PERMUTATION-EQUIVARIANT
QUANTUM K-THEORY V.
TORIC $q$-HYPERGEOMETRIC FUNCTIONS

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Abstract. We first retell in the K-theoretic context the heuristics of $S^1$-equivariant Floer theory on loop spaces which gives rise to $D_q$-module structures, and in the case of toric manifolds, vector bundles, or super-bundles to their explicit $q$-hypergeometric solutions. Then, using the fixed point localization technique developed in Parts II–IV, we prove that these $q$-hypergeometric solutions represent K-theoretic Gromov-Witten invariants.

$S^1$-equivariant Floer theory

We recall here our old (1994) heuristic construction [5, 6] which highlights the role of $D$-modules in quantum cohomology theory, and adjust the construction to the case of quantum K-theory and $D_q$-modules, following the more recent exposition [8].

Let $X$ be a compact symplectic (or Kähler) target space, which for simplicity is assumed simply-connected, and such that $\pi_2(X) = H_2(X) \cong \mathbb{Z}^K$. Let $d = (d_1, \ldots, d_K)$ be integer coordinates on $H_2(X)$, and $\omega_1, \ldots, \omega_K$ be closed 2-forms on $X$ with integer periods, representing the corresponding basis of $H^2(X, \mathbb{R})$.

On the space $L_0X$ of contractible parameterized loops $S^1 \to X$, as well as on its universal cover $\tilde{L}_0X$, one defines closed 2-forms $\Omega_i$, which associates to two vector fields $\xi$ and $\eta$ along a given loop the value

$$\Omega_i(\xi, \eta) := \oint \omega_i(\xi(t), \eta(t)) \, dt.$$ 

A point $\gamma \in \tilde{L}_0X$ is a loop in $X$ together with a homotopy type of a disk $u : D^2 \to X$ attached to it. One defines the action functionals
$H_i : \tilde{\mathcal{L}}_0 \tilde{X} \to \mathbb{R}$ by evaluating the 2-forms $\omega_i$ on such disks:

$$H_i(\gamma) := \int_{D^2} u^* \omega_i.$$ 

Consider the action of $S^1$ on $\tilde{\mathcal{L}}_0 \tilde{X}$, defined by the rotation of loops, and let $V$ denote the velocity vector field of this action. It is well-known that $V$ is $\Omega_i$-hamiltonian with the Hamilton function $H_i$, i.e.:

$$i_V \Omega_i + dH_i = 0, \quad i = 1, \ldots, K.$$ 

Denote by $z$ the generator of the coefficient ring $H^*(BS^1)$ of $S^1$-equivariant cohomology theory. The $S^1$-equivariant de Rham complex (of $\tilde{\mathcal{L}}_0 \tilde{X}$ in our case) consists of $S^1$-invariant differential forms with coefficients in $\mathbb{R}[z]$, and is equipped with the differential $D := d + zi_V$. Then the degree-2 elements

$$p_i := \Omega_i + zH_i, \quad i = 1, \ldots, K,$$

are $S^1$-equivariantly closed: $Dp_i = 0$. This is standard in the context of Duistermaat–Heckman’s formula.

Furthermore, the lattice $\pi_2(X)$ acts by deck transformations on the universal covering $\tilde{\mathcal{L}}_0 \tilde{X} \to \mathcal{L}_0 X$. Namely, an element $d \in \pi_2(X)$ acts on $\gamma \in \tilde{\mathcal{L}}_0 \tilde{X}$ by replacing the homotopy type $[u]$ of the disk with $[u]+d$. We denote by $Q^d = Q_1^{d_1} \cdots Q_K^{d_K}$ the operation of pulling-back differential forms by this deck transformation. It is an observation from [5, 6] that the operations $Q_i$ and the operations of exterior multiplication by $p_i$ do not commute:

$$p_i Q_{i'} - Q_{i'} p_i = -zQ_i \delta_{ii'}.$$ 

These are commutation relations between generators of the algebra of differential operators on the $K$-dimensional torus:

$$[-zQ_{i} \partial_{Q_{i}}, Q_{i'}] = -zQ_{i} \delta_{ii'}.$$ 

Likewise, if $P_i$ denotes the $S^1$-equivariant line bundle on $\tilde{\mathcal{L}}_0 \tilde{X}$ whose Chern character is $e^{-p_i}$, then tensoring vector bundles by $P_i$ and pulling back vector bundles by $Q_i$ do not commute:

$$P_i Q_{i'} = ((q-1)\delta_{ii'} + 1)Q_{i'} P_i.$$ 

These are commutation relations in the algebra of finite-difference operators, generated by multiplications and translations:

$$Q_i \mapsto Q_i \times, \quad P_i \mapsto e^{zQ_i \partial_{Q_i}} = q^{Q_i \partial_{Q_i}}, \quad \text{where} \quad q = e^z.$$ 

