GLOBAL PHASE PORTRAITS AND BIFURCATION DIAGRAMS
FOR REVERSIBLE EQUIVARIANT HAMILTONIAN SYSTEMS
OF LINEAR PLUS QUARTIC HOMOGENEOUS POLYNOMIALS

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Abstract. This paper is devoted to the complete classification of global phase portraits for reversible equivariant Hamiltonian systems of linear plus quartic homogeneous polynomials. Such system is affinely equivalent to one of five normal forms by an algebraic classification of its infinite singular points. Then, we classify the global phase portraits of these normal forms on the Poincaré disc. There are exactly 13 different global topological structures on the Poincaré disc. Finally, we provide the bifurcation diagrams for the corresponding global phase portraits.

1. Introduction. It is well known that Poincaré in [28] provided the qualitative theory method to study differential systems. Nowadays the qualitative theory of differential systems plays an increasingly important role in applications, especially in biology [19], chemistry [26], control theory [2] and physics [17]. In order to obtain the global dynamics behavior of differential systems, one of the classical methods is to characterize its global phase portraits. Since Hilbert D. posed the 16th problem in 1900, one of the main subjects is to investigate the global phase portraits of polynomial differential systems in the qualitative theory, see for example [30,31,34] and references therein. Many mathematicians have researched the global phase portraits of various polynomial differential systems including the homogeneous [8, 25, 33], semi-homogeneous [4–6] and quasi-homogeneous [21, 29] polynomial systems.

During these years, the global dynamical behavior of the polynomial Hamiltonian systems with high degree have been extensively investigated, see [7, 13, 16, 22]. The authors of [18] provided an algorithm to obtain the phase portraits of separable polynomial Hamiltonian $H(x,y) = F(x) + G(y)$. The complete classification of global phase portraits of quadratic polynomial Hamiltonian systems were given by Artés et al. in [3]. There are a few results on the global phase portraits of polynomial Hamiltonian systems of degree larger than 2. Recently, the characterization of global phase portraits for linear plus homogeneous polynomial Hamiltonian systems has been attracting a lot of attentions. For instance the global phase portraits of a certain class of linear plus cubic homogeneous polynomial were characterized by

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Colak et al. in [9, 10]. The bifurcation diagrams for the global phase portraits of these systems is given in [11, 12].

In this paper, we consider the linear plus quartic homogeneous polynomials Hamiltonian systems

\begin{align}
\dot{x} &= -hx + dy - gx^4 - 2ax^3y - 3kxy^2 - 4bxy^3 - 5fy^4 \triangleq P(x, y), \\
\dot{y} &= cx + hy - cx^4 + 4gxy^3 + 3axy^2 + 2kxy^3 + hy^4 \triangleq Q(x, y),
\end{align}

where \( a, b, c, d, e, f, g, h, k \in \mathbb{R}, d^2 + e^2 + h^2 \neq 0 \) (otherwise the system is homogeneous) and \( a^2 + b^2 + c^2 + f^2 + g^2 + k^2 \neq 0 \) (otherwise the system is linear). Denoted by \( X = (P, Q) \) the polynomial vector field associated to system (1). Up to now there are a few results about the global topological phase portraits of system (1). The main difficulty is that system (1) has very large number parameters and higher degree. So the numerous researchers restrict their attention to the class of reversible system (1). On the other hand, the study of reversible systems is also a classical subject. In [32] and [23], the authors provided a complete classification of global phase portraits of system (1) reversible with respect to the \( y \)-axis.

In this work, we are going to give a complete classification of the global phase portraits of another reversible class of system (1) which is called reversible equivariant. The polynomial vector field \( X: \mathbb{R}^2 \to \mathbb{R}^2 \) is called reversible equivariant if

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} X(x, y) = X(-x, y),
\]

see [20, 24], etc. The system (1) satisfying (2) becomes

\begin{align}
\dot{x} &= -hx - 2ax^3y - 4bxy^3, \\
\dot{y} &= hy + 3ax^2y^2 + by^4 - cx^4,
\end{align}

where \( a, b, c, h \in \mathbb{R}, h \neq 0 \) and \( a^2 + b^2 + c^2 \neq 0 \). We denote \( \Delta = 5a^2 + 4c \) and \( \delta = a^2 - 12c \). Our main result is the following.

