Domains in $\mathbb{C}^{n+1}$ with Noncompact Automorphism Group. II

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§1. Introduction

We consider here smoothly bounded domains with noncompact automorphism groups. Examples of such domains may be obtained as follows. To the variables $z_1, \ldots, z_n$ we assign weights $\delta_1, \ldots, \delta_n$ with $\delta_j = 1/2m_j$ for $m_j$ a positive integer. If $J = (j_1, \ldots, j_n)$ and $K = (k_1, \ldots, k_n)$ are multi-indices, we set $\text{wt}(J) = j_1\delta_1 + \ldots + j_n\delta_n$ and $\text{wt}(z^J \bar{z}^K) = \text{wt} J + \text{wt} K$. We consider real polynomials of the form

$$p(z, \bar{z}) = \sum_{\text{wt} J = \text{wt} K = \frac{1}{2}} a_{JK} z^J \bar{z}^K. \quad (1.1)$$

The reality condition is equivalent to $a_{JK} = \bar{a}_{KJ}$. The balance of the weights, i.e. $\text{wt} J = \text{wt} K$ for each monomial in $p$, implies that the domain

$$G = \{(w, z_1, \ldots, z_n) \in \mathbb{C} \times \mathbb{C}^n : |w|^2 + p(z, \bar{z}) < 1\} \quad (1.2)$$

is invariant under the $T^2$-action

$$(\varphi, \theta) \mapsto (e^{i\varphi} w, e^{i\delta_1\theta} z_1, \ldots, e^{i\delta_n\theta} z_n) \quad (1.3)$$

In addition, the weighted homogeneity of $p$ allows the transform $(w, z) \mapsto (w^*, z^*)$ defined by

$$w = \left(1 - \frac{iw^*}{4}\right) \left(1 + \frac{iw^*}{4}\right)^{-1}, \quad z_j = z_j^* \left(1 + \frac{iw^*}{4}\right)^{-2\delta_j} \quad (1.4)$$

to map the domain $G$ biholomorphically onto

$$D = \{(w, z_1, \ldots, z_n) \in \mathbb{C} \times \mathbb{C}^n : \Re m w + p(z, \bar{z}) < 1\}. \quad (1.5)$$

This unbounded representation of $G$ shows that the automorphism group is noncompact, since it is invariant under translation in the $\Re w$-direction. Since $p$ is homogeneous, there is also a 1-parameter family dilations; so with (1.3), $\text{Aut}(G)$ has dimension at least 3.

Presumably, every smoothly bounded domain $\Omega \subset \mathbb{C}^{n+1}$ with noncompact automorphism group is equivalent to a domain of the form (1.2). Several papers have been written on the general subject of describing such domains, starting with the work of R. Greene and S. Krantz (see [BP] for a more complete list of references). Our approach to this
problem may be thought of as involving two steps. The first is to show that \( \Omega \) is biholomorphically equivalent to one of a more special class of model domains \( D \). Such a model domain would have a nontrivial holomorphic vector field. The second step is to transport this vector field back to \( \Omega \), analyze such a vector field tangent to \( \partial \Omega \) at the parabolic fixed point, and use this information determine the original domain.

The first progress in this direction was obtained for the first step. S. Frankel [F] (see also Kim [K]) showed that if \( \Omega \) is a (possibly nonsmooth) convex domain, then \( \Omega \) has an unbounded representation which is independent of the variable \( u \). This particular model domain, however, is not well enough behaved for us to carry out step 2 of our program. In [BP] we used the approach described above in the case where the Levi form of \( \partial \Omega \) had at most one zero eigenvalue, and we obtain model domains with polynomial boundaries.

Here we consider the case of convex domains, which makes the scaling arguments of [BP] easier. Thus we are able to complete step 1 more quickly and make the transition to step 2. Now we focus on the second step and show that if a domain has a vector field of positive weight (defined below), then the domain must be of the form (1.2). We note that our main result, Theorem 3.7, is proved for domains considerably more general than convex.

By holomorphic vector field we mean a vector field of the form \( H = \sum_{j=0}^{n} H_j \frac{\partial}{\partial z_j} \), where \( H_j \) is holomorphic on \( \Omega \), and we use the notation \( z_0 = w \). If the coefficients of \( H \) extend smoothly to \( \overline{\Omega} \), and if the real vector field \( \Re H = \frac{1}{2}(H + \overline{H}) \) is tangent to \( \partial \Omega \), then we will say that \( H \) is tangent to \( \partial \Omega \). If \( H \) is a holomorphic vector field which is tangent to \( \partial \Omega \), then the exponential \( \exp(tH) \) defines an automorphism of \( \Omega \) for any \( t \in \mathbb{R} \).

We assign weight 1 to the variable \( w = z_0 \), i.e. \( \delta_0 = 1 \). Thus we set \( wt(z^J \frac{\partial}{\partial z_k}) = wt(J) - \delta_k \) for any multi-index \( J = (j_0, \ldots, j_n) \), and \( 0 \leq k \leq n \). If \( H \) is a holomorphic vector field (not necessarily homogeneous), we let \( wtH \) denote the smallest weight of a nonzero homogeneous term in the Taylor expansion of \( H \) at 0. If \( H \) vanishes to infinite order, we set \( wtH = \infty \). The one-parameter subgroup of translations in the \( \Re w \)-direction (automorphisms of \( D \)), are generated by the vector field \( 2 \frac{\partial}{\partial w} \), which has a fixed point at \( \infty \). If \( p \geq 0 \), then \( w \mapsto -1/w \) is well-defined on \( D \), and the vector field \( 2 \frac{\partial}{\partial w} \) is taken into \( 2w^2 \frac{\partial}{\partial w} \), which has a fixed point at 0 and weight 1 there.

Let us fix a point \( (w^0, z^0) \in \partial \Omega \). After a translation and rotation of coordinates, we may assume that \( (w^0, z^0) = (0, 0) \) and that near \( (0, 0) \) we have \( \Omega = \{ v + f(u, z, \overline{z}) \} \) with \( f(0, 0) = \nabla f(0, 0) = 0 \). An assignment of weights \( \delta_1, \ldots, \delta_n \) is admissible if there is a homogeneous polynomial \( p(z, \overline{z}) \) (consisting of all the terms in the Taylor expansion of \( f \) weight 1) such that all the terms in the Taylor expansion of \( f - p \) have weight strictly greater than 1. In this case we define

\[
\Omega_{hom} = \{ v + p(z, \overline{z}) < 0 \}
\]

to be the homogeneous model of \( \Omega \) at \( (0, 0) \). (In general the homogeneous model depends on the choice of weights.)

**Theorem 1.** Let \( \Omega \) be a domain with smooth boundary, and suppose that there is an assignment of weights at a point \( (0, 0) \in \partial \Omega \) such that in the homogeneous model \( p \geq 0 \),
and $p$ does not vanish on any complex variety passing through 0. If there is a tangential holomorphic vector field $H$ for $\Omega$ with $H(0) = 0$ and $0 < wt H < \infty$, then $\Omega$ is biholomorphically equivalent to a domain of the form (1.2).

A tangential holomorphic vector field $H$ generates a 1-parameter group of automorphisms via the exponential map $t \mapsto \exp(tH)$. The hypothesis that $wt H > 0$ in Theorem 1 serves to eliminate the possibility that $H$ generates a local translation at $(0,0)$ (in which case $wt H < 0$) or a local dilation or rotation (in both cases $wt H = 0$). Our motivation in proving Theorem 1 is that with the addition of the results and techniques of [BP] we obtain the following.

**Theorem 2.** Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded, convex set with smooth, finite type boundary. If $\text{Aut}(\Omega)$ is noncompact, then $\Omega$ is biholomorphically equivalent to a convex domain of the form (1.2).

Theorem 1 does not make any assumption of pseudoconvexity. Our arguments need some sort of convexity hypothesis, however, to pass from the noncompactness of $\text{Aut}(\Omega)$ to the existence of a vector field $H$.

It is worth noting that Theorem 2 gives a classification, up to biholomorphism, of the convex, finite type domains with noncompact automorphism groups. For by (1.3), a domain of the form (1.2) is a Cartan domain, and by [KU] it follows that two domains of the form (1.2) are biholomorphically equivalent if and only if they are linearly equivalent.

Most of the following paper is devoted to the analysis of homogeneous vector fields tangent to $\partial \Omega$. Only certain vector fields can arise. In §2 we show that there can be no tangent vector field of the form (2.5), i.e. independent of the variable $z_0 = w$. The next major case to analyze is the case of vector fields of positive weight. This is done in §3. In §4 we show how to apply these algebraic results to domains with noncompact automorphism groups. For the most part, this is a reiteration of results of [BP]. In the convex case, we have more flexibility in our normal family arguments. Thus we give Lemma 4.1, which makes the transition between steps one and two more natural and understandable than the treatment in [BP, Lemma 7].

