Abstract
The sums and maxima of non-stationary random length sequences of regularly varying random variables may have the same tail and extremal indices, Markovich and Rodionov (2020). The main constraint is that there exists a unique series in a scheme of series with the minimum tail index. The result is now revised allowing a random bounded number of series to have the minimum tail index. This new result is applied to random networks.

Keywords: Random length sequence, Tail index, Extremal index, Random network

1. Introduction
Random length sequences and distribution tails of their sums and maxima attract the interest of many researchers due to numerous applications including queues, branching processes and random networks [1], [10], [15], [16], [18], [20], [21], [22].

Let \( \{Y_{n,i} : n, i \geq 1\} \) be a doubly-indexed array of nonnegative random variables (r.v.s) in which the "row index" \( n \) corresponds to time, and the "column index" \( i \) enumerates the series. On the same probability space, the existence of a sequence of non-negative integer-valued r.v.s \( \{N_n : n \geq 1\} \) is assumed. Let \( \{Y_{n,i} : n \geq 1\} \) be a strict-sense stationary sequence with extremal index \( \theta_i \) having a regularly varying tail

\[
P\{Y_{n,i} > x\} = \ell_i(x)x^{-k_i}
\]

with tail index \( k_i \) > 0 and a slowly varying function \( \ell_i(x) \). There are no assumptions on the dependence structure in \( i \). In [18], the weighted sums and maxima

\[
Y^*_n(z, N_n) = \max(z_1Y_{n,1}, ..., z_{N_n}Y_{n,N_n}), \quad Y_n(z, N_n) = z_1Y_{n,1} + ... + z_{N_n}Y_{n,N_n}
\]

for positive constants \( z_1, z_2, ... \) were considered. A similar result was obtained in [13] for random sequences of a fixed length \( l \geq 1 \) and when \( \{Y_{n,i} : n \geq 1\} \) has a power-type tail, i.e. \( P(Y_{n,i} > x) \sim c^{(i)}x^{-k_i} \) as \( x \to \infty \), where \( c^{(i)} \) is a real-valued positive constant.

Definition 1. A stationary sequence \( \{Y_n\}_{n \geq 1} \) with distribution function \( F(x) \) and \( M_n = \max_{j=1}^{n} Y_j \) is said to have extremal index \( \theta \in [0, 1] \) if for each \( 0 < \tau < \infty \) there is a sequence of real numbers \( u_\tau = u_\tau(\tau) \) such that

\[
\lim_{n \to \infty} n(1 - F(u_\tau)) = \tau \quad \text{and}
\]
\begin{equation}
\lim_{n \to \infty} P\{M_n \leq u_n\} = e^{-\tau \theta}
\end{equation}

hold ([14], p.63).

I.i.d. r.v.s \{Y_n\} give \( \theta = 1 \). The converse may be incorrect. An extremal index that is close to zero implies a kind of a strong local dependence.

Let us recall Theorem 4 derived in [18]. It is assumed that the "column" sequences \( \{Y_{n,i} : i \geq 1\} \) have stationary distributions [1] in \( n \) with positive tail indices \( \{k_1, k_2, \ldots\} \) and extremal indices \( \{\theta_1, \theta_2, \ldots\} \) for each fixed \( i \), where \( \{\ell_i(x)\} \) are restricted by the condition: for all \( A > 1, \delta > 0 \) there exists \( x_0(A, \delta) \) such that for all \( i \geq 1 \)

\begin{equation}
\ell_i(x) \leq Ax^\delta, \quad x > x_0(A, \delta)
\end{equation}

holds. \( N_n \) has a regularly varying distribution with tail index \( \alpha > 0 \), that is

\begin{equation}
P(N_n > x) = x^{-\alpha} \ell_n(x).
\end{equation}

There is a minimum tail index \( k_1 \) and \( k := \lim_{n \to \infty} \inf_{2 \leq i \leq l_n} k_i \),

\begin{equation}
l_n = [n^\chi],
\end{equation}

and \( \chi \) satisfies

\begin{equation}
0 < \chi < \chi_0, \quad \chi_0 = \frac{k - k_1}{k_1(k + 1)}.
\end{equation}

An arbitrary dependence structure between \( \{Y_{n,i}\} \) and \( \{N_n\} \) is allowed. The tail of \( N_n \) does not dominate the tail of the most heavy-tailed term \( Y_{n,1} \). Let \( u_n = yn^{1/k_1} \ell_1^i(n), \ y > 0, \) where \( \ell_1^i(n) \) is the de Bruijn conjugate of \( \ell(x) = (\ell_1(x))^{-1/k_1} \), and the positive weights \( \{z_i\} \) are assumed to be bounded as in [18].

**Theorem 1.** [18] Let the sets of slowly varying functions \( \{\ell_n(x)\}_{n \geq 1} \) in [4] and \( \{\ell_i(x)\}_{i \geq 1} \) in [1] satisfy the condition [9]. Suppose that \( k_1 < k \) and

\begin{equation}
P\{N_n > l_n\} = o\left( P\{Y_{n,1} > u_n\} \right), \quad n \to \infty
\end{equation}

hold, where the sequence \( l_n \) satisfies [3] and [9]. Then the sequences \( Y_{n}^*(z, N_n) \) and \( Y_{n}(z, N_n) \) have the same tail index \( k_1 \) and the same extremal index \( \theta_1 \).

Theorem [1] is based on Theorem [2] we recall further. Let us denote \( Y_{n}^*(z) = Y_{n}^*(z, l_n) \) and \( Y_{n}(z) = Y_{n}(z, l_n) \).