**Theorem 1.1.** After a linear change of variables and a scaling of the time variable, system (3) can be written in one of the following five forms:

1. System (I.1) the unique phase portrait is 1.1.
2. For system (I.2) the phase portrait is 1.2 when \( c < 0, 1.3 \) when \( c = 0 \) and 1.4 when \( c > 0 \), respectively.
3. For system (I.3) the phase portrait is 1.5 when \( a > 0, 1.6 \) when \( a = 0 \) and 1.7 when \( a < 0 \), respectively.
4. For system (I.4) the phase portrait is 1.8 when \( a > 0 \) and 1.9 when \( a < 0 \), respectively.
5. For system (I.5) the phase portrait is 1.8 when \( (a, c) \in \{ \Delta < 0 \} \cup \mathcal{R}_1 \); 1.10 when \( (a, c) \in \mathcal{R}_2 \); 1.11 when \( (a, c) \in \mathcal{R}_3 \); 1.12 when \( (a, c) \in \mathcal{R}_4 \); 1.13 when \( (a, c) \in \mathcal{R}_5 \).
The bifurcation diagram of system (I.5) is given in Figure 2. (see the subsection 3.5.2 for the definitions of $\mathcal{R}_1$-$\mathcal{R}_5$)

The paper is organized as follows. In section 2 we present some preliminary results. The proof of Theorem 1.1 is given in section 3.
2. Preliminary results. In order to provide Theorem 1.1, we introduce some preliminary results in this section.

2.1. Poincaré compactification. We use the Poincaré compactification to study the behavior of the trajectories of the system near infinity, see for instance Chapter 5 of [15]. Roughly speaking, Poincaré compactification can be described as follows.

In the Poincaré compactification we prefer to work on a Poincaré sphere $S^2 = \{ y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1 \}$. The induced vector field on the Poincaré sphere $S^2$ is called the Poincaré compactification of the vector field $X$, denoted by $p(X)$. More precisely, the vector field $X$ can be induced to the Poincaré sphere by projecting each point $Y \in \mathbb{R}^2 = (y_1, y_2, 1) \in \mathbb{R}^3$ onto the Poincaré sphere applying a straight line through $Y$ and the origin of $\mathbb{R}^3$. As we know, the equator $S^1 = \{ (y_1, y_2, y_3) \in S^2 : y_3 = 0 \}$ corresponds to the infinity of $\mathbb{R}^2$. Usually, we use the local charts to make calculations on the Poincaré sphere. On the Poincaré sphere $S^2$, we consider the local charts $(U_k, \phi_k)$ and $(V_k, \psi_k)$ defined as follows

$U_k = \{ Y \in S^2 : y_k > 0 \}, \quad V_k = \{ Y \in S^2 : y_k < 0 \},$

where

$\phi_k : U_k \to \mathbb{R}^2, \quad \psi_k : V_k \to \mathbb{R}^2$

satisfying $\phi_k(Y) = -\psi_k(Y) = (y_m/y_k, y_l/y_k)$ for $m < l$ and $m, l \neq k$, for $k = 1, 2, 3$.

If polynomial vector field $X$ has degree $n$ ($n = \max\{\deg P, \deg Q\}$), the expression for $p(X)$ in the local chart $U_1$ is given by

$\dot{u} = v^n \left[ -uP \left( \frac{1}{v}, \frac{u}{v} \right) + Q \left( \frac{1}{v}, \frac{u}{v} \right) \right], \quad \dot{v} = -v^{n+1}P \left( \frac{1}{v}, \frac{u}{v} \right).$  \hspace{1cm} (4)

Similarly, in the local chart $U_2$ we get

$\dot{u} = v^n \left[ P \left( \frac{u}{v}, 1 \right) - uQ \left( \frac{u}{v}, 1 \right) \right], \quad \dot{v} = -v^{n+1}Q \left( \frac{u}{v}, 1 \right).$  \hspace{1cm} (5)

Finally, in the chart $U_3$ we have

$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$

The expression for $p(X)$ in the chart $V_k$ is the same as for $U_k$ multiplied by $(-1)^{n-1}$. Infinite singular points of $X$ are the singular points of the corresponding
polynomial systems (4) and (5) on the Poincaré disk lying on $\mathbb{S}^1$. To investigate the infinite singular points, we only need to study the singular points at $U_1|v=0$ and the origin of $U_2$.