§2. Holomorphic Tangent Vector Fields

A vector field on $\mathbb{C}^n$ may be written as $Q = \sum q_j(z) \frac{\partial}{\partial z_j}$, and $Q$ is homogeneous of weight $\mu$ if $wt q_j = \mu - \delta_j$ holds for $1 \leq j \leq n$. An integral curve of $Q$ is a holomorphic function $\varphi : D \to \mathbb{C}^n$ for some domain $D \subset \mathbb{C}$ such that $\dot{\varphi}(t) = Q(\varphi(t))$ for all $t \in D$. If $D$ is a maximal domain of analyticity of an integral curve $\varphi$, then the image $\varphi(D)$ is a complex orbit of $Q$.

A complex orbit is necessarily unbounded unless it is a constant (i.e. a critical point). For if $\varphi : D \to \mathbb{C}^n$ is an integral curve, and if $|\varphi(t)| < M$ on $D$ then for any point $t_0 \in D$, the solution of $\dot{\varphi}(t) = Q(\varphi(t))$ may be analytically extended to a disk $\{|t-t_0| < r\}$ where $r$ is independent of $t_0$. Thus $\varphi$ extends to an integral curve $\varphi : \mathbb{C} \to \mathbb{C}^n$. Since $\varphi$ is bounded, it must be constant, and so $\varphi(D)$ must be a critical point of $Q$.

For $\tau \in \mathbb{C}$ we define the dilation $D_\tau : \mathbb{C}^n \to \mathbb{C}^n$ by

$$D_\tau(z) = (\tau^{\delta_1} z_1, \ldots, \tau^{\delta_n} z_n),$$
where \( \tau^\delta \) denotes an arbitrary but fixed choice of fractional power. If \( \varphi \) is a solution of the vector field \( Q \), then so is \( \varphi_\tau(t) = D_\tau(\varphi(\tau^\mu t)) \). Thus if \( S \) is an orbit of \( Q \), then so is \( D_\tau(S) \).

We start with some observations about homogeneous vector fields on \( \mathbb{C}^n \).

**Lemma 2.1.** Let \( Q \) be a nonzero homogeneous vector field of weight \( \mu \neq 0 \), and set \( A := \{Q = 0\} \). If \( A = \{0\} \), then there is an orbit \( S \) of \( Q \) which contains 0 in its closure.

More generally, let \( \Gamma \subset \mathbb{C}^n \) be a subvariety such that \( A \cap \Gamma = \{0\} \) and \( \Gamma \) is invariant under \( D_\tau \) for all \( \tau \in \mathbb{C} \). If \( Q \) is tangent to \( \Gamma \) at all regular points, then there exists an orbit \( S \) of \( Q \) with \( S \subset \Gamma \) and which contains 0 in its closure.

**Proof.** We assume first that \( A = \{0\} \). Let \( S' \) be a nontrivial orbit of \( Q \) passing through a point \( b_0 \), and take a sequence \( a_k \in S' \), \( |a_k| \to \infty \). For each \( k \), we choose a dilation \( D_\epsilon \) with \( \epsilon = \epsilon_k \) chosen such that \( |D_\epsilon(a_k)| = 1 \). For each \( k \), \( S_k := D_\epsilon(S') \) is an orbit of \( Q \) and contains the point \( b_k := D_\epsilon(b_0) \). Since \( b_k \to 0 \) as \( k \to \infty \), we may assume that \( |b_k| < 1 \). Thus there exists a point \( a_k \in S_k \) with \( |a_k| = 1 \). Passing to a subsequence, we may assume that \( a_k \to a_0 \) with \( |a_0| = 1 \). Since \( Q(a_0) \neq 0 \), there is a nontrivial orbit \( S \) passing through \( a_0 \), and it is a basic property of systems of ordinary differential equations that the orbits \( s_k \) converge to the orbit \( S \). Since the \( b_k \in S_k \) converge to 0, it follows that 0 is in the closure of \( S \).

The proof of the second assertion is identical; we just observe that \( S_k \) and \( S \) remain inside \( \Gamma \).

**Lemma 2.2.** Let \( Q \) be a nonzero homogeneous vector field of weight \( \mu \neq 0 \) which is not identically zero. Then there is a nontrivial set \( \Sigma \) which is invariant under \( D_\tau \) for all \( \tau \in \mathbb{C} \) such that any solution \( f \) of \( Qf = 0 \) satisfies \( f = f(0) \) on \( \Sigma \).

**Proof.** Let us fix a point \( c \) with \( Q(c) \neq 0 \), and let \( S \) denote the orbit passing through \( c \). Let \( \Sigma \) denote the set of all limit points of sequences \{\( a_\nu \)\} with \( a_\nu \in D_\tau S \) and \( \tau_\nu \to 0 \). We note that \( \Sigma \cap \{|z| = 1\} \neq \emptyset \). It follows that \( \Sigma \) is invariant under \( D_\tau \). Further if \( a_0 \in \Sigma \), then there exists a sequence \( a_\nu \in D_\tau S \) with \( a_\nu \to a_0 \) and \( \tau_\nu \to 0 \). Thus \( b_\nu := D_\tau c \to 0 \). If \( Qf = 0 \), then \( f \) constant on \( D_\tau S \), so \( f|_{D_\tau S} = f(b_\nu) \). Thus \( f(a) = \lim_{\nu \to \infty} f(b_\nu) = f(0) \). Since this holds for all \( a \in \Sigma \), \( f|_{\Sigma} = f(0) \).

Any polynomial of \( z \) and \( \bar{z} \) may be written in the form \( p = \sum_{A,B} c_{A,B} z^A \bar{z}^B \). We define the signature \( \delta(A,B) = \text{wt } A - \text{wt } B \). Thus we may write \( p \) as a sum of monomials of fixed signature

\[
p = p^{(\nu_1, \ldots, \nu_{L+1})} + p^{(\nu_1, \ldots, \nu_{L})} + \ldots + p^{(\nu_1, \ldots, \nu_0)} + p^{(\nu_1)},
\]

where \( \nu_1 = -\nu_1, \nu_0 = 0 \), and \( \nu_1 < \nu_{L+1} \). Thus \( p^{(\nu_1, \ldots, \nu_{L+1})} = \overline{p^{(\nu_1, \ldots, \nu_{L})}} \). We may define holomorphic functions \( f_{\nu,B} \) by summing first over the indices \( A \):

\[
p^{(\nu)} = \sum_{\delta(A,B) = \nu} c_{A,B} z^A \bar{z}^B = \sum_B f_{\nu,B}(z) \bar{z}^B.
\]

We say that \( p \) is balanced if \( p = p^0 \). A homogeneous polynomial of weight 1 is balanced if and only if it has the form (1.1). We will assume throughout the rest of this paper that \( p \) is homogeneous of weight 1. Thus each of the functions \( f_{\nu,B} \) is homogeneous of weight \( (\nu + 1)/2 \).
Lemma 2.3. Let $Q$ be a homogeneous vector field with $Q(0) = 0$, and suppose that $S$ is a nontrivial orbit of $Q$ with $0 \in \overline{S}$, and such that $Q \neq 0$ on $S$. If $p^{(\nu)}$ is in the form (2.2) and if $Q^\alpha p^{(\nu)} = 0$ on $\mathbb{C}^n$, then $p^{(\nu)}|_S = 0$.

Proof. Since the vector field $Q$ is tangent to $S$, it follows that $Q^\alpha p^{(\nu)}|_S = (Q^\alpha|_S)(p^{(\nu)}|_S)$. Thus from the equation

$$Q^\alpha p^{(\nu)} = \sum Q^\alpha f_{\nu,B}(z)\bar{z}^B = 0$$

we deduce that $Q^\alpha f_{\nu,B} = 0$ on $\mathbb{C}^n$ for all $\nu$ and $B$, and thus this holds on $S$. Since $Q \neq 0$ on $S$, $Q^{-1}f_{\nu,B}$ is constant on $S$. But since $Q(0) = 0$, we have $Q^{-1}f_{\nu,B}(0) = 0$, and since $0 \in \overline{S}$ it follows that $Q^{-1}f_{\nu,B}|_S = 0$. Proceeding in this way, we have $Q^\alpha f_{\nu,B}|_S = \ldots = Q f_{\nu,B}|_S = f_{\nu,B}|_S = 0$. Since this holds for all $\nu$ and $B$, we conclude that $p^{(\nu)} = 0$ on $S$. \qed

We let $\mathcal{A}(\mu)$ denote the set of holomorphic vector fields which are homogeneous of weight $\mu$ and which are tangential to the domain $\Omega_{hom}$. With the notation

$$Q = q_0 \frac{\partial}{\partial w} + \sum_{j=1}^n q_j \frac{\partial}{\partial z_j}$$

the tangency condition is given by

$$\Re \left( -\frac{i}{2} q_0 + \sum_{j=1}^n q_j \frac{\partial p}{\partial z_j} \right) = 0 \quad (2.3)$$

for all $(w, z) \in \partial\Omega_{hom}$.

