Exact $U(N_c) \rightarrow U(N_1) \times U(N_2)$ factorization of Seiberg-Witten curves and $\mathcal{N} = 1$ vacua

Romuald A. Janik*  
M. Smoluchowski Institute of Physics,  
Jagellonian University,  
Reymonta 4,  
30-059 Kraków, Poland

November 23, 2018

Abstract

$\mathcal{N} = 2$ gauge theories broken down to $\mathcal{N} = 1$ by a tree level superpotential are necessarily at the points in the moduli space where the Seiberg-Witten curve factorizes. We find exact solution to the factorization problem of Seiberg-Witten curves associated with the breaking of the $U(N_c)$ gauge group down to two factors $U(N_1) \times U(N_2)$. The result is a function of three discrete parameters and two continuous ones. We find discrete identifications between various sets of parameters and comment on their relation to the global structure of $\mathcal{N} = 1$ vacua and their various possible dual descriptions. In an appendix we show directly that integrality of periods leads to factorization.

1 Introduction

A breakthrough in our understanding of nonperturbative behaviour of $\mathcal{N} = 2$ supersymmetric gauge theories occurred in 1994 with the work [1, 2] which showed that the low energy dynamics of the theory was encoded in the properties of an associated hyperelliptic Seiberg-Witten curve.

*e-mail: ufrjanik@if.uj.edu.pl
Recently there has been much renewed interest in this circle of problems associated with a set of completely new methods and insights into the structure of $\mathcal{N} = 1$ theories obtained by breaking $\mathcal{N} = 2$ theories to $\mathcal{N} = 1$ by an inclusion of a superpotential for the adjoint $\mathcal{N} = 1$ chiral superfield. Calabi-Yau constructions of the gauge theories, together with geometric transitions [3, 4] gave rise to formulas for the exact superpotential which where later reinterpreted in terms of a computable random matrix theory [5]. This correspondence was subsequently proven in a purely field-theoretic context [6, 7].

Since the $\mathcal{N} = 1$ theories which arise in the above way sit in the region of $\mathcal{N} = 2$ moduli space where (some) monopoles become massless i.e. where the Seiberg-Witten curves factorize this new circle of ideas could be used as a new tool for studying the $\mathcal{N} = 2$ phenomena and vice-versa. These interrelations were studied in [8, 9, 10, 11, 12, 13, 14]. In [15] matrix model methods were used to obtain a solution of the complete factorization of Seiberg-Witten curves for theories with fundamental flavours (see also [16] for an application of these results).

The above dealt predominantly with the case when the gauge group was not broken. In the opposite case e.g. for the breaking $U(N_c) \rightarrow U(N_1) \times U(N_2)$ it was found [17, 18] that the space of $\mathcal{N} = 1$ vacua exhibits a very complex structure of various connected components, each of which allows for multiple dual descriptions of the same physics but with different patterns of breaking. The analysis was later extended to the case of other gauge groups and matter fields in the fundamental representation [19, 20].

However in the preceding in order to map out the structure of the vacua one had to factorize the Seiberg-Witten curves for low $N_c$ on a case by case basis. The aim of this paper is to provide an exact solution of the factorization problem for arbitrary $N_c$ and any pattern of breaking $U(N_c) \rightarrow U(N_1) \times U(N_2)$.

The plan of this paper is as follows. In section 2 we give a short introduction to the relation of factorized curves with $\mathcal{N} = 1$ vacua. In section 3 we summarize the equations for the meromorphic 1-form, the solution of which gives rise to the solution of the factorization problem. In section 4, which is the main part of the paper, we construct our solution. Then we proceed, in section 5, to study its properties relevant for the uncovering of the global structure of $\mathcal{N} = 1$ vacua. We close the paper with a discussion and two appendices. In appendix A we give a mathematical self-contained proof that factorization follows from integrality of periods. We do this since
the condition of integrality of periods as a criterion for factorization was obtained using *gauge-theoretical* considerations. In appendix B we list the *Mathematica* code which implements our construction.

## 2 Factorization of Seiberg-Witten curves and $\mathcal{N} = 1$ vacua

Since the celebrated work of [1, 2] it is known that $\mathcal{N} = 2$ $U(N_c)$ SYM theories develop an $N_c$-dimensional moduli space of vacua parametrized by vacuum expectation values of the adjoint scalar field

$$
u_p = \left\langle \frac{1}{p} \text{tr} \Phi_p \right\rangle$$

with $1 \leq p \leq N_c$. The properties of the low energy dynamics of the theory around each such vacuum is encoded in the geometry of the Seiberg-Witten curve [1, 2, 21, 22]

$$y^2 = P_{N_c}(x, u_p)^2 - 4\Lambda^2 N_c$$

where the polynomial $P_{N_c}(x, u_p) = \langle \det(x I - \Phi) \rangle \equiv \sum_{\alpha=0}^{N_c} s_\alpha x^{N_c - \alpha}$ depends on the moduli $u_p$'s parameterizing the vacua through

$$\alpha s_\alpha + \sum_{k=0}^{\alpha} ks_{\alpha-k} u_k = 0$$

$$s_0 = 1, \quad u_0 = 0$$

The $\mathcal{N} = 2$ theory admits monopoles which at a generic point in the moduli space are massive. However at special submanifolds of the moduli space of vacua, $n$ of these monopoles ($n < N_c$) may become massless and condense. At these points the Seiberg-Witten curve *factorizes* i.e. the rhs of (2) has $n$ double roots and $2(N_c - n)$ single roots:

$$P_{N_c}(x, u_p^{\text{fact.}})^2 - 4\Lambda^2 N_c = F_{2(N_c-n)}(x) \cdot H_n^2(x).$$

The sets of moduli $\{u_p^{\text{fact.}}\}$ then depend generically on $N_c - n$ *continuous* parameters and some discrete ones.