2.2. Quasi-homogeneous directional blow up. The singular point $q$ is called degenerate if the linear part at $q$ is identically zero. We are going to use the method of quasi-homogeneous directional blow up (or $(\alpha, \beta)$-blow up) to investigate the local phase portraits of a degenerate singular point, see for instance [14, 15]. In general a quasi-homogeneous directional blow up is a change of variables of the form

positive $x$-direction: $(x, y) \mapsto (\tilde{u}^\alpha, \tilde{v}^\beta)$, negative $x$-direction: $(x, y) \mapsto (-\tilde{u}^\alpha, \tilde{v}^\beta)$,

positive $y$-direction: $(x, y) \mapsto (\tilde{u}^\alpha, \tilde{v}^\beta)$, negative $y$-direction: $(x, y) \mapsto (\tilde{u}^\alpha, -\tilde{v}^\beta)$,

where $(\alpha, \beta) \in \mathbb{N}^+ \times \mathbb{N}^+$. By the Newton diagram, we can determine the values of $\alpha$ and $\beta$, see [1] for more details.

2.3. Topological index and Neumann’s Theorem. Next, we give two theorems which characterize the index of the isolated singular points.

**Theorem 2.1 (Poincaré Index Formula [15]).** Let $q$ be an isolated singular point having the finite sectorial decomposition property. Let $e$, $h$ and $p$ denote the number of elliptic, hyperbolic and parabolic sectors of $q$, respectively. Then the index of $q$ is $(e - h) / 2 + 1$.

**Theorem 2.2 (Poincaré-Hopf Theorem [15]).** For every tangent vector field on the sphere $\mathbb{S}^2$ with a finite number of singular points, the sum of the indices of the singular points is 2.

From Theorem 2.1, we have immediately a proposition below.

**Proposition 1.** The indices of a saddle, a center and a cusp are $-1$, $1$ and $0$, respectively.

**Remark 1.** Nilpotent singular points of Hamiltonian planar polynomial systems are either saddles, centers, or cusps (see Chapters 2 and 3 of [15], specifically sections 2.6 and 3.5).

The following theorem was given in [27].

**Theorem 2.3. (Neumann’s Theorem).** Two continuous flows in $\mathbb{S}^2$ with isolated singular points are topologically equivalent if and only if their separatrix configurations are equivalent.

Neumann’s Theorem tell us that the global phase portraits on the Poincaré disc of polynomial systems are determined by its separatrix configuration.

3. Proof of Theorem 1.1. Our main aim of this section is to prove Theorem 1.1. By a rescaling of the time variable $t \mapsto -t/h$, system (3) can be written as

$$\dot{x} = x \left(1 + 2ax^2y + 4by^3\right), \quad \dot{y} = -y - 3ax^2y - by^4 + cx^4$$

with Hamiltonian $H(x, y) = xy + by^4 + ax^3y^2 - cx^5/5$. Note that the origin is a saddle and $x = 0$ is an invariant straight line of system (6).

In the local chart $U_1$ system (6) can be written as

$$\dot{u} = -5bu^4 - 5au^2 - 2uv^3 + c, \quad \dot{v} = -v \left(4bu^3 + 2au + v^3\right).$$

(7)
On \{v = 0\}, the singular points of system (7) are given by the solution of \(\omega(z) \triangleq 5bz^2 + 5az - c = 0\) with \(z = u^2\). Therefore we can obtain an algebraic classification of system (6) given by the coefficients of \(\omega(z)\) in Table 1. As a result, we get systems (I.1)-(I.5) of Theorem 1.1.

### 3.1. Global phase portraits of system (I.1)

In the local chart \(U_1\), system (I.1) can be rewritten as

\[
\begin{align*}
\dot{u} &= 1 - 2uv^3, \\
\dot{v} &= -v^4.
\end{align*}
\]

On the \(u\)-axis, there are no singular points. In the local chart \(U_2\), we apply the equation (5) to obtain

\[
\begin{align*}
\dot{u} &= u(2v^3 - u^4), \\
\dot{v} &= v(v^3 - u^4).
\end{align*}
\]

The origin of system (8) is a singular point and its linear part is identically zero. We perform \((u,v) \mapsto (\bar{u}^3, \bar{u}^4\bar{v})\) in the positive \(u\)-direction. After division by \(\bar{u}^{12}\), we obtain

\[
\begin{align*}
\bar{\dot{u}} &= \frac{\bar{u}}{3} (-1 + 2\bar{v}^3), \\
\bar{\dot{v}} &= \frac{\bar{v}}{3} (1 - 5\bar{u}^3).
\end{align*}
\]

On the \(\bar{u}\)-axis, the singular points of system (9) are \(0, 0\) and \(0, 1/\sqrt[4]{5}\), which are saddle and stable node, respectively.

