SCALING LIMIT RESULTS FOR THE SUM OF MANY INVERSE LÉVY SUBORDINATORS

INGEMAR KAJ AND ANDERS MARTIN-LÖF

Abstract. The first passage time process of a Lévy subordinator with heavy-tailed Lévy measure has long-range dependent paths. The random fluctuations that appear under two natural schemes of summation and time scaling of such stochastic processes are shown to converge weakly. The limit process is fractional Brownian motion in one case and a non-Gaussian and non-stable process in the other case. The latter appears to be of independent interest as a random process that arises under the influence of coexisting Gaussian and stable domains of attraction and is known from other applications to provide a bridge between fractional Brownian motion and stable Lévy motion.

1. Introduction and statement of results

A Lévy subordinator \( \{X_t, t \geq 0\} \) is a real-valued random process with independent and stationary increments and increasing pure-jump trajectories. The inverse process \( \{T_x, x \geq 0\} \) defined by the first passage times \( T_x = \inf \{ t \geq 0 : X_t > x \} \) has non-decreasing trajectories, where the lengths of the flat pieces of \( \{T_x\} \) correspond to the jump sizes of \( \{X_t\} \). The dependence structure in the paths of the inverse process is entirely different from that of the Lévy subordinator, since big jumps in the Lévy process may cause strong dependencies that last over a considerable period of evolution of the path of its inverse. In this paper we take a scaling approach to study the nature of the random fluctuations that build up as a result of such long-memory effects. By superposing a large number of paths of the inverse Lévy process and simultaneously scale the time parameter of the process, we obtain scaling limit results for the centered and normalized superposition process.

In somewhat more detail, our starting point is a Lévy subordinator with Lévy measure \( \nu(dx) \) of regularly varying tail with index \( 1 + \beta, 0 < \beta < 1 \). In particular, \( \mu := \int x \nu(dx) < \infty \). The initial distribution of the subordinator process is chosen such that the resulting inverse process has stationary increments and expected value \( E(T_x) = x/\mu \). Letting \( \{T_x^i\}_{i \geq 1} \) be a collection of independent copies of \( \{T_x\} \), our main result is the derivation of a limit process for the summation scheme

\[
\frac{1}{a} \sum_{i=1}^{m} (T_{ax}^{i} - \frac{1}{\mu}ax), \quad x \geq 0,
\]

as both \( m \) and \( a = a_m \) tend to infinity in such a way that \( m \) is of the same order of magnitude as \( a^\beta \), modulo slowly varying functions. The reason for this choice of scaling is to attempt to trace the superposition process on a time scale that captures the size of the fluctuations around its mean. In the asymptotic limit appears a non-Gaussian, non-stable process with long-range dependence, which is known to arise also in other related models and has been called fractional Poisson motion, \([9]\), \([8]\), \([13]\), \([5]\). The general study \([6]\) of
higher-order moment measures for heavy-tailed renewal point processes, provides a unified framework of [8] and the present work.

To give a heuristic context for the topics of interest in this work, let us recall the following limit result for Lévy processes. Writing \( \alpha = 1 + \beta \), the centered and scaled process \( (X_t - \mu t)^{1/\alpha} \) converges in distribution as \( t \to \infty \) to a random variable \( Z_\alpha \), having a stable distribution with stable index \( \alpha \). If we write \( \Gamma_x \) for the overshoot at \( x \), so that \( X_{T_x} = x + \Gamma_x \), then

\[
\frac{T_x - x}{x^{1/\alpha}} = -\frac{X_{T_x} - \mu T_x}{\Gamma_x} \left( \frac{T_x}{x} \right)^{1/\alpha} \frac{1}{\mu} + \frac{\Gamma_x}{\mu x^{1/\alpha}}.
\]

In this relation, \( T_x/x \to 1/\mu \) as \( x \to \infty \) by the law of large numbers. It can be shown moreover that the second term on the right hand side is a remainder term with \( \Gamma_x/x^{1/\alpha} \to 0 \) as \( x \to \infty \). Therefore \( (T_x - x)/x^{1/\alpha} \) converges in distribution to \( -Z_\alpha/\mu^{1+1/\alpha} \) as \( x \to \infty \). Proceeding heuristically, with \( m \sim a^\beta \) we may rewrite the superposition process either as

\[
\frac{1}{a} \sum_{i=1}^m \left( T^i_{ax} - \frac{1}{a} ax \right) \sim \frac{1}{a^{1-\beta}} \frac{1}{\mu} \int_0^x T^i_a - \frac{1}{a} du \]

or

\[
\frac{1}{a} \sum_{i=1}^m \left( T^i_{ax} - \frac{1}{a} ax \right) \sim \frac{1}{m^{1/\beta}} \frac{1}{\mu^{1/(1+\beta)}} \sum_{i=1}^m T^i_{ax} - \frac{ax}{a^{1/(1+\beta)}}.
\]

The first representation emphasizes a sequence of random variables in the domain of attraction of a Gaussian law \( (m \to \infty \text{ with } a \text{ fixed}) \). The second representation highlights a sequence in the domain of attraction of a stable law with index \( 1 + \beta \) \( (a \to \infty \text{ with } m \text{ fixed}) \), which is the type of convergence just discussed above. For the limit regime of interest in our case Gaussian and stable attraction appear to coexist and both influence the resulting limit process.

The main result (Theorem 2 below) is a scaling limit theorem for the intermediate type rescaling regime indicated above. In parallel to this we discuss the scaling regime of Gaussian predominance, leading to fractional Brownian motion in the limit (Theorem 1). Scaling limit results with fractional Brownian fluctuations are known for a variety of models, such as modeling random variation in aggregated data traffic streams. For an introduction and overview of these topics and discussion of the modeling context, as well as detailed statements and derivations of such results, see e.g. [16][17][13].

The model is introduced in detail and all results are stated in Section 1 of the paper. We then focus on the proof of Theorem 2 for the intermediate scaling regime, starting with the analysis of marginal distributions in Section 2. The main technique we use for the study of the one-dimensional distributions of the scaled processes and their limit behavior is that of double transforms in the sense of taking Laplace transforms in the time variable of the logarithmic moment generating function of the random variables. In Section 3 we continue with a study of the finite-dimensional distributions, which are obtained from recursive sets of integral equations for the finite-dimensional cumulant functions. Finally in Section 4 we provide a summary of the proof of Theorem 1 for Gaussian scaling, where each parallel step turns out to be simpler, and the proof of tightness.

1.1. A Lévy subordinator and its inverse. Let \( \{ \tilde{X}_t, t \geq 0 \} \), \( \tilde{X}_0 = 0 \), denote a Lévy subordinator with right-continuous paths, having drift zero and Lévy measure \( \nu(a, b) = \int_a^b \nu(dx) \) with no atom at zero, such that

\[
\int_0^\infty (1 \wedge x) \nu(dx) < \infty \quad \text{and} \quad \mu = \int_0^\infty x \nu(dx) < \infty,
\]
which implies that the first moment is finite, $E(\tilde{X}_t) = \mu t < \infty$. The Laplace transform is given by $-\ln E(e^{-uX_t}) = t\Phi(u)$, $u \geq 0$, with Laplace exponent
\[ \Phi(u) = \int_0^\infty (1 - e^{-ux}) \nu(dx). \]

Let $X_t = X_0 + \tilde{X}_t$ denote the corresponding delayed subordinator process with general initial distribution $X_0$ assumed to be independent of $\{\tilde{X}_t\}$. We will study the case when $X_0 > 0$ has distribution function
\[ (2) \quad P(X_0 \leq x) = \frac{1}{\mu u} \int_0^x \int_y^\infty \nu(ds) \, dy, \]
for which $E(e^{-uX_0}) = \frac{1}{\mu u} \Phi(u)$ and so
\[ (3) \quad E(e^{-uX_t}) = \frac{1}{\mu u} \Phi(u) \exp\{-t\Phi(u)\} \quad u \geq 0. \]

Next we introduce the first passage process of the subordinator. Useful references are Bertoin [1], [2]. Van Harn and Steutel, [11], investigate stationarity properties of delayed subordinators and derive closely related results to those in Lemma 1 and Lemma 3 below.

The entrance time of the Lévy process \{subordinators\} and derive closely related results to those in Lemma 1 and Lemma 3 below.

Bertoin [1], [2]. Van Harn and Steutel, [11], investigate stationarity properties of delayed subordinators and derive closely related results to those in Lemma 1 and Lemma 3 below.

The entrance time of the Lévy process \{subordinators\} and derive closely related results to those in Lemma 1 and Lemma 3 below.

Bertoin [1], [2]. Van Harn and Steutel, [11], investigate stationarity properties of delayed subordinators and derive closely related results to those in Lemma 1 and Lemma 3 below.

The entrance time of the Lévy process \{subordinators\} and derive closely related results to those in Lemma 1 and Lemma 3 below.

Bertoin [1], [2]. Van Harn and Steutel, [11], investigate stationarity properties of delayed subordinators and derive closely related results to those in Lemma 1 and Lemma 3 below.

We will prove below the following

\[ \text{for } x \geq 0, \]
Lemma 1. The inverse subordinator process \( \{T_x, x \geq 0\} \) has stationary increments.

1.2. Scaling limit theorem. Our basic assumption is that the Lévy measure \( \nu \) is regularly varying at infinity with index \( 1 + \beta \), \( 0 < \beta < 1 \), i.e.

\[
\int_x^\infty \nu(dy) \sim \frac{1}{x^{1+\beta}} L(x), \quad x \to \infty,
\]

where \( L \) is a slowly varying function and we write \( f(x) \sim g(x) \) if \( f \) and \( g \) are positive functions and \( f(x)/g(x) \to 1 \) as \( x \to \infty \). The summation schemes to be applied involve speeding up the time parameter using a rescaling sequence \( a_m \to \infty \), either such that

\[
mL(a_m)/a_m^\beta \to \infty, \quad m \to \infty,
\]

or such that

\[
mL(a_m)/a_m^\beta \to c^\beta \mu, \quad m \to \infty,
\]

where \( c \), \( 0 < c < \infty \), is an additional parameter that signifies the relative change of scales of size and time. In addition, we assume that the lower index of the Lévy measure \( \nu \) satisfies

\[
\sigma = \sup\{\alpha > 0 : \lim_{\lambda \to \infty} \lambda^{-\alpha} \Phi(\lambda) = \infty\} > \beta.
\]

Theorem 1. Under assumptions (4) and (7), let \( a_m \) be a sequence such that \( a_m \to \infty \) as \( m \to \infty \) and (5) holds, and define \( b_m \) by

\[
b_m^2 = ma_m^{2-\beta}L(a_m)/\mu.
\]

Then, in the sense of weak convergence of random processes in \( C \),

\[
\left\{ \frac{1}{b_m} \sum_{i=1}^m (T_{a_m}x - \frac{1}{\mu} a_m x), \ x \geq 0 \right\} \Rightarrow \left\{ \mu^{-1}\sigma^\beta B_H(x), \ x \geq 0 \right\},
\]

where

\[
\sigma^2 = \frac{2}{\beta(1-\beta)(2-\beta)}, \quad H = 1 - \beta/2,
\]

and \( B_H \) is standard fractional Brownian motion with Hurst index \( H \), i.e. the Gaussian process with stationary increments, variance \( V(B_H(t)) = t^{2H} \) and continuous sample paths.

\[
\log E \exp \left\{ \sum_{i=1}^n \theta_i B_H(x_i) \right\} = \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n \theta_i \theta_j (x_i^{2-\beta} + x_j^{2-\beta} - (x_i - x_j)^{2-\beta}),
\]

where \( 0 = x_0 \leq x_1 \leq \cdots \leq x_n, \ n \geq 1 \).

