Moduli of certain wild covers of curves

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Abstract

A fine moduli space (cf. Section 2) is constructed, for cyclic-by-$p$ covers of an affine curve over an algebraically closed field $k$ of characteristic $p > 0$. An intersection (cf. Definition 31) of finitely many fine moduli spaces for cyclic-by-$p$ covers of affine curves gives a moduli space for $p'$-by-$p$ covers of an affine curve. A local moduli space is also constructed, for cyclic-by-$p$ covers of $Spec(k((x)))$, which is the same as the global moduli space for cyclic-by-$p$ covers of $\mathbb{P}^1 - \{0\}$ tamely ramified over $\infty$ with the same Galois group. Then it is shown that a restriction morphism (cf. Lemma 60) is finite etale: There are finitely many deleted points (cf. Figure 1) of an affine curve from its smooth completion. A cyclic-by-$p$ cover of an affine curve gives a product of local covers with the same Galois group of the punctured infinitesimal neighbourhoods of the deleted points. So there is a restriction morphism from the global moduli space to a product of local moduli spaces. A stack version of the global moduli space is also given.

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1 Introduction

The paper mainly generalizes the results in [H80] for $p$-groups to cyclic-by-$p$ groups defined in Section 2. Cf. Section 2 for notations and terminology below. Since [H80] is frequently cited, the statements of its main results are given in the Introduction.

In [H80], it is shown that (Theorem 1.2) when $P$ is a finite $p$-group, there exists a fine moduli space for pointed principal $P$-covers (cf. Section 2 for the definition) of an affine curve over an algebraically closed field $k$ of characteristic $p > 0$, which is an inductive limit of affine spaces. When $P'$ is a finite group whose order is prime to $p$, there are only finitely many pointed principal $P'$-covers of an affine curve. The wild case, where $p$ divides the order of the Galois group of the cover, and the tame case, where $p$ does not divide the order of the Galois group of the cover, are very different. The fine moduli space for pointed principal $P$-covers of $\mathbb{P}^1 - \{0\}$ gives a coarse moduli space for local pointed principal $P$-covers of $Spec(k((x)))$ (Proposition 2.1). This is a special case of
the next result since the finite etale morphism there becomes an isomorphism here. Finally it is shown that a restriction morphism is finite etale (Proposition 2.7), where the restriction morphism is described in the Abstract with the cyclic-by-$p$ group there replaced by $P$ here. The result can be interpreted as a local-global principle: Given a $P$-local cover at each of the deleted points of the affine curve from its smooth completion, there are only finitely many global $P$-covers of the affine curve, whose restrictions at the deleted points are the ones given.

In Figure 1, points 0 and $\infty$ are the deleted points of $\mathbb{A}^1 - \{0\}$ from its smooth completion $\mathbb{P}^1$. The infinitesimal neighborhood of 0 is $\text{Spec}(k((x)))$ and the infinitesimal neighborhood of $\infty$ is $\text{Spec}(k((x^{-1})))$. $f$ gives a trivial $\mathbb{Z}/2$-cover of $\mathbb{A}^1 - \{0\}$. The restriction of the global cover at 0 is a trivial $\mathbb{Z}/2$-cover of the infinitesimal neighborhood. Similarly for $\infty$.

The fine moduli space for pointed principal $P$-covers of an affine curve in [H80] is constructed in an inductive way with the base case for $P = \mathbb{Z}/p$.

Cyclic-by-$p$ groups are the next simplest after $p$-groups in the wild case. In the local situation, the Galois group of a connected Galois cover of $\text{Spec}(k((x)))$ is a cyclic-by-$p$ group over an algebraically closed field.

The fine moduli space for pointed principal cyclic-by-$p$ covers of an affine curve is also constructed in an inductive way. The fine moduli space is a disjoint union of finitely many direct limits of affine spaces. Its relation to the the fine moduli space for pointed principal $P$-covers of an affine curve constructed in [H80] is shown in Section 4.

The next simplest groups after cyclic-by-$p$ groups are $p'$-by-$p$ groups defined in Section 2. A disjoint union of finitely many unions of several irreducible components in an intersection of finitely many fine moduli spaces for cyclic-by-$p$ covers of affine curves gives a moduli space for $p'$-by-$p$ covers of an affine curve.

Two local-global principal results similar to those in [H80] described above are obtained based on the construction of the fine moduli space for pointed principal cyclic-by-$p$ covers of an affine curve, using similar methods to those in [H80].

Finally a global analogue over a quite general field of characteristic $p > 0$, of the moduli stack for local cyclic-by-$p$ covers of $\text{Spec}(k((x)))$ constructed in [TY17], can be constructed.

Here is the structure of the paper.

In Section 2 notations and terminology are given, which are used throughout Sections 3, 4, and 5 without explanation again. In Section 3 a fine moduli space for
pointed principal $G$-covers of an affine curve (Theorem 24), where $G$ is a cyclic-by-$p$ group, is constructed. In Section 4, it is shown that a disjoint union of finitely many unions of several irreducible components in an intersection of finitely many fine moduli spaces for cyclic-by-$p$ covers of some affine curves gives a moduli space for $p'$-by-$p$ covers of an affine curve (Corollary 38). In Section 5, a global fine moduli space is constructed (Proposition 42) for cyclic-by-$p$ covers of an affine curve tamely ramified over finitely many closed points, as well as a parameter space for local cyclic-by-$p$ covers of Spec$(k((x)))$ (Proposition 53). Then it is shown that a restriction morphism is finite etale, which is from the global moduli space to a product of the local parameter spaces (Proposition 61). [TY17] constructs a moduli stack for local cyclic-by-$p$ torsors of Spec$(k((x)))$, which is an inductive limit of Deligne-Mumford stacks, ind DM-stack for short. In Section 6, a parallel construction to the one in [TY17] gives a moduli stack for global cyclic-by-$p$ torsors of an affine curve over a quite general field of characteristic $p > 0$, which is also an ind DM-stack (Theorem $A'$).

Leitfaden:

Section 2 ↓

Section 3 ↓

Section 4 ▶

Section 5 ▶

Section 6

Besides [TY17], similar work can also be found in [K86] and [P02]. In [K86], Main Theorem 1.4.1 is essentially the version over a general field of characteristic $p > 0$ of Proposition 54. In [P02], a configuration space $C(I,j)$ is constructed in Section 2.2, which is for $I = \mathbb{Z}/p \times \mathbb{Z}/n$-covers of Spec$(k[[u^{-1}]]))$ with jump $j$. This is related to the parameter space given in Proposition 53.

For the results in [H80] below, $(U,u_g), P$ and $k$ are defined in Section 2.

**Theorem 1.** (Theorem 1.2, [H80]) There is a fine moduli space $M_{U,P}$ (denoted by $M_G$ there with $G = P$) for pointed families of principal $P$-covers of $U$, namely a direct limit of affine spaces $A^N_k$.

Cf. Section 2 for the definition of a fine moduli space. Theorem 1 means that $M_{U,P}$ represents the moduli functor $F_{U,P}: S \to \text{(Sets)}; (S,s_0) \mapsto [\tilde{\varphi}]$ where $\tilde{\varphi} : \pi_1(S \times U,(s_0,u_g)) \to P$ is a group homomorphism and $[\tilde{\varphi}]$ is the equivalence class (cf. Remark 6) of $\tilde{\varphi}$.

The local case, moduli problem for $P$-covers of Spec$(k((x)))$, is simpler than the global case. A parallel construction to the one in the global case gives a fine moduli space of pointed $P$-covers of Spec$(k((x)))$.

**Proposition 2.** (Proposition 2.1, [H80]) The fine moduli space $M_{\mathbb{P}^1-\{0\},P}$ for pointed principal $P$-covers of $\mathbb{P}^1-\{0\}$ is also a coarse moduli space for pointed principal $P$-covers of Spec$(k((x)))$, compatibly with the inclusion Spec$(k((x))) \subseteq \mathbb{P}^1-\{0\}$.

3
Proposition 2 means that $M_{P^1} - \{0\}, P$ represents the moduli functor $F^w_{U_0, P}: S_1 \to \text{(Sets)}$; $(S, s_0) \mapsto [\tilde{\varphi}]^w$ where $\tilde{\varphi}: \pi_1(S \times U_0, (s_0, u_0)) \to P$ is a group homomorphism and $[\tilde{\varphi}]^w$ is the w-equivalence class (cf. Definition 44) of $\tilde{\varphi}$.

Proposition 3. (Proposition 2.7, [H80]) Let $M_{U, P^1} \to \Pi_1 M^1_{U_0, P}$ be the restriction morphism described in the Abstract. It is an etale cover. Its degree is a power of $p$, and is equal to the number of pointed principal $P$-covers of the completion $\bar{U}$.

Theorem 4. (Theorem 1.12, [H80]) Let $M_{U, P}$ be the fine moduli space for pointed principal $P$-covers of $(U, u_g)$. There is a natural action of $\text{Aut}(P)$ on $M_{U, P}$, and a dense open subset $M^0_{U, P}$ of $M_{U, P}$ parameterizing connected principal covers, such that $\bar{M}^0_{U, P} := M^0_{U, P}/\text{Aut}(P)$ is a fine moduli space for pointed families of Galois covers of $(U, u_g)$ with group $P$.

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2 Notations and Terminology

Terms and symbols are defined here which will be used in Sections 3, 4, and 5 without being explained again.

2.1 General settings

Groups of the form $P \rtimes_{\rho} \mathbb{Z}/n$ are called cyclic-by-$p$ groups, where $p$ is a prime number, $P$ a finite $p$-group and $\rho: \mathbb{Z}/n \to \text{Aut}(P)$ an action of $\mathbb{Z}/n$ on $P$ with $n$ and $p$ coprime.

Groups of the form $P \rtimes_{\rho'} P'$ are called $p'$-by-$p$ groups, where $P'$ is a finite group whose order is prime to $p$ and $\rho': P' \to \text{Aut}(P)$ an action of $P'$ on $P$.

Let $n_t$ be a factor of $n$, $x_t = n/n_t$, $t_{n_t}$ the embedding of $\mathbb{Z}/n_t$ into $\mathbb{Z}/n$ sending $1 \in \mathbb{Z}/n_t$ to $\bar{x_t} \in \mathbb{Z}/n$, and $\rho_{n_t} = \rho \circ t_{n_t}$.

Let $k$ be an algebraically closed field of characteristic $p > 0$ and fix a primitive $n$-th root of unity $\zeta_n$ in $k$. Write $U_0 = \text{Spec}(k((x)))$ and $\bar{U}_0 = \text{Spec}(k[[x]])$. Denote the fiber product $S \times_k X$ by $S \times X$, where $S$ and $X$ are $k$-schemes.

Denote by $S$ (resp. $S_1$) the full subcategory of the category (Pointed $k$-schemes) of all pointed $k$-schemes, whose objects are connected affine pointed finite type $k$-schemes (resp. connected affine pointed $k$-schemes). Denote by $S'$ (resp. $S'_1$) the non pointed version of $S$ (resp. $S_1$).

Pointed means geometrically pointed unless otherwise stated. A geometric point of a scheme $X$ is a morphism from $\text{Spec}(\Omega)$ to $X$ with $\Omega$ an algebraically closed field. Curve means a connected smooth integral affine 1 dimensional scheme of finite type over $k$.

Gr always represents an arbitrary finite group, $G$ represents $P \rtimes_{\rho} \mathbb{Z}/n$ and $(U, u_g)$ represents a pointed curve.

The word “lift” has two meanings in the paper. The first meaning is expressed in diagram (3.1). The second meaning is to lift a morphism $\phi$ mapping to a quotient group
$ar{P}$, to some morphism $\phi$ mapping to the original group $P$:

$$
\begin{array}{ccc}
\pi_1(U, u_g) & \xrightarrow{\phi} & \bar{P} \\
\downarrow & & \downarrow \\
\phi & & \phi
\end{array}
$$

### 2.2 Covers

Let $X$ be a connected scheme and $(Fet_X)$ the category of finite etale covers of $X$. For every point $x$ of $X$, the fiber functor $\text{Fib}_x: (Fet_X) \to (\text{Sets})$ sends a finite etale cover $Y \twoheadrightarrow X$ to $Y_x$, the geometric fiber (pullback of $Y$ to $x$) of $Y$ at $x$. The fundamental group $\pi_1(X, x)$ is defined as the automorphism group of the fiber functor $\text{Fib}_x$. Then $Y_x$ is a left $\pi_1(X, x)$-set.

A principal $Gr$-cover of a connected scheme $X$ not necessarily over $k$, is a finite etale cover $Z \to X$ together with an embedding of $Gr$ in the group $\text{Aut}(Z/X)$, such that $Gr$ acts simply transitively on every geometric fiber of $Z \to X$. A $Gr$-cover means a principal $Gr$-cover. $Z$ is pointed over $x_0$ means $Z$ is pointed at some point $z_0$ that maps to $x_0$ under $Z \to X$. Two pointed $Gr$-covers of $(X, x_0)$ are isomorphic if there is an isomorphism between them:

$$
\begin{array}{ccc}
(Z, z_0) & \xrightarrow{f \cong} & (Z', z'_0) \\
\downarrow & & \downarrow \\
(X, x_0), & & (X, x_0)
\end{array}
$$

such that the triangle diagram commutes and the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{f} & Z' \\
\downarrow g & & \downarrow g \\
Z & \xrightarrow{f} & Z'
\end{array}
$$

commutes for each $g \in Gr$.

There is a natural bijection between the set of isomorphism classes of $Gr$-covers of $X$ pointed over a fixed base point $x_0$, and the set of homomorphisms from $\pi_1(X, x_0)$ to $Gr$, hence below a pointed $Gr$-cover is often identified with the homomorphism corresponding to it.

If $Gr$ is abelian, the set of homomorphisms from $\pi_1(X, x_0)$ to $Gr$ is a group. It may be identified with the etale cohomology group $H^1(X, Gr)$ [SGA 1, XI 5], or in terms of group cohomology with $H^1(\pi_1(X, x_0), Gr)$.

**Remark 5.** Let $Gr$ be an abelian group and $W \to U$ be a $Gr$-cover of $U$. Then $W$ pointed at a point $w_g$ over $u_g$ is isomorphic to, as pointed $Gr$-covers of $(U, u_g)$, $W$ pointed at any other point $w'_g$ over $u_g$. 

5
A point \((s_0, v_g)\) on a fiber product \(S \times V\) of \(k\)-schemes means a commutative diagram by the universal property of a fiber product:

\[
\begin{array}{ccc}
\text{Spec}(\Omega) & \xrightarrow{v_g} & V \\
\downarrow s_0 & & \downarrow \\
S & \xrightarrow{\pi} & \text{Spec}(k),
\end{array}
\]

where \(\Omega\) is some algebraically closed field.

A pointed family of \(\text{Gr}\)-covers of a pointed connected \(k\)-scheme \(X\), parametrized by a pointed connected affine \(k\)-scheme \(S\), means an equivalence class of pointed \(\text{Gr}\)-covers of \(S \times X\), two being equivalent if they become isomorphic after being pulled back by some finite etale cover \((T, t_0) \to (S, s_0)\).

Remark 6. Two elements \(\tilde{\phi}\) and \(\tilde{\phi}'\) in \(\text{Hom}(\pi_1(S \times U, (s_0, u_g)), \text{Gr})\) are equivalent if their corresponding pointed \(\text{Gr}\)-covers of \((S \times U, (s_0, u_g))\) are equivalent. Denote the equivalence class of \(\tilde{\phi}\) by \([\tilde{\phi}]\).

Remark 7. Using equivalence classes (cf. Definition 11, Definition 15, Definition 17, Definition 23), rather than isomorphism classes, a fine moduli space can be constructed. The definition of equivalence, using finite etale covers instead of etale covers, arises naturally in the proof of Theorem 12.

If \(\text{Gr}\) is abelian, then the set of such pointed families may be identified with \(H^1(S \times X, \text{Gr})/H^1(S, \text{Gr})\) (standard etale cohomology groups).

Suppose \(X\) is a connected scheme and \(x, x'\) two geometric points on \(X\). A chemin \(x' \to x\) means an isomorphism from the fiber functor \(\text{Fib}_x\) to the fiber functor \(\text{Fib}_{x'}\) ([T09], Remark 5.5.3, p171). The fiber functor \(\text{Fib}_x\) is the functor: \((\text{Finite etale covers of } X) \to \text{(Sets)}; Y \mapsto Y_x\), where \(Y_x\) is the geometric fiber of \(Y\) over \(x\). Since the fundamental group \(\pi_1(X, x)\) is defined as the automorphism group of the fiber functor \(\text{Fib}_x\), a chemin \(x' \to x\) : \(\text{Fib}_x \xrightarrow{i} \text{Fib}_{x'}\) induces an isomorphism \(\pi_1(X, x) \simeq \pi_1(X, x'); \alpha \mapsto i\alpha i^{-1}\).

### 2.3 Ind schemes and moduli space

An ind scheme means, in the paper, a direct system of \(k\)-schemes \(\{X_i\}\) indexed by natural numbers with compatible transition \(k\)-morphisms \(\{X_i \xrightarrow{x_i} X_{i+1}\}\).

Remark 8. Every moduli space ind scheme \(M\) in the paper is a disjoint union of finitely many direct limits of affine spaces, then each direct limit of affine spaces is called a connected component of \(M\).

No attempt is made to give the most general definitions about ind schemes. Rather, definitions enough for use below about them are given.

A morphism from an ind scheme \(\{X_m\}\) with transition morphisms \(\{x_m\}\) to another ind scheme \(\{Y_m\}\) with transition morphisms \(\{y_m\}\) is a system of compatible \(k\)-morphisms \(\{f_m\}\) between schemes. The system \(\{f_m\}\) is compatible means that for
every $m$ the following diagram is commutative:

\[
\begin{array}{c}
X_m \xrightarrow{f_m} X_{m+1} \\
\downarrow \quad \quad \downarrow \\
Y_m \xrightarrow{y_m} Y_{m+1}.
\end{array}
\]

The morphism $\{f_m\}$ above is surjective, if there exists some natural number $m_0$ such that for every $m \geq m_0$ the morphism $f_m$ is surjective.

The morphism $\{f_m\}$ above is finite etale, if there exists some natural number $m_0$ such that for every $m \geq m_0$ the morphism $f_m$ is finite etale.

A morphism from a $k$-scheme $X$ to an ind scheme $\{X_m\}$ is a system of $k$-morphisms between schemes $\{g_m : X \to X_m | m \geq m_0 \in \mathbb{N}\}$ compatible in the obvious way.

A fine moduli space $M$ for a contravariant functor $F$ from the category $\mathcal{S}_1$ to the category (Sets), is an ind scheme such that $F$ is isomorphic to the functor $\text{Hom}(\bullet, M) : \mathcal{S}_1 \to \text{Sets}; (S, s_0) \mapsto \{\text{morphisms from } S \text{ to } M\}$.

Comments about several subtle concepts are collected below for reference convenience.

Comments concerning ind schemes include Remark 8.
Comments concerning universal families include Remark 14, Definition 18, Remark 20, Remark 21, and Definition 30.
Comments concerning fine moduli spaces include Remark 6, Remark 13, Remark 8, Remark 21, Remark 22, Remark 29, and Definition 30.

2.4 Table of symbols

Below is a table of symbols, which are used in Sections 3, 4 and 5 without explanation again after their definitions. It gives meanings of symbols and places where they are defined.

| Symbol | Definition |
|--------|-----------|
| $c$ | a fixed element in $\pi_1(U, u_g)$ that maps to $\bar{1}$ under $\theta$; Section 3, beginning |
| $c_i$ | similar to $c$ |
| $c_i'$ | a fixed element in $\pi_1(V_i', v_{i g})$ that maps to $p_i'$ under $\theta_i'$; Section 4, beginning |
| $H$ | an elementary abelian group of order a $p$-power; Lemma 9 |
| $\{pr_i : (V_i, v_i) \to (U, u_g)\}$ | the set of all connected pointed $\mathbb{Z}/n_i$-covers of $(U, u_g)$ with $n_i$ running over factors of $n$; Section 3, beginning |
| $T$ | a finite set of closed points on $U$ not including $u_g$; Section 5, beginning |
| $U^0, U$ | $U - T$; Section 5, beginning |
| $(U_0, u_0)$ | pointed $\text{Spec}(k((x)))$; Section 5, Notation 43 |
| $(V, v_g)$ | a fixed connected pointed $\mathbb{Z}/n$-cover of $(U, u_g)$; Section 3, beginning |
| $(V', v'_g)$ | a fixed connected pointed $P'$-cover of $(U, u_g)$; Section 4, beginning |
| $(V_i', v'_{i g})$ | quotient of $(V', v'_g)$ by $\langle p_i' \rangle$; Section 4, beginning |
| $\{(V_0^0, v_1)\}$ | the set of all connected pointed $\mathbb{Z}/n_l$-covers of $(U^0, u_g)$ with $n_l$ running over factors of $n$; Section 5, beginning |
$V_i$; extension of $V_i^0$ by putting back in the close points over $T$ to $V_i^0$ that are originally missing from $V_i^0$’s smooth completion; Section 5 beginning

$(V_0, v_0)$: a connected pointed $\mathbb{Z}/n_i$-cover of $(U_0, u_0)$ with $n_i$ a factor of $n$; Section 5

Notation 43

$\rho$: an action of $\mathbb{Z}/n$ on $P$; Section 2

$[\hat{\phi}]$: the equivalence class of $\hat{\phi}$; Remark 6

$\rho'$: an action of $P'$ on $P$; Section 2

$\rho'_i$: an action of $\langle p'_i \rangle$ on $P$ given by restriction of $\rho'$; Section 4

$\theta$: the group homomorphism $\pi_1(U, u_g) \to \mathbb{Z}/n$ corresponding to $(V, v_g) \to (U, u_g)$; Section 3 beginning

$\theta_i$: similar to $\theta$

$\theta'$: the group homomorphism $\pi_1(U, u_g) \to P'$ corresponding to $(V', v'_g) \to (U, u_g)$; Section 4 beginning

$\theta'_i$: the group homomorphism $\pi_1(V'_i, v'_g) \to \langle p'_i \rangle$ corresponding to $(V', v'_g) \to (V'_i, \overline{v'_g})$; Section 4 beginning

3 Existence of moduli space for cyclic-by-$p$ covers

In Section 3, a fine moduli space which represents the functor $F_{U, G}$ defined above Theorem 24 for pointed $G$-covers of the pointed affine curve $(U, u_g)$, where $G$ is a cyclic-by-$p$ group, is constructed. The construction is done in 3 steps: Theorem 12 $\Rightarrow$ Theorem 19 $\Rightarrow$ Theorem 24. Theorem 12 is the base case of an induction, and Theorem 19 is the inductive step. Theorem 24 collects building blocks given in Theorem 19 to build the target fine moduli space.