Consider the blow-up \((u,v) \mapsto (-\bar{u}^3, \bar{u}^4\bar{v})\) in the negative \(u\)-direction. After cancelling a common factor \(\bar{u}^{12}\), we have

\[
\begin{align*}
\hat{u} &= \frac{\bar{u}}{3} (-1 + 2\bar{v}^3), \\
\hat{v} &= \frac{\bar{v}}{3} (1 - 5\bar{u}^3).
\end{align*}
\]

On \(\{\bar{u} = 0\}\) system (10) has a saddle at \(0, 0\) and a stable node at \(0, 1/\sqrt[4]{5}\).

We apply blow-up \((u,v) \mapsto (\bar{u}\bar{v}^3, \bar{v}^4)\) in the positive \(v\)-direction as well as \((u,v) \mapsto (\bar{u}^3, -\bar{v}^4)\) in the negative \(v\)-direction. After division by \(v^9\), we get, respectively,

\[
\begin{align*}
\bar{\dot{u}} &= \frac{\bar{u}}{4} (5 - \bar{u}^4), \\
\bar{\dot{v}} &= \frac{\bar{v}}{4} (1 - \bar{v}^4),
\end{align*}
\]

and

\[
\begin{align*}
\hat{u} &= -\frac{\bar{u}}{4} (5 + \bar{u}^4), \\
\hat{v} &= -\frac{\bar{v}}{4} (\bar{u}^4 + 1).
\end{align*}
\]

The origin of systems (11) and (12) are singular points, which is an unstable and a stable node, respectively. The blow-up procedure and the local phase portrait of the system (8) at the origin are shown in Figure 3.

For system (I.1), only the origin is finite singular point, which is a saddle.
3.2. Global phase portraits of system (I.2). In the local chart $U_1$ the compactified system (I.2) is

$$
\dot{u} = c - 5u^2 - 2uv^3, \quad \dot{v} = -v \left(2u + v^3\right).
$$

On the $u$-axis, the two possible singular points of system (13) other than its origin are $p_1^+ = \left(\pm \sqrt{c/5}, 0\right)$. The eigenvalues of $p_1^+$ are $-2\sqrt{5c}$ and $-2\sqrt{c/5}$. For the singular point $p_1^-$ the associated eigenvalues are $2\sqrt{5c}$ and $2\sqrt{c/5}$. Thus, we have the following results.

(i) If $c > 0$, then $p_1^+$ is a stable node and $p_1^-$ is an unstable node.

(ii) If $c = 0$, then $p_1^\pm = (0, 0)$ is degenerate. Applying (3, 1)-blow up, we obtain the local phase portrait of system (13) at the origin for $c = 0$, see Figure 5.

(iii) If $c < 0$, then $p_1^\pm$ do not exist.
In the local chart $U_2$ the compactified system (I.2) is written as
\[
\dot{u} = u\left(5u^2 - cu^4 + 2v^3\right), \quad \dot{v} = v\left(3u^2 - cu^4 + v^3\right).
\] (14)

The origin of system (14) is a degenerate singular point. With the help of (3,2)-blow up technique, we get the local phase portrait of system (14) at the origin, see Figure 6.

**Figure 5.** The local phase portrait of system (13) at origin for $c = 0$.

**Figure 6.** The local phase portrait of the system (14) at the origin.

**Figure 7.** All the local phase portraits of system (I.2) on the Poincaré disk.

About the finite singular points of system (I.2), we have the following statements.

(A) If $c \leq 0$, then only the origin is finite singular point, which is a saddle.
If $c > 0$, then the finite singular points of system (I.2) other than its origin are $e_1^\pm = 1/\sqrt{2}(\pm 1/\sqrt{c}, -\sqrt{c})$. The points $e_1^\pm$ are saddles since their eigenvalues are $\pm \sqrt{3}$.

