using again Lemma 3.1 we find that the dual lattice $\bar{L}' = \bar{G}' = (Q^*)^{-1}G' \subset \mathbb{R}^d$. Observe that for each nonzero $\zeta \in \bar{L}'$ the vector $\xi = Q^*\zeta \in G''$, and

$$\tilde{\varphi}(v) = \zeta \cdot Q\varphi(v) = Q^*\zeta \cdot \varphi(v) = \xi \cdot \varphi(v),$$

$$\tilde{a}(v) \cdot \zeta = QA(v)Q^*\zeta \cdot \zeta = a(v)Q^*\zeta \cdot \zeta = a(v)\xi \cdot \xi$$

By condition (2.7) we claim that in any vicinity of $m$ for every $\zeta \in \bar{L}'$ either the function $v \to \zeta \cdot \tilde{\varphi}(v)$ is not affine or the function $v \to \tilde{a}(v)\zeta$ is not zero on a set of positive measure. Observe that $\zeta \in \bar{L}' \subset \mathbb{R}^d$, and we may replace $\tilde{\varphi}(v)$, $\tilde{a}(v)$ by $P\tilde{\varphi}(v)$, $P\tilde{a}(v)P^*$, respectively, where $P : \mathbb{R}^n \to \mathbb{R}^d$ is the orthogonal projection. This means that equation (2.10) satisfies the nonlinearity-diffusivity condition subject to the lattice $\bar{L}$ in $\mathbb{R}^d$. By the decay property [1, Theorem 1.1] applied to the e.s. $v(t, y')$ of (2.10) we claim that

$$\text{ess lim}_{t \to +\infty} v(t, \cdot) = \bar{m} = \int_{\bar{T}^d} v_0(y')dy' \text{ in } L^1(\bar{T}^d), \quad (2.11)$$

where $\bar{T}^d = \mathbb{R}^d/\bar{L} = \mathbb{R}^n/\bar{G}$ is a torus corresponding to the lattice $\bar{L}$, and $dy'$ denotes the normalized Lebesgue measure on this torus. Making the change of variables $y = Qx$, which induces an isomorphism $Q : \mathbb{T}^d \to \bar{T}^d$, we find that

$$\bar{m} = \int_{\bar{T}^d} u_0(x)dx = m,$$

and relation (2.8) follows from (2.11). The proof is complete. \qed
Remark also that Theorem 2.2 follows from result [11] on decay of almost periodic (in Besicovitch sense) e.s. In [11] the decay property (in the Besicovitch space) was established under the same assumption as in Theorem 2.2 but with the set \( G \) (in Besicovitch sense) e.s. In [11] the decay property (in the Besicovitch space) was contained in the lattice \( M \) by the additive group \( \mathbb{Z} \). In particular, the spectrum \( Sp(u_0) = \{ \lambda \in \mathbb{R}^n | a_\lambda \neq 0 \} \) as well as the group \( M(u_0) \) are contained in the lattice \( G' \). This implies that \( M(u_0) \) is a lattice too. By the duality, in order to prove the equality \( M(u_0) = G' \) it is sufficient to verify the inclusion \( M(u_0)' \subset G \). If \( e \in M(u_0)' \) then \( \lambda \cdot e \in \mathbb{Z} \) for each \( \lambda \in Sp(u_0) \). This implies that the Fourier series of \( u_0(x + e) \) is the same as of \( u_0(x) \). Hence, \( u_0(x + e) = u_0(x) \) a.e. in \( \mathbb{R}^n \) and \( e \in G \). We see that \( M(u_0)' \subset G \) and, therefore, \( M(u_0) = G' \). Thus, the assumption of [11] reduces to our assumption. Taking also into account that in periodic case the Besicovitch norm is equivalent to the norm of \( L^1(\mathbb{T}^d) \), we conclude that the decay property stated in Theorem 2.2 holds.

Let us show that condition (2.7) is precise, that is, if this condition fails, then there exists a periodic function exactly with the group of periods \( G \) such that the corresponding e.s. does not satisfy the decay property. In fact, if (2.7) fails, we can find a nonzero vector \( \xi \in G' \) and constants \( \delta > 0 \), \( k \in \mathbb{R} \) such that \( \xi \cdot \varphi(u) - ku = \text{const} \) on the segment \( |u - m| \leq \delta \) while \( a(u)\xi \cdot \xi = 0 \) a.e. on this segment. Since the matrix \( a(u) \geq 0 \), the equality \( a(u)\xi \cdot \xi = 0 \) holds if and only if \( a(u)\xi = 0 \). We define the hyperspace \( E = \{ x \in \mathbb{R}^n \mid \xi \cdot x = 0 \} \). The linear functional \( \xi \) is a homomorphism of the group \( G \) into \( \mathbb{Z} \). The range of this homomorphism is a subgroup \( r\mathbb{Z} \subset \mathbb{Z} \) for some \( r \in \mathbb{N} \). Denote by \( G_1 = E \cap G \) the kernel of \( \xi \). There exists an element \( e_0 \in G \) such that \( \xi \cdot e_0 = r \). Then, as is easy to verify, the map \( (e, m) \to e + me_0 \) forms a group isomorphism of \( G_1 \oplus \mathbb{Z} \) onto \( G \). We choose \( n - 1 \) independent vectors \( \zeta_i \) such that \( \zeta_i \cdot e_0 = 0 \), \( i = 1, \ldots, n - 1 \), and complement them to a basis attaching the vector \( \zeta_n = \xi \). We make the linear change \( y = y(t, x) \)

\[
y_i = \zeta_i \cdot x, \quad i = 1, \ldots, n - 1, \quad y_n = \zeta_n \cdot x - kt, \quad (2.12)
\]

which reduces (1.1) to the equation

\[
u_t + \text{div}_y(\bar{\varphi}(u) - \bar{a}(u)\nabla_y u) = 0 \quad (2.13)
\]

such that \( \bar{\varphi}_n(u) = \xi \varphi(u) - ku = \text{const} \), \( \bar{a}_{in} = \bar{a}_{ni} = a(u)\xi \cdot \zeta_i = 0 \), \( i = 1, \ldots, n \), a.e. on the segment \( |u - m| \leq \delta \). If an initial data \( \bar{u}_0(y) \) satisfies the condition \( |\bar{u}_0 - m| \leq \delta \) then a corresponding e.s. \( u = \tilde{u}(t, x) \) of (2.13) also satisfies the condition \( |\tilde{u}(t, x) - m| \leq \delta \) a.e. on \( \Pi \), by the maximum-minimum principle [10 Corollary 2.2]. Since \( \bar{\varphi}_n(u) \) is constant on the segment \( |u - m| \leq \delta \), while the diffusion coefficients \( \bar{a}_{ij}(u) \) with \( \text{max}(i, j) = n \).
vanish a.e. on this segment, then \( \tilde{u}(t, y) \) is an e.s. of the equation