The submanifolds where the monopoles condense are also important due to the fact that they are relevant when the $\mathcal{N} = 2$ theory is broken down to
$\mathcal{N} = 1$ through an addition of a tree level superpotential

$$W_{\text{tree}} = \sum_{p=1}^{n+1} g_p \cdot \frac{1}{p} \text{tr} \Phi^p.$$  \hspace{1cm} (6)

Then the gauge symmetry $U(N_c)$ may be either unbroken, broken down to $U(N_1) \times U(N_2)$ with $N_1 + N_2 = N_c$ etc. This is easy to understand clasically as it corresponds to redistributing $N_c$ eigenvalues of $\Phi$ among respectively one, two or more minima of the potential. On the quantum level the exact description is furnished by the factorized form of the Seiberg-Witten curve (5) with respectively $n = N_c - 1$, $n = N_c - 2$ etc.

The general solution for $n = N_c - 1$ (complete factorization) has been found in [23]. The solution is expressed in terms of Chebyshev polynomials

$$P_{N_c}(x, u^{\text{fact}}_p) = 2\Lambda^{N_c} \eta^{N_c} T_{N_c} \left( \frac{x + x_0}{2\eta\Lambda} \right)$$  \hspace{1cm} (7)

where $\eta = \exp(\pi i k/N_c)$. Note that this solution depends both on a continuous parameter $x_0$ and on a discrete one $k = 0...N_c - 1$ (counting vacua in an $\mathcal{N} = 1$ $U(N_c)$ theory).

Some solutions for low $N_c$ in the case of breaking $U(N_c) \rightarrow U(N_1) \times U(N_2)$ were found on a case by case basis in [17]. The aim of this paper is to provide a general solution in this case for any pattern of breaking characterized by $N_1$ and $N_2$ and the appropriate discrete parameters $k_1$ and $k_2$.

Such a general solution is interesting especially as a very rich structure of the vacua was uncovered for small $N_c$ in [17] (see also [18]). In particular the space of vacua turns out to have multiple connected components. Moreover within each component one may describe the same physics (the same Seiberg-Witten curve) in terms of different labels $(N_1, N_2, k_1, k_2)$ and $(N'_1, N'_2, k'_1, k'_2)$ corresponding to different patterns of breaking of the gauge group. Note that one can define the $N_i$’s and the appropriate glueball superfields $S_i$’s not only in the semiclassical limit ($\Lambda \rightarrow 0$) but also consistently at strong coupling through $N_i = (1/2\pi i) \oint_{\alpha_i} \omega$ where $\omega$ is a meromorphic 1-form (see next section) while $\alpha_i$ are some cycles. Dual descriptions correspond to different choices of the fundamental cycles $\alpha_i$ but lead obviously to equivalent physics. The main motivation for this work was the fact that the knowledge of an exact solution of the factorization problem with different patterns of

\footnote{The number of parameters is just $N_c$ since the solution depends only on $\eta^2$.}
breaking of the gauge group might help to map out a global picture of the vacua and their possible dual descriptions.

3 Integrality of periods

A key object in the Seiberg-Witten theory is the meromorphic 1-form

\[ \omega = T(x)dx \equiv \left\langle \text{tr} \frac{dx}{x - \Phi} \right\rangle \]

which is explicitly given for any Seiberg-Witten curve (2) through [8, 10, 7]

\[ \omega(x) = \frac{P_N'(x)dx}{\sqrt{P_N^2(x) - 4\Lambda^2 N_c}} = \frac{d}{dx} \log \left( P_N(x) + \sqrt{P_N^2(x) - 4\Lambda^2 N_c} \right) \]

From the definitions it is clear that

1. \( T(x) \) has residue \( N_c \) at infinity.

2. The moduli of the vacuum \( u_p \) can be reconstructed through

\[ u_p = \frac{1}{p} \frac{1}{2\pi i} \oint_{c_\infty} x^p \cdot \omega = \frac{1}{p} \text{res}_{x=\infty} x^p \cdot \omega \]

3. The scale of the theory \( \Lambda \) can be reconstructed from the regularized integral

\[ \left\{ \int_a^\infty \omega \right\}_{\text{reg}} \equiv \lim_{x \to \infty} \left( \int_a^x \omega - N_c \log x \right) = -\log \Lambda^{N_c} \]

where \( a \) is a branch point of the Seiberg-Witten curve.