As always, we follow notations and terminology defined in Section 2. For example $G$ represents a cyclic-by-$p$ group.

Here are necessary settings for Theorem 12

Since $(n, p) = 1$, for every factor $n'$ of $n$, there are only finitely many connected pointed $\mathbb{Z}/n'$-covers of $(U, u_g)$, up to isomorphism. Cf. also Remark 5.

Denote these covers by $pr_i : (V_i, v_i) \to (U, u_g)$, for all $n'$’s. For each $i$, $pr_i : (V_i, v_i) \to (U, u_g)$ is of some degree $n_i | n$ and corresponds to some group homomorphism $\pi_1(U, u_g) \to \mathbb{Z}/n_i$; fix a $c_i \in \pi_1(U, u_g)$ which maps to $\bar{1} \in \mathbb{Z}/n_i$ under $\theta_i$. Pick a $pr : (V, v_g) \to (U, u_g)$ which is a $\mathbb{Z}/n$-cover. Suppose it corresponds to $\pi_1(U, u_g)$ which is a $\mathbb{Z}/n$-cover. There is a short exact sequence of groups

$$1 \to \pi_1(V, v_g) \to \pi_1(U, u_g) \xrightarrow{\theta} \mathbb{Z}/n \to 1.$$

Let $\text{Hom}(\pi_1(V, v_g), P)$ be the set of group homomorphisms from $\pi_1(V, v_g)$ to $P$. A group homomorphism $\phi \in \text{Hom}(\pi_1(V, v_g), P)$ is $\rho$-liftable if there exists a group homomorphism $\hat{\phi}$ such that the diagram

$$\begin{array}{ccc}
\pi_1(V, v_g) & \xrightarrow{\phi} & P \\
\downarrow & & \downarrow \\
\pi_1(U, u_g) & \xrightarrow{\hat{\phi}} & G \\
\end{array} \xrightarrow{Qr} \mathbb{Z}/n$$

\[\text{(3.1)}\]
commutes and the bottom horizontal arrow $\pi_1(U,u_g) \to \mathbb{Z}/n$ is $\theta$, where $Q_P$ is the projection map. We also say that $\hat{\phi}$ lifts $\phi$. There are two different meanings of “lift” in the paper, cf. Section 2. In this situation, the pointed $P$-cover of $(V,v_g)$ corresponding to $\phi$ is called a pointed $\rho$-liftable cover of $(V,v_g)$. If $\hat{\phi}(c) = (p,\bar{1})$ for some $p \in P$ then $(\phi,p)$ is called a $\rho$-liftable pair.

Let $(S,s_0) \in S_1$. When $(V,U)$ is replaced by $(S \times V, S \times U)$, similarly a $\rho$-liftable $\hat{\phi} \in \text{Hom}(\pi_1(S \times V, (s_0, v_g)), P)$ is defined; the pointed family of $P$-covers of $V$ parameterized by $S$ corresponding to $\hat{\phi}$ is called a pointed $\rho$-liftable family.

Denote by $c_\ast$ the image of $c$ under the group homomorphism $\pi_1(U,u_g) \to \pi_1(S \times U, (s_0, u_g))$ induced by $U \hookrightarrow S \times U$. Similarly a $\rho$-liftable pair $(\hat{\phi}, p)$ is defined. A pointed $\rho$-liftable family pair means a pair whose first entry is the pointed $\rho$-liftable family corresponding to $\hat{\phi}$ and the second entry $p$, for some $(\phi,p)$ a $\rho$-liftable pair. The pair is also denoted by $(\hat{\phi}, p)$.

Below are two Lemmas for Theorem 12.

Irreducible linear representations of $\mathbb{Z}/n$ over the field $\mathbb{F}_p$ correspond to the direct summands in $\mathbb{F}_p[x]/(x^n - 1) = \oplus_i \mathbb{F}_p[x]/(f_i(x))$, where $f_i(x)$'s are irreducible factors of $x^n - 1$ over $\mathbb{F}_p$. The action of $\bar{1} \in \mathbb{Z}/n$ on $\mathbb{F}_p[x]/(f_i(x))$ is multiplication by $[x]$, where $[x]$ means the equivalence class of $x$. Thus for a map $(H, \rho)$ in the case of Lemma 9 there is a group isomorphism $H \xrightarrow{\sim} \mathbb{F}_q$, where $q = p^m$, such that the induced action of $\rho(-\bar{1})$ on $\mathbb{F}_q$ is the multiplication by some $e_\rho \in \mathbb{F}_q$ with $e_\rho = 1$.

**Lemma 9.** Let $P = H = (\mathbb{Z}/p)^m$, an elementary abelian group, and suppose the action $\rho$ on $H$ is irreducible (i.e. $\rho$ can not be an action on any subgroup of $H$). A group homomorphism $\phi \in \text{Hom}(\pi_1(V,v_g), H)$ is $\rho$-liftable iff for every $b \in \pi_1(V,v_g)$

\[
\phi(c^{-1}bc) = \rho(-\bar{1})(\phi(b)).
\]

Moreover, if $\rho = 1$, there is only one $\hat{\phi}$ which can lift $\phi$, and in this case $\hat{\phi}(c) = (\bar{n}_1 \phi(c^n), \bar{1})$, where $\bar{n}_1$ is a natural number such that $\bar{n}_1 n \equiv 1 \mod \rho$. If $\rho \neq 1$, there is a set $\{\hat{\phi}_h | h \in H\}$ consisting of $|H|$ elements which can all lift $\phi$ and in this case $\hat{\phi}_h(c) = (h, \bar{1})$.

**Proof.** Only if : If there is a $\hat{\phi}$ fitting in the diagram of (3.1), then $\phi(c^{-1}bc) = \hat{\phi}(c)^{-1} \phi(b) \hat{\phi}(c)$. Since $\hat{\phi}(c) = (h, \bar{1})$ for some $h \in H$, $\phi(c^{-1}bc) = (h, \bar{1})^{-1} \phi(b)(h, \bar{1}) = \rho(-\bar{1})(\phi(b))$.

If : Suppose $(\ast)$ holds. For every element $h \in H$ define a map $\hat{\phi}_h : \pi_1(U,u_g) \to H \times_\rho \mathbb{Z}/n$ by $\hat{\phi}_h(bc^i) = \phi(b)(h,\bar{1})^i$. The map is well defined since every element in $\pi_1(U,u_g)$ can be written uniquely in the form $bc^i$ with $b \in \pi_1(V,v_g)$ and $0 \leq i \leq n-1$. Such $\hat{\phi}_h$’s are not necessarily homomorphisms; they make the diagram commute. The map $\hat{\phi}_h$ is a homomorphism iff $\phi(c^n) = (h,\bar{1})^n$. If $\rho = 1$, $(h,\bar{1})^n = (nh,0)$. Then there is a unique $h_0 = n^{-1} \phi(c^n) \in H$ such that $\hat{\phi}_{h_0}$ is a homomorphism. If $\rho \neq 1$, the condition automatically holds since both sides equal 0. One can compute $(h,\bar{1})^n = 0$ using $\rho(-\bar{1})(h) = e_\rho h$. Hence for every $h \in H$, $\hat{\phi}_h$ is a homomorphism. \(\square\)
Here is the second lemma needed in the proof of Theorem 12.
Let $\sigma$ be the automorphism in $\text{Gal}(V/U)$ corresponding to $1 \in \mathbb{Z}/n$. Since $U$ and $V$ are affine, $U = \text{Spec}(A)$ and $V = \text{Spec}(B)$ for some rings $A$ and $B$. Then $\sigma$ corresponds to a ring automorphism $\sigma \in \text{Gal}(B/A)$.

**Lemma 10.** A group homomorphism $\phi \in \text{Hom}(\pi_1(V, v_g), H)$ satisfies condition $(\ast)$ of Lemma 9 iff $\phi$ makes the diagram commutative:

$$
\begin{array}{ccc}
\pi_1(V, v_g) & \xrightarrow{\phi} & H \\
\sigma \downarrow & & \downarrow \rho(-1) \\
\pi_1(V, v_{g_1}) & \xrightarrow{\phi_1} & H,
\end{array}
$$

where $v_{g_1}$ is the image of $v_g$ under $\sigma$, $\sigma \ast$ induced by $\sigma$ and $\phi_1$ induced from $\phi$ using any chemin $v_g \rightarrow v_{g_1}$.

**Proof.** Since $H$ is abelian, any chemin $v_g \rightarrow v_{g_1}$ gives the same isomorphism $\pi_1(V, v_{g_1}) \simeq \pi_1(V, v_g)$, thus induces the same $\phi_1$ from $\phi$.

Denote by $\text{Fib}_{b_0}$ (resp. $\text{Fib}_{b_1}$) the fiber functor from (Finite etale covers of $V$) to (Sets) at $v_g$ (resp. $v_{g_1}$). Similarly denote by $\text{Fib}_{u_0}$ the fiber functor from (Finite etale covers of $U$) to (Sets) at $u_g$. Denote by $\text{PL}_V$ the pullback functor from (Finite etale covers of $V$) to (Finite etale covers of $V$) using $V \to U$. There are canonical isomorphisms $i_0$ from $\text{Fib}_{b_0} \circ \text{PL}_V$ to $\text{Fib}_{b_0}$ and $i_1$ from $\text{Fib}_{b_1} \circ \text{PL}_V$ to $\text{Fib}_{b_0}$.

The chemin $c \in \pi_1(U, u_g)$ maps to $\bar{1}$ under $\theta$, and $\bar{1} \in \mathbb{Z}/n$ corresponds to $\sigma \in \text{Gal}(V/U)$, which sends $v_g$ to $v_{g_1}$. And since every finite etale cover of $V$ composed with $V \to U$ is a finite etale cover of $U$, $c \in \pi_1(U, u_g)$ induces $c_{01}$ a chemin $v_{g_1} \rightarrow v_g$. So the first and the last squares of the diagram below commute:

$$
\begin{array}{ccc}
\text{Fib}_{b_0} \circ \text{PL}_V & \xrightarrow{\text{Fib}_{b_1} \circ \text{PL}_V} & \text{Fib}_{b_1} \circ \text{PL}_V \\
\downarrow i_0 & & \downarrow i_1 \\
\text{Fib}_{b_0} & \xrightarrow{\pi \ast} & \text{Fib}_{b_0} \\
\downarrow c & & \downarrow \pi_1 \circ i_1 \\
\text{Fib}_{b_0} & \xrightarrow{\pi \ast} & \text{Fib}_{b_0} \\
\downarrow \pi_1 \circ i_0 & & \downarrow \pi_1 \circ i_1 \\
\text{Fib}_{b_0} & \xrightarrow{\pi \ast} & \text{Fib}_{b_0} \\
\end{array}
$$

Since $\pi \ast (b) = \pi \ast (\sigma \ast) (b)$ for every $b \in \pi_1(V, v_g)$,

$$
\begin{array}{ccc}
\pi_1(V, v_g) & \xrightarrow{\sigma \ast} & \pi_1(V, v_{g_1}) \\
\downarrow \pi_1 \circ i_0 & & \downarrow \pi_1 \circ i_1 \\
\pi_1(U, u_g). & & \\
\end{array}
$$

The middle square of the diagram above commutes.

Hence the whole 2 by 4 diagram is commutative which shows $\phi(c_{01}^{-1} bc) = \phi(c_{01}^{-1} \sigma \ast (b)c_{01})$. Since $\phi(c_{01}^{-1} \sigma \ast (b)c_{01}) = \phi_1(\sigma \ast (b))$, the lemma follows. \qed

The two lemmas above are used to prove Theorem 12, the first step in the three step construction of the fine moduli space in Theorem 24.
**Definition 11.** Define $F_{V,H}^\rho: \mathcal{S}_1 \to (\text{Sets})$ as the contravariant functor given by $F_{V,H}^\rho(S, s_0) = \{[\phi] \mid \phi: \pi_1(S \times V, (s_0, v_g)) \to H \text{ is } \rho\text{-liftable}\}$, the set of $\rho$-liftable families of $H$-covers of $V$ parameterized by $S$ pointed over $(s_0, v_g)$.

Let $S = \text{Spec}(k)$ with $s_0$ determined by $v_g$ using diagram (2.1). Then $F_{V,H}^\rho(S, s_0)$ is the set of all $\rho$-liftable pointed $H$-covers of $(V, v_g)$.

**Theorem 12.** Let $H$ be an elementary abelian group $(\mathbb{Z}/p)^n$, $\rho: \mathbb{Z}/n \to \text{Aut}(H)$ an irreducible action of $\mathbb{Z}/n$ on $H$, and $V \to U$ as above in this section. There is a fine moduli space $M_{V,H}^0$ representing $F_{V,H}^\rho$, the functor for pointed $\rho$-liftable $H$-covers of $(V, v_g)$, which is a direct limit of affine spaces

$$\lim_{i \in \mathbb{N}} \mathbb{A}^{N_i}.$$ 

**Proof.** In the proof, we will pass between $\mathbb{F}_q$ and $H$ freely using the isomorphism $\tau$ between them given above Lemma 9.

Let $F = F_{V,H}^\rho$.

The Artin-Schreier short exact sequence $0 \to \mathbb{F}_q \to G_a \xrightarrow{\ell} G_a \to 0$, where $\phi(f) = f^q - f$, yields $H^0(V, \mathcal{O}) \xrightarrow{\pi} H^0(V, \mathcal{O}) \to H^1(V, \mathbb{F}_q) \to 0$, where $0 = H^1(V, \mathcal{O})$. This is a short exact sequence of $\mathbb{F}_q$-vector spaces. Let $X$ be the subset of $\text{Hom}(\pi_1(V, v_g), H)$ of all the pointed $\rho$-liftable $H$-covers of $(V, v_g)$.

By Lemma 10 $\phi \in X$ iff $\phi_1 \circ \sigma_\phi = e_\rho \phi$. Let $\sigma^* : H^1(V, \mathbb{F}_q) \to H^1(V, \mathbb{F}_q)$ be induced by $\sigma$; it is a homomorphism of $\mathbb{F}_q$-vector spaces. Identify $H^1(V, \mathbb{F}_q)$ with $\text{Hom}(\pi_1(V, v_g), \mathbb{F}_q)$. By definition of $\sigma^*$, $\sigma^*(\phi) = \phi_1 \circ \sigma_\phi$. So $\phi \in X$ iff

$$\sigma^*(\phi) = e_\rho \phi,$$  

which shows that $X$ is an $\mathbb{F}_q$-subspace of $H^1(V, \mathbb{F}_q)$.

There is a commutative diagram consisting of two short exact sequences of $\mathbb{F}_q$-vector spaces with every symbol already defined above:

$$B = H^0(V, \mathcal{O}) \xrightarrow{\phi} H^0(V, \mathcal{O}) \xrightarrow{\pi} H^1(V, \mathbb{F}_q) \xrightarrow{\varphi} 0$$ (12.1)

which comes from a commutative diagram consisting of two Artin-Schreier short exact sequences of sheaves:

$$0 \to \mathbb{F}_q \xrightarrow{\sigma} G_a \xrightarrow{\varphi} G_a \xrightarrow{\sigma} 0$$

where $\mathbb{F}_q \xrightarrow{\sigma} G_a$ is induced by $V \xrightarrow{\sigma} V$ and similarly for $G_a$.

Let $b \in B$. By the right square of diagram (12.1), (1) implies

$$\phi := \pi b \in X \iff \sigma b = e_\rho b + \varphi B.$$  

(2)
Define $D = \sigma - e_\rho : B \to B$, an $A$-module endomorphism of $B$, where $e_\rho$ acts on $B$ by multiplication. Similarly to the proof of Theorem 1.2 in [H80], there is an exact sequence $KerD \xrightarrow{\sigma} KerD \xrightarrow{\phi} \mathbb{X} \to 0$, of $\mathbb{F}_q$-vector spaces. (Denote the restriction of $\phi$ (resp. $\pi$) to $KerD$ also by $\phi$ (resp. $\pi$).

Now construct $M^\rho_{V,H}$ using the $Ker$ short exact sequence above. Let $(KerD)_n = KerD \cap H^0(V, q^n Div_V)$, where $Div_V = \Sigma P_i$ the sum of all the closed points in $V - V$ and $V$ is the smooth completion of $V$. There is a $k$-vector space filtration $(KerD)_n \leq (KerD)_{n-1} \leq ... \leq (KerD)_{1} \leq ...$. Let $\mathbb{X}_n = \pi((KerD)_n)$. There is a short exact sequence $(KerD)_{n-1} \xrightarrow{\sigma} (KerD)_n \to \mathbb{X}_n \to 0$ obtained from the similar one above. Inductively choose bases $K_n$ of each $(KerD)_n$ as a finite dimensional $k$-vector space, such that $K_{n+1}$ includes both $K_n$, and a maximal linearly independent subset of $\{q^\sigma f \in K_n - K_{n-1}\}$. This is the way to choose bases inductively in a similar situation in the proof of Theorem 1.2 in [H80]. The restriction of $\pi$ to the $k$-linear span $(K_i - K_{i-1})_k$ of $K_i - K_{i-1}$ is an isomorphism of $\mathbb{F}_q$-vector spaces $(K_i - K_{i-1})_k \xrightarrow{\pi} \mathbb{X}_i$, which gives a $k$-vector space structure to $\mathbb{X}_i$.

Let $(S, s_0) \in S$ with $S = Spec(R)$. Similarly there is a commutative diagram consisting of two short exact sequences of $\mathbb{F}_q$-vector spaces:

\[
\begin{array}{cccc}
H^0(S \times V, \mathcal{O}) & \xrightarrow{\rho} & H^0(S \times V, \mathcal{O}) & \xrightarrow{\Pi} H^1(S \times V, \mathbb{F}_q) \to 0 \\
\sigma & & & \sigma \\
\downarrow & & & \downarrow \\
\hat{\sigma} & & & \hat{\sigma}
\end{array}
\]

where $\hat{\sigma}$ is an $R \otimes_k A$-module endomorphism: $R \otimes_k B \to R \otimes_k B, r \otimes b \mapsto r \otimes (\sigma(b))$ and $\phi : r \otimes b \mapsto (r \otimes b)^\sigma - r \otimes b$. Let $\hat{D} = \hat{\sigma} - e_\rho$. As above, there is a short exact sequence of $\mathbb{F}_q$-vector spaces $KerD \xrightarrow{\sigma} KerD \xrightarrow{\Pi} \widehat{\mathbb{X}} \to 0$, where $\widehat{\mathbb{X}}$ denotes $\{\hat{\phi} \in H^1(S \times V, \mathbb{F}_q) | \hat{\sigma}^* (\hat{\phi}) = e_\rho \hat{\phi}\}$ and $\hat{\sigma} \in Gal(S \times V/S \times U)$ corresponds to $\hat{\sigma}$. One can check that $Ker\hat{D} = R \otimes_k KerD$.