Without loss of generality we may assume that

$$p = \sum_{A,B} c_{A,B} z^A \bar{z}^B$$

contains no holomorphic (or antiholomorphic) monomials, \hspace{1cm} (2.4)

i.e. neither multi-index $A$ or $B$ in the summation is equal to $(0, \ldots, 0)$. Since $p$ is real, this is equivalent to the condition $f_{\nu,B}(0) = 0$ for all $f_{\nu,B}$ in (2.2).

Our first step will be to eliminate tangent vector fields of the special form (independent of the variable $w$)

$$Q = \sum_{j=1}^n q_j(z_1, \ldots, z_n) \frac{\partial}{\partial z_j} \quad (2.5)$$

from the context of Theorems 1 and 2. Such a vector field can belong to $\mathcal{A}(\mu)$ in the degenerate case $Qp = 0$. Vector fields of the form (2.5) arise, too, as rotations $\sum c_{ij} z_i \frac{\partial}{\partial z_j}$ in the case of weight zero. Another possibility with positive weight is as follows.

Example. Let $f = z_1^3 z_2^2$, $g = z_1 z_2$, $p = 2\Re f \bar{g}$, and $Q = iz_1^2 z_2(2z_1 \frac{\partial}{\partial z_1} - 3z_2 \frac{\partial}{\partial z_2})$. Then $Qp = -if$, and $\Re Qp = 0$.
In order to eliminate the possibility of nonzero vector fields of the form (2.5) with weight \( \mu \neq 0 \), we will make the hypothesis

\[ \{ p = 0 \} \text{ contains no nontrivial complex manifold.} \]  

(2.6)

We will say that \((\tilde{z}_1, \ldots, \tilde{z}_n)\) is a \textit{weighted change of coordinates} if \(\tilde{z}_j\) is a homogeneous polynomial of \((z_1, \ldots, z_n)\), and its weight is equal to \(wtz_j\). This leads to a new polynomial \(\tilde{p}\), defined by \(\tilde{p}(\tilde{z}) = p(z)\). It is evident that \(\tilde{p}\) satisfies (2.4) and (2.6) if and only if \(p\) does. Similarly, \(\tilde{p}\) is balanced if and only if \(p\) is.

**Lemma 2.4.** If \(Q \in A(\mu)\), then \(Q(0) = 0\).

**Proof.** For otherwise, \(q_j = c \neq 0\) for some \(j\). In this case we may make a homogeneous change of coordinates to bring \(Q\) into the form \(\frac{\partial}{\partial z_j}\). This means that \(p\) is independent of the variable \(x_j\). However, setting \(z_k = 0\) for \(k \neq j\), we must have a nontrivial homogeneous polynomial of weight \(\mu\) in the variable \(z_j\) alone, by (2.6). On the other hand, the only polynomial independent of \(x_j\) is \((i(z_j - \bar{z}_j))^{m_j}\), which violates (2.4). Thus we must have \(Q(0) = 0\).

**Proposition 2.5.** Let \(Q \in A(\mu), \mu \neq 0\), be a vector field of the form (2.5). If (2.6) holds, then \(Q = 0\).

**Proof.** Let us suppose that \(\mu > 0\). (The case \(\mu < 0\) is similar.) The tangency condition is

\[ \Re Q_p = \sum_{l = -L}^{L} \Re Q_p^{(\nu_l)} = 0. \]  

(2.7)

We note that \(Q_p^{(\nu_L)}\) is the only term in (2.7) with signature \(\mu + \nu_L\), which is the largest possible, and there are no terms of signature \(-\mu - \nu_L\) which might cancel with it upon taking the real part. Thus

\[ Q_p^{(\nu_L)} = 0. \]

The terms of signature \(\nu_{L-1} + \mu\) in (2.7) must vanish, so this condition is given either by

\[ Q_p^{(\nu_{L-1})} \]

alone or by

\[ Q_p^{(\nu_{L-1})} + \overline{Q_p^{(-\nu_L)}} = 0; \]  

(2.8)

the occurrence of the second case depends on whether \(\nu_{L-1} + \mu = -(\nu_L + \mu)\) or not.

We show now that in case (2.8) we have \(Q^2 p^{(\nu_{L-1})} = 0\). We recall that

\[ p^{(-\nu_L)} = p^{(\nu_L)} = \sum f^{\nu_L,B}(z)z^B \]

which gives

\[ \overline{Q p^{(-\nu_L)}} = \sum f^{\nu_L,B}(z)\overline{Q z^B}. \]  

(2.9)
Since \( Q p^{(\nu L)} = 0 \), we have \( Q f_{\nu L, B} = 0 \) for all \( B \). Combining this with (2.9), we have

\[
Q Q p^{(-\nu L)} = \sum Q f_{\nu L, B} Q z^B = 0.
\]

From (2.8), then, we have \( Q^2 p^{(\nu L-1)} = 0 \). Continuing in this fashion, we have

\[
Q^\alpha p^{(\nu_l)} = 0 \quad \text{for some} \quad \alpha \leq L - l + 1.
\]

Now set \( A := \{ Q = 0 \} \). In case \( A = \{ 0 \} \), we let \( S \) denote the orbit given by Lemma 2.1. By the remarks above, we must have \( Q(0) = 0 \), so that by Lemma 2.3 it follows from (2.4) that \( p^{(\nu_l)}|_S = 0 \) for \( -L \leq l \leq L \). Thus we have \( S \subset \{ p = 0 \} \) which contradicts (2.6).

The other case is \( A \neq \{ 0 \} \). If \( Q \neq 0 \), then we may let \( \Sigma \) denote the set given by Lemma 2.2. We may assume that \( \Sigma \subset \{ Q = 0 \} \), for otherwise if \( c \in \Sigma \cap \{ Q \neq 0 \} \), then as in the proof of Lemma 2.1, the orbit \( S \) passing through \( c \) contains \( 0 \) in its closure, and we derive a contradiction as in the previous case.

Now we show that

\[
p^{(\nu_l)}|_\Sigma = 0 \quad \text{for all} \quad -L \leq l \leq L.
\]  

Comparing terms of signature \( \nu_l \) in (2.7), we must have either \( Q p^{(\nu_l)} = 0 \) or

\[
Q p^{(\nu_l)} + Q p^{(\nu_k)} = 0 \quad \text{(2.11)}
\]

if there is a \( k \) such that \( \nu_l + \mu = -(-\nu_k + \mu) \). From (2.9) and (2.11) we obtain

\[
Q p^{(\nu_l)} = -\sum_B f_{\nu_k, B}(z) Q z^B.
\]  

(2.12)

Let \( \kappa = (1 - \nu_l)/2 \) denote the weight of the indices \( B \) appearing in (2.12), and let \( P_\kappa(z) \) denote the holomorphic polynomials in \( z \) of weight \( \kappa \). Let \( \varphi_1, \ldots, \varphi_N \) denote a basis for the space \( P_{\kappa+\mu}(z)/Q P_\kappa(z) \). It follows that \( \{ z^I \varphi_j \} \) forms a basis for the space

\[
\mathcal{P}_{\lambda, \kappa+\mu}(z, \bar{z})/(\mathcal{P}_\lambda(z) Q P_\kappa(z))
\]

where \( \mathcal{P}_{\lambda, \kappa+\mu}(z, \bar{z}) = \mathcal{P}_\lambda(z) P_{\kappa+\mu}(z) \). Since \( \{ \varphi_j \} \) is a basis, there exist holomorphic polynomials \( g_B \) and \( g_j \) such that \( p^{(\nu_l)} = \sum g_B Q z^B + \sum g_j \varphi_j \). This yields

\[
Q p^{(\nu_l)} = \sum Q g_B Q z^B + \sum Q g_j \varphi_j.
\]  

(2.13)

The difference between (2.12) and (2.13) must vanish, and \( \{ \varphi_j \} \) is a basis, so \( \sum Q g_j \varphi_j = 0 \). Thus

\[
Q \left( p^{(\nu_l)} - \sum g_B Q z^B \right) = 0.
\]

By Lemma 2.2, we conclude that

\[
p^{(\nu_l)} - \sum g_B Q z^B = 0 \quad \text{on} \quad \Sigma.
\]

Finally, \( Q = 0 \) on \( \Sigma \) since \( \Sigma \subset A \), so \( p^{(\nu_l)}|_\Sigma = 0 \). Thus we have \( \Sigma \subset \{ p = 0 \} \), which contradicts (2.6).
Lemma 2.6. If $wt Q \geq 0$, and $Q$ is nonzero, then $q_0$ is not identically zero and is divisible by $w$.

Proof. If $q_0$ is not divisible by $w$, then it contains a holomorphic polynomial in the variables $z_1, \ldots z_n$. Since $wt Q \geq 0$, we must have $wt q_j > 0$ for all $j$. Thus $q_j \frac{\partial p}{\partial z_j}$ can contain no pure holomorphic or antiholomorphic terms. By equation (2.3), we must have $\Re iq_0 = 0$, so that $q_0 = 0$. It follows, then, by Lemma 2.5 that $Q = 0$.