**Theorem 2.** [18] Let \( k_1 < k \), [3], [9] and [6] hold. Then the sequences \( Y_{n}^*(z) \) and \( Y_{n}(z) \) have the same tail index \( k_1 \) and the same extremal index \( \theta_1 \).

Our objectives are twofold. At first, we revise Theorem [1] for the case when a random number of "column" series may have the minimum tail index. At second, we modify Theorem [1] with regard to random networks. Each node pair in a random network is connected with some probability \( \delta \). Let \( G_n = (V_n, E_n) \) be a directed graph with a set of vertices \( V_n = \{1, \ldots, n\} \), and a set of directed edges \( E_n \). Google’s PageRank vector \( R = \{R_1, \ldots, R_n\} \) is the unique solution to the following system of linear equations:

\begin{equation}
R_i = c \sum_{j: (j,i) \in E_n} \frac{R_j}{D_j} + (1 - c)q_i, \quad i = 1, \ldots, n,
\end{equation}
where the summation is taken over a number of pages \( j \) that link to page \( i \) (in-degree), \( D_j \) is the number of outgoing links of page \( j \) (out-degree), \( c \in (0, 1) \) is a damping factor, \( q = (q_1, q_2, \ldots, q_n) \) is a personalization probability vector or user’s preferences such that \( q_i \geq 0 \) and \( \sum_{i=1}^{n} q_i = 1 \), and \( n \) is the total number of pages. The World Wide Web (Web) is a very large interconnected graph where nodes correspond to pages. The PageRank was designed to rank pages on the Web in such a way that a page is important if many important pages have a hyperlink to it [6].

A stochastic approach to analyze (8) is the following. The PageRank of a randomly chosen Web page (i.e. a vertex on a Web graph) considered as a root node of a Galton-Watson tree with random in- and out-degrees may be modeled as a r.v. \( R \) which is the solution of the fixed-point problem

\[
\begin{align*}
R &= D \sum_{j=1}^{N} A_j R_j + Q, \\
\end{align*}
\]

assuming that \( \{R_j\} \) are independent identically distributed (i.i.d.) copies of \( R \) and \( E(Q) < 1 \) holds. \((Q, N, \{A_j\})\) is a real-valued vector. \( N \) denotes the in-degree. \( Q \) is a personalization value of the vertex [23]. \( =D \) means equality in distribution. Under the assumptions (we shall call it Assumptions A) that \( \{R_j\} \) are i.i.d. and independent of \((Q, N, \{A_j\})\) with \( \{A_j\} \) independent of \((N, Q)\), and that \( N \) and \( Q \) are allowed to be dependent, it is stated in [23], [8] that the stationary distribution of \( R \) in (9) is regularly varying and its tail index is determined by the most heavy-tailed distributed term in the regularly varying distributed pair \((N, Q)\).

This result was generalized by [1], and the unique solution of (9) is proved to be intermediate regularly varying \(^1\) if \( Q \) or \( N \) has an intermediate regularly varying distribution, or \((Q, N)\) has a two-dimensional regularly varying distribution. The multivariate version of (9)

\[
\begin{align*}
R(i) = D \sum_{k=1}^{K} \sum_{m=1}^{N^{(k)}(i)} R_m(k) + Q(i), \\
\end{align*}
\]

where \( R_m(k) = D R(k) \) holds, is considered with similar assumptions and regularly varying statements by thinking that \( N^{(k)}(i) \) is a number of type-\( k \) children of a type-\( i \) ancestor and considering a multi-type Galton-Watson tree.

A Max-Linear Model \([12]\) is obtained by substitution sums in (8) by maxima, i.e.

\[
\begin{align*}
R(i) &= \bigvee_{j \to i} A_j R(j) \lor Q_i, \quad i = 1, \ldots, n. \\
\end{align*}
\]

Under Assumptions A the power law tail \( P\{|R| > x\} \sim H x^{-\alpha} \) \( \alpha > 0 \), \( H > 0 \) as \( x \to \infty \) of the so-called ‘minimal/endogeneous’ solution of the equation

\[
\begin{align*}
R &= D \left( \bigvee_{j=1}^{N} A_j R_j \right) \lor Q, \\
\end{align*}
\]

\(^1\)The class of intermediate regularly varying distributions such that \( \lim_{\alpha \uparrow 1} \limsup_{x \to \infty} F(\alpha x)/F(x) = 1 \) includes regularly varying distributions.

\(^2\)The symbol \( \sim \) means asymptotically equal to or \( f(x) \sim g(x) \iff f(x)/g(x) \to 1 \) as \( x \to a \), \( x \in M \) where the functions \( f(x) \) and \( g(x) \) are defined on some set \( M \) and \( a \) is a limit point of \( M \).
is derived in \[9\]. We propose to apply Theorem 1 to the right-hand sides of (9) and (11) under weaker assumptions than the Assumptions A. Namely, the conditions that PageRanks \(\{R_j\}\) of the first generation are i.i.d. and the mutual independence of \(\{R_j\}\) and \(N\) are omitted. By a modified Theorem 1 (Theorem 5) we obtain that the most heavy-tailed terms \(\{R_j\}\) determine the heaviness of the tail and the extremal index of both the sum and maximum.

The paper is organized as follows. The revision of Theorems 1 and 2 by Theorems 3 and 4 is given in Section 2. The modification of Theorem 4 by Theorem 5 to find the extremal and tail indices of PageRank and the Max-Linear Model as influence measures of nodes in random direct graphs is presented in Section 3. Conclusions and a discussion are presented in Section 4. The proofs are given in Section 5.