If two $S$-parametrized $P$-covers of $V$ pointed over $(s_0, v_g)$ are equivalent, they are considered the same element in $F(S, s_0)$, by definition of $F$. Hence $F(S, s_0) = \frac{\mathbb{X} + H^1(S, \mathbb{F}_q)}{H^1(S, \mathbb{F}_q)}$, cf. Section 3. The automorphism $\hat{\sigma}$ of $S \times V$ does not change the $S$-factor, thus for any $\hat{\phi} \in H^1(S, \mathbb{F}_q), \hat{\sigma}^* (\hat{\phi}) = \hat{\phi}$. If $e_\rho \neq 1$, $\widehat{\mathbb{X}} \cap H^1(S, \mathbb{F}_q) = 0$ and $F(S, s_0) = \mathbb{X} = \Pi(Ker\hat{D}) = \Pi(R \otimes_k KerD)$. Let the transition map from $R \otimes_k (K_n - K_{n-1})_k$ to $R \otimes_k (K_{n+1} - K_n)_k$ be Frobenius $(r \otimes b \mapsto (r \otimes b)^\eta)$. Then $\lim_{\to} R \otimes_k (K_n - K_{n-1})_k$ is an $\mathbb{F}_q$-vector space. There is an $\mathbb{F}_q$-vector space isomorphism $\lim_{\to} R \otimes_k (K_n - K_{n-1})_k \xrightarrow{\Pi} (R \otimes_k KerD)$. Hence $F(S, s_0) = \lim_{\to} R \otimes_k (K_n - K_{n-1})_k$. Write out elements in $K_n - K_{n-1}$ as $\{x_1, \ldots, k_d\}$. Now $R \otimes_k (K_n - K_{n-1})_k = Hom_k((K_n - K_{n-1})_k, R) = Hom_k(S, \mathbb{A}((K_n - K_{n-1})_k))$, where $\mathbb{A}((K_n - K_{n-1})_k) = Spec(k[x_1, \ldots, x_d])$ with $\langle x_1, \ldots, x_d \rangle_k$ the dual space of $\langle x_1, \ldots, x_d \rangle_k$. Therefore $M^\rho_{V,H} := \lim_{\to} \mathbb{A}((K_n - K_{n-1})_k)$, where the transition morphism between $\mathbb{A}((K_n - K_{n-1})_k)$ and $\mathbb{A}((K_n - K_{n-1})_k)$ is given by Frobenius, represents $F$.

If $e_\rho = 1$, then $F(S, s_0) = \frac{H^1(S \times U, \mathbb{F}_q)}{H^1(S, \mathbb{F}_q)}$. This is the case, if $H = \mathbb{Z}/p$, of the base step in the proof of Theorem 1.2 in [H80]; the proof there also works for any elementary abelian group $H$. Hence $F$ is represented by $M^\rho_{U,H} := M_{U,H}$, which is denoted by $M_G$.
there with $G = H$, a direct limit of affine spaces with transition morphisms given by Frobenius as well. Since now $G$ is a product $H \times \mathbb{Z}/n$, it can be derived directly that $F$ is represented by $M_{U,H}$.

**Remark 13.** By Theorem[12] for any pointed affine connected $k$-scheme $(S, s_0)$, there is a bijection between $F(S, s_0)$ and $M^0_{V,H}(S)$, where the latter set is the set of $k$-morphisms from $S$ to $M^0_{V,H}$.

Let $S = Spec(k)$ with $s_0$ determined by $v_\rho$ using diagram (21). Then $F(S, s_0)$, the set of all $\rho$-liftable pointed $H$-covers of $(V, v_\rho)$, are in bijection with $M^0_{V,H}(S)$, the set of $k$-points of $M^0_{V,H}$, same as the set of closed points of $M^0_{V,H}$.

**Remark 14.** (Remark/Definition)

With the same notations of Theorem[12]

Let $M^0_{V,H,n}$ be the $n$-th piece of $M^0_{V,H}$. A compatible system of covers of $V$ over $M^0_{V,H}$ means a collection of covers $\{H$-covers $\tilde{Z}_n$ of $M^0_{V,H,n} \times V \mid n \geq 1\}$ such that $\tilde{Z}_n$ pulled back to $M^0_{V,H,n-1} \times V$ is isomorphic to $\tilde{Z}_{n-1}$. Since $H$ is abelian, given any point $m$ on $M^0_{V,H,n}$, where we point $\tilde{Z}_n$ over $(m, v_\rho)$ does not matter by Remark 5.

A universal family representative over the moduli space $M^0_{V,H}$ means, a compatible system of covers $\{H$-covers $\tilde{Z}_n$ of $M^0_{V,H,n} \times V \mid n \geq 1\}$ of $V$ over $M^0_{V,H}$, which can be used to give the isomorphism of functors $M^0_{V,H} \xrightarrow{\approx} F^0_{V,H}$ given in Theorem 12 as follows: Sending a $k$-morphism from a $k$-scheme $S$ with $(S, s_0) \in S_1$ to $M^0_{V,H}$, to the equivalence class of the pullback of $\tilde{Z}_n$ using the morphism to $S \times V$ pointed arbitrarily over $(s_0, v_\rho)$, is the isomorphism of functors $M^0_{V,H} \xrightarrow{\approx} F^0_{V,H}$ given in Theorem 12. Since the $\tilde{Z}_n$'s are compatible any $n$ can be used.

It is derived from definitions that any two universal family representatives are equivalent. There must be a universal family representative: Let $S$ be $M^0_{V,H,n}$. Identity morphism of $M^0_{V,H,n}$ determines a morphism $S \to M^0_{V,H}$. The morphism gives an equivalence class in $F^0_{V,H}(S, m)$ using $M^0_{V,H} \xrightarrow{\approx} F^0_{V,H}$ given in Theorem 12 for any point $m$ on $S$. Then $\rho$-liftable representatives in the equivalence class are candidates for the $n$-th element of a universal family representative. Use the same kind of argument as in Lemma 4.25 of [TY17], a compatible system of covers can be chosen.

If $\rho \neq 1$, a universal family representative over the moduli space $M^0_{V,H}$ can be given by $\{H$-cover of $M^0_{V,H,n} \times V$ given by $z^q - z = \sum k_i \otimes k_i \mid n \geq 1\}$, by the construction. The $H$-covers are compatible for different $n$'s.

If $\rho = 1$, similarly a universal family representative over $M^0_{V,H}$ can be given explicitly: Replace $z^q - z = \sum k_i \otimes k_i$ above by $z^q - z = \sum l_i \otimes l_i$. $l_i$ an analogue of $K_i$, is the basis chosen inductively for $A_n/k^+$ in the proof of Theorem 1.2 in [H80]; here $A_n = H^0(U, q^n Div_U)$ with $p$ there replaced by $q$ and $B_n$ there is denoted by $K$ here.

The universal family representative over $M^0_{V,H}$ given above is the canonical universal family representative over $M^0_{V,H}$. Every other universal family representative over $M^0_{V,H}$ differs from the canonical one by an element in $H^1(M^0_{V,H}, H)$, cf. the proof of Theorem 12.

The corollary below is a version of Theorem 12 for pairs, which will be used in the proof of Theorem[19].
Definition 15. Define $F^\bullet_{V,H}: S_1 \to \text{(Sets)}$ as the contravariant functor given by $F^\bullet_{V,H}(S, s_0) = \{([\tilde{\phi}], h) \mid \tilde{\phi} : \pi_1(S \times V, (s_0, v_g)) \to H \text{ and } (\tilde{\phi}, h) \text{ is a } \rho\text{-liftable pair}\}$, the set of $\rho$-liftable family pairs of $H$-covers of $V$ parameterized by $S$, pointed over $(s_0, v_g)$.

Let $S = \text{Spec}(k)$ with $s_0$ determined by $v_g$ using diagram (2.1). Then $F^\bullet_{V,H}(S, s_0)$ is the set of $\rho$-liftable pairs of $V$.

Corollary 16. Under the same setting of Theorem 12, there is a fine moduli space $M^\bullet_{V,H}$ representing $F^\bullet_{V,H}$, the functor for $\rho$-liftable pairs of $V$. It is a disjoint union of finitely many copies of $M^\rho_{V,H}$ in Theorem 12.

Proof. Let $M^\bullet_{V,H}$ be $M^\rho_{V,H}$ if $\rho = 1$ and $\Pi_{h \in H} M^\rho_{V,H,h}$ if $\rho \neq 1$, where $M^\rho_{V,H,h}$ means a copy of $M^\rho_{V,H}$ indexed by $h$.

Let $\phi \in \text{Hom}(\pi_1(V, v_g), H)$ and $(\phi, h_0)$ be a $\rho$-liftable pair. The map $h_0$, as defined in the proof of Lemma 9, is in fact a homomorphism. As stated in Lemma 9 if $\rho = 1$ then $h_0$ is the only element in $H$ such that $(\phi, h_0)$ is a $\rho$-liftable pair. If $\rho \neq 1$ then for every $h \in H$ the pair $(\phi, h)$ is $\rho$-liftable. Using this fact and Theorem 12 $F^\bullet_{V,H}$ is represented by $M^\bullet_{V,H}$.

By Corollary 16 $M^\bullet_{V,H}$ is a connected component of the ind scheme $M^\bullet_{V,H}$. Cf. Remark 8.

Here is the 2nd step of the 3 step construction of the fine moduli space in Theorem 24. Let $P$ be an arbitrary finite $p$-group now.

Definition 17. Let $F^\bullet_{V,P}: S_1 \to \text{(Sets)}$ be the contravariant functor given by $F^\bullet_{V,P}(S, s_0) = \{([\tilde{\phi}], p) \mid \tilde{\phi} : \pi_1(S \times V, (s_0, v_g)) \to P \text{ and } (\tilde{\phi}, p) \text{ is a } \rho\text{-liftable pair}\}$, the set of $\rho$-liftable family pairs of $P$-covers of $V$ parameterized by $S$, pointed over $(s_0, v_g)$.

Let $S = \text{Spec}(k)$ with $s_0$ determined by $v_g$ using diagram (2.1). Then $F^\bullet_{V,P}(S, s_0)$ is the set of $\rho$-liftable pairs of $V$.

Definition 18. A similar definition for pairs to a universal family representative over $M^\bullet_{V,P}$ in Remark 14 will be given.

Assume there is an ind scheme $M^\bullet_{V,P}$, consisting of finitely many connected components (cf. Remark 8), representing $F^\bullet_{V,P}$ with an isomorphism between functors $M^\bullet_{V,P} \to F^\bullet_{V,P}$.

Below $M^\bullet_{V,P}$ is viewed as a scheme instead of an ind scheme, cf. Remark 21. Connected components of $M^\bullet_{V,P}$ are denoted by $\{M^\rho_{V,P,j}\}$.

A system of universal family pair representatives over $M^\bullet_{V,P}$, means a collection of a $\rho$-liftable pair $(\tilde{\phi}_{0,j,m_j} : \pi_1(M^\rho_{V,P,j} \times V, (m_j, v_g)) \to (\tilde{P}, p_{\tilde{\phi}_{0,j,m_j}})$ for every base point $m_j$ over each $M^\rho_{V,P,j}$, which can be used to give the isomorphism of functors $M^\bullet_{V,P} \to F^\bullet_{V,P}$ as follows: Sending a $k$-morphism $S \to M^\rho_{V,P,j}$ with $(S, s_0) \in S_1$ and $s_0$ mapped to $m_j$ under $c$, to the pair $([\tilde{\phi}_{0,j,m_j} \circ \tilde{c}], p_{\tilde{\phi}_{0,j,m_j}})$ with $S \times V \to M^\rho_{V,P,j} \times V$ and $\tilde{c}$, the homomorphism between fundamental groups induced by $\tilde{c}$, is the isomorphism of functors $M^\rho_{V,H} \to F^\rho_{V,H}$.

It can be derived from definition that any two $\tilde{\phi}_{0,j,m_j}$ and $\tilde{\phi}_{0,j,m_j}$ in two different systems of universal family pair representatives over $M^\bullet_{V,P}$ are equivalent and $p^{-}_{\tilde{\phi}_{0,j,m_j}} p_{\tilde{\phi}_{0,j,m_j}}$. 

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Similarly to Remark 14, there must be a system of universal family pair representatives over $M_{V,P}^{\rho}$. 

**Theorem 19.** With the notations above, there exists a fine moduli space $M_{V,P}^{\rho}$ representing $F_{V,P}$, the functor for $\rho$-liftable pairs of $V$. It is a disjoint union of finitely many direct limit of affine spaces $A_N$. 

**Proof.** Induct on $|P|$. 

Let $F = F_{V,P}$. 

Take a minimal normal subgroup $H$ of $G$ inside $G(P)$, the nontrivial center of $P$. It is a product of copies of some simple group $S$. Hence $H \cong (\mathbb{Z}/p)^m$ for some $m \geq 1$. Let $\rho_0 : \mathbb{Z}/n \rightarrow Aut(H)$ be the $\mathbb{Z}/n$-action induced by $\rho$; $\rho_0$ is irreducible by the minimality of $H$. If $H = P$, then this is in the case of Corollary 16. Hence assume $H < P$ below. 

Let $\bar{P} : \mathbb{Z}/n \rightarrow Aut(\bar{P})$ be the $\mathbb{Z}/n$-action induced by $\rho$, where $\bar{P} = P/H$. By the inductive hypothesis and Corollary 16 respectively $\bar{M} := M_{V,P}^{\rho}$ and $M^0 := M_{V,H}^{\rho}$ exist.

It will be shown that $\bar{M} \times M^0$ is the moduli space desired. 

First need to lift a system of universal family pair representatives over $\bar{M}$. For every point $\bar{m}$ of $\bar{M}$, there is a $\rho$-liftable pair $(\bar{\mu}_0 : \pi_1(\bar{M} \times V, (\bar{m}, v_g)) \rightarrow \bar{P}, \psi_0)$ in the system, which is the counterpart of $(\phi_0 : \pi_1(M \times V, (m, v_g)) \rightarrow P, \psi_0)$ in Definition 18. Denote by $c_\rho$ the image of $c$ under the group homomorphism $\pi_1(U, u_g) \rightarrow \pi_1(\bar{M} \times U, (\bar{m}, u_g))$ induced by $U \hookrightarrow \bar{M} \times U$. The pair gives a $\bar{\mu}_0 : \pi_1(\bar{M} \times U, (\bar{m}, u_g)) \rightarrow \bar{P} \times_{\rho} \mathbb{Z}/n$ with $\mu_0(c_\rho) = (\bar{p}_0, \bar{1})$, similar to the diagram (3.1). As $\pi_1(\bar{M} \times U, (\bar{m}, u_g))$ has $\text{cd}_\rho \leq 1$ ([H80], p1101), $\bar{\mu}_0$ lifts (a different meaning of “lift”, cf. Section 2) to a $\bar{\psi}_0 : \pi_1(\bar{M} \times U, (\bar{m}, u_g)) \rightarrow P \times_{\rho} \mathbb{Z}/n$ ([Serre], I Prop. 16) with $\bar{\psi}_0(c_\rho) = (p_0, \bar{1})$, for some $p_0 \in P$ mapping to $\bar{p}_0 \in \bar{P}$. Denote the restriction of $\bar{\psi}_0$ on $\pi_1(\bar{M}, V, (\bar{m}, v_g))$ by $\bar{\psi}_0$.

Then use the lift $\bar{\psi}_0$ obtained above to separate a $\rho$-liftable pair of $S \times V$ into two parts. Let $(S, s_0) \in S_1$ and suppose $(\bar{\phi} : \pi_1(S \times V, (s_0, v_g)) \rightarrow P, p_1)$ is a $\rho$-liftable pair. Its quotient $(\tilde{\phi} : \tilde{\pi}_1(S \times V, (s_0, v_g)) \rightarrow \tilde{P}, \tilde{p}_1)$ is a $\rho$-liftable pair. By the inductive hypothesis, the quotient pair corresponds to a morphism $\beta : S \rightarrow \bar{M}$. Denote $\beta \times Id_V$ by $\bar{\beta}$. Denote the induced homomorphism $\pi_1(S \times V, (s_0, v_g)) \rightarrow \pi_1(\bar{M} \times V, (\beta(s_0), v_g))$ by $\tilde{\beta}_s$, and let $\tilde{\psi} := \tilde{\psi}_0 \circ \tilde{\beta}_s$ (letting $\tilde{m}$ above be $\beta(s_0)$ here). Then define a “quotient homomorphism” $\tilde{\eta} : \tilde{\pi}_1(S \times V, (s_0, v_g)) \rightarrow H$ by $\tilde{\phi} \tilde{\psi}^{-1}$. Since $\bar{M}$ is a fine moduli space for $\bar{F}$ which involves equivalence classes, $\tilde{\phi}$ and $\tilde{\psi}$ only agree pulled back to some finite etale cover $T$ of $S$, by definition of $\bar{F}$. Pick a point $t_0$ on $T$ mapping to $s_0$. Define $\tilde{\eta}_T(a) = \tilde{\phi}_T(a) \tilde{\psi}_T(a^{-1})$ for every $a \in \pi_1(T \times V, (t_0, v_g))$, where $\tilde{\phi}_T$ means $\tilde{\phi}$ pulled back to $T$ and similarly for $\tilde{\psi}_T$. Actually $\tilde{\eta}_T$ maps to $H$ and the centrality of $H$ in $P$ implies that $\tilde{\eta}_T$ is a homomorphism. Let $h_1 = p_1 \bar{p}_0^{-1}$. Then $(\tilde{\eta}_T, h_1)$ is a $\rho_0$-liftable pair and hence corresponds to a morphism $\alpha_T : T \rightarrow M^0$. By etale descent ([H80], p1109, second paragraph) $\alpha_T$ descends to a morphism $\alpha : S \rightarrow M^0$. Hence get $(\alpha, \beta) : S \rightarrow M$, where $M = M_{V,P}^{\rho} := M^0 \times \bar{M}$.

It is straightforward to verify that the assignment $(\tilde{\phi}, p_1) \mapsto (\alpha, \beta)$ is well defined on pointed $\rho$-liftable family pairs (i.e. is independent of the choice of $\tilde{\phi}$ in its equivalence
class), and yields a bijection between \( F(S, s_0) \) and \( \text{Hom}(S, M) \). As the bijection is compatible with pullback, it follows that \( M \) represents \( F \).

**Remark 20.** Keeping track of universal family representatives in the inductive construction, a \( \rho \)-liftable universal family representative over (each connected component of) \( M^\bullet_{V,P} = M^0 \times \bar{M} \) can be given by the product of a \( \rho_0 \)-liftable universal family representative over \( M^0 \) and a \( \rho \)-liftable lift of a \( \bar{\rho} \)-liftable universal family representative over \( \bar{M} \), using the inclusions \( M^0 \hookrightarrow M^0 \times \bar{M} \) and \( \bar{M} \hookrightarrow M^0 \times \bar{M} \).

**Remark 21.** Since the moduli spaces are ind schemes, strictly speaking the argument above needs to be carried out for each \( n \) and check compatibility for different \( n \)'s. The argument given above has the advantage of being more concise, which follows the way of presentation in Theorem 1.2 of [H80].

**Remark 22.** In the inductive proof of Theorem 19, \((P, \rho)\) is said to be decomposed to \((H, \rho_0)\) and \((\bar{P}, \bar{\rho})\). If \((\bar{P}, \bar{\rho})\) is not in the case considered in Theorem 12, then it can be further decomposed similarly. Repeat the inductive step in Theorem 19 until the last pair got is in the case of Theorem 12. The pairs got in the process are denoted by \((H_t, \rho_t)_t\), which are all in the case of Theorem 12 and the first of which is \((H, \rho_0)\). Then \( M^\bullet_{V,P} = \Pi_t M^\bullet_{V,H_t} \), which consists of finitely many connected components of the form \( \Pi_t M^\bullet_{V,H_t} \).

Here is the main theorem of this section, on moduli for covers with a given cyclic-by-\( p \) Galois group.

**Definition 23.** Let \( F_{U,G} : \mathcal{S}_1 \rightarrow (\text{Sets}) \) be the contravariant functor given by \( F_{U,G}(S, s_0) = \{ [\hat{o}] | \hat{o} : \pi_1(S \times U, (s_0, u_g)) \rightarrow G \} \), the set of families of \( G \)-covers of \( U \) parametrized by \( S \), pointed over \((s_0, u_g)\).

Let \( S = \text{Spec}(k) \) with \( s_0 \) determined by \( u_g \) using diagram (2.1). Then \( F_{U,G}(S, s_0) \) is the set of pointed \( G \)-covers of \((U, u_g)\).

**Theorem 24.** There exists a fine moduli space \( M_{U,G} \) representing \( F_{U,G} \), the functor for pointed \( G \)-covers of \((U, u_g)\), which is a disjoint union of finitely many direct limits of affine spaces.

**Proof.** It will be shown that \( F_{U,G} \) is isomorphic to \( \Pi_{V_i} F^\bullet_{V_i, P} \) (see “Table of symbols” for \( V_i \)). The disjoint union of functors means taking disjoint union of sets, since the functors map to the category of sets. Hence it is represented by \( \Pi_{V_i} M^\bullet_{V_i,P} \), by Theorem 19.

First the left to right direction map is given in the isomorphism wanted.

Let \((S, s_0) \in \mathcal{S}_1 \) and \((\tilde{W}, \tilde{u}_g) \rightarrow (S \times U, (s_0, u_g))\) be a pointed \( G \)-cover corresponding to some \( \tilde{o} \in \text{Hom}(\pi_1(S \times U, (s_0, u_g)), G) \). Let \((\tilde{W}_m, \tilde{w}_g)\) be the pointed connected component of \( \tilde{W} / P \), a \((\mathbb{Z}/n')\)-cover of \((S \times U, (s_0, u_g))\) with \((\mathbb{Z}/n')\) the order \( n' \) subgroup in \( \mathbb{Z}/n \) for some \( n'|n \). The diagram commutes:

\[
\begin{array}{ccc}
\pi_1(\tilde{W}_m, \tilde{w}_g) & \xrightarrow{\tilde{o}_m} & P \\
\downarrow & & \downarrow \\
\pi_1(S \times U, (s_0, u_g)) & \xrightarrow{\tilde{o}} & G.
\end{array}
\]
Let $T$ be a connected component of the inverse image in $\overline{W}_m$ of $S \times \{u'_g\}$, where $u'_g$ is any $k$-point on $U$. The fibers of $\overline{W}_m$ over $k$-points of $U$ does not vary since the degree of the cover is prime to $p$. The $k$-scheme $T$ is a finite etale cover of $S$ and pick any base point $t_0$ that maps to $s_0$. The cover $\overline{W}_m$ pulled back to $T \times U$ is isomorphic to a disjoint union of copies of a product $T \times V_i$ for some $V_i$ a $\mathbb{Z}/n_i$-cover of $U$:

$$\coprod (T \times V_i, (t_0, v_i)) \to (\overline{W}_m, \overline{w}_g)$$

as $(\mathbb{Z}/n')$-covers of $T \times U$, using the canonical embedding $\iota_{n_i}$ of $\mathbb{Z}/n_i$ in $\mathbb{Z}/n$ given in Section 2.