4. If the Seiberg-Witten curve factorizes as in (5), the 1-form \( \omega = T(x)dx \) defines a 1-form on the reduced curve

\[ y^2 = F_{2(N_c-n)}(x) \]
From the above properties we see that once we have found the meromorphic 1-form for the factorized curve we can reconstruct the moduli $u_p$ and thus the solution to the factorization problem. We need therefore a criterion for the 1-form to come from a factorized curve.

In \cite{19} and in \cite{18} (using different notation – see eqn. (A27) in the first paper of \cite{18}) it was shown from the Konishi anomaly perspective and the Dijkgraaf-Vafa superpotential respectively that the periods of $\omega = T(x)dx$ should be integer:\footnote{We quote the equations for the case of $n = N_c - 2$.}

\begin{align}
\frac{1}{2\pi i} \int_{\alpha} \omega &= N_1 \tag{13} \\
\frac{1}{2\pi i} \int_{\beta} \omega &= \Delta k \equiv k_1 - k_2 \tag{14}
\end{align}

The cycles $\alpha$ and $\beta$ in (13)-(14) are ordinary compact cycles. As mentioned above $\omega$ can be understood to be a meromorphic 1-form on the elliptic curve

$$y^2 = F_4(x) \equiv (x - a)(x - b)(x - c)(x - d) \tag{15}$$

The main aim of this paper is to give an explicit construction of the general solution to these equations.

Note that the equations (13)-(14) were obtained from gauge-theoretical considerations. In appendix A we give a purely mathematical proof which shows that factorization indeed follows from the integrality of periods. That construction also shows a strong structural similarity between the standard Chebyshev polynomials appearing in (7) and the polynomials constructed here which solve the factorization problem.

\section{Construction of the meromorphic 1-form}

Before we proceed to solve equations (13)-(14) let us note some obvious symmetries of the problem. Firstly, given a solution $P(x)$, the shifted polynomial $\tilde{P}(x) = P(x + x_0)$ is also a solution. Secondly, a rescaling $x \to \alpha x$ corresponds to a rescaling $\Lambda^{2N_c} \to \Lambda^{2N_c}/\alpha^{2N_c}$. This allows us to change $\Lambda$ and also generate distinct solutions for fixed $\Lambda$ by picking $\alpha$ to be an appropriate root of unity.
The nontrivial part of the problem is then to find solutions parametrized by $N_1$ and $\Delta k$ and one continuous parameter.

Let us note that in the original variables the problem seems intractable. Namely we have to solve the equations

$$\frac{1}{\pi} \int_{a}^{b} \frac{N_c x + \mu}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} dx \sim N_1 \tag{16}$$

$$\frac{1}{\pi} \int_{b}^{c} \frac{N_c x + \mu}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} dx \sim \Delta k \tag{17}$$

for two of the five unknowns $a, b, c, d$ and $\mu$. These are coupled very nonlinear transcendental equations. And there seem to be no way to obtain an explicit solution from the above expressions.

**The torus with marked points**

The key to our solution is to use the representation of the elliptic curve (15) as a torus. However since the points at infinity play a special role in our setup we have to pick two points $\tilde{a}_1$ and $\tilde{a}_2$ on the torus (see fig. 1). These points will get mapped to the $x = \infty$ infinities on the two branches of the curve (15).

The advantage of this representation is that holomorphic (and meromorphic) 1-forms have a very simple form and periods may be calculated quite explicitly. Indeed using the general representation of a meromorphic function

![Figure 1: The torus with modular parameter $\tau$ and two marked points $\tilde{a}_1$ and $\tilde{a}_2$ which will be mapped to infinity.](image)
with prescribed poles and residues on a torus [24] we may write \( \omega \) as

\[
\omega = \left( N_c \frac{d}{dz} \log \frac{\theta(z - a_1)}{\theta(z - a_2)} + C \right) dz
\]  

where

\[
\theta(z) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau + 2\pi i n z}
\]

is the standard theta function and \( \tau \) is the modular parameter of the torus. Then \( \omega \) has poles with residues \( N_c, -N_c \) at \( \tilde{a}_1 = \frac{1+i\tau}{2} + a_1 \) and \( \tilde{a}_2 = \frac{1+i\tau}{2} + a_2 \) respectively.

Now it is trivial to calculate the periods. Using the periodicity properties of the \( \theta \) function we get immediately

\[
\frac{1}{2\pi i} \int_0^1 \omega = \frac{C}{2\pi i}
\]

\[
\frac{1}{2\pi i} \int_0^\tau \omega = N_c(a_1 - a_2) + \frac{C\tau}{2\pi i}
\]

Of course due to the fact that \( \omega \) has poles with residue \( \pm N_c \), the above expressions are only defined up to a multiple of \( N_c \). We may now solve for the free parameters in \( \omega \) to get

\[
C = 2\pi i N_1
\]

\[
a_2 - a_1 \equiv \Delta a = \frac{N_1 \tau - \Delta k}{N_c}
\]

In fact the final result will not depend on \( a_1 \) so we will set it to zero.

