Existence of positive solutions of a superlinear boundary value problem with indefinite weight

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Abstract

We deal with the existence of positive solutions for a two-point boundary value problem associated with the nonlinear second order equation
\[ u'' + a(x)g(u) = 0. \]
The weight \( a(x) \) is allowed to change its sign. We assume that the function \( g: [0, +\infty[ \to \mathbb{R} \) is continuous, \( g(0) = 0 \) and satisfies suitable growth conditions, so as the case \( g(s) = s^p \), with \( p > 1 \), is covered. In particular we suppose that \( g(s)/s \) is large near infinity, but we do not require that \( g(s) \) is non-negative in a neighborhood of zero. Using a topological approach based on the Leray-Schauder degree we obtain a result of existence of at least a positive solution that improves previous existence theorems.

1 Introduction

In this paper we are interested in the study of positive solutions for the nonlinear two-point boundary value problem
\[
\begin{align*}
    u'' + a(x)g(u) &= 0 \\
    u(0) &= u(L) = 0,
\end{align*}
\]
where \( a: [0, L] \to \mathbb{R} \) is a Lebesgue integrable function and \( g: \mathbb{R}^+ \to \mathbb{R} \) is a continuous function, where \( \mathbb{R}^+ := [0, +\infty[ \) denotes the set of non-negative real numbers. We recall that a positive solution of (1.1) is an absolutely continuous function \( u: [0, L] \to \mathbb{R}^+ \) such that its derivative \( u'(x) \) is absolutely continuous, \( u(x) \) satisfies \( (1.1) \) for a.e. \( x \in [0, L] \) and \( u(x) > 0 \) for every \( x \in [0, L] \).

This issue has been considered by many authors. As classical examples, we mention [1 2 3 5 8 11] (see also the references therein), where different techniques are used to face this type of problem. Our work benefits from a new approach based on the Leray-Schauder topological degree, so, to obtain a
positive solution, our goal is to prove that the degree of a suitable operator is non-zero on an open domain of $C([0, L])$ not containing the trivial solution.

Our assumptions allow the weight function $a(x)$ to change its sign a finite number of times and, concerning the nonlinearity, we suppose that $g(s)$ can change its sign, even an infinite number of times, and that, roughly speaking, it has a superlinear growth at zero and at infinity. More in detail, with respect to the growth of $g(s)/s$ at zero, we assume a very general condition which depends on the sign of $g(s)$ in a right neighborhood of zero.

Our main result states that, under the conditions just presented, problem (1.1) has at least a positive solution. This theorem clearly covers the case $g(s) = s^p$, with $p > 1$. Moreover, the results concerning the BVP (1.1) where is assumed that $a(x)g(s) \geq 0$ for a.e. $x \in [0, L]$ and for all $s \geq 0$ (see [5, 8, 11]) or that $g(s) > 0$ for all $s > 0$, when $a(x)$ is allowed to change sign (see [3, 6, 7]), do not contain our result and, in some cases, are easy consequences of it.

Figure 1 and Figure 2 show examples of nonlinearities $g(s)$ satisfying our assumptions and which are not covered by previous results.

Figure 1: A numerical simulation obtained by setting $I = [0, 1]$, $a(x) = \sin(3\pi x)$ and $g(s) = \min\{20s^{6/5} - 6s^3 + s^4, 400s\arctan(s)\}$. On the left we have shown the graph of $g(s)$. We underline that $g(s)$ changes sign and $g(s)/s \nrightarrow +\infty$ as $s \to +\infty$. On the right we have represented the image of the segment $\{0\} \times [0, 12]$ through the Poincaré map in the phase-plane $(u, u')$. It intersects the negative part of the $u'$-axis in a point, hence there is a positive initial slope at $x = 0$ from which departs a solution which is positive on $[0, 1]$ and vanishes at $x = 1$.