Let $\phi_T$ be the composition $\pi_1(T \times V_i, \{t_0, v_i\}) \to \pi_1(\overline{W}_m, \overline{w}_g) \to P$ induced by $T \times V_i \to \overline{W}_m$. Let $c_{i*}$ be the image of $c_i$ under $\pi_1(U, u_g) \to \pi_1(S \times U, (s_0, u_g))$. Let $p_0$ be the first entry of $\phi(c_{i*}) \in P = \mathbb{Z} \ast_p \mathbb{Z}/n$. Then $(\phi_T, p_0)$ is a $\rho_{n_i}$-liftable pair. It corresponds to a morphism $c_T : T \to M_{V_i,P}^{\rho_{n_i}}$. By etale descent again $c_T$ descends to a morphism $c_S : S \to M_{V_i,P}^{\rho_{n_i}}$. The morphism $c_S$ corresponds to an element in $F_{V_i,P}^{\rho_{n_i}}(S, s_0)$. In fact a morphism $\delta : F_{U,G} \to \Pi_{V_i} F_{V_i,P}^{\rho_{n_i}}$ is got.

Conversely, suppose $(\tilde{\phi}, p_0)$ is a $\rho_{n_i}$-liftable pair with $\pi_1(S \times V_i, (s_0, v_i)) \to P$. The diagram commutes:

$$\begin{array}{ccc}
\pi_1(S \times V_i, (s_0, v_i)) & \xrightarrow{\tilde{\phi}} & P \\
\downarrow & & \downarrow \\
\pi_1(S \times U, (s_0, u_g)) & \xrightarrow{\tilde{\phi}} & P \ast_{\rho_{n_i}} \mathbb{Z}/n_i \xrightarrow{\iota_{n_i}} G,
\end{array}$$

where $\tilde{\phi}$ sends $c_{i*}$ to $(p_0, 1)$, and $\iota_{n_i}$ is the group embedding induced by $\iota_{n_i}$. Hence a pointed family of $G$-covers of $U$ parametrized by $S$ corresponding to $\tilde{\iota}_{n_i} \circ \tilde{\phi}$ is got. In fact a morphism $\gamma : \Pi_{V_i} F_{V_i,P}^{\rho_{n_i}} \to F_{U,G}$ is got, which is inverse to $\delta$.

## 4 Moduli for $p'$-by-$p$ covers

In Section 3 it is shown that an intersection of finitely many fine moduli spaces for cyclic-by-$p$ covers of some affine curves gives a moduli space for $p'$-by-$p$ covers of the curve (Corollary 38).

The next simplest groups after cyclic-by-$p$ groups are $p'$-by-$p$ groups. The first idea on how to get a moduli space for $p'$-by-$p$ covers out of fine moduli spaces for cyclic-by-$p$ covers constructed in Section 3 is to intersect them.

The fine moduli spaces for cyclic-by-$p$ covers intersect in some fixed fine moduli space $M_{V',P,0}$, which is given first below.

Lemma 27 and Lemma 28 show how to embed a fine moduli space for cyclic-by-$p$ covers in $M_{V',P,0}$. The first lemma is the base case for the induction in the proof of the 2nd lemma.
Then an intersection gives a target moduli space $M_{V',P}^{0\rho'}$. However, it is not a moduli space for covers with Galois group the $p'$-by-$p$ group given, because pieces do not patch together well when $P$ is not abelian, cf. Remark \[33\]. It is a moduli space for something else, cf. Proposition \[37\]. Similarly pieces may not patch together well for a disconnected $P$-cover. Therefore $M_{V',P}^{0\rho'}$ only contains connected covers. The moduli space result desired is a corollary of Proposition \[37\].

One final thing for the intersection idea work, is to use a weaker definition of equivalence. A new ER-equivalence is introduced below in the definition of $F_{V',P}^{\text{er},\text{Gal}/\rho'}$, the functor to present, and that of $M_{V',P}^{\text{er}0}$, a functor related to the moduli space $M_{V',P}^{0\rho'}$. Using ER-equivalence $F_{V',P}^{\text{er},\text{Gal}/\rho'}$ and $M_{V',P}^{\text{er}0}$ are proven isomorphic in Proposition \[37\].

As always, we follow notations and terminology defined in Section \[2\].

First the space where intersections take place is given.

Let $(V', v_g') \to (U, u_g)$ be a pointed connected $P'$-cover of $(U, u_g)$, which corresponds to $\theta': \pi_1(U, u_g) \to P'$.\[Remark \[25\].\] Since $P$ can be decomposed in different ways in the construction of $M_{V',P}$ (cf. proof of Theorem 1.2 in [H80]; cf. Remark \[22\] for a $p$-liftable version), there are different forms of $M_{V',P}$. Since they are all fine moduli spaces of $F_{V',P}$, it is derived from the definition that they are isomorphic. Fix a fine moduli space $M_{V',P,0}$ for $F_{V',P}$ below, where intersections take place.

Now the objects which intersect later are given.

Let $(V', v_g')$ be the quotient of $(V', v_g')$ by $\langle p_i' \rangle$, the subgroup generated by $p_i'$, and let $\rho_i' : \langle p_i' \rangle \to \text{Aut}(P)$ be the restriction of $\rho'$. There is a short exact sequence of groups.

\[1 \to \pi_1(V', v_g') \to \pi_1(V_i', v_g') \to \langle p_i' \rangle \to 1.\]

Let $\iota_i'$ be the homomorphism between fundamental groups induced by $\iota_i' : V_i' \to U$. The following diagram commutes.

\[
\begin{array}{ccc}
\pi_1(V_i', v_g') & \xrightarrow{\theta_i'} & \langle p_i' \rangle \\
\iota_i' \downarrow & & \downarrow c \\
\pi_1(U, u_g) & \xrightarrow{\theta'} & P'
\end{array}
\]

For every $p_i' \in P'$, fix a $c_i'$ in $\pi_1(V_i', v_g')$ that maps to $p_i'$ under $\theta_i'$. The pointed $\langle p_i' \rangle$-cover $(V', v_g') \to (V_i', v_g')$ is the counterpart of the pointed $\mathbb{Z}/n$-cover $(V, v_g) \to (U, u_g)$ in Theorem \[19\] of Section \[3\]. Apply Theorem \[19\] on $(V', v_g') \to (V_i', v_g')$ and a fine moduli space $M_{V',P,0}$ for $\rho_i'$-liftable pairs of $(V', v_g')$ is got.

For every $p_i'$ denote by $\{M_{V',P,i,j}^{\rho_i'}\}$ the set of finitely many connected components of $M_{V',P,0}^{\rho_i'}$. Denote by $(M_{V',P,i,j}^{\rho_i'})_i$ a tuple of connected components indexed by $i$, an element in $\Pi_i\{M_{V',P,i,j}^{\rho_i'}\}$. For each tuple $(M_{V',P,i,j}^{\rho_i'})_i$ do their intersection in $M_{V',P,0}$, the way of which will be defined below. Then take the disjoint union of intersections belonging to.
different tuples. The disjoint union is almost $M^{0,q}_{V',P}$.

Below are two lemmas to embed every $M^{0,q}_{V',P,j}$ in $M_{V',P,0}$ for intersection purpose.

The base case is for $(\rho, H)$ in the case of Theorem [12]. With the same setting as in

Theorem [12], let the map $M^\rho_{V,H} \to M_{V,H}$ be given by the canonical universal family representative over $M^\rho_{V,H}$ (cf. Remark [14]). The morphism $\iota$ can be given explicitly by tracking the construction of both moduli spaces in Lemma [27].

**Example 26.** Here is an example that is a prototype for the morphism $M^\rho_{V,H} \to M^\rho_{V,H}$ in the diagram of Lemma [27] below. The subring $k[X^p]$ of $k[X]$ is also a polynomial ring. The inclusion $k[X^p] \subset k[X]$ induces a bijection between closed points in $Spec(k[X])$ and those in $Spec(k[X^p])$, given explicitly by $(X - \lambda) \leftrightarrow (X^p - \lambda^p)$.

**Lemma 27.** There is a closed subscheme $M^\rho_{V,H}$ of $M_{V,H}$ which $\iota$ factors through and whose closed points are in bijection with those of $M^\rho_{V,H}$ under $\iota$.

\[ \begin{array}{ccc}
M^\rho_{V,H} & \overset{\iota}{\longrightarrow} & M^\rho_{V,H} \\
\downarrow & & \downarrow \\
M_{V,H} & \overset{\iota}{\longrightarrow} & M_{V,H}
\end{array} \]

**Proof.** Theorem [12], Remark [22] and the base step for induction in Theorem 1.2 in [H80] are the references for this proof. Every fact used here can be found in one of the three places.

The explicit expression of $\iota$ on each $n$-th piece of $M^\rho_{V,H}$ will be given, using which the statements in the Lemma can be shown.

Denote by $M^\rho_{V,H,n}$ the $n$-th piece of $M^\rho_{V,H}$. The affine space $M^\rho_{V,H,n}$ can be identified with $Spec(k[K^\vee_n - K^\vee_{n-1}])$, where $K_n$, containing $K_{n-1}$, is the basis chosen for the $k$-vector space $(KerD)_n = KerD \cap H^0(V, q^nDiv_V)$ in the proof of Theorem [12]. Denote by $K^\vee_n$ the set of the dual’s of vectors in $K_n$. Write out elements in $K_n - K_{n-1}$ as $\{k_i, 1 \leq i \leq d_k\}$.

Similarly denote by $M_{V,H,n}$ the $n$-th piece of $M_{V,H}$. The affine space $M_{V,H,n}$ can be identified with $Spec(k[L^\vee_n - L^\vee_{n-1}])$, where $L_n$, containing $L_{n-1}$, is the basis chosen for $H^0(V, q^nDiv_V)/k^\times$. Denote by $L^\vee_n$ the set of the dual’s of vectors in $L_n$. Write out elements in $L_n - L_{n-1}$ as $\{l_j, 1 \leq j \leq d_L\}$. The way to choose $L_n$ is described in the base step for induction in the proof of Theorem 1.2 in [H80], analogous to the way to choose $K_n$. Only need to change the symbol $U$ there to $V$, $B_n$ there to $L_n$, and $A_n = H^0(U, q^nDiv_U)$ there to $B_n = H^0(V, q^nDiv_V)$. Recall that $U = Spec(A)$ and $V = Spec(B)$. $A_n$ and $B_n$ denote $k$-subspaces of $A$ and $B$.

Denote by $\iota_n$ the restriction of $\iota$ on $M^\rho_{V,H,n}$. The morphism $\iota_n$ maps every closed point in $M^\rho_{V,H,n}$ to the closed point in $M_{V,H,n}$ that represents the same pointed $H$-cover as it. Denote the $k$-algebra homomorphism that corresponds to $\iota_n$ by $\iota_n^*: k[L^\vee_n - L^\vee_{n-1}] \to k[K^\vee_n - K^\vee_{n-1}]$. It turns out that $\iota_n^*$ has the form: $l^\vee_j \mapsto \Sigma_{i \in X_j}(\lambda_{i,j}k^\vee_i)^{q_{ij}}$, where $X_j$ some finite set, $\lambda_{i,j} \in k$ and $q_{ij}$ is some $p$-power.

The form of $\iota_n^*$ is obtained as follows. All pointed $H$-covers of $(V, v_q)$ can be given by elements in $B$ using Artin-Schreier equations $z^q - z = b$ with $b \in B$. Elements in
the $k$-linear span of $L_n - L_{n-1}$ give bijectively all the pointed $H$-covers of $(V, v_g)$ that can be given by $z^q - z = b$ with $b \in B_n$. Every element $\sum_i \lambda_i k_i$ in the $k$-linear span of $K_n - K_{n-1}$ is in $B_n$. Hence the pointed $H$-cover of $(V, v_g)$ given by $z^q - z = \sum_i \lambda_i k_i$ is isomorphic to the pointed $H$-cover of $(V, v_g)$ given by $z^q - z = \sum_j \lambda_j l_j$, for some unique $\sum_j \lambda_j l_j$ in the $k$-linear span of $L_n - L_{n-1}$. The correspondence $\sum_i \lambda_i k_i \leftrightarrow \sum_j \lambda_j l_j$ is what is used to get the form of $t_t^*$. A closed point in $\Spec(k[L_n - L_{n-1}])$ has the form $(k_1^\gamma - \lambda_1, ..., k_d^\gamma - \lambda_d)$. The maximal ideal represents the pointed $H$-cover of $(V, v_g)$ given by $z^q - z = \sum_i \lambda_i k_i$, pointed anywhere above $v_g$. There is a unique $k$-algebra homomorphism $k[L_n - L_{n-1}] \rightarrow k[K_n^\gamma - K_{n-1}^\gamma]$ such that the inverse image of $(k_1^\gamma - \lambda_1, ..., k_d^\gamma - \lambda_d)$ is $((l_1^\gamma - \lambda_1, ..., l_d^\gamma - \lambda_d))$, which represents the pointed $H$-cover of $(V, v_g)$ given by $z^q - z = \sum_j \lambda_j l_j$, for every closed point $(k_1^\gamma - \lambda_1, ..., k_d^\gamma - \lambda_d)$ in $\Spec(k[K_n^\gamma - K_{n-1}^\gamma])$. Hence the homomorphism is $t_t^*$, by the definition of $t_t^*$. It is left as an exercise to the reader to write out the precise formula of the homomorphism, which has the form given above.

Let $M_{V,H,n}^\rho = \Spec(Im u_n^*)$, which is a closed subscheme of $M_{V,H,n}$. After simplification by elimination $Im u_n^*$ turns out a polynomial ring $k[\{k_i', 1 \leq i \leq d_K\}]$, where $k_i'$ is a sum of powers $\sum_{i \leq i \leq d_K} k_i'^{n_i}$ and $n_i$ is a $p$-power. Moreover for every $i$ the polynomial ring $Im u_n^*$ contains a $k_i'^{n_i}$ with $q_i$, a $p$-power. Similar to Example 26, $t_n$ gives a bijection between the closed points of $M_{V,H,n}^\rho$ and those of $M_{V,H,n}^\rho$.

The $t_n$’s for different $n$’s are compatible.

Here are some necessary settings to prove the 2nd lemma for embedding $M_{V',P,j}^\rho$ in $M_{V',P,0}$.

With the same setting as in Theorem 19, the ind scheme $M_{V,P}^\rho \bullet$ consists of finitely many connected components $\{M_{V,P,j}^\rho\}$. For every $j$, the universal family (cf. Remark 18 for more precise terminology) over $M_{V,P,j}^\rho$ determines a morphism $\rho_{V,P,j} \rightarrow M_{V,P}$, since $M_{V,P}$ is the fine moduli space for $F_{V,P}$.

If $(\rho, H)$ is a decomposition of $(\rho, P)$ (cf. Remark 22), then $M_{V,P}^\rho = \Pi_t M_{V,H,t}^\rho$, and $M_{V,P} = \Pi_t M_{V,H,t}$. Hence $M_{V,P,j}^\rho$ has the form $\Pi_t M_{V,H,t}^\rho$ for every $j$. The morphism $\iota$ can be given componentwise for each $t$.

**Lemma 28.** With the notations above, the morphism $M_{V,P,j}^\rho = \Pi_t M_{V,H,t}^\rho \rightarrow M_{V,H,t}$ is given by $\Pi_t$, where $M_{V,H,t}^\rho \rightarrow M_{V,H,t}$ is the morphism given in Lemma 27.

**Proof.** Theorem 12, Remark 22 and the base step for induction in Theorem 1.2 in [H80] are the references for this proof. Every fact used here can be found in one of the three places.

Induct on $|P|$. Moreover for every $t$ since $M_{V,H,t}$ is the fine moduli space for $F_{V,H,t}$, the canonical universal family representative over $M_{V,H,t}$ given in 1.9 Rmk of [H80] pulled back to $M_{V,H,t}^\rho$ via $t_t$, differs from the canonical universal family representative over $M_{V,H,t}^\rho$ given in Remark 14 by some element in $H^1(M_{V,H,t}^\rho, H)$, by tracking definitions and cf. Remark 14. Lemma 27 shows that $t_t$ gives a bijection on closed points of $M_{V,H,t}^\rho$ and $M_{V,H,t}^\rho$. Using this fact and the same kind of argument in Lemma 4.25 of [TY17], a universal family representative over $M_{V,H,t}$ can be chosen such that it pulls back to the canonical universal family representative over $M_{V,H,t}^\rho$. 

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Below is the inductive step.

In Theorem \[19\] an \( H \) inside the center \( C(P) \) of \( P \) is taken, and then the inductive process is carried out, which gives \( M_{V,P}^{\ast} \) as \( M_{V,P}^{\ast} \times M_{V,H}^{\ast} \). The notation \( \Pi_1 M_{V,H}^{\alpha} \) means that \((H, \rho_0)\) is denoted by \((H_1, \rho_1)\) here and the inductive step there is carried out for some finite steps until the induction ends, cf. Remark \[22\]. Thus a connected component of \( M_{V,P}^{\ast} \) can be denoted by \( \Pi_{t \geq 2} M_{V,H}^{\alpha} \) and \( M_{V,P} \) by \( \Pi_{t \geq 2} M_{V,H} \).

By inductive hypothesis, the universal family over \( \Pi_{t \geq 2} M_{V,H}^{\alpha} \) determines a morphism \( \Pi_{t \geq 2} M_{V,H}^{\alpha} \rightarrow \Pi_{t \geq 2} M_{V,H} \) with \( \iota_t \) given in Lemma \[27\] and there is a universal family representative over \( \Pi_{t \geq 2} M_{V,H} \) such that it pulls back to a \( \tilde{\rho} \)-liftable universal family representative over \( \Pi_{t \geq 2} M_{V,H}^{\alpha} \). Then lift of the \( \tilde{\rho} \)-liftable universal family representative over \( \Pi_{t \geq 2} M_{V,H} \) as in Theorem \[19\] and pick any lift of the universal family representative over \( \Pi_{t \geq 2} M_{V,H} \). Again using the same kind of argument in Lemma 4.25 of \[TY17\], the latter lift can be modified such that its pullback to \( \Pi_{t \geq 2} M_{V,H}^{\alpha} \) is the previous lift. (Strictly speaking, Definition \[18\] needs to be used and pairs should be dealt with, which however will make the proof unnecessarily longer.)

A \( \rho \)-liftable universal family representative over \( \Pi_{t} M_{V,H}^{\alpha} \) can be given by the product of a \( \rho_1 \)-liftable universal family representative over \( M_{V,H}^{\alpha} \) and a \( \rho \)-liftable lift of a \( \tilde{\rho} \)-liftable universal family representative over \( \Pi_{t \geq 2} M_{V,H}^{\alpha} \), cf. Remark \[20\]. A universal family representative over \( \Pi_{t} M_{V,H} \) is a similar product. The 2nd paragraph and the paragraph right above together show that the pullback of some universal family representative over \( \Pi_{t} M_{V,H} \), via \( \Pi_{t} M_{V,H}^{\alpha} \rightarrow \Pi_{t} M_{V,H} \) is a \( \rho \)-liftable universal family representative over \( \Pi_{t} M_{V,H}^{\alpha} \). By the definition of the fine moduli space \( M_{V,P} \), this means that \( \Pi_{t \iota} \) is the morphism \( \iota \) determined by the universal family over \( M_{V,P}^{\alpha} = \Pi_{t} M_{V,H}^{\alpha} \).

**Remark 29.** By Lemma \[27\] and Lemma \[28\] \( \Pi_{t \iota t} \) factors through \( \Pi_{t} M_{V,H}^{\alpha} \), a closed subscheme of \( M_{V,P} \).

\[
\begin{array}{ccc}
\Pi_{t} M_{V,H}^{\alpha} & \xrightarrow{\iota} & \Pi_{t} M_{V,H}^{\alpha} \\
\downarrow & & \downarrow \\
\Pi_{t} M_{V,H} & & \Pi_{t} M_{V,H}^{\alpha}
\end{array}
\]

**Definition 30.** If a \( M_{V,P} = \Pi_{t} M_{V,H} \), a \( M_{V,P}^{\alpha} = \Pi_{t} M_{V,H}^{\alpha} \) and their respective universal family representatives are constructed together, using the inductive process in the proof of Lemma \[28\]. Then the \( M_{V,P} \) is called attached to the \( M_{V,P}^{\alpha} \).