Next we classify the vector fields of negative weight. It is obvious that $A^{(\mu)}$ is trivial if $\mu < -1$, and $A^{(-1)}$ is generated by $\frac{\partial}{\partial w}$. It turns out that the only other possible weights are the $-\delta_j$ themselves.

Lemma 2.7. Let a nonzero vector field $Q \in A^{(\mu)}$, $\mu < 0$, be given. Then $\mu = -\delta_j$ for some $1 \leq j \leq n$, and there is a weighted homogeneous change of coordinates $(\tilde{z}_1, \ldots \tilde{z}_n)$, depending on $(z_1, \ldots, z_n)$ such that after possibly relabeling the indices $Q$ is given in the new coordinates as $s_0 \frac{\partial}{\partial w} + \frac{\partial}{\partial \tilde{z}_1}$. The new defining function $\tilde{p}$ satisfies (2.15) and (2.16) below, and

$$s_0(z_1, 0, \ldots, 0) = mc \left( \frac{z_1}{2i} \right)^{m-1}$$

for some real constant $c$.

Proof. It is evident that if $wt Q < 0$, then each coefficient $q_j$ depends only on $z_1, \ldots, z_n$, i.e. it is independent of $w = z_0$. If $q_0 = 0$, then $Q = 0$ by Lemma 2.5. Thus we suppose that $q_0 \neq 0$. If $q_0$ is a real constant, then $wt Q = -1$, and $Q = c \frac{\partial}{\partial w}$, which completes the proof in this case.

Otherwise, $q_0$ is a holomorphic polynomial of positive weight, which will need to be cancelled by pluriharmonic terms in (2.3). By (2.4) $\frac{\partial p}{\partial z_j}$ cannot contain any purely holomorphic monomials. And if $q_j \neq 0$ and $q_j(0) = 0$ (i.e. if $wt q_j > 0$), then $q_j \frac{\partial p}{\partial z_j}$ contains no pluriharmonic monomials, so there is nothing pluriharmonic to cancel $q_0$ in (2.3). Thus there must be a $j$ such that $q_j(0) \neq 0$, i.e. $q_j$ is constant, and thus $wt Q = -\delta_j$.

For convenience of notation, we may assume that $j = 1$. Let us define homogeneous polynomials $h_2(z), \ldots, h_n(z)$ by $\frac{\partial h_j}{\partial z_1} = q_j$. It is immediate that $wt h_j = wt q_j + \delta_1$; and since $wt Q = -\delta_1$, we have $wt q_j = -\delta_1 + \delta_j$. Thus $wt h_j = \delta_j$, and the coordinate change defined by

$$w = \tilde{w}, \quad z_1 = \tilde{z}_1, \quad z_j = \tilde{z}_j + h_j(\tilde{z}), \quad 2 \leq j \leq n$$

is weighted homogeneous. Since $\frac{\partial}{\partial z_1} = \sum \frac{\partial z_j}{\partial \tilde{z}_1} \frac{\partial}{\partial z_j}$, we have $Q = s_0 \frac{\partial}{\partial w} + \frac{\partial}{\partial \tilde{z}_1}$ if we define $s_0$ by $s_0(\tilde{z}) = q_0(z)$.

We define a homogeneous polynomial $S$ by requiring that it be divisible by $\tilde{z}_1$ and that $\frac{\partial S}{\partial \tilde{z}_1} = s_0$. We define coordinates $(\tilde{w}, \tilde{z})$ by setting

$$\tilde{w} = \tilde{w} + S(\tilde{z}), \quad \tilde{z} = \tilde{z}.$$

The surface $\{ \tilde{p}(\tilde{z}) + \tilde{v} = 0 \}$ becomes $\{ \tilde{p}(\tilde{z}) + 3m S(\tilde{z}) + \tilde{v} = 0 \}$. In the $\tilde{z}$-coordinates we have $Q = \frac{\partial}{\partial \tilde{z}_1}$.
Let us now drop the hats from the coordinates. The condition that \( Q \) is tangential to \( \partial \Omega \) is equivalent to the condition that the function \( \tilde{p}(z) + 3mS(z) \) is independent of the variable \( \Re z_1 \). Thus we may write

\[
\tilde{p}(z) + \frac{1}{2i}(S(z) - \bar{S}(z)) = c \left( \frac{z_1 - \bar{z}_1}{2i} \right)^m + \frac{2\Re}{k=2}^{n} \alpha_k z_k \left( \frac{z_1 - \bar{z}_1}{2i} \right)^{m_k} + O \left( \sum_{k=2}^{n} |z_k|^2 \right).
\]

with \( 1 \leq m_k \leq m \). We claim that, in addition, we have

\[
m/2 \leq m_k \leq m - 2. \quad (2.14)
\]

Since \( p \) has weight 1, we must have \( m\delta_1 = 1 \) and \( \delta_k + m_k\delta_1 = 1 \). Since \( \delta_k \) is the reciprocal of an integer, we have \( 1 - m_k/m \leq 1/2 \), which gives the lower bound on \( m_k \). For the upper bound, it is obvious that \( m_k \leq m - 1 \). If equality holds, then \( \delta_1 = \delta_k \), i.e. \( z_1 \) and \( z_k \) have the same weight. Thus the coordinate change \( \tilde{z}_1 = z_1 - i\alpha_k z_k \) is homogeneous, and in it the monomial \( z^m k \) does not appear.

By (2.4), the pure holomorphic terms are inside \( S \), so we have

\[
\tilde{p}(z_1, 0, \ldots, 0) = c \left[ \left( \frac{z_1 - \bar{z}_1}{2i} \right)^m - 2\Re \left( \frac{z_1}{2i} \right)^m \right] \quad (2.15)
\]

\[
\frac{\partial \tilde{p}}{\partial z_j}(z_1, 0, \ldots, 0) = \alpha_j \left[ \left( \frac{z_1 - \bar{z}_1}{2i} \right)^{m_j} - \left( \frac{z_1}{2i} \right)^{m_j} \right] \quad (2.16)
\]

and

\[
\frac{1}{2i}S(z_1, 0, \ldots, 0) = c \left( \frac{z_1}{2i} \right)^m.
\]

The derivative, \( s_0 \), thus has the form stated above.

\[\square\]

§3. Balanced Domains

\( \mathcal{A}^{(0)} \) contains the vector field

\[
\mathcal{D} = w \frac{\partial}{\partial w} + \sum \delta_j z_j \frac{\partial}{\partial z_j},
\]

which corresponds to homogeneous dilation. For any vector field \( Q \in \mathcal{A}^{(\mu)} \), \( [\mathcal{D}, Q] = \mu Q \).

If \( p \) is weighted homogeneous of weight \( \mu \), then the familiar Euler identity may be recast in the form

\[2\Re \mathcal{D} p = (\mathcal{D} + \overline{\mathcal{D}}) p = \mu p.\]

An analogue of this which will be useful later is

\[2\mathcal{D} p = \mu p \text{ if and only if } p \text{ is balanced}.
\]

If \( p \) is not balanced, then we may consider \( p^{(\nu)} \) as in (2.1). In this case we have

\[
\mathcal{D} p^{(\nu)} = \frac{\mu + \nu}{2} p^{(\nu)}.
\]

Lemma 3.1. If \( Q \in \mathcal{A}^{(0)} \), then \( Q = c\mathcal{D} + \mathcal{L} \) for some \( c \in \mathbb{R} \), and \( \mathcal{L} \) is of the form (2.5).

Proof. By Lemma 4, we must have \( q_0 = cw \) for some constant \( c \). It is evident that \( \mathcal{L} := Q - c\mathcal{D} \in \mathcal{A}^{(0)} \) and has the form (2.5). \[\square\]
Lemma 3.2. If $Q \in \mathcal{A}(\mu)$, $0 < \mu < 1$, and if $Q \neq 0$, then $\mu = \frac{1}{2}$.

Proof. Let us define $S := [\partial / \partial w, Q] = \sum \frac{\partial q_j}{\partial w} \frac{\partial}{\partial z_j}$. Then $S \in \mathcal{A}(\mu-1)$, and by Lemma 2.6, $S \neq 0$. By Lemma 2.7, then, $\mu - 1 = -\delta_j$ for some $j$. In particular, if $\mu \neq \frac{1}{2}$, then $\frac{1}{2} < \mu = 1 - \delta_j < 1$. By Lemma 3.2, we may assume that $S = s_0(z) \frac{\partial}{\partial w} + \frac{\partial}{\partial z_1}$. By Lemma 2.6, $q_0$ is divisible by $w$, so

$$Q = w s_0 \frac{\partial}{\partial w} + (w + r_1(z)) \frac{\partial}{\partial z_1} + \sum_{j=2}^n r_j(z) \frac{\partial}{\partial z_j}$$

for some homogeneous polynomials $r_j$.