Thinking of these operations acting on $S^1$-equivariant Floer theory of the loop space, one arrives at the conclusion that $S^1$-equivariant Floer cohomology (K-theory) should carry the structure of a module over the
algebra of differential (respectively finite-difference) operators. We will elucidate this conclusion with toric examples after giving a convenient description of toric manifolds.

**Toric manifolds**

Fans and momentum polyhedra are two the most popular languages in algebraic and symplectic geometry of toric manifolds [1]. In symplectic topology, a third framework, where toric manifolds are treated as symplectic reductions or GIT quotients of a linear space, turns out to be more convenient [4].

Let $\triangle$ be the momentum polyhedron of a compact symplectic toric manifold (we remind that it lives in the dual of the Lie algebra of a compact torus, and is therefore equipped with the integer lattice), and $N$ be the number of its hyperplane faces. The corresponding $N$ supporting affine linear functions with the minimal integer slopes canonically embed $\triangle$ into the first orthant $\mathbb{R}_+^N$ in $\mathbb{R}^N$, and thereby represent the toric manifold as the symplectic quotient of $\mathbb{C}^N$.

Indeed, the torus $T^N$ acts by diagonal matrices on $\mathbb{C}^N$ with the momentum map $(z_1, \ldots, z_N) \mapsto (|z_1|^2, \ldots, |z_N|^2)$: $\mathbb{C}^N \to \mathbb{R}_+^N \subset \text{Lie}^* T^N$. For a subtorus $T^K \subset T^N$, the momentum map is obtained by further projection $m : \text{Lie}^* T^N \to \text{Lie}^* T^K = \mathbb{R}^K$. The last equality uses a basis, $(p_1, \ldots, p_K)$, which we will assume integer. In fact one only needs to look at the picture $m(\mathbb{R}_+^N) \subset \mathbb{R}^K$ of the first orthant (see example on Figure 1), i.e. to know the images $u_1, \ldots, u_N$ in $\mathbb{R}^K$ of the unit coordinate vectors from $\mathbb{R}^N$: 

$$u_j = p_1 m_{1j} + \cdots + p_K m_{Kj}, \quad j = 1, \ldots, N.$$  

When $\triangle$ is the fiber $m^{-1}(\omega)$ in the first orthant over some regular value $\omega$, the initial toric manifold is identified with the symplectic reduction $X = \mathbb{C}^N //_\omega T^K$. Alternatively, removing from $\mathbb{C}^N$ all coordinate subspaces whose moment images do not contain $\omega$, one identifies $X$ with the quotient $\mathbb{C}^N / T^K$ of the rest by the action of the complexified torus (GIT quotient), and thereby equips $X$ with a complex structure.

Here is how basic topological information about $X$ can be read off the picture. The space $\mathbb{R}^K$ and the lattice spanned by $p_i$ are identified with $H^2(X, \mathbb{R}) \supset H^2(X, \mathbb{Z})$. The vectors $u_1, \ldots, u_N$ represent cohomology classes of the toric divisors of complex codimension 1 (they correspond to the hyperplane faces of the momentum polyhedron), and $c_1(T_X)$ is their sum. In the example of Figure 1, $u_1 = u_2 = p_1, u_3 = p_2, u_4 = p_2 - p_1$. The chamber (connected component) of the set of regular values of the moment map, which contains $\omega$ (it is the darkest region
on Figure 1) becomes the Kähler cone of $X$. It is the intersection of the images of those $K$-dimensional walls $\mathbb{R}_+^K$ of the first orthant $\mathbb{R}_+^N$ which contain $\omega$ in their image. In the example, there are 4 of these: spanned by $(u_1, u_3)$, $(u_2, u_3)$, $(u_1, u_4)$, and $(u_2, u_4)$. They are in one-to-one correspondence with the vertices of the momentum polyhedron, and hence with fixed points of $T^N$ in $X$. By the way, $X$ is non-singular if and only if the determinants of these maps $\mathbb{R}_+^K \to \mathbb{R}_+^K$ (i.e. appropriate $K \times K$ minors of the $K \times N$ matrix $m$) are equal to $\pm 1$.