**Definition 31.** With the preparation of the two lemmas above, the moduli space \( M_{V,P}^{\alpha} \) for Proposition \[37\] can be defined. There is a closed subscheme image (like \( \Pi_{t} M_{V,H}^{\alpha} \) in Remark \[29\]) of \( M_{V,P}^{\alpha} \) in the \( M_{V,P} \) attached (cf. Definition \[30\]) to \( M_{V,P}^{\alpha} \). Using the isomorphism (cf. Remark \[25\]) from the \( M_{V,P} \) attached to \( M_{V,P}^{\alpha} \), to the fixed \( M_{V,P,0} \), the closed subscheme image has its isomorphic image in \( M_{V,P,0} \), which is denoted by \( M_{V,P,ij}^{\alpha} \). Denote the morphism \( M_{V,P,ij}^{\alpha} \rightarrow M_{V,P,ij}^{\alpha} \) by \( \iota_{t,ij}^{\alpha} \). Let \( M_{V,P}^{\alpha} = \coprod_{(M_{V,P,ij}^{\alpha})} \cap \bigcap_{M_{V,P,ij}^{\alpha}} \) (see above in this section for the tuple \( (M_{V,P,ij}^{\alpha})_i \)). Let \( M_{V,P}^{\alpha} \) be
the dense open subset of $M_{V',P,0}$ which parameterizes all connected pointed $P$-covers of $(V', v'_g)$ ([H80], Theorem 1.12). Let $M^0_{V',P} = \coprod_{(M^0_{V',P},i)} (\bigcap_i M^0_{V',P,i} \cap M^0)$.

**Remark 32.** What does the space $M^0_{V',P}$ parameterize?

Every closed point in $M^0_{V',P}$ represents a connected pointed $P$-cover $(W, w_g) \to (V', v'_g)$ corresponding to some homomorphism $\pi_1(V', v'_g) \to \hat{\pi}$ that is $\rho'$-liftable for every $i$. In fact, for every $i$ the closed point gives a $\rho_i'$-liftable pair $(\phi_i, p_i)$ for some $p_i \in P$, by the definition of $M^0_{V',P}$ and the fact that every point in $M_{V,P}$ represents a $\rho$-liftable pair as shown in Theorem 19.

See above in this section for $c'_i$ and $\theta'_i$. The cover $(V', v'_g) \to (V'_i, \overline{v'_{gi}})$ corresponds to some homomorphism $\pi_1(V'_i, \overline{v'_{gi}}) \to \langle p'_i \rangle$ that maps $c'_i$ to $p'_i$. There is a similar diagram as in (3.1) with $\widehat{\phi_i(c'_i)} = (p_i, p'_i)$:

$$
\pi_1(V', v'_g) \xrightarrow{\phi} P \\
\pi_1(V'_i, \overline{v'_{gi}}) \xrightarrow{\widehat{\phi_i}} P \rtimes \rho'_i \langle p'_i \rangle \xrightarrow{Q'_{\rho}} \langle p'_i \rangle,
$$

where the composition of the bottom two arrows $\pi_1(V'_i, \overline{v'_{gi}}) \to \langle p'_i \rangle$ is $\theta'_i$.

Hence the cover $(W, w_g)/(V'_i, \overline{v'_{gi}})$ is a pointed $P \rtimes \rho'_i \langle p'_i \rangle$-cover. Denote by $\gamma_i$ the element in the Galois group of $W/V'_i$ corresponding to $(1, p'_i)$ with 1 the identity of $P$. Denote by $\rho'_i\rho_i$ the automorphism in the Galois group of $V' \to U$ that corresponds to $p_i \in P$. Denote by $\gamma_p$ the element in the Galois group of $W/V'$ that corresponds to $p \in P$. $\gamma_i$ lies over $\gamma_p$, and satisfies $\text{ord}(\gamma_i) = \text{ord}(p'_i)$ and $\gamma_i\gamma_p\gamma_i^{-1} = \gamma_{p'(p_i)(p)}$ for every $p \in P$. These three conditions are called condition (**i)).

So a closed point in $M^0_{V',P}$ gives a pair $((W, w_g) \to (V', v'_g), \{\gamma_i\})$ with the first entry a connected pointed $P$-cover of $(V', v'_g)$ and the second entry a subset of the Galois group of $W/U$ with cardinality $|P'|$, the $i$-th element of which satisfies condition (**i)). The set of such pairs is denoted by $\text{Gal}_{V'/\rho'}$.

Reading backwards the discussion above, every pair in $\text{Gal}_{V'/\rho'}$ has a unique closed point in $M^0_{V',P}$ which represents the pair. Hence there is a canonical bijection between closed points in $M^0_{V',P}$ and $\text{Gal}_{V'/\rho'}$.

**Remark 33.** There is a finite partition of closed points in $M^0_{V',P}$ by covers’ Galois groups over $U$.

With the same notations as in the previous remark, the cover $W/U$ is Galois by a group order counting argument.

Denote $\text{Gal}(W/V')$ by $\Gamma_{\rho'}$, $\text{Gal}(W/U)$ by $\Gamma$, and $\text{Gal}(V'/U)$ by $\Gamma_{\rho'}$. The isomorphism $\Gamma_p \simeq P$ is already given since $(W, w_g) \to (V', v'_g)$ is a $P$-cover. Similarly for $\Gamma_{\rho'} \simeq P'$. Fix a subgroup $\widehat{\Gamma}_{\rho'}$ in $\Gamma$ which maps isomorphically to $\Gamma_{\rho'}$ under the canonical quotient map $\Gamma \to \Gamma_{\rho'}$. The existence of such a subgroup is given by Schur-Zassenhaus since $(p, |P'|) = 1$. Then $\Gamma$ is canonically isomorphic to $\Gamma_p \rtimes \widehat{\Gamma}_{\rho'}$, an inner semiproduct.

The isomorphism $\Gamma_{\rho'} \simeq P'$ induces an isomorphism $\widehat{\Gamma}_{\rho'} \simeq P'$. Substituting $\Gamma_p$ by $P$ and $\widehat{\Gamma}_{\rho'}$ by $P'$ in $\Gamma_p \rtimes \widehat{\Gamma}_{\rho'}$, an induced semiproduct $P \rtimes \rho', P'$ and an induced isomorphism
\[ \Gamma \cong P \rtimes_{\rho''} P' \] are got. The diagram is commutative:

\[
\begin{array}{c}
1 \longrightarrow \Gamma_p \longrightarrow \Gamma \longrightarrow \Gamma_{p'} \longrightarrow 1 \\
\approx \quad \quad \quad \quad \quad \approx \quad \quad \quad \quad \quad \approx \\
1 \longrightarrow P \longrightarrow P \rtimes_{\rho''} P' \longrightarrow P' \longrightarrow 1.
\end{array}
\]

For every \( p' \in P' \), the action of \( \rho'(p') \) on \( P \) differs from that of \( \rho''(p') \) by the conjugation of some element \( p_{p'} \in P \), since \( \gamma_i \) and its counterpart in \( \Gamma_{p'} \) differ by some element in \( \Gamma_p \).

When \( P \) is abelian, the two groups \( P \rtimes_{\rho'} P' \) and \( P \rtimes_{\rho''} P' \) are the same. The Galois group over \( U \) for any element in \( \text{Gal}_{V'/U} \) is \( P \rtimes_{\rho'} P' \). If \( P \) is not abelian, the two groups \( P \rtimes_{\rho'} P' \) and \( P \rtimes_{\rho''} P' \) may not be the same. The Galois group can not be nailed down.

Define \( G_{p_{p'}} \) as the finite set \( \{ P \rtimes_{\rho}_{p''} P' \mid \text{ for every } p' \text{ there exists a } (p, p') \in P \rtimes_{\rho''} P' \text{ such that } \text{ord}(p, p')=\text{ord}(p') \text{ and } (p, p')p(p, p')^{-1} = \rho'(p') \} \), where \( \rho'_{p''} : P' \rightarrow \text{Aut}(P) \) is an action of \( P' \) on \( P \).

By the end of Remark 32, the closed points of \( M_{0}/p_{p'} \) are in canonical bijection with \( \text{Gal}_{V'/\rho'} \). Every pair in \( \text{Gal}_{V'/\rho} \) gives a pointed \( P \rtimes_{\rho''} P' \)-cover (a similar diagram to diagram (31)):

\[
\begin{array}{c}
\pi_1(V', v_{q}') \quad \phi \quad P \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\pi_1(U, u_g) \quad \quad \tilde{\phi} \quad P \rtimes_{\rho''} P' \quad Q_p \quad P',
\end{array}
\]

for some \( \rho''_s \) using the process given above diagram (31), where the composition of the bottom two arrows is \( \theta' \). The group \( P \rtimes_{\rho''_s} P' \) is said to belong to the pair or belong to the closed point corresponding to \( \theta' \). A different \( P \rtimes_{\rho''_{s_1}} P' \) can belong to the same pair, if a different section \( \Gamma_{p'} \) is chosen in the process. If two \( P \rtimes_{\rho''_{s_1}} P' \) and \( P \rtimes_{\rho''_{s_1}} P' \) belong to the same pair, then a similar diagram to (32)

\[
\begin{array}{c}
\pi_1(V', v_{q}') \quad \phi \quad P \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\pi_1(U, u_g) \quad \quad \tilde{\phi}_1 \quad P \rtimes_{\rho''_{s_1}} P' \quad Q_p \quad P'
\end{array}
\]

is also commutative. It together with diagram (32) gives

\[
\begin{array}{c}
1 \longrightarrow P \longrightarrow P \rtimes_{\rho''_{s_1}} P' \longrightarrow P' \longrightarrow 1 \\
\approx \quad \quad \quad \quad \quad \approx \quad \quad \quad \quad \quad \approx \\
1 \longrightarrow P \rtimes_{\rho''_{s_1}} P' \longrightarrow P' \longrightarrow 1.
\end{array}
\]

Hence \( P \rtimes_{\rho''_{s_1}} P' \) and \( P \rtimes_{\rho''_{s_1}} P' \) are isomorphic extensions. Pick a representative (pick \( P \rtimes_{\rho'} P' \) in its class) in each isomorphism class of extensions and denote the subset obtained in this way of \( G_{p_{p'}} \) by \( G_{p_{p'}} \). The set \( \text{Gal}_{V'/\rho'} \) has a finite partition by elements in \( G_{p_{p'}} = \{ P \rtimes_{\rho''_{s}} P' \} \) by discussion above.
For any $P \times \rho' \rightarrow P' \in G_{P_{\rho'}}$ and any pointed $P \times \rho_i P'$-cover $(W, w_g) \rightarrow (U, u_g)$ corresponding to some $\pi_1(U, u_g) \xrightarrow{\hat{\phi}} P \times \rho_i P'$, a pointed $P \times \rho_i' \langle p'_i \rangle$-cover $(W, w_g) \rightarrow (V_i', \overline{v}_{gi})$ can be got for every $i$:

$$
\begin{array}{ccc}
\pi_1(V', v_g') & \xrightarrow{\hat{\phi}} & P \\
\downarrow & & \downarrow \\
\pi_1(V_i', \overline{v}_{gi}) & \xrightarrow{\rho_i' \langle p_i' \rangle} & P \\
\downarrow & & \downarrow \\
\pi_1(U, u_g) & \xrightarrow{\hat{\phi}} & P \times \rho_i P',
\end{array}
$$

(33.3)

where $\rho_i''$ is the restriction of $\rho_i''$ on $\langle p_i' \rangle$. The pointed $P \times \rho_i' \langle p_i' \rangle$-cover $(W, w_g) \rightarrow (V_i', \overline{v}_{gi})$ is also a pointed $P \times \rho_i' \langle p_i' \rangle$-cover, as shown in diagram (33.3), where the group isomorphism $P \times \rho_i' \langle p_i' \rangle \rightarrow P \times \rho_i' \langle p_i' \rangle$ sends $(p_i, p_i')$ to $(1, p_i')$ and every $p \in P$ to $p$. Then Remark 32 shows that $(W, w_g) \rightarrow (U, u_g)$ gives a pair in $Gal_{V'/\rho'}$ corresponding to some closed point in $M_{V', P}$, which can be used to discover the original $(W, w_g) \rightarrow (U, u_g)$ using diagram (33.3). If a closed point in $M_{V', P}$ is used to discover, using diagram (33.3), a pointed $P \times \rho_i P'$-cover of $(U, u_g)$ for some $P \times \rho_i P'$ in $G_{P_{\rho'}}$, there are several possibilities for $\rho_i''$.

To define the functor $F_{er,Gal/\rho'}^V$ in Proposition 37, several new definitions are needed. The inclusion of polynomial rings $Im{t_i}^* \subset k\{\{k_i, 0 \leq i \leq d_{K}\}\}$ in the proof of Lemma 27 and the morphism $\Pi_i M_{V', H_i} \rightarrow \Pi_i M_{V', H_i}$ in Remark 29 motivate the first two definitions given below respectively.

**Definition 34.** Let $k[X_1, ..., X_d]$ be a polynomial ring. Suppose $k[X'_1, ..., X'_d]$ is a subring where each $X_i'$ is a sum of powers $\Sigma_{i \leq \ell \leq d} X_i^{n_{i\ell}}$ with $n_{i\ell}$ a $p$-power, such that the subring also contains $X_i^{n_i}$ with some $l_i \geq 0$ for every $i$. Let $P'$ be a polynomial ring with an injective $k$-algebra homomorphism $f: P' \hookrightarrow k[X_1, ..., X_d]$. If $f$ gives an isomorphism between $P'$ and $k[X'_1, ..., X'_d]$, then $f: P' \hookrightarrow k[X_1, ..., X_d]$ is an $R$-extension.

Let $\{P_i \hookrightarrow P'_i\}$ be a collection of finitely many $R$-extensions, with different $P_i$’s and $P'_i$’s. Tensoring over $k$ gives a morphism $Spec(\otimes_i P_i) \rightarrow Spec(\otimes_i P'_i)$. For any $(S', s'_0) \in S$ and $(S', s'_0) \rightarrow (Spec(\otimes_i P'_i), x'_0)$ a morphism in $S$, the pullback $(S, s_0) \rightarrow (S', s'_0)$ of $(Spec(\otimes_i P_i), x_0) \rightarrow (Spec(\otimes_i P'_i), x'_0)$, for some $x_0$ mapping to $x'_0$, is called a morphism of type $R$.

$$
\begin{array}{ccc}
S & \xrightarrow{f'} & Spec(\otimes_i P_i) \\
\downarrow & & \downarrow \\
S' & \xrightarrow{f} & Spec(\otimes_i P'_i).
\end{array}
$$

A morphism $(T, t_0) \rightarrow (S, s_0)$ in $S$ is of type $ER$, if it can be decomposed into a finite sequence of finite etale covers and morphisms of type $R$.

**Remark 35.** The morphism $M_{V', P, i, j}^{\rho_i, \rho_j} \xrightarrow{\varphi_{ij}} M_{V', P, ij}^{\rho_i}$ in Definition 31 is an example of the right column in the square diagram in Definition 34.
Below is the definition for the functor in Proposition 37, which is motivated by the discussion in Remark 32. Let \((S, s_0) \in \mathcal{S}\) and let \(\text{Gal}(S, s_0)\) be the set of \(T\)-parameterized \(P\)-covers \((\tilde{W}, w_0) \to (T \times V', (t_0, v'_g))\) of \(V'\) pointed over \((t_0, v'_g)\) for some \((T, t_0) \to (S, s_0)\) of type \(ER\) with connected fibers over the closed points of \(T\), such that the composition \((\tilde{W}, w_0) \to (T \times V', (t_0, v'_g)) \to (T \times U, (t_0, u_g))\) is Galois. For a pointed \(P\)-cover \((\tilde{W}, w_0) \to (T \times V', (t_0, v'_g))\), and an element \(p \in P\), denote by \(\tilde{\gamma}_p\) the automorphism in its Galois group which corresponds to \(p\). Denote by \(\tilde{\gamma}_p\) the automorphism in the Galois group of \(T \times V' \to T \times U\) which corresponds to \(p'_i \in P\). Let \(\text{Gal}/p'(S, s_0)\) be the set of pairs \(((\tilde{W}, w_0) \to (T \times V', (t_0, v'_g)), \{\tilde{\gamma}_i\})\), where \((\tilde{W}, w_0) \to (T \times V', (t_0, v'_g))\) is in \(\text{Gal}(S, s_0)\) and \(\tilde{\gamma}_i\) is in the Galois group of the cover \(\tilde{W} \to T \times U\) which lies over \(\tilde{\gamma}_p\) and satisfies \(\text{ord}(\tilde{\gamma}_i) = \text{ord}(p'_i)\) and \(\tilde{\gamma}_i \tilde{\gamma}_p \tilde{\gamma}_i^{-1} = \tilde{\gamma}_{p'(p_i)(p)}\) for every \(p \in P\). The set \(\text{Gal}/p'(S, s_0)\) is an \(S\)-parameterized version of \(\text{Gal} V'/\rho'\); by Remark 32, the set \(\text{F}_{\text{er}, \text{Gal}/\rho'}(\text{Spec}(k), s_0)\), with \(s_0\) determined by \(v'_g\) using diagram (21), is the set \(\text{Gal} V'/\rho'\). Two elements \(((\tilde{W}_j, w_{j0}) \to (T_j \times V', (t_{j0}, v'_{jg})), \{\tilde{\gamma}_{j1}\}, \{\tilde{\gamma}_{j2}\})\) \((j = 1, 2)\) in \(\text{Gal}/p'(S, s_0)\) are \(ER\)-equivalent if there exists a morphism \((T_d, t_{d0}) \to (S, s_0)\) of type \(ER\), where \((T_d, t_{d0})\) also maps to \((T_j, t_{j0})\) \((j = 1, 2)\) in the category \(\mathcal{S}\), such that the two pointed \(P\)-covers, together with \(\{\tilde{\gamma}_{j1}\}\) and \(\{\tilde{\gamma}_{j2}\}\), pulled back to \(T_d\) are become isomorphic. Let \(\text{F}_{\text{er}, \text{Gal}/\rho'}\) be the functor: \(\mathcal{S} \to (\text{Sets})\); \((S, s_0) \mapsto \{\text{ER-equivalence classes of }((\tilde{W}, w_{g}) \to (T \times V', (t_0, v'_g)), \{\tilde{\gamma}_i\})\}\) \(\in \text{Gal}/p'(S, s_0)\).

Here is the last definition involved in the statement of Proposition 37. Two morphisms \(T_j \xrightarrow{f_j} M_{V', \rho'}^{0} (j = 1, 2)\), where \((T_j, t_{j0}) \to (S, s_0)\) of type \(ER\) are \(ER\)-equivalent, if there exists a morphism \((T_d, t_{d0}) \to (S, s_0)\) of type \(ER\) with \((T_d, t_{d0})\) also mapping to \((T_j, t_{j0})\) \((j = 1, 2)\) in the category \(\mathcal{S}\), such that the \(f_j\)'s pulled back to \(T_d\) are the same. Let \(M_{V', \rho'}^{\text{tor}^0}\) be the functor: \(\mathcal{S} \to (\text{Sets})\); \((S, s_0) \mapsto \{\text{ER-equivalence classes of }T \to M_{V', \rho'}^{0}\},\) where \((T, t_0) \to (S, s_0)\) runs over all morphisms to \(S\) of type \(ER\).

Remark 36. The two \(ER\)-equivalences in the definitions of functors \(\text{F}_{\text{er}, \text{Gal}/\rho'}\) and \(M_{V', \rho'}^{\text{tor}^0}\) arise naturally in the proof of Proposition 37 based on the intersection idea.

Proposition 37. With the same notations as above, the ind scheme \(M_{V', \rho'}^{0}\) is the moduli space for \(\text{F}_{\text{er}, \text{Gal}/\rho'}\) in the sense that there exists an isomorphism \(\text{F}_{\text{er}, \text{Gal}/\rho'} \simeq M_{V', \rho'}^{\text{tor}^0}\).

Moreover, on each of the finitely many irreducible components of \(M_{V', \rho'}^{0}\), there is a unique \(P \rtimes_{\rho'} P'\) in \(G_{P', \rho'}\) which belongs to (defined in Remark 33) all the closed points. Conversely, for every \(P \rtimes_{\rho'} P'\) in \(G_{P', \rho'}\), there is an irreducible component, such that \(P \rtimes_{\rho'} P'\) belongs to all the closed points of the component.

Proof. Proof of the first statement:

Let \((S, s_0) \in \mathcal{S}\) and \(((\tilde{W}, w_{g}) \to (T \times V', (t_0, v'_g)), \{\tilde{\gamma}_i\})\) be a representative in an \(ER\)-equivalence class of \(\text{F}_{\text{er}, \text{Gal}/\rho'}(S, s_0)\). Then \((\tilde{W}, w_{g}) \to (T \times V', (t_0, v'_g))\) is Galois, cf. Remark 32. Letting \(\gamma_i\) correspond to \((1, p'_i) \in P \rtimes_{\rho'} (p'_i)\), \((\tilde{W}, w_{g}) \to (T \times V', (t_0, v'_g))\) is a pointed \(P \rtimes_{\rho'} (p'_i)\)-cover. By the definition of \(M_{V', \rho'}^{0}\), the pointed \(P \rtimes_{\rho'} (p'_i)\)-cover corresponds to a morphism \(T \xrightarrow{c_i} M_{V', \rho'}^{0}\). Since \(T\) is connected, the morphism \(c_i\) lands in a
connected component \(M_{V',P,ij}^{\ell_i'}\) of \(M_{V',P}^\bullet\). Embedding \(M_{V',P,ij}^{\ell_i'}\) in \(M_{V',P,0}\) as in Remark 35, a morphism \(T \xrightarrow{\xi_i} M_{V',P,0}\) is got. The morphism \(\xi_i\) is the same as the morphism from \(T\) to \(M_{V',P,0}\) determined by the pointed \(P\)-cover \((\tilde{W}, \tilde{w}_g) \to (T \times V', (t_0, v'_g))\), using that \(M_{V',P,0}\) is the fine moduli space for pointed families of \(P\)-covers of \((V', v'_g)\) ([H80], Theorem 1.2). The above discussion applies for every \(i\). Hence a morphism \(T \xrightarrow{\xi} M_{V',P}^{0}\) is got, by the definition of \(M_{V',P}^{0}\).