Figure 2: A numerical simulation obtained by setting $I = [0, 1]$, $a(x) = \sin(7\pi x)$ and $g(s) = s^3 + s^2\sin(1/s)$. On the left we have shown the graph of $g(s)$. The nonlinearity $g(s)$ changes sign an infinite number of times in every neighborhood of zero. On the right we have represented the image of the segment $\{0\} \times [0, 16]$ through the Poincaré map in the phase-plane $(u, u')$. 
The plan of the paper is as follows. In Section 2, we present some basic facts. More in detail, we list the hypotheses and we introduce an equivalent fixed point problem that permits to face the problem with a topological approach. In fact, using the technical assumptions, we are able to compute the degree on suitable small and large balls, in the same spirit of [6].

In Section 3, we present our main result. The theorem we state is an immediate corollary of the results exhibited in the previous section. In particular, we prove that the topological degree is non-zero on an annular domain. Therefore, a nontrivial fixed point exists, this corresponds to a positive solution (using a standard maximum principle). Straightforward corollaries are then obtained.

Section 4 shows an important existence result of radially symmetric solutions on annular domains.

2 Preliminaries

In this section, we state the hypotheses on \(a(x)\) and on \(g(s)\), we recall some classical results and we prove two preliminary lemmas that are then employed in Section 3 for the main result.

Consider the nontrivial closed interval \([0, L]\), pointing out that different choices of a nontrivial compact interval contained in \(\mathbb{R}\) can be made. Let \(a : [0, L] \to \mathbb{R}\) be a \(L^1\)-weight function. Clearly the case of a continuous function can be treated as well. We assume that

\((H1)\) there exist \(m \geq 1\) intervals \(I_1, \ldots, I_m\), closed and pairwise disjoint, such that

\[
a(x) \geq 0, \quad \text{for a.e. } x \in \bigcup_{i=1}^{m} I_i;
\]

\[
a(x) \leq 0, \quad \text{for a.e. } x \in [0, L] \setminus \bigcup_{i=1}^{m} I_i.
\]

We underline that assumption \((H1)\) trivially includes the case where \(a(x) \geq 0\) for a.e. \(x \in [0, L]\), taking \(m = 1\) and \(I_1 = [0, L]\). As standard notation, we define

\[
a^+(x) := \max\{a(x), 0\}, \quad a^-(x) := \max\{-a(x), 0\}.
\]

Concerning the nonlinearity, we suppose that \(g : \mathbb{R}^+ \to \mathbb{R}\) is a continuous function such that

\((H2)\) \hspace{1cm} \(g(0) = 0\) \hspace{0.5cm} and \hspace{0.5cm} \(g \not\equiv 0\).

We set

\[
g_0^{inf} := \liminf_{s \to 0^+} \frac{g(s)}{s} \geq -\infty, \quad g_0^{sup} := \limsup_{s \to 0^+} \frac{g(s)}{s} < +\infty
\]

and

\[
g_\infty := \liminf_{s \to +\infty} \frac{g(s)}{s} > 0.
\]

We stress that we do not suppose \(g(s) \geq 0\) on \(\mathbb{R}^+\) and, in particular, it is not required that \(g(s) > 0\) for all \(s > 0\) (cf. [5, 6, 7, 8]). Consequently, the
nonlinearity $g(s)$ could be non-negative, non-positive or it could change sign, even an infinite number of times, on a compact neighborhood of zero.

Now we show how the superlinearity of $g$ is expressed at zero and at infinity. As first step we impose a condition on the growth of $g(s)/s$ at 0, depending on the sign of $g(s)$. Precisely we assume that