Theorem 2. Under the assumptions (4) and (7), if \( a_m \) is a sequence such that \( a_m \to \infty \) and (6) holds for some constant \( c > 0 \) as \( m \to \infty \), then

\[
\left\{ \frac{1}{a_m} \sum_{i=1}^m (T_{a_m}x - \frac{1}{\mu} a_m x), \ x \geq 0 \right\} \Rightarrow \left\{ -\mu^{-1}c Y_\beta(x/c), \ x \geq 0 \right\},
\]

in the sense of weak convergence in \( C \). Here \( \{Y_\beta(x), x \geq 0\} \) is a zero mean stochastic process with continuous paths and finite-dimensional distributions characterized by the
cumulant generating function

\[
\log E \exp \left\{ \sum_{i=1}^{n} \theta_i (Y_\beta(x_i) - Y_\beta(x_{i-1})) \right\} = \frac{1}{\beta} \sum_{i=1}^{n} \theta_i^2 \int_{0}^{\Delta x_i} \int_{0}^{v} e^{\theta_i u - \beta u} du dv
\]

\[
+ \frac{1}{\beta} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \theta_i \theta_j \exp \left\{ \sum_{k=i+1}^{j-1} \theta_k \Delta x_k \right\}
\times \int_{0}^{\Delta x_i} \int_{0}^{\Delta x_j} e^{\theta_i u} e^{\theta_j v} (x_{j-1} - x_i + u + v)^{-\beta} du dv,
\]

(11)

where \(0 = x_0 \leq x_1 \leq \cdots \leq x_n\), and \(\Delta x_i = x_i - x_{i-1}\), \(i = 1, \ldots, n\).

**Remarks.**

a) It is known that \(\{Y_\beta\}\) has a simple representation as the stochastic integral

\[
Y_\beta(x) = \int_{R \times (0, \infty)} \int_{0}^{x} 1_{\{s < y < s + v\}} dy \tilde{N}(ds, dv)
\]

where \(\tilde{N}(ds, dv) = N(ds, dv) - n(ds, dv)\) and \(N(ds, dv)\) is a Poisson random measure on \(R \times (0, \infty)\) with intensity measure \(n(ds, dv) = (1 + \beta) ds v^{-2-\beta} dv\), \[12, 8, 13\]. We have not been able to find however a method of proof of the present results which utilizes more directly this inherent Poisson structure of the model.

b) The process \(\{Y_\beta(x)\}\) has been derived in \[9\] as a limit process in the setting of a superposition of independent renewal processes with stationary increments and heavy-tailed inter-renewal distribution, and in \[12\] and \[13\] for an infinite source Poisson process with heavy-tailed activity periods. The motivation is partly from modeling the total traffic load generated by many independent sources at an arrival point in a data traffic network. In these references condition (5) is called *fast connection rate* and (6) *intermediate connection rate*. They are compared to an alternative third scaling regime of *slow connection rate*, for which the limit process turns out to be a stable Lévy process with stable index \(\alpha = 1 + \beta\), see also et al. \[14\] or et al. \[17\].

c) Proofs of the following properties among others can be found in Gaigalas and Kaj \[9\]. The process \(\{Y_\beta\}\) has stationary increments and continuous trajectories. The process is not self-similar. The higher moments are of the order \(E(Y_\beta^k(x)) \sim \text{const } x^{k-\beta}\), \(k \geq 2\), for large \(x\). Specifically, the second-order properties (mean, variance, covariance) are the same (modulo constants) as those for fractional Brownian motion, whereas higher order moments are different. For example, \(\{Y_\beta\}\) is positively skewed. The paths are \(\gamma\)-Hölder continuous for all \(\gamma < 1 - \beta/2\) (not \(\gamma < 1\) as claimed in \[9\]).

d) The renewal processes studied in \[9\] can be viewed as discrete local time processes of discrete regenerative sets (ranges of compound Poisson subordinators). In this light, the present situation is the natural analogue for continuous local time processes of perfect regenerative sets (ranges of subordinators that are not compound Poisson). One can expect the scaling limits to transfer since they are large-time asymptotics which should not depend on the local structure. Some relevant references for the connections of regenerative sets and subordinators are \[7\], and \[10\].

2. Analysis of the marginal distribution

As a preliminary for the proof of Theorem 2 we observe the following properties of the functions introduced in \[11\], which are straightforward to verify.
Lemma 3. Theorem 49.2 in Sato [15]. X subordinator. The following Lemma is a special case of a result valid for general Lévy \( x \) and represents at time \( 0 \) such that for any \( n \geq y \) the following inequality holds:

\[
\Gamma_x > \theta \Rightarrow \Gamma_x \leq \frac{\Phi(v) - \Phi(u)}{\Phi(u) - \Phi(\theta)}.
\]

Relation (11) defines a consistent family of finite-dimensional distributions, such that for any \( c > 0 \)

\[
\log E \exp \left\{ \sum_{i=1}^{n} \theta_i (cY_\beta(x_i/c) - cY_\beta(x_{i-1}/c)) \right\}
\]

\[
= \exp \left\{ \sum_{i=1}^{n} \theta_i (Y_\beta(x_i) - Y_\beta(x_{i-1})) \right\}.
\]

The main part of the proofs of Theorem 1 and Theorem 2 consists in establishing convergence of the scaled \( n \)-point cumulant functions

\[
\log E \exp \left\{ \sum_{i=1}^{n} \frac{1}{b_m} \sum_{k=1}^{m} \left( T_{a_m x_i}^{(k)} - \frac{1}{\mu} a_m x_i \right) \right\}
\]

\[
= mE \left[ \exp \left\{ \sum_{i=1}^{n} \frac{1}{b_m} \left( T_{a_m x_i}^{(k)} - \frac{1}{\mu} a_m x_i \right) \right\} - 1 \right] + O(1/m)
\]

(12) toward the corresponding functionals of the limit processes. As a preparation we study the joint distribution \((T_x, \Gamma_x)\), where \( \{\Gamma_x\} \) is the overshoot process, and other properties of the one-dimensional marginal distributions of \( T_x \).

2.1. Marginal distributions and the overshoot process. The overshoot process \( \{\Gamma_x, x \geq 0\} \) associated with the first passage time \( \{T_x\} \) is defined for \( x \geq 0 \) by

\[
\Gamma_x = X_{T_x} - x,
\]

and represents at time \( x \) the remaining time until the next point of increase of the inverse subordinator. The following Lemma is a special case of a result valid for general Lévy processes adapted to the case of a general initial distribution \( X_0 \). For a proof see Theorem 49.2 in Sato [15].

Lemma 2. Relation (11) defines a consistent family of finite-dimensional distributions, such that for any \( c > 0 \)

\[
\int_0^\infty u e^{-ux} E(e^{\theta T_x - v \Gamma_x}) \, dx = \frac{u}{u - v} \left( \frac{\Phi(v)}{\mu v} - \frac{\Phi(v) - \Phi(u)}{\Phi(u) - \Phi(\theta)} \right).
\]

Proof of Lemma In Lemma 3 take \( u > 0 \) and \( v > 0 \), \( u \neq v \), and let \( \theta = 0 \). We obtain

\[
\int_0^\infty u e^{-ux} E(e^{-v \Gamma_x}) \, dx = \frac{u}{u - v} \left( \frac{\Phi(v)}{\mu v} - \frac{\Phi(v) - \Phi(u)}{\Phi(u)} \right) = \frac{\Phi(v)}{\mu v}.
\]

Hence, for any \( x \geq 0 \), \( \Gamma_x \stackrel{d}{=} X_0 \). Consequently, for each \( x \) the increment process \( T_{x+y} - T_x \), \( y \geq 0 \), begins with a flat period for a duration of time having the distribution \( X_0 \), which is just the same behavior as the original process \( T_x \), \( x \geq 0 \). To formalize the argument, note

\[
P(T_{x+y} - T_x > t) = P(\Gamma_x < y, T_{x+y} - T_x > t) = P(\Gamma_x < y, X_{T_x+t} - X_{T_x} < y - \Gamma_x).
\]

Since \( \Gamma_x = X_{T_x} - x \) is independent of \( X_{T_{x+t}} - X_{T_x} \) and \( X_{T_{x+t}} - X_{T_x} \stackrel{d}{=} X_t \) it follows that

\[
P(T_{x+y} - T_x > t) = P(X_0 < y, X_{T_{x+t}} - X_{T_x} < y - X_0) = P(X_t < y) = P(T_y > t).
\]

Lemma 4. For \( u > 0 \) and \( \theta < \Phi(u) \),

\[
\int_0^\infty u e^{-ux} E(e^{\theta T_x}) \, dx = 1 + \frac{\theta}{\Phi(u) - \Phi(\theta)} \frac{\Phi(u)}{\mu u}.
\]
Also, for \( u > 0 \) and \( \theta > -\mu u \),

\[
\int_0^\infty u e^{-ux} E(e^{\theta(T_x-x/\mu)} - 1) \, dx = \frac{\theta^2}{(\mu u + \theta)^2} \left[ \frac{\mu u}{\Phi(u + \theta/\mu) - \theta} - 1 \right]
\]

and

\[
\int_0^\infty u e^{-ux} E(e^{\theta(\tilde{T}_x-x/\mu)} - 1) \, dx = \frac{\theta}{\mu u + \theta} \left[ \frac{\mu u}{\Phi(u + \theta/\mu) - \theta} - 1 \right].
\]

**Proof.** Relation (13) follows by letting \( v \rightarrow 0 \) in Lemma 3 and using that \( \Phi(v)/v \rightarrow \mu \) in this limit.

The remaining calculations, involving the random variables \( T_x - x/\mu \) and \( \tilde{T}_x - x/\mu \), follow from (13) and the analogous expression

\[
\int_0^\infty u e^{-ux} E(e^{\theta\tilde{T}_x}) \, dx = \frac{\Phi(u)}{\Phi(u) - \theta},
\]

where we note \( \phi(u + \theta/\mu) < \mu u + \theta \) for all \( \theta \), such that \( u + \theta/\mu > 0 \).

\[\Box\]

**Lemma 5.** The function

\[ E(e^{\theta(T_x-x/\mu)} - 1), \quad x \geq 0, \]

is nonnegative for any real parameter \( \theta \), and differentiable and nondecreasing with respect to the variable \( x \). The derivative with respect to \( x \) is given by

\[ \frac{d}{dx} E(e^{\theta(T_x-x/\mu)} - 1) = \frac{\theta e^{-\theta x/\mu}}{\mu u} E(e^{\theta T_x} - e^{\theta T_x})/\mu \geq 0. \]

**Proof.** The nonnegativity follows from Jensen’s inequality. It follows from (14), (15) and the uniqueness property of Laplace transforms that \( E(e^{\theta(T_x-x/\mu)} - 1) \) is obtained as the convolution of \( E(e^{\theta(T_x-x/\mu)} - 1) \) with the exponential \( e^{-\theta x/\mu} \). Hence

\[ E(e^{\theta(T_x-x/\mu)} - 1) = \frac{\theta}{\mu} \int_0^\infty e^{-\theta(x-y)/\mu} E(e^{\theta(\tilde{T}_y-y/\mu)} - 1) \, dy. \]

The left hand side is differentiable in \( x \) with derivative

\[ \frac{d}{dx} E(e^{\theta(T_x-x/\mu)} - 1) = -\frac{\theta}{\mu} e^{\theta(T_{x-} - x/\mu)} - 1 + \frac{\theta}{\mu} e^{\theta(T_x-x/\mu)} - 1 
\]

\[ = \frac{\theta}{\mu} e^{-\theta x/\mu} E(e^{\theta T_x} - e^{\theta T_x}). \]

Now we observe that the processes \( T_x \) and \( \tilde{T}_x \) can be constructed on the same probability space by a shift of size \( X_0 \) so that \( T \) is a copy of \( \tilde{T} \) with the first point of increase in \( X_0 \) rather than in 0. In particular \( P(T_x \geq T_x) = 1 \). Hence \( \theta E(e^{\theta T_x} - e^{\theta T_x}) \geq 0 \) for any \( \theta \).

\[\Box\]