**Embedding of the torus**

In order to complete the solution of the factorization problem we need to represent the torus with the modular parameter \( \tau \) and the marked points \( a_1 = 0 \) and \( a_2 = \Delta a \) as the curve (15). Once this is done we will have at our disposal the function \( x(z) \) which will allow us to reconstruct the moduli parameters \( u_p \) and the scale of the theory \( \Lambda \). Comparing the general forms of the meromorphic 1-form in both representations:

\[
\frac{N_c x + \mu}{y} dx = (N_c f(z) + C) dz
\]
leads to the choice $x = f(z)$ and $y = dx/dz = f'(z)$. Although this choice is not unique, we will use the freedom of rescaling $x \rightarrow ax + x_0$ at the end of the construction. In principle the knowledge of $x(z)$ should be enough to calculate the $u_p$’s through (10), however it is more convenient for numerical calculation to reconstruct explicitly the curve (15) and to calculate $u_p$’s from

$$u_p = \frac{1}{p} \text{res}_{x=\infty} x^p \cdot \frac{N_c x + \mu}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}$$  \hspace{1cm} (25)$$

In order to reconstruct the equation satisfied by $y = f'(z)$ and $x = f(z)$ one can use standard reasoning and look at the Laurent expansion of $f(z)$ around it’s pole at $z = \tilde{a}_1$:

$$f(z) \sim \frac{1}{z - \tilde{a}_1} + f_0 + f_1 (z - \tilde{a}_1) + f_2 (z - \tilde{a}_1)^2 + f_3 (z - \tilde{a}_1)^3$$ \hspace{1cm} (26)$$

Then one fixes the coefficients of the quartic polynomial (15) by requiring that in the Laurent expansion of the difference

$$f'(z)^2 - (f^4(z) + S_1 f^3(z) + S_2 f^2(z) + S_3 f(z) + S_4)$$ \hspace{1cm} (27)$$

all poles in $(z - \tilde{a}_1)$ and the constant term cancel. Then the function (27) could have at most one pole at $z = \tilde{a}_2$ but on a torus that is impossible. Therefore (27) is holomorphic hence constant (and equal to zero). The elliptic curve (15) corresponding to our solution is then

$$y^2 = x^4 + S_1 x^3 + S_2 x^2 + S_3 x + S_4$$ \hspace{1cm} (28)$$

with the $S_i$’s given by

$$S_1 = -4 f_0$$ \hspace{1cm} (29)$$
$$S_2 = 6 f_0^2 - 6 f_1$$ \hspace{1cm} (30)$$
$$S_3 = 12 f_0 f_1 - 4 f_0^3 - 8 f_2$$ \hspace{1cm} (31)$$
$$S_4 = f_0^4 + 7 f_1^2 - 6 f_0^2 f_1 + 8 f_0 f_2 - 10 f_3$$ \hspace{1cm} (32)$$

The parameter $\mu$ in (25) is just $\mu = 2 \pi i N_1$.

**The scale $\Lambda$ of the theory**

It remains to evaluate the regularized integral related to $\Lambda$. It is convenient to calculate the equivalent quantity

$$- \log \Lambda^{2 N_c} = \lim_{\epsilon \to 0} \left[ \int_{\tilde{a}_2 - \epsilon}^{\tilde{a}_1 + \epsilon} \omega + 2 \log \epsilon \right]$$ \hspace{1cm} (33)$$
The result is
\[ \log \Lambda^2 = 2 \log \theta'(\tilde{\tau}) - \log (\theta(\tilde{\tau} - \Delta a)\theta(\tilde{\tau} + \Delta a)) - \frac{C\Delta a}{N_c} - i\pi \] (34)

where \( \tilde{\tau} = (1 + \tau)/2 \).

This completes our solution. We have constructed a solution parameterized by two discrete parameters \( N_1, \Delta k \) and the modular parameter of the elliptic curve \( \tau \) in the upper half plane. From this the standard rescaling \( x \rightarrow \alpha x + x_0 \) provides the second continuous parameter \( x_0 \), allows us to fix \( \Lambda \) to the desired physical value and introduces the third discrete parameter through the appropriate root of unity in \( \alpha \).

Explicitly we set
\[ \alpha = -i\Lambda^{-1} \cdot \theta'(\tilde{\tau}) (\theta(\tilde{\tau} - \Delta a)\theta(\tilde{\tau} + \Delta a))^{-\frac{1}{2}} e^{-\frac{C\Delta a}{2N_c}} \cdot e^{-i\frac{\pi}{N_c}} \] (35)

where \( \Lambda \) is the physical scale of the theory, \( C = 2\pi i N_1 \), \( \Delta a = (N_1 \tau - \Delta k)/N_c \) and \( k \) is the new discrete parameter.

In appendix B we give the Mathematica code which implements the above construction.

5 Some properties of the solution and the global structure of \( \mathcal{N} = 1 \) vacua

The factorization solution constructed in the previous section depends, for a given \( N_c \), on the following set of parameters:
\[ [N_1, k, \Delta k, \tau, x_0] \] (36)

where \( N_1, k \) and \( \Delta k \) are discrete, \( \tau \) lies in the upper half plane, while \( x_0 \) is an arbitrary complex number.