Conversely, given \(T \xrightarrow{\xi} M_{V',P}^{0}\) for some \((T, t_0) \to (S, s_0)\) of type \(ER\), a morphism \(T \xrightarrow{\xi} M_{V',P,0}\) is got by the definition of \(M_{V',P}^{0}\) and the connectedness of \(T\). For each \(i\), there is a morphism \(T \xrightarrow{\xi_i} M_{V',P,ij}^{\ell_i}\), for some \(M_{V',P,ij}^{\ell_i}\) (see above in this section for \(M_{V',P,ij}^{\ell_i}\) ) which \(\xi\) factors through

\[
\begin{array}{ccc}
T & \xrightarrow{\xi_i} & M_{V',P,ij}^{\ell_i} \\
\downarrow & & \downarrow \\
M_{V',P,0} & & \\
\end{array}
\]

After pointing \(M_{V',P,ij}^{\ell_i} \xrightarrow{\xi_i,ij} M_{V',P,ij}^{\ell_i}\) (cf. Definition 31) properly, the pullback morphism \((T_i, t_{i0}) \to (T, t_0)\) is of type \(R\), by Remark 35 and the lower square of the diagram, in which both squares are pullbacks:

\[
\begin{array}{ccc}
T_{di} & \xrightarrow{\xi_i} & M_{ij} \\
\downarrow & & \downarrow \\
T_i & \xrightarrow{\xi_i} & M_{V',P,ij}^{\ell_i} \\
\downarrow & & \downarrow \\
T & \xrightarrow{\xi_i} & M_{V',P,ij}^{\ell_i}. \\
\end{array}
\]

The (ind) (cf. Remark 21) scheme \(M_{ij}\) in the upper right corner is a finite etale cover of \(M_{V',P,ij}^{\ell_i}\), such that a fixed universal family representative over \(M_{V',P,0}\) pulled back to \(M_{ij}\), is the same as the pullback to \(M_{ij}\) of a \(\rho_i\)-liftable universal family representative over \(M_{V',P,ij}^{\ell_i}\), cf. the discussion between Lemma 27 and Lemma 28.

The relationship between \(M_{ij}\) and \(M_{V',P,ij}^{\ell_i}\) is the same as that between \(T\) and \(S\) in the proof of Theorem 19. Hence the pullback \((T_{di}, t_{d0}) \to (T_i, t_{i0})\) is a finite etale cover.

The two diagrams together imply that the fixed universal family representative over \(M_{V',P,0}\), which is a pointed \(P\)-cover of \((M_{V',P,0} \times V', (\tilde{c}(t_0), v'_g))\), pulled back to \(T_{di}\) is \(\rho_{i'}\)-liftable. Let \((T_{d}, t_{d0})\) be the common pullback of the \((T_{di}, t_{d0})'s\) over \((T, t_0)\). The pullback to \(T_d\) of the fixed universal family representative over \(M_{V',P,0}\) is \(\rho_{i'}\)-liftable for every \(i\). Consider \(\rho_{i'}\)-liftable pairs at some places in the discussion above. Then a pair \((\tilde{W}, \tilde{w}_g) \to (T_d \times V', (t_{d0}, v'_g))\) is got, a pointed \(P\)-cover together with \(|P'|\) elements in \(Gal(\tilde{W}/T_d \times U)\), whose \(ER\)-equivalence class is in \(F_{V',P}^{er, Gal/\rho}(S, s_0)\).

The two maps are well defined for equivalence classes and inverse to each other.
Proof of the statements after “Moreover”: Every component $\bigcap_i M_{V',P_{ij}}^{0,P}$ of $M_{V',P}^{0,P}$ is a dense open of $\bigcap_i M_{V',P_{ij}}^{0,P}$, which itself is an affine closed subscheme of $M_{V',P_0}$. Pick any irreducible component of $\bigcap_i M_{V',P_{ij}}^{0,P}$ and a covering of it consisting of connected affine open subsets of finite type over $k$, which are all dense and intersect each other. Denote any of the affine open subsets by $M$. The inclusion of $M$ in $M_{V',P}$, with any base point $m_g$, gives a pointed $P$-cover of $(M' \times V', (m'_g, v'_g))$ for some $(M', m'_g) \to (M, m_g)$ of type-$ER$. The pointed $P$-cover satisfies a $M'$-parameterized version of $(**i)$ for every $i$ and thus gives a pointed $P \times_{\rho'_i} P'$-cover of $(M' \times U, (m'_g, u_g))$ for some $P \times_{\rho'_i} P'$ in $G_{p'}$, cf. Remark 32 and Remark 33. Hence for every closed point $m$ (need to use chemins for base point issues) in $M$, $P \times_{\rho'_i} P'$ belongs to the pair in $Gal_{V'/p'}$ that $m$ corresponds. Then Remark 32 and Remark 33 suffice to give all the statements.

Denote the maximal union of irreducible components of $M_{V',P}^{0,P}$, to all of whose closed points $P \times_{\rho_i} P'$ belongs as shown in Proposition 37 by $M_{U,V',P_{\rho_i}}^{ER}$. Denote by $M_{U,V',P_{\rho_i}}^{ER}$ the disjoint union of $M_{U,V',P_{\rho_i}}^{ER}$’s over all possible $(V', v'_g)$’s pointed connected $P'$-covers of $(U, u_g)$.

Define a functor $M_{U,V',P_{\rho_i}}^{ER}$: $S \to (Sets)$; $(S, s_0) \mapsto \{ER$-equivalence classes of $T \to M_{U,V',P_{\rho_i}}^{ER}$, where $(T, t_0) \to (S, s_0)$ runs over all morphisms to $S$ of type ER}. Similarly to $M_{V',P}^{ER}$ and $M_{V',P}^{0,P}$ above.

Define a functor $F_{U,V',P_{\rho_i}}^{ER}$: $S \to (Sets)$; $(S, s_0) \mapsto \{ER$-equivalence classes of pointed $P \times_{\rho_i} P'$-covers $(\tilde{W}, \tilde{w}_g) \to (T \times U, (t_0, u_g))$ whose fibers over closed points of $T$ are all connected, where $(T, t_0) \to (S, s_0)$ runs over all morphisms to $S$ of type ER}. The definition of $ER$-equivalence classes here is obvious, cf. definition of the functor $F_{V',P}^{ER,Gal/p'}$.

**Corollary 38.** The functor $F_{U,V',P_{\rho_i}}^{ER}$ is isomorphic to the functor $M_{U,V',P_{\rho_i}}^{ER}$, which shows that $M_{U,V',P_{\rho_i}}^{ER}$ is a moduli space for $P \times_{\rho_i} P'$-covers of $(U, u_g)$.

**Proof.** Directly from Proposition 37 and its proof. Cf. also proof of Theorem 24.

## 5 Local vs. global moduli

In Section 3, a fine moduli space (Proposition 42) for cyclic-by-$p$ covers of an affine curve tamely ramified over finitely many closed points, is constructed. The new type of fine moduli space is obtained by modifying the proof for the previous global fine moduli space constructed in Theorem 24 Section 3, and is constructed in similar 3 steps. The new type of fine moduli space is the global side of a local-global principal Proposition 54. There is a different phenomenon for cyclic-by-$p$ covers from that for $p$-covers. In [H80] the similar local-global principal for $p$-groups stated in Proposition 2.1 does not involve (tamely) ramified global covers; there the global covers are etale. The local-global principal Proposition 54 has a version over a general field of characteristic $p > 0$, which is Main Theorem 1.4.1 in [K86].

A parameter space for local cyclic-by-$p$ covers of $Spec(k((x)))$ is constructed in Proposition 53, which is the local side of the local-global principal Proposition 54. The construction is also by modifying the one in Section 3 and has similar 3 steps.

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Finally it is shown that a restriction morphism (a general case of the local-global principal Proposition 54 with the isomorphism there replaced by a finite etale morphism now) is finite etale, which is from the new type of global moduli space to a product of the local parameter spaces (Proposition 61), an analogue to Proposition 2.7 in [H80]. It is proved by a similar argument.

As always, we follow notations and terminology defined in Section 2. For example $G$ represents a cyclic-by-$p$ group.

Here are some necessary settings for the construction of the fine moduli space (Proposition 12).

Let $T$ be a finite set of closed points on $U$ not including $u_g$ and $U^0 = U - T$. Denote by $\{(V_i^0, v_i)\}$ the set of all the finitely many connected pointed $\mathbb{Z}/n_i$-covers of $(U^0, u_g)$, where $n_i$ can be any factor of $n$. Let $(V_i, v_i) \to (U, u_g)$ be the extension of $(V_i^0, v_i) \to (U^0, u_g)$, obtained by putting back in some deleted closed points from the smooth completions of both curves.

Let $F^T_{V,G}$ be the functor: $\mathcal{S}_1 \to (\text{Sets})$, $(S, s_0) \mapsto \{\text{equivalence classes of possibly ramified } G\text{-covers } \tilde{W} \to S \times U \text{ pointed over } (s_0, u_g), \text{ where the restriction of } \tilde{W} \text{ over } S \times U^0 \text{ is a } G\text{-cover and } \tilde{W} \to \tilde{W}/P \text{ is finite etale}\}$. Let $S = \text{Spec}(k)$ with $s_0$ determined by $u_g$ using diagram (2.1). Then $F^T_{V,G}(S, s_0)$ is the set of possibly ramified pointed $G$-covers $(W, w_g) \to (U, u_g)$ whose restriction $W^0$ over $U^0$ is a $G$-cover and $W \to W/P$ is finite etale.

A group homomorphism $\bar{\phi}^0 : \pi_1(S \times V_i^0, (s_0, v_i)) \to P$ factors through $V_i$ if $\bar{\phi}^0 = (\pi_1(S \times V_i^0, (s_0, v_i)))^{(\text{Id}_S \times (V_i^0 \subseteq V_i))}_{(s_0, v_i)} \pi_1(S \times V_i, (s_0, v_i)) \xrightarrow{\bar{\phi}} P$ for some $\bar{\phi}$.

Let $F^\rho_{\nu_i^0,T}$ be the functor: $\mathcal{S}_1 \to (\text{Sets})$, $(S, s_0) \mapsto \{\{[\bar{\phi}^0], p), \text{ where } (\bar{\phi}^0, p) \text{ is a } \rho_{n_i}\text{-liftable pair with } \bar{\phi}^0 \text{ factoring through } V_i, \}$]. Let $S = \text{Spec}(k)$ with $s_0$ determined by $v_i$ using diagram (2.1). Then $F^\rho_{\nu_i^0,T}(S, s_0)$ is the set of $\rho_{n_i}$-liftable pairs of $(V_i^0, v_i)$ the first entries of which can all extend to $P$-covers of $V_i$.

Below is the first of the 3 steps in constructing the fine moduli space in Proposition 12.

**Definition 39.** (Remark/Definition)

The cover $V_i \to U$ above is ramified at finitely many closed points on $U$. Hence the old definition of $\rho$-liftable can not apply here. New definition: A group homomorphism $\phi : \pi_1(V_i, v_i) \to H$ with $(\rho, H)$ in the case of Theorem 12 is $\rho$-liftable if $\phi$ makes the diagram in Lemma 10 commutative. Using the new definition of $\rho$-liftable in the definition of $F^\rho_{V, H}$, Theorem 12 still holds and $F^\rho_{V, H}$ is presented by the same fine moduli space $M^\rho_{V, H}$.

The new definition of $\rho$-liftable is used below for $V_i \to U$.

**Lemma 40.** With the notations above, suppose $(\rho_{n_i}, P)$ is in the case of Theorem 12 and write $H$ for $P$ in this case. The functor $F^\rho_{\nu_i^0,T}$ for $\rho_{n_i}$-liftable pairs of $(V_i^0, v_i)$ the first entries of which can all extend to $H$-covers of $V_i$, has a fine moduli space, a disjoint union of finitely many direct limits of affine spaces.

**Proof.** It is to be shown that $F^\rho_{\nu_i^0,T} = \bigoplus_h F^\rho_{V_i, H}$, copies of $F^\rho_{V_i, H}$ indexed by $h$, where depending on $(\rho_{n_i}, H)$ the $h$ runs over $H$ or it is just 1 (cf. Corollary 16). Suppose
a $\rho_l$-liftable pair $(\tilde{\phi}^0 : \pi_1(S \times V_i^0, (s_0, v_i)) \to H, h_0)$ with $\tilde{\phi}^0$ factoring through $V_i$ is a representative for an equivalence class $([\tilde{\phi}^0], h_0)$ in $F_{V_i,H}^{\rho_l,*,/T}(S, s_0)$. Since $(Id_S \times (V_i^0 \subset V_i))_*$ is surjective, $\tilde{\phi}$ is also $\rho_{nl}$-liftable (cf. Definition 39) and see above for $\tilde{\phi})$. The left to right map sends $([\tilde{\phi}^0], h_0)$ to $[\tilde{\phi}]$ in the copy of $F_{V_i,H}^{\rho_l,*,/T}(S, s_0)$ indexed by $h_0$. The inverse map is obvious. By Theorem 12 (cf. also Definition 39), $F_{V_i,H}^{\rho_l,*,/T}$ is represented by a direct limit of affine spaces $M_{V_i,H}^{\rho_l,*,/T}$.

Here is the 2nd of the 3 steps in constructing the fine moduli space in Proposition 42.

**Lemma 41.** The functor $F_{V_i,H}^{\rho_l,*,/T}$ for $\rho_{nl}$-liftable pairs of $(V_i^0, v_i)$ the first entries of which can all extend to $P$-covers of $V_i$, has a fine moduli space, a disjoint union of finitely many direct limits of affine spaces.

**Proof.** The proof is by simply replacing with their obvious counterparts symbols in and slightly modifying the proof of Theorem 19.

Denote $V_i$ by $V$ and $\rho_{nl}$ by $\rho$ in the proof. Replace $F_{V,H}^{*,/T}$ with $F_{V,0,P}^{*,/T}$, Corollary 16 with Lemma 40. $M_{V,P}^{*,/T}$ with $M_{V,0,P}^{*,/T}$, and $M_{V,H}^{*,/T}$ with $M_{V,0,H}^{*,/T}$.

In the 3rd paragraph of the proof of Theorem 19, since the first entry of some universal pair $(\tilde{\mu}_0 : \pi_1(\tilde{M} \times V^0, (m, v_i)) \to \tilde{P}, \tilde{h}_0)$ can factor through $V$: $\tilde{\mu}_0 = (\pi_1(\tilde{M} \times V^0, (m, v_i)) \to \pi_1(\tilde{M} \times V, (m, v_i)) \to \tilde{P})$ for some $\tilde{h}_0$, lift $(\tilde{\mu}_0, \tilde{h}_0)$ first to get $(\tilde{\psi}_0, \tilde{h}_0)$ with restriction $\tilde{\psi}_0$. The last paragraph in the proof of Theorem 19 carries over with some obvious modification involving the property of factoring through $V$.

Below is the last of the 3 steps in constructing the fine moduli space in Proposition 42.

**Proposition 42.** There is a fine moduli space representing $F_{U,G}^{T}$, the functor for pointed $G$-covers of $(U, u_g)$ tamely ramified over finitely many closed points $T$ on $U$, which is a disjoint union of finitely many direct limits of affine spaces.

**Proof.** By the same argument for Theorem 24 with slight modification, $F_{U,G}^{T} = \Pi_{U,H} F_{V_i,H}^{\rho_l,*,/T}$ which has a fine moduli space by Lemma 41.

Above is the construction of the global side of the local-global principal Proposition 54 whose local side is the local parameter space in Proposition 53 constructed below in similar 3 steps.

**Notation 43.** Recall $U_0 = \text{Spec}(k((x)))$ and point $U_0$ at $u_0$. Let $(V_{0t}, v_{0t})$ run over all the finitely many connected pointed $\mathbb{Z}/n_1$-covers of $(U_0, u_0)$, where $n_1$ can be any factor of $n$. Let $V_0$ be a connected $\mathbb{Z}/n$-cover of $U_0$ given by $k((x))[Y]/(Y^n - x) = k((y))$ with $\bar{t} \in \mathbb{Z}/n$ acting on $k((y))$ as $y \mapsto \zeta_n y$. Since $\mathbb{Z}/n$ is abelian, $V_0$ can be pointed at any $v_0$ over $u_0$, by Remark 5.
**Definition 44.** Two pointed $Gr$-covers of $(S \times U_0, (s_0, u_0))$ with $(S, s_0) \in S_1$ are $w$-equivalent if they become isomorphic pulled back to $(\tilde{T}_0, \tilde{t}_0)$, which is the restriction over $S \times U_0$ of some finite etale cover $(\tilde{T}, \tilde{t}) \to (S \times U_0, (s_0, u_0))$.

Let $\tilde{\phi}_i : \pi_1(S \times \tilde{U}_0, (s_0, u_0)) \to Gr$ $(i=1,2)$ be two group homomorphisms. They are $w$-equivalent if their corresponding pointed $Gr$-covers of $(S \times U_0, (s_0, u_0))$ are $w$-equivalent. Denote the $w$-equivalence class of $\tilde{\phi}_1$ by $[\tilde{\phi}_1]^w$.

**Remark 45.** The definition of $w$-equivalence is taken from the 2nd paragraph in the proof of Proposition 2.1 in [H80], which is the right definition of equivalence in the local case to make the proof work.

Let $F_{V_0,H}^{\text{wp}}$ be the functor: $S_1 \to \text{(Sets)}$, $(S, s_0) \mapsto \{$$w$-equivalence classes of $\rho$-liftable $P$-covers of $S \times V_0$ pointed over $(s_0, v_0)\}$. Below is the building block needed in the first of the 3 steps in constructing the local parameter space in Proposition 53.

**Proposition 46.** Let $(V_0, v_0)$ be given in Notation 43. Suppose $(\rho, H)$ is in the case of Theorem 13. Then there exists a fine moduli space $M_{V_0,H}^{\text{wp}}$ for $F_{V_0,H}^{\text{wp}}$, the functor for $w$-equivalence classes of pointed $\rho$-liftable $H$-covers of $(V_0, v_0)$.

**Proof.** The proof is parallel to that of Theorem 12.

Similarly to the proof of Theorem 12, start with a short exact sequence $k((y)) \xrightarrow{\rho} k((y)) \to H^1(V_0, H) \to 0$ given by the Artin-Schreier sequence. It can be simplified to $y^{-1}k[y^{-1}] \xrightarrow{\rho} y^{-1}k[y^{-1}] \xrightarrow{\sigma_0} H^1(V_0, H) \to 0$. (46.1)

Denote by $\sigma_0$ the automorphism in $\text{Gal}(k((y))/k((x)))$ given by $y \mapsto \zeta_n y$. The action of $\rho$ on $H$ is given by multiplication by $\epsilon_\rho \in \mathbb{F}_q$ $(q = |H|)$, cf. proof of Theorem 12. Similarly let $D_0$ be the $x^{-1}k[x^{-1}]$-module endomorphism $\sigma_0 - \epsilon_\rho$ of $y^{-1}k[y^{-1}]$.

Similarly extract from (46.1) an $\mathbb{F}_q$-vector space short exact sequence $Ker D_0 \xrightarrow{\rho} Ker D_0 \xrightarrow{\pi} \mathbb{X}_0 \to 0$, where $\mathbb{X}_0$ is the set of $\rho$-liftable pointed $H$-covers of $(V_0, v_0)$.

From the Ker exact sequence construct the fine moduli space $M_{V_0,H}^{\text{wp}}$ same as before, which is also a direct limit of affine spaces. Choose basis $K_0$, an analogue to $K_i$ in the proof of Theorem 12 inductively for $i \in \mathbb{N}$. The affine space $\text{Spec}(k[K_{0i+1} - K_{0i}])$ can be identified with the $(i+1)$-th piece of $M_{V_0,H}^{\text{wp}}$: the transition morphism from $\text{Spec}(k[K_{0i} - K_{0i-1}])$ to $\text{Spec}(k[K_{0i+1} - K_{0i}])$ is given by Frobenius as before. Finally, with slight modification to the last two paragraphs of the proof of Theorem 12, $M_{V_0,H}^{\text{wp}}$ can be shown to represent $F_{V_0,H}^{\text{wp}}$.

**Remark 47.** Similar to Remark 14 a canonical universal family representative over $M_{V_0,H}^{\text{wp}}$ can be given by $z^n - z = \sum_{k_0, k_{0n} - k_{0n-1}} k_{00} \otimes \mathbb{X}_0$ $(n \geq 1)$. Precise description can be got by some obvious replacement of symbols in Remark 14.

**Remark 48.** The remark is the base case in the proof for the local-global principal Proposition 54. Let $A' = \text{Spec}(k[x^{-1}])$. Suppose $A'$ is pointed at $u_0$ such that the map $U_0 \to A'$ sends $u_0$ to $a_g$.