The ring $H^*(X)$ is multiplicatively generated by $u_1, \ldots, u_N$, which besides the $N-K$ linear relations (given by the above expressions in terms of $p_1, \ldots, p_K$,) satisfy multiplicative Kirwan’s relations. Namely, $\prod_{j \in J} u_j = 0$ whenever the toric divisors $u_j$ with $j \in J \subset \{1, \ldots, N\}$ have empty geometric intersection. In minimalist form, for each maximal subset $J \subset \{1, \ldots, N\}$ such that the cone spanned by the vectors $u_j$ on the picture misses the Kähler cone, there is one Kirwan’s relation $\prod_{j \notin J} u_j = 0$. In our example, there are two Kirwan’s relations: $u_1 u_2 = 0$ and $u_3 u_4 = 0$, i.e. the complete presentation of $H^*(X)$ is $p_1^2 = p_2(p_2 - p_1) = 0$.

The spectrum of the algebra defined by Kirwan’s relations (we call it hedgehog, after Czech, or anti-tank hedgehogs), is described geometrically as follows. For each $\alpha \in X^T$, consider the corresponding $K$-dimensional wall of $\mathbb{R}_+^N$ whose picture contains $\omega$, and let $J(\alpha)$ denote the corresponding cardinality-$K$ subset of $\{1, \ldots, N\}$. In the complex space with coordinates $u_1, \ldots, u_N$, consider the $N-K$-dimensional coordinate subspace (rail) $\mathbb{C}^{N-K}_\alpha$ given by the equations $u_j = 0, j \in J(\alpha)$. The hedgehog is the union of the rails. Respectively $H^*(X, \mathbb{C})$ is the algebra of functions on the “thick point”, obtained by intersecting the hedgehog with the $K$-dimensional range of the map $\mathbb{C}^K \to \mathbb{C}^N: p \mapsto u = m^T p$. In the $T^N$-equivariant version of the theory, this subspace is
deformed into

\[ u_j(p) = \sum_{i=1}^{K} p_i m_{ij} - \lambda_j, \quad j = 1, \ldots, N, \]

and for generic \( \lambda \) intersects the hedgehog at isolated points corresponding to the fixed points \( \alpha X^T \). Here \( \lambda_j \) are the generators of the coefficient ring \( H^\ast(BT^N) \). Finally, the operation of integration \( H^*_{T,N}(X) \to H^*_{T,N}(pt) \) can be written (under some orientation convention) in the form of the residue sum over these intersection point:

\[
\int_X \phi(p, \lambda) = \sum_{\alpha \in X_T} \text{Res}_{p,u(p) \in C_{N-K}^\alpha} \frac{\phi(p, \lambda) dp_1 \wedge \cdots \wedge dp_K}{u_1(p) \cdots u_N(p)}.
\]

This follows from fixed point localization.

In K-theory, let \( P_i \) and \( U_j \) be the \((T^N\text{-equivariant})\) line bundles whose Chern characters are \( e^{-u_j} \) and \( e^{-p_i} \) respectively. The ring \( K^0_{T,N}(X) \) is described by Kirwan’s relations

\[
\prod_{j \in J} (1 - U_j) = 0 \text{ whenever } \bigcap_{j \in J} u_j = \emptyset \text{ for corresponding toric divisors,}
\]

together with the multiplicative relations

\[
U_j = \prod_{i=1}^{k} P_i^{m_{ij}} \Lambda_j^{-1}, \text{ where } \Lambda_j = e^{-\lambda_j} \text{ are generators of } \text{Repr}(T^N),
\]

the coefficient ring of \( T^N\text{-equivariant K-theory}. \) The K-theoretic hedgehog, defined by Kirwan’s relations, lives in the complex torus \((\mathbb{C}^\times)^N \) with coordinates \( U_1, \ldots, U_N \), and is the union of subtori \((\mathbb{C}^\times)^{N-K}_\alpha \) given by the equations \( U_j = 1, j \in J(\alpha) \). The trace operation \( \text{tr}_{T^N} : K^0_{T,N}(X) \to K^0_{T,N}(pt) \) takes on the residue form

\[
\text{tr}_{T^N}(X; \Phi(P)) = \sum_{\alpha \in X_T} \text{Res}_{p \in U(P) \in (\mathbb{C}^\times)^{N-K}} \frac{\Phi(P) dp_1 \wedge \cdots \wedge dp_K}{\prod_{j=1}^{N}(1 - U_j(P))} P_1 \cdots P_K.
\]