It may happen that two different sets of parameters describe the same factorized Seiberg-Witten curve. This may happen for two reasons. Firstly, as mentioned above the periods are defined only modulo \( N_c \), so we may expect that there are transformation rules that allow us to reduce the parameters \( \text{mod } N_c \). Secondly one expects e.g. that \( \Delta k \) has simple periodicity w.r.t. \( N_1 \) and not only with \( N_c \). Thirdly, and this is the most interesting possibility, the different sets of (discrete) parameters for a single Seiberg-Witten curve
represent different dual descriptions of the same physics — this is part of the structure uncovered in [17]. Indeed the description of the identifications between the sets of discrete parameters describes precisely the global structure of $\mathcal{N} = 1$ vacua.

If the Seiberg-Witten curve stays invariant, then necessarily the elliptic curve also. Therefore identifications can only occur between sets of parameters where $\tau$ and $\tau'$ are linked by a $SL(2,\mathbb{Z})$ transformation. We will first consider identifications for fixed $\tau$ and later deal with the cases $\tau' = \tau + 1$ and $\tau' = -1/\tau$.

**Identifications with** $\tau' = \tau$

The most trivial property is periodicity in $k$ (see (35)):

$$[N_1, k + 2N_c, \Delta k, \tau, x_0] \equiv [N_1, k, \Delta k, \tau, x_0]$$

Let us now consider a shift $\Delta k \rightarrow \Delta k + N_c$. Then $\Delta a \rightarrow \Delta a - 1$ and due to the invariance $\theta(z) = \theta(z + 1)$ of the $\theta$ functions the embedding remains unchanged. The only modification is in $\alpha$ which gets rescaled by a phase $e^{i\pi N_1/N_c}$. Hence we have

$$[N_1, k, \Delta k + N_c, \tau, x_0] \equiv [N_1, k - N_1, \Delta k, \tau, x_0]$$

The shift $N_1 \rightarrow N_1 + N_c$ is more complicated. Then $C \rightarrow C + 2\pi i N_c$, $\Delta a \rightarrow \Delta a + \tau$. Using the transformation law $\theta(z + \tau) = e^{-\pi \tau} e^{-2\pi i z} \theta(z)$ we get $x \rightarrow x - 2\pi i$ and $\alpha \rightarrow \pm \alpha \cdot e^{-i\pi \Delta k/N_c}$. The sign ambiguity is due to the branch cut of the square root in (35). Hence we get

$$[N_1 + N_c, k, \Delta k, \tau, x_0] \equiv [N_1, k + \Delta k(+N_c), \Delta k, \tau, x_0 + 2\pi i]$$

**Identification with** $\tau' = \tau + 1$

Under the modular transformation $\tau \rightarrow \tau + 1$ the theta function transforms as $\theta(z, \tau + 1) = \theta(z + 1/2, \tau)$. Therefore the elliptic curve remains the same, the only thing that changes is the identification of the periods namely

$$\Delta a = \frac{N_1(\tau + 1) - \Delta k}{N_c} = \frac{N_1 \tau - (\Delta k - N_1)}{N_c}$$

Hence we have

$$[N_1, k, \Delta k, \tau + 1, x_0] \equiv [N_1, k, \Delta k - N_1, \tau, x_0]$$
This transformation law has a clear physical meaning. In our construction
\( \Delta k = k_1 - k_2 \) was periodic only modulo \( N_c \), while we know from the gauge
theory perspective that \( k_1 \), which labels vacua of \( U(N_1) \) should have peri-
odicity \( N_1 \). This identification is furnished by the modular transformation
\( \tau \rightarrow \tau + 1 \).

**Identification with** \( \tau' = -1/\tau \)

Here the modular transformation law of the theta function is more involved:

\[
\theta \left( w, \frac{-1}{\tau'} \right) = (-i\tau)^{\frac{1}{2}} e^{\pi i w^2 \tau} \theta(w\tau, \tau)
\]  

(42)

The meromorphic function defining \( x \) transforms as

\[
\frac{d}{dw} \log \frac{\theta(w)}{\theta(w - \Delta a')} \rightarrow \tau \left[ 2\pi i \Delta a' + \frac{d}{dz} \log \frac{\theta(z)}{\theta(z - \Delta a' \tau)} \right]
\]  

(43)

where \( z = w\tau \) and \( \Delta a' = (-N_1'/\tau - \Delta k')/N_c \). The periods may be calculated
to be

\( N_1 = -\Delta k' \quad \Delta k = N_1' \)  

(44)

Finally \( \alpha \) transforms simply as \( \alpha \rightarrow \pm \tau \cdot \alpha \). This factor exactly cancels the \( \tau \)
in (43) when constructing the torus embedding. Putting all these ingredieints
together we get

\[
\left[ N_1, k, \Delta k, \frac{-1}{\tau}, x_0 \right] \equiv \left[ -\Delta k, k(1 + N_c), N_1, \tau, \frac{x_0}{\tau} + \frac{2\pi i}{N_c} \left( \frac{N_1}{\tau} + \Delta k \right) \right]
\]  

(45)

This is the key transformation which allows us to obtain a dual description
with a different pattern of breaking \( U(N_c) \rightarrow U(N_1) \times U(N_2) \).