**Lemma 6.** For \( x > 0 \),

i) \( x/\mu \leq E(\tilde{T}_x) \leq \frac{e}{(1-x/\mu)} \),

ii) \( E(\tilde{T}_x) \leq \frac{e^2(e - 1)^{-1}}{\nu(x, \infty)} \),

iii) \( \frac{d}{dx} \text{Var}(T_x) = \frac{2}{\mu} e^2 \int_0^\infty E(\tilde{T}_x - x/\mu) \geq 0, \)

iv) \( \text{Var}(T_x) \leq \frac{2}{\mu} e \int_0^\infty \Phi(1/y) \, dy - (x/\mu)^2. \)
Proof. For i), it was noticed in the proof of Lemma 5 that the processes $T_x$ and $\tilde{T}_x$ could be constructed such that $\tilde{T}_x \geq T_x$ almost surely. Hence $E(\tilde{T}_x) \geq E(T_x) = x/\mu$. Moreover,

$$E(\tilde{T}_x) = \int_0^\infty P(\tilde{X}_t \leq x) \, dt \leq \int_0^\infty e^{\tilde{X}_t} \, dt = e \int_0^\infty e^{-t\Phi(1/x)} \, dt = e/\Phi(1/x).$$

Inequality ii) follows from

$$\Phi(1/x) = \frac{1}{x} \int_0^\infty e^{-u/x} \nu(u, \infty) \, du \geq \frac{1}{x} \int_0^x e^{-u/x} \nu(u, \infty) \, du \geq \nu(x, \infty)(1 - e^{-1}).$$

To prove iii) and iv), differentiate twice with respect to $\theta$ in (14) to obtain

$$\int_0^\infty u e^{-ux} \text{Var}(T_x) \, dx = \frac{2}{(\mu u)^2} \left( \frac{\mu u}{\Phi(u)} - 1 \right).$$

Similarly, using (15),

$$\int_0^\infty u e^{-ux} E(\tilde{T}_x - x/\mu) \, dx = \frac{1}{\mu u} \left( \frac{\mu u}{\Phi(u)} - 1 \right),$$

hence by partial integration

$$\int_0^\infty u e^{-ux} \int_0^x E(\tilde{T}_y - y/\mu) \, dy \, dx = \frac{1}{\mu u^2} \left( \frac{\mu u}{\Phi(u)} - 1 \right).$$

By identification of the Laplace transforms,

$$\text{Var}(T_x) = \frac{2}{\mu} \int_0^x E(\tilde{T}_y - y/\mu) \, dy.$$

The two inequalities in (i) now imply iii) and iv).

$$\blacksquare$$

2.2. The marginal distribution under scaling. We will need the weak law of large numbers and an elementary renewal type theorem for $\tilde{T}_x$. Such results are well-known. The first property below follows from the law of large numbers for $\tilde{X}_t$. The second from the formula

$$E(\tilde{T}_{ax})/a = \int_0^\infty P(\tilde{X}_{at}/a \leq x) \, dt \to x/\mu,$$

Lemma 7. As $a \to \infty$, we have

i) $\frac{1}{a} \tilde{T}_{ax} \to \frac{x}{\mu}$ in distribution

ii) $\frac{1}{a} E(\tilde{T}_{ax}) \to \frac{x}{\mu}$.

We are now prepared to prove a limit property of the centered variable $T_x - x/\mu$ under scaling, which is crucial for the distributional convergence in Theorem 2.

Lemma 8. If the sequence $a = a_m$ is such that (12) holds for some $c > 0$, then as $m \to \infty$,

$$m E(e^{\theta(T_{ax} - ax)/a} - e^{\theta(T_{ax} - ax)/\mu}) \to \frac{c^3}{\mu \beta} \int_0^x \theta e^{-\theta t/\mu} t^{-\beta} \, dt$$

and

$$m \frac{d}{dx} E(e^{\theta(T_{ax} - ax)/a} - 1) \to \frac{c^3}{\mu^2 \beta} \int_0^x \theta^2 e^{-\theta t/\mu} t^{-\beta} \, dt.$$
Proof. It is enough to prove (17) since (18) then follows directly from Lemma 5.

Recall from (2) the relation $P(X_0 \leq x) = \frac{1}{\mu} \int_0^x \nu(u, \infty) \, du$ where we use the notation $\nu(y, \infty) = \int_y^\infty \nu(dt)$. For fixed $x$ condition on $X_0$ to get

$$P(T_x < t < \bar{T}_x) = P(X_0 > x)P(t < \bar{T}_x) + \frac{1}{\mu} \int_0^x P(\bar{T}_{x-y} < t < \bar{T}_x) \nu(y, \infty) \, dy.$$ 

Multiply this identity by $\theta e^{\beta t}$ and integrate over $t \geq 0$ to obtain

$$E(\theta \bar{T}_x - \theta T_x) = P(X_0 > x)E(\theta \bar{T}_x - 1) + \frac{1}{\mu} \int_0^x E(\theta \bar{T}_x - \theta T_{x-y}) \nu(y, \infty) \, dy.$$ 

Hence

$$m E(\theta \bar{T}_{ax}/a - \theta T_{ax}/a) = m P(X_0 > ax)E(\theta \bar{T}_{ax}/a - 1)$$

$$+ \frac{1}{\mu} \int_0^x E(\theta \bar{T}_{x/a} - \theta T_{(x-y)/a}) am \nu(ay, \infty) \, dy.$$ 

By (4),

$$\frac{1}{\mu} am \nu(ay, \infty) \to e^{\beta y^{-1}}.$$ 

By (3) and (4), and using the direct half of Karamata’s theorem,

$$m P(X_0 > ax) \to \beta^{-1} c^\beta x^{-\beta},$$

cf. Bingham et al. (1987) Thm. 1.5.11 ii) (using in their notation $f(x) = \nu(x, \infty)$, $\rho = -(1 + \beta)$, $\sigma = 0$). If we assume for the moment that the order can be interchanged in which we integrate over $y$ and take the limit $m, a \to \infty$, then applying the above asymptotic results as well as Lemma 7(i),

$$m E(\theta \bar{T}_{ax}/a - \theta T_{ax}/a)$$

$$\to \beta^{-1} c^\beta x^{-\beta} (e^{\theta x/\mu} - 1) + \int_0^x (e^{\theta x/\mu} - e^{\theta (x-y)/\mu}) c^\beta y^{-1-\beta} \, dy$$

$$= e^{\theta x/\mu} \frac{c^\beta}{\mu \beta} \int_0^x \theta e^{-\theta t/\mu} t^{-\beta} \, dt,$$

which is the desired relation (17). In the remaining part of the proof we verify the validity of this limit operation by deriving an upper bound for the integrand $E(\theta \bar{T}_{ax}/a - \theta T_{ax-y}/a) am \nu(ay, \infty)$ in (19), which is $dy$-integrable over $(0, x]$.

Using

$$|E(\theta \bar{T}_x - \theta T_{x-y})| \leq |\theta| E[(\theta \bar{T}_x \vee 1) \bar{T}_x - \bar{T}_{x-y}]$$

and Hölder’s inequality we have, for each integer $k \geq 2$,

$$|E(\theta \bar{T}_x - \theta T_{x-y})| \leq |\theta| E[(\theta \bar{T}_x \vee 1)^{k/(k-1)}]^{1-1/k} E[(\bar{T}_x - \bar{T}_{x-y})^k]^{1/k}.$$ 

Now,

$$E[|\bar{T}_x - \bar{T}_{x-y}|^k] = E \int_0^\infty \cdots \int_0^\infty 1_{\{T_{x-y} < t_1, \ldots, t_k < T_x\}} \, dt_1 \cdots \, dt_k$$

$$= k! \int_0^\infty \cdots \int_0^\infty P(\bar{T}_{x-y} < t_1, \ldots, t_k < T_x) \, dt_1 \cdots \, dt_k$$

$$= k! \int_0^\infty \cdots \int_0^\infty P(\rho < x < \bar{X}_{t_k} < \cdots < \bar{X}_{t_1} < x) \, dt_1 \cdots \, dt_k.$$
For the event $x - y < \tilde{X}_{t_1} < \cdots < \tilde{X}_{t_k} < x$ to occur it is necessary, in addition to $X_{t_1} \leq x$, that all increments $\tilde{X}_{t_j} - \tilde{X}_{t_{j-1}}$, $2 \leq j \leq k$ are less than $y$ in size. Hence the right hand side is at most

$$k! \int_{t_1} \cdots \int_{t_1 < \cdots < t_k} P(\tilde{X}_{t_1} < x, \tilde{X}_{t_j} - \tilde{X}_{t_{j-1}} < y, 2 \leq j \leq k) \, dt_1 \cdots dt_k,$$

which equals

$$k! \int_0^\infty dt_1 P(\tilde{X}_{t_1} < x) \int_{t_1}^\infty dt_2 P(\tilde{X}_{t_2-t_1} < y) \cdots \int_{t_{k-1}}^\infty dt_k P(\tilde{X}_{t_k-t_{k-1}} < y) = k! E(\tilde{T}_x) E(\tilde{T}_y)^{k-1},$$

since the increments of $X(t)$ are independent and stationary. By (20),

$$|E(e^\theta \tilde{T}_{ax} - e^\theta \tilde{T}_{a(x-y)})|^k \leq k! |\theta|^k E[(e^\theta \tilde{T}_{ax} \vee 1)^{k/(k-1)}] E(\tilde{T}_{ax}) E(\tilde{T}_{ay})^{k-1}.$$

By Lemma 4 we may assume

$$E[(e^\theta \tilde{T}_{ax} \vee 1)^{k/(k-1)}] E(\tilde{T}_{ax}) \leq 2(e^\theta / \mu \vee 1)^k (x / \mu),$$

and thus

$$|E(e^\theta \tilde{T}_{ax} - e^\theta \tilde{T}_{a(x-y)})| \leq C_{\theta,k}(x) E(\tilde{T}_{ay})^{1-1/k}$$

for $a \geq a_0$ and sufficiently large $a_0$, with $C_{\theta,k}(x) = |\theta|(2k!)^{1/k} (e^\theta / \mu \vee 1) (x / \mu)^{1/k}$. For the integrand in (19) we have obtained

$$|E(e^\theta \tilde{T}_{ax} - e^\theta \tilde{T}_{a(x-y)})| a \nu(ay, \infty) \leq C_{\theta,k}(x) E(\tilde{T}_{ay})^{1-1/k} a \nu(ay, \infty), \quad 0 < y \leq x, \quad a \geq a_0.$$

We split the further task of estimating the right hand side in the above expression in the two cases $ay > a_0$ and $ay \leq a_0$.

By (14) and (7), there exist a constant $C_1$ such that for any $q < \sigma$ we have $\Phi(\lambda) \geq C_1 (\lambda \wedge \lambda^a)$, $\lambda > 0$. The lower bound in (7) ensures, moreover, that we may take $q$ such that $\beta < q < \sigma$. In combination with Lemma 6 i), this yields, for such $q$, $E(\tilde{T}_x) \leq C_2(x \vee x^q)$. Thus

$$E(\tilde{T}_{ay}) \leq C_2(y \vee (y^q / a_0^{1-q})) \leq C_3 y^q, \quad 0 \leq y \leq x, \quad a \geq a_0.$$

Furthermore, since the function $\nu(x, \infty)$ is regularly varying at infinity with index $-(1+\beta)$, we have for $ay > a_0$ and $\epsilon > 0$ the Potter type bound

$$a \nu(ay, \infty) \leq C_4 y^{-1-\beta} \max(y^\epsilon, y^{-\epsilon}).$$

(Bingham et al. (1987), Ch. 1.5). Thus, for some constant $C$,

$$E(\tilde{T}_{ay})^{1-1/k} a \nu(ay, \infty) \leq C y^{q(1-1/k)} y^{-1-\beta-\epsilon}.$$

Since $q > \beta$ we may take $k$ so large that $q(1-1/k) > \beta$ and then $\epsilon$ so small that $\epsilon < q(1-1/k) - \beta$ to obtain a dominating function for the integrand in (21) which is integrable in $y$ over $[0, x]$.