This follows from Lefschetz’ fixed point formula. In the example, we have: \( U_1 = P_1/\Lambda_1, U_2 = P_1/\Lambda_2, U_3 = P_2/\Lambda_3, U_4 = P_2/P_1 \Lambda_4 \). There are four intersections with the hedgehog: \( U_1 = U_3 = 1, U_2 = U_3 = 1, U_1 = U_4 = 1, \) and \( U_2 = U_4 = 1 \). The respective residues take the form:

\[
\frac{\Phi(\Lambda_1, \Lambda_3)}{(1 - \Lambda_1/\Lambda_2)(1 - \Lambda_3/\Lambda_1 \Lambda_4)} + \frac{\Phi(\Lambda_2, \Lambda_3)}{(1 - \Lambda_2/\Lambda_1)(1 - \Lambda_3/\Lambda_2 \Lambda_4)} + \frac{\Phi(\Lambda_1, \Lambda_4)}{(1 - \Lambda_1/\Lambda_2)(1 - \Lambda_1 \Lambda_4/\Lambda_3)} + \frac{\Phi(\Lambda_2, \Lambda_4)}{(1 - \Lambda_2/\Lambda_1)(1 - \Lambda_2 \Lambda_4/\Lambda_3)}.
\]
Returning to the heuristics based on loop spaces, we replace the universal cover $\tilde{L}_0X$ of the space of contractible loops in $X = \mathbb{C}^N//\omega T^K$ with the infinite dimensional toric manifold $\mathbb{C}^N//\omega T^K$. Note that the group $LT^K$ of loops in $T^K$ is homotopically the same as $T^K \times \pi_1(T^K)$, and that neglecting to factorize by $\pi_1(T^K) = \mathbb{Z}^K$ is equivalent to passing to the universal cover of $L_0X$. We consider the model of the loop space $L\mathbb{C}^N = \mathbb{C}^N[\zeta, \zeta^{-1}]$ as equivariant with respect to $T^N \times S^1$, where $T^N$ acts as before on $\mathbb{C}^N$, and $S^1$ acts by rotation of the loop’s parameter: $\zeta \mapsto e^{it}\zeta$. The picture in $\mathbb{R}^K$ corresponding to our infinite dimensional toric manifold consists of countably many copies of each of the vectors $u_j$. The copies represent the Fourier modes of the loops, and the equivariant classes of the corresponding toric divisors in terms of the basis $p_1, \ldots, p_K$ have the form

$$\sum_{i=1}^{K} p_i m_{ij} - \lambda_j - rz = u_j(p) - rz, \quad j = 1, \ldots, N, \quad r = 0, \pm1, \pm2, \ldots$$

In K-theory of the loop space, the line corresponding line bundles are

$$\prod_{i=1}^{K} P_i^{m_{ij}} \Lambda_j^{-1} q^r = U_j(P)q^r, \quad j = 1, \ldots, N, \quad r = 0, \pm1, \pm2, \ldots$$

The Floer fundamental cycle $Fl_X$ in the loop space, by definition, consists of those loops which bound holomorphic disks. In our model of the loop space, $Fl_X = \mathbb{C}^N[\zeta]//\omega T^K$. This gives rise to the following formula for the trace over $Fl_X$:

$$\text{tr}_{T^N \times S^1}(Fl_X; \Phi(P)) = \frac{1}{(2\pi i)^K} \oint \Phi(P) \prod_{j=1}^{N} \prod_{r=1}^{\infty} (1 - U_j(P)q^r) \frac{dP_1 \wedge \cdots \wedge dP_K}{P_1 \cdots P_K}.$$ 