\begin{itemize}
  \item if there exists $\delta > 0$ such that $g(s) \geq 0$, for all $s \in [0, \delta]$, it holds that
    \begin{align*}
      a^+(x) \not\equiv 0 \text{ on } [0, L] \quad \text{and} \quad g_{0}^{\text{up}} < \lambda_0^+,
    \end{align*}
    where $\lambda_0^+ > 0$ is the first eigenvalue of the eigenvalue problem
    \begin{align*}
      \varphi'' + \lambda a^+(x) \varphi = 0, \quad \varphi(0) = \varphi(L) = 0;
    \end{align*}
  \item if there exists $\delta > 0$ such that $g(s) \leq 0$, for all $s \in [0, \delta]$, it holds that
    \begin{align*}
      a^-(x) \not\equiv 0 \text{ on } [0, L] \quad \text{and} \quad g_{0}^{\text{inf}} > -\lambda_0^-,
    \end{align*}
    where $\lambda_0^- > 0$ is the first eigenvalue of the eigenvalue problem
    \begin{align*}
      \varphi'' + \lambda a^-(x) \varphi = 0, \quad \varphi(0) = \varphi(L) = 0;
    \end{align*}
  \item if $g(s)$ changes sign an infinite number of times in every neighborhood of zero, it holds that
    \begin{align*}
      a(x) \not\equiv 0 \text{ on } [0, L] \quad \text{and} \quad -\lambda_0 < g_{0}^{\text{inf}} \leq g_{0}^{\text{sup}} < \lambda_0,
    \end{align*}
    where $\lambda_0 > 0$ is the first eigenvalue of the eigenvalue problem
    \begin{align*}
      \varphi'' + \lambda |a(x)| \varphi = 0, \quad \varphi(0) = \varphi(L) = 0.
    \end{align*}
\end{itemize}

The functions $a(x)$ and $g(s)$ introduced in Figure 1 satisfy the first condition of hypothesis (H3), while the example shown in Figure 2 corresponds to the third case.

As second step we define the superlinear behavior at infinity. We suppose that

\begin{itemize}
  \item for all $i \in \{1, \ldots, m\}$
    \begin{align*}
      a(x) \not\equiv 0 \text{ on } I_i \quad \text{and} \quad g_{\infty} > \lambda_i^1,
    \end{align*}
    where $\lambda_i^1 > 0$ is the first eigenvalue of the eigenvalue problem
    \begin{align*}
      \varphi'' + \lambda a^+(x) \varphi = 0, \quad \varphi|_{\partial I_i} = 0.
    \end{align*}
\end{itemize}

Now we describe the topological approach we adopt to face problem (1.1). Our first goal is to introduce a completely continuous operator and to define an equivalent fixed point problem.

Let $\tilde{g} : \mathbb{R} \to \mathbb{R}$ be the standard extension of $g(s)$ defined as

\begin{align*}
  \tilde{g}(s) = \begin{cases}
    g(s), & \text{if } s \geq 0; \\
    0, & \text{if } s \leq 0.
  \end{cases}
\end{align*}
We deal with the boundary value problem
\[
\begin{aligned}
  u'' + a(x)\tilde{g}(u) &= 0 \\
  u(0) = u(L) &= 0.
\end{aligned}
\]
(2.1)

From conditions \((H2)\) and \((H3)\) and by a classical maximum principle (cf. \([6, 9]\)), it follows that all possible solutions of \((2.1)\) are non-negative. Moreover, if these solutions are nontrivial, then they are strictly positive on \([0, L]\) and hence positive solutions of \((1.1)\).

The next step is to define the classical operator \(\Phi: C([0, L]) \to C([0, L])\) by
\[
(\Phi u)(x) := \int_0^L G(x, \xi)a(\xi)\tilde{g}(u(\xi)) d\xi,
\]
(2.2)
where \(G(x, s)\) is the Green function associated to the equation \(u'' + u = 0\) with the two-point boundary condition. The operator \(\Phi\) is completely continuous in \(C([0, L])\), endowed with the sup-norm \(\|\cdot\|_{\infty}\), and such that \(u\) is a fixed point of \(\Phi\) if and only if \(u\) is a solution of \((2.1)\). Therefore we have transformed problem \((1.1)\) into an equivalent fixed point problem.

We close this section by proving two technical lemmas that allow us to find a nontrivial fixed point of \(\Phi\), hence a positive solution of \((1.1)\). The approach we use now is based on the Leray-Schauder topological degree and it is in the same spirit of \([6]\).