For the remaining case $ay \leq a_0$, Lemma 6 i) implies

$$E(\tilde{T}_{ay})^{1-1/k} a \nu(ay, \infty) \leq (e^2 / (e-1))^{1-1/k} m a^{1/k} a \nu(ay, \infty)^{1/k}.$$

Using a property of slowly varying functions (Bingham et al. (1987), Prop 1.3.6), for any $\epsilon > 0$, $L(a) a^\epsilon \to \infty$ as $a \to \infty$. Hence we may assume $a^{-\epsilon} \leq L(a)$. Also,

$$\nu(ay, \infty) \leq \frac{1}{ay} \int_{ay}^\infty u \nu(du) \leq \frac{\mu}{ay}.$$
Thus, using (23),
\[ m a^{1/k} \nu(ay, \infty)^{1/k} \leq \frac{m L(a)}{a^\beta} a^{\epsilon+\beta+1/k} (\mu/ay)^{1/k} \leq 2e^{\beta} a^{1+1/k} a^{\epsilon+\beta} y^{-1/k}. \]

Now apply \( a \leq a_0/y \) to obtain from (21) a constant \( C \) for which
\[ |E(e^{\theta(T_{ax} - ax/\mu)} - e^{\theta(ax - u/\mu)})| \alpha\nu(ay, \infty) \leq C a^{\epsilon+\beta} y^{-1/k-\epsilon}. \]

This is again integrable if we make the same choice of \( k \) and \( \epsilon \) as above. This concludes the proof that the limit in (19) can be carried out under the integral sign and hence the proof of the lemma.

\[ \square \]

**Lemma 9.** For \( a = a_m \) such that (7) holds for some \( c > 0 \),
\[ m E(e^{\theta(T_{ax} - ax/\mu)/a} - 1) \rightarrow \frac{e^\beta}{\beta \mu^3} \int_0^x \int_0^y \theta^2 e^{-\theta s/\mu} s^{-\beta} ds dy, m \rightarrow \infty. \]

**Proof.** By Lemma 5, \( m E(e^{\theta(T_{ax} - ax/\mu)/a} - 1) \) is nonnegative and increasing in \( x \). The limit function on the right hand side is also nonnegative and increasing. Hence the lemma follows from weak convergence of measures if we can prove
\[ \int_0^\infty e^{-ux} \frac{d}{dx} m E(e^{\theta(T_{ax} - ax/\mu)/a} - 1) dx \rightarrow \int_0^\infty e^{-ux} \left( \frac{e^\beta}{\beta \mu^3} \int_0^x \int_0^y \theta^2 e^{-\theta s/\mu} s^{-\beta} ds \right) dx. \]

To find the Laplace transform on the right hand side note that
\[ \frac{\theta^2 \Gamma(1-\beta)}{(\mu+\theta)^{1-\beta}} = \frac{\theta^2}{\beta \mu^3} \int_0^\infty e^{-ux} x^{-\beta} dx, \quad \theta < u. \]

Multiplication of the transform by \( 1/u \) corresponds to integration of \( e^{\theta x} x^{-\beta} \). Hence
\[ \int_0^\infty e^{-ux} \left( \frac{1}{\beta} \int_0^x \theta^2 e^{\theta s} s^{-\beta} ds \right) dx = \frac{\Gamma(1-\beta) e^{\beta^2 \theta^2}}{\beta \mu^3}, \quad \theta < u, \]
and hence (22) is equivalent to
\[ \int_0^\infty u e^{-ux} m E(e^{\theta(T_{ax} - ax/\mu)/a} - 1) dx \rightarrow \frac{\Gamma(1-\beta) e^{\beta^2 \theta^2}}{\beta \mu^3}, \quad \theta > -\mu u. \]

To help analyze the Laplace transform in (23) we introduce the additional notation
\[ I(u) = \mu u - \Phi(u) = \int_0^\infty (e^{-ux} - 1 + ux) \nu(dx) \geq 0. \]

Writing \( I(u) = u^2 \int_0^\infty e^{-ux} U(x) dx \) with \( U(x) = \int_0^\infty \nu(y, \infty) dy \), it follows from Kar-\nmatata’s Tauberian Theorem (Thm. 1.7.6 in Bingham et al. (1987)) that
\[ a I(u/a) \sim \frac{\Gamma(1-\beta) L(a/u) a^{1+\beta}}{\beta a^\beta}, \quad a \rightarrow \infty. \]

Relation (24) of Lemma 4 now shows
\[ \int_0^\infty u e^{-ux} m E(e^{\theta(T_{ax} - ax/\mu)/a} - 1) dx \]
\[ = \frac{m \theta^2}{(\mu u + \theta)^2} \frac{a I((u + \theta/\mu)/a)}{\mu u - a I((u + \theta/\mu)/a)} \]
\[ \sim \frac{m L(a)}{a^\beta} \frac{a I((u + \theta/\mu)/a)}{(u + \theta/\mu)^{(1-\beta)} \Gamma(1-\beta) \theta^2}{\beta \mu^3 u} \]
\[ \sim \frac{\Gamma(1-\beta) e^{\beta^2 \theta^2}}{\beta u (u + \theta/\mu)^{1-\beta} \mu^2}, \quad \theta > -\mu u, \]
which proves \( \mathbb{E} \) and hence the lemma. \( \square \)

We are now able to conclude convergence of the marginal distributions.

**Lemma 10.** Under the assumptions of Theorem 2, for any \( x \geq 0 \)

\[
\frac{1}{a_m} \sum_{i=1}^{m} \left( T_{a_m i} - \frac{1}{\mu} \theta_m x \right) \xrightarrow{d} - \frac{1}{\mu} c Y_\beta(x/c)
\]

**Proof.** Writing

\[
\Lambda^{(m)}(\theta; x) = m E(e^{\theta(T_{ax} - ax/\mu)}/a - 1),
\]

Lemma 3 shows that

\[
\log E \left\{ \frac{1}{a_m} \sum_{k=1}^{m} (T^{(k)}_{a_m x} - \frac{1}{\mu} a_m x) \right\} dx = \log \left( 1 + \frac{1}{m} \Lambda^{(m)}(\theta; x) \right) \rightarrow \frac{c}{\beta \mu^2} \int_0^x \int_0^y \theta^2 e^{-\theta s/\mu s^{-\beta}} ds dy.
\]

This proves the lemma since the limit process \( Y_\beta \) has the property

\[
\log E(e^{\theta Y_\beta(x)}) = \frac{1}{\beta} \int_0^x \int_0^y \theta^2 e^{\theta s - s^{-\beta}} ds dy
\]

and so, as noticed in Lemma 2

\[
\log E(e^{-\theta c Y_\beta(x/c)}) = \frac{c}{\beta \mu^2} \int_0^x \int_0^y \theta^2 e^{-\theta s/\mu s^{-\beta}} ds dy.
\]

\( \square \)

3. Multivariate distributions

The proofs of convergence of the finite-dimensional distributions are based on the following recursive equations for moment generating functions.

**Lemma 11.** Fix \( n \geq 2 \) and a sequence of time points \( 0 \leq x_1 \leq \cdots \leq x_n \). The moment generating function of the finite-dimensional distributions of the stationary inverse Lévy subordinator process \( \{ T_x \} \) satisfies the recurrence relation

\[
E \exp \left\{ \sum_{i=1}^{n} \theta_i T_{x_i} \right\} = E \exp \left\{ \sum_{i=2}^{n} \theta_i T_{x_i} \right\}
+ \frac{\theta_1}{\sum_{i=1}^{n} \theta_i} \int_{0}^{x_1} E \left[ \exp \left\{ \sum_{i=2}^{n} \theta_i \bar{T}_{x_1 - x} \right\} \right] d_x E \left[ \exp \left\{ T_{x} \sum_{i=1}^{n} \theta_i \right\} \right],
\]

where \( \bar{T}_x \) is the corresponding pure inverse Lévy process. Moreover,

\[
E \exp \left\{ \sum_{i=1}^{n} \theta_i \bar{T}_{x_i} \right\} = E \exp \left\{ \sum_{i=2}^{n} \theta_i \bar{T}_{x_i} \right\}
+ \frac{\theta_1}{\sum_{i=1}^{n} \theta_i} \int_{0}^{x_1} E \left[ \exp \left\{ \sum_{i=2}^{n} \theta_i \bar{T}_{x_1 - x} \right\} \right] d_x E \left[ \exp \left\{ \bar{T}_{x} \sum_{i=1}^{n} \theta_i \right\} \right],
\]

**Proof.** We have

\[
E \exp \left\{ \sum_{i=1}^{n} \theta_i T_{x_i} \right\} - E \exp \left\{ \sum_{i=2}^{n} \theta_i T_{x_i} \right\} = E \left[ \exp \left\{ \sum_{i=1}^{n} \theta_i T_{x_i} \right\} \left( e^{\theta_1 T_{x_1}} - 1 \right) \right].
\]

Since

\[
e^{\theta_1 T_{x_1}} = 1 - \int_{0}^{\infty} 1_{\{u \leq T_{x_1}\}} \theta_1 e^{\theta_1 u} du = \int_{0}^{\infty} 1_{\{x_1 \leq x\}} \theta_1 e^{\theta_1 u} du,
\]

...
it follows that
\[
E \left[ \exp \left\{ \sum_{i=1}^{n} \theta_i T_{x_i} \right\} \left( e^{\theta_1 T_{x_1}} - 1 \right) \right] \\
= E \left[ \int_0^\infty 1_{\{X_u \leq x_1\}} \exp \left\{ \sum_{i=2}^{n} \theta_i T_{x_i} \right\} \theta_1 e^{\theta_1 u} \, du \right] \\
= E \left[ \int_0^\infty 1_{\{X_u \leq x_1\}} \exp \left\{ \sum_{i=2}^{n} \theta_i (T_{x_i} - T_{x_n}) \right\} \theta_1 \exp \left\{ \left( \sum_{i=1}^{n} \theta_i \right) T_{x_n} \right\} \, du \right].
\]
Here, \( T_{x_n} = u \). For any \( u > 0 \) and \( i \geq 2 \), on the set \( \{ X_u \leq x_1 \} \) we have
\[
\{ T_{x_i} - T_{x_n} \leq \theta \} = \{ T_{x_i} \leq u + \theta \} = \{ X_{u+\theta} > x_i \}.
\]
Since \( \{ X_i \} \) has independent increments the rightmost event has the same probability as
\[
\{ X_u + \tilde{X}_i > x_i \} = \{ \tilde{T}_{x_i} - X_u \leq \theta \},
\]
where \( X_u \leq x_1 \) is assumed independent of \( \tilde{X}_i \). Thus, on \( \{ X_u \leq x_1 \} \) the increment \( T_{x_i} - T_{x_n} \) has the same distribution as \( \tilde{T}_{x_i} - X_u \). It follows that
\[
E \exp \left\{ \sum_{i=1}^{n} \theta_i T_{x_i} \right\} - E \exp \left\{ \sum_{i=2}^{n} \theta_i T_{x_i} \right\} \\
= \theta_1 E \left[ \int_0^\infty 1_{\{X_u \leq x_1\}} E \left[ \exp \left\{ \sum_{i=2}^{n} \theta_i \tilde{T}_{x_i} - X_u \right\} | X_u \right] \exp \left\{ \left( \sum_{i=1}^{n} \theta_i \right) u \right\} \, du \right] \\
= \theta_1 E \left[ \int_0^{x_1} E \left[ \exp \left\{ \sum_{i=2}^{n} \theta_i \tilde{T}_{x_i} - X_u \right\} \right] \exp \left\{ \left( \sum_{i=1}^{n} \theta_i \right) T_{x_n} \right\} \, dT_{x_n} \right],
\]
where the integration after variable substitution \( x = X_u \) is with respect to the increasing function of bounded variation \( \{ T_{x_i}, x \geq 0 \} \). (Intuitively, the time-change \( X_u \) picks out the rightmost point of each flat piece of \( T_{x_i} \).) Moreover, if we change to the measure
\[
dx \left( \exp \left\{ T_x \sum_{i=1}^{n} \theta_i \right\} \right) = \left( \sum_{i=1}^{n} \theta_i \right) \exp \left\{ T_x \sum_{i=1}^{n} \theta_i \right\} \, dT_x
\]
we obtain
\[
E \exp \left\{ \sum_{i=1}^{n} \theta_i T_{x_i} \right\} - E \exp \left\{ \sum_{i=2}^{n} \theta_i T_{x_i} \right\} \\
= \frac{\theta_1}{\sum_{i=1}^{n} \theta_i} E \left[ \int_0^{x_1} E \left[ \exp \left\{ \sum_{i=2}^{n} \theta_i \tilde{T}_{x_i} - x_1 \right\} \right] \, dx \left( \exp \left\{ T_x \sum_{i=1}^{n} \theta_i \right\} \right) \right] \\
= \frac{\theta_1}{\sum_{i=1}^{n} \theta_i} \int_0^{x_1} E \left[ \exp \left\{ \sum_{i=2}^{n} \theta_i \tilde{T}_{x_i} - x_1 \right\} \right] \, dx E \left[ \exp \left\{ T_x \sum_{i=1}^{n} \theta_i \right\} \right],
\]
which is (26). Start with \( \tilde{T} \) rather than \( T \) to get (26).