2. Revision of Theorem 1

We revise Theorem 1 by Theorem 4 allowing a random bounded number \(d \geq 1\) of series to have a minimum tail index. To this end, we extend Theorem 2 by Theorem 3. We assume in Theorems 3 and 4 that \(k_i = k_1, i \in \{1, ..., d\}, 1 \leq d \leq l_n - 1, k_1 < k\), where

\[
\kappa := \lim_{n \to \infty} \inf_{d+1 \leq i \leq l_n} k_i,
\]

and that (3), (4) hold.

We introduce the following independence condition:

(A1) The stationary sequences \(\{Y_{n,i}\}_{n \geq 1}, i \in \{1, ..., d\}\) are mutually independent, and independent of the sequences \(\{Y_{n,i}\}_{n \geq 1}, i \in \{d + 1, ..., l_n\}\).

Denote \(M_n^{(i)} = \max\{Y_{1,i}, Y_{2,i}, ..., Y_{n,i}\}, i \in \{1, ..., l_n\}\).

**Theorem 3.** Let (3) hold for all \(d + 1 \leq i \leq l_n\). Then the sequences \(Y_n^{*}(z, l_n)\) and \(Y_n(z, l_n)\) have the same tail index \(k_1\).

1. If, in addition, (A1) holds, then \(Y_n^{*}(z, l_n)\) and \(Y_n(z, l_n)\) have the same extremal index

\[
\theta(z) = \sum_{j=1}^{d} \theta j z_j^{k_j} / \sum_{j=1}^{d} z_j^{k_j}.
\]

2. If, instead, (A1) does not hold and \(d = 1\), then their extremal index is equal to \(\theta_1\), but it may not exist, if \(d > 1\). Assuming for \(d > 1\) that all elements in pairs of the "column" sequences \(\{Y_{n,i}\}_{n \geq 1}, i \in \{1, ..., d\}\) have the same mutual dependence, and

\[
\sum_{j=1}^{d-1} P\{z_j M_n^{(j)} > u_n, z_{j+1} M_n^{(j+1)} \leq u_n, ..., z_d M_n^{(d)} \leq u_n\} = o\{P\{z_d M_n^{(d)} \leq u_n\}\}
\]

holds as \(n \to \infty\), \(Y_n^{*}(z, l_n)\) and \(Y_n(z, l_n)\) have the extremal index \(\theta_d\).

**Remark 1.** Since an enumeration of the first \(d\) "column" sequences is not significant, one can rewrite the condition (14) with regard to the \(i\)th column and obtain the same statement for any \(\theta_i\), \(i \in [1, d]\).

**Corollary 1.** Let \(Y_{n,i} = Y_{n,1}, n \geq 1, i \in \{1, ..., d\}\). Then \(Y_n^{*}(z, l_n)\) and \(Y_n(z, l_n)\) have the same tail index \(k_1\) and the same extremal index \(\theta_1\).
Remark 2. If there are in total $d+1$ stationary mutually independent "column" sequences having the same tail index $k_1$ and extremal indices $\theta_1, ..., \theta_{d+1}$, then $Y_n^*(z,l_n)$ and $Y_n(z,l_n)$ have the tail index $k_1$ and the extremal index that is a superposition of $\theta_1, ..., \theta_{d+1}$ as derived in [13], see Theorem 2 in [13].

Example 1. In case of an arbitrary dependence among elements of the $d$ "columns" series with the minimum tail index, the extremal index of maxima $Y_n^*(z,l_n)$ and $Y_n(z,l_n)$ may not exist. Suppose elements of odd "row" sequences coincide and elements of the even rows are i.i.d.. Then sums and maxima over rows are differently distributed and the sequences $Y_n^*(z,l_n)$ and $Y_n(z,l_n)$ are nonstationary.

Example 2. The assumption (13) is valid for the following $d$ "column" sequences. Let elements of each "column" sequence be sums of corresponding elements of all previous "columns", i.e. $Y_{n,i} = \sum_{j=1}^{d+1} Y_{n,j}$ and $z_1 \leq z_2 \leq ... \leq z_d$. Each of $d$ "column" sequences has the tail index $k_1$ that follows from the first statement of Theorem 3. Since $M_n^{(1)} = M_n^{(2)} < M_n^{(3)} < ... < M_n^{(d)}$ holds, then $\sum_{j=1}^{d+1} P\{z_n M_n^{(j)} > u_n, z_{j+1} M_n^{(j+1)} \leq u_n, ..., z_d M_n^{(d)} \leq u_n\} = 0$ holds.

Now we reformulate Theorem 4.

Theorem 4. Let the sets of slowly varying functions \{\tilde{\ell}_n(x)\}_{n \geq 0} in 4 and \{\ell_i(x)\}_{d+1 \leq i \leq t_n}$ in \ref{equation:small variation} satisfy the condition \ref{equation:small variation}. Assume that $d$ is a bounded discrete r.v. such that $d < d_n = \min(C, l_n)$, $C > 1$ holds, and $d$ and $\{Y_{n,i}\}$ are mutually independent. Let \ref{equation:sums} hold. Then $Y_n^*(z,N_n)$ and $Y_n(z,N_n)$ have the same tail index $k_1$, but their extremal indices do not exist.

Corollary 2. The results of Theorems 3 and 4 remain true if the tail indices $\{k_{n,i}\}$ of elements in the "columns" $\{Y_{n,i}: n \geq 1\}$ are different, apart of those columns with the minimum tail index. The elements of the columns with non-minimum tail indices may be partly light-tailed distributed.