**Global structure of vacua**

The above identifications allow us to study the global structure of \( N = 1 \)
vacua. The connected components could be obtained by finding the orbits of the *three* discrete labels \([N_1, k, \Delta k] \) under the group of transformations generated by (37), (38), (39), (41) and (45), assuming that there are no further identifications. We leave the detailed investigation of this structure to a subsequent work [25].
6 Discussion

In this paper we constructed an exact solution of the factorization problem of Seiberg-Witten curves with \( n = N_c - 2 \) massless monopoles. This is relevant for the breaking of \( \mathcal{N} = 2 \) theories down to \( \mathcal{N} = 1 \) with the gauge group \( U(N_c) \) broken down to a product of two factors.

The solution was obtained by constructing a meromorphic 1-form with integral periods on an elliptic curve. The solution depends on three discrete parameters and two continuous ones. In principle one could generalize the structure to lower \( n \) by considering the construction of meromorphic functions on hyperelliptic curves of higher genus. Another direction in generalization could be the extension to other gauge groups and/or addition of matter fields. A more complicated but interesting setting would be the consideration of quiver gauge theories, where nonhyperelliptic curves appear, and where it was shown that integrality of periods also plays an important role [26].

We also studied certain discrete identifications between the parameters and in appendix A we gave a proof that integrality of periods, a condition obtained in [19, 18] on physical grounds, indeed leads to a solution of the factorization problem.

The physical interest of an exact solution lies in the possibility of studying in detail the global structure of \( \mathcal{N} = 1 \) vacua for any \( N_c \) along the lines of [17]. The main features of the analysis in [17], like the appearance of connected components of vacua and possible dual descriptions are related to discrete identifications between the parameters labeling the factorization solution. We leave a detailed investigation of this question for future work [25].

Acknowledgments RJ would like to thank the Niels Bohr Institute for hospitality while this work was carried out. This work was supported in part by KBN grants 2P03B09622 (2002-2004), 2P03B08225 (2003-2006) and by “MaPhySto”, the Center of Mathematical Physics and Stochastics financed by the National Danish Research Foundation.

Appendix A. A mathematical proof of the factorization property of our solution

The main object of this paper was to construct a solution of the factorization problem by constructing explicitly a meromorphic 1-form \( \omega \) on an elliptic
curve
\[ y^2 = (x - a)(x - b)(x - c)(x - d) \]  
with the following properties:
\[ \frac{1}{2\pi i} \oint_{\gamma_i} \omega \in \mathbb{Z} \]  
\[ \text{res}_{x=\pm \infty} \omega = \pm N_c \]  
These properties followed from the expressions for the superpotential in the Dijkgraaf-Vafa framework. Then we reconstructed the parameters of the factorized curve from the 'physical' formulas
\[ u_p = \frac{1}{p} \text{res}_{x=\infty} x^p \cdot \omega \]  
In this appendix we show directly, without any physical input, that the conditions (47)-(48) give rise to a solution of the factorization problem, namely they allow us to construct a polynomial \( P(x) = x^{N_c} + \ldots \) such that the Seiberg-Witten curve factorizes:
\[ P^2 - 1 = (x - a)(x - b)(x - c)(x - d) \cdot H(x)^2 \]  
with \( H(x) \) a polynomial of degree \( N_c - 2 \). From the construction in the main text we see that we may assume without loss of generality that
\[ \int_a^\infty \omega \equiv \lim_{x \to \infty} \left( \int_a^x \omega - N_c \log x \right) = \log 2 \]  
Let us define the holomorphic function \( P(x) \) on the complex plane outside the cuts defined by (46) through
\[ P(x) = \cosh \left( \int_a^x \omega \right) \]
where \( a \) is a branching point of the curve (double covering) (46). Since the periods of \( \omega \) are integer this is a well-posed definition which does not depend on the path of integration.
One can then show that \( P(x) \) (and its derivative) is continuous across the cuts. This follows from the fact that across the cut \( \omega \) changes just by a
Thus $P(x)$ defined by (52) can be extended to an entire function on the complex plane.

Then the requirement that the residue of $\omega$ is $N_c$ and the normalization integral (51) leads to the asymptotic behaviour

$$\int_a^x \omega \sim N_c \log x + \log 2 + \ldots$$

which shows that $P(x)$ must be a polynomial of order $N_c$ with unit coefficient.

We are now ready to show that factorization occurs. Let us denote $F_4 = (x-a)(x-b)(x-c)(x-d)$. Then the preceding footnote shows that we have

$$P^2 - 1 = F_4 \cdot G$$

We have to show that $G$ has zeroes with even multiplicity. To this end consider the 1-form

$$\sqrt{F_4} \cdot \frac{dP}{\sqrt{P^2 - 1}} = \frac{P'(x)dx}{\sqrt{G}}$$

but this is exactly equal to

$$\sqrt{F_4} \cdot \frac{\omega \sinh \int_a^x \omega}{\sinh \int_a^x \omega} = \sqrt{F_4} \omega$$

which is a polynomial 1-form. Hence the roots of $G$ have to appear in pairs which proves factorization

$$P^2 - 1 = F_4 \cdot H^2$$

Note that the above reasoning works for any number of cuts and hence for any meromorphic 1-form with integer periods and fixed residues at infinity on an hyperelliptic curve.