For \( n \geq 1 \) and \( 1 \leq k \leq n \), put \( \theta_{k,n} = (\theta_k, \ldots, \theta_n) \) and \( \bar{x}_{k,n} = (x_k, \ldots, x_n) \), where \( 0 = x_0 \leq x_1 \leq \cdots \leq x_n \) and let
\[
\Phi_{n-k+1}(\bar{\theta}_{k,n}; \bar{x}_{k,n}) = E \exp \left\{ \sum_{i=k}^{n} \theta_i (T_{x_i} - x_i/\mu) \right\}
\]
denote the multivariate moment generating functions for the centered process \( \{ T_{x_i} - x/\mu \}_{x \geq 0} \). Here, the subindex \( n - k + 1 \) is the number of elements of the argument vectors.
\[ \tilde{\theta}_{k,n}, \tilde{x}_{k,n} \]. Similarly, let \( \tilde{\Phi}_{n-k+1}(\tilde{\theta}_{k,n}; \tilde{x}_{k,n}) \), \( 1 \leq k \leq n \), denote the corresponding functions for the pure process \( \{T_x - x/\mu\}_{x \geq 0} \). The subtraction \( \tilde{x}_{k,n} - u = (x_k - u, \ldots, x_n - u) \) is interpreted component-wise in the next statement and in the sequel.

**Lemma 12.** The moment generating functions defined in (27) satisfy the integral equation

\[
\Phi_n(\tilde{\theta}_{1,n}; \tilde{x}_{1,n}) = \Phi_{n-1}(\tilde{\theta}_{2,n}; \tilde{x}_{2,n}) e^{-\theta_1 x_1/\mu} \\
+ \frac{\theta_1}{\sum_{i=1}^n \theta_i} \int_0^{x_1} e^{-\theta_1 (x_1 - x)/\mu} \tilde{\Phi}_{n-1}(\tilde{\theta}_{2,n}; \tilde{x}_{2,n} - x) \Phi_1 \left( \sum_{i=1}^n \theta_i; dx \right) \\
+ \frac{\theta_1}{\mu} \int_0^{x_1} e^{-\theta_1 (x_1 - x)/\mu} \tilde{\Phi}_{n-1}(\tilde{\theta}_{2,n}; \tilde{x}_{2,n} - x) \Phi_1 \left( \sum_{i=1}^n \theta_i; x \right) dx.
\]

**Proof.** By Lemma 11

\[
\Phi_n(\tilde{\theta}_{1,n}; \tilde{x}_{1,n}) = \Phi_{n-1}(\tilde{\theta}_{2,n}; \tilde{x}_{2,n}) e^{-\theta_1 x_1/\mu} + \frac{\theta_1}{\sum_{i=1}^n \theta_i} \\
\times \int_0^{x_1} e^{-\theta_1 (x_1 - x)/\mu} \tilde{\Phi}_{n-1}(\tilde{\theta}_{2,n}; \tilde{x}_{2,n} - x) \exp \left\{-\frac{x}{\mu} \sum_{i=1}^n \theta_i \right\} d_x E \left[ \exp \left\{ T_x \sum_{i=1}^n \theta_i \right\} \right],
\]

which, by observing

\[
\exp \left\{-\frac{x}{\mu} \sum_{i=1}^n \theta_i \right\} d_x E \left[ \exp \left\{ T_x \sum_{i=1}^n \theta_i \right\} \right] \\
= d_x E \left[ \exp \left\{ (T_x - x/\mu) \sum_{i=1}^n \theta_i \right\} \right] + \frac{1}{\mu} \sum_{i=1}^n \theta_i E \left[ \exp \left\{ (T_x - x/\mu) \sum_{i=1}^n \theta_i \right\} \right] dx \\
= \Phi_1 \left( \sum_{i=1}^n \theta_i; dx \right) + \frac{1}{\mu} \sum_{i=1}^n \theta_i \Phi_1 \left( \sum_{i=1}^n \theta_i; x \right) dx,
\]

may be rewritten in the form stated in the lemma. \( \Box \)

According to (12) we must find the limits of the scaled function

\[
m(\Phi_n(\tilde{\theta}_{1,n}/b; a\tilde{x}_{1,n}) - 1) = m E \left[ \exp \left\{ \sum_{i=1}^n \theta_i (T_{ax_i} - ax_i/\mu)/b \right\} - 1 \right]
\]

as \( m, a \) and \( b \) tend to infinity, when \( a \) and \( b \) satisfy either (5) together with (8) or condition (6). The first case is \( FBM \) scaling leading to fractional Brownian motion in the limit, as in Theorem 1, and the second case (with \( a = b \)) is the intermediate scaling studied in Theorem 2.

For \( n \geq 1, m \geq 1 \) and \( a, b > 0 \) we introduce

\[
\Lambda_n^{(m)}(\tilde{\theta}_{1,n}; \tilde{x}_{1,n}) = m(\Phi_n(\tilde{\theta}_{1,n}/b; a\tilde{x}_{1,n}) - 1),
\]

as well as

\[
\bar{\Lambda}_n^{(m)}(\tilde{\theta}_{1,n}; \tilde{x}_{1,n}) = \frac{am}{b} (\Phi_n(\tilde{\theta}_{1,n}/b; a\tilde{x}_{1,n}) - 1)
\]

and

\[
\Xi_n^{(m)}(\tilde{\theta}_{1,n}; \tilde{x}_{1,n}) = \bar{\Lambda}_n^{(m)}(\tilde{\theta}_{1,n}; \tilde{x}_{1,n}) - \Lambda_n^{(m)}(\tilde{\theta}_{1,n}; \tilde{x}_{1,n}).
\]

Our strategy for finding the corresponding limit functions is to derive for fixed \( m \) sequences of integral equations, which are recursive in \( n \). As already pointed out we give the detailed
proof only for Theorem 2. To simplify notation during this analysis we will also use a special notation for the sums
\[ \eta_n = \sum_{i=1}^{n} \theta_i, \quad n \geq 1, \]
which appear frequently as evident in Lemma 12.

3.1. Multivariate distribution under the intermediate scaling. We study the asymptotic limits of \( \Lambda_{n}(m) \) and \( \tilde{\Lambda}_{n}(m) \) as \( m \to \infty \) under assumption (5). For simplicity the constant in (6) is set to \( c = 1 \). The general case \( c \neq 1 \) then follows from Lemma 2. We begin with a system of equations for the functions \( \Xi_{n}(m) \) defined in (29), which will be used to determine corresponding limit functions as \( m \to \infty \).

**Lemma 13.** We have
\[
\Xi_{n}(m)(\bar{\theta}_{1,n}; \bar{x}_{1,n}) = \Xi_{1}(m)(\eta_{n}; x_{1}) + e^{-\theta_{1}x_{1}/\mu} \Xi_{n-1}(m)(\bar{\theta}_{2,n}; x_{2,n})
\]
\[ + (\theta_{1}/\eta_{n} - 1) \int_{0}^{x_{1}} e^{-\theta_{1}(x_{1} - x)/\mu} \Xi_{1}(m)(\eta_{n}; dx) + \frac{1}{m} R_{1}(m), \]
where
\[ R_{1}(m) = \frac{\theta_{1}}{\eta_{n}} \int_{0}^{x_{1}} e^{-\theta_{1}(x_{1} - x)/\mu} \tilde{\Xi}_{n-1}(m)(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) \tilde{\Lambda}_{1}(m)(\eta_{n}; dx). \]

**Proof.** After inserting the scaling parameters \( m \) and \( a \) into the equation obtained in Lemma 12 and sorting the terms appropriately, it is seen that the scaled functions \( \Lambda_{n}(m) \) and \( \tilde{\Lambda}_{n}(m) \) satisfy
\[
\Lambda_{n}(m)(\bar{\theta}_{1,n}; \bar{x}_{1,n}) = \Lambda_{n-1}(m)(\bar{\theta}_{2,n}; \bar{x}_{2,n}) e^{-\theta_{1}x_{1}/\mu} + I_{1}(m) + I_{2}(m) + I_{3}(m) + \frac{1}{m} R_{1}(m),
\]
where
\[ I_{1}(m) = \frac{\theta_{1}}{\eta_{n}} \int_{0}^{x_{1}} e^{-\theta_{1}(x_{1} - x)/\mu} \Lambda_{1}(m)(\eta_{n}; dx) \]
\[ I_{2}(m) = \frac{\theta_{1}}{\mu} \int_{0}^{x_{1}} e^{-\theta_{1}(x_{1} - x)/\mu} \tilde{\Lambda}_{n-1}(m)(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) dx \]
\[ I_{3}(m) = \frac{\theta_{1}}{\mu} \int_{0}^{x_{1}} e^{-\theta_{1}(x_{1} - x)/\mu} \Lambda_{1}(m)(\eta_{n}; x) dx \]
\[ R_{1}(m) = \frac{\theta_{1}}{\eta_{n}} \int_{0}^{x_{1}} e^{-\theta_{1}(x_{1} - x)/\mu} \tilde{\Lambda}_{n-1}(m)(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) \Lambda_{1}(m)(\eta_{n}; x) dx \]

By Lemma 6
\[
\frac{d}{dx} \Lambda_{1}(m)(\theta; x) = \frac{d}{dx} m E(e^{\theta(T_{ax} - ax)/\mu}) = \frac{\theta}{\mu} m E(e^{\theta(T_{ax} - ax)/\mu}) = \frac{\theta}{\mu} \Xi_{1}(m)(\theta; x).
\]
(29)

Thus,
\[ R_{1}(m) = \frac{\theta_{1}}{\mu} \int_{0}^{x_{1}} e^{-\theta_{1}(x_{1} - x)/\mu} \tilde{\Lambda}_{n-1}(m)(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) \tilde{\Lambda}_{1}(m)(\eta_{n}; x) dx. \]

By partial integration,
\[ I_{3}(m) = \Lambda_{1}(m)(\eta_{n}; x_{1}) - \int_{0}^{x_{1}} e^{-\theta_{1}(x_{1} - x)/\mu} \Lambda_{1}(m)(\eta_{n}; dx). \]
Hence
\[
\Lambda_n^{\text{(m)}}(\bar{\theta}_{1,n}; \bar{x}_{1,n}) = \Lambda_{n-1}^{\text{(m)}}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) e^{-\theta_1 x_1/\mu} + \Lambda_1^{\text{(m)}}(\eta_n; x_1) + I_2^{(m)} + I_4^{(m)} + \frac{1}{m} R_1^{(m)},
\]
where now
\[
I_4^{(m)} = \left( \frac{\theta_1}{\eta_n} - 1 \right) \int_0^{x_1} e^{-\theta_1 (x_1-x)/\mu} \Lambda_1^{\text{(m)}}(\eta_n; dx).
\]
Similarly,
\[
\Lambda_n^{\text{(m)}}(\bar{\theta}_{1,n}; \bar{x}_{1,n}) = \Lambda_{n-1}^{\text{(m)}}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) e^{-\theta_1 x_1/\mu} + \Lambda_1^{\text{(m)}}(\eta_n; x_1) + I_2^{(m)} + \bar{I}_4^{(m)} + \frac{1}{m} \bar{R}_2^{(m)},
\]
with
\[
\bar{I}_4^{(m)} = \left( \frac{\theta_1}{\eta_n} - 1 \right) \int_0^{x_1} e^{-\theta_1 (x_1-x)/\mu} \bar{\Lambda}_1^{\text{(m)}}(\eta_n; dx)
\]
and
\[
\bar{R}_2^{(m)} = \frac{\theta_1}{\eta_n} \int_0^{x_1} e^{-\theta_1 (x_1-x)/\mu} \bar{\Lambda}_1^{\text{(m)}}(\eta_n; dx)
\]
By subtracting (30) from (31) and using \( R^{(m)} = R_2^{(m)} - R_1^{(m)} \) we obtain the desired equation for \( \Xi_n^{\text{(m)}}. \)

\[\square\]

The next result generalizes Lemma 5 to the multivariate distributions.