\[
\begin{align*}
  u_t + \nabla_y (\tilde{\varphi}(u) - \tilde{a}(u) \nabla_y u) &= u_t + \sum_{i=1}^{n-1} (\tilde{\varphi}_i(u))_{y_i} - \left( \sum_{i,j=1}^{n-1} \tilde{a}_{ij}(u)u_{y_j} \right)_{y_i} = 0, \quad (2.14)
\end{align*}
\]

where \( y' = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} \). This readily implies that for a.e. fixed \( y_n \in \mathbb{R} \) the function \( \tilde{u}(t, y', y_n) \) is an e.s. of the Cauchy problem for the low-dimensional equation \((2.14)\), considered in the domain \( \mathbb{R}_+ \times \mathbb{R}^{n-1} \), with the corresponding initial function \( \tilde{u}_0(y', y_n) \). Assume that the function \( \tilde{u}_0(y', y_n) \) is \( y' \)-periodic (with some group of periods) with the mean value \( m(y_n) = \frac{1}{|P|} \int_P u_0(y', y_n)dy' \), \( P \subset \mathbb{R}^{n-1} \) being the periodicity cell (or, the same, the corresponding torus). Then for a.e. \( y_n \in \mathbb{R} \) the mean value of \( \tilde{u}(t, \cdot, y_n) \) does not depend on \( t \) and equals \( m(y_n) \) (see, for instance, [9, Corollary 2.1]). If this function \( m(y_n) \) is not constant (a.e. in \( \mathbb{R} \)) then the e.s. \( \tilde{u}(t, y) \) cannot satisfy the decay property. In fact, if \( \tilde{u}(t, \cdot) - m \to 0 \) as \( t \to +\infty \) in \( L_{loc}^1(\mathbb{R}^n) \), then for each interval \( I \subset \mathbb{R} \)

\[
\left| \int_I (m(y_n) - m)dy_n \right| = \frac{1}{|P|} \left| \int_I (\tilde{u}(t, y) - m)dy \right| \leq \frac{1}{|P|} \int_{P \times I} |\tilde{u}(t, y) - m|dy,
\]

which implies, in the limit as \( t \to +\infty \), that \( \int_I (m(y_n) - m)dy_n = 0 \). Since \( I \) is an arbitrary interval, we find that \( m(y_n) = m \) a.e. in \( \mathbb{R} \), which contradicts to our assumption.

Now we choose a function \( v(x) \in C(E) \) such that \( \|v\|_\infty \leq \delta/2 \) and that \( v(x) \) is periodic with the group of periods \( G_1 \) and with zero mean value. Since \( G_1 = H \oplus (E \cap L_0) \), such \( v(x) \) actually exists, this function is constant in the direction \( H \) and is periodic in \( E \cap H^\perp \) with exactly the lattice of periods \( E \cap L_0 \). We set

\[
u_0(x) = m + v(\text{pr}(x)) + \frac{\delta}{2} \sin(2\pi \xi \cdot x/r),
\]

where \( \text{pr}(x) \in E \) is the projection of \( x \) on \( E \) along the vector \( e_0 \) (so that \( x - \text{pr}(x) \parallel e_0 \)). Then \( \|u_0 - m\|_\infty \leq \delta \). Let us show that \( u_0(x) \) is periodic with the group of periods \( G \). Since vectors \( e \in G_1 \) and \( e_0 \) are periods of \( u_0 \), then the group \( G = G_1 + Z e_0 \) consists of period of \( u_0 \). On the other hand, if \( e \in \mathbb{R}^n \) is a period of \( u_0 \) then it can be decomposed into a sum \( e = e_1 + \lambda e_0 \), where \( e_1 \in E \), \( \lambda \in \mathbb{R} \). For \( x = x' + se_0 \), \( x' \in E \), we have

\[
u_0(x + e) = m + v(x' + e_1) + \frac{\delta}{2} \sin(2\pi (s + \lambda)) = u_0(x) = m + v(x') + \frac{\delta}{2} \sin(2\pi s).
\]

Averaging this equality over \( x' \), we obtain that \( \sin(2\pi (s + \lambda)) = \sin(2\pi s) \) for all \( s \in \mathbb{R} \), which implies that \( \lambda \in Z \) and that \( v(x' + e_1) = v(x') \) for all \( x' \in E \). Therefore, \( e_1 \in G_1 \) (remind that \( G_1 \) is the group of periods of \( v \)). Hence \( e = e_1 + \lambda e_0 \in G_1 + Z e_0 = G \). We
proved that the group of periods of \( u_0 \) is exactly \( G \). It is clear that \( m \) is the mean value of \( u_0 \).

Now, we are going to show that an e.s. \( u(t, x) \) of (1.1), (1.4) with the chosen initial data does not satisfy decay property (2.8). After the change (2.12) the initial function \( u_0(x) \) transforms into \( \tilde{u}_0(y) = m + \tilde{\nu}(y') + \frac{\delta}{2} \sin(2\pi y_n/r) \), the function \( \tilde{\nu}(y') \in C(\mathbb{R}^{n-1}) \) is determined by the identity \( \nu(x) = \tilde{\nu}(y(x)) \), where \( y(x) = y(0, x), \ x \in E \), is a linear isomorphism \( E \rightarrow \mathbb{R}^{n-1} \). Obviously, the function \( \tilde{\nu}(y') \) is periodic with the group of periods \( y(G_1) \) and zero mean value. Therefore, the mean value of initial data over the variables \( y' \) equals \( m(y_n) = m + \frac{\delta}{2} \sin(2\pi y_n/r) \) and it is not constant. In this case it has been already demonstrated that an e.s. \( \tilde{u}(t, y(t, x)) \) of the Cauchy problem for equation (2.13) does not satisfy the decay property. Due to the identity \( u(t, x) = \tilde{u}(t, y(t, x)) \), we see that an e.s. of original problem does not satisfy (2.8) either.

3 Proof of the main results

3.1 Auxiliary lemmas

**Lemma 3.1.** Let \( G \) be the group of periods of a periodic function \( p(x) \in L^\infty(\mathbb{R}^n) \), and let, as in Introduction, \( H \) be a maximal linear subspace of \( G \), \( L_0 = G \cap H^\perp \), and let \( G' \) be a dual group to \( G \). Then

(i) \( L_0 \) is a lattice of dimension \( d = \dim H^\perp \);
(ii) \( G' \) is a lattice in \( H^\perp \) of dimension \( d \), and \( G' = L'_0 \) in \( H^\perp \).

**Proof.** (i) We have to prove that the group \( L_0 \) is discrete, that is, all its points are isolated. Since \( L_0 \) is a group it is sufficient to show that 0 is an isolated point of \( L_0 \). Assuming the contrary, we find a sequence \( h_k \in L_0 \), such that \( h_k \neq 0, h_k \rightarrow 0 \) as \( k \rightarrow \infty \). By compactness of the unit sphere \( |x| = 1 \), we may suppose that the sequence \( |h_k|^{-1}h_k \rightarrow \xi \) as \( k \rightarrow \infty \), where \( \xi \in H^\perp, |\xi| = 1 \). Let \( w(x) \in C^1(\mathbb{R}^n) \) and \( v(x) be the convolution \( v = p * w(x) = \int_{\mathbb{R}^n} p(x-y)w(y)dy \). By known property of convolution \( v(x) \in C^1(\mathbb{R}^n) \). Further, for each \( e \in G \) and \( x \in \mathbb{R}^n \)

\[
v(x + e) = \int_{\mathbb{R}^n} p(x - y + e)w(y)dy = \int_{\mathbb{R}^n} p(x - y)w(y)dy = v(x)
\]

and in particular \( v(x + h_k) = v(x) \) \( \forall k \in \mathbb{N} \). Since \( v \) is differentiable,

\[
0 = |h_k|^{-1}(v(x + h_k) - v(x)) = \nabla v(x) \cdot |h_k|^{-1}h_k + \varepsilon_k, \quad (3.1)
\]
where \( \varepsilon_k \to 0 \) as \( k \to \infty \). Passing in (3.1) to the limit as \( k \to \infty \), we obtain that \( \frac{\partial v(x)}{\partial \xi} = \nabla v(x) \cdot \xi = 0 \) for all \( x \in \mathbb{R}^n \). Therefore, \( v \) is constant in the direction \( \xi \); \( v(x + s \xi) = v(x) \) for all \( s \in \mathbb{R} \), \( x \in \mathbb{R}^n \). We choose a nonnegative function \( w(x) \in C_0^1(\mathbb{R}^n) \) such that \( \int_{\mathbb{R}^n} w(x)dx = 1 \) and set \( \omega_r(x) = r^n \omega(rx) \), \( r \in \mathbb{N} \). This sequence (an approximate unity) converges to Dirac \( \delta \)-function as \( r \to \infty \) weakly in the space of distributions \( \mathcal{D}'(\mathbb{R}^n) \). The corresponding sequence of averaged functions \( v_r = p * w_r(x) \) converges to \( p \) as \( r \to \infty \) in \( L_{1 loc}^1(\mathbb{R}^n) \). As was already established, the functions \( v_r \) are constant in the direction \( \xi \). Therefore, for each \( \alpha \in \mathbb{R} \), \( R > 0 \)

\[
\int_{|x|<R} |v_r(x + \alpha \xi) - v_r(x)|dx = 0.
\]

Passing in this relation to the limit as \( r \to \infty \), we obtain that

\[
\int_{|x|<R} |p(x + \alpha \xi) - p(x)|dx = 0 \quad \forall R > 0,
\]

which implies that \( p(x + \alpha \xi) = p(x) \) a.e. in \( \mathbb{R}^n \), that is, \( \alpha \xi \in G \). Thus, the linear subspace \( H_1 = \{ x + \alpha \xi \mid x \in H, \alpha \in \mathbb{R} \} \subset G \). Since \( \xi \in H^\perp \), \( \xi \neq 0 \), then \( H \subsetneq H_1 \). But this contradicts to the maximality of \( H \). This contradiction proves that \( L_0 \) is a discrete additive subgroup of \( \mathbb{R}^n \), i.e., a lattice. By the construction, \( G = H \oplus L_0 \). Since \( G \) generates the entire space \( \mathbb{R}^n \), then \( L_0 \) must generate \( H^\perp \), that is, \( \dim L_0 = d \).

(ii) Since \( \dim G = n \), we can choose a basis \( e_k, k = 1, \ldots, n \), of the linear space \( \mathbb{R}^n \) lying in \( G \). We define \( R = \max_{k=1,\ldots,n} |e_k| \delta = 1/R \). Let \( \xi \in G^\perp \), \( |\xi| < \delta \). Then

\[
|\xi \cdot e_k| \leq |\xi||e_k| \leq |\xi|R < 1.
\]

Since \( \xi \cdot e_k \in \mathbb{Z} \) we claim that \( \xi \cdot e_k = 0 \) for all \( k = 1, \ldots, n \). Since \( e_k, k = 1, \ldots, n \) is a basis, this implies that \( \xi = 0 \). We obtain that the ball \( |\xi| < \delta \) contains only zero element of the group \( G^\perp \). This means that this group is discrete and therefore it is a lattice. If \( \xi \in G^\perp \), \( e \in H \) then \( \alpha \xi \cdot e = \xi \cdot \alpha e \in \mathbb{Z} \) for all \( \alpha \in \mathbb{R} \). This is possible only if \( \xi \cdot e = 0 \). This holds for every \( e \in H \), that is, \( \xi \in H^\perp \).

Obviously, for such \( \xi \), the requirement \( \xi \in G^\perp \) reduces to the condition \( \xi \cdot e \in \mathbb{Z} \) for all \( e \in L_0 \). We conclude that \( G' = L_0 \). Since \( L_0 \) is a lattice, this in particular implies that \( \dim G' = \dim L_0 = d \).

\[ \square \]

**Lemma 3.2.** The norms \( \| \cdot \|_V \) defined in (2.5) are mutually equivalent.

**Proof.** Let \( V_1, V_2 \) be open bounded sets in \( \mathbb{R}^n \), and \( K_1 = \text{Cl} V_1 \) be the closure of \( V_1 \). Then \( K_1 \) is a compact set while \( y + V_2, y \in K_1 \), is its open covering. By the compactness there is a finite set \( y_i, i = 1, \ldots, m \), such that \( K_1 \subset \bigcup_{i=1}^m (y_i + V_2) \). This implies that for every
$y \in \mathbb{R}^n$ and $u = u(x) \in L^\infty(\mathbb{R}^n)$

$$
\int_{y+V_1} |u(x)| dx \leq \sum_{i=1}^{m} \int_{y+y_i+V_2} |u(x)| dx \leq m\|u\|_{V_2}.
$$

Hence, $\forall u = u(x) \in L^\infty(\mathbb{R}^n)$

$$
\|u\|_{V_1} = \sup_{y \in \mathbb{R}^n} \int_{y+V_1} |u(x)| dx \leq m\|u\|_{V_2}.
$$

Changing the places of $V_1$, $V_2$, we obtain the inverse inequality $\|u\|_{V_2} \leq l\|u\|_{V_1}$ for all $u \in L^\infty(\mathbb{R}^n)$, where $l$ is some positive constant. This completes the proof. \(\square\)

**Proposition 3.1.** Let $G$ be a lattice and values $\alpha^+, \alpha^- \in F$ be such that $\alpha^- < m < \alpha^+$. Then an e.s. $u(t, x)$ of (1.1), (1.4) satisfies the property

$$
\text{ess lim sup}_{t \to +\infty} \|u(t, \cdot) - m\|_X \leq 2^n(\alpha^+ - \alpha^-).
$$

**Proof.** Let $e_k, k = 1, \ldots, n,$ be a basis of the lattice $G$. We define for $r \in \mathbb{N}$ the parallelepiped

$$
P_r = \left\{ x = \sum_{k=1}^{n} x_k e_k : -r/2 \leq x_k < r/2, k = 1, \ldots, n \right\}.
$$

It is clear that $P_r$ is a fundamental parallelepiped for a lattice $rG \subset G$. We introduce the functions

$$
v_r^+(x) = \sup_{e \in G} v(x + re), \quad v_r^-(x) = \inf_{e \in G} v(x + re), \quad V_r(x) = \sup_{e \in G} |v(x + re)|.
$$

Since $G$ is countable, these functions are well-defined in $L^\infty(\mathbb{R}^n)$, and $|v_r^\pm| \leq V_r(x) \leq C_0 = \|v\|_\infty$ for a.e. $x \in \mathbb{R}^n$. It is clear that $v_r^\pm(x)$ are $rG$-periodic and

$$
v_r^-(x) \leq v(x) \leq v_r^+(x).
$$

(3.2)

Let us show that under condition (2.1)

$$
M_r = \frac{1}{|P_r|} \int_{P_r} V_r(x) dx \to 0 \quad \text{as } r \to +\infty.
$$

(3.3)
For that we fix \( \varepsilon > 0 \) and define the set \( A = \{ x \in \mathbb{R}^n : |v(x)| > \varepsilon \} \). In view of (2.1) the measure of this set is finite, \( \text{meas} A = q < +\infty \). We also define the sets

\[
A^e_r = \{ x \in P_r : x + re \in A \} \subset P_r, \quad r > 0, \quad e \in G, \quad A_r = \bigcup_{e \in G} A^e_r.
\]

By the translation invariance of Lebesgue measure and the fact that \( \mathbb{R}^n \) is the disjoint union of the sets \( re + P_r, e \in G \), we have

\[
\sum_{e \in G} \text{meas} A^e_r = \sum_{e \in G} \text{meas}(re + A^e_r) = \sum_{e \in G} \text{meas}(A \cap (re + P_r)) = \text{meas} A = q.
\]

This implies that

\[
\text{meas} A_r \leq \sum_{e \in G} \text{meas} A^e_r = q. \tag{3.4}
\]

If \( x \notin A_r \) then \( |v(x + re)| \leq \varepsilon \) for all \( e \in G \), which implies that \( V_r(x) \leq \varepsilon \). Taking (3.4) into account, we find

\[
\int_{P_r} V_r(x)dx = \int_{A_r} V_r(x)dx + \int_{P_r \setminus A_r} V_r(x)dx \leq C_0 \text{meas} A_r + \varepsilon \text{meas} P_r \leq C_0q + \varepsilon |P_r|.
\]

It follows from this estimate that

\[
\limsup_{r \to +\infty} M_r \leq \lim_{r \to +\infty} \left( \frac{C_0q}{|P_r|} + \varepsilon \right) = \varepsilon
\]

and since \( \varepsilon > 0 \) is arbitrary, we conclude that (3.3) holds. Let

\[
\varepsilon_r^\pm = \frac{1}{|P_r|} \int_{P_r} v_r^\pm(x)dx
\]

be mean values of \( rG \)-periodic functions \( v_r^\pm(x) \). In view of (3.3)

\[
|\varepsilon_r^\pm| \leq M_r \to 0. \tag{3.5}
\]

By (3.5) we claim that \( |\varepsilon_r^\pm| < \min(\alpha^+ - m, m - \alpha^-) \) for sufficiently large \( r \in \mathbb{N} \). We introduce for such \( r \) the \( rG \)-periodic functions

\[
u_0^+(x) = p(x) + v_r^+(x) + \alpha^+ - m - \varepsilon_r^+, \quad u_0^-(x) = p(x) + v_r^-(x) - (m - \alpha^- + \varepsilon_r^-)
\]

with the mean values \( \alpha^+, \alpha^- \), respectively. In view of (3.2) and the conditions \( \alpha^+ - m - \varepsilon_r^+ > 0, m - \alpha^- + \varepsilon_r^- > 0 \), we have

\[
u_0^-(x) \leq u_0(x) \leq u_0^+(x). \tag{3.6}
\]
Let $u^\pm$ be unique (by [10, Theorem 1.3]) e.s. of (1.1), (1.4) with initial functions $u^\pm_0$, respectively. Taking into account that $(rG)' = \frac{1}{r}G'$, we see that condition (2.7), corresponding to the lattice $rG$ and the mean values $\alpha^-, \alpha^+$, is satisfied. By Theorem 2.2 (or [9, Theorem 1.1]) we find that

$$\text{ess lim } t \to +\infty \int_{P_r} |u^\pm(t, x) - \alpha^\pm| \, dx = 0. \quad (3.7)$$

By the periodicity, for each $y \in \mathbb{R}^n$

$$\int_{y+P_r} |u^\pm(t, x) - \alpha^\pm| \, dx = \int_{P_r} |u^\pm(t, x) - \alpha^\pm| \, dx,$$

which readily implies that for $V = \text{Int } P_r$

$$\|u^\pm(t, x) - \alpha^\pm\|_V = \int_{P_r} |u^\pm(t, x) - \alpha^\pm| \, dx.$$

In view of Lemma 3.2 we have the estimate

$$\|u^\pm(t, x) - \alpha^\pm\|_X \leq C \int_{P_r} |u^\pm(t, x) - \alpha^\pm| \, dx, \ C = C_r = \text{const.}$$

By (3.7) we claim that

$$\text{ess lim } t \to +\infty \|u^\pm(t, \cdot) - \alpha^\pm\|_X = 0. \quad (3.8)$$

Let $u = u(t, x)$ be an e.s. of the original problem (1.1), (1.4) with initial data $u_0(x)$. Since the functions $u^\pm_0$ are periodic, then it follows from (3.6) and the comparison principle [10, Theorem 1.3] that $u^- \leq u \leq u^+$ a.e. in $\Pi$. This readily implies the relation

$$\|u(t, \cdot) - m\|_X \leq \|u^-(t, \cdot) - m\|_X + \|u^+(t, \cdot) - m\|_X \leq \|u^-(t, x) - \alpha^-\|_X + \|u^+(t, x) - \alpha^+\|_X + c(\alpha^+ - m + m - \alpha^-), \quad (3.9)$$

where $c \leq 2^n$ is Lebesgue measure of the unit ball $|x| < 1$ in $\mathbb{R}^n$. In view of (3.8) it follows from (3.9) in the limit as $t \to +\infty$ that

$$\text{ess lim sup } t \to +\infty \|u(t, \cdot) - m\|_X \leq c(\alpha^+ - \alpha^-) \leq 2^n(\alpha^+ - \alpha^-),$$

as was to be proved.
3.2 Proof of Theorem 2.1

We are going to establish that the statement of Proposition 3.1 remains valid in the case of arbitrary $G$. We will suppose that $\dim H < n$, otherwise $p \equiv \text{const}$ and this case has been already considered in Introduction.

**Proposition 3.2.** Assume that $m \in (\alpha^-, \alpha^+)$, where $\alpha^\pm \in F$, and $u = u(t,x)$ is an e.s. of (1.1), (1.4). Then,

$$\limsup_{t \to +\infty} \|u(t,\cdot) - m\|_X \leq 2^{n+3}(\alpha^+ - \alpha^-).$$

**Proof.** Suppose firstly that $v(x) \in L^\infty(\mathbb{R}^n)$ is a finite function, that is, its closed support $\text{supp } v$ is compact. Let $A = H + \text{supp } v = H \oplus K$, where $K$ is the orthogonal projection of $\text{supp } v$ on the space $H^\perp$. We define the functions $v^\pm(x) = \pm \|v\|_\infty \chi_A(x)$, where $\chi_A(x)$ is the indicator function of the set $A$. We define functions $u_0^\pm(x) = p(x) + v^\pm(x)$ and let $u^-(t,x)$ be the smallest e.s. of (1.1), (1.4) with initial data $u_0^-(x)$, $u^+(t,x)$ be the largest e.s. of (1.1), (1.4) with initial data $u_0^+(x)$, existence of such e.s. was established in [10, Theorems 1.1]. Since $u_0^-(x) \leq u_0(x) \leq u_0^+(x)$ a.e. on $\mathbb{R}^n$, we derive that $u^- \leq u \leq u^+$ by the property of monotone dependence of the smallest and the largest e.s. on initial data, cf. [10, Theorem 1.2]. Observe that the initial functions $u_0^\pm(x)$ are constant in direction $H$. Therefore, for any $e \in H$, the functions $u^\pm(t, x + e)$ are the largest and the smallest e.s. of the same problems as $u^\pm(t,x)$. By the uniqueness we claim that $u^\pm(t, x + e) = u^\pm(t,x)$ a.e. in $\Pi$. Hence, $u^\pm(t,x) = u^\pm(t,x')$, $x' = \text{pr}_H x$. As is easy to see, $u^\pm(t,x')$ are the largest and the smallest e.s. of $d$-dimensional problem

$$u_t + \text{div}_{x'}(\tilde{\varphi}(u) - \tilde{a}(u) \nabla_{x'} u) = 0, \quad u(0,x') = u_0^\pm(x')$$

on the subspace $H^\perp$, where $\tilde{\varphi}(u) = P\varphi(u)$, $\tilde{a}(u) = Pa(u)P_s$, $P = \text{pr}_H$ being the orthogonal projection on the space $H^\perp$. In the same way as in the proof of Theorem 2.2 this problem can be written in the standard way (like (2.9)) by an appropriate change of the space variables. Since the initial functions $u_0^\pm(x') = p(x') + v^\pm(x')$, where $p(x')$ is periodic with the lattice of periods $L_0$, while $v^\pm(x')$ are bounded functions with compact support $K$ (so that $v^\pm(x') \in L^\infty(H^\perp)$), we may apply Proposition 3.1. By this proposition,

$$\limsup_{t \to +\infty} \|u^\pm(t,x) - m\|_X \leq \limsup_{t \to +\infty} 2^{n-d}\|u^\pm(t, x') - m\|_X \leq 2^{n-d}2^d(\alpha^+ - \alpha^-) = 2^n(\alpha^+ - \alpha^-),$$

where the first $X$-norm is taken in $L^\infty(\mathbb{R}^n)$ while the second $X$-norm is in $L^\infty(H^\perp)$. Since the e.s. $u(t,x)$ is situated between $u^-$ and $u^+$, then

$$\|u(t,\cdot) - m\|_X \leq \|u^+(t,\cdot) - m\|_X + \|u^-(t,\cdot) - m\|_X$$

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and in view of (3.11)

$$\text{ess lim sup}_{t \to +\infty} \|u(t, \cdot) - m\|_X \leq 2^{n+1}(\alpha^+ - \alpha^-).$$

(3.12)

Now we suppose that \(v \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\). For fixed \(\varepsilon > 0\) we can find a function \(\tilde{v} \in L^\infty(\mathbb{R}^n)\) with compact support such that \(\|v - \tilde{v}\|_1 \leq \varepsilon\). We denote by \(u^+ = u^+(t, x)\), \(u^- = u^-(t, x)\) the largest and the smallest e.s. of (1.1), (1.4) with initial function \(u_0 = p(x) + v(x)\). Similarly, by \(\tilde{u}^+ = \tilde{u}^+(t, x)\), \(\tilde{u}^- = \tilde{u}^-(t, x)\) we denote the largest and the smallest e.s. of (1.1), (1.4) with initial function \(\tilde{u}_0 = p(x) + \tilde{v}(x)\). It is known, cf. \([10, \text{Theorems 1.2}]\), that the largest and the smallest e.s. exhibit the \(L^1\)-contraction property. In particular, for a.e. \(t > 0\)

$$\int_{\mathbb{R}^n} |u^\pm(t, x) - \tilde{u}^\pm(t, x)|\,dx \leq \int_{\mathbb{R}^n} |u_0(x) - \tilde{u}_0(x)|\,dx = \|v - \tilde{v}\|_1 < \varepsilon.$$  

(3.13)

Since the function \(\tilde{v}\) has finite support, relation (3.12) holds for the e.s. \(\tilde{u}^\pm(t, x)\), i.e.,

$$\text{ess lim sup}_{t \to +\infty} \|\tilde{u}^\pm(t, \cdot) - m\|_X \leq 2^{n+1}(\alpha^+ - \alpha^-).$$

(3.14)

In view of (3.13)

$$\|u^\pm(t, \cdot) - \tilde{u}^\pm(t, \cdot)\|_X \leq \|u^\pm(t, \cdot) - \tilde{u}^\pm(t, \cdot)\|_1 < \varepsilon$$

and (3.14) implies the estimates

$$\text{ess lim sup}_{t \to +\infty} \|u^\pm(t, \cdot) - m\|_X \leq 2^{n+1}(\alpha^+ - \alpha^-) + \varepsilon,$$

and since \(\varepsilon > 0\) is arbitrary, we find that

$$\text{ess lim sup}_{t \to +\infty} \|u^\pm(t, \cdot) - m\|_X \leq 2^{n+1}(\alpha^+ - \alpha^-).$$

(3.15)

Since \(u^- \leq u \leq u^+\), then \(\|u(t, \cdot) - m\|_X \leq \|u^+(t, \cdot) - m\|_X + \|u^-(t, \cdot) - m\|_X\) and it follows from (3.15) that

$$\text{ess lim sup}_{t \to +\infty} \|u(t, \cdot) - m\|_X \leq 2^{n+2}(\alpha^+ - \alpha^-).$$

(3.16)

In the general case \(v \in L^\infty(\mathbb{R}^n)\) we choose such \(\delta > 0\) that \(\alpha^- < m - \delta < m + \delta < \alpha^+\) and set \(v_+(x) = \max(v(x) - \delta, 0)\), \(v_-(x) = \min(v(x) + \delta, 0)\). Observe that these functions vanish outside of the set \(|v(x)| > \delta\) of finite measure. Therefore, \(v_\pm \in L^1(\mathbb{R}^n)\). Obviously,

$$u_0(x) \leq u_{0+}(x) = p(x) + \delta + v_+(x), \quad u_0(x) \geq u_{0-}(x) = p(x) - \delta + v_-(x).$$

(3.17)
Notice that \( p(x) \pm \delta \) are periodic functions with the same group of periods \( G \) as \( p(x) \) and with the mean values \( m \pm \delta \in (\alpha_- , \alpha_+ ) \). Let \( u_+(t,x) \) be the largest e.s. of problem (1.1), (1.4) with initial function \( u_{0+}(x) \), and \( u_-(t,x) \) be the smallest e.s. of this problem with initial data \( u_{0-}(x) \). In view of (3.17), we have \( u_-(t,x) \leq u(t,x) \leq u_+(t,x) \). As we have already established, the e.s. \( u_{\pm}(t,x) \) satisfy relation (3.16):

\[
\text{ess lim sup}_{t \to +\infty} \|u_+(t, \cdot) - (m \pm \delta)\|_X \leq 2^{n+2}(\alpha^+ - \alpha^-).
\]

This implies that

\[
\text{ess lim sup}_{t \to +\infty} \|u(t, \cdot) - m\|_X \leq 2^{n+2}(\alpha^+ - \alpha^-) + 2^n \delta, \tag{3.18}
\]

where we use again the fact that measure of a unit ball in \( \mathbb{R}^n \) is not larger than \( 2^n \). Since the e.s. \( u \) is situated between \( u_- \) and \( u_+ \), we derive from (3.18) that

\[
\text{ess lim sup}_{t \to +\infty} \|u(t, \cdot) - m\|_X \leq 2^{n+3}(\alpha^+ - \alpha^-) + 2^{n+1} \delta,
\]

and to complete the proof it only remains to notice that a sufficiently small \( \delta > 0 \) is arbitrary.

Under the assumption of Theorem 2.1 the value \( \alpha^+ - \alpha^- \) in (3.10) may be arbitrarily small. Therefore, (2.6) follows from (3.10). This completes the proof of our main Theorem 2.1. Remark that in the case when the perturbation \( v \geq 0 \) (\( v \leq 0 \)) we can weaken the nonlinearity-diffusivity assumption in Theorem 2.1 by the requirement \( \forall b > m \ (m,b) \cap F \neq \emptyset \) (respectively, \( \forall a < m \ (a,m) \cap F \neq \emptyset \)). However, it is not possible, to weaken this assumption by the condition \( m \in F \), as in the periodic case \( v \equiv 0 \). In the hyperbolic case \( a \equiv 0 \) this was confirmed by [12, Example 2.6]. In the next section we construct another example for the parabolic equation \( u_t - A(u)_{xx} = 0 \).

### 4 Exactness of the nonlinearity-diffusivity assumption for a parabolic equation.

In the one-dimensional case \( n = 1 \) we consider the following purely parabolic equation

\[
u_t - A(u)_{xx} = 0 \tag{4.1}
\]

with the diffusion function \( A(u) = u^+ = \max(0,u) \), so that \( a(u) = A'(u) \) is the Heaviside function (in fact, the problem (4.1), (1.4) is the Stefan problem). Since the flux vector
is absent, the set $F$ in the nonlinearity-diffusivity condition consists of such points $u_0$ that $A(u)$ is not constant in any vicinity $|u - u_0| < \delta$ of $u_0$. It is clear that $F = [0, +\infty)$. First, we are going to construct a $x$-periodic e.s. $u(t, x)$. It is known that e.s. of equation (4.1) is characterized by the Carrillo conditions [1]:

$$x_t - (\text{sign}(u - k)A(u)_x)x = |u - k|t - |A(u) - A(k)|_{xx} \leq 0 \text{ in } D'(\Pi).$$

(4.2)

For piecewise $C^2$-smooth e.s the above conditions imply the following restrictions on discontinuity lines $x = x(t)$ similar to the jump conditions in [13, Theorem 1.1]:

$$[A(u)] = 0,$$

(4.3)

$$\forall k \in \mathbb{R} \quad [u - k]\nu_0 - [\text{sign}(u - k)A(u)_x]\nu_1 \leq 0,$$

(4.4)

where $[w] = w_+ - w_-, \ w_\pm = w_\pm(t) = w(t, x(t)\pm)$, is the jump of a function $w = w(t, x)$ across the line $x = x(t)$, and $\nu = (\nu_0, \nu_1)$ is a normal vector on this line, directed from $w_-$ to $w_+$ (we may take $\nu = (-x'(t), 1)$). Conditions (4.3), (4.4) follow from entropy relation (4.2) by application to a nonnegative test function and integration by parts (with the help of Green’s formula). It is not difficult to verify that these conditions together with the requirement that $u$ is a classical solution in domains of its smoothness are equivalent to the statement that $u$ is an e.s. of (4.1). Let us make condition (4.4) more precise. Taking $k > \max(u_-, u_+)$, $k < \min(u_-, u_+)$, we readily arrive at the Rankine-Hugoniot relation

$$[u]\nu_0 - [A(u)_x]\nu_1 = 0.$$  

(4.5)

Assuming (for fixed $t$) that $u_+ > u_-$ and putting relation (4.4) together with (4.5), we arrive at the inequality

$$(u_+ - k)\nu_0 - (A(u)_x)_+\nu_1 \leq 0 \quad \forall k \in [u_-, u_+].$$

This inequality is valid whenever it holds in the end-points $k = u_\pm$. Taking $k = u_+$, we obtain $(A(u)_x)_+ \geq 0$. For $k = u_-$ we have $(u_+ - u_-)\nu_0 - (A(u)_x)_+\nu_1 \leq 0$, and subtracting (4.5), we obtain that $(A(u)_x)_- \geq 0$, Similarly, in the case $u_+ < u_-$ we derive the inequalities $(A(u)_x)_\pm \leq 0$. Hence,

$$\text{sign}(u_+ - u_-)(A(u)_x)_\pm \geq 0$$

(4.6)

The obtained conditions are different from the jump conditions of [13, Theorem 1.1]. This connected with the fact that our diffusion function $A(u)$ is merely Lipschitz and the derivative $A(u)_x$ may have nontrivial traces at a discontinuity line.

To construct the desired solution, we need to solve the following initial-boundary value problem for the heat equation $u_t = u_{xx}$ in the domain $t > 0, |x| < r(t) = 2 - e^{-\alpha t}$, where
the positive constant \( \alpha \) will be indicated later, with homogeneous Dirichlet condition \( u(t, \pm r(t)) = 0 \) and with the initial condition \( u(0, x) = \varphi(x) \in C^\infty((-1, 1)) \), where the initial function is even, \( \varphi(-x) = \varphi(x) \).

Making the change \( y = x/r(t) \) and denoting \( v = v(t, y) = u(t, yr(t)) \), we reduce our problem to the standard problem in the fixed segment \( |y| \leq 1 \)

\[
v_t = \frac{1}{r^2} v_{yy} + \frac{r'}{r} y v_y, \quad v(t, \pm 1) = 0, \quad v(0, y) = \varphi(y). \tag{4.7}
\]

Notice that the coefficient \( 1/4 < \frac{1}{r^2} \leq 1 \) and in correspondence with the classic results [7], this problem admits a unique classical solution \( v(t, y) \). Since \( v(t, -y) \) is a solution of the same problem then, by the uniqueness, \( v(t, -y) = v(t, y) \). Moreover, this solution is bounded by the maximum principle. Differentiating (4.7) with respect to the space variable \( y \), we derive that the functions \( w = v_y \) and \( p = v_{yy} \) satisfy the equations

\[
w_t = \frac{1}{r^2} w_{yy} + \frac{r'}{r} (yw_y + w), \tag{4.8}
\]

\[
p_t = \frac{1}{r^2} p_{yy} + \frac{r'}{r} (yp_y + 2p). \tag{4.9}
\]

It follows from equations (4.7), (4.8) and the boundary condition \( u(t, \pm 1) = 0 \) that at the boundary points \( y = \pm 1 \)

\[
p = v_{yy} = -rr'yv_y = -rr'yw, \tag{4.10}
\]

\[
\frac{1}{r^2} p_y = \frac{1}{r^2} w_{yy} = w_t - \frac{r'}{r} (yw_y + w) = w_t - \frac{r'}{r} (yp + w). \tag{4.11}
\]

We multiply (4.9) by \( 2p \) and integrate over the variable \( y \). This yields the relation

\[
\frac{\partial}{\partial t} \int_{-1}^{1} p^2 dy = \frac{2}{r^2} \int_{-1}^{1} pp_y dy + \frac{r'}{r} \int_{-1}^{1} (p^2)_y dy + \frac{4r'}{r} \int_{-1}^{1} p^2 dy. \tag{4.12}
\]

Integrating by parts, we obtain

\[
\int_{-1}^{1} pp_{yy} dy = - \int_{-1}^{1} (p_y)^2 dy + pp_y|_{y=1}, \quad \int_{-1}^{1} (p^2)_y dy = - \int_{-1}^{1} p^2 dy + p^2|_{y=1}.
\]

Placing these equalities into (4.12) and taking into account relations (4.11), we arrive at
the relation
\[
\frac{\partial}{\partial t} \int_{-1}^{1} p^2 dy = -\frac{2}{r^2} \int_{-1}^{1} (p_y)^2 dy + 2(w_t - \frac{r'}{r} w)p|_{y=-1}^{y=1} - \frac{2r'}{r} p^2 y|_{y=-1}^{y=1} + \frac{3r'}{r} \int_{-1}^{1} p^2 dy +
\]
\[
\frac{r'}{r} p^2 y|_{y=-1}^{y=1} = -\frac{2}{r^2} \int_{-1}^{1} (p_y)^2 dy + \frac{3r'}{r} \int_{-1}^{1} p^2 dy - 2rr'(w_t - \frac{r'}{r} w)w|_{y=-1}^{y=1} - \frac{r'}{r} p^2 y|_{y=-1}^{y=1} \leq
\]
\[
-\frac{2}{r^2} \int_{-1}^{1} (p_y)^2 dy + \frac{3r'}{r} \int_{-1}^{1} p^2 dy - rr' \sum_{y=\pm 1} (w^2)_t(t, y) + 2(r')^2 \sum_{y=\pm 1} w^2(t, y),
\]
where we drop the non-positive term $-\frac{r'}{r} p^2 y|_{y=-1}^{y=1}$. Since $rr'(w^2)_t = (rr'w^2)_t - (rr'')w^2 = (rr'w^2)_t - (rr'' + (r')^2)w^2$, this relation can be written as
\[
\frac{\partial}{\partial t} \left( \int_{-1}^{1} p^2 dy + rr' \sum_{y=\pm 1} w^2(t, y) \right) \leq -\frac{2}{r^2} \left( \int_{-1}^{1} (p_y)^2 dy + \sum_{y=\pm 1} p^2(t, y) \right) + \frac{2}{r^2} \sum_{y=\pm 1} p^2(t, y) + \frac{3r'}{r} \int_{-1}^{1} p^2 dy + (rr'' + 3(r')^2) \sum_{y=\pm 1} w^2(t, y) \leq
\]
\[
-\frac{2}{r^2} \left( \int_{-1}^{1} (p_y)^2 dy + \sum_{y=\pm 1} p^2(t, y) \right) + \frac{3r'}{r} \int_{-1}^{1} p^2 dy + 5(r')^2 \sum_{y=\pm 1} w^2(t, y), \tag{4.13}
\]
where we used that $r'' = -\alpha^2 e^{-\alpha t} < 0$ and that $\frac{2}{r^2} \sum_{y=\pm 1} p^2(t, y) = 2(r')^2 \sum_{y=\pm 1} w^2(t, y)$, by (4.10). Now we apply the Poincare inequality
\[
\int_{-1}^{1} p^2 dy \leq c \left( \int_{-1}^{1} (p_y)^2 dy + \sum_{y=\pm 1} p^2(t, y) \right), \quad c = \text{const},
\]
and derive
\[
\frac{\partial}{\partial t} \left( \int_{-1}^{1} p^2 dy + rr' \sum_{y=\pm 1} w^2(t, y) \right) \leq \left( -\frac{2}{cr^2} + \frac{3r'}{r} \right) \int_{-1}^{1} p^2 dy + 5(r')^2 \sum_{y=\pm 1} w^2(t, y) = \left( -\frac{1}{cr^2} + \frac{3r'}{r} \right) \int_{-1}^{1} p^2 dy - \frac{1}{cr^2} \int_{-1}^{1} p^2 dy + 5(r')^2 \sum_{y=\pm 1} w^2(t, y). \tag{4.14}
\]
Since \( w = u_y \) is an odd function and \( p = w_y \), then \( w(t,0) = 0 \) and \( w(t,y) = \int_0^y p(t,s)ds \). By Jensen’s inequality, this implies the estimate \( (w(t,y))^2 \leq \int_0^y (p(t,s))^2ds \) for all \( y \in [-1,1] \). In particular,

\[
\sum_{y=\pm1} w^2(t,y) \leq \int_{-1}^1 (p(t,y))^2dy
\]

and it follows from (4.14) that

\[
\frac{\partial}{\partial t} \left( \int_{-1}^1 p^2dy + rr' \sum_{y=\pm1} w^2(t,y) \right) \leq -\left( \frac{1}{cr^2} - \frac{3r'}{r} \right) \int_{-1}^1 p^2dy - \left( \frac{1}{cr^2} - 5(r')^2 \right) \sum_{y=\pm1} w^2(t,y). \tag{4.15}
\]

Observe that \( \frac{1}{cr^2} > \frac{1}{4e} \) while \( \frac{3r'}{r} \leq 3\alpha, \ 5(r')^2 \leq 5\alpha^2 \). Therefore, we can choose \( \alpha > 0 \) so small that \( \frac{1}{cr^2} - \frac{3r'}{r} > 3\alpha, \ \frac{1}{cr^2} - 5(r')^2 > 3\alpha, \ rr' < 2\alpha < 1 \). Then, it follows from (4.15) that

\[
\frac{\partial}{\partial t} \left( \int_{-1}^1 p^2dy + rr' \sum_{y=\pm1} w^2(t,y) \right) \leq -3\alpha \left( \int_{-1}^1 p^2dy + rr' \sum_{y=\pm1} w^2(t,y) \right),
\]

which implies the estimate

\[
\int_{-1}^1 p^2dy + rr' \sum_{y=\pm1} w^2(t,y) \leq Ce^{-3\alpha t}, \ C = \text{const}. \tag{4.16}
\]

Since \( rr' = \alpha re^{-\alpha t} > \alpha e^{-\alpha t} \), we conclude that

\[
|v_y(t,y)| = |w(t,y)| \leq C_1 e^{-\alpha t} \leq C_2 r'(t) \ \forall t > 0, \ y = \pm1, \ C_1, C_2 = \text{const}. \tag{4.17}
\]

It also follows from (4.16) that

\[
v(t,y) \leq \text{const} \left( \int_{-1}^1 p^2dy \right)^{1/2} \leq \text{const} \cdot e^{-3\alpha t/2} \rightarrow 0. \tag{4.18}
\]

We are going to construct a periodic e.s. of (4.1) with period 5 such that on the segment \([-5/2, 5/2]\) it has the following structure: \( u(t,x) \) is the described above solution of the Cauchy-Dirichlet problem for the heat equation in the domain \(|x| < r(t)\), so that \( u(t,x) = v(t,x/r(t)) \), where \( v(t,y) \) is the solution of problem (4.7). For \( r(t) < |x| < 5/2 \)
our e.s. \( u = u(x) \leq 0 \) satisfies the equations \( u_t = 0 \), for \( 1 < |x| < 2 \) the values \( u(x) = -\psi(|x|) \) uniquely determined by Rankine-Hugoniot relation (4.5) on the line \( x = r(t) \): 
\[ \psi(r)r' + u_x(t, r) = 0, \] 
so that 
\[ \psi(r(t)) = -u_x(t, r(t))r'(t) = -v_y(t, 1)/(r(t)r'(t)) > 0. \] 
In view of estimate (4.17) \( \psi(x) \in L^\infty((1, 2)) \). Finally, in the strips \( 2 < |x| < 5/2 \) we set \( u = 0 \). By the periodicity, the e.s. \( u(t, x) \) is then extended on the whole half-plane \( \Pi \), see figure 1. By the construction, \( u(t, x) \) satisfies all requirements (4.3), (4.5), (4.6) on the discontinuity lines \( x = \pm r(t) \), Hence, it is an e.s. of the Cauchy problem for equation (4.1) with 5-periodic initial data
\[
\begin{cases}
\phi(x), & |x| < 1, \\
-\psi(|x|), & 1 < |x| < 2, \\
0, & 2 < |x| < 5/2.
\end{cases}
\]
Notice that
\[
2 \int_1^2 \psi(x)dx = 2 \int_0^\infty \psi(r(t))r'(t)dt = - \int_0^\infty (u_x(t, r(t)) - u_x(t, -r(t)))dt = \\
- \int_0^\infty dt \int_{|x|<r(t)} u_{xx}(t, x)dx = - \int_{-2}^2 dx \int_{|x|<r(t)} u_t(t, x)dt = \int_{-1}^1 \phi(x)dx,
\]
where we use that \( u = 0 \) on the set \( |x| = r(t) \). This equality implies that
\[
m = \frac{1}{5} \int_{|x|<5/2} u(0, x)dx = \frac{1}{5} \left( \int_{-1}^1 \phi(x)dx - 2 \int_1^2 \psi(x)dx \right) = 0.
\]
Using (4.18), we conclude that \( u(t, x) \) satisfies the decay property
\[
\lim_{t \to +\infty} u(t, \cdot) = 0 \text{ in } L^1((-5/2, 5/2)).
\]
This is consistent with the statement of Theorem 2.2 since \( m = 0 \in F \). But \( (a, 0) \cap F = \emptyset \) for each \( a < 0 \) and the nonlinearity-diffusivity condition of Theorem 2.1 is violated. Let us show that after a perturbation \( p(x) + v(x) \), \( v(x) \in L_0^\infty(\mathbb{R}) \), of initial data the decay property can fail. In fact, taking the perturbation \( v(x) < 0 \) supported on the segment \([2, 3]\), we find that the corresponded e.s. is \( u(t, x) + v(x) \) and it does not satisfy the decay property, see figure 2.

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Figure 1: The solution $u(t, x)$ with the periodic initial data.

Figure 2: The solution $u(t, x)$ with the perturbed initial data.

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