Thus, the structure sheaf of the semi-infinite cycle $Fl_X$ is Poincaré-dual to the semi-infinite product

$$\hat{I}_X := \prod_{j=1}^{N} \prod_{r=1}^{\infty} (1 - U_j(P)q^r).$$

Similarly, for a toric bundle $E \to X$ or super-bundle $\Pi E$ (see Part IV), endowed with the fiberwise scalar action of $T^1$, $\hat{I}_E$ and $\hat{I}_{\Pi E}$ are obtained from $\hat{I}_X$ by respectively division and multiplication by the K-theoretic
Euler class of the obvious semi-infinite vector bundle:
\[ \hat{\iota}_E := \hat{\iota}_X / \prod_{a=1}^{L} \prod_{r=0}^{\infty} (1 - \lambda V_a(P)q^{-r}), \quad \text{and} \quad \hat{\iota}_{IE} = \hat{\iota}_X \prod_{a=1}^{L} \prod_{r=0}^{\infty} (1 - \lambda V_a(P)q^{-r}). \]

Here \( \lambda \in T^1 \), and \( V_a \) are toric line bundles,
\[ V_a(P) = P_1^{l_1} \cdots P_K^{l_K}, \quad E = \bigoplus_{a=1}^{L} V_a. \]

Our aim is to compute the left \( D_q \)-module generated by \( \hat{\iota}_X \). The deck transformation \( Q^d \) corresponding to a homology class \( d \in H_2(X) \) acts in our model by \( Q^d(P_i) = P_i q^{-d_i} \), where \( (d_1, \ldots, d_K) \) are coordinates on \( H_2(X) \) in the basis dual to \( (p_1, \ldots, p_K) \). We find that \( \hat{\iota}_X \) satisfies the following relations:
\[ Q^d \hat{\iota}_X = \prod_{j=1}^{N} \prod_{r=-\infty}^{D_j(d)-1} (1 - U_j(P)q^{-r}) \hat{\iota}_X^j. \]

Of course, the relations for all \( Q^d \) follow from the basis relations with \( Q^d = Q_i, i = 1, \ldots, K \). For instance, in our example, after some rearrangements, we obtain a system of two finite-difference equations (for \( d = (1, 0) \) and \( (0, 1) \)):
\[ (1 - P_1 \Lambda_1^{-1})(1 - P_2 \Lambda_2^{-1}) \hat{\iota}_X = Q_1(1 - P_2 P_1^{-1} \Lambda_1^{-1}) \hat{\iota}_X \]
\[ (1 - P_2 \Lambda_2^{-1})(1 - P_2 P_1^{-1} \Lambda_1^{-1}) \hat{\iota}_X = Q_2 \hat{\iota}_X. \]

To save space, we refer the reader to [6] for an explanation (though given in the cohomological context) of how to mechanically pass from this “momentum” representation of the Floer fundamental class (i.e. expresses as a function of \( P \)) to the “coordinate” representation in the form of the hypergeometric \( Q \)-series with vector coefficients in \( K^0(X) \). In that representation, \( Q^d \) acts as multiplication by \( Q_1^{d_1} \cdots Q_K^{d_K} \), and \( P_i \) acts as \( P_i q^{Q_i \partial Q_i} \), i.e. as the change \( Q_i \mapsto qQ_i \) accompanied with multiplication by \( P_i \) in \( K^0(X) \). With these conventions, we have:
\[ I_X = \sum_{d \in \mathbb{Z}^K} Q^d \prod_{j=1}^{N} \prod_{r=-\infty}^{D_j(d)} (1 - U_j(P)q^r). \]

This is just another way to describe the same \( D_q \)-module, and so \( I_X \) satisfies the system of finite-difference equations:
\[ \prod_{j=1}^{N} \prod_{r=-\infty}^{m_{ij}-1} (1 - q^{-r} U_j(P q^{Q_j \partial Q_j})) \prod_{r=-\infty}^{m_{ij}-1} (1 - q^{-r} U_j(P q^{Q_j \partial Q_j}))^{i-1} I_X = Q_i I_X, \quad i = 1, \ldots, K. \]
The toric $q$-hypergeometric function $I_X$, though comes from heuristic manipulation, has something to do with real life.

**Theorem.** The series $(1 - q) I_X$ is a value of the big $J$-function in symmetrized $T^N$-equivariant quantum $K$-theory of toric manifold $X$.