Lemma 14. For any \( 0 \leq s \leq x_1 \) and \( n \geq 1 \),
\[- \frac{d}{ds} \Lambda_n^{\text{(m)}}(\bar{\theta}_{1,n}; \bar{x}_{1,n}-s) = \frac{\eta_n}{\mu} \Xi_n^{\text{(m)}}(\bar{\theta}_{1,n}; \bar{x}_{1,n}-s).\]

Proof. For \( n = 1 \) this is (29). For \( n \geq 2 \) and \( 0 < s < x_1 \), using (30),
\[
\frac{d}{ds} \Lambda_n^{\text{(m)}}(\bar{\theta}_{1,n}; \bar{x}_{1,n}-s) = \frac{1}{m} \frac{d}{ds} R_1^{(m)}(\bar{x}_{1,n}-s) + \frac{\theta_1}{\eta_n} \frac{d}{ds} \Lambda_1^{\text{(m)}}(\eta_n; x_1-s)
\]
\[+ \left( \frac{\theta_1}{\eta_n} - 1 \right) \frac{\theta_1}{\mu} \int_0^{x_1-s} e^{-\theta_1 (x_1-s-x)/\mu} \Lambda_1^{\text{(m)}}(\eta_n; dx)
\]
\[+ e^{-\theta_1 (x_1-s)/\mu} \frac{d}{ds} \Lambda_{n-1}^{\text{(m)}}(\bar{\theta}_{2,n}; \bar{x}_{2,n}-s) - \frac{\theta_1}{\mu} e^{-\theta_1 (x_1-s)/\mu} \Xi_{n-1}^{\text{(m)}}(\bar{\theta}_{2,n}; \bar{x}_{2,n}-s)
\]
and hence by (29),
\[
- \frac{d}{ds} \Lambda_n^{\text{(m)}}(\bar{\theta}_{1,n}; \bar{x}_{1,n}-s) = - \frac{1}{m} \frac{d}{ds} R_1^{(m)}(\bar{x}_{1,n}-s) + \frac{\theta_1}{\mu} \Xi_n^{\text{(m)}}(\eta_n; x_1-s)
\]
\[+ e^{-\theta_1 (x_1-s)/\mu} \frac{1}{\mu} (\eta_n - \theta_1) \frac{\theta_1}{\mu} \int_0^{x_1-s} e^{\theta_1 x/\mu} \Xi_1^{\text{(m)}}(\eta_n; x) dx
\]
\[- e^{-\theta_1 (x_1-s)/\mu} \left( \frac{d}{ds} \Lambda_{n-1}^{\text{(m)}}(\bar{\theta}_{2,n}; \bar{x}_{2,n}-s) - \frac{\theta_1}{\mu} \Xi_{n-1}^{\text{(m)}}(\bar{\theta}_{2,n}; \bar{x}_{2,n}-s) \right).\]
so
\[
\frac{d}{ds} \Lambda_n^{(m)}(\bar{\theta}_1, n; \bar{x}_{1,n} - s) = \frac{1}{m} \sum_{i=1}^{n} \theta_i \Xi_n^{(m)}(\bar{\theta}_1, n; \bar{x}_{1,n} - s) - \frac{1}{m} \mu \Xi_{n-1}^{(m)}(\bar{\theta}_1, n; \bar{x}_{1,n} - s)
\]

By replacing the integral term using Lemma 13 this implies
\[
\frac{d}{ds} \Lambda_n^{(m)}(\bar{\theta}_1, n; \bar{x}_{1,n} - s) = \frac{1}{\mu} \sum_{i=1}^{n} \theta_i \Xi_n^{(m)}(\bar{\theta}_1, n; \bar{x}_{1,n} - s)
\]

Since
\[
R_1^{(m)}(\bar{x}_{1,n} - s) = \frac{\theta_1}{\mu} \int_{s}^{x_1} e^{-\theta_1(x_1-x)/\mu} \Lambda_{n-1}^{(m)}(\bar{\theta}_2, n; \bar{x}_{2,n} - x) \Lambda_1^{(m)}(\eta_n; x-s) \, dx
\]
we have the identity
\[
\frac{d}{ds} R_1^{(m)}(\bar{x}_{1,n} - s) = \frac{\theta_1}{\mu} \int_{s}^{x_1} e^{-\theta_1(x_1-x)/\mu} \Lambda_{n-1}^{(m)}(\bar{\theta}_2, n; \bar{x}_{2,n} - x) \, dx \Lambda_1^{(m)}(\eta_n; x-s)
\]
\[
= \frac{1}{\mu} \sum_{i=1}^{n} \theta_i R_i^{(m)}(\bar{x}_{1,n} - s).
\]

Thus,
\[
\frac{d}{ds} \Lambda_n^{(m)}(\bar{\theta}_1, n; \bar{x}_{1,n} - s) + \frac{\eta_n}{\mu} \Xi_n^{(m)}(\bar{\theta}_1, n; \bar{x}_{1,n} - s)
\]
\[
= e^{-\theta_1(x_1-s)/\mu} \left( \frac{d}{ds} \Lambda_{n-1}^{(m)}(\bar{\theta}_2, n; \bar{x}_{2,n} - s) + \frac{1}{\mu} \sum_{i=2}^{n} \theta_i \Xi_{n-1}^{(m)}(\bar{\theta}_2, n; \bar{x}_{2,n} - s) \right)
\]

The statement of the lemma now follows by induction. \qed

**Lemma 15.** For each \( n \geq 1 \), the limit functions
\[
\Xi_n(\bar{\theta}_1, n; \bar{x}_{1,n}) = \lim_{m \to \infty} \Xi_n^{(m)}(\bar{\theta}_1, n; \bar{x}_{1,n})
\]
exist and are given by
\[
\Xi_n(\tilde{\theta}_{1, n}; \tilde{x}_{1, n}) = \sum_{j=1}^{n} \exp \left\{- \sum_{i=1}^{j-1} (\theta_i + \cdots + \theta_n)(x_i - x_{i-1})/\mu \right\}
\]
(32)
\[
\times \frac{1}{\mu} \int_{x_{j-1}}^{x_j} (\theta_j + \cdots + \theta_n)e^{-(\theta_j + \cdots + \theta_n)(u-x_{j-1})/\mu} \beta^{-1} u^{-\beta} \, du.
\]
For \( n \geq 2 \) they solve the recursive system
\[
\Xi_n(\tilde{\theta}_{1, n}; \tilde{x}_{1, n}) - \Xi_1(\eta_n; x_1) = e^{-\theta_1 x_1/\mu} \left( \Xi_{n-1}(\tilde{\theta}_{2, n}; \tilde{x}_{2, n}) - \Xi_1(\eta_n - \theta_1; x_1) \right).
\]

**Proof.** As \( m \to \infty \), by Lemmas \( 5 \) and \( 8 \),
\[
\Xi_1^{(m)}(\theta; x) = \frac{\mu}{d} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Xi_1^{(m)}(\alpha; dx)
= \Xi_1^{(m)}(\alpha; x_1) - \frac{\theta_1}{\mu} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Xi_1^{(m)}(\alpha; dx)
= \Xi_1^{(m)}(\alpha; x_1) - \frac{\theta_1}{\alpha} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Lambda_1^{(m)}(\alpha; dx)
= \Xi_1^{(m)}(\alpha; x_1) - \frac{\theta_1}{\alpha} \Lambda_1^{(m)}(\alpha; x_1) + \frac{\theta_2}{\mu \alpha} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Lambda_1^{(m)}(\alpha; dx).
\]

Using (29), for arbitrary \( \alpha \),
\[
\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Xi_1^{(m)}(\alpha; dx) = \Xi_1^{(m)}(\alpha; x_1) - \frac{\theta_1}{\mu} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Xi_1^{(m)}(\alpha; dx)
= \Xi_1^{(m)}(\alpha; x_1) - \frac{\theta_1}{\alpha} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Lambda_1^{(m)}(\alpha; dx)
= \Xi_1^{(m)}(\alpha; x_1) - \frac{\theta_1}{\alpha} \Lambda_1^{(m)}(\alpha; x_1) + \frac{\theta_2}{\mu \alpha} \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Lambda_1^{(m)}(\alpha; dx).
\]

Here, (33) shows that \( \Xi_1^{(m)}(\theta; x_1) \) converges to \( \Xi_1 \), and Lemma \( 9 \) shows that \( \Lambda_1^{(m)}(\alpha; x) \) converges to a limit function \( \Lambda_1 \) which satisfies \( \frac{d}{dx} \Lambda_1(\alpha; x) = \alpha \Xi_1(\alpha, x)/\mu \). Since \( 0 \leq \Lambda_1^{(m)}(\alpha; x) \leq \Lambda_1^{(m)}(\alpha; x_1) \) for \( 0 \leq x \leq x_1 \) by Lemma \( 5 \), we can find a dominating function for \( \Lambda_1^{(m)}(\alpha; x) \) on \([0, x_1]\) and conclude that the last integral term also converges. By reversing the partial integrations this yields with \( \alpha = \eta_n \)
\[
\int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Xi_1^{(m)}(\eta_n; dx) \to \int_0^{x_1} e^{-\theta_1(x_1-x)/\mu} \Xi_1(\eta_n; dx).
\]

Let us now consider the remainder terms \( R^{(m)} \) in Lemma \( 13 \). Because of (29) we may rewrite using
\[
\tilde{\Lambda}_1^{(m)}(\alpha; dx) = \Lambda_1^{(m)}(\alpha; dx) + \Xi_1^{(m)}(\alpha; dx)
= \Lambda_1^{(m)}(\alpha; dx) + e^{-\alpha x/\mu} d_x (e^{\alpha x/\mu} \Xi_1^{(m)}(\alpha; x)) - \frac{\alpha}{\mu} \Xi_1^{(m)}(\alpha; x) dx
= e^{-\alpha x/\mu} d_x (e^{\alpha x/\mu} \Xi_1^{(m)}(\alpha; x)),
\]
and obtain, again with \( \alpha = \sum_{i=1}^{\eta_n} \theta_i \),
\[
R^{(m)} = \frac{\theta_1}{\alpha} e^{-\theta_1 x_1/\mu} \int_0^{x_1} e^{-(\alpha - \theta_1) x/\mu} \tilde{\Lambda}_1^{(m)}(\tilde{\theta}_{2, n}; \tilde{x}_{2, n} - x) d_x (e^{\alpha x/\mu} \Xi_1^{(m)}(\alpha; x)).
\]

The above integration is carried out with respect to the function
\[
P^{(m)}(x) = e^{\alpha x/\mu} \Xi_1^{(m)}(\alpha; x) = mE(e^{\alpha \tilde{T}_x/a} - e^{\alpha T_x/a}).
\]