It is amusing to see that the same construction generates the classical Chebyshev polynomials appearing in the solution of the complete factorization problem by Douglas and Shenker [23]. There, the relevant formula would be

$$P_{N_c}(x) = \cosh \left( \int_1^x \frac{N_c dx}{\sqrt{x^2 - 1}} \right)$$

The above formula indeed gives exactly the Chebyshev polynomial $T_{N_c}(x)$. Our formula (52) thus gives a very natural generalization of Chebyshev polynomials from the sphere to elliptic and hyperelliptic curves.

---

3 And consequently one can show that $P$ is $\pm 1$ at the branch points of (46).
4 Here for simplicity we relaxed the normalization constraint (51).
Appendix B. Mathematica code which implements the construction of the factorized Seiberg-Witten curve

In this appendix we enclose the Mathematica code for generating the factorized Seiberg-Witten curve. The calling sequence is
generate[nc,n1,k,dk,tau,x0,dl]
where dl stands for $\Lambda^{2Nc}$. All calculations have to be performed with high numerical accuracy.

prec=30 (* numerical precision *)

mer[a_,z_]:=Pi*EllipticThetaPrime[3,Pi(z-a),q]/EllipticTheta[3, Pi(z-a),q]
(* torus embedding *)

setpa:=({pa,pb,pc,pd,pe} =
{(D[EllipticTheta[3, Pi(z-a1),q],z] /. z->tt+a1),
 ((1/2)D[EllipticTheta[3, Pi(z-a1),q],z,z] /. z->tt+a1),
 ((1/6)D[EllipticTheta[3, Pi(z-a1),q],z,z,z] /. z->tt+a1),
 ((1/24)D[EllipticTheta[3, Pi(z-a1),q],z,z,z,z] /. z->tt+a1),
 ((1/120)D[EllipticTheta[3, Pi(z-a1),q],z,z,z,z,z] /. z->tt+a1))}

setaa:=({aa,bb,cc,dd} ={ pb/pa, -((pb^2 - 2*pa*pc)/pa^2),
 (pb^3 - 3*pa*pb*pc + 3*pa^2*pd)/pa^3,
 -((pb^4 - 4*pa*pb^2*pc + 4*pa^2*pb*pd +
 2*pa^2*(pc^2 - 2*pa*pe))/pa^4) } )

seta:=({f0,f1,f2,f3} =
{ (aa-mer[a2,z]) /. z->tt+a1,
 (bb-D[mer[a2,z],z]) /. z->tt+a1,
 (cc-(1/2)D[mer[a2,z],z,z]) /. z->tt+a1,
 (dd-(1/6)D[mer[a2,z],z,z,z]) /. z->tt+a1})

sets:= ({s1,s2,s3,s4} = {-4f0, 6(f0^2-f1), 12 f0 f1 -4f0^3-8f2,
 f0^4+7f1^2-6f0^2 f1+8f0 f2-10 f3})
defcurve := setpa; setaa; seta; sets

(* Seiberg-Witten curves *)

u[i_] := data[[i]]

s[0] = 1
s[r_] := N[-Sum[s[r-a] a u[a], {a, 1, r}]/r, prec]

swpol[nc_] := Sum[s[a] x^(nc-a), {a, 0, nc}]

swc[nc_, dl_] := (swpol[nc]^2 - 4 dl)

(* main procedure *)

generate[nc_, n1_, k_, dk_, tau_, x0_, dl_] := Block[{},
  nu = n1/nc; deltak = dk/nc;
  a1 = 0;
  a2 = nu tau - deltak;
  q = Exp[Pi I tau];
  tt = (1 + tau)/2;
  defcurve;
  Print["Generating..."];
  thetafncs = -I*Pi EllipticThetaPrime[3, Pi tt, q] * 
    (EllipticTheta[3, Pi (a2 - a1 + tt), q] * EllipticTheta[3, Pi (a1 - a2 + tt), q])^(-1/2) * 
    Exp[-2 Pi I n1 (n1 tau - dk)/(2*nc^2)];
  al = dl^(-1/(2*nc)) thetafncs Exp[-I*Pi*k/nc];
  forminit = (x + 2 Pi I nu)/Sqrt[x^4 + s1 x^3 + s2 x^2 + s3 x + s4];
  form = N[al*forminit /. x -> al*x + x0, prec];
  data = Table[-nc Residue[form*x^i, {x, Infinity}]/i, {i, 1, nc}];
  data = data/(-Residue[form, {x, Infinity}]);
  Chop[swc[nc, dl]]]

References

[1] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills the-
ory,” Nucl. Phys. B 426 (1994) 19 [Erratum-ibid. B 430 (1994) 485 [arXiv:hep-th/9407087].

[2] N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD,” Nucl. Phys. B 431 (1994) 484 [arXiv:hep-th/9408099].

[3] C. Vafa, “Superstrings and topological strings at large N,” J. Math. Phys. 42 (2001) 2798 [arXiv:hep-th/0008142].