Using this first lemma we are able to compute the degree of \(Id - \Phi\) on small balls.

**Lemma 2.1.** There exists \(r_0 > 0\) such that
\[
\deg(Id - \Phi, B(0, r), 0) = 1, \quad \forall 0 < r \leq r_0.
\]

**Proof.** We divide the proof in two steps.

**Step 1.** We prove that there exists \(r_0 > 0\) such that every solution \(u(x) \geq 0\) of the two-point BVP
\[
\begin{aligned}
  u'' + \vartheta a(x)g(u) &= 0, \quad 0 \leq \vartheta \leq 1, \\
  u(0) = u(L) &= 0
\end{aligned}
\]
(2.3)
satisfying \(\max_{x \in [0, L]} u(x) \leq r_0\) is such that \(u(x) = 0\), for all \(x \in [0, L]\).

The proof of this first step is given only when there exists \(\delta > 0\) such that \(g(s) \geq 0\), for all \(s \in [0, \delta]\). The two remaining cases can be treated in an analogous way.

Using condition \((H3)\), we fix \(0 < r_0 < \delta\) such that
\[
\frac{g(s)}{s} < \lambda_0^+ \quad \forall 0 < s \leq r_0.
\]

Now, suppose by contradiction that there exist \(\delta \in [0, 1]\) and a positive solution \(u(x) \neq 0\) of \((2.3)\) such that \(\max_{x \in [0, L]} u(x) = r\) for some \(0 < r \leq r_0\). The choice of \(r_0\) and the maximum principle imply that
\[
0 \leq \vartheta g(u(x)) < \lambda_0^+ u(x), \quad \text{for all} \ x \in [0, L] ,
\]
Let \( \varphi \) be a positive eigenfunction of
\[
\begin{cases}
\varphi'' + \lambda_0^+ a^+(x)\varphi = 0 \\
\varphi(0) = \varphi(L) = 0.
\end{cases}
\]
We stress that \( \varphi(x) > 0 \), for all \( x \in [0, L] \). Using a Sturm comparison argument, we attain
\[
0 = \left[ u'(x)\varphi(x) - u(x)\varphi'(x) \right]_0^L
= \int_0^L \frac{d}{dx} \left[ u'(x)\varphi(x) - u(x)\varphi'(x) \right] dx
= \int_0^L u''(x)\varphi(x) - u(x)\varphi''(x) \right] dx
= \int_0^L \left[ -\vartheta a(x)g(u(x))\varphi(x) + u(x)\lambda_0^+ a^+(x)\varphi(x) \right] dx
\geq \int_0^L \left[ \lambda_0^+ u(x) - \vartheta g(u(x)) \right] a^+(x)\varphi(x) dx
> 0,
\]
a contradiction.

Step 2. Computation of the degree. Let us fix \( 0 \leq \vartheta \leq 1 \). As remarked when we have introduced the operator \( \Phi \), the maximum principle ensures that every fixed point in \( C([0, L]) \) of the operator \( \vartheta \Phi \) is non-negative and, moreover, \( u \in C([0, L]) \) satisfies \( u = \vartheta \Phi(u) \) if and only if \( u \) is a solution of the equation (2.3). Therefore, setting \( r \in [0, r_0] \), Step 1 implies that \( \|u\|_\infty \neq r \) and hence
\[
u \neq \vartheta \Phi(u), \quad \forall \vartheta \in [0, 1], \forall u \in \partial B(0, r).
\]
By the homotopic invariance property of the topological degree, we obtain that
\[
\text{deg}(\text{Id} - \Phi, B(0, r), 0) = \text{deg}(\text{Id}, B(0, r), 0) = 1.
\]

Now we compute the degree on large balls.