It was observed in the proof of Lemma \( 5 \) that \( T \) can be viewed as a shift of \( \tilde{T} \) with the first point of increase in \( X_0 \). In particular, \( \tilde{T}_x - T_x \) equals \( \tilde{T}_{X_0} \) almost surely on the set
{X_0 < x}. Hence the increment

\[ E(e^{\alpha T_x + b} - e^{\alpha T_x}) - E(e^{\alpha T_x - e^{\alpha T_x}}) \]

\[ = E((e^{\alpha T_x} - e^{\alpha T_x})(e^{\alpha T_x} - 1), X_0 < x) \]

\[ + E(e^{\alpha T_x + b} - (e^{\alpha T_x} - 1), x < X_0 < x + h) \]

\[ + E(e^{\alpha T_x + b} - e^{\alpha T_x}, x + h < X_0) \]

is positive for \( \alpha > 0 \) and negative for \( \alpha < 0 \). Thus, \( F(m) \) is a monotone measure with limit \( e^{\alpha x}/\mu \Xi(\alpha; x) \). Using the variation measure \( |F(m)| \) we obtain a constant \( C_n(x) \) uniform in \( m \) such that

\[ |R^{(m)}| \leq \left| \frac{\theta_1}{\alpha} \right| \sup_{0 \leq x \leq x_1} \left| e^{-(\theta_1 x_1 + (\alpha - \theta_1)x)/\mu} \Lambda^{(m)}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) \right| |F(m)| \]

\[ \leq C_n(x) \sup_{0 \leq x \leq x_1} \left| \Lambda^{(m)}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) \right| \]

The same arguments apply to the remainder terms \( R^{(m)}_i \) in (30) and \( R^{(m)}_2 \) in (31). Hence, taking \( C_n(x_1) \) sufficiently large,

\[ R^{(m)}_i \leq C_n(x_1) \sup_{0 \leq x \leq x_1} \left| \Lambda^{(m)}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) \right|, \quad i = 1, 2. \]

We are now prepared to carry out an induction on \( n \) in Lemma 13 and equation (32). Assume that \( \Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) \) converges to \( \Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) \) and \( \Lambda^{(m)}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) \) is such that

\[ \sup_{0 \leq x \leq x_1} \left| \Lambda^{(m)}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) \right| < C_{n-1}(x_1). \]

Then, applying (33), (34) and the induction hypothesis to Lemma 13 it follows that all limit functions \( \Xi_n \) exist and satisfy

\[ \Xi_n(\bar{\theta}_1, \bar{x}_1) - \Xi_1(\bar{\theta}_n; x_1) = e^{-\theta_1 x_1 / \mu} \Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) \]

\[ + \left( \frac{\theta_1}{\eta_1} - 1 \right) \int_0^{x_1} e^{-\theta_1 (x_1 - x) / \mu} \Xi_1(\eta_1; dx) \]

\[ = e^{-\theta_1 x_1 / \mu} \left( \Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) - \frac{1}{\mu} (\eta_1 - \theta_1) \int_0^{x_1} e^{-x(\eta_1 - \theta_1) / \mu} \beta^{-1} x^{-\beta} dx \right), \]

which is the desired relation. Moreover, observing that the convergence of \( \tilde{I}_1^{(m)} \) is a byproduct of the proof of (34), it follows from (34) that

\[ \sup_{0 \leq u \leq x_0} \left| \Lambda^{(m)}_n(\bar{\theta}_1; \bar{x}_1 - u) \right| < C_n(x_0), \quad 0 < x_0 < x_1 < \ldots < x_n. \]

To verify the explicit form (32) of the solution, assume that the claim is correct for index \( n - 1 \). Then

\[ \Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) = \frac{1}{\mu} \int_0^{x_2} (\theta_2 + \ldots + \theta_n) e^{-(\theta_2 + \ldots + \theta_n)u/\mu} \beta^{-1} u^{-\beta} du \]

\[ + \sum_{j=2}^{n-1} \exp \left\{ - (\theta_2 + \ldots + \theta_n)x_2/\mu - \sum_{i=2}^{j-1} (\theta_{i+1} + \ldots + \theta_n)(x_{i+1} - x_i)/\mu \right\} \]

\[ \times \frac{1}{\mu} \int_{x_j}^{x_j+1} (\theta_{j+1} + \ldots + \theta_n) e^{-(\theta_{j+1} + \ldots + \theta_n)(u-x_j)/\mu} \beta^{-1} u^{-\beta} du. \]
This implies

\[
\Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) - \Xi_1\left(\sum_{i=2}^{n} \theta_i; x_1\right) = \frac{1}{\mu} \int_{x_1}^{x_2} (\theta_2 + \cdots + \theta_n) e^{-(\theta_2 + \cdots + \theta_n)u/\mu} \beta^{-1} u^{-\beta} du
\]

\[+ \sum_{j=3}^{n} \exp\left\{ - (\theta_2 + \cdots + \theta_n)x_2/\mu - \sum_{i=3}^{j-1} (\theta_i + \cdots + \theta_n)(x_i - x_{i-1})/\mu \right\} \times \frac{1}{\mu} \int_{x_{j-1}}^{x_j} (\theta_j + \cdots + \theta_n) e^{-(\theta_j + \cdots + \theta_n)(u-x_{j-1})/\mu} \beta^{-1} u^{-\beta} du.
\]

Hence

\[e^{-\theta_1 x_1/\mu} \left( \Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) - \Xi_1\left(\sum_{i=2}^{n} \theta_i; x_1\right) \right) = e^{-(\theta_1 + \cdots + \theta_n)x_1/\mu} \frac{1}{\mu} \int_{x_1}^{x_2} (\theta_2 + \cdots + \theta_n) e^{-(\theta_2 + \cdots + \theta_n)(u-x_1)/\mu} \beta^{-1} u^{-\beta} du
\]

\[+ \sum_{j=3}^{n} \exp\left\{ - \sum_{i=1}^{j-1} (\theta_i + \cdots + \theta_n)(x_i - x_{i-1})/\mu \right\} \times \frac{1}{\mu} \int_{x_{j-1}}^{x_j} (\theta_j + \cdots + \theta_n) e^{-(\theta_j + \cdots + \theta_n)(u-x_{j-1})/\mu} \beta^{-1} u^{-\beta} du
\]

\[= \sum_{j=2}^{n} \exp\left\{ - \sum_{i=1}^{j-1} (\theta_i + \cdots + \theta_n)(x_i - x_{i-1})/\mu \right\} \times \frac{1}{\mu} \int_{x_{j-1}}^{x_j} (\theta_j + \cdots + \theta_n) e^{-(\theta_j + \cdots + \theta_n)(u-x_{j-1})/\mu} \beta^{-1} u^{-\beta} du.
\]

\[\Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) - \Xi_1 \left( \sum_{i=2}^{n} \theta_i; x_1 \right)
\]

\[= \frac{1}{\mu} \int_{x_1}^{x_2} (\theta_2 + \cdots + \theta_n) e^{-(\theta_2 + \cdots + \theta_n)u/\mu} \beta^{-1} u^{-\beta} du
\]

\[+ \sum_{j=3}^{n} \exp\left\{ - (\theta_2 + \cdots + \theta_n)x_2/\mu - \sum_{i=3}^{j-1} (\theta_i + \cdots + \theta_n)(x_i - x_{i-1})/\mu \right\} \times \frac{1}{\mu} \int_{x_{j-1}}^{x_j} (\theta_j + \cdots + \theta_n) e^{-(\theta_j + \cdots + \theta_n)(u-x_{j-1})/\mu} \beta^{-1} u^{-\beta} du.
\]

\[\Xi_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) - \Xi_1 \left( \sum_{i=2}^{n} \theta_i; x_1 \right)
\]

\[= \frac{1}{\mu} \int_{x_1}^{x_2} (\theta_2 + \cdots + \theta_n) e^{-(\theta_2 + \cdots + \theta_n)u/\mu} \beta^{-1} u^{-\beta} du
\]

\[+ \sum_{j=3}^{n} \exp\left\{ - (\theta_2 + \cdots + \theta_n)x_2/\mu - \sum_{i=3}^{j-1} (\theta_i + \cdots + \theta_n)(x_i - x_{i-1})/\mu \right\} \times \frac{1}{\mu} \int_{x_{j-1}}^{x_j} (\theta_j + \cdots + \theta_n) e^{-(\theta_j + \cdots + \theta_n)(u-x_{j-1})/\mu} \beta^{-1} u^{-\beta} du.
\]
Proof. For \( n = 1 \) this follows from Lemma 9 and for \( n \geq 2 \) from 30 and a further partial integration of the term \( I_2^{(m)} \), which gives

\[
\Lambda_n^{(m)}(\bar{x}_1,n; \bar{x}_{1,n}) = \Lambda_{n-1}^{(m)}(\bar{x}_{2,n}; \bar{x}_{2,n} - x_1) + \Lambda_1^{(m)}(\eta_n; x_1) \\
+ \left( \frac{\theta_i}{\eta_n} - 1 \right) \int_0^{x_1} e^{-\theta_i(x_1-x)/\mu} \Lambda_1^{(m)}(\eta_n; dx) \\
- \int_0^{x_1} e^{-\theta_i(x_1-x)/\mu} d_x \Lambda_{n-1}^{(m)}(\bar{x}_{2,n}; \bar{x}_{2,n} - x) \\
+ \frac{\theta_i}{\mu} \int_0^{x_1} e^{-\theta_i(x_1-x)/\mu} \Xi_{n-1}^{(m)}(\bar{x}_{2,n}; \bar{x}_{2,n} - x) dx + \frac{1}{m} R_1^{(m)} \\
= \Lambda_{n-1}^{(m)}(\bar{x}_{2,n}; \bar{x}_{2,n} - x_1) + \Lambda_1^{(m)}(\eta_n; x_1) \\
+ \left( \frac{\theta_i}{\eta_n} - 1 \right) \int_0^{x_1} e^{-\theta_i(x_1-x)/\mu} \Lambda_1^{(m)}(\eta_n; dx) \\
- \left( \frac{\theta_i}{\eta_n} - \theta_1 \right) + \frac{1}{m} \frac{1}{n} \int_0^{x_1} e^{-\theta_i(x_1-x)/\mu} d_x \Lambda_{n-1}^{(m)}(\bar{x}_{2,n}; \bar{x}_{2,n} - x) + \frac{1}{m} R_1^{(m)},
\]

where we apply Lemma 14 for the last equality. The arguments which justify that we are allowed to exchange the order of integration and taking limits in \( m \), as well as controlling the remainder terms, are parallel to those in the proof of Lemma 15 again based on Lemma 14.

In view of (12), we conclude from Lemma 16 the convergence of the finite-dimensional distributions in Theorem 2.

Lemma 17. The finite-dimensional distributions of the sequence of random processes studied in Theorem 3 (with \( c = 1 \)) converge to those of a limit process \( Y_{\beta} \), such that the collection of logarithmic moment generating functions

\[
\Lambda_n(\bar{\theta}_1,n; \bar{x}_{1,n}) = \log E \exp \left\{ \sum_{i=1}^{n} \theta_i Y_{\beta}(x_i) \right\}, \quad n \geq 1
\]

is the unique solution to the closed system of linear integral equations

\[
\Lambda_n(\bar{\theta}_1,n; \bar{x}_{1,n}) = \Lambda_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x_1) + \Lambda_1 \left( \sum_{i=1}^{n} \theta_i; x_1 \right) \\
+ \left( \frac{\theta_i}{\sum_{i=1}^{n} \theta_i} - 1 \right) \int_0^{x_1} e^{-\theta_i(x_1-x)/\mu} \Lambda_1 \left( \sum_{i=1}^{n} \theta_i; dx \right) \\
- \left( \frac{\theta_i}{\sum_{i=2}^{n} \theta_i} + \frac{1}{m} \right) \int_0^{x_1} e^{-\theta_i(x_1-x)/\mu} d_x \Lambda_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x), \quad n \geq 2,
\]

with \( \Lambda_1 \) as in (35).

Lemma 18. The cumulant function for the increments of \( Y_{\beta} \),

\[
\Gamma_n(\bar{\theta}_1,n; \bar{x}_{1,n}) = \log E \exp \left\{ \sum_{i=1}^{n} \theta_i (Y_{\beta}(x_i) - Y_{\beta}(x_{i-1})) \right\}
\]

has the explicit form given in (71).

