Two-Dimensional Decaying Turbulence in Different Shaped Containers

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Abstract. Several interesting phenomena have been observed simulating two-dimensional decaying turbulence in bounded domains. In this paper an overview is given about our observations obtained by simulating freely decaying turbulence in different regular polygon shaped containers with no-slip walls. For these simulations the lattice Boltzmann method has been used as a numerical approach. The initial Reynolds number based on the container dimension was in the order of 1000. The initial condition was the same in each simulation, therefore, we were able to compare the effect of geometrical constraints on the evolution of relevant physical quantities such as the kinetic energy and the enstrophy.

1. Introduction

Two-dimensional turbulence in bounded domains plays an important role in geosciences, meteorology and oceanography. One of the most impressive example has been taken from the nature; the vortex formation in coastal currents, a relevant phenomenon in geophysical fluid dynamics. In the last two decades numerical computational studies of decaying two-dimensional turbulence in a container have revealed some interesting results. It was pointed out, for instance, that the decay process of two-dimensional turbulence in a container differs significantly from the one which takes place in a double-periodic domain [1]. Clercx et al. [2] studied the evolution of 2D turbulence in square and circle shaped containers. They demonstrated that the presence of walls acts on the evolution of enstrophy due to the interactions between walls and vortices. In contrast to the evolution of 2D turbulence in a double periodic domain, the enstrophy is not a monotone decaying function of time in a container, but there can be local maxima in the enstrophy as a consequence of interactions between walls and vortices. The final state of the evolution is also different in a container and in a double periodic domain. While a simple dipole is formed at the end of the decay in a double periodic domain, in a container the flow spontaneously spin-up forming a large vortex in the center with small satellites around this structure.

In this paper we present simulation results of 2D decaying turbulence in regular polygon shaped (circle, octagon, hexagon, square) containers. Our aim is to learn more about the spontaneous spin-up process and the effect of the walls in 2D turbulence. Therefore we have performed several simulations starting them from the same initial condition, i.e. the same
Reynolds number and the same initial vortex arrangement were used for each geometry. Then the flow was left to evolve and the relevant physical quantities were recorded. Finally, the simulation results were compared with each other as it is presented in the paper. The outline of the paper is follows. In Section 2 our numerical approach, the lattice Boltzmann method (LBM) is briefly introduced. In Section 3 the problem description is outlined and in Section 4 our results are presented. Finally, the conclusions are drawn in Section 5.

2. Lattice Boltzmann method

This section is devoted to briefly summarize some basic information on the lattice Boltzmann method [3],[4], which had been applied first for the simulation of two-dimensional turbulence in the pioneering work of Benzi and Succi [5].

Using the lattice Boltzmann method one solves a discrete kinetic equation for the one-particle velocity distribution functions \( f_i \):

\[
f_i(x + c_i \delta t, t + \delta t) - f_i(x, t) = \Omega_i, \tag{1}
\]

where \( c_i \) is the lattice vector, \( \delta t \) is the time step and \( \Omega_i \) is the collision operator. In this paper we use the simplest form of the latter i.e. the BGK operator [6], [7]:

\[
\Omega_i = -\frac{1}{\tau} \left[ f_i(x, t) - f_i^{eq}(x, t) \right], \tag{2}
\]

where \( \tau \) is the relaxation time and \( f_i^{eq} \) is a local equilibrium distribution function, which can take the following form:

\[
f_i^{eq} = w_i \rho \left[ 1 + \frac{c_{i\alpha} u_{\alpha}}{c_s^2} + \frac{u_{\alpha} u_{\beta}}{2c_s^4} \left( c_{i\alpha} c_{i\beta} - c_s^2 \delta_{\alpha\beta} \right) \right], \tag{3}
\]

where \( w_i \) is the lattice weight, \( u \) is the hydrodynamic velocity and \( c_s \) is the speed of sound (repeated Greek indices imply summation).

Solving Eq. (1) one can obtain the macroscopic quantities by taking the suitable moments of the distribution functions:

\[
\rho = \sum_i f_i, \quad \rho u_{\alpha} = \sum_i c_{i\alpha} f_i. \tag{4}
\]

The pressure can be obtained through the equation of state of an ideal gas, i.e. \( p = \rho c_s^2 \).

For a specific model, the lattice vector and the lattice weights need to be selected. In our analysis we will use the parameters of a two-dimensional nine-velocity model (D2Q9).