The corollary follows from the proof of Theorem 3 and Lemma 1 in [18].

3. Application to Random Networks

Each node in a random network may be considered as a root of some directed graph of its followers (i.e. nodes with incoming links to the root node) which may contain cycles. Thus, generations of followers may be overlapping. The $r$th generation implies the set of nodes in the tree at distance $r$ from the root node. There may exist links between nodes within generations. Influen characteristics of nodes such as PageRanks or the Max-Linear Models calculated by these generations may be dependent.

Generations may be non-stationary distributed since their nodes may belong to communities with different distributions and the graph is not necessarily fully connected, Fig[11] A community structure is "the organization of vertices in clusters, with many edges joining vertices of the same cluster and comparatively few edges joining vertices of different clusters." [12]. There are methods to extract the community structure of large networks, see, for instance, [4], [19].

Theorem 4 is further reformulated in the context of PageRank and the Max-Linear Model. Denote an in-degree of the node $i$ as $N_i$. $N_i$ determines the random size of the generation of a one-link neighborhood from the node $i$. Let us denote $A_j R(j) = c R(j)/D_j$, $j \in \{1, ..., N_i\}$ in [18] and [10].
as \( z_j Y_{i,j} \) with \( z_j = c \). Then one can rewrite the right-hand sides of (9) and (11) in notations of Theorems 1 and 4 as

\[
Y_i(c, N_i) = c \sum_{j=1}^{N_i} Y_{i,j} + Q_i, \quad Y_i^*(c, N_i) = c \bigvee_{j=1}^{N_i} Y_{i,j} \vee Q_i, \quad i \in \{1, \ldots, n\},
\]

where \( \{Y_{i,j}\} \) are not i.i.d. r.v.s. \( \{Y_{i,j}\} \) and \( Q_i \) as well as \( \{Y_{i,j}\} \) and \( N_i \) are not necessarily mutually independent. \( N_i \) satisfies (7). In the context of PageRank \( Q_i = (1-c)q_i \) holds.

Let us consider the matrix (16) of the array \( \{Y_{n,i} : n, i \geq 1\} \) corresponding to (15), and the tail and extremal indices of its columns are shown in matrix (17):

\[
\begin{pmatrix}
  cY_{1,1} & cY_{1,2} & cY_{1,N_1} & \ldots & 0 & 0 & Q_1 \\
  cY_{2,1} & 0 & cY_{2,3} & \ldots & cY_{n,N_2} & 0 & Q_2 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & cY_{n,2} & cY_{n,3} & \ldots & 0 & cY_{n,N_n} & Q_n \\
\end{pmatrix},
\]

\[
\left( \begin{array}{cccccccc}
  k_1 & k_2 & k_3 & \ldots & k_{N_1} & \beta \\
  \theta_1 & \theta_2 & \theta_3 & \ldots & \theta_{N_n} & 1 \\
\end{array} \right).
\]

The sequence \( \{N_n\} \) is equal to the number of non-zero elements in the rows and it is not necessarily increasing. Row elements in (16) (excluding \( \{Q_i\} \)) correspond to the first generations of followers of nodes that can be arbitrary enumerated as \( 1, 2, \ldots, n \). The column \( i \) contains a community and has the extremal index \( \theta_i \). Zeroes in the rows imply that the first generations do not contain followers from the corresponding communities. "Column" series may be mutually dependent and have tails (11) with tail indices \( \{k_i\} \). The latter columns may (partly) coincide. In random networks the column dependence reflects the fact that communities may be overlapping.

The last column in (16) contains i.i.d. r.v.s of user preferences \( \{Q_i\} \). Theorem 4 is valid if \( \{Q_i\} \) are heavy- or light-tailed distributed. In- and out-degrees of nodes are available statistics in practice, but the user preference has to be simulated [23]. The preference is often expected to be uniformly distributed, i.e. an arbitrary page can be selected by a user uniformly among the nodes of the network. In [23] Pareto distributed \( \{Q_i\} \) were studied by real data. The tail of \( \{N_i\} \) is assumed...
here to be lighter than the regularly varying tails of \{Y_{i,j}\} and \{Q_i\}. We assume that there exist columns with a minimum tail index \(k_1, k_1 < k\).

Let the graph be partitioned into communities of nodes whose PageRanks are stationary regularly varying distributed with tail indices \(\{k_1, k_2, \ldots\}\) and extremal indices \(\{\theta_1, \theta_2, \ldots\}\). We assume that at least one neighbor of a root-node \(i\) belongs to one of the \(d\) most heavy-tailed (dominating) communities (i.e. there exists a type-\(k_1\) child of the ancestor \(i\)). The number of such neighbors is random and they may belong to different dominating communities. It means that all row sequences \(\{Y_{n,i} : n \geq 1\}\) in (16) must have at least one element with the smallest tail index.

**Theorem 5.** PageRanks and the Max-Linear Models of \(n\) root-nodes have the same minimum tail index as one of the \(d\) most heavy-tailed distributed communities within a random graph associated with the roots as \(n \to \infty\). Their extremal index is equal to the extremal index of the most heavy-tailed distributed community if the latter is unique, and it is calculated by (13) using extremal indices of the \(d\) most heavy-tailed communities if the latter are mutually independent and independent of the rest of communities, or it may not exist if the latter independence condition is not valid.