Now let us calculate the commutator

$$[S, Q] = \left( s_0^2 - \sum r_j \frac{\partial s_0}{\partial z_j} \right) \frac{\partial}{\partial w} + \left( s_0 + \frac{\partial r_1}{\partial z_1} \right) \frac{\partial}{\partial z_1} + \sum_{j=2}^n r_j \frac{\partial}{\partial z_j}.$$

Since $\delta_1 < \frac{1}{2}$, the commutator has weight $1 - 2\delta_1 > 0$. Further, since the coefficient of $\frac{\partial}{\partial w}$ is not divisible by $w$ unless it is 0, we see that $[S, Q] = 0$ by Lemma 2.6. In particular, $\partial r_j / \partial z_1 = 0$ for $2 \leq j \leq n$. If we set $z_2 = \ldots = z_n = 0$, then $r_j = 0$ for $2 \leq j \leq n$. By homogeneity, $s_0(z_1, 0, \ldots, 0) = \alpha z_1^{m-1}$ and $r_1(z_1, 0, \ldots, 0) = \beta z_1^m$. Since the coefficient of $\frac{\partial}{\partial z_1}$ vanishes, we must have $\alpha + m\beta = 0$. But since the coefficient of $\frac{\partial}{\partial w}$ vanishes, too, we have $\alpha^2 - (m-1)\alpha \beta = 0$. Thus $\alpha = \beta = 0$. But by Lemma 2.7, we have $\alpha = cm = 0$, so $c = 0$. This contradiction proves the Lemma.

We note that a polynomial $p$ can be nondegenerate in the sense of satisfying (2.6), but $p^{(0)}$ may be degenerate. We consider a different nondegeneracy condition on a homogeneous polynomial $\varphi$:

There is no holomorphic vector field $R \neq 0$ such that $R \varphi = 0$. \hspace{1cm} (3.2)

If $\varphi$ is strictly psh at some point, then (3.2) holds.

Lemma 3.3. Suppose that the balanced part $p^{(0)}$ of $p$ satisfies (3.2), and let $\mu = \text{wt } p$. If $l_k$, $1 \leq k \leq n$ are weighted homogeneous of weight $\delta_k$, and if $p + \sum_{k=1}^m l_k \frac{\partial p}{\partial z_k} = 0$, then $l_k = -2\mu^{-1} \delta_k z_k$, and $p$ is balanced.

Proof. By (3.1) the operator $L = \sum l_k \frac{\partial}{\partial z_k}$ preserves the splitting (2.1) in the sense that $\delta(Lp^{(\nu)}) = \nu$. It follows that $p^{(0)} = -Lp^{(0)}$. Since $p^{(0)}$ is balanced, we have

$$\sum_{k=1}^m (l_k + 2\mu^{-1} \delta_k z_k) \frac{\partial p^{(0)}}{\partial z_k} = 0.$$

Thus the integral curves of the vector field $L + 2\mu^{-1} \partial$ lie in the level sets of $p^{(0)}$, which contradicts (2.6), unless $l_k = -2\mu^{-1} \delta_k z_k$. It follows now from (3.1) that $p$ is balanced. \hfill \Box
Lemma 3.4. Suppose that \( p \) satisfies (2.6) and that \( p^{(0)} \) satisfies (3.2). If there is a nonzero vector field \( Q \in \mathcal{A}^{(\frac{1}{2})} \), then \( p \) is a balanced polynomial; and after a homogeneous change of coordinates and a permutation of variables,

\[
Q = \lambda \left( -2iwz_1 \frac{\partial}{\partial w} + w \frac{\partial}{\partial z_1} - \sum_{j=1}^{n} 2i\delta_j z_1 z_j \frac{\partial}{\partial z_j} \right) \tag{3.3}
\]

for some \( \lambda \in \mathbb{R} \).

Proof. Let us use the notation \( z_1, \ldots, z_d \) for the variables of weight \( \frac{1}{2} \), and let \( \zeta_1, \ldots, \zeta_e \) for the variables with weight \( < \frac{1}{2} \). Let \( Q \) be a nonzero vector field of weight \( \frac{1}{2} \), and let \( Q^{(-\frac{1}{2})} := [\frac{\partial}{\partial w}, Q] \). By Lemma 2.7 we may assume that \( Q^{(-\frac{1}{2})} = -2iz_1 \frac{\partial}{\partial w} + \frac{\partial}{\partial z_1} \). Thus by Lemma 2.5 we have

\[
Q = -2iz_1 w \frac{\partial}{\partial w} + w \frac{\partial}{\partial z_1} + \sum_{j=1}^{d} q_j(z, \zeta) \frac{\partial}{\partial z_j} + \sum_{j=1}^{e} \tilde{q}_j(z, \zeta) \frac{\partial}{\partial \zeta_j}.
\]

We may write

\[
p = \sum z_j \bar{z}_j + \sum (z_j \varphi_j(\zeta, \bar{\zeta}) + \bar{z}_j \bar{\varphi}_j(\zeta, \bar{\zeta})) + \tilde{p}(\zeta, \bar{\zeta}). \tag{3.4}
\]

After a change of coordinates of the form \( z_j \mapsto z_j + \psi_j(\zeta) \), we may assume that \( \varphi_j \) contains no anti-holomorphic terms. Then after \( w \mapsto w + \chi(\zeta) \), we may assume that \( \varphi_j \) contains no holomorphic terms, i.e. that (2.4) is satisfied. Thus (2.3) takes the form

\[
u \Re \varphi_1 + \Re \left[ -2iz_1 v + iv\varphi_1 + \sum_{j=1}^{d} q_j(\bar{z}_j + \varphi_j) + \sum_{k=1}^{e} \tilde{q}_k \left( \sum_{j=1}^{d} z_j \frac{\partial \varphi_j}{\partial \zeta_k} + \sum_{j=1}^{d} \bar{z}_j \frac{\partial \bar{\varphi}_j}{\partial \zeta_k} + \frac{\partial \tilde{p}}{\partial \zeta_k} \right) \right] = 0. \tag{3.5}
\]

We note immediately that the coefficient of \( u \) must vanish, i.e. that \( \Re \varphi_1 = 0 \).

Now we set \( u = 0 \) and \( \zeta = 0 \) in equation (2.3). Since \( \varphi_j \) has weight 1, \( \frac{\partial \varphi_j}{\partial \zeta_k} = 0 \), so we have

\[
\Re(2iz_1 \sum z_j \bar{z}_j + \sum q_j(z, 0) \bar{z}_j) = \Re \sum \bar{z}_j (2iz_1 z_j + q_j(z, 0)) = 0.
\]

We conclude, then, that

\[
q_j(z, \zeta) = -2iz_1 z_j + \sum z_k q_j^{(k)}(\zeta) + q_j^j(\zeta)
\]

for \( 1 \leq j \leq d \). Similarly, since \( \tilde{q}_j \) has weight \( < 1 \), we may write

\[
\tilde{q}_j = \sum z_k \tilde{q}_j^{(k)}(\zeta) + \tilde{p}_j(\zeta).
\]
Now we observe that there are no coefficients (i.e. functions of $\zeta$ and $\bar{\zeta}$) of $\bar{z}_1^2$ inside the term in square brackets in (3.5). Thus the coefficient of $z_1^2$ must vanish, i.e.

$$\sum_{k=1}^{e} q_k^{(1)} \frac{\partial \varphi_1}{\partial \zeta_k} = 0.$$ 

Similarly, taking the coefficient of $z_1 \bar{z}_1$ we have

$$Re \left( q_1^{(1)} + i \varphi_1 + \sum_{k=1}^{e} \bar{q}_k^{(1)} \frac{\partial \bar{\varphi}_1}{\partial \zeta_k} \right) = 0.$$ 

The only purely holomorphic or antiholomorphic terms come from $q_1^{(1)}$ so $q_1^{(1)} = 0$. Now since $\varphi_1$ is pure imaginary, these two equations give $\varphi_1 = 0$.

There is no function of $\zeta, \bar{\zeta}$ as multiple of $\bar{z}_1 \bar{z}_m$ for $2 \leq m \leq d$ in the bracketed term in (3.5). Thus the coefficient of $z_1 z_m$ must vanish:

$$\sum_{k=1}^{e} q_k^{(1)} \frac{\partial \varphi_m}{\partial \zeta_k} = 0.$$ 

The coefficient of $z_1 \bar{z}_m$ plus the conjugate of the coefficient of $z_m \bar{z}_1$ must also vanish, so

$$q_m^{(1)} + \bar{q}_1^{(m)} + \sum_{k=1}^{e} \bar{q}_k^{(1)} \frac{\partial \bar{\varphi}_m}{\partial \zeta_k} = 0.$$ 

It follows upon comparing pure terms that $q_m^{(1)} = q_1^{(m)} = 0$.