[4] F. Cachazo, K. A. Intriligator and C. Vafa, “A large N duality via a geometric transition,” Nucl. Phys. B 603 (2001) 3 [arXiv:hep-th/0103067].

[5] R. Dijkgraaf and C. Vafa, “A perturbative window into non-perturbative physics,” arXiv:hep-th/0208048.

[6] R. Dijkgraaf, M. T. Grisaru, C. S. Lam, C. Vafa and D. Zanon, “Perturbative computation of glueball superpotentials,” Phys. Lett. B 573 (2003) 138 [arXiv:hep-th/0211017].

[7] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, “Chiral rings and anomalies in supersymmetric gauge theory,” JHEP 0212 (2002) 071 [arXiv:hep-th/0211170].

[8] F. Cachazo and C. Vafa, “N = 1 and N = 2 geometry from fluxes,” arXiv:hep-th/0206017.

[9] F. Ferrari, “On exact superpotentials in confining vacua,” Nucl. Phys. B 648 (2003) 161 [arXiv:hep-th/0210135].

[10] R. Gopakumar, “N = 1 theories and a geometric master field,” JHEP 0305 (2003) 033 [arXiv:hep-th/0211100].

[11] S. G. Naculich, H. J. Schnitzer and N. Wyllard, “The N = 2 U(N) gauge theory prepotential and periods from a perturbative matrix model calculation,” Nucl. Phys. B 651 (2003) 106 [arXiv:hep-th/0211123].

[12] S. G. Naculich, H. J. Schnitzer and N. Wyllard, “Matrix model approach to the N = 2 U(N) gauge theory with matter in the fundamental representation,” JHEP 0301 (2003) 015 [arXiv:hep-th/0211254].
[13] R. A. Janik and N. A. Obers, “SO(N) superpotential, Seiberg-Witten curves and loop equations,” Phys. Lett. B 553 (2003) 309 [arXiv:hep-th/0212069].

[14] V. Balasubramanian, J. de Boer, B. Feng, Y. H. He, M. x. Huang, V. Jejjala and A. Naqvi, “Multi-trace superpotentials vs. matrix models,” Commun. Math. Phys. 242 (2003) 361 [arXiv:hep-th/0212082].

[15] Y. Demasure and R. A. Janik, “Explicit factorization of Seiberg-Witten curves with matter from random matrix models,” Nucl. Phys. B 661 (2003) 153 [arXiv:hep-th/0212212].

[16] Y. Demasure, “Affleck-Dine-Seiberg from Seiberg-Witten,” arXiv:hep-th/0307082.

[17] F. Cachazo, N. Seiberg and E. Witten, “Phases of N = 1 supersymmetric gauge theories and matrices,” JHEP 0302 (2003) 042 [arXiv:hep-th/0301006].

[18] F. Ferrari, “Quantum parameter space and double scaling limits in N = 1 super Yang-Mills theory,” Phys. Rev. D 67 (2003) 085013 [arXiv:hep-th/0211069];
F. Ferrari, “Quantum parameter space in super Yang-Mills. II,” Phys. Lett. B 557 (2003) 290 [arXiv:hep-th/0301157].

[19] F. Cachazo, N. Seiberg and E. Witten, “Chiral Rings and Phases of Supersymmetric Gauge Theories,” JHEP 0304 (2003) 018 [arXiv:hep-th/0303207].

[20] C. h. Ahn and Y. Ookouchi, “Phases of N = 1 supersymmetric SO / Sp gauge theories via matrix model,” JHEP 0303 (2003) 010 [arXiv:hep-th/0302150]; V. Balasubramanian, B. Feng, M. x. Huang and A. Naqvi, “Phases of N = 1 supersymmetric gauge theories with flavors,” arXiv:hep-th/0303065; C. h. Ahn, B. Feng and Y. Ookouchi, “Phases of N = 1 SO(N(c)) gauge theories with flavors,” arXiv:hep-th/0306068; P. Merlatti, “Gaugino condensate and phases of N = 1 super Yang-Mills theories,” arXiv:hep-th/0307115; C. h. Ahn, B. Feng and Y. Ookouchi, “Phases of N = 1 USp(2N(c)) gauge theories with flavors,” arXiv:hep-th/0307190.
[21] P. C. Argyres and A. E. Faraggi, “The vacuum structure and spectrum of $N=2$ supersymmetric SU(n) gauge theory,” Phys. Rev. Lett. 74 (1995) 3931 [arXiv:hep-th/9411057].

[22] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, “Simple singularities and N=2 supersymmetric Yang-Mills theory,” Phys. Lett. B 344 (1995) 169 [arXiv:hep-th/9411048].

[23] M. R. Douglas and S. H. Shenker, “Dynamics of SU(N) supersymmetric gauge theory,” Nucl. Phys. B 447 (1995) 271 [arXiv:hep-th/9503163].

[24] D. Mumford, “Tata lectures on theta. I”, Progress in Mathematics 28, Birkhauser Boston, 1983.

[25] Work in progress.

[26] R. Casero and E. Trincherini, “Phases and geometry of the $N = 1$ A(2) quiver gauge theory and matrix models,” JHEP 0309 (2003) 063 [arXiv:hep-th/0307054].