**Remark 3.** There are three important practical assumptions in Theorem 5.

1. By Corollary 2 the node influence indices within non-dominating communities are allowed to be non-stationary distributed. This property allows us to use non-stationary distributed communities if they are not tail dominant.
2. All elements of the dominating communities have to be regularly varying distributed. To our best knowledge, PageRanks are proved to be power law distributed for branching trees in [23], the directed configuration model in [12] and directed generalized random graphs in [5] under Assumptions A. The existence of an asymptotic PageRank distribution for directed graphs is proved assuming a local weak convergence of a sequence of directed graphs to a limiting graph [17]. The conditions under which the PageRank of the root in the limiting graph shows a power-law tail was not found in [11].
3. The dependence between communities impacts the extremal index of PageRanks and the Max-Linear Models of the root-nodes.

Our approach can be used to simulate an enlargement of graphs with given values of the tail and extremal indices of the nodes. To this end, one can simulate communities of nodes with given tail and extremal indices connected to a sequence of root-nodes which do not belong to the communities. If there is a unique community with the minimum tail index among them, then the sequence of the root nodes inherits its tail and extremal indices. Selecting \(d\) mutually independent communities with a minimum tail index \(k_1\) and extremal indices \(\theta_1, \ldots, \theta_d\), PageRank and the Max-Linear Model sequences of the associated root nodes will have the same tail index \(k_1\) and the extremal index \(\theta(z)\) calculated by (13) using \(\theta_1, \ldots, \theta_d\) and taking weights \(z_j = c, j \in \{1, \ldots, d\}\). Adding the sequence of roots as a new community does not change the dependence structure and the heaviness of tail of PageRank and the Max-Linear Model of the next sequence of roots, namely, it will be \((k_1, \theta(z))\).

Since the reciprocal of the extremal index approximates the mean cluster size [14], the extremal index can also be used as the node influence. Namely, a node with the extremal index close to zero may be considered as an influential one since there are generations (the clusters) of followers in its coupled graph with a large number of nodes whose influence measures exceed a sufficiently high
threshold $u$. One can consider the generations as random length blocks that can be overlapping. Theorem 3 is valid both for directed and undirected graphs. The calculation of PageRank by undirected graphs is considered in [2], for instance.

4. Conclusions and discussion

This paper makes a fundamental step forward by extending the analysis of PageRank and the Max-Linear Model to its extremes and to graphs that are not necessarily trees. Our approach to find the asymptotic distributions and local cluster properties of PageRank and the Max-Linear Model is based on results derived for the sums and maxima of non-stationary random length sequences in [18]. The main constraint of these results is that a unique series with the minimum tail index in a scheme of series with regularly varying tails is assumed. This assumption may not be realistic in practice since a node in a random network may have a random number of followers (not necessarily a unique follower) belonging to the most heavy-tailed communities. Another important assumption is that the mutually dependent series are allowed to be regularly varying. Regarding PageRank and the Max-Linear Model the regularly varying assumption of the latter characteristics is crucial, since the power-law tails are proved to our best knowledge only for mutually independent series.

Our results are twofold. At first, we extend Theorems 3 and 4 in [18] by Theorems 3 and 4. Secondly, we interpret these results to PageRank and the Max-Linear Model. Theorem 5 is the analogue of Theorem 4 for random networks.

To our best knowledge, the extremal index of PageRank and the Max-Linear Model is considered at first time. The extremal index may serve as a new influence measure of nodes.

5. Proofs

5.1. Proof of Theorem 3

The proof is similar to that of Theorem 3 [18]. We just indicate the modifications. The numeration of Theorem 3 [18] is preserved throughout the proof. Let us take the same sequence of thresholds $u_n = y n^{1/k} l_t(n)$, $y > 0$ as in [18].

Tail index. At first, we show that the tail index of $Y_n(z)$ and $Y_n^*(z)$ is the same. The right-hand side of (12) in [18] can be rewritten as

$$P\{Y_n(z) > u_n\} \leq P\{\sum_{i=1}^{d} z_i Y_{n,i} > u_n(1 - \varepsilon)\} + \sum_{i=d+1}^{l_n} P\{z_i Y_{n,i} > u_n \varepsilon_i\},$$

where $\sum_{i=1}^{l_n} \varepsilon_i = 1$ holds and $\{\varepsilon_i\}$ is a sequence of positive elements. Let us denote $\varepsilon = \sum_{i=d+1}^{l_n} \varepsilon_i$. One may take $\varepsilon_i, i \in \{d+1, ..., l_n\}$, in such a way to satisfy $\varepsilon_i \to 0$ and $\varepsilon \to 0$ as $n \to \infty$. Moreover, choosing $\{\varepsilon_i = 1/n^{\eta+1}\}, \eta > 0$ as in Lemma 1 in [18], one can derive (13) in [18] substituting 2 by $d + 1$, namely, the following

$$\sum_{i=d+1}^{l_n} P\{z_i Y_{n,i} > u_n \varepsilon_i\} = o(1/n)$$

(19)
as \( n \to \infty \). To prove the latter, we need to assume (3) for \( d + 1 \leq i \leq l_n \). For simplicity, let us consider the case \( d = 2 \). Then by (7) in [18]

\[
P\{z_1 Y_{n,1} > u_n\} = (z_1/y)^{k_1} n^{-1}(1 + o(1)), \quad n \to \infty
\]

and (13) in [18] we obtain

\[
P\{z_1 Y_{n,1} + z_2 Y_{n,2} > u_n(1 - \varepsilon)\}
\leq \frac{n^{-1}}{(y(1 - \varepsilon))^{k_1}} \left[ \left( \frac{z_1}{\varepsilon_1} \right)^{k_1} + \left( \frac{z_2}{\varepsilon_2} \right)^{k_1} \right] (1 + o(1)),
\]

where \( \varepsilon_1^2 + \varepsilon_2^2 = 1 \) holds. Let us denote

\[
(z^*)^{k_1} = \left( \frac{z_1}{\varepsilon_1} \right)^{k_1} + \left( \frac{z_2}{\varepsilon_2} \right)^{k_1}.
\]