In this work Eq. (1) was solved by a simple streaming and collision procedure and to model no-slip walls we used the bounce-back rules [9], i.e. distribution functions encountering a solid surface were reflected back in the direction they came from.

3. Initial field and problem description

We consider a circle, an octagon, a hexagon and a square container with rigid no-slip walls and divide the domain into lattices using the resolution \( N \times N = 1024 \times 1024 \). The initial vorticity field contains forty-eight equal-size shielded Gaussian vortices [10]:

\[
\omega = -\omega_0 \left( \frac{r^2}{r_0^2} - 1 \right) e^{-r^2/\sigma^2}, \tag{5}
\]

placed on a regular lattice in the center of the domain, with some small perturbations in their individual positions \((x_k, y_k)\) in order to break the symmetry of the problem. The corresponding velocity field of a vortex is given by
\[ u_x = \frac{1}{2} \omega_0 (y - y_k) e^{-\frac{r^2}{r_0^2}}, \quad u_y = -\frac{1}{2} \omega_0 (x - x_k) e^{-\frac{r^2}{r_0^2}}, \] (6)

where \( r_0 \) is the initial vortex radius. Half of the vortices have positive and the rest have negative circulation.

4. Results
In order to characterize the evolution process we define the kinetic energy \( E(t) \) and enstrophy \( \Omega(t) \) as follows [11]:

\[ E(t) = \frac{1}{2} \int_{\Omega} |u|^2 dA, \] (7)

\[ Z(t) = \frac{1}{2} \int_{\Omega} |\omega|^2 dA. \] (8)

These quantities were continuously calculated and recorded during each simulation and here we briefly present how they evolve in the various geometry.

First the decay of the kinetic energy is shown in Fig. 1 for each geometry and for two different initial Reynolds numbers. As one can see the trend of the decay is almost the same in each case, it depends strongly on the initial Reynolds number, but apparently, the shape of the geometry does not influence the evolution of the kinetic energy. This observation might be surprising, since there is inverse energy cascade in 2D turbulence, which means that the energy is dissipated in large, integral scales. So one might expect that the shape of the container have some influence on the decaying process. However, the evolution of the flow is very similar in each geometry.

In the initial stage, vortices interact with each other forming larger and larger structures, while they also interact with the wall producing small but strong counter-rotating vortices. This fact can be seen in Fig. 2 and Fig. 3, where snapshots of vorticity are shown at the same time instant for the different containers. In Fig. 2 the difference is almost negligible, and the size of the relevant vortices is very similar in the later stage, too as one can see in Fig. 3.

Figure 1. Evolution of kinetic energy between \( t = 0 \) and \( t = 700000 \) step at \( Re = 5000 \) and \( Re = 10000 \) Reynolds number
Figure 2. Vorticity fields at $t = 65000$ time step in four different geometries.
So even if the evolution is chaotic and differs locally in the various containers, the global quantities like the kinetic energy erodes in a very similar manner. The evolution of enstrophy follows similar behaviour as one can see in Fig. 4 for each geometry. However, here the situation is somewhat different. In 2D turbulence there is a direct enstrophy cascade, so enstrophy is generated in large scales and dissipated in small scales. The first interactions between the walls and the vortices can be clearly recognized in Fig. 4 as a local maxima in the enstrophy around 65000 steps. It is worth emphasizing that the enstrophy is a monotone decreasing function of time in a double periodic domain, since there is no physical mechanism to produce enstrophy in such geometry. Here, however, enstrophy can be produced by wall shear stress and although our containers have different shapes, we cannot see significant differences in the vorticity production, the enstrophy functions follow more or less the same trend. The reason of this agreement is obvious, the corners of the shapes do not have significant effect on the initial evolution as we have already seen in Fig. 2. The first interaction between the walls and the vortices take place in a very similar manner, therefore the enstrophy functions do not differ from each other.
Figure 4. Evolution of enstrophy between $t = 0$ and $t = 700000$ step at $Re = 5000$ and $Re = 10000$ Reynolds number

5. Conclusion
Studying the decay of 2D turbulence in four different shaped containers We have shown that the relevant physical quantities such as the kinetic energy and enstrophy evolve almost independently on the geometry of the containers. Enstrophy has a local maxima when the initial vortices reach the walls, then it decays with the same trend in each container. The decay of the kinetic energy also follows the same trend independently from the shape of the container.

6. References
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