So, the corresponding local phase portraits of system (I.2) are described in Figure 7. It is obvious that the global phase portrait is topologically equivalent to $1.2$ of Figure 1 and $1.3$ of Figure 1 if $c < 0$ and $c = 0$, respectively. Finally, we consider the case $c > 0$. The unstable manifolds of the origin cannot cross the coordinate $x$-axis since $H|_{y=0} = -cx^5/5$. There do not exist heteroclinic orbits connecting the origin and $e_1^\pm$ due to the fact that $H(O) \neq H(e_1^\pm) = \pm 3\sqrt{2}\sqrt{c}/10$. Thus, one unstable manifold of $e_1^\pm$ must intersect the $x$-axis. Together with the local behavior at infinity, we obtain that the global phase portrait of system (I.2) with $c > 0$ is described in $1.4$ of Figure 1.

3.3. Global phase portraits of system (I.3). In the local chart $U_1$ system (I.3) can be written as

$$
\dot{u} = -u \left(5u^3 + 5au + 2v^3\right), \quad \dot{v} = -v \left(2au + 4u^3 + v^3\right).
$$

(15)

For system (15) the following statements hold.

(i) If $a \geq 0$, the singular point of system (15) on the $u$-axis is the origin, which is a degenerate. We perform a $(1,1)$ and $(3,1)$-blow up to system (15) if $a = 0$ and $a > 0$, respectively. The local phase portrait of system (15) at the origin is given in (2) of Figure 8 and (2) of Figure 5 if $a = 0$ and $a > 0$, respectively.

(ii) If $a < 0$, system (15) on the $u$-axis has three singular points $p_1^+ = (\pm \sqrt{-a}, 0)$ and the origin. The eigenvalues of $p_1^+$ are $10a\sqrt{-a}$ and $2a\sqrt{-a}$, so $p_1^+$ is...
a stable node. The point $p_1^-$ is an unstable node since its eigenvalues are $-10a\sqrt{-a}$ and $-2a\sqrt{-a}$. Doing (3,1)-blow up, we obtain the local phase portrait of system (15) at the origin, see (4) of Figure 8.

In the local chart $U_2$ system (I.3) can be written as

$$
\dot{u} = 5u + 2uv^3 + 5au^3, \quad \dot{v} = v + 3au^2v + v^4.
$$

(16)

The origin of system (16) is an unstable node.

Next we study the finite singular points. The two possible finite singular points of system (I.3) are $e_1^\pm = 10^{-1/3}(\pm \sqrt{3}a/a, -1)$ in addition to the origin and $e_1 = (0, -1)$. The eigenvalues of $e_1$ are $\pm 3$. Thus, $e_1$ is a saddle. We get the following results.

(A) If $a \leq 0$, then $e_2^\pm$ do not exist.

(B) If $a > 0$, then $e_2^\pm$ are centers since their eigenvalues are $\pm 3\sqrt{5}/5$.

![Figure 9](image)

Figure 9. All the local phase portraits of system (I.3) on the Poincaré disk.

Note that system (I.3) has the two invariant straight lines $x = 0$ and $y = 0$. From the previous analysis, the local phase portraits of system (I.3) are given in Figure 9. Combining with the local behavior at infinity, we can conclude that the global phase portrait of system (I.3) is topologically equivalent to 1.5 of Figure 1 if $a > 0$, 1.6 of Figure 1 if $a = 0$ and 1.7 of Figure 1 if $a < 0$, respectively.

3.4. Global phase portraits of system (I.4). System (I.4) becomes

$$
\dot{u} = -5u^4 - 5au^2 - 2uv^3 - \frac{5a^2}{4}, \quad \dot{v} = -v \left(2au + 4u^3 + v^3\right),
$$

(17)

in the local chart $U_1$. On the $u$-axis, the two possible singular points of system (17) are $p_1^\pm = \left(\pm \sqrt{-a}/2, 0\right)$. Therefore the following results hold.

(i) If $a > 0$, then $p_1^\pm$ do not exist.

(ii) If $a < 0$, then $p_1^\pm$ are degenerate. Moving the singular point $p_1^+$ to the origin through the change of coordinates $(u, v) \mapsto \left(u + \sqrt{-a}/2, v\right)$, and doing the (3,2)-blow up, we get the local phase portrait of system (17) at $p_1^+$ for this case, see (2) of Figure 10. Analogously, we obtain the local phase portrait at $p_1^-$, see (4) of Figure 10.
(1) Blow up.

\[ \dot{u} = 5u + 2uv^3 + \frac{5}{4}a^2u^5 + 5au^3, \quad \dot{v} = v + \frac{5}{4}a^2u^4v + 3au^2v + v^4. \] (18)

Figure 10. The local phase portraits of system (17) at \( p_1^\pm \) for \( a < 0 \).