By (14) in [18] it follows

\[
P\{Y_n(z) > u_n\} \leq \left( \frac{z^*}{y(1 - \varepsilon)} \right)^{k_1} n^{-1}(1 + o(1)) + o(1/n).
\]

From another side, we have

\[
P\{Y_n(z) > u_n\} \geq P\{z_1 Y_{n,1} > u_n\} = \left( \frac{z_1}{y} \right)^{k_1} n^{-1}(1 + o(1))
\]

and the left-hand side of (9) in [18] does not change. The heaviness of tail of both \( Y_n(z) \) and \( Y^*_n(z) \) coincides. The proof is the same for \( d > 2 \).

**Extremal index for \( d \) independent "column" sequences.** We assume the condition (A1). Denoting

\[
M^*_n(z) = \max\{Y^*_1(z), ..., Y^*_n(z)\} = \max\{z_1 M_n^{(1)}, ..., z_n M_n^{(l_n)}\}, \ n \geq 1,
\]

where \( Y^*_n(z) = Y^*_n(z, l_n) \) as in (18) [18], we have due to independence

\[
P\{M^*_n(z) \leq u_n\} = \prod_{j=1}^{d} P\{z_j M_n^{(j)} \leq u_n\} P\{z_{d+1} M_n^{(d+1)} \leq u_n, ..., z_{l_n} M_n^{(l_n)} \leq u_n\}.
\]

Since the extremal index of the sequence \( \{Y_{n,j}\}_{n \geq 1}, 1 \leq j \leq d \) is assumed to be equal to \( \theta_j \), by (2) and (20) we get

\[
\lim_{n \to \infty} \prod_{j=1}^{d} P\{z_j M_n^{(j)} \leq u_n\} = \exp(- \sum_{j=1}^{d} \theta_j (z_j/y)^{k_1}).
\]
By (19) and (21) in 18 and by (2), we obtain
\[ P\{z_{d+1}M^{(d+1)}_n \leq u_n, \ldots, z_{l_n}M^{(l_n)}_n \leq u_n \} = P\{z_{d+1}M^{(d+1)}_n \leq u_n \}(1 + o(1)) \rightarrow 1, \]
since
\[ nP\{z_{d+1}Y_{n,d+1} > u_n \} \sim n^{-k_{d+1}/k_1} \ell(n) \rightarrow 0 \]
as \( n \rightarrow \infty \) due to \( k_{d+1} > k_1 \). We obtain
\[ \lim_{n \rightarrow \infty} nP\{Y^*_n(z) > u_n \} = \sum_{j=1}^{d} \left( \frac{z_j}{y} \right)^{k_1} \]
since it holds
\[ P\{Y^*_n(z) > u_n \} = P\{\max(z_1Y_{1,1}, \ldots, z_dY_{d,d}, z_{d+1}Y_{n,d+1}, \ldots, z_{l_n}Y_{n,l_n}) > u_n \} \]
\[ = \sum_{i=1}^{d} P\{z_iY_{i,i} > u_n \} + P\{\max(z_1Y_{1,1}, \ldots, z_dY_{d,d}) \leq u_n \} P\{\max(z_{d+1}Y_{n,d+1}, \ldots, z_{l_n}Y_{n,l_n}) > u_n \} \]
\[ = \sum_{j=1}^{d} \left( \frac{z_j}{y} \right)^{k_1} \cdot n^{-1}(1 + o(1)) + o(1/n) \]
due to (9), (12), (13) and Lemma 1 all in 18. Then from
\[ P\{M^*_n(z) \leq u_n \} = \exp(-\sum_{j=1}^{d} d_j(z_j/y)^{k_1})(1 + o(1)) \]  
we obtain that the extremal index of \( Y^*_n(z) \) is equal to 13. In the same way as in the proof of Theorem 3 in 18, one can show that \( Y_n(z) \) has the same extremal index.

**Extremal index for \( d \) arbitrary dependent "column" sequences.** We show that the extremal indices of the sequences \( Y^*_n(z) \) and \( Y_n(z) \) may not exist if the condition (A1) is not valid. If \( d = 1 \), i.e. there is a unique "column" series with the minimum tail index, we are in the conditions of Theorem 2. Let us consider the case \( d > 1 \).