The coefficient of $\bar{z}_1$ in the bracketed term in (3.5) is $q_1'$. Adding the conjugate of this to the coefficient of $z_1$, we obtain

$$q_1' + 2i \bar{\rho} + \sum_{k=1}^{e} \bar{q}_k^{(1)} \frac{\partial \bar{\rho}}{\partial \zeta_k} = 0.$$ 

The only pure terms come from $q_1'$, so we must have $q_1' = 0$. By Lemma 3.3, then, we conclude that

$$\bar{q}_k^{(1)} = -4i \delta_k \zeta_k \quad (3.6)$$

and that $\bar{\rho}$ is balanced.

We note that there are no terms of the form $\bar{z}_1 \bar{z}_j$ in the bracketed term in (3.5) for $2 \leq j \leq d$, so the coefficient of $z_1 z_j$ must vanish, i.e.

$$\sum_{k=1}^{e} q_k^{(1)} \frac{\partial \varphi_j}{\partial \zeta_k} = 0.$$ 

But by (3.6), this gives $\varphi_j = 0$, since $\varphi_j$ contains no purely anti-holomorphic terms.
Now we may inspect the coefficients of $z_j\,\bar{z}_m$ for $2 \leq j, m \leq d$, and since $\varphi_j = 0$, we get $q_j^{(m)} = 0$.

Finally, the coefficients of $z_j$ and $\bar{z}_j$ for $2 \leq j \leq d$ give

$$q'_j + \sum_{k=1}^e q_j^{(j)} \frac{\partial \tilde{p}}{\partial \xi_k} = 0.$$  

Again, the pure terms vanish, so $q'_j = 0$. By Proposition 2.5, we have $q_j^{(j)} = 0$. Setting $z = 0$ in (3.5), we have

$$\Re \sum_{k=1}^e q'_k \frac{\partial \tilde{p}}{\partial \xi_k} = 0,$$

so that $q'_k = 0$ by Proposition 2.5. This completes the proof. 

\textbf{Remark.} We have in fact shown that under the hypotheses of Lemma 3.4 we have

$$p = \sum_{j=1}^d z_j \bar{z}_j + \bar{p}(\zeta, \bar{\zeta}).$$

\textbf{Lemma 3.5.} Suppose that $p$ satisfies (2.6), and that $p^{(0)}$, and $(p^2)^{(0)}$ satisfy (3.2). If there is a nonzero vector field $Q \in A^{(1)}$, then $p$ is a balanced polynomial; and after a homogeneous change of coordinates and a permutation of variables,

$$Q = \lambda \left( w^2 \frac{\partial}{\partial w} + \sum_{j=1}^n 2\delta_j w z_j \frac{\partial}{\partial z_j} \right)$$

for some $\lambda \in \mathbb{R}$.

\textbf{Proof.} By Lemmas 2.6 and 2.7, $q_0$ is a real multiple of $w^2$. Thus

$$Q = w^2 \frac{\partial}{\partial w} + \sum_{j=1}^n (w q_j(z) + r_j(z)) \frac{\partial}{\partial z_j}$$

where $w q_j = \delta_j$, and $w r_j = 1 + \delta_j$. The coefficient of $u$ in (2.3) is then

$$\Re \left( -p + \sum_{j=1}^n q_j \frac{\partial p}{\partial z_j} \right) = 0.$$

Multiplying by $2p$, we have

$$\Re \left( -2p^2 + \sum_{j=1}^n q_j \frac{\partial (p^2)}{\partial z_j} \right) = 0.$$  \hfill (3.8)
Now we set \( u = 0 \) in (2.3) and obtain
\[
\Re \left[ -i \sum p q_j \frac{\partial p}{\partial z_j} + \sum r_j \frac{\partial p}{\partial z_j} \right] = 0
\]
or, after doubling,
\[
\Im \left( \sum q_j \frac{\partial (p^2)}{\partial z_j} \right) + 2 \Re \left( \sum r_j \frac{\partial p}{\partial z_j} \right) = 0. \tag{3.9}
\]
Every monomial in \( r_j \frac{\partial p}{\partial z_j} \) is of the form \( z^A \bar{z}^B \) with \( wt A \geq 1 + \delta_j \) and \( wt B \leq 1 - \delta_j \). Thus the second term in (3.9) can have no balanced monomials. The operator \( \sum q_j \frac{\partial}{\partial z_j} \) preserves balanced polynomials, so we may add (3.8) and (3.9) and take the balanced part to obtain
\[
\sum_{j=1}^{n} q_j \frac{\partial (p^2)^{(0)}}{\partial z_j} = 2(p^2)^{(0)}
\]
where \( (p^2)^{(0)} \) denotes the balanced part of \( p^2 \). By Lemma 3.3, we have \( q_j = 2\delta_j z_j \).

Now we add (3.8) and (3.9) to obtain
\[
2D(p^2) = \sum_{j=1}^{n} 2\delta_j z_j \frac{\partial (p^2)^{(0)}}{\partial z_j} = 2p^2 - 2i \Re \sum_{j=1}^{n} r_j \frac{\partial p}{\partial z_j}. \tag{3.10}
\]

Now we write
\[
p^2 = (p^2)^{(-\mu_1)} + \ldots + (p^2)^{(\mu_1)}
\]
as in (2.1). By (3.1) we have
\[
\sum_{j=1}^{n} 2\delta_j z_j \frac{\partial (p^2)^{(\mu_j)}}{\partial z_j} = \sum_{j=1}^{n} (2 + \mu_j)(p^2)^{(\mu_j)}.
\]
Thus if we can show that \( r_j = 0 \) for \( 1 \leq j \leq n \), then from (3.10) we will have
\[
(1 + \frac{\mu_j}{2})(p^2)^{(\mu_j)} = (1 - \frac{\mu_j}{2})(p^2)^{(-\mu_j)} = (1 - \frac{\mu_j}{2})(p^2)^{(\mu_j)}.
\]
We conclude that the only terms that can appear correspond to \( \mu_j = 0 \). Thus \( p^2 \) is a balanced polynomial. It follows that \( p \) is balanced, too. With Lemma 3.6, then, the proof will be complete.

\[\square\]

**Lemma 3.6.** With the notation of Lemma 3.5, \( R := \sum r_j \frac{\partial}{\partial z_j} = 0 \).

**Proof.** From (3.1) and (3.10) we obtain
\[
i(\nu + 1)(p^2)^{(\nu+1)} = Rp^{(\nu)} \tag{3.11}
\]
for all indices \( \nu = \nu_L, \nu_{-L+1}, \ldots, \nu_L \).
The largest signature in \((p^2)^{(\nu)}\) is \(2\nu_L < 1 + \nu_L\), so setting \(\nu = \nu_L\) in (3.11) we obtain

\[ Rp^{(\nu_L)} = 0. \]

Now proceed by induction to show that

\[ R^{2^j} p^{(\nu_{L-j})} = 0 \quad \text{for} \quad 0 \leq j \leq 2L. \tag{3.12} \]

We consider (3.11) for \(\nu = \nu_j\). Each element of \((\nu + 1)(p^2)^{(\nu + 1)}\) is made up from a product of the form \((\nu + 1)p^{(\nu_a)}p^{(\nu_b)}\) with \(\nu_a + \nu_b = \nu_j + 1\). Since \(\nu_a, \nu_b < 1\), it follows that \(\nu_a, \nu_b > \nu_j\). By induction, \(R^{2^{j-1}} p^{(\nu)} = 0\) for \(\nu = \nu_a\) and \(\nu = \nu_b\); and so

\[ R^{2^{j-1}}(p^{(\nu_a)}p^{(\nu_b)}) = 0. \]

This proves (3.12).

Next we consider the ordered family \(\mathcal{H} = \{h_1, h_2, \ldots, h_N\}\) of homogeneous polynomials defined as follows. We list out \(\mathcal{H}\) in groups in the order \(G_L, G_{L-1}, \ldots, G_{-L}\). For fixed \(l, -L \leq l \leq L\) we let \(B_1, B_2, \ldots\) be an arbitrary ordering of the (finitely many) multi-indices appearing in \(p^{(\nu)} = \sum f_{\nu,B}z^B\). Then we define \(G_l\) to be the (finite) ordered set

\[ \{R^{2L-l-1}f_{\nu_1,B_1}, R^{2L-2}f_{\nu_2,B_1}, \ldots, f_{\nu_l,B_1}, R^{2L-l-2}f_{\nu_2,B_2}, R^{2L-l-1}f_{\nu_1,B_2}, \ldots, f_{\nu_l,B_2}, \ldots\}. \]

Let us define \(V_0 = \mathbb{C}^n\) and \(V_m = \{h_1 = \ldots = h_m = 0\}\) for \(1 \leq m \leq N\). Clearly \(V_0 \supset V_1 \supset \ldots \supset V_N\). Since each \(f_{\nu,B}\) vanishes on \(V_N\), it follows that \(V_N \subset \{p = 0\}\). Thus \(\dim V_N = 0\). And since \(V_N\) is invariant under \(D_\tau\) for all \(\tau \in \mathbb{C}\), we have \(V_N = \{0\}\). Further, by the choice of ordering of \(\mathcal{H}\), thogether with (3.12), we see that

\[ Rh_{j+1} = 0 \quad \text{on} \quad V_j \quad \text{for every} \quad j. \tag{3.13} \]

Thus \(R\) is tangent to \(V_m\) at the regular points of \(V_m\) where \(R \neq 0\).