In the local chart \( U_2 \), we have

The origin of system (18) is an unstable node.

The two possible finite singular points of system (I.4) other than its origin and
\( e_1 = (0, -1) \) are \( e_2^\pm = -\sqrt{10}/4 \left( \pm \sqrt{30a}/5a, 1 \right) \). The eigenvalues of \( e_1 \) are \( \pm 3 \). So, \( e_1 \) is a saddle. We have the following statements.

(A) If \( a < 0 \), then the points \( e_2^\pm \) do not exist.

(B) If \( a > 0 \), then \( e_2^\pm \) are centers since their eigenvalues are \( \pm 3\sqrt{2i}/2 \).

We have the local phase portraits of system (I.4) on the Poincaré disk, see Figure 11. In the case \( a > 0 \), the origin and \( e_1 \) must be on the boundary of the period annulus of the centers \( e_2^\pm \). Therefore the global phase portraits of system (I.4) for \( a > 0 \) is topologically equivalent to 1.8 of Figure 1. For the case \( a < 0 \), it is easy to prove that the global phase portrait is topologically equivalent to 1.9 of Figure 1.

Figure 11. The local phase portraits of system (I.4) on the Poincaré disk.
3.5. Global phase portraits of system (I.5). Note that the system (I.5) has the Hamiltonian \( H(x, y) = xy + ax^2y^2 + xy^4 - cx^5/5 \).

Firstly, we consider the infinite singular points of system (I.5). In the local chart \( U_1 \), system (I.5) becomes

\[
\dot{u} = c - 5au^2 - 5u^4 - 2uv^3, \quad \dot{v} = -2auv - 4u^3v - v^4.
\] (19)

On the \( u \)-axis, the four possible singular points of system (19) are

\[
p_1^\pm = \left( \frac{\pm \sqrt{-5a - 5\Delta}}{10}, 0 \right) \text{ and } p_2^\pm = \left( \frac{\pm \sqrt{-5a + 5\Delta}}{10}, 0 \right),
\]

where \( \Delta = 5a^2 + 4c \). The eigenvalues of \( p_1^\pm \) are

\[
\lambda_1 = -2\Delta \left( 5a + \sqrt{5\Delta} \right) \text{ and } \lambda_2 = -2\Delta \left( 5a + \sqrt{5\Delta} \right)/25.
\]

The singular point \( p_1^- \) has eigenvalues \( -\lambda_1 \) and \( -\lambda_2 \). The eigenvalues of \( p_2^+ \) are

\[
\lambda_3 = -\sqrt{2\Delta \left( 5\Delta - 5a \right)} \text{ and } \lambda_4 = -\sqrt{2\Delta \left( 5\Delta - 5a \right)}/25.
\]

For the singular point \( p_2^- \) the associated eigenvalues are \( -\lambda_3 \) and \( -\lambda_4 \).

In the local chart \( U_2 \), system (I.5) is expressed as

\[
\dot{u} = 5u + 5au^3 - cu^2 + 2u^3v, \quad \dot{v} = 3u^2v - cv^4v + v^4,
\] (20)

having the origin as an unstable node. Consequently, we only need to investigate the local chart \( U_1 \) for system (I.5).

Now we analyze the finite singular points of system (I.5). The four possible finite singular points of system (I.5) other than its origin and \( e_1 = (0, -1) \) are

\[
e_2^\pm = \sqrt{\frac{a^2 + 4c + a\sqrt{\delta}}{4\Delta}} \left( \frac{\pm \sqrt{a - \sqrt{\delta}}}{2c}, -1 \right) \text{ and } e_3^\pm = \sqrt{\frac{a^2 + 4c - a\sqrt{\delta}}{4\Delta}} \left( \frac{\pm \sqrt{a + \sqrt{\delta}}}{2c}, -1 \right)
\]

with \( \delta = a^2 - 12c \). The eigenvalues of \( e_1 \) are \( \pm 3 \). Thus, \( e_1 \) is a saddle. The singular points \( e_2^\pm \) has eigenvalues \( \lambda_2^\pm = \pm \sqrt{3\delta \left( \sqrt{\delta} - 4a \right)/\Delta} \). For the singular points \( e_3^\pm \) the associated eigenvalues are \( \lambda_3^\pm = \pm \sqrt{3\delta \left( \sqrt{\delta} + 4a \right)/\Delta} \).