Let $V \to A'$ be the $\mathbb{Z}/n$-cover given by $k[x^{-1}][y^{-1}]/((Y^{-1})^n - x^{-1}) = k[y^{-1}]$ ramified at $\infty$, with $\tilde{1} \in \mathbb{Z}/n$ acting as $y^{-1} \mapsto \zeta_n^{-1} y^{-1}$. Point $V$ at $v_y$ such that $V \to A'$ sends
v_g to a_g. Its restriction (pullback) at 0 gives \((V_0, v'_0)\) and let \(v_0\) above be \(v'_0\):

\[
\begin{array}{ccc}
(V_0, v'_0) & \rightarrow & (V, v_g) \\
\downarrow & & \downarrow \\
(U_0, u_0) & \rightarrow & (A^1', a_g).
\end{array}
\]

The constructions show that \(M^{wp}_{V_0,H} = M^p_{V,H}\):

The short exact sequence \(y^{-1}k[y^{-1}] \rightarrow y^{-1}k[y^{-1}] \rightarrow H^1(V_0, H) \rightarrow 0\) in the proof of Proposition 46 is similar to the one \(k[y^{-1}] \rightarrow k[y^{-1}] \rightarrow H^1(V, H) \rightarrow 0\) for \(V\) in the proof of Theorem 12 after modding \(k[y^{-1}]\) by \(k\). The short exact sequence \(KerD_0 \rightarrow KerD_0 \rightarrow X_0 \rightarrow 0\) above, is similar to the short exact sequence \(KerD \rightarrow KerD \rightarrow X \rightarrow 0\) for \(V\) in the proof of Theorem 12 and \(KerD_0 = KerD\). Then the constructions of the two moduli spaces out of the \(Ker\) short exact sequences are the same, which shows that \(M^{wp}_{V_0,H}\) is the same ind scheme as \(M^p_{V,H}\).

Moreover, there is a triangle compatibility diagram. For any pointed \(\rho\)-liftable \(H\)-cover \((W, w_g)\) of \((V, v_g)\) corresponding to some \(k\)-morphism \(Spec(k) \rightarrow M^p_{V,H}\), its restriction \((W_0, w_0)\) over \((V_0, v_0)\) is a pointed \(\rho\)-liftable \(H\)-cover of \(V_0\):

\[
\begin{array}{ccc}
(W_0, w_0) & \rightarrow & (W, w_g) \\
\downarrow & & \downarrow \\
(V_0, v_0) & \rightarrow & (V, v_g).
\end{array}
\]

The local cover corresponds to some \(k\)-morphism \(Spec(k) \rightarrow M^{wp}_{V_0,H}\). The following diagram commutes:

\[
\begin{array}{ccc}
Spec(k) & \rightarrow & M^{wp}_{V_0,H} \\
\downarrow & & \downarrow \\
Spec(k) & \rightarrow & M^p_{V,H}.
\end{array}
\]  

\(\text{(48.1)}\)

\textbf{Remark 49.} All the \(\mathbb{Z}/n_i\)-covers of \(U_0\), where \(n_i\) can be any factor of \(n\), correspond bijectively to all the \(\mathbb{Z}/n_i\)-covers of \(A^1\) ramified at \(\infty\), since these covers can be given by explicit equations of the type in Proposition 46 and that in Remark 48.

Below is the first of the 3 steps in constructing the local parameter space in Proposition 53 using the building block in Proposition 46.

\textbf{Definition 50.} For the local case, a pointed \(\rho\)-liftable \(P\)-cover of \((V_0, v_0)\) is defined in the obvious similar way to the global case in diagram (3.1). Similarly for a \(\rho\)-liftable pair.

The \(k\)-points of an ind scheme \(M\) parameterize certain covers, if there is a bijection \(\chi\) together given with \(M\) between the set of \(k\)-points of \(M\) and the set of these certain covers.

\textbf{Lemma 51.} Suppose \((\rho, H)\) is in the case of Theorem 12. There exists a parameter space \(M^{wp}_{\rho,H}\), a disjoint union of finitely many inductive limits of affine spaces, whose \(k\)-points parameterize (cf. Definition 50) all the \(\rho\)-liftable pairs of \((V_0, v_0)\).
\textbf{Proof.} Let }S = \text{Spec}(k)\text{ pointed at }s_0\text{ which is determined by }v_0\text{, using diagram (21). Since }M_{V_0,H}^{\varphi}\text{ represents }F_{V_0,H}^{\varphi,\text{ep}}\text{, there is a bijection }\chi_{V_0,H}\text{ between }F_{V_0,H}^{\varphi,\text{ep}}(\text{Spec}(k), s_0)\text{ and }M_{V_0,H}^{\varphi}(\text{Spec}(k)). \ F_{V_0,H}^{\varphi,\text{ep}}(\text{Spec}(k), s_0)\text{ is the set of pointed }\rho\text{-liftable }H\text{-covers of }\langle V_0, v_0 \rangle. \text{ Let }M_{V_0,H}^{\varphi,\bullet} = M_{V_0,H}^{\varphi,\text{ep}}, \text{ an analogue of Corollary 16. Depending on }\rho, H\text{, }h \text{ runs over }H \text{ or it is just }1. \text{ By the same kind of argument of Corollary 16, there is a bijection }\chi_{V_0,H}^{\varphi,\bullet}\text{ between the set of }\rho\text{-liftable pairs of pointed }H\text{-covers of }\langle V_0, v_0 \rangle, \text{ and }M_{V_0,H}^{\varphi,\bullet}(\text{Spec}(k)). \ \blacksquare

Here is the 2nd of the 3 steps in constructing the local parameter space in Proposition 53.

\textbf{Lemma 52.} \text{There exists a parameter space }M_{V_0,P}^{\varphi,\bullet}, \text{ a disjoint union of finitely many inductive limits of affine spaces, whose }k\text{-points parameterize (cf. Definition 56) all the }\rho\text{-liftable pairs of }\langle V_0, v_0 \rangle.

\textbf{Proof.} \text{The proof is parallel to that of Theorem 19 but simpler. It simply replace some symbols in and do a little modification to the proof of Theorem 19. First of all there is no longer an }F, \text{ instead there is }C_{V_0,P}^{\varphi,\bullet}\text{ the set of }\rho\text{-liftable pairs of pointed }P\text{-covers of }\langle V_0, v_0 \rangle. \text{ In the 2nd paragraph replace }M_{V,P}^{\beta,\bullet} \text{ by }M_{V_0,P}^{\varphi,\bullet}\text{ and }M_{V,H}^{\beta,\bullet} \text{ by }M_{V_0,H}^{\varphi,\bullet}. \text{ In the 3rd paragraph replace }V \text{ by }V_0. \text{ In the last paragraph, there is no longer an }S. \text{ Replace every }V \text{ by }V_0, \text{ and }v_g \text{ by }v_0. \text{ Replace }\tilde{\phi},p_1 \text{ by an element }\varphi: \pi_1(V_0, v_0) \rightarrow P,p_1 \text{ in }C_{V_0,P}^{\varphi,\bullet} \text{ and }\tilde{\varphi},\tilde{p}_1 \text{ by }\tilde{\varphi}: \pi_1(V_0, v_0) \rightarrow \tilde{P},\tilde{p}_1. \text{ Then replace }\beta \text{ by a }k\text{-morphism }Spec(k) \rightarrow \tilde{M} \text{ and }\tilde{\beta}_s \text{ by }\tilde{\beta}_s. \text{ There is no need for etale descent now and one directly gets a }c_{\alpha} : Spec(k) \rightarrow M^0. \text{ Then replace }M_{V,P}^{\beta,\bullet} \text{ by }M_{V_0,P}^{\varphi,\bullet}. \text{ Finally the assignment }\varphi \mapsto (c_{\alpha},c_{\beta}) \text{ is a bijection between }C_{V_0,P}^{\varphi,\bullet} \text{ and }M(\text{Spec}(k)), \text{ which gives the bijection }\chi_{V_0,P}^{\varphi,\bullet} \text{ desired.} \ \blacksquare

Here is the last of the 3 steps in constructing the local parameter space in Proposition 53.

\textbf{Proposition 53.} \text{There exists a parameter space }M_{U_0,G}^{\varphi,\bullet}, \text{ a disjoint union of finitely many direct limits of affine spaces, whose }k\text{-points parameterize (cf. Definition 56) all the pointed }G\text{-covers of }\langle U_0, u_0 \rangle.

\textbf{Proof.} \text{Let }M_{U_0,G}^{\varphi} = \Pi_{V_0}M_{V_0,P}^{\varphi,\bullet}, \text{ an analogue of Theorem 24. Using the argument of Theorem 24 with some obvious modification and Lemma 52, there is a bijection }\chi_{U_0,G}^{\varphi} \text{ between }k\text{-points of }M_{U_0,G}^{\varphi} \text{ and pointed }G\text{-covers of }\langle U_0, u_0 \rangle. \ \blacksquare

Let }M_{A',G}^{\infty} \text{, with }A' \text{ defined in Remark 48, be the short hand notation for }M_{A',G}^{\varphi,\bullet}. \text{ the fine moduli space for }F_{A',G}^{\infty} \text{ given by Proposition 42. Below is the local-global principal which involves the global moduli space in Proposition 42 and the local parameter space in Proposition 53.

\textbf{Proposition 54.} \text{The fine moduli space }M_{A',G}^{\infty} \text{, is the same ind scheme as the parameter space }M_{U_0,G}^{\varphi,\bullet}, \text{ compatibly with the inclusion of }U_0 \text{ in }A' \text{ (cf. diagram (48.1)).}
**Proof.** In the construction of both spaces, there are similar 3 steps to the global case, i.e. Theorem 12⇒Theorem 19⇒Theorem 24. Hence the equality wanted will be proven in similar 3 steps. The bijections χ’s given in Lemma 51, Lemma 52 and Proposition 53 will be used but not written out unnecessarily.

First both spaces have as building blocks an analogue of the moduli space in Theorem 12. Let \( V_0 \) and \( V \) be the same as in Remark 48 which shows that the local building block is the same as the global one and a triangle compatibility diagram holds. Then using the same kind of argument as in Corollary 16 \( M^{\rho}_{V_0,H} = M^{\rho}_{V,H} \) and a triangle compatibility diagram similar to that in Remark 48 holds. Moreover, by Remark 48 the canonical \( \rho \)-liftable universal family representative of \( H \)-covers of \( V_0 \) over each connected component of \( M^{\rho}_{V_0,H} \), is the restriction of the canonical \( \rho \)-liftable universal family representative of \( H \)-covers of \( V \) over the corresponding connected component of \( M^{\rho}_{V,H} \), which is \( M^{\rho}/(\infty) \) by Lemma 46. Here \( U = A^1 \), \( T = \{ \infty \} \), and \( V^0 \) is defined at the beginning of this section.

Next \( M^{\rho}_{V_0,P} = M^{\rho}_{V_0,H} \times M^{\rho}_{V_0,P} \) and \( M^{\rho}/(\infty) = M^{\rho}_{V,H}/(\infty) \times M^{\rho}_{V,P}/(\infty) \) given respectively in Lemma 52 and Lemma 41. By inductive hypothesis \( M^{\rho}_{V_0,P} = M^{\rho}_{V,P}/(\infty) \), a triangle compatibility diagram similar to that in Remark 48 holds, and a \( \tilde{\rho} \)-liftable universal family representative of pointed \( \tilde{\rho} \)-covers of \((V_0,v_0)\) over each connected component of \( M^{\rho}_{V,P} \), is the restriction of the \( \tilde{\rho} \)-liftable universal family representative of pointed \( \tilde{\rho} \)-covers of \((V,v_0)\) over the corresponding connected component of \( M^{\rho}_{V,P}/(\infty) \). (Strictly speaking, Definition 18 needs to be used and pairs should be dealt with, which however will make the proof unnecessarily longer.) A \( \rho \)-liftable lift of the previous representative can be got from the restriction of a \( \rho \)-liftable lift of the latter representative. By this fact and the paragraph above \( M^{\rho}_{V_0,P} = M^{\rho}_{V,P}/(\infty) \) and a triangle compatibility diagram similar to that in Remark 48 holds.

Finally by Remark 49, Proposition 42 and Proposition 53 the proposition follows and there is a triangle compatibility diagram similar to that in Remark 48.

**Corollary 55.** Any pointed \( G \)-cover of \((U_0,u_0)\) extends uniquely to a pointed \( G \)-cover of \((A^1',a_g)\) which is tamely ramified at \( \infty \).

**Proof.** By the compatibility assertion in Proposition 54.

Here are some necessary settings for the last result in Section 5, Proposition 61.

**Notation 56.** Let \( U_0 \) be the spectrum of the fraction field of the complete local ring at the \( i \)-th closed point of \( \bar{U} - U \), which is an infinitesimal neighborhood of that point. Let \( n' \) be a factor of \( n \). Let \( V_0,n' \) be a fixed connected \( \mathbb{Z}/n' \)-cover of \( U_0 \). All the connected \( \mathbb{Z}/n' \)-covers of \( U_0 \) are isomorphic to \( V_0,n' \) (cf. Remark 19); they only differ by the action of \( \mathbb{Z}/n' \). Any two actions differ by an element in \( Aut(\mathbb{Z}/n') \).

For any \( (n_i)_i \), where \( n_i \) is a factor of \( n \), there exists a possibly ramified connected \( \mathbb{Z}/n \)-cover \( V \) of \( U \) which may ramify at a finite set of closed points \( T \) on \( U \), such that its ramification index at \( U_0 \) is \( n_i \). The cover \( V \) can be obtained as follows: Suppose \( U = Spec(A) \) and denote the fraction field of \( A \) by \( K \). Pick \( a_0 \in A \), such that \( a_0 \) has poles \( \sum_i N_i Q_i \), where \( Q_i \) is the \( i \)-th closed point of \( \bar{U} - U \) and \( N_i >> 0 \) with \( (N_i,n) = n/n_i \). By Riemann-Roch, such an \( a_0 \) exists. By adding a constant in \( K \) to \( a_0 \), \( Y^n - a_0 \)
can be assumed an irreducible polynomial in $K[Y]$. Denote $K[Y]/(Y^n-a_0) = K(y)$ by $F$. The normalization of $U$ in $F$ gives $V$, which may ramify over the zeros of $a_0$ on $U$.

Suppose $U_0$ is pointed at $u_0$ and $(U_0, u_0)$ maps to $(U, u_{gi})$. Choose $v_{gi}$ such that $(V, v_{gi}) \rightarrow (U, u_{gi})$. Let the pointed connected component of $V$’s restriction (pullback) over $U_0$ be $(V_{i0}, v_{i0})$:

$$(V_{i0}, v_{i0}) \longrightarrow (V, v_{gi})$$

$$(U_0, u_{i0}) \longrightarrow (U, u_{gi}).$$

Then $V_{i0}$ is isomorphic to $V_{0,n_i}$ which is one of those fixed above, as covers of $U_0$. Choose $v_g$ such that $(V, v_g) \rightarrow (U, v_g)$ and chemins $\omega_i$ from $v_{gi}$ to $v_g$ which induce chemins $\varpi_i$ from $u_{gi}$ to $u_g$.

Let $U^0 = U - T$ and $V^0$ be $V$’s restriction over $U^0$, same as the beginning of this section. A $\rho$-liftable pointed $P$-cover of $(V^0, v_g)$ gives a $\rho_{n_i}$-liftable (cf. proof of Theorem 24) pointed $P$-cover of $(V_{i0}, v_{i0})$ for each $i$ using the following diagram:

$$\pi_1(V_{i0}, v_{i0}) \longrightarrow \pi_1(V^0, v_{gi}) \longrightarrow \pi_1(V^0, v_g) \longrightarrow P$$

$$\pi_1(U_{i0}, u_{i0}) \longrightarrow \pi_1(U^0, u_{gi}) \longrightarrow \pi_1(U^0, u_g) \longrightarrow P \rtimes_\rho \mathbb{Z}/n,$$

where $\tau_{\omega_i}$ is the isomorphism induced by the chemin $\omega_i$ and similarly for $\tau_{\varpi_i}$.

Here is a definition involved in the statement of Proposition 61.

**Definition 57.** For every $(V_{0,n_i}, v_{0})$ a degree $n_i$ cover of $(U_0, u_0)$, denote by $M_{V_{0,n_i}, P}^{ppn_i} a$ connected component (cf. Remark 8) of $M_{V_{0,n_i}, P}^{ppn_i}$. Let $i$ be the index for the $i$-th closed point of $\bar{U} - U$ and $(n_i)_i$ the same notation in Notation 56. A morphism from an ind scheme that is a disjoint union of finitely many direct limits of affine spaces, to $\Pi M_{P, G}^\rho$ is essentially surjective, if for any $(n_i)_i$ there is a connected component of the source ind scheme which maps surjectively to a connected component of the target ind scheme, whose $i$-th factor for each $i$ is $M_{V_{0,n_i}, P}^{ppn_i}$ for some $(V_{0,n_i}, v_{0})$ a degree $n_i$ cover of $(U_{0}, u_{0})$.

**Remark 58.** The definition of essentially surjective is needed because: Suppose $(V_i)^0$ is a tuple whose $i$-th component is the restriction of a possibly ramified $\mathbb{Z}/n$-cover $V$ of $U$ to and of degree $n_i$ over $U_0$. The Galois actions of $\mathbb{Z}/n_i$’s on the $V_{i0}$’s are related to each other as shown in the example. Thus not every tuple $(V_{0,n_i})_i$ (same notation as in Definition 57) could be the image of the restrictions of some $V$. Hence the restriction morphism in Proposition 61 below is not surjective. However in some sense it is surjective, which motivates the definition of essential surjectivity.

Suppose $p = 3$. The $\mathbb{Z}/3$-cover of $U = Spec(k[x, x^{-1}])$, the affine line with 0 deleted, given by $V = Spec(k[x, x^{-1}][y]/(Y^3 - x)) = Spec(k[x, x^{-1}][y])$ with $\bar{y} \in \mathbb{Z}/3$ acting on $V$ over $U$ as $y \mapsto \zeta_3 y$, has restrictions at 0 and $\infty$. At 0, its restriction is a $\mathbb{Z}/3$-cover of $Spec(k((x)))$ given by $V_{00} = Spec(k((x))[y]/(Y^3 - x)) = Spec(k((x))[y])$ with $\bar{y} \in \mathbb{Z}/3$ acting on $V_{00}$ over $Spec(k((x)))$ as $y \mapsto \zeta_3 y$. At $\infty$, its restriction is a $\mathbb{Z}/3$-cover of
Since $\bar{1} \in \mathbb{Z}/3$ acting on $V_{0,\infty}$ over $\text{Spec}(k((x^{-1})))$ as $y \mapsto \zeta_3 y$.

Changing the $\mathbb{Z}/3$ actions on the two local $\mathbb{Z}/3$-covers at 0 and $\infty$ above, the pair of local $\mathbb{Z}/3$-covers $(\text{Spec}(k((x)))[Y]/(Y^3-x)) \rightarrow \text{Spec}(k((x)))$, $\bar{1}: y \mapsto \zeta_3 y$ , $(\text{Spec}(k((x^{-1})))[Y]/(Y^3-x)) \rightarrow \text{Spec}(k((x^{-1})))$, $\bar{1}: y \mapsto \zeta_3^{-1} y$) got can not come from restrictions of a global $\mathbb{Z}/3$-cover of $\text{Spec}(k[x, x^{-1}])$.

Below is another ingredient involved in the statement of Proposition 61.

For any $(n_i)_i$, as shown in Notation 56, there exists a $V_{(n_i)_i}$, which may ramify at a finite set of closed points on $U$, denoted by $T_{V_{(n_i)_i}}$, such that its ramification index at $U_0$ is $n_i$. Let $T = \cup_{(n_i)_i} T_{V_{(n_i)_i}}$.

The last ingredient involved in the statement of Proposition 61, the restriction morphism, is given in two steps in Lemma 59 and Lemma 60. First a map $\tau$ involved in the statements of Lemma 59 and Lemma 60 is defined.

A pointed $G$-cover of $(U^0, u_0)$ gives a local cover of $(U_0, u_0)$ for each $i$: $\pi_1(U_0, u_0) \rightarrow \pi_1(U^0, u_{g_i}) \rightarrow \pi_1(U^0, u_0) \rightarrow G$. Thus there is a map $\tau$ from the closed points (same as $k$-points) of $M^p_{V,H}$, which parameterize certain pointed $G$-covers of $(U^0, u_0)$, to the closed points of $\Pi, M^p_{V_0, G}$, which parameterize tuples each of which consists of covers indexed by $i$ with the $i$-th entry a pointed $G$-cover of $(U_0, u_0)$. Similarly there is a map $\tau_0$ from the closed points of $M^p_{V,H}$ to those of $\Pi, M^w_{V_0,H}$.

Lemma 59. Suppose $(\rho, H)$ is in the case of Theorem 12. With the notations above, there is a restriction morphism $r_0 : M^p_{V,H} \rightarrow \Pi, M^p_{V_0,H}$ such that every closed point of $M^p_{V,H}$ maps to the same closed point of $\Pi, M^p_{V_0,H}$ under $r_0$ or $\tau_0$.

Proof. $r_0$ is given by giving for every $i$ its $i$-th factor using $M^p_{V_0,H} = M^w_{V_0,H}$ is a fine moduli space.