Proof. We have

\[
\Gamma_n(\bar{\theta}_1,n; \bar{x}_{1,n}) = \Lambda_n((\theta_1 - \theta_2, \ldots, \theta_{n-1} - \theta_n, \theta_n), \bar{x}_{1,n})
\]
so by Lemma 17
\begin{align*}
\Gamma_n(\bar{\theta}_{1,n}; \bar{x}_{1,n}) &= \Gamma_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x_1) + \Lambda_1(\theta_1; x_1) \\
&= \frac{\theta_2}{\theta_1} \int_0^{x_1} e^{-(\theta_1 - \theta_2)(x_1-x)/\mu} \Lambda_1(\theta_1; dx) \\
&- \frac{\theta_1}{\theta_2} \int_0^{x_1} e^{-(\theta_1 - \theta_2)(x_1-x)/\mu} d\Gamma_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x_1), \quad n \geq 2.
\end{align*}

It may now be checked that the functions in (11) solve the above system of equations. For details, see Gaigalas, Kaj [9], Section 6.3.

\section{4. Remaining Proofs}

\subsection*{4.1. Limiting distribution under FBM scaling.}

In this section we discuss briefly the convergence of the finite-dimensional distributions in Theorem 1. Recall that for standard fractional Brownian motion $B_H$,

\begin{align*}
\log E \exp \left\{ \sum_{i=1}^n \theta_i \sigma_i B_H(x_i) \right\} &= \frac{1}{2} \sigma_i^2 \sum_{i=1}^n \sum_{j=1}^n \theta_i \theta_j \text{Cov}(B_H(x_i), B_H(x_j))
\end{align*}

with

\begin{align*}
\text{Cov}(B_H(x), B_H(y)) &= \frac{1}{2} (x^{2-\beta} + y^{2-\beta} - (x - y)^{2-\beta}).
\end{align*}

In the scaling regime defined by (5) and (8) we have

\begin{align*}
a \sigma_i^2 &= \sigma_i^2 \mu \frac{mL(a)}{\mu} \to 0, \quad \frac{am}{b} = \sqrt{\frac{a^2 \mu \sigma_i^2}{mL(a)}} \to \infty.
\end{align*}

By analyzing in this case the recursive equations for $\tilde{\Lambda}_n^{(m)}(\bar{\theta}_{1,n}; \bar{x}_{1,r})$ it follows that

\begin{align*}
\tilde{\Lambda}_n(\bar{\theta}_{1,n}; \bar{x}_{1,r}) &= \lim_{m \to \infty} \tilde{\Lambda}_n^{(m)}(\bar{\theta}_{1,n}; \bar{x}_{1,r}) = \frac{\delta_1^2}{\mu} \sum_{i=1}^n \theta_i x_{1,i}^{1-\beta}, \quad \delta_1^2 = \frac{1}{\beta(1-\beta)}.
\end{align*}

Moreover, the limit functions $\Lambda_n(\bar{\theta}_{1,n}; \bar{x}_{1,r}) = \lim_{m \to \infty} \Lambda_n^{(m)}(\bar{\theta}_{1,n}; \bar{x}_{1,r})$ satisfy, in analogy to the result of Lemma 17

\begin{align*}
\Lambda_1(\theta; x) &= \lim_{m \to \infty} \Lambda_1^{(m)}(\theta; x) = \frac{1}{2} \sigma_i^2 \mu x^{2-\beta}
\end{align*}

and for $n \geq 2$,

\begin{align*}
\Lambda_n(\bar{\theta}_{1,n}; \bar{x}_{1,n}) &= \Lambda_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) + \frac{\theta_1}{\mu} \sum_{i=1}^n \theta_i \left( \sum_{i=1}^n \theta_i x_1 \right) \\
&+ \frac{\theta_1}{\mu} \int_0^{x_1} \tilde{\Lambda}_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n} - x) dx.
\end{align*}

Thus, using (36),

\begin{align*}
\Lambda_n(\bar{\theta}_{1,n}; \bar{x}_{1,n}) &= \Lambda_{n-1}(\bar{\theta}_{2,n}; \bar{x}_{2,n}) + \frac{\sigma_i^2}{2\mu^2} \sum_{j=1}^n \theta_i \theta_j \left[ x_1^{2-\beta} + x_j^{2-\beta} - (x_j - x_1)^{2-\beta} \right].
\end{align*}

Hence

\begin{align*}
\Lambda_n(\bar{\theta}_{1,n}; \bar{x}_{1,n}) &= \frac{\sigma_i^2}{2\mu^2} \sum_{i=1}^n \sum_{j=1}^n \theta_i \theta_j \left[ x_i^{2-\beta} + x_j^{2-\beta} - (x_i - x_j)^{2-\beta} \right].
\end{align*}
4.2. Proof of tightness in $C$. To complete the proofs of our results we establish tightness of the sequences

$$Y^{(m)}(x) = \frac{1}{b^m} \sum_{i=1}^{m} (T_{amx} - \frac{a_{mx}}{\mu})$$

studied in Theorems 1 and 2, by applying a standard moment criterion. Since $Y^{(m)}(x)$ has stationary increments, to prove that $\{Y^{(m)}\}$ is tight in $C$ it is enough to find $\gamma > 1$, an integer $m_0$ and a constant $K$ such that for fixed $T$,

$$\text{Var}(Y^{(m)}(x)) = \frac{m}{b^m} \text{Var}(T_{amx}) \leq Kx^\gamma$$

for $0 < x < T$ and $m \geq m_0$ (Billingsley (1968), Thm. 12.3).

By Lemma [3], the variance of $T_x$ is a non-decreasing function in $x$. Hence we may apply Karamata’s Tauberian theorem (Bingham et al. [4] Theorem 1.7.1) to show that $\text{Var}(T_x)$ is regularly varying in infinity with index $2 - \beta$. Indeed, recalling the previously used notation $I(u) = \mu u - \Phi(u)$, the asymptotic property [4] implies

$$\int_0^\infty u e^{-ux} \text{Var}(T_x) dx = \frac{2}{(\mu u)^2} \sum_{n=1}^{\infty} \left( \frac{I(u)}{\mu u} \right)^n \sim \frac{2\Gamma(1-\beta)L(1/u)}{\beta \mu^2 u^{2-\beta}}, \quad u \to 0,$$

hence

$$\text{Var}(T_x) \sim \frac{2\Gamma(1-\beta)x^{2-\beta}L(x)}{\Gamma(3-\beta)\beta \mu^3} = \frac{\sigma_x^2}{\mu^3} L(x) x^{2-\beta}, \quad x \to \infty.$$ The next step is to apply the Potter bounds for regularly varying functions (Bingham et al., Ch 1.5) to obtain for any $\epsilon > 0$ an $a_0$, such that

$$\frac{\text{Var}(T_{ax})}{\text{Var}(T_a)} \leq (1 + \epsilon) \max(x^{2-\beta+\epsilon}, x^{2-\beta-\epsilon}), \quad a \geq a_0, \ ax \geq a_0.$$ Hence for $m \geq m_0$ so large that $a \geq a_0, \ ax \geq a_0$,

$$m \text{Var}(T_{ax})/b^2 \leq (1 + \epsilon) m \text{Var}(T_a)b^{-2} \max(x^{2-\beta+\epsilon}, x^{2-\beta-\epsilon})$$

But in either case of the FBM scaling [5], [8] or the intermediate scaling [6], the asymptotic relation [8] yields

$$m \text{Var}(T_a)/b^2 \to \sigma_x^2/\mu^2, \quad m \to \infty,$$

and so, eventually choosing a larger $m_1 \geq m_0$,

$$m \text{Var}(T_{ax})/b^2 \leq (1 + \epsilon)(\sigma_x^2 + \epsilon) \max(x^{2-\beta+\epsilon}, x^{2-\beta-\epsilon}), \quad m \geq m_1.$$ With $\epsilon < 1 - \beta$ this yields [8] for $ax \geq a_0$.

It remains to prove [8] for $a \geq a_0$ and $ax < a_0$. By Lemma [6 iv],

$$m \text{Var}(T_{ax})/b^2 \leq \frac{2e}{\mu} \frac{m}{b^2} \int_0^{ax} \Phi(1/y)^{-1} dy \leq \frac{2e}{\mu} \frac{m}{b^2} \Phi(1/ax)$$

By [7] we can find a constant $C_1$ and $q > \beta$, such that $\phi(1/ax) \leq C_1 (ax)^q$. As in the proof of Lemma [8] we may take $a^{-\epsilon} \leq L(a)$. In Theorem 1, $ma^{2-\beta}L(a)/\mu b^2 = 1$. In Theorem 2, $ma^{2-\beta}L(a)/\mu b^2 \to c^\beta$. Thus,

$$m \text{Var}(T_{ax})/b^2 \leq C_2 \frac{ma^{2-\eta}L(a)}{b^2} \frac{(ax)^{1+q}}{a^{2-\beta-\epsilon}} \leq C_3 a_0^{\beta} x^{1+q-\beta} \frac{1}{a^{1-q-\epsilon}}$$

Since $\beta < q < \sigma \leq 1$, we may take $\epsilon < 1 - q$ to obtain [8] for $\gamma = 1 + q - \beta > 1$. 

$\square$
REFERENCES

[1] J. Bertoin, Lévy Processes, Cambridge University Press, Cambridge, 1996.
[2] J. Bertoin, Subordinators: Examples and Applications, in: Ecole d’été de Probabilités de St-Flour XXVII, Lecture Notes in Mathematics 1717, Springer-Verlag, Berlin, 1997, pp. 4-91.
[3] P. Billingsley, Convergence of probability measures. John Wiley and Sons, New York, 1968.
[4] N. H. Bingham, C. M. Goldie, J. L. Teugels, Regular Variation, Cambridge University Press, Cambridge, 1997.
[5] C. Dombry, I. Kaj. The on-off network traffic model under intermediate scaling. Queuing Systems, vol. 69, no. 1, pp. 29-44, 2011.
[6] C. Dombry, I. Kaj. Moment measures of heavy-tailed renewal point processes: asymptotics and applications.
[7] B.E. Fristedt, Intersections and limits of regenerative sets, in: Random Discrete Structures, Eds. D. Aldous and R. Pemantle, Springer-Verlag, Berlin, 1996, pp. 121-151.
[8] R. Gaigalas, A Poisson bridge between fractional Brownian motion and stable Lévy motion, Stochastic Process. Appl. 116 (2006) 447-462.
[9] R. Gaigalas, I. Kaj, Convergence of scaled renewal processes and a packet arrival model, Bernoulli 9 (2003) 671-703.
[10] A. Gnedin, J. Pitman, Regenerative composition structures, Ann. Probab. 33 (2005) 445-479.
[11] K. van Harn, F.W. Steutel, Stationarity of delayed subordinators, Stochastic Models 17 (2001) 369-374.
[12] I. Kaj, Limiting fractal random processes in heavy-tailed systems, in: Fractals in Engineering, New Trends in Theory and Applications, Springer, London, 2005, pp. 199-218.
[13] Kaj, I., Taqqu, M.S., 2008. Convergence to fractional Brownian motion and to the Telecom process: the integral representation approach. In an Out of Equilibrium 2, Eds. M.E. Vares, V. Sidoravicius. Progress in Probability, Vol 60, 383-427. Birkhauser 2008.
[14] Th. Mikosch, S. Resnick, H. Rootzén, A. Stegeman, Is network traffic approximated by stable Lévy motion or fractional Brownian motion?, Ann. Appl. Probab. 12 (2002) 23-68.
[15] K.-I. Sato, Lévy Processes and Infinitely Divisible Processes, Cambridge University Press, Cambridge, 1999.
[16] M.S. Taqqu, The modeling of Ethernet data and of signals that are heavy-tailed with infinite variance, Scandinavian Journal of Statistics 29 (2002) 273-295.
[17] W. Willinger, V. Paxson, R.H. Riedi, M.S. Taqqu, Long-range dependence and data network traffic, in: Theory and Applications of Long-Range Dependence, Eds. P. Doukhan, G. Oppenheim, M.S. Taqqu, Birkhäuser, Basel, 2003.

IK, DEPT. OF MATHEMATICS, UPPSALA UNIVERSITY, BOX 480, SE 751 06 UPPSALA, SWEDEN
E-mail address: ikaj@math.uu.se

AM-L, MATHEMATICAL STATISTICS, STOCKHOLM UNIVERSITY, SE 106 91 STOCKHOLM, SWEDEN
E-mail address: andersml@math.su.se