Note, that \( Y^*_n(z) \) and \( Y_n(z) \) may be non-stationary distributed if the pair-wise dependence between elements of "column" sequences is different as in Example 1. Assume further that \( Y^*_n(z) \) and \( Y_n(z) \) are stationary distributed. Let us rewrite (19) in 18 as
\[ P\{M_{1,d} \leq u_n \} - \sum_{i=d+1}^{l_n} P\{z_iM^{(i)}_n \leq u_n \} \leq P\{z_1M^{(1)}_n \leq u_n, \ldots, z_{l_n}M^{(l_n)}_n \leq u_n \} = P\{M^*_n(z) \leq u_n \} \leq P\{M_{1,d} \leq u_n \} \leq P\{z_dM^{(d)}_n \leq u_n \}, \]
\[ \sum_{i=d+1}^{l_n} P\{z_iM^{(i)}_n \leq u_n \} = o(1), \quad n \rightarrow \infty. \]
We get
\[
P\{M_{1,d} \leq u_n\} = 1 - P\{M_{1,d} > u_n\} = 1 - P\{z_d M_{n}^{(d)} > u_n\} - \sum_{j=1}^{d-1} P\{z_j M_{n}^{(j)} > u_n, z_{j+1} M_{n}^{(j+1)} \leq u_n, \ldots, z_d M_{n}^{(d)} \leq u_n\}.
\]

Assuming that (14) holds, the expression
\[
P\{M_{n}^{*}(z) \leq u_n\} = P\{z_d M_{n}^{(d)} \leq u_n\}(1 + o(1)) = \exp(-\theta_d(z_d/y)^{k_1})(1 + o(1))
\]
that is required to obtain the extremal index \(\theta_d\) for \(Y_{n}^{*}(z)\) follows, otherwise not. The same holds for \(Y_n(z)\) since
\[
P\{M_{n}^{*}(z) \leq u_n\} = P\{M_{n}(z) \leq u_n\}(1 + o(1))
\]
can be derived similarly to (23) in [18] due to the common tail index of \(Y_{n}^{*}(z)\) and \(Y_n(z)\).

5.2. Proof of Corollary

The existence of a common tail index \(k_1\) for \(Y_{n}^{*}(z, l_n)\) and \(Y_n(z, l_n)\) can be shown the same way as in the proof of Theorem 3. Since \(Y_{n}^{*}(z, d) = \max_{i=1}^{d} z_i Y_n, 1 = z^{**} Y_{n, 1}\) and \(Y_n(z, d) = \sum_{i=1}^{d} z_i Y_n, 1 = z^{*} Y_{n, 1}\) hold and by (18), (19), (20) we get
\[
P\{Y_n(z, l_n) > u_n\} = (z^{*}/y)^{k_1} n^{-1}(1 + o(1)),
\]
as \(n \to \infty\). The same is valid for \(Y_{n}^{*}(z, l_n)\) since \(z^{**} < z^{*}\). Due to \(M_{1,d} = M_{n}^{(1)} z^{**}\) and by (25), (26) we obtain
\[
P\{M_{n}^{*}(z) \leq u_n\} = P\{M_{n}^{(1)} z^{**} \leq u_n\}(1 + o(1)) = \exp(-\theta_1(z^{**} z_1/y)^{k_1})(1 + o(1))
\]
due to (20). The same is valid for \(M_{n}(z)\) as in Theorem 3 in [18]. Thus, the statement follows.

5.3. Proof of Theorem

Denote \(S_{n,d} = \sum_{i=1}^{d} z_i Y_{n, i}\) and \(S_{n,l_n-d} = \sum_{i=d+1}^{l_n} z_i Y_{n, i}\). Since \(d\) is bounded by \(d_n = \min(C, l_n)\) we get
\[
P\{Y_n(z) > u_n\} = P\{S_{n,d} + S_{n,l_n-d} > u_n\} = \sum_{m=1}^{d_n-1} P\{S_{n,m} + S_{n,l_n-m} > u_n\} P\{d = m\} + o(1), \quad n \to \infty.
\]
In the same way it follows
\[
P\{Y_n(z) > u_n\} \geq P\{S_{n,d} > u_n, 1 \leq d \leq |d_n - 1|\} = \sum_{m=1}^{d_n-1} P\{S_{n,m} > u_n\} P\{d = m\}.
\]
Similarly, one can obtain the lower bound of \(P\{Y_{n}^{*}(z) > u_n\}\) replacing sums by maxima. Applying the same steps of the proof to \(P\{S_{n,m} + S_{n,l_n-m} > u_n\}\) and \(P\{S_{n,m} > u_n\}\) as in Theorem 3 we
obtain that the sequences $Y^*_n(z, l_n)$ and $Y_n(z, l_n)$ have the same tail index $k_1$.

We investigate the extremal indices of $Y^*_n(z, l_n)$ and $Y_n(z, l_n)$ in case of the independence condition (A1). If $d_n = C$ holds, then by (23) and (24) it follows

$$P\{M^*_n(z) \leq u_n\} = \sum_{m=1}^{[C-1]} \exp \left( -\sum_{j=1}^m \theta_j(z_j/y)^{k_1} \right) P\{d = m\}(1 + o(1)) = A(y),$$

$$\lim_{n \to \infty} nP\{Y^*_n(z) > u_n\} = \sum_{m=1}^{[C-1]} \sum_{j=1}^m (z_j/y)^{k_1} P\{d = m\} = \tau(y).$$

The expression

$$\theta(z) = -\ln (A(y))/\tau(y)$$

cannot be considered as an extremal index of $Y^*_n(z)$ due to the presence of the arbitrary constant $y > 0$ which is included in $u_n$. The same result is valid for $Y_n(z)$ due to (27). The same proof follows for $d_n = l_n < C$ because of the majorant converging series.

If (A1) does not hold, then the extremal indices of $Y^*_n(z)$ and $Y_n(z)$ do not exist by similar reasons. In case that (27) is fulfilled, one can indicate only bounds for $P\{Y_n > u_n\}$ and $P\{Y^*_n > u_n\}$ by (21) and (22).