**Lemma 2.2.** There exists \( R^* > 0 \) such that
\[
\text{deg}(\text{Id} - \Phi, B(0, R), 0) = 0, \quad \forall R \geq R^*.
\]

**Proof.** We divide the proof in two steps.

**Step 1. A priori bounds for \( u \) on each \( I_i \).** For each \( i \in \{1, \ldots, m\} \), we prove that there exists \( R_i > 0 \) such that for each \( L^1 - \text{Carathéodory function} \ h \colon [0, L] \times \mathbb{R}^+ \to \mathbb{R} \) with
\[
h(x, s) \geq a(x)g(s), \quad \text{a.e.} \ x \in I_i, \forall s \geq 0,
\]
every solution \( u(x) \geq 0 \) of the two-point BVP
\[
\begin{cases}
  u'' + h(x, u) = 0 \\
  u(0) = u(L) = 0
\end{cases}
\] (2.4)
satisfies \( \max_{x \in I_i} u(x) < R_i \).

We fix an index \( i \in \{1, \ldots, m\} \) and set \( I_i := [\sigma_i, \tau_i] \). Let \( 0 < \varepsilon < (\tau_i - \sigma_i)/2 \) be fixed such that
\[
a^+(x) \neq 0 \quad \text{on } I_i^c,
\]
where \( I_i^c := [\sigma_i + \varepsilon, \tau_i - \varepsilon] \), and such that the first positive eigenvalue \( \hat{\lambda} \) of the eigenvalue problem
\[
\begin{align*}
\varphi'' + \lambda a^+(x) \varphi &= 0 \\
\varphi|_{\partial I_i^c} &= 0
\end{align*}
\] (2.5)
is such that
\[
0 < \hat{\lambda} < g_\infty.
\] The existence of \( \varepsilon \) is ensured by the continuity of the eigenvalue as function of the boundary condition (see \( [12] \)) and by hypothesis (\( H4 \)). From the previous inequality it follows that there exists a constant \( \hat{R} > 0 \) such that
\[
g(s) > \hat{\lambda}s, \quad \forall s \geq \hat{R}.
\]

By contradiction, suppose there is not a constant \( R_i > 0 \) with the properties listed above. So, for each integer \( n > 0 \) there exists a solution \( u_n \geq 0 \) of (2.4) with \( \max_{x \in I_i} u_n(x) =: \tilde{R}_n > n \).

We claim that there exists an integer \( N \geq \hat{R} \) such that \( u_n(x) > \hat{R} \) for every \( x \in I_i^c \) and \( n \geq N \). If it is not true, for every integer \( n \geq \hat{R} \) there is an integer \( \hat{n} \geq n \) and \( x_\hat{n} \in I_i^c \) such that \( u_{\hat{n}}(x_\hat{n}) = \hat{R} \). We note that the solution \( u_n(x) \) is concave on each subinterval of \( I_i \) where \( u_n(x) \geq \hat{R} \), since \( a(x) g(s) \geq 0 \) for a.e. \( x \in I_i \) and for all \( s \geq \hat{R} \). Then, without loss of generality, we can assume that there exists a maximum point \( \tilde{x}_n \in I_i \) of \( u_n \) such that \( u_n(x) > \hat{R} \) for all \( x \) between \( x_n \) and \( \tilde{x}_n \). If necessary, we change the choice of \( x_n \). From the assumptions, it follows that
\[
\hat{n} < \tilde{R}_n \leq u_n(\tilde{x}_n) = u_n(x_\hat{n}) + \int_{x_n}^{\tilde{x}_n} u_n'(\xi) \, d\xi \leq \hat{R} + (\tau_i - \sigma_i)|u_n'(x_\hat{n})|. \tag{2.6}
\]
Since \( h(x, s) \) is a \( L^1 \)-Carathéodory function, there exists \( \gamma_\hat{R} \in L^1([0, L], \mathbb{R}^+) \) such that \( |h(x, s)| \leq \gamma_\hat{R}(x) \), for a.e. \( x \in [0, L] \) and for all \( |s| \leq \hat{R} \). Then, we fix a constant \( C > 0 \) such that
\[
C > \frac{\hat{R}}{\varepsilon} + \|\gamma_\hat{R}\|_{L^1}.
\]
Using (2.6), we have that for every \( n \geq (\tau_i - \sigma_i)C + \hat{R} \) there exists \( \hat{n} \geq n \) and \( x_\hat{n} \in I_i^c \) such that \( u_{\hat{n}}(x_\hat{n}) = \hat{R} \) and \( |u_{\hat{n}}'(x_\hat{n})| > C \). Let us fix \( n \geq (\tau_i - \sigma_i)C + \hat{R} \), \( \hat{n} \geq n \) and \( x_\hat{n} \in I_i^c \) with the properties just listed. Suppose that \( u_{\hat{n}}'(x_\hat{n}) > C \) and consider the interval \([\sigma_i, x_\hat{n}]\). If \( u_{\hat{n}}'(x_\hat{n}) < -C \) we proceed similarly dealing with the interval \([x_\hat{n}, \tau_i]\). For every \( x \in [\sigma_i, x_\hat{n}] \)
\[
u_{\hat{n}}'(x) = u_{\hat{n}}'(x_\hat{n}) + \int_x^{x_n} u_{\hat{n}}''(\xi) \, d\xi,
\]
then
\[
u_{\hat{n}}'(x) > C - \int_x^{x_n} |h(\xi, u_{\hat{n}}(\xi))| \, d\xi.
\]
From this inequality we obtain that $u_{\lambda}(x) \leq \tilde{R}$, for all $x \in [\sigma_i, x_0]$, and therefore