Since we are trying to prove that \(R = 0\), we may assume that \(A := \{R = 0\} \neq \mathbb{C}^n\). Thus we may choose \(m\) such that \(V_{m+1} \cap A\) has a component which has codimension 1 in \(V_m\). Passing to irreducible components, we may assume that \(V_m\) and \(V_{m+1}\) are irreducible, and \(V_{m+1} \subset A\). There are now two cases to consider. The first case is that \(V_{m+1} = \{0\}\). Thus \(R \neq 0\) on \(V_m - \{0\}\), and by Lemma 2.1 there is an orbit \(S\) of \(R\) with 0 in its closure. It follows from (3.12) and Lemma 2.3 that \(p^{(\nu)} = 0\) on \(S\). Thus \(p = 0\) on \(S\), which contradicts (2.6).

In the other case, \(dim(V_m) \geq 1\), and we may choose a (constant) tangent vector \(T = \sum \alpha_j \frac{\partial}{\partial z_j}\) which is tangent to \(V_m\) at some point, and we define \(\varphi := T^kh_{m+1}\). We may choose \(k\) such that \(V_{m+1} \subset \{\varphi = 0\} \cap V_m\) and \(d\varphi|_{V_m}\) does not vanish identically on \(V_{m+1}\). Then there is a holomorphic vector field \(\tilde{R}\) on \(V_m\) such that \(\varphi^d\tilde{R} = R\) for some \(d \geq 1\) and such that \(\tilde{R}\) does not vanish identically on \(V_{m+1}\). By (3.13) we have \(\varphi^d\tilde{R} = Rh_{m+1} = 0\) on \(V_m - V_{m+1}\). Thus \(\tilde{R}h_{m+1} = 0\) on \(V_{m+1}\). It follows that \(\tilde{R}\) is tangential to the regular points of \(V_{m+1}\) on the (nonempty) set where \(\tilde{R} \neq 0\).
For each \( \mu \), we let \( s_\mu \geq 0 \) denote the largest integer such that
\[
p^{(\mu)} = O(|\varphi|^{s_\mu})
\]
holds on \( V_m \). By (3.11) and the definition of \( \tilde{R} \) it follows that \( s_\mu \geq d \) for \( \mu > 0 \). Since \( p^{(-\mu)} = p^{(\mu)} \), we have \( s_\mu \geq d \) for \( \mu < 0 \). Let \( s \) be the minimum value of \( s_\mu \) for \( \mu \neq 0 \). Since \( (p^2)^{(1)} \) consists of products \( p^{(\nu_a)}p^{(\nu_b)} \) with \( 0 < \nu_a, \nu_b < 1 \) and \( \nu_a + \nu_b = 1 \), it follows that
\[
(p^2)^{(1)} = O(|\varphi|^{2s}).
\]
From (3.10) we have \( R\theta^0 = i(p^2)^{(1)} \), so we have
\[
\varphi^d \tilde{R}\theta^0 = O(|\varphi|^{2d}).
\]
It follows, then, that \( \tilde{R}\theta^0 = 0 \) on \( V_m \). By Lemma 2.2, then, \( \theta^0 \) vanishes on a variety passing through 0. Thus \( \theta \) vanishes on the same variety, which contradicts (2.6).

We may summarize the work of §3 by the following.

**Theorem 3.7.** Let \( \theta \) be homogeneous of weight 1, let \( \theta \) satisfy (2.6), and let \( \theta^0 \) and \( (p^2)^{(0)} \) both satisfy (3.2). If there is a tangential holomorphic vector field \( Q \) for \( \{v + \theta(z) < 0\} \) with \( \text{wt} Q > 0 \), then \( \theta \) is balanced.

§4. Domains with Noncompact Automorphism Groups

In this Section we give the proofs of Theorems 1 and 2.

**Proof of Theorem 1.** Let \( \theta \) be homogeneous. We may obtain the balanced part \( \theta^0 \) as
\[
\theta^0(z) = \frac{1}{2\pi} \int_0^{2\pi} \theta(e^{i\theta}z_1, \ldots, e^{i\theta}z_n) d\theta.
\]
If \( \theta \geq 0 \), then \( \theta^0 \geq 0 \) and \( (p^2)^{(0)} \geq 0 \). Since \( \theta^0 \) is symmetrized, it follows that \( \theta^0 \) is invariant under \( D_\tau \) for all \( \tau \in \mathbb{C} \). Thus, for \( z_0 \neq 0 \), \( \theta^0(z_0) > 0 \), since otherwise \( \theta^0 \) (and thus \( \theta \)) would vanish on the \( D_\tau \)-orbit of \( z_0 \). It follows that the level sets \( \{\theta^0 = c\} \) are compact, and thus there are points where \( \theta^0 \) is strongly psh. So (3.2) holds. Similarly, (3.2) holds for \( (p^2)^{(0)} \).

Now let \( Q \) denote the homogeneous part of \( H \) of lowest weight. If \( Q \) has weight \( \mu > 1 \), then the commutator \( [Q, \frac{\partial}{\partial \varphi}] \) has weight \( \mu - 1 \) and is not the zero vector field by Lemma 2.6. Taking further commutators, we may assume that \( Q \) has weight \( 0 < \mu \leq 1 \). By Lemma 3.2, then \( \mu \) is either \( \frac{1}{2} \) or 1. Applying Lemmas 3.4 and 3.5, we may assume that \( Q \) has the form of either (3.3) or (3.7). In either case, we may apply the argument of the final rescaling in §5 of [BP] to conclude that \( \Omega \) is biholomorphically equivalent to \( \{v + \theta < 0\} \).
Admissible Assignment of Weights. For the rest of this Section we let \( \Omega \) denote a smooth, convex surface with finite type boundary. Let us fix a point \( 0 \in \partial \Omega \), and assume that the tangent plane to \( \partial \Omega \) at \( 0 \) is given by \( \{ v = 0 \} \). Let us begin by showing how to make an admissible assignment of weights. We assign weight \( 1 \) to the variable \( w \), and we write \( \partial \Omega \) as \( \{ v + f(u, z) = 0 \} \) in a neighborhood of \( 0 \). For a tangent vector \( T = \sum a_j \frac{\partial}{\partial z_j} \), we let \( \text{Ord}(f(0, z), T) = m \) be the smallest positive integer such that \( T^k(f(0,0)) = 0 \) for \( 1 \leq k \leq m-1 \) and \( T^m f(0,0) \neq 0 \). Since \( \Omega \) has finite type, each vector \( T \neq 0 \) has a finite order. We let \( L_1 \) be the set of complex tangent vectors such that \( \text{Ord}(f(0, z), T) = m_1 \) is maximum. Then we define numbers \( m_1 > \cdots > m_k \) and complex subspaces \( L_1 \subset L_2 \subset \cdots \subset L_k = \mathbb{C}^n \) by the condition that \( \text{Ord}(f(0, z), T) = m_j \) for \( T \in L_j - L_{j-1} \).

After a complex linear change of coordinates, we may assume that there are integers \( n_1 < n_2 < \cdots < n_k \) such that the coordinate system \( \{ z_1, \ldots, z_n \} \) has the property that \( L_j \) is spanned by \( \{ \frac{\partial}{\partial z_s} : n_{j-1} + 1 \leq s \leq n_j \} \). We assign weight \( m_j^{-1} \) to the variables \( \{ z_s : n_{j-1} + 1 \leq s \leq n_j \} \). By the convexity of \( f \), the \( m_j \) are all even.

Now let \( p \) denote the terms of weight \( 1 \) in the Taylor expansion of \( f \) at \( z = 0 \). We must show that all monomials in the Taylor expansion of \( f - p \) at the origin have weight greater than or equal to one. If not, there is a term of minimal weight \( \mu < 1 \) in \( f \). Let \( q(z) \) denote the terms of weight \( \mu \). If we perform the scaling of coordinates \( \chi_t(w, z_1, \ldots, z_n) = (t^u w, t^{\delta_1} z_1, \ldots, t^{\delta_u} z_n) \), then the surface \( \Omega \) is transformed to the surface \( \Omega_t \), which converges to \( \{ v + q(z) < 0 \} \) as \( t \to 0 \). Since \( \Omega \) is convex, it follows that \( q \) is a convex function. By the construction of the weights, however, there are no monomials of the form \( x_j^m \) or \( y_j^m \) appearing in \( q \). By convexity, then, \( q = 0 \). We conclude that this assignment of weights is admissible.