5.4. Proof of Theorem 5

The statement is based on the proof of Theorems 3 and 4. According to (1) and (2) sums $\{Y_i(c, N_i)\}$ and maxima $\{Y^*_i(c, N_i)\}$, $i \in \{1, ..., n\}$ determine PageRanks and the Max-Linear Models of $n$ nodes in the graph, where $Y_i(c, N_i)$ and $Y^*_i(c, N_i)$ are built by PageRanks of neighbors of node $i$ considered as the root of the coupled tree. These neighbors belong to some communities. In the same way as in the proof of Theorem 4 one can get that $Y_n(c, N_n)$ and $Y^*_n(c, N_n)$ have the minimum tail index as the most heavy-tailed distributed communities, let’s say $k_1$, as $n \to \infty$. Following the proof of Theorem 4 in (1) and considering the maxima of sequences $\{Y_i(c, N_i)\}$ and $\{Y^*_i(c, N_i)\}$, $i \in \{1, ..., n\}$ one can confirm that these maxima have the same asymptotic distribution as $n \to \infty$ and thus, PageRanks and the Max-Linear Models of $n$ root-nodes have the extremal index $\theta_1$ of the most heavy-tailed distributed community in case the latter is unique. If the number of the most heavy-tailed distributed communities related to $n$ root-nodes is fixed and the PageRanks of the neighbors from these communities are mutually independent, then according to Theorem 3 the extremal index of the PageRanks and the Max-Linear Models of the roots is calculated by (13). Otherwise, the extremal index of the characteristics of the roots may not exist.

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References

[1] Asmussen, S., Foss, S. (2018). Regular variation in a fixed-point problem for single- and multi-class branching processes and queues. Branching Processes and Applied Probability. Papers in Honour of Peter Jagers. *Adv. Appl. Prob.*, 50A, 47-61.

[2] Avrachenkov, K., Kadavankandy, A. & Litvak, N. (2018). Mean Field Analysis of Personalized PageRank with Implications for Local Graph Clustering. *J. Stat. Phys.*, 173, 895-916.

[3] Barabási, A. L. Network science, Cambridge University Press, 2016

[4] Blondel, V.D., Guillaume, J.-L., Lambiotte, R., Lefebvre, E. Fast unfolding of communities in large networks, Journal of Statistical Mechanics: Theory and Experiment, P10008, 10 (2008)

[5] Chen, N., Litvak, N., Olvera-Cravioto, M. (2014). PageRank in Scale-Free Random Graphs. *WAW 2014, LNCS 8882*, ed. A. Bonato et al. (pp. 120-131). Switzerland: Springer.

[6] Chen, N., Litvak, N., Olvera-Cravioto, M. Ranking algorithms on directed configuration networks (2014)

[7] Fortunato, S. Community detection in graphs. Physics Reports, 486, 3, 75 - 174 (2010)

[8] Jelenkovic, P. R., Olvera-Cravioto, M. (2010). Information ranking and power laws on trees. *Adv. Appl. Prob.*, 42(4), 1057-1093.

[9] Jelenkovic, P. R., Olvera-Cravioto, M. (2015). Maximums on trees. *Stoch. Process. Appl.*, 125, 217-232.

[10] Jessen, A. H., Mikosch, T. (2006). Regularly varying functions. *Publ. Inst. Math. (Beograd) (N.S.)*, 80, 171-192.

[11] Garavaglia, A., van der Hoogstad, R. & Litvak, N. (2020). Local weak convergence for PageRank. *Ann. Appl. Prob.*, 30(1), 40-79.

[12] Gissibl, N., Klüppelberg, C. (2018). Max-linear models on directed acyclic graphs. *Bernoulli*, 24(4A), 2693-2720.

[13] Goldshtik, A. A. (2013). Indices of multivariate recurrent stochastic sequences. In Shiryaev A. N. (ed.): Modern problem of mathematics and mechanics VIII(3), Moscow State University, 42-51. ISBN 978-5-211-05652-7 (in Russian)

[14] Leadbetter, M.R., Lingren, G. & Rootzén, H. (1983). *Extremes and Related Properties of Random Sequence and Processes*. ch.3, New York: Springer.

[15] Lebedev, A. V. (2015). Activity maxima in some models of information networks with random weights and heavy tails. *Problems of Information Transmission*, 51(1), 66-74.

[16] Lebedev, A. V.: Extremal indices in the series scheme and their applications. Informatics Appl. 9(3), 39-54. (in Russian) (2015b)

[17] Lee, J., Olvera-Cravioto, M. PageRank on inhomogeneous random digraphs, Stochastic Processes and their Applications, 130, 4, 2312-2348 (2020)

[18] Markovich, N.M., Rodionov I.V. (2020a). Maxima and sums of non-stationary random length sequences. *Extremes*, 23(3), 451-464. DOI: 10.1007/s10687-020-00372-5

[19] Newman, M. E. J., Girvan, M. Finding and evaluating community structure in networks, Phys. Rev. E, 69, 026113 (2004)

[20] Newman, M. E. J., Girvan, M. Finding and evaluating community structure in networks, Phys. Rev. E, 69, 026113 (2004)

[21] Robert, C.Y., Segers, J. Tails of random sums of a heavy-tailed number of light-tailed terms. *Insurance: Mathematics and Economics*, 43, 85-92 (2008)

[22] Tillier, C., Wintenberger, O.: Regular variation of a random length sequence of random variables and application to risk assessment. *Extremes*, 21, 27-56 (2018)

[23] Volkovich, Y. V., Litvak, N. Asymptotic analysis for personalized web search. *Adv. Appl. Prob.*, 42(2), 577-604 (2010)