$$u_\lambda'(x) > \frac{\tilde{R}}{\varepsilon}, \quad \text{for all } x \in [\sigma_i, x_0].$$

Then, we obtain

$$\tilde{R} \leq \frac{\tilde{R}}{\varepsilon} (x_0 - \sigma_i) < \int_{\sigma_i}^{x_0} u_\lambda'(\xi) d\xi = u_\lambda(x_0) - u_\lambda(\sigma_i) \leq u_\lambda(x_0) = \tilde{R},$$

which is a contradiction. Hence the claim is proved. So, we can fix an integer $N \geq \tilde{R}$ such that $u_\lambda(x) > \tilde{R}$ for every $x \in I_i^*$ and for $n \geq N$.

We denote by $\varphi$ the positive eigenfunction of the eigenvalue problem \(\text{(2.5)}\) with $\|\varphi\|_\infty = 1$. Then $\varphi(x) > 0$, for every $x \in [\sigma_i + \varepsilon, \sigma_i - \varepsilon]$, and $\varphi'(\sigma_i + \varepsilon) > 0 > \varphi'(\sigma_i - \varepsilon)$. We remark that $u_\lambda(\sigma_i + \varepsilon) > 0$ and $u_\lambda(\sigma_i - \varepsilon) > 0$, for every integer $n$, employing the maximum principle.

Using a Sturm comparison argument, for each $n \geq N$, we obtain

$$0 > u_\lambda(\tau_i - \varepsilon)\varphi'(\tau_i - \varepsilon) - u_\lambda(\sigma_i + \varepsilon)\varphi'(\sigma_i + \varepsilon) = \left[u_\lambda(\sigma_i + \varepsilon)u_\lambda'(\sigma_i + \varepsilon) - u_\lambda'(\sigma_i + \varepsilon)\varphi(\sigma_i + \varepsilon)\right]_{\sigma_i + \varepsilon}^{\tau_i - \varepsilon}
= \int_{\tau_i - \varepsilon}^{\sigma_i + \varepsilon} \frac{d}{dx} \left[u_\lambda(x)\varphi'(x) - u_\lambda'(x)\varphi(x)\right] dx
= \int_{I_i^*} \left[u_\lambda(x)\varphi''(x) - u_\lambda''(x)\varphi(x)\right] dx
= \int_{I_i^*} \left[-u_\lambda(x)\hat{\lambda}a^+(x)\varphi(x) + h(x, u_\lambda(x))\varphi(x)\right] dx
= \int_{I_i^*} \left[h(x, u_\lambda(x)) - \hat{\lambda}a^+(x)u_\lambda(x)\right] \varphi(x) dx
\geq \int_{I_i^*} \left[g(u_\lambda(x)) - \hat{\lambda}a^+(x)u_\lambda(x)\right] \varphi(x) dx
\geq 0,$$

which is a contradiction.