**Proof.** We follow the plan based on fixed point localization and explained in detail in Part II and Part IV in the example of complex projective spaces.

We write $I_X = \sum_{\alpha \in X \cap N} I_X^{(\alpha)} \phi_\alpha$ is components in the basis $\{ \phi_\alpha \}$ of delta-functions of fixed points. Denote by $U_j(\alpha)$ the restriction of $U_j(P)$ to the fixed point $\alpha$. We have $U_j(P) = 1$ for each of the $K$ values of $j \in J(\alpha)$, i.e.

$$P_1^m \cdots P_K^m = \Lambda_j, \quad j \in J(\alpha).$$

This determines expressions for $P_i$, and consequently for $U_j(P)$ with $j \notin J(\alpha)$ as Laurent monomials in $\Lambda_1, \ldots, \Lambda_N$. We have

$$I_X^{(\alpha)}(q) = \sum_{d \in \mathbb{Z}_+^K(\alpha)} \frac{Q_d}{\prod_{j \in J(\alpha)} \prod_{r=1}^{D_j(d)} (1 - q^r)} \prod_{j \notin J(\alpha)} \prod_{r=-\infty}^0 (1 - q^r U_j(\alpha)).$$

The summation range $\mathbb{Z}_+^K(\alpha)$ is over $d \in \mathbb{Z}^K$ such that $D_j(d) \geq 0$ for all $j \in J(\alpha)$, because outside this range, there is a factor $(1 - q^0)$ in the numerator.\(^1\)

(i) Temporarily encode degrees $d$ by $D_j(d), j \in J(\alpha)$, i.e. introduce Laurent monomials $Q_j(\alpha)$ in Novikov’s variables $Q_1, \ldots, Q_K$ such that

$$Q_1^{d_1} \cdots Q_K^{d_K} = \prod_{j \in J(\alpha)} Q_j(\alpha)^{D_j(d)}$$

for all $d$.

We have:

$$\sum_{d \in \mathbb{Z}_+^K(\alpha)} \frac{Q_d}{\prod_{j \in J(\alpha)} \prod_{r=1}^{D_j(d)} (1 - q^r)} = \sum_{k>0} \sum_{j \in J(\alpha)} Q_j(\alpha)^k / k(1 - q^k).$$

According to Part I (or Part III), the right hand side is $\sum_{d \in \mathbb{Z}_+^K(\alpha)} Q_j(\alpha)^{D_j(d)}$ of the $K$-dimensional face of $\mathbb{R}_+^N$ corresponding to $\alpha$. Note that the intersection of all $\mathbb{R}_+^K(\alpha)$ is the closure of the Kähler cone, and respectively the convex hull of all $\mathbb{Z}_+^K(\alpha)$ is the Mori cone of possible degrees of holomorphic curves.

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\(^1\)It is dual in $\mathbb{Z}^K = H_2(X, \mathbb{Z})$ to the image $\mathbb{R}_+^K(\alpha)$ on the picture $\mathbb{R}^K = H^2(X, \mathbb{R})$ of the $K$-dimensional face of $\mathbb{R}_+^N$ corresponding to $\alpha$. Note that the intersection of all $\mathbb{R}_+^K(\alpha)$ is the closure of the Kähler cone, and respectively the convex hull of all $\mathbb{Z}_+^K(\alpha)$ is the Mori cone of possible degrees of holomorphic curves.
value of the big J-function $J_{pt}$. Namely, recall from Part IV that by
$\Gamma$-operators we mean $q$-difference operators with symbol defined by

$$
\Gamma_q(x) := e^{\sum_{k>0} x^k/(1-q^k)} \sim \prod_{r=0}^{\infty} \frac{1}{1-xq^r}.
$$

Expressing each $u_j$ as a linear combination $u_j = \sum_{i \in J(\alpha)} u_{n_{ij}}$, we find for multi-variable $\Gamma$-operators:

$$
\frac{\Gamma_{q^{-1}}(\lambda)}{\Gamma_{q^{-1}}(\lambda)q^{\sum_{i \in J(\alpha)} n_{ij}Q_{i}(\alpha)\partial Q_{i}(\alpha)}} Q^d = Q^d \prod_{r=-\infty}^{0} (1 - \lambda q^r) \prod_{r=-\infty}^{D_j(d)} (1 - \lambda q^r).
$$

Thus, applying to $J_{pt}/(1-q)$ such operators (one for each $j = 1, \ldots, N$) and setting $\lambda = U_j(\alpha)$, we obtain $I_X^{(\alpha)}$. Due to the invariance of the big J-function $J_{pt}$ with respect to the $q$-difference operators (as explained in Part IV), we conclude that $(1-q)I_X^{(\alpha)}$ is a value of $J_{pt}$.