Next, we will divide the proof into two cases \( \Delta < 0 \) and \( \Delta > 0 \).

3.5.1. Global phase portraits of system (I.5) with \( \Delta < 0 \). For system (19), the infinite singular points \( p_1^\pm \) and \( p_2^\pm \) do not exist. The origin of system (20) is an unstable node.

The condition \( \Delta < 0 \) implies \( c < 0 \), \( \delta > 0 \) and \( -\sqrt{\delta} < a < \sqrt{\delta} \). So the finite singular points of system (I.5) with \( \Delta < 0 \) are \( e_1 \) and \( e_2^\pm \) in addition to origin. We have \( \left( \sqrt{\delta} - 4a \right) \left( \sqrt{\delta} + 4a \right) = -3\Delta > 0 \), which means that \( \sqrt{\delta} - 4a > 0 \). Thus \( e_2^\pm \) are centers.
The corresponding local phase portrait of system (I.5) in this case is characterized in (1) of Figure 11. The global phase portrait is topologically equivalent to 1.8 of Figure 1.

3.5.2. Global phase portraits of system (I.5) with $\Delta > 0$. We study the local behavior at infinite singular points of (I.5) with $\Delta > 0$. The origin of system (20) is an unstable node. For system (19), the following statements hold.

(i) If $a > 0$ and $c < 0$, then the points $p_1^+$ and $p_2^+$ do not exist.
(ii) If $a \in \mathbb{R}$ and $c > 0$, then $p_1^+$ do not exist. The point $p_2^+$ is a stable node and $p_2^-$ is an unstable node.
(iii) If $a < 0$ and $c < 0$, then $p_1^+$ and $p_2^-$ are unstable nodes, and $p_1^-$ and $p_2^+$ are stable nodes.

\[ \text{Figure 12. The local phase portraits of system (I.5) with } \Delta > 0 \text{ on the Poincaré disk.} \]

About the finite singular points of system (I.5) with $\Delta > 0$, we have the following results.

(A) If $a > 0$ and $c < 0$, then $\delta > 0$ and $-\sqrt{\delta} < a < \sqrt{\delta}$. The points $e_3^\pm$ do not exist. We have $\left( \sqrt{\delta} - 4a \right) \left( \sqrt{\delta} + 4a \right) = -3\Delta < 0$, so $\sqrt{\delta} - 4a < 0 < \sqrt{\delta} + 4a$.

Since $\lambda_2^\pm$ are purely imaginary, the points $e_2^\pm$ are centers.

(B) If $a \in \mathbb{R}$ and $c > 0$, then the following statements hold.

(B.1) If $\delta < 0$, then the points $e_2^+$ and $e_2^-$ do not exist.
(B.2) If $\delta = 0$, then $c = a^2/12$. We obtain that when $a < 0$, the points $e_2^+$ and $e_2^-$ do not exist, and when $a > 0$, the points $e_2^+ = e_2^- = 1/2 \sqrt{2} (\pm \sqrt{6a}/a, -1)$ are nilpotent singular points due to the fact that their eigenvalues are 0 and their linear parts are not identically zero. By Theorem 2.1, the total index of the infinite singular points are 6. Applying Theorem 2.2, we get that the index of $e_2^\pm$ are 0, which means that $e_2^\pm$ are cusps.
(B.3) If $\delta > 0$, then $a \neq 0$. If $a < 0$, then $a \pm \sqrt{\delta} < 0$ and the points $e^a_2$ and $e^b_2$ do not exist. If $a > 0$, then $a \pm \sqrt{\delta} > 0$ and $\sqrt{\delta} - 4a < 0 < \sqrt{\delta} + 4a$.

From $\lambda_2^a$ and $\lambda_2^b$, we have that $e^a_2$ are centers and $e^b_2$ are saddles.