**Step 2. Computation of the degree.** We stress that the constant $R_i$, $i \in \{1, \ldots, m\}$, does not depend on the function $h(x, s)$. Define

$$R^* := \max_{i=1, \ldots, m} R_i + \tilde{R} > 0$$

and fix a radius $R \geq R^*$.

We denote by $1_A$ the characteristic function of the set $A := \bigcup_{i=1}^m I_i$. Let us define $v(x) := \int G(x, s)1_A(s) ds$. Using a classical result (see [4, Theorem 3.1] or [10, Lemma 1.1]), if we show that

$$u \neq \Phi(u) + \alpha v, \quad \text{for all } u \in \partial B(0, R) \text{ and } \alpha \geq 0,$$

the theorem is proved.
Let $\alpha \geq 0$. The maximum principle ensures that any nontrivial solution $u \in C([0, L])$ of $u = \Phi(u) + \alpha v$ is a non-negative solution of $u'' + a(x)\tilde{g}(u) + \alpha A(x) = 0$ with $u(0) = u(L) = 0$. Hence, $u$ is a non-negative solution of (2.4) with

$$h(x, s) = a(x)g(s) + \alpha A(x).$$

By definition, we have that $h(x, s) \geq a(x)g(s)$, for a.e. $x \in A$ and for all $s \geq 0$, and $h(x, s) = a(x)g(s)$, for a.e. $x \in [0, L] \setminus A$ and for all $s \geq 0$. By the convexity of the solution $u$ on the intervals of $[0, L] \setminus A$ where $u(x) \geq \tilde{R}$, we obtain that

$$\|u\|_{\infty} = \max_{x \in [0, L]} u(x) \leq \max\left\{ \max_{x \in A} u(x), \tilde{R} \right\}.$$

From Step 1 and the definition of $\tilde{R}$ we deduce that $\|u\|_{\infty} < R^* \leq R$. Then (2.7) is proved and the theorem follows.

3 The main result

In this section we apply the two technical lemmas just proved to obtain the existence of a positive solution to the two-point boundary value problem (1.1). More in detail, we use the additivity of the topological degree to provide the existence of a nontrivial fixed point of the operator $\Phi$ defined in (2.2).

A first immediate consequence of Lemma 2.1 and Lemma 2.2 is our main theorem.

**Theorem 3.1.** Let $a: [0, L] \to \mathbb{R}$ be a $L^1$-function and $g: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function satisfying $(H1)$, $(H2)$, $(H3)$ and $(H4)$. Then there exists at least a positive solution of the two-point boundary value problem (1.1).

**Proof.** Let $r_0$ be as in Lemma 2.1 and $R^*$ be as in Lemma 2.2. We observe that $0 < r_0 < R^* < +\infty$. From the additivity property and the two preliminary lemmas it follows that

$$\deg(\text{Id} - \Phi, B(0, R^*) \setminus B[0, r_0], 0) =$$

$$= \deg(\text{Id} - \Phi, B(0, R^*), 0) - \deg(\text{Id} - \Phi, B(0, r_0), 0) =$$

$$= 0 - 1 = -1 \neq 0.$$

Then there exists a nontrivial fixed point of $\Phi$ and hence a corresponding positive solution of (1.1), as already remarked.

From Theorem 3.1 we easily achieve the following two results.