(ii) All poles of $I_X^{(\alpha)}$ away from roots of unity are simple for generic values of $\Lambda_1, \ldots, \Lambda_N$. We compute the residues at such poles. The pole is specified by the choice in the denominators of one of the factors $1 - q^n U_{j_0}$ with a $j_0 \notin J(\alpha)$, and by the choice of one of the $m$th roots $\lambda^{1/m}$ of $\lambda := U_{j_0}^{(\beta)}$. The choice of $j_0 \notin J(\alpha)$ determines a 1-dimensional orbit of $T_C^N$ in $X$, connecting the fixed point $\alpha$ with another fixed point, $\beta$. The closure of this orbit is a holomorphic sphere $\mathbb{C}P^1 \subset X$, represented on the picture by a collection of $u_j$ of cardinality $K + 1$: the union $J(\alpha) \sqcup \{j_0\} = J(\beta) \sqcup \{j_0'\}$, where $j_0'$ is a unique element of $J(\alpha)$ missing in $J(\beta)$. The torus $T_N$ acts on the cotangent lines to this $\mathbb{C}P^1$ at the fixed points by the characters $\lambda = U_{j_0}(\alpha)$ and $\lambda^{-1} = U_{j_0}^{(\beta)}$ respectively (which are therefore inverse to each other). Moreover, denote by $d_{\alpha\beta}$ the degree of this $\mathbb{C}P^1$. Then $U_j(\alpha)/U_j(\beta) = \lambda^{D_j(d_{\alpha\beta})}$, and in particular $D_{j_0}(d_{\alpha\beta}) = D_j^{(d_{\alpha\beta})} = 1$. Indeed, by cohomological fixed point localization on this $\mathbb{C}P^1$,

$$
D_j(d_{\alpha\beta}) := -\int_{d_{\alpha\beta}} \ln U_j = -\frac{\ln U_j(\alpha)}{-\ln \lambda} + \frac{\ln U_j(\beta)}{-\ln \lambda} = \frac{\ln U_j(\alpha)/U_j(\beta)}{-\ln \lambda}.
$$

Consequently, at $q = \lambda^{-1/m}$ we have for all $r$ and $j$:

$$
1 - q^r U_j(\alpha) = 1 - q^{-mD_j(d_{\alpha\beta})} U_j(\beta).
$$

Under these constraints,

$$
\frac{\prod_{r=-\infty}^{mD_j(d_{\alpha\beta})}(1 - q^r U_j(\alpha))}{\prod_{r=-\infty}^{D_j(d)}(1 - q^r U_j(\alpha))} = \frac{\prod_{r=-\infty}^{0}(1 - q^r U_j(\beta))}{\prod_{r=-\infty}^{D_j(d)-mD_j(d_{\alpha\beta})}(1 - q^r U_j(\beta))}.
$$
It follows that at $q = \lambda^{-1/m}$,

$$(1 - q^m\lambda)I^{(\alpha)}_X(q) = (1 - q^mU_{j_0}(\alpha)) \sum_{d \in \mathbb{Z}^K} Q^d \prod_{j=1}^N \prod_{r=\infty}^{q^{mD_j}} (1 - q^rU_j(\alpha)) =$$

$$Q^{md_{\alpha\beta}} (1 - q^mU_{j_0}(\alpha)) \prod_{j=1}^N \prod_{r=\infty}^{q^{mD_j(d_{\alpha\beta})}} (1 - q^rU_j(\alpha)) \times$$

$$\sum_{d \in \mathbb{Z}^K} Q^{d-md_{\alpha\beta}} \prod_{j=1}^N \prod_{r=\infty}^{q^{mD_j(d_{\alpha\beta})}} (1 - q^rU_j(\beta)).$$