(C) If $a < 0$ and $c < 0$, then $a - \sqrt{\delta} < 0 < a + \sqrt{\delta}$ and $\sqrt{\delta} - 4a > 0$. So, the points $e^c_2$ do not exist. The singular points $e^c_2$ are saddles due to the fact that $\lambda_2^c < 0$.

The local phase portraits of system (I.5) with $\Delta > 0$ on the Poincaré disk depend on the parameters $a$ and $c$, that is,

(a) (1) of Figure 11: If $\{\Delta > 0 \cap a > 0 \cap c < 0\} \triangleq R_1$.

(b) (1) of Figure 12: If $\{\Delta > 0 \cap c > 0 \cap \delta < 0\} \cup \{\Delta > 0 \cap c > 0 \cap a < 0 \cap \delta > 0\} \triangleq R_2$.

(c) (2) of Figure 12: If $\{\Delta > 0 \cap c > 0 \cap a > 0 \cap \delta = 0\} \triangleq R_3$.

(d) (3) of Figure 12: If $\{\Delta > 0 \cap c > 0 \cap a > 0 \cap \delta > 0\} \triangleq R_4$.

(e) (4) of Figure 12: If $\{\Delta > 0 \cap a < 0 \cap c < 0\} \triangleq R_5$.

It is not difficult to prove that the global phase portrait of system (I.5) with $\Delta > 0$ is topologically equivalent to 1.8 of Figure 1 if $(a, c) \in R_1$ and 1.10 of Figure 1 if $(a, c) \in R_2$. Next, we investigate the remaining cases.

(c) $(a, c) \in R_3$: The Hamiltonian values of the cusps $e^c_2$ are

$$H(e^c_2) = \pm \frac{3\sqrt{2}\sqrt{6a}}{40a}.$$ 

There do not exist heteroclinic orbits connecting the origin and $e_2^c$ due to the fact that $H(e^c_2) \neq H(O)$. Similarly, there do not exist heteroclinic orbits connecting $e_1$ and $e_2^c$. Since $\dot{y}|_{y=0} = cx^4 > 0$, the stable manifolds of saddle $e_1$ cannot cross the $x$-axis. Together with the local behavior at infinity, the global phase portrait is topologically equivalent to 1.11 of Figure 1.

(d) $(a, c) \in R_4$: The Hamiltonian values of the saddles $e^c_3$ are

$$H(e^c_3) = \pm \frac{3\sqrt{2}\sqrt{a^2 + 4c - a\sqrt{\delta}} \sqrt{4a + \sqrt{\delta}}}{10\Delta^+}.$$ 

Note that $\left(a^2 + 4c - a\sqrt{\delta}\right)\left(a^2 + 4c + a\sqrt{\delta}\right) = 4c\Delta > 0$. Thus $H(e^c_3) \neq H(O) = H(e_1) = 0$. We can get that there do not exist heteroclinic orbits connecting the origin and $e^c_3$, and there do not exist heteroclinic orbits connecting $e_1$ and $e^c_3$. Suppose that the origin and $e_1$ form the boundary of the period annulus of the centers $e^c_3$. From the local phase portrait (3) of Figure 12, we can observe that the invariant manifolds of $e^c_3$ must respectively intersect the $x$-axis two times, which is in contradiction with the invariant manifolds of $e^c_3$ respectively cross the $x$-axis only one time (because $H|_{y=0} = -cx^5/5$). Therefore the boundary of the period annulus of the centers $e^c_2$ are formed by a homoclinic loop associated to saddles $e^c_3$, respectively.

In summary, we obtain that the global phase portrait is characterized in 1.12 of Figure 1.

(e) $(a, c) \in R_5$: We have

$$H(O) = H(e_1) \neq H(e^c_2) = \pm \frac{3\sqrt{2}\left(a^2 + 4c + a\sqrt{\delta}\right) \sqrt{\delta} - 4a}{10\Delta^+}.$$
Thus there do not exist heteroclinic orbits connecting the origin and \( e_1^\pm \), and there do not exist heteroclinic orbits connecting \( e_1 \) and \( e_2^\pm \). Since \( \dot{y}|_{y=0} = cx^4 < 0 \), the stable manifolds of saddles \( e_2^\pm \) cannot cross the \( x \)-axis. Combining with the local behavior at infinity, we conclude that the global phase portrait is topologically equivalent to 1.13 of Figure 1.

Based on the above previous analysis, we obtain the bifurcation diagram of system (I.5) shown in Figure 2.

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