**Corollary 3.1.** Let $a: [0, L] \to \mathbb{R}$ be a $L^1$-function and $g: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function satisfying $(H1)$ and $(H2)$. Assume that

$$g'(0) = \lim_{s \to 0^+} \frac{g(s)}{s} = 0,$$

and, for each $i \in \{1, \ldots, m\}$, suppose that $a(x) \neq 0$ on $I_i$ and

$$g'(\infty) := \lim_{s \to +\infty} \frac{g(s)}{s} = +\infty.$$

Then there exists at least a positive solution of the two-point BVP (1.1).
Corollary 3.2. Let $a: [0, L] \to \mathbb{R}$ be a $L^1$-function satisfying (H1) and such that $a(x) \not\equiv 0$ on $I_i$, for each $i \in \{1, \ldots, m\}$. Let $g: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function satisfying (H2) and such that $g'(0) = 0$ and $g'(\infty) = \Lambda > 0$. Then there exists $\lambda^* > 0$ such that, for each $\lambda > \lambda^*$, the two-point BVP

$$
\begin{cases}
  u'' + \lambda a(x) g(u) = 0 \\
  u(0) = u(L) = 0
\end{cases}
$$

has at least a positive solution.

Although hypothesis (H1) is more interesting when the set $[0, L] \setminus \bigcup_{i=1}^m I_i$ is not negligible, we can consider a weight $a(x) \geq 0$ for a.e. $x \in [0, L]$, as previously observed. In that situation Corollary 3.1 ensures the existence of a positive solution in the superlinear case (i.e. $g'(0) = 0$ and $g'(\infty) = +\infty$), provided that $a \not\equiv 0$. No sign condition on the function $g(s)$ is required. Thus we have extended [5, Theorem 1], attained as an application of Krasnosel’$\overset{\sim}{\text{s}}$kiĭ fixed point Theorem.

Remark 3.1. Our approach is based on the definition of a fixed point problem which is equivalent to the boundary value problem considered. It is clear that we could deal with different conditions at the boundary of $[0, L]$ like $u'(0) = u(L) = 0$ or $u(0) = u'(L) = 0$, since a suitable maximum principle and a Green function (cf. [5]) are available to define an equivalent fixed point problem and to adapt the scheme shown in this paper.

4 Radially symmetric solutions

We denote by $\| \cdot \|$ the Euclidean norm in $\mathbb{R}^N$ (for $N \geq 2$). Let

$$
\Omega := B(0, R_2) \setminus B[0, R_1] = \{ x \in \mathbb{R}^N : R_1 < \| x \| < R_2 \}
$$

be an open annular domain, with $0 < R_1 < R_2$. Let $a: [R_1, R_2] \to \mathbb{R}$ be a continuous function. In this section we consider the Dirichlet boundary value problem

$$
\begin{cases}
  -\Delta u = a(\| x \|) g(u) & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega
\end{cases}
$$

(4.1)

and we are interested in the existence of positive solutions of (4.1), namely classical solutions such that $u(x) > 0$ for all $x \in \Omega$.

Since we look for radially symmetric solutions of (4.1), our study can be reduced to the search of positive solutions of the two-point boundary value problem

$$
w''(r) + \frac{N-1}{r} w'(r) + a(r) g(w(r)) = 0, \quad w(R_1) = w(R_2) = 0. \quad (4.2)
$$

Indeed, if $w(r)$ is a solution of (4.2), then $u(x) := w(\| x \|)$ is a solution of (4.1). Using the standard change of variable

$$
t = h(r) := \int_{R_1}^r \xi^{1-N} \, d\xi
$$

10
and defining
\[ L := \int_{R_1}^{R_2} \xi^{1-N} \, d\xi, \quad r(t) := h^{-1}(t) \quad \text{and} \quad v(t) = w(r(t)), \]
we transform (4.2) into the equivalent problem
\[ v''(t) + r(t)^2(N-1)a(r(t))g(v(t)) = 0, \quad v(0) = v(L) = 0. \] (4.3)
Consequently, the two-point boundary value problem (4.3) is of the same form of (1.1) considering \( r(t)^2(N-1)a(r(t)) \) as weight function.

Clearly the following result holds.

**Theorem 4.1.** Let \( a: [R_1, R_2] \to \mathbb{R} \) and \( g: \mathbb{R}^+ \to \mathbb{R} \) be continuous functions satisfying (H1), (H2), (H3) and (H4). Then problem (4.1) has at least a positive solution.

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