Denote by $\bar{Z}$ the canonical $\rho$-liftable universal family representative of $H$-covers of $V$ over $M^p_{V,H}$, which corresponds to, for every point $m$ on $M^p_{V,H}$, some group homomorphism $\pi_1(M^p_{V,H} \times V, (m, v_0)) \rightarrow H$. The composition $\pi_1(M^p_{V,H} \times V_0, (m, v_0)) \rightarrow \pi_1(M^p_{V,H} \times V, (m, v_0)) \rightarrow \pi_1(M^p_{V,H} \times V, (m, v_0))$ gives a pointed $H$-cover of $(M^p_{V,H} \times V_0, (m, v_0))$ the non pointed version of which is denoted by $\bar{Z}_0$. By Remark 3 there are base points here do not matter. It is the restriction (pullback) of $\bar{Z}$ to $M^p_{V,H} \times V_0$:

$$\begin{array}{ccc}
\bar{Z}_0 & \longrightarrow & \bar{Z} \\
\downarrow & & \downarrow \\
M^p_{V,H} \times V_0 & \longrightarrow & M^p_{V,H} \times V.
\end{array}$$

Since $M^w_{V_0,H} = M^w_{V_0,H}$ and $M^w_{V_0,H}$ represents $F^w_{V_0,H}$ by Proposition 46, there is a morphism $r_{0i} : M^p_{V,H} \rightarrow M^w_{V_0,H}$ given by $\bar{Z}_0$. A different base point $m'$ gives the same $r_{0_i}$. Then define $r_0 := (r_{0i})_i$.

Now it is enough to verify that a closed point $m'$ of $M^p_{V,H}$ maps to the same closed point under $r_{0i}$ or $r_0$, where $r_{0i}$ is the $i$-th factor of $r_0$. Tracking definitions, $r_{0i}(m')$
represents the restriction to \((V_{i_0}, v_{i_0})\) of the pointed \(H\)-cover of \((V, v_g)\) represented by \(m'\). And \(r_i\) does the same thing by its definition. So \(r_{i_0}\) and \(r_{0i}\) agree. \(\square\)

**Lemma 60.** Let \(G = H \times_\rho \mathbb{Z}/n\) for some \((\rho, H)\) in the case of Theorem 13. There is a restriction morphism \(r : M^T_{U,G} \rightarrow \Pi_i M^p_{U_{i_0},G}\) such that every closed point of \(M^T_{U,G}\) maps to the same closed point of \(\Pi_i M^p_{U_{i_0},G}\) under \(r\) or \(r^*\), where \(r^*\) is defined above Lemma 59.

**Proof.** By construction, \(M^T_{U,G}\) and \(\Pi_i M^p_{U_{i_0},G}\) are both a disjoint union of finitely many direct limits of affine spaces. The morphism \(r\) will be given for each connected component of \(M^T_{U,G}\).

Proposition 42 and Lemma 40 give \(M^T_{U,G} = \Pi_i M^{p_{ni}}_{V_{i_0}}\) respectively. A connected component of \(M^T_{U,G}\) is of the form \(M^{p_{ni}}_{V_{i_0}}\).

Let the pointed connected component of \(V_i\) over \(U_0, \text{ be } V_{i_0}\), a \(\mathbb{Z}/n_i\)-cover of \(U_0\). Using Notation 56, \(r\) maps \(M^{p_{ni}}_{V_{i_0}}\) to \(\Pi_i M^{p_{ni}}_{V_{i_0}}\) where \(r\) is required to agree with the map \(r\) on closed points. Similarly the target connected component of each connected component of \(M^{p_{ni}}_{V_{i_0}}\) under \(r\) can be identified. Denote by \(M^{p_{ni}}_{V_{i_0},H}\) a connected component of \(M^{p_{ni}}_{V_{i_0}}\) by \(\Pi_i M^{p_{ni}}_{V_{i}}\) its target connected component, and by \(r_{ij}\) (suppose \(M^{p_{ni}}_{V_{i_0},H}\) is the \(j\)-th component) the restriction of \(r\) on \(M^{p_{ni}}_{V_{i_0},H}\).

Finally let \(r_{ij}\) be the restriction morphism given in Lemma 59 for every index \(l_j\). One can check that the morphism \(r\) satisfies the requirement. \(\square\)

With the preparation from Notation 56 to Lemma 60 the last result in Section 5 can be given.

**Proposition 61.** Let \(G = H \times_\rho \mathbb{Z}/n\) for some \((\rho, H)\) in the case of Theorem 13. The restriction morphism \(M^T_{U,G} \rightarrow \Pi_i M^p_{U_{i_0},G}\) given in Lemma 60 is essentially surjective and finite etale. And the degrees of \(r\) on different connected components of \(M^T_{U,G}\) are all powers of \(p\).

**Proof.** The proof follow the points of the proof of Proposition 2.7 in [H80]. A calculation of the dimensions of the \(n\)-th pieces of both source and target shows that they are the same. By this fact the map \(r\) restricted on each connected component of the source can be proven surjective. Then all the three statements follow.

With the same notations as in the proof of Lemma 60 denote a connected component of \(M^{p_{ni}}_{V_{i_0}}\) by \(M^{p_{ni}}_{V_{i_0},H}\), whose \(n\)-th piece is \(M^{p_{ni}}_{V_{i_0},H_{ni}}\). Denote the connected component of \(\Pi_i M^{p_{ni}}_{V_{i_0}}\), which \(M^{p_{ni}}_{V_{i_0},H}\) maps to under \(r_{ij}\), by \(\Pi_i M^{p_{ni}}_{V_{i}}\), whose \(n\)-th piece is \(\Pi_i M^{p_{ni}}_{V_{i_0},H_{ni}}\).

For \(n >> 0\), Riemann-Roch shows that the dimension of \(M^{p_{ni}}_{V_{i_0},H_{ni}}\) is at least that of \(\Pi_i M^{p_{ni}}_{V_{i_0}}\) for a subsequence \(\{n_k\}\) of \(\mathbb{N}\): A simpler but similar computation is done in the 1st paragraph of the proof of Proposition 2.7 in [H80]. Here pass from \(V_i\) to \(U\) first using ramification indices and then do a similar computation to [H80]. Denote by \(\Sigma d_{0n}\) the dimension of \(\Pi_i M^{p_{ni}}_{V_{i_0}}\). For \(n >> 0\) Riemann-Roch gives \(\Sigma d_{0n} = \Sigma \left( p^{a_{ni} - 3a_{ni}} - \left\lfloor \frac{p^{a_{ni} - 3a_{ni}}}{n_i} \right\rfloor \right)\) with some natural number \(i_0\) between 0 and \(n_i\). Denoted by \(d_{ni}\) the dimension of \(M^{p_{ni}}_{V_{i_0},H_{ni}}\). For \(n >> 0\) a similar computation gives \(d_n = \left\lfloor p^\delta \right\rfloor - \left\lfloor \frac{p^{a_{ni} - 3a_{ni}}}{n_i} + \delta \right\rfloor\) for
some $\delta \in \mathbb{Q}$. Since the remainder of $q^n$ divided by $n_i$ is periodic for $n \in \mathbb{N}$ there is a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $d_{n_k} \geq \Sigma d_{0n_k}$.

$r_{lj}$ is quasi finite of degree a $p$-power: The restriction of $r_{lj}$ on the $n$-th piece of $M^p_{V_i,H}$ gives $M^{p_{n_k}}_{V_i,H} \xrightarrow{r_{ljn}} \Pi_i M^{p_{n_k}}_{V_i,H,n}$ using Lemma 59, which is in fact a homomorphism between $\mathbb{F}_p$-vector spaces (The closed points of any moduli space involved here form a $\mathbb{F}_p$-vector space, by the definitions of the moduli spaces.) Since there are, up to isomorphism, only finitely many pointed (etale) $P$-covers of the completion $X_l = \bar{V}_l$ [SGA1, X 2.12], the kernel of $r_{ljn}$ is finite (and equal to this number when $n >> 0$). Thus every non empty fiber of $r_{ljn}$ (hence of $r_{lj}$) contains the same finite number of points. This number is a power of $p$, being the cardinality of a $\mathbb{F}_p$-vector space.

The 2nd paragraph in the proof of Proposition 2.7 in [H80] shows that since $d_{n_k} \geq \Sigma d_{0n_k}$ for every $k$ large enough and $r_{ljn}$ is quasi finite, the morphism $M^{p_{n_k}}_{V_i,H,n_k} \to \Pi_i M^{p_{n_k}}_{V_i,H,n_k}$ is surjective. Thus for every $n$ large enough, using Lemma 59, the morphism $M^{p_{n_k}}_{V_i,H,n} \to \Pi_i M^{p_{n_k}}_{V_i,H,n}$ is surjective. Hence the map $r$ restricted on every connected component of $M^{p_{n_k}}_{V_i,H}^{\bullet\mp}$ maps surjectively to a connected component of $\Pi_i M_{V_i,H}^{p_{n_k}T}$.

Then the last paragraph there carries over which shows that the map $r$ is etale. The choice of $T$ shows that $r$ is essentially surjective.

\[ \square \]

6 Global cyclic-by-$p$ moduli stack

[TY17] constructs a moduli space for local cyclic-by-$p$ torsors of $Spec(k((x)))$, which is an inductive limit of Deligne-Mumford stacks, ind DM-stack for short. In Section 6 a parallel construction to the one in [TY17] gives a moduli for global cyclic-by-$p$ torsors of an affine curve, which is also an ind DM-stack (Theorem $A'$).

Theorem $A'$ is the global analogue of Theorem A in [TY17]; similarly for other numberings.

The parallel construction is not written out in full detail. The only two concrete proofs in the parallel construction are those of Theorem 4.16' and Theorem B'. One can get the complete construction by reading [TY17], replacing symbols and terms with their obvious global counterparts, and checking similar results to those in [TY17] hold in the global case.

Notation 62. Terms defined here and in the first section “Notation and Terminology” of [TY17] will be used within this section without being explained again.

Let $k$ be a field of characteristic $p > 0$ which contains $\mu_n$, the group of all the $n$-th roots of unity with $n$ and $p$ coprime. Let $C_u = Spec(A)$ be a connected smooth affine 1 dimensional scheme of finite type over $k$, $F$ the fraction field of $A$ and $A_\infty$ the finite set of places in $\mathbb{P}(F/k)$ which are not in $C_u$. Assume that $k$ is the full constant field of $F$ and all places in $A_\infty$ are of degree 1. Let $C$ be the site of affine schemes over $k$, with fppf topology. For a $k$-algebra $R$ write $R \in C$ instead of $Spec(R) \in C$. Denote by $\mathcal{C}_R$ the subsite of $C$ whose objects are over $Spec(R)$. For any finite group $Gr$, let $\Delta_{Gr,C_U}$ be the category fibered in groupoids over the category $C$, such that for an $R \in C$, $\Delta_{Gr,C_U}(R)$ is the category of $Gr$-torsors over $Spec(R) \times_k C_U$. Let $H$ be a finite $p$-group.
Remark 63. The above assumptions on the field are made so that the arguments are simpler without involving too many technical details. For the same reason, only finite constant groups are considered.

Theorem 64. (Theorem A’) Let $G$ be a semidirect product $H \rtimes \mu_n$. With the notations above, there exists a direct system $X_\ast$, with the index set the natural numbers, of separated DM stacks with finite and universally injective transition maps, with a direct system of finite and etale atlases $X_n \to X_n$ from affine schemes and with an equivalence $\varinjlim \Delta_{G,C_U}^n$.

For the group $H$, the stacks $X_n$ are smooth and integral. More precisely there is a strictly increasing sequence $v: \mathbb{N} \to \mathbb{N}$ such that $X_n = \mathbb{A}^v_n$, the maps $\mathbb{A}^v_n \to X_n$ are finite and etale and the transition maps $\mathbb{A}^v_n \to \mathbb{A}^{v+1}$ are composition of the inclusion $\mathbb{A}^v_n \to \mathbb{A}^{v+1}$ and the Frobenius $\mathbb{A}^{v+1} \to \mathbb{A}^{v+1}$.

Remark 65. The first statement of Theorem A’ is an analogue of Theorem 24. The 2nd statement is an analogue of Theorem 19. In Theorem 19, a $p$-group is denoted by $P$ while in Theorem A’ by $H$, which denotes an elementary abelian group in Theorem 12.

Denote by $\{C_{V_i}\}$ the set of all the finitely many $\mu_n$-torsors of $C_U$.

Theorem 66. (Theorem B’) With the notations above, we have an equivalence

$$\bigsqcup_q B(\mu_n) \to \Delta^\prime_{\mu_n,C_U}$$

where the map $B(\mu_n) \to \Delta^\prime_{\mu_n,C_U}$ in the index $q$ maps the trivial $\mu_n$-torsor to the $\mu_n$-torsor $C_{V_q} \to C_U \in \Delta^\prime_{\mu_n,C_U}(k)$.

Remark 67. Theorem B’ is a version of the fact used in Section 3 that there are only finitely many $\mathbb{Z}/n$-covers of $U$. Any representative in a family of $\mathbb{Z}/n$-covers of $U$ parameterized by some scheme $S \in S^1_i$ becomes a product after a finite etale pullback of $S$, which is used in the proof of Theorem 24. The symbols $C_{V_q}$ and $C_U$ are the counterparts of $V_i$ and $U$ given at the beginning of Section 3.

4.1’ The group $G = \mathbb{Z}/p$.

In this section we consider $G = \mathbb{Z}/p$. Let $C$ be an $\mathbb{F}/p$-algebra. By Artin-Schreier a $\mathbb{Z}/p$-torsor is of the form $C[X]/(X^p - X - c)$, where $c \in C$ and the action is induced by $X \mapsto X + f$ for $f \in \mathbb{F}_p$.

Proposition 68. (Proposition 4.12’) Same as Proposition 4.12.

Notation 4.13’ Same as Notation 4.13.

In particular we see that if $c \in B(\mathbb{Z}/p)(C)$ then $c \simeq c^p$.

Define $\text{Div}_{C_U} = \Sigma_{P_i \in A_\infty} P_i$. Inductively choose bases $B_n$ of each $H^0(C_U, p^n\text{Div}_{C_U})$ as a finite dimensional $k$-vector space, such that $B_{n+1}$ includes both $B_n$ and a maximal linearly independent subset of $\{f^p|f \in B_n - B_{n-1}\}$, exactly the same as in the proof of Theorem 12.

Let $\Delta B_n = B_n - B_{n-1}$ for $n \geq 1$ and $\Delta B_0 = B_0 - \{1\}$. Let $S_k$ be the index set for elements in $\Delta B_k$. Write out elements in $\cup_k \Delta B_k$ as $\{b_j, j \in \Pi_k S_k\}$.
Notation 4.15' Set $A^{(S_k)}: C \to (\text{Sets})$ where $A^{(S_k)}(R)$ is the set of maps $r: S_k \to R$. Define $F_k: A^{(S_k)} \to A^{(S_{k+1})}$ by $r \mapsto (r|_p)$ where $(r|_p)(j) = r(i)_p$ for any $b_i \in \Delta B_k$ with $j \in S_{k+1}$ the index for $b_i^j$ and $r|i$ is zero at all other elements in $S_{k+1}$. We set $\phi_k: A^{(S_k)} \to \Delta^{l}_{Z/p,CU}, \phi_k(r) = \Sigma_{i \in S_k} r(i) \otimes b_i$, where $b_i \in \Delta B_k$. And we set $\psi_k: A^{(S_k)} \times B(\mathbb{Z}/p) \to \Delta^{l}_{Z/p,CU}$, $\psi_k(r, r_0) = \phi_k(r) + r_0 \otimes 1$. Notice that for any $r \in A^{(S_k)}(R)$, $\psi_k(r, r_0)$ and $\psi_{k+1} \circ (F_k \times Id_{B(\mathbb{Z}/p)})(r, r_0)$ are isomorphic in $\Delta^{l}_{Z/p,CU}(R) = R \otimes A$. We have a functor $\lim_{\to_k} A^{(S_k)} \times B(\mathbb{Z}/p) \to \Delta^{l}_{Z/p,CU}$.

Theorem 69. (Theorem 4.16') The functor $\lim_{\to_k} A^{(S_k)} \times B(\mathbb{Z}/p) \to \Delta^{l}_{Z/p,CU}$ is an equivalence of fibered categories.

Proof. Essential surjectivity: For any $\Sigma_j r_j \otimes b_j + r_0 \otimes 1 \in \Delta^{l}_{Z/p,CU}(R)$, since $r_j \otimes b_j \simeq (r_j \otimes b_j)^p$, is isomorphic to a $\Sigma_j \epsilon S_k r_j' \otimes b_j + r_0 \otimes 1$ for some large $k$, which is in the image of the functor.

Fully faithfulness: Suppose $([r, k], r_0)$ and $([r', k'], r_0')$ $(k \geq k')$ are mapped to $\Sigma_{j \in S_k} r_j \otimes b_j + r_0 \otimes 1$ and $\Sigma_{j \in S_k} r_j' \otimes b_j + r_0' \otimes 1$ respectively. If $([r, k], r_0)$ and $([r', k'], r_0')$ are isomorphic, then clearly $\Sigma_{j \in S_k} r_j \otimes b_j + r_0 \otimes 1$ and $\Sigma_{j \in S_k} r_j' \otimes b_j + r_0' \otimes 1$ are isomorphic. Conversely, suppose $\Sigma_{j \in S_k} r_j \otimes b_j + r_0 \otimes 1 = (\Sigma_{j \in S_k} r_j'' \otimes b_j + r_0'' \otimes 1)$ for some $\Sigma_{j \in S_k} r_j'' \otimes b_j + r_0'' \otimes 1$ in $R \otimes_k B$. Then $r_0 - r_0'' = (r_0''')^p - r_0''$ and $[r, k] = [r', k']$. \qed

Remark 70. Theorem 4.16' is an analogue of the base step in Theorem 1.2 of [H80]. Theorem 12 is also an analogue of the base step in Theorem 1.2 of [H80].

Corollary 71. (Corollary 4.19') Let $G$ be a $p$-group, then $\Delta_{G, CU}$ is a stack in fppqc topology.

Proof. Taken from [TY17]. If $G = p^l$ we proceed by induction on $l$. If $l = 1$ then $\Delta_{Z/p,CU} \simeq \lim_{\to_k} A^{(S_k)} \times B(\mathbb{Z}/p)$ which is a product of stacks. For a general $G$ let $H$ be a non-trivial central subgroup. By induction $\Delta'_{G,H,CU}$ is a stack and it is enough to show that all base change of $\Delta_{G,CU} \to \Delta'_{G,H,CU}$ along a map $Spec(R) \to \Delta'_{G,H,CU}$ is a stack. This fiber product is $Spec(R) \times \Delta_{H,CU}$ thanks to (Proposition 4.11') and 4.18 in [TY17], which is a stack by inductive hypothesis. \qed

4.2' Tame cyclic case

Proof. (Proof of Theorem $B'$)

Let $R \in C$ and $W \to Spec(R)$ be in $B(\mu_n)(R)$, indexed by $q$. The functor sends it to $W \boxtimes C_{V_q} \to Spec(R) \times_k C_U$, where $W \boxtimes C_{V_q} = (W \times_k C_{V_q})/D'(\mu_n)$ with $D'(\mu_n) = \{(\xi, \xi^{-1}) | \xi \in \mu_n\}$.

Essential surjectivity: For any $\tilde{W} \to Spec(R) \times_k C_U$ in $\Delta'_{\mu_n,CU}(R)$, take a $k$-point $u_0$ of $C_U$ and let the inverse image of $Spec(R) \times \{u_0\}$ in $\tilde{W}$ be $Spec(R_1)$. The $Spec(R_1) \to Spec(R)$ is finite etale, along which the pull back of $\tilde{W}$ is a product $Spec(R_1) \times_k C_{V_q}$ for some $C_{V_q}$. The image of $Spec(R_1) \to Spec(R)$, in the copy of $B(\mu_n)(R)$ indexed by $q$, also pulls back along $Spec(R_1) \to Spec(R)$ to $Spec(R_1) \times C_{V_q}$. Since $\Delta'_{\mu_n,CU}$ is a prestack by 4.6', $\tilde{W} \to Spec(R) \times_k C_U$ is isomorphic to the image of $Spec(R_1) \to Spec(R)$.

Fully faithfulness: Let $W_i \to Spec(R)$ be in $B(\mu_n)(R)$, $i = 1, 2$. And let $W_i \boxtimes C_{V_q} \to Spec(R) \times_k C_U$ be their images. Suppose the two images are isomorphic then $q_1 = q_2$. 39
Choose a covering \( \{ R \to R_i \} \) such that both \( W_i \)'s are trivial over \( \text{Spec}(R_i) \) for every \( i \).

We have the commutative diagram

\[
\begin{array}{ccc}
B(\mu_n)(R) & \to & \Delta'_{\mu_n, C_U}(R) \\
\downarrow & & \downarrow \\
B(\mu_n)(\{ R \to R_i \}) & \to & \Delta'_{\mu_n, C_U}(\{ R \to R_i \}),
\end{array}
\]

where \( B(\mu_n)(\{ R \to R_i \}) \) contains descent data and similarly for \( \Delta'_{\mu_n, C_U}(\{ R \to R_i \}) \).

The two columns are fully faithful since \( B(\mu_n) \) and \( \Delta'_{\mu_n, C_U} \) are both prestacks. We can verify easily that the bottom row gives a bijection from the Hom set of images of \( W_i \to \text{Spec}(R) \)'s \( (i = 1, 2) \) to its obvious target Hom set. The commutativity of the diagram finishes the proof.

\[\square\]

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