Equivalently,

$$\text{Res}_{q=\lambda^{-1/m}} I^{(\alpha)}_X(q) \frac{dq}{q} = -Q^{md_{\alpha\beta}} \frac{\phi\alpha}{m} \frac{C_{\alpha\beta}(m)}{\phi\alpha} I^{(\beta)}_X(\lambda^{-1/m}),$$

where $\phi\alpha = \prod_{j \notin J(\alpha)} (1 - U_j(\alpha)) = \text{Euler}_{F^K}^X(T^*_pX)$, and

$$\frac{C_{\alpha\beta}(m)}{\phi\alpha} = \phi\alpha \prod_{r=1}^{m-1} (1 - \lambda^{-r/m}) \prod_{j \notin J_0} \prod_{r=\infty}^{q^{mD_j(d_{\alpha\beta})}} (1 - \lambda^{-r/m}U_j(\alpha)) \prod_{r=\infty}^{q^{mD_j(d_{\alpha\beta})}} (1 - \lambda^{-r/m}U_j(\alpha)).$$

Thus, the residues at the simple poles satisfy the recursion relations derived by fixed point localization arguments in Part II. More precisely, it remains to check that $C_{\alpha\beta}(m) = \text{Euler}_{F^K}^X(T^*_p\mathcal{M}_{0,2}(X, md_{ab}))$, where $T^*_p$ is the virtual cotangent space to the moduli space at the point $p$ represented by the $m$-multiple cover of the 1-dimensional orbit connecting fixed points $\alpha$ and $\beta$. This verification is straightforward. In $K^0_{T^*X}(X)$, we have $T^*X = U_1 + \cdots + U_N - K$ (as follows from the quotient description of $X$, or by localization to $X^T$). Therefore the cotangent $T^*_p$ is identified with the dual of $\oplus_j \left[H^0(\mathbb{C}P^1; U^{-m}_j) \odot H^1(\mathbb{C}P^1; U^{-m}_j)\right] - K - 1$ (the last $-1$ stands for reparameterizations of $\mathbb{C}P^1$ with 2 marked points), which is easily described in terms of spaces of binary forms of degrees $mD_j(d_{\alpha\beta})$. The factors in the formula for $C_{\alpha\beta}(m)$ correspond to $T^N$-weight of the monomials in such binary forms.

From (i) and (ii) it follows that $(1 - q)I_X$ is a value of the big J-function in permutation-equivariant quantum K-theory of $X$. Since $I_X$ is defined over the $\lambda$-algebra $\mathbb{Z}[\Lambda^{\pm 1}][[Q]]$ involving only Novikov’s variables and functions on $T^N$, the value actually belongs to the symmetrized theory, i.e. carries information only about multiplicities the part of sheaf cohomology, invariant under permutations of marked points. $\square$
In the case of bundle $E$ or super-bundle $\Pi E$, where $E = \oplus_{a=1}^{L} V_a$ is the sum of toric line bundles $V_a = \prod_{i} P_{i}^{l_{ia}}$, one similarly obtains $q$-hypergeometric series

$$I_E = \sum_{d \in \mathbb{Z}^K} Q^d \prod_{j=1}^{N} \prod_{r=-\infty}^{0} (1 - q^r U_j(P)) \prod_{a=1}^{L} \prod_{r=-\infty}^{0} (1 - \lambda q^r V_a(P)),$$

$$I_{\Pi E} = \sum_{d \in \mathbb{Z}^K} Q^d \prod_{j=1}^{N} \prod_{r=-\infty}^{0} (1 - q^r U_j(P)) \prod_{a=1}^{L} \prod_{r=-\infty}^{0} (1 - \lambda q^r V_a(P)),$$

where $\Delta_a(d) = \sum_i d_i l_{ia}$.

**Theorem.** Functions $(1-q)I_E$ and $(1-q)I_{\Pi E}$ represent some values of the big $J$-functions in symmetrized quantum $K$-theories of toric bundle space $E$ and super-bundle $\Pi E$ respectively.

This theorem is proved the same way as the previous one.

The above results are $K$-theoretic analogues of cohomological “mirror formulas” [7, 3, 9]. The strategy we followed is due to J. Brown [2]. Some special cases were obtained in [8] by a different strategy. Some further results and applications can be found in the recent preprint [10].

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