The polynomial \( p \) itself is obtained as the limit of \( f \) under the scalings above with \( \mu = 1 \). Thus \( p \) is convex, and \( \{ p = 0 \} \) is a real linear subspace of \( \mathbb{C}^n \). Since the order of \( p \) in any direction \( T \) is finite, \( p \) cannot vanish on a complex line. Thus \( \{ p = 0 \} \) is a totally real linear subspace of \( \mathbb{C}^n \), and hence (2.6) is satisfied.

We will invoke several results which were proved under slightly different hypotheses in [BP]. In fact, since \( \Omega \) is convex, several technicalities in [BP] can be avoided here. First, we apply Lemma 2 of [BP]: if \( \text{Aut}(\Omega) \) is noncompact, then \( \Omega \) is biholomorphically equivalent to a domain \( D = \{ v + \rho(z_1, \ldots, z_n) < 0 \} \), where \( \rho \) is a convex polynomial (not necessarily homogeneous). By construction, \( \rho \) satisfies (2.4). Let \( g : D \to \Omega \) denote this biholomorphism, and let \( H = g_*(2\frac{\partial}{\partial \bar{w}}) \). Since \( \Omega \) is convex and finite type, \( H \) extends smoothly to \( \bar{\Omega} \) and induces a tangential holomorphic vector field. The mapping \( g \) extends to a homeomorphism between \( D \cup \{ \infty \} \) and \( \bar{\Omega} \). We translate coordinates so that \( g(\infty) = 0 \in \partial \Omega \).

Since \( \Omega \) is convex, we assign weights as above, and let \( Q \) denote the part of \( H \) with minimal weight. Since \( H \) vanishes to finite order, it follows that \( Q \neq 0 \). Since \( Q(0) = 0 \), it follows from Lemma 2.7 that \( Q \) has weight \( \geq 0 \). If \( \text{wt} Q = 0 \), then by Lemma 3.1, \( Q = cD + L \). However, if \( c \neq 0 \), then \( 0 \) is either a source or a sink, and in neither case can it be parabolic.

Uniform Hyperbolicity. We will consider a family of convex domains in \( \mathbb{C}^{n+1} \), and we wish to have an estimate for the Kobayashi metric \( F \), which is uniform over the family of domains. We let \( F(q, \xi, \Omega) \) denote either the Kobayashi metric for the domain \( \Omega \) at a
We follow this with a scaling of coordinates \( G^{(\nu)} = \{ v + \rho^{(\nu)}(z) < 0 \} \), where \( \rho^{(\nu)} \) is a sequence of convex polynomials of degree \( \leq d \), which converge to a convex polynomial \( \rho^{(\infty)} \) which is nondegenerate in the sense that \( \{ z : \rho^{(\infty)}(z) = 0 \} \) is a totally real linear subspace of \( \mathbb{C}^{n+1} \). It follows that there is a nondegenerate convex polynomial \( \tilde{\rho} \) such that \( \rho^{(\nu)} \geq \tilde{\rho} \), and so we can use \( \tilde{G} = \{ v + \tilde{\rho} < 0 \} \) to obtain the estimate \( F(q, \xi, G^{(\nu)}) \geq F(q, \xi, \tilde{G}) \) for all \( \xi \in \Omega^{(\nu)} \) and \( q \in G^{(\nu)} \cap \tilde{G} \). Since the limit domain \( \tilde{G} \) is Kobayashi hyperbolic, it follows that for a compact \( K \subset \tilde{G} \), there exists \( \epsilon > 0 \) such that
\[
F(q, \xi, G^{(\nu)}) \geq \epsilon|\xi|
\]
holds for all \( q \in K \) if \( \nu \) is sufficiently large.

The second is a family of scalings \( \Omega^{(\nu)} := \chi^{(\nu)}(\Omega) \). As was observed above, the domains \( \Omega^{(\nu)} \) converge to the domain \( \Omega_{\text{hom}} \). Further, there exists an \( \epsilon > 0 \) such that for any compact \( K \subset \Omega_{\text{hom}} \)
\[
F(q, \xi, \Omega^{(\nu)}) \geq \epsilon|\xi|
\]
holds for \( q \in K \) and \( t \) sufficiently large.

**Erratum.** Similar estimates were used in the proof of Lemma 7 in [BP, p. 181]. It was incorrectly stated there that \( \tilde{\rho}^* \) was strictly psh on \( \{ z_1 \neq 0 \} \). However, we may set \( c = 0 \), so that \( \tilde{\rho}^* \) is strictly psh on \( \{ \psi z_1 \neq 0 \} \), and the proof of Lemma 7 works without change if the cases \( \gamma = 0 \) and \( \gamma \neq 0 \) are considered separately. The Remark after Lemma 7 is incorrect. The domain \( \{ v + P(z_1, z_2) + z_2 \bar{z}_2 + \ldots + z_n \bar{z}_n < 0 \} \) is in fact Carathéodory complete (which follows from [BF]). But the weaker property of being Kobayashi hyperbolic is not as trivially proved as was asserted in [BP].

**Lemma 4.1.** There is a biholomorphic mapping \( h : D \to \Omega_{\text{hom}} \) such that \( h_*(\frac{\partial}{\partial w}) = cQ \) for some real number \( c \neq 0 \). Further, \( Q \) has weight > 0.

**Proof.** We let \( g : D \to \Omega \) be as above, and we consider a sequence of mappings \( h^{(\nu)} : G^{(\nu)} \to \Omega^{(\nu)} \) defined as follows. We let \( \Omega^{(\nu)} := \chi^{(\nu)}(\Omega) \) and we let \( q^{(\nu)} = (w^{(\nu)}, z_1^{(\nu)}, \ldots, z_n^{(\nu)}) := g^{-1} \chi^{-1}(w^0, z^0) \), for some point \( (w^0, z^0) \in \Omega \). Now we make a coordinate change
\[
\tilde{z}_j = z_j - z_j^{(\nu)}, \quad 1 \leq j \leq n
\]
\[
\tilde{w} = w - \Re w^{(\nu)} - a_0^{(\nu)} i + \Re \sum a_j^{(\nu)} (z_j - z_j^{(\nu)}).
\]
We follow this with a scaling of coordinates
\[
\hat{w} = \lambda^{(\nu)} \tilde{w}, \quad \hat{z}_j = \mu_j^{(\nu)} \tilde{z}_j, \quad 1 \leq j \leq n.
\]
Thus the domain \( D \) takes the form \( G^{(\nu)} = \{ v + \rho^{(\nu)}(z) < 0 \} \). We choose the \( \lambda^{(\nu)} \) and \( \mu^{(\nu)} \) such that the coordinates of \( q^{(\nu)} \) are \((-i, 0, \ldots, 0)\), and for \( 1 \leq j \leq n \) the modulus of the largest coefficient of \( \rho^{(\nu)}(0, \ldots, z_j, \ldots, 0) \) is 1. The mapping \( h^{(\nu)} \) is then defined by applying the change of coordinates which takes \( G^{(\nu)} \) to \( D \) and following this with \( \chi^{(\nu)} \circ g \).

The \( G^{(\nu)} \) are a family of convex domains which, if we pass to a subsequence, will converge to a domain \( G := \{ v + \rho^{(\infty)}(z) < 0 \} \). By the uniform hyperbolicity condition
(4.2), \( \{ h^{(\nu)} \} \) is a normal family, with a limit function \( h : G \to \Omega_{\text{hom}} \). However, since \( h^{(\nu)}(-i, 0, \ldots, 0) = (w^0, z^0) \in \Omega_{\text{hom}} \) and since we have estimate (4.1), it follows that \( h \) is a biholomorphism.

Finally, we observe that the change of coordinates on the domain \( D \) dilates the vector field \( \frac{\partial}{\partial w} \) by a factor of \( 1/\lambda^{(\nu)} \). Applying \( g_\nu \), we obtain a scalar multiple of the vector field \( H \) on \( \Omega \). Finally, the scalings \( \chi_{\nu_*} \), applied to a multiple of \( H \), converge to a multiple of \( Q \) as \( \nu \to \infty \). Since \( h \) is a biholomorphism, we must have \( c \neq 0 \).

Now we wish to show that \( wt Q > 0 \). By the remarks above, \( wt Q > 0 \); and if \( wt Q = 0 \), the \( Q = L \), as in Lemma 3.1. But in this case, the orbits of \( \Re L \) lie inside the level sets \( \{ p = c \} \). Since such orbits do not occur for a parabolic fixed point, we must have \( wt Q > 0 \). □

**Proof of Theorem 2.** Applying Lemma 4.1, we have \( h : G \to \Omega_{\text{hom}} \). Since \( \frac{\partial}{\partial w} \) is a parabolic vector field on \( G \), it follows that \( Q \) must be a parabolic vector field on \( \Omega_{\text{hom}} \) with weight \( > 0 \). By Theorem 1, then, it follows that \( \Omega \) is biholomorphically equivalent to a domain of the